Noise-induced dynamical phase transitions in long-range systems

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In the thermodynamic limit, the time evolution of isolated long-range interacting systems is properly described by the Vlasov equation. This equation admits non-equilibrium dynamically stable stationary solutions characterized by a zero order parameter. We show that the presence of external noise sources, like for instance a heat bath, can induce at a specific time a dynamical phase transition moved by a non-zero order parameter. This transition corresponds to a restoring of the full ergodic properties of the system and may be used as a distinctive experimental signature of the existence of non-equilibrium Vlasov-stable states. In particular, we evidence for the first time a regime characterized by an order parameter pulse. Our analytical results are corroborated by numerical simulations of a paradigmatic long-range model.

The study of the nonequilibrium behavior of long-range interacting systems recently triggered an intense research activity [1]. This is partly a consequence of the variety of physical situations characterized by long-range interactions (including gravitational, plasma and nuclear physics, hydrodynamics, wave-matter interactions, ... ) [1], which makes the topic amenable to interdisciplinary approaches and cross-fields fertilization. However, in most of the literature in this area the influence of an external environment that may act as a thermal bath has not been taken into account. Especially for laboratory-scale systems, such an influence is expected to be relevant. Here we show that the existence of external noise sources can induce, on long-range systems characterized by different equilibrium phases, a dynamical phase transition from a homogeneous Vlasov-stable phase to an inhomogeneous one. This dynamical transition corresponds to a restoring of ergodicity realized by the action of the noise. By extending the linear stability analysis of the Vlasov equation in order to take into account the influence of the noise on the system’s dynamics, we identify the time $t_c$, at which the dynamical phase transition occurs. At $t_c$, in the thermodynamic limit, phase functions like the order parameter are not analytic. We thus provide a “dynamical phase diagram” in which time replaces the role of a thermodynamic parameter in ordinary equilibrium phase diagrams. For specific ranges of initial energies, the system undergoes a dynamical phase transition even if the final equilibrium phase is homogeneous. This behavior corresponds to an order-parameter pulse occurring at $t = t_c$. Our analytical results are corroborated by numerical simulations of a paradigmatic model, which fully agree with the analytical predictions.

A system is considered “long-range interacting” if the interparticle potential $V$ decays at large distances $r$ slower than $1/r^d$, where $d$ is the system’s spatial dimension. Under these circumstances, phase transitions have to be addressed within a proper definition of the thermodynamic limit, which is attained by rescaling the interaction strength with a function of the system’s size in order to make the energy formally extensive [1] (Kac’s prescription) and by taking the number of particles $N \to \infty$. In this limit, the interparticle correlations (due to finite-$N$ effects) are completely negligible, and the system’s time-evolution is correctly described by the Vlasov equation [1]. We have recently demonstrated [2] that a Hamiltonian thermal bath [2] and a stochastic Langevin bath [2] produce the same equilibrium and nonequilibrium effects when in contact with a long-range system, a conclusion not obvious a priori due to the character of the system’s interactions. Assuming $d = 1$ for simplicity, the time evolution of the one-particle distribution function $f(x,v,t)$ [normalized such that $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv f(x,v,t) = N$, $\forall t \in \mathbb{R}$] is thus conveniently described by adding to the Vlasov equation a diffusion and a damping term. This gives the so-called mean-field Kramers’ equation [3]:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \Phi(x) \frac{\partial f}{\partial v} = \gamma \frac{\partial}{\partial v} \left( D \frac{\partial f}{\partial v} + \gamma f v \right), \quad (1)$$

where $\Phi(x,t) \equiv \int_{-\infty}^{\infty} dx' V(x-x') \int_{-\infty}^{\infty} dv f(x',v,t)$, and $D$ and $\gamma$ are respectively the diffusion and damping coefficients. The former measures the strength of the noise disturbing the Vlasov dynamics, whereas the latter quantifies the dissipation to the external environment. Below, we will assume that these two effects have the same physical origin, like when the system interacts with an heat bath. In particular, we suppose that the temperature $T$ of the bath satisfies the Einstein relation $D = \gamma T$ (we work in natural adimensional units in which the Boltzmann constant and the mass of the particles are equal to one). One can however conceive more general situations in which the noise strength is not directly linked to the dissipation rate. An interesting recent example [4] is the consideration of a noise term representing energy-preserving inter-particles fast collision processes that are superposed to the long-range in-
teractions. We are interested in situations in which the potential $V$ is such that there exist a critical temperature $T_c$ that separates a high-temperature equilibrium homogeneous phase from a low-temperature inhomogeneous one. These phases can be described in terms of an order parameter, $\varphi$, which is zero at and above $T_c$, and non-zero below $T_c$. Furthermore, we assume that the interparticle potential is even in the spatial coordinate: $V(-x) = V(x)$.

Let us consider a spatially homogeneous initial condition $f(x,v,0) = f_0(v)$ that is Vlasov stable. For instance, this could be the result of a violent relaxation, as in Lynden-Bell’s theory [4]. Hence, $\partial \Phi / \partial x = 0$ ($\varphi = 0$) and the system evolves under the sole effect of the interaction with the thermal bath according to

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left( D \frac{\partial f}{\partial v} + \gamma f v \right). \quad (2)$$

Notice that for $\gamma = D = 0$ (microcanonical setup), we recover the Vlasov stationarity condition $\partial f / \partial t = 0$, which would last for infinite time in this thermodynamic limit. In contrast, for $\gamma \neq 0$, the distribution will change with time due to the interaction with the bath. Equation (2) remains valid only as long as $f(v,t)$ is dynamically stable.

If, at time $t = t_c$, it becomes Vlasov unstable, a dynamical phase transition occurs from the homogeneous $\varphi = 0$ phase to an inhomogeneous $\varphi \neq 0$ phase [3]. The time $t_c$ can be identified by performing a linear (Landau) stability analysis [1] of the Vlasov equation for the $f(v,t)$ which is solution of Eq. (2) at time $t$ (in this analysis, $f(v,t)$ is regarded as a function of the time $t$ being fixed adiabatically). While a general $f(v,t)$ can be studied within the Landau formalism, in the following we will restrict ourselves, for the sake of simplicity, to distributions with a single maximum at $v = 0$. In this case, one then shows [3] that $t_c$ is the minimum time $t$ for which the equation

$$1 - \tilde{V}(k) \int_{-\infty}^{+\infty} \frac{\partial f(t,v)}{\partial v} dv = 0 \quad (3)$$

admits a solution for at least a $k \in \mathbb{R}$, $k \neq 0$. This expresses marginal stability. Here, $\tilde{V}(k) \in \mathbb{R}$ is the Fourier transform of the potential: $\tilde{V}(k) \equiv \int_{-\infty}^{+\infty} dx \ e^{ikx} \ V(x)$. In order to find $t_c$, we can analytically solve Eq. (2) for a given uniform initial condition $f(x,v,0) = f_0(v)$, plug the result in Eq. (3), and invert the solution corresponding to the minimum time.

The Green function of Eq. (2) is

$$W(v,t|v_0,t=0) = \frac{\exp \left[ -\left( v - a(t) v_0 \right)^2 / 2 \sigma^2(t) \right]}{\sqrt{2 \pi} \sigma^2(t)}, \quad (4)$$

where $a(t) \equiv e^{-\gamma t}$, and $\sigma^2(t) \equiv T \left( 1 - e^{-2\gamma t} \right)$. Formally, the expression for $f(v,t)$ is thus

$$f(v,t) = \int_{-\infty}^{+\infty} dv_0 \ W(v,t|v_0,t=0) f_0(v_0). \quad (5)$$

Since the time evolution can be expressed in adimensional units through the mapping $t \rightarrow \gamma t$, below we implicitly adopt this convention. Our problem can be solved in Fourier space. First we take the Fourier transform in the velocities, $\tilde{f}(v) \equiv \int_{-\infty}^{+\infty} dv \ e^{ivw} f(v)$, of Eq. (4), to get

$$\tilde{f}(w,t) = \tilde{f}_0(a(t) w) \ e^{-\sigma^2(t) w^2 / 2}. \quad (6)$$

Recalling the expressions for $a(t)$ and $\sigma^2(t)$, it is clear that the short-time behavior of $\tilde{f}(w,t)$ is dominated by the initial condition $\tilde{f}_0(w)$ (e.g., by the initial system’s energy), while its long-time behavior is determined by the heat bath temperature. Inserting the Fourier representation of $f(v,t)$ in Eq. (3), we get that the transition time is equivalently identified by the zeroes of the function

$$I(t,k) \equiv 1 + \frac{\tilde{V}(k)}{2} \int_{-\infty}^{+\infty} dw \ |w| \tilde{f}_0(a(t) w) \ e^{-\sigma^2(t) w^2 / 2}, \quad (7)$$

for some $k \neq 0$.

The analysis outlined so far is valid under the following general assumptions: (i) One-dimensional system; (ii) Kac’s thermodynamic limit; (iii) Symmetric potential; (iv) Spatially homogeneous distribution function $f(v,t)$ with a single maximum at $v = 0$. We now specify the above general formulas to the Hamiltonian Mean Field (HMF) model, a long-range interacting system that has been widely used in the recent literature as a paradigmatic case of study [1]. The HMF model, which can be thought of as a set of $N$ globally coupled rotators or XY-spins, is described by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{v_i^2}{2} + \frac{1}{N} \sum_{i<j} \left[ 1 - \cos (\theta_i - \theta_j) \right], \quad (8)$$

where $\theta_i \in [0,2\pi]$ is an angular coordinate, $V(\theta) \equiv (1 - \cos \theta) / N$ is the (cosine) potential, and the order parameter $\varphi$ amounts to the system’s magnetization:

$$\varphi \equiv \left[ \sum_{i=1}^{N} \cos x_i, \sin x_i \right] / N. \quad \text{Notice that, following the Kac’s prescription, the interaction strength has been rescaled by a factor 1/N. In this way the total energy of the system, } E = \sum_{i=1}^{N} v_i^2 / 2 + N (1 - \varphi^2) / 2, \text{ becomes extensive, and we can consistently introduce the specific energy } e = E / N. \quad \text{Once coupled with a Langevin thermal bath as in Eq. (4), the system has also been named Brownian Mean Field (BMF) model [4].}$$

To begin with, we consider uniform initial conditions:

$$f_0(v) = \frac{N}{4\pi v^2} \left[ \vartheta(v + \overline{v}) - \vartheta(v - \overline{v}) \right], \quad (9)$$

($\vartheta$ being the Heaviside step function), where $\overline{v} = \sqrt{6(v_0 - 1/2)}$ and $v_0$ is the initial specific energy of the system. In Fourier space we thus have $\tilde{f}_0(w) = \frac{N}{2\pi \overline{v}^2} \sin(\overline{v}w) / 2\pi \overline{v} \overline{w}$. $\tilde{V}(k) = \pi (2\delta_{k,0} - \delta_{k,1} - \delta_{k,-1}) / N,$
and we are led to examine the zeroes of Eq. (10)

\[ I(t, \pm 1) = 1 - \frac{1}{2a(t)} \int_0^{+\infty} dw \sin(a(t)\tau w) e^{-\sigma^2(t)\tau^2/2}. \]

If \( e_0 < e^* \leq 7/12, I(0, \pm 1) > 0 \), which means that at \( t = 0 \), \( f_0(v) \) is dynamically unstable (11). Since our analysis applies to states which are initially stable, in the following, we will restrict ourselves to cases with \( e_0 > e^* \). According to the values \( e_0 \) and \( T \), there are three possible qualitative behaviors of \( I(t, \pm 1) \), as depicted in Fig. 1.

In Fig. 1a, \( I(t, \pm 1) < 0 \) for all \( T \), and the system magnetization \( \varphi \) is always zero (no dynamical phase transition occurs). Correspondingly, \( t_c = +\infty \). This happens if \( T \) is above the critical value \( 10 T_c = 1/2 \) \([\lambda(\infty,\pm 1) < 0]\) and the initial energy \( e_0 \) is sufficiently high. An alternative is that \( I(t, \pm 1) \) crosses zero for a single value \( t = t_c \) (Fig. 1b). For \( t < t_c \), \( \varphi \) is zero, whereas for \( t > t_c \), the system becomes and remains magnetized. This occurs for \( T < T_c \). There is however a third possibility: \( I(t, \pm 1) \) may cross zero twice (Fig. 1c). This occurs for \( T > T_c \) and if \( e_0 \) is below a critical value \( e_c(T) \), which in general also depends on the shape of \( f_0(v) \). The first zero corresponds to a dynamical phase transition occurring at \( t = t_c \), which drives the system to \( \varphi > 0 \). Since \( T > T_c \), the system must eventually come back to zero magnetization in order to reach equilibrium. Under these conditions, one observes a “magnetization pulse” in the behavior of \( \varphi(t) \) (Fig. 2a). Notice that we cannot interpret the second zero of \( I(t, \pm 1) \) as the second transition time, since our stability analysis is only valid under the assumption of spatially homogeneous conditions, which breaks at \( t = t_c \). However, the system eventually goes back to \( \varphi = 0 \) to equilibrate with the thermal bath. Notice also that in this thermodynamic limit description, whenever the dynamic phase transition occurs, \( \varphi(t) \) is non-analytic at \( t = t_c \). This can be checked numerically with the BMF model, recalling that \( \varphi \) in a homogeneous state scales with the system size \( N \) as \( \varphi(N) \sim 1/\sqrt{N} \) (in the inhomogeneous phase, \( \varphi \sim 1 \)). Rescaling by \( \sqrt{N} \), the curves obtained at different \( N \)'s, we indeed find that all the curves collapse for \( t < t_c \) and develop the same non-regular behavior at \( t = t_c \) (Fig. 2b).

All the previous findings can be summarized in a dynamical phase transition diagram representing \( t_c \) vs \( e_0 \), for different values of \( T \) (solid lines in Fig. 3). If \( T < T_c \), a dynamical phase transition always occurs and, correspondingly, \( 0 < t_c < +\infty \) \( \forall e_0 > e^* \). For \( T \geq T_c \), the transition line ends at a critical value \( e_c(T) \); if \( e_0 > e_c(T) \), \( t_c \) becomes infinite and the dynamical phase transition does not occur anymore. In the range \( e^* < e_0 \leq e_c(T) \), the dynamical evolution of \( \varphi \) corresponds in fact to a “pulse” starting at \( t = t_c \) (see Fig. 2a). Fig. 3 also reports the results of numerical simulations of the BMF model (1), which fully agree with the analytical curves. It is also possible to obtain two asymptotic expressions for \( t_c \) in a closed form (11): for \( T < T_c \) and \( e_0 \gg 1 \), \( t_c \) grows logarithmically as

\[ t_c \sim 1 - \frac{1}{2} \ln \left[ \frac{12e_0 - 6(1 + T)}{6T(1 - T/T_c)} \right]. \]

For \( T \gg T_c \) and \( e_0 \rightarrow e^{*+} \), \( t_c \) goes to zero linearly as

\[ t_c \sim \frac{6}{\gamma} \frac{e_0 - e^*}{1 + T/(6e_0 - 3)}. \]

We have also checked the validity of our conclusions for a class of (spatially uniform) initial distributions \( f_0(v) \).
tial conditions, we have found that \( \kappa \) by the form-factor cutoff effects than the water-bag, which are parametrized of modeling the action of a velocity-selector with milder inverse velocity, and the most stable distribution of the class. However, all given \( \epsilon_0 \) the water-bag initial conditions correspond to a monotonically decreasing function, meaning that for a inverse velocity, and \( \epsilon_0 \)(0.55) = 0.584\ldots \). Above this energy, \( t_c \) becomes infinite. Points and error bars refer to simulations with \( N = 10^5 \), averaged over 20 runs.

wider than the water-bag, namely:

\[
\begin{align*}
    f_0(v) & = \frac{N}{2\pi} \left[ \kappa + \left( \frac{\pi}{4\pi} - \frac{\pi\kappa}{2} \right) \cos \left( \frac{\pi v}{2\pi} \right) \right] \\
    & \times [\vartheta(v + \pi) - \vartheta(v - \pi)].
\end{align*}
\]  

(13)

Here, \( 0 \leq \kappa \leq 1/2\pi \) is a form factor with dimension of inverse velocity, and \( \vartheta \) depends now on both \( \epsilon_0 \) and \( \kappa \). For \( \kappa = 1/2\pi \), Eq. (13) reduces to Eq. (9), whereas for \( \kappa = 0 \) \( f_0(v) \) becomes a bump-like distribution that touches the \( v \)-axis. Eq. (13) can be understood as a way of modeling the action of a velocity-selector with milder cutoff effects than the water-bag, which are parametrized by the form-factor \( \kappa \). With respect to this class of initial conditions, we have found that \( t_c(\kappa) \) is in general a monotonically decreasing function, meaning that for a given \( \epsilon_0 \) the water-bag initial conditions correspond to the most stable distribution of the class. However, all the qualitative features of Fig. 3 are confirmed for any value of allowed \( \kappa \).

The dynamical evolution of long-range interacting systems is dominated by a global interaction which erases the interparticles correlations when \( N \to +\infty \). As a consequence, the system gets easily trapped in non-equilibrium states which correspond to the occupation of a small fraction of the available system’s phase space. We have shown that the action of a reservoir or of an external environment, renders these non-equilibrium states dynamically unstable, restoring the full (ergodic) occupation of the system’s phase space. Correspondingly, the system manifests a dynamical phase transition, which can be detected by monitoring the time evolution of the order parameter. The transition time \( t_c \) can be exactly identified by extending the Landau stability analysis to take into account the influence of the heat bath. The result is a dynamical phase diagram, in which time replaces the role ordinarily played by a thermodynamic parameter. The dynamical transition may occur even if the external temperature is above \( T_c \), and under these circumstances it is revealed by a peak in the time evolution of the order parameter. We have studied situations in which the external noise satisfies a fluctuation-dissipation relation which enables the definition of the heat-bath temperature. However, it is also possible to extend the Landau stability analysis to more general cases. This could be important to reproduce experimental conditions in which many sources of noise that do not necessarily satisfy a fluctuation-dissipation relation act on the system. The generality of our approach and the robustness of our results under different initial conditions suggest that these dynamical phase transitions should be common and marked enough, to overcome experimental test. In particular, a pulse in the time evolution of the order parameter may be used as a signature of the existence of Vlasov stable out-of-equilibrium states in experiments with long-range interacting systems.

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FIG. 3: Phase diagram of the dynamical phase transitions. At \( T < T_c = 1/2 \), \( t_c \) is always finite (plots refer to \( T = T_c = 0.5 \), \( T = 0.45 \), \( T = 0.4 \)). Inset: At \( T > T_c \) the curves end at a critical value \( \epsilon_c(T) \) [plot refers to \( T = 0.55 \), \( \epsilon_c(0.55) = 0.584 \ldots \)]. Above this energy, \( t_c \) becomes infinite. Points and error bars refer to simulations with \( N = 10^5 \), averaged over 20 runs.

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