In the paper we give the structural (regularity) theorem and kernel theorem for Gelfand-Shilov spaces, of Roumieu and Beurling type.

1. Introduction

The Gelfand-Shilov spaces of Roumieu and Beurling type, denoted by \( \mathcal{S}^{(M_p)} \) and \( \mathcal{S}^{(M_p)} \), are subclasses of Denjoy-Carleman classes \( C^{(M_p)} \) and \( C^{(M_p)} \), which are invariant under Fourier transform, closed under differentiation and multiplication by polynomials, and which contain Schwartz space of tempered distributions as a subspace. This makes the Gelfand Shilov spaces appropriate domains for harmonic analysis. The space \( \mathcal{S}^{(M_p)} \) (resp. \( \mathcal{S}^{(M_p)} \)) is defined as the inductive (resp. projective) limit of Banach spaces \( \mathcal{S}^{(M_p,m)}, m > 0 \), where by \( \mathcal{S}^{(M_p,m)} \), we denote the space of smooth functions \( \varphi \) on \( \mathbb{R}^d \), such that for some \( C > 0 \) and \( m > 0 \)

\[
\| \varphi \|_{\mathcal{S}^{(M_p,m)}} = \sup_{\alpha, \beta \in \mathbb{N}^d} \frac{m^{\alpha + \beta}}{M^{[\alpha]} M^{[\beta]}} \| (1 + x^2)^{\beta/2} \varphi^{(\alpha)}(x) \|_{L^\infty} < \infty,
\]

and is equipped with the norm \( \| \cdot \|_{\mathcal{S}^{(M_p,m)}} \).

Examples of classes of sequences, which generate the spaces and satisfy the conditions, we assume in the paper, are:

\[
M_p = e^{pr}, \quad p \in \mathbb{N}, \quad r \in (1, 2].
\]

and

\[
M_p = p^{sp}(\log p)^tp, \quad p \in \mathbb{N}, \quad s \geq 1/2, \quad t \geq 0,
\]

with additional condition (only) in Beurling case: \( s + t > 1/2 \).

So the classes of sequences \( \{M_p\} \), which we consider, are much wider than just the class of Gevery sequences.

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Many classical spaces of analysis are Gelfand Shilov spaces. In the special case when \( \{M_p\} \) is a Gevrey sequence \( p!^{\alpha} \), the space is equal to \( S_{\alpha}^2 \), \( \alpha \geq 1/2 \) ([6]). The dual of this space has been successfully used in differential operators theory, spectral analysis, and more recently in theory of pseudodifferential operators ([11]) and in quantum field theory. It was shown ([3], [10]) that the space isomorphic with \( S(\{M_p\}) = S_1^1 \) is well adapted for the use in quantum field theory with a fundamental length. It is our believe that all Gelfand Shilov spaces, especially those with quasianalytic test function spaces, are good domains for the quantum field theory. The theory requires technical results from the theory of generalized functions and not merely differential calculus and well defined Fourier transform, but also the kernel theorem and the structural theorem.

The aim of this paper is to present simple proofs of the kernel and structural theorem for the Gelfand Shilov spaces (in quasianalytic and nonquasianalytic case in an uniform way) using the minimal amount of real analysis. Recently published results giving characterizations of Fourier Hermite coefficients of Gelfand Shilov spaces, enabled us to use Simons ideas ([15]), and use the harmonic oscillator wavefunctions, as a basic tool for proving kernel and structural theorem. The characterization has been given by Langenbruch in [9]. Without knowing for the result, and unfortunately after the Langenbruch paper was published online, we (Perisic and Lozanov-Crvenkovic) have obtained ([4]) the same characterization, but using different techniques.

In Section 2 we prove, that every element \( T \) of the dual space \( S(\{M_p\})' \) (resp. \( S(\{M_p\})' \)) is of the form:

\[
T = \sum_{\beta=0}^{\infty} \frac{\mu^{2\beta}}{M_{2\beta} 2^{3\beta}} \left( -\frac{d^2}{dx^2} + x^2 - 1 \right)^\beta f
\]

for every (resp. some) \( \mu > 0 \) and some continuous bounded function \( f \), (i.e. structural theorem).

In the same Section we prove the kernel theorem, which states every continuous linear map \( K \) on the space \( (S^{(M_p)}(\mathbb{R}^l))_x \) of test functions in some variable \( x \), into the space \( (S^{(M_p)})'(\mathbb{R}^s))_y \) of in a second variable \( y \), is given by a unique \( K \in (S^{(M_p)})'(\mathbb{R}^{l+s}) \) in both variables \( x \) and \( y \).

An immediate consequence of the Kernel theorem is that the representation of the Heisenberg group and the Weyl transform can be extended to the Gelfand-Shilov spaces generated by a sequences \( \{M_p\} \) from a large class of sequences, where the Gevrey sequence (either in quasianalytic or nonquasianalytic case) is just one of the examples. This gives the possibility to introduce the \( \Psi \)DO theory in the framework of the spaces. In [11] it was
demonstrated how it can be done by using Gelfand-Shilov spaces of Beurling type generated by Gevrey sequences \( \{p!^\alpha\}_p \), for \( \alpha \geq 1 \), that is in strong nonquasianalytic case.

In Section 3 we list several examples of Gelfand-Shilov spaces, discuss conditions we assume on the sequences \( \{M_p\} \) and their consequences on the structure of the Gelfand-Shilov spaces.

1.1. **Notations and basic notions.** Throughout the paper by \( C \) we denote a positive constant, not necessarily the same at each occurrence. Also we use \( * \) instead of \( \{M_p\} \) or \( (M_p) \).

Let \( \{M_p\}_{p \in \mathbb{N}_0} \) be a sequence of positive numbers, where \( M_0 = 1 \). We also assume that the sequence \( \{M_p\}_{p \in \mathbb{N}_0} \) satisfies:

\[
(M.1) \quad M_p^2 \leq M_{p-1}M_{p+1}, \quad p = 1, 2, \ldots
\]

(logarithmic convexity)

\[
(M.2) \quad \text{There exist constants } A, H > 0 \text{ such that}
M_p \leq AH^p \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p = 0, 1, \ldots
\]

(separativity condition or stability under ultradifferential operators)

\[
(M.3)'' \quad \text{There exist constants } C, L > 0 \text{ such that}
p^2 \leq CL^p M_p, \quad p = 0, 1, \ldots
\]

(non triviality condition for the spaces \( S^{(M_p)}(\mathbb{R}^d) \))

When we discuss spaces of Beurling type, instead of \( (M.3)'' \) we assume moreover:

\[
(M.3)''' \quad \text{For every } L > 0, \text{ there exists } C > 0 \text{ such that}
p^2 \leq CL^p M_p, \quad p = 1, 2, \ldots
\]

(non triviality condition for the spaces \( S^{(M_p)}(\mathbb{R}^d) \)).

We will denote by \( S^{(M_p)}'(\mathbb{R}^d) \) (resp. \( S^{(M_p)}'(\mathbb{R}^d) \)) the strong dual of the space \( S^{(M_p)}(\mathbb{R}^d) \) (resp. \( S^{(M_p)}(\mathbb{R}^d) \)).

By

\[
\mathcal{H}_n(x) = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{x^2/2} \frac{d^n}{dx^n} \left(e^{-x^2}\right), \quad n \in \mathbb{N},
\]

we denote the **Hermite functions**. The functions are eigenvalues of harmonic oscillator:

\[
\left(-\frac{d^2}{dx^2} + x^2 - 1\right)\mathcal{H}_n = n\mathcal{H}_n,
\]

and that satisfy estimation estimation:

\[
\|\mathcal{H}_n\|_{\infty} \leq Cn^k,
\]
for some $C$ and $k$ independent of $n$. They are elements of the Gelfand-Shilov spaces (see [9] and [14]).

In multidimensional case the Hermite functions are defined by

$$H_n(x) = H_{n_1}(x_1)H_{n_2}(x_2)\cdots H_{n_d}(x_d), \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d,$$

where $n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$. By $H_{(n,k)}$, $(n, k) \in \mathbb{N}_0^d \times \mathbb{N}_0^s$, we denote

$$H_{(n,k)} = H_{n_1}(x_1)H_{n_2}(x_2)\cdots H_{n_l}(x_l)H_{k_1}(x_{l+1})H_{k_2}(x_{l+2})\cdots H_{k_s}(x_{l+s}).$$

The Fourier-Hermite coefficients of an element $\varphi$ of Gelfand-Shilov space are numbers defined by:

$$a_n(\varphi) = \int_{\mathbb{R}^d} \varphi(x)H_n(x)dx, \quad n \in \mathbb{N}_0^d,$$

and the Fourier-Hermite coefficients of an element $f$ of the dual space Gelfand-Shilov space is defined by:

$$b_n(f) = \langle f, H_n \rangle.$$

The sequence of the Fourier-Hermite coefficients $\{a_n(\varphi)\}_{n \in \mathbb{N}_0^d}$ and $\{b_n(f)\}_{n \in \mathbb{N}_0^d}$ of $\varphi$ and of $f$ are the Hermite representation of $\varphi$ and $f$.

The proof of our theorems heavily rely on the characterization of Fourier-Hermite coefficients of elements of Gelfand-Shilov spaces and their duals. Let us give a brief account on the characterization of Gelfand-Shilov spaces of Roumieu type (resp. Gelfand-Shilov space of Beurling) can be identified with the space of multisequences of ultrafast falloff, i.e. of multisequences $\{a_n\}_{n \in \mathbb{N}_0^d}$ of complex numbers which satisfies that which satisfy that for some (resp. each) $\theta = (\theta_1, \ldots, \theta_n)$, where $\theta_k > 0$, for $k = 1, \ldots, d$,

$$\left(\sum_{n \in \mathbb{N}_0^d} |a_n|^2 \exp \left[\sum_{k=1}^{d} M(\theta_k n_k)\right]\right)^{1/2} < \infty,$$

Here, $M(\cdot)$ is the associated function for the sequence $\{M_p\}_{p \in \mathbb{N}_0}$ defined by

$$(1.3) \quad M(p) = \sup_{p \in \mathbb{N}_0} \log \frac{p^p}{M_p}, \quad p > 0.$$

In the special case [12], one have $M(p) = p^{\frac{1}{2}}(\log p)^{-\frac{1}{2}}, \quad p \gg 0$.

Since the space of sequences of ultrafast falloff is a Frechet space, the spaces $S^{(M_p)}$ and $S^{(M_p)'}$ are Frechet spaces too.

The dual space $S^{(M_p)'}(\mathbb{R}^d)$ (resp. $S^{(M_p)'}(\mathbb{R}^d)$) can be identified with the space of multisequences of ultrafast growth. This implies that $f$ belongs to
the space if and only if is Hermite representation \( \{b_n\} \) satisfies that for every (resp. some) \( \theta = (\theta_1, ... , \theta_d) \in \mathbb{R}^d_+ \) holds:

\[
|b_n(f)| \leq \exp \left[ \sum_{k=1}^{d} M(\theta_k \sqrt{n_k}) \right], \quad n = (n_1, ... , n_d)
\]

One of the immediate consequences of the characterization is the Parseval equation: For \( f \in S^{s'} \) and \( \varphi \in S^s \) we have

\[
\langle f, \varphi \rangle = \sum_{n \in \mathbb{N}_0^d} b_n(f)a_n(\varphi), \quad \varphi \in S^s(\mathbb{R}^d),
\]

where the multisequences \( \{a_n(\varphi)\}_{n \in \mathbb{N}_0^d} \) and \( \{b_n(\varphi)\}_{n \in \mathbb{N}_0^d} \) are the Hermite representatives of \( \varphi \) and \( f \).

Let us now prove the other important consequences of the characterization.

2. Main results

2.1. Regularity theorem.

**Theorem 2.1.** If \( T \) belongs to the space \( S^{(M_p)'} \), then for every \( \mu > 0 \) there exists a continuous and bounded function \( f \) such that in the space \( S^{(M_p)'} \):

\[
T = \sum_{\beta = 0}^{\infty} \frac{\mu^{2\beta}}{M_{2\beta}} \frac{1}{2^\beta} \left( - \frac{d^2}{dx^2} + x^2 - 1 \right)^\beta f.
\]

**Proof.** Let \( T \) be an element of the space \( S^{(M_p)'}(\mathbb{R}) \) and let \( \{b_n\}_{n=0}^{\infty} \) be its Hermite representation. It follows that for every \( \theta > 0 \) we have:

\[
|b_n| \leq C \exp[M(\theta \sqrt{n})].
\]

Let \( \mu \) be an arbitrary positive number and let \( \{a_n\}_{n=0}^{\infty} \) be a sequence defined by:

\[
a_n = b_n \left( \sum_{\alpha=0}^{\infty} \frac{\mu^{2\alpha} n^{\alpha}}{M_{2\alpha}} \right)^{-1}.
\]

Note that the sequence \( \{a_n\}_{n=0}^{\infty} \) is well defined sequence since by \((M.3)''\) we have:

\[
1 \leq \sum_{\alpha=0}^{\infty} \frac{\mu^{2\alpha} n^{\alpha}}{M_{2\alpha}} \leq C \sum_{\alpha=0}^{\infty} \frac{\mu^{2\alpha} n^{\alpha} L^{2\alpha}}{(2\alpha)^{\alpha}} \leq C \sum_{\alpha=0}^{\infty} \frac{1}{\alpha^2} < \infty.
\]

First we will prove that

\[
|a_n \mathcal{H}_n| \leq C \frac{1}{n^2},
\]

which will imply (see for example [13, Th 7.10 and Th.7.12]) that the sum \( \sum_{n=0}^{\infty} a_n \mathcal{H}_n \) uniformly converges to a bounded and continuous function \( f \).
Inequality (2.2), the estimate for Hermite functions and the condition 
(M.2) imply that for $\theta = \frac{\mu}{H}$ we have:

$$|a_n \mathcal{H}_n| = |b_n| \cdot \left( \sum_{\alpha=0}^{\infty} \frac{\mu^{2\alpha} \alpha^\alpha}{M_{2\alpha}^2} \right)^{-1} \cdot |\mathcal{H}_n|$$

$$\leq C \exp[M(\theta \sqrt{n})] \cdot \left( \sum_{\alpha=0}^{\infty} \frac{\mu^{2\alpha} \alpha^\alpha}{H^{2\alpha} M_{\alpha}^2} \right)^{-1} \cdot n^k$$

$$\leq C \exp[M(\theta \sqrt{n})] \cdot \left( \sup_{\alpha} \left( \frac{\mu}{H} \right)^{2\alpha} \alpha^\alpha \frac{1}{M_{\alpha}^2} \right)^{-1} \cdot n^k$$

$$= C \exp[M(\theta \sqrt{n}) - 2M(\mu/H \sqrt{n})] \cdot n^k$$

$$= C \exp[-M(\mu/H \sqrt{n})] \cdot n^k = C \frac{n^k}{\sup_{\alpha} \left( \frac{\mu}{H} \alpha^\alpha \sqrt{n} \right)^{-2}}$$

$$\leq C \frac{n^k M_{2k+4}^{2k+4} n^{k+2}}{(\mu/H)^{2k+4} n^{k+2}} \leq C \frac{1}{n^2}.$$

Next we show that in the space $\mathcal{S}^{(M_p)^{\prime}}(\mathbb{R})$ holds:

$$T = \sum_{\beta=0}^{\infty} \frac{\mu^{2\beta}}{M_{2\beta}} \frac{1}{2^\beta} \left( -\frac{d^2}{dx^2} + x^2 - 1 \right)^\beta f.$$ 

But first let us note that $\left( -\frac{d^2}{dx^2} + x^2 - 1 \right)^\beta f = \sum_{n=0}^{\infty} a_n n^\beta \mathcal{H}_n$. This follows from the fact that $f$ is continuous and bounded function (thus $f \in \mathcal{S}^{(M_p)^{\prime}}(\mathbb{R})$, so $\left( -\frac{d^2}{dx^2} + x^2 - 1 \right)^\beta f \in \mathcal{S}^{(M_p)^{\prime}}(\mathbb{R})$), and from equality:

$$\langle \left( -\frac{d^2}{dx^2} + x^2 - 1 \right)^\beta f, \mathcal{H}_n \rangle = \langle f, \left( -\frac{d^2}{dx^2} + x^2 - 1 \right)^\beta \mathcal{H}_n \rangle$$

$$= \langle f, n \mathcal{H}_n \rangle = n \langle f, \mathcal{H}_n \rangle = na_n.$$

So, in the space $\mathcal{S}^{(M_p)^{\prime}}(\mathbb{R})$ we have:

$$\sum_{\beta=0}^{\infty} \frac{\mu^{2\beta}}{M_{2\beta}} \left( -\frac{d^2}{dx^2} + x^2 - 1 \right)^\beta f = \sum_{\beta=0}^{\infty} \frac{\mu^{2\beta}}{M_{2\beta}} \left( -\frac{d^2}{dx^2} + x^2 - 1 \right)^\beta \sum_{n=0}^{\infty} a_n \mathcal{H}_n$$

$$= \sum_{\beta=0}^{\infty} \frac{\mu^{2\beta}}{M_{2\beta}} \sum_{n=0}^{\infty} a_n n^\beta \mathcal{H}_n$$

$$= \sum_{\beta=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mu^{2\beta}}{M_{2\beta}} a_n n^\beta \mathcal{H}_n.$$
and

\[ T = \sum_{n=0}^{\infty} b_n \mathcal{H}_n = \sum_{n=0}^{\infty} a_n \left( \sum_{\beta=0}^{\infty} \frac{\mu^{2\beta}}{M^{2\beta}} n^\beta \right) \mathcal{H}_n \]

\[ = \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{\mu^{2\beta}}{M^{2\beta}} a_n n^\beta \mathcal{H}_n. \]

It only remains to show that the above two sequences of equations are equal. For every \( \varphi \in \mathcal{S}(\mathbb{M}_p) \) we have:

\[ \sum_{\beta=0}^{\infty} \left| \frac{\mu^{2\beta}}{M^{2\beta}} a_n n^\beta \langle \mathcal{H}_n, \varphi \rangle \right| = |a_n| \left( \sum_{\beta=0}^{\infty} \frac{\mu^{2\beta}}{M^{2\beta}} n^\beta \right) |\langle \mathcal{H}_n, \varphi \rangle| \]

\[ = |b_n| \cdot |\langle \mathcal{H}_n, \varphi \rangle| \]

and therefore, since \( \{b_n\}_{n=0}^{\infty} \) is the Hermite representation of \( T \in \mathcal{S}(\mathbb{M}_p)' \) and \( \langle \mathcal{H}_n, \varphi \rangle_{n=0}^{\infty} \) is the Hermite representation of \( \varphi \), we have:

\[ \sum_{n=0}^{\infty} |b_n| |\langle \mathcal{H}_n, \varphi \rangle| \leq \left( \sum_{n=0}^{\infty} |b_n|^2 \exp[-2M(\theta \sqrt{n})] \right)^{1/2} \cdot \left( \sum_{n=0}^{\infty} \left| \langle \mathcal{H}_n, \varphi \rangle \right| \cdot \exp[2M(\theta \sqrt{n})] \right)^{1/2} < \infty. \]

So by [13, Th.8.3.], we have that (6) and (8) are equal.

\[ \square \]

2.2. Kernel Theorems. In the section we will prove the kernel theorem for Gelfand-Shilov spaces. As a consequence of the Kernel theorem we have that the Weyl transform can be extended on a spaces of Gelfand-Shilov and this gives the possibility to introduce the \( \Psi \)DO theory in the framework of the spaces.

**Theorem 2.2 (Kernel theorem).** Every jointly continuous bilinear functional \( K \) on \( \mathcal{S}(\mathbb{M}_p)(\mathbb{R}^l) \times \mathcal{S}(\mathbb{M}_p)(\mathbb{R}^s) \) defines a linear map \( \mathcal{K} : \mathcal{S}(\mathbb{M}_p)(\mathbb{R}^s) \rightarrow \mathcal{S}(\mathbb{M}_p)'(\mathbb{R}^l) \) by

\[ \langle \mathcal{K} \varphi, \psi \rangle = K(\psi \otimes \varphi), \quad \text{for } \psi \in \mathcal{S}(\mathbb{M}_p)(\mathbb{R}^l), \varphi \in \mathcal{S}(\mathbb{M}_p)(\mathbb{R}^s) \]

and \( (\varphi \otimes \psi)(x,y) = \varphi(x)\psi(y) \), which is continuous in the sense that \( \mathcal{K} \varphi_j \rightarrow 0 \) in \( \mathcal{S}(\mathbb{M}_p)'(\mathbb{R}^l) \) if \( \varphi_j \rightarrow 0 \) in \( \mathcal{S}(\mathbb{M}_p)(\mathbb{R}^s) \).

Conversely, for every such linear map \( \mathcal{K} \) there is a unique \( K \in \mathcal{S}(\mathbb{M}_p)'(\mathbb{R}^{l+s}) \) such that (2.4) is valid. The \( K \) is called the kernel of operator \( \mathcal{K} \).

**Proof.** If \( K \) is a jointly continuous bilinear functional \( \in \mathcal{S}(\mathbb{M}_p)(\mathbb{R}^l) \times \mathcal{S}(\mathbb{M}_p)(\mathbb{R}^s) \), then (2.4) defines an element \( \langle \mathcal{K} \varphi \rangle \in \mathcal{S}(\mathbb{M}_p)'(\mathbb{R}^l) \) since \( \psi \mapsto \).
$K(\psi \otimes \varphi)$ is continuous. The mapping $K : \mathcal{S}^{(M_p)}(\mathbb{R}^l) \times \mathcal{S}^{(M_p)'}(\mathbb{R}^s)$ is continuous since the mapping $\varphi \mapsto K(\psi \otimes \varphi)$ is continuous.

Let us prove the converse. To prove the existence we define a bilinear form $B$ on $\mathcal{S}^{(M_p)'}(\mathbb{R}^l) \otimes \mathcal{S}^{(M_p)'}(\mathbb{R}^s)$ by

$$B(\varphi, \psi) = \langle K\psi, \varphi \rangle, \quad \psi \in \mathcal{S}^{(M_p)}(\mathbb{R}^l), \quad \varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^s).$$

The form $B$ is a separately continuous bilinear form on the product $\mathcal{S}^{(M_p)}(\mathbb{R}^l) \times \mathcal{S}^{(M_p)}(\mathbb{R}^s)$ of Fréchet spaces and therefore it is jointly continuous, see [?].

Let $C > 0, \theta \in \mathbb{R}^l_+, \nu \in \mathbb{R}^s_+$ be chosen so that

$$|B(\varphi, \psi)| \leq C||\varphi||_\theta ||\psi||_\nu, \quad (2.5)$$

and let

$$t_{(n,k)} = B(\mathcal{H}_n, \mathcal{H}_k), \quad n \in \mathbb{N}^l, k \in \mathbb{N}^s.$$

Since $B$ is jointly continuous on $\mathcal{S}^{(M_p)}(\mathbb{R}^l) \times \mathcal{S}^{(M_p)}(\mathbb{R}^s)$, for $\varphi = \sum a_n \mathcal{H}_n$ and $\psi = \sum b_k \mathcal{H}_k$ we have that

$$B(\varphi, \psi) = \sum t_{(n,k)} a_n b_k.$$

On the other hand, for $(n,k) \in \mathbb{N}^l \times \mathbb{N}^s$ and $(\theta, \nu) \in \mathbb{R}^l \times \mathbb{R}^s$, by (2.5) we have

$$|t_{(n,k)}| \leq C||\mathcal{H}_n||_\theta ||\mathcal{H}_k||_\nu = ||\mathcal{H}_{n_1}||_\theta_1 ||\mathcal{H}_{n_2}||_\theta_2 \cdots ||\mathcal{H}_{n_l}||_\theta_1 ||\mathcal{H}_{k_1}||_\nu_1 ||\mathcal{H}_{k_2}||_\nu_2 \cdots ||\mathcal{H}_{k_s}||_\nu_s =$$

$$= \exp \left[ 2 \sum_{i=1}^l M\left(\theta_i \sqrt{\eta_i}\right) \right] \exp \left[ 2 \sum_{j=1}^s M\left(\nu_j \sqrt{\kappa_j}\right) \right].$$

Thus, from Hermite characterizations of Gelfand-Shilov spaces, it follows that the sequence $\{t_{(n,k)}\}_{(n,k)}$ is a Hermite representation of an element $K \in \mathcal{S}^{(M_p)'}(\mathbb{R}^l \times \mathbb{R}^s)$. Thus

$$\langle K, \varphi \rangle = \sum t_{(n,k)} c_{(n,k)}, \quad (2.6)$$

for $\varphi = \sum c_{(n,k)} \mathcal{H}_{n,k} \in \mathcal{S}^{(M_p)}(\mathbb{R}^{l+s})$.

If $\varphi = \sum a_n \mathcal{H}_n \in \mathcal{S}^{(M_p)}(\mathbb{R}^l)$ and $\psi = \sum b_k \mathcal{H}_k \in \mathcal{S}^{(M_p)}(\mathbb{R}^s)$ then $\varphi \otimes \psi$ has the Hermite representation $\{a_n b_k\}_{(n,k)}$ and we have that for $K$ defined by (2.6)

$$K(\varphi \otimes \psi) = \sum_{(n,k)} t_{(n,k)} a_n b_k = B(\varphi, \psi),$$

so $K = B$. This proves the existence.
The uniqueness follows from the fact that $K$ is completely determined by its Hermite representation $\{(K, \mathcal{H}_{(n,k)})\}_{(n,k)}$ and the fact that for every $(n, k) \in \mathbb{N}^l \times \mathbb{N}^s$

$$
\langle K, \mathcal{H}_{(n,k)} \rangle = \langle K, \mathcal{H}_n \otimes \mathcal{H}_k \rangle = B(\mathcal{H}_n, \mathcal{H}_k) = t_{(n,k)}.
$$

\[\square\]

3. Remarks

3.1. Examples of Gelfand-Shilov spaces. The Gelfand Shilov spaces generated by the sequences $\{M_p\}$, which belong to the classes of sequences considered in the paper, are generalizations of many spaces known in the literature.

- If $M_p = p^{\alpha_p}$, the space $S^{(M_p)}(\mathbb{R}^d)$ is the Gelfand-Shilov space $S^\alpha(\mathbb{R})$ and $S^{(M_p)}(\mathbb{R}^d)$ is the Pilipović space $\sum^\alpha(\mathbb{R}^d)$.
- If $M_p = p^p$ then the space $S^{(M_p)}(\mathbb{R}^d)$ is isomorphic with the Sato space $\mathcal{F}$, the test space for Fourier hyperfunctions $\mathcal{F}'$, and $S^{(M_p)}(\mathbb{R}^d)$ is the Silva space $\mathcal{G}$, the test space for extended Fourier hyperfunctions $\mathcal{G}'$.
- The space considered in [3] and [10] is isomorphic with $S^{(p^p)}$.
- Braun-Meise-Taylor space $S^{(\omega)}$, $\omega \in W$, studied in the series of papers by the same authors (see [1] and references in there), is the space $S^{(M_p)}(\mathbb{R}^d)$, where

$$
M_p = \sup_{\rho > 0} \rho^p e^{-\omega(\rho)}.
$$

The sequence satisfies the conditions (M.1), (M.2) and (M.3)', and it is in general different from a Gevrey sequence.
- Beurling-Björk space $S^{(\omega)}$, $\omega \in \mathcal{M}$, introduced in [2], is equal to the space $S^{(M_p)}(\mathbb{R}^d)$, where

$$
M_p = \sup_{\rho > 0} \rho^p e^{-\omega(\rho)}.
$$

The sequence satisfies the conditions (M.1) and (M.3)', and it is in general different from a Gevrey sequence. If we assume additionally that $\omega(\rho) \geq C (\log \rho)^2$ for some $C > 0$, then (M.2) is satisfied.
- In [8] Korevaar developed a very general theory of Fourier transforms, based on a set of original and well motivated ideas. In order to obtain a formal class of objects which contain functions of exponential growth and which is closed under Fourier transform he introduced objects called pansions of exponential growth. From characterization theorem [8, Theorem 92.1] and [9] it follows that exponential pansions are exactly Gelfand Shilov spaces of Roumieu type, generated by the sequence $M_p = p^{p/2}$.  

3.2. About conditions we assume on \( \{M_p\} \). Let us give few remarks about the conditions we assume on the sequence \( \{M_p\} \).

The condition (M.1) is of technical nature, which simplify the work and involve no loss of generality. This is the well known fact for the Denjoy-Carleman classes of functions. See for example [14].

The condition (M.2) is standard in the theory of generalized functions (see [7]). It implies that the class \( C^{(M_p)} \) is closed under the (ultra)differentiation, and is important in characterization of Denjoy-Carleman classes in multidimensional case.

To be able to discuss our results in the context of Komatsu’s ultradistributions, let us state condition (which we do not assume):

\[
(M.3)' \quad \sum_{p=1}^{\infty} \frac{M_p}{M_p^{1}} < \infty.
\]

(Non-quasi-analyticity)

The non-triviality conditions \((M.3)''\) and \((M.3)'''\) are weaker than the condition \((M.3)'\). Under the conditions \((M.3)''\) and \((M.3)'''\). For example, the sequence \([12]\) satisfies conditions (M.1), (M.2), (M.3)'', and if \( t > 0 \) also the condition \((M.3)'''\) but not \((M.3)'\); and for \( s > 1 \) it satisfies the stronger condition \((M.3)'\).

The condition \((M.3)'\) is necessary and sufficient condition that the classe \( C^{(M_p)} \) has a nontrivial subclass of functions with compact support, i.e. that \( C^{(M_p)} \) is non-quasianalytic class of functions. This is standard nontrivially condition in the ultradistribution theory. If it is satisfied Gelfand-Shilov spaces are the proper subspaces of the Komatsu’s spaces of ultradistributions ([7]) of Roumieu or Beurling type, so-called tempered ultradistributions. If it is not satisfied the Komatsu’s spaces are trivial and elements of Gelfand-Shilov spaces are hyperfunctions.

The smallest nontrivial Gelfand-Shilov space is \( S_{1/2}^{1/2} \). Condition \((M.3)''\) essentially means that the space \( S_{1/2}^{1/2} \) is a subset of \( S^{(M_p)} \).

The smallest nontrivial Pilipović space \( \sum_{\alpha}^{a} \) ([12]) does not exist. Note, \( \sum_{1/2}^{1/2} = \{0\} \), but the space \( \sum_{\alpha}^{a}, \alpha > 1/2 \), is nontrivial. Moreover, every nontrivial Pilipović space \( \sum_{\alpha}^{a} \) contains as a proper subspace one Gelfand Shilov space of Beurling type, for example, the space \( S^{(M_p)} \), where \( M_p = p^{p/2}(\log p)^{p}, t > 0 \).

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