Direct Algebraic Restoration of Slavnov-Taylor Identities in the Abelian Higgs-Kibble Model

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Abstract

A purely algebraic method is devised in order to recover Slavnov-Taylor identities (STI), broken by intermediate renormalization. The counterterms are evaluated order by order in terms of finite amplitudes computed at zero external momenta. The evaluation of the breaking terms of the STI is avoided and their validity is imposed directly on the vertex functional. The method is applied to the abelian Higgs-Kibble model. An explicit mass term for the gauge field is introduced, in order to check the relevance of nilpotency. We show that, since there are no anomalies, the imposition of the STI turns out to be equivalent to the solution of a linear problem. The presence of ST invariants implies that there are many possible solutions, corresponding to different normalization conditions. Moreover, we find more equations than unknowns (over-determined problem). This leads us to the consideration of consistency conditions, that must be obeyed if the restoration of STI is possible.

PACS codes: 11.10.G, 11.15.B

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1 Introduction

In quantum field theory, many essential physical requirements can be expressed by means of Ward identities, valid for the vertex functional $\Gamma$. Translating the invariance of the theory under a certain symmetry in a functional form, Ward identities impose several constraints on the possible structure of $\Gamma$. In perturbation theory, renormalization schemes must fulfill these constraints (we speak in this case of an invariant action $\Gamma$), in order to get a finite, physically relevant theory. But very often regularization and subtraction procedures are unable to respect all the symmetries of the theory, producing Green functions that (although finite) are not invariant and hence not correct.

In the absence of anomalies \cite{1}, one can recover exact Green functions, generated by the correct effective action (henceforth denoted by $\Pi$), by introducing counterterms, designed to compensate the breaking of Ward identities occurring for $\Gamma$.

In a preceding paper \cite{2} a strategy was proposed to construct these counterterms in gauge theories for the case of BRST symmetry \cite{3, 4} and related Ward identities (STI). The strategy was applied to the abelian Higgs-Kibble (HK) model \cite{4, 5} and was based on the knowledge of the breaking terms of the STI. The STI for the HK model are (appendix A)

$$S(I) = \int d^4 x \left[ \partial^\mu c \frac{\delta I}{\delta A^\mu} + \left( \partial^\mu A_\mu + \frac{e v}{\alpha} \phi_2 \right) \frac{\delta I}{\delta \bar{c}} \right] + (I(I), I)$$

The parenthesis is defined as

$$(X, Y) = \int d^4 x \left[ \frac{\delta X}{\delta J_1} \frac{\delta Y}{\delta \phi_1} + \frac{\delta X}{\delta J_2} \frac{\delta Y}{\delta \phi_2} - \frac{\delta X}{\delta \psi} \frac{\delta Y}{\delta \bar{\eta}} + \frac{\delta X}{\delta \bar{\psi}} \frac{\delta Y}{\delta \eta} \right]$$

$J_1, J_2, \eta, \bar{\eta}$ are external sources coupled to non-linear BRST variations of the fields $\phi_1, \phi_2, \bar{\psi}, \psi$. It is convenient to introduce the linearized ST operator \cite{3}:

$$S_0(\Pi) = \int d^4 x \left[ \partial^\mu c \frac{\delta \Pi}{\delta A^\mu} + \left( \partial^\mu A_\mu + \frac{e v}{\alpha} \phi_2 \right) \frac{\delta \Pi}{\delta \bar{c}} \right] + \left( \Pi^{(0)}, \Pi \right) + \left( \Pi, \Pi^{(0)} \right)$$

In order to get the counterterms at the $n$-th order of perturbation theory, one can construct a functional $\Xi^{(n)}$ whose image under the linearized Slavnov-Taylor operator is a quantity expressed by means of the lower-order symmetric Green functions $\Pi^{(j)}, j < n$, and the renormalized Green functions $\Gamma^{(n)}$ at the $n$-th order.

In this paper we perform the construction of $\Pi$ by the direct imposition of the STI in the Algebraic Renormalization framework. Thus we avoid the explicit calculation of the breaking terms. The resulting recursive construction turns out to be much simpler. The problem consists in the solution of a linear set of equations. In general, the number of equations exceeds the number of unknowns (over-determined problem) and moreover many solutions are possible, due to the existence of ST invariants. The last property allows the imposition of the normalization conditions. The former property yields a set of consistency conditions. Their origin is ascribable

\footnote{Notice that $\Pi^{(0)} = \Gamma^{(0)}$ (the classical action) is a known quantity.}
to the Quantum Action Principle (QAP) \[1, 6\] and to the STI. The rôle of the nilpotency of \( S_0 \) is not very clear in the present model. To illustrate this peculiar aspect of the abelian model we introduce an explicit mass term. Such a theory is BRST invariant and moreover STI are valid due to the absence of any anomaly. However the nilpotency of \( S_0 \) is broken by the mass term. We verify that the consistency conditions are still valid, although modified by the presence of the explicit mass term. Thus, in some sense, one is tempted to conclude that the consistency conditions are valid even if \( S_0 \) is not nilpotent. The last conclusion is stated in a conditional form, because the model under consideration violates physical unitarity \[7\], and therefore the conclusion could not be pertinent for physically relevant models.

Finally we stress that the present procedure of the direct imposition of the STI turns out to be very efficient for deriving an algorithm which is implementable on computer. Therefore the aim of this paper is to provide a preparatory work which can be translated to the Standard Model and its extensions. There more involved functional identities and the big amount of possible candidates for breaking terms and counterterms require an analysis by means of symbolic manipulation.

## 2 Direct algebraic imposition of STI

The QAP implies that the breaking terms which spoil the symmetries are, at the first non-trivial order, local and compatible with the power-counting. Therefore they can be removed, in absence of anomalies, by non-invariant counterterms. Since we are concerned only in the construction of the counterterms we can replace \( \Pi \) with its effective part (i.e. the Taylor expansion of any amplitude in the independent external momenta around zero \[5\]). In this way we associate to \( \Gamma^{(n)} \) a formal series given by an infinite sum of local Lorentz-invariant functionals. Without possibility of confusion we can denote the series by \( \Pi^{(n)} \) itself

\[
\Pi^{(n)} = \sum_j \int d^4x m_j^{(n)} \mathcal{M}_j(x) \tag{4}
\]

The Lorentz-scalar monomials \( \mathcal{M}_j(x) \) in the fields and external sources (and their derivatives) have to comply with all unbroken symmetries of the theory. In the HK model, we require them to be \( C \)-even and with zero FP-charge. We choose them linearly independent. Notice that the expansion of \( \Gamma \) in eq. \[4\] may contain monomials \( \mathcal{M}_j \) with arbitrary positive dimension.

The coefficients \( m_j^{(n)} \) are uniquely determined once the basis \( \{\mathcal{M}_j\}_{j \in \mathbb{N}} \) has been fixed. The vector space spanned by \( \{\mathcal{M}_j\}_{j \in \mathbb{N}} \) is denoted by \( \mathcal{V} \).

We have to impose recursively on \( \Pi \) the validity of the STI

\[
S(\Pi) = 0 \tag{5}
\]

We expand the above equation in powers of \( \hbar \). The contribution to the \( n \)-th order is

\[
[S(\Pi)]^{(n)} = S_0(\Pi^{(n)}) + \sum_{i=1}^{n-1} (\Pi^{(n-i)}, \Pi^{(i)}) = 0 \tag{6}
\]

\[5\] In the absence of IR problems, as it is in the HK model.
The second term is given by $\Gamma^{(j)}$ with $j < n$. We assume that $\Pi^{(j)}$, $j < n$ satisfy STI. The unknown quantities are the action-like parts of $\Pi^{(n)}$, (i.e. monomials with dimensions less or equal to four with the correct symmetry properties) which we denote by $\Xi^{(n)}$. $\Gamma^{(n)}$ is constructed by using the counterterms $\Xi^{(j)}$ with $j < n$ and it is finite. In the expansion of eq. (8), $\Xi^{(n)}$ is given in terms of $M_k(x)$ with $\dim M_k(x) \leq 4$ and their coefficients are denoted by $\gamma_k^{(n)}$. In the same way, the superficially convergent part of $\Gamma^{(n)}$ is given in terms of $M_k(x)$ with $\dim M_k(x) > 4$ and their coefficients are denoted by $\gamma_k^{(n)}$. We maintain the notation $m_j^{(n)}$ to indicate collectively both $\gamma_j^{(n)}$ and $\gamma_j^{(n)}$. In the HK model, $[S(\Pi)]^{(n)}$ is an element of a vector space $W$ spanned by all possible linearly independent Lorentz-invariant, $C$-even monomials in the fields and external sources, with FP charge $+1$.

We choose a basis $\{N_i(x)\}_{i=1,2,3,...}$ for $W$. We insert decomposition (3) in eq. (3):

$$[S(\Pi)]^{(n)} = \sum_j m_j^{(n)} S_0 \left( \int d^4x M_j \right) + \sum_{i=1}^{n-1} \sum_{jj'} m_j^{(i)} m_{j'}^{(n-i)} \left( \int d^4x M_j(x), \int d^4x' M_{j'}(x') \right)$$

(7)

There are coefficients $a^j_r$, $b^j_{kr}$ (uniquely fixed by the choice of $N_i(x)$ and by the ST transformations) such that

$$S_0 \left( \int d^4x M_j \right) = \int d^4x \sum_r a^j_r N_r(x)$$

$$\left( \int d^4x M_j(x), \int d^4x' M_{j'}(x') \right) = \int d^4x \sum_r b^{j}_{kr} N_r(x)$$

(8)

Then eq. (7) becomes

$$\sum_j a^j_r m_j^{(n)} + \sum_{i=1}^{n-1} \sum_{jj'} m_j^{(i)} m_{j'}^{(n-i)} b^j_{kr} = 0 \quad r = 0, 1, 2, \ldots$$

(9)

For $r$ such that $\dim N_r(x) > 5$, eq. (9) is an identity by the virtue of the QAP. In the absence of anomalies (as it is in the HK model), one can solve eq. (9) expressing coefficients $\gamma_j^{(n)}$ in terms of the coefficients $\gamma_j^{(l)}$, $l < n$. That is, we construct $\Xi^{(n)}$ from the superficially convergent part of $\Gamma^{(n)}$, and from lower order contributions $\Pi^{(l)}$, $l < n$.

If we can reach this goal, then STI can be restored. Notice that at every step one does not need to consider coefficients $m_j^{(n)}$ whose associated monomials have dimension $> 6$ (for the HK model): they will never contribute to eqs. (3) for $r$ such that $\dim N_r(x) \leq 5$. This allows the recursive algebraic construction of $\Gamma$. However the procedure requires the fixing of the normalization conditions (associated to the existence of ST invariants) and the use of the consistency conditions. Both items have been discussed at length in Ref. [2]. The normalization conditions are used in the solution of the linear problem given by eqs. (8) and (9), in order to simplify the algebra. The evaluation of the physical S-matrix elements requires an analysis of the two-point functions; in particular one must evaluate the location of the poles and their residua. The present approach makes use of the normalization conditions in the same manner. However the consistency conditions show up in a different way. In Ref. [2] the action counter-terms $\Xi^{(n)}$ are obtained by solving the equation

$$S_0(\Xi^{(n)}) = -\Psi^{(n)}$$

(10)
where \( \Psi^{(n)} \) is given in terms of finite amplitudes. A solution exists only if

\[
S_0(\Psi^{(n)}) = 0 \tag{11}
\]

and this provides the consistency conditions (the use of the ghost equation might be necessary if nilpotency of the BRST is only on-shell). In the present approach the counter-terms are evaluated by imposing STI directly on the effective vertex functional and one doesn’t need to evaluate \( \Psi^{(n)} \). Moreover the method makes no use of the nilpotency of the BRST. Therefore the consistency conditions in the present approach must be the consequence of a more general property than \( \Xi^{(1)} \). In the next section the use of the normalization conditions is briefly recalled. In order to illustrate the problem of the consistency conditions we consider the HK model with an explicit mass term \( M \). The model is BRST symmetric but nilpotency is broken by the explicit mass term. Furthermore it has an unpleasant feature: physical unitarity is violated (as it can be checked by an explicit calculation). However this fact is not relevant in our discussion.

## 3 The massive HK model

We study the HK massive model, whose classical action is given in Appendix A. \( \Gamma \) satisfies the ghost equation

\[
\frac{\delta \Gamma}{\delta \bar{c}} = (\alpha \Box + M^2)c + ev \frac{\delta \Gamma}{\delta J_2} \tag{12}
\]

Eq.\( (12) \) implies

\[
ev \xi^{(n)}_{J_2 c \phi} = \xi^{(n)}_{c \phi}, \quad ev \xi^{(n)}_{J_2 c \phi_1} = \xi^{(n)}_{c \phi_1}, \quad ev \gamma^{(n)}_{J_2 c \phi_1} = \xi^{(n)}_{c \phi_1},
\]

\[
ev \gamma^{(n)}_{J_2 c \phi_1} = \xi^{(n)}_{c \phi_1}, \quad ev \gamma^{(n)}_{J_2 c A_\mu} = \xi^{(n)}_{c A_\mu}, \quad ev \gamma^{(n)}_{J_2 c \square} = \xi^{(n)}_{c \square} \tag{13}
\]

so it fixes the following counterterms to all orders \( n \geq 1 \)

\[
\xi_{c \Box}, \xi_{c \phi_1}, \xi_{c \phi_1}, \xi_{c \phi_1}, \xi_{c \phi_1}, \xi_{c \phi_1}, \xi_{c \phi_1} \tag{14}
\]

while \( \xi_{c \Box}, \xi_{c \phi_1} \) depend on the normalization conditions imposed on external sources counterterms \( \xi_{J_2 c \phi}, \xi_{J_2 c \phi_1} \). We notice that the solution of eq.\( (14) \) is not unique. Consider eq.\( (13) \) and suppose that \( \Xi^{(n)} \) is a solution. Then, for arbitrary c-numbers \( v_j, \Xi^{(n)} + \sum_j v_j \mathcal{I}_j \) is again a solution, provided that \( S_0[\mathcal{I}_j] = 0 \). As in \( \Xi^{(n)} \) we look for action-like \( S_0 \)-invariants preserving unbroken symmetries of the model. We find 11 linearly independent \( S_0 \)-invariants \( \mathcal{I}_1 - \mathcal{I}_{11} \), listed in Appendix B. For the HK massive model, after taking into account the ghost equation \( (12) \) the \( S_0 \) operator becomes (\( n \geq 1 \))

\[
\hat{S}_0(\Pi^{(n)}) = \int d^4x \left\{ -c \Box_\mu \Pi^{(n)}_{\alpha \mu} + c c \phi_2 \Pi^{(n)}_{\phi_2} + c c (\phi_1 + v) \Pi^{(n)}_{\phi_1} \right\} + c \Pi^{(0)}_{\phi_2} - \frac{c}{2} \Pi^{(0)}_{\phi_2} \gamma_5 \Pi^{(0)}_{\phi_2}
\]

\[
\tag{15}
\]

\^6If not explicitly shown, the order of counterterms is understood to be \( n \).
The transformation
\[ \Xi^{(n)} \to \Xi^{(n)} + \sum_j v_j I_j \]  
must be compatible with eq. (12). This requirement entails the constraints
\[ \begin{align*}
    v_7 &= 0 \\
    v_8 + [(ev)^2 + M^2]v_9 + e^2 v v_{11} &= 0.
\end{align*} \]  

All other invariants \( I_j \) can be used to put to zero some of the counterterms in \( \Xi \), to all orders in perturbation theory. Invariants \( I_9 - I_{11} \) are used to set to zero the bosonic external sources counterterms
\[ \xi_{J_2c \phi_1}, \xi_{J_2c}, \xi_{J_1c}. \]

On the contrary, it is not possible to put to zero fermionic external sources counterterms. Then one gets contributions to \( \Xi^{(n)} \) from the fermionic mass term. The remaining \( I_j \) are used to put to zero the following counterterms:
\[ \begin{align*}
    \xi_{\phi_1}, \xi_{\phi_2 \phi_1}, \xi_{A_{\mu} \phi_1}, \xi_{F_{\mu \nu}^2}, \\
    \xi_{\bar{\psi}_\gamma \gamma_5 \psi \phi_2}, \xi_{i \bar{\psi}_\gamma \gamma_5 \psi \phi_2}.
\end{align*} \]

\( \Xi^{(n)} \) is then completely determined from the set of linear equations (16), once we adopt the normalization conditions fixed in eqs. (18) and (19). In particular, conditions in eq. (19) are obtained according to the introduction of a hierarchy in the counterterms, designed to get maximal decoupling among equations (18). Consider two action-like monomials \( \mathcal{M}_j(x) \) and \( \mathcal{M}_{j'}(x) \) such that their \( S_0 \)-images do not contain common terms; then their coefficients \( \xi_j \) and \( \xi_{j'} \) can never appear in the same equation in (18): they do not couple. We decompose the counterterms \( \Xi^{(n)} \) in disjoint sectors, thanks to normalization conditions (18) and (19). Next we find it convenient to solve eqs. (18) for amplitudes with lower number of external legs and higher derivatives in the external momenta. This reduces the number of graphs that must be evaluated (at the cost of a higher number of derivatives). Notice that the analysis of counterterms sectors is indeed equivalent to the construction of coefficients \( a_{r}^j \) in eq. (8), by taking into account the normalization conditions in eqs. (18), (19). Since the difference in \( S_0 \) between the massive and massless HK models only involves terms depending on \( J_1, J_2 \), the same counterterms sectors can be used in the massive case as well as in the massless one. We recover the conventional HK model by taking the limit \( M \to 0 \) in the expression of \( \Xi^{(n)} \). This limit is smooth.

## 4 Consistency conditions

In the Higgs-Kibble boson-gauge sector there are more equations than unknowns (over-determined problem). We then find a set of consistency conditions associated to the linear problem. For the massive case they are:
\[ \hat{m}_1^2 \left[ e v \left( \gamma_{\partial_{\mu} J_1} c \partial_{\mu} \phi_2 - \gamma_{J_1} c \square \phi_2 \right) + e \gamma \square J_2 c + \left( \gamma_{\partial_{\mu} J_1} c A_{\mu} - \gamma_{J_1} c \partial_{\mu} A_{\mu} \right) \right] \]
\[+e v \frac{M^2}{\alpha} (\gamma_{\partial_{\mu} J_2 c \partial_{\nu}\phi_1} - \gamma_{\partial_{\mu} J_2 c \phi_1}) = 0 \quad (20)\]

\[6 \lambda v \left( \gamma_{J_1 c \partial_{\mu} A^\mu} - \gamma_{\partial_{\mu} J_1 c A^\mu} \right) - 3 \tilde{m}_1^2 \gamma_{J_1 c A^\mu \partial_{\nu}\phi_1} - 3 \tilde{m}_1^2 \gamma_{\partial_{\mu} J_1 c A^\mu \phi_1} + 2 \tilde{m}_1^2 \gamma_{J_1 c \partial_{\mu} A^\mu \phi_1} \]

\[-e m_1^2 \gamma_{\partial_{\mu} J_1 c \partial_{\mu} \phi_2} - 2 e^2 v \gamma_{\partial_{\mu} \phi_2 \partial_{\nu} \phi_2 \phi_1} - 4 \lambda e v \gamma_{\partial_{\mu} J_1 c A^\mu \phi_1} - e \frac{M^2}{\alpha} \gamma_{\partial_{\mu} J_2 c \partial_{\nu}\phi_1} \]

\ [+e v \left( \gamma_{\phi_2 \partial_{\mu} \phi_2 A^\mu} - \gamma_{\phi_2 \partial_{\mu} \phi_2 A^\mu} \right) + 6 \sum_{j=1}^{n-1} \left( \gamma_{\partial_{\mu} J_1 c A^\mu} - \gamma_{\partial_{\mu} J_1 c A^\mu A^\mu} \phi_1 \right) \xi_{\phi_1}^{(n-j)} = 0 \quad (21)\]

\[-2 \tilde{m}_1^2 \gamma_{J_1 c \phi_2 \phi_2 A^2} - 4 e \lambda v \gamma_{J_1 c A^\mu} + 2 e^2 v \gamma_{J_1 c \phi_2 \phi_2} - 4 e \lambda v \gamma_{J_1 c A^\mu} - e^2 v \gamma_{\phi_2 \phi_1} \partial_{\mu} \phi_1 A^\mu \]

\[+e m_1^2 \gamma_{J_1 c A^\mu \partial_{\mu} \phi_1} + e m_1^2 \gamma_{\partial_{\mu} J_1 c A^\mu \phi_1} + 2 \sum_{j=1}^{n-1} \gamma_{\partial_{\mu} J_1 c \phi_2 \phi_2} \xi_{\phi_1}^{(n-j)} - 6 e \sum_{j=1}^{n-1} \gamma_{\partial_{\mu} J_1 c A^\mu A^\mu} \xi_{\phi_1}^{(n-j)} = 0 \quad (22)\]

\[e \frac{M^2}{\alpha} \gamma_{\partial_{\mu} J_2 c \partial_{\nu}\phi_1} + 2 e^2 v \gamma_{\partial_{\mu} \phi_2 \partial_{\nu} \phi_2 \phi_1} + 2 \lambda v (\gamma_{J_1 c \partial_{\mu} A^\mu} - \gamma_{\partial_{\mu} J_1 c A^\mu}) \]

\[+e m_1^2 \gamma_{\partial_{\mu} J_1 c \partial_{\mu} \phi_2} - 6 e v \gamma_{\phi_2 \partial_{\mu} \phi_1 A^\mu} - 2 e v \gamma_{J_2 c \phi_2^2} + 4 e \lambda v \gamma_{\partial_{\mu} J_2 c} = 0 \quad (23)\]

\[e v \left[ \gamma_{\phi_2 \partial_{\mu} A^\mu A^2} + e \gamma_{\partial_{\mu} J_1 c \partial_{\mu} A^\mu} \right] + \sum_{j=1}^{n-1} \gamma_{\partial_{\mu} J_1 c A^\mu} \xi_{\phi_1}^{(n-j)} = 0 \quad (24)\]

In the Fermi sector

\[e v \gamma_{\phi_2 \bar{\psi} \gamma^\mu \psi A_{\mu}} + G v \left( \gamma_{c \bar{\psi} \gamma^\mu \gamma^\alpha A_{\mu}} - \gamma_{c \gamma^\alpha \gamma^\mu \psi A_{\mu}} \right) + \sum_{j=1}^{n-1} \xi_{\psi \gamma}^{(j)} \left( \gamma_{c \bar{\psi} \gamma^\mu \gamma^\alpha A_{\mu}} - \gamma_{c \gamma^\alpha \gamma^\mu \psi A_{\mu}} \right) = 0. \quad (25)\]

\[\tilde{m}_1^2\] denotes the quantity \(2 \lambda v^2 + \frac{M^2}{\alpha}\). The above consistency conditions can be obtained from a quite general equation. Let us introduce the linearized version \(S_F\) of the ST operator \(S(F)\) defined in eq.[11] for a generic functional \(F\):

\[S_F(\cdot) = \int d^4x \left[ \partial^\mu c \frac{\delta(\cdot)}{\delta A_{\mu}} + \left( \partial^\mu A_{\mu} + \frac{e v}{\alpha} \phi_2 \right) \frac{\delta(\cdot)}{\delta c} \right] + (F, \cdot) + (\cdot, F). \quad (26)\]

By straightforward algebra one obtains the identity

\[S_F S(F) = \int d^4x \left( \Box c + \frac{e v}{\alpha} \frac{\delta F}{\delta J_2} \right) \frac{\delta F}{\delta c}. \quad (27)\]

If \(F\) obeys the ghost equation \([12]\), then

\[S_F S(F) = -\frac{M^2}{\alpha} \int d^4x \frac{\delta F}{\delta c}. \quad (28)\]

Suppose that we have restored the STI up to the \((n-1)\)-th order in perturbation theory, i.e. \(S(\Pi)^{(j)} = 0\) for \(j = 0, 1, \ldots, n-1\). At the \(n\)-th order of perturbation theory equation \([28]\) implies

\[\Sigma_0 [S(\Gamma)^{(n)}] = -\frac{M^2}{\alpha} \int d^4x \frac{\delta \Gamma^{(n)}}{\delta c}. \quad (29)\]
Now we prove that in the abelian HK model the r.h.s. is zero. Let us consider the breaking terms of the STI at the order \( n \)

\[
\Delta \Gamma^{(n)} \equiv S(\Gamma)^{(n)}.
\]  

From the QAP we know that \( \Delta \Gamma^{(n)} \) is a local functional with dimension less or equal five and FP-charge equal one. Then by construction

\[
- \frac{M^2}{\alpha} \int d^4x \ c \delta \frac{\delta \Gamma^{(n)}}{\delta c} = S_0[S(\Gamma)^{(n)}] = S_0[\Delta \Gamma^{(n)}]
\]

i.e \( c \frac{\delta \Gamma^{(n)}}{\delta x} \) is a local functional and has dimension less or equal four and FP-charge equal two. There are no terms with these properties (\( \int d^4x c \Box cc = 0, \int d^4x cc = 0, \int d^4x A^\mu \partial_\mu c \) is not allowed by C-conjugation, etc.). This implies that

\[
S_0[S(\Gamma)^{(n)}] = 0.
\]

On account of the results obtained in sect. 2, \( S(\Gamma)^{(n)} \) can be expanded on monomials \( N_r(x) \) (see eqs.(7),(8)) with dimension \( \leq 5 \), whose coefficients are constructed according to eq.(9). The imposition of eq.(32) then yields a set of consistency conditions equivalent to eqs.(20)-(24).

We illustrate this procedure for the consistency condition in eq.(24). We denote by \( n_c \partial_\mu A^\mu A^2 \) the coefficient of the monomial \( c \partial_\mu A^\mu A^2 \) in the expansion of \( S(\Gamma)^{(n)} \), and by \( n_{cA^\mu \partial_\mu A^2} \) the analogous coefficient for \( cA^\mu \partial_\mu A^2 \). The coefficient of the monomial \( A^2 \Box cc \) in \( S_0[S(\Gamma)^{(n)}] \) turns out to be equal to \( n_c \partial_\mu A^\mu A^2 - n_{cA^\mu \partial_\mu A^2} \). This must be zero according to eq.(32):

\[
n_c \partial_\mu A^\mu A^2 - n_{cA^\mu \partial_\mu A^2} = 0 \quad (33)
\]

The method of sect.2 allows the explicit evaluation of \( n_c \partial_\mu A^\mu A^2 \) and \( n_{cA^\mu \partial_\mu A^2} \). According to eq.(3), we get

\[
n_c \partial_\mu A^\mu A^2 = ev \left[ \gamma_{\phi_2 \partial^\mu A^\mu} A^2 + e \gamma_{J_1 \partial_\mu A^\mu} + \gamma_{J_2 cA^2} \right] - 4 \xi A^4 + \sum_{j=1}^{n-1} \gamma_{J_1 cA^\mu} \xi^{(n-j)}_{\phi_1 A^2} \quad (34)
\]

and

\[
n_{cA^\mu \partial_\mu A^2} = ev \left[ \gamma_{\phi_2 A^\mu \partial^\mu A^2} + e \gamma_{\partial_\mu J_1 cA^\mu} \right] - 4 \xi A^4 + \sum_{j=1}^{n-1} \gamma_{\partial_\mu J_1 cA^\mu} \xi^{(n-j)}_{\phi_1 A^2} \quad (35)
\]

Inserting eqs.(34) and (35) in eq.(33) we finally recover the consistency condition (24).

5 Conclusions

In the present paper we propose a method for imposing STI directly on the effective vertex functional (its formal Taylor series expansion at zero external momenta), without going through the explicit evaluation of the breaking terms. The method can be applied to any (not symmetrically) renormalized gauge theory where the renormalization procedure (regularization and
subtraction of divergent parts) violates STI. The algorithm amounts to find a basis of all action-like local Lorentz-invariant monomials and their ST transforms. Preserved symmetries have to be imposed on the allowed monomials (C-conjugation, FP charge neutrality, etc.). BRST sources have to be considered in the construction of the monomials. The counter-terms are constructed order by order by solving a linear problem where the input data are a set of finite zero-momenta amplitudes of dimension five and six. The existence of a set of ST invariants allows to fix an equivalent number of normalization conditions. These are chosen in order to simplify the solution and the evaluation of the finite amplitudes (normalization conditions at zero momenta). There is a number of consistency conditions associated to the linear problem. We suggest the use of a general property of the STI (\(S_0[S(\Gamma)^{(n)}] = 0\) for the example considered here, i.e. an abelian gauge model) in order to find the necessary consistency conditions that should be satisfied by the finite amplitudes relevant for the linear problem.

6 Acknowledgment

We acknowledge a partial financial support by MURST.

A Classical action

The classical action for the massive HK model is

\[
\Gamma^{(0)} = \int d^4x \left[ -\frac{1}{4} F_{\mu \nu}^2 + \frac{e^2 v^2}{2} A_\mu^2 \\
-\frac{\alpha}{2} \partial A^2 + \alpha \bar{c}c + e^2 v^2 \bar{c}c + e^2 v \bar{c}c \phi_1 \\
+ \frac{1}{2}((\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2) - \lambda v^2 \phi_1^2 - \frac{e^2 v^2}{2\alpha} \phi_2^2 \\
+ e A_\mu (\phi_2 \partial_\mu \phi_1 - \partial_\mu \phi_2 \phi_1) + e^2 v \phi_1 A^2 + \frac{e^2}{2} (\phi_1^2 + \phi_2^2) A^2 \\
- \lambda v \phi_1 (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4} (\phi_1^3 + \phi_2^3)^2 \\
+ \bar{\psi} i \gamma_\mu \psi + G v \bar{\psi} \psi + \frac{e}{2} \bar{\psi} \gamma_\mu \gamma_5 \psi A^\mu \\
+ G \bar{\psi} \psi \phi_1 - i G \bar{\psi} \gamma_5 \psi \phi_2 \\
+ J_1 [-e c \phi_2] + J_2 e c (\phi_1 + v) + i \frac{e}{2} \bar{\psi} \gamma_5 \psi c + i \frac{e}{2} c \bar{\psi} \gamma_5 \eta \\
+ \frac{M^2}{2} A_\mu^2 + M^2 \bar{c}c - \frac{M^2}{2\alpha} (\phi_1^2 + \phi_2^2) \right] \tag{36}
\]

BRST transformations

\[
s A_\mu = \partial_\mu c, \quad s \phi_1 = -e c \phi_2, \quad s \phi_2 = e c (\phi_1 + v) \\
s \bar{\psi} = -i \frac{e}{2} \gamma_5 \psi c, \quad s \bar{\psi} = i \frac{e}{2} c \bar{\psi} \gamma_5 \\
sc = \partial A + \frac{e v}{\alpha} \phi_2, \quad sc = 0 \tag{37}
\]
B  $S_0$ invariants

$$I_1 = \int d^4x (\phi_1^2 + \phi_2^2 + 2v\phi_1)$$

$$I_2 = \int d^4x (\phi_1^4 + \phi_2^4 + 2\phi_1^2\phi_2^2 + 4v\phi_1^3 + 4v\phi_1\phi_2^2 + 4v^2\phi_2^2)$$

$$I_3 = \int d^4x |D_\mu \phi|^2$$

$$I_4 = \int d^4x (F_{\mu\nu})^2$$

$$I_5 = \int d^4x \bar{\psi} i\gamma_\mu D_\mu \psi$$

$$I_6 = \int d^4x \bar{\psi} [(\phi_1 + v) - i\gamma_5\phi_2] \psi$$

$$I_7 = \int d^4x (\frac{1}{2} F^2 + \bar{c} \delta_{\text{BRST}} F)$$

$$I_8 = \int d^4x (\frac{1}{2} A^2 + \bar{c} c + \frac{v}{\alpha} \phi_1)$$

$$I_9 = \int d^4x [A^\mu \Gamma^{(0)}_{A\mu} + \bar{c} \Gamma^{(0)}_c + \alpha (F\partial^\mu A_\mu - \bar{c} \Box c) + \frac{M^2}{2\alpha} (\phi_1^2 + \phi_2^2)]$$

$$I_{10} = S_0(\int d^4x J_1), \quad I_{11} = S_0(\int d^4x J_1 \phi_1) \quad (38)$$

$$D_\mu = \partial_\mu - ieA_\mu, \quad D_\mu = \partial_\mu - i\frac{e}{2}\gamma_5 A_\mu$$

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