GROUND STATES FOR PSEUDO-RELATIVISTIC HARTREE EQUATIONS OF CRITICAL TYPE

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Abstract. We study the existence of ground state solutions for a class of non-linear pseudo-relativistic Schrödinger equations with critical two-body interactions. Such equations are characterized by a nonlocal pseudo-differential operator closely related to the square-root of the Laplacian. We investigate such a problem using variational methods after transforming the problem to an elliptic equation with a nonlinear Neumann boundary conditions.

1. Introduction

The relativistic Hamiltonian for $N$ identical particles of mass $m$, position $x_i$ and momentum $p_i$ interacting through the two body potential $\alpha W(|x_i - x_j|)$ is given by

$$H = \sum_{i=1}^{N} \left( \sqrt{p_i^2 c^2 + m^2 c^4} - mc^2 \right) - \alpha \sum_{i \neq j} W(|x_i - x_j|).$$

where $c$ is the speed of light and $\alpha > 0$ is a coupling constant.

According to the usual quantization rules the dynamics of the corresponding system of $N$-identical quantum spinless particles (a Bose gas) is described by the complex wave function $\Psi_N = \Psi_N(t,x_1,\ldots,x_N)$ governed by the Schrödinger equation

$$i\hbar \partial_t \Psi_N = H_N \Psi_N$$

where $\hbar$ is the Planck’s constant. Here $H_N: \mathcal{D} \subset L^2(\mathbb{R}^3)^{\otimes N} \to L^2(\mathbb{R}^3)^{\otimes N}$ is the quantum mechanics Hamiltonian operator, obtained from the classical Hamiltonian with the usual quantization rule $p \mapsto -i\hbar \nabla$, and defined in a suitable dense domain $\mathcal{D}$. In the case we are interested in $H_N$ is

$$H_N = (\sum_{j=1}^{N} \sqrt{-\hbar^2 c^2 \Delta_j + m^2 c^4} - mc^2) - \alpha \sum_{i \neq j} W(|x_i - x_j|).$$

where $W$ is the multiplication operator corresponding to the two body interaction potential, (e.g. $W(|x|) = |x|^{-1}$ for gravitational interactions).

The operator (from now on we will take $\hbar = 1, c = 1$)

$$\sqrt{-\Delta + m^2}$$

(1.1)

can be defined for all $f \in H^1(\mathbb{R}^N)$ as the inverse Fourier transform of the $L^2$ function $\sqrt{|k|^2 + m^2 F[f](k)}$ (here $F[f]$ denotes the Fourier transform of $f$) and it is also associated to the quadratic form

$$Q(f,g) = \int_{\mathbb{R}^N} \sqrt{|k|^2 + m^2 F[f] F[g]} \, dk$$

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recent development on models involving the pseudorelativistic operator

\[ H^{1/2}(\mathbb{R}^N) = \{ f \in L^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |k| |\mathcal{F}[f](k)|^2 \, dk < +\infty \} \]

(see e.g. [10] for more details).

In the mean field limit approximation (i.e. \( \alpha N \approx O(1) \) as \( N \to +\infty \)) of a quantum relativistic Bose gas, one is lead to study the nonlinear mean field equation — called the pseudorelativistic Hartree equation — given by

\[ i\partial_t \psi = (\sqrt{-\Delta + m^2} - m)\psi - (W * |\psi|^2)\psi. \]

(1.2)

where \(*\) denotes convolution. We will take attractive two body interaction, and hence \( W \) will always be a nonnegative function.

See [11] for the study of this equation when \( W \) is the gravitational interaction, and [4] for a rigorous derivation of the mean field equation (1.2) as a limit as \( N \to +\infty \) of the Schrödinger equation for \( N \) quantum particles, and [3] for more recent development on models involving the pseudorelativistic operator \( \sqrt{-\Delta + m^2} \).

It has recently been proved that for Newton or Yukawa-type two body interactions (i.e. \( W(|x|) = |x|^{-1} \) or \( |x|^{-1} e^{-|x|} \) in \( \mathbb{R}^3 \)) such an equation is locally well-posed in \( H^s, s \geq 1/2, \) and that the solution is global in time for small initial data in \( L^2 \) (see [3]). Blow up has been proved in [6, 7].

Due to the focusing nature of the nonlinearity (attractive two-body interaction) there exist solitary waves solutions given by

\[ \psi(t, x) = e^{i\mu t} \varphi(x) \]

where \( \varphi \) satisfy the nonlinear eigenvalue equation

\[ \sqrt{-\Delta + m^2} \varphi - m \varphi - (W * |\varphi|^2)\varphi = -\mu \varphi. \]

(1.3)

In [11] the existence of such solutions (in the case \( W(x) = |x|^{-1} \)) has been proved provided that \( M < M_c, \) \( M_c \) being the Chandrasekhar limit mass. More precisely the Authors have shown the existence in \( H^{1/2}(\mathbb{R}^3) \) of a radial, real-valued nonnegative minimizer (ground state) of

\[ \mathcal{E}(|\psi|) = \frac{1}{2} \int_{\mathbb{R}^3} \psi(\sqrt{-\Delta + m^2} - m)\psi \, dx - \frac{1}{4} \int_{\mathbb{R}^3} (|x|^{-1} * |\psi|^2)|\psi|^2 \, dx. \]

(1.4)

with given fixed “mass-charge” \( M = \int_{\mathbb{R}^3} |\psi|^2 \, dx < M_c. \) We call mass-critical the potentials \( W \) whose associated functional \( \mathcal{E} \) exhibits this kind of phenomenon.

More recently in [5] it has been proved that the ground state solution is regular \( (H^s(\mathbb{R}^3), \) for all \( s \geq 1/2, \) strictly positive and that it decays exponentially. Moreover the solution is unique, at least for small \( L^2 \) norm (9).

Let us remark that these last results are heavily based on the specific form (Newton or Yukawa type) of the two body interactions in the Hartree nonlinearity. Indeed in these cases the estimates of the nonlinearity relies on the following facts

- for this class of potentials one has that

\[ \frac{e^{-\mu |x|}}{4\pi |x|} * f = (\mu^2 - \Delta)^{-1} f \quad \text{for} \ f \in \mathcal{S}(\mathbb{R}^3), \mu \geq 0 \]

- the use of a generalized Leibnitz rule for Riesz and Bessel potentials
- the following estimate holds

\[ \| \frac{1}{|x|} * |u|^2 \|_{L^\infty} \leq \frac{\pi}{2} \| (-\Delta)^{1/4} u \|_{L^2}^2. \]
In [2] it has been proved an existence and regularity result for the solutions of (1.3) for a wider class of nonlinearities by exploiting the relation of equation (1.3) with an elliptic equation on $\mathbb{R}^{N+1}$ with a nonlinear Neumann boundary condition. Such a relation has been recently used to study several problems involving fractional powers of the laplacian (see e.g. [1] and references therein) and it is based on an alternative definition of the operator (1.1) that can be described as follows. Given any function $u \in \mathcal{S}(\mathbb{R}^N)$ there is a unique function $v \in \mathcal{S}(\mathbb{R}^{N+1}_+)$ (here $\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R} \times \mathbb{R}^N \mid x > 0\}$) such that

$$\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
v(0,y) = u(y) & \text{for } y \in \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+.
\end{cases}$$

Setting

$$Tu(y) = -\frac{\partial v}{\partial x}(0,y)$$

we have that the equation

$$\begin{cases}
-\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
w(0,y) = Tu(y) = -\frac{\partial w}{\partial x}(0,y) & \text{for } y \in \mathbb{R}^N
\end{cases}$$

has the solution $w(x,y) = -\frac{\partial w}{\partial x}(x,y)$. From this we have that

$$T(Tu)(y) = -\frac{\partial w}{\partial x}(0,y) = \frac{\partial^2 v}{\partial x^2}(0,y) = (-\Delta v + m^2 v)(0,y)$$

and hence $T^2 = (-\Delta + m^2)^2$.

In [2] we have studied the equation

$$(1.5) \quad \sqrt{-\Delta + m^2} v = \mu v + \nu |v|^{p-2} v + \sigma(W*|v|^2)v \quad \text{in } \mathbb{R}^N$$

where $p \in (2, \frac{2N}{N-2})$, $\mu < m$ is fixed, $\nu$, $\sigma \geq 0$ (but not equal 0 both), $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $W \geq 0$, $(r > N/2)$, $W(x) = W(|x|) \to 0$ as $|x| \to +\infty$.

The results are obtained, following the approach outlined above, by studying the following equivalent elliptic problem with nonlinear boundary condition

$$(1.6) \quad \begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
\frac{\partial v}{\partial x} = \mu v + \nu |v|^{p-2} v + \sigma(W*|v|^2)v & \text{on } \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+
\end{cases}$$

and the associated functional on $H^1(\mathbb{R}^{N+1}_+)$. Let us point out that in [2] the $L^2$ norm of the solution is not prescribed. In such a case existence of a (positive, radially symmetric) solution can be proved for a certain class of potentials $W$ and exponents $p$ which is larger then the one we can deal with here.

When the $L^2$ norm is prescribed to be $M$ (the most relevant problem from a physical point of view), as in [11], then the Newton potential $(|x|^{-1}$ in $\mathbb{R}^3$) is critical, in the sense that minimization of $\mathcal{E}$ given by (1.4) is possible only when $M < M_c$ (see theorem [13]).

The main purpose of this paper is to exploit this approach also for the problem of finding minimizer of the static energy

$$E[u] = \frac{1}{2} \int_{\mathbb{R}^N} u(\sqrt{-\Delta} + m^2 - m)u \, dx + \frac{\eta}{p} \int_{\mathbb{R}^N} |u|^p \, dx - \frac{\sigma}{4} \int_{\mathbb{R}^N} (W*|u|^2)|u|^2 \, dx.$$ 

with prescribed $L^2$ norm, for a wider class of attractive two-body potential including the critical case.
To be more precise, we consider a class of two-body potential $W \in L^q_0(\mathbb{R}^N)$, with $q \geq N$. We recall that $L^q_0(\mathbb{R}^N)$, the weak $L^q$ space, is the space of all measurable functions $f$ such that

$$\sup_{a>0} a^{1/q} \mu \{ x \in \mathbb{R}^N : |f(x)| > a \}^{1/q} < +\infty,$$

where $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{R}^N$. Note that $W(x) = |x|^{-1}$ does not belong to any $L^q$-space but it belongs to $L^q_0(\mathbb{R}^N)$. We say that a potential $W$ is critical if $W \in L^N(\mathbb{R}^N)$.

Our main result is the following

**Theorem 1.8.** Let $W \in L^q_0(\mathbb{R}^N)$, $q \geq N \geq 2$, $W(y) \geq 0$ for all $y \in \mathbb{R}^N$ and such that

$$W(\lambda^{-1}y) \geq \lambda^a W(y), \quad \text{for all } \lambda \in (0, 1) \text{ and for some } a > 0. \quad (1.9)$$

We also assume that $W(r) = W(|r|)$ is rotationally symmetric and that $W(r) \to 0$ as $r \to +\infty$.

Take $\eta \geq 0$, $\sigma > 0$ and $p \in \{2 + \frac{2}{q}, 2 + \frac{2}{N-q}\}$. Then

- if $\eta > 0$ or $\eta = 0$ and $q > N$, for all $M > 0$ there is a strictly positive minimizer $u \in H^{1/2}(\mathbb{R}^N)$ of $\mathcal{E}[u]$ such that $\int_{\mathbb{R}^N} u^2 = M$.
- (mass-critical case) if $\eta = 0$ and $q = N$ there is a critical value $M_c > 0$ such that for all $0 < M < M_c$ there is a strictly positive minimizer $u \in H^{1/2}(\mathbb{R}^N)$ of $\mathcal{E}[u]$ such that $\int_{\mathbb{R}^N} u^2 = M$.

Moreover there exists $\mu > 0$ such that $u$ is a smooth, exponentially decaying at infinity, solution of

$$-\Delta u + m^2 u - m u = -\mu u - \eta |u|^{p-2} u + \sigma(W(u^2))u$$

in $\mathbb{R}^N$, and $u$ is radial if $W = W(r)$ is a decreasing function of $r > 0$.

**Remark 1.10.** The nonlinear term $|u|^{p-2} u$ is a defocusing nonlinearity, the convolution term is a focusing nonlinearity. An open problem is to understand if solitons exist also for other ranges of $p$, in particular for $2 < p \leq 2 + \frac{2}{q}$ and $W \in L^q_0$.

**Remark 1.11.** If $W \in L^q_0$ and $(1.9)$ holds for some $\alpha > 0$, then necessarily $\alpha \in (0, N/q]$. If $W(x) = |x|^{-\alpha}$, then $W \in L^q_0$ if and only if $\alpha = N/q$.

**Remark 1.12.** $\mu$ is a Lagrange multiplier.

2. Preliminaries

Let $(x, y) \in \mathbb{R} \times \mathbb{R}^N$. We have already introduced $\mathbb{R}^{N+1}_+ = \{ (x, y) \in \mathbb{R}^{N+1} \mid x > 0 \}$. With $\|u\|_p$, we will always denote the norm of $u \in L^p(\mathbb{R}^{N+1}_+)$, with $\|u\|$ the norm of $u \in H^1(\mathbb{R}^{N+1}_+)$ and with $\|v\|_p$ the norm of $v \in L^p(\mathbb{R}^N)$.

We recall that for all $v \in H^1(\mathbb{R}^{N+1}_+) \cap C_0^{\infty}(\mathbb{R}^{N+1})$

$$\int_{\mathbb{R}^N} |v(0, y)|^p \, dy = \int_{\mathbb{R}^N} \frac{\partial}{\partial x} v(x, y)|^p \, dx$$

$$\leq p \int_{\mathbb{R}^{N+1}_+} |v(x, y)|^{p-1} |\partial_x v(x, y)| \, dx \, dy$$

$$\leq p \left( \int_{\mathbb{R}^{N+1}_+} |v(x, y)|^{2(p-1)} \, dx \, dy \right)^{1/2} \left( \int_{\mathbb{R}^{N+1}_+} |\partial_x v(x, y)|^2 \, dx \, dy \right)^{1/2}\]$$

that is

$$|v(0, \cdot)|_p^p \leq p \|v\|_{2(p-1)}^{p-1} \|\partial_x v\|_2.$$
which, by Sobolev embedding, is finite for all $2 \leq 2(p-1) \leq 2(N+1)/(N+1-2)$, that is $2 \leq p \leq 2^*$, where we have set $2^* = 2N/(N-1)$. By density of $H^1(\mathbb{R}^{N+1}) \cap C_0^\infty(\mathbb{R}^{N+1})$ in $H^1(\mathbb{R}^{N+1})$ such an estimates allows us to define the trace $\gamma(v)$ of $v$ for all the functions $v \in H^1(\mathbb{R}^{N+1})$. The inequality

$$\left| \gamma(v) \right|_p^p \leq p \|v\|_2^{p-1} \|\frac{\partial v}{\partial x}\|_2,$$

holds then for all $v \in H^1(\mathbb{R}^{N+1})$.

It is known that the traces of functions in $H^1(\mathbb{R}^{N+1})$ belong to $H^{1/2}(\mathbb{R}^N)$ and that every function in $H^{1/2}(\mathbb{R}^N)$ is the trace of a function in $H^1(\mathbb{R}^{N+1})$. Then (2.2) is in fact equivalent to the well known fact that $\gamma(v) \in H^{1/2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ provided $q \in [2, 2^*)$. We also recall here that

$$\|w\|_{H^{1/2}}^2 = \inf \{ \|u\|^2 | u \in H^1(\mathbb{R}^{N+1}), \gamma(u) = w \} = \int_{\mathbb{R}^N} \left( 1 + |\xi| \right) |\mathcal{F}u(\xi)|^2 \, d\xi.$$

Let us also introduce the norm of the weak $L^q$-space as follows

$$\|f\|_{q,w} = \sup_A |A|^{-1/r} \int_A |f(x)| \, dx$$

where $1/q + 1/r = 1$ and $A$ denotes any measurable set of finite measure $|A|$ (see e.g. [10] for more details). Now using this norm we can state the weak Young inequality. If $g \in L^q_0(\mathbb{R}^N)$, $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$ where $1 < q, p, r < +\infty$ and $1/q + 1/p + 1/r = 2$ then

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(y)g(y-z)h(y) \, dy \, dz \leq C_{p,q,r} \|g\|_{q,w} \|f\|_{p,w} |h|_r.$$

We consider the class of two-body interactions $W \in L^q_0(\mathbb{R}^N)$ for $q \geq N$. By weak Young inequality and Hölder inequality we have for $r = 4q/(2q-1) \in (2, 2^*)$ since $q \geq N$ and for all $p \in (4q/(2q-1), 2^*[2]

$$\int_{\mathbb{R}^N} (W * |w|^2) |w|^2 \, dy \leq C \|W\|_{q,w} |w|^4 \|w\|_{2^*}^{2^*-\frac{4}{q}} |w|^\frac{2^*}{2^*-\frac{4}{q}}.$$

For $p = 2^*$ we get

$$\int_{\mathbb{R}^N} (W * |w|^2) |w|^2 \, dy \leq C \|W\|_{q,w} |w|^4 \|w\|_{2^*}^{2^*-\frac{4}{q}} |w|^\frac{2^*}{2^*-\frac{4}{q}}.$$

In the (critical) case $q = N$ this gives

$$\int_{\mathbb{R}^N} (W * |w|^2) |w|^2 \, dy \leq C \|W\|_{N,w} |w|^2 \|w\|_{2^*}^{2^*}.$$

Let us point out that one cannot deduce (2.4) from the weak Young’s inequality (2.3) directly, and that it is not true, in general, that $L^\infty$ norm of $W * |w|^2$ can be bounded by the $L^{2^*}$ norm of $u$ if $W \in L^N_w$.

For all $v \in H^1(\mathbb{R}^{N+1})$, we consider the functional given by

$$\mathcal{I}(v) = \frac{1}{2} \left( \int_{\mathbb{R}^{N+1}} (|\nabla v|^2 + m^2 |v|^2) \, dx \, dy - \int_{\mathbb{R}^N} m |\gamma(v)|^2 \, dy \right) + \frac{\eta}{p} \int_{\mathbb{R}^N} |\gamma(v)|^p \, dy - \sigma \frac{1}{4} \int_{\mathbb{R}^N} (W * |\gamma(v)|^2) |\gamma(v)|^2 \, dy.$$

In view of (2.2) and (2.3) all the terms in the functional $\mathcal{I}$ are well defined if $p \in (2, 2^*)$ and $W \in L^q_w(\mathbb{R}^N)$ with $q \geq N$. 


Remark that from (2.1) with \( p = 2 \) follows that

\[
(2.7) \quad m \int_{\mathbb{R}^N} |\gamma(v)|^2 \, dy \leq 2(m\|v\|_2^2 \|\nabla v\|_2 \leq \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 |v|^2) \, dx \, dy
\]

demonstrating that the quadratic part in the functional \( I \) is nonnegative.

Moreover the following property can be easily verified

Lemma 2.8. For \( u \in H^1(\mathbb{R}^{N+1}_+) \), let \( w = \gamma(u) \in H^{1/2}(\mathbb{R}^N) \), \( \hat{w} = F(w) \) and

\[
v(x, y) = F^{-1}(e^{-x\sqrt{m^2 + |y|^2}} \, \hat{w}) = \int_{\mathbb{R}^N} e^{-x\sqrt{m^2 + |\xi|^2}} \hat{w}(\xi)e^{i\xi y} \, d\xi.
\]

Then \( v \in H^1(\mathbb{R}^{N+1}_+) \), \( \|v\| = \|w\|_{H^{1/2}} \), \( \mathcal{I}(v) \leq \mathcal{I}(u) \) and \( \mathcal{I}(v) = E[w] \).

3. Minimization problem

We consider the following minimization problem

\[
(3.1) \quad I(M) = \inf\{ \mathcal{I}(v) \mid v \in \mathcal{M}_M \}
\]

where the manifold \( \mathcal{M}_M \) is given by

\[
\mathcal{M}_M = \{ v \in H^1(\mathbb{R}^{N+1}_+) \mid \int_{\mathbb{R}^N} |\gamma(v)|^2 = M \}
\]

Remark 3.2. The term \( m \int_{\mathbb{R}^N} |\gamma(v)|^2 \) in the functional \( \mathcal{I}(v) \) is constant for all \( v \in \mathcal{M}_M \). The presence of such a term will allow us to show that the infimum of the functional \( \mathcal{I} \) on \( \mathcal{M}_M \) is negative.

Concerning the existence of a minimizer for problem (3.1) we start by proving, in the following lemmas, boundedness from below of functional \( \mathcal{I} \) on \( \mathcal{M}_M \) and some properties of the infimum \( I(M) \).

Lemma 3.3. The functional \( \mathcal{I} \) is bounded from below and coercive on \( \mathcal{M}_M \subset H^1(\mathbb{R}^{N+1}_+) \) for all \( M > 0 \) if \( \eta > 0 \) or \( q > N \) and for all \( M \) small enough if \( \eta = 0 \) and \( q = N \).

Proof. Let us examine first the convolution term.

If \( \eta > 0 \), from (2.1) and \( |\gamma(u)|_2^2 = M \) we have

\[
(3.4) \quad 0 \leq \int_{\mathbb{R}^N} \left( W \ast |\gamma(u)|^2 \right) |\gamma(u)|^2 \leq C\|W\|_{q,w} \|\gamma(u)\|_2^{4 - \frac{2p}{(p-2)}} \|\gamma(u)\|_p^{\frac{2p}{p-2}}
\]

\[
= C\|W\|_{q,w} M^{2 - \frac{2p}{p-2}} \|\gamma(u)\|_p^{\frac{2p}{p-2}}.
\]

Since \( \frac{2p}{(p-2)} < p \) by assumption, this is enough to prove coercivity if \( \eta > 0 \). Indeed we have in such a case that

\[
\mathcal{I}(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}mM + C_1 |\gamma(u)|_p^p - C_2 |\gamma(u)|_p^{\frac{2p}{p-2}} \geq \frac{1}{2}\|u\|^2 - C_3.
\]

In case \( \eta = 0 \) we deduce from (2.6) and \( |\gamma(u)|_{2^*} \leq C\|u\| \) that

\[
\mathcal{I}(u) \geq \|u\|^2 - mM - C\|W\|_{q,w} M^{2 - N/q} \|u\|^{2N/q}.
\]

It is then clear that the functional is bounded below and coercive whenever \( q > N \) and, in case \( q = N \), if \( \|W\|_{N,w} M \) is small enough.

Lemma 3.5. \( I(M) < 0 \) for all \( M > 0 \).
Proof. Take any function $u \in C_0^\infty(\mathbb{R}^N)$, $|u|^2 = M$, and let $w(x,y) = e^{-mx}u(y)$.

Then

$$I(M) = \inf_{v \in \mathcal{M}_M} \mathcal{I}(v) \leq \mathcal{I}(u) = \frac{1}{2} \iint_{\mathbb{R}^{N+1}_+} (|\partial_x u|^2 + |\nabla_y u|^2 + m^2 |u|^2) \, dx \, dy - \frac{m}{2} \int_{\mathbb{R}^N} |u|^2 \, dy + G(u)$$

$$= \frac{m}{4} \int_{\mathbb{R}^N} |u|^2 \, dy + \frac{1}{4m} \int_{\mathbb{R}^N} |\nabla_y u|^2 \, dy + \frac{m}{4} \int_{\mathbb{R}^N} |u|^2 \, dy - \frac{m}{2} \int_{\mathbb{R}^N} |u|^2 \, dy + G(u)$$

$$= \frac{1}{4m} \int_{\mathbb{R}^N} |\nabla_y u|^2 \, dy + G(u)$$

where

$$G(u) = \frac{\eta}{p} \int_{\mathbb{R}^N} |u|^p \, dy - \frac{\sigma}{4} \int_{\mathbb{R}^N} (W * |u|^2) |u|^2 \, dy$$

Take now, for $\lambda > 0$, $u_\lambda(y) = \lambda^{N/2} u(\lambda y)$ and $w_\lambda(x,y) = e^{-m_\lambda^2} u_\lambda(y) \in \mathcal{M}_M$ for all $\lambda > 0$. We find that

$$I(M) \leq \inf_{\lambda > 0} \mathcal{I}(u_\lambda)$$

$$\leq \inf_{\lambda \in (0,1)} \left[ \frac{\lambda^2}{4m} \int_{\mathbb{R}^N} |\nabla_y u_\lambda|^2 + \frac{\eta}{p} \int_{\mathbb{R}^N} |u_\lambda|^p - \frac{\sigma}{4} \int_{\mathbb{R}^N} (W * |u_\lambda|^2) |u_\lambda|^2 \right]$$

and since $\alpha < N(p^{-\alpha} - 1) < 2$ we have that the infimum is negative. \(\square\)

**Lemma 3.6.** For all $M > 0$ and $\beta \in (0, M)$ we have that $I(M) < I(M - \beta) + I(\beta)$.

Moreover $\frac{I(M)}{M}$ is a concave function of $M$ and hence $I(M)$ is a continuous function of $M$.

**Proof.** The subadditivity is a consequence of the fact that for all $\theta > 1$

$$I(\theta M) < \theta I(M) \quad \text{which implies} \quad \frac{1}{\theta} I(M) < I(M/\theta).$$

Indeed, taking $\theta_1 = \frac{M}{\beta}$ and $\theta_2 = \frac{M - \beta}{M - \beta}$ we have that

$$I(M) = \frac{\beta}{M} I(M) + \frac{M - \beta}{M} I(M) < I(\beta) + I(M - \beta)$$

To prove that (3.7) holds, we remark that for all $v \in \mathcal{M}_M$ and $\lambda = \theta^{1/2} > 1$ we have, thanks to (2.7)

$$\mathcal{I}(\lambda v) = \frac{\lambda^2}{2} \left[ \iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 |v|^2) \, dx \, dy - m \int_{\mathbb{R}^N} |\gamma(v)|^2 \, dy \right]$$

$$+ \frac{\eta \lambda^p}{p} \int_{\mathbb{R}^N} |\gamma(v)|^p \, dy - \frac{\sigma \lambda^4}{4} \int_{\mathbb{R}^N} (W * |\gamma(v)|^2) |\gamma(v)|^2 \, dy \leq \lambda^2 \mathcal{I}(v)$$

Hence, since $I(M) < 0$

$$I(\theta M) = \inf_{|\gamma(v)|^2 = \theta M} \mathcal{I}(v) = \inf_{|\gamma(v)|^2 = \theta M} \mathcal{I}(\theta^{1/2} v)$$

$$\leq \theta^2 \inf_{|\gamma(v)|^2 = M} \mathcal{I}(v) = \theta^2 I(M) < \theta I(M) < I(M)$$

To prove the concavity of $\frac{I(M)}{M}$, we remark that

$$\frac{I(M)}{M} = \frac{1}{M} \inf_{u \in \mathcal{M}_M} \mathcal{I}(u) = \inf_{u \in \mathcal{M}_1} \frac{\mathcal{I}(\sqrt{M} u)}{M}.$$
Lemma 3.9. We now show that, for all \( u \in \mathcal{M}_1 \), \( M \rightarrow I(\sqrt{M}u)/M \) is a concave function of \( M \).

This will immediately prove that \( I(M)/M \) is a concave function.

Since
\[
\frac{I(\sqrt{M}v)}{M} = \frac{1}{2} \left( \int_{\mathbb{R}^N} \left( |\nabla v|^2 + m^2 v^2 \right) \, dx \, dy - \int_{\mathbb{R}^N} m |\gamma(v)|^2 \, dy \right) + \frac{\eta M^{p/2} - 1}{p} \int_{\mathbb{R}^N} |\gamma(v)|^p \, dy - \frac{\sigma M}{4} \int_{\mathbb{R}^N} (W * |\gamma(v)|^2) |\gamma(v)|^2 \, dy
\]

It is then immediate to check that the second derivative with respect to the variable \( M \) is negative for all \( M > 0 \) when \( p/2 < 2 \) and that the function is linear when \( p = 4 \) (namely the critical exponent for \( N = 2 \)). \( \square \)

We are now ready to prove existence of a minimizer for the functional \( I \) on \( \mathcal{M}_M \).

**Proposition 3.8.** For every \( M > 0 \) there is a function \( u \in H^1(\mathbb{R}^N_+ \times [0,1]) \) such that
\[
\begin{cases}
I(u) = I(M) \\
\int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy = M
\end{cases}
\]
i.e. a minimizer for \( I \) in \( \mathcal{M}_M \).

**Proof.** Let \( \{u_n\} \subset \mathcal{M}_M \) be a minimizing sequence. Follows from lemma 2.8 that also the sequence
\[
v_n(x, y) = F^{-1} \left( e^{-2 \sqrt{m^2 + 1} |\gamma(u_n)|} \right)
\]
is a minimizing one. From lemma 3.3 we deduce that \( v_n \) is bounded in \( H^1(\mathbb{R}^N_+ \times [0,1]) \) and that \( v_n \equiv \gamma(v_n) = \gamma(u_n) \) is bounded in \( H^{1/2}(\mathbb{R}^N) \) and \( \int_{\mathbb{R}^N} |v_n|^2 \, dy = M \).

We will now use the concentration-compactness method of P.L. Lions [12].

Namely, one of the following cases must occur

- **vanishing:** for all \( R > 0 \)
  \[
  \lim_{n \to +\infty} \sup_{z \in \mathbb{R}^N} \int_{z + B_R} |w_n|^2 \, dy = 0;
  \]

- **dichotomy:** for a subsequence \( \{n_k\} \)
  \[
  \lim_{R \to +\infty} \lim_{k \to +\infty} \sup_{z \in \mathbb{R}^N} \int_{z + B_R} |w_{n_k}|^2 \, dy = \alpha \in (0, M);
  \]

- **compactness:** for all \( \epsilon > 0 \) there is \( R > 0 \), a sequence \( \{y_k\} \) and a subsequence \( \{w_{n_k}\} \) such that
  \[
  \int_{y_k + B_R} |w_{n_k}|^2 \, dy \geq M - \epsilon.
  \]

Following the usual strategy we will show that vanishing and dichotomy cannot occur.

**Lemma 3.9.** If vanishing occurs, then
\[
\int_{\mathbb{R}^N} (W * |w_n|^2) |w_n|^2 \, dy \to 0.
\]

**Proof of lemma 3.9.** Take any \( \delta > 0 \) and \( R > 0 \). Let define \( W_\delta = W_\delta(\cdot |y|) \) and
\[
W_\delta^R(|y|) = (W_\delta(|y|) - R)^+ \mathbb{I}_{|y| < R} + W_\delta(|y|) \mathbb{I}_{|y| \geq R},
\]
where \( \mathbb{I}_A \) is the characteristic function of the set \( A \). Then it easy to check that \( W \in L^q_m(\mathbb{R}^N) \) implies that \( W_\delta \in L^s(\mathbb{R}^N) \) for any \( s \in [1, q] \) and moreover that
\[|W^R_\delta|_s \to 0\] as \(R \to +\infty\) for any \(\delta > 0\). Let us define also \(\Gamma^R_\delta = W_\delta - W^R_\delta\). It is clear that
\[
0 \leq (W - W_\delta)(|y|) \leq \delta, \quad 0 \leq \Gamma^R_\delta(|y|) \leq R \quad \forall y \in \mathbb{R}^N
\]
Then for any given \(\delta > 0\) and \(R > 0\) and for some \(s \geq N/2\) (which implies that \(2 < 4s/(2s-1) \leq 2N/(N-1)\)) we get from the Young inequality (also taking into account that by Sobolev embedding the sequence \(\{w_n\}\) is bounded in \(L^p\) for \(p \in [2, 2N/(N-1)]\))
\[
\int_{\mathbb{R}^N} (W * |w_n|^2) |w_n|^2 \leq \int_{\mathbb{R}^N} ((W - W_\delta) * |w_n|^2) |w_n|^2 + \int_{\mathbb{R}^N} (W^R_\delta * |w_n|^2) |w_n|^2
\]
\[
+ \int_{\mathbb{R}^N} (\Gamma^R_\delta * |w_n|^2) |w_n|^2
\]
\[
\leq \delta |w_n|^4 + |W^R_\delta|_s |w_n|^4_{4s/(2s-1)} + R \int_{\mathbb{R}^N} |w_n(y)|^2 |w_n(z)|^2 \delta_{|z-y| \leq R} dy dz
\]
\[
\leq \delta M^2 + C|W^R_\delta|_s + RM \sup_{z \in \mathbb{R}^N} \int_{z+B_R} |w_n|^2 dy.
\]
Now, letting first \(n \to +\infty\), then \(R \to +\infty\) and finally \(\delta \to 0^+\) we conclude the proof of the lemma. \(\square\)

\textbf{Lemma 3.10.} \textit{If dichotomy occurs, then for any \(\alpha \in (0, M)\) we have}
\[
I(M) \geq I(\alpha) + I(M - \alpha).
\]

\textit{Proof of lemma 3.10.} If dichotomy occurs then there is a sequence \(\{u_k\} \subset \mathbb{N}\) such that for any \(\epsilon > 0\) there exists \(R > 0\) and a sequence \(\{z_k\} \subset \mathbb{R}^N\) such that
\[
\lim_{k \to +\infty} \int_{z_k+B_R} |w_{n_k}|^2 dy \in (\alpha - \epsilon, \alpha + \epsilon).
\]
Let define \(\tilde{w}_k = w_{n_k}(\cdot + z_k)\) and
\[
\tilde{u}_k(x, y) = F^{-1}(e^{-\frac{1}{2}m^2|y|^2} \mathcal{F}(\tilde{w}_k))
\]
so that \(\{\tilde{u}_k\}\) is a minimizing sequence for \(I\) on \(\mathcal{M}_M\) such that
\[
\lim_{k \to +\infty} \int_{B_R} |\gamma(\tilde{u}_k)|^2 dy \in (\alpha - \epsilon, \alpha + \epsilon).
\]
Since \(\{\tilde{u}_k\}\) is a bounded sequence in \(H^1(\mathbb{R}^{N+1})\) then \(\tilde{u}_k \to u\) weakly in \(H^1(\mathbb{R}^{N+1})\) and \(\tilde{w}_y = \gamma(\tilde{u}_k) \to w = \gamma(u)\) weakly in \(H^{1/2}\) and strongly in \(L^p_{\text{loc}}(\mathbb{R}^N)\) for \(p \in [2, 2N/(N-1)]\). Hence for all \(\epsilon > 0\) there is \(R > 0\) such that
\[
\int_{B_R} |\gamma(u)|^2 dy = \lim_{k \to +\infty} \int_{B_R} |\gamma(\tilde{u}_k)|^2 dy \in (\alpha - \epsilon, \alpha + \epsilon).
\]
and
\[
\int_{\mathbb{R}^N} |\gamma(u)|^2 dy = \lim_{R \to +\infty} \int_{B_R} |\gamma(u)|^2 dy = \alpha.
\]
We set \(v_k = \tilde{u}_k - u\) and \(\beta_k = \int_{\mathbb{R}^N} |\gamma(v_k)|^2 dy\), by weak convergence in \(L^2\) of the sequence \(\{\gamma(\tilde{u}_k)\}\) we get \(\lim_{k \to +\infty} \beta_k = M - \alpha\).
Now we claim that
\[
I(M) = \lim_{k \to +\infty} I(\tilde{u}_k) = I(u) + \lim_{k \to +\infty} I(v_k) \geq I(\alpha) + \lim_{k \to +\infty} I(\beta_k)
\]
and by the continuity of the function \(I\), as stated in lemma 3.9 the lemma follows.
Now let us prove the claim. We will show that
\[
\lim_{k \to +\infty} (I(\tilde{u}_k) - I(v_k)) \to I(u)
\]
Indeed by weak convergence in $H^1(\mathbb{R}^{N+1}_+)$ we immediately get

$$\lim_{k \to +\infty} \left( \int_{\mathbb{R}^{N+1}_+} |\nabla \tilde{u}_k|^2 - \int_{\mathbb{R}^{N+1}_+} |\nabla v_k|^2 \right) = \int_{\mathbb{R}^{N+1}_+} |\nabla u|^2$$

$$\lim_{k \to +\infty} \left( \int_{\mathbb{R}^{N+1}_+} |\tilde{u}_k|^2 - \int_{\mathbb{R}^{N+1}_+} |v_k|^2 \right) = \int_{\mathbb{R}^{N+1}_+} |u|^2$$

and by the Brezis-Lieb lemma

$$\lim_{k \to +\infty} \left( \int_{\mathbb{R}^N} |\gamma(\tilde{u}_k)|^p - \int_{\mathbb{R}^N} |\gamma(v_k)|^p \right) = \int_{\mathbb{R}^N} |\gamma(u)|^p$$

for $2 \leq p \leq 2N/(N - 1)$.

Hence we have to investigate the last nonlinear term. We will show in the Appendix A that

$$\lim_{k \to +\infty} \left( \int_{\mathbb{R}^N} (W * |\tilde{u}_k|^2) |\tilde{u}_k|^2 - \int_{\mathbb{R}^N} (W * |\gamma(v_k)|^2) |\gamma(v_k)|^2 \right) = \int_{\mathbb{R}^N} (W * |w|^2) |w|^2,$$

from which the claim follows. \[\square\]

Finally, since we have ruled out both vanishing and dichotomy, then we may conclude that indeed compactness occurs, namely that for all $\epsilon > 0$ there is $R > 0$, a sequence $\{y_k\}$ and a subsequence $\{w_{nk}\}$ such that

$$\int_{y_k + B_R} |w_{nk}|^2 \, dy \geq M - \epsilon.$$ 

So let us define as before $\tilde{w}_k = w_{nk} (\cdot + y_k)$ and

$$\tilde{u}_k(x, y) = F^{-1}(e^{-x\sqrt{w_{nk}^2 + 2^2}} F(\tilde{w}_k)).$$

Then $\tilde{u}_k$ is a minimizing sequence for $I$ on $\mathcal{M}_M$ such that

$$\int_{B_R} |\gamma(\tilde{u}_k)|^2 \geq M - \epsilon.$$ 

Since $\{\tilde{u}_k\}$ is a bounded sequence in $H^1(\mathbb{R}^{N+1}_+)$ then $\tilde{u}_k \to u$ weakly in $H^1(\mathbb{R}^{N+1}_+)$ and $\tilde{w}_k = \gamma(\tilde{u}_k) \to w = \gamma(u)$ weakly in $H^{1/2}$ and strongly in $L^p_{\text{loc}}(\mathbb{R}^N)$ for $p \in [2, 2N/(N - 1)]$. As in the proof of lemma 3.10 we deduce that $\int_{\mathbb{R}^N} |\gamma(u)|^2 = M$.

Moreover we claim that as $k \to +\infty$

$$\int_{\mathbb{R}^N} (W * |\tilde{w}_k|^2) |\tilde{w}_k|^2 \to \int_{\mathbb{R}^N} (W * w^2) w^2.$$

Indeed, by the weak Young inequality and by Hölder inequality we have

$$\left| \int_{\mathbb{R}^N} (W * \tilde{w}_k^2) \tilde{w}_k^2 - \int_{\mathbb{R}^N} (W * w^2) w^2 \right| \leq \int_{\mathbb{R}^N} (W * (\tilde{w}_k^2 + w^2)) |\tilde{w}_k^2 - w^2|$$

$$\leq C ||W||_{q,w} |\tilde{w}_k^2 + w^2|_{s} |\tilde{w}_k^2 - w^2|_{s} \leq C |\tilde{w}_k - w|_{2s} \to 0$$

since $2 < 2s < 4q/(2q - 1) < 2N/(N - 1)$.

Hence finally by weakly lower semicontinuity of $H^1$ and $L^p$ norms (the positive terms of the functional $I$) we may conclude that

$$I(u) \leq \liminf_{k \to +\infty} I(\tilde{u}_k) = I(M)$$

which implies the $u$ is a minimizer for $I$ in $\mathcal{M}_M$. \[\square\]

Now we collect all the results obtained to conclude the proof of Theorem 1.8.
Proof of Theorem 1.8. By proposition 3.8 there exists a function $u \in H^1(\mathbb{R}^{N+1})$ which minimizes $I$ in $\mathcal{M}_M$. Therefore $u$ can always be assumed nonnegative and, by lemma 2.8, of the form

$$ u(x, y) = F^{-1}(e^{-\frac{y}{m+1} F(w)}) $$

where $w = \gamma(u) \in H^{1/2}(\mathbb{R}^N)$.

If $W$ is a nonincreasing radial function, then $w$ can be assumed to be a radial nonincreasing function. Indeed let $w^*$ be the spherically symmetric decreasing rearrangement of $w$ and define

$$ u^*(x, y) = F^{-1}(e^{-\frac{y}{m+1} F(w^*)}). $$

Then $I(u^*) = E[w^*]$ (also this follows from lemma 2.8). We can then use the properties of the spherically symmetric decreasing rearrangement, namely

(i) $w^*$ is a nonincreasing, radial function;

(ii) $w \in L^p(\mathbb{R}^N)$ implies $w^* \in L^p(\mathbb{R}^N)$ and $|w^*|_p = |w|_p$;

(iii) symmetric decreasing rearrangement decreases kinetic energy (Lemma 7.17 in [10]), that is

$$ \int_{\mathbb{R}^N} w^*(\sqrt{-\Delta + m^2} - m)w^* \, dy \leq \int_{\mathbb{R}^N} w(\sqrt{-\Delta + m^2} - m)w \, dy; $$

(iv) Riesz’s rearrangement inequality (see Theorem 3.7 in [10]), namely

$$ \int_{\mathbb{R}^N} (W * |w^*|^2)w^* \, dy \geq \int_{\mathbb{R}^N} (W * |w|^2)|w|^2 \, dy $$

if $W(y) = W^*(|y|)$ (in particular if $W$ is radial and nonincreasing)

to deduce that

$$ I(u^*) = E[w^*] \leq E[w] = I(u) = I(M). $$

Moreover, by the theory of Lagrange multipliers, any minimizer $u \in H^1(\mathbb{R}^{N+1})$ of the functional $I$ on $\mathcal{M}_M$ is such that

$$ \begin{aligned}
\int_{\mathbb{R}^{N+1}} (\nabla u \nabla w + m^2 uw) \, dx \, dy &- \int_{\mathbb{R}^N} m\gamma(u)\gamma(w) \, dy + \mu \int_{\mathbb{R}^N} \gamma(u)\gamma(w) \, dy \\
+ \eta \int_{\mathbb{R}^N} |\gamma(u)|^{p-2} \gamma(u)\gamma(w) \, dy - \sigma \int_{\mathbb{R}^N} (W * |\gamma(u)|^2)\gamma(u)\gamma(w) \, dy &\leq 0
\end{aligned} \tag{3.11} $$

for all $w \in H^1(\mathbb{R}^{N+1})$, i.e. $u$ is a weak solution of the following nonlinear Neumann boundary condition problem

$$ \begin{aligned}
-\Delta u + m^2 u &= 0 \quad \text{in } \mathbb{R}^{N+1} \\
-\frac{\partial u}{\partial \nu} + \mu u &= \mu u - \eta |u|^{p-2} u + \sigma (W * |u|^2)u \quad \text{on } \partial \mathbb{R}^{N+1}
\end{aligned} \tag{3.12} $$

for some Lagrange multiplier $\mu \in \mathbb{R}$. To prove that $\mu > 0$ we take $w = u$ in (3.11) to get

$$ \begin{aligned}
0 &= \int_{\mathbb{R}^{N+1}} (|\nabla u|^2 + m^2 |u|^2) \, dx \, dy - \int_{\mathbb{R}^N} m|\gamma(u)|^2 \, dy + \mu \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy \\
+ \eta \int_{\mathbb{R}^N} |\gamma(u)|^{p-2} \gamma(u) \, dy - \sigma \int_{\mathbb{R}^N} (W * |\gamma(u)|^2) |\gamma(u)|^2 \, dy \\
= 2I(u) + \mu \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy + \eta(1 - \frac{2}{p}) \int_{\mathbb{R}^N} |\gamma(u)|^p \, dy \\
- \frac{\sigma}{2} \int_{\mathbb{R}^N} (W * |\gamma(u)|^2) |\gamma(u)|^2 \, dy.
\end{aligned} $$
Since $I(u) < 0$ we have in particular that
\[
\frac{2}{p} \int_{\mathbb{R}^N} |\gamma(u)|^p \, dy < \frac{\sigma}{4} \int_{\mathbb{R}^N} (W \ast |\gamma(u)|^2) |\gamma(u)|^2 \, dy
\]
and hence, since $p \leq 2N/(N - 1) \leq 4$, for $N \geq 2$, we get
\[
\mu \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy
\]
\[
= -2I(u) - \eta(1 - \frac{2}{p}) \int_{\mathbb{R}^N} |\gamma(u)|^p + \frac{\sigma}{2} \int_{\mathbb{R}^N} (W \ast |\gamma(u)|^2) |\gamma(u)|^2 \, dy
\]
\[
> \eta(\frac{4}{p} - 1) \int_{\mathbb{R}^N} |\gamma(u)|^p \, dy \geq 0.
\]

Finally the regularity, the strictly positivity and the exponential decay at infinity of the weak nonnegative solutions of (3.12) follow straightforwardly from Theorems 3.14 and 5.1 in [2].

\[\square\]

4. APPENDIX A

We prove that
\[
\int_{\mathbb{R}^N} |(W \ast w\gamma(v_k))w\gamma(v_k)| + \int_{\mathbb{R}^N} |(W \ast \gamma(v_k)^2)w^2| + \int_{\mathbb{R}^N} |W \ast w\gamma(v_k)w| + \int_{\mathbb{R}^N} |(W \ast \gamma(v_k)^2)w\gamma(v_k)| \to 0 \quad \text{as } k \to +\infty.
\]
as claimed in the proof of lemma 3.10. Indeed we have the following result.

Lemma 4.1. For any $w \in H^{1/2}([\mathbb{R}^N])$ and for sequences $\{f_n, g_n, h_n\}$ bounded in $H^{1/2}([\mathbb{R}^N])$ and such that $f_n \to 0$ in $L^2_{\text{loc}}$ we have
\[
\int_{\mathbb{R}^N} (W \ast |f_n g_n|) |w h_n| \to 0 \quad \text{as } n \to +\infty.
\]

Proof. It is convenient to introduce for any given $\delta > 0$ and $R > 0$, $W_\delta = W \mathbb{I}_{W \geq \delta}$ and
\[
W_\delta^R(y) = (W_\delta - R)^+ \mathbb{I}_{|y| < R} + W_\delta^1 \mathbb{I}_{|y| \geq R}.
\]
Then for $W \in L^p_{\text{loc}}([\mathbb{R}^N])$ we have $W_\delta \in L^p([\mathbb{R}^N])$ for any $p \in [1, \infty)$ and moreover that $|W_\delta^R|^p \to 0$ as $R \to +\infty$ for any $\delta > 0$. Let introduce again also $\Gamma_\delta^R = \delta W_\delta - W_\delta^R$. Note that $\supp \Gamma_\delta^R \subset B_R$ and $0 \leq \Gamma_\delta^R \leq R$.

From Young inequality (with $p = N/2$, $r = 2/(2p - 1) = N/(N - 1)$), Hölder inequality and Sobolev embedding we have
\[
\int_{\mathbb{R}^N} (W \ast |f_n g_n|) |w h_n| \leq \int_{\mathbb{R}^N} ((W - W_\delta) \ast |f_n g_n|) |w h_n| + \int_{\mathbb{R}^N} (W_\delta^R \ast |f_n g_n|) |w h_n|
\]
\[
+ \int_{\mathbb{R}^N} (W_\delta \ast |f_n g_n|) |w h_n|
\]
\[
\leq \delta |f_n g_n|_1 |w h_n|_1 + |W_\delta^R|_{N/2} |f_n g_n|_r |w h_n|_r + \int_{\mathbb{R}^N} (\Gamma_\delta^R \ast |f_n g_n|) |w h_n|
\]
\[
\leq C(\delta + |W_\delta^R|_{N/2}) + \int_{\mathbb{R}^N} (\Gamma_\delta^R \ast |f_n g_n|) |w h_n|.
\]

First of all we claim that
\[
\int_{\mathbb{R}^N} (\Gamma_\delta^R \ast |f_n g_n|) |w h_n| \to 0 \quad \text{as } n \to +\infty.
\]
Indeed, for any $\epsilon > 0$ we fix $R_1 > 0$ such that $|\mathcal{I}_{\mathbb{R}^N \setminus B_1} w|_2 < \epsilon$, where $B_1 = B_{R_1}$.

We introduce also $R_2 = R_1 + R$ and $B_2 = B_{R_2}$ so that for any $y \in B_1$ and $z \in \mathbb{R}^N \setminus B_2$, we have $|z - y| \geq R$ and hence $\Gamma^R_{\delta}(z - y) = 0$.

Now we estimate the term as follows

$$
\int_{\mathbb{R}^N} (\Gamma^R \ast |f_n g_n|) |w h_n| = \int_{B_1} (\Gamma^R \ast (|B_2| f_n g_n)) |w h_n| + \int_{\mathbb{R}^N \setminus B_1} (\Gamma^R \ast |f_n g_n|) |w h_n| \\
\leq R(|B_2| f_n g_n) |w h_n| + |\Gamma^R |(f_n g_n)|_{\infty} |\mathcal{I}_{\mathbb{R}^N \setminus B_1} h_n|_2 |\mathcal{I}_{\mathbb{R}^N \setminus B_1} w|_2 \\
\leq R|g_n|_2 |h_n|_2 (|B_2 f_n|_2 |w|_2 + R|f_n|_2 |\mathcal{I}_{\mathbb{R}^N \setminus B_1} w|_2) \\
\leq CR(|B_2 f_n|_2 + |\mathcal{I}_{\mathbb{R}^N \setminus B_1} w|_2)
$$

and since $f_n \to 0$ as $n \to +\infty$ in $L^2(B_2)$ the claim is proved.

Then we conclude the proof of the lemma sending first $n \to +\infty$, then $R \to +\infty$ and finally $\delta \to 0$.

\[\square\]

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