MAXIMAL ACCELERATION IS NONROTATING *

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Abstract
In a stationary axisymmetric spacetime, the angular velocity of a stationary observer that Fermi-Walker transports its acceleration vector is also the angular velocity that locally extremizes the magnitude of the acceleration of such an observer, and conversely if the spacetime is also symmetric under reversing both $t$ and $\varphi$ together. Thus a congruence of Nonrotating Acceleration Worldlines (NAW) is equivalent to a Stationary Congruence Accelerating Locally Extremely (SCALE). These congruences are defined completely locally, unlike the case of Zero Angular Momentum Observers (ZAMOs), which requires knowledge around a symmetry axis. The SCALE subcase of a Stationary Congruence Accelerating Maximally (SCAM) is made up of stationary worldlines that may be considered to be locally most nearly at rest in a stationary axisymmetric gravitational field. Formulas for the angular velocity and other properties of the SCALEs are given explicitly on a generalization of an equatorial plane, infinitesimally near a symmetry axis, and in a slowly rotating gravitational field, including the far-field limit, where the SCAM is shown to be counterrotating relative to infinity. These formulas are evaluated in particular detail for the Kerr-Newman metric. Various other congruences are also defined, such as a Stationary Congruence Rotating at Minimum (SCRAM), and Stationary Worldlines Accelerating Radially Maximally (SWARM), both of which coincide with a SCAM on an equatorial plane of reflection symmetry. Applications are also made to the gravitational fields of maximally rotating stars, the Sun, and the Solar System.

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1 Introduction

A stationary axisymmetric spacetime has various preferred congruences of stationary observers, such as those whose four-velocities are parallel to the Killing vector field that is timelike at radial infinity (wordlines nonrotating relative to infinity), or those whose four-velocities are perpendicular to the Killing vector field which has closed orbits and which vanishes on the symmetry axis (Zero Angular Momentum Observers, or ZAMOs) [1, 2]. Here a new preferred congruence is defined (SCAM, a special case of SCALE = NAW) in terms of the purely local properties of the commuting Killing vector fields, without reference to what they do elsewhere (e.g., at radial infinity or around the symmetry axis).

Using the MTW sign conventions [2] — in particular, the metric sign convention (++++) — and the same boldface symbols for vectors and for the corresponding one-forms that have components obtained by using the metric tensor to lower the vector components, consider a region of spacetime with two independent Killing vector fields, vector fields

\[ k = k^\alpha \partial / \partial x^\alpha, \quad l = l^\alpha \partial / \partial x^\alpha, \]  

(1)

that are independent (not obeying \( a k + b l = 0 \) for any constants \( a \) and \( b \) not both zero) and whose corresponding 1-form components

\[ k_\alpha = g_{\alpha \beta} k^\beta, \quad l_\alpha = g_{\alpha \beta} l^\beta, \]  

(2)

obey Killing’s equation,

\[ k_{\alpha;\beta} = -k_{\beta;\alpha}, \quad l_{\alpha;\beta} = -l_{\beta;\alpha}. \]  

(3)

Assume that these two Killing vector fields \( k \) and \( l \) also have the following three additional properties (though only the first two properties are necessary for the first part of the theorem to be proved):

1. The 2-form

\[ A = k \wedge l \]  

(4)

corresponding to the Killing bivector is timelike, obeying

\[ A^{\alpha \beta} A_{\alpha \beta} = 2(k \cdot k)(l \cdot l) - 2(k \cdot l)^2 < 0. \]  

(5)

Then in each sufficiently small neighborhood one can redefine, if necessary, \( k \) and \( l \) to be two new independent linear combinations of the original Killing vectors such that \( l \) is spacelike and the orthogonal vector field \((l \cdot l)k - (k \cdot l)l\), which by Eq. (4) is necessarily timelike, is future pointing (by choosing the appropriate sign for \( k \)). Then the linear combination

\[ K = k + \Omega l \]  

(6)

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is future-pointing timelike at each point in the neighborhood for some finite range of the constant $\Omega$, say $\Omega_{-1} < \Omega < \Omega_{+1}$.

(2) The two Killing vector fields commute,

$$[k, l] = 0,$$

or, in component form,

$$k^\alpha l^\beta_{\alpha} = l^\alpha k^\beta_{\alpha}.$$  \hfill (8)

This implies that one can choose two of the four coordinates, say $x^0 = t$ and $x^1 = \varphi$, such that

$$k = \partial/\partial t, \quad l = \partial/\partial \varphi.$$  \hfill (9)

(3) The 2-form $A = k \wedge l$ obeys

$$* A \wedge *dA = 0,$$  \hfill (10)

which, when $A \neq 0$ holds, as is implied by Eq. (5), is equivalent to the orthogonally transitive or circularity condition $[3]$

$$k \wedge l \wedge dk = k \wedge l \wedge dl = 0.$$  \hfill (11)

In component form it is equivalent to

$$A_{\mu[\alpha A_{\beta\gamma;\delta}]} = 0$$  \hfill (12)

or

$$k_{[\alpha l_{\beta k_{\gamma;\delta}]} = l_{[\alpha k_{\beta l_{\gamma;\delta}]} = 0.$$  \hfill (13)

This condition is equivalent to the condition that one may construct in a local neighborhood a family of two-surfaces orthogonal to both Killing vector fields $[3, 4]$. One may define two coordinates, say $x^a = (r, \theta)$ for $a = 2, 3$, on these two-surfaces such that orbits of the Killing vectors $k$ and $l$, and hence of all stationary observers, each stay at fixed $x^a$. Then $\partial/\partial r$ and $\partial/\partial \theta$ are both orthogonal to $k = \partial/\partial t$ and to $l = \partial/\partial \varphi$, so in this coordinate basis the metric tensor has no components mixing the first two (0 or 1) and the last two (2 or 3) indices. I.e., it is block diagonal. As a result, $A$ may be written as

$$A = -D dt \wedge d \varphi,$$  \hfill (14)

where

$$- D \equiv g_{00}g_{11} - g_{01}g_{10} \equiv g_{tt}g_{\varphi\varphi} - g_{t\varphi}g_{\varphi t} = 1/2 A^{\alpha\beta} A_{\alpha\beta} = (k \cdot k)(l \cdot l) - (k \cdot l)^2 < 0.$$  \hfill (15)

is the determinant of the first two-dimensional block of the metric.
Another simple way to state this third condition is to say that the spacetime is invariant under the simultaneous reversal of both coordinates $t$ and $\varphi$. This follows from the block diagonality of the metric, and it implies that each of the quantities in Eq. (11) are zero, since they are odd under this transformation.

The most important examples of spacetimes with two independent commuting Killing vectors obeying these two properties are asymptotically flat stationary axisymmetric spacetimes with the Ricci tensor obeying

$$k_{\mu}R_{\mu[\alpha}k_{\beta]\gamma] = l_{\mu}R_{\mu[\alpha}k_{\beta]\gamma] = 0,$$

which implies that property (3) above holds, though Eq. (16) just by itself does not imply property (3). In such spacetimes one may uniquely choose the Killing vector fields such that $k = \partial/\partial t$ is a unit timelike vector field at radial infinity and $l = \partial/\partial \varphi$ is a spacelike vector that vanishes on the symmetry axes (e.g., at $\theta = 0$ or $\theta = \pi$) and has closed orbits with period $\Delta \varphi = 2\pi$. However, the results below apply more generally, assuming only that $k$ and $l$ have $k \wedge l$ timelike and obey Killing’s Eq. (\ref{eq:killing-eq}), the commutativity condition $[k, l] = 0$, and the two-surface-orthogonality condition $*(k \wedge l) \wedge *d(k \wedge l) = 0$.

An observer whose four-velocity is

$$u = (-K \cdot K)^{-1/2}K$$

with $K = k + \Omega l$ with fixed $\Omega = d\varphi/dt$ (which shall be called the angular velocity, since that is what it for a stationary axisymmetric spacetime) may be defined to be a stationary observer (SO). A stationary congruence of observers (SCO) is a space-filling family of observers (one crossing each point of each local spatial hypersurface in the region of spacetime under consideration) with four-velocities

$$u_C = \frac{(k + \Omega(x^a)l)}{|k + \Omega(x^a)l|}$$

that have the angular velocity $\Omega$ depending only on the two $x^a$ coordinates that stay fixed along each worldline.

In order to calculate the acceleration vector

$$a = \nabla uu$$

with $\Omega$ constant along the worldline, it is convenient to consider the Killing vector field $K = k + \Omega l$ with this same $\Omega$ fixed as a constant everywhere (and not having the spatial variation with $x^a$ that a stationary congruence of different worldlines at different $x^a$ might have). If for this fixed ($x^a$-independent) $\Omega$ one defines the scalar field

$$\Phi \equiv \Phi(\Omega) \equiv \frac{1}{2} \ln(-K \cdot K).$$
then

\[ u = e^{-\Phi}K \]  

is, over the region of spacetime where \( K \) is (future-pointing) timelike, the four-velocity of a rigidly rotating stationary congruence of observers, differing from the four-velocities \( u_C \) of the congruence given by Eq. (18), where \( \Omega \) is allowed to be a function of \( x^a \) (i.e., different angular velocities for different stationary observers within the congruence, though I am always taking the angular velocity \( \Omega \) to be fixed for a given observer).

Then the antisymmetry of the covariant derivative of the Killing vector field \( K \) implies that the covariant components of the acceleration vector are

\[
a_\alpha = u^\beta u_{\alpha;\beta} = e^{-\Phi}K^\beta(e^{-\Phi}K_\alpha;\beta)
= e^{-2\Phi}(K^\beta K_{\alpha;\beta} - K^\beta K_\alpha \Phi;\beta)
= e^{-2\Phi}[-K^\beta K_{\beta;\alpha} - K^\beta K_\alpha \frac{1}{2}(-K^{\mu}K_{\mu};\beta)/(\Omega K_{\nu})]
= e^{-2\Phi}[-\frac{1}{2}(K^\beta K_{\beta})_{;\alpha} + K_\alpha K^\beta K^{\mu}K_{\mu;\beta}/(\Omega K_{\nu})]
= e^{-2\Phi}[-\frac{1}{2}(-e^{2\Phi})_{;\alpha} + 0]
= \Phi_{;\alpha},
\]

which are nonzero only for \( \alpha = a = 2 \) or 3. That is,

\[ a = \nabla \Phi \]  

is perpendicular to both \( k \) and \( l \). This fact requires property (2) but not property (3) above.

2 Stationary Congruence Accelerating Locally Extremely (SCALE)

Now for each value of the pair of coordinates \( x^a \), we would like to find the value of \( \Omega \) that extremizes the magnitude of the acceleration of the corresponding stationary observer. For this purpose, it is convenient to define the following scalar fields (functions of \( x^a \)):

\[
A \equiv -k \cdot k = -g_{00} \equiv -g_{tt},
\]

\[
B \equiv -k \cdot l = -g_{01} \equiv -g_{t\phi},
\]

\[
C \equiv -l \cdot l = -g_{11} \equiv -g_{\phi\phi},
\]

\[
F \equiv F(\Omega) \equiv e^{2\Phi} \equiv -k \cdot K = A + 2\Omega B + \Omega^2 C,
\]

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\[ G \equiv G(\Omega) \equiv \frac{1}{2} \frac{\partial}{\partial \Omega} F(\Omega) \equiv -\mathbf{K} \cdot \mathbf{l} = B + \Omega C, \]  

where the partial derivative with respect to \( \Omega \) is at fixed \( x^a \) and so at fixed \( A, B, \) and \( C \). Since we have chosen \( \mathbf{l} \) to be spacelike, \( C < 0 \). If the source of the axisymmetric gravitational field is entirely rotating in the same direction, \( B \) will typically have the sign of this direction everywhere, which by the appropriate choice of the sign of the coordinate \( \phi \) can be chosen to be positive. \( A \) will be positive outside any ergospheres but negative inside, if any exist. However, property (2) above implies that we are outside any Killing horizon, so

\[ D \equiv B^2 - AC = -\frac{1}{2} A^{\alpha\beta} A_{\alpha\beta} > 0. \]  

Then

\[ \Phi = \frac{1}{2} \ln F, \]  

so

\[ a = \nabla \Phi = \frac{\nabla F}{2F} = \frac{\nabla A + 2\Omega \nabla B + \Omega^2 \nabla C}{2(A + 2\Omega B + \Omega^2 C)}, \]  

and thus

\[ \frac{\partial a}{\partial \Omega} = \nabla \left( \frac{G}{F} \right) = \frac{(A \nabla B - B \nabla A) + \Omega(A \nabla C - C \nabla A) + \Omega^2(B \nabla C - C \nabla B)}{(A + 2\Omega B + \Omega^2 C)^2}, \]  

using \([\mathbf{K}, \mathbf{l}] = 0\).

Setting \( \partial a^2/\partial \Omega \) to zero at each point gives an extremum of the magnitude of the acceleration of a stationary observer there, with \( \Omega \) thus obeying the equation

\[ [\nabla A + 2\Omega \nabla B + \Omega^2 \nabla C] \cdot [(A \nabla B - B \nabla A) + \Omega(A \nabla C - C \nabla A) + \Omega^2(B \nabla C - C \nabla B)] = 0. \]  

Expanded out in powers of the angular velocity \( \Omega \), this is a quartic equation for \( \Omega(x^a) \), with coefficients that are combinations of \( A \equiv -g_{00}, B \equiv -g_{01}, C \equiv -g_{11}, \) and dot products of their gradients.

A congruence of stationary worldlines corresponding to one of the roots of Eq. (34) for \( \Omega(x^a) \) at each \( x^a \) might be called a Stationary Congruence Accelerating Locally Extremely (SCALE), and if the local extremum of the acceleration (as a
function of $\Omega$) is a (local) maximum, the congruence might be called a Stationary Congruence Accelerating Maximally (SCAM). Since in the frame of the device (e.g., a rocket) accelerating the observer, the magnitude of the acceleration may be interpreted as the apparent weight or heaviness of the observer (e.g., as in saying that an observer in free fall along a geodesic is "weightless"), one might say that an observer moving along a SCALE is "extremely heavy," taking extreme to mean either a local maximum (for a SCAM) or a minimum (for the other SCALEs).

(Generically there is no global maximum for the acceleration, since it can be made arbitrarily large by making $\Omega$ arbitrarily near one of the two endpoints $\Omega_{-1}$ and $\Omega_{+1}$ of its allowed range,

$$\Omega_{\pm 1} = \frac{B \pm \sqrt{B^2 - AC}}{-C},$$

where $F$ goes to zero and hence $K = k + \Omega l$ becomes null, unless this endpoint corresponds to a null geodesic where $\nabla F$ also goes to zero, in which case the acceleration stays finite. This last fact uses property (1) in the form of Eq. (29), $D = B^2 - AC > 0$, so that the two endpoints have a nonzero separation in $\Omega$, and hence $F$, as a quadratic polynomial of $\Omega$ given by Eq. (27), has only a simple zero at each end and cannot give infinity when divided into $\nabla F$ if the latter also has a zero at the corresponding endpoint.)

Although typically $a^2$ thus has no global maximum within the allowed range of $\Omega$ where $K$ is timelike, there are usually (at least in weak gravitational fields) two local minima for $a^2$ and one local maximum between these two minima, though it is also possible in strong gravitational fields to have only one local minimum and no local maximum.

For a stationary axisymmetric spacetime in a region of weak gravity, one of the roots of the quartic Eq. (34) in $\Omega(x^a)$ corresponds to an imaginary spatial four-velocity $u = i(K \cdot K)^{-1/2}K$ (unphysical), and the other three roots correspond to real timelike four-velocities, with the two outer roots (say $\Omega_-$ and $\Omega_+$) giving local minima of the acceleration and the root in between (say $\Omega_0$) giving a local maximum. In a static spacetime, the local maximum of the acceleration occurs for a static worldline, at $\Omega_0 = 0$, accelerating against the pull of gravity to stay at a fixed position. In a Newtonian description in which $l = \partial/\partial \varphi$ vanishes along the $z$-axis, the two local minima of the acceleration occur at the angular velocities $\Omega_{\pm}$ at which the centrifugal acceleration (in the $x$-$y$ plane) balances the component of the gravitational acceleration that is anti-parallel to it, leaving only an unbalanced $z$-component of the gravitational acceleration. For a stationary axisymmetric spacetime that has a reflection symmetry about an equatorial plane, in that plane there is no other component of the gravitational acceleration, so in the equatorial plane the local minima actually have zero acceleration and correspond to stationary geodesics or circular Keplerian orbits at the corresponding $\Omega_{\pm}$. 

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For a nonstatic (e.g., rotating) stationary axisymmetric spacetime, the value \( \Omega_0 \) of the angular velocity that gives the local maximum of the acceleration \((\Omega_- < \Omega_0 < \Omega_+)\) gives a local definition of a congruence, the SCAM, that in a local sense can be considered to be the most nearly at rest. Any slightly different rotation rate \( \Omega \) would give a change in the centrifugal acceleration and/or gravitational acceleration that would reduce the total acceleration. (In the Newtonian limit in which the gravitational acceleration is independent of the velocity, the reduction of the acceleration needed to balance gravity would be provided purely by the centrifugal acceleration, and in that limit, \( \Omega_0 = 0 \) is the angular velocity giving no centrifugal acceleration.)

As one enters regions of strong gravity (e.g., near a black hole, \( r < 3M \) for a Schwarzschild black hole), one of the roots \( \Omega_- \) or \( \Omega_+ \) may reach the corresponding endpoint \( \Omega_{-1} \) or \( \Omega_{+1} \), or it may merge with \( \Omega_0 \) and thence go complex, in either case disappearing from the allowed region of the real \( \Omega \) line (or all three roots may merge simultaneously in the nonrotating case), leaving only a single physical extremum (a minimum) for the acceleration. The fact that the acceleration then increases for a change in the velocity can be attributed to a reversal of the direction of the centrifugal acceleration \([11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]\) if one assumes (or defines) the gravitational acceleration to be independent of the velocity, or it can be attributed to a greater increase in the gravitational acceleration than that in the centrifugal acceleration \([23, 24, 25, 26, 27, 28, 29, 30, 31, 32]\) if one assumes (or defines) the gravitational acceleration to increase with velocity in the way (i.e., proportional to the square of the relativistic gamma factor, \( \propto \gamma^2 = 1/(1 - v^2) \)) that one would get for an object moving transversely across a spatially flat horizontal floor in a rocket whose vertical acceleration in flat spacetime simulates gravity inside by the equivalence principle.

3 Local definitions of stationary congruences

The definition of the SCAM has the advantage of locality over the definition of the congruence with \( \Omega = 0 \) (nonrotating with respect to infinity), which requires the definition of which linear combination (ignoring the overall normalization, which is irrelevant for the present purpose) of the two Killing vector fields is \( k \) (usually made by choosing the combination that remains timelike at spatial infinity), a definition that cannot be made locally but instead requires a knowledge of the behavior of the Killing vector fields out to spatial infinity.

The SCAM also has this same advantage, though to a lesser degree, over the ZAMOs (Zero Angular Momentum Observers), which are defined to be orthogonal to the Killing vector field and so require that that vector be uniquely picked out, again only up to normalization (typically by choosing the Killing vector field with closed orbits, which usually vanishes on a symmetry axis). This again requires
nonlocal knowledge, unless one is at the symmetry axis where \( l \) vanishes. Since property (1) implies that \( l \) not vanish, I am explicitly assuming that one is not at a symmetry axis, except for some discussions below where I take the limit of going there.

ZAMOs were originally called “locally nonrotating observers” \([1, 2]\), because they have angular velocities midway between \( \Omega_{-1} \) and \( \Omega_{+1} \), so that if two photons (in null but generically nongeodesic stationary orbits, say skimming along mirrors) were sent around both opposite directions from a ZAMO, they would both return to the ZAMO at the same time. This definition (essentially equivalent to defining \( l \) to be the combination of the Killing vector fields with closed orbits) is quasilocal in that it does not require a knowledge of the Killing vector fields out to spatial infinity, but it is still nonlocal in that it requires a knowledge of the fields along the stationary null orbits until they return to the ZAMO (i.e., all the way around the symmetry axis). On the other hand, the definition of the SCAM (when it exists) is completely local.

To state more precisely what the conditions are for a quantity to be local if it depends on the two Killing vector fields \( k \) and \( l \), note that locally (away from spatial infinity and from a symmetry axis) one has nothing that determines which vector in the entire \( k \wedge l \) plane of vectors is \( k \) and which other vector is \( l \). In fact, one can make a global redefinition by the constant linear transformation

\[
\begin{pmatrix} k \\ l \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{k} \\ \tilde{l} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} \alpha k + \beta l \\ \gamma k + \delta l \end{pmatrix}
\]

with constants \( \alpha, \beta, \gamma, \delta \).

If the four-velocity \( u \) given in Eqs. (17), (18), and (21) is to remain invariant under this redefinition of the Killing vector fields, the angular velocity \( \Omega \) must transform by the fractional linear transformation

\[
\Omega \rightarrow \tilde{\Omega} = \frac{\alpha \Omega - \beta}{\delta - \gamma \Omega}.
\]

Conversely, any four-velocity defined in terms of an angular velocity \( \Omega \) which does not transform by Eq. (37), when the Killing vectors are transformed by Eq. (36), is not locally determined. For example, the congruence that is nonrotating relative to infinity has \( \Omega = 0 \) when \( k \) is chosen to be the Killing vector field that remains timelike at spatial infinity. But local information does not determine this \( k \), and when one considers the transformation given by Eq. (36), \( \Omega = 0 \) is not invariant under Eq. (37) as it would need to be to be a locally determined condition.

The ZAMO is defined by the condition that

\[
0 = u \cdot l = e^{-\Phi}(k + \Omega_Z l) \cdot l = e^{-\Phi}(-B - \Omega_Z C),
\]

(38)
or
\[
\Omega_Z = -\frac{B}{C} \equiv -\frac{g_{01}}{g_{11}} \equiv -\frac{g_{\theta\phi}}{g_{\phi\phi}}.
\] (39)

However, this angular velocity, like \( \Omega = 0 \), does not transform according to Eq. (37) when \( l \) is transformed to a linear combination of itself and of \( k \), so it also does not locally determine a unique four-velocity \( u \).

To see this in detail and to prepare the way for recognizing which four-velocities \( u \) determined by the behavior of \( A, B, \) and \( C \) are locally determined and independent of the transformation (36), one should note that Eqs. (24) - (26) defining these quantities imply that they transform as
\[
\begin{pmatrix} A & B \\ B & B \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{C} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} A & B \\ B & B \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}
\]
\[
= \begin{pmatrix} \alpha^2 A + 2\alpha \beta B + \beta^2 C \\ \alpha \gamma A + (\alpha \delta + \beta \gamma)B + \beta \delta C \end{pmatrix} \begin{pmatrix} \gamma^2 A + 2\gamma \delta B + \delta^2 C \end{pmatrix}.
\] (40)

Therefore, if one makes a linear transformation of the Killing vector fields by Eq. (36) and then defines \( \tilde{\Omega}_Z \equiv -\tilde{B}/\tilde{C} \), one finds that when \( \gamma \neq 0 \), this \( \tilde{\Omega}_Z \) does not generally agree with the \( \tilde{\Omega}_Z \) one would get from applying Eq. (37) to \( \Omega_Z = -B/C \).

In contrast, other quantities and conditions are invariant under the transformations of Eq. (36) and (37), such as \( u \) (by construction, since the transformation of \( \Omega \) was designed to keep \( u \) invariant), \( a \) given by Eq. (23), the normalized two-form \(-\frac{1}{2}A^{\alpha\beta}A_{\alpha\beta}\) obtained from the two-form \( A = k \wedge l \) (which itself, like \( D^{1/2} \) from Eq. (15), gets multiplied by the determinant of the transformation matrix, \( \alpha \delta - \beta \gamma \)), and the condition given by Eq. (34) for the SCALEs. Thus the SCALEs are indeed locally determined congruences, unlike the nonrotating (relative to infinity) observers and the ZAMOs.

Incidentally, I might point out that there are other quantities, analogous to \( A \) and \( D^{1/2} \), that are not quite invariant under the transformations (36) and (37) but which for constant \( \Omega \) get multiplied by the constant \((\alpha \delta - \beta \gamma)/(\delta - \gamma \Omega)\), namely \( K \) and \( e^\Phi \equiv F^{1/2} \). Thus \( \Phi = (1/2) \ln (-K \cdot K) \) is not quite locally determined, but for a Killing vector field \( K \) that is fixed up to normalization (e.g., by being set parallel at some location to a stationary four-velocity \( u \) that is locally determined), \( \nabla \Phi \) (which is the acceleration of \( u \)) is locally determined.

4 Nonrotating Acceleration Worldlines (NAW) form a SCALE

Another way of locally selecting a preferred stationary congruence of observers is to require that their acceleration vectors be Fermi-Walker transported and hence be nonrotating relative to an ideal system of gyroscopes carried by the observers. In
other words, the direction that each nonweightless observer feels to be up stays fixed in his or her Fermi-Walker transported frame, so that one might state Delphically that each such observer is “fixed up” or “stays up.” Such a congruence might be called Nonrotating Acceleration Worldlines (NAW). Here I shall show that a NAW is equivalent to a SCALE (or SCAM in the case that the local extremum of the acceleration is a local maximum).

This equivalence is what one would expect in the Newtonian limit, since there, as discussed above, the SCAM has zero angular velocity ($\Omega_0 = 0$) and hence has an acceleration vector that stays constant. For observers that rotate, generically the acceleration vector also rotates, but for the SCALEs that are the two minima of the acceleration at angular velocities $\Omega_{\pm}$, the acceleration vector is purely in the $z$-direction, which does not rotate.

The Fermi-Walker derivative of a vector $\mathbf{v}$ along a worldline with four-velocity $\mathbf{u} = d/d\tau$ and acceleration $\mathbf{a} = \nabla \mathbf{u}$ is \[ \mathbf{v}' \equiv F_{\mathbf{u}}[\mathbf{v}] \equiv \nabla_{\mathbf{u}} \mathbf{v} - (\mathbf{u} \wedge \mathbf{a}) \cdot \mathbf{v} \equiv D \mathbf{v} / d\tau - \mathbf{u}(\mathbf{a} \cdot \mathbf{v}) + \mathbf{a}(\mathbf{u} \cdot \mathbf{v}), \] or, in component form, \[ v'_\alpha = u^\beta v_{\alpha;\beta} - u_\alpha a^\beta v_\beta + a_\alpha u^\beta v_\beta. \] This is zero by construction when $\mathbf{v}$ is the four-velocity $\mathbf{u}$. When $\mathbf{v}$ is the acceleration $\mathbf{a}$, the Fermi-Walker derivative is \[ a' \equiv F_{\mathbf{u}}[\mathbf{a}] = \nabla \mathbf{u} \mathbf{a} - a^2 \mathbf{u} \] (since $\mathbf{u} \cdot \mathbf{a} = 0$). A Nonrotating Acceleration Worldline or NAW (more strictly, in view of my definition above of a NAW, a member of a NAW congruence) is a worldline in which $a'$ is either zero or is parallel to $\mathbf{a}$ (i.e., $\mathbf{a} \wedge a' = 0$), so that in a frame carried along with the observer by Fermi-Walker transport, the acceleration vector $\mathbf{a}$ does not change direction. For a stationary observer, the magnitude of $\mathbf{a}$ stays fixed, so $a' = 0$ if the stationary observer is a member of a NAW.

Using the explicit formulas (17) and (22) for $\mathbf{u}$ and $\mathbf{a}$ with fixed $\Omega$, which imply \[ u_{(\alpha;\beta)} + u_{(\alpha} a_{\beta)} = 0, \] one can write this Fermi-Walker derivative of the acceleration of a stationary observer as \[ a' = \nabla \mathbf{u} \] or \[ a'_\alpha = a^\beta u_{\alpha;\beta} = e^{-5\Phi} K^\beta K^\gamma K^{\gamma;\delta}(K_{\beta K_{\alpha;\delta}} - K_{\alpha K_{\beta;\delta}}). \]
Now a comparison with Eq. (33) shows that
\[
\frac{\partial a^2}{\partial \Omega} = -4e^{-\Phi} l^\alpha a_\alpha' = -4e^{-\Phi} l \cdot a'.
\] (47)

Thus \( a' = 0 \) implies that the acceleration is an extremum with respect to \( \Omega \), so NAW \( \Rightarrow \) SCALE (stationary worldlines whose acceleration is nonrotating also have their acceleration a local extremum). The proof of the implication in this direction used only the assumptions that the four-velocity is a linear combination \( e^{-\Phi} K = e^{-\Phi} (k + \Omega l) \) of two Killing vector fields \( k \) and \( l \) that commute. Semerák [34], after announcing that he had heard of my result by personal communication, has independently given a concise proof in this direction, NAW \( \Rightarrow \) SCALE.

A mnemonic for remembering this result, using the somewhat cryptic shorthand phrases given above, is to say, “A stationary observer fixed up is extremely heavy,” or, “A stationary observer who stays up is extremely heavy.”

5 A SCALE is a NAW

To show the converse, that SCALE \( \Rightarrow \) NAW (stationary worldlines with locally extreme acceleration also have the acceleration nonrotating, which was not proved in [34]), we need to invoke not only the assumed commutativity of the two Killing vector fields \( k \) and \( l \), but also the assumed property (3), \( k \wedge l \wedge dk = k \wedge l \wedge dl = 0 \) for the corresponding 1-forms.

For this purpose it is convenient to define the Killing vector field \( L \) (a linear combination of \( k \) and \( l \)) that at the position of the stationary worldline under consideration has unit length and is orthogonal to the observer’s four-velocity \( u = e^{-\Phi} K \) there,
\[
L = \gamma k + \delta l,
\] (48)
where \( \gamma \) and \( \delta \) are constants that at the position of the stationary observer have the values
\[
\gamma = -\frac{e^{-\Phi}}{\sqrt{D}} G = \frac{-B - \Omega C}{\sqrt{(B^2 - AC)(A + 2\Omega B + \Omega^2 C)}},
\] (49)
\[
\beta = \frac{e^{-\Phi}}{\sqrt{D}} (F - \Omega G) = \frac{A + \Omega B}{\sqrt{(B^2 - AC)(A + 2\Omega B + \Omega^2 C)}}.
\] (50)

One can readily check from these formulas that at the position of the stationary observer, \( L \cdot L = 1 \) and \( L \cdot u = 0 \). Thus \( u \) and \( L \) are orthonormal vectors (or 1-forms, since I am using the same symbols for both vectors and the corresponding 1-forms, with the distinction, if necessary, being clear from the context) in the \( k \wedge l \) plane. Also, clearly \([k, l] = 0 \Rightarrow [K, L] = 0\), or
\[
K^\alpha L^\beta_{\gamma\alpha} = L^\alpha K^\beta_{\gamma\alpha}.
\] (51)

\[12\]
A 1-form in the (2,3) plane orthogonal to the \( k \land l \) or \( u \land L \) or (0,1) plane is \( a \) (which I will now assume is nonzero; otherwise it trivial that the acceleration is nonrotating). One may normalize it to get the unit 1-form

\[
\hat{a} = (a \cdot a)^{-1/2}a
\]  

(52)

An orthonormal 1-form in this same (2,3) plane is

\[
b = *(u \land L \land \hat{a}).
\]

(53)

Thus \((u, L, \hat{a}, b)\) form an orthonormal frame or basis of 1-forms at the position of the stationary observer that is defined purely in terms of the two commuting Killing vector fields and hence is Lie transported by the action of either one of them. However, this orthonormal frame is generically rotating relative to a frame that is Fermi-Walker transported along the stationary observer’s worldline. In fact, the Fermi-Walker derivative of any of the basis 1-forms above, or of any linear combination of them with constant coefficients, say \( v \), is given by

\[
v' = *(u \land \omega \land v),
\]

(54)

where

\[
\omega = \frac{1}{2} *(u \land du) = \frac{1}{2} e^{-2\Phi} *(K \land dK)
\]

(55)

is the normalized twist or rotation or vorticity 1-form of the Killing vector field \( K \) along one of whose integral curves the stationary observer moves. In particular,

\[
a' = *(u \land \omega \land a).
\]

(56)

The normalization of \( \omega \) makes it independent of the constant linear transformation (36) of the Killing vector fields and corresponding fractional linear transformation (37) of the angular velocity \( \Omega \), so long as the direction of \( K \) is kept fixed, as it indeed will be if one uses both transformations (36) and (37). In particular, \( \omega \) is constructed to be independent of the normalization of \( K \). Therefore, if the four-velocity \( u \) is locally determined, so is \( \omega \) and hence also \( a' \).

Eq. (54) is the analogue for 1-forms of the four-dimensional form of the equation that in the subspace of the tangent space orthogonal to the four-velocity \( u \), the Fermi-Walker derivative \( v' \) is the three-dimensional cross-product of the rotation vector \( \omega \) with \( v \). In component notation,

\[
v'_\alpha = -\epsilon_{\alpha\beta\gamma\delta} u^\beta \omega^\gamma v^\delta,
\]

(57)

where

\[
\omega_\alpha = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} u^\beta u^\gamma v^\delta = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} e^{-2\Phi} K^\beta K^{\gamma\delta}.
\]

(58)
The Fermi-Walker rotation is thus in the two-plane orthogonal to the $\mathbf{u} \wedge \omega$ plane, so both $\mathbf{u}$ and $\omega$ have zero Fermi-Walker derivative. Also, it is clear from Eq. (55) or (58) that $\omega$ is orthogonal to $\mathbf{u}$.

Note that in the definition (55) or (58) of $\omega$, I am implicitly assuming that the four-velocity $\mathbf{u}$ which is differentiated there has a fixed angular velocity $\Omega$ and hence is proportional to a fixed combination $\mathbf{K} = \mathbf{k} + \Omega \mathbf{l}$ of the two Killing vector fields, with $\Omega$ held constant during the exterior or covariant differentiation. In other words, $\omega$ is the rotation 1-form for a congruence that has four-velocity everywhere parallel to a single Killing vector field $\mathbf{K}$ and hence may be considered to be rigidly rotating, since its shear and expansion are zero.

When one considers a congruence, such as a SCALE or a NAW, that has $\Omega$ varying as a function of the two $x^a$ coordinates ($a = 2$ or 3), then $\omega$ as I am generally using it in this paper is not the rotation 1-form that one would get from inserting that $\mathbf{u}(x^a)$ into Eq. (55), but rather the 1-form that one gets at each $x^a$ by instead using the auxiliary congruence with constant $\Omega$ (and hence with four-velocities parallel to a single Killing vector field $\mathbf{K}$) that is chosen to match that of the original congruence at that point. (Of course, for a stationary congruence $\Omega$ does not depend on the proper time along each worldline. Unless the context makes it clear that I am assuming otherwise, I actually make the stronger assumption that the congruences I am considering have $\Omega$ and other scalar quantities independent of both of the (0,1) coordinates, so their derivatives in both the $\mathbf{k}$ and $\mathbf{l}$ directions vanish.)

The stationarity of the worldlines tangent to the Killing vector field $\mathbf{K}$ imply that any covariantly determined vector $\mathbf{v}$ that is orthogonal to the worldline and determined by the local properties of the worldline and curvature (i.e., contractions of derivatives of the four-velocity and of the curvature tensor and its derivatives) is Lie transported by the Killing vector field $\mathbf{K}$ and so has itself a Fermi-Walker derivative that obeys Eq. (54) or (57). In other words, the stationarity of metric and of the worldline under translations along the integral curves of $\mathbf{K}$ imply that all locally determined vectors have components that stay constant in the locally rotating frame with basis 1-forms obeying Eq. (54) or (57) with respect to a locally nonrotating (Fermi-Walker transported) frame.

If $\omega = 0$, then one would obviously get $\alpha' = 0$ by inserting $\mathbf{a}$ instead of $\mathbf{v}$ in Eq. (54) or (57), so to show that $\text{SCALE} \Rightarrow \text{NAW}$, it remains only to check the case in which $\omega \neq 0$, which will henceforth be assumed.

Now $\mathbf{K} \wedge \mathbf{L} \propto \mathbf{k} \wedge \mathbf{l}$, so Eq. (51) implies the corresponding equation for $\mathbf{K}$ and $\mathbf{L}$,

$$\mathbf{K} \wedge \mathbf{L} \wedge d\mathbf{K} = \mathbf{K} \wedge \mathbf{L} \wedge d\mathbf{L} = 0.$$  \hspace{1cm} (59)

In terms of the rotation or normalized twist 1-form, the first of these equations is
simply
\[ \omega \cdot L = 0. \quad (60) \]

Therefore, \( u, \omega, L \), and
\[ L' = *(u \wedge \omega \wedge L) \quad (61) \]
form an orthogonal set of 1-forms at the position of the stationary observer, with \( u \) and \( L \) having unit magnitude and hence with \( \omega \) and \( L' \) having the same magnitude. Thus one may write
\[ \omega = *(u \wedge L' \wedge L), \quad (62) \]
and then using this in Eq. \( (56) \) gives
\[ a' = *(u \wedge \omega \wedge a) = -(a \cdot L')L. \quad (63) \]

Now inserting \( \omega \) from Eq. \( (55) \) into Eq. \( (61) \) for \( L' \), evaluating, and comparing with Eq. \( (32) \) shows that \( L' = e^{2\Phi} \frac{\partial a}{\partial \Omega} = (A + 2\Omega B + \Omega^2 C) \frac{\partial a}{\partial \Omega} \quad (64) \)

Putting this expression into Eq. \( (55) \) and comparing with Eq. \( (53) \), or comparing with Eq. \( (47) \), one gets
\[ a' = -e^{2\Phi} \frac{\partial a^2}{4\sqrt{D} \partial \Omega} L = -(A + 2\Omega B + \Omega^2 C) \frac{\partial a^2}{4\sqrt{B^2 - AC} \partial \Omega} L. \quad (65) \]

Thus, for a spacetime with two independent Killing vector fields with properties (1)-(3) above, \( \partial a^2/\partial \Omega = 0 \Leftrightarrow a' = 0 \). That is, a Stationary Congruence Acceleration Locally Extremely (SCALE) is equivalent to a Nonrotating Acceleration Worldline (NAW). This proof implies the title statement, “Maximal Acceleration Is Nonrotating” (so it might be called the “MAIN Page proof.”)

In contrast, one might be tempted to say of the converse result given earlier, “NAW, it’s a SCAM.” But this would be a scam, since actually a NAW was proved to be a SCALE, and not all SCALEs are SCAMs.

A Delphic way of stating the result, using phrases explained above, and substituting “permanent” and “always” for “stationary,” would be, “Someone permanently fixed up is extremely heavy, and someone always extremely heavy stays up.”

6 Acceleration and rotation of observers

6.1 Stationary congruence parallel to a Killing field

As an aside, one can make an analogy between the acceleration and rotation vectors for a stationary observer corresponding to a particular timelike Killing vector field,
and the electric and magnetic fields of an electromagnetic field as seen by a particular observer. In particular, if the timelike Killing vector field is $K$ with squared magnitude

$$F \equiv e^{2\Phi} \equiv -K \cdot K,$$  \hspace{1cm} (66)

then one may regard the normalized 2-form

$$f \equiv -\frac{1}{2} e^{-\Phi} dK = -\frac{1}{2} du + \frac{1}{2} u \wedge a$$  \hspace{1cm} (67)

with antisymmetric tensor components

$$f_{\alpha \beta} \equiv e^{-\Phi} K_{\alpha \beta} = u_{[\alpha ; \beta]} + u_{[\alpha \alpha \beta]}$$  \hspace{1cm} (68)

as being algebraically analogous to an electromagnetic field 2-form with tensor components $F_{\alpha \beta}$, though it will not in general obey the analogues $df = 0$ and $d \ast f = 0$ of the vacuum Maxwell equations.

(The normalization of $f$ is chosen to make it independent of the constant transformations (36) and (37) of the Killing vector fields $k$ and $l$ and of $\Omega$ that keep $K$ pointing in the same spacetime direction but which may change its overall magnitude.)

Then in the frame of the stationary observer with normalized four-velocity $u = e^{-\Phi} K$, the acceleration 1-form with components

$$a_{\alpha} = f_{\alpha \beta} u^\beta$$  \hspace{1cm} (69)

is analogous to the electric field in a frame with four-velocity $u$, which has components

$$E_{\alpha} = F_{\alpha \beta} u^\beta.$$  \hspace{1cm} (70)

Similarly, the rotation 1-form with components

$$\omega_{\alpha} = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} f_{\beta \gamma} u^\delta$$  \hspace{1cm} (71)

is analogous to the negative of the magnetic field in the observer’s frame, which has components

$$B_{\alpha} = -\frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} F_{\beta \gamma} u^\delta.$$  \hspace{1cm} (72)

One can then write the Fermi-Walker derivative of the components of any 1-form $v$ which is Lie transported by the Killing vector field $K$ as

$$v'_\alpha = \omega_{\alpha \beta} v^\beta.$$  \hspace{1cm} (73)
where
\[
\omega_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma\delta} u^\gamma \omega^\delta \\
= (\delta_\alpha^\mu + u_\alpha u^\mu) u_{\mu\nu}(\delta_\nu^\beta + u^\nu u_\beta) \\
= (\delta_\alpha^\mu + u_\alpha u^\mu) f_{\mu\nu}(\delta_\nu^\beta + u^\nu u_\beta) \\
= u_{\alpha\beta} + a_\alpha u_\beta \\
= f_{\alpha\beta} - u_\alpha a_\beta + a_\alpha u_\beta
\] (74)
are the components of the rotation 2-form
\[
\tilde{\omega} = -\ast (u \wedge \omega) = -\frac{1}{2} e^\Phi d(e^{-2\Phi} K) = -\frac{1}{2} e^\Phi d(e^{-\Phi} u)
\] (75)
which is the part of the 2-form \(-\frac{1}{2} du\), or of the 2-form \(f = -\frac{1}{2} e^{-\Phi} dK\), that is orthogonal to \(u = e^{-\Phi} K\). As usual, the normalization is chosen to make \(\tilde{\omega}\) independent of the transformations (36) and (37).

In particular, one can see from Eq. (56) that the Fermi-Walker derivative of the acceleration 1-form, \(a'\), is analogous to \(4\pi\) times the Poynting vector flux of the electromagnetic field that is analogous to \(f\). In other words, if one defines a symmetric second rank tensor
\[
4\pi t_{\alpha\beta} = f_{\alpha\mu} f_{\beta}^\mu - \frac{1}{4} g_{\alpha\beta} f_{\mu\nu} f^{\mu\nu},
\] (76)
that is the analogue of the electromagnetic stress-energy tensor (generically not conserved here, since \(f\) does not obey the vacuum Maxwell equations), then
\[
a'_\alpha = 4\pi p_\alpha = [\ast(u \wedge \omega \wedge a)]_\alpha = -4\pi(\delta_\alpha^\beta + u_\alpha u^\beta) t_{\beta\gamma} u^\gamma = (\delta_\alpha^\beta + u_\alpha u^\beta) f_{\beta}^\gamma f_{\gamma}^\delta u^\delta. \tag{77}
\]

One can easily see that there are precisely three independent Lorentz-invariant scalars at a point that depend only on the Killing vector field \(K\) and its first covariant derivative and are invariant under the transformations (36) and (37) that change its normalization:
\[
a^2 \equiv a \cdot a = -f_{\alpha\beta} f_{\gamma}^\beta u^\gamma u_\alpha,
\] (78)
\[
\omega^2 \equiv \omega \cdot \omega = f_{\alpha\beta} f_{\gamma}^\beta (\frac{1}{2} \delta_\gamma^\alpha - u^\gamma u_\alpha),
\] (79)
and
\[
a \cdot \omega = -\frac{1}{8} \epsilon_{\alpha\beta\gamma\delta} f_{\alpha\beta} f_{\gamma}^\delta u^\delta \pm \frac{1}{4} f_{\alpha\beta} f_{\gamma}^\beta f_{\delta}^\gamma f_{\delta}^\gamma - \frac{1}{8} (f_{\alpha\beta} f_{\gamma}^\beta)^{1/2}. \tag{80}
\]

For example, dot products of \(\omega\), \(a\) and of all of its Fermi-Walker derivatives \(a^{(n)}\) of order \(n\) may be expressed algebraically in terms of these three scalars. For positive integers \(m\) and \(n\),
\[
a^2 \equiv a' \cdot a' \equiv a^{(1)} \cdot a^{(1)} = (a \cdot a)(\omega \cdot \omega) - (a \cdot \omega)^2 = a^2 \omega^2 - (a \cdot \omega)^2, \tag{81}
\]
\[ \omega \cdot a^{(n)} = 0, \]  
\[ a \cdot a^{(2n-1)} = 0, \]  
\[ a \cdot a^{(2n)} = \omega^{2n-2} a'^2, \]  
\[ a^{(n)} \cdot a^{(n+2m-1)} = 0, \]

and
\[ a^{(n)} \cdot a^{(n+2m)} = (-1)^{m} \omega^{2n+2m-2} a'^2. \]

One can also get analogous relations between the vectors or 1-forms themselves:

If one defines
\[ a_\perp \equiv a - (a \cdot \omega) \omega / \omega^2, \]
the part of the acceleration vector or 1-form \( a \) that is orthogonal to the rotation vector or 1-form \( \omega \), which has squared length
\[ a_\perp^2 \equiv a_\perp \cdot a_\perp = a'^2 / \omega^2 = a^2 - (a \cdot \omega)^2 / \omega^2, \]
then \( u, \omega, a_\perp \), and \( a' \) form an orthogonal set of vectors or 1-forms, and each Fermi-Walker derivative \( a^{(n)} \) is parallel (or anti-parallel, in alternate cases) either to \( a_\perp \) or to \( a' \):
\[ a^{(2n)} = (-1)^n \omega^{2n} a_\perp, \]  
\[ a^{(2n+1)} = (-1)^n \omega^{2n} a'. \]

When \( a' \neq 0 \), one can solve for the rotation 1-form
\[ \omega = *(u \wedge a' \wedge a'') / a'^2 \]
in terms of the four-velocity \( u \) and the first two Fermi-Walker derivatives \( a' \) and \( a'' \) of the acceleration \( a \). In particular, the squared magnitude of this rotation 1-form is then
\[ \omega^2 \equiv \omega \cdot \omega = \frac{a'' \cdot a''}{a' \cdot a'} = \frac{[(A \nabla B - B \nabla A) + \Omega(A \nabla C - C \nabla A) + \Omega^2(B \nabla C - C \nabla B)]^2}{4(B^2 - AC)(A + 2\Omega B + \Omega^2 C)^2}. \]

One can compute directly from Eq. (91) that this formula for \( \omega^2 \) in terms of \( A \), \( B \), and \( C \) is invariant under the transformations (36) and (37).

When one has a stationary observer in a curved spacetime, one can also get Lorentz-invariant scalars from contracting combinations of the Killing vector field \( \mathbf{K} \) and its covariant derivatives with combinations of the Riemann curvature tensor and its covariant derivatives. As is well known, one can use Killing’s equation, Eq.
to eliminate all covariant derivatives of the Killing vector field of order higher than two. Therefore, since

$$f \equiv -\frac{1}{2} e^{-\Phi} dK = u \wedge a + \bar{\omega} = u \wedge a - *(u \wedge \omega),$$

(93)

any of these Lorentz-invariant scalars can be obtained by contracting appropriate combinations of $u$, $\omega$, and $a$ with appropriate combinations of the Riemann curvature tensor and its covariant derivatives, up to powers of the squared magnitude $e^{2\Phi} = -K \cdot K$ of the Killing vector field that get multiplied by constants when one performs the transformations (36) and (37).

Just as a SCALE extremizes, and a SCAM locally maximizes, the scalar $a^2$ as a function of the angular velocity $\Omega$, so one could also define other congruences that extremize, maximize, minimize, or set to zero other combinations of the scalars $a^2$, $\omega^2$, or $a \cdot \omega$. For example, one might define a Stationary Congruence Rotating At Minimum (SCRAM) as a stationary congruence that at each point, as a function of the angular velocity $\Omega$, minimizes the squared magnitude $\omega^2$ of the rotation $\omega$. (Remember that $\omega$ is actually the rotation at each point, not of the original congruence being considered, but of an auxiliary rigid congruence, parallel to a single Killing vector field $K$ with $\Omega$ having a constant value that matches the $x^a$-dependent $\Omega$ of the original congruence at the point where $\omega$ is being evaluated.) One can see from extremizing Eq. (92) that the equation for this minimum, like that for a SCAM, is generically a messy quartic equation in $\Omega$. The resulting four-velocity is independent of the transformations (36) and (37) and hence is determined purely locally.

6.2 Nonstationary observers

Many of the formulas above apply or have generalizations to the case of a nonstationary congruence with $u$ defined over the region of spacetime under consideration, even when it is not proportional to any Killing vector field. For example, the acceleration vector $a$ is still given by Eq. (19), and the rotation 1-form $\omega$ by the first expression on the right hand side of Eq. (75). Then one can use the last expression on the right hand side of Eq. (93) to define a 2-form $f$ that is algebraically analogous to an electromagnetic field and which gives back the acceleration and rotation 1-forms by Eqs. (69) and (71). One can also use various other definitions, such as Eq. (87) for $a_\perp$, the part of the acceleration vector that is orthogonal to $\omega$.

For a nonstationary congruence, the components of quantities defined in terms of the four-velocity and its covariant derivatives (e.g., the components of $a$ and of $\omega$) are no longer constant in a frame which accelerates and rotates with the congruence, as such components are for a stationary congruence. Therefore, the Fermi-Walker derivative is no longer given by Eq. (54). In particular, $a'$ is no longer given by Eq.
Nevertheless, one can still define an analogue of $4\pi$ times the Poynting flux,

$$4\pi \mathbf{p} \equiv \ast (\mathbf{u} \wedge \mathbf{\omega} \wedge \mathbf{a}),$$  \hspace{1cm} (94)$$

whose components are given by the right hand side of Eq. (77). When $4\pi \mathbf{p}$ is nonzero, it may be normalized to define the unit 1-form

$$\mathbf{L} \equiv \mathbf{p}/(\mathbf{p} \cdot \mathbf{p})^{1/2} \equiv [a^2 \mathbf{\omega}^2 - (\mathbf{a} \cdot \mathbf{\omega})^2]^{-1/2} \ast (\mathbf{u} \wedge \mathbf{\omega} \wedge \mathbf{a}).$$  \hspace{1cm} (95)$$

Then $\mathbf{u}$, $\mathbf{\omega}/|\mathbf{\omega}|$, $\mathbf{a}_\perp/|\mathbf{a}_\perp|$, and $\mathbf{L}$ form an orthonormal basis defined by the congruence.

In general for a nonstationary congruence (or even for a stationary congruence in a metric which does not have a second Killing vector field obeying properties (2) and (3) above), $\mathbf{L}$ will not be parallel to any Killing vector. However, one can start from an original congruence with the four-velocity field $\mathbf{u}$, construct the unit orthogonal vector $\mathbf{L}$ by the procedure above (when $4\pi \mathbf{p} \neq 0$), and thereby define a new set of congruences with four-velocities

$$\mathbf{U} = \frac{\mathbf{u} + v \mathbf{L}}{\sqrt{1 - v^2}},$$  \hspace{1cm} (96)$$

where $v$ is the velocity of the new congruence relative to the old one, a parameter analogous to $\Omega$ in the stationary axisymmetric case.

Then one can play the same game of choosing the velocity $v$ at each point of the region of spacetime under consideration to extremize, maximize, minimize, or set to zero various combinations of such scalars as $a^2$, $\omega^2$, $\mathbf{a} \cdot \mathbf{\omega}$, $\mathbf{a} \cdot \mathbf{a}'$, $\mathbf{a}' \cdot \mathbf{a}'$, $\mathbf{a} \cdot \mathbf{a}''$, $\mathbf{\omega} \cdot \mathbf{a}''$, $\mathbf{a}' \cdot \mathbf{a}''$, $\mathbf{a}'' \cdot \mathbf{a}''$, etc., an arbitrarily large number of which can be independent for a nonstationary congruence, unlike the case of a stationary congruence, for which these scalars are all algebraically dependent on the first three.

For example, one might define $v$ to give a local extremum of $a^2$, which might be called a Congruence Accelerating Locally Extremely (CALE), or a Congruence Accelerating Maximally (CAM) if the extremum is a local maximum. Or, one might minimize $\omega^2$, giving a Congruence Rotating At Minimum (CRAM). One could still have a Nonrotating Acceleration Worldline (NAW) if $\mathbf{a}'$ is parallel to $\mathbf{a}$, but since this requirement is equivalent to solving two equations, it generically cannot be satisfied by any choice of the single parameter $v$ for a nonstationary congruence, as it could in the stationary axisymmetric case. One could instead choose $v$ at each point to minimize the squared length of the Fermi-Walker derivative of the direction of the acceleration,

$$\left(\frac{\mathbf{a}}{|\mathbf{a}|}\right)' \cdot \left(\frac{\mathbf{a}}{|\mathbf{a}|}\right)' = \frac{a^2 \mathbf{a}^2 - (\mathbf{a} \cdot \mathbf{a})^2}{a^4},$$ \hspace{1cm} (97)$$

giving a Congruence Having Acceleration Rotating Minimally (CHARM). Clearly a NAW is a special case of a CHARM.
Because of the independence of the higher covariant derivatives of the four-velocity of a nonstationary congruence, in contrast to the case for a stationary congruence, it appears that one would not in general be able to conclude that a CAM is a CHARM, for instance. It might be of interest to see when this is so, but I shall not pursue this issue further here.

Some of the formulas above can be applied even when there is no congruence but only a single worldline. In this case one cannot define a rotation 1-form $\omega$ by Eq. (55), but one can still define the acceleration $a$ and its various Fermi-Walker derivatives $a', a''$, etc. If one liked, one could make up a definition for a rotation 1-form $\omega$, such as Eq. (91), which agrees with Eq. (55) for a stationary congruence when $a'^2 \neq 0$. It might be somewhat better for a general stationary congruence to replace $a'$ by its part that is orthogonal to $a$, namely

$$a'_{\perp} \equiv a' - a(a \cdot a')/a^2,$$

with squared magnitude

$$(a'_{\perp})^2 \equiv a'_{\perp} \cdot a'_{\perp} = a' \cdot a' - (a \cdot a')^2/a^2,$$  

thus leaving out a possible change in the magnitude of the acceleration that is automatically absent for a stationary worldline, and to replace $a''$ by $(a'_{\perp})'$, the Fermi-Walker derivative of $a'_{\perp}$. Then one could define a rotation 1-form to be

$$\omega = *(u \wedge a'_{\perp} \wedge (a'_{\perp})')/(a'_{\perp})^2$$

One might prefer also to take only the part of $(a'_{\perp})'$ that is orthogonal to $a'_{\perp}$, but this makes no difference in Eq. (100).

Thus for a single worldline one can say that it has the property of being a member of a NAW if and only if $a'_{\perp} = 0$, but without having the rest of a congruence, one cannot say whether or not it is a member of a SCALE, SCAM, SCRAM, CALE, CAM, or CHARM (except that it must have the latter property if it is a NAW).

7 Stationary Congruence Accelerating Maximally (SCAM)

For a generic location (of the two $x^a$ coordinates parametrizing the two-surfaces orthogonal to the two Killing vector fields $k$ and $l$) in a generic axisymmetric metric, the quartic Eq. (34) giving the angular velocity $\Omega_0$ of the SCAM (Stationary Congruence Accelerating Maximally) is messy to solve, and the explicit form of the solution is presumably not very perspicuous (though I have not bothered to write it out in gory detail). However, there are at least three situations in which one can
get simpler explicit solutions. Below I shall give an example where the quartic factorizes into two fairly simple quadratics. But first I shall give the angular velocity of a SCAM in a spacetime with relatively slow rotation.

### 7.1 In a spacetime with slow rotation or weak fields

For a spacetime with relatively slow rotation, after a suitable constant linear transformations (36) of the Killing vector fields \( k \) and \( l \) if necessary, the part of the metric in the (0,1) plane is nearly diagonal, and so are its derivatives. In particular, \( B^2 \ll -AC \), and \( \nabla B \cdot \nabla B \ll -\nabla A \cdot \nabla C \). Then one can readily write down the solution of Eq. (34) that is first order in \( B \) and \( \nabla B \), ignoring higher order corrections in these quantities (which will start with terms cubic in these quantities, which I shall label simply as \( O(B^3) \)):

\[
\Omega_0 = -\frac{\nabla A \cdot (A\nabla B - B\nabla A)}{\nabla A \cdot (A\nabla C - C\nabla A)} + O(B^3).
\]

(101)

In the far-field limit outside an isolated source, typically \( A \equiv -g_{tt} \) is just slightly less than 1 (e.g., roughly \( 1 - 2M/r \) at \( r \gg M \)) and has a gradient nearly in the positive radial direction with coordinate \( r = x^2 \). For simplicity, in the following discussion let the derivative with respect to \( r \) be denoted by a prime. (The context will keep it clear that here this prime does not mean a Fermi-Walker derivative, which will usually be zero for the quantities that I shall be applying the prime to, such as the stationary scalar quantities \( A, B, \) and \( C \)). \( C \equiv -g_{\varphi \varphi} \) will be negative, typically going roughly as the negative square of the distance from the axis of rotation, so at angle \( \theta = x^3 \) from the positive axis, \( C \) will go like \( -r^2 \sin^2 \theta \), and hence off the axes will have \( C' < 0 \). Furthermore, \( AC' - CA' \) will be dominated by its first term and hence be negative. Then if the source is rotating in the positive \( \varphi \) direction, then in the far-field limit \( B \equiv -g_{t\varphi} \) will be positive, typically going as \( (2J/r) \sin^2 \theta \), where \( J \) is the intrinsic angular momentum of the source [2]. Thus \( B \) will have a negative radial gradient \( B' \) as it tends to zero at infinity, and \( AB' - BA' \) will be dominated by its first term and hence be negative.

Therefore, in the far-field limit outside an isolated source centered at the origin of standard spherical polar coordinates, Eq. (101) gives the approximate angular velocity of the SCAM as

\[
\Omega_0 \approx -\frac{B'}{C'} \approx -\frac{J}{r^3}.
\]

(102)

This is in contrast to a ZAMO, which has

\[
\Omega_Z = -\frac{B}{C} \approx +\frac{2J}{r^3}.
\]

(103)

That is, a ZAMO corotates with the source by what is usually called the dragging of inertial frames, but a SCAM counterrotates. Why is this?
A simple physical argument that applies in the far-field limit is the following: Consider the source being a ring of nonrelativistic matter in the the equatorial plane rotating with $v \ll 1$ in the positive $\varphi$-direction, and consider a stationary observer just outside this ring. In the observer’s frame, the energy density of the nearby portion of the ring will be a minimum if the observer is corotating with the ring. If the observer is not rotating relative to infinity ($\Omega = 0$), she will see a higher energy density for the source, because of a relativistic $\gamma$-factor ($\gamma \equiv 1/\sqrt{1 - v^2}$) for the energy of each particle of the source (increase of its kinetic energy), and because of another $\gamma$-factor from the Lorentz-contraction of the source. If the observer is counterrotating, she will have an even greater velocity relative to the nearby portion of the ring, and so she will see an even higher energy density (increasing roughly linearly with $-\Omega$ when it is small compared with the magnitude of the rotation rate of the ring itself).

Because energy density is the main source of the gravitational field in the nonrelativistic limit, a counterrotating stationary observer will thus have a greater gravitational attraction to the ring (greater acceleration radially outward as seen in a freely falling frame).

As the observer increases her counterrotation rate, she will also have a counteracting centripetal acceleration inward, but it will increase at first only quadratically with her counterrotating angular velocity $-\Omega$. Thus a small counterrotation rate will increase the experienced outward acceleration against gravity more than the increase of the centripetal acceleration inward, so the net acceleration (which is outward, primarily against the gravitational attraction of the ring) will at first increase with increasing the magnitude of the counterrotation angular velocity $-\Omega$. (De Felice [35] previously noticed this effect on the equatorial plane of the Kerr metric.)

Eventually, the (roughly quadratic) increase in the inward centripetal acceleration will balance the (roughly linear) increase in the outward acceleration against gravity, and one will reach a local maximum of the acceleration, thus at a counterrotating angular velocity ($\Omega_0 < 0$) of the SCAM.

This is the picture in a weak gravitational field, but in a strong field ($M \sim r$), the increase of the gravitational attraction with velocity (by the two $\gamma$-factors, i.e., as $1/(1 - v^2)$) can dominate over the increase in the centripetal acceleration (which goes as $v^2/(1 - v^2)$, but typically with a different coefficient, roughly $1/r$ rather than roughly $M/r^2$), so that there is no local maximum of the acceleration and hence no SCAM at that location. By taking out the $\gamma$-factor dependence of the gravitational acceleration and including it instead with the centripetal acceleration (which is not strictly forbidden, since only the sum of the two is a gauge-invariant observable quantity), Abramowicz and his collaborators [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] label this latter effect as the reversal of the sign of the centripetal acceleration, though to me it seems more intuitively understandable to recognize it
as the greater increase of the gravitational acceleration with velocity than has the centripetal acceleration [23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

7.2 At an orbital

Another situation in which the SCAM has properties that are simpler than those given by the solution of the generic quartic Eq. (34), is at a location which I shall call an orbital. This I define to be a location where there are two stationary time-like geodesics at different values of $\Omega$ (e.g., corotating and counterrotating circular Keplerian orbits). There the angular velocity of the SCAM is given by a quadratic equation rather than a quartic.

In terms of the (reversed-sign) $(t, \varphi)$ metric components $A \equiv -g_{tt}$, $B \equiv -g_{t\varphi}$, and $C \equiv -g_{\varphi\varphi}$ defined above, Eq. (31) gave

$$a = \frac{\nabla A + 2\Omega \nabla B + \Omega^2 \nabla C}{2(A + 2\Omega B + \Omega^2 C)}.$$  \hspace{1cm} (104)

At a generic location, $\nabla A$, $\nabla B$, and $\nabla C$ are rather arbitrary vectors in the 2-dimensional $(2,3)$ tangent plane, so for no value of $\Omega$ is $a = 0$.

Only if $-2\nabla B$ is on the two-branched hyperbola $\Omega^{-1} \nabla A + \Omega \nabla C$ is there a value of $\Omega$ that gives $a = 0$ (a stationary geodesic). If there are two stationary geodesics at different values of $\Omega$, then the two branches of the hyperbola must degenerate to a single straight line (i.e., $\nabla A$ and $\nabla C$ must point in the same or opposite direction). For $\nabla B$ to be on this degenerate hyperbola (at two different values of $\Omega$), it must also point in that same direction. In other words, at an orbital, $\nabla A$, $\nabla B$, and $\nabla C$ must all be parallel (or anti-parallel). However, this is a sufficient condition for an orbital only if both values of $\Omega$ that give $a = 0$ give timelike four-velocities.

Since $\nabla A$, $\nabla B$, and $\nabla C$ lie in the tangent $(2,3)$ plane that orthogonal to the plane of the two-form $A = k \wedge l$ (which is not to be confused with the scalar $A = -k \cdot k$), the condition that these three gradients are all parallel is equivalent to the two scalar equations

$$\star (A \wedge \nabla A \wedge \nabla B) = 0,$$ \hspace{1cm} (105)

$$\star (A \wedge \nabla B \wedge \nabla C) = 0.$$ \hspace{1cm} (106)

In the $(2,3)$ coordinate space, these two equations generically have solutions only at isolated points, if at all, so it is by no means guaranteed that a stationary axisymmetric metric will have any orbitals. But if the space is symmetrical with respect to reflection about an equatorial plane containing $l$, then on that plane $\nabla A$, $\nabla B$, and $\nabla C$ will all lie within its tangent plane and be parallel (since they are all orthogonal to $l$). Therefore, if a spacetime has such an equatorial plane, and if both values of $\Omega$ that give $a = 0$ give timelike four-velocities for some region of that plane, then this region consists of orbitals.
Assume that the common direction of $\nabla A$, $\nabla B$, and $\nabla C$ is not orthogonal to the radial direction with coordinate $r = x^2$. (I.e., choose the radial direction so that it is not orthogonal to that common direction.) As above, let the derivative with respect to $r$ be denoted by a prime. Then clearly $\nabla A$, $\nabla B$, and $\nabla C$ will all be proportional to $A'$, $B'$, and $C'$ respectively, with the same constant of proportionality (at a fixed location).

Therefore, at an orbital, Eq. (34) for the angular velocity $\Omega$ of a SCALE becomes

$$[A' + 2B'\Omega + C'\Omega^2][(AB' - BA') + (AC' - CA')\Omega + (BC' - CB')\Omega^2] = 0. \tag{107}$$

This thus factorizes into two quadratic equations. One is the equation for the geodesic condition, $a = 0$, or

$$A' + 2B'\Omega + C'\Omega^2 = 0. \tag{108}$$

Its solutions are the angular velocities of the Stationary Congruences Accelerating Locally Extremely (SCALEs) that have local minima of the magnitude of the acceleration (here both at the global minimum of 0). I have been calling the angular velocities of the SCALEs that are local minima $\Omega_\pm$, but here I shall call them $\Omega_\pm K$, since for a stationary axisymmetric spacetime they are the angular velocities of circular Keplerian orbits:

$$\Omega_{\pm K} = \frac{B' \pm \sqrt{B'^2 - A'C'}}{-C'}. \tag{109}$$

The other quadratic equation,

$$(AB' - BA') + (AC' - CA')\Omega + (BC' - CB')\Omega^2 = 0, \tag{110}$$

is the equation at an orbital for the Stationary Congruence Accelerating Maximally (SCAM), the SCALE with a local maximum of the magnitude of the acceleration (and for an unphysical root). The physical root has angular velocity

$$\Omega_0 = \frac{AC' - CA' + \sqrt{(AC' - CA')^2 - 4(AB' - BA')(BC' - CB')}}{2(CB' - BC')} \tag{111}.$$

One can see from Eq. (12) that at an orbital, a SCAM has zero rotation, $\omega^2 = 0$, so it is obviously also a SCRAM there. (This fact was previously noted [27] for the Kerr metric, and now we see that it is general.) Therefore, at an orbital, a SCAM has the same four-velocity as a rigid congruence with four-velocity parallel to the Killing vector field $K$ with constant $\Omega$ that at that location has zero vorticity. (The value of $\Omega$ giving $\omega = 0$ on the equatorial plane of Kerr was first given in [24], and later [27] de Felice discovered this is also the angular velocity that extremizes the acceleration there.)
Again one should be reminded that I have defined the rotation $\omega$ so that it is the rotation of a congruence moving along the orbits of a single Killing vector field $K$ with $\Omega$ constant (with the constant matching the angular velocity $\Omega$ of the original congruence at that location, but not necessarily at other locations). That is, if one inserted the $x^a$-dependent four-velocity $u(x^a)$ of a SCAM at an orbital into Eq. (55), $\omega = \frac{1}{2} \ast (u \wedge du)$, one would generically not get zero, but only if one used in that formula the four-velocity $u = e^{-\phi}(k + \Omega l)$ with constant $\Omega$ chosen to make this $u$ match that of the SCAM at the position where $\omega$ is being evaluated. Thus a SCAM at an orbital is not generically part of a congruence that itself has zero vorticity. Only the rigid congruence moving along the orbits of $K$ has zero vorticity there (and that congruence generically has nonzero vorticity at other locations).

(When we have two Killing vector fields obeying the properties (1) - (3) above, stationary congruences with zero vorticity are those with a constant ratio of angular momentum to energy,

$$\frac{\text{angular momentum}}{\text{energy}} = \frac{u_\phi}{-u_0} = \frac{-B - C\Omega}{A + B\Omega} = \text{const.},$$

such as the ZAMOs with $\Omega$ obeying Eq. (103), $\Omega = -B/C$, so that its angular momentum, and hence the ratio above, is zero. Thus a congruence of ZAMOs has the local property of zero vorticity, though it generically does not have $\omega = 0$ by my indirect method of defining $\omega$ in terms of an auxiliary congruence for each location that rotates rigidly with constant $\Omega$. Incidentally, though the zero vorticity of a ZAMO is a local property of that congruence, it is not sufficient to distinguish locally that particular congruence from the other zero-vorticity stationary congruences with different constant values of the ratio of angular momentum to energy. The problem is that locally there is no way to distinguish the angular momentum connected with the Killing vector field $l$ with closed orbits from a combination of angular momentum and energy connected with a different Killing vector field, since the property only that $l$ has, of having closed orbits, is not a local property that can be determined without knowing the metric in a loop around the symmetry axis.)

At an orbital one has, for a fixed choice of the Killing vector fields $k$ and $l$, seven special values (at least) of the angular velocity $\Omega$: $\Omega_{+1}$ and $\Omega_{-1}$ given by Eq. (107) (those of the speed of light in the forward and backward directions respectively), $\Omega_{+K}$ and $\Omega_{-K}$ given by Eq. (109) (those of the stationary geodesics or circular Keplerian orbits in the forward and backward directions respectively), $\Omega_0$ given by Eq. (111) (that of the Stationary Congruence Accelerating Maximally, or SCAM, which has the local maximum of the magnitude of the acceleration as a function of $\Omega$, and which, like the geodesics with $\Omega_{\pm K}$ that are members of the other Stationary Congruences Accelerating Locally Extremely, or SCALEs, are also Nonrotating
Acceleration Worldlines, or members of a NAW congruence that Fermi-Walker transports the acceleration vector), $\Omega_Z$ given by Eq. (39) (that of the ZAMO, which has zero angular momentum), and $\Omega_{NR} \equiv 0$ (that of a congruence nonrotating relative to infinity).

The four-velocities corresponding to the first five of these angular velocities are determined locally and are invariant under the linear transformations (36) of the Killing vector fields $k$ and $l$, but, as discussed above, that is not true of the last two, since they require the nonlocal knowledge of which Killing vector field is $k$ (e.g., the one that is timelike at infinity, usually normalized to have unit timelike length there) and of which one is $l$ (e.g., the one with closed orbits, usually normalized to give period $2\pi$ around the orbit). (For the first two angular velocities $\Omega_{\pm 1}$, there are no normalized four-velocities with those angular velocities, so for them I mean instead the corresponding null vectors, which are locally determined only up to normalization.) Of course, the particular values of all but the last of these angular velocities $\Omega$ depends on the particular choice of $k$ and $l$, but my point is that the corresponding $K$ (up to normalization) at each point does not, for the first five angular velocities.

Now, as one might expect, there are a number of algebraic relations between these angular velocities and between the corresponding four-velocities. For example, it is well known [2] that the ZAMO has the average of the angular velocities of the two null orbits,

$$\Omega_Z = \frac{1}{2}(\Omega_{+1} + \Omega_{-1}),$$

which implies that if a ZAMO sent two photons around an orbital in opposite direction (using, say, a tube to deflect the photons into these nongeodesic null orbits by an infinite number of glancing collisions that each transfer an infinitesimal momentum to the corresponding photon), they would return to the ZAMO at the same time [2]. Of course, this procedure requires the nonlocal information of the metric around the orbital, the same nonlocal information that is required to pick out $l$ as the Killing vector field with closed orbits, a choice that is necessary before one can define the angular velocities and get formulas such as Eq. (113), which is not invariant under the transformations (36) and (37) if there the transformation constant $\gamma \neq 0$. However, one may at least note that Eq. (113) is independent of the choice of the Killing vector $k$ (determined by the transformation constants $\alpha$ and $\beta$) or of the normalization of the Killing vector $l$ (determined by the transformation constant $\delta$ if $\gamma = 0$), even though changing these will change the $\Omega$’s appearing in Eq. (113).

Another more complicated relation one may find is

$$\Omega_0 = \frac{\sqrt{(\Omega_{+1} - \Omega_{-K})(\Omega_{-K} - \Omega_{-1})}\Omega_{+K} + \sqrt{(\Omega_{+1} - \Omega_{+K})(\Omega_{+K} - \Omega_{-1})}\Omega_{-K}}{\sqrt{(\Omega_{+1} - \Omega_{-K})(\Omega_{-K} - \Omega_{-1})} + \sqrt{(\Omega_{+1} - \Omega_{+K})(\Omega_{+K} - \Omega_{-1})}}. \quad (114)$$
Thus the SCAM angular velocity $\Omega_0$ at an orbital is a weighted average of the angular velocities $\Omega_{\pm K}$ of the two circular Keplerian orbits. This relationship is invariant under the transformations (36) and (37).

This relation simplifies greatly if one considers instead the relative velocities between these various observers. One readily finds that in the frame of an observer having four-velocity $u_1$ with angular velocity $\Omega_1$, the three-velocity of a second observer having four-velocity $u_2$ with angular velocity $\Omega_2$ is

$$v_2 \equiv \frac{u_2}{-u_1 \cdot u_2} - u_1 = v(\Omega_2, \Omega_1)\mathbf{L}_1,$$  \hspace{1cm} (115)

where the (signed) relative speed of observer 2 relative to observer 1 is [34]

$$v(\Omega_2, \Omega_1) = \frac{\sqrt{B^2 - AC}(\Omega_2 - \Omega_1)}{A + B(\Omega_1 + \Omega_2) + C\Omega_1\Omega_2} = \frac{(\Omega_{+1} - \Omega_{-1})(\Omega_2 - \Omega_1)}{(\Omega_{+1} + \Omega_{-1})(\Omega_2 + \Omega_1) - 2\Omega_{+1}\Omega_{-1} - 2\Omega_1\Omega_2},$$  \hspace{1cm} (116)

which is invariant under the transformations (36) and (37).

Then one can calculate [34] that at an orbital, the two circular Keplerian orbits have equal and opposite speeds in the frame of the SCAM, the magnitude of which may be called the Keplerian orbital speed $v_K$:

$$v_K = v(\Omega_{+K}, \Omega_0) = -v(\Omega_{-K}, \Omega_0) = \frac{\sqrt{(\Omega_{+1} - \Omega_{-1})(\Omega_{+K} - \Omega_{-1})} - \sqrt{(\Omega_{+1} - \Omega_{+K})(\Omega_{-K} - \Omega_{-1})}}{\sqrt{(\Omega_{+1} - \Omega_{-K})(\Omega_{+K} - \Omega_{-1}) + \sqrt{(\Omega_{+1} - \Omega_{+K})(\Omega_{-K} - \Omega_{-1})}}}$$

$$= \frac{\Omega_{+1} - \Omega_{-1}(\Omega_{+K} - \Omega_{-1})}{(\Omega_{+1} - \Omega_{-K})(\Omega_{+K} - \Omega_{-1}) - \sqrt{(B^2 - AC)^2 - 4(B^2 - AC)(B^2 - A'C')}}$$

$$= \frac{D' - \sqrt{\sigma}}{2\sqrt{DH}} = \frac{2\sqrt{DH}}{D' + \sqrt{\sigma}},$$  \hspace{1cm} (117)

where Eqs. (17) and (29) give $D \equiv B^2 - AC$ (minus the determinant of the first two-dimensional block of the metric),

$$H \equiv B'^2 - A'C',$$  \hspace{1cm} (118)

and

$$\sigma \equiv D'^2 - 4DH \equiv (B^2 - AC)^2 - 4(B^2 - AC)(B^2 - A'C')$$

$$= (AC' - CA')^2 - 4(AB' - BA')(BC' - CB')$$

$$= C^2C'^2(\Omega_{+1} - \Omega_{+K})(\Omega_{+1} - \Omega_{-K})(\Omega_{+K} - \Omega_{-1})(\Omega_{-K} - \Omega_{-1}).$$  \hspace{1cm} (119)
One may check from Eq. (40) for the transformations of the quantities $A$, $B$, and $C$ that the final three expressions on the right hand side of Eq. (117) are indeed invariant under the transformations (31) and (37) and hence are locally defined.

Therefore, a SCAM at an orbital has a four-velocity $u_0$ that is a normalized average between the four-velocities $u_{+K}$ and $u_{-K}$ of the two circular Keplerian orbits:

$$u_0 = \frac{u_{+K} + u_{-K}}{|u_{+K} + u_{-K}|} \equiv \frac{u_{+K} + u_{-K}}{\sqrt{2 - 2u_{+K} \cdot u_{-K}}}$$  \hspace{1cm} (120)$$

This means that if one has two equal-mass point particles moving along opposite stationary Keplerian orbits, and they collide to form a single particle in a totally inelastic collision, the velocity of this single particle immediately after the collision will be that of the SCAM at that location. This relation was discovered by Semarák first in the special case of the Kerr metric [28], and then later in general [34]. Thus we see that it is a feature at an orbital of any stationary axisymmetric metric invariant under reversing both $t$ and $\varphi$.

One can see that even if $\nabla A$, $\nabla B$, and $\nabla C$ are all parallel (or anti-parallel), the SCAM is not defined as a real congruence when $\sigma$ defined by Eq. (119) goes negative. It reaches zero when one of the two circular Keplerian orbits reaches the speed of light (usually when $\Omega_{-K}$ becomes as negative as $\Omega_{-l}$ for a positively rotating source, so that it is usually the last factor in the last expression of Eq. (119) that goes to zero first). Then there are no longer two timelike stationary geodesics, so one is not really at an orbital as defined above, even if $\nabla A$, $\nabla B$, and $\nabla C$ are all parallel (or anti-parallel).

At an orbital, the two SCALEs that are not the SCAM give the two circular Keplerian orbits. Away from an orbital, these two SCALEs have locally minimal, but not zero, magnitude of acceleration. One might think that Eq. (120) would generalize to this case, with $u_{+K}$ being replaced by $u_{\pm}$, the four-velocities of the SCALEs that are not the SCAM, but this is not generically the case. In other words, Eq. (120) applies only at an orbital, the only place where there are stationary geodesics, the circular Keplerian orbits with four-velocities parallel to combinations of the two Killing vector fields $k$ and $l$. (I am always implicitly excluding stationary geodesics that have four-velocities parallel to any other possible Killing vector fields that might be present, such as in a spherically symmetric spacetime.)

Another difference from the case of of an orbital, where the solution of the quadratic equation for the SCAM breaks down (becomes complex) after one of the circular Keplerian orbits reaches the speed of light as one enters deeper into a strong gravitational field, is that away from an orbital, the solution of the quartic equation for the SCAM becomes complex after the four-velocity for the SCAM merges with that of one of the other SCALEs as one enters deeper into a strong gravitational field. (Deeper in the field there is only one extremum, a minimum, for the magnitude of the acceleration as a function of the angular velocity for timelike worldlines.) Thus
one can have the case in which the relative velocity between the SCAM and one of the other SCALEs goes to zero (at the boundary of the region where the SCAM is defined as a real congruence), whereas the relative velocity between the the SCAM and the third SCALE can remain nonzero there.

In the frame of the SCAM at an orbital, one can readily show that the 3-velocity of an observer nonrotating relative to infinity (i.e., nonrotating relative to the \( \mathbf{k} \) Killing vector, so \( \Omega_{NR} = 0 \)), and of a member of a ZAMO are, respectively,

\[
v_{NR} = -\frac{\Omega_0 \sqrt{B^2 - AC}}{A + B\Omega_0}, \tag{121}
\]

\[
v_{Z} = \frac{B + C\Omega_0}{\sqrt{B^2 - AC}}. \tag{122}
\]

However, these quantities are not locally defined and hence are not invariant under the transformations (34) and (37).

For a corotating source, usually \( A, B, -C, \) and \(-\Omega_0\) are positive, so both \( v_{NR} \) and \( v_{Z} \) are then positive. In the far-field limit outside an isolated source centered at the origin of standard spherical polar coordinates, one gets, using Eqs. (102) and (103),

\[
v_{NR} \approx \frac{\sqrt{-C} B'}{C'} \approx \frac{J}{r^2} \sin \theta, \tag{123}
\]

\[
v_{Z} \approx \frac{B C' - C B'}{C' \sqrt{-C}} \approx \frac{3J}{r^2} \sin \theta \approx 3v_{NR}. \tag{124}
\]

The orbitals of an isolated source, if any exist, are on or near the approximate equatorial plane \( \theta = 0 \), but Eqs. (123) and (124) apply at arbitrary \( \theta \) outside an isolated source in the far-field limit.

The stationary observer of the SCAM at an orbital has an acceleration that, in an orthogonal \( (g_{23} = 0) \) coordinate system for the \((2,3)\) plane in which the gradients of \( A, B, \) and \( C \) are purely in the direction of the coordinate \( r = x^2 \), has magnitude

\[
a_0 = \frac{D' - \sqrt{\sigma}}{4D \sqrt{g_{rr}}} = \frac{(B^2 - AC)' - \sqrt{(B^2 - AC)^2 - 4(B^2 - AC)(B'^2 - AC')}}{4(B^2 - AC)\sqrt{g_{rr}}}. \tag{125}
\]

(Here the subscript 0 does not denote the time component of the acceleration, which is zero, but rather the acceleration at the SCAM, for which the subscript 0 has been used.) The acceleration 1-form of the SCAM is then \( a = a_0 e^r \), with

\[
e^r = \sqrt{g_{rr}} \, dr = \sqrt{g_{rr}} \nabla r \tag{126}
\]
being the unit 1-form in the outward radial direction. This acceleration 1-form $a$ is clearly locally defined and invariant under the transformations (36) and (37).

Now one can readily calculate that for any stationary observer (one moving in the $\mathbf{k} \wedge \mathbf{l}$ plane along the orbits of a fixed Killing vector field $\mathbf{K}$) that has speed $v$ in the frame of the SCAM at an orbital, the acceleration 1-form is simply

$$a = a_0 \frac{1-v^2/v_K^2}{1-v^2} \mathbf{e}^r. \quad (127)$$

In particular, the acceleration is a symmetric function of the velocity in the SCAM frame.

One can regard $a_0/(1-v^2)$ as being the (radial outward) gravitational part of the acceleration, which is indeed proportional to $\gamma^2 = 1/(1-v^2)$ as one would get by the equivalence principle for an object moving horizontally across the flat floor of a rocket that has constant acceleration perpendicular to the floor. Then $(a_0/v_K^2)v^2/(1-v^2)$ can be regarded as the centripetal part of the acceleration, pointing radial inward, and also having the $\gamma^2$ dependence that it does in flat spacetime.

For example, suppose that one had an idealized (fictitious) static, spherically symmetric metric with constant $m$,

$$ds^2 = -e^{-2m/R}dt^2 + R^2 \sin^2 \theta d\phi^2 + dR^2 + R^2d\theta^2, \quad (128)$$

which is spatially flat but has a spherically symmetric gravitational potential

$$\Phi = -\frac{m}{R} \quad (129)$$

for a static observer with four-velocity $u = e^{-\Phi} \mathbf{K} = e^{-\Phi} \mathbf{k}$. (I use the radial coordinate $R$ instead of $r$ so that later I can compare with a different coordinate $r$ in a different metric, such as Kerr-Newman. The radial metric component is $g_{RR} = 1$, so the unit radial 1-form $\mathbf{e}^r$ is simply $dR$ or $\nabla R$.)

Here $B \equiv g_{t\phi} = 0$, so the SCAM consists of nonrotating (static) observers, which have acceleration

$$a = \nabla \Phi = \frac{m}{R^2} \mathbf{e}^r = a_0 \mathbf{e}^r. \quad (130)$$

In the equatorial plane, $\theta = \pi/2$, which has orbitals everywhere, a stationary observer orbiting with velocity $v$ relative to that of the static SCAM has acceleration

$$a = a_g + a_c = \left( \frac{m}{R^2(1-v^2)} - \frac{v^2}{R(1-v^2)} \right) \mathbf{e}^r = a_0 \frac{1-v^2/v_K^2}{1-v^2} \mathbf{e}^r \quad (131)$$

with $a_0 = m/R^2$ and $v_K = \sqrt{a_0 R} = \sqrt{m/R}$, so it is natural to split up the total acceleration into a gravitational piece

$$a_g = \frac{m}{R^2(1-v^2)} \mathbf{e}^r = \frac{a_0}{1-v^2} \mathbf{e}^r \quad (132)$$

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and a centripetal piece
\[ a_c = -\frac{v^2}{R(1-v^2)} \mathbf{e}^r = -\frac{a_0 v^2/v_K^2}{1-v^2} \mathbf{e}^r. \] (133)

One might use this analysis with the idealized metric (128) to define an ‘effective orbital radius of curvature’
\[ R \equiv \frac{v_K^2}{a_0} = \frac{(D' - \sqrt{\sigma}) \sqrt{g_{rr}}}{H} = \frac{4D \sqrt{g_{rr}}}{D' + \sqrt{\sigma}} \] (134)
and an ‘effective gravitational mass’
\[ m \equiv \frac{v_K^4}{a_0} = R v_K^2 = \frac{R^3 H}{g_{rr} D} = \frac{(D' - \sqrt{\sigma})^3 \sqrt{g_{rr}}}{H} \] (135)
for an orbital in any stationary axisymmetric metric. These formulas are chosen to make Eqs. (131) - (133) true for a particular orbital in any stationary axisymmetric metric. Then one can write the total acceleration (127) as
\[ a = \left(\frac{v_K^2 - v^2}{1-v^2}\right) \mathbf{e}^r \] (136)
which is thus much simpler in the SCAM frame than the corresponding formula in the ZAMO frame [30].

Following [30], we can also note that there is a simple geometrical description of the effective orbital radius of curvature $R$: If the Killing vector field $\mathbf{l}$ has period $\Delta \varphi = 2\pi$, then the circumferential radius (circumference divided by $2\pi$) measured by a stationary congruence with constant $\Omega$ is
\[ \hat{r} = \sqrt{\frac{D}{F}} = e^{-\Phi} \sqrt{D}. \] (137)
Then if
\[ ds_r \equiv \sqrt{g_{rr}} dr \] (138)
is an infinitesimal element of proper distance in the radial direction (an element of proper radius), one can readily calculate that if $\Omega = \Omega_0$ (to be held constant during the spatial differentiation, as usual in this paper),
\[ R = \hat{r} \frac{ds_r}{d\hat{r}}, \] (139)
which is what one would calculate the proper radial distance to be to a center (where $\hat{r}$ should vanish) if one assumed that $\hat{r}$ varied linearly with proper distance. E.g., for a circle of latitude in the northern hemisphere on the surface of an axisymmetric earth in flat space, $\hat{r}$ would be the cylindrical radial distance from the circle to the axis.
inside the earth (intersecting it orthogonally), $ds_r$ would be an infinitesimal proper distance along a meridian (line of constant longitude) in the southward direction along the earth’s surface, and $R$ would be the distance along a cone tangent to the surface of the earth at the circle, from the circle to the apex of the cone over the north pole.

Then if one uses Eqs. (126), (136), (138), (139), and the fact that $d\hat{r} \equiv \nabla \hat{r} = (d\hat{r}/ds_r)e^r$, one gets an alternative simple formula for the acceleration of a stationary worldline of speed $v$ in the frame of the SCAM,

$$a = \left(\frac{v_K^2 - v^2}{1 - v^2}\right) \frac{\nabla \hat{r}}{\hat{r}} = \frac{\nabla \Phi - v^2 \nabla \ln \hat{r}}{1 - v^2},$$  \quad (140)

Of course, unlike the case for the idealized metric (128), for a more general metric not only the effective orbital radius of curvature $R$, but also the effective gravitational mass $m$, may vary from orbital to orbital. For example, for the Schwarzschild metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + r^2 \sin^2 \theta d\phi^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\theta^2$$  \quad (141)

with constant ADM mass $M$, one has

$$a_0 = \frac{M}{r^2} (1 - \frac{2M}{r})^{-1/2}$$  \quad (142)

and

$$v_K = \sqrt{\frac{M}{r}} (1 - \frac{2M}{r})^{-1/2} = \sqrt{\frac{M}{r - 2M}},$$  \quad (143)

so one gets for the effective radius $R$ and mass $m$

$$R = r(1 - \frac{2M}{r})^{-1/2}$$  \quad (144)

and

$$m = M(1 - \frac{2M}{r})^{-3/2}. \quad (145)$$

In the Schwarzschild metric, one can compensate for this dependence of the effective mass $m$ on the radial coordinate $r$ by defining an ‘effective Scharzschildian mass’

$$\hat{M} = \frac{m}{(1 + 2m/R)^{-3/2}} = \frac{v_K^4}{a_0(1 + 2v_K^2)^{3/2}}$$

$$= \frac{16\sqrt{g_{rr}}D^2H}{[(D' + \sqrt{\sigma})^2 + 8DH]^{3/2}}, \quad (146)$$

which for the Schwarzschild metric is designed to give precisely $M$.

Since only the total acceleration $a$ is observable, one is free to divide it up into ‘gravitational’ and ‘centripetal’ contributions any way one wishes. Although the
split given by Eqs. (131) - (133) above seems most natural to me (and to several others [23, 24, 25, 26, 27, 28, 29, 30, 31, 32]), Abramowicz and his collaborators [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] have advocated an alternative split

\[ \mathbf{a} = \tilde{\mathbf{a}}_g + \tilde{\mathbf{a}}_c \]  

with

\[ \tilde{\mathbf{a}}_g = a_0 \mathbf{e}^r \]  

being independent of velocity and all the velocity dependence being put into the centripetal piece

\[ \tilde{\mathbf{a}}_c = -\frac{a_0 (1 - v^2)}{v_K^2 (1 - v^2)} \mathbf{e}^r. \]

If one wished, with this alternative split one could define an alternative effective radius \( \tilde{R} \) and effective gravitational mass \( \tilde{m} \) so that

\[ \tilde{\mathbf{a}}_g = a_0 \mathbf{e}^r = \frac{\tilde{m}}{\tilde{R}^2} \mathbf{e}^r \]  

and

\[ \tilde{\mathbf{a}}_c = -\frac{v^2}{\tilde{R}(1 - v^2)} \mathbf{e}^r. \]

Then the alternative effective radius is

\[ \tilde{R} \equiv \frac{v_K^2}{a_0(1 - v^2)} = \frac{R}{1 - v^2} = \frac{R^2}{R - m}, \]

and the alternative effective gravitational mass is

\[ \tilde{m} \equiv \frac{v_K^4}{a_0(1 - v^2)^2} = \frac{m R^2}{(R - m)^2}. \]

One can also give the same geometrical interpretation for \( \tilde{R} \) as for \( R \) if one uses instead of \( \tilde{r} \) the optical circumferential radius

\[ \tilde{r} = e^{-\Phi} \tilde{r} = e^{-2\Phi} \sqrt{D} \]

from the optical metric

\[ ds^2 = e^{-2\Phi} ds^2. \]

Then one has

\[ \tilde{R} = \tilde{r} \frac{ds_r}{d\tilde{r}}, \]

and

\[ \mathbf{a} = a_0 \mathbf{e}^r - \left( \frac{v^2}{1 - v^2} \right) \frac{d\tilde{r}}{\tilde{r}} = \nabla \Phi - \left( \frac{v^2}{1 - v^2} \right) \nabla \ln \tilde{r}. \]
When the circular Keplerian orbital velocity $v_K$ exceeds 1 (the speed of light), $\tilde{a}_c$, $\tilde{d}$, and $\tilde{R}$ reverse sign, which is what Abramowicz calls the reversal of the sign of the centrifugal acceleration. However, this interpreted reversal depends on the particular splitting of the total acceleration $a$ into gravitational and centripetal parts given by Eqs. (147) - (149) so that the gravitational part $\tilde{a}_g$ is defined to be independent of the orbital velocity $v$. To me it seems more natural to use the split given by Eqs. (131) - (133) so that the gravitational part $a_g$ has the $\gamma^2$ velocity dependence that one would expect if gravity couples to energy rather than, say, to rest mass. In an equivalence-principle argument, one can readily calculate that the acceleration for objects skimming horizontally over a spatially flat floor of a rocket having constant acceleration vertically in flat spacetime indeed has this $\gamma^2$ velocity dependence.

Nevertheless, even in the rocket example, one could follow Abramowicz and say that part of the acceleration is centripetal acceleration from the curvature of the floor world membrane (the timelike three-surface that is the world history of the accelerating floor). Indeed, the floor looks curved as seen by photon geodesics (which the floor will intercept twice if they are not traveling purely vertically), even though at each moment of time the instantaneous two surface of the floor is flat and contains tachyon geodesics that move at infinite speed. However, one cost of Abramowicz’s interpretation is that no gravitational acceleration would be ascribed to photons, so that they would be interpreted as having no weight, even though they have energy.

### 7.3 Near a rotation axis

A third location in which the properties of the SCAM are more simple than in the generic location (besides the two discussed above, first of the slow-rotation and/or far-field limit that one would expect to find far from an isolated source, and second at the location of orbitals where both corotating and counterrotating circular Keplerian geodesic orbits exist) is infinitesimally near a rotation axis where $\ell$ vanishes. If $\ell$ is given its standard normalization of having closed orbits with period $2\pi$ (i.e., if $\ell = \partial/\partial \varphi$ with $\varphi$ being periodic with period $2\pi$), then regularity of the metric near the axis implies that $C = -\ell \cdot \ell$ must go as $-\varpi^2 + O(\varpi^4)$ (with unit coefficient), to lowest order being the negative square of the distance $\varpi$ from the axis along a spatial geodesic that is orthogonal to $k$ and $\ell$ and which intersects the axis orthogonally. We can let $\varpi$ be one of the cylindrical coordinates for the (2,3) plane orthogonal to $k$ and $\ell$ and let $z$ be the other, with the property that $z$ is proper distance along the axis and that each spatial geodesic mentioned above that intersects the axis orthogonally has constant $z$, so that $\nabla \varpi$ and $\nabla z$ are orthogonal everywhere and the former is normalized to have unit magnitude.
Then in a neighborhood of the axis where these two coordinates are well behaved, the metric may be written

\[
d s^2 = -A(\varpi, z) dt^2 - 2B(\varpi, z) dtd\varphi - C(\varpi, z) d\varphi^2 + d\varpi^2 + g_{33}(\varpi, z) dz^2
\]

\[
= -[A_0(z) + A_2(z) \varpi^2 + O(\varpi^4)] dt^2 - 2[\Omega_Z(z) \varpi^2 + O(\varpi^4)] dtd\varphi + [\varpi^2 + O(\varpi^4)] d\varphi^2
\]

\[
+ d\varpi^2 + [1 + O(\varpi^2)] dz^2.
\]

(158)

On the axis the ZAMO is locally determined, since \( l \) is uniquely determined by the requirements of the previous paragraph if one has access to the axis, unlike the case away from the axis, where one does not have local access to the axis. In particular,

\[
\Omega_Z(z) \equiv \lim_{\varpi \to 0} \frac{B(\varpi, z)}{C(\varpi, z)} = -\frac{1}{2} k_{\alpha;\beta} l^{\alpha;\beta},
\]

where the last expression is to be evaluated on the axis, at \( \varpi = 0 \). However, note that although the ZAMO is locally defined at the axis, the angular velocity \( \Omega_Z \) ascribed to it is not, since the latter depends on the definition of \( k \), which is not locally determined but can be redefined by multiplying it by a constant and by adding any constant multiple of \( l \).

Similarly,

\[
A_0(z) \equiv A(\varpi = 0, z),
\]

and

\[
A_2(z) \equiv \lim_{\varpi \to 0} \frac{\nabla A(\varpi, z) \cdot \nabla C(\varpi, z)}{4C(\varpi, z)} = \frac{1}{8}(2\nabla^2 A - 2A''(z) - A^0(z)^2/A_0(z)),
\]

where in the last expression the (four-dimensional) Laplacian of \( A \) is to be evaluated on the axis, and I am here using the notation that a prime denotes a partial derivative with respect to \( z \). Here I shall also use the convention that when I give the functional dependence of any quantity as \( (z) \), I mean that the quantity is to be evaluated at \( \varpi = 0 \).

If \( e^z = d\varpi \) is the unit 1-form in the \( z \)-direction on the axis, then one can readily calculate that the 'nonrotating' \( (\Omega = \Omega_{NR} \equiv 0) \) congruence with four-velocity \( u = A^{-1/2}k \) has, on the axis, an acceleration 1-form

\[
a(z) \equiv a(z)e^z = \frac{A'_0(z)}{2A_0(z)} e^z
\]

(162)

and a normalized rotation 1-form

\[
\omega_{NR}(z) \equiv \omega_{NR} e^z = -A^{-1/2}_0(z) \Omega_Z e^z.
\]

(163)
This is negatively rotating, since it is the ZAMO (and not, for example, the SCAM) which has the angular velocity $\Omega_Z$ of the Killing vector field $K = k + \Omega_Z l$, which, when $\Omega_Z$ is held constant, has zero rotation or vorticity precisely on the axis, $\omega_Z(z) = 0$. (This is rather opposite to the case at an orbital, where it is the SCAM rather than the ZAMO which has the angular velocity of the Killing vector field that has zero rotation or vorticity there. Thus a SCRAM, which has $\Omega$ chosen at each location to minimize $\omega^2$ there, should interpolate between the SCAM at any orbitals and the ZAMO on an axis of symmetry.)

One should also note that precisely on the axis where $l$ vanishes, $u = A_0^{-1/2}(z)k$ for any $\Omega$, so the acceleration $a(z)$ there is independent of $\Omega$. Thus, strictly speaking, the definition of a SCALE does not work precisely on the axis, but it does if one moves infinitesimally off, and then one can define the SCALEs on the axis to have the limit of $\Omega(\varpi)$ as $\varpi \to 0$.

Now when one evaluates the quantity on the left hand side of Eq. (34) for a SCALE near the axis of symmetry, one finds that it goes as $\varpi^2$ plus higher-order terms in $\varpi$, as the acceleration goes as $a(z)$ plus a correction term that to lowest order is proportional to $\varpi^2$ with a coefficient that depends on $\Omega$. If one divides the left hand side of Eq. (34) by $\varpi^2$ and takes the limit of $\varpi$ going to zero, one gets on the axis not a quartic but a cubic for the SCAM $\Omega_0$ and the other two SCALEs $\Omega_{\pm}$:

$$
(\Omega - \Omega_Z)^3 - (A_2 + \Omega_Z - \frac{A'^2}{4A})(\Omega - \Omega_Z) + \frac{1}{4} A'\Omega'Z = 0.
$$

Here all the quantities are to be evaluated on the axis $\varpi = 0$ and so are functions purely of $z$, and for compactness I have used $A$ and its $z$-derivative $A'$ instead of $A_0$ and $A_0'$, since they are the same quantities on the axis.

There are three distinct real solutions of this equation, giving one local maximum (the angular velocity $\Omega_0$ of the SCAM) and two local minima (the angular velocities $\Omega_{\pm}$ of the remaining two SCALEs) of the magnitude of the acceleration, if the discriminant of the cubic, which is proportional to

$$
27A'^2\Omega_Z^2 - 16(A_2 + \Omega_Z^2 - \frac{A'^2}{4A})^3,
$$

is negative. If, on the other hand, this quantity is positive, there is only one real solution, corresponding to the minimum value of the acceleration, and thus the SCAM does not exist there.

One notes that the coefficient of the quadratic term in $\Omega - \Omega_Z$ is zero. Thus when all three solutions are real and hence denote the angular velocities of the three SCALEs,

$$
\Omega_0 + \Omega_- + \Omega_+ = 3\Omega_Z,
$$

so the ZAMO angular velocity is the average of that of the three SCALEs on the axis.

37
Instead of writing the cubic Eq. (164) for the SCALEs in terms of their angular velocities (which depend on the choice of \( k \)), it may be more illuminating to write it in terms of the orthonormal \( z \)-component \( \omega \), on the axis, of the normalized rotation 1-form \( \omega \) of the rigidly rotating congruence with the corresponding angular velocity \( \Omega \) at the location in question on the axis,

\[
\omega(z) \equiv \omega z = A_0^{-1/2}(z)(\Omega - \Omega_Z)e^z. \tag{167}
\]

Then a bit of algebra, including the use of the standard formula \( [5] \)

\[
\xi_{\alpha;\beta\gamma} = R_{\alpha\beta\gamma\delta} \xi^\delta \tag{168}
\]

for any Killing vector field \( \xi \) to eliminate its covariant derivatives of order higher than one, gives the following simple cubic for the orthonormal rotation component of the limit of a SCALE on the axis:

\[
\omega^3 - p\omega - q = 0, \tag{169}
\]

where

\[
p = R_{t\omega t\omega} - a^2 = \frac{1}{2}(R_{ii} - 3a^2 - a') \tag{170}
\]

and

\[
q = \frac{1}{2}aR_{\phi\omega z\iota} = \frac{1}{2}a(\omega'_R + a\omega_R). \tag{171}
\]

Here all of the quantities (such as the orthonormal Riemann and Ricci curvature components, and the orthonormal \( z \)-component \( a \) of the acceleration—its only nonzero component) are to evaluated in the limit of going onto the axis, the prime denotes a derivative with respect to proper length \( z \) along the axis, and

\[
\omega_R = A^{-1/2}(\Omega_R - \Omega_Z) \tag{172}
\]

is the \( z \)-dependent rotation of any rigidly rotating (\( \Omega_R = \text{const.} \)) congruence, e.g., the nonrotating congruence with \( \Omega = 0 \), though to define this particular congruence requires a specific choice of \( k \) that is not required in Eq. (172). One can easily see that \( (\omega'_R + a\omega_R) \) is invariant under changing from one allowed rigidly rotating congruence to another by changing the constant \( \Omega_R \) in Eq. (172), so the solutions of the cubic Eq. (169) do not depend on this choice. In other words, the coefficients \( p \) and \( q \) are both invariant under the allowed transformations (36) and (37), which on the axis are restricted to have \( \gamma = 0 \) and \( \delta = 1 \) so that only \( k \), but not \( l \), may be changed.

The explicit solutions of the cubic Eq. (169) may be written as

\[
\omega_0 = -2\sqrt{\frac{2}{3}} \sin \left[ \frac{1}{3} \sin^{-1} \left( \frac{27}{4p^3} \right) \right] \tag{173}
\]
for the normalized rotation rate of the SCAM, and

\[ \omega_{\pm} = 2 \sqrt{\frac{p}{3}} \sin \left[ \pm \frac{2\pi}{3} - \frac{1}{3} \sin^{-1} \left( q \sqrt{\frac{27}{4p^3}} \right) \right]. \quad (174) \]

for the rotations of the other two SCALEs on the axis.

After solving the cubic Eq. (169) for the invariant normalized rotation rates \( \omega \), one can use Eq. (167) to solve for the angular velocities \( \Omega_0 \) and \( \Omega_{\pm} \), if one wishes, for a particular choice of \( k \) and hence of \( A \) and of \( \Omega_z \). However, these angular velocities \( \Omega \) are not invariant under the allowed transformations (36) and (37) that change \( k \), whereas the normalized local rotation rates \( \omega \) (of congruences with constant angular velocities \( \Omega \) that match those of the SCALEs at the chosen location on or infinitesimally near the axis) are invariant under these transformations of the Killing vector field \( k \).

To first order in \( q \left( 27/4p^3 \right)^{1/2} \) when \( q^2 \ll p^3 \), the explicit solutions (173) and (174) reduce to the approximations

\[ \omega_0 \approx -\frac{q}{p} = \frac{aR_{\hat{\varphi}z\hat{t}}}{R_{\hat{t}\hat{t}z} - a^2}, \quad (175) \]

\[ \omega_{\pm} \approx \pm \sqrt{p} + \frac{q}{2p}. \quad (176) \]

For example, in the far-field limit outside an isolated stationary source of mass \( M \) and intrinsic angular momentum \( J \) at rest at the origin, with \( z \gg M + \sqrt{J} \) being the positive proper distance along the axis from the source, in the direction of the angular momentum vector, and with the stress-energy tensor (and hence \( R_{\hat{t}\hat{t}} \), assuming Einstein’s equations) being negligible there, one has on the axis \( A = -g_{tt} \approx 1 - 2M/z, \ a = A'/2A \approx M/z^2, \ a' \equiv da/dz \approx -2M/z^3 \ll -a^2, \ R_{\hat{t}\hat{t}z} \approx \frac{1}{2}(R_{\hat{tt}} - a^2 - a') \approx M/z^3, \) so \( p \approx M/z^3 \), and then choosing \( \Omega_R = 0 \) gives \( \omega_R = -A^{-1/2} \Omega_Z \approx -\Omega_Z \approx -2J/z^3 \), \( \omega' \equiv d\omega_R/dz \approx 6J/z^4 \gg |a\omega_R|, \) so \( R_{\hat{\varphi}z\hat{t}} = \omega' + a\omega_R \approx 6J/z^4 \), and \( q \approx 3MJ/z^6 \). Then Eqs. (175) and (176) give

\[ \omega_0 \approx -3J/z^3 \quad (177) \]

and

\[ \omega_{\pm} \approx \pm \sqrt{M/z^3 + (3/2)J/z^3} \approx \pm \sqrt{R_{\hat{t}\hat{t}z}}. \quad (178) \]

Inserting Eq. (177) into Eq. (167) then gives \( \Omega_0 \approx -J/z^3 \), which agrees with the general far-field Eq. (102) with \( r = z \).

One can iterate the approximations of Eqs. (175) and (176) by rewriting the cubic Eq. (169) as

\[ \omega_0 = -\frac{q + \omega_0^3}{p} \quad (179) \]
and as
\[ \omega_{\pm} = \pm \sqrt{p + \frac{q}{\omega_{\pm}}}. \] (180)

Then one starts with setting the terms involving \( \omega_0 \) or \( \omega_{\pm} \) equal to zero on the right hand sides, evaluates the right hand sides to get the first approximations for the left hand sides, enters these approximations back into the right hand sides to get the second approximations, and iterates to get the desired accuracy. This procedure is often a better way to proceed when \( q^2 \ll p^3 \) (particularly when one has functional expressions for \( p \) and \( q \) in terms of some coordinate like \( z \) along the axis) than to use the explicit solutions (173) and (174) of the cubic Eq. (169).

The discriminant of the cubic Eq. (169), which is proportional to
\[ 108q^2 - 16p^3 = 27a^2(R_{\varphi z t})^2 - 16(R_{t z t} - a^2)^3, \] (181)
is negative when the magnitude of the argument of the inverse sine in the exact solutions (173) and (174), \((27q^2/4p^3)^{1/2}\), is less than unity, leading to three real solutions for \( \omega \). When one gets sufficiently deep into a strong rotating gravitational field that \( 27q^2 > 4p^3 \), the roots \( \omega_0 \) and \( \omega_{\pm} \) merge at \(-\sqrt{p/3}\) and then go off into the complex plane, leaving no real \( \omega_0 \) for a SCAM but only the single remaining real root for the SCALE that has a global minimum of the acceleration, with normalized rotation
\[ \omega_+ = (\frac{q}{2})^{1/3}[(1 + \sqrt{1 - \frac{4p^3}{27q^2}})^{1/3} + (1 - \sqrt{1 - \frac{4p^3}{27q^2}})^{1/3}]. \] (182)

8 The SCAM and other SCALEs in the Kerr-Newman metric

Now let us evaluate the properties discussed above of the SCAM (Stationary Congruence Accelerating Maximally) and other SCALEs (Stationary Congruences Accelerating Locally Extremely) in the Kerr-Newman metric
\[ ds^2 = -\frac{\Delta}{\rho^2}[dt - a(1 - c^2)d\varphi]^2 + \frac{1 - c^2}{\rho^2}[(r^2 + a^2)d\varphi - adt]^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{1 - c^2}dc^2, \] (183)
where, to keep everything algebraic, I have used \( c \equiv \cos \theta \) instead of \( \theta \) in what are otherwise Boyer-Lindquist coordinates. Here, as usual (see, e.g., [2, 3, 5, 6, 36])
\[ a \equiv J/M \] (184)
is such a standard Kerr parameter that I shall continue to use it in expressions directly involving the metric, even though elsewhere I use it for the acceleration,
\[ \Delta \equiv r^2 - 2Mr + a^2 + Q^2, \] (185)
which is $D/(1-c^2)$ in terms of $D = g_{01}^2 - g_{00}g_{11}$ defined by (15) and (29) above, and

$$\rho^2 \equiv r^2 + a^2c^2.$$  \hspace{1cm} (186)

The Kerr-Newman metric thus has

$$A \equiv -g_{tt} = 1 - \frac{2Mr - Q^2}{\rho^2} = 1 - 2U,$$  \hspace{1cm} (187)

$$B \equiv -g_{t\varphi} = \frac{a(2Mr - Q^2)(1-c^2)}{\rho^2} = 2aU(1-c^2),$$  \hspace{1cm} (188)

$$C \equiv -g_{\varphi\varphi} = -(r^2 + a^2)(1-c^2) - \frac{a^2(2Mr - Q^2)(1-c^2)^2}{\rho^2}$$
$$= -(r^2 + a^2)(1-c^2) - 2Ua^2(1-c^2)^2,$$  \hspace{1cm} (189)

where

$$U \equiv \frac{2Mr - Q^2}{2\rho^2} = \frac{Mr - Q^2/2}{r^2 + a^2c^2}$$ \hspace{1cm} (190)

is a particular generalization of the Newtonian potential (with the sign reversed to make it positive for $r > Q^2/(2M)$). In fact, another way to write the Kerr-Newman metric above is

$$ds^2 = -dt^2 + (r^2 + a^2)(1-c^2)d\varphi^2 + \frac{r^2 + a^2c^2}{r^2 + a^2}dr^2 + \frac{r^2 + a^2c^2}{1-c^2}dc^2$$
$$+ 2U\{[dt - a(1-c^2)d\varphi]^2 + \frac{(r^2 + a^2c^2)^2dr^2}{(r^2 + a^2)(r^2 + a^2 - 2Mr + Q^2)}\},$$ \hspace{1cm} (191)

where the first line is simply flat spacetime in spheroidal coordinates \[5\]

$$r \equiv \sqrt{\frac{1}{2}[x^2 + y^2 + z^2 - a^2 + \sqrt{(x^2 + y^2 + z^2 - a^2)^2 + 4a^2z^2}]},$$ \hspace{1cm} (192)

$$c \equiv \cos \theta \equiv z/r.$$ \hspace{1cm} (193)

One can in principle obtain the angular velocity $\Omega$ of the SCAM in the Kerr-Newman metric as a function of the coordinates $r$ and $c$ by evaluating Eq. \[14\] and setting it equal to zero, which gives a quartic equation for $\Omega$. However, when this equation is rationalized and expanded out in powers of $a$, $M$, $Q$, $r$, $c$, and $\Omega$, one gets literally hundreds of terms. Thus it seems likely that the explicit solution would take more space to print than the entire rest of this paper, so I have not bothered to do that.

In this way a SCAM, though it is simple to specify implicitly, is not nearly so simple to specify explicitly (e.g., by $\Omega(r, c)$) as a Zero Angular Momentum Observer.
(ZAMO [1, 2]) or even as an Extremely Accelerated Observer (EAO [33, 28]), which is a Stationary Observer whose angular velocity extremizes the cylindrical radial component of the acceleration, and which can be given by a seven-line explicit expression for the Kerr metric ($Q = 0$).

However, one can specify explicitly the SCAM in the three limiting cases described above for a general stationary axisymmetric metric.

### 8.1 In the slow-rotation limit

First, consider the case in which the Kerr parameter $a$ is much smaller than $M$. This is the slow-rotation limit, and evaluating Eq. (101) to first order in $a$ (but including the lowest-order term in $1/r$ which is cubic in the Kerr parameter $a \equiv J/M$, the first term that depends on the angular variable $c \equiv \cos \theta$) gives the angular velocity of the SCAM as

$$\Omega_0 = \frac{-a(Mr^2 - Q^2r + 3a^2c^2)}{r^3(r^2 - 3Mr + 2Q^2)} + O\left(\frac{a^3M^2}{r^6}, \frac{a^3Q^2}{r^6}\right)$$

$$= - \frac{aM}{r^3} \left[1 + \frac{3M^2 - Q^2}{Mr} + \frac{9M^2 - 5Q^2 + 3a^2c^2}{r^2} + O\left(\frac{a^2M}{r^3}, \frac{a^2Q^2}{Mr^3}\right)\right]. \quad (194)$$

This requires that

$$r > \frac{1}{2} (3M + \sqrt{9M^2 - 8Q^2}) \quad (195)$$

in order that the denominator of the first term on the right hand side of Eq. (194) not have changed sign as one comes in from infinity, but the denominator may be arbitrarily small, so long as $a$ is sufficiently small (e.g., not only small compared with $M$, but also much smaller than the square root of the denominator divided by $Mr^2$).

This has the consequence that, in the slow-rotation limit, the SCAM for Kerr-Newman is defined for values of the radial variable $r$ obeying the inequality (195), and its angular velocity is given by Eq. (194) to good accuracy so long as the magnitude of this expression for $\Omega_0$ is much smaller than $1/a$.

By comparison, in the Kerr-Newman metric the ZAMO has angular velocity given explicitly by

$$\Omega_Z = \frac{a(2Mr - Q^2)}{(r^2 + a^2)^2 - a^2\Delta(1 - c^2)} = \frac{a(2Mr - Q^2)}{r^4} + O(a^3). \quad (196)$$

Obviously, in the far-field limit, $r^2 \gg M^2 + Q^2$, Eqs. (194) and (196) agree with Eqs. (102) and (103). However, outside the far-field limit but still within the slow-motion limit, one does not have the simple relation $\Omega_Z = -2\Omega_0$ that one has in the far-field limit.
8.2 At the orbitals (on the equatorial plane)

Second, consider the case of the orbitals (locations of pairs of stationary geodesic observers, corotating and counterrotating timelike circular Keplerian orbits). These all occur on the equatorial plane ($c = 0$) of the Kerr-Newman metric. There the angular velocities of the speed of light, given in general by Eq. (35), are

$$\Omega_{\pm 1} = \pm \frac{r^2 \sqrt{r^2 - 2Mr + a^2 + Q^2 + a(2Mr - Q^2)}}{r^2(r^2 + a^2) + a^2(2Mr - Q^2)},$$

and the angular velocities of the SCALEs that are the Keplerian orbiting stationary geodesics, given in general by Eq. (107), are

$$\Omega_{\pm K} = \pm \frac{r^2 - a^2(\sqrt{Mr - Q^2})}{r^2 \pm a\sqrt{Mr - Q^2}}.$$

On the equatorial plane the other SCALE, the SCAM which gives a local maximum of the acceleration, reduces to what Smererák [33, 28] calls an Extremely Accelerated Observer (EAO), with Eq. (111) giving its angular velocity as

$$\Omega_0 = -\frac{r^2(r^2 - 3Mr + 2Q^2) - 2a^2(Mr - Q^2) - r^2 \sqrt{(r^2 - 3Mr + 2Q^2)^2 - 4a^2(Mr - Q^2)}}{2a[r^2(3Mr - 2Q^2) + a^2(Mr - Q^2)]}
= \frac{-2a(Mr - Q^2)}{r^2(r^2 - 3Mr + 2Q^2) - 2a^2(Mr - Q^2) + r^2 \sqrt{(r^2 - 3Mr + 2Q^2)^2 - 4a^2(Mr - Q^2)}}
= \frac{-a(Mr - Q^2)}{r^2(r^2 - 3Mr + 2Q^2)} \left[ 1 + \frac{a^2(Mr - Q^2)(2r^2 - 3Mr + 2Q^2)}{r^2(r^2 - 3Mr + 2Q^2)^2} + O(a^4) \right],
= \frac{-a(Mr - Q^2)}{r^2 - 3Mr + 2Q^2}
= \frac{r^2[r^2 + 9M^2 - 5MQ^2 + 27M^4 - 21MQ^2 + 2Q^4 + 2a^2M^2]}{r^3} + O(\frac{1}{r^4}).$$

a straightforward extension to $Q \neq 0$ of the result in the Kerr metric [33, 28, 25].

The velocities of the circular Keplerian orbits in the SCAM frame (both of equal magnitudes but of opposite signs, as Smererák found was the case in the Kerr equatorial plane [28] and later found in general [34]) have magnitude given by Eq. (117) specialized to the Kerr-Newman metric:

$$v_K = \frac{r(r - M) - \sqrt{(r^2 - 3Mr + Q^2)^2 - 4a^2(Mr - Q^2)}}{2\sqrt{(Mr - Q^2)(r^2 - 2Mr + a^2 + Q^2)}}
= \frac{r(r - M) + \sqrt{(r^2 - 3Mr + Q^2)^2 - 4a^2(Mr - Q^2)}}{2\sqrt{(Mr - Q^2)(r^2 - 2Mr + a^2 + Q^2)}}
= \sqrt{\frac{Mr - Q^2}{r^2 - 2Mr + Q^2}} \left[ 1 + \frac{a^2r(r - M)}{2(r^2 - 2Mr + Q^2)(r^2 - 3Mr + 2Q^2)^2} + O(a^4) \right].$$
Similarly, Eq. (125) specialized to the Kerr-Newman metric gives the magnitude of the acceleration of the equatorial SCAM as

\[
a_0 = \frac{\sqrt{Mr - Q^2}}{r^2}v_K
\]

\[
= \frac{r(r - M) - \sqrt{(r^2 - 3Mr + Q^2)^2 - 4a^2(Mr - Q^2)}}{2r^2\sqrt{r^2 - 2Mr + a^2 + Q^2}}
\]

\[
= \frac{2(Mr - Q^2)\sqrt{r^2 - 2Mr + a^2 + Q^2}}{r^2[r(r - M) + \sqrt{(r^2 - 3Mr + Q^2)^2 - 4a^2(Mr - Q^2)}]}
\]

\[
= \frac{Mr - Q^2}{r^2\sqrt{r^2 - 2Mr + Q^2}}[1 + \frac{a^2r(r - M)}{2(r^2 - 2Mr + Q^2)(r^2 - 3Mr + 2Q^2)^2} + O(a^4)]
\]

(201)

Then one can use Eq. (127) to get the acceleration of a stationary observer at any speed \(v\) relative to that of the SCAM on the equatorial plane.

Many of these formulas for the properties of the SCAM on the equatorial plane of the Kerr-Newman metric are simpler in the case of extreme Kerr-Newman, \(Q^2 = M^2 - a^2\), so that \(\Delta = (r - M)^2\) and the event horizon is at \(r = M\). Then, taking \(a\) to be positive, one can calculate that the SCAM exists for values of \(r\) down to \(2M + 2a\), where \(\sigma\) defined by Eq. (119) passes through zero and goes negative, making \(\Omega_0\) and various other quantities complex for smaller \(r\). Hence I shall first give the value at general \(r\) when \(Q^2 = M^2 - a^2\), and then, after the first arrow in each equation, the limiting value for each quantity when one sets \(r = 2M + 2a\). Finally, after a second arrow, I shall give the limiting value in extreme Kerr \(a = M\) (so \(Q = 0\)) at \(r = 4M\), the radial inner boundary on the equatorial plane of the region where the SCAM exists as a real congruence:

\[
\Omega_{+1} = \frac{r^2(r - M) + a(2Mr - M^2 + a^2)}{r^2(r^2 + a^2) + a^2(2Mr - M^2 + a^2)} \rightarrow \frac{M + 3a}{4M^2 + 9aM + 7a^2} \rightarrow \frac{1}{5M^3}
\]

(202)

\[
\Omega_{-1} = \frac{-r^2(r - M) + a(2Mr - M^2 + a^2)}{r^2(r^2 + a^2) + a^2(2Mr - M^2 + a^2)} \rightarrow \frac{-1}{4M + 3a} \rightarrow \frac{1}{7M^3}
\]

(203)

\[
\Omega_{+K} = \frac{\sqrt{Mr - M^2 + a^2}}{r^2 + a\sqrt{Mr - M^2 + a^2}} \rightarrow \frac{1}{4M + 5a} \rightarrow \frac{1}{9M}
\]

(204)

\[
\Omega_{-K} = \frac{-\sqrt{Mr - M^2 + a^2}}{r^2 - a\sqrt{Mr - M^2 + a^2}} \rightarrow \frac{-1}{4M + 3a} \rightarrow \frac{1}{7M}
\]

(205)

\[
\Omega_0 = \frac{-2a(Mr - M^2 + a^2)}{r^2(r - M)(r - 2M) - 2a^2(r^2 + Mr - M^2 + a^2) + r^2(r - M)\sqrt{(r - 2M)^2 - 4a^2}}
\]

\[
\rightarrow \frac{-1}{4M + 3a} \rightarrow \frac{1}{7M^3}
\]

(206)
\[ \Omega_z = \frac{a(2Mr - M^2 + a^2)}{(r^2 - ar + aM + a^2)(r^2 + ar - aM + a^2)} \rightarrow \frac{a(3M + a)}{(4M + 3a)(4M^2 + 9aM + 7a^2)} \rightarrow \frac{1}{35M}. \]  
\[ (207) \]

\[ v_K = \frac{r - \sqrt{(r - 2M - 2a)(r - 2M + 2a)}}{2\sqrt{Mr - M^2 + a^2}} = \frac{2\sqrt{Mr - M^2 + a^2}}{r + \sqrt{(r - 2M - 2a)(r - 2M + 2a)}} \rightarrow 1 \rightarrow 1, \quad (208) \]

\[ a_0 = \frac{r - \sqrt{(r - 2M - 2a)(r - 2M + 2a)}}{2r^2} \rightarrow \frac{1}{4(M + a)} \rightarrow \frac{1}{8M}. \quad (209) \]

One can thus see that at the inner boundary of the SCAM, at \( r = 2(M + a) \) on the equatorial plane of extreme Kerr-Newman (\( Q^2 = M^2 - a^2 \)), both the counterrotating circular Keplerian orbit and the SCAM have angular velocities, \( \Omega_{-K} \) and \( \Omega_0 \) respectively, that approach the angular velocity \( \Omega_{-1} \) of the counterrotating speed of light. Since at this inner boundary \( v_K = 1 \), Eq. (127) says that the acceleration there is independent of the (signed) speed \( v \) of a stationary observer relative to the SCAM if \( |v| < 1 \). However, one must take care, since this formula is then degenerate at \( |v| = 1 \), and since Eq. (116) with \( \Omega_1 = \Omega_0 = \Omega_{-1} \) gives \( v(\Omega_2, \Omega_1) = 1 \) for any angular velocity \( \Omega_2 > \Omega_1 \). For \( \Omega > \Omega_0 = \Omega_{-1} \), one should instead return to Eq. (31), which gives at \( r = 2(M + a) \) in extreme Kerr-Newman the acceleration

\[ a = \frac{(M + 2a)[1 - (4M + 5a)\Omega]}{4(M + a)((M + 3a) - (4M^2 + 9aM + 7a^2)\Omega)}e^r \rightarrow \frac{3}{32M} \left( \frac{1 - 9M\Omega}{1 - 5M\Omega} \right)e^r, \]  
\[ (210) \]

with the expression after the arrow being that at \( r = 4M \) when \( a = M \) (and hence \( Q = 0 \)). Note that Eq. (210) is one where the \( a \) on the left hand side is the acceleration, whereas all the \( a \)'s on the right hand side denote the Kerr parameter \( a = J/M \), as I have warned.

### 8.3 On the axis of rotation

The third limiting case where one can give the SCAM explicitly for the Kerr-Newman metric without solving a very messy quartic equation is on one of the axes of symmetry, say, for concreteness, the one at \( \theta = 0 \) (where \( c \equiv \cos \theta = 1 \)). There the rotation \( \omega_0 \) of the SCAM is given by the solution (173) of the cubic Eq. (169), \( \omega^3 - p\omega - q = 0 \), where the coefficients \( p \) and \( q \), given by Eqs. (170) and (171) in the general case, take on, in the Kerr-Newman metric, the values

\[ p = \frac{Mr^2 - Q^2r^2 - 3a^2Mr + a^2Q^2}{(r^2 + a^2)^3} - \frac{(Mr^2 - Q^2r - a^2M)^2}{(r^2 + a^2)^3(r^2 - 2Mr + a^2 + Q^2)} \]  
\[ (211) \]

and

\[ q = \frac{a(Mr^2 - Q^2r - a^2M)(3Mr^2 - 2Q^2r - a^2M)}{(r^2 + a^2)^{9/2}(r^2 - 2Mr + a^2 + Q^2)^{1/2}}. \]  
\[ (212) \]
Once one has the normalized rotation $\omega_0$, which is independent of the choice of the Killing vector field $k$ (though near the axis $l$ is uniquely determined, up to sign, by its property of having a magnitude which goes as the proper distance from the axis), one can calculate, for $k = \partial/\partial t$, the angular velocity $\Omega_0$ of the SCAM arbitrarily near the axis by solving Eq. (167). On the axis of the Kerr-Newman metric, this gives

$$\Omega_0 = \Omega_Z + \sqrt{A} \omega_0 = \frac{q(2Mr - Q^2)}{r^2(r^2 + a^2)} - 2 \sqrt{\frac{r^2 - 2Mr + a^2 + Q^2}{r^2 + a^2}} \sqrt{\frac{p}{3}} \sin \left( \frac{1}{3} \sin^{-1} \left( q\sqrt{\frac{27}{4p^2}} \right) \right). \quad (213)$$

In the case of the extreme Kerr metric, $a = M$ and $Q = 0$, if we let $x \equiv r/M$ and $y \equiv (r^2 + M^2)^{3/2}\omega/M^2$, then the cubic Eq. (169) becomes

$$y^3 - (x^3 - x^2 - 5x - 1)y - (3x^3 + 3x^2 - x - 1) = 0. \quad (214)$$

The discriminant of this cubic is proportional to $31 + 114x + 115x^2 + 56x^3 + 12x^4 - 4x^5$, which is negative for

$$x \equiv \frac{r}{M} \gtrsim 6.158629999016071260705,$$  

the region where the cubic has three real roots (one for the SCAM and one for each of the other two SCALEs that are local minima of the acceleration). Therefore, near the axis of an extreme Kerr black hole, the SCAM exists only outside $r \approx 6.15862999M$, whereas on the equatorial plane it exists outside $r = 4M$. 

Although it is rather messy to do for general $a$ and $Q$, for extreme Kerr one can also readily compare expansions of $\Omega_0$ in inverse powers of $r$ in the equatorial plane and on the axis. In the equatorial plane for $a = M$ and $Q = 0$ one gets

$$\Omega_0 = -\frac{M^2}{r^3} \left[ 1 + \frac{3M}{r} + \frac{9M^2}{r^2} + \frac{29M^3}{r^3} + O\left( \frac{M^4}{r^4} \right) \right], \quad (216)$$

whereas on the axis one gets

$$\Omega_0 = -\frac{M^2}{\rho^3} \left[ 1 + \frac{3M}{\rho} + \frac{12M^2}{\rho^2} + \frac{53M^3}{\rho^3} + O\left( \frac{M^4}{\rho^4} \right) \right]. \quad (217)$$

The difference in the two expressions, starting at the second-order correction, persists even when one changes the radial variable from $r$ to $\rho \equiv \sqrt{r^2 + a^2 \cos^2 \theta}$ (which is the same as $r$ on the axis), since then the series on the equatorial plane becomes

$$\Omega_0 = -\frac{M^2}{\rho^3} \left[ 1 + \frac{3M}{\rho} + \frac{21M^2}{2\rho^2} + \frac{35M^3}{\rho^3} + O\left( \frac{M^4}{\rho^4} \right) \right], \quad (218)$$

whereas on the axis it has the same form as Eq. (217) but with $r$ replaced by $\rho$. 

The first three terms of Eqs. (216) and (217) can be readily be seen to agree with Eq. (194) when $a = M$, $Q = 0$, and either $c = 0$ ($\theta = \pi/2$) or $c = 1$ ($\theta = 0$).
9 Stationary Worldlines Accelerating Radially Maximally (SWARM)

Because the explicit expression for the angular velocity $\Omega_0$ of the SCAM (a root of a messy quartic equation) appears to be generally rather intractable except in the special cases discussed above (slow rotation, at an orbital, and on an axis), it might be useful to define other congruences that have simpler explicit expressions and which have some of the properties of the SCAM. For example, one might define Stationary Worldlines Accelerating Radially Maximally (SWARM) as those that have $\Omega$ chosen to maximize the radial component of the acceleration.

Then the question arises as to how to define the radial component (or direction, since the radial component is needed only up to a positive constant of proportionality at each location in order to be able to define its maximum as a function of the angular velocity). Semerák [33, 28] defined a maximum of the cylindrical radial component of the acceleration in the Kerr metric as an Extremely Accelerated Observer (EAO), but I prefer to define something different here and leave that name for what he defined there. (And to avoid confusion I propose that the EAO retain its original definition in [28], rather than being redefined to be the SCAM that I invented, despite Semerák’s proposal to do that in [34].)

Alternatively, if one defined the radial direction to be that of the acceleration of the SCAM, then of course the SWARM would simply be the SCAM, but then finding the radial direction would involve solving a quartic equation, and there would be no advantage to defining a SWARM.

Therefore, I propose that a SWARM be defined so that in the Kerr-Newman metric the radial direction is that of $\nabla r$, the gradient of the Boyer-Lindquist radial coordinate $r$. This direction has several remarkable properties in Kerr-Newman, connected with the existence of Carter’s ‘fourth constant of motion’ [37, 2].

For example, there is the fact that for a timelike geodesic, out of the set of four parameters that govern the orbit in the (2,3) or $(r, c \equiv \cos \theta)$ coordinates (e.g., the value of $r$ at $c = 0$, the value of $dr/dc$ there, the value of the conserved energy per rest mass $-u_0 \equiv -u_t$, and the value of the conserved angular momentum per rest mass $u_1 \equiv u_\phi$), a two-parameter subset leads to orbits with constant $r$ (e.g., $r$ and $u_1$, setting $dr/dc = 0$ at $c = 0$ and choosing $-u_0$ as a function of $u_1$ so that $d^2r/dc^2 = 0$ there). For a generic stationary axisymmetric metric, one could define a radial coordinate so that a one-parameter set of geodesics have constant $r$ (e.g., $r$ along some one-dimensional line in the $(2,3)$ plane analogous to the $c = 0$ line, by defining $r$ so that it is constant along the orbit that one gets with a specific choice of $-u_0$ and $u_1$), but if one tried varying a second parameter, e.g. $u_1$, then there would be no choice of $-u_0$ for different values of $u_1$ that would give other orbits along the same constant $r$ line.
In the Kerr-Newman metric, the geodesic orbits with constant $r$ typically oscillate in $c$ (unless they are circular Keplerian orbits in the equatorial plane, which stay at fixed $c = 0$ as well as constant $r$ and hence are stationary geodesics), going between a positive maximum where $dc/d\tau = 0$ as well as $dr/d\tau = 0$ (so that the velocity is momentarily zero in the $(2,3)$ plane at that turning point, though of course the velocity is not zero in the $(0,1)$ plane), and a negative minimum of equal magnitude (because of the symmetry with respect to reflections about the equatorial plane) where also the velocity in the $(2,3)$ plane is momentarily zero, $dc/d\tau = 0$ as well as $dr/d\tau = 0$.

However, in a generic stationary axisymmetric metric, an orbit that starts with zero velocity somewhere in the $(2,3)$ plane will not generically have zero velocity elsewhere on its orbit in that plane. Nevertheless, requiring zero velocity in the $(2,3)$ plane at one point there puts only one restriction on the conserved quantities $-u_0$ and $u_1$, so by choosing $u_1$, say, appropriately (which then leaves $-u_0$ determined as a function of $u_1$ so that the velocity in the $(2,3)$ plane is zero there), one has the right number of free parameters to be able to get zero velocity in the $(2,3)$ plane at some other location. If this requirement of a second turning point of zero velocity in the $(2,3)$ plane uniquely fixes $u_1$ (and hence also $-u_0$) at the original turning point, then the direction in which the orbit starts out (determined by $dx^a/d\tau = 0$ at the turning point where $dx^a/d\tau = 0$) could be defined as the angular direction, the direction of constant radius $r$ (after defining the radius $r$ suitably). Then the orthogonal direction could be defined as the radial direction.

This is admittedly a nonlocal definition of the radial direction, and it might not always give a unique answer, but it is a procedure that would give the $\nabla r$ direction in the Kerr-Newman metric and presumably would give a unique direction for small perturbations of that metric. So let me define the radial direction by this procedure when it works, and then define a prime as denoting a partial derivative in that direction.

Then Eq. (111) gives the angular velocity of what is now the SWARM, which maximizes the radial component (rather than the entire magnitude) of the acceleration. Eq. (109) gives the angular velocities, not of circular Keplerian orbits, but of stationary worldlines with zero radial component of the acceleration. These will also be the angular velocities at that location of the nonstationary geodesic orbits that have a turning point in the $(2,3)$ plane at that location and also have $d^2r/d\tau^2 = 0$ there. By the same argument as before, these have equal but opposite velocities in the frame of the SWARM, of magnitude given by Eq. (117). Furthermore, Eq. (125) gives, not the total acceleration, but the radial component of the acceleration of the SWARM at that location, and Eq. (127) gives the part of the acceleration in the radial direction for a stationary worldline at speed $v$ in the SWARM frame.

In the Kerr-Newman metric (183), one can write out all these expressions explic-
itly as functions of $r$ and $c$, but they don’t easily fit into single lines, so I won’t bother doing that straightforward calculation here. One can see that in the Kerr-Newman metric, which has a two-parameter family of timelike geodesics with constant $r$ as mentioned above, the SWARM four-velocity is the normalized average of the four-velocities of the two constant-$r$ geodesics that have their turning points in the $(2,3)$ plane at that location (i.e., have their maximum of $|c|$ there). In other words, if one takes two equal-mass particles moving along constant-$r$ geodesics that at a certain location in the $(2,3)$ plane have their four-velocity entirely in the $(0,1)$ plane, with one moving forward in $\varphi$ and the other moving backward, then if one makes a totally inelastic collision between these particles, immediately after the collision the resulting particle will have the four-velocity of the SWARM at that location.

One can also say that on the equatorial plane in the Kerr-Newman metric, as at any orbital in a generic stationary axisymmetric metric, the total acceleration is in the radial direction, so the SWARM has the same four-velocity as the SCAM there. The four-velocity (and acceleration) also agrees on an axis, trivially, since there the four-velocity is independent of the angular velocity, but the angular velocity of the SWARM is not the same as that of the SCAM near the axis of the Kerr-Newman metric. In the limit of going onto the axis itself, Eq. (213) gives the angular velocity of the SCAM, and the angular velocity of the SWARM there is

$$\tilde{\Omega}_0 = \frac{-a(Mr^2 - Q^2r - a^2M)}{(r^2 + a^2)(r^3 - 3Mr + 2Q^2r + a^2r + a^2M)}$$

$$= -\frac{aM}{r^3} \left[1 + \frac{3M^2 - Q^2}{Mr} + \frac{9M^2 - 5Q^2 - 2a^2}{r^2} + O\left(\frac{a^2M}{r^3}, \frac{a^2Q^2}{Mr^3}\right)\right]. \quad (219)$$

By comparing with the series expansion of Eq. (194) at the axis ($c^2 = 1$), one sees that, to lowest order in $1/r$, the angular velocity of the SWARM on the axis is less negative than that of the SCAM by $5a^3M/r^5$, so if $a/M$ is small, the SWARM is a very good approximation for the SCAM on the axis, and, presumably, at all other angles or values of $c$. This agreement is a consequence of the fact, that for small $a/M$ at least, the direction of the acceleration of the SCAM is nearly radial in the Kerr-Newman metric, so maximizing the radial component of the acceleration for the SWARM (Stationary Congruence Accelerating Radially Maximally) gives very nearly the same angular velocity as finding the local maximum of the magnitude of the total acceleration for the SCAM (Stationary Congruence Accelerating Maximally).

10 Application to Maximally Rotating Stars

One application of the fact that the SCAM is usually counterrotating relative to a rotating source (e.g., a black hole or star) is an explanation of the fact that corotating
Keplerian orbits in an equatorial plane of the source may have periods that are longer than one would get from a naïve application of Kepler’s third law. In certain cases Kepler’s third law gives a better approximation to the orbital frequency $\Omega - \Omega_0$ relative to the SCAM, rather than the orbital frequency $\Omega$ relative to a nonrotating observer with $\Omega_{NR} = 0$. Since $\Omega_0$ is typically negative (when the coordinate system is chosen so that the source is rotating positively), the orbital frequency relative to a nonrotating observer will be slower than that relative to the SCAM, so its period will be greater.

Physically, this effect may be explained by the same mechanism used above to explain the counterrotation of the SCAM: A corotating orbiting observer will see the part of the source nearest her as partially moving with her and hence as having a lower energy density in her frame than the part of the source farthest away from her, which is moving in the opposite direction. Hence she will see the energy distribution shifted slightly away from her, where it will have a weaker net gravitational attraction on her than a source of the same energy density distribution (as seen in a nonrotating frame) that is not rotating. Therefore, she will orbit more slowly by this relativistic effect that can be ascribed to the angular momentum of the source.

Realistic situations are further complicated by the fact that the source (e.g., a star) will not generally have the same energy density distribution in a nonrotating frame when it is rotating as when it is nonrotating. One effect is that if a source is spun up, it will gain rotational energy and hence total mass-energy. However, this effect can easily be compensated for by removing from the source an amount of energy equal to that given it in spinning it up, say by removing an appropriate number of baryons. In any case, if one is using Kepler’s third law, the mass in that formula should be the total mass-energy in the source, so a change in the total mass is already taken into account.

However, another effect that is not taken into account by Kepler’s third law is that the shape of the source generally changes when it is spun up. For example, if the source is a self-gravitating fluid, such as a star, the centrifugal forces of the rotation will generally cause the star to become oblate. Then as the energy becomes more concentrated upon the equatorial plane, where it is on average nearer to the observer orbiting in that plane, it will exert a greater gravitational attraction upon the observer, causing her to orbit faster than she would have in the absence of the oblateness. Equivalently, the greater gravitational attraction at a fixed radius in the equatorial plane is caused by the quadrupole moment of the source.

This oblateness or quadrupole effect on the orbital frequency is opposite in sign to the relativistic effect of the source angular momentum discussed above. At low source angular velocities, the relativistic effect is linear in the source angular velocity, whereas the oblateness effect is quadratic. However, the quadrupole effect...
persists even in the Newtonian limit, so it can be larger than the relativistic angular momentum effect.

As an example in which the corotating orbital period in the equatorial plane can be calculated exactly and compared with Kepler’s third law, consider the Kerr metric with zero charge. Then from Eq. (198), and temporally restoring Newton’s constant \( G \) and the speed of light \( c \) that have been set equal to unity, one can readily get the period as

\[
P = \frac{2\pi}{\Omega_{+K}} = 2\pi \sqrt{\frac{r^3}{GM}} + \frac{2\pi J}{Mc^2}.
\]

The second term on the right hand side can be identified with the linear increase in the period with the angular momentum \( J \), and the square of the speed of light in the denominator shows that it is a relativistic effect. (It is interesting that Newton’s constant does not appear in this term. If one takes the quantum-mechanical phase of a system with energy \( E = Mc^2 \) and angular momentum \( J \) to be \( e^{(-iEt+iJ\phi)/\hbar} \), then the time period for this phase to rotate around once is precisely this second term. However, Newton’s constant does appear in the ratio of the second term to the first term, so the increase in the period with the angular momentum really does involve both special relativity and gravity and is thus a general relativistic effect rather than purely a special relativistic effect.)

It is tempting to identify the first of the two terms on the right hand side of Eq. (220) as being precisely Kepler’s third law for a circular orbit of radius \( r \), but there is the question of whether \( r \) is the most natural measure of the the radius. In the nonrotating limit (Schwarzschild, \( J = Ma = 0 \)), \( r \) is the circumferential radius, \( 1/2\pi \) times the proper circumference of a closed circle in the equatorial plane with fixed \( r \) and \( t \), which is a fairly simple geometric definition of a radius that makes Kepler’s third law exact for circular orbits in the Schwarzschild metric. If in this section one takes \( R \) to be the circumferential radius in the equatorial plane around a source of total mass \( M \), then one can define

\[
\Gamma \equiv \frac{R^3\Omega_{+K}^2}{GM}
\]

as a measure of how closely Kepler’s third law holds, which would state that \( \Gamma = 1 \).

Although \( \Gamma = 1 \) for the Schwarzschild metric, in the equatorial plane of the Kerr metric with \( J \neq 0 \) the circumferential radius is

\[
R = \sqrt{-C} = \sqrt{r^2 + \frac{J^2}{M^2c^2} + \frac{2GJ^2}{c^4Mr}},
\]

so

\[
\Gamma = \left(1 + \frac{J}{c^2\sqrt{\frac{G}{Mr^3}}}\right)^{-2} \left(1 + \frac{J^2}{M^2c^2r^2} + \frac{2GJ^2}{Mc^2r^3}\right)^{3/2}
\]
\[ = 1 - \frac{2J}{c^2} \sqrt{\frac{G}{Mr^3}} + \frac{3J^2}{2M^2c^2r^2} + O(r^{-3}). \tag{223} \]

Alternatively, one can solve Eq. (222) for \( r \) as a function of \( R \) and insert this into Eq. (220) to get the orbital period (as seen from radial infinity) as

\[ P = \frac{2\pi}{\Omega_{+K}} = 2\pi \sqrt{\frac{R^3}{GM} + \frac{2\pi J}{Mc^2} - \frac{3\pi J^2}{2M^2c^2\sqrt{GM}} + O(R^{-3/2}).} \tag{224} \]

The third term on the right hand side represents the effect of the quadrupole moment of the Kerr metric.

Since in the Kerr metric a stationary observer must have \( r > GM/c^2 \), and since the dimensionless Kerr rotation parameter,

\[ \alpha \equiv \frac{a}{M} \equiv \frac{cJ}{GM^2}, \tag{225} \]

is less than or equal to unity, the negative of the ratio of the second term to the third term in the last expression for \( \Gamma \) in Eq. (223), or in the last expression for \( P \) in Eq. (224), is

\[ \sqrt{\frac{16GM^3r}{9J^2}} = \frac{4}{3\alpha} \sqrt{\frac{c^2r}{GM}} > 1. \tag{226} \]

Therefore, at least at large \( r \) where one can drop the \( O(r^{-3}) \) term in Eq. (223) or the \( O(R^{-3/2}) \) term in Eq. (224), the second term of either of these equations, which represents the relativistic period-increasing effect that is linear in the angular momentum \( J \), dominates over the third term, which represents the effect of the period-decreasing quadrupole moment that is quadratic in the angular momentum.

However, if the dimensionless Kerr rotation parameter \( \alpha \) is larger than about 0.952518, then there are stable circular corotating orbits in Kerr (which exist for \( \]

\[ r^2 - 6Mr + 8a\sqrt{Mr} - 3a^2 \geq 1, \tag{227} \]

temporarily reverting to units in which \( G = c = 1 \)) for which the effect of the quadrupole moment dominates so that \( \Gamma > 1 \). For example, for the extreme Kerr metric (\( \alpha = 1 \) or \( a = M \)), \( \Gamma > 1 \) for \( r \) less than about 2.01186\( GM/c^2 \), and at the smallest innermost stable circular corotating orbit at \( r = GM/c^2 \), one has \( R = 2GM/c^2, \Omega_{+K} = c^3/2GM \), and hence \( \Gamma = 2 \).

If we turn to models of maximally rotating stars (stars with the equatorial surface rotating at the Keplerian velocity and hence just marginally bound), typically the quadrupole moments are larger than they are for the Kerr metric with the same stellar mass and angular momentum, because stars are not so gravitationally concentrated. Thus one gets a larger radius at which the relativistic period-increasing effect (linear in the angular momentum) balances the effect of the period-decreasing
quadrupole moment. For nonrelativistic maximally rotating stars this radius tends to be outside the surface of the star, so that one usually gets $\Gamma > 1$ from Eq. (221) when $R$ is set to be the circumferential equatorial radius of the star in its rotating frame, and $\Omega_{+K}$ is both the angular velocity of the corotating circular equatorial orbit at the surface of the star, and the angular velocity of the star itself.

However, for certain relativistic maximally rotating stars, the relativistic period-increasing effect can exceed the period-decreasing effect of the quadrupole moment, making $\Gamma < 1$. For example, in the numerical models of rapidly rotating polytropes in general relativity by Cook, Shapiro, and Teukolsky [39], the maximum uniform rotation models of the “supramassive” sequence given in their Table 2 have $\Gamma < 1$ for values of the polytropic index $n \leq 1.5$, namely $\Gamma = 0.958$ for $n = 0.5$, $\Gamma = 0.988$ for $n = 1.0$, and $\Gamma = 0.999$ for $n = 1.5$. Higher values of the polytropic index, $n \geq 2$, generally seem to give $\Gamma > 1$, namely $\Gamma = 1.005$ for $n = 2.0$ and $\Gamma = 1.009$ for $n = 2.5$. The data from [39] for $n = 2.9$ naively gives $\Gamma = 0.999 < 1$, but since the data are given only to three places, and since this large value of $n$ gives a highly nonrelativistic model ($2GM/Rc^2 = 0.0109$, as opposed to the highly relativistic value $2GM/Rc^2 = 0.578$ for $n = 0.5$, for example), I would be sceptical that actually $\Gamma < 1$ for this large value of $n$.

Similarly, one might doubt that $\Gamma < 1$ for $n = 1.5$, where the three-place data give a result also just slightly below unity, but presumably there is a value of the polytropic index $n$ fairly near 1.5 such that $\Gamma \equiv R^3\Omega^2_{+K}/(GM) < 1$ for maximally uniformly rotating supramassive polytropic models with smaller $n$ but such that $\Gamma > 1$ for similar models with larger $n$. The small values of $n$ give relativistic stellar models for which the counterrotation of the SCAM is sufficient to make it so that the corotating Keplerian orbital velocity (which matches the stellar rotation rate at the equatorial surface in these maximally rotating models) relative to a nonrotating observer at infinity is slowed down by this relativistic effect more than it is sped up by the quadrupole moment.

11 Application to the gravitational field of the Sun and Solar System

Another application of the stationary congruences defined above and of the related deviations from Kepler’s third law is to the gravitational field of the Sun. Since the Sun is nearly spherical and is not very relativistic, the metric for its external gravitational field may be given, to an accuracy of about one part in $10^{15}$, and using units in which $G = c = 1$, as [40]

$$ds^2 = - \left[1 - \frac{2M_\odot}{r} + \frac{2Q_\odot}{r^3} P_2(\cos \theta)\right]dt^2 + \left[1 - \frac{2M_\odot}{r} + \frac{2Q_\odot}{r^3} P_2(\cos \theta)\right]^{-1}dr^2$$
\[ + \left[1 - \frac{2Q_\odot}{r^3} P_2(\cos \theta)\right] r^2 [d\theta^2 + \sin^2 \theta (d\varphi - \frac{2J_\odot}{r^3} dt)^2]. \] (228)

Here \( P_2(\cos \theta) = (3 \cos^2 \theta - 1)/2 \) is the standard second-order Legendre polynomial, and \( Q_\odot \) is the quadrupole moment of the Sun (not to be confused with the previous use of \( Q \) to denote the charge of the Kerr-Newman black hole; in this section the charge will always taken to be zero and \( Q \) will always denote a quadrupole moment, with its sign chosen so that an oblate spheroid, such as a rotating body, has positive \( Q \)),

\[ Q_\odot \equiv J_2 M_\odot R_\odot^2 = - \int \rho r^2 P_2(\cos \theta) dr \sin \theta d\theta d\varphi = \int \rho \left( \frac{1}{2} x^2 + \frac{1}{2} y^2 - z^2 \right) dxdydz. \] (229)

Here \( R_\odot \) is the radius of the Sun, and \( J_2 \) is its dimensionless quadrupole moment parameter.

In metric length units (meters), the 1996 Review of Particle Physics [41] gives the mass and radius of the Sun as

\[ GM_\odot/c^2 = 1 476.625 04 \text{ m} = 0.9137 \times 10^{38} \ell_P, \] (230)

\[ R_\odot = 6.96 \times 10^8 \text{ m} = 4.31 \times 10^{43} \ell_P, \] (231)

where \( \ell_P \equiv \sqrt{\hbar G/c^3} = (1.616 05 \pm 0.000 10) \times 10^{-35} \text{ m} \) is the Planck length, using data from the same source [41].

A recent helioseismic determination of the solar gravitational quadrupole moment [42], which is consistent with a less-precise direct measurement of the solar oblateness [43], gives the dimensionless solar quadrupole parameter as

\[ J_2 = (2.18 \pm 0.06) \times 10^{-7}. \] (232)

Therefore, the solar quadrupole moment is

\[ Q_\odot = J_2 M_\odot R_\odot^2 = (2.10 \pm 0.06) \times 10^{41} \text{ kg m}^2, \] (233)

or in length units (cubic meters) it is

\[ GQ_\odot/c^2 = G J_2 M_\odot R_\odot^2/c^2 = (1.60 \pm 0.04) \times 10^{14} \text{ m}^3 = 160 000 \pm 4 000 \text{ km}^3 \]
\[ = (53.8 \pm 0.5 \text{ km})^3 = (3.69 \pm 0.11) \times 10^{17} \ell_P^3. \] (234)

From this one can define an effective quadrupole radius of the Sun as

\[ r_{Q_\odot} = \sqrt{2Q_\odot/M_\odot} = 460 \pm 6 \text{ km} \]
\[ = (3.07 \pm 0.04) \times 10^{-6} \text{ AU} = (2.84 \pm 0.04) \times 10^{40} \ell_P, \] (235)

the radius of a solar-mass ring of the same quadrupole moment as the Sun. Here 1 AU = 149 597 870 660 \pm 20 \text{ m} = 0.9257 \times 10^{40} \ell_P \text{ is the astronomical unit [41].}
The same helioseismic measurements \[42\] also give the angular momentum of the Sun as

\[ J_\odot = (1.900 \pm 0.015) \times 10^{41} \text{ kg m}^2 \text{ s}^{-1} = (1.801 \pm 0.014) \times 10^{25} \hbar, \] (236)

which when converted to length and area units gives

\[ GJ_\odot/c^3 = (4.705 \pm 0.037) \times 10^5 \text{ m}^2 = (686 \pm 3 \text{ m})^2 = 47.05 \pm 0.37 \text{ hectares} = 116 \pm 1 \text{ acres}. \] (237)

From this and the solar mass one can readily calculate that the Kerr parameter \( a \) of Eq. (184) and the dimensionless Kerr rotation parameter \( \alpha \) of Eq. (225) have the values

\[ a_\odot \equiv \frac{J_\odot}{M_\odot c} = 318.65 \pm 2.52 \text{ m} = (1.972 \pm 0.016) \times 10^{37} \ell_P, \] (238)

\[ \alpha_\odot \equiv \frac{a_\odot c^2}{GM_\odot} = \frac{cJ_\odot}{GM_\odot^2} = 0.2158 \pm 0.0017. \] (239)

The fact that \( \alpha_\odot < 1 \) means that if the Sun were able to undergo gravitational collapse to become a black hole (which it is not, since it is too light, except for some extremely tiny tunneling probability or possibly some artificial compressional procedure), it would not need to lose any angular momentum to do so.

By comparing the metric (228) with the uncharged Kerr metric (183), one may deduce \[40\] that the quadrupole moment of the Kerr metric is

\[ Q = \frac{J^2}{Mc^2} = Ma^2, \] so

\[ \frac{M_\odot c^2 Q}{J^2_\odot} = (1.04 \pm 0.05) \times 10^6, \] (240)

so the quadrupole moment of the Sun is about a million times larger than that of a Kerr metric with the same mass and angular momentum. This is reasonable, since the dimensionless quadrupole moment \( J_2 \) is two-thirds of the solar oblateness \[42\], which one expects to be of the order of the square of the angular velocity of the Sun divided by the Kepler orbital velocity at the surface of the Sun, which is \( \Omega^2_\odot/\Omega^2_{+K} \approx \Omega^2_\odot R^3_\odot /GM_\odot \). Therefore, one expects (to order of magnitude) \( Q_\odot = J_2 M_\odot R^2_\odot \approx R^5_\odot \Omega^2_\odot /G \). Then since \( J_\odot \sim M_\odot R^2 \Omega_\odot \), one gets \( M_\odot c^2 Q/J^2_\odot \approx R^5_\odot c^2/GM_\odot = 0.471 \times 10^6 \), within about a factor of two of the correct answer.

From the angular momentum \( J_\odot \) of the Sun, one can calculate that the Stationary Congruence Accelerating Maximally (SCAM) in the equatorial plane rotates around the Sun with an angular velocity (as seen by a nonrotating observer at infinity)

\[ \Omega_0 \approx -\frac{GJ_\odot}{c^2 r^3} = -(4.213 \pm 0.033) \times 10^{-20} \left( \frac{\text{AU}}{r} \right)^3 \text{ s}^{-1}. \]
\[ (241) \]

\[ \frac{d}{dt} = (4.184 \pm 0.033) \times 10^{-13} \left( \frac{R_{\odot}}{r} \right)^3 \text{s}^{-1} \]

\[ = -\frac{2\pi}{476,000 \pm 4,000 \text{ yr}} \left( \frac{R_{\odot}}{r} \right)^3, \]

and a linear velocity

\[ v_0 \approx -\frac{G J_{\odot}}{c^2 r^2} = (6.303 \pm 0.050) \times 10^{-9} \left( \frac{\text{AU}}{r} \right)^2 \text{ m s}^{-1} \]

\[ = -(2.932 \pm 0.023) \times 10^{-4} \left( \frac{R_{\odot}}{r} \right)^2 \text{ m s}^{-1} \]

\[ = -(9.253 \pm 0.073) \left( \frac{R_{\odot}}{r} \right)^2 \text{ km/yr.} \]

This rotation rate is extremely slow, though not quite glacially slow, but in the 4.6 \pm 0.1 billion year age of the Solar System, the member of the SCAM at the surface of the Sun would have made almost ten thousand backward revolutions relative to the distant stars, assuming that the far-field Eq. (102) applies all the way down to the surface of the Sun.

Since the SCAM maximizes the magnitude of the acceleration for all stationary observers at a given location, it has a larger acceleration than that of a nonrotating observer. For the weak field of the Sun, the difference is very tiny:

\[ \Delta a \equiv a_0 - a_{NR} \approx \frac{v_0^2}{r} \approx -\frac{G^2 J_{\odot}^2}{c^4 r^5} \]

\[ = (2.656 \pm 0.042) \times 10^{-28} \left( \frac{\text{AU}}{r} \right)^5 \text{ m s}^{-2} \]

\[ = (4.775 \pm 0.075) \times 10^{-80} \left( \frac{\text{AU}}{r} \right)^5 c^2 \ell_p^2. \]

Over the lifetime of the Solar System, this tiny acceleration at \( r = 1 \text{ AU} \) would, starting from rest, give a spatial motion \( \frac{1}{2} \Delta a t^2 \) of about 2800 km.

If, despite its rotation, the Sun were perfectly spherical and hence had no quadrupole moment, then Kepler’s third law would apply to high accuracy to the orbital angular velocity relative to the SCAM. Since the SCAM is counterrotating, the corotating Keplerian orbit has a lower angular frequency, and hence a longer period, relative to a static observer than relative to the SCAM. However, for the realistic Sun, the quadrupole moment from the oblateness, also caused by the rotation, increases the equatorial gravitational attraction, and hence also the orbital angular frequency there. This effect thus decreases the orbital period as measured by a nonrotating observer at infinity. These two changes in the period may be calculated as follows:
In the metric Eq. (228) for the gravitational field of the Sun, in the equatorial plane ($\theta = \pi/2$ or $P_2(\cos \theta) = -1/2$) the circumferential radius is

$$R \equiv \sqrt{-C} \equiv g_{\varphi\varphi} = r \sqrt{1 + \frac{Q_\odot}{r^3}} \approx r + \frac{Q_\odot}{2r^2}. \quad (244)$$

The corotating Keplerian orbital velocity is then

$$\Omega_{+K} = \sqrt{\frac{M_\odot}{R^3}} \left(1 - \frac{J_\odot}{\sqrt{M_\odot R^3}} + \frac{3Q_\odot}{4M_\odot R^2} + O(R^{-3})\right), \quad (245)$$

and so the orbital period (as seen by a nonrotating observer at infinity) is

$$P \equiv \frac{2\pi}{\Omega_{+K}} = P_K + \Delta P_{J_\odot} + \Delta P_{J_2} + O(R^{-3/2})$$

$$= 2\pi \sqrt{\frac{R^3}{GM_\odot}} + 2\pi J_\odot \frac{M_\odot}{M_\odot c^2} - \frac{3\pi Q_\odot}{2\sqrt{GM_\odot^3 R}} + O(R^{-3/2}), \quad (246)$$

where Newton’s gravitational constant $G$ and the speed of light $c$ have been restored in Eq. (246).

Here

$$P_K = 2\pi \sqrt{\frac{R^3}{GM_\odot}} = 31558196.0 \left(\frac{R}{\text{AU}}\right)^{3/2} \text{ s} \quad (247)$$

is Kepler’s third law for the period of a test body in the field of the Sun (ignoring the gravitational effects of the planets for now),

$$\Delta P_{J_\odot} = \frac{2\pi J_\odot}{M_\odot c^2} = 2\pi a_\odot/c = (2002 \pm 16 \text{ m})/c$$

$$= (6.678 \pm 0.053) \times 10^{-6} \text{ s} = 6.678 \pm 0.053 \mu\text{s} \quad (248)$$

is the increase in the period due to the linear effect of the Sun’s angular momentum, and

$$\Delta P_{J_2} = -\frac{3\pi Q_\odot}{2\sqrt{GM_\odot^3 R}} = -(1.117 \pm 0.032) \times 10^{-4} \left(\frac{\text{AU}}{R}\right)^{1/2} \text{ s} \quad (249)$$

is the decrease in the period due to the quadrupole moment of the Sun.

One can see that these corrections to Kepler’s third law in the gravitational field of the Sun are very small and currently unmeasurable, but it is amusing to calculate them as an academic exercise. It is also amusing to note that the effect of the quadrupole moment (which is essentially quadratic in the angular momentum) dominates over that linear in the angular momentum for radii $r < r_{K_\odot}$, where

$$r_{K_\odot} = \frac{9c^4Q_\odot^2}{16GJ_\odot^2 M_\odot} = 280 \pm 20 \text{ AU} \quad (250)$$
is the orbital radius at which Kepler’s third law would be exact, at about seven times the orbital radius (semimajor axis) of Pluto.

One can use the order-of-magnitude estimates above for the angular momentum $J_\odot \sim M_\odot R_\odot^2 \Omega_\odot$ and the quadrupole moment $Q_\odot \sim R_\odot^5 \Omega_\odot^2 / G$ to estimate

$$r_{K\odot} \sim \left( \frac{v_r}{c} \right)^2 \left( \frac{c}{v_e} \right)^6 R_\odot,$$

where $v_r = R_\odot \Omega_\odot$ is the linear rotation velocity of the equatorial surface of the Sun and $v_e = \sqrt{2GM_\odot/R_\odot} = 2.060 \times 10^{-3} c$ is the escape velocity from the surface of the Sun. The Sun is not rotating rigidly, so its angular velocity $\Omega_\odot$ is not constant, but one can take, as a sort of averaged value for $\Omega_\odot$, twice $T_\odot$, the total kinetic energy in rotation of the Sun, divided by the Sun’s angular momentum $J_\odot$. Since the total kinetic energy in rotation is

$$T_\odot = (2.534 \pm 0.072) \times 10^{35} \text{ kg } m^2 \text{ s}^{-2},$$

one gets an effective averaged angular velocity of the Sun as

$$\Omega_\odot = \frac{2T_\odot}{J_\odot} = (2.67 \pm 0.10) \times 10^{-6} \text{ s}^{-1} = \frac{2\pi}{27.3 \pm 1.0 \text{ days}}.$$

(Here I have simply linearly added the relative errors given for $J_\odot$, about 0.0079, and for $T_\odot$, about 0.0284, to get a conservative relative error estimate of 0.0363 for $\Omega_\odot$.) Multiplying $\Omega_\odot$ by the radius $R_\odot = 6.96 \times 10^8 \text{ m}$ of the Sun gives $v_r = 1860 \pm 70 \text{ m/s} = (6.19 \pm 0.22) \times 10^{-6} c$. Inserting this linear surface velocity $v_r$ and the escape velocity $v_e$ above into Eq. (251) gives the order-of-magnitude estimate of Eq. (251) $r_{K\odot}$ as roughly $500\,000 R_\odot$, which is approximately $3.5 \times 10^{14} \text{ m}$ or 2300 AU. This is about a factor of eight larger than what Eq. (250) gives, which is not too surprising, because of the neglect of all numerical factors and details of the structure of the Sun in Eq. (251), and because of the high powers of the velocities that enter into that estimate.

For a self-gravitating rotating fluid object (e.g., a star) which has its linear rotational velocity $v_r/c$ at its surface (in units of the speed of light) greater than roughly the cube of the escape velocity $v_e/c$ from its surface (again in units of the speed of light), as indeed is the case for the Sun, we can see from the estimate of Eq. (251) that the radius $r_K$, where Kepler’s third law is exact for corotating circular orbits in the equatorial plane, is greater than the radius $R$ of the object. If the object is maximally rotating, so that its rotational velocity $v_r$ is comparable to its escape velocity $v_e$, then $r_K$ will be outside the object (leading to $\Gamma \equiv R^3 \Omega_{r,K}^2 / (GM) > 1$ and hence a shorter orbital period at the surface than what Kepler’s third law would give), unless possibly the rotational and escape velocities are close to the velocities of light (i.e., unless possibly the object is highly relativistic, as we found was necessary for the polytropic models with $\Gamma < 1$ in the previous Section).
In particular, one might ask what different effective angular velocity the Sun would need in order that its new \( r_K \) would then coincide with the solar radius \( R_\odot \).

To calculate this, instead of using Eq. (251), use the precise formula (250) (precise to the extent that it gives, as it does for the Sun, an \( r_K \) that is in the far-field region where the field is both weak and is entirely dominated by the monopole and quadrupole contributions). Suppose that the effective moment of inertia of the Sun about its axis,

\[
I_\odot = \frac{J_\odot^2}{2T_\odot} = (7.12 \pm 0.31) \times 10^{48} \text{ kg m}^2 = \left(\frac{c^2}{G}\right)(5.29 \pm 0.23) \times 10^{21} \text{ m}^3
\]

would stay constant as its angular velocity were changed. More precisely, assume that the angular momentum of the Sun would be linearly proportional to its effective angular velocity, and that the quadrupole moment of the Sun would be proportional to the square of this angular velocity, with the proportionality constants staying fixed. Then in order to get \( r_K = R_\odot \), one would need to change the Sun’s angular velocity from \( \Omega_\odot \) to

\[
\Omega = \left(\frac{R_\odot}{r_K}\right)^{1/2} \Omega_\odot = \frac{8T_\odot \sqrt{GM_\odot R_\odot}}{3c^2 Q_\odot} = (1.088 \pm 0.061) \times 10^{-8} \text{ s}^{-1} = \frac{2\pi}{18.3 \pm 1.0 \text{ yr}}.
\]

In other words, if the Sun were rotating with a period of greater than about 18.3 years, then the linear term in the angular velocity (the relativistic effect linear in the angular momentum \( J \)) would dominate over the quadratic term in the angular velocity (the Newtonian quadrupole effect) at all radii outside the Sun, and so the corotating circular orbital period would be slightly increased everywhere outside such a slowly rotating Sun.

Of course, the numerical result of Eq. (250) for the actual radius \( r_K \), where Kepler’s third law would be exact for a corotating test body in a circular equatorial orbit in the gravitational field of the Sun as it is actually rotating, is entirely hypothetical, since the planets would exert perturbations on the orbital period far larger than those of \( \Delta P_J \) and \( \Delta P_{J_z} \) of Eq. (248). For orbits at \( r_K \sim 280 \text{ AU} \) or greater, one might suppose that a reasonable estimate for some sort of averaged corotating period in the equatorial plane would be to use Eq. (246) but with the solar mass \( M_\odot \), angular momentum \( J_\odot \), and quadrupole moment \( Q_\odot \) replaced by the analogous quantities \( M_{SS} \), \( J_{SS} \), and \( Q_{SS} \) for the entire Solar System.

Combining the data in [44] with that in [41] and in [42] gives directly

\[
GM_{SS}/c^2 = 1.001346 \text{ and } \frac{J_{SS}}{\ell_P} = 1478.612 \text{ m} = 0.9138 \times 10^{38} \ell_P
\]

and

\[
J_{SS} = 3.148 \times 10^{43} \text{ kg m}^2 \text{ s}^{-1} = 2.985 \times 10^{77} \bar{h} = (165.7 \pm 1.3) J_\odot.
\]
which when converted to length and area units gives
\[
G J_{SS}/c^3 = 7.796 \times 10^7 \text{ m}^2 = (8829 \text{ m})^2 = 30.10 \text{ mi}^2
= 7800 \text{ hectares} = 19260 \text{ acres}.
\]

(258)

From these data one can readily calculate that the Kerr rotational length parameter \(a\), and the corresponding dimensionless rotation parameter \(\alpha\), take on the values for the entire Solar System of
\[
a_{SS} \equiv \frac{J_{SS}}{M_{SS} c} = 52.72 \text{ km} = 3.263 \times 10^{39} \ell_p = (165.5 \pm 1.3) a_\odot,
\]

(259)

\[
\alpha_{SS} \equiv \frac{a_{SS} c^2}{GM_{SS}} \equiv \frac{c J_{SS}}{G M_{SS}^2} = 35.66 = (165.3 \pm 1.3) \alpha_\odot.
\]

(260)

The fact that \(\alpha_{SS} > 1\), unlike the case for the Sun, means that the Solar System would have to give up angular momentum (in fact, give up more than 97% of its angular momentum) before it could possibly become a black hole.

By adding up the time-averaged quadrupole moments of each planet and the Sun, around their common center of mass, from the data on the planetary masses (including their moons) and the semimajor axes and eccentricities of their orbits in [44], I obtained a quadrupole moment for the Solar System of
\[
Q_{SS} = 2.576 \times 10^{51} \text{ kg m}^2 = 1.23 \times 10^{10} Q_\odot,
\]

(261)

of which about 40% came from Neptune, 23% came from Saturn, 22% came from Jupiter, 14% came from Uranus, 6.7% came from Pluto, 0.0026% came from Earth, 0.0011% came from Venus, 0.00066% came from Mars, and 0.000023% came from Mercury. There is a positive error in my estimate from neglecting the fact that the orbits are not all in the same plane, and a negative error from neglecting the quadrupole moment contributions of the asteroids and comets, but I have not attempted to estimate these errors. Almost certainly not all of the four digits given above are correct, but I have given them just to show the answer I got for the planets and Sun if their orbits were coplanar.

Converting the quadrupole moment of the Solar System to length units gives
\[
G Q_{SS}/c^2 = 1.91 \times 10^{24} \text{ m}^3 = (124000 \text{ km})^3 = 4.53 \times 10^{128} \ell_p^3,
\]

(262)

The effective quadrupole radius of the Solar System is then
\[
r_{QSS} = \sqrt{2 Q_{SS}/M_{SS}} = 111000 r_{Q\odot} = 5.09 \times 10^{10} \text{ m} = 0.340 \text{ AU} = 3.15 \times 10^{45} \ell_p,
\]

(263)

meaning that one would get the same quadrupole moment if one placed all the mass (of which 99.8656% comes from the Sun) into a ring at radius \(r_{QSS} = 0.34\ \text{ AU}\),
about 88% of the semimajor axis of the orbit of Mercury \[44\]. Incidentally, for such a ring to give the angular momentum of the Solar System, it would have to rotate around with a period of

\[ P_{SS} = \frac{4\pi Q_{SS}}{J_{SS}} = 1.03 \times 10^9 \text{ s} = 32.6 \text{ yr}. \]  

(264)

The corresponding period for the 460 km solar-mass ring that gives the quadrupole moment of the Sun in Eq. \(235\) is about 14 seconds to give the solar angular momentum given in Eq. \(236\).

As we did for the Sun in Eqs. \(240\), we can calculate that for the Solar System

\[ \frac{Q_{SS}}{J_{SS}^2/M_{SS}c^2} = 4.65 \times 10^{11}, \]  

(265)

so the quadrupole moment of the Solar System is about 465 billion times larger than that of a Kerr metric with the same mass and angular momentum.

Now if we insert these data for the Solar System in place of the corresponding data for the Sun alone in Eqs. \(246\) - \(249\), we get some sort of averaged deviations from Kepler’s third law for very distant orbits around the Solar System as follows:

\[ P \equiv \frac{2\pi}{\Omega + K} = P_{KSS} + \Delta P_{JSS} + \Delta P_{QSS} + O(R^{-3/2}) \]

\[ = 2\pi \sqrt{\frac{R^3}{GM_{SS}}} + \frac{2\pi J_{SS}}{M_{SS}c^2} - \frac{3\pi Q_{SS}}{2\sqrt{GM_{SS}^3 R}} + O(R^{-3/2}), \]  

(266)

\[ P_{KSS} = 2\pi \sqrt{\frac{R^3}{GM_{SS}}} = 31 536 986 \left( \frac{R}{\text{AU}} \right)^{3/2} \text{ s}, \]  

(267)

\[ \Delta P_{JSS} = \frac{2\pi J_{SS}}{M_{SS}c^2} = 2\pi a_{SS}/c = (331 \text{ km})/c \]

\[ = 1.105 \times 10^{-3} \text{ s} = 1.105 \text{ ms}, \]  

(268)

due to the linear effect of the angular momentum of the Solar System, and

\[ \Delta P_{QSS} = -\frac{3\pi Q_{SS}}{2\sqrt{GM_{SS}^3 R}} = -1.367 \times 10^6 \left( \frac{\text{AU}}{R} \right)^{1/2} \text{ s} = -15.82 \left( \frac{\text{AU}}{R} \right)^{1/2} \text{ days}, \]  

(269)

due to the quadrupole moment of the Solar System, e.g., about -2.5 days for the orbit of Pluto at 39.481 686 77 AU \[45\], assuming that the effect of the planets is merely to provide mass, angular momentum, and quadrupole moment for the Solar System. This is actually not a very good approximation for Pluto, since the formula above gives an orbital period of about 90 550 days for Pluto, whereas the sidereal period is actually 90 465 days \[44, 45\], about 85 days shorter.
One can then calculate that the effect of the quadrupole moment dominates over that linear in the angular momentum for the Solar System for

\[ r < r_{KSS} = \frac{9e^4 Q^2_{SS}}{16GJ^2_{SS}M_{SS}} = 2.29 \times 10^{29} \text{m} = 1.53 \times 10^{18} \text{AU} = 7.42 \times 10^6 \text{Mpc}. \] (270)

Since \( r_{K\odot} \) defined by Eq. (250) for the Sun alone is larger than the Solar System, and since the radius \( r_{KSS} \) at which Kepler’s third law would be exact for a test body orbiting the entire Solar System in otherwise flat spacetime is far larger than the presently observable universe, we can conclude that for realistic orbits (i.e., at orbital distances less than that to the next nearest star), the quadrupole moment of the Sun or of the planets always dominates over the linear effect of the angular momentum in changing the period of circular orbits from the value given by Kepler’s third law, so that the period is always smaller than that given by Kepler’s third law. (This is under the approximation that the effect of the planets is merely to give a quadrupole moment to the Solar System and ignores more complicated effects when the orbital radii, and hence periods, of the planets are non-negligible fractions of the radius and period of the orbit of the test body.)

However, we saw in the previous Section that for certain relativistic polytropic star models with polytropic index not too large (not too soft an equation of state), it is possible to have the linear effect of the angular momentum dominate over that of the quadrupole moment, even at the surface of the star, so that the corotating Keplerian orbits at the equatorial surface of the star can have a longer period (as seen by a nonrotating observer at infinity) than what Kepler’s third law would give. As discussed above, this effect is related to the fact that the angular momentum of a source generally causes the Stationary Congruence Accelerating Maximally (SCAM) to be counterrotating, so that for a given positive orbital angular velocity relative to the SCAM, the angular velocity relative to infinity is less, giving rise to a longer period.

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