Melnikov Analysis for Singly Perturbed DSII Equation

Yanguang (Charles) Li

Abstract. Rigorous Melnikov analysis is accomplished for Davey-Stewartson II equation under singular perturbation. Unstable fibre theorem and center-stable manifold theorem are established. The fact that the unperturbed homoclinic orbit, obtained via a Darboux transformation, is a classical solution, leads to the conclusion that only local well-posedness is necessary for a Melnikov measurement.

1. Introduction

To build a Melnikov analysis for high dimensional nonlinear wave equations, we consider the Davey-Stewartson II equation (DSII) under a singular perturbation

\[
\begin{align*}
    iq_t &= \Upsilon q + 2(|q|^2 - \omega^2) + u_y|q| + i\epsilon|\Delta q - \alpha q + \beta|, \\
    \Delta u &= -4\partial_y|q|^2,
\end{align*}
\]

where \(q\) is a complex-valued function of the three variables \((t, x, y)\), \(u\) is a real-valued function of the three variables \((t, x, y)\), the external parameters \(\omega, \alpha, \) and \(\beta\) are all positive constants, and \(\epsilon > 0\) is the perturbation parameter,

\(\Upsilon = \partial_{xx} - \partial_{yy}\), \(\Delta = \partial_{xx} + \partial_{yy}\), \(i = \sqrt{-1}\).

Periodic boundary condition is imposed,

\[
q(t, x + 2\pi/\kappa_1, y) = q(t, x, y) = q(t, x, y + 2\pi/\kappa_2),
\]

\[
u(t, x + 2\pi/\kappa_1, y) = u(t, x, y) = u(t, x, y + 2\pi/\kappa_2),
\]

where \(\kappa_1\) and \(\kappa_2\) are positive constants. Even constraint is also imposed,

\[
q(t, -x, y) = q(t, x, y) = q(t, x, -y),
\]

\[
u(t, -x, y) = u(t, x, y) = u(t, x, -y).
\]

1991 Mathematics Subject Classification. Primary 35Q55, 35Q30; Secondary 37L10, 37L50.

Key words and phrases. Melnikov integral, Davey-Stewartson equation, Darboux transformations, invariant manifolds, fibers.
Further constraints are placed upon $\omega$, $\alpha$, $\beta$, $\kappa_1$, and $\kappa_2$. The first one $0<\alpha\omega<\beta$ is the condition for the existence of a saddle, and the second one is the condition for the existence of only two unstable modes,

$$
\begin{cases}
\kappa_2 < \kappa_1 < 2\kappa_2, \\
\kappa_1^2 < 4\omega^2 < \min\{\kappa_1^2 + \kappa_2^2, 4\kappa_1^2\},
\end{cases}
$$

or

$$
\begin{cases}
\kappa_1 < \kappa_2 < 2\kappa_1, \\
\kappa_2^2 < 4\omega^2 < \min\{\kappa_1^2 + \kappa_2^2, 4\kappa_1^2\}.
\end{cases}
$$

Davey-Stewartson II equation can be regarded as a generalization of the 1D cubic nonlinear Schrödinger equation (NLS). In fact, it is a nontrivial generalization in the sense that the spatial part of the Lax pair of the DSII is a system of two first order partial differential equations, for which there is no Floquet discriminant to describe the isospectral property, in contrast to the case for NLS. It turns out that Melnikov vectors can still be obtained through quadratic products of Bloch eigenfunctions, instead of the gradient of the Floquet discriminant as in the NLS case.

At the moment, there is no global well-posedness for DSII in Sobolev spaces. In fact, DSII has finite-time blow-up solutions in $H^s(\mathbb{R}^2)$, $0 < s < 1$. Of course, DSII has local well-posedness in Sobolev spaces. The Melnikov measurement is built upon an unperturbed homoclinic orbit of the unperturbed DSII. Explicit expression of such a homoclinic orbit can be obtained through Darboux transformation. The homoclinic orbit is a classical solution. This enables us to iterate the local well-posedness result in time, and complete a Melnikov measurement. Unstable fiber theorem and center-stable manifold theorem are of course needed, and established along the same line as in [1]. Novelities in regularity are introduced by the singular perturbation $\epsilon\Delta$ which generates the semigroup $e^{t\epsilon\Delta}$.

The article is organized as follows: section 2 deals with local theory which includes unstable fiber theorem and center-stable manifold theorem, and we handle global theory in section 3 which includes integrable theory and Melnikov analysis.

2. Local Theory

One can view the perturbed DSII (1.1) as an evolution equation in the $q$ variable. First, one can define the spatial mean as

$$
\langle q \rangle = \frac{\kappa_1\kappa_2}{4\pi^2} \int_0^{2\pi/\kappa_2} \int_0^{2\pi/\kappa_1} q \, dx \, dy.
$$

Then one may introduce the space $\dot{H}^s$ as

$$
\dot{H}^s = \{ q \in H^s \mid \langle q \rangle = 0 \}.
$$

The inverse Laplacian $\Delta^{-1} : \dot{H}^s \mapsto \dot{H}^{s+2}$ is an isomorphism. The perturbed DSII (1.1) can be rewritten as

$$
iqt = \Upsilon q + 2|\Delta^{-1}\Upsilon||q|^2 + \langle |q|^2 \rangle - \omega^2|q| + i\epsilon(\Delta q - \alpha q + \beta).
$$
2.1. Change of Coordinates. Denote by $\Pi$ the 2D subspace
\begin{equation}
\Pi = \{ q \mid \partial_x q = \partial_y q = 0 \}.
\end{equation}
Dynamics in $\Pi$ is the same as that given in (2.1). Denote by $S_\omega$ the circle
\begin{equation}
S_\omega = \{ q \in \Pi \mid |q| = \omega \}.
\end{equation}
When $\alpha \omega < \beta$, there is a saddle $Q_\varepsilon$ near $S_\omega$ in $\Pi$, which is located at $q = I e^{i \theta}$ where
\begin{equation}
\begin{align*}
I &= \omega^2 - \epsilon \frac{1}{2} \sqrt{\beta^2 - \alpha^2 \omega^2} + \cdots, \\
\cos \theta &= \frac{\alpha \sqrt{\beta \omega}}{\beta}, \quad \theta \in \left(0, \frac{\pi}{2}\right).
\end{align*}
\end{equation}
Its eigenvalues are
\begin{equation}
\mu_{1,2} = \pm \sqrt{\epsilon \left(4 \sqrt{7} \sin \theta - \epsilon \left(\frac{\beta \sin \theta}{\sqrt{I}}\right)^2\right)} - \epsilon \alpha,
\end{equation}
where $I$ and $\theta$ are given in (2.4). In the entire phase space, $Q_\varepsilon$ is still a saddle. Local theory will be built in a tubular neighborhood of $S_\omega$. Let
\begin{equation}
q(t, x, y) = [\rho(t) + f(t, x, y)] e^{i \theta(t)}, \quad \langle f \rangle = 0.
\end{equation}
Let
\begin{equation}
I = \langle |q|^2 \rangle = \rho^2 + \langle |f|^2 \rangle, \quad J = I - \omega^2.
\end{equation}
In terms of the new variables $(J, \theta, f)$, Equation (2.1) can be rewritten as
\begin{align*}
\dot{J} &= \epsilon \left[ -2 \alpha (J + \omega^2) + 2 \beta \sqrt{J + \omega^2} \sin \theta \right] + \epsilon R_2^J, \\
\dot{\theta} &= -2J - \epsilon \beta \frac{\sin \theta}{\sqrt{J + \omega^2}} + R_2^\theta, \\
f_t &= L_c f + V_c f - i N_2 - i N_3,
\end{align*}
where
\begin{align*}
L_c f &= -i Y f + \epsilon (\Delta - \alpha) f - 2i \omega^2 \Delta^{-1} Y (f + \bar{f}), \\
V_c f &= -i 2J \Delta^{-1} Y (f + \bar{f}) + i \beta \frac{\sin \theta}{\sqrt{J + \omega^2}} f, \\
R_2^J &= -2 \langle \nabla f \cdot \nabla f \rangle + 2 \beta \cos \theta \left( \sqrt{J + \omega^2} - \langle f \rangle^2 \right) - \sqrt{J + \omega^2}, \\
R_2^\theta &= -(\langle f + \bar{f} \rangle \Delta^{-1} Y (f + \bar{f})) - \rho^{-1} \langle (f + \bar{f}) \Delta^{-1} Y |f|^2 \rangle - \epsilon \beta \sin \theta \left( \frac{1}{\sqrt{J + \omega^2} - \langle |f|^2 \rangle} - \frac{1}{\sqrt{J + \omega^2}} \right), \\
N_2 &= 2 \rho \left( \Delta^{-1} Y |f|^2 + f \Delta^{-1} Y (f + \bar{f}) - \langle f \Delta^{-1} Y (f + \bar{f}) \rangle \right), \\
N_3 &= -\langle (f + \bar{f}) \Delta^{-1} Y (f + \bar{f}) \rangle - 2 \langle |f|^2 \rangle \Delta^{-1} Y (f + \bar{f}) + 2 \left[ f \Delta^{-1} Y |f|^2 - \langle f \Delta^{-1} Y |f|^2 \rangle \right] - \rho^{-1} \langle (f + \bar{f}) \Delta^{-1} Y |f|^2 \rangle f - \epsilon \beta \sin \theta \left( \frac{1}{\sqrt{J + \omega^2} - \langle |f|^2 \rangle} - \frac{1}{\sqrt{J + \omega^2}} \right) f.
\end{align*}
Since \( H^s (s \geq 2) \) is a Banach algebra, we have
\[
|\mathcal{R}_2^\perp| \sim O(\|f\|_s^2), \quad |\mathcal{R}_3^\perp| \sim O(\|f\|_s^2), \\
\|\mathcal{N}_2\|_s \sim O(\|f\|_s^2), \quad \|\mathcal{N}_3\|_s \sim O(\|f\|_s^2), \quad (s \geq 2).
\]

2.2. Unstable Fibers. On \( \Pi (2.2) \), the saddle \( Q_e \) has an unstable and a stable curves which lie in an annular neighborhood of \( S_{\omega} \). The width of this annular neighborhood is of order \( O(\sqrt{\tau}) \).

**Definition 2.1.** For any \( \delta > 0 \), we define the annular neighborhood of the circle \( S_{\omega} \) in \( \Pi (2.2) \) as
\[
\mathcal{A}(\delta) = \{(J, \theta) \mid |J| < \delta\}.
\]

Unstable fibers with base points in \( \mathcal{A}(\delta) \) for some \( \delta > 0 \) persist, even under the singular perturbation.

The spectrum of \( L_e \) consists of only point spectrum. The eigenvalues of \( L_e \) are:
\[
\mu^\pm_e = \alpha + \epsilon \pm |\xi|^2 \pm |\xi|^2 \sqrt{4\omega^2 - |\xi|^2},
\]
where \( \xi = (\xi_1, \xi_2) \), \( \xi_j = k_j \kappa_j \), \( k_j = 0, 1, 2, \ldots \), \( (j = 1, 2) \), \( k_1 + k_2 > 0 \), \( |\xi|^2 = \xi_1^2 + \xi_2^2 \), and \( \kappa_1, \kappa_2 \), and \( \omega \) satisfy the constraint (1.2) or (1.3).

Denote \( \mu^+_1 \) by \( \mu^+ \) and \( \mu^+_{(k_1, k_2)} \) by \( \mu^+ \). The eigenfunctions corresponding to \( \mu^x \) and \( \mu^y \) are
\[
u^x = e^{\pm i\beta x} \cos \kappa_1 x, \quad e^{\pm i\beta x} = \frac{\kappa_1 \mp i \sqrt{4\omega^2 - \kappa_1^2}}{2\omega},
\]
\[
u^y = e^{\pm i\beta y} \cos \kappa_2 y, \quad e^{\pm i\beta y} = \frac{\kappa_2 \pm i \sqrt{4\omega^2 - \kappa_2^2}}{2\omega}.
\]
Notice also that the singular perturbation \( -\epsilon |\xi|^2 \) breaks the gap between the center spectrum and the stable spectrum. Nevertheless, the gap between the unstable spectrum and the center spectrum survives. This leads to the following unstable fiber theorem.

**Theorem 2.2 (Unstable Fiber Theorem).** For any \( s \geq 2 \), there exists a \( \delta > 0 \) such that for any \( p \in \mathcal{A}(\delta) \), there is an unstable fiber \( \mathcal{F}^u_p \) which is a 2D surface. \( \mathcal{F}^u_p \) has the following properties:

1. \( \mathcal{F}^u_p \) is a \( C^1 \) smooth surface in \( \| \|_s \) norm.
2. \( \mathcal{F}^u_p \) is also \( C^1 \) smooth in \( \epsilon, \alpha, \beta, \omega, \) and \( p \) in \( \| \|_s \) norm, \( \epsilon \in (0, \epsilon_0) \) for some \( \epsilon_0 > 0 \) depending on \( s \).
3. \( p \in \mathcal{F}^u_p \), \( \mathcal{F}^u_p \) is tangent to \( \text{span} \{u^x, u^y\} \) at \( p \) when \( \epsilon = 0 \).
4. \( \mathcal{F}^u_p \) has the exponential decay property: Let \( S^t \) be the evolution operator of (2.4)-(2.8). \( \forall p_1 \in \mathcal{F}^u_p \),
\[
\|S^t p_1 - S^t p\|_s \leq C e^{\mu^+ t} \|p_1 - p\|_s, \quad \forall t \leq 0,
\]
where \( \mu^+ = \min \{\mu^+_x, \mu^+_y\} \).
5. \( \{\mathcal{F}^u_p\}_{p \in \mathcal{A}(\delta)} \) forms an invariant family of unstable fibers,
\[
S^t \mathcal{F}^u_p \subset \mathcal{F}^u_{S^t p}, \quad \forall t \in [-T, 0],
\]
and \( \forall T > 0 \) (\( T \) can be \( +\infty \)), such that \( S^t p \in \mathcal{A}(\delta), \forall t \in [-T, 0] \).
The proof of this theorem follows from the same arguments as in [1]. Notice, in particular, that $F^u_p \subset H^s$ for any $s \geq 2$. It is this fact that leads to the $C^1$ smoothness of $F^u_p$ in $\epsilon$. Denote by $W^u(Q_\epsilon)$ the unstable manifold of the saddle $Q_\epsilon$ (2.4), which is 3-dimensional. Denote by $W^u_{II}(Q_\epsilon)$ the unstable curve of $Q_\epsilon$ in $\Pi (2.2)$. $W^u_{II}(Q_\epsilon) = \Pi \cap W^u(Q_\epsilon)$, and $W^u_{II}(Q_\epsilon) \subset A(\delta)$. $W^u(Q_\epsilon)$ has the fiber representation

\[(2.11) \quad W^u(Q_\epsilon) = \bigcup_{p \in W^u_{II}(Q_\epsilon)} F^u_p.\]

Thus $W^u(Q_\epsilon) \subset H^s$ for any $s \geq 2$.

2.3. Center-Stable Manifold. Also due to the fact that the gap between unstable spectrum and center spectrum survives under the singular perturbation (2.10), a center-stable manifold persists.

**Theorem 2.3 (Center-Stable Manifold Theorem).** There exists a $C^1$ smooth codimension 2 locally invariant center-stable manifold $W^{cs}_n$ in $H^n$ for any $n \geq 2$.

1. At points in the subset $W^{cs}_{n+4}$ of $W^{cs}_n$, $W^{cs}_n$ is $C^1$ smooth in $\epsilon$, in $H^n$ norm, for $\epsilon \in [0, \epsilon_0)$ and some $\epsilon_0 > 0$.
2. $W^{cs}_n$ is $C^1$ smooth in $(\alpha, \beta, \omega)$.
3. The annular neighborhood $A(\delta)$ in Theorem 2.2 is included in $W^{cs}_n$.

The proof of this theorem follows from the same arguments as in [1].

Regularity of $W^{cs}_n$ in $\epsilon$ is crucial in Melnikov analysis. Melnikov integrals are the leading order terms in $\epsilon$ of the signed distances between $W^u(Q_\epsilon)$ (2.11) and $W^{cs}_n$. The signed distances are set up along an unperturbed homoclinic orbit, and the regularity of $W^{cs}_n$ in $\epsilon$ at $\epsilon = 0$ determines the order of the signed distances in $\epsilon$. Due to the singular perturbation, $W^{cs}_n$ is not $C^1$ in $\epsilon$ at every point rather at points in the subset $W^{cs}_{n+4}$. Here one may be able to replace $W^{cs}_{n+4}$ by $W^{cs}_{n+2}$. But we are not interested in sharper results, and the current result is sufficient for our purpose.

2.4. Local Well-Posedness. Following a much easier argument than that in e.g. [7, 8], one can prove the following local well-posedness theorem.

**Theorem 2.4.** For any $q_0 \in H^n$ ($n \geq 2$), there exists $\tau = \tau(\|q_0\|_n) > 0$, such that the perturbed DSII (2.4) has a unique solution $q(t) = S^t(q_0; \epsilon, \alpha, \beta, \omega) \in C^0([0, \tau], H^n)$, $q(0) = q_0$, where $S^t$ denotes the evolution operator. $S^t(\cdot; \epsilon, \alpha, \beta, \omega) : H^n \mapsto H^n$ is $C^1$ in $q_0$ and $(\alpha, \beta, \omega)$. $S^t(\cdot; \epsilon, \alpha, \beta, \omega) : H^{n+4} \mapsto H^n$ is $C^1$ in $t$ and $\epsilon$, $\epsilon \in [0, \epsilon_0)$, $\epsilon_0 > 0$.

Here “$C^1$ in $q_0$ and $(\alpha, \beta, \omega)$” can be replaced by “$C^\infty$ in $q_0$ and $(\alpha, \beta, \omega)$”. $H^{n+4}$ can be replaced by $H^{n+2}$. But we are not interested in sharper results.

3. Global Theory

Global theory is referred to a theory global in phase space, which includes integrable theory and Melnikov analysis. Integrable theory provides two ingredients for a Melnikov analysis: (1) An explicit expression of the unperturbed homoclinic orbit, (2) Melnikov vectors with explicit expressions.
3.1. Integrable Theory. Calculations in this subsection are essentially the same with those in [5]. The minor differences are introduced by the spatial periods $2\pi/\kappa_1$ and $2\pi/\kappa_2$ in contrast to $2\pi$ and $2\pi$ in [3]. Proofs of theorems and lemmas can be found in [5].

The DSII [$\epsilon = 0$ in (1.1)] is an integrable system with the Lax pair

$$L\psi = \lambda \psi, \quad (3.1)$$

$$\partial_t \psi = A\psi, \quad (3.2)$$

where $\psi = (\psi_1, \psi_2)^T$, and

$$L = \begin{pmatrix} D^- & q \\ \bar{q} & D^+ \end{pmatrix},$$

$$A = i \begin{pmatrix} -\partial_x^2 & q\partial_x \\ \bar{q}\partial_x & \partial_x^2 \end{pmatrix} + \begin{pmatrix} r_1 \\ -(D^- q) \end{pmatrix} \begin{pmatrix} (D^+ q) \\ r_2 \end{pmatrix},$$

where

$$D^+ = \alpha \partial_y + \partial_x, \quad D^- = \alpha \partial_y - \partial_x, \quad \alpha^2 = -1,$$

$r_1$ and $r_2$ have the expressions,

$$r_1 = \frac{1}{2}[-2(|q|^2 - \omega^2) - u_y + i\tilde{u}], \quad r_2 = \frac{1}{2}[2(|q|^2 - \omega^2) + u_y + i\tilde{u}],$$

where $\tilde{u}$ is also a real-valued function satisfying

$$\Delta \tilde{u} = 4i\alpha \partial_x \partial_y |q|^2.$$

Notice that DSII is invariant under the transformation $\sigma$:

$$\sigma(q, \bar{q}, r_1, r_2; \alpha) = (q, \bar{q}, -r_2, -r_1; -\alpha). \quad (3.5)$$

Applying the transformation $\sigma$ to the Lax pair (3.1, 3.2), we have a congruent Lax pair for which the compatibility condition gives the same DSII. The congruent Lax pair is given as:

$$\hat{L}\hat{\psi} = \lambda \hat{\psi}, \quad (3.6)$$

$$\partial_t \hat{\psi} = \hat{A}\hat{\psi}, \quad (3.7)$$

where $\hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2)$, and

$$\hat{L} = \begin{pmatrix} -D^+ & q \\ \bar{q} & -D^- \end{pmatrix},$$

$$\hat{A} = i \begin{pmatrix} -\partial_x^2 & q\partial_x \\ \bar{q}\partial_x & \partial_x^2 \end{pmatrix} + \begin{pmatrix} r_1 \\ -(D^- q) \end{pmatrix} \begin{pmatrix} (D^+ q) \\ r_2 \end{pmatrix}.$$

The Bäcklund-Darboux transformation can be formulated as follows. Let $(q, u)$ be a solution to the DSII, and let $\lambda_*$ be any value of $\lambda$. Let $\psi = (\psi_1, \psi_2)^T$ be a solution to the Lax pair (3.1, 3.2) at $(q, \bar{q}, r_1, r_2; \lambda_*)$. Define the matrix operator:

$$\Gamma = \begin{pmatrix} \wedge + a & b \\ c & \wedge + d \end{pmatrix},$$

where $a = 0$ and $d = 0$. Then, the Bäcklund-Darboux transformation is given by

$$\sigma(q, \bar{q}, r_1, r_2; \lambda) \rightarrow \sigma(\Gamma q, \Gamma \bar{q}, \Gamma r_1, \Gamma r_2; \lambda_*)$$

and

$$\psi \rightarrow \psi_* = \Gamma \psi.$$
where $\land = \alpha \partial_y - \lambda$, and $a, b, c, d$ are functions defined as:

$$
\begin{align*}
a &= \frac{1}{\Delta} \left[ \psi_2 \land_2 \psi_2 + \bar{\psi}_1 \land_1 \psi_1 \right], \\
b &= \frac{1}{\Delta} \left[ \bar{\psi}_2 \land_1 \psi_1 - \psi_1 \land_2 \bar{\psi}_2 \right], \\
c &= \frac{1}{\Delta} \left[ \bar{\psi}_1 \land_1 \psi_2 - \psi_2 \land_2 \bar{\psi}_1 \right], \\
d &= \frac{1}{\Delta} \left[ \bar{\psi}_2 \land_1 \psi_2 + \psi_1 \land_2 \bar{\psi}_1 \right],
\end{align*}
$$

in which $\land_1 = \alpha \partial_y - \lambda_1$, $\land_2 = \alpha \partial_y + \lambda_2$, and

$$
\Delta = - \left[ |\psi_1|^2 + |\psi_2|^2 \right].
$$

Define a transformation as follows:

$$
\begin{align*}
(q, r_1, r_2) &\rightarrow (Q, R_1, R_2), \\
\phi &\rightarrow \Phi; \\
Q &= q - 2b, \\
R_1 &= r_1 + 2(D^+a), \\
R_2 &= r_2 - 2(D^-d), \\
\Phi &= \Gamma \phi;
\end{align*}
$$

where $\phi$ is any solution to the Lax pair (3.1), (3.2) at $(q, \bar{q}, r_1, r_2; \lambda)$, $D^+$ and $D^-$ are defined in (3.3), we have the following theorem [5].

**Theorem 3.1.** The transformation (3.8) is a Bäcklund-Darboux transformation. That is, the function $Q$ defined through the transformation (3.8) is also a solution to the DSII. The function $\Phi$ defined through the transformation (3.8) solves the Lax pair (3.1), (3.2) at $(Q, \bar{Q}, R_1, R_2; \lambda)$.

Consider the spatially independent solution,

$$
q_c = \eta \exp\{-2i[\eta^2 - \omega^2]t + i\gamma\},
$$

where $\eta$ satisfies the constraint (1.2) and (1.3) with $\omega$ replaced by $\eta$. The dispersion relation for the linearized DSII at $q_c$ is

$$
\Omega = \pm \frac{|\xi_1^2 - \xi_2^2|}{\sqrt{\xi_1^2 + \xi_2^2}} \sqrt{4\eta^2 - (\xi_1^2 + \xi_2^2)}, \quad \text{for } \delta q \sim q_c \exp\{i(\xi_1 x + \xi_2 y) + \Omega t\},
$$

where $\xi_1 = k_1 \kappa_1$, $\xi_2 = k_2 \kappa_2$, and $k_1$ and $k_2$ are integers. There are only two unstable modes $(k_1, 0)$ and $(0, k_2)$ under even constraint.

The Bloch eigenfunction of the Lax pair (3.1) and (3.2) is given as,

$$
\tilde{\psi} = c(t) \left[ \begin{array}{c} -q_c \\ \chi \end{array} \right] \exp\{i(\xi_1 x + \xi_2 y)\},
$$

where $\chi$ is a complex constant.
where
\[ c(t) = c_0 \exp \left\{ \left[ 2\xi_1 (i\alpha \xi_2 - \lambda) + ir_2 \right] t \right\}, \]
\[ r_2 - r_1 = 2(|q_c|^2 - \omega^2), \]
\[ \chi = (i\alpha \xi_2 - \lambda) - i\xi_1, \]
\[ (i\alpha \xi_2 - \lambda)^2 + \xi_1^2 = \eta^2. \]

For the iteration of the Bäcklund-Darboux transformations, one needs two sets of eigenfunctions. First, we choose \( \xi_1 = \pm \frac{1}{2} \kappa_1, \xi_2 = 0, \lambda_0 = \sqrt{\eta^2 - \frac{1}{4} \kappa_1^2} \) (for a fixed branch),

\[ (3.11) \quad \psi^\pm = c^\pm \begin{bmatrix} -q_c \\ 1 \end{bmatrix} \exp \left\{ \pm i \frac{1}{2} \kappa_1 x \right\}, \]

where
\[ c^\pm = c_0^\pm \exp \left\{ \left[ \mp \kappa_1 \lambda_0 + ir_2 \right] t \right\}, \]
\[ \chi^\pm = -\lambda_0 \mp i \frac{1}{2} \kappa_1 = \eta e^{\mp i(\frac{1}{2} \phi_1)}, \quad \text{i.e.} \quad \eta e^{\pm i\phi_1} = \frac{1}{2} \kappa_1 \pm i \lambda_0. \]

We apply the Bäcklund-Darboux transformations with \( \psi = \psi^+ + \psi^-, \) which generates the unstable foliation associated with the \((\kappa_1, 0)\) linearly unstable mode. Then, we choose \( \xi_2 = \pm \frac{1}{2} \kappa_2, \lambda = 0, \xi_1^0 = \sqrt{\eta^2 - \frac{1}{4} \kappa_2^2} \) (for a fixed branch),

\[ (3.12) \quad \phi^\pm = c^\pm \begin{bmatrix} -q_c \\ \chi^\pm \end{bmatrix} \exp \left\{ i(\xi_1^0 x \pm \frac{1}{2} \kappa_2 y) \right\}, \]

where
\[ c^\pm = c_0^\pm \exp \left\{ \left[ \pm i\alpha \kappa_2 \xi_1^0 + ir_2 \right] t \right\}, \]
\[ \chi^\pm = \pm i\alpha \frac{1}{2} \kappa_2 - i\xi_1^0 = \pm i\eta e^{\mp i\phi_2}, \quad \text{i.e.} \quad \eta e^{\pm i\phi_2} = \pm i \frac{1}{2} \kappa_2 \pm i \xi_1^0. \]

We start from these eigenfunctions \( \phi^\pm \) to generate \( \Gamma \phi^\pm \) through Bäcklund-Darboux transformations, and then iterate the Bäcklund-Darboux transformations with \( \Gamma \phi^+ + \Gamma \phi^- \) to generate the unstable foliation associated with all the linearly unstable modes \((\kappa_1, 0)\) and \((0, \kappa_2)\). It turns out that the following representations are convenient,

\[ (3.13) \quad \psi^\pm = \sqrt{c_0^+ c_0^-} e^{ir_2 t} \begin{bmatrix} v_1^\pm \\ v_2^\pm \end{bmatrix}, \]
\[ (3.14) \quad \phi^\pm = \sqrt{c_0^+ c_0^-} e^{-i\xi_1^0 x + ir_2 t} \begin{bmatrix} w_1^\pm \\ w_2^\pm \end{bmatrix}, \]

where
\[ v_1^\pm = -q c e^{\mp i\tilde{t} z}, \quad v_2^\pm = \eta c e^{\mp i\tilde{t} z}, \]
\[ w_1^\pm = -q e^{\mp i\tilde{t} x}, \quad w_2^\pm = \pm \eta e^{\mp i\tilde{t} x}, \]

and
\[ c_0^+/c_0^- = e^{\rho + i\phi}, \quad \tau = 2\kappa_1 \lambda_0 t - \rho, \quad \tilde{x} = \frac{1}{2} \kappa_1 x + \frac{\vartheta}{2}, \quad \tilde{z} = \tilde{x} - \frac{\pi}{2} - \vartheta_1, \]
\( c_0^+ / c_0^- = e^{\hat{\rho} + i \hat{\vartheta}}, \quad \hat{\tau} = 2i\alpha\kappa_2 \xi_0^+ t + \hat{\rho}, \quad \hat{y} = \frac{1}{2} \kappa_2 y + \frac{\hat{\vartheta}}{2}, \quad \hat{z} = \hat{y} - \vartheta_2. \)

The following representations are also very useful,

\[
\psi = \psi_+ + \psi_- = 2\sqrt{c_0^+ c_0^-} e^{i t^2} (v_1 + i v_2), \tag{3.15}
\]

\[
\phi = \phi_+ + \phi_- = 2\sqrt{c_0^+ c_0^-} e^{i \xi_0^+ x + i t^2} (w_1 + i w_2), \tag{3.16}
\]

where

\[
v_1 = -q_c [\cosh \frac{\tau}{2} \cos \tilde{x} - i \sinh \frac{\tau}{2} \sin \tilde{x}], \quad v_2 = \eta [\cosh \frac{\tau}{2} \cos \tilde{\vartheta} - i \sinh \frac{\tau}{2} \sin \tilde{\vartheta}],
\]

\[
w_1 = -q_c [\cosh \frac{\hat{\tau}}{2} \cos \hat{y} + i \sinh \frac{\hat{\tau}}{2} \sin \hat{y}], \quad w_2 = \eta [\cosh \frac{\hat{\tau}}{2} \cos \hat{\vartheta} + i \sinh \frac{\hat{\tau}}{2} \sin \hat{\vartheta}].
\]

Applying the Bäcklund-Darboux transformations (3.8) with \( \psi \) given in (3.15), we have the representations,

\[
a = -\lambda_0 \sech \tau \sin(\tilde{x} + \tilde{\vartheta}) \sin(\tilde{x} - \tilde{\vartheta})
\]

\[
\times \left[ 1 + \sech \tau \cos(\tilde{x} + \tilde{\vartheta}) \cos(\tilde{x} - \tilde{\vartheta}) \right]^{-1}, \tag{3.17}
\]

\[
b = -q_c \hat{b} = -\frac{\lambda_0 q_c}{\eta} \left[ \cos(\tilde{x} - \tilde{\vartheta}) - i \tanh \tau \sin(\tilde{x} - \tilde{\vartheta})
\right.

\[
+ \sech \tau \cos(\tilde{x} + \tilde{\vartheta}) \left[ 1 + \sech \tau \cos(\tilde{x} + \tilde{\vartheta}) \cos(\tilde{x} - \tilde{\vartheta}) \right]^{-1}, \tag{3.18}
\]

\[
c = b, \quad d = -a. \tag{3.19}
\]

The evenness of \( b \) in \( x \) is enforced by the requirement that \( \vartheta - \vartheta_1 = \pm \frac{\pi}{2} \), and

\[
a^\pm = \mp \lambda_0 \sech \tau \sin(\tilde{x} + \tilde{\vartheta}) \sin(\tilde{x} - \tilde{\vartheta})
\]

\[
\times \left[ 1 \mp \sech \tau \sin \vartheta_1 \cos(\kappa_1 x) \right]^{-1}, \tag{3.20}
\]

\[
b^\pm = -q_c \hat{b}^\pm = -\frac{\lambda_0 q_c}{\eta} \left[ -\sin \vartheta_1 - i \tanh \tau \cos \vartheta_1
\right.

\[
\pm \sech \tau \cos(\kappa_1 x) \left[ 1 \mp \sech \tau \sin \vartheta_1 \cos(\kappa_1 x) \right]^{-1}, \tag{3.21}
\]

\[
c = b, \quad d = -a. \tag{3.22}
\]

Notice also that \( a^\pm \) is an odd function in \( x \). Under the above Bäcklund-Darboux transformations, the eigenfunctions \( \phi_\pm (3.13) \) and \( \phi \) are transformed into

\[
\varphi^\pm = \Gamma \phi_\pm, \quad \varphi = \Gamma \phi = \Gamma \phi_+ + \Gamma \phi_-, \tag{3.23}
\]

where

\[
\Gamma = \begin{pmatrix}
\Lambda + a & b \\
\bar{b} & \Lambda - a
\end{pmatrix}.
\]
and \( \Lambda = \alpha \partial_y - \lambda \) with \( \lambda \) evaluated at 0. Then

\[
\varphi^\pm = \sqrt{c^0_\pm c^0_\mp e^{i \xi_1 x + i \tau_2 t}} \begin{bmatrix} -q_e W_1^\pm \\ \eta W_2^\pm \end{bmatrix},
\]

where

\[
W_1^\pm = \left[ \pm i \frac{1}{2} \alpha \kappa_2 + a \pm \eta \bar{b} e^{\mp i \theta_2} \right] e^{\mp \hat{\tau}_2 \pm i \bar{y}},
\]

\[
W_2^\pm = \pm e^{\mp i \theta_2} \left[ \pm i \frac{1}{2} \alpha \kappa_2 - a \pm \eta \bar{b} e^{\pm i \theta_2} \right] e^{\mp \hat{\tau}_2 \pm i \bar{y}};
\]

\[
\varphi = 2 \sqrt{c^0_\pm c^0_\mp e^{i \xi_1 x + i \tau_2 t}} \begin{bmatrix} -q_e W_1 \\ \eta W_2 \end{bmatrix},
\]

where

\[
W_1 = \cosh \frac{\hat{\tau}_2}{2} [a \cos \bar{y} - \frac{1}{2} \alpha \kappa_2 \sin \bar{y} + i \eta \bar{b} \sin \bar{z}] + \sinh \frac{\hat{\tau}_2}{2} \left[ i \alpha \kappa_2 \cos \bar{y} + i a \sin \bar{y} + \eta \bar{b} \cos \bar{z} \right],
\]

\[
W_2 = \cosh \frac{\hat{\tau}_2}{2} [-ia \sin \bar{z} + \frac{1}{2} i \alpha \kappa_2 \cos \bar{z} + \eta \bar{b} \cos \bar{y}] + \sinh \frac{\hat{\tau}_2}{2} \left[ -\frac{1}{2} \alpha \kappa_2 \sin \bar{z} - a \cos \bar{z} + i \eta \bar{b} \sin \bar{y} \right].
\]

We generate the coefficients in the Bäcklund-Darboux transformations (3.8) with \( \varphi \) (the iteration of the Bäcklund-Darboux transformations),

\[
(a^{(I)}) = - \left[ W_2 (\alpha \partial_y W_2^{\dagger}) + \overline{W_1}(\alpha \partial_y W_1^{\dagger}) \right] \left[ |W_1|^2 + |W_2|^2 \right]^{-1},
\]

\[
(b^{(I)}) = \frac{q_e}{\eta} \left[ \overline{W_2}(\alpha \partial_y W_1^{\dagger}) - W_1 (\alpha \partial_y W_2^{\dagger}) \right] \left[ |W_1|^2 + |W_2|^2 \right]^{-1},
\]

\[
c^{(I)} = b^{(I)}, \quad d^{(I)} = -a^{(I)}.
\]
where

\[
W_2(\alpha \partial_y W_2) + W_1(\alpha \partial_y W_1) \\
= \frac{1}{2}\alpha \kappa_2 \left\{ \cosh \tau \left[ -\alpha \kappa_2 a + i \alpha \eta \hat{y} + i \alpha \kappa_2 \eta \hat{y} \cos \vartheta \right] \right. \\
+ \left[ \frac{1}{4} \kappa_2^2 - a^2 - \eta^2 \hat{b} \right] \cos (\hat{y} + \hat{z}) \sin \vartheta + \sinh \tau \left[ \alpha \kappa_2 \eta \hat{y} \sin \vartheta \right] \right\},
\]

\[
|W_1|^2 + |W_2|^2 \\
= \cosh \tau \left[ a^2 + \frac{1}{4} \kappa_2^2 + \eta^2 \hat{b}^2 + i \alpha \kappa_2 \eta \hat{y} \hat{b} \cos \vartheta \right] \\
+ \left[ \frac{1}{4} \kappa_2^2 - a^2 - \eta^2 \hat{b} \right] \sin (\hat{y} + \hat{z}) \sin \vartheta + \sinh \tau \left[ \alpha \kappa_2 \eta \hat{y} \sin \vartheta \right],
\]

\[
\bar{W}_2(\alpha \partial_y W_1) - W_1(\alpha \partial_y \bar{W}_2) \\
= \frac{1}{2}\alpha \kappa_2 \left\{ \cosh \tau \left[ -\alpha \kappa_2 \hat{b} + i(-a^2 + \frac{1}{4} \kappa_2^2 + \eta^2 \hat{b}^2) \cos \vartheta \right] \right. \\
+ \sinh \tau \left[ a^2 - \frac{1}{4} \kappa_2^2 + \eta^2 \hat{b} \right] \sin \vartheta \right\}.
\]

The new solution to the DSII is given by

\[
(3.27) \quad Q = q_c - 2b - 2b^{(I)}.
\]

The evenness of \(b^{(I)}\) in \(y\) is enforced by the requirement that \(\hat{\vartheta} - \vartheta_2 = \pm \frac{\pi}{2}\). In fact, we have

**Lemma 3.2.** Choosing the Bäcklund parameters \(\vartheta\) and \(\hat{\vartheta}\) as follows: \(\vartheta = \vartheta_1 \pm \frac{\pi}{2}\), and \(\hat{\vartheta} = \vartheta_2 \pm \frac{\pi}{2}\),

\[
(3.28) \quad b(-x) = b(x), \quad b^{(I)}(-x, y) = b^{(I)}(x, y) = b^{(I)}(x, -y),
\]

and \(Q = q_c - 2b - 2b^{(I)}\) is even in both \(x\) and \(y\).

The asymptotic behavior of \(Q\) can be computed directly. In fact, we have the asymptotic phase shift lemma.

**Lemma 3.3 (Asymptotic Phase Shift Lemma).** For \(\lambda_0 > 0\), \(\xi_0 > 0\), and \(\alpha = -i\); as \(t \to \pm \infty\),

\[
(3.29) \quad Q = q_c - 2b - 2b^{(I)} \to q_c e^{i \pi / 2 (\vartheta_1 - \vartheta_2)}.
\]

In comparison, the asymptotic phase shift of the first application of the Bäcklund-Darboux transformations is given by

\[
q_c - 2b \to q_c e^{i \pi / 2 \vartheta_1}.
\]

Next we generate the Melnikov vectors. Starting from \(\psi^\pm\) and \(\phi^\pm\) given in (3.11) and (3.12), we generate the following eigenfunctions corresponding to the solution \(Q\) given in (3.27) through the iterated Bäcklund-Darboux transformations,

\[
(3.30) \quad \Psi^\pm = \Gamma^{(I)} \Gamma \psi^\pm, \quad \text{at} \quad \lambda = \lambda_0 = \sqrt{\eta^2 - \frac{1}{4} \kappa_2^2},
\]

\[
(3.31) \quad \Phi^\pm = \Gamma^{(I)} \Gamma \phi^\pm, \quad \text{at} \quad \lambda = 0,
\]
where
\[
\Gamma = \begin{bmatrix}
\Lambda + a & b \\
\frac{b}{c} & \Lambda - a
\end{bmatrix}, \quad \Gamma^{(I)} = \begin{bmatrix}
\Lambda + a^{(I)} & b^{(I)} \\
\frac{b^{(I)}}{c^{(I)}} & \Lambda - a^{(I)}
\end{bmatrix},
\]
where \( \Lambda = \alpha \partial_y - \lambda \) for general \( \lambda \).

**Lemma 3.4.** The eigenfunctions \( \Psi_\pm \) and \( \Phi_\pm \) defined in (3.30) and (3.31) have the representations,
\[
\begin{align*}
\Psi_\pm &= \pm i \lambda_0 \kappa_1 \eta^{-1} \sqrt{c_0^2 e^{i r_2 t} [v_1^2 + |v_2|^2]}^{-1} \left[ -q_c \left( \lambda_0 - a^{(I)} \right) \frac{v_2}{\bar{v}_2} + \eta \frac{b^{(I)}}{\bar{v}_1} \right] \\
&\quad \times \left[ \eta \left[ -q_c \frac{\bar{\Sigma}_1}{\eta} - \left( \lambda_0 + a^{(I)} \right) \frac{v_1}{\bar{v}_1} \right] \right], \quad (3.32)
\end{align*}
\]
\[
\Phi_\pm = \pm \frac{1}{4} \alpha \kappa_1 \sqrt{c_0^2 e^{i (r_1 + r_2) t} [W_1^2 + |W_2|^2]}^{-1} \left[ -q_c \frac{\bar{\Sigma}_1}{\eta} \right], \quad (3.33)
\]
where \( b^{(I)} = -q_c \frac{\bar{b}^{(I)}}{c} \), and
\[
\begin{align*}
\bar{\Sigma}_1 &= 2W_1 (W_1^+ W_1^-) + \frac{1}{2} (W_1^+ W_1^-) + \frac{1}{2} (W_1^+ W_1^-), \\
\bar{\Sigma}_2 &= 2W_2 (W_2^+ W_2^-) + \frac{1}{2} (W_2^+ W_2^-) + \frac{1}{2} (W_2^+ W_2^-).
\end{align*}
\]
If we take \( r_2 \) to be real in the Melnikov vectors, \( r_2 \) appears in the form \( r_2 - r_1 = 2(|q_c|^2 - \omega^2) \), then
\[
\Psi_\pm \rightarrow 0, \quad \Phi_\pm \rightarrow 0, \quad \text{as } t \rightarrow \pm \infty. \quad (3.34)
\]

Next we generate eigenfunctions solving the corresponding congruent Lax pair (3.6, 3.7) with the potential \( Q \), through the iterated Bäcklund-Darboux transformations and the symmetry transformation (3.5).

**Lemma 3.5.** Under the replacement
\[
\begin{align*}
&\alpha \rightarrow -\alpha, \quad \text{(then \( \varphi_2 \rightarrow \pi - \varphi_2 \)),} \quad r_1 \rightarrow -r_2, \\
&r_2 \rightarrow -r_1, \quad \hat{\varphi} \rightarrow \hat{\varphi} + \pi - 2\varphi_2, \quad \hat{\rho} \rightarrow -\hat{\rho}, \quad (3.35)
\end{align*}
\]
the potentials are transformed as follows,
\[
\begin{align*}
Q &\rightarrow Q, \\
R_1 &\rightarrow -R_2, \\
R_2 &\rightarrow -R_1.
\end{align*}
\]

The eigenfunctions \( \Psi_\pm \) and \( \Phi_\pm \) given in (3.32) and (3.33) depend on the variables in the replacement (3.35):
\[
\Psi_\pm = \Psi_\pm (\alpha, r_1, r_2, \hat{\varphi}, \hat{\rho}), \quad \Phi_\pm = \Phi_\pm (\alpha, r_1, r_2, \hat{\varphi}, \hat{\rho}).
\]
Under replacement (3.33), $\Psi^\pm$ and $\Phi^\pm$ are transformed into

\begin{align}
\tilde{\Psi}^\pm &= \Psi^\pm(-\alpha, -r_2, -r_1, \hat{\theta} + \pi - 2\hat{\theta}_2, -\hat{\rho}), \\
\tilde{\Phi}^\pm &= \Phi^\pm(-\alpha, -r_2, -r_1, \hat{\theta} + \pi - 2\hat{\theta}_2, -\hat{\rho}).
\end{align}

**Lemma 3.6.** $\tilde{\Psi}^\pm$ and $\tilde{\Phi}^\pm$ solve the congruent Lax pair (3.6, 3.7) at $(Q, Q, R_1, R_2; \lambda_0)$ and $(Q, Q, R_1, R_2; 0)$, respectively.

Notice that as a function of $\eta$, $\xi_1^0$ has two (plus and minus) branches. In order to construct Melnikov vectors, we need to study the effect of the replacement $\xi_1^0 \rightarrow -\xi_1^0$.

**Lemma 3.7.** Under the replacement

\begin{equation}
\xi_1^0 \rightarrow -\xi_1^0 \quad \text{then} \quad \hat{\theta}_2 \rightarrow -\hat{\theta}_2, \quad \hat{\vartheta} \rightarrow \hat{\vartheta} + \pi - 2\hat{\vartheta}_2, \quad \hat{\rho} \rightarrow -\hat{\rho},
\end{equation}

the potentials are invariant,

\begin{equation}
Q \rightarrow Q, \quad R_1 \rightarrow R_1, \quad R_2 \rightarrow R_2.
\end{equation}

The eigenfunction $\Phi^\pm$ given in (3.33) depends on the variables in the replacement (3.38):

\begin{equation}
\Phi^\pm = \Phi^\pm(\xi_1^0, \hat{\vartheta}, \hat{\rho}).
\end{equation}

Under the replacement (3.38), $\Phi^\pm$ is transformed into

\begin{equation}
\tilde{\Phi}^\pm = \Phi^\pm(-\xi_1^0, \hat{\vartheta} + \pi - 2\hat{\vartheta}_2, -\hat{\rho}).
\end{equation}

**Lemma 3.8.** $\tilde{\Phi}^\pm$ solves the Lax pair (3.1, 3.2) at $(Q, Q, R_1, R_2; 0)$.

In the construction of the Melnikov vectors, we need to replace $\Phi^\pm$ by $\tilde{\Phi}^\pm$ to guarantee the periodicity in $x$ of period $2\pi/\kappa_1$.

The Melnikov vectors for the Davey-Stewartson II equations are given by,

\begin{align}
U^+ &= \left( \begin{array}{c}
\Psi_2^+ \tilde{\Psi}_2^+ \\
\Psi_1^+ \tilde{\Psi}_1^+
\end{array} \right)^- + S \left( \begin{array}{c}
\Psi_2^+ \tilde{\Psi}_2^+ \\
\Psi_1^+ \tilde{\Psi}_1^+
\end{array} \right), \\
U_+ &= \left( \begin{array}{c}
\tilde{\Phi}_2^+ \tilde{\Phi}_2^+ \\
\tilde{\Phi}_1^+ \tilde{\Phi}_1^+
\end{array} \right)^- + S \left( \begin{array}{c}
\tilde{\Phi}_2^+ \tilde{\Phi}_2^+ \\
\tilde{\Phi}_1^+ \tilde{\Phi}_1^+
\end{array} \right),
\end{align}

where "-" denotes complex conjugate, and $S = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$. In fact, the even parts of $U^+$ and $U_+$ are the Melnikov vectors in our phase space. Nevertheless, the Melnikov integral formulas end up the same, as shown in [3]. For simplicity, we just use $U^+$ and $U_+$. 
\[ \textbf{3.2. Melnikov Analysis.} \] The main difficulty in a rigorous Melnikov measurement is due to the lack of global well-posedness. The main idea in resolving this difficulty is to iterate the small time interval in local well-posedness by virtue of the fact that the unperturbed homoclinic orbit is a classical solution.

Let \( p \) be any point on \( W_{1}^{u}(Q_{e}) \), the unstable curve of \( Q_{e} \) in \( \Pi \). By the Unstable Fiber Theorem 2.2, \( F_{P}^{u} \) is \( C^{1} \) in \( \epsilon \) for \( \epsilon \in [0, \epsilon_{0}) \), \( \epsilon_{0} > 0 \); thus, there are two points \( q_{e}(0) \) and \( q_{0}(0) \) on the unstable fibers \( F_{P}^{u} \) and \( F_{P}^{u}|_{\epsilon=0} \), such that

\[
\| q_{e}(0) - q_{0}(0) \|_{n} \leq C_{n}^{(1)} \epsilon , \quad (n \geq 2) .
\]

The key point here is that \( F_{P}^{u} \subset H^{s} \) for any fixed \( s \geq 2 \). The expression of the unperturbed homoclinic orbit \( q_{0}(t) \) has been given in (3.27) which represents a classical solution to the DSII. Let

\[
D_{s}^{*} = \sup_{t \in (-\infty, +\infty)} \{ \| q_{0}(t) \|_{s} \} , \quad (s \geq 2) .
\]

By the Local Well-Posedness Theorem 2.3, there exists \( \tau = \tau(D_{n}^{*}) > 0 \), such that

\[
\| q_{e}(t) - q_{0}(t) \|_{n} \leq C_{n}^{(2)} \epsilon , \quad t \in [0, \tau] ,
\]

where \( C_{n}^{(2)} = C_{n}^{(2)}(D_{n+4}^{*}) \). There is an integer \( N > 0 \) such that

\[
q_{0}(N\tau) \in W_{n}^{cs}|_{\epsilon=0} ,
\]

where \( W_{n}^{cs} \) is given by the Center-Stable Manifold Theorem 2.3. Iterating the Local Well-Posedness Theorem \( N \) times, one gets

\[
\| q_{e}(t) - q_{0}(t) \|_{n} \leq C_{n}^{(3)} \epsilon , \quad t \in [0, N\tau] ,
\]

where \( C_{n}^{(3)} = C_{n}^{(3)}(D_{n+4}^{*}) \). Our goal is to determine when \( q_{e}(N\tau) \in W_{n}^{cs} \) through Melnikov measurement. The two Melnikov vectors \( U^{+} \) and \( U_{+} \) (3.40)-(3.41) are transversal to \( W_{n}^{cs} \). There is a unique point \( \hat{q}_{e}(N\tau) \in W_{n}^{cs} \) such that

\[
q_{e}(N\tau) - \hat{q}_{e}(N\tau) \in \text{span} \{ U^{+}, U_{+} \} ;
\]

thus, \( \hat{q}_{e}(N\tau) \in W_{n+4}^{cs} \). By the Center-Stable Manifold Theorem 2.3,

\[
\| \hat{q}_{e}(N\tau) - q_{0}(N\tau) \|_{n} \leq C_{n}^{(4)} \epsilon ,
\]

where \( C_{n}^{(4)} = C_{n}^{(4)}(D_{n+4}^{*}) \). Thus

\[
\| q_{e}(N\tau) - \hat{q}_{e}(N\tau) \|_{n} \leq C_{n} \epsilon ,
\]

where \( C_{n} = C_{n}(D_{n+4}^{*}) \). To determine when \( q_{e}(N\tau) = \hat{q}_{e}(N\tau) \), one can define the signed distances

\[
d_{1} = \langle U^{+}, \hat{q}_{e}(N\tau) - q_{e}(N\tau) \rangle , \quad d_{2} = \langle U_{+}, \hat{q}_{e}(N\tau) - q_{e}(N\tau) \rangle ,
\]

where \( \vec{q} = (q, \hat{q})^{T} \), and

\[
\langle A, B \rangle = \int_{0}^{2\pi/\kappa_{2}} \int_{0}^{2\pi/\kappa_{1}} \{ \overline{A_{1}}B_{1} + \overline{A_{2}}B_{2} \} \; dx \; dy .
\]

The rest of the derivation for Melnikov integrals is completely standard. For details, see e.g. (9)-(10),

\[
d_{k} = \epsilon M_{k} + o(\epsilon) , \quad k = 1, 2 ,
\]

where

\[
M_{1} = \int_{-\infty}^{\infty} \langle U^{+}, G \rangle \; dt , \quad M_{2} = \int_{-\infty}^{\infty} \langle U_{+}, G \rangle \; dt ,
\]
where $G = (f, \dot{f})^T$, $f = \Delta Q - \alpha Q + \beta$. That is,

$$M_1 = \int_{-\infty}^{\infty} \int_{0}^{2\pi/\kappa_2} \int_{0}^{2\pi/\kappa_1} \text{Re} \left\{ (\Psi^{(1)}_+ \dot{\Psi}^{(1)}_+) f + (\Psi^{(2)}_+ \dot{\Psi}^{(2)}_+) f \right\} \, dxdydt ,$$

$$M_2 = \int_{-\infty}^{\infty} \int_{0}^{2\pi/\kappa_2} \int_{0}^{2\pi/\kappa_1} \text{Re} \left\{ (\Phi^{(1)}_+ \dot{\Phi}^{(1)}_+) f + (\Phi^{(2)}_+ \dot{\Phi}^{(2)}_+) f \right\} \, dxdydt ,$$

where $\eta = \omega$, and we divide $\Psi^+$ by the constant $i\lambda_0\kappa_1 \sqrt{c_0^2 c_0^2 e^{i\gamma/2}}$, and $\Phi^+$ by $\frac{1}{4} i\alpha\kappa_2\eta \sqrt{c_0^2 c_0^2 e^{i\gamma/2}}$. It has been verified numerically that multiplication of $\Psi^+$ and $\Phi^+$ by a complex constant leads to equivalent results. It turns out that

$$M_j = M_j^{(1)} + \alpha M_j^{(2)} + \beta \cos \gamma M_j^{(3)} + \beta \sin \gamma M_j^{(4)} , \quad (j = 1, 2) ,$$

where $M_j^{(1)} = M_j^{(1)}(\omega, \Delta \rho)$, $(j = 1, 2; 1 \leq l \leq 4)$, $\Delta \rho = \hat{\rho} + i\alpha\kappa_2 c_0 \omega \lambda_0^{-1} \rho$, $\hat{\tau} = i\alpha\kappa_2 c_0 \omega \lambda_0^{-1} \tau + \Delta \rho$. $M_j = 0$ $(j = 1, 2)$ imply that

$$\alpha = \alpha(\omega, \Delta \rho, \gamma) = \left\{ M_1^{(1)} [\cos \gamma M_2^{(3)} \sin \gamma M_2^{(4)}] \right\} ,$$

$$\beta = \beta(\omega, \Delta \rho, \gamma) = \left\{ -M_1^{(1)} [\cos \gamma M_2^{(3)} \sin \gamma M_2^{(4)}] \right\}^{-1} ,$$

$$\beta = \beta(\omega, \Delta \rho, \gamma) = \left\{ M_1^{(2)} [\cos \gamma M_2^{(3)} \sin \gamma M_2^{(4)}] \right\}^{-1} .$$

(3.42)

(3.43)

Theorem 3.9. There exists $\epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0)$, there exists a domain $D_\epsilon \subset \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ where $\omega$ satisfies the constraint (3.2) or (3.3), and $\alpha \omega < \beta$. For any $(\alpha, \beta, \omega) \in D_\epsilon$, there exists another orbit in $W_c(\Pi_{\epsilon}) \cap W^{cs}_{\infty}$ other than the unstable curve $W^{u}_{\Pi}(\Pi_{\epsilon})$ of $\Pi_{\epsilon}$ in $\Pi$, for the perturbed DSII (3.4).}

Proof. The zeros of $M_j$ $(j = 1, 2)$ are given by (3.42) and (3.43). We need $\alpha > 0$ and $\beta > 0$ which define a region in the external parameter space, parametrized by $\Delta \rho$ and $\gamma$. Then the theorem follows from the implicit function theorem. Q.E.D.

For example, when $\kappa_1 = 1$ and $\kappa_2 = \sqrt{2}$,

$$\alpha\left(\frac{\sqrt{2}}{2} + 0.11, 1.1, \frac{\pi}{2}\right) = 5.645 , \quad \beta\left(\frac{\sqrt{2}}{2} + 0.11, 1.1, \frac{\pi}{2}\right) = 11.336 .$$

4. Appendix

The main obstacle toward proving the existence of a homoclinic orbit for the perturbed DSII (4.4) comes from a technical difficulty in the normal form transform [4]. In this appendix, we will present the difficulty.
4.1. The Technical Difficulty in the Normal Form Transform. To locate a homoclinic orbit to \( Q_\ast \) (2.4), we need to estimate the size of the local stable manifold of \( Q_\ast \). The size of the variable \( J \) is of order \( \mathcal{O}(\sqrt{\epsilon}) \). The size of the variable \( \theta \) is of order \( \mathcal{O}(1) \). To be able to track a homoclinic orbit, we need the size of the variable \( f \) to be of order \( \mathcal{O}(\epsilon^s) \), \( \mu < 1 \). Such an estimate can be achieved, if the quadratic term \( \mathcal{N}_2 \) in (2.8) can be removed through a normal form transformation. In fact, it is enough to remove its leading order part

\[
\hat{\mathcal{N}}_2 = 2\omega \left[ \Delta^{-1} \Upsilon |f|^2 + f \Delta^{-1} \Upsilon (f + \bar{f}) - (f \Delta^{-1} \Upsilon (f + \bar{f})) \right].
\]

That is, our goal is to find a normal form transform \( g = f + K(f, f) \) where \( K \) is a bilinear form, that transforms the equation

\[
f_t = L_\epsilon f - i\hat{\mathcal{N}}_2,
\]

into an equation with a cubic nonlinearity

\[
g_t = L_\epsilon g + \mathcal{O}(|g|^3), \quad (s \geq 2),
\]

where \( L_\epsilon \) is given in (2.8). In terms of Fourier transforms,

\[
f = \sum_{k \neq 0} \hat{f}(k)e^{ik\xi}, \quad \bar{f} = \sum_{k \neq 0} \bar{f}(-k)e^{ik\xi},
\]

where \( k = (k_1, k_2) \in \mathbb{Z}^2, \xi = (\kappa_1 x, \kappa_2 y) \). The terms in \( \hat{\mathcal{N}}_2 \) can be written as

\[
\Delta^{-1} \Upsilon |f|^2 = \frac{1}{2} \sum_{k+\ell \neq 0} a(k + \ell) \left[ \hat{f}(k)\bar{\hat{f}}(-\ell) + \hat{f}(\ell)\bar{\hat{f}}(-k) \right] e^{i(k+\ell)\xi},
\]

\[
f \Delta^{-1} \Upsilon f - \langle f \Delta^{-1} \Upsilon \bar{f} \rangle = \frac{1}{2} \sum_{k+\ell \neq 0} [a(k) + a(\ell)] \hat{f}(k)\bar{\hat{f}}(\ell) e^{i(k+\ell)\xi},
\]

\[
f \Delta^{-1} \Upsilon \bar{f} - \langle f \Delta^{-1} \Upsilon \bar{f} \rangle = \frac{1}{2} \sum_{k+\ell \neq 0} \left[ a(\ell) \hat{f}(k)\bar{\hat{f}}(-\ell) + a(k) \hat{f}(\ell)\bar{\hat{f}}(-k) \right] e^{i(k+\ell)\xi},
\]

where \( a(k) = \frac{k_1^2 \kappa_1^2 - k_2^2 \kappa_2^2}{k_1^2 \kappa_1^2 + k_2^2 \kappa_2^2} \).

We will search for a normal form transform of the general form,

\[
g = f + K(f, f),
\]

where

\[
K(f, f) = \sum_{k+\ell \neq 0} \left[ \hat{K}_1(k, \ell)\hat{f}(k)\bar{\hat{f}}(\ell) + \hat{K}_2(k, \ell)\hat{f}(\ell)\bar{\hat{f}}(-k) \right. \\
\left. + \hat{K}_3(\ell, k)\bar{\hat{f}}(-k)\hat{f}(\ell) + \hat{K}_4(\ell, k)\hat{f}(\ell)\hat{f}(\ell) \right] e^{i(k+\ell)x},
\]

where \( \hat{K}_j(k, \ell), \ (j = 1, 2, 3) \) are the unknown coefficients to be determined, and \( \hat{K}_j(k, \ell) = \hat{K}_j(\ell, k), \ (j = 1, 3) \). To eliminate the quadratic terms, we first need to set

\[
iL_\epsilon K(f, f) - iK(L_\epsilon f, f) - iK(f, L_\epsilon f) = \hat{\mathcal{N}}_2,
\]
which takes the explicit form:

\[(4.1) \quad (\sigma_1 + i\sigma)\tilde{K}_1(k, \ell) + B(\ell)\tilde{K}_2(k, \ell) + B(k)\tilde{K}_2(\ell, k) + B(k + \ell)\tilde{K}_3(\ell, k) = \frac{1}{2\omega} [B(k + \ell) + B(\ell)],
\]

\[(4.2) \quad -B(\ell)\tilde{K}_1(k, \ell) + (\sigma_2 + i\sigma)\tilde{K}_2(k, \ell) + B(k + \ell)\tilde{K}_2(\ell, k) + B(k)\tilde{K}_3(\ell, k) = \frac{1}{2\omega} [B(k + \ell) + B(\ell)],
\]

\[(4.3) \quad +B(\ell)\tilde{K}_1(k, \ell) + (\sigma_3 + i\sigma)\tilde{K}_2(k, \ell) + B(k + \ell)\tilde{K}_2(\ell, k) + B(k)\tilde{K}_3(\ell, k) = \frac{1}{2\omega} [B(k + \ell) + B(\ell)],
\]

\[(4.4) \quad + (\sigma_4 + i\sigma)\tilde{K}_3(k, \ell) = 0,
\]

where \(B(k) = 2\omega^2 a(k),\) and

\[
\sigma = \epsilon \left[ a - 2(k_1\ell_1\kappa_1^2 + k_2\ell_2\kappa_2^2) \right],
\]

\[
\sigma_1 = 2(k_2\ell_2\kappa_2^2 - k_1\ell_1\kappa_1^2) + B(k + \ell) - B(k) - B(\ell),
\]

\[
\sigma_2 = 2[(k_2 + \ell_2)k_2\kappa_2^2 - (k_1 + \ell_1)\kappa_1^2] + B(k + \ell) - B(k) - B(\ell),
\]

\[
\sigma_3 = 2[(k_2 + \ell_2)k_2\kappa_2^2 - (k_1 + \ell_1)k_1\kappa_1^2] + B(k + \ell) + B(k) - B(\ell),
\]

\[
\sigma_4 = 2[(k_2^2 + k_2\ell_2 + \ell_2^2)\kappa_2^2 - (k_1^2 + k_1\ell_1 + \ell_1^2)\kappa_1^2] + B(k + \ell) + B(k) + B(\ell).
\]

Since these coefficients are even in \((k, \ell),\) we will search for even solutions, i.e.

\[
\tilde{K}_j(-k, -\ell) = \tilde{K}_j(k, \ell), \quad j = 1, 2, 3.
\]

The technical difficulty in the normal form transform comes from not being able to answer the following two questions in solving the linear system \((4.1)-(4.4):\)

1. Is it true that for all \(k, \ell \in \mathbb{Z}^2/\{0\},\) there is a solution ?
2. What is the asymptotic behavior of the solution as \(k\) and/or \(\ell \to \infty\) ? In particular, is the asymptotic behavior like \(k^{-m}\) and/or \(\ell^{-m}\) \((m \geq 0)\) ?

**4.2. A Formal Calculation.** Formally conducting the calculation for the second measurement to locate a homoclinic orbit \(|\bullet|\), one gets the formulas

\[
M_j = 0 \quad (j = 1, 2), \quad \beta \cos \gamma = -\frac{\alpha\omega\Delta\gamma}{2 \sin \frac{\Delta\gamma}{2}},
\]

where \(\Delta\gamma = -4(\vartheta_1 - \vartheta_2).\) Thus we have \(\alpha = 1/\chi,\)

\[
\chi = \chi(\omega, \Delta\rho) = (M_2^{(1)} M_1^{(4)} - M_1^{(1)} M_2^{(4)})^{-1} \left[ M_1^{(2)} M_2^{(4)} - M_2^{(2)} M_1^{(4)} \right] - \omega \Delta\gamma [2 \sin \frac{\Delta\gamma}{2}]^{-1} (M_1^{(3)} M_2^{(4)} - M_2^{(3)} M_1^{(4)})
\]

\[
\beta = \beta(\omega, \Delta\rho) = \left[ (\alpha\omega\Delta\gamma)^2 [2 \sin \frac{\Delta\gamma}{2}]^{-2} + (M_2^{(4)})^{-2} (M_1^{(1)} + \alpha M_1^{(2)} - M_1^{(3)} \omega\Delta\gamma [2 \sin \frac{\Delta\gamma}{2}]^{-1})^2 \right]^{1/2}.
\]
For example, when $\kappa_1 = 1$ and $\kappa_2 = \sqrt{2}$,
\[
\chi\left(\frac{\sqrt{2}}{2} + 0.11, 1.1\right) = 0.4326 .
\]

References

[1] Y. Li, Persistent homoclinic orbits for nonlinear Schrödinger equation under singular perturbation, Submitted (2001).
[2] T. Ozawa, Exact blow-up solutions to the Cauchy problem for the Davey-Stewartson systems, Proc. R. Soc. Lond. 436 (1992), 345.
[3] J. M. Ghidaglia and J. C. Saut, On the initial value problem for the Davey-Stewartson systems, Nonlinearity 3 (1990), 475.
[4] L. Y. Sung, An inverse scattering transform for the Davey-Stewartson II equations, part I, II, III, J. Math. Anal. Appl. 183 (1994), 121, 289, 477.
[5] Y. Li, Bäcklund-Darboux transformations and Melnikov analysis for Davey-Stewartson II equations, Journal of Nonlinear Sciences 10, no.1 (2000), 103.
[6] R. Adams, Sobolev Space, Academic Press, New York, 1975.
[7] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, Journal of Functional Analysis 9 (1972), 296.
[8] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, Lecture Notes in Math., Springer 448 (1975), 25.
[9] Y. Li and D. W. McLaughlin, Homoclinic orbits and chaos in discretized perturbed NLS system, part I, homoclinic orbits, Journal of Nonlinear Sciences 7 (1997), 211.
[10] Y. Li et al., Persistent homoclinic orbits for a perturbed nonlinear Schrödinger equation, Comm. Pure Appl. Math. XLIX (1996), 1175.

Department of Mathematics, University of Missouri, Columbia, MO 65211
E-mail address: cli@math.missouri.edu