ON THE FAMILY OF PENTAGONAL CURVES OF GENUS 6 AND ASSOCIATED MODULAR FORMS ON THE BALL

KENJI KOIKE

ABSTRACT. In this article we study the inverse of the period map for the family $F$ of complex algebraic curves of genus 6 equipped with an automorphism of order 5. This is a family with 2 parameters, and is fibred over a certain type of Del Pezzo surface. The period satisfies the hypergeometric differential equation for Appell’s $F_1\left(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}; \frac{6}{5}\right)$ of two variables after a certain normalization of the variable parameter.

This differential equation and the family $F$ are studied by G. Shimura (1964), T. Terada (1983, 1985), P. Deligne - G.D. Mostow (1986) and T. Yamazaki - M. Yoshida (1984). Recently M. Yoshida presented a new approach using the concept of configuration space. Based on their results we show the representation of the inverse of the period map in terms of Riemann theta constants. This is the first variant of the work of H. Shiga (1981) and K. Matsumoto (1989, 2000) to the co-compact case.

0. INTRODUCTION

Let $F$ be the family of algebraic curves given by

$$C(\lambda) : w^5 = \prod_{i=1}^{5}(z - \lambda_i),$$

here the parameter $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ lives on the domain $(\mathbb{P}^1)^5 - \Delta$, where

$$\Delta = \{(\lambda_i) \in (\mathbb{P}^1)^5 : \lambda_i = \lambda_j \text{ for some } i \neq j\}.$$

By putting $(\lambda_1, \lambda_2, \lambda_3) = (0, 1, \infty)$ we can normalize $C(\lambda)$ in the form

$$C'(x, y) : w^5 = z(z - 1)(z - x)(z - y)$$

where the parameter $(x, y)$ lives in

$$\Lambda = \{(x, y) \in \mathbb{C}^2 : xy(x - 1)(y - 1)(x - y) \neq 0\}$$

The period of $C'(x, y)$

$$\eta(x, y) = \int_{\gamma} \frac{dz}{w^2}$$

satisfies the system of differential equation

$$x(1 - x) \frac{\partial^2 u}{\partial x^2} + y(1 - y) \frac{\partial^2 u}{\partial y^2} + \left(\frac{6}{5} - \frac{11}{5} x\right) \frac{\partial u}{\partial x} - \frac{3}{5} y \frac{\partial u}{\partial y} - \frac{9}{5} u = 0$$

$$y(1 - y) \frac{\partial^2 u}{\partial y^2} + x(1 - x) \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{6}{5} - 2y\right) \frac{\partial u}{\partial y} - \frac{2}{5} x \frac{\partial u}{\partial x} - \frac{6}{5} u = 0$$

It is the hypergeometric differential equation for the Appell’s $F_1\left(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}; \frac{6}{5}; x, y\right)$. The dimension of the solution space is equal to 3. If it holds $\lambda' = g \circ \lambda$ for a certain projective...
transformation \( g \in \text{PGL}_2(\mathbb{C}) \), then we have the biholomorphic equivalence \( C(\lambda) \cong C(\lambda') \).

So we consider the quotient space

\[
X^0(2, 5) = ((\mathbb{P}^1)^5 - \Delta)/\text{PGL}_2(\mathbb{C}).
\]

as the parameter space for \( \mathcal{F} \), that is biholomorphically equivalent with \( \Lambda \).

According to the work of T. Terada (13), P. Deligne - G.D. Mostow (1) and T. Yamazaki - M. Yoshida (15) we have the following properties:

1. Let \( \{ \eta_1, \eta_2, \eta_3 \} \) be the basis of the solutions of (1.1). The image of the Schwarz map \( (x, y) \mapsto [\eta_1(x, y) : \eta_2(x, y) : \eta_3(x, y)] \in \mathbb{P}^2 \) is an open dense subset of a 2-dimensional ball \( \mathbb{B}_2 \).

2. The monodromy group for (1.1) is characterized as a certain congruence subgroup of the Picard modular group for \( k = \mathbb{Q}(e^{2\pi i/5}) \).

3. Let \( S_5 \) be the symmetric group of permutations of \( \{ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \} \), it has a natural action on \( X^0(2, 5) \). There is a compactification \( X(2, 5) \) of \( X^0(2, 5) \) so that we have \( S_5 \subset \text{Aut}(X(2, 5)) \). Yoshida showed \( X(2, 5) \) is a Del Pezzo surface of degree 5.

4. We obtain a single valued modular map on \( \mathbb{B}_2 \) as the inverse of the Schwarz map.

By the so-called Picard principle we can reduce the period map for \( \mathcal{F} \) to the Schwarz map for (1.1). So we proceed our study by the following steps.

In first 4 sections we make up the explicit realization of the above properties (1) - (4):

Section 1. We describe the parameter space \( X^0(2, 5) \) for \( \mathcal{F} \) and its compactification \( X(2, 5) \). We list up certain divisors those become to be essential in our study.

Section 2. We construct the period map for \( \mathcal{F} \). And we show how it reduces to the map \( \Phi : X^0(2, 5) \to \mathbb{B}_2 \).

Section 3. We list up the generator system of the monodromy group for \( \Phi \) in terms of the unitary reflections.

Section 4. We observe the degeneration of the Schwarz map for (1.1).

Section 5 is the main part of the article. There we study the 0 values of the Riemann theta functions of genus 6 with the characteristic \( (a, b) \in (\frac{1}{10}\mathbb{Z})^6 \times (\frac{1}{10}\mathbb{Z})^6 \) (Theorem 5.3). These are considered to be a certain kind of automorphic form on \( \mathbb{B}_2 \). Many of the above theta constants identically vanish on \( \mathbb{B}_2 \). At first it is proved that there are only 25 among them those are invariant under the action of the monodromy group. We show they are not identically zero on \( \mathbb{B}_2^4 \). Then we determine the vanishing locus on \( \mathbb{B}_2 \) of every theta constant in question.

In Section 6 we state the main theorem (Theorem 6.1), that is the representation of \( \Phi^{-1} \) via the theta constants. As the direct consequence we show the representation of the inverse Schwarz map for the Gauss hypergeometric differential equation \( E_{2,1}\left(\frac{1}{5}, \frac{2}{5} \right) \). In this case we have the arithmetic triangle group of co-compact type \( \Delta(5, 5, 5) \) as the monodromy group, and it is the case mentioned by Shimura (11). As the another application we show the explicit generator system for the graded ring of the automorphic forms with respect to the unitary group \( U(2, 1; \mathcal{O}_k) \) over \( \mathcal{O}_k \) (Theorem 6.2).

1. The configuration space \( X(2, 5) \)

Here we summarize the fundamental facts of \( X(2, 5) \). For precise arguments, see [17, Chapter V]. Let \( [a : b] \) be a point on \( \mathbb{P}^1 \), and let \( \lambda = b/a \) be its representative on \( \mathbb{C} \cup \{ \infty \} \). Always we use the notation \( \lambda_i \in \mathbb{P}^1 \). Let us consider ordered distinct five points on \( \mathbb{P}^1 \):

\[
\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in (\mathbb{P}^1)^5 - \Delta.
\]
where, $\Delta$ is degenerate locus:

$$\Delta = \{(\lambda_i) \in (\mathbb{P}^1)^5 : \lambda_i = \lambda_j \text{ for some } i \neq j\}.$$ 

A projective transformation $g \in \text{PGL}_2(\mathbb{C})$ acts on $(\mathbb{P}^1)^5$ as

$$g \cdot (\lambda_1, \cdots, \lambda_5) = (g(\lambda_1), \cdots, g(\lambda_5)).$$

The configuration space $X^0(2, 5)$ is defined by the quotient space

$$X^0(2, 5) = ((\mathbb{P}^1)^5 - \Delta)/\text{PGL}_2(\mathbb{C}).$$

It has a good compactification

$$X(2, 5) = \overline{X^0(2, 5)} = ((\mathbb{P}^1)^5 - \Delta')/\text{PGL}_2(\mathbb{C})$$

where

$$\Delta' = \{(\lambda_i) \in (\mathbb{P}^1)^5 : \lambda_i = \lambda_j = \lambda_k \text{ for some } i \neq j \neq k \neq i\}.$$

There exist ten lines on $X(2, 5)$ of the form

$$L(ij) = \{\text{the orbit of the form } \lambda_i = \lambda_j\}/\text{PGL}_2(\mathbb{C}) \cong \mathbb{P}^1.$$

Notice that $L(ij) \cap L(jk) = \emptyset (i \neq j \neq k \neq i)$ by the definition, and the degenerate locus $X(2, 5) - X^0(2, 5)$ is just the union of these ten lines. $X(2, 5)$ is isomorphic to the blow-up of $\mathbb{P}^2$ at four points. We can see the blow down $\pi : X(2, 5) \to \mathbb{P}^2$ by the following way.

Let us specialize $\lambda_4 = 0$, $\lambda_5 = \infty$ and regard $[\lambda_1 : \lambda_2 : \lambda_3]$ as a point in $\mathbb{P}^2$, then we obtain the following correspondence;

- $P_1 = [1 : 0 : 0] = \pi(L(15)), \quad P_2 = [0 : 1 : 0] = \pi(L(25)),$
- $P_3 = [0 : 0 : 1] = \pi(L(35)), \quad P_4 = [1 : 1 : 1] = \pi(L(45)),$

and

$$\pi(X^0(2, 5)) = \{[\lambda_1 : \lambda_2 : \lambda_3] \in \mathbb{P}^2 : \lambda_i \neq \lambda_j (i \neq j), \quad i, j = 1, 2, 3, 4\}.$$

For five distinct numbers $i, j, k, l, m$ in $\{1, 2, 3, 4, 5\}$, We define a divisor $D(ijklm)$ on $X(2, 5)$ by

$$D(ijklm) = L(ij) + L(jk) + L(kl) + L(lm) + L(mi).$$

Such a divisor is understood as a “juzu sequence” (see [17]). A 5-juzu sequence $(ijklm)$ is the pentagon with vertices $i, j, k, l, m$ in this cyclic order. The divisor $D(ijklm)$ is given by the edges of this pentagon. We identify $(ijklm)$ and $(imlkj)$ since $L(ij) = L(ji)$.

![Figure 1](image_url)

There are twelve different divisors of this form. Let $H$ be a line on $\mathbb{P}^2$. As easily shown, $D(ijklm)$ are linearly equivalent to the divisor

$$3\pi^*H - L(15) - L(25) - L(35) - L(45).$$
By the general theory of Del Pezzo surfaces (for example, see [4, Chapter 5]), this is anti-canonical class and very ample. In fact, we have the following proposition by direct calculations.

**Proposition 1.1.** Set
\[ J(ijklm)(\lambda) = (\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_l)(\lambda_l - \lambda_m)(\lambda_m - \lambda_i) \]
for twelve \((ijklm)\). Then the map
\[ J : X(2,5) \rightarrow \mathbb{P}^{11}, \quad J(\lambda) = [\cdots : J(ijklm)(\lambda) : \cdots] \]
is an embedding.

**Remark 1.1.** It is necessary to make precise the above notation for \(J(ijklm)\). By using the homogeneous coordinate \([a_i : b_i]\) for \(\lambda_i \in \mathbb{P}^1\), we set \(d(ij) = a_ib_i - a_ib_j\). So \(\lambda_i - \lambda_j\) stands for \(d(ij)\). The ratio \([J(ijklmn) : J(i'j'k'l'm')]\) defines a rational function on \(X(2,5)\).

We shall give the correspondence between these divisors and theta functions in a later section.

## 2. The Family of Pentagonal Curves and the Periods

Let us consider the algebraic curve
\[ C_\lambda : y^5 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)(x - \lambda_4)(x - \lambda_5), \quad \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in (\mathbb{P}^1)^5 - \Delta. \]
If it holds \(\lambda' = g \cdot \lambda\) for some \(g \in \text{PGL}_2(\mathbb{C})\), then we have a biholomorphic equivalence \(C(\lambda) \cong C(\lambda')\). So we identify \(C(\lambda)\) and \(C(\lambda')\) in this case. Set \(\mathcal{F} = \{C_\lambda : \lambda \in X^0(2,5)\}\). We regard \(C_\lambda\) as a five sheeted cyclic covering over \(\mathbb{P}^1\) branched at \(\lambda_i\) via the projection
\[ \pi : C_\lambda \rightarrow \mathbb{P}^1, \quad (x,y) \mapsto x. \]
By the Hurwitz formula, the genus of \(C_\lambda\) is six. We have the following basis of \(H^0(C_\lambda, \Omega^1)\):
\[
(2.1) \quad \varphi_1 = \frac{dx}{y^2}, \quad \varphi_2 = \frac{dx}{y^3}, \quad \varphi_3 = \frac{x}{y^3}, \quad \varphi_4 = \frac{dx}{y^4}, \quad \varphi_5 = \frac{x^2}{y^4}, \quad \varphi_6 = \frac{dx}{y^4}.
\]
Let \(\rho\) denotes the automorphism of order five:
\[ \rho : C_\lambda \rightarrow C_\lambda, \quad (x,y) \mapsto (x,\zeta y) \quad (\zeta = \exp(2\pi \sqrt{-1}/5)) \]
on \(C_\lambda\).

**Remark 2.1.** Throughout this article always \(\zeta\) stands for \(\exp(2\pi \sqrt{-1}/5)\).

Next, we construct a symplectic basis of \(H_0(C_\lambda, \mathbb{Z})\). Let \(\lambda^0 = (\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0, \lambda_5^0) \in X^0(2,5)\) be a real point such that \(\lambda_1 < \cdots < \lambda_5\) and \(C_0\) be the corresponding curve. Take a point \(x_0 \in \mathbb{P}^1\) such that \(\text{Im}(x_0) < 0\), and make line segments \(l_i\) connecting \(x_0\) and \(\lambda_i\). Then \(\Sigma = \mathbb{P}^1 - \cup l_i\) is simply connected and \(\pi^{-1}(\Sigma)\) is isomorphic to \(\Sigma \times \mathbb{Z}/5\mathbb{Z}\). Here, we choose the fiber coordinates \(k \in \mathbb{Z}/5\mathbb{Z}\) such that \(\rho(x,k) = (x, k + 1)\). Let (\(i,j\)) be the oriented arc from \(\lambda_i\) to \(\lambda_j\) in \(\Sigma\). We obtain the following five oriented arcs \(\alpha_k(i,j)\) \((k = 1, \cdots, 5)\) in \(C_0\):
\[
(2.2) \quad \alpha_k(i,j) = (\alpha(i,j), k) \subset \Sigma \times \mathbb{Z}/5\mathbb{Z}.
\]
We define cycles $\gamma_1$, $\gamma_2$, $\gamma_3$ on $C_0$ (Figure 2) using this notation:

$$
\begin{align*}
\gamma_1 &= \alpha_1(1, 2) + \alpha_2(2, 1), \\
\gamma_2 &= \alpha_1(3, 4) + \alpha_2(4, 3), \\
\gamma_3 &= \alpha_1(1, 3) + \alpha_2(3, 4) + \alpha_3(4, 2) + \alpha_2(2, 1).
\end{align*}
$$

(2.3)

The intersection numbers of these cycles are given by

$$
\begin{align*}
(2.4) \quad B_1 &= \rho(\gamma_1) + \rho^3(\gamma_1), \\
B_2 &= \rho(\gamma_2) + \rho^3(\gamma_2), \\
B_3 &= \rho(\gamma_3) + \rho^2(\gamma_3), \\
B_4 &= \rho^3(\gamma_1), \\
B_5 &= \rho^3(\gamma_2), \\
B_6 &= \rho(\gamma_3).
\end{align*}
$$

The intersection numbers of these cycles are given by

$$
A_i \cdot A_j = B_i \cdot B_j = 0, \quad A_i \cdot B_j = \delta_{ij}.
$$

So, $\{A_i, B_i\}$ is a symplectic basis of $H_1(C_0, \mathbb{Z})$. Let $\lambda$ be a point on $X^0(2, 5)$, and suppose an arc $r$ from $\lambda^0$ to $\lambda$. Since the family $\mathcal{F}$ is locally trivial as a topological fiber space over $X^0(2, 5)$, by using this trivialization along $r$, we obtain the systems $\{\alpha_k(i, j)(\lambda)\}$, $\{\gamma_i(\lambda)\}$ and the symplectic basis $\{A_i(\lambda), B_i(\lambda)\}$ on $C_\lambda$. We have the relation (2.4) between $\{\gamma_i(\lambda)\}$ and $\{A_i(\lambda), B_i(\lambda)\}$ also. We note that $\{A_i(\lambda), B_i(\lambda)\}$ depend on the homotopy class of $r$.

Now, we consider the period matrix of $C_\lambda$:

$$
\Pi(\lambda) = \Pi = (Z_1, Z_2) = \begin{pmatrix}
\int_{A_1} \varphi_1 & \cdots & \int_{A_6} \varphi_1 \\
\int_{B_1} \varphi_1 & \cdots & \int_{B_6} \varphi_1 \\
\int_{A_1} \varphi_6 & \cdots & \int_{A_6} \varphi_6 \\
\int_{B_1} \varphi_6 & \cdots & \int_{B_6} \varphi_6
\end{pmatrix}.
$$

The normalized period matrix $\Omega(\lambda) = \Omega = Z_1^{-1}Z_2$ belongs to the Siegel upper half space of degree 6:

$$
\mathcal{S}_6 = \{\Omega \in \text{GL}_6(\mathbb{C}) : \Omega = \Omega, \ \text{Im}(\Omega) \text{ is positive definite}\}.
$$

The automorphism $\rho$ acts on $H^0(C_\lambda, \Omega^1)$ and $H_1(C_\lambda, \mathbb{Z})$. So we have the representation matrices $R \in \text{GL}_6(\mathbb{C})$ and $M \in \text{GL}_{12}(\mathbb{Z})$ of $\rho$ with respect to the basis $\{\varphi_i\}$ and $\{A_i, B_i\}$, respectively. It holds $R\Pi = \Pi M$. Put

$$
(2.5) \quad M = \begin{pmatrix}
^tD & ^tB \\
^tC & ^tA
\end{pmatrix}, \quad \sigma = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}.
$$
Then the matrix $\sigma$ belongs to the symplectic group
\[
\text{Sp}_{12}(\mathbb{Z}) = \{ g \in \text{GL}_{12}(\mathbb{Z}) : {}^t g J g = J \}, \quad J = \begin{pmatrix} 0 & I_6 \\ -I_6 & 0 \end{pmatrix}
\]
and it holds
\[
\Omega = (A\Omega + B)(C\Omega + D)^{-1}.
\]
As easily shown, $\varphi_i (i = 1, \cdots, 6)$ is eigenvectors of $\rho$ and we have
\[
R = \begin{pmatrix}
\zeta^3 & \zeta^2 & 0 \\
\zeta^2 & \zeta & 0 \\
0 & \zeta & \zeta
\end{pmatrix}.
\]
By the relation (2.4) of $A_i, B_i$, we have
\[
(2.6) \quad \Pi = (a, b, c, R^2 a, R^2 b, R^4 c, (R + R^3) a, (R + R^3) b, (R + R^2) c, R^3 a, R^3 b, R c),
\]
where we denote
\[
a = {}^t(\int_{\gamma_1} \varphi_1, \cdots, \int_{\gamma_1} \varphi_6), \quad b = {}^t(\int_{\gamma_2} \varphi_1, \cdots, \int_{\gamma_2} \varphi_6), \quad c = {}^t(\int_{\gamma_3} \varphi_1, \cdots, \int_{\gamma_3} \varphi_6).
\]
According to (2.4),
\[
\rho(A_1) = \rho(\gamma_1) = (\rho(\gamma_1) + \rho^3(\gamma_1)) - \rho^3(\gamma_1) = B_1 - B_4.
\]
By the same way, we can describe $\rho(A_2), \cdots, \rho(B_6)$ in terms of $\{A_i, B_i\}$. So we can determine $M$, and obtain
\[
(2.7) \quad \sigma = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Put
\[
\eta(\lambda) = [\eta_1(\lambda) : \eta_2(\lambda) : \eta_3(\lambda)] \in \mathbb{P}^2, \quad \eta_1(\lambda) = \int_{\gamma_1} \varphi_1, \quad \eta_2(\lambda) = \int_{\gamma_2} \varphi_1, \quad \eta_3(\lambda) = \int_{\gamma_3} \varphi_1.
\]
These are multi-valued analytic functions of $\lambda$. Applying the Riemann positive condition
\[
(\int_{A_1} \varphi_1, \cdots, \int_{B_6} \varphi_1) J ({}^t(\int_{A_1} \varphi_1, \cdots, \int_{B_6} \varphi_1)) > 0
\]
for (2.6), we obtain
\[
|\eta_1|^2 + |\eta_2|^2 + \frac{1 - \sqrt{5}}{2} |\eta_3|^2 < 0.
\]
So, \( \eta = (\eta_1, \eta_2, \eta_3) \) belongs to the complex ball

\[
B^A_2 = \{ \eta \in \mathbb{P}^2 : \tau \eta A \eta < 0 \}, \quad A = \text{diag}(1, 1, \frac{1 - \sqrt{5}}{2}).
\]

Next, we determine \( \Omega \) explicitly. Write \( a = (a_i), \ b = (b_i) \) and \( c = (c_i) \). Then, the Riemann bilinear relation \( \Pi^H \Pi = 0 \) induces the following equations:

\[
c_2 = -(\zeta^2 + \zeta^4)(a_1a_2 + b_1b_2)/c_1, \quad c_3 = -(\zeta^2 + \zeta^4)(a_1a_3 + b_1b_3)/c_1.
\]

By substituting them for \( Z_1, \ Z_2 \) in \( \Pi \) we can proceed the calculation of \( \Omega = Z_1^{-1}Z_2 \) (using a computer). Hence we have the following:

**Lemma 2.1.** Let \( \Delta = \eta_1^2 + \eta_2^2 - \zeta^3(1 + \zeta)\eta_3^2 \). The period matrix \( \Omega = (\Omega_{ij}) \) is given by

\[
\begin{align*}
\Omega_{11} &= (\zeta^3 - 1)(\eta_1^2 + (1 + \zeta^3)\eta_2^2 + \eta_3^2)/\Delta, \\
\Omega_{12} &= (\zeta^3 - \zeta)\eta_1\eta_2/\Delta, \\
\Omega_{13} &= (\zeta^4 - \zeta)\eta_1\eta_3/\Delta, \\
\Omega_{21} &= (\zeta^3 - \zeta)\eta_1\eta_2/\Delta, \\
\Omega_{22} &= (\zeta^3 - 1)((1 + \zeta^3)\eta_1^2 + \eta_2^2 + \eta_3^2)/\Delta, \\
\Omega_{23} &= (1 - \zeta^2)\eta_1\eta_3/\Delta, \\
\Omega_{31} &= (\zeta^4 - \zeta)\eta_1\eta_3/\Delta, \\
\Omega_{32} &= (1 - \zeta^2)\eta_1\eta_3/\Delta, \\
\Omega_{33} &= (\zeta^3 - \zeta)\eta_1\eta_3/\Delta, \\
\Omega_{41} &= (1 - \zeta)\eta_1\eta_3/\Delta, \\
\Omega_{42} &= (1 - \zeta)\eta_1\eta_3/\Delta, \\
\Omega_{43} &= (\zeta^3((1 + \zeta)\eta_1^2 + (1 + \zeta^3)\eta_2^2 + \eta_3^2)/\Delta, \\
\Omega_{51} &= (\zeta^3((1 + \zeta)\eta_1^2 + (1 + \zeta^3)\eta_2^2 + \eta_3^2)/\Delta, \\
\Omega_{52} &= (\zeta^3((1 + \zeta)\eta_1^2 + (1 + \zeta^3)\eta_2^2 + \eta_3^2)/\Delta, \\
\Omega_{53} &= (\zeta^3((1 + \zeta)\eta_1^2 + (1 + \zeta^3)\eta_2^2 + \eta_3^2)/\Delta, \\
\Omega_{61} &= (\zeta + \zeta^2)(\eta_1^2 + \eta_2^2 - \zeta^3(1 + \zeta^2)\eta_3^2)/\Delta.
\end{align*}
\]

Now we define our period map

\[
\Phi : X^\circ(2, 5) \longrightarrow \mathbb{B}^A_2, \quad \lambda \mapsto [\eta_1(\lambda) : \eta_2(\lambda) : \eta_3(\lambda)],
\]

that is multi-valued analytic. The above Lemma says that the original period map \( \lambda \mapsto \Omega(\lambda) \) factors as

\[
X^\circ(2, 5) \longrightarrow \mathbb{B}^A_2 \longrightarrow \mathcal{S}_6.
\]

Throughout this paper, we denote matrices of the form in Lemma 2.1 by \( \Omega(\eta) \).

### 3. The Monodromy Group and Reflections

The multi-valuedness of \( \Phi \) induces a unitary representation with respect to \( A \) in (2.8) of the fundamental group \( \pi_1(X^\circ(2, 5)) \). We call it the monodromy group of \( \Phi \). The structures of our monodromy group is studied in [13]. Set

\[
\Gamma = \{ g \in \text{GL}_3(\mathbb{Z}[\zeta]) : \tau g A g = A \}, \quad \Gamma(1 - \zeta) = \{ g \in \Gamma : \ g \equiv I_3 \mod 1 - \zeta \}.
\]

The group \( \Gamma \) acts on \( \mathbb{B}_2^A \) (left action).

**Theorem 3.1** (T. Yamazaki, M. Yoshida [13]). (1). The monodromy group of the period map \( \Phi \) coincides with \( \Gamma(1 - \zeta) \) and the quotient \( \Gamma/(\pm I)\Gamma(1 - \zeta) \) is isomorphic to the symmetric group \( S_6 \).

(2). The quotient \( \mathbb{B}_2^A/\Gamma(1 - \zeta) \) is biholomorphically equivalent to the blow up of \( \mathbb{P}^2 \) at four points.
Remark 3.1 (see [15]). There are ten $(-1)$-curves on $\mathbb{P}^2_2/\Gamma(1-\zeta)$, and $S_5$ acts transitively on them.

According to [13] and [15], it is proved that $\Gamma$ and $\Gamma(1-\zeta)$ are reflection groups and the generator systems are given also. We expose those generator system in a form adapted for our calculation in the later sections.

Let us consider the reference point $\lambda_0^0 \in X^\circ(2,5)$ again. Now we define the half way monodromy transformation $g_{12}$ induced from the permutation of $\lambda_1^0$ and $\lambda_2^0$. Let us consider a continuous arc $R_{12}$ starting from $\lambda_0^0$:

$$\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3^0, \lambda_4, \lambda_5^0), \quad (0 \leq t \leq 1)$$

such that (Figure 3)

$$\lambda_2(1) = \lambda_1^0, \quad \lambda_1(1) = \lambda_2^0, \quad \text{Im}(\lambda_1(t)) < 0 < \text{Im}(\lambda_2(t)) \quad (0 < t < 1).$$

![Figure 3.](image)

Let $\eta(t) = \eta(\lambda(t))$ be the corresponding periods. Recall the definition (2.3). It is apparent that $\gamma_2$ and $\gamma_3$ are invariant after this deformation process. Describing $\gamma_1(t)$ for any $0 \leq t \leq 1$, we get $\gamma_1(1) = -\rho(\gamma_1(0))$. Namely,

$$
\begin{pmatrix}
\eta_1(1) \\
\eta_2(1) \\
\eta_3(1)
\end{pmatrix} = 
\begin{pmatrix}
-\zeta^3 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\eta_1(0) \\
\eta_2(0) \\
\eta_3(0)
\end{pmatrix}.
$$

The matrix in the right hand side belongs to $\Gamma$, and we denote it by $g_{12}$. We define $g_{23}$, $g_{34}$, $g_{45}$ by the same manner. Set

$$h_{12} = (g_{12})^2, \quad h_{23} = (g_{23})^2, \quad h_{34} = (g_{34})^2
$$

(3.2)

$$h_{13} = (g_{23})^{-1}(g_{12})^2g_{23}, \quad h_{14} = (g_{23}g_{34})^{-1}(g_{12})^2g_{23}g_{34}.
$$

Proposition 3.1 (see [16]). The monodromy group is generated by $h_{ij}$ in (3.2).

Let $T_\alpha$ be the reflection on $\mathbb{P}^2_2$ with respect to a root $\alpha$;

$$T_\alpha(\eta) = \eta - (1 + \zeta^3)\frac{t_\alpha\eta}{t_\alpha A\alpha},$$

and $R_\beta$ be the reflection on $\mathbb{P}^2_2$ with respect to a root $\beta$;

$$R_\beta(\eta) = \eta - (1 - \zeta)\frac{t_\beta\eta}{t_\beta A\beta}.$$
Then we see that

**Lemma 3.1.** Set

\[ \alpha_{12} = (1, 0, 0), \quad \alpha_{23} = (\zeta^3, 1, -(1 + \zeta)), \quad \alpha_{34} = (0, 1, 0), \quad \alpha_{45} = (0, 1, \zeta^3) \]

and set

\[ \beta_{12} = (1, 0, 0), \quad \beta_{13} = (1, -1, 1 + \zeta), \quad \beta_{14} = (1, \zeta^3, 1 + \zeta), \]
\[ \beta_{23} = (\zeta^3, 1, -(1 + \zeta)), \quad \beta_{34} = (0, 1, 0). \]

Then it holds \( g_{ij} = T_{\alpha_{ij}}, \quad h_{kl} = R_{\beta_{ij}} \). And \( g_{ij} \) is of order five, \( h_{kl} \) is of order ten.

**Remark 3.2.** The group \( \Gamma' \) generated by \( \{g_{ij}\} \) has a representation to \( S_5 \). According to Theorem 3.1, \( \Gamma \) is generated \( \Gamma' \) and \( \pm I \).

The deformation of the curve \( C_{\lambda(t)} \) along \( R_{12} \) in (3.1) induces a symplectic basis \( \{A_i(t), B_i(t)\} \) on it. So \( \{A_i(1), B_i(1)\} \) is again a symplectic basis on \( C_{\lambda(0)} \). Hence we obtain a symplectic transformation

\[ t(B_1(1), \cdots, B_6(1), A_1(1) \cdots, A_6(1)) = \hat{g}_{12}(B_1(0), \cdots, B_6(0), A_1(0) \cdots, A_6(0)). \]

For \( \hat{g}_{12} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), we have

\[ \Omega(\eta_1) = (A\Omega(\eta_0) + B)(C\Omega(\eta_0) + D)^{-1}. \]

Recall \( R_{12} \) induces the change of cycles

\[ (\gamma_1, \gamma_2, \gamma_3) \rightarrow (-\rho(\gamma_1), \gamma_2, \gamma_3). \]

Together with (2.4), we obtain:

\[ \hat{g}_{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}. \]
By same consideration, we obtain following:

\[
\hat{g}_{23} = \begin{bmatrix}
1 & 1 & 0 & 1 & -1 & 1 & 2 & -1 & -1 & 1 & -1 & 1 \\
-1 & 1 & -1 & -1 & 1 & 0 & -2 & 2 & 0 & -1 & 1 & -2 \\
1 & -1 & 2 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 1 \\
0 & 1 & -1 & 1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & -1 & 1 & 0 & -1 & 1 & -1 \\
1 & 0 & 1 & 1 & -1 & 1 & 2 & -2 & 0 & 1 & 1 & -2 \\
-1 & 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\
1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\
-1 & 1 & -2 & -1 & 1 & 1 & -1 & 2 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\
1 & -1 & 2 & 0 & -1 & -2 & 0 & -1 & 1 & -1 & 0 & 2 
\end{bmatrix},
\]

\[
\hat{g}_{34} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix},
\]

\[
\hat{g}_{45} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 
\end{bmatrix}.
\]

4. Degenerate Loci

According to Theorem 3.1, P. Deligne - G.D. Mostow\(^\text{(1)}\) and T. Terada\(^\text{(3)}\) the period map \(\Phi\) induces the biholomorphic equivalence

\[
\tilde{\Phi} : X^\circ(2, 5) \approx \mathbb{B}_2^2/\Gamma(1 - \zeta),
\]

where \(\mathbb{B}_2^2 = \text{Im}\Phi\). Moreover we have the unique extension

\[
\tilde{\Phi} : X(2, 5) \approx \mathbb{B}_2^A/\Gamma(1 - \zeta),
\]
and \( \cup L(ij) = X(2,5) - X^0(2,5) \) corresponds to \((\mathbb{B}^A_2 - \mathbb{B}^B_2) / \Gamma(1 - \zeta)\). Let \( \pi \) be the projection \( \mathbb{B}^A_2 \to \mathbb{B}^B_2 / \Gamma(1 - \zeta) \), and let \( \ell(ij) \) denote \( \pi^{-1}(\Phi(L(ij))) \).

Now we consider a degenerate curve
\[
y^5 = (x - \lambda_1)^2(x - \lambda_3)(x - \lambda_4)(x - \lambda_5)
\]
with \( (\lambda_1, \lambda_1, \lambda_3, \lambda_4, \lambda_5) \in L(12) \), and putting \( \lambda' = (\lambda_1, \lambda_3, \lambda_4, \lambda_5) \) we denote it by \( C_{\lambda'} \). Let \( \tilde{C}_{\lambda'} \) denote the non-singular model of \( C_{\lambda'} \). It is a curve of genus 4. Set \( \mathcal{F}_{12} \) be the totality of \( \tilde{C}_{\lambda'} \). For the parameter \( (\lambda' \gamma) \) the cycle \( \gamma_1 \) vanishes on \( \tilde{C}_{(\lambda' \gamma)} \), but \( \gamma_2 \) and \( \gamma_3 \) are still alive. So we can define \( A_i, B_i \) \( (i = 2, 3, 5, 6) \) on \( \tilde{C}_{\lambda'} \) by the same argument as for \( C_{\lambda} \). Hence we obtain a basis \( \{A_i, B_i\} \) \( (i = 2, 3, 5, 6) \) of \( H_1(\tilde{C}_{\lambda'}, \mathbb{Z}) \). By putting \( \lambda' = (0, 1, t, \infty) \) the period
\[
\int_\gamma x^{-\frac{1}{2}}(x - 1)^{-\frac{1}{2}}(x - t)^{-\frac{1}{2}}dx,
\]
on \( \tilde{C}_{\lambda'} \) gives a solution for the Gauss hypergeometric differential equation \( E_{2,1}(\frac{1}{5}, \frac{2}{5}, \frac{4}{5}) \):
\[
t(1-t)\frac{d^2u}{dt^2} + (\frac{4}{5} - \frac{8}{5}t)\frac{du}{dt} - \frac{2}{5}u = 0
\]
The corresponding monodromy group is the triangle group \( \Delta(5,5,5) \) (see [12], [10], [17], p.138). Set
\[
\mathbb{B}_1 = \{ \eta \in \mathbb{B}^A_2 : \eta_1 = 0 \},
\]
it is the mirror of the reflection \( g_{12} \). By using the system \( \{\gamma_2, \gamma_3\} \) we define a multi-valued map
\[
\Phi_{12} : L(12) \to \mathbb{B}_1,
\]
It induces the restriction \( \tilde{\Phi}_{|L(12)} \). By the same manner we obtain that \( \tilde{\Phi}_{|[\ell(ij)]} \) is the mirror of the reflection \( g_{ij} \). Suppose \( \lambda \in L(12) \) and set \( \eta = \eta(\lambda) = \Phi_{12}(\lambda) \). By putting \( \eta_1 = 0 \) in Lemma 2.3, we see that
\[
\Omega(\eta) = \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} \oplus \Omega'(\eta), \quad \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} = \tau_0 = \begin{pmatrix} \zeta - 1 & \zeta^2 + \zeta^3 \\ -\zeta^4 & -\zeta^2 \end{pmatrix}
\]
with a certain element \( \Omega'(\eta) \in \mathfrak{S}_4 \). Moreover, in case \( \eta_0 = [0 : 0 : 1] \in \ell(12) \cap \ell(34) \) we have
\[
\Omega(\eta_0) = \begin{pmatrix} \Omega_{11} & \Omega_{14} \\ \Omega_{41} & \Omega_{44} \end{pmatrix} \oplus \begin{pmatrix} \Omega_{22} & \Omega_{25} \\ \Omega_{52} & \Omega_{55} \end{pmatrix} \oplus \begin{pmatrix} \Omega_{33} & \Omega_{36} \\ \Omega_{63} & \Omega_{66} \end{pmatrix} = \tau_0 \oplus \tau_0 \oplus \tau_0
\]
We use the above matrix to numerical evaluation of theta functions in later section.

5. Theta functions

5.1. Invariant theta characteristics. We recall basic facts on the Riemann theta functions. For a characteristic \( (a, b) \in (\mathbb{R}^2)^2 \), the theta function \( \Theta_{(a,b)}(z, \Omega) \) on \( \mathbb{C}^g \times \mathfrak{S}_g \) is defined by the series
\[
\Theta_{(a,b)}(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp[\pi \sqrt{-1} t(n + a) \Omega(n + a) + 2\pi \sqrt{-1} t(n + a)(z + b)].
\]
These functions satisfy the following period relations
\[
\Theta_{(a,b)}(z + m, \Omega) = \exp(2\pi \sqrt{-1} t ma)\Theta_{(a,b)}(z, \Omega),
\]
(5.1)
(5.2) \[ \Theta_{(a,b)}(z + \Omega m, \Omega) = \exp(-\pi \sqrt{-1} m \Omega m - 2\pi \sqrt{-1} m(z + b)) \Theta_{(a,b)}(z, \Omega) \]
for \( m \in \mathbb{Z}^9 \). For the characteristics, we have
(5.3) \[ \Theta_{(a+n,b+m)}(z, \Omega) = \exp(2\pi \sqrt{-1} \Omega m) \Theta_{(a,b)}(z, \Omega), \]
for \( n, m \in \mathbb{Z}^9 \) and
(5.4) \[ \Theta_{(-a,-b)}(z, \Omega) = \Theta_{(a,b)}(z, \Omega). \]

Theta constants \( \Theta_{(a,b)}(\Omega) = \Theta_{(a,b)}(0, \Omega) \) satisfy following transformation formula (see [4, p176]) as function on \( \mathfrak{F}_g \). For \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(\mathbb{Z}) \), set
(5.5) \[ g\Omega = (A\Omega + B)(C\Omega + D)^{-1} \]
(5.6) \[ g(a,b) = (Da - Cb, -Ba + Ab) + \frac{1}{2} (C^tD)_0, (A^tB)_0 \]
(5.7) \[ \phi_{(a,b)}(g) = -\frac{1}{2} ((a^t D B a - 2 \Phi^t A B C b + 4 \Phi^t C A b) + \frac{1}{2} (t^t a D - t^t b^t C) (A^t B)_0 \]
where \( (A)_0 \) stands for the diagonal vector of a matrix \( A \). Then we have
(5.8) \[ \Theta_{g(a,b)}(g\Omega) = \kappa(g) \exp(2\pi \sqrt{-1} \phi_{a,b}(g)) \det(C\Omega + D)^{\frac{1}{2}} \Theta_{(a,b)}(\Omega) \]
where, \( \kappa(g) \) is a certain 8-th root of 1 depending only on \( g \).

**Remark 5.1.** By the definition, we have
(5.9) \[ \Theta_{(a,b)}(z, \Omega) = \exp(\pi \sqrt{-1} \Omega a + 2\pi \sqrt{-1} a(z + b)) \Theta_{(0,0)}(z + \Omega a + b, \Omega), \]
so we often identify a characteristic \((a,b) \in (\mathbb{R}^9)^2 \) with \( \Omega a + b \in \mathbb{C}^9 \). For \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_{2g}(\mathbb{Z}) \), we have
\( \Omega'(Da - Cb) + (-Ba + Ab) = t(C\Omega + D)^{-1}(\Omega a + b), \)
where \( \Omega' = (A\Omega + B)(C\Omega + D)^{-1} \).

For these formulas, see [4] and [8].

Henceforth we suppose the characteristics \((a,b) \) satisfy \( a, b \in (\frac{1}{10}\mathbb{Z})^6 \).

**Lemma 5.1.** Let \( \sigma \) be the matrix in (2.3) and write \( a = (a_i), \ b = (b_i) \).
1. We have
\( \sigma(a,b) \equiv (a,b) \mod \mathbb{Z} \)
if and only if
\( 5a_1 \equiv \frac{1}{2}, \ a_4 \equiv a_1, \ b_1 \equiv -2a_1, \ b_4 \equiv -a_1 \)
\( 5a_2 \equiv \frac{1}{2}, \ a_5 \equiv a_2, \ b_2 \equiv -2a_2, \ b_5 \equiv -a_2 \mod \mathbb{Z} \)
\( 5a_3 \equiv \frac{1}{2}, \ a_6 \equiv a_3, \ b_3 \equiv -2a_3, \ b_6 \equiv -a_3 \)
(2) Let \((a,b) \) be the characteristic with the above condition. Then we have
\( \hat{g}(a,b) \equiv (a,b) \mod \mathbb{Z} \) for all \( g \in \Gamma(1 - \zeta) \).
Proof. (1) Using the exact form (2.7) we can describe \( \sigma(a, b) \). Then we deduce the assertion.

(2) The transformation \( g(a, b) \) in (5.6) define a group action of the symplectic group on \((\mathbb{R}/\mathbb{Z})^{2g}\) (see [3]). We can check that the equality for every member of the generator system \( \{h_{ij}\} \) of \( \Gamma(1 - \zeta) \).

**Definition 5.1.** Let \((a, b)\) be the characteristic satisfying the condition Lemma 5.1 (1). Then we can put

\[
(5.9) \quad a = \frac{1}{10} t(a_1, a_2, a_3, a_1, a_2, a_3), \quad b = \frac{1}{10} t(-2a_1, -2a_2, -2a_3, -a_1, -a_2, -a_3).
\]

Let \((a_1, a_2, a_3)\) denote this characteristic. We call \((a, b) = (a_1, a_2, a_3)\) “\( \sigma \)-invariant” if \(a_1, a_2, a_3\) are odd integers. For a characteristic of this type, we denote the zero locus of \(\Theta_{(a_1, a_2, a_3)}\) on \(\mathbb{B}_2^A\) by \(\vartheta(a_1, a_2, a_3)\):

\[
\vartheta(a_1, a_2, a_3) = \{\eta \in \mathbb{B}_2^A : \Theta_{(a_1, a_2, a_3)}(\Omega(\eta)) = 0\}.
\]

**Remark 5.2.** By the transformation formula (5.8) and Lemma 5.1, we see that

\[
\Theta_{(a_1, a_2, a_3)}(g\Omega(\eta)) = (a \text{ unit function}) \times \Theta_{(a_1, a_2, a_3)}(\Omega(\eta))
\]

for a invariant characteristic \((a_1, a_2, a_3)\) and \(g \in \Gamma(1 - \zeta)\). Hence if we have \(\eta \in \vartheta(a_1, a_2, a_3)\), then \(\Gamma(1 - \zeta)\)-orbit of \(\eta\) contained in \(\vartheta(a_1, a_2, a_3)\).

**Lemma 5.2.** Let \((a_1, a_2, a_3)\) be a \( \sigma \)-invariant characteristic. If \(2a_1^2 + 2a_2^2 + a_3^2 \notin 5\mathbb{Z}\), then \(\vartheta(a_1, a_2, a_3) = \mathbb{B}_2^A\). Namely, \(\Theta_{(a_1, a_2, a_3)}\) vanishes on \(\mathbb{B}_2^A\).

Proof. We apply the transformation formula (5.8) for \(g = \sigma^4\).

For it we proceed the preparatory calculations. At first, get the explicit form of \(g = \sigma^4 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\) by using (2.7). So we obtain

\[
\phi_{(a_1, a_2, a_3)}(\sigma^4) = \frac{1}{40}(2a_1^2 + 2a_2^2 + a_3^2).
\]

Using the explicit form of \(\Omega(\eta)\) in Lemma 2.1, we get

\[
\det(C\Omega(\eta) + D) = 1
\]

for all \(\eta \in \mathbb{B}_2^A\) by a computer and calculation. By (5.3), we may put

\[
\Theta_{\sigma^4(a_1, a_2, a_3)}(\Omega) = \exp[2\pi \sqrt{-1}am] \Theta_{(a_1, a_2, a_3)}(\Omega)
\]

for a certain \(m \in \mathbb{Z}^6\). Returning to the explicit form of \(\sigma^4(a_1, a_2, a_3)\) we should get \(m\). We check that \(\exp[2\pi \sqrt{-1}am] = 1\) by a computer aided calculation. Hence we have

\[
\Theta_{(a_1, a_2, a_3)}(\Omega(\eta)) = \kappa(\sigma^4) \exp\left(\frac{1}{20}\pi \sqrt{-1}(2a_1^2 + 2a_2^2 + a_3^2)\right) \Theta_{(a_1, a_2, a_3)}(\Omega(\eta))
\]

for all \(\eta \in \mathbb{B}_2^A\). This implies our assertion since \(\kappa(\sigma^4)\) is an \(8\)-th root of 1.

We consider odd integers \(a_1, a_2, a_3\) modulo \(10\mathbb{Z}\). There exist 25 representatives of the \( \sigma \)-invariant characteristic \((a_1, a_2, a_3)\) satisfying the condition \(2a_1^2 + 2a_2^2 + a_3^2 \in 5\mathbb{Z}\);

\[
(5.10) \quad (1, 1, 1), (1, 1, 9), (1, 9, 1), (9, 1, 1), (1, 3, 5), (1, 7, 5),
\]

\[
(3, 1, 5), (7, 1, 5), (3, 3, 3), (3, 3, 7), (3, 7, 3), (7, 3, 3),
\]
and
\[
(5, 9, 9), (9, 9, 1), (9, 1, 9), (1, 9, 9), (9, 7, 5), (9, 3, 5),
(7, 9, 5), (3, 9, 5), (7, 7, 7), (7, 7, 3), (7, 3, 7), (3, 7, 7),
\]
and \((5, 5, 5)\).

**Remark 5.3.** (1) The characteristic \((5, 5, 5)\) is an odd half integer characteristic (see [8], p149), hence \(\Theta_{(5,5,5)}(\Omega)\) vanishes identically.

(2) By \((5.3)\) and \((5.4)\), we see that \(\Theta_{(a_1,a_2,a_3)}(\Omega)\) is a scalar multiple of \(\Theta_{(b_1,b_2,b_3)}(\Omega)\) if \(a_1 + b_1, a_2 + b_2, a_3 + b_3 \in 10\mathbb{Z}\). So the system in \((5.10)\) and the system in \((5.11)\) are essentially the same.

**Lemma 5.3.** Let \((a_1, a_2, a_3)\) be a member of the system \((5.10)\) (equivalently \((5.11)\)). The group \(\Gamma\) acts on the set of twelve \(\hat{\theta}(a_1, a_2, a_3)\) transitively.

**Proof.** We have an explicit form of \(\hat{g}_{ij}\) in \((5.3) - (5.7)\). We use it and obtain
\[
\hat{g}_{12}(a_1, a_2, a_3) \equiv (-a_1, a_2, a_3), \quad \hat{g}_{34}(a_1, a_2, a_3) \equiv (a_1, -a_2, a_3),
\]

\[
\begin{array}{ccc|ccc}
(a_1, a_2, a_3) & g_{23}(a_1, a_2, a_3) & g_{45}(a_1, a_2, a_3) \\
(1,1,1) & (3,3,7) & (1,9,9) \\
(1,1,9) & (7,7,7) & (1,7,5) \\
(1,9,1) & (9,7,5) & (1,3,5) \\
(9,1,1) & (9,7,9) & (9,9,9) \\
(1,3,5) & (9,1,9) & (1,9,1) \\
(1,7,5) & (7,3,3) & (1,1,9) \\
(7,1,5) & (1,9,9) & (3,3,7) \\
(7,3,3) & (9,9,1) & (3,7,7) \\
(3,3,7) & (1,1,1) & (3,1,5) \\
(3,7,3) & (7,1,5) & (3,9,5) \\
(7,3,3) & (1,7,5) & (7,7,7) \\
\end{array}
\]

According to \((5.3)\),
\[
g(\hat{\theta}(a_1, a_2, a_3)) = \hat{\theta}(\hat{g}(a_1, a_2, a_3))
\]

So the assertion follows.

5.2. **The zero loci of twelve theta functions.** Here we state Riemann’s theorem. Let \(C\) be an algebraic curve of genus \(g\), let \(\{A_i, B_i\}\) be a symplectic basis of \(H_1(C, \mathbb{Z})\) such that \(A_i \cdot B_j = \delta_{ij}\), and let \(\{\omega_i\}\) be the basis of \(H^0(C, \Omega^1)\) such that \(\int_{A_i} \omega_j = \delta_{ij}\). Then \(\Omega = (\int_{B_i} \omega_j)\) belongs to \(\mathcal{G}_g\). We denotes \(\int_\gamma \omega_1, \cdots, \int_\gamma \omega_g\) by \(\int_\gamma \omega\).

**Theorem 5.1** (see [8], p149). Let us fix a point \(P_0 \in C\). Then there is a vector \(\Delta \in \mathbb{C}^g\), such that for all \(z \in \mathbb{C}^g\), multi-valued function
\[
f(P) = \Theta_{(0,0)}(z + \int_{P_0}^P \omega, \Omega) \quad (P \in C)
\]
on \(C\) either vanishes identically, or has \(g\) zeros \(Q_1, \cdots, Q_g\) with
\[
\sum_{i=1}^g \int_{P_0}^{Q_i} \omega \equiv -z + \Delta \mod \Omega \mathbb{Z}^g + \mathbb{Z}^g.
\]
Remark 5.4 (see §). (1) The vector $\Delta$ in the theorem is called the Riemann constant, and depends on the symplectic basis $\{A_i, B_i\}$ and the base point $P_0$. For the fixed $\{A_i, B_i\}$ and $P_0$, $\Delta$ is uniquely determined as the point of the Jacobian $J(C) = \mathbb{C}^g/(\Omega \mathbb{Z}^g + \mathbb{Z}^g)$ by the property of the theorem.

(2) If we take $P_0$ such that the divisor $(2g - 2)P_0$ is linearly equivalent to the canonical divisor, then we have $\Delta \in \frac{1}{2} \Omega \mathbb{Z} + \frac{1}{2} \mathbb{Z}$.

Corollary 5.1 (see §). Under same situation as the theorem, $\Theta_{\alpha, \beta}(\Omega) = 0$ if and only if there exist $Q_1, \cdots, Q_g \in C$ such that

$$\Delta - (\Omega a + b) \equiv \sum_{i=1}^{g-1} \int_{P_0}^{Q_i} \omega.$$

Now, let us return to our case. Let $\lambda \in X^0(2,5)$ and $C_0$ be as in section § and $\omega_1, \cdots, \omega_6$ be the basis of $H^0(C_0, \Omega^1)$ such that $\int_{A_i} \omega_j = \delta_{ij}$. We denote the ramified points over $\lambda_i \in \mathbb{P}^1$ by $P_i \in C_0$. Let us take the base point $P_0$ arbitrary among $\{P_1, \cdots, P_5\}$ and $\Delta_0$ be the Riemann constant with respect to $\{A_i, B_i\}$ and $P_0$.

Lemma 5.4. The Riemann constant $\Delta_0$ corresponds to the characteristic $(5, 5, 5)$.

Proof. The divisor of the holomorphic 1-form $(x - \lambda_i)^2 dx/y^4$ is $10P_i$. Hence $\Delta_0$ is a half integer characteristic (see Remark §). For $z = \Omega a + b$ ($a, b \in \mathbb{R}^6$) and $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, applying (5.8) we have

$$\Theta_{\sigma(\Delta_0 - z)}(\Omega) = (\text{a unit function}) \times \Theta_{\Delta_0 - z}(\Omega)$$

since $\sigma \Omega = \Omega$. By (5.4) and Remark §, we have

$$\sigma(\Delta_0 - z) = \sigma \Delta_0 - t(C\Omega + D)^{-1}z.$$

Hence it holds

$$\Theta_{\sigma(\Delta_0 - t(C\Omega + D)^{-1}z)}(\Omega) = 0 \iff \Theta_{\Delta_0 - z}(\Omega) = 0$$

$$\iff z \equiv \sum_{i=1}^{5} \int_{P_0}^{Q_i} \omega \quad \text{for} \quad \exists Q_1, \cdots, Q_5 \in C_0$$

by Corollary §. Namely, putting $w = t(C\Omega + D)^{-1}z$ we have

$$\Theta_{\sigma \Delta_0 - w}(\Omega) = 0 \iff t(C\Omega + D)w \equiv \sum_{i=1}^{5} \int_{P_0}^{Q_i} \omega \quad \text{for} \quad \exists Q_1, \cdots, Q_5 \in C_0.$$

Let us recall that $\sigma$ is the symplectic representation matrix of $\rho$ with respect to the basis $\{A_i, B_i\}$ of $H_1(C_0, \mathbb{Z})$. And we have

$$\begin{pmatrix} I & \Omega \\ C & D \end{pmatrix} \begin{pmatrix} tD \\ tC \end{pmatrix} = \begin{pmatrix} t(C\Omega + D) \\ t(A\Omega + B) \end{pmatrix} = \begin{pmatrix} t(C\Omega + D) \\ \Omega \end{pmatrix},$$

so $t(C\Omega + D)$ is the representation matrix of $\rho$ with respect to the basis $\{\omega_1, \cdots, \omega_6\}$ of $H^0(C_0, \Omega^1)$. Hence it holds

$$\Theta_{\sigma \Delta_0 - w}(\Omega) = 0 \iff w \equiv \sum_{i=1}^{5} \int_{P_0}^{Q_i} (\rho^{-1})^* \omega \equiv \sum_{i=1}^{5} \int_{P_0}^{\rho^{-1}(Q_i)} \omega \equiv \sum_{i=1}^{5} \int_{P_0}^{\rho^{-1}(Q_i)} \omega.$$
Recalling Remark 5.4 (1), this implies that $\sigma \Delta_0$ is the Riemann constant, that is $\sigma \Delta_0 \equiv \Delta_0$. Hence we have $\Delta_0 \equiv (5, 5, 5)$ since $(5, 5, 5)$ is the unique $\sigma$-invariant half integer characteristic.

Next, let us consider the oriented arcs $\alpha_k(i, j)$ defined by (2.2) and the integrals $\int_{\alpha_k(i,j)} \omega \in \mathbb{C}^6$.

**Lemma 5.5.** The integral $\int_{\alpha_k(i,j)} \omega$ is a five torsion point $\Omega a + b$ on $\mathbb{C}^6/(\Omega \mathbb{Z}^6 + \mathbb{Z}^6)$ of the form

\[
a = \frac{1}{10} (a_1, a_2, a_3, a_1, a_2, a_3), \quad b = \frac{1}{10} (-2a_1, -2a_2, -2a_3, -a_1, -a_2, -a_3)
\]

with $a_1, a_2, a_3 \in 2\mathbb{Z}$. In explicit way, it holds

\[
\int_{\alpha_k(1,2)} \omega \equiv (6, 0, 0), \quad \int_{\alpha_k(1,3)} \omega \equiv (8, 2, 6), \quad \int_{\alpha_k(1,4)} \omega \equiv (8, 8, 6), \quad \int_{\alpha_k(1,5)} \omega \equiv (8, 0, 8)
\]

mod $\Omega \mathbb{Z}^6 + \mathbb{Z}^6$

with the same notation in Definition 5.4 and identification referred in Remark 5.1 (Note that any $\alpha_k(i, j)$ is written as a combination of $\alpha_k(1, 2), \alpha_k(1, 3), \alpha_k(1, 4)$ and $\alpha_k(1, 5)$).

**Proof.** Since $D_{ij} = \alpha_i(1, 5) - \alpha_j(1, 5)$ is a cycle, we see that $\int_{\alpha_i(1,5)} \omega \equiv \int_{\alpha_j(1,5)} \omega \mod \Omega \mathbb{Z}^6 + \mathbb{Z}^6$. And we have

\[
\int_{D_{12} + D_{15}} \varphi_1 = \int_{2\alpha_1(1,5) - \alpha_2(1,5) - \alpha_5(1,5)} \varphi_1 = (2 - \zeta^2 - \zeta^3) \int_{\alpha_1(1,5)} \varphi_1.
\]

By the same calculation, we see that

\[
\int_{\alpha_1(1,5)} \varphi_k = \begin{cases} 
\frac{1}{5} (2 - \zeta - \zeta^4) \int_{D_{12} + D_{15}} \varphi_k & (k = 1, 2, 3) \\
\frac{1}{5} (2 - \zeta^2 - \zeta^3) \int_{D_{12} + D_{15}} \varphi_k & (k = 4, 5, 6)
\end{cases}
\]

Calculating intersection numbers, we have the following equality

\[
[2 - \rho^2 - \rho^3] (D_{12} + D_{15}) = 2A_1 + 2A_3 + A_4 + A_6 + 4B_1 + 4B_3 - B_4 - B_6
\]

as homology classes. Hence it holds

\[
\int_{\alpha_1(1,5)} \omega \equiv \frac{1}{5} \int_{2A_1 + 2A_3 + A_4 + A_6 + 4B_1 + 4B_3 - B_4 - B_6} \omega \\
\equiv \frac{1}{10} \int_{-6A_1 - 6A_3 - 8A_4 - 8A_6 + 8B_1 + 8B_3 + 8B_4 + 8B_6} \omega \equiv (8, 0, 8).
\]

By the same way, we obtain the results for $\alpha_k(1, 2), \alpha_k(1, 3)$ and $\alpha_k(1, 4)$. 

Let $C_\lambda (\lambda \in X^\circ(2, 5))$ be any element of our family $\mathcal{F}$. We defined in Section 2 the system \{\(\alpha_k(i,j)(\lambda)\), \(\gamma_i(\lambda)\)\} and \{\(A_i(\lambda), B_i(\lambda)\)\} on $C_\lambda$ depending on the arc $r$. The point $P_0$ has always the same meaning. So Lemma 5.4 and 5.5 are true for $C_\lambda$ using these notations. Let $\Delta \equiv (5, 5, 5)$ denote the Riemann constant on $C_\lambda$.

Now, recall that $\mathbb{H}^0_2$, $\ell(ij)$ stands for $\Phi(X^\circ(2, 5))$ and $\pi^{-1}(\Phi(L(ij)))$ respectively (see Section 4).
Proposition 5.1. \( \vartheta(1, 1, 1) \cap \mathbb{B}^2_2 = \phi \).

Proof. Let us consider a curve \( C = C_\lambda (\lambda \in X^\varnothing(2, 5)) \) and its period \( \Omega = \Omega_\lambda \). We assume that \( \Theta_{(1,1,1)}(\Omega) = 0 \). According to Corollary 5.1, there exist points \( Q_1, \ldots, Q_5 \in C \) such that
\[
\sum_{i=1}^{5} \int_{P_i} Q_i \omega \equiv \Delta - (1, 1, 1) \equiv (4, 4, 4).
\]
On the other hand, by Lemma 5.5, we have
\[
\int_{P_3} P_4 \omega \equiv (0, 4, 0), \quad \int_{P_1} P_5 \omega \equiv (2, 0, 2).
\]
Hence it holds
\[
\sum_{i=1}^{5} \int_{P_i} Q_i \omega \equiv 2 \int_{P_5} \omega + \int_{P_4} \omega.
\]
By Abel’s theorem, the divisor \( \sum_{i=1}^{9} Q_i \) is linearly equivalent to the divisor \( D = 2P_1 + P_3 - P_4 + 3P_5 \), and we have
\[
(5.13) \quad \dim H^0(C, \mathcal{O}(D)) = \dim H^0(C, \mathcal{O}(\sum_{i=1}^{9} Q_i)) \geq 1
\]
For the effective divisor \( D' = D + P_0 \), we have
\[
\dim H^0(C, \mathcal{O}(D')) = \dim H^0(C, \Omega^1(-D')) + 1
\]
by the Riemann-Roch. We claim that \( \dim H^0(C, \Omega^1(-D')) = 0 \). In fact, the basis \( \{ \varphi_i \} \) is written as
\[
\varphi_1 = y^2 \varphi, \quad \varphi_2 = y \varphi, \quad \varphi_3 = x^2 \varphi, \quad \varphi_4 = x \varphi, \quad \varphi_5 = xy \varphi, \quad \varphi_5 = \varphi
\]
\[
(\varphi = 4 \frac{dy}{f'(x)}, \quad f(x) = \prod_{i=1}^{5} (x - \lambda_i)),
\]
and we have following vanishing orders;
\[
\text{ord}_{P_i}(y) = 1, \quad \text{ord}_{P_i}(x - \lambda_j) = 5 \delta_{ij}, \quad \text{ord}_{P_i}(\varphi) = 0 \quad (i, j = 1, \ldots, 5).
\]
Because any holomorphic 1-form is written in the form
\[
(\text{inhomogeneous quadratic polynomial of } x, y) \times \varphi,
\]
we see that there is no holomorphic 1-form \( \xi \) such that
\[
\text{ord}_{P_i}(\xi) \geq 2, \quad \text{ord}_{P_5}(\xi) \geq 1, \quad \text{ord}_{P_5}(\xi) \geq 3.
\]
Hence we have \( \dim H^0(C, \mathcal{O}(D')) = 1 \), that is, \( H^0(C, \mathcal{O}(D')) \) contains only constant functions. This contradicts to \( (5.13) \) since \( H^0(C, \mathcal{O}(D')) \subset H^0(C, \mathcal{O}(D')) \) and \( D \) is not effective.

Corollary 5.2. Let \( (a_1, a_2, a_3) \) be a \( \sigma \)-invariant characteristic in \( (5.10) \). Then we have \( \vartheta(a_1, a_2, a_3) \cap \mathbb{B}^2_2 = \phi \).

Proof. This follows from Lemma 5.3

Hence \( \vartheta(a_1, a_2, a_3) \) is the union of certain \( \ell(ij) \)'s.
Lemma 5.6. Let \( \eta_0 \) be the point \([0 : 0 : 1] \in \mathbb{B}_2^4 \), and let \((a_1, a_2, a_3)\) be a member of \((5.10)\). If \(a_1, a_2, a_3 \in \{1, 9\} \), then we have \( \Theta(a_1, a_2, a_3)(\Omega(\eta_0)) \neq 0 \).

Proof. Let

\[
(a', b') = \left( \frac{1}{10} t(\alpha, \alpha), \frac{1}{10} t(-2\alpha, -\alpha) \right)
\]

be a characteristic in \((\mathbb{Q}^2)^2\). Let \( \Theta_\alpha(\tau) \) denote the theta constant \( \Theta_{(a', b')}(\tau) \) \((\tau \in \mathfrak{S}_2)\). Using this notation, we have

\[
\Theta_{(a_1, a_2, a_3)}(\Omega(\eta_0)) = \Theta_{a_1}(\tau_0)\Theta_{a_2}(\tau_0)\Theta_{a_3}(\tau_0), \quad \tau_0 = \left( \frac{1}{\zeta^2 + \zeta^3} \right)
\]

(see \((4.4)\)). So our assertion is reduced to the inequality \( \Theta_1(\tau_0) \neq 0 \), since \( \Theta_0 \) is a constant multiple of \( \Theta_1 \). Set

\[
a = t(\frac{1}{10}, \frac{1}{10}), \quad b = t(-\frac{2}{10}, -\frac{1}{10}), \quad n = t(n_1, n_2),
\]

and set

\[
f(n_1, n_2) = \exp\left[ \pi \sqrt{-1} (n + a) \tau_0 (n + a + 2t(n + b) b) \right].
\]

By definition, \( \Theta_1(\tau_0) = \sum_{n_1, n_2 \in \mathbb{Z}} f(n_1, n_2) \). For simplicity, we denote \( n + a \) by \( m = (m_1, m_2) \). By elementary calculations, we see that

\[
|f(n_1, n_2)| = \exp\left[ -\pi \sin\left( \frac{2\pi}{5} \right) \{m_1^2 + (3 - \sqrt{5})m_1m_2 + m_2^2\} \right].
\]

In case \( m_1m_2 > 0 \), we have

\[
|f(n_1, n_2)| < \exp\left[ -\pi \sin\left( \frac{2\pi}{5} \right) \{m_1^2 + m_2^2\} \right].
\]

In case \( m_1m_2 < 0 \), we have

\[
|f(n_1, n_2)| < \exp\left[ -\pi \sin\left( \frac{2\pi}{5} \right) \{m_1^2 + m_1m_2 + m_2^2\} \right]
= \exp\left[ -\pi \sin\left( \frac{2\pi}{5} \right) \{ \frac{1}{2}(m_1^2 + m_2^2) + \frac{1}{2}(m_1 + m_2)^2 \} \right]
< \exp\left[ -\frac{\pi}{2} \sin\left( \frac{2\pi}{5} \right) \{m_1^2 + m_2^2\} \right].
\]

Consequently,

\[
|f(n_1, n_2)| < \alpha^{m_1^2 + m_2^2}, \quad (\alpha = \exp\left[ -\frac{\pi}{2} \sin\left( \frac{2\pi}{5} \right) \right])
\]

for any \( n_1, n_2 \in \mathbb{Z} \). Set

\[
D_1 = \{(n_1, n_2) \in \mathbb{Z}^2 : -10 \leq n_1, n_2 \leq 10\}, \quad D_2 = \mathbb{Z}^2 - D_1,
\]

and consider the summations

\[
S_1 = \sum_{D_1} f(n_1, n_2), \quad S_2 = \sum_{D_2} f(n_1, n_2).
\]

Using a computer, we can evaluate \( |S_1| \) and \( |S_2| \). We have a approximate value

\[
|S_1| \approx 1.13746 \cdots,
\]

by Mathematica. On the other hand, we have

\[
|S_2| < \sum_{D_2} |f(n_1, n_2)| < \sum_{D_2} \alpha^{m_1^2 + m_2^2}.
\]
The last term is very small. For example,
\[ \sum_{n_1 \geq 10, n_2 \geq 0} \alpha^{n_1^2 + n_2^2} < \left( \sum_{n_1 \geq 10} \alpha^{n_1} \right) \left( \sum_{n_2 \geq 0} \alpha^{n_2} \right) = \left( \frac{\alpha^{10}}{1 - \alpha} \right) \approx 5.40545 \times 10^{-7}, \]
and the same calculations show \( |S_1| \gg |S_2| \). This implies \( \Theta_1(\tau_0) = S_1 + S_2 \neq 0 \).  

Lemma 5.7.  
(1) If we have \( a_1 \equiv 3, 7 \mod 10 \), then \( \Theta_{(a_1, a_2, a_3)} \) vanishes on \( \ell(12) \).  
(2) If we have \( a_2 \equiv 3, 7 \mod 10 \), then \( \Theta_{(a_1, a_2, a_3)} \) vanishes on \( \ell(34) \).  

Proof. Set \( g = \hat{g}_{12} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) and set \( \Omega = \Omega(\eta) \) with \( \eta = [0 : \eta_2 : \eta_3] \in \mathbb{B}^A_2 \). By the computation same as the one in the proof of Lemma 5.2, we have
\[ g \Omega = \Omega, \quad \det(C \Omega + D) = \zeta, \quad \phi_{(a_1, a_2, a_3)}(g) = \frac{1}{40} a_1^2, \quad \Theta_{g(a_1, a_2, a_3)}(\Omega) = \Theta_{(a_1, a_2, a_3)}(\Omega). \]
Hence it holds
\[ \Theta_{(a_1, a_2, a_3)}(\Omega)^{\delta} = \exp \left[ 2 \frac{\pi \sqrt{-1}}{5} (a_1^2 - 1) \right] \Theta_{(a_1, a_2, a_3)}(\Omega)^{\delta} \]
Therefore \( \Theta_{(a_1, a_2, a_3)}(\Omega(\eta)) \) vanishes on the mirror of \( g_{12} \) provided \( a_1 \equiv 3, 7 \mod 10 \). This implies assertion (1). The assertion (2) follows by the same argument with \( g = g_{34} \) and \( \eta = [\eta_1 : 0 : \eta_3] \in \mathbb{B}^A_2 \).  

Proposition 5.2. We have the Table 1 for the vanishing loci of twelve theta constants coming from the system \((5.10)\). In the table, “v” implies that \( \Theta_{(a_1, a_2, a_3)} \) vanishes there, and the blank implies \( \Theta_{(a_1, a_2, a_3)} \) is not identically zero there. For example, \( \Theta_{(1,1,1)} \) vanishes on \( \ell(13) \) and is not identically zero on \( \ell(12) \).

| \((a_1, a_2, a_3)\) | \(\ell(12)\) | \(\ell(13)\) | \(\ell(14)\) | \(\ell(15)\) | \(\ell(23)\) | \(\ell(24)\) | \(\ell(25)\) | \(\ell(34)\) | \(\ell(35)\) | \(\ell(45)\) |
|---|---|---|---|---|---|---|---|---|---|---|
| (1, 1, 1) | v | v | v | v | v | v | v | v | v | v |
| (1, 1, 9) | v | v | v | v | v | v | v | v | v | v |
| (1, 9, 1) | v | v | v | v | v | v | v | v | v | v |
| (9, 1, 1) | v | v | v | v | v | v | v | v | v | v |
| (1, 3, 5) | v | v | v | v | v | v | v | v | v | v |
| (1, 7, 5) | v | v | v | v | v | v | v | v | v | v |
| (3, 1, 5) | v | v | v | v | v | v | v | v | v | v |
| (7, 1, 5) | v | v | v | v | v | v | v | v | v | v |
| (3, 3, 3) | v | v | v | v | v | v | v | v | v | v |
| (3, 3, 7) | v | v | v | v | v | v | v | v | v | v |
| (3, 7, 3) | v | v | v | v | v | v | v | v | v | v |
| (7, 3, 3) | v | v | v | v | v | v | v | v | v | v |

Table 1.

Proof. By Lemma 5.7, \( \Theta_{(3,1,5)}, \Theta_{(7,1,5)}, \Theta_{(3,3,3)}, \Theta_{(3,3,7)}, \Theta_{(3,7,3)}, \Theta_{(7,3,3)} \) vanish on \( \ell(12) \), and
\[ \Theta_{(1,3,5)}, \Theta_{(1,3,5)}, \Theta_{(3,3,3)}, \Theta_{(3,3,7)}, \Theta_{(3,7,3)}, \Theta_{(7,3,3)} \]
vanish on $\ell(34)$. By Lemma 5.3,

$$\Theta_{(1,1,1)}, \quad \Theta_{(1,1,9)}, \quad \Theta_{(1,9,1)}, \quad \Theta_{(9,1,1)}$$

are not identically zero on $\ell(12)$ and on $\ell(34)$, since $\eta_0 = [0 : 0 : 1] \in \ell(12) \cap \ell(34)$. The result is obtained by applying the transformation formula (5.8) for above theta constants and $\hat{g}_{ij}$. For example, we have

$$\Theta_{g_1 g_2} (\hat{g}_1 \hat{g}_2 \Omega) = (\text{a unit function}) \times \Theta_{(a_1, a_2, a_3)} (\Omega).$$

Since $\hat{g}_1 \hat{g}_2 (1, 3, 5) \equiv (9, 9, 1)$ (see Lemma 5.3) and $g_1 g_2 (\ell(12)) = \ell(12)$, we see that $\Theta(1, 3, 5)$ is not identically zero on $\ell(12)$. 

5.3. Automorphic Factor. We study the automorphic factor appeared in the transformation formula (5.8) with respect to $\Gamma(1 - \zeta)$ and $\Omega = \Omega(\eta)$. Let $H$ be the diagonal matrix $\text{diag}(1, 1, -\zeta^3(1 + \zeta))$. We denote $i\eta H \eta$ by $\langle \eta, \eta \rangle$. Set

$$F_g (\eta) = \frac{\langle g\eta, g\eta \rangle}{\langle \eta, \eta \rangle}$$

for $g \in \Gamma$ and $\eta \in \mathbb{B}_2^4$. Obviously, we have the following lemma.

**Lemma 5.8.** $F_g (\eta)$ satisfies the cocycle condition with respect to $\Gamma$. That is,

$$F_{g_1 g_2} (\eta) = F_{g_1} (g_2 \eta) F_{g_2} (\eta), \quad g_1, g_2 \in \Gamma.$$

**Proposition 5.3.** There exist the non trivial character

$$\chi : \Gamma \rightarrow \mu_5 = \{1, \zeta, \cdots, \zeta^4\}$$

such that

$$\det(C \Omega(\eta) + D) = \chi(g) F_g (\eta) \quad (\eta \in \mathbb{B}_2^4)$$

for $g \in \Gamma$, where the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the symplectic representation $\hat{g}$ of $g$.

**Proof.** According to the case by case calculation, we have

$$\det(C \Omega(\eta) + D) = \zeta^3 F_g (\eta) \quad (\eta \in \mathbb{B}_2^4)$$

for $g = g_{12}, g_{23}, g_{34}, g_{45}$. Since $\det(C \Omega(\eta) + D) / F_g (\eta)$ satisfies the cocycle condition, we obtain the result. 

Now let $(a, b)$ be a invariant characteristic $(a_1, a_2, a_3)$, and $(a_g, b_g)$ be $\hat{g} (a, b)$ for $g \in \Gamma(1 - \zeta)$. Since

$$(a_g, b_g) \equiv (a, b) \mod \mathbb{Z},$$

we have

$$\Theta_{\hat{g} (a, b)} (\Omega) = \Theta_{(a_g, b_g)} (\Omega) = \Theta_{(a_g - a + a, b_g - b + b)} (\Omega) = \exp[2\pi \sqrt{-1} a(b_g - b)] \Theta_{(a, b)} (\Omega)$$

by (5.3). Set

$$\phi'_{(a_1, a_2, a_3)} (\hat{g}) = \phi_{(a_1, a_2, a_3)} (\hat{g}) - i a(b_g - b).$$

Then we can write the transformation formula (5.8) as

$$(5.15) \quad \Theta_{(a_1, a_2, a_3)} (\Omega(g\eta)) = \kappa (\hat{g}) \exp(2\pi \sqrt{-1} \phi'_{(a_1, a_2, a_3)} (\hat{g})) [\chi(g) F_g (\eta)]^{\frac{1}{2}} \Theta_{(a_1, a_2, a_3)} (\Omega(\eta)),$$

where $\kappa(\hat{g})$ is a 8-th root of 1 depending only on $\hat{g}$. 

Lemma 5.9. Let \( g \) be in \( \Gamma(1 - \zeta) \). Then, the values
\[
[\exp(2\pi\sqrt{-1}\phi'_{(a_1,a_2,a_3)}(\hat{g}))]^5
\]
are the same for all twelve characteristics \((a_1,a_2,a_3)\) in \( (5.10) \).

Proof. By direct calculations, we have
\[
5\phi'_{\lambda}(a_1,a_2,a_3)(\hat{h}_{12}) \equiv \frac{1}{8}, \quad 5\phi'_{\lambda}(a_1,a_2,a_3)(\hat{h}_{13}) \equiv \frac{3}{4}, \quad 5\phi'_{\lambda}(a_1,a_2,a_3)(\hat{h}_{14}) \equiv \frac{1}{2},
\]
\[
5\phi'_{\lambda}(a_1,a_2,a_3)(\hat{h}_{23}) \equiv \frac{1}{2}, \quad 5\phi'_{\lambda}(a_1,a_2,a_3)(\hat{h}_{34}) \equiv \frac{3}{4} \pmod{\mathbb{Z}}
\]
for the twelve \((a_1,a_2,a_3)\). According to Lemma 5.8, the equality \( (5.13) \) shows that
\[
\kappa(\hat{g}) \exp[2\pi\sqrt{-1}\phi'_{(a_1,a_2,a_3)}(\hat{g})]
\]
is a character on \( \Gamma(1 - \zeta) \). So we obtain the result for any \( g \in \Gamma(1 - \zeta) \).

\[\square\]

Corollary 5.3. Let \((a_1,a_2,a_3)\) and \((b_1,b_2,b_3)\) be in \( (5.10) \). Then, the function
\[
\frac{\Theta_{(a_1,a_2,a_3)}(\Omega(\eta))}{\Theta_{(b_1,b_2,b_3)}(\Omega(\eta))}
\]
is well-defined as meromorphic function on \( \mathbb{B}_2^A/\Gamma(1 - \zeta) \).

Let \( \Omega = \Omega_\lambda \) be the period matrix of a curve \( C_\lambda \) \((\lambda \in X^0(2,5))\), \( P_0 \) be a ramified point of \( C \to \mathbb{P}^1 \).

Proposition 5.4. Let \((a_1,a_2,a_3)\) and \((b_1,b_2,b_3)\) be in \( (5.10) \). The function
\[
f(P) = \frac{\Theta_{(a_1,a_2,a_3)}(\int_{P_0}^P \omega, \Omega)^5}{\Theta_{(b_1,b_2,b_3)}(\int_{P_0}^P \omega, \Omega)^5} \quad (P \in C_\lambda)
\]
is a single-valued meromorphic function on \( C_\lambda \), where the paths of integrations in the numerator and the denominator are chosen as same.

Proof. Note that Corollary 5.2 asserts
\[
\Theta_{(a_1,a_2,a_3)}(\int_{P_0}^P \omega, \Omega) = \text{const.} \times \Theta_{(a_1,a_2,a_3)}(0, \Omega) \neq 0,
\]
where the constant depends on the path of integration. So the numerator is not identically zero, and it is same for the denominator. By the assumption we have
\[
(a_1,a_2,a_3) - (b_1,b_2,b_3) \in \left(\frac{1}{3}\mathbb{Z}^6\right)^2.
\]
By using the formula \( (5.1) \) and \( (5.2) \) we can check that
\[
\frac{\Theta_{(a_1,a_2,a_3)}(\int_{P_0}^P \omega + \Omega m + n, \Omega)^5}{\Theta_{(b_1,b_2,b_3)}(\int_{P_0}^P \omega + \Omega m + n, \Omega)^5} = \frac{\Theta_{(a_1,a_2,a_3)}(\int_{P_0}^P \omega, \Omega)^5}{\Theta_{(b_1,b_2,b_3)}(\int_{P_0}^P \omega, \Omega)^5}
\]
for \( m, n \in \mathbb{Z}^6 \). This implies single-valuedness of \( f \).

\[\square\]
Let us consider the meromorphic function

\[ f(P) = \frac{\Theta_{(1,1,1)}(\int_{P_2} P, \Omega)^5}{\Theta_{(3,3,7)}(\int_{P_1} P, \Omega)^5} \]

on \( C_\lambda \). By Lemma 5.3, we have

\[ \Delta - (1, 1, 1) \equiv (4, 4, 4) \equiv 2 \int_{P_1} P + 3 \int_{P_1} P + \int_{P_1} P, \]

\[ \Delta - (3, 3, 7) \equiv (2, 2, 8) \equiv 3 \int_{P_1} P + 2 \int_{P_1} P + \int_{P_1} P. \]

By Corollary 5.1, the zero divisor of \( \Theta_{(1,1,1)}(\int_{P_2} P, \Omega) \) and \( \Theta_{(3,3,7)}(\int_{P_1} P, \Omega) \) are \( 2P_2 + 3P_3 + P_4 \) and \( 3P_2 + 2P_3 + P_4 \) respectively. Hence we can write

\[ f(P) = c \frac{x(P) - \lambda_3}{x(P) - \lambda_2} \]

where \( x(P) \) is the coordinate function \( x \in \mathbb{C}[x, y]/(y^5 - \prod (x - \lambda_i)) \) and \( c \neq 0 \) is a certain constant. By Lemma 5.3,

\[ \int_{P_1} P \equiv (0, 0, 0), \quad \int_{P_1} P \equiv (8, 0, 8). \]

Substitutes \( P = P_1, P_5 \) in the above form, then we obtain

\[ \Theta_{(1,1,1)}((0, 0, 0), \Omega)^5 = c \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}, \quad \Theta_{(3,3,7)}((0, 0, 0), \Omega)^5 = c \frac{\lambda_5 - \lambda_3}{\lambda_5 - \lambda_2}. \]

Set \((8, 0, 8) = \Omega \varepsilon' + \varepsilon''\). By elementary and patient calculation, we have

\[ \Theta_{(1,1,1)}((8, 0, 8), \Omega)^5 = -c^2 \exp[-5\pi \sqrt{-1} \varepsilon' \Omega \varepsilon' - 10\pi \sqrt{-1} \varepsilon' \varepsilon'' \Omega] \chi_{(1,9,1)}(\Omega)^5 \]

\[ \Theta_{(3,3,7)}((8, 0, 8), \Omega)^5 = \exp[-5\pi \sqrt{-1} \varepsilon' \Omega \varepsilon' - 10\pi \sqrt{-1} \varepsilon' \varepsilon'' \Omega] \chi_{(3,5,5)}(\Omega)^5 \]

Eliminating \( c \), we have the following equality

\[ \frac{\Theta_{(1,1,1)}(\Omega)^5 \Theta_{(3,5,5)}(\Omega)^5}{\Theta_{(3,3,7)}(\Omega)^5 \Theta_{(1,9,1)}(\Omega)^5} = -c^2 \frac{(\lambda_1 - \lambda_3)(\lambda_5 - \lambda_2)}{(\lambda_1 - \lambda_2)(\lambda_5 - \lambda_3)}. \]

Note that we can regard the above equality as that of meromorphic functions on \( \mathbb{B}^2_2 / \Gamma (1 - \zeta) \cong X(2, 5) \). By the above equality and Proposition 5.2, we see that

1. The vanishing order of \( \Theta_{(1,1,1)}(\Omega(\eta))^5 \) on \( \Phi(L(13)) \) is 1,
2. The vanishing order of \( \Theta_{(1,3,5)}(\Omega(\eta))^5 \) on \( \Phi(L(25)) \) is 1,
3. The vanishing order of \( \Theta_{(3,3,7)}(\Omega(\eta))^5 \) on \( \Phi(L(12)) \) is 1,
4. The vanishing order of \( \Theta_{(1,9,1)}(\Omega(\eta))^5 \) on \( \Phi(L(35)) \) is 1.

Because \( \Gamma \) acts transitively on the set of \( \sigma \)-invariant characteristics (see Lemma 5.3), we obtain the following result.

**Proposition 5.5.** Let \((a_1, a_2, a_3)\) be a \( \sigma \)-invariant characteristic. If the multi-valued function \( \Theta_{(a_1,a_2,a_3)}(\Omega(\eta))^5 \) on \( \mathbb{B}^2_2 / \Gamma (1 - \zeta) \) identically vanishes on \( \Phi(L(ij)) = \ell(ij)/\Gamma (1 - \zeta) \), then the vanishing order is 1.
6. Conclusion

Now we state our results.

— The Schwarz inverse for the Appell HGDE $F_1(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{6}{5})$ —

Recall the embedding of $J : X(2,5) \to \mathbb{P}^{11}$ with

$$J(ijklm)(\lambda) = (\lambda_i - \lambda_j)(\lambda_j - \lambda_k)(\lambda_k - \lambda_l)(\lambda_l - \lambda_m)(\lambda_m - \lambda_i)$$

in Proposition [1.3], and the extended period map $\tilde{\Phi}$ in section [4].

**Theorem 6.1.** We have a commutative diagram:

\[
\begin{array}{ccc}
X(2,5) & \xrightarrow{\tilde{\Phi}} & \mathbb{B}^4_2/\Gamma(1 - \zeta) \\
J & & \\
\mathbb{P}^{11} & \xrightarrow{\Theta} & \\
\end{array}
\]

**Figure 4.**

by putting

\[
\Theta = \begin{bmatrix}
\Theta_{(1,1,1)}(\Omega(\eta))^5 \\
\Theta_{(1,1,9)}(\Omega(\eta))^5 \\
\Theta_{(1,9,1)}(\Omega(\eta))^5 \\
\Theta_{(9,1,1)}(\Omega(\eta))^5 \\
\Theta_{(1,3,5)}(\Omega(\eta))^5 \\
\Theta_{(1,7,5)}(\Omega(\eta))^5 \\
\Theta_{(3,3,3)}(\Omega(\eta))^5 \\
\Theta_{(3,3,7)}(\Omega(\eta))^5 \\
\Theta_{(3,7,3)}(\Omega(\eta))^5 \\
\Theta_{(3,7,5)}(\Omega(\eta))^5 \\
\Theta_{(7,3,3)}(\Omega(\eta))^5 \\
\Theta_{(7,3,5)}(\Omega(\eta))^5 \\
\Theta_{(7,1,5)}(\Omega(\eta))^5 \\
\Theta_{(3,1,5)}(\Omega(\eta))^5 \\
\end{bmatrix}, \quad J = \begin{bmatrix}
c_1 J(13245)(\lambda) \\
c_2 J(13524)(\lambda) \\
c_3 J(15324)(\lambda) \\
c_4 J(13254)(\lambda) \\
c_5 J(15234)(\lambda) \\
c_6 J(13425)(\lambda) \\
d_1 J(12534)(\lambda) \\
d_2 J(12345)(\lambda) \\
d_3 J(14352)(\lambda) \\
d_4 J(15342)(\lambda) \\
d_5 J(12453)(\lambda) \\
d_6 J(12354)(\lambda)
\end{bmatrix},
\]

with constants

$$[c_1 : \cdots : c_6 : d_1 : \cdots : d_6] = [1 : -1 : 1 : 1 : \zeta^3 : \zeta^3 : -\zeta : \zeta : -\zeta : -1 : -1] \in \mathbb{P}^{11}.$$

Moreover the map $\Theta$ is an embedding.

**Proof.** By Proposition [5,2] and Proposition [5,3], the zero divisor of the i-th component of $\Theta$ coincides with that of the i-th component of $J$ via the isomorphism $\tilde{\Phi}$. So we can write as

\[
\frac{\Theta_{(1,1,1)}(\Omega(\tilde{\Phi}(\lambda)))^5}{J(13245)(\lambda)} = c_1, \quad \cdots, \quad \frac{\Theta_{(3,1,5)}(\Omega(\tilde{\Phi}(\lambda)))^5}{J(12354)(\lambda)} = d_6
\]

with certain constants $c_1, \cdots, d_6$. It shows the diagram in question is commutative. Since $J$ is an embedding and $\tilde{\Phi}$ is an isomorphism, we see that $\Theta$ is an embedding.

Next we determine the ratios of the constants $c_1, d_i$.

Let $\eta \in \mathbb{B}^4_2$ be a point of the form $[0 : \eta_2 : \eta_3]$. For such $\eta$, we have the decomposition

$$\Omega(\eta) = \tau_0 \oplus \Omega(\eta)'$$

PENTAGONAL CURVES 23
(see (4.3)). So we have the splitting
\[ \Theta_{(a_1,a_2,a_3)}(\Omega(\eta)) = \Theta_{a_1}(\tau_{0})\Psi_{(a_2,a_3)}(\Omega(\eta)'), \]
where \( \Theta_{a_1}(\tau_{0}) \) is same as in the proof of Lemma (5.3), and \( \Psi_{(a_2,a_3)}(\Omega(\eta)') \) is the theta function \( \Theta_{(a,b)}(\Omega(\eta)') \) of degree 4 with the characteristic
\[ a = \frac{1}{10}(-2a_2, -2a_3, -a_2, -a_3), \quad b = \frac{1}{10}(-2a_2, -2a_3, -a_2, -a_3). \]
By the same way, we have
\[ \Theta_{(a_1,a_2,a_3)}(\Omega(\eta)) = \Theta_{a_2}(\tau_{0})\Psi_{(a_1,a_3)}(\Omega(\eta)'). \]
for \( \eta = [\eta_1 : 0 : \eta_3] \in \mathbb{H}^3 \). We can check that
\[ \Theta_1(\tau_{0})^5 = -\Theta_9(\tau_{0})^5, \quad \Psi_{(1,9)}(\Omega(\eta)')^5 = \Psi_{(9,1)}(\Omega(\eta)')^5, \quad \Psi_{(3,5)}(\Omega(\eta)')^5 = \Psi_{(7,5)}(\Omega(\eta)')^5 \]
by (5.3) and (5.4). So we have
\[ \Theta_{(1,1,1)}(\Omega(\eta))^5 = \frac{\Theta_{(1,1,1)}(\Omega(\eta))^5}{\Theta_{(9,1,1)}(\Omega(\eta))^5} = \frac{\Theta_{(1,1,1)}(\Omega(\eta))^5}{\Theta_{(9,1,1)}(\Omega(\eta))^5} = -1 \quad \text{on } \ell(12). \]
By the same way, we see
\[ \Theta_{(1,1,9)}(\Omega(\eta))^5 = \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,9,1)}(\Omega(\eta))^5} = 1, \quad \Theta_{(1,3,5)}(\Omega(\eta))^5 = \frac{\Theta_{(1,3,5)}(\Omega(\eta))^5}{\Theta_{(1,7,5)}(\Omega(\eta))^5} = 1 \quad \text{on } \ell(12), \]
and
\[ \Theta_{(1,1,1)}(\Omega(\eta))^5 = -1, \quad \Theta_{(9,1,1)}(\Omega(\eta))^5 = 1, \quad \Theta_{(3,1,5)}(\Omega(\eta))^5 = \frac{\Theta_{(3,1,5)}(\Omega(\eta))^5}{\Theta_{(7,1,5)}(\Omega(\eta))^5} = 1 \quad \text{on } \ell(34). \]
Moreover, we have
\[ \Theta_{(1,1,9)}(\Omega(\eta))^5 = \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5} = -1 \quad \text{for } \eta \in \ell(12) \cap \ell(34) \]
(see (5.4)). By the transformation formula (5.8), we have
\[ \Theta_{g(a_1,a_2,a_3)}(\Omega(g\eta))^5 = \exp[2\pi\sqrt{-1}\{\phi_{(a_1,a_2,a_3)}(\hat{g}) - \phi_{(b_1,b_2,b_3)}(\hat{g})\}] \Theta_{(a_1,a_2,a_3)}(\Omega(\eta))^5 \]
for any pair of \( \sigma \)-invariant characteristics \( (a_1,a_2,a_3), (b_1,b_2,b_3), \) and \( g \in \Gamma \). By explicit calculation of the above formula, we obtain
\[ \Theta_{(1,1,1)}(\Omega(\eta))^5 = -\zeta^4 \Theta_{(3,3,7)}(\Omega(g_{23}\eta))^5 \Theta_{(1,1,9)}(\Omega(\eta))^5 = \zeta^2 \Theta_{(3,3,3)}(\Omega(g_{23}\eta))^5, \]
\[ \Theta_{(1,3,5)}(\Omega(\eta))^5 = \zeta \Theta_{(1,9,1)}(\Omega(g_{23}\eta))^5 \Theta_{(3,1,5)}(\Omega(\eta))^5 = \zeta^2 \Theta_{(3,7,3)}(\Omega(g_{23}\eta))^5, \]
\[ \Theta_{(3,1,5)}(\Omega(\eta))^5 = \zeta \Theta_{(1,19,1)}(\Omega(g_{23}\eta))^5 \Theta_{(1,3,5)}(\Omega(\eta))^5 = \zeta^2 \Theta_{(3,7,3)}(\Omega(g_{23}\eta))^5. \]
Comparing these with (4.3), (4.4), (4.5) and (4.6), we have
\[ \Theta_{(3,3,7)}(\Omega(\eta))^5 = \zeta, \quad \Theta_{(3,3,3)}(\Omega(\eta))^5 = \zeta^2, \quad \Theta_{(1,9,1)}(\Omega(\eta))^5 = \zeta^4 \quad \text{on } \ell(13), \]
\[\frac{\Theta_{(9,1,1)}(\Omega(\eta))^5}{\Theta_{(3,7,3)}(\Omega(\eta))^5} = \zeta^4 \quad \text{on } \ell(24),\]

\[\frac{\Theta_{(3,3,7)}(\Omega(\eta))^5}{\Theta_{(3,7,3)}(\Omega(\eta))^5} = -1, \quad \text{on } \ell(35),\]

\[\frac{\Theta_{(1,7,5)}(\Omega(\eta))^5}{\Theta_{(9,1,1)}(\Omega(\eta))^5} = \zeta^3 \quad \text{on } \ell(12) \cap \ell(35),\]

since it holds
\[g_{23}(\ell(12)) = \ell(13), \quad g_{23}(\ell(34)) = \ell(24), \quad g_{45}(\ell(12)) = \ell(12), \quad g_{45}(\ell(34)) = \ell(35).\]

Because the commutativity of diagram is established, by using (6.3), we see that
\[-1 = \frac{\Theta_{(1,1,1)}(\Omega(\eta))^5}{\Theta_{(9,1,1)}(\Omega(\eta))^5}|_{\ell(12)} = \frac{c_1 J(13245)}{c_4 J(13254)}|_{\ell(12)} = \frac{c_1 (\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_4)(\lambda_4 - \lambda_5)(\lambda_5 - \lambda_1)}{c_4 (\lambda_1 - \lambda_3)(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_5)(\lambda_5 - \lambda_4)(\lambda_4 - \lambda_1)},\]

that is \(c_1 = c_4\). By the same calculation using (6.4)–(6.10), we obtain
\[c_1 = c_4, \quad c_2 = -c_3, \quad c_3 = c_6, \quad c_1 = c_3, \quad c_2 = -c_4, \quad d_5 = d_6, \quad c_1 = -c_2,\]
\[d_2 = -\zeta d_6, \quad d_1 = -\zeta^3 c_5, \quad c_3 = -\zeta^4 d_4, \quad c_4 = \zeta^4 d_3, \quad d_2 = d_3, \quad c_6 = \zeta^3 c_4.\]

These equalities gives the ratios in the assertion. \(\square\)

**Remark 6.1.** We have the following equalities;
\[J(ijklm) = -J(mlkj), \quad \Theta_{(a_1,a_2,a_3)}(\Omega(\eta))^5 = -\Theta_{(10-a_2,10-a_1,10-a_3)}(\Omega(\eta))^5.\]

Let us denote \(X(2, 5)\) by \(X\), and let \(K_X\) be the canonical class of \(X\).

**Corollary 6.1.** We have an isomorphism of \(\mathbb{C}\)-algebras
\[\mathbb{C}[\Theta_{(a_1,a_2,a_3)}(\Omega(\eta))^5] \cong \oplus_{n=0}^{\infty} \mathbb{H}^0(X, \mathcal{O}_X(-nK_X)),\]

where the left hand side is the \(\mathbb{C}\)-algebra of the functions on \(\mathbb{B}_2^4\) generated by the twelve theta functions in Theorem 6.4. Especially the \(\mathbb{C}\)-vector space spanned by \(\{\Theta_{(a_1,a_2,a_3)}(\Omega(\eta))^5\}\) coincides with \(\mathbb{H}^0(X, \mathcal{O}_X(-K_X))\).

**Proof.** The map \(J\) is essentially anti-canonical map (see Section 5). Hence it follows from Theorem 6.1. \(\square\)

**Remark 6.2.** By the Riemann-Roch theorem, we obtain
\[\dim \mathbb{H}^0(X, \mathcal{O}_X(-nK_X)) = \frac{1}{2}(-nK_X) \cdot (-nK_X - K_X) + 1\]
\[= \frac{5}{2}n(n+1) + 1\]
since \((-K_X) \cdot (-K_X) = 5\). So we have \(\dim \mathbb{H}^0(X, \mathcal{O}_X(-K_X)) = 6\), and twelve \(\Theta_{(a_1,a_2,a_3)}(\Omega(\eta))^5\) satisfy 6 linear equations. It is known that the image of \(X\) in \(\mathbb{P}^5\) by the anti-canonical map is determined by the system of quadratic equations (see [2], Chapter 5].
The graded ring of Automorphic forms

Recall the automorphic factor \( F_g(\eta) \) in Lemma 5.8. We consider the automorphic function \( f(\eta) \) on \( \mathbb{B}^2 \) in the sense that we have

\[
(6.11) \quad f(g\eta) = F_g(\eta)^k f(\eta) \quad \text{for } g \in \Gamma(1-\zeta),
\]

where \( k \) is a non negative integer. Let us denotes the vector space of holomorphic functions satisfying (6.11) by \( A_k(F_g) \).

Proposition 6.1. Let \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\) be the member of the system in (5.11), then it holds

\[
\Theta_{(a_1,a_2,a_3)}(\Omega(\eta)) \Theta_{(b_1,b_2,b_3)}(\Omega(\eta))^5 \in A_5(F_g).
\]

Proof. By Proposition 6.1, a function \( f \) satisfies (6.11) by \( \Theta_{(a_1,a_2,a_3)}(\Omega(\eta)) \Theta_{(b_1,b_2,b_3)}(\Omega(\eta))^5 \). Hence we have the assertion (1).

Theorem 6.2. (1) We have the isomorphism of the \( \mathbb{C} \)-algebras:

\[
\bigoplus_{n=0}^{\infty} A_5n(F_g) = \mathbb{C}[\Theta_{(a_1,a_2,a_3)}(\Omega(\eta)) \Theta_{(b_1,b_2,b_3)}(\Omega(\eta))^5]
\]

\[
\cong \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(-2nK_X)).
\]

(2) \( A_n(F_g) = \{0\} \) for \( n \in \mathbb{N} \), \( n \equiv 1, 2, 3, 4 \) mod 5.

Proof. By Proposition 6.1, a function \( f \in A_5(F_g) \) defines the meromorphic function

\[
\frac{f(\eta)}{\Theta^{(1,1,1)}(\Omega(\eta))^{10}}
\]

on \( \mathbb{B}^2/\Gamma(1-\zeta) \). So, by Theorem 5.1, we have the isomorphism of \( \mathbb{C} \)-vector space:

\[
A_{5n}(F_g) \cong H^0(X, \mathcal{O}_X(-2nK_X)) \quad \text{for } n \in \mathbb{N}.
\]

Hence we have the assertion (1).

Next let us recall that \( X \) is the blow up of \( \mathbb{P}^2 \) at 4 points. We denote this blow up by \( \pi : X \rightarrow \mathbb{P}^2 \). Then the Neron–Severi group \( NS(X) \) has the free generator \( E_1, E_2, E_3, E_4 \) and \( \pi^*H \), where \( \{E_i\} \) are the exceptional curves with respect to \( \pi \), and \( H \) is a general line on \( \mathbb{P}^2 \). For \( n \notin \mathbb{Z} \), there is no divisor \( D \) on \( X \) such that \( 5D = -2nK_X \) since \( -K_X = 3\pi^*H - E_1 - E_2 - E_3 - E_4 \). This implies the assertion (2) since

\[
A_n(F_g)^5 \subset A_{5n}(F_g) \cong H^0(X, \mathcal{O}_X(-2nK_X)).
\]
--- The Schwarz inverse for the Gauss HGDE $E_{2,1}(\frac{1}{5}, \frac{2}{5}, \frac{4}{5})$ ---

Let us consider the 1-dimensional disk
\[ \mathbb{B}_1 = \{ \eta \in \mathbb{B}^A_2 : \eta_1 = 0 \}, \]
and the degenerate period map
\[ \Phi_{12} : L(12) \cong \mathbb{P}^1 \longrightarrow \mathbb{B}_1, \quad t \mapsto [0 : \int_{\gamma_2} \omega : \int_{\gamma_3} \omega], \]
\[ \omega = x^{-\frac{1}{2}}(x-1)^{-\frac{3}{2}}(x-t)^{-\frac{5}{2}}dx, \]
as in Section 4 (the parameter $\lambda$ is specialized as $(\lambda_1, \cdots, \lambda_5) = (0, 0, 1, t, \infty)$). Set
\[ \Gamma(1 - \zeta)_1 = \{ g \in \Gamma(1 - \zeta) : g(\mathbb{B}_1) = \mathbb{B}_1 \}. \]
As we mentioned in Section 3, this is the triangle group $\Delta(5, 5, 5)$ up to the center. Recall those are the Schwarz map and the monodromy group for Gauss hypergeometric differential equation $E_{2,1}(\frac{1}{5}, \frac{2}{5}, \frac{4}{5})$ (see (1.2)). We have the explicit description of the inverse:

**Theorem 6.3.** The map
\[ \Theta_{12} : \mathbb{B}_1/\Gamma(1 - \zeta)_1 \longrightarrow \mathbb{P}^1, \quad \eta \mapsto [\Theta_{(1,1,1)}(\Omega(\eta))^5 : -\Theta_{(1,1,9)}(\Omega(\eta))^5] \]
is an isomorphism, and this is the inverse map of the Schwarz map
\[ \Phi_{12} : \mathbb{P}^1 \longrightarrow \mathbb{B}_1/\Gamma(1 - \zeta)_1, \quad [1 : t] \mapsto [0 : \int_{\gamma_2} \omega : \int_{\gamma_3} \omega]. \]

**Proof.** By Theorem 5.1, the restriction of meromorphic function
\[ \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5} \]
on $L(12)$ is of order 1. In fact, $L(12) \cap L(13) = L(12) \cap L(14) = L(12) \cap L(15) = \phi$, so the numerator vanishes at only $L(12) \cap L(35)$ with order 1, the denominator vanishes at only $L(12) \cap L(45)$ with order 1, and $L(12) \cap L(35) \neq L(12) \cap L(45)$ (see Section 3). Hence the map $\Theta_{12}$ is an isomorphism. Moreover, by Theorem 6.1, we have the equality
\[ \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5} = \frac{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_5)(\lambda_5 - \lambda_2)(\lambda_2 - \lambda_4)(\lambda_4 - \lambda_1)}{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_4)(\lambda_4 - \lambda_5)(\lambda_5 - \lambda_1)}, \]
on $\mathbb{B}_2^A/\Gamma(1 - \zeta) \cong X(2, 5)$, and this induces the equality
\[ \frac{\Theta_{(1,1,9)}(\Omega(\eta))^5}{\Theta_{(1,1,1)}(\Omega(\eta))^5} = \frac{(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_1)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_5)} \]
on $L(12)$. Putting $(\lambda_1, \lambda_3, \lambda_4, \lambda_5) = (0, 1, t, \infty)$, we obtain
\[ \frac{(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_1)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_5)} = t. \]

Let us consider a holomorphic function $f$ on $\mathbb{B}_1$ satisfying the condition:
\[ f(g\eta) = F_g(\eta)^k f(\eta) \quad \text{for } g \in \Gamma(1 - \zeta)_1, \]
and we denote the $\mathbb{C}$-vector space of such functions by $M_k(F_g)$. □
Corollary 6.2. (1). We have an isomorphism of \(C\)-algebras:
\[
\bigoplus_{n=0}^\infty M_5(F_g) = \mathbb{C}[\Theta_{(1,1,1)}(\Omega(\eta))^{10}, \Theta_{(1,1,1)}(\Omega(\eta))^5, \Theta_{(1,1,1)}(\Omega(\eta))^5, \Theta_{(1,1,1)}(\Omega(\eta))^5] \\
\cong \mathbb{C}[x_0^2, x_0 x_1, x_1^2] \\
\cong \bigoplus_{n=0}^\infty H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-nK_{\mathbb{P}^1}))
\]
where \([x_0 : x_1]\) is homogeneous coordinates of \(\mathbb{P}^1\).

(2). \(M_n(F_g) = \{0\}\) for \(n \in \mathbb{N}, n \equiv 1, 2, 3, 4 \mod 5\).

Proof. The assertion (1) is a direct consequence of Corollary 6.2 and Theorem 6.3. The assertion (2) is obtained by the same argument as the proof of Theorem 6.2.

Acknowledgments. I express my sincere thanks to Professor Hironori Shiga for advices during the prepartion of this paper.

References

[1] P. Deligne and G. D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, Publ. Math I.H.E.S. 63(1986), 5-88.
[2] R. Friedmann, Algebraic surfaces and holomorphic vector bundles, Springer(1998).
[3] R. P. Holzapfel, Ball and surface arithmetics. Aspects of Mathematics, E29. Friedr. Vieweg & Sohn, Braunschweig(1998).
[4] J. Igusa, Theta functions, Springer, Heidelberg, New-York(1972).
[5] K. Matsumoto, On Modular Functions in 2 Variables Attached to a Family of Hyperelliptic Curves of Genus 3, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 (1989), no. 4, 557-578.
[6] K. Matsumoto, Theta functions on the bounded symmetric domain of type \(I_2,2\) and the period map of a 4-parameter family of K3 surfaces, Math. Ann. 295(1993), 383-409.
[7] K. Matsumoto, Theta constants associated with the cyclic triple coverings of the complex projective line ranging at six points, math.AG/0008022.
[8] D. Mumford, Tata Lectures on Theta I, Birkhäuser, Boston-Basel-Stuttgart(1983).
[9] H. Shiga, On the representation of the Picard modular function by \(\theta\) constants I-II, Pub. R.I.M.S. Kyoto Univ. 24(1988), 311-360.
[10] H. Shiga, One attempt to the K3 modular function I-II, Ann. Scuola Norm. Pisa, Ser. IV-Vol. VI(1979), 609-635, Ser. IV-Vol. VIII(1981), 157-182.
[11] G. Shimura, On purly transcendental fields of automorphic functions of several complex variables. Osaka J. Math. 1(1964), 1-14.
[12] K. Takeuchi, Arithmetic triangle group, J. Math. Soc. Japan, Vol. 29-No. 1(1977), 91-106.
[13] T. Terada, Fonctions hypergéométriques \(F_1\) et fonctions automorphes I-II, J. Math. Soc. Japan 35(1983), 451-475, 37(1985), 173-185.
[14] J. Wolfart, Graduierte algebraen automorpher formen zu dreiecksgruppen, Analysis 1, no. 3(1981), 177-190.
[15] T. Yamazaki and M. Yashida, On Hirzebruch’s Examples of Surfaces with \(c_1^2 = 3c_2\), Math. Ann. 266(1984), 421-431.
[16] M. Yoshida, Fuchsian differential equations, Aspects of Mathematics, E11. Friedr. Vieweg & Sohn, Braunschweig(1987).
[17] M. Yoshida, Hypergeometric functions, my love, Aspects of Mathematics, E32. Friedr. Vieweg & Sohn, Braunschweig(1997).

Department of Mathematics & Informatics, Faculty of Science, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan
E-mail address: mkoike@math.s.chiba-u.ac.jp