Bound state in the vector channel of the extended Nambu-Jona-Lasinio model at fixed $f_\pi$

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Abstract

We show that, as a consequence of fixing $f_\pi = 93$ MeV: (1) a bound state pole in the the $J^P = 1^-$ scattering amplitude of the ENJL model exists for arbitrarily weak (positive) vector coupling $G_2$ so long as the constituent quark mass is sufficiently large; (2) there is a bound state for any quark mass when $G_2 \geq 0.6/(8f_\pi^2)$; (3) this bound state becomes massless at $G_2 = 1/(8f_\pi^2)$ and a tachyon for $G_2$ exceeding it. We show by way of an example that the model has no trouble fitting the $\rho$ meson mass simultaneously with other observables.

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**Introduction.** Extension of the Nambu and Jona-Lasinio [NJL] model to include vector and axial-vector mesons can be traced back to the original paper [1]. The results of this extension concerning vector mesons have undergone considerable change with time: it has long been understood that bound states exist for sufficiently strong vector coupling $G_2$, but it is also believed that such high values of $G_2$ are incompatible with the phenomenology. In the early 90’s Takizawa et al. [2] found a new solution to the Bethe-Salpeter (BS) equation at lower values of $G_2$, that lay, however, on the the second lower Riemann sheet of the $J^\pi = 1^-$ elastic scattering matrix element. This pole was interpreted as a “virtual bound state”, in analogy with the nonrelativistic situation, which, however, involves only two Riemann sheets vs. $\infty$ many present here. The extended Nambu-Jona-Lasinio [ENJL] model and its solutions have recently come under renewed scrutiny [3,4]. In these papers, however, the solutions to the BS equation at different values of $G_2$ have different values of the pion decay constant $f_\pi$.

It is the purpose of this Letter to show that the ENJL model results regarding the vector- and axial-vector states undergo a drastic change when the “sliding” pion decay constant $f_\pi$ is replaced by a fixed one. In particular, we show that, as a consequence of keeping $f_\pi$ fixed, there is a bound state pole in the the $J^P = 1^-$ scattering amplitude of the ENJL model for any vector coupling $G_2 \geq 0$, so long as the constituent quark mass $m$ is large enough. We exhibit the dependence of the minimal necessary quark mass $m_{\text{min}}$ on the vector coupling $G_2$. When $G_2$ exceeds $0.6/(8f_\pi^2)$ the vector bound state exists for all values of the quark mass. This bound state becomes massless at $G_2 = 1/(8f_\pi^2)$ and a tachyon for $G_2$ exceeding this value. We find no other “resonances” in this or the axial-vector channel [4]. Our results ought to have significant consequences in ENJL-based models of electroweak interactions, such as “technicolour, top-colour” etc., since there also the scalar bilinear v.e.v. must be kept fixed.

**Conventions and preliminaries** We shall work in the chiral limit throughout this letter for the sake of clarity. Both vector and axial-vector (isovector) currents are conserved in the chiral limit and the pion is massless. The extension to the nonchiral case is straightforward. The chirally symmetric field theory described by $L_{\text{NJL}}$

\[
L_{\text{NJL}} = \bar{\psi} [i \not\partial + G_1 (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \tau \psi)^2] - G_2 [(\bar{\psi}\gamma_{\mu} \tau \psi)^2 + (\bar{\psi}\gamma_\mu \gamma_5 \tau \psi)^2] ;
\]

in both its original ($G_2 = 0$) and extended versions ($G_2 \neq 0$) exhibits spontaneous symmetry breakdown into a nontrivial ground state with constituent quark mass generation and a finite quark condensate, when dealt with non-perturbatively. The non-perturbative dynamics of the model to leading order in $1/N_c$ are described by two Schwinger-Dyson [SD] equations: the gap equation and the BS equation.

The original NJL model has two free parameters: the positive coupling constant $G_1$ of dimension (mass)$^{-2}$ and a regulating cutoff $\Lambda$ that determines the mass scale. The gap equation establishes a relation between the constituent quark mass $m$ and the two free parameters $G_1$ and $\Lambda$. This relation is not one-to-one, however: there is a (double) continuum of allowed $G_1$ and $\Lambda$ values that yield the same nontrivial solution $m$ to the gap equation. Even under the assumption that we know the precise value of $m$, which we don’t, there is still a great deal of freedom left in the $(G_1, \Lambda)$ parameter space.
Blin, Hiller and Schaden [5] showed how one can eliminate one of the two continuum degeneracies by fixing the $G_2 = 0$ value of the pion decay constant $f_\pi = f_\pi(G_2 = 0)$ at the observed value 93 MeV. Starting from the Goldberger-Treiman (GT) relation $f_\pi g_\rho = m$ one finds

$$\left(\frac{f_\pi}{m}\right)^2 = g_\rho^2 = \frac{3}{(2\pi)^2} \sum_{s=0}^{2} C_s \log(M_s^2/m^2), \quad (2)$$

where the $C_s$ and $M_s^2 = m^2 + \alpha_s \Lambda^2$ are the standard parameters of the Pauli-Villars (PV) regularization scheme [6]. The result of solving the constraint Eq. (2) is a quark mass $m$ vs. cutoff $\Lambda$ curve, shown in Fig. 1, all points on which satisfy $f_\pi = 93$ MeV. One can now select a single point on this curve by calculating an observable that is sensitive to the quark mass $m$, but not very sensitive to non-chiral corrections, and then fitting the aforementioned observable to its experimental value. One such calculation was carried out in Ref. [7] with the result $m = 225$ MeV. Such a procedure completely determines the free parameters of the NJL model.

Now let $G_2 \neq 0$: This implies a finite renormalization of the “bare” ($G_2 = 0$) pion decay constant $f_\pi$ to $f_\pi$ and of the constituent quark axial coupling $g_A$ [3] according to

$$g_A = \left(1 + 8G_2 f_\pi^2\right)^{-1} = \left(\frac{f_\pi}{f_\rho}\right)^2. \quad (3)$$

This leads to the relation

$$g_A = 1 - 8G_2 f_\pi^2 \quad (4)$$

between $g_A$ and $G_2$ and $f_\pi$, the last of which is kept constant. An $f_\pi$-fixing procedure analogous to the one described above now yields a separate $m$ vs. $\Lambda$ curve for every value of $g_A$, see Fig. 1. An important consequence of the relation (4) and of the second line of Eq. (3) is the inequality $0 \leq g_A \leq 1$. This imposes a new upper bound on $G_2$: $G_2 \leq 1/(8f_\pi^2)$, (5)

apart from the trivial lower bound $G_2 \geq 0$. $G_2$ values exceeding the bound imply imaginary values of $g_\rho$ and $f_\rho$, which in turn imply complex cutoff $\Lambda$ and/or mass $m$. Physical interpretation of such complex objects is lacking.

We see from Eq. (4) that $G_2$ can be determined from the value of the constituent quark axial coupling constant $g_A$, at constant $f_\pi$. One common prescription for estimating $g_A$ is based on the SU(6) symmetric nucleon wave function and impulse approximation result for the nucleon axial coupling

$$g_A^N = \frac{5}{3} g_A = 1.25|_{\text{expt.}}, \quad (6)$$

which yields $g_A = 0.75$. This procedure is subject to the assumption that there are no two-quark axial current contributions to the nucleon axial current matrix element, which assumption is known, however, to be in conflict with the chiral symmetry of the model [8]. Hence we shall use $g_A = 0.75$ only as an order of magnitude guide.
Perhaps the most important consequence of the fixed \( f_\pi \) is the fact that the unrenormalized pseudoscalar \( \pi qq \) coupling \( g_p \) is a function of \( G_2 \):

\[
\left( \frac{g_p}{g_\pi} \right)^2 = \left( \frac{f_\pi}{f_p} \right)^2 = g_A = 1 - 8G_2f_\pi^2.
\] (7)

This fact is the source of the changes in the vector-channel spectrum of the ENJL model, to be discussed next.

**Solutions to the BS equation**

The BS equation in the vector/axial-vector channel reads

\[
1 + 2G_2\Pi_{V,A}(s_{V,A}) = 0
\] (8)

In order to find the bound state roots \( 0 \leq s_{V,A} \leq 4m^2 \) to these equations we require the polarization functions \( \Pi_{V,A} \) [3]

\[
\Pi_V(s) = -\frac{2}{3}g_p^{-2} \left[ 2m^2[F(s) - 1] + sF(s) \right]
\]

\[
\Pi_A(s) = \Pi_V(s) + 4f_p^2F(s)
\] (9)

where

\[
F(s) = 1 - \frac{3g_p^2}{2\pi^2} \{ \sqrt{-f}\text{Arccot}\sqrt{-f} - 1 \} \text{PV}
\] (10)

and \( f = 1 - 4m^2/s \). Pauli-Villars (PV) regularization of \( F(s) \) has been used. These \( \Pi_{V,A} \) are appropriate when \( m \) and \( \Lambda \), and hence also \( f_p \) and \( g_p \) are fixed. That is the parameter-fixing procedure that was used in previous solutions of the vector BS Eq. extant in the literature. But, then Eq. (3) implies that the physical pion decay constant \( f_\pi \) changes with varying \( G_2 \), as noticed in Ref. [4].

If, on the other hand, we insist on keeping \( m \) and \( f_\pi \) (hence also \( g_\pi \)) fixed, then \( \Pi_{V,A}(s), F(s) \) in Eqs. (3), (10) are implicit functions of \( G_2 \). This implicit \( G_2 \) dependence can be easily made explicit by using Eq. (3):

\[
\Pi_V(s,G_2) = -\frac{2}{3g_A}g_\pi^{-2} \left[ s + (s + 2m^2)[F(s) - 1] \right]
\]

\[
\Pi_A(s,G_2) = \Pi_V(s) + 4f_\pi^2F(s)
\]

\[
F(s,G_2) = 1 - \frac{3g_Ag_\pi^2}{2\pi^2} \{ \sqrt{-f}\text{Arccot}\sqrt{-f} - 1 \} \text{PV}
\] (11)

where we kept \( g_A \) as an abbreviation for \( 1 - 8G_2f_\pi^2 \), for the sake of conciseness. These \( \Pi_{V,A} \) lead to solutions to the BS Eq. (3) that are rather different from what they were with a sliding \( f_\pi \).

In Fig. 2 we show the numerical solutions to the vector channel BS Eq. (8) on the physical sheet of the S-matrix for both sliding- and fixed- \( f_\pi \). There we also show the Takizawa-Kubodera-Myhrer (TKM) “virtual bound state” mass, for both the fixed- and the sliding- \( f_\pi \). One sees that: (a) the onset of the vector bound state is at substantially lower values of \( G_2 \) than with a sliding \( f_\pi \); (b) the bound state mass drops sharply with increasing
In Fig. 2 we have also shown the analytic approximation to the vector bound state mass

\[ m_V^2 = \frac{3g_AG^2}{4G_2} = 6m^2 \left( \frac{g_A}{1 - g_A} \right) . \]  

(12)

It is manifest from Fig. 2 that Eq. (12) is a good approximation to the exact result as \( g_A \to 0 \), i.e., as \( G_2 \to (8f_\pi^2)^{-1} \), but otherwise consistently overestimates the bound state mass. According to Eq. (12) the bound state ought to dissolve for \( g_A \geq 0.4 \), but the exact solution shows that the bound state may exist at even higher values of \( g_A \), i.e., at lower values of \( G_2 \), depending on the value of the constituent quark mass \( m \). In the next section we shall find the range of values of \( m = m(\Lambda) \) in which a bound state exists for a given \( G_2 \).

But first, for the sake of completeness we discuss the properties of the solutions to the axial-vector BS Eq. (8). There is only one solution to this equation, at \( G_2 = 1/(8f_\pi^2) \), on the physical sheet, and none on the “second” lower sheet. The reason for this is that \( F(s) \), and hence also \( \Pi_A(s) \) has an imaginary part that does not vanish in the region of interest, i.e., for \( s \geq 4m^2 \). Solutions to the real part of the axial-vector BS Eq. (8) are plotted in Fig. 2, together with the analytic approximation

\[ m_A^2 = m_V^2 + 6m^2 = \frac{6m^2}{1 - g_A} . \]  

(13)

This \( m_A^2 \) must not be interpreted as the real part of the resonance pole position, because the imaginary part of \( 1 + 2G_2\Pi_A(s) \) does not vanish anywhere in the mentioned quadrant of the complex \( s \) plane, i.e., there is actually no pole in the S-matrix element. It is curious that although the axial-vector BS equation does not have the “virtual bound state” solution on the second lower sheet, there is one such solution on the second-, as well as on each of infinitely many upper Riemann sheets. \[ \text{[The branch point } s = 4m^2 \text{ is a logarithmic one.]} \]

Physical interpretation of these solutions, if it exists at all, remains obscure. We have not found any other solutions either in the vector- or in the axial-vector channels, in particular we have not found the new “resonance solutions” of Ref. [4]. Our present results do not change the results and conclusions of Ref. [3] regarding the spectral sum rules.

**Minimal quark mass necessary for a vector bound state**  
In order to determine the values of \( G_2, m \) for which the vector-channel BS Eq. (8) has bound state solutions it is sufficient to consider the inequality

\[ 1 + 2G_2\Pi_V(4m^2) \leq 0 . \]  

(14)

Using Eq. (11) to find

\[ \Pi_V(4m^2) = \frac{4f_\pi^2}{3g_A} \left[ 3F(4m^2) - 1 \right] \]

\[ = - \frac{8f_\pi^2}{3g_A} \left[ 1 + g_A \left( \frac{3g_\pi}{2\pi} \right)^2 \right] . \]  

(15)

This and the inequality (14) lead to

\[ \left( \frac{3g_\pi}{2\pi} \right)^2 \geq \frac{5g_A - 2}{2g_A(1 - g_A)} , \]  

(16)
which is our vector bound state criterion. For $g_A \leq 0.4$ the r.h.s. of this inequality is non-positive, i.e., the inequality is trivially satisfied and there is a vector bound state for all real values of $m$. For $g_A \geq 0.4$ this turns into a lower bound on the constituent quark mass $m$:

$$m \geq m_{\text{min}}(g_A) = f_\pi \frac{2\pi}{3} \sqrt{\frac{5g_A - 2}{2g_A(1 - g_A)}}, \quad (17)$$

which for $g_A = 0.75$ yields the minimal constituent quark mass of 420 MeV. That, in turn gives $m_\rho \simeq 840$ MeV, not far from the empirical 770 MeV. Fig. 1 then determines the cutoff $\Lambda = 750$ MeV. This example shows that the ENJL model can easily accomodate a bound $q\bar{q}$ state in the $\rho$ channel with realistic constituent quark mass and reasonable $g_A$.

**Conclusion**  We have shown that the ENJL model binds $q\bar{q}$ states in the vector channel for arbitrarily small positive values of the coupling constant $G_2$ provided the constituent quark mass $m$ is large enough, and for all values of $m$ with $G_2 \geq 0.6(8f_\pi^2)^{-1}$, as a consequence of keeping $f_\pi$ consistently fixed at 93 MeV. This implies that bound vector states can be found at substantially lower values of $G_2$ than previously believed and the $\rho$ meson mass easily reproduced. The axial-vector state remains unbound for all allowed values of $G_2$.

Further, we have found that $G_2$ must not exceed $(8f_\pi^2)^{-1}$ if the vector bound state is not to become a tachyon. This “critical” value of the coupling constant determines a phase transition point, the nature of the “second” phase being unclear at the moment. The “first” phase, with vector coupling below the critical one, corresponds to a gauged chiral linear sigma model with massive gauge bosons $\rho, A_1$ [9]. The exact local gauge symmetry is recovered at the critical point at which the vector gauge boson ($\rho$) becomes massless, whereas $A_1$ keeps a mass of $\sqrt{6}m$ due to the Higgs mechanism.

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FIG. 1. The constituent quark mass $m$ as a function of the Pauli-Villars [PV] cutoff $\Lambda$ (in units of MeV) in the NJL ($g_A = 1$ - the far left h.s. curve), and ENJL models for $g_A = 3/4, 2/5$, (the middle and the far right h.s. curves, respectively) at fixed $f_\pi = 93$ MeV.
FIG. 2. Solutions to the BS equation [vector-, or axial-vector state mass squared $m_{V,A}^2$ rescaled by the constituent quark mass squared $m^2$] as functions of the rescaled vector interaction coupling constant $G_2m^2$, with $m = 313$ MeV in the ENJL model. [The continuum threshold is at 4.] (1) vector bound state with sliding $f_\pi$ [solid line denoted by $\rho_{old}$] continuing into the Takizawa-Kubodera-Myhrer [TKM] “virtual bound state” [long dashes] at lower values of $G_2$; (2) vector bound state with fixed $f_\pi = 93$ MeV [lower solid line denoted by $\rho$] continuing into the TKM “virtual bound state” with fixed $f_\pi$ [short dashes] at lower values of $G_2$; (3) root of the real-part of the axial-vector BS Eq. with fixed $f_\pi$ [solid line denoted by $A_1$]; (4) analytic approximations to the vector bound state, and the axial-vector state at fixed $f_\pi$ [dot-dashes].