A New Results of Injective Module with Divisible Property

Fawzi N. Hammad¹, Majid Mohammed Abed²

¹²Department of Mathematics, College of Education for pure Sciences, University of Anbar

Emails: faw19u2014@uoanbar.edu.iq
majid_math@uoanbar.edu.iq

Abstract: In this paper, we will present a new result that clarify the relationship between divisible module and Injective module. But before starting to give the most important results, we need to present the following lemma with some basic definitions that help us reach the desired goal.

Keywords: Injective module, Divisible module, Exact sequence, Artinian module, Torsion-free module.

1. Introduction

All rings and all modules in this article have a unity. In [1], “if r.m=0, r∈R and m∈M , either r=0 or m=0”. Then author, M is a torsion free. The author [2], “defined coherent ring R is coherent ring if and only if every direct product of flat module is flat and P-coherent(P-coh) if every P.I is f. presented”. By [3],”any module M satisfy (Baer’s criterion) is injective if every ideal I and every morphism f: I------->M, ∃m∈M with f(x)=mx, x∈I”. By [4], “M is called divisible if r.m=m, 0≠r∈R”. Enochs in [5],” explained the relationships between injective module and flat covers property”. In [4],” R is a left P-coh iff any direct pro. of torsion-free right R-modules is torsion-free”.

In this paper, we study strong relationship between injective module and divisible module. The main result is every injective module gives divisible but the converse need another condition P.I.D.

2. Main Results

Now we present the meaning of injective module in a more in depth way:

Definition (2.1). Let M be an R – module. If the next conditions are true, then M is called injective module.
1- M is a sub-module of Q s.t Q is a module.
2- K ⊆ M ⊃ M + K = Q and M ∩ K = \{0\} s.t K + M is an internal direct.

**Equivalent Definitions:**

**Definition (2.2).** Any module M is injective if the short exact

\[ 0 \longrightarrow Q \longrightarrow M \longrightarrow K \longrightarrow 0 \]

is split s.t the meaning of exact sequence and split can explain by the following:

\(^(*)\) A pair of module homomorphism

\[ M_1 \longrightarrow f \longrightarrow M_2 \longrightarrow g \longrightarrow M_3 \]

Is called exact at M2 if ker(g) = Img(f). In general, we can say a finite sequence of module homomorphism

\[ M_0 \longrightarrow f_1 \longrightarrow M_1 \longrightarrow f_2 \cdots M_2 \longrightarrow f_3 \longrightarrow f_{n-1} \longrightarrow f_n \longrightarrow M_n \]

Is exact if ker(f_i) = Img(f_i) : \forall i=1,2,3,\ldots,n-1

\(^(**)\) An infinite sequence of module homomorphism

\[ \ldots \longrightarrow f_{i-1} \longrightarrow M_{i-1} \longrightarrow f_i \longrightarrow M_i \longrightarrow f_{i+1} \longrightarrow M_{i+1} \longrightarrow \ldots \]

Is exact if ker(f_i) = Img(f_i) ; \forall i \in \mathbb{Z}.

**Remark (2.3).** The exact sequence is homomorphism if and only if (f) is module monomorphism

\[ 0 \longrightarrow f \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0 \]

is exact sequence of homo. iff \ g is module epi.

**Definition (2.4).** An exact sequence

\[ 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \ldots \]

and if for 0 ≠ M ∈ divisor in the ring R with M = mM , then M is divisible.

**Lemma (2.5).** “Let R be a P.I. D. If M is a divisible R-module, then M is injective”[4].

**Example (2.6).** If Z is P.I.D. injective Z-module; Z is divisible module over Z.

**Corollary (2.7).** Let R be a P.I.D if M is an injective R-module and N ⊆ M, then \( \frac{M}{N} \) is injective.

**Proof:** M is divisible so \( \frac{M}{N} \) is divisible with R P.I.D imply \( \frac{M}{N} \) is injective.

**Theorem (2.8).** For any field K; every divisible M on K – module, then M is injective.

**Proof:** Suppose that an ideal a normal of K (I ⊆ K) and 0 ≠ a s.t every k ∈ K and K = (Ki) i ∈ I . Therefore I = K and hence \{0\} = \{0\} and K = (1) only Ideal’s in K. Hence K is a P.I.D. But M is divisible module. Then M is an injective.

**Remark (2.9).** We Know that there is a relationship between projective module and divisible module.

**Theorem (2.10).** Let M1 be a divisible module. If M2 is a projective module, then the homomorphism from M2 into M1 is also divisible module.

**Proof:** Suppose a ∈ Hom_R(M_2,M_1) and b ∈ Hom_R(M_2,M_1) . IF 0 ≠ r ∈ R ∃ divisor (non – Zero divisor in R). In order to prove that Hom_R(M_2,M_1) is a divisible we need to show β = ra. Suppose
that $\chi$ be a homomorphism from $M_1$ into $M_1$ and define by: $\chi(m)=rm$. But we have $M_1$ is divisible. So $\chi$ is onto . Also we have $M_2$ is a projective module. Then $M_2$ is injective. Let $m_2\in M_2$. So $\beta(M_2)=\chi(\alpha(m_2))=\alpha(\alpha(m_2))$ and hence $\beta=r$. Thus $\text{Hom}_R(M_2, M_1)$ is a divisible.

Corollary (2.11). Let $M$ satisfy (d.c.c) over P.I.D. If $Z(M)$ subset $Z(R)$, then $M$ is injective $R$–module.

Proof: If $Z(R)$ be the zero divisors of $R$; $Z(M)$ is a zero divisors of $M$. Suppose that $\Gamma_1=R\setminus Z(R)$ and $\Gamma_2=R\setminus Z(M)$.

We know $\Gamma_1\supseteq\Gamma_2$. Let $\alpha\in\Gamma_1\subseteq\Gamma_2$. Since $M$ satisfy (d.c.c), then $M$ is Artinian module. So $\alpha M\supseteq\alpha^2 M\supseteq\ldots$. Hence $\alpha^n M=\alpha^{n+1} M$; $n\in\mathbb{Z}^+$. Now if $m_1\in M$, then $\alpha^n m_1=\alpha^{n+1} m_2=\alpha m_2 M$. Then $\alpha^n(m_1-\alpha m_2)=0$.

Since $\alpha^n \Gamma_2$ and $m_1-\alpha m_2=0$, then $m_1=\alpha m_2$. Therefore.

$M=\alpha M, \forall \alpha\in\Gamma_1$.

Thus $M$ is a divisible. But $R$ is a P.I.D, then $M$ is injective.

Corollary (2.12). Let $M$ be a divisible $R$–module. If $M$ has no zero divisors, then $M$ is injective.

Proof: Suppose that $a$ be non zero element in $R$. so $\exists m_1 M \ni am=1$. Hence:

$$mam=m$$

and then $(ma-1)m=0$. But $M$ has no zero divisors so $ma=1$. Hence $M$ is injective module ($0\neq a \in R \exists a$ invertible) then $M$ is injective. Because if $0\neq b \in I(M)$, $\exists a R \exists ab \in M$ and hence $b= a^{-1}(a b)\in M$.

Theorem (2.13). Let $a$ be $0\neq a$ invertible element of $R$ in $M_1$ If $M$ is torsion–free module, then it is divisible, $M$ is injective.

Proof: Suppose that $M$ is divisible module and $0\neq b \in I(M)$. “Then $\exists 0\neq a \in R \exists ab \in M$”. [1]. So $b= a^{-1}(a b)\in M$ thus $M$ is injective.

Remark (2.14). We know that a direct of divisible over $R$ is also divisible module. Therefore it easy say the following statement is true:

$$(****)\text{ Let }0\neq a\in R\text{ be invertible. So the direct sum of torsion-free modules is also injective.}$$

From [2], “recall that a ring $R$ is coherent If $I$ is a f.g ideal of $R$ is a presented and we call $R$ is P-coherent ring if $I$ is principal ideal of $R$ presented and if we have a direct product of copies of a ring $R$ is a torsion free, then $R$ is a p-coherent ring”.

Theorem (2.15). Suppose that a direct of P.I.D $R$ is a torsion.

If $\alpha$: $\text{Ext}(M, M) \rightarrow \text{Hom}(M, M)$ is an isom. (epi) , then $M$ is injective.

Proof. Since a direct of a P.I.D is torsion-free , $R$ is a P- coherent. But $\alpha$ is an isom.(epi ). So $\exists\beta \in \text{Hom}(D, M) \ni \alpha(\beta^+ \text{ im(}f^*))=\beta \text{ g } = I_m$.

Hence $M=\oplus F$. Then $M$ is a divisible with $R$ is P.I.D imply that $M$ is injective.

Corollary (2.16). Let $R$ be a ring with unity If $M_1$ and $M_2$ are two $R$-modules then there exists a one to one mapping $\alpha: M_1 \rightarrow M_2$.

Corollary (2.17). Let $G_1$ be a comm. group. If $G_2$ abelian group, then there exists a mapping $\beta$ ; $G_1 \rightarrow G_2$ is one to one such that $G_2$ is a divisible.

Proposition (2.18). An abelian group $G$ embedded in a divisible comm. group. Moreover; If $D$ is a divisible comm. group; $\text{Hom}(Z(R); D)$ is an injective.

Example (2.19). We have $Z$ is a P.I.D injective $Z$-modules are divisible $Z$-modules (divisible abelian groups).
Remark (2.20). "Consider \( D \) being a f. subset of a vector space \( V \) s.t \( D \) is a basis of \( V \), so every non trivial, f.g torsion-free is not divisible and we can find a \( T: V \rightarrow \text{inj}(V) \) \( \exists T \) is injective and surjective"[7].

Corollary (2.21). If we have a torsion-free \( \mathbb{Z} \)-module \( V \). Then \( \text{inj}(V) \) is a submodule of divisible Module \( (V) \).

Proof: Set \( E = \text{inj}(V) \). Set \( D = \text{divisible of the module}(V) \). If \( a \) object in the \( E \) and \( a \) in the \( D \) by [6], " mult \( E = \text{mult} D^+"."

Remark (2.22). [2]." a left \( \mathbb{R} \)-module \( M \) is called D-injective if \( \text{Ext}(G, M) = 0 \) for every divisible left \( \mathbb{R} \)-module \( G \) and \( N \) is D-flat if \( \text{Tor}(N, G) = 0 \) for every divisible left \( \mathbb{R} \)-modules \( N \) and \( G \).

Proposition (2.23). "By Wakamutsu’s Lemma, any kernel of a D-cover is D-injective and \( M \) is D-flat iff \( M^+ \) is D-injective by \( \text{Ext}(N, M^+) \cong \text{Tor}(M, N)^+ \) for every divisible left \( \mathbb{R} \)-module \( N \)[7].

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