In this paper we show how to write any variety over a field of characteristic zero as the difference of two pure motives, thus answering a question asked by Serre (in arbitrary characteristic, [Se] p.341).

The Grothendieck theory of (pure effective Chow) motives starts with the category $\mathbf{V}$ of smooth projective varieties over a field $k$. A motive is a pair $(X,p)$, where $X$ is in $\mathbf{V}$ and $p$ is a projector in the ring of algebraic correspondences from $X$ to itself (algebraic cycles on $X \times X$ modulo linear equivalence). One can add two motives $M_1 = (X_1,p_1)$ and $M_2 = (X_2,p_2)$ by considering $M_1 \oplus M_2 = (X_1 \amalg X_2, p_1 + p_2)$.

Let $K_0(\mathbf{M})$ be the abelian group associated to this monoid. Assuming the field $k$ has characteristic zero, for any variety $X$ over $k$ (i.e. any reduced scheme of finite type over $k$) we define a class $[X]$ in $K_0(\mathbf{M})$, which is characterized by the following two properties: if $X$ is connected and lies in $\mathbf{V}$, $[X]$ is the class of the motive $(X,\Delta_X)$, where $\Delta_X$ is the diagonal in $X \times X$; if $Y$ is a closed subset in $X$, with its reduced scheme structure, the following identity holds in $K_0(\mathbf{M})$:

$$[X] = [Y] + [X - Y].$$

This result (Theorem 4) is derived from a stronger one. Namely, to any variety $X$ we associate a cochain complex $W(X)$ in the additive category of complexes of motives, which is well-defined up to canonical homotopy equivalence. Our construction of $W(X)$ and the proof of its properties (Theorem 2) use higher algebraic $K$-theory and, more specifically, the Gersten complexes of schemes. These complexes are made out of the $K$-theory of all residue fields of a given scheme, and among their homology groups are precisely the Chow groups of algebraic cycles modulo linear equivalence (see Section 1.1). To prove Theorem 2, we first extend Manin’s identity principle [M1], by showing that a complex $C$ of varieties is contractible as a complex of motives if and only if a certain family of Gersten complexes associated to $C$ are acyclic (Theorem 1). We then use a variant of the theory of cohomological descent which applies to Gersten complexes (and to $K'$-theory) of simplicial schemes, and was developed by the first author in [G2] (Proposition 1). Here the proper surjective maps used in the conventional theory of cohomological descent ([SD], [D]) are replaced by envelopes in the sense of [F-G] and [G2]: a proper map of schemes $f : X' \to X$ is an envelope when, for every field $F$, all $F$-valued points of $X$ lift to $X'$. When $X$ is a variety over a field $k$ of characteristic zero,
Chow’s lemma and Hironaka’s theorem imply that $X$ admits such an envelope $X'$ which is both smooth and projective (resolution of singularities is the only reason for us to ask that the ground field has characteristic zero). This fact, together with simplicial arguments similar to those of Deligne’s theory of mixed Hodge structures [D], lead to the existence of the complex $W(X)$. This complex $W(X)$ is homotopy equivalent to a bounded complex, and its class in $K_0(M)$ is the virtual motive $[X]$. The proof of (0.1) follows (using Theorem 1 and descent) from the fact that the Gersten complexes of $Y$, $X$ and $X - Y$ fit into short exact sequences, since points of $X$ lie either in $Y$ or in $X - Y$.

We want to emphasize that we only consider pure motives, and that mixed motives never enter our discussion. This is made possible by Theorem 1, which enables us to avoid the usual difficulties arising from the homotopy theory of complexes by giving us an acyclicity criterion for a complex of motives to be contractible. If $k = \mathbb{C}$ say, one cannot recover from the complex $W(X)$ the full mixed Hodge structure on the rational cohomology of $X$, but only the graded quotients of its weight filtration; for this reason we call $W(X)$ the weight complex of $X$. We indicate in Section 3.2.4 how $[X]$ relates to some constructions of Voevodsky in the derived category of mixed motives [V]. Hanamura told us that he had, independently, used cohomological descent for higher Chow groups tensored with $\mathbb{Q}$ to associate mixed motives to arbitrary varieties. Concerning Grothendieck’s work on Serre’s question and virtual motives, see [Gr3] pp.185 and 191.

Our construction gives a new proof of the existence of virtual Betti numbers for complex varieties, which does not use Hodge theory (Section 3.3.1). Furthermore we prove that, from $E_2$ on, the weight spectral sequence of singular cohomology with compact supports and arbitrary constant coefficients is independent of choices when defined by means of (hyper)envelopes (Theorem 3); for rational coefficients and arbitrary proper hypercoverings this is a result of Deligne [D]. In particular, we obtain a canonical weight filtration on singular cohomology with compact supports and arbitrary coefficients.

An example of descent comes from the standard square diagram of varieties associated to a monoidal transform. When the center is regularly embedded, one can show directly that the corresponding Chow groups and higher $K$-groups fit into short exact sequences (Theorem 5). This gives an alternative approach to some of the results in the paper.

If linear equivalence is replaced by any other adequate relation on algebraic cycles (for instance homological or numerical equivalence), the same results are clearly also valid. Furthermore, motives can also be defined by taking for correspondences from $X$ to $Y$ the Grothendieck group $K_0(X \times Y)$ (when $X$ and $Y$ are smooth and projective). For any variety $X$ over a field $k$ of characteristic zero, we get a complex $KW(X)$ of such motives, well-defined up to homotopy (4.3). We then define the higher $K$-groups with compact support of $X$ (Theorem 7). These groups $K'_m(X)$, $m \geq -\dim(X)$, which coincide with the usual $K$-theory groups for projective non-singular $X$, admit a weight filtration and a pairing with the $K'$-theory of $X$ (Proposition 8).

The paper is organized as follows. In Section 1 we introduce the Gersten complexes, our version of Manin’s identity principle (Theorem 1), motives and descent.
In Section 2 we prove the existence of $W(X)$ and derive its main properties: functoriality, Mayer-Vietoris property and multiplicativity (Theorem 2). In Section 3 we use $W(X)$ to define $[X]$ as well as other invariants of varieties, we define the weight filtration on cohomology with compact supports (Theorem 3), and we give some examples. In Section 4 (which is almost entirely independent of the previous ones) we study the behaviour of Chow and $K$-groups under blow up. In Section 5 we define the $K$-theory spectrum with compact support of varieties (Theorem 7). The Appendix contains some facts about (co)homotopy limits and pairings which are needed in Section 5.

The results of this paper were presented at the International Conference on Algebraic K-theory in Paris, July 1994. A new construction of $W(X)$ has been given by Guillen and Navarro in [G-N], where it is denoted $h_c(X)$. They construct also a canonical complex of motives $h(X)$, corresponding to cohomology without support.

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1. Motivic Descent

1.1 The Gersten complex.

Let $X$ be a scheme of type over a field and $q \geq 0$ an integer. The Gersten complex is a chain complex of abelian groups $R_{q,*}(X)$ such that

$$R_{q,i}(X) = \bigoplus_{x \in X, \text{dim}\{x\} = q+i} K_i(k(x)),$$

where $K_i(k(x))$ is the $i$-th higher K-group of the residue field at the point $x$ of $X$, and $\{x\}$ denotes the Zariski closure of the set $\{x\}$. When $q$ varies, these complexes form the $E_1$-term of the spectral sequence associated to the $K$-theory of the category of coherent sheaves on $X$, filtered by dimension of support [Q] §7,Theorem 5.4 and [G1], and the differential is the corresponding $d_1$. It is proved in [G1] Th. 7.22 that, if $f : X \to Y$ is a proper map of schemes, then we have a map of chain complexes:

$$f_* : R_{q,*}(X) \to R_{q,*}(Y)$$

$$\bigoplus_{x} f_x : \bigoplus_{x \in X, \text{dim}\{x\} = q+i} K_i(k(x)) \to \bigoplus_{y \in Y, \text{dim}\{y\} = q+i} K_i(k(y)),$$

where $f_x = 0$ if $\text{dim}\{f(x)\} < \text{dim}\{x\}$, and $f_x$ is the norm map associated to the finite field extension $k(x)/k(y)$ if $\text{dim}\{f(x)\} = \text{dim}\{x\}$. The 0-th homology group of the complex $R_{q,*}$ is the Chow homology group of dimension $q$ cycles, $CH_q(X)$, while the other homology groups are the higher $K$-theory type Chow homology groups $CH_{q,p}(X) = H_p(R_{q,*}(X))$, which are related to $K_{q-p}(X)$. As shown in [G1] p.276, the direct sum of these groups for all $p$ and $q$ is a graded module over the ring $\bigoplus_{p,q} H^p(X,K_q)$, where $K_q$ is the Zariski sheaf associated to the presheaf $K_q$. In particular, for a non-singular $X$ of finite type over a field, it is a graded module over the Chow ring.
1.2 Universal acyclicity.

Throughout this paper, given a field $k$, a variety over $k$ means a reduced scheme of finite type over $\text{Spec}(k)$, and we denote by $\mathbf{V}$ either the category of smooth proper varieties over $k$ or the category of smooth projective varieties over $k$: all statements will be valid for both definitions of $\mathbf{V}$ (except in section 5 where we shall require that varieties in $\mathbf{V}$ are projective). Let $Z\mathbf{V}$ be the category with the same objects as $\mathbf{V}$, but with $\text{Hom}_{Z\mathbf{V}}(X, Y)$ equal to the free abelian group $\text{ZHom}_{\mathbf{V}}(X, Y)$ on $\text{Hom}_{\mathbf{V}}(X, Y)$. For varieties $X$, $Y$ and $Z$, the composition pairing

$$\text{Hom}_{Z\mathbf{V}}(X, Y) \times \text{Hom}_{Z\mathbf{V}}(Y, Z) \to \text{Hom}_{Z\mathbf{V}}(X, Z)$$

is bilinear and induced by the usual composition of morphisms in $\mathbf{V}$. Notice that, for each $q \geq 0$, $X \mapsto R^q, * (X)$ is a covariant functor from $\mathbf{V}$ to the category of chain complexes of abelian groups, and therefore factors through $Z\mathbf{V}$.

Let $C$ be the category of correspondences in $\mathbf{V}$, having the same objects as $\mathbf{V}$, but with $\text{Hom}_C(X, Y) = \bigoplus_{i \in I} CH^{\text{dim}(Y_i)}(X \times Y_i)$, where $Y_i, i \in I$, are the connected components of $Y$. The composition law is defined as follows:

$$(\alpha, \beta) \mapsto \pi_{XZ}^* (\pi_{XY}^*(\alpha) \pi_{YZ}^*(\beta))$$

where $\pi_{XZ}$, $\pi_{XY}$ and $\pi_{YZ}$ are the projections from $X \times Y \times Z$ to $X \times Z$, $X \times Y$ and $Y \times Z$ respectively. This composition is bi-additive, and for any $X \in \text{Ob}\mathbf{V}$ the class $1_X = [\Delta_X]$ of the diagonal is the identity in the ring $\text{End}_C(X)$. There is a covariant functor

$$\Gamma : \mathbf{V} \to C$$

mapping a morphism $f$ to the class $[\Gamma_f] \in \text{Hom}_C(X, Y)$ of the graph of $f$. This functor factors through $Z\mathbf{V}$. The Chow homology type functors $CH^{p, q}$ all factor through $C$, with a correspondence acting on a class $x$ by $\alpha_* : x \mapsto \pi_Y^*(\alpha \pi^*(x))$ ([So1] Section 1.2).

Recall from [M] p.448 the following lemma:

Lemma 1. Let $f : X \to Y$ be a map in $\mathbf{V}$, and $\theta \in CH^*(Z \times X)$ a correspondence. Then

$$(1_Z \times f)_* \theta = [f] \circ \theta \in CH^{\text{dim}(Y)}(Z \times Y)$$

Proof.

$$[f] \circ \theta = \pi_{ZY}^*(\pi_{ZX}^*(\theta) \pi_{XY}^*(\pm))$$

$$= \pi_{ZY}^*(\pi_{ZX}^*(\theta) \pi_{XY}^*(\pm))$$

$$= \pi_{ZY}^*(\pi_{ZX}^*(\theta) \pi_{XY}^*(\pm))$$

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Here $j : Z \times X \to Z \times X \times Y$ is the graph of $f \circ \pi_X : Z \times X \to Y$. □
Theorem 1. Suppose that
\[ \ldots \to X_2 \xrightarrow{\delta_2} X_1 \xrightarrow{\delta_1} X_0 \]
is a chain complex in \( \mathbb{Z}V \), such that for all \( V \in V \) and all \( q \geq 0 \), the total complex of the double complex
\[ \ldots \to R_{q,*}(V \times X_2) \to R_{q,*}(V \times X_1) \to R_{q,*}(V \times X_0) \]
is acyclic. Then the complex
\[ \ldots \to X_2 [\delta_2] \to X_1 [\delta_1] \to X_0 \]
in \( C \) has a contracting homotopy.

Proof. We proceed by induction on \( n \geq 0 \) to construct correspondences \( h_n : X_n \to X_{n+1} \) such that \( h_{n-1} \delta_n + \delta_{n+1} h_n = 1_{X_n} \).

We start with the case of \( n = 0 \). Consider the chain complex
\[ \ldots \to X_0 \times X_2 \xrightarrow{\delta_2} X_0 \times X_1 \xrightarrow{\delta_1} X_0 \times X_0 \]
and write \( d = \text{dim}(X_0) \). By hypothesis, the total complex associated to the double complex
\[ \ldots \to R_{d,*}(X_0 \times X_2) \xrightarrow{\delta_2} R_{d,*}(X_0 \times X_1) \xrightarrow{\delta_1} R_{d,*}(X_0 \times X_0) \]
is acyclic. In particular
\[ R_{d,1}(X_0 \times X_0) \oplus R_{d,0}(X_0 \times X_1) \to R_{d,0}(X_0 \times X_0) \]
is surjective. Hence
\[ (1_{X_0} \times \delta_1)_* : R_{d,0}(X_0 \times X_1) \to CH_{d}(X_0 \times X_0) = CH^d(X_0 \times X_0) \]
is surjective. In particular there exists a cycle \( \eta_0 \) on \( X_0 \times X_1 \) such that
\[ (1_{X_0} \times \delta_1)_*(\eta_0) = [1_{X_0}](= [\Delta_{X_0}]). \]

By Lemma 1, if we set \( h_0 = [\eta_0] \), the correspondence represented by \( \eta_0 \), then
\[ \delta_1 \circ h_0 = 1_{X_0}. \]

Suppose now that \( n \geq 1 \) and that correspondences \( h_i : X_i \to X_{i+1} \) for \( i = 0 \ldots n-1 \) have been constructed, such that
\[ h_{i-1} \circ \delta_i + \delta_{i+1} \circ h_i = 1_{X_i}. \]

Since, for all \( p, q \geq 0 \), \( CH_{q,p} \) factors through \( C \), it follows that for all varieties \( V \) we have maps:
\[ (1_{X_0} \times \delta_1)_* : CH_{q,p}(V \times X_0) \to CH_{q,p}(V \times X_0) \]
for all \( i \), and maps
\[
(1_V \times h_i)_*: CH_{q,p}(V \times X_i) \to CH_{q,p}(V \times X_{i+1})
\]
for \( i \leq n-1 \), such that, for \( i \leq n-1 \),
\[
(1_V \times h_{i-1})_* \circ (1_V \times \delta_i)_* + (1_V \times \delta_{i+1})_* \circ (1_V \times h_i)_* = 1_{CH_{q,p}(V \times X_i)}.
\]
It follows that, for \( i \leq n-1 \),
\[
H_i(* \mapsto CH_{q,p}(V \times X_*)) = 0
\]
Now by hypothesis, for any \( q \geq 0 \), the total complex \( \int R_{q,*}(V \times X_*) \) associated to the double complex \( R_{q,*}(V \times X_*) \) is acyclic. Consider the spectral sequence
\[
E_{m,p}^2 = H_m(* \mapsto CH_{q,p}(V \times X_*)) \Rightarrow H_{m+p}(\int R_{q,*}(V \times X_*)).
\]
From the induction hypothesis and the fact that the functors \( CH_{q,p} \) factor through \( C \), we deduce that \( E_{m,p}^2 = 0 \) when \( 0 \leq m \leq n-1 \) and \( p \geq 0 \). Therefore \( E_{m,p}^r = 0 \) whenever \( r \geq 2 \), \( 0 \leq m \leq n-1 \) and \( p \geq 0 \), and the only part of \( \bigoplus_{m+p=0} E_{m,p}^2 \) which can possibly be non-zero is \( E_{n,0}^2 = H_n(* \mapsto CH_q(V \times X_*)) \). Using the induction hypothesis again, we have that all the differentials out of \( E_{n,0}^2 \) are zero. On the other hand, since \( p = 0 \), there are no differentials into \( E_{n,0}^2 \). Hence
\[
E_{n,0}^2 = E_{n,0}^{\infty} = 0,
\]
since \( H_n(\int R_{q,*}(V \times X_*)) = 0 \). Therefore, for all varieties \( V \), and all \( q \geq 0 \), we have that
\[
CH_q(V \times X_{n-1}) \leftarrow CH_q(V \times X_n) \leftarrow CH_q(V \times X_{n+1})
\]
is exact in the middle.

Now take \( V = X_n \) and \( q = \dim(X_n) \). Consider the element
\[
1_{X_n} - h_{n-1} \circ \delta_n \in CH_q(X_n \times X_n) \simeq CH^0(X_n \times X_n).
\]
By Lemma 1, the image of this class under
\[
(1_{X_n} \times \delta_n)_*: CH_q(X_n \times X_n) \to CH_q(X_n \times X_{n-1})
\]
is the correspondence
\[
\delta_n \circ (1_{X_n} - h_{n-1} \circ \delta_n) = \delta_n - \delta_n \circ h_{n-1} \circ \delta_n
\]
\[
= \delta_n - (1_{X_{n-1}} - h_{n-2} \circ \delta_{n-1}) \circ \delta_n
\]
\[
= 0 \quad \text{since} \quad \delta_{n-1} \circ \delta_n = 0.
\]
Hence, by the exactness above, there exists a cycle
\[
\eta_n \in CH_q(X_n \times X_{n+1})
\]
such that
\[
(1_{X_n} \times \delta_{n+1})_* \eta_n = 1_{X_n} - h_{n-1} \circ \delta_n.
\]
This cycle represents a correspondence
\[
h_n: X_n \to X_{n+1}
\]
which satisfies, by Lemma 1, the identity
\[
\delta_{n+1} \circ h_n + h_{n-1} \circ \delta_n = 1_{X_n}
\]
and we are done. \( \Box \)
1.3 Motives.

Let \( M \) be the category of (pure effective) Chow motives over \( k \). It is obtained from the category \( C \) of correspondences by inverting the arrows and adding the images of projectors (cf. [M1], [Kl] ). An object of \( M \) is a pair \((X, p)\) where the variety \( X \) lies in \( V \) (i.e. it is smooth and either projective or proper over \( k \)) and \( p \in \text{End}_C(X) \) satisfies \( p^2 = p \). A morphism from \((X, p)\) to \((Y, q)\) is an element in \( q\text{Hom}_C(Y, X)p \) (see [J] or [Sc] for this definition). There is a contravariant functor from \( V \) to \( M \) mapping a variety \( X \) to the Chow motive \( M(X) = (X, 1_X) \) and a morphism \( f \) to the transpose of \([\Gamma_f]\). The category \( M \) is pseudo-abelian, i.e. it is additive and projectors have images. Disjoint union and product of varieties can be extended to motives where they are denoted \( \oplus \) and \( \otimes \) respectively.

More generally, given any equivalence relation \( \sim \) on algebraic cycles which is adequate in the sense of [Kl], one may substitute to Chow groups the groups of cycles modulo \( \sim \) in the above construction, getting a category of motives \( M_{\sim} \). Basic examples of adequate equivalence relation are linear equivalence, homological equivalence and numerical equivalence, written \( \text{num} \) in what follows. Notice that for any \( \sim \) there exist covariant functors \( M \rightarrow M_{\sim} \) and \( M_{\sim} \rightarrow M_{\text{num}} \) ([Kl] Proposition 3.5).

Given any chain complex \( X. \) in \( ZV \), we shall denote by \( M(X. ) \) (resp. \( M_{\sim}(X. ) \)) the corresponding cochain complex in \( M \) (resp. \( M_{\sim} \)).

**Corollary 1.** Let \( f : X. \rightarrow Y. \) be a morphism of chain complexes in \( ZV \) such that for all \( V \in V \) and all \( q \geq 0 \), the induced map of double complexes

\[
R_{q,*}(V \times X. ) \rightarrow R_{q,*}(V \times Y. )
\]

induces a quasi-isomorphism of total complexes. Then the corresponding map \( X. \rightarrow Y. \) of chain complexes in \( C \) is a homotopy equivalence.

Consequently, for any choice of an adequate equivalence relation, the induced map \( f^* : M_{\sim}(Y. ) \rightarrow M_{\sim}(X. ) \) is a homotopy equivalence.

**Proof.** To prove this corollary we shall use a general theorem of Verdier [V]. Let \( A \) be an arbitrary additive category. For any map \( f : X. \rightarrow Y. \) of complexes in \( A \), denote by \( \text{Cone}(f) = C(f) \) the mapping cone of \( f \). According to [V] II, Proposition 1.3.2, the category \( \text{Hot}(A) \) of complexes in \( A \) up to homotopy is a triangulated category, in which the triangles are the diagrams isomorphic in \( \text{Hot}(A) \) to diagrams of the form

\[
X. \xrightarrow{f} Y. \rightarrow C(f). \rightarrow X.[1]
\]

where \( f : X. \rightarrow Y. \) is any morphism of complexes in \( A \).

It follows from this that if \( f : A. \rightarrow B. \) is a map between chain complexes in \( A \) the following two statements are equivalent:

- The map is a chain homotopy equivalence (i.e. there exist a chain map \( g : B. \rightarrow A. \) and homotopies \( g \circ f \sim 1_A. \) and \( f \circ g \sim 1_B. \));
- The mapping cone \( C(f) \) is contractible.

Indeed, if \( f \) is a homotopy equivalence and if \( \text{id} : X. \rightarrow X. \) is the identity map, there exists a morphism of triangles in \( \text{Hot}(A) \).
\[ X \xrightarrow{\text{id}} X \xrightarrow{\text{id}} C(\text{id}) \xrightarrow{\text{id}} X.\]  
\[ X \xrightarrow{f} Y \xrightarrow{\phi} C(f) \xrightarrow{\text{id}} X.\]  

(property TRIII in [V] II 1.1.1). From Cor. 1.2.5 in [V] II, it follows that \( \phi \) is an isomorphism in \( \text{Hot}(A) \). Since \( C(\text{id}) \) is contractible, it follows that \( C(f) \) is contractible.

Conversely, if \( C(f) \) is contractible it is isomorphic to 0 in \( \text{Hot}(A) \) and we get a triangle \( X \xrightarrow{f} Y \xrightarrow{0} X.\), hence a triangle \( 0 \xrightarrow{f} Y \xrightarrow{0} X.\) (TRII in [V] II 1.1.1). Therefore \( f \) is an isomorphism in \( \text{Hot}(A) \), by TRI in [V] II 1.1.1 and the uniqueness of the cone, [V] II Cor. 1.2.6.

To prove the corollary, notice that \( R_{q_*} \) commutes with the formation of mapping cones of complexes in \( ZV \). So, under our assumptions, the complex \( C(f) \) in \( ZV \) satisfies the hypotheses of Theorem 1. Therefore \( C(f) \) is contractible when viewed as a complex in \( C \), and hence the map \( f : X. \to Y. \) is a homotopy equivalence of complexes in \( C \). \( \square \)

### 1.4 Envelopes.

**1.4.1.** An *envelope* \( p : X \to Y \) is a proper map of schemes such that for all fields \( F \), the induced map \( X(F) \to Y(F) \) is surjective. A *hyperenvelope* is a map \( p : X. \to Y. \) of simplicial schemes which is proper in each degree and which is a hypercovering for the Grothendieck topology in which the covering maps are envelopes. Specifically, for all \( i \geq 0 \) the map

\[ X_i \to (\cosk_{i-1} Y, \sk_{i-1}(X_i)) \]

is an envelope (see [G2]). Equivalently, a hyperenvelope is a proper map of simplicial schemes such that for each field \( F \) the map of simplicial sets \( X.(F) \to Y.(F) \) is a trivial Kan fibration [Ma].

Let \( f : Y. \to X. \) be a morphism of simplicial schemes. A *hyperenvelope* \( h : \tilde{f} \to f \) is a map in the category of arrows of simplicial schemes, i.e. a commutative square

\[ \begin{array}{ccc}
\tilde{Y}. & \xrightarrow{\tilde{f}} & \tilde{X}.\\
Y. & \xrightarrow{f} & X. \\
\end{array} \]

such that \( h_Y \) and \( h_X \) are hyperenvelopes.

If \( k \) is a field of characteristic zero, any projective variety \( X \) over \( k \) has a *non-singular envelope*, i.e. an envelope lying in \( V \). This can be shown by induction on the dimension of \( X \): by Chow’s lemma and Hironaka’s resolution of singularities one may find \( X' \in V \) and a proper map \( X' \to X \), which is an isomorphism over a dense open subset \( U \subset X \); the disjoint union of \( X' \) with a non-singular envelope of \( X - U \) is a non-singular envelope of \( X \). From this one gets by induction on the simplicial degree that, when viewed as a constant simplicial scheme, \( X \) admits a hyperenvelope \( Z \), which is non-singular in the sense that each \( Z_i, i \geq 0 \) lies in \( V \).

More generally, the following holds:
Lemma 2. If $k$ is a field of characteristic zero any simplicial variety over $k$ has a non-singular envelope. Furthermore, given a morphism of simplicial varieties $f : Y \to X$ over $k$, there exists a non-singular hyperenvelope $h : \tilde{f} \to f$.

Proof. To prove the first assertion consider a simplicial variety $X$. One defines a non-singular hyperenvelope $f : Y \to X$ by induction on $n \geq 0$, as follows (see [SD] 5.1.3 and [D] Section 6.3 for the analogous result for cohomological descent). Start by choosing a non-singular envelope $f_1 : Y_0 \to X_0$. Suppose now that $n > 0$, and that the $(n-1)$-skeleton of $Y$. has been constructed together with maps $f_i : Y_i \to X_i$ for $i = 0 \ldots n - 1$ which define a hyperenvelope of $(n-1)$-truncated simplicial schemes. Let $Z_n \to \cosk_{n-1}^X(sk_{n-1}Y)_n$ be a non-singular envelope. Set

$$Y_n = Z_n \coprod_{j=0}^{n-1} Y_{n-1}.$$ 

For $i = 0 \ldots n - 1$, let the degeneracy map $s_i : Y_{n-1} \to Y_n$ be the identity map onto the $i$-th summand in the coproduct. The face maps $d_i : Y_n \to Y_{n-1}$ are then defined as follows. On $Z_n$ they are the composition of the envelope $Z_n \to \cosk_{n-1}^X(sk_{n-1}(Y))_n$ with the face maps $d_i : \cosk_{n-1}^X(sk_{n-1}Y)_n \to Y_{n-1}$. On the $j$-th summand in $\coprod_{j=0}^{n-1} Y_{n-1}$, we set $d_i = s_j d_l$ where $k = j \pm 1$ or $k = j$ and $l = i - 1$ or $l = i$, according to the formula in definition 1.1 (iii) of [Ma]. Let $f_n : Y_n \to X_n$ be the map which on $Z_n$ is the composition of the envelope $Z_n \to \cosk_{n-1}^X(sk_{n-1}Y)_n$ with the natural map $\cosk_{n-1}^X(sk_{n-1}Y)_n \to X_n$. On the $i$-th summand of the coproduct $Z_n \to \cosk_{n-1}^X(sk_{n-1}Y)_n$ we set $f_n = s_j f_{n-1}$. We define the $n$-skeleton of $Y$, to be the $n$-truncated simplicial scheme obtained by this process, and the sequence of maps $f_i$ for $i = 0 \ldots n$ are a hyperenvelope $sk_n(Y) \to sk_n(X)$.

To prove the second assertion, let $\tilde{X}$. be a non-singular hyperenvelope of $X$. Now choose $\tilde{Y}$. to be a non-singular hyperenvelope of the pull-back $Y \times_X \tilde{X}.$, with $\tilde{f} : \tilde{Y} \to \tilde{X}$. the obvious composition. We get a non-singular envelope $h : \tilde{f} \to f$, i.e. a commutative square

$$\begin{array}{ccc}
\tilde{Y}. & \xrightarrow{\tilde{f}} & \tilde{X}.\\
h_{\tilde{Y}} & & \downarrow h_{\tilde{X}} \\
Y. & \xrightarrow{f} & X.
\end{array}$$

where $h_{\tilde{X}}$ and $h_{\tilde{Y}}$ are hyper-envelopes and, for all $i \geq 0$, $\tilde{Y}_i$ and $\tilde{X}_i$ lie in $V$. \qed

The following are some basic properties of hyperenvelopes, which follow in a straightforward fashion from the definitions:

- The composition of two hyperenvelopes is again a hyperenvelope.
- If $f : X \to Y$ is a hyperenvelope, and $g : Z \to X$ is a map of simplicial schemes, then the pull-back of $f$ along $g$ is again a hyper-envelope. The same is true for envelopes of arrows of simplicial schemes.
- The fibre product of two hyperenvelopes is again a hyperenvelope (this follows from the previous two assertions).
1.4.2. If \( X \) is a simplicial scheme with proper face maps, then for each \( q \geq 0 \) we have the Gersten complexes \( R_{q,*}(X) \), by which we mean the total complex associated to the double complex \((i,j) \mapsto R_{q,i}(X_j)\). Given a proper map \( f : X \rightarrow Y \) of simplicial schemes (with proper face maps), there is a push-forward map \( f_* : R_{q,*}(X) \rightarrow R_{q,*}(Y) \), making \( R_{q,*}(X) \) a covariant functor from the category of simplicial schemes with proper face maps to the category of chain complexes. Envelopes then have the following descent property.

**Proposition 1.** If \( f : X \rightarrow Y \) is a hyperenvelope of simplicial schemes with proper face maps, then \( f_* : R_{q,*}(X) \rightarrow R_{q,*}(Y) \) is a quasi-isomorphism.

**Proof.** For the case when \( Y \) is the constant simplicial scheme associated to a scheme \( Y \) (which is the only case that we need in this paper) this is proved in [G2] Theorem 4.3. The general case follows by an extension of the argument of op. cit. Specifically, if we let \( X[i] = cosk_i^Y(\text{sk}_i(X)) \) when \( i \geq 0 \) and \( X[-1] = Y \), one can show that the map \( X[i] \rightarrow X[i-1] \) induces a quasi-isomorphism of Gersten complexes for all \( i \geq 0 \): the argument in [SD], proof of Theorem 3.3.3, reduces this assertion to the descent theorem in the case \( Y \) is constant and \( X = cosk_0^Y(X_0) \), i.e. to Step II in [G2] Theorem 4.3. \( \square \)

1.4.3. Given a simplicial object \( X \) in \( V \), write \( ZX \) for the complex

\[
\ldots X_i \sum_{j=0}^{i} (-1)^j d_j \rightarrow X_{i-1} \ldots
\]

which we view as a complex in \( ZV \), and \( M(X) \) for the corresponding complex of motives.

**Proposition 2.** Let \( p : X \rightarrow Y \) be a map of simplicial objects in \( V \) which is a hyperenvelope. Then the associated map of complexes in \( M \):

\[
p^* : M(Y) \rightarrow M(X)
\]

is a homotopy equivalence.

**Proof.** Since \( p \) is a hyper-envelope, for any variety \( V \) so is \( 1_V \times p : V \times X \rightarrow V \times Y \), see 1.4.1. or [G2], Lemma 3.1 i). Therefore, by Proposition 1, for any \( q \geq 0 \) the map of Gersten complexes

\[
p_* : R_{q,*}(V \times X) \rightarrow R_{q,*}(V \times Y)
\]

is a quasi-isomorphism. Consider the map of chain complexes \( ZX \rightarrow ZY \) induced by \( p \). Since \( Z(V \times X) = V \times ZX \), and the formation of \( R_{q,*}(X) \) can be done using either \( X \) or \( ZX \), we conclude from Corollary 1 that \( p^* : M(Y) \rightarrow M(X) \) is a homotopy equivalence. \( \square \)
2. The weight complex of arbitrary varieties

2.1. Let \( k \) be a field of characteristic zero and \( X \) a (possibly singular) variety over \( k \). Choose a compactification \( \bar{X} \) of \( X \), by which we mean any complete variety containing \( X \) as an open subvariety, with complement \( Y = \bar{X} - X \). Then for any non-singular hyper-envelope \( p : \bar{X} \to X \) there exists a non-singular hyper-envelope \( q : \tilde{Y} \to Y \), and a map \( \tilde{j} : \tilde{Y} \to \bar{X} \) such that we have a commutative diagram, where \( j : Y \to \bar{X} \) is the inclusion:

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{j}} & \bar{X} \\
\downarrow q & & \downarrow p \\
Y & \xrightarrow{j} & \bar{X}
\end{array}
\]

We associate to this data the cochain complex \( S(\tilde{j}) = \text{Cone}(\tilde{j}^*)[-1] \), where \( \tilde{j}^* \) is the induced map of cochain complexes \( M(\bar{X}) \to M(\tilde{Y}) \); in particular, for all \( n \geq 0 \), \( S(\tilde{j})^n = M(\bar{X}_n) \oplus M(\tilde{Y}_{n-1}) \). Let \( W(X) \) be the class of \( S(\tilde{j}) \) in \( \text{Hot}(\mathcal{M}) \) of cochain complexes in \( \mathcal{M} \).

**Theorem 2.** In \( \text{Hot}(\mathcal{M}) \) the complex \( W(X) \) is independent, up to canonical isomorphism, of the choices made. It has the following properties:

i) \( W(X) \) is isomorphic to a bounded complex of the form

\[
M(X_0) \to M(X_1) \to \cdots \to M(X_k)
\]

where, for all \( i, 0 \leq i \leq k \), \( X_i \) is a variety lying in \( V \) and \( \dim(X_i) \leq \dim(X) - i \); in particular \( k \leq \dim(X) \).

ii) Any proper map \( f : X \to X' \) of varieties induces a morphism \( f^* : W(X') \to W(X) \) in \( \text{Hot}(\mathcal{M}) \). Given two composable proper maps \( f \) and \( g \), one has \( (fg)^* = g^*f^* \). Any open immersion \( i : X \to X' \) induces a morphism \( i_* : W(X) \to W(X') \) in \( \text{Hot}(\mathcal{M}) \) and, given two composable open immersions \( i \) and \( j \), \( (ij)_* = i_*j_* \).

iii) Let \( X \) be a variety, and suppose that \( i : U \to X \) is an open immersion, with complement the closed immersion \( f : T \to X \). Then there is a canonical triangle in \( \text{Hot}(\mathcal{M}) \):

\[
W(U) \xrightarrow{i^*} W(X) \xrightarrow{f^*} W(T) \to W(U)[-1].
\]

iv) Assume \( X \) is the union of two closed subvarieties \( A \) and \( B \). Then there is a canonical triangle in \( \text{Hot}(\mathcal{M}) \)

\[
W(X) \to W(A) \oplus W(B) \to W(A \cap B) \to W(X)[-1].
\]

If \( X \) is the union of two open subvarieties \( U \) and \( V \), there is a canonical triangle

\[
W(X)[-1] \to W(U \cap V) \to W(U) \oplus W(V) \to W(X).
\]

v) If \( X \) and \( Y \) are quasi-projective varieties, then

\[
W(X \times Y) = W(X) \otimes W(Y).
\]

The complex \( W(X) \) in \( \text{Hot}(\mathcal{M}) \) will be called the weight complex of the variety \( X \). As the proof in sections 2.2 to 2.6 will show, Theorem 2 is already true in the homotopy category of bounded complexes \( \text{i}^\prime \text{n} \mathcal{C} \) instead of \( \text{Hot}(\mathcal{M}) \).
2.2. Let $\mathbf{P}$ be the category of proper varieties over $k$ and $\text{Ar}(\mathbf{P})$ the category of arrows in $\mathbf{P}$, where a morphism $g : f_1 \to f_2$ is defined to be a commutative square

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{f_1} & Z_1 \\
\downarrow{g_Y} & & \downarrow{g_Z} \\
Y_2 & \xrightarrow{f_2} & Z_2
\end{array}
$$

in $\mathbf{P}$. As a preliminary to the proof of Theorem 2, we shall define a functor $T$ from $\text{Ar}(\mathbf{P})$ to $\text{Hot}(\mathbf{M})$.

First suppose that a morphism $g : f_1 \to f_2$ as above is given and that we have two non-singular hyperenvelopes $h_i : \tilde{f}_i \to f_i$, $i = 1, 2$, i.e. commutative squares

$$
\begin{array}{ccc}
\tilde{Y}_i & \xrightarrow{f_i} & \tilde{Z}_i \\
\downarrow{h_{Y,i}} & & \downarrow{h_{Z,i}} \\
Y_i & \xrightarrow{f_i} & Z_i
\end{array}
$$

in which $h_{Y,i}$ and $h_{Z,i}$ are non-singular hyperenvelopes. Consider the fiber product $\tilde{f}_1 \times_{f_2} \tilde{f}_2$ and the projections $p_i : \tilde{f}_1 \times_{f_2} \tilde{f}_2 \to \tilde{f}_i$, $i = 1, 2$. By 1.4.1, $p_1$ is an hyperenvelope; however it may not be non-singular. So let $\pi : \tilde{f} \to \tilde{f}_1 \times_{f_2} \tilde{f}_2$ be a non-singular hyperenvelope. If $a = p_1 \circ \pi$ and $b = p_2 \circ \pi$ we get a commutative diagram

$$
\begin{array}{ccc}
\tilde{f} & \xrightarrow{b} & \tilde{f}_2 \\
\downarrow{a} & & \downarrow{h_2} \\
\tilde{f}_1 & \xrightarrow{h_1} & \tilde{f}_2
\end{array}
$$

Since $a$ is an hyperenvelope we deduce from Proposition that $S(a) : S(\tilde{f}_1) \to S(\tilde{f})$ is an isomorphism in $\text{Hot}(\mathbf{M})$. Therefore we get a map $\theta_{\tilde{f}} = S(a)^{-1} S(b)$ from $S(\tilde{f}_2)$ to $S(\tilde{f}_1)$.

This map does not depend on the choice of $\pi$. Indeed, assume

$$
\pi' : \tilde{f}' \to \tilde{f}_1 \times_{f_2} \tilde{f}_2
$$

is another non-singular hyperenvelope. Choose a non-singular hyperenvelope $\tilde{f}'' \to \tilde{f} \times_{(\tilde{f}_1 \times_{f_2} \tilde{f}_2)} \tilde{f}'$. We have commutative squares

$$
\begin{array}{ccc}
\tilde{f}'' & \xrightarrow{k'} & \tilde{f}' \\
\downarrow{k} & & \downarrow{a'} \\
\tilde{f} & \xrightarrow{\tilde{f}} & \tilde{f}'
\end{array}
$$
and

\[
\begin{array}{ccc}
\tilde{f}'' & \xrightarrow{k'} & \tilde{f}' \\
\downarrow & & \downarrow \beta \\
\tilde{f} & b & \tilde{f}_2
\end{array}
\]

where \( k, k', a \) and \( a' \) are hyperenvelopes. It follows that

\[
\theta_{\tilde{f}'} = S(a')^{-1} S(b') = S(a)^{-1} S(k) S(k') S(b') = S(a)^{-1} S(b) = \theta_{\tilde{f}}.
\]

Thus we have a canonical map

\[
T(g) : S(\tilde{f}_2) \rightarrow S(\tilde{f}_1).
\]

When \( g \) is the identity map, \( p_2 \) is also an hyperenvelope, therefore \( S(b) \) and \( T(g) \) are isomorphisms in \( \text{Hot}(\mathcal{M}) \).

We shall now check that \( T(g) \) is compatible with composition. Consider morphisms \( g : f_1 \rightarrow f_2 \) and \( k : f_2 \rightarrow f_3 \) in \( \text{Ar}(\mathcal{P}) \) as well as non-singular hyperenvelopes \( h_i : \tilde{f}_i \rightarrow f_i, i = 1, 2, 3 \). From the argument above, we get a commutative diagram

\[
\begin{array}{ccc}
\tilde{f}'' & \xrightarrow{\beta} & \tilde{f} \\
\downarrow & & \downarrow b \\
\tilde{f}' & b' & \tilde{f}_2 \\
\downarrow & & \downarrow \\
\tilde{f}_1 & & \\
\downarrow & & \\
\tilde{f}_1 & \xrightarrow{g} & \tilde{f}_2 \\
\downarrow & & \downarrow k \\
f_1 & & f_2 \\
\downarrow & & \downarrow \\
f_1 & & f_3
\end{array}
\]

(2.3)

where all vertical maps are hyperenvelopes, \( \tilde{f}'' \) being defined as a non-singular hyperenvelope of \( \tilde{f}' \times_{\tilde{f}_2} \tilde{f} \); since the map \( a \) is a hyperenvelope, the same is true of \( \alpha : \tilde{f}'' \rightarrow \tilde{f}' \). The obvious morphism \( \tilde{f}'' \rightarrow \tilde{f}_1 \times_{f_3} \tilde{f}_3 = \left( \tilde{f}_1 \times_{f_2} \tilde{f}_2 \right) \times_{f_3} \left( \tilde{f}_2 \times_{f_3} \tilde{f}_3 \right) \) is the composition of a hyperenvelope and the fiber product of two hyperenvelopes and is therefore also a hyperenvelope. So, from the commutativity of (2.3) we deduce that

\[
T(kg) = \theta_{\tilde{f}''} = S(a' \alpha)^{-1} S(b \beta) = S(a')^{-1} S(b') S(a)^{-1} S(b) = \theta_{\tilde{f}} \theta_{\tilde{f}} = T(k) T(g).
\]

We conclude from this discussion that, once we fix a choice \( h = (h_f) \) of a non-singular hyperenvelope \( h_f : \tilde{f} \rightarrow f \) for each arrow \( f : Y \rightarrow Z \) in \( \text{Ar}(\mathcal{P}) \), there is a contravariant functor

\[
T : \text{Ar}(\mathcal{P}) \rightarrow \text{Hot}(\mathcal{M}).
\]
with \( T_h(f) = S(\tilde{f}) \) and \( T_h(g : f_1 \to f_2) = T(g) \). Given two different choices \( h = (h_f) \) and \( h' = (h'_f) \), the maps \( T(1_f) : S(\tilde{f}) \to S(\tilde{f}') \) define an isomorphism of functors \( T_h \cong T_{h'} \). We will therefore suppress the choice \( h \) from the notation and just write \( T \) for \( T_h \) for some fixed \( h \).

The functor \( T \) has the following property. Consider a morphism \( g : f_1 \to f_2 \) in \( \text{Ar}(\mathbf{P}) \), i.e. a commutative square like (2.1). We say that \( g \) is Gersten acyclic when the following property holds: if \( C \) is the complex of varieties in \( \mathbf{P} \)

\[
Y_1 \xrightarrow{(g,Y,f_1)} Y_2 \oplus Z_1 \xrightarrow{f_2,gZ} Z_2,
\]

for any integer \( q \geq 0 \) and any variety \( V \) in \( \mathbf{V} \) the total complex of corresponding Gersten complexes \( R_q(V \times C) \) is acyclic. Now we claim that, when \( g \) is Gersten acyclic, the map \( T(g) : T(f_2) \to T(f_1) \) is an isomorphism in \( \text{Hot}(\mathbf{M}) \). Indeed, if we consider the product of the diagram (2.2) above by the identity map \( 1_V \) on \( V \), all morphisms \( 1_V \times a, 1_V \times h_1, 1_V \times g \) and \( 1_V \times h_2 \) induce quasi-isomorphisms of Gersten complexes, so the same is true for \( 1_V \times b \), hence, by Theorem 1, \( S(b) \) is an isomorphism in \( \text{Hot}(\mathbf{M}) \) and \( T(g) = S(a)^{-1} S(b) \) is also an isomorphism.

2.3. Given a variety \( X \) and a complete variety \( \overline{X} \) containing \( X \) as a Zariski open set, if we write \( j : Y \to \overline{X} \) for the inclusion of the complement of \( X \), we define

\[
W(X) = T(j) = S(\tilde{j}),
\]

for \( \tilde{j} \) any non-singular hyperenvelope of \( j \). We shall prove that \( W(X) \) is, up to canonical isomorphism, independent of choices and contravariant for proper morphisms.

So let \( f : X_1 \to X_2 \) be a proper morphism of varieties, each \( X_i \) being equipped with a compactification \( X_i : \overline{X}_i \) with complement \( j_i : Y_i = \overline{X}_i - X_i \to \overline{X}_i \). Consider the Zariski closure \( \overline{X}_f \) of the graph of \( f \) in \( \overline{X}_1 \times \overline{X}_2 \) and \( j_f : \overline{X}_f - Y_f \to \overline{X}_f \) the inclusion of the complement. The projections \( \overline{X}_1 \times \overline{X}_2 \to \overline{X}_i \) induce maps \( \pi_i : j_f \to j_i \). Since the map \( \overline{X}_f \to \overline{X}_1 \) induces an isomorphism \( \overline{X}_f - Y_f \to \overline{X}_1 - Y_1 \), the morphism \( \pi_1 : j_f \to j_1 \) in \( \text{Ar}(\mathbf{P}) \) is Gersten acyclic. Indeed, for all \( q \geq 0 \) and \( V \) in \( \mathbf{V} \), there is a commutative diagram of Gersten complexes

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & R_{q,*}(V \times Y_f) & \longrightarrow & R_{q,*}(V \times \overline{X}_f) & \longrightarrow & R_{q,*}(V \times (\overline{X}_f - Y_f)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & || \\
0 & \longrightarrow & R_{q,*}(V \times Y_1) & \longrightarrow & R_{q,*}(V \times \overline{X}_1) & \longrightarrow & R_{q,*}(V \times (\overline{X}_1 - Y_1)) & \longrightarrow & 0,
\end{array}
\]

where the exactness of the rows follows easily from the definition of Gersten complexes: the horizontal maps are push-forward for closed immersions and pull-back for open immersions, the exactness degreewise comes from the fact that, given a scheme and a closed subset, any point lies either in the closed subset or in its open complement. It follows that \( T(\pi_1) \) is an isomorphism. We now define

\[
W(f) : W(X_1) = T(\pi_1) \to W(X_2) = T(\pi_2),
\]
to be $T(\pi_1)^{-1} T(\pi_2)$ (note that $W(f)$ depends on the choice of $j_1$ and $j_2$). When $f$ is an isomorphism, $\pi_2 : j_f \to j_2$ is also Gersten acyclic and $W(f)$ is an isomorphism.

To check that $W(f)$ is compatible with composition, consider proper morphisms $f : X_1 \to X_2$ and $g : X_2 \to X_3$, together with compactifications $X_i \to \overline{X}_i$, $i = 1, 2, 3$, of complements $j_i : \overline{X}_i - X_i \to \overline{X}_i$.

Let $\overline{X}_{(f,g)}$ be the Zariski closure of the image of $X_1$ under the map $(1, f, gf)$ into $\overline{X}_1 \times \overline{X}_2 \times \overline{X}_3$ and $j_{(f,g)} : \overline{X}_{(f,g)} - X_1 \to \overline{X}_{(f,g)}$ its complement. Then we have a commutative diagram in $\Ar(\mathbb{P})$

\[
\begin{array}{ccc}
j_{(f,g)} & \xrightarrow{b} & j_g \\
\downarrow a & & \downarrow \pi_2' \\
j_f & \xrightarrow{\pi_2} & j_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
j_1 & \xrightarrow{} & j_3
\end{array}
\]

induced by the obvious projections. All vertical maps in this diagram are Gersten acyclic, hence $T$ turn them into isomorphisms. Furthermore the composite maps $j_{(f,g)} \to j_1$ and $j_{(f,g)} \to j_3$ in this diagram factor though the projection $j_{(f,g)} \to j_{gf}$, which is also Gersten acyclic. Therefore $u_1 : j_{gf} \to j_1$ is Gersten acyclic and we have a map $u_3 : j_{gf} \to j_3$. Now we compute

\[
W(f) W(g) = (T(\pi_1)^{-1} T(\pi_2))(T(\pi_2')^{-1} T(\pi_3))
\]

\[
= T(\pi_1 a)^{-1} T(\pi_3 b) = T(u_1)^{-1} T(u_3) = W(gf)
\]

(from $W(j_3)$ to $W(j_1)$).

Thus, once we fix a choice $j = (j_X)$ of compactifications $X \leftrightarrow \overline{X}$ with complement $j_X : \overline{X} - X \leftrightarrow \overline{X}$ for all varieties $X$ of finite type over $k$, we get a contravariant functor $W_j$ from varieties and proper morphisms to $\Hot(\mathcal{M})$ which maps $X$ to $W_j(X) = T(j_X)$ and $f : X \to Y$ to $W(f)$. If $j' = (j'_X)$ is another choice of compactifications, the maps $W(1_X) : T(j_X) \to T(j'_X)$ define an isomorphism of functors from $W_j$ to $W_{j'}$. In that sense, $X \mapsto W(X)$ is independent of choices and contravariant for proper morphisms.

To check that $W(X)$ is covariant for open immersions, notice that given a compactification $\overline{X}$ of $X$ there is a contravariant equivalence of categories between open subschemes of $X$ and closed subschemes of $\overline{X}$ containing $\overline{X} - X$, mapping $(U \leftrightarrow X)$ to $(\overline{X} - U \leftrightarrow \overline{X})$. Since $W(U) = T(\overline{X} - U \leftrightarrow \overline{X})$ and $T$ is contravariant on $\Ar(\mathbb{P})$, we conclude that any open immersion $i : U \leftrightarrow X$ induces $i_* : W(U) \to W(X)$ and that, given $i : U \leftrightarrow V$ and $j : V \leftrightarrow X$ two open immersions, the identity $(ji)_* = j_* i_*$ holds. This concludes the proof of Theorem 2 ii).
2.3. To prove Theorem 2 iii), let $\bar{X}$ be a compactification of $X$, and write $Y = \bar{X} - X$ and $Z = \bar{X} - U$, so that $T = Z - Y$. Choose non-singular hyper-envelope $\bar{X} \to \bar{X}$, $\bar{Z} \to Z$, and $Y \to Y$ such that there are maps

$$\tilde{Y} \xrightarrow{f} \tilde{Z} \xrightarrow{g} \tilde{X}.$$ 

lifting the inclusions $Y \to Z \to \bar{X}$. These induce maps of complexes of motives:

$$M(\tilde{X}) \xrightarrow{g^*} M(\tilde{Z}) \xrightarrow{f^*} M(\tilde{Y})$$

and a corresponding triangle of mapping cones:

$$C(g^*)[-1] \to C((g \circ f)^*)[-1] \to C(f^*)[-1] \to C(g^*).$$

which by definition may be rewritten:

$$W(U) \to W(X) \to W(T) \to W(U)[1].$$

Property iii) indicates that $W(X)$ behaves like cohomology with compact supports; see also Theorem 3 below.

2.4. To prove Theorem 2 i), let $U \subset X$ be a smooth dense open subset and $T = X - U$ its closed complement. From 2.3 we know that there is a triangle

$$W(T)[-1] \to W(U) \to W(X)$$

in $Hot(M)$. Assume we know that Theorem 2 i) is true for $U$ and $T$, i.e. there exist homotopy equivalences $A_i \to W(T)$ and $B_i \to W(U)$ where $A_i$ and $B_i$ are motives of varieties lying in $V$ such that $\dim(A_i) \leq \dim(T) - i$ and $\dim(B_i) \leq \dim(X) - i$. In the triangulated category $Hot(M)$, $W(X)$ is then isomorphic to the cone $C$ of a map

$$A_i[-1] \to B_i.$$ 

(again by [V] II Cor. 1.). Since $C_i = A_i \oplus B_i$, we have $\dim(C_i) \leq \dim(X) - i$, and hence i) is true for $X$.

By noetherian induction we can therefore assume that $X$ is smooth and quasi-projective. In that case, let $\overline{X}$ be a smooth compactification of $X$ lying in $V$, with complement $T = \overline{X} - X = D^{\text{red}}$, where $D$ is a divisor with normal crossing in $\overline{X}$. Since $W(\overline{X})$ is represented by $M(\overline{X})$ in degree zero, Theorem 2 i) holds for $\overline{X}$. Using 2.3 again, $W(X)$ is the cone of a map

$$W(\overline{X})[-1] \to W(T)[-1]$$

and, since $\dim(T) \leq \dim(X) - 1$, we get i) for $X$ by induction on dimensions. See 2.7 for a more explicit description of $W(X)$. 


2.5. Let us prove Theorem 2 iv) when \( X \) is complete (the general case is left to the reader). First assume that \( X = A \cup B \), where \( A \) and \( B \) are closed in \( X \). From our discussion in 2.2, diagram (2.2), we know that there exists a commutative square

\[
\begin{array}{ccc}
A \cap B & \xrightarrow{v} & \tilde{B} \\
\downarrow & & \downarrow \beta \\
\tilde{A} & \xrightarrow{\alpha} & \tilde{X} \\
\end{array}
\]

of simplicial varieties in \( \mathbf{V} \) mapping by hyperenvelopes to the commutative square

\[
\begin{array}{ccc}
A \cap B & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \rightarrow & X \\
\end{array}
\]

The triangle

\[
W(X) \rightarrow W(A) \oplus W(B) \rightarrow W(A \cap B) \rightarrow W(X)[1]
\]

is a consequence of the fact that the total complex of the bicomplex in \( \mathbf{M} \)

\[
M \left( \tilde{X} \right) \xrightarrow{\alpha^* \oplus \beta^*} M \left( \tilde{A} \right) \oplus M \left( \tilde{B} \right) \xrightarrow{u^* \oplus v^*} M \left( \tilde{A} \cap \tilde{B} \right)
\]

is contractible. But this complex is equal, up to a shift, to the mapping cone of the map of complexes

\[
\varphi : S \left( \tilde{A} \hookrightarrow \tilde{X} \right) \rightarrow S \left( \tilde{A} \cap \tilde{B} \hookrightarrow \tilde{B} \right).
\]

Since \( X - A = B - (A \cap B) \), \( \varphi \) is the canonical homotopy between two representatives of \( W(X - A) \) (see 2.3), and this proves our claim.

Assume now that the complete variety \( X \) is the union of two open subvarieties \( U \) and \( V \). Let \( A = X - U \) and \( B = X - V \). Choose hyperenvelopes \( \tilde{A} \rightarrow \tilde{X} \) and \( \tilde{B} \rightarrow \tilde{X} \) of the inclusions \( A \hookrightarrow X \) and \( B \hookrightarrow X \). Since \( B = U - (U \cap V) \) we know from 2.4 that \( W(U \cap V) \) is represented by the complex of motives

\[
C_1 = C \left( C \left( M \left( \tilde{X} \right) \rightarrow M \left( \tilde{A} \right) \right)[-1] \rightarrow M \left( \tilde{B} \right) \right)[-1]
\]

On the other hand, \( W(U) \oplus W(V) \) is represented by

\[
C_2 = C \left( M \left( \tilde{X} \right) \oplus M \left( \tilde{X} \right) \rightarrow M \left( \tilde{A} \right) \oplus M \left( \tilde{B} \right) \right)[-1],
\]

and \( W(X) \) is represented by \( M \left( \tilde{X} \right) \). The diagonal \( M \left( \tilde{X} \right) \rightarrow M \left( \tilde{X} \right) \oplus M \left( \tilde{X} \right) \) and the difference map \( M \left( \tilde{X} \right) \oplus M \left( \tilde{X} \right) \rightarrow M \left( \tilde{X} \right) \) lead to a complex

\[
C_1 \rightarrow C_2 \rightarrow M \left( \tilde{X} \right)
\]

which is contractible, and this proves the existence of a triangle

\[
W(X)[-1] \rightarrow W(U \cap V) \rightarrow W(U) \oplus W(V) \rightarrow W(X).
\]
The equality $W(X \times Y) = W(X) \otimes W(Y)$ in $Hot(M)$ follows from the fact that the product of two envelopes is an envelope. Indeed, we first assume that $X$ and $Y$ are projective and we let $\tilde{X} \to X$ and $\tilde{Y} \to Y$ be two hyperenvelopes. Since coskeleta commute with products, we deduce from this fact that the product simplicial scheme $(\tilde{X} \times \tilde{Y}) = (n \mapsto \tilde{X}_n \times \tilde{Y}_n)$ is an envelope of $X \times Y$. By the Eilenberg-Zilber theorem ([D-P] 2.9 and 2.16) the associated complex of motives $M((\tilde{X} \times \tilde{Y}))$ is homotopy equivalent to the total complex of the tensor product $M(\tilde{X}) \otimes M(\tilde{Y})$, and the equality $W(X \times Y) = W(X) \otimes W(Y)$ follows.

If we do not assume that $X$ and $Y$ are projective, we let $\bar{X}$ and $\bar{Y}$ be compactifications, $\tilde{X}$, $\tilde{Y}$, $\tilde{S}$, and $\tilde{T}$ be hyperenvelopes of $X$, $Y$, $S = X - X$ and $T = Y - Y$ respectively, with maps $\tilde{S} \to \tilde{X}$ and $\tilde{T} \to \tilde{Y}$ the cones of which represent $W(X)$ and $W(Y)$. Then $\bar{X} \times \bar{Y}$ is a compactification of $X \times Y$, and

$$(\bar{X} \times \bar{Y}) - (X \times Y) = (S \times \bar{Y}) \cup (\bar{X} \times T).$$

By 2.2, diagram (2.2), there is a commutative square of simplicial varieties in $V$

$$
\begin{array}{ccc}
\tilde{S} \times \tilde{T} & \longrightarrow & \bar{X} \times \tilde{T} \\
\downarrow & & \downarrow \\
\tilde{S} \times \bar{Y} & \longrightarrow & \bar{X} \times \bar{Y}.
\end{array}
$$

mapping by hyperenvelopes to the square

$$
\begin{array}{ccc}
S \times T & \longrightarrow & \bar{X} \times T \\
\downarrow & & \downarrow \\
S \times \bar{Y} & \longrightarrow & \bar{X} \times \bar{Y}.
\end{array}
$$

According to the Mayer-Vietoris property iv), we can represent $W((S \times \bar{Y}) \cup (\bar{X} \times T))$ by the cone of the map

$$M(\tilde{S} \times \bar{Y}) \oplus M(\bar{X} \times \tilde{T}) \to M(\tilde{S} \times \tilde{T}),$$

therefore $W(X \times Y)$ is represented by the total complex of the bicomplex

$$M(\bar{X} \times \bar{Y}) \to M(\tilde{S} \times \bar{Y}) \oplus M(\bar{X} \times \tilde{T}) \to M(\tilde{S} \times \tilde{T}).$$

By the previous step and the uniqueness proved in 2.3, this bicomplex of motives is homotopy equivalent to

$$M(\tilde{X}) \otimes M(\tilde{Y}) \to M(\tilde{S}) \otimes M(\tilde{Y}) \oplus M(\tilde{X}) \otimes M(\tilde{T}) \to M(\tilde{S}) \otimes M(\tilde{T}),$$

which is the tensor product of the cone of $M(\tilde{X}) \to M(\tilde{S})$ with the cone of $M(\tilde{Y}) \to M(\tilde{T})$. □
2.7. For future use, we shall give a more precise description of $W(X)$ when $X$ is smooth and equipped with a compactification $\overline{X}$ lying in $\mathcal{V}$. We assume that $T = \overline{X} - X = D^{\text{red}}$, where $D$ is a divisor with normal crossing in $\overline{X}$. Let $Y_1, \ldots, Y_n$ be the irreducible components of $T$. For any subset $I \subset \{1, \ldots, n\}$ we let $Y_I = \cap_{i \in I} Y_i$ and, for any integer $r \geq 1$, we define

$$Y^{(r)} = \coprod_{\text{card}(I)=r} Y_I.$$ 

Also we let $Y^{(0)} = Y_0 = \overline{X}$. Clearly $\dim Y^{(r)} = \dim(X) - r$.

If $1 \leq k \leq r$ we let $\delta_k : Y^{(r)} \to Y^{(r-1)}$ be the disjoint union of the inclusions $Y_I \to Y_J$ where $I$ is the ordered set $\{i_1, \ldots, i_r\}$ and $J = \{i_1, \ldots, \hat{i}_k, \ldots, i_r\}$. In $\mathbb{ZV}$ we define

$$\partial = \sum_{k=1}^{r} (-1)^k \delta_k : Y^{(r)} \to Y^{(r-1)}.$$ 

One checks that $\partial \circ \partial = 0$ (notice that $\partial : Y^{(1)} \to Y^{(0)} = \overline{X}$ factors via $T$).

Proposition 3. The complex

$$M\left(\overline{X}\right) \xrightarrow{\delta^*} M\left(Y^{(1)}\right) \xrightarrow{\delta^*} M\left(Y^{(2)}\right) \xrightarrow{\delta^*} \cdots \xrightarrow{\delta^*} M\left(Y^{(\dim X)}\right)$$

is a representative of $W(X)$.

Proof. Let $\tilde{Y} = \cosk^T_0 \left(Y^{(1)}\right)$ be the coskeleton of $Y^{(1)}$ over $T$. For any $r \geq 0$, $\tilde{Y}_r$ is the disjoint union of the varieties $Y_{\sigma}$ in $\mathcal{V}$, where $\sigma$ runs over all maps $\sigma : \{1, \ldots, r-1\} \to \{1, \ldots, n\}$ and $Y_\sigma = Y_{\im(\sigma)}$. Since the canonical map $\tilde{Y} \to T$ is a non-singular hyperenvelope, $M\left(\tilde{Y}\right)$ is a representative of $W(T)$ and $W(X)$ is represented by $C \left( M\left(\overline{X}\right) \xrightarrow{f} M\left(\tilde{Y}\right) \right)[-1]$, where $f : \tilde{Y} \to \overline{X}$ is defined using the inclusions $Y_\sigma \subset \overline{X}$. Denote by $M$ the complex of motives

$$M\left(Y^{(1)}\right) \xrightarrow{\delta^*} M\left(Y^{(2)}\right) \xrightarrow{\delta^*} \cdots \xrightarrow{\delta^*} M\left(Y^{(\dim X)}\right).$$

We shall prove Proposition 3 by exhibiting a homotopy equivalence $\varphi$ from $M\left(\tilde{Y}\right)$ to $M$. Given any subset $I \subset \{1, \ldots, n\}$ of cardinality $r \geq 1$, let $\sigma_I : \{1, \ldots, r\} \to \{1, \ldots, n\}$ be the unique map of ordered sets with image $I$. The disjoint union of the identity maps $Y_I \to Y_{\sigma_I}$, $\text{card}(I) = r$, defines an inclusion $Y^{(r)} \to \tilde{Y}_{r+1}$ and a morphism of complexes $\varphi : M\left(\tilde{Y}\right) \to M$. Now $M\left(\tilde{Y}\right)$ is a representative of $W(T)$ and $M\left(Y^{(r)}\right)$ represents $W\left(Y^{(r)}\right)$. The Mayer-Vietorvis property iv) in Theorem 2 and induction on the number of components of $T$ prove that $\varphi$ is a homotopy equivalence. $\square$
3. SOME MOTIVIC INVARIANTS OF VARIETIES

Let $k$ be a field of characteristic zero. In this paragraph, given a variety $X$ over $k$, we shall describe several invariants of $X$ which depend only of the associated weight complex $W(X)$.

3.1 Weights.

3.1.1. Let $\sim$ be any adequate equivalence relation on algebraic cycles, $M_\sim$ be the associated category of motives (see 1.3), and $\Gamma : M_\sim \rightarrow A$ a covariant (resp. contravariant) functor from $M_\sim$ to an abelian category $A$. If $X$ is any variety over $k$, we may consider the image $W(X)_\sim$ of $W(X)$ in $M_\sim$ and the associated complex $\Gamma(W(X)_\sim)$ in $A$. For any integer $i \geq 0$, we define

$$R^i\Gamma(X) \in \text{Ob}(A) \quad \text{(resp. } L_i\Gamma(X) \in \text{Ob}(A)),$$

where $W(X)_\sim$ is the $i$-th cohomology (resp. homology) of $\Gamma(W(X)_\sim)$.

From Theorem 2 we conclude that $R^i\Gamma(X)$ is well defined, contravariant in $X$, and equal to zero if $i > \dim(X)$. Furthermore, when $X$ lies in $V$, $R^0\Gamma(X) = \Gamma(M(X))$ and $R^i\Gamma(X) = 0$ if $i > 0$. When $T \subset X$ is a closed subvariety with complement $U = X - T$, there is a long exact sequence

$$\cdots \rightarrow R^i\Gamma(U) \rightarrow R^i\Gamma(X) \rightarrow R^i\Gamma(T) \rightarrow R^{i+1}\Gamma(U) \rightarrow \cdots$$

Similar properties hold for $L_iF$.

3.1.2. A basic example comes when $k = \mathbb{C}$, $\sim$ is homological equivalence, and $\Gamma(X,p) = p_* H^n(X(\mathbb{C}), A)$ is the singular (= Betti) cohomology of the motive $M = (X,p)$ with constant coefficients in a given ring $A$, for a fixed value of the integer $n \geq 0$, correspondences acting on cohomology in the usual way. Clearly $\Gamma$ defines a contravariant functor $H^n$ from $M_\sim$ to the category of finitely generated $A$-modules.

Choose a compactification $\overline{X}$ of $X$, let $j : Y = \overline{X} - X \rightarrow \overline{X}$ be its complement and let $\tilde{j} : \tilde{Y} \rightarrow \tilde{X}$ be a non-singular hyperenvelope of $j$. By definition $W(X)$ is represented by $S(\tilde{j})$. Therefore

$$R^iH^n(X) = H^i \left( * \mapsto H^n(S(\tilde{j})^*, A) \right).$$

Since envelopes are proper and surjective they define morphisms “de descente cohomologique universelle” by [D], 5.3.5 (II). Therefore, the hypercohomology of $S(\tilde{j})$ is the relative cohomology $H^*(\overline{X}, \overline{Y}, A) = H^*_e(X(\mathbb{C}), A)$, and $R^iH^n(X)$ coincides with the term $E_2^{i,n}$ of the corresponding descent spectral sequence, which is thus independent of choices.

When $A = \mathbb{Q}$, this weight spectral sequence degenerates at $E_2$ (op.cit. Proposition (8.1.20)) and we get (loc.cit.)

$$H^i \left( * \mapsto H^n(S(\tilde{j})^*, \mathbb{Q}) \right) = E_2^{i,n} = E_\infty^{i,n} = \text{gr}_n W_H^{i+n}(X(\mathbb{C}), \mathbb{Q}).$$

In other words, using Theorem 2, we have the following result
**Theorem 3.** The cohomological descent spectral sequence

\[ E_2^{i,n} \Rightarrow H_c^{i+n}(X(\mathbb{C}), A) \]

is independent of choices when it comes from hyperenvelopes as above. It defines a canonical increasing weight filtration \( F^W_n \) on the cohomology with compact support of \( X(\mathbb{C}) \) with constant coefficients in the ring \( A \). This filtration has length at most \( \dim(X) + 1 \). It is compatible with products, pull-back by proper maps, and push-forward by open immersions.

When \( A = \mathbb{Q} \), the filtration coincides with the weight filtration defined by Deligne in [D] and

\[ R^i H^n(X) \otimes \mathbb{Q} = \text{gr}_n^W H_c^{i+n}(X(\mathbb{C}), \mathbb{Q}), \]

where \( \text{gr}_n^W \) is the subquotient of weight \( n \).

When \( X \) is projective and \( k > 0 \), \( F^W_{k-1} H_c^k(X(\mathbb{C}), A) \) is the kernel of the map

\[ \pi^* : H_c^k(X(\mathbb{C}), A) \to H_c^k(X'(\mathbb{C}), A) \]

for any resolution of singularities \( \pi : X' \to X \). That this kernel is independent of choices was first observed by Grothendieck [Gr2].

Theorem 5.13 in [G-N] leads similarly to a canonical weight filtration on the cohomology without support and with arbitrary coefficients.

### 3.1.3

In order to check that the weight filtration on \( H_c^k(X(\mathbb{C}), \mathbb{Z}) \) is non-trivial and cannot be defined in a simple way from the cohomology with rational coefficients, let us consider the following example.

Let \( T \) be an abelian surface, and let \( i : T \to T \) be the involution defined by \( i(x) = -x \). The quotient surface \( S = T/ \langle 1, i \rangle \) is projective with sixteen ordinary double points \( \{p_1, ..., p_{16}\} \) of type \( A_1 \). On resolving these singularities, we obtain the Kummer surface \( \tilde{S} \) associated to \( T \), which is a \( K_3 \)-surface. If \( \pi : \tilde{S} \to S \) is the resolution, the sixteen exceptional curves \( W_i = \pi^{-1}(p_i) \) are all isomorphic to \( \mathbb{P}^1 \). We may view \( S \) as the pushout in the following diagram

\[
\begin{array}{c}
\coprod E_i \\
\downarrow \\
\coprod p_i \\
\end{array} \longrightarrow 
\begin{array}{c}
\tilde{S} \\
\downarrow \\
S.
\end{array}
\]

In this case one may check that the weight spectral sequence of Theorem 3 is simply the corresponding Mayer-Vietoris exact sequence:

\[ \cdots \to H^n(S, \mathbb{Z}) \to H^n(\tilde{S}, \mathbb{Z}) \oplus H^n(\coprod p_i, \mathbb{Z}) \to H^n(\coprod E_i, \mathbb{Z}) \to H^{n+1}(S, \mathbb{Z}) \to \cdots \]

In particular we find that the sequence

\[ \cdots \to H^2(S, \mathbb{Z}) \to H^2(\tilde{S}, \mathbb{Z}) \oplus \mathbb{Z} \to H^3(S, \mathbb{Z}) \to \cdots \]
is exact, where $W = \{W_1, ..., W_6\}$, so that
\[ H^3(S, \mathbb{Z}) = F_2^W H^3(S, \mathbb{Z}) = Gr_2^W H^3(S, \mathbb{Z}) \]
while $H^1(S, \mathbb{Z}) = 0$.

In Proposition 5.5 of Chapter VIII of [B-P-V], it is shown that the sublattice of $H_2(S, \mathbb{Z})$ generated by the classes of the divisors $W_i$ is of index 32 in the smallest primitive sublattice containing it, with the quotient being 2-torsion; dualizing we find
\[ Gr_2^W H^3(S, \mathbb{Z}) = H^3(S, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^5. \]

On the other hand, if $X$ is an Enriques surface (op. cit. Chapter VIII, Section 15), $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$, and so by Poincaré duality $H^3(X, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $H^1(X, \mathbb{Z}) = 0$. Since $X$ is smooth and projective
\[ Gr_3^W H^3(X, \mathbb{Z}) = H^3(X, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}. \]

Taking the cartesian product $S \times X$ we obtain a four-fold with
\[ H^3(S \times X, \mathbb{Z}) = H^3(S, \mathbb{Z}) \oplus H^3(X, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^6 \]
(since $H^1(S, \mathbb{Z}) = H^1(X, \mathbb{Z}) = 0$ and $H^2(S, \mathbb{Z})$ is torsion free), and
\[ Gr_2^W H^3(S \times X, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \]
\[ Gr_3^W H^3(S \times X, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^5. \]

3.1.4. When $M = (X, \pi)$ is a Chow motive and $p \geq 0$ an integer, we may consider the Chow cohomology (resp. homology) group $\pi_* CH^p(X)$ (resp. $\pi_* CH_p(X)$).

**Proposition 4.** When $X$ is a (possibly singular) complete variety over $k$, the group $L_0 CH_p(X)$ coincides with $CH_p(X)$ and $R^0 CH^p(X)$ is the operational Chow group $A^p(X \xrightarrow{id} X)$ (see [F]).

**Proof.** This follows from the descent properties of Chow homology and from a result of Kimura [Ki], see [B-G-S] Appendix. \(\square\)

More generally, if $Z$ is a non-singular hyperenvelope of a projective variety $X$, for any $p \geq 0$ there is a canonical weight spectral sequence
\[ E_2^{st}(X) = H^s (n \mapsto H^t(Z_n, K_p)) \Rightarrow H^{s+t}(Z, K_p) \]
converging to the hypercohomology of the simplicial scheme $Z$ with coefficients in the Zariski sheaf $K_p$. Up to canonical isomorphism, this spectral sequence is independent of the choice of $Z$. From $E_2$ on (indeed, a map of hyperenvelopes induces a morphism of spectral sequences and $E_2^{st}(X)$ depends only on $W(X)$, so we can argue as in 2.2). From Proposition 3 and the Gersten conjecture [Q] we conclude that
\[ R^0 CH^p(X) = E_2^{0,p}(X). \]
3.1.5. Assume $X$ is a smooth variety and that $\overline{X}$ is a compactification of $X$ lying in $V$, with complement $T = D_{\text{red}}$, where $D$ is a divisor with normal crossing in $\overline{X}$ as in 2.7. From Proposition 3 it follows that $R^iCH^p(X)$ is the $i$-th cohomology of the complex

\[
(3.1) \quad CH^p(\overline{X}) \overset{\partial^*}{\to} CH^p(Y^{(1)}) \overset{\partial^*}{\to} CH^p(Y^{(2)}) \overset{\partial^*}{\to} \cdots
\]

when $L_iCH_p(X)$ is the $i$-th homology of

\[
\cdots \overset{\partial^*}{\to} \overset{\partial^*}{\to} \overset{\partial^*}{\to} CH_p(Y^{(2)}) \overset{\partial^*}{\to} \overset{\partial^*}{\to} CH_p(Y^{(1)}) \overset{\partial^*}{\to} CH_p(\overline{X}),
\]

where $\partial^* = \Sigma (-1)^k \delta^*_k$ and $\partial_* = \Sigma (-1)^k \delta^*_k$ (with the same notation as in 2.7). So these groups are independent of the choice of the compactification $\overline{X}$.

Let $i : Y^{(1)} \to \overline{X}$ be the disjoint union of the inclusions $Y_i \to \overline{X}, 1 \leq i \leq n$, $CH^p(D) = A^p \left(D \overset{\text{Id}}{\to} D\right)$ the operational Chow group of $D$, $d = \dim(X)$ and

\[
i^*i_* : CH_{d-p}(Y^{(1)}) \to CH^p(Y^{(1)})
\]

the composite of the maps

\[
CH_{d-p}(Y^{(1)}) \overset{i^*}{\to} CH_{d-p}(\overline{X}) = CH^p(\overline{X}) \overset{i_*}{\to} CH^p(Y^{(1)}).
\]

It follows from the result above that the cohomology of the $\mathbb{Z}$-graded complex

\[
\cdots \to CH_{d-p}(Y^{(2)}) \overset{\partial^*}{\to} CH_{d-p}(Y^{(1)}) \overset{i^*i_*}{\to} CH^p(Y^{(1)}) \to CH^p(Y^{(2)}) \to \cdots
\]

is independent of choices. In particular we recover Theorem 2.2.1 of [B-G-S]. If we follow the analogy of that paper with Kähler geometry, we may view the complex above as analogous to the following one, defined for any smooth projective complex manifold $M$:

\[
\cdots \overset{d}{\to} A^{p-3,p-1}(M) \oplus A^{p-2,p-2}(M) \oplus A^{p-1,p-3}(M) \overset{d}{\to} A^{p-2,p-1}(M) \oplus A^{p-1,p-2}(M) \overset{d}{\to} A^{p-1,p-1}(M) \overset{dd^c}{\to} A^{p,p}(M) \overset{d}{\to} A^{p+1,p}(M) \oplus A^{p,p+1}(M) \overset{d}{\to} \cdots,
\]

where $A^{pq}(M)$ are complex forms of type $(p,q)$. This complex is known to compute the Deligne cohomology groups $H_D^p(M, \mathbb{R}(p))$ when appropriate reality conditions are imposed on forms (see [B] Theorem 1.10). We conclude from this that $R^iCH^p(X)$ is analogous to $H_D^{2p+i-1}(M, \mathbb{R}(p))$ and $L_iCH_{d-p}(X)$ is analogous to $H_D^{2p-i}(M, \mathbb{R}(p))$ when $i \geq 1$. 
3.1.6. Assume now that $\sim$ is numerical equivalence. It was shown by Jannsen [J] that $M_{\text{num}}$ is an abelian semi-simple category. If we apply the discussion of 3.1.1 to the identity functor, we get canonical motives $W^i(X)$ attached to $X$, namely the cohomology groups of $W(X)_{\text{num}}$. These satisfy the properties of 3.1.1 as well as the Künneth formula

$$W^i(X \times Y) = \bigoplus_{j+k=i} W^j(X) \otimes W^k(Y).$$

If one knew that numerical equivalence implies homological equivalence, for any $n \geq 0$, the group $\text{gr}^W_i H_{\text{c}}^{n+i}(X(\mathbb{C}), \mathbb{Q})$ would depend only on $W^i(X)$ (when the ground field is $\mathbb{C}$).

3.2. The Grothendieck group of motives.

3.2.1. Let $\sim$ be any adequate equivalence relation on algebraic cycles and $M_\sim$ the associated category of motives. The Grothendieck group of this category is the quotient $K_0(M_\sim)$ of the free abelian group on the isomorphism classes $[M]$ of objects $M$ in $M$ by the subgroup generated by elements of the form $[M] - [M'] - [M'']$ whenever $M \sim M' \oplus M''$. On the other hand, if $\text{Hot}^b(M_\sim)$ denotes the category of bounded cochain complexes in $M_\sim$ up to homotopy, we may consider its Grothendieck group $K_0(\text{Hot}^b(M_\sim))$, which is generated by objects in $\text{Hot}^b(M_\sim)$ with the relation $[Y'] = [X'] + [Z']$ whenever there exists a triangle

$$X' \to Y' \to Z' \to X'[1].$$

**Lemma 3.** The obvious functor $M_\sim \to \text{Hot}^b(M_\sim)$ induces a group isomorphism $K_0(M_\sim) \to K_0(\text{Hot}^b(M_\sim))$.

**Proof.** This fact is true for any pseudo-abelian category instead of $M_\sim$. Indeed, when $M \sim M' \oplus M''$ there is a triangle

$$M' \to M \to M'' \to M'[1],$$

[V] Cor. 1.2.3, so we get a morphism $\phi : K_0(M_\sim) \to K_0(\text{Hot}^b(M_\sim))$.

Given a bounded complex $M.$ in $M_\sim$ we let

$$\chi(M') = \sum_i (-1)^i [M^i] \in K_0(M_\sim).$$

When $f : M' \to N'$ is a morphism of complexes and $C(f)$ is its mapping cone, we have

$$\chi(C(f)) = \chi(N') - \chi(M').$$

So to prove that $\chi$ induces a morphism from $K_0(\text{Hot}^b(M_\sim))$ to $K_0(M_\sim)$ all we need to check is that $\chi(M') = 0$ when $M'$ is contractible. For this we proceed by induction on the length $k$ of $M'$. Let $h : M^{i+1} \to M^i$ be such that

$$dh + hd = id_M.$$

It follows that $(hd)^2 = hd$. Let $A^{k-1}$ be the image of the projector $hd$ in $M^{k-1}$ and $B^{k-1}$ its complement. The map $d : M^{k-1} \to M^k$ is zero on $B^{k-1}$ and induces an isomorphism $B^{k-1} \to A^{k-1}$, and

$$\text{gr}^W_i H_{\text{c}}^{n+i}(X(\mathbb{C}), \mathbb{Q}) \cong H_{\text{c}}^{n+i}(X(\mathbb{C}), \mathbb{Q}) [1].$$

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an isomorphism from $A^{k-1}$ to $M^k$. Let $N^\cdot$ be the complex obtained from $M^\cdot$ by replacing $M^{k-1}$ by $B^{k-1}$ and $M^k$ by zero. We get
$$\chi(M^\cdot) = \chi(N^\cdot) + [A^{k-1}] - [X^k] = \chi(N^\cdot).$$
Since $N^\cdot$ is contractible, this proves that $\chi(M^\cdot) = 0$ by induction. It is now easy to check that $\phi$ and $\chi$ are isomorphisms inverse to each other. $\square$

Using this lemma and Theorem 2 we get

**Theorem 4.** Any quasi-projective variety $X$ has a class $[X] \in K_0(M_\sim)$ characterized by the following properties:

i) If $X$ lies in $V$, $[X]$ is the class of the motive $(X, 1_X)$;

ii) If $Y \subset X$ is a closed subvariety in $X$,

$$[X] = [Y] + [X - Y].$$

**Proof.** Define $[X]$ to be the class of $W(X)$ in $K_0(\text{Hot}^b(M_\sim)) = K_0(M_\sim)$. Then property i) is clear and ii) follows from Theorem 2 iii). To see that $[X]$ is uniquely characterized by i) and ii) we proceed by induction on dimensions. If $U \subset X$ is a smooth dense open subset and $\overline{U}$ a smooth compactification of $U$ lying in $V$ (which exist by resolution of singularities), we get

$$[X] = [U] + [X - U] = [\overline{U}] - [\overline{U} - U] + [X - U],$$

which fixes $[X]$ uniquely. $\square$

Theorem 4 answers positively a question of Serre, [Se] p.341.

3.2.2.

**Proposition 5.** The class $[X] \in K_0(M_\sim)$ has the following properties:

i) If $U$ and $V$ are two locally closed subvarieties in $X$, then

$$[U \cap V] + [U \cup V] = [U] + [V];$$

ii) If $f : X \to B$ is a fibration of fiber $F$ which is locally trivial for the Zariski topology of $B$, then

$$[X] = [F] \cdot [B].$$

**Proof.** For i) notice that $V - (U \cap V) = (U \cup V) - V$, therefore, if $U$ and $V$ are open or if $U$ and $V$ are closed, we deduce from ii) in Theorem 4 (or from Theorem 2 iv)) that

$$[V] - [U \cap V] = [V - (U \cap V)] = [U \cup V] - [V].$$

The general case follows from these two.

To prove ii), cover $B$ by a finite collection of open subsets $U_\alpha, \alpha \in A$, such that $f$ is trivial over $U_\alpha$. Using i) and induction on the cardinality of $A$ one is reduced to proving ii) when $f$ is trivial, i.e.

$$[X \times Y] = [X] \cdot [Y] \text{ in } K_0(M_\sim).$$

Such an equality follows from Theorem 2 v) or can be deduced directly from the case where $X$ and $Y$ are smooth and projective by induction on $\dim(X) + \dim(Y)$, as in the proof of Theorem 4 above. $\square$
3.2.3. It follows from Theorem 4 that the class of the affine line in $K_0(M_\sim)$ is class of the Tate motive $L$:

$$[\mathbb{A}^1] = [\mathbb{P}^1] - [1] = [L].$$

If $X$ is the affine cone with base a variety $Y$ lying in $\mathbb{V}$, then, as in [Se],

$$[X] = [1] + [Y] \otimes [L] - [Y].$$

Much more elaborate cases can be found in the recent paper by Manin [M2].

3.2.4. Theorem 4 is related to the theory of mixed motives as follows. In [Vo], Voevodsky associates to any perfect field $k$ a triangulated category $\text{DM}^\text{eff}_{gm}$ of “effective geometrical motives over $k$”. When $\text{char}(k) = 0$, any variety $X$ of finite type over $k$ has classes $M(X)^c$ and $M(X)$ in $\text{DM}^\text{eff}_{gm}$. Their properties are listed in [Vo] 2.2. In particular, $M(X) = M(X)^c$ when $X$ is smooth and proper over $k$, and the restriction of $M$ to $\mathbb{V}$ can be factorized through an additive functor from the category of Chow motives $\mathbf{M}$ to $\text{DM}^\text{eff}_{gm}$. Therefore we get a group morphism

$$\varphi : K_0(\mathbf{M}) \rightarrow K_0 \left( \text{DM}^\text{eff}_{gm} \right),$$

from $K_0(\mathbf{M})$ to the Grothendieck group of Voevodsky’s triangulated category.

For any $X$ of finite type over $k$, $\varphi([X])$ is the class of $M(X)^c$. Indeed, this is true by definition when $X$ lies in $\mathbb{V}$, and the general case follows using Theorem 4 ii) and Property 2 of $M(X)^c$ in [Vo] 2.2.

It seems a hard problem to decide whether $\varphi$ is a group isomorphism or not.

3.3. Numerical invariants.

3.3.1. From Theorem 4 it follows that any additive map $h : \text{Ob}(\mathbf{M}_\sim) \rightarrow A$, where $A$ is an abelian group defines for each variety $X$ over $k$ a class $h(X) \in A$. This class has the property that $h(X) = h(Y) + h(X - Y)$ when $Y$ is closed in $X$.

For instance, if $k = \mathbb{C}$, $\sim$ is homological equivalence, $A = \mathbb{Z}$, $n \geq 0$ is a fixed integer and

$$h(M) = \dim_{\mathbb{Q}} p_* H^n_c(X(\mathbb{C}), \mathbb{Q})$$

if $M = (X, p)$, we deduce from Theorem 3 that, for an arbitrary variety $X$, $h(X) \in \mathbb{Z}$ is the “$n$-th virtual Betti number”

$$h^n(X) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} gr^W_n H^{n+i}_c(X(\mathbb{C}), \mathbb{Q}) \in \mathbb{Z}.$$ 

A much finer invariant than the virtual Betti number, which includes torsion data, may be found by taking $A$ to be the Grothendieck group of all finitely generated abelian groups with respect to direct sum. Thus:

$$A = \mathbb{Z} \oplus \bigoplus Y.$$
where \( p \) runs over all prime integers and \( Y \) is the free abelian group on the set \( \mathbb{N} \) of natural numbers. That is, \( Y \) is the group of functions \( \phi : \mathbb{N} \to \mathbb{Z} \) such that \( \phi(n) = 0 \) for all but finitely many \( n \).

If \( G \) is an abelian group the corresponding class \( h(G) \in A \) is

\[
\text{rk}(G) \oplus \bigoplus_p \phi_p
\]

where

\[
G \simeq \mathbb{Z}^{\text{rk}G} \oplus \bigoplus_p (\mathbb{Z}/p^n)^{\phi_p(n)}
\]

Thus from the class \( h(G) \) we recover the isomorphism class of \( G \). Now we may take, for \( X \) a smooth projective variety, \( h(X) \) to be the class of \( H^n(X, \mathbb{Z}) \) in this class group. The corresponding invariant of singular varieties seems to be new, and combines information about the torsion in integral cohomology with the weight structure. From this invariant one can deduce several numerical invariants. For example the invariant obtained by taking \( \text{card}(\text{tors}(H^n(X, \mathbb{Z}))) \) for \( X \) smooth and projective, which was first suggested by Totaro.

Given \( p, q \geq 0 \) two integers, we may also consider algebraic De Rham cohomology and, for any motive \( M = (X, p) \), the integer

\[
h(M) = \dim_k p_* H^p(X, \Omega^q),
\]

where correspondences act via their De Rham fundamental class. The corresponding invariant \( h^{p,q}(X) \in \mathbb{Z} \) (for \( X \) any variety over \( k \)) is mentioned by Grothendieck in [Gr3] p.191. In terms of the mixed Hodge structure on cohomology [D] one gets, if \( k \subset \mathbb{C} \) say,

\[
h^{p,q}(X) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} \text{gr}^q_i \text{gr}^W \text{gr}^{p+q+i} \text{H}^{p+q+i}_c(X(\mathbb{C}), \mathbb{C}).
\]

It follows from the axiomatic definition of these numbers that, for every \( X \), one has

\[
h^n(X) = \sum_{p+q=n} h^{p,q}(X)
\]

and \( h^{p,q}(X) = h^{q,p}(X) \).

**3.3.2.** If \( \sim \) is numerical equivalence and \( k = \mathbb{C} \) the usual Euler characteristic

\[
\chi(X) = \sum_{n \geq 0} (-1)^n \dim H^n(X(\mathbb{C}), \mathbb{Q})
\]

extends to all motives by the formula

\[
\chi(M) = \Delta \cdot p
\]

where \( M = (X, p) \), \( \Delta \in \text{End}_\mathbb{C}(X) \) is the diagonal and \( \Delta \cdot p \) denotes its intersection number with \( p \) in the Chow ring of \( X \times X \). From Theorem 3 and 3.1.5 we get

\[
\chi(W^i X) = \sum_{n \geq 0} (-1)^n \dim \text{gr}_i^W H^{n+i}_c(X(\mathbb{C}), \mathbb{Q})
\]

for any complex variety \( X \).
4. Blow ups

4.1. For any integer \( m \geq 0 \) and any noetherian (separated) scheme \( S \), denote by \( K_m(S) \) the higher \( K \)-group of perfect complexes on \( S \) (\cite{T-T} 3.1) and by \( K'_m(S) \) the higher \( K \)-group of coherent \( \mathcal{O}_S \)-modules on \( S \). If \( S \) is noetherian and regular and if \( X/S \) is any scheme of finite over \( S \) we denote by \( A_p(X/S) \) the homological Chow group of cycles of relative dimension \( p \) on \( X \) and by \( A^p(X/S) \) the cohomological (operational) Chow group, as in \cite{F}, Appendix.

Now let \( X \) be a noetherian scheme, \( i : Y \to X \) a closed immersion. Consider the blow up \( f : X' \to X \) of \( X \) along \( Y \), and let \( Y' \) be the inverse image of \( Y \) so that we get a cartesian square of proper maps

\[
\begin{array}{ccc}
Y' & \xrightarrow{j} & X' \\
g \downarrow & & \downarrow f \\
Y & \xrightarrow{i} & X
\end{array}
\]

Theorem 5.

i) Assume \( i \) is regular. Then, for any \( m \geq 0 \), the following sequence is exact:

\[
0 \to K_m(X) \xrightarrow{(i^*, -f^*)} K_m(Y) \oplus K_m(X') \xrightarrow{g^* + j^*} K_m(Y') \to 0.
\]

ii) Assume \( i \) is regular and there exists an ample line bundle on \( X \). Then, for any \( m \geq 0 \), the following sequence is exact

\[
0 \to K'_m(Y') \xrightarrow{(g^*, j_*)} K'_m(Y) \oplus K'_m(X') \xrightarrow{i^* - f^*} K'_m(X) \to 0.
\]

iii) Assume that \( X \) is of finite type over a regular noetherian scheme \( S \) and that \( i \) is regular. Then, for any \( p \geq 0 \), the following sequences are exact:

\[
0 \to A_p(Y'/S) \xrightarrow{(g_*, j_*)} A_p(Y/S) \oplus A_p(X'/S) \xrightarrow{i^* - f^*} A_p(X/S) \to 0
\]

and

\[
0 \to A^p(X'/S) \xrightarrow{(i^*, -f^*)} A^p(Y'/S) \oplus A^p(X'/S) \xrightarrow{g^* + j^*} A^p(Y'/S) \to 0.
\]

iv) Assume that \( S = \text{Spec}(k) \) where \( k \) is any field of characteristic zero, and that \( X \) is smooth and projective over \( k \). Then, for any closed immersion \( i \) (regular or not), the sequences (4.3) and (4.4) are exact.

4.2. In order to prove Theorem 5, we first remark that in cases i), ii) and iii) the closed immersion \( i : Y \to X \) is regular, therefore the same is true for \( j : Y' \to X' \), and the maps \( g \) and \( f \) are locally complete intersection morphisms. Consequently all the maps \( i, j, f \) and \( g \) are perfect morphisms and in particular of finite Tor-dimensions. It follows that together with the morphisms \( i^*, j^*, f^*, g^* \) on \( K_m \) and \( A^p \) there are push-forward morphisms \( i_*, j_*, f_*, g_* \) (which do not preserve the degree), and similarly \( K'_m \) and \( A^p \) are both covariant and contravariant for
these maps (for $K'_m$ this uses the fact that $X$ has an ample line bundle). If $N$ is the normal sheaf of $Y$ in $X$, $Y'$ is the projective bundle $\mathbf{P}(N)$ (in the sense of Grothendieck). Let $F = \ker(g^*N \to \mathcal{O}(1))$ and $\Lambda^iF$ the exterior powers of this locally free sheaf on $Y'$.

To prove Theorem 5 i) we first consider the map $g^!: K_m(Y) \to K_m(Y')$ defined by $g^!(x) = \lambda_{-1}(F) \; g^*(x)$. Since $f_*f^*(x) = f^*f_*(1) = x$ for any $x \in K_m(X)$ ([Gr1] VII Lemme 3.5), the morphism $f^*$ is injective, and since $g^*g^!(x) = g^*(\lambda_{-1}(F)) = x$ ([Gr1] VI.5.9 or (6) below), the exactness of (4.1) is equivalent to saying that $j^*$ induces an isomorphism

$$\frac{K_m(X')}{f^* K_m(X)} \to \frac{K_m(Y')}{g^* K_m(Y)}.$$ 

According to [T2] (2.2.1) the following sequence is exact

$$0 \to K_m(Y) \xrightarrow{(g^!, -i^*)} K_m(Y') \oplus K_m(X) \xrightarrow{j^* + f^*} K_m(X') \to 0.$$ 

Therefore $j_*: K_m(Y') \to K_m(X')$ induces an isomorphism

$$\frac{K_m(Y')}{g^! K_m(Y)} \to \frac{K_m(X')}{f^* K_m(X)}$$

and all we need to show is that $j^* j_*$ induces an isomorphism

$$(4.5) \quad \frac{K_m(Y')}{g^! K_m(Y)} \to \frac{K_m(Y')}{g^* K_m(Y)}.$$ 

Now $\mathcal{O}(1)$ is the conormal bundle of $Y'$ in $X'$, and the composite $j^* j_*$ is the product by the element $\xi = 1 - L \in K_0(Y')$, with $L = [\mathcal{O}(-1)]$ being the class of the dual of $\mathcal{O}(1)$ ([T2] (3.1.4)). Any element $x$ in $K_m(Y')$ can be written uniquely as a sum

$$x = \sum_{i=0}^{d-1} g^*(y_i) \; L^i$$

where $y_i \in K_m(Y)$ and $d$ is the rank of $N$, since $Y' = \mathbf{P}(N)$. It follows that $x$ can be written uniquely as

$$x = \sum_{i=0}^{d-1} g^*(z_i) \; \xi^i$$

with $z_i \in K_m(Y)$. Therefore $x - g^*(z_0)$ is a multiple of $\xi$, and the map (4.5) is surjective.

To prove that (4.5) is injective we first get from the definition of $F$ the formulae

$$[g^* \Lambda^i N] = [\Lambda^i F] + [\Lambda^{i-1} F] \; [\mathcal{O}(1)] \quad \text{for } i \geq 1.$$
in $K_0(Y')$. By induction on $i$ it follows that

$$[\Lambda^i F] = (-1)^i [O(1)]^i + \sum_{j<i} \alpha_j [O(1)]^i$$

with $\alpha_j \in g^*K_0(Y)$ and

$$\lambda_{-1}(F) = \xi^{d-1} + \sum_{i=0}^{d-1} \beta_i \xi^i$$

with $\beta_i \in g^*K_0(Y)$. Therefore any element $x$ in $K_0(Y')$ can be written uniquely as

$$x = \lambda_{-1}(F) g^*(y) + \sum_{i=0}^{d-2} g^*(y_i) \xi^i$$

with $y, y_i \in K_m(Y)$. If we assume that $\xi x = g^*(z)$ we get

$$\sum_{i=0}^{d-2} g^*(y_i) \xi^{i+1} + g^*(y \lambda_{-1}(N) - z) = 0$$

(since $\xi \lambda_{-1}(F) = g^* \lambda_{-1}(N)$), therefore $y_i = 0, 0 \leq i \leq d-2$ and $x = \lambda_{-1}(F) g^*(y)$. This proves that (4.5) is injective.

4.3. To prove Theorem 5 ii) we first notice that the open complement $U = X' - Y'$ is isomorphic to $X - Y$. From [Q] §7 Proposition 3.2 and Proposition 2.11 we get a commutative diagram

$$\begin{array}{cccccc}
K'_{m+1}(U) & \longrightarrow & K'_m(Y') & \longrightarrow & K'_m(U) & \longrightarrow & K'_{m-1}(Y') \\
\downarrow & & \downarrow j^* & & \downarrow f_* & & \downarrow \\
K'_{m+1}(U) & \longrightarrow & K'_m(Y) & \longrightarrow & K'_m(U) & \longrightarrow & K'_{m-1}(Y)
\end{array}$$

Therefore the sequence (4.2) is exact in the middle and all we need to show is that $f_* : K'_m(X') \to K'_m(X)$ is surjective. This is where we use that $X$ has an ample line bundle, since we get then a morphism $f^* : K'_m(X) \to K'_m(X')$ such that $f_* f^*(x) = x f_*(1)$ ( [Q] Proposition 2.10). Since $f_*(1) = 1$ in $K_0(X)$ ( [Gr1] VII Lemme 3.5), this concludes the proof.

4.4. We prove Theorem 5 iii) by using the fact that the groups $A_p(X/S)$ and $A^p(X/S)$ satisfy all the properties stated in [F] when $S = \text{Spec}(k)$, except those involving external products ( [F] 20.1).

To simplify the notations, we write $A_p(X), A_p(Y), A^p(X) \ldots$ instead of $A_p(X/S), A_p(Y/S), A^p(X/S) \ldots$.

We first prove that (4.3) is exact. Let $c_{d-1}(F) \in A^{d-1}(Y')$ be the top Chern class of $F$ and $g^* : A_p(Y) \to A_p(Y')$ be the map sending $x$ to $g^*(x) c_{d-1}(F)$. According to [F] Proposition 6.7 we have $f_* f^* = \text{id}$, and, by [F] Example 3.2.3, $g_* g^* = \text{id}$. Therefore
Therefore $f_*$ and $g_*$ are both surjective and the exactness of (4.3) is equivalent to the fact that

$$\ker(g_* : A_p(Y') \to A_p(Y)) \xrightarrow{j^*} \ker(f_* : A_p(X') \to A_p(X))$$

is an isomorphism, hence to the fact that

$$\text{coker}(A_p(Y) \xrightarrow{g^!} A_p(Y')) \xrightarrow{j^*} \text{coker}(A_p(X) \xrightarrow{f^*} A_p(X'))$$

is an isomorphism. But this follows from the usual exact sequence computing $A_p(X')$, [F], Proposition 6.7 c).

To prove that (4.4) is exact we define once more $g^! : A_p(Y) \to A_p(Y')$ by the formula $g^!(x) = g^*(x) c_{d-1}(F)$, and we deduce from the equalities $f_* f^* = \text{id}$ ([F], Proposition 17.5 a)) and $g_* g^! = \text{id}$ ([F] Example 3.3.3) and from the exact sequence

$$0 \to A^{p-d}(Y) \to A^{p-1}(Y') \oplus A^{p}(X) \to A^{p}(X') \to 0$$

([F] Example 17.5.1 c)) that we just need to show that $j^* j_*$ induces an isomorphism

$$(4.7) \quad \frac{A^{p-1}(Y')}{g^! A^{p-d}(Y)} \to \frac{A^{p}(Y')}{g^* A^{p}(Y)}.$$

Now $j^* j_*$ is the product by $\xi = c_1(O(1))$ ([F] Proposition 17.4.1) and any element $x \in A^{p}(Y')$ can be written uniquely as

$$x = \sum_{i=0}^{d-1} g^*(y_i) \xi^i,$$

[F] Example 17.5.1 b), hence, also uniquely, as

$$x = \sum_{i=0}^{d-2} g^*(z_i) \xi^i + g^!(z),$$

since $c_{d-1}(F) = \xi^{d-1} + \sum_{i=0}^{d-2} g^*(\alpha_i) \xi^i$.

This implies that (4.7) is an isomorphism as in the proof of i).

**4.5.** To prove Theorem 5 iv) we let $U = X' - Y' = X - Y$ and we consider the obvious morphism of exact sequences from

$$0 \to R^0CH^p(U) \to R^0CH^p(X) \to R^0CH^p(Y) \to R^1CH^p(U) \to R^1CH^p(X)$$

to

$$0 \to R^0CH^p(U) \to R^0CH^p(X') \to R^0CH^p(Y') \to R^1CH^p(U) \to R^1CH^p(X')$$

(see 3.1.1). Since $X$ is smooth and projective we have $R^1CH^p(X) = 0$ (loc. cit.), therefore, by diagram chase one concludes that (4.4) is exact. The same argument using $L_iCH_p$ instead of $R^iCH^p$ proves that (4.3) is exact. This ends the proof of Theorem 5.
4.6. By applying Theorem 5, we can analyse the effect of a blow up on the complex (3.1) considered in section 3.1.5.

Indeed, let $S$ be a regular noetherian scheme, $X$ a regular scheme of finite type over $S$, $D \subset X$ a relative Cartier divisor with normal crossings, and $Y_1, \ldots, Y_n$ the components of the associated reduced scheme $D^\text{red}$. As in 2.4 and 3.1.4 we may define a cochain complex of abelian groups

$$ C^\bullet(X) : A^p(X/S) \xrightarrow{\partial^*} A^p(Y^{(1)}/S) \xrightarrow{\partial^*} A^p(Y^{(2)}/S) \to \ldots $$

Now let $W \subset D^\text{red}$ be a regular irreducible closed subscheme meeting the components $Y_i$ normally, in the sense of [H], and let $f : X' \to X$ be the blow up of $X$ along $W$. The components of $f^*(D)^\text{red}$ are the proper transforms $Y'_i$ of $Y_i$, $1 \leq i \leq n$, together with the exceptional divisor $Y'_0 = f^{-1}(Y)$. We define from these a complex $C^\bullet(X')$ by the same method as $C^\bullet(X)$ (see 3.1.4).

**Proposition 6.** The complexes $C^\bullet(X)$ and $C^\bullet(X')$ are quasi-isomorphic.

**Sketch of proof.** We first define a new complex $B^\bullet$ as follows. If $I \subset \{0, \ldots, n\}$ we let

$$ \Omega_I = Y_I \quad \text{when} \quad 0 \notin I $$

and

$$ \Omega_I = W \cap Y_J = W_J \quad \text{if} \quad I = \{0\} \cup J. $$

Define

$$ \Omega^{(r)} = \coprod_{\text{card} I = r} \Omega_I $$

and $B^r = A^p(\Omega^{(r)}/S)$. The differential $\partial^*$ on $B^\bullet$ are defined as for $C^\bullet(X)$ from the inclusions $\Omega_I \subset \Omega_J$ if $J \subset I$.

There is a natural projection $B^\bullet \to C^\bullet(X)$. Its kernel $K^\bullet$ is acyclic. Indeed we have $K^0 = 0$ and

$$ K^r = \bigoplus_{\text{card} J = r-1} A^p(W_J/S) $$

when $r > 0$. Let $P \subset \{1, \ldots, n\}$ be the set of indices $i$ such that $Y_i$ does not contain $W$. For any $I \subset P$ we get a subcomplex $K^r_I$ of $K^r$ by considering only those $A^p(W_J/S)$'s such that $J \cap P = I$. Since $W_J = W_I$ for all such $J$, this complex $K^r_I$ is acyclic. Using that $K^r = \bigoplus_{I \subset P} K^r_I$ for all $r \geq 1$, we conclude that $K^\bullet$ itself is acyclic by applying the following lemma, the proof of which is left to the reader:

**Lemma 4.** Let $A$ be a finite ordered set, let $C^\bullet$ and $C^\bullet_\alpha, \alpha \in A$, be bounded cochain complexes of abelian groups, and let

$$ f : \bigoplus_{\alpha \in A} C^r_\alpha \to C^r $$
be group isomorphisms such that, for all $x \in C^r_\alpha$, $(f_*, d - df_*)(x)$ lies in $\oplus_{\alpha < \beta} C^{r+1}_\beta$.

Assume that, for all $\alpha \in A$, the complex $C^*_\alpha$ is acyclic. Then $C^*$ is acyclic.

On the other hand one checks that for any non empty subset $I \subset \{1, \ldots, n\}$ the intersection $Y'_I = \cap_{i \in I} Y'_i$ is the blow up of $Y_I$ along $W_I = W \cap Y_I$, and that $Y'_0 \cap Y'_I$ is the exceptional divisor of that blow up (this uses the fact that $W$ meets $D^{\text{red}}$ normally). In particular $f : X' \to X$ maps $Y'_I$ onto $\Omega_I$ for any $I \subset \{0, \ldots, n\}$ and induces a map of complexes

$$f^* : B^* \to C^*(X').$$

One can prove that this is a quasi-isomorphism by considering its cone $C(f^*)$: and the subcomplexes

$$0 \to A^p(Y_I/S) \to A^p(W_I/S) \oplus A^p(Y'_I/S) \to A^p(Y_0 \cap Y'_I/S) \to 0$$

of $C(f^*)$, for all subsets $I \subset \{1, \ldots, n\}$. We know from Theorem 5 iii) that these complexes are acyclic, and, by applying Lemma 4 again, we conclude that $f^*$ is a quasi-isomorphism.

4.7. One can use Proposition 6 to get another proof of the results in 3.1.4. We know from [H] that when $S = \text{Spec}(k)$ and $\text{char}(k) = 0$, given two smooth compactifications $X$ and $X'$ of a quasi-projective variety $U$ over $k$, such that $X - U$ and $X' - U$ are divisors with normal crossing, there exists a third one $X''$ and maps $X'' \to X$, $X'' \to X'$ which are the identity on $U$, one of them being the composite of blow ups of the kind considered in Proposition 6. This can be used to show that, up to quasi-isomorphism, $C_\ast(X)$ depends only on $U$.

To come back to motives, Theorem 5 implies that, given a closed immersion $i : Y \to X$ of smooth proper varieties over a field $k$, if $X'$ is the blow up of $X$ along $Y$ and $Y'$ its exceptional divisor as in 4.1, the sequence of motives

$$(4.8) \quad 0 \to M(X) \xrightarrow{(i^*, -f^*)} M(Y) \oplus M(X') \xrightarrow{g^* + j^*} M(Y') \to 0$$

is contractible. This follows from Theorem 5 iii) by Manin’s identity principle, as stated in [Sc] 2.3 ii).

This fact is the starting point of the alternative construction of $W(X)$ due to Guillen and Navarro [G-N], (5.4), my means of cubic hyperresolutions [G-N-P-P].

5. $K$-Theory

5.1 Preliminaries.

5.1.1. We start by discussing the construction and some of the properties of $K$-theory. Recall that given an exact category $\mathbf{E}$ in the sense of [Q], we can associate to it a category $Q\mathbf{E}$, such that the classifying space $BQ\mathbf{E}$. Here by “space” we mean fibrant simplicial set, and for a category $\mathbf{C}$, $BC$ denotes the result of applying Kan’s $\text{Ex}^\infty$ functor to the nerve $N.\mathbf{C}$ of $\mathbf{C}$. We shall assume that the zero objects of all exact categories that we deal with are unique. (If necessary, given an arbitrary


exact category $\mathcal{E}$, we may form an equivalent exact category $\tilde{\mathcal{E}}$ with a unique zero object by replacing the subcategory of zero objects of $\mathcal{E}$ by a single zero object."

The classifying space $BQ\mathcal{E}$ is therefore canonically pointed. We furthermore assume that this space is the zeroth space of a spectrum $K(\mathcal{E})$, in the sense of Appendix A.1, with the following properties:

- $\mathcal{E} \mapsto K(\mathcal{E})$ is strictly (i.e. not just up to homotopy) functorial;
- Given exact categories $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$, and a biexact functor $\mu: \mathcal{E} \times \mathcal{F} \to \mathcal{G}$ there is a canonical, functorial, pairing:
  $$K(\mu): K(\mathcal{E}) \land K(\mathcal{F}) \to K(\mathcal{G}).$$

For example we can use the multiple $Q$-construction; see [G1]. In particular, as in [T1] Appendix A, we shall assume that these spectra are fibrant, and that they are cofibrant as prespectra (i.e. the map of the suspension of the $i$-th space to the $(i+1)$-st space is injective). The K-theory groups are then defined as the homotopy groups of these spectra:

$$K_m(\mathcal{E}) = \pi_{m+1}(K(\mathcal{E})),\]$$

and are functorial with respect to exact functors between exact categories. They are also compatible with products in the sense that, given a biexact functor

$$\mu: \mathcal{E} \times \mathcal{F} \to \mathcal{G}$$

as above, we get a functorial pairing of graded groups

$$K_s(\mu): K_m(\mathcal{E}) \otimes K_n(\mathcal{F}) \to K_{m+n}(\mathcal{G}).$$

5.1.2. Given a (noetherian) scheme $X$, we can consider two exact categories: the category $\mathcal{M}(X)$ of coherent sheaves of $\mathcal{O}_X$-modules and the sub-category $\mathcal{P}(X)$ of $\mathcal{M}(X)$ consisting of locally free modules. We then obtain the groups:

$$K_m(X) = K_m(\mathcal{P}(X))$$

and

$$K'_m(X) = K_m(\mathcal{M}(X)).$$

These groups are functorial with respect to pull-back and proper push-forward respectively. However the K-theory spectra themselves are not strictly functorial, but rather are functorial only up to homotopy, and similarly the projection formula is also only true up to homotopy; this is because the underlying functors from the category of varieties to the category of categories are “lax” rather than “strict”. To remedy to this we must rigidify the underlying category valued functors. The standard constructions for rigidifying lax functors are due to Street, [S]. We now give a description of Street’s (second) construction in the case of locally free sheaves. We then describe, following Thomason [T-T], an intrinsically rigid construction of $K'$-theory; we then modify this construction in order to make the projection formula true exactly rather than up to a natural isomorphism.

Given a scheme we let $\mathcal{P}^{\text{Big}}(X)$ be the category of locally free sheaves in the big Zariski site over $X$. An object in $\mathcal{P}^{\text{Big}}(X)$ consists of a locally free sheaf $\mathcal{F}_f$ on $Y$ for each map of schemes $f: Y \to X$, and of an isomorphism $g^*(\mathcal{F}_{f_2}) \to \mathcal{F}_{f_1}$ for each morphism $g: (f_1: Y_1 \to X) \to (f_2: Y_2 \to X)$ in the category of schemes over $X$ (i.e. a map $g: Y_1 \to Y_2$ such that $f_2 \circ g = f_1$), with the obvious compatibility with respect to composition. We omit the proof of the following proposition, since it is straightforward.
Lemma 5. The forgetful functor from $P^{\text{Big}}(X)$ to the category of locally free sheaves on (the small Zariski site of) $X$ is an equivalence of categories.

If $f : X \to Y$ is a map of schemes, then we have a restriction functor from $f^* : P^{\text{Big}}(Y) \to P^{\text{Big}}(X)$ which takes a family $(g : Z \to Y) \to F_g$ to the family $(h : Z \to X) \to F_{h \circ f}$. Clearly if $g : V \to X$ is another map, then $(fg)^* = g^*f^*$ is an equality of functors, not just a natural equivalence. Under the previous equivalence of categories this functor is compatible with the usual pull-back map on vector bundles. Therefore if we define $K(X)$ to be the spectrum $K(P^{\text{Big}}(X))$ we obtain a (strictly) contravariant functor from schemes to spectra such that

$$K_m(X) = \pi_{m+1}K(P^{\text{Big}}(X))$$

for all $m \geq 0$.

5.1.3. In order to make $K'$ covariant functorial, we use the construction of Thomason [T-T]. Given a scheme $X$ let $C^b(X)$ be the category of complexes of flasque quasi-coherent sheaves of $O_X$-modules, having cohomology that is coherent and bounded. Taking the category $w$ of quasi-isomorphisms of sheaves to be the weak equivalences, and the standard notion of exact sequence, the pair $(C^b(X), w)$ is a category with cofibrations and weak equivalences. The following is due to Thomason, op. cit:

Proposition 7.

i) $K_*(C^b(X), w) \simeq K'(X)$

ii) If $f : X \to Y$ is a proper morphism of schemes, $A \mapsto f_!A$ is an exact functor, preserving weak equivalences, $f_* : (C^b(X), w) \to (C^b(Y), w)$. Furthermore, if $g : Y \to Z$, then we have an identity (not just a natural equivalence) of functors $g_*f_* = (gf)_*$

To make the projection formula itself, and not just its constituent functors, an identity, we must still rigidify further. We use a construction similar, but not identical, to Street’s first construction in [S]. Let $\hat{C}(X)$ denote the category with objects pairs $(f : Y \to X, A^i \in C^b(Y)) = (f, A^i)$ and morphisms

$$\theta \in \text{Hom}_{\hat{C}(X)}((f, A^i), (g, B^i))$$

given simply by maps $\theta^i : f_*A^i \to g_*B^i$ of complexes of sheaves on $X$. We say that $\theta$ is a weak equivalence if $\theta^i$ is a quasi-isomorphism. Similarly we say that $\theta$ is a cofibration if $\theta^i$ is a monomorphism equal to the kernel of a map of complexes in $C^b(X)$.

Lemma 6. The obvious inclusion functor $j : C^b(X) \to \hat{C}(X)$ is an equivalence of categories with cofibrations and weak equivalences. In particular, it induces a homotopy equivalence of $K$-theory spectra.

Proof. Clearly the inclusion functor preserves cofibrations and weak equivalences. Now consider the functor $p : \hat{C}^b(X) \to C(X)$ given by

$$p : (f, A^i) \mapsto f_!A^i$$
on objects, and by the identity on Hom-sets. The composition \( p \circ j \) is clearly the identity. In the other direction, the composition \( j \circ p \) is isomorphic to the identity functor on \( \check{C}^b(X) \), mapping \((1_X, f_\ast A)\) to \((f, A)\).

\[\square\]

Now, given a map of schemes \( g : X \to Y \), we have an exact functor \( g_\ast : \check{C}^b(X) \to \check{C}^b(Y) \), given on objects by:

\[g_\ast : (f, A) \mapsto (g \circ f, A)\]

and on morphisms by the natural action of \( g_\ast \) on the underlying Hom-sets of complexes on \( X \). One can check that, via the equivalence of categories of Lemma 6, this functor is compatible with the usual direct image.

The usual cap-product: \( K_m(X) \otimes K'_n(X) \to K'_{m+n}(X) \) is usually viewed as being induced by the bi-exact functor

\[\otimes_{\mathcal{O}_X} : \mathcal{P}(X) \times \mathcal{M}(X) \to \mathcal{M}(X).\]

Given a proper morphism \( f : X \to Y \), we have the projection formula, for \( \alpha \in K_\ast(Y) \), and \( \beta \in K'_\ast(X) \):

\[\alpha \cap f_\ast(\beta) = f_\ast(f^\ast(\alpha)) \cap \beta\]

This formula is usually derived from the isomorphism of functors:

\[\mathcal{F} \otimes_{\mathcal{O}_Y} f_\ast \mathcal{M} \simeq f_\ast(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M})\]

Now we can describe the cap-product as follows: there is a biexact functor

\[\mu : \mathcal{P}^{\text{Big}}(X) \times \mathcal{C}^b(X) \to \mathcal{C}^b(X)\]

\[\mu : ((g : Z \to X) \to \mathcal{F}_g) \times (f : Y \to X, \mathcal{A}) \mapsto (f : Y \to X, \mathcal{F}_f \otimes \mathcal{A}).\]

On morphisms this acts as follows. A morphism of locally free sheaves in the big Zariski site \((f \mapsto \phi_f) : (f \mapsto \mathcal{F}_f) \to (f \mapsto \mathcal{G}_f)\) induces the morphism

\[(\phi_f \otimes 1) : (f : Y \to X, \mathcal{F}_f \otimes \mathcal{A}) \to (f : Y \to X, \mathcal{G}_f \otimes \mathcal{A})\]

in \( \mathcal{C}^b(X) \). A morphism in the category \( \mathcal{C}^b(X) \)

\[\theta : (f, \mathcal{A}) \to (g, \mathcal{B})\]

i.e. a morphism of complexes of sheaves on \( X \)

\[\theta' : f_\ast \mathcal{A} \to g_\ast \mathcal{B}\]

gives a map

\[(1, \theta') : ((h \mapsto \mathcal{F}_h), (f, \mathcal{A})) \to ((h \mapsto \mathcal{F}_h), (g, \mathcal{B}))\]

in the product category \( \mathcal{P}^{\text{Big}}(X) \times \mathcal{C}^b(X) \) and its image by the functor \( \mu \) is the unique map:

\[f_\ast(\mathcal{F}_f \otimes \mathcal{A}) \to g_\ast(\mathcal{F}_g \otimes \mathcal{B})\]
which makes the diagram

\[
\begin{array}{ccc}
F_{id_X} \otimes f_*A & \longrightarrow & F_{id_X} \otimes g_*B \\
\downarrow & & \downarrow \\
F_{id_X} \otimes f_*A & \longrightarrow & F_{id_X} \otimes g_*B
\end{array}
\]

commutative, where the vertical maps are the isomorphisms induced by the projection formula at the level of modules.

Given a map of varieties \( h : X \to Y \), an object \( A = (g \mapsto F_g \in P^{big}(Y)) \) and an object \( B = (f : Z \to Y, A) \) in \( C^b(Y) \) we find that:

\[
h_* (\mu (h^* A, B)) = h_* (f : Z \to Y, F_{hf} \otimes A) = (h \cdot f : Z \to Y, F_{hf} \otimes A) = \mu (A, h_* B)
\]

One may also check compatibility for morphisms.

To summarize, we have functors \( K \) and \( K' \) from the category of projective varieties to \( \text{Spectra} \), the first contravariant, the second covariant, together with products which satisfy the projection formula exactly rather than up to homotopy. For example, if \( X : \Delta^{op} \to \text{Pr} \) is a simplicial projective variety, we get from \( X \) by the construction above a simplicial spectrum \( K' : \Delta^{op} \to \text{Spectra} \) and we can define the \( K' \)-prespectrum of \( X \) as being the corresponding homotopy colimit:

\[
K'(X.) = \text{hocolim}_{\Delta^{op}} K'(X_p).
\]

Its homotopy groups \( K'_m(X.) = \pi_{m+1} K'(X.) \) are the abutment of a first-quadrant convergent spectral sequence

\[
E^2_{pq} = H_p (\ast \mapsto K'_q (X_*)) \Rightarrow K'_{p+q}(X.)
\]

([B-K] XII 5.7 and [T1] Proposition 5.7).

When dealing with the \( K \)-theory of simplicial schemes one gets a similar definition by replacing homotopy colimits with homotopy limits. However the associated spectral sequences need not be convergent in general (see [B-K] XII 7 or [T1] 5.44). The consideration of \( K_0 \)-motives in the next paragraph will help us to solve this difficulty in Section 5.3.

5.1.4 Remark. When \( X \) is smooth \( K_m (X) = K'_m (X) \) for all \( m \geq 0 \) [Q]. In [G1] Lemma 4.5 it is asserted that in general any element in \( K_m (X) \) is the inverse image of an element in \( K_m (M) \), where \( M \) is a smooth variety. This is used in [So2] 6.2 to define operations on the \( K \)-theory of singular varieties. However the proof of [G1] Lemma 4.5 is incorrect since the compatibility statements (c) in loc.cit., p.247, are not enough to describe an arbitrary diagram in \( QP(X) \).

5.2 \( K_0 \)-motives.

5.2.1. We shall now give analogs of the results in Section 1 for \( K \)-theory instead of Chow groups. From now on \( V \) will denote the category of smooth projective varieties over a fixed field \( k \).
First remark that one gets a theory of motives by replacing Chow groups by $K_0$ in the definition of correspondences (see also [M1]). Namely, let $\mathbf{KC}$ be the category with the same objects as $\mathbf{V}$, and with morphisms

$$\text{Hom}_{\mathbf{KC}}(X, Y) = K_0(X \times Y).$$

The composition law is defined as in $\text{Hom}_{\mathbf{C}}(X, Y)$ (see 1.2) and there is a covariant functor $\mathbf{V} \to \mathbf{KC}$ mapping a morphism $f : X \to Y$ to the class $[O_f] \in K_0(X \times Y)$ of the structure sheaf of the graph of $f$. Let $\mathbf{KM}$ be the associated category of motives, defined as in 1.3; we call them $K_0$-motives.

Notice that for all $m \geq 0$, the functor $K_m$ from $\mathbf{V}$ to abelian groups can be factored through $\mathbf{KC}$, and hence $\mathbf{KM}$, both as a covariant and a contravariant functor.

We obtain results similar to those in Section 1 by replacing the Gersten complexes $R_q, q \geq 0$, with the $K'$-theory spectrum. However, since the functor $X \mapsto K'(X)$ does not factor through $\mathbb{Z}\mathbf{V}$ we need to modify some of the arguments. Instead of complexes in $\mathbb{Z}\mathbf{V}$ we must work with simplicial schemes. The following will play the role of Theorem 1:

**Theorem 6.** Let

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
S & \longrightarrow & T
\end{array}
\]

be a commutative square of maps between simplicial objects in $\mathbf{V}$. Suppose that, for all varieties $V$ in $\mathbf{V}$, the associated square of spectra

\[
\begin{array}{ccc}
K'(V \times X_i) & \longrightarrow & K'(V \times Y_i) \\
\downarrow & & \downarrow \\
K'(V \times S_i) & \longrightarrow & K'(V \times T_i)
\end{array}
\]

(5.1)

is homotopy cartesian. Then the associated square of complexes of $K_0$-motives

\[
\begin{array}{ccc}
KM(S_i) & \leftarrow & KM(T_i) \\
\downarrow & & \downarrow \\
KM(X_i) & \leftarrow & KM(Y_i)
\end{array}
\]

is homotopy cartesian (i.e. the associated total complex is contractible).

**Proof.** The square (5.1) is homotopy cartesian if and only if the homotopy colimit of the diagram of simplicial spectra

\[
\begin{array}{ccc}
K'(i \mapsto V \times X_i) & \longrightarrow & K'(i \mapsto V \times Y_i) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
K'(i \mapsto V \times S_i) & \longrightarrow & K'(i \mapsto V \times T_i)
\end{array}
\]

is homotopy cartesian.\n
is contractible. By [B-K] XII 3.3., this iterated homotopy colimit is isomorphic to the homotopy colimit of the associated $I \times \Delta^{\text{op}}$-diagram of spectra, where $I$ is the finite category

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet \\
\end{array}
\]

If $A.$ denotes this diagram, there is an associated spectral sequence

\[
E^2_{pq} = H_p(I \times \Delta^{\text{op}}, \pi_q(A.)) \Rightarrow \pi_{p+q} \hocolim_{I \times \Delta^{\text{op}}}(A.)
\]  
([B-K] XII 5.7 and [T1] Proposition 5.17). Given any functor $\Phi$ from $I \times \Delta^{\text{op}}$ to abelian groups which vanishes on $* \times \Delta^{\text{op}}$, the homology groups $H_p(I \times \Delta^{\text{op}}, \Phi)$ are those of the homotopy push-out of the diagram of chain complexes

\[
\begin{array}{ccc}
\Phi(a,.) & \rightarrow & \Phi(b,.) \\
\downarrow & & \downarrow \\
\Phi(c,.) & \rightarrow & \Phi(d,.) \\
\downarrow & & \\
0 & & \\
\end{array}
\]

i.e. the total complex of the square

\[
\begin{array}{ccc}
\Phi(a,.) & \rightarrow & \Phi(b,.) \\
\downarrow & & \downarrow \\
\Phi(c,.) & \rightarrow & \Phi(d,.) \\
\end{array}
\]

Thus the groups $E^2_{pq}$, $p \geq 0$, are the homology groups of the complex $K_q(V \times C.)$, where $C.$ is the total complex in $\mathbb{Z}V$ associated to the commutative square of simplicial varieties

\[
\begin{array}{ccc}
X. & \rightarrow & Y. \\
\downarrow & & \downarrow \\
S. & \rightarrow & T. \\
\end{array}
\]

It follows that, for all $V$, we have a spectral sequence

\[
E^2_{pq} = H_p(K_q(V \times C_*)) \Rightarrow 0.
\]

As in the proof of Theorem 1, we may now prove by induction on $n \geq 1$ that $C.$ is contractible as a complex of $K_0$-motives in degrees less than $n$, i.e. that there exist $K_0$-correspondences $h_i \in \text{Hom}_{\mathbf{KC}}(C_i, C_{i+1})$, such that

\[
h_{i-1} \circ \delta_i + \delta_{i+1} \circ h_i = 1_{C_i}, \quad 0 \leq i \leq n - 1.
\]

Since $K_q$ factors via $\mathbf{KC}$ we get that $E^2_{pq} = 0$ for all $q \geq 0$ and $p \leq n - 1$. The end of the proof is then parallel to that of Theorem 1. $\Box$
5.2.2. Using Theorem 6 and the descent Theorem 4.1 in [G2] for $K'$-theory, we can associate to any variety $X$ over a field $k$ of characteristic zero a cochain complex of $K_0$-motives $KW(X)$ in the homotopy category $\text{Hot}(\text{KM})$, which is well-defined up to canonical isomorphism and enjoys the same properties as $W(X)$ in Theorem 2 above. If $\overline{X}$ is a compactification of $X$ and $\tilde{j} : \overline{Y} \to \overline{X}$, a non-singular hyperenvelope of the inclusion $j : \overline{X} - X \to \overline{X}$, $KW(X)$ is represented by the complex

$$C \left( KM \left( \overline{X} \right) \right) \to KM \left( \overline{Y} \right) \left[ -1 \right] \text{ in } \text{Hot}(\text{KM}).$$

The proof of the properties of $KW(X)$ is the same as in Theorem 2. For instance let $\tilde{j} \to \tilde{j}'$ be a map of non-singular hyperenvelopes of $j$. To check that these define the same complex $KW(X)$ up to homotopy equivalence, notice that for any $V$ in $\mathbf{V}$ the associated square of $K'$-theory spectra $K' \left( 1_V \times \tilde{j} \right) \to K' \left( 1_V \times j \right)$, $K' \left( 1_V \times \tilde{j}' \right) \to K' \left( 1_V \times j \right)$ and hence $K' \left( 1_V \times \tilde{j}' \right) \to K' \left( 1_V \times \tilde{j} \right)$ are homotopy cartesian. Therefore, by Theorem 6, the associated square of complexes of $K_0$-motives $KM \left( \tilde{j} \right) \to KM \left( \tilde{j}' \right)$ is homotopy cartesian, as desired.

Similarly, the same proof as in 2.4 tells us that $KW(X)$ is represented by a bounded complex in $\text{KM}$, of length at most $\dim(X) + 1$.

5.3 $K$-theory with compact support.

5.3.1. We are now able to define the $K$-theory with compact support of any variety over a field $k$ of characteristic zero.

First consider the case of a complete variety $X$ over $k$. Let $\overline{X} \to X$ be a non-singular hyperenvelope of $X$. By applying the contravariant $K$-theory functor $K : \mathbf{V} \to \text{Spectra}$ we get a cosimplicial spectrum Following [T1], section 5.6, we can form the spectrum $K(X.) := \text{holim}_n K(\overline{X}_n)$. By op. cit. Proposition 5.13, the spectral sequences of Bousfield and Kan in the unstable case ([B-K] IX 5 and XII 7.1) give rise to an unfringed spectral sequence abutting to $\pi_*(K(X.))$

We claim that this spectral sequence is strongly convergent. Indeed its $E_2$-term is

$$E_2^{pq} = H^p \left( \ast \mapsto \pi_{-q} K \left( \overline{X}_* \right) \right) = H^p \left( \ast \mapsto K_{-q} \left( \overline{X}_* \right) \right),$$

and we know from the last section that the complex of $K_0$-motives $KM \left( \overline{X} \right)$ is homotopy equivalent to a bounded complex of length at most $\dim(X) + 1$. Therefore $E_2^{pq} = 0$ when $p \geq \dim(X) + 1$, and, for all $r \geq 2$, $E_r^{pq} = 0$ unless $q + i \leq 0$, $0 \leq p \leq \dim(X)$, and $p + q \leq 0$. The spectral sequence therefore converges strongly to $\pi_{-p-q} \text{holim}_n K \left( \overline{X}_n \right)$ (see [B-K] loc. cit. or [T1] Lemma 5.48 i)).

In general, we get a functor from simplicial schemes to the category of spectra

$$X. \mapsto K(X.) := \text{holim}_n K(X_n).$$

We then define $K_*(X.) := \pi_*(K(X.))$. Note that for $\overline{X}$ a hyperenvelope of a variety $X$, the spectral sequence above will be concentrated in the strip $0 \leq p \leq \dim(X)$ and that therefore $K_m \left( \overline{X} \right)$ will in general be non-zero for negative values of $m$, $m > -\dim(X)$. 


Now let $X$ be an arbitrary variety over $k$, $X \subset \overline{X}$ a compactification, $j : Y = \overline{X} - X \hookrightarrow \overline{X}$ its complement, and $\tilde{j} : \tilde{Y} \rightarrow \tilde{X}$ a non-singular hyperenvelope of $j$. Consider the homotopy fiber $\mathbf{K}(\tilde{j})$ of the map of spectra $\tilde{j}^* : \mathbf{K}(\tilde{X}) \rightarrow \mathbf{K}(\tilde{Y})$. We view it as the homotopy limit of the diagram

$$
\begin{array}{ccc}
\ast & \rightarrow & \mathbf{K}(\tilde{Y}) \\
\downarrow & & \downarrow \\
\mathbf{K}(\tilde{X}) & \leftarrow & \mathbf{K}(\tilde{X}),
\end{array}
$$

which is isomorphic by [B-K] XI 4.3. to the homotopy limit of the corresponding $I \times \Delta$ diagram of spectra, where $I$ is the small category

$$
\begin{array}{ccc}
\ast & \rightarrow & a \\
\downarrow & & \downarrow \\
b & \leftarrow & b.
\end{array}
$$

Therefore we get a convergent spectral sequence

$$
H^p \left( \ast \mapsto C \left( K_{-q} \left( \tilde{X} \right) \rightarrow K_{-q} \left( \tilde{Y} \right) \right)^{*-1} \right) \Rightarrow K_{-p-q} \left( \tilde{j} \right),
$$

where $K_m \left( \tilde{j} \right) = \pi_m \mathbf{K} \left( \tilde{j} \right)$.

The spectrum $\mathbf{K} \left( \tilde{j} \right)$ is our definition of the $K$-theory with compact support of $X$. To see that is independent of choices up to canonical homotopy equivalence, consider once more two compactifications $\overline{X}_1$ and $\overline{X}_2$ of $X$, $j_1 : \overline{X}_1 - X \rightarrow \overline{X}_1$ and $j_2 : \overline{X}_2 - X \rightarrow \overline{X}_2$ their complements, $\pi : \overline{X}_1 \rightarrow \overline{X}_2$ a morphism which is the identity on $X$, and $\tilde{\pi} : \tilde{j}_1 \rightarrow \tilde{j}_2$ a map of non-singular hyperenvelopes of $j_1$ and $j_2$, compatible with the morphism $\pi : j_1 \rightarrow j_2$ in $\text{Ar}(\mathbf{P})$ in the obvious way (compare 2.2 and 2.3 above). There is then a map of spectral sequences from

$$
H^p \left( \ast \mapsto C \left( K_{-q} \left( \tilde{X}_2 \right) \rightarrow K_{-q} \left( \tilde{Y}_2 \right) \right)^{*-1} \right) \Rightarrow K_{-p-q} \left( \tilde{j}_2 \right)
$$

to

$$
H^p \left( \ast \mapsto C \left( K_{-q} \left( \tilde{X}_1 \right) \rightarrow K_{-q} \left( \tilde{Y}_1 \right) \right)^{*-1} \right) \Rightarrow K_{-p-q} \left( \tilde{j}_1 \right)
$$

induced by $\tilde{\pi}$. Since the map of complexes of motives

$$
C \left( KM \left( \tilde{X}_2 \right) \rightarrow KM \left( \tilde{Y}_2 \right) \right) \Rightarrow C \left( KM \left( \tilde{X}_1 \right) \rightarrow KM \left( \tilde{Y}_1 \right) \right)
$$

is a homotopy equivalence, the map of $E_2$-terms in the above spectral sequences is an isomorphism and therefore $\tilde{\pi}^* : \mathbf{K} \left( \tilde{j}_2 \right) \rightarrow \mathbf{K} \left( \tilde{j}_1 \right)$ is a homotopy equivalence.

As in Section 2.3 we can then show that, given any proper map $f : X_1 \rightarrow X_2$ of varieties, together with compactifications $X_1 \hookrightarrow \overline{X}_1 \rightarrow \overline{X}_1$ and $X_2 \hookrightarrow \overline{X}_2 \rightarrow \overline{X}_2$, and
non-singular hyperenvelopes \( \pi_i : \tilde{j}_i \to j_i \), there is a canonical map in the homotopy category of spectra

\[
\mathbf{K}(f) : \mathbf{K}(\tilde{j}_2) \to \mathbf{K}(\tilde{j}_1),
\]

which is a homotopy equivalence when \( f \) is an isomorphism. If \( f : X_1 \to X_2 \) and \( g : X_2 \to X_3 \) are two such maps, and if we choose compactifications and hyperenvelopes \( \tilde{j}_i, i = 1, 2, 3 \), for all three varieties, then

\[
\mathbf{K}(gf) = \mathbf{K}(f) \mathbf{K}(g) : \mathbf{K}(\tilde{j}_3) \to \mathbf{K}(\tilde{j}_1).
\]

It follows that, if we choose a compactification and a non-singular hyperenvelope \( \tilde{j}_X \) of its complement for every variety \( X \), we obtain a contravariant functor \( \mathbf{K}^c \) from varieties to the homotopy category of spectra (i.e. the stable homotopy category) by sending \( X \) to \( \mathbf{K}(\tilde{j}_X) \) and any proper morphism \( f \) to \( \mathbf{K}(f) \). Two families of choices give rise to canonically isomorphic functors.

The properties of this functor \( \mathbf{K}^c \) are summarized in the following theorem:

**Theorem 7.** To each variety \( X \) over a field \( k \) of characteristic zero is associated a spectrum \( \mathbf{K}^c(X) \), which is well defined up to canonical homotopy equivalence and enjoys the following properties:

i) If \( X \) is complete and non-singular, \( \mathbf{K}^c(X) \) is the usual Quillen \( K \)-theory of vector bundles.

ii) Any proper map \( f : X \to X' \) of varieties induces a pull-back map of spectra \( f^* : \mathbf{K}^c(X') \to \mathbf{K}^c(X) \). Given two composable proper maps \( f \) and \( g \), then \( (fg)^* = g^*f^* \).

iii) Any immersion \( i : U \to X \) induces a map of spectra \( i_* : \mathbf{K}^c(U) \to \mathbf{K}^c(X) \). Given two composable open immersions \( i \) and \( j \), then \( (ji)_* = j_*i_* \).

iv) If \( i : U \to X \) is an open immersion with complement \( j : Y = X - U \to X \), there is a fibration sequence of spectra

\[
\mathbf{K}^c(U) \xrightarrow{i_*} \mathbf{K}^c(X) \xrightarrow{j^*} \mathbf{K}^c(Y).
\]

v) If \( K^c_m(X) = \pi_m \mathbf{K}^c(X) \) denote the homotopy groups of \( \mathbf{K}(X) \) then \( K^c_m(X) = 0 \) if \( m < -\dim(X) \). There is a strongly convergent weight spectral sequence

\[
E^p_{r} \Rightarrow K^c_{p-q}(X), \quad r \geq 2,
\]

which is equal to the spectral sequence (5.2) above for any choice of a compactification of \( X \) and of a non-singular hyperenvelope \( \tilde{j} \). The associated filtration \( F^pK^c_m(X) \), called the weight filtration, is increasing, finite, and independent of choices.

vi) If \( \mathbb{A}^1 \) is the affine line on \( k \), the inclusion of \( X \) as \( X \times \{0\} \) in \( X \times \mathbb{A}^1 \) induces isomorphisms \( K^c_m(X) \simeq K^c_m(X \times \mathbb{A}^1) \).
5.3.2. Finally, we shall describe pairings of spectra between $K$-theory with compact support and $K'$-theory of varieties.

Recall from 5.1 that $K'$ and $K$ are covariant and contravariant functors from the category of projective varieties to the category of spectra and that the cap-product $K(X) \wedge K'(X) \rightarrow K'(X)$ satisfies the projection formula exactly (and not up to homotopy). Therefore, if $X$ is a simplicial object in $V$, there is a pairing of $\Delta^{op}$-diagrams

$$
K(X) \wedge K'(X) \rightarrow K'(X)
$$

in the sense of the Appendix A.2.1, and hence, by Proposition 9, a pairing of prespectra

$$
\text{holim}_n K(X_n) \wedge \text{hocolim}_n K'(X_n) \rightarrow \text{hocolim}_n K'(X_n).
$$

For any map $f : X \rightarrow Y$ of simplicial objects, this pairing satisfies the projection formula exactly.

Now let $X$ be a variety over a field $k$ of characteristic zero, $\overline{X}$ a compactification of $X$, $j : \overline{X} - X \rightarrow \overline{X}$ the complementary inclusion, and $\tilde{j} : \tilde{Y} \rightarrow \tilde{X}$ a non-singular hyperenvelope of $j$. We obtain a commutative diagram of pairings of spectra

$$
\begin{array}{ccc}
K(\tilde{X}) \wedge K'(\tilde{Y}) & \xrightarrow{j^* \wedge \text{Id}} & K(\tilde{Y}) \wedge K'(\tilde{Y}) \\
\text{Id} \wedge \tilde{j} & & \text{Id} \wedge \tilde{j} \\
K(\tilde{X}) \wedge K'(\tilde{X}) & \longrightarrow & \longrightarrow & \longrightarrow K'(\tilde{X}) \\
\end{array}
$$

Therefore, by the Appendix A.2.2, we have a pairing

$$
\text{Fiber}(\tilde{j}^*) \wedge \text{Cofiber}(\tilde{j}_*) \rightarrow K'(\tilde{X}) \approx K'(\overline{X}),
$$

and hence a pairing of spectra

$$
\mu : K^c(X) \wedge K'(X) \rightarrow K'(\overline{X})
$$

for which one can check the following projection formulae:

**Proposition 8.** Assume $f : X \rightarrow Y$ is a proper map of varieties, and $\overline{f} : \overline{X} \rightarrow \overline{Y}$ extends $f$ to compactifications of $X$ and $Y$. Then the following diagram is commutative

$$
\begin{array}{ccc}
K^c(Y) \wedge K'(X) & \longrightarrow & K^c(X) \wedge K'(X) \\
& \downarrow & \downarrow \\
K^c(Y) \wedge K'(\overline{Y}) & \longrightarrow & K'(\overline{X})
\end{array}
$$


If \( i : U \to X \) is an open immersion and \( \overline{X} \) a compactification of \( X \), the following diagram is commutative
\[
\begin{array}{ccc}
K_c(U) \land K'(X) & \longrightarrow & K_c(X) \land K'(X) \\
\downarrow & & \downarrow \\
K_c(U) \land K'(U) & \longrightarrow & K'(\overline{X}).
\end{array}
\]

Finally, let us remark that we can define the weight filtration \( F_q K_n'(X) \) as the filtration coming from the spectral sequence
\[
H_q \left( \ast \mapsto C \left( K'_r \left( \widetilde{Y} \right) \to K'_r \left( \widetilde{X} \right) \right) \right) \Rightarrow K'_{q+r}(X)
\]
as in (5.2) using the descent theorem for \( K' \)-theory [G2] Theorem 4.1. This descending filtration is finite and independent of choices. We expect the pairing of spectra considered above to induce a pairing on the weight filtrations:
\[
F^p K^c_m(X) \otimes F_q K_n'(X) \to F_{q-p} K_{m+n}'(\overline{X}).
\]

**Appendix: Pairings and homotopy (co-)limits**

**A.1 Spectra and homotopy (co-)limits.** Recall [A],[T1] that a simplicial pre-spectrum \( X \) is a sequence \( X_n, n \in \mathbb{N} \) of pointed simplicial sets, together with maps \( SX_n = S^1 \land X_n \to X_{n+1} \). Its homotopy groups are defined, for all \( m \in \mathbb{N} \), as the inductive limit
\[
\pi_m(X) = \lim_n \pi_{m+n}(X_n).
\]
By \( S^1 \) we mean the simplicial circle obtained by identifying the two vertices of the standard one simplex. Note that the structure maps are adjoint to maps \( X_n \to \Omega X_{n+1} \). We say that \( X \) is a spectrum if these maps are weak homotopy equivalences for all \( n \) and the \( X_n \) are all fibrant. We assume furthermore that all spectra are cofibrant as prespectra, i.e. that the maps \( SX_n = S^1 \land X_n \to X_{n+1} \) are inclusions.

We shall also need smash products of (pre-)spectra. These are only fully developed in the existing literature for topological spectra; however we are using simplicial spectra, because there is a fully developed theory of homotopy limits in the simplicial situation. Fortunately we shall need only elementary properties of the smash product, in particular we make no use of associativity and commutativity of the smash product. We therefore will use the handicrafted smash product of Boardman, as described in the book of Adams [A], replacing the topological spectra in that book by simplicial prespectra. On passing to the geometric realization we get the smash product of Boardman on topological spectra.

Let \( I \) be a small category, and \( A^* : i \mapsto A^i \), a contravariant functor from \( I \) to the category of spectra. Then the homotopy limit of the functor \( A^* \) is the spectrum defined, following [B-K] XI 3.2, and [T1] 5.6 as
\[
\text{holim}_i(A^i) := \text{Hom}_{\text{Top}}(B(I \setminus -), A^*),
\]
where \( \text{Hom}_{\text{Top}} \) means morphisms of contravariant functors on \( I \). Similarly we can define, as in [B-K] XII 2.1, and [T1] 5.6 and 5.10 the homotopy colimit \( \text{hocolim}_i(A_i) \) of a covariant functor \( A_i : I \to \text{Prespectra} \), to be the difference cokernel of:
\[
\coprod B(I \setminus j) \triangleright A_i \longrightarrow \coprod B(I \setminus k) \triangleright A_k
\]
in the category of prespectra. Here $X \triangleright Y$ for $X$ a simplicial set and $Y$ a prespectrum denotes the prespectrum $X_+ \wedge Y$, where $X_+$ denotes the simplicial set $X$ equipped with a disjoint basepoint.

A.2 Pairings.

A.2.1. The following discussion is part of the general theory of homotopy coends; we make no claim of originality. See for example [H-V] for a discussion in the case of diagrams of spaces. Let $A^*$ be an $I^{op}$-diagram of spectra, and $B_*$ and $C_*$ be $I$-diagrams of prespectra. A pairing

$$A^* \wedge B_* \to C_*$$

consists of pairings

$$A^i \wedge B_i \to C_i$$

for all $i$, such that for all maps $f : i \to j$ in $I$ we have a commutative diagram:

$$
\begin{array}{ccc}
A^j \wedge B_i & \xrightarrow{A(f) \wedge \text{Id}} & A^i \wedge B_i \\
\| & & \downarrow{\text{C}(f)} \\
A^j \wedge B_i & \xrightarrow{\text{Id} \wedge B(f)} & A^j \wedge B_j
\end{array}
$$

**Proposition 9.** With the notation of the preceding definition, a pairing of $I$-diagrams $A^* \wedge B_* \to C_*$ gives rise to a pairing

$$\text{holim}_i(A^i) \wedge \text{hocolim}_i(B_i) \to \text{hocolim}_i(C_i).$$

**Proof.** We start by considering a slight generalization of the homotopy colimit. Given an $I^{op}$-diagram of pointed spaces or prespectra $X_*$ and an $I$-diagram of pointed spaces or prespectra $Y_*$, we can form the pointed space or prespectrum $X_* \triangleright Y_*$ defined as the difference cokernel:

$$
\coprod_{f : i \to j} X^j \wedge Y_i \xrightarrow{X(f) \wedge \text{Id}} \prod_k X^k \wedge Y_k.
$$

In particular the homotopy colimit of a diagram of prespectra $Y_*$ may be defined as

$$B(\Gamma^* \triangleright Y_*).$$

Hence the theorem follows from the more general statement that given a pairing $\mu : A^* \wedge B_* \to C_*$ as above, there exists for any $I^{op}$-diagram $Z^*$ of pointed simplicial sets, a pairing of prespectra

$$\text{Hom}_{I^{op}}(Z^*,A^*) \wedge (Z^* \triangleright B_*) \to (Z^* \triangleright C_*).$$

Given a simplex $\phi \in \text{Hom}_{I^{op}}(Z^*,A^*)$, for each $i \in I$ we write $\phi_i \in \text{Hom}_{I^{op}}(Z^i,A^i)$ for its projection into the $i$-th factor of the product. Then for each $i$ we have a map

$$\phi_i : \text{Hom}_{I^{op}}(Z^*,A^*) \wedge (Z^i \wedge B_i) \to (Z^i \wedge C_i).$$
where \( \mu_i : A^i \wedge B_i \to C_i \) is the pairing \( \mu \) evaluated at \( i \in I \). We must verify that these pairings induce a map between the difference cokernels in the construction of \( Z^* \to B* \) and \( Z^* \to C* \). That is, given a map \( f : k \to l \) in \( I \), we need to know that the following diagrams commute.

\[
\begin{array}{ccc}
\text{Hom}_{I^{op}}(Z^*, A^*) \wedge (Z^l \wedge B_k) & \xrightarrow{1 \wedge Z(f) \wedge 1} & \text{Hom}_{I^{op}}(Z^*, A^*) \wedge (Z^k \wedge B_k) \\
\beta_{l,k} & & \gamma_k \\
Z^l \wedge C_k & \xrightarrow{Z(f) \wedge 1} & Z^k \wedge C_k,
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Hom}_{I^{op}}(Z^*, A^*) \wedge (Z^l \wedge B_k) & \xrightarrow{1 \wedge 1 \wedge B(f)} & \text{Hom}_{I^{op}}(Z^*, A^*) \wedge (Z^l \wedge B_l) \\
\beta_{l,k} & & \gamma_l \\
Z^l \wedge C_k & \xrightarrow{1 \wedge C(f)} & Z^l \wedge C_l,
\end{array}
\]

where

\[
\beta_{k,l} : \phi \wedge z^l \wedge b_k \mapsto z^l \wedge \mu_k(A(f)(\phi_l(z^l)) \wedge b_k). \]

For the first diagram, we have

\[
\gamma_k((1 \wedge Z(f) \wedge 1)(\phi \wedge z^l \wedge b_k)) = Z(f)(z^l) \wedge \mu_k(\phi_k(Z(f)(z^l)) \wedge b_k) = Z(f)(z^l) \wedge \mu_k(A(f)(\phi_l(z^l)) \wedge b_k) \quad \text{(since } A(f) \cdot \phi_l = \phi_k \cdot Z(f)\text{)} = (Z(f) \wedge 1)(\beta_{k,l}(\phi \wedge z^l \wedge b_k)).
\]

While for the second square

\[
\gamma((1 \wedge 1 \wedge B(f))(\phi \wedge z^l \wedge b_k)) = \gamma(\phi \wedge z^l \wedge B(f)(b_k)) = z^l \wedge \mu_l(\phi_l(z^l) \wedge B(f)(b_k)) = z^l \wedge C(f)(\mu_k(A(f)(\phi_l(z^l)) \wedge b_k)) \quad \text{(by the “projection formula” for the pairing } A^* \wedge B* \to C*) = (1 \wedge C(f))(\beta_{k,l}(\phi \wedge z^l \wedge b_k)) .
\]

and we are done. \( \square \)

**A.2.2.** Suppose we have maps of prespectra \( A \xrightarrow{f} B \) and \( E \xrightarrow{h} F \), and of spectra \( C \xrightarrow{g} D \), and pairings

\[
\phi : C \wedge A \to E
\]

and

\[
\psi : D \wedge B \to E
\]
such that the following diagram commutes

\[
\begin{array}{ccc}
D \wedge A \xrightarrow{g \wedge \text{Id}} C \wedge A & \xrightarrow{} & E \\
\downarrow \text{Id} \wedge f & & \downarrow h \\
D \wedge B & \xrightarrow{} & F.
\end{array}
\]

We may then define as follows a natural pairing

\[\mu : \text{Fiber}(g) \wedge \text{Cofiber}(f) \to F.\]

Note that Fiber\((g)\) is a spectrum, while Cofiber\((f)\) is a prespectrum.

For \(C\) and \(D\) spaces rather than spectra, the homotopy fiber Fiber\((g)\) of \(g\) is, by definition, the subcomplex of the product \(D \times (C, \ast)(\Delta[1], \ast)\) which makes the square

\[
\begin{array}{ccc}
\text{Fiber}(g) & \xrightarrow{} & (C, \ast)(\Delta[1], \ast) \\
\downarrow & & \downarrow \\
D & \xrightarrow{} & C
\end{array}
\]

become cartesian. Here \((C, \ast)(\Delta[1], \ast)\) is the function space of pointed maps. Since \(C\) is supposed fibrant, one knows that the evaluation map \((C, \ast)(\Delta[1], \ast) \to C\) is a Kan fibration. It follows that the same is true for the map Fiber\((g)\) \to \(D\) induced by projection onto the first factor. Hence Fiber\((g)\) is fibrant. For \(C\) and \(D\) spectra, we apply the above construction degreewise; since \(C\) and \(D\) are fibrant spectra, the same is true for Fiber\((g)\). On the other hand, the mapping cone Cofiber\((f)\) is the disjoint union of \(A \wedge \Delta[1]\) with \(B\), modulo the identification \((a, 1) = f(a)\); note that this is only a prespectrum even if \(A\) and \(B\) are spectra. The pairing \(\mu\) is defined as follows, we have

\[\text{Cofiber}(f) = B \cup (\Delta[1] \wedge A).\]

We therefore define the pairing separately on two pieces:

\[\mu : \text{Fiber}(g) \wedge B \to F\]

is induced by the projection Fiber\((g)\) \to \(D\) and the product \(\psi : D \wedge B \to F\), while

\[\mu : \text{Fiber}(g) \wedge \Delta[1] \wedge A \to F\]

is induced by the adjunction map

\[\text{Fiber}(g) \wedge \Delta[1] \to C\Delta[1] \wedge \Delta[1] \to C\]

followed by the map \(h \circ \phi : C \wedge A \to E\). Using the commutativity of diagram (A1), one may then check that on the intersection \(CA \cap B \subset \text{Cofiber}(f)\) these two maps agree.
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