Construction of perfect crystals conjecturally corresponding to Kirillov-Reshetikhin modules over twisted quantum affine algebras

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Abstract

Assuming the existence of the perfect crystal bases of Kirillov-Reshetikhin modules over simply-laced quantum affine algebras, we construct certain perfect crystals for twisted quantum affine algebras, and also provide compelling evidence that the constructed crystals are isomorphic to the conjectural crystal bases of Kirillov-Reshetikhin modules over twisted quantum affine algebras.

0 Introduction.

The finite-dimensional irreducible modules over quantum affine algebras $U'_q(\mathfrak{g})$ with (classical) weight lattice $P_{cl}$ have been extensively studied from various viewpoints, but the study of these modules from the viewpoint of the crystal base theory still seems to be insufficient. This is mainly because unlike (infinite-dimensional) integrable highest weight modules over quantum affine algebras $U_q(\mathfrak{g})$ with (affine) weight lattice $P$, finite-dimensional irreducible $U'_q(\mathfrak{g})$-modules do not have a crystal base in general. (It is not even known which finite-dimensional irreducible $U'_q(\mathfrak{g})$-modules have a crystal base.) However, it is conjectured (see Conjecture 1.5.1 below, and also [HKOTY] §2.3, [HKOTT] §2.3) that a certain important class of finite-dimensional irreducible $U'_q(\mathfrak{g})$-modules, called Kirillov-Reshetikhin modules (KR modules for short), do have a crystal base. In this paper, assuming the existence of the perfect crystal bases $\mathcal{B}^{i,s}$ of the KR modules over the simply-laced quantum affine algebras $U'_q(\mathfrak{g})$, we construct certain perfect crystals $\hat{\mathcal{B}}^{i,s}$ for twisted quantum affine algebras $U'_q(\hat{\mathfrak{g}})$. Furthermore, we explicitly describe, in almost all cases, how the crystals $\hat{\mathcal{B}}^{i,s}$ decompose, when regarded as a $U_q(\hat{\mathfrak{g}}_{f_0})$-module by restriction,
into a direct sum of the crystal bases of irreducible highest weight $U_q(\widehat{\mathfrak{g}}_{f_0})$-modules, where $U_q(\widehat{\mathfrak{g}}_{f_0})$ is the quantized universal enveloping algebra of the (canonical) finite-dimensional, reductive Lie subalgebra $\widehat{\mathfrak{g}}_{f_0}$ of $\widehat{\mathfrak{g}}$. These results motivate a conjecture that the crystals $\overline{B}^{i,s}$ are isomorphic to the crystal bases of certain KR modules with specified Drinfeld polynomials (see §1.5 below) over the twisted quantum affine algebras $U_q'(\widehat{\mathfrak{g}})$, since they agree with the conjectural branching rules (with $q = 1$) in [HKOTT, Appendix A].

We now describe our results more precisely. Let $\mathfrak{g}$ be a simply-laced affine Lie algebra over $\mathbb{C}$, i.e., let $\mathfrak{g}$ be the Kac-Moody algebra $\mathfrak{g}(A)$ over $\mathbb{C}$ associated to the generalized Cartan matrix (GCM for short) $A = (a_{ij})_{i,j \in I}$ of type $A^{(1)}_{2n-1} (n \geq 2)$, $A^{(1)}_{2n} (n \geq 1)$, $D^{(1)}_{n+1} (n \geq 3)$, $D^{(1)}_4$, or $E^{(1)}_6$, where $I$ is an index set for the simple roots (numbered as in §2.1 below). We denote by $U_q(\mathfrak{g})$ (resp., $U_q'(\mathfrak{g})$) the associated quantum affine algebra over $\mathbb{C}(q)$ with $P$ (resp., $P_\mathfrak{g}$) as the weight lattice. For each $i \in I_0 := I \setminus \{0\}$, $s \in \mathbb{Z}_{\geq 1}$, and $\zeta \in \mathbb{C}(q)^\times := \mathbb{C}(q) \setminus \{0\}$, let $W^{(i)}_s(\zeta)$ be the finite-dimensional irreducible module over the quantum affine algebra $U_q'(\mathfrak{g})$ whose Drinfeld polynomials $P_j(u) \in \mathbb{C}(q)[u]$ for $j \in I_0$ are given by:

$$P_j(u) = \begin{cases} \prod_{k=1}^s (1 - \zeta q^{s+2-2k}u) & \text{if } j = i, \\
1 & \text{otherwise.} \end{cases}$$

(Here we are using the Drinfeld realization of $U_q'(\mathfrak{g})$, and the classification of its finite-dimensional irreducible modules of “type 1” by Drinfeld polynomials; see [CP] for details.) We call this $U_q'(\mathfrak{g})$-module $W^{(i)}_s(\zeta)$ a Kirillov-Reshetikhin module (KR module for short) over $U_q'(\mathfrak{g})$. It is conjectured (see Conjecture 1.5.1 below, and also [HKOTY] §2.3)) that for every $i \in I$ and $s \in \mathbb{Z}_{\geq 1}$, there exists some $\zeta^{(i)}_s \in \mathbb{C}(q)^\times$ such that the KR module $W^{(i)}_s(\zeta^{(i)}_s)$ has a crystal base.

Let $\omega : I \to I$ be a nontrivial diagram automorphism such that $\omega(0) = 0$, and $\widehat{\mathfrak{g}}$ the corresponding orbit Lie algebra of $\mathfrak{g}$ (see §2.2 below for the definition). Note that the orbit Lie algebra $\widehat{\mathfrak{g}}$ is a twisted affine Lie algebra, i.e., $\widehat{\mathfrak{g}}$ is the Kac-Moody algebra $\mathfrak{g}(\widehat{A})$ over $\mathbb{C}$ associated to the GCM $\widehat{A} = (\widehat{a}_{ij})_{i,j \in I}$ of type $D^{(2)}_{n+1} (n \geq 2)$, $A^{(2)}_{2n} (n \geq 1)$, $A^{(2)}_{2n-1} (n \geq 3)$, $D^{(3)}_4$, or $E^{(2)}_6$, where $\widehat{I} \subset I$ is a certain complete set (containing $0 \in I$) of representatives of the $\omega$-orbits in $I$, and is also an index set for the simple roots of $\widehat{\mathfrak{g}}$. In this paper, we assume that for (arbitrarily) fixed $i \in \widehat{I}_0 := \widehat{I} \setminus \{0\}$ and $s \in \mathbb{Z}_{\geq 1}$, there exists some $\zeta^{(i)}_s \in \mathbb{C}(q)^\times$ such that for every $0 \leq k \leq N_i - 1$, the KR module $W^{(\omega \omega(i))}_{\zeta^{(i)}_s}(\zeta^{(i)}_s)$ over $U_q'(\mathfrak{g})$ has a crystal base, denoted by $B^{\omega \omega(i),s}$, where $N_i \in \mathbb{Z}_{\geq 1}$ is the number of elements of the $\omega$-orbit of $i \in \widehat{I}_0$ in $I$; here we note that the $\zeta^{(i)}_s \in \mathbb{C}(q)^\times$ is assumed to be independent of $0 \leq k \leq N_i - 1$. We further assume that the $B^{\omega \omega(i),s}$, $0 \leq k \leq N_i - 1$, are all perfect $U_q'(\mathfrak{g})$-crystals of level $s$ (in the sense of Definition 1.4.12). Now, for the (fixed) $i \in \widehat{I}_0$ and $s \in \mathbb{Z}_{\geq 1}$, we define the tensor product $U_q'(\mathfrak{g})$-crystal $\overline{B}^{i,s}$ equipped with the Kashiwara
operators $e_j$ and $f_j$, $j \in I$, by:

$$
\tilde{B}^{i,s} = B^{i,s} \otimes B^{\omega (i),s} \otimes \cdots \otimes B^{\omega N_{i-1} (i),s},
$$
on which the diagram automorphism $\omega : I \to I$ acts in a canonical way (see §2.3 below). Also, we define $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$ on $\tilde{B}^{i,s}$ for $j \in \tilde{I}$ by (see §2.4.1), and also Remark 2.2.1:

$$
\tilde{x}_j = \begin{cases} 
  x_j x_{\omega(j)}^2 x_j & \text{if } N_j = 2 \text{ and } a_{j,\omega(j)} = a_{\omega(j),j} = -1, \\
  x_j x_{\omega(j)} \cdots x_{\omega N_j - 1 (j)} & \text{if } a_{j,\omega^k(j)} = 0 \text{ for all } 1 \leq k \leq N_j - 1,
\end{cases}
$$

where $x$ is either $e$ or $f$. Then, the $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$, $j \in \tilde{I}$, stabilize the fixed point subset $\tilde{B}^{i,s}$ of $\tilde{B}^{i,s}$ under the action of the diagram automorphism $\omega$, and hence equip the $\tilde{B}^{i,s}$ with a structure of $U'_q (\hat{g})$-crystal, where $U'_q (\hat{g})$ denotes the quantum affine algebra associated to $\hat{g}$. Furthermore, we prove that the $\tilde{B}^{i,s}$ is a perfect $U'_q (\hat{g})$-crystal of level $s$ (in the sense of Definition 1.4.12), thereby establishing Theorem 2.4.1.

Because the $U'_q (\hat{g})$-crystal $\tilde{B}^{i,s}$ for the (fixed) $i \in \tilde{I}_0 = \tilde{I} \setminus \{0\}$ and $s \in \mathbb{Z}_{\geq 1}$ is perfect (and hence regular), it decomposes, under restriction, into a direct sum of the crystal bases of integrable highest weight modules over the quantized universal enveloping algebra $U_q (\hat{g}_{\tilde{I}_0})$ of the finite-dimensional, reductive Lie subalgebra $\hat{g}_{\tilde{I}_0}$ of $\hat{g}$ corresponding to $\tilde{I}_0 \subset \tilde{I}$. In fact, we can give, in almost all cases, an explicit description (see §5.2 – §5.6) of the branching rule with respect to the restriction to $U_q (\hat{g}_{\tilde{I}_0})$, i.e., how the $\tilde{B}^{i,s}$ decomposes into a disjoint union of connected components as a $U_q (\hat{g}_{\tilde{I}_0})$-crystal. This result deserves to be supporting evidence that the $\tilde{B}^{i,s}$ is isomorphic as a $U'_q (\hat{g})$-crystal to the conjectural crystal base of a certain KR module with specified Drinfeld polynomials (see §1.3 below), denoted by $\tilde{W}_s (i) (\tilde{\zeta}_s (i))$, over $U'_q (\hat{g})$, since our branching rule for the $\tilde{B}^{i,s}$ indeed agrees with the “branching rule” (with $q = 1$) conjectured in [HKOTT, Appendix A] for the KR module $\tilde{W}_s (i) (\tilde{\zeta}_s (i))$ over $U'_q (\hat{g})$, regarded as a $U_q (\hat{g}_{\tilde{I}_0})$-module by restriction.

Finally, we should mention the relationship between the $U'_q (\hat{g})$-crystals $\tilde{B}^{i,s}$, $i \in \tilde{I}_0$, $s \in \mathbb{Z}_{\geq 1}$, and “virtual” crystals defined in [OSS1, OSS2]. Since the $U'_q (\hat{g})$-crystals $\tilde{B}^{i,s}$, $i \in \tilde{I}_0$, $s \in \mathbb{Z}_{\geq 1}$, are perfect (and hence simple), their crystal graphs are connected. Therefore, the $U'_q (\hat{g})$-crystals $\tilde{B}^{i,s}$ coincide with virtual crystals at least in the cases where $\hat{g}$ is of type $A_{2n-1}^{(1)}$ and $\hat{g}$ is of type $D_{n+1}^{(2)}$ for $n \geq 2$, $\hat{g}$ is of type $D_{n+1}^{(1)}$ and $\hat{g}$ is of type $A_{2n-1}^{(2)}$ for $n \geq 3$, $\hat{g}$ is of type $D_{n+1}^{(1)}$ and $\hat{g}$ is of type $D_{n+1}^{(3)}$, and where $\hat{g}$ is of type $E_6^{(1)}$ and $\hat{g}$ is of type $E_6^{(2)}$. This clarifies the representation-theoretical meaning of virtual crystals.

The organization of this paper is as follows. In §1 we briefly review some basic notions in the theory of crystals for quantum affine algebras, and recall a conjecture about the existence of the crystal bases of KR modules. In §2 we first fix the notation for diagram automorphisms of simply-laced affine Lie algebras $\hat{g}$, and also for corresponding orbit Lie
algebras $\widehat{\mathfrak{g}}$. Then we define the $U'_q(\widehat{\mathfrak{g}})$-crystals $\widehat{B}^{i,s}$, $i \in \widehat{I}_0$, $s \in \mathbb{Z}_{\geq 1}$, and then state our main result (Theorem 2.4.1). In §3 we show some technical propositions and lemmas about the fixed point subsets of regular crystals, or of tensor products of regular crystals under the action of the diagram automorphism $\omega$, which will be needed later. In §4 we prove the perfectness of the $U'_q(\widehat{\mathfrak{g}})$-crystals $\widehat{B}^{i,s}$, $i \in \widehat{I}_0$, $s \in \mathbb{Z}_{\geq 1}$, thereby establishing Theorem 2.4.1. In §5 we give explicit descriptions of the branching rules for the $U'_q(\widehat{\mathfrak{g}})$-crystals $\widehat{B}^{i,s}$, $i \in \widehat{I}_0$, $s \in \mathbb{Z}_{\geq 1}$, with a few exceptions, and then propose a conjecture that for each $i \in \widehat{I}_0$ and $s \in \mathbb{Z}_{\geq 1}$, the $\widehat{B}^{i,s}$ is isomorphic as a $U'_q(\widehat{\mathfrak{g}})$-crystal to the conjectural crystal base of a certain KR module over the twisted quantum affine algebra $U'_q(\widehat{\mathfrak{g}})$.

1 Crystals for quantum affine algebras.

1.1 Cartan data. Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix (GCM for short). A Cartan datum for the GCM $A$ is, by definition, a quintuplet $(A, P, P^\vee, \Pi, \Pi^\vee)$ consisting of the GCM $A$, a free $\mathbb{Z}$-module $P^\vee$ of finite rank, its dual $P := \text{Hom}_\mathbb{Z}(P^\vee, \mathbb{Z})$, a subset $\Pi^\vee := \{h_j\}_{j \in I}$ of $P^\vee$, and a subset $\Pi := \{\alpha_j\}_{j \in I}$ of $P$ satisfying $\alpha_k(h_j) = a_{jk}$ for $j, k \in I$. Further, we assume that the elements $h_j$, $j \in I$, of $\Pi^\vee \subset P^\vee$ are linearly independent; however, we do not assume that the elements $\alpha_j$, $j \in I$, of $\Pi \subset P$ are linearly independent.

1.2 Crystals. Let us briefly recall some basic notions in the theory of crystals from [HK] Chap. 4, §4.5] (see also [Kas2 §7]). Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable GCM, and let $(A, P, P^\vee, \Pi, \Pi^\vee)$ be a Cartan datum for the GCM $A$. A crystal associated to the Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$ is a set $\mathcal{B}$ equipped with maps $\text{wt} : \mathcal{B} \to P$, $e_j$, $f_j : \mathcal{B} \cup \{\theta\} \to \mathcal{B} \cup \{\theta\}$, $j \in I$, and $\varepsilon_j, \varphi_j : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\}$, $j \in I$, satisfying Conditions (1) – (7) of [HK Definition 4.5.1] (with $\widehat{e}_j$, $\widehat{f}_j$, 0 replaced by $e_j$, $f_j$, $\theta$, respectively). We call the map $e_j$ (resp., $f_j$) the raising (resp., lowering) Kashiwara operator with respect to $\alpha_j \in \Pi$, and understand that $e_j\theta = f_j\theta = \theta$ for all $j \in I$. A crystal $\mathcal{B}$ is said to be semiregular if

\[
\varepsilon_j(b) = \max\{m \geq 0 \mid e_j^m b \neq \theta\}, \quad \varphi_j(b) = \max\{m \geq 0 \mid f_j^m b \neq \theta\},
\]

for all $b \in \mathcal{B}$ and $j \in I$; every crystal treated in this paper is semiregular.

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be crystals associated to the Cartan datum $(A, P, P^\vee, \Pi, \Pi^\vee)$ above. We define the tensor product crystal $\mathcal{B}_1 \otimes \mathcal{B}_2$ of $\mathcal{B}_1$ and $\mathcal{B}_2$ as in [HK] Definition 4.5.3] (see also [Kas2 §7.3]). Note that the definition of tensor product crystals in [OSS1] and [OSS2] is different from [HK] Definition 4.5.3], and hence from ours; the roles of $e_j$ and $f_j$ are interchanged for each $j \in I$ (see, for example, [OSS1] (2.10) and (2.11)).
1.3 Quantum affine algebras. From now throughout this paper, we assume that a GCM \( A = (a_{ij})_{i,j \in I} \) is of affine type. Take a special vertex \( 0 \in I \) as in [Kac, §4.8, Tables Aff 1 – Aff 3], and set \( I_0 := I \setminus \{0 \} \). Let \( g = g(A) \) be the affine Lie algebra over the field \( \mathbb{C} \) of complex numbers associated to the GCM \( A \) of affine type. Then

\[
\mathfrak{h} = \left( \bigoplus_{j \in I} \mathbb{C} h_j \right) \oplus \mathbb{C} d
\]

(1.3.1)

is a Cartan subalgebra of \( g \), with \( h_j, j \in I \), the simple coroots, and \( d \) the scaling element. The simple roots \( \alpha_j \in \mathfrak{h}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C}), \ j \in I \), and fundamental weights \( \Lambda_j \in \mathfrak{h}^* \), \( j \in I \), are defined by (see [HK, Chap. 10, §10.1]):

\[
\alpha_k(h_j) = a_{jk}, \quad \alpha_k(d) = \delta_{k,0}, \\
\Lambda_k(h_j) = \delta_{k,j}, \quad \Lambda_k(d) = 0,
\]

(1.3.2)

for \( j, k \in I \). Let \( E_j, F_j, j \in I \), be the Chevalley generators of \( g \), where \( E_j \) (resp., \( F_j \)) corresponds to the simple root \( \alpha_j \) (resp., \( -\alpha_j \)), and let

\[
\delta = \sum_{j \in I} a_j \alpha_j \in \mathfrak{h}^*, \quad \text{and} \quad c = \sum_{j \in I} a_j^\vee h_j \in \mathfrak{h}
\]

(1.3.3)

be the null root and the canonical central element of \( g \), respectively. We take a dual weight lattice \( P^\vee \) and a weight lattice \( P \) as follows:

\[
P^\vee = \left( \bigoplus_{j \in I} \mathbb{Z} h_j \right) \oplus \mathbb{Z} d \subset \mathfrak{h} \quad \text{and} \quad P = \left( \bigoplus_{j \in I} \mathbb{Z} \Lambda_j \right) \oplus \mathbb{Z} \left( \frac{1}{a_0} \delta \right) \subset \mathfrak{h}^*.
\]

(1.3.4)

Clearly, we have \( P \cong \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z}) \). Here we should note that \( a_0 = 1 \) except the case where \( g \) is of type \( A_{2n}^{(1)} \), and \( a_0 = 2 \) in the case where \( g \) is of type \( A_{2n}^{(2)} \). It is easily seen that the quintuplet \((A, P, P^\vee, \Pi, \Pi^\vee)\) is a Cartan datum for the affine type GCM \( A = (a_{ij})_{i,j \in I} \). Let \( U_q(g) = \langle E_j, F_j, q^h \mid j \in I, h \in P^\vee \rangle \) be the quantized universal enveloping algebra of \( g \) over the field \( \mathbb{C}(q) \) of rational functions in \( q \) (with complex coefficients) with weight lattice \( P \), and Chevalley generators \( E_j, F_j, j \in I \).

Now, we set

\[
\mathfrak{h}_{cl} := \bigoplus_{j \in I} \mathbb{C} h_j \subset \mathfrak{h} \quad \text{and} \quad P^\vee_{cl} := \bigoplus_{j \in I} \mathbb{Z} h_j \subset P^\vee
\]

(1.3.5)

For each \( \lambda \in \mathfrak{h}^* \), we define \( \text{cl}(\lambda) \in \mathfrak{h}_{cl}^* := (\mathfrak{h}_{cl})^* \) to be the restriction \( \lambda|_{\mathfrak{h}_{cl}} \) of \( \lambda \in \mathfrak{h}^* \) to \( \mathfrak{h}_{cl} \) (we simply write \( \lambda \) for \( \text{cl}(\lambda) \) if there is no fear of confusion). It follows that \( \mathfrak{h}_{cl}^* = \text{cl}(\mathfrak{h}^*) \cong \mathfrak{h}^*/\mathbb{C} \delta \) as \( \mathbb{C} \)-vector spaces, and

\[
\mathfrak{h}_{cl}^* = \bigoplus_{j \in I} \mathbb{C} \text{cl}(\Lambda_j).
\]

(1.3.6)
Then we define the classical weight lattice $P_{cl}$ to be $\mathfrak{cl}(P) \subset \mathfrak{h}_{cl}$. We have $P_{cl} \cong \text{Hom}_\mathbb{Z}(P_{cl}', \mathbb{Z}) \cong P/(\mathbb{C}\delta \cap P)$ as (free) $\mathbb{Z}$-modules, and

$$P_{cl} = \bigoplus_{j \in I} \mathbb{Z}\Lambda_j,$$

where $\mathfrak{cl}(\Lambda_j) \in \mathfrak{h}_{cl}^*$ is simply denoted by $\Lambda_j$ for $j \in I$. Further, we set

$$(P_{cl})_0 := \{ \mu \in P_{cl} \mid \mu(c) = 0 \}, \quad P_{cl}^+ := \sum_{j \in I} \mathbb{Z}_{\geq 0} \Lambda_j,$$

$$(P_{cl}^+_s) := \{ \mu \in P_{cl}^+ \mid \mu(c) = s \} \text{ for each } s \in \mathbb{Z}_{\geq 0}.$$  \hfill(1.3.7)

It is easily seen that the quintuplet $(A, P_{cl}, P_{cl}', \Pi, \Pi')$ is also a Cartan datum for the affine type GCM $A = (a_{ij})_{i,j \in I}$. For simplicity, a crystal associated to the Cartan datum $(A, P_{cl}, P_{cl}', \Pi, \Pi')$ is called a $U_q'(\mathfrak{g})$-crystal, where $U_q'(\mathfrak{g})$ denotes the $\mathbb{C}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $E_j, F_j, j \in I$, and $q^h$, $h \in P_{cl}'$ (which is the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{C}(q)$ with weight lattice $P_{cl}$).

1.4 Perfect crystals for quantum affine algebras. We keep the notation of [AK, §1.3]. Let us fix a proper subset $J$ of $I$. We set $A_J := (a_{ij})_{i,j \in J}, \Pi_J := \{ \alpha_j \}_{j \in J} \subset P_{cl}, \Pi_J' := \{ h_j \}_{j \in J} \subset P_{cl}'$, and denote by $\mathfrak{g}_J$ the Lie subalgebra of the affine Lie algebra $\mathfrak{g}$ generated by $E_j, F_j, j \in J$, and $\mathfrak{h}_{cl}$. Then the quintuplet $(A_J, P_{cl}, P_{cl}', \Pi_J, \Pi_J')$ is a Cartan datum for the GCM $A_J$. For simplicity, a crystal associated to this Cartan datum $(A_J, P_{cl}, P_{cl}', \Pi_J, \Pi_J')$ is called a $U_q(\mathfrak{g}_J)$-crystal, where $U_q(\mathfrak{g}_J)$ denotes the $\mathbb{C}(q)$-subalgebra of $U_q'(\mathfrak{g})$ generated by $E_j, F_j, j \in J$, and $q^h$, $h \in P_{cl}'$ (which is the quantized universal enveloping algebra of $\mathfrak{g}_J$ over $\mathbb{C}(q)$ with weight lattice $P_{cl}$). If $B$ is a $U_q'(\mathfrak{g})$-crystal and $J$ is a (proper) subset of $I$, then the set $B$ equipped with the Kashiwara operators $e_j, f_j, j \in J$, and the maps $\text{wt}: B \to P_{cl}, \varepsilon_j, \varphi_j: B \to \mathbb{Z} \cup \{ -\infty \}, j \in J$, is a $U_q(\mathfrak{g}_J)$-crystal.

Definition 1.4.1 (see [AK, §1.4]). A $U_q'(\mathfrak{g})$-crystal $B$ is said to be regular if for every proper subset $J \subsetneq I$, the $B$, regarded as a $U_q(\mathfrak{g}_J)$-crystal in the way above, is isomorphic to the crystal base of an integrable $U_q(\mathfrak{g}_J)$-module (for details about crystal bases, see, for example, [HK, Chap. 4, §4.2] and [Kas2, §4]).

Let $W := \langle r_j \mid j \in I \rangle \subset \text{GL}(\mathfrak{h}^*)$ be the Weyl group of $\mathfrak{g}$, where $r_j \in \text{GL}(\mathfrak{h}^*)$ is the simple reflection in $\alpha_j \in \mathfrak{h}^*$. Note that the weight lattice $P \subset \mathfrak{h}^*$ is stable under the action of the Weyl group $W$, and that there exists an action of $W$ on $P_{cl}$ induced from that on $P$, since $W\delta = \delta$.

We can define an action of the Weyl group $W$ on a regular $U_q'(\mathfrak{g})$-crystal $B$ as follows (see [Kas1, §7]). For each $j \in I$, we define $S_j : B \to B$ by:

$$S_j b = \begin{cases} f_j^m b & \text{if } m := (\text{wt } b)(h_j) \geq 0 \\ e_j^{-m} b & \text{if } m := (\text{wt } b)(h_j) < 0 \end{cases} \quad \text{for } b \in B.$$  \hfill(1.4.1)
Definition 1.4.3 (see [AK, §W]). Let $B$ be a regular $U_q'(g)$-crystal. An element $b \in B$ is said to be extremal (or more accurately, $W$-extremal) if for every $w \in W$, either $e_j s_w b = \theta$ or $f_j s_w b = \theta$ holds for each $j \in I$.

Remark 1.4.4. It immediately follows from the definition above that if $b \in B$ is an extremal element, then $s_w b \in B$ is an extremal element of weight $w \cdot b$ for each $w \in W$.

We know the following lemma from [AK, Lemma 1.6 (1)] and its proof (see also Lemma 1.2.1 below).

Lemma 1.4.5. Let $B_1, B_2$ be regular $U_q'(g)$-crystals of finite cardinality such that the weights of their elements are all contained in $(P_{cl})_0$. Let $b_1 \in B_1$ and $b_2 \in B_2$ be extremal elements whose weights are contained in the same Weyl chamber with respect to the simple coroots $h_j$, $j \in I_0 = I \setminus \{0\}$. Then, $b_1 \otimes b_2 \in B_1 \otimes B_2$ is an extremal element. Also, $s_w (b_1 \otimes b_2) = s_w b_1 \otimes s_w b_2$ holds for all $w \in W$.

Definition 1.4.6. A regular $U_q'(g)$-crystal $B$ is said to be simple if it satisfies the following conditions:

(S1) The set $B$ is of finite cardinality, and the weights of elements of $B$ are all contained in $(P_{cl})_0$.

(S2) The set of all extremal elements of $B$ coincides with a Weyl group orbit in $B$.

(S3) Let $b \in B$ be an extremal element, and set $\mu := w t b \in (P_{cl})_0$. Then the subset $B_\mu \subset B$ of all elements of weight $\mu$ consists only of the element $b$, i.e., $B_\mu = \{b\}$.

Remark 1.4.7. Let $B$ be a regular $U_q'(g)$-crystal satisfying condition (S1) of Definition 1.4.6. Then we see that there exists at least one extremal element in $B$ (see the comment after the proof of [Kas1, Proposition 9.3.2]).

Lemma 1.4.8. Let $B$ be a simple $U_q'(g)$-crystal. Then there exists a unique extremal element $u \in B$ such that $(w t u)(h_j) \geq 0$ for all $j \in I_0$.

Proof. The existence of an extremal element with the desired property immediately follows from Remark 1.4.4. So, it remains to show the uniqueness. Let $u_1, u_2$ be extremal elements whose weights are both dominant with respect to the simple coroots $h_j$, $j \in I_0$, and set $\mu_1 := w t u_1$ and $\mu_2 := w t u_2$. By condition (S2) of Definition 1.4.6, there exists some $w \in W$ such that $s_w u_2 = u_1$. Then it follows that $\mu_1 = w \mu_2 \in W \mu_2$. We recall from [Kac, Proposition 6.5] that the Weyl group $W$ decomposes into the semidirect product.
follows that the weights of elements of $B$ are all contained in the convex hull of the $\mathbb{W}$-orbit of $\mu$, which shows the uniqueness.

Remark 1.4.9. Let $B$ be a simple $U'_q(\mathfrak{g})$-crystal, and let $u \in B$ be the unique extremal element such that $(\text{wt } u)(h_j) \geq 0$ for all $j \in I_0$. We can show by the same argument as in the proof of [Kas3, Corollary 5.2], using [AK] Lemma 1.5, that the weights of elements of $B$ are all contained in the convex hull of the $W$-orbit of $u$. Hence, by Lemma 1.4.8, it follows that the weights of elements of $B$ are all contained in the set $\text{wt } u - \sum_{j \in I_0} \mathbb{Z}_{\geq 0} \alpha_j$.

We know from [NS3, Remark 2.5.7] that the definition of simple $U'_q(\mathfrak{g})$-crystals above is equivalent to [AK] Definition 1.7 and [Kas3] Definition 4.9]. Thus we know the following from [AK] Lemmas 1.9 and 1.10].

Proposition 1.4.10. (1) The crystal graph of a simple $U'_q(\mathfrak{g})$-crystal is connected.

(2) Let $B_1, B_2$ be simple $U'_q(\mathfrak{g})$-crystals. Then, the tensor product $B_1 \otimes B_2$ is also a simple $U'_q(\mathfrak{g})$-crystal. In particular, the crystal graph of $B_1 \otimes B_2$ is connected.

Let $B$ be a simple $U'_q(\mathfrak{g})$-crystal. We define maps $\varepsilon, \varphi : B \to P_{cl}^+$ by:

$$\varepsilon(b) = \sum_{j \in I} \varepsilon_j(b) \Lambda_j \quad \text{and} \quad \varphi(b) = \sum_{j \in I} \varphi_j(b) \Lambda_j \quad \text{for } b \in B.$$  \hfill (1.4.2)

Further, we define a positive integer $\text{lev } B$ (called the level of $B$) and a subset $B_{\text{min}}$ of $B$ by:

$$\text{lev } B = \min \{ (\varepsilon(b))(c) \mid b \in B \} \in \mathbb{Z}_{>0}, \quad \text{(1.4.3)}$$

$$B_{\text{min}} = \{ b \in B \mid (\varepsilon(b))(c) = \text{lev } B \} \subset B. \quad \text{(1.4.4)}$$

Remark 1.4.11. It can easily be seen from the definition of crystals that $\varphi(b) - \varepsilon(b) = \text{wt } b$ for every $b \in B$. Since $\text{wt } b \in (P_{cl})_0$ by condition (S1) of Definition 1.4.6, we have $(\varepsilon(b))(c) = (\varphi(b))(c)$ for all $b \in B$, and hence $\text{lev } B = \min \{ (\varphi(b))(c) \mid b \in B \}$.

Definition 1.4.12. A simple $U'_q(\mathfrak{g})$-crystal $B$ is said to be perfect if the restrictions of the maps $\varepsilon, \varphi : B \to P_{cl}^+$ to $B_{\text{min}}$ induce bijections $B_{\text{min}} \to (P_{cl}^+)_s$, where $s := \text{lev } B$.

Remark 1.4.13. (1) In the definition of perfect $U'_q(\mathfrak{g})$-crystals $B$, it is often required that $B$ is isomorphic to the crystal base of a finite-dimensional $U'_q(\mathfrak{g})$-module as a $U'_q(\mathfrak{g})$-crystal (see, for example, Condition (1) of [HK] Definition 10.5.1]); but we do not require it in this paper.
Proposition 1.4.15. (2) Our definition of perfect $U'_q(g)$-crystals seems to be slightly different from the ones in [HK] Definition 10.5.1 and [HKOTT] §2.2: We can deduce from Remark 1.4.9 condition (S3) of Definition 1.4.6 and Proposition 1.4.10 that if $B$ is a perfect $U'_q(g)$-crystal in the sense of Definition 1.4.12, then $B$ satisfies Conditions (2) – (5) of [HK] Definition 10.5.1. But, in [HKOTT], it is required that the perfect $U'_q(g)$-crystal $B$ is “finite” (in the sense of [HKKOT] Definition 2.5); we do not require this “finiteness” condition in our definition of perfect $U'_q(g)$-crystals, since it does not seem to be essential for our purposes.

(3) If $B$ is a perfect $U'_q(g)$-crystal in the sense of Definition 1.4.12 and is isomorphic to the crystal base of a finite-dimensional $U'_q(g)$-module as a $U'_q(g)$-crystal, then $B$ is perfect in the sense of [OSS1] §2.10.

Lemma 1.4.14. Let $B_1$, $B_2$ be perfect $U'_q(g)$-crystals of the same level $s$. Then, the tensor product $B_1 \otimes B_2$ is also a perfect $U'_q(g)$-crystal of level $s$.

Proof. We know from Proposition 1.4.10(2) that the tensor product $B_1 \otimes B_2$ is a simple $U'_q(g)$-crystal. In addition, by elementary arguments using the tensor product rule for crystals, we can easily show that the $B_1 \otimes B_2$ is of level $s$, and the maps $\varepsilon, \varphi : (B_1 \otimes B_2)_{\text{min}} \to (P^+_s) \otimes B_1 \otimes B_2$ are bijective.

We know the following proposition from [OSS1] Theorem 2.4.

Proposition 1.4.15. (1) Let $B_1$, $B_2$ be perfect $U'_q(g)$-crystals isomorphic to the crystal bases of finite-dimensional $U'_q(g)$-modules as a $U'_q(g)$-crystal. Then, there exists a unique isomorphism (called a combinatorial $R$-matrix) $R : B_1 \otimes B_2 \to B_2 \otimes B_1$ of $U'_q(g)$-crystals.

(2) Let $B$ be a perfect $U'_q(g)$-crystal isomorphic to the crystal base of a finite-dimensional $U'_q(g)$-module as a $U'_q(g)$-crystal. Then, there exists a $\mathbb{Z}$-valued function $H : B \otimes B \to \mathbb{Z}$ (called an energy function) satisfying

$$H(e_j(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) + 1 & \text{if } j = 0 \text{ and } \varphi_0(b_1) \geq \varepsilon_0(b_2), \\ H(b_1 \otimes b_2) - 1 & \text{if } j = 0 \text{ and } \varphi_0(b_1) < \varepsilon_0(b_2), \\ H(b_1 \otimes b_2) & \text{if } j \neq 0, \end{cases}$$

for all $j \in I$ and $b_1 \otimes b_2 \in B \otimes B$ such that $e_j(b_1 \otimes b_2) \neq 0$.

Remark 1.4.16. With the notation and assumption of Proposition 1.4.15(2), we have

$$H(f_j(b_1 \otimes b_2)) = \begin{cases} H(b_1 \otimes b_2) - 1 & \text{if } j = 0 \text{ and } \varphi_0(b_1) > \varepsilon_0(b_2), \\ H(b_1 \otimes b_2) + 1 & \text{if } j = 0 \text{ and } \varphi_0(b_1) \leq \varepsilon_0(b_2), \\ H(b_1 \otimes b_2) & \text{if } j \neq 0, \end{cases}$$

for all $j \in I$ and $b_1 \otimes b_2 \in B \otimes B$ such that $f_j(b_1 \otimes b_2) \neq 0$. 

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1.5 A conjectural family of perfect crystals. In this subsection, let $\mathfrak{g}$ be either a simply-laced affine Lie algebra, or a twisted affine Lie algebra. More specifically, let $\mathfrak{g}$ be the affine Lie algebra of type $A_n^1 (n \geq 2), D_n^1 (n \geq 4), E_6^1$, or of type $D_{n+1}^2 (n \geq 2), A_{2n}^2 (n \geq 1), A_{2n-1}^2 (n \geq 3), D_4^3, E_8^2$, with the index set $I$ for the simple roots numbered as in [Kac, §4.8, Tables Aff 1 – Aff 3]. For each $i \in I_0 = I \setminus \{0\} = \{1, 2, \ldots, n\}$, $s \in \mathbb{Z}_{\geq 1}$, and $\zeta \in \mathbb{C}(q)^\times := \mathbb{C}(q) \setminus \{0\}$, we denote by $W_s^i(\zeta)$ the finite-dimensional irreducible module over the quantum affine algebra $U'_q(\mathfrak{g})$ whose Drinfeld polynomials $P_j(u) \in \mathbb{C}(q)[u]$ are specified as follows (see [KNT, Definition 5.3]):

**Case 1:** the case of type $A_n^1 (n \geq 2), D_n^1 (n \geq 4), E_6^1$. In this case, the Drinfeld polynomials $P_j(u), j \in I_0 = \{1, 2, \ldots, n\}$, are given by:

$$P_j(u) = \begin{cases} \prod_{k=1}^{s} (1 - \zeta q^{s+2-2k}u) & \text{if } j = i, \\ 1 & \text{otherwise.} \end{cases}$$

**Case 2:** the case of type $D_{n+1}^2 (n \geq 2)$ (resp., $A_{2n-1}^2 (n \geq 3)$). In this case, the Drinfeld polynomials $P_j(u), j \in I_0 = \{1, 2, \ldots, n-1, n\}$, are given by:

$$P_j(u) = \begin{cases} \prod_{k=1}^{s} (1 - \zeta q^{d_i(s+2-2k)}u) & \text{if } j = i, \\ 1 & \text{otherwise,} \end{cases}$$

where $d_i = 1$ if $i = n$ (resp., $i \neq n$), and $d_i = 2$ otherwise.

**Case 3:** the case of type $D_4^3$ (resp., $E_6^2$). In this case, the Drinfeld polynomials $P_j(u), j \in I_0 = \{1, 2\}$ (resp., $j \in I_0 = \{1, 2, 3, 4\}$), are given by:

$$P_j(u) = \begin{cases} \prod_{k=1}^{s} (1 - \zeta q^{d_i(s+2-2k)}u) & \text{if } j = i, \\ 1 & \text{otherwise,} \end{cases}$$

where $d_i = 1$ if $i = 1$ (resp., $i = 1, 2$), and $d_i = 3$ (resp., $d_i = 2$) otherwise.

**Case 4:** the case of type $A_{2n}^2 (n \geq 1)$. In this case, we should remark that the index set for the Drinfeld polynomials $P_j(u)$ is not $I_0 = \{1, 2, \ldots, n\}$, but $\{0, 1, \ldots, n-1\}$, and that the Drinfeld polynomials $P_j(u), j \in I_0 = \{0, 1, \ldots, n-1\}$, are given by:

$$P_j(u) = \begin{cases} \prod_{k=1}^{s} (1 - \zeta q^{2(s+2-2k)}u) & \text{if } j = n - i, \\ 1 & \text{otherwise.} \end{cases}$$
(In all cases above, we used the Drinfeld realization of $U'_q(\mathfrak{g})$, and the classification of its finite-dimensional irreducible modules of type 1 by Drinfeld polynomials; see [CP1], [CP2] for details.) We call this $U'_q(\mathfrak{g})$-module $W^{(i)}_s(\zeta)$ a Kirillov-Reshetikhin module (KR module for short) over $U'_q(\mathfrak{g})$.

Now, let us fix (arbitrarily) $i \in I_0$ and $s \in \mathbb{Z}_{\geq 1}$.

**Conjecture 1.5.1 (cf. [HKOTT, Conjecture 2.1 (1)]).** For some $\zeta^{(i)}_s \in \mathbb{C}(q)^{\times} = \mathbb{C}(q) \setminus \{0\}$, the KR module $W^{(i)}_s(\zeta^{(i)}_s)$ over $U'_q(\mathfrak{g})$ has a crystal base $B^{i,s}$ that is a perfect $U'_q(\mathfrak{g})$-crystal of level $s$.

**Remark 1.5.2.** Conjecture [1.5.1] has already been proved in some cases (see [HKOTY, Remark 2.3] and [HKOTT, Remark 2.6]; see also Remark 2.3.3 below).

**Lemma 1.5.3.** Assume that Conjecture [1.5.1] holds for the fixed $i \in I_0$ and $s \in \mathbb{Z}_{\geq 1}$. Let $L^{i,s} \subset W^{(i)}_s(\zeta^{(i)}_s)$ be the crystal lattice corresponding to the crystal base $B^{i,s}$. Suppose that $(L, B)$ is another crystal lattice and crystal base of $W^{(i)}_s(\zeta^{(i)}_s)$ such that the crystal graph of $B$ is connected. Then, $L = f(q)L^{i,s}$ holds for some $f(q) \in \mathbb{C}(q) \setminus \{0\}$, and $B$ is isomorphic to $B^{i,s}$ as a $U'_q(\mathfrak{g})$-crystal. Namely, the crystal base of $W^{(i)}_s(\zeta^{(i)}_s)$ is unique, up to a nonzero constant multiple.

**Proof.** Let $b \in B^{i,s}$ be an extremal element, and set $\mu := \text{wt} b \in (P_1)_0$. Note that the set $(B^{i,s})_{\mu}$ consists only of the element $b$ by condition (S3) of Definition 1.4.6 and hence that the $\mu$-weight space of $W^{(i)}_s(\zeta^{(i)}_s)$ is one-dimensional. Let $v \in L^{i,s}$ be an element of weight $\mu$ corresponding to the $b$ under the canonical projection $L^{i,s} \twoheadrightarrow L^{i,s}/qL^{i,s}$. Here we remark that the crystal graph of $B^{i,s}$ is connected by Proposition 1.4.10(1). By using Nakayama’s lemma, we can show that $L^{i,s}$ is equal to the $A$-module generated by all elements of the form $x_{j_1}x_{j_2} \cdots x_{j_k}v$, $j_1, j_2, \ldots, j_k \in I$, $k \geq 0$, where $A := \{f(q) \in \mathbb{C}(q) \mid f(q) \text{ is regular at } q = 0\}$, and $x_j$ is either the raising Kashiwara operator $e_j$ or the lowering Kashiwara operator $f_j$ on $W^{(i)}_s(\zeta^{(i)}_s)$ for each $j \in I$:

$$L^{i,s} = \sum_{j_1, j_2, \ldots, j_k \in I; k \geq 0} A x_{j_1}x_{j_2} \cdots x_{j_k}v. \quad (1.5.1)$$

Similarly, take an element $v' \in L$ of weight $\mu$ corresponding to a unique element $b' \in B$ of weight $\mu$. Because the crystal graph of $B$ is connected by the assumption, we obtain that

$$L = \sum_{j_1, j_2, \ldots, j_k \in I; k \geq 0} A x_{j_1}x_{j_2} \cdots x_{j_k}v' \quad (1.5.2)$$

in the same way as above.

Since the $\mu$-weight space of $W^{(i)}_s(\zeta^{(i)}_s)$ is one-dimensional as mentioned above, it follows that $v' = f(q)v$ for some $f(q) \in \mathbb{C}(q) \setminus \{0\}$. Combining this fact with (1.5.1) and (1.5.2),
we have $L = f(q)L^i$; hence, we have a $\mathbb{C}$-linear isomorphism $\Psi : L^i/qL^i \cong L/qL$ induced from the transformation $v \mapsto f(q)v$ on $W^i(q^i)$. Furthermore, we can deduce from the connectedness of the crystal bases $B^i$ and $B$ that the restriction $\Psi|_{B^i}$ of $\Psi$ to $B^i$ gives an isomorphism of $U'_q(g)$-crystals between $B^i$ and $B$. This proves the lemma.

2 Construction of perfect crystals for twisted quantum affine algebras.

From now on throughout this paper, let $g = g(A)$ be the affine Lie algebra of type $A^{(1)}_n$ ($n \geq 2$), $D^{(1)}_n$ ($n \geq 4$), or $E^{(1)}_6$, and let $\omega : I \to I$ be a nontrivial diagram automorphism satisfying the (additional) condition that $\omega(0) = 0$.

2.1 Diagram automorphisms of simply-laced affine Lie algebras. Here we give all pairs $(g, \omega)$ of an affine Lie algebra $g$ and a nontrivial diagram automorphism $\omega : I \to I$ satisfying the condition that $\omega(0) = 0$, after introducing our numbering of the index set $I$. (For the definition of the matrix $\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}$, see §2 below.)

Case (a). The affine Cartan matrix $A = (a_{ij})_{i,j \in I}$ is of type $A^{(1)}_{2n-1}$ ($n \geq 2$), and the diagram automorphism $\omega : I \to I$ is given by: $\omega(0) = 0$ and $\omega(j) = 2n - j$ for $j \in I_0 = I \setminus \{0\}$ (note that the order of $\omega$ is 2). Then the matrix $\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}$ is the affine Cartan matrix of type $D^{(2)}_{n+1}$:

\[
\begin{array}{c}
\begin{array}{c}
A: \\
0 \\
1 \\
2 \\
\vdots \\
n-1 \\
n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\hat{A}: \\
0 \\
1 \\
2 \\
\vdots \\
n-1 \\
n
\end{array}
\end{array}
\]

Case (b). The affine Cartan matrix $A = (a_{ij})_{i,j \in I}$ is of type $A^{(1)}_{2n}$ ($n \geq 1$), and the diagram automorphism $\omega : I \to I$ is given by: $\omega(0) = 0$ and $\omega(j) = 2n + 1 - j$ for $j \in I_0 = I \setminus \{0\}$ (note that the order of $\omega$ is 2). Then the matrix $\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}$ is the affine Cartan matrix of type $A^{(2)}_{2n}$.

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If $n \geq 2$, then

$$A: \begin{array}{cccccccc}
0 & 1 & 2 & \cdots & n-1 & n \\
2n & 2n-1 & \cdots & n+2 & n+1 \\
\end{array}$$

$$\hat{A}: \begin{array}{cccccccc}
0 & 1 & 2 & \cdots & n-1 & n \\
0 & 1 & \cdots & n & 0 \\
\end{array}$$

Case (c). The affine Cartan matrix $A = (a_{ij})_{i,j \in I}$ is of type $D_{n+1}^{(1)} (n \geq 3)$, and the diagram automorphism $\omega : I \to I$ is given by: $\omega(j) = j$ for $j \in I \setminus \{n, n+1\}$, and $\omega(n) = n+1$, $\omega(n+1) = n$ (note that the order of $\omega$ is 2). Then the matrix $\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}$ is the affine Cartan matrix of type $A_{2n-1}^{(2)}$:

$$A: \begin{array}{cccccccc}
0 & 1 & 2 & \cdots & n-1 & n \\
1 & 2 & 3 & \cdots & n & n+1 \\
\end{array}$$

$$\hat{A}: \begin{array}{cccccccc}
0 & 1 & 2 & \cdots & n-1 & n \\
0 & 1 & 2 & 3 & \cdots & n \\
\end{array}$$

Case (d). The affine Cartan matrix $A = (a_{ij})_{i,j \in I}$ is of type $D_{4}^{(1)}$, and the diagram automorphism $\omega : I \to I$ is given by: $\omega(0) = 0$, $\omega(1) = 1$, $\omega(2) = 3$, $\omega(3) = 4$, and $\omega(4) = 2$ (note that the order of $\omega$ is 3). Then the matrix $\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}$ is the affine Cartan matrix of type $D_{4}^{(3)}$:

$$A: \begin{array}{cccc}
0 & 1 & 3 & 4 \\
1 & 2 & 3 & 4 \\
\end{array}$$

$$\hat{A}: \begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3 \\
\end{array}$$

Case (e). The affine Cartan matrix $A = (a_{ij})_{i,j \in I}$ is of type $E_{6}^{(1)}$, and the diagram automorphism $\omega : I \to I$ is given by: $\omega(0) = 0$, $\omega(1) = 1$, $\omega(2) = 2$, $\omega(3) = 5$, $\omega(4) = 6$, $\omega(5) = 6$, $\omega(6) = 6$.
\(\omega(5) = 3,\) and \(\omega(6) = 4\) (note that the order of \(\omega\) is 2). Then the matrix \(\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}}\) is the affine Cartan matrix of type \(E_6^{(2)}\):

\[
\begin{array}{c}
A: \\
(0, 1, 2) \\
(3, 4) \\
(5, 6, 0, 1, 2, 3, 4)
\end{array}
\]

\[
\begin{array}{c}
\hat{A}: \\
(0, 1, 2, 3, 4) \\
(5, 6, 0, 1, 2, 3, 4)
\end{array}
\]

We define \(C\)-linear isomorphisms \(\omega : \mathfrak{h} \rightarrow \mathfrak{h}\) and \(\omega^* : \mathfrak{h}^* \rightarrow \mathfrak{h}^*\) by:

\[
\begin{align*}
\omega(h) &= h_{\omega(j)} \quad \text{for } j \in I, \quad \omega(d) = d, \\
(\omega^*(\lambda))(h) &= \lambda(\omega^{-1}(h)) \quad \text{for } \lambda \in \mathfrak{h}^* \text{ and } h \in \mathfrak{h}.
\end{align*}
\]

It follows that the subsets \(P^\vee, \mathfrak{h}_{\text{cl}}\), and \(P^\vee_{\text{cl}}\) of \(\mathfrak{h}\) are all stable under \(\omega \in \text{GL}(\mathfrak{h})\), and that \(P \subset \mathfrak{h}^*\) is stable under \(\omega^* \in \text{GL}(\mathfrak{h}^*)\). In addition,

\[
\begin{align*}
\omega(c) &= c, \quad \omega^*(\delta) = \delta, \\
\omega^*(\alpha_j) &= \alpha_{\omega(j)}, \quad \omega^*(\Lambda_j) = \Lambda_{\omega(j)} \quad \text{for } j \in I.
\end{align*}
\]

Also, we have a \(C(q)\)-algebra automorphism \(\omega \in \text{Aut}(U_q(\mathfrak{g}))\) such that \(\omega(E_j) = E_{\omega(j)}\), \(\omega(F_j) = F_{\omega(j)}\) for \(j \in I\), and \(\omega(q^h) = q^{\omega(h)}\) for \(h \in P^\vee\). Since \(P^\vee_{\text{cl}}\) is stable under \(\omega \in \text{GL}(\mathfrak{h})\), we see that the \(C(q)\)-subalgebra \(U'_q(\mathfrak{g})\) is stable under \(\omega \in \text{Aut}(U_q(\mathfrak{g}))\), thus obtaining a \(C(q)\)-algebra automorphism \(\omega\) of \(U'_q(\mathfrak{g})\). Further, we define a \(C\)-linear automorphism \(\omega^* : \mathfrak{h}_{\text{cl}}^* \rightarrow \mathfrak{h}_{\text{cl}}^*\) by: \(\omega^*(\Lambda_j) = \Lambda_{\omega(j)}\). Note that this \(C\)-linear automorphism of \(\mathfrak{h}_{\text{cl}}^* = (\mathfrak{h}_{\text{cl}})^*\) can be thought of as the one induced from \(\omega^* \in \text{GL}(\mathfrak{h}^*)\) since \(\mathfrak{h}_{\text{cl}}^* \cong \mathfrak{h}^*/C\delta\), as well as the contragredient map of the restriction of \(\omega \in \text{GL}(\mathfrak{h})\) to \(\mathfrak{h}_{\text{cl}}\). Then we set

\[
(\mathfrak{h}_{\text{cl}})^0 := \{ h \in \mathfrak{h}_{\text{cl}} \mid \omega(h) = h \}, \quad (\mathfrak{h}_{\text{cl}}^*)^0 := \{ \lambda \in \mathfrak{h}_{\text{cl}}^* \mid \omega^*(\lambda) = \lambda \}.
\]

2.2 Orbit Lie algebras. We choose (and fix) a complete set \(\widehat{I}\) (containing \(0 \in I\)) of representatives of the \(\omega\)-orbits in \(I\) in such a way that if \(j \in \widehat{I}\), then \(j \leq \omega^k(j)\) for all \(k \in \mathbb{Z}_{\geq 0}\) (see the figures in \S2.1). Now we set

\[
c_{ij} := \sum_{k=0}^{N_j-1} a_{i,\omega^k(j)} \quad \text{for } i, j \in \widehat{I}, \quad \text{and} \quad c_j := c_{jj} \quad \text{for } j \in \widehat{I},
\]

where \(N_j\) is the number of elements of the \(\omega\)-orbit of \(j \in \widehat{I}\) in \(I\). (In fact, \(N_j\) is equal to 1, 2, or 3.)
Remark 2.2.1 (cf. [FSS §2.2]). We see that $c_j = 2$ except the case where the pair $(\mathfrak{g}, \omega)$ is in Case (b) and $j = n$; if $c_j = 2$, then the subdiagram of the Dynkin diagram of $A$ corresponding to the $\omega$-orbit of the $j$ is of type $A_1 \times \cdots \times A_1$ ($N_j$ times). On the other hand, if the pair $(\mathfrak{g}, \omega)$ is in Case (b) and $j = n$, then $c_j = 1$; in this case, the subdiagram of the Dynkin diagram of $A$ corresponding to the $\omega$-orbit of the $j$ is of type $A_2$.

Further, we set $\tilde{a}_{ij} := 2c_{ij}/c_j$ for $i, j \in \tilde{I}$.

Lemma 2.2.2 (see [FSS §2.2]). The matrix $\hat{A} := (\hat{a}_{ij})_{i,j \in \tilde{I}}$ is a generalized Cartan matrix of (twisted) affine type. Moreover, the explicit type of the GCM $\hat{A}$ is as in §2.1

Let $\hat{\mathfrak{g}} := \mathfrak{g}(\hat{A})$ be the (affine) Kac-Moody algebra over $\mathbb{C}$ associated to the GCM $\hat{A}$ above, which is called the orbit Lie algebra (corresponding to the diagram automorphism $\omega$). Then, $\hat{\mathfrak{h}} = \left( \bigoplus_{j \in \tilde{I}} \mathbb{C} \hat{h}_j \right) \oplus \mathbb{C} \hat{d}$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$, with $\hat{\Pi}^\vee := \{ \hat{h}_j \}_{j \in \tilde{I}}$ the set of simple coroots, and $\hat{d}$ the scaling element. Denote by $\hat{\Pi} := \{ \hat{a}_j \}_{j \in \tilde{I}} \subset \hat{\mathfrak{h}}^* := (\hat{\mathfrak{h}})^*$ the set of simple roots, and $\hat{\Lambda}_j \in \hat{\mathfrak{h}}^*$, $j \in \tilde{I}$, the fundamental weights for the orbit Lie algebra $\hat{\mathfrak{g}}$ (of affine type); note that $\hat{a}_j(\hat{d}) = \delta_{j,0}$ and $\hat{\Lambda}_j(\hat{d}) = 0$ for $j \in \tilde{I}$. Let

$$\hat{\delta} := \sum_{j \in \tilde{I}} \hat{a}_j \hat{\Lambda}_j \in \hat{\mathfrak{h}}^* \quad \text{and} \quad \hat{c} := \sum_{j \in \tilde{I}} \hat{a}_j^\vee \hat{h}_j \in \hat{\mathfrak{h}}$$

be the null root and the canonical central element of $\hat{\mathfrak{g}}$, respectively. We take a dual weight lattice $\hat{P}^\vee \subset \hat{\mathfrak{h}}$ and a weight lattice $\hat{P} \subset \hat{\mathfrak{h}}^*$ as follows:

$$\hat{P}^\vee = \left( \bigoplus_{j \in \tilde{I}} \mathbb{Z} \hat{h}_j \right) \oplus \mathbb{Z} \hat{d} \subset \hat{\mathfrak{h}}, \quad \hat{P} = \left( \bigoplus_{j \in \tilde{I}} \mathbb{Z} \hat{\Lambda}_j \right) \oplus \mathbb{Z} \left( \frac{1}{\hat{a}_0} \hat{\delta} \right) \subset \hat{\mathfrak{h}}^*.$$ (2.2.3)

Define $\hat{\mathfrak{h}}_{cl}, \hat{P}^\vee_{cl} \subset \hat{\mathfrak{h}}$, and $\hat{P}_{cl} \subset \hat{\mathfrak{h}}^*_{cl} := (\hat{\mathfrak{h}}_{cl})^*$ for the orbit Lie algebra $\hat{\mathfrak{g}}$, and also define subsets $(\hat{P}_{cl})_0, \hat{P}_{cl}^+, \text{ and } (\hat{P}_{cl}^+)_s, s \in \mathbb{Z}_{\geq 0}$, of $\hat{P}_{cl}$ as in §1.3. Note that $(\hat{A}, \hat{P}_{cl}, \hat{P}^\vee_{cl}, \hat{\Pi}, \hat{\Pi}^\vee)$ is a Cartan datum for the GCM $\hat{A}$. Let $U_q(\hat{\mathfrak{g}})$ be the quantized universal enveloping algebra of the orbit Lie algebra $\hat{\mathfrak{g}}$ over $\mathbb{C}(q)$ with weight lattice $\hat{P}$, and define its $\mathbb{C}(q)$-subalgebra $U'_q(\hat{\mathfrak{g}})$ as in §1.3 (which is the quantized universal enveloping algebra of $\hat{\mathfrak{g}}$ over $\mathbb{C}(q)$ with weight lattice $\hat{P}_{cl}$). We call a crystal associated to the Cartan datum $(\hat{A}, \hat{P}_{cl}, \hat{P}^\vee_{cl}, \hat{\Pi}, \hat{\Pi}^\vee)$ a $U'_q(\hat{\mathfrak{g}})$-crystal. Further, for a proper subset $\tilde{J}$ of $\tilde{I}$, let us define the Lie subalgebra $\hat{\mathfrak{g}}_{\tilde{J}}$ of $\hat{\mathfrak{g}}$, and the $\mathbb{C}(q)$-subalgebra $U_q(\hat{\mathfrak{g}}_{\tilde{J}})$ of $U'_q(\hat{\mathfrak{g}})$ corresponding to the subset $\tilde{J}$ as in §1.3. We call a crystal associated to the Cartan datum $(\hat{A}_{\tilde{J}}, \hat{P}_{cl}, \hat{P}^\vee_{cl}, \hat{\Pi}_{\tilde{J}}, \hat{\Pi}^\vee_{\tilde{J}})$ for the GCM $\hat{A}_{\tilde{J}} := (\hat{a}_{ij})_{i,j \in \tilde{J}}$ a $U_q(\hat{\mathfrak{g}}_{\tilde{J}})$-crystal, where $\hat{\Pi}_{\tilde{J}} := \{ \hat{a}_j \}_{j \in \tilde{J}} \subset \hat{P}_{cl}$ and $\hat{\Pi}^\vee_{\tilde{J}} := \{ \hat{h}_j \}_{j \in \tilde{J}} \subset \hat{P}^\vee_{cl}$.

Now, let us define $\mathbb{C}$-linear isomorphisms $P_\omega : (\hat{\mathfrak{h}}_{cl})^0 \to \hat{\mathfrak{h}}_{cl}$ from the fixed point subspace $(\hat{\mathfrak{h}}_{cl})^0$ onto $\hat{\mathfrak{h}}_{cl} = \bigoplus_{j \in \tilde{I}} \mathbb{C} \hat{h}_j$, and $P_\omega^* : \hat{\mathfrak{h}}_{cl}^* \to (\hat{\mathfrak{h}}_{cl}^*)^0$ from $\hat{\mathfrak{h}}_{cl}^* = \bigoplus_{j \in \tilde{I}} \mathbb{C} \hat{\Lambda}_j$ onto the fixed point subspace $(\hat{\mathfrak{h}}_{cl}^*)^0$ by:

$$P_\omega \left( \frac{1}{N_j} \sum_{k=0}^{N_j-1} h_{\omega_k(j)} \right) = \hat{h}_j \quad \text{and} \quad P_\omega^*(\hat{\Lambda}_j) = \sum_{k=0}^{N_j-1} \Lambda_{\omega_k(j)} \quad \text{for } j \in \tilde{I} \quad (2.2.4)$$
(here note that \((\mathfrak{h}_c^0)^*\) can be identified with \((\mathfrak{h}_c^*)^0\) in natural way). Then it is easily seen that \((P^*_\omega(\tilde{\lambda}))(h) = \tilde{\lambda}(P(\omega))(h)\) for \(\tilde{\lambda} \in \mathfrak{h}_c^*\) and \(h \in (\mathfrak{h}_c)^0\), and that

\[
P(\omega)(c) = \tilde{c}, \quad P^*_\omega(\tilde{\alpha}_j) = \frac{2}{c_j} \sum_{k=0}^{N_j-1} \alpha_{\omega^k(j)} \quad \text{for} \ j \in \tilde{I}.
\]  

(2.2.5)

Furthermore, we can identify the Weyl group \(\widehat{W} := \langle \tilde{r}_j \mid j \in \tilde{I} \rangle\) of the orbit Lie algebra \(\widehat{\mathfrak{g}}\) with the subgroup \(\widehat{W} := \{w \in W \mid \omega^*w = w\omega^*\}\) of the Weyl group \(W = \langle r_j \mid j \in I \rangle\) of \(\mathfrak{g}\) as follows. Define \(w_j \in W\) by:

\[
w_j = \begin{cases} r_j r_{\omega(j)} r_j & \text{if} \ c_j = 1, \\ r_j r_{\omega(j)} \cdots r_{\omega^{N_j-1}(j)} & \text{if} \ c_j = 2, \end{cases}
\]  

(2.2.6)

for each \(j \in \tilde{I}\) (see Remark 2.2.1). Then it follows that \(w_j \in \tilde{W}\) for all \(j \in \tilde{I}\). Also, we see from [FRS, §3] that there exists a group isomorphism \(\Theta : \tilde{W} \rightarrow \tilde{W}\) such that \(\Theta(\tilde{w})|_{(\mathfrak{h}_c)^0} = P^*_\omega \circ \tilde{w} \circ (P(\omega))^{-1}\) for each \(\tilde{w} \in \tilde{W}\), and \(\Theta(\tilde{r}_j) = w_j\) for all \(j \in \tilde{I}\).

### 2.3 Fixed point subsets

Let \(\tilde{I} \subset I\) be the index set (chosen as in §2.2) for the orbit Lie algebra \(\widehat{\mathfrak{g}}\) corresponding to the \(\omega\). Let us fix (arbitrarily) \(i \in \tilde{I}_0 := \tilde{I} \setminus \{0\}\) and \(s \in \mathbb{Z}_{\geq 1}\). For the rest of this section, we make the following assumption (cf. Conjecture 1.5.1):

**Assumption 2.3.1.** There exists some \(\zeta^{(i)}_s \in \mathbb{C}(q)^\times\) (independent of \(0 \leq k \leq N_i - 1\)) such that for every \(0 \leq k \leq N_i - 1\), the KR module \(W^s(\omega^k(i))(\zeta^{(i)}_s)\) over \(U'_q(\mathfrak{g})\) has a crystal base, denoted by \(B^{\omega^k(i), s}\). Further, the \(B^{\omega^k(i), s}, 0 \leq k \leq N_i - 1\), are all perfect \(U'_q(\mathfrak{g})\)-crystals of level \(s\).

**Remark 2.3.2.** Let \(0 \leq k \leq N_i - 1\). Then, since the \(U'_q(\mathfrak{g})\)-crystal \(B^{\omega^k(i), s}\) is perfect (and hence simple) by Assumption 2.3.1 it follows from Lemma 1.4.8 that there exists a unique extremal element of \(B^{\omega^k(i), s}\), denoted by \(u_{\omega^k(i), s}\), such that \((\text{wt } u_{\omega^k(i), s})(h_j) \geq 0\) for all \(j \in I_0\). In addition, we can show that \(\text{wt } u_{\omega^k(i), s} = s \omega_{\omega^k(i)}\), where \(\omega_{\omega^k(i)} := \Lambda_{\omega^k(i)} - a_{\omega^k(i)}^\vee \Lambda_0 \in P_c\).

**Remark 2.3.3** (see [HKOTY, Remark 2.3]). For Cases (a) and (b), we know from [KMN] that Assumption 2.3.1 is satisfied for all \(i \in \tilde{I}_0\) and \(s \in \mathbb{Z}_{\geq 1}\). For Case (c) (resp., Case (d)), we know from [KMN] that Assumption 2.3.1 is satisfied if \(i = 1, n\) (resp., \(i = 2\)), and from [Ko] that Assumption 2.3.1 is satisfied if \(i \neq n\) and \(s = 1\) (resp., if \(s = 1\)).

First, we define a bijection \(\tau_\omega : B^{i,s} \rightarrow B^{\omega(i), s}\) such that \(\tau_\omega \circ e_j = e_{\omega(j)} \circ \tau_\omega\) and \(\tau_\omega \circ f_j = f_{\omega(j)} \circ \tau_\omega\) for all \(j \in I\) (\(\tau_\omega(\theta)\) is understood to be \(\theta\)), and such that \(\text{wt } (\tau_\omega(b)) = \omega^*(\text{wt } b)\) for each \(b \in B^{i,s}\) as follows. Let \(\rho : U'_q(\mathfrak{g}) \rightarrow \text{End}_{\mathbb{C}(q)}(W^i(\omega^k(i))(\zeta^{(i)}_s))\) be the representation map affording the KR module \(W^i(\omega^k(i))(\zeta^{(i)}_s)\) over \(U'_q(\mathfrak{g})\). It immediately follows that the the representation of \(U'_q(\mathfrak{g})\) on the (same) \(\mathbb{C}(q)\)-vector space \(W^i(\omega^k(i))(\zeta^{(i)}_s)\) given by \(\rho \circ \omega^{-1}\), denoted
by \((\rho \circ \omega^{-1}, W_s^{(i)}(\zeta_s^{(i)}))\), is finite-dimensional and irreducible. In addition, we can easily check that if \(P_j(u) \in \mathbb{C}(q)[u], j \in I_0\), are the Drinfeld polynomials of the KR module \(W_s^{(i)}(\zeta_s^{(i)})\) over \(U'_q(\mathfrak{g})\) (see [135]), then the Drinfeld polynomials \(P_{j}^{\omega}(u) \in \mathbb{C}(q)[u], j \in I_0\), of the representation \((\rho \circ \omega^{-1}, W_s^{(i)}(\zeta_s^{(i)}))\) of \(U'_q(\mathfrak{g})\) are given by: \(P_{j}^{\omega}(u) = P_{\omega^{-1}(j)}(u)\) for each \(j \in I_0\). (Here we have used Assumption 2.3.1 that the \(\zeta_s^{(i)} \in \mathbb{C}(q)^{\times}\) is independent of \(0 \leq k \leq N_i - 1\).) Because the finite-dimensional irreducible \(U'_q(\mathfrak{g})\)-modules (of type 1) are parametrized by their Drinfeld polynomials up to \(U'_q(\mathfrak{g})\)-module isomorphism, it follows that the representation \((\rho \circ \omega^{-1}, W_s^{(i)}(\zeta_s^{(i)}))\) of \(U'_q(\mathfrak{g})\) is equivalent to the KR module \(W_s^{(i)}(\zeta_s^{(i)})\) over \(U'_q(\mathfrak{g})\). We denote by \(\tau_\omega : W_s^{(i)}(\zeta_s^{(i)}) \to W_s^{(i)}(\zeta_s^{(i)})\) an intertwining map between these two representations of \(U'_q(\mathfrak{g})\). Namely, \(\tau_\omega : W_s^{(i)}(\zeta_s^{(i)}) \to W_s^{(i)}(\zeta_s^{(i)})\) denotes a \(\mathbb{C}(q)\)-linear isomorphism such that

\[
\tau_\omega(xv) = \omega(x)\tau_\omega(v) \quad \text{for } x \in U'_q(\mathfrak{g}) \text{ and } v \in W_s^{(i)}(\zeta_s^{(i)}).
\]

It immediately follows from 2.3.1 that for each \(\mu \in P_d\), the \(\mu\)-weight space of \(W_s^{(i)}(\zeta_s^{(i)})\) is sent to the \(\omega^*(\mu)\)-weight space of \(W_s^{(i)}(\zeta_s^{(i)})\) under \(\tau_\omega\). Also, we can easily deduce that

\[
\tau_\omega \circ e_j = e_{\omega(j)} \circ \tau_\omega, \quad \text{and} \quad \tau_\omega \circ f_j = f_{\omega(j)} \circ \tau_\omega \quad \text{for all } j \in I,
\]

where \(e_j\) (resp., \(f_j\), \(j \in I\), denote the raising (resp., lowering) Kashiwara operators on \(W_s^{(i)}(\zeta_s^{(i)})\), and also those on \(W_s^{(i)}(\zeta_s^{(i)})\). Let us denote by \(\mathcal{L}^{i,s}\) the crystal lattice of \(W_s^{(i)}(\zeta_s^{(i)})\). By 2.3.2, we see that the image \(\tau_\omega(\mathcal{L}^{i,s})\) of the \(\mathcal{L}^{i,s}\) is stable under the action of the Kashiwara operators \(e_j\) and \(f_j\), \(j \in I\), on \(W_s^{(i)}(\zeta_s^{(i)})\). Hence these Kashiwara operators on \(W_s^{(i)}(\zeta_s^{(i)})\) induces operators, denoted also by \(e_j\) and \(f_j\), \(j \in I\), on the \(\mathbb{C}\)-vector space \(\tau_\omega(\mathcal{L}^{i,s})/q\tau_\omega(\mathcal{L}^{i,s})\). If \(\tau_\omega : \mathcal{L}^{i,s}/q\mathcal{L}^{i,s} \to \tau_\omega(\mathcal{L}^{i,s})/q\tau_\omega(\mathcal{L}^{i,s})\) denotes the induced \(\mathbb{C}\)-linear map, then it follows from 2.3.2 that the set \(\tau_\omega(\mathcal{B}^{i,s}) \cup \{0\}\) is stable under the Kashiwara operators \(e_j\) and \(f_j\), \(j \in I\), on \(\tau_\omega(\mathcal{L}^{i,s})/q\tau_\omega(\mathcal{L}^{i,s})\), which means that \((\tau_\omega(\mathcal{L}^{i,s}), \tau_\omega(\mathcal{B}^{i,s}))\) is a crystal base of \(W_s^{(i)}(\zeta_s^{(i)})\). Therefore, it follows from Lemma 5.5.3 that \(\tau_\omega(\mathcal{B}^{i,s}) \cong \mathcal{B}^{i,s}\) as \(U'_q(\mathfrak{g})\)-crystals. Thus we have obtained a bijection \(\tau_\omega : \mathcal{B}^{i,s} \to \mathcal{B}^{i,s}\) such that

\[
\tau_\omega \circ e_j = e_{\omega(j)} \circ \tau_\omega, \quad \text{and} \quad \tau_\omega \circ f_j = f_{\omega(j)} \circ \tau_\omega \quad \text{for all } j \in I,
\]

\[
\text{wt}(\tau_\omega(b)) = \omega^*(\text{wt } b) \quad \text{for each } b \in \mathcal{B}^{i,s}.
\]

Here (and below) we understand that \(\tau_\omega(\theta) = \theta\). Similarly, for each \(1 \leq k \leq N_i - 1\), we obtain a bijection \(\tau_\omega : \mathcal{B}^{k(i),s} \to \mathcal{B}^{k+1(i),s}\) such that \(\tau_\omega \circ e_j = e_{\omega(j)} \circ \tau_\omega\) and \(\tau_\omega \circ f_j = f_{\omega(j)} \circ \tau_\omega\) for all \(j \in I\), and such that \(\text{wt}(\tau_\omega(b)) = \omega^*(\text{wt } b)\) for each \(b \in \mathcal{B}^{k(i),s}\).

Next, we set

\[
\mathcal{B}^{i,s} := \begin{cases} 
\mathcal{B}^{i,s} & \text{if } N_i = 1, \\
\mathcal{B}^{i,s} \otimes \mathcal{B}^{(i),s} & \text{if } N_i = 2, \\
\mathcal{B}^{i,s} \otimes \mathcal{B}^{(i),s} \otimes \mathcal{B}^{(2),s} & \text{if } N_i = 3.
\end{cases}
\]
Since the $B^{\omega^k(i),s}$, $0 \leq k \leq N_i - 1$, are perfect $U_q'(g)$-crystals of (the same) level $s$ by Assumption 2.3.1, it follows from Lemma 1.4.14 that $\tilde{B}^{i,s}$ is a perfect $U_q'(g)$-crystal of level $s$. We define an action of the diagram automorphism $\omega$ on $\tilde{B}^{i,s}$ as follows. If $N_i = 1$, then $\omega : \tilde{B}^{i,s} \rightarrow \tilde{B}^{i,s}$ is defined to be $\tau_\omega$. If $N_i = 2$, then we first define a bijection from $B^{i,s} \otimes B^{\omega(i),s}$ onto $B^{\omega(i),s} \otimes B^{i,s}$ by: $b_1 \otimes b_2 \mapsto \tau_\omega(b_1) \otimes \tau_\omega(b_2)$ for $b_1 \otimes b_2 \in B^{i,s} \otimes B^{\omega(i),s}$. By Proposition 1.4.15 (1), we have an isomorphism (a combinatorial $R$-matrix) $B^{\omega(i),s} \otimes B^{i,s} \sim \rightarrow B^{i,s} \otimes B^{\omega(i),s}$ of $U_q'(g)$-crystals. We now define $\omega : \tilde{B}^{i,s} \rightarrow \tilde{B}^{i,s}$ to be the composition of these maps:

$$\omega : \tilde{B}^{i,s} = B^{i,s} \otimes B^{\omega(i),s} \overset{\tau_\omega \otimes \tau_\omega}{\sim} B^{\omega(i),s} \otimes B^{i,s} \sim \rightarrow B^{i,s} \otimes B^{\omega(i),s} = \tilde{B}^{i,s}. \quad (2.3.5)$$

Similarly, if $N_i = 3$, then we define an action of $\omega$ on $\tilde{B}^{i,s}$ to be the composition of the map $\tau_\omega \otimes \tau_\omega \otimes \tau_\omega$ with combinatorial $R$-matrices:

$$\omega : \tilde{B}^{i,s} = B^{i,s} \otimes B^{\omega(i),s} \otimes B^{\omega^2(i),s} \overset{\tau_\omega \otimes \tau_\omega \otimes \tau_\omega}{\sim} B^{\omega(i),s} \otimes B^{\omega^2(i),s} \otimes B^{i,s} \sim \rightarrow B^{i,s} \otimes B^{\omega(i),s} \otimes B^{\omega^2(i),s} = \tilde{B}^{i,s}. \quad (2.3.6)$$

In all cases above, we can deduce from the tensor product rule for crystals, (2.3.5), and the comment after (2.3.3) that

$$\omega \circ e_j = e_{\omega(j)} \circ \omega \quad \text{and} \quad \omega \circ f_j = f_{\omega(j)} \circ \omega \quad \text{on } \tilde{B}^{i,s} \text{ for all } j \in I,$$

$$\text{wt}(\omega(b)) = \omega^*(\text{wt } b) \quad \text{for each } b \in \tilde{B}^{i,s},$$

where $\omega(\theta)$ is understood to be $\theta$. Finally, we set

$$\tilde{B}^{i,s} := \{ b \in \tilde{B}^{i,s} | \omega(b) = b \}. \quad (2.3.8)$$

Note that the weights of elements of $\tilde{B}^{i,s}$ are all contained in $(P_{cl})_0 \cap (\mathfrak{h}_{cl})^0$ by condition (S1) of Definition 1.4.6 and the second equality of (2.3.7).

2.4 Main result. For each $j \in \tilde{I}$, we define $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$ on $\tilde{B}^{i,s} \cup \{ \theta \}$ by:

$$\tilde{e}_j = \begin{cases} x_j x_{\omega(j)}^2 x_j & \text{if } c_j = 1, \\
 x_j x_{\omega(j)} \cdots x_{\omega^{N_j-1}(j)} & \text{if } c_j = 2, \end{cases} \quad (2.4.1)$$

where $x$ is either $e$ or $f$. The main result of this paper is the following theorem.

**Theorem 2.4.1.** Let $i \in \tilde{I}_0 = \tilde{I} \setminus \{ 0 \}$ and $s \in \mathbb{Z}_{\geq 1}$ (fixed as in (2.3.3)). We keep Assumption 2.3.1. Then, the subset $\tilde{B}^{i,s} \cup \{ \theta \}$ of $\tilde{B}^{i,s} \cup \{ \theta \}$ is stable under the $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$ on $\tilde{B}^{i,s} \cup \{ \theta \}$ for all $j \in \tilde{I}$. Moreover, if we set

$$\tilde{\text{wt}} b := (P_{\omega})^{-1}(\text{wt } b) \in \tilde{P}_{cl} \quad \text{for } b \in \tilde{B}^{i,s},$$

$$\tilde{e}_j(b) := \max \{ m \geq 0 | (\tilde{e}_j)^m b \neq \theta \} \quad \text{for } b \in \tilde{B}^{i,s} \text{ and } j \in \tilde{I}, \quad (2.4.2)$$

$$\tilde{f}_j(b) := \max \{ m \geq 0 | (\tilde{f}_j)^m b \neq \theta \} \quad \text{for } b \in \tilde{B}^{i,s} \text{ and } j \in \tilde{I},$$

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then the set $\tilde{B}^{i,s}$ equipped with the $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$, $j \in \tilde{I}$, the maps $\tilde{\varpi} : \tilde{B}^{i,s} \to \hat{P}_{cl}$, and $\tilde{\varepsilon}_j, \tilde{\varphi}_j : \tilde{B}^{i,s} \to \mathbb{Z}_{\geq 0}$, $j \in \tilde{I}$, becomes a perfect $U'_q(\hat{g})$-crystal of level $s$.

We will establish Theorem 2.4.1 under the following plan. First, in §3.2 we show that $\tilde{B}^{i,s}$ is a regular $U'_q(\hat{g})$-crystal. Next, in §4.2 we prove that the $U'_q(\hat{g})$-crystal $\tilde{B}^{i,s}$ is simple. Finally, in §1.3 we show that the level of $\tilde{B}^{i,s}$ is equal to $s$, and that the restrictions of the maps $\tilde{\varepsilon}, \tilde{\varphi}$ to $(\tilde{B}^{i,s})_{\min}$ induce bijections $(\tilde{B}^{i,s})_{\min} \to (\hat{P}_{cl}^+)_{s}$, where the maps $\tilde{\varepsilon}, \tilde{\varphi} : \tilde{B}^{i,s} \to \hat{P}_{cl}^+$ are defined as in (1.4.2), and the set $(\tilde{B}^{i,s})_{\min}$ is defined as in (1.4.4).

3 Fixed point subsets of crystals under the action of $\omega$.

Let $g = g(A)$ be the affine Lie algebra of type $A_n^{(1)}$ ($n \geq 2$), $D_n^{(1)}$ ($n \geq 4$), or $E_6^{(1)}$, and let $\omega : I \to I$ be a nontrivial diagram automorphism satisfying the condition that $\omega(0) = 0$.

3.1 Fixed point subsets of crystal bases. Let us fix a proper subset $J$ of $I$ such that $\omega(J) = J$. For an integral weight $\lambda \in P_{cl}$ that is dominant with respect to the simple coroots $h_j$, $j \in J$, which we call a $J$-dominant integral weight, we denote by $V_J(\lambda)$ the integrable highest weight $U_q(g_J)$-module of highest weight $\lambda$. Further, let us denote by $B_J(\lambda)$ the crystal base of $V_J(\lambda)$ with raising Kashiwara operators $e_j$, $j \in J$, and lowering Kashiwara operators $f_j$, $j \in J$.

Let us take (and fix) a $J$-dominant integral weight $\lambda \in P_{cl}$ such that $\omega^*(\lambda) = \lambda$. Then, as in [NS1 §3.2], we obtain an action $\omega : B_J(\lambda) \to B_J(\lambda)$ of the diagram automorphism $\omega$ on the crystal base $B_J(\lambda)$ satisfying the condition:

$$
\omega \circ e_j = e_{\omega(j)} \circ \omega \quad \text{and} \quad \omega \circ f_j = f_{\omega(j)} \circ \omega \quad \text{for all } j \in I,
$$

$$
\text{wt}(\omega(b)) = \omega^*(\text{wt } b) \quad \text{for each } b \in B_J(\lambda),
$$

where $\omega(\theta)$ is understood to be $\theta$. We set

$$
B_J^\omega(\lambda) := \{ b \in B_J(\lambda) \mid \omega(b) = b \}. \quad (3.1.2)
$$

It immediately follows from (3.1.1) that $\text{wt } b \in P_{cl} \cap (h_{\text{cl}}^*)^0$ for all $b \in B_J^\omega(\lambda)$. Set $\tilde{J} := J \cap \tilde{I} \subseteq \tilde{I}$. For an integral weight $\tilde{\lambda} \in \hat{P}_{cl}$ that is dominant with respect to the simple coroots $\tilde{h}_j$, $j \in \tilde{J}$, which we call a $\tilde{J}$-dominant integral weight, we denote by $\hat{V}_J(\tilde{\lambda})$ the integrable highest weight $U_q(\hat{g}_J)$-module of highest weight $\tilde{\lambda}$. Also, $\hat{B}_J(\tilde{\lambda})$ denotes the crystal base of $\hat{V}_J(\tilde{\lambda})$. Further, for each $j \in \tilde{J}$, we define the $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$ on $B_J(\lambda) \cup \{ \theta \}$ by:

$$
\tilde{e}_j = \begin{cases} 
  x_j x_{\omega(j)}^2 x_j & \text{if } c_j = 1, \\
  x_j x_{\omega(j)} \cdots x_{\omega^{n_j-1}(j)} & \text{if } c_j = 2,
\end{cases} \quad (3.1.3)
$$

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where $x$ is either $e$ or $f$.

We know the following theorem from [NS2 Theorem 2.2.1 (1) – (3)]: note that the restriction $\omega|_J$ of $\omega$ to the subset $J$ of $I$ is a diagram automorphism for the finite-dimensional, reductive Lie subalgebra $g_J$ of $g$, and the Lie subalgebra $\hat{g}_J$ of $\hat{g}$ can be thought of as the orbit Lie algebra of $g_J$ corresponding to the $\omega|_J$.

**Theorem 3.1.1.** Let $\lambda \in P_{cl}$ be a $J$-dominant integral weight such that $\omega^*(\lambda) = \lambda$, and set $\hat{\lambda} := (P^*_\omega)^{-1}(\lambda)$. Then, the subset $B^*_J(\lambda) \cup \{\emptyset\}$ of $B_J(\lambda) \cup \{\emptyset\}$ is stable under the $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$ on $B_J(\lambda) \cup \{\emptyset\}$ for all $j \in \hat{J}$. Moreover, the set $B^*_J(\lambda)$ equipped with the $\omega$-Kashiwara operators $\tilde{e}_j$, $\tilde{f}_j$, $j \in \hat{J}$, and the maps

\[
\begin{align*}
\wt b &:= (P^*_\omega)^{-1}(\wt b) \in \hat{P}_{cl} \quad \text{for } b \in B^*_J(\lambda), \\
\tilde{e}_j(b) &:= \max\{m \geq 0 \mid (\tilde{e}_j)^m b \neq \emptyset \} \in \mathbb{Z}_{\geq 0} \quad \text{for } b \in B^*_J(\lambda) \text{ and } j \in \hat{J}, \\
\tilde{f}_j(b) &:= \max\{m \geq 0 \mid (\tilde{f}_j)^m b \neq \emptyset \} \in \mathbb{Z}_{\geq 0} \quad \text{for } b \in B^*_J(\lambda) \text{ and } j \in \hat{J},
\end{align*}
\]

becomes a $U_q(\hat{g}_J)$-crystal isomorphic to the crystal base $\hat{B}_J(\hat{\lambda})$ of the integrable highest weight $U_q(\hat{g}_J)$-module $\hat{V}_J(\hat{\lambda})$ of highest weight $\hat{\lambda}$.

For each $m \in \mathbb{Z}_{\geq 1}$ and $j \in \hat{J}$, we define operators $\tilde{e}(m)_j$ and $\tilde{f}(m)_j$ on $B(\lambda) \cup \{\emptyset\}$ by:

\[
\tilde{x}(m)_j = \begin{cases} 
  x_j^m x_{\omega(j)}^m x_j^m & \text{if } c_j = 1, \\
  x_j^m x_{\omega(j)}^m \cdots x_{\omega N_j - 1(j)}^m & \text{if } c_j = 2,
\end{cases}
\]

where $x$ is either $e$ or $f$. We know the following from [NS2 Theorem 2.2.1 (4)].

**Proposition 3.1.2.** Let $\lambda \in P_{cl}$ be a $J$-dominant integral weight such that $\omega^*(\lambda) = \lambda$. Then, for every $m \in \mathbb{Z}_{\geq 1}$ and $j \in \hat{J}$, we have $\tilde{e}(m)_j = (\tilde{e}_j)^m$ and $\tilde{f}(m)_j = (\tilde{f}_j)^m$ on $B^*_J(\lambda) \cup \{\emptyset\}$.

**Proposition 3.1.3.** Let $\lambda \in P_{cl}$ be a $J$-dominant integral weight such that $\omega^*(\lambda) = \lambda$. Then, for each $b \in B^*_J(\lambda)$, we have $\tilde{e}_j(b) = \varepsilon_{\omega^k(j)}(b)$ and $\tilde{f}_j(b) = \varphi_{\omega^k(j)}(b)$ for all $j \in \hat{J}$ and $0 \leq k \leq N_j - 1$.

**Proof.** Let $b \in B^*_J(\lambda)$. Then we see from (3.1.1) that $\varepsilon_{\omega^k(j)}(b) = \varepsilon_j(b)$ and $\varphi_{\omega^k(j)}(b) = \varphi_j(b)$ for all $j \in \hat{J}$ and $0 \leq k \leq N_j - 1$. So, we need only show that $\tilde{e}_j(b) = \varepsilon_j(b)$ and $\tilde{f}_j(b) = \varphi_j(b)$. But, these equalities follow from [NS2 Lemma 2.1.3 and Theorem 2.2.1 (2)].

### 3.2 Fixed point subsets of regular crystals

Let $B$ be a regular $U'_q(g)$-crystal with an action $\omega : B \to B$ of the diagram automorphism $\omega$ satisfying the condition:

\[
\omega \circ e_j = e_{\omega(j)} \circ \omega \quad \text{and} \quad \omega \circ f_j = f_{\omega(j)} \circ \omega \quad \text{for all } j \in I,
\]

\[
\wt(\omega(b)) = \omega^*(\wt b) \quad \text{for each } b \in B.
\]
Here (and below) we understand that $\omega(\theta) = \theta$. We set

$$\mathcal{B}^\omega := \{ b \in \mathcal{B} \mid \omega(b) = b \}, \quad (3.2.2)$$

and assume that $\mathcal{B}^\omega \neq \emptyset$. Note that $\text{wt} b \in P_{cl} \cap (\mathfrak{h}^{\ast}_{cl})^0$ for all $b \in \mathcal{B}^\omega$ by the second equality of (3.2.1). For each $j \in \tilde{I}$, we define $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$ on $\mathcal{B} \cup \{ \theta \}$ by:

$$\tilde{x}_j = \begin{cases} x_j x_{\omega(j)}^2 x_j & \text{if } c_j = 1, \\ x_j x_{\omega(j)} \cdots x_{\omega^{N_j-1}(j)} & \text{if } c_j = 2, \end{cases} \quad (3.2.3)$$

where $x$ is either $e$ or $f$. Further, we define maps $\hat{\omega} : \mathcal{B}^\omega \to P_{cl}$ and $\tilde{\omega}_j, \tilde{\varphi}_j : \mathcal{B}^\omega \to \mathbb{Z}_{\geq 0}$, $j \in \tilde{I}$, by:

$$\begin{cases} \hat{\omega} b := (P^{\ast}_{cl})^{-1}(\text{wt} b) \in \tilde{P}_{cl} & \text{for } b \in \mathcal{B}^\omega, \\ \tilde{\omega}_j(b) := \max\{ m \geq 0 \mid (\tilde{e}_j)^m b \neq \theta \} \in \mathbb{Z}_{\geq 0} & \text{for } b \in \mathcal{B}^\omega \text{ and } j \in \tilde{I}, \\ \tilde{\varphi}_j(b) := \max\{ m \geq 0 \mid (\tilde{f}_j)^m b \neq \theta \} \in \mathbb{Z}_{\geq 0} & \text{for } b \in \mathcal{B}^\omega \text{ and } j \in \tilde{I}. \end{cases} \quad (3.2.4)$$

**Proposition 3.2.1.** Let $\mathcal{B}$ be a regular $U'_q(\mathfrak{g})$-crystal with an action $\omega : \mathcal{B} \to \mathcal{B}$ of the diagram automorphism $\omega$ satisfying (3.2.1). Then, the subset $\mathcal{B}^\omega \cup \{ \theta \}$ of $\mathcal{B} \cup \{ \theta \}$ is stable under the $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$ on $\mathcal{B} \cup \{ \theta \}$ for all $j \in \tilde{I}$. Moreover, the the fixed point subset $\mathcal{B}^\omega$ equipped with the maps $\hat{\omega}, \tilde{\omega}_j, \tilde{\varphi}_j, j \in \tilde{I}$, becomes a regular $U'_q(\mathfrak{g})$-crystal.

**Proof.** **Step 1.** For a proper subset $\tilde{J}$ of $\tilde{I}$, we set $J := \{ \omega^k(j) \mid 0 \leq k \leq N_j - 1, j \in \tilde{J} \} \subseteq I$. Since $\mathcal{B}$ is a regular $U'_q(\mathfrak{g})$-crystal, it follows that $\mathcal{B}$ is isomorphic as a $U_q(\mathfrak{g}_J)$-crystal to the crystal base of an integrable $U_q(\mathfrak{g}_J)$-module, and hence to a direct sum of the crystal bases of integrable highest weight $U_q(\mathfrak{g}_J)$-modules. Namely, there exists an isomorphism of $U_q(\mathfrak{g}_J)$-crystals:

$$\Psi_J : \mathcal{B} \tilde{\rightarrow} \mathcal{B}_J(\lambda_1) \sqcup \mathcal{B}_J(\lambda_2) \sqcup \cdots \sqcup \mathcal{B}_J(\lambda_p), \quad (3.2.5)$$

for some $J$-dominant integral weights $\lambda_1, \lambda_2, \ldots, \lambda_p \in P_{cl}$. Put $\mathcal{B}_t := \Psi_J^{-1}(\mathcal{B}_J(\lambda_t))$, and $b_t := \Psi_J^{-1}(v_{\lambda_t})$ for $1 \leq t \leq p$, where $v_{\lambda_t}$ is the highest weight element of $\mathcal{B}_J(\lambda_t)$. Assume that $\mathcal{B}^\omega \cap \mathcal{B}_t \neq \emptyset$ for any $1 \leq t \leq p'$, and $\mathcal{B}^\omega \cap \mathcal{B}_t = \emptyset$ for all $p' + 1 \leq t \leq p$ ($1 \leq p' \leq p$). Then the highest weight elements $b_t \in \mathcal{B}_t$, $1 \leq t \leq p'$, are all fixed by $\omega : \mathcal{B} \to \mathcal{B}$, i.e., $\omega(b_t) = b_t$ for all $1 \leq t \leq p'$. Indeed, if $1 \leq t \leq p'$ and $b \in \mathcal{B}^\omega \cap \mathcal{B}_t \neq \emptyset$, then there exist $j_1, j_2, \ldots, j_t \in J$ such that $b_t = e_{j_1} e_{j_2} \cdots e_{j_t} b$. So, it follows from (3.2.1) that $\omega(b_t) = e_{\omega(j_1)} e_{\omega(j_2)} \cdots e_{\omega(j_t)} b$, since $\omega(b) = b$. Here we note that $\omega(j_1), \omega(j_2), \ldots, \omega(j_t) \in J$, since $J$ is stable under $\omega$. Thus, because $\mathcal{B}_t$ is a connected component of $\mathcal{B}$ regarded as a $U_q(\mathfrak{g}_J)$-crystal, it follows that $\omega(b_t) = e_{\omega(j_1)} e_{\omega(j_2)} \cdots e_{\omega(j_t)} b$ is also contained in $\mathcal{B}_t$. In addition, we see from (3.2.1) that $\omega(b_t) \in \mathcal{B}_t$ is the highest weight element with respect
to $e_j$, $j \in J$. Therefore, we conclude that $\omega(b_t) = b_t$ by the uniqueness of the highest weight element in $B_t$, which is isomorphic to $B_j(\lambda_t)$ as a $U_q(\mathfrak{g}_J)$-crystal, and hence that $\omega^*(\lambda_t) = \lambda_t$. Note also that $\omega(B_t) = B_t$ since $B_t$ is a connected $U_q(\mathfrak{g}_J)$-crystal. Moreover, since $B_t$ is connected as a $U_q(\mathfrak{g}_J)$-crystal, it follows from (3.2.6) that the following diagram commutes:

$\begin{align*}
B_t & \xrightarrow{\sim} B_j(\lambda_t) \\
\omega & \downarrow \hspace{3cm} \downarrow \omega \\
B_t & \xrightarrow{\sim} B_j(\lambda_t).
\end{align*}$

Hence we deduce that

$$\Psi_J(B^\omega) = B^\omega_j(\lambda_1) \sqcup B^\omega_j(\lambda_2) \sqcup \cdots \sqcup B^\omega_j(\lambda_{p^*}).$$

(3.2.6)

Note that by Theorem 3.1.1, the set on the right-hand side of (3.2.6), equipped with the $\omega$-Kashiwara operators, becomes a $U_q(\mathfrak{g}_J)$-crystal isomorphic to the crystal base $\hat{B}_j(\lambda_1) \sqcup \hat{B}_j(\lambda_2) \sqcup \cdots \sqcup \hat{B}_j(\lambda_{p^*})$ of the integrable $U_q(\mathfrak{g}_J)$-module $\hat{V}_j(\lambda_1) \oplus \hat{V}_j(\lambda_2) \oplus \cdots \oplus \hat{V}_j(\lambda_{p^*})$, where $\hat{\lambda}_t := (P^*_{\lambda_t}^{-1}(\lambda_t))$ for $1 \leq t \leq p'$.

**Step 2.** First we show that the set $B^\omega \cup \{\theta\}$ is stable under the $\omega$-Kashiwara operators $\tilde{e}_j$, $j \in \hat{I}$. Let us fix (arbitrarily) $j \in \hat{I}$, and let $b \in B^\omega$ be such that $\tilde{e}_j b \neq \theta$. Set $\hat{J} := \{j\} \subseteq \hat{I}$, $J := \{\omega^k(j) \mid 0 \leq k \leq N_j - 1\} \subseteq I$, and let $\Psi_J$ be the isomorphism (3.2.5) of $U_q(\mathfrak{g}_J)$-crystals. Then, from the definitions (3.1.3) and (3.2.6) of the $\omega$-Kashiwara operator $\tilde{e}_j$ on $(B_j(\lambda_1) \sqcup B_j(\lambda_2) \sqcup \cdots \sqcup B_j(\lambda_{p^*})) \cup \{\theta\}$ and the one on $B \cup \{\theta\}$, respectively, we see that $\tilde{e}_j \circ \Psi_J = \Psi_J \circ \tilde{e}_j$. Namely, we have $\tilde{e}_j b = \Psi_J^{-1}(\tilde{e}_j(\Psi_J(b)))$. Also, we know from (3.2.6) that the image $\Psi_J(B^\omega)$ is a disjoint union of the fixed point subsets $B^\omega_j(\lambda_t)$ under the action of $\omega$ of the crystal bases $B_j(\lambda_t)$ of integrable highest weight $U_q(\mathfrak{g}_J)$-modules. Hence, we see by Theorem 3.1.1 that $\tilde{e}_j(\Psi_J(b))$ is contained in $\Psi_J(B^\omega)$. Consequently, $\tilde{e}_j b = \Psi_J^{-1}(\tilde{e}_j(\Psi_J(b)))$ is contained in $B^\omega$. Similarly, we can show that the set $B^\omega \cup \{\theta\}$ is stable under the $\omega$-Kashiwara operators $\tilde{f}_j$, $j \in \hat{I}$. This proves the first assertion.

Next, let us prove the second assertion. We show only the equality $\tilde{\varphi}_j(b) = \tilde{\varphi}_j(b) + (\hat{\omega}(b))$ for each $b \in B^\omega$ and $j \in \hat{I}$ (i.e., Condition (1) of [HK] Definition 4.5.1); the other conditions immediately follow from the definitions (3.2.3) of the $\omega$-Kashiwara operators $\tilde{e}_j$, $\tilde{f}_j$, $j \in \hat{I}$, the definitions (3.2.4) of the maps $\hat{\omega}$, $\tilde{e}_j$, $\tilde{\varphi}_j$, $j \in \hat{I}$, for $B^\omega$, and equality (2.2.5). Let us fix (arbitrarily) $j \in \hat{I}$, and set $\hat{J} := \{j\} \subseteq \hat{I}$, $J := \{\omega^k(j) \mid 0 \leq k \leq N_j - 1\} \subseteq I$ as above. Then we deduce from (3.2.6) and Theorem 3.1.1 that for each $b \in B^\omega$,

$$\tilde{\varphi}_j(\Psi_J(b)) = \tilde{\varphi}_j(\Psi_J(b)) + (\hat{\omega}(\Psi_J(b))) = \tilde{\varphi}_j(\Psi_J(b)).$$

(3.2.7)

In addition, since $\Psi_J$ is an isomorphism of $U_q(\mathfrak{g}_J)$-crystals (see (3.2.4)), we see from the definitions (3.2.1) and (3.2.4) of the maps $\hat{\omega}$, $\tilde{e}_j$, $\tilde{\varphi}_j$ for $B^\omega_j(\lambda_1) \sqcup B^\omega_j(\lambda_2) \sqcup \cdots \sqcup B^\omega_j(\lambda_{p^*})$
and for \( \mathcal{B}^\omega \) that

\[
\tilde{\text{wt}}(\Psi_f(b)) = \text{wt} b, \quad \tilde{\varepsilon}_j(\Psi_f(b)) = \varepsilon_j(b), \quad \tilde{\varphi}_j(\Psi_f(b)) = \varphi_j(b)
\]  

(3.2.8)

for each \( b \in \mathcal{B}^\omega \). Therefore, by combining (3.2.7) and (3.2.8), we obtain that \( \tilde{\varphi}_j(b) = \varepsilon_j(b) + (\text{wt} b)(h_j) \), as desired.

Finally, the regularity of the \( U'_q(\hat{g}) \)-crystal \( \mathcal{B}^\omega \) follows from (3.2.6) and the comment after it. This proves the second assertion. \( \square \)

By the same argument as in the proof of Proposition 3.2.1, we also obtain the following Lemmas 3.2.2 and 3.2.3 using Propositions 3.1.2 and 3.1.3 respectively.

**Lemma 3.2.2.** Let \( \mathcal{B} \) be a regular \( U'_q(g) \)-crystal with an action \( \omega : \mathcal{B} \to \mathcal{B} \) of the diagram automorphism \( \omega \) satisfying (3.2.1). Then, we have \( \tilde{e}(m)_j = (\tilde{\varepsilon}_j)^m \) and \( \tilde{f}(m)_j = (\tilde{\varphi}_j)^m \) on the fixed point subset \( \mathcal{B}^\omega \) for every \( m \geq 0 \) and \( j \in \hat{I} \), where \( \tilde{e}(m)_j \) and \( \tilde{f}(m)_j \) are defined by:

\[
\tilde{x}(m)_j = \begin{cases} 
    x_j^m x_{\omega(j)}^m & \text{if } c_j = 1, \\
    x_j^m x_{\omega(j)}^m \cdots x_{\omega^{N_j-1}(j)}^m & \text{if } c_j = 2,
\end{cases}
\]

(3.2.9)

where \( x \) is either \( e \) or \( f \).

**Lemma 3.2.3.** Let \( \mathcal{B} \) be a regular \( U'_q(g) \)-crystal with an action \( \omega : \mathcal{B} \to \mathcal{B} \) of the diagram automorphism \( \omega \) satisfying (3.2.1). Then, for every element \( b \) of the fixed point subset \( \mathcal{B}^\omega \), we have \( \tilde{\varepsilon}_j(b) = \varepsilon_{\omega^k(j)}(b) \) and \( \tilde{\varphi}_j(b) = \varphi_{\omega^k(j)}(b) \) for all \( j \in \hat{I} \) and \( k \geq 0 \).

Let \( \mathcal{B} \) be a regular \( U'_q(g) \)-crystal with an action \( \omega : \mathcal{B} \to \mathcal{B} \) of the diagram automorphism \( \omega \) satisfying (3.2.1). Then it follows from Proposition 3.2.1 that the fixed point subset \( \mathcal{B}^\omega \) becomes a regular \( U'_q(\hat{g}) \)-crystal. Hence there exists a unique action \( \hat{S} : \hat{W} \to \text{Bij}(\mathcal{B}^\omega) \), \( \hat{w} \mapsto \hat{S}_{\hat{w}} \), of the Weyl group \( \hat{W} \) of the orbit Lie algebra \( \hat{g} \) on the set \( \mathcal{B}^\omega \) such that \( \hat{S}_{\hat{r}_j} = \hat{S}_j \) for all \( j \in \hat{I} \), where \( \hat{S}_j \) is defined as in (1.4.1) (see Proposition 1.4.2).

**Lemma 3.2.4.** We have \( \hat{S}_{\hat{w}} = S_{\Theta(\hat{w})} \) on \( \mathcal{B}^\omega \) for all \( \hat{w} \in \hat{W} \), where \( \Theta : \hat{W} \to \tilde{W} \) is the isomorphism of groups introduced at the end of 2.2. In particular, the equality \( \hat{S}_j = S_{\hat{w}_j} \) holds on \( \mathcal{B}^\omega \) for all \( j \in \hat{I} \).

**Proof.** We need only show that the equality \( \hat{S}_j = S_{\hat{w}_j} \) holds on \( \mathcal{B}^\omega \) for all \( j \in \hat{I} \). Fix an
arbitrary \( j \in \widehat{I} \). Let \( b \in B^\omega \), and set \( m := (\hat{w}t b)(\hat{h}_j) \). Here we note that

\[
(\hat{w}t b)(h_j) = \frac{1}{N_j} \sum_{k=0}^{N_j-1} ((\omega^*)^k(\hat{w}t b))(h_j) \quad \text{since } \hat{w}t b \in (\hat{h}_j^*)^0
\]


\[
= \frac{1}{N_j} \sum_{k=0}^{N_j-1} (\hat{w}t b)(\omega^{-k}(h_j)) = \frac{1}{N_j} \sum_{k=0}^{N_j-1} (\hat{w}t b)(h_{\omega^{-k}(j)})
\]


\[
= (\hat{w}t b)((P_\omega^{-1}(h_j))) \quad \text{by the definition (2.2.4) of } P_\omega
\]


\[
= ((P_\omega^*)^{-1}(\hat{w}t b))(h_j) = (\hat{w}t b)(h_j).
\]

So, we have \( (\hat{w}t b)(h_j) = m \). In addition, since \( \hat{w}t b \in (\hat{h}_j^*)^0 \), it follows that

\[
(\hat{w}t b)(h_{\omega^k(j)}) = m \quad \text{for all } 0 \leq k \leq N_j - 1.
\]

Now, assume that \( m \geq 0 \). Then we obtain that

\[
\tilde{S}_j b = (\tilde{f}_j)^m b = \tilde{f}(m) b = \begin{cases} \ f^m_j \ f^m_{\omega(j)} \ f^m_j b & \text{if } c_j = 1, \\ \ f^m_j \ f^m_{\omega(j)} \cdots f^m_{\omega^{N_j-1}(j)} b & \text{if } c_j = 2, \end{cases}
\]

from the definition of \( \tilde{S}_j \) and Lemma 3.2.2. On the other hand, it follows that

\[
S_{w_j} b = \begin{cases} \ f^m_j \ f^m_{\omega(j)} \ f^m_j b & \text{if } c_j = 1, \\ \ f^m_j \ f^m_{\omega(j)} \cdots f^m_{\omega^{N_j-1}(j)} b & \text{if } c_j = 2. \end{cases}
\]

Indeed, if \( c_j = 1 \), then we have \( w_j = r_j r_{\omega(j)} r_j \) by Remark 2.2.1 and the definition of \( w_j \), and hence \( S_{w_j} b = S_j S_{\omega(j)} S_j b = S_j S_{\omega(j)} f^m_j b \) by definition. Since \( c_j = 1 \), we deduce from Remark 2.2.1 and (3.2.11) that

\[
(\hat{w}t f^m_j b)(h_{\omega(j)}) = (\hat{w}t b - m\alpha_j)(h_{\omega(j)}) = (\hat{w}t b)(h_{\omega(j)}) - m\alpha_j(h_{\omega(j)})
\]

\[
= m - m \times (-1) = 2m.
\]

Therefore, it follows that \( S_j S_{\omega(j)} f^m_j b = S_j f^m_{\omega(j)} f^m_j b \). Similarly, it follows that \( S_j f^m_{\omega(j)} f^m_j b = f^m_j f^m_{\omega(j)} f^m_j b \). Thus, we obtain that \( S_{w_j} b = f^m_j f^m_{\omega(j)} f^m_j b \). The proof for the case where \( c_j = 2 \) is easier.

Consequently, we obtain that \( \tilde{S}_j b = S_{w_j} b \) if \( m \geq 0 \). The proof for the case \( m \leq 0 \) is similar. This proves the lemma.

3.3 Fixed point subsets of tensor products of regular crystals. Let \( B_1 \) (resp., \( B_2 \)) be a regular \( U'_q(\mathfrak{g}) \)-crystal with an action of the diagram automorphism \( \omega \) satisfying (3.2.1), with \( B = B_1 \) (resp., \( B = B_2 \)). Define an action \( \omega : B_1 \otimes B_2 \rightarrow B_1 \otimes B_2 \) of the diagram automorphism \( \omega \) by: \( \omega(b_1 \otimes b_2) = \omega(b_1) \otimes \omega(b_2) \) for \( b_1 \otimes b_2 \in B_1 \otimes B_2 \).
Then, we can easily check by the tensor product rule for $U'_q(\mathfrak{g})$-crystals that this action

$$\omega : B_1 \otimes B_2 \to B_1 \otimes B_2$$

satisfies condition (3.2.1), with $B = B_1 \otimes B_2$. Hence it follows from Proposition 3.2.1 that the fixed point subset $(B_1 \otimes B_2)^\omega$ of $B_1 \otimes B_2$ under this action becomes a regular $U'_q(\widehat{\mathfrak{g}})$-crystal, when equipped with the $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$, $j \in \tilde{T}$.

On the other hand, we obviously have

$$(B_1 \otimes B_2)^\omega = \{ b_1 \otimes b_2 \mid b_1 \in B_1^\omega, b_2 \in B_2^\omega \},$$

(3.3.1)

where $B_1^\omega$ and $B_2^\omega$ are the fixed point subsets of $B_1$ and $B_2$ under the action of $\omega$, respectively.

We know from Proposition 3.2.1 that both of the fixed point subsets $B_1^\omega$ and $B_2^\omega$ become regular $U'_q(\widehat{\mathfrak{g}})$-crystals, when equipped with the $\omega$-Kashiwara operators $\tilde{e}_j$ and $\tilde{f}_j$, $j \in \tilde{T}$. Denote by $B_1^\omega \otimes B_2^\omega$ the tensor product $U'_q(\widehat{\mathfrak{g}})$-crystal of $B_1^\omega$ and $B_2^\omega$; we use the symbol $\otimes$ instead of $\otimes$ to emphasize that it means the tensor product of $U'_q(\widehat{\mathfrak{g}})$-crystals. Also, we denote by $\tilde{e}_j, j \in \tilde{T}$, and $\tilde{f}_j, j \in \tilde{T}$, the raising Kashiwara operators and lowering Kashiwara operators, respectively, on the tensor product $U'_q(\widehat{\mathfrak{g}})$-crystal $B_1^\omega \otimes B_2^\omega$.

**Lemma 3.3.1.** Let $\Phi : B_1^\omega \otimes B_2^\omega \to (B_1 \otimes B_2)^\omega$ be the map defined by: $\Phi(b_1 \otimes b_2) = b_1 \otimes b_2$ for $b_1 \otimes b_2 \in B_1^\omega \otimes B_2^\omega$. Then, $\Phi$ is an isomorphism of $U'_q(\widehat{\mathfrak{g}})$-crystals between $B_1^\omega \otimes B_2^\omega$ and $(B_1 \otimes B_2)^\omega$.

**Proof.** It is obvious by 3.3.1 that $\Phi$ is bijective. In addition, it immediately follows that $\Phi$ preserves weights. So, it remains to show show that $\Phi \circ \tilde{e}_j = \tilde{e}_j \circ \Phi$ and $\Phi \circ \tilde{f}_j = \tilde{f}_j \circ \Phi$ for all $j \in \tilde{T}$ since both of $B_1^\omega \otimes B_2^\omega$ and $(B_1 \otimes B_2)^\omega$ are semiregular $U'_q(\widehat{\mathfrak{g}})$-crystals.

We show only the equality $\Phi \circ \tilde{e}_j = \tilde{e}_j \circ \Phi$ in the case where $c_j = 1$ (see Remark 2.2.1), since the proof of the equality $\Phi \circ \tilde{f}_j = \tilde{f}_j \circ \Phi$ is similar, and the proof for the case where $c_j = 2$ is easier (cf. [OSS1] Proposition 6.4] in the case where $c_j = 2$). Let $b_1 \otimes b_2 \in B_1^\omega \otimes B_2^\omega$. We deduce from Lemma 3.2.3 and the tensor product rules for the $U'_q(\mathfrak{g})$-crystal $B_1 \otimes B_2$ and for the $U'_q(\widehat{\mathfrak{g}})$-crystal $B_1^\omega \otimes B_2^\omega$, together with a computation similar to 3.2.10, that $\tilde{e}_j(b_1 \otimes b_2) = \varepsilon_j(b_1 \otimes b_2)$ (note that $b_1 \otimes b_2 = \Phi(b_1 \otimes b_2)$ by definition), where $\varepsilon_j(b_1 \otimes b_2) := \max\{ n \geq 0 \mid \varepsilon_j'(b_1 \otimes b_2) \neq \theta \}$. Also, we see from Lemma 3.2.3 that $\tilde{e}_j(b_1 \otimes b_2) = \varepsilon_j(b_1 \otimes b_2)$. Combining these equalities, we obtain that for $b_1 \otimes b_2 \in B_1^\omega \otimes B_2^\omega$,

$$\tilde{e}_j(b_1 \otimes b_2) = \varepsilon_j(b_1 \otimes b_2).$$

(3.3.2)

Therefore, we conclude that $\tilde{e}_j(b_1 \otimes b_2) = \theta$ if and only if $\varepsilon_j(\Phi(b_1 \otimes b_2)) = \varepsilon_j(b_1 \otimes b_2) = \theta$.

Now, we assume that $\tilde{e}_j(b_1 \otimes b_2) \neq \theta$; note that $\tilde{e}_j(b_1 \otimes b_2) \neq \theta$, as seen above. We further assume that $\tilde{e}_j(b_1) \geq \varepsilon_j(b_2)$, since the proof for the case where $\tilde{e}_j(b_1) < \varepsilon_j(b_2)$ is similar. Then we have $\tilde{e}_j(b_1 \hat{\otimes} b_2) = \tilde{e}_j b_1 \hat{\otimes} b_2$ by the definition of $\tilde{e}_j$. Set $b'_1 \otimes b'_2 := \tilde{e}_j(b_1 \otimes b_2)$. Because $\tilde{e}_j(b_1) \geq \varepsilon_j(b_2)$ by the assumption, it follows from Lemma 3.2.3 that
\( \varphi_j(b_1) \geq e_j(b_2) \). Hence we deduce that

\[
b_1' \otimes b_2' = \widehat{e}_j(b_1 \otimes b_2) = (e_je^2_\omega(j)e_j)(b_1 \otimes b_2) = e_je^2\omega(j)(e_jb_1 \otimes b_2).
\]

Here, we obviously have \( b_1 \in \mathcal{B}^\sigma_1, b_2 \in \mathcal{B}^\sigma_2 \), since \( b_1 \otimes b_2 \in \mathcal{B}_1^\sigma \otimes \mathcal{B}_2^\sigma \). In addition, since \( b_1' \otimes b_2' = \widehat{e}_j(b_1 \otimes b_2) = (\mathcal{B}_1 \otimes \mathcal{B}_2)^\omega \) by Proposition \( 3.2.1 \), it follows from \( 3.3.1 \) that \( b_1' \in \mathcal{B}^\sigma_1, b_2' \in \mathcal{B}^\sigma_2 \). Thus, we have \( \text{wt } b_1, \text{wt } b_1', \text{wt } b_2, \text{wt } b_2' \in (\mathfrak{h}_0^\ast)'^0 \). Consequently, by the tensor product rule for crystals, we obtain that

\[
b_1' \otimes b_2' = e_je^2\omega(j)(e_jb_1 \otimes b_2)
= (e_je^2\omega(j)e_jb_1) \otimes b_2 \quad \text{or} \quad (e_\omega(j)e_jb_1) \otimes (e_\omega(j)b_2).
\]

Indeed, the \( b_1' \otimes b_2' \) cannot be equal to \( (e_\omega(j)e_jb_1) \otimes (e_\omega(j)b_2) \), for example, since \( \text{wt } (e_\omega b_2) = \text{wt } b_2 + \alpha_j \) is not contained in \( (\mathfrak{h}_0^\ast)' \). Moreover, we have the following claim.

**Claim.** The \( b_1' \otimes b_2' \) cannot be equal to \( (e_\omega(j)e_jb_1) \otimes (e_\omega(j)b_2) \).

**Proof of Claim.** Suppose that \( b_1' \otimes b_2' = (e_\omega(j)e_jb_1) \otimes (e_\omega(j)b_2) \). Then we have \( b_1' = e_\omega(j)e_jb_1 \). Let \( J := \{ j, \omega(j) \} \subsetneq I \), and denote by \( \mathcal{B}_1(b_1) \) the connected component of \( \mathcal{B}_1 \), regarded as a \( U_q(\mathfrak{g}_j) \)-crystal, containing the \( b_1 \in \mathcal{B}_1 \). Note that \( b_1' \) is also contained in \( \mathcal{B}_1(b_1) \). We see that \( \mathcal{B}_1(b_1) \) is isomorphic as a \( U_q(\mathfrak{g}_j) \)-crystal to the crystal base \( \mathcal{B}_f(\lambda) \) for some \( J \)-dominant integral weight \( \lambda \in P_\mathfrak{cl} \) (see (3.2.5)). Because \( \mathcal{B}_f(\lambda) \cap \mathcal{B}_1(b_1) \) contains \( b_1 \) and \( b_1' \) (and hence is nonempty), we deduce as in Step 1 of the proof of Proposition \( 3.2.1 \) that \( \omega_\ast(\lambda) = \lambda \), and that \( \mathcal{B}_f(\lambda) \cap \mathcal{B}_1(b_1) \) equipped with \( \omega \)-Kashiwara operators becomes a \( U_q(\mathfrak{g}_j) \)-crystal isomorphic to the crystal base \( \mathcal{B}_f(\lambda) \), where \( \hat{J} := \{ j \} \) and \( \hat{\lambda} := (P_\omega)^{-1}(\lambda) \). Since \( \mathcal{B}_f(\lambda) \) is connected, it follows that both of \( \text{wt } b_1 \) and \( \text{wt } b_1' \) are contained in the set \( \hat{\lambda} + \mathbb{Z}\hat{\alpha}_j \), and hence that \( \text{wt } b_1 - \text{wt } b_1' \in \mathbb{Z}\hat{\alpha}_j \). But, we obtain by (2.2.5) that

\[
\text{wt } b_1' = \text{wt } (e_\omega(j)e_jb_1) = \text{wt } b_1 + \frac{1}{2}\hat{\alpha}_j \notin \text{wt } b_1 + \mathbb{Z}\hat{\alpha}_j,
\]

which is a contradiction. This proves the claim.

Therefore, we conclude that \( \widehat{e}_j(b_1 \otimes b_2) = (e_je^2_\omega(j)e_jb_1) \otimes b_2 = \widehat{e}_j(b_1 \otimes b_2) \), from which the equality \( \Phi(\widehat{e}_j(b_1 \otimes b_2)) = \widehat{e}_j(\Phi(b_1 \otimes b_2)) \) immediately follows. This completes the proof of the lemma.

Because the fixed point subsets \( \mathcal{B}^\sigma_1, \mathcal{B}^\sigma_2 \), and \( (\mathcal{B}_1 \otimes \mathcal{B}_2)^\sigma \), equipped with the \( \omega \)-Kashiwara operators \( \widehat{e}_j \) and \( \widehat{f}_j, j \in I \), are all regular \( U_q(\hat{\mathfrak{g}}) \)-crystals by Proposition \( 3.2.1 \) there exist actions of the Weyl group \( \widehat{W} \) of \( \hat{\mathfrak{g}} \) on them by Proposition \( 1.1.2 \) all of which are denoted by \( \widehat{S}_\mathfrak{w}, \mathfrak{w} \in \widehat{W} \). With this notation, we have the following lemma.

**Lemma 3.3.2.** Let \( b_1 \in \mathcal{B}^\sigma_1 \) and \( b_2 \in \mathcal{B}^\sigma_2 \) be \( \widehat{W} \)-extremal elements whose weights are contained in the same Weyl chamber with respect to the simple coroots \( \widehat{h}_j, j \in I_0 \). Then, \( b_1 \otimes b_2 \in (\mathcal{B}_1 \otimes \mathcal{B}_2)^{\sigma} \) is also a \( \widehat{W} \)-extremal element, and the equality \( \widehat{S}_\mathfrak{w}(b_1 \otimes b_2) = \widehat{S}_\mathfrak{w}b_1 \otimes \widehat{S}_\mathfrak{w}b_2 \) holds for all \( \mathfrak{w} \in \widehat{W} \).
Proof. Since the tensor product \( U'_q(\mathfrak{g}) \)-crystal \( B_1^\otimes B_2^\otimes \) is regular, there exists an action of the Weyl group \( \hat{W} \) on \( B_1^\otimes B_2^\otimes \) (see Proposition 1.4.2). In addition, since \( \Phi : B_1^\otimes B_2^\otimes \to (B_1 \otimes B_2)^\otimes \) is an isomorphism of \( U'_q(\mathfrak{g}) \)-crystals by Lemma 3.3.1 it follows from the definitions of the actions of the Weyl group \( \hat{W} \) on \( B_1^\otimes B_2^\otimes \) and on \( (B_1 \otimes B_2)^\otimes \) that \( \Phi \circ \hat{S}_{\hat{w}} = \hat{S}_{\hat{w}} \circ \Phi \) for all \( \hat{w} \in \hat{W} \). Hence, we see that \( b_1 \otimes b_2 \in (B_1 \otimes B_2)^\otimes \) is a \( \hat{W} \)-extremal element since \( \Phi^{-1}(b_1 \otimes b_2) = b_1 \otimes b_2 \) is a \( \hat{W} \)-extremal element of the tensor product \( U'_q(\mathfrak{g}) \)-crystal \( B_1^\otimes B_2^\otimes \) by Lemma 1.4.5. Also, the equality \( \hat{S}_{\hat{w}}(b_1 \otimes b_2) = \hat{S}_{\hat{w}}b_1 \otimes \hat{S}_{\hat{w}}b_2 \) immediately follows from Lemma 1.4.5 and the equality \( \Phi \circ \hat{S}_{\hat{w}} = \hat{S}_{\hat{w}} \circ \Phi \).

4 Proof of the main result.

In this section, we use the notation of 2.3 and keep Assumption 2.3.1. Recall that we fixed (arbitrarily) \( i \in \hat{I}_0 = \hat{I} \setminus \{0\} \) and \( s \in \mathbb{Z}_{\geq 1} \). By 2.3.1, \( B_i^s \) is a regular \( U'_q(\mathfrak{g}) \)-crystal with an action \( \omega : B_i^s \to B_i^s \) of the diagram automorphism \( \omega \) satisfying condition (3.2.1), with \( B = B_i^s \). Hence we can apply the results in 3.2 to the case where \( B = B_i^s \), \( B^\omega = B_i^s \), and those in 3.3 to the case where \( B_1 = B_2 = B_i^s \), \( B_1^\otimes = B_2^\otimes = B_i^s \).

4.1 Proof of regularity. The following proposition immediately follows from Proposition 3.2.1 applied to the case \( B = B_i^s \) (note that \( B^\omega = B_i^s \)).

**Proposition 4.1.1.** The subset \( \hat{B}_i^s \cup \{\theta\} \) of \( \hat{B}_i^s \cup \{\theta\} \) is stable under the \( \omega \)-Kashiwara operators \( \hat{e}_j \) and \( \hat{f}_j \) on \( \hat{B}_i^s \cup \{\theta\} \) for all \( j \in \hat{I} \). In addition, the \( \hat{B}_i^s \) equipped with the maps \( \hat{w}t, \hat{e}_j, \hat{f}_j, j \in \hat{I} \), and \( \hat{e}_j, \hat{f}_j, j \in \hat{I} \), becomes a regular \( U'_q(\mathfrak{g}) \)-crystal.

4.2 Proof of simplicity. In this subsection, we prove that the \( U'_q(\mathfrak{g}) \)-crystal \( \hat{B}_i^s \) is simple. First we show the following proposition.

**Proposition 4.2.1.** The set \( \hat{B}_i^s \) is of finite cardinality, and the weights of elements of \( \hat{B}_i^s \) are all contained in \((\hat{P}_c)_0\). Namely, the \( U'_q(\mathfrak{g}) \)-crystal \( \hat{B}_i^s \) satisfies condition (S1) of Definition 1.4.6.

**Proof.** Since \( \hat{B}_i^s \) is the tensor product of the crystal bases \( B^{\omega^k(i),s} \), \( 0 \leq k \leq N_i - 1 \), of finite cardinality, it follows that \( \hat{B}_i^s \) is a finite set, and hence so is \( \hat{B}_i^s \). Therefore, it suffices to show that \( \hat{w}t b \in (\hat{P}_c)_0 \) for all \( b \in \hat{B}_i^s \). Let \( b \in \hat{B}_i^s \). Then we deduce that

\[
(\hat{w}t b)(\hat{c}) = ((P^s_\omega)^{-1}(\hat{w}t b))(\hat{c})
= (w t b)(P^{-1}_\omega(\hat{c})) \quad \text{(see the comment after (2.2.4))}
= (w t b)(c) \quad \text{by (2.2.5).} \quad (4.2.1)
\]

Because the \( U'_q(\mathfrak{g}) \)-crystal \( \hat{B}_i^s \) is perfect (and hence simple), we have \( (w t b)(c) = 0 \), and hence \( (w t b)(\hat{c}) = 0 \). This proves the proposition. \( \square \)
The rest of this subsection is devoted to proving that the $U_q'(\mathfrak{g})$-crystal $\tilde{B}^{i,s}$ satisfies condition (S2), and also condition (S3), of Definition 1.4.6. Since $\tilde{B}^{i,s}$ is a perfect (and hence simple) $U_q'(\mathfrak{g})$-crystal, it follows from Lemma 1.4.8 that there exists a unique $W$-extremal element of $\tilde{B}^{i,s}$, denoted by $\tilde{u}$, such that $wt(\tilde{u})$ is $I_0$-dominant, i.e., $(wt(\tilde{u}))(h_j) \geq 0$ for all $j \in I_0 = I \setminus \{0\}$ (this $\tilde{u}$ is equal to the element $\tilde{u}_{i,s} \in \tilde{B}^{i,s}$ in Remark 5.1.3).

Lemma 4.2.2. The $W$-extremal element $\tilde{u} \in \tilde{B}^{i,s}$ is contained in the fixed point subset $\tilde{B}^{i,s}$. Moreover, the element $\tilde{u}$ is also a $\tilde{W}$-extremal element of the $U_q'(\mathfrak{g})$-crystal $\tilde{B}^{i,s}$. Namely, for every $\hat{w} \in \hat{W}$, either $\hat{e}_j \hat{S}_\theta \hat{u} = \theta$ or $\hat{f}_j \hat{S}_\theta \hat{u} = \theta$ holds for each $j \in \hat{I}$.

Proof. We first show that $\omega(\tilde{u}) = \tilde{u}$. Note that $wt(\omega(\tilde{u}))$ is $I_0$-dominant. In addition, for each $w \in W$, we have

$$S_w(\omega(\tilde{u})) = \omega(S_w \tilde{u})$$

for some $w' \in W$. Indeed, we see from the definition of the action of $W$ on $\tilde{B}^{i,s}$ and (2.3.7) that $S_j(\omega(b)) = \omega(\omega^{-1}(j)b)$ for all $j \in I$ and $b \in \tilde{B}^{i,s}$. Therefore, if $w = r_{j_1}r_{j_2}\cdots r_{j_p}$ is a reduced expression of $w \in W$, then we have

$$S_w(\omega(\tilde{u})) = S_{j_1}S_{j_2}\cdots S_{j_p}(\omega(\tilde{u})) = \omega(S_{\omega^{-1}(j_1)}S_{\omega^{-1}(j_2)}\cdots S_{\omega^{-1}(j_p)}\tilde{u}) = \omega(S_{w'}\tilde{u}),$$

where $w' := r_{\omega^{-1}(j_1)}r_{\omega^{-1}(j_2)}\cdots r_{\omega^{-1}(j_p)} \in W$. Since $S_w \tilde{u} \in \tilde{B}^{i,s}$ is a $W$-extremal element, we deduce from (2.3.7) and (1.2.2) that $\omega(\tilde{u})$ is also a $W$-extremal element, whose weight $wt(\omega(\tilde{u}))$ is $I_0$-dominant. Consequently, it follows from the uniqueness of $\tilde{u}$ (see Lemma 1.4.8) that $\omega(\tilde{u}) = \tilde{u}$. Thus we obtain that $\tilde{u} \in \tilde{B}^{i,s}$.

Now, let $\hat{w} \in \hat{W}$. We know from Lemma 3.2.2 that $\hat{S}_\theta \hat{u} = S_{\theta(\hat{w})} \tilde{u}$. Since $\tilde{u}$ is a $W$-extremal element of the $U_q'(\mathfrak{g})$-crystal $\tilde{B}^{i,s}$, either $\hat{e}_j \hat{S}_\theta \hat{u} = e_j S_{\theta(\hat{w})} \tilde{u} = \theta$ or $\hat{f}_j \hat{S}_\theta \hat{u} = f_j S_{\theta(\hat{w})} \tilde{u} = \theta$ holds for each $j \in \hat{I}$. Therefore, it follows from the definition of the $\omega$-Kashiwara operators $\omega_j$, $\hat{f}_j$, $j \in \hat{I}$, that $\omega_j \hat{S}_\theta \hat{u} = \theta$ or $\hat{f}_j \hat{S}_\theta \hat{u} = \theta$ for each $j \in \hat{I}$. This proves the lemma. 

Let $b' \in \tilde{B}^{i,s}$ be a $\tilde{W}$-extremal element of the regular $U_q'(\mathfrak{g})$-crystal $\tilde{B}^{i,s}$. Then there exists some $\hat{w} \in \hat{W}$ such that $\hat{w}t(S_{\hat{w}}b')$ is $\hat{I}_0$-dominant, i.e., $(\hat{w}t(S_{\hat{w}}b'))(\hat{h}_j) \geq 0$ for all $j \in \hat{I}_0 = \hat{I} \setminus \{0\}$. Hence, in order to prove that the $U_q'(\mathfrak{g})$-crystal $\tilde{B}^{i,s}$ satisfies condition (S2) of Definition 1.4.6 it suffices to show the next proposition.

Proposition 4.2.3. Let $b \in \tilde{B}^{i,s}$ be a $\tilde{W}$-extremal element of the regular $U_q'(\mathfrak{g})$-crystal $\tilde{B}^{i,s}$ such that $\hat{w}t b$ is $\hat{I}_0$-dominant. Then we have $b = \tilde{u}$.

In the proof below of Proposition 4.2.3 we need some lemmas.
Lemma 4.2.4. Let $\hat{\mu}, \hat{\nu}$ be elements of $(\hat{P}_{cl})_0$ that are contained in the same Weyl chamber with respect to $\hat{h}_j$, $j \in \hat{I}_0$. Then, for each $\hat{w} \in \hat{W}$, $\hat{w}(\hat{\mu})$ and $\hat{w}(\hat{\nu})$ are contained in the same Weyl chamber with respect to the simple coroots $\hat{h}_j$, $j \in \hat{I}_0$. Moreover, for each $\hat{w} \in \hat{W}$, we have either $(\hat{w}(\hat{\mu}))(\hat{h}_0) \leq 0$ and $(\hat{w}(\hat{\nu}))(\hat{h}_0) \leq 0$, or $(\hat{w}(\hat{\mu}))(\hat{h}_0) \geq 0$ and $(\hat{w}(\hat{\nu}))(\hat{h}_0) \geq 0$.

Proof. Recall that the Weyl group $\hat{W}$ of $\hat{g}$ decomposes into the semidirect product $\hat{W}_{I_0} \ltimes \hat{T}$ of the Weyl group $\hat{W}_{I_0} := \langle \hat{r}_j \mid j \in \hat{I}_0 \rangle$ (of finite type) and the abelian group $\hat{T}$ of translations. Hence, for each $\hat{w} \in \hat{W}$, there exists $\hat{z} \in \hat{W}_{I_0}$ and $\hat{t} \in \hat{T}$ such that $\hat{w} = \hat{z}\hat{t}$. Then, since $\hat{\mu}, \hat{\nu} \in (\hat{P}_{cl})_0$ by the assumption, it follows from [Kac, Chap. 6, formula (6.5.5)] that $\hat{t}(\hat{\mu}) = \hat{\mu}$, $\hat{t}(\hat{\nu}) = \hat{\nu}$, and hence that $\hat{w}(\hat{\mu}) = \hat{z}(\hat{\mu})$, $\hat{w}(\hat{\nu}) = \hat{z}(\hat{\nu})$. Consequently, for each $\hat{w} \in \hat{W}$, $\hat{w}(\hat{\mu})$ and $\hat{w}(\hat{\nu})$ are contained in the same Weyl chamber with respect to the simple coroots $\hat{h}_j$, $j \in \hat{I}_0$.

Now, we fix an arbitrary $\hat{w} \in \hat{W}$. Note that $\hat{h}_0 = \hat{c} - \hat{\theta}^\vee$, where $\hat{\theta}^\vee$ is the highest coroot of the dual root system of $\hat{g}_{I_0}$. Since $\hat{w}(\hat{\mu})$ and $\hat{w}(\hat{\nu})$ are contained in $(\hat{P}_{cl})_0$, we have

\[
(\hat{w}(\hat{\mu}))(\hat{h}_0) = (\hat{w}(\hat{\mu}))(\hat{c} - \hat{\theta}^\vee) = -(\hat{w}(\hat{\mu}))(\hat{\theta}^\vee),
\]

\[
(\hat{w}(\hat{\nu}))(\hat{h}_0) = (\hat{w}(\hat{\nu}))(\hat{c} - \hat{\theta}^\vee) = -(\hat{w}(\hat{\nu}))(\hat{\theta}^\vee).
\]

(4.2.3)

Because $\hat{w}(\hat{\mu})$ and $\hat{w}(\hat{\nu})$ are contained in the same Weyl chamber as shown above, we conclude from [Kac] that either $(\hat{w}(\hat{\mu}))(\hat{h}_0) \leq 0$ and $(\hat{w}(\hat{\nu}))(\hat{h}_0) \leq 0$, or $(\hat{w}(\hat{\mu}))(\hat{h}_0) \geq 0$ and $(\hat{w}(\hat{\nu}))(\hat{h}_0) \geq 0$. This proves the lemma.

Lemma 4.2.5. Let $\hat{\mu}, \hat{\nu}$ be elements of $(\hat{P}_{cl})_0$ that are $\hat{I}_0$-dominant, and assume that $\hat{\mu} \neq \hat{\nu}$. Then, there exists some $\hat{z} \in \hat{W}_{I_0}$ such that $(\hat{z}(\hat{\mu}))(\hat{h}_0) \neq (\hat{z}(\hat{\nu}))(\hat{h}_0)$, and $(\hat{z}(\hat{\mu}))(\hat{h}_0) \leq 0$, $(\hat{z}(\hat{\nu}))(\hat{h}_0) \leq 0$.

Proof. First, let us show that there exists some $\hat{w} \in \hat{W}$ such that $(\hat{w}(\hat{\mu}))(\hat{h}_0) \neq (\hat{w}(\hat{\nu}))(\hat{h}_0)$, and such that $(\hat{w}(\hat{\mu}))(\hat{h}_0) \leq 0$, $(\hat{w}(\hat{\nu}))(\hat{h}_0) \leq 0$. Suppose that $(\hat{w}(\hat{\mu}))(\hat{h}_0) = (\hat{w}(\hat{\nu}))(\hat{h}_0)$ for all $\hat{w} \in \hat{W}$, or equivalently

\[
\hat{\mu}(\hat{h}) = \hat{\nu}(\hat{h}) \quad \text{for all } \hat{h} \in \hat{W}\hat{h}_0.
\]

(4.2.4)

It is easy to show (cf. the argument in the hint for [Kac, Exercise 6.9]) that $\hat{h}_{cl}$ is generated by $\hat{W}\hat{h}_0$ as a $\mathbb{C}$-vector space: $\hat{h}_{cl} = \sum_{\hat{h} \in \hat{W}\hat{h}_0} \mathbb{C}\hat{h}$. Therefore, it immediately follows from [Kac1] that $\hat{\mu}(\hat{h}) = \hat{\nu}(\hat{h})$ for all $\hat{h} \in \hat{h}_{cl}$, and hence that $\hat{\mu} = \hat{\nu}$, which contradicts the assumption. This shows that $(\hat{w}(\hat{\mu}))(\hat{h}_0) \neq (\hat{w}(\hat{\nu}))(\hat{h}_0)$ for some $\hat{w} \in \hat{W}$. Furthermore, for this $\hat{w} \in \hat{W}$, we see from Lemma 4.2.4 that either $(\hat{w}(\hat{\mu}))(\hat{h}_0) \leq 0$ and $(\hat{w}(\hat{\nu}))(\hat{h}_0) \leq 0$, or $(\hat{w}(\hat{\mu}))(\hat{h}_0) \geq 0$ and $(\hat{w}(\hat{\nu}))(\hat{h}_0) \geq 0$. If $(\hat{w}(\hat{\mu}))(\hat{h}_0) \geq 0$ and $(\hat{w}(\hat{\nu}))(\hat{h}_0) \geq 0$, then $(\hat{r}_0\hat{w}(\hat{\mu}))(\hat{h}_0) \leq 0$ and $(\hat{r}_0\hat{w}(\hat{\nu}))(\hat{h}_0) \leq 0$, with $(\hat{r}_0\hat{w}(\hat{\mu}))(\hat{h}_0) \neq (\hat{r}_0\hat{w}(\hat{\nu}))(\hat{h}_0)$. Thus, we have obtained an element $\hat{w} \in \hat{W}$ such that $(\hat{w}(\hat{\mu}))(\hat{h}_0) \neq (\hat{w}(\hat{\nu}))(\hat{h}_0)$, and $(\hat{w}(\hat{\mu}))(\hat{h}_0) \leq 0$, $(\hat{w}(\hat{\nu}))(\hat{h}_0) \leq 0$. 29
If we write the $\tilde{w} \in \widehat{W}$ in the form $\tilde{w} = \tilde{x} \tilde{t}$, with $\tilde{x} \in \widehat{W}_0$ and $\tilde{t} \in \widehat{T}$, then we have $\tilde{w}(\tilde{u}) = \tilde{x}(\tilde{u})$ and $\tilde{w}(\tilde{v}) = \tilde{x}(\tilde{v})$ (see the proof of Lemma 4.2.4), and hence that $(\tilde{x}(\tilde{u}))\langle \widehat{h}_0 \rangle \neq (\tilde{x}(\tilde{v}))\langle \widehat{h}_0 \rangle$, and $(\tilde{x}(\tilde{u}))\langle \widehat{h}_0 \rangle \leq 0$, $(\tilde{x}(\tilde{v}))\langle \widehat{h}_0 \rangle \leq 0$. This proves the lemma. 

\textbf{Proof of Proposition 4.2.3.} Recall from 2.3 that $\widehat{B}^{i,s}$ is a perfect $U_q'(\mathfrak{g})$-crystal isomorphic to the crystal base of a tensor product of finite-dimensional irreducible $U_q'(\mathfrak{g})$-modules. By Proposition 1.4.15 (2), we have an energy function $B$ on the tensor product $U_q'(\mathfrak{g})$-crystal $\widehat{B}^{i,s} \otimes \widehat{B}^{i,s}$. We show that if $b$ were not equal to $\tilde{u}$, then the energy function $H$ would attain infinitely many different values on the finite set $\widehat{B}^{i,s} \otimes \widehat{B}^{i,s}$. This proves the proposition by contradiction.

Now, suppose that $b \neq \tilde{u}$. Since $\tilde{u} \in \widehat{B}^{i,s}$ by Lemma 4.2.2 and $b \in \widehat{B}^{i,s}$ by assumption, it follows that $\mu := \text{wt } \tilde{u} \in (P_c)_0 \cap (h^*_c)^0$ and $\nu := \text{wt } b \in (P_c)_0 \cap (h^*_c)^0$. Note that $\mu \neq \nu$ since $(\widehat{B}^{i,s})_\mu = \{ \tilde{u} \}$ by condition (S3) of Definition 1.4.6. If we set $\hat{\mu} := (P^*_\omega)^{-1}(\mu)$ and $\hat{\nu} := (P^*_\omega)^{-1}(\nu)$, then by Proposition 4.2.4 we have $\hat{\mu}, \hat{\nu} \in (\widehat{P}_c)_0$. Let us take $\tilde{z} \in \widehat{W}_0$ such that $(\tilde{z}(\hat{\mu}))\langle \widehat{h}_0 \rangle \neq (\tilde{z}(\hat{\nu}))\langle \widehat{h}_0 \rangle$, and such that $(\tilde{z}(\hat{\mu}))\langle \widehat{h}_0 \rangle \leq 0$, $(\tilde{z}(\hat{\nu}))\langle \widehat{h}_0 \rangle \leq 0$ (see Lemma 4.2.5).

\textbf{Step 1.} Define an action of $\omega : \widehat{B}^{i,s} \otimes \widehat{B}^{i,s} \to \widehat{B}^{i,s} \otimes \widehat{B}^{i,s}$ of the diagram automorphism $\omega$ by: $\omega(b_1 \otimes b_2) = \omega(b_1) \otimes \omega(b_2)$ for $b_1 \otimes b_2 \in \widehat{B}^{i,s} \otimes \widehat{B}^{i,s}$, and let $(\widehat{B}^{i,s} \otimes \widehat{B}^{i,s})^\omega$ be the fixed point subset of $\widehat{B}^{i,s} \otimes \widehat{B}^{i,s}$ under this action of $\omega$ as in 3.3. Obviously, $\tilde{u} \otimes b$ is contained in the fixed point subset $(\widehat{B}^{i,s} \otimes \widehat{B}^{i,s})^\omega$. Furthermore, we deduce from Lemma 3.3.2 that $\tilde{u} \otimes b$ is a $\widehat{W}$-extremal element of the regular $U_q'(\mathfrak{g})$-crystal $(\widehat{B}^{i,s} \otimes \widehat{B}^{i,s})^\omega$ equipped with $\omega$-Kashiwara operators, and that $\widehat{S}_z(\tilde{u} \otimes b) = \widehat{S}_z \tilde{u} \otimes \widehat{S}_z b$. We set $p := -(\tilde{z}(\hat{\mu}))\langle \widehat{h}_0 \rangle$, and $q := -(\tilde{z}(\hat{\nu}))\langle \widehat{h}_0 \rangle$; note that $p \neq q$, and $p \geq 0$, $q \geq 0$. Then, because

$$
\left(\text{wt}(\widehat{S}_z \tilde{u} \otimes \widehat{S}_z b)\right)\langle \widehat{h}_0 \rangle = (\tilde{z}(\hat{\mu}) + \tilde{z}(\hat{\nu}))\langle \widehat{h}_0 \rangle = -(p + q) \leq 0,
$$

we have by the definition of $\widehat{S}_0$,

$$
\widehat{S}_0 \widehat{S}_z(\tilde{u} \otimes b) = (\tilde{e}_0)^{p+q}(\widehat{S}_z \tilde{u} \otimes \widehat{S}_z b) = e_0^{p+q}(\widehat{S}_z \tilde{u} \otimes \widehat{S}_z b)
$$

(note that $\tilde{e}_0 = e_0$). In addition, since $\widehat{S}_z \tilde{u}$ and $\widehat{S}_z b$ are $\widehat{W}$-extremal elements of the $U_q'(\mathfrak{g})$-crystal $\widehat{B}^{i,s}$, and since $\left(\text{wt} \widehat{S}_z \tilde{u}\right)\langle \widehat{h}_0 \rangle = (\tilde{z}(\hat{\mu}))\langle \widehat{h}_0 \rangle = -p \leq 0$, $(\text{wt} \widehat{S}_z b)\langle \widehat{h}_0 \rangle = (\tilde{z}(\hat{\nu}))\langle \widehat{h}_0 \rangle = -q \leq 0$, we obtain that

$$
\varepsilon_0(\widehat{S}_z \tilde{u}) = \tilde{e}_0(\widehat{S}_z \tilde{u}) = 0 \quad \text{and} \quad \varepsilon_0(\widehat{S}_z b) = \tilde{e}_0(\widehat{S}_z b) = 0,
$$

$$
\varphi_0(\widehat{S}_z \tilde{u}) = \tilde{e}_0(\widehat{S}_z \tilde{u}) = p \quad \text{and} \quad \varphi_0(\widehat{S}_z b) = \tilde{e}_0(\widehat{S}_z b) = q
$$

(note that $\tilde{e}_0 = e_0$, $\tilde{f}_0 = f_0$). Using these equalities, we can easily show by the tensor product rule for the $U_q'(\mathfrak{g})$-crystal $\widehat{B}^{i,s} \otimes \widehat{B}^{i,s}$ that

$$
e_0(\widehat{S}_z \tilde{u} \otimes \widehat{S}_z b) = \begin{cases} e_0(\widehat{S}_z \tilde{u}) \otimes e_0(\widehat{S}_z b) & \text{for } 0 \leq l \leq q, \\ e_0(l-q)(\widehat{S}_z \tilde{u}) \otimes e_0^q(\widehat{S}_z b) & \text{for } q \leq l \leq p + q. \end{cases} \quad (4.2.5)$$

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Thus, by using (1.4.5) (applied to the case $\mathcal{B} = \tilde{\mathcal{B}}^{i,*}$) successively, we obtain that
\[
H\left(\tilde{S}_0\tilde{S}_z(\widetilde{u} \otimes b)\right) = H\left(\tilde{S}_z\widetilde{u} \otimes e_0^q(\tilde{S}_z b)\right) + p. \tag{4.2.6}
\]

Indeed, we deduce that
\[
H\left(\tilde{S}_0\tilde{S}_z(\widetilde{u} \otimes b)\right) = H\left(e_0^{p+q}(\tilde{S}_z\widetilde{u} \otimes \tilde{S}_z b)\right)
\]
\[
= H\left(e_0(e_0^{p-1}(\tilde{S}_z\widetilde{u}) \otimes e_0^q(\tilde{S}_z b))\right) \quad \text{by } (4.2.5)
\]
\[
= H\left(e_0\left(e_0^{p-1}(\tilde{S}_z\widetilde{u}) \otimes e_0^q(\tilde{S}_z b)\right) + 1 \quad \text{since } \varepsilon_0(e_0^q(\tilde{S}_z b)) = 0
\]
\[
= H\left(e_0\left(e_0^{p-2}(\tilde{S}_z\widetilde{u}) \otimes e_0^q(\tilde{S}_z b)\right) + 1 \quad \text{by } (4.2.5)
\]
\[
= H\left(e_0\left(e_0^{p-2}(\tilde{S}_z\widetilde{u}) \otimes e_0^q(\tilde{S}_z b)\right) + 2 \quad \text{since } \varepsilon_0(e_0^q(\tilde{S}_z b)) = 0.
\]

Continuing in this way, we finally obtain (4.2.6). Furthermore, again by using (1.4.6) successively, we obtain that
\[
H\left(\tilde{S}_z\widetilde{u} \otimes e_0^q(\tilde{S}_z b)\right) = H(\tilde{S}_z\widetilde{u} \otimes \tilde{S}_z b) - q. \tag{4.2.7}
\]

Indeed, we deduce that
\[
H\left(\tilde{S}_z\widetilde{u} \otimes e_0^q(\tilde{S}_z b)\right) = H\left(e_0(\tilde{S}_z\widetilde{u} \otimes e_0^{q-1}(\tilde{S}_z b))\right) \quad \text{by } (4.2.5)
\]
\[
= H\left(\tilde{S}_z\widetilde{u} \otimes e_0^{q-1}(\tilde{S}_z b)\right) - 1 \quad \text{since } \varphi_0(\tilde{S}_z\widetilde{u}) = 0 \text{ and } \varepsilon_0(e_0^{q-1}(\tilde{S}_z b)) > 0
\]
\[
= H\left(e_0\left(\tilde{S}_z\widetilde{u} \otimes e_0^{q-2}(\tilde{S}_z b)\right)\right) \quad \text{by } (4.2.5)
\]
\[
= H\left(\tilde{S}_z\widetilde{u} \otimes e_0^{q-2}(\tilde{S}_z b)\right) - 2 \quad \text{since } \varphi_0(\tilde{S}_z\widetilde{u}) = 0 \text{ and } \varepsilon_0(e_0^{q-2}(\tilde{S}_z b)) > 0.
\]

Continuing in this way, we finally obtain (4.2.7). Hence, by combining (4.2.6) and (4.2.7), we have
\[
H\left(\tilde{S}_0\tilde{S}_z(\widetilde{u} \otimes b)\right) = H\left(\tilde{S}_z\widetilde{u} \otimes \tilde{S}_z b\right) + p - q.
\]

Also, we deduce from (1.4.3) and (1.4.6) (applied to the case $\mathcal{B} = \tilde{\mathcal{B}}^{i,*}$) that
\[
H\left(\tilde{S}_z\widetilde{u} \otimes \tilde{S}_z b\right) = H\left(\tilde{S}_z(\widetilde{u} \otimes b)\right) = H(\tilde{u} \otimes b),
\]

since $\tilde{S}_z$ is defined by using only $\tilde{e}_j$ and $\tilde{f}_j$, $j \in \tilde{I}_0$, and hence by using only $e_j$ and $f_j$, $j \in I_0$. Thus we conclude that
\[
H\left(\tilde{S}_0\tilde{S}_z(\widetilde{u} \otimes b)\right) = H(\tilde{u} \otimes b) + p - q. \tag{4.2.8}
\]
Step 2. We write $\hat{r}_0 \in \hat{W}$ in the form $\hat{r}_0 = \zeta' \hat{r}$, with $\zeta' \in \hat{W}_{\hat{I}_0}$ and $\hat{r} \in \hat{T}$. Then we obtain that

$$\zeta'(\hat{r}_0 \zeta(\hat{\mu})) = \zeta' \hat{r}(\hat{r}_0 \zeta(\hat{\mu})) = \hat{r}(\hat{r}_0 \zeta(\hat{\mu})) = \zeta(\hat{\mu}),$$

and similarly that $\zeta' \hat{r}_0 \zeta(\hat{\mu}) = \zeta(\hat{\mu})$. Set $b_1 \otimes b_2 := \hat{S}_{\zeta'} \hat{S}_0 \hat{S}_z (\tilde{u} \otimes b)$. It follows from Lemma 3.3.2 that $b_1 = \hat{S}_{\zeta'} \hat{S}_0 \hat{S}_z (\tilde{u})$ and $b_2 = \hat{S}_{\zeta'} \hat{S}_0 \hat{S}_z (b)$. Hence, by (4.2.9), we have $\hat{w}t_1 = \hat{z}(\hat{\mu})$ and $\hat{w}t_2 = \hat{z}(\hat{\nu})$. Now, by the same argument as in Step 1 above, we can deduce that

$$H(\hat{S}_0 (b_1 \otimes b_2)) = H(b_1 \otimes b_2) + p - q.$$  

Note that since $\zeta' \in \hat{W}_{\hat{I}_0}$, the $\hat{S}_{\zeta'}$ is defined by using only $\tilde{e}_j, \tilde{f}_j, j \in \hat{I}_0$, and hence by using only $e_j, f_j, j \in I_0$. Therefore, by (1.4.5) and (1.4.6), we obtain that

$$H(\hat{S}_0 (b_1 \otimes b_2)) = H(b_1 \otimes b_2) + q - H(\hat{S}_{\zeta'} \hat{S}_0 \hat{S}_z (\tilde{u} \otimes b)) + p - q$$

$$= H(\hat{S}_0 \hat{S}_z (\tilde{u} \otimes b)) + p - q$$

$$= H(\tilde{u} \otimes b) + 2(p - q) \text{ by (4.2.8)}.$$  

Repeating this argument, we can show that for every $k \in \mathbb{Z}_{\geq 0}$, there exists some $x \in (\hat{B}^{i,s} \otimes \hat{B}^{i,s})^\omega \subset \hat{B}^{i,s} \otimes \hat{B}^{i,s}$ such that $H(x) = H(\tilde{u} \otimes b) + 2k(p - q)$. This contradicts the fact that $\hat{B}^{i,s} \otimes \hat{B}^{i,s}$ is a finite set. Thus we have proved the proposition.

Finally, we show that the $\hat{B}^{i,s}$ satisfies condition (S3) of Definition 1.4.6.

Proposition 4.2.6. Let $b \in \hat{B}^{i,s}$ be a $\hat{W}$-extremal element of the $U_q'(\hat{g})$-crystal $\hat{B}^{i,s}$, and set $\hat{\mu} := \hat{w}t b$. Then, the subset $(\hat{B}^{i,s})_{\hat{\mu}} \subset \hat{B}^{i,s}$ of all elements of weight $\hat{\mu}$ consists only of the element $b$, i.e., $(\hat{B}^{i,s})_{\hat{\mu}} = \{b\}$.

Proof. By Proposition 4.2.3 together with the comment before it, we see that $b$ is contained in the $\hat{W}$-orbit of $\tilde{u} \in \hat{B}^{i,s}$: $b = \hat{S}_{\tilde{u}} \tilde{u}$ for some $\tilde{u} \in \hat{W}$. Furthermore, it follows from Lemma 3.2.4 that $b = S_{\theta(\tilde{u})} \tilde{u}$, and hence that $b$ is contained also in the $W$-orbit of $\tilde{u}$. Note that since $\tilde{u} \in \hat{B}^{i,s}$ is a $\hat{W}$-extremal element of the simple $U_q'(\hat{g})$-crystal $\hat{B}^{i,s}$, so is the $b \in \hat{B}^{i,s}$. In particular, by condition (S3) of Definition 1.4.6, we have $(\hat{B}^{i,s})_{\hat{w}t b} = \{b\}$.

Therefore, we obtain that $(\hat{B}^{i,s})_{\hat{\mu}} = \{b\}$, as desired. $\Box$

Combining Propositions 4.2.1, 4.2.3 and 4.2.6 we now establish the simplicity of the $U_q'(\hat{g})$-crystal $\hat{B}^{i,s}$.
4.3 Proof of bijectivity.

**Proposition 4.3.1.** The level of the simple $U'_q(\mathfrak{g})$-crystal $\hat{B}^{i,s}$ is equal to $s$. Moreover, the maps $\tilde{\epsilon}, \tilde{\varphi}: (\hat{B}^{i,s})_{\min} \to (\hat{P}^+_{cl})_s$ are bijective.

**Proof.** We see from Lemma 3.2.3 applied to the case where $\mathcal{B} = \tilde{B}^{i,s}$ and $\omega = \hat{B}^{i,s}$ that $P'_\omega(\epsilon(b)) = \epsilon(b) \forall b \in \hat{B}^{i,s}$. Therefore, we deduce that

$$\epsilon(b)(\tilde{c}) = (P'_\omega)^{-1}(\epsilon(b))(\tilde{c}) = (\epsilon(b))(P'_{\omega}^{-1}(\tilde{c})) \quad \text{(see the comment after (2.2.3))}$$

and hence that $(\tilde{\epsilon}(b))(\tilde{c}) = (\epsilon(b))(c) \geq s \forall b \in \hat{B}^{i,s} \subseteq \hat{B}^{i,s}$ since $\hat{B}^{i,s}$ is a perfect $U'_q(\mathfrak{g})$-crystal of level $s$. This implies that the level of $\hat{B}^{i,s}$ is greater than or equal to $s$.

So, to prove that the level of $\hat{B}^{i,s}$ is equal to $s$, it suffices to show that there exists some $b \in \hat{B}^{i,s}$ such that $\epsilon(b)(\tilde{c}) = s$. Let $\hat{\mu} \in (\hat{P}^+_{cl})_s$; note that $(\hat{P}^+_{cl})_s \neq \emptyset$ for all $s \in \mathbb{Z}_{\geq 1}$ since $s\lambda_0 \in (\hat{P}^+_{cl})_s$. Then we deduce that $(P'_\omega(\hat{\mu}))(c) = \hat{\mu}(P_\omega(c)) = \hat{\mu}(\tilde{c}) = s$, and hence that $P'_\omega(\hat{\mu}) \in (P^+_{cl})_s$. Because $\hat{B}^{i,s}$ is a perfect $U'_q(\mathfrak{g})$-crystal of level $s$, there exists a (unique) $b \in (\hat{B}^{i,s})_{\min}$ such that $\epsilon(b) = P'_\omega(\hat{\mu}) \in (P^+_{cl})_s$. It follows that this $b$ is contained in $\hat{B}^{i,s}$, i.e., that $\omega(b) = b$. Indeed, we see from (2.3.7) that $\epsilon_j(\omega(b)) = \epsilon_{\omega^{-1}(j)}(b)$ for all $j \in I$, and hence that $\epsilon(\omega(b)) = \omega^*(\epsilon(b))$. But, since $\epsilon(b) = P'_\omega(\hat{\mu}) \in (P^+_{cl})_s \cap (b^*_{cl})^0$, we have $\omega^*(\epsilon(b)) = \epsilon(b)$. So, we have $\epsilon(\omega(b)) = \epsilon(b)$. Because $\epsilon : (\hat{B}^{i,s})_{\min} \to (P^+_{cl})_s$ is bijective, we conclude that $\omega(b) = b$. Also, it follows from the formulas $P'_\omega(\epsilon(b)) = \epsilon(b)$ and $P'_\omega(\hat{\mu}) = \epsilon(b)$ that $\tilde{\epsilon}(b) = \hat{\mu}$. Consequently, we have $(\tilde{\epsilon}(b))(\tilde{c}) = \hat{\mu}(\tilde{c}) = s$ since $\hat{\mu} \in (\hat{P}^+_{cl})_s$. Thus, we have shown that $\text{lev} \hat{B}^{i,s} = s$.

Next, we prove that the map $\tilde{\epsilon} : (\hat{B}^{i,s})_{\min} \to (\hat{P}^+_{cl})_s$ is bijective. The argument above shows that for each $\hat{\mu} \in (\hat{P}^+_{cl})_s$, there exists some $b \in (\hat{B}^{i,s})_{\min}$ such that $\tilde{\epsilon}(b) = \hat{\mu}$, which means that the map $\tilde{\epsilon} : (\hat{B}^{i,s})_{\min} \to (\hat{P}^+_{cl})_s$ is surjective. Let us show that $\tilde{\epsilon} : (\hat{B}^{i,s})_{\min} \to (\hat{P}^+_{cl})_s$ is injective. Assume that $\tilde{\epsilon}(b) = \tilde{\epsilon}(b')$ for $b, b' \in (\hat{B}^{i,s})_{\min}$. Note that, by Lemma 3.2.3, $\epsilon(b) = P'_\omega(\tilde{\epsilon}(b)) = P'_\omega(\tilde{\epsilon}(b')) = \epsilon(b')$, and hence that $\epsilon(b)(c) = \epsilon(b')(c) = s$, $(\epsilon(b'))(c) = (\tilde{\epsilon}(b'))(\tilde{c}) = s$, i.e., that $b, b' \in (\hat{B}^{i,s})_{\min}$. Therefore, we conclude from the equality $\epsilon(b) = \epsilon(b')$ that $b = b'$ since $\epsilon : (\hat{B}^{i,s})_{\min} \to (P^+_{cl})_s$ is bijective. Thus, we have shown that the map $\tilde{\epsilon} : (\hat{B}^{i,s})_{\min} \to (\hat{P}^+_{cl})_s$ is injective, and hence bijective. Similarly, we can show that the map $\tilde{\varphi} : (\hat{B}^{i,s})_{\min} \to (\hat{P}^+_{cl})_s$ is bijective. This proves the proposition.

By Proposition 4.3.1 we can conclude that $\hat{B}^{i,s}$ is a perfect $U'_q(\mathfrak{g})$-crystal of level $s$, thereby completing the proof of Theorem 2.4.1.

4.4 Relation to virtual crystals. Let $\tilde{u}$ be the unique $W$-extremal element of the simple $U'_q(\mathfrak{g})$-crystal $\hat{B}^{i,s}$ whose weight is $I_0$-dominant. Then, by Lemma 4.2.2, the element
\( \tilde{u} \) is contained in the fixed point subset \( \tilde{B}^{i,s} \). In addition, since \( \tilde{B}^{i,s} \) is a perfect (and hence simple) \( U_q(\hat{g}) \)-crystal by Theorem 2.4.1 we see by Proposition 1.4.10(1) that \( \tilde{B}^{i,s} \) is equal to the set of elements of \( \tilde{B}^{i,s} \) obtained by applying the \( \omega \)-Kashiwara operators \( \tilde{e}_j, \tilde{f}_j, j \in \tilde{I}, \) successively to \( \tilde{u} \). Therefore, if the pair \( (g, \omega) \) is in Case (a) (resp., (c), (d), (e)) in \( \tilde{B}^{i,s} \) then \( (\tilde{B}^{i,s}, \tilde{B}^{i,s}) \) agrees with the virtual crystal for \( (D_{n+1}^{(2)}, A_{2n-1}^{(1)}) \) defined in [OSS1] 6.7] (resp., \( (A_{2n-1}^{(2)}, D_{n+1}^{(1)}), (D_{4}^{(3)}, D_{4}^{(1)}), (E_6^{(2)}, E_6^{(1)}) \) defined in [OSS2] Definition 2.6).

As for combinatorial \( R \)-matrices, we have the following proposition (see also [OSS2] Definition–Conjecture 3.4) and the comment after it).

**Proposition 4.4.1.** Let us fix \( i_1 \in \hat{I}_0, s_1 \in \mathbb{Z}_{\geq 1} \), and \( i_2 \in \hat{I}_0, s_2 \in \mathbb{Z}_{\geq 1} \), for which Assumption 2.3.1 is satisfied. Then, there exists an isomorphism \( \tilde{R} : \tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2} \rightarrow \tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1} \) of \( U_q(\hat{g}) \)-crystals.

**Proof.** As in 3.3 we define an action \( \omega : \tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2} \rightarrow \tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2} \) of the diagram automorphism \( \omega \) by: \( \omega(b_1 \otimes b_2) = \omega(b_1) \otimes \omega(b_2) \) for \( b_1 \otimes b_2 \in \tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2} \), and let \( (\tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2})^\omega \) be the fixed point subset under this action of \( \omega \). Similarly, we define an action \( \omega : \tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1} \rightarrow \tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1} \) of \( \omega \) as above, and let \( (\tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1})^\omega \) be the fixed point subset under this action of \( \omega \).

**Claim.** Let \( R : \tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2} \rightarrow \tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1} \) be the combinatorial \( R \)-matrix in Proposition 1.4.15(1). Then we have \( R((\tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2})^\omega) \subset (\tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1})^\omega \).

**Proof of Claim.** Let \( \tilde{u}_1 \) (resp., \( \tilde{u}_2 \)) be the unique \( W \)-extremal element of the simple \( U_q'(g) \)-crystal \( \tilde{B}^{i_1,s_1} \) (resp., \( \tilde{B}^{i_2,s_2} \)) whose weight is \( I_0 \)-dominant (see Lemma 1.4.8). Then we see from Lemma 1.2.2 and 3.3.1 that \( \tilde{u}_1 \otimes \tilde{u}_2 \in (\tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2})^\omega \) and \( \tilde{u}_2 \otimes \tilde{u}_1 \in (\tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1})^\omega \). Now, let \( b \in (\tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2})^\omega \). Because \( (\tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2})^\omega \) is isomorphic to \( \tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1} \) as a \( U_q'(\hat{g}) \)-crystal by Lemma 3.3.1 it follows from Theorem 2.4.1 and Proposition 1.4.10(2) that the \( U_q'(\hat{g}) \)-crystal \( (\tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2})^\omega \) is simple. Therefore, the \( b \) in \( (\tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2})^\omega \) can be written as: \( b = \tilde{x}_j_1 \tilde{x}_j_2 \cdots \tilde{x}_j_j(\tilde{u}_1 \otimes \tilde{u}_2) \), where \( \tilde{x}_j \) is either \( \tilde{e}_j \) or \( \tilde{f}_j \) for each \( j \in \tilde{I} \). Since \( R \) is an isomorphism of \( U_q'(g) \)-crystals, we see from the definition of the \( \omega \)-Kashiwara operators \( \tilde{e}_j \) and \( \tilde{f}_j \), \( j \in \tilde{I} \), that \( R(b) = \tilde{x}_j_1 \tilde{x}_j_2 \cdots \tilde{x}_j_j(\tilde{u}_1 \otimes \tilde{u}_2) \). Furthermore, it follows from Lemma 1.4.3 that \( \tilde{u}_1 \otimes \tilde{u}_2 \) (resp., \( \tilde{u}_2 \otimes \tilde{u}_1 \)) is a \( W \)-extremal element of the simple \( U_q'(g) \)-crystal \( \tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2} \) (resp., \( \tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1} \)) whose weight is \( I_0 \)-dominant. Because such an element of the simple \( U_q'(g) \)-crystal is unique by Lemma 1.4.8 we conclude that \( R(\tilde{u}_1 \otimes \tilde{u}_2) = \tilde{u}_2 \otimes \tilde{u}_1 \), and hence that \( R(b) = \tilde{x}_j_1 \tilde{x}_j_2 \cdots \tilde{x}_j_j(\tilde{u}_2 \otimes \tilde{u}_1) \). In addition, we deduce from Proposition 3.2.1 applied to the case \( B = \tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1} \) that \( R(b) \in (\tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1})^\omega \) since \( \tilde{u}_2 \otimes \tilde{u}_1 \in (\tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1})^\omega \). This proves the claim.

Now we define a map \( \tilde{R} : \tilde{B}^{i_1,s_1} \otimes \tilde{B}^{i_2,s_2} \rightarrow \tilde{B}^{i_2,s_2} \otimes \tilde{B}^{i_1,s_1} \) by the following commutative
diagram:
\[
\begin{align*}
\hat{B}^{i_1,s_1} \otimes \hat{B}^{i_2,s_2} & \xrightarrow{\sim} (\hat{B}^{i_1,s_1} \otimes \hat{B}^{i_2,s_2})^\omega \\
\hat{R} \downarrow & \quad \downarrow R \\
\hat{B}^{i_2,s_2} \otimes \hat{B}^{i_1,s_1} & \xrightarrow{\sim} (\hat{B}^{i_2,s_2} \otimes \hat{B}^{i_1,s_1})^\omega,
\end{align*}
\] (4.4.1)

where the isomorphism \(\Phi_1\) (resp., \(\Phi_2\)) of \(U'_q(\hat{\mathfrak{g}})\)-crystals on the top (resp., on the bottom) is given by Lemma \ref{lem:branching}. It immediately follows from this commutative diagram that \(\hat{R} : \hat{B}^{i_1,s_1} \otimes \hat{B}^{i_2,s_2} \to \hat{B}^{i_2,s_2} \otimes \hat{B}^{i_1,s_1}\) is an embedding of \(U'_q(\hat{\mathfrak{g}})\)-crystals. In addition, since the \(U'_q(\hat{\mathfrak{g}})\)-crystals \(\hat{B}^{i_1,s_1}\) and \(\hat{B}^{i_2,s_2}\) are perfect (and hence simple) by Theorem \ref{thm:branching}, we see from Proposition \ref{prop:branching}(2) that \(\hat{B}^{i_2,s_2} \otimes \hat{B}^{i_1,s_1}\) is a simple \(U'_q(\hat{\mathfrak{g}})\)-crystal, and that \(\hat{B}^{i_2,s_2} \otimes \hat{B}^{i_1,s_1}\) is connected. Consequently, we deduce that \(\hat{R} : \hat{B}^{i_1,s_1} \otimes \hat{B}^{i_2,s_2} \to \hat{B}^{i_2,s_2} \otimes \hat{B}^{i_1,s_1}\) is surjective, and hence bijective. Thus we have proved the proposition.

Similarly, using the isomorphism \(\Phi : \hat{B}^{i,s} \otimes \hat{B}^{s'} \to (\hat{B}^{i,s} \otimes \hat{B}^{s'})^\omega \subset \hat{B}^{i,s} \otimes \hat{B}^{s'}\) of \(U'_q(\hat{\mathfrak{g}})\)-crystals in Lemma \ref{lem:branching} we can prove the following proposition.

**Proposition 4.4.2.** We define a \(\mathbb{Z}\)-valued function \(\hat{H} : \hat{B}^{i,s} \otimes \hat{B}^{i,s} \to \mathbb{Z}\) by

\[
\hat{H}(b_1 \otimes b_2) := H(\Phi(b_1 \otimes b_2)) = H(b_1 \otimes b_2) \quad \text{for } b_1 \otimes b_2 \in \hat{B}^{i,s} \otimes \hat{B}^{i,s},
\] (4.4.2)

where \(H : \hat{B}^{i,s} \otimes \hat{B}^{s'} \to \mathbb{Z}\) is the energy function in Proposition \ref{prop:branching}(2). Then, the function \(\hat{H}\) enjoys the following property:

\[
\hat{H}(\hat{e}_j(b_1 \otimes b_2)) = \begin{cases} 
\hat{H}(b_1 \otimes b_2) + 1 & \text{if } j = 0 \text{ and } \hat{\varphi}_0(b_1) \geq \hat{\varepsilon}_0(b_2), \\
\hat{H}(b_1 \otimes b_2) - 1 & \text{if } j = 0 \text{ and } \hat{\varphi}_0(b_1) < \hat{\varepsilon}_0(b_2), \\
\hat{H}(b_1 \otimes b_2) & \text{if } j \neq 0,
\end{cases}
\] (4.4.3)

for all \(j \in \hat{I}\) and \(b_1 \otimes b_2 \in \hat{B}^{i,s} \otimes \hat{B}^{i,s}\) such that \(\hat{e}_j(b_1 \otimes b_2) \neq \theta\).

## 5 Branching rules with respect to \(U_q(\hat{\mathfrak{g}})\).

In this section, we use the notation of \ref{rem:branching} and keep Assumption \ref{assump:branching} for the (arbitrarily) fixed \(i \in \hat{I}_0 = \hat{I} \setminus \{0\}\) and \(s \in \mathbb{Z}_{\geq 1}\). Since \(\hat{B}^{i,s}\) is a perfect (and hence regular) \(U'_q(\hat{\mathfrak{g}})\)-crystal for \(i \in \hat{I}_0 = \hat{I} \setminus \{0\}\) and \(s \in \mathbb{Z}_{\geq 1}\) by Theorem \ref{thm:branching} it decomposes, as a \(U_q(\hat{\mathfrak{g}})\)-crystal, into a direct sum of the crystal bases of integrable highest weight \(U_q(\hat{\mathfrak{g}})\)-modules. In this section, we explicitly describe the branching rule, i.e., how the \(\hat{B}^{i,s}\) decomposes, with respect to the restriction to \(U_q(\hat{\mathfrak{g}})_{\hat{I}_0}\) for almost all \(i \in \hat{I}_0\) and \(s \in \mathbb{Z}_{\geq 1}\).

### 5.1 Preliminary results.

Let us set

\[
(\hat{B}^{i,s})_{h.w.} := \{ b \in \hat{B}^{i,s} \mid \hat{e}_j b = \theta \text{ for all } j \in \hat{I}_0 \}
\] (5.1.1)

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for \( i \in \hat{I}_0 \) and \( s \in \mathbb{Z}_{\geq 1} \). Then, by the regularity of the \( U'_q(\mathfrak{g}) \)-crystal \( \tilde{B}^{i,s} \), we have

\[
\tilde{B}^{i,s} \cong \bigoplus_{b \in (\tilde{B}^{i,s})_{h.w.}} B_{I_0}(\text{wt } b) \quad \text{as } U_q(\mathfrak{g}_{I_0}) \text{-crystals},
\]

(5.1.2)

where \( \tilde{B}_{I_0}(\tilde{\lambda}) \) denotes the crystal base of the integrable highest weight \( U_q(\mathfrak{g}_{I_0}) \)-module of \((\hat{I}_0\text{-dominant}) \) highest weight \( \tilde{\lambda} \in \hat{P}_c \). Similarly, we set

\[
(\tilde{B}^{i,s})_{h.w.} := \{ b \in \tilde{B}^{i,s} \mid e_j b = \theta \text{ for all } j \in I_0 \}
\]

(5.1.3)

for \( i \in \hat{I}_0 \) and \( s \in \mathbb{Z}_{\geq 1} \). Then, by the regularity of the \( U'_q(\mathfrak{g}) \)-crystal \( \tilde{B}^{i,s} \), we have

\[
\tilde{B}^{i,s} \cong \bigoplus_{b \in (\tilde{B}^{i,s})_{h.w.}} B_{I_0}(\text{wt } b) \quad \text{as } U_q(\mathfrak{g}_{I_0}) \text{-crystals},
\]

(5.1.4)

where \( B_{I_0}(\lambda) \) denotes the crystal base of the integrable highest weight \( U_q(\mathfrak{g}_{I_0}) \)-module of \((I_0\text{-dominant}) \) highest weight \( \lambda \in P_c \).

**Lemma 5.1.1.** We have

\[
(\tilde{B}^{i,s})_{h.w.} = (\tilde{B}^{i,s})_{h.w.} \cap \tilde{B}^{i,s} = \{ b \in (\tilde{B}^{i,s})_{h.w.} \mid \omega(b) = b \}.
\]

(5.1.5)

**Proof.** The inclusion \((\tilde{B}^{i,s})_{h.w.} \supset (\tilde{B}^{i,s})_{h.w.} \cap \tilde{B}^{i,s}\) is obvious from the definition of the raising \( \omega \)-Kashiwara operators \( \tilde{e}_j \), \( j \in \hat{I} \). For the reverse inclusion, let \( b \in (\tilde{B}^{i,s})_{h.w.} \) (note that \( \omega(b) = b \) by definition). Then it follows that \( \tilde{e}_j(b) = 0 \) for all \( j \in \hat{I}_0 \), and hence by Lemma 3.2.3 that \( \varepsilon_j(b) = 0 \) for all \( j \in I_0 \). This implies that \( e_j b = \theta \) for all \( j \in I_0 \). Hence we have \( b \in (\tilde{B}^{i,s})_{h.w.} \cap \tilde{B}^{i,s} \). This proves the lemma. \( \square \)

**Lemma 5.1.2.** Assume that the decomposition (5.1.4) of \( \tilde{B}^{i,s} \) as a \( U_q(\mathfrak{g}_{I_0}) \)-crystal is multiplicity-free. Then we have

\[
(\tilde{B}^{i,s})_{h.w.} = \{ b \in (\tilde{B}^{i,s})_{h.w.} \mid \omega^*(\text{wt } b) = \text{wt } b \}.
\]

(5.1.6)

**Proof.** The inclusion \( \subset \) is obvious by Lemma 5.1.1. For the reverse inclusion, let \( b \in (\tilde{B}^{i,s})_{h.w.} \) be such that \( \omega^*(\text{wt } b) = \text{wt } b \). It immediately follows from (2.3.7) that \( \omega(b) \) is also contained in \((\tilde{B}^{i,s})_{h.w.} \) (note that \( \omega(I_0) = I_0 \)). In addition, we have \( \text{wt}(\omega(b)) = \omega^*(\text{wt } b) = \text{wt } b \). Therefore, we deduce that \( \omega(b) = b \), since the decomposition (5.1.4) of \( \tilde{B}^{i,s} \) as a \( U_q(\mathfrak{g}_{I_0}) \)-crystal is multiplicity-free by the assumption. Thus, the \( b \in (\tilde{B}^{i,s})_{h.w.} \) is contained in the set \((\tilde{B}^{i,s})_{h.w.} \cap \tilde{B}^{i,s} \), and hence in the set \((\tilde{B}^{i,s})_{h.w.} \) by Lemma 5.1.1. This proves the lemma. \( \square \)

In \( \S 5.2 \) -- \( \S 5.6 \) below, by using Lemma 5.1.2, we give an explicit description of the branching rule for \( \tilde{B}^{i,s} \) for \( i \in \hat{I}_0 \) and \( s \in \mathbb{Z}_{\geq 1} \) with respect to the restriction to \( U_q(\mathfrak{g}_{I_0}) \),
except a few cases where the decomposition (5.1.4) of \( \widehat{B}^{i,s} \) as a \( U_q(\mathfrak{g}_{I_0}) \)-crystal is not multiplicity-free. In the following, we use the notation:

\[
\varpi_i := \Lambda_i - \alpha_i^\vee \Lambda_0 \in P_{cl} \quad \text{for } i \in I_0,
\]

(5.1.7)

\[
\tilde{\varpi}_i := \sum_{k=0}^{N_i-1} \varpi_{\omega^k(i)} \in P_{cl} \quad \text{for } i \in \hat{I}_0, \quad \text{and} \quad \tilde{\varpi}_0 := 0,
\]

(5.1.8)

\[
\hat{\varpi}_i := \hat{\Lambda}_i - \hat{\alpha}_i^\vee \hat{\Lambda}_0 \in \hat{P}_{cl} \quad \text{for } i \in \hat{I}_0, \quad \text{and} \quad \hat{\varpi}_0 := 0.
\]

(5.1.9)

Note that \( P^*_\omega(\tilde{\varpi}_i) = \tilde{\varpi}_i \) for all \( i \in \hat{I}_0 \), and also for \( i = 0 \).

**Remark 5.1.3.** For \( i \in \hat{I}_0 \) and \( s \in \mathbb{Z}_{\geq 1} \), we set \( \tilde{u}_{i,s} = u_{i,s} \otimes u_{\omega(i),s} \otimes \cdots \otimes u_{\omega^{N_i-1}(i),s} \), where the \( u_{\omega^k(i),s} \in B^{\omega^k(i),s} \), \( 0 \leq k \leq N_i - 1 \), are the \( W \)-extremal elements in Remark 2.3.2. Then, by Lemma 1.4.3, this \( \tilde{u}_{i,s} \) is the \( W \)-extremal element of the simple \( U_q(\mathfrak{g}) \)-crystal \( \widehat{B}^{i,s} \) such that \( (wt \tilde{u}_{i,s})(\lambda_j) \geq 0 \) for all \( j \in I_0 \). In fact, we have \( wt \tilde{u}_{i,s} = s \tilde{\varpi}_i \). Therefore, it follows from Remark 1.4.9 that the weights of elements of \( \widehat{B}^{i,s} \) are all contained in the set \( s \tilde{\varpi}_i - \sum_{j \in I_0} \mathbb{Z}_{\geq 0} \alpha_j \). Also, the weights of elements of \( \widehat{B}^{i,s} \) are all contained in the set \( s \tilde{\varpi}_i - \sum_{j \in \hat{I}_0} \mathbb{Z}_{\geq 0} \alpha_j \).

**5.2 Branching rule for Case (a).** We know from [KMN] that

\[
B^{i,s} \cong B_{I_0}(s\tilde{\varpi}_i) \quad \text{as } U_q(\mathfrak{g}_{I_0}) \text{-crystals}
\]

(5.2.1)

for \( i \in I_0 \) and \( s \in \mathbb{Z}_{\geq 1} \). Here we recall the (well-known) fact that the integrable highest weight \( U_q(\mathfrak{g}_{I_0}) \)-module \( V_{I_0}(\lambda) \) of \( I_0 \)-dominant highest weight \( \lambda \in P_{cl} \) has the same character as the integrable highest weight \( \mathfrak{g}_{I_0} \)-module of the same highest weight, and this character is equal to \( \sum_{b \in B_{I_0}(\lambda)} e^{\text{wt } b} \). By using this fact, we deduce from Decomposition Rule on page 145 of [L] that if \( 1 \leq i \leq n - 1 \), then as \( U_q(\mathfrak{g}_{I_0}) \)-crystals,

\[
\widehat{B}^{i,s} = B^{i,s} \otimes B^{\omega(i),s} \cong B_{I_0}(s\tilde{\varpi}_i) \otimes B_{I_0}(s\tilde{\varpi}_{\omega(i)}) \quad \text{by (5.2.1)}
\]

\[
\cong \bigoplus_{s_0+s_1+\cdots+s_i=s \atop s_0, s_1, \ldots, s_i \in \mathbb{Z}_{\geq 0}} B_{I_0}(s_0\tilde{\varpi}_0 + s_1\tilde{\varpi}_1 + \cdots + s_i\tilde{\varpi}_i)
\]

\[
= \bigoplus_{s_1+\cdots+s_i \leq s \atop s_1, \ldots, s_i \in \mathbb{Z}_{\geq 0}} B_{I_0}(s_1\tilde{\varpi}_1 + \cdots + s_i\tilde{\varpi}_i)
\]

for \( s \in \mathbb{Z}_{\geq 1} \). If \( i = n \), then \( \widehat{B}^{n,s} = B^{n,s} \cong B_{I_0}(s\tilde{\varpi}_n) \) as \( U_q(\mathfrak{g}_{I_0}) \)-crystals for \( s \in \mathbb{Z}_{\geq 1} \). Note that for every \( i \in \hat{I}_0 \) and \( s \in \mathbb{Z}_{\geq 1} \), the decomposition of \( \widehat{B}^{i,s} \) above is multiplicity-free, and that the highest weights appearing in the decomposition are all fixed by \( \omega^* \). Consequently, by using Lemma 5.1.2 we obtain the branching rule as follows.
Proposition 5.2.1. For $i \in \widehat{I}_0$ and $s \in \mathbb{Z}_{\geq 1}$, we have

$$\widehat{B}^{i,s} \cong \begin{cases} \bigoplus_{s_1 + \cdots + s_i \leq s} \widehat{B}_{I_0} (s_1 \widehat{\omega}_1 + \cdots + s_i \widehat{\omega}_i) & \text{if } 1 \leq i \leq n - 1, \\ \widehat{B}_{I_0} (s \widehat{\omega}_n) & \text{if } i = n, \end{cases} \quad (5.2.2)$$

as $U_q(\widehat{g})$-crystals.

5.3 Branching rule for Case (b). In exactly the same way as in Case (a), we have the following proposition.

Proposition 5.3.1. For $i \in \widehat{I}_0$ and $s \in \mathbb{Z}_{\geq 1}$, we have

$$\widehat{B}^{i,s} \cong \bigoplus_{s_1 + \cdots + s_i \leq s} \widehat{B}_{I_0} (s_1 \widehat{\omega}_1 + \cdots + s_i \widehat{\omega}_i) \quad (5.3.1)$$

as $U_q(\widehat{g})$-crystals.

5.4 Branching rule for Case (c). We know from [3] that the KR module $W_s^{(i)} (\zeta_s^{(i)})$ over $U_q(\mathfrak{g})$ for $i \in I_0$, $s \in \mathbb{Z}_{\geq 1}$, and $\zeta_s^{(i)} \in \mathbb{C}(q)^\times$, decomposes under the restriction to $U_q(\mathfrak{g})_0$ as follows:

$$W_s^{(i)} (\zeta_s^{(i)}) \cong \begin{cases} \bigoplus_{s_{p_1} + s_{p_1+2} + \cdots + s_i = s} V_{I_0} (s_{p_1} \overline{\omega}_{p_1} + s_{p_1+2} \overline{\omega}_{p_1+2} + \cdots + s_i \overline{\omega}_i) & \text{if } 1 \leq i \leq n - 1, \\ V_{I_0} (s \overline{\omega}_i) & \text{if } i = n, n + 1, \end{cases} \quad (5.4.1)$$

as $U_q(\mathfrak{g})_0$-modules, where the $p_i \in \{0, 1\}$ for $1 \leq i \leq n - 1$ is defined to be 0 (resp., 1) if $i$ is even (resp., odd), and $V_{I_0} (\lambda)$ is the integrable highest weight $U_q(\mathfrak{g})_0$-module of $(I_0$-dominant) highest weight $\lambda \in P_0$. Accordingly, from (5.4.1), we obtain the following decomposition of the crystal base $B^{i,s}$ of $W_s^{(i)} (\zeta_s^{(i)})$, regarded as a $U_q(\mathfrak{g})_0$-crystal by restriction:

$$B^{i,s} \cong \begin{cases} \bigoplus_{s_{p_1} + s_{p_1+2} + \cdots + s_i = s} B_{I_0} (s_{p_1} \overline{\omega}_{p_1} + s_{p_1+2} \overline{\omega}_{p_1+2} + \cdots + s_i \overline{\omega}_i) & \text{if } 1 \leq i \leq n - 1, \\ B_{I_0} (s \overline{\omega}_i) & \text{if } i = n, n + 1, \end{cases} \quad (5.4.2)$$
for $s \in \mathbb{Z}_{\geq 1}$. Consequently, as in Case (a), we deduce from Decomposition Rule on page 145 of \cite{L} that as $U_q(\mathfrak{g}_{\tilde{I}_0})$-crystals,

$$\tilde{B}^{n,s} = B^{n,s} \otimes B^{n+1,s} \cong B_{\tilde{I}_0}(s\varpi_n) \otimes B_{\tilde{I}_0}(s\varpi_{n+1}) \quad \text{by (5.4.2)}$$

$$\cong \bigoplus_{s_{p_n} + s_{p_n+2} + \cdots + s_n = s, s_{p_n}, s_{p_n+2}, \ldots, s_n \in \mathbb{Z}_{\geq 0}} B_{\tilde{I}_0}(s_{p_n} \varpi_{p_n} + s_{p_n+2} \varpi_{p_n+2} + \cdots + s_n \varpi_n)$$

for $s \in \mathbb{Z}_{\geq 1}$. Also, if $1 \leq i \leq n-1$, then we have

$$\tilde{B}^{i,s} = B^{i,s} \cong \bigoplus_{s_{p_i} + s_{p_i+2} + \cdots + s_i = s, s_{p_i}, s_{p_i+2}, \ldots, s_i \in \mathbb{Z}_{\geq 0}} B_{\tilde{I}_0}(s_{p_i} \varpi_{p_i} + s_{p_i+2} \varpi_{p_i+2} + \cdots + s_i \varpi_i) \quad \text{by (5.4.2)}$$

$$= \bigoplus_{s_{p_i} + s_{p_i+2} + \cdots + s_i = s, s_{p_i}, s_{p_i+2}, \ldots, s_i \in \mathbb{Z}_{\geq 0}} B_{\tilde{I}_0}(s_{p_i} \varpi_{p_i} + s_{p_i+2} \varpi_{p_i+2} + \cdots + s_i \varpi_i).$$

Because in all cases, the decomposition of $\tilde{B}^{i,s}$ above is multiplicity-free, and the highest weights appearing in the decomposition are all fixed by $\omega^s$, we obtain the following proposition by using Lemma \ref{Lemma5.1.2}.

**Proposition 5.4.1.** For $i \in \tilde{I}_0$ and $s \in \mathbb{Z}_{\geq 1}$, we have

$$\tilde{B}^{i,s} \cong \bigoplus_{s_{p_i} + s_{p_i+2} + \cdots + s_i = s, s_{p_i}, s_{p_i+2}, \ldots, s_i \in \mathbb{Z}_{\geq 0}} B_{\tilde{I}_0}(s_{p_i} \varpi_{p_i} + s_{p_i+2} \varpi_{p_i+2} + \cdots + s_i \varpi_i) \quad (5.4.3)$$

as $U_q(\mathfrak{g}_{\tilde{I}_0})$-crystals.

### 5.5 Branching rule for Case (d).

First, we should remark that our numbering of the index set $I$ is different from the ones in \cite{HKOTY} and \cite{L}. As in Case (c), we obtain that as $U_q(\mathfrak{g}_{I_0})$-crystals,

$$\tilde{B}^{1,s} \cong \bigoplus_{0 \leq s_1 \leq s} B_{I_0}(s_1 \varpi_1), \quad (5.5.1)$$

$$\tilde{B}^{2,s} \cong B_{I_0}(s \varpi_2) \otimes B_{I_0}(s \varpi_3) \otimes B_{I_0}(s \varpi_4), \quad (5.5.2)$$

for $s \in \mathbb{Z}_{\geq 1}$. Note that in general, the irreducible decomposition of the tensor product $V_{I_0}(s \varpi_2) \otimes V_{I_0}(s \varpi_3) \otimes V_{I_0}(s \varpi_4)$ as a $U_q(\mathfrak{g}_{I_0})$-module is not multiplicity-free, and hence that the decomposition \cite{5.1.4} of $\tilde{B}^{2,s}$, regarded as a $U_q(\mathfrak{g}_{I_0})$-crystal by restriction, is not multiplicity-free. Hence we exclude this case, i.e., the case where $i = 2$. Then, by using Lemma \ref{Lemma5.1.2} as above, we obtain the following proposition.

**Proposition 5.5.1.** For $s \in \mathbb{Z}_{\geq 1}$, we have

$$\tilde{B}^{1,s} \cong \bigoplus_{0 \leq s_1 \leq s} \tilde{B}_{I_0}(s_1 \varpi_1) \quad (5.5.3)$$

as $U_q(\mathfrak{g}_{\tilde{I}_0})$-crystals.
5.6 Branching rule for Case (e). First, we should remark that our numbering of the index set $I$ is different from the ones in [HKOTY] and [L]. Assume that $i \neq 2$ (note that the decomposition of $B^{2,s}$ is known not to be multiplicity-free in general; cf. the formula for $\mathcal{W}_{s}^{(3)}$ in the case where $X_n = E_6$ on page 278 of [HKOTY Appendix A]). Then, as in Case (c), we obtain from (5.6.1) the following decomposition of $B^{i,s}$, regarded as a $U_q(\mathfrak{g}_{I_0})$-crystal by restriction:

$$B^{i,s} \cong \begin{cases}
B_{I_0}(s \varpi_i) & \text{if } i = 4, 6, \\
\bigoplus_{0 \leq s_1 \leq s} B_{I_0}(s_1 \varpi_1) & \text{if } i = 1, \\
\bigoplus_{s_3 + s_6 = s \atop s_3, s_6 \geq 0} B_{I_0}(s_3 \varpi_3 + s_6 \varpi_6) & \text{if } i = 3, \\
\bigoplus_{s_4 + s_5 = s \atop s_4, s_5 \geq 0} B_{I_0}(s_4 \varpi_4 + s_5 \varpi_5) & \text{if } i = 5,
\end{cases}$$

for $s \in \mathbb{Z}_{\geq 1}$. Consequently, as in Cases (a), (c), we deduce from Decomposition Rule on page 145 of [L] that as $U_q(\mathfrak{g}_{I_0})$-crystals,

$$\widehat{B}^{1,s} = B^{1,s} \cong \bigoplus_{0 \leq s_1 \leq s} B_{I_0}(s_1 \varpi_1), \quad (5.6.1)$$

$$\widehat{B}^{4,s} = B^{4,s} \otimes B^{6,s} \cong B_{I_0}(s \varpi_4) \otimes B_{I_0}(s \varpi_6) \cong \bigoplus_{s_1 + s_4 \leq s \atop s_1, s_4 \in \mathbb{Z}_{\geq 0}} B_{I_0}(s_1 \varpi_1 + s_4 \varpi_4), \quad (5.6.2)$$

for $s \in \mathbb{Z}_{\geq 1}$. Here we have exclude the case $i = 3$, since the decomposition of $\widehat{B}^{3,s} = B^{3,s} \otimes B^{5,s}$, regarded as a $U_q(\mathfrak{g}_{I_0})$-crystal by restriction, is known not to be multiplicity-free in general. Then, by using Lemma 5.1.2 as above, we obtain the following proposition.

**Proposition 5.6.1.** For $s \in \mathbb{Z}_{\geq 1}$, we have

$$\widehat{B}^{1,s} \cong \bigoplus_{0 \leq s_1 \leq s} \widehat{B}_{I_0}(s_1 \widehat{\varpi}_1), \quad \widehat{B}^{4,s} \cong \bigoplus_{s_1 + s_4 \leq s \atop s_1, s_4 \in \mathbb{Z}_{\geq 0}} \widehat{B}_{I_0}(s_1 \widehat{\varpi}_1 + s_4 \widehat{\varpi}_4), \quad (5.6.3)$$

as $U_q(\mathfrak{g}_{I_0})$-crystals.

5.7 Comments. In addition to Assumption 2.3.1 for $\mathfrak{g}$, let us assume that Conjecture 1.5.1 with $\mathfrak{g}$ replaced by $\widehat{\mathfrak{g}}$ holds for the (fixed) $i \in \widehat{I}_0$ and $s \in \mathbb{Z}_{\geq 1}$. Denote the KR module over $U_q'(\widehat{\mathfrak{g}})$ having a crystal base by $\widehat{\mathcal{W}}_s^{(i)}(\zeta^{(i)}_s)$, where $\zeta^{(i)}_s \in \mathbb{C}(q)^\times$. Then we make the following observations.
Observations. By comparing (5.2.2) (resp., (5.3.1), (5.4.3)) with the formula for $W_s^{(a)}$ in the case where $g = D_{n+1}^{(2)}$ (resp., $A_{2n}^{(2)}, A_{2n-1}^{(2)}$) on page 247 (resp., page 247, page 246) of [HKOTT Appendix A] (with “a” replaced by “i”, “Λ̂_i” replaced by “\(\hat{\sigma}_i\)”, and “q” equal to “1”), we can check that they indeed agree. Similarly, by comparing (5.5.3) (resp., (5.6.3)) with the formula for $W_s^{(1)}$ (resp., $W_s^{(1)}$ and $W_s^{(4)}$) in the case where $g = D_{4}^{(3)}$ (resp., $E_{6}^{(2)}$) on page 249 (resp., page 248) of [HKOTT Appendix A] (with “\(\Lambda_i\)” replaced by “\(\hat{\sigma}_i\)” and “q” equal to “1”), we can check that they also agree. Here the formula for $W_s^{(a)}$ (with $q = 1$) describes the branching rule for the KR module $\hat{W}_s^{(a)}(\hat{\zeta}_s^{(a)})$ over $U_q(\widehat{g})$ with respect to the restriction to $U_q(\widehat{g}_{I_0})$.

Motivated by the observations above, we propose the following conjecture.

Conjecture 5.7.1. Let us fix (arbitrarily) $i \in \hat{I}_0$ and $s \in \mathbb{Z}_{\geq 1}$, and keep Assumption [2.3.1].

Then, the perfect $U'_q(\widehat{g})$-crystal $\hat{B}_{i,s}$ is isomorphic to the (conjectural) crystal base of the KR module $\hat{W}_s^{(i)}(\hat{\zeta}_s^{(i)})$ over the twisted quantum affine algebra $U'_q(\hat{g})$.

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