Tetrahedron equation and generalized quantum groups

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Abstract

We construct $2^n$-families of solutions of the Yang–Baxter equation from $n$-products of three-dimensional $R$ and $L$ operators satisfying the tetrahedron equation. They are identified with the quantum $R$ matrices for the Hopf algebras known as generalized quantum groups. Depending on the number of $R$’s and $L$’s involved in the product, the trace construction interpolates the symmetric tensor representations of $U_q(A_n^{(1)})$ and the antisymmetric tensor representations of $U_{-q}^{-1}(A_{n-1}^{(1)})$, whereas a boundary vector construction interpolates the $q$-oscillator representation of $U_q(D_{n+1}^{(2)})$ and the spin representation of $U_{-q}^{-1}(D_{n+1}^{(2)})$. The intermediate cases are associated with an affinization of quantum superalgebras.

Keywords: tetrahedron equation, Yang–Baxter equation, generalized quantum groups

1. Introduction

The tetrahedron equation [37] is a generalization of the Yang–Baxter equation [6] and serves as a key to quantum integrability in three dimensions (3D). It represents a factorization
condition on the scattering of straight strings in \((2 + 1)\)-dimensions and also a sufficient condition for the commutativity of layer-to-layer transfer matrices in 3D lattice models. Among several versions of the tetrahedron equation we are concerned with the following two types in this paper:

\[ R_{1,2,4}R_{1,3,5}R_{2,3,6}R_{4,5,6} = R_{4,5,6}R_{2,3,6}R_{1,3,5}R_{1,2,4}, \]

\[ L_{1,2,4}L_{1,3,5}L_{2,3,6}R_{4,5,6} = R_{4,5,6}L_{2,3,6}L_{1,3,5}L_{1,2,4}. \]

Here \( R \) and \( L \) are linear operators on \( F \otimes F \otimes F \) and \( V \otimes V \otimes F \), respectively, with some vector spaces \( F \) and \( V \). The indices signify the components on which these operators act nontrivially. The first equation is to hold in \( \text{End}(F^{\otimes 6}) \) and the second one in \( \text{End}(V^{\otimes 3} \otimes F^{\otimes 3}) \). We refer to \( R \) and \( L \) as 3D \( R \) and 3D \( L \) for short.

The first remarkable example of 3D \( R \) was proposed in [38]. It was referred to as ‘an extraordinary feat of intuition’ by Baxter, who proved that it indeed satisfies the tetrahedron equation [7]. It was actually another extraordinary feat which pioneered the subject and inspired subsequent developments that still continue in earnest.

The tetrahedron equation reduces to the Yang–Baxter equation

\[ R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}, \]

if the spaces 4, 5, 6, which we call the auxiliary spaces, are evaluated away suitably. It implies a certain connection between a class of solvable models in 2D and 3D by regarding the third direction in the latter as the internal degrees of freedom of local spins in the former. Such a correspondence between 2D and 3D theories has been studied in a variety of contexts, e.g. [13, 24, 32], and highlighted by the celebrated interpretation/extension of the 2D chiral Potts model [2, 8] and its generalizations [10, 14] in the 3D picture [9, 34].

In this paper we study reductions of the tetrahedron equations to the Yang–Baxter equation for the distinguished example of the 3D \( R \) and the 3D \( L \) associated with the quantized algebra of functions on \( SL_3 \) [23] and the \( q \)-oscillator algebra [12]. The 3D \( R \)’s in these works are known to coincide [25], contain a parameter \( q \) and correspond to choosing \( F \) to be the \( q \)-bosonic Fock space \( \bigoplus_{m \geq 0} \mathbb{C}\langle m \rangle \) and \( V = \mathbb{C}^2 \). See section 2 and the literature cited therein for more description.

There are three kinds of freedom that one can introduce in performing the reduction. First, the elimination of the auxiliary spaces can be done either by taking the trace [12] or matrix elements with respect to special boundary vectors [30]. Curiously, this freedom is known to reflect the boundary shape of the Dynkin diagram relevant to the final result as observed in [30, remark 7.2] and [27, remark 14]. Second, the reduction can be applied to the \( n \)-layer version of the tetrahedron equations for any \( n \geq 1 \). Third, the resulting product of \( n \) operators may consist of a mixture of \( R \)’s and \( L \)’s in any order. This last freedom, which was pointed out in [26, 33] but hitherto remained almost intact, is the theme of systematic investigation in this paper. It leads to \( 2^n \)-families of solutions to the Yang–Baxter equation corresponding to \( (R \text{ or } L)^n \). They act on \( \mathcal{W} \otimes \mathcal{W} \) where \( \mathcal{W} = (F \text{ or } V)^{\otimes n} \) is an arbitrary \( n \)-fold tensor product of \( F \) and \( V \) (section 2.5). There is a similarity transformation exchanging \( F \otimes V \) and \( V \otimes F \) locally in \( \mathcal{W} \), hence there are essentially \( (n + 1) \)-tuple of solutions for each \( n \) (section 2.7). Our principal result is theorem 4.1, which clarifies their origin as the quantum \( R \) matrices for the Hopf algebras that we will also introduce in section 3. They include an affinization of quantum superalgebras [17, 35, 36] as well as a class of quantum affine algebras [16, 22]. In general, they offer examples of generalized quantum groups. This notion emerged through the classification of pointed Hopf algebras [1, 19] and was first introduced in [20]. For recent developments of generalized quantum groups, see for instance [3–5, 21].
By changing the portion of $R$ and $L$ or equivalently $V$ and $F$ in the $n$-product, the trace construction interpolates the quantum $R$ matrices for the symmetric tensor representations of the quantum affine algebra $U_q(A_n^{(1)})$ and the antisymmetric tensor representations of $U_q(D_n^{(2)})$. Similarly, the boundary vector construction interpolates the $q$-oscillator representation of $U_q(D_n^{(2)})$ and the spin representation of $U_q(D_n^{(2)})$. The intermediate cases are related to the quantum superalgebras (section 3.3). These results generalize and synthesize the previous works [12, 26–30, 33]. They indicate hidden quantum group structures in 3D integrable lattice models, or put another way, hidden 3D structures in the quantum group theory.

The layout of the paper is as follows. In section 2 we recall the 3D $R$, the 3D $L$ and the construction of the $2^n$-families of spectral parameter dependent solutions $S(e_1, ..., e_n)(e_i = 0, 1)$ of the Yang–Baxter equation by various $2D$ reductions of their mixed $n$-products. We explain the equivalence of $S(e_1, ..., e_n)$ under permutations of $e_1, ..., e_n$ and summarize the known results in section 2.8. This part serves as an extended version of the introduction. In section 3 we introduce the generalized quantum groups $U_A = U_A(e_1, ..., e_n)$, $U_B = U_B(e_1, ..., e_n)$ and their irreducible representations $\pi_x$. They are relevant to the trace and a boundary vector construction, respectively. Precise relations to the quantum superalgebras $A_n(m, m')$ and $B_n(m, m')$ [17] (see also [35]) are explained in section 3.3. The quantum $R$ matrices are defined via the commutativity with $U_A$ or $U_B$ and a normalization condition. In section 4 the main result of the paper, theorem 4.1, is presented which identifies the $S(e_1, ..., e_n)$ constructed in section 2 with the quantum $R$ matrices introduced in section 3.4.

The remaining sections 5–7 are devoted to a proof of theorem 4.1. Our strategy is to establish that $S(e_1, ..., e_n)$ satisfies the same characterization as the quantum $R$ matrices given in section 3.4. In section 5 we prove the commutativity of $S(e_1, ..., e_n)$ with $U_A$ or $U_B$. It is vital to also ensure the irreducibility of the tensor product representation $\pi_x \otimes \pi_x$ in order to characterize the $R$ matrices as their commutant. Since no relevant result was found in the literature, we include a self-contained proof of the irreducibility for the $U_A(e_1, ..., e_n)$-module in section 6 and the $U_B(e_1, ..., e_n)$-module in section 7 for $(e_1, ..., e_n)$ of the form $(F, 0^{n-s}) = (1, ..., 1, 0, ..., 0)$. In the course of the proof, we obtain the spectral decomposition of the quantum $R$ matrices for the $U_A$ and $U_B$ explicitly. In particular (6.13), (6.16) and proposition 7.3 are new results, which lead to the explicit formulas as in examples 3.5–3.7.

Throughout the paper we assume that $q$ is generic and use the following notations:

\[
(z; q)_m = \prod_{k=1}^{m} \left( 1 - zq^{k-1} \right), \quad (q)_m = (q; q)_m, \quad \binom{m}{k}_q = \frac{(q)_m}{(q)_k(q)_{m-k}},
\]

\[
[m] = [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_q ! = \prod_{k=1}^{m} [k]_q, \quad \left[ \frac{m}{k} \right] = \frac{[m]!}{[k]! [m-k]!},
\]

where both the $q$-binomials are to be understood as zero unless $0 \leq k \leq m$.

2. Families of solutions to the Yang–Baxter equation

This section may still be regarded as a continuation of the introduction, where we formulate our problem and list the preceding results precisely.
2.1. 3D R

Let \( F \) and \( F^* \) be a Fock space and its dual

\[
F = \bigoplus_{m \geq 0} \mathbb{C} \langle m \rangle, \quad F^* = \bigoplus_{m \geq 0} \mathbb{C} | m \rangle
\]

whose pairing is given by \( \langle m | m' \rangle = \delta_{m,m'} (q^2)^m \). Define a linear operator \( \mathcal{R} \) on \( F \otimes F \otimes F \) by

\[
\mathcal{R}(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c \geq 0} \mathcal{R}_{i,j,k}^{a,b,c} |a\rangle \otimes |b\rangle \otimes |c\rangle,
\]

(2.1)

\[
\mathcal{R}_{i,j,k}^{a,b,c} = \delta_{i+j}^{a+b} \delta_{j+k}^{b+c} \sum_{\lambda+\mu = b} (-1)^q q^{(i-j)+(k+1)j+\mu} (q^2)^{i+\mu} (q^2)^{j} \delta_{\lambda}(i,j,k),
\]

(2.2)

where \( \delta_{i}^{j} = \delta_{i,j} \) just to save the space. The sum (2.2) is over \( \lambda, \mu \geq 0 \) satisfying \( \lambda + \mu = b \), which is also bounded by the condition \( \mu \leq i \) and \( \lambda \leq j \). The \( \mathcal{R} \) will simply be called 3D \( R \) in this paper. It satisfies the tetrahedron equation:

\[
\mathcal{R}_{1,2,4} \mathcal{R}_{1,3,5} \mathcal{R}_{2,3,6} \mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6} \mathcal{R}_{2,3,6} \mathcal{R}_{1,3,5} \mathcal{R}_{1,2,4},
\]

(2.3)

which is an equality in \( \text{End}(F^8) \). Here \( \mathcal{R}_{i,j,k} \) acts as \( \mathcal{R} \) on the \( i, j, k \)th components from the left in the tensor product \( F^8 \). With the space \( F \) denoted by an arrow, the relation (2.3) is depicted as follows:

![Diagram](image_url)

The 3D \( R \) was obtained as the intertwiner of the quantized coordinate ring \( \mathcal{A}_q(sl_3) \) [23] \(^6\). It was also found from a quantum geometry consideration in a different gauge [12]. They were shown to be the same object in [25 equation (2.29)]. Appendix A in [27] contains the recursion relations characterizing \( \mathcal{R} \) and useful corollaries which will also be utilized in the present paper. Here we note

\[
\mathcal{R} = \mathcal{R}^{-1}, \quad \mathcal{R}_{i,j,k}^{a,b,c} = \mathcal{R}_{a,b,c}^{i,j,k}, \quad \mathcal{R}_{i,j,k}^{a,b,c} = \left( \frac{q^2}{a} \right) \left( \frac{q^2}{b} \right) \left( \frac{q^2}{c} \right) \mathcal{R}_{i,j,k}^{a,b,c}.
\]

(2.4)

The last property makes it consistent to define the action of 3D \( R \) on \( F^* \otimes F^* \otimes F^* \) via

\[
(\langle i | \otimes | j | \otimes | k |) \mathcal{R} = \sum_{a,b,c \geq 0} \mathcal{R}_{i,j,k}^{a,b,c} \langle a | \otimes \langle b | \otimes \langle c |.
\]

\[^6\] The formula for it on p194 in [23] contains a misprint unfortunately. Equation (2.2) here is a correction of it.
Let $h$ be the linear operator on $F$ and $F^*$ such that $h|m\rangle = m|m\rangle$ and $\langle m|h = \langle m|m\rangle$. The factor $\delta_{\nu_+\rho_+}\delta_{\nu_-\rho_-}$ in (2.2) implies

$$\left[R, x^{h_i(x)}h_j^{b_i}h_j^{b_i}\right] = 0,$$

where $h_i$ denotes the one acting nontrivially on $F$ in $F \otimes F \otimes F$.

### 2.2. Boundary vectors

Let us introduce the following vectors in $F$ and $F^*$:

$$|k_1\rangle = \sum_{m \geq 0} \frac{|m\rangle}{(q)_m}, \quad \langle k_1| = \sum_{m \geq 0} \frac{(m)}{(q)_m},$$

$$|k_2\rangle = \sum_{m \geq 0} \frac{|2m\rangle}{(q^4)_m}, \quad \langle k_2| = \sum_{m \geq 0} \frac{(2m)}{(q^4)_m}.$$

We further set $|\chi_{s}(z)\rangle = z^{h_i}|\chi_{s}\rangle$ and $\langle \chi_{s}(z)| = \langle \chi_{s}\rangle z^{b_i}$ for $s = 1, 2$, where the factor $1/s$ is just a matter of normalization of the spectral parameter $z$. They are called **boundary vectors**.

The following property [30], which actually reduces to the $x = y = 1$ case by (2.5), will play a key role:

$$R\left( |k_1(x)\rangle \otimes |\chi_{s}(xy)\rangle \otimes |k_1(y)\rangle \right) = |k_1(x)\rangle \otimes |\chi_{s}(xy)\rangle \otimes |k_1(y)\rangle \in F \otimes F \otimes F,$

$$\left( \langle k_1(x)| \otimes \langle \chi_{s}(xy)| \otimes \langle k_1(y)| \right)R = \langle k_1(x)| \otimes \langle \chi_{s}(xy)| \otimes \langle k_1(y)| \in F^* \otimes F^* \otimes F^*.$$

### 2.3. 3D L

Now we proceed to the 3D L [12]. Let $V = C v_0 \oplus C v_1$ and define $\mathcal{L}$ by

$$\mathcal{L} = \left( L^x_{\alpha,\beta} \right) \in \text{End}(V \otimes V \otimes F),$$

$$\mathcal{L}\left( v_\alpha \otimes v_\beta \otimes |m\rangle \right) = \sum_{\gamma,\delta} v_\gamma \otimes v_\delta \otimes L_{\gamma,\delta}^x |m\rangle,$$

where $L^x_{\alpha,\beta} \in \text{End}(F)$ are zero except the following six cases:

$$L^x_{0,0} = L^x_{1,1} = 1, \quad L^x_{0,1} = -qk, \quad L^x_{1,0} = k, \quad L^x_{0,1} = a^-, \quad L^x_{1,0} = a^+.$$

The operators $a^\pm, k \in \text{End}(F)$ are called $q$-oscillators and act on $F$ by

$$a^+|m\rangle = |m + 1\rangle, \quad a^-|m\rangle = (1 - q^{2m})|m - 1\rangle, \quad k|m\rangle = q^m|m\rangle.$$

Thus $k = q^h$ in terms of $h$ defined around (2.5). They satisfy the relations

$$ka^\pm = q^{\pm 1}a^\pm k, \quad a^+a^- = 1 - k^2, \quad a^-a^+ = 1 - q^2k^2.$$

The $\mathcal{L}$ will simply be called 3D $L$ in this paper. It may be regarded as a six-vertex model having the $q$-oscillator valued Boltzmann weights. From this viewpoint the last two relations

...
are viewed as a quantization of the so-called free fermion condition \(|\omega\omega = 0\). [6, equation (10.16.4)] We will also use the notation similar to (2.1) to express (2.8) as
\[
\mathcal{L} \left( {v}_a \otimes {v}_\beta \otimes |m\rangle \right) = \sum_{\tau, \delta, j} \mathcal{L}_{\tau, \delta, j}^{\alpha, \beta, m} {v}_\alpha \otimes {v}_\beta \otimes |j\rangle,
\]
\[
\mathcal{L}_{0,0,0}^{0,0,0} = \mathcal{L}_{1,1,1}^{0,1,0} = \delta_0^j, \quad \mathcal{L}_{0,1,0}^{0,1,0} = -\delta_0^j q^{m+1}, \quad \mathcal{L}_{1,0,0}^{1,0,0} = \delta_0^j q^m.
\]
\[
\mathcal{L}_{0,1,0}^{0,1,0} = \delta_0^j \left( 1 - q^{2m} \right), \quad \mathcal{L}_{0,1,0}^{1,0,0} = \delta_{m+1}^j. \quad (2.12)
\]
The other \(\mathcal{L}_{\tau, \delta, j}^{\alpha, \beta, m}\) are zero. The 3D \(L\) satisfies the \(RLLL\) type tetrahedron equation [12]:
\[
\mathcal{L}_{1,2,4} \mathcal{L}_{1,3,5} \mathcal{L}_{2,3,6} \mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6} \mathcal{L}_{1,3,5} \mathcal{L}_{1,2,4}. \quad (2.13)
\]
This is an equality in \(\text{End}(V \otimes V \otimes V \otimes F \otimes F \otimes F)\), where \(V, V, V\) are copies of \(V\) and \(F, F, F\) are the ones for \(F\). The indices of \(\mathcal{R}\) and \(\mathcal{L}\) signify the components of the tensor product on which these operators act nontrivially. With the space \(V\) denoted by a dotted arrow, the relation (2.13) is depicted as follows:

Viewed as an equation on \(\mathcal{R}\), (2.13) is equivalent to the intertwining relation of the irreducible representations of the quantized coordinate ring \(A_q(sl_2)\) [25, equation (2.15)] in the sense that they both lead to the same solution given in (2.2) up to an overall normalization.

2.4. \(n\)-layer version of the tetrahedron equation

In order to treat \(\mathcal{R}\) and \(\mathcal{L}\) on an equal footing we introduce the notation
\[
\mathcal{W}^{(0)} = F, \quad \mathcal{W}^{(1)} = V, \\
\mathcal{S}^{(0)} = \mathcal{R}, \quad \mathcal{S}^{(1)} = \mathcal{L}, \quad \mathcal{S}^{(0)}_{i,j,k} = \mathcal{R}_{i,j,k}^{a,b,c}, \quad \mathcal{S}^{(1)}_{i,j,k} = \mathcal{L}^{a,b,c}_{i,j,k}.
\]
Then \(\mathcal{S}^{(e)} \in \text{End}(\mathcal{W}^{(e)} \otimes \mathcal{W}^{(e)} \otimes \mathcal{F})\) for \(e = 0, 1\). From (2.2) and (2.12) it obeys the conservation law:
\[
\mathcal{S}^{(e)}_{i,j,k} = 0 \text{ unless } (a + b, b + c) = (i + j, j + k). \quad (2.14)
\]
The tetrahedron equations of \(RRRR\) type (2.3) and \(RLLL\) type (2.13) are summarized as
\[
\mathcal{S}^{(e)}_{1,2,4} \mathcal{S}^{(e)}_{1,3,5} \mathcal{S}^{(e)}_{2,3,6} \mathcal{R}_{4,5,6} = \mathcal{R}_{4,5,6} \mathcal{S}^{(e)}_{1,3,5} \mathcal{S}^{(e)}_{1,2,4}, \quad (e = 0, 1), \quad (2.15)
\]
which is an equality in \(\text{End}(\mathcal{W}^{(e)} \otimes \mathcal{W}^{(e)} \otimes \mathcal{W}^{(e)} \otimes \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F})\).

Let \(n\) be a positive integer. Given an arbitrary sequence \((\epsilon_1, ..., \epsilon_n) \in \{0,1\}^n\), we set
\[
\mathcal{W} = \mathcal{W}^{(\epsilon_1)} \otimes ... \otimes \mathcal{W}^{(\epsilon_n)}. \quad (2.16)
\]
Regarding (2.15) as a one-layer relation, we extend it to the $n$-layer version. Let $W^{(c)}$, $W'^{(c)}$, $W''^{(c)}$ be copies of $W^{(c)}$, where $a_i, \beta_i$ and $\gamma(i = 1, \ldots, n)$ are just distinct labels. Replacing the spaces 1, 2, 3 by them in (2.15) we have

$$S_1^{(c)} S_2^{(c)} S_3^{(c)} R_{4,5,6} = R_{4,5,6} S_4^{(c)} S_5^{(c)} S_6^{(c)}$$

for each $i$. Thus for any $i$ one can carry $R_{4,5,6}$ through $S_3^{(c)} S_5^{(c)} S_7^{(c)}$ to the left converting it into the reverse order product $S_2^{(c)} S_4^{(c)} S_6^{(c)}$. Repeating this $n$ times leads to

$$R_{4,5,6} = R_{4,5,6} S_3^{(c)} S_5^{(c)} S_7^{(c)} = R_{4,5,6} S_3^{(c)} S_5^{(c)} S_7^{(c)} S_2^{(c)} S_4^{(c)} S_6^{(c)}$$

This is an equality in $\text{End}(W \otimes W \otimes \cdots \otimes W \otimes F \otimes F \otimes F)$, where $a = (a_1, \ldots, a_n)$ is the array of labels and $W = W^{(c)} \otimes \cdots \otimes W^{(c)}$. The notations $W$ and $W'$ should be understood similarly. They are just copies of $W$ defined in (2.16).

Here the broken arrows represent either usual arrow or dotted arrow depending on whether the corresponding $e_i$ is 0 or 1 and accordingly whether $S^{(c)}$ is $R$ or $L$.

The argument so far is just a 3D analogue of the well known fact in 2D that a single site relation $RLL = LLR$ for a local $L$ operator implies a similar relation for the $n$-site monodromy matrix in the quantum inverse scattering method.

2.5. Reduction to Yang–Baxter equation

In the $n$-layer tetrahedron equation (2.17), the space $F \otimes F \otimes F$ will be referred to as auxiliary space. One can reduce (2.17) to the Yang–Baxter equation by evaluating the auxiliary space away appropriately. One natural way is to take the trace of (2.17) over the auxiliary space after left multiplication of $a^{\hbar_i}(xy)^{\hbar_i} h_{\beta_i}$ and right multiplication of $R_{4,5,6}^{(3)}$ [12]. Another way is to evaluate (2.17) between the boundary vectors $(x_i(x)) \otimes (x_i(y)) \otimes (x_i(s))$ and $[x_i(x)] \otimes [x_i(y)](s, t = 1, 2)$ in (2.7) by regarding them as belonging to the auxiliary space and its dual $[30]$. By using (2.5) and (2.7) it is easy to see that the result reduces to the Yang–Baxter equation

$$S_{\alpha, \beta}(x) S_{\alpha, \gamma}(xy) S_{\beta, \gamma}(y) = S_{\beta, \gamma}(y) S_{\alpha, \gamma}(xy) S_{\alpha, \beta}(x) \in \text{End}(W \otimes W \otimes W)$$

(2.18)

Using $[x_i(x')] \otimes [x_i(y')] \otimes [x_i(y')]$ just leads to a redefinition of $x, y$ due to (2.5) and (2.14).
for the matrix $S_{\alpha\beta}(z) \in \text{End}(\mathcal{W} \otimes \mathcal{W})$ constructed as (cf. [33, section VIII] and [26, section 5])

$$
S_{\alpha\beta}(z) = q(z) \text{Tr}_3 \left( z^{h_{\alpha}} S_{\alpha_1\beta_1} z^{h_{\alpha_2}} \cdots S_{\alpha_{n} \beta_{n}} z^{h_{\beta}} \right) \quad \text{(trace),}
$$

$$
= q(z) \left\{ x_i \left( z^{h_{\alpha}} S_{\alpha_1\beta_1} z^{h_{\alpha_2}} \cdots S_{\alpha_{n} \beta_{n}} z^{h_{\beta}} \right) \right\} \quad \text{(evaluation by boundary vectors),}
$$

where the scalar $q(z)$ is inserted to control the normalization. The trace or evaluation by boundary vectors are done with respect to the auxiliary Fock space $F = F^3$ signified by 3.

To express the matrix elements of $S_{\alpha\beta}(z)$ uniformly, we introduce the following notation for the basis of $\mathcal{W}$ (2.16):

$$
\mathcal{W} = \bigoplus_{m_1, \ldots, m_n} \mathbb{C} \left[ m_1, \ldots, m_n \right], \quad \left[ m_1, \ldots, m_n \right] = \left[ m_1 \right]^{(\epsilon_1)} \otimes \cdots \otimes \left[ m_n \right]^{(\epsilon_n)},
$$

$$
\left[ m \right]^{(0)} = m \in F \quad \left( m \in \mathbb{Z}_{\geq 0} \right), \quad \left[ m \right]^{(1)} = v_m \in V \quad \left( m \in \{0, 1\} \right). \quad \text{(2.22)}
$$

The range of the indices $m_i$ are to be understood as $\mathbb{Z}_{\geq 0}$ or $\{0, 1\}$ according to $\epsilon_i = 0$ or 1 as in (2.22). It will crudely be denoted by $0 \leq m_i \leq 1/\epsilon_i$. We use the shorthand $[m] = [m_1, \ldots, m_n]$ for $m = (m_1, \ldots, m_n)$ and write (2.21) as $\mathcal{W} = \bigoplus_m \mathbb{C} \left[ m \right]$. In particular $[0]$ with $0 := (0, \ldots, 0)$ denotes the vacuum vector. We set $[m] = m_1 + \cdots + m_n$. In the later sections (e.g., sections 6 and 7) where the distinction between $F$ and $V$ is clear from the context, we will denote $v_0$ and $v_1$ also by $[0]$ and $[1]$.

Let $S^{\alpha}(z)$ and $S^{\alpha \beta}(z) \in \text{End}(\mathcal{W} \otimes \mathcal{W})$ denote the solutions (2.19) and (2.20) of the Yang–Baxter equation, where the inessential labels $\alpha, \beta$ are now suppressed. Their actions are described as

$$
S^{\alpha}(z) (|i\rangle \otimes |j\rangle) = \sum_{a,b} S^{\alpha}(z)_{ab} \langle a | \otimes | b \rangle,
$$

$$
S^{\alpha \beta}(z) (|i\rangle \otimes |j\rangle) = \sum_{a,b} S^{\alpha \beta}(z)_{ab} \langle a | \otimes | b \rangle, \quad \text{(2.23)}
$$

with the matrix elements constructed as

$$
S^{\alpha \beta}(z)_{ab} = q(z) \sum_{c_1, \ldots, c_{n-1}} c_{c_1}^{a} S_{c_1 c_2}^{c_2} \cdots S_{c_{n-1} c_{n}}^{c_n} \langle c_{n} | \otimes | a \rangle, \quad c_{c_1}^{a} \equiv \langle c_{1} | a \rangle,
$$

$$
S^{\alpha \beta}(z)_{ab} = q(z) \sum_{c_1, \ldots, c_{n-1}} c_{c_1}^{a} \sum_{s_{c_1 c_2}} q^{s_{c_1 c_2}} S_{c_2 c_3}^{c_3} \cdots S_{c_{n-1} c_{n}}^{c_n} \langle c_{n} | \otimes | a \rangle. \quad \text{(2.24)}
$$

The factor $(q^{3})_{m_0}$ in (2.25) originates in $\langle m | m' \rangle = \delta_{m,m'} (q^{3})_{m}$. See section 2.1. From (2.14) it follows that

$$
S^{\alpha \beta}(z)_{ab} = 0 \quad \text{unless} \quad a + b = i + j \quad \text{and} \quad |a\rangle = |i\rangle, \quad |b\rangle = |j\rangle. \quad \text{(2.26)}
$$
\[ S^{s,t}(z)^{a,b}_{i,j} = 0 \text{ unless } a + b = i + j \quad (1 \leq s, t \leq 2), \quad (2.27) \]
\[ S^{2,2}(z)^{a,b}_{i,j} = 0 \text{ unless } |a| \equiv |i|, |b| \equiv |j| \mod 2. \quad (2.28) \]

Given such \( a, b, i \) and \( j \), (2.14) effectively reduces the sums over \( c_l \in \mathbb{Z}_{\geq 0} \) in both (2.24) and (2.25) into a single sum. The latter property in (2.26) implies the direct sum decomposition:

\[ S^\nu(z) = \bigoplus_{l,m \geq 0} S^\nu_{lm}(z), \quad S^\nu_{lm}(z) \in \text{End}(\mathcal{W}_l \otimes \mathcal{W}_m), \]

\[ \mathcal{W}_l = \bigoplus_{m \mid |m|=l} \mathbb{C}[m] \subset \mathcal{W}, \quad (2.29) \]

where the former sum ranges over \( 0 \leq l, m \leq n \) if \( e_1 \cdots e_n = 1 \) and \( l, m \in \mathbb{Z}_{\geq 0} \) otherwise. Similarly, \( S^{2,2}(z) \) decomposes into four components due to (2.28). The normalization factor \( q(z) \) can be taken depending on the components and will be specified in section 2.6.

Assign a solid arrow to \( F \) and a dotted arrow to \( V \), and depict the matrix elements of 3D \( R \) and 3D \( L \) as

\[ \mathcal{R}_{a,b,c}^{i,j,k} = \]
\[ \mathcal{L}_{a,b,c}^{i,j,k} = \]

Then the constructions (2.24) and (2.25) are depicted as

Here the broken arrows designate either solid or dotted arrows according to \( c_i = 0 \) or 1 at the corresponding site. Thus (2.24) and (2.25) may be regarded as a ‘matrix product construction’ of \( S^\nu(z) \) and \( S^{s,t}(z) \) in terms of 3D \( R \) and 3D \( L \) with the auxiliary space \( F \).

### 2.6. Examples: normalization of \( S^\nu(z) \) and \( S^{1,1}(z) \)

Set

\[ e_i = (0, \ldots, 0, i, 0, \ldots, 0) \in \mathbb{Z}^n, \]
\[ e_{r,m} = e_{m+1} + \cdots + e_n \quad (1 \leq m \leq n). \]

We calculate some typical matrix elements from (2.24) and (2.25). As the first example we consider \( S^\nu_{lm}(z) (0 \leq l, m \leq n) \) with \( e_1 \cdots e_n = 1 \). We normalize it as

\[ S^\nu_{lm}(z) \left( |e_{r,n-1} \rangle \otimes |e_{r,n-m} \rangle \right) = |e_{r,n-1} \rangle \otimes |e_{r,n-m} \rangle. \quad (2.30) \]
By the definition (2.24) the relevant matrix element is calculated as

$$S_{n,m}^H(z) = q(z)$$

$$\sum_{c \geq 0} c^c \left( \ell^{0,0,c}_{n,m} \right)^{(l-m)} \left( \ell^{0,1,c}_{n,m} \right)^{(m-l)}$$

where \((x)_+ = \max(x, 0)\). Thus from (2.12) we find that the condition (2.30) leads to the choice \(q(z) = (-q)^{m-l}/(1 - q^{m-l})\).

As the second example we consider \(S_{n,m}^H(z) (l, m \in \mathbb{Z}_{>0})\) with \(e_1 \cdots e_n = 0\). We pick any i such that \(e_i = 0\) and normalize it as

$$S_{n,m}^H(z) (|e_i \rangle \otimes |me_i \rangle) = |e_i \rangle \otimes |me_i \rangle.$$  \hspace{1cm} (2.31)

The relevant matrix element reads

$$\sum_{\rho} \subseteq \geq \leq$$

$$\underbrace{\cdots}_{\text{max}(0,0,0,0)} \subseteq \leq$$

$$\underbrace{\cdots}_{\text{min}(0,0,0,0)} \subseteq \leq$$

where \(x \in \{0,1\}\) is a permutation of \(e_1, \ldots, e_n\). Let \(W\) and \(W'\) be the spaces (2.16) associated to them. Then there is an invertible linear map \(\Phi : W \rightarrow W'\) such that

$$\sum_{\rho} \subseteq \geq \leq$$

$$\underbrace{\cdots}_{\text{max}(0,0,0,0)} \subseteq \leq$$

$$\underbrace{\cdots}_{\text{min}(0,0,0,0)} \subseteq \leq$$

$$\Phi S^u(z) = S^u(z) \Phi, \hspace{1cm} \Phi S^{t'}(z) = S^{t'}(z) \Phi.$$  \hspace{1cm} (2.32)

2.7. Equivalence relations

Let us write \(S^u(z) = S^u(z|e_i, \ldots, e_n\rangle\) and \(S^{t'}(z) = S^{t'}(z|e_i, \ldots, e_n\rangle\) when their dependence on \(e_1, \ldots, e_n\) is to be emphasized. The following fact was briefly mentioned in [33, section VIII].

**Proposition 2.1.** Suppose \(e_1, \ldots, e_n \in \{0,1\}^n\) is a permutation of \(e_1', \ldots, e_n'\). Let \(W\) and \(W'\) be the spaces (2.16) associated to them. Then there is an invertible linear map \(\Phi : W \rightarrow W'\) such that

$$\Phi S^u(z|e_i, \ldots, e_n\rangle) = S^u(z|e_i, \ldots, e_n\rangle) \Phi, \hspace{1cm} \Phi S^{t'}(z|e_i, \ldots, e_n\rangle) = S^{t'}(z|e_i, \ldots, e_n\rangle) \Phi.$$  \hspace{1cm} (2.32)

**Proof.** In the RLLL relation (2.13), take the trace over the space \(V\). The result reads

$$\Phi_{2,3,4,5} = L_{2,3,4,5} \otimes R_{2,3,4,5} \otimes L_{2,3,4,5},$$

where \(\Phi_{2,3,4,5} = \text{Tr}_{\{L_{1,2,4}L_{1,3,5}\}}\). It may be regarded as a linear map \(\Phi : (V \otimes F) \otimes (V \otimes F) \rightarrow (F \otimes V) \otimes (F \otimes V)\). When represented as \(\Phi (v_{\alpha} \otimes v_{\beta}) = \sum_{\alpha' \beta'} v_{\alpha' \beta'} \otimes v_{\alpha' \beta'}\) in terms of the four by four matrix \((\Phi_{\alpha', \beta'}^{\alpha, \beta})\) with elements from \(\text{End}(F \otimes F)\), it looks as
where $(\alpha, \beta) = (0, 0), (0, 1), (1, 0), (1, 1)$ from the left to the right and similarly for $(\alpha', \beta')$ from the top to the bottom. It is easy to check that the square of this matrix equals \( \text{diag}(1 + k_4 k_5)^2, (1 - q k_4 k_5)^2, (1 - q k_4 k_5)^2, (1 + q^2 k_4 k_5)^2) \). Since the spectrum of \( k \) is \( q^{2\rho_0}, \phi \) is invertible. It follows that reversing the product of \( \mathcal{L} \) and \( \mathcal{R} \) in \( F \) is equivalent to a similarity transformation in the other spaces by \( \phi \). Applying this observation to (2.19) and (2.20), we have

\[
\left( \phi_{\alpha, \beta}^{\alpha', \beta'} \right) = \begin{pmatrix}
1 + k_4 k_5 & 0 & 0 & 0 \\
0 & k_4 - q k_5 & a_4^2 a_5^- & 0 \\
0 & a_4^2 a_5^- & k_5 - q k_4 & 0 \\
0 & 0 & 0 & 1 + q^2 k_4 k_5
\end{pmatrix},
\]

whenever there are consecutive 1, 0 or 0, 1 in \((\epsilon_1, \ldots, \epsilon_n)\). Repeating this transposition one can convert \((\epsilon_1, \ldots, \epsilon_n)\) into \((\epsilon'_1, \ldots, \epsilon'_n)\). By denoting the composition of the corresponding \( \phi \)'s by \( \Phi \), the assertion follows.

Proposition 2.1 reduces the study of the \( 2^n \)-families \( S^u(z|\epsilon_1, \ldots, \epsilon_n) \) and \( S^{s,t}(z|\epsilon_1, \ldots, \epsilon_n) \) to the \( (n+1) \)-families

\[
S^u(z|1, \ldots, 1, 0, 0), \quad S^u(z|1, \ldots, 0, 1, 0), \quad S^{s,t}(z|1, \ldots, 1, 0, 0), \quad S^{s,t}(z|1, \ldots, 0, 1, 0)
\]

(0 \( \leq \kappa \leq n \)).

2.8. Results on homogeneous cases and present work

Let us temporarily suppress \( z \) in \( S^u(z) \) and \( S^{s,t}(z) \) in (2.23) and write them as \( S^u(\epsilon_1, \ldots, \epsilon_n) \) and \( S^{s,t}(\epsilon_1, \ldots, \epsilon_n) \). Let \( U_q(\mathfrak{g}) \) be a quantum affine algebra and let \( R_{U_q(\mathfrak{g})}(M \otimes M') \) denote the quantum \( R \) matrix acting on the \( U_q(\mathfrak{g}) \)-module \( M \otimes M' \). Known results concern the homogeneous cases \( \epsilon_1 = \cdots = \epsilon_n = 0, 1 \). Leaving minor technical remarks aside\(^8\), they are stated in the present convention as follows ('rep' means representation).

\[
S^u(0, \ldots, 0) = \bigoplus_{l,m \geq 0} R_{U_q(\mathfrak{g})}(\lambda^{(l)}_{\epsilon_1}) V_l \otimes V_m, \quad V_l = l\text{-symmetric tensor rep},
\]

(2.33)

\[
S^u(1, \ldots, 1) = \bigoplus_{0 \leq l, m \leq n} R_{U_q(\mathfrak{g})}(\lambda^{(l)}_{\epsilon_1}) \left( V^l \otimes V^m \right), \quad V^l = l\text{-antisymmetric tensor rep},
\]

(2.34)

\[
S^{s,t}(1, \ldots, 1) = R_{U_q(\mathfrak{g})}(\lambda^{(l)}_{\epsilon_1}) V_{sp} \otimes V_{sp}, \quad V_{sp} = \text{spin rep}.
\]

(2.35)

\(^8\) For example, a slight gauge adjustment is necessary as in (5.1).
\[ S^{2,1}(1, \ldots, 1) = R_{U_q \rightarrow U_q}(D^{(1)})(V_p \otimes V_p), \quad V_p = \text{spin rep}, \] (2.36)

\[ S^{2,2}(1, \ldots, 1) = R_{U_q \rightarrow U_q}(D^{(1)})(V_p \otimes V_p), \quad V_p = (\text{spin rep}) \oplus \sigma(\text{spin rep}), \] (2.37)

\[ S^{1,1}(0, \ldots, 0) = R_{U_q \rightarrow U_q}(A^{(1)}_{n,1})(V_{\text{osc}} \otimes V_{\text{osc}}), \quad V_{\text{osc}} = q\text{-oscillator rep}, \] (2.38)

\[ S^{1,2}(0, \ldots, 0) = R_{U_q \rightarrow U_q}(A^{(2)}_{n,1})(V_{\text{osc}} \otimes V_{\text{osc}}), \quad V_{\text{osc}} = q\text{-oscillator rep}, \] (2.39)

\[ S^{2,2}(0, \ldots, 0) = R_{U_q \rightarrow U_q}(c^{(1,1)})(V_{\text{osc}} \otimes V_{\text{osc}}), \quad V_{\text{osc}} = \left( (q - \text{osc. rep}) \oplus (q - \text{osc. rep})^\ast \right). \] (2.40)

The result (2.33) is stated in [12] where \( \mathcal{V} \simeq \mathcal{W} \) (2.29) as a vector space. See also [27, appendix B] for a proof. The result (2.34) where \( \mathcal{V}' \simeq \mathcal{W} \) (2.29) has not been stated explicitly in the literature and it will be covered as a special case in this paper. The results (2.35)–(2.37) are due to [30] where \( V_p \simeq \mathcal{W} = V^{\otimes n} \). The \( \sigma \) in (2.37) is the order 2 Dynkin diagram automorphism of \( D_n \). See [30, remark 7.2] for \( S^{1,2}(1, \ldots, 1) \). The results (2.38)–(2.40) and the \( q \)-oscillator representations are obtained in [27] where \( V_{\text{osc}} \simeq \mathcal{W} = F^{\otimes n} \). In (2.40) \( (q - \text{osc. rep})^\ast \) denotes the even and odd irreducible subrepresentations of \( V_{\text{osc}} \) in [27, equation (2.20)]. Such a parity decomposition can also be inferred from (2.28). As for \( S^{2,1}(0, \ldots, 0) \), it is reducible to \( S^{1,2}(0, \ldots, 0) \) by [27, equation (2.16)].

Inhomogeneous cases of \( \epsilon_i \)'s, i.e. a mixture of \( R \) and \( L \), was first proposed in [33, section VIII] for the trace construction and in [26, section 5] including the boundary vector construction. These works manifested that the full problem is much larger than the homogeneous case and indicated possible connections to quantum superalgebras.

This paper is the first systematic study on \( S^u(\epsilon_1, \ldots, \epsilon_n) \) and \( S^{1,1}(\epsilon_1, \ldots, \epsilon_n) \) for general inhomogeneous case \( (\epsilon_1, \ldots, \epsilon_n) \in (0,1]^n \). The latter is a representative example of the boundary vector construction \( S^u(\epsilon_1, \ldots, \epsilon_n) \). The other cases \( (s, t) \neq (1, 1) \) are not included in this paper to avoid complexity of the presentation.

It is not the most essential problem nor our primary concern to seek a closed formula for the matrix elements (2.24) and (2.25) by manipulating the multiple sum therein. (See section 2.6 and examples 3.5–3.7 however.) Our main interest lies in the characterization of \( S^u(\epsilon_1, \ldots, \epsilon_n) \) and \( S^{1,1}(\epsilon_1, \ldots, \epsilon_n) \) by a quantum group-like object in the sense similar to the usual \( R \) matrices characterized by \( U_q(\mathfrak{g}) \) [16, 22]. As a guide to what will happen, compare the two homogeneous cases of \( S^{1,1}(\epsilon) \) in (2.35) and (2.38) where the spin representation of \( U_{\mathbb{C}}(D^{(1)}_{n+1}) \) and the \( q \)-oscillator representations of \( U_q(D^{(2)}_{n+1}) \) are linked. Thus it is not only the representation but also the algebra itself that are interpolated with various choices of \( (\epsilon_1, \ldots, \epsilon_n) \). We will show that the resulting family of algebras offer examples of generalized quantum groups [20, 21] which include a class of quantum superalgebras.
3. Generalized quantum groups and quantum $R$ matrices

3.1. Hopf algebras $\mathcal{U}_A(e_1, \ldots, e_n)$ and $\mathcal{U}_B(e_1, \ldots, e_n)$

Set

$$p = iq^{-1}, \quad q_i = \begin{cases} q & e_i = 0, \\ -q^{-1} & e_i = 1, \end{cases} \quad (1 \leq i \leq n),$$

$$\tilde{n} = \begin{cases} n - 1 & \text{for } \mathcal{U}_A(e_1, \ldots, e_n), \\ n & \text{for } \mathcal{U}_B(e_1, \ldots, e_n), \end{cases}$$

where $i = \sqrt{-1}$ and $e_i = 0, 1$ according to (2.16). We assume $\tilde{n} \geq 1$ and often write $\mathcal{U}_A(e_1, \ldots, e_n)$ and $\mathcal{U}_B(e_1, \ldots, e_n)$ as $\mathcal{U}_A$ and $\mathcal{U}_B$ for short. When considering $\mathcal{U}_A$ all the indices (like $i$ in (3.1)) are to be understood as belonging to $\mathbb{Z}/n\mathbb{Z}$. We prepare the constants $(D_{ij})_{0 \leq i,j \leq \tilde{n}}$ and $(r_i)_{0 \leq i \leq \tilde{n}}$:

$$D_{ij} = D_{ji} = \prod_{k \in \{i\} \cap \{j\}} q_k^{2\delta_{ij} - 1},$$

$$\langle i \rangle = \begin{cases} \{i, i + 1\} & \text{for } \mathcal{U}_A, \\ \{i, i + 1\} \cap [1, n] & \text{for } \mathcal{U}_B, \end{cases}$$

$$r_i = q \quad \text{for } \mathcal{U}_A, \quad r_i = \begin{cases} p & i = 0, n, \\ q & 0 < i < n \end{cases} \quad \text{for } \mathcal{U}_B.$$  \hfill (3.2)

**Example 3.1.** For $\mathcal{U}_A(e_1, e_2)$ and $\mathcal{U}_A(e_1, e_2, e_3)$ one has

$$\left( D_{ij} \right)_{0 \leq i,j \leq 1} = \begin{pmatrix} q_1 q_2 & q_1^{-1} q_2^{-1} \\ q_1^{-1} q_2^{-1} & q_1 q_2 \end{pmatrix},$$

$$\left( D_{ij} \right)_{0 \leq i,j \leq 2} = \begin{pmatrix} q_1 q_2 q_3^{-1} & q_1^{-1} q_2^{1} q_3^{-1} \\ q_1^{-1} q_2^{-1} q_3 & q_1 q_2 q_3^{-1} \end{pmatrix},$$

where the top left element is $D_{0,0}$. Similarly for $\mathcal{U}_B(e_1, e_2, e_3)$ one has

$$\left( D_{ij} \right)_{0 \leq i,j \leq 3} = \begin{pmatrix} q_1 & q_1^{-1} & 1 & 1 \\ q_1^{-1} & q_1 q_2 & q_2^{-1} & 1 \\ 1 & q_2 q_3 & q_3^{-1} & 1 \\ 1 & 1 & q_3^{-1} & q_3 \end{pmatrix},$$

$$(r_i)_{0 \leq i \leq 3} = (p, q, q, p).$$

Let $\mathcal{U}_A$ and $\mathcal{U}_B$ be the $\mathbb{C}(q^\mathbb{Z})$-algebras generated by $e_i, f_i, k_i^{\pm 1} (0 \leq i \leq \tilde{n})$ obeying the relations
They are Hopf algebras with coproduct $\Delta$, counit $\varepsilon$ and antipode $\sigma$ given by

\begin{align*}
\Delta k_i^{\pm 1} &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\
\Delta e_i &= 1 \otimes e_i + e_i \otimes k_i, \\
\Delta f_i &= f_i \otimes 1 + k_i^{-1} \otimes f_i, \\
\varepsilon(k_i) &= 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \\
S(k_i^{\pm 1}) &= k_i^{\mp 1}, \\
S(e_i) &= -e_i k_i^{-1}, \quad S(f_i) = -k_i f_i.
\end{align*}

With a supplement of appropriate Serre relations, the homogeneous cases are identified with the quantum affine algebras [16, 22] as

\begin{align*}
U_A(0, ..., 0) &= U_q(A_{n-1}^{(1)}), \\
U_A(1, ..., 1) &= U_{-q^{-1}}(A_{n-1}^{(1)}), \\
U_B(0, ..., 0) &= U_q(D_{n+1}^{(2)}), \\
U_B(1, ..., 1) &= U_{-q^{-1}}(D_{n+1}^{(2)}).
\end{align*}

In the bottom left case, one actually needs to rescale $f_0$ and $e_0$ by the factor $\frac{q^{\pm 1}}{q^{-1}}$. In the bottom right case, the choice of the branch $(-q^{-1})^{1/2} = p$ is assumed. In general $U_A(e_1, ..., e_n)$ and $U_B(e_1, ..., e_n)$ are examples of generalized quantum groups [20, 21]. We let $U_A(e_1, ..., e_n)$ and $U_B(e_1, ..., e_n)$ denote the subalgebras of $U_A(e_1, ..., e_n)$ and $U_B(e_1, ..., e_n)$ without involving $e_0, f_0$ and $k_0^{\pm 1}$.  

### 3.2. Representation $\pi_x$

Recall that $\mathcal{W}_l$ is defined by (2.29), (2.21) and (2.22). Let $x$ be a parameter.

**Proposition 3.2.** The map $\pi_x : U_A(e_1, ..., e_n) \to \text{End}(\mathcal{W}_l)$ defined by

\begin{align*}
e_i |m\rangle &= x^{\delta_{i,0}} |m_i\rangle |m - e_i + e_{i+1}\rangle, \\
f_i |m\rangle &= x^{-\delta_{i,0}} |m_{i+1}\rangle |m + e_i - e_{i+1}\rangle, \\
k_i |m\rangle &= (q_i)^{m_i} (q_i)^{m_{i+1}} |m\rangle
\end{align*}

for $i \in \mathbb{Z}/n\mathbb{Z}$ is an irreducible representation, where $0 \leq l \leq n$ if $e_1 \cdots e_n = 1$ and $l \in \mathbb{Z}_{\geq 0}$ otherwise.

**Proposition 3.3.** The map $\pi_x : U_B(e_1, ..., e_n) \to \text{End}(\mathcal{W})$ defined by

\begin{align*}
e_0 |m\rangle &= x |m + e_1\rangle, \\
f_0 |m\rangle &= x^{-1} |m_1\rangle |m - e_1\rangle, \\
k_0 |m\rangle &= p^{-1} (q_i)^{m_0} |m\rangle, \\
e_i |m\rangle &= |m_i\rangle |m - e_i + e_{i+1}\rangle \quad (0 < i < n),
\end{align*}

$^9$ Image $\pi_x(q)$ is denoted by $g$ for simplicity.
\[ f_i |\mathbf{m}\rangle = [m_{i+1}]|\mathbf{m} + e_i - e_{i+1}\rangle \quad (0 < i < n), \]
\[ k_i |\mathbf{m}\rangle = (q_i)^{-m_i}(q_{i+1})^{-m_{i+1}}|\mathbf{m}\rangle \quad (0 < i < n), \]
\[ e_i |\mathbf{m}\rangle = [m_i]|\mathbf{m} - e_i\rangle, \]
\[ f_i |\mathbf{m}\rangle = |\mathbf{m} + e_n\rangle, \]
\[ k_n |\mathbf{m}\rangle = p(q_n)^{-m_n}|\mathbf{m}\rangle \quad (3.7) \]
is an irreducible representation.

We define \( \pi_i(k_i)^{-1} \) to be \( \pi_i(k_i)^{-1} \). In the rhs of (3.6) and (3.7), vectors \( |\mathbf{m}'\rangle = |m'_1, \ldots, m'_n\rangle \) are to be understood as zero unless \( 0 \leq m'_i \leq 1/e_i \) for all \( 1 \leq i \leq n \). Thus for example in (3.7) one has \( e_0 |e_i\rangle = 0 \) if \( e_i = 1 \). Similarly when \( (e_i, e_{i+1}) = (1, 1), e_i |\mathbf{m}\rangle \) with \( 0 < i < n \) is non-vanishing if and only if \( (m_i, m_{i+1}) = (1, 0) \).

Propositions 3.2 and 3.3 can be directly checked. The irreducibility of (3.7) is seen from \( \mathcal{W} = \mathcal{U}_B(\theta) \) and \( |\theta\rangle \in \mathcal{U}_B(\mathbf{m}) \) for any \( \mathbf{m} \).

**Remark 3.4.** Up to the remark after (3.5), the representations in propositions 3.2 and 3.3 reduce to the known ones in the homogeneous case \( e_1 = \cdots = e_n \):

\[ \mathcal{W}_l \simeq l\text{-fold symmetric tensor rep. of } \mathcal{U}_q(A_{n-1}^{(1)}) \text{ for } e_1 = \cdots = e_n = 0, \]
\[ \mathcal{W}_l \simeq l\text{-fold antisymmetric tensor rep. of } \mathcal{U}_q(D_{\alpha+1}^{(2)}) \text{ for } e_1 = \cdots = e_n = 1, \]
\[ \mathcal{W} \simeq q\text{-oscillator rep. of } \mathcal{U}_q(D_{\alpha+1}^{(2)}) \text{ for } e_1 = \cdots = e_n = 0, \]
\[ \mathcal{W} \simeq \text{spin rep. of } \mathcal{U}_q(D_{\alpha+1}^{(2)}) \text{ for } e_1 = \cdots = e_n = 1. \]

### 3.3. Relation with quantum superalgebras

We adopt the convention in [17] (see also [35]) for the quantum superalgebras \( A_q(\kappa, \kappa') \) and \( B_q(m, m') \), which are related to the \( q \)-deformations of Lie superalgebras \( sl(m+1, m'+1) \) and \( osp(2m+1, 2m') \), respectively.

#### 3.3.1. \( A_q \) and \( \mathcal{U}_A \)

We compare \( A_q^{(\kappa, \kappa')} = (\kappa - 1, \kappa' - 1) \) and \( \mathcal{U}_A(\theta' , 1') = \mathcal{U}_A(0, \ldots, 0, 1, \ldots, 1) \) with \( \kappa + \kappa' = n \). We assume \( 0 < \kappa < n \). As an illustration, consider \( A_q^{(1, 2)} \) generated by \( \tilde{e}_i, \tilde{f}_j, \tilde{k}_i^{\pm 1} \) (\( 1 \leq i \leq 4 \)). (Tilde is assigned for distinction.) Replacing \( \tilde{k}_i^{2} \) with \( \tilde{k}_i \) they satisfy [17, equation (3.2)]

\[
\tilde{k}_i \tilde{e}_j = G_{ij} \tilde{e}_j \tilde{k}_i, \quad \tilde{k}_i \tilde{f}_j = G_{ij}^{-1} \tilde{f}_j \tilde{k}_i, \]

\[
(G_{ij})_{1 \leq i, j \leq 4} = \begin{pmatrix}
q^2 & q^{-1} & 1 & 1 \\
q^{-1} & q & 1 & 1 \\
1 & q^{-1} & q & q^{-2} \\
1 & 1 & q & q^{-2}
\end{pmatrix},
\]

\[
[\tilde{e}_i, \tilde{f}_j] = \frac{\tilde{k}_j - \tilde{k}_i^{-1}}{q - q^{-1}}, \quad [\tilde{e}_i, \tilde{f}_j] = \frac{\tilde{k}_j - \tilde{k}_i^{-1}}{q - q^{-1}}.
\]
\[
\left[ \hat{e}_i, \hat{f}_j \right] = -\frac{\hat{k}_i - \hat{k}_j^{-1}}{q - q^{-1}} \quad (i = 3, 4), \quad \left[ \hat{e}_i, \hat{f}_j \right] = 0 \quad (i \neq j),
\]
(3.8)

where \([a, b]_q = ab + ba\). They are also to obey the so-called Serre relations including, e.g.
\[
\hat{e}_i^2 = 0, \quad \hat{e}_i \hat{e}_j - \hat{e}_j \hat{e}_i = -[2] \hat{e}_i \hat{e}_j \hat{e}_i + \hat{e}_j \hat{e}_i \hat{e}_j - 1 = [2] \hat{e}_j \hat{e}_i \hat{e}_j \hat{e}_i + \hat{e}_i \hat{e}_j \hat{e}_i \hat{e}_j = 0 \quad [35 \text{ equation (3.4)}].
\]

On the other hand the subalgebra \(\mathcal{U}_A(0^2, \mathbb{1}')\) of \(\mathcal{U}_A(0^2, \mathbb{1}')\) (see the end of section 3.1 for the definition) has the generators \(e_i, f_j, k_i^\pm\) \((1 \leq i \leq 4)\) satisfying the relations (3.3):
\[
\begin{align*}
& k_i e_j = D_{ij} e_j k_i, \quad k_i f_j = D_{ij}^{-1} f_j k_i, \\
& (D_{ij})_{1 \leq i, j \leq 4} = \begin{pmatrix}
q^{-1} & 1 & 0 \\
-q & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
& \left[ e_i, f_j \right] = \delta_{ij} k_i - k_i^{-1} \quad q - q^{-1}.
\end{align*}
\]
(3.9)

The difference of \(G_{ij}\) and \(D_{ij}\) are just by signs. To compare \(\mathcal{U}_A(0^2, \mathbb{1}')\) with \(\mathcal{U}_A(1, \mathbb{2})\) we add involutive elements \(\theta_i\) \((i = 2, 3, 4)\) to \(\mathcal{U}_A(0^2, \mathbb{1}')\) that (anti-)commute with \(e_i, f_i\) as
\[
\begin{align*}
& \left[ e_i, \theta_j \right] = \left[ f_i, \theta_j \right] = 0 \iff G_{ij} = D_{ij}, \\
& \left[ e_i, \theta_j \right]_+ = \left[ f_i, \theta_j \right]_+ = 0 \iff G_{ij} = -D_{ij}.
\end{align*}
\]
(3.10)

We assume that \(\theta_i\)’s commute with \(k_j\)’s. Set
\[
\begin{align*}
& \hat{e}_1 = e_1, \quad \hat{f}_1 = f_1, \quad \hat{k}_1 = k_1, \quad \hat{e}_2 = e_2, \quad \hat{f}_2 = f_2 \theta_2, \quad \hat{k}_2 = k_2 \theta_2, \\
& \hat{e}_3 = e_3 \theta_3, \quad \hat{f}_3 = f_3, \quad \hat{k}_3 = -k_3 \theta_3, \quad \hat{e}_4 = e_4, \quad \hat{f}_4 = f_4 \theta_4, \quad \hat{k}_4 = -k_4 \theta_4.
\end{align*}
\]

Then all the relations (3.9) are transferred to (3.8). On the relevant space \(W = F \otimes \mathbb{2} \otimes V \otimes \mathbb{3}, \theta_i (i = 2, 3, 4)\) satisfying (3.10) is realized as
\[
\theta_2 | m \rangle = (-1)^{m_1} | m \rangle, \quad \theta_3 | m \rangle = (-1)^{m_2 + m_3} | m \rangle, \quad \theta_4 | m \rangle = (-1)^{m_4 + m_3} | m \rangle.
\]
(3.11)

Moreover, the Serre relations [35, equation (3.4)] are all valid. Hence \(W\) is also an \(A(1, 2)\) \(-\)module.

A similar correspondence holds for the general case. Let \(\hat{e}_1, \hat{f}_1, \hat{k}_1^{-1} (\hat{k}_i \text{ corresponds to } k_i^2\) in [17]) denote the generators of \(A(\mathbb{1}', \mathbb{1}')\) with \(\kappa + \kappa' = n\) and \(0 < \kappa < n\). Add involutive elements \(\theta_1, \ldots, \theta_{n-1}\) to \(\mathcal{U}_A(0', \mathbb{1}')\) that (anti-)commute with \(e_1 (1 \leq i \leq n)\) as
\[
\begin{align*}
& e_i \theta_k = (-1)^{\delta_{ik} + \delta_{ik'} + k} \theta_k e_i, \\
& e_i \theta_j = (-1)^{k' \delta_{ij} + \delta_{ik} + \delta_{ik'}} \theta_j e_i \quad (k < j < n - 1), \\
& e_i \theta_{n-1} = (-1)^{\delta_{ik} + \delta_{ik'}} \theta_{n-1} e_i.
\end{align*}
\]
with \(f_i\) similarly and commute with \(k_j\)'s. Set

\[
\tilde{e}_i = e_i \theta_i x^{(\Xi \Xi \Xi-1)}, \quad \tilde{f}_i = f_i \theta_i x^{(\Xi \Xi \Xi)}, \\
\tilde{k}_i = (-1)^{1+k_i} k_i \theta_i, \quad \text{for } \kappa \leq i < n
\]

and \((\tilde{e}_i, \tilde{f}_i, \tilde{k}_i) = (e_i, f_i, k_i)\) for \(1 \leq i < \kappa\), where \(\chi\) (true) = 1, \(
\chi\) (false) = 0 and \(\equiv\) means the equality mod 2. Then all the relations of \(\overline{\mathcal{A}}_{\kappa}(0^\kappa, 1^\kappa')\) are transferred to those of \(A_{\kappa}(\kappa-1, \kappa'-1)\). On \(\mathcal{W} = F^{\otimes \kappa} \otimes V^{\otimes \kappa'}\), such \(\theta_\kappa, \ldots, \theta_{n-1}\) are realized as \(\theta_j | m\rangle = (-1)^{\chi^j(\Xi \Xi \Xi)^m_j + \chi^j(\Xi \Xi \Xi)^{m_j}} \cdot (m)\). Then the relations of \(A_{\kappa}(\kappa-1, \kappa'-1)\) including the Serre ones are all valid. Thus we conclude that \(\mathcal{W}\) is also an \(A_{\kappa}(\kappa-1, \kappa'-1)\)-module.

### 3.3.2. \(B_q\) and \(U_B\)

As opposed to the \(A_q\) case, we need to take the deformation parameter \(q\) of \(B_q\) to be \(p^{-1}\) to adjust conventions. Thus we compare \(B_{p^{-1}}(\kappa', \kappa)\) and \(U_B(0^\kappa, 1^\kappa')\) with \(\kappa + \kappa' = n\) for \(0 < \kappa < n\). As an illustration, consider \(B_{p^{-1}}(2, 2)\) generated by \(\tilde{e}_i, \tilde{f}_i, \tilde{k}_i^{1\pm 1}\) (\(1 \leq i \leq 4\)). Replacing \(\tilde{k}_i^2\) with \(\tilde{k}_i\), again they satisfy \([17, \text{equation (3.2)}]\):

\[
\tilde{k}_i \tilde{e}_j = G_{ij} \tilde{e}_j \tilde{k}_i, \quad \tilde{k}_i \tilde{f}_j = G_{ij}^{-1} \tilde{f}_j \tilde{k}_i,
\]

\[
(G_{ij})_{1 \leq i < j < 4} = \begin{pmatrix}
  q^2 & -q^{-1} & 1 & 1 \\
  -q^{-1} & 1 & -q & 1 \\
  1 & -q & q^{-2} & -q \\
  1 & 1 & -q & -q^{-1}
\end{pmatrix}
\]

\[
\left[ \tilde{e}_1, \tilde{f}_2 \right] = \frac{\tilde{k}_1 - \tilde{k}_i^{-1}}{q - q^{-1}}, \quad \left[ \tilde{e}_2, \tilde{f}_3 \right] = \frac{-\tilde{k}_2 - \tilde{k}_2^{-1}}{q - q^{-1}},
\]

\[
\left[ \tilde{e}_3, \tilde{f}_4 \right] = \frac{\tilde{k}_3 - \tilde{k}_4^{-1}}{q - q^{-1}}, \quad \left[ \tilde{e}_4, \tilde{f}_1 \right] = 0 \ (i \neq j)
\]  \hspace{1cm} (3.12)

and the Serre relations \([35, \text{equation (3.4)}]\).

On the other hand, the subalgebra \(\overline{U}_B(0^2, 1^2)\) of \(U_B(0^2, 1^2)\) has the generators \(e_i, f_i, k_i^{\pm 1}\) (\(1 \leq i \leq 4\)) satisfying the relations (3.3):

\[
k_i e_j = D_{ij} e_j k_i, \quad k_i f_j = D_{ij}^{-1} f_j k_i, \quad (D_{ij})_{1 \leq i < j < 4} = \begin{pmatrix}
  q^2 & q^{-1} & 1 & 1 \\
  q^{-1} & 1 & -q & 1 \\
  1 & -q & q^{-2} & -q \\
  1 & 1 & -q & -q^{-1}
\end{pmatrix},
\]

\[
\left[ e_i, f_j \right] = \delta_{ij} \frac{k_i - k_j^{-1}}{r_j - r_i^{-1}}, \quad (r_1, \ldots, r_4) = (q, q, q, p).
\]  \hspace{1cm} (3.13)

To compare \(\overline{U}_B(0^2, 1^2)\) with \(B_{p^{-1}}(2, 2)\) we add involutive elements \(\theta_j (i = 1, 2)\) to \(\overline{U}_B(0^2, 1^2)\) that (anti-)commute with \(e_i, f_j\) as in (3.10). We assume that \(\theta_j\)'s commute with \(k_j\)'s. Set
\[ \tilde{e}_1 = e_1 \theta_1, \quad \tilde{f}_1 = f_1, \quad \tilde{k}_1 = -k_1 \theta_1, \quad \tilde{e}_2 = e_2, \quad \tilde{f}_2 = f_2 \theta_2, \quad \tilde{k}_2 = -k_2 \theta_2, \]
\[ \tilde{e}_3 = e_3, \quad \tilde{f}_3 = f_3, \quad \tilde{k}_3 = k_3, \quad \tilde{e}_4 = e_4, \quad \tilde{f}_4 = f_4, \quad \tilde{k}_4 = k_4. \]

Then all the relations (3.13) are transferred to (3.12). On \( \mathcal{W} = F^\otimes 2 \otimes V^\otimes 2, \theta_1, \theta_2, \theta_3 \) satisfying (3.10) are realized as
\[ \theta_1 |m\rangle = (-1)^{m_1 + m_2} |m\rangle, \quad \theta_2 |m\rangle = (-1)^{m_1} |m\rangle. \]

Moreover the Serre relations [35, equation (3.4)] are all valid. Hence \( \mathcal{W} \) is also a \( B_{\nu^{-1}}(2, 2) \)-module.

The general case is similar. Let \( \tilde{e}_i, \tilde{f}_i, \tilde{k}_i \) (\( \tilde{k}_i \) corresponds to \( k_i^2 \) in [17]) denote the generators of \( B_{\nu^{-1}}(\kappa', \kappa) \). Let \( e_i, f_i, k_i^\pm (1 \leq i \leq n) \) denote the generators of \( \mathcal{U}_B(0, \nu') \) with \( \kappa + \kappa' = n \) and \( 0 < \kappa < n \). Add elements \( \theta_1, \ldots, \theta_n \) to \( \mathcal{U}_B(0, \nu') \) that (anti-)commute with \( e_i(1 \leq i \leq n) \) as
\[ e_i \theta_j = (-1)^{\delta_{ij}+\delta_{ij}+1} \theta_j e_i \quad (j \leq \kappa), \quad e_i \theta_k = (-1)^{\delta_{ij}+\delta_{ij}+1} \theta_k e_i, \]
with \( f_j \) similarly and commute with \( k_i \)'s. Set
\[ \tilde{e}_i = e_i \theta_j^{(i \in \mathbb{Z})}, \quad \tilde{f}_i = f_i \theta_j^{(i \in \mathbb{Z})}, \quad \tilde{k}_i = k_i \theta_i \quad \text{for} \quad 1 \leq i \leq \kappa \]
\[ (\equiv \text{ is again mod } 2) \text{ and } (\tilde{e}_i, \tilde{f}_i, \tilde{k}_i) = (e_i, f_i, k_i) \text{ for } \kappa < i \leq n. \]

Then all the relations of \( \mathcal{U}_B(0, \nu') \) are transferred to those of \( B_{\nu^{-1}}(\kappa', \kappa) \). On \( \mathcal{W} = F^\otimes \nu \otimes V^\otimes \nu' \), such \( \theta_1, \ldots, \theta_n \) are realized as \( \theta_j |m\rangle = (-1)^{j(\nu \otimes \mu_1 + \nu' \otimes \mu_2)} |m\rangle \). Then the relations of \( B_{\nu^{-1}}(\kappa', \kappa) \) including the Serre ones are all valid. Thus we conclude that \( \mathcal{W} \) is also a \( B_{\nu^{-1}}(\kappa', \kappa) \)-module.

### 3.4. Quantum R matrices

Consider the linear equation on \( R \in \text{End} (\mathcal{W}_i \otimes \mathcal{W}_m) \) for \( \mathcal{U}_A \) and \( R \in \text{End} (\mathcal{W} \otimes \mathcal{W}) \) for \( \mathcal{U}_B \):

\[ \Delta'(g) R = R \Delta(g) \quad \forall g \in \mathcal{U}_A \text{ or } \mathcal{U}_B, \]

where \( \pi_2 \otimes \pi_2 \) has been omitted on both sides and \( \Delta' \) is the coproduct opposite to (3.4). Namely \( \Delta' = P \circ \Delta \circ P \) where \( P(u \otimes v) = v \otimes u \) is the exchange of the components. A little inspection of the representations \( \pi_2, \pi_2 \) tells us that \( R \) depends on \( x \) and \( y \) only via the ratio \( z = x/y \). Henceforth we write \( R \) as \( R(z) \).

Suppose \( (e_1, \ldots, e_n) = (1, 0^{n-1}). \) For \( 0 < \kappa < n \) we will show that the \( \mathcal{U}_A \)-module \( \mathcal{W}_i \otimes \mathcal{W}_m \) and the \( \mathcal{U}_B \)-module \( \mathcal{W} \otimes \mathcal{W} \) are both irreducible in propositions 6.7, 6.11 and 7.7. (The same fact holds also for \( \kappa = 0, n \) due to the earlier results mentioned in remark 3.4.) Therefore \( R \) is determined (if it exists) by postulating (3.14) for \( g = k_r, e_r, f_r, \) with \( 0 \leq r \leq n \) up to an overall scalar. Explicitly these conditions read

\[ R(k_r \otimes k_r) R(z) = R(z)(k_r \otimes k_r), \]

\[ R(e_r \otimes 1 + k_r \otimes e_r) R(z) = R(z)(1 \otimes e_r + e_r \otimes k_r), \]

\[ (1 \otimes f_r + f_r \otimes k_r^{-1}) R(z) = R(z)(f_r \otimes 1 + k_r^{-1} \otimes f_r) \]

for \( 0 \leq r \leq n \), where \( \pi_2 \otimes \pi_2 \) is again omitted. We call the intertwiner \( R(z) \) the quantum R matrix. Its existence will be established in theorem 5.1, which also provides an explicit construction. From (2.18) and theorem 4.1 it satisfies the Yang–Baxter equation

\[ R_{12}(x) R_{13}(xy) R_{23}(y) = R_{23}(y) R_{13}(xy) R_{12}(x), \]

which is an equality in \( \text{End} (\mathcal{W}_i \otimes \mathcal{W}_i \otimes \mathcal{W}_m) \) for some \( k, l, m \) for \( \mathcal{U}_A \) and \( \text{End} (\mathcal{W} \otimes \mathcal{W} \otimes \mathcal{W}) \) for \( \mathcal{U}_B \).
For $\mathcal{U}_B$ we introduce a gauge transformed quantum $R$ matrix by

$$\tilde{R}(z) = \left( K^{-1} \otimes 1 \right) R(z)(1 \otimes K), \quad K|\mathbf{m}\rangle = p^{-m_1, \ldots, -m_l} |\mathbf{m}\rangle.$$  \hfill (3.19)

It is easy to see that $\tilde{R}(z)$ also satisfies the Yang–Baxter equation. We fix the normalization of $R(z)$ by

$$R(z)\left(\{e_{>n-l}\} \otimes |e_{>n-m}\rangle\right) = |e_{>n-l}\rangle \otimes |e_{>n-m}\rangle \text{ for } \mathcal{U}_A(1, \ldots, 1),$$  \hfill (3.20)

$$R(z)(\{|e_i\} \otimes |me_i\rangle) = |e_i\rangle \otimes |me_i\rangle \text{ for } \mathcal{U}_A(e_1, \ldots, e_n) \text{ with } e_1 \cdots e_n = 0,$$  \hfill (3.21)

$$\tilde{R}(z)(\{0\} \otimes \{0\}) = |0\rangle \otimes |0\rangle \text{ for } \mathcal{U}_B(e_1, \ldots, e_n),$$  \hfill (3.22)

where $i$ in (3.21) is taken to be the same as that in (2.31).

In view of section 3.3, these $R$ matrices with $0 < \kappa < n$ are related to quantum superalgebras. However they do not fall in the known examples, e.g. [11, 18, 31] since the structure of the space $\mathcal{W} = V^\otimes \kappa \otimes F^{\otimes n-\kappa}$ is distinct from them.

**Example 3.5.** Consider $\mathcal{U}_A(1, 0)$. For $l, m \geq 1$, one has $\mathcal{W}_m = \mathbb{C}|0, m\rangle \oplus \mathbb{C}|1, m - 1\rangle \subset \mathcal{W} = V \otimes F$ and similarly for $\mathcal{W}_l$. The action of $R(z)$ on $\mathcal{W}_l \otimes \mathcal{W}_m$ is given by

$$R(z)(\{0\} \otimes \{0\}) = |0\rangle \otimes |0\rangle,$$

$$R(z)(\{l\} \otimes \{0\}) = \frac{1 - q^{2m}}{z - q^{4m}} \{l\} \otimes \{m\},$$

$$R(z)(\{0\} \otimes \{l\}) = \frac{q^m - q^{-m}}{z - q^{4m}} \{0\} \otimes \{m\},$$

$$R(z)(\{l\} \otimes \{l\}) = \frac{1 - q^{2l}}{z - q^{4l}} \{l\} \otimes \{l\}.$$  

These formulas are deduced from the spectral decomposition (6.13). Equating them to $S^u(z|1, 0)$ by theorem 4.1 already leads to a highly nontrivial identity on the sum (2.24) involving the 3D $R$.

**Example 3.6.** Consider $\mathcal{U}_A(1, 1, 0)$. For $l, m \geq 2$, one has $\mathcal{W}_m = \mathbb{C}|0, 0, m\rangle \oplus \mathbb{C}|0, m - 1\rangle \oplus \mathbb{C}|1, 0, m - 1\rangle \oplus \mathbb{C}|1, m - 2\rangle \subset \mathcal{W} = V \otimes V \otimes F$ and similarly for $\mathcal{W}_l$. The action of $R(z)$ on $\mathcal{W}_l \otimes \mathcal{W}_m$ is given by
\[
R(z)(|0, i_2, i_3 \rangle \otimes |0, j_2, j_3 \rangle) = \left[ R(z)(|i_2, i_3 \rangle \otimes |j_2, j_3 \rangle) \right]_{|0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle},
\]
\[
R(z)(|i_1, 0, i_3 \rangle \otimes |j_1, 0, j_3 \rangle) = \left[ R(z)(|i_1, i_3 \rangle \otimes |j_1, j_3 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle},
\]
\[
R(z)(|1, i_2, i_3 \rangle \otimes |1, j_2, j_3 \rangle) = \frac{1 - q^{l + m}z}{z - q^l + m} \left[ R(z)(|i_2, i_3 \rangle \otimes |j_2, j_3 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle},
\]
\[
R(z)(|i_1, 1, i_3 \rangle \otimes |j_1, 1, j_3 \rangle) = \frac{1 - q^{l + m}z}{z - q^l + m} \left[ R(z)(|i_1, i_3 \rangle \otimes |j_1, j_3 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle},
\]

\[
R(z)(|0, 0, l \rangle \otimes |1, 1, m - 2 \rangle) = \frac{q^l z - q^m l}{z - q^l + m} \left[ R(z)(|0, 0, l \rangle \otimes |1, 1, m - 2 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle}\]
\[
+ \frac{1 - q^{2l}z}{z - q^l + m} \left[ R(z)(|0, l - 1 \rangle \otimes |1, m - 1 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle},
\]
\[
R(z)(|1, 0, l - 2 \rangle \otimes |0, 0, m \rangle) = \frac{q(1 - q^{2m})}{z - q^l + m} \left[ R(z)(|1, 0, l - 2 \rangle \otimes |0, 0, m \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle}\]
\[
+ \frac{q^m z - q^l}{z - q^l + m} \left[ R(z)(|1, l - 2 \rangle \otimes |0, m \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle},
\]
\[
R(z)(|1, 0, l - 1 \rangle \otimes |0, 1, m - 1 \rangle) = \frac{q^{m-1}(1 - q^2)}{z - q^l + m} \left[ R(z)(|1, 0, l - 1 \rangle \otimes |0, m - 1 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle}\]
\[
+ \frac{q^m z - q^l}{z - q^l + m} \left[ R(z)(|0, 0, l - 1 \rangle \otimes |1, m - 1 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle}\]
\[
+ \frac{1 - q^{2m-2}z}{z - q^l + m} \left[ R(z)(|1, l \rangle \otimes |1, m - 2 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle},
\]
\[
R(z)(|0, 1, l - 1 \rangle \otimes |1, 0, m - 1 \rangle) = \frac{q(1 - q^{2l-2}z)}{z - q^l + m} \left[ R(z)(|0, 1, l - 2 \rangle \otimes |0, m \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle}\]
\[
+ \frac{q^l z - q^m}{z - q^l + m} \left[ R(z)(|0, l - 1 \rangle \otimes |0, m - 1 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle}\]
\[
+ \frac{q^{m-1}(1 - q^2)}{z - q^l + m} \left[ R(z)(|0, l \rangle \otimes |1, m - 2 \rangle) \right]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle},
\]

where \(R(z)\) on the rhs is the one in example 3.5 for appropriate \(l\) and \(m\). The notation \([X]_{|\psi_0\rangle \otimes |\psi_0\rangle \otimes |\psi_0\rangle}\) for instance stands for the replacement \([a, b] \otimes [c, d] \mapsto [a, 0, b] \otimes [c, 0, d]\) for all the monomials contained in \(X\). Thus for example,
\[
R(z)(|1, 0, l - 1 \rangle \otimes |0, 1, m - 1 \rangle) \\
= \left( q^{2m} - q^2 \right) \left( q^m - q^2 z \right) (q^{l+m} - z)(q^{l+m} - q^2 z) (|0, 0, l \rangle \otimes |1, 1, m - 2 \rangle) \\
+ \left( q^2 - 1 \right) q^{l+m} + \left( q^2 - q^{2+2l} - q^{2+2m} + q^{2+2m} z \right) (q^{l+m} - z)(q^{l+m} - q^2 z) \\
\times |0, 1, l - 1 \rangle \otimes |1, 0, m - 1 \rangle \\
+ q \left( q^m - q^2 z \right)(q^l - q^m z) (q^{l+m} - z)(q^{l+m} - q^2 z) (|1, 0, l - 1 \rangle \otimes |0, 1, m - 1 \rangle) \\
+ \left( q^{2l} - q^2 \right) \left( q^l - q^m z \right) (q^{l+m} - q^2 z) (|1, 1, l - 2 \rangle \otimes |0, 0, m \rangle).
\]

Again these formulas are derived from the spectral decomposition (6.13) and comparison with example 3.5.

**Example 3.7.** Consider \(U_q(1, 0)\). \(\hat{R}(z)\) (3.19) acts on \(W \otimes W\) with \(W = V \otimes F\), satisfies (2.27) and contains integer powers of \(q\) only. Proposition 7.3 leads to the following \((i = 0, 1)\):

\[
\hat{R}(z)(|i, 0 \rangle \otimes |i, 0 \rangle) = |i, 0 \rangle \otimes |i, 0 \rangle, \\
\hat{R}(z)(|0, 0 \rangle \otimes |1, 0 \rangle) = \frac{1 + qz}{1 + qz} |1, 0 \rangle \otimes |0, 0 \rangle - \frac{q(1 - z)}{1 + qz} |0, 0 \rangle \otimes |1, 0 \rangle, \\
\hat{R}(z)(|1, 0 \rangle \otimes |0, 0 \rangle) = \frac{1 + q}{1 + qz} |0, 0 \rangle \otimes |1, 0 \rangle + \frac{1 - z}{1 + qz} |1, 0 \rangle \otimes |0, 0 \rangle, \\
\hat{R}(z)(|i, 1 \rangle \otimes |i, 0 \rangle) = \frac{1 + q}{1 + qz} |i, 0 \rangle \otimes |i, 1 \rangle + \frac{1 - z}{1 + qz} |i, 1 \rangle \otimes |i, 0 \rangle, \\
\hat{R}(z)(|1, 1 \rangle \otimes |0, 0 \rangle) \\
= \frac{(1 + q)(1 + q^2)}{(1 + qz)(1 + q^2 z)} |0, 0 \rangle \otimes |1, 1 \rangle + \frac{q(1 + q)(1 - z)}{(1 + qz)(1 + q^2 z)} |0, 1 \rangle \otimes |1, 0 \rangle \\
+ \frac{(1 + q)(1 - z)}{(1 + qz)(1 + q^2 z)} |1, 0 \rangle \otimes |0, 1 \rangle \\
\hat{R}(z)(|0, 1 \rangle \otimes |1, 0 \rangle) \\
= - \frac{q(1 + q)(1 - z)}{(1 + qz)(1 + q^2 z)} |0, 0 \rangle \otimes |1, 1 \rangle - \frac{q(1 - z)(1 - qz)}{(1 + qz)(1 + q^2 z)} |0, 1 \rangle \otimes |1, 0 \rangle \\
+ \frac{(1 + q)(1 - z)z}{(1 + qz)(1 + q^2 z)} |1, 0 \rangle \otimes |0, 1 \rangle \\
+ \frac{(1 + q)(1 - z)z}{(1 + qz)(1 + q^2 z)} |1, 1 \rangle \otimes |0, 0 \rangle.
\]
\[ \hat{R}(z)(|i, 2 \rangle \otimes |i, 0 \rangle) = \frac{(1 + q)(1 + q^2)}{(1 + qz)(1 + qz^2)} |i, 0 \rangle \otimes |i, 2 \rangle + \frac{(1 + q)(1 + q^2)(1 - z)}{(1 + qz)(1 + qz^2)} |i, 1 \rangle \otimes |i, 1 \rangle \]
\[ + \frac{(1 - z)(1 - qz)}{(1 + qz)(1 + qz^2)} |i, 2 \rangle \otimes |i, 0 \rangle, \]
\[ \hat{R}(z)(|i, 1 \rangle \otimes |i, 1 \rangle) = -\frac{q(1 + q)(1 - z)}{(1 + qz)(1 + qz^2)} |i, 0 \rangle \otimes |i, 2 \rangle \]
\[ + \frac{(1 + q)(1 + q + q^2)z - q(1 + z^2)}{(1 + qz)(1 + qz^2)} |i, 1 \rangle \otimes |i, 1 \rangle \]
\[ + \frac{(1 + q)(1 - z)z}{(1 + qz)(1 + qz^2)} |i, 2 \rangle \otimes |i, 0 \rangle, \]
\[ \hat{R}(z)(|i, 0 \rangle \otimes |i, 2 \rangle) = \frac{q^2(1 - z)(1 - qz)}{(1 + qz)(1 + qz^2)} |i, 0 \rangle \otimes |i, 2 \rangle - \frac{q(1 + q)(1 + q^2)(1 - z)z}{(1 + qz)(1 + qz^2)} |i, 1 \rangle \otimes |i, 1 \rangle \]
\[ + \frac{(1 + q)(1 + q^2)z^2}{(1 + qz)(1 + qz^2)} |i, 2 \rangle \otimes |i, 0 \rangle. \]

4. Main result: \( S^{ir}(z) \) and \( S^{1,1}(z) \) as quantum R matrices

Denote the \( R(z) \) for \( \mathcal{U}_A(e_1, ..., e_n) \) by \( R_A(z|e_1, ..., e_n) \) and \( \hat{R}(z) \) (3.19) for \( \mathcal{U}_B(e_1, ..., e_n) \) by \( \hat{R}_B(z|e_1, ..., e_n) \). The main result of this paper is the following.

**Theorem 4.1.** Suppose \((e_1, ..., e_n) = (\{0, 1\}^n)\). For any \( 0 \leq \kappa \leq n \) the following identification holds:
\[ S^{ir}(z|e_1, ..., e_n) = R_A(z|e_1, ..., e_n), \]
\[ S^{1,1}(z|e_1, ..., e_n) = \hat{R}_B(z|e_1, ..., e_n). \]

The two equalities hold in \( \text{End}(\mathcal{W}_l \otimes \mathcal{W}_m) \) for each \( l, m \) and in \( \text{End}(\mathcal{W} \otimes \mathcal{W}) \), respectively. Combined with proposition 2.1, theorem 4.1 tells us that \( S^{ir}(z|e_1, ..., e_n) \) and \( S^{1,1}(z|e_1, ..., e_n) \) with arbitrary \((e_1, ..., e_n) \in \{0, 1\}^n\) are equivalent to the quantum \( R \) matrices of the generalized quantum groups. For \((e_1, ..., e_n) \) not of the above form, the right hand sides are yet to be characterized uniquely. See also the comments below.

The rest of the paper is devoted to a proof of theorem 4.1. It consists of three parts. In part I (section 5) we prove that \( S^{ir} \) and \( S^{1,1} \) possess the same commutativity (3.14) with \( \mathcal{U}_A \) and \( \mathcal{U}_B \) as the quantum \( R \) matrices (theorem 5.1). This will be done for an arbitrary sequence \((e_1, ..., e_n) \in \{0,1\}^n\). In part II (section 6) and part III (section 7) we show that the relevant \( \mathcal{U}_A \)-module \( \mathcal{W}_l \otimes \mathcal{W}_m \) and the \( \mathcal{U}_B \)-module \( \mathcal{W} \otimes \mathcal{W} \) are irreducible for the choice \((e_1, ..., e_n) = (\{0, 1\}^n)\). This is an indispensable claim to guarantee that the \( R \) matrices are
characterized as the commutant of $\mathcal{U}_A$ and $\mathcal{U}_B$ up to a normalization. Finally the agreement of the normalization is assured by (2.30)–(2.32) and (3.20)–(3.22). We have not proved the irreducibility of $\mathcal{U}_A$-module $\mathcal{W}_1 \otimes \mathcal{W}_m$ and the $\mathcal{U}_B$-module $\mathcal{W} \otimes \mathcal{W}$ for $(e_i, \ldots, e_n)$ not of the above form although we expect they are so.

In parts II and III we will utilize the following fact.

**Proposition 4.2.** Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $U_q(\mathfrak{g})$ be its quantized enveloping algebra with the standard Drinfeld–Jimbo generators $e_i, f_i, k_i^{\pm1}$. Let $V, V'$ be irreducible $U_q(\mathfrak{g})$-modules. Let $u_i, u'_i$ be highest and lowest weight vectors of $V$. Then each of the vectors $u_i \otimes u'_i$ and $u'_i \otimes u_i$ generates $V \otimes V'$.

**Proof.** Set $M = U_q(\mathfrak{g})(u_i \otimes u'_i)$. It is easy to see that $M$ contains $u_i \otimes u'_i$. We are left to show that for a weight vector $u \otimes u'$, if $u \otimes u'$ is contained in $M$, then $f_i u \otimes u'$ is also contained in $M$. Note that for any $u' \in V'$, $\Delta(f_i)(u \otimes u') = f_i u \otimes u' + k_i^{-1}u \otimes f_i u'$. The lhs belongs to $M$ by the definition of $M$ and so does the second term of the rhs by the assumption. Hence $f_i u \otimes u' \in M$. The other case is similar. \[\square\]

**5. Proof part I: commutativity with $\mathcal{U}_A$ and $\mathcal{U}_B$.**

For $S^{1,1}(z)$ we introduce a slight gauge transformation

$$ \tilde{S}^{1,1}(z) = (K \otimes 1)S^{1,1}(z)(1 \otimes K^{-1}), \tag{5.1} $$

where $K$ is defined in (3.19). The main property of $S^u(z)$ and $S^{1,1}(z)$ is the commutativity with the generalized quantum groups $\mathcal{U}_A$ and $\mathcal{U}_B$ identical to (3.14).

**Theorem 5.1.** For an arbitrary sequence $(e_1, \ldots, e_n) \in \{0,1\}^n$, the following commutativity holds:

$$ \Delta'(g)S^u(z|e_1, \ldots, e_n) S^u(z|e_1, \ldots, e_n) \Delta(g) \quad \forall g \in \mathcal{U}_A(e_1, \ldots, e_n), \quad \Delta'(g)\tilde{S}^{1,1}(z|e_1, \ldots, e_n) S^u(z|e_1, \ldots, e_n) \Delta(g) \quad \forall g \in \mathcal{U}_B(e_1, \ldots, e_n), \tag{5.2} $$

where $\Delta(g)$ and $\Delta'(g)$ stand for the tensor product representation $(\pi_x \otimes \pi_y)\Delta(g)$ and $(\pi_x \otimes \pi_y)\Delta'(g)$ of those in proposition 3.2 and 3.3 and $z = x/y$.

**Proof.** It suffices to show that $S(z) = S^u(z)$ and $S^{1,1}(z)$ satisfy

$$ (k_r \otimes k_r)S(z) = S(z)(k_r \otimes k_r), \tag{5.2} $$

$$ (\tilde{e}_r \otimes 1 + k_r \otimes e_r)S(z) = S(z)(1 \otimes \tilde{e}_r + e_r \otimes k_r), \tag{5.3} $$

$$ (1 \otimes f_r + \tilde{f}_r \otimes k^{-1}_r)S(z) = S(z)(f_r \otimes 1 + k^{-1}_r \otimes \tilde{f}_r) \tag{5.4} $$

for $0 \leq r \leq \bar{n}$, where $\tilde{e}_r = e_r, \tilde{f}_r = f_r$ for $S^u(z)$ and $\tilde{e}_r = K^{-1}e_r, \tilde{f}_r = K^{-1}f_r$ for $S^{1,1}(z)$. $\pi_x \otimes \pi_y$ is again omitted. It is easy to see that (5.2) is guaranteed by (2.26)–(2.27). In what follows we demonstrate a proof of (5.3). The relation (5.4) can be verified similarly. It unifies the earlier proofs for $S^{1,1}(z|1, \ldots, 1)$ in [30], $S^{1,1}(z|0, \ldots, 0)$ in [27], and $S^u(z|0, \ldots, 0)$ in [27, Prop.17].
Consider the action of both sides of (5.3) on a base vector $|i\rangle \otimes |j\rangle \in W \otimes W$:

\[
y^{-\delta_{ij}}(e_r \otimes 1 + k_r \otimes e_r)S(z)|i\rangle \otimes |j\rangle = \sum_{a,b} A_{ij}^{ab}(z)|a\rangle \otimes |b\rangle, \tag{5.5}
\]

\[
y^{-\delta_{ij}}S(z)(1 \otimes e_r + e_r \otimes k_r)|i\rangle \otimes |j\rangle = \sum_{a,b} B_{ij}^{ab}(z)|a\rangle \otimes |b\rangle, \tag{5.6}
\]

where we have multiplied $y^{-\delta_{ij}}$ to confine the dependence on $x$ and $y$ to the ratio $z = x/y$. We are to show the equality of the matrix elements $A_{ij}^{ab}(z) = B_{ij}^{ab}(z)$.

(i) Case $0 \leq r < n$ for $S^u(z)$ and Case $0 < r < n$ for $S^{1,1}(z)$. $\xi_r = e_r$ holds also for $S^{1,1}(z)$. For $S^u(z)$, the index 0 is to be identified with $n$. The action of $e_r$ and $k_r$ in (3.6) and (3.7) only concerns the $r$th and $(r+1)$th components $|m_r\rangle^{(c_l)} \otimes |m_{r+1}\rangle^{(c_{l+1})}$ of $W$. Denoting them simply by $|m_r, m_{r+1}\rangle$, we depict (5.5) by the following diagram:

\[
\begin{align*}
& \left\{i_r, i_{r+1}\right\} \otimes \left\{j_r, j_{r+1}\right\} \\
& \xrightarrow{S(z)} \left\{a_r + 1, a_{r+1} - 1\right\} \otimes \left\{b_r, b_{r+1}\right\} \\
& \xrightarrow{z^{k_r} e_r} \left\{a_r, a_{r+1}\right\} \otimes \left\{b_r + 1, b_{r+1} - 1\right\} \\
& \xrightarrow{k_r \otimes e_r} \left\{a_r, a_{r+1}\right\} \otimes \left\{b_r, b_{r+1}\right\} \\
& \xrightarrow{(q_r)^{-a_r}(q_{r+1})^{a_{r+1}}[b_r + 1]} \left\{a_r, a_{r+1}\right\} \otimes \left\{b_r, b_{r+1}\right\}
\end{align*}
\]

Thus we have

\[
A_{ij}^{ab}(z) = \sum_{c_0, \ldots, c_n} z^{c_0 + b_{r+1,0}} U(c_0, \ldots, c_{r-1}, c_r+1, \ldots, c_n) \times \left( \left[ a_r + 1 \right] S^{(c_r)}_{c_r, c_r+1, b_r, b_{r+1}} S^{(c_{r+1})}_{c_{r+1}, c_{r+1}+1, b_{r+1}, b_{r+1}+1} \right) + (q_r)^{-a_r}(q_{r+1})^{a_{r+1}}[b_r + 1] \left[ a_r + 1 \right] S^{(c_r)}_{c_r, c_r+1, b_r, b_{r+1}} S^{(c_{r+1})}_{c_{r+1}, c_{r+1}+1, b_{r+1}, b_{r+1}+1} \right) \tag{5.7}
\]

for some $U(c_0, \ldots, c_{r-1}, c_r+1, \ldots, c_n)$ which is independent of $z$. In the second term we have shifted the dummy summation variable $c_r$ to $c_r + 1$. This has the effect of letting the two terms have the identical constraints $b_l + c_{l-1} = j_l + c_l$ ($l = r, r + 1$) and the common $z$-dependence $z^{c_r + b_{r+1,0}}$. Similarly the diagram for (5.6) looks like

\[
\begin{align*}
& \left\{i_r, i_{r+1}\right\} \otimes \left\{j_r, j_{r+1}\right\} \\
& \xrightarrow{S(z)} \left\{a_r, a_{r+1}\right\} \otimes \left\{b_r, b_{r+1}\right\}
\end{align*}
\]
This leads to the expression

\[ B_{C}^{ab}(z) = \sum_{c_{0}, \ldots, c_{n}} z^{c_{0} + \delta_{ab}} U(c_{0}, \ldots, c_{r-1}, c_{r+1}, \ldots, c_{n}) \]

\[ \left( [k] S_{ij}^{(c_{0})} a_{i} b_{j} c_{i-1} \right) \delta_{i+1 j+1} c_{i+1} \delta_{i+1 j+1} c_{i+1} + (q_{r+1})^{-1} \left( i_{r} \right) S_{ij}^{(c_{0})} a_{i} b_{j} c_{i-1} \delta_{i+1 j+1} c_{i+1} \delta_{i+1 j+1} c_{i+1} \right) \]  

(5.8)

with the same \( U(c_{0}, \ldots, c_{r-1}, c_{r+1}, \ldots, c_{n}) \) as (5.7). This time \( c_{r} \) has been shifted to \( c_{r} + 1 \) in the first term for the same reason as in (5.7). Comparing (5.7) and (5.8) and noting the conservation law (2.14), we find that \( A_{C}^{ab}(z) = B_{C}^{ab}(z) \) is reduced to the equality of the quantities in the parenthesis. Writing \((a_{r}, b_{r}, i_{r}, j_{r})\) as \((a, b, i, j)\), \((a_{r+1}, b_{r+1}, i_{r+1}, j_{r+1})\) as \((a', b', i', j')\), \((c_{r-1}, c_{r}, c_{r+1})\) as \((c, k, c')\), \((c, k, c')\) as \((c, e')\) and \((q_{r}, q_{r+1})\) as \((\rho, \rho')\), it reads

\[ [a + 1] S^{(a)}_{ij} + b_{c} \delta^{(a)'}_{ij} \delta^{(a)'}_{c} = \left( -q^{-1} \right)^{a'} \delta^{(a)'}_{ij} \delta^{(a)'}_{c} \]  

(5.9)

Note that \((\rho, \rho') = ((-1)^{y}, q^{1-2e}, (-1)^{y'}, q^{1-2e'})\) by the definition (3.1). For \((e, e') = (0, 0)\), (5.9) coincides with [27, equation (A.16)]. For \((e, e') = (1, 1)\), (5.9) reads

\[ \delta^{(a)b}_{ij} \vdash^{(a)'}_{ij} \delta^{(a)b}_{c} = \delta^{(a)b}_{ij} \vdash^{(a)'}_{ij} \delta^{(a)b}_{c} \]  

(5.9)

where all the indices are in \((0, 1)\) and \(\delta^{(a)b}_{ij} = \delta^{(a)b}_{ij} \delta^{(a)b}_{c}\). These 28 relations can be checked directly by substituting (2.9) and using (2.11). For \((e, e') = (0, 0)\), (5.9) reads

\[ \delta^{(a)b}_{ij} \vdash^{(a)'}_{ij} \delta^{(a)b}_{c} = \delta^{(a)b}_{ij} \vdash^{(a)'}_{ij} \delta^{(a)b}_{c} \]  

(5.9)

Among 29 choices of \((a', b', i', j')\), there are four that lead to nontrivial relations for some values of \(k' = k\). They are given by \((a', b', i', j', k') = (1, 0, 0, 0, 0\), \((1, 1, 0, 0, 0, k + 1)\) and \((1, 1, 1, 0, 0, k + 1)\). The corresponding relations read

\[ [a + 1] R^{(a)}_{ij,k} = [j] R^{(a)}_{j,i,k+1} = q^{-i-j} \delta^{j}_{ij} R^{(a)}_{i,j,k} = 0, \]  

(5.10)

\[ q^{a+1}\left[ a + 1 \right] R^{(a)}_{ij,k+1} = q^{-a-1}\left[ a + 1 \right] R^{(a)}_{ij,k} = 0, \]  

(5.11)

\[ q^{-a}\left[ b + 1 \right] R^{(a)}_{ij,k+1} = q^{a+2}\left[ j \right] R^{(a)}_{j,i,k+1} = q^{-j} \left( 1 - q^{2k+2} \right)^{2} \delta^{j}_{ij} R^{(a)}_{i,j,k} = 0 \]  

(5.12)

\[ \left( 1 - q^{2k+2} \right) \left[ a + 1 \right] R^{(a)}_{ij,k} = q^{-a+k}\left[ b + 1 \right] R^{(a)}_{ij,k+1} = \left[ j \right] R^{(a)}_{j,i,k+1} = 0, \]  

(5.13)

Equations (5.10) and (5.12) are equivalent to the known identities named \( t_{1} \) and \( t_{1} \) appearing before [27 equation (A.2)]. Any element of \( R \) in (5.11) and (5.13) can be converted into the form \( R^{ab\delta}_{ijk} \) by using (5.10) to decrease \( a \) and (5.12) to decrease \( b \). The resulting expressions turn out to be identically zero because of \((a + b, b + c) = (i + j - 1, j + k)\). See (2.2). For \((e, e') = (1, 0)\), (5.9) reads...
Among 2^k choices of \((a, b, i, j)\), there are four that lead to nontrivial relations for some values of \(c - k\). They are given by \((a, b, i, j, c) = (0, 0, 0, 1, k + 1), (1, 0, 1, 1, k + 1), (0, 0, 1, 0, k)\) and \((0, 1, 1, 1, k)\). The corresponding relations, after removing primes, read

\[
\mathcal{R}^a_{i,j,k} = q^a \mathcal{R}^{a}_{i,j,k} + q^b \mathcal{R}^{a}_{i,j,k} + q^c \mathcal{R}^{a}_{i,j,k} - q \mathcal{R}^{a}_{i,j,k} = 0.
\]

Equations (5.14) and (5.16) are equivalent to (A.3) and (A.2) in [27], respectively. Any element of \(\mathcal{R}\) in (5.15) (resp. (5.17)) can be converted into the form \(\mathcal{R}^{a}_{i,j,k}\) (resp. \(\mathcal{R}^{a}_{i,j,k}\)) by using (5.16) to decrease \(i\) and (5.14) to decrease \(j\). The resulting expressions are identically zero.

(ii) Case \(r = n\) for \(S^{1,1}(z)\). The action of \(e_n\) and \(k_n\) in (3.7) only concern the \(n\)th component \(|m_n\rangle\) of \(|m\rangle\). Denoting it simply by \(|m_n\rangle\), we depict (5.5) as

\[
|i_n\rangle \otimes |j_n\rangle \quad (S(z)) \quad |a_n+1\rangle \otimes |b_n\rangle,
\]

\[
p^{-1}[a_n + 1] \quad \tilde{e}_n \otimes 1 \quad k_n \otimes e_n \quad p(q_n)^{-a_n}[b_n + 1]
\]

where \(\tilde{e}_n = p^{-1}e_n\) has been used. Thus we have

\[
p^{-1}\mathcal{A}^{a,b}_{\epsilon_{n+1}}(z) = \sum_{c_0, \ldots, c_{n-1}} \frac{z_{c_0}}{(q)_{c_n}} X(c_0, \ldots, c_{n-1}) \times \left( p^{-2}[a_n + 1](1 - q^{c_n}) S^{c_n}_{\epsilon_{n+1},a_n+1,b_n+1} + (q_n)^{-a_n}[b_n + 1] S^{c_n}_{\epsilon_{n+1},a_n+1,b_n+1} \right).
\]

where \(c_n\) has been shifted to \(c_n - 1\) in the first term. \(X(c_0, \ldots, c_{n-1})\) is independent of \(z\). Similarly (5.6) with \(r = n\) is depicted as

\[
p^{-1}[j_n] \quad \tilde{e}_n \otimes e_n \quad |i_n\rangle \otimes |j_n\rangle \quad |i_n\rangle \otimes |j_n - 1\rangle \quad |i_n - 1\rangle \otimes |j_n\rangle.
\]
This leads to

\[
p^{-1}B_{ij}^{ab}(z) = \sum_{c_0, \ldots, c_{n-1}} z_{c_0}^{e_0}X(c_0, \ldots, c_{n-1}) \times \left(p^{-2}\left[\frac{i_0}{i_a}\right]S^{(c_0)i_0b_i_0c_{i-1}}_{i_0,i_0,i-1} + (q_{c_0})^{-1}\left[\frac{i_0}{i_a}\right](1 - q^{c_0})S^{(c_0)i_0b_i_0c_{i-1}}_{i_0,i_0,i-1}\right),
\]

(5.19)

where \(c_n\) has been shifted to \(c_{n-1}\) in the second term. \(X(c_0, \ldots, c_{n-1})\) is the same as in (5.18). From (5.18), (5.19) and (2.14), \(A_{ij}^{ab}(z) = B_{ij}^{ab}(z)\) is reduced to the equality of the quantities in the parenthesis:

\[
-q[a + 1](1 - q^k)S^{(c_0)a_1, b_i, c_{i-1}}_{i_0, i_0, j} + \rho^{-a}[b + 1]S^{(c_0)a_1, b_i, c_{i-1}}_{i_0, i_0, j} + q[j]S^{(c_0)a_1, b_i, c_{i-1}}_{i_0, i_0, j} - \rho^{-j}[i](1 - q^k)S^{(c_0)a_1, b_i, c_{i-1}}_{i_0, i_0, j-1} = 0,
\]

(5.20)

where we have set \((a_n, b_n, c_n-1, i_n, j_n, e_n, q_n) = (a, b, c, i, j, k, e, \rho)\). Thus \(\rho = (-1)^{q-2c}\) by (3.1). For \(e = 1\), (5.20) reads

\[
-q\left(1 - q^k\right)\delta^a \nabla^{(c_0)a_1, b_i, c_{i-1}}_{i_0, i_0, j} + (-q)^j\delta^b \nabla^{(c_0)a_1, b_i, c_{i-1}}_{i_0, i_0, j} + q\delta^j \nabla^{(c_0)a_1, b_i, c_{i-1}}_{i_0, i_0, j} - (-q)^i\left(1 - q^k\right)\delta^a \nabla^{(c_0)a_1, b_i, c_{i-1}}_{i_0, i_0, j-1} = 0,
\]

which can directly be verified by using (2.12). For \(e = 0\), (5.20) represents the relation obtained by replacing \(S^{(c_0)}\) by \(R^{(c_0)}\) and \(\rho\) by \(q\). All the elements of \(R^{(c_0)}\) can be converted to the form \(R_{abc}^{(c_0)}\) by decreasing \(a\) by (5.10) and \(b\) by (5.12). The resulting expression turns out to be identically zero.

(iii) Case \(r = 0\) for \(S^{(c_0)}(z)\). The action of \(e_0\) and \(k_0\) in (3.7) only concern the first component \(m_{i_0}\) of \(|m\rangle\). Denoting it simply by \(|m_{i_0}\rangle\), we depict (5.5) and (5.6) as

\[
|a_1 - 1\rangle \otimes |b_1\rangle \quad \rightarrow \quad |a_1\rangle \otimes |b_1 - 1\rangle \quad \rightarrow \quad |a_1\rangle \otimes |b_1\rangle
\]

\[
|a_1\rangle \otimes |b_0\rangle \quad \rightarrow \quad |a_1\rangle \otimes |b_1\rangle \quad \rightarrow \quad |a_1\rangle \otimes |b_0\rangle
\]

where \(\tilde{e}_0 = p e_0\) has been used. From this and \(z = x/y\) we have

\[
p^{-1}A_{ij}^{ab}(z) = \sum_{c_0, \ldots, c_n} z_{c_0}^{e_0+1}(-q; q)c_{a_0} Y(c_0, \ldots, c_n) \left(S^{(c_0)i_0b_i_0c_{i-1}}_{i_0,i_0,i-1} + p^{-2}(q_{c_0})^{a_0}(1 + q^{c_0+1})S^{(c_0)i_0b_i_0c_{i-1}}_{i_0,i_0,i-1}\right),
\]

\[
p^{-1}B_{ij}^{ab}(z) = \sum_{c_0, \ldots, c_n} z_{c_0}^{e_0+1}(-q; q)c_{a_0} Y(c_0, \ldots, c_n) \left(1 + q^{c_0+1})S^{(c_0)i_0b_i_0c_{i-1}}_{i_0,i_0,i-1} + p^{-2}(q_{c_0})^{a_0}S^{(c_0)i_0b_i_0c_{i-1}}_{i_0,i_0,i-1}\right)
\]

with a common \(Y(c_0, \ldots, c_n)\) independent of \(z\). We have shifted \(c_0\) to \(c_0 + 1\) in the second term of \(p^{-1}A_{ij}^{ab}(z)\) and in the first term of \(p^{-1}B_{ij}^{ab}(z)\). Now \(A_{ij}^{ab}(z) = B_{ij}^{ab}(z)\) is reduced to
for $\rho = (-1)^e q^{1-2\epsilon}$. For $\epsilon = 1$, this can be verified directly from (2.12). For $\epsilon = 0$, all the elements of $R$ can be expressed in the form $R_{ij,kl}^{abc}$ by using (5.14) to decrease $j$ and (5.16) to decrease $i$. The result turns out to be identically zero. The proof of (5.3) is completed. □

6. Proof part II: irreducibility of $\mathcal{W}_l \otimes \mathcal{W}_m$ for $\mathcal{U}_A$

Here we consider $\mathcal{U}_A$ of the form $\mathcal{U}_A(0, 0^{n-k})(0 \leq \kappa \leq n)$ and show that the $\mathcal{U}_A$-module $\mathcal{W}_l \otimes \mathcal{W}_m$ is irreducible. See (2.29) for the definition of $\mathcal{W}_l \subset \mathcal{W}$. We assume that $0 \leq l, m \leq n$ if $\kappa = n$ and $l, m \in \mathbb{Z}_{\geq 0}$ otherwise. We will flexibly write $|a_1, a_2, \ldots, a_n\rangle \otimes |b_1, b_2, \ldots, b_n\rangle \in \mathcal{W} \otimes \mathcal{W}$ as

$$
\left(|a_1\rangle \otimes |b_1\rangle\right) \boxtimes \left(|a_2\rangle \otimes |b_2\rangle\right) \boxtimes \cdots \boxtimes \left(|a_n\rangle \otimes |b_n\rangle\right)
$$

or

$$
\left(|a_1, \ldots, a_j\rangle \otimes |b_1, \ldots, b_j\rangle\right) \boxtimes \left(|a_{j+1}, \ldots, a_n\rangle \otimes |b_{j+1}, \ldots, b_n\rangle\right)
$$

for some $j$ and so on. The vectors $v_0 = |0⟩^{(1)}$, $v_1 = |1⟩^{(1)} \in V$ (2.22) will simply be denoted by $|0⟩$, $|1⟩$. They are to be distinguished from $|0⟩ = |0⟩^{(0)}$, $|1⟩ = |1⟩^{(0)} \in F$ from the context. (See the remark after (2.22).) We treat the cases $\kappa = n, n - 1$ and $1 \leq \kappa \leq n - 2$ separately. We include the results on the spectral decompositions although the concrete forms of the eigenvalues in (6.10), (6.13) and (6.16) are not necessary for our main issue, namely, the proof of the irreducibility.

6.1. Case $\kappa = n$

As mentioned in (3.5), the relevant algebra $\mathcal{U}_A(1, \ldots, 1)$ supplemented with the Serre relation is $U_{-q^{-1}}(\mathfrak{sl}_n)$. The representation $\mathcal{W}_l$ in proposition 3.2 is the $(-q^{-1})$-analogue of the $l$-fold antisymmetric tensor representation. Thus we assume $0 \leq l, m \leq n$. It is known that $\mathcal{W}_l \otimes \mathcal{W}_m$ is an irreducible $U_{-q^{-1}}(\mathfrak{sl}_n)$-module, and the quantum $R$ matrix is given for example in [15]. We recall it as a preparation for the next case $\kappa = n - 1$. Note that proposition 3.2 with $\mathcal{W} = V^{\otimes n}$ and $q = -q^{-1}$ gives

$$
(x_1 \otimes x_i)A(e_i)(|m_i, m_{i+1}, \ldots\rangle \otimes |m_i', m_{i+1}', \ldots\rangle)
$$

$$
= y^{x_i} \delta_{m_i, m_i'} |m_i, m_{i+1}, \ldots\rangle \otimes |0, 1, \ldots\rangle
$$

$$
+ \delta_{m_i, m_i'} |m_i, m_{i+1}, \ldots\rangle \otimes |0, 1, \ldots\rangle
$$

$$
A(e_i)(|m_i, m_{i+1}, \ldots\rangle \otimes |m_i', m_{i+1}', \ldots\rangle)
$$

(6.1)
6.1.1. Singular vectors. For \( r \geq 1 \) we define

\[
\mathcal{J}_{r,j} = \sum_{\{i_1, \ldots, i_r\} \in \{0, 1\}^r} q^{\text{inv}(i_1, \ldots, i_r)} |i_1, \ldots, i_r\rangle \otimes |\bar{i}_1, \ldots, \bar{i}_r\rangle
\]

\( \in V^{\otimes r} \otimes V^{\otimes r} \) (0 \( \leq j \leq r)\),

\[
\text{inv}(i_1, \ldots, i_r) = \sum_{1 \leq j \leq r} i_j \bar{i}_j, \quad \bar{i} = 1 - i.
\]

(6.2)

It is characterized by the recursion relations

\[
\mathcal{J}_{r,j} = \mathcal{J}_{r-1,j-1} \otimes (|1\rangle \otimes |0\rangle) + q^j \mathcal{J}_{r-1,j} \otimes (|0\rangle \otimes |1\rangle)
\]

(6.3)

\[
= (|0\rangle \otimes |1\rangle) \otimes \mathcal{J}_{r-1,j} + q^{-j} (|1\rangle \otimes |0\rangle) \otimes \mathcal{J}_{r-1,j-1}
\]

(6.4)

with the initial condition \( \mathcal{J}_{0,0} = |0\rangle \otimes |1\rangle \) and \( \mathcal{J}_{1,1} = |1\rangle \otimes |0\rangle \), where \( \mathcal{J}_{r,j} \) with \( j \not\in [0, r] \) is to be understood as 0. For example, the \( r = 2 \) case reads

\[
\mathcal{J}_{2,0} = |0, 0\rangle \otimes |1, 1\rangle,
\]

\[
\mathcal{J}_{2,1} = |0, 1\rangle \otimes |1, 0\rangle + q |1, 0\rangle \otimes |0, 1\rangle,
\]

\[
\mathcal{J}_{2,2} = |1, 1\rangle \otimes |0, 0\rangle.
\]

We also understand that \( \mathcal{J}_{0,0} \) is the object that formally makes the above recursion relations valid for \( r = 1 \). Note that \( \mathcal{J}_{n,j} \in \mathcal{W}_j \otimes \mathcal{W}_{n-j} \).

Lemma 6.1. \((\pi_x \otimes \pi_y)\Delta(e_t)\mathcal{J}_{n,j} = 0\) holds for \( 1 \leq i \leq n - 1 \) and \( 0 \leq j \leq n \).

Proof. By using (6.1) and the above example, the case \( n = 2 \) can be directly checked. Then the assertion follows by induction on \( n \) thanks to (6.3) and (6.4). \( \square \)

For \( 0 \leq l, m \leq n \) define the following vector in \( \mathcal{W}_l \otimes \mathcal{W}_m \):

\[
\xi_s = \zeta^{l,m}_s = (|0_{n-s-l}\rangle \otimes |0_{n-s-l}\rangle) \otimes \mathcal{J}_{s,l-t} \otimes (|1_s\rangle \otimes |1_s\rangle)
\]

(6.5)

for \( l + m - n \) \( \leq t \leq \min(l, m) \), where the symbol \((x)_+ \) is defined after (2.30). We have set \( |i_s\rangle = |i\rangle^{0_s} \in V^{\otimes s} \) for \( i = 0, 1 \). Note that \( \zeta^{l,m}_{\min(l,m)} = |e_{n-s-1}\rangle \otimes |e_{s-n-1}\rangle \). Using lemma 6.1 one can show

Proposition 6.2. The weight vectors in \( \mathcal{W}_l \otimes \mathcal{W}_m \) annihilated by \((\pi_x \otimes \pi_y)\Delta(e_t)\) for \( 1 \leq i \leq n - 1 \) are given by \( \xi_s \) with \( l + m - n \) \( \leq t \leq \min(l, m) \) up to an overall scalar.

6.1.2. Spectral decomposition. A direct calculation shows
Lemma 6.3. For \((l + m - n)_+ \leq t < \min(l, m)\) the following relations hold:
\[
\left( \pi_x \otimes \pi_y \right) \Delta \left( e_{n-l-\ldots-m+1} \ldots e_{n-l-1} e_0 \right) \xi_t
\]
\[
= \left( -[2] \right)^{\delta_{l-t}} q^{-1} \left( q^{l+m-2} y - x \right) \xi_{t+1},
\]
\[
\left( \pi_x \otimes \pi_y \right) \Delta \left( f_0 f_1 \ldots f_{n-l-1-m+2} \ldots f_{n-l} \right) \xi_t
\]
\[
= (q x y)^{-1} \left( q^{l+m-2} y - x \right) \xi_{t+1}.
\]

Set \(z = x/y\) and let \(R(z) \in \text{End}(\mathcal{W}_l \otimes \mathcal{W}_m)\) be the quantum \(R\) matrix satisfying (3.14) normalized as (3.20). Due to proposition 6.2 it has the spectral decomposition
\[
PR(z) = \sum_{l=\min(l,m)} P(z) \mathcal{P}_z^{l,m},
\]
where \(P\) is defined after (3.14) and \(\mathcal{P}_z^{l,m} : \mathcal{W}_l \otimes \mathcal{W}_m \rightarrow \mathcal{W}_m \otimes \mathcal{W}_l\) is the projector characterized by
\[
\mathcal{P}_z^{l,m} : \xi_s' \mapsto \delta_{s,s'} \xi_z^{m,l},
\]
(6.7)
\[
(\pi_x \otimes \pi_y) \Delta (g) \mathcal{P}_z^{l,m} = \mathcal{P}_z^{l,m} (\pi_x \otimes \pi_y) \Delta (g)
\]
for all \(g \in \mathcal{U}_A(1, \ldots, 1)\). See the end of section 3.1 for the definition of \(\mathcal{U}_A(1, \ldots, 1)\). The combination \(PR(z)\) is the intertwiner of \(\pi_x \otimes \pi_y\) and \(\pi_y \otimes \pi_x\) denoted by \(\tilde{R}(z)\) in [22].

Substituting (6.6) into either (3.16) or (3.17) with \(r = 0\) one gets
\[
\frac{\rho_{s+1}(z)}{\rho(z)} = \frac{1 - q^{l+m-2} z}{z - q^{l+m-2} z}.
\]
(6.9)

From (3.20) it follows that \(\rho_{\min(l,m)}(z) = 1\) and
\[
PR(z) = \sum_{l=\min(l,m)}^{\min(l,m)} \left( \prod_{i=s+1}^{\min(l,m)} \frac{z - q^{l+m-2 i + 2}}{1 - q^{l+m-2 i + 2} z} \right) \mathcal{P}_z^{l,m}.
\]
(6.10)

6.2 Case \(\kappa = n-1\)

Consider \(\mathcal{U}_A\) of the form \(\mathcal{U}_A(1, \ldots, 1, 0)\). We show that the \(\mathcal{U}_A\)-module \(\mathcal{W}_l \otimes \mathcal{W}_m\) with \(\mathcal{W} = \mathcal{V}^{\otimes m-1} \otimes I\) is irreducible and present the spectral decomposition of the associated quantum \(R\) matrix. We assume \(l, m \in \mathbb{Z}_{\geq 0}\).

6.2.1. Singular vectors. For \(0 \leq s \leq \min(n-1, l, m)\) define the following vector in \(\mathcal{W}_l \otimes \mathcal{W}_m\):
\[
\xi_s = \xi_s^{l,m} = (\ket{0_{n-s-1}} \otimes \ket{0_{n-s-1}}) \mathcal{X}
\]
\[
\sum_{j=0}^{s} (-1)^j q^{j(m-s+1)} \mathcal{J}_{s-s-j} \mathcal{X} \left( \ket{l + j} \otimes \ket{m - j} \right).
\]
(6.11)

The \(\mathcal{J}_{s-s-j}\) is defined by (6.2). Note that \(\xi_0 = \ket{l e_n} \otimes \ket{m e_n}\).

Proposition 6.4. The weight vectors in \(\mathcal{W}_l \otimes \mathcal{W}_m\) annihilated by \((\pi_x \otimes \pi_y) \Delta (e_i)\) for \(1 \leq i \leq n - 1\) are given by \(\xi_s\) with \(0 \leq s \leq \min(n-1, l, m)\) up to an overall scalar.
6.2.2. Spectral decomposition

Lemma 6.5. For $1 \leq s \leq \min(n-1, l, m)$ the following relations hold:

$$
\begin{align*}
\left( \pi_x \otimes \pi_y \right) \Delta(e_{n-1} \cdots e_{n-s} \cdots e_0) \xi_s &= (y - q^{l-1} + \cdots + q^{m-1})(-2)^{j_{n-1}-1} \xi_s, \\
\left( \pi_x \otimes \pi_y \right) \Delta(f_{i_0} f_{i_1} \cdots f_{i_{s-1}}) \xi_s &= (xy)^{-1}(y - q^l + m-2 + \cdots + 2) \xi_{s-1}.
\end{align*}
$$

Set $z = x/y$ and let $R(z) \in \text{End}(\mathcal{W}_l \otimes \mathcal{W}_m)$ be the quantum $R$ matrix satisfying (3.14) normalized as (3.21). Due to proposition 6.4 it has the spectral decomposition

$$
PR(z) = \sum_{s=0}^{\min(n-1,l,m)} \rho_s(z) \mathcal{P}^l_m,
$$

where the projector $\mathcal{P}^l_m: \mathcal{W}_l \otimes \mathcal{W}_m \to \mathcal{W}_m \otimes \mathcal{W}_l$ is characterized by (6.7) and (6.8) for all $g \in \mathbb{P}_A(1, \ldots, 1, 0)$. Substituting (6.12) into either (3.16) or (3.17) with $r = 0$ one formally gets the same relation as (6.9). In (3.21) (and also (2.31)), the $i$ should be taken as $n$, which leads to $PR(z)(|e_n\rangle \otimes |me_{n'}\rangle) = |me_n\rangle \otimes |e_n\rangle$. This implies $\rho_0(z) = 1$ and

$$
PR(z) = \sum_{s=0}^{\min(n-1,l,m)} \left( \frac{1}{z} - \frac{q^{l+m-2s} + 2}{z} \right) \mathcal{P}^l_m.
$$

This formally coincides with (6.10) up to an overall factor and the range of $s$.

6.2.3. Irreducibility of $\mathcal{W}_l \otimes \mathcal{W}_m$. Consider the direct sum decomposition

$$
\mathcal{W}_l \otimes \mathcal{W}_m = \bigoplus_{0 \leq i \leq j \leq k \leq \min(k, n-1)} X_{i,k} \quad (s_k = \min(k, n-1)),
$$

where the latter direct sum is over $i, \ldots, i_{n-1}, i'_1, \ldots, i'_{n-1} \in \{0, 1\}$ such that $(i_1 + \cdots + i_{n-1}, i'_1 + \cdots + i'_{n-1}) = (j, k)$. The subalgebra of $\mathcal{U}_A(1, \ldots, 1, 0)$ generated by $e_i, f_i, k^{\pm 1}$ with $1 \leq i \leq n-2$ (with the Serre relations) is isomorphic to $U_{q^{-1}}(A_{n-2})$. Let the same symbol denote its coproduct action. Then we have

Lemma 6.6.

$$
X_{i,k} = U_{q^{-1}}(A_{n-2}) u_{i,k},
$$

$$
u_{i,k} := [1, \ldots, 1, 0, \ldots, 0, l-j] \otimes [0, \ldots, 0, 1, \ldots, 1, m-k].
$$

Proof. As an element of a $U_{q^{-1}}(A_{n-2})$-module, $u_{i,k}$ is (lowest wt. vec.) $\otimes$ (highest wt. vec.) in the tensor product of the antisymmetric tensor representations of order $j$ and $k$. Thus the assertion follows from proposition 4.2.

We use the notation $e_{[i,j]} = e_i + e_{i+1} + \cdots + e_j$. The notation $|e_n\rangle + \cdots + |e_{n'}\rangle + \cdots \otimes |e_{i_1}\rangle + \cdots + |e_{i_1'}\rangle + \cdots$ will be used only when $\max(i_1, \ldots, i_{n'}, i'_1, \ldots, i'_{n'}) < n$ and is to be
understood as \( |e_i + \cdots + e_n + (l - a)e_n \rangle \otimes |e_{j'_1} + \cdots + e_{j'_n} + (m - b)e_n \rangle \). Thus \( u_{j,k} = |e_{[1,j]} + \cdots + e_{[n-k,n-1]} + \cdots + e_{[1,j]} + \cdots + e_{[n-k,n-1]} + \rangle \).

Proposition 6.7. The \( U_q(1, \ldots, 1, 0) \)-module \( \mathcal{W}_l \otimes \mathcal{W}_m \) is irreducible.

**Proof.** Let \( W \) be a nonzero submodule of \( \mathcal{W}_l \otimes \mathcal{W}_m \). Due to lemma 6.6 it suffices to show that all the \( u_{j,k} \) are generated from a vector in \( W \). We show this by induction on \( j + k \geq 0 \). By proposition 6.4 and lemma 6.5, we can generate \( u_{0,0} = \xi_0 \) by applying \( e_i \)'s and \( f_j \)'s appropriately to any nonzero vector in \( W \). Thus \( j + k = 0 \) case is true. Set \( X_{j,k} = \oplus_{j,k=0}^{\infty} X_{j,k} \).

Let us show that all the \( u_{j,k} \) with \( j + k = s \) are generated by assuming that \( X_{s-1} \) has already been generated.

**(i) Case \( s \leq n - 1 \).** Step 1. We show that \( X_{s,0} \) is generated. Set
\[
\zeta_s = u_{s,0}, \quad \zeta_j = \left( |e_{[1,j]} + \cdots + e_{[j+2,s]} + \rangle \otimes |e_{j+1} + \rangle \right) \quad (0 \leq j \leq s - 1).
\]
They are vectors in \( X_s \). We have \((\pi_\alpha \otimes \pi_\beta)\Delta(e_i)\) simply denoted by \( e_i \) and similarly for \( f_j \).

\[
c. f_{j+1} f_{j+2} \cdots f_{n-1} \left( |e_{[1,j]} + \cdots + e_{[j+2,s]} + \rangle \otimes |e_{n} + \rangle \right) = \zeta_{j+1} = q \zeta_j \quad (0 \leq j \leq s - 2),
\]
\[
c. f_{s+1} f_{s+2} \cdots f_{n-1} u_{s,0} = [s + 1] \zeta_s + q^{s-l-1} [m] \zeta_{s-1},
\]
\[
c. e_0 \left( |e_{[2,s]} + \cdots + \rangle \otimes |e_{n} + \rangle \right) = xq^{-m}[s + 1] \zeta_s + y[m] \zeta_0
\]
where \( c. \) means multiplication by a nonzero rational function of \( q \) which does not involve \( x \) and \( y \). Regarding them as the \( s + 1 \) linear equations on \( \zeta_0, \ldots, \zeta_s \), one finds that the coefficient matrix is invertible for generic \( x \) and \( y \). Moreover all the lhs belong to \( X_{s-1} \). Thus \( u_{s,0} = \zeta_s \) is generated. Then by lemma 6.6, \( X_{s,0} \) is generated.

Step 2. Set \( s = j + k \). We show that \( X_{j+k+1} \) is generated assuming that \( X_{j,k} \) (and \( X_{j+1} \)) are already generated. This claim follows from (\( 1 \leq j \leq s \))
\[
f_{j+1} f_{j+2} \cdots f_{n-1} \left( |e_{[1,j-1]} + \cdots + \rangle \otimes |e_{[j+k-1]} + \rangle \right) = q^{j+k-l} [m-k] \left( |e_{[1,j-1]} + \cdots + \rangle \otimes |e_{[j+k]} + \rangle \right)
\]
\[
+ \left( [s + 1] |e_{[1,j-1]} + \cdots + e_{[j+k]} + \rangle \otimes |e_{[j+k]} + \rangle \right).
\]
The \( \text{lhs} \) belongs to \( X_{j+1} \subseteq X_{s-1} \) and the second term on the rhs to \( X_{j,k} \). Therefore \( |e_{[1,j-1]} + \cdots + \rangle \otimes |e_{[j+k]} + \rangle \) is generated. Applying \( e_{n-2} e_{n-3} \cdots e_{j+k} \) to it we get \( u_{j+k+1} \). Then \( X_{j+1,k+1} \) is generated by lemma 6.6.

By step 1 and applying step 2 repeatedly in the order \( j = s, s - 1, \ldots, 1 \), we get \( X_s \).

**(ii) Case \( s \geq n \).** By the induction we assume that \( u_{j,k} \in X_{s-1} \) is already generated. From \( j' + k' = s - 1 \geq n - 1 \), we have
\[
c. f_{j'+1} f_{j'+2} f_{n-1} u_{j',k'} = u_{j'+1,k'}, \quad c. e_{k-1} \cdots e_0 u_{j',k'} = y u_{j',k'+1}.
\]
Thus \( X_s \) is generated. By (i) and (ii) the induction step has been proved. \( \square \)
6.3. Case $0 \leq \kappa \leq n-2$

Consider $\mathcal{U}_A$ of the form $\mathcal{U}_A(F, 0^{n-k})$ with $0 \leq \kappa \leq n - 2$. We show that the $\mathcal{U}_A$-module $W_l \otimes W_m$ with $W = V^{\otimes \kappa} \otimes F^{\otimes n-\kappa}$ is irreducible and present the spectral decomposition of the associated quantum $R$ matrix. We assume $l, m \in \mathbb{Z}_{\geq 0}$ and $n \geq 2$. Due to $\kappa \leq n - 2$, the rightmost two components in $W$ is $F \otimes F$.

6.3.1. Singular vectors. For $0 \leq s \leq \min(l, m)$ introduce the following vector in $W_l \otimes W_m$:

$$
\xi_s = \xi_{s,m} = \left( |0_{n-2}\rangle \otimes |0_{n-2}\rangle \right) \otimes \sum_{j=0}^{s} (-1)^{j} q^{j(2s - m - j + 1 + m)} \left( \begin{array}{c} s \\ j \end{array} \right) |j, l - j\rangle \otimes |s - j, m - s + j\rangle,
$$

(6.14)

where $|0_{n-2}\rangle = |0\rangle \otimes \cdots \otimes |0\rangle \in V^{\otimes \kappa} \otimes F^{\otimes n-2}$. Note that $\xi_0 = |l e_0\rangle \otimes |m e_0\rangle$.

**Proposition 6.8.** The weight vectors in $W_l \otimes W_m$ annihilated by $(\pi_x \otimes \pi_y)\Delta(e_i)$ for $1 \leq i \leq n - 1$ are given by $\xi_s$ with $0 \leq s \leq \min(l, m)$ up to an overall scalar.

6.3.2. Spectral decomposition

**Lemma 6.9.** For $0 \leq s \leq \min(l, m)$ the following relations hold:

$$
\begin{align*}
A &= q^{-s} - \frac{1}{s} [l - s][l + m + 1 - s][m - s] \left( x - q^{l + m - 2s} y \right), \\
B &= q^{-s} - \frac{1}{s} [l + m + 1 - s][l + m - 2s] \left( q^l + y + 2s x - y \right), \\
C &= \frac{1}{s} [l - s][l + m + 2 - s][m - s][l + m - s] \left( x \leftrightarrow m, x \leftrightarrow y \right), \\
\end{align*}
$$

$$
\begin{align*}
(\pi_x \otimes \pi_y)\Delta(f_0 \cdots f_{n-2})\xi_s &= (xy)^{-1} \left( q^{l + m - 2s} - q^{2s - 2} \right) [s] \xi_{s-1}.
\end{align*}
$$

Set $z = x/y$ and let $R(z) \in \text{End}(W_l \otimes W_m)$ be the quantum $R$ matrix satisfying (3.14) normalized as (3.21). Due to proposition 6.8 it has the spectral decomposition

$$
PR(z) = \sum_{s=0}^{\min(l, m)} \rho_s(z) P_{s,m}^{l,m},
$$

(6.15)

where the projector $P_{s,m}^{l,m}$: $W_l \otimes W_m \to W_m \otimes W_l$ is characterized by (6.7) and (6.8) for all $g \in \mathcal{U}_A(e_1, \ldots, e_{n-2}, 0, 0)$. From lemma 6.9 one formally gets the same relation as (6.9). The normalization condition (3.21) tells us that $PR(z)(|l e_0\rangle \otimes |m e_0\rangle) = |m e_0\rangle \otimes |l e_0\rangle$. Thus we have $\rho_0(z) = 1$ and

$$
PR(z) = \sum_{s=0}^{\min(l, m)} \left( \prod_{j=1}^{s} \frac{1 - q^{l + m - 2s + 2z}}{z - q^{l + m - 2s + 2z}} \right) P_{s,m}^{l,m},
$$

(6.16)

which is formally identical with (6.13) except for the range of $s$. 

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6.3.3. Irreducibility of $\mathcal{W}_l \otimes \mathcal{W}_m$. Consider the direct sum decomposition

$$\mathcal{W}_l \otimes \mathcal{W}_m = \bigoplus_{0 \leq k \leq l} Y_{l,k} \ (t_k = \min(\kappa, k)),
$$

$$Y_{l,k} = \bigoplus C(q)[i_1, \ldots, i_l] \otimes [i'_1, \ldots, i'_l],$$

where the latter direct sum is over $i_1, \ldots, i_l, i'_1, \ldots, i'_l \in \{0, 1\}$ and $i_1, \ldots, i_{k+1}, \ldots, i_l, i'_{k+1}, \ldots, i'_l \in \mathbb{Z}_{\geq 0}$ such that $(i_1 + \cdots + i_k, i'_1 + \cdots + i'_k) = (j, k)$ and $(i_{k+1} + \cdots + i_l, i'_{k+1} + \cdots + i'_l) = (l - j, m - k).$ The subalgebra of $U_q(A_{r-1})$ denoted by $e_i, f_i, k^{\pm 1}$ with $1 \leq i \leq \kappa - 1$ (resp. $\kappa + 1 \leq i \leq n - 1$) with the Serre relations is isomorphic to $U_q(A_{r-1})$ (resp. $U_q(A_{n-r-1})$). Let the same symbols denote their coproduct action. Then $U_q(A_{r-1})$ and $U_q(A_{n-r-1})$ are commuting and we have

**Lemma 6.10.**

$$Y_{j,k} = U_{-q^{-1}}(A_{r-1})U_q(A_{n-r-1})v_{j,k},$$

$$v_{j,k} = [1, \ldots, 1, 0, \ldots, 0, l - j]$$

$$\otimes [0, \ldots, 0, 1, \ldots, 1, m - k, 0, \ldots, 0].$$

**Proof.** As an element of a $U_{-q^{-1}}(A_{r-1})$-module, $v_{j,k}$ is the (lowest wt. vec.) $\otimes$ (highest wt. vec.) in the tensor product of the antisymmetric tensor representations of order $j$ and $k$. As an element of a $U_q(A_{n-r-1})$-module, $v_{j,k}$ is the (highest wt. vec.) $\otimes$ (lowest wt. vec.) in the tensor product of the symmetric tensor representations of order $l - j$ and $m - k$. Thus the assertion follows from proposition 4.2. □

**Proposition 6.11.** The $U_q(A_{r-1})$-module $\mathcal{W}_l \otimes \mathcal{W}_m$ is irreducible.

**Proof.** Let $\mathcal{W}$ be a nonzero submodule of $\mathcal{W}_l \otimes \mathcal{W}_m$. Due to lemma 6.10 it suffices to show that all the $v_{j,k}$ are generated from a vector in $\mathcal{W}$. By proposition 6.8 and lemma 6.9, we can generate all the $\xi_i$ (6.14). By applying $U_q(A_{n-r-1})$ to them further we can generate $v_{0,0}$. It contains the vector $[\xi_n] \otimes [\xi_{e_{k+1}}]$. Then $v_{j,k}$ is generated as $v_{j,k} = c_i F_i F_{e_{k+1}} \cdots F_{-e_{k+1}} E_0 E_1 \cdots E_{j-1} [\xi_n] \otimes [\xi_{e_{k+1}}]$, where $E_i = x^{-1}e_{i+1} \cdots e_0$ and $F_i = f_i f_{e_{k+1}} \cdots f_{e_{k+1}}$. □

7. Proof part III: irreducibility of $\mathcal{W} \otimes \mathcal{W}$ for $U_B$

Consider $U_B = U_B(\mathbf{F}^r, 0^r)$ ($\kappa' = n - \kappa$). In this section we show the irreducibility of the $U_B$-module $\mathcal{W} \otimes \mathcal{W}$ and present the spectral decomposition of the associated quantum $R$ matrix. We assume $\kappa, \kappa' \geq 1$, since the $\kappa' = 0$ case was treated in [30] and the $\kappa = 0$ case in [27, 28]. We follow the convention for the vector in $\mathcal{W} \otimes \mathcal{W}$ in the beginning of section 6. For a subset $J$ of $\{0, 1, \ldots, n\}$ define the subalgebra $U_{B,J}$ by the one generated by $e_i, f_i, k^{\pm 1}$ for $i \in J$. 

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7.1. Singular vectors and spectral decomposition

Although our algebra $\mathcal{B}$ and module $\mathcal{W}$ are different from $U_q(D^{(2)}_{n+1})$ and $F^{\otimes n}$ treated in [27], the action of generators in proposition 3.3 is quite similar to [27, Prop.1], and consequently, the following propositions remain valid.

**Proposition 7.1.** The weight vectors in $\mathcal{W} \otimes \mathcal{W}$ annihilated by $(\pi_x \otimes \pi_y)\Delta(e_i)$ for $1 \leq i \leq n$ are given by

$$
\xi_i = \sum_{m=0}^{l} (-p)^{-m} q^{m(l-(m+1)/2)} \left[ \begin{array}{c} l \\ m \end{array} \right] m \mathbf{e}_n \otimes (l-m)\mathbf{e}_n
$$

for some $l \in \mathbb{Z}_{\geq 0}$ up to an overall scalar.

**Proposition 7.2.** We have

$$
\left( \pi_x \otimes \pi_y \right)\Delta(e_{n-1} \cdots e_1 e_0) \xi_0 = \frac{1}{1 - q^{1/2}} \left( q x + y \right) \xi_0 - \frac{1}{1 - q^{1/2}} \left( q x^{-1} + y^{-1} \right) \xi_0
$$

for some $l \in \mathbb{Z}_{\geq 0}$ up to an overall scalar.

**Proposition 7.3.** For $\mathcal{B}$, $PR(z)$ has the following spectral decomposition.

$$
PR(z) = \sum_{i=0}^{\infty} \prod_{j=1}^{i} \frac{z + q^j}{1 + q^j} \mathcal{P}_i,
$$

where $\mathcal{P}_i$ is the projector on the space generated from $\xi_i$ over $\mathcal{B}_{\{1,...,n\}}$.

7.2. Irreducibility of $\mathcal{W} \otimes \mathcal{W}$

We prove the irreducibility of the $\mathcal{B}$-module $\mathcal{W} \otimes \mathcal{W}$. Set $K = \{ 1, \ldots, \kappa \}$, $K' = \{ \kappa + 1, \ldots, n \}$. The subalgebra $\mathcal{B}_{K^c,1}$ ($K^c = \{ 0, \ldots, \kappa - 1 \}$) (resp. $\mathcal{B}_{K',1}$) is isomorphic to $U_{-q^{-1}}(B_\kappa)$ (resp. $U_q(B_{\kappa})$). We use the same symbol $\mathcal{P}_i$: $i^{\otimes t} \in V^{\otimes (t \leq \kappa)} \otimes F^{\otimes (t > \kappa)}$ as in section 6. For instance, one can write

$$
\xi_0 = \left( \mathcal{P}_0 \right) \otimes \left( \mathcal{P}_0 \right)
$$

We also use a notation $\xi_i^{(k)} = \left( \mathcal{P}_0 \right) \otimes \left( \mathcal{P}_0 \right)$.

**Lemma 7.4.** Let $a \leq \kappa$, $a' = \sum_{i=1}^{a'} a_i \mathbf{e}_i + \sum_{i=1}^{a-1} a_i' \mathbf{e}_i$. For $a$, set $W_{a,K'} = \langle \mathbf{a} + \sum_{i=a+1}^{\kappa} \mathbf{e}_i \rangle \in \mathcal{Z}(0)$, where $\langle Y \rangle$ means the linear span of the set $Y$ of the vectors. Then we have

$$
W_{a,K'} \otimes W_{a',K'} = \sum_{i=0}^{\infty} \mathcal{U}_{a,K'} \left( \langle a_1, \ldots, a_{a-1} \rangle \otimes \langle a'_1, \ldots, a'_{a-1} \rangle \right) \otimes \xi_i^{(a+a')}. 
$$
Proof. Follow carefully the proof of [28, Prop.7]. The proof also works in the present setting.

A direct calculation shows

Lemma 7.5. For $a \leq \kappa$, $a = \sum_{i=1}^{a-1} a_i e_i$, $a' = \sum_{i=1}^{a-1} a'_i e_i$ we have

$$\left( \pi_x \otimes \pi_y \right) \Delta \left( \sum_{i=1}^{a-1} a_i e_i \right) \otimes \left( \sum_{i=1}^{a-1} a'_i e_i \right) \otimes \xi_{lq}^{(a-a')} = \left( \left| a, \ldots, a_{a-1} \right\rangle \otimes \left| a', \ldots, a'_{a-1} \right\rangle \right) \otimes \left( \left| 0 \right\rangle \otimes \left| 1 \right\rangle \right) \otimes \xi_{lq}^{(a-a')}.$$

Lemma 7.6. For any $l \in \mathbb{Z}_{\geq 0}$, $(V_{\otimes q} \otimes V_{\otimes q}) \otimes \xi_{lq}^{(a-k)}$ is generated from $(\left( |0\right\rangle \otimes |0\right\rangle) \otimes \xi_{lq}^{(a-k)}$ over $U_B$.

Proof. We prove $(V_{\otimes q} \otimes V_{\otimes q}) \otimes \xi_{lq}^{(a-k)}$ is generated from $(\left| 0\right\rangle \otimes |0\right\rangle) \otimes \xi_{lq}^{(a-k)}$ by induction on $k$ ($0 \leq k \leq \kappa$). When $k = 0$, there is nothing to prove. Suppose the statement is valid with $k - 1$, that is, $(V_{\otimes q} \otimes V_{\otimes q}) \otimes \xi_{lq}^{(a-k)}$ is generated. Then the following vectors are generated.

By lemma 7.5 with $a = k$, $a = 0$, $a' = e_1 + \cdots + e_{k-1}$ and $l$ replaced with $l + 1$, we can also generate

$$\left( \left| 0\right\rangle \otimes |1\right\rangle \right) \otimes \xi_{lq}^{(a-k)}.$$

Take the coefficients of $\left| 0\right\rangle \otimes |1\right\rangle$, $\left| 0\right\rangle \otimes |0\right\rangle \otimes |1\right\rangle$, $\left| 0\right\rangle \otimes |1\right\rangle$ (written dropping $\otimes \xi_{lq}^{(a-k)}$ from (7.1), (7.2), (7.3) and make a matrix $C = (c_{ij})$ \( j \leq k + 1 \) where

$$c_{ij} = \begin{cases} \frac{1}{q} & (i = j = 1) \\ q & (i = 1 & j = 2) \\ 1 - q & (i = j & 2 \leq i \leq k \text{ or } i = k + 1 & j = 1) \\ -q & (i = j - 1 & 2 \leq i \leq k) \\ ig^{q+1/2} & (i = j = k + 1). \end{cases}$$

Since $det C = ig^{q+1/2} - p^{-1} q^{-k-1} x \neq 0$, $(\left| 0\right\rangle \otimes |1\right\rangle) \otimes \xi_{lq}^{(a-k)}$ is also generated. Now notice that $U_B(\mathbb{Z}_{\geq 0})$ is isomorphic to $U_{-q^{-1}}(B_k)$ with the opposite ordering of Dynkin indices. With this identification, $V_{\otimes q}$ is the spin representation with $|1\rangle$ as a highest weight vector. Hence, $|0\rangle \otimes |1\rangle$ is a tensor product of a lowest weight vector and a highest one with respect to $U_{-q^{-1}}(B_k)$. By proposition 4.2 $(V_{\otimes q} \otimes V_{\otimes q}) \otimes \xi_{lq}^{(a-k)}$ is generated and the induction proceeds.

Proposition 7.7. $\mathcal{W} \otimes \mathcal{W}$ is irreducible.
Proof. Let $W$ be a nonzero submodule of $V^\otimes 2$. By applying $e_i$ ($1 \leq i \leq n$) on a nonzero vector in $W$, we arrive at some singular vector $\xi_l$. By proposition 7.2, one can generate all $\xi_m$ for $m \in \mathbb{Z}_{\geq 0}$. Then lemma 7.6 shows $(V^\otimes 2) \otimes \xi_l \subseteq W$ for any $l$. The claim follows from lemma 7.4.

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