Abstract. This paper defines a faithful action of the virtual braid group $\mathcal{VB}_n$ on certain planar diagrams called virtual curve diagrams. Our action is similar in spirit to the Artin action of the braid group $B_n$ on the free group $F_n$ and it provides an easy combinatorial solution to the word problem in $\mathcal{VB}_n$.

1. Introduction

While we are primarily interested in the virtual braid group $\mathcal{VB}_n$ (good references on virtual knots and braids are [K] and [MI]), we start with a brief review of the ordinary braid group and its “Artin representation”. Our work on $\mathcal{VB}_n$ is then a generalization of the ordinary case.

Recall the braid group $B_n$ on $n$ strands. The standard presentation of $B_n$ has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$. Let $F_n$ be the free group on $n$ symbols $x_1, x_2, \ldots, x_n$. Artin [Ar] studied a homomorphism $\phi$ from $B_n$ to the automorphism group $\text{Aut}(F_n)$ defined by

$$\phi(\sigma_i)(x_j) = \begin{cases} x_j & j \neq i, i+1 \\ x_ix_{i+1}x_i^{-1} & j = i \\ x_i & j = i + 1 \end{cases}$$

If $\beta$ is a braid on $n$ strands then there is a permutation $\pi$ of $\{1, 2, \ldots, n\}$ and an element $U_j \in F_n$ for each $1 \leq j \leq n$ such that $\phi(\beta)(x_j) = U_j x_{\pi(j)} U_j^{-1}$. The elements $U_j$ are uniquely determined up to multiplication on the right by $x_{\pi(j)}^{\pm 1}$. Thus the cosets

$$(U_1 x_{\pi(1)}, U_2 x_{\pi(2)}, \ldots, U_n x_{\pi(n)})$$

determine the map $\phi(\beta)$, and vice versa. The cosets will be referred to as the braid coordinates of $\beta$.

Let $H_n$ be the upper half-plane $\mathbb{R} \times [0, \infty)$ with $n$ punctures lying on some horizontal line above the boundary. Choose a base point on
the boundary. Consider a closed path starting at the base point and winding once clockwise around the $j^{th}$ puncture from the left as in Figure 1. By associating this path to the free group generator $x_j$, the fundamental group of $H_n$ can be identified with $F_n$. The coordinates of a braid can then be depicted as a collection of $n$ curves, each curve connecting the base point to one of the punctures in $H_n$.

For example, Figure 2 shows the curve diagram of the braid $\sigma_2^{-1}\sigma_1\sigma_2 \in B_3$. For the sake of neatness, we have $n$ different base points ordered from left to right. The curve starting at the $i^{th}$ base point from the left represents the $i^{th}$ braid coordinate. The curves are taken up to homotopy relative to the base points and puncture points. One verifies that the diagram in Figure 2 is correct by calculating the action of $\phi(\sigma_2^{-1}\sigma_1\sigma_2)$ on the free group generators $x_1$, $x_2$ and $x_3$. The braid coordinates of $\sigma_2^{-1}\sigma_1\sigma_2$ are

$$(x_1(x_3), x_1x_3^{-1}(x_2), (x_1)),$$

and this agrees with the diagram.

If the braid coordinates of $\beta$ are known, then one can determine the braid coordinates of $\sigma_i^{\pm 1}\beta$ and $\beta\sigma_i^{\pm 1}$. This induces both a left action and a right action of $B_n$ on curve diagrams with $n$ curves. Both actions can be described by a “pushing off” procedure, as in Gaifullin-Manturov [GM]. If $D$ is a curve diagram then the curve diagram $D \cdot \sigma_i^{\pm 1}$ is obtained by stacking $D$ on top of the braid diagram for $\sigma_i^{\pm 1}$ and pushing the upper strand off of the lower strand. This is shown in Figure 3. The curve diagram $\sigma_i^{\pm 1} \cdot D$ is obtained by stacking the curve diagram for $\sigma_i^{\pm 1}$ on top of the curve diagram $D$ and pushing off the
Figure 3. The right action of $B_n$ on curve diagrams.

\[
\sigma_1 \cdot \sigma_1 = \sigma_1 \quad \sim 
\]

Figure 4. The left action of $B_n$ on curve diagrams.

\[
\sigma_2 \cdot \sigma_2 = \sigma_2 \quad \sim 
\]

Figure 5. The right action of $\sigma_2^{-1} \sigma_1 \sigma_2 \in pB_3$ on $I_3$, the trivial curve diagram with three curves.

The stacking is done in such a way that the bottom endpoints of the diagram of $\sigma_i^{\pm 1}$ are extended below the curves of $D$ to connect to the puncture points of $D$. This is shown in Figure 4.

In Figure 5, the pushing off procedure is used to compute the right action of the braid word $\sigma_2^{-1} \sigma_1 \sigma_2$ on $I_3$, the trivial curve diagram with 3 curves. The procedure returns the curve diagram in Figure 2. This is expected, since for any braid $\beta$ we have $\beta \cdot I_n = I_n \cdot \beta$ and both are equal to the curve diagram for that braid. Artin proved that $\phi$ is injective, so a braid is determined by its braid coordinates or equivalently by its curve diagram.
All of this can be repeated, with some modifications, in the setting of virtual braids. Let \( VB_n \) be the virtual braid group on \( n \) strands. This group has a presentation with generators \( \sigma_1, \ldots, \sigma_{n-1} \) and \( \tau_1, \ldots, \tau_{n-1} \). The generators \( \sigma_1, \ldots, \sigma_{n-1} \) satisfy the ordinary braid group relations. The generators \( \tau_1, \ldots, \tau_{n-1} \) generate the symmetric group on \( n \) symbols and satisfy the relations
\[
\tau_i \tau_j = \tau_j \tau_i \quad \text{for} \quad |i - j| > 1 \quad \text{and} \quad \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 2 \quad \text{and} \quad \tau_i^2 = 1 \quad \text{for} \quad 1 \leq i \leq n - 1.
\]
Additionally, there are mixed relations \( \sigma_i \tau_j = \tau_j \sigma_i \) for \( |i - j| > 1 \) and \( \tau_{i+1} \sigma_i \tau_{i+1} = \tau_i \tau_{i+1} \tau_i \) for \( 1 \leq i \leq n - 2 \). Virtual braid words can be represented by virtual braid diagrams. Each generator has a virtual braid diagram as in Figure 6. The diagram of a product \( \alpha \beta \) is obtained by stacking the diagram for \( \alpha \) on top of the diagram for \( \beta \).

We start with the naive extension of the Artin representation to the virtual case. Note that it is weaker than the action we will ultimately adopt, in Section 2. Define a homomorphism \( \psi : VB_n \rightarrow \text{Aut}(F_n) \) by the formula
\[
\psi(\sigma_i) = \phi(\sigma_i) \quad \text{and} \quad \psi(\tau_i)(x_j) = \begin{cases} 
  x_j & j \neq i, i + 1 \\
  x_{i+1} & j = i \\
  x_i & j = i + 1
\end{cases}.
\]
For any virtual braid \( \beta \), there is a permutation \( \pi \) of \( \{1, 2, \ldots, n\} \) and an element \( U_j \in F_0 \) for each \( 1 \leq j \leq n \) such that \( \psi(\beta)(x_j) = U_j x_{\pi(j)} U_j^{-1} \).

The elements \( U_j \) are uniquely determined up to multiplication on the right by \( x_{\pi(j)}^+ \). To any virtual braid \( \beta \in VB_n \) we assign the \( n \)-tuple of free group cosets
\[
(U_1<x_{\pi(1)}>, U_2<x_{\pi(2)}>, \ldots, U_n<x_{\pi(n)}>),
\]
which we will again refer to as the coordinates of \( \beta \). The coordinates of a virtual braid may be represented as curves in a punctured half plane in the same way as the coordinates of an ordinary braid. The curves are taken up to homotopy fixing the base points and the punctures. For example, the coordinates of the two virtual braids \( \tau_2 \sigma_1 \sigma_2 \) and \( \sigma_1 \sigma_2 \tau_1 \) in \( VB_3 \) are
\[
(x_1<x_3>, x_1<x_2>, <x_1>),
\]
and the corresponding curve diagram is shown in Figure 7 (up to a homotopy of the curves). As we will see, these two virtual braids are distinct and $\psi$ is not injective.

In search of a virtual braid invariant that is stronger than $\psi$, we alter the definition of a curve diagram. Recall that the $n$ punctures of $H_n$ lie on a horizontal line above the boundary. We call this line the upper line. The different ways that a homotopy of the curves in a curve diagram can interact with the upper line are shown in Figure 8. By restricting the allowed homotopies to exclude the homotopy labelled $F$, one gets a finer equivalence relation on curve diagrams. These finer equivalence classes of curve diagrams will be called \textbf{virtual curve diagrams} (see Section 2 for a more detailed definition). We denote the set of virtual curve diagrams with $n$ curves by $\mathcal{VCD}_n$.

There are well-defined left and right actions of $\mathcal{VB}_n$ on $\mathcal{VCD}_n$ via the pushing off procedure. Returning to the example of $\tau_2\sigma_1\sigma_2$ and $\sigma_1\sigma_2\tau_1$ in $\mathcal{VB}_3$, Figure 9 shows that the right action of both braids on $I_3$ yields distinct virtual curve diagrams, thanks to the absence of the forbidden homotopy $F$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7}
\caption{The curve diagram of $\tau_2\sigma_1\sigma_2$ and $\sigma_1\sigma_2\tau_1$ in $\mathcal{VB}_3$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8}
\caption{Homotopies of curve diagrams interacting with the upper line.}
\end{figure}
We will show that the map from $\mathcal{VB}_n$ to $\mathcal{VCD}_n$ given by $\beta \mapsto \beta \cdot I_n$ is injective ($I_n$ is the trivial virtual curve diagram with $n$ curves). Virtual curve diagrams are easy to tell apart, hence we solve the word problem for $\mathcal{VB}_n$. We note that in [GP], another solution to the word problem in $\mathcal{VB}_n$ is found.

In Section 2 we introduce virtual curve diagrams, give an example, and define equivalence of virtual curves diagrams. In Section 3 we describe the left action of $\mathcal{VB}_n$ on $\mathcal{VCD}_n$ and show that it is well-defined. In Section 4 we prove that the left action of $\mathcal{VB}_n$ on $\mathcal{VCD}_n$ is faithful. In Section 5 we mention another representation of $\mathcal{VB}_n$ into the automorphisms of a free group, whose faithfulness is still open.

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2. Virtual curve diagrams

In this section we give a formal definition of a virtual curve diagram. Refer to Figure 10 for an example.

Definition 2.1. A virtual curve diagram (or vcd) $D$ with $n$ curves is a tuple $(P, <_P, <_C)$ where:

- $P$ is a finite set whose elements are called points.
- The pair $(P, <_P)$ is a partially ordered set isomorphic to two disjoint chains $U$ and $B$. The elements of $U$ are upper points and the elements of $B$ are base points.
The pair \( (P, C) \) is a partially ordered set that is isomorphic to \( n = |B| \) disjoint chains called curves. Every curve's minimal point is a base point and every curve ends at an upper point, the terminal point. Pairs of points \((a, b)\) where \(b < C\)-covers \(a\) will be called arcs. Note that since terminal points are not base points, every curve contains at least two points.

Assume some curve consists of the points \(a_0 < C a_1 < C \ldots < C a_r\), where \(a_0\) is a base point and \(a_r\) is a terminal point. The arc \((a_0, a_1)\) is the base arc. For \(k \geq 1\), any arc of the form \((a_{2k-1}, a_{2k})\) is an over arc and any arc of the form \((a_{2k}, a_{2k+1})\) is an under arc. The arc \((a_{r-1}, a_r)\) is the terminal arc, and it may happen that the terminal arc is a base arc, or an over arc, or an under arc. If \((x, y)\) and \((z, w)\) are any two over arcs (potentially from different curves) then

\[
\min_P \{ x, y \} \prec_P \min_P \{ z, w \} \prec_P \max_P \{ x, y \} \prec_P \max_P \{ z, w \},
\]

must not occur, as shown in Figure 11. Note that the min and max functions are taken with respect to the \(\prec_P\)-order.

**Remark 2.2.** We draw curve diagrams with a few conventions. The upper and base points lie on two horizontal lines oriented left to right. The upper line is called the upper line and contains the upper points and the lower line is called the base line and contains the base points. The upper line is dotted. The \(\prec_P\)-order agrees with the order of the
The configuration of over arcs \((x, y), (z, w)\) that may not occur in a virtual curve diagram.

points along the oriented lines. The over and under arcs of \(\prec_C\) are represented by arcs connecting the corresponding points. Base arcs will be represented by line segments connecting the base point to the corresponding upper point. Base points and terminal points will be emphasized with big dots. Note that in Definition 2.1, the inequality condition on over arcs is equivalent to the curves not intersecting above the upper line. Figure 10 depicts the curve diagram of Example 2.3.

Example 2.3. Let \(U\) be the set \(\{a_1, b_1, b_2, c_1, c_2, c_3\}\) and \(B\) the set \(\{a_0, b_0, c_0\}\). Let \(a_0 \prec_C a_1\) and \(b_0 \prec_C b_1 \prec_C b_2 \prec C b_3\) and \(c_0 \prec_C c_1 \prec C c_2 \prec C c_3\), and let \(\prec_P\) be given by \(a_0 \prec_P b_0 \prec_P c_0\) and \(c_3 \prec_P b_1 \prec_P a_1 \prec_P b_2 \prec_P c_1 \prec_P b_3 \prec_P c_2\).

The over arcs are \((b_1, b_2)\) and \((c_1, c_2)\) and one can see that the condition on over arcs is satisfied, so this is indeed a virtual curve diagram. This example is shown in Figure 10 following the conventions in Remark 2.2.

Example 2.4. The trivial curve diagram \(I_n\) is the only vcd with \(n\) curves and \(n = |U| = |B|\).

We now define an equivalence relation \(\sim\) on \(VCD_n\). This equivalence permits the moves \(T\) and \(B\) described in the introduction, as well as relabellings of the points of a curve diagram. If \(x\) and \(y\) are points, we say they are adjacent if there are no points strictly between them in the \(\prec_P\) order.

Definition 2.5. Refer to Figure 12. Let \(D = (P, \prec_P, \prec_C)\) and \(E = (Q, \prec_Q, \prec_R)\) be two vcds. Then \(D\) and \(E\) are related by a relabelling of their points if there exists a bijection \(P \to Q\) that induces order isomorphisms \((P, \prec_P) \to (Q, \prec_Q)\) and \((P, \prec_C) \to (Q, \prec_R)\).

We say \(D\) and \(E\) are related by a \(T\)-move, if there exists a curve in \(D\) containing a terminal arc \((a, b)\) where \(a\) and \(b\) are adjacent, and \(Q = P \setminus \{b\}\), and the orders \(\prec_Q\) and \(\prec_R\) are the restrictions of \(\prec_P\) and \(\prec_C\) to \(Q\). In this case, the point \(a\) is a terminal point in \(E\).
We say $D$ and $E$ are related by a $B$-move, if there exists a curve in $D$ containing three $<_{C}$-consecutive arcs $(a, b), (b, c), (c, d)$ where $b$ and $c$ are adjacent, and $Q = P \setminus \{b, c\}$, and the orders $<_{Q}$ and $<_{R}$ are the restrictions of $<_{P}$ and $<_{C}$ to $Q$.

If there exists a sequence $D_1, D_2, \ldots, D_n$ of vcds such that $D_k$ is related to $D_{k+1}$ by a relabelling or a $T$ or $B$-move for each $1 \leq k < n$, then we say $D_1$ is equivalent to $D_n$, and write $D_1 \sim D_n$.

Remark 2.6. Any diagram is equivalent to a unique diagram with a minimal amount of points. Such a diagram has no $T$ or $B$-moves possible which reduce the number of points. If a diagram is the minimal representative of its equivalence class we will say that the diagram is reduced. Figure [13] gives a pair of equivalent vcds, one of which is reduced.

3. THE ACTION OF VIRTUAL BRAIDS ON VIRTUAL CURVE DIAGRAMS

In the introduction we claimed there are left and right actions of $\mathcal{VB}_n$ on $\mathcal{VC}_{D_n}$. From now on we will only be interested in the left action, of which Figure [14] gives two examples. Note that the left action of braid generators, is equivalent to the interpretation of $B_n$ as the mapping class group of $H_n$. In the first row of Figure [14], the second puncture
effectively travelled over and to the right of the third puncture, pulling any curves in the way along with it.

In the next section we will prove that the left action is faithful. To check that the action is well-defined, it is only necessary to check that \( R_1 \cdot D = R_2 \cdot D \) for any \( D \in VCD_n \) and any defining relation \( R_1 = R_2 \) in the presentation for \( V\mathcal{B}_n \).

Figure 14 shows how one would prove this equality for the easiest relation \( \sigma_1 \cdot (\sigma_1^{-1} \cdot D) = D \). We have simplified the situation in that the generic diagram \( D \) could have many more over arcs than depicted in Figure 15, however this fact won’t change the overall proof. Note that the contents of the box labelled \( D \) are irrelevant to the proof as well. In a similar way one would prove the relation \( \tau_1 \cdot (\sigma_2 \cdot (\tau_1 \cdot D)) = \tau_2 \cdot (\sigma_1 \cdot (\tau_2 \cdot D)) \) as in Figure 16. Finally, Figure 17 shows the Reidemeister three relation \( \sigma_1 \cdot (\sigma_2 \cdot (\sigma_1 \cdot D)) = \sigma_2 \cdot (\sigma_1 \cdot (\sigma_2 \cdot D)) \).

**Example 3.1.** In this example we describe the effect a permutation \( \pi \), acting on a reduced vcd, has on an over arc in that vcd. Suppose, as in Figure 18, a reduced vcd \( D \) contains a non-terminal over arc \((a, b)\) (with \( a <_p b \)) enclosing the terminal points \( t_i \) to \( t_l \) (with \( i < l \)). For \( i \leq j < k < l \) assume \( \pi \) maps the closed interval \([i, j]\) to \([\pi(i), \pi(j)]\), \([j + 1, k]\) to \([\pi(j + 1), \pi(k)]\) and \([k + 1, l]\) to \([\pi(k + 1), \pi(l)]\), preserving the order and length of the intervals in each case. Assume also that \( \pi(k) < \pi(i) \) and \( \pi(j) < \pi(k + 1) \). Then the reduced representative of \( \pi \cdot D \) will contain three consecutive over arcs (delimited by under arcs, as in Figure 18), with the first over arc enclosing at least the terminal points \( t_{\pi(i)} \) to \( t_{\pi(j)} \), the second enclosing \( t_{\pi(j + 1)} \) to \( t_{\pi(k)} \) and the last enclosing at least the terminal points \( t_{\pi(k + 1)} \) to \( t_{\pi(l)} \).
Figure 15. The relation $\sigma_1 \cdot (\sigma_1^{-1} \cdot D) = D$ holds.

Figure 16. The relation $\tau_1 \cdot (\sigma_2 \cdot (\tau_1 \cdot D)) = \tau_2 \cdot (\sigma_1 \cdot (\tau_2 \cdot D))$ holds.

Figure 17. The relation $\sigma_1 \cdot (\sigma_2 \cdot (\sigma_1 \cdot D)) = \sigma_2 \cdot (\sigma_1 \cdot (\sigma_2 \cdot D))$ holds.

Consider also the situation where $(a, b) = (a, t_k)$ (with $a < t_k$) is a terminal over arc enclosing the terminal points $t_i$ to $t_{k-1}$. Suppose for $i \leq j < k - 1$ that $\pi$ maps the closed interval $[i, j]$ to $[\pi(i), \pi(j)]$ and $[j + 1, k - 1]$ to $[\pi(j + 1), \pi(k - 1)]$, whilst preserving the order and length of the intervals. Assume also that $\pi(k - 1) < \pi(i)$ and $\pi(j) < \pi(k)$. Then the reduced representative of $\pi \cdot D$ will have an over arc enclosing at least the terminal points $t_{\pi(i)}$ to $t_{\pi(j)}$, followed by an under arc, followed by an over arc enclosing the terminal points $t_{\pi(j+1)}$ to $t_{\pi(k-1)}$, followed by an under arc terminating at $t_{\pi(k)}$ as shown in Figure 19.
In this section we show that $\beta \mapsto \beta \cdot I_n$ is injective. We will define a binary “simplification” relation $\to$ on pairs $(\beta, D)$ where $\beta \in \mathcal{VB}_n$, $D \in \mathcal{VCD}_n$ and $\beta \cdot I_n = D$. We will define a complexity measure $c(D)$ for a diagram $D$ and show that if $(\beta, D) \to (\alpha, E)$ then $c(D) > c(E)$. We will also show that $\to$ satisfies the conditions of the diamond lemma (see Lemma 4.3), and so has a unique minimal element $(1_{\mathcal{VB}_n}, I_n)$. This will imply that if $\beta \cdot I_n = I_n$ then $\beta = 1_{\mathcal{VB}_n}$, or else $(\beta, I_n)$ would be another minimal element with respect to $\to$.

In the above explanation we have simplified the actual situation slightly by ignoring the fact that we will be taking all virtual braids and
virtual curve diagrams “up to” permutations, in a sense to be made precise shortly.

Let $D$ be a reduced virtual curve diagram. If $(a, b)$ is an over or under arc and $c$ is an upper point such that $a <_P c <_P b$ or $b <_P c <_P a$ then the arc $(a, b)$ encloses the point $c$. If $(a, b)$ encloses both points of some arc $(c, d)$ then $(a, b)$ will be said to enclose $(c, d)$.

Assume that the terminal points of the curves in $D$ are $t_1 <_P t_2 <_P \ldots <_P t_n$. If a terminal over arc terminates at the terminal point $t_{i+1}$ and encloses $t_i$ we will say the terminal over arc is of type $(i, i + 1)$ and if it terminates at $t_i$ and encloses $t_{i+1}$ we will say it is of type $(i + 1, i)$. Any terminal over arc is of one of these types for some $i$. This is depicted in Figure 20. Note that Figure 10 in Section 2 depicts a vcd with no terminal over arcs. Note also that in any vcd with terminal over arcs as in Figure 20 there certainly may be arcs between $t_i$ and $t_{i+1}$. They are not depicted in the figure, and in many figures in this section we simply do not draw all the arcs that may be present, only those being referenced.

In our proof of injectivity, we will need a measure of complexity of a virtual curve diagram. For each $i$ let $o_i(D)$ equal the number of over arcs strictly enclosing $t_i$. Let the complexity $c(D)$ equal $\sum_{i=1}^n o_i(D)$. If $D$ is not reduced, define $o_i(D) = o_i(E)$ and $c(D) = c(E)$ where $E$ is the reduced diagram equivalent to $D$.

**Definition 4.1.** Define a binary relation $\rightarrow$ on $\mathcal{VCD}_n$ as follows:

1. If a reduced diagram $D$ contains a terminal over arc of type $(i, i + 1)$ then let $D \rightarrow \sigma_i^{-1} \cdot D$.
2. If a reduced diagram $D$ contains a terminal over arc of type $(i + 1, i)$ then let $D \rightarrow \sigma_i \cdot D$.

**Remark 4.2.** Note that if $D_1 \rightarrow D_2$ then $c(D_1) > c(D_2)$. If $D_1$ had a terminal over arc of type $(i, i + 1)$ (resp. $(i + 1, i)$) then multiplying by $\sigma_i^{-1}$ (resp. $\sigma_i$) would have the effect in Figure 21.

There is a similar binary relation $\rightarrow$ on the set of pairs

$$\mathcal{P} = \{(\beta, D) | \beta \in \mathcal{VB}_n, D \in \mathcal{VCD}_n\},$$
Figure 21. The $\to$ relation reduces complexity.

given by $(\beta, D) \to (\sigma_i^{\pm 1} \beta, \sigma_i^{\pm 1} \cdot D)$ for the two corresponding cases above.

Define an equivalence relation $\leftrightarrow$ on virtual curve diagrams by $D \leftrightarrow E$ if there is a permutation $\pi$ such that $D = \pi \cdot E$. Denote the equivalence class of $D$ with square brackets, $[D]$. Similarly let $[\beta]$ be the left coset $S_n \beta \subset VB_n$, where $S_n$ is the symmetric group generated by the $\tau_i$’s.

Consider the set

$$P = \{ ([\beta], [D]) | \beta \in VB_n, D \in VCD_n \}.$$ 

There is also a binary relation on $P$ given by $(D_1, D_2) \to (D_2, D_2)$ if there exists $\beta_1 \in D_1, D_1 \in D_1, \beta_2 \in D_2, D_2 \in D_2$ such that $(\beta_1, D_1) \to (\beta_2, D_2)$.

If there is a sequence $X \to Y \to \ldots \to Z$ we will write $X \Rightarrow Z$ (for $X, Y, \ldots Z$ in $VCD_n$ or $P'$ or $P$).

Later we will need to apply the diamond lemma to the relation $\to$ on the connected components of $P$. We remind the reader now of the precise statement of the diamond lemma.

**Lemma 4.3** (Diamond lemma). If $\to$ is a connected Noetherian binary relation (Noetherian: an infinite $a_1 \to a_2 \to \cdots$ is ultimately stationary), and if whenever $a \to b$ and $a \to c$ there is $d$ with $b \Rightarrow d$ and $c \Rightarrow d$ where $\Rightarrow$ is the reflexive transitive closure of $\to$, then there is a unique $m$ such that $\forall a, a \Rightarrow m$.

The following proposition will tell us that for any vcd $D$, either $\sigma_i \cdot D \to D$ or $D \to \sigma_i \cdot D$, with a similar statement holding for $\sigma_i^{-1}$. This will be needed when showing that $\{ ([\beta], [D]) \in P | \beta \cdot I_n = D \}$ is connected with respect to $\to$. 
Figure 22. Creating a terminal over arc of type \((i, i+1)\).

Figure 23. A special case in Proposition 4.4.

**Proposition 4.4.** Let \(D\) be a reduced diagram and fix an \(i\) with \(1 \leq i \leq n - 1\). If \(D\) has no terminal over arc of type \((i+1, i)\) then \(\sigma_i \cdot D\) has a terminal over arc of type \((i, i+1)\). Likewise, if \(D\) has no terminal over arc of type \((i, i+1)\) then \(\sigma_i^{-1} \cdot D\) has a terminal over arc of type \((i+1, i)\).

**Proof.** Assume \(D\) does not have a terminal over arc of type \((i+1, i)\). The possible configurations before and after multiplying by \(\sigma_i\) are shown in Figure 22. The right hand side of each equation has a terminal over arc of type \((i, i+1)\) as required.

However, one configuration has a special case which behaves differently than the generic case after multiplying by \(\sigma_i\). It is shown in Figure 23. In the special case, the under arc terminating at \(t_{i+1}\) encloses only the terminal point \(t_i\) and a possibly empty collection of parallel non-crossing under arcs. After multiplying by \(\sigma_i\) the diagram will be as in the “special” column of Figure 23 as opposed to the generic column. The right hand side still has a terminal over arc of type \((i, i+1)\). The proof of the second half of the proposition, for \(\sigma_i^{-1}\), is similar.
Figure 24. The two configurations of a terminal over arc of type \((i, i + 1)\) not left intact by a permutation.

\[\square\]

**Definition 4.5.** Let \(D\) be a reduced diagram. Let \(\pi\) be some permutation. Assume there is a terminal over arc of type \((i, i + 1)\) or \((i + 1, i)\) in \(D\). We say it is left intact by \(\pi\) or that the terminal over arc is intact in \(\pi \cdot D\) if \(\pi(i + 1) = \pi(i) + 1\).

**Remark 4.6.** If \(\pi\) leaves a terminal over arc of type \((i, i + 1)\) (resp. \((i + 1, i)\)) intact, then \(\pi \cdot D\) will contain a terminal over arc of type \((\pi(i), \pi(i) + 1)\) (resp. \((\pi(i) + 1, \pi(i))\)). If a terminal over arc is not left intact, then \(\pi \cdot D\) will contain one of the two configurations in Figure 24 (in the figure, we are depicting the case where the terminal over arc was of type \((i, i + 1)\)). In both configurations, the dotted gap might contain other terminal points. Note that if \(\pi(i + 1) = \pi(i) + 1\) then \(\pi \sigma_{i}^{\pm 1} = \sigma_{\pi(i)}^{\pm 1} \pi\).

The next proposition will tell us that one terminal over arc being intact does not interfere with another terminal over arc being intact. This will be convenient when proving the second part of Theorem 4.8.

**Proposition 4.7.** Let \(D\) be a reduced diagram. Assume \(D\) contains a terminal over arc of type \((i, i + 1)\) and \(\pi \cdot D\) contains a terminal over arc of type \((j, j + 1)\) (or type \((j + 1, j)\)), where \(\pi\) is some permutation. Then there is a permutation \(\gamma\) such that both terminal over arcs are intact in \(\gamma \cdot D\).

**Proof.** Note that our goal is to construct a permutation \(\gamma\) such that \(\gamma(i + 1) = \gamma(i) + 1\) and \(\gamma \pi^{-1}(j + 1) = \gamma \pi^{-1}(j) + 1\). Combinatorially speaking, this is not always possible. For example, if \(\pi(i + 1) \neq \pi(i) + 1\) and \(\pi(i) = j\) there can be no such \(\gamma\). We shall see, however, that that cannot occur in a vcd.

If \(\pi\) leaves \((i, i + 1)\) intact then we take \(\gamma = \pi\). Assume \(\pi\) does not leave \((i, i + 1)\) intact. First we assume the terminal over arc in \(\pi \cdot D\) is of type \((j, j + 1)\). Since \(\pi\) does not leave \((i, i + 1)\) intact, \(\pi \cdot D\) contains one of the two configurations of Figure 24. Assume it contains the first configuration. Then \(\pi(i + 1) > \pi(i) + 1\). If \(j \neq \pi(i), \pi(i) + 1, \pi(i + 1)\)
then let $\gamma$ be the composition $t\pi$ where $t$ is the transposition $(\pi(i + 1)\pi(i) + 1)$. Then $\gamma(i + 1) = t\pi(i + 1) = \pi(i) + 1 = \gamma(i) + 1$ so $\gamma$ leaves the terminal over arc of type $(i, i + 1)$ in $D$, intact in $\gamma \cdot D$. Now $j + 1$ cannot equal $\pi(i + 1)$ (since there is already an under arc terminating at $t\pi(i + 1)$ in $\pi \cdot D$). Thus $\{\pi(i) + 1, \pi(i + 1)\} \cap \{j, j + 1\} = \emptyset$ and so $t$ fixes both $j$ and $j + 1$ and so the terminal over arc of type $(j, j + 1)$ in $\pi \cdot D$ is intact in $\gamma \cdot D$. We consider now separately each of the cases $j = \pi(i), \pi(i) + 1, \pi(i + 1)$.

Consider now the case $j = \pi(i)$ (this is one case that is impossible). Then clearly $j + 1 = \pi(i) + 1$ and the configuration present in $\pi \cdot D$ looks as in the left side of Figure 25. However after applying $\pi^{-1}$ we will get the right side of Figure 25 which cannot have a terminal over arc of type $(i, i + 1)$, a contradiction. Thus $j = \pi(i)$ is impossible.

Consider the case $j = \pi(i) + 1$. Then since $j \leq \pi(i + 1)$ and there is already a terminal under arc terminating at $t\pi(i + 1)$, it follows that $j + 1 < \pi(i + 1)$. The configuration is depicted in Figure 26. Let $t$ be a map that takes $\pi(i + 1)$ to $\pi(i) + 1$ and the pair $(\pi(i) + 1, \pi(i) + 2)$ to some pair $(k, k + 1)$ disjoint from $(\pi(i), \pi(i) + 1)$ and let $\gamma = t\pi$. Then both terminal over arcs are intact in $\gamma \cdot D$.

Consider the case $j = \pi(i + 1)$. Then $\pi \cdot D$ contains a configuration as in the left side of Figure 27. By applying the transposition $(\pi(i + 1)\pi(i) + 1)$ followed by $(\pi(i + 1) + 1 \pi(i) + 2)$ we get a diagram in which both terminal over arcs are intact and as in the right side of Figure 27.
Figure 27. The case $j = \pi(i + 1)$ in Proposition 4.7.

Similar arguments can be made when $(i, i + 1)$ is not intact in $\pi \cdot D$ and it is in the second configuration of Figure 24. As well similar arguments can be made when the terminal over arc is of type $(j + 1, j)$ instead of $(j, j + 1)$.

This theorem demonstrates that Lemma 4.3 is applicable to $\rightarrow$ on $P$.

**Theorem 4.8.** The relation $\rightarrow$ on $P$ satisfies the following properties:

1. There is no infinite sequence $(d_1, D_1) \rightarrow (d_2, D_2) \rightarrow (d_3, D_3) \rightarrow \ldots$

2. If $(d, D) \rightarrow (e, E)$ and $(d, D) \rightarrow (f, F)$ then there is a pair $(g, G)$ such that $(e, E) \Rightarrow (g, G)$ and $(f, F) \Rightarrow (g, G)$.

3. The set $\{(\beta, [D]) \in P | \beta \cdot \Pi_n = D\} \subset P$ is a connected component of $\rightarrow$.

**Proof.** The first part follows from Remark 4.2 if $D_1 \rightarrow D_2$ then $c(D_1) > c(D_2)$.

For the second part let $(d, D) \rightarrow (e, E)$ and $(d, D) \rightarrow (f, F)$. Assume that $\beta \in d, D \in D$, and $\pi$ a permutation such that $D$ has a terminal over arc of type $(i, i + 1)$ and $\pi \cdot D$ has a terminal over arc of type $(j, j + 1)$. Let $E = \sigma_{i+1}^{-1} \cdot D$ and $F = \sigma_{j+1}^{-1} \cdot (\pi \cdot D)$.

By Proposition 4.7, there is a permutation $\gamma$ such that the terminal over arc of type $(i, i + 1)$ in $D$ and the terminal over arc of type $(j, j + 1)$ in $\pi \cdot D$ are intact in $\gamma \cdot D$. These two terminal over arcs will appear in one of the configurations in Figure 28.

Assume it is in the first configuration of the first row. Then $\gamma(i + 1) = \gamma(i) + 1$ and $\pi^{-1}(j) = i + 1$ and $\gamma \pi^{-1}(j + 1) = \gamma(i) + 2$. The squares of the commutative diagram in Figure 29 are commutative due to the last statement in Remark 4.6. Thus we are lead to a common vcd $G$. Similar arguments work for the other configurations in Figure 28. As well, similar arguments work if the terminal over arc in $\pi \cdot D$ is of type $(j + 1, j)$ instead of $(j, j + 1)$. In all these cases we have a commutative diagram as in Figure 29.
Finally we show that \{([\beta], [D]) \in \mathcal{P}|\beta \cdot I_n = D\} is a connected component of $\rightarrow$. By Proposition 4.4 if $D$ does not have a terminal over arc of type $(i+1,i)$ then $\sigma_i \cdot D$ does so in this case $\sigma_i \cdot D \rightarrow D$. If $D$ does have a terminal over arc of type $(i+1,i)$ then $D \rightarrow \sigma_i \cdot D$. Thus for any $D$ either $D \rightarrow \sigma_i \cdot D$ or $\sigma_i \cdot D \rightarrow D$ (with a similar statement holding for $\sigma_i^{-1}$).

Now any $\hat{D} \in \mathcal{VB}_n \cdot I_n$ is of the form $\beta \cdot I_n$ for some virtual braid word $\beta$ so there is a sequence of diagrams

$$I_n = D_1, D_2, \ldots, D_r = D,$$

such that for all $k \geq 1$ either $D_k \leftrightarrow D_{k+1}$ or $D_k \rightarrow D_{k+1}$ or $D_{k+1} \rightarrow D_k$. Thus \{([\beta], [D]) \in \mathcal{P}|\beta \cdot I_n = D\} is a connected component of $\rightarrow$. \qed
We can now prove injectivity.

**Theorem 4.9.** Let $\beta \in \mathcal{VB}_n$ such that $\beta \cdot I_n = I_n$. Then $\beta = 1_{\mathcal{VB}_n}$.

**Proof.** By Lemma 4.3 and Theorem 4.8, there is a unique minimal element in $\mathcal{P}$ with respect to $\rightarrow$. Clearly that element is $([1_{\mathcal{VB}_n}], [I_n])$.

If $\beta$ was not trivial, it certainly wouldn’t be a permutation since $\pi \cdot I_n \neq I_n$ unless $\pi = 1_{\mathcal{VB}_n}$. Thus $([\beta], [I_n])$ would be a minimal element in $\mathcal{P}$ distinct from $([1_{\mathcal{VB}_n}], [I_n])$, a contradiction. $\square$

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**5. Future work**

In the introduction we gave an example of a non-injective map $\phi : \mathcal{VB}_n \to \text{Aut}(F_n)$ by extending Artin’s $\phi : \mathcal{B}_n \to \text{Aut}(F_n)$ to the generators $\tau_i \in \mathcal{VB}_n$. There is another extension $\psi : \mathcal{VB}_n \to \text{Aut}(F_{n+1})$, in [B], into the automorphism group of the free group on generators $x_1, x_2, \ldots, x_n, q$ satisfying

\[
\psi(\sigma_i)(x_j) = \begin{cases} 
  x_j & j \neq i, i + 1 \\
  x_i x_{i+1} x_i^{-1} & j = i \\
  x_i & j = i + 1
\end{cases},
\]

and

\[
\psi(\tau_i)(x_j) = \begin{cases} 
  x_j & j \neq i, i + 1 \\
  qx_{i+1} q^{-1} & j = i \\
  q^{-1} x_i q & j = i + 1
\end{cases},
\]

and $\psi(\sigma_i)(q) = \psi(\tau_i)(q) = q$.

Do our methods shed any light on the injectivity of this map?

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