GENERALIZED COSINE TRANSFORMS AND CLASSES OF STAR BODIES

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Abstract. The spherical Radon transform on the unit sphere in \( \mathbb{R}^n \) can be regarded as a member of the analytic family of suitably normalized generalized cosine transforms. We derive new formulas for these transforms and apply them to study classes of intersections bodies in convex geometry. In particular, we show that some known classes of intersection bodies are subclasses of a more general class \( K_{\alpha,n} \) of origin-symmetric star bodies in \( \mathbb{R}^n \) that can be defined and characterized in terms of the generalized cosine transforms.

1. Introduction

This article has two sources. The first one is the theory of the spherical Radon transforms, that amounts to classical works by H. Minkowski, P. Funk, and S. Helgason [He]. The simplest example is the Minkowski-Funk transform, which integrates functions on the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \) over great circles of codimension 1. The spherical Radon transform can be regarded as a member of the analytic family of suitably normalized generalized cosine transforms. The latter, without naming, first appeared in the paper by V.I. Semyanisty [Se] (codimension 1) and extended by the author [R3] (codimension \( \geq 1 \)).

The analytic family of generalized cosine transforms (usually, without naming) has proved to be important in PDE, harmonic analysis, and other areas; see [Es], [Pl], [R1], [R2], [Sa1], [Sa2], [Str], and references therein. Higher rank modifications of cosine transforms were considered in [Al], [AB], [GH1], [GH2], [OR]. We note that the name “spherical” or “circular” Radon transform is attributed in some publications to Radon-like transforms of different type (see, e.g., [Al], [AK], [Q]).

Another source of our article is convex geometry, related problems in probability, stochastic geometry, and Banach space theory; see [BL], [C], [GZ], [K], [Schn], and references therein. Here the name cosine

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transform was adopted for the integral operator

\[(Cf)(\theta) = \int_{S^{n-1}} f(u)|\theta \cdot u|\,du\]

according to pioneering works by W. Blaschke, A.D. Alexandrov, and P. Lévy. A more general \(p\)-cosine transform

\[(C_pf)(\theta) = \int_{S^{n-1}} f(u)|\theta \cdot u|^p\,du, \quad \theta \in S^{n-1}, \quad p \in \mathbb{R}, \quad p > 0,
\]

is also commonly in use and reflects a number of important geometric concepts.

In fact, both sources are intimately connected, and analytic families associated to the spherical Radon transform include \(p\)-cosine transforms up to normalization.

In the present paper, we continue our study of the generalized cosine transforms started in [R1]–[R4], keeping in mind applications to convex geometry. Section 2 contains preliminaries. In Section 3, we derive new formulas, which reveal interrelation of different analytic families of intertwining operators on the sphere. One of such formulas is a factorization of the Minkowski-Funk transform as a product of mutually orthogonal spherical Radon transforms of codimension greater than 1; see Theorem 3.7. Section 4 deals with applications. Using results of Section 3, we give alternative proof to some known facts in convex geometry with the main focus on classes of intersection bodies. We show that some of these classes, studied separately in a series of publications, are, in fact, subclasses of a certain more general class \(K_{\alpha,n}\) of origin-symmetric star bodies that can be characterized in terms of the generalized cosine transforms.

Our approach can also be applied to a series of problems related to projection bodies, \(p\)-centroid bodies, and their polars; see [YY] and [K4] regarding this circle of problems and further references.

One should also mention important works of J. Bourgain, S. Campi, R. Gardner, P. Goodey, E. Grinberg, H. Groemer, S. Helgason, A. Koldobsky, E. Lutwak, R. Schneider, R. Strichartz, W. Weil, G. Zhang, and many others, containing substantial contribution to harmonic analysis on the sphere in the context of its application to integral geometry. Our list of references is far from being complete and gives only key directions. Our interest to this research was stimulated in part by the Busemann-Petty type problems [BZ], [K4], [RZ], which reveal a remarkable interplay between harmonic analysis, convex geometry, and Radon transforms.
2. Preliminaries

2.1. Harmonic analysis on the sphere. The main references are [Mü], [Ne], [SW], and a survey article [Sa2]. We use the following notation: $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $C(S^{n-1})$ the spaces of continuous functions on $S^{n-1}$, $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ the area of $S^{n-1}$. For $\theta \in S^{n-1}$, $d\theta$ denotes the normalized induced Lebesgue measure on $S^{n-1}$ and $d(\cdot, \cdot)$ stands for the geodesic distance on $S^{n-1}$. We denote by $e_1, e_2, \ldots, e_n$ the coordinate unit vectors; $SO(n)$ is the special orthogonal group of $\mathbb{R}^n$; $SO(n-1)$ is the subgroup of $SO(n)$ preserving $e_n$. For $\gamma \in SO(n)$, we denote by $d\gamma$ the normalized $SO(n)$-invariant measure on $SO(n)$ with total mass 1. We use the notation $\mathcal{D} = \mathcal{D}(S^{n-1})$ for the space of infinitely differentiable test functions on $S^{n-1}$ equipped with the standard topology, and denote by $\mathcal{D}' = \mathcal{D}'(S^{n-1})$ the corresponding dual space of distributions. The subspace of even test functions (distributions) is denoted by $\mathcal{D}_e$ ($\mathcal{D}'_e$). The notation $\mathcal{M}(S^{n-1})$ is adopted for the space of finite Borel measures on $S^{n-1}$. If $i$ is an integer, $1 \leq i \leq n-1$, then $G_{n,i}$ denotes the Grassmann manifold of $i$-dimensional linear subspaces $\xi$ of $\mathbb{R}^n$; $d\xi$ stands for the normalized $SO(n)$-invariant measure on $G_{n,i}$; $\mathcal{D}(G_{n,i})$ is the space of infinitely differentiable functions on $G_{n,i}$.

Let $\{Y_{j,k}(\theta)\}$ be an orthonormal basis of spherical harmonics on $S^{n-1}$. Here $j = 0, 1, 2, \ldots$, and $k = 1, 2, \ldots, d_n(j)$ where $d_n(j)$ is the dimension of the subspace of spherical harmonics of degree $j$. Each test function $\omega \in \mathcal{D}$ admits a decomposition $\omega(\theta) = \sum_{j,k} \omega_{j,k} Y_{j,k}(\theta)$ with the Fourier-Laplace coefficients $\omega_{j,k} = \int_{S^{n-1}} \omega(\theta) Y_{j,k}(\theta) d\theta$, which decay rapidly as $j \to \infty$. Each distribution $f \in \mathcal{D}'$ can be defined by $(f, \omega) = \sum_{j,k} f_{j,k} \omega_{j,k}$ where $f_{j,k} = (f, Y_{j,k})$ grow not faster than $j^m$ for some integer $m$.

The Poisson integral of a function $f \in L^1(S^{n-1})$ is defined by

\[
(\Pi_t f)(\theta) = (1 - t^2) \int_{S^{n-1}} f(u)[\theta - tu]^{-n}du, \quad 0 < t < 1,
\]

with the Fourier-Laplace decomposition $\Pi_t f = \sum_{j,k} t^j f_{j,k} Y_{j,k}$. For $f \in \mathcal{D}'$, this decomposition serves as a definition of $\Pi_t f$. The space $\mathcal{D}(\mathcal{D}_e)$ is dense in $\mathcal{D}'(\mathcal{D}_e)$ because each distribution $f$ can be approximated in the weak sense by its Poisson integral $\Pi_t f$, when $t \to 1$.

A distribution $f \in \mathcal{D}'$ is nonnegative if $(f, \omega) \geq 0$ for every nonnegative test function $\omega$. Given a certain space $A(X)$, consisting of functions, measures, or distributions on $X$, we denote by $A_+(X)$ the relevant subspace of all nonnegative elements of $A(X)$.
The following statement is a spherical analog of the well-known fact for distributions on $\mathbb{R}^n$ [Schw]. For the sake of completeness, we present it with proof.

**Theorem 2.1.** A distribution $f \in \mathcal{D}'(S^{n-1})$ is nonnegative if and only if it is a nonnegative finite measure on $S^{n-1}$, i.e., $f \in \mathcal{M}_+(S^{n-1})$.

**Proof.** The “if” part is obvious. The proof of the “only if” part relies on the following

**Proposition.** A distribution $f \in \mathcal{D}'$ is a finite measure on $S^{n-1}$ if and only if the order of $f$ equals 0, i.e.,

$$(2.2) \quad |(f, \omega)| \leq c ||\omega||_{C(S^{n-1})} \quad \forall \omega \in \mathcal{D}.$$ 

**Proof of the Proposition.** The “only if” part is obvious. Conversely, let $(2.2)$ hold. Since $D$ is dense in $C(S^{n-1})$, then $f$ extends as a linear continuous functional on $C(S^{n-1})$. By the Riesz theorem, there is a measure $\mu$ on $S^{n-1}$ such that $(f, \omega) = \int_{S^{n-1}} \omega(\theta) d\mu(\theta)$ for every $\omega \in C(S^{n-1})$. This gives the statement.

Now we conclude the proof of the theorem. For $\omega \in \mathcal{D}$,

$$-(||\omega||_{C(S^{n-1})}) \leq \omega \leq ||\omega||_{C(S^{n-1})}.$$ 

Hence, if $f \in \mathcal{D}'$ is nonnegative, then

$$-(f, 1) ||\omega||_{C(S^{n-1})} \leq (f, \omega) \leq (f, 1) ||\omega||_{C(S^{n-1})},$$

i.e., $|(f, \omega)| \leq (f, 1) ||\omega||_{C(S^{n-1})}$ for every $\omega \in \mathcal{D}$. This means that $f$ has order 0 and, by Proposition, $f$ is a (nonnegative) finite measure.  

2.2. **Spherical Radon transforms.** For continuous functions $f(\theta)$ on $S^{n-1}$ and $\varphi(\xi)$ on $G_{n,i}$, the totally geodesic Radon transform $R_i f$ and its dual $R_i^* \varphi$ are defined by

$$(2.3) \quad (R_i f)(\xi) = \int_{S^{n-1} \cap \xi} f(\theta) d_\xi \theta, \quad (R_i^* \varphi)(\theta) = \int_{\xi \ni \theta} \varphi(\xi) d_\theta \xi,$$

where $d_\xi \theta$ and $d_\theta \xi$ denote the normalized induced measures on the corresponding manifolds $S^{n-1} \cap \xi$ and $\{\xi \in G_{n,i}: \xi \ni \theta\}$; see [He], [R3]. The precise meaning of the second integral is

$$\int_{SO(n-1)} \varphi(r_\theta e_{n}) d\gamma,$$

where $p_0 = \mathbb{R} e_{n-i+1} + \ldots + \mathbb{R} e_n$ is the coordinate $i$-plane and $r_\theta \in SO(n)$ is a rotation satisfying $r_\theta e_n = \theta$. The corresponding duality relation
has the form

\[(2.5) \quad \int_{G_{n,i}} (R_i f)(\xi) \varphi(\xi) d\xi = \int_{S^{n-1}} f(\theta)(R_i^* \varphi)(\theta) d\theta.\]

It is applicable when the integral in either side is finite for \(f\) and \(\varphi\) replaced by \(|f|\) and \(|\varphi|\), respectively.

The Radon transform \(R_i\) and its dual extend as linear bounded operators from \(L^1(S^{n-1})\) to \(L^1(G_{n,i})\) and from \(L^1(G_{n,i})\) to \(L^1(S^{n-1})\), respectively. For finite Borel measures \(\mu\) on \(S^{n-1}\) and \(\nu\) on \(G_{n,i}\), owing to \(2.5\), we define \(R_i \mu \in \mathcal{M}(G_{n,i})\) and \(R_i^* \nu \in \mathcal{M}(S^{n-1})\) by the following equalities:

\[(2.6) \quad \int_{G_{n,k}} (R_i \mu)(\xi) \varphi(\xi) d\xi = \int_{S^{n-1}} (R_i^* \varphi)(\theta) d\mu(\theta), \quad \varphi \in C(G_{n,i});\]

\[(2.7) \quad \int_{S^{n-1}} (R_i^* \nu)(\theta) f(\theta) d\theta = \int_{G_{n,i}} (R_i f)(\xi) d\nu(\xi), \quad f \in C(S^{n-1}).\]

We also write \(2.5\), \(2.6\), and \(2.7\) briefly as

\((R_i f, \varphi) = (f, R_i^* \varphi), \quad (R_i \mu, \varphi) = (\mu, R_i^* \varphi), \quad (R_i^* \nu, f) = (\nu, R_i f).\]

If \(i = n - 1\), \(u \in S^{n-1}\), and \(\xi = u^\perp \in G_{n,n-1}\), it is convenient to use another notation \((R_{n-1} f)(u^\perp) = (M f)(u)\) where

\[(2.8) \quad (M f)(u) = \int_{\{\theta: \theta \cdot u = 0\}} f(\theta) d_u \theta, \quad u \in S^{n-1},\]

is the Minkowski-Funk transform of \(f\). Here \(d_u \theta\) denotes the corresponding normalized measure.

### 3. Analytic families

**3.1. Definitions and basic properties.** We start by reviewing some facts from [R3, R4]. Given a subspace \(\xi \in G_{n,i}\), we denote by \(Pr_{\xi \perp} \theta\) the orthogonal projection of \(\theta \in S^{n-1}\) onto \(\xi^\perp\), the orthogonal complement of \(\xi\). Then \(|Pr_{\xi \perp} \theta| = \sin[d(x,S^{n-1} \cap \xi)]\) is the length of \(Pr_{\xi \perp} \theta\). We consider analytic families of intertwining operators defined for \(f \in L^1(S^{n-1})\) and \(\varphi \in L^1(G_{n,i})\) by

\[(3.1) \quad (R^\alpha f)(\xi) = \gamma_{n,i}(\alpha) \int_{S^{n-1}} |Pr_{\xi \perp} \theta|^{\alpha + i - n} f(\theta) d\theta,\]
\[
(R^\alpha \varphi)(\theta) = \gamma_{n,i}(\alpha) \int_{G_{n,i}} |Pr_{\xi}^+ \theta|^{\alpha+i-n} \varphi(\xi) \, d\xi,
\]
\[
\gamma_{n,i}(\alpha) = \frac{\sigma_{n-1} \Gamma((n-\alpha-i)/2)}{2\pi^{(n-1)/2} \Gamma(\alpha/2)}, \quad \text{Re} \alpha > 0, \quad \alpha+i-n \neq 0, 2, 4, \ldots.
\]

For \(i = n-1\), we write (3.1) as
\[
(M^\alpha f) (u) = \gamma_n(\alpha) \int_{S^{n-1}} f(\theta) |\theta \cdot u|^{\alpha-1} \, d\theta,
\]
\[
\gamma_n(\alpha) = \frac{\sigma_{n-1} \Gamma((1-\alpha)/2)}{2\pi^{(n-1)/2} \Gamma(\alpha/2)}, \quad \text{Re} \alpha > 0, \quad \alpha \neq 1, 3, 5, \ldots.
\]

Operators (3.1) and (3.2) were introduced in [R3] as generalizations of (3.3). The latter was introduced by Semyanistyi [Se] and studied in numerous publications; see [R4], [Sa1], [Sa2], and references therein. All these operators are intimately related to the Radon transform (2.3) and (2.8). Namely, if \(f\) and \(\varphi\) are continuous functions, then (3.4)
\[
\lim_{\alpha \to 0} R_i^\alpha f = R_i^0 f = c_i R_i f, \quad c_i = \frac{\sigma_{i-1}}{2\pi^{(i-1)/2}};
\]
\[
\lim_{\alpha \to 0} \hat{R}_i^\alpha \varphi = \hat{R}_i^0 \varphi = c_i \hat{R}_i^\varphi,
\]
\[
\lim_{\alpha \to 0} M_i^\alpha f = M_i^0 f = c_{n-1} M_i f, \quad c_{n-1} = \frac{\sigma_{n-2}}{2\pi^{(n-2)/2}}.
\]

This means that the Radon transform, its dual, and the Minkowski-Funk transform can be regarded (up to a constant multiple) as members of the corresponding analytic families \(\{R_i^\alpha\}, \{\hat{R}_i^\alpha\}, \{M_i^\alpha\}\).

Integrals (3.1) - (3.3) are absolutely convergent if \(\text{Re} \alpha > 0\) for any integrable functions \(f\) and \(\varphi\). When \(f\) and \(\varphi\) are infinitely differentiable, these integrals extend to all \(\alpha \in \mathbb{C}\) as meromorphic functions of \(\alpha\). For (3.3), this extension can be realized in terms of spherical harmonic decomposition. Namely (see, e.g., [R1], [R4]), if \(f \in \mathcal{D}(S^{n-1})\), then
\[
M^\alpha f = \sum_{j,k} m_{j,\alpha} f_{j,k} Y_{j,k}
\]
where \(f_{j,k} = \int_{S^{n-1}} f(\theta) Y_{j,k}(\theta) \, d\theta\),
\[
m_{j,\alpha} = \begin{cases} (-1)^{j/2} \frac{\Gamma(j/2+(1-\alpha)/2)}{\Gamma(j/2+(n-1+\alpha)/2)} & \text{if } j \text{ is even}, \\ 0 & \text{if } j \text{ is odd}. \end{cases}
\]
If \( f \in \mathcal{D}' \), then, \( M^\alpha f \) is a distribution defined by
\[
(M^\alpha f, \omega) = (f, M^\alpha \omega) = \sum_{j,k} m_{j,\alpha} f_{j,k} \omega_{j,k}, \quad \omega \in \mathcal{D}; \quad \alpha \neq 1, 3, 5, \ldots.
\]

**Remark 3.1.** The normalization in (3.3) is motivated by the asymptotic relation
\[
m_{j,\alpha} \sim (-1)^{j/2} (j/2)^{-(\alpha+n/2-1)}, \quad j \to \infty, \quad j \text{ even},
\]
according to which \( M^\alpha \) is a smoothing operator of order \( \Re \alpha + n/2 - 1 \). It means that in many aspects, \( M^\alpha \) acts as an integral operator if \( \Re \alpha \geq 1 - n/2 \) (even outside of the domain of absolute convergence) and as a differential operator otherwise. Action of \( M^\alpha \) in different scales of function spaces (Hölder spaces, \( L^p \)-spaces, Sobolev spaces) was studied in [Str1], [Str2], [Sa1], [R2, Section 2]. In numerous publications related to integral geometry, operators (3.3) with \( \alpha - 1 \) replaced by \( p \) are called the (normalized) \( p \)-cosine transforms (cf. (1.1)) in view of close connection with isometric embeddings of normed spaces (\( \mathbb{R}^n, ||\cdot|| \)) into \( L^p \)-spaces.

The following obvious consequence of (3.8) was widely used in diverse publications related to the analytic family \( \{M^\alpha\} \); see [R4].

**Lemma 3.2.** Let \( \alpha, \beta \in \mathbb{C}; \alpha, \beta \neq 1, 3, 5, \ldots \). If \( \alpha + \beta = 2 - n \), then
\[
M^\alpha M^\beta = I \quad \text{(the identity operator)}.
\]

If \( \alpha, 2-n-\alpha \neq 1, 3, 5, \ldots \), then \( M^\alpha \) is an authomorphism of the spaces \( \mathcal{D}_e(S^{n-1}) \) and \( \mathcal{D}'_e(S^{n-1}) \).

**Proof.** The equality \( M^\alpha M^\beta = I \) is equivalent to \( m_{j,\alpha} m_{j,\beta} = 1, \quad \alpha + \beta = 2 - n \). The latter immediately follows from (3.8). The second statement is a consequence of the standard theory of spherical harmonics [Ne], because the Fourier-Laplace multiplier \( m_{j,\alpha} \) has a power behavior as \( j \to \infty \). \( \square \)

**Corollary 3.3.** The Minkowski-Funk transform \( M \) on the space \( \mathcal{D}_e(S^{n-1}) \) can be inverted by the formula
\[
(M)^{-1} = c_{n-1} M^{2-n}, \quad c_{n-1} = \frac{\sigma_{n-2}}{2\pi^{(n-2)/2}}.
\]

Note that there is a wide variety of diverse inversion formulas for the Minkowski-Funk transform (see [He], [R4] and references therein), but all of them are, in fact, different realizations of (3.11), depending on classes of functions.
Lemma 3.4. Let $\alpha, \beta \in \mathbb{C}$; $\alpha, \beta \neq 1, 3, 5, \ldots$. If $Re \alpha > Re \beta$, then $M^\alpha = M^\beta A_{\alpha, \beta}$, where $A_{\alpha, \beta}$ is a smoothing operator of order $Re (\alpha - \beta)$ with the Fourier-Laplace multiplier

$$a_{\alpha, \beta}(j) = \frac{\Gamma(j/2 + (1 - \alpha)/2)}{\Gamma(j/2 + (n - 1 + \alpha)/2)} \frac{\Gamma(j/2 + (n - 1 + \beta)/2)}{\Gamma(j/2 + (1 - \beta)/2)},$$

so that $a_{\alpha, \beta}(j) \sim (j/2)^{\beta-\alpha}$ as $j \to \infty$. If $\alpha$ and $\beta$ are real numbers satisfying $\alpha > \beta > 1 - n$, $\alpha + \beta < 2$, then $A_{\alpha, \beta}$ is an integral operator with the property $A_{\alpha, \beta} f \geq 0$ for any nonnegative $f \in L^1(S^{n-1})$.

Proof. The first statement follows from (3.8). To prove the second one, we consider integral operators

$$Q^\mu_+(f)(x) = \frac{2}{\Gamma(\mu/2)} \int_0^1 (1 - t^2)^{\mu/2-1} (\Pi_t f)(x) t^{n-\nu} dt,$$

$$Q^\mu_-(f)(x) = \frac{2}{\Gamma(\mu/2)} \int_1^\infty (t^2 - 1)^{\mu/2-1} (\Pi_1/t f)(x) t^{1-\nu} dt,$$

containing the Poisson integral (2.1). The Fourier-Laplace multipliers of $Q^\mu_+$ and $Q^\mu_-$ are

$$\hat{q}^\mu_+(j) = \frac{\Gamma((j+n-\nu+1)/2)}{\Gamma((j+n-\nu+1+\mu)/2)}, \quad \hat{q}^\mu_-(j) = \frac{\Gamma((j+\nu-\mu)/2)}{\Gamma((j+\nu)/2)}.$$

They can be easily calculated by taking into account that $\Pi_t \sim t^j$ in the Fourier-Laplace terms [SW, p. 145]. Integrals (3.13) and (3.14) are absolutely convergent for $f \in L^1(S^{n-1})$ provided $0 < Re \mu < Re \nu < n$ and preserve positivity of $f$. Comparing (3.15) and (3.12), we obtain a factorization $A_{\alpha, \beta} = Q^\alpha_+ - \beta, 1-\beta) Q^\alpha_- - \beta, 1-\beta)$ (set $\mu = \alpha - \beta$, $\nu = 1 - \beta$), which implies the second statement of the lemma. \hfill \Box

The next analytic family we deal with is the family of the generalized sine transforms

$$Q^\alpha f(\theta) = \frac{\sigma_{n-1} \Gamma((n-1-\alpha)/2)}{2\pi^{(n-1)/2} \Gamma(\alpha/2)} \int_{S^{n-1}} (1 - |u \cdot \theta|^2)^{(\alpha-n+1)/2} f(u) du,$$

$$= \frac{\sigma_{n-1} \Gamma((n-1-\alpha)/2)}{2\pi^{(n-1)/2} \Gamma(\alpha/2)} \int_{S^{n-1}} |\sin[d(u, \theta)]|^{\alpha-n+1} f(u) du,$$

where $Re \alpha > 0$, $\alpha - n \neq 0, 2, 4, \ldots$. Operators $Q^\alpha$ serve as analogues of Riesz potentials in the theory of spherical Radon transforms [He, Str2]. Detailed investigation of operators (3.16), including inversion formulas, can be found in [R3]. The Fourier-Laplace multiplier of $Q^\alpha$
has the form

\[(3.17) \quad \hat{q}_\alpha(j) = \frac{\Gamma\left(\frac{j+n-1-\alpha}{2}\right) \Gamma\left(\frac{j+1}{2}\right)}{\Gamma\left(\frac{j+\alpha+1}{2}\right) \Gamma\left(\frac{j+n-1}{2}\right)} \quad (\sim (j/2)^{-\alpha}, \quad j \to \infty)\]

for \(j\) even, and \(\hat{q}_\alpha(j) = 0\) for \(j\) odd. For \(Re \alpha \leq 0\), \(Q^\alpha f\) is defined by analytic continuation. The following equalities play an important role in the theory of Radon transforms on \(S^{n-1}\). If \(f \in D_e(S^{n-1})\) and \(\alpha \in \mathbb{C}, \quad \alpha + i - n \neq 1, 3, 5, \ldots\), then

\[(3.18) \quad R^\alpha_i R_i f = \lambda_1 Q^{\alpha+i-1} f, \quad R^\alpha_i R_i f = \lambda_2 Q^{\alpha+i-1} f,\]

\(\lambda_1 = \frac{\Gamma((n-1)/2)}{\sigma_{n-1} \Gamma((n-i)/2)}, \quad \lambda_2 = \frac{\Gamma((n-1)/2)}{\Gamma((n-i)/2)}\).

In particular, by \((3.17)\),

\[(3.19) \quad R^\alpha_i R_i f = c Q^{i-1} f, \quad c = \frac{2\pi^{(i-1)/2} \Gamma((n-1)/2)}{\sigma_{i-1} \Gamma((n-i)/2)} \left(\frac{1}{\hat{q}_{i-1}(0)}\right),\]

and

\[(3.20) \quad M^\alpha M^0 f = Q^{\alpha+n-2} f.\]

The latter follows directly from \((3.17)\) and \((3.18)\). Furthermore, by \((3.17)\), \(Q^0 f = f\), and \((3.18)\) yields the following inversion formula:

\[(3.21) \quad R^{*-\alpha}_i R_i f = \lambda_1 f.\]

The next statements represent main results of the “analytic part” of the paper. We establish new connections between operator families defined above. For \(\xi \in G_{n,i}\), we denote

\[(3.22) \quad (R_{n-i,\perp})_f(\xi) = (R_{n-i})_f(\xi), \quad (R^\alpha_{n-i,\perp})_f(\xi) = (R^\alpha_{n-i})_f(\xi).\]

**Lemma 3.5.** Let \(f \in L^1(S^{n-1})\), \(Re \alpha > 0; \quad \alpha \neq 1, 3, 5, \ldots\). Then

\[(3.23) \quad (R_i M^\alpha f)(\xi) = \frac{2\pi^{(i-1)/2}}{\sigma_{i-1}} (R^{\alpha+i-1}_n f)(\xi), \quad \xi \in G_{n,i},\]

or (replace \(i\) by \(n - i\))

\[(3.24) \quad (R_{n-i,\perp} M^\alpha f)(\xi) = \frac{2\pi^{(n-i-1)/2}}{\sigma_{n-i-1}} (R^{\alpha+n-i-1}_i f)(\xi).\]

If \(f \in D_e(S^{n-1})\), then \((3.23)\) and \((3.24)\) extend to \(Re \alpha \leq 0\) by analytic continuation. In particular,

\[(3.25) \quad (R_i M^{1-i} f)(\xi) = \tilde{c} (R_{n-i,\perp} f)(\xi), \quad \tilde{c} = \frac{\sigma_{n-i-1} \pi^{i-n/2}}{\sigma_{i-1}}.\]
Proof. For $\Re \alpha > 0$,
\[(R_i M^\alpha f)(\xi) = \gamma_n(\alpha) \int_{S^{n-1} \cap \xi} d\xi u \int_{S^{n-1}} f(\theta) |\theta \cdot u|^{\alpha-1} d\theta.\]

Since $|\theta \cdot u| = |Pr_\xi \theta| |v_\theta \cdot u|$ for some $v_\theta \in S^{n-1} \cap \xi$, by changing the order of integration, we obtain
\[(R_i M^\alpha f)(\xi) = \gamma_n(\alpha) \int_{S^{n-1}} f(\theta) |Pr_\xi \theta|^{\alpha-1} d\theta \int_{S^{n-1} \cap \xi} |v_\theta \cdot u|^{\alpha-1} d\xi u.\]

The inner integral is independent on $v_\theta$ and can be easily evaluated:
\[\int_{S^{n-1} \cap \xi} |v_\theta \cdot u|^{\alpha-1} d\xi u = \frac{\sigma_{i-2}}{\sigma_{i-1}} \frac{1}{\sigma_i-1} \int_{-1}^1 |t|^{\alpha-1} (1-t^2)^{(i-3)/2} dt = \frac{2\pi (i-1)/2 \Gamma(\alpha/2)}{\sigma_i-1 \Gamma((i+\alpha-1)/2)}.\]

This implies (3.23). Formula (3.25) then follows by analytic continuation (just set $\alpha = 1 - i$ and make use of (3.4) with $i$ replaced by $n - i$).

Equality (3.25), written in terms of Fourier transforms, was actually obtained by Koldobsky; see [K2, Lemma 7] and [K3, Corollary 1]. The argument in these works essentially differs from ours.

Lemma 3.6. For all $\alpha \in \mathbb{C}$ such that $\alpha, 2 - n - \alpha \neq 1, 3, 5, \ldots$,
\[(3.26)\]
\[R_i^n(D_e) = R_i(D_e).\]

Specifically, $R_i^\alpha f = R_i f_1$ for functions $f$ and $f_1$ in $D_e$ connected by
\[(3.27)\]
\[f = \frac{2\pi (i-1)/2}{\sigma_i-1} M^{1-n+i} M^{1-\alpha-i} f_1,\]
\[(3.28)\]
\[f_1 = \frac{\pi (i-1)/2 \sigma_i-1}{2} M^{1-\alpha} M^{\alpha+i-1-n} f.\]

Proof. We use (3.24) with $\alpha$ replaced by $\alpha + 1 + i - n$ and then apply (3.25). This gives
\[R_i^\alpha f = \frac{\pi (i-1)/2 \sigma_{n-i-1}}{2} R_{n-i,\perp} M^{\alpha+i+1-n} f = \frac{\pi (i-1)/2 \sigma_{i-1}}{2} R_i M^{1-i} M^{\alpha+i+1-n} f = R_i f_1.\]

Equality (3.27) then follows from (3.10).
The next statement contains an intriguing factorization of the Minkowski-Funk transform in terms of Radon transforms associated to mutually orthogonal subspaces.

**Theorem 3.7.** For \( f \in L^1(S^{n-1}) \) and \( 0 < i < n \),
\[
Mf = R_i^* R_{n-i, \perp} f.
\]

**Proof.** By (2.4),
\[
(R_i^* R_{n-i, \perp} f)(\theta) = \int_{so(n-1)} (R_{n-i, \perp} f)(r_\theta \gamma \mathbb{R}^i) \, d\gamma
\]
\[
= \int_{so(n-1)} (R_{n-i} f)(r_\theta \gamma \mathbb{R}^{n-i}) \, d\gamma
\]
\[
= \int_{so(n-1)} d\gamma \int_{S^{n-1} \cap r_\theta \gamma \mathbb{R}^{n-i}} f(v) \, dv
\]
\[
= \int_{S^{n-1} \cap \mathbb{R}^{n-i}} dw \int_{so(n-1)} f(r_\theta \gamma w) \, d\gamma.
\]
The inner integral is independent on \( w \in S^{n-1} \cap \mathbb{R}^{n-i} \) and equals \((Mf)(\theta)\). This gives (3.29). \( \square \)

The following statement reveals a remarkable interplay between different analytic families and gives a series of explicit representations of the right inverse of the dual Radon transform \( R_i^* \) (note that \( R_i^* \) is non-injective on \( \mathcal{D}(G_{n,i}) \) when \( 1 < i < n - 1 \)).

**Lemma 3.8.** For \( 0 < i < n \), every function \( f \in \mathcal{D}_e(S^{n-1}) \) is represented by the dual Radon transform \( f = R_i^* g \), \( g = Af \), where the operator \( A : \mathcal{D}_e(S^{n-1}) \to \mathcal{D}(G_{n,i}) \) has the following forms:

\[
(3.30) \quad (Af)(\xi) = \frac{\sigma_{n-2}}{2\pi^{n/2-1}} (R_{n-i, \perp} M^{2-n} f)(\xi)
\]
\[
(3.31) \quad = \frac{\pi^{(1-i)/2} \sigma_{n-2}}{\sigma_{n-i-1}} (R_{1-i}^{-1} f)(\xi)
\]
\[
(3.32) \quad = \frac{\pi^{1-i} \sigma_{n-2} \sigma_{i-1}}{2 \sigma_{n-i-1}} (R_{i}(Q^{i-1})^{-1} f)(\xi), \quad \xi \in G_{n,i}.
\]

**Proof.** We first show that expressions (3.30)-(3.32) coincide. The coincidence of (3.30) and (3.31) follows from (3.24). To show that (3.31) coincides with (3.32), we set \( f = Q^{i-1} f_1 \). Since \( Q^{i-1} \) is injective, it suffices to check the equality \( 2\pi^{(i-1)/2} R_{1-i}^{-1} Q^{i-1} f_1 = \sigma_{i-1} R_i f_1 \). The latter
holds by Lemma \(3.6\). Indeed, for \(\alpha = 1 - i\), equalities \(3.27\) and \(3.20\) yield
\[\sigma_{i-1} R_i f_1 = 2\pi^{(i-1)/2} R_i^{1-i} M_{1-n+i}^0 f_1 = 2\pi^{(i-1)/2} R_i^{1-i} Q^{i-1} f_1.\]
It remains to note that the representation \(f = R_i^* g\) with \(g = A f\) defined by \(3.31\) follows from the second equality in \((3.18)\), if we apply it with \(\alpha = 1 - i\) and take into account that \(Q^0 f = f\).

\[\square\]

\textbf{Remark 3.9.} As it was mentioned above, the map \(R_i^* : \mathcal{D}(G_{n,i}) \to \mathcal{D}_e(S^{n-1})\) is non-injective. In fact, every function \(\varphi \in \mathcal{D}(G_{n,i})\) is represented as a sum \(\varphi = \varphi_R + \varphi_0\) where \(\varphi_R\) belongs to the range \(\mathcal{R}_i(\mathcal{D}_e(S^{n-1}))\) and \(\varphi_0 \in \ker\mathcal{R}_i^*.\) Indeed, let \(\varphi_R = c^{-1} R_i[(Q^{i-1})^{-1} R_i^* \varphi],\) \(c\) being a constant from \(3.19\), \(\varphi_0 = \varphi - \varphi_R.\) Then, by \(3.19,\)
\[\begin{align*}
R_i^* \varphi_0 &= R_i^* \varphi - c^{-1} R_i^* \varphi_R = R_i^* \varphi - c^{-1} R_i^* R_i[(Q^{i-1})^{-1} R_i^* \varphi] \\
&= R_i^* \varphi - Q^{i-1}(Q^{i-1})^{-1} R_i^* \varphi = R_i^* \varphi - R_i^* \varphi = 0,
\end{align*}\]
i.e., \(\varphi_0 \in \ker R_i^*.\)

The following statement is dual to Lemma \(3.6\).

\textbf{Lemma 3.10.} Let \(\varphi \in L^1(G_{n,i}), \varphi^\perp(\eta) = \varphi(\eta^\perp), \ \eta \in G_{n,n-i}.\) If \(\Re \alpha > 0, \ \alpha \neq 1, 3, 5, \ldots,\) then
\begin{equation} \tag{3.33}
M^{\alpha} R_i^* \varphi = c R_{n-i-1}^* \varphi^\perp, \quad c = \frac{2\pi^{(i-1)/2}}{\sigma_{i-1}}.
\end{equation}
If \(\varphi \in \mathcal{D}(G_{n,i}),\) then \(3.33\) extends to all complex \(\alpha \neq 1, 3, 5, \ldots\) by analytic continuation. In particular, by \(3.34,\)
\begin{equation} \tag{3.34}
M^{1-i} R_i^* \varphi = \bar{c} R_{n-i}^* \varphi^\perp, \quad \bar{c} = \frac{\pi^{i-n/2} \sigma_{n-i-1}}{\sigma_{i-1}}.
\end{equation}

\textbf{Proof.} Let \(\Re \alpha > 0.\) Owing to \(2.30\) and \(3.28,\) for any test function \(\omega \in \mathcal{D}_e(S^{n-1})\) we have
\[
\int_{S^{n-1}} (M^{\alpha} R_i^* \varphi)(\theta) \omega(\theta) \ d\theta = \int_{S^{n-1}} (R_i^* \varphi)(\theta) \ (M^{\alpha} \omega)(\theta) \ d\theta
\]
\[
= \int_{G_{n,i}} \varphi(\xi) (R_i M^\alpha \omega)(\xi) \ d\xi = \frac{2\pi^{(i-1)/2}}{\sigma_{i-1}} \int_{G_{n,i}} \varphi(\xi) (R_{n-i-1}^{\alpha+i-1} \omega)(\xi) \ d\xi
\]
\[
= \frac{2\pi^{(i-1)/2}}{\sigma_{i-1}} \int_{G_{n,n-i}} \varphi^\perp(\eta) (R_{n-i}^{\alpha+i-1} \omega)(\eta) \ d\eta
\]
\[
= \frac{2\pi^{(i-1)/2}}{\sigma_{i-1}} \int_{S^{n-1}} (R_{n-i}^* \varphi^\perp)(\theta) \omega(\theta) \ d\theta.
\]
This gives the desired result. □

Corollary 3.11. If \( \varphi \in L^1(G_{n,i}) \), then for any \( \text{Re}\alpha > 0, \ \alpha \neq 1, 3, 5, \ldots \),
\[(3.35) \quad f = R_i^*\varphi \quad \text{if and only if} \quad M^{\alpha} f = c R^{\alpha+i-1}_{n-i} \varphi^\perp, \]
c being a constant from (3.33). If \( \varphi \in D(G_{n,i}) \), then (3.35) extends to all complex \( \alpha \neq 1, 3, 5, \ldots \) by analytic continuation. In particular, for \( \alpha = 1 - i \),
\[(3.36) \quad f = R_i^*\varphi \quad \text{if and only if} \quad M^{1-i} f = c R^{i}_{n-i} \varphi^\perp. \]

Proof. The statement is a consequence of (3.33) and injectivity of \( M^{\alpha} \). □

4. Positive definite homogeneous distributions and star bodies

In this section we invoke the Fourier transform of homogeneous distributions and apply some results of Section 3 to study classes of star bodies arising in convex geometry.

4.1. The Fourier transform of homogeneous distributions. This is one of the oldest topics in the theory of distributions, and there is a vast literature on this subject; see, e.g., [GS], [Se], [Le]. Let \( S(\mathbb{R}^n) \) be the Schwartz space of rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^n \) and \( S' = S'(\mathbb{R}^n) \) the corresponding space of tempered distributions. The Fourier transform of \( F \in S' \) is defined by
\[
\langle \hat{F}, \hat{\phi} \rangle = (2\pi)^n \langle F, \phi \rangle, \quad \hat{\phi}(y) = \int_{\mathbb{R}^n} \phi(x) e^{-ix \cdot y} dx, \quad \phi \in S(\mathbb{R}^n). 
\]

A distribution \( F \in S' \) is positive definite if the Fourier transform \( \hat{F} \) is a positive distribution, i.e., \( \langle \hat{F}, \hat{\phi} \rangle \geq 0 \) for every nonnegative test function \( \phi \). Given a distribution \( F \in S' \) and a complex number \( \lambda \), we say that \( F \) is a homogeneous distribution of degree \( \lambda \) if for any \( \phi \in S(\mathbb{R}^n) \) and any \( a > 0 \),
\[
\langle F, \phi(x/a) \rangle \equiv a^{\lambda+n} \langle F, \phi \rangle. 
\]
Homogeneous distributions on \( \mathbb{R}^n \) are intimately related to distributions on the unit sphere. We define a homogeneous continuation of a function \( \varphi \) on \( S^{n-1} \) by
\[(4.1) \quad (E_\lambda \varphi)(x) = |x|^\lambda \varphi(x/|x|), \quad x \in \mathbb{R}^n \setminus \{0\}. 
\]
If \( \lambda \neq -n, -n-1, -n-2, \ldots \), then the operator \( E_\lambda \) extends to distributions \( f \in D' \) by the formula
\[(4.2) \quad \langle E_\lambda f, \phi \rangle = (f, \phi_\lambda), 
\]
where \( \phi \in \mathcal{S}(\mathbb{R}^n) \), 
\[ \phi_\lambda(\theta) = \int_0^\infty r^{\lambda+n-1} \phi(r\theta) \, dr \in \mathcal{D}(S^{n-1}). \]
For \( \text{Re} \, \lambda \leq -n \), \( \lambda \neq -n, -n-1, -n-2, \ldots \), the last integral is understood in the sense of analytic continuation.

**Lemma 4.1.** Let \( \lambda \in \mathbb{C}; \lambda \neq -n, -n-1, -n-2, \ldots \). Then \( E_\lambda \) is a linear continuous operator from \( \mathcal{D}' \) to \( \mathcal{S}' \).

**Proof.** If \( f_j \in \mathcal{D}' \) and \( \lim_{j \to \infty} (f_j, \omega) = (f, \omega) \) for any \( \omega \in \mathcal{D} \), then for any \( \phi \in \mathcal{S}(\mathbb{R}^n) \) we have
\[ \lim_{j \to \infty} \langle E_\lambda f_j, \phi \rangle = \lim_{j \to \infty} (f_j, \phi_\lambda) = (f, \phi_\lambda) = \langle E_\lambda f, \phi \rangle. \]

The following theorem due to Lemoine [Le] characterizes the structure of homogeneous distributions.

**Theorem 4.2.** Let \( \tau \) be a homogeneous distribution of degree \( \lambda \in \mathbb{C} \).

a) If \( \lambda \) is not an integer \( \leq -n \), there is an \( f \in \mathcal{D}' \) such that \( \tau = E_\lambda f \).

b) If \( \lambda = -n, -n-1, \ldots \), there are \( f \in \mathcal{D}' \) and a polynomial \( P_{-\lambda-n} \) homogeneous of degree \( -\lambda-n \) such that \( \tau = E_\lambda f + P_{-\lambda-n}(D)\delta \), where \( D = (\partial/\partial x_1, \ldots, \partial/\partial x_n) \) and \( \delta \) is the Dirac measure.

The operator family \( \{M^\alpha\} \) arises in the Fourier analysis of homogeneous distributions in a natural way thanks to the formula
\[
\langle E_{1-n-\alpha} f, \phi \rangle = 2^{1-\alpha} \pi^{n/2} E_{\alpha-1} M^\alpha f.
\]
This holds for a \( C^\infty \) even function \( f \) and any complex \( \alpha \) satisfying
\[ \alpha \notin \{1, 2, 3, \ldots \} \cup \{1-n, -n, -n-1, \ldots \}. \]

Formula (4.3) is understood in the \( \mathcal{S}' \)-sense. Namely,
\[ \langle E_{1-n-\alpha} f, \phi \rangle = 2^{1-\alpha} \pi^{n/2} \langle E_{\alpha-1} M^\alpha f, \hat{\phi} \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^n), \]
where both sides are interpreted in the sense of analytic continuation. This formula (and a more general one for arbitrary, not necessarily even, functions) is known in analysis for many years and has many applications; see, e.g., [Sc], [Es], [Pl], [Sa1], [Sa2], [R1], [K4], and references therein. Generalizations of (4.3) to functions on Stiefel and Grassmann manifolds were obtained in [OR]. Since \( \mathcal{D}_e \) is dense in \( \mathcal{D}' \), then, owing to Lemmas 3.2 and 4.1, equalities (4.3) and (4.5) extend to all even distributions on \( S^{n-1} \).

The following lemma makes a bridge between positive definite homogeneous distributions and operators \( M^\alpha \).

**Lemma 4.3.** Let \( \alpha \) be a real number satisfying (4.4). If \( f \in \mathcal{D}_e \), then \( E_{1-n-\alpha} f \) is a positive definite distribution if and only if \( (M^\alpha f)(\theta) \geq 0 \) for every \( \theta \in S^{n-1} \). If \( f \in \mathcal{D}_e \), then \( E_{1-n-\alpha} f \) is a positive definite distribution if and only if \( M^\alpha f \) is a nonnegative finite measure on \( S^{n-1} \).
get a pointwise inequality (\(M\omega\) lemma, we choose \((M_S\text{nonnegative test function on})\). 

4.2. Classes of star bodies. Below we introduce classes of origin-symmetric star bodies in \(\mathbb{R}^n\) associated with the analytic family \(M^\alpha\) of the generalized cosine transforms. These classes include intersection bodies and some other classes of bodies commonly used in convex geometry. Let \(K\) be a compact subset of \(\mathbb{R}^n\), \(n \geq 2\), star-shaped with respect to the origin. For \(x \in \mathbb{R}^n \setminus \{0\}\), the radial function of \(K\) is defined by \(\rho_K(x) = \sup\{\lambda \geq 0 : \lambda x \in K\}\). If \(\theta \in S^{n-1}\), then \(\rho_K(\theta)\) is the Euclidean distance from the origin to the boundary of \(K\) in the direction of \(\theta\). If \(\rho_K\) is a positive continuous function on \(S^{n-1}\), then \(K\) is said to be a star body. We denote by \(\mathbb{K}^n\) the class of all origin-symmetric star bodies in \(\mathbb{R}^n\). The Minkowski functional of \(K \in \mathbb{K}^n\) is defined by \(||x||_K = \min\{a \geq 0 : x \in aK\}\) so that \(||\theta||_K = \rho_K^{-1}(\theta)\). A body \(K \in \mathbb{K}^n\) is called infinitely smooth if \(\rho_K \in \mathcal{D}_e(S^{n-1})\). We say that a sequence of bodies \(K_j \in \mathbb{K}^n\) converges to \(K \in \mathbb{K}^n\) in the radial metric if \(\lim_{j \to \infty} ||\rho_{K_j} - \rho_K||_{C(S^{n-1})} = 0\). Given a subset \(\mathcal{K} \subset \mathbb{K}^n\), we denote by \(\text{cl}\mathcal{K}\) the closure of \(\mathcal{K}\) in the radial metric.

**Definition 4.4.** Let \(\alpha\) be a real number,

\[
\alpha \notin \{0, -2, -4, \ldots\} \cup \{n, n + 2, n + 4, \ldots\}.
\]

We define the following classes of origin-symmetric star bodies:

\[
\mathcal{K}_{\alpha,n} = \{K \in \mathbb{K}^n : \rho_K^\alpha = M^{1-\alpha}\mu \text{ for some } \mu \in \mathcal{M}_{e+}(S^{n-1})\};
\]

\[
\mathcal{KB}_{\alpha,n} = \{K \in \mathbb{K}^n : \rho_K^\alpha = M^{1-\alpha}\rho_L^{n-\alpha} \text{ for some body } L \in \mathbb{K}^n\};
\]

\[
\mathcal{KB}_{\alpha,n}^\infty = \{K \in \mathcal{KB}_{\alpha,n} : \rho_K \in \mathcal{D}_e(S^{n-1})\}.
\]
Some comments are in order:
1. The equality \( \rho_K^\alpha = M^{1-\alpha} \mu \) in (4.10) means that
   \[
   (\rho_K^\alpha, \omega) = (\mu, M^{1-\alpha} \omega) \quad \forall \omega \in \mathcal{D}(S^{n-1}).
   \]
2. If \( K \in \mathcal{KB}_{\alpha,n}^\infty \), then \( \rho_K^\alpha = M^{1-\alpha} \rho_L^{n-\alpha} \) where, by (3.10), \( \rho_L^{n-\alpha} = M^{1-n+\alpha} \rho_K^\alpha \in \mathcal{D}_{e+}(S^{n-1}) \).
3. For some \( \alpha \), the class \( \mathcal{K}_{\alpha,n} \) looks somewhat artificial and does not contain such a nice body as the unit ball \( B \). Indeed, since \( \rho_B = 1 \), then, owing to (3.8), \( M^{1-n+\alpha} \rho_B^\alpha = \Gamma((n-\alpha)/2)/\Gamma(\alpha/2) \). This is negative if
   \[
   \alpha \in \left( \bigcup_{k=0}^\infty (-4k-2,-4k) \cup \left( \bigcup_{k=0}^\infty (n+4k,n+4k+2) \right) \right).
   \]
Thus \( B \notin \mathcal{K}_{\alpha,n} \) for all such \( \alpha \).

**Theorem 4.5.** Let \( \alpha \) be a real number satisfying (4.10). Then
   \[
   \mathcal{K}_{\alpha,n} = \text{cl} \mathcal{KB}_{\alpha,n} = \text{cl} \mathcal{KB}_{\alpha,n}^\infty.
   \]

*Proof.* STEP 1. We first prove that \( \mathcal{K}_{\alpha,n} \subseteq \text{cl} \mathcal{KB}_{\alpha,n}^\infty \). Let \( K \in \mathcal{K}_{\alpha,n} \), i.e., \( \rho_K^\alpha = M^{1-\alpha} \mu, \mu \in \mathcal{M}_{e+}(S^{n-1}) \). Our aim is to define a sequence \( K_j \in \mathcal{KB}_{\alpha,n}^\infty \) such that \( \rho_{K_j} \to \rho_K \) in the \( C \)-norm. Consider the Poisson integral \( \Pi_t \rho_K^\alpha \) that converges to \( \rho_K^\alpha \) in the \( C \)-norm when \( t \to 1 \). For any test function \( \omega \in \mathcal{D} \), we have
   \[
   (\Pi_t \rho_K^\alpha, \omega) = (\rho_K^\alpha, \Pi_t \omega) = (\mu, M^{1-\alpha} \Pi_t \omega) = (M^{1-\alpha} \Pi_t \mu, \omega),
   \]
where \( \Pi_t \mu \in \mathcal{D}_{e+} \). Hence, one can choose \( K_j \in \mathcal{KB}_{\alpha,n}^\infty \) so that \( \rho_{K_j} = \Pi_t \rho_K^\alpha = M^{1-\alpha} \Pi_t \mu \), where \( t_j \) is a sequence in \((0,1)\) approaching 1. Clearly, \( K_j \) converges to \( K \) in the radial metric.

Conversely, let \( K \in \text{cl} \mathcal{KB}_{\alpha,n}^\infty \). It means that there is a sequence of bodies \( K_j \in \mathcal{KB}_{\alpha,n}^\infty \) such that \( \lim_{j \to \infty} ||\rho_K - \rho_{K_j}||_C = 0 \) and \( \rho_{K_j} = M^{1-\alpha} \rho_{L_j}^{n-\alpha}, \rho_{L_j} \in \mathcal{D}_{e+} \). Then \( \rho_{K_j}^\alpha \) approaches \( \rho_K^\alpha \) in the \( C \)-norm, and for any \( \omega \in \mathcal{D}_{e+} \), the expression \( (\rho_{K_j}^\alpha, M^{1-n+\alpha} \omega) \) is nonnegative because
   \[
   (\rho_{K_j}^\alpha, M^{1-n+\alpha} \omega) = (M^{1-\alpha} \rho_{L_j}^{n-\alpha}, M^{1-n+\alpha} \omega) = (\rho_{L_j}^{n-\alpha}, \omega) \geq 0.
   \]
If \( j \to \infty \), then
   \[
   (\rho_{K_j}^{n-\alpha} M^{1-n+\alpha} \omega) \to (\rho_K^{n-\alpha} M^{1-n+\alpha} \omega) = (M^{1-n+\alpha} \rho_K^\alpha, \omega) \geq 0.
   \]
Hence, by Theorem 2.1, \( M^{1-n+\alpha} \rho_K^\alpha \) is a nonnegative measure. Let \( \mu = M^{1-n+\alpha} \rho_K^\alpha \). By (3.11), for any \( \omega \in \mathcal{D} \) we have
   \[
   (\rho_K^\alpha, \omega) = (M^{1-\alpha} \rho_K^\alpha, M^{1-\alpha} \omega) = (\mu, M^{1-\alpha} \omega) = (M^{1-\alpha} \mu, \omega),
   \]
i.e., \( K \in \mathcal{K}_{\alpha,n} \). This gives \( \mathcal{K}_{\alpha,n} = \text{cl} \mathcal{KB}_{\alpha,n}^\infty \).
STEP 2. Let us prove that $\mathcal{K}_{\alpha,n} = \text{cl} \mathcal{KB}_{\alpha,n}$. Since $\mathcal{KB}_{\alpha,n}^\infty \subset \mathcal{K}_{\alpha,n}$, then, by Step 1, $\mathcal{K}_{\alpha,n} \subset \text{cl} \mathcal{KB}_{\alpha,n}$. The proof of the reverse inclusion coincides with the second part in Step 1 with the only difference that now $\rho_{L_j}$ are only continuous and not necessarily smooth. □

**Theorem 4.6.** A body $K \in \mathbb{K}^n$ belongs to $\mathcal{K}_{\alpha,n}$ if and only if $|| \cdot ||^\alpha_K$ is a positive definite distribution.

**Proof.** By Lemma 4.3, $|| \cdot ||^\alpha_K \equiv E_{-\alpha} \rho^\alpha_K$ is a positive definite distribution if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1})$ such that

$$(M^{1-n+\alpha} \rho^\alpha_K, \omega) = (\mu, \omega) \quad \forall \omega \in \mathcal{D}_e.$$ 

Owing to (3.10), the latter is equivalent to $(\rho^\alpha_K, \tilde{\omega}) = (\mu, M^{1-\alpha} \tilde{\omega}) \forall \tilde{\omega} \in \mathcal{D}_e$ (choose $\omega = M^{1-\alpha} \tilde{\omega}$). This is what we need. □

Below we consider some examples when known geometric objects are members of the class $\mathcal{K}_{\alpha,n}$. If $K \in \mathbb{K}^n$ and $\xi$ is an $i$-dimensional subspace of $\mathbb{R}^n$, i.e., $\xi \in G_{n,i}$, $1 \leq i < n$, then the volume of the cross-section $K \cap \xi$ can be evaluated by

\begin{equation}
\text{vol}_i(K \cap \xi) = \frac{\sigma_{i-1}}{i} \int_{S^{n-1} \cap \xi} \rho^i_{K}(\theta) d\xi \theta = \frac{\sigma_{i-1}}{i} (R_i \rho^i_K)(\xi).
\end{equation}

This can be easily obtained by passing to polar coordinates in $\xi$.

**Example 1.** According to Lutwak [Lu], the class $IB_n$ of intersection bodies of star bodies in $\mathbb{R}^n$ is defined as the range of the map $IB : \mathbb{K}^n \to \mathbb{K}^n$ by the rule

$$\rho_{IB(L)}(\theta) = \text{vol}_{n-1}(L \cap \theta^\perp), \quad \theta \in S^{n-1}, \quad L \in \mathbb{K}^n.$$ 

By (4.13), it means that $K = IB(L)$ if and only if

\begin{equation}
\rho_K = \frac{\sigma_{n-2}}{n-1} M\rho_{L}^{n-1},
\end{equation}

where $M$ is the Minkowski-Funk transform (2.8). We denote by $IB_n^\infty$ the subclass of $IB_n$ consisting of infinitely smooth bodies. Intersection bodies of centered convex bodies first appeared in the work of Busemann [Bu] who did not give them a particular name. A more general class $I_n$ of star bodies, which was called the class of intersection bodies (without wording “of star bodies”) was defined in [GLW] as a collection of all bodies $K \in \mathbb{K}^n$ with the property

\begin{equation}
\rho_K = M\mu \quad \text{for some} \quad \mu \in \mathcal{M}_+(S^{n-1}).
\end{equation}

By Definition 4.4, the class $I_n$ is a member of the family $\{\mathcal{K}_{\alpha,n}\}$ corresponding to $\alpha = 1$. Thus, Theorems 4.5 and 4.6 imply the following known statement.
**Theorem 4.7.** A body $K \in \mathbb{K}^n$ is an intersection body, i.e., $K \in \mathcal{I}_n$ if and only if $\| \cdot \|_K^{-1}$ is a positive definite distribution. The class $\mathcal{I}_n$ is the closure of the classes $\mathcal{IB}_n$ and $\mathcal{IB}_n^\infty$ of intersection bodies of star bodies in the radial metric.

The first part of this theorem can be found in [K4, Theorem 4.1]. Regarding the second part, see [GZ, Theorem 5.5], [GLW], and references therein.

**Example 2.** The following extension of the definitions in Example 1 to sections of arbitrary dimension $0 < i < n$ was suggested by Koldobsky [K2]. According to [K2], a body $K \in \mathbb{K}^n$ is an $i$-intersection body of a body $L \in \mathbb{K}^n$ (we write $K = IB_i(L)$) if

$$\text{vol}_i(K \cap \xi) = \text{vol}_{n-i}(L \cap \xi^\perp) \quad \forall \xi \in G_{n,i},$$

or, in other words,

$$\frac{\sigma_{i-1}}{i}(R_i \rho_K^i)(\xi) = \frac{\sigma_{n-i-1}}{n-i}(R_{n-i,\perp} \rho_L^{n-i})(\xi) \quad \forall \xi \in G_{n,i}.$$

We denote by $\mathcal{IB}_{i,n}$ the class of all star bodies with this property and by $\mathcal{IB}_{i,n}^\infty$ the subclass of $\mathcal{IB}_{i,n}$ consisting of infinitely smooth bodies. The above definition has a remarkable symmetry:

$$K = IB_i(L) \iff L = IB_{n-i}(K).$$

We generalize this definition as follows.

**Definition 4.8.** A body $K \in \mathbb{K}^n$ is an $i$-intersection body if there is a measure $\mu \in \mathcal{M}_+^i(S^{n-1})$ such that

$$R_i \rho_K^i = R_{n-i,\perp} \mu.$$

We denote by $\mathcal{I}_{i,n}$ the class of $i$-intersection bodies in $\mathbb{R}^n$.

The concept of the $i$-intersection body based on (4.16) was introduced by Koldobsky [K3, Definition 3]. His definition is given in terms of the Fourier transforms. Our Definition 4.8 agrees with (4.15) and uses the language of Radon transforms. As we shall see below, both definitions are equivalent.

**Theorem 4.9.**

(i) For $\alpha = i$, the classes $\{K_{\alpha,n}\}$, $\{K\mathcal{B}_{\alpha,n}\}$, and $\{K\mathcal{B}_{\alpha,n}^\infty\}$ coincide with $\{\mathcal{I}_{i,n}\}$, $\{\mathcal{IB}_{i,n}\}$, and $\{\mathcal{IB}_{i,n}^\infty\}$, respectively.

(ii) A body $K \in \mathbb{K}^n$ is an $i$-intersection body, i.e., $K \in \mathcal{I}_{i,n}$, if and only if $\| \cdot \|_K^{-i}$ is a positive definite distribution.

(iii) The class $\mathcal{I}_{i,n}$ is the closure of the classes $\{\mathcal{IB}_{i,n}\}$ and $\{\mathcal{IB}_{i,n}^\infty\}$ of $i$-intersection bodies of star bodies in the radial metric.
(iv) If a body $K \in \mathbb{K}^n$ is infinitely smooth and $K = \mathcal{I}B_i(L)$, then

\begin{equation}
\rho^{n-i}_L = \frac{\pi^{i-n/2}(n-i)}{i} M^{1-n+i} \rho^i_K.
\end{equation}

Proof. (i) Let us prove that equality (4.19) is equivalent to

\begin{equation}
\rho^i_K = \bar{c}^{-1} M^{1-i} \mu, \quad \bar{c} = \frac{\pi^{i-n/2} \sigma_{n-i-1}}{\sigma_{i-1}}.
\end{equation}

Indeed, (4.19) means that for any test function $\varphi \in \mathcal{D}(G_{n,i})$, we have

\begin{equation}
\langle \rho^i_K, R^i_{n-i} \varphi \rangle = \langle \mu, R^i_{n-i} \varphi \rangle = \bar{c}^{-1} \langle \mu, M^{1-i} R^i_{n-i} \varphi \rangle.
\end{equation}

Since any function $\omega \in \mathcal{D}(S^{n-1})$ can be expressed as $\omega = R^i_{n-i} \varphi$ for some $\varphi \in \mathcal{D}(G_{n,i})$ (see Lemma 3.8), we are done, i.e., \{$K_{i,n}$\} = \{$I_{i,n}$\}. The equalities \{$K_{B_i,n}$\} = \{$I_{B_i,n}$\} and \{$K_{B_{\alpha,n}}$\} = \{$I_{B_{\alpha,n}}$\} can be proved similarly: just use (4.17) and replace $\mu$ by $c \rho^{n-i}_L$ with $c = i \sigma_{n-i-1}/(n-i) \sigma_{i-1}$.

Statements (ii) and (iii) follow from Theorems 4.6 and 4.5 respectively.

(iv) We make use of (3.24) with $\alpha = 1 - n + i$ and $f = \rho^i_K$. Owing to (3.4), it can be written in the form

\begin{equation}
R^i_K \rho^i_K = \bar{c} R^i_{n-i,1} M^{1-n+i} \rho^i_K
\end{equation}

with constant $\bar{c}$ as in (4.21). On the other hand, if $K$ is an infinitely smooth body and $K = \mathcal{I}B_i(L)$, then, by (4.17),

\begin{equation}
R^i_K \rho^i_K = \frac{i \sigma_{n-i-1}}{(n-i) \sigma_{i-1}} R^i_{n-i,1} \rho^{n-i}_L.
\end{equation}

Comparing (4.22) and (4.23), we obtain (4.20). $\square$

Some comments are in order.

1. Statement (ii) of Theorem 4.9 was proved by Koldobsky (see [K2], Theorem 4) for the case, when $K$ is an $i$-intersection bodies of a star body with $C^\infty$-boundary. In our notation it means that $K \in \mathcal{I}B_{i,n}$. This result was extended in [K3] to a more general class of $i$-intersection bodies, which coincides with $\mathcal{I}_{i,n}$. Our argument essentially differs from that in cited works.

2. Unlike the case $i = 1$, when a body $K = \mathcal{I}B(L)$ can be constructively realized for any origin-symmetric star body $L$, this is not so if $i > 1$, when the definition of $\mathcal{I}B_i(L)$ is purely analytic. It is known that for $i > 3$, owing to Theorem 4.9 and the symmetry (4.18), the $i$-intersection body $\mathcal{I}B_i(L)$ is not defined if $L$ is the unit ball \{$x \in \mathbb{R}^n : |x_1|^4 + \ldots + |x_n|^4 < 1$\} of the space $\ell_1^n$. The reason is that $||x||_{\ell_1^n}$ is not a positive definite distribution; see Theorem 2 in [K1].
**Problem.** The last remark provokes the following question: Is there a reasonable way to generalize the original Busemann’s construction $\mathcal{B}_n$ to sections of codimension $>1$ so that it would be applicable to every origin-symmetric star body? Clearly, the normal unit vector in this construction should be replaced by the orthonormal frame, the element of the corresponding Stiefel manifold.

**Example 3.** For each origin-symmetric convex body $K$ in $\mathbb{R}^n$, the Minkowski functional $|| \cdot ||_K$ is a norm in $\mathbb{R}^n$. A well known result going back to P. Lévy [K4, Section 6.1] says that the space $(\mathbb{R}^n, || \cdot ||_K)$ embeds isometrically into $L_p$, $p > 0$, if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1})$ such that

$$||\theta||^p_K = \int_{S^{n-1}} |\theta \cdot u|^p d\mu(u).$$

For $p \neq 2, 4, \ldots$, equality (4.24) is obviously equivalent to $K \in \mathcal{K}_{\alpha,n}$ with $\alpha = -p$. Passing from $\alpha = -p$ to arbitrary $\alpha$ in Definition 4.4 allows us to extend the wording “embeds isometrically into $L_p$” to negative $p$; see [K4]. For $p \neq -n, -n - 2, \ldots$, this is equivalent to $K \in \mathcal{K}_{-p,n}$.

4.3. **Zhang’s class of intersection bodies.** The following class of bodies was introduced by Zhang [Z3] in his research related to the lower dimensional Busemann-Petty problem.

**Definition 4.10.** An origin-symmetric star body $K$ in $\mathbb{R}^n$ is called an $i^*$-intersection body if there is a measure $\nu \in \mathcal{M}_+(G_{n,i})$ such that $\rho_K^{n-i} = R_i^* \nu$.

Here abbreviation $i^*$ has been chosen to distinguish this class of bodies from that in Definition 4.8 and to indicate implementation of the dual Radon transform $R_i^*$. Another notation for both classes of bodies was utilized in [M1].

Connection between the class of $i^*$-intersection bodies and that of $i$-intersection bodies is an important problem intimately related to the lower dimensional Busemann-Petty problem for sections of convex bodies (the case of 2 and 3-dimensional sections is of primary interest). To the best of our knowledge, both problems are still open; see [K4] for details. The fact that each $(n-i)^*$-intersection body is an $i$-intersection body is due to Koldobsky [K3, Corollary 3], who proved it using the Fourier transform technique and isometric embedding in $L^p$-spaces. The theorem below includes this statement and provides additional information. Our proof differs from that in [K3] and does not involve the Fourier transform.
Given a measure \( \nu \in \mathcal{M}_+(G_{n,n-i}) \), we define a measure \( \nu^\perp \in \mathcal{M}_+(G_{n,i}) \) by \( (\nu^\perp, \varphi) = (\nu, \varphi^\perp) \) where \( \varphi \in C'(G_{n,n-i}) \), \( \varphi^\perp(\xi) = \varphi(\xi^\perp) \), \( \xi \in G_{n,i} \).

**Theorem 4.11.** Each \((n-i)^*\)-intersection body \( K \) is an \( i \)-intersection body. Specifically, if \( \rho_K^i = R_{n-i}^* \nu \), \( \nu \in \mathcal{M}_+(G_{n,n-i}) \), then \( R_i \rho_K^i = R_{n-i,\perp} \mu \) where \( \mu = R_i^* \nu^\perp \). Conversely, if \( K \) is an \( i \)-intersection body, i.e., \( R_i \rho_K^i = R_{n-i,\perp} \mu \) for some \( \mu \in \mathcal{M}_+(S^{n-1}) \), then \( K \) is an \((n-i)^*\)-intersection body provided that \( \mu \) is represented in the form \( \mu = R_{n-i}^* \nu^\perp \) for some \( \nu \in \mathcal{M}_+(G_{n,n-i}) \). If the latter is true, then \( \rho_K^i = R_{n-i}^* \nu \).

**Proof.** Let \( K \) be an \((n-i)^*\)-intersection body, i.e., \( \rho_K^i = R_{n-i}^* \nu \), \( \nu \in \mathcal{M}_+(G_{n,n-i}) \). For any function \( \psi \in \mathcal{D}(G_{n,i}) \), we have

\[
(R_i \rho_K^i, \psi) = (\rho_K^i, R_i^* \psi) = (R_{n-i}^* \nu, R_i^* \psi) = (\nu, R_{n-i} R_i^* \psi)
\]

(3.34)

\[
= (\nu^\perp, R_{n-i,\perp} R_i^* \psi)
\]

(3.25)

\[
= \tilde{c}^{-1}(\nu^\perp, R_{n-i,\perp} M^{1-i} R_i^* \psi).
\]

If we set \( \mu = R_i^* \nu^\perp \), then the last expression becomes \( \tilde{c}^{-1}(\mu, M^{1-i} R_i^* \psi) \) which coincides with \( (\mu, R_{n-i} R_i^* \psi^\perp) = (R_{n-i,\perp} \mu, \psi) \) by (3.34). This gives the result.

Conversely, let \( R_i \rho_K^i = R_{n-i,\perp} \mu \), \( \mu \in \mathcal{M}_+(S^{n-1}) \). We take a test function \( \omega \in \mathcal{D}_\omega(S^{n-1}) \) and represent it in the form \( \omega = R_i^* \psi \), \( \psi \in \mathcal{D}(G_{n,i}) \) (this is possible by Lemma 3.8). Then

\[
(\rho_K^i, \omega) = (\rho_K^i R_i^* \psi) = (R_i \rho_K^i, \psi) = (R_{n-i,\perp} \mu, \psi).
\]

Since \( \mu = R_i^* \nu^\perp \), we continue

\[
= (R_{n-i,\perp} R_{n-i}^* \psi^\perp) = (\nu, R_{n-i,\perp} R_{n-i}^* \psi^\perp)
\]

(3.25) with \( i \) replaced by \( n-i \)

\[
= \tilde{c}^{-1}(\nu, R_{n-i} M^{1-n+i} R_{n-i}^* \psi^\perp) = \tilde{c}^{-1}(R_{n-i}^* \nu, M^{1-n+i} R_{n-i}^* \psi^\perp).
\]

By (3.34), this coincides with \( (R_{n-i}^* \nu, R_{n-i} \psi) = (R_{n-i}^* \nu, \omega) \). Hence \( \rho_K^i = R_{n-i}^* \nu \), and the proof is complete. \( \square \)

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