On Conformally Compactified Phase Space

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Abstract

Conformally compactified phase space is conceived as an automorphism space for the global action of the extended conformal group. Space time and momentum space appear then as conformally dual, that is conjugate with respect to conformal reflections. If now the former, as generally agreed, is appropriate for the description of classical mechanics in euclidean geometrical form, then the latter results appropriate for the description of quantum mechanics in spinor geometrical form. In such description, fermion multiplets will naturally appear as consequence of higher symmetries and furthermore, the euclidean geometry, bilinearly resulting from that of spinors, will a priori guarantee the absence of ultraviolet divergences when dealing with quantum field theories. Some further possible consequences of conformal reflections of interest for physics, are briefly outlined.

1 Introduction

The discovery of Maxwell’s equations covariance with respect to the conformal group $C = \{L, D, P_4, S_4\}$ (where $L, D, P_4, S_4$ mean Lorentz-, Dilatation-, Poincarè-, and special conformal transformations respectively building up $C$) [1] have induced several authors [2] to conjecture that Minkowski space-time $\mathbb{M} = \mathbb{R}^{3,1}$ may be densely contained in conformally compactified space-time $\mathbb{M}_c$:

$$\mathbb{M}_c = \frac{S^3 \times S^1}{\mathbb{Z}_2}$$ (1)

From which Robertson-Walker space-time $\mathbb{M}_{RW} = S^3 \times R^1$ is obtained, familiar to cosmologists (being at the origin of the Cosmological Principle [3]), where $R^1$ is conceived as the infinite covering of $S^1$. 

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often conceived as the homogeneous space of the conformal group:

\[ \mathcal{M}_c = \frac{C}{c_1} \]

where \( c_1 = \{L, D, S_4\} \) is the stability group of the origin \( x_\mu = 0 \).

As well known \( C \) may be linearly represented by \( SO(4, 2) \), acting in \( \mathbb{R}^{4,2} \), containing the Lorentz group \( SO(3, 1) \) as a subgroup which, because of the relevance of space-time reflections for natural phenomena, should be extended to \( O(3, 1) \). But then \( SO(4, 2) \) should be also extended to \( O(4, 2) \) including conformal reflections \( I \) (with respect to hyperplanes orthogonal to the 5th and 6th axis), whose relevance for physics should then be expected as well. To start with, in fact, in this case \( \mathcal{M}_c = C/c_1 \) seems not to be the only automorphism space of \( O(4, 2) \), since:

\[ IM_c I^{-1} = I \frac{C}{c_1} I^{-1} = \frac{C}{c_2} = \mathbb{P}_c = \frac{S^3 \times S^1}{\mathbb{Z}_2} \]

where \( c_2 = \{L, D, P_4\} \) is the stability group of infinity. Therefore, being \( c_1 \) and \( c_2 \) conjugate, \( \mathcal{M}_c \) and \( \mathbb{P}_c \) are two copies of the same homogeneous space \( \mathcal{H} \) of the conformal group including reflections, and, as we will see, both are needed to represent the group linearly. Because of eq. (2) we will call \( \mathcal{M}_c \) and \( \mathbb{P}_c \) conformally dual.

There are good arguments \[4\] (see also footnote \[4\]) in favor of the hypothesis that \( \mathbb{P}_c \) may represent conformally compactified momentum space \( \mathbb{P} = \mathbb{R}^{3,1} \). In this case \( \mathcal{M}_c \) and \( \mathbb{P}_c \) build up conformally compactified phase space, which then is a space of automorphism of the conformal group \( C \) including reflections.

2 Conformally compactified phase space

For simultaneous compactification of \( \mathcal{M} \) and \( \mathbb{P} \) in \( M_c \) and \( \mathbb{P}_c \) no exact Fourier transform is known. It can be only approximated by a finite lattice phase space \[5\].

An exact Fourier transform may be defined instead in the 2-dimensional space time when \( M = \mathbb{R}^{1,1} = \mathbb{P} \) and for which

\[ \mathcal{M}_c = \frac{S^1 \times S^1}{\mathbb{Z}_2} = \mathbb{P}_c \]
Then inscribing in each $S_1$, of radius $R$, of $\mathbb{M}_c$ and in each $S_1$ of radius $K$ of $\mathbb{P}_c$ a regular polygon with

$$2N = 2\pi RK$$

vertices any function $f(x_{mn})$ defined in the resulting lattice $M_L \subset \mathbb{M}_c$ is correlated to the Fourier-transformed $F(k_{\rho\tau})$ on the $P_L \subset \mathbb{P}_c$ lattice by the finite Fourier series:

$$f(x_{nm}) = \frac{1}{2\pi R^2} \sum_{\rho,\tau = -N}^{N-1} \varepsilon^{(n\rho - m\tau)} F(k_{\rho\tau})$$

$$F(k_{\rho\tau}) = \frac{1}{2\pi K^2} \sum_{n,m = -N}^{N-1} \varepsilon^{-(n\rho - m\tau)} f(x_{nm})$$

where $\varepsilon = e^{i\frac{\pi}{N}}$ is the $2N$-root of unit. They may be called Fourier transforms since either for $R \to \infty$ or $K \to \infty$, (or both) they coincide with the standard ones. This further confirms the identification of $\mathbb{P}_c$ with momentum space which is here on purpose characterized geometrically rather than algebraically (Poisson bracket). On this model the action of the conformal group $O(2,2)$ may be easily operated and tested.

### 3 Conformal duality

The non linear, local action of $I$ on $\mathbb{M}$ is well known; for $x_{\mu} \in \mathbb{M}$ we have:

$$I : x_{\mu} \to I(x_{\mu}) = \pm \frac{x_{\mu}}{x^2}$$

For $x^2 \neq 0$ and $x_{\mu}$ space-like, if $x$ indicates the distance of a point from the origin, we have:

$$I : x \to I(x) = \frac{1}{x}$$

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\footnote{I maps every point, at a distance $x$ from the center, in the sphere $S^2$, to a point outside of it at a distance $x^{-1}$. For $\mathbb{M} = \mathbb{R}^{2,1}$ the sphere $S^2$ reduces to a circle $S^1$ and then \footnote{\text{\textsuperscript{2}}} reminds Target Space duality in string theory \footnote{\text{\textsuperscript{3}}}, which then might be the consequence of conformal inversion. For $\mathbb{M} = \mathbb{R}^{1,1}$, that is for the two dimensional model $I$ may be locally represented through quotient rational transformation by means of $I = i\sigma_2$ and the result is \footnote{\text{\textsuperscript{4}}}, which in turn represents the action of $I$ for the conformal group $G = \{D,P_1,T_1\}$ on a straight line $\mathbb{R}^1$.}
Since $\mathbb{M}$ is densely contained in $\mathbb{M}_c$, $x_\mu$ defines a point of the homogeneous space $\mathcal{H}$, of automorphisms for $C$; as such $x_\mu$ must then be conceived as dimensionless in (3), as usually done in mathematics. Therefore for physical applications, to represent space-time we must substitute $x_\mu$ with $x_\mu/l$, where $l$ represents an (arbitrary) unit of lengths, then from (3) and (7) we obtain:

**Proposition $P_1$:** Conformal reflections determine a map in space of the microworld to the macroworld (with respect to $l$) and vice-versa.

The conformal group may be well represented in momentum space $\mathbb{P} = \mathbb{R}^{3,1}$, densely contained in $\mathbb{P}_c$, where the action of $I$ induces non linear transformation like (3) and (4), where $x_\mu$ and $x$ are substituted by $k_\mu \in \mathbb{P}$ and $k$. If we then take (7) and the corresponding for $\mathbb{P}$ we obtain (see also footnote 2):

$$I : xk \rightarrow I(xk) = \frac{1}{xk}$$

Now physical momentum $p$ is obtained after multiplying the wave-number $k$ by an (arbitrary) unit of action $H$ by which (8) becomes:

$$I : \frac{xp}{H} \rightarrow I\left(\frac{xp}{H}\right) = \frac{H}{xp}$$

from which we obtain:

**Proposition $P_2$:** Conformal reflections determine a map, in phase space of the world of micro actions to the one of macro actions (with respect to $H$), and vice-versa.

Now if we choose for the arbitrary unit $H$ the Planck's constant $\hbar$ then from propositions $P_1$ and $P_2$ we have:

**Corollary $C_2$:** Conformal reflections determine a map between classical and quantum mechanics.

Let us now remind the identifications $\mathbb{M}_c \equiv C/c_1$ and $\mathbb{P}_c \equiv C/c_2$ to be conceived as two copies of the homogeneous space $\mathcal{H}$, representing conformally compactified space-time and momentum space respectively, and that $I\mathbb{M}_cI^{-1} = \mathbb{P}_c$ and then $I$ represents a map of every point $x_\mu$ of $\mathbb{M}$ to a point $k_\mu$ of $\mathbb{P}$: $I \rightarrow I(x_\mu) = p_\mu$ and we have:

**Proposition $P_3$:** Conformal reflections determine a map between space-time and momentum space.

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3 The action of $I$ may be rigorously tested in the two dimensional model where $I(x_{nm}) = k_{nm}$ and the action of $I$ is linear in the compactified phase space at difference with its local non linear action in $\mathbb{M}$ and $\mathbb{P}$, as will be further discussed elsewhere.
Let us now assume, as the history of celestial mechanics suggests, that space-time $\mathbb{M}$ is the most appropriate for the description of classical mechanics then, as a consequence of propositions $P_1$, $P_2$ and of corollary $C_2$, momentum space should be the most appropriate for the description of quantum mechanics. The legitimacy of this conjecture seems in fact to be supported by spinor geometry as we will see.

4 Quantum mechanics in momentum space

Notoriously the most elementary constituents of matter are fermions, represented by spinors, whose geometry, as formulated by its discoverer E. Cartan \[7\] has already the form of equations of motions for fermions in momentum space.

In fact given a pseudo euclidean, $2n$-dimensional vector space $V$ with signature $(k, l)$; $k + l = 2n$ and the associated Clifford algebra $\mathbb{C}\ell (k, l)$ with generators $\gamma_a$ a Dirac spinor $\psi$ is an element of the endomorphism space of $\mathbb{C}\ell (k, l)$ and is defined by Cartan’s equation

$$\gamma_a p^a \psi = 0$$  \hspace{1cm} (10)

where $p_a$ are the components of a vector $p \in V$.

Now it may be shown that for the signatures $(k, l) = (3, 1), (4, 1)$, the Weyl, (Maxwell), Majorana, Dirac equations, respectively may be naturally obtained \[8\], from (10), precisely in momentum space. For the signature $(4, 2)$ eq. (10) contains twistors equations and, for $\psi \neq 0$, the vector $p$ is null: $p_a p^a = 0$ and the directions of $p$ form the projective quadric \[\mathbb{P}^c (S^3 \times S^1)/\mathbb{Z}_2\] identical to conformally compactified momentum space $\mathbb{P}_c$ given in (2).

For the signature $(5, 3)$ instead one may easily obtain \[8\] from (10) the equation

$$ (\gamma_\mu p^\mu + \bar{\pi} \cdot \bar{\sigma} \otimes \gamma_5 + M) N = 0 $$  \hspace{1cm} (11)

where $\bar{\pi} = \langle \bar{N}, \bar{\sigma} \otimes \gamma_5 \rangle$ and $N = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$; $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$; $\bar{N} = \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{bmatrix}$.

\[\text{Since Weyl equation in } \mathbb{P} = \mathbb{R}^{3,1}, \text{ out of which Maxwell’s equation for the electromagnetic tensor } F_{\mu\nu} \text{ (expressed bilinearly in term of Weyl spinors) may be obtained, is contained in twistor equation in } \mathbb{P} = \mathbb{R}^{4,2}, \text{ defining the projective quadric: } \mathbb{P}_c = (S^3 \times S^1)/\mathbb{Z}_2. \text{ This is a further argument why momentum space } \mathbb{P} \text{ should be densely contained in } \mathbb{P}_c.\]
\[ \tilde{\psi}_j = \psi_j^\dagger \gamma_0, \text{ with } \psi_1, \psi_2 - \text{space-time Dirac spinors, and } \bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \text{ Pauli matrices.} \]

Eq. (11) represents the equation, in momentum space, of the proton-neutron doublet interacting with the pseudoscalar pion triplet \( \vec{\pi} \).

Also the equation for the electroweak model may be easily obtained \[ \text{in the frame of} \ \mathbb{C} \ell(5, 3). \]

All this may further justify the conjecture that spinor geometry in conformally compactified momentum space is the appropriate arena for the description of quantum mechanics of fermions.

It is remarkable that for signatures \((3, 1), (4, 1)\) and \((5, 3), (7, 1)\) (while not for \((4, 2)\)) the real components \(p_a\) of \(p\) may be bilinearly expressed in terms of spinors \[8\].

If spinor geometry in momentum space is at the origin of quantum mechanics of fermions, then their multiplicity, observed in natural phenomena, could be a natural consequence of the fact that a Dirac spinor associated with \(\mathbb{C} \ell(2n)\) has \(2^n\) components, which naturally splits in multiplets of space-time spinors (as it already appears in eq. (11)). Since in this approach vectors appear as bilinearly composed by spinors, some of the problematic aspects of dimensional reduction could be avoided dealing merely with spinors \[10\].

Also rotations naturally arise, in spinor spaces as products of reflections operated by the generators \(\gamma_a\) of the Clifford algebras. These could then be at the origin of the so-called internal symmetry. This appears in eq. (11) where the isospin symmetry of nuclear forces arises from conformal reflections which appear there as the units of the quaternion field of numbers of which, proton-neutron equivalence, for strong interactions, could be a realization in nature. For \(\mathbb{C} \ell(8)\) and higher Clifford algebras, and the associated spinors, octonions could be expected to play a role as recently advocated \[10\].

5 Some further consequences of conformal duality

Compact phase space implies, for field theories, the absence of the concept of infinity in both space-time and momentum and, provided we may rigorously define Fourier dual manifolds, where fields may be defined, it would imply also the absence of both infrared and ultraviolet divergences in perturbation expansions. This is, for the moment, possible only in the four-dimensional
phase space-model, where such manifold restricts to the lattice $M_L$ and $P_L$ on a double torus, where Fourier transforms (5) hold. One could call $M_L$ and $P_L$ the physical spaces which are compact and discrete, to distinguish them from the mathematical spaces $M_c$ and $P_c$ which are compact and continuous, the latter are only conformally dual while the former are both conformally and Fourier dual.

In the realistic eight dimensional phase space one would also expect to find physical spaces represented by lattices [5] as non commutative geometry seems also to suggest [11].

Let us now consider as a canonical example of a quantum system: the hydrogen atom in stationary states. According to our hypothesis it should be appropriated to deal with it in momentum space (as a non relativistic limit of the Dirac equation for an electron subject to an external e.m. field). This is possible as shown by v. Fock [12] through the integral equation:

$$\phi(p) = \frac{\lambda}{2\pi^2} \int_{S^3} \frac{\phi(q)}{(p-q)^2} d^3q$$

where $S^3$ is the one point compactification of momentum space $\mathbb{P} = \mathbb{R}^3$, and

$$\lambda = \frac{e^2}{\hbar c} \sqrt{\frac{p_0^2}{2mE}},$$

where $p_0$ is a unit of momentum and $E$ the (negative) energy of the H-atom.

For $\lambda = n + 1$, $\phi(p) = Y_{nlm}(\alpha\beta\gamma)$ which are the harmonics on $S^3$, and for $p_0 = mc$, we obtain:

$$E_n = -\frac{me^4}{2\hbar^2 (n+1)^2}$$

which are the energy eigenvalues of the H-atom.

It is interesting to observe that eq. (12) is a purely geometrical equation, where the only quantum parameter is the dimensionless fine structure constant

$$\frac{e^2}{\hbar c} = \frac{1}{137},...$$

According to this equation the stationary states of the H-atom may be represented as eigenvibrations of the $S^3$ sphere (of radius $p_0$) in conformally compactified momentum space and out of which the quantum numbers $n,l,m$...

\[\text{The geometrical determination of this parameter in eq. (12) (through harmonic analysis, say) could furnish a clue to understanding the geometrical origin of quantum mechanics. This was a persistent hope of the late Wolfgang Pauli.}\]
result. If conformal duality is realized in nature then there should exist a classical system represented by eigenvibration of $S^3$ in $\mathbb{M}_c$ or $\mathbb{M}_{RW}$.

In fact this system could be the universe since recent observations on distant galaxies (in the direction of N-S galactic poles) have revealed that their distribution may be represented $^{[13]}$ by the $S^3$ eigenfunction

$$Y_{n,0,0} = k_n \frac{\sin(n + 1) \chi}{\sin \chi}$$

with $k_n$ a constant, $\chi$ the geodetic distance from the center of the corresponding eigenvibration on the $S^3$ sphere of the $\mathbb{M}_{RW}$ universe. Now $Y_{n,0,0}$ is exactly equal to the eigenfunction of the H-atom however in momentum space. If the astronomical observations will confirm eq. (13) then the universe and the H-atom would represent a realization in nature of conformal duality. Here we have in fact that $Y_{n,0,0}$ on $\mathbb{P}_c$ represents the (most symmetric) eigenfunction of the (quantum) H-atom and the same $Y_{n,0,0}$ on $\mathbb{M}_c$ may represent the (visible) mass distribution of the (classical) universe. They could then be an example $^6$ of conformally dual systems: one classical and one quantum mechanical.

There could be another important consequence of conformal duality. In fact suppose that $\mathbb{M}_c$ and $\mathbb{P}_c$ are also Fourier dual $^7$. Then to the eigenfunctions: $Y_{nlm}(\alpha\beta\gamma)$ on $\mathbb{P}_c$ of the H-atom there will correspond on $\mathbb{M} = \mathbb{R}^{3,1}$ (densely contained in $\mathbb{M}_c$) their Fourier transforms; that is the known eigenfunctions $\Psi_{nlm}(x_1,x_2,x_3)$ in $\mathbb{M} = \mathbb{R}^{3,1}$ of the H-atom stationary states.

Now according to propositions $P_1$, $P_3$ and corollary $C_2$ for high values of the action of the system; that is for high values of $n$ in $Y_{nlm}$, the system should be identified with the corresponding classical one, and in space-time $\mathbb{M}$ where it is represented by $\Psi_{nlm}(x_1,x_2,x_3)$.

$^6$There could be other examples of conformal duality represented by our planetary system. In fact observe that in order to compare with the density of matter the square of the $S^3$ harmonic $Y_{nlm}$ has to be taken. Now $Y^2_{n00}$ presents maxima for $r_n = (n + \frac{1}{2}) r_0$, and it has been shown by Y.K. Gulak $^{[14]}$ that the values of the large semi axes of the 10 major solar planets satisfy this rule, which could then suggest that they arise from a planetary cloud presenting the structure of a $S^3$ eigenvibrations, as will be discussed elsewhere.

$^7$ Even if, for defining rigorously Fourier duality for $\mathbb{M}_c$ and $\mathbb{P}_c$ one may have to abandon standard differential calculus, locally it may be assumed to be approximately true, with reasonable confidence since the spacing of the possible lattice will be extremely small. In fact taking for $K$ the Planck’s radius one could have of the order of $10^{30}$ points of the lattice per centimeter.
In fact this is what postulated by the correspondence principle: for high values of the quantum numbers the wavefunction $\Psi_{nlm}(x_1, x_2, x_3)$ identifies with the Kepler orbits; that is the same system (with potential proportional to $1/r$) dealt in the frame of classical mechanics. In this way, at least in this particular example, the correspondence principle appears as a consequence of conformal duality\footnote{Also the property of Fourier transforms play a role. Consider in fact a classical system with fixed orbits: a massive point particle on $S^1$, say. Its quantization appears in the Fourier dual momentum space $\mathbb{Z}$: $(m = 0, \pm 1, \pm 2, \ldots)$ and for the large $m$ the eigenfunctions identifies with $S^1$.} and precisely of propositions $P_1$, $P_3$ and corollary $C_2$. Obviously at difference with the previous case of duality, in this case it is the same system (the two body problem) dealt once quantum mechanically in $\mathbb{P}_c$ and once classically in $\mathbb{M}_c$. What they keep in common is the $SO(4)$ group of symmetry, named accidental symmetry when discovered by W. Pauli for the Kepler orbits, while here it derives from the properties of conformal reflections, which preserve $S^3$, as seen from $[2]$.  

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