Parafermions: A New Conformal Field Theory

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Abstract

A new parafermionic algebra associated with the homogeneous space $A_2^{(2)}/U(1)$ and its corresponding $Z$-algebra have been recently proposed. In this paper, we give a free boson representation of the $A_2^{(2)}$ parafermion algebra in terms of seven free fields. Free field realizations of the parafermionic energy-momentum tensor and screening currents are also obtained. A new algebraic structure is discovered, which contains a $W$-algebra type primary field with spin two.

1 Introduction

The notion of parafermions \cite{1} was introduced in the context of statistical models and conformal field theory \cite{2}. Parafermions generalize the Majorana fermions and have found important applications in many areas of physics. From statistical mechanics point of view, parafermions are related to the exclusion statistics introduced by Haldane \cite{3}. In particular, the $Z_k$ parafermion models offer various extensions of the Ising model which corresponds to the $k = 2$ case \cite{4, 5, 6, 7, 8, 9, 10}.

The category for parafermions (nonlocal operators) is the generalized vertex operator algebra \cite{11, 12}. The $Z_k$ parafermion algebra was referred to as $Z$-algebra in \cite{11, 12}, and it was proved that the $Z$-algebra is identical with the $A_1^{(1)}$ parafermions.

The $Z_k$ parafermions proposed in \cite{1} are basically related to the simplest $A_1^{(1)}$ algebra. Various extensions have been considered by many researchers. Gepner proposed a parafermion algebra associated with any given untwisted affine Lie algebra $G^{(1)}$ \cite{13, 14}, which has been subsequently used in the study of $D$-branes. The operator product expansions (OPEs) and the corresponding $Z$-algebra of the untwisted parafermions were studied in \cite{15}, and a $W_3$-algebra was constructed from the $SU(3)_k$ parafermions. In \cite{16} a $W_5$-algebra was constructed by using the $SU(2)_k$ parafermions.

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More recently, Camino et al extended the $Z_k$ parafermion algebra and investigated graded parafermions associated with the $osp(1|2)^{(1)}$ Lie superalgebra [17]. The central charge of the graded parafermions algebra is $c = -\frac{3}{2k+3}$, which implies that for $k$ a positive integer, $c$ is always negative. Thus the graded parafermion theory is not unitary.

In [20], we found a new type of nonlocal currents (quasi-particles), which were referred to as twisted parafermions. The nonlocal currents take values in the homogeneous space $A_2^{(2)}/U(1)$. To distinguish our theory from the twisted versions of [1, 11, 12], in this paper we refer our nonlocal currents to as $A_2^{(2)}$ parafermions.

The $A_2^{(2)}$ parafermion system contains a boson-like spin-1 field and six nonlocal fields with fractional spins $1 - \frac{1}{4k}$ and $1 - \frac{1}{k}$, and leads to a new conformal field theory which is different from the known ones. Let us remark that, while fields with conformal dimensions of $1 - \frac{1}{4k}$ and $1 - \frac{1}{k}$ also appeared in the graded parafermion algebra [17], our $A_2^{(2)}$ parafermion theory is quite different. In particular, our theory is unitary and has a different central charge.

In this paper, we investigate the theory further. We give a free field representation of the $A_2^{(2)}$ parafermion theory in terms of seven free bosons. We also obtain the free field realization of the highest weight state and screening currents. We discover a new algebraic structure which contains a spin-2 primary field. This is similar to a $W$-algebra structure but now the spin is two.

The layout of the paper is as follows. In section 2, we briefly review the $A_2^{(2)}$ parafermion algebra obtained in [20], which will be extensively used in this paper. In section 3, we give some results concerning the $A_2^{(2)}$ parafermion Hilbert space. In section 4, we give a free field representation of the $A_2^{(2)}$ parafermion currents, highest weight state and screening currents. We present a new $W$-algebra structure in section 5.

2 Twisted parafermions: a brief review

The parafermion current algebra proposed in [21] is related to the twisted affine Lie algebra $A_2^{(2)}/U(1)$. This $A_2^{(2)}$ parafermion theory is not an extension of the usual $Z_k$ parafermions, but a twisted version of the Gepner construction [13]. To begin with, let $g$ be a simple finite-dimensional Lie algebra and $\sigma$ be an automorphism of $g$ satisfying $\sigma^r = 1$ for a positive integer $r$, then $g$ can be decomposed into the form [21]:

$$g = \bigoplus_{j \in \mathbb{Z}/r} g_j,$$

where $g_j$ is the eigenspace of $\sigma$ with eigenvalue $e^{2j\pi i/r}$, and $[g_i, g_j] \subset g_{(i+j) \bmod r}$, then $r$ is called the order of the automorphism. For $A_2^{(2)}$ algebra, we have $g = A_2$ so that

$$A_2 = g_0 \oplus g_1,$$

where $g_0$ is the fixed point subalgebra of $A_2$ under the automorphism and $g_1$ is the five dimensional representation of $g_0$. $g_0$ and $g_1$ satisfy $[g_0, g_1] \subset g_{(i+j) \bmod 2}$. We associate a
generating parafermion to a vector in the root lattice $M \mod kM_L$, where $M_L$ is the long root lattice and $k$ is a constant identified with the level of the corresponding twisted affine algebra. Generating parafermions are defined by projecting out the affine currents corresponding to the Cartan subalgebra $\mathfrak{g}$. Denote by $\psi_a$ the $A_2^{(2)}$ generating parafermions with $a = \tilde{0}, \pm \alpha, \pm \tilde{\alpha}, \pm \tilde{a}_2$ and $\tilde{\alpha}_2 = \alpha + \tilde{\alpha}$, where $\tilde{\alpha}^2 = \tilde{0} \cdot \alpha = \tilde{0} \cdot \tilde{\alpha} = \tilde{0} \cdot \tilde{\alpha}_2 = 0$ and $\alpha^2 = \tilde{\alpha}^2 = \alpha \cdot \tilde{\alpha} = 1$, $\tilde{\alpha}_2^2 = 4$. Then $\psi_{\pm a}$ are currents in $g_0$ and $\psi_{\tilde{0}, \psi_{\pm \tilde{\alpha}}, \psi_{\pm \tilde{a}_2}$ are currents in $g_1$. Note that $\tilde{0}$ is a "null" root vector in the sense that both its length and its dot product with all other root vectors are zero. So parafermion currents associated with $A_2^{(2)}$ live in either Ramond sector (i.e. the currents in $g_0$) or Neveu-Schwarz sector (i.e. the currents in $g_1$).

Due to the mutually semilocal property between two parafermions, the radial ordering products are multivalued functions. So we define the radial ordering product of (generating) $A_2^{(2)}$ parafermions (TPFs) as

$$\psi_a(z)\psi_b(w)(z - w)^{a-b/2k} = \psi_b(w)\psi_a(z)(w - z)^{a-b/2k},$$

where $a = \tilde{0}, \pm \alpha, \pm \tilde{\alpha}, \pm \tilde{a}_2$. From the above definition, we have

$$T_\psi(z)\psi_a(w) = \frac{\Delta_a}{(z-w)^2}\psi_a(w) + \frac{1}{z-w}\partial\psi_a(w) + \ldots.$$  

where $T_\psi$ is the energy-momentum tensor of the theory, and $\Delta_a = 1 - \frac{a_{\alpha}^2}{4k}$ is the conformal dimension of current $\psi_a$. It follows that the conformal dimensions of the $A_2^{(2)}$ parafermion currents are $1 (a = \tilde{0}), 1 - \frac{1}{4k} (a = \pm \alpha, \pm \tilde{\alpha})$ and $1 - \frac{k}{4} (a = \pm \tilde{a}_2)$, for a given level $k$. The values of the dimensions determine the leading singularity in the OPE of the currents. In fact, we have

$$\psi_a(z)\psi_b(w) = (z-w)^{\Delta_{a+b} - \Delta_a - \Delta_b} (\varepsilon_{a,b}\psi_{a+b}(w) + \ldots),$$

$$\psi_a(z)\psi_{-a}(w) = (z-w)^{-2\Delta_a}I_0(z) + \ldots,$$

where $\varepsilon_{a,b}$ are the constants, $I_0(z)$ will be determined below and $\Delta_a, \Delta_b$ and $\Delta_{a+b}$ are the conformal dimensions of $\psi_a, \psi_b$ and $\psi_{a+b}$, respectively. If $\Delta_{a+b} - \Delta_a - \Delta_b \geq 0$, then two operators $\psi_a, \psi_b$ commute. The nontrivial OPEs are given by $\Delta_{a+b} - \Delta_a - \Delta_b < 0$. For example,

$$\Delta_{\tilde{a}_2} - \Delta_{\tilde{\alpha}} - \Delta_{\alpha} = 1 - \frac{1}{k} - 2 \left(1 - \frac{1}{4k}\right) = -1 - \frac{1}{2k}.$$  

In the OPE form, this corresponds to

$$\psi_a(z)\psi_{\tilde{\alpha}}(w) = \frac{\varepsilon_{a,\tilde{\alpha}}}{(z-w)^{1+1/2k}}\psi_{\tilde{a}_2}(w) + \ldots,$$

or equivalently,

$$\psi_a(z)\psi_{\tilde{\alpha}}(w)(z-w)^{1/2k} = \frac{\varepsilon_{a,\tilde{\alpha}}}{z-w}\psi_{\tilde{a}_2}(w) + \ldots.$$  

Similarly,
\[ -\Delta_\alpha - \Delta_{-\alpha} = -2 \left( 1 - \frac{1}{4k} \right) = -2 + \frac{1}{2k}. \]  

(2.10)

From the dimensional analysis, the conformal dimension of \( I_0(z) \) on the r.h.s. of the 2nd equation of (2.9) is zero. It follows that \( I_0(z) \) is a constant, which will be set to 1 in the sequel.

We have seen that \( \psi_0 \) is a spin-1 current in \( g_1 \) that corresponds to the “null” root vector \( \vec{0} \). Keeping in mind the fact that \( \psi_{\pm \alpha} \in g_0 \) and \( \psi_{\pm \bar{\alpha}} \in g_1 \), we have \( \psi_\alpha \psi_{-\bar{\alpha}} = \psi_{\alpha-\bar{\alpha}} \in g_1 \), \( \psi_{\bar{\alpha}} \psi_\alpha \sim \psi_{\bar{\alpha}-\alpha} \in g_1 \), \( \psi_0 \psi_{\alpha} \sim \psi_{0+\alpha} \in g_1 \) and \( \psi_0 \psi_{\bar{\alpha}} \sim \psi_{0+\bar{\alpha}} \in g_0 \). From the dimensional analysis and noting that \((0+\alpha)^2 = 1 = (0+\bar{\alpha})^2 \) and \((\alpha-\bar{\alpha})^2 = 0 = (\bar{\alpha}-\alpha)^2 \), we have the identifications: \( \psi_{0+\alpha} \sim \psi_\alpha \), \( \psi_{0+\bar{\alpha}} \sim \psi_{\bar{\alpha}} \), \( \psi_{\alpha-\bar{\alpha}} \sim \psi_0 \) and \( \psi_{\bar{\alpha}-\alpha} \sim \psi_{\bar{0}} \). Thus we have

\[
\begin{align*}
\psi_\alpha(z)\psi_{-\bar{\alpha}}(w)(z-w)^{-1/2k} &= \frac{\epsilon_{\alpha,\bar{\alpha}}}{z-w}\psi_0(w) + \cdots, \\
\psi_0(z)\psi_\alpha(w) &= \frac{\epsilon_{0,\alpha}}{z-w}\psi_\alpha(w) + \cdots, \\
\psi_0(z)\psi_{\bar{\alpha}}(w) &= \frac{\epsilon_{0,\bar{\alpha}}}{z-w}\psi_\alpha(w) + \cdots.
\end{align*}
\]

We see that \( \psi_0(z) \) is a spin-1 primary field which transforms the fields in the Ramond sector to those in the Neveu-Schwarz sector or vice versa. This is easy to understand from the theory of the \( A_2 \) current algebra [22].

Summarizing, we may write the OPEs in the general form:

\[
\begin{align*}
\psi_\mu(z)\psi_\nu(w)(z-w)^{\mu,\nu/2k} &= \frac{\delta_{\mu+\nu,0}}{(z-w)^2} + \frac{\epsilon_{\mu,\nu}}{z-w}\psi_{\mu+\nu}(w) + \cdots, \\
\psi_{\bar{\mu}}(z)\psi_{\bar{\nu}}(w)(z-w)^{\bar{\mu},\bar{\nu}/2k} &= \frac{\delta_{\bar{\mu}+\bar{\nu},0}}{(z-w)^2} + \frac{\epsilon_{\bar{\mu},\bar{\nu}}}{z-w}\psi_{\bar{\mu}+\bar{\nu}}(w) + \cdots, \\
\psi_\mu(z)\psi_{\bar{\nu}}(w)(z-w)^{\mu,\bar{\nu}/2k} &= \frac{\epsilon_{\mu,\bar{\nu}}}{z-w}\psi_{\mu+\bar{\nu}}(w) + \cdots, \tag{2.11}
\end{align*}
\]

where \( \mu, \nu = \pm \alpha, \pm \bar{\alpha}, \bar{0}, \pm \alpha_2; \epsilon_{\mu,\nu}, \epsilon_{\bar{\mu},\bar{\nu}} \) and \( \epsilon_{\mu,\bar{\nu}} \) are structure constants. For consistency, \( \epsilon_{a,b} \) must have the properties: \( \epsilon_{a,b} = -\epsilon_{b,a} = -\epsilon_{-a,-b} = \epsilon_{-a,a+b} \) and \( \epsilon_{a,-a} = 0 \). The following structure constants have been found in [20]:

\[
\begin{align*}
\epsilon_{\alpha,-\bar{\alpha}} = \epsilon_{\alpha,\bar{\alpha}} = \epsilon_{-\alpha,\bar{\alpha}} = \frac{1}{\sqrt{k}}, \\
\epsilon_{\alpha,-\bar{\alpha}} = \epsilon_{0,\bar{0}} = \frac{\sqrt{3}}{2k}.
\end{align*}
\]

(2.12)

and other constants can be obtained by using the symmetry properties. The above structure constants are compatible with the associativity requirement of the OPEs. This is seen as follows. Let \( A_a \) and \( B_b \) be two arbitrary functions of the \( A_2^{(2)} \) parafermions with charges \( a \) and \( b \), respectively. We write their OPEs as

\[
A_a(z)B_b(w)(z-w)^{a-b/2k} = \sum_{n=-[h_A+h_B]}^{\infty} [A_aB_b]_{-n}(w)(z-w)^n, \tag{2.13}
\]

(2.13)
where \([h_A] \) stands for the integral part of the dimension of \(A\). Hence we have \([A_{a}B_{b}]_{n}(w) = \int_{w} dz \ A_{a}(z)B_{b}(w)(z - w)^{n-1+a-b/2k}\). It is easy to find the following relation between the three-fold radial ordering products

\[
\left\{ \int_{w} du \int_{w} dz \ R(A(u)R(B(z)C(w))) \\
- \int_{w} dz \int_{w} du (-)^{a-b/2k} R(B(z)R(A(u)C(w))) \\
- \int_{w} dz \int_{z} du R(R(A(u)B(z))C(w)) \right\} \\
(z - w)^{p-1+b-c/2k}(u - w)^{q-1+a-c/2k}(u - z)^{r-1+a-b/2k} = 0,
\]  
\[(2.14)\]

where integers \(p, q, r\) are in the regions: \(-\infty < p \leq [h_B + h_C]\), \(-\infty < q \leq [h_C + h_A]\), \(-\infty < r \leq [h_A + h_B]\); \(a, b, c\) are parafermionic charges of the fields \(A, B\) and \(C\), respectively. Performing the binomial expansions, we obtain the following identity:

\[
\sum_{i=p}^{[h_B+h_C]} C_{r-1+a-b/2k}^{(i-p)}[A[BC]]_{i}Q^{-i}(w) + \sum_{j=q}^{[h_C+h_A]} C_{r-1+a-b/2k}^{(j-q)}[B[AC]]_{j}Q^{-j}(w) \\
= \sum_{l=r} (-)^{(l-r)} C_{q-1+a-c/2k}^{(l-r)}[AB]_{l}C[Q^{-l}(w)],
\]  
\[(2.15)\]

where \(Q = p + q + r - 1\). This identity is the twisted Jacobi identity of the parafermion algebra.

Now the structure constants of the currents can be derived from the (2.13). For example, we calculate the OPEs of \([\psi_{a}\psi_{-a}]_{0}\) with \(\psi_{a}\) and \([\psi_{b}\psi_{-b}]_{0}\). Setting \(Q = 2, q = 1\) and \(r = 0\) in (2.15), and comparing with (2.4), then we have

\[
\sum_{a} \varepsilon_{a,b}\varepsilon_{-a,a+b} = \frac{6 - b^{2}}{k}, \quad \sum_{a} a \cdot b = 0, \quad \sum_{a} (a \cdot b)^{2} = 12b^{2}.
\]  
\[(2.16)\]

The structure constants (2.12) satisfy all these equations.

A special case of the parafermion OPEs is

\[
\psi_{a}(z)\psi_{-a}(w)(z - w)^{-a^{2}/2k} = \frac{1}{(z - w)^{2}} + \left(2 + \frac{24(1 - \Delta_{a})}{a^{2}}\right)t_{a} + \cdots,
\]  
\[(2.17)\]

where terms \(t_{a}\) in the right hand side are fields with conformal dimension 2. From the conformal field theory, they should be related to the energy-momentum tensor. Indeed, if we define,

\[
T_{\psi}(z) = \sum_{a} t_{a}
\]  
\[(2.18)\]

then

\[
T_{\psi}(z)T_{\psi}(w) = \frac{c_{\psi}/2}{(z - w)^{4}} + \frac{2T_{\psi}(w)}{(z - w)^{2}} + \frac{\partial T_{\psi}(w)}{z - w} + \cdots,
\]  
\[(2.19)\]
where
\[ c_\psi = 2 \sum_a \frac{a^2 \Delta_a}{2a^2 + 24(1 - \Delta_a)} = 7 - \frac{24}{k + 3} = \frac{8k}{k + 3} - 1, \] (2.20)
is the central charge of the \( A_2^{(2)} \) parafermion theory.

### 3 \( A_2^{(2)} \) parafermion Hilbert space

we now analyze the structure of the Hilbert representation \( \mathcal{H} \) space of the parafermion theory. For every field in the parafermion theory there are a pair of charges \((\lambda, \bar{\lambda})\), which take values in the weight lattice. We denote such a field by \( \phi_{\lambda, \bar{\lambda}}(z, \bar{z}) \) [1, 13, 15]. Let \( \mathcal{H}_{\lambda, \bar{\lambda}} \) be the subspace of \( \mathcal{H} \) with the indicated charges. Then \( \mathcal{H} \) is the direct sum of the form:

\[ \mathcal{H} = \oplus \mathcal{H}_{\lambda, \bar{\lambda}}, \] (3.1)

The non-locality \( \gamma \) of two fields \( \phi_{\lambda, \bar{\lambda}} \) and \( \phi_{\lambda', \bar{\lambda}'} \) is defined by

\[ \gamma \left( \phi_{\lambda, \bar{\lambda}}, \phi_{\lambda', \bar{\lambda}'} \right) = \frac{1}{2k} \left( \lambda \cdot \lambda' - \bar{\lambda} \cdot \bar{\lambda}' \right). \] (3.2)

Note that the parafermionic current \( \psi_b(z) \) has the left charge \( \lambda = b \) and the right charge \( \bar{\lambda} = 0 \). So for the currents with zero right charges, the exponent is,

\[ \gamma \left( \psi_a, \psi_b \right) = \frac{a \cdot b}{2k}, \] (3.3)

which is exactly the exponent appeared in (2.11). If we rewrite the (2.11) as

\[ \psi_a(z) \psi_b(w)(z - w)^{a \cdot b/2k} \equiv \sum_{n=-\infty}^{\infty} (z - w)^n [\psi_a \psi_b]_n, \] (3.4)

then we have \([\psi_a \psi_b]_l = 0 \; (l \geq 3)\), \([\psi_a \psi_b]_2 = \delta_{a+b,0}\) and \([\psi_a \psi_b]_1 = \varepsilon_{a,b} \psi_{a+b}\).

The OPE of \( \psi_a \) with \( \phi_{\lambda, \bar{\lambda}}(z, \bar{z}) \) is given by

\[ \psi_a(z) \phi_{\lambda, \bar{\lambda}}(w, \bar{w}) = \sum_{m=-\infty}^{\infty} (z - w)^{-m-a \cdot \lambda/2k} A_{m}^{a,\lambda} \phi_{\lambda, \bar{\lambda}}(w, \bar{w}), \] (3.5)

where \( m \in \mathbb{Z} \) (Ramond sector) for \( a = \pm \alpha \) and \( m \in \mathbb{Z} + \frac{1}{2} \) (Neveu-Schwarz sector) for \( a = 0, \pm \tilde{\alpha}, \pm \tilde{\alpha}_2 \). \( A_{m}^{a,\lambda} \) are modes of the nonlocal parafermion field \( \psi_a(z) \) on \( \phi_{\lambda, \bar{\lambda}} \), and its action on \( \phi_{\lambda, \bar{\lambda}}(z) \) is defined by the integration

\[ A_{m}^{a,\lambda} \phi_{\lambda, \bar{\lambda}}(w, \bar{w}) = \oint_{c_w} (z - w)^{m-a \cdot \lambda/2k} \psi_a(z) \phi_{\lambda, \bar{\lambda}}(w, \bar{w}), \] (3.6)

where \( c_w \) is a contour around \( w \). From the OPEs, we know that the dimension of \( A_{m}^{a,\lambda} \phi_{\lambda, \bar{\lambda}}(w, \bar{w}) \) is

\[ \Delta(A_{m}^{a,\lambda} \phi_{\lambda, \bar{\lambda}}) = h - m - \frac{a \cdot \lambda}{2k} - \frac{a^2}{4k}, \] (3.7)

where \( h \) is the dimension of the parafermionic field \( \phi_{\lambda, \bar{\lambda}}(w, \bar{w}) \). We define \([\psi_a \psi_b]_{\lambda, \bar{\lambda}}^{m,n} \) by
\[ [\psi_a \psi_b]^{\lambda}_{-m} \phi_{\lambda, \bar{\lambda}}(w, \bar{w}) = \oint_{c_w} dz \ (z - w)^{m+n+(a+b)\cdot \lambda/2k} [\psi_a \psi_b]_{-n} (z) \phi_{\lambda, \bar{\lambda}}(w, \bar{w}), \quad (3.8) \]

Following the standard procedure in conformal field theory (for the parafermion theory, see [1]), we multiply the last equation on both sides by \( z^{m+a\cdot \lambda/2k} w^{n+b\cdot \lambda/2k} \) and integrate the resulting equation by choosing appropriate contours, to get

\[ \sum_{l=0}^{\infty} C_{a+b/2l}^{(l)} [A_{m-l}^{a+b+\lambda} A_{n+l}^{b+\lambda} - A_{n-l}^{b,a+\lambda} A_{m+l}^{a+\lambda}] = \left( m + \frac{a \cdot \lambda}{2k} \right) \delta_{a+b,0} \delta_{m+n,0} + \varepsilon_{a,b} A_{m+n}^{a+b+\lambda}, \quad (3.9) \]

where we have chosen the powers of \( z \) and \( w \) such that the integrands are single valued, and the binomial coefficients \( C_x^{(l)} \) are

\[ C_x^{(l)} = \frac{(-1)^l x(x-1) \cdots (x-l+1)}{l!} \quad (3.10) \]

and \( C_0^{(l)} = C_0^{(0)} = C_{-1}^{(l)} = 1, \quad C_0^{(p)} = 0, \quad \text{for} \quad l > p > 0. \) Similarly, we get

\[ \sum_{l=0}^{\infty} C_{-1+a-b/2l}^{(l)} [A_{m-l}^{a,\lambda-a} A_{n+l}^{-a,\lambda} + A_{n-l}^{-a,a+\lambda} A_{m+l}^{a,\lambda}] \]

\[ = \sum_a \frac{1}{2} \left( m + 1 + \frac{a \cdot \lambda}{2k} \right) \left( m + \frac{a \cdot \lambda}{2k} \right) \delta_{m+n,0} + \frac{2(k+3)}{k} L_{m+n}^{\lambda}, \quad (3.11) \]

where \( L_{m+n}^{\lambda} \) represents the action of the energy-momentum tensor \( L_{m+n} \) on the field \( \phi_{\lambda, \bar{\lambda}}. \) Using the above identity, we can show that \( L_m \) satisfies the Virasoro algebra with central charge \( c_\psi. \) We refer the above identity to as \( A_{2}^{(2)} \) Z-algebra. Generally, we have

\[ \sum_{p=0}^{\infty} C_{-p+1+a-b/2k}^{(p+2)} [A_{m-l-p+q}^{a+b+\lambda} A_{n+l+p-q}^{b+\lambda} + (-1)^{p} A_{n-l-q-1}^{b,a+\lambda} A_{m+l+q+1}^{a+\lambda}] \]

\[ = C_{m+q+1+a-b/2k}^{(p+1)} \delta_{a,-b} \delta_{m,-n} + C_{m+q+1+a-b/2k}^{(p+1)} \varepsilon_{a,b} A_{m+n}^{a+b+\lambda} \]

\[ + \sum_{r=0}^{\infty} C_{m+q+1+a-b/2k}^{(p-r)} [\psi_a \psi_b]^{\lambda}_{-r,m+n}; \quad p = 2q \quad \text{or} \quad 2q+1, \quad (3.12) \]

The above results will be extensively used in the sequel. Now let \( \phi_{\Lambda}^{\lambda} \) be a state in the Hilbert space \( \mathcal{H}, \) where \( \Lambda \) takes value on the weight lattices. The condition for \( \phi_{\Lambda}^{\lambda} \) to be a highest weight state is defined by
\[ A_n^{a,\Lambda} \phi_{\Lambda}^A = 0, \text{ for } n > 0, \text{ or } n = 0 \text{ if } a = \alpha, \tilde{\alpha}, \bar{\alpha}_2, \tag{3.13} \]
\[ L_0 \phi_{\Lambda}^A = \Delta_\Lambda \phi_{\Lambda}^A, \tag{3.14} \]
where \( \Delta_\Lambda \) is the conformal dimension of \( \phi_{\Lambda}^A \), which is found by using the identity (3.11) to be
\[ \Delta_\Lambda = \left[ \Lambda(\Lambda + 4) \frac{4}{4(k+3)} + \frac{\Lambda^2}{12(k+3)} - \frac{\Lambda^2}{4k} \right]. \tag{3.15} \]
This result agrees with that obtained from the free field calculation given in next section. All other state of \( \mathcal{H} \) can be obtained from \( \phi_{\Lambda}^A \) by applying \( A_n^{a,\Lambda} \) with \( n \geq 0 \) repeatedly.

We define the state
\[ \phi_{\Lambda}^A = \prod_j A_{-n_j}^{a_j,\lambda_j} \phi_{\Lambda}^A, \text{ } n_j \geq 0. \tag{3.16} \]
It is easy to show
\[ [L_m, A_{-n}^{a,\lambda}] = \left[ (m+1)(\Delta_a - 1) + (n - \frac{a \cdot \lambda}{2k}) \right] A_{m-n}^{a,\lambda}, \tag{3.17} \]
from which we find the conformal dimension of the field \( \phi_{\Lambda}^A \):
\[ \Delta_{\Lambda}^{\lambda} = \Delta_\Lambda + \frac{\Lambda^2}{4k} - \frac{\lambda^2}{4k} + \sum_i n_i, \tag{3.18} \]
where \( \lambda = \Lambda + \sum_i a_i \), and \( n_i \in \mathbb{Z} \) if \( a_i = \mu \), or \( n_i \in \mathbb{Z} + \frac{1}{2} \) if \( a_i = \tilde{\mu} \).

In the usual \( \mathbb{Z}_k \) parafermion theory, highest weight state \( \Phi^l \) exists only if the condition \( l \leq k \) is satisfied. This is also the unitarity condition of the representation. To obtain the unitarity condition for our theory, we define the hermiticity condition:
\[ (A_m^{a,\Lambda})^\dagger \phi_{\Lambda+a}^A = A_{-m}^{-a, a+\Lambda} \phi_{\Lambda+a}^A, \text{ } k^\dagger = k, \tag{3.19} \]
If \( v_{\Lambda} \) is a vacuum state, i.e
\[ A_n^{a,\Lambda} v_{\Lambda} = 0, \text{ } n > 0, \tag{3.20} \]
thus we have the norm
\[ (A_m^{a,\Lambda} v_{\Lambda}, A_m^{a,\Lambda} v_{\Lambda}) = (v_{\Lambda}, A_{-m}^{-a,a+\Lambda} A_m^{a,\Lambda} v_{\Lambda}) \\
= (v_{\Lambda}, [A_m^{-a,a+\Lambda} A_m^{a,\Lambda} - A_m^{-a,a+\Lambda} A_{-m}^{-a,a+\Lambda}] v_{\Lambda}), \\
= (v_{\Lambda}, (m - \frac{a \cdot \Lambda}{2k}) v_{\Lambda}). \tag{3.21} \]
This requires \( a \cdot \Lambda \leq 2km \) for the norm to be positive. Considering that the minimal value \( m \) can take is \( m = 1/2 \) in the Neveu-Schwarz sector and \( m = 1 \) in the Ramond sector, we must have \( a \cdot \Lambda \leq k \) for the representation to be unitary.
In general, the Verma module with highest weight $\Lambda$ and level $k$ contains infinite many singular vectors \([18, 19]\), and thus the representations obtained above are not irreducible. To determine irreducible modules, an BRST analysis \([18]\) in the Hilbert space is necessary, which we will not discuss here. Nevertheless, we will in the next section give a free field realization of screen currents of the parafermion theory, which is important ingredient in the BRST method.

4 Free field representation of the twisted parafermions

It was shown in \([20]\) that the twisted affine current algebra $A^{(2)}_2$ allows the following representation in terms of the twisted parafermionic currents:

$$
\begin{align*}
    j^+(z) &= 2\sqrt{k\psi_\alpha(z)}e^{\frac{i\sqrt{2}}{2k}\phi_0(z)}, \\
    j^0(z) &= 2\sqrt{2ki}\partial\phi_0(z), \\
    j^-(z) &= 2\sqrt{k\psi_{-\alpha}(z)}e^{-\frac{i\sqrt{2}}{2k}\phi_0(z)}, \\
    J^+(z) &= 2\sqrt{k\psi_\tilde{\alpha}_2(z)}e^{i\sqrt{2}k\phi_0(z)}, \\
    J^0(z) &= 2\sqrt{6k}\psi_0(z), \\
    J^-(z) &= 2\sqrt{k\psi_{-\tilde{\alpha}_2}(z)}e^{-i\sqrt{2}k\phi_0(z)}, \\
    J^{++}(z) &= 2\sqrt{k\psi_\alpha_2(z)}e^{i\sqrt{2}k\phi_0(z)}, \\
    J^{--}(z) &= 2\sqrt{k\psi_{-\tilde{\alpha}_2}(z)}e^{-i\sqrt{2}k\phi_0(z)}, \\
    J^0(z) &= 2\sqrt{6k}\psi_0(z).
\end{align*}
$$

(4.1)

where $\phi_0(z)$ is an $U(1)$ current obeying $\phi_0(z)\phi_0(w) = -ln(z - w)$, and has the modes expansion of $\partial\phi_0(z) = \sum_{n \in \mathbb{Z}} \phi_n z^{n+1}$.

On the other hand, we know that the twisted affine current algebra $A^{(2)}_2$ allows a free field representation in terms of three $(\beta, \gamma)$ pairs and one 2-component scalar field \([22]\). So to get a free field representation of the twisted parafermionic currents, one need to projecting out a $U(1)$ current, as is seen from (4.1), and regarding the parafermion currents as operators in the space $A^{(2)}_2/U(1)$. This implies that seven independent scalar fields are needed to realize the $A^{(2)}_2$ parafermion algebra. So, we introduce seven scalar fields, $\phi(z)$ and $\xi_j(z), \eta_j(z)$ ($j = 0, 1, 2$), which satisfy the following relations:

$$
\begin{align*}
    \xi_i(z)\xi_j(w) &= -\delta_{ij}ln(z - w), \\
    \eta_i(z)\eta_j(w) &= -\delta_{ij}ln(z - w), \\
    \phi(z)\phi(w) &= -ln(z - w).
\end{align*}
$$

The modes expansions are

$$
\begin{align*}
    \partial\chi(z) &= \sum_{n \in \mathbb{Z}} \chi_n z^{n+1}, \quad \chi = \xi_0, \quad \eta_0 \\
    \partial\chi(z) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} \chi_n z^{n+1}, \quad \chi = \xi_j, \quad \eta_j \quad (j = 1, 2) \quad \text{or} \quad \phi.
\end{align*}
$$

The conformal dimension of $\psi_b$ is $1 - \frac{\beta^2}{4k}$. So we make the ansatz about the free field representation of the twisted parafermionic currents:
\[\psi_a(z) = f_a(\xi_i(z), \eta_j(z), \phi(z))e^{\sqrt{\frac{a \Delta}{4k}} \phi(z)},\]
\[\psi_{-a}(z) = f_{-a}(\xi_i(z), \eta_j(z), \phi(z))e^{-\sqrt{\frac{a \Delta}{4k}} \phi(z)},\]

where \(a = 0, \alpha, \tilde{\alpha}, \tilde{\alpha}_2\), the factor \(e^{\pm \sqrt{\frac{a \Delta}{4k}} \phi(z)}\) will contribute \(-\frac{a^2}{4k}\) to the conformal dimension of \(\psi_{\pm a}(z)\), and \(f_{\pm a}(\xi_i(z), \eta_j(z), \phi(z))\) are operators with conformal dimension one.

From dimensional analysis, no term of the form \(e^{\sqrt{\frac{a \Delta}{4k}} \phi(z)}\) will appear in \(f_{\pm a}(\xi_i(z), \eta_j(z), \phi(z))\). Recall that the dimensions of \(f_{\pm a}(\xi_i(z), \eta_j(z), \phi(z))\) are 1, while the dimensions of \(\xi_i(z), \eta_j(z), \phi(z)\) are zero, so polynomials of \(f_{\pm a}\) are the functions of \(\partial \xi_i(z), \partial \eta_j(z), \partial \phi(z)\). Notice that \((\xi_i(z) - i\eta_i(z))(\xi_j(w) - i\eta_j(w))\) have no contribution to the OPE. We find, after a long and tedious calculation,

\[
f_a(z) = \frac{1}{2\sqrt{2k}} \left[ -\alpha_0 \partial \xi_0(z) - 2\partial \xi_1(z) + \partial \xi_2(z) + \partial \phi(z) \right] \times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -\alpha_0 (\xi_0(z) - i\eta_0(z)) \right. \right.
\]
\[
\left. \left. - 2(\xi_1(z) - i\eta_1(z)) + (\xi_2(z) - i\eta_2(z)) \right] \right\},
\]
\[
f_{-a}(z) = \frac{1}{2\sqrt{2k}} \left\{ - [(4k + 1) [\alpha_0 (\partial \xi_0(z) - i\partial \eta_0(z)) + 2(\partial \xi_1(z) - i\partial \eta_1(z))
\]
\[
\left. - (\partial \xi_2(z) - i\partial \eta_2(z)) - \partial \phi(z) \right] + i [\alpha_0 \partial \eta_0(z) + 2\partial \eta_1(z) - \partial \eta_2(z)] \right\} \times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ \alpha_0 (\xi_0(z) - i\eta_0(z)) + 2(\xi_1(z) - i\eta_1(z)
\]
\[
\left. - (\xi_2(z) - i\eta_2(z)) \right] \right\} + 3 [\alpha_0 \partial \xi_0(z) + 2\partial \xi_1(z) - \partial \xi_2(z) - \partial \phi(z)]
\]
\[
\times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -(4 + \alpha_0) (\xi_0(z) - i\eta_0(z)) - 2(1 - \alpha_0)(\xi_1(z) - i\eta_1(z)) + 3(\xi_2(z) - i\eta_2(z)) \right] \right\}
\]
\[
+ 2 [2\partial \xi_0(z) - \alpha_0 \partial \xi_1(z) - \partial \xi_2(z) + \partial \phi(z)]
\]
\[
\times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ 3(\xi_0(z) - i\eta_0(z)) - (\alpha_0 + 1)(\xi_1(z) - i\eta_1(z)
\]
\[
\left. - (1 - \alpha_0)(\xi_2(z) - i\eta_2(z)) \right] \right\} - 4 [\partial \xi_0(z) - \partial \xi_1(z) + \alpha_0 \partial \xi_2(z) - 2\partial \phi(z)]
\]
\[
\times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -7(\xi_0(z) - i\eta_0(z)) + (1 + 3\alpha_0)(\xi_1(z) - i\eta_1(z)
\]
\[
\left. + (3 - \alpha_0)(\xi_2(z) - i\eta_2(z)) \right] \right\} + 2\sqrt{3}i\alpha_0 [\partial \xi_0(z) + \partial \xi_1(z) + 2\partial \xi_2(z) + \alpha_0 \partial \phi(z)]
\]
\[
\times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -2(\xi_0(z) - i\eta_0(z)) + \alpha_0 (\xi_1(z) - i\eta_1(z)
\]
\[
\left. + (\xi_2(z) - i\eta_2(z)) \right] \right\},
\]
\[
f_{\tilde{a}}(z) = \frac{1}{2\sqrt{2k}} \left[ -\partial \xi_0(z) + \partial \xi_1(z) - \alpha_0 \partial \xi_2(z) + 2\partial \phi(z) \right]
\]
\[
\begin{align*}
    f_{\bar{a}}(z) &= \frac{1}{2\sqrt{2k}} \{ [2\partial \xi_0(z) - \alpha_0 \partial \xi_1(z) - \partial \xi_2(z) + \partial \phi(z)] \\
    & \quad \times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -\alpha_0(\xi_0(z) - i\eta_0(z)) + (\xi_1(z) - i\eta_1(z)) \right] \right\}, \\
    f_{\bar{0}}(z) &= \frac{\sqrt{3}}{2k} \{ [-\alpha_0 \partial \xi_0(z) - 2\partial \xi_1(z) + \partial \xi_2(z) + \partial \phi(z)] \\
    & \quad \times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -2(2 + \alpha_0)(\xi_0(z) - i\eta_0(z)) - (2 - \alpha_0)(\xi_1(z) - i\eta_1(z)) + 2(\xi_2(z) - i\eta_2(z))] \right] \right\}, \\
    f_{-\bar{a}}(z) &= \frac{1}{2\sqrt{2k}} \{ -[(4k + 5) [2\alpha_0(\partial \xi_0(z) - i\partial \eta_0(z)) - \alpha_0(\partial \xi_1(z) - i\partial \eta_1(z)) \\
    & \quad \left( \partial \xi_2(z) - i\partial \eta_0(z) \right) + \partial \phi(z))] \\
    & \quad + (2 - 4\alpha_0)i\partial \eta_0(z) + (12 - \alpha_0)i\partial \eta_1(z) - (5 + 4\alpha_0)i\partial \eta_2(z)] \\
    & \quad \times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -2(\xi_0(z) - i\eta_0(z)) + \alpha_0(\xi_1(z) - i\eta_1(z)) + (\xi_2(z) - i\eta_2(z))] \right] \right\}, \\
    & \quad + 2 [\alpha_0 \partial \xi_0(z) - 2\partial \xi_1(z) - \partial \xi_2(z) - \partial \phi(z)] \\
    & \quad \times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ (1 - \alpha_0)(\xi_0(z) - i\eta_0(z)) - 3(\xi_1(z) - i\eta_1(z)) \\
    & \quad + (1 + \alpha_0)(\xi_2(z) - i\eta_2(z))] \right\}, \\
    & \quad + 3 [-2\partial \xi_0(z) + \alpha_0 \partial \xi_1(z) + \partial \xi_2(z) - \partial \phi(z)] \\
    & \quad \times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ 2(1 + \alpha_0)(\xi_0(z) - i\eta_0(z)) \\
    & \quad + 3(\xi_1(z) - i\eta_1(z)) + 2\xi_2(z) - i\eta_2(z))] \right\} \right\}. 
\end{align*}
\]
\[(4 - \alpha_0)\left(\xi_1(z) - i\eta_1(z)\right) - 3(\xi_2(z) - i\eta_2(z))\] \[
- 2 [\partial_0(z) - \partial_1(z) + \alpha_0 \partial_2(z) - 2\partial \phi(z)]
\times \left\{ \frac{1}{\sqrt{2k}} \left[ -(1 - 3\alpha_0)(\xi_0(z) - i\eta_0(z)) + 7(\xi_1(z) - i\eta_1(z)) - (3 + \alpha_0)(\xi_2(z) - i\eta_2(z))] \right. \\
+ 3 \exp\left\{ \frac{1}{\sqrt{2k}} \left[ -(5 - \alpha_0)(\xi_0(z) - i\eta_0(z)) + (3 + 2\alpha_0)(\xi_1(z) - i\eta_1(z)) + (1 - \alpha_0)(\xi_2(z) - i\eta_2(z))] \right. \\
+ 2\sqrt{3i\alpha_0} [\partial_0(z) + \partial_1(z) + 2\partial_2(z) + \alpha_0 \partial \phi(z)] \\
\times \exp\left\{ \frac{1}{\sqrt{2k}} \left[ \alpha_0(\xi_0(z) - i\eta_0(z)) + 2(\xi_1(z) - i\eta_1(z)) - (\xi_2(z) - i\eta_2(z))] \right. \\
\left. \right. \right. \right. }

f_{-\alpha_0}(z) = \frac{1}{2\sqrt{2k}} \left\{ -4[(k + 1)] [\partial_0(z) - i\partial \eta_0(z)] - (\partial_1(z) - i\partial \eta_1(z)) - 2\partial \phi(z)] \\
\left[ \frac{1}{\sqrt{2k}} \left[ -(6 + \alpha_0)(\xi_0(z) - i\eta_0(z)) - (2 - 3\alpha_0)(\xi_1(z) - i\eta_1(z)) + (\xi_2(z) - i\eta_2(z))] \right. \\
- 2 [(2 - \alpha_0)i\partial \eta_0(z) - (6 - \alpha_0)i\partial \eta_1(z) + (2 + 4\alpha_0)i\partial \eta_2(z)] \\
+ (4k + 2) [\partial_0(z) - i\partial \eta_0(z)] + \alpha_0(\partial_1(z) - i\partial \eta_1(z)) + (\partial_2(z) - i\partial \eta_2(z))] - 4(k + 2)\partial \phi(z) \\
- 3(2 - \alpha_0)(\partial_0(z) - i\partial \eta_0(z)) + 3(2 + \alpha_0)\alpha_0(\partial_1(z) - i\partial \eta_1(z)] \\
\left[ \frac{1}{\sqrt{2k}} \left[ -(2 - \alpha_0)(\xi_0(z) - i\eta_0(z)) + (2 + \alpha_0)(\xi_1(z) - i\eta_1(z))] \right. \\
+ 2 \left[ -2\partial_0(z) + \alpha_0 \partial_1(z) + \partial_2(z) - \partial \phi(z)] \\
\left[ \frac{1}{\sqrt{2k}} \left[ (2 + 3\alpha_0)(\xi_0(z) - i\eta_0(z)) + (6 - \alpha_0)(\xi_1(z) - i\eta_1(z)) - 4(\xi_2(z) - i\eta_2(z))] \right. \\
- [\partial_0(z) - \partial_1(z) + \alpha_0 \partial_2(z) - 2\partial \phi(z)] \\
\times \exp\left\{ \frac{1}{\sqrt{2k}} \left[ -(1 - 4\alpha_0)(\xi_0(z) - i\eta_0(z)) + 9(\xi_1(z) - i\eta_1(z)) - (4 + \alpha_0)(\xi_2(z) - i\eta_2(z))] \right. \\
\left. \right. \right. \right. \}.
After a long computation, we find the following two screen operators:

\[ S_+(z) = \frac{1}{\sqrt{2k}} \{ 2 [-\partial \xi_0(z) + \partial \xi_1(z) - \alpha_0 \partial \xi_2(z) + 2 \partial \phi(z)] \times \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -3(\xi_0(z) - i\eta_0(z)) + (1 + \alpha_0)(\xi_1(z) - i\eta_1(z)) + (1 - \alpha_0)(\xi_2(z) - i\eta_2(z)) \right] \right\} \]

\[ + \exp \left\{ \frac{1}{\sqrt{2k}} \left[ -(5 - 2\alpha_0)(\xi_0(z) - i\eta_0(z)) + (5 + 2\alpha_0)(\xi_1(z) - i\eta_1(z)) - \alpha_0(\xi_2(z) - i\eta_2(z)) \right] \right\} \]

where \( \alpha_0 = \sqrt{-2(k+3)} \). It can be checked that \( \psi_a(z) \) indeed satisfy all of the OPE relations of our parafermion algebra. By the above representation of the \( A_2^{(2)} \) parafermionic currents, we can bosonize the parafermion energy-momentum tensor \( T_\psi \). The result is

\[
T_\psi(z) = -\frac{1}{2} : \{ (\partial \xi_0(z))^2 + (\partial \xi_1(z))^2 + (\partial \xi_2(z))^2 \\
+ (\partial \eta_0(z))^2 + (\partial \eta_1(z))^2 + (\partial \eta_2(z))^2 + (\partial \phi(z))^2 \} : \\
- \frac{1}{2\sqrt{2k}} \{ -15(1 - \alpha_0)(\partial^2 \xi_0(z) - i\partial^2 \eta_0(z)) + (1 + \alpha_0)(\partial^2 \xi_1(z) - i\partial^2 \eta_1(z)) + \alpha_0(\partial^2 \xi_2(z) - i\partial^2 \eta_2(z)) - 4\partial^2 \phi(z) \} \\
+ \frac{1}{\sqrt{k(k+3)}} (\partial^2 \eta_0(z) + \partial^2 \eta_1(z) + \partial^2 \eta_2(z)).
\]

It is easy to check that the central charge of this operator is indeed \( c_\psi \) given in (2.20).

Using the free field realization of the parafermionic currents, we obtain a free field representation of the highest weight state \( V^j_j(z) \) of the parafermion algebra:

\[
V^j_j(z) = \exp \left\{ \frac{-j}{2\sqrt{k(k+3)}} [\eta_0(z) + \eta_1(z) + 2\eta_2(z)] \right\} \\
\times \exp \left\{ \frac{-j}{2\sqrt{3k(k+3)}} [\xi_0(z) + \xi_1(z) + 2\xi_2(z)] - \frac{ij}{\sqrt{6k}} \phi(z) \right\},
\]

where \( j \) is the spin of the highest weight state.

Now we come to the free field realization of screen currents of the parafermion algebra. After a long computation, we find the following two screen operators:
+ \{\alpha_0 \partial \xi_0(z) + 2 \partial \xi_1(z) - \partial \xi_2(z) - \partial \phi(z)\}
\times \exp\left\{\frac{-\alpha_0(\xi_0(z) - i \eta_0(z))}{\sqrt{2k}}\right\}
- 2(\xi_1(z) - i \eta_1(z)) + (\xi_2(z) - i \eta_2(z))\})
+ [2 \partial \xi_0(z) - \alpha_0 \partial \xi_1(z) - \partial \xi_2(z) + \partial \phi(z)]
\times \exp\left\{\frac{-\alpha_0(\xi_0(z) - i \eta_0(z))}{\sqrt{2k}}\right\}
- (\xi_2(z) - i \eta_2(z))\})
\times \exp\left\{\frac{i}{\sqrt{2k\alpha_0}}\right\}
[(\sqrt{3} \xi_0(z) + \eta_0(z)) + (\sqrt{3}\xi_1(z) + \eta_1(z))
+ (\sqrt{3}\xi_2(z) + \eta_2(z))]\}$exp\left\{\frac{1}{\sqrt{2k}}\left(\sqrt{3}\phi(z) + i \phi(z)\right)\right\} (4.6)

and

$S_-(z) = \frac{1}{\sqrt{2k}}\{2 [-\partial \xi_0(z) + \partial \xi_1(z) - \alpha_0 \partial \xi_2(z) + 2 \partial \phi(z)]
\times \exp\left\{\frac{-\alpha_0(\xi_0(z) - i \eta_0(z))}{\sqrt{2k}}\right\}
- 2(\xi_1(z) - i \eta_1(z)) + (\xi_2(z) - i \eta_2(z))\})
+ [2 \partial \xi_0(z) - \alpha_0 \partial \xi_1(z) - \partial \xi_2(z) + \partial \phi(z)]
\times \exp\left\{\frac{-\alpha_0(\xi_0(z) - i \eta_0(z))}{\sqrt{2k}}\right\}
- (\xi_2(z) - i \eta_2(z))\})
\times \exp\left\{\frac{i}{\sqrt{2k\alpha_0}}\right\}
[(\sqrt{3} \xi_0(z) - \eta_0(z)) + (\sqrt{3}\xi_1(z) - \eta_1(z))
+ (\sqrt{3}\xi_2(z) - \eta_2(z))]\}$exp\left\{\frac{1}{\sqrt{2k}}\left(\sqrt{3}\phi(z) - i \phi(z)\right)\right\}. (4.7)

The OPEs of these two screen currents with the energy-momentum tensor and parafermion currents are given by

$T_\psi(z)S_\pm(w) = \partial_w \left(\frac{1}{z - w}S_\pm(w)\right) + \ldots$,
$\psi_\alpha(z)S_\pm(w) = \ldots$, $\psi_\tilde{\alpha}(z)S_\pm(w) = \ldots$,
$\psi_-\alpha(z)S_\pm(w) = \partial_w \left(\frac{2 \alpha_0^2}{z - w}S_\pm(w)\right) + \ldots$,
$\psi_\tilde{\alpha}_2(z)S_\pm(w) = \ldots$, $\psi_0(z)S_\pm(w) = \ldots$,
$\psi_-\tilde{\alpha}(z)S_\pm(w) = \partial_w \left(\frac{2 \alpha_0^2}{z - w}S_\pm(w)\right) + \ldots$,
$\psi_-\tilde{\alpha}_2(z)S_\pm(w) = \partial_w \left(\frac{-4 \alpha_0^2}{z - w}S_\pm(w)\right) + \ldots$, (4.8)
where,

\[
\tilde{S}_\pm(z) = \exp\left\{ \pm \frac{i}{\sqrt{2k\alpha_0}} \left[ (\sqrt{3}\xi_0(z) \pm \eta_0(z)) + (\sqrt{3}\xi_1(z) \pm \eta_1(z)) + (\sqrt{3}\xi_2(z) \pm \eta_2(z)) \right] \right\} \exp\left\{ \frac{1}{\sqrt{2k}} (\sqrt{3}\phi(z) \pm i\phi(z)) \right\},
\]

\[
\tilde{S}_2(z) = \frac{1}{\sqrt{2k}} [ -i(2 + \alpha_0)\xi_0(z) - i(2 - \alpha_0)\xi_1(z) + 2i\xi_2(z) ] \tilde{S}_+(z),
\]

\[
\tilde{S}_{-2}(z) = \frac{1}{\sqrt{2k}} [ -i(2 - \alpha_0)\xi_0(z) - i(2 + \alpha_0)\xi_1(z) + 2i\phi(z) ] \tilde{S}_-(z),
\]

(4.9)

5 Spin-2 Primary field and novel algebraic structure

In conformal field theory, primary fields are fundamental objects. The descendant fields can be obtained from primary fields.

It is well known that, the energy-momentum stress is not primary field (unless the central charge is zero). The lowest spin of the primary field obtained by the Hamiltonian reduction approach is three [23, 24]. This agrees with the other methods, such as high order Casimir, coset model [25], free field realization [26], or the parafermion construction [15]. For more references about \( W \)-algebras see the reviews [24, 27], and their applications in string and gravity theory see [28, 29]. In the following, we use the \( A_{(2)}^2 \) parafermionic currents to construct a primary field of spin two. Define the spin-2 currents:

\[
\tilde{w}_2(z) = \frac{k(4k - 1)}{4(2k + 3)(k - 1)} \sum_{a = \pm \alpha, \pm \tilde{\alpha}} [\psi_a \psi_{-\tilde{a}}]_0.
\]

(5.1)

The action of \( \tilde{w}_2(z) \) on \( \psi_b(z) \) is given by the following OPE:

\[
\tilde{w}_2(z) \psi_b(w) = \frac{\Delta_b}{(z - w)^2} \psi_b(w) + \frac{1}{z - w} \partial \psi_b(w) + \ldots, \quad b = \pm \alpha, \pm \tilde{\alpha},
\]

\[
\tilde{w}_2(z) \psi_{\pm \alpha_2}(w) = \ldots, \quad \tilde{w}_2(z) \psi_0(w) = \ldots,
\]

(5.2)

It should be be understood that \( \tilde{b} = b \) in the first line of (5.2). This OPE fixes the normalization of \( \tilde{w}_2(z) \). (5.2) suggests that \( \tilde{w}_2(z) \) behaves similar to an energy-momentum tensor except that it transforms \( \psi_b \) into \( \psi_{\tilde{b}} \) and vice versa. Recall that \( \psi_b \) and \( \psi_{\tilde{b}} \) have the same conformal dimensions. The OPEs of \( \tilde{w}_2(z) \) with itself and the stress tensor are

\[
\tilde{w}_2(z) \tilde{w}_2(w) = \tilde{c}/2 + \frac{2U(w)}{(z - w)^2} + \frac{\partial U(w)}{z - w} + \ldots,
\]

\[
T_{\psi}(z) \tilde{w}_2(w) = \frac{2\tilde{w}_2(w)}{(z - w)^2} + \frac{\partial \tilde{w}_2(w)}{z - w} + \ldots,
\]

(5.3)

where,
\[
\tilde{c} = \frac{(4k - 1)^2}{2(k - 1)(2k + 3)} = \frac{(13c_\psi + 5)^2}{24(c_\psi + 9)(c_\psi - 1)},
\]

is the central charge of \(\tilde{w}_2(z)\), and \(c_\psi\) is the central charge of the \(A_2^{(2)}\) parafermionic energy-momentum tensor \((5.2)\), and \(U(z)\) is a spin two field given by

\[
U(z) = \frac{(k + 1)(k + 3)(4k - 1)^2}{4(2k + 3)^2(k - 1)^2} \left( T_\psi(z) - \frac{1}{2} \Omega_0(z) - \frac{k(k + 2)}{(k + 1)(k + 3)} \Omega_2(z) \right),
\]

\[
\Omega_0(z) = \frac{1}{2} [\tilde{\psi}_0 \tilde{\psi}_{0,0}](z), \quad \Omega_2(z) = \frac{k}{k + 2} [\tilde{\psi}_{\tilde{\alpha}_2} \tilde{\psi}_{-\tilde{\alpha}_2}](z).
\]

The OPEs of \(\Omega_0(z)\) and \(\Omega_2(z)\) with the parafermion currents are

\[
\begin{align*}
\Omega_0(z) \tilde{\psi}_0(w) &= \frac{1}{(z - w)^2} \tilde{\psi}_0(w) + \frac{1}{z - w} \partial \tilde{\psi}_0(w) + \ldots, \\
\Omega_0(z) \tilde{\psi}_b(w) &= \frac{3}{4k - 1} \left( \frac{\Delta_b}{(z - w)^2} \tilde{\psi}_b(w) + \frac{1}{z - w} \partial \tilde{\psi}_b(w) + \ldots \right), \\
\Omega_0(z) \tilde{\psi}_{\pm \tilde{\alpha}_2}(w) &= \ldots, \\
\Omega_2(z) \tilde{\psi}_0(w) &= \ldots, \\
\Omega_2(z) \tilde{\psi}_b(w) &= \frac{k - 1}{2k(k + 2)} \left( \frac{\Delta_b}{(z - w)^2} \tilde{\psi}_b(w) + \frac{1}{z - w} \partial \tilde{\psi}_b(w) + \ldots \right), \\
\Omega_2(z) \tilde{\psi}_{\pm \tilde{\alpha}_2}(w) &= \frac{\Delta_{\pm \tilde{\alpha}_2}}{(z - w)^2} \tilde{\psi}_{\pm \tilde{\alpha}_2}(w) + \frac{1}{z - w} \partial \tilde{\psi}_{\pm \tilde{\alpha}_2}(w) + \ldots,
\end{align*}
\]

From \((5.3)\), we see that the field \(\tilde{w}_2\) introduced above is a primary field with conformal spin two. The modes expansion of \(\tilde{w}_2(z)\) is

\[
\tilde{w}_2(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \tilde{w}_{2,n} z^{-n - 2}.
\]

So like the spin-1 primary field \(\tilde{\psi}_0(z)\), \(\tilde{w}_2(z)\) lives in the Neveu-Schwarz sector, and it transforms fields in the Ramond sector into those in the Neveu-Schwarz sector or vice versa. If we regard the spin-2 currents as affine connections, then the energy-momentum tensor is a project connection that transforms fields in one sector, while field \(\tilde{w}_2(z)\) is a project connection which transforms a field in one sector into a field in a different sector. The mode expansions of \(\Omega_0(z)\) and \(\Omega_2(z)\) are

\[
\Omega_0(z) = \sum_{n \in \mathbb{Z}} \Omega_{0,n} z^{-n - 2}, \quad \Omega_2(z) = \sum_{n \in \mathbb{Z}} \Omega_{2,n} z^{-n - 2},
\]

respectively. So \(\Omega_0(z)\) and \(\Omega_2(z)\) live in the Ramond sector.

From the above, we see that \(\tilde{w}_2(z)\), \(T(z)\) do not close to an algebra. It can be checked that the field \(U(z)\) with itself also cannot form a closed algebra. But if we decompose the field \(U(z)\) as above, then the algebra generated by \(\tilde{w}_2(z)\), \(T(z)\), \(\Omega_0(z)\), \(\Omega_2(z)\) is closed, as can be seen from the following additional OPEs,
\[ \Omega_0(z)\Omega_0(w) = \frac{c_0/2}{(z-w)^4} + \frac{2\Omega_0(w)}{(z-w)^2} + \frac{\partial\Omega_0(w)}{(z-w)} + \cdots, \quad (5.8) \]
\[ \Omega_2(z)\Omega_2(w) = \frac{c_2/2}{(z-w)^4} + \frac{2\Omega_2(w)}{(z-w)^2} + \frac{\partial\Omega_2(w)}{(z-w)} + \cdots, \quad (5.9) \]
\[ \Omega_0(z)\Omega_0(w) = \cdots, \quad (5.10) \]
\[ T_{\psi}(z)\Omega_0(w) = \frac{c_0/2}{(z-w)^4} + \frac{2\Omega_0(w)}{(z-w)^2} + \frac{\partial\Omega_0(w)}{(z-w)} + \cdots, \quad (5.11) \]
\[ T_{\psi}(z)\Omega_2(w) = \frac{c_2/2}{(z-w)^4} + \frac{2\Omega_2(w)}{(z-w)^2} + \frac{\partial\Omega_2(w)}{(z-w)} + \cdots, \quad (5.12) \]
\[ \Omega_0(z)\tilde{w}_2(w) = \cdots, \quad (5.13) \]
\[ \Omega_2(z)\tilde{w}_2(w) = \frac{1}{k+2} \left\{ \frac{2\tilde{w}_2(w)}{(z-w)^2} + \frac{\partial\tilde{w}_2(w)}{(z-w)} + \cdots \right\}. \quad (5.14) \]

where
\[ c_0 = 1, \quad c_2 = \frac{2(k-1)}{(k+2)} \quad (5.15) \]

The situation here is similar to that in \( W \)-algebra, where for instance \( W_3(z), T(z) \) are not closed since a spin-4 field will appear in the OPE of \( W_3(z) \) with itself. The spin-4 field is also decomposed into two fields which together with \( T(z) \) and \( W_3(z) \) do close to the so-called \( W_3 \) algebra [23].

Some remarks are in order. To our knowledge, the algebraic structure we found is new. It is similar to a \( W \)-algebra structure, but now all the fields have conformal dimension 2. The primary field \( \tilde{w}_2(z) \) is a “spin-2 analog” of \( W \)-currents, and it transforms between fields \( \psi_a \) (fields in Ramond sector) and \( \psi_a^{\bar{\nu}} \) (fields in Neveu-Schwaz sector). In a sense, the primary field \( \tilde{w}_2(z) \), together with the other two spin-2 fields \( \Omega_1(z) \) and \( \Omega_2(z) \), “non-abelianizes” the Virasoro algebra generated by \( T(z) \). Possible applications of this new algebraic structure are under investigation.

In [30], A.B. Zamolodchikov considered an algebra having \( N + 1 \) spin-2 primary fields with OPEs given by
\[ T^i(z)T^j(w) = \frac{c\delta_{i,j}/2}{(z-w)^4} + \frac{2\delta_{i,j}T^k(w)}{(z-w)^2} + \frac{\delta_{i,j}\partial T^k(w)}{(z-w)} + \cdots, \quad i, j = 0, 1, \cdots N, \quad (5.16) \]
where \( T^0 = T \) is the usual energy-momentum tensor. In this algebra there is only one central charge, and \( T^i, i = 0, 1, \cdots, N, \) are decoupled so that the algebra is reduced to \( N + 1 \) copies of the Virasoro algebra. Our algebra is different. Firstly, in our theory, there are four (independent) different central charges. So our theory is a conformal field theory with multi-centers. Secondly, the four generators in our algebra are coupled to each other and can not be reduced to copies of the Virasoro algebra.
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