Topological derivatives of eigenvalues and neural networks in identification of imperfections

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Abstract. Numerical method for identification of imperfections is devised for elliptic spectral problems. The neural networks are employed for numerical solution. The topological derivatives of eigenvalues are used in the learning procedure of the neural networks. The topological derivatives of eigenvalues are determined by the methods of asymptotic analysis in singularly perturbed geometrical domains. The convergence of the numerical method in a probabilistic setting is analysed. The method is presented for the identification of small singular perturbations of the boundary of geometrical domain, however the framework is general and can be used for numerical solutions of inverse problems in the presence of small imperfections in the interior of the domain. Some numerical results are given for elliptic spectral problem in two spatial dimensions.

1. Introduction

We present a numerical method for the identification of small holes in the geometrical domain $\Omega$ based upon the observation of a finite number of eigenvalues of an elliptic spectral problem defined in the same domain $\Omega$.

In the present paper we are going to investigate the identification problems which can be formulated in the following way. Given geometrical domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, which contains a finite number of imperfections, in the form of cracks, holes or cavities, and inclusions. We want to identify the number and the properties of imperfections. The imperfections are defined in a specific geometrical form, a family of circles $B_j(x_j) = \{ |x - x_j| < \rho_j \}, j = 1, \ldots, N$, with $B_j(x_j) \subset \Omega$, in the simplest case, and some material properties inside of the sets $B_j(x_j)$ are assumed to be known. We are going to consider the holes or cavities, so the material properties are specified by the Neumann boundary conditions on the boundaries $\partial B_j(x_j)$. Our strategy is to observe the eigenvalues of the spectral problem defined in the
domain $\Omega$ and detect the imperfections from the inverse mapping
\[ [\text{geometrical parameters of imperfections}] \mapsto [\text{finite number of eigenvalues}] \,.
\]
In the simplest case the above mapping takes the form
\[ [(X, \rho) = (x_1, \ldots, x_N, \rho_1, \ldots, \rho_N) \in \mathbb{R}^{N(d+1)}] \mapsto [\lambda = (\lambda_1, \ldots, \lambda_M) \in \mathbb{R}^M] \, ,
\]
where the vectors $X \in \mathbb{R}^{Nd}, \rho \in \mathbb{R}^N$, are to be determined by the solution procedure of the inverse problem, and the vector $\lambda \in \mathbb{R}^M$ is supposed to be given or to be known. It seems reasonable to assume that $M > N(d + 1)$ i.e., we need enough data in order to solve the inverse problem.

In order to determine the inverse mapping the neural networks are applied. Therefore, we need an efficient learning procedure for the network. The procedure is constructed under the assumption that there is a small parameter in the spectral problem under considerations, therefore, we can use the asymptotic analysis in order to establish appropriate learning set, which is crucial observation from the point of view of the efficiency of the numerical method proposed here. Actually, we assume that the size of imperfections
\[ B_j(x_j) \subseteq \Omega \]
\[ |\rho|^2 = \sum \rho_j^2 \to 0 \]
is a small parameter, the asymptotic analysis can be applied in order to determine the asymptotics of the observation vector $\lambda$, which includes the finite number of eigenvalues of the spectral problem under considerations, in the singularly perturbed geometrical domain $\Omega_\rho = \Omega \setminus \bigcup_{j=1}^N \big( B_j(x_j) \big)$. The asymptotic formula which is useful for our learning procedure reads
\[ \lambda = \lambda + \sum_{j=1}^N \rho_j^2 T_{\Omega_j}(x_j) + O(|\rho|^{2+}) \, , \tag{1} \]
with the first term determined by solution of the spectral problem in unperturbed domain, with no imperfections or holes inside. The precise formula for e.g., the topological derivative of a simple eigenvalue in the case of the boundary imperfection is given by (15). In general, the second term in the asymptotic expansion of the eigenvalues vector $\lambda$ can be determined from the asymptotics and it is called the topological derivative. We refer the reader to the forthcoming paper [10] for complete proofs in the asymptotic analysis of spectral problems for Laplacian and the related formulae for the topological derivatives of eigenvalues.

Let $d = 2$ and $N = 1$. In this particular case the mapping $\mathcal{G}$ is defined by following formula
\[ \mathcal{G} : \mathbb{R}^3 \ni (x_1, \rho_1) \mapsto (\lambda = (\lambda_1, \ldots, \lambda_M)) \in \mathbb{R}^M \, . \tag{2} \]
Mapping $\mathcal{G}$, for one imperfection calculates the vector of eigenvalues. The neural networks are used to minimization of the following functional
\[ \min_{(x, \rho)} \text{dist}(\mathcal{G}(x, \rho), \lambda) \tag{3} \]
where $\lambda = (\lambda_1, \ldots, \lambda_M) \in \mathbb{R}^M$ is the vector of values determined for the real object. This is definition of the inverse problem. Neural network is used as an approximator of inverse mapping. The learning data for the neural networks are constructed using the asymptotic analysis which is very useful from the numerical point of view but unfortunately restricts the validity of the proposed approach only for small radii of imperfections. We assume also that the given data for the inverse problem are exact in the sense, that there are some unknown imperfections (which furnish the prescribed values of the observation shape functionals). Therefore, the inverse mapping $\mathcal{G}^{-1}(\lambda) = (x, \rho)$ can be defined and is given by the neural network. In fact, the situation is more complicated, since we can only model the conditional expectations. The positive conclusion for our approach is the „probabilistic” convergence of the numerical procedure which, to the best of our knowledge, is an original contribution to the field of inverse problems.
2. Topological Derivatives of Shape Functionals

One of the most important applications of topological derivatives of shape functionals is the field of numerical solution for the inverse problems. The mathematical theory of asymptotic analysis in singularly perturbed domains [3], [8], leads to the asymptotic expansions of solutions and to the topological derivatives. This approach is described in details in the paper [7]. The concept of topological derivatives is based on the method of compound asymptotic expansions [8].

The topological derivative \( T_\Omega \) of a shape functional \( J(\Omega) \) is introduced in the the first paper on the subject [12] published in 1997, in order to characterize the infinitesimal variation of \( J(\Omega) \) with respect to the infinitesimal variation of the topology of the domain \( \Omega \). The topological derivative allows us to derive the new optimality condition for the shape optimization problem:

\[
J(\Omega^*) = \inf_\Omega J(\Omega).
\]

The optimal domain \( \Omega^* \) is characterized by the first order condition [12] defined on the boundary of the optimal domain \( \Omega^* \), \( dJ(\Omega^*; V) \geq 0 \) for all admissible vector fields \( V \), and by the following optimality condition defined in the interior of the domain \( \Omega^* \):

\[
T_{\Omega^*}(x) \geq 0 \quad \text{in} \quad \Omega^*.
\]

The other use of the topological derivative is connected with approximating the influence of the holes in the domain on the values of integral functionals of solutions, what allows us to solve a class of shape inverse problems.

In general terms the notion of the topological derivative (TD) has the following meaning. Assume that \( \Omega \subset \mathbb{R}^N \) is an open set and that there is given a shape functional

\[
\mathcal{J} : \Omega \setminus K \to \mathbb{R}
\]

for any compact subset \( K \subset \overline{\Omega} \). We denote by \( B_\rho(x), x \in \Omega \), the ball of radius \( \rho > 0 \), \( B_\rho(x) = \{ y \in \mathbb{R}^N : ||y-x|| < \rho \} \), \( \overline{B_\rho(x)} \) is the closure of \( B_\rho(x) \), and assume that there exists the following limit

\[
\mathcal{I}(x) = \lim_{\rho \downarrow 0} \frac{J(\Omega \setminus \overline{B_\rho(x)}) - J(\Omega)}{\overline{B_\rho(x)}}.
\]

The function \( \mathcal{I}(x), x \in \Omega \), is called the topological derivative of \( J(\Omega) \), and provides the information on the infinitesimal variation of the shape functional \( J \) if a small hole is created at \( x \in \Omega \). This definition is suitable for Neumann–type boundary conditions on \( \partial B_\rho \).

In several cases this characterization is constructive, i.e. TD can be evaluated for shape functionals depending on solutions of partial differential equations defined in the domain \( \Omega \).

3. Asymptotic analysis of spectral problems

Let \( \Omega \subset \mathbb{R}^2 \) be a domain with the smooth boundary \( \Gamma \), the boundary is a simple, regular, and closed contour. In the neighbourhood of \( \Gamma \) a curvilinear system of coordinates \((n, s)\) is defined, where \( s \) is the length of the curve measured along \( \Gamma \), \( n \) denotes the oriented distance to \( \Gamma \), while \( n > 0 \) in \( \Omega^c = \mathbb{R}^2 \setminus \Omega \). By \( \omega \subset \mathbb{R}^2 = (-\infty, 0) \times \mathbb{R} \) is denoted a domain with the compact closure \( \overline{\omega} = \omega \cup \partial \omega \). The boundary \( \partial \Xi \) of the infinite domain \( \Xi = \mathbb{R}^2 \setminus \overline{\omega} \) is assumed to be piecewise-smooth, which means that there is a finite set of points \( P_1, \ldots, P_N \) on \( \partial \Xi \), such that each curvilinear interval \( P_i P_{i+1} \) is smooth and the angles between tangents at \( P_i, i = 1, \ldots, N \) are strictly positive. In other words, peaks directed outside are forbidden.

Introduce a family of domains, depending on the small parameter \( h > 0 \),

\[
\omega_h = \{ x = (x_1, x_2) : \xi = (\xi_1, \xi_2) := (h^{-1}n, h^{-1}s) \in \omega \},
\]

\[
\Omega(h) = \Omega \setminus \overline{\omega_h}
\]
Here and in the sequel, a point on the contour $\Gamma$ is identified with its coordinate $s$, with the convention that $s < 0$ for the points which are located on $\Gamma$ on the left-hand side of the origin $O$.

Let us consider the spectral Neumann problem

$$ -\Delta_x u^h(x) = \lambda^h u^h(x), \quad x \in \Omega(h), \quad \lambda^h \in \mathbb{R}^+ $$

(6)

$$ \partial_n u^h(x) = 0, \quad x \in \Gamma(h) \equiv \partial \Omega(h), \quad (\partial_n = n^h \cdot \nabla) $$

(7)

for the Laplace operator $\Delta_x$, where $\partial_n = n^h \cdot \nabla$ is the normal derivative along the outer normal $n^h$. Note that (6), with the fixed $\lambda^h$, is but the Helmholtz equation. Conditions (7) are prescribed along $\Gamma(h)$ except for the points $P_i(h), i = 1, \ldots, N$, which are images of the points $P_i, i = 1, \ldots, N$, on the contour $\Gamma(h)$. Problem (6), (7) admits the sequence of eigenvalues

$$ 0 = \lambda_0^h < \lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_m^h \leq \cdots \rightarrow +\infty $$

(8)

where the multiplicity is explicitly indicated. The corresponding eigenfunctions $u_0^h, u_1^h, u_2^h, \ldots, u_m^h, \ldots$ can be subject to the orthogonality and normalisation conditions

$$ \langle u_p^h, u_m^h \rangle_{\Omega(h)} = \delta_{p,m}, \quad p, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}, $$

(9)

where $(\cdot, \cdot)_\Omega$ is the scalar product in the Lebesgue space $L_2(\Omega)$, and $\delta_{p,m}$ the Kronecker symbol.

Our aim is the derivation of asymptotic formulae for the eigenvalues and eigenfunctions of problem (6), (7). It is not difficult to conclude (cf. §3), that for a fixed index $m$ and with $h \rightarrow 0$ the entry $\lambda_m^h$ of the sequence (8) converges to the appropriate element of the sequence

$$ 0 = \lambda_0^0 < \lambda_1^0 \leq \lambda_2^0 \leq \cdots \leq \lambda_m^0 \leq \cdots \rightarrow +\infty $$

(10)

of the eigenvalues for the limit spectral Neumann problem

$$ -\Delta_x \psi^0(x) = \lambda^0 \psi^0(x), \quad x \in \Omega; \quad \partial_n \psi^0(x) = 0, \quad x \in \Gamma. $$

(11)

The eigenfunctions of (11) are smooth functions in $\overline{\Omega}$ and are subject to the orthogonality and normalisation conditions

$$ \langle \psi_p^0, \psi_m^0 \rangle_{\Omega} = \delta_{p,m}, \quad p, m \in \mathbb{N}. $$

(12)

3.1. Asymptotic ansätze and procedures.

In the paper the following ansätze are used to construct the asymptotics of eigenvalues and eigenfunctions

$$ \lambda_m^h = \lambda_m^0 + h^2 \lambda_m' + \cdots, $$

(13)

$$ u_m^h(x) = \psi_m^0(x) + h \chi(x)w_m^0(\xi) + h^2 \chi(x)w_m^2(\xi) + h^2 \psi_m^2(x) + \cdots $$

(14)

Here $\psi_m^0$ and $\psi_m^2$ are terms of regular type, a smooth and a continuous function, respectively on the set $\overline{\Omega}$, and $w_m^0$, $w_m^2$ are terms of the boundary layer type, which depend on the rapid variables $\xi = (\xi_1, \xi_2)$, defined in (4), and are given by the solutions of the Neumann problem in the domain $\Xi$. Finally, $\chi \in C^\infty(\overline{\Omega})$ is the cut-off function, equal to zero outside of a neighbourhood of the origin $O$ and equal to one in the vicinity of the point $O$.

The procedures of constructions of asymptotic ansätze of the type (13), (14) as well as the determination of theirs terms are not of an particular interest. During the years 70-80 of the last century the subject was fully investigated in the framework of two methods, of matched [3] and compound [8] asymptotic expansions for examining solutions in domains with singularly perturbed boundaries. In addition, if the domain $\Omega$ is included in the half-space bounded by the tangent $L$ to the contour $\Gamma$ at the point $O$, and $\Gamma$ coincides with $L$ in the vicinity of $O$, then by means of even extension over the part of
the boundary (odd for the Dirichlet boundary conditions) we obtain from (6), (7) the spectral Neumann problem in the domain with small hole. For such a problem the asymptotics are obtained e.g., in [7] and in [9].

We find [9] the formula for asymptotic correction term in representation (13) of the simple eigenvalue

$$\lambda_{n}^0 = m(\Xi)\partial_1 v_{m}^0(\Omega) - \lambda^0 \text{mes}_2(\omega) v_{m}^0(\Omega|^2. \quad (15)$$

Here we denote

$$m(\Xi) = \int_{\Xi} |\nabla_\xi W(\xi)|^2 d\xi + \text{mes}_2(\omega), \quad (16)$$

where $W$ is a canonical solution of the Neumann problem

$$-\Delta_\xi W(\xi) = 0, \quad \xi \in \Xi, \quad \partial_\nu W(\xi) = -\nu_2(\xi), \quad \xi \in \partial\Xi, \quad (17)$$

and $\nu = (\nu_1, \nu_2)$ is the outward unit vector on the boundary $\partial \Xi \subset \mathbb{R}^2$.

Assume now, that $\lambda_{0}^0$ is an eigenvalue of the multiplicity $\nu_m$, i.e.,

$$\lambda_{m-1}^0 < \lambda_{m}^0 = \cdots = \lambda_{m+\nu_m-1}^0 < \lambda_{m+\nu_m}^0. \quad (18)$$

In such a case ansätze (13) and (14) are valid for $p = m, \ldots, m + \nu_m - 1$, however, the principal terms of expansions for the eigenfunctions $u_{m}^0, \ldots, u_{m+\nu_m-1}^0$ in problem (6), (7) are predicted if the form of linear combinations

$$v^{p0} = a_{1}^{p} v_{m}^0 + \cdots + a_{\nu_m}^{p} v_{m+\nu_m-1}^0 \quad (19)$$

of eigenfunctions, for the eigenvalue $\lambda_{m}^0$, subject to the orthogonality and normalization conditions. The coefficients in (19) are given by the eigenvectors in the following spectral problem

$$\lambda^p a^p = Ma^p \quad (20)$$

with the matrix $M = (M_{jk})_{j,k=0}^{\nu_m-1}$ of the dimension $\nu_m \times \nu_m$,

$$M_{jk} = m(\Xi)\partial_1 v_{m+k}^0(\Omega)\partial_1 v_{m+j}^0(\Omega) - \lambda_{m}^0 v_{m+k}^0(\Omega) v_{m+j}^0(\Omega) \text{mes}_2(\omega)$$

4. Neural networks

We are going to present the method of approximation of the inverse mapping using neural networks. We assume that the distance between the observation from real object and from the mathematical model equals to zero, it means that there is $(x, \rho)$ such that

$$G(x, \rho) = \lambda \in \mathbb{R}^M \quad (21)$$

thus, it makes sense to consider the inverse mapping $G^{-1}$ (we know that mapping $G^{-1}$ exists). Therefore, the neural network is constructed to approximate the inverse mapping $G^{-1}$. In practice we approximate conditional expectation of location and radius of one hole instead of mapping $G^{-1}$. Our procedure uses formula (1) to construct data for learning of the network so it is formula that based on the mathematical models and on the asymptotic expressions of the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_M)$. In other words we are going to determine the size $\rho > 0$ and the location $x \in \mathbb{R}^2$ of one hole based on vector of eigenvalues but in effect we obtain conditional expectation of these values.

Roughly speaking, given a vector of actual observations $\lambda \in \mathbb{R}^M$ which describes the unknown properties of imperfections, we want to find $(x, \rho) \in \mathbb{R}^3$ such that the vector $G(x, \rho) \in \mathbb{R}^M$ coincides with the given vector of actual observations.

Fundamental element of artificial neural networks is artificial neuron. Construction of artificial neuron is on the ground of human neuron. Artificial neuron has many inputs and single output. Each of inputs has
some real number named weight. Output signal is calculated inside neuron based on input information. Each artificial neuron has activation function. This function decides about kind of transformation of input information. Additionally every neuron has one extra input named bias. This input is always equals one and has own weight. We assume that some neuron has \( n \) inputs and one bias. We denote the input vector by \( x = [x_0, x_1, \ldots, x_n] \), where \( x_0 = 1 \) and weight vector by \( w = [w_0, w_1, \ldots, w_n] \), as well as activation function by \( f \).

There exist many kinds of activation function. The most popular are sigmoidal and linear activation functions. Depending on where we use neuron we define appropriate activation function.

Multilayer feedforward neural network is a set of neurons that are in separate groups named layers. Input vector represents input vector for all neurons in the first layer. Each connection between input and neuron has own weight. Each neuron of the first layer generates output signal. Vector of output signals from the first layer represents input vector to the second layer.

4.1. Inverse problem

We consider particular case of mapping \( G \). Let \( d = 2 \) and \( N = 1 \). It means that we consider domain with single hole \( B_1 \).

Mapping \( G : \mathbb{R}^3 \rightarrow \mathbb{R}^M \) is defined by

\[
G : (x_1, \rho_1) \rightarrow (\lambda_1, \ldots, \lambda_M),
\]

where

\( (y_1, \rho_1) \) – center and radius of hole,
\( \lambda_1, \ldots, \lambda_M \) – eigenvalues.

This mapping for domain with one holes calculates vector of eigenvalues.

Let \( G^{-1} : \mathbb{R}^M \rightarrow \mathbb{R}^3 \) be inverse mapping defined by

\[
G^{-1} : (\lambda_1, \ldots, \lambda_M) \rightarrow (y_1, \rho_1).
\]

The inverse mapping \( G^{-1} \) for vector of eigenvalues calculates location and size of single hole in our domain. Any formula for mapping \( G^{-1} \) is unknown. Our aim is approximation of this mapping using artificial neural networks.

Multilayer feedforward neural network is capable of arbitrarily accurate approximations to arbitrary mapping. To approximate mapping \( G^{-1} \) by neural network we have to construct network and learning set. Appropriate neural network consider 1 hidden layer. \( q \) denotes the numbers of neurons in hidden layer. \( \phi \) denotes sigmoidal activation function for hidden layer. What is more our network has 3 output layer with 3 neurons and linear activation function. Input vector has \( M \) coordinates and represents vector of eigenvalues. Number of neurons in hidden layer is not fixed, it increases to infinity when the size of learning set increases to infinity. Output vector represents vector of one hole. First and second coordinates represent the center of the hole, third coordinate represents the radius of the hole.

To construct learning set we use probability theory. We consider a sequence \( \{Z^k \}, k = 1, 2, \ldots \) of vectors, where \( Z^k = (Y^k, X^k) \). \( X^k = (X_1^k, X_2^k, X_3^k) \) – describes the one hole. Its coordinates denote:

\( (X_1^k, X_2^k) \) – the center of hole,
\( X_3^k \) – the radius of hole.

We generate this vector randomly by choosing one hole in our domain.

\( Y^k = (Y_1^k, Y_2^k, Y_3^k) \) – describes unknown value of function \( G^{-1} \). Using notation of previous section we have that

\( (Y_1^k, Y_2^k) = x_1 \),
\( Y_3^k = \rho_1 \).
We generate vectors $X^k, k = 1, 2, \ldots$, in a random way. For $k \to \infty$ the whole domain can be covered.

We have generated vectors $X^k, k = 1, 2, \ldots$. For them we calculate, using the mapping $G$, eigenvalues $\lambda^k = (\lambda_1^k, \ldots, \lambda_M^k), k = 1, 2, \ldots$. It is our information about the graph of the inverse mapping $G^{-1}$ which is used for the construction of the learning set. Values of mapping $G$ are not precise because we have used asymptotic formula (1) so our network is an approximator of the conditional expectation defined by the following formula

$$\theta_\omega(X^k) = E(Y^k \mid X^k).$$ (24)

It is the conditional expectation of $Y^k$ provided that $X^k$ is known. We use an artificial neural network as an approximator of $\theta_\omega$.

Let $K$ be the size of the learning set. The learning set is composed of the following vectors

$$\left(\lambda_1^k, \ldots, \lambda_M^k, X_1^k, X_2^k, X_3^k\right), \quad k = 1, \ldots, K,$$

where

$$\left(\lambda_1^k, \ldots, \lambda_M^k\right)$$ is input vector for neural network,

$$\left(X_1^k, X_2^k, X_3^k\right)$$ is output vector (required).

Furthermore we have dependence

$$(\lambda_1^k, \ldots, \lambda_M^k) = G(X_1^k, X_2^k, X_3^k), \quad k = 1, \ldots, K.$$ (25)

To solve our problem we use artificial neural networks. Each of networks is some mapping $f^q : \mathbb{R}^M \to \mathbb{R}^3$. Parameter $q$ describes number of neurons in hidden layer. We construct a sequence of networks as a sequence of approximation to $\theta_\omega$ by letting networks where $q$ grows with $K$ ($q \to \infty$ as $K \to \infty$) at an approximate rate. For given $K$ learning networks provides an approximation to the unknown regression function $\theta_\omega$.

Set $\Theta$ as a function space containing $\theta, \theta(\cdot) := f^q(\cdot, \delta^q)$ where $\delta^q$ is a set parameters of networks. The function space $\Theta$ contains $\theta_\omega$. We construct a sequence of „sieves“ $\{\Theta_K\}$ for $K = 1, 2, \ldots$ where $\Theta_K$ is a function space containing networks learned by $K$ elements learning sets.

The sieve „connectionist sieve estimator“ $\tilde{\theta}_K$ is defined as a solution to the least squares problem (appropriate for learning $E(Y^K \mid X^k)$)

$$\min_{\theta \in \Theta_K} \sum_{k=1}^{K} [Y^k - \theta(X^k)]^2, \quad (25)$$

for $K = 1, 2, \ldots$.

**Theorem 1** Under some technical assumptions there exists connectionist sieve estimator $\tilde{\theta}_K$ such that

$$\frac{1}{K} \sum_{k=1}^{K} [Y^k - \tilde{\theta}_K(X^k)]^2 = \min_{\theta \in \Theta_K} \frac{1}{K} \sum_{k=1}^{K} [Y^k - \theta(X^k)]^2, \quad (26)$$

for $K = 1, 2, \ldots$. Furthermore, $d(\tilde{\theta}_K, \theta_0) \to 0$ (i.e., for all $\varepsilon > 0 P_{\omega \in \text{domain}} \left( d(\tilde{\theta}_K(\omega), \theta_0) > \varepsilon \right) \to 0$ as $K \to \infty$), where $d$ measures the distance between functions and the convergence is in measure.

4.2. Conclusion

In this paper we use artificial neural networks as an approximator of some mapping which for vector of eigenvalues calculates location and radius of one hole. In [4] authors consider the same problem but for different mapping. Instead of shape functionals our mapping calculates vector of eigenvalues. We consider different mapping but not only. Our learning data are not precise because to calculate them we
use asymptotic formula (1). So network, that we obtained, is an approximator of conditional expectation $\theta_0$.

Our aim is finding the solution of problem (25). This problem is defined for precise but unknown values $Y_k$ whereas network is learned based on not precise data. In Theorem 1 we present existence of the solution  $\tilde{\theta}_K$ of problem (25) as well as the convergence of the used method. We have that $d(\tilde{\theta}_K, \theta_0) \overset{P}{\to} 0$. It means that distance between:  $\tilde{\theta}_K$ – the solution of problem (25) and $\theta_0$ – the conditional expectation tends to zero in probability measure.

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