Method of Covariant Calculation of the Amplitudes of Processes with the Polarized Dirac Particles

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Abstract

General scheme for covariant calculation of the amplitudes of processes with the polarized Dirac particles is considered. It is so concretized that the obtained expressions can be used for calculation of the amplitudes of processes with interfering diagrams. As an illustration the expressions for the amplitudes of processes with massless particles are presented.

1 Introduction

The serious difficulties take place when we calculate the cross sections of processes for high order diagrams (especially when we take into account polarizations of particles, involved in reaction), because it is necessary to evaluate

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traces of products of the great number of Dirac $\gamma$-matrices. In this case the problem often arises with obtaining of analytical expressions for the different physical quantities because of their cumbersomeness.

One of the ways to avoid this problem is to calculate the amplitudes of the processes directly. In particular, the expressions, obtained by multiplication of $\gamma$-matrices and bispinors which are written in components in concrete frames, are given in [1]. Authors of some later articles had to use such a method on account of calculating difficulties (see, e.g. [2]). The obvious defects of this method are complicated calculations, bulky and noncovariant form of obtained results.

Different authors made attempts of the covariant calculation of amplitudes (see [3] – [8]). However, expressions obtained in their papers are unsuitable for calculations with interfering diagrams. The scheme which is the extension of the results of these authors is considered below. It is concretized to avoid problems with interference.

2 General scheme of covariant calculation of amplitudes

There is even number ($2N$) of fermions in initial and final state for any reaction with Dirac particles. Therefore every diagram contains $N$ nonclosed fermion lines. The expression

$$M_{if} = \bar{u}_f Qu_i$$

corresponds to every line in the amplitude of process, where $u_i$, $u_f$ are Dirac bispinors for free particles. (For definiteness, we anticipate that fermions are particles. However the obtained results are true in case if both fermions are antiparticles or one fermion is particle and another one is antiparticle.)

$$\bar{u} = u^+ \gamma^0 .$$

$Q$ is matrix operator which characterizes interaction. It is expressed as linear combination of the products of Dirac $\gamma$-matrices (or of its contractions with 4-vectors) and can have any number of free Lorentz indexes.

For calculating $M_{if}$ we use the following scheme:

$$M_{if} = \bar{u}_f Qu_i = (\bar{u}_f Qu_i) \cdot \frac{\bar{u}_i Zu_f}{u_i Zu_f} = \frac{Tr(Qu_i \bar{u}_i Zu_f \bar{u}_f)}{\bar{u}_i Zu_f}$$

$$\simeq \frac{Tr(Qu_i \bar{u}_i Zu_f \bar{u}_f)}{\mid \bar{u}_i Zu_f \mid} = \frac{Tr(Qu_i \bar{u}_i Zu_f \bar{u}_f)}{Tr(\bar{Z}u_i Zu_f \bar{u}_f)}^{1/2} = M_{if}$$

where $Z$ is an arbitrary $4 \times 4$-matrix ,

$$\bar{Z} = \gamma^0 Z \gamma^0$$
(the symbol \(\simeq\) stands for "an equality to within a phase factor sign").

The projection operators are substituted for \(u\bar{u}\) in (1). For particle with mass \(m\):

\[
\bar{u}(p, n)u(p, n) = \frac{1}{4m}(\hat{p} + m)(1 + \gamma_5\hat{n}) = \mathcal{P}
\]

where \(\hat{p} = \gamma_\mu p^\mu\), \(p^2 = m^2\), \(n^2 = -1\), \(pn = 0\bar{u}u = 1\), \(\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\).

[We use the metric \(a_\mu = (a^0, \vec{a})\), \(a_\mu = (a^0, -\vec{a})\), \(ab = a_\mu b^\mu = a_0b_0 - \vec{a}\vec{b}\).]

For massless particle the projection operator is the following:

\[
\bar{u}_\pm(q)u_\pm(q) = \frac{1}{2}(1 \pm \gamma_5)\hat{q} = \mathcal{P}_\pm
\]

where \(q^2 = 0\), \(\bar{u}_\pm\gamma_\mu u_\pm = 2q_\mu\) (signs \(\pm\) correspond to helicity of particle).

Notice that

\[
(M_{if})^* = \frac{[(\bar{u}_fQu_i)(\bar{u}_iZu_f)]^*}{[Tr(Zu_i\bar{u}_iZu_f\bar{u}_f)]^{1/2}} = \frac{(\bar{u}_fZu_i)(\bar{u}_iQu_f)}{[Tr(Zu_i\bar{u}_iZu_f\bar{u}_f)]^{1/2}} = \frac{Tr(Zu_i\bar{u}_iQu_f\bar{u}_f)}{[Tr(Zu_i\bar{u}_iZu_f\bar{u}_f)]^{1/2}}.
\]

In the articles [3], [4] one chooses

\[Z = 1\]

The results of article [5] come to the same approach for the processes with massless particles.

In [6] one proposes

\[Z = \gamma_5\]

too. The results of article [6] come to the same choice.

The results of article [7] correspond to the choice

\[Z = 1 + \gamma_0\]

The results of [8] come to

\[Z = m + \hat{r}\]

(Where \(r\) is arbitrary 4-momentum, such as \(r^2 = m^2\)). In this paper for 4-vectors, which determine axes of spin projections, one uses

\[
n_i = \frac{m^2p_f - (p_ip_f)p_i}{m[(p_ip_f)^2 - m^4]^{1/2}}, \quad n_f = -\frac{m^2p_i - (p_ip_f)p_f}{m[(p_ip_f)^2 - m^4]^{1/2}}.
\]

However all expressions for the amplitudes [as it follows from (1)] are known to within a phase factor. Really

\[
\mathcal{M}_{if} = M_{if} \cdot \frac{\bar{u}_iZu_f}{|\bar{u}_iZu_f|}.
\]
It is obvious that this circumstance creates no problems, when we calculate amplitude for one diagram. However, in the general case, the presence of unknown phase factor does not enable formula (1) to be used for calculation of amplitudes of the processes which proceed in a few channels since the expressions for amplitudes which corresponding to different channels are multiplied by different phase factors.

3 Calculation of the amplitudes of processes with interfering diagrams

Let us consider in the general form the process, which proceeds in two different channels (see Fig.1):

![Diagram](image.png)

Figure 1: The diagrams of the process, which proceeds in 2 different channels in general form.

The expression
\[ M = (\bar{u}_3 Qu_1) \cdot (\bar{u}_4 Ru_2) = M_{13} \cdot M_{24} \]
corresponds to the first diagram. The expression
\[ M' = (\bar{u}_4 Su_1) \cdot (\bar{u}_3 Tu_2) = M_{14} \cdot M_{23} \]
corresponds to the second one, where \( Q, R, S, T \) are the arbitrary matrix operators characterizing interaction.

There are three possibilities for the correct calculation of the amplitude of process (see Fig.1):

1. It is possible to make Fierz arrangement for the second diagram: 3 ↔ 4. But this method requires a large volume of additional calculations.

2. It is possible to multiply both amplitudes by the same phase factors, for example by
\[ \frac{\bar{u}_1 Z u_3}{|\bar{u}_1 Z u_3|} \cdot \frac{\bar{u}_2 X u_4}{|\bar{u}_2 X u_4|} \].

4
We have expression for the first diagram in this case

\[ \frac{Tr(Qu_1 \bar{u}_1 Zu_3 \bar{u}_3)}{[Tr(Zu_1 \bar{u}_1 Zu_3 \bar{u}_3)]^{1/2}} \cdot \frac{Tr(Ru_2 \bar{u}_2 Xu_4 \bar{u}_4)}{[Tr(Xu_2 \bar{u}_2 Xu_4 \bar{u}_4)]^{1/2}} \]

and that for the second diagram

\[ \frac{Tr(Su_1 \bar{u}_1 Zu_3 \bar{u}_3 Tu_2 \bar{u}_2 Xu_4 \bar{u}_4)}{[Tr(Zu_1 \bar{u}_1 Zu_3 \bar{u}_3)]^{1/2}} \cdot \frac{Tr(Xu_2 \bar{u}_2 Xu_4 \bar{u}_4)}{[Tr(Xu_2 \bar{u}_2 Xu_4 \bar{u}_4)]^{1/2}} . \]

It is obvious that calculation of the amplitude for the second diagram is very complicated. Besides, it is very inconvenient to use expressions with the different structure for calculation of amplitudes for the different diagrams.

3. The third possibility is to calculate the relative phase for the first and second diagrams and to use expression the obtained for the phase correction of one of two amplitudes.

We will use the third possibility. Interference term has the form

\[ M \cdot (M')^* = M_{13} \cdot M_{24} \cdot (M_{14})^* \cdot (M_{23})^* = (\bar{u}_3 Qu_1)(\bar{u}_4 Ru_2)(\bar{u}_1 Su_4)(\bar{u}_2 Tu_3) . \]

Let us multiply the interference term by

\[ \frac{(\bar{u}_1 Zu_3)(\bar{u}_2 Y u_4)(\bar{u}_4 \bar{X} u_1)(\bar{u}_3 \bar{V} u_2)}{(\bar{u}_1 Zu_3)(\bar{u}_2 Y u_4)(\bar{u}_4 \bar{X} u_1)(\bar{u}_3 \bar{V} u_2)} \cdot \frac{(\bar{u}_3 \bar{Z} u_1)(\bar{u}_1 X u_4)(\bar{u}_4 \bar{Y} u_2)(\bar{u}_2 \bar{V} u_3)}{(\bar{u}_3 Zu_1)(\bar{u}_1 X u_4)(\bar{u}_4 \bar{Y} u_2)(\bar{u}_2 \bar{V} u_3)} \equiv 1 \]

where \( X, Y, Z, V \) are as yet arbitrary matrix operators.

We have as the result in this case:

\[
M \cdot (M')^* \\
\equiv \frac{Tr(Qu_1 \bar{u}_1 Zu_3 \bar{u}_3)}{[Tr(Zu_1 \bar{u}_1 Zu_3 \bar{u}_3)]^{1/2}} \cdot \frac{Tr(Ru_2 \bar{u}_2 Y u_4 \bar{u}_4)}{[Tr(Y u_2 \bar{u}_2 Y u_4 \bar{u}_4)]^{1/2}} \cdot \frac{Tr(\bar{X} u_1 \bar{u}_1 Su_4 \bar{u}_4)}{[Tr(X u_1 \bar{u}_1 X u_4 \bar{u}_4)]^{1/2}} \cdot \frac{Tr(\bar{V} u_2 \bar{u}_2 Tu_3 \bar{u}_3)}{[Tr(V u_2 \bar{u}_2 V u_3 \bar{u}_3)]^{1/2}} \\
\times Tr(\bar{Z} u_1 \bar{u}_1 X u_4 \bar{u}_4 \bar{Y} u_2 \bar{u}_2 V u_3 \bar{u}_3) \\
= \frac{Tr(Qu_1 \bar{u}_1 Zu_3 \bar{u}_3)}{[Tr(Zu_1 \bar{u}_1 Zu_3 \bar{u}_3)]^{1/2}} \cdot \frac{Tr(Ru_2 \bar{u}_2 Y u_4 \bar{u}_4)}{[Tr(Y u_2 \bar{u}_2 Y u_4 \bar{u}_4)]^{1/2}} \cdot \frac{Tr(\bar{X} u_1 \bar{u}_1 Su_4 \bar{u}_4)}{[Tr(X u_1 \bar{u}_1 X u_4 \bar{u}_4)]^{1/2}} \cdot \frac{Tr(\bar{V} u_2 \bar{u}_2 Tu_3 \bar{u}_3)}{[Tr(V u_2 \bar{u}_2 V u_3 \bar{u}_3)]^{1/2}} \\
\times \frac{Tr(\bar{Z} u_1 \bar{u}_1 X u_4 \bar{u}_4 \bar{Y} u_2 \bar{u}_2 V u_3 \bar{u}_3)}{[Tr(Zu_1 \bar{u}_1 Zu_3 \bar{u}_3) Tr(X u_1 \bar{u}_1 X u_4 \bar{u}_4) Tr(Y u_2 \bar{u}_2 Y u_4 \bar{u}_4) Tr(V u_2 \bar{u}_2 V u_3 \bar{u}_3)]^{1/2}} \\
= M_{13} \cdot M_{24} \cdot (M_{14})^* \cdot (M_{23})^* \cdot K.
\]
where $M_{13}$, $M_{24}$, $(M_{14})^*$, $(M_{23})^*$ are given by expressions analogous to (1), (2); the coefficient $K$ is given by formula

$$K = \frac{Tr(\bar{Z}u_1 X u_4 \bar{u}_4 Y u_2 \bar{u}_2 V u_5 \bar{u}_3)}{[Tr(\bar{Z}u_1 X u_4 \bar{u}_4)Tr(Y u_2 \bar{u}_2 V u_5 \bar{u}_3)Tr(V u_2 \bar{u}_2 V u_5 \bar{u}_3)]^{1/2}}. \quad (5)$$

It is obvious that $|K| = 1$.

Thus we have to calculate the amplitude of process with interfering diagrams in the form

$$M + M' = K \cdot M_{13} \cdot M_{24} + M_{14} \cdot M_{23}. \quad (6)$$

Let us require for the maximum simplicity of calculations

$$K \equiv 1.$$ 

This requirement is satisfied if we choose $Z = X = Y = V = P$ [see (2)] or $Z = X = Y = V = P_\pm$ [see (3)], since the projection operators have the following properties

$$\bar{P} = P, \quad PAP = Tr[PA] \cdot P, \quad (7)$$

$$\bar{P}_\pm = P_\pm, \quad P_{\pm} A P_{\pm} = Tr[P_{\pm} A] \cdot P_{\pm}. \quad (8)$$

Really

$$\bar{P} = \gamma_0 P^+ \gamma_0 = \gamma_0 (u \bar{u})^+ \gamma_0 = \gamma_0 (u u^+ \gamma_0)^+ \gamma_0 = \gamma_0 [(\gamma_0^0)^+ (u^+)^+ u^+] \gamma_0$$

$$= \gamma_0^0 [u u^+] \gamma_0 = u u^+ \gamma_0 = u \bar{u} = P,$$

$$PAP = (u)_{\alpha} (\bar{u})_{\beta} (A)^{\beta \rho} (u)_{\rho} (\bar{u})_{\delta} = [(\bar{u})_{\beta} (A)^{\beta \rho} (u)_{\rho}] (u)_{\alpha} (\bar{u})_{\delta}$$

$$= [(u)_{\rho} (\bar{u})_{\beta} (A)^{\beta \rho}] (u)_{\alpha} (\bar{u})_{\delta} = Tr[PA] \cdot P.$$ 

Therefore we can calculate the amplitude

$$M + M' = \frac{Tr(QP_1 PP_3) \cdot Tr(RP_2 PP_4)}{[Tr(PP_1)Tr(PP_2)Tr(PP_3)Tr(PP_4)]^{1/2}}$$

$$\quad + \frac{Tr(SP_1 PP_4) \cdot Tr(TP_2 PP_3)}{[Tr(PP_1)Tr(PP_2)Tr(PP_3)Tr(PP_4)]^{1/2}}. \quad (9)$$

This expression enables to calculate the amplitude numerically. Complex numbers being obtained under calculation are used for calculation of the process cross section.
Notice that the calculation of amplitude for the alone diagram is easier than the calculation of squared amplitude, if operators, characterizing interaction, contain the product of greater number of $\gamma$-matrices, than the projection operator.

Really, when the number of $\gamma$-matrices in operator $Q$ increases by $I$ [see (1)], their number in the numerator of (1) increases only by $I$ (denominator does not change), but in construction of $Tr(Qu_i\bar{u}_iQu_f\bar{u}_f)$, which appears, when we calculate the squared matrix element, the number of $\gamma$-matrices increases by $2I$. We take into account that trace of product of $2J$ $\gamma$-matrices contains $1\cdot3\cdot5\cdot\ldots\cdot(2J-1)$ terms and we obtain that the more complicated is a process the bigger are the advantage given under calculation of this process by the method of direct amplitudes calculation.

However, for processes with the interfering diagrams method of amplitudes calculation is easier in any case, because we need not calculate the interference terms.

As it was mentioned before, it is simple to make a generalization of this method for the reactions with participation of antiparticles. It is sufficient for it to substitute the projection operators of antiparticles in place of operators of particles. Let, for example, we are interested in value $\bar{v}_fQu_i$, where $v_f$ is bispinor for a free antiparticle. Then

$$\bar{v}_fQu_i = \frac{Tr(Qu_i\bar{u}_iZv_f\bar{v}_f)}{Tr(Zu_i\bar{u}_iZv_f\bar{v}_f)^{1/2}}$$

(10)

where

$$v(p, n)\bar{v}(p, n) = \frac{1}{4m}(-m + \hat{p})(1 + \gamma_5\hat{n})$$

for massive antiparticle, or

$$v_{\pm}(q)\bar{v}_{\pm}(q) = \frac{1}{2}(1 + \gamma_5)\hat{q}$$

for massless antiparticle. As always, we use $\equiv$ or $\cong$ instead of $Z$.

Notice that in (11) and in the further consideration we shall use equality sign instead of symbol $\simeq$, since there exists not any trouble with the phase factors already.

It is significant that the methods of calculation of the amplitudes with $Z = 1, \gamma_5, 1 + \gamma^0, m + \hat{p}$ for the processes with interfering diagrams require to calculate the expression not only for amplitudes of separate diagrams, but also for coefficient $K$ by the formula (5). It is necessary to use formula (6) in this case.
4 Application of method to calculations of the amplitudes of processes with massless Dirac particles

Formulas for calculation of amplitudes of processes with the massless Dirac particles are very simple. In this case formula (1) takes the following form

\[
\bar{u}_\pm(p_3)Qu_\pm(p_1) = \frac{Tr[Q\hat{p}_1\hat{q}\hat{p}_3(1 \mp \gamma_5)]}{4[(qp_1)(qp_3)]^{1/2}} .
\] (11)

Here \( Z = \frac{1}{2}(1 \mp \gamma_5)\hat{q} = \mathcal{P}_\mp \), \( q^2 = 0 \). Massless 4-vector \( q \) can be arbitrary, but it must be the same for all considered nonclosed fermion lines of diagrams.

\[
\bar{u}_\pm(p_3)Qu_\mp(p_1) = \frac{Tr[Q\hat{p}_1(m \pm \hat{n}\hat{p})\hat{p}_3(1 \mp \gamma_5)]}{4\{(pp_1) \pm m(np_1)\}(pp_1 \pm m(np_3))^{1/2}} .
\] (12)

Here \( Z = \frac{1}{4m}(m + \hat{p})(1 + \gamma_5\hat{n}) = \mathcal{P}, \quad p^2 = m^2, \quad n^2 = -1, \quad pn = 0 \).

As regards 4-vectors \( p \) and \( n \) the same observation as the one for vector \( q \) in (11) is right. We may require for maximum simplicity of calculations \( m = 0, \quad p^2 = 0 \) in (12).

In the last case we can not use easier operator \( \mathcal{P}_\mp \) for \( Z \), since in this case numerator and denominator are identical with 0.

If under numerical calculations denominator in (11) or in (12) is equal to 0 for some values \( p_1 \) and \( p_3 \), it is sufficient to change values of arbitrary 4-vectors \( q \) or \( p, n \) being contained by these formulae (simultaneously for all lines of diagrams being considered).

Another approach may be useful under calculation of value \( \bar{u}_\pm(p_3)Qu_\mp(p_1) \):

\[
\bar{u}_\pm(p_3)Qu_\mp(p_1) \simeq \frac{Tr[Q\hat{p}_1\hat{q}\hat{p}_3(1 \mp \gamma_5)]}{2[2(p_1p_3)]^{1/2}} .
\] (13)

Here \( Z = 1 \). However, it is necessary to use formula (13) if we have interfering amplitudes. In this case [see (3)]

\[
K = \frac{Tr[\hat{p}_1\hat{q}\hat{p}_2\hat{p}_3(1 \mp \gamma_5)]}{8[(p_1p_3)(p_1p_4)(p_2p_3)(p_2p_4)]^{1/2}} .
\] (14)

Formulae (13), (14) generalize the method of calculation of matrix elements offered in [3].
References

[1] R.P. Feynman, Quantum Electrodynamics (Benjamin, New York, 1961) p.64

[2] F.A. Berends, P.De Causmaecker, R. Gastmans, R. Kleiss, W. Troost, T.T. Wu, Nucl.Phys. B264 (1986) 243.

[3] E. Bellomo, Il Nuovo Cimento Ser.X., 21 (1961) 730.

[4] H.W. Fearing, R.R. Silbar, Phys.Rev.D6 (1972) 471.

[5] P.De Causmaecker, R. Gastmans, W. Troost, T.T. Wu, Nucl.Phys. B206 (1982) 53.

[6] F.I. Fedorov, Sov.Phys.J. 23 (1980) 100.

[7] F.I. Fedorov, Theor.and Math.Phys. 18 (1974) 233.

[8] S.M. Sikach, Institute of Physics of Academy of Science of Belarus, Preprints No. 658, 659 (1992)