THOMPSON’S GROUP $F$ IS ALMOST $\frac{3}{2}$-GENERATED

GILI GOLAN POLAK

Abstract. Recall that a group $G$ is said to be $\frac{3}{2}$-generated if every non-trivial element of $G$ belongs to a generating pair of $G$. Thompson’s group $V$ was proved to be $\frac{3}{2}$-generated by Donoven and Harper in 2019. It was the first example of an infinite finitely presented non-cyclic $\frac{3}{2}$-generated group. Recently, Bleak, Harper and Skipper proved that Thompson’s group $T$ is also $\frac{3}{2}$-generated. In this paper, we prove that Thompson’s group $F$ is “almost” $\frac{3}{2}$-generated in the sense that every element of $F$ whose image in the abelianization forms part of a generating pair of $\mathbb{Z}^2$ is part of a generating pair of $F$. We also prove that for every non-trivial element $f \in F$ there is an element $g \in F$ such that the subgroup $\langle f, g \rangle$ contains the derived subgroup of $F$. Moreover, if $f$ does not belong to the derived subgroup of $F$, then there is an element $g \in F$ such that $\langle f, g \rangle$ has finite index in $F$.

1. Introduction

A group $G$ is said to be $\frac{3}{2}$-generated if every non-trivial element of $G$ is part of a generating pair of $G$. In 2000, settling a problem of Steinberg from 1962 [21], Guralnick and Kantor proved that all finite simple groups are $\frac{3}{2}$-generated [18]. In 2008, Breuer, Guralnick and Kantor [4] observed that if a group $G$ is $\frac{3}{2}$-generated then every proper quotient of it must be cyclic. They conjectured that for finite groups this is also a sufficient condition. The conjecture was proved in 2021 by Burness, Guralnick and Harper [7].

Note that for an infinite group $G$ every proper quotient being cyclic is not a sufficient condition for the group being $\frac{3}{2}$-generated. Indeed, the infinite alternating group $A_\infty$ is simple but not finitely generated. Moreover, there are finitely generated infinite simple groups which are not 2-generated (see [17]) and in particular, are not $\frac{3}{2}$-generated. Recently, Cox [10] constructed an example of an infinite 2-generated group $G$ such that every proper quotient of $G$ is cyclic and yet $G$ is not $\frac{3}{2}$-generated.

Obvious examples of infinite $\frac{3}{2}$-generated groups are the Tarski monsters constructed by Olshanskii [20]. Recall that Tarski monsters are infinite finitely generated non-cyclic groups where every proper subgroup is cyclic$^1$. In particular, if $T$ is a Tarski monster, then $T$ is generated by any pair of non-commuting elements of $T$. Since the center of $T$ is trivial, every non-trivial element of $T$ is part of a generating pair of $T$.

In 2019, Donoven and Harper gave the first examples of infinite non-cyclic $\frac{3}{2}$-generated groups, other than Tarski monsters. Indeed, they proved that Thompson’s group $V$ is

---

1There are two types of Tarski monsters. One where every proper subgroup is infinite cyclic and one where every proper subgroup is cyclic of order $p$ for some fixed prime $p$. 

The research was supported by ISF grant 2322/19.
\( \frac{3}{2} \)-generated. More generally, they proved that all Higman–Thompson groups \( V_n \) (see [19]) and all Brin–Thompson groups \( nV \) (see [5]) are \( \frac{3}{2} \)-generated. In 2020, Cox constructed two more examples of infinite \( \frac{3}{2} \)-generated groups with some special properties (see [10]).

Quite recently (in 2022), Bleak, Harper and Skipper proved that Thompson’s group \( T \) is also \( \frac{3}{2} \)-generated [2].

In this paper, we study Thompson’s group \( F \). Recall that Thompson’s group \( F \) is the group of all piecewise-linear homeomorphisms of the interval \([0, 1]\) with finitely many breakpoints where all breakpoints are dyadic fractions (i.e., numbers from \( \mathbb{Z} \left[ \frac{1}{2} \right] \cap (0, 1) \)) and all slopes are integer powers of 2. Thompson’s group \( F \) is 2-generated. The derived subgroup of \( F \) is infinite and simple and can be characterized as the subgroup of \( F \) of all functions \( f \) with slope 1 both at 0+ and at 1− (see [9]). The abelianization \( F/[F, F] \) is isomorphic to \( \mathbb{Z}^2 \). The standard abelianization map \( \pi_{ab} : F \to \mathbb{Z}^2 \) maps every function \( f \in F \) to \((\log_2(f(0^+)), \log_2(f(1^-))) \) (see [9]). Since the abelianization of \( F \) is \( \mathbb{Z}^2 \), Thompson’s group \( F \) cannot be \( \frac{3}{2} \)-generated. However, we prove that it is “almost” \( \frac{3}{2} \)-generated in the sense that the following theorem holds.

**Theorem 1.** Every element of \( F \) whose image in the abelianization \( \mathbb{Z}^2 \) is part of a generating pair of \( \mathbb{Z}^2 \) is part of a generating pair of \( F \).

In fact, we have the following more general theorem.

**Theorem 2.** Let \((a, b), (c, d) \in \mathbb{Z}^2\) be such that \{a, c\} \( \neq \{0\} \) and \{b, d\} \( \neq \{0\} \). Let \( f \in F \) be a non-trivial element such that \( \pi_{ab}(f) = (a, b) \), then there is an element \( g \in F \) such that \( \pi_{ab}(g) = (c, d) \) and such that

\[ \langle f, g \rangle = \pi_{ab}^{-1}(\langle (a, b), (c, d) \rangle). \]

Recall that a subgroup \( H \) of \( F \) is normal if and only if it contains the derived subgroup of \( F \) [9] (in particular, all finite index subgroups of \( F \) are normal subgroups of \( F \)). It follows that the normal subgroups of \( F \) are exactly the subgroups \( \pi_{ab}^{-1}(\langle (a, b), (c, d) \rangle) \) for \((a, b), (c, d) \in \mathbb{Z}^2\). Note that if \( a = c = 0 \) or \( b = d = 0 \), then \( \pi_{ab}^{-1}(\langle (a, b), (c, d) \rangle) \) is not finitely generated (see Remark 6 below). Theorem 2 implies that all other normal subgroups of \( F \) are 2-generated. Hence, we have the following.

**Corollary 3.** Every finitely generated normal subgroup of \( F \) is 2-generated.

In particular, every finite index subgroup of \( F \) is 2-generated. Recall that in [3] (see also [6]), the finite index subgroups of \( F \) that are isomorphic to \( F \) were characterized. Let \( p, q \in \mathbb{N} \). We denote by \( F_{p, q} \) the subgroup \( \pi_{ab}^{-1}(p\mathbb{Z} \times q\mathbb{Z}) \) of \( F \). Then for every \( p, q \in \mathbb{N} \) the subgroup \( F_{p, q} \) is isomorphic to \( F \) and these are the only finite index subgroups of \( F \) that are isomorphic to \( F \) [3]. In particular, the finite index subgroups \( F_{p, q} \) of \( F \) are known to be 2-generated. It was conjectured in [14, Conjecture 12.6] that all finite index subgroups of \( F \) are 2-generated.

Note that for every non-trivial \((a, b) \in \mathbb{Z}^2\), there exists \((c, d) \in \mathbb{Z}^2\) such that \( \langle (a, b), (c, d) \rangle \) is a finite index subgroup of \( \mathbb{Z}^2 \) of the form \( p\mathbb{Z} \times q\mathbb{Z} \) for some \( p, q \in \mathbb{N} \) (see Lemma 7 below). Hence, Theorem 2 implies the following.
Corollary 4. Let \( f \in F \) be an element whose image in the abelianization of \( F \) is non-trivial. Then there is an element \( g \in F \) such that the subgroup \( \langle f, g \rangle \) is isomorphic to \( F \) and has finite index in \( F \).

Note that Theorem 1 shows that in some sense it is “easy” to generate Thompson’s group \( F \). Several other results demonstrate (in different ways) the abundance of generating pairs of Thompson’s group \( F \). Recall that in [15], we prove that in the two natural probabilistic models studied in [8], a random pair of elements of \( F \) generates \( F \) with positive probability. In [12], Gelander, Juschenko and the author proved that Thomson’s group \( F \) is invariably generated by 3 elements (i.e., there are 3 elements \( f_1, f_2, f_3 \in F \) such that regardless of how each one of them is conjugated, together they generate \( F \).) Using results from [16], one can show that in fact, there is a pair of elements in \( F \) which invariably generates \( F \).

2. Preliminaries

2.1. \( F \) as a group of homeomorphisms. Recall that Thompson group \( F \) is the group of all piecewise linear homeomorphisms of the interval \([0, 1]\) with finitely many breakpoints where all breakpoints are dyadic fractions and all slopes are integer powers of 2. The group \( F \) is generated by two functions \( x_0 \) and \( x_1 \) defined as follows [9].

\[
x_0(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{4} \\ t + \frac{1}{4} & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2} & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \quad x_1(t) = \begin{cases} t & \text{if } 0 \leq t \leq \frac{1}{2} \\ 2t - \frac{1}{2} & \text{if } \frac{1}{2} \leq t \leq \frac{5}{8} \\ t + \frac{1}{8} & \text{if } \frac{5}{8} \leq t \leq \frac{3}{4} \\ \frac{1}{2} + \frac{1}{2} & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases}
\]

The composition in \( F \) is from left to right.

Every element of \( F \) is completely determined by how it acts on the set \( Z_{[\frac{1}{2}]} \). Every number in \((0, 1)\) can be described as \( .s \) where \( s \) is an infinite word in \( \{0, 1\} \). For each element \( g \in F \) there exists a finite collection of pairs of (finite) words \((u_i, v_i)\) in the alphabet \( \{0, 1\} \) such that every infinite word in \( \{0, 1\} \) starts with exactly one of the \( u_i \)'s. The action of \( F \) on a number \( .s \) is the following: if \( s \) starts with \( u_i \), we replace \( u_i \) by \( v_i \). For example, \( x_0 \) and \( x_1 \) are the following functions:

\[
x_0(t) = \begin{cases} .0\alpha & \text{if } t = .00\alpha \\ .10\alpha & \text{if } t = .01\alpha \\ .11\alpha & \text{if } t = .1\alpha \end{cases} \quad x_1(t) = \begin{cases} .0\alpha & \text{if } t = .0\alpha \\ .10\alpha & \text{if } t = .100\alpha \\ .110\alpha & \text{if } t = .101\alpha \\ .111\alpha & \text{if } t = .11\alpha \end{cases}
\]

where \( \alpha \) is any infinite binary word.

The group \( F \) has the following finite presentation [9].

\[
F = \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_1 x_0] = 1, [x_0 x_1^{-1}, x_1 x_0^2] = 1 \rangle,
\]

where \( a^b \) denotes \( b^{-1} a b \).
2.2. Elements of $F$ as pairs of binary trees. Often, it is more convenient to describe elements of $F$ using pairs of finite binary trees (see [9] for a detailed exposition). The considered binary trees are rooted full binary trees; that is, each vertex is either a leaf or has two outgoing edges: a left edge and a right edge. A branch in a binary tree is a simple path from the root to a leaf. If every left edge in the tree is labeled “0” and every right edge is labeled “1”, then a branch in $T$ has a natural binary label. We rarely distinguish between a branch and its label.

Let $(T_+, T_-)$ be a pair of finite binary trees with the same number of leaves. The pair $(T_+, T_-)$ is called a tree-diagram. Let $u_1, \ldots, u_n$ be the (labels of) branches in $T_+$, listed from left to right. Let $v_1, \ldots, v_n$ be the (labels of) branches in $T_-$, listed from left to right. We say that the tree-diagram $(T_+, T_-)$ has the pair of branches $u_i \rightarrow v_i$ for $i = 1, \ldots, n$. The tree-diagram $(T_+, T_-)$ represents the function $g \in F$ which takes binary fraction $.u_i\alpha$ to $.v_i\alpha$ for every $i$ and every infinite binary word $\alpha$. We also say that the element $g$ takes the branch $u_i$ to the branch $v_i$. For a finite binary word $u$, we denote by $[u]$ the dyadic interval $[.u, .u]1^\mathbb{N}$. If $u \rightarrow v$ is a pair of branches of $(T_+, T_-)$, then $g$ maps the interval $[u]$ linearly onto $[v]$.

A caret is a binary tree composed of a root with two children. If $(T_+, T_-)$ is a tree-diagram and one attaches a caret to the $i$th leaf of $T_+$ and the $i$th leaf of $T_-$ then the resulting tree diagram is equivalent to $(T_+, T_-)$ and represents the same function in $F$. The opposite operation is that of reducing common carets. A tree diagram $(T_+, T_-)$ is called reduced if it has no common carets; i.e., if there is no $i$ for which the $i$ and $i + 1$ leaves of both $T_+$ and $T_-$ have a common father. The reduced tree-diagrams of the generators $x_0$ and $x_1$ of $F$ are depicted in Figure 1.

![Figure 1](image.png)

**Figure 1.** (A) The reduced tree-diagram of $x_0$. (B) The reduced tree-diagram of $x_1$. In both figures, $T_+$ is on the left and $T_-$ is on the right.

When we say that a function $f \in F$ has a pair of branches $u_i \rightarrow v_i$, the meaning is that some tree-diagram representing $f$ has this pair of branches. In other words, this is equivalent to saying that $f$ maps the dyadic interval $[u_i]$ linearly onto $[v_i]$. Clearly, if $u \rightarrow v$ is a pair of branches of $f$, then for any finite binary word $w$, $uw \rightarrow vw$ is also a pair of branches of $f$. Similarly, if $f$ has the pair of branches $u \rightarrow v$ and $g$ has the pair of branches $v \rightarrow w$ then $fg$ has the pair of branches $u \rightarrow w$. 

Let $B$ be the set of all finite binary words and let $u, v \in B$. We say that $v$ is a descendant of $u$ if $u$ is a strict prefix of $v$. We say that $u$ and $v$ are incomparable if $u$ is not a prefix of $v$ and $v$ is not a prefix of $u$. Note that if $u$ and $v$ are incomparable then the interiors of $[u]$ and $[v]$ have empty intersection. In that case, we will write $[u] < [v]$ if for every $x$ in the interior of $[u]$ and every $y$ in the interior of $[v]$ we have $x < y$. The following lemma will be useful.

**Lemma 5.** Let $f$ be a non-trivial element of $F$. Then there exist $u, v, w \in B$ such that $[u] < [v] < [w]$ and such that $f$ or $f^{-1}$ has the pairs of branches $u \rightarrow v$ and $v \rightarrow w$.

**Proof.** Let $\alpha$ be the maximal number in $[0,1]$ such that $f$ fixes the interval $[0, \alpha]$ pointwise. Since $f$ is non-trivial, $\alpha < 1$. Note that $f'(\alpha^+) \neq 1$. If $f'(\alpha^+) > 1$, let $g = f$, otherwise, let $g = f^{-1}$, so that $g'(\alpha^+) > 1$. Since $\alpha$ is a breakpoint of $f$, it must be dyadic. Hence, there exists a finite binary word $s$ such that $\alpha = .s$. Since $g'(\alpha^+) > 1$, there exist $n > m$ in $\mathbb{N}$ such that $g$ has the pair of branches $s0^n \rightarrow s0^m$ (see [14, Lemma 2.6]). Now, let $u \equiv s0^{n+1}1, v \equiv s0^m1$ and $w \equiv s0^n1$. Then $[u] < [v] < [w]$ and the function $g$ has the pairs of branches $u \rightarrow v$ and $v \rightarrow w$, as necessary. \hfill $\Box$

2.3. The derived subgroup and the abelianization of $F$. Recall that the derived subgroup of $F$ is an infinite simple group which can be characterized as the subgroup of $F$ of all functions $f$ with slope 1 both at $0^+$ and at $1^-$ (see [9]). That is, a function $f \in F$ belongs to $[F,F]$ if and only if the reduced (equiv. any) tree-diagram of $f$ has pairs of branches of the form $0^n \rightarrow 0^m$ and $1^n \rightarrow 1^m$ for some $m, n \in \mathbb{N}$.

As noted above, a subgroup $H$ of $F$ is normal if and only if it contains the derived subgroup of $F$. In particular, every finite index subgroup of $F$ is a normal subgroup of $F$ [9]. Recall that the abelianization of $F$ is isomorphic to $\mathbb{Z}^2$ and that the standard abelianization map $\pi_{ab} : F \rightarrow \mathbb{Z}^2$ maps an element $f \in F$ to $(\log_2(f'(0^+)), \log_2(f'(1^-)))$.

We make the following observation.

**Remark 6.** Let $(a, b), (c, d) \in \mathbb{Z}^2$ be such that $a = c = 0$ or $b = d = 0$. Then the subgroup $\pi_{ab}^{-1}((a, b), (c, d))) \leq F$ is not finitely generated.

**Proof.** We prove it in the case where $a = c = 0$. Note that in that case, $(a, b), (c, d) = ((0, g)), g = \gcd(b, d)$. Let $H = \pi_{ab}^{-1}((0, g)))$. We claim that $H$ is not finitely generated. Assume by contradiction that $H$ is generated by $f_1, \ldots, f_n$, for $n \in \mathbb{N}$. Then for each $i = 1, \ldots, n$ we have $f'_i(0^+) = 1$. Hence, for each $i$, $f_i$ fixes a right neighborhood of $0$. Let $\alpha \in (0, 1)$ be a small enough dyadic fraction such that for each $i = 1, \ldots, n$, $f_i$ fixes the interval $[0, \alpha]$ pointwise. Since $H$ is generated by $f_1, \ldots, f_n$, it follows that $H$ fixes the interval $[0, \alpha]$ pointwise. But it is easy to construct an element $f \in F$ such that $\pi_{ab}(f) = (0, g)$ and such that $f$ does not fix $\alpha$, a contradiction. \hfill $\Box$

Recall that for every pair of elements $(a, b), (c, d) \in \mathbb{Z}^2$ the index of the subgroup $\langle (a, b), (c, d) \rangle$ of $\mathbb{Z}^2$ is the absolute value of the determinant $ad - bc$ (where if $ad - bc = 0$, the index is infinite). The following simple lemma was referred to in the introduction. As we were unable to find a reference for it, we provide a proof.
Lemma 7. Let \((a, b) \in \mathbb{Z}^2\) be non-trivial. Then there exists \((c, d) \in \mathbb{Z}^2\) and \(p, q \in \mathbb{N}\) such that \((\langle a, b \rangle, \langle c, d \rangle) = p\mathbb{Z} \times q\mathbb{Z}\) and such that \(pq = \gcd(a, b)\).

Proof. If \(a = 0\) then for \((c, d) = (1, 0)\) and \(p = 1, q = |b|\) we have the result. Indeed,

\[
\langle (a, b), (c, d) \rangle = \langle (0, b), (1, 0) \rangle = \mathbb{Z} \times |b|\mathbb{Z}.
\]

Similarly, if \(b = 0\), we are done. Hence, we can assume that \(a, b \neq 0\). Let \(g = \gcd(a, b)\) and let \(a', b' \in \mathbb{Z}\) be such that \(a = ga'\) and \(b = gb'\). Clearly, \(\gcd(a', b') = 1\). Let \(g = p_1^{n_1} \cdots p_m^{n_m}\) be the prime factorization of \(g\). Since \(\gcd(a', b') = 1\), each of the primes \(p_i\) for \(i = 1, \ldots, m\) divides at most one of the numbers \(a'\) or \(b'\). Let \(q\) be the product of all \(p_i^{n_i}, i = 1, \ldots, m\) such that \(p_i\) divides \(a'\) (if there are no such \(p_i\)’s, we let \(q = 1\)). Let \(p \in \mathbb{N}\) be such that \(g = pq\) (in other words, \(p\) is the product of all \(p_i^{n_i}\) such that \(p_i\) does not divide \(a'\)). By construction, \(q\) and \(b'\) are co-prime, \(p\) and \(a'\) are co-prime and \(p\) and \(q\) are co-prime. It follows that \(q a'\) and \(p b'\) are co-prime. Hence, by the Euclidean algorithm there exist \(m, n \in \mathbb{Z}\) such that \(m(qa') - n(pb') = 1\). Let \((c, d) = (np, mq)\). Then the index of the subgroup \(\langle (a, b), (c, d) \rangle\) in \(\mathbb{Z}^2\) is

\[
|ad - bc| = |(ga')(mq) - (gb')(np)| = g|mq a' - np b'| = g = pq.
\]

On the other hand, since \(p\) divides both \(a = ga'\) and \(c = np\), we have that \(p|\gcd(a, c)\).

Similarly, \(q\) divides \(\gcd(b, d)\). Hence,

\[
\langle (a, b), (c, d) \rangle \leq \gcd(a, c)\mathbb{Z} \times \gcd(b, d)\mathbb{Z} \leq p\mathbb{Z} \times q\mathbb{Z}.
\]

Since \(p\mathbb{Z} \times q\mathbb{Z}\) is an over-group of \(\langle (a, b), (c, d) \rangle\) and they both have index \(pq\) in \(\mathbb{Z}^2\), they must coincide. Hence,

\[
\langle (a, b), (c, d) \rangle = p\mathbb{Z} \times q\mathbb{Z},
\]

as necessary. \(\square\)

2.4. Generating normal subgroups of \(F\). Let \(H\) be a subgroup of \(F\). A function \(f \in F\) is said to be a piecewise-\(H\) function if there is a finite subdivision of the interval \([0, 1]\) such that on each interval in the subdivision, \(f\) coincides with some function in \(H\). Note that since all break-points of elements in \(F\) are dyadic fractions, a function \(f \in F\) is a piecewise-\(H\) function if and only if there is a dyadic subdivision of the interval \([0, 1]\) into finitely many pieces such that on each dyadic interval in the subdivision, \(f\) coincides with some function in \(H\).

Following \([13, 14]\), we define the closure of a subgroup \(H\) of \(F\), denoted \(\text{Cl}(H)\), to be the subgroup of \(F\) of all piecewise-\(H\) functions. A subgroup \(H\) of \(F\) is closed if \(H = \text{Cl}(H)\). In \([14]\), we gave the following characterization of subgroups of \(F\) which contain the derived subgroup of \(F\) (as well as an algorithm for determining if a finitely generated subgroup of \(F\) contains the derived subgroup of \(F\)).

Theorem 8. \([14, \text{Theorem 7.10}]\) Let \(H\) be a subgroup of \(F\). Then \(H\) contains the derived subgroup of \(F\) (equiv. \(H\) is a normal subgroup of \(F\)) if and only if the following conditions hold.

1. \(\text{Cl}(H)\) contains the derived subgroup of \(F\).
(2) There is an element $h \in H$ and a dyadic fraction $\alpha \in (0, 1)$ such that $h$ fixes $\alpha$, $h'(\alpha^-) = 1$ and $h'(\alpha^+) = 2$.

Note that if a subgroup $H$ of $F$ contains the derived subgroup of $F$, then the image of $H$ in the abelianization of $F$ completely determines the subgroup $H$.

3. Subgroups $H$ of $F$ whose closure is a normal subgroup of $F$

In the next section, we apply Theorem 8 to prove that a given subset of $F$ generates a normal subgroup $H$ of $F$. To do so, we will have to prove in particular that $H$ satisfies Condition (1) of Theorem 8, i.e., that the closure of $H$ contains the derived subgroup of $F$. To that end, with each subgroup of $F$, we associate an equivalence relation on the set of finite binary words $B$.

Definition 9. Let $H$ be a subgroup of $F$. The equivalence relation induced by $H$ on the set of finite binary words $B$, denoted $\sim^H$, is defined as follows. For every pair of finite binary words $u, v \in B$ we have $u \sim^H v$ if and only if there is an element in $H$ with the pair of branches $u \to v$.

Note that if $u \sim^H v$ then for every finite binary word $w$, we have $uw \sim^H vw$. Moreover, if $H$ is closed, we have the following.

Lemma 10. Let $H$ be a closed subgroup of $F$. Then for every pair of finite binary words $u, v \in B$ we have $u \sim^H v$ if and only if $u0 \sim^H v0$ and $u1 \sim^H v1$.

Proof. Assume that $u0 \sim^H v0$ and $u1 \sim^H v1$. We claim that there is an element in $H$ with the pair of branches $u \to v$. Since $u0 \sim^H v0$, there is an element $h_1 \in H$ with the pair of branches $u0 \to v0$. Similarly, since $u1 \sim^H v1$, there is an element $h_2 \in H$ with the pair of branches $u1 \to v1$. Let $\alpha = .u01^N .u1$ and $\beta = .v01^N .v1$. Then $h_1(\alpha) = \beta$ and $h_2(\alpha) = \beta$. Let $h_3 \in F$ be the following function

$$h_3(x) = \begin{cases} h_1(x) & x \in [0, \alpha] \\ h_2(x) & x \in [\alpha, 1] \end{cases}$$

(Since $\alpha$ is dyadic, $h_3$ is indeed in $F$.) By construction, $h_3$ is a piecewise-$H$ function. Hence, since $H$ is closed, the element $h_3 \in H$. But the element $h_3$ has the pairs of branches $u0 \to v0$ and $u1 \to v1$. Hence, $h_3$ maps the interval $[u0]$ linearly onto $[v0]$ and the interval $[u1]$ linearly onto $[v1]$. It follows that $h_3$ maps the interval $[u]$ linearly onto the interval $[v]$. Hence, it has the pair of branches $u \to v$. Therefore, $u \sim^H v$. \qed

Remark 11. An equivalence relation $\sim$ on the set of finite binary words $B$ such that for every $u, v \in B$ we have $u \sim v$ if and only if $u0 \sim v0$ and $u1 \sim v1$ is called a coherent equivalence relation in $[1]$. Coherent equivalence relations were used in $[1]$ to study maximal subgroups of Thompson’s group $V$.

The following corollary follows from Lemma 10 by induction on $k$. 
Corollary 12. Let $H$ be a closed subgroup of $F$. Let $u,v \in B$ and let $k \in \mathbb{N}$. Assume that for every finite binary word $w$ of length $k$ we have $uw \sim_H vw$. Then $u \sim_H v$.

Let $B'$ be the set of all finite binary words which contain both digits “0” and “1”. In other words, $B' = B \setminus \{\emptyset, 0^n, 1^n \mid n \in \mathbb{N}\}$.

Lemma 13. Let $H$ be a closed subgroup of $F$. Assume that for every pair of finite binary words $u,v \in B'$ we have $u \sim_H v$. Then $H$ contains the derived subgroup of $F$.

Proof. Let $f \in [F,F]$. Then the reduced tree-diagram of $f$ consists of pairs of branches

$$f : \begin{cases} 0^m \to 0^m \\ u_i \to v_i \text{ for } i = 1, \ldots, k \\ 1^n \to 1^n \end{cases}$$

where $k, m, n \in \mathbb{N}$ and where for each $i = 1, \ldots, k$, the binary words $u_i$ and $v_i$ contain both digits “0” and “1” and as such belong to $B'$. By assumption, for each $i = 1, \ldots, k$ we have $u_i \sim_H v_i$ and as such there is an element $h_i \in H$ with the pair of branches $u_i \to v_i$. Then $h_i$ coincides with $f$ on the interval $[u_i]$. We note also that $f$ coincides with the identity function $1 \in H$ on $[0^m]$ and on $[1^n]$. Since $[0^m], [u_1], \ldots, [u_k], [1^n]$ is a subdivision of the interval $[0,1]$ and on each of these intervals $f$ coincides with a function in $H$, $f$ is a piecewise-$H$ function. Since $H$ is closed, $f \in H$. \hfill \Box

Lemma 14. Let $H$ be a subgroup of $F$. Assume that there is $k \in \mathbb{N}$ such that for every $u,v \in B'$ and every $s \in B$ such that the length $|s| = k$ we have $us \sim_H vs$. Then $\text{Cl}(H)$ contains the derived subgroup of $F$.

Proof. Since $H$ is contained in $\text{Cl}(H)$, the equivalence relation $\sim_H$ is contained in $\sim_{\text{Cl}(H)}$. By Lemma 13, to prove that $\text{Cl}(H)$ contains the derived subgroup of $F$ it suffices to prove that for every $u,v \in B'$ we have $u \sim_{\text{Cl}(H)} v$. Let $u,v \in B'$. By assumption, there exists $k \in \mathbb{N}$ such that for every finite binary word $s$ of length $k$ we have $us \sim_{\text{Cl}(H)} vs$. But then, since $\text{Cl}(H)$ is closed, by Corollary 12, we have $u \sim_{\text{Cl}(H)} v$, as necessary. \hfill \Box

The next lemma is used in the next section to show that for certain subgroups $H$ of $F$, the closure of $H$ contains the derived subgroup of $F$.

Lemma 15. Let $H$ be a subgroup of $F$ and let $T$ be a non-empty finite binary tree with branches $u_1, \ldots, u_n$. Let $w \in B$ and assume that the following assertions hold.

1. $w \sim_H w0 \sim_H w1$.
2. For every $i = 2, \ldots, n - 1$ we have $u_i \sim_H w$.
3. For every $i \geq 0$, we have $u_i 0^i 1 \sim_H w$.
4. For every $i \geq 0$, we have $u_n 1^i 0 \sim_H w$.

Then $\text{Cl}(H)$ contains the derived subgroup of $F$.

Proof. First, note that since $w \sim_H w0 \sim_H w1$, it follows by induction that for every finite binary word $q$, we have $wq \sim_H w$. In other words, every descendant $v$ of $w$ is $\sim_H$-equivalent
to \( w \). It follows, that for every \( p \in \mathcal{B} \) such that \( p \sim_H w \), every descendant of \( p \) is also \( \sim_H \)-equivalent to \( w \). Now, let \( k \) be the maximum length of a branch of \( T \) and let \( u, v \in B' \).

By Lemma 14, to prove that \( \text{Cl}(H) \) contains the derived subgroup of \( F \) it suffices to prove that for every \( s \in \mathcal{B} \) of length \( k \) we have \( us \sim_H vs \). Let \( s \in \mathcal{B} \) of length \( k \). We claim that \( us \sim_H w \). Indeed, since \( |us| > k \), there is \( i \in \{1, \ldots, n\} \) such that the branch \( u_i \) of \( T \) is a strict prefix of \( us \). If \( i \in \{2, \ldots, n\} \), then since \( u_i \sim_H w \) and \( us \) is a descendant of \( u_i \), we have that \( us \sim_H w \) as necessary. Otherwise, either \( u_1 \) or \( u_n \) is a strict prefix of \( us \). We consider the case where \( u_1 \) is a strict prefix of \( us \), the other case being similar. Since \( u_1 \) is a string of zeros and \( u \) contains the digit 1, the word \( u_1 \) must be a strict prefix of \( u \). In fact, there exists \( j \geq 0 \) such that \( u_10^j1 \) is a prefix of \( u \). Since \( u_10^j1 \sim_H w \) by assumption and since \( us \) is a descendant of \( u_10^j1 \), we get that \( us \sim_H w \), as necessary. In a similar way, one can show that \( vs \sim_H w \). Hence, \( us \sim_H vs \), as required. \( \square \)

4. PROOF OF THE MAIN THEOREMS

To prove Theorem 2, we will need the following proposition.

**Proposition 16.** Let \( f \) be a non-trivial element of \( F \) and let \( c, d \in \mathbb{Z} \setminus \{0\} \). Then the following assertions hold.

1. There is an element \( g \in F \) such that \( \pi_{ab}(g) = (c, d) \) and such that \( \langle f, g \rangle \) contains the derived subgroup of \( F \).
2. If \( f \) has non-trivial slope at \( 1^- \) then there is an element \( g \in F \) such that \( \pi_{ab}(g) = (c, 0) \) and such that \( \langle f, g \rangle \) contains the derived subgroup of \( F \).
3. If \( f \) has non-trivial slope at \( 0^+ \) then there is an element \( g \in F \) such that \( \pi_{ab}(g) = (0, d) \) and such that \( \langle f, g \rangle \) contains the derived subgroup of \( F \).
4. If \( f \) has non-trivial slope both at \( 0^+ \) and at \( 1^- \) then there is an element \( g \in F \) such that \( \pi_{ab}(g) = (0, 0) \) (i.e., such that \( g \in [F, F] \)) and such that \( \langle f, g \rangle \) contains the derived subgroup of \( F \).

**Proof.** We prove parts (1) and (2). The proofs of parts (3) and (4) are similar.

(1) Since \( f \) is non-trivial, by Lemma 5, there exist finite binary words \( u, v \) and \( w \) such that \( |u| < |v| < |w| \) and such that \( f \) or \( f^{-1} \) has the pairs of branches \( u \rightarrow v \) and \( v \rightarrow w \). It is easy to see that \( u, v \) and \( w \) must belong to \( B' \). Note that if (1) holds for \( (c, d) \) it also holds for \( (-c, -d) \). Hence, we can assume without loss of generality that \( c > 0 \). We will also assume that \( d > 0 \), and we will explain below how to modify the proof for the case where \( c > 0 \) and \( d < 0 \).

Let \( T \) be a finite binary tree such that \( u, v_0, v_1, w_0, w_{10} \) and \( w_{11} \) are branches of \( T \) and such that \( T \) has at least three branches after the branch \( w_{11} \). Let \( (S_+, S_-) \) be the reduced tree-diagram of \( x_1 \) (see Figure 1(B)). Let \( T_1 \) be the minimal binary tree with branch \( 0^{c+1} \) and let \( T_2 \) be the minimal binary tree with branch \( 1^c \). Let \( C_1 \) and \( C_2 \) be trees which consist of a single caret each. Let \( T_3 \) be the minimal binary tree with branch \( 1^{d+1} \) and \( T_4 \) be the minimal binary tree with branch \( 0^d \).
Let \( u_1, \ldots, u_n \) be the branches of \( T \). We will use two copies of the tree \( T \) to construct a new tree-diagram in \( F \). Let \( R_+ \) be the tree obtained from the first copy of \( T \) by performing the following operations:

1. Attaching the tree \( T_1 \) to the end of the branch \( u_1 \);
2. Attaching the tree \( S_1 \) to the end of the branch \( w_{10} \);
3. Attaching the tree \( T_3 \) to the end of the branch \( u_n \).

Let \( R_- \) be the tree obtained from the second copy of \( T \) by performing the following operations:

1. Attaching the tree \( T_2 \) to the end of the branch \( u_1 \).
2. Attaching the caret \( C_1 \) to the end of the branch \( w_0 \).
3. Attaching the tree \( S_2 \) to the end of the branch \( w_{10} \).
4. Attaching the caret \( C_2 \) to the end of the branch \( w_{11} \).
5. Attaching the tree \( T_4 \) to the end of the branch \( u_n \).

Note that in the construction of \( R_+ \), we added \((c + 1) + 3 + (d + 1)\) carets to the tree \( T \). Similarly, in the construction of \( R_- \), \( c + 1 + 3 + 1 + d \) carets were added to the tree \( T \). Hence \( R_+ \) and \( R_- \) have the same number of carets. Let \( g \) be the element represented by the tree-diagram \((R_+, R_-)\). We claim that the assertion in Proposition 16(1) holds for \( g \).

Recall that \( w_0, w_{10}, w_{11} \) are (necessarily consecutive) branches of \( T \). In addition, \( w_0 \) cannot be one of the first 4 branches of \( T \), since \( u_1, u, v_0 \) and \( v_1 \) precede it. By assumption, there are at least three branches in \( T \) to the right of the branch \( w_{11} \). Hence, there exists \( 5 \leq k \leq n - 5 \) such that \( u_k \equiv w_0 \), \( u_{k+1} \equiv w_{10} \) and \( u_{k+2} \equiv w_{11} \). By construction, the tree-diagram \((R_+, R_-)\) has the following sets of pairs of branches (we recommend to the reader verifying it to read separately the list of branches of \( R_+ \) (the branches on the left hand-side of (A),(B) and (C)) and the list of branches of \( R_- \) (the branches on the right hand-side of (A),(B) and (C))):

\[
\begin{align*}
(A) & \quad \left\{ \begin{array}{l}
10 i + 1 \rightarrow u_{10} \\
10 i + 1 - i \rightarrow u_{11} i 0, \quad \text{for } 1 \leq i \leq c - 1 \\
u_1 01 \rightarrow u_1 1 c \\
u_1 1 \rightarrow u_2 \\
u_i \rightarrow u_{i+1} \\
u_{k-1} \rightarrow w_{00} \\
u_k \equiv w_0 \rightarrow w_{01}
\end{array} \right.
\end{align*}
\]

\[
(B) \quad \left\{ \begin{array}{l}
w_{10} \rightarrow w_{10} \\
w_{10100} \rightarrow w_{1010} \\
w_{10101} \rightarrow w_{10110} \\
w_{1011} \rightarrow w_{10111}
\end{array} \right.
\]
from the attachment of the trees $S$ that the pairs of branches $w_{10}$ of both copies of $T$ (see Figure 1(B) for the branches of $S_1$ and $S_2$). The preceding pairs of branches of $(R_+, R_-)$ are all in (A). In particular, (A) contains all the pairs of branches of $(R_+, R_-)$ which result from the attachment of the tree $T_1$ to the first copy of $T$ and the trees $T_2$ and $C_1$ to the second copy of $T$. Similarly, (C) contains all the pairs of branches of $(R_+, R_-)$ which result from the attachment of the tree $T_3$ to the first copy of $T$ and the trees $T_4$ and $C_2$ to the second copy of $T$.

Now, the first and last pairs of branches of $(R_+, R_-)$ show that $\pi_{ab}(g) = (c, d)$ as required. Let $H = \langle f, g \rangle$. It suffices to prove that $H$ contains the derived subgroup of $F$. First, note that the pairs of branches $w_{101} \to w_{101}$ and $w_{10100} \to w_{1010}$ in (B) show that $g$ fixes the dyadic fraction $\alpha = .w_{101} = .w_{1010}^{11}$ and that $g'((\alpha^{-})) = 1$ and $g'((\alpha^{+})) = 2$. Hence, $H$ satisfies condition (2) of Theorem 8. Therefore, to prove that $H$ contains $[F, F]$ it suffices to prove that $\text{Cl}(H)$ contains $[F, F]$. For that, we will make use of Lemma 15. Let us denote the equivalence relation $\sim_H$ by $\sim$. To prove that $\text{Cl}(H)$ contains the derived subgroup of $F$ it suffices to prove that for the tree $T$ (with branches $u_1, \ldots, u_n$) and the word $w$, conditions (1)-(4) of Lemma 15 hold.

First, note that since $f$ or $f^{-1}$ has the pairs of branches $u \to v$ and $v \to w$, we have $u \sim v \sim w$. Now, let us consider the pairs of branches of $g$. The pairs of branches $u_i \to u_{i+1}$, $2 \leq i \leq k - 2$ imply that $u_2 \sim u_3 \sim \cdots \sim u_{k-1}$. Recall that $u, v_0$ and $v_1$ are branches of $T$ (distinct from $u_1$) which precede $u_k \equiv w_0$. As such, $u, v_0, v_1 \in \{u_2, \ldots, u_{k-1}\}$, so that $u \sim v_0 \sim v_1$. Since $u \sim v$, we get that $v \sim v_0 \sim v_1$. It follows that every descendant of $v$ is $\sim$-equivalent to $v$. Since $w \sim v$, every descendant of $w$ is $\sim$-equivalent to $w$ (and as such, Condition (1) of Lemma 15 holds). In particular, since $u_k \equiv w_0, u_{k+1} \equiv w_{10}$ and $u_{k+2} \equiv w_{11}$, we have $u_k \sim u_{k+1} \sim u_{k+2} \sim w$. Since $u_2 \sim \cdots \sim u_{k-1} \sim u \sim w$, we have $u_2 \sim \cdots \sim u_{k-1} \sim u_k \sim u_{k+1} \sim u_{k+2} \sim w$.

Now, the pair of branches $u_{k+3} \to w_{111}$ in (C) implies that $u_{k+3} \sim w_{111} \sim w$. The pairs of branches $u_i \to u_{i-1}$ for $k + 4 \leq i \leq n - 1$ imply that $u_{k+3} \sim u_{k+4} \sim \cdots \sim u_{n-1}$. Since $u_{k+3} \sim w$, we have that $u_2 \sim \cdots \sim u_{k+2} \sim u_{k+3} \sim \cdots \sim u_{n-1} \sim u \sim w$. That is, for every $2 \leq i \leq n - 1$ we have $u_i \sim w$. Hence, Condition (2) of Lemma 15 holds.

Next, we show that Condition (3) of Lemma 15 holds. That is, we claim that for every $i \geq 0$, we have $u_i 0^1 \sim w$. Indeed, for $i = 0$ it holds since the pair of branches $u_1 \to u_2$ of the element $g$ implies that $u_1 1 \sim w$. Moreover, since $u_1 1 \sim w$, every descendant
of \( u1 \) is \( \sim \)-equivalent to \( w \). Hence, the pair of branches \( u101 \rightarrow u11^c \) of \( g \) implies that \( u101 \sim u11^c \sim w \), so the claim holds for \( i = 1 \) as well. Now, the pairs of branches \( u10^{c+1-j}1 \rightarrow u11^0 \) for \( 1 \leq j \leq c - 1 \) of \( g \) imply that for all \( 1 \leq j \leq c - 1 \) we have \( u10^{c+1-j}1 \sim w \). Letting \( i = c + 1 - j \), we get that for all \( i = 2, \ldots, c \) we have \( u10^i1 \sim w \). Hence, the claim holds for every \( i = 0, \ldots, c \). We prove by induction that the claim also holds for every \( i > c \). Indeed, let \( i \geq c + 1 \). Then the pair of branches \( u10^{c+1} \rightarrow u10 \) of \( g \) implies that

\[
u10^i \equiv u10^{c+1}0^{i-c+1} \sim u100^{i-c} \equiv u10^{i-c}.
\]

That implies that \( u10^i1 \sim u10^{i-c}1 \sim w \), by induction. Hence, for all \( i \geq 0 \), we have \( u10^i1 \sim w \) and Condition (3) of Lemma 15 holds.

Note that in the proof of Condition (3) we have made use of branches in (A) (more specifically, of the branches written in the first 4 rows in (A)). In an almost identical manner, using branches of (C) (more specifically, the branches written in the last 4 rows of (C)), one can show that Condition (4) from Lemma 15 holds. It follows that \( Cl(H) \) contains the derived subgroup of \( F \). Hence, by Theorem 8, \( H = \langle f, g \rangle \) contains the derived subgroup of \( F \), as required.

It remains to note that one can modify the proof for the case where \( c > 0 \) and \( d < 0 \) as follows. First, wherever \( d \) appears in the above proof, we replace it by \( |d| \). Second, in the construction of \( R_+ \) and \( R_- \), instead of performing operation (3a) on the first copy of \( T \) (used in the construction of \( R_+ \)) and operations (4b) and (5b) on the second copy of \( T \) (used in the construction of \( R_- \)), we perform operation (3a) on the second copy of \( T \) and operations (4b) and (5b) on the first copy of \( T \). The result is a tree-diagram \( (R'_+, R'_-) \), whose pairs of branches coincide with the pairs of branches of \( (R_+, R_-) \), other than the ones listed in (C): for every pair of branches \( p \rightarrow q \) in (C), the tree-diagram \( (R'_+, R'_-) \) has the “opposite” pair of branches \( q \rightarrow p \). Note that this change, does not affect the proof that the subgroup generated by \( f \) and the element represented by \( (R'_+, R'_-) \) contains the derived subgroup of \( F \), as the equivalence relation \( \sim \) is not affected.

(2) The proof is similar to the proof of (1). First, let \( u, v \) and \( w \) be as in part (1). By assumption, the element \( f \) has non-trivial slope at \( 1^- \). Replacing \( f \) by \( f^{-1} \) if necessary, we can assume that \( f'(1) > 1 \). Hence, there exist \( m > \ell \) in \( \mathbb{N} \) such that \( f \) has the pair of branches \( 1^m \rightarrow 1^{m-\ell} \).

Let \( T' \) be a finite binary tree such that \( u, v0, v1, w0, w10, w11 \) are branches of \( T' \). Let \( T \) be the tree obtained from \( T' \) by attaching the minimal binary tree with branch \( 1^m \) to the right-most leaf of \( T' \). Let \( r \) be the length of the last branch of \( T' \) and note that \( T \) has the branches \( 1^0, 1^r+1, \ldots, 1^r+m-1, 1^{r+m} \). (In particular, since \( m \geq 2 \), there are at least three branches in \( T \) after the branch \( w11 \)).

Let \( (S_+, S_-), T_1, T_2, C_1 \) and \( C_2 \) be as in part (1). Let \( T_3 \) be the minimal binary tree with branch 00 and \( T_4 \) be a tree which consists of a single caret.

Let \( u_1, \ldots, u_n \) be the branches of \( T \). We use two copies of the tree \( T \) to construct a new tree-diagram \( (R'_+, R'_-) \) in \( F \), by performing the operations (1a) – (3a) on the first copy of \( T \) and (1b) – (5b) on the second copy of \( T \), as in the proof of part (1). Note that there
exists $5 \leq k \leq n - 5$ such that the branch $u_k \equiv w_0$. By construction, the tree-diagram $(R_+, R_-)$ has the sets (A) and (B) of pairs of branches as in the proof of part (1) as well as the following set

\[
(C') \begin{cases}
  w11 \to w110 \\
  u_{k+3} \to w111 \\
  u_i \to u_{i-1} & \text{for } k + 4 \leq i \leq n - 1 \\
  u_n00 \to u_{n-1} \\
  u_n01 \to u_n0 \\
  u_n1 \to u_n1
\end{cases}
\]

Let $g \in F$ be the element represented by $(R_+, R_-)$. Then $\pi_{ab}(g) = (c, 0)$. Let $H = \langle f, g \rangle$. The proof that $H$ contains the derived subgroup of $F$ is almost identical to the proof in part (1) (note that the first 3 rows in $(C')$ also coincide with the first three rows in $(C)$). The only difference is in the proof that Condition (4) of Lemma 15 holds for $H$ (with the tree $T$ and the word $w$). Let us prove that the condition holds. That is, we claim that for every $i \geq 0$, we have $u_n1i0 \sim w$ (where $\sim$ stands for $\sim_H$). Recall that by construction $u_n \equiv 1^{r+m}$. Hence, we need to prove that for every $i \geq 0$, we have $1^{r+i}0 \sim w$. Clearly, it suffices to prove that for every $i \geq 0$, we have $1^{r+i}0 \sim w$. By Condition (1) of Lemma 15 (which holds here, as in part (1)), for every $j \in \{2, \ldots, n - 1\}$ we have $u_j \sim w$. Since $1^0, 1^{r+1}0, \ldots, 1^{r+m-1}0$ are branches of $T$ (which are clearly not the first nor the last branch of $T$), for every $0 \leq i \leq m - 1$, we have $1^{r+i}0 \sim w$. We prove by induction that the claim also holds for every $i \geq m$. Indeed, let $i \geq m$. Since $f$ has the pair of branches $1^m \to 1^{m-\ell}$, we have $1^m \sim 1^{m-\ell}$. It follows that $1^{r+i}0 \sim 1^{r+(i-\ell)}0$. Since $i - \ell < i$, we are done by induction. Hence, Condition (4) of Lemma 15 holds for $H$, as required.

Theorem 2 is a corollary of Proposition 16. Recall the statement of the theorem.

**Theorem 2.** Let $(a, b), (c, d) \in \mathbb{Z}^2$ be such that $\{a, c\} \neq \{0\}$ and $\{b, d\} \neq \{0\}$. Let $f \in F$ be a non-trivial element such that $\pi_{ab}(f) = (a, b)$. Then there exists an element $g \in F$ such that $\pi_{ab}(g) = (c, d)$ and such that $\langle f, g \rangle = \pi_{ab}^{-1}(\langle (a, b), (c, d) \rangle)$.

**Proof.** It suffices to prove that there exists an element $g \in F$ such that $\pi_{ab}(g) = (c, d)$ and such that $\langle f, g \rangle$ contains the derived subgroup of $F$. Indeed, in that case, $\pi_{ab}(\langle f, g \rangle) = \langle (a, b), (c, d) \rangle$ and since $\langle f, g \rangle$ contains the derived subgroup of $F$, we have that

$$\langle f, g \rangle = \pi_{ab}^{-1}(\langle (a, b), (c, d) \rangle),$$

as necessary.

If $c, d \neq 0$, then by Proposition 16(1), there is an element $g \in F$ such that $\pi_{ab}(g) = (c, d)$ and such that $\langle f, g \rangle$ contains the derived subgroup of $F$, as required.

If $c \neq 0$ and $d = 0$, then by assumption $b \neq 0$. Hence, $f$ has non-trivial slope at $1^-$. Then by Proposition 16(2), there is an element $g \in F$ such that $\pi_{ab}(g) = (c, 0) = (c, d)$ and such that $\langle f, g \rangle$ contains the derived subgroup of $F$, as necessary.

If $c = 0$ and $d \neq 0$, we are done in a similar way, using Proposition 16(3).
Finally, if \( c = d = 0 \), then \( a \neq 0 \) and \( b \neq 0 \). Then \( f \) has non-trivial slope both at \( 0^+ \) and at \( 1^- \). Hence, by Proposition 16(4), there is an element \( g \in F \) such that \( \pi_{ab}(g) = (0,0) = (c,d) \) and such that \( \langle f, g \rangle \) contains the derived subgroup of \( F \), as necessary. \( \square \)

Finally, Theorem 1 is a corollary of Theorem 2. Indeed, let \( f \in F \) be such that \( \pi_{ab}(f) = (a,b) \) forms part of a generating pair of \( \mathbb{Z}^2 \). Then there exists \( (c,d) \in \mathbb{Z}^2 \) such that \( \{(a,b), (c,d)\} \) is a generating set of \( \mathbb{Z}^2 \). Since \( (a,b) \) and \( (c,d) \) generate \( \mathbb{Z}^2 \), we have that \( \{a,c\} \neq \{0\} \) and \( \{b,d\} \neq \{0\} \). Hence, by Theorem 2, there is an element \( g \in F \) such that \( \pi_{ab}(g) = (c,d) \) and such that

\[
\langle f, g \rangle = \pi_{ab}^{-1}(\langle (a,b), (c,d) \rangle) = \pi_{ab}^{-1}(\mathbb{Z}^2) = F.
\]

Hence, \( f \) is part of a 2-generating set of \( F \). \( \square \)

References

[1] J. Belk, C. Bleak, M. Quick and R. Skipper, Type systems and maximal subgroups of Thompson’s group \( V \), arXiv:2206.12631.
[2] C. Bleak, S. Harper and R. Skipper, Thompson’s group \( T \) is \( \frac{3}{2} \)-generated, arXiv:2206.05316
[3] C. Bleak and B. Wassink, Finite index subgroups of R. Thompson’s group \( F \). arXiv:0711.1014
[4] T. Breuer, R. Guralnick and W. Kantor, Probabilistic generation of finite simple groups, II, J. Algebra 320 (2008), 443–494.
[5] M. Brin, Higher dimensional Thompson groups, Geom. Dedicata 108 (2004), 163–192.
[6] J. Burillo, S. Cleary and E. Röver, Commensurations and Subgroups of Finite Index of Thompson’s Group \( F \), Geom. Topol. 12(3): 1701-1709 (2008).
[7] T. Burness, R. Guralnick and S. Harper, The spread of a finite group, Ann. of Math. (2) 193 (2021), 619–687.
[8] S. Cleary, M. Elder, A. Rechnitzer, J. Taback, Random subgroups of Thompson’s group \( F \), Groups Geom. Dyn. 4 (2010), no. 1, 91-126.
[9] J. Cannon, W. Floyd, and W. Parry, Introductory notes on Richard Thompson’s groups. L’Enseignement Mathematique, 42 (1996), 215–256.
[10] C. Cox, On the spread of infinite groups. Proceedings of the Edinburgh Mathematical Society. 2022 Feb;65(1):214-28.
[11] C. Donoven and S. Harper, Infinite \( \frac{3}{2} \)-generated groups, Bull. Lond. Math. Soc. 52 (2020), 657–673.
[12] T. Gelander, G. Golan and K. Juschenko, Invariable generation of Thompson groups, Journal of Algebra 478 (2017), 261–270.
[13] G. Golan and M. Sapir, On subgroups of R. Thompson group \( F \), Trans. Amer. Math. Soc. 369 (2017), 8857–8878.
[14] G. Golan, The generation problem in Thompson group \( F \), arXiv:1608.02572, to appear in Memoirs of the AMS.
[15] G. Golan Polak, Random Generation of Thompson’s group \( F \), Journal of Algebra, 593 (2022), 507-524.
[16] G. Golan Polak, On Maximal subgroups of Thompson’s group \( F \), arXiv:2209.03244.
[17] V. Guba, A finite generated simple group with free 2-generated subgroups, Siberian Mat. Zh. 27 (1986), 50–67.
[18] R. Guralnick and W. Kantor, Probabilistic generation of finite simple groups, J. Algebra 234 (2000), 743–792.
[19] G. Higman, Finitely presented infinite simple groups, Notes on Pure Mathematics, vol. 8, Department of Mathematics, I.A.S., Australia National University, Canberra, 1974.
[20] A. Olshanskii, *Geometry of defining relations in groups*, (Nauka, Moscow, 1989) (in Russian); (English translation by Kluwer Publications 1991).

[21] R. Steinberg, *Generators for simple groups*, Canadian J. Math. 14 (1962), 277–283.

Gili Golan
Department of Mathematics,
Ben Gurion University of the Negev,
golangi@bgu.ac.il