EMBEDDINGS OF $\mathbb{C}^*$-SURFACES INTO WEIGHTED PROJECTIVE SPACES

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Abstract. Let $V$ be a normal affine surface which admits a $\mathbb{C}^*$- and a $\mathbb{C}_+$-action. Such surfaces were classified e.g., in [FlZa1, FlZa2], see also the references therein. In this note we show that in many cases $V$ can be embedded as a principal Zariski open subset into a hypersurface of a weighted projective space. In particular, we recover a result of D. Daigle and P. Russell, see Theorem A in [DR].

1. Introduction

If $V = \text{Spec} A$ is a normal affine surface equipped with an effective $\mathbb{C}^*$-action, then its coordinate ring $A$ carries a natural structure of a $\mathbb{Z}$-graded ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$. As was shown in [FlZa1], such a $\mathbb{C}^*$-action on $V$ has a hyperbolic fixed point if and only if $C = \text{Spec} A_0$ is a smooth affine curve and $A_{\pm 1} \neq 0$. In this case the structure of the graded ring $A$ can be elegantly described in terms of a pair $(D_+, D_-)$ of $\mathbb{Q}$-divisors on $C$ with $D_+ + D_- \leq 0$. More precisely, $A$ is the graded subring $A = A_0[\frac{D_+}{D_+}, \frac{D_-}{D_-}] \subseteq K_0[u, u^{-1}]$, $K_0 := \text{Frac} A_0$, where for $i \geq 0$

\begin{align*}
A_i &= \{ f \in K_0 \mid \text{div} f + iD_+ \geq 0 \} u^i \quad \text{and} \quad A_{-i} = \{ f \in K_0 \mid \text{div} f + iD_- \geq 0 \} u^{-i}.
\end{align*}

This presentation of $A$ (or $V$) is called in [FlZa1] the DPD-presentation. Furthermore two pairs $(D_+, D_-)$ and $(D'_+, D'_-)$ define equivariantly isomorphic surfaces over $C$ if and only if they are equivalent that is,

\begin{align*}
D_+ = D'_+ + \text{div} f \quad \text{and} \quad D_- = D'_- - \text{div} f \quad \text{for some} \ f \in K_0^\times.
\end{align*}

In this note we show that if such a surface $V$ admits also a $\mathbb{C}_+$-action then it can be $\mathbb{C}^*$-equivariantly embedded (up to normalization) into a weighted projective space as a hypersurface minus a hyperplane; see Theorem 2.3 and Corollary 2.5 below. In particular we recover the following result of Daigle and Russell [DR].

Theorem 1.1. Let $V$ be a normal Gizatullin surface\(^1\) with a finite divisor class group. Then $V$ can be embedded into a weighted projective plane $\mathbb{P}(a, b, c)$ minus a hypersurface. More precisely:

(a) If $V = V_{d,e}$ is toric\(^2\) then $V$ is equivariantly isomorphic to the open part\(^3\) $D_+(z)$ of the weighted projective plane $\mathbb{P}(1, e, d)$ equipped with homogeneous coordinates $(x : y : z)$ and with the 2-torus action $(\lambda_1, \lambda_2).(x : y : z) = (\lambda_1 x : \lambda_2 y : z)$.

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\(^1\)That is, $V$ admits a completion by a linear chain of smooth rational curves; see Section 3 below.

\(^2\)See 3.1(a) below.

\(^3\)We use the standard notation $V_+(f) = \{ f = 0 \}$ and $D_+(f) = \{ f \neq 0 \}$. 

(b) If \( V \) is non-toric then \( V \cong \mathbb{D}_+ (xy - zm) \subseteq \mathbb{P}(a, b, c) \) for some positive integers \( a, b, c \) satisfying \( a + b = cm \) and \( \gcd(a, b) = 1 \).

2. Embeddings of \( \mathbb{C}^* \)-surfaces into weighted projective spaces

According to Proposition 4.8 in [FlZa1] every normal affine \( \mathbb{C}^* \)-surface \( V \) is equivariantly isomorphic to the normalization of a weighted homogeneous surface \( V' \) in \( \mathbb{A}^4 \). In some cases (described in loc.cit.) \( V' \) can be chosen to be a hypersurface in \( \mathbb{A}^3 \). Cf. also [Du] for affine embeddings of some other classes of surfaces.

In Theorem 2.3 below we show that any normal \( \mathbb{C}^* \)-surface \( V \) with a \( \mathbb{C}_+ \)-action is the normalization of a principal Zariski open subset of some weighted projective hypersurface.

In the proofs we use the following observation from [Fl].

**Proposition 2.1.** Let \( R = \bigoplus_{i \geq 0} R_i \) be a graded \( R_0 \)-algebra of finite type containing the field of rational numbers \( \mathbb{Q} \). If \( z \in R_d, d > 0 \), is an element of positive degree then the group of \( d \)th roots of unity \( E_d \cong \mathbb{Z}/d \) acts on \( R \) and then also on \( R/(z - 1) \) via 

\[
\zeta \cdot a = \zeta^i \cdot a \quad \text{for} \quad a \in R_i, \quad \zeta \in E_d,
\]

with ring of invariants \( (R/(z - 1))/E_d \cong (R[1/z])_0. \) Consequently 

\[
(\text{Spec } R/(z - 1))/E_d \cong \mathbb{D}_+(z)
\]

is isomorphic to the complement of the hyperplane \( \{ z = 0 \} \) in \( \text{Proj}(R) \).

Let us fix the notations.

2.2. Let \( V = \text{Spec } A \) be a normal \( \mathbb{C}^* \)-surface with DPD-presentation

\[
A = \mathbb{C}[t][D_+, D_-] \subseteq \mathbb{C}(t)[u, u^{-1}].
\]

If \( V \) carries a \( \mathbb{C}_+ \)-action then according to [FlZa2], after interchanging \( (D_+, D_-) \) and passing to an equivalent pair, if necessary, we may assume that

\[
D_+ = -\frac{e_+}{d}[0] \quad \text{with} \quad 0 < e_+ \leq d, \\
D_- = -\frac{e_-}{d}[0] - \frac{t}{k}D_0
\]

with an integral divisor \( D_0 \), where \( D_0(0) = 0 \). We choose a polynomial \( Q \in \mathbb{C}[t] \) with \( D_0 = \text{div}(Q) \); so \( Q(0) \neq 0 \).

**Theorem 2.3.** Let \( F \) be the polynomial

\[
F = x^ky - s^{k(e_+ + e_-)}Q(s^d/z)z^{\deg Q} \in \mathbb{C}[x, y, z, s],
\]

which is weighted homogeneous of degree \( k(e_+ + e_-) + d \deg Q \) with respect to the weights

\[
\deg x = e_+, \quad \deg y = ke_- + d \deg Q, \quad \deg z = d, \quad \deg s = 1.
\]

Then the surface \( V \) as in 2.2 above is equivariantly isomorphic to the normalization of the principal Zariski open subset \( \mathbb{D}_+(z) \) of the hypersurface \( \mathbb{V}_+(F) \) in the weighted projective 3-space

\[
\mathbb{P} = \mathbb{P}(e_+, ke_- + d \deg Q, d, 1).
\]

\[\text{We note that } e_+ + e_- = d(-D_+(0) - D_-(0)) \geq 0.\]
Example 4.10 in [FlZa]

The cyclic group $E_1$.

Proposition 4.12 in [FlZa]

with invariant ring $A$ with respect to the Galois group is the group of $d$th roots of unity $E_d$ acting on $L$ via the identity on $K$ and by $\zeta \cdot s = \zeta \cdot s$ if $\zeta \in E_d$. Let $A'$ be the normalization of $A$ in $L$. According to Proposition 4.12 in [FlZa],

$$A' = \mathbb{C}[s][D'_+, D'_-] \subseteq \mathbb{C}(s)[u, u^{-1}]$$

with $D'_\pm = \pi_d^*(D_{\pm})$, where $\pi_d : \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the covering $s \mapsto s^d$. Thus

$$(D'_+, D'_-) = \left( -e_+[0], -e_-[0] - \frac{1}{k} \pi_d^*(D_0) \right) = \left( -e_+[0], -e_-[0] - \frac{1}{k} \text{div}(Q(s^d)) \right).$$

The element $x = s^{e_+}u \in A'_1$ is a generator of $A'_1$ as a $\mathbb{C}[s]$-module. According to Example 4.10 in [FlZa], the graded algebra $A'$ is isomorphic to the normalization of $B = \mathbb{C}[x, y, s]/(x^k y - s^{k(e_+ + e_-)} Q(s^d))$.

The cyclic group $E_d$ acts on $A'$ via

$$\zeta \cdot x = \zeta^{e_+} x, \quad \zeta \cdot y = \zeta^{e_-} y, \quad \zeta \cdot s = \zeta s$$

with invariant ring $A$. Clearly this action stabilizes the subring $B$. Assigning to $x, y, z, s$ the degrees as in (4), $F$ as in (3) is indeed weighted homogeneous. Since $F(x, y, 1, s) = x^k y - s^{k(e_+ + e_-)} Q(s^d)$, the graded algebra

$$R = \mathbb{C}[x, y, z, s]/(F)$$

satisfies $R/(z - 1) \cong B$. Applying Proposition 2.1, $V = \text{Spec } A$ is isomorphic to the normalization of $\mathbb{D}_+(\mathbb{A}) \cap \mathbb{V}_+(F)$ in the weighted projective space $\mathbb{P}$.

Remark 2.4. In general not all weights of the weighted projective space $\mathbb{P}$ in (5) are positive. Indeed it can happen that $ke_- + d \deg Q \leq 0$. In this case we can choose $\alpha \in \mathbb{N}$ with $ke_- + d(\deg Q + \alpha) > 0$ and consider instead of $F$ the polynomial

$$(7) \quad \tilde{F} = x^k y - s^{k(e_+ + e_-)} Q(s^d/z)^\alpha \in \mathbb{C}[x, y, z, s],$$

which is now weighted homogeneous of degree $k(e_+ + e_-) + d(\deg Q + \alpha)$ with respect to the positive weights

$$(8) \quad \deg x = e_+, \quad \deg y = ke_- + d(\deg Q + \alpha), \quad \deg z = d, \quad \deg s = 1.$$

As before $V = \text{Spec } A$ is isomorphic to the normalization of the principal open subset $\mathbb{D}_+(\mathbb{A})$ of the hypersurface $\mathbb{V}_+(F)$ in the weighted projective space

$$\mathbb{P} = \mathbb{P}(e_+, ke_- + d(\deg Q + \alpha), d, 1).$$

In certain cases it is unnecessary in Theorem 2.3 to pass to normalization.

Corollary 2.5. Assume that in (2) one of the following conditions is satisfied.

(i) $k = 1$;
(ii) $e_+ + e_- = 0$, and $D_0$ is a reduced divisor.

Then $V = \text{Spec } A$ is equivariantly isomorphic to the principal open subset $\mathbb{D}_+(\mathbb{A})$ of the weighted projective hypersurface $\mathbb{V}_+(F)$ as in (3) in the weighted projective space $\mathbb{P}$ from (4).
Applying Theorem 2.3 with \( e \) \( C \) is normal. In other words, the quotient \( R/(z-1) \) of the graded ring \( R = \mathbb{C}[x, y, z, s]/(F) \) is normal and so is its ring of invariants \( (R/(z-1))^{E_d} \). Comparing with Theorem 2.3 the result follows.

Similarly, in case (ii)

\[
F(x, y, 1, s) = x^k y - Q(s^d) .
\]

Since the divisor \( D_0 \) is supposed to be reduced and \( D_0(0) = 0 \), the polynomials \( Q(t) \) and then also \( Q(s^d) \) both have simple roots. Hence the hypersurface \( F(x, y, 1, s) = 0 \) in \( A^3 \) is again normal, and the result follows as before. \( \square \)

**Remark 2.6.** The surface \( V \) as in 2.2 is smooth if and only if the divisor \( D_0 \) is reduced and \(-m_+m_-(D_+(0) + D_-(0)) = 1\), where \( m_+ > 0 \) is the denominator in the irreducible representation of \( D_+(0) \), see Proposition 4.15 in [FKZ]. It can happen, however, that \( V \) is smooth but the surface \( V_+(F) \cap D_+(z) \subseteq \mathbb{P} \) has non-isolated singularities. For instance, if in 2.2 \( D_0 = 0 \) (and so \( Q = 1 \)), then \( V \) is an affine toric surface. In fact, every affine toric surface different from \((A^1)^2\) or \( A^1 \times A^1 \) appears in this way, see Lemma 4.2(b) in [FKZ].

In this case the integer \( k > 0 \) can be chosen arbitrarily. For any \( k > 1 \), the affine hypersurface \( V_+(F) \cap D_+(z) \subseteq \mathbb{P} \) with equation \( x^k y - s^{k(e_+ + e_-)} = 0 \) has non-isolated singularities and hence is non-normal. Its normalization \( V = \text{Spec} A \) can be given as the Zariski open part \( D_+(z) \) of the hypersurface \( V_+(xy' - s^{e_+ + e_-}) \in \mathbb{P}' = \mathbb{P}(e_+, e_-, d, 1) \) (which corresponds to the choice \( k = 1 \)). Indeed, the element \( y' = s^{e_+ + e_-}/x \in K \) with \( y'^k = y \) is integral over \( A \). However cf. Theorem 1.1(a).

**Example 2.7.** (Danilov-Gizatullin surfaces) We recall that a Danilov-Gizatullin surface \( V(n) \) of index \( n \) is the complement to a section \( S \) in a Hirzebruch surface \( \Sigma_d \), where \( S^2 = n > d \). By a remarkable result of Danilov and Gizatullin up to an isomorphism such a surface only depends on \( n \) and neither on \( d \) nor on the choice of the section \( S \), see e.g., [DaGi], [CN], [FKZ].

According to [FKZ] §5, up to conjugation \( V(n) \) carries exactly \( (n-1) \) different \( \mathbb{C}^* \)-actions. They admit DPD-presentations

\[
(D_+, D_-) = \left( -\frac{1}{d}[0], -\frac{1}{n-d}[1] \right), \quad \text{where} \quad d = 1, \ldots, n-1 .
\]

Applying Theorem 2.3 with \( e_+ = 1, e_- = 0 \), and \( k = n-d \), the \( \mathbb{C}^* \)-surface \( V(n) \) is the normalization of the principal open subset \( D_+(z) \) of the hypersurface \( V_+(F_{n,d}) \subseteq \mathbb{P}(1, d, d, 1) \) of degree \( n \), where

\[
F_{n,d}(x, y, z, s) = x^{n-d}y - s^{n-d}(s^d - z) .
\]

Taking here \( d = 1 \) it follows that \( V(n) \) is isomorphic to the normalization of the hypersurface \( x^{n-1}y - (s-1)s^{n-1} = 0 \) in \( A^3 \).

As our next example, let us consider yet another remarkable class of surfaces. These were studied from different viewpoints in [MM] Theorem 1.1, [FKZ] Theorem 1.1(iii), [GMMR] 3.8-3.9, [KK] Theorem 1.1. and Example 1, [Za] Theorem 1(b) and Lemma

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\(5\) See 3.1(a) below.
Theorem 2.8. For a smooth affine surface \( V \), the following conditions are equivalent.

(i) \( V \) is not Gizatullin and admits an effective \( \mathbb{C}^* \)-action and an \( \mathbb{A}^1 \)-fibration \( V \to \mathbb{A}^1 \) with exactly one degenerate fiber, which is irreducible\(^6\).

(ii) \( V \) is \( \mathbb{Q} \)-acyclic, \( \overline{k}(V) = -\infty \) and \( V \) carries a curve \( \Gamma \cong \mathbb{A}^1 \) with \( \overline{k}(V \setminus \Gamma) \geq 0 \).

(iii) \( V \) is \( \mathbb{Q} \)-acyclic and admits an effective \( \mathbb{C}^* \)- and \( \mathbb{C}_+ \)-actions. Furthermore, the \( \mathbb{C}^* \)-action possesses an orbit closure \( \Gamma \cong \mathbb{A}^1 \) with \( \overline{k}(V \setminus \Gamma) \geq 0 \).

(iv) The universal cover \( \tilde{V} \to V \) is isomorphic to a surface \( x^k y - (s^d - 1) = 0 \) in \( \mathbb{A}^3 \), with the Galois group \( \pi_1(V) \cong E_d \) acting via \( \zeta \cdot (x,y,s) = (\zeta x, \zeta^{-k} y, \zeta^e s) \), where \( k > 1 \) and \( \gcd(e,d) = 1 \).

(v) \( V \) is isomorphic to the \( \mathbb{C}^* \)-surface with DPD presentation \( \text{Spec} \mathbb{C}[t][D_+, D_-] \), where

\[
(D_+, D_-) = \left( -\frac{e}{d} [0], \frac{e}{d} [0] - \frac{1}{k} [1] \right) \quad \text{with} \quad 0 < e \leq d \quad \text{and} \quad k > 1.
\]

(vi) \( V \) is isomorphic to the Zariski open subset

\[
\mathbb{D}_+(x^k y - s^d) \subseteq \mathbb{P}(e, d - ke, 1), \quad \text{where} \quad 0 < e \leq d \quad \text{and} \quad k > 1.
\]

Proof. In view of the references cited above it remains to show that the surfaces in (v) and (vi) are isomorphic. By Corollary 2.5(ii) with \( e_+ = -e_- = e \), the surface \( V \) as in (v) is isomorphic to the principal open subset \( \mathbb{D}_+(z) \) in the weighted projective hypersurface

\[
V_+(x^k y - (s^d - z)) \subseteq \mathbb{P}(e, d - ke, d, 1).
\]

Eliminating \( z \) from the equation \( x^k y - (s^d - z) = 0 \) yields (vi). \( \square \)

These surfaces admit as well a constructive description in terms of a blowup process starting from a Hirzebruch surface, see [GMR, 3.8] and [KK, Example 1].

An affine line \( \Gamma \cong \mathbb{A}^1 \) on \( V \) as in (ii) is distinguished because it cannot be a fiber of any \( \mathbb{A}^1 \)-fibration of \( V \). In fact there exists a family of such affine lines on \( V \), see [Za].

Some of the surfaces as in Theorem 2.8 can be properly embedded in \( \mathbb{A}^3 \) as Bertin surfaces \( x^e y - x - s^d = 0 \), see [FlZa, Example 5.5] or [Za, Example 1].

3. Gizatullin surfaces with a finite divisor class group

A Gizatullin surface is a normal affine surface completed by a zigzag i.e., a linear chain of smooth rational curves. By a theorem of Gizatullin [Gi] such surfaces are characterized by the property that they admit two \( \mathbb{C}_+ \)-actions with different general orbits.

In this section we give an alternative proof of the Daigle-Russell Theorem 1.1 cited in the Introduction. It will be deduced from the following result proven in [FKZ2, Corollary 5.16].

Proposition 3.1. Every normal Gizatullin surface with a finite divisor class group is isomorphic to one of the following surfaces.

\(^6\)Since \( V \) is not Gizatullin there is actually a unique \( \mathbb{A}^1 \)-fibration \( V \to \mathbb{A}^1 \). A surface \( V \) as in (i) is necessarily a \( \mathbb{Q} \)-homology plane (or \( \mathbb{Q} \)-acyclic) that is, all higher Betti numbers of \( V \) vanish.

\(^7\)As usual, \( \overline{k} \) stands for the logarithmic Kodaira dimension.
(a) The toric surfaces $V_{d,e} = \mathbb{A}^2/E_d$, where the group $E_d \cong \mathbb{Z}_d$ of $d$-th roots of unity acts on $\mathbb{A}^2$ via
$$\zeta(x,y) = (\zeta x, \zeta^e y).$$

(b) The non-toric $\mathbb{C}^*$-surfaces $V = \text{Spec} \mathbb{C}[t][D_+, D_-]$, where
$$D_+, D_- = \left(-\frac{e}{m}[p], \frac{e}{m}[p] - c[q]\right) \quad \text{with} \quad c \geq 1, \ p, q \in \mathbb{A}^1, \ p \neq q,$$
and with coprime integers $e, m$ such that $1 \leq e < m$.

Conversely, any normal affine $\mathbb{C}^*$-surface $V$ as in (a) or (b) is a Gizatullin surface with a finite divisor class group.

Let us now deduce Theorem 1.1.

Proof of Theorem 1.1. To prove (a), we note that according to 2.1 the cyclic group $E_d$ acts on the ring $\mathbb{C}[x, y, z]/(z-1) \cong \mathbb{C}[x, y]$ via $\zeta x = \zeta x, \zeta y = \zeta^e y$, and $\zeta z = z$, where $\deg x = 1, \ \deg y = e, \ \text{and} \ \deg z = d$.

Hence $D_+(z) = \text{Spec} \mathbb{C}[x, y]^{E_d} = V_{d,e}$, as required in (a).

To show (b) we consider $V = \text{Spec} A$ as in 3.1(b), where
$$A = \mathbb{C}[t][D_+, D_-] \subseteq \mathbb{C}[t][u, u^{-1}].$$

By definition (1) the homogeneous pieces $A_{\pm 1}$ of $A$ are generated as $\mathbb{C}[t]$-modules by the elements
$$u_+ = tu \quad \text{and} \quad u_- = (t-1)^eu^{-1},$$
and similarly $A_{\pm m}$ by
$$v_+ = t^e u_m \quad \text{and} \quad v_- = t^{-e}(t-1)^cm u^{-m}.$$ Thus
$$u_+^m = t^{m-e}v_+, \quad u_-^m = t^e v_-, \quad \text{and} \quad u_+u_-=t(t-1)^e.$$ The algebra $A$ is the integral closure of the subalgebra generated by $u_+, v_+$ and $t$.

Consider now the normalization $A'$ of $A$ in the field $L = \text{Frac}(A)[u'_+], u'_-, \text{where}$
$$u'_+ = \sqrt{v_+} \quad \text{with} \quad d = cm.$$ Clearly the elements $\sqrt{v_+} = t^{e-m}u_+$ and then also $t^{e-m}$ both belong to $L$. Since $e$ and $m$ are coprime we can choose $\alpha, \beta \in \mathbb{Z}$ with $\alpha(e-m) + \beta m = 1$. It follows that the element $\tau := \frac{1}{m} = t^{\alpha(e-m)}$ is as well in $L$ whence being integral over $A$ we have $\tau \in A'$.

The element $u'_+$ as in (10) also belongs to $A'$ and as well $u'_- = \sqrt{v_-} \in A'$. Now $v_+v_- = (t-1)^cm$, so taking $d$th roots we get for a suitable choice of the root $u'_-$,
$$u'_+u'_- = \tau^m - 1.$$ We note that $u_+, v_+$ and $t$ are contained in the subalgebra $B = \mathbb{C}[u'_+, u'_-, \tau] \subseteq A'$. The equation (11) defines a smooth surface in $\mathbb{A}^3$. Hence $B$ is normal and so
$$A' = B \cong \mathbb{C}[u'_+, u'_-, \tau]/(u'_+u'_- - (\tau^m - 1)).$$

By Lemma [3.2] below, for a suitable $\gamma \in \mathbb{Z}$ the integers $a = e - \gamma m$ and $d$ are coprime. We may assume as well that $1 \leq a < d$. We let $E_d$ act on $A'$ via $\zeta.u'_+ = \zeta^a u'_+$ and...
On log coprime. However, the latter is evident since the residue classes of $CNR$ P. Cassou-Noguès, P. Russell:

Birational morphisms to a transposition and up to replacing $(a, b)$ by $(a', b') = (a - sm, b + sm)$, while keeping $\gcd(a', b') = 1$.

The algebra $B = \mathbb{C}[u'_+, u'_-, \tau]$ is naturally graded via

$$\deg u'_+ = a, \quad \deg u'_- = b, \quad \text{and} \quad \deg \tau = c.$$ 

According to Proposition [2.1] Spec $A = \text{Spec } A^{E_d}$ is the complement of the hypersurface $V_+(f)$ of degree $d = a + b$ in the weighted projective plane

$$\text{Proj}(B) = \mathbb{P}(a, b, c), \quad \text{where} \quad f = u'_+ u'_- - \tau^m,$$

proving (b).

To complete the proof we still have to show the following elementary lemma.

**Lemma 3.2.** Assume that $c, m \in \mathbb{Z}$ are coprime. Then for every $c \geq 2$ there exists $\gamma \in \mathbb{Z}$ such that $\gamma m - e$ and $c$ are coprime.

**Proof.** Write $c = c' \gamma$ such that $c'$ and $m$ have no common factor and every prime factor of $\gamma$ occurs in $m$. Then for every $\gamma \in \mathbb{Z}$ the integers $\gamma m - e$ and $\gamma$ have no common prime factor. Indeed, such a prime must divide $m$ and then also $e = \gamma m - (\gamma m - e)$. Hence it is enough to establish the existence of $\gamma \in \mathbb{Z}$ such that $\gamma m - e$ and $c'$ are coprime. However, the latter is evident since the residue classes of $\gamma m, \gamma \in \mathbb{Z}$, in $\mathbb{Z}/c'$ cover this group. \hfill \Box

**Remark 3.3.** 1. Two triples $(1, e, d)$ and $(1, e', d)$ as in Theorem [1.1](a) define the same affine toric surface if and only if $ee' \equiv 1 \mod d$, see [FZa] Remark 2.5].

2. As follows from Theorem 0.2 in [FKZa], the integers $c, m$ in Theorem [1.1](b) are invariants of the isomorphism type of $V$. Indeed, the fractional parts of both divisors $D_+$ as in (9) being nonzero and concentrated at the same point, there is a unique DPD presentation for $V$ up to interchanging $D_+$ and $D_-$, passing to an equivalent pair and applying an automorphism of the affine line $\mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$.

Furthermore, from the proof of Theorem [1.1] one can easily derive that

$$a \equiv e \mod m \quad \text{and} \quad b = mc - a \equiv -e \mod m.$$

Therefore also the pair $(a, b)$ is uniquely determined by the isomorphism type of $V$ up to a transposition and up to replacing $(a, b)$ by $(a', b') = (a - sm, b + sm)$, while keeping $\gcd(a', b') = 1$.

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