PARTIAL REGULARITY FOR MINIMIZERS OF QUASICONVEX FUNCTIONALS WITH GENERAL GROWTH

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Abstract. We prove a partial regularity result for local minimizers of quasiconvex variational integrals with general growth. The main tool is an improved $A$-harmonic approximation, which should be interesting also for classical growth.

1. Introduction

In this paper we study partial regularity for vector-valued minimizers $u : \Omega \to \mathbb{R}^N$ of variational integrals:

\begin{equation}
\mathcal{F}(u) := \int_{\Omega} f(\nabla u) \, dx,
\end{equation}

where $\Omega \subset \mathbb{R}^n$ is a domain and $f : \mathbb{R}^{N \times n} \to \mathbb{R}$ is a continuous function.

Let us recall Morrey’s notion of quasiconvexity \cite{26}:

\begin{definition}
$f$ is called quasiconvex if and only if
\begin{equation}
\int_{B_1} f(A + \nabla \xi) \, dx \geq f(A)
\end{equation}
holds for every $A \in \mathbb{R}^{nN}$ and every smooth $\xi : B_1 \to \mathbb{R}^N$ with compact support in the open unit ball $B_1$ in $\mathbb{R}^n$.
\end{definition}

By Jensen’s inequality, quasiconvexity is a generalization of convexity. It was originally introduced as a notion for proving the lower semicontinuity and the existence of minimizers of variational integrals. In fact, assuming a power growth condition, quasiconvexity is proved to be a necessary and sufficient condition for the sequential weak lower semicontinuity on $W^{1,p}(\Omega, \mathbb{R}^N)$, $p > 1$, see \cite{25} and \cite{1}. For general growth condition see \cite{20} and \cite{31}. In the regularity issue, a stronger definition comes into play. In the fundamental paper \cite{19} Evans considered strictly quasi-convex integrands $f$ in the quadratic case and proved that if $f$ is of class $C^2$ and has bounded second derivatives then any minimizing function $u$ is of class $C^{1,\alpha}(\Omega \setminus \Sigma)$ where $\Sigma$ has $n$-dimensional Lebesgue measure zero. In \cite{1}, this result was generalized to integrands $f$ of $p$-growth with $p \geq 2$ while the subquadratic growth was considered in \cite{6}.

In order to treat the general growth case, we introduce the notion of strictly $W^{1,p}$-quasiconvex function, where $\varphi$ is a suitable N-function, see Assumption \cite{27}.

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**Definition 1.2.** The function $f$ is strictly $W^{1,p}$-quasiconvex if and only if

$$
\int_B f(Q + \nabla w) - f(Q) \, dx \geq k \int_B \varphi_a(|\nabla w|) \, dx,
$$

for all balls $B \subset \Omega$, all $Q \in \mathbb{R}^{N \times n}$ and all $w \in C^1_0(B)$, where $\varphi_a(t) \sim \varphi''(a + t) t^2$ for $a, t \geq 0$. A precise definition of $\varphi_a$ is given in Section 2.

We will work with the following set of assumptions:

(H1) $f \in C^1(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$,

(H2) for all $Q \in \mathbb{R}^{N \times n}$ it holds

$$
|f(Q)| \leq K \varphi(|Q|),
$$

(H3) the function $f$ is strictly $W^{1,p}$-quasiconvex,

(H4) for all $Q \in \mathbb{R}^{N \times n} \setminus \{0\}$

$$
|(D^2f)(Q)| \leq c \varphi''(|Q|)
$$

(H5) the following Hölder continuity of $D^2f$ away from 0

$$
|D^2f(Q) - D^2f(Q + P)| \leq c \varphi''(|Q|)|Q|^{-\beta}|P|^\beta
$$

holds for all $P, Q \in \mathbb{R}^{N \times n}$ such that $|P| \leq \frac{1}{2}|Q|$.

Due to (H2), $\mathcal{F}$ is well defined on the Sobolev-Orlicz space $W^{1,p}(\Omega, \mathbb{R}^N)$, see section 2. Let us observe that assumption (H5) has been used to show everywhere regularity of radial functionals with $\varphi$-growth, [12]. Following the argument given in [22] it is possible to prove that (H3) implies the following strong Legendre-Hadamard condition

$$
(D^2 f)(Q)(\eta \otimes \xi, \eta \otimes \xi) \geq c \varphi''(|Q|)|\eta|^2|\xi|^2
$$

for all $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{N \times n} \setminus \{0\}$. Furthermore, (H3) implies that the functional

$$
\mathcal{J}(t) := \int_B f(Q + t\nabla w) - f(Q) - k|\nabla w| \, dx
$$

attains its minimal value at $t = 0$. Hence $\mathcal{J}''(0) \geq 0$, that is

$$
\int_B (D^2 f)(Q)(\nabla w, \nabla w) \, dx \geq k \int_B \varphi''(0)|\nabla w|^2 \, dx \geq c \varphi''(|Q|) \int |\nabla w|^2 \, dx.
$$

As usual, the strategy for proving partial regularity consists in showing an excess decay estimate, where the excess function is

$$
\Phi_s(B, u) := \left( \int_B |V(\nabla u) - (V(\nabla u))_B|^{2s} \, dx \right)^{\frac{1}{s}}
$$

with $V(Q) = \frac{\sqrt{\varphi(|Q|)}}{\varphi(|Q|)} Q$ and $s \geq 1$. We write $\Phi := \Phi_1$. Note that $\Phi_{s_1}(B, u) \leq \Phi_{s_2}(B, u)$ for $1 \leq s_1 \leq s_2$ and $|V(Q)|^2 \sim \varphi(|Q|)$.

Our regularity theorem states:
Theorem 1.3 (Main theorem). Let $u$ be a local minimizer of the quasiconvex functional $W$, with $f$ satisfying $(H1), (H5)$ and fix some $\beta \in (0, 1)$. Then there exists $\delta = \delta(\beta) > 0$ such that the following holds: If

\begin{equation}
\Phi(2B, u) \leq \delta \int_{2B} |V(\nabla u)|^2 \, dx
\end{equation}

for some ball $B \subset \mathbb{R}^n$ with $2B \subset \Omega$, then $V(\nabla u)$ is $\beta$-Hölder continuous on $B$.

The proof of this theorem can be found at the end of Section 4. We define the set of regular points $R(u)$ by

\begin{equation}
R(u) = \{ x_0 \in \Omega : \liminf_{r \to 0} \Phi(B(x_0, r), u) = 0 \}.
\end{equation}

As an immediate consequence of Theorem 1.3 we have:

Corollary 1.4. Let $u$ be as in Theorem 1.3 and let $x_0 \in R(u)$ with $\nabla u \neq 0$. Then for every $\beta \in (0, 1)$ the function $V(\nabla u)$ is $\beta$-Hölder continuous on a neighborhood of $x_0$.

Note that the Hölder continuity of $V(\nabla u)$ implies the Hölder continuity of $\nabla u$ with a different exponent depending on $\varphi$. Consider for example the situation $\varphi(t) = t^p$ with $1 < p < \infty$. Therefore, $\beta$-Hölder continuity of $V(\nabla u)$ implies for $p \leq 2$ that $\nabla u$ is $\beta$-Hölder continuous and for $p > 2$ that $\nabla u$ is $\beta^2$-Hölder continuous.

The proofs of the regularity results for local minimizers in [19,11,6], are based on a blow-up technique originally developed by De Giorgi [7] and Almgren [3,4] in the setting of the geometric measure theory, and by Giusti and Miranda for elliptic systems, [29].

Another more recent approach for proving partial regularity for local minimizers is based on the so-called $A$-harmonic approximation method. This technique has its origin in Simon’s proof of the regularity theorem [50] (see also Allard [2]). The technique has been successfully applied in the framework of the geometric measure theory, and to obtain partial-regularity results for general elliptic systems in a series of papers by Duzaar, Grotowski, Kronz, Mingione [15,16,17,18]. More precisely, we consider a bilinear form on $\text{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ which is (strongly) elliptic in the sense of Legendre-Hadamard, i.e. if for all $a, b \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ it holds

\[ A_{ij} a^i b_\alpha a^\alpha b_\beta \geq \kappa_A |a|^2 |b|^2 \]

for some $\kappa_A > 0$. The method of $A$-harmonic approximation consists in obtaining a good approximation of functions $u \in W^{1,2}(B)$, which are almost $A$-harmonic (in the sense of Theorem 1.1) by $A$-harmonic functions $h \in W^{1,2}(B)$, in both the $L^2$-topology and in the weak topology of $W^{1,2}$. Let us recall that $h \in W^{1,2}(B)$ is called $A$-harmonic on $B$ if

\begin{equation}
\int_B A(Dh, D\eta) \, dx = 0, \forall \eta \in C_0^\infty(B)
\end{equation}

holds. Here, in order to prove the result, we will follow the second approach.

As in the situations considered in the above-mentioned papers, the required approximate $A$-harmonicity of a local minimizer $u \in W^{1,2}(\Omega \setminus \Sigma)$ is a consequence of the minimizing property and of the smallness of the excess.

Next, having proven the $A$-harmonic approximation lemma and the corresponding approximate $A$-harmonicity of the local minimizer $u$, the other steps are quite standard. We prove a Caccioppoli-type inequality for minimizers $u$ and thus we compare $u$ with the
a-harmonic approximation $h$ to obtain, via our Caccioppoli-type inequality, the desired excess decay estimate.

Thus, the main difficulty is to establish a suitable version of the $a$-harmonic approximation lemma in this general setting. However, let us point out that our $a$-harmonic approximation lemma differs also in the linear or $p$-growth situation from the classical one in [17]. Firstly, we use a direct approach based on the Lipschitz truncation technique which requires no contradiction argument. This allows for a precise control of the constants, which will only depend on the $\Delta_2$-condition for $\varphi$ and its conjugate. In fact, we will apply the approximation lemma to the family of shifted N-functions that inherit the same $\Delta_2$ constants of $\varphi$. Secondly, we are able to preserve the boundary values of our original function, so $u - h$ is a valid test function. Thirdly, we show that $h$ and $u$ are close with respect to the gradients rather than just the functions. The main tools in the proof is a Lipschitz approximation of the Sobolev functions as in [11, 5]. However, since $a$ is only strongly elliptic in the sense of Legendre-Hadamard, we will not be able to apply the Lipschitz truncation technique directly to our almost $a$-harmonic function $u$. Instead, we need to use duality and apply the Lipschitz truncation technique to the test functions.

Let us conclude by observing that here we are able to present a unified approach for both cases: superquadratic and subquadratic growth.

2. Notation and preliminary results

We use $c, C$ as generic constants, which may change from line to line, but does not depend on the crucial quantities. Moreover we write $f \sim g$ iff there exist constants $c, C > 0$ such that $c f \leq g \leq C f$. For $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a ball $B \subset \mathbb{R}^n$ we define

$$\langle w \rangle_B := \frac{1}{|B|} \int_B w(x) \, dx,$$

where $|B|$ is the $n$-dimensional Lebesgue measure of $B$. For $\lambda > 0$ we denote by $\lambda B$ the ball with the same center as $B$ but $\lambda$-times the radius. For $U, \Omega \subset \mathbb{R}^n$ we write $U \Subset \Omega$ if the closure of $U$ is a compact subset of $\Omega$.

The following definitions and results are standard in the context of N-functions, see for example [21, 28]. A real function $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be an N-function if it satisfies the following conditions: $\varphi(0) = 0$ and there exists the derivative $\varphi'$ of $\varphi$. This derivative is right continuous, non-decreasing and satisfies $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t \to \infty} \varphi'(t) = \infty$. Especially, $\varphi$ is convex.

We say that $\varphi$ satisfies the $\Delta_2$-condition, if there exists $c > 0$ such that for all $t \geq 0$ holds $\varphi(2t) \leq c \varphi(t)$. We denote the smallest possible constant by $\Delta_2(\varphi)$. Since $\varphi(t) \leq \varphi(2t)$ the $\Delta_2$ condition is equivalent to $\varphi(2t) \sim \varphi(t)$.

By $L^\varphi$ and $W^{1,\varphi}$ we denote the classical Orlicz and Sobolev-Orlicz spaces, i.e. $f \in L^\varphi$ iff $\int \varphi(|f|) \, dx < \infty$ and $f \in W^{1,\varphi}$ iff $f, \nabla f \in L^\varphi$. By $W_0^{1,\varphi}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{1,\varphi}(\Omega)$.

By $(\varphi')^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ we denote the function

$$(\varphi')^{-1}(t) := \sup \{s \in \mathbb{R}^{\geq 0} : \varphi'(s) \leq t\}.$$
If \( \varphi' \) is strictly increasing then \( (\varphi')^{-1} \) is the inverse function of \( \varphi' \). Then \( \varphi^* : \mathbb{R}^+ \to \mathbb{R}^+ \) with

\[
\varphi^*(t) := \int_0^t (\varphi')^{-1}(s) \, ds
\]
is again an N-function and \( (\varphi^*)'(t) = (\varphi')^{-1}(t) \) for \( t > 0 \). It is the complementary function of \( \varphi \). Note that \( \varphi^*(t) = \sup_{s \geq 0} (st - \varphi(s)) \) and \( (\varphi^*)^* = \varphi \). For all \( \delta > 0 \) there exists \( c_\delta \) (only depending on \( \Delta_2(\varphi, \varphi^*) \)) such that for all \( t, s \geq 0 \) holds

\[
(2.9) \quad t s \leq \delta \varphi(t) + c_\delta \varphi^*(s),
\]
For \( \delta = 1 \) we have \( c_\delta = 1 \). This inequality is called Young’s inequality. For all \( t \geq 0 \)

\[
(2.10) \quad \varphi \left( \frac{\varphi^*(t)}{t} \right) \leq \varphi^*(t) \leq \varphi \left( \frac{2 \varphi^*(t)}{t} \right).
\]
Therefore, uniformly in \( t \geq 0 \)

\[
(2.11) \quad \varphi(t) \sim \varphi'(t) t, \quad \varphi^*(\varphi'(t)) \sim \varphi(t),
\]
where the constants only depend on \( \Delta_2(\varphi, \varphi^*) \).

We say that a N-function \( \psi \) is of type \((p_0, p_1)\) with \( 1 \leq p_0 \leq p_1 < \infty \), if

\[
(2.12) \quad \psi(st) \leq C \max \{ s^{p_0}, s^{p_1} \} \psi(t) \quad \text{for all } s, t \geq 0.
\]
We also write \( \psi \in \mathcal{I}(p_0, p_1, C) \).

**Lemma 2.1.** Let \( \psi \) be an N-function with \( \psi \in \Delta_2 \) together with its conjugate. Then \( \psi \in \mathcal{I}(p_0, p_1, C_1) \) for some \( 1 < p_0 < p_1 < \infty \) and \( C_1 > 0 \), where \( p_0, p_1 \) and \( C_1 \) only depend on \( \Delta_2(\psi, \psi^*) \). Moreover, \( \psi \) has the representation

\[
(2.13) \quad \psi(t) = t^{p_0} (h(t))^{p_1 - p_0} \quad \text{for all } t \geq 0,
\]
where \( h \) is a quasi-concave function, i.e.

\[
(2.14) \quad h(\lambda t) \leq C_2 \max \{ 1, \lambda \} h(t) \quad \text{for all } \lambda, t \geq 0,
\]
where \( C_2 \) only depends on \( \Delta_2(\psi, \psi^*) \).

**Proof.** Let \( K := \Delta_2(\psi) \) and \( K_* := \max \{ \Delta_2(\psi^*), 3 \} \). Then \( \psi^*(2t) \leq K_* \psi^*(t) \) for all \( t \geq 0 \) implies \( \psi(t) \leq K_* \psi(2t/K_*) \) for all \( t \geq 0 \). Now, choose \( p_0, p_1 \) such that \( 1 < p_0 < p_1 < \infty \) and \( K \leq 2^{p_0} \) and \( (K_*/2)^{p_0} \leq K_* \). We claim that

\[
(2.15) \quad \psi(st) \leq C \max \{ s^{p_0}, s^{p_1} \} \psi(t) \quad \text{for all } s, t \geq 0,
\]
where \( C \) only depends on \( K \) and \( K_* \). Indeed, if \( s \geq 1 \), then choose \( m \geq 0 \) such that \( 2^m \leq s \leq 2^{m+1} \). Using \( \psi \in \Delta_2 \), we get

\[
(2.16) \quad \psi(st) \leq \psi(2^{m+1} t) \leq K^{m+1} \psi(t) \leq K(2^{p_1})^{m} \psi(t) \leq K s^{p_1} \psi(t).
\]
If \( s \leq 1 \), then we choose \( m \in \mathbb{N}_0 \) such that \( (K_*/2)^m s \leq 1 \leq (K_*/2)^{m+1} s \), so that

\[
\psi(st) \leq K^m s \psi \left( \left( \frac{2}{K_*} \right)^m s \right) \leq K_* \left( \frac{K_*}{2} \right)^{p_0(m-1)} \psi(t) \leq K_* s^{p_0} \psi(t).
\]
This proves (2.14).
Now, let us define
\[ h(u) := \psi\left( u^{\frac{1}{p_1-p_0}} \right) u^{-\frac{p_0}{p_1-p_0}}, \]
then \( \psi \) satisfies (2.13). It remains to show that \( h \) is quasi-concave. We estimate with (2.14)
\[ h(su) \leq K \psi\left( u^{\frac{1}{p_1-p_0}} \right) \max\left\{ s^{-\frac{p_1-p_0}{p_1-p_0}}, s_{\frac{p_1}{p_1-p_0}} \right\} (su)^{-\frac{p_0}{p_1-p_0}} = K \psi(u) \max\{s, 1\} \]
for all \( s, u \geq 0 \).

Throughout the paper we will assume that \( \varphi \) satisfies the following assumption.

**Assumption 2.2.** Let \( \varphi \) be an \( N \)-function such that \( \varphi \) is \( C^1 \) on \([0, \infty)\) and \( C^2 \) on \((0, \infty)\). Further assume that
\[ \varphi'(t) \sim t \varphi''(t) \]
uniformly in \( t > 0 \). The constants in (2.10) are called the characteristics of \( \varphi \).

We remark that under these assumptions \( \Delta_2(\varphi, \varphi^*) < \infty \) will be automatically satisfied, where \( \Delta_2(\varphi, \varphi^*) \) depends only on the characteristics of \( \varphi \).

For given \( \varphi \) we define the associated \( N \)-function \( \psi \) by
\[ \psi(t) := \sqrt{\varphi'(t)t}. \]

It is shown in [8, Lemma 25] that if \( \varphi \) satisfies Assumption 2.2 then also \( \varphi^* \), \( \psi \), and \( \psi^* \) satisfy this assumption.

Define \( A, V : \mathbb{R}^{N \times n} \rightarrow \mathbb{R}^{N \times n} \) in the following way:
\begin{align*}
(2.18a) & \quad A(Q) = \varphi'(|Q|) \frac{Q}{|Q|}, \\
(2.18b) & \quad V(Q) = \psi'(|Q|) \frac{Q}{|Q|}.
\end{align*}

Another important set of tools are the shifted \( \varphi \)-functions \( \{\varphi_a\}_{a \geq 0} \) introduced in [8], see also [10, 29]. We define for \( t \geq 0 \)
\[ \varphi_a(t) := \int_0^t \varphi_a(s) \, ds \quad \text{with} \quad \varphi_a'(t) := \varphi'(a + t) \frac{t}{a + t}. \]

Note that \( \varphi_a(t) \sim \varphi'(a)t \). Moreover, for \( t \geq a \) we have \( \varphi_a(t) \sim \varphi(t) \) and for \( t \leq a \) we have \( \varphi_a(t) \sim \varphi''(a)t^2 \). This implies that \( \varphi_a(s,t) \leq c s^2 \varphi_a(t) \) for all \( s \in [0,1] \), \( a \geq 0 \) and \( t \in [0,a] \). The families \( \{\varphi_a\}_{a \geq 0} \) and \( \{(\varphi_a)^*\}_{a \geq 0} \) satisfy the \( \Delta_2 \)-condition uniformly in \( a \geq 0 \).

The connection between \( A, V \) and the shifted functions of \( \varphi \) is best reflected in the following lemma [12, Lemma 2.4], see also [8].

**Lemma 2.3.** Let \( \varphi \) satisfy Assumption 2.2 and let \( A \) and \( V \) be defined by (2.18). Then
\[ (A(P) - A(Q)) \cdot (P - Q) \sim |V(P) - V(Q)|^2 \sim \varphi\alphaQ|(|P - Q|), \]
\[ |A(P) - A(Q)| \sim \varphi\alphaP|(|P - Q|), \]
uniformly in \( P, Q \in \mathbb{R}^{N \times n} \). Moreover,
\[ A(Q) \cdot Q \sim |V(Q)|^2 \sim \varphi\alpha|Q|, \]
uniformly in $Q \in \mathbb{R}^{N \times n}$.

We state a generalization of Lemma 2.1 in [1] to the context of convex functions $\varphi$.

**Lemma 2.4** (Lemma 20, [8]). Let $\varphi$ be an $N$-function with $\Delta_2(\varphi, \varphi^*) < \infty$. Then uniformly for all $P_0, P_1 \in \mathbb{R}^{N \times n}$ with $|P_0| + |P_1| > 0$ holds

\[
\int_0^1 \frac{\varphi'(|P_{\theta}|)}{|P_{\theta}|} d\theta \sim \frac{\varphi'(|P_0| + |P_1|)}{|P_0| + |P_1|},
\]

where $P_{\theta} := (1 - \theta)P_0 + \theta P_1$. The constants only depend on $\Delta_2(\varphi, \varphi^*)$.

Note that $(H5)$ and the previous Lemma imply that

\[
\left| (Df)(Q) - (Df)(P) \right| = \left| \int_0^1 (D^2f)(P + t(Q - P))(Q - P) dt \right|
\]

\[
\leq c \int_0^1 \varphi''(|P + t(Q - P)|) dt |P - Q|
\]

\[
\leq c \varphi''(|P| + |Q|)|P - Q|
\]

\[
\leq c \varphi'_{|Q|}(|P - Q|).
\]

The following version of Sobolev-Poincaré inequality can be found in [8, Lemma 7].

**Theorem 2.5** (Sobolev-Poincaré). Let $\varphi$ be an $N$-function with $\Delta_2(\varphi, \varphi^*) < \infty$. Then there exist $0 < \alpha < 1$ and $K > 0$ such that the following holds. If $B \subset \mathbb{R}^n$ is some ball with radius $R$ and $w \in W^{1, \varphi}(B, \mathbb{R}^N)$, then

\[
\int_B \varphi \left( \frac{|w - (w)_B R|}{R} \right) dx \leq K \left( \int_B \varphi^\alpha(|\nabla w|) dx \right)^{1/\alpha},
\]

where $(w)_B := \int_B w(x) dx$.

3. **Caccioppoli estimate**

We need the following simple modification of lemma 3.1, (Chap. 5) from [21].

**Lemma 3.1.** Let $\psi$ be an $N$-function with $\psi \in \Delta_2$, let $r > 0$ and $h \in L^0(B_{2r}(x_0))$. Further, let $f : [r/2, r] \to [0, \infty)$ be a bounded function such that for all $\frac{r}{2} < s < t < r$

\[
f(s) \leq \theta f(t) + A \int_{B_t(x_0)} \psi \left( \frac{|h(y)|}{t - s} \right) dy
\]

where $A > 0$ and $\theta \in [0, 1)$. Then

\[
f \left( \frac{r}{2} \right) \leq c(\theta, \Delta_2(\psi)) A \int_{B_{2r}(x_0)} \psi \left( \frac{|h(y)|}{2r} \right) dy.
\]
Proof. Since $\psi \in \Delta^2$, there exists $C_2 > 0$ and $p_1 < \infty$ (both depending only on $\Delta^2(\psi)$) such that $\psi(\lambda u) \leq C_2 \lambda^{p_1} \psi(u)$ for all $\lambda \geq 1$ and $u \geq 0$ (compare (2.15) of Lemma 2.1). This implies

$$f(t) \leq \theta f(s) + A \int_{B_s(x_0)} \psi\left(\frac{|h(y)|}{2r}\right) dy \leq C_2^2 \lambda^{p_1} \psi(u)$$

for all $\lambda \geq 1$ and $u \geq 0$. Now Lemma 3.1 in [21], with $\alpha := p_1$ implies

$$f \left( \frac{r}{2} \right) \leq c(\theta, p_1) A \int_{B_s(x_0)} \psi\left(\frac{|h(y)|}{2r}\right) dy \leq C_2^2 \lambda^{p_1} \psi(u),$$

which proves the claim. \hfill \Box

**Theorem 3.2.** Let $u \in W^{1,\psi}_{\text{loc}}(\Omega)$ be a local minimizer of $F$ and $B$ be a ball with radius $R$ such that $2B \subset \subset \Omega$. Then

$$\int_B \varphi(|\nabla u - Q|) \, dx \leq c \int_{2B} \varphi\left(\frac{|u - q|}{R}\right) \, dx$$

for all $Q \in \mathbb{R}^{N \times n}$ and all linear polynomials $q$ on $\mathbb{R}^n$ with values in $\mathbb{R}^N$ and $\nabla q = Q$, where $c$ only depends on $n, N, k, K$ and the characteristics of $\varphi$.

**Proof.** Let $0 < s < t$. Further, let $B_s$ and $B_t$ be balls in $\Omega$ with the same center and with radius $s$ and $t$, respectively. Choose $\eta \in C_0^\infty(B_t)$ with $\chi_{B_s} \leq \eta \leq \chi_{B_t}$ and $|\nabla \eta| \leq c/(t-s)$. Now, define $\xi := \eta(u - q)$ and $z := (1-\eta)(u - q)$. Then $\nabla \xi + \nabla z = \nabla u - Q$. Consider

$$I := \int_{B_t} f(Q + \nabla \xi) - f(Q) \, dx.$$

Then by the quasi-convexity of $f$, see (H3) follows

$$I \geq c \int_{B_t} \varphi|Q|(|\nabla \xi|) \, dx.$$

On the other hand since $\nabla \xi + \nabla z = \nabla u - Q$ we get

$$I = \int_{B_t} f(Q + \nabla \xi) - f(Q) \, dx$$

$$= \int_{B_t} f(Q + \nabla \xi) - f(Q + \nabla \xi + \nabla z) \, dx$$

$$+ \int_{B_t} f(\nabla u) - f(\nabla u - \nabla \xi) \, dx$$

$$+ \int_{B_t} f(Q + \nabla z) - f(Q) \, dx$$

$$=: II + III + IV.$$
Since $u$ is a local minimizer, we know that $(III) \leq 0$. Moreover,

$$II + IV = \int_{B_t} \int_0^1 (\langle Df(\mathbf{Q} + t\nabla z) - (Df)(\mathbf{Q} + \nabla \xi - t\nabla z) \rangle \nabla z \, dt \, dx$$

$$= \int_{B_t} \int_0^1 (\langle Df(\mathbf{Q} + t\nabla z) - (Df)(\mathbf{Q}) \rangle \nabla z \, dt \, dx$$

$$- \int_{B_t} \int_0^1 (\langle Df(\mathbf{Q} + \nabla \xi - t\nabla z) - (Df)(\mathbf{Q}) \rangle \nabla z \, dt \, dx.$$

This proves

$$|II| + |IV| \leq c \int_{B_t} \int_0^1 \phi_{|\mathbf{Q}|}(|t|\nabla \mathbf{z}|) \, dt \, |\nabla \mathbf{z}| \, dx$$

$$+ c \int_{B_t} \int_0^1 \phi_{|\mathbf{Q}|}(\nabla \xi - t\nabla \mathbf{z}|) \, dt \, |\nabla \mathbf{z}| \, dx.$$

Using $\phi_{|\mathbf{Q}|}(\nabla \xi - t\nabla \mathbf{z}|) \leq c \phi_{|\mathbf{Q}|}(\nabla \xi|) + c \phi_{|\mathbf{Q}|}(\mathbf{z}|)$, we get

$$|II| + |IV| \leq c \int_{B_t} \phi_{|\mathbf{Q}|}(\nabla \mathbf{z}|) \, dx + c \int_{B_t} \phi_{|\mathbf{Q}|}(\nabla \xi|) \, |\nabla \mathbf{z}| \, dx$$

$$\leq c \int_{B_t} \phi_{|\mathbf{Q}|}(\nabla \mathbf{z}|) \, dx + \frac{1}{2}(I),$$

where we have used Young’s inequality in the last step. Overall, we have shown the a priori estimate

(3.23) \[ \int_{B_t} \phi_{|\mathbf{Q}|}(\nabla \xi|) \, dx \leq c \int_{B_t} \phi_{|\mathbf{Q}|}(\nabla \mathbf{z}|) \, dx. \]

Note that $\nabla \mathbf{z} = (1 - \eta)(\nabla u - \mathbf{Q}) - \nabla \eta(u - \mathbf{q})$, which is zero outside $B_t \setminus B_s$. Hence,

$$\int_{B_t \setminus B_s} \phi_{|\mathbf{Q}|}(\nabla \xi|) \, dx \leq c \int_{B_t \setminus B_s} \phi_{|\mathbf{Q}|}(\nabla u - \mathbf{Q})| \, dx + c \int_{B_t \setminus B_s} \phi_{|\mathbf{Q}|}\left(\frac{|u - \mathbf{q}|}{t - s}\right) \, dx.$$

Since $\eta = 1$ on $B_s$, we get

$$\int_{B_s} \phi_{|\mathbf{Q}|}(\nabla u - \mathbf{Q})| \, dx \leq c \int_{B_s} \phi_{|\mathbf{Q}|}(\nabla u - \mathbf{Q})| \, dx + c \int_{B_t \setminus B_s} \phi_{|\mathbf{Q}|}\left(\frac{|u - \mathbf{q}|}{t - s}\right) \, dx.$$

The hole-filling technique proves

$$\int_{B_s} \phi_{|\mathbf{Q}|}(\nabla u - \mathbf{Q})| \, dx \leq \lambda \int_{B_t} \phi_{|\mathbf{Q}|}(\nabla u - \mathbf{Q})| \, dx + c \int_{B_t} \phi_{|\mathbf{Q}|}\left(\frac{|u - \mathbf{q}|}{t - s}\right) \, dx$$

for some $\lambda \in (0, 1)$, which is independent of $\mathbf{Q}$ and $\mathbf{q}$. Now Lemma 3.1 proves the claim. \qed
Corollary 3.3. There exists $0 < \alpha < 1$ such that for all local minimizers $u \in W_{loc}^{1,\varphi}(\Omega)$ of $F$, all balls $B$ with $2B \subset\subset \Omega$ and all $Q \in \mathbb{R}^{N \times n}$

\[
\int_B |V(\nabla u) - V(Q)|^2 \, dx \leq c \left( \int_{2B} |V(\nabla u) - V(Q)|^{2\alpha} \, dx \right)^{\frac{1}{\alpha}}
\]

Proof. Apply Theorem 3.2 with $q$ such that $\langle u - q \rangle_{2B} = 0$. Then use Theorem 2.5 with $w(x) = u(x) - Qx$. □

Using Gehring’s Lemma we deduce the following assertion.

Corollary 3.4. There exists $s_0 > 1$ such that for all local minimizers $u \in W_{loc}^{1,\varphi}(\Omega)$ of $F$, all balls $B$ with $2B \subset\subset \Omega$ and all $Q \in \mathbb{R}^{N \times n}$

\[
\left( \int_B |V(\nabla u) - V(Q)|^{2s_0} \, dx \right)^{\frac{1}{s_0}} \leq c \int_{2B} |V(\nabla u) - V(Q)|^2 \, dx.
\]

4. The $A$-harmonic approximation

In this section we present a generalization of the $A$-harmonic approximation lemma in Orlicz spaces. Basically it says that if a function locally “almost” behaves like an $A$-harmonic function, then it is close to an $A$-harmonic function. The proof is based on the Lipschitz truncation technique, which goes back to Acerbi-Fusco [1] but has been refined by many others.

Originally the closeness of the function to its $A$-harmonic approximation was stated in terms of the $L^2$-distance and later for the non-linear problems in terms of the $L^p$-distance. Based on a refinement of the Lipschitz truncation technique [11], it has been shown in [13] that also the distance in terms of the gradients is small.

Let us consider the following elliptic system

\[
-\partial_\alpha (A^{\alpha\beta}_{ij} D_{\beta} u^j) = -\partial_\alpha H^\alpha_{\alpha}
\]

in $B$, where $\alpha, \beta = 1, \ldots, n$ and $i, j = 1, \ldots, N$. We use the convention that repeated indices are summed. In short we write $-\text{div}(A\nabla u) = -\text{div} G$. We assume that $A$ is constant. We say that $A$ is strongly elliptic in the sense of Legendre-Hadamard if for all $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$

\[
A^{\alpha\beta}_{ij} a^i b^j \geq \kappa_A |a|^2 |b|^2
\]

for some $\kappa_A > 0$. The biggest possible constant $\kappa_A$ is called the ellipticity constant of $A$. By $|A|$ we denote the Euclidean norm of $A$. We say that a Sobolev function $w$ on a ball $B$ is $A$-harmonic, if it satisfies $-\text{div}(A\nabla w) = 0$ in the sense of distributions.

Given a Sobolev function $u$ on a ball $B$ we want to find an $A$-harmonic function $h$ which is close to our function $u$. The way to find $h$ is very simple: it will be the $A$-harmonic function with the same boundary values as $u$. In particular, we want to find a Sobolev function $h$ which satisfies

\[
(4.24) \quad -\text{div}(A\nabla h) = 0 \quad \text{on } B
\]

\[
h = u \quad \text{on } \partial B
\]

in the sense of distributions.
Let $w := h - u$, then (1.23) is equivalent to finding a Sobolev function $w$ which satisfies

$$\begin{align*}
- \text{div}(A \nabla w) &= - \text{div}(A \nabla u) & \text{on } B \\
w &= 0 & \text{on } \partial B
\end{align*}$$

(4.25)

in the sense of distributions.

Our main approximation result is the following.

**Theorem 4.1.** Let $B \subset \Omega$ be a ball with radius $r_B$ and let $\tilde{B} \subset \Omega$ denote either $B$ or $2B$. Let $A$ be strongly elliptic in the sense of Legendre-Hadamard. Let $\psi$ be an $N$-function with $\Delta_2(\psi, \psi^*) < \infty$ and let $s > 1$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ only depending on $n$, $N$, $\kappa_A$, $|A|$, $\Delta_2(\psi, \psi^*)$ and $s$ such that the following holds: let $u \in W^{1, \psi}(\tilde{B})$ be almost $A$-harmonic on $B$ in the sense that

$$\left| \int_B A \nabla u \cdot \nabla \xi \, dx \right| \leq \delta \int_B |\nabla u| \, \|\nabla \xi\|_{L^\infty(B)}$$

for all $\xi \in C_0^\infty(B)$. Then the unique solution $w \in W^{1, \psi}_0(B)$ of (1.25) satisfies

$$\int_B \psi \left( \frac{|w|}{r_B} \right) \, dx + \int_B \psi(|\nabla w|) \, dx \leq \varepsilon \left( \int_B \left( \psi(|\nabla u|) \right)^s \, dx \right)^{\frac{1}{s}} + \int_B \psi(|\nabla u|) \, dx \right).$$

The proof of this theorem can be found at the end of this section. The distinction between $B$ and $\tilde{B}$ on the right-hand side of (1.27) allows a finer tuning with respect to the exponents. If $B = \tilde{B}$, then only the term involving $s$ is needed.

The following result on the solvability and uniqueness in the setting of classical Sobolev spaces $W^{1,q}_0(B, \mathbb{R}^N)$ can be found in [12] Lemma 2.

**Lemma 4.2.** Let $B \subset \subset \Omega$ be a ball, let $A$ be strongly elliptic in the sense of Legendre-Hadamard and let $1 < q < \infty$. Then for every $G \in L^q(B, \mathbb{R}^{N\times n})$, there exists a unique weak solution $u = T_A G \in W^{1,q}_0(B, \mathbb{R}^N)$ of

$$\begin{align*}
- \text{div}(A \nabla u) &= - \text{div} G & \text{on } B \\
u &= 0 & \text{on } \partial B
\end{align*}$$

(4.28)

The solution operator $T_A$ is linear and satisfies

$$\|\nabla T_A G\|_{L^q(B)} \leq c \|G\|_{L^q(B)},$$

where $c$ only depends on $n$, $N$, $\kappa_A$, $|A|$ and $q$.

**Remark 4.3.** Note that our constants do not depend on the size of the ball, since the estimates involved are scaling invariant.

Let $T_A$ be the solution operator of Lemma 4.2. Then by the uniqueness of Lemma 4.2 the operator $T_A : L^q(B, \mathbb{R}^{N\times n}) \to W^{1,q}_0(B, \mathbb{R}^N)$ does not depend on the choice of $q \in (1, \infty)$. Therefore, $T_A$ is uniquely defined from $\bigcup_{1 < q < \infty} L^q(B, \mathbb{R}^{N\times n})$ to $\bigcup_{1 < q < \infty} W^{1,q}_0(B, \mathbb{R}^N)$.

We need to extend Lemma 4.2 to the setting of Orlicz spaces. We will do so by means of the following real interpolation theorem of Peetre [27] Theorem 5.1] which states, that whenever $\psi$ is of the form (2.13), then $L^\psi$ is an interpolation space between $L^{p_0}$ and $L^{p_1}$.
Theorem 4.4. Let \( \psi \) be an N-function with \( \Delta_2(\psi, \psi^*) \) and \( p_0, p_1 \) as in Lemma 2.1. Moreover let \( S \) be a linear, bounded operator from \( L^{p_j} \to L^{p_j} \) for \( j = 0, 1 \). Then there exists \( K_2 \), which only depends on \( \Delta_2(\psi, \psi^*) \), and the operator norms of \( S \) such that

\[
\|Sf\|_\psi \leq K_2\|f\|_\psi
\]

\[
\int \psi(|Sf|/K_2) \, d\mu \leq \int \psi(|f|) \, d\mu
\]

for every \( f \in L^\psi \).

This interpolation result and Lemma 4.2 immediately imply:

Theorem 4.5. Let \( B \subset \Omega \) be a ball, let \( A \) be strongly elliptic in the sense of Legendre-Hadamard and \( \psi \) be an N-function with \( \Delta_2(\psi, \psi^*) \). Then the solution operator \( T_A \) of Lemma 4.2 is continuous from \( L^\psi(B, \mathbb{R}^{N \times n}) \) to \( W_0^{1,\psi}(B, \mathbb{R}^n) \) and

\[
\|\nabla T_A G\|_{L^\psi(B)} \leq c \|G\|_{L^\psi(B)},
\]

\[
\int_B \psi(|\nabla T_A G|) \, dx \leq c \int_B \psi(|G|) \, dx,
\]

for all \( G \in L^\psi(B, \mathbb{R}^{N \times n}) \), where \( c \) only depends on \( n, N, \kappa_A, |A|, \Delta_2(\psi, \psi^*) \).

Remark 4.6. Since \( \psi \) satisfies (2.12) for some \( 1 < p_0 < p_1 < \infty \) it follows easily that \( L^\psi(B) \to L^{p_0}(B) \) for every ball \( B \subset \Omega \). From this and the uniqueness in Lemma 4.2 the solution of (4.28) is also unique in \( W_0^{1,\psi}(B, \mathbb{R}^N) \).

Since \( A \) is only strongly elliptic in the sense of Legendre-Hadamard, we will not be able to apply the Lipschitz truncation technique directly to our almost \( A \)-harmonic function \( u \). Instead, we need to use duality and apply the Lipschitz truncation technique to the test functions. For this reason, we prove the following variational inequality.

Lemma 4.7. Let \( B \subset \Omega \) be a ball and let \( A \) be strongly elliptic in the sense of Legendre-Hadamard. Then it holds for all \( u \in W_0^{1,\psi}(B) \) that

\[
\|\nabla u\|_\psi \sim \sup_{\xi \in C_0^\infty(B)} \int_B A \nabla u \cdot \nabla \xi \, dx,
\]

\[
\int_B \psi(|\nabla u|) \, dx \sim \sup_{\xi \in C_0^\infty(B)} \left[ \int_B A \nabla u \cdot \nabla \xi \, dx - \int_B \psi^*(|\nabla \xi|) \, dx \right].
\]

The implicit constants only depend on \( n, N, \kappa_A, |A|, \Delta_2(\psi, \psi^*) \).

Proof. We begin with the proof of (4.30a). The \( \gtrsim \) estimate is a simple consequence of Hölder’s inequality, so let us concentrate on \( \lesssim \). Since \( (L^\psi)^* \cong L^{(\psi^*)} \) (with constants bounded by 2) and \( C_0^\infty(B) \) is dense in \( L^{(\psi^*)}(\Omega) \), we have

\[
\|\nabla u\|_\psi \leq 2 \sup_{H \in C_0^\infty(B, \mathbb{R}^{N \times n})} \|H\|_{\psi^*} \int_B \nabla u \cdot H \, dx,
\]
Define $\mathbf{A}$ by $A^\alpha_{ij} := A^\beta_{ji}$, then $-\text{div}(\mathbf{A} \nabla u)$ is the formal adjoint operator of $-\text{div}(\mathbf{A} \nabla u)$. In particular, using (4.28)

$$\int_B \nabla u \cdot \mathbf{H} = \int_B \nabla u \cdot \mathbf{A} \nabla T_\mathbf{A} \mathbf{H} \, dx$$

(4.31)

$$= \int_B A \nabla u \cdot \nabla T_\mathbf{A} \mathbf{H} \, dx.$$ 

Hence,

$$\|\nabla u\|_\psi \leq 2 \sup_{\mathbf{H} \in C^\infty_0(B, \mathbb{R}^{N \times n})} \int_B A \nabla u \cdot \nabla T_\mathbf{A} \mathbf{H} \, dx$$

$$\leq 4 \sup_{\mathbf{H} \in C^\infty_0(B, \mathbb{R}^{N \times n})} \|A \nabla u\|_{L^\psi(B)} \|\nabla T_\mathbf{A} \mathbf{H}\|_{\psi^*}$$

$$\leq c \|A \nabla u\|_{L^\psi(B)},$$

where we used in the last step Theorem 4.5 for $T_\mathbf{A}$ and $\psi^*$. This proves (4.30a).

Let us now prove (4.30b). The estimate $\gtrsim$ just follows from

$$\int_B A \nabla u \cdot \nabla \xi \, dx - \int_B \psi^* (|\nabla \xi|) \, dx \leq \int_B \psi(|A| |\nabla u|) \, dx$$

$$\leq c(|A|) \int_B \psi(|\nabla u|) \, dx,$$

where we used $|A \nabla u \cdot \nabla \xi| \leq |A| |\nabla u| |\nabla \xi|$, Young’s inequality and $\psi \in \Delta_2$.

We turn to $\lesssim$ of (4.30b). Recall that

$$\psi^{**}(t) = \psi(t) = \sup_{u \geq 0} (ut - \psi^*(u)),$$

where the supremum is attained at $u = \psi'(t)$. Thus the choice $\mathbf{H} := \psi'(|\nabla u|) \frac{\nabla u}{|\nabla u|}$ (with $\mathbf{H} = 0$ where $\nabla u = 0$) implies

$$\int_B \psi(|\nabla u|) \, dx \leq \sup_{\mathbf{H} \in L^{\psi^*}(B, \mathbb{R}^{N \times n})} \left[ \int_B \nabla u \cdot \mathbf{H} \, dx - \int_B \psi^* (|\mathbf{H}|) \, dx \right].$$

Using $T_\mathbf{A}$ we estimate with (4.31)

$$\int_B \psi(|\nabla u|) \, dx \leq \sup_{\mathbf{H} \in L^{\psi^*}(B, \mathbb{R}^{N \times n})} \left[ \int_B A \nabla u \cdot \nabla T_\mathbf{A} \mathbf{H} \, dx - \int_B \psi^* (|\mathbf{H}|) \, dx \right].$$

By Theorem 4.5 there exists $c \geq 1$ such that

$$\int_B \psi^* (|\nabla T_\mathbf{A} \mathbf{H}|) \, dx \leq c \int_B \psi^* (|\mathbf{H}|) \, dx.$$
This proves the following:
\[
\int_B \psi(|\nabla u|) \, dx \leq \sup_{H \in \mathcal{L}^\infty(B,\mathbb{R}^N)} \left[ \int_B A \nabla u \cdot \nabla T_\lambda H \, dx - c \int_B \psi^*(|\nabla T_\lambda H|) \, dx \right]
\]
\[
\leq \sup_{\xi \in \mathcal{L}^\infty(B,\mathbb{R}^N)} \left[ \int_B A \nabla u \cdot \nabla \xi \, dx - c \int_B \psi^* (|\nabla \xi|) \, dx \right].
\]
We replace \(u\) by \(cu\) to get
\[
\int_B \psi (c|\nabla u|) \, dx \leq c \sup_{\xi \in \mathcal{L}^\infty(B,\mathbb{R}^N)} \left[ \int_B A \nabla u \cdot \nabla \xi \, dx - \int_B \psi^* (|\nabla \xi|) \, dx \right].
\]
Now the claim follows using \(\psi \in \Delta_2\) on the left-hand side and the density of \(C_0^\infty (B, \mathbb{R}^N)\) in \(L^{\psi^*}(B, \mathbb{R}^N)\) (using \(\psi^* \in \Delta_2\)).

Moreover, we need the following result of [13, Theorem 3.3] about Lipschitz truncations in Orlicz spaces.

**Theorem 4.8** (Lipschitz truncation). Let \(B \subset \Omega\) be a ball and let \(\psi\) be an \(N\)-function with \(\Delta_2 (\psi, \psi^*) < \infty\). If \(w \in W^{1,\psi}_0 (B, \mathbb{R}^N)\), then for every \(m_0 \in \mathbb{N}\) and \(\gamma > 0\) there exists \(\lambda \in [\gamma, 2^{m_0} \gamma]\) and \(w_\lambda \in W^{1,\infty}_0 (B, \mathbb{R}^N)\) (called the Lipschitz truncation) such that
\[
\|\nabla w_\lambda\|_\infty \leq c \lambda,
\]
\[
\int_B \psi (|\nabla w_\lambda| \chi_{\{w_\lambda \neq w\}}) \, dx \leq c \psi (\lambda) \frac{|\{w_\lambda \neq w\}|}{|B|} \leq \frac{c}{m_0} \int_B \psi (|\nabla w|) \, dx
\]
\[
\int_B \psi (|\nabla w_\lambda|) \, dx \leq c \int_B \psi (|\nabla w|) \, dx.
\]
The constant \(c\) depends only on \(\Delta_2 (\psi, \psi^*), n\) and \(N\).

We are ready to prove Theorem 4.1.

**Proof of Theorem 4.1** We begin with an application of Lemma 4.1
\[
\int_B \psi (|\nabla u|) \, dx \leq c \sup_{\xi \in \mathcal{C}_0^\infty (B, \mathbb{R}^N)} \left[ \int_B A \nabla u \cdot \nabla \xi \, dx - \int_B \psi^* (|\nabla \xi|) \, dx \right].
\]

In the following let us fix \(\xi \in \mathcal{C}_0^\infty (B)\). Choose \(\gamma \geq 0\) such that
\[
\psi^* (\gamma) = \int_B \psi^* (|\nabla \xi|) \, dx.
\]
and let \(m_0 \in \mathbb{N}\). Due to Theorem 4.8 applied to \(\psi^*\) we find \(\lambda \in [\gamma, 2^{m_0} \gamma]\) and \(\xi_\lambda \in W^{1,\infty}_0 (B)\) such that
\[
\|\nabla \xi_\lambda\|_\infty \leq c \lambda,
\]
\[
\psi^* (\lambda) \frac{|\{\xi_\lambda \neq \xi\}|}{|B|} \leq \frac{c}{m_0} \int_B \psi^* (|\nabla \xi|) \, dx
\]
\[
\int_B \psi^* (|\nabla \xi_\lambda|) \, dx \leq c \int_B \psi^* (|\nabla \xi|) \, dx.
\]
Let us point out that the use of the Lipschitz truncation is not a problem of the regularity of $\xi$ as it is $C^0_\infty$. It is the precise estimates above that we need.

We calculate
\[
\int_B A\nabla u \cdot \nabla \xi \, dx = \int_B A\nabla u \cdot \nabla \xi_\lambda \, dx + \int_B A\nabla u \cdot \nabla (\xi - \xi_\lambda) \, dx =: I + II.
\]

Using Young’s inequality and (4.36) we estimate
\[
II = \int_B A\nabla u \cdot \nabla (\xi - \xi_\lambda) \chi_{\{\xi \neq \xi_\lambda\}} \, dx
\leq c \int_B \psi(|\nabla u|) \chi_{\{\xi \neq \xi_\lambda\}} \, dx + \frac{1}{2} \int_B \psi^*(|\nabla \xi|) \, dx =: II_1 + II_2,
\]
where $c$ depends on $|A|, \Delta_2(\psi, \psi^*)$. With Hölder’s inequality we get
\[
II_1 \leq c \left( \int_B (\psi(|\nabla u|)) \, dx \right)^{\frac{1}{s}} \left( \frac{|\{\xi_\lambda \neq \xi\}|}{|B|} \right)^{1-\frac{1}{s}}.
\]

If follows from (4.35), (4.33) and $\lambda \geq \gamma$ that
\[
\frac{|\{\xi_\lambda \neq \xi\}|}{|B|} \leq \frac{c \psi^*(\gamma)}{m_0 \psi^*(\lambda)} \leq \frac{c}{m_0}.
\]
Thus
\[
II_1 \leq c \left( \int_B (\psi(|\nabla u|)) \, dx \right)^{\frac{1}{s}} \left( \frac{c}{m_0} \right)^{1-\frac{1}{s}}.
\]

We choose $m_0$ so large such that
\[
II_1 \leq \varepsilon \frac{1}{2} \left( \int_B (\psi(|\nabla u|)) \, dx \right)^{\frac{1}{s}}.
\]

Since $u$ is almost $A$-harmonic and $\|\nabla \xi_\lambda\|_\infty \leq c \lambda \leq c 2^{m_0} \gamma$ we have
\[
|I| \leq \delta \int_B |\nabla u| \, dx \|\nabla \xi_\lambda\|_\infty \leq \delta \int_B |\nabla u| \, dx c 2^{m_0} \gamma.
\]

We apply Young’s inequality and (4.33) to get
\[
|I| \leq \delta 2^{m_0} c \left( \int_B \psi(|\nabla u|) \, dx + \psi^*(\gamma) \right)
\leq \delta 2^{m_0} c \int_B \psi(|\nabla u|) \, dx + \delta 2^{m_0} c \int_B \psi^*(|\nabla \xi|) \, dx.
\]

Now, we choose $\delta > 0$ so small such that $\delta 2^{m_0} c \leq \varepsilon/2$. Thus
\[
|I| \leq \varepsilon \frac{1}{2} \int_B \psi(|\nabla u|) \, dx + \frac{1}{2} \int_B \psi^*(|\nabla \xi|) \, dx.
\]
Combining the estimates for $I$, $II$ and $II_1$ we get
\[ \int_B A \nabla u \cdot \nabla \xi \, dx \leq \varepsilon \left( \int_B \left( \frac{1}{\beta} \int_B \psi(|\nabla u|) \, dx \right)^{\frac{1}{2}} + \int_B \psi(|\nabla u|) \, dx \right) + \int_B \psi^*(|\nabla \xi|) \, dx. \]

Now taking the supremum over all $\xi \in C_0^\infty(B)$ and using (4.32) we get
\[ \int_B \psi(|\nabla w|) \, dx \leq \varepsilon \left( \int_B \left( \frac{1}{\beta} \int_B \psi(|\nabla u|) \, dx \right)^{\frac{1}{2}} + \int_B \psi(|\nabla u|) \, dx \right). \]

The claim follows by Poincaré inequality, see Theorem 2.5. \hfill \Box

5. Almost $A$-harmonicity

The following result is a special case of [9, Lemma A.2].

Lemma 5.1. Let $B \subset \mathbb{R}^n$ be a ball and $w \in W^{1,\varphi}(B)$. Then
\[ \int_B |\nabla w| - \langle \nabla w \rangle_B \, dx \sim \int_B |\nabla w| - \langle \nabla w \rangle_B \, dx. \]

The constants are independent of $B$ and $w$; they only depend on the characteristics of $\varphi$.

Lemma 5.2. There exists $\delta > 0$, which only depends on the characteristics of $\varphi$, such that for every ball $B$ with $B \subset \subset \Omega$ and every $u \in W^{1,\varphi}(B)$ the estimate
\[ \int_B |\nabla u| - \langle \nabla u \rangle_B \, dx \leq \delta \int_B |\nabla u| \, dx \]
implies
\[ \int_B |\nabla u| \, dx \leq 4 |\nabla \langle \nabla u \rangle_B| \]
and
\[ \int_B |\nabla u| - \langle \nabla u \rangle_B \, dx \leq 4 \delta |\nabla \langle \nabla u \rangle_B|. \]

Proof. It follows from (5.37) and Lemma 5.1 that
\[ \int_B |\nabla u| \, dx \leq 2 \int_B |\nabla u| - \langle \nabla u \rangle_B \, dx + 2 |\nabla \langle \nabla u \rangle_B| \]
and
\[ \int_B |\nabla u| - \langle \nabla u \rangle_B \, dx \leq 2 |\nabla \langle \nabla u \rangle_B|. \]

For small $\delta$ we absorb the first term of the right-hand side to get (5.38). The remaining estimate (5.39) is a combination of (5.37) and (5.38). \hfill \Box
Lemma 5.3. Let $u$ be a local minimizer of $F$. Then for every ball $B$ with $2B \subset \subset \Omega$ and every $Q \in \mathbb{R}^{N \times n}$ it holds
\[\int_{B} \varphi_{Q}(|\nabla u - Q|) \, dx \leq c \varphi_{Q} \left( \int_{2B} |\nabla u - Q| \, dx \right).\]

Proof. From Corollary 3.3 we get
\[\int_{B} \varphi_{Q}(|\nabla u - Q|) \, dx \leq c \left( \int_{2B} \varphi_{Q}(|\nabla u - Q|^\alpha) \, dx \right)^{\frac{1}{\alpha}}.
\]
We can apply then Corollary 3.4 in [9] to conclude.

Lemma 5.4. For all $\varepsilon > 0$ there exists $\delta > 0$, which only depends on $\varepsilon$ and the characteristics of $\varphi$, such that for every ball $B$ with $2B \subset \subset \Omega$ and every $u \in \text{W}^{1,\varphi}(B)$
\[\int_{B} |V(\nabla u) - \langle V(\nabla u) \rangle_{B}|^2 \, dx \leq \delta \int_{B} |V(\nabla u)|^2 \, dx\]
implies
\[\int_{B} |\nabla u - \langle \nabla u \rangle_{B}| \, dx \leq \varepsilon |\langle \nabla u \rangle_{B}|.
\]

Proof. Let $Q = \langle \nabla u \rangle_{B}$. Then, by Jensen’s inequality and Lemma 5.2 we get
\[\varphi_{|\nabla u|_{B}} \left( \int_{B} |\nabla u - \langle \nabla u \rangle_{B}| \, dx \right) \leq \int_{B} \varphi_{|\nabla u|_{B}}(|\nabla u - \langle \nabla u \rangle_{B}|) \, dx \]
\[\leq c \int_{B} |V(\nabla u) - V(\langle \nabla u \rangle_{B})|^2 \, dx \]
\[\leq \delta c |V(\langle \nabla u \rangle_{B})|^2 \]
\[\leq \delta c |\langle \nabla u \rangle_{B}| \]
\[\leq \delta c \varphi_{|\nabla u|_{B}}(|\nabla u|_{B}).
\]
For the last inequality we used the fact that $\varphi(a) \sim \varphi(a)$ for $a \geq 0$. Using the $\Delta_{2}$-condition of $\varphi_{|\nabla u|_{B}}$ it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that
\[\int_{B} |\nabla u - \langle \nabla u \rangle_{B}| \, dx \leq \varepsilon |\langle \nabla u \rangle_{B}|.
\]

Note that the smallness assumption in (5.40) automatically implies that $\langle \nabla u \rangle_{B} \neq 0$ (unless $\nabla u = 0$ on $B$). So the smallness assumption ensures that in some sense in the non-degenerate situation.

Lemma 5.5. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every local minimizer $u \in \text{W}^{1,\varphi}_{\text{loc}}(\Omega)$ of $F$ and every ball $B$ with $2B \subset \subset \Omega$ and
\[\int_{2B} |V(\nabla u) - \langle V(\nabla u) \rangle_{2B}|^2 \, dx \leq \delta \int_{2B} |V(\nabla u)|^2 \, dx
\]
there holds
\[
|\int_B D^2 f(Q)(\nabla u - Q, \nabla \xi)\, dx| \leq \varepsilon \varphi''(|Q|) \int_{2B} |\nabla u - Q|\, dx \|\nabla \xi\|_{\infty},
\]
for every \( \xi \in C_0^\infty(B) \), where \( Q := \langle \nabla u \rangle_{2B} \). In particular, \( u \) is almost \( A \)-harmonic (in the sense of Theorem 4.1), with \( A = D^2 f(Q)/\varphi''(|Q|) \).

Proof. Let \( \varepsilon > 0 \). Without loss of generality we can assume that \( \delta > 0 \) is so small that the Lemmas 5.2 and 5.4 give
\[
\int_{2B} |V(\nabla u)|^2 \, dx \leq 4 |V(Q)|^2, \quad (5.44)
\]
\[
\int_{2B} |\nabla u - Q|\, dx \leq \varepsilon |Q|. \quad (5.45)
\]
From the last inequality we deduce
\[
\varphi''(|Q|) \left( \int_{2B} |\nabla u - Q|\, dx \right)^2 \sim \varphi_{|Q|} \left( \int_{2B} |\nabla u - Q|\, dx \right). \quad (5.46)
\]
Since the estimate \( (5.43) \) is homogeneous with respect to \( \|\nabla \xi\|_{\infty} \), it suffices to show that \( (5.43) \) holds for all \( \xi \in C_0^\infty(B) \) with \( \|\nabla \xi\|_{\infty} = \int_{2B} |\nabla u - Q|\, dx \). Hence, because of \( (5.46) \) it suffices to prove
\[
|\int_B D^2 f(Q)(\nabla u - Q, \nabla \xi)\, dx| \leq \varepsilon c_{|Q|} \int_{2B} |\nabla u - Q|\, dx \quad (5.47)
\]
for all such \( \xi \). We define
\[
B^\geq := \{ x \in B : |\nabla u - Q| \geq \frac{1}{2} |Q| \}, \\
B^< := \{ x \in B : |\nabla u - Q| < \frac{1}{2} |Q| \}.
\]
From the Euler-Lagrange equation we get \( \int_B (Df(\nabla v) - Df(Q)) : \nabla \xi \, dx = 0 \), and therefore
\[
\int_B D^2 f(Q)(\nabla u - Q, \nabla \xi)\, dx \quad (5.48) \]
\[
= \int_0^1 \int_B \left( D^2 f(Q) - D^2 f(Q + \theta(\nabla u - Q)) \right)(\nabla u - Q, \nabla \xi) \, d\theta \, dx.
\]
We split the right-hand side into the integral $I$ over $B^\geq$ and the integral $II$ over $B^<$. Using \textbf{(H4)} we get

$$|I| \leq c \int_B \chi_{B^\geq} \left[ \frac{1}{2} \theta'(\theta - \theta) \right] d\theta \int_B |\nabla u - Q| |\nabla \xi| dx$$

$$\leq c \int_B \chi_{B^\geq} \left( \frac{1}{2} \theta' + \frac{1}{2} \theta' \right) dx \int_B |\nabla u - Q| |\nabla \xi| dx$$

$$\leq c \int_B \chi_{B^\geq} \left( |\nabla u - Q| + |\nabla u - Q| |\nabla \xi| \right) dx \int_B \frac{|\nabla \xi|}{|Q|} dx$$

$$\leq \varepsilon c \int_B \chi_{B^\geq} \left( |\nabla u - Q| + |\nabla u - Q| |\nabla \xi| \right) dx.$$  

We used Lemma 2.2 for the second, Assumption 2.2 for the third and (5.45) for the last estimate. Now, using $|Q| \leq 2 |\nabla u - Q|$ on $B^\geq$ and $\varphi(t) \sim \varphi(t)$ for $0 \leq a \leq t$ we get

$$|I| \leq \varepsilon c \int_B \chi_{B^\geq} \left( |\nabla u - Q| + |\nabla u - Q| \right) dx$$

$$\leq \varepsilon c \int_B \varphi(|Q|) |\nabla u - Q| dx.$$  

Let us estimate the modulus of $II$. Using \textbf{(H5)} and $|\nabla u - Q| < \frac{1}{2} |Q|$ on $B^<$ we get

$$|II| \leq c \int_B \chi_{B^<} \varphi(|Q|) |\nabla u - Q|^{1 + \beta_1} |\nabla \xi| dx$$

where $\beta_1 := \min \{ s_0, \beta \}$ with the constant $s_0$ from Corollary 3.3. Using Young’s inequality we get

$$|II| \leq \gamma \varphi''(|Q|) |\nabla \xi|^2 + c_\gamma \int_B \chi_{B^<} \varphi''(|Q|) |Q|^{-2\beta_1} |\nabla u - Q|^{2(1 + \beta_1)} dx$$

$$\leq \gamma c \varphi(|Q|) \chi_{B^<} |\nabla \xi|^2 + c_\gamma \left( \frac{1}{2} \theta' \right) \int_B \chi_{B^<} \left( \frac{1}{2} \theta' + \frac{1}{2} \theta' \right) dx$$

$$\leq \gamma c \int_B \varphi(|Q|) |\nabla u - Q| dx + c_\gamma \left( \frac{1}{2} \theta' \right) \int_B \chi_{B^<} \left( \frac{1}{2} \theta' + \frac{1}{2} \theta' \right)$$

Here we used (5.46) for the second and Jensen’s inequality, $\varphi''(t^2) \sim \varphi'(t)$ for $0 \leq t \leq a$ and $|\nabla u - Q| < \frac{1}{2} |Q|$ on $B^<$ for the third estimate. With the help of Corollary 3.4 we get

$$|II| \leq \gamma c \int_{2B} |V(\nabla u) - V(Q)|^2 dx + c_\gamma \left( \frac{1}{2} \theta' \right) \int_{2B} |V(\nabla u) - V(Q)|^{2(1 + \beta_1)} dx.$$
Using the assumption \((5.42)\), Lemma \(5.1\) and \((5.44)\) it follows that 
\[
|II| \leq \gamma c \int_{2B} |V(\nabla u) - V(Q)|^2 \, dx + c_\gamma \delta \int_{2B} |V(\nabla u) - V(Q)|^2 \, dx.
\]
Choosing \(\gamma > 0\) and then \(\delta > 0\) small enough we get the assertion. \(\square\)

6. Excess decay estimate

In this section we will focus on the excess decay estimate. Therefore, we compare the almost harmonic solution with its harmonic approximation.

Proposition 6.1. For all \(\varepsilon > 0\), there exists \(\delta = \delta(\varphi, \varepsilon) > 0\) such that the following is true: if for some ball \(B\) with \(2B \subset \subset \Omega\) the smallness assumption \((5.42)\) holds true, then for every \(\tau \in (0,1]\)
\[
\Phi(\tau B, u) \leq c \tau^2 (1 + \varepsilon \tau^{-n-2}) \Phi(2B, u),
\]
where \(c\) depends only on the characteristics of \(\varphi\) and is independent of \(\varepsilon\).

Proof. It suffices to consider the case \(\tau \leq \frac{1}{2}\). Let \(s_0\) be as in Corollary 3.4. Let \(q\) be a linear function such that \(\langle u - q \rangle_{2B} = 0\) and \(Q := \nabla q = \langle \nabla u \rangle_{2B}\). Define \(z := u - q\). Let \(h\) be the harmonic approximation of \(z\) with \(h = z\) on \(\partial B\). It follows from Lemma 5.5 that \(z\) is almost \(A\)-harmonic with \(A = D^2 f(Q) / \varphi''(|Q|)\). Thus by Theorem 4.1 for suitable \(\delta = \delta(\varphi, \varepsilon)\) and by Theorem 4.1 the \(A\)-harmonic approximation \(h\) satisfies
\[
\int_{B} \varphi|Q|(|\nabla z - \nabla h|)|\, dx \leq \varepsilon \left( \int_{2B} \varphi_{s_0}|Q|(|\nabla u - Q|) \, dx \right)^{\frac{1}{2}} + \int_{2B} \varphi|Q|(|\nabla u - Q|) \, dx.
\]
Now, it follows by Corollary 3.4 that
\[
\int_{B} \varphi|Q|(|\nabla z - \nabla h|) \, dx \leq c \varepsilon \Phi(2B, u).
\]
Since \(\nabla z = \nabla u - Q\) and \(\langle \nabla z \rangle_{\tau B} = \langle \nabla u \rangle_{\tau B} - Q\), we get
\[
\Phi(\tau B, u) \leq c \int_{\tau B} \varphi|Q|(|\nabla z - \langle \nabla z \rangle_{\tau B}|) \, dx
\]
\[
\leq c \int_{\tau B} \varphi|Q|(|\nabla h - \langle \nabla h \rangle_{\tau B}|) \, dx + c \int_{\tau B} \varphi|Q|(|\nabla z - \nabla h|) \, dx
\]
\[
=: I + II.
\]
For the second estimate we used Jensen’s inequality. Using \((6.49)\) we obtain
\[
II \leq \tau^{-n} c \int_{B} \varphi_{s_0}|Q|(|\nabla z - \nabla h|) \, dx \leq \tau^{-n} c \varepsilon \Phi(2B, u).
\]
By the interior regularity of the \(A\)-harmonic function \(h\), \([21]\), and \(\tau \leq \frac{1}{2}\) it holds that
\[
\sup_{\tau B} |\nabla h - \langle \nabla h \rangle_{\tau B}| \leq c \tau \int_{B} |\nabla h - \langle \nabla h \rangle_{B}| \, dx.
\]
Due to our assumption, we can apply Proposition 6.1 for any Proof.

Using the estimate \( \psi(t) \leq s \psi(t) \) for any \( s \in [0, 1] \), \( t \geq 0 \) and any N-function \( \psi \), we would get a factor \( \tau \) in the estimate of \( I \). However, to produce a factor \( \tau^2 \), we have to work differently and use the improved estimate \( \varphi_a(s t) \leq c s^2 \varphi_a(t) \) for all \( s \in [0, 1] \), \( a \geq 0 \) and \( t \in [0, a] \). We begin with

\[
\int_B |\nabla h - \langle \nabla h \rangle_B| \, dx \leq \int_B |\nabla z - \langle \nabla z \rangle_B| \, dx + 2 \int_B |\nabla z - \nabla h| \, dx
\]

which implies

\[
I \leq c \varphi_{\mathcal{Q}} \left( \int_B |\nabla u - \langle \nabla u \rangle_B| \, dx \right) + c \tau \varphi_{\mathcal{Q}} \left( \int_B |\nabla z - \nabla h| \, dx \right).
\]

Due to (5.45), we can use for the first term the improved estimate \( \varphi_a(s t) \leq c s^2 \varphi_a(t) \), which gives

\[
I \leq c \tau^2 \varphi_{\mathcal{Q}} \left( \int_B |\nabla u - \langle \nabla u \rangle_B| \, dx \right) + c \tau \varphi_{\mathcal{Q}} \left( \int_B |\nabla z - \nabla h| \, dx \right)
\]

\[
\leq c \tau^2 \int_B \varphi_{\mathcal{Q}}(|\nabla u - \langle \nabla u \rangle_B|) \, dx + c \tau \int_B \varphi_{\mathcal{Q}}(|\nabla z - \nabla h|) \, dx.
\]

Thus using (6.49) we get

\[
I \leq c \tau^2 \Phi(B, u) + c \tau \varepsilon \Phi(2B, u) \leq c (\tau^2 + \varepsilon \tau) \Phi(2B, u).
\]

Combining the estimates for \( I \) and \( II \) we get the claim. \( \Box \)

It follows now, by a series of standard arguments, that for any \( \beta \in (0, 1) \), there exists a suitable small \( \delta \) that ensures local \( C^{0, \beta} \)-regularity of \( V(\nabla u) \), which implies Hölder continuity of the gradients as well.

**Proposition 6.2** (Decay estimate). For \( 0 < \beta < 1 \) there exists \( \delta = \delta(\varphi, \beta) > 0 \) such that the following is true. If for some ball \( B \subset \Omega \) the smallness assumption (6.42) holds true, then

\[
\Phi(\rho B, u) \leq c \rho^{2 \beta} \Phi(2B, u)
\]

for any \( \rho \in (0, 1] \), where \( c = c(\varphi) \) depends only on the characteristics of \( \varphi \).

**Proof.** Due to our assumption, we can apply Proposition 6.1 for any \( \tau \). Let \( \gamma(\varepsilon, \tau) := c \tau^2 (1 + \varepsilon^{-n-2}) \) as in (6.48). Let us fix \( \tau > 0 \) and \( \varepsilon > 0 \), such that \( \gamma(\varepsilon, \tau) \leq \min \{(\tau/2)^{2 \beta}, 1/4\} \). Let \( \delta = \delta(\varphi, \varepsilon) \) chosen accordingly to Proposition 6.1 and also so small that \( (1 + \tau^{-n/2})\delta^{1/2} \leq \frac{1}{4\tau} \). By Proposition 6.1 we have

\[
\Phi(\tau B, u) \leq \min \{(\tau/2)^{2 \beta}, 1/4\} \Phi(2B, u).
\]
We claim that the smallness assumption is inherited from $2B$ to $\tau B$, so that we can iterate (6.51). For this we estimate with the help of our smallness assumption
\[
\left( \frac{1}{2B} \int |V(\nabla u)|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \Phi(2B, u) \right)^{\frac{1}{2}} + \left( \frac{1}{\tau B} \int |V(\nabla v)|^2 \, dx \right)^{\frac{1}{2}}
\]
\[
 \leq \left( \Phi(2B, u) \right)^{\frac{1}{2}} + \tau^{-n/2} \left( \Phi(2B, u) \right)^{\frac{1}{2}} + \left( \frac{1}{\tau B} \int |V(\nabla u)|^2 \, dx \right)^{\frac{1}{2}}
\]
\[
 \leq \left( 1 + \tau^{-n/2} \right) \delta^{1/2} \left( \frac{1}{2B} \int |V(\nabla u)|^2 \, dx \right)^{\frac{1}{2}} + \left( \frac{1}{\tau B} \int |V(\nabla u)|^2 \, dx \right)^{\frac{1}{2}}.
\]
Using \( (1 + \tau^{-n/2}) \delta^{1/2} \leq \frac{4}{\tau} \), we get
\[
\frac{1}{2B} \int |V(\nabla u)|^2 \, dx \leq 4 \frac{1}{\tau B} \int |V(\nabla v)|^2 \, dx.
\]
Now (6.51) and the previous estimate imply
\[
\Phi(\tau B, u) \leq \frac{1}{4} \Phi(2B, u) \leq \frac{1}{4} \delta \frac{1}{2B} \int |V(\nabla u)|^2 \, dx \leq \delta \frac{1}{\tau B} \int |V(\nabla u)|^2 \, dx.
\]
In particular, the smallness assumption is also satisfied for $\tau B$. So by induction we get
\[
(6.52) \quad \Phi((\tau/2)^k 2B, u) \leq \min \{ (\tau/2)^{23k}, 4^{-k} \} \Phi(2B, u),
\]
which is the desired claim. \( \square \)

Having the decay estimate, it is easy to prove our Main Theorem.

\textit{Proof of the Main Theorem} \( \square \) We can assume that (5.42) is satisfied with a strict inequality. By continuity, (5.42), holds for $B = B(x)$ and all $x$ in some neighborhood of $x_0$. By Proposition (6.2) and Campanato’s characterisation of Hölder continuity we deduce that $V(\nabla u)$ is $\beta$-Hölder continuous in a neighbourhood of $x_0$. \( \square \)

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