Comment on “The Kiselev black hole is neither perfect fluid, nor is it quintessence”

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Abstract
The article being commented on points out that the expressions “perfect fluid” and “quintessence” are used improperly in most of the articles that build on the Kiselev spacetime; hence that spacetime is not as relevant as many of those authors might think. Here we point out that the original derivation of the Kiselev spacetime involves the use of too many coordinate conditions as well, strongly restricting its relevance. Hence the conclusions of the article being commented on are strengthened.

In [1], the author expounds the title above, that is, points out that most of the many articles citing [2] use the expressions perfect fluid and quintessence in a manner that is incompatible with the terminology as understood in the cosmology community. These articles seem to assume, incorrectly, that the Kiselev spacetime represents a black hole in a universe pervaded by the cosmological quintessence. Of course, since the discovery of the accelerated expansion of the universe tells us that it might be dominated by quintessence, such a black hole would be more realistic than the asymptotically flat Schwarzschild black hole; hence the attractiveness of the concept.

However, any perfect fluid, including quintessence, must be isotropic, and the source supporting the Kiselev spacetime is not. Together with this fact, [1] points out that even though the original paper [2] did state this anisotropy, the inappropriate uses of these two expressions start with that paper. [1] also analyzes efforts in the literature to use the Kiselev spacetime(s) together with Rastall gravity ideas, and concludes that no new physics is really involved. Here we would like to point out an alternative way of seeing that the Kiselev spacetime is not as general or as relevant as most articles citing it imply; focusing on the metric ansatz used to derive it.

The problem is with the “principle of additivity and linearity”, eq.(9) of [2], equivalent to

\[ g_{tt} g_{rr} = -1, \]  

which the author claims is possible to adopt without loss of generality, in the statement immediately following eq.(9). This is incorrect:

It is well known [3] that the general diagonal static spherically symmetric metric can be written as

\[ ds^2 = -B(r)dt^2 + A(r)dr^2 + y(r)^2d\Omega^2, \]  

if one has not yet specified the choice for the radial coordinate. One such choice can be made now, e.g. specifying one of the functions in [2] [only one choice, since there is only one relevant coordinate, namely, \( r \)]. By far the most popular choice is \( y(r) = r \), called Schwarzschild coordinates or curvature coordinates. Another popular choice is isotropic coordinates, corresponding to \( y(r)^2 = A(r)r^2 \). Some choices, including these, are shown in Table [4] in terminology used by Visser and collaborators (e.g. see [1]).
Table 1: Some coordinate choices for static spherically symmetric 3+1 dimensional spacetimes. The names are those used by Visser and collaborators [4]

| Name             | Choice                  | Comment/Explanation (r chosen such that...)                                      |
|------------------|-------------------------|-----------------------------------------------------------------------------------|
| Schwarzschild,   | $y(r) = r$              | the area of a sphere of symmetry is $4\pi r^2$                                    |
| Curvature        |                         |                                                                                    |
| Isotropic        | $(y(r))^2 = A(r)r^2$    | the spatial part of the metric is conformal to 3-D flat (Euclidean) space          |
| Gaussian polar   | $A(r) = 1$              | $\Delta r =$ proper distance (on a radial path)                                  |
| Synge isothermal | $A(r) = B(r)$           | radial null trajectories satisfy $\Delta r = \pm \Delta t$                        |
| Buchdahl         | $A(r) = B(r)^{-1}$      | correct signature is guaranteed for all $r$                                       |

Note that a choice leaves two independent functions in the metric to be found for a solution. For example, if the spacetime is filled with a perfect fluid, this also brings two functions, namely the energy density $\rho$ and the pressure $p$. For this metric ansatz, the Einstein Equations give three independent components, and together with an equation-of-state of the form $f(\rho, p) = 0$, we have four equations with four functions to be determined; i.e. a well-defined problem. Therefore it is entirely appropriate for the static spherically symmetric metric to contain two independent functions at the beginning of an analysis.

It is this feature that eq. (9) of [2] violates: The ansatz, eq.(1) of [2], already makes the Schwarzschild choice, and there is no more coordinate freedom left! Adopting eq.(9) of [2], equivalent to (1) or $A(r) B(r) = 1$, on top of eq.(1) of [2] decreases the number of metric functions to one; and far from “no loss of generality”, severely restricts the range of possible solutions. As also stated in [1], it leads to a very special family of metrics [5]; in particular, forcing anisotropy on the stress-energy-momentum tensor. Another way to state this loss of generality is that [2] is attempting to use Schwarzschild and Buchdahl coordinates simultaneously.

On the other hand, the Kiselev spacetimes are interesting in that they allow superposition of sources (see the last equation on p.1189, and eq.(18) of [2]); a rare feature in General Relativity.

To summarize and reiterate, the Kiselev spacetime(s) are not as generic as most of the articles that cite it and build on it assume. The assumption (9) of the original article [2] makes it a very particular solution. To continue the itemized list in the conclusion of [1],

- Do not take the Kiselev spacetime to represent the generic, realistic black hole in a universe dominated by quintessence; it doesn’t.

References

References

[1] M Visser, “The Kiselev black hole is neither perfect fluid, nor is it quintessence” *Classical and Quantum Gravity* **37**, 045001 (2020).

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[4] P Boonserm and M Visser, “Buchdahl-like transformations for perfect fluid spheres” *International Journal of Modern Physics* **17**, 135 (2008).

[5] T Jacobson, “When is $g_{tt} g_{rr} = -1$?” *Classical and Quantum Gravity* **24**, 5717 (2007).