MODEL-CATEGORIES OF COALGEBRAS OVER OPERADS

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Abstract. This paper constructs model structures on the categories of coalgebras and pointed irreducible coalgebras over an operad whose components are projective, finitely generated in each dimension, and satisfy a condition that allows one to take tensor products with a unit interval. The underlying chain-complex is assumed to be unbounded and the results for bounded coalgebras over an operad are derived from the unbounded case.

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1. INTRODUCTION

Although the literature contains several papers on homotopy theories for algebras over operads — see [14], [17], and [18] — it is more sparse when one pursues similar results for coalgebras. In [20], Quillen developed a model structure on the category of 2-connected cocommutative coalgebras over the rational numbers. V. Hinich extended this in [13] to coalgebras whose underlying chain-complexes were unbounded (i.e., extended into negative dimensions). Expanding on Hinich’s
methods, K. Lefèvre derived a model structure on the category of coassociative coalgebras — see [15]. In general, these authors use indirect methods, relating coalgebra categories to other categories with known model structures.

Our paper finds model structures for coalgebras over any operad fulfilling a basic requirement (condition 4.2). Since operads uniformly encode many diverse coalgebra structures (coassociative-, Lie-, Gerstenhaber-coalgebras, etc.), our results have wide applicability.

The author’s intended application involves investigating the extent to which Quillen’s results in rational homotopy theory ([20]) can be generalized to integral homotopy theory.

Several unique problems arise that require special techniques. For instance, constructing injective resolutions of coalgebras naturally leads into infinitely many negative dimensions. The resulting model structure — and even that on the underlying chain-complexes — fails to be cofibrantly generated (see [5]). Consequently, we cannot easily use it to induce a model structure on the category of coalgebras.

We develop the general theory for unbounded coalgebras, and derive the bounded results by applying a truncation functor.

In § 2, we define operads and coalgebras over operads. We also give a basic condition (see 4.2) on the operad under consideration that we assume to hold throughout the paper. Cofibrant operads always satisfy this condition and every operad is weakly equivalent to one that satisfies this condition.

In § 3, we briefly recall the notion of model structure on a category and give an example of a model structure on the category of unbounded chain-complexes.

In § 4, we define a model structure on categories of coalgebras over operads. When the operad is projective and finitely-generated in all dimensions, we verify that nearly free coalgebras satisfy Quillen’s axioms of a model structure (see [19] or [11]).

Section 4.1 describes our model-structure — classes of cofibrations, fibrations and weak equivalences. Section 4.2 proves the first few axioms of a model-structure (CM 1 through CM 3, in Quillen’s notation). Section 4.3 proves axiom CM 5, and section 4.4 proves CM 4.

A key step involves proving the existence of cofibrant and fibrant replacements for objects. In our model structure, all coalgebras are cofibrant (solving this half of the problem) and the hard part of is to find fibrant replacements.

We develop resolutions of coalgebras by cofree coalgebras — our so-called rug-resolutions — that solves the problem: see lemma 4.16 and corollary 4.17. This construction naturally leads into infinitely many negative dimensions and was the motivation for assuming underlying chain-complexes are unbounded.

All coalgebras are cofibrant and fibrant coalgebras are characterized as retracts of canonical resolutions called rug-resolutions (see corollary 4.18 and corollary 4.18) — an analogue to total spaces of Postnikov towers.

In the cocommutative case over the rational numbers, the model structure that we get is not equivalent to that of Hinich in [13]. He gives an example (9.1.2) of a coalgebra that is acyclic but not contractible. In our theory it would be contractible, since it is over the rational numbers and bounded.

In § 4.5, we discuss the (minor) changes to the methods in § 4 to handle coalgebras that are bounded from below. This involves replacing the cofree coalgebras by their truncated versions.
In § 5, we consider two examples over the rational numbers. In the rational, 2-connected, cocommutative, coassociative case, we recover the model structure Quillen defined in [21] — see example 5.2.

In appendix A, we study nearly free $\mathbb{Z}$-modules. These are modules whose countable submodules are all $\mathbb{Z}$-free. They take the place of free modules in our work, since the cofree coalgebra on a free modules is not free (but is nearly free).

In appendix B, we develop essential category-theoretic constructions, including equalizers (§ B.2), products and fibered products (§ B.3), and colimits and limits (§ B.4). The construction of limits in § B.4 was this project’s most challenging aspect and consumed the bulk of the time spent on it. This section’s key results are corollary B.24, which allows computation of inverse limits of coalgebras and theorem B.26, which shows that these inverse limits share a basic property with those of chain-complexes.

I am indebted to Professor Bernard Keller for several useful discussions.

2. Notation and conventions

Throughout this paper, $R$ will denote a field or $\mathbb{Z}$.

Definition 2.1. An $R$-module $M$ will be called nearly free if every countable submodule is $R$-free.

Remark. This condition is automatically satisfied unless $R = \mathbb{Z}$.

Clearly, any $\mathbb{Z}$-free module is also nearly free. The Baer-Specker group, $\mathbb{Z}^{\mathbb{N}}$, is a well-known example of a nearly free $\mathbb{Z}$-module that is not free — see [10], [1], and [29]. Compare this with the notion of $\aleph_1$-free groups — see [4].

By abuse of notation, we will often call chain-complexes nearly free if their underlying modules are (ignoring grading).

Nearly free $\mathbb{Z}$-modules enjoy useful properties that free modules do not. For instance, in many interesting cases, the cofree coalgebra of a nearly free chain-complex is nearly free.

Definition 2.2. We will denote the closed symmetric monoidal category of unbounded, nearly free $R$-chain-complexes with $R$-tensor products by $\text{Ch}$. We will denote the category of $R$-free chain-complexes that are bounded from below in dimension 0 by $\text{Ch}_0$.

The chain-complexes of $\text{Ch}$ are allowed to extend into arbitrarily many negative dimensions and have underlying graded $R$-modules that are

- arbitrary if $R$ is a field (but they will be free)
- nearly free, in the sense of definition 2.1 if $R = \mathbb{Z}$.

We make extensive use of the Koszul Convention (see [12]) regarding signs in homological calculations:

Definition 2.3. If $f: C_1 \to D_1$, $g: C_2 \to D_2$ are maps, and $a \otimes b \in C_1 \otimes C_2$ (where $a$ is a homogeneous element), then $(f \otimes g)(a \otimes b)$ is defined to be $(-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b)$.

Remark 2.4. If $f_i$, $g_i$ are maps, it isn’t hard to verify that the Koszul convention implies that $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2) \cdot \deg(g_1)} (f_1 \circ f_2) \otimes g_1 \circ g_2)$. 

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Definition 2.5. The symbol $I$ will denote the unit interval, a chain-complex given by

\begin{align*}
I_0 &= R \cdot p_0 \oplus R \cdot p_1 \\
I_1 &= R \cdot q \\
I_k &= 0 \text{ if } k \neq 0, 1 \\
\partial q &= p_1 - p_0
\end{align*}

Given $A \in \text{Ch}$, we can define $A \otimes I$ and $\text{Cone}(A) = A \otimes I / A \otimes p_1$.

The set of morphisms of chain-complexes is itself a chain complex:

Definition 2.6. Given chain-complexes $A, B \in \text{Ch}$ define $\text{Hom}_R(A, B)$ to be the chain-complex of graded $R$-morphisms where the degree of an element $x \in \text{Hom}_R(A, B)$ is its degree as a map and with differential

$$\partial f = f \circ \partial_A - (-1)^{\deg f} \partial_B \circ f$$

As a $R$-module $\text{Hom}_R(A, B)_k = \prod_j \text{Hom}_R(A_j, B_{j+k})$.

Remark. Given $A, B \in \text{Ch}^{S_n}$, we can define $\text{Hom}_{RS_n}(A, B)$ in a corresponding way.

Definition 2.7. If $G$ is a discrete group, let $\text{Ch}_G^0$ denote the category of chain-complexes equipped with a right $G$-action. This is again a closed symmetric monoidal category and the forgetful functor $\text{Ch}_G^0 \rightarrow \text{Ch}^0$ has a left adjoint, $(-)[G]$. This applies to the symmetric groups, $S_n$, where we regard $S_1$ and $S_0$ as the trivial group. The category of collections is defined to be the product

$$\text{Coll} (\text{Ch}_0) = \prod_{n \geq 0} \text{Ch}_0^{S_n}$$

Its objects are written $\mathcal{V} = \{ \mathcal{V}(n) \}_{n \geq 0}$. Each collection induces an endofunctor (also denoted $\mathcal{V}$) $\mathcal{V} : \text{Ch}_0 \rightarrow \text{Ch}_0$

$$\mathcal{V}(X) = \bigoplus_{n \geq 0} \mathcal{V}(n) \otimes_{RS_n} X^\otimes n$$

where $X^\otimes n = X \otimes \cdots \otimes X$ and $S_n$ acts on $X^\otimes n$ by permuting factors. This endofunctor is a monad if the defining collection has the structure of an operad, which means that $\mathcal{V}$ has a unit $\eta : R \rightarrow \mathcal{V}(1)$ and structure maps

$$\gamma_{k_1, \ldots, k_n} : \mathcal{V}(n) \otimes \mathcal{V}(k_1) \otimes \cdots \otimes \mathcal{V}(k_n) \rightarrow \mathcal{V}(k_1 + \cdots + k_n)$$

satisfying well-known equivariance, associativity, and unit conditions—see [22], [14].

We will call the operad $\mathcal{V} = \{ \mathcal{V}(n) \}$ $\Sigma$-cofibrant if $\mathcal{V}(n)$ is $RS_n$-projective for all $n \geq 0$. 


Remark. The operads we consider here correspond to symmetric operads in [22].

The term “unital operad” is used in different ways by different authors. We use it in the sense of Kriz and May in [14], meaning the operad has a 0-component that acts like an arity-lowering augmentation under compositions. Here \( V(0) = \mathbb{R} \).

The term \( \Sigma \)-cofibrant first appeared in [2].

A simple example of an operad is:

Example 2.8. For each \( n \geq 0 \), \( C(n) = \mathbb{Z} S_n \), with structure-map induced by
\[
\gamma_{\alpha_1, \ldots, \alpha_n} : S_n \times S_{\alpha_1} \times \cdots \times S_{\alpha_n} \to S_{\alpha_1 + \cdots + \alpha_n}
\]
defined by regarding each of the \( S_{\alpha_i} \) as permuting elements within the subsequence \( \{ \alpha_1 + \cdots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_i \} \) of the sequence \( \{1, \ldots, \alpha_1 + \cdots + \alpha_n \} \) and making \( S_n \) permute these \( n \)-blocks. This operad is denoted \( S_0 \). In other notation, its \( n \)th component is the symmetric group-ring \( \mathbb{Z} S_n \). See [21] for explicit formulas.

Another important operad is:

Example 2.9. The Barratt-Eccles operad, \( \mathcal{S} \), is given by
\[
\mathcal{S}(n) = \{ C^*(\widetilde{K}(S_n, 1)) \}
\]
where \( C^*(\widetilde{K}(S_n, 1)) \) is the normalized chain complex of the universal cover of the Eilenberg-MacLane space \( K(S_n, 1) \). This is well-known (see [?] or [21]) to be a Hopf-operad, i.e. equipped with an operad morphism
\[
\delta : \mathcal{S} \to \mathcal{S} \otimes \mathcal{S}
\]
and is important in topological applications. See [21] for formulas for the structure maps.

For the purposes of this paper, the main example of an operad is

Definition 2.10. Given any \( C \in \text{Ch} \), the associated coendomorphism operad, \( \text{CoEnd}(C) \) is defined by
\[
\text{CoEnd}(C)(n) = \text{Hom}_R(C, C \otimes^n)
\]
Its structure map
\[
\gamma_{\alpha_1, \ldots, \alpha_n} : \text{Hom}_R(C, C \otimes^n) \otimes \text{Hom}_R(C, C \otimes^{\alpha_1}) \otimes \cdots \otimes \text{Hom}_R(C, C \otimes^{\alpha_n}) \to \text{Hom}_R(C, C \otimes^{\alpha_1 + \cdots + \alpha_n})
\]
simply composes a map in \( \text{Hom}_R(C, C \otimes^n) \) with maps of each of the \( n \) factors of \( C \).

This is a non-unital operad, but if \( C \in \text{Ch} \) has an augmentation map \( \varepsilon : C \to R \) then we can regard \( \varepsilon \) as the generator of \( \text{CoEnd}(C)(0) = R \cdot \varepsilon \subset \text{Hom}_R(C, C \otimes^0) = \text{Hom}_R(C, R) \).

Given \( C \in \text{Ch} \) with subcomplexes \( \{D_1, \ldots, D_k\} \), the relative coendomorphism operad \( \text{CoEnd}(C; \{D_j\}) \) is defined to be the sub-operad of \( \text{CoEnd}(C) \) consisting of maps \( f \in \text{Hom}_R(C, C \otimes^n) \) such that \( f(D_j) \subseteq D_j \otimes^n \subseteq C \otimes^n \) for all \( j \).

We use the coendomorphism operad to define the main object of this paper:

Definition 2.11. A coalgebra over an operad \( V \) is a chain-complex \( C \in \text{Ch} \) with an operad morphism \( \alpha : V \to \text{CoEnd}(C) \), called its structure map. We will sometimes want to define coalgebras using the adjoint structure map
\[
\alpha : C \to \prod_{n \geq 0} \text{Hom}_{RS_n}(V(n), C \otimes^n)
\]
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(in Ch) or even the set of chain-maps

\[ \alpha_n : C \to \text{Hom}_{RS_n}(V(n), C^{\otimes n}) \]

for all \( n \geq 0 \).

We will sometimes want to focus on a particular class of \( V \)-coalgebras: the
pointed, irreducible coalgebras. We define this concept in a way that extends the
conventional definition in [24].

**Definition 2.12.** Given a coalgebra over a unital operad \( V \) with adjoint structure-
map

\[ \alpha_n : C \to \text{Hom}_{RS_n}(V(n), C^{\otimes n}) \]

an element \( c \in C \) is called group-like if \( \alpha_n(c) = f_n(c^{\otimes n}) \) for all \( n > 0 \). Here
\( c^{\otimes n} \in C^{\otimes n} \) is the \( n \)-fold \( R \)-tensor product,

\[ f_n = \text{Hom}_R(\epsilon_n, 1) : \text{Hom}_R(R, C^{\otimes n}) = C^{\otimes n} \to \text{Hom}_{RS_n}(V(n), C^{\otimes n}) \]

and \( \epsilon_n : V(n) \to V(0) = R \) is the augmentation (which is \( n \)-fold composition with
\( V(0) \)).

A coalgebra \( C \) over an operad \( V \) is called pointed if it has a unique group-like
element (denoted \( 1 \)), and pointed irreducible if the intersection of any two sub-
coalgebras contains this unique group-like element.

**Remark.** Note that a group-like element generates a sub \( V \)-coalgebra of \( C \) and must
lie in dimension 0.

Although this definition seems contrived, it arises in “nature”: The
chain-complex of a pointed, simply-connected reduced simplicial set is naturally a
pointed irreducible coalgebra over the Barratt-Eccles operad, \( \mathcal{S} = \{ C(K(S_n, 1)) \} \)
(see [21]). In this case, the operad action encodes the chain-level effect of Steenrod
operations.

**Proposition 2.13.** Let \( D \) be a pointed, irreducible coalgebra over an operad \( V \). Then
the augmentation map

\[ \varepsilon : D \to R \]

is naturally split and any morphism of pointed, irreducible coalgebras

\[ f : D_1 \to D_2 \]

is of the form

\[ 1 \oplus f : D_1 = R \oplus \ker \varepsilon_{D_1} \to D_2 = R \oplus \ker \varepsilon_{D_2} \]

where \( \varepsilon_i : D_i \to R \), \( i = 1, 2 \) are the augmentations.

**Proof.** The definition (2.12) of the sub-coalgebra \( R \cdot 1 \subseteq D_i \) is stated in an invariant
way, so that any coalgebra morphism must preserve it. Any morphism must also
preserve augmentations because the augmentation is the 0th-order structure-map.
Consequently, \( f \) must map \( \ker \varepsilon_{D_i} \) to \( \ker \varepsilon_{D_2} \). The conclusion follows. \( \square \)

**Definition 2.14.** We denote the category of nearly free coalgebras over \( V \) by \( \mathcal{S}_0 \).
The terminal object in this category is \( 0 \), the null coalgebra.

The category of nearly free pointed irreducible coalgebras over \( V \) is denoted \( \mathcal{S}_0 \) —
this is only defined if \( V \) is unital. Its terminal object is the coalgebra whose
underlying chain complex is \( R \) concentrated in dimension 0 with coproduct that
sends \( 1 \in R \) to \( 1^{\otimes n} \in R^{\otimes n} \).

It is not hard to see that these terminal objects are also the initial objects of
their respective categories.
We also need:

**Definition 2.15.** If \( A \in \mathcal{C} = \mathcal{I}_0 \) or \( \mathcal{S}_0 \), then \([A]\) denotes the underlying chain-complex in \( \text{Ch} \) of

\[
\text{ker } A \to \bullet
\]

where \( \bullet \) denotes the terminal object in \( \mathcal{C} \) — see definition \ref{definition13}. We will call \([\_]\) the *forgetful functor* from \( \mathcal{C} \) to \( \text{Ch} \).

We can also define the analogue of an ideal:

**Definition 2.16.** Let \( C \) be a coalgebra over the operad \( \mathcal{U} \) with adjoint structure map

\[
\alpha: C \to \prod_{n \geq 0} \text{Hom}_{\text{RS}_n}(\mathcal{U}(n), C^\otimes n)
\]

and let \( D \subseteq [C] \) be a sub-chain complex that is a direct summand. Then \( D \) will be called a *coideal* of \( C \) if the composite

\[
\alpha|_D: D \to \prod_{n \geq 0} \text{Hom}_{\text{RS}_n}(\mathcal{U}(n), C^\otimes n) \xrightarrow{\text{Hom}_R(1_\mathcal{U}, p^\otimes)} \prod_{n \geq 0} \text{Hom}_{\text{RS}_n}(\mathcal{U}(n), (C/D)^\otimes n)
\]

vanishes, where \( p: C \to C/D \) is the projection to the quotient (in \( \text{Ch} \)).

**Remark.** Note that it is easier for a sub-chain-complex to be a coideal of a coalgebra than to be an ideal of an algebra. For instance, all sub-coalgebras of a coalgebra are also coideals. Consequently it is easy to form quotients of coalgebras and hard to form sub-coalgebras. This is dual to what occurs for algebras.

We will use the concept of cofree coalgebra cogenerated by a chain complex:

**Definition 2.17.** Let \( C \in \text{Ch} \) and let \( \mathcal{V} \) be an operad. Then a \( \mathcal{V} \)-coalgebra \( G \) will be called the *cofree coalgebra cogenerated by \( C \)* if

1. there exists a morphism of DG-modules \( \varepsilon: G \to C \)
2. given any \( \mathcal{V} \)-coalgebra \( D \) and any morphism of DG-modules \( f: D \to C \),
   there exists a *unique* morphism of \( \mathcal{V} \)-coalgebras, \( \hat{f}: D \to G \), that makes the diagram

\[
\begin{array}{ccc}
D & \xrightarrow{f} & G \\
\downarrow{\hat{f}} & & \downarrow{\varepsilon} \\
C & & \\
\end{array}
\]

commute.

This universal property of cofree coalgebras implies that they are unique up to isomorphism if they exist. The paper \cite{22} gives a constructive proof of their existence in great generality (under the unnecessary assumption that chain-complexes are \( R \)-free). In particular, this paper defines cofree coalgebras \( L_{\mathcal{V}}C \) and pointed irreducible cofree coalgebras \( P_{\mathcal{V}}C \) cogenerated by a chain-complex \( C \). There are several ways to define them:

1. \( L_{\mathcal{V}}C \) is essentially the largest submodule of

\[
C \oplus \prod_{k=1}^\infty \text{Hom}_{\text{RS}_k}(\mathcal{V}(k), C^\otimes k)
\]
on which the coproduct defined by the dual of the composition-operations of $\mathcal{V}$ is well-defined.

(2) If $C$ is a coalgebra over $\mathcal{V}$, its image under the structure map

$$C \to C \oplus \bigoplus_{k=1}^{\infty} \text{Hom}_{\text{RS}_k}(\mathcal{V}(k), C^{\otimes k})$$

turns out to be a sub-coalgebra of the target — with a coalgebra structure that vanishes on the left summand $(C)$ and is the dual of the structure-map of $\mathcal{V}$ on the right. We may define $L_\mathcal{V}C$ to be the sum of all coalgebras in $C \oplus \bigoplus_{k=1}^{\infty} \text{Hom}_{\text{RS}_k}(\mathcal{V}(k), C^{\otimes k})$ formed in this way. The classifying map of a coalgebra

$$C \to L_\mathcal{V}C$$

is just the structure map of the coalgebra structure.

3. Model categories

We recall the concept of a model structure on a category $\mathcal{G}$. This involves defining specialized classes of morphisms called cofibrations, fibrations, and weak equivalences (see [19] and [11]). The category and these classes of morphisms must satisfy the conditions:

CM 1: $\mathcal{G}$ is closed under all finite limits and colimits
CM 2: Suppose the following diagram commutes in $\mathcal{G}$:

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{h} & & \downarrow{f} \\
Z & \xleftarrow{f} & \end{array}
$$

If any two of $f, g, h$ are weak equivalences, so is the third.

CM 3: These classes of morphisms are closed under formation of retracts:

Given a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{f} \\
C & \xrightarrow{g} & D \\
\downarrow{f} & & \downarrow{f} \\
C & \xleftarrow{f} & \end{array}
$$

whose horizontal composites are the identity map, if $g$ is a weak equivalence, fibration, or cofibration, then so is $f$.

CM 4: Given a commutative solid arrow diagram

$$
\begin{array}{ccc}
U & \xrightarrow{i} & A \\
\downarrow{p} & & \downarrow{p} \\
W & \xrightarrow{j} & B \\
\end{array}
$$

where $i$ is a cofibration and $p$ is a fibration, the dotted arrow exists whenever $i$ or $p$ are trivial.

CM 5: Any morphism $f: X \to Y$ in $\mathcal{G}$ may be factored:

(1) $f = p \circ i$, where $p$ is a fibration and $i$ is a trivial cofibration

(2) $f = q \circ j$, where $q$ is a trivial fibration and $j$ is a cofibration

We also assume that these factorizations are functorial — see [19].
Definition 3.1. An object, $X$, for which the map $\bullet \to X$ is a cofibration, is called cofibrant. An object, $Y$, for which the map $Y \to \bullet$ is a fibration, is called fibrant.

The properties of a model category immediately imply that:

Lemma 3.2. Let $f: A \to B$ be a trivial fibration in which $B$ is cofibrant. Then $f$ is a retraction of $A$ onto $B$, i.e., there exists a morphism $g: B \to A$ such that $f \circ g = 1: B \to B$.

Proof. Consider the diagram

\[ \begin{array}{ccc}
\bullet & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
B & \xleftarrow{} & B \\
\end{array} \]

Property CM 4 implies that we can complete this to a diagram

\[ \begin{array}{ccc}
\bullet & \xrightarrow{g} & A \\
\downarrow & & \downarrow \\
B & \xleftarrow{f} & B \\
\end{array} \]

\[ \square \]

3.1. A model-category of chain-complexes. Let $\mathbf{Ch}$ denote the category of unbounded chain-complexes over the ring $R$. The absolute model structure of Christensen and Hovey in [8], and Cole in [6] is defined via:

1. Weak equivalences are chain-homotopy equivalences: two chain-complexes $C$ and $D$ are weakly equivalent if there exist chain-maps: $f: C \to D$ and $g: D \to C$ and chain-homotopies $\varphi_1: C \to C$ and $\varphi_2: D \to D$ such that $d\varphi_1 = g \circ f - 1$ and $d\varphi_2 = f \circ g - 1$.

2. Fibrations are surjections of chain-complexes that are split (as maps of graded $R$-modules).

3. Cofibrations are injections of chain-complexes that are split (as maps of graded $R$-modules).

Remark. All chain complexes are fibrant and cofibrant in this model.

In this model structure, a quasi-isomorphism may fail to be a weak equivalence. It is well-known not to be cofibrantly generated (see [8]).

Since all chain-complexes are cofibrant, lemma [3.2] implies that all trivial fibrations are retractions — i.e., they are split as chain-maps. We will need the following relative version of lemma [3.2] in the sequel:

Lemma 3.3. Let

\[ \begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow f & & \downarrow g \\
C & \xleftarrow{v} & D \\
\end{array} \]

be a commutative diagram in $\mathbf{Ch}$ such that

- $f$ and $g$ are trivial fibrations
- $v$ is a cofibration

Then there exists maps $\ell: C \to A$ and $m: D \to B$ such that

1. $f \circ \ell = 1: C \to C$, $g \circ m = 1: D \to D$
(2) the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\downarrow{\ell} & & \downarrow{m} \\
C & \xrightarrow{v} & D
\end{array}
\]

commutes.

If \( u \) is injective and split, there exist homotopies \( \varphi_1: A \otimes I \to A \) from \( 1 \) to \( \ell \circ f \) and \( \varphi_2: B \otimes I \to B \) from \( 1 \) to \( m \circ g \), respectively, such that the diagram

\[
\begin{array}{ccc}
A \otimes I & \xrightarrow{u \otimes 1} & B \otimes I \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} \\
A & \xrightarrow{u} & B
\end{array}
\]

commutes.

Proof. We construct \( \ell \) exactly as in lemma 3.2 and use it to create a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{u \otimes f} & B \\
\downarrow{v} & & \downarrow{g} \\
D & = & D
\end{array}
\]

Property CM 4 implies that we can complete this to a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{u \otimes \ell} & B \\
\downarrow{v} & & \downarrow{g} \\
D & = & D
\end{array}
\]

Let \( \varphi_1': A \otimes I \to A \) and \( \varphi_2': B \otimes I \to B \) be any homotopies from the identity map to \( \ell \circ f \) and \( m \circ g \), respectively. If there exists a chain-map \( p: B \to A \) \( p \circ u = 1: A \to A \) then \( \varphi_1 = \varphi_1' \) and \( \varphi_2 = u \circ \varphi_1' \circ (p \otimes 1) + \varphi_2' \circ (1 - (u \circ p) \otimes 1) \) have the required properties. \( \square \)

4. Model-categories of coalgebras

4.1. Description of the model-structure. We will base our model-structure on that of the underlying chain-complexes in \( \text{Ch} \). Definition 4.4 and definition 4.5 describe how we define cofibrations, fibrations, and weak equivalences.

We must allow non-\( R \)-free chain-complexes (when \( R = \mathbb{Z} \)) because the underlying chain complexes of the cofree coalgebras \( P_V(*) \) and \( L_V(*) \) are not known to be \( R \)-free. They certainly are if \( R \) is a field, but if \( R = \mathbb{Z} \) their underlying abelian groups are subgroups of the Baer-Specker group, \( \mathbb{Z}^{\aleph_0} \), which is \( \mathbb{Z} \)-torsion free but well-known not to be a free abelian group (see [23], [3] or the survey [7]).

Proposition 4.1. The forgetful functor (defined in definition 2.15) and cofree coalgebra functors define adjoint pairs

\[
P_V(*): \text{Ch} \xrightarrow{} \mathcal{J}_0: [*] \\
L_V(*): \text{Ch} \xrightarrow{} \mathcal{J}_0: [*]
\]
Remark. The adjointness of the functors follows from the universal property of cofree coalgebras — see [22].

Condition 4.2. Throughout the rest of this paper, we assume that \( \mathcal{V} \) is an operad equipped with a morphism of operads

\[
\delta: \mathcal{V} \to \mathcal{V} \otimes \mathcal{G}
\]

where \( \mathcal{G} \) is the Barratt-Eccles operad (see example 2.9) — that makes the diagram

\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\delta} & \mathcal{V} \otimes \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{V} & \xrightarrow{} & \mathcal{V} \otimes T = \mathcal{V}
\end{array}
\]

commute. Here, the operad structure on \( \mathcal{V} \otimes \mathcal{G} \) is just the tensor product of the operad structures of \( \mathcal{V} \) and \( \mathcal{G} \), and the vertical map is projection:

\[
\mathcal{V} \otimes \mathcal{G} \xrightarrow{\mathcal{V} \otimes \epsilon} \mathcal{V} \otimes T = \mathcal{V}
\]

where \( T \) is the operad that is \( R \) in all arities and \( \epsilon: \mathcal{G} \to T \) is defined by the augmentations:

\[\epsilon_n: RS_n \to R\]

In addition, we assume that, for each \( n \geq 0 \), \{\( \mathcal{V}(n) \)\} is an \( RS_n \)-projective chain-complex of finite type.

We also assume that the arity-1 component of \( \mathcal{V} \) is equal to \( R \), generated by the unit.

Remark 4.3. Free and cofibrant operads (with each component of finite type) satisfy this condition. The condition that the chain-complexes are projective corresponds to the Berger and Moerdijk’s condition of \( \Sigma \)-cofibrancy in [2].

Now we define our model structure on the categories \( \mathcal{I}_0 \) and \( \mathcal{S}_0 \).

Definition 4.4. A morphism \( f: A \to B \) in \( \mathcal{C} = \mathcal{I}_0 \) or \( \mathcal{S}_0 \) will be called

(1) a weak equivalence if \([f]: [A] \to [B]\) is a chain-homotopy equivalence in \( \text{Ch} \). An object \( A \) will be called contractible if the augmentation map

\[
A \to \bullet
\]

is a weak equivalence, where \( \bullet \) denotes the terminal object in \( \mathcal{C} \) — see definition 2.14.

(2) a cofibration if \([f]\) is a cofibration in \( \text{Ch} \).

(3) a trivial cofibration if it is a weak equivalence and a cofibration.

Remark. A morphism is a cofibration if it is a degreewise split monomorphism of chain-complexes. Note that all objects of \( \mathcal{C} \) are cofibrant.

Our definition makes \( f: A \to B \) a weak equivalence if and only if \([f]: [A] \to [B]\) is a weak equivalence in \( \text{Ch} \).

Definition 4.5. A morphism \( f: A \to B \) in \( \mathcal{I}_0 \) or \( \mathcal{S}_0 \) will be called

(1) a fibration if the dotted arrow exists in every diagram of the form

\[
\begin{array}{ccc}
U & \xrightarrow{i} & A \\
\downarrow f & & \downarrow f \\
W & \xrightarrow{} & B
\end{array}
\]
in which \( i: U \rightarrow W \) is a trivial cofibration.

(2) a trivial fibration if it is a fibration and a weak equivalence.

Definition 4.4 explicitly described cofibrations and definition 4.5 defined fibrations in terms of them. We will verify the axioms for a model category (part of CM 4 and CM 5) and characterize fibrations.

We will occasionally need a stronger form of equivalence:

**Definition 4.6.** Let \( f, g: A \rightarrow B \) be a pair of morphisms in \( \mathcal{S}_0 \) or \( \mathcal{I}_0 \). A strict homotopy between them is a coalgebra-morphism (where \( A \otimes I \) has the coalgebra structure defined in condition 4.2)

\[
F: A \otimes I \rightarrow B
\]

such that \( F|A \otimes p_0 = f: A \otimes p_0 \rightarrow B \) and \( F|A \otimes p_1 = g: A \otimes p_{01} \rightarrow B \). A strict equivalence between two coalgebras \( A \) and \( B \) is a pair of coalgebra-morphisms

\[
f: A \rightarrow B \\
g: B \rightarrow A
\]

and strict homotopies from \( f \circ g \) to the identity of \( B \) and from \( g \circ f \) to the identity map of \( A \).

**Remark.** Strict equivalence is a direct translation of the definition of weak equivalence in \( \text{Ch} \) into the realm of coalgebras. Strict equivalences are weak equivalences but the converse is not true.

The reader may wonder why we didn’t use strict equivalence in place of what is defined in definition 4.4. It turns out that in we are only able to prove CM 5 with the weaker notion of equivalence used here.

In a few simple cases, describing fibrations is easy:

**Proposition 4.7.** Let

\[
f: A \rightarrow B
\]

be a fibration in \( \text{Ch} \). Then the induced morphisms

\[
P_\gamma f: P_\gamma A \rightarrow P_\gamma B \\
L_\gamma f: L_\gamma A \rightarrow L_\gamma B
\]

are fibrations in \( \mathcal{S}_0 \) and \( \mathcal{I}_0 \), respectively.

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{P_\gamma f} & P_\gamma B \\
\downarrow & & \downarrow \\
V & \xrightarrow{P_\gamma f} & P_\gamma B
\end{array}
\]

where \( U \rightarrow V \) is a trivial cofibration — i.e., \([U] \rightarrow [V]\) is a trivial cofibration of chain-complexes. Then the dotted map exists by the the defining property of cofree coalgebras and by the existence of the lifting map in the diagram

\[
\begin{array}{ccc}
[U] & \xrightarrow{A} & B \\
\downarrow & \downarrow & \downarrow \\
[V] & \xrightarrow{B} & B
\end{array}
\]
of chain-complexes. □

**Corollary 4.8.** All cofree coalgebras are fibrant.

**Proposition 4.9.** Let $C$ and $D$ be objects of $\text{Ch}$ and let $f_1, f_2: C \to D$

be chain-homotopic morphisms via a chain-homotopy

(4.1) $F: C \otimes I \to D$

Then the induced maps

$P_{\gamma} f_i: P_{\gamma} C \to P_{\gamma} D$

$L_{\gamma} f_i: L_{\gamma} C \to L_{\gamma} D$

$i = 1, 2$, are left-homotopic in $\mathcal{I}_0$ and $\mathcal{S}_0$, respectively via a strict chain homotopy

$F': P_{\gamma} f_i: (P_{\gamma} C) \otimes I \to P_{\gamma} D$

If we equip $C \otimes I$ with a coalgebra structure using condition 4.2 and if $F$ in (4.1) is 

strict then the diagram

\[
\begin{array}{ccc}
C \otimes I & \xrightarrow{F} & D \\
\downarrow{\alpha_C \otimes 1} & & \downarrow{\alpha_D} \\
P_{\gamma}(C) \otimes I & \xrightarrow{F'} & P_{\gamma} D
\end{array}
\]

commutes in the pointed irreducible case and the diagram

\[
\begin{array}{ccc}
C \otimes I & \xrightarrow{F} & D \\
\downarrow{\alpha_C \otimes 1} & & \downarrow{\alpha_D} \\
L_{\gamma}(C) \otimes I & \xrightarrow{F'} & L_{\gamma} D
\end{array}
\]

commutes in the general case. Here $\alpha_C$ and $\alpha_D$ are classifying maps of coalgebra structures.

**Remark.** In other words, the cofree coalgebra functors map homotopies and weak equivalences in $\text{Ch}$ to strict homotopies and strict equivalences, respectively, in $\mathcal{I}_0$ and $\mathcal{S}_0$.

If the homotopy in $\text{Ch}$ was the result of applying the forgetful functor to a strict homotopy, then the generated strict homotopy is compatible with it.

**Proof.** We will prove this in the pointed irreducible case. The general case follows by a similar argument. The chain-homotopy between the $f_i$ induces

$P_{\gamma} F: P_{\gamma} (C \otimes I) \to P_{\gamma} D$

Now we construct the map

$H: (P_{\gamma} C) \otimes I \to P_{\gamma} (C \otimes I)$

using the universal property of a cofree coalgebra and the fact that the coalgebra structure of $(P_{\gamma} C) \otimes I$ extends that of $P_{\gamma} C$ on both ends by condition 4.2. Clearly

$P_{\gamma} F \circ H: (P_{\gamma} C) \otimes I \to P_{\gamma} D$

is the required left-homotopy.
If we define a coalgebra structure on $C \otimes I$ using condition 4.2, we get diagram

$$
\begin{array}{ccc}
C \otimes I & \xrightarrow{F} & D \\
\downarrow{\alpha_C \otimes 1} & & \downarrow{\alpha_D} \\
\alpha_C \otimes 1 & \xrightarrow{P_V(C) \otimes I} & P_V D \\
\downarrow{\alpha_C \otimes 1} & & \downarrow{\alpha_D} \\
C \otimes I & \xrightarrow{H} & P_V(C \otimes I) \\
\end{array}
$$

where $\alpha_{C \otimes I}$ is the classifying map for the coalgebra structure on $C \otimes I$.

We claim that this diagram commutes. The fact that $F$ is a coalgebra morphism implies that the upper right square commutes. The large square on the left (bordered by $C \otimes I$ on all four corners) commutes by the property of co-generating maps (of cofree coalgebras) and classifying maps. The two smaller squares on the left (i.e., the large square with the map $H$ added to it) commute by the universal properties of cofree coalgebras (which imply that induced maps to cofree coalgebras are uniquely determined by their composites with co-generating maps). The diagram in the statement of the result is just the outer upper square of this diagram, so we have proved the claim.

This result implies a homotopy invariance property of the categorical product, $A_0 \boxtimes A_1$, defined explicitly in definition B.13 of appendix B.

**Lemma 4.10.** Let $g: B \rightarrow C$ be a fibration in $\mathcal{I}_0$ and let $f: A \rightarrow C$ be a morphism in $\mathcal{I}_0$. Then the projection $A \boxtimes C B \rightarrow A$ is a fibration.

**Remark.** The notation $A \boxtimes C B$ denotes a fibered product — see definition B.15 in appendix B.3 for the precise definition. In other words, pullbacks of fibrations are fibrations.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
U & \xrightarrow{n} & A \boxtimes C B \\
\downarrow{i} & & \downarrow{p_A} \\
V & \xrightarrow{v} & A
\end{array}
$$

where $U \rightarrow V$ is a trivial cofibration. The defining property of a categorical product implies that any map to $A \boxtimes C B \subseteq A \boxtimes B$ is determined by its composites with the projections

$$
p_A: A \boxtimes B \rightarrow A \\
p_B: A \boxtimes B \rightarrow B
$$
Consider the composite $p_B \circ w: U \to B$. The commutativity of the solid arrows in diagram (4.2) implies that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{p_B \circ u} & B \\
\downarrow^i & & \downarrow^g \\
V & \xrightarrow{f \circ v} & C
\end{array}
\]

commutes and this implies that the solid arrows in the diagram

(4.3)

\[
\begin{array}{ccc}
U & \xrightarrow{p_B \circ u} & B \\
\downarrow^i & & \downarrow^g \\
V & \xrightarrow{f \circ v} & C
\end{array}
\]

commute. The fact that $g: B \to C$ is a fibration implies that the dotted arrow exists in diagram (4.3), which implies the existence of a map $V \to A \boxtimes B$ whose composites with $f$ and $g$ agree. This defines a map $V \to A \boxtimes C$ that makes all of diagram (4.2) commute. The conclusion follows. □

4.2. Proof of CM 1 through CM 3. CM 1 asserts that our categories have all finite limits and colimits.

The results of appendix B prove that all countable limits and colimits exist — see theorem B.2 and theorem B.6.

CM 2 follows from the fact that we define weak equivalence the same way it is defined in $\text{Ch}$ — so the model structure on $\text{Ch}$ implies that this condition is satisfied on our categories of coalgebras. A similar argument verifies condition CM 3.

4.3. Proof of CM 5. We begin with:

**Corollary 4.11.** Let $A \in \mathcal{I}_0$ be fibrant and let $B \in \mathcal{I}_0$. Then the projection

$A \boxtimes B \to B$

is a fibration.

This allows us to verify CM 5, statement 2:

**Corollary 4.12.** Let $f: A \to B$ be a morphism in $\mathcal{C} = \mathcal{I}_0$ or $\mathcal{S}_0$, and let

$Z = \begin{cases} 
P_Y \text{Cone}(\lceil A \rceil) \boxtimes B & \text{when } \mathcal{C} = \mathcal{I}_0 \\
L_Y \text{Cone}(\lceil A \rceil) \boxtimes B & \text{when } \mathcal{C} = \mathcal{S}_0
\end{cases}$

Then $f$ factors as

$A \to Z \to B$

where

1. $\text{Cone}(\lceil A \rceil)$ is the cone on $\lceil A \rceil$ (see definition 2.3) with the canonical inclusion $i: \lceil A \rceil \to \text{Cone}(\lceil A \rceil)$
2. the morphism $i \boxtimes f: A \to Z$ is a cofibration
3. the morphism $Z \to B$ is projection to the second factor and is a fibration.

Consequently, $f$ factors as a cofibration followed by a trivial fibration.
Proof. We focus on the pointed irreducible case. The general case follows by essentially the same argument. The existence of the (injective) morphism \( A \to P_{\nu}(\{A\}) \square B \) follows from the definition of \( \square \). We claim that its image is a direct summand of \( P_{\nu}(\{A\}) \square B \) as a graded \( R \)-module (which implies that \( i \square f \) is a cofibration). We clearly get a projection

\[
P_{\nu}(\{A\}) \square B \to P_{\nu}(\{A\})
\]

and the composite of this with the co-generating map \( [P_{\nu}(\{A\})] \to \text{Cone}(\{A\}) \) gives rise a morphism of chain-complexes

\[
(4.4) [P_{\nu}(\{A\}) \square B] \to \text{Cone}(\{A\})
\]

Now note the existence of a splitting map

\[
\text{Cone}(\{A\}) \to \{A\}
\]

of graded \( R \)-modules (not coalgebras or even chain-complexes). Combined with the map in equation (4.4), we conclude that \( A \to P_{\nu}(\{A\}) \square B \) is a cofibration.

There is a weak equivalence \( c: \text{Cone}(\{A\}) \to \bullet \) in \( \mathbf{Ch} \), and (4.4) implies that it induces a strict equivalence \( P_{\nu}c: P_{\nu}(\{A\}) \to \bullet \). Proposition 4.14 implies that

\[
c \square 1: P_{\nu}(\{A\}) \square B \to \bullet \square B = B
\]

is a strict equivalence. \( \Box \)

The first part of CM 5 will be considerably more difficult to prove.

Definition 4.13. Let \( \text{pro} - \mathcal{I}_0 \) and \( \text{pro} - \mathcal{J}_0 \) be the categories of inverse systems of objects of \( \mathcal{I}_0 \) and \( \mathcal{J}_0 \), respectively and let \( \text{ind} - \mathcal{I}_0 \) and \( \text{ind} - \mathcal{J}_0 \) be corresponding categories of direct systems. Morphisms are defined in the obvious way.

Now we define the rug-resolution of a cofibration:

Definition 4.14. Let \( \mathcal{V} = \{\mathcal{V}(n)\} \) be a \( \Sigma \)-cofibrant (see definition 2.7) operad such that \( \mathcal{V}(n) \) is of finite type for all \( n \geq 0 \). If \( f: C \to D \) is a cofibration in \( \mathcal{I}_0 \) or \( \mathcal{J}_0 \), define

\[
\begin{align*}
G_0 &= D \\
f_0 = f: C &\to G_0 \\
G_{n+1} &= G_n \square L_{\mathcal{V}}[H_n] L_{\mathcal{V}}H_n \\
p_{n+1}: G_{n+1} &\to G_n
\end{align*}
\]

for all \( n \), where

1. \( c: C \to \bullet \) is the unique morphism.
2. \( H_n \) is the cofiber of \( f_n \) in the push-out

3. \( G_n \to L_{\mathcal{V}}[H_n] \) is the composite of the classifying map

\[
G_n \to L_{\mathcal{V}}[G_n]
\]

with the map

\[
L_{\mathcal{V}}[G_n] \to L_{\mathcal{V}}[H_n]
\]
(4) $\bar{H}_n = \Sigma^{-1} \text{Cone}([H_n])$ — where $\Sigma^{-1}$ denotes desuspension (in $\text{Ch}$). It is contractible and comes with a canonical $\text{Ch}$-fibration

$$v_n : \bar{H}_n \to [H_n]$$

inducing the fibration

$$L_Y v_n : L_Y \bar{H}_n \to L_Y [H_n]$$

(5) $p_{n+1} : G_{n+1} = G_n \boxtimes_{L_Y [H_n]} L_Y \bar{H}_n \to G_n$ is projection to the first factor,

(6) The map $f_{n+1} : C \to G_n \boxtimes_{L_Y [H_n]} L_Y \bar{H}_n$ is the unique morphism that makes the diagram

\[
\begin{array}{ccc}
G_n \boxtimes L_Y \bar{H}_n & \rightarrow & L_Y \bar{H}_n \\
\downarrow & & \downarrow \\
G_n & \leftarrow & \bar{H}_n \\
\uparrow & & \uparrow \\
f_n & & \epsilon \\
\end{array}
\]

commute, where the downwards maps are projections to factors. The map $\epsilon : C \to L_Y \bar{H}_n$ is

(a) the map to the basepoint if the category is $\mathcal{I}_0$ (and $L_Y \bar{H}_n$ is replaced by $P_Y \bar{H}_n$),

(b) the zero-map if the category is $\mathcal{I}_0$.

The commutativity of the diagram

\[
\begin{array}{ccc}
L_Y H_n & \rightarrow & L_Y \bar{H}_n \\
\downarrow & & \downarrow \\
G_n & \leftarrow & \bar{H}_n \\
\uparrow & & \uparrow \\
f_n & & \epsilon \\
\end{array}
\]

implies that the image of $f_{n+1}$ actually lies in the fibered product, $G_n \boxtimes_{L_Y [H_n]} L_Y \bar{H}_n$.

The rug-resolution of $f : C \to D$ is the map of inverse systems \( \{ f_i \} : \{ C \} \to \{ G_i \} \to D \), where \( \{ C \} \) denotes the constant inverse system.

Remark. Very roughly speaking, this produces something like a “Postnikov resolution” for $f : C \to D$. Whereas a Postnikov resolution’s stages “push the trash upstairs,” this one’s “push the trash horizontally” or “under the rug” — something feasible because one has an infinite supply of rugs.

**Proposition 4.15.** Following all of the definitions of 4.14 above, the diagrams

\[
\begin{array}{ccc}
G_{n+1} & \rightarrow & \bar{H}_n \\
\downarrow & & \downarrow \\
f_{n+1} & \leftarrow & \epsilon \\
\end{array}
\]

\[
\begin{array}{ccc}
C & \rightarrow & G_n \\
\downarrow & \rightarrow & \downarrow \\
f_n & \leftarrow & p_{n+1} \\
\end{array}
\]

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commute and induce maps $H_{n+1} \xrightarrow{p_{n+1}} H_n$ that fit into commutative diagrams of chain-complexes

$$
\begin{array}{ccc}
H_{n+1} & \xrightarrow{p_{n+1}} & H_n \\
\uparrow \, u_n & & \downarrow \, \varepsilon_n \\
\tilde{H}_{n+1} & \rightarrow & \tilde{H}_n
\end{array}
$$

It follows that the maps $p_{n+1}$ are nullhomotopic for all $n$.

**Proof.** Commutativity is clear from the definition of $f_{n+1}$ in terms of $f_n$ above.

To see that the induced maps are nullhomotopic, consider the diagram

$$
\begin{array}{ccc}
G_n \boxtimes^L V \left[ H_n \right] & \xrightarrow{\alpha} & L_V H_n \\
\downarrow \, p_{n+1} & & \downarrow \, \varepsilon \\
G_n & \xrightarrow{v_n} & L_V H_n
\end{array}
$$

where $v_n$ is defined in equation 4.5 both $\varepsilon$-maps are cogenerating maps — see definition 2.17 — and $\alpha: H_n \rightarrow L_V H_n$ is the classifying map.

The left square commutes by the definition of the fibered product, $G_n \boxtimes^L V \left[ H_n \right] L_V H_n$ — see definition B.15. The right square commutes by the naturality of cogenerating maps.

Now, note that the composite $H_n \xrightarrow{\beta} L_V H_n \xrightarrow{\varepsilon} H_n$ is the identity map (a universal property of classifying maps of coalgebras). It follows that, as a chain-map, the composite

$$
G_n \boxtimes^L V \left[ H_n \right] L_V \tilde{H}_n \xrightarrow{p_{n+1}} G_n \rightarrow H_n
$$

coincides with a chain-map that factors through the contractible chain-complex $\tilde{H}_n$. □

Our main result is:

**Lemma 4.16.** Let $f: C \rightarrow D$ be a cofibration as in definition 4.14 with rug-resolution $\{ f_i \}: \{ C \} \rightarrow \{ G_i \} \rightarrow D$. Then

$$
f_{\infty} = \lim_{\leftarrow} f_n: C \rightarrow \lim_{\leftarrow} G_n
$$

is a trivial cofibration.

**Proof.** We make extensive use of the material in appendix B.4 to show that the cofiber of

$$
f_{\infty}: C \rightarrow \lim_{\leftarrow} G_n
$$

is contractible. We focus on the category $S_0$ — the argument in $S_0$ is very similar. In this case, the cofiber is simply the quotient. We will consistently use the notation $H_n = \Sigma^{-1} \text{Cone}(\left[ H_n \right])$

First, note that the maps

$$
G_{n+1} \rightarrow G_n
$$

induce compatible maps

$$
L_V [H_{n+1}] \rightarrow L_V [H_n] \\
L_V H_{n+1} \rightarrow L_V H_n
$$
so proposition B.18 implies that
\[
\lim_{\leftarrow} G_n = (\lim_{\leftarrow} G_n) \otimes_{L_V(\lim_{\leftarrow} H_n)} (\lim_{\leftarrow} L_V H_n)
\]
and theorem B.2 implies that
\[
\lim_{\leftarrow} L_V [H_n] = L_V(\lim_{\leftarrow} [H_i])
\]
\[
\lim_{\leftarrow} L_V \tilde{H}_n = L_V(\lim_{\leftarrow} \tilde{H}_i) = L_V(\Sigma^{-1}\text{Cone}(\lim_{\leftarrow} [H_i]))
\]
from which we conclude
\[
\lim_{\leftarrow} G_n = (\lim_{\leftarrow} G_n) \otimes_{L_V(\lim_{\leftarrow} [H_i])} L_V(\Sigma^{-1}\text{Cone}(\lim_{\leftarrow} [H_i]))
\]

We claim that the projection

(4.7) \[ h: (\lim_{\leftarrow} G_n) \otimes_{L_V(\lim_{\leftarrow} [H_i])} L_V(\Sigma^{-1}\text{Cone}(\lim_{\leftarrow} [H_i])) \to \lim_{\leftarrow} G_n \]

is split by a coalgebra morphism. To see this, first note that, by proposition 4.15, each of the maps

\[ H_{n+1} \to H_n \]

is nullhomotopic via a nullhomotopy compatible with the maps in the inverse system \( \{H_n\} \). This implies that

\[ \lim_{\leftarrow} [H_n] \]

— the inverse limit of chain complexes — is contractible. It follows that the projection

\[ \Sigma^{-1}\text{Cone}(\lim_{\leftarrow} [H_n]) \to \lim_{\leftarrow} [H_n] \]

is a trivial fibration in Ch, hence split by a map

(4.8) \[ j: \lim_{\leftarrow} [H_n] \to \Sigma^{-1}\text{Cone}(\lim_{\leftarrow} [H_n]) \]

This, in turn, induces a coalgebra morphism

\[ L_V j : L_V(\lim_{\leftarrow} [H_n]) \to L_V(\Sigma^{-1}\text{Cone}(\lim_{\leftarrow} [H_n])) \]

splitting the canonical surjection

\[ L_V(\Sigma^{-1}\text{Cone}(\lim_{\leftarrow} [H_n])) \to L_V(\lim_{\leftarrow} [H_n]) \]

and induces a map, \( g \)

(4.9) \[ \lim_{\leftarrow} G_n = (\lim_{\leftarrow} G_n) \otimes_{L_V(\lim_{\leftarrow} [H_i])} L_V(\lim_{\leftarrow} [H_i]) \]
\[
\cong_{L_V j} (\lim_{\leftarrow} G_n) \otimes_{L_V(\lim_{\leftarrow} [H_i])} L_V(\Sigma^{-1}\text{Cone}(\lim_{\leftarrow} [H_i]))
\]

splitting the projection in formula 4.7. Since the image of \( f_\infty(C) \) vanishes in \( L_V(\lim_{\leftarrow} [H_i]) \), it is not hard to see that \( 1 \otimes L_V j \) is compatible with the inclusion of \( C \) in \( \lim_{\leftarrow} G_i \).
Now consider the diagram

\[
\begin{array}{c}
\lim G_n / f_\infty(C) \\
\downarrow q \\
\lim H_n
\end{array}
\]

where:

1. The map
   \[
   q : \left( \lim G_n \boxtimes^L \Sigma^{-1} \text{Cone}(\lim H_i) \right) / f_\infty(C) \\
   \rightarrow \left( \lim G_n / f_\infty(C) \boxtimes^L \Sigma^{-1} \text{Cone}(\lim H_i) \right)
   \]
   is induced by the projections

2. The equivalence
   \[
   \lim G_n / f_\infty(C) = \lim G_n / f_n(C)
   \]
   follows from theorem B.26.

3. The vertical map on the left is the identity map because \( g \) splits the map \( h \) in formula 4.7.

We claim that the map (projection to the left factor)

\[
[p] : \left[ \lim H_n \boxtimes^L \Sigma^{-1} \text{Cone}(\lim H_i) \right] \rightarrow \left[ \lim H_n \right]
\]

is nullhomotopic (as a Ch-morphism). This follows immediately from the fact that

\[
\lim H_n \hookrightarrow L_V(\lim H_i)
\]
by corollary \[B.25\] so that
\[
(\lim_{\leftarrow} H_n) \boxtimes_{L_V(\Sigma^{-1} \text{Cone}(\lim_{\leftarrow} H_i))} L_V(\Sigma^{-1} \text{Cone}(\lim_{\leftarrow} H_i))
\subseteq L_V(\lim_{\leftarrow} H_i) \boxtimes_{L_V(\Sigma^{-1} \text{Cone}(\lim_{\leftarrow} H_i))} L_V(\Sigma^{-1} \text{Cone}(\lim_{\leftarrow} H_i))
= L_V(\Sigma^{-1} \text{Cone}(\lim_{\leftarrow} H_i))
\]
and \(L_V(\Sigma^{-1} \text{Cone}(\lim_{\leftarrow} H_i))\) is contractible, by proposition \[4.9\] and the contractibility of \(\Sigma^{-1} \text{Cone}(\lim_{\leftarrow} H_i)\).

We conclude that
\[
(\lim_{\leftarrow} G_n)/f_{\infty}(C) \xrightarrow{\text{id}} (\lim_{\leftarrow} G_n)/f_{\infty}(C)
\]
is nullhomotopic so \((\lim_{\leftarrow} G_n)/f_{\infty}(C)\) is contractible and
\[
[f_{\infty}]: [C] \to [\lim_{\leftarrow} G_n]
\]
is a weak equivalence in \(\text{Ch}\), hence (by definition \[4.4\]) \(f_{\infty}\) is a weak equivalence. \(\square\)

**Corollary 4.17.** Let \(V = \{V(n)\}\) be a \(\Sigma\)-cofibrant operad such that \(V(n)\) is of finite type for all \(n \geq 0\). Let
\[
f: A \to B
\]
be a morphism in \(\mathcal{S}_0\) or \(\mathcal{S}_0\). Then there exists a functorial factorization of \(f\)
\[
A \to Z(f) \to B
\]
where
\[
A \to Z(f)
\]
is a trivial cofibration and
\[
Z(f) \to B
\]
is a fibration.

**Remark.** This is condition CM5, statement 1 in the definition of a model category at the beginning of this section. It, therefore, proves that the model structure described in \[4.4\] and \[4.5\] is well-defined.

By abuse of notation, we will call the \(\{f_i\}: \{A\} \to \{G_i\} \to L_V([A]) \boxtimes B \to B\) the rug-resolution of the morphism \(A \to B\) (see the proof below), where \(\{f_i\}: \{A\} \to \{G_i\} \to L_V([A]) \boxtimes B\) is the rug-resolution of the cofibration \(A \to L_V([A]) \boxtimes B\).

See proposition \[B.2\] and corollary \[B.21\] for the definition of inverse limit in the category \(\mathcal{C}\).

**Proof.** Simply apply definition \[4.14\] and lemma \[4.16\] to the cofibration
\[
A \to L_V([A]) \boxtimes B
\]
and project to the second factor. \(\square\)

We can characterize fibrations now:

**Corollary 4.18.** If \(V = \{V(n)\}\) is a \(\Sigma\)-cofibrant operad such that \(V(n)\) is of finite type for all \(n \geq 0\), then all fibrations are retracts of their rug-resolutions.

**Remark.** This shows that rug-resolutions of maps contain canonical fibrations and all others are retracts of them.
Proof. Suppose \( p: A \to B \) is some fibration. We apply corollary 4.17 to it to get a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i} & \bar{A} \\
\downarrow & & \downarrow \\
A & \xrightarrow{u_\infty} & B
\end{array}
\]

where \( i: A \to \bar{A} \) is a trivial cofibration and \( u_\infty: \bar{A} \to B \) is a fibration. We can complete this to get the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{p} & \bar{A} \\
\downarrow & & \downarrow \\
A & \xrightarrow{u_\infty} & B
\end{array}
\]

The fact that \( p: A \to B \) is a fibration and definition 4.5 imply the existence of the dotted arrow making the whole diagram commute. But this splits the inclusion \( i: A \to \bar{A} \) and implies the result. \( \square \)

The rest of this section will be spent on trivial fibrations — with a mind to proving the second statement in CM 4 in theorem 4.20. Recall that the first statement was a consequence of our definition of fibrations in \( \mathcal{I}_0 \) and \( \mathcal{S}_0 \).

4.4. Proof of CM 4. The first part of CM 4 is trivial: we have defined fibrations as morphisms that satisfy it — see definition 4.5. The proof of the second statement of CM 4 is more difficult and makes extensive use of the Rug Resolution defined in definition 4.14.

We begin by showing that a fibration of coalgebras becomes a fibration in Ch under the forgetful functor:

**Proposition 4.19.** Let \( p: A \to B \) be a fibration in \( \mathcal{C} = \mathcal{I}_0 \) or \( \mathcal{S}_0 \). Then

\[
[p]: [A] \to [B]
\]

is a fibration in \( \text{Ch}^+ \) or \( \text{Ch} \), respectively.

**Proof.** In the light of corollary 4.18 it suffices to prove this for rug-resolutions of fibrations.

Since they are iterated pullbacks of fibrations with contractible total spaces, it suffices to prove the result for something of the form

\[
A \boxtimes^{L_\mathcal{V}B} L_\mathcal{V}(\Sigma^{-1}\text{Cone}(B)) \to A
\]

where \( f: A \to L_\mathcal{V}B \) is some morphism. The fact that all morphisms are coalgebra morphisms implies the existence of a coalgebra structure on

\[
Z = [A] \oplus^{[L_\mathcal{V}B]} [L_\mathcal{V}(\Sigma^{-1}\text{Cone}(B))] \subset [A \boxtimes L_\mathcal{V}(\Sigma^{-1}\text{Cone}(B))] \Rightarrow [L_\mathcal{V}B]
\]

where \( [A] \oplus^{[L_\mathcal{V}B]} [L_\mathcal{V}(\Sigma^{-1}\text{Cone}(B))] \) is the fibered product in \( \text{Ch} \). Since \( L_\mathcal{V}(\Sigma^{-1}\text{Cone}(B)) \to L_\mathcal{V}B \) is surjective, (because it is induced by the surjection, \( \Sigma\text{Cone}(B) \to B \)) it follows that the equalizer

\[
[A \boxtimes L_\mathcal{V}(\Sigma^{-1}\text{Cone}(B))] \Rightarrow [L_\mathcal{V}B]
\]

surjects onto \([A]\). Since \( Z \) has a coalgebra structure, it is contained in the core,

\[
\langle [A \boxtimes L_\mathcal{V}(\Sigma^{-1}\text{Cone}(B))] \Rightarrow [L_\mathcal{V}B] \rangle = A \boxtimes^{L_\mathcal{V}B} L_\mathcal{V}(\Sigma^{-1}\text{Cone}(B))
\]
which also surjects onto \( A \) — so the projection
\[
[A \boxtimes_{L^B} L^B(\Sigma^{-1}\text{Cone}(B))] \rightarrow [A]
\]
is surjective and — as a map of graded \( R \)-modules — split. This is the definition of a fibration in \( \text{Ch} \).

We are now in a position to prove the second part of CM 4:

**Theorem 4.20.** Given a commutative solid arrow diagram

\[
\begin{array}{ccc}
U & \xrightarrow{f} & A \\
\downarrow{i} & & \downarrow{p} \\
W & \xrightarrow{g} & B
\end{array}
\]

where \( i \) is any cofibration and \( p \) is a trivial fibration, the dotted arrow exists.

**Proof.** Because of corollary 4.18, it suffices to prove the result for the rug-resolution of the trivial fibration \( p: A \rightarrow B \). We begin by considering the diagram

\[
\begin{array}{ccc}
[U] & \xrightarrow{[f]} & [A] \\
\downarrow{[i]} & & \downarrow{[p]} \\
[W] & \xrightarrow{[g]} & [B]
\end{array}
\]

Because of proposition 4.19 \([p]\) is a trivial fibration and the dotted arrow exists in \( \text{Ch} \).

If \( \alpha: A \rightarrow L^A \) is the classifying map of \( A \), \( \hat{\ell}: W \rightarrow L^A \) is induced by \( \ell: [W] \rightarrow [A] \), and \( p_2: L^A \boxtimes B \rightarrow B \) is projection to the second factor, we get a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{(\alpha \circ f) \boxtimes (p \circ f)} & L^A \boxtimes B \\
\downarrow{i} & & \downarrow{p_2} \\
W & \xrightarrow{\hat{\ell} \boxtimes g} & B
\end{array}
\]

It will be useful to build the rug-resolutions of \( A \rightarrow L^A \boxtimes B = G_0 \) and \( B \rightarrow L^B \boxtimes B = \tilde{G}_0 \) in parallel — denoted \( \{G_n\} \) and \( \{\tilde{G}_n\} \), respectively. Clearly the vertical morphisms in

\[
\begin{array}{ccc}
A & \xrightarrow{f_0} & L^A \boxtimes B \\
\downarrow{p} & & \downarrow{q_0} \\
B & \xrightarrow{f_0} & L^B \boxtimes B
\end{array}
\]

are trivial fibrations via strict homotopies — see propositions 4.19 and B.14.

We prove the result by an induction that:

1. lifts the map \( \hat{\ell}_1 = \hat{\ell} \boxtimes g: W \rightarrow L^A \boxtimes B \) to morphisms \( \hat{\ell}_k: W \rightarrow G_k \) to successively higher stages of the rug-resolution of \( p \). Diagram (4.11) implies the base case.
(2) establishes that the vertical morphisms in

\[
\begin{array}{ccc}
A & \xrightarrow{f_n} & G_n \\
p & & q_n \\
\downarrow & & \downarrow \\
B & \xrightarrow{f_n} & \tilde{G}_n
\end{array}
\]

are trivial fibrations for all \(n\), via strict homotopies.

Lemma 3.3 implies that we can find splitting maps \(u\) and \(v\) such that

\[
\begin{array}{ccc}
& [A] & \xrightarrow{f_n} & [G_n] \\
& \uparrow u & & \uparrow v \\
[B] & \xrightarrow{f_n} & \tilde{G}_n
\end{array}
\]

\(p \circ u = 1: [B] \to [B], \ q_n \circ v: [\tilde{G}_n] \to [\tilde{G}_n]\) and contracting homotopies \(\Phi_1\) and \(\Phi_2\) such that

\[
\begin{array}{ccc}
[\ast] \otimes I & \xrightarrow{f_n \otimes 1} & [G_n] \otimes I \\
\downarrow \Phi_1 & & \downarrow \Phi_2 \\
[\ast] & \xrightarrow{f_n} & [G_n]
\end{array}
\]

commutes, where \(d\Phi_1 = u \circ p - 1\), and \(d\Phi_2 = v \circ q_n - 1\) — where \(\Phi_1\) can be specified beforehand. Forming quotients gives rise to a commutative diagram

\[
\begin{array}{ccc}
[\ast] & \xrightarrow{f_n} & [G_n] \\
p & & q_n \\
\downarrow & & \downarrow \\
[B] & \xrightarrow{f_n} & \tilde{G}_n \\
\end{array} \quad \begin{array}{ccc}
& & \rightarrow H_n \\
& & \downarrow q_n \\
& & \tilde{H}_n
\end{array}
\]

Furthermore the commutativity of diagrams 4.12 and 4.13 implies that \(v\) induces a splitting map \(w: \tilde{H}_n \to H_n\) and \(\Phi_2\) induces a homotopy \(\Xi: H_n \otimes I \to H_n\) with \(d\Xi = v \circ q_n - 1\) — so \(\hat{q}_n\) is a weak equivalence in \(\text{Ch}\) — even a trivial fibration.

If we assume that the lifting has been carried out to the \(n\)th stage, we have a map

\[
\ell_n: W \to G_n
\]

making

\[
\begin{array}{ccc}
[W] & \xrightarrow{\ell_n} & [G_n] & \rightarrow H_n \\
g & & \downarrow q_n & & \downarrow \hat{q}_n \\
[B] & \xrightarrow{f_n} & [\tilde{G}_n] & \rightarrow \tilde{H}_n
\end{array}
\]

commute. Since the image of \(B\) in \(\tilde{H}_n\) vanishes (by the way \(H_n\) and \(\tilde{H}_n\) are constructed — see statement 2 in definition 4.14), it follows that the image of \(W\) in \(H_n\) lies in the kernel of \(\hat{q}_n\) — a trivial fibration in \(\text{Ch}\). We conclude that the
inclusion of $W$ in $H_n$ is null-homotopic, hence lifts to $\tilde{H}_n = \Sigma^{-1}\text{Cone}(H_n)$ in such a way that

$$
\begin{array}{ccc}
[W] & \xrightarrow{\ell_n} & [G_n] \\
\downarrow g & & \downarrow q_n \\
[B] & \xrightarrow{f_n} & [\tilde{G}_n] \\
\end{array}
\quad
\begin{array}{ccc}
\tilde{H}_n & \xrightarrow{t} & H_n \\
\downarrow & & \downarrow \tilde{q}_n \\
\tilde{G}_n & \xrightarrow{\ell_n} & \tilde{H}_n \\
\end{array}
$$

commutes — as a diagram of chain-complexes. Now note that $G_{n+1}$ is the fibered product $G_n \boxtimes L\Sigma^{-1}\text{Cone}(G_n/A)$ and that the chain-maps $r$ and $t \circ \ell_n$ induce coalgebra morphisms making the diagram

$$
\begin{array}{ccc}
G_{n+1} & \xrightarrow{t} & \tilde{H}_n \\
\downarrow p_{n+1} & & \downarrow \tilde{q}_n \\
G_n & \xrightarrow{\ell_n} & \tilde{H}_n \\
\end{array}
$$

commute — thereby inducing a coagebra-morphism

$$
\ell_{n+1}: W \to G_n \boxtimes L\Sigma^{-1}\text{Cone}(H_n) = G_{n+1}
$$

that makes the diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\ell_{n+1}} & G_{n+1} \\
\downarrow & & \downarrow \phi_{n+1} \\
W & \xrightarrow{\ell_n} & G_n \\
\end{array}
$$

commute (see statement 5 of definition 4.14). This proves assertion 1 in the induction step.

To prove assertion 2 in our induction hypothesis, note that the natural homotopy in diagram 4.13 induces (by passage to the quotient) a natural homotopy, $\Phi'$, that makes the diagram of chain-complexes

$$
\begin{array}{ccc}
[G_n] \otimes I & \xrightarrow{\ell \otimes 1} & H_n \otimes I \\
\downarrow \phi_2 & & \downarrow \Phi' \\
[G_n] & \xrightarrow{t} & H_n \\
\end{array}
$$

commute. This can be expanded to a commutative diagram

$$
\begin{array}{ccc}
[G_n] \otimes I & \xrightarrow{\ell \otimes 1} & H_n \otimes I & \xrightarrow{\Phi'} & \tilde{H}_n \otimes I \\
\downarrow \phi_2 & & \downarrow \Phi' & & \downarrow \Phi \\
[G_n] & \xrightarrow{t} & H_n & \xrightarrow{\Phi'} & \tilde{H}_n \\
\end{array}
$$
The conclusion follows from the fact that $\Phi'$ and $\bar{\Phi}$ induce strict homotopies (see definition 4.6) after the cofree coalgebra-functor is applied (see proposition 4.9) and proposition B.16.

We conclude that $G_{n+1} \to \bar{G}_{n+1}$ is a trivial fibration.

Induction shows that we can define a lifting

$$\iota_{\infty}: W \to \lim_{\leftarrow} G_n$$

that makes the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\iota \circ f} & \lim_{\leftarrow} G_n \\
\downarrow i & & \downarrow p_{\infty} \\
W & \xrightarrow{g} & B \\
\end{array}
\]

commute. □

4.5. The bounded case. In this section, we develop a model structure on a category of coalgebras whose underlying chain-complexes are bounded from below.

**Definition 4.21.** Let:

1. $\mathbf{Ch}_0$ denote the subcategory of $\mathbf{Ch}$ bounded at dimension 1. If $A \in \mathbf{Ch}_0$, then $A_i = 0$ for $i < 1$.
2. $\mathcal{S}_0^+$ denote the category of pointed irreducible coalgebras, $C$, over $\mathcal{V}$ such that $[C] \in \mathbf{Ch}_0$. This means that $C_i = 0$, $i < 1$. Note, from the definition of $[C]$ as the kernel of the augmentation map, that the underlying chain-complex of $C$ is equal to $R$ in dimension 0.

There is clearly an inclusion of categories

$$\iota: \mathbf{Ch}_0 \to \mathbf{Ch}$$

compatible with model structures.

Now we define our model structure on $\mathcal{S}_0^+$:

**Definition 4.22.** A morphism $f: A \to B$ in $\mathcal{S}_0^+$ will be called

1. a weak equivalence if $[f]: [A] \to [B]$ is a weak equivalence in $\mathbf{Ch}_0$ (i.e., a chain homotopy equivalence). An object $A$ will be called contractible if the augmentation map

   $$A \to R$$

   is a weak equivalence.

2. a cofibration if $[f]$ is a cofibration in $\mathbf{Ch}_0$.

3. a trivial cofibration if it is a weak equivalence and a cofibration.

**Remark.** A morphism is a cofibration if it is a degreewise split monomorphism of chain-complexes. Note that all objects of $\mathcal{S}_0^+$ are cofibrant.

If $R$ is a field, all modules are vector spaces therefore free. Homology equivalences of bounded free chain-complexes induce chain-homotopy equivalence, so our notion of weak equivalence becomes the same as homology equivalence (or quasi-isomorphism).

**Definition 4.23.** A morphism $f: A \to B$ in $\mathcal{S}_0^+$ will be called
(1) a fibration if the dotted arrow exists in every diagram of the form

\[
\begin{array}{ccc}
U & \xrightarrow{i} & A \\
\downarrow & & \downarrow \\
W & \xrightarrow{f} & B
\end{array}
\]

in which \(i: U \rightarrow W\) is a trivial cofibration.

(2) a trivial fibration if it is a fibration and a weak equivalence.

Corollary 4.24. If \(\mathcal{V} = \{\mathcal{V}(n)\}\) is an operad satisfying condition 4.2, the description of cofibrations, fibrations, and weak equivalences given in definitions 4.22 and 4.23 satisfy the axioms for a model structure on \(\mathcal{F}_0^+\).

Proof. We carry out all of the constructions of §4 and appendix B while consistently replacing cofree coalgebras by their truncated versions (see [22]). This involves substituting \(M_{\mathcal{V}}(\ast)\) for \(L_{\mathcal{V}}(\ast)\) and \(F_{\mathcal{V}}(\ast)\) for \(P_{\mathcal{V}}(\ast)\).

\[\square\]

5. Examples

We will give a few examples of the model structure developed here. In all cases, we will make the simplifying assumption that \(R\) is a field (this is not to say that interesting applications only occur when \(R\) is a field). We begin with coassociative coalgebras over the rationals:

Example 5.1. Let \(\mathcal{V}\) be the operad with component \(n\) equal to \(\mathbb{Q}S_n\) with the obvious \(S_n\)-action — and we consider the category of pointed, irreducible coagebras, \(\mathcal{F}_0\). Coalgebras over this \(\mathcal{V}\) are coassociative coalgebras. In this case \(P_{\mathcal{V}}C = T(C)\), the graded tensor algebra with coproduct

\[
c_1 \otimes \cdots \otimes c_n \mapsto \sum_{k=0}^{n} (c_1 \otimes \cdots \otimes c_k) \otimes (c_{k+1} \otimes \cdots \otimes c_n)
\]

where \(c_1 \otimes \cdots \otimes c_0 = c_{n+1} \otimes \cdots \otimes c_n = 1 \in C^0 = \mathbb{Q}\). The \(n\)-fold coproducts are just composites of this 2-fold coproduct and the “higher” coproducts vanish identically. We claim that this makes

\[
A \boxdot B = A \otimes B
\]

(5.2)

This is due to the well-known identity \(T([A] \oplus [B]) = T([A]) \otimes T([B])\). The category \(\mathcal{F}_0^+\) is a category of 1-connected coassociative coalgebras where weak equivalence is equivalent to homology equivalence.

If we assume coalgebras to be cocommutative we get:

Example 5.2. Suppose \(R = \mathbb{Q}\) and \(\mathcal{V}\) is the operad with all components equal to \(\mathbb{Q}\), concentrated in dimension 0, and equipped with trivial symmetric group actions. Coalgebras over \(\mathcal{V}\) are just cocommutative, coassociative coalgebras and \(\mathcal{F}_0^+\) is a category of 1-connected coassociative coalgebras similar to the one Quillen considered in [20]. Consequently, our model structure for \(\mathcal{F}_0^+\) induces the model structure defined by Quillen in [20] on the subcategory of 2-connected coalgebras.

In this case, \(P_{\mathcal{V}}C\) is defined by

\[
P_{\mathcal{V}}C = \bigoplus_{n \geq 0} \left(\mathbb{Q}^\otimes n\right)^{S_n}
\]
where \((C \otimes n)^{S_n}\) is the submodule of
\[
\underbrace{C \otimes \cdots \otimes C}_n \text{ factors}
\]
invariant under the \(S_n\)-action. The assumption that the base-ring is \(\mathbb{Q}\) implies a canonical isomorphism
\[
P_C = \bigoplus_{n \geq 0} (C \otimes n)^{S_n} \cong S(C)
\]

Since \(S([A] \oplus [B]) \cong S([A]) \otimes S([B])\), we again get \(A \boxtimes B = A \otimes B\).

Appendix A. Nearly free modules

In this section, we will explore the class of nearly free \(\mathbb{Z}\)-modules — see definition 2.1. We show that this is closed under the operations of taking direct sums, tensor products, countable products and cofree coalgebras. It appears to be fairly large, then, and it would be interesting to have a direct algebraic characterization.

Remark A.1. A module must be torsion-free (hence flat) to be nearly free. The converse is not true, however: \(\mathbb{Q}\) is flat but not nearly free.

The definition immediately implies that:

**Proposition A.2.** Any submodule of a nearly free module is nearly free.

Nearly free modules are closed under operations that preserve free modules:

**Proposition A.3.** Let \(M\) and \(N\) be \(\mathbb{Z}\)-modules. If they are nearly free, then so are \(M \oplus N\) and \(M \otimes N\).

Infinite direct sums of nearly free modules are nearly free.

**Proof.** If \(F \subseteq M \oplus N\) is countable, so are its projections to \(M\) and \(N\), which are free by hypothesis. It follows that \(F\) is a countable submodule of a free module.

The case where \(F \subseteq M \otimes N\) follows by a similar argument: The elements of \(F\) are finite linear combinations of monomials \(\{m_\alpha \otimes n_\alpha\}\) — the set of which is countable. Let
\[
A \subseteq M
\]
\[
B \subseteq N
\]
be the submodules generated, respectively, by the \(\{m_\alpha\}\) and \(\{n_\alpha\}\). These will be countable modules, hence \(\mathbb{Z}\)-free. It follows that
\[
F \subseteq A \otimes B
\]
is a free module.

Similar reasoning proves the last statement, using the fact that any direct sum of free modules is free.

**Proposition A.4.** Let \(\{F_n\}\) be a countable collection of \(\mathbb{Z}\)-free modules. Then
\[
\prod_{n=1}^\infty F_n
\]
is nearly free.
Proof. In the case where \( F_n = \mathbb{Z} \) for all \( n \)

\[
B = \prod_{n=1}^{\infty} \mathbb{Z}
\]

is the Baer-Specker group, which is well-known to be nearly free — see [1], [10] vol. 1, p. 94 Theorem 19.2], and [4]. It is also well-known not to be \( \mathbb{Z} \)-free — see [23] or the survey [7].

First suppose each of the \( F_n \) are countably generated. Then

\[
F_n \subseteq B
\]

and

\[
\prod F_n \subseteq \prod B = B
\]

which is nearly-free.

In the general case, any countable submodule, \( C \), of \( \prod F_n \) projects to a countably-generated submodule, \( A_n \), of \( F_n \) under all of the projections

\[
\prod F_n \to F_n
\]

and, so is contained in

\[
\prod A_n
\]

which is nearly free, so \( C \) must be \( \mathbb{Z} \)-free. \( \square \)

Corollary A.5. Let \( \{ N_k \} \) be a countable set of nearly free modules. Then

\[
\prod_{k=1}^{\infty} N_k
\]

is also nearly free.

Proof. Let

\[
F \subseteq \prod_{k=1}^{\infty} N_k
\]

be countable. If \( F_k \) is its projection to factor \( N_k \), then \( F_k \) will be countable, hence free. It follows that

\[
F \subseteq \prod_{k=1}^{\infty} F_k
\]

and the conclusion follows from proposition A.4. \( \square \)

Corollary A.6. Let \( A \) be nearly free and let \( F \) be \( \mathbb{Z} \)-free of countable rank. Then

\[
\text{Hom}_\mathbb{Z}(F, A)
\]

is nearly free.

Proof. This follows from corollary A.5 and the fact that

\[
\text{Hom}_\mathbb{Z}(F, A) \cong \prod_{k=1}^{\text{rank}(F)} A
\]

\( \square \)
Corollary A.7. Let \( \{F_n\} \) be a sequence of countably-generated \( ZS_n \)-projective modules and let \( A \) be nearly free. Then
\[
\prod_{n=1}^{\infty} \text{Hom}_{ZS_n}(F_n, A^\otimes n)
\]
is nearly free.

Proof. This is a direct application of the results of this section and the fact that
\[
\text{Hom}_{ZS_n}(F_n, A^\otimes n) \subseteq \text{Hom}_{Z}(F_n, A^\otimes n) \subseteq \text{Hom}_{Z}({\hat{F}}_n, A^\otimes n)
\]
where \( {\hat{F}}_n \) is a \( ZS_n \)-free module of which \( F_n \) is a direct summand. \( \square \)

Theorem A.8. Let \( C \) be a nearly free \( Z \)-module and let \( \mathcal{V} = \{\mathcal{V}(n)\} \) be a \( \Sigma \)-finite operad with \( \mathcal{V}(n) \) of finite type for all \( n \geq 0 \). Then
\[
[LV C], [MV C], [PV C], [\mathcal{F}_C]
\]
are all nearly free.

Proof. This follows from theorem B.7 which states that all of these are submodules of
\[
\prod_{n \geq 0} (\mathcal{V}(n), A^\otimes n)
\]
and the fact that near-freeness is inherited by submodules. \( \square \)

Appendix B. Category-Theoretic Constructions

In this section, we will study general properties of coalgebras over an operad. Some of the results will require coalgebras to be pointed irreducible. We begin by recalling the structure of cofree coalgebras over operads in the pointed irreducible case.

B.1. Cofree-coalgebras. We will make extensive use of cofree coalgebras over an operad in this section — see definition 2.17.

If they exist, it is not hard to see that cofree coalgebras must be unique up to an isomorphism.

The paper \[22\] gave an explicit construction of \( LV C \) when \( C \) was an \( R \)-free chain complex. When \( R \) is a field, all chain-complexes are \( R \)-free, so the results of the present paper are already true in that case.

Consequently, we will restrict ourselves to the case where \( R = Z \).

Proposition B.1. The forgetful functor (defined in definition 2.15) and cofree coalgebra functors define adjoint pairs
\[
P_\mathcal{V}(\ast) : \text{Ch} \leftrightarrows \mathcal{A}_0 : [\ast]
\]
\[
L_\mathcal{V}(\ast) : \text{Ch} \leftrightarrows \mathcal{A}_0 : [\ast]
\]

Remark. The adjointness of the functors follows from the universal property of cofree coalgebras — see \[22\].

The Adjoint and Limits Theorem in \[10\] implies that:
**Theorem B.2.** If \( \{A_i\} \in \text{ind} - \text{Ch} \) and \( \{C_i\} \in \text{ind} - \mathcal{I}_0 \) or \( \text{ind} - \mathcal{S}_0 \) then
\[
\lim_{\leftarrow} P_V(A_i) = P_V(\lim_{\leftarrow} A_i) \\
\lim_{\leftarrow} L_V(A_i) = L_V(\lim_{\leftarrow} A_i) \\
\big[\lim_{\rightarrow} C_i\big] = \lim_{\rightarrow} \big[ C_i \big]
\]

**Remark.** This implies that colimits in \( \mathcal{I}_0 \) or \( \mathcal{S}_0 \) are the same as colimits of underlying chain-complexes.

**Proposition B.3.** If \( C \in \text{Ch} \), let \( G(C) \) denote the lattice of countable subcomplexes of \( C \). Then
\[
C = \lim_{\rightarrow} G(C)
\]

**Proof.** Clearly \( \lim_{\rightarrow} G(C) \subseteq C \) since all of the canonical maps to \( C \) are inclusions. Equality follows from every element \( x \in C_k \) being contained in a finitely generated subcomplex, \( C_x \), defined by
\[
(C_x)_i = \begin{cases} 
R \cdot x & \text{if } i = k \\
R \cdot \partial x & \text{if } i = k - 1 \\
0 & \text{otherwise}
\end{cases}
\]

**Lemma B.4.** Let \( n > 1 \) be an integer, let \( F \) be a finitely-generated projective (non-graded) \( RS_n \)-module, and let \( \{C_\alpha\} \) a direct system of modules. Then the natural map
\[
\lim_{\rightarrow} \text{Hom}_{RS_n}(F, C_\alpha) \rightarrow \text{Hom}_{RS_n}(F, \lim_{\rightarrow} C_\alpha)
\]
is an isomorphism.

If \( F \) and the \( \{C_\alpha\} \) are graded, the corresponding statement is true if \( F \) is finitely-generated and \( RS_n \)-projective in each dimension.

**Proof.** We will only prove the non-graded case. The graded case follows from the fact that the maps of the \( \{C_\alpha\} \) preserve grade.

In the non-graded case, finite generation of \( F \) implies that the natural map
\[
\bigoplus_{\alpha} \text{Hom}_{RS_n}(F, C_\alpha) \rightarrow \text{Hom}_{RS_n}(F, \bigoplus_{\alpha} C_\alpha)
\]
is an isomorphism, where \( \alpha \) runs over any indexing set. The projectivity of \( F \) implies that \( \text{Hom}_{RS_n}(F, \ast) \) is exact, so the short exact sequence defining the filtered colimit is preserved.

**Proposition B.5.** Let \( V = \{V(n)\} \) be an operad satisfying condition 4.2 and let \( C \) be a chain-complex with \( \mathcal{G}(C) = \{ C_\alpha \} \) a family of flat subcomplexes ordered by inclusion that is closed under countable sums. In addition, suppose
\[
C = \lim_{\rightarrow} C_\alpha
\]

Then
\[
\prod_{n \geq 0} \text{Hom}_{RS_n}(V(n), C^{\otimes n}) = \lim_{\rightarrow} \prod_{n \geq 0} \text{Hom}_{RS_n}(V(n), C^{\otimes n})
\]
Proof. Note that $C$, as the limit of flat modules, it itself flat.

The $\mathbb{Z}$-flatness of $C$ implies that any $y \in C^\otimes n$ is in the image of

$$C^\otimes n \hookrightarrow C^\otimes n$$

for some $C_\alpha \in \mathcal{G}(C)$ and any $n \geq 0$. The finite generation and projectivity of the \{\mathcal{V}(n)\} in every dimension implies that any map

$$x_i \in \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)_j$$

lies in the image of

$$\text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n) \hookrightarrow \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)$$

for some $C_{\alpha_i} \in \mathcal{G}(C)$. This implies that

$$x \in \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)$$

lies in the image of

$$\text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n) \hookrightarrow \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)$$

where $C_\alpha = \sum_{i=0}^\infty C_{\alpha_i}$, which is still a subcomplex of the lattice $\mathcal{G}(C)$.

If

$$x = \prod_{n \geq 0} x_n \in \prod_{n \geq 0} \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)$$

then each $x_n$ lies in the image of

$$\text{Hom}_{RS_n}(\mathcal{V}(n), C_{\alpha_n}^\otimes n) \hookrightarrow \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)$$

where $C_{\alpha_n} \in \mathcal{G}(C)$ and $x$ lies in the image of

$$\prod_{n \geq 0} \text{Hom}_{RS_n}(\mathcal{V}(n), C_{\alpha_n}^\otimes n) \hookrightarrow \prod_{n \geq 0} \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)$$

where $C_\alpha = \sum_{n \geq 0} C_{\alpha_n}$ is countable.

The upshot is that

$$\prod_{n \geq 0} \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n) = \lim_{n \geq 0} \prod_{n \geq 0} \text{Hom}_{RS_n}(\mathcal{V}(n), C_{\alpha_n}^\otimes n)$$

as $C_\alpha$ runs over all subcomplexes of the lattice $\mathcal{G}(C)$. □

**Theorem B.6.** Let $\mathcal{V} = \{\mathcal{V}(n)\}$ be an operad satisfying condition 4.2.

If $C$ is a $\mathcal{V}$-coalgebra whose underlying chain-complex is nearly free, then

$$C = \lim_{\to} C_\alpha$$

where $\{C_\alpha\}$ ranges over all the countable sub-coalgebras of $C$.

**Proof.** To prove the statement, we show that every

$$x \in C$$

is contained in a countable sub-coalgebra of $C$.

Let

$$a: C \to \prod_{n \geq 0} \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)$$

be the adjoint structure-map of $C$, and let $x \in C_1$, where $C_1$ is a countable sub-chain-complex of $[C]$.

Then $a(C_1)$ is a countable subset of $\prod_{n \geq 0} \text{Hom}_{RS_n}(\mathcal{V}(n), C^\otimes n)$, each element of which is defined by its value on the countable set of $RS_n$-projective generators of
\{V_n\} for all \(n > 0\). It follows that the targets of these projective generators are a countable set of elements
\[\{x_j \in C^{\otimes n}\}\]
for \(n > 0\). If we enumerate all of the \(c_{i,j}\) in \(x_j = c_{1,j} \otimes \cdots \otimes c_{n,j}\), we still get a countable set. Let
\[C_2 = C_1 + \sum_{i,j} R \cdot c_{i,j}\]
This will be a countable sub-chain-complex of \([C]\) that contains \(x\). By an easy induction, we can continue this process, getting a sequence \(\{C_n\}\) of countable sub-chain-complexes of \([C]\) with the property
\[a(C_i) \subseteq \prod_{n \geq 0} \text{Hom}_{RS_n}(V(n), C^{\otimes n}_{i+1})\]
arriving at a countable sub-chain-complex of \([C]\)
\[C_\infty = \bigcup_{i=1}^\infty C_i\]
that is closed under the coproduct of \(C\). It is not hard to see that the induced coproduct on \(C_\infty\) will inherit the identities that make it a \(V\)-coalgebra. □

**Corollary B.7.** Let \(V = \{V(n)\}\) be a \(\Sigma\)-cofibrant operad such that \(V(n)\) is of finite type for all \(n \geq 0\). If \(C\) is nearly-free, then the cofree coalgebras
\[L_V C, P_V C, M_V C, F_V C\]
are well-defined and
\[L_V C = \lim_{\longrightarrow} L_V C_\alpha\]
\[P_V C = \lim_{\longrightarrow} P_V C_\alpha\]
\[M_V C = \lim_{\longrightarrow} M_V C_\alpha\]
\[F_V C = \lim_{\longrightarrow} F_V C_\alpha\]
where \(C_\alpha\) ranges over the countable sub-chain-complexes of \(C\).

**Proof.** The near-freeness of \(C\) implies that the \(C_\alpha\) are all \(\mathbb{Z}\)-free when \(R = \mathbb{Z}\), so the construction in [22] gives cofree coalgebras \(L_V C_\alpha\).

Since (by theorem B.6)
\[C = \lim_{\longrightarrow} C_\alpha\]
where \(C_\alpha\) ranges over countable sub-coalgebras of \(C\), we get coalgebra morphisms
\[b_\alpha: C_\alpha \to L_V [C_\alpha]\]
inducing a coalgebra morphism
\[b: C \to \lim_{\longrightarrow} L_V [C_\alpha]\]

We claim that \(L_V [C] = \lim_{\longrightarrow} L_V [C_\alpha]\). We first note that \(\lim_{\longrightarrow} L_V [C_\alpha]\) depends only on \([C]\) and not on \(C\). If \(D\) is a \(V\)-coalgebra with \([C] = [D]\) then, by theorem B.6
\[D = \lim_{\longrightarrow} D_\beta\]
where the \(D_\beta\) are the countable sub-coalgebras of \(D\).

We also know that, in the poset of sub-chain-complexes of \([C] = [D]\), \(\{[C_\alpha]\}\) and \(\{[D_\beta]\}\) are both cofinal. This implies the cofinality of \(\{L_V [C_\alpha]\}\) and \(\{L_V [D_\beta]\}\), hence
\[\lim_{\longrightarrow} L_V [C_\alpha] = \lim_{\longrightarrow} L_V [D_\beta]\]
This unique $\mathcal{V}$-coalgebra has all the categorical properties of the cofree coalgebra $L_\mathcal{V}[C]$ which proves the first part of the result. The statement that

$$L_\mathcal{V}[C] \subseteq \prod_{n \geq 0} \text{Hom}_{\mathcal{RS}_n}(\mathcal{V}(n), C \otimes n)$$

follows from

1. The canonical inclusion

$$L_\mathcal{V}C_{\alpha} \subseteq \prod_{n \geq 0} \text{Hom}_{\mathcal{RS}_n}(\mathcal{V}(n), C_{\alpha} \otimes n)$$

in [22], and

2. the fact that the hypotheses imply that

$$\prod_{n \geq 0} \text{Hom}_{\mathcal{RS}_n}(\mathcal{V}(n), C_{\alpha} \otimes n) = \lim_{\longrightarrow} \prod_{n \geq 0} \text{Hom}_{\mathcal{RS}_n}(\mathcal{V}(n), C_{\alpha} \otimes n)$$

— see proposition B.5.

Similar reasoning applies to $P_\mathcal{V}C$, $M_\mathcal{V}C$, $F_\mathcal{V}C$.

B.2. Core of a module.

Lemma B.8. Let $A, B \subseteq C$ be sub-coalgebras of $C \in \mathcal{C} = \mathcal{S}_0$ or $\mathcal{I}_0$. Then $A + B \subseteq C$ is also a sub-coalgebra of $C$.

In particular, given any sub-DG-module $M \subseteq [C]$ there exists a maximal sub-coalgebra $\langle M \rangle$ — called the core of $M$ — with the universal property that any sub-coalgebra $A \subseteq C$ with $[A] \subseteq M$ is a sub-coalgebra of $\langle M \rangle$.

This is given by

$$\alpha(\langle M \rangle) = \alpha(C) \cap P_\mathcal{V}M \subseteq P_\mathcal{V}C$$

where

$$\alpha: C \to P_\mathcal{V}C$$

is the classifying morphism of $C$.

Proof. The first claim is clear — $A + B$ is clearly closed under the coproduct structure. This implies the second claim because we can always form the sum of any set of sub-coalgebras contained in $M$.

The second claim follows from:

The fact that

$$\langle M \rangle = \alpha^{-1}(\alpha(C) \cap P_\mathcal{V}M)$$

implies that it is the inverse image of a coalgebra (the intersection of two coalgebras) under an injective map ($\alpha$), so it is a subcoalgebra of $C$ with $[\langle M \rangle] \subseteq M$.

Given any subcoalgebra $A \subseteq C$ with $[A] \subseteq M$, the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & C \\
\downarrow \alpha & & \downarrow e \\
C & \xrightarrow{\epsilon} & C
\end{array}$$

implies that $\alpha$ is a well-defined surjective map. Therefore, $\alpha$ is surjective and $\alpha(C) \cap P_\mathcal{V}M$.
where \( \epsilon: P\gamma C \to C \) is the cogeneration map, implies that
\[
\alpha(A) \subseteq P\gamma M \subseteq \epsilon(P\gamma M)
\]
which implies that \( A \subseteq \langle M \rangle \), so \( \langle M \rangle \) has the required universal property. \( \square \)

**Corollary B.9.** Let \( C \in \mathcal{C} = \mathcal{S}_0 \) or \( \mathcal{S}_0 \) and \( M \subseteq [C] \) a sub-DG-module and suppose
\[
\Phi: C \otimes I \to C
\]
is a coalgebra morphism with the property that \( \Phi(M \otimes I) \subseteq M \). Then
\[
\Phi(\langle M \rangle \otimes I) \subseteq \langle M \rangle
\]
\[\text{Proof.}\] The hypotheses imply that the diagrams
\[
\begin{array}{c}
C \otimes I \xrightarrow{\alpha \otimes 1} (L\gamma C) \otimes I \\
| \downarrow \Phi \downarrow \downarrow L\gamma \Phi \\
C \otimes I \xrightarrow{\alpha} L\gamma C
\end{array}
\]
and
\[
\begin{array}{c}
(L\gamma M) \otimes I \xrightarrow{L\gamma(\Phi|M \otimes I)} (L\gamma C) \otimes I \\
| \downarrow L\gamma \Phi \downarrow \downarrow L\gamma \Phi \\
L\gamma M \xrightarrow{L\gamma \Phi} L\gamma C
\end{array}
\]
commute. Lemma B.8 implies the result. \( \square \)

This allows us to construct equalizers in categories of coalgebras over operads:

**Corollary B.10.** If
\[
f_i: A \to B
\]
with \( i \) running over some index set, is a set of morphisms in \( \mathcal{C} = \mathcal{S}_0 \) or \( \mathcal{S}_0 \), then the equalizer of the \( \{f_i\} \) is
\[
\langle M \rangle \subseteq A
\]
where \( M \) is the equalizer of \( [f_i]: [A] \to [B] \) in \( \text{Ch} \).

**Remark.** Roughly speaking, it is easy to construct coequalizers of coalgebra morphisms and hard to construct equalizers — since the kernel of a morphism is not necessarily a sub-coalgebra. This is dual to what holds for algebras over operads.

**Proof.** Clearly \( f_i|\langle M \rangle = f_j|\langle M \rangle \) for all \( i, j \). On the other hand, any sub-DG-algebra with this property is contained in \( \langle M \rangle \) so the conclusion follows. \( \square \)
Proposition B.11. Let \( C \in \mathcal{I}_0 \) and let \( \{A_i\}, i \geq 0 \) be a descending sequence of sub-chain-complexes of \( [C] \) — i.e., \( A_{i+1} \subseteq A_i \) for all \( i \geq 0 \). Then
\[
\left( \bigcap_{i=0}^{\infty} A_i \right) = \bigcap_{i=0}^{\infty} \langle A_i \rangle
\]

**Proof.** Clearly, any intersection of coalgebras is a coalgebra, so
\[
\bigcap_{i=0}^{\infty} \langle A_i \rangle \subseteq \left( \bigcap_{i=0}^{\infty} A_i \right)
\]
On the other hand
\[
\left[ \left( \bigcap_{i=0}^{\infty} A_i \right) \right] \subseteq \bigcap_{i=0}^{\infty} A_i \subseteq A_n
\]
for any \( n > 0 \). Since \( \langle \bigcap_{i=0}^{\infty} A_i \rangle \) is a coalgebra whose underlying chain complex is contained in \( A_n \), we must actually have
\[
\left[ \left( \bigcap_{i=0}^{\infty} A_i \right) \right] \subseteq [\langle A_n \rangle]
\]
which implies that
\[
\left( \bigcap_{i=0}^{\infty} A_i \right) \subseteq \bigcap_{i=0}^{\infty} \langle A_i \rangle
\]
and the conclusion follows. \( \square \)

Definition B.12. Let \( A \) and \( B \) be objects of \( \mathcal{C} = \mathcal{I}_0 \) or \( \mathcal{I}_0 \) and define \( A \vee B \) to be the push out in the diagram
\[
\begin{array}{c}
\bullet \\
\downarrow \\
A \\
\end{array}
\quad
\begin{array}{c}
\rightarrow \\
\downarrow \\
A \vee B
\end{array}
\]
where \( \bullet \) denotes the initial object in \( \mathcal{C} \) — see definition [2.13].

B.3. **Categorical products.** We can use cofree coalgebras to explicitly construct the categorical product in \( \mathcal{I}_0 \) or \( \mathcal{I}_0 \):

**Definition B.13.** Let \( A_i, i = 0, 1 \) be objects of \( \mathcal{C} = \mathcal{I}_0 \) or \( \mathcal{I}_0 \). Then
\[
A_0 \boxtimes A_1 = \langle M_0 \cap M_1 \rangle \subseteq Z = \begin{cases} 
L_V([A_0] \oplus [A_1]) & \text{if } \mathcal{C} = \mathcal{I}_0 \\
P_V([A_0] \oplus [A_1]) & \text{if } \mathcal{C} = \mathcal{I}_0
\end{cases}
\]
where
\[
M_i = p_i^{-1}(\text{im } A_i)
\]
under the projections
\[
p_i: Z \rightarrow \begin{cases} 
L_V[A_i] & \text{if } \mathcal{C} = \mathcal{I}_0 \\
P_V([A_i]) & \text{if } \mathcal{C} = \mathcal{I}_0
\end{cases}
\]
induced by the canonical maps \( [A_0] \oplus [A_1] \rightarrow [A_i] \). The \( \text{im } A_i \) are images under the canonical morphisms
\[
A_i \rightarrow \begin{cases} 
L_V[A_i] & \text{if } \mathcal{C} = \mathcal{I}_0 \\
P_V([A_i]) & \text{if } \mathcal{C} = \mathcal{I}_0
\end{cases}
\rightarrow Z
\]
classifying coalgebra structures — see definition 2.17.

**Remark.** By identifying the $A_i$ with their canonical images in $Z$, we get canonical projections to the factors

$$A_0 \boxtimes A_1 \to A_i$$

**Proposition B.14.** Let $F: A \otimes I \to A$ and $G: B \otimes I \to B$ be strict homotopies in $C = \mathcal{F}_0$ or $\mathcal{S}_0$ (see definition 4.6). Then there is a strict homotopy

$$(A \boxtimes B) \otimes I \xrightarrow{F \boxtimes G} A \boxtimes B$$

that makes the diagrams

\[
\begin{array}{ccc}
(A \boxtimes B) \otimes I & \xrightarrow{F \boxtimes G} & A \boxtimes B \\
\downarrow & & \downarrow \\
A \otimes I & \xrightarrow{F} & A \\
\end{array}
\]

and

\[
\begin{array}{ccc}
(A \boxtimes B) \otimes I & \xrightarrow{F \boxtimes G} & A \boxtimes B \\
\downarrow & & \downarrow \\
B \otimes I & \xrightarrow{G} & B \\
\end{array}
\]

commute. If

$$f_1, f_2: A \to A'$$

$$g_1, g_2: B \to B'$$

are strictly homotopic morphisms with respective strict homotopies

$$F: A \otimes I \to A'$$

$$G: B \otimes I \to B'$$

then $F \boxtimes G$ is a strict homotopy between $f_1 \boxtimes g_1$ and $f_2 \boxtimes g_2$.

Consequently, if $f: A \to A'$ and $g: B \to B'$ are strict equivalences, then

$$f \boxtimes g: A \boxtimes B \to A' \boxtimes B'$$

is a strict equivalence.

**Proof.** The projections

$$A \boxtimes B \to B$$

$$\downarrow$$

$$A$$

induce projections

$$\begin{array}{ccc}
(A \boxtimes B) \otimes I & \to & B \otimes I \\
\downarrow & & \downarrow \\
A \otimes I & & A \otimes I \\
\end{array}
$$

and the composite

$$(A \boxtimes B) \otimes I \to (A \otimes I) \boxtimes (B \otimes I) \xrightarrow{F \boxtimes G} A \boxtimes B$$
satisfies the first part of the statement.

Note that diagrams B.1 and B.2 — and the fact that maps to $A' \boxtimes B'$ are uniquely determined by their composites with the projections $A' \boxtimes B' \Rightarrow A', B'$ (the defining universal property of $\boxtimes$) — implies that $F \boxtimes G$ is a strict homotopy between $f_1 \boxtimes g_1$ and $f_2 \boxtimes g_2$. The final statement is also clear. □

In like fashion, we can define categorical fibered products of coalgebras:

**Definition B.15.** Let

$$
\begin{array}{ccc}
F & \rightarrow & A \\
i & \downarrow & f \\
B & \rightarrow & C
\end{array}
$$

be a diagram in $\mathcal{S}_0$ or $\mathcal{I}_0$. Then the fibered product with respect to this diagram, $A \boxtimes C B$, is defined to be the equalizer

$$
F \rightarrow A \boxtimes B \Rightarrow C
$$

by the maps induced by the projections $A \boxtimes B \rightarrow A$ and $A \boxtimes B \rightarrow B$ composed with the maps in the diagram.

We have an analogue to proposition B.14:

**Proposition B.16.** Let $A \rightarrow B \leftarrow C$, $A' \rightarrow B' \leftarrow C'$ be diagrams in $\mathcal{C} = \mathcal{S}_0$ or $\mathcal{I}_0$ and let

$$
\begin{array}{ccc}
A \otimes I & \stackrel{f \otimes 1}{\rightarrow} & B \otimes I & \leftarrow & C \otimes I \\
H_A & \downarrow & & \downarrow & H_B \\
A' & \stackrel{f'}{\rightarrow} & B' & \leftarrow & C
\end{array}
$$

commute, where the $H_\alpha$ are strict homotopies. Then there exists a strict homotopy

$$(A \boxtimes \boxtimes C) \otimes I \xrightarrow{H_A \boxtimes H_B \boxtimes H_C} A' \boxtimes B' \boxtimes C'$$

between the morphisms

$$(H_A|A \otimes p_i) \boxtimes (H_C \otimes p_i): A \boxtimes B \boxtimes C \rightarrow A' \boxtimes B' \boxtimes C'$$

for $i = 0, 1$.

**Proof.** The morphism $H_A \boxtimes H_B \boxtimes H_C$ is constructed exactly as in proposition B.14. The conclusion follows by the same reasoning used to prove the final statement of that result. □

**Proposition B.17.** Let $U, V$ and $W$ be objects of $\text{Ch}$ and let $Z$ be the fibered product of

$$
\begin{array}{ccc}
V & \rightarrow & W \\
g & \downarrow & \\
U & \rightarrow & W
\end{array}
$$

in $\text{Ch}$ — i.e., $W$ is the equalizer

$$
Z \rightarrow U \oplus V \Rightarrow W
$$
in Ch. Then \( P_VZ \) is the fibered product of

\[
\begin{array}{ccc}
P_VU & \xrightarrow{P_Vf} & P_VW \\
p_Vg & & \\
\end{array}
\]

in \( \mathcal{I}_0 \) and \( L_VZ \) is the fibered product of

\[
\begin{array}{ccc}
L_VU & \xrightarrow{L_Vf} & L_VW \\
L_Vg & & \\
\end{array}
\]

in \( \mathcal{I}_0 \).

**Proof.** We prove this in the pointed irreducible case. The other case follows by an analogous argument.

The universal properties of cofree coalgebras imply that \( P_V(U \oplus V) = P_VU \boxtimes P_VV \). Suppose \( F \) is the fibered product of diagram B.3. Then

\[
P_VZ \subseteq F
\]

On the other hand, the composite

\[
F \to P_VU \boxtimes P_VV = P_V(U \oplus V) \to U \oplus V
\]

where the rightmost map is the co-generating map, has composites with \( f \) and \( g \) that are equal to each other — so it lies in \( Z \subseteq U \oplus V \). This induces a *unique* coalgebra morphism

\[
j: F \to P_VZ
\]

left-inverse to the inclusion

\[
i: P_VZ \subseteq F
\]

The uniqueness of induced maps to cofree coalgebras implies that \( j \circ i = i \circ j = 1 \). \( \square \)

**B.4. Limits and colimits.** We can use cofree coalgebras and adjointness to the forgetful functors to define categorical limits and colimits in \( \mathcal{I}_0 \) and \( \mathcal{S}_0 \).

Categorical reasoning implies that

**Proposition B.18.** Let

\[
\begin{array}{ccc}
\{B_i\} & \xrightarrow{b_i} & \{C_i\} \\
\{A_i\} & \xrightarrow{a_i} & \{C_i\}
\end{array}
\]

be a diagram in \( \text{pro-} \mathcal{I}_0 \) or \( \text{pro-} \mathcal{S}_0 \). Then

\[
\lim \left( A_i \boxtimes C_i, B_i \right) = \left( \lim A_i \right) \boxtimes \left( \lim C_i \right) \left( \lim B_i \right)
\]

See definition B.13 for the fibered product notation.

Theorem B.2 implies that colimits in \( \mathcal{I}_0 \) or \( \mathcal{S}_0 \) are the same as colimits of underlying chain-complexes. The corresponding statement for limits is not true except in a special case:
Proposition B.19. Let \( \{C_i\} \in \text{pro-}\mathcal{I}_0 \) or \( \text{pro-}\mathcal{I}_0 \) and suppose that all of its morphisms are injective. Then
\[
\varprojlim C_i = \varprojlim [C_i]
\]

Remark. In this case, the limit is an intersection of coalgebras. This result says that to get the limit of \( \{C_i\} \), one

1. forms the limit of the underlying chain-complexes (i.e., the intersection) and
2. equips that with the coalgebra structure in induced by its inclusion into any of the \( C_i \).

That this constructs the limit follows from the uniqueness of limits.

Definition B.20. Let \( A = \{A_i\} \in \text{pro-}\mathcal{I}_0 \). Then define the normalization of \( A \), denoted \( \hat{A} = \{\hat{A}_i\} \), as follows:

1. Let \( V = P_V(\varprojlim [A_i]) \) with canonical maps
\[
q_n : P_V(\varprojlim [A_i]) \rightarrow P_V([A_n])
\]
for all \( n > 0 \).
2. Let \( f_n : A_n \rightarrow P_V([A_n]) \) be the coalgebra classifying map — see definition 2.17.

Then \( \hat{A}_n = \langle q_n^{-1}(f_n(A_n)) \rangle \), and \( \hat{A}_{n+1} \subseteq \hat{A}_n \) for all \( n > 0 \). Define \( \hat{A} = \{\hat{A}_n\} \), with the injective structure maps defined by inclusion.

If \( A = \{A_i\} \in \text{pro-}\mathcal{I}_0 \) then the corresponding construction holds, where we consistently replace \( P_V(\ast) \) by \( L_V(\ast) \).

Normalization reduces the general case to the case dealt with in proposition B.19.

Corollary B.21. Let \( C = \{g_i : C_i \rightarrow C_{i-1}\} \in \text{pro-}\mathcal{I}_0 \) or \( \text{pro-}\mathcal{I}_0 \). Then
\[
\varprojlim C_i = \varprojlim \hat{C}_i
\]
where \( \{\hat{C}_i\} \) is the normalization of \( \{C_i\} \). In particular, if \( C \) is in \( \mathcal{I}_0 \)
\[
\varprojlim C_i = \left\langle \bigcap_{i=0}^{\infty} [p_i]^{-1}[\alpha_i](\lim [C_i]) \right\rangle \subseteq P_V(\varprojlim [C_i])
\]
where \( p_i : P_V(\varprojlim [C_i]) \rightarrow P_V([C_i]) \) and \( \alpha_n : C_n \rightarrow P_V([C_i]) \) are as in definition B.20, and the corresponding statement holds if \( C \) is in \( \text{pro-}\mathcal{I}_0 \) with \( P_V(\ast) \) replaced by \( L_V(\ast) \).

Proof. Assume the notation of definition B.20. Let
\[
f_i : C_i \rightarrow \left\{ \begin{array}{c} P_V([C_i]) \\ L_V[C_i] \end{array} \right\}
\]
be the classifying maps in \( \mathcal{I}_0 \) or \( \mathcal{I}_0 \), respectively — see definition 2.17. We deal with the case of the category \( \mathcal{I}_0 \) — the other case is entirely analogous. Let
\[
q_n : P_V(\varprojlim [C_i]) \rightarrow P_V([C_n])
\]
be induced by the canonical maps \( \lim [C_i] \rightarrow [C_n] \).

We verify that
\[
X = \left\langle \bigcap_{i=0}^{\infty} [q_i]^{-1}[f_i]([C_i]) \right\rangle = \varprojlim \hat{C}_i
\]
has the category-theoretic properties of an inverse limit. We must have morphisms
\[ p_i: X \rightarrow C_i \]
making the diagrams
\[ \xymatrix{ X \ar[r]^{p_i} & C_i \\ C_{i-1} \ar[u]^{g_i} \ar[r]_{p_{i-1}} & C_i } \]
commute for all \( i > 0 \). Define \( p_i = f_i^{-1} \circ q_i: X \rightarrow C_i \) — using the fact that the classifying maps \( f_i: C_i \rightarrow \text{PV}[C_i] \) are always injective (see \[22\] and the definition \[2.17\]). The commutative diagrams
\[ \xymatrix{ C_i \ar[r]^{\alpha_i} & \text{PV}[C_i] \\ C_{i-1} \ar[u]^{g_i} \ar[r]_{\alpha_{i-1}} & \text{PV}[C_{i-1}] } \]
and
\[ \lim_{\leftarrow} \text{PV}[C_i] \ar[r]^{p_i} & \text{PV}[C_i] \\ \ar[r]_{p_{i-1}} & \text{PV}[C_{i-1}] } \]
together imply the commutativity of the diagram with the diagrams \[\text{B.5}\]. Consequently, \( X \) is a candidate for being the inverse limit, \( \lim_{\leftarrow} C_i \).

We must show that any other candidate \( Y \) possesses a unique morphism \( Y \rightarrow X \), making appropriate diagrams commute. Let \( Y \) be such a candidate. The morphism of inverse systems defined by classifying maps (see definition \[2.17\])
\[ C_i \rightarrow \text{PV}([C_i]) \]
implies the existence of a unique morphism
\[ Y \rightarrow \lim_{\leftarrow} \text{PV}[C_i] = \text{PV} \lim_{\leftarrow} [C_i] \]
The commutativity of the diagrams
\[ \xymatrix{ Y \ar[r] & \text{PV} \lim_{\leftarrow} [C_i] \\ C_i \ar[u] \ar[r]^{\alpha_i} & \text{PV}[C_i] } \]
for all \( i \geq 0 \) implies that \( \lim Y \subseteq [p_i]^{-1} [\alpha_i]([C_i]). \) Consequently
\[ \lim Y \subseteq \bigcap_{i=0}^{\infty} [p_i]^{-1} [\alpha_i]([C_i]) \]
Since \( Y \) is a coalgebra, its image must lie within the maximal sub-coalgebra contained within \( \bigcap_{i=0}^{\infty} [p_i]^{-1} [\alpha_i]([C_i]) \), namely \( X = \langle \bigcap_{i=0}^{\infty} [p_i]^{-1} [\alpha_i]([C_i]) \rangle \). This proves the first claim. Proposition \[B.11\] implies that \( X = \bigcap_{i=0}^{\infty} C_i = \lim_{\leftarrow} C_i \). \( \square \)
**Lemma B.22.** Let \( \{g_i: C_i \to C_{i-1}\} \) be an inverse system in \( \text{Ch} \). If \( n > 0 \) is an integer, then the natural map

\[
\left( \lim_i C_i \right)^{\otimes n} \to \lim_i C_i^{\otimes n}
\]

is injective.

**Proof.** Let \( A = \lim_i C_i \) and \( p_i: A \to C_i \) be the natural projections. If

\[
W_k = \ker p_k^{\otimes n}: \left( \lim_i C_i \right)^{\otimes n} \to C_k^{\otimes n}
\]

we will show that

\[
\bigcap_{k=1}^{\infty} W_k = 0
\]

If \( K_i = \ker p_i \), then

\[
\bigcap_{i=1}^{\infty} K_i = 0
\]

and

\[
W_i = \sum_{j=1}^{n} A \otimes \cdots \otimes K_i \otimes \cdots \otimes A
\]

Since all modules are nearly-free, hence, flat (see remark [A.1]), we have

\[
W_{k+1} \subseteq W_k
\]

for all \( k \), and

\[
\bigcap_{i=1}^{m} W_i = \sum_{j=1}^{n} A \otimes \cdots \otimes \left( \bigcap_{i=1}^{m} K_i \right) \otimes \cdots \otimes A
\]

from which the conclusion follows. \( \square \)

**Proposition B.23.** Let \( \{C_i\} \in \text{pro} - \mathcal{K}_0 \), and suppose \( V = \{V(n)\} \) is a \( \Sigma \)-cofibrant operad with \( V(n) \) of finite type for all \( n \geq 0 \). Then the projections

\[
[P_V(\lim_i [C_i])] \to [P_V([C_n])]
\]

for all \( n > 0 \), induce a canonical injection

\[
\mu: [P_V(\lim_i [C_i])] \hookrightarrow \lim_i [P_V([C_i])]
\]

In addition, the fact that the structure maps

\[
\alpha_i: C_i \to P_V([C_i])
\]

of the \( \{C_i\} \) are coalgebra morphisms implies the existence of an injective \( \text{Ch} \)-morphism

\[
\hat{\alpha}: \lim_i [C_i] \hookrightarrow \lim_i [P_V([C_i])]
\]

Corresponding statements hold for \( \text{pro} - \mathcal{K}_0 \) and the functors \( L_V(*) \).
Proof. We must prove that
\[ \mu: \lim \left[ P_{\mathcal{V}}(\lim \left[ \mathcal{C}_i \right]) \right] \to \lim \left[ P_{\mathcal{V}}([C_i]) \right] \]
is injective. Let \( K = \text{ker} \mu \). Then
\[ K \subset \left[ P_{\mathcal{V}}(\lim \left[ \mathcal{C}_i \right]) \right] \subseteq \prod_{n \geq 0} \text{Hom}_{R\mathcal{S}}(\mathcal{V}(n), D^{\otimes n}) \]
where \( D = \lim \left[ C_i \right] \) (see [22]). If \( n \geq 0 \), let
\[ p_n: \prod_{n \geq 0} \text{Hom}_{R\mathcal{S}}(\mathcal{V}(n), D^{\otimes n}) \to \text{Hom}_{R\mathcal{S}}(\mathcal{V}(n), D^{\otimes n}) \]
denote the canonical projections. The diagrams
\[
\begin{array}{ccc}
\prod_{n \geq 0} \text{Hom}_{R\mathcal{S}}(\mathcal{V}(n), D^{\otimes n}) & \xrightarrow{\prod \text{Hom}_{R}(1, b_k^{\otimes n})} & \prod_{n \geq 0} \text{Hom}_{R\mathcal{S}}(\mathcal{V}(n), C_k^{\otimes n}) \\
p_n & & q_n \\
\text{Hom}_{R\mathcal{S}}(\mathcal{V}(n), D^{\otimes n}) & \xrightarrow{\text{Hom}_{R}(1, b_k^{\otimes n})} & \text{Hom}_{R\mathcal{S}}(\mathcal{V}(n), C_k^{\otimes n})
\end{array}
\]
commute for all \( k \) and \( n \geq 0 \), where \( q_n \) is the counterpart of \( p_n \) and \( b_k: \lim \left[ C_i \right] \to [C_k] \) is the canonical map. It follows that
\[ p_k(K) \subseteq \text{ker} \text{Hom}_{R}(1, b_k^{\otimes n}) \]
for all \( n \geq 0 \), or
\[ p_k(K) \subseteq \bigcap_{k > 0} \text{ker} \text{Hom}_{R}(1, b_k^{\otimes n}) \]
We claim that
\[ \bigcap_{n > 0} \text{ker} \text{Hom}_{R}(1, b_k^{\otimes n}) = \text{Hom}_{R}(1, \bigcap_{k > 0} \text{ker} b_k^{\otimes n}) \]
The equality on the left follows from the left-exactness of \( \text{Hom}_{R} \) and filtered limits (of chain-complexes). The equality on the right follows from the fact that
(1) \( \bigcap_{k > 0} \text{ker} b_k = 0 \)
(2) the left exactness of \( \otimes \) for \( R\)-flat modules (see remark A.1).
(3) Lemma B.22
It follows that \( p_n(K) = 0 \) for all \( n \geq 0 \) and \( K = 0 \).
The map
\[ \hat{\alpha}: \lim \left[ C_i \right] \to \lim \left[ P_{\mathcal{V}}([C_i]) \right] \]
is induced by classifying maps of the coalgebras \( \{C_i\} \), which induce a morphism of limits because the structure maps \( C_k \to C_{k-1} \) are coalgebra morphisms, making the diagrams
\[
\begin{array}{ccc}
C_k & \to & C_{k-1} \\
\downarrow & & \downarrow \\
P_{\mathcal{V}}([C_k]) & \to & P_{\mathcal{V}}([C_{k-1}])
\end{array}
\]
commute for all \( k > 0 \). \qed
Corollary B.24. Let \( C = \{ q_i: C_i \to C_{i-1} \} \in \text{pro} - \mathcal{I}_0 \), and suppose \( \mathcal{V} = \{ \mathcal{V}(n) \} \) is a \( \Sigma \)-cofibrant operad with \( \mathcal{V}(n) \) of finite type for all \( n \geq 0 \). Then

\[
\lim \left( \bigoplus \right) C_i = \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) \subseteq L_V(\lim \left( \bigoplus \right) [C_i])
\]

with the coproduct induced from \( L_V(\lim \left( \bigoplus \right) [C_i]) \), and where

\[
q_i: L_V(\lim \left( \bigoplus \right) [C_i]) \to L_V([C_i])
\]

is the projection and

\[
\alpha_i: C_i \to L_V([C_i])
\]

is the classifying map, for all \( i \). In addition, the sequence

\[
0 \to \left[ \lim \left( \bigoplus \right) C_i \right] \to \left[ \lim \left( \bigoplus \right) C_i \right] \xrightarrow{\hat{\alpha}} \frac{\lim \left[ L_V([C_i]) \right]}{\mu([L_V(\lim \left( \bigoplus \right) [C_i]))]} \to \frac{\lim \left( \left[ L_V([C_i]) / \alpha_i(C_i)] \right) \right]}{\lim \mu([L_V(\lim \left( \bigoplus \right) [C_i])])} \to \lim 1 [C_i] \to 0
\]

is exact in \( \text{Ch} \), where the injection

\[
\left[ \lim \left( \bigoplus \right) C_i \right] \to \left[ \lim \left( \bigoplus \right) C_i \right]
\]

is induced by the projections

\[
p_i: \lim \left( \bigoplus \right) C_i \to C_i
\]

and

\[
\hat{\alpha}: \left[ \lim \left( \bigoplus \right) C_i \right] \to \left[ \lim \left( \bigoplus \right) L_V([C_i]) \right]
\]

is induced by the \( \{ \alpha_i \} \) in equation B.7. The map

\[
\mu: [L_V(\lim \left( \bigoplus \right) [C_i])] \to [L_V([C_i])]
\]

is constructed in Proposition B.23.

If \( C \in \text{pro} - \mathcal{I}_0 \), then the corresponding statements apply, where \( L_V(*) \) is replaced by \( P_{V(*)} \).

Remark. The first statement implies that the use of the \( \langle * \rangle \)-functor in corollary B.21 is unnecessary — at least if \( \mathcal{V} \) is projective in the sense defined above.

The remaining statements imply that \( \lim \left( \bigoplus \right) C_i \) is the largest sub-chain-complex of \( \left[ \lim \left( \bigoplus \right) C_i \right] \) upon which one can define a coproduct that is compatible with the maps

\[
\lim \left( \bigoplus \right) C_i \to C_i
\]

Proof. First, consider the projections

\[
q_i: L_V(\lim \left( \bigoplus \right) [C_i]) \to L_V([C_i])
\]

The commutativity of the diagram

\[
\begin{array}{ccc}
L_V(\lim \left( \bigoplus \right) [C_i]) & \xrightarrow{\alpha} & \lim \left( \bigoplus \right) L_V(C_i) \\
\downarrow n & & \downarrow \lim \downarrow \\
L_V(C_i) & \xrightarrow{q_i} & L_V(C_i)
\end{array}
\]

implies that

\[
\lim \ker q_i = \bigcap_{i=1}^{\infty} \ker q_i = 0
\]
Now, consider the exact sequence

$$0 \to \ker q_i \to q_i^{-1}(\alpha_i(C_i)) \to [C_i] \to 0$$

and pass to inverse limits. We get the standard 6-term exact sequence for inverse limits (of \(\mathbb{Z}\)-modules):

$$(\text{B.9}) \quad 0 \to \lim_{\leftarrow} \ker q_i \to \lim_{\leftarrow} q_i^{-1}(\alpha_i([C_i])) \to \lim_{\leftarrow} [C_i]$$

$$\quad \to \lim_{\leftarrow} 1 \ker q_i \to \lim_{\leftarrow} 1 q_i^{-1}(\alpha_i(C_i)) \to \lim_{\leftarrow} 1 [C_i] \to 0$$

which, with the fact that \(\lim_{\leftarrow} \ker q_i = 0\), implies that

$$\bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) = \lim_{\leftarrow} q_i^{-1}(\alpha_i(C_i)) \to \lim_{\leftarrow} [C_i]$$

The conclusion follows from the fact that

$$\lim_{\leftarrow} C_i = \left(\bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i))\right) \subseteq \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i))$$

It remains to prove the claim in equation \(\text{B.6}\), which amounts to showing that

$$J = \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) \subseteq L_{V}(\lim_{\leftarrow} [C_i])$$

is closed under the coproduct of \(L_{V}(\lim_{\leftarrow} [C_i])\) — i.e., it is a coalgebra even \textit{without} applying the \((\ast)\)-functor. If \(n \geq 0\), consider the diagram

\[\text{Diagram with arrows and mappings}\]

where:

1. the \(\delta_i\) and \(\hat{\delta}\)-maps are coproducts and the \(\alpha_i\) are coalgebra morphisms.
2. \(r_{n,j} = \text{Hom}_R(1, q_j^\otimes n)\),
3. The map \(\hat{\mu}_n\) is defined by

\[
\hat{\mu}_n = \text{Hom}_R(1, \mu^\otimes n): \text{Hom}_{RS_n}(V(n), (L_{V}(\lim_{\leftarrow} [C_i])^\otimes n)) \rightarrow \text{Hom}_{RS_n}(V(n), (L_{V}(\lim_{\leftarrow} [C_i])^\otimes n))
\]
where \( \mu: L_V(\lim [C_i]) \to \lim L_V([C_i]) \) is the map defined in Proposition B.23.

4. \( s_{n,j} = \text{Hom}_R(1, \alpha_j^{\otimes n}) \) and \( \alpha_j: C_j \to L_V([C_j]) \) is the classifying map.

5. \( p_j: \lim L_V([C_i]) \to L_V([C_j]) \) is the canonical projection.

6. \( c_{n,j}: C_j \to \text{Hom}_{RS_\ast}(V(n), C^{\otimes n}_j) \) is the coproduct.

This diagram and the projectivity of \( \{V(n)\} \) and the near-freeness of \( L_V(\lim [C_i]) \)
(and flatness: see remark A.1) implies that

\[ \hat{\delta}_n \left( q_j^{-1}(\alpha_j(C_j)) \right) \subseteq \text{Hom}_{RS_\ast}(V(n), L_{n,j}) \]

where \( L_{n,j} = q_j^{-1}(\alpha_j(C_j))^\otimes n + \text{ker} r_{n,j} \) and

\[ \bigoplus_{j=1}^{\infty} L_{n,j} = \left( \bigoplus_{j=1}^{\infty} q_j^{-1}(\alpha_j(C_j)) \right)^\otimes n + \text{ker} \mu_n = J^\otimes n \]

so \( J \) is closed under the coproduct for \( L_V(\lim [C_i]) \).

Now, we claim that the exact sequence B.8 is just B.9 in another form — we have expressed the \( \lim^{-1} \) terms as quotients of limits of other terms.

The exact sequences

\[ 0 \to \text{ker} q_k \to [L_V(\lim [C_i])] \xrightarrow{q_k} [L_V([C_k])] \to 0 \]

for all \( k \), induces the sequence of limits

\[ 0 \to \lim^{-1} \text{ker} q_k \to [L_V(\lim [C_i])] \to \lim L_V([C_i]) \to \lim^{-1} \text{ker} q_k \to 0 \]

which implies that

\[ \lim^{-1} \text{ker} q_k = \frac{\lim [L_V([C_i])]}{[L_V(\lim [C_i])]} \]

In like fashion, the exact sequences

\[ 0 \to q_k^{-1}(\alpha_k(C_k)) \to [L_V(\lim [C_i])] \to [L_V([C_k])]/\alpha_k(C_k)] \to 0 \]

imply that

\[ \lim^{-1} q_i^{-1}(\alpha_i(C_i)) = \frac{\lim ([L_V([C_i])]/[\alpha_i(C_i)])}{\text{im}[L_V(\lim [C_i])]} \]

\[ \square \]

**Corollary B.25.** Let \( \{C_i\} \in \text{pro-} \mathcal{A}_0 \), and suppose \( V = \{V(n)\} \) is a \( \Sigma \)-cofibrant operad with \( V(n) \) of finite type for all \( n \geq 0 \). If

\[ \alpha_i: C_i \to P_V([C_i]) \]

are the classifying maps with

\[ \hat{\alpha}: \lim [C_i] \to \lim [P_V([C_i])] \]

the induced map, and if

\[ \mu: [P_V(\lim [C_i])] \to \lim [P_V([C_i])] \]

then...
is the inclusion defined in proposition B.23, then

\[ \mu \left( \lim \left\{ C_i \right\} \right) = \mu \left( \left[ P_V \left( \lim_{\left\{ C_i \right\}} \right) \right] \right) \cap \hat{\alpha} \left( \lim \left\{ C_i \right\} \right) \subseteq \lim \left[ P_V \left( [C_i] \right) \right] \]

A corresponding result holds in the category pro–\( S_0 \) after consistently replacing the functor \( P_V(*) \) by \( L_V(*) \).

**Remark.** The naive way to construct \( \lim \left\{ C_i \right\} \) is to try to equip \( \lim_{\left\{ C_i \right\}} \left[ C_i \right] \) with a coproduct — a process that fails because we only get a map

\[ \lim_{\left\{ C_i \right\}} \left[ C_i \right] \to \prod_{n \geq 0} \Hom_{\text{RS}_n}(V(n), \lim_{\left\{ C_i \right\}} \left( C_i \otimes^n \right)) \neq \prod_{n \geq 0} \Hom_{\text{RS}_n}(V(n), \lim_{\left\{ C_i \right\}} \left( C_i \otimes^n \right)) \]

which is not a true coalgebra structure.

Corollary B.25 implies that this naive procedure almost works. Its failure is precisely captured by the degree to which

\[ \left[ P_V \left( \lim_{\left\{ C_i \right\}} \left[ C_i \right] \right) \right] \neq \lim_{\left\{ C_i \right\}} \left[ P_V \left( [C_i] \right) \right] \]

**Proof.** This follows immediately from the exact sequence B.8. \( \square \)

Our main result

**Theorem B.26.** Let \( \{ f_i \} : \{ A \} \to \{ C_i \} \) be a morphism in pro–\( S_0 \) over a \( \Sigma \)-cofibrant operad \( V = \{ V(n) \} \) with \( V(n) \) of finite type for all \( n \geq 0 \). Let

1. \( \{ A \} \) be the constant object
2. the \( \{ f_i \} \) be cofibrations for all \( i \)

Then \( \{ f_i \} \) induces an inclusion \( f = \lim \{ f_i \} : A \to \lim \{ C_i \} \) and the sequence

\[ 0 \to [A] \xrightarrow{\lim f_i} \left[ \lim \left\{ C_i \right\} \right] \to \left[ \lim \left\{ C_i/A \right\} \right] \to 0 \]

is exact. In particular, if \( \lim \left\{ C_i/A \right\} \) is contractible, then \( \lim f_i \) is a weak equivalence.

**Proof.** We will consider the case of \( S_0 \) — the other case follows by a similar argument.

The inclusion

\[ \left[ \lim C_i \right] \subseteq \left[ \lim \left\{ C_i \right\} \right] \]

from corollary B.24, and the left-exactness of filtered limits in Ch implies the left-exactness of the filtered limits in pro–\( S_0 \), and that the inclusion

\[ \left[ A \right] \hookrightarrow \left[ \lim C_i \right] \]

is a cofibration in Ch.

The fact that

\[ \left[ \lim C_i \right] = \left[ \bigcap_{i=1}^{\infty} q_i^{-1}(\alpha_i(C_i)) \right] \subseteq \left[ L_V(\lim \left\{ C_i \right\}) \right] \]
shows that the map $h$ is surjective. The conclusion follows.  

\begin{flushright}
\Box
\end{flushright}

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