NUMERICALLY FLAT HOLOMORPHIC BUNDLES OVER NON KÃ¶HLER MANIFOLDS

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Abstract. In this paper, we study numerically flat holomorphic vector bundles over a compact non-Kähler manifold \((X, \omega)\) with the Hermitian metric \(\omega\) satisfying the Gauduchon and Astheno-Kähler conditions. We prove that numerically flatness is equivalent to numerically effectiveness with vanishing first Chern number, semistability with vanishing first and second Chern numbers, approximate Hermitian flatness and the existence of a filtration whose quotients are Hermitian flat. This gives an affirmative answer to the question proposed by Demailly, Peternell and Schneider.

1. Introduction

The notion of positivity plays an important role in algebraic geometry and complex geometry. Let \(L\) be a line bundle over a compact complex manifold \(X\). \(L\) is said to be positive (semipositive) if there is a Hermitian metric \(h\) on \(L\) such that the curvature \(\sqrt{-1} \Theta(L, h) > 0\) \((\geq 0)\). A natural generalization and more flexible notion is numerically effective (nef for short). When \(X\) is projective, \(L\) is said to be nef if \(L \cdot C \geq 0\) for every compact curve \(C \subset X\). However, when \(X\) is just a general compact complex manifold, there maybe no compact curves over \(X\). Motived by the following property of nef line bundles over projective manifolds,

Lemma 1.1. (\cite[Lemma 1.1]{4}) Let \(A\) be an ample line bundle over a projective manifold \(X\). Then a line bundle \(L\) is nef if and only if \(L^k \otimes A\) is ample for every integer \(k \geq 0\).

Demailly, Peternell and Schneider (\cite{4}) generalized this definition to general compact complex manifolds in terms of curvature, that is

Definition 1.2. Let \((X, \omega)\) be an \(n\)-dimensional compact Hermitian manifold. A line bundle \(L\) over \((X, \omega)\) is said to be numerically effective, if for every \(\epsilon > 0\), there exists a smooth metric \(h_\epsilon\) on \(L\) such that the curvature \(\sqrt{-1} \Theta(L, h_\epsilon) = \sqrt{-1} \partial \bar{\partial} \log h_\epsilon \geq -\epsilon \omega\).

This means the curvature of \(L\) can have arbitrary small negative part. It is obvious that a Hermitian flat line bundle is nef. A vector bundle \(E\) of rank \(r \geq 2\) is said to be nef if the anti tautological line bundle \(O_E(1)\) on the projective bundle \(PE\) is nef. \(E\) is said to be numerically flat (nflat for short), if both \(E\) and its dual \(E^*\) are nef.

In \cite{4}, the authors established the relationship between Hermitian flatness and nflatness. For line bundles, nflatness is equivalent to Hermitian flatness(\cite[Corollary 1.5]{4}). As to vector bundles of higher rank, they showed that a holomorphic vector bundle over a compact Kähler manifold is nflat if and only if it admits a filtration by sub-bundles such that the quotients are Hermitian flat. Based on the above results, they raised an interesting question whether the above result holds in non-Kähler case and pointed out the difficulty is to show the second
Chern number of a numerically flat vector bundle is zero. In [4], they obtained it by the Fulton-Lazarsfeld inequalities for Chern classes of nef vector bundles ([4, Theorem 2.5]) which only hold over compact Kähler manifolds. Under the assumption of $c_2(E) = 0$, Biswas and Pingali ([1]) obtained a characterization of numerically flat bundle $E$ on a compact complex manifold $(X, \omega)$ with $\omega$ satisfying Gauduchon ($\partial \bar{\partial} \omega^{n-1} = 0$) and Astheno-Kähler ($\partial \bar{\partial} \omega^{n-2} = 0$) conditions which make the first and second Chern numbers well-defined ([1, Theorem 3.2]).

In this paper, we consider nflat vector bundles over compact non-Kähler manifolds without the assumption of $c_2(E) = 0$. In fact, we obtain the equivalence between nflatness, nefness with vanishing first Chern number, semistability with vanishing first and second Chern numbers, approximate Hermitian flatness and the existence of the filtration by sub-bundles whose quotients are Hermitian flat. That is

**Theorem 1.3.** Let $(X, \omega)$ be a compact Hermitian manifold of dimension $n$ with $\omega$ satisfying $\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0$. Let $(E, \bar{\partial} E)$ be a holomorphic vector bundle over $X$. Then the following statements are equivalent:

1. Numerically flat;
2. Numerically effective with $ch_1(E) \cdot [\omega^{n-1}] = 0$;
3. Semistable with $ch_1(E) \cdot [\omega^{n-1}] = ch_2(E) \cdot [\omega^{n-2}] = 0$;
4. Approximately Hermitian flat;
5. There exists a filtration $0 \subset E_0 \subset E_1 \cdots E_l = E$

by sub-bundles whose quotients are Hermitian flat.

**Remark 1.4.** It has been proved by Gauduchon ([3]) that if $X$ is compact, then there exists a Gauduchon metric in the conformal class of every Hermitian metric. So any compact complex manifold $X$ admits a Hermitian metric $\omega$ satisfying Gauduchon condition. If in addition $X$ is a complex surface, i.e. $\dim X = 2$, then $\omega$ automatically also satisfies Astheno-Kähler condition. By this fact, it is easy to check that, if $(X_1, \omega_1)$ is a Kähler manifold and $(X_2, \omega_2)$ is a Gauduchon surface, then $(X, \omega) = (X_1 \times X_2, \omega_1 + \omega_2)$ also satisfies Gauduchon and Astheno-Kähler conditions. For another examples of Gauduchon Astheno-Kähler manifolds, see [8, 14–16], for instance.

**Corollary 1.5.** Let $X$ be a compact complex surface, and $(E, \bar{\partial} E)$ be a holomorphic vector bundle over $X$. Then the following statements are equivalent:

1. Numerically flat;
2. Approximately Hermitian flat;
3. There exists a filtration $0 \subset E_0 \subset E_1 \cdots E_l = E$

by sub-bundles whose quotients are Hermitian flat.

We give an overview of the proof. The key points are to show a numerically effective holomorphic bundle with vanishing first Chern number is semi-stable with vanishing first and second Chern numbers and a semistable holomorphic bundle with vanishing first and second Chern numbers is approximate Hermitian flat.

As to the former, first following the argument of Step 1 in the proof of [4, Theorem 1.18] and Lemma 2.9, we have nef vector bundles with vanishing first Chern number are semi-stable and the determinant line bundles are Hermitian flat. Then using the push forward formula of Segre forms by the first Chern form of the anti tautological line bundle $\mathcal{O}_E(1)$ on $P \mathcal{E}$ and Bogomolov type inequality (Proposition 2.6), we obtain that the second Chern number of nflat vector bundles is zero.
And as to the later, we have the following theorem:

**Theorem 1.6.** Let \((X, \omega)\) be an \(n\)-dimensional compact Hermitian manifold with \(\omega\) satisfying \(\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0\). Let \((E, \bar{\partial}_E)\) be a holomorphic vector bundle of rank \(r\). If \((E, \bar{\partial}_E)\) is semistable with \(ch_1(E) \cdot [\omega^{n-1}] = ch_2(E) \cdot [\omega^{n-2}] = 0\), then it is approximate Hermitian flat.

In [20], by using the Hermitian-Yang-Mills flow, we derived that a semistable holomorphic vector bundle with vanishing first and second Chern numbers over a compact Kähler manifold is approximate Hermitian flat. Let \((E, \bar{\partial}_E)\) be a holomorphic vector bundle over compact complex manifold \((X, \omega)\) and \(H\) be an arbitrary Hermitian metric on \(E\). When \(\omega\) is Kähler, by the Chern-Weil theory, we have

\[
\int_X |F_{H, \bar{\partial}_E}|^2 H \frac{\omega^n}{n!} = \int_X |\sqrt{-1} \Lambda \omega F_{H, \bar{\partial}_E} - \lambda \cdot \text{Id}_E|_H^2 \frac{\omega^n}{n!} - 8\pi^2 \int_X ch_2(E, H) \wedge \frac{\omega^{n-2}}{(n-2)!} + \lambda^2 \text{rank}(E) \text{Vol}(X, \omega),
\]

where \(\lambda = \frac{2\pi \mu(E)}{\text{Vol}(X, \omega)}\). In fact, (1.1) also holds when \(\omega\) satisfies \(\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0\). And when \(ch_1(E, H) \cdot [\omega^{n-1}] = ch_2(E, H) \cdot [\omega^{n-2}] = 0\), we have

\[
\int_X |F_{H, \bar{\partial}_E}|^2 H = \int_X |\sqrt{-1} \Lambda \omega F_{H, \bar{\partial}_E}|_H^2.
\]

Since \(\text{deg}\omega(E) = 0\), consider the following Hermitian-Yang-Mills flow

\[
\begin{cases}
H(t)^{-1} \frac{\partial H(t)}{\partial t} = -2\sqrt{-1} \Lambda \omega F_{H(t), \bar{\partial}_E}, \\
H(0) = H_0,
\end{cases}
\]

where \(H_0\) is an arbitrary Hermitian metric. It has been proved ([17, Equation 3.14]) that when \((E, \bar{\partial}_E)\) is semi-stable, along the flow (1.3), it holds

\[
\int_X |F_{H(t), \bar{\partial}_E}|^2 H(t) = \int_X |\sqrt{-1} \Lambda \omega F_{H(t), \bar{\partial}_E}|_H(t) \to 0, \quad \text{as} \quad t \to \infty.
\]

Then by the small energy regularity theorem of Yang-Mills flow, we ([20, Equation 3.3]) obtained that

\[
\sup_X |F_{H(t), \bar{\partial}_E}|_H(t) \to 0, \quad \text{as} \quad t \to \infty.
\]

However, when \(\omega\) is not Kähler, we do not know whether (1.4) still holds along the Hermitian-Yang-Mills flow and can not generalize the argument in [20] directly. In this paper, we combine the continuity method and the Hermitian-Yang-Mills flow to construct the approximate Hermitian flat structure. Fixing a proper Hermitian metric \(K\) with \(\text{tr}(\sqrt{-1} \Lambda \omega F_{K(t), \bar{\partial}_E} - \lambda \cdot \text{Id}_E) = 0\), we consider the following perturbed equation

\[
\sqrt{-1} \Lambda \omega F_{H_\varepsilon(t), \bar{\partial}_E} - \lambda \cdot \text{Id}_E + \varepsilon \log f_\varepsilon = 0, \quad \varepsilon \in (0, 1],
\]

where \(f_\varepsilon = K^{-1} H_\varepsilon \in \text{Herm}^+(E, K)\). It has been proved that (1.5) is solvable for \(\varepsilon \in (0, 1]\). Then for each \(\varepsilon \in (0, 1]\), consider the Hermitian-Yang-Mills flow with \(H_\varepsilon\) as the initial metric

\[
\begin{cases}
H_\varepsilon^{-1}(t) \frac{\partial H_\varepsilon(t)}{\partial t} = -2\sqrt{-1} \Lambda \omega F_{H_\varepsilon(t), \bar{\partial}_E}, \\
H_\varepsilon(0) = H_\varepsilon.
\end{cases}
\]

From [21, Theorem 3.2], we have that when \((E, \bar{\partial}_E)\) is semi-stable, it holds

\[
\sup_X |\Lambda \omega F_{H_\varepsilon, \bar{\partial}_E}|_{H_\varepsilon} \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]
By Equality (1.2) and Lemma 3.2 it holds
\[ \int_X |F_{A_\varepsilon(t)}|^2 \frac{\omega^n}{n!} \leq \int_X |F_{A_\varepsilon(0)}|^2 \frac{\omega^n}{n!} = \int_X |F_{H_\varepsilon, \partial_E}|^2 \frac{\omega^n}{n!} = \int_X |\sqrt{-1} \Lambda \omega F_{H_\varepsilon, \partial E}|^2 \frac{\omega^n}{n!} \rightarrow 0, \quad \varepsilon \rightarrow 0, \]
where \( A_\varepsilon(t) \) is the solution of (3.11). This implies that when \( \varepsilon \) is small enough, the \( L^2 \)-norm of the curvature \( F_{A_\varepsilon(t)} \) is also small. Then we use the small energy regularity to obtain the approximate Hermitian flat structure.

This paper is organized as below. In Section 2, we will recall some basic notions and related properties. In Section 3, we give a detailed proof of Theorem 1.3. In Section 4, we give a detailed proof of Theorem 1.3.

2. Preliminary

In this section, we recall some definitions and properties of nef vector bundles needed in this paper.

2.1. Some definitions. Let \( X \) be an \( n \)-dimensional compact complex manifold and \( g \) be a Hermitian metric with associated \((1,1)\)-form \( \omega \). \( g \) is called Gauduchon if \( \omega \) satisfies \( \partial \bar{\partial} \omega^{n-1} = 0 \). If \( \partial \bar{\partial} \omega^{n-2} = 0 \), the Hermitian metric \( g \) is said to be Astheno-Kähler which was introduced by Jost and Yau in [12]. In this paper, we assume \( \omega \) satisfies \( \partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0 \).

Let \((L, h)\) be a Hermitian line bundle over \( X \). The \( \omega \)-degree of \( L \) is defined by
\[ \deg_{\omega}(L) := \int_X c_1(L, A_h) \wedge \frac{\omega^{n-1}}{(n-1)!}, \]
where \( c_1(L, A_h) \) is the first Chern form of \( L \) associated with the Chern connection \( A_h \) with respect to the Hermitian metric \( h \). Since \( \partial \bar{\partial} \omega^{n-1} = 0 \), \( \deg_{\omega}(L) \) is well defined and independent of the choice of metric \( h \) ([13, p. 34-35]).

Now given a coherent analytic sheaf \( \mathcal{F} \) of rank \( s \), we consider the determinant line bundle \( \text{det} \mathcal{F} = (\wedge^s \mathcal{F})^* \). Define the \( \omega \)-degree of \( \mathcal{F} \) by
\[ \deg_{\omega}(\mathcal{F}) := \deg_{\omega}(\text{det} \mathcal{F}). \]
If \( \mathcal{F} \) is non-trivial and torsion free, the \( \omega \)-slope of \( \mathcal{F} \) is defined by
\[ \mu_{\omega}(\mathcal{F}) = \frac{\deg_{\omega}(\mathcal{F})}{\text{rank}(\mathcal{F})}. \]

Let \((E, \bar{\partial}_E, H)\) be a rank \( r \) holomorphic Hermitian vector bundle. Denote \( D_{H, \bar{\partial}_E} \) the Chern connection of \((E, \bar{\partial}_E, H)\) and \( F_{H, \bar{\partial}_E} = D_{H, \bar{\partial}_E}^2 \) the Chern curvature. Then the corresponding Chern forms \( c_k(E, H) \in A^{k,k}(X) \) are computed by
\[ \det \left( \text{Id}_E + \frac{\sqrt{-1}}{2\pi} F_{H, \bar{\partial}_E} \right) = \sum_{i=0}^{\min\{r, n\}} c_i(E, H) t^i. \]

Let \( H_1 \) and \( H_2 \) be two Hermitian metrics on \( E \). It has been proved by Donaldson [6, Proposition 6] that for every \( 1 \leq k \leq \min\{r, n\} \), there exists \( R_{k-1}(H_1, H_2) \in A^{k-1,k-1}(X) \) such that
\[ c_k(E, H_1) - c_k(E, H_2) = \sqrt{-1} \partial \bar{\partial} R_{k-1}(H_1, H_2). \]
So when $\bar{\partial}\partial\omega^{n-1} = \partial\bar{\partial}\omega^{n-2} = 0$,

$$c_1(E) \cdot [\omega^{n-1}] = \int_X c_1(E, H) \land \omega^{n-1}$$

and

$$c_2(E) \cdot [\omega^{n-2}] = \int_X c_2(E, H) \land \omega^{n-2}, \quad c_2^2(E) \cdot [\omega^{n-2}] = \int_X c_2^2(E, H) \land \omega^{n-2}$$

are well-defined and independent of the Hermitian metrics on $E$, where $[\omega^{n-1}] \in H_{n-1}^{n-1}(X)$, $[\omega^{n-2}] \in H_{n-2}^{n-2}(X)$ and

$$c_1(E) \in H_{BC}^{1}(X), \quad c_2(E), c_2^2(E) \in H_{BC}^{2}(X).$$

**Remark 2.1.** The Bott-Chern cohomology and Aeppli cohomology are defined by

$$H_{BC}^{\bullet, \bullet}(X) = \frac{\text{Ker} \partial \cap \text{Ker} \bar{\partial}}{\text{Imm} \partial \cap \text{Imm} \bar{\partial}}$$

and

$$H_{A}^{\bullet, \bullet}(X) = \frac{\text{Ker} \partial \bar{\partial}}{\text{Imm} \partial \cap \text{Imm} \bar{\partial}}.$$ 

**Definition 2.2.** Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle over $X$. We say $E$ is $\omega$-stable ($\omega$-semi-stable) in the sense of Mumford-Takemoto if for every proper coherent sub-sheaf $F \hookrightarrow E$, it holds

$$\mu_{\omega}(F) < \mu_{\omega}(E)(\mu_{\omega}(F) \leq \mu_{\omega}(E)).$$

**Definition 2.3.** A Hermitian metric $H$ on $E$ is said to be $\omega$-Hermitian-Einstein if the Chern curvature $F_{H,\bar{\partial}_E}$ satisfies the Einstein condition

$$\sqrt{-1}\Lambda_{\omega} F_{H,\bar{\partial}_E} = \lambda \cdot \text{Id}_E,$$

where $\lambda = \frac{2\pi \mu_{\omega}(E)}{\text{Vol}(X, \omega)}$.

**Remark 2.4.** By [3, Proposition 1.5 or Lemma 2.1], when checking the stability of a holomorphic vector bundle, we only need to consider proper saturated sub-sheaves, i.e. sub-sheaves with torsion free quotients.

The classic Donaldson-Uhlenbeck-Yau theorem ([6, 7, 19, 23]) tells us there exist Hermitian-Einstein metrics on holomorphic vector bundles over compact Kähler manifolds if they are stable and was generalized by Li and Yau ([15]) for general compact Gauduchon manifolds. When the Kähler form is understood, we omit the subscript $\omega$ in the above definitions.

**Definition 2.5.** A holomorphic vector bundle $(E, \bar{\partial}_E)$ is said to admit an approximate Hermitian-Einstein structure, if for every $\epsilon > 0$, there exists a Hermitian metric $H_\epsilon$, such that

$$\sup_X \sqrt{-1}\Lambda_{\omega} F_{H_\epsilon,\bar{\partial}_E} - \lambda \cdot \text{Id}_E|_{H_\epsilon} < \epsilon.$$ 

Kobayashi ([13]) introduced this notion for a holomorphic vector bundle. Similar with the relationship between stability and the existence of Hermitian-Einstein metrics, a holomorphic vector bundle admits an approximate Hermitian-Einstein structure if it is semistable. It was proved by Kobayashi ([13]) for projective manifolds, by Jacob ([11]), Li and Zhang ([17]) for compact Kähler manifolds and by Nie and Zhang ([21]) for general compact Gauduchon manifolds. Furthermore, if $\omega$ is both Gauduchon and Astheno-Kähler, we have the following Bogomolov
type inequality, which was first obtained by Bogomolov ([2]) for semi-stable holomorphic vector bundles on complex algebraic surfaces.

**Proposition 2.6.** Let \((X, \omega)\) be an \(n\)-dimensional compact complex manifold with \(\omega\) satisfying \(\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0\) and \((E, \bar{\partial}_E)\) be a rank \(r\) holomorphic vector bundle. If \(E\) is semi-stable, then we have the following Bogomolov type inequality

\[
4\pi^2 \left(2c_2(E) - \frac{r-1}{r} c_1^2(E) \right) \cdot [\omega^{n-2}] \geq 0.
\]

**Proof.** From the above, when \(\omega\) satisfies \(\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0\), we have

\[
4\pi^2 \left(2c_2(E) - \frac{r-1}{r} c_1^2(E) \right) \cdot [\omega^{n-2}]
\]

is well-defined and independent of the choice of the Hermitian metrics on \((E, \bar{\partial}_E)\). Endowed \(E\) with an arbitrary Hermitian metric \(H\), we have

\[
4\pi^2 \left(2c_2(E) - \frac{r-1}{r} c_1^2(E) \right) \cdot \frac{[\omega^{n-2}]}{(n-2)!} = 4\pi^2 \int_X \left(2c_2(E, H) - \frac{r-1}{r} c_1(E, H) \wedge c_1(E, H) \right) \wedge \frac{[\omega^{n-2}]}{(n-2)!}
\]

\[
\geq - \int_X |\sqrt{-1} \Lambda_\omega F_{H, \bar{\partial}_E} - \lambda \cdot \text{Id}_E - \frac{1}{r} \text{tr}(\sqrt{-1} \Lambda_\omega F_{H, \bar{\partial}_E} - \lambda \cdot \text{Id}_E) \text{Id}_E| H, [\omega^n]_n!,
\]

where \(F_{H, \bar{\partial}_E} = F_{H, \bar{\partial}_E} - \frac{\text{tr} F_{H, \bar{\partial}_E} \cdot \text{Id}_E}{r} \cdot \text{Id}_E\) is the trace free part of \(F_{H, \bar{\partial}_E}\).

Since \((E, \bar{\partial}_E)\) is semi-stable, \((E, \bar{\partial}_E)\) admits an approximate Hermitian-Einstein structure ([21]), that is for every \(\epsilon > 0\), there exists \(H_\epsilon\) such that

\[
\sup_X |\sqrt{-1} \Lambda_\omega F_{H_\epsilon, \bar{\partial}_E} - \lambda \cdot \text{Id}_E| H_\epsilon < \epsilon.
\]

Then

\[
\int_X |\sqrt{-1} \Lambda_\omega F_{H_\epsilon, \bar{\partial}_E} - \lambda \cdot \text{Id}_E - \frac{1}{r} \text{tr}(\sqrt{-1} \Lambda_\omega F_{H_\epsilon, \bar{\partial}_E} - \lambda \cdot \text{Id}_E) \text{Id}_E| H, [\omega^n]_n! \to 0, \quad \epsilon \to 0.
\]

Therefore, by Equation (2.7) and (2.8), we have

\[
4\pi^2 \left(2c_2(E) - \frac{r-1}{r} c_1^2(E) \right) \cdot \frac{[\omega^{n-2}]}{(n-2)!} \geq 0.
\]

\[
\square
\]

**Definition 2.7.** A holomorphic vector bundle \((E, \bar{\partial}_E)\) is said to be approximate Hermitian flat, if for every \(\epsilon > 0\), there exists a Hermitian metric \(H_\epsilon\) such that

\[
\sup_X |F_{H_\epsilon, \bar{\partial}_E}| H_\epsilon < \epsilon.
\]
2.2. Basic properties of nef vector bundles. In this subsection, we will present some basic properties of nef vector bundles. For the detailed proof, please see reference [4].

Proposition 2.8. ([4, Corollary 1.5]) $L$ is numerically flat if and only if it is Hermitian flat.

Lemma 2.9. Let $(X, \omega)$ be an $n$-dimensional compact Hermitian manifold with $\omega$ satisfying $\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0$. Let $L$ be a holomorphic line bundle on $X$. If $L$ is nef and $c_1(L) \cdot [\omega^{n-1}] = 0$, then $L$ is Hermitian flat.

Proof. From the above, we have when $\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0$, $c_1(L) \cdot [\omega^{n-1}]$ and $c_1(L)^2 \cdot [\omega^{n-2}]$ are well-defined and can be computed by the Chern form $\Theta(L, h)$ of an arbitrary Hermitian metric $h$ on $L$. On one hand, since $L$ is nef, for every $\epsilon > 0$, there exists a Hermitian metric $h_\epsilon$ such that $\sqrt{-1} \Theta(L, h_\epsilon) \geq -\epsilon \omega$.

So

$$0 \leq \int_X \left( \sqrt{-1} \Theta(L, h_\epsilon) + \epsilon \omega \right)^2 \wedge \omega^{n-2}$$

$$= \int_X \left( \sqrt{-1} \Theta(L, h_\epsilon) \right)^2 \wedge \omega^{n-2} + 2\epsilon \int_X \sqrt{-1} \Theta(L, h_\epsilon) \wedge \omega^{n-1} + \epsilon^2 \int_X \omega^n$$

$$\to c_1(L)^2 \cdot [\omega^{n-2}] , \quad \epsilon \to 0.$$  \hfill (2.9)

This implies

$$c_1(L)^2 \cdot [\omega^{n-2}] \geq 0.$$  \hfill (2.10)

And on the other hand, since $c_1(L) \cdot [\omega^{n-1}] = 0$, we can find a Hermitian metric $h$ on $E$ such that

$$\sqrt{-1} \Lambda \omega \Theta(L)_h = 0.$$  \hfill (2.11)

From (1.1), we have

$$c_1(L)^2 \cdot [\omega^{n-2}/(n-2)!] = 2ch_2(L) \cdot [\omega^{n-2}/(n-2)!]$$

$$= \frac{1}{4\pi^2} \left( \int_X |\sqrt{-1} \Lambda \omega \Theta(L, h)|^2 - \int_X |\Theta(L, h)|^2 \right)$$

$$= -\frac{1}{4\pi^2} \int_X |\Theta(L, h)|^2.$$  \hfill (2.11)

Combining (2.9) and (2.11), we have $\Theta(L, h) = 0$. This concludes the proof. \hfill \square

Let $m$ be a positive integer and let $S^m E$ be the $m$-th symmetric power of $E$, then $\det S^m E = (\det E)^N$ where

$$N = \frac{m \rank(S^m E)}{\rank(E)}.$$  

Together with ([4, Theorem 1.12], we can easily check that

Proposition 2.10. Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle over $(X, \omega)$. If $E$ is nef, then $\det E$ is nef.

By Proposition 2.10 and Proposition 2.8 we have

Proposition 2.11. Let $(E, \bar{\partial}_E)$ be a holomorphic vector bundle over $(X, \omega)$. If $E$ is nflat, then $\det E$ is Hermitian flat.
Proposition 2.12. ([4, Proposition 1.14]) Let \( E \) and \( F \) be two holomorphic vector bundles over \( X \). If \( E \) and \( F \) are nef, then \( E \otimes F \) is nef.

Proposition 2.13. ([4, Proposition 1.15]) Let \( 0 \to F \to E \to Q \to 0 \) be an exact sequence of holomorphic vector bundles. Then

1. If \( E \) is nef, then \( Q \) is nef;
2. If \( F \) and \( Q \) are nef, then \( E \) is nef;
3. If \( E \) and \((\det Q)^{-1}\) are nef, then \( F \) is nef.

2.3. Segre forms. In this subsection, we will introduce the push forward formula of Segre forms which was proved by Guler([10]) for projective manifolds and by Diverio([5]) for general compact complex manifolds.

Let \((E, \bar{\partial}_E) \to X\) be a rank \( r \) holomorphic vector bundle and \( c_\bullet(E) = 1 + c_1(E) + \cdots + c_r(E) \in H^*(X, \mathbb{Z})\) the total Chern class of \( E \). The inverse of \( c_\bullet(E) \) is by definition the total Segre class \( s_\bullet(E) = 1 + s_1(E) + \cdots + s_n(E) \in H^*(X, \mathbb{Z}) \). Endow \((E, \bar{\partial}_E)\) with a Hermitian metric \( H \). Then from the Chern-Weil theory, the Segre forms \( s_k(E, H) \) can be defined inductively by the relation

\[
(2.12) \quad s_k(E, H) + c_1(E, H)s_{k-1}(E, H) + \cdots + c_k(E, H) = 0, \quad 0 \leq k \leq n.
\]

For example,

\[
(2.13) \quad s_1(E, H) = -c_1(E, H),
\]

\[
(2.14) \quad s_2(E, H) = c_1(E, H)^2 - c_2(E, H).
\]

Push forward of forms. Let \( M, N \) be oriented differential manifolds of dimension \( m, n \) \((m > n)\) and \( f : M \to N \) be a proper submersion. Set \( s = m - n \). Then for any smooth \((p+s)\)-form \( \eta \) on \( M \), there exists a unique smooth \( p\)-form \( \xi \) on \( N \) such that the equality

\[
(2.15) \quad \int_M \eta \wedge f^* \phi = \int_N \xi \wedge \phi
\]

holds for any smooth \((n-p)\)-form \( \phi \) on \( N \) with compact support.

Given a Hermitian metric \( H \) on \( E \), denote \( h \) the induced metric on \( \mathcal{O}_E(1) \to PE \) and \( \Xi = \sqrt{-1} \Theta(\mathcal{O}_E(1), h) \). Then we have the following push forward formula of Segre forms:

Lemma 2.14. ([3, Proposition 1.1]) For each \( k = 0, \cdots, n \), the equality

\[
(2.16) \quad \pi_* (\Xi^{r-1+k}) = s_k(E, H),
\]

holds, where \( s_0(E, H) \) is the function on \( X \) and constantly equal to 1.
**Lemma 3.1.** ([21, Theorem 3.2]) If \((E, \bar{\partial}_E)\) is semi-stable, then
\[
\sup_X |\sqrt{-1}\Lambda_{\omega} F_{H,\bar{\partial}_E} - \lambda \cdot \text{Id}_E|_{H^0} \to 0, \quad \varepsilon \to 0.
\]

Given an arbitrary metric \(H_0\) on \((E, \bar{\partial}_E)\), consider the Hermitian-Yang-Mills flow,
\[
\begin{aligned}
H(t)^{-1}\frac{\partial H(t)}{\partial t} &= -2(\sqrt{-1}\Lambda_{\omega} F_{H(t),\bar{\partial}_E} - \lambda \cdot \text{Id}_E), \\
H(0) &= H_0.
\end{aligned}
\]

Denote the space of connections of \(E\) compatible with \(H_0\) by \(A_{H_0}\), the space of unitary integrable connections of \(E\) by \(A^1_{H_0}\) and the complex gauge group (resp. unitary gauge group) of \((E, H_0)\) by \(G^C\) (resp. \(G\), where \(G = \{\sigma \in G^C | \sigma^* H_0 \sigma = \text{Id} \}\)). \(G^C\) acts on the space \(A_{H_0}\) as follows: for \(\sigma \in G^C\) and \(A \in A_{H_0}\),
\[
(\sigma \circ A) = \sigma \circ \bar{\partial}_A \circ \sigma^{-1}, \quad \partial_{\sigma(A)} = (\sigma^* H_0)^{-1} \circ \partial_A \circ \sigma^* H_0.
\]

From [22], we have the heat flow (3.3) is equivalent to the following flow
\[
\begin{aligned}
\frac{\partial A(t)}{\partial t} &= \sqrt{-1}(\bar{\partial}_A - \partial A)\Lambda_{\omega} F_A, \\
A(0) &= (\bar{\partial}_E, H_0).
\end{aligned}
\]

The global existence and uniqueness of (3.5) has been given in [22]. In fact, \(A(t) = \sigma(t)(A_0)\), where \(\sigma(t) \in G^C\) satisfies \(\sigma(t)^* H_0 \sigma(t) = H_0^{-1} H(t)\) and \(H(t)\) is the long time solution of (3.3). It is easy to check the following relations:
\[
F_{A(t)} = \sigma(t) \circ F_{H(t),\bar{\partial}_E} \circ \sigma(t)^{-1},
\]
\[
|F_{A(t)}|_{H_0}^2 = |F_{H(t),\bar{\partial}_E}|_{H(t)}^2.
\]

Along the flow (3.6), we have the following Bochner type inequality
\[
(\triangle_g - \frac{\partial}{\partial t})|F_A|_{H_0}^2 \geq 2|\nabla_A F_A|_{H_0}^2 - C_1(1 + |F_A|_{H_0} + |Ric|_g)|F_A|_{H_0}^2 - C_1|F_A|_{H_0} |\nabla_A F_A|_{H_0},
\]
where \(C_1 > 0\) depends on the geometry of \((X, \omega)\). And denoting
\[
\text{YM}(t) = \int_X |F_A|_{H_0}^2 \frac{\omega^n}{n!},
\]
we have the energy inequality

**Lemma 3.2.** ([22, Lemma 2.3]) Let \((X, \omega)\) be an \(n\)-dimensional compact Hermitian manifold with \(\omega\) satisfying \(\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0\). Suppose \(A(t)\) is a solution of the heat flow (3.5) with initial data \(A_0\). Then
\[
\text{YM}(t) + 2 \int_0^t \int_X |\frac{\partial A}{\partial t}|^2 = \text{YM}(0).
\]

Let \(i_X\) be the infimum of the injective radius over \(X\). For any \((x_0, t_0) \in X \times \mathbb{R}^+\) and \(r \leq i_X\), denote \(P_t(x_0, t_0) = B_r(x_0) \times [t_0 - r^2, t_0 + r^2]\). We have the small energy regularity theorem

**Lemma 3.3.** ([22, Theorem 2.10]) Suppose that \(A(t)\) is a smooth solution of the heat flow (3.5) over \((X, \omega)\) with \(\omega\) satisfying \(\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0\). Then there exist positive constants
$\epsilon_0$ and $\delta_0$ depending on the geometry of $(X, \omega)$ and YM(0), such that if for some $0 < R < \min\{i_X/2, \sqrt{t_0}/2\}$, the inequality
\[
R^{2-2n} \int_{P_R(x_0, t_0)} |F_{A_\epsilon}|^2_{H_0} < \epsilon_0
\]
holds, then for any $\delta \in (0, \min\{\delta_0, 1/4\})$, we have
\[
\sup_{P_{3R}(x_0, t_0)} |F_{A_\epsilon}|^2_{H_0} < 16(\delta R)^{-4}.
\]

**Proof of Theorem 1.6**

Since $ch_1(E, H) \cdot [\omega^{n-1}] = 0$, we have $\lambda = 0$. Fix a proper Hermitian metric $K$ on $(E, \bar{\partial}_E)$ with $\text{tr}(\sqrt{-1} A_\omega F_{K, \bar{\partial}_E}) = 0$ and consider the following perturbed equation
\[
(3.9) \quad \sqrt{-1} A_\omega F_{H_\epsilon, \bar{\partial}_E} + \varepsilon \log f_\epsilon = 0, \quad \varepsilon \in (0, 1],
\]
where $f_\epsilon = K^{-1} H_\epsilon$. It has been proved in [15, 18] that (4.8) is solvable for all $\varepsilon \in (0, 1)$. Then for every $\varepsilon$, we consider the following Hermitian-Yang-Mills flow with $H_\epsilon$ as the initial metric
\[
(3.10) \quad \begin{cases}
H_\varepsilon(t)^{-1} \partial H_\varepsilon(t) = -2\sqrt{-1} A_\omega F_{H_\varepsilon(t), \bar{\partial}_E}, \\
H_\varepsilon(0) = H_0,
\end{cases}
\]
and its gauge equivalent flow
\[
(3.11) \quad \begin{cases}
\partial A_\varepsilon(t) = \sqrt{-1}(\bar{\partial}_E A_\varepsilon(t) - \partial A_\varepsilon(t)) A_\omega F_{A_\varepsilon(t)}, \\
A_\varepsilon(0) = (\bar{\partial}_E, H_0).
\end{cases}
\]
Since $(E, \bar{\partial}_E)$ is semi-stable, by Lemma 3.1, we have
\[
\sup_X |\sqrt{-1} A_\omega F_{H_\varepsilon, \bar{\partial}_E}|_{H_\varepsilon} \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]
By (1.1), (1.2) and Lemma 3.2, we have
\[
(3.12) \quad \int_X |F_{A_\varepsilon(t)}|^2_{H_\varepsilon} \omega^n n! \leq \int_X |F_{A_\varepsilon(0)}|^2_{H_\varepsilon} \omega^n n! = \int_X |F_{H_\varepsilon, \bar{\partial}_E}|^2_{H_\varepsilon} \omega^n n! = 0, \quad \varepsilon \to 0.
\]
(3.13)
\[
(3.13) \quad \int_X |\sqrt{-1} A_\omega F_{H_\varepsilon, \bar{\partial}_E}|^2_{H_\varepsilon} \omega^n n! \to 0, \quad \varepsilon \to 0.
\]
This implies for every $\epsilon > 0$, there exists $\varepsilon(\epsilon) > 0$, such that when $\varepsilon < \varepsilon(\epsilon)$, it holds
\[
(3.14) \quad \int_X |F_{A_\varepsilon(t)}|^2_{H_\varepsilon} \omega^n n! \leq \int_X |F_{A_\varepsilon(0)}|^2_{H_\varepsilon} \omega^n n! < \epsilon.
\]
Particularly, there exists $\varepsilon_0 > 0$, such that when $\varepsilon < \varepsilon_0$, it holds
\[
\int_X |F_{A_\varepsilon(0)}|^2_{H_\varepsilon} \omega^n n! < 1.
\]
So by the small energy regularity theorem (Lemma 3.3), there exist uniform positive constants $\epsilon_0$ and $\delta_0$ depending only on the geometry of $(X, \omega)$, such that for any $(x_0, t_0) \in X \times \mathbb{R}^+$, if for some $0 < R < \min\{i_X/2, \sqrt{t_0}/2\}$,
\[
R^{2-2n} \int_{P_R(x_0, t_0)} |F_{A_\varepsilon(0)}|^2_{H_\varepsilon} < \epsilon_0
\]
holds, then for any $\delta \in (0, \min\{\delta_0, 1/4\})$, we have
\[
\sup_{P_{R}(x_0, t_0)} |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}} < 16(\delta R)^{-4}.
\]
In addition, by (3.14), setting $\epsilon = \frac{1}{2}(\frac{iX}{2})^{2n-4} \epsilon_0$, we can find a positive constant $\epsilon_1$, such that when $\epsilon < \epsilon_1$, it holds
\[
\int_X |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(x, t) < \frac{1}{2} \left( \frac{iX}{2} \right)^{2n-4} \epsilon_0,
\]
for any $t > 0$. Choose $R = iX/2$. We have for any $x_0 \in X$ and $\tau \geq iX^2$, when $\epsilon < \epsilon_1$, it holds (3.15)
\[
R^{2-2n} \int_{P_{R}(x_0, \tau)} |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(x, t) \leq R^{2-2n} \int_{\tau-R^2} \int_X |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(x, t) \leq 2R^{4-2n} \times \left( \frac{1}{2} \epsilon_0 R^{2n-4} \right) = \epsilon_0.
\]
Then from the small energy regularity theorem, we have
\[
(3.16) \quad |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(x_0, \tau) \leq \sup_{P_{R/2}(x_0, \tau)} |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(x, t) \leq 16 \left( \frac{1}{2} \delta_0 R \right)^{-4} = 256(\delta_0 R)^{-4}, \quad \epsilon < \epsilon_1.
\]
This implies that when $\epsilon < \epsilon_1$, $\sup_X |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(\cdot, t)$ is uniformly bounded in $[iX^2, \infty)$. From the Bochner type inequality (3.7), when $\epsilon < \epsilon_1$, there exists a positive constant $C_2$ independent of $\epsilon$, such that
\[
(3.17) \quad (\triangle g - \frac{\partial}{\partial t})|F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}} \geq -C_2 |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}.
\]
Using the parabolic mean value inequality, we can find a positive constant $C_3$ independent of $\epsilon(\epsilon < \epsilon_1)$, such that for $t > iX^2 + 1$, it holds
\[
(3.18) \quad \sup_X |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(\cdot, \cdot) \leq C_3 \int_X |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(x, t - 1) \leq C_3 \int_X |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(\cdot, 0)
\]
\[
= C_3 \int_X |F_{H_{\epsilon}, \delta_{\epsilon}}^2|_{H_{\epsilon}} = C_3 \int_X |\sqrt{-1}A_{\omega}F_{H_{\epsilon}, \delta_{\epsilon}}^2|_{H_{\epsilon}}.
\]
Choosing $t = iX^2 + 2$, from the above, we have when $\epsilon < \epsilon_1$.
\[
(3.19) \quad \sup_X |F_{H_{\epsilon}(iX^2+2), \delta_{\epsilon}}^2|_{H_{\epsilon}(iX^2+2)} = \sup_X |F_{A_{x_{\epsilon}}}^2|_{H_{\epsilon}}(\cdot, iX^2 + 2) \leq C_3 \int_X |\sqrt{-1}A_{\omega}F_{H_{\epsilon}}^2|_{H_{\epsilon}} \to 0, \quad \epsilon \to 0.
\]
This implies the existence of approximate Hermitian flat structure on semistable vector bundles with $ch_1(E, H) \cdot [\omega^{n-1}] = ch_2(E, H) \cdot [\omega^{n-2}] = 0$.

\[\Box\]

4. Proof of Theorem 1.3

In this section, we will give a detailed proof of Theorem 1.3.

**Proof of Theorem 1.3**

We first prove that (1) $\Rightarrow$ (2). We need only to show $\det E$ is Hermitian flat. By Proposition 2.11, we have $\det E$ is Hermitian flat. This implies
\[
(4.1) \quad \det E \cdot [\omega^{n-1}] = c_1(E) \cdot [\omega^{n-1}] = c_1(\det E) \cdot [\omega^{n-1}] = 0.
\]
Then we prove (2) \(\Rightarrow\) (3). Equality (4.1) implies \(\deg_\omega E = 0\). For any proper saturated subsheaf \(F \hookrightarrow E\), \(E/F\) is torsion free and \((E/F)^*\) is a proper subsheaf of \(E^*\). Following the argument in [4, Theorem 1.18], it holds

\[
(4.2) \quad \deg_\omega (E/F)^* \leq 0.
\]

Together with \(\deg_\omega E = 0\), we have

\[
\deg_\omega F \leq 0.
\]

This implies \((E, \bar{\partial}_E)\) is semi-stable. Then it remains to show \(ch_2(E, H) \cdot [\omega^{n-2}] = 0\). Since \((E, \bar{\partial}_E)\) is nef with \(ch_1(E, H) : [\omega^{n-1}] = 0\), 

\[
\det E \leq 0.
\]

Equality (4.1) implies \(\deg_\omega E \geq 0\). Following the argument in [4, Theorem 1.18], it holds

\[
(4.3) \quad c_1(E)^2 \cdot [\omega^{n-2}] = c_1(\det E)^2 \cdot [\omega^{n-2}] = 0.
\]

Let \(h_1\) and \(h_2\) be two Hermitian metrics on \(O_E(1)\). It is easy to check that

\[
(4.4) \quad \Theta(O_E(1), h_1) - \Theta(O_E(1), h_2) = \bar{\partial} \log h_2^{-1} h_1.
\]

Since \(\partial \bar{\partial} \omega^{-1} = \partial \bar{\partial} \omega^{-2} = 0\) and \(h_2^{-1} h_1\) is a well-defined smooth function,

\[
(4.5) \quad \int_{PE} \Theta(O_E(1), h_1)^r \land \pi^* \omega^{n-1} - \int_{PE} \Theta(O_E(1), h_2)^r \land \pi^* \omega^{n-1} = 0
\]

and

\[
(4.6) \quad \int_{PE} \Theta(O_E(1), h_1)^{r+1} \land \pi^* \omega^{n-2} - \int_{PE} \Theta(O_E(1), h_2)^{r+1} \land \pi^* \omega^{n-2} = 0.
\]

This implies that \(\int_{PE} \Theta(O_E(1), h)^r \land \pi^* \omega^{n-1}\) and \(\int_{PE} \Theta(O_E(1), h)^{r+1} \land \pi^* \omega^{n-2}\) are independent of the choice of Hermitian metrics on \(O_E(1)\).

Endow \(PE\) with a Hermitian metric \(\omega_{PE}\). Since \((E, \bar{\partial}_E)\) is nflat, \(O_E(1)\) is nef. This means for every \(\epsilon > 0\), there exists a Hermitian metric \(h_\epsilon\) on \(O_E(1)\), such that

\[
(4.7) \quad \sqrt{-1} \Theta(O_E(1), h_\epsilon) \geq -\epsilon \omega_{PE}.
\]

So

\[
0 \leq \int_{PE} (\sqrt{-1} \Theta(O_E(1), h_\epsilon) + \epsilon \omega_{PE})^{r+1} \land \pi^* \omega^{n-2} = \int_{PE} (\sqrt{-1} \Theta(O_E(1), h_\epsilon))^{r+1} \land \pi^* \omega^{n-2}
\]

\[
+ \sum_{i=1}^{r+1} \int_{PE} (\sqrt{-1} \Theta(O_E(1), h_\epsilon))^{r+1-i} \land (\epsilon \omega_{PE})^i \land \pi^* \omega^{n-2} \to \int_{PE} (\sqrt{-1} \Theta(O_E(1), h))^{r+1} \land \pi^* \omega^{n-2}, \quad \text{as} \quad \epsilon \to 0.
\]
So by Lemma 2.14,

\[ (4.8) \quad s_2(E) \cdot [\omega^{n-2}] = \int_{PE} \Xi^{r+1} \wedge \pi^*\omega^{n-2} \geq 0, \]

where \( \Xi = \sqrt{-1} \Theta(O_E(1), h) \) and \( h \) is an arbitrary metric on \( O_E(1) \). From (4.3), (4.8) and (2.14), we have

\[ (4.9) \quad c_2(E) \cdot [\omega^{n-2}] = c_1(E)^2 \cdot [\omega^{n-2}] - s_2(E) \cdot [\omega^{n-2}] \leq 0. \]

On the other hand, since \( E \) is semi-stable, by Bogomolov inequality (Proposition 2.6), we have

\[ (4.10) \quad c_2(E) \cdot [\omega^{n-2}] \geq \frac{r-1}{2r} c_1(E)^2 \cdot [\omega^{n-1}] = 0. \]

Combining (4.9) and (4.10), we have

\[ (4.11) \quad c_2(E) \cdot [\omega^{n-2}] = 0 \]

and consequently \( ch_2(E, H) \cdot [\omega^{n-2}] = \frac{1}{2}(c_1^2(E) - 2c_2(E)) \cdot [\omega^{n-2}] = 0. \)

(3) \( \Rightarrow \) (4) is just Theorem 1.6.

(4) \( \Rightarrow \) (5). This can be proved by the result of the existence of Harder-Narasimhan filtration on non-Kähler manifolds (3) and the argument of Step 2 and Step 3 in the proof of Theorem 1.1 in [20]. Here we omit the proof.

At last, we prove (5) \( \Rightarrow \) (1). It is obvious that Hermitian flat vector bundles are nflat. And by Proposition 2.13 we get that (5) implies (1).

\[ \square \]

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