Berry curvature induced anisotropic magnetotransport in a quadratic triple-component fermionic system

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Abstract

Triple-component fermions (TCFs) are pseudospin-1 quasiparticles hosted by certain three-band semimetals in the vicinity of their band-touching nodes (2019 Phys. Rev. B 100 235201). The excitations comprise of a flat band and two dispersive bands. The energies of the dispersive bands are $E_{\pm} = \pm \sqrt{\alpha^2 k_n^2 + v_z^2 k_{\perp}^2}$ with $k_{\perp} = \sqrt{k_x^2 + k_y^2}$ and $n = 1, 2, 3$. In this work, we obtain the exact expression of Berry curvature, approximate form of density of states and Fermi energy as a function of carrier density for any value of $n$. In particular, we study the Berry curvature induced electrical and thermal magnetotransport properties of quadratic ($n = 2$) TCFs using semiclassical Boltzmann transport formalism. Since the energy spectrum is anisotropic, we consider two orientations of magnetic field ($B$): (i) $B$ applied in the $x$–$y$ plane and (ii) $B$ applied in the $x$–$z$ plane. For both the orientations, the longitudinal and planar magnetoelectric/magnetothermal conductivities show the usual quadratic-$B$ dependence and oscillatory behavior with respect to the angle between the applied electric field/temperature gradient and magnetic field as observed in other topological semimetals. However, the out-of-plane magnetoconductivity has an oscillatory dependence on angle between the applied fields for the second orientation but is angle-independent for the first one. We observe large differences in the magnitudes of transport coefficients for the two orientations at a given Fermi energy. A noteworthy feature of quadratic TCFs which is typically absent in conventional systems is that certain transport coefficients and their ratios are independent of Fermi energy within the low-energy model.

Keywords: magnetotransport, electrical conductivity, thermal conductivity, Boltzmann transport formalism, Berry curvature, topological semimetals

(Some figures may appear in colour only in the online journal)

1. Introduction

Topologically nontrivial gapless materials termed as topological semimetals have unearthed vast research interests in condensed matter physics in the past few years. Some well-known examples are topological Dirac [1–5] and Weyl [6–13] semimetals which can host emergent quasiparticle excitations analogous to the fermionic elementary particles in high-energy physics. They are characterized by symmetry-protected crossings of two energy bands at the Fermi level. In close vicinity of such band-touching points, the fermionic system can be effectively expressed in terms of pseudospin degrees of freedom, where the distinct eigenvalues of the pseudospin projection corresponds to different energy bands.

Apart from the above mentioned half-integer spin Weyl and Dirac semimetals, recent studies [14] suggested the existence of another class of topological semimetals with three-, six-,
or eight-fold degenerate points at the Fermi level that exhibit fermionic quasiparticles having no counterparts in high-energy physics. Furthermore, it was observed that various possible symmetries existing in symmorphic crystals, for instance, mirror and discrete rotational symmetries may lead to topologically protected three-fold degenerate crossing points [15, 16]. It was predicted by first-principles calculations [17–20] that several material candidates categorized under symmorphic space groups, such as TaN [18], MoP [21], WC [22], RhSi [23], RhGe [24] and ZrTe [25] can host crossings of three bands in the neighborhood of the Fermi level. The low energy excitations in such systems behave like pseudospin-1 quasiparticles but obey fermionic statistics and are named as triple-component fermions (TCFs) [26]. The touching of these energy bands occurs only in pairs at specific points in the Brillouin zone whose location is decided by the underlying symmetry of the system. Such band-touching points are called triple-component points or nodes.

The general low-energy Hamiltonian of such pseudospin-1 fermionic system in the vicinity of band-touching points can be effectively described as \( H = d(\mathbf{k}) \cdot \mathbf{S} \), where \( d(\mathbf{k}) \) is a function of momenta and \( \mathbf{S} \) are the three spin-1 matrices. The low-energy dispersion of the quasiparticle near each node appears as \( m \sqrt{\alpha_n^2 (k_x^2 + k_y^2) + \nu z_k^2} \) with \( m = \pm 1 \) and 0 representing the dispersive and flat bands respectively. Here, the different combination of crystal symmetries [27] imposes a restriction on \( n \), such that \( n \leq 3 \). TCFs are predicted to occur in time-reversal symmetric (TRS) materials such as Pd$_3$Bi$_2$S$_2$ and Ag$_2$Se$_2$Au with space group 199 and 214 respectively [14]. The experimental realization of materials having TRS broken TCFs is still missing.

The band touching nodes of topological semimetals are sources and drains of Berry curvature [28, 29] which acts as a fictitious magnetic field in the momentum space and leads to several intriguing transport phenomena such as anomalous Hall effect [29–33], anomalous thermal Hall effect [34–36], negative longitudinal magnetoresistance (MR) [37–47], planar Hall effect (PHE) [48–52] and planar thermal Hall effect (PTHE) [53–55]. The Berry curvature induced magnetotransport phenomena in type-I multi-Weyl semimetals has been studied recently [56]. The experimental realization of materials having TRS broken TCFs is still missing.

The effective low-energy Hamiltonian for TRS broken generalized TCFs can be expressed as [26]

\[
H_{n,\tau}(\mathbf{k}) = d_{1}^\tau(\mathbf{k}) S_x + d_{2}^\tau(\mathbf{k}) S_y + \tau d_{3}(\mathbf{k}) S_z, \quad (1)
\]

where \( \tau = \pm 1 \) is the valley index representing the two valleys or triple-component points and \( S_x, S_y, \) and \( S_z \) are the components of spin-1 matrix operator given by

\[
S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\]

\[
S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Here, the different combinations of symmetries in pseudospin-1 fermionic systems impose a restriction on \( n \), i.e., \( n \leq 3 \). The coefficients \( d_{1,2,3}(\mathbf{k}) \) are functions of momenta which vary according to the given choice of \( n \) [26]. The Hamiltonian in equation (1) can be rewritten compactly as

\[
H_{n,\tau}(\mathbf{k}) = \alpha_n \mathbf{k}^\mu (S_x \cos n\phi + S_y \sin n\phi) + \tau \nu \mathbf{r} \cdot \mathbf{k} S_z, \quad (3)
\]

where \( \mathbf{k} = \sqrt{k_x^2 + k_y^2}, \nu = v/c, \mathbf{r} = (\alpha_1, \alpha_2, \alpha_3) \) have the dimension of velocity, \( \alpha_2/\hbar \) has the dimension of inverse mass and \( \alpha_3 \) has
the dimension of inverse of density of states (DOS). The diagonalization of the Hamiltonian leads to the energy spectra in the neighborhood of each valley as
\[ e_n^0(k) = m \sqrt{\alpha_n^2 k^2 + v_0^2 k^2}, \]
where \( m = -1, 0, 1 \) is the pseudospin projection. The dispersive bands with \( m = \pm 1 \) indicate the conduction and valence bands respectively and \( m = 0 \) represents the flat band at zero energy. For \( n = 1 \), the two dispersive bands scale linearly with all the three components of momentum. These excitations are called linear TCFs. For \( n = 2 \), the excitations are named as quadratic TCFs since the energy scales linearly with \( k_z \) but quadratically with \( k_x \). Cubic TCFs refer to the \( n = 3 \) case, where the energy scales linearly with \( k_x \) but cubically with \( k_z \).

The eigenstates corresponding to the three bands can be written as
\[ |\psi_{n,r,k}^+\rangle = \left( \begin{array}{c} \frac{1 + \tau \cos \gamma_n}{2} \\
\frac{\sin \gamma_n e^{i\phi}}{\sqrt{2}} \\
\frac{1 - \tau \cos \gamma_n}{2} e^{2i\phi} \end{array} \right), \]
\[ n = 1,2,3 \] (5)

The Berry curvature of the flat band is identically zero for all momenta, but is finite for the two dispersive bands carrying opposite signs. The Berry curvature of linear TCFs contains only the radial component while that of quadratic and cubic TCFs have angular components apart from the radial ones. In the \( k_x - k_y \) plane, the Berry curvature varies as \( \sim m n r \tau \nu \gamma k^{n+1} k \hat{k} \). Integrating the Berry curvature over a unit sphere enclosing the triple-component point defines the monopole charge, given by
\[ N = \frac{1}{2\pi} \int_A \Omega_k \cdot d\mathbf{A} = 2n. \]
\[ \Omega_{n,r,k}^m = m n r \tau \nu \gamma k^{2n-4} \left[ -\frac{(\sin \theta)^{2n-2}[1 + (n - 1) \cos^2 \theta]k + [(n - 1) \cos \theta \sin \theta]^{2n-1}\theta}{\alpha_n^2 k^{2n-2}(\sin^2 \theta + v_0^2 \cos^2 \theta)^{3/2}} \right]. \]
\[ n = 1,2,3 \] (8)

The vector plots of Berry curvature of conduction band for TCFs with \( n = 1,2,3 \) and \( \tau = 1 \) in \( k_x - k_y \) plane and \( k_x - k_z \) plane are shown in figure 1. The angular dependence of Berry curvature for \( n = 2 \) and \( n = 3 \) can be seen in the plots for \( k_z - k_x \) plane. The converging nature of arrows represents the sink of Berry curvature at the origin.

The DOS for linear TCFs including valley degeneracy is given by
\[ D_1(\mu) = D_1' \frac{\mu^2}{2\pi^2}, \]
\[ \mu = \mu_1', \]
\[ \mu_1' = k' \nu \gamma, \]
where \( D_1' = 2k'^2 \nu_\gamma / \alpha^2 \) and \( \mu = \mu_1' k' \nu \gamma \) being an energy scale and \( k' \) an arbitrary wavenumber scale. The DOS for quadratic TCFs can be obtained as
\[ D_2(\mu) = D_2' \frac{\mu^2}{2\pi^2} \sqrt{\cos^2 \theta + 4\mu^2} + \frac{\mu^2}{2\pi^2} \sin^2 \theta \]
\[ \left( \frac{\mu_1^2}{2\pi^2} \right) \]
\[ \mu = \mu_2', \]
\[ \mu_2' = \nu_0^2 / \alpha^2, \]
\[ \nu_0^2 = \nu_0^2 / \alpha^2. \]
\[ D_2(\mu) = D_2' \frac{\mu^2}{2\pi^2} \sqrt{\cos^2 \theta + 4\mu^2} + \frac{\mu^2}{2\pi^2} \sin^2 \theta \]
\[ \left( \frac{2\pi^2}{\mu_2^2} \right) \]
\[ \mu = \mu_2', \]
\[ \mu_2' = \nu_0^2 / \alpha^2. \]
\[ D_2(\mu) = D_2' \frac{\mu^2}{2\pi^2} \sqrt{\cos^2 \theta + 4\mu^2} + \frac{\mu^2}{2\pi^2} \sin^2 \theta \]
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\[ \left( \frac{2\pi^2}{\mu_2^2} \right) \]
\[ \mu = \mu_2', \]
\[ \mu_2' = \nu_0^2 / \alpha^2. \]
Figure 1. Vector plots of the Berry curvature of the conduction band for triple-component semimetals—upper row: $n = 1$, middle row: $n = 2$ and lower row: $n = 3$ in $k_x-k_y$ plane and $k_x-k_z$ plane respectively.

The quadratic and cubic TCFs show the variation with electron density as $\mu/\mu'_{2} \sim (n_e/n'_{2})^{1/2}$ and $\mu/\mu'_{3} \sim (n_e/n'_{3})^{3/5}$ with $n'_{2} = (\nu_{e}/\alpha_{2})^{2}$ and $n'_{3} = (\nu_{e}/\alpha_{3})^{3/2}$ respectively. In figure 3, we have plotted the Fermi energy as a function of electron density for generalized TCFs which scales as $\mu \sim (n_e)^{n_{n}+2}$.

3. Theoretical formulation

We will use Boltzmann transport formalism to obtain the magnetotransport properties of quadratic TCFs. The semiclassical Boltzmann transport approach is valid for weak magnetic fields such that $\omega_c \tau \ll 1$, where $\omega_c$ is the cyclotron frequency and $\tau$ is the average time between two successive collisions. In this limit, the Landau quantization gets wiped out by disorder effects.

The quadratic and cubic TCFs show the variation with electron density as $\mu/\mu'_{2} \sim (n_e/n'_{2})^{1/2}$ and $\mu/\mu'_{3} \sim (n_e/n'_{3})^{3/5}$ with $n'_{2} = (\nu_{e}/\alpha_{2})^{2}$ and $n'_{3} = (\nu_{e}/\alpha_{3})^{3/2}$ respectively. In figure 3, we have plotted the Fermi energy as a function of electron density for generalized TCFs which scales as $\mu \sim (n_e)^{n_{n}+2}$.

The phenomenological Boltzmann transport equation (BTE) for the non-equilibrium distribution function $f_{r,k,r}^{m}$ is given by [59]

$$
\left( \frac{\partial}{\partial t} + \hat{v}^{m} \cdot \nabla_{r} + \hat{k}^{m} \cdot \nabla_{k} \right) f_{r,k,r}^{m} = I_{\text{coll}}\{ f_{r,k,r}^{m} \},
$$

where right-hand side represents the collision integral and $m$ is the band index. Under the relaxation-time approximation, the collision integral takes the form, $I_{\text{coll}}\{ f_{b}^{m} \} = -\frac{f_{b}^{m} - f_{eq}^{m}}{\tau_{b}^{m}}$, where the scattering time scale $\tau_{b}^{m}$ is the intranode relaxation time and $f_{eq}^{m} = [1 + e^{\beta (\epsilon_{m} - \mu)}]^{-1}$ is the equilibrium Fermi–Dirac distribution function. We will ignore the momentum dependence of relaxation time and assumed it to be constant throughout the paper. In the steady-state condition, the BTE takes the...
Dynamics as the invariant phase space volume for Berry curvature modified following form

\[
\partial f_k^m / \partial \epsilon = \left[ -eE / T - \nabla T \right] - \left( n_k^m - \mu_k^m \right) \nabla \mu_k^m.
\]

(17)

The second term in the above expression denotes the deviation from equilibrium. The term \((v_k^m \cdot \Omega_k^m)\) represents the Lorentz force contribution whose analytical expressions are given in the appendix A [48,55]. Since the prime focus of this paper is to study the effect of Berry curvature in transport coefficients, we have not considered the Lorentz force contribution to the conductivity. The linear response equations for the charge current \(J^c\) and thermal current \(J^T\) to the external fields can be written as [59]

\[
J^c_a = L_{a1}^{11} E_b + L_{ab}^{12} (\nabla_b T),
\]

(18)

and

\[
J^T_a = L_{a1}^{21} E_b + L_{ab}^{22} (\nabla_b T),
\]

(19)

where \(a\) and \(b\) are the spatial indices running over \(x, y\) and \(z\). The set of \(L\) represents the transport coefficients, for instance, \(L^{11}\) and \(L^{22}\) are the electrical and thermal conductivities respectively. The thermopower is defined as \(S = L^{12}/L^{11}\). The Onsager’s relation relates the \(L^{12}\) and \(L^{21}\) as \(L^{21} = TL^{12}\).

In the subsequent sections, we study the Berry curvature driven Boltzmann transport phenomena for quadratic TCFs.

4. Electrical transport

In the absence of thermal gradient, the charge current is defined as

\[
j = -e \sum_m \int [dk] |D_k^m| (v_k^m) f_{eq}^m.
\]

(20)

Substituting equations (15) and (17) in equation (20), we obtain the general expression of Berry curvature dependent electric conductivity as:

\[
\sigma_{ij} = -e^2 \sum_m \int [dk] \epsilon_{ij}(\Omega_k^{m}) f_{eq}^m
\]

\[
+ e^2 \sum_m \int [dk] (D_k^m)^{-1} \left( v_k^m + \frac{e}{h} B_j (v_k^m \cdot \Omega_k^m) \right)
\]

\[
\times \left( v_k^l + \frac{e}{h} B_l (v_k^m \cdot \Omega_k^m) \right) \left( -\frac{\partial f_{eq}^m}{\partial \epsilon_k} \right),
\]

(21)

where \(\epsilon_{ij}\) is the Levi-Civita tensor and \(\Omega_k^{m}\) denotes the \(l\)-component of Berry curvature. We expand the term \((D_k^m)^{-1}\) up to quadratic order in \(B\)-field as \((1 + (e/h)B \cdot \Omega_k^m)^{-1} = 1 - (e/h)B \cdot \Omega_k^m + \left( e^2 / h^2 \right)(B \cdot \Omega_k^m)^2\) in order to express the conductivity in power of magnetic field by separating various \(B\)-contributions. For the above expansion to converge, we must have \(\left( \frac{\partial \Omega_k^m}{\partial \epsilon_k} \right) \ll 1\). We find that the above condition

![Figure 2. DOS as a function of the Fermi energy for triple-component semimetals with \(n = 1, 2, 3\).](image)

![Figure 3. Plots of variation of the Fermi energy with electron density in the \(T \to 0\) limit for linear, quadratic and cubic TCFs.](image)
is satisfied when \( \tilde{\mu} \gg 3.5 \) for \( B = 2 \), where \( B = B_0/\tilde{B} \) and \( \tilde{\mu} = \mu/\mu_0 \) with \( B_0 = v_F^2 \hbar/e\alpha^2 \) and \( \mu_0 = v_F^2/\alpha_1 \) being the magnetic field and energy scales respectively, of the given system. Therefore, for the rest of our calculations, we choose \( B = 2 \) and \( \tilde{\mu} = 10 \).

We emphasize that out of three bands, only the two dispersive bands participate in the transport properties. In this work, we choose \( \mu > 0 \) and thus equation (21) will have contribution only from conduction band \( (m = 1) \). Therefore, we drop the band index \( m \) for rest of the paper to investigate these quantities. The first term of equation (21) represents the Berry curvature induced intrinsic anomalous Hall conductivity. Since we are interested in calculation of transport coefficients in presence of magnetic field, we neglect the anomalous term.

It is to be noted that the magnetotransport results will be qualitatively same for both low-energy model and lattice model as long as Fermi energy is close to the nodes. However, the calculation of anomalous Hall conductivity using low-energy model gives completely different result as obtained from the tight-binding model. In low-energy model, anomalous Hall conductivity trivially vanishes, whereas for lattice model it is finite and proportional to the distance between the nodes [26].

**Drude conductivities.** In the absence of magnetic field, the diagonal components of conductivity matrix are called Drude conductivities \( \sigma_{ii}^{(0)} \). We compute the Drude conductivities of quadratic TCFs in this section and discuss their dependence on Fermi energy. In the wake of anisotropic energy spectrum of quadratic TCFs, we have \( \sigma_{xx}^{(0)} = \sigma_{yy}^{(0)} \neq \sigma_{zz}^{(0)} \). The general form of Drude conductivity is given by

\[
\sigma_{ii}^{(0)} = e^2 \tau \frac{d^3k}{(2\pi)^3} \nu_i^2 \left( -\frac{\partial f_{eq}}{\partial \epsilon_k} \right),
\]

where we know that \( (-\partial f_{eq}/\partial \epsilon_k) = \delta(\mu - \epsilon_k) \) in the zero temperature limit. The semiclassical band velocities of the system can be obtained as

\[

v_{x} = \frac{2\alpha_1^2 k^2 \sin \theta \cos \phi}{\hbar \sqrt{\alpha_2^2 k^2 \sin^2 \theta + \nu_2^2 \cos^2 \theta}},
\]

\[

v_{y} = \frac{2\alpha_1^2 k^2 \sin \theta \sin \phi}{\hbar \sqrt{\alpha_2^2 k^2 \sin^2 \theta + \nu_2^2 \cos^2 \theta}},
\]

and

\[

v_{z} = \frac{\nu_2^2 \cos \theta}{\hbar \sqrt{\alpha_2^2 k^2 \sin^2 \theta + \nu_2^2 \cos^2 \theta}}.
\]

The Drude conductivity along the \( x \) direction can be obtained as:

\[
\sigma_{xx}^{(0)} = \frac{e^2 \tau}{(2\pi)^3} \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi \int_0^{\infty} k^2 dk \nu_x^2 \delta(\mu - \epsilon_k).
\]

We have used the property of Dirac delta function to evaluate the above integrand which is given by

\[
\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|},
\]

where \( x_i \) are the roots or zeroes of function \( g(x) \). For the case of quadratic TCFs, using the above equation, \( \delta(\mu - \epsilon_k) \equiv \delta[g(k)] \) can be written as

\[
\delta \left[ \mu - \sqrt{\alpha_1^2 k^4 \sin^4 \theta + \nu_2^2 k^2 \cos^2 \theta} \right] = \frac{\delta(k - k_F)}{|g'(k_F)|},
\]

where the root of \( g(k) \) for the conduction band is obtained as

\[
k_F = -\nu_2^2 \cos^2 \theta + \sqrt{\nu_2^4 \cos^4 \theta + 4\alpha_1^2 \nu_2^2 \sin^4 \theta} \quad \left( \frac{1}{2\nu_2^2 \sin^2 \theta} \right).
\]

Using the expression of \( k_F \) in equation (26), we get

\[
\sigma_{xx}^{(0)} = 2 \left( \frac{e^2 \tau \nu_x^2}{8\sqrt{2\pi^3} \alpha_2^2 \hbar^2} \right) \int_0^{\pi} d\theta \left( \frac{-\cos^2 \theta + \sqrt{\cos^2 \theta + 4\mu^2 \sin^4 \theta}^3}{\sin^5 \theta \left( \sqrt{\cos^2 \theta + 4\mu^2 \sin^2 \theta} \right) \left( \cos^2 \theta + \sqrt{\cos^2 \theta + 4\mu^2 \sin^2 \theta} \right)} \right),
\]

and \( \sigma_{yy}^{(0)} = \sigma_{xx}^{(0)} \). The Drude conductivity along the \( z \) direction is obtained as

\[
\sigma_{zz}^{(0)} = 2 \left( \frac{e^2 \tau \nu_z^3}{8\sqrt{2\pi^3} \alpha_2^2 \hbar^2} \right) \int_0^{\pi} d\theta \left( \frac{-\cos^2 \theta + \sqrt{\cos^2 \theta + 4\mu^2 \sin^2 \theta}}{\sqrt{\cos^2 \theta + 4\mu^2 \sin^2 \theta}} \right) \times \left( \frac{2 \cos^2 \theta}{\sqrt{\cos^2 \theta + \sqrt{\cos^2 \theta + 4\mu^2 \sin^2 \theta}}} \right),
\]

\[
\sigma_{zz}^{(0)}.
\]
In the above expressions, an overall factor of 2 is multiplied, covering the contributions from two triple-component nodes. We have plotted the numerically calculated Drude conductivities as a function of the Fermi energy in figures (a) and (b). The large difference in the values of $\sigma_{xx}$ and $\sigma_{zz}$ illustrates the strong anisotropic nature of the system. The variation of $\sigma_{zz}^{(2)}$ is more linear in Fermi energy than $\sigma_{xx}^{(0)}$.

Next, we consider the two orientations of magnetic field i.e., $\mathbf{B}$ in the $x$-$y$ plane and $\mathbf{B}$ in the $x$-$z$ plane. We have computed numerically the different components of electrical conductivity for the low-energy model of the quadratic TCFs (including the contribution of two triple-component nodes) in the following subsections.

4.1. Magnetic field applied in the $x$-$y$ plane

We consider a planar Hall geometry where the applied electric field is along the $x$ axis ($\mathbf{E} = E\hat{x}$) and the magnetic field is confined in the $x$-$y$ plane making an angle $\beta$ with the $x$ axis, i.e., $\mathbf{B} = (B\cos \beta, B\sin \beta, 0)$ as shown in figure 5(a). The electric and magnetic fields are parallel to each other at $\beta = 0$ resulting in maximum value of $\mathbf{E} \cdot \mathbf{B}$.

The expression for the LMC using equation (21) can be written as

$$\sigma_{xx} = e^2 \tau \int \frac{d^3k}{(2\pi)^3} D_k^{-1} \left[ v_x + \frac{eB \cos \beta}{h} (\mathbf{v}_k \cdot \mathbf{\Omega}_k) \right]^2 \times \left( -\frac{\partial f_{sa}}{\partial \varepsilon_k} \right).$$

(32)

Further, the magnetococonductivity in the planar configuration $\sigma_{xx}$ can be expressed explicitly in terms of $\sigma_{\parallel}$ and $\sigma_{\perp}$ as

$$\sigma_{xx} = \sigma_{\perp} + \Delta \sigma \cos^2 \beta,$$

(33)

where $\Delta \sigma = \sigma_{\parallel} - \sigma_{\perp}$ with $\sigma_{\parallel} = \sigma_{xx}(\beta = 0) = \sigma_{xx}^{(0)} + \sigma_{xx}^{(2)}$ and $\sigma_{\perp} = \sigma_{xx}(\beta = \pi/2) = \sigma_{xx}^{(0)} + \sigma_{xx}^{(2)}$. The $\cos^2 \beta$ results in oscillations in the LMC when the magnetic field is rotated in the $x$-$y$ plane, as shown in figure 5(b). The $B^2$ dependence of LMC is depicted in figure 5(c) for $\beta = 0$.

**Planar Hall effect:** we observe PHE in the quadratic TCFs for the given planar geometry. The PHE [48, 49] refers to a phenomenon where a voltage is induced perpendicular to the applied electric field in presence of a magnetic field which is coplanar with the applied field and the induced voltage. The expression for PHC is obtained as

$$\sigma_{xx} = e^2 \tau \int \frac{d^3k}{(2\pi)^3} D_k^{-1} \left[ v_x + \frac{eB \cos \beta}{h} (\mathbf{v}_k \cdot \mathbf{\Omega}_k) \right]^2 \times \left[ v_y + \frac{eB \sin \beta}{h} (\mathbf{v}_k \cdot \mathbf{\Omega}_k) \right] \left( -\frac{\partial f_{sa}}{\partial \varepsilon_k} \right).$$

(34)

Using the above expression, the PHC can be written as

$$\sigma_{xx} = \Delta \sigma \sin \beta \cos \beta,$$

(35)

where $\Delta \sigma$ has been defined after equation (33). The angular dependence of PHC is shown in figure 6(a). The PHC reaches the maximum value at an odd multiple of $\pi/4$. The amplitude of PHC ($\Delta \sigma$) shows a quadratic dependence on $B$ for any value of $\beta$ (except for $\beta = 0$ and $\beta = \pi/2$ where it trivially vanishes) as shown in figure 6(b). Since its origin is associated with Berry curvature and not the Lorentz force, the PHC obeys a symmetric relation $\sigma_{xx} = \sigma_{yy}$ unlike the regular Hall conductivity.

The other diagonal elements of conductivity for different directions of electric field are calculated as: $\sigma_{xy}(\beta) = \sigma_{xx}(\pi/2 - \beta)$ and the $zz$-component of conductivity is given by

$$\sigma_{zz} = e^2 \tau \int \frac{d^3k}{(2\pi)^3} D_k^{-1} \left[ v_z + \frac{eB \cos \beta}{h} (\mathbf{v}_k \cdot \mathbf{\Omega}_k) \right]^2 \times \left( -\frac{\partial f_{sa}}{\partial \varepsilon_k} \right).$$

(36)

It also follows quadratic-$B$ dependence as shown in figure 5(d) and no angular dependence. It is to be noted that the lowest-order magnetic field correction to the conductivity comes quadratic and the $B$-linear dependence of conductivity is zero in our system. All the other off-diagonal components of conductivity are calculated to be zero.

**Dependence on the Fermi energy ($\mu/\mu_B$):** in figures 7(a) and (b), the variation of quadratic $B$ component of the LMC ($\sigma_{xx}^{(2)}$) and PHC ($\sigma_{xx}^{(2)}$) is plotted with respect to the Fermi energy. Both of them decrease with the Fermi energy. This is expected because the magnitude of Berry curvature decreases as the Fermi energy shifts away from the band touching node.

4.2. Magnetic field applied in the $x$-$z$ plane

Now, we consider another planar Hall setup where the electric field is applied along the $z$ direction ($\mathbf{E} = E\hat{z}$) and the magnetic field is rotated in the $x$-$z$ plane such that it makes an angle $\beta$ with respect to the $z$ axis, i.e., $\mathbf{B} = (B \sin \beta, 0, B \cos \beta)$.

The LMC for the above configuration using equation (21) can be written as

$$\sigma_{zz} = e^2 \tau \int \frac{d^3k}{(2\pi)^3} D_k^{-1} \left[ v_z + \frac{eB \cos \beta}{h} (\mathbf{v}_k \cdot \mathbf{\Omega}_k) \right]^2 \times \left( -\frac{\partial f_{sa}}{\partial \varepsilon_k} \right).$$

(37)

The above expression takes the following form

$$\sigma_{zz} = \sigma_{zz}^{(0)} + \sigma_{\parallel} + \sigma_{\perp} \cos^2 \beta,$$

(38)

where $\sigma_{zz}^{(0)}$ is the Drude conductivity given by equation (31) and the conductivity coefficients $\sigma_{\parallel}$ and $\sigma_{\perp}$ are proportional to $B^2$. All the three terms of $\sigma_{zz}$ show significant dependence on Fermi energy. The LMC shows the angular dependence of $\cos^2 \beta$ as shown in figure 8(a). Its $B^2$ dependence (except at $\beta = \pi/2$) is depicted in figure 8(b).

The simplified expression of the PHC for $B$ applied in the $x$-$z$ plane can be written as

$$\sigma_{zz} = e^2 \tau \int \frac{d^3k}{(2\pi)^3} D_k^{-1} \left[ v_z + \frac{eB \cos \beta}{h} (\mathbf{v}_k \cdot \mathbf{\Omega}_k) \right]^2 \times \left[ v_x + \frac{eB \sin \beta}{h} (\mathbf{v}_k \cdot \mathbf{\Omega}_k) \right] \left( -\frac{\partial f_{sa}}{\partial \varepsilon_k} \right).$$

(39)
We can also write the PHC as

$$\sigma_{xy} = \sigma_3 \sin \beta \cos \beta. \quad (40)$$

Its angular and quadratic-B dependences are shown in figures 8(c) and (d) respectively. The maximum value is obtained at $\beta = \pi/4$. The variations of quadratic in $B$ component of the LMC ($\sigma_{zz}^{(2)}$) and PHC ($\sigma_{yz}$) with respect to the Fermi energy are plotted in figure 9(a). Similar to the previous orientation, both the quantities decrease with Fermi energy.

The other diagonal components of the conductivity are obtained as

$$\sigma_{xx} = \sigma_{xx}^{(0)} + \sigma_4 \cos^2 \beta + \sigma_5 \sin^2 \beta, \quad (41)$$

and

$$\sigma_{yy} = \sigma_{yy}^{(0)} + \sigma_4 \cos^2 \beta + \sigma_6 \sin^2 \beta. \quad (42)$$

Here, $\sigma_{xx}^{(0)}$ and $\sigma_{yy}^{(0)}$ calculated numerically from equation (30). Apart from the Drude parts, there are quadratic-$B$ corrections in conductivity viz $\sigma_4$, $\sigma_5$ and $\sigma_6$. Here, $\sigma_4$ denotes the quadratic-$B$ correction in both $\sigma_{xx}$ and $\sigma_{yy}$ when $B \parallel \hat{z}$ while $\sigma_5$ and $\sigma_6$ denotes the quadratic-$B$ correction in $\sigma_{xx}$ and $\sigma_{yy}$ respectively when $B \parallel \hat{x}$. The variation of these correction terms with Fermi energy is shown in figure 9(b). We observe that $\sigma_4$ is independent of Fermi energy while $\sigma_5$ and $\sigma_6$ show a decrease.

An important feature which distinguishes this orientation of planar Hall geometry from the previous one is that the out-of-plane component of conductivity $\sigma_{yy}$ also has an oscillatory behavior as the magnetic field is rotated in the $x$–$z$ plane.
5. Magnetoresistance

We have evaluated the magnetoconductivity matrix numerically from the low-energy model of the quadratic TCFs for two different planar geometries in the last section. Next, we calculate the MR which is defined as

$$\text{MR}_{ij} = \frac{\rho_i(B) - \rho_i(0)}{\rho_i(0)},$$

(43)

where \(\rho_i\) denotes the diagonal components of the resistivity tensor with \(i = x, y, z\). The Drude resistivity \(\rho_i(0)\) is simply the inverse of Drude conductivity \(\sigma_i^{ij}(0)\). In this section also, we study and compare MR for the two orientations of magnetic field mentioned earlier.

For the case of magnetic field applied in the \(x-y\) plane making an angle \(\beta\) with the \(x\) axis, the planar MR [\(\text{MR}_{xx}(\beta)\)] is anisotropic as it shows an angular variation of \(\cos^2\beta\) as shown in figure 10(a). The decrease in magnetoresistivity is maximum at \(\beta = 0\) and \(\pi\) and minimum at \(\beta = \pi/2\). It is evident from figure 10(a), when \(\mu/\mu_0 = 10\) and \(B/B_0 = 2\), the negative MR resulting from the Berry curvature is about \(-1.8\%\) at \(\beta = 0\) and \(\pi\). We have plotted the magnetic field dependence of longitudinal MR (\(\text{MR}_{xx}\)) at \(\beta = 0\) in figure 10(b).

For the second orientation where applied magnetic field is rotated in the \(x-z\) plane by making an angle \(\beta\) with the \(z\) axis, the planar MR is denoted by [\(\text{MR}_{zz}(\beta)\)] which also follows the angular dependence of \(\cos^2\beta\) as shown in figure 10(c).

Here, the MR reaches about \(-29\%) and thus the Berry curvature effects on MR are considerably large in this orientation. The increasing nature of absolute longitudinal MR (\(\text{MR}_{zz}\)) at \(\beta = 0\) with the magnetic field is shown in figure 10(d).

Dependence on the Fermi energy (\(\mu/\mu_0\)): the longitudinal MRs \(\text{MR}_{xx}\) and \(\text{MR}_{zz}\) corresponding to their respective magnetic field configurations are plotted as function of Fermi energy in figures 11(a) and (b). The absolute value of MR in both the cases decrease with the Fermi energy.

6. Thermal conductivity

The thermal current is defined as [53, 59]

$$\bar{j}_q = \int [dk] D_k(\epsilon_k - \mu) \bar{r}_k f_k,$$

(44)

Substituting \(\bar{r}\) and \(f_k\) from equations (15) and (17) in equation (44) yields the following general expression of Berry curvature dependent thermal conductivity:

$$\kappa_{ij} = \frac{-1}{\hbar} \int [dk] c_k \Omega_k \frac{(\epsilon_k - \mu)^2}{T} f_{eq}$$

$$+ \tau \int [dk] [D_k]^{-1} \left( v_i + \frac{e}{\hbar} B_i (v_k \cdot \Omega_k) \right)$$

$$\times \left( v_j + \frac{e}{\hbar} B_j (v_k \cdot \Omega_k) \right) \frac{(\epsilon_k - \mu)^2}{T} \left( -\frac{\partial f_{eq}}{\partial \epsilon_k} \right),$$

(45)
Figure 8. (a) and (c) Shows the angular dependence of LMC ($\sigma_{zz} \propto \cos^2 \beta$) and PHC ($\sigma_{zx} \propto \sin \beta \cos \beta$) calculated numerically for the quadratic TCFs, when a magnetic field is rotated in the $x$–$z$ plane at $B/B_0 = 2$. Here, the y axes of (a) and (c) are normalized by $\sigma_{zz}$ ($\beta = 0$) and $\sigma_{zx}$ ($\beta = \pi/4$) respectively. (b) and (d) Depicts the variation of LMC at $\beta = 0$ (normalized by $\sigma_{zz}$ for $B/B_0 = 2$) and the amplitude of PHC (normalized by $\sigma_{zx}$ for $B/B_0 = 2$) as a function of magnetic field. The calculations are performed at $\mu/\mu_0 = 10$.

Figure 9. (a) Plots of variation of quadratic $B$-correction to LMC ($\sigma_{zz}^{(2)}$ at $\beta = 0$) and PHC ($\sigma_{zx}^{(2)}$ at $\beta = \pi/4$) with the Fermi energy for $B/B_0 = 2$ respectively. In both the cases, the quadratic in $B$ component of LMC and PHC shows decrease with the Fermi energy. (b) Dependence of the quadratic-$B$ correction in $\sigma_{xx}$ and $\sigma_{yy}$ when $B \parallel \hat{x}$, i.e., ($\sigma_4$ and $\sigma_6$) with the Fermi energy. Both the conductivity coefficients decreases with the Fermi energy. The quadratic-$B$ correction in both $\sigma_{xx}$ and $\sigma_{yy}$ when $B \parallel \hat{z}$ denoted by $\sigma_4$ is independent of the Fermi energy. In the plot, $\sigma_4$ is divided by a factor of 10.

where the first term corresponds to the anomalous thermal Hall conductivity without magnetic field. The longitudinal thermal conductivity in the absence of magnetic field is obtained as

$$\kappa_{xx}^{(0)} = \tau \int \frac{d^3k}{(2\pi)^3} \frac{(\epsilon_k - \mu)^2}{T} \left[ -\frac{\partial f_{\epsilon_k}}{\partial \epsilon_k} \right].$$

(46)

6.1. Magnetic field is applied in the $x$–$y$ plane

We will now investigate the LMTC and PTHC for a configuration where temperature gradient is applied along the $x$ axis and magnetic field is in the $x$–$y$ plane making an angle $\beta$ from the $x$ axis such that $\nabla T = \nabla T \hat{x}$, $B = (B \cos \beta, B \sin \beta, 0)$ and $E = 0$.

Next, we consider the temperature gradient and magnetic field are parallel in order to calculate the LMTC which is given by the following expression as

$$\kappa_{xx} = \tau \int \frac{(\epsilon_k - \mu)^2}{T} \left[ v_x + \frac{eB \cos \beta}{h} (v_k \cdot \Omega_k) \right]^2 \times \left( -\frac{\partial f_{\epsilon_k}}{\partial \epsilon_k} \right).$$

(47)

We can further express the magneto-thermal conductivity in the planar configuration $\kappa_{xx}$ in terms of $\kappa_{||}$ and $\kappa_{\perp}$ describing the cases of thermal current flowing parallel and perpendicular to the magnetic field as

$$\kappa_{xx} = \kappa_{\perp} + \Delta \kappa \cos^2 \beta.$$  

(48)

Here $\Delta \kappa = \kappa_{||} - \kappa_{\perp}$ with $\kappa_{||} = \kappa_{xx}(\beta = 0) = \kappa_{xx}^{(0)} + \kappa_{xx}^{(2)}$ and $\kappa_{\perp} = \kappa_{xx}(\beta = \pi/2) = \kappa_{xx}^{(0)} + \kappa_{xx}^{(2)}$. The LMTC has the angular dependence of $\cos^2 \beta$ which is depicted in figure 12(a) and thus shows anisotropic behavior. The LMTC shows the quadratic-$B$...
dependence on the magnetic field (except at $\beta = \pi / 2$) for the quadratic TCFs at $T = 300$ K as shown in figure 12(b).

**Planar thermal Hall effect:** the PTHC [55] is defined as the appearance of in-plane transverse temperature gradient when the applied temperature gradient and magnetic field are coplanar. It is the thermal analog of PHE. The expression for PTHC is obtained as

$$\kappa_{yx} = \tau \int \frac{d^3 k}{(2\pi)^3 \hbar^2} \left[ v_x + \frac{eB \cos \beta}{\hbar} (v_k \cdot \Omega_k) \right] \left[ v_y + \frac{eB \sin \beta}{\hbar} (v_k \cdot \Omega_k) \left( -\frac{\partial f_{eq}}{\partial \epsilon_k} \right) \right],$$

(49)

The PHTC can be further expressed as

$$\kappa_{yx} = \Delta \kappa \sin \beta \cos \beta.$$

(50)

The PTHC follows the angular dependence of $\sin \beta \cos \beta$. The amplitude of PTHC ($\Delta \kappa$) shows a $B^2$ dependence for any value of $\beta$ except for $\beta = 0$ and $\beta = \pi / 2$. We have plotted the angular and magnetic field dependence of $\kappa_{yx}$ at $T = 300$ K in figures 12(c) and (d) respectively.

6.2. Magnetic field is applied in the $x$–$z$ plane

Next, we fix the direction of temperature gradient along the $z$ axis and magnetic field is rotated in the $x$–$z$ plane making an angle $\beta$ from the $z$ axis and thus we have $\nabla T = \nabla T_z$, $B = (B \sin \beta, 0, B \cos \beta)$ and $E = 0$. We have evaluated the LMTC $\kappa_{zz}$ and PTHC $\kappa_{zx}$ and discuss the numerical results of the above thermal conductivities. The LMTC for this configuration can be expressed as

$$\kappa_{zz} = \kappa_{zz}^{(0)} + \kappa_1 + \kappa_2 \cos^2 \beta.$$

(51)
The PTHC is given by

\[ \kappa_{zx} = \kappa_3 \sin \beta \cos \beta. \]  

Here, the thermal conductivity coefficients (\(\kappa_1\), \(\kappa_2\) and \(\kappa_3\)) are proportional to \(B^2\). The LMTC \(\kappa_{zz}\) has the angular dependence of \(\cos^2 \beta\) as depicted in figure 13(a), whereas the PTHC follows the standard \(\sin \beta \cos \beta\) dependence which is shown in figure 13(c). Figures 13(b) and (d) depicts the variation of \(\kappa_{zz}\) and \(\kappa_{zx}\) with the magnetic field for the quadratic TCFs at \(T = 300\) K. Both the LMTC and PTHC shows quadratic dependence on the magnetic field.

**Validity of Wiedemann–Franz law:** The Wiedemann–Franz law states that the ratio of electronic contribution of thermal conductivity to electrical conductivity for metal is proportional
to the temperature and can be expressed as \( \kappa_{ij}/(\sigma_{ij}T) = L_0 \), where \( L_0 = \frac{e^2}{2\pi^2\hbar} \) is the Lorentz number. The Wiedemann–Franz law is valid in our case of TRS broken quadratic TCFs for the numerically computed electrical and thermal conductivities at \( T = 300 \text{ K} \).

7. Quantitative comparison of the transport coefficients

In this section, we compare the magnitudes of transport coefficients for the two orientations by plotting their ratios with respect to magnetic field and Fermi energy. In figure 14(a), the ratio of LMCs \( \sigma_{zz}/\sigma_{xx} \) and \( \sigma_{zx}/\sigma_{xy} \) of the respective orientations is plotted as a function of \( B \) for \( \mathbf{E} \parallel \mathbf{B} \). As \( B \rightarrow 0 \), the ratio refers to that of the Drude conductivities. The value of \( \sigma_{zz}^{(0)} \) is \( \sim6\% \) of that \( \sigma_{xx}^{(0)} \) for the chosen set of parameters, which implies a highly reduced Drude conductivity along \( z \) axis. The ratio \( \sigma_{zz}/\sigma_{xx} \) is a slowly increasing function of \( B \). The increase is only \( \sim2\% \) up to \( B/B_0 = 2 \). Figure 14(b) shows that the ratio of the MRs is a decreasing function of \( B \). At very low \( B \), MR_{zz} is nearly 22 times of MR_{xx}. The ratio drops to \( \sim16 \) at \( B/B_0 = 2 \). Thus, the direction in which the energy scales linearly with momentum in quadratic TCFs favors higher magnitude of MR and lower value of LMC. The measurements of MR and resistivity in presence of small magnetic field may hence act as probes for detecting different symmetry axes of the system. We also observe that the ratio of quadratic-\( B \) correction to LMCs \( \sigma_{zz}^{(2)}/\sigma_{xx}^{(2)} \) and PHCs \( \sigma_{zx}^{(2)}/\sigma_{xy}^{(2)} \) for the two orientations are independent of Fermi energy as shown in figure 14(c). The ratio \( \sigma_{zz}^{(2)}/\sigma_{xx}^{(2)} \) is higher than \( \sigma_{zx}^{(2)}/\sigma_{xy}^{(2)} \). The ratio of longitudinal MR \( \text{MR}_{zz}/\text{MR}_{xx} \) is almost a linearly increasing function of Fermi energy as depicted in figure 14(d).

8. Conclusions

Triple-component semimetals host crossings of three energy bands near which the quasiparticles behave like pseudospin-1 fermions. In this work, we have dealt with the low-energy model for generalized TCFs and obtained the Berry curvature, DOS and Fermi energy as function of carrier density. The DOS of dispersive bands varies with the Fermi energy as \( D_\beta(\mu) \sim \mu^{3/2n} \), where \( n \) = 1, 2, 3. The Fermi energy as a function of electron density scales as \( \mu \sim n_e^{6/(n+2)} \). These exponents are independent of the coefficients \( \nu_{\alpha} \) and \( \alpha_{\alpha} \).

We have specifically considered quadratic TCFs i.e., \( n = 2 \) case and studied its Berry curvature induced electrical and thermal magnetotransport properties using semiclassical Boltzmann transport formalism. On the account of anisotropic energy spectrum of quadratic TCFs, we have obtained the transport coefficients for two different orientations of magnetic field—\( B \) applied in \( x-y \) and \( x-z \) plane. PHE is observed in both the orientations due to Berry curvature-modified magnetotransport. The PHC shows \( B^2 \) dependence in the lowest order in \( B \) and an angular dependence of \( \sin \beta \cos \beta \) as observed in other topological semimetals, where \( \beta \) is the angle between applied electric and magnetic fields. The LMC also follows \( B^2 \) dependence due to Berry curvature, resulting in negative longitudinal MR. The longitudinal MRs corresponding to their respective \( B \)-field configurations \( (\text{MR}_{xx}(\beta) \) and \( \text{MR}_{zz}(\beta) \) vary as \( \cos^2 \beta \). We find that the out-of-plane magnetoconductivity shows an oscillatory dependence with \( \beta \) for the second
orientation of magnetic field, but it is angle-independent for the first one.

For the case of $B$ applied in $x$–$y$ plane, negative MR has the maximum value of about $-1.8\%$ whereas for $B$ confined in $x$–$z$ plane, the maximum MR reaches about $-29\%$. Hence it is evident that Berry curvature effects are considerably large for the latter case. Thus, the direction in which the energy scales linearly with momentum in quadratic TCFs favors higher magnitude of longitudinal MR.

The Drude conductivities $\sigma^{(0)}_{ij}$ and $\sigma^{(0)}_{jj}$ are increasing functions of Fermi energy. The second-order corrections in longitudinal and planar magnetoconductivities and the absolute MRs for both the orientations decrease with the Fermi energy. This is attributed to the fact that Berry curvature effects decrease as the Fermi energy moves away from the band-touching nodes. Also, in such systems the triple-component coefficients will be enhanced due to contribution from a greater number of 4 just like the case of inversion symmetry broken TCFs, the band touching nodes should appear in multiple of 4 just like the case of inversion symmetry broken Weyl semimetals. So, the magnitudes of the magnetotransport coefficients will be enhanced due to contribution from a greater number of nodes. Also, in such systems the triple-component modes may not be at the same energy. This may result in different values of DOS, velocities and Berry curvatures at the Fermi surface of different nodes, unlike the TRS broken case. Moreover, in such systems, it is possible that the Fermi surface may intersect conduction band near one node (electron-like) and valence-band near the other (hole-like). So, we have to calculate the total magnetotransport coefficients considering low-energy Hamiltonians around each node separately.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Lorentz-force contribution

The components of $\Gamma_m$ in equation (17) for electrical and thermal transport are given below [48, 55]. For brevity, we have dropped the band index $m$.

(a) For the planar Hall setup: $E = Ex$, $B = (B \cos \beta, B \sin \beta, 0)$ and $\nabla T = 0$,

$$\Gamma_x = \frac{-N(M_1 M_2 + M_2 M_4)}{(D \tau)^2 - \left(\frac{eb \cos \beta}{m_{ec}} - \frac{eb \sin \beta}{m_{ec}}\right)^2 - \frac{eb}{m_{ec}} M_4},$$

$$\Gamma_y = -\cos \beta \left(N M_1 + \Gamma_x \frac{eb}{m_{ec}}\right) M_2,$$

$$\Gamma_z = -\tan \beta \Gamma_y,$$

where $M_1 = -\frac{\sin \beta}{m_{ec}} + \cos \beta \frac{eb}{m_{ec}}, M_2 = \frac{eb}{m_{ec}} C_2, C_1 = \frac{eb}{m_{ec}} C_3 + \frac{1}{m_{ec}}$, $M_3 = \frac{eb \sin \beta}{m_{ec}} + \frac{eb \cos \beta}{m_{ec}} - \frac{eb \sin \beta}{m_{ec}}$, $C_1 = \frac{eb}{m_{ec}} C_3 + \frac{1}{m_{ec}}$, $C_2 = \frac{eb}{m_{ec}} + \frac{eb}{m_{ec}} C_1$ and $C_3 = \frac{eb}{m_{ec}} C_1 + \frac{1}{m_{ec}}$.

(b) For the planar Hall setup: $B = (B \cos \beta, B \sin \beta, 0), \nabla T = \nabla T \hat{x}$ and $E = 0$,

$$\Gamma_x = \frac{-N_0 (\alpha_1 \alpha_2 - \alpha_3 \alpha_4)}{(D \tau)^2 - \left(\frac{eb \cos \beta}{m_{ec}} - \frac{eb \sin \beta}{m_{ec}}\right)^2 - \frac{eb}{m_{ec}} \alpha_4},$$

$$\Gamma_y = -\cos \beta \left(N_0 \left(\frac{1}{m_{ec}} + \frac{eb}{m_{ec}} C_3 \cos \beta\right) - \Gamma_x \frac{eb}{m_{ec}}\right),$$

$$\Gamma_z = -\tan \beta \Gamma_y,$$

where $N_0 = eb \nabla T \frac{\alpha_3 (e \beta / m_{ec})}{m_{ec}}, \alpha_1 = \frac{eb \sin \beta}{m_{ec}} - \frac{eb \cos \beta}{m_{ec}} + \frac{eb}{m_{ec}} \alpha_3, \alpha_2 = \frac{eb \cos \beta}{m_{ec}} C_1 + \frac{1}{m_{ec}}, \alpha_3 = \frac{eb \cos \beta}{m_{ec}} \left(C_2 + C_1 \sin \beta\right)$, $\alpha_4 = \frac{eb \sin \beta}{m_{ec}}$.

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