Self-organized criticality in a discrete model for Smoluchowski’s equation

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Abstract

We study a discrete model of coagulation, involving a large number $N$ of particles. Pairs of particles are given i.i.d exponential clocks with parameter $1/N$. When a clock rings, a link between the corresponding pair of particles is created only if its two ends belong to small clusters, i.e. of size less than $\alpha(N)$, with $1 \ll \alpha(N) \ll N$. The concentrations of clusters of size $m$ in this model are known to converge to the solution to Smoluchowski’s equation with a multiplicative kernel. Under the additional assumption $N^{2/3} \log^{\gamma}(N) \leq \alpha(N)$, for some $\gamma > 1/3$, we study finer properties of this model, namely the combinatorial structure of the graph consisting of small clusters. We prove that this graph is essentially an Erdős-Rényi random graph, which is subcritical before time 1, and remains (near-)critical after time 1. In particular, we show that our model exhibits self-organized criticality at a microscopic level: the limiting distribution of a typical finite cluster is that of a critical Galton-Watson tree.

Keywords: Erdős-Rényi random graph, gelation, hydrodynamic limit, self-organized criticality, Smoluchowski’s coagulation equation

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1 Introduction

1.1 Motivation

Smoluchowski’s equation, introduced in [35], is used to describe the coagulation of particles in a mean-field setting. It is characterized by a symmetric kernel $\kappa(m,m')$ describing at which rate clusters of size $m$ and $m'$ coalesce. The most interesting may be the multiplicative kernel $\kappa(m,m') = mm'$, where two clusters coalesce at a rate which is proportional to the number of potential links between them. In this setting, Smoluchowski’s equation reads

$$\frac{d}{dt}c_t(m) = \frac{1}{2} \sum_{m' = 1}^{m-1} m'(m - m')c_t(m')c_t(m - m') - \sum_{m' \geq 1} mm'c_t(m)c_t(m'),$$

with $m \geq 1$, where $c_t(m)$ is meant to represent the concentration of clusters of size $m$. For general references, see [2, 23, 31]. What makes the multiplicative kernel intriguing is that it exhibits gelation. Let us explain this phenomenon, which will be fundamental in our analysis. One could perhaps expect that the mass

$$n_t := \sum_{m \geq 1} mc_t(m)$$
is preserved over time. At least it seems so by formally differentiating the above term by term and using (1). However this formal differentiation does not actually hold: it is well-known [12, 15, 22, 24] that in fact, for any non-zero initial conditions, there is a time $T_{gel}$ such that the mass is constant on $[0, T_{gel})$ and decreasing afterwards. The usual interpretation of this phenomenon is that one or several clusters of infinite mass are created (and they account for a positive proportion of the total mass). These clusters are called the gel. The words concentrations and gel all come from the chemical interpretation of (1) [35]: one can think of a solution (more precisely a colloid) where clusters of particles interact by coagulating at a multiplicative rate. In the pre-gelation phase, clusters of particles are small enough that they all remain in colloidal suspension. When a cluster becomes too massive, it cannot stay in suspension anymore and goes into a precipitate, or a gel. The time at which the precipitate starts forming is called the gelation time and denoted $T_{gel}$. At times past $T_{gel}$, clusters of particles which are small enough remain in colloidal suspension, whereas the largest ones form the gel. Of course, more and more particles precipitate as time goes by, which is why the proportion of particles in solution decreases after $T_{gel}$. From now on, we slightly abuse the chemical interpretation, and replace the more accurate in colloidal suspension with the simpler in solution to refer to particles which do not belong to the gel.

Interestingly, some other kernels, such as the constant kernel $\kappa(m, m') = 1$ or the additive kernel $\kappa(m, m') = m + m'$ do not exhibit gelation. In those cases, particles do not coalesce quickly enough to allow infinite clusters to form in finite time, see e.g. [7] and references therein.

The main issue with gelation is that it makes the system very challenging to study. Physically, this is due to the presence of two phases in the system. Mathematically, this raises a large number of technical issues, due to the presence of an infinite sum in (1). This sum is not well-behaved, and does not allow the typical sum-integral inversions: indeed, as we already discussed below (1), if these inversions were allowed, the mass would remain constant. Consequently, fewer post-gelation results are known. Statements proven in the literature include the following.

- For well-behaved kernels, such as constant or additive, there is no gelation, and there exists a unique solution to (1), see [7, 30] and references therein.

- Existence of gelation for large enough kernels, typically $\kappa(m, m') \geq (mm')^\alpha$ for $\alpha > 1/2$ [12, 13, 22, 24].

- For gelling kernels, existence and uniqueness of a solution before gelation [15, 26, 30].

- For gelling or non-gelling kernels, global existence of solutions through analytic methods [21, 22, 34].

- For gelling or non-gelling kernels, convergence (up to a subsequence) of discrete random models to a solution of Smoluchowski’s equation [16, 19, 30], which also gives a probabilistic proof of global existence.

The reader will notice that one central point is missing here: the global uniqueness of gelling solutions. As far as we know, the first such post-gelation result appears in [20], for Smoluchowski’s equation with a multiplicative kernel (1) and initial conditions $c_0(m) = 1$ if $m=1$. The extension to general initial conditions was obtained in [29]. We are not aware of any post-gelation uniqueness results for other kernels.
One fundamental fact about the multiplicative kernel is that it has a very natural probabilistic interpretation. Recall indeed that the rate of coagulation between two clusters corresponds to the number of potential links between these two clusters. The simplest probabilistic model which exhibits a multiplicative coagulation rate is the following: associate i.i.d. exponential clocks with parameter $1/N$ to pairs of particles, and create a link between particles when the corresponding clock rings. Obviously, two clusters of size $m, m'$ coagulate at a rate proportional to $mm'$. One could perhaps then hope that the concentrations $(c_t^{(N)}(m))$ of clusters of size $m$ in this model converge to a solution of Smoluchowski’s equation, but it turns out that this is not the case past time $1$.

In this (pure coagulation) model, the configuration of clusters at time $t$ is exactly an edge percolation on the complete graph with parameter $1 - \exp(-t/N) \sim_{N \to \infty} t/N$. In other words, the configuration of clusters is exactly that of an Erdős-Rényi model with $N$ vertices and connection probability $p_t^{(N)} := 1 - \exp(-t/N) \sim t/N$, which we denote by $ER(N, p_t^{(N)})$.

As is well-known [11, 10, 6, 18, 33], when $t \leq 1$, $ER(N, p_t^{(N)})$ is in its subcritical phase, and in particular, all but a vanishing proportion of the particles belong to finite-size clusters. This shows that

$$
\sum mc_t(m) = 1 \quad \text{for all } t \leq 1.
$$

When $t > 1$ on the other hand, the model is in its supercritical phase, more precisely there exists, with probability tending to one as $N \to \infty$, a giant cluster containing a positive fraction $\zeta(t)$ (which is the survival probability of a Galton-Watson process with Poisson $t$ offspring distribution) of the particles. Also, all but a vanishing proportion of the particles not in the giant belong to finite-size clusters. This shows that for such a model, any limiting concentration sequence satisfies

$$
\sum mc_t(m) = 1 - \zeta(t) \quad \text{for all } t \geq 0
$$

(of course $\zeta(t) = 0$ for $t \leq 1$).

At time $t$, a cluster of mass $m$ links to the giant at a rate proportional to $m\zeta(t)$, and therefore, instead of Smoluchowski’s limit we find that for any $m \geq 1$,

$$
\frac{d}{dt} c_t(m) = \frac{1}{2} \sum_{m' = 1}^{m-1} m'(m - m')c_t(m')c_t(m - m') - \sum_{m' \geq 1} mm'c_t(m)c_t(m') - m\zeta(t),
$$

and the above is known as Flory’s equation, see [16, 29, 30]. Observe that it coincides with (1) up to time 1 since $\zeta(t) = 0$ at those times, but the additional term $m\zeta(t)$ becomes positive past time 1. In fact, though Flory’s equation is not as nice looking as Smoluchowski’s equation, it is much easier to study, in particular it was earlier known to be well-posed, and many quantities and asymptotics can be studied [12, 29, 31].

This is physically reasonable, since as we explained above, Flory’s equation comes from a very simple discrete model, which in fact does not distinguish between gel and solution. Indeed, any two clusters, regardless of their sizes, always interact. By contrast, in Smoluchowski’s equation, if a positive proportion of particles is in an infinite-size cluster, it should no longer interact with the finite-size clusters. In other words, gel should be inert. Comparing Flory’s equation with (1), it is obvious that the loss of mass happens faster in Flory. Precisely, one can show that the total concentration decreases exponentially fast for Flory’s equation, but as $1/t$ for Smoluchowski’s, see [29]. Even though the above model is not what we are looking for, it will be useful for later comparisons, and we will refer to it as Flory’s discrete model.

Now, the obvious question is: how to modify the microscopic model to instead have convergence to Smoluchowski’s equation? It should be clear from the previous discussion that one should prevent coalescence of finite-size clusters with large clusters. The idea of [16] is to simply introduce a threshold $\alpha(N)$, which merely needs to verify

$$
1 \ll \alpha(N) \ll N
$$

(2)
below which clusters are considered small, and above which they are considered large and become inert. We will sometimes need the stronger assumption on \((\alpha(N)):\)

\[
N^{2/3} \log^\gamma N \ll \alpha(N) \ll N, \quad \gamma > 1/3.
\] (3)

Let us now be more precise with the definition of our model.

### 1.2 Definition of the model

Consider \(N \geq 2\) particles \([N] := \{1, \ldots, N\}\) and a sequence \(\alpha(N)\) verifying (2). On each pair of particles, we attach an independent exponential clock with parameter \(1/N\). When a clock rings, we create the corresponding link, except if one of its ends belongs to a large\(^1\) cluster, i.e. a cluster of size \(\alpha(N)\) or greater. In the latter case, this link is not created, and will never be. In other words, the large clusters become inactive. From now on we will refer to this algorithm as *Smoluchowski’s discrete model*, see an example in Figure 1. At a time \(t\), the links whose clock have rung are called *activated* (and thus may or may not be created).

\[
\begin{array}{|c|c|c|}
\hline
\text{Order of activation} & t_1 & t_2 & t_3 \\
\hline
\text{1} & \text{2} & \text{3} & \text{4} \\
\hline
\text{5} & \text{6} & \text{7} & \text{8} \\
\hline
\text{9} & \text{10} & \text{11} & \text{12} \\
\hline
\end{array}
\]

**Figure 1:** An example of Smoluchowski’s discrete model, where \(N = 8\), \(\alpha(N) = 3\) and link \(i\) is activated at time \(t_i\), with \(t_1 < \cdots < t_6\).

Clusters of size \(\geq \alpha(N)\) will be called large clusters, and the ones of size \(< \alpha(N)\) called small. We will say that particles in small clusters are *in solution*, whereas particles in large clusters are in the *gel*. When a large cluster is created, we say that it falls into the gel, expressing the fact that it “precipitates” and does not interact with particles in solution anymore.

\(^1\)We use *large* in contrast to *giant*, which is the term usually used in the Random Graphs literature to describe clusters of size of order \(N\).
We aim at studying the configuration in solution, that is, the graph whose vertices are the particles in solution and whose edges are the created links. To be precise, we will always consider graphs up to graph isomorphism, but this is a minor technical issue that we will not mention again. As is traditional, we will always consider that our processes are càdlàg.

Let us introduce the following notation.

- $S(t)$ is the set of particles in solution at time $t$;
- $C(t)$ is the configuration in solution at time $t$;
- for $S \subset [N]$, $\overline{S} = [N] \setminus S$;
- $n_t^{(N)} = \#S(t)/N$ is the concentration of particles in solution at time $t$;
- $p_t^{(N)} = 1 - e^{-t/N}$ is the probability for a link to be activated at time $t$;
- $(\tau_i)_{i \geq 1}$ is the sequence of gelation times, i.e. successive times when a large component falls into the gel (we adopt the convention $\tau_i = +\infty$ when the total number of falling components ends up smaller than $i$);
- w.h.p. stands for with high probability, and means that a sequence of events has probability tending to 1 as $N \to \infty$;
- for $S, S' \subset [N]$, a link $S \leftrightarrow S'$ is a link with an end in $S$ and an end in $S'$;
- the Poisson distribution with parameter $\lambda$ is denoted by $\mathcal{P}(\lambda)$.

The dependency in $N$ of certain quantities such as $p^{(N)}$ and $n^{(N)}$ is reminded with the exponent $(N)$, but with a slight abuse, we do a bit of bookkeeping and do not write this dependency for quantities such as $S(t)$, $C(t)$ or $\tau_i$.

### 1.3 Results

Theorem 1.2, stated below and due to [16], concerns the convergence of concentrations to the solution of Smoluchowski’s equation.

We denote by $c_t^{(N)}(m)$ the concentration of clusters of size $m$ at time $t$, i.e. there are $Nc_t^{(N)}(m)$ such clusters. As usual (see e.g. [4]), $\mathbb{D}(\mathbb{R}^+, E)$ denotes the Skorokhod space of càdlàg functions on $[0, +\infty)$ with values in a Polish space $E$. We defer to Section 2 for more details on Smoluchowski’s equation: the precise definition of a solution is given in Definition 2.1, its existence and uniqueness is stated in Theorem 2.4.

**Theorem 1.1.** Assume that $c_0^{(N)} \to c_0$ in $\ell^1$, with $\langle c_0, m^2 \rangle < +\infty$, and that (2) holds. Then the process $(c^{(N)})$ converges in distribution in $\mathbb{D}(\mathbb{R}^+, \ell^1)$ to the unique solution of Smoluchowski’s equation (1) started from $c_0$.

In our discrete model, we actually have $c_0(m) = 1_{\{m=1\}}$, but we could obviously modify the initial conditions $c_0^{(N)}$ to have convergence to any $c_0$.

As mentioned above, Theorem 1.2 was proved in [16], up to two minor differences:

- the coagulation rates are not exactly the same;
the convergence in [16] is stated only up to a subsequence, due to the fact that the uniqueness result for (1) of [29] was not yet known.

For the sake of completeness, and because it can be done quickly enough, we give in Section 2.2 a proof of this result.

The main goal of the paper however, is to study finer properties of our model, and ultimately to know about the shape of the clusters in solution. The main results are the following.

**Theorem 1.2.** For any $t \geq 0$, conditionally on $n_t^{(N)}$, the configuration in solution is that of a

$$
\text{ER}\left(n_t^{(N)}, p_t^{(N)}\right)
$$

graph, conditioned on having no large component.

Moreover, if (3) holds, then the sequence $(n^{(N)})$ converges in the Skorokhod space $\mathbb{D}(\mathbb{R}^+, \mathbb{R})$ to the deterministic function

$$
n_t := \begin{cases} 
1 & t \leq 1; \\
1/t & t \geq 1.
\end{cases}
$$

Since, as $N \to +\infty$, $p_t^{(N)} \sim t/N$, while $n_t^{(N)} \sim n_t$, we observe two distinct regimes. First, when $t < 1$, $Nn_t^{(N)} \times p_t^{(N)} \to t < 1$, and thus, at such times, we are dealing with a strictly subcritical conditioned ER graph. It is well-known (see e.g. [33], Chapter 4) that the unconditioned graph’s largest component is of logarithmic size w.h.p, and therefore the above conditioning is asymptotically trivial. When $t \geq 1$ on the other hand, $Nn_t^{(N)} \times p_t^{(N)} \to 1$, and thus we are dealing with a critical, or at least near-critical conditioned ER graph, and it seems much harder to deal with the conditioning. Thankfully, in the course of proving Theorem 1.2, we will incidentally establish the following Proposition (precisely, it turns out to be a direct consequence of Lemma 5.2 and Proposition 5.5 below).

**Proposition 1.3.** For any fixed $T \geq 1$, there exists a sequence of gelation times $(T_N := \tau_{i_N}^{(N)})$ which converges in probability to $T$ as $N \to \infty$.

Moreover, as $N \to \infty$, the probability that $\text{ER}\left(n_{T_N}^{(N)}, p_{T_N}^{(N)}\right)$ possesses a large component tends to 0.

The above implies that, at least for quantities evolving in a continuous manner in the limit as $N \to \infty$, we may simply neglect the effect of the conditioning at all times. In particular, the proposition yields the following result. Recall that we consider the graphs and trees up to isomorphism.

**Theorem 1.4.** At time $t \geq 0$, pick uniformly at random a particle in solution $i$, and let $C(t)$ denote the graph rooted at $i$ consisting of the connected component of $i$. Then, for any rooted finite tree $\mathcal{T}$,

$$
\lim_{N \to \infty} P(C(t) = \mathcal{T}) = P(\text{GW}(\mathbb{P}(\lambda)) = \mathcal{T}),
$$

where $\text{GW}(\mathbb{P}(\lambda))$ denotes a Galton-Watson tree with offspring distribution $\mathbb{P}(\lambda)$.

In particular, at any time $t \geq 1$, the limiting distribution of a typical cluster is that of a critical Galton-Watson tree with $\mathbb{P}(1)$ offspring distribution.

Theorem 1.4 shows that Smoluchowski’s discrete model exhibits self-organized criticality (SOC), and that criticality is exactly reached at the gelation time 1. For a more precise description of what happens in solution, see Section 7.
1.4 Organization of the paper

After recalling some results on Smoluchowski’s equation, we prove Theorem 1.1 in Section 2.2. The proofs of Theorem 1.2, Proposition 1.3 and Theorem 1.4 take up the remainder of the paper. It will be organized in the following main steps.

- In Section 3, we prove some precise estimates on the largest component of a slightly supercritical Erdős-Rényi graph (Theorem 3.1). Later in that section, we see how these translate into information about the first gelation event (Proposition 3.3).

- In Section 4, we describe the combinatorial structure of particles in solution, which yields a proof for the first part of Theorem 1.2 (Lemmas 4.1 and 4.3).

- Section 5 introduces an alternative model constructed from a family of Erdős-Rényi graphs. We prove that it has, with high probability, the same distribution as our original model (Lemma 5.2). This allows us to give precise estimates on the gelation times (Proposition 5.5). Note that, as already mentioned, Proposition 1.3 simply is a consequence of these two results.

- These estimates are used in Section 6 to prove the tightness of \( (n^{(N)}) \), and show that it has a unique limit point given by (4).

Finally, in Section 7, we explain how Theorem 1.4 follows from Theorem 1.2 and Proposition 1.3, along with other concluding comments.

1.5 Related works

1.5.1 Convergence to Smoluchowski’s equation

As we already mentioned above, Theorem 1.1, up to minor differences, was proven in [16]. In this paper, \( (c^{(N)}) \) is considered as a pure-jump Markov process, and its dynamics is given by the jump rates. Note however that the model we consider contains much more information about the system: for instance, the shape of the clusters (in terms of their graph structure) is lost in the mere knowledge of \( (c^{(N)}) \). In some sense, keeping track of the graph structure allows to look at phenomena on a microscopic scale.

1.5.2 SOC in a forest-fire model

Self-organized-criticality for a closely related forest-fire model was obtained in [32]. In this work, edges appear (or re-appear) at rate \( 1/N \) (regardless of the fact that any of the corresponding vertices belongs to a small or large cluster), while lightning strikes particles at a fixed rate \( \alpha(N)/N \), (with \( \alpha(N) \) satisfying (2) for the most interesting behavior). The effect of lightning is that edges in the corresponding connected components simply vanish (but vertices remain). Although the two models present obvious similarities, they also have important differences.

- In [32], only the edges are affected by the coagulation-fragmentation dynamics, so that the total mass remains constant. This is why, in the limit past gelation, instead of Smoluchowski’s equation, the authors obtain a system of constrained ODE’s. The constraint simply is that the total mass remains 1. The system of ODE’s is closely related to Smoluchowski’s equation: on the one hand the equations remain unchanged for the evolution
of concentrations of clusters of mass \( m \geq 2 \). On the other hand, the evolution of clusters of mass 1 is modified, as they not only disappear due to coagulation with other clusters, but also appear due to the effect of lightning (in such a way that the total mass remains 1). Importantly, a solution to the constrained system of ODE’s found in [32] is, past time 1, very different from a solution to Smoluchowski’s equation: indeed, the presence of a greater number of isolated vertices also affects the growth of larger clusters.

- The characterization of SOC in [32] is through the fact that the tail of the concentrations satisfies \( \sum_{m \geq k} c_t(m) \sim C(t) k^{-1/2} \), with \( C(t) > 0 \) for \( t \geq 1 \), instead of having an exponential decay. In fact, such “macroscopic” characterization of SOC can be directly established for the explicit solution to Smoluchowski’s equation with monodisperse initial conditions, see Section 7.2 below.

Our characterization may be a more striking way of exhibiting SOC, and in a sense, Theorem 1.4 describes the SOC phenomenon at a microscopic level: it asserts that the limiting distribution of a typical cluster is that of a finite, critical tree. In particular, as we detail in Section 7.2, the characterization of SOC in the sense of [32] can be easily explained through Theorem 1.4: the probability for a critical Galton-Watson tree to be of size greater than \( k \) decays as \( C k^{-1/2} \).

It would be interesting to investigate what is, at time \( t \geq 1 \) the limiting distribution of a typical cluster in the model of [32]. We conjecture that it is, once again, a critical tree. Note however that its distribution can not be stationary, because of the increasing proportion of isolated vertices past time 1.

### 1.5.3 Self-organized criticality in a discrete model of limited aggregation

In a forthcoming paper [27], we study a similar model, but where particles initially have a certain number of arms, used to perform coagulations.

Specifically, arms are paired uniformly at random at some rate, but only when they both belong to small clusters. Some of the techniques of the present paper can be applied, and others have to be introduced. We manage to prove again self-organized criticality: past the gelation time, a typical cluster in solution is a delayed critical Galton-Watson tree, and the distribution of the reproduction law can be given.

The main differences between the present model and that of [27] are the following.

- The convergence of concentrations is towards a solution of Smoluchowski’s equation with limited aggregations (see [29]).
- The model of the present paper at a given time is related to a (conditioned) Erdős-Rényi random graph, while the model in [27] is related to a (conditioned) configuration model.
- An important parameter of the model in [27] is the initial distribution of arms. The behavior of the model depends heavily on it: for instance, gelation occurs only when there are sufficiently many arms to begin with. Also, for most initial distribution of arms (except the Poisson ones), the reproduction law of the delayed critical Galton-Watson tree is no longer stationary past gelation.

For the readers interested in [27], the present paper should be an appropriate and more accessible introduction.
2 Convergence to Smoluchowski’s equation

2.1 Definition and well-posedness of Smoluchowski’s equation

Smoluchowski’s equation with a multiplicative kernel is given by (1). In this paragraph, we recall results about this equation, and fix some inaccuracies from [29]. To begin with, we need to define what we mean by a solution. Let, for \( f, g : \mathbb{N} \to \mathbb{R}_+ \),

\[
\langle f, g \rangle = \sum_{m \geq 1} f(m)g(m).
\]

With a slight abuse of notation, we will, in this paragraph, write \( m \) for the function \( f : m \to m \), \( m^2 \) for the function \( f : m \to m^2 \), and so on.

**Definition 2.1.** We call a family \( (c_t(m), m \geq 1)_{t \geq 0} \) of nonnegative continuous functions a solution to Smoluchowski’s equation with initial conditions \( c_0 \in [0, +\infty)^\mathbb{N} \) if

- for every \( t \geq 0 \), \( \int_0^t \langle m, c_s \rangle^2 \, ds < +\infty \);
- for every \( t \geq 0 \) and \( f : \mathbb{N} \to [0, +\infty) \) with compact support,

\[
\langle c_t, f \rangle - \langle c_0, f \rangle = \frac{1}{2} \int_0^t \sum_{m, m' = 1}^{+\infty} mm'c_s(m)c_s(m') (f(m + m') - f(m) - f(m')) \, ds. \tag{5}
\]

Note that the RHS of (5) is well-defined by the first part of the definition, and, with \( f = 1_{\{m\}} \), it is just the integrated form of (1). Extending to \( f \) with bounded support is simply the linearity of the equation. Unlike the definition given in [29], we do not assume that \( \langle c_t, m^2 \rangle \) is bounded in a neighborhood of 0 whenever it holds at time 0. It turns out that this was an unnecessary assumption, as we shall now prove.

**Lemma 2.2.** Assume that \( (c_t) \) is a solution to (1) with \( \langle c_0, m \rangle < +\infty \). Then \( \langle c_t, m \rangle \leq \langle c_0, m \rangle \) for all \( t \geq 0 \). In particular, (5) extends to all bounded \( f : \mathbb{N} \to [0, +\infty) \).

**Proof.** The first part of the above is exactly Lemma 2.4 in [29]. Now, fix \( f \) as in the statement, take \( t > 0 \) and write \( f^b(m) = f(m)1_{\{m \leq b\}} \). Consider (5) with \( f^b \). By monotone convergence,

\[
\lim_{b \to +\infty} \langle c_0, f^b \rangle = \langle c_0, f \rangle, \quad \lim_{b \to +\infty} \langle c_t, f^b \rangle = \langle c_t, f \rangle.
\]

Moreover,

\[
\left| \sum_{m, m' \geq 1} mm'c_s(m)c_s(m') (f^b(m + m') - f^b(m) - f^b(m')) \right| \leq 3 \sup_{m \in \mathbb{N}} f(m) \times \langle m, c_s \rangle^2 \\
\leq 3 \sup_{m \in \mathbb{N}} f(m) \times \langle m, c_0 \rangle^2,
\]

and we may then pass to the limit in (5) as \( b \to +\infty \) by bounded convergence. \( \square \)

The following result shows that our definition of a solution and the one given in [29] coincide.

**Lemma 2.3.** Assume that \( (c_t) \) is a solution to (1) with \( \langle c_0, m^2 \rangle < +\infty \). Then \( \langle c_t, m^2 \rangle \) is bounded in a neighborhood of 0.
Proof. From the previous result, we may take, for \( b \geq 0, f^b(m) = (m \wedge b)^2 \) in (5). One readily checks that
\[
f^b(m + m') - f^b(m) - f^b(m') \leq 2mm'1_{\{m \leq b\}}1_{\{m' \leq b\}}.
\]
Plugging this in (5) shows that
\[
\langle c_t, f^b \rangle - \langle c_0, f^b \rangle \leq \int_0^t \left( \sum_{m=1}^b m^2 c_s(m) \right) ds \leq \int_0^t \langle c_s, f^b \rangle^2 ds.
\]
This is an integral inequality à la Gronwall of the type \( u_t \leq a + \int_0^t u_s^2 ds \). If one denotes by \( v \) the RHS, then \( v' = u^2 \leq v^2 \), whence one deduces by integrating that \( v_t \leq (1/a - t)^{-1} \), for \( t \leq 1/a \), and the same inequality holds for \( u \) since \( u \leq v \). Hence, the previous inequality implies that
\[
\langle c_t, f^b \rangle \leq \frac{1}{1/\langle c_0, f^b \rangle - 1}, \quad t \leq 1/\langle c_0, f^b \rangle.
\]
By monotone convergence, we thus get
\[
\langle c_t, m^2 \rangle \leq \frac{1}{1/\langle c_0, m^2 \rangle - 1}, \quad t \leq 1/\langle c_0, m^2 \rangle,
\]
which proves the result. \( \square \)

This being done, we may recall the well-posedness result obtained in Theorem 2.2 and Proposition 2.6 of [29]. For convenience, we rephrase it in our context.

**Theorem 2.4.** Consider initial conditions \((c_0(m))\) with
\[
m_0 = \langle m, c_0 \rangle \in (0, +\infty), \quad K = \langle m^2, c_0 \rangle \in (0, +\infty),
\]
and define
\[
g_0(x) = \langle mx^m, c_0 \rangle, \quad x \in [0,1].
\]
Let
\[
T_{gel} = 1/K \in [0, +\infty).
\]
Then the following hold.
1. Smoluchowski’s equation (1) has a unique solution defined on \( \mathbb{R}^+ \).
2. For \( t > T_{gel} \), the equation
\[
\ell_t g(\ell_t) = \frac{1}{t}
\]
has a unique solution \( \ell_t \in (0,1) \). The mass in solution \( n_t = \langle m, c_t \rangle \) is given by
\[
n_t = \begin{cases} 1, & t \leq T_{gel}, \\ g_0(\ell_t), & t > T_{gel}. \end{cases}
\]
3. In particular \((n_t)\) is continuous, constant on \([0, T_{gel}]\), strictly decreasing on \([T_{gel}, +\infty)\).

In other words, (1) is well-posed, and by the third point above, some mass starts to disappear at time \( T_{gel} \), i.e. there is gelation at \( T_{gel} \).
2.2 Convergence

Recall that we denote by \( c_t^{(N)}(m) \) the concentration of clusters of size \( m \) at time \( t \), i.e. there are \( Nc_t^{(N)}(m) \) such clusters at time \( t \). Let us call a link between particles relevant (at time \( t \)) if

- it is not activated;
- both its ends are in small clusters;
- its ends are in two different clusters.

The point of that notion is that a coagulation occurs if and only if a relevant link is activated.

How many such links are there? Consider a small cluster of size \( m \). Each particle in this cluster has \( c_t^{(N)}(m) \) relevant links, namely the links with particles in solution not belonging to the same cluster. Hence, there are \( m(Nc_t^{(N)}(m) - m) \) relevant links starting from this cluster. When we sum these quantities over all clusters in solution, each relevant link ends up being counted twice, we therefore obtain a total of

\[
\frac{1}{2} \sum_{m=1}^{\alpha(N)-1} Nc_t^{(N)}(m)m(c_t^{(N)}(N) - m)
\]

relevant links at time \( t \). Each is activated at rate \( 1/N \), so the next coagulation event happens at rate

\[
\lambda^{(N)}(c_t^{(N)}) = \frac{1}{2} \sum_{m=1}^{\alpha(N)-1} c_t^{(N)}(m)m(c_t^{(N)}(N) - m).
\]

Only these events make \( (c_t^{(N)}(m)) \) change, which should make it clear that \( (c_t^{(N)}(m), m \geq 1)_{t \geq 0} \) is a pure-jump Markov process, with values in \( \ell^1 \), say. By the same reasoning, it is easy to compute the rates. For \( m \neq m' \), there are

\[
mNc_t^{(N)}(m)m'Nc_t^{(N)}(m')
\]

relevant links between small clusters of size \( m \) and \( m' \), and

\[
\frac{1}{2}(mNc_t^{(N)}(m) - m)mNc_t^{(N)}(m)
\]

relevant links between small clusters of the same size \( m \). Assume then that the system is in a state \( \eta \in \ell^1 \). Define \( 1_{\{m\}} \in \ell^1 \) to consist only of zeroes, except for 1 at \( m \). Let

\[
\Delta^{(N)}(m, m') = \frac{1}{N} \left( 1_{\{m+m'\}} - 1_{\{m\}} - 1_{\{m'\}} \right).
\]

Then, if the system is in a state \( \eta \in \ell^1 \), it jumps to \( \eta + \Delta^{(N)}(m, m') \) at rate

\[
\lambda^{(N)}_{m, m'}(\eta) = \begin{cases} 
mm'\eta(m)\eta(m')N & \text{if } m \neq m' \\
mm\eta(m)(\eta(m) - 1/N)N/2 & \text{if } m = m'. 
\end{cases}
\]

Hence, the process \( (c^{(N)}) \) is a pure-jump Markov process on \( \ell^1 \) with generator

\[
G^{(N)}(\eta) = \frac{1}{2} \sum_{m, m'=1}^{\alpha(N)-1} \left( \mathcal{F}\left(\eta + \Delta^{(N)}(m, m')\right) - \mathcal{F}(\eta) \right) \lambda^{(N)}_{m, m'}(\eta),
\] (6)
with $\mathcal{F} : \ell^1 \to \mathbb{R}$. In order to prove Theorem 1.1, we classically proceed in two steps, first proving tightness, then the uniqueness of the limit points. The technical results all appear in [14], see also [4].

**Lemma 2.5.** Assume that $c_0^{(N)} \to c_0$ in $\ell^1$, with $\langle c_0, m \rangle < +\infty$. Then the process $(c^{(N)})$ is tight in $\mathbb{D}(\mathbb{R}^+, \ell^1)$, and any limit point is a continuous function from $\mathbb{R}^+$ to $\ell^1$.

**Proof.** For the tightness, we can use Aldous’s criterion. The norm we consider is the $\ell^1$-norm $| \cdot |$. There are two points to prove. First, that

$$
\lim_{a \to +\infty} \limsup_{N \to +\infty} \mathbb{P} \left( \sup_{t \leq T} |c_t^{(N)}| > a \right) \to 0
$$

for all $T \geq 0$, which is obvious since $|c_t^{(N)}| \leq |c_0^{(N)}| \to |c_0| < +\infty$. Secondly, we need to check that for $\varepsilon, \eta, T > 0$, there is a $\delta_0$ and $N_0$ such that for any $\delta < \delta_0$, $N \geq N_0$ and $\tau$ a discrete $c^{(N)}$-stopping time with $\tau \leq T$, we have

$$
\mathbb{P} \left( |c_\tau^{(N)} - c_\tau^{(N)}| \geq \varepsilon \right) \leq \eta.
$$

But any jump of $c^{(N)}$ has size bounded (in $\ell^1$) by $3/N$. Moreover, at any $c^{(N)}$ stopping time $\sigma$, the next jump occurs after an exponential time with rate $\lambda^{(N)}(c_\sigma^{(N)})$. But

$$
\lambda^{(N)}(c_\sigma^{(N)}) \leq \frac{1}{2} c_\sigma^{(N)} \sum_{m=1}^{\alpha(N)-1} c_\sigma^{(N)}(m)mN \leq \frac{1}{2} Nc_\sigma^{(N)} \sum_{m=1}^{\alpha(N)-1} c_\sigma^{(N)}(m)m \leq KN
$$

for some constant $K$ depending only on $(c_0^{(N)})$. The expectation of number of jumps on the interval $[\tau, \tau + \delta]$ is thus bounded by $KN\delta$, so the result just follows from Markov’s inequality for small enough $\delta$. The continuity of the limit points is just that the size of the jumps, at most $3/N$, tends to 0. \hfill \Box

We then prove that any limit point of this sequence solves Smoluchowski’s equation. Up to a subsequence, and using Skorokhod’s representation theorem, we may assume that $c^{(N)} \to c$ a.s. in $\mathbb{D}(\mathbb{R}^+, \ell^1)$. Since $c$ is continuous, there is even uniform convergence on compact sets.

We now fix any $T > 0$, and prove below that $c$ is a.s. a solution to (1) on $[0, T]$. This is of course enough to ensure that $c$ is a.s. a solution to (1), in the sense of Definition 2.1. Since (1) has a unique solution by Theorem 2.4, it will suffice to ensure that $(c^{(N)})$ does converge to this solution.

To begin with, for verifying (5), it suffices to take $\mathcal{F}$ linear in (6): for such $\mathcal{F}$, $\mathcal{F}(\eta) = \sum_m \eta(m)\mathcal{F}(1_{\{m\}})$, so letting $f : \mathbb{N} \to \mathbb{R}$ such that $f(m) = \mathcal{F}(1_{\{m\}})$ we get $\mathcal{F}(\eta) = \langle f, \eta \rangle$. With a slight abuse of notation we will now confuse $\mathcal{F}$ and $f$, and we may write

$$
G^{(N)} f(\eta) = \frac{1}{2} \sum_{m,m'=1}^{\alpha(N)-1} (f(m + m') - f(m) - f(m')) m m' \eta(m) \eta(m')
$$

$$
- \frac{1}{N} \sum_{m=1}^{\alpha(N)-1} (f(2m) - 2f(m)) m^2 \eta(m).
$$

(7)

Recall as well that

$$
M_t^{(N),f} := f(c_t^{(N)}) - f(c_0^{(N)}) - \int_0^t G^{(N)} f(c_s^{(N)}) \, ds
$$

(8)
is a martingale for any bounded \( f \).

Take \( f^b(m) = m \wedge b \) for some \( b > 0 \) in (7). It is easy to check that

\[
-b \leq f^b(m + m') - f^b(m) - f^b(m') \leq -b \mathbb{1}_{\{m \geq b\}} \mathbb{1}_{\{m' \geq b\}},
\]

so that

\[
G^{(N)} f^b(c^{(N)}_s) \leq -\frac{b}{2} \left( \sum_{m=b}^{\alpha(N)-1} mc^{(N)}_s(m) \right)^2 + b \frac{\alpha(N)}{N} \sum_{m=1}^{\alpha(N)-1} mc^{(N)}_s(m)
\]

\[
\leq -\frac{b}{2} \left( \sum_{m=b}^{\alpha(N)-1} mc^{(N)}_s(m) \right)^2 + b \frac{\alpha(N)}{N} \sum_{m=1}^{\alpha(N)-1} mc^{(N)}_0(m)
\]

\[
\leq -\frac{b}{2} \left( \sum_{m=b}^{\alpha(N)-1} mc^{(N)}_s(m) \right)^2 + K b \frac{\alpha(N)}{N}
\]

for \( K = \sup_N \{c^{(N)}_0, m\} \). Taking into account that \( f^b(c^{(N)}_t) \geq 0 \) and \( f^b(c^{(N)}_t) \leq \sum_m mc^{(N)}_0 \leq K \), we obtain from the martingale (8) that

\[
\mathbb{E} \left( \int_0^t \left( \sum_{m=b}^{\alpha(N)-1} mc^{(N)}_s(m) \right)^2 \, ds \right) \leq 2K \left( \frac{1}{b} + T \frac{\alpha(N)}{N} \right).
\]

To conclude, take any \( f \) with compact support and \( C \) be a constant that may change from line to line, but which depends only on \( \sup |f|, (c^{(N)}_t) \) and \( T \). Denote

\[
\|g\| = \mathbb{E} \left( \int_0^T |g(s)| \, ds \right),
\]

the norm on \( L^1(\mathbb{P} \otimes 1_{[0,T]} \, dt) \). To begin with, (7) and (9) show that

\[
\left\| G^{(N)} f(c^{(N)}_s) - \frac{1}{2} \sum_{m,m'=1}^b (f(m + m') - f(m) - f(m')) mm' c^{(N)}_s(m)c^{(N)}_s(m') \right\| \leq C \left( \frac{\alpha(N)}{N} + \frac{1}{b} \right).
\]

Since \( c^{(N)} \to c \) uniformly on compact sets, then

\[
\frac{1}{2} \sum_{m,m'=1}^b (f(m + m') - f(m) - f(m')) mm' c^{(N)}(m)c^{(N)}(m')
\]

converges to

\[
\frac{1}{2} \sum_{m,m'=1}^b (f(m + m') - f(m) - f(m')) mm' c(m)c(m')
\]

for \( \| \cdot \| \). On the other hand, the quadratic variation of \( M^{(N)}_t f \) is

\[
\langle M^{(N)}_t f \rangle = \sum_{t \leq T} \left( \Delta M^{(N)}_t f \right)^2
\]
where $\Delta M_t^{(N),f}$ is the jump of $M^{(N),f}$ at $t$, which is clearly bounded by $3 \sup_m |f(m)|/N$. The number of jumps on $[0, T]$ is of order $N$ as in the proof of Lemma 2.5, so that $\mathbb{E}(\langle M_T^{(N),f} \rangle) \to 0$. By Doob’s inequality, we thus have

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} M_t^{(N),f} \right)^2 \to 0$$

so that

$$\|M^{(N),f}\| \to 0.$$ 

Passing to the limit for $\| \cdot \|$ in (8) and using (10) ensure that

$$\|f(c \cdot) - f(c_0) - \int_0^t \frac{1}{2} \sum_{m,m'=1}^b (f(m + m') - f(m) - f(m')) mm' c_s(m)c_s(m') ds\| \leq C \frac{1}{b}.$$ 

Having $b \to \infty$ shows that $c$ solves (5) a.s., for almost every $t \in [0, T]$. By continuity, $c$ solves (5) a.s. on $[0, T]$, and the proof is complete.

### 2.3 About the total concentration of particles

Let us insist that Theorem 1.1 does not allow to bypass the proof of the second part of Theorem 1.2. Indeed, the concentrations $(c_t(m))$ converge to the solution of Smoluchowski’s equation by Theorem 1.1, but Theorem 1.1 only states a convergence in $\ell^1$. Therefore, at this point, the convergence of the total mass $\sum_{m=1}^{\alpha(N)-1} mc_t^{(N)}(m)$ towards $\sum_{m \geq 1} mc_t(m) = n_t$, given in (4), remains an open question.

Even though we do not have yet a precise estimate on the number of particles in solution past gelation time, we are however at least able to show that on any compact interval, w.h.p this total mass is uniformly bounded below. This will turn out to be useful later on when having to use some asymptotic results.

**Lemma 2.6.** For any $t \geq 0$, there exists $\nu_t > 0$ such that

$$\mathbb{P}(n_t^{(N)} \leq \nu_t) \to 0.$$ 

**Proof.** Fix a time $t \geq 0$, and denote $X_i^{(N)}$ the indicator function of the event that no link with an end in $i$ is activated at time $t$. Since $N - 1$ links start from $i$, and each is activated independently with probability $1 - e^{-t/N}$, $X_i^{(N)}$ is 1 with probability $e^{-t(N-1)/N}$ and 0 with probability $1 - e^{-t(N-1)/N}$. Moreover, the $X_i^{(N)}$ have a small covariance, since indeed, $X_i^{(N)}$ and $X_j^{(N)}$ are both 0 when the $2N - 3$ links with an end in $i$ or $j$ are not activated. These $2N - 3$ activations are independent, so

$$\text{Cov}(X_i^{(N)}, X_j^{(N)}) = e^{-t(2N-3)/N} - e^{-2t(N-1)/N} \sim \frac{t}{N} e^{-2t}.$$ 

It is then a simple application of Chebyshev’s inequality to check that, for $\nu_t < e^{-t}$,

$$\mathbb{P} \left( \sum_{i=1}^N X_i^{(N)} < \nu_t N \right) \to 0.$$ 

Hence, w.h.p., more than $\nu_t N$ particles are linked to no other particle, and are thus in solution.
3 Erdős-Rényi random graph

The configuration of particles in solution and the Erdős-Rényi random graph with appropriate parameters, conditioned on having no large component, are very closely linked. Until the first gelation event, this is obvious; afterwards it is much less so. However, we will see in Sections 4 and 5 that a precise relationship between the two holds, at least under assumption (3). This is why we will start by recalling and proving properties of Erdős-Rényi graphs.

3.1 Known results, consequences on the first gelation time

Recall from the introduction that we denote by $ER(N, p_N)$ the Erdős-Rényi random graph with $N$ vertices and connection probability $p_N$. We define $|C_{\text{max}}|$ to be the largest size of its components. As is famously known, $ER(N, p_N)$ exhibits a phase transition.

- If $Np_N \to \lambda < 1$, then w.h.p. $|C_{\text{max}}| = O(\log N)$ (subcritical regime).
- If $Np_N \to \lambda > 1$, then there is w.h.p. a unique largest component of size $\zeta(\lambda)N + o(N)$, with $\zeta(\lambda)$ the survival probability of a Galton-Watson process with $P(\lambda)$ offspring distribution; and, w.h.p, all other components have size $O(\log N)$ (supercritical regime).
- If $Np_N \to 1$, then $|C_{\text{max}}|$ depends more precisely on the behavior of $Np_N$. Loosely speaking, this is the critical regime. More precisely, the critical window corresponds to $p_N = 1/N + t/N^{4/3} + o(1/N^{4/3}), t \in \mathbb{R}$, and for such $p_N$, the largest components are of size of order $N^{2/3}$. On the other hand when $p_N \to 1$ but $|Np_N - 1| \gg N^{-1/3}$, we will rather speak of a near-critical regime. We will also sometimes use the wording slightly subcritical when $Np_N \to 1$ but $N^{1/3}(Np_N - 1) \to -\infty$ (in such regime w.h.p. $\log N \ll |C_{\text{max}}| \ll N^{2/3}$); and slightly supercritical when $Np_N \to 1$ but $N^{1/3}(Np_N - 1) \to +\infty$ (in such regime w.h.p. $N^{2/3} \ll |C_{\text{max}}| \ll N$).

The transition phase was first proved in the seminal paper by Erdős and Rényi [10], while the critical window was exhibited in the beautiful paper [1]. More detailed results and modern proofs can be found e.g. in [6, 18, 33].

The transition phase result already provides a very interesting result about our model, namely that, under the assumption $\log(N) \ll \alpha(N) \ll N$,

$$
\tau_{1(N)}^{(p)} \xrightarrow{p_{\text{w.h.p.}}} 1, \tag{11}
$$

where we recall that $\tau_{1(N)}^{(p)}$ is the first gelation time, that is, the time when the largest component reaches a size $\geq \alpha(N)$.

To see why this holds, let us compare Smoluchowski’s discrete model with Flory’s. Recall that in Flory’s, the configuration at time $t$ is exactly that of $ER(N, p_t^{(N)})$. Further observe that the two algorithms start behaving differently at the first time when a link is activated between a small and a large component, so that the two models exactly coincide at least up to time $\tau_{1(N)}^{(p)}$. Because $Np_t^{(N)} \to t$, by the aforementioned transition phase results, at a time $t < 1$, all the components in Flory’s discrete model are w.h.p. of size $O(\log N) \ll \alpha(N)$; conversely at a time $t > 1$, w.h.p. there exists at least a component of size $\zeta(t)N + o(N) \gg \alpha(N)$. Therefore a component of size $\alpha(N)$ is created w.h.p. in the interval $(1 - \varepsilon, 1 + \varepsilon)$ in Flory’s discrete model, and thus in Smoluchowski’s discrete model as well. This just means (11).
Knowing more precisely when the first gelation occurs, and what happens at this time, depends in fact on our choice of threshold. Whenever \( \log(N) \ll \alpha(N) \ll N^{2/3} \), the first gelation event happens before the critical window is reached. In this near-subcritical regime there are many components of size comparable to the largest. Not only does it become challenging to control precisely the size of the falling component, but it also turns out to be harder to deal with the combinatorial structure of the remaining ones.

When, on the other hand, \( N^{2/3} \ll \alpha(N) \ll N \), the first gelation event occurs after we passed the critical window, when a near-supercritical regime is reached. This regime is easier and much better understood than the near-subcritical one. Indeed, in the near-supercritical regime, there already is w.h.p. a unique component of maximal size (the emerging giant), whereas other components are much smaller. Also, it is easy to control precisely the size of the emerging giant. In fact we have the following result, first proved in [5, 25] (see also [28]) : if \( p_N = (1 + \varepsilon_N)/N \) with \( N^{-1/3} \ll \varepsilon_N \ll 1 \) (so that we are in the slightly supercritical regime), then \( ER(N, p_N) \) has a unique largest component of size \( 2N\varepsilon_N + o(N\varepsilon_N) \) w.h.p.

By the exact same reasoning as for proving (11), the above result tells us not only that
\[
\tau_1^{(N)} = 1 + \frac{\alpha(N)}{2N} + o\left(\frac{\alpha(N)}{N}\right),
\]
but also that the first falling component is of size \( \alpha(N) + o(\alpha(N)) \), w.h.p.

Later on, we will need to carry out this argument over several gelation events. We thus need to control the probabilities involved in a uniform manner, and the price to pay is that we have to replace the assumption on \( N^{2/3} \ll \alpha(N) \ll N \) by (3).

### 3.2 Largest component of a slightly supercritical ER graph: precise bounds

#### 3.2.1 Result

Let \( C_1(n, p) \) and \( C_2(n, p) \) be the largest and second largest components of a \( ER(n, p) \) graph (chosen uniformly at random if there are several choices), and denote \( |C_1(n, p)| \geq |C_2(n, p)| \) their sizes.

For simplicity, in the whole Section 3.2, we fix two sequences
\[
n^{-1/3} \ll \varepsilon_n^\pm \ll 1.
\]
(13)

These sequences are just technical artifacts to allow for universal constants. We will then consider \( (\varepsilon_n), (p_n) \) and \( \gamma \) with
\[
\varepsilon_n^- \leq \varepsilon_n \leq \varepsilon_n^+, \quad \gamma \in [-1, 1], \quad p_n = \frac{1}{n}(1 + \gamma\varepsilon_n).
\]
(14)

The main result of this section is the following.

**Theorem 3.1.** Let \( (p_n) \) be as in (14). Then, for any \( \delta > 0 \), there exists a constant \( \kappa > 0 \), depending only on \( \delta \) and \( (\varepsilon_n^\pm) \), such that, with probability greater than
\[
1 - n \exp(-\kappa n \varepsilon_n^3),
\]
it holds that:
• if $\gamma > 0$,
\[
(2\gamma - \delta)n\varepsilon_n < |\mathcal{C}_1(n, p_n)| < (2\gamma + \delta)n\varepsilon_n, \quad |\mathcal{C}_2(n, p_n)| < \delta n\varepsilon_n;
\]

• if $\gamma \leq 0$,
\[
|\mathcal{C}_1(n, p_n)| < \delta n\varepsilon_n.
\]

Of course, this result is only useful when $n\varepsilon \to 0$. This is why we shall assume (3), so that, with $\varepsilon_n = o(n)/n$, we indeed have $n\varepsilon \to 0$, even faster than any power of $n$.

The results concerning the size of the second largest component, or the case $\gamma \leq 0$ can certainly be significantly improved, but they suffice for our purposes, as we only need to know whether the size of a component exceeds the threshold.

The proof essentially relies on getting good estimates for the exploration process. This is detailed in the following paragraph.

### 3.2.2 Exploration process

The exploration process $(S_k)$ of the ER($n, p$) random graph is defined in [33], Chapter 4. Its distribution is given by
\[
S_0 = 0, \quad S_k = S_{k-1} + X_k - 1, \quad X_k \sim \text{Bin}(n - k - S_{k-1}, p), \quad k \geq 1.
\]

Clearly, it can only go up, stay put, or go down by $-1$. An excursion of this process will mean an excursion above the current minimum, i.e. the parts of the process between 0 and $-1$, $-1$ and $-2$, and so on. The important property of the exploration process is that the size of these excursions is exactly the size of the connected components (and a fortiori, the number of excursions is the number of connected components). We will show the following result.

**Proposition 3.2.** Define $(S_k)$ by
\[
S_0 = 0, \quad S_k = S_{k-1} + X_k - 1, \quad X_k \sim \text{Bin}(n - k - S_{k-1}, p), \quad k \geq 1,
\]
with $(p_n)$ as in (14). Then, for any $\eta > 0$, there is a constant $\kappa > 0$, depending only on $\eta$ and $(\varepsilon_n^\pm)$, such that
\[
\mathbb{P}\left(\sup_{0 \leq k \leq 3n\varepsilon_n} \frac{1}{n\varepsilon_n^2} \left| S_k - k \left( \gamma \varepsilon_n - \frac{k}{2n} \right) \right| > \eta \right) \leq \exp(-\kappa n\varepsilon_n^3).
\]

If we rescale $S$ by letting $S_u(n) = S_{\lfloor u n\varepsilon_n \rfloor} / (n\varepsilon_n^2)$, then this can be reformulated as
\[
\mathbb{P}\left(\sup_{u \in [0,3]} \left| S_u(n) - u \left( \gamma - \frac{u}{2} \right) \right| > \eta \right) \leq \exp(-\kappa n\varepsilon_n^3).
\]

In other words, with extremely high probability, $(S_u(n))$ remains in a tube of small vertical section $2\eta$ around the parabola $u(\gamma - u/2)$. This will be the main ingredient of the proof of Theorem 3.1. The force of this result is that it allows to control the exploration process uniformly. In the notation of the proof, one could also decide to optimize the size of the tube $(T(n))$, so that it tends to 0. Keeping uniform probabilities would then require stronger assumptions on $(\varepsilon^\pm)$. The above result will prove the most convenient to us.
Proof. 1. To begin with, replace $(S_k)$, defined in (15), by an actual random walk: fix a sequence $(\beta(n))$ and define

$$S_0 = 0, \quad S_k = S_{k-1} + X_k - 1, \quad X_k \sim \text{Bin}(n - k - \beta(n), p_n), \quad k \geq 1.$$  

(16)

If $\mathcal{F}_k$ is the $\sigma$-algebra generated by $X_1, \ldots, X_k$, it is easy to compute that, for $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left(e^{\lambda S_k} \mid \mathcal{F}_{k-1}\right) = e^{\lambda S_{k-1}} e^{-\lambda} \left(1 - p_n + p_ne^\lambda\right)^{n-\beta(n)-k}.$$  

One then readily checks that $(M_k(\lambda))$ is a martingale, where

$$M_k(\lambda) := e^{\lambda S_k/m_k(\lambda)}, \quad m_k(\lambda) := e^{-\lambda k} \left(1 - p_n + p_ne^\lambda\right)^{(n-\beta(n))k-(k+1)/2}.$$  

Recall that $(p_n)$ is as in (14), which is far too strong for most computations, but weaker assumptions would not be much more useful. We also assume that $\lambda := \lambda_n$ changes with $n$, with $|\lambda_n| \leq 1$ and $\lambda_n \to 0$. These assumptions ensure that, in the following computations, the constants hidden into the $O(\cdot)$ only come from the Taylor expansion of usual functions, or the relationship between the different sequences, and thus depend only on $(\varepsilon_n^\pm)$. It is first easy to check that

$$\log \left(1 - p_n + p_ne^\lambda\right) = p_n \lambda_n \left(1 + O(\lambda_n)\right)$$  

and that, for $k \leq 3n \varepsilon_n$, say, we have

$$\frac{1}{\lambda_n} \log m_k(\lambda_n) = k \left(\gamma \varepsilon_n - \frac{k}{2n}\right) + O(\beta(n)\varepsilon_n) + O(n\varepsilon_n^3) + O(n \varepsilon_n \lambda_n).$$  

(17)

Assume now that $\lambda_n \geq 0$. For any nonnegative sequence $T(n)$, Doob’s inequality implies that

$$\mathbb{P}\left(\sup_{k=0,\ldots,3n \varepsilon_n} M_k(\pm \lambda_n) > e^{\lambda_n T(n)}\right) \leq \mathbb{E}(M_{3n \varepsilon_n}(\pm \lambda_n)) e^{-\lambda_n T(n)} = e^{-\lambda_n T(n)}.$$  

Hence, with probability at least $1 - e^{-\lambda_n T(n)}$, we have

$$\frac{1}{\lambda_n} \log M_k(\pm \lambda_n) \leq T(n), \quad k = 0, \ldots, 3n \varepsilon_n.$$  

In addition,

$$\frac{1}{\lambda_n} \log M_k(\pm \lambda_n) = \pm S_k - \frac{1}{\lambda_n} \log(m_k(\pm \lambda_n))$$  

$$= \pm \left(S_k - k \left(\gamma \varepsilon_n - \frac{k}{2n}\right) + O(\beta(n)\varepsilon_n) + O(n\varepsilon_n^3) + O(n \varepsilon_n \lambda_n)\right),$$  

where the second equality comes from (17). Hence for some constant $C$ depending only on $(\varepsilon_n^\pm)$, with probability greater than $1 - 2e^{-\lambda_n T(n)}$, for $k \leq 3n \varepsilon_n$,

$$\left|S_k - k \left(\gamma \varepsilon_n - \frac{k}{2n}\right)\right| \leq C \left(\beta(n)\varepsilon_n + n\varepsilon_n^3 + n \varepsilon_n \lambda_n\right) + T(n).$$  

(18)

Let us now fix $\eta \in (0, 1/2)$ and take

$$T(n) = \frac{\eta}{4} n \varepsilon_n^2, \quad \lambda_n = \frac{\eta}{4C} \varepsilon_n \land 1.$$  

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Then (18) reads: there is a constant $C$, depending only $(\varepsilon_+^n)$ such that, for $k \leq 3n\varepsilon_n$,

$$\frac{1}{n\varepsilon_n^2} \left| S_k - k \left( \gamma \varepsilon_n - \frac{k}{2n} \right) \right| \leq C \left( \frac{\beta(n)}{n\varepsilon_n} + \varepsilon_n \right) + \eta/2,$$

with probability greater than $1 - 2e^{-\kappa n\varepsilon_n}$, where $\kappa = \eta^2/(16C) > 0$. At least, this holds for $n$ large enough, uniformly in the parameters, and we may take a smaller $\kappa$ to make the result true for all $n$.

2. Define $(S_k^\pm)$ as in (16) with $\beta(n) = 0$. From (19), with probability greater than $1 - 2e^{-\kappa n\varepsilon_n}$,

$$\frac{1}{n\varepsilon_n^2} \sup_{k=0,\ldots,3n\varepsilon_n} S_k^\pm \leq \frac{1}{n\varepsilon_n^2} \sup_{k=0,\ldots,3n\varepsilon_n} k \left( \gamma \varepsilon_n - \frac{k}{2n} \right) + C\varepsilon_n + \eta/2 \leq \frac{\gamma^2}{2} + C + \eta/2.$$ 

Hence, with probability greater than $1 - 2e^{-\kappa n\varepsilon_n}$,

$$\sup_{k=0,\ldots,3n\varepsilon_n} S_k^\pm \leq (C+1)n\varepsilon_n^2.$$ 

(20)

3. Consider finally the real exploration process $(S_k)$ defined by (15). Define $(S_k^\pm)$ as in (16) with $\beta(n) = 0$, and $(S_k^-)$ similarly with

$$\beta(n) = (C+1)n\varepsilon_n^2.$$ 

It is clear that we can couple $S$, $S^+$, $S^-$, using for instance Bernoulli variables, in a way that

$$S_k^- \leq S_k \leq S_k^+, \quad k = 0,\ldots,3n\varepsilon_n,$$

as long as (20) holds. But (20) holds with probability greater than $1 - 2e^{-\kappa n\varepsilon_n}$, so (19) shows that, with probability at least $1 - 4e^{-\kappa n\varepsilon_n}$, we have

$$\frac{1}{n\varepsilon_n^2} \left| S_k - k \left( \gamma \varepsilon_n - \frac{k}{2n} \right) \right| \leq C ((C+1)\varepsilon_n + \varepsilon_n) + \eta/2 \leq C(C+2)\varepsilon_n^+ + \eta/2,$$

for $k \leq 3n\varepsilon_n$. For $n$ large enough, depending only on $(\varepsilon_+^n)$, the right hand side above becomes smaller than $\eta$. Choosing a smaller $\kappa$ allows to take care of smaller values of $n$ and get rid of the constant 4, which yields Proposition 3.2.

3.2.3 Proof of Theorem 3.1

Assume $\gamma \geq 0$, fix $\delta > 0$ and $\eta > 0$, and let $P^\pm(u) = u(\gamma - u/2)\pm\eta$. Define $u^- \leq U^-$ the zeros of $P^-$, with the convention that $u^- = U^- = 0$ whenever $P^-$ has no roots. Take also $U^+$ the largest solution to $P^+(u) = -\eta$, see Figure 2. Note that, always, $0 \leq u^- \leq U^- < U^+$.

Clearly, we may choose $\eta$ small enough, independently of $\gamma \in [0, 1]$, such that $2\gamma - \delta/2 < U^- \leq 2\gamma < U^+ < 2\gamma + \delta/2$ and $u^- < \delta/2$. Let $(S_k)$ be the exploration process of $ER(n, p_n)$. 

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Figure 2: The exploration process is wedged between $P^-$ and $P^+$. According to Proposition 3.2, there is a $\kappa > 0$, depending only on $\eta$ and $(\varepsilon_n^\pm)$ such that, with probability greater than $1 - 2e^{-\kappa n \varepsilon_n^3}$, we have

$$\sup_{0 \leq k \leq 3n \varepsilon_n} \frac{1}{n \varepsilon_n^2} \left| S_k - k \left( \gamma \varepsilon_n - \frac{k}{2n} \right) \right| < \eta.$$ 

Extending the trajectory of $S$ to $\mathbb{R}^+$ by linear interpolation and defining

$$S_u^{(n)} = \frac{1}{n \varepsilon_n^2} S_{u \varepsilon_n},$$

this can be rewritten as

$$P_u^- < S_u^{(n)} < P_u^+, \quad u \in [0, 3],$$

with probability greater than $1 - e^{-\kappa n \varepsilon_n^3}$. Clearly, this implies that there is an excursion (above the current minimum) of $S^{(n)}$, starting in $[0, u^-]$, and ending in $[U^-, U^+]$, which has thus a size between $U^- - u^+ > 2\gamma - \delta$ and $U^+ < 2\gamma + \delta$. Hence, $S$ itself has an excursion of size between $(2\gamma - \delta)n \varepsilon_n$ and $(2\gamma + \delta)n \varepsilon_n$, and thus the random graph has a component $C_0$ of this size.

To bound the size of the other components, note first that any component explored before $C_0$ has size less than $u^- n \varepsilon_n \leq \delta n \varepsilon_n / 2$. Let $V$ be the (random) set of vertices explored after $C_0$, let $C(v)$ be the component containing a vertex $v$, and $|C(v)|$ its size. Consider

$$M = \sum_{v \in V} 1_{\{|C(v)| > \delta n \varepsilon_n\}}.$$ 

By exchangeability, all $|C(v)|$, $v \in V$ have the same distribution, so that $\mathbb{E}(M) \leq n \mathbb{P}(|C(v_0)| > \delta n \varepsilon_n)$, where $v_0$ is the first vertex explored after $C_0$. The corresponding excursion starts somewhere in $[U^-, U^+]$. It is clear that the largest excursion that can fit between $P^-$ and $P^+$ has length less than $U^+ - U^- < \delta$. Hence,

$$\mathbb{P}(|C(v_0)| > \delta n \varepsilon_n) \leq 2 \exp(-\kappa n \varepsilon_n^3).$$
The probability that there is an excursion of size greater than \( \delta n \varepsilon_n \) explored after the large component is thus
\[
\mathbb{P}(M \geq 1) \leq \mathbb{E}(M) \leq 2n \exp(-\kappa n \varepsilon_n^3),
\]
and the result follows, taking again a smaller \( \kappa \) if necessary to absorb the constants.

Finally, the result for \( \gamma \leq 0 \) is a direct consequence of the case \( \gamma = 0 \). Indeed the size of the largest component is stochastically increasing in \( \gamma \), so that, for \( \gamma \leq 0 \),
\[
\mathbb{P}(|\mathcal{C}_1(n, (1 + \gamma \varepsilon_n)/n)| \geq \delta n \varepsilon_n) \leq \mathbb{P}(|\mathcal{C}_1(n, 1/n)| \geq \delta n \varepsilon_n) \leq 1 - n \exp(-\kappa n \varepsilon_n^3)
\]
and this completes the proof.

### 3.3 First gelation event in a dynamic graph: precise bounds

Observe that in Theorem 3.1 the probability of presence of an edge is fixed, and we control precisely the size of the largest (and sometimes second-largest) component.

Until the first gelation event in our model, the probability of presence of an edge increases exactly like \( p_t^{(N)} \). Hence, until gelation, we are exactly dealing with the usual dynamic version of an ER graph \( (\text{ER}(N, p_t^{(N)}))_{t \geq 0} \), and can deduce approximately at which time a large component is formed. This is exactly how we argued to obtain (12).

But as we already mentioned, ER graphs and our model will turn out to remain closely related even at times past gelation. After gelation, edges are still created at rate \( 1/N \), the threshold remains \( \alpha(N) \), but we should take into account that one or several components have fallen so the total number of vertices is, say, \( n \), less than \( N \).

It therefore makes sense to consider the càdlàg random graph process \( (\mathcal{G}_t^{(N)}(n))_{t \geq 0} \) on \([n]\) obtained by creating each link independently at rate \( 1/N \). We define

- \( \sigma^{(N)}(n) \) as the time when a large component i.e. a component of size \( \geq \alpha(N) \), appears;
- \( g^{(N)}(n) \) as the size of that large component;
- \( s^{(N)}(n) = n - g^{(N)}(n) \) as the number of particles remaining in solution at \( \sigma^{(N)}(n) \).

We claim the following. Beware in the formula of the difference between \( \sigma^{(N)}_\pm \) and \( \zeta^{(N)}_\pm \), so that \( \sigma^{(N)}_- \leq \sigma^{(N)}_+ \), and \( \zeta^{(N)}_- \leq \zeta^{(N)}_+ \).

**Proposition 3.3.** Assume that \( (\alpha(N)) \) verifies (3) holds. Let \( \nu \in (0, 1] \), \( \delta > 0 \),
\[
\sigma^{(N)}_\pm(n) = -N \log \left( 1 - \frac{1}{n} \left( 1 \pm \frac{1}{2} \left( 1 \mp \delta \right) \frac{\alpha(N)}{n} \right) \right),
\]
and
\[
\zeta^{(N)}_\pm(n) = -N \log \left( 1 - \frac{1}{n} \left( 1 - \frac{1}{2} \left( 1 \mp \delta \right) \frac{\alpha(N)}{n} \right) \right).
\]
Then there is a constant \( \kappa > 0 \), depending only on \( \delta, \nu \) and \( (\alpha(N)) \), such that, for all \( N \geq 2 \) and \( \nu N \leq n \leq N \), the following statements hold with probability greater than
\[
1 - N \exp\left(-\kappa \frac{\alpha(N)^2}{N^2} \right).
\]
1. The gelation time verifies
\[ \sigma_-(N)(n) \leq \sigma(N)(n) \leq \sigma_+(N)(n). \]

2. The size of the large component enjoys
\[ \alpha(N) \leq g(N)(n) \leq (1 + \delta)\alpha(N). \]

3. The gelation time also verifies
\[ \varsigma_-(N) \left( s(N)(n) \right) \leq \sigma(N)(n) \leq \varsigma_+(N) \left( s(N)(n) \right). \]

4. Conditionally on \( s(N)(n) \) and \( \sigma(N)(n) \), the graph \( \text{ER} \left( s(N)(n), p_{\sigma(N)(n)} \right) \) has no large component.

This result gives precise bounds on the gelation time for \( G(N)(n) \). Point 1 gives a bound in terms of \( n \), which is natural, whereas the bound in Point 3 is in terms of \( s(N)(n) \), the number of particles in solution right after the creation of the large component. Both results will turn out to be useful for us. Point 2 allows to control the size of the falling components, whereas Point 4 will allow us to compare our original model to an alternative model constructed from the graphs \( G(N)(n) \), see Section 5.

**Proof.** We may obviously assume \( \delta < 1/2 \). Let
\[ \varepsilon_n^- = \frac{1}{n} \inf_{n \leq k \leq n/\nu} \alpha(k), \quad \varepsilon_n^+ = \frac{1}{n} \sup_{n \leq k \leq n/\nu} \alpha(k), \quad \varepsilon_n = \frac{\alpha(N)}{n}. \]

Since \((\alpha(N))\) verifies (2), then (13) holds, and we may take \( \kappa \) as in Theorem 3.1, which depends only on \( \delta \) and \((\varepsilon_n^\pm)\), and thus on \( \delta, \nu \) and \((\alpha(N))\). We shall write w.h.p. below to mean with probability greater than
\[ 1 - Cn \exp(-\kappa n\varepsilon_n^3) \]
where \( C \) is just some large enough constant to take into account the union bounds below.

At any time \( t \), the distribution of \( G_t(N)(n) \) is that of a \( \text{ER} \left( n, p_t(N) \right) \). Hence \( \sigma_t(N)(n) < \sigma_-(N)(n) \) means that \( \text{ER} \left( n, p_{\sigma_t(N)(n)} \right) \) has a large component, with
\[ p_{\sigma_-(N)(n)}^{(N)} = \frac{1}{n} \left( 1 + \frac{1}{2} \left( 1 - \delta \right) \frac{\alpha(N)}{n} \right) = \frac{1}{n} \left( 1 + \gamma \varepsilon_n \right) \]
for \( \gamma = 1/2(1 - \delta) \). But Theorem 3.1 implies that
\[ \left| \mathcal{C}_1 \left( n, p_{\sigma_-(N)(n)}^{(N)} \right) \right| < (2\gamma + \delta)n\varepsilon_n = \alpha(N) \]
w.h.p. Hence \( \sigma(N)(n) > \sigma_-(N)(n) \) w.h.p. Conversely, \( \sigma(N)(n) > \sigma_+(N)(n) \) means that the graph \( \text{ER} \left( n, p_{\sigma_+(N)(n)}^{(N)} \right) \) has no large component, what, for the same reason, does not happen w.h.p. This proves Point 1.
For Point 2, notice that \( C_1(n,p) \) is stochastically increasing with \( p \). Hence, on the event of high probability \( \{\sigma^{(N)}(n) < \sigma^{(N)}_+(n)\} \), the size of the large component at \( \sigma^{(N)}(n) \) is bounded by \( |C_1(n,p^{\sigma^{(N)}_+(n)}(n))| \), which, by Theorem 3.1, is bounded by \((1 + 2\delta)n\varepsilon_n \) w.h.p., and Point 2 follows.

Now, using the two results just proved, let us write that, still w.h.p.,

\[
P^{(N)}_{\sigma^{(N)}(n)} \leq p^{(N)}_{\sigma^{(N)}_+(n)}(n)
= \frac{1}{n} \left( 1 + \frac{1}{2} (1 + \delta) \frac{\alpha(N)}{n} \right)
\leq \frac{1}{s^{(N)}(n)} \left( 1 - \frac{\alpha(N)}{n} \right) \left( 1 + \frac{1}{2} (1 + \delta) \frac{\alpha(N)}{n} \right)
\leq \frac{1}{s^{(N)}(n)} \left( 1 - \frac{1}{2} (1 - \delta) \frac{\alpha(N)}{n} \right)
\leq \frac{1}{s^{(N)}(n)} \left( 1 - \frac{1}{2} (1 - \delta) \frac{\alpha(N)}{n} \right).
\]

But obviously, \( s^{(N)}(n) \geq n - 2\alpha(N) \), so that

\[
\frac{\alpha(N)}{n} = \frac{\alpha(N)}{s^{(N)}(n)} \geq \frac{\alpha(N)}{s^{(N)}(n)} \left( 1 - 2\frac{\alpha(N)}{n} \right) \geq \frac{\alpha(N)}{s^{(N)}(n)} \left( 1 - 2\frac{\alpha(N)}{n} \right) \geq \frac{\alpha(N)}{s^{(N)}(n)} (1 - \delta),
\]

for \( N \) large enough, since \( \alpha(N)/N \to 0 \). Finally

\[
p^{(N)}_{\sigma^{(N)}(n)} \leq \frac{1}{s^{(N)}(n)} \left( 1 - \frac{1}{2} (1 - \delta) \frac{\alpha(N)}{s^{(N)}(n)} \right)
\leq \frac{1}{s^{(N)}(n)} \left( 1 - \frac{1}{2} (1 - 2\delta) \frac{\alpha(N)}{s^{(N)}(n)} \right),
\]

and the result follows after reordering. The other direction is similar.

For the last point, let us condition on \( s^{(N)}(n) \) and \( \sigma^{(N)}(n) \). On the event of high probability \( \{\sigma^{(N)}(n) \leq \zeta^{(N)}_+(s^{(N)}(n))\} \), we have

\[
P^{(N)}_{\sigma^{(N)}(n)} \leq p^{(N)}_{\sigma^{(N)}_+(s^{(N)}(n))} \leq \frac{1}{s^{(N)}(n)} \left( 1 - \frac{1}{2} (1 - \delta) \frac{\alpha(N)}{s^{(N)}(n)} \right) \leq \frac{1}{s^{(N)}(n)}.
\]

By the second point of Theorem 3.1,

\[
|C_1 \left( s^{(N)}(n), \frac{1}{s^{(N)}(n)} \right) | \leq \delta n\varepsilon_n < \alpha(N)
\]

with probability at least

\[
1 - s^{(N)}(n) \exp(-\kappa s^{(N)}(n)\varepsilon_n^3).
\]

Hence, by stochastic domination again,

\[
P \left( |C_1 \left( s^{(N)}(n), p^{(N)}_{\sigma^{(N)}_+(n)} \right) | \geq \alpha(N) \right) \leq s^{(N)}(n) \exp(-\kappa s^{(N)}(n)\varepsilon_n^3) + P \left( \sigma^{(N)}(n) > \zeta^{(N)}_+(s^{(N)}(n)) \right)
\leq s^{(N)}(n) \exp(-\kappa s^{(N)}(n)\varepsilon_n^3) + Cn \exp(-\kappa n\varepsilon_n^3).
\]

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To conclude, notice \( N \geq sN(n) \geq n - 2\alpha(N) \geq \nu N/2 \) for \( N \) large enough, and that for \( \nu N/2 \leq r \leq N \),

\[
r \exp(-\kappa r \varepsilon_n^3) = r \exp\left(-\kappa r \frac{\alpha(N)^3}{n^3}\right) \leq N \exp\left(-\frac{\kappa \nu \alpha(N)^3}{2 N^2}\right).
\]

As usual, it suffices to take \( \kappa \) smaller to get rid of the “\( N \) large enough” and the constants. \( \square \)

## 4 Combinatorial structure

The two results below explain the combinatorial structure of particles in solution in Smoluchowski’s discrete model, at a fixed time (Lemma 4.1), or at some particular stopping times (Lemma 4.3).

Let us start with an important preliminary remark. The natural filtration \((\mathcal{F}_t)\) of Smoluchowski’s (resp. Flory’s) discrete model is the one generated by the clocks, i.e. for any \( t \geq 0 \),

\[
\mathcal{F}_t := \sigma\left(e_{ij} \mathbb{1}_{\{e_{ij} \leq t\}}, \ i, j \in [N]\right),
\]

where \( e_{ij} \) is the clock on the link between \( i \) and \( j \).

In Flory’s discrete model the presence of the link between any two particles \( i \) and \( j \) at time \( t \) is exactly equivalent to the fact that the corresponding clock has rung, and in particular it is independent of the presence of any other link. Hence, the configuration in Flory’s discrete model at time \( t \) is exactly that of \( \text{ER}(N, p_t(N)) \). Similarly, for any \( S \subset [N] \), the configuration of particles of \( S \) at time \( t \) is exactly \( \text{ER}(|S|, p_t(N)) \). In other words, the configuration of particles in \( S \) at time \( t \) in Flory’s model is \( \mathcal{F}_t^S := \sigma\{e_{ij} \mathbb{1}_{\{e_{ij} \leq t\}}, \ i, j \in S\} \)-measurable.

In Smoluchowski’s discrete model on the other hand, for a subset \( S \subset [N] \), the knowledge of all the clocks \( S \leftrightarrow S \), or even the knowledge of all clocks \( S \leftrightarrow [N] \) is not enough to decide which links \( S \leftrightarrow S \) are created or not. In fact we need the information on all clocks up to time \( t \). For instance, in Figure 1, knowing all clocks but the one on link 3 does not allow to tell whether 5 is activated or not when it rings. In some sense, the model is “non-local”. In other words the configuration of particles in \( S \) at time \( t \) in Smoluchowski’s model is no longer \( \mathcal{F}_t^S \)-measurable.

Fortunately, there is a nice consistency property. For \( S \subset [N] \), let us define a graph \( G_S(t) \) by applying Smoluchowski’s algorithm up to time \( t \), but only to the subset \( S \) (i.e. perform the algorithm when only activating the links \( S \leftrightarrow S \)). As we just mentioned, \( G_S(t) \) has a priori nothing to do with the configuration on \( S \) at \( t \). It is however easy to check that, conditionally given that no link in \( S \leftrightarrow S \) exists at time \( t \), the configuration on \( S \) is indeed given by \( G_S(t) \).

This observation is the main tool in obtaining the following result. Recall that \( S(t) \) is the set of particles in solution at time \( t \), and let \( \text{ER}'(n, p) \) be the \( \text{ER}(n, p) \) random graph conditioned on having no large cluster, i.e. conditioned on having no cluster of size \( \geq \alpha(N) \).

**Lemma 4.1.** For any \( t \geq 0 \), conditionally given \( S(t) \), the configuration on \( S(t) \) is that of a

\[
\text{ER}'(|S(t)|, p_t(N))
\]

random graph, i.e. a \( \text{ER}(|S(t)|, p_t(N)) \) graph conditioned on having no large component.
Proof. Recall that for \( S \subset [N] \), and \( t \geq 0 \), \( \mathcal{F}^S_t = \sigma(\mathbf{e}_{ij} \mathbf{1}_{\{e_{ij} \leq t\}}, i \in S \text{ and } j \in S) \), and introduce as well
\[
\mathfrak{F}^S_t := \sigma(\mathbf{e}_{ij} \mathbf{1}_{\{e_{ij} \leq t\}}, i \in S \text{ or } j \in S)
\]
Observe in particular that for any \( S, t \), the \( \sigma \)-fields \( \mathcal{F}^S_t \) and \( \mathfrak{F}^S_t \) are independent, since they are generated by disjoint sets of clocks.

Furthermore, it is easily seen that \( \{ S(t) = S \} = E_1^S(t) \cap E_2^S(t) \), where
\[
E_1^S(t): \text{ } G_S(t) \text{ has no large component};
\]
\[
E_2^S(t): \text{ } G_{\overline{S}}(t) \text{ consists only of large components}; \text{ and any link } i \leftrightarrow j, \text{ with } i \in S, j \in \overline{S}, \text{ is activated after } j \text{ has become part of a large cluster in } G_{\overline{S}}(t).
\]

Obviously \( E_1^S(t) \in \mathcal{F}^S_t \), while \( E_2^S(t) \in \mathfrak{F}^S_t \), so these two events are independent.

Now, take \( G \) a graph on \( S \) with no large component. Recall from the introduction to the section that
\[
\mathbb{P}(C(t) = G|S(t) = S) = \mathbb{P}(G_S(t) = G|S(t) = S).
\]
Clearly \( \{ G_S(t) = G \} \in \mathcal{F}^S_t \) so it is also independent from \( E_2^S(t) \), and it follows that
\[
\mathbb{P}(G_S(t) = G|S(t) = S) = \mathbb{P}(G_S(t) = G|E_1(t), E_2(t)) = \mathbb{P}(G_S(t) = G|E_1(t)).
\]

This is the result, since conditionally on \( E_1(t) \), \( G_S(t) \) is just given by creating all activated links, as long as no large component is formed, i.e. it is a \( ER'(S, p^{(N)}_t) \) graph.

A simple but useful extension of this result can be obtained.

**Definition 4.2.** We say that \( \tau \) is a gelation stopping time if
\begin{itemize}
  \item \( \tau \) is a (\( \mathcal{F}_t \))-stopping time,
  \item for any \( S \subset [N] \) and \( t \geq 0 \), conditionally given \( \{ S(t) = S \} \), \( \tau \mathbf{1}_{\{\tau \leq t\}} \) is independent of \( \mathcal{F}^S_t \).
\end{itemize}

The most important and only gelation stopping times that we will consider are the gelation times \( (\tau_k) \). To check that they are indeed gelation stopping times, note that, conditionally given \( \{ S(t) = S \} \), \( \tau_k \mathbf{1}_{\{\tau \leq t\}} \) is simply determined by the \( k \)-th gelation time in \( (G_{\overline{S}}(s))_{s \leq t} \), and is therefore independent of \( \mathcal{F}^S_t \). On the other hand, for instance, the first time after \( \tau_k \) that a cluster of mass \( \alpha(N)/2 \) appears in solution is not a gelation stopping time.

**Lemma 4.3.** For any gelation stopping time \( \tau \), conditionally on \( \tau \) and \( S(\tau) \), the configuration on \( S(\tau) \) is that of a
\[
ER'(|S(\tau)|, p^{(N)}_\tau)
\]
random graph.

**Proof.** Informally, we essentially have the same proof as for a deterministic time:
\[
\mathbb{P}(C(\tau) = G|\tau = t, S(\tau) = S) = \mathbb{P}(G_S(t) = G|\tau = t, S(t) = S)
= \frac{\mathbb{P}(G_S(t) = G, \tau = t|S(t) = S)}{\mathbb{P}(\tau = t|S(t) = S)}
= \frac{\mathbb{P}(G_S(t) = G|S(t) = S) \mathbb{P}(\tau = t|S(t) = S)}{\mathbb{P}(\tau = t|S(t) = S)}
= \mathbb{P}(G_S(t) = G|S(t) = S),
\]
where, for the third equality, we use that both $G_S(t)$ and $\{\tau = t\}$ are independent conditionally on $\{S(t) = S\}$. Then the conclusion is as in the proof of the first result.

More rigorously, one needs to justify the degenerate conditioning with respect to the event $\{\tau = t\}$, and we thus need to study

$$\lim_{\varepsilon \to 0} \mathbb{P}(C(\tau) = G|S(\tau) = S, t - \varepsilon \leq \tau \leq t) = \lim_{\varepsilon \to 0} \mathbb{P}(G_S(\tau) = G|S(\tau) = S, t - \varepsilon \leq \tau \leq t).$$

But

$$\mathbb{P}(G_S(\tau) = G|S(\tau) = S, t - \varepsilon \leq \tau \leq t) = \frac{\mathbb{P}(G_S(\tau) = G, t - \varepsilon \leq \tau \leq t|S(\tau) = S)}{\mathbb{P}(t - \varepsilon \leq \tau \leq t|S(\tau) = S)}.$$

Recall that we work with $N$ fixed in this section, so that the probability that no clock rings between $\tau$ and $\tau + \varepsilon$ goes to one as $\varepsilon \to 0$. By using the Markov property at $\tau$, it is easily seen that, as $\varepsilon \to 0$,

$$\mathbb{P}(G_S(\tau) = G, t - \varepsilon \leq \tau \leq t|S(\tau) = S) \sim \mathbb{P}(G_S(t) = G, t - \varepsilon \leq \tau \leq t|S(t) = S)$$

and

$$\mathbb{P}(t - \varepsilon \leq \tau \leq t|S(\tau) = S) \sim \mathbb{P}(t - \varepsilon \leq \tau \leq t|S(t) = S).$$

It follows that as $\varepsilon \to 0$,

$$\mathbb{P}(C(\tau) = G|S(\tau) = S, t - \varepsilon \leq \tau \leq t) \sim \frac{\mathbb{P}(G_S(t) = G, t - \varepsilon \leq \tau \leq t|S(t) = S)}{\mathbb{P}(t - \varepsilon \leq \tau \leq t|S(t) = S)}$$

and we conclude as in the first informal computation, using that $\{t - \varepsilon \leq \tau \leq t\}$ is independent of $G_S(t) = G$ conditionally on $\{S(t) = S\}$, since $\tau$ is a gelation stopping time. \hfill \Box

5 Alternative model

The goal of this section is to present an alternative model which is easier to study than Smoluchowski’s since it has a much nicer combinatorial structure. We will prove however that w.h.p. the two models have the same distribution on compact time intervals.

A direct consequence of Lemma 4.3 is that, conditionally on $\tau_i$ and $n = |S(\tau_i)|$, the configuration in solution is that of conditioned ER graph $ER'(n, p_{\tau_i})$. Informally speaking, our goal is simply to somehow get rid of the “no large component” conditioning part.

We introduce below a slightly more complicated process, which is however merely built from dynamic ER graphs. Encoded in that process are two processes: our original one (at least, its state in solution $(C(t))$), and a process $(D(t))$ which is much easier to study. In other words, we have a coupling between our process and the more simple $(D(t))$: this is the purely combinatorial Lemma 5.1. The fundamental fact is that the processes are actually equal w.h.p., as proved in Lemma 5.2. This second statement makes heavy use of what was proven in Proposition 3.3. To explain our strategy, we will first start with a simple example.

5.1 A warming-up example

Consider two variables $X$ and $Y$ on $[N]$, constructed as follows:
\begin{itemize}
  \item X has a uniform distribution on \([N]\);
  \item conditionally on X, Y has a uniform distribution on \(\{X, \ldots, N\}\).
\end{itemize}

Note that the distribution of \((X, Y)\) is not the distribution of an independent couple of variables \((U, V)\) conditioned on \(V \geq U\). For instance \(\mathbb{P}(X = Y = N) \neq \mathbb{P}(U = V = N)\). More generally, one cannot obtain the distribution of \((X, Y)\) from a mere conditioning of \((U, V)\). To obtain this distribution from independent variables, it is common to use rejection sampling: consider \(U\) uniform and an i.i.d. family of uniform variables \((V_k)_{k \geq 1}\), independent from \(U\). Denote

\[ K = \inf\{k \geq 1, V_k \geq U\}. \]

Then, conditionally on \(U\), the distribution of \(V_K\) is that of a uniform variable on \(\{U, \ldots, N\}\). In other words, \((U, V_K)\) has the distribution of \((X, Y)\). Note also that, clearly, the uniform distribution here is not specific.

The whole point of this construction is that it provides a coupling between \((X, Y)\) and \((U, V)\), which allows us to compare them easily. For instance, if the distributions we consider are such that \(K = 1\) with high probability, then for any measurable set \(E\),

\[ \mathbb{P}((X, Y) \in E) = \mathbb{P}((U, V_K) \in E) \approx \mathbb{P}((U, V) \in E) \]

where \(\approx\) means “up to \(\mathbb{P}(K \neq 1)\)”, which is small. Studying \((X, Y)\) then essentially amounts to studying the much more tractable \((U, V)\). We shall carry out this construction in the following section, but in a slightly more elaborate setting.

### 5.2 Definition

Consider a family \((\mathcal{G}^{(N)}(n, k))\) indexed by \([N] \times \mathbb{N}\) of independent dynamic random graph processes, with \(n \in [N]\) and \(k \geq 1\), such that, for each \(n\) and \(k\), \(\mathcal{G}^{(N)}(n, k)\) is a random graph process on \([n]\) whose edges appear at rate \(1/N\). In other words, \(\mathcal{G}^{(N)}(n, k)\) has the distribution of \(\mathcal{G}^{(N)}(n)\) introduced in Section 3.3. We shall use the same notation \(\sigma, g, s\), but writing \(\sigma^{(N)}(n, k), g^{(N)}(n, k)\) and \(s^{(N)}(n, k)\) to insist that we refer to the process \(\mathcal{G}^{(N)}(n, k)\). We construct the process \((B(t))\) as follows.

**Step 0** Let \(N_0 = N\), \(\sigma_0 = 0\) and \(K(0) = 1\). Consider \(\mathcal{G}^{(N)}(N_0, K(0))\), up to the time \(t_1 = \sigma(N_0, K(0))\) when a large component, of exact size \(g_1 = g(N_0, K(0))\), appears. Define \(B(t) = \mathcal{G}_{t_1}^{(N)}(N_0, K(0))\) for \(t \in [0, t_1]\), let \(N_1 = N_0 - g_1 = s(N_0, K(0))\) and go to step 1.

**Step i** Consider here the graph processes \(\mathcal{G}^{(N)}(N_i, k)\) for \(k \geq 1\). Define \(K(i)\) to be the first \(k\) such that \(\mathcal{G}_{t_1}^{(N)}(N_i, k)\) has no large component. Then take \(t_{i+1} = \sigma(N_i, K(i))\) and \(g_{i+1} = g(N_i, K(i))\). Finally, define \(B(t) = \mathcal{G}_{t_1}^{(N)}(N_i, K(i))\) for \(t \in [t_i, t_{i+1})\), let \(N_{i+1} = N_i - g_{i+1} = s(N_i, K(i))\) and go to step \(i + 1\).

Obviously, we stop and let \(t_{i+1} = +\infty\) when no more large component can be created.

We will prove the following result, and, in doing so, explain this construction in more details. Recall that \((C(t))\) is the configuration in solution of our process.

**Lemma 5.1.** The processes \((C(t), t \geq 0)\) and \((B(t), t \geq 0)\) have the same distribution.
Proof. Let $M_0 = N$ and $M_i = c_i N$, $i \geq 1$ to be the mass in solution right after the gelation times. Our process is, before gelation, just a random graph process on $[N]$, so has the distribution of $G(N, 1) = G(N, 0, K(0))$. In particular, the time of the appearance of a large component and its size are the same for both processes, i.e.

$$\{(\tau_1, M_1, (C(t), t \in [0, \tau_1]) = (t_1, N_1, (B(t), t \in [0, t_1])\}.$$  

Let us then prove by induction that for every $i \geq 1$, $H_i$ holds, where $H_i$ is the assumption

$$\{(\tau_i, M_i, (C(t), t \in [0, \tau_i]) = (t_i, N_i, (B(t), t \in [0, t_i])\}.$$

We just checked $H_1$, so assume that $H_i$ holds for some $i \geq 1$. On the one hand, we know from Lemma 4.3 that

$$C(\tau_i) \overset{d}{=} ER'(M_i, p_{\tau_i}).$$

Now, how do we get $B(t_i)$? From the construction, $B(t_i)$ is constructed from the graphs $G_1^{(N)}(N_i, k)$. Each has distribution $ER(N_i, p_{\tau_i})$, and we choose the $K(i)$-th graph, the first which has no large component. As in the previous section, it is thus distributed as an $ER'(N_i, p_{\tau_i})$ graph. By $H_i$, this has the same distribution as a $ER'(M_i, p_{\tau_i})$ graph, and thus

$$B(t_i) \overset{d}{=} ER'(N_i, p_{\tau_i}) \overset{d}{=} ER'(M_i, p_{\tau_i}) \overset{d}{=} C(\tau_i).$$

Then, by Markov property, $C(\tau_i + t)_{t \geq 0}$ and $B(t_i + t)_{t \geq 0}$ evolve in the same way until the next gelation event, and $H_i+1$ follows. 

Hence, from now on, we will merely forget about $(B(t))$, and assume that $(C(t))$ is constructed as above. In particular, it is coupled with the process $(D(t))$ that we introduce now. Recall that, as mentioned in the previous section, the goal of this construction is to couple our process with a more simple process made of independent variables, so as to be able to compare them. This more simple process $(D(t))$ will be defined as follow, where we write $G^{(N)}(n) = G^{(N)}(n, 1)$.

**Step 0** Let $N_0 = N$ and $\sigma_0 = 0$. Consider $G^{(N)}(N_0)$, up to the time $\sigma_1 = \sigma(N_0)$ when a large component, of size $g_1 = g(N_0)$, appears. Define $D(t) = G^{(N)}(N_0)$ for $t \in [0, \sigma_1)$, let $N_1 = N_0 - g_1 = s(N_0)$ and go to step 1.

**Step i** Consider here the graph processes $G^{(N)}(N_i)$. Take $\sigma_{i+1} = \sigma(N_i) \lor \sigma_i$, $g_{i+1} = g(N_i)$ and define $D(t) = G^{(N)}(N_i)$ for $t \in [\sigma_i, \sigma_{i+1})$, where this interval might be empty if $\sigma(N_i) \leq \sigma_i$. Then let $N_{i+1} = N_i - g_{i+1} = s(N_i)$ and go to step $i+1$.

In other words, we consider the same model as above, but taking always $K(i) = 1$, in the spirit of what we hinted at at the end of the previous section. It should be clear that $(D(t))$ is easier to study than $(C(t))$. Our next claim is that under (3), $D$ and $C$ are barely any different, at least on compact intervals. To this end, define

$$I = \inf\{i \geq 1, K(i) \neq 1\}, \quad E(T) = \{t_T > T\},$$

so that $E(T)$ is the event that $K(i) = 1$ for all the $G^{(N)}$ we consider before time $T$. Hence, on $E(T)$, the processes $(C(t), t \in [0, T])$ and $(D(t), t \in [0, T])$ are equal. As usual, we will write $E^{(N)}(T)$ when we want to insist on the dependence on $N$. We will prove the following.
Lemma 5.2. If \((\alpha(N))\) is such that (3) holds, then, for all \(T \geq 0\), \(\mathbb{P}(E^{(N)}(T)) \to 1\).

Proof. Take \(\nu = \nu_T\) as in Lemma 2.6. Let also \(\kappa\) as in Proposition 3.3 (with \(\delta = 1/2\) say, but it does not matter). Now note that, \(K(i) = 1\) means that \(G^{(N)}_{t_i}(N_i, 1)\) has no large component. But
\[
G^{(N)}_{t_i}(N_i, 1) \overset{d}{=} ER(N_i, p_{t_i}) = ER(s(N_i-1), p_{\sigma(N_i-1)}),
\]
and we know by Point 4 of Proposition 3.3 that this graph has no large component with probability greater than
\[
1 - N \exp \left( -\kappa \frac{\alpha(N)^3}{N^2} \right),
\]
provided \(N_i-1 \geq \nu N\). Then, since \(t_i \leq T\) and \(n_T \geq \nu\) implies \(N_i \geq \nu N\), we have
\[
1 - \mathbb{P}(E^{(N)}(T)) = \mathbb{P}(t_i \leq T)
\leq \mathbb{P}(t_i \leq T, n_T \geq \nu) + \mathbb{P}(n_T < \nu)
= \mathbb{P} \left( \bigcup_i \{t_i \leq T, K(i) \neq 1, n_T \geq \nu\} \right) + \mathbb{P}(n_T < \nu)
\leq \mathbb{P} \left( \bigcup_i \{K(i) \neq 1, N_i \geq \nu N\} \right) + \mathbb{P}(n_T < \nu)
\leq N \times N \exp \left( -\kappa \frac{\alpha(N)^3}{N^2} \right) + \mathbb{P}(n_T < \nu),
\]
where we use at the last line that there are at most \(N\) (even \(N/\alpha(N)\)) gelation events. The first part on the last RHS tends to 0 by (3), the second as well by choice of \(\nu\) as in Lemma 2.6. \(\square\)

Recalling that \((C(t), t \in [0, T])\) and \((D(t), t \in [0, T])\) are equal on \(E(T)\), we may thus state our conclusion as follows.

Lemma 5.3. If \((\alpha(N))\) is such that (3) holds, then, for all \(T > 0\),
\[
\mathbb{P}((D(t), t \in [0, T]) = (B(t), t \in [0, T])) \to 1.
\]

Hence, anything that happens w.h.p. for \((D(t))\) also happens w.h.p. for \((C(t))\), at least on compact intervals. We will thus only have to study \((D(t))\) from now on. Its main properties are summarized in the following section.

5.3 Gelation times

Recall that \((\sigma_i)\) is the sequence of (possibly equal) gelation times in the alternative model \((D(t))\).

Lemma 5.4. Assume (3). Then for all \(\nu > 0\) and \(\delta > 0\), there exists \(\kappa > 0\) such that, for all \(i\) such that \(N_i \geq \nu N\), with probability greater than
\[
1 - N \exp \left( -\kappa \frac{\alpha(N)^3}{N^2} \right),
\]
the following hold.
1. The gelation time is related to the mass in solution by

\[
\frac{N}{N_i} \left( 1 - \left( \frac{1}{2} + \delta \right) \frac{\alpha(N)}{N_i} \right) \leq \sigma_i \leq \frac{N}{N_i} \left( 1 - \left( \frac{1}{2} - \delta \right) \frac{\alpha(N)}{N_i} \right)
\]

and

\[
\frac{N}{N_i} \left( 1 + \left( \frac{1}{2} - \delta \right) \frac{\alpha(N)}{N_i} \right) \leq \sigma_i \leq \frac{N}{N_i} \left( 1 - \left( \frac{1}{2} - \delta \right) \frac{\alpha(N)}{N_i} \right)
\]

2. The time before the next gelation event verifies

\[
\frac{N}{N_i} \alpha(N)(1 - \delta) \leq \sigma_{i+1} - \sigma_i \leq \frac{N}{N_i} \alpha(N)(1 + \delta).
\]

3. The mass lost at a gelation event enjoys

\[
\alpha(N) \leq N_i - N_{i+1} \leq (1 + \delta)\alpha(N).
\]

**Proof.** Let \( \kappa \) be as in Proposition 3.3, and let us use the same notation. All the events below are implied to happen with probability greater than \( 1 - N \exp(-\kappa\alpha(N)^3/N^2) \).

By definition, \( \sigma_i = \sigma_i^{(N)}(N_{i-1}) \) and \( N_i = s^{(N)}(N_{i-1}) \). Point 1 and 3 of Proposition 3.3 then ensure that

\[
\sigma^{(N)}_{i}(N_{i-1}) \leq \sigma_i \leq \sigma^{(N)}_{i}(N_{i-1}) \quad \text{and} \quad \varsigma^{(N)}_{i}(N_i) \leq \sigma_i \leq \varsigma^{(N)}_{i}(N_i)
\]

when \( N_i \geq \nu N \). A simple Taylor expansion provides the first part of the result, for \( N \) large enough, since \( \alpha(N)/N \to 0 \). From this, the second part is a straightforward computation, and the last inequality is just Point 2 of Proposition 3.3. As usual, the constants can be absorbed by taking a smaller \( \kappa \).

This results concerns the gelation times in the alternative model. This allows to study the gelation times in our original model, as summarized below.

**Proposition 5.5.** Assume (3). Then for any \( T > 0 \) and \( \delta > 0 \), with probability tending to one, it holds that

\[
\frac{1}{n_{\tau_i}^2} \frac{\alpha(N)}{N} (1 - \delta) \leq \tau_{i+1} - \tau_i \leq \frac{1}{n_{\tau_i}^2} \frac{\alpha(N)}{N} (1 + \delta)
\]

and

\[
\frac{\alpha(N)}{N} \leq n_{\tau_i} - n_{\tau_{i+1}} \leq (1 + \delta) \frac{\alpha(N)}{N}
\]

for all \( i \) with \( \tau_i \leq T \).

**Proof.** First, note that Points 2 and 3 in Lemma 5.4 hold for all \( i \) with \( N_i \geq \nu N \) with probability at least

\[
1 - N \times N \exp \left( -\frac{\kappa \alpha(N)^3}{N^2} \right)
\]

since there are at most \( N \) gelation events. Since \( (\alpha(N)) \) verifies (3), this probability tends to 1.

Now, denote by \( d_i = \#D(t)/N \) the concentration in the alternative model. By Lemma 5.3, w.h.p., \( \tau_i = \sigma_i \) and \( n_{\tau_i} = d_{\sigma_i} \) for all \( i \leq T \). Finally, by construction, if \( K(i) = 1 \), then \( N_i = d_{\sigma_i} N \). In particular, this holds on the event of high probability \( E^{(N)}(T) \), as long as \( \tau_i \leq T \). The result is then just replacing \( N_i \) by \( n_{\tau_i} N \) in Points 2 and 3 of Lemma 5.4. \( \square \)
6 Convergence

In this section we finish the proof of Theorem 1.2. We will classically proceed in two steps: first prove the tightness, and then prove that the limit is uniquely given by the result.

**Lemma 6.1.** Assume (3). Then the sequence \((n^{(N)})\) is tight in \(D([0, +\infty))\), and any limiting point is Lipschitz-continuous.

**Proof.** Fix \(T > 0\) and define
\[
w^{(N)}(\delta) = \sup_{s,t \leq T, |s-t| \leq \delta} |n_s^{(N)} - n_t^{(N)}|.
\]
According to Theorem 16.8 and (12.7) in [4], to check the tightness, it suffices to show that
\[
\lim_{a \to +\infty} \limsup_{N \to +\infty} P \left( \sup_{t \in [0,T]} |n_t^{(N)}| > a \right) = 0
\]
and for all \(\varepsilon > 0\),
\[
\lim_{\delta \to 0} \limsup_{N \to +\infty} P \left( w^{(N)}(\delta) > \varepsilon \right) = 0.
\]
The first point is obvious since \((n^{(N)})\) is bounded by 1. For the second point, taking \(\delta = 1/2\) in Proposition 5.5, implies that w.h.p., all the gelation times \((\tau_i)\) before \(T\) are more than \(\alpha(N)/2N\) apart. But a gelation event makes at most \(2\alpha(N)\) particles fall into the gel, so that the concentration lost on an interval of size \(\delta\) in \([0,T]\) is at most
\[
\frac{1}{N} \frac{2N}{\alpha(N)} \delta^2 \alpha(N) = 4\delta.
\]
There is merely a fencepost error here, and we thus have
\[
w^{(N)}(\delta) \leq 4\delta + \frac{2\alpha(N)}{N}
\]
w.h.p., and the tightness, as well as the Lipschitz-continuity, follow. \(\square\)

Let us now finish the proof of Theorem 1.2. We can assume, by the previous result, that \((n^{(N)})\) converges to some continuous \(m\) in \(D([0, +\infty))\). We shall prove that there is a unique limit point given by (4), which suffices to conclude.

**Lemma 6.2.** Almost surely, any limit point of \((n^{(N)})\) verifies (4).

**Proof.** We already know, say by (11), that no component has fallen into the gel at a time \(t < 1\) w.h.p., so that \(m_t = 1\) for \(t < 1\).

Now, consider \(t > 1, s > 0\). Fix \(\delta > 0\). All the following events will happen w.h.p. At least one gelation event has happened at time \(t\) and moreover, by Proposition 5.5 with \(T = t + s\), if \(t \leq \tau_i \leq t + s\), then
\[
\frac{\alpha(N)}{N(n_i^{(N)})^2} (1 - \delta) \leq \frac{\alpha(N)}{N(n_{\tau_i}^{(N)})^2} (1 - \delta) \leq \tau_{i+1} - \tau_i \leq \frac{\alpha(N)}{N(n_{\tau_i}^{(N)})^2} (1 + \delta) \leq \frac{\alpha(N)}{N(n_{i+s}^{(N)})^2} (1 + \delta).
\]
Consequently, the number of gelation events between $t$ and $t+s$ is at least
\[ \frac{N}{\alpha(N)} \frac{1}{1 + \delta} (n_t^{(N)})^2 s \]
and at most
\[ 1 + \frac{N}{\alpha(N)} \frac{1}{1 - \delta} (n_t^{(N)})^2 s. \]
Still by Proposition 5.5, a gelation events makes at most $(1 + \delta)\alpha(N)$ particles fall in the gel. Hence, the quantity that falls in the gel between $t$ and $t+s$ is at least
\[ N \frac{1}{1 + \delta} (n_t^{(N)})^2 s \]
and at most
\[ (1 + \delta)\alpha(N) + N \frac{1 + \delta}{1 - \delta} (n_t^{(N)})^2 s. \]
Passing to the limit shows that, for all $\delta$, almost surely,
\[ \frac{1}{1 + \delta} m_{t+s}^2 \times s \leq m_t - m_{t+s} \leq \frac{1 + \delta}{1 - \delta} m_t^2 \times s. \]
By continuity, this extends to all $t \geq 1$ and $s \geq 0$, and we may let $\delta \to 0$, to obtain
\[ m_{t+s}^2 \times s \leq m_t - m_{t+s} \leq m_t^2 \times s \]
for all $t \geq 1$, $s \geq 0$, so that
\[ \frac{d}{dt} m_t = -m_t^2, \quad t \geq 1. \]
Since $m_t = 1$ for $t < 1$, by continuity $m_1 = 1$, and the result follows. \qed

7 Conclusion

7.1 Microscopic description: proof of the last results

Let $K_N$ a complete graph on $N$ vertices, whose edges are given i.i.d mean $N$ exponential lengths. In [3] it is proven (see Theorem 4.1 and (4.11) of that reference) that one has weak local convergence of $K_N$ towards the Poisson Weighted Infinite Tree (PWIT) with $d = 1$. We refer to [3], Section 2 of for a precise definition of weak local convergence\(^2\), and Section 4.2 for a precise definition of the PWIT.

Letting $\tilde{p}_N = -\ln(1 - p_N)$, it is straightforward that if $\tilde{K}_N$ is obtained by performing an edge percolation on $K_N$ in keeping only edges whose length lies below $N\tilde{p}_N$, then the graph structure of $\tilde{K}_N$ is exactly that of a $ER(N, \tilde{p}_N)$ random graph. Moreover, observe that when $Np_N \to c$, $N\tilde{p}_N \to c$ as well.

Now, in the PWIT with $d = 1$, edges from a given vertex are assigned lengths according to a Poisson point process with intensity one. Thus, when suppressing all edges in the PWIT whose

\(^2\)In this definition of [3], graphs are seen as metric spaces, and edge lengths play the obvious role in defining the distance. For graphs whose edges lengths have not been precised — as for the graphs of this article — we adopt the usual convention that they are identically equal to 1.
length is above \(c\), we end up with a graph structure which is exactly that of a Galton-Watson tree with \(\mathcal{P}(c)\) reproduction law.

To conclude\(^3\), under the assumption \(Np_N \to c\), the result of Aldous and Steele implies the weak local convergence of \(ER(N, p_N)\) towards a Galton-Watson tree with offspring distribution \(\mathcal{P}(c)\).

Further, observe that if we were to consider \(G_n\), the subgraph of \(\tilde{K}_N\) obtained from \(\tilde{K}_N\) by keeping only \(n\) uniformly chosen vertices out of the \(N\) original ones, then \(G_n\) simply is a \(ER(n, p_N)\) random graph. Under the assumption \(np_N \to \lambda\), Aldous and Steele’s result implies weak local convergence of \(G_n\) towards a Galton-Watson tree with offspring distribution \(\mathcal{P}(\lambda)\).

Finally recall that, whenever \(\lambda \leq 1\), a Galton-Watson tree with offspring distribution \(\mathcal{P}(\lambda)\) is almost surely finite. In that case, weak local convergence of \(G_n\) towards a \(GW(\mathcal{P}(\lambda))\) exactly says that, if \(\mathcal{C}_{ER}(n, p_N)\) denotes the cluster rooted at some randomly chosen vertex of a \(ER(n, p_N)\) graph, and \(np_N \to \lambda\), then for any finite rooted tree \(\mathcal{T}\)

\[
P(\mathcal{C}_{ER}(n, p_N) = \mathcal{T}) \to P(GW(\mathcal{P}(\lambda)) = \mathcal{T}).
\]

Theorem 1.4 is therefore obvious for \(t < 1\): at such time \(C(t)\) is a \(ER'(n_t^{(N)} N, p_t^{(N)})\). But \(n_t^{(N)} N p_t^{(N)} \to t < 1\) by Theorem 1.2, so we are dealing with a subcritical \(ER\) graph conditioned on having no large component. The conditioning is therefore asymptotically trivial, and we conclude.

Now fix \(t \geq 1\). Let \(i_N\) be such that the \(i_N\)-th gelation time is close to \(t\): by Proposition 5.5 it suffices to take

\[
i_N = \max\{\tau_i : \tau_i \leq t\}.
\]

Indeed, if we write \(T_N = \tau_{i_N}\), then

\[
T_n \leq t \leq T_N + \frac{2}{\nu_t^2} \frac{\alpha(N)}{N}
\]

w.h.p. In particular \(T_N\) converges in probability to \(t\). Now, by Lemma 4.3, given \(T_N\) and \(n_{T_N}^{(N)}\), we are dealing at time \(T_N\) with a \(ER'(N n_{T_N}^{(N)}, p_{T_N}^{(N)})\) graph. But by Lemma 5.2, the conditioning on having no large component is asymptotically trivial at this time. This proves Proposition 1.3.

By Theorem 1.2, and the fact that \(T_N \to t\), we find that \(N n_{T_N}^{(N)} p_{T_N}^{(N)} \to 1\), in probability, as \(N \to \infty\). It follows by our initial discussion that, if \(i\) is uniformly chosen amongst the particles in solution at time \(T_N\), and \(\mathcal{C}(T_n)\) denotes the graph rooted at \(i\) consisting of its connected component, then for any finite rooted tree \(\mathcal{T}\),

\[
P(\mathcal{C}(T_n) = \mathcal{T}) \to P(GW(\mathcal{P}(1)) = \mathcal{T}).
\]

We finally need to look at a typical component at time \(t\). Observe first that particles in solution at \(T_N\) are exactly the same than those at \(t\), so our choice of \(i\) is also uniform amongst particles in solution at \(t\), and we may as well do the same choice at both times. Now, conditionally given that \(|C_{T_N}(i)| = |\mathcal{T}| =: k\), there are at most \(k(N - k)\) edges that may cause the size of the component of \(i\) to increase between times \(T_N\) and \(t\). Edges appear at rate \(1/N\)

---

\(^3\)This weak local convergence result for \(ER\) random graphs may not have been stated directly in [3], but it has been well-known since. A direct statement and a detailed proof may be found in [8], Proposition 2.6.
but there is only a time interval of length $\frac{2 \alpha(N)}{N}$. In the end, w.h.p none of those edges is activated between times $T_N$ and $t$, and therefore
\[ P(\mathcal{C}(t) = \mathcal{T}) \sim P(\mathcal{C}(T_N) = \mathcal{T}) \rightarrow P(\text{GW}(\mathcal{P}(1)) = \mathcal{T}), \]
which concludes the proof of Theorem 1.4.

7.2 Recovering the macroscopic results

When the initial conditions to Smoluchowski’s equation are assumed monodisperse, i.e. given by $c_0(m) = 1_{\{m=1\}}$, there is an earlier version of well-posedness obtained in [20], and the solution in that case is truly explicit. More precisely, let us define
\[ B(\lambda, m) = \frac{\left(\frac{\lambda m}{m!}\right)^{m-1}}{m!} e^{-\lambda m} \]
to be the Borel distribution with parameter $\lambda$. It is the distribution of the total progeny of a Galton-Watson process with reproduction law $\mathcal{P}(\lambda)$ (see e.g. Dwass’ formula [9]).

According to [20, 29], the solution to (1) with monodisperse initial conditions is given by
\[ c_t(m) = \begin{cases} \frac{1}{m} B(t, m) & t \leq 1, \\ \frac{1}{mt} B(1, m) & t \geq 1, \end{cases} \quad (22) \]

Theorem 1.4 therefore provides a better understanding of this explicit solution. Indeed, since the distribution of the size of the cluster to which a particle chosen uniformly at random belongs is size-biased, we find
\[ c_t^{(N)}(m) = n_t^{(N)} \frac{1}{m} \mathcal{P}(|\mathcal{C}(t)| = m). \]
Consequently, for $t \geq 1$,
\[ c_t(m) = \lim_{N \to \infty} c_t^{(N)}(m) = \frac{1}{m} n_t^{(N)} \lim_{N \to \infty} \mathcal{P}(|\mathcal{C}(t)| = m) = \frac{1}{m} \frac{1}{t} P(|\text{GW}(\mathcal{P}(1))| = m) = \frac{1}{mt} B(1, m) \]
Clearly, the same reasoning would also work for $t < 1$, only instead the Galton-Watson tree would have $\mathcal{P}(t)$ offspring distribution.

Observe further, as we mentioned at the end of the introduction, that the “macroscopic” SOC in the sense of [32] follows in fact directly from (22). Indeed, for $t \geq 1$, we have
\[ \sum_{m \geq k} m c_t(m) = \frac{1}{t} \sum_{m \geq k} B(1, m) \sim_{k \to \infty} \frac{1}{t^{\sqrt{2}/2}} \sum_{m \geq k} m^{-3/2} \sim_{k \to \infty} \frac{\sqrt{2}}{t^{\sqrt{2}/2}} k^{-1/2}. \]
where we used Stirling’s formula to get the first equivalent above.

Here again, Theorem 1.4 provides a better understanding: the quantity $\frac{1}{t} \sum_{m \geq k} B(1, m)$ above is the probability for a critical $\mathcal{P}(1)$ Galton-Watson tree to have size greater than $k$, which is well-known to behave in $k^{-1/2}$, as compared to an exponential decay in the strictly subcritical case, corresponding to $t < 1$. 

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7.3 Particles in the gel

It is reasonable to ask what happens for particles in the gel. As should be clear from the alternative model, gel is w.h.p a collection of clusters of size $\alpha(N) + o(\alpha(N))$. Also, w.h.p, for any $u \in [0,1)$, the $\lfloor uN/\alpha(N) \rfloor$-th one is the first large cluster which forms in a $(ER(N(1-u),t/N))_{t \geq 0}$ dynamic graph. Note that in this dynamic graph, such an emerging giant appears at time

$$\frac{1}{1-u} \left( 1 + \frac{1}{2} \frac{\alpha(N)}{(1-u)N} \right) + o\left( \frac{\alpha(N)}{N} \right),$$

according to Proposition 3.3.

Of course, the way we decided to deal with the gel is somewhat arbitrary. If any links were to be allowed between particles in the gel, this would not change anything to our results, but this would of course cause the emergence of a single giant. A more physically relevant picture would be to introduce some spatial structure for gel components, in which case the problem would become much harder to deal with.

7.4 Assumption on the threshold

One may rightfully question the assumption (3) on $(\alpha(N))$. With the techniques of this paper, it could be slightly relaxed. Going through the proofs shows that we only need

$$\frac{N^2}{\alpha(N)} \exp \left( -\kappa \frac{\alpha(N)^3}{N^2} \right) \to 0,$$

for any $\kappa > 0$, what can be achieved with logarithmic factors.

But our techniques can not be extended to the case of smaller thresholds. The main reason is that, even though Lemma 4.3 still holds, the conditioning becomes asymptotically non trivial. Think for instance of the case $\alpha(N) = cN^{2/3}$ which brings the model at its first gelation time right inside the critical window, as studied by Aldous [1]. In the limit $N \to \infty$ the conditioning would corresponds to conditioning a parabolically drifted BM to have no excursion above its infimum of size greater than $c$, an event of probability $p_c \in (0,1)$. In fact, in the case $\alpha(N) \ll N^{2/3}$ the conditioning could even be degenerate.

Hence, for smaller thresholds, the configuration in solution at some time past gelation is far from obvious, and it may strongly affect at least the distribution of the largest clusters in solution.

We conjecture however that both Theorem 1.2 and 1.4 remain true under (2). The intuition is that only the largest clusters are affected by the asymptotically non-trivial conditioning. Since those largest clusters asymptotically account for a vanishing proportion of vertices, and since they live very briefly in solution, we believe that the typical microscopic evolution remains asymptotically the same.

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