On embedding theorems for \( \mathcal{X} \)-subgroups

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Abstract. Let \( \mathcal{X} \) be a class of finite groups closed under subgroups, homomorphic images, and extensions. We study the question which goes back to the lectures of H. Wielandt in 1963–1964: Given an \( \mathcal{X} \)-subgroup \( K \) and a maximal \( \mathcal{X} \)-subgroup \( H \), is it possible to detect embeddability of \( K \) in \( H \) (up to conjugacy) by their projections into the factors of a fixed subnormal series? On the one hand, we construct examples where \( K \) has the same projections as some subgroup of \( H \) but is not conjugate to any subgroup of \( H \). On the other hand, we prove that if \( K \) normalizes the projections of a subgroup \( H \), then \( K \) is conjugate to a subgroup of \( H \) even in the more general case when \( H \) is a submaximal \( \mathcal{X} \)-subgroup.

Mathematics Subject Classification. 20E28, 20D20, 20D35.

Keywords. Finite group, Complete class, Maximal \( \mathcal{X} \)-subgroup, Submaximal \( \mathcal{X} \)-subgroup, Embedding theorems.

1. Main concepts and results. Throughout this paper, all groups that we consider are by convention finite. Let \( G \) be such a group. We fix a series of subgroups

\[
G = G_0 \geq G_1 \geq \cdots \geq G_n = 1.
\]

As usual, we say that the series (\( \ast \)) is normal or subnormal if, respectively, \( G \geq G_i \) or \( G_{i-1} \geq G_i \) for each \( i = 1, \ldots, n \). Given a subgroup \( H \) of \( G \), put

\[
H_i = H \cap G_i \quad \text{and} \quad H^i = H_{i-1}G_i/G_i \quad \text{for} \quad i = 1, \ldots, n.
\]

In particular, \( G^i = G_{i-1}/G_i \) is the \( i \)th factor of the series (\( \ast \)). The subgroup \( H^i \) of \( G^i \) is said to be the projection of \( H \) into the factor \( G^i \).

The authors were supported by the Natural Science Foundation of China (No. 12171126). The second and third authors were supported by the Program of Fundamental Research RAS, project FWNF-2022-0002.
In general, two subgroups having the same projections into the factors of $\ast$ need not even be isomorphic. The situation turns out to be essentially different if we consider not arbitrary subgroups, but restrict to those which are maximal among subgroups from some complete class. Recall that, according to Wielandt [5,6], a nonempty class of groups $\mathcal{X}$ is called complete if it is closed under taking subgroups, homomorphic images, and extensions. The groups in $\mathcal{X}$ are called $\mathcal{X}$-groups. A $\mathcal{X}$-subgroup $H$ of a group $G$ is said to be $\mathcal{X}$-maximal if $H \leq K \leq G$ and $K \in \mathcal{X}$ always imply $H = K$. In this case, $H$ is also called a maximal $\mathcal{X}$-subgroup and written $H \in m_\mathcal{X}(G)$. We emphasize that, if $G$ itself is an $\mathcal{X}$-group, then $G$ is the unique maximal $\mathcal{X}$-subgroup of $G$ (like a Sylow $p$-subgroup of a $p$-group $G$ is $G$ itself). Wielandt proved [5, Theorem 14.1(3)] that a maximal $\mathcal{X}$-subgroup is uniquely determined up to conjugacy by its projections into the factors of a normal series $\ast$.

A similar assertion for a subnormal series is also true. Moreover, it is true even in a stronger and (as far as the proof goes) more convenient form proposed by Wielandt in [6]. According to [6], a subgroup $H$ of $G$ is called a submaximal $\mathcal{X}$-subgroup (also $\mathcal{X}$-submaximal, $H \in \mathcal{X}m(G)$) if there exists an embedding $\varphi : G \hookrightarrow G_*$ of $G$ into some group $G_*$ such that $G_\varphi$ is subnormal in $G_*$ and $H_\varphi = G_\varphi \cap H_*$ for some $H_* \in m_\mathcal{X}(G_*)$.

Wielandt announced the assertion [6, 5.4(c)]: if $H, K \in \mathcal{X}m(G)$ and the projections of these subgroups into factors of a subnormal series $\ast$ coincide, then $H$ and $K$ are conjugate in $G$. A proof was presented in [4, Corollary 1], apparently the first published proof. In the present paper, we strengthen this theorem to some degree, and hence also the original assertion about maximal $\mathcal{X}$-subgroups.

**Theorem 1.** Let $G$ be a finite group with a subnormal series $\ast$ and let $H \in \mathcal{X}m(G)$. Then every subgroup $K$ of $G$ whose projections into factors of the series $\ast$ coincide with those of $H$ is conjugate to $H$ in $\langle H, K \rangle$ and, in particular, $K \in \mathcal{X}m(G)$.

A natural question coming next is whether it is possible to see from the projections that an arbitrary subgroup $K$ of $G$ can be embedded into a given maximal or submaximal $\mathcal{X}$-subgroup $H$. In the case of a normal series $\ast$, Wielandt pointed out some sufficient condition under which this is possible [5, Theorem 14.1]: up to conjugacy, an $\mathcal{X}$-subgroup $K$ of $G$ is embedded into $H \in m_\mathcal{X}(G)$ if $G$ has a normal series $\ast$ and $K \leq N_G(H \cap G_{i-1})G_i$ for each $i = 1, \ldots, n$. Further, in [5, Open Problems to Theorem 14.1], he formulated the following general problem.

**Wielandt’s problem.** Let a subgroup $K$ have the same projections into the factors of a normal series $\ast$ as some subgroup $K_* \leq H$, where $H \in m_\mathcal{X}(G)$. Is it true that $K$ is conjugate to a subgroup of $H$?

We show (see the examples in Section 2) that in general the answer to this question is negative.

Later, see [6, 5.4(b)], Wielandt announced an analogue of his assertion [5, Theorem 14.1] for the case of a subnormal series: if an $\mathcal{X}$-subgroup $K$ of $G$ with
a subnormal series \((\ast)\) normalizes every projection \(H^i\) of \(H \in \text{sm}_X(G)\), then \(K\) is conjugate in \((H,K)\) to a subgroup of \(H\). The problem with the formulation of the latter assertion is that the condition \(K\) normalizes the projection \(H^i\) obviously needs to be clarified since \(N_G(H \cap G_{i-1})G_i\) is no longer necessarily a subgroup. The main goal of this note is to propose an exact formulation and to prove Wielandt’s assertion [6, 5.4(b)]. To this end, we need the following definitions.

We say that subgroups \(H\) and \(K\) of a group \(G\) with a subnormal series \((\ast)\) are congruent modulo the series \((\ast)\) and write \(H \equiv K \mod (\ast)\) if \(H_i = K_i\) for each \(i = 1, \ldots, n\). This defines an equivalence relation on the set of subgroups of \(G\).

Moreover, if an automorphism \(\alpha \in \text{Aut}(G)\) stabilizes the series \((\ast)\), i.e., \(G^\alpha_i = G_i\) for each \(i = 1, \ldots, n\), then for all \(H,K \leq G\),

\[
H \equiv K \mod (\ast) \Rightarrow H^\alpha \equiv K^\alpha \mod (\ast).
\]

We say that an element \(x\) of \(G\) normalizes a subgroup \(H\) modulo the series \((\ast)\) if it stabilizes the series \((\ast)\) under conjugation and \(H \equiv H^x \mod (\ast)\). The subset of all such elements of \(G\) is called the normalizer of \(H\) modulo the series \((\ast)\) and denoted by \(N_G^{(\ast)}(H)\). Obviously, \(N_G^{(\ast)}(H)\) is a subgroup of \(G\).

**Theorem 2.** Let \(G\) be a finite group with a subnormal series \((\ast)\) and let \(H \in \text{sm}_X(G)\). If an \(X\)-subgroup \(K\) lies in \(N_G^{(\ast)}(H)\), then \(K\) is conjugate in \(\langle H,K \rangle\) to a subgroup of \(H\).

The proofs of Theorems 1 and 2 exploit the concept of an \(X\)-separable group which goes back to the works of Chunikhin (see, e.g., [1]). Recall that a group is called an \(X'\)-group if it does not contain any nontrivial \(X\)-subgroups. According to [5, Definition 12.8], a group is said to be \(X\)-separable if it has a normal or, equivalently, subnormal series such that each factor is an \(X\)- or \(X'\)-group. The least possible number of \(X\)-factors in such series is called the \(X\)-length of an \(X\)-separable group \(G\) and is denoted by \(l_X(G)\). For example, if \(X\) is the class of \(p\)-groups for some prime \(p\), then an \(X\)-separable group is \(p\)-solvable and its \(X\)-length is the \(p\)-length as in the famous Hall and Higman paper [3].

We refer to an \(X\)-subgroup \(H\) of \(G\) as a \textit{Hall} \(X\)-subgroup (\textit{X-Hall subgroup}, \(H \in \text{Hall}_X(G)\)) if the index \(|G : H|\) is not a multiple of any prime \(p\) such that \(X\) contains a group of order \(p\). Every \(X\)-separable group \(G\) enjoys an analogue of Sylow’s theorem for their Hall \(X\)-subgroups, that is: they exist, are conjugate to one another, and every \(X\)-subgroup of \(G\) is included in some \(X\)-Hall subgroup, in particular, \(m_X(G) = \text{Hall}_X(G)\) (cf. Lemma 2.2 below).

We prove Theorems 1 and 2 by embedding submaximal \(X\)-subgroups into some \(X\)-separable subgroups, where they turn out to be \(X\)-Hall subgroups. We do this using induction on the length of the series \((\ast)\) and the following assertion, which is of independent interest.
Theorem 3. Let $G$ be a finite group with a normal series $(*)$ of length $n$, let $H \in \text{sm}_{\mathfrak{X}}(G)$, and let $\Delta = \{K \leq G \mid K \equiv H \pmod{(*)}\}$. Then the subgroup $W = \langle \Delta \rangle$ is $\mathfrak{X}$-separable, $l_{\mathfrak{X}}(W) \leq n$, and $H \in \text{Hall}_{\mathfrak{X}}(W)$.

In fact, even a little more can be proved.

Corollary 4. Let $G$ be a finite group with a normal series $(*)$ of length $n$ and let $H \in \text{sm}_{\mathfrak{X}}(G)$. Then the subgroup $N^*(G)(H)$ is $\mathfrak{X}$-separable, $l_{\mathfrak{X}}(N^*(G)(H)) \leq n$, and $H \in \text{Hall}_{\mathfrak{X}}(N^*(G)(H))$.

It would be interesting to know whether the assertions on $\mathfrak{X}$-separability in Theorem 3 and Corollary 4 remain true under the weaker assumption that $(*)$ be subnormal.

2. Examples and proofs. First, we give examples showing that the answer to Wielandt’s problem in Section 1 is negative.

Example 1. Let $G = S_n$ be the symmetric group on the set $\Omega = \{1, 2, \ldots, n\}$, where $n \geq 5$. Suppose that the class $\mathfrak{X}$ consists of all finite groups whose nonabelian composition factors have orders less than $n!/2$. It is clear that in this case every maximal $\mathfrak{X}$-subgroup of $G$ is simply a maximal subgroup distinct from the alternating group $A_n$, and vice versa. As such a maximal subgroup $H$ of $G$, we take the point stabilizer of $n \in \Omega$, isomorphic to $S_{n-1}$, and consider the normal series

$$G = G_0 > G_1 > G_2 = 1,$$

(\text{**) where } G_1 = A_n. We indicate $\mathfrak{X}$-subgroups $K$ and $K_*$ such that $K \equiv K^* \pmod{(**)}$, $K_* \leq H$, but $K$ is not conjugate to any subgroup of $H$.

To this end, consider the stabilizer $H_*$ of the set $\{n-1, n\}$, which is isomorphic to $S_{n-2} \times S_2$. Denote by $K_*$ the intersection of $H$ and $H_*$, which is clearly isomorphic to $S_{n-2}$. Let $K$ be the subgroup consisting of those elements in $H_*$ that induce even permutations on $\Omega \setminus \{n-1, n\}$. Clearly, $K \cong A_{n-2} \times S_2$. It is easy to check that the projections of $K$ and $K_*$ into the factors of the series $(**)$ are the same. Furthermore, $K_* \leq H$, but $K$ is not conjugate to any subgroup of $H$ since $K$ contains elements acting on $\Omega$ without fixed points.

In the above example, the subgroups $K$ and $K_*$ are not isomorphic. It is not hard to give an example of isomorphic $\mathfrak{X}$-subgroups that are congruent modulo the series $(**)$, one is contained in the maximal $\mathfrak{X}$-subgroup $H$ and the other is not conjugate to any subgroup of $H$.

Example 2. In the situation described in Example 1, suppose that $n = 2m$, where $m \geq 3$ is odd. As before, $H$ is the point stabilizer of $n \in \Omega$. Let

$$T_* = \langle (1,2) \rangle \leq H \quad \text{and} \quad T = \langle (1,2) \ldots (2m-1, 2m) \rangle.$$

Note that $T$ acts semiregularly on $\Omega$, and hence is not conjugate to any subgroup of $H$. Meanwhile, the projections of $T$ and $T^*$ into factors of the series $(**)$ coincide since any nontrivial permutation in them is odd.

We start our proofs with some auxiliary statements.
Lemma 2.1 ([5, 13.5]). Every subgroup of an $\mathfrak{X}$-separable group is also $\mathfrak{X}$-separable.

Lemma 2.2. Let $G$ be an $\mathfrak{X}$-separable group. Then

(i) every two elements of $m_{\mathfrak{X}}(G)$ are conjugate;
(ii) $m_{\mathfrak{X}}(G) = \text{Hall}_{\mathfrak{X}}(G)$.

Proof. Claim (i) is [5, Theorem 12.10]. Since $\mathfrak{X}$ is complete, the following three conditions are equivalent for every prime $p$: $\mathfrak{X}$ contains a group of order divisible by $p$, $\mathfrak{X}$ contains a group of order $p$, and $\mathfrak{X}$ contains all $p$-groups.

Assume that $p$ satisfies any of these conditions. Claim (i) and Sylow’s theorem imply that every $H \in m_{\mathfrak{X}}(G)$ contains a Sylow $p$-subgroup of $G$. Therefore, $H \in \text{Hall}_{\mathfrak{X}}(G)$. The reverse inclusion $\text{Hall}_{\mathfrak{X}}(G) \subseteq m_{\mathfrak{X}}(G)$ is clear. □

Lemma 2.2 allows us to use the following argument which is similar to the well-known Frattini argument for Sylow subgroups.

Lemma 2.3 (Frattini argument). Let a group $G$ contain a normal $\mathfrak{X}$-separable subgroup $V$ and let $H \in \text{Hall}_{\mathfrak{X}}(V)$. Then $G = VN_G(H)$.

Proof. If $g \in G$, then $Hg \in \text{Hall}_{\mathfrak{X}}(V)$. By Lemma 2.2(i), there exists an element $x \in V$ such that $Hg = Hx$. Hence $gx^{-1} \in N_G(H)$ and $g \in N_G(H)x \subseteq N_G(H)V = VN_G(H)$. □

Hall $\mathfrak{X}$-subgroups of an arbitrary group lend themselves well to study, thanks to the remarkable inductive property presented in the next lemma.

Lemma 2.4. If $H \in \text{Hall}_{\mathfrak{X}}(G)$ and if a subgroup $N$ is subnormal in $G$, then $H \cap N \in \text{Hall}_{\mathfrak{X}}(N)$.

Proof. Clearly, $H \cap N \in \mathfrak{X}$. Arguing by induction on the length of a subnormal series from $G$ to $N$, we may assume that $N$ is normal in $G$, in particular, $HN$ is a subgroup of $G$. Now the lemma follows from the definition of a Hall $\mathfrak{X}$-subgroup and the basic equality $|G : H| = |G : HN||N : (H \cap N)|$. □

Unfortunately, maximal $\mathfrak{X}$-subgroups do not enjoy the same property (see examples in [2, Section 5]), while $\mathfrak{X}$-submaximal subgroups do.

Lemma 2.5. If $H \in \text{sm}_{\mathfrak{X}}(G)$ and a subgroup $N$ is subnormal in $G$, then $H \cap N \in \text{sm}_{\mathfrak{X}}(N)$.

This property immediately follows from the definition. In contrast, another fundamental property of $\mathfrak{X}$-submaximal subgroups, first formulated by Wielandt in [6, 5.4(a)], requires a justification, a proof can be found in [4, Theorem 2].

Lemma 2.6. If $H \in \text{sm}_{\mathfrak{X}}(G)$, then $N_G(H)/H$ is an $\mathfrak{X}'$-group.

If a group $G$ has a series $(*i)$, then for carrying out induction arguments, it is useful to denote by $(*i)$ the segment of $*$ starting at $G_i$, that is,

$$G_i \supseteq G_{i+1} \supseteq \cdots \supseteq G_n = 1. \quad (*i)$$

It is easy to see that the statements below are equivalent:
(i) $H \equiv K \pmod{\ast}$;
(ii) $H_i \equiv K_i \pmod{\ast}$ for each $i = 1, \ldots, n$;
(iii) $H_i \equiv K_i \pmod{\ast i}$ for each $i = 1, \ldots, n$.

Recall that the normalizer $N_G^G(H)$ of a subgroup of $H$ modulo $\ast$ is the set of elements $x \in G$ such that

(N1) $x$ stabilizes $\ast$, i.e., $G_i^x = G_i$ for all $i$, and
(N2) $H \equiv H^x \pmod{\ast}$.

If the series $\ast$ is normal, then the condition (N1) is satisfied automatically. In the general case, the set of elements stabilizing the series $\ast$ coincides with $G^{(\ast 0)}$, where for every $i = 0, \ldots, n$, we refer to

$$G^{(\ast i)} := \bigcap_{j=i}^n N_G(G_j)$$

as the stabilizer of the series $\ast i$. We also define the normalizer (in $G$) of $H_i$ modulo the series $\ast i$ as the set

$$N_G(H_i) = \{ x \in G^{(\ast i)} \mid H_i^x \equiv H_i \pmod{\ast i} \}.$$

The following properties of these normalizers are easily verified.

Lemma 2.7. Let $G$ be a finite group with a series $\ast$, let $H \trianglelefteq G$, and let $i \in \{1, \ldots, n\}$.

(i) $N_G^G(H)$ and $N_G(H_i)$ are subgroups of $G$.
(ii) If $H \equiv K \pmod{\ast}$, then $N_G^G(H) = N_G^G(K)$; if $H_i \equiv K_i \pmod{\ast i}$, then $N_G^G(H_i) = N_G^G(K_i)$.
(iii) Let $\Delta = \{ K \leq G \mid K \equiv H \pmod{\ast} \}$ and $\Delta_i = \{ L \leq G_i \mid L \equiv H_i \pmod{\ast i} \}$. Then $N_G^G(H)$ acts by conjugation on $\Delta$ and $N_G(H_i)$ acts by conjugation on $\Delta_i$.
(iv) $N_G^G(H) = N_G^{G^{(0)}}(H_0) \leq N_G^{G^{(1)}}(H_1) \leq \cdots \leq N_G^{G^{(n)}}(H_n) = G$.
(v) If the series $\ast$ is normal, then $H \trianglelefteq N_G^G(H) \trianglelefteq N_G^G(H)$.

Note that in the case of a subnormal series $\ast$, the normalizer $N_G^G(H)$ of a subgroup $H$ modulo $\ast$ need not include $H$.

We are now ready to prove our main results.

Proof of Theorem 1. We argue by induction on $n$. For $n = 1$, we have $H = K$ and everything is proved.

Let $n > 1$. Put $J = \langle H, K \rangle$. By Lemma 2.5, we have $H_1 = H \cap G_1 \in \text{sm}_X(G_1)$ and $H_1 \equiv K_1 \pmod{\ast 1}$. By the induction hypothesis, $H_1 = K_1^x$ for some $x \in \langle H_1, K_1 \rangle \leq J \cap G_1$. Moreover, $H^1 = K^1$ implies $HG_1 = KG_1 = K^x G_1$. Hence $K^x \equiv H \pmod{\ast}$. Since $x \in J$, it follows that $\langle H, K^x \rangle \leq J$. Replacing $K$ by $K^x$, we may thus assume without loss of generality that $H_1 = K_1$.

For brevity, we set $G^* = HG_1 = KG_1$ and $T = H_1 = K_1$. Now $H, K \leq N_G^G(T)$, whence $J \leq N_G^G(T)$. Note that $N_G^G(T) = G_1 \cap N_G^G(T)$ is a normal
subgroup of $N_{G^*}(T) = HN_{G_1}(T)$. The factor group
\[
N_{G^*}(T)/N_{G_1}(T) = HN_{G_1}(T)/N_{G_1}(T) \cong H/(H \cap N_{G_1}(T))
\]
is an $\mathcal{X}$-group. Furthermore, $T \in \text{sm}_X(G_1)$ and, by Lemma 2.6, the factor $N_{G_1}(T)/T$ is an $\mathcal{X}'$-group. Finally, $T = H_1$ is an $\mathcal{X}$-group. Thus, all factors of the normal series
\[
N_{G^*}(T) \triangleright N_{G_1}(T) \triangleright T \triangleright 1
gives
\]
lie either in $\mathcal{X}$ or $\mathcal{X}'$, so the group $N_{G^*}(T)$ is $\mathcal{X}$-separable. Notice that the projections of $H$ into the factors of the series ($\dagger$) are $\mathcal{X}$-Hall in these factors. Therefore, $H \in \text{Hall}_X(N_{G^*}(T))$. By Lemma 2.1, the subgroup $J$ of $N_{G^*}(T)$ is also $\mathcal{X}$-separable, so $H \in \text{Hall}_X(J)$.

Since $H \equiv K \pmod{\ast}$ and $\mathcal{X}$ is complete, it follows that $K \in \mathcal{X}$ and $|H| = |K|$. Therefore, $K \in \text{Hall}_X(J)$. Thus, $H$ and $K$ are conjugate in $J$ by Lemma 2.2. $\square$

**Proof of Theorem 3.** We proceed by induction on $n$. If $n = 1$, then $\Delta = \{H\}$ and $W = H \in \mathcal{X}$, so there is nothing to prove.

Let $n > 1$. Consider the series
\[
G_1 \triangleright \cdots \triangleright G_n = 1
\]
of length $n - 1$. Let
\[
\Gamma = \{K \cap G_1 \mid K \in \Delta\} \quad \text{and} \quad \Delta_1 = \{L \leq G_1 \mid L \equiv H_1 \pmod{\ast 1}\}.
\]

We set $V = \langle \Delta_1 \rangle$ and claim that
(a) $V$ is $\mathcal{X}$-separable, $l_X(V) \leq n - 1$, and $H_1 = H \cap G_1 \in \text{Hall}_X(V)$;
(b) if $K \in \Delta$, then $K \leq N_G(V)$;
(c) $\Gamma = \Delta_1$.

Lemma 2.5 yields $H_1 \in \text{sm}_X(G_1)$, so claim (a) holds by the inductive hypothesis.

Let $K \in \Delta$. Since the series ($\ast$) is normal, Lemma 2.7(v) implies that $K \leq N_G^{(\ast)}(K)$. By statements (ii) and (iv) of the same lemma,
\[
N_G^{(\ast)}(K) = N_G^{(\ast)}(H) \leq N_G^{(\ast 1)}(H_1).
\]
Applying Lemma 2.7(iii), we conclude that $K$ acts by conjugation on $\Delta_1$, so claim (b) holds.

It is clear that $\Gamma \subseteq \Delta_1$. Let us prove the reverse inclusion. Take $L \in \Delta_1$. Since $H_1 \in \text{Hall}_X(V)$ by (a) and $L \equiv H_1 \pmod{\ast 1}$, it follows that $L \in \text{Hall}_X(V)$. The subgroup $V$ is $\mathcal{X}$-separable by (a) and $H \in \Delta$ normalizes $V$ by (b), so the subgroup $U = HV$ is also $\mathcal{X}$-separable. Moreover, $H_1$ coincides with $H \cap V$ as a Hall $\mathcal{X}$-subgroup of $V$ (cf. claim (a)), so $H \in \text{Hall}_X(U)$ because $|U : H| = |V : H_1|$.

Take $M \in \text{m}_X(N_U(L))$. Then $L \leq M$, so $L \leq M \cap V$. Moreover, $L = M \cap V$ because $L \in \text{Hall}_X(V)$. The normalizer $N_U(L)$ is $\mathcal{X}$-separable as a subgroup of the $\mathcal{X}$-separable group $U$, in view of Lemma 2.1. Lemma 2.2(ii) yields $M \in \text{Hall}_X(N_U(L))$. 

According to the Frattini argument (Lemma 2.3), \( U = VN_U(L) \). Since
\[ |U : M| = |VN_U(L) : M| = |V : N_V(L)||N_U(L) : M| \]
and \( |V : N_V(L)| \) divides \( |V : L| \), the index \( |U : M| \) is coprime to the order of any \( \mathcal{X} \)-group. Therefore, \( M \in \text{Hall}_{\mathcal{X}}(U) \).

Since \( H, M \in \text{Hall}_{\mathcal{X}}(U) \) and \( H_1, L \in \text{Hall}_{\mathcal{X}}(V) \), we have \( |H| = |M| \) and \( |H \cap V| = |H_1| = |L| = |M \cap V| \). It follows that \( |MV| = |HV| \), so \( MV = U = HV \). The latter equality yields \( MG_1 = MVG_1 = HVG_1 = HG_1 \) because \( V \leq G_1 \).

The equalities \( |H| = |M| \) and \( |HG_1| = |MG_1| \) imply that \( |L| = |H_1| = |M_1| \), where \( M_1 = M \cap G_1 \). Furthermore, \( L = M \cap V \leq M \cap G_1 = M_1 \), so \( L = M_1 \).

Since \( M_1 = L \equiv H_1 \pmod{1} \) and \( MG_1 = HG_1 \), we have \( M \equiv H \pmod{1} \). Finally, \( M \in \Delta \) and \( L = M = M_1 \in \Gamma \), which proves (c).

We conclude that \( V \leq W \) by claim (c) and \( V \not\leq W \) by claim (b). Put \( W_1 = W \cap G_1 \) and consider the normal series
\[ W \triangleright W_1 \triangleright V \triangleright 1. \]
Since \( HG_1 = K \cap G_1 \) holds for all \( K \in \Delta \), we have \( WG_1 = HG_1 \). Hence
\[ W/W_1 \cong WG_1/G_1 = HG_1/G_1 \cong H/H_1 \]
is an \( \mathcal{X} \)-group. Further, the Frattini argument yields \( W_1 = VN_{W_1}(H_1) \), so \( W_1/V \cong N_{W_1}(H_1)/N_V(H_1) \). Since \( N_{W_1}(H_1) \leq N_{G_1}(H_1) \) and \( H_1 \leq N_V(H_1) \), the factor group \( W_1/V \) is isomorphic to a section of the group \( N_{G_1}(H_1)/H_1 \), which, in turn, is an \( \mathcal{X}' \)-group due to Lemma 2.6. Claim (a) implies that \( V \) is a normal \( \mathcal{X} \)-separable subgroup of \( W \), \( l_{\mathcal{X}}(V) \leq n - 1 \), and \( H_1 \in \text{Hall}_{\mathcal{X}}(V) \).

Thus, \( W \) is \( \mathcal{X} \)-separable and \( l_{\mathcal{X}}(W) \leq n \). Moreover, \( H_1 \in \text{Hall}_{\mathcal{X}}(W_1) \) because \( W_1/V \) is an \( \mathcal{X}' \)-group. Finally, \( W/W_1 \cong H/H_1 \) implies \( |W : H| = |W_1 : H_1| \) and \( H \in \text{Hall}_{\mathcal{X}}(W) \), as required.

\textbf{Remark.} The idea to consider the subgroup \( W \) defined in Theorem 3 came to the authors while studying Wielandt’s diaries [7].

\textit{Proof of Corollary 4.} Let \( \Delta = \{ K \leq G \mid K \equiv H \pmod{1} \} \) and \( W = \langle \Delta \rangle \).

From claims (ii), (iii), and (v) of Lemma 2.7, it follows that \( W \) is the normal subgroup of \( N_G^{\langle \ast \rangle}(H) \). By Theorem 3, the subgroup \( W \) is \( \mathcal{X} \)-separable, \( l_{\mathcal{X}}(W) \leq n \), and \( H \in \text{Hall}_{\mathcal{X}}(W) \). Thus, it suffices to show that \( N_G^{\langle \ast \rangle}(H)/W \) is an \( \mathcal{X}' \)-group. This follows from the Frattini argument and Lemma 2.6.

\textbf{Remark.} The idea to consider the subgroup \( W \) defined in Theorem 3 came to the authors while studying Wielandt’s diaries [7].

\textit{Proof of Corollary 4.} Let \( \Delta = \{ K \leq G \mid K \equiv H \pmod{1} \} \) and \( W = \langle \Delta \rangle \).

From claims (ii), (iii), and (v) of Lemma 2.7, it follows that \( W \) is the normal subgroup of \( N_G^{\langle \ast \rangle}(H) \). By Theorem 3, the subgroup \( W \) is \( \mathcal{X} \)-separable, \( l_{\mathcal{X}}(W) \leq n \), and \( H \in \text{Hall}_{\mathcal{X}}(W) \). Thus, it suffices to show that \( N_G^{\langle \ast \rangle}(H)/W \) is an \( \mathcal{X}' \)-group. This follows from the Frattini argument and Lemma 2.6.

Note that Theorem 2 does not follow from Corollary 4 since in the setting of Theorem 2 the series \( \langle \ast \rangle \) is subnormal (and need not be normal as in the hypothesis of the corollary). However, using induction and Theorem 3, we are able to prove the following proposition and then Theorem 2.

\textbf{Proposition 2.8.} Let \( G \) be a finite group with a subnormal series \( \langle \ast \rangle \) and let \( K \) be an \( \mathcal{X} \)-subgroup of \( G \). Let \( i \in \{ 0, 1, \ldots n \} \) and suppose that \( H_i \in \text{sm}_{\mathcal{X}}(G_i) \) and \( K \leq N_G^{\langle \ast \rangle}(H_i) \). Then there exists an element \( x \in G_i \cap \langle H_i, K \rangle \) such that \( K \leq N_G^{\langle \ast \rangle}(H_i^x) \) and \( H_i \equiv H_i^x \pmod{1} \).

\textbf{Proof.} We argue by induction on \( m = n - i \). If \( m = 0 \), then \( i = n \) and \( H_i = G_i = 1 \), so \( x = 1 \) fits.
Let $m > 0$. Then, by Lemma 2.5,

$$H_{i+1} = H_i \cap G_{i+1} \in \text{sm}_\mathcal{X}(G_{i+1})$$

and, by Lemma 2.7(iv),

$$K \leq N^\ast_G(H_i) \leq N^\ast_G(H_{i+1}).$$

By the inductive hypothesis, there exists an element $y \in G_{i+1} \cap \langle H_{i+1}, K \rangle$ such that $K \leq N_G(H^y_{i+1})$ and $H^y_{i+1} \equiv H_{i+1}$ (mod $\ast (i + 1)$).

Since $y \in G_{i+1}$, it follows that $H^y_i G_{i+1} = H_i G_{i+1}$, so

$$H^y_i \equiv H_i \pmod {\ast i}.$$  

By Lemma 2.7(ii), $K \leq N^\ast_G(H_i) = N^\ast_G(H^y_i)$.

Then $K$ normalizes $H^y_i$ modulo the normal series

$$G_i \triangleright G_{i+1} \triangleright 1$$

of the group $G_i$. Moreover, setting

$$\Delta = \{L \leq G_i \mid L \equiv H^y_i \pmod {\ast i} \},$$

we see that $K$ normalizes the subgroup $W = \langle \Delta \rangle \leq G_i$ due to Lemma 2.7(iii).

Since the series $(\dagger)$ is normal, Theorem 3 implies that the subgroup $W$ is $\mathcal{X}$-separable and $H^y_i \in \text{Hall}_\mathcal{X}(W)$. The subgroup $KW$ is also $\mathcal{X}$-separable because the $\mathcal{X}$-subgroup $K$ normalizes $W$. By Lemma 2.1, the subgroup $J = \langle H^y_i, K \rangle \leq KW$ is $\mathcal{X}$-separable too.

Take $T \in \text{m}_\mathcal{X}(J)$ such that $K \leq T$. By Lemma 2.2(ii), $T \in \text{Hall}_\mathcal{X}(J)$. Consider the subgroup $U = T \cap W$. Since $J \cap W$ is a normal subgroup of $J$, it follows that $U \in \text{Hall}_\mathcal{X}(J \cap W)$ in view of Lemma 2.4. On the other hand, $H^y_i \in \text{Hall}_\mathcal{X}(J \cap W)$ because $H^y_i \in \text{Hall}_\mathcal{X}(W)$. By Lemma 2.2(i), there is an element $z \in J \cap W$ such that $U = H^y_i \cap z$. Put $x = yz$. Since $K \leq T$, we have $K \leq N_G(U) = N_G(H^y_i)$.

Now $y \in G_{i+1} \cap \langle H_{i+1}, K \rangle \leq G_i \cap \langle H_i, K \rangle$, in particular, $J = \langle H^y_i, K \rangle \leq \langle H_i, K \rangle$. Furthermore, $z \in W \leq G_i$ and $z \in J \leq \langle H_i, K \rangle$. Thus, $x = yz \in G_i \cap \langle H_i, K \rangle$, as required.

It remains to show that $U = H^x_i \equiv H_i \pmod {\ast i}$. As $H^y_i \equiv H_i \pmod {\ast i}$, it suffices to show that $U \equiv H^y_i \pmod {\ast i}$. Since $H^y_i$ is $\mathcal{X}$-Hall in both $J \cap W$ and $W$, it follows that $\text{Hall}_\mathcal{X}(J \cap W) \subseteq \text{Hall}_\mathcal{X}(W)$ and $U \in \text{Hall}_\mathcal{X}(W)$. Therefore, $H^y_i G_{i+1} = U G_{i+1} = UG_i$, i.e., $(H^y_i)^i = U^i$.

Both $H^y_i$ and $K$ normalize $H^y_{i+1}$, whence $H^y_{i+1}$ is a normal $\mathcal{X}$-subgroup of $J$. Hence $H^y_{i+1}$ lies in a $\mathcal{X}$-subgroup $T$ of $J$. Since $H^y_{i+1} \leq H^y_i \leq W$, it follows that $H^y_{i+1} \leq T \cap W = U$ and $H^y_{i+1} \cap G_{i+1} = H^y_{i+1} \cap U \cap G_{i+1}$. Now the equalities $H^y_i G_{i+1} = U G_{i+1}$ and $|H^y_i| = |U|$ imply that $H^y_i \cap G_{i+1} = U \cap G_{i+1}$. The latter equality and the fact that $(H^y_i)^i = U^i$ together imply that $U \equiv H^y_i \pmod {\ast i}$, and we are done.

**Proof of Theorem 2.** Applying Proposition 2.8 in the case $i = 0$, we conclude that there exists $x \in \langle H, K \rangle$ such that $K \leq N_G(H^x)$. Since $H^x \in \text{sm}_\mathcal{X}(G)$, Lemma 2.6 implies that $K H^x = H^x$. It follows that $K \leq H^x$ and $K$ is conjugate to the subgroup $K^{x^{-1}}$ of $H$.  

\[ \square \]
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References

[1] Chunikhin, S.A.: Subgroups of Finite Groups. Wolters-Noordhoff Publishing, Groningen (1969)
[2] Guo, W., Revin, D.O.: Pronormality and submaximal $\mathcal{X}$-subgroups in finite groups. Comm. Math. Stat. 6(3), 289–317 (2018)
[3] Hall, P., Higman, G.: On the $p$-length of $p$-solvable group and reduction theorem for Burnside’s problem. Proc. Lond. Math. Soc. 3(6), 1–42 (1956)
[4] Revin, D.O., Skresanov, S.V., Vasil’ev, A.V.: The Wielandt–Hartley theorem for submaximal $\mathcal{X}$-subgroups. Monatsh. Math. 193(1), 143–155 (2020)
[5] Wielandt, H.: Zusammengesetzte Gruppen endlicher Ordnung. Lecture notes, Math. Inst. Univ. Tübingen (1963/1964). In: Wielandt, H. (ed.) Mathematische Werke. Mathematical Works, vol. 1 (Group Theory), pp. 607–655. De Gruyter, Berlin (1994)
[6] Wielandt, H.: Zusammengesetzte Gruppen: Höliders Programm heute. In: The Santa Cruz Conference on Finite Groups, Santa Cruz (1979). Proceedings of Symposia in Pure Mathematics, vol. 37, pp. 161–173. American Mathematical Society, Providence (1980)
[7] Wielandt, H.: Die mathematischen Tagebücher. https://www3.math.tu-berlin.de/numerik/Wielandt/

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