Quantum mechanics in de Sitter space

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Abstract

We consider some possible phenomenological implications of the extended uncertainty principle, which is believed to hold for quantum mechanics in de Sitter spacetime. The relative size of the corrections to the standard results is however of the order of the ratio between the length scale of the quantum mechanical system and the de Sitter radius, and therefore exceedingly small. Nevertheless, the existence of effects due to the large scale curvature of spacetime in atomic experiments has a theoretical relevance.

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1. Introduction

The possibility that quantum gravity induces the deformation of the commutation relations of quantum mechanics has been widely discussed in recent years [1]. The corrections should be proportional to the square of the Planck length \( l_P \sim 10^{-35} \text{ m} \). For example, one may assume [2]

\[
\Delta x_i \Delta p_j \geq \frac{\delta_{ij}}{2} \left( 1 + l_P^2 \Delta p_i^2 \right) .
\]

A relation of this kind has been called generalized uncertainty principle (GUP).

The implications of this hypothesis on quantum mechanical systems have been considered in several papers [3-5]. Corrections of relative size \( l_P^2 / a^2 \sim 10^{-48} \), where \( a = 1 / me^2 \) is the Bohr radius, are for example expected to arise in the spectrum of the hydrogen atom [4,5]. Although these effects are too small to be experimentally detectable, observations can fix limits on the value of the deviations from the Heisenberg formula.

On the other hand, similar effects can also derive from more classical settings. For example, it has been argued that in a de Sitter background the Heisenberg uncertainty principle should be modified by introducing corrections proportional to the cosmological constant \( \Lambda = 3 \lambda^2 \) [6],

\[
\Delta x_i \Delta p_j \geq \frac{\delta_{ij}}{2} \left( 1 - \lambda^2 \Delta x_i^2 \right) .
\]

This modification of the Heisenberg relation was named extended uncertainty principle (EUP). It has been motivated either by analogy with the GUP, or by gedanken experiments in which the expansion of the universe during a measurement is taken into account [7]. More recently, it has been shown that it can also be derived from the definition of quantum mechanics on a de Sitter background, with a suitably chosen parametrization [8].

In this letter, we discuss the implications of the EUP on quantum mechanical systems, in analogy with the investigation made in refs. [3-5] for the GUP. In particular, we define through a nonlinear transformation new variables, that obey canonical commutation relations, and calculate perturbatively their effect on the spectrum of the harmonic oscillator and of the hydrogen atom.

2. Perturbations of the spectrum

The spatial part of the deformed Heisenberg algebra leading to the extended uncertainty principle, studied in [8], is given by

\[
\begin{align*}
[x_i, x_j] &= 0, \\
[p_i, p_j] &= -\lambda^2 J_{ij}, \\
[x_i, p_j] &= \delta_{ij} - \lambda^2 x_i x_j,
\end{align*}
\]

\footnote{We use units such that \( \hbar = c = 1 \).}
where $J_{ij} = x_ip_j - x_jp_i$. The uncertainty principle (2) follows from (3) if $< x_i > = 0$. For spherical symmetric systems this is true for all states, provided the origin of the coordinates is put in the center of symmetry, which is always possible because of the homogeneity of de Sitter spacetime.

A representation of the commutation relations (3) can be obtained from operators $\bar{x}_i$ and $\bar{p}_i$ satisfying canonical commutation relations, through the nonlinear transformations [9]

$$x_i = \frac{\bar{x}_i}{\sqrt{1 + \lambda^2 r^2}}, \quad p_i = \sqrt{1 + \lambda^2 x^2} \bar{p}_i. \quad (4)$$

In particular, in a position representation $\bar{x}_i$ acts as a multiplication operator, while $\bar{p}_i = -i \frac{\partial}{\partial \bar{x}_i}$. The Schrödinger equation for the variables $x_i, p_i$,

$$\left[ \frac{p^2}{2m} + V(x) \right] \psi = E \psi, \quad (5)$$

can then be obtained by substituting (4) into (5). Since the exact form of the transformations (4) is not easy to handle, we shall consider an expansion at first order in the small parameter $\lambda^2$. In the following, we shall consider central potentials for which $V = V(r)$, with $r = \sqrt{x_i^2}$. We shall therefore expand

$$r = \frac{\bar{r}}{\sqrt{1 + \lambda^2 r^2}} \sim (1 - \frac{\lambda^2}{2} \bar{r}^2) \bar{r}, \quad \bar{p}^2 = (1 + \lambda^2 \bar{r}^2) \bar{p}^2, \quad (6)$$

and the Schrödinger equation becomes at first order in $\lambda^2$,

$$\left[ \frac{\bar{p}^2}{2m} + V(\bar{r}) + \frac{\lambda^2}{2} \left( \bar{r}^2 \frac{\bar{p}^2}{m} - \bar{r}^3 \frac{dV(\bar{r})}{d\bar{r}} \right) \right] \psi \equiv \left[ \bar{H} + \frac{\lambda^2}{2} \Delta \bar{H} \right] \psi = E \psi, \quad (7)$$

where $\bar{H}$ is the original Hamiltonian, but written in terms of the barred operators. We can consider $\Delta \bar{H}$ as a small perturbation to the Hamiltonian $\bar{H}$. Notice that the first term in $\Delta \bar{H}$ may give rise to ordering problems. When necessary we shall adopt the symmetric ordering $\frac{1}{2} (\bar{r}^2 \bar{p}^2 + \bar{p}^2 \bar{r}^2)$.

The spectrum of the Hamiltonian $H$ can be obtained through perturbation theory as

$$E_k = \bar{E}_k + \frac{\lambda^2}{2} \Delta \bar{E}_k, \quad (8)$$

where $k$ denotes the energy levels of the unperturbed Hamiltonian $\bar{H}$, and $\Delta \bar{E}_k$ are the eigenvalues of the matrix

$$< k | \Delta \bar{H} | k' > = \left< k \left| \frac{\bar{r}^2 \bar{p}^2}{m} - \bar{r}^3 \frac{dV(\bar{r})}{d\bar{r}} \right| k' \right>, \quad (9)$$

calculated on degenerate states of the given energy level.
2.1. Harmonic oscillator

The simplest example is given by a three-dimensional harmonic oscillator. Its Hamiltonian is

\[ H = \frac{p^2}{2m} + \frac{k}{2} r^2. \] (10)

Its energy eigenvalues are parametrized by the quantum numbers \( n \) and \( l \), while the quantum number \( m \) does not enter the calculations:

\[ \bar{E}_{n,l} = \sqrt{\frac{k}{m}} \left( 2n + l + \frac{3}{2} \right). \] (11)

The perturbation is at first order,

\[ \Delta H = \frac{\bar{r}^2 \bar{p}^2}{2m} - \frac{k}{2} \bar{r}^4, \] (12)

and the calculation of the energy shift is simplified by the use of the identity [4]

\[ \frac{\bar{p}^2}{2m} = \bar{H} - \frac{k}{2} \bar{r}^2, \] (13)

which gives

\[ <n, l | \Delta H | n, l> = 2\bar{E}_{n,l} <n, l | \bar{r}^2 | n, l> - 2k <n, l | \bar{r}^4 | n, l>. \] (14)

The matrices \( <n, l | \bar{r}^2 | n, l> \) and \( <n, l | \bar{r}^4 | n, l> \) are diagonal, and an explicit calculation gives [4]

\[ <n, l | \bar{r}^2 | n, l> = \frac{\bar{E}_{n,l}}{2k}, \]
\[ <n, l | \bar{r}^4 | n, l> = \frac{1}{km} \left( 6n^2 + 9n + 6nl + \frac{6l + 15}{4} \right). \] (15)

The shift in the energy levels due to the extended uncertainty principle is therefore

\[ \Delta E_{n,l} = -\frac{1}{m} \left( 8n^2 + 12n + 8nl + l^2 + 5l + \frac{21}{4} \right). \] (16)

For \( l = 0 \), the relative magnitude of the corrections is then

\[ \frac{\lambda^2}{2} \frac{\Delta E_{n,0}}{\bar{E}_{n,0}} \sim -\left( n + \frac{3}{2} \right) \frac{\lambda^2}{\sqrt{km}}. \] (17)

2.2. Hydrogen atom

The Hamiltonian of the hydrogen atom is

\[ H = \frac{p^2}{2m} + \frac{e^2}{r} \sim \bar{H} + \frac{\lambda^2}{2} \Delta H, \] (18)
with
\[ \Delta H = \frac{\vec{r}^2\bar{p}^2}{m} - e^2\vec{r}. \] (19)

The energy spectrum of the unperturbed Hamiltonian is given by
\[ E_n = -\frac{me^4}{2n^2} \] (20)

In analogy with the harmonic oscillator, one can write
\[ \frac{\bar{p}^2}{2m} = \bar{H} - \frac{e^2}{\vec{r}}, \] (21)

obtaining a diagonal matrix with (again the quantum number \( m \) is not relevant)
\[ \langle n,l | \Delta H | n,l \rangle = 2E_n \langle n,l | \vec{r}^2 | n,l \rangle - 3e^2 \langle n,l | \vec{r} | n,l \rangle. \] (22)

Using [10]
\[ \langle n,l | \vec{r} | n,l \rangle = \frac{1}{2me^2} [3n^2 - l(l + 1)], \]
\[ \langle n,l | \vec{r}^2 | n,l \rangle = \frac{n^2}{2m^2e^4} [5n^2 - 3l(l + 1) + 1], \]

one gets
\[ \Delta E_{n,l} = -\frac{1}{m} \left[ 7n^2 - 3l(l + 1) + \frac{1}{2} \right]. \] (23)

Hence, for \( l = 0 \), the relative strength of the corrections is
\[ \frac{\lambda^2}{2} \frac{\Delta E_{n,0}}{E_n} \sim \frac{7n^4\lambda^2}{2m^2e^4} = \frac{7}{2} n^4a^2\lambda^2, \] (24)

where \( a \) is the Bohr radius. The corrections are therefore of the order of the square of the ratio between the Bohr radius and the de Sitter radius \( 1/\lambda \), i.e. \( 10^{-72} \). This effect is even tinier than the one due to the generalized uncertainty principle, and could be detectable experimentally only if the parameter \( \lambda^2 \) in our formulae were much bigger than the observed value of the cosmological constant.

### 3. Lamb shift

A slightly different calculation can be performed to obtain corrections to the Lamb shift effect in the hydrogen atom, in analogy with the investigation of ref. [5] in the case of the generalized uncertainty principle. The shift in the wave function is at first order [10]
\[ \psi_{nlm} = \bar{\psi}_{nlm} + \sum_{n' \neq n'} \langle n',l',m' | \Delta H | n,l,m \rangle \frac{E_n - E_{n'}}{\bar{E}_n - \bar{E}_{n'}} \bar{\psi}_{n'lm'}, \] (25)
where a bar indicates the unperturbed wave function. Using the standard expression of the wave function and the orthogonality relations of the spherical harmonics, the shift in the ground state wave function yields

$$\Delta \psi_{100} = \psi_{100} - \bar{\psi}_{100} = \frac{\langle 2, 0, 0 | \Delta H | 1, 0, 0 \rangle}{E_1 - E_2} \bar{\psi}_{200}. \quad (26)$$

Substituting the expression (19) for $\Delta H$, the explicit calculation of the matrix element gives

$$\langle 2, 0, 0 | \Delta H | 1, 0, 0 \rangle = \langle 2, 0, 0 | H \bar{r}^2 + \bar{r}^2 H - 3e^2 \bar{r} | 1, 0, 0 \rangle = \frac{608\sqrt{2}}{243} a e^2, \quad (27)$$

where we have used a symmetric ordering for the first term. It follows that

$$\Delta \psi_{100} = \frac{2432\sqrt{2}}{729} \lambda^2 a^2 \bar{\psi}_{200}. \quad (28)$$

On the other hand, the Lamb shift for the ground state of the hydrogen atom is given by

$$\Delta E_1 = -\frac{4e^2 \ln(e^2)}{3m^2} |\psi_{100}(0)|^2. \quad (29)$$

The contribution due to the modification of the commutation relations is therefore

$$\frac{\Delta E_1^{EUP}}{\Delta E_1} = \frac{2|\Delta \psi_{100}(0)|}{\psi_{100}(0)} = \frac{2432}{729} \lambda^2 a^2. \quad (30)$$

Also in this case the effect is of order $\lambda^2 a^2$ and hence not detectable experimentally.

4. Conclusions

The corrections to the spectra of quantum mechanical systems due to the EUP are qualitatively similar to those associated to the GUP, discussed in ref. [3-5]. However, their size is different. While those associated to GUP are of relative size $(l_P/L)^2$, $L$ being the typical length scale of the system, in the case of EUP they are of order $(\lambda L)^2$, as could have been expected from dimensional arguments. For systems like the hydrogen atom they are of order $10^{-48}$ and $10^{-72}$ respectively, and therefore much smaller in the EUP case. Of course they are not detectable experimentally, unless some mechanism which fixes a value for $\lambda$ in the uncertainty principle greater than the one associated with the cosmological constant is available. The same calculations hold in the case of anti-de Sitter spacetime, but the corrections have opposite sign.

The main result of this paper is that it is possible, at least in principle, to detect effects due to the large scale curvature of spacetime in atomic experiments. However, with the accuracy of present day experimental setup, this is still of theoretical interest.
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