THE FINITE RAT-SPLITTING FOR COALGEBRAS

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Abstract. Let $C$ be a coalgebra. We investigate the problem of when the rational part of every finitely generated $C^*$-module $M$ is a direct summand $M$. We show that such a coalgebra must have at most countable dimension, $C$ must be artinian as right $C^*$-module and injective as left $C^*$-module. Also in this case $C^*$ is a left Noetherian ring. Following the classic example of the divided power coalgebra where this property holds, we investigate a more general type of coalgebras, the chain coalgebras, which are coalgebras whose lattice of left (or equivalently, right, two-sided) coideals form a chain. We show that this is a left-right symmetric concept and that these coalgebras have the above stated splitting property. Moreover, we show that this type of coalgebras are the only infinite dimensional colocal coalgebras for which the rational part of every finitely generated left $C^*$-module $M$ splits off in $M$, so this property is also left-right symmetric and characterizes the chain coalgebras among the colocal coalgebras.

Introduction

Let $R$ be a ring and $T$ be a torsion preradical on the category of left $R$-modules $R\mathcal{M}$. Then $R$ is said to have splitting property provided that $T(M)$, the torsion submodule of $M$, is a direct summand of $M$ for any $M \in R\mathcal{M}$. More generally, if $\mathcal{C}$ is a Grothendieck category and $\mathcal{A}$ is a subcategory of $\mathcal{C}$, then $\mathcal{A}$ is called closed if it is closed under subobjects, quotient objects and direct sums. To every such subcategory we can associate a preradical $t$ (also called torsion functor) if for every $M \in \mathcal{C}$ we denote by $t(M)$ the sum of all subobjects of $M$ that belong to $\mathcal{A}$. We say that $\mathcal{C}$ has the splitting property with respect to $\mathcal{A}$ if $t(M)$ is a direct summand of $M$ for all $M \in \mathcal{C}$. In the case of the category of $R$-modules, the splitting property with respect to some closed subcategory is a classical problem which has been considered by many authors. In particular, when $R$ is a commutative ring, the question of when the (classical) torsion part of an $R$ module splits off is a well known problem. J. Rotman has shown in [Rot] that for a commutative domain the torsion submodule splits off in every $R$-module if and only if $R$ is a field. I. Kaplansky proved in [K1], [K2] that for a commutative integral domain $R$ the torsion part of every finitely generated $R$-module $M$ splits in $M$ if and only if $R$ is a Prufer domain. While complete results have been obtained for commutative rings, the problem still remains wide open for the non-commutative case. In this paper we investigate the situation when the ring $R$ arises as the dual algebra of a $K$-coalgebra $C$, $R = C^*$. Then the category of the left $R$-modules naturally contains the category $\mathcal{M}^C$ of all right $C$-comodules as a full subcategory. In fact, $\mathcal{M}^C$ identifies with the subcategory $Rat(C^* \mathcal{M})$ of all rational left $C^*$-modules, which is generally a closed subcategory of $C^* \mathcal{M}$. Then two questions regarding the splitting property with respect to $Rat(C^* \mathcal{M})$ naturally arise: first when is the rational part of every left $C^*$-module $M$ a direct summand of $M$ and when does the rational part of every finitely generated $C^*$-module $M$ split in $M$. The first problem, the splitting of $C^* \mathcal{M}$ with respect with the

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closed subcategory $Rat(C, \mathcal{M})$ has been treated by C. Năstăsescu and B. Torrecillas in [NT] where it is proved that if all $C^*$-modules split with respect to $Rat$ then the coalgebra $C$ must be finite dimensional. The techniques used involve some amount of category theory (localization in categories) and strongly relies on some general results of M.L.Teply from [T1], [T2], [T3].

We consider the more general problem of when $C$ has the splitting property only for finitely generated modules, that is, the problem of when is the rational part $Rat(M)$ of $M$ a direct summand in $M$ for all finitely generated left $C^*$-modules $M$. We call these coalgebras left finite $Rat$-splitting coalgebras (or we say that they have the left finite $Rat$-splitting property). If the coalgebra $C$ is finite dimensional, then every left $C^*$-module is rational so $\mathcal{M}^C$ is equivalent to $C, \mathcal{M}$ and $Rat(M) = M$ for all $C^*$-modules $M$ and in this case $Rat(M)$ trivially splits in any $C^*$-module. Therefore we will deal with infinite dimensional coalgebras, as generally the infinite dimensional coalgebras produce examples essentially different from the ones in algebra theory. We first prove some general properties for left finite $Rat$-splitting coalgebras, namely such a coalgebra $C$ is artinian as right $C^*$-module and injective as left $C^*$-module, it has at most countable dimension and has finite dimensional coradical. Also $C^*$ is a left Noetherian ring. We look at a very simple example of a coalgebra where this property holds, namely, the divided power coalgebra (see [DNR], Example 1.1.4), which has $K[[X]]$ as its dual algebra. This is in some sense the simplest possible example of infinite dimensional coalgebra that has the left (and right) finite $Rat$-splitting property. We introduce and study (left) chain coalgebras to be the coalgebras for which every two left subcomodules $M, N$ satisfy either $M \subseteq N$ or $N \subseteq M$.

We see that this is a left-right symmetric concept and we give a simple characterization of these coalgebras as being exactly those having each factor of the coradical series a simple comodule. Moreover, this gives a complete characterization of these coalgebras in the case when the base field is algebraically closed: the divided power coagebra and its subcoalgebras are the only ones of this type. We show that chain coalgebras have the (left and right) finite $Rat$-splitting property. Then we investigate the colocal finite $Rat$-splitting coalgebras. In the main result of the paper we show that a colocal coalgebra satisfying the left finite $Rat$-splitting property must be a chain coalgebra, and therefore it also has the right finite $Rat$-splitting property. This provides a characterization of the divided power coalgebra over an algebraically closed field (or more generally of chain coalgebras) among local coalgebras, namely they are exactly those coalgebras $C$ for which the rational part of every finitely generated left (or right) $C^*$-module splits off.

1. Splitting Problem

Let $C$ be a coalgebra with counit $\varepsilon$ and comultiplication $\Delta$. We use the Sweedler convention $\Delta(c) = c_1 \otimes c_2$ where we omit the summation symbol. For a vector space $V$ and a subspace $W$ of $V$ denote by $V^\perp = \{f \in V^* \mid f(x) = 0, \forall x \in W\}$ and for a subspace $X \subseteq V^*$ denote by $X^\perp = \{x \in V \mid f(x) = 0, \forall f \in X\}$. If $M$ is a right (or left) $R$-module denote by $\mathcal{L}_R(M)$ (or $\mathcal{L}_L(M)$) the lattice of the submodules of $M$. Also, if $S$ is another ring and $Q$ is a fixed $R-S$-bimodule, for any left $R$-module $M$ we have applications

$$\mathcal{L}_R(M) \ni N \to N^\perp = \{f \in \text{Hom}_R(M, Q) \mid f(x) = 0, \forall x \in N\} \in \mathcal{L}_S(\text{Hom}(M, Q))$$

$$\mathcal{L}_S(\text{Hom}(M, Q)) \ni X \to X^\perp = \{x \in M \mid f(x) = 0, \forall f \in X\} \in \mathcal{L}_R(M)$$

forming a Galois pair (see [AN]).

**Lemma 1.1.** Let $C$ be a coalgebra. Then for any finitely generated right (or left, or two-sided) submodule $X$ of $C^*$, $(X^\perp)^\perp = X$. 
Proof. Put $R = \text{End}(C^C, C^C)$. Then $R$ is a ring with multiplication "." equal to opposite composition of morphisms. Let $M = C^*C$ and $Q = C^*C_R$ where the right $R$-module structure on $C$ is $c \cdot f = f(c)$. It is not difficult to see that the isomorphism of rings $C^* \simeq \text{End}(C^C, C^C) = R$, $c^* \mapsto (c \mapsto c'(c_1)c_2)$, transposes the problem to the Galois correspondence $X \rightarrow X^\perp$ between the left $R$ module $C$ and the right $R$ module $\text{End}^C(C, C) = \text{Hom}(C^*C, C^*C)$). That is, it is enough to prove the statement for finitely generated right ideals of $R$. Suppose $X \subseteq R$ is a right ideal generated by $f_1, \ldots, f_n$ as right $R$ module and let $f \in R$ such that $f|_{X^\perp} = 0$. Then we have $X = \sum f_i \cdot R$ so $X^\perp = \bigcap_{i=1}^n (f_i \cdot R)^\perp = \bigcap_{i=1}^n \ker f_i$. Then $f$ induces a morphism $\overline{f} : \bigcap_{i=1}^n C_{f_i} \rightarrow C$ and as $C^C$ is injective, the canonic monomorphism $0 \rightarrow \bigcap_{i=1}^n C_{f_i} \rightarrow \bigoplus_{i=1}^n C_{\ker f_i}$ gives rise to the exact sequence

$$\bigoplus_{i=1}^n \text{Hom}^C\left(\frac{C}{\ker f_i}, C\right) \simeq \text{Hom}^C\left(\bigoplus_{i=1}^n \frac{C}{\ker f_i}, C\right) \varphi \rightarrow \text{Hom}^C\left(\frac{C}{\bigcap_{i=1}^n \ker f_i}, C\right) \rightarrow 0$$

Let $(g_i)_{i=1}^n \in \bigoplus_{i=1}^n \text{Hom}^C\left(\frac{C}{\ker f_i}, C\right)$ be such that $\varphi(\sum_{i=1}^n g_i) = \overline{f}$. As for any $i$ we have a monomorphism $\overline{f_i} : \frac{C}{\ker f_i} \rightarrow C$ induced by $f_i$ and as the right $C$-comodule $C$ is injective, any the diagram

$$\begin{array}{ccc}
0 & \rightarrow & C \\
\downarrow & & \downarrow \varphi \\
\ker f_i & \rightarrow & C \\
\overline{f_i} & \rightarrow & C \\
\end{array}$$

can be completed commutatively by a morphism of right $C$-comodules $h_i$. Then we have $\varphi(\sum_{i=1}^n \overline{f_i} \cdot h_i) = \varphi(\sum_{i=1}^n h_i \circ \overline{f_i}) = \overline{f}$ and composing this with the cannonical projection $p : C \rightarrow \bigcap_{i=1}^n C_{\ker f_i}$ it is not difficult to see that we get $f = \sum_{i=1}^n f_i \cdot h_i$ so $f \in X$. \hfill \Box

**Proposition 1.2.** Let $C$ be a coalgebra such that $\text{Rat}(M)$ splits off in any finitely generated left $C^*$-module $M$. Then any indecomposable injective left $C$-comodule $E$ contains only finite dimensional proper subcomodules.

**Proof.** Let $T$ be the socle of $E$; then $T$ is simple and $E = E(T)$ is the injective envelope of $T$. We show that if $K \subseteq E(T)$ is an infinite dimensional subcomodule then $K = E(T)$. Suppose $K \subsetneq E(T)$. Then there is a left $C$-subcomodule (right $C^*$-submodule) $K \subset L \subset E(T)$ such that $L/K$ is finite dimensional. We have an exact sequence of left $C^*$-modules:

$$0 \rightarrow (L/K)^* \rightarrow L^* \rightarrow K^* \rightarrow 0$$

As $L/K$ is a finite dimensional left $C$-comodule, we have that $(L/K)^*$ is a rational left $C^*$-module; thus $\text{Rat}(L^*) \neq 0$. Also $L^*$ is finitely generated as it is a quotient of $E(T)^*$ which is a direct summand of $C^*$. We have $L^* = \text{Rat}(L^*) \oplus X$ for some left $C^*$-submodule $X$ of $L^*$. Then $\text{Rat}(L^*)$ is finitely generated because $L^*$ is, so it is finite dimensional.
As \( L \) is infinite dimensional by our assumption, we have \( X \neq 0 \). This shows that \( L^* \) is decomposable and finitely generated, thus it has at least two maximal submodules, say \( M, N \). We have an epimorphism \( E(T)^* \xrightarrow{f} L^* \to 0 \) and then \( f^{-1}(M) \) and \( f^{-1}(N) \) are distinct maximal \( C^* \)-submodules of \( E(T)^* \). But by \([I]\), Lemma 1.4, \( E(T)^* \) has only one maximal \( C^* \)-submodule which is \( T^\perp \), so we have obtained a contradiction. □

Let \( C_0 \) be the coradical of \( C \), the sum of all simple subcomodules of \( C \). By \([DNR]\), Section 3.1, \( C_0 \) semisimple coalgebra that is a direct sum of simple subcoalgebras \( C_0 = \bigoplus C_i \) and each simple subcoalgebra \( C_i \) contains only one type of simple left (or right) \( C \)-comodule; moreover, any simple left (or right) \( C \)-comodule is isomorphic to one contained in a \( C_i \).

**Proposition 1.3.** Let \( C \) be a coalgebra such that the rational part of every finitely generated left \( C^* \) module splits off. Then there is only a finite number of isomorphism types of simple left \( C \)-comodules, equivalently, \( C_0 \) is finite dimensional.

**Proof.** By the above considerations, if \( S_i \) is a simple left \( C \)-subcomodule of \( C_i \), we have that \((S_i)_{i \in I}\) forms a set of representatives for the isomorphism types of simple left \( C \)-comodules. Let \( S \) be a set of representatives for the simple right \( C \)-comodules. Let \( E(C_i) \) be an injective envelope of the left \( C \)-comodule \( C_i \) included in \( C \); then as \( C_0 \) is essential in \( C \) we have \( C = \bigoplus E(C_i) \) as left \( C \)-comodules or right \( C^* \)-modules. Then \( C^* = \prod_{i \in I} E(C_i)^* \) as left \( C^* \)-modules. As \( S_i \subseteq E(C_i) \), we have epimorphisms of left \( C^* \)-modules \( E(C_i)^* \to S_i^* \to 0 \) and therefore we have an epimorphism of left \( C^* \)-modules \( C^* \to \prod_{i \in I} S_i^* \to 0 \). But there is a one-to-one correspondence between left and right simple \( C \)-comodules given by \( \{S_i \mid i \in I\} \ni S \mapsto S^* \in S \). Hence there is an epimorphism \( C^* \to \prod_{S \in S} S^* \to 0 \), which shows that the left \( C^* \)-module \( P = \prod_{S \in S} S \) is finitely generated (actually generated by a single element). But then as \( \text{Rat}(C^* \text{-} P) \) is a direct summand in \( P \), we must have that \( \text{Rat}(C^* \text{-} P) \) is finitely generated, so it is finite dimensional. Therefore, as \( \Sigma = \bigoplus_{S \in S} S \) is a rational left \( C^* \)-module which is naturally included in \( P \), we have \( \Sigma \subseteq \text{Rat}(P) \). This shows that \( \bigoplus_{S \in S} S \) is finite dimensional so \( I \) must be finite. This is equivalent to the fact that \( C_0 \) is finite dimensional, because each \( C_i \) is a simple coalgebra, thus a finite dimensional one. □

We shall say that a coalgebra is left (right) finite Rat-splitting if the rational part of any finitely generated left (right) \( C^* \)-module splits off.

**Proposition 1.4.** Let \( C \) be a left finite Rat-splitting coalgebra. Then the following assertions hold:

(i) \( C \) is artinian as left \( C \)-comodule (equivalently, as right \( C^* \)-module).

(ii) \( C^* \) is left Noetherian.

(iii) \( C \) has at most countable dimension.

(iv) \( C \) is injective as left \( C^* \)-module.

**Proof.** (i) We have a direct sum decomposition \( C = \bigoplus_{i \in F} E(S_i) \) where \( C_0 = \bigoplus_{i \in F} S_i \) is the decomposition of \( C_0 \) into simple left \( C \)-comodules and \( E(S_i) \) are injective envelopes of \( S_i \) contained in \( C \). Then \( F \) is finite as \( C_0 \) is finite dimensional. Also, by Proposition \([22]\) the \( E(S_i)'s \) are artinian as they contain only finite dimensional proper subcomodules, thus \( C \) is an artinian left \( C \)-comodule.
(ii) Take $I$ a left ideal of $C^*$ and suppose it is not finitely generated; then we can find a sequence $(x_k)_k$ of elements of $I$ such that denoting $I_k = C^* < x_1, x_2, \ldots, x_k >$, $x_{k+1} \notin I_k$.

Then, corresponding to the ascending chain of left $C^*$ submodules of $C^*$, $I_1 \subset I_2 \subset \cdots \subset I_k \subset \cdots$ we have a descending chain of right $C^*$ submodules of $C$, $I_1^+ \supseteq I_2^+ \supseteq \cdots \supseteq I_k^+ \supseteq \cdots$, which must be stationary as $C_{C^*}$ is artinian, so $I_{k+1}^+ = I_{k+1} = \ldots$.

Then $(I_{k+1}^+)^\perp = (I_k^+)^\perp = \cdots$ and then by Lemma 1.1 we get that $I_k = I_{k+1}$, and then $x_{k+1} \in I_{k+1} = I_k$ which is a contradiction.

(iii) For any $i \in F$, if $E(S_i)$ is infinite dimensional, we may inductively build a sequence $(x_k)_k$ of elements of $E(S_i)$ such that $x_{k+1} \notin \sum_{j=1}^k x_j \cdot C^*$ for any $k$ because $\sum_{j=1}^k x_j \cdot C^*$ is always a finite dimensional comodule. Then $E(S_i) = \sum_k x_k \cdot C^*$, because $\sum_k x_k \cdot C^*$ is an infinite dimensional subcomodule of $E(S_i)$ and one can apply Lemma 1.2. Now, as each $x_k \cdot C^*$ is finite dimensional, the conclusion follows.

(iv) As $C$ is a finite coproduct of $E(S_i)$’s it is enough to prove that each $E(S_i)$ is injective and by [10] Lemma 2, it is enough to prove that $E = E(S_1)$ splits off in any left $C^*$ module $M$ such as $M/E$ is 1-generated, that is, it is generated by an element $\hat{x} \in M/E$. Let $H = \text{Rat}(C^* \cdot \hat{x})$; then there is $X < C^* \cdot \hat{x}$ such that $H \oplus X = C^* \cdot \hat{x}$. Then $E + H$ is a rational $C^*$ module so $(E + H) \cap X = 0$; also $M = C^* \cdot \hat{x} + E$ so $(E + H) + X = M$, showing that $E + H$ is a direct summand in $M$. But as $E$ is an injective comodule, we have that $E$ splits off in $E + H$, thus $E$ must split in $M$ and the proof is finished. \quad \square

2. Chain Coalgebras

Let $C$ be a coalgebra and denote by $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ the coradical filtration of $C$, that is, $C_0$ is the coradical of $C$, and $C_{n+1} \subseteq C$ such that $C_{n+1}/C_n$ is the socle of the right (or left) $C$-comodule $C/C_n$ for all $n \in \mathbb{N}$. Then $C_n$ is a subcoalgebra of $C$ for all $n$, and the same $C_n$ is obtained whether we take the socle of the left $C$-comodule $C/C_n$ or of the right $C$-comodule $C/C_n$. Put $C_{-1} = 0$ and $R = C^*$. By [10] we have $\bigcup_{n \in \mathbb{N}} C_n = C$.

Definition 2.1. We say that a coalgebra $C$ is a left (right) chain coalgebra if and only if the lattice of the left (right) subcomodules of $C$ is a chain, that is, any two subcomodules $M, N$ of $C$ are comparable (given two subsets $A, B$ of a set $X$ we say that $A$ and $B$ are comparable if either $A \subseteq B$ or $B \subseteq A$).

The following result shows that this definition is left-right symmetric and also characterizes all chain coalgebras.

Proposition 2.2. The following assertions are equivalent for a coalgebra $C$:
(i) $C$ is a right chain coalgebra.
(ii) $C_{n+1}/C_n$ is either 0 or a simple (right) comodule for all $n \geq -1$.
(iii) $C$ is a left chain coalgebra. In this case $C$ and $C_n$, $n \geq -1$ are the only subcomodules (left, right, two-sided) of $C$ and $C_n$ is finite dimensional for all $n$.

Proof. (ii)$\Rightarrow$(i) We prove that any subcomodule of $C$ must be equal either to one of the $C_n$’s or to $C$. Let $M$ be a right subcomodule of $C$ and suppose $M \neq C$ and $M \neq 0$. Then there is $n \geq 0$ such that $C_n \nsubseteq M$ and let $n$ the minimal natural number with this property. Then we must have $C_{n-1} \subseteq M$ by the minimality of $M$ and we show that $C_{n-1} = M$. Indeed, if $C_{n-1} \nsubseteq M$ we can find a simple subcomodule of $M/C_{n-1}$. But then $C_{n-1} \neq C$ so $C_{n-1} \neq C_n$ and as $C_n/C_{n-1}$ is the only simple subcomodule of $C/C_n$ we find $C_n/C_{n-1} \subseteq M/C_{n-1}$, that is $C_n \subseteq M$, a contradiction. This also proves the last
statement of the proposition.

(i) ⇒ (ii) If \( C_{n+1}/C_n \) is nonzero and it is not simple then we can find \( S_1 \) and \( S_2 \) two distinct simple modules contained in \( C/C_n \). Then \( S_1 = M_1/C_n, S_2 = M_2/C_n \) and \( M_1 \cap M_2 = C_n, M_1 \neq C_n, M_2 \neq C_n \) because \( S_1 \cap S_2 = 0 \) and \( S_1 \) and \( S_2 \) are distinct simple subcomodules of \( C/C_n \). But this shows that neither \( M_1 \subseteq M_2 \) nor \( M_2 \subseteq M_1 \) which is a contradiction.

(ii) ⇔ (iii) is proved similarly.

Denote \( J = C_0^\perp \); by [DNR] Lemma 2.5.7 and Corollary 3.1.10 we have that \( J = J(C^*) \) (the Jacobson radical of \( C^* \)) and \( (J^n)^\perp = C_{n-1} \), so \( J^n \subseteq ((J^n)^\perp)^\perp = C_{n-1}^\perp \). As \( \bigcup_{n \in \mathbb{N}} C_n = C \), we see that \( \bigcap_{n \in \mathbb{N}} J^n = 0 \).

**Definition 2.3.** We say that a coalgebra \( C \) is left almost finite if the left regular comodule \( C \) has only finite dimensional proper subcomodules.

By Proposition 2.2 a chain coalgebra is left and right almost finite.

**Proposition 2.4.** Let \( C \) be an left almost finite coalgebra. Then \( C^* \) is left Noetherian; moreover all nonzero left ideals of \( C^* \) have finite codimension.

**Proof.** Then if \( I \) is a nonzero left ideal of \( C^* \) take \( f \neq 0, f \in I \). Then \( F := (C^* \cdot f)^\perp \) is a left coideal of \( C \) so \( F \) is finite dimensional. Then \( F^\perp \) has finite codimension; but \( C^* \cdot f = F^\perp \) by Proposition 1.1. This shows that \( I \) has finite codimension. Consequently, \( C^* \) is left Noetherian.

**Proposition 2.5.** If \( C \) is a chain coalgebra, then \( J^n = C_{n-1}^\perp \) for all \( n \) and \( 0 \) and \( J^n, n \geq 0 \) are the only ideals (left, right, two-sided) of \( C^* \). Consequently, \( C^* \) is a chain algebra.

**Proof.** If \( I \) is a left ideal of \( C^* \), then by Lemma 1.1 and Proposition 2.4 we have \( (I^\perp)^\perp = I \). By Proposition 2.2 \( I^\perp = C \) or \( I^\perp = C_{n-1} \) for some \( n \geq 1 \) and therefore \( I = (I^\perp)^\perp = C_{n-1}^\perp \) or \( I = 0 \). We prove by induction on \( n \) that \( J^n = C_{n-1}^\perp \), for all \( n \geq 1 \). For \( n = 1 \), \( J = C_0^\perp \) and assume \( J^n = C_{n-1}^\perp \) for some \( n \geq 2 \). We again have \( J^{n+1} = ((J^n)^\perp)^\perp \) and as \( (J^{n+1})^\perp = C_n \), we get \( J^{n+1} = C_n^\perp \) and the proof is finished.

For a left \( C^* \)-module \( M \) denote by \( T(M) \) the set of all torsion elements of \( M \), that is, \( T(M) = \{ x \in M \mid \text{ann}_{C^*} x \neq 0 \} \). If \( C \) is a finite dimensional coalgebra, then obviously any right \( C \)-comodule is rational and the category of right comodules coincides to that of the left \( C^* \)-modules. Then it is interesting to investigate the infinite dimensional case. We first consider a special kind of coalgebra:

**Proposition 2.6.** Let \( C \) be an infinite dimensional left almost finite coalgebra and let \( R = C^* \). Then for any left \( R \) module \( M \) we have \( \text{Rat}(M) = T(M) \); moreover, \( x \in \text{Rat}(M) \) if and only if \( R \cdot x \) is finite dimensional.

**Proof.** If \( x \in T(M) \) then \( R \cdot x \) is finite dimensional and then \( \text{ann}_{R \cdot x} \) must be of finite codimension, thus nonzero as \( R \) is infinite dimensional. Conversely, if \( x \in T(M) \) and \( x \neq 0 \) then \( I = \text{ann}_{R \cdot x} \) is a nonzero left ideal of \( R \) so it must have finite codimension by Proposition 2.3. Then as \( R/I \simeq R \cdot x \) we get that \( R \cdot x \) is finite dimensional. Also \( I^\perp \subseteq C \) is a finite dimensional left submodule of \( C \) and thus the subcoalgebra \( C' \) of \( C \) generated by \( I^\perp \) is finite dimensional. Taking Proposition 1.1 into account, \( I = (I^\perp)^\perp \subseteq C'^\perp \). Then \( C'^\perp \cdot x = 0 \), and \( R \cdot x \) becomes a left \( R/C'^\perp \)-module. Now note that as \( C' \) is a finite dimensional coalgebra with \( C'^* \simeq C^*/C'^\perp \), \( R \cdot x \) has a structure of right \( C' \)-module. So we have a map \( \rho : R \cdot x \to R \cdot x \otimes C' = c_0 \otimes c_1 \) such that \( h \cdot c = h(c_1)c_0 \) for \( h \in C'^* \).
But then if \( \pi : C^* \to C'^* \) is the canonical projection \( \pi(r) = r|_{C'^*} \), for \( r \in R \) and \( c \in R \cdot x \) we have \( r \cdot c = \pi(r) \cdot c = \pi(r)(c_1)c_0 = r(c_1)c_0 \), so we may regard \( \rho \) as a \( C \) comultiplication of \( R \cdot x \), thus \( x \in \text{Rat}(M) \).

We show that a chain coalgebra is a (left and right) finite splitting coalgebra. The proof of this can be done by a standard extension to the noncommutative case of the proof of the theorem on the structure of finitely generated modules over a PID, and obtain as a consequence the fact that \( T(M) \) is a direct summand in \( M \) for every finitely generated module \( M \). However we can do this in a more direct way.

**Theorem 2.7.** If \( C \) is a chain coalgebra, then \( C \) a left and right finite splitting coalgebra.

**Proof.** First notice that every torsion-free \( R \) finitely generated module \( M \) is free: indeed if \( x_1, \ldots, x_n \) is a minimal system of generators, then if \( \lambda_1 x_1 + \cdots + \lambda_n x_n = 0 \) with \( \lambda_i \) not all zero, we may assume that \( \lambda_1 \neq 0 \). Without loss of generality we may also assume that \( \lambda_1 R \supseteq \lambda_i R, \forall i \) as any two ideals of \( R \) are comparable by Proposition 2.5. Therefore we have \( \lambda_1 = \lambda_1 s_i \) for some \( s_i \in R \). Then \( \lambda_1 x_1 + \lambda_1 s_2 x_2 + \cdots + \lambda_1 s_n x_n = 0 \) implies \( x_1 + s_2 x_2 + \cdots + s_n x_n = 0 \) as \( M \) is torsionfree and \( \lambda_1 \neq 0 \). Hence \( x_1 \in R < x_2, \ldots, x_n > \), contradicting the minimality of \( n \).

Now if \( M \) is any left \( R \) module and \( T = T(M) = \text{Rat}(M) \) then \( T(M/T(M)) = 0 \). Indeed take \( \hat{x} \in T(M/T(M)) \) and put \( I = \text{ann}_C \hat{x} \neq 0 \) so \( I \) has finite codimension and \( I \) is a two-sided ideal by Proposition 2.5. By Proposition 2.2 \( I \) is generated by some \( h_1, \ldots, h_k \in I \). Then if \( y \in Ix \) we have \( y = f \cdot x \), \( f \in I \) so \( f = \sum_{i=1}^{n} r_i \cdot h_i \) and \( y = f \cdot x = \sum_{i=1}^{n} r_i \cdot h_i x \). Therefore \( Ix \) is generated by \( h_1 x, \ldots, h_n x \). Because \( I = \text{ann}_R \hat{x} \) we have \( Ix \subseteq T = \text{Rat}(M) \) (we use Proposition 2.6) and as \( Ix \) is finitely generated rational we get that \( Ix \) has finite dimension. We obviously have an epimorphism \( \frac{R}{I} \to \frac{Rx}{Ix} \) which shows that \( Rx/Ix \) is finitely dimensional because \( I \) has finite codimension in \( R \). Therefore we get that \( \dim(Rx) = \dim(Rx/Ix) + \dim(Ix) < \infty \) so then by Proposition 2.6 we have that \( Rx \) is rational, thus \( x \in T \) so \( \hat{x} = 0 \).

Now as \( M/T \) is torsion-free, there are \( x_1, \ldots, x_n \in M \) whose images \( \hat{x}_1, \ldots, \hat{x}_n \) in \( M/T \) form a basis. Then it is easy to see that \( x_1, \ldots, x_n \) are linearly independent in \( M \). Then if \( X = Rx_1 + \cdots + Rx_n \) we have \( X + T = M \) and \( X \cap T = 0 \), because if \( a_1 x_1 + \cdots + a_n x_n \in T \) we get \( a_1 \hat{x}_1 + \cdots + a_n \hat{x}_n = 0 \) so \( a_i = 0, \forall i \) because \( \hat{x}_1, \ldots, \hat{x}_n \) are independent in \( M/T \). Thus \( T(M) \) splits off in \( M \) and the theorem is proved, as \( T(M) = \text{Rat}_R(M) \) by 2.6. \( \square \)

We will denote by \( K_n \) the coalgebra with a basis \( c_0, c_1, \ldots, c_{n-1} \) and comultiplication \( c_k \mapsto \sum_{i+j=k} c_i \otimes c_j \) and counit \( \varepsilon(c_i) = \delta_{0,i} \). The coalgebra \( \bigcup_{n \in \mathbb{N}} K_n \) having a basis \( c_n, n \in \mathbb{N} \) and comultiplication and counit given by these equations is called the divided power coalgebra (see [DNR]).

**Lemma 2.8.** Let \( C \) be a finite dimensional chain coalgebra over an algebraically closed field. Then \( C \) is isomorphic to \( K_n \) for some \( n \in \mathbb{N} \).

**Proof.** Let \( A = C^* \); we have \( \dim C_0 = 1 \) because \( K \) is algebraically closed (thus \( \text{End}_A C_0 \) is a skewfield containing \( K \)). Thus \( \dim C_k = k \) for all \( k \) for which \( C_k \neq 0 \). As \( C^* \) is finite dimensional \( J^n = 0 \) for some \( n \) and let \( n \) be minimal with this property. By Proposition 2.5 \( J^k = C_{k-1}^+ \). Then \( J^k/J^{k+1} \) has dimension equal to the dimension of \( C_k/C_{k-1} \) which is 1 for \( k < n \), because \( C_{k+1}/C_k \) it is a simple comodule isomorphic to \( C_0 \). We then have that \( J^k/J^{k+1} \) is generated by any of its nonzero elements. Choose \( x \in J \setminus J^2 \).
We prove that $x^{n-1} \neq 0$. Suppose the contrary holds and take $y_1, \ldots, y_{n-1} \in J$. As $x$ generates $J/J^2$, there is $\lambda \in K$ such that $y_1 - \lambda x \in J^2$ and then $y_1 x^{n-2} - \lambda x^{n-1} \in J^n$, so $y_1 x^{n-2} \in J^n = 0$ because $x^{n-1} = 0$. Again, there is $\mu \in K$ such that $y_2 - \mu x \in J^2$ and then $y_1 y_2 - \mu y_1 x \in J^3$ so $y_1 y_2 x^{n-3} \in J^n$ ($y_1 x^{n-2} = 0$). By continuing this procedure, one gets that $y_1 y_2 \cdots y_{n-2} x = 0$ and then we again find $\alpha \in K$ with $y_{n-1} - \alpha x \in J^2$, thus $y_1 \cdots y_{n-1} - \alpha y_1 \cdots y_{n-2} x \in J^n = 0$. This shows that $y_1 \cdots y_{n-1} = 0$ for all $1 \cdots y_{n-1} = 0$.

Thus $J^{n-1} = 0$, a contradiction.

As $x^{n-1} \neq 0$ we see that $x^k \in J^k \setminus J^{k+1}$ for all $k = 0, \ldots, n-1$, so $J^k/J^{k+1}$ is generated by the class of $x^k$. Now if $y \in A$, there is $\lambda_0 \in K$ such that $y - \lambda_0 \cdot 1_A \in J$ (either $y \in J$ or $y$ generates $A/J$). As $J/J^2$ is 1 dimensional and generated by the image of $x$, there is $\lambda_1 \in K$ such that $y - \lambda_0 - \lambda_1 x \in J^2$. Again, as $J^2/J^3$ is 1 dimensional generated by the image of $x^2$, there is $\lambda_2 \in K$ such that $y - \lambda_0 - \lambda_1 x - \lambda_2 x^2 \in J^3$. By continuing this procedure we find $\lambda_0, \ldots, \lambda_{n-1} \in K$ such that $y - \lambda_0 - \lambda_1 x - \cdots - \lambda_{n-1} x^{n-1} \in J^n = 0$, so $y = \lambda_0 + \lambda_1 x + \cdots + \lambda_{n-1} x^{n-1}$. This obviously gives an isomorphism between $A$ and $K[X]/(X^n)$. Therefore $C$ is isomorphic to $K_n$, because there is an isomorphism of $K$-algebras $K_n \cong K[X]/(X^n)$.

\[\square\]

**Theorem 2.9.** If $K$ is an algebraically closed field and $C$ is an infinite dimensional chain coalgebra, then $C$ is isomorphic to the divided power coalgebra.

**Proof.** By the previous Lemma we have that $C_n \cong K_n$ for all $n$. If $e \in C_0$, $\Delta(e) = \lambda e \otimes e$, $\lambda \in K$, then for $c_0 = \lambda e$ we get $\Delta(c_0) = c_0 \otimes c_0$. Suppose we constructed a basis $c_0, c_1, \ldots, c_{n-1}$ for $C_{n-1}$ with $\Delta(c_k) = \sum_{i+j=k} c_i \otimes c_j$, $\epsilon(c_i) = \delta_{0,i}$. Denote by $A_n = C_n^*$ the dual of $C_n$; for the rest of this proof, if $V \subseteq C_n$ is a subspace of $C_n$ we write $V^\perp$ for the set of the functions of $A_n$ which are 0 on $V$. Choose $E_1 \in C_0^+ \setminus C_1^+$; then $E_1^n \neq 0$ and $E_1^{n+1} = 0$ as in the proof of Lemma 2.8 ($E_1 \in A_n$). This shows that $E_1 \subseteq C_{k-1}^+ \setminus C_k^+$, that $\epsilon(c_n, E_1, \ldots, E_n)$ exhibits a basis for $A_n$ and that there is an isomorphism of algebras $A_n \cong K[X]/(X^{n+1})$ taking $E_1$ to $X$. We can easily see that $E_1(c_j) = \delta_{ij}$, $\forall k = 0, 1, \ldots, n-1$ and then by a standard linear algebra result we can find $c_n \in C_n$ such that $E_1^n(c_n) = 1$ and $E_1^n(c_i) = 0$ for $i < n$. Then by dualization, the relations $E_1(c_j) = \delta_{ij}$, $\forall i, j = 0, 1, \ldots, n$ become $\Delta(c_k) = \sum_{i+j=k} c_i \otimes c_j$, $\forall k = 0, 1, \ldots, n$. Therefore we may inductively build the basis $(c_n)_{n \in \mathbb{N}}$ with $\epsilon(c_k) = \delta_{0k}$ and $\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j$, $\forall n$.

\[\square\]

In the following we construct an example of a chain coalgebra that is not cocommutative and thus different of the divided power coalgebra over $K$. Recall that if $A$ is a $k$ algebra, $\varphi : A \to A$ is a morphism and $\delta : A \to A$ is a $\varphi$-derivation (that is a linear map such that $\delta(ab) = \delta(a)b + \varphi(a)\delta(b)$ for all $a, b \in A$), we may consider the Ore extension $A[X, \varphi, \delta]$ which is $A[X]$ as a vector space and with multiplication induced by $Xa = \varphi(a)X + \delta(a)$. Let $K$ be a subfield of $\mathbb{R}$, the field of real numbers. Let $D$ be the subalgebra of Hamilton’s quaternion algebra having the set $B = \{1, i, j, k\}$ as a vector space basis over $K$. Recall that multiplication is given by the rules $i \cdot j = -j \cdot i = k; j \cdot k = -i; k \cdot i = -j; i \cdot k = j$; $i^2 = j^2 = k^2 = -1$. Denote by $\sigma : D \to D$ the linear map defined on the basis of $D$ by

$$\sigma = \begin{pmatrix} 1 & i & j & k \\ i & j & k & i \end{pmatrix}$$

It is not difficult to see then that $\sigma$ is an algebra automorphism, and that $D$ is a division algebra (skewfield). Our example will be such an Ore extension constructed with a trivial
derivation: denote by $D_\sigma[X] = D[X, \sigma, 0]$ the Ore extension of $D$ constructed by $\sigma$ with the derivation $\varphi$ equal to 0 everywhere. Then a basis for $D_\sigma[X]$ over $K$ consists of the elements $uX^k$, with $u \in B$ and $k \in \mathbb{N}$. Also denote by $A_n = D_\sigma[X]/ < X^n >$ the algebra obtained by factoring out the two-sided ideal generated by $X^n$ from $D_\sigma[X]$.

**Proposition 2.10.** The two sided ideal $< X^n >$ of $D_\sigma[X]$ consists of elements of the form $f = \sum_{l=0}^{n+m} a_l X^l$. Moreover, the only (left, right, two-sided) ideals containing $< X^n >$ are the ideals $< X^l >$, $l = 0, \ldots, n$ and consequently $A_n$ is a chain $K$ algebra.

**Proof.** It is clear by the multiplication rule $Xa = \sigma(a)X$ for $a \in B$ that elements of $D_\sigma[X]$ are of the type $\sum_{l=0}^{n} a_l X^l$ and that every element of $A_n$ is a "polynomial" of the form $f = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$, with $a_l \in D$ and where $x$ represents the class of $X$. Such an element $f$ is invertible if and only if $a_0 \neq 0$. To see this, first note that if $a_0 = 0$ then $f$ is nilpotent, as $x$ is nilpotent and one has $f^l \in < x^l >$ by successively using the relation $xa = \sigma(a)x$. Conversely write $f = a_0 \cdot (1 + a_0^{-1} a_1 x + \cdots + a_0^{-1} a_{n-1} x^{n-1})$ and note that the element $g = a_0^{-1} a_1 x + \cdots + a_0^{-1} a_{n-1} x^{n-1}$ is nilpotent as before, so $1 + g$ must be invertible in $A_n$ and therefore $f$ must be invertible. Thus we may write every element $f = a_0 x^l + \cdots a_{n-1} x^{n-1}$ of $A_n$ as the product $f = (a_l + a_{l+1} x + \cdots + a_{n-1} x^{n-1}) \cdot x^l = g \cdot x^l$ with invertible $g$. Then if $I$ is a left ideal of $A_n$ and $f \in I$, we have $f = g \cdot x^l$ for an invertible element $g$ and some $l \leq n$. Hence it follows that $x^l \in I$. Taking the smallest number $l$ with the property $x^l \in I$, we obviously have that $I = < x^l >$. □

Let $C_n$ denote the coalgebra dual to $A_n$. Note that $A_n$ has a $K$ basis $B = \{ax^l | a \in B, l \in 0, 1, \ldots, n-1\}$ and we have the relations $(ax^l)(bx^j) = a\sigma^l(b)x^{l+j}$. Let $(E_i^p)_{a \in B, i \in \mathbb{N}, n-1}$ the basis of $C_n$ which is dual to $B$, that is, $E_i^p(bx^j) = \delta_{ij} \delta_{ab}$ for all $a, b \in B$ and $i, j \in \mathbb{N}$. Also, for $i \in \mathbb{N}$ and $a \in B$ denote by $i \cdot a = \sigma^i(a)$, the action of $\mathbb{N}$ on $B$ induced by $\sigma$.

**Proposition 2.11.** With the above notations, denoting by $\Delta_n$ and $\varepsilon_n$ the comultiplication and respectively, the counit of $C_n$ we have

$$\Delta_n(E_p^c) = \sum_{i+j=p; \ a(i-b) = \pm c} c^{-1} a(i \cdot b) E_i^a \otimes E_j^b$$

and

$$\varepsilon_n(E_p^c) = \delta_{p,0} \delta_{c,1}.$$

**Proof.** For $u, v \in B$ and $k, l \in \mathbb{N}$ we have $E_p^c(ux^k \cdot vx^l) = E_p^c(u(k \cdot v)x^{k+l})$ and as $k \cdot v \in B$ by the formulas defining $D$ we have that if $d = u(k \cdot v)$ then either $d \in B$ or $-d \in B$. Then $E_p^c(ux^k \cdot vx^l) = E_p^c(dx^{k+l}) = \delta_{k+l,p} \delta_{u(k-v), \pm c} c^{-1} u(k \cdot v)$ as the sign of this expression must be 1 if $d \in B$ and $-1$ if $d \notin B$, and this is exactly $c^{-1} u(k \cdot v)$ when $u(k \cdot v) = \pm c$. We also have

$$\sum_{i+j=p; \ a(i-b) = \pm c} c^{-1} a(i \cdot b) E_i^a (ux^k) E_j^b (vx^l) = \sum_{i+j=p; \ a(i-b) = \pm c} \delta_{k,i} \delta_{u,a} \delta_{l,j} \delta_{v,b} c^{-1} a(i \cdot b)$$

and therefore we get

$$\sum_{i+j=p; \ a(i-b) = \pm c} c^{-1} a(i \cdot b) E_i^a (ux^k) E_j^b (vx^l) = E_p^c(ux^k \cdot vx^l)$$
As this is true for all $ux^k, vx^t \in B$, by the definition of the comultiplication of the coalgebra dual to an algebra, we get the first equality in the statement of the proposition. The second one is obvious, as $\varepsilon_n(E^n_0) = E^n_p(1 \cdot X^0) = \delta_{p,0}\delta_{i,1}$.

Now notice that there is an injective map $C_n \subset C_{n+1}$ taking $E^n_i$ from $C_n$ to $E^n_i$ from $C_{n+1}$. Therefore we can regard $C_n$ as subcoalgebra of $C_{n+1}$. Denote by $C = \bigcup_{n \in \mathbb{N}} C_n$; it has a basis formed by the elements $E^n_c$, $n \in \mathbb{N}$, $c \in B$ and comultiplication $\Delta$ and counit $\varepsilon$ given by

$$\Delta(E^n_c) = \sum_{i+j=n; \ a(i)b) = \pm c} c^{-1}a(i)E^a_i \otimes E^b_j$$

and

$$\varepsilon(E^n_c) = \delta_{n,0}\delta_{i,1}.$$ 

By Proposition 2.10 we have that $A_n$ is a chain algebra and therefore $C_n = A^n_n$ is a chain coalgebra. Therefore, we get that the coradical filtration of $C$ is $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ and that this is a chain coalgebra which is obviously non-cocommutative.

3. The co-local case

Throughout this section we will assume (unless otherwise specified) that $C$ is left finite Rat-splitting and that it is a colocal coalgebra, that is, $C_0$ is a simple left (and consequently simple right) $C^*$-module. Then as $J = C^+_0$, $C^*$ is a local algebra. We will also assume that $C$ is not finite dimensional, thus by Proposition 1.4 $C$ has a countable basis. We have that $C$ is the injective envelope of $C_0$ as left comodules, thus by Proposition 1.2 we have that every left subcomodule of $C$ is finite dimensional (all $C_n$ are finite dimensional). Then if $I$ is a left ideal of $C^*$ different from $C^*$, by Proposition 1.4 and Proposition 2.3 $I$ is finitely generated and of finite codimension. Denote again $R = C^*$. Also for a left $R$-module $M$ denote by $J(M)$ the Jacobson radical of $M$.

**Proposition 3.1.** With the above notations, $R$ is a domain.

**Proof.** Let $S = \text{End}^C(C, C)$. Note that $S$ is a ring with multiplication equal to the composition of morphisms and that $S$ is isomorphic to $R$ by an isomorphism that takes every morphism of left $C$-comodules $f \in S$ to the element $\varepsilon \circ f \in R$. Then it is enough to show that $S$ is a domain. If $f : C \to C$ is a nonzero morphism of left $C$ comodules, then $\ker(f) \not\subseteq C$ is a proper left subcomodule of $C$ so it must be finite dimensional. Then as $C$ is not finite dimensional we see that $\text{Im}(f) \simeq C/\ker(f)$ is an infinite dimensional subcomodule of $C$. Thus $\text{Im}(f) = C$, and therefore every nonzero morphism of left comodules from $C$ to $C$ must be surjective. Now if $f, g \in S$ are nonzero then they are surjective so $f \circ g$ is surjective and thus $f \circ g \neq 0$. 

**Proposition 3.2.** $R$ satisfies ACCP on right ideals and also on left ideals.

**Proof.** Suppose there is an ascending chain of right ideals $x_0 \cdot R \subseteq x_1 \cdot R \subseteq x_2 \cdot R \subseteq \ldots$ that is not stationary. Then there are $(\lambda_n)_{n \in \mathbb{N}}$ in $R$ such that $x_n = x_{n+1} \cdot \lambda_{n+1}$. Note that $\lambda_{n+1} \in J$, because otherwise $\lambda_{n+1}$ would be invertible in $R$ as $R$ is local and then we would have $x_{n+1} = x_n \cdot \lambda_n^{-1}$. This would yield $x_n \cdot R = x_{n+1} \cdot R$, a contradiction. Then $x_1 = x_{n+1} \cdot \lambda_{n+1} \lambda_n \ldots \lambda_2$, so $x_1 \in J^n$ for all $n \in \mathbb{N}$, showing that $x_1 \in \bigcap_{n \in \mathbb{N}} J^n = 0$. Thus we obtain a contradiction: $x_0 \cdot R \subseteq x_1 \cdot R = 0$. The statement is obvious for left ideals as $R$ is Noetherian.

The next proposition contains the main idea of the result.
Proposition 3.3. Suppose \( \alpha R \) and \( \beta R \) are two right ideals that are not comparable. Then any two principal right ideals of \( R \) contained in \( \alpha R \cap \beta R \) are comparable.

Proof. Take \( aR, bR \subseteq \alpha R \cap \beta R \), so \( a = \alpha x = \beta y \) and \( b = \alpha u = \beta v \); we may obviously assume that \( a, b \neq 0 \) as otherwise the assertion is obvious. Then \( \alpha, \beta, x, y, u, v \) are nonzero. Denote by \( L \) the left submodule of \( R \times R \) generated by \((x, u)\) and by \( M \) the quotient module \( \frac{R \times R}{L} \). We write \((s, t)\) for the image of the element \((s, t)\) through the canonical projection \( \pi : R \times R \to M \). We have \((y, v) \neq (0, 0)\) as otherwise \((y, v) = \lambda (x, u)\) for some \( \lambda \in R \); then we would have \( y = \lambda x, v = \lambda u \) so \( \beta y = \beta \lambda x = \alpha x \) and then \( \beta \lambda = \alpha \) (because \( R \) is a domain), a contradiction to \( \alpha R \subseteq \beta R \). Also \( \beta \cdot (y, v) = \alpha \cdot (x, u) = (0, 0) \) with \( \beta \neq 0 \). This shows that \((0, 0) \neq (y, v) \in T = T(M)\), so \( T(M) \neq 0 \). Take \( X < M \) such that \( M = T \oplus X \).

We must have \( X \neq 0 \), as otherwise \((1, 0) \in T \) so there would be a nonzero \( \lambda \in R \) and a \( \mu \in R \) such that \( \lambda \cdot (1, 0) = \mu \cdot (x, u) \in L \). But then \( \lambda = \mu x, \mu = 0, \mu = 0 \) \((u \neq 0)\) showing that \( \lambda = 0 \), a contradiction.

Now note that \( x \) and \( u \) are not invertible, as otherwise, for \( x \) invertible, \( \alpha x = \beta y \) implies \( \alpha \in \beta R \) so \( \alpha R \subseteq \beta R \); the same can be inferred if \( u \) is invertible. Therefore \( x, u \in J \) as \( R \) is local so \( L \subseteq J \times J \). Hence \( J(M) = J \times J/L \) so \( M/J(M) = \frac{R \times R/L}{J \times J/L} \cong R \times R/J \times J \) which has dimension 2 as a module over the skewfield \( R/J \). Since \( M = T \oplus X \) and \( X \) is finite generated, then so are \( T \) and \( X \) and therefore \( J(X) \neq X \) and \( J(T) \neq T \). Then as \( J(x, u) = \frac{R \times R}{L} = \frac{R \times R}{J \times J} \) has dimension 2 over \( R/J \), it follows that both \( T/J(T) \) and \( X/J(X) \) are simple. Hence \( T \) and \( X \) are local, and as they are finitely generated, it follows that they are generated by any element not belonging to their Jacobson radical. Let \( T' \) (respectively \( X' \)) be the inverse images of \( T \) (and \( X \) respectively) in \( R \times R \) and \( t \in T' \) be such that \( Rt + L = T' \) and \( Rs + L = X' \). We have \( R \times R = T' \times X' = Rt + L + Rs + L = (Rt + Rs) + L \subseteq (Rt + Rs) + J \subseteq R \times R \) so \( (Rt + Rs) + J \subseteq R \times R \). Therefore we obtain \( Rt + Rs = R \times R \) because \( J \times J \) is small in \( R \times R \).

Write \( t = (p, q) \in T' \). Then \( \overline{t} = t + L \in T \) implies that there is \( \lambda \neq 0 \) in \( R \) such that \( \lambda \overline{t} = 0 \in M \) and therefore there is \( \mu \in R \) with \( \lambda p, q = \mu (x, u) \). We show that either \( p \notin J \) or \( q \notin J \). Indeed assume otherwise: \( t = (p, q) \in J \times J \). Then we get \( Rt \subseteq J \times J \). Because \( Rt + Rs = R \times R \) we see that \( R \times R/J \times J \) must be generated over \( R \) by the image of \( s \). This shows that the \( R/J \) module \( R \times R/J \times J = (R/J)^2 \) has dimension 1 and this is obviously a contradiction.

Finally, suppose \( p \notin J \) so \( p \) is invertible; then the equations \( \lambda p = \mu x, \lambda q = \mu u \) imply \( \lambda = \mu x p^{-1} \) and \( \mu x p^{-1} q = \mu u \). But \( \mu \neq 0 \) because \( p \) is invertible and \( \lambda \neq 0 \). Therefore we obtain \( u = xp^{-1}q \); but \( b = \alpha u = \alpha xp^{-1}q = \alpha p^{-1}q \) showing that \( b \in aR \) i.e. \( bR \subseteq aR \).

Similarly if \( q \) is invertible, we get \( aR \subseteq bR \). \( \square \)

Theorem 3.4. If \( C \) is a left finite splitting (infinite dimensional) local coalgebra, then \( C \) is a chain coalgebra.

Proof. We first show that every two principal left ideals of \( C \) are comparable. Suppose there are two left ideals of \( R \), \( R \cdot x_0 \) and \( R \cdot y_0 \) that are not comparable. Then as they have finite codimension and \( C^* \) is infinite dimensional, we have \( Rx_0 \cap Ry_0 \neq 0 \) and take \( 0 \neq \alpha x_0 = \beta y_0 \in Rx_0 \cap Ry_0 \). Then the right ideals \( \alpha R \) and \( \beta R \) are not comparable, as otherwise, if for example \( \alpha R \subseteq \beta R \), we would have a relation \( \alpha = \beta x \) so \( \alpha x_0 = \beta x x_0 = \beta y_0 \). As \( \beta \neq 0 \) we get \( \lambda x_0 = y_0 \) because \( R \) is a domain, and then \( Ry_0 \subseteq Rx_0 \), a contradiction. By Proposition 3.2 the set \( \{ \lambda R \mid \lambda R \subseteq aR \cap \beta R \} \) is Noetherian (relative to inclusion) and let \( \lambda R \) be a maximal element. If \( x \in aR \cap \beta R \) then by Proposition 3.3 we have that \( xR \) and \( \lambda R \) are comparable and by the maximality of \( \lambda R \) it follows that \( xR \subseteq \lambda R \), so \( x \in \lambda R \). Therefore \( aR \cap \beta R = \lambda R \). Note that \( \lambda \neq 0 \), because \( aR \) and \( \beta R \) are nonzero
ideals of finite codimension. Then we see that $\lambda R \simeq R$ as right $R$ modules, because $R$ is a domain, and again by Proposition 3.3 any two principal right ideals of $\lambda R = \alpha R \cap \beta R$ are comparable, so the same must hold in $R_R$. But this is in contradiction with the fact that $\alpha R$ and $\beta R$ are not comparable, and therefore the initial assertion is proved.

Now we prove that $J^n/J^{n+1}$ is a simple right module for all $n$. As $R/J$ is semisimple (it is a skewfield) and $J^n/J^{n+1}$ has an $R/J$ module structure, it follows that $J^n/J^{n+1}$ is a semisimple left $R/J$-module and then $J^n/J^{n+1}$ is semisimple also as $R$-module. If we assume that it is not simple, then there are $f, g \in J^n \setminus J^{n+1}$ such that $Rf = (Rf + J^{n+1})/J^{n+1}$ and $Rg = (Rg + J^{n+1})/J^{n+1}$ are different simple $R$-modules, so $Rf \cap Rg = \hat{0}$ in $J^n/J^{n+1}$. Then $(Rf + J^{n+1}) \cap (Rg + J^{n+1}) = J^{n+1}$ which shows that $Rf$ and $Rg$ cannot be comparable, a contradiction. As $J^n = C_{n-1}^{n}$, we see that $\dim(C_{n-1}) = \text{codim}(J^n)$. Then for $n \geq 1$, $\dim(C_n/C_{n-1}) = \dim(C_n) - \dim(C_{n-1}) = \text{codim}_R(J^n) - \text{codim}_R(J^{n+1}) = \dim(J^n/J^{n+1}) = \dim(C_0)$. Because $C_0$ is the only type of simple right $C$-comodule, this last relation shows that the right $C$-comodule $C_n/C_{n-1}$ must be simple. Therefore $C$ must be a chain coalgebra. □

We may now combine the results of Sections 2 and 3 and obtain

**Corollary 3.5.** Let $C$ be a co-local coalgebra. Then $C$ is a left (right) finite splitting coalgebra if and only if $C$ is a chain coalgebra. Moreover, if the base field $K$ is algebraically closed then this is further equivalent to the fact that $C$ is isomorphic to the divided power coalgebra.

**Proof.** This follows from Theorems 2.7, 2.9 and 3.4 □

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WHEN DOES THE RATIONAL TORSION SPLIT OFF FOR FINITELY
GENERATED MODULES

MIODRAG CRISTIAN IOVANOV

Dedicated to Fred Van Oystaeyen for his sixtieth birthday

Abstract. It is well known that the torsion part of any finitely generated module over
the formal power series ring $K[[X]]$ is a direct summand. In fact, $K[[X]]$ is an algebra
dual to the divided power coalgebra over $K$ and the torsion part of any $K[[X]]$-module
actually identifies with the rational part of that module. More generally, for a certain
general enough class of coalgebras - those having only finite dimensional subcomodules
- we see that the above phenomenon is preserved: the set of torsion elements of any
$C^*$-module is exactly the rational submodule. With this starting point in mind, given
a coalgebra $C$ we investigate when the rational submodule of any finitely generated left
$C^*$-module is a direct summand. We prove various properties of coalgebras $C$
having
this splitting property. Just like in the $K[[X]]$ case, we see that standard examples of
coalgebras with this property are the chain coalgebras which are coalgebras whose lattice
of left (or equivalently, right, two-sided) coideals form a chain. We give some representa-
tion theoretic characterizations of chain coalgebras, which turn out to make a left-right
symmetric concept. In fact, in the main result of this paper we characterize the colo-
cal coalgebras where this splitting property holds non-trivially (i.e. infinite dimensional
coalgebras) as being exactly the chain coalgebras. This characterizes the cocommutative
coalgebras of this kind. Furthermore, we give characterizations of chain coalgebras in
particular cases and construct various and general classes of examples of coalgebras with
this splitting property.

Introduction

Let $R$ be a ring and $T$ be a torsion preradical on the category of left $R$-modules $R\mathcal{M}$. Then
$R$ is said to have splitting property provided that $T(M)$, the torsion submodule of $M$, is a
direct summand of $M$ for any $M \in R\mathcal{M}$. More generally, if $\mathcal{C}$ is a Grothendieck category
and $\mathcal{A}$ is a subcategory of $\mathcal{C}$, then $\mathcal{A}$ is called closed if it is closed under subobjects, quotient
objects and direct sums. To every such subcategory we can associate a preradical $t$ (also
called torsion functor) if for every $M \in \mathcal{C}$ we denote by $t(M)$ the sum of all subobjects of
$M$ that belong to $\mathcal{A}$. We say that $\mathcal{C}$ has the splitting property with respect to $\mathcal{A}$ if $t(M)$ is a
direct summand of $M$ for all $M \in \mathcal{C}$. In the case of the category of $R$-modules, the splitting
property with respect to some closed subcategory is a classical problem which has been
considered by many authors. In particular, when $R$ is a commutative domain, the question
of when the (classical) torsion part of an $R$ module splits off is a well known problem. J.
Rotman has shown in [Rot] that for a commutative domain the torsion submodule splits
off in every $R$-module if and only if $R$ is a field. I. Kaplansky proved in [K1], [K2] that for

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a commutative integral domain $R$ the torsion part of every finitely generated $R$-module $M$ splits in $M$ if and only if $R$ is a Prüfer domain. While complete or partial results have been obtained for different cases of subcategories of $R\mathcal{M}$ - such as the Dickson subcategory - or for commutative rings (see also [1], [2], [3]), the general problem remains open for the non-commutative case and the general categorical setting.

In this paper we investigate a special and important case of rings (algebras) $R$ arising as the dual algebra of a $K$-coalgebra $C$, $R = C^*$. We are thus situated in the realm of the theory of coalgebras and their dual algebras, a theory intensely studied over the last two decades. Then the category of the left $R$-modules naturally contains the category $\mathcal{M}^C$ of all right $C$-comodules as a full subcategory. In fact, $\mathcal{M}^C$ identifies with the subcategory $R\text{at}(C,\mathcal{M})$ of all rational left $C^*$-modules, which is generally a closed subcategory of $C\mathcal{M}$.

Then it is natural to study splitting properties with respect to this subcategory, and two questions regarding this splitting property with respect to $R\text{at}(C,\mathcal{M})$ naturally arise: first, when is the rational part of every left $C^*$-module $M$ a direct summand of $M$ and second, when does the rational part of every finitely generated $C^*$-module $M$ split in $M$. The first problem, the splitting of $C\mathcal{M}$ with respect to the closed subcategory $R\text{at}(C,\mathcal{M})$ has been treated by C. Năstăsescu and B. Torrecillas in [NT] where it is proved that if all $C^*$-modules split with respect to $R\text{at}$ then the coalgebra $C$ must be finite dimensional. The techniques used involve some amount of category theory (localization in categories) and strongly rely on some general results of M.L.Teply from [1], [2], [3]; another proof of this fact also based on the general results of Teply is found in [4]; see also [5] for a direct approach.

We consider the more general problem of when $C$ has the splitting property only for finitely generated modules, that is, the problem of when is the rational part of every left $C^*$-module $M$ a direct summand of $M$ for all finitely generated left $C^*$-modules $M$. We say that such a coalgebra has the left f.g. $R\text{at}$-splitting property (or we say that it has the $R\text{at}$-splitting property for finitely generated left modules). If the coalgebra $C$ is finite dimensional, then every left $C^*$-module is rational so $\mathcal{M}^C$ is equivalent to $C\mathcal{M}$ and $R\text{at}(M) = M$ for all $C^*$-modules $M$ and in this case $R\text{at}(M)$ trivially splits in any $C^*$-module. Therefore we will deal with infinite dimensional coalgebras, as generally the infinite dimensional coalgebras produce examples essentially different from the ones in algebra theory.

The starting and motivating point of our research is the fact that over the ring of formal power series over a field $R = K[[X]]$ (or a division algebra), any finitely generated module splits into its torsion part and a complementary module. In this case, $R$ is the dual of the so called divided power coalgebra, and the torsion part of any module identifies with the rational submodule. Here the analogue with classical torsion splitting problems becomes obvious. In fact, what turns out to be essential in this example is the structure of ideals of $K[[X]]$, and that is, they are linearly ordered. This suggests the consideration of more general coalgebras, those whose left subcomodules form a chain. This turns out to be a left-right symmetric concept, and the most basic example of infinite dimensional coalgebra having the f.g. $R\text{at}$-splitting property (Proposition 2.3 and Theorem 2.5). One key observation in this study is that if $C$ has the f.g. $R\text{at}$-splitting property, then the indecomposable left injectives have only finite dimensional proper subcomodules, and this motivates the introduction of comodules and coalgebras $C$ having only finite dimensional proper left subcomodules, which we call almost finite (or almost finite dimensional) comodules. This proves to be the proper generalization of the phenomenon found in the case of $K[[X]]$,
i.e. the set of torsion elements of a left \( C^* \)-module \( M \) forms a submodule which coincides exactly with the rational submodule of \( M \) (Proposition [12]). Before turning to the study of chain comodules and coalgebras, we give several general results for coalgebras \( C \) with the f.g. \( \text{Rat} \)-splitting property: they are artinian as right \( C^* \)-module and injective as left \( C^* \)-module, have at most countable dimension and \( C^* \) is a left Noetherian ring. Moreover, such coalgebras have finite dimensional coradical and the f.g. \( \text{Rat} \)-splitting property is preserved by subcoalgebras.

The f.g. \( \text{Rat} \)-splitting property has been studied before in [C] where the last two of the above statements were proven, but with the use of very strong results of M.L. Teply; we also include alternate direct proofs. Chain coalgebras were also studied recently in [LS] and also briefly in [C] and [CGT]. However, our interest in chain coalgebras is of a different nature; it is a representation theoretic one and is directed towards our main result of this paper, that generalizes a result previously obtained [C] in the commutative case: we characterize the coalgebras having the f.g. \( \text{Rat} \)-splitting property and that are colocal, and show that they are exactly the chain coalgebras (Section 3, Theorem [3.4]), a result that will involve quite technical arguments. In fact, our characterizations of chain coalgebras are done as a consequence of more general discussions such as the study chain of comodules and more generally almost finite comodules and coalgebras. For example, we show that almost finite coalgebras are reflexive, and that chain coalgebras are almost finite, and thus obtain the fact that chain coalgebras are reflexive (a result also found in the recent [LS]) from our more general framework.

We provide several nontrivial examples. One will be the construction of a noncocommutative chain coalgebra with coradical isomorphic to the dual of the Hamilton algebra of quaternions. However, we see that when the base field \( K \) is algebraically closed or the coalgebra is pointed, then a chain coalgebra is isomorphic to the divided power coalgebra if it is infinite dimensional or to one of its subcoalgebras otherwise. This also characterizes the divided power coalgebra over an algebraically closed field as the only local coalgebra having the above mentioned splitting property. As an application of the main result, we obtain the structure of cocommutative coalgebras having the f.g. \( \text{Rat} \)-splitting property from [C] in a more precise form: they are finite coproducts of finite dimensional coalgebras and infinite dimensional chain coalgebras. Moreover, following this model, our results allow us to generalize to the noncommutative case and show that a coalgebra that is a finite direct sum of infinite dimensional left chain comodules (serial coalgebra) has the left f.g. \( \text{Rat} \)-splitting property; moreover, this is again a left-right symmetric concept. More generally, a coproduct of such a coalgebra and a finite dimensional one again has the f.g. \( \text{Rat} \)-splitting property. We conclude by constructing a class of explicit examples of noncocommutative coalgebras of this type over an arbitrary field, which will depend on a positive integer \( q \) and a permutation \( \sigma \) of \( q \) elements.

1. General Considerations

Let \( C \) be a coalgebra with counit \( \epsilon \) and comultiplication \( \Delta \). We use the Sweedler convention \( \Delta(c) = c_1 \otimes c_2 \) where we omit the summation symbol. For general facts about coalgebras and comodules we refer to [A], [DNR] or [Sw]. For a vector space \( V \) and a subspace \( W \) of \( V \) denote by \( W^\perp = \{ f \in V^* \mid f(x) = 0, \forall x \in W \} \) and for a subspace \( X \in V^* \) denote by \( X^\perp = \{ x \in V \mid f(x) = 0, \forall f \in X \} \) (it will be understood from the context what is the space \( V \) with respect to which the orthogonal is considered). Various properties of this correspondence between subspaces of \( V \) and \( V^* \) are well known and studied in more
general topology on the dual $V^*$ of a vector space $V$: a basis of 0 for this linear topology is given by the sets $W^\perp$ with $W$ a finite dimensional subspace of $V$. Any topological consideration will refer to this topology. We usually denote the following: a subspace $X$ of $V^*$ is closed (in the finite topology) if and only if $(X^\perp)^\perp = X$; also, if $W$ is a subspace of $V$, then $(W^\perp)^\perp = W$. (see [DNR], Chapter 1)

For a coalgebra $C$ denote by $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ the coradical filtration of $C$, that is, $C_0$ is the coradical of $C$, and $C_{n+1} \subseteq C$ such that $C_{n+1}/C_n$ is the socle of the right (or left) $C$-comodule $C/C_n$ for all $n \in \mathbb{N}$. Then $C_n$ is a subcoalgebra of $C$ for all $n$, and the same $C_n$ is obtained whether we take the socle of the left $C$-comodule $C/C_n$ or of the right $C$-comodule $C/C_n$. Put $C_{-1} = 0$ and $R = C^*$. Denote $J = J(C^*)$ the Jacobson radical of $C^*$. By [DNR] we have $\bigcup_{n \in \mathbb{N}} C_n = C$, $J = C_0$ and $(J^{n+1})^\perp = C_n$. Then $J^n \subseteq (J^n)^\perp = C_{n-1}$ and since $\bigcup_{n \in \mathbb{N}} C_n = C$, we see that $\bigcap_{n \in \mathbb{N}} J^n = 0$.

For a left (right) $C$-comodule $M$ with comultiplication $\rho : M \to C \otimes M$ ($\rho : M \to M \otimes C$), the Sweedler notation writes $\rho(m) = m_{-1} \otimes m_0$ (respectively $\rho(m) = m_0 \otimes m_1$). Moreover, the dual $M^*$ of $M$ becomes a left (right) $C^*$-module by the action induced by the right (left) $C^*$-action on $M$ by duality: for $m^* \in M^*$, $m \in M$ and $c^* \in C^*$, $(c^* \cdot m^*)(m) = m^*(m \cdot c^*) = c^*(m_{-1})m^*(m_0)$ (respectively $(m^* \cdot c^*)(m) = m^*(m_0)c^*(m_1)$).

**Lemma 1.1.** Let $C$ be a coalgebra over a field $K$ and $M$ be a left $C$-comodule. Then for any finitely generated left submodule $X$ of $M^*$, $(X^\perp)^\perp = X$, that is, $X$ is closed in the finite topology on $M^*$.

**Proof.** It is enough to prove this for cyclic submodules: if $(C^*f_i)^\perp = C^*f$ for all $f \in M^*$ and $X = C^*f_1 + \ldots + C^*f_n$ then $(X^\perp)^\perp = \bigcap_{i=1}^n (C^*f_i)^\perp = \bigcap_{i=1}^n C^*f_i = X$ (since $(\bigcap_{i=1}^n M_i)^\perp = \bigcap_{i=1}^n M_i^\perp$ for $M_i \subseteq M$; see, for example [I0], Proposition 3 or [DNR], Chapter 1, also [AN] and [AF]).

Let $X = C^*f$ and $u : M \to C$, $u(m) = m_{-1}f(m_0)$, where for $m \in M$, $m_{-1} \otimes m_0 \in C \otimes M$ denotes the comultiplication of $m \in M$; then $L = (C^*f)^\perp = \{m \in M \mid (h f)(m) = 0, \forall h \in C^*\} = \{m \in M \mid f(h(m_{-1})m_0) = h(m_{-1}f(m_0)) = 0, \forall h \in C^*\} = \{m \in M \mid m_{-1}f(m_0) = 0\}$, so $L = \ker(u)$ (the left $C^*$-module structure on $M^*$ is induced from the right $C^*$-module structure on $M$ by duality). If $g \in L^\perp \subseteq M^*$, then $\ker(u) \subseteq \ker(g)$ we can factor $g$ as $g = p \circ u$ with $p : \text{Im}(u) \to K$, and then defining $h \in C^*$ as $h = p$ on $\text{Im}(u) \subseteq C$ and 0 on some complement of $\text{Im}(u)$ we get $(h f)(m) = f(m \cdot h) = f(h(m_{-1})m_0) = h(m_{-1}f(m_0)) = h(u(m)) = p \circ u(m) = g(m)$, i.e. $g \in C^*f$. This shows that $(C^*f)^\perp = C^*f$. \qed

1.1. ”Almost finite” coalgebras and comodules.

**Definition 1.2.** A $C$-comodule $M$ will be called almost finite (or almost finite dimensional) if it has only finite dimensional proper subcomodules. Call a coalgebra $C$ left almost finite if $C C$ is almost finite.

**Proposition 1.3.** Let $M$ be a left almost finite (dimensional) $C$-comodule. Then:

(i) $M$ is artinian as left $C$-comodule (equivalently, as right $C^*$-module).

(ii) Any nonzero submodule of $M^*$ has finite codimension; consequently $M^*$ is (left) Noetherian. Moreover, all submodules of $M^*$ are closed in the finite topology of $M^*$.

(iii) $M$ has at most countable dimension.
Proof. (i) Obvious.
(ii) Let \( 0 \neq I < M^* \) be a submodule, \( 0 \neq f \in M^* \). Then \( X = (C^* \cdot f)^\perp \) is a submodule of \( M \) which is finite dimensional and \( C^* \cdot f = X^\perp \) from Lemma 1.1. So \( C^* \cdot f \) has finite codimension, and so does \( I \supseteq C^* \cdot f \). Thus \( M^* \) is Noetherian and the last assertion of (ii) follows now from Lemma 1.1.
(iii) Assume \( M \) is infinite dimensional and define inductively a sequence \( \langle m_k \rangle _{k \geq 0} \) such that \( m_{k+1} \notin M_k = m_1 \cdot C^* + \ldots + m_k \cdot C^* \). This can be done since the \( M_k \)'s are finite dimensional, and then \( \bigcup _{k \geq 0} M_k \subseteq M \) is infinite-countable dimensional, and thus cannot be a proper submodule of \( M \). Thus \( \bigcup _{k \geq 0} M_k = M \), and the proof is finished. \( \square \)

The above Proposition shows that a left almost finite coalgebra \( C \) is coreflexive by [DNR], Exercise 1.5.14, since every ideal of finite codimension in \( C \) is closed (Also, by a result of Radford, \( C \) is coreflexive if and only any finite dimensional \( C^* \)-module is rational). Thus we have:

**Corollary 1.4.** Let \( C \) be a left almost finite coalgebra. Then any nonzero left ideal of \( C^* \) is closed in the finite topology on \( C^* \) and has finite codimension, \( C^* \) is Noetherian and \( J^n = C_{n-1} \). Moreover, \( C \) is coreflexive.

For a left \( C^* \)-module \( M \) denote by \( T(M) \) the set of all torsion elements of \( M \), that is, \( T(M) = \{ x \in M \mid \text{ann}_{C^*} x \neq 0 \} \). If \( C \) is a finite dimensional coalgebra, it is well known that the categories of right \( C \)-comodules and left \( C^* \)-modules are equivalent. Thus, we are interested in the infinite dimensional case. As mentioned above, for coreflexive coalgebras, \( \text{Rat}(M) = \{ x \in M \mid C^* \cdot x \text{ is finite dimensional} \} = \{ x \in M \mid \text{ann}_{C^*}(x) \text{ has finite codimension} \} \). For (infinite dimensional) almost finite coalgebras, we see that the rational submodule of a \( C^* \)-module has an even more special form:

**Proposition 1.5.** Let \( C \) be an infinite dimensional left almost finite coalgebra and let \( R = C^* \). Then for any left \( R \)-module \( M \) we have \( \text{Rat}(M) = T(M) \); moreover, \( x \in \text{Rat}(M) \) if and only if \( R \cdot x \) is finite dimensional.

**Proof.** If \( x \in \text{Rat}(M) \) then \( R \cdot x \) is finite dimensional and then \( \text{ann}_R(x) \) must be of finite codimension, thus nonzero as \( R \) is infinite dimensional. Conversely, if \( x \in T(M) \) and \( x \neq 0 \) then \( I = \text{ann}_R(x) \) is a nonzero left ideal of \( R \) so it is closed by Corollary 1.1, thus \( I = X^\perp \) with \( X \neq C \) a finite dimensional subcomodule of \( C \). Then \( R \cdot x \simeq R/\text{ann}_R(x) = C^*/X^\perp \simeq X^* \) which is a rational left \( C^* \)-module, being the dual of a finite dimensional subcomodule of \( C \). \( \square \)

1.2. The Splitting Property.

**Definition 1.6.** We shall say that a coalgebra \( C \) has the left (right) f.g. Rat-splitting property, or that it has the left (right) Rat-splitting property for all finitely generated modules if the rational part of any finitely generated left (right) \( C^* \)-module splits off.

The following key observation, together with the succeeding study of chain coalgebras, motivates our previous introduction of almost finite comodules and coalgebras.

**Proposition 1.7.** Let \( C \) be a coalgebra such that \( \text{Rat}(M) \) splits off in any finitely generated left \( C^* \)-module \( M \). Then any indecomposable injective left \( C \)-comodule \( E \) is an almost finite comodule.

**Proof.** Let \( T \) be the socle of \( E \); then \( T \) is simple and \( E = E(T) \) is the injective envelope of \( T \). We show that if \( K \subseteq E(T) \) is an infinite dimensional subcomodule then \( K = E(T) \).
Suppose $K \subseteq E(T)$. Then there is a left $C$-subcomodule (right $C^*$-submodule) $K \subseteq L \subset E(T)$ such that $L/K$ is finite dimensional. We have an exact sequence of left $C^*$-modules:

$$0 \rightarrow (L/K)^* \rightarrow L^* \rightarrow K^* \rightarrow 0$$

As $L/K$ is a finite dimensional left $C$-comodule, we have that $(L/K)^*$ is a rational left $C^*$-module; thus $\text{Rat}(L^*) \neq 0$. Also $L^*$ is finitely generated as it is a quotient of $E(T)^*$ which is a direct summand of $C^*$. We have $L^* = \text{Rat}(L^*) \oplus X$ for some left $C^*$-submodule $X$ of $L^*$. Then $\text{Rat}(L^*)$ is finitely generated because $L^*$ is, so it is finite dimensional. As $L$ is infinite dimensional by our assumption, we have $X \neq 0$. This shows that $L^*$ is decomposable and finitely generated, thus it has at least two maximal submodules, say $M, N$. We have an epimorphism $E(T)^* \xrightarrow{f} L^* \rightarrow 0$ and then $f^{-1}(M)$ and $f^{-1}(N)$ are distinct maximal $C^*$-submodules of $E(T)^*$. But by \cite{I}, Lemma 1.4, $E(T)^*$ has only one maximal $C^*$-submodule which is $T^*$, so we have obtained a contradiction. \hfill \Box

Let $C_0$ be the coradical of $C$, the sum of all simple subcomodules of $C$. By \cite{DNR}, Section 3.1, $C_0$ is a cosemisimple coalgebra that is a direct sum of simple subcoalgebras $C_0 = \bigoplus_{i \in I} C_i$ and each simple subcoalgebra $C_i$ contains only one type of simple left (or right) $C$-comodule; moreover, any simple left (or right) $C$-comodule is isomorphic to one contained in some $C_i$. A coalgebra $C$ with $C_0$ finite dimensional is called almost connected coalgebra.

The following two Propositions have also been observed in \cite{C} (Lemma 3.2 and Lemma 3.3), but general powerful techniques from \cite{T3} are used there. We provide here direct simple arguments.

**Proposition 1.8.** Let $C$ be a coalgebra with the left f.g. Rat-splitting property. Then there is only a finite number of isomorphism types of simple left $C$-comodules, equivalently, $C_0$ is finite dimensional.

**Proof.** By the above considerations, if $S_i$ is a simple left $C$-subcomodule of $C_i$, we have that $(S_i)_{i \in I}$ forms a set of representatives for the isomorphism types of simple left $C$-comodules. Let $S$ be a set of representatives for the simple right $C$-comodules. Let $E(C_i)$ be an injective envelope of the left $C$-comodule $C_i$ included in $C$; then as $C_0$ is essential in $C$ we have $C = \bigoplus_{i \in I} E(C_i)$ as left $C$-comodules or right $C^*$-modules. Then $C^* = \prod_{i \in I} E(C_i)^*$ as left $C^*$-modules. As $S_i \subseteq E(C_i)$, we have epimorphisms of left $C^*$-modules $E(C_i)^* \rightarrow S_i^* \rightarrow 0$ and therefore we have an epimorphism of left $C^*$-modules $C^* \rightarrow \prod_{i \in I} S_i^* \rightarrow 0$. But there is a one-to-one correspondence between left and right simple $C$-comodules given by $\{S_i \mid i \in I\} \ni S \mapsto S^* \in S$. Hence there is an epimorphism $C^* \rightarrow \prod_{S \in S} S \rightarrow 0$, which shows that the left $C^*$-module $P = \prod_{S \in S} S$ is finitely generated (actually generated by a single element). But then as $\text{Rat}(C^*P)$ is a direct summand in $P$, we must have that $\text{Rat}(C^*P)$ is finitely generated, so it is finite dimensional. Therefore, as $\Sigma = \bigoplus_{S \in S} S$ is a rational left $C^*$-module which is naturally included in $P$, we have $\Sigma \subseteq \text{Rat}(P)$. This shows that $\bigoplus_{S \in S} S$ is finite dimensional, so $S$ (and also $I$) must be finite. This is equivalent to the fact that $C_0$ is finite dimensional, because each $C_i$ is a simple coalgebra, thus a finite dimensional one. \hfill \Box

**Proposition 1.9.** If $C$ is has the left f.g. Rat-splitting property then so does any subcoalgebra $D$ of $C$. 


Proof. Let $M$ be a finitely generated left $D^*$-module. Since $C^*/D^\perp \simeq D^*$, $M$ has an induced left $C^*$-module structure and is annihilated by $D^\perp$ (that is, $D^\perp \cdot x = 0$ for all $x \in M$). Then a subspace of $M$ is a $C^*$-submodule if and only if it is a $D^*$-submodule. There is $M = T \oplus X$ a direct sum of $C^*$-modules (equivalently $D^*$-submodules, since $D^\perp$ annihilates the elements in both $T$ and $X$) with $T$ the rational $C^*$-submodule of $M$. It will suffice to show that a submodule of $M$ is rational as $C^*$-module if and only if it is rational as $D^*$-module. Indeed, let $m \in T = \text{Rat}_{C^*}(M)$; then there is $\sum_i m_i \otimes c_i \in T \otimes C$ such that $c^* \rho m = \sum_i c^*(c_i)m_i$; we may assume that the $m_i$‘s are linearly independent. Then for $c^* \in D^\perp \subseteq C^*$ we get $0 = c^* \cdot m = \sum_i c^*(c_i)m_i$ and so $c^*(c_i) = 0$ since the $m_i$‘s are independent, showing that $c_i \in (D^\perp)^\perp = D$. Therefore $\rho(m) = \sum m_i \otimes c_i \in T \otimes D$, where $\rho$ is the comultiplication of $T$, and thus $m \in \text{Rat}_{D^*}(M)$. The converse inclusion $\text{Rat}_{D^*}(M) \subseteq \text{Rat}_{C^*}(M)$ is obvious, since the $D$-comultiplication $\text{Rat}_{D^*}(M) \otimes D \subseteq \text{Rat}_{D^*}(M) \otimes C$ induces a $C$-comultiplication through the canonical inclusion $D \subseteq C$, compatible with the $C^*$-multiplication of $M$. □

Proposition 1.10. Let $C$ be a coalgebra that has the left f.g. Rat-splitting property. Then the following assertions hold:
(i) $C$ is artinian as left $C$-comodule (equivalently, as right $C^*$-module).
(ii) $C^*$ is left Noetherian.
(iii) $C$ has at most countable dimension.
(iv) $C$ is injective as left $C^*$-module.

Proof. (i) We have a direct sum decomposition $C = \bigoplus_{i \in F} E(S_i)$ where $C_0 = \bigoplus_{i \in F} S_i$ is the decomposition of $C_0$ into simple left $C$-comodules and $E(S_i)$ are injective envelopes of $S_i$ contained in $C$. Since $F$ is finite as $C_0$ is finite dimensional, the result follows from Propositions 1.7 and 1.3.
(ii) Since $C^* = \bigoplus_{i \in F} E(S_i)^*$, this also follows from 1.3.
(iii) Similar to (i).
(iv) By Lemma 2, it is enough to prove that $E = C^C$ splits off in any left $C^*$-module $M$ in which it embeds ($E \subseteq M$) and such that $M/E$ is cyclic generated by an element $\hat{x} \in M/E$. Let $H = \text{Rat}(C^* \cdot x) \subseteq M$; then there is $X < C^* \cdot x$ such that $H \cap X = C^* \cdot x$. Then $E + H$ is a rational $C^*$-module so $(E + H) \cap X = 0$; also $M = C^* \cdot x + E$ so $(E + H) + X = M$, showing that $E + H$ is a direct summand in $M$. But as $E$ is an injective comodule, we have that $E$ splits off in $E + H$, thus $E$ must split in $M$ and the proof is finished. □

2. Chain Coalgebras

Definition 2.1. We say that a left (right) $C$-comodule $M$ is a chain (or uniserial) comodule if and only if the lattice of the left (right) subcomodules of $C$ is a chain, that is, for any two subcomodules $X, Y$ of $M$ either $X \subseteq Y$ or $Y \subseteq X$. We say a coalgebra $C$ is a left (right) chain coalgebra (or uniserial coalgebra) if $C$ is a left (right) chain $C$-comodule.

In other words, a left $C$-comodule $M$ is a chain comodule if $M$ is uniserial as a right $C^*$-module. Part of the following proposition is a somewhat different form of Lemma 2.1 from [CGT]. However, we will need to use some of the other equivalent statements below.

Proposition 2.2. Let $M$ be a left (right) $C$-comodule. The following assertions are equivalent:
(i) $M$ is a chain comodule.
(ii) $M^*$ is a chain (uniserial) left (right) $C^*$-module.
(iii) $M$ and $M_n = "the n'th Loewy term in the Loewy series of M for n ≥ −1", are the only subcomodules of $M/ι(M_{−1} = 0)$.
(iv) $M_n^\perp = \{u \in M^* | u(x) = 0, \forall x \in M_n\}$ for $n ≥ −1$ and 0 are the only submodules of $M^*$.
(v) $M_n/M_{n−1}$ is either simple or 0 for all $n ≥ −1$. (If $M_n/M_{n−1}$ is 0 for some $n$ then $M_k/M_{k−1}$ is 0 for all $k ≥ n$.)

**Proof.** (iv)$\Rightarrow$(ii) is obvious.
(ii)$\Rightarrow$(i) If $M^*$ is uniserial, then for any two submodules $X, Y$ of $C$ we have $X^\perp \subseteq Y^\perp$, say. Thus we get $X = (\bigwedge X)^\perp \supseteq (\bigwedge Y)^\perp = Y$.
(i)$\Rightarrow$(iii) is obvious (note that (iii) this does not exclude the possibility that $M = M_n$ from some $n$ onward)
(i)$\Rightarrow$(iv) If $M$ is a chain comodule, it is enough to assume that $M$ is infinite dimensional, because of the duality of categories between finite dimensional left comodules and finite dimensional right comodules. We note that each $M_n^\perp$ is generated by any $u_n \in M_n^\perp \setminus M_n$.
Let $f ∈ M_n^\perp$ and denote $\overline{u_n}, f : M → C$, $\overline{u_n}(m) = m−1u_n(m_0)$ $(f(m) = m−1u_n(m_0))$.
Then $u_n ∈ M_n^\perp \setminus M_{n+1}$ shows that $\overline{u_n}$ is a morphism of left $C$-comodules that factors to a morphism $M/M_n → C$ which does not cancel on $M_n/M_n$ - the only simple subcomodule of $M/M_n$. Therefore $\ker (\overline{u_n} : M/M_n → C) = 0$ and we have a diagram

$$
\begin{array}{ccc}
0 & \rightarrow & M/M_n \\
\downarrow & \Phi & \downarrow \\
C & \rightarrow & \overline{u_n}
\end{array}
$$

that is completed commutatively by $\Phi$ (as $C/C$ is injective), so that we get $\Phi \circ \overline{u_n} = f$ and then if $g = \epsilon \circ \overline{u_n}$ we have, for $m ∈ M$, $g(m_0)u_n(m_0) = g(m_1u_n(m_1)) = \epsilon(\overline{u_n}(m_0)) = \epsilon(f(m_0)) = (m−1)f(m_0) = f(m)$. Thus $g \cdot u_n = f$. This shows that any cyclic submodule of $M^*$ coincides to one of the $M_n^\perp$, because for any $0 \neq f ∈ M^*$ there is some $n$ such that $f ∈ M_n^\perp \setminus M_{n+1}$, since $M = \bigcup M_n$. It therefore follows that for any nonzero submodule $I$ of $M^*$ there is $M_n^\perp ⊆ I$; since the $M_n$’s are (obviously) finite dimensional, $M_n^\perp$ and $I$ have finite codimension and it now easily follow from the above considerations that $I = M_k^\perp$, where $k$ is the smallest number such that $M_k^\perp ⊆ I$.
(v)$\Rightarrow$(iii) Let $X$ be a right submodule of $M$ and suppose $X \neq M$ and $X \neq 0$. Then there is $n ≥ 0$ such that $M_n \not= X$ and let $n$ be minimal with this property. Then we must have $M_{n−1} \subseteq X$ by the minimality of $n$ and we show that $M_{n−1} = X$. Indeed, if $M_{n−1} \subseteq X$ we can find a simple submodule of $X/M_{n−1}$. But then $M_{n−1} \neq M$, so $M_{n−1} \neq M_n$ and as $M_n/M_{n−1}$ is the only simple subcomodule of $M/M_n$, we find $M_n/M_{n−1} ⊆ X/M_{n−1}$, that is $M_n ⊆ X$, a contradiction.
(i)$\Rightarrow$(v) If $M_{n+1}/M_n$ is nonzero and it is not simple then we can find $S_1 = X_1/M_n$ and $S_2 = X_2/M_n$ ($X_1, X_2 ⊆ M$) two distinct simple modules contained in $M/M_n$. Then $X_1 \cap X_2 = M_n$, $X_1 \neq M_n$, $X_2 \neq M_n$. But this shows that neither $X_1 \subseteq X_2$ nor $X_2 \subseteq X_1$ which is a contradiction. □

The following result shows that chain coalgebra is a left-right symmetric notion and also characterizes chain coalgebras.

**Proposition 2.3.** The following assertions are equivalent for a coalgebra $C$:
(i) $C$ is a right chain coalgebra.
(ii) $C_{n+1}/C_n$ is either 0 or a simple (right) comodule for all $n \geq -1$.
(iii) $C_n$, $n \geq -1$ and $C$ are the only right subcomodules of $C$.
(iv) $J^n$, $n \geq 0$ and 0 are the only right ideals of $C^*$.
(v) $C^*$ is a right (or left) uniserial ring (chain algebra).
(vi) The left hand side version of (i)-(iv).
(vii) $C_1$ has length less or equal to 2.

**Proof.** The equivalence of (i)-(vi) follows from Proposition 2.2 and also by Corollary 1.4. (i)⇒(vii) is obvious and (vii)⇒(i) is a result from [C]. We note a direct argument for this case: it is enough to deal with the case when $C_1$ has length 2; by induction, assume $C_k/C_{k-1}$ is simple or 0 for $k \leq n$. Assume $C_n \neq C_{n-1}$ and note that since $C_n/C_{n-1}$ is the socle of $C/C_{n-1}$, then $C/C_{n-1}$ embeds in $C$ and therefore if $C_{n+1}/C_{n-1}$ has length at most 2, since it embeds in $C_1$. Thus $C_{n+1}/C_n$ is simple or 0. □

**Remark 2.4.** The above Proposition includes many of the results in [LS] sections 5.1-5.3. By Proposition 2.2 a chain module is almost finite and by 2.3 a chain coalgebra is left and right almost finite, so the results of the first section apply here. Therefore we also obtain that a chain coalgebra is coreflexive.

Next we show that a chain coalgebra is both a left and a right f.g. Rat-splitting property coalgebra. Although this follows in a more general setting as in Section 4, we also provide a direct proof that does not involve the tools used in there, but makes use of the interesting fact that for a left almost finite coalgebra $C$ and any left $C^*$-comodule $M$, $T(M)$ is a submodule of $M$ and is exactly the rational submodule of $M$.

**Theorem 2.5.** If $C$ is a chain coalgebra, then $C$ has the left and right f.g. Rat-splitting property.

**Proof.** Of course, we only need to consider the case when $C$ is infinite dimensional. First notice that every torsion-free $R$-finitely generated module $M$ is free: indeed if $x_1, \ldots, x_n$ is a minimal system of generators, then if $\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$ with $\lambda_i$ not all zero, we may assume that $\lambda_1 \neq 0$. Without loss of generality we may also assume that $\lambda_1 R \supseteq \lambda_i R, \forall i$ as any two ideals of $R$ are comparable by Proposition 2.3. Therefore we have $\lambda_1 = \lambda_1 s_1$ for some $s_1 \in R$. Then $\lambda_1 x_1 + \lambda_1 s_2 x_2 + \cdots + \lambda_1 s_n x_n = 0$ implies $x_1 + s_2 x_2 + \cdots + s_n x_n = 0$ as $M$ is torsion-free and $\lambda_1 \neq 0$. Hence $x_1 \in R < x_2, \ldots, x_n >$, contradicting the minimality of $n$.

Now if $M$ is any left $R$-module and $T = T(M) = \text{Rat}(M)$ (by Proposition 1.5) then $T(M/T(M)) = 0$. Indeed take $\hat{x} \in T(M/T(M))$ and put $I = \text{ann}_C \hat{x} \neq 0$ so $I$ has finite codimension and $I$ is a two-sided ideal by Proposition 2.3. By Corollary 1.4 and Remark 2.4, $I$ is finitely generated and therefore $Ix$ is also finitely generated. Also, since $I = \text{ann}_C \hat{x}$, we get $Ix \subseteq T = \text{Rat}(M)$. Thus $Ix$ is finitely generated rational, so $Ix$ has finite dimension. We obviously have an epimorphism $\frac{R}{I} \to \frac{R}{Ix}$ which shows that $R/Ix$ is finite dimensional because $I$ has finite codimension in $R$. Therefore we get that $\dim(Rx) = \dim(Rx/Ix) + \dim(Ix) < \infty$, so then by Proposition 1.5 we have that $Rx$ is rational, thus $x \in T$ so $\hat{x} = 0$.

Now as $M/T$ is torsion-free, there are $x_1, \ldots, x_n \in M$ whose images $\bar{x}_1, \ldots, \bar{x}_n$ in $M/T$ form a basis. Then it is easy to see that $x_1, \ldots, x_n$ are linearly independent in $M$. Then if $X = Rx_1 + \cdots + Rx_n$ we have $X + T = M$ and $X \cap T = 0$, because if $a_1 x_1 + \cdots + a_n x_n \in T$ we get $a_1 \bar{x}_1 + \cdots + a_n \bar{x}_n = 0$ so $a_i = 0, \forall i$ because $\bar{x}_1, \ldots, \bar{x}_n$ are independent in $M/T$. Thus $T(M)$ splits off in $M$ and the theorem is proved, as $T(M) = \text{Rat}_R(M)$ by 1.3 □
We will denote by $K_n$ the coalgebra with a basis $c_0, c_1, \ldots, c_{n-1}$ and comultiplication $c_k \mapsto \sum_{i+j=k} c_i \otimes c_j$ and counit $\varepsilon(c_i) = \delta_{0,i}$. The coalgebra $\bigcup_{n \in \mathbb{N}} K_n$ having a basis $c_n, n \in \mathbb{N}$ and comultiplication and counit given by these equations is called the divided power coalgebra (see [DNR]). Part of the following Lemma is discussed in [CGT] Theorem 3.2; also part of it in the cocommutative case is observed in [C], 3.5 and 3.6. The same result appears in [LS], but with a different proof. Also Theorem 2.7 below can be obtained as a consequence of the general theory of serial coalgebras developed in [CGT] (Theorem 2.10 (iii) and Remark 2.12); in this respect, Lemma 2.6 could then be obtained as a consequence of Theorem 2.7. We provide here a direct argument.

**Lemma 2.6.** Let $C$ be a finite dimensional chain coalgebra over a field $K$ and suppose that either $K$ is algebraically closed or $C$ is pointed. Then $C$ is isomorphic to $K_n$ for some $n \in \mathbb{N}$.

**Proof.** Let $A = C^*$; we have $\text{dim} C_0 = 1$ because $K$ is algebraically closed (thus $\text{End}_A C_0$ is a skewfield containing $K$). Thus $\text{dim} C_k = k$ for all $k$ for which $C_k \neq 0$. As $C^*$ is finite dimensional $J^n = 0$ for some $n$ and let $n$ be minimal with this property. By Corollary 1.4 $J^k = C_{k-1}$. Then $J^k/J^{k+1}$ has dimension equal to the dimension of $C_k/C_{k-1}$ which is 1 for $k < n$, because $C_{k+1}/C_k$ it is a simple comodule isomorphic to $C_0$. We then have that $J^k/J^{k+1}$ is generated by any of its nonzero elements. Choose $x \in J \setminus J^2$. We prove that $x^{n-1} \neq 0$. Suppose the contrary holds and take $y_1, \ldots, y_{n-1} \in J$. As $x$ generates $J/J^2$, there is $\lambda \in K$ such that $y_1 - \lambda x \in J^2$ and then $y_1 x^{n-2} - \lambda x^{n-1} \in J^n$, so $y_1 x^{n-2} \in J^n = 0$ because $x^{n-1} = 0$. Again, there is $\mu \in K$ such that $y_2 - \mu x \in J^2$ and then $y_1 y_2 - \mu y_1 x \in J^3$ so $y_1 y_2 x^{n-3} \in J^n$ ($y_1 x^{n-2} = 0$). By continuing this procedure, one gets that $y_1 y_2 \cdots y_{n-2} x = 0$ and then we again find $\alpha \in K$ with $y_{n-1} - \alpha x \in J^2$, thus $y_1 \cdots y_{n-1} - \alpha y_1 \cdots y_{n-2} \in J^n = 0$. This shows that $y_1 \cdots y_{n-1} = 0$ for all $y_1, \ldots, y_{n-1} \in J$. Thus $J^{n-1} = 0$, a contradiction.

As $x^{n-1} \neq 0$ we see that $x^k \in J^k \setminus J^{k+1}$ for all $k = 0, \ldots, n-1$, so $J^k/J^{k+1}$ is generated by the class of $x^k$. Now if $y \in A$, there is $\lambda_0 \in K$ such that $y - \lambda_0 \cdot 1_A \in J$ (either $y \in J$ or $y$ generates $A/J$). As $J/J^2$ is 1 dimensional and generated by the image of $x$, there is $\lambda_1 \in K$ such that $y - \lambda_0 - \lambda_1 x \in J^2$. Again, as $J^2/J^3$ is 1 dimensional generated by the image of $x^2$, there is $\lambda_2 \in K$ such that $y - \lambda_0 - \lambda_1 x - \lambda_2 x^2 \in J^3$. By continuing this procedure we find $\lambda_0, \ldots, \lambda_{n-1} \in K$ such that $y - \lambda_0 - \lambda_1 x - \cdots - \lambda_{n-1} x^{n-1} \in J^n = 0$, so $y = \lambda_0 + \lambda_1 x + \cdots + \lambda_{n-1} x^{n-1}$. This obviously gives an isomorphism between $A$ and $K[X]/(X^n)$. Therefore $C$ is isomorphic to $K_n$, because there is an isomorphism of $K$-algebras $K_n^* \simeq K[X]/(X^n)$.

**Theorem 2.7.** If $K$ is an algebraically closed field and $C$ is an infinite dimensional chain coalgebra, then $C$ is isomorphic to the divided power coalgebra. The same conclusion holds provided the infinite dimensional chain coalgebra $C$ is pointed.

**Proof.** By the previous Lemma we have that $C_n \simeq K_n$ for all $n$. If $e \in C_0$, $\Delta(e) = \lambda e \otimes e, \lambda \in K$, then for $c_0 = \lambda e$ we get $\Delta(c_0) = c_0 \otimes c_0$. Suppose we constructed a basis $c_0, c_1, \ldots, c_{n-1}$ for $C_{n-1}$ with $\Delta(c_k) = \sum_{i+j=k} c_i \otimes c_j, \varepsilon(c_i) = \delta_{0,i}$. Denote by $A_n = C_n^*$ the dual of $C_n$; for the rest of this proof, if $V \subseteq C_n$ is a subspace of $C_n$ we write $V^\perp$ for the set of the functions of $A_n$ which are 0 on $V$. Choose $E_1 \in C_0^+ \setminus C_1^+$; then $E_1^\perp \neq 0$ and $E_1^{n+1} = 0$ as in the proof of Lemma 2.6 ($E_1 \in A_n$). This shows that $E_1^\perp \subseteq C_{k-1}^+ \setminus C_k^+$, that $\varepsilon|_{C_n, E_1, \ldots, E_n}$ exhibits a basis for $A_n$ and that there is an isomorphism of algebras $A_n \simeq K[X]/(X^{n+1})$ taking $E_1$ to $X$. We can easily see that $E_1(c_j) = \delta_{ij}, \forall k = 0, 1, \ldots, n-1$ and then by a standard linear algebra result we can find
\[ c_n \in C_n \text{ such that } E_1^n(c_n) = 1 \text{ and } E_1^n(c_j) = 0 \text{ for } i < n. \] Then by dualization, the relations \[ E_1^j(c_j) = \delta_{ij}, \forall i, j = 0, 1, \ldots, n \] become \[ \Delta(c_k) = \sum_{i+j=k} c_i \otimes c_j, \forall k = 0, 1, \ldots, n. \] Therefore we may inductively build the basis \( (c_n)_{n \in \mathbb{N}} \) with \( \varepsilon(c_k) = \delta_{0k} \) and \( \Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j, \forall n. \)

\[ \square \]

**A non-trivial example.** In the following we construct an example of a chain coalgebra that is not cocommutative and thus different of the divided power coalgebra over \( K \). Recall that if \( A \) is a \( k \)-algebra, \( \varphi : A \to A \) is a morphism and \( \delta : A \to A \) is a \( \varphi \)-derivation (that is a linear map such that \( \delta(ab) = \delta(a)b + \varphi(a)\delta(b) \) for all \( a, b \in A \)), we may consider the Ore extension \( A[X, \varphi, \delta] \) which is \( A[X] \) as a vector space and with multiplication induced by \( Xa = \varphi(a)X + \delta(a) \). Let \( K \) be a subfield of \( \mathbb{R} \), the field of real numbers. Let \( D \) be the subalgebra of Hamilton’s quaternion algebra having the set \( B = \{1, i, j, k\} \) as a vector space basis over \( K \). Recall that multiplication is given by the rules \( \bar{i} \cdot \bar{j} = -\bar{j} \cdot \bar{i} = k; \bar{j} \cdot k = -\bar{k} \cdot \bar{j} = i; \bar{k} \cdot i = -\bar{i} \cdot \bar{k} = j; \bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = -1 \). Denote by \( \sigma : D \to D \) the linear map defined on the basis of \( D \) by

\[ \sigma = \begin{pmatrix} 1 & \bar{k} & \bar{j} & \bar{i} \\ \bar{i} & 1 & \bar{k} & \bar{j} \\ \bar{j} & \bar{k} & 1 & \bar{i} \\ \bar{k} & \bar{j} & \bar{i} & 1 \end{pmatrix} \]

It is not difficult to see then that \( \sigma \) is an algebra automorphism, and that \( D \) is a division algebra (skewfield). Our example will be constructed with the aid of such an Ore extension constructed with a trivial derivation: denote by \( D_{\sigma}[X] = D[X, \sigma, 0] \) the Ore extension of \( D \) constructed by \( \sigma \) with the derivation \( \varphi \) equal to 0 everywhere. Then a basis for \( D_{\sigma}[X] \) over \( K \) consists of the elements \( uX^k \), with \( u \in B \) and \( k \in \mathbb{N} \). Also denote by

\[ A_n = D_{\sigma}[X]/\langle X^n \rangle \]

the algebra obtained by factoring out the two-sided ideal generated by \( X^n \) from \( D_{\sigma}[X] \).

**Proposition 2.8.** The two sided ideal \( \langle X^n \rangle \) of \( D_{\sigma}[X] \) consists of elements of the form \( f = \sum_{l=0}^{n+m} a_l X^l \). Moreover, the only (left, right, two-sided) ideals containing \( \langle X^n \rangle \) are the ideals \( \langle X^l \rangle \), \( l = 0, \ldots, n \) and consequently \( A_n \) is a chain \( K \)-algebra.

**Proof.** It is clear by the multiplication rule \( Xa = \sigma(a)X \) for \( a \in B \) that elements of \( D_{\sigma}[X] \) are of the type \( \sum_{l=0}^{n} a_l X^l \) and that every element of \( A_n \) is a "polynomial" of the form \( f = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \), with \( a_i \in D \) and where \( x \) represents the class of \( X \). Such an element \( f \) is invertible if and only if \( a_0 \neq 0 \). To see this, first note that if \( a_0 = 0 \) then \( f \) is nilpotent, as \( x \) is nilpotent and one has \( f^l \in \langle x^l \rangle \) by successively using the relation \( xa = \sigma(a)x \). Conversely write \( f = a_0 \cdot (1 + a_0^{-1}a_1 x + \cdots + a_0^{-1}a_{n-1} x^{n-1}) \) and note that the element \( g = a_0^{-1}a_1 x + \cdots + a_0^{-1}a_{n-1} x^{n-1} \) is nilpotent as before, so \( 1 + g \) must be invertible in \( A_n \) and therefore \( f \) must be invertible. Thus we may write every element \( f = a_0 x^l + \cdots + a_{n-1} x^{n-1} \) of \( A_n \) as the product \( f = (a_0 + a_{l+1} x + \cdots + a_{n-1} x^{n-1-l}) \cdot x^l = g \cdot x^l \) with invertible \( g \). Then if \( I \) is a left ideal of \( A_n \) and \( f \in I \), we have \( f = g \cdot x^l \) for an invertible element \( g \) and some \( l \leq n \). Hence it follows that \( x^l \in I \). Taking the smallest number \( l \) with the property \( x^l \in I \), we obviously have that \( I = \langle x^l \rangle \).

Let \( C_n \) denote the coalgebra dual to \( A_n \). Note that \( A_n \) has a \( K \)-basis \( \mathcal{B} = \{ax^l \mid a \in B, l \in 0, 1, \ldots, n-1 \} \) and we have the relations \( (ax^l)(bx^j) = a\sigma^i(b)x^{l+j} \). Let \( (E_i^a)_{a \in B, i \in 0, \ldots, n-1} \) be the basis of \( C_n \) which is dual to \( \mathcal{B} \), that is, \( E_i^a(bx^j) = \delta_{ij}\delta_{ab} \) for all \( a, b \in B \) and \( i, j \in \mathbb{N} \). Also, for \( i \in \mathbb{N} \) and \( a \in B \) denote by \( i \cdot a = a \sigma^i(a) \) the action of \( \mathbb{N} \) on \( B \) induced by \( \sigma \).
Proposition 2.9. With the above notations, denoting by $\Delta_n$ and $\varepsilon_n$ the comultiplication and respectively, the counit of $C_n$ we have

$$\Delta_n(E^c_p) = \sum_{i+j=p; \; a(i-b)=c} c^{-1}a(i \cdot b)E_i^a \otimes E_j^b$$

and

$$\varepsilon_n(E^c_p) = \delta_{p,0}\delta_{c,1}.$$ 

Proof. For $u, v \in B$ and $k, l \in \mathbb{N}$ we have $E^c_p(ux^k \cdot vx^l) = E^c_p(u(k \cdot v)x^{k+l})$ and as $k \cdot v \in B$ by the formulas defining $D$ we have that if $d = u(k \cdot v)$ then either $d \in B$ or $-d \in B$. Then $E^c_p(ux^k \cdot vx^l) = E^c_p(dx^{k+l}) = \delta_{k+l,p}\delta_{u(k \cdot v),\pm c}c^{-1}u(k \cdot v)$ as the sign of this expression must be 1 if $d \in B$ and $-1$ if $d \notin B$, and this is exactly $c^{-1}u(k \cdot v)$ when $u(k \cdot v) = \pm c$. We also have

$$\sum_{i+j=p; \; a(i-b)=c} c^{-1}a(i \cdot b)E_i^a(ux^k)E_j^b(vx^l) = \sum_{i+j=p; \; a(i-b)=c} \delta_{k,i}\delta_{u,a}\delta_{l,j}\delta_{v,b}c^{-1}a(i \cdot b)$$

$$= \delta_{k+l,p}\delta_{u(k \cdot v),\pm c}c^{-1}u(k \cdot v)$$

and therefore we get

$$\sum_{i+j=p; \; a(i-b)=c} c^{-1}a(i \cdot b)E_i^a(ux^k)E_j^b(vx^l) = E^c_p(ux^k \cdot vx^l)$$

As this is true for all $ux^k, vx^l \in B$, by the definition of the comultiplication of the coalgebra dual to an algebra, we get the first equality in the statement of the proposition. The second one is obvious, as $\varepsilon_n(E^c_p) = E^c_p(1 \cdot X^0) = \delta_{p,0}\delta_{c,1}$. 

Now notice that there is an injective map $C_n \subset C_{n+1}$ taking $E^c_n$ from $C_n$ to $E^c_n$ from $C_{n+1}$. Therefore we can regard $C_n$ as subcoalgebra of $C_{n+1}$. Denote by $C = \bigcup_{n \in \mathbb{N}} C_n$; it has a basis formed by the elements $E^c_n$, $n \in \mathbb{N}, c \in B$ and comultiplication $\Delta$ and counit $\varepsilon$ given by

$$\Delta(E^c_n) = \sum_{i+j=n; \; a(i-b)=c} c^{-1}a(i \cdot b)E_i^a \otimes E_j^b$$

and

$$\varepsilon(E^c_n) = \delta_{n,0}\delta_{c,1}.$$ 

By Proposition 2.8 we have that $A_n$ is a chain algebra and therefore $C_n = A^*_n$ is a chain coalgebra. Therefore, we get that the coradical filtration of $C$ is $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ and that this is a chain coalgebra which is obviously non-cocommutative.

3. The Co-local Case

Throughout this section we will assume (unless otherwise specified) that $C$ has the left f.g. Rat-splitting property and that it is a colocal coalgebra, that is, $C_0$ is a simple left (and consequently simple right) $C^*$-module. Then as $J = C_0^\perp$, $C^*$ is a local algebra. We will also assume that $C$ is not finite dimensional, thus by Proposition 1.10 $C$ has a countable basis. We have that $C$ is the injective envelope of $C_0$ as left comodules, thus by Proposition 1.17 we have that every left submodule of $C$ is finite dimensional (all $C_n$ are finite dimensional). Then if $I$ is a left nonzero ideal of $C^*$ different from $C^*$, by Corollary 1.4 $I$ is finitely generated and of finite codimension. Denote again $R = C^*$. Also for a left $R$-module $M$ denote by $J(M)$ the Jacobson radical of $M$.

Proposition 3.1. With the above notations, $R$ is a domain.
Proof. Let $S = \text{End}(C, C)$. Note that $S$ is a ring with multiplication equal to the composition of morphisms and that $S$ is isomorphic to $R$ by an isomorphism that takes every morphism of left $C$-comodules $f \in S$ to the element $\varepsilon \circ f \in R$. Then it is enough to show that $S$ is a domain. If $f : C \to C$ is a nonzero morphism of left $C$-comodules, then $\text{Ker}(f) \subset C$ is a proper left submodule of $C$ so it must be finite dimensional. Then as $C$ is not finite dimensional we see that $\text{Im}(f) \simeq C/\text{Ker}(f)$ is an infinite dimensional submodule of $C$. Thus $\text{Im}(f) = C$, and therefore every nonzero morphism of left comodules from $C$ to $C$ must be surjective. Now if $f, g \in S$ are nonzero then they are surjective so $f \circ g$ is surjective and thus $f \circ g \neq 0$. □

Proposition 3.2. $R$ satisfies ACCP on right ideals and also on left ideals.

Proof. Suppose there is an ascending chain of right ideals $x_0 \cdot R \subseteq x_1 \cdot R \subseteq x_2 \cdot R \subseteq \ldots$ that is not stationary. Then there are $(\lambda_n)_{n \in \mathbb{N}}$ in $R$ such that $x_n = x_{n+1} \cdot \lambda_{n+1}$. Note that $\lambda_{n+1} \in J$, because otherwise $\lambda_{n+1}$ would be invertible in $R$ as $R$ is local and then we would have $x_{n+1} = x_n \cdot \lambda_n^{-1}$. This would yield $x_n \cdot R = x_{n+1} \cdot R$, a contradiction. Then $x_1 = x_{n+1} \cdot \lambda_{n+1} \lambda_n \ldots \lambda_2$, so $x_1 \in J^n$ for all $n \in \mathbb{N}$, showing that $x_1 \in \bigcap_{n \in \mathbb{N}} J^n = 0$. Thus we obtain a contradiction: $x_0 \cdot R \subsetneq x_1 \cdot R = 0$. The statement is obvious for left ideals as $\alpha R$ is Noetherian. □

The next proposition together with the following theorem contain the main ideas of the result.

Proposition 3.3. Suppose $\alpha R$ and $\beta R$ are two right ideals that are not comparable, i.e. neither one is a subset of the other. Then any two principal right ideals of $R$ contained in $\alpha R \cap \beta R$ are comparable.

Proof. Take $aR, bR \subseteq \alpha R \cap \beta R$, so $a = \alpha x = \beta y$ and $b = \alpha u = \beta v$; we may obviously assume that $a, b \neq 0$ as otherwise the assertion is obvious. Then $\alpha, \beta, x, y, u, v$ are nonzero. Denote by $L$ the left submodule of $R \times R$ generated by $(x, u)$ and by $M$ the quotient module $R \times R / L$. We write $(s, t)$ for the image of the element $(s, t)$ through the canonical projection $\pi : R \times R \to M$. We have $(y, v) \neq (0, 0)$ as otherwise $(y, v) = (\lambda x, u)$ for some $\lambda \in R$; then we would have $y = \lambda x, v = \lambda u$ so $\beta y = \beta \lambda x = \alpha x$ and then $\beta \lambda = \alpha$ (because $R$ is a domain), a contradiction to $\alpha R \subsetneq \beta R$. Also $\beta \cdot (y, v) = \alpha \cdot (x, u) = (0, 0)$ with $\beta \neq 0$. This shows that $(0, 0) \neq (y, v) \in T = T(M)$, so $T(M) \neq 0$. Take $X < M$ such that $M = T \oplus X$. We must have $X \neq 0$, as otherwise $(1, 0) \in T$ so there would be a nonzero $\lambda \in R$ and a $\mu \in R$ such that $\lambda \cdot (1, 0) = \mu \cdot (x, u) \in L$. But then $\lambda = \mu x, 0 = \mu u$, so $\mu = 0$ ($u \neq 0$) showing that $\lambda = 0$, a contradiction.

Now note that $x$ and $u$ are not invertible, as otherwise, for $x$ invertible, $\alpha x = \beta y$ implies $\alpha \in \beta R$ so $\alpha R \subseteq \beta R$; the same can be inferred if $u$ is invertible. Therefore $x, u \in J$ as $R$ is local so $L \subseteq J \times J$. Hence $J(M) = J \times J / L$ so $M/J(M) = \frac{R \times R / L}{J \times J / L} \simeq R \times R / J \times J$ which has dimension 2 as a module over the skewfield $R/J$. Since $M = T \oplus X$ and $M$ is finitely generated, then so are $T$ and $X$ and therefore $J(X) \neq X$ and $J(T) \neq T$. Then as $\frac{M}{J(M)} = \frac{T}{J(T)} \oplus \frac{X}{J(X)}$ has dimension 2 over $R/J$, it follows that both $T/J(T)$ and $X/J(X)$ are simple. Hence $T$ and $X$ are local, and as they are finitely generated, it follows that they are generated by any element not belonging to their Jacobson radical. Let $T'$ (respectively $X'$) be the inverse images of $T$ (and $X$ respectively) in $R \times R$ and $t \in T'$ and $s \in X'$ be such that $R t + L = T'$ and $R s + L = X'$. We have $R \times R = T' + X' = R t + L + R s + L = (R t + R s) + L \subseteq (R t + R s) + J \times J \subseteq R \times R$ so $(R t + R s) + J \times J = R \times R$. Therefore we obtain $R t + R s = R \times R$ because $J \times J$ is small in $R \times R$. 

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Write $t = (p, q) \in T'$. Then $\overline{t} = t + L \in T$ implies that there is $\lambda \neq 0$ in $R$ such that $\lambda \overline{t} = \overline{t} \in M$ and therefore there is $\mu \in R$ with $\lambda(p, q) = \mu(x, u)$. We show that either $p \notin J$ or $q \notin J$. Indeed assume otherwise: $t = (p, q) \in J \times J$. Then we get $Rt \subseteq J \times J$. Because $Rt + Rs = R \times R$ we see that $R \times R/J \times J$ must be generated over $R$ by the image of $s$. This shows that the $R/J$ module $R \times R/J \times J = (R/J)^2$ has dimension 1 and this is obviously a contradiction.

Finally, suppose $p \notin J$ so $p$ is invertible; then the equations $\lambda p = \mu x$, $\lambda q = \mu u$ imply $\lambda = \mu x p$ and $\mu x p q = \mu u$. But $\mu \neq 0$ because $p$ is invertible and $\lambda \neq 0$. Therefore we obtain $u = x p q$; thus $b = \alpha u = \alpha x p q = \alpha x p q$ showing that $b \in aR$ i.e. $bR \subseteq aR$. Similarly if $q$ is invertible, we get $aR \subseteq bR$.

\begin{proof}
We first show that every two principal left ideals of $R$ are comparable. Suppose there are two left ideals of $R$, $R \cdot x_0$ and $R \cdot y_0$ that are not comparable. Then as they have finite codimension and $C^*$ is infinite dimensional, we have $Rx_0 \cap Ry_0 \neq 0$ and take $0 \neq ax_0 = \beta y_0 \in Rx_0 \cap Ry_0$. Then the right ideals $aR$ and $bR$ are not comparable, as otherwise, if for example $aR \subseteq bR'$, we would have a relation $\alpha = \beta \lambda$ so $\alpha x_0 = \beta \lambda x_0 = \beta y_0$. As $\beta \neq 0$ we get $\lambda x_0 = y_0$ because $R$ is a domain, and then $Ry_0 \subseteq Rx_0$, a contradiction.

By Proposition 3.2 the set $\{\lambda R \mid \lambda R \subseteq aR \cap bR\}$ is Noetherian (relative to inclusion) and let $\lambda R$ be a maximal element. If $x \in aR \cap bR$ then by Proposition 3.3 we have that $xR$ and $\lambda R$ are comparable and by the maximality of $\lambda R$ it follows that $xR \subseteq \lambda R$, so $x \in \lambda R$. Therefore $aR \cap bR = \lambda R$. Note that $\lambda \neq 0$, because $aR$ and $bR$ are nonzero ideals of finite codimension. Then we see that $\lambda R \cong R$ as right $R$ modules, because $R$ is a domain, and again by Proposition 3.3 any two principal right ideals of $\lambda R = aR \cap bR$ are comparable, so the same must hold in $R_R$. But this is in contradiction with the fact that $aR$ and $bR$ are not comparable, and therefore the initial assertion is proved.

Now we prove that $J^n/J^{n+1}$ is a simple right module for all $n$. As $R/J$ is semisimple (it is a skewfield) and $J^n/J^{n+1}$ has an $R/J$-module structure, it follows that $J^n/J^{n+1}$ is a semisimple left $R/J$-module and then $J^n/J^{n+1}$ is semisimple also as $R$-module. If we assume that it is not simple, then there are $f,g \in J^n \setminus J^{n+1}$ such that $R \hat{f} = (Rf + J^{n+1})/J^{n+1}$ and $R \hat{g} = (Rg + J^{n+1})/J^{n+1}$ are different simple $R$-modules, so $R \hat{f} \cap R \hat{g} = 0$ in $J^n/J^{n+1}$. Then $(Rf + J^{n+1}) \cap (Rg + J^{n+1}) = J^n$ which shows that $Rf$ and $Rg$ cannot be comparable, a contradiction. As $J^n = C_{n-1}^\perp$, we see that $\dim(C_{n-1}) = \dim(J^n)$. Then for $n \geq 1$, $\dim(C_n/C_{n-1}) = \dim(J_n) - \dim(C_{n-1}) = \dim(J^n) - \dim(J^{n+1}) = \dim(J^n/J^{n+1}) = \dim(C_0)$. Because $C_0$ is the only type of simple right $C$-comodule, this last relation shows that the right $C$-comodule $C_n/C_{n-1}$ must be simple. Therefore $C$ must be a chain coalgebra.

We may now combine the results of Sections 2 and 3 and obtain

\begin{corollary}
Let $C$ be a co-local (infinite dimensional) coalgebra. Then $C$ is a left (right) finite splitting coalgebra if and only if $C$ is a chain coalgebra. Moreover, if the base field $K$ is algebraically closed or the coalgebra $C$ is pointed, then this is further equivalent to the fact that $C$ is isomorphic to the divided power coalgebra.
\end{corollary}

\begin{proof}
This follows from Theorems 2.5, 2.7 and 3.4.
\end{proof}
4. Serial coalgebras and General Examples

In this section we provide some nontrivial general examples of non-colocal coalgebras for which this splitting property holds.

Lemma 4.1. Let \( C = D \oplus E \) be coproduct of two coalgebras \( D \) and \( E \). Then \( C \) has the left f.g. Rat-splitting property if and only if \( D \) and \( E \) have the Rat-splitting property.

Proof. Assume \( C \) has the left f.g. Rat-splitting property. It is well known that the category of modules over \( C^* \cong D^* \times E^* \) is isomorphic to the product of the category of \( D^* \)-modules with that of \( E^* \)-modules; in this respect, if \( M \) is a left \( C^* \)-module, then \( M = N \oplus P \) where \( N = E^\perp \cdot M \), \( P = D^\perp \cdot M \) are \( C^* \) submodules that have an induced \( D^* = C^*/D^\perp \)- and respectively \( E^* = C^*/E^\perp \)-module structure (since \( D^\perp \cdot N = E^\perp \cdot P \)). Also, one can check that a \( D^* \)-module \( X \) is rational if and only if it is rational as \( C^* \)-module with its induced \( C^* \)-module structure; if \( \rho : X \to X \otimes C \) is a \( C \)-comultiplication then we must have \( \rho(X) \subseteq X \otimes D \) since \( D^\perp \) cancels \( X \), and \( \rho \) becomes a \( D \)-comultiplication. Indeed, if \( \rho(x) = \sum_i x_i \otimes y_i + \sum_j x'_j \otimes y'_j \) with \( x_i, x'_j \in X \) assumed linearly independent, \( y_i \in D \) and \( y'_j \in E \), then for any \( e^* \in C^* \) such that \( e^*|_D = 0 \), we have \( 0 = e^* \cdot x = \sum_j e^*(y'_j)x'_j \), so \( e^*(y'_j) = 0 \) by linear independence. This shows that \( x'_j \in (D^\perp)^1 = D \) so \( x'_j = 0 \) for all \( j \). Thus, we obtain that \( \text{Rat}(D \cdot N) = \text{Rat}(C \cdot N) \) and \( \text{Rat}(E \cdot P) = \text{Rat}(C \cdot P) \), and we have direct sums \( N = \text{Rat}(N) \oplus N' \) and \( P = \text{Rat}(P) \oplus P' \) in \( D^* \cdot M \) and \( E^* \cdot M \); but \( N' \) and \( P' \) also have an induced \( C^* \)-module structure with \( E^* = D^\perp \) acting as 0, and we finally observe that this yields a direct sum of \( C^* \) modules \( M = \text{Rat}(C \cdot N) \oplus N' \oplus \text{Rat}(C \cdot P) \oplus P' = \text{Rat}(C \cdot M) \oplus (N' \oplus P') \).

The other implication follows from Proposition \( \text{I.9} \). \( \square \)

We note now the following proposition which was also proved in \( \text{C} \), but with techniques involving general results of M. Teply from \( \text{I}1 \) and \( \text{I}3 \).

Proposition 4.2. Assume \( C \) is a cocommutative coalgebra. Then \( C \) is is a f.g. Rat-splitting coalgebra if and only if it is a finite coproduct of finite dimensional coalgebras and infinite dimensional chain coalgebras. Moreover, these chain coalgebras are isomorphic to the divided power coalgebra in any of the cases:

(i) the base field is algebraically closed;

(ii) \( C \) is pointed.

Proof. Since \( C \) is cocommutative, \( C = \bigoplus_{i=1}^n C_i \), where \( C_i \) are colocal subcoalgebras of \( C \). Now each of the \( C_i \) must have the splitting property for finitely generated modules by Proposition \( \text{I.9} \) and therefore they must be either finite dimensional or be chain coalgebras. The converse follows from the previous Lemma and the results of Section 2. The final assertion comes from Theorem \( \text{I.7} \). \( \square \)

Recall, for example from \( \text{E} \), 25.1.12 that a module \( M \) is called serial if it is a direct sum of uniserial (chain) modules; a ring \( R \) is said to be left (right) serial if \( R \) is serial when regarded as left (right) \( R \)-module, and serial when \( R \) is both left and right serial. In analogy to these definitions, for a \( C \)-comodule \( M \) we say that \( M \) is serial if it is serial when regarded as \( C^* \)-module (so it is a direct sum of serial -or chain- comodules). A coalgebra will be called left (right) serial if and only if it is serial as a right (left) \( C^* \)-module, i.e. as a left (right) \( C \)-comodule, and serial if it is both left and right serial. These definitions coincide with those in \( \text{CGT} \). We note at this point that in our definitions, a
uniserial coalgebra is the same as a chain coalgebra, while a uniserial coalgebra in \([\text{CGT}]\) is understood as a homogeneous uniserial coalgebra, that is, a coalgebra \(C\) that is serial and the composition factors of each indecomposable injective comodule are isomorphic (see Definition 1.3 \([\text{CGT}]\)). The following is a generalization of Proposition 1.6, \([\text{CGT}]\).

**Proposition 4.3.** Let \(C\) be a coalgebra. Then the following are equivalent:

(i) \(C\) is a right serial coalgebra and \(C_0\) is finite dimensional.

(ii) \(C^*\) is a right serial algebra.

Consequently \(C^*\) is serial if an only if \(C\) is serial and \(C_0\) is finite dimensional, equivalently, \(C\) is serial and \(C^*\) is semilocal.

**Proof.** (i)⇒(ii) Let \(C_0 = \bigoplus_{i=1}^{k} S_i\) be a decomposition of \(C_0\) into simple right comodules, \(E(S_i)\) be an injective envelope of \(S_i\) contained in \(C\); then \(C = \bigoplus_{i=1}^{k} E(S_i)\) in \(\mathcal{M}^C\) and \(C^*\)-\(\mathcal{M}\). Since any other decomposition of \(C\) in \(\mathcal{M}^C\) is equivalent to this one, we have that \(E(S_i)\) are chain comodules and then \(E(S_i)^*\) are chain modules by Proposition 2.2. As \(C^* = \bigoplus_{i=1}^{n} E(S_i)^*\) in \(\mathcal{M}_{C^*}\) we get that \(C^*\) is right serial.

(ii)⇒(i) If \(C^*\) is right serial, it is a direct sum of uniserial modules \(C^* = \bigoplus M_i\), each of which has to be cyclic; then we easily see that these modules have to be local (for example by [F], 25.4.1B) and indecomposable (a finitely generated local module is indecomposable). Since there can be only a finite number of \(M_i\)'s in a decomposition of \(C^*\), and each of the \(M_i\)'s are local we get that \(C^*\) is semilocal, and then \(C^*/J\) is semisimple \((J = C_0^\perp)\). But \(C^*/J = C^*/C_0^\perp = C_0^\ast\) and thus \(C_0\) is cosemisimple finite dimensional. Then \(C^* = \bigoplus_{i=1}^{k} M_i\) with \(M_i\) local uniserial. Let \(E_i = (\bigoplus_{j \neq i} M_j)^\perp\); since \(\bigoplus M_j\) is finitely generated, it is closed in the finite topology of \(C^*\) and therefore \(E_i^\perp = \bigoplus_{j \neq i} M_j\), so \(E_i^\ast \simeq C^*/E_i^\perp = C^*/(\bigoplus_{j \neq i} M_j) \simeq M_i\). Then by Proposition 2.2 we get that \(E_i\) is a right chain \(C\)-comodule; also because of the anti-isomorphism of lattices between the right subcomodules of \(C\) and closed right \(C^*\)-modules of \(C^*\) (see [DNR] or [10], Theorem 1), we get that \(C = \bigoplus_{i=1}^{k} E_i\), with \(E_i\) right chain comodules. Thus \(C\) is a left serial coalgebra. \(\Box\)

We say that a coalgebra \(C\) is purely infinite dimensional serial if it is serial and the uniserial left (and also the uniserial right) comodules into which it decomposes are infinite dimensional. Equivalently, one can say that injective envelopes of any left (and also every right) simple \(C\)-comodule is infinite dimensional. It is not difficult to see that for an almost connected coalgebra it is enough to ask that only left injective envelopes are infinite dimensional: let \(C = \bigoplus_{i=1}^{k} E(S_i)\) be a decomposition of \(C\) with \(S_i\) simple left comodules and \(E(S_i)\) an injective envelope for each \(S_i\). Assume \(C\) is serial; then each \(E(S_i)\) is uniserial.

Then if \(L_n E(S_i)\) is the \(n\)-th term in the Loewy series of \(E(S_i)\) then \(C_n = \bigoplus_{i=1}^{k} L_n E(S_i)\) and \(E(S_i)\) is infinite dimensional for all \(i\) if and only if \(L_n E(S_i) \neq L_{n-1} E(S_i)\) for all \(i\) and all \(n \geq 0\) \((L_{-1} = 0)\), equivalently, \(C_n/C_{n-1} \simeq \bigoplus_{i=1}^{k} L_n E(S_i)/L_{n-1} E(S_i)\) has length \(k\).
(as a module) for all \( n \). Since this last condition is a left-right symmetric condition, the assertion follows. The next proposition provides the general example of this section:

**Proposition 4.4.** Let \( C \) be a purely infinite dimensional serial coalgebra which is almost connected. Then \( C \) has the left (and also the right) \( f.g. \) Rat-splitting property.

**Proof.** By the previous proposition, \( C^* \) is serial. Let \( M \) be a left finitely generated \( C^* \)-module. Let \( C = \bigoplus_{i=1}^{k} E(S_i) \) be a decomposition as above, in \( C\mathcal{M} \), with \( E(S_i) \) chain comodules; then \( C^* = \bigoplus_{i \in I} E(S_i)^* \) in \( C^*\mathcal{M} \). By Remark 2.3 and Proposition 1.3 the \( E(S_i)^* \)'s are noetherian. Hence \( C^* \) is Noetherian (both left and right, since \( C \) is left and right serial). This shows that every finitely generated \( C^* \)-module is also finitely presented. Then, by \( [\mathcal{E}] \), Corollary 25.3.4, \( M = \bigoplus_{j=1}^{n} M_j \) with \( M_j \) cyclic uniserial left \( C^* \)-modules. For each \( j \) there are two possibilities:

- **\( M_j \) is finite dimensional.** Let \( m_j \) be a generator of the left \( C^* \)-module \( M_j \), and then let \( I = \text{ann}_{C^*}(m_j) \). Then \( I \) is a left ideal of \( C^* \) and it is finitely generated (\( C^* \) is Noetherian), so \( I = X^\perp \), \( X \subseteq C \) (Lemma 1.1). Moreover, \( C^*/I \simeq C^* \cdot m_j = M_j \) and so \( I \) has finite codimension since \( M_j \) is finite dimensional. Hence \( X \) is finite dimensional and is a left subcomodule of \( C \). Then \( M_j \simeq C^*/X^\perp \simeq X^* \), following that \( M_j \) is rational as a dual of the rational right \( C^* \)-module \( X \). So \( \text{Rat}(M_j) = M_j \).

- **\( M_j \) is infinite dimensional.** Let \( m_j \) be a generator of \( M_j \) as before, and \( S = M_j/J(M_j) \) which is a simple module because \( M_j \) is local since it is cyclic and uniserial. Let \( P_i = E(S_i)^* \); since \( C^*/J = \bigoplus_{i=1}^{k} P_i/JP_i \) and \( J \) is local, there is some \( i \) such that \( P_i/JP_i \simeq S \). Then we have a diagram

\[
\begin{array}{ccc}
M_j & \to & S \\
\pi \downarrow & & \downarrow p \\
P_i & \to & 0
\end{array}
\]

completed commutatively by \( u \) since \( P_i \) is projective, and \( p, \pi \) are the canonical maps. Note that \( u \) is surjective, since otherwise \( \text{Im}(u) \subseteq \text{Ker}(\pi) \) because \( \text{Ker}(\pi) \) is the only maximal submodule of the finitely generated module \( M_i \). This cannot happen since \( \pi u = p \neq 0 \). By Remark 2.3 and Proposition 1.3 we see that any nonzero submodule of \( P_i = E(S_i)^* \) has finite codimension. Then if \( \text{Ker}(u) \neq 0 \), \( M_j = \text{Im}(u) \simeq P_i/\text{Ker}(u) \) would be finite dimensional, which is excluded by the hypothesis on \( M_j \). This shows that \( u \) is an isomorphism so \( M_j \simeq E(S_j)^* \) and we now get that \( M_j \) has no finite dimensional submodules besides \( 0 \) (again by Remark 2.3 and Proposition 1.3). This shows that \( \text{Rat}(M_j) = 0 \).

Finally, if we set \( \mathcal{F} = \{ j \mid M_j \text{ finite dimensional} \} \), we see that \( \text{Rat}(M) = \bigoplus_{j=1}^{n} \text{Rat}(M_j) = \bigoplus_{j \in \mathcal{F}} M_j \), and this shows that \( \text{Rat}(M) \) is a direct summand in \( M = \bigoplus_{j=1}^{n} M_j \). \( \square \)

**Example 4.5.** Let \( K \) be a field, \( q \geq 1 \) and \( \sigma \in S_q \) be a permutation of \( \{1, 2, \ldots, q\} \). Denote by \( K[x] \) the vector space with basis \( x_{p,n} \) with \( p \in \{1, 2, \ldots, q\} \) and \( n \geq 0 \). Define
a comultiplication $\Delta$ and a counit $\varepsilon$ on $K^q_\sigma[X]$ as follows:

$$\Delta(x_{p,n}) = \sum_{i+j=n} x_{p,i} \otimes x_{\sigma^i(p),j}$$

$$\varepsilon(x_{p,n}) = \delta_{n,0}, \quad \forall p \in \{1, 2, \ldots, q\}, \; n \geq 0$$

It is easy to see that $\Delta$ is coassociative and $\varepsilon$ becomes a counit, so $K^q_\sigma[X]$ becomes a coalgebra:

$$(\Delta \otimes I)\Delta(x_{p,n}) = (\Delta \otimes I)(\sum_{i+j=n} x_{p,i} \otimes x_{\sigma^i(p),j})$$

$$= \sum_{i+j=n} \sum_{s+t=i} x_{p,s} \otimes x_{\sigma^s(p),t} \otimes x_{\sigma^t(p),j}$$

$$= \sum_{s+t+j=n} x_{p,s} \otimes x_{\sigma^s(p),t} \otimes x_{\sigma^s(p),j}$$

$$= \sum_{s+u=n} x_{p,s} \otimes \sum_{t+j=u} x_{\sigma^s(p),t} \otimes x_{\sigma^t(p),j}$$

$$= (I \otimes \Delta)(\sum_{s+u=n} x_{p,s} \otimes x_{\sigma^s(p),u})$$

$$= (I \otimes \Delta)\Delta(x_{p,n})$$

Also, we have

$$\sum_{i+j=n} \varepsilon(x_{p,i}) x_{\sigma^i(p),j} = \sum_{i=0}^{n} \delta_{i,0} x_{\sigma^i(p),n-i} = x_{p,n} \quad \text{and} \quad \sum_{i+j=n} x_{p,i} \varepsilon(x_{\sigma^i(p),j}) = \sum_{i+j=n} x_{p,i} \varepsilon(x_{\sigma^i(p),j}) = x_{p,n},$$

showing that $K^q_\sigma[X]$ together with these morphisms is a coalgebra.

Let $E_p$ be the vector subspace of $K^q_\sigma[X]$ with basis $x_{p,n}$, $n \geq 0$. Note that the $E_p$'s are right subcomodules of $K^q_\sigma[X]$ (obviously by the definition of $\Delta$ and $\varepsilon$). We show $E_p$ are chain comodules in several steps:

(i) Let $E_{p,n} = \langle x_{p,0}, x_{p,1}, \ldots, x_{p,n} \rangle$ be the space with basis $\{x_{p,0}, x_{p,1}, \ldots, x_{p,n}\}$; it is actually a right subcomodule of $E_p$. We note that $E_p/E_{p,n} \simeq E_{\sigma^{n+1}(p)}$. Indeed, if $\overline{x}$ denotes the image of $x \in E_p$ in $E_p/E_{p,n}$, we have the following formulas for the comultiplication of $E_p/E_{p,n}$

$$\overline{x_{p,m}} \rightarrow \sum_{i+j=m,n+1} \overline{x_{p,i}} \otimes \overline{x_{\sigma^i(p),j}} = \sum_{i+j=m-n-1} \overline{x_{p,i+n+1}} \otimes \overline{x_{\sigma^i(\sigma^{n+1}(p)),j}}$$

for $m \geq n + 1$. The comultiplication of $E_{\sigma^{n+1}(p)}$ is given by the formulas:

$$x_{\sigma^{n+1}(p),s} \rightarrow \sum_{i+j=s} x_{\sigma^{n+1}(p),i} \otimes x_{\sigma^i(\sigma^{n+1}(p)),j}$$

These relations show that the correspondence $\overline{x_{p,i+n+1}} \rightarrow x_{\sigma^{n+1}(p),i}$ is an isomorphism of $K^q_\sigma[X]$-comodules.

(ii) Let $x = \lambda_0 x_{p,0} + \lambda_1 x_{p,1} + \cdots + \lambda_n x_{p,n} \in E_p$ and assume $\lambda_n \neq 0$. Let $f \in K^q_\sigma[X]^*$ be equal to 1 on $x_{p,n}$ and 0 on the rest of the elements of the basis $x_{t,i}$. Then one easily sees that $f \cdot x = \sum_{i+j \leq n} \lambda_i x_{p,i} f(x_{\sigma^i(p),j}) = \lambda_n x_{p,0}$ (the only terms remaining are the ones having $j = n$, $i = 0$, and such a term occurs only once in this sum). Since $\lambda_n \neq 0$, we get that $x_{p,0}$ belongs to the submodule generated by $x$. This shows that $E_{p,0}$ is contained in any submodule of $E_p$. This shows that $E_p$ is colocal and $E_{p,0}$ is its socle (which is a simple comodule).

(iii) An inductive argument now shows that $E_{p,n}$ are chain comodules for all $n$. Indeed, by
the isomorphism in (i) and by (ii), we have that $E_{p,n+1}/E_{p,n} \simeq E_{\sigma^{n+1}(p),0}$. This shows that $E_p$ is a chain comodule by Proposition 2.2.

Since $K^q_\sigma[X] = \bigoplus_{p=1}^q E_p$ as right $K^q_\sigma[X]$-comodules, we see that $K^q_\sigma[X]$ is right serial, so it is serial by Proposition 4.3 and even purely infinite dimensional, and thus constitutes an example of a left and right f.g. Rat-splitting coalgebra by Proposition 4.4.

More examples can be obtained by

**Corollary 4.6.** If $C = D \oplus E$ where $D$ is a finite dimensional coalgebra and $E$ is a purely infinite serial dimensional coalgebra, then $C$ has both the left and the right f.g. Rat-splitting property.

**Remark 4.7.** The fact that $K^q_\sigma[X]$ is also left serial (and then purely infinite dimensional) can also follow by noting that $K^q_\sigma[X]^{op} \simeq K^q_{\sigma^{-1}}[X]$ as coalgebras. It is also interesting to note that if $\sigma = \sigma_1 \ldots \sigma_r$ is a decomposition of $\sigma$ into disjoint cycles of respective lengths $q_1, \ldots, q_r$ (or, more generally, into mutually commuting permutations), then there is an isomorphism of coalgebras

$$K^q_\sigma[X] \simeq \bigoplus_{i=1}^r K^{q_{\sigma_i}}[X]$$

We omit the proofs here. As a final comment, we note that by the above results, some natural questions arise: is the concept of f.g. Rat-splitting left-right symmetric? That is, does the left f.g. Rat-splitting property of a coalgebra also imply the right f.g. Rat-splitting property? One should note that all the above examples have both the left and the right Rat-splitting property. Also, it would be interesting to know whether a generalization of the results in the local case hold in the general non-cocommutative case as the cocommutative case of this section and the above non-cocommutative examples seem to suggest: if $C$ has the left f.g. Rat-splitting property, can it be written as a direct sum of finite dimensional injectives and infinite dimensional chain injectives (likely in $\mathcal{CM}$), or maybe a decomposition of coalgebras as in Corollary 4.6, and to what extent such a decomposition would characterize this property?

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