A minimal model for vertical shear instability in protoplanetary accretion disks

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ABSTRACT
The vertical shear instability is an axisymmetric effect suggested to drive turbulence in the magnetically inactive zones of protoplanetary accretion disks. Here we examine its physical mechanism in analytically tractable “minimal models” in three settings that include a uniform density fluid, a stratified atmosphere, and a shearing-box section of a protoplanetary disk. Each of these analyses show that the vertical shear instability’s essence is similar to the slantwise convective symmetric instability in the mid-latitude Earth atmosphere, in the presence of vertical shear of the baroclinic jet stream, as well as mixing in the top layers of the Gulf Stream. We show that in order to obtain instability, the fluid parcels’ slope should exceed the slope of the mean absolute momentum in the disk radial-vertical plane. We provide a detailed and mutually self-consistent physical explanation from three perspectives: in terms of angular momentum conservation, as a dynamical interplay between a fluid’s radial and azimuthal vorticity components, and from an energy perspective involving a generalised Solberg-Høiland Rayleigh condition. Furthermore, we explain why anelastic dynamics yields oscillatory unstable modes and isolate the oscillation mechanism from the instability one.

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1. Introduction
Fifty years of spectroscopic observations of young stellar objects (YSO) reveal them to exhibit strong UV excess indicative of some type of anomalous torque inducing process driving transport of mass and angular momentum within the YSO’s surrounding accretion disk. There are a variety of processes that might give rise to this phenomenon, but highest on the list are winds and turbulence. In the latter case, the source of turbulence depends upon the temperature and ionisation state of the accretion disk gas. Around relatively cold planet-forming “protoplanetary” disks, where the degree of ionisation is practically zero, identifying candidate turbulence generating mechanisms had remained stubbornly elusive until the last 5–10 years where three viable linear instability processes have been identified (for detailed discussion of all mechanisms and their context in

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planet-forming disks, see, Lyra and Umurhan 2019). Of these, the vertical shear instability (“VSI” hereafter) (Urpin and Brandenburg 1998, Arlt and Urpin 2004, Nelson et al. 2013, Stoll and Kley 2014) appears to be most relevant inside the 5–50 AU “Ohmic Zone” (OZ hereafter) of a protoplanetary disk like our own solar nebula (Malygin et al. 2017) – the zone where the cores of the gas and ice giants, as well as the Kuiper Belt objects, were likely assembled from 20–100 km sized planetesimals. Turbulence affects the way planetesimals are manufactured (e.g. Chen and Lin 2020, Hartlep and Cuzzi 2020, Umurhan et al. 2020) and given that the VSI appears likely to be central in driving turbulence in the outer solar system, understanding how it operates from a mechanistic view is essential for understanding how it interacts with planetesimal formation scenarios.

The VSI is the zero Prandtl number, rapid thermal relaxation time limit of the Goldreich-Schubert-Fricke instability (Goldreich and Schubert 1967, Fricke 1968, “GSF” hereafter), which has been examined as a possible mixing-agent in radiative zones of differentially rotating stars. Knobloch and Spruit (1982) show that the velocity displacement of unstable GSF modes are inclined, lying between the rotation axis and surfaces of constant angular momentum (see also Latter and Papaloizou 2018). Knobloch and Spruit (1982) further explain that while buoyancy acts to stabilise these modes, sufficiently small enough radial scales can become GSF unstable by buoyancy suppression brought about by increased thermal diffusivity (see also Lin and Youdin 2015). VSI is essentially an instability of inertial modes that arises in a strongly rotating disk flow supporting a radial temperature gradient. This temperature profile induces a vertical shear in the azimuthal (Keplerian) flow of the gas around the central star. This vertical shear arises in the same manner as does the thermal wind solution for the Earth. Furthermore, the VSI appears to be analogous to the long-studied Symmetric Instability thought to be an important mixing-agent for mesoscale and sub-mesoscale dynamics in both the Earth’s oceans (like the Gulf Stream) and atmosphere (mid-latitudes) (e.g. Thomas et al. 2013, Stamper and Taylor 2017, and other studies cited within). An important ingredient for the VSI’s operation is that the thermal relaxation times are very short compared to the local orbital times (nominally quantifying the frequency timescales of the unstable carrier inertial modes, see also, Nelson et al. 2013). In this sense, the instability is thermally driven and its energy is ultimately derived by the star’s light and is stabilised with increased vertical stable stratification Lin and Youdin 2015. The exact location in a disk with conditions favourable for the VSI to dominate over other magnetic and/or purely hydrodynamic instabilities is still under debate, as these criteria depend open both the disk model and the thermodynamic response of the disk, both of which are still not fully understood (Malygin et al. 2017, Lyra and Umurhan 2019, Pfeil and Klahr 2019). While VSI surface modes exhibit approximately similar – if not slightly weaker – growth rates compared to the body modes (Barker and Latter 2015), it was argued in Umurhan et al. (2016) that if one were to project the energy of a nonlinearly developed state of the VSI onto the set of eigenmodes of the system (as, for example, observed in simulations), one would find that majority of that energy would project onto the lowest order body modes. In this work, we therefore relate mostly to the body modes which are semi-global in the sense that they span the disk’s full vertical extent, i.e. several pressure scale heights ($H$), while its radial scales are fractions of a scale height ($\sim \epsilon H$, where $\epsilon$ is the disk opening angle).

1 i.e. Ohmic zones are regions in the disk where Ohmic resistivity is so large that MHD effects are effectively inoperative.
Despite the VSI appearing to be central for OZ dynamics across large swaths of the solar nebula, its physical mechanism is not fully understood. Identifying how the VSI works and understanding it in relation to other similar processes in planetary atmospheres – and perhaps establishing that it may be a member of a super-class of processes – is in its own right a desirable physics-oriented intellectual end. The instability is often explained in terms of the semi-quantitative Solberg-Høiland generalised Rayleigh-like condition (Umurhan et al. 2013, Stoll and Kley 2014, Barker and Latter 2015, Manger and Klahr 2018), which examines the net energy obtained when pairs of fluid parcels exchange positions – which might place the process into the same Taylor-Couette family of instabilities.

Our aim here is to clarify the mechanistic understanding of VSI by analysing an analytically tractable minimal model (section 2). Similar to the Symmetric Instability mechanism (e.g. Holton and Hakim 2012) in the Earth mid-latitudinal atmosphere, the VSI can be viewed from angular momentum (section 3), vorticity (section 4) and energetic perspectives (section 5). We show that all perspectives boil down to the condition that the fluid parcel’s slope trajectory must exceed the mean absolute momentum slope in the disk’s radial-vertical plane. Only then the gain in the azimuthal velocity via vertical advection of the mean shearing overcomes the loss via the Coriolis deceleration due to radial motion.

2. Minimal model for VSI

Our starting point is the shearing box linearised asymptotic reduced set of equations, introduced by Nelson et al. (2013). For the detailed careful steps leading to the reduced model, the reader is kindly referred to the appendices of that paper. The equation set is derived by assuming that the spatial and temporal scales of motion are related to one another according to the following: temporal dynamics are given by $O(1/\epsilon \Omega_0)$, radial dynamic scales $x$ are $O(\epsilon^2 R_0)$, and the vertical scales $z$ are on the scale height $H_0 = O(\epsilon R_0)$, in which the small parameter $\epsilon \equiv H_0 / R_0$ measures the disk opening angle. Note specifically that in this reduced model the stability effect of vertical buoyancy is neglected. Written in the slightly more general form the reduced model is governed by the following set of equations

$$0 = C_x v - \frac{\partial \Pi}{\partial x}, \quad (1a)$$
$$\dot{v} = -C_y u + \bar{S}(z)w, \quad (1b)$$
$$\dot{w} = -\frac{\partial \Pi}{\partial z}, \quad (1c)$$
$$\frac{\partial}{\partial x} [\bar{\rho}(z)u] + \frac{\partial}{\partial z} [\bar{\rho}(z)w] = 0, \quad (1d)$$

where dots over quantities denote $\partial / \partial t$. Here we look at a box within the upper part of the disk where the Cartesian coordinates $(x, y, z)$ correspond approximately to the (radial, azimuthal, vertical) directions (figure 1(a)). $(u, v, w)$ are the corresponding perturbation velocity components and $\Pi$ is the pressure perturbation, scaled by a reference density. $(C_x, C_y) = \Omega_0(2, 1/2)$ are the local $(x, y)$ components of the Coriolis parameter, taken to be constant within the box, where $\Omega_0$ is the Keplerian rotation frequency at the box centre so that $C_y$ is equal to the negative value of the radial Keplerian shear within the box. $\bar{S}(z) = -\partial \bar{V} / \partial z > 0$ represents the negative vertical shear of the azimuthal mean flow.
(denoted by overbar) in the upper part of the disk (as the Keplerian flow decays away from the mid-plane) and $\bar{\rho}(z)$ is the mean density profile. We consider dynamics without variations in the azimuthal direction ($\partial / \partial y(\cdot) = 0$).

The careful scale analysis of Nelson et al. (2013) suggests the approximation in which the azimuthal perturbation velocity, $v$, is in geostrophic balance (1a). $v$ can change both due to radial and vertical motions (1b). Inner radial flow ($u < 0$) generates azimuthal acceleration due to the Coriolis force and upward motion ($w > 0$) advects large Keplerian azimuthal flow from the mid-plane. Equation (1c) states that vertical acceleration results from the vertical pressure gradient deviation from the mean hydrostatic balance. Finally, the continuity equation (1d) is assumed anelastic where only the vertical variations in the mean density are taking into account.

The simplest solution of (1a–d), admitting the simplest instance of the VSI, is obtained when both $\bar{\rho}$ and $\bar{S}$ are taken as constants ($\rho_0$, $S_0$). Straightforward manipulations reduces the equation set (1) into a single constant coefficient partial differential equation for the pressure perturbation

$$\frac{\partial^2 \tilde{\Pi}}{\partial x^2} = -C_x \left( S_0 \frac{\partial^2 \Pi}{\partial x \partial z} + C_y \frac{\partial^2 \Pi}{\partial z^2} \right). \quad (2)$$

Substituting a plane-wave solution of the form of $\exp[i(kx + mz - \omega t)]$ into the above yields the dispersion relation

$$\omega^2 = \frac{S_0 C_x}{\alpha^2} (\alpha_c - \alpha), \quad (3)$$

with $\alpha \equiv -k/m$. Equation (1d) indicates then that $\alpha = w/u$ is the fluid parcel slope in the $(x, z)$ plane. Hence, instability is obtained when:

$$\frac{w}{u} = \alpha > \alpha_c = \frac{C_y}{S_0} > 0, \quad (4)$$

i.e. when the parcel trajectory slopes are tilted upward and outward with a tilt that is larger than the critical slope of $C_y/S_0 = \Omega_0/2S_0$, which is the ratio between the absolute values of the azimuthal Keplerian shear and the vertical mean shear (figure 1(b))²

This critical slope is the slope of the mean azimuthal absolute momentum surfaces, as explained in the following subsection.

² From reflection symmetry, this condition holds as well for the lower part of the disk where $S_0$ is generally negative.
3. Absolute momentum perspective

Denote the axisymmetric dynamic total azimuthal velocity (deviating from the basic Keplerian disk flow) as \( V(x, z) = \bar{V}(z) + \nu(x, z) \), then the linearised momentum equation in the radial direction (1b) reads

\[
\frac{D}{Dt} V = -C_y u = -C_y \frac{D}{Dt} x,
\]

where \( \frac{D}{Dt} = \partial/\partial t + u \cdot \nabla \) is the material time derivative. Define \( M \equiv V + C_y x \) as the absolute azimuthal linear momentum (per unit mass), equation (5) results then from the material conservation of \( M \)

\[
\frac{D}{Dt} M = 0.
\]

Define the mean absolute momentum as \( \bar{M}(x, z) = \bar{V}(z) + C_y x \) and assess variations of its constituent terms along surfaces of constant \( \bar{M} \). Thus

\[
\delta \bar{M} = 0 = \frac{\partial \bar{M}}{\partial x} \delta x + \frac{\partial \bar{M}}{\partial z} \delta z = C_y \delta x - S_0 \delta z
\]

yields the critical slope

\[
\left( \frac{\delta z}{\delta x} \right)_{\bar{M}} = \alpha_c.
\]

Furthermore, writing \( M = \bar{M} + \text{Green} m \), then the perturbation azimuthal absolute momentum is simply \( m = \nu \). Equation (1b) is then the linearised version of (6), namely

\[
\dot{\nu} = \dot{m} = -u \cdot \nabla \bar{M} = - \left( u \frac{\partial \bar{M}}{\partial x} + w \frac{\partial \bar{M}}{\partial z} \right) = -C_y u + S_0 w = \left( -\frac{\alpha_c}{\alpha} + 1 \right) S_0 w,
\]

where for the plane-wave solution (1c) becomes

\[
\dot{w} = \frac{C_x}{\alpha} \nu.
\]

Hence, mutual amplification between \( \nu \) and \( w \) is obtained when the gain in the azimuthal velocity \( \nu \), via vertical advection of the mean shear (the term \( S_0 w \) in (9)), overcomes its loss via the Coriolis deceleration due to radial motion (the term \( -C_y u \)). This is equivalent to negative advection of the mean absolute momentum by the perturbation (figure 2) as fluid parcels from an initial absolute momentum surface of \( \bar{M} \), are displaced to the absolute momentum surface of \( \bar{M} - \delta \bar{M} \), thus generating a positive anomaly \( m \). We note as well that for instability the three components of the velocity should have the same sign (recall that \( u = w/\alpha \) and \( \alpha > 0 \)).

Note that for an anomalous case where the vertical shear is reversed (\( S < 0 \)), the slope of \( \bar{M} \) remains as in figure 2; however, now \( \partial \bar{M}/\partial z > 0 \). In this case (9) and (10) indicate that instability may be achieved for negative parcel’s slope (\( \alpha < 0 \)), under the condition \( |S| > C_y/|\alpha| \).
Figure 2. Stability diagram on the $x-z$ (radial-vertical) plane. Instability is obtained when the mean particles trajectory slope, $\alpha = w/u$ (black line), exceeds the critical slope $\alpha_c = \Omega_0/(2S_0)$, of constant azimuthal absolute momentum (blue lines). The wedge of stability (between zero slope to the absolute momentum slope) is shaded in grey. The blue arrow corresponds to the advection of the mean absolute momentum by the perturbation velocity. (Colour online)

4. Vorticity perspective

A different perspective on the amplification mechanism may be obtained when considering the interplay between the perturbation vorticity components in the $x$ direction, $\omega_x = -\partial v/\partial z$ (recall that $\partial / \partial y = 0$) and in the $y$ direction, $\omega_y = \partial u/\partial z - \partial w/\partial x$. We first note that equations (1a,c) yield

$$\frac{\partial \dot{w}}{\partial x} = C_x \omega_x. \quad (11)$$

In figure 3(a), it is shown how the geostrophic balance in (1a) leaves the positive (negative) pressure anomaly to the right (left) of $v$. Thus, negative vertical shear of the geostrophic velocity (positive $\omega_x$) imposes vertical pressure gradient structure which is pointing upward (downward) to the right (left) of the shear and therefore accelerates the vertical velocity in the opposite directions of the vertical pressure gradients. Since $u$ is tied to $w$ by continuity, together with the assumption of plane-wave solutions, equation (11) may be expressed in terms of the interplay of component vorticities

$$\dot{\omega}_y = -C_x \left(1 + \alpha^{-2}\right) \omega_x. \quad (12)$$

On the other hand, we have

$$\dot{\omega}_x = C_y \frac{\partial u}{\partial z} - S_0 \frac{\partial w}{\partial z}. \quad (13)$$

The Coriolis acceleration to the right explains how the shear ($\partial u/\partial z$) generates $\omega_x$ (figure 3(b)), whereas the vertical advection of the mean shear by the vertical perturbation velocity explains how vertical convergence ($-\partial w/\partial z$) generates as well $\omega_x$ (figure 3(c)). Using
Figure 3. Different mechanisms for the generation of the different vorticity terms (see details in the main text). (a). Negative azimuthal vertical shear $-\partial v/\partial z$ generates positive vertical radial shear $\partial w/\partial x$. (b) Radial vertical shear $\partial u/\partial z$ generates negative azimuthal vertical shear $-\partial v/\partial z$. (c). Vertical convergence $-\partial w/\partial z$ generates as well negative azimuthal vertical shear $-\partial v/\partial z$. In all sub-figures, solid line arrows represent the generating mechanisms and dashed or dot arrows represent the responding dynamics. (Colour online)

continuity again we may write (13) for the plane-wave solution as

$$\dot{\omega}_x = -S_0 \left( \frac{\alpha - \alpha_c}{1 + \alpha^2} \right) \omega_y.$$  

(14)

It is clear from (12) and (14) that mutual amplification between the two vorticity components can be obtained only if $\alpha > \alpha_c$ and when $\omega_x$ and $\omega_y$ are of opposite sign. Such scenario is demonstrated in figure 4(a) for the case where $\omega_x < 0$ and $\omega_y > 0$. The pressure
structure, resulting from the geostrophic balance associated with $-\omega_x$, amplifies $\omega_y$ according to figure 3(a) and (12). In turn, however, $\omega_y$ acts to diminish $\omega_x$ via the term $C_y(\partial u/\partial z)$ (figure 4(b)), as described in figure 3(b). Thus, in order to maintain mutual amplification a strong enough vertical divergence is required (figure 4(c)) so that $|S_0(\partial w/\partial z)| > |C_y(\partial u/\partial z)|$, and for plane-waves this condition is indeed satisfied for $\alpha > \alpha_c$. As a result, unstable eddies in the $(x,z)$ plane are elongated ellipses with a tilt that is steeper than the absolute momentum isolines slope (figure 4(d)). Since $\omega_z = \alpha \omega_x$, similar reasoning leads to unstable interplay between $\omega_y$ and $\omega_z$.

5. Energy perspective

The condition for instability ($\alpha > \alpha_c$) agrees as well with the Solberg-Høiland Rayleigh-like condition (Umurhan et al. 2013, Barker and Latter 2015, Lin and Youdin 2017, Latter and Papaloizou 2018).

Suppose we exchange two closed-looped fluid filaments located initially at positions $(x_1, z_1)$, $(x_2, z_2)$, so that $(\delta x = x_2 - x_1, \delta z = z_2 - z_1)$ and each hoop’s slope trajectory is $\alpha = \delta z/\delta x$. Denoting the initial state (before the flow is perturbed) and the final state (after the parcels exchanged places) by the superscripts $i, f$, respectively, material conservation of the absolute momentum implies that

$$V^f_{1,2} = V_{2,1} \pm C_y \delta x.$$  \hspace{1cm} (15)

The corresponding change in the azimuthal energy of the two parcels due to the exchange is given by

$$\Delta E = \frac{1}{2} \left[ (V^f_1)^2 + (V^f_2)^2 \right] - \frac{1}{2} \left[ (V_1)^2 + (V_2)^2 \right].$$  \hspace{1cm} (16)

Substituting equation (15) and the definitions of $(\alpha, \alpha_c)$ into the above and sorting through the algebra we find

$$\Delta E = C_y S_0 (\delta x)^2 (\alpha_c - \alpha).$$  \hspace{1cm} (17)

On the basis of energy minimisation, linear instability is expected when $\Delta E < 0$. Given that both $C_y$ and $S_0$ are defined as positive constants, inspection of equation (17) demonstrates that we recover the previously identified condition for instability, $\alpha > \alpha_c$. We note that the derivation of this Rayleigh-like condition is somewhat heuristic: As in the Rayleigh conditions for centrifugal or inertial instabilities the hidden assumption is that fluid parcels/filaments instantaneously reach pressure equilibrium. This means that the work performed by the perturbation pressure gradient, which would normally contribute to the energy budget but actually contributes nothing to it because the fluid density is assumed constant, is here ignored. The consequence of this assumption is that only the azimuthal component of the energy is taken into account as the work performed by the vertical component of the pressure gradient force (hereinafter, PGF) is not considered and the radial velocity is assumed to be tied to the vertical one by continuity. The strength of this rests on the robustness of the assumption of incompressibility, which is more or less valid if the timescales of fluid motions of a given length scale are much longer than the corresponding sound propagation timescales.

3 Because this analysis is axisymmetric, we imagine infinitesimally thin annuli orbiting the central star.
Figure 4. VSI instability from vorticity component interaction. (a) For unstable setups $\omega_x$ and $\omega_y$ are in opposite signs. Here negative $\omega_x$ generates positive $\omega_y$ by the mechanism described in figure 3(a). (b) Generation of positive $\omega_x$ by positive $\omega_y$ via the mechanism described in figure 3(b). (which leads to stability). (c) Generation of negative $\omega_x$ by vertical divergence, via the mechanism described in figure 3(c). (which leads to instability). (d) Superposition of the mechanisms in sub-figures (a–c). When the slantwise motion in the $(x,z)$ plane is steeper than the mean absolute momentum slope the amplifying mechanism (c) overwhelms the decaying mechanism (b). In all sub-figures solid line arrows represent the generating mechanisms and dashed or dot arrows represent the responding dynamics. (Colour online)

Therefore, the Solberg-Høiland condition is equivalent to the condition to obtain growth in the azimuthal component of the perturbation kinetic energy. From (9), we get

$$\frac{\partial}{\partial t} \left( \frac{v^2}{2} \right) = \left( 1 - \frac{\alpha_c}{\alpha} \right) S_0 v w,$$

indicating that positive Reynolds stress (where $v$ and $w$ are positively correlated in the $(y,z)$ plane, as $S_0 = -\partial \mathbf{V} / \partial z > 0$) leads to energy growth when $\alpha > \alpha_c$. 

\[ \text{(18)} \]
Since the source of the instability is the vertical mean shear we may define the non-dimensionalised growth rate as $\lambda_{\text{inc}} \equiv -i\omega/S_0$ (where the subscript $\text{inc}$ denotes the incompressible solution). Then for the disk values, $C_x = 4C_y$, we obtain from (3) that the growth rate depends solely on the ratio between the fluid parcel’s slope trajectory and the mean absolute momentum slope, $\chi \equiv \alpha/\alpha_c$, i.e.

$$\lambda_{\text{inc}} = \frac{2}{\chi} \sqrt{\chi - 1}. \quad (19)$$

In figure 5, we plot $[\lambda, (u, v, w), (\omega_x, \omega_y)]$ as a function of $\chi$ for the instability regime ($\chi > 1$). The largest growth rate is obtained at $\chi = 2$ ($\lambda_{\max}(\chi = 2) = 1$), that is when the parcel’s slope at the $(x, z)$ plane is twice of the critical slope ($w/u = 2\alpha_c = \Omega_0/S_0$). Since $w = 2v$ for $\chi = 2$, the parcel slope in the $(x, y)$ plane is $\alpha_c$. It is evident from figures 5(b),(c) that for the most unstable mode the fluid motion in the radial-vertical plane is more prominent than in the azimuthal-vertical plane.

A quick inspection indicates that the largest possible instantaneous growth rate cannot be maintained. Consider the perturbation kinetic energy growth for the azimuthal and vertical components

$$\frac{\partial}{\partial t} \left( \frac{v^2 + w^2}{2} \right) = \left( 1 + \frac{3}{\chi} \right) S_0 v w. \quad (20)$$

Applying the Cauchy-Schwarz inequality indicates that the largest possible instantaneous normalised growth rate is obtained when $v = w$ with $\lambda_{\text{inst}} = [1 + (3/\chi)]/2$. For the modal
instability regime it is maximised to a value of 2 for $\chi = 1$; however, this instantaneous growth cannot be sustained since it amplifies $w$ while leaving $v$ unchanged as the right-hand side of (1b) vanishes for $\chi = 1$. Similar non-modal growth dynamics is obtained in atmospheric symmetrically unstable flows (Heifetz and Farrell 2008). The major difference between these two problems is the lack of explicit dependence in the isentropes’ slope in the disk asymptotic equation set (1a), and consequently the lack of the condition for mean negative Ertel potential vorticity. The latter is equivalent to the condition for the isentropes’ slope to exceed the slope of constant absolute momentum surfaces (e.g. Holton and Hakim 2012). The differences between those two similar problems are discussed in the appendices where the governing equations of the mesoscale atmospheric symmetric instability (e.g. equation set (1) in Heifetz and Farrell 2008) are translated to the disk shearing box configuration.

6. Oscillatory instability resulting from anelasticity

Detailed numerical simulations of protoplanetary disk “dead zones” show the existence of oscillatory unstable VSI dynamics (Stoll and Kley 2014, Barker and Latter 2015, Richard et al. 2016, Stoll and Kley 2016) suggested to be rooted in anelasticity (Umurhan et al. 2016). Typically speaking, overstable dynamics cannot be described by the incompressible analysis like what we have employed up to now. Indeed, equations (3) and (19), based purely on an assumption of incompressibly, indicate that the modes can be either exponentially growing or purely oscillatory, but never as oscillations with exponentially growing amplitude (overstability). Here we show that if we replace incompressibility with anelasticity, we recover overstable behaviour.

An analytically tractable solution is obtained by assuming a neutrally-stratified exponential density profile in the form of $\bar{\rho} = \rho_m \exp(-z/H)$, where $\rho_m$ is the mid-plane density and the density scale height, $H$, is assumed constant. Substituting this form for $\bar{\rho}$ into equation (1d) we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0,$$

where consequently, equation (2) has now an additional term in the right-hand side

$$\frac{\partial^2 \Pi}{\partial x^2} = -C_x \left[ S_0 \frac{\partial^2 \Pi}{\partial x \partial z} + C_y \left( \frac{\partial^2 \Pi}{\partial z^2} - \frac{1}{H} \frac{\partial \Pi}{\partial z} \right) \right].$$

In order to properly account for stratification in the perturbation energy, we introduce the modified plane wave solution of the form of $\exp[i(kx + mz - \omega t) + z/2H]$. Inserting into equation (22), we obtain the complex dispersion relation

$$\omega^2 = \frac{S_0 C_x}{\alpha^2} \left\{ \alpha_c \left[ 1 + \left( \frac{\alpha}{2kH} \right)^2 \right] - \alpha \left[ 1 + i \frac{\alpha}{2kH} \right] \right\}.$$

The equivalent VSI condition to equation (4) for the “exponential atmosphere” is derived in appendix B

$$|\frac{w}{u}| = \frac{\alpha}{\sqrt{1 + \beta^2}} > \alpha_c \sqrt{1 + \beta^2} \quad \implies \quad \text{Re} \left\{ \frac{w}{u} \right\} > \alpha_c$$
(see particularly (B.1), (B.8) and (B.9)), where \( \alpha = -k/m \), and \( \beta = 1/(2mH) \). For infinite scale height \( H \), equation (4) is recovered. For finite \( H \) however, the mean parcel’s slope is smaller than the ratio \( k/m \) and by the same time the minimal critical slope for instability increases. Therefore, the anelastic effect tends to inhibit VSI.

Using the disk values of \( C_x = 4C_r \) we obtain a complex normalised growth rate wherein \( \lambda_{\text{ane}} \equiv \lambda_r + i\lambda_i = -i\omega/S_0 \) (where the ‘ane’ subscript refers to the anelastic solution), in which \( \lambda_{\text{ane}} \) satisfies the following

\[
\lambda_{\text{ane}}^2 = \left[ \lambda_{\text{inc}}^2 - \left( \frac{\alpha_c}{kH} \right)^2 \right] + 2i\left( \frac{\alpha_c}{kH} \right) = \left( \lambda_r^2 - \lambda_i^2 \right) + 2i\lambda_r\lambda_i. \tag{25}
\]

Equation (25) admits solutions in which perturbations may grow while oscillating (over-stability). Note that for positive growth rate, \( \lambda_r > 0 \), the imaginary part of (25) indicates that \( \lambda_i > 0 \) as well. Normalising time by \( S_0^{-1} \) the general structure of a growing modal perturbation is of the form \( \exp(|\lambda|r) \exp(z/2H) \exp[i(kx + mz + |\lambda_i|t)] \). A straightforward derivation (data not shown here) indicates that \( \lambda_r < 1 \) for finite scale height, thus, in agreement with (24) and with McNally and Pessah (2015), the anelastic effect tends to decrease the instability. We focus now on the mechanism by which anelasticity generates oscillations. To isolate the latter we can look at the untitled limit where \( m = 0 \) so that \( \lambda_{\text{inc}} = 0 \). Then the oscillation is due to propagation in the radial direction with the normalised phase speed \( c_x = -|\lambda_i|/k \). A further simplification of the algebra is obtained when considering the wavenumber \( k \), satisfying \( \alpha_c/(kH) \equiv 1 \). Then \( \lambda_{\text{ane}}^2 = -1 + 2i \), with \( (\lambda_r, \lambda_i) = (0.786, 1.27) \). The continuity equation (21) dictates that the divergence in the \( (x, z) \) plane is proportional to \( w \) and for \( m = 0 \), \( u = -iw/(2kH) \). Furthermore, writing \((v, w)\) in terms of their amplitude and phases

\[
v = \hat{v}(t) \exp[i(kx + \epsilon_v(t))], \quad w = \hat{w}(t) \exp[i(kx + \epsilon_w(t))], \tag{26}
\]

the eigen-structure satisfies \( \hat{v} \approx 0.75\hat{w} ; (\epsilon_w - \epsilon_v) \approx 0.176\pi \). By geostrophy \( v \) lags the pressure perturbation \( \Pi \) by a quarter of a wavelength. This normal mode structure is plotted schematically in figure 6(a).

We wish to explain now how this mode is growing while propagating in concert. Substitute it into equation (1b) to obtain

\[
\dot{v} = \left( \frac{1}{2}i + 1 \right) w \approx 1.12 e^{0.15\pi i} w, \tag{27}
\]

indicating that the generation of \( v \) is to the left of \( w \) (figure 6(b)) since both positive \( w \) and negative \( u \) generate positive \( v \), and negative values of \( u \) are located a quarter of wavelength to the left of positive values of \( w \) (figure 6(a)). On the other hand, using the geostrophy conditions of equations (1a) and (1c) we find that

\[
\dot{w} = 2 e^{0.5\pi i} v, \tag{28}
\]

i.e. the vertical pressure gradient anomaly associated with the geostrophic balance of \( v \), shifts \( w \) a quarter of wavelength to the left of \( v \) (figure 6(c)). As a result, both \( v \) and \( w \) are growing while propagating to the left. Put more rigorously, we can substitute \( v \) and \( w \) into equations (27) and (28). Then we solve for the real and the imaginary parts. This gives for
Figure 6. Propagation and growth mechanisms of untilted ($m = 0$), anelastic modes. (a). Structure of the modal velocity components presented in the $(x, z)$ plane. (b). Generation of azimuthal velocity due to the combined effects of the vertical advection of mean azimuthal velocity and the Coriolis force acting on the radial velocity. (c). Generation of vertical velocity due to the pressure structure imposed by geostrophy and the radial velocity which is tied to it via anelastic continuity. (d). Inward propagation combined with growth, due to the combined effects described in subplots (a–c). For clarity only, the azimuthal and vertical velocities are plotted. In all sub-figures solid line arrows represent the generating mechanisms and dashed or dot arrows represent the responding dynamics. (Colour online)

modal instability (where all fields experience the same growth rate and propagation speed)

$$
\lambda_r = \dot{\hat{\nu}} = \dot{\hat{w}} = 2 \hat{v} \sin (\epsilon_w - \epsilon_v), \quad \lambda_i = \dot{\epsilon}_v = \dot{\epsilon}_w = 2 \hat{v} \cos (\epsilon_w - \epsilon_v). \quad (29)
$$

For completeness, we note that in this untilted solution $\omega_y = [1 + (2kH)^2]u/(2H)$ and $\omega_x = -v/(2H)$. Hence the normalised vorticity equations become

$$
\dot{\omega}_y = -\left[ \frac{1 + (2\alpha_c)^2}{\alpha_c} \right] \omega_x, \quad (30)
$$

and

$$
\dot{\omega}_x = -(2i - 1) \left[ \frac{1 + (2\alpha_c)^2}{\alpha_c} \right]^{-1} \omega_y. \quad (31)
$$
This is similar to the incompressible case, when $\omega_x$ and $\omega_y$ are anti-phased the two vorticity components amplify each other, but here in the anelastic case they also shift each other in the $x$ direction to propagate in concert.

7. VSI in a realistic disk setup

We further investigate the VSI in a more relevant setting for a planet-forming disk. We therefore assume a locally isothermal atmosphere (i.e. $T$ is a function of radius only) with a Gaussian density profile in the form of $\bar{\rho} = \rho_m \exp(-z^2/2H^2)$ (Nelson et al. 2013, Umurhan et al. 2016). Substituting $\bar{\rho}$ into equation set (1a) results in the following equation set in dimensional form:

\begin{align*}
0 &= 2\Omega v - \frac{\partial \Pi}{\partial x}, \\
\frac{\partial v}{\partial t} &= -\frac{1}{2} \Omega u - \frac{1}{2} q\Omega \epsilon z \frac{w}{H}, \\
\frac{\partial w}{\partial t} &= -\frac{\partial \Pi}{\partial z}, \\
0 &= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - \frac{z}{H^2} w,
\end{align*}

where for $S(z)$ we have used $-\partial \tilde{v}/\partial z = -qz\Omega \epsilon/2H$. This set is properly non-dimensionalised by assuming the following scalings for space and time as $x \rightarrow \epsilon H\tilde{x}$, $z \rightarrow H\tilde{z}$, $t \rightarrow \Omega \tau/\epsilon$, while for the other depended quantities we assume $u \rightarrow \epsilon \tilde{u}$, $w \rightarrow \tilde{w}$, $v \rightarrow \tilde{v}$ and $\Pi \rightarrow \epsilon \tilde{\Pi}$, in which all quantities with tildes over them are non-dimensional. We then rewrite equations in terms of those quantities and we drop all tildes except for $\tilde{u}$ and $\tilde{\Pi}$ (in order to visually track that these dependent quantities are necessarily small by a factor of $\epsilon$ in comparison to the other quantities).

After dropping the tilde symbol from all independent variables so that $t$, $x$, $z$ are in terms of the above introduced scalings, we combine equations (32a–d) into a single partial differential equation governing the response of $\Pi$, i.e.

\begin{equation}
-\frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 \Pi}{\partial x^2} \right) - \frac{\partial^2 \Pi}{\partial z^2} + \left( 1 + q \frac{\partial}{\partial x} \right) z \frac{\partial \Pi}{\partial z} = 0.
\end{equation}

We assume the usual normal mode ansatz

\begin{equation}
\Pi = \hat{\Pi}(x, z) e^{-i\omega t} + c.c.,
\end{equation}
where we consider solutions for \( \hat{\Pi} \) that both filters out surface modes and retains the body modes by adopting the following ansatz

\[
\hat{\Pi} = P_m(z, x) = \begin{cases} 
\sum_{n=0,2,\ldots}^{m} P_{n,m}(x) z^n, & m = \text{even}, \\
\sum_{n=1,3,\ldots}^{m} P_{n,m}(x) z^n, & m = \text{odd},
\end{cases}
\]

(as in Umurhan et al. 2016). As most of the kinetic energy of the instability is contained in the low order body modes, we only consider modes with \( n = m = 1, 2 \) (to avoid confusion please note that in this section \( m \) is not the vertical wavenumber), subject to the no normal flow boundary condition at the inner and outer edges, which in terms of this formulation becomes

\[
\frac{\partial P_{n,m}}{\partial x} + \frac{q n}{\omega^2} P_{n,m} = 0 \quad \text{(at } x = \pm L_x \text{)},
\]

in which

\[
\frac{\omega^2}{m} = \frac{1 \pm \sqrt{1 - k_j^2 q^2}}{2 k_j^2}.
\]

The boundary condition in equation (36) dictates that this model of the VSI is vertically global but radially local and therefore only appropriate for studying narrow radial disk patches. While a more appropriate boundary condition might be to ignore radial boundaries and seek (suitably slanted) locally wave-like perturbations, we impose the no normal flow boundary condition as these are in the same spirit as those of published simulations. In addition, the choice of the boundary condition is important for the details of the manifestation of the instability, but it does not affect its underlying mechanism, for further justification (see Umurhan et al. 2016).

For \( m = 1 \) the function sum \( P_1 \) consists of the single separable function \( P_{1,1} z \), while for \( m = 2 \), \( P_2 \) is given by the inseparable form \( P_{2,2} z^2 + P_{0,2} \). In general the “top” functional form for any value \( m \) is given by

\[
P_m = \left[ A_j \sin(k_j x) + B_j \cos(k_j x) \right] \exp \left( -\frac{q m}{2 \omega^2} x \right).
\]

where \( k = k_j = j \pi / 2L_x \), where \( j \) is any integer including zero. When \( j \) is an odd integer, then \( A_j = -k_j \), \( B_j = -q m / 2 \omega^2 \). However, when \( j \) is an even integer (including zero) \( A_j = -q m / 2 \omega^2 \), \( B_j = -k_j \).

For \( P_2 \) the additional term \( P_{0,2} \) is calculated by solving

\[
\omega^2 \frac{\partial^2 P_{n,m}}{\partial x^2} + n \left( P_{n,m} + q \frac{\partial P_{n,m}}{\partial x} \right) = -(n + 2)(n + 1) P_{n+2,m}
\]

---

4 Note, equation (5) of Umurhan et al. (2016) contains a typographical error: the sign in front of the second partial derivative with respect to \( z \) term should be negative, cf. equation (33) of this study. All results quoted in that paper, especially those pertaining to the eigenvalues calculated, are unaffected the sign of this term as it does not enter into its determination.
subject to the boundary condition (36). It results in
\[ P_{0,2} = \frac{1}{2} (-1)^{2/3} \left( \sqrt{3} + i \right) \exp(2\sqrt{-1} \pi z) \left[ \left( \sqrt{3} + i \right) \sin(2x) + 2i \cos(2x) \right] \] (40)
for \( k = 2 \).

We note that only the \( m = 1 \) fundamental corrugation mode is a separable function of \( x \) and \( z \). However, this feature does not carry over to the structure functions for all overtones (i.e. \( P_m(z,x) \forall m \geq 2 \)) – these will always be functions of \( x \) and \( z \) that are not fundamentally separable in the usual sense of it's definition. However, we note that given that these are a polynomial series in \( z \), these functions are easily factorisable.

To calculate the ratio \( w/u \) we explicitly use equations (32a–c) and (38), to produce
\[ \frac{w}{u} = \frac{\tilde{w}}{\tilde{u}} = \frac{1}{2\epsilon} \left[ \frac{\partial_z \hat{N}}{i\epsilon} + \text{c.c.} \right] \left[ \left( i\omega/2 \right) \frac{\partial_x \hat{N}}{\partial x} - (1/2i\omega) qz \partial_z \hat{N} \right] + \text{c.c.} \] (41)

Inserting the expression for \( \hat{N} \), found in equations (34), (35), into equation (41) reveals that \( w/u \) is dependent on \( m \) because of this inseparability of the individual eigenfunctions describing \( u \) and \( w \). While overtone modes (i.e. \( \forall m > 2 \)) are of academic interest, our concern here is just with the fundamental modes as these are the ones that appear prominent in disk simulations of the VSI (e.g. Nelson et al. 2013, Richard et al. 2016, Manger and Klahr 2018, and others).

In figure 7, we plot the inverse tangent of the radial average of \( w/u \) – here as \( \langle w/u \rangle \) – for \( P_1 \) and \( P_2 \), together with the inverse tangent of \( \alpha_c \), as functions of \( z \) for the maximum growth rate wavenumber
\[ |k_{\text{max}}| = \frac{2}{|q|} \quad q = -1 \quad k_{\text{max}} = 2 \] (42)
(in dimensional units the fastest growing modes scale as \( \lambda_{\text{max}} = \pi \epsilon H \)). As indicated from figure 7(a), between \( 0 < z < 1 \), the corrugation mode is potentially unstable while the breathing mode is predicted to be stable. Where \( z > 1 \) we observe \( \pi/2 > \langle w/u \rangle > \alpha_c \) for both the corrugation and the breathing modes, suggesting that this region supports unstable dynamics for the two modes. \( \langle w/u \rangle \) of the two modes decrease with height and converge at \( z \approx 5 \) to the approximate angle of \( \pi/5 \), where the threshold stability slope, \( \alpha_c = 1/z \), decreases with height as well. Consequently, one might speculate that the VSI should initially manifest itself at \( 1 < z < 2 \) by the breathing mode – whose growth rate is also a bit larger than the corrugation mode, as seen from equation (37) – while the corrugation mode can spread/initiate the instability in the lower regions of the disk. This seems to be consistent with simulations showing the VSI initially taking shape around 1 scale height via a breathing mode and then evolves into a corrugation mode which spread into regions above and below (Nelson et al. 2013, Stoll and Kley 2014, Lyra and Umurhan 2019). Figure 7(b) is the same as figure 7(a) but plotted up to \( z = 100 \). While typical disks in the Ohmic Zone do not span further than a few scale heights it can be seen how at high values of \( z \) the slopes converge to the stable threshold of \( \alpha_c \).
Figure 7. The R.H.S of equation (41) plotted for the corrugation $P_1$ (orange), and breathing $P_2$ (blue) modes and the inverse tangent of the critical slope $\alpha_c = 1/z$ (green), all as a function of $z$. (a, top panel) $0 < z < 5$ (b, bottom panel) $0 < z < 100$, the plots are in the $z-\theta$ surface where $q = -1$ and $k = k_{\text{max}} = 2$, for the setup of Gaussian density profile. (Colour online)
8. Summary and conclusions

Detailed numerical simulations of the dead zone dynamics of cold protoplanetary accretion disks have demonstrated that VSI plays an essential role in driving turbulent gas motions that can stir solids and affect the planet formation process. As the simulations are highly complex, with several processes occurring during various stages of their development, identifying the underlying physical mechanisms responsible for the wide phenomena manifest in them are not always clear. This is especially valid for the linear instability itself. For this reason, we have devised a physically meaningful minimal model of the VSI that captures the essence of the instability mechanism while remaining analytically tractable.

Since 2D (barotropic) Keplerian shear flows are stable for small perturbations, the VSI can be regarded as a mechanism that overcomes this obstacle by exploiting the vertical shear within the disk that emerges as a baroclinic response to a radial temperature gradient. With respect to the disk’s radial-vertical planes, a fluid parcel’s motion should be particularly slantwise: On one hand, it should be high enough to obtain more energy from the vertical shear than it loses to the Keplerian shear by radial motions. On the other hand, it should not be too high for otherwise radial motions are suppressed which is important to foster radial redistribution of the azimuthal angular momentum and maximise energetic growth during the accretion process – cf. equation (18). Therefore, parcel motions ought be at an angle which is larger than the critical slope while not being too steep for otherwise the rate of energy extraction starts becoming inefficient. In this minimal model, the critical slope is determined directly by the ratio between the Keplerian and the vertical shear magnitudes and is also aligned with the surfaces of the mean absolute momentum (angular momentum in a frame of rest) surfaces.

Here we have shown explicitly how motions exceeding the critical slope accelerates the flow away from its initial position, not only in the radial-vertical plane but also in the azimuthal direction. Furthermore, we have shown that the VSI can be explained in terms of the mutual amplification mechanism between the radial-vertical plane oriented and the azimuthal-vertical plane oriented circulations (quantified by their respective vorticity components). We have also made concrete the energetic perspective of VSI, commonly rationalised in terms of fulfilling the semi-quantitative condition of spontaneous energy release when pairs of fluid parcels exchange positions. This argument is based (like all the generalised Rayleigh-like instability conditions) on instantaneous pressure adjustment of the fluid parcel and consequently on vanishing of the pressure perturbation. This is not the case however for VSI, as is evident from the azimuthal and vertical momentum equations (1aa,c). We showed that the Solberg-Høiland condition is indeed valid but it is the Reynolds stress mechanism, in the azimuthal-vertical plane, that yanks kinetic energy from the vertical mean shear to the perturbation. Furthermore, by the same time the Reynolds stress in the azimuthal-radial plane acts to stabilise the perturbations by transferring energy from the perturbation back to the Keplerian flow. Hence, only when the parcel slope is larger than the critical slope the Reynolds stress mechanism yields instability. Despite of its simplicity, the minimal model enables super-modal transient growth, that is a growth mechanism which is more efficient than the one obtained by the most unstable normal mode. Here we showed how this mechanism operates but also why it cannot be sustained.

The minimal model in its simplest incompressible guise cannot describe the oscillatory instability dynamics observed in disk numerical simulations. However, we show that it is
enough to add neutrally stable anelastic stratification to enable oscillatory instability. The simplicity of the model allows one then to disentangle the oscillatory mechanism, due to the stratification, from the instability one when considering untilted modal structures.

It is interesting to note the parallels between the VSI and the mesoscale Symmetric Instability slantwise convection, the latter of which operates in the mid-latitudinal Earth atmosphere as well as being responsible for mixing in the Gulf Stream (e.g. Thomas et al. 2013). In appendix A, we converted the equations from its atmospheric context into the disk shearing box setup. The similarity between the equations of the two instability mechanisms becomes transparent as well as the main difference between the two – in Symmetric Instability buoyancy dynamics is explicit, thus consequently, the most unstable mode parcels’ slope coincides with the mean isentropes’ slope. In contrast, in VSI, the buoyancy force is smaller by at least one order of magnitude than the vertical component of the vertical component of the pressure gradient force perturbation. Consequently, the buoyancy force does not play an effective role as a restoring force to inhibit the instability mechanism. For future work it could be interesting to generalise the VSI minimal model by assuming the mean flow vertical shear to be in a thermal wind balance that is maintained by the slanted mean isentropes. In such a case the isentropes may be tilted in a different slope from the absolute momentum surfaces. This setup is expected to invoke richer potential vorticity-like dynamical behaviour, though its relevancy to the Ohmic Zone should be carefully examined.

It remains a particularly outstanding question how dust loading influences the progress of any given primary instability mechanism that leads to turbulence. Two recent studies (Lin 2019, Schäfer et al. 2020) examined the role that particle-loading has on the emergence and development of the VSI in model disks. The numerical simulations of Lin (2019) suggest that particles can reduce the efficacy of the VSI. The corresponding simulations of Schäfer et al. (2020), however, indicate that the particle accumulation mechanism known as the streaming instability (Youdin and Goodman 2005, Johansen et al. 2007) can operate in the face of the unsteady motions emerging from the VSI, in contradiction to several theoretical predictions (e.g. Chen and Lin 2020, Gole et al. 2020, Umurhan et al. 2020). This controversy requires properly understanding how dust interacts with the fluid motions that gives rise to the VSI and can be a subject for future studies.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendices

Appendix A VSU versus atmospheric symmetric instability

Here we translate the governing equations of mesoscale symmetric instability (e.g. equation set (1) in Heifetz and Farrell 2008) to the disk shearing box setup in order to highlight the similarities and differences between them and equation set (1a) in this paper.

First we note that the mean flow in both cases is in thermal wind balance (i.e. geostrophic in the radial direction and hydrostatic in the vertical one)

\[ \nabla = \frac{1}{C_x} \frac{\partial \Pi}{\partial x}, \quad \bar{B} = \frac{\partial \Pi}{\partial z} \quad \Rightarrow \quad S = -\frac{\partial \nabla}{\partial z} = -\frac{1}{C_x} \frac{\partial \bar{B}}{\partial x}, \]  

\( \text{(A.1)} \)

where \( \bar{B} \) is the mean buoyancy. Then the symmetric instability equations in the shearing box become

\[ \dot{u} = C_x v - \frac{\partial \Pi}{\partial x}, \]  

\( \text{(A.2)} \)

\[ \dot{v} = -C_y u + S w, \]  

\( \text{(A.3)} \)

\[ 0 = -\frac{\partial \Pi}{\partial z} + b, \]  

\( \text{(A.4)} \)

\[ \dot{b} = -N^2 w - \frac{\partial \bar{B}}{\partial x} u = -N^2 w + C_x S u, \]  

\( \text{(A.5)} \)

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \]  

\( \text{(A.6)} \)

where \( N \) is the buoyancy frequency and \( b \) is the buoyancy perturbation. Comparing the incompressible version of equation set (1a) to (A.2)–(A.6), we find that in the atmospheric symmetric instability \( \nu \) is not assumed to be in geostrophic balance but instead \( w \) is in quasi-hydrostatic balance. The buoyancy dynamics becomes explicit and consequently, for the most unstable mode, the parcels’ slope coincides with the mean istentorpes’s slope.

Appendix B Derivation of the VSI criteria for exponential disk atmosphere

Substitute the modal solution form \( \exp[i(kx + mz - \omega t) + z/2H] \) into equation (21) for the exponential density profile yields

\[ \frac{w}{u} = \frac{\alpha}{1 + \beta^2} (1 - i\beta) = \frac{\alpha}{\sqrt{1 + \beta^2}} \tan^{-1}(-\beta), \]  

\( \text{(B.1)} \)

where \( \beta = 1/(2mH) = -\alpha/(2kH) \) and \( \alpha = -k/m \). Rewrite the dispersion relation of equation (23) as

\[ \omega^2 = D[\alpha_c (1 + \beta^2) - \alpha (1 - i\beta)] = G \left( \alpha_c - \frac{w}{u} \right), \]  

\( \text{(B.2)} \)

where \( D = S_0 C_x / \alpha^2 \) and \( G = S_0 C_x (1 + \beta^2) / \alpha^2 \). We separate the equation into its real and imaginary parts to obtain

\[ \omega_r^2 - \omega_i^2 = D[\alpha_c (1 + \beta^2) - \alpha], \]  

\( \text{(B.3)} \)

\[ 2\omega_r \omega_i = D \alpha \beta \quad \Rightarrow \quad \omega_r = \frac{D \alpha \beta}{2\omega_i}. \]  

\( \text{(B.4)} \)
Insert now (B.4) into (B.3) and define \( F \equiv D/2 \) and \( s \equiv \omega_i^2 \) to obtain a quadratic equation for \( s \), namely

\[
s^2 + 2F[\alpha_c (1 + \beta^2) - \alpha]s - F^2 \beta^2 \alpha^2 = 0, \quad (B.5)
\]

whose solutions are

\[
s_{1,2} = \frac{-F[\alpha_c (1 + \beta^2) - \alpha] \pm \sqrt{\left(\frac{\beta \alpha}{\alpha_c (1 + \beta^2) - \alpha}\right)^2 + 1}}{A_2 A_3} \left\{ 1 \pm \left[ 1 + \left(\frac{\beta \alpha}{\alpha_c (1 + \beta^2) - \alpha}\right)^2 \right]^{1/2} \right\}.
\]

(B.6)

Hence, in terms of \( \omega_i \), we have

\[
\omega_{\pm} = \pm \sqrt{-F(\alpha_c (1 + \beta^2) - \alpha)} \sqrt{1 + \left[ 1 + \left(\frac{\beta \alpha}{\alpha_c (1 + \beta^2) - \alpha}\right)^2 \right]^{1/2}}.
\]

(B.7)

Since \( s \) and \( F \) are positive, (B.6) implies that the product \( A_2 \times A_3 < 0 \), meaning the equation distinguishes only between neutral and non-neutral modes but not between growing and decaying modes. In order to determine between growth and decay, one needs to examine the modal relations between the fields \((u, v, w, \hat{\Pi})\), where the growing modes are the ones in which all four fields have the same sign (as mentioned at the end of section 3). Consequently, the condition corresponding to the growing modes is

\[
\alpha_c - \frac{\alpha}{1 + \beta^2} < 0 \quad \implies \quad \frac{\alpha}{\sqrt{1 + \beta^2}} > \frac{|w|}{u} > \alpha_c \sqrt{1 + \beta^2}.
\]

(B.8)

The condition in equation (B.8) can be also written as

\[
\frac{\alpha}{1 + \beta^2} = \text{Re} \left\{ \frac{w}{u} \right\} > \alpha_c.
\]

(B.9)