THE GABRIEL-ROITER MEASURE FOR \( \tilde{A}_n \) II

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Abstract. Let \( Q \) be a tame quiver of type \( \tilde{A}_n \) and \( \text{Rep}(Q) \) the category of finite dimensional representations over an algebraically closed field. A representation is simply called a module. It will be shown that a regular string module has, up to isomorphism, at most two Gabriel-Roiter submodules. The quivers \( Q \) with sink-source orientations will be characterized as those, whose central parts do not contain preinjective modules. It will also be shown that there are only finitely many (central) Gabriel-Roiter measures admitting no direct predecessors. This fact will be generalized for all tame quivers.

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1. Introduction

Let \( \Lambda \) be an Artin algebra and \( \text{mod} \Lambda \) the category of finitely generated left \( \Lambda \)-modules. For each \( M \in \text{mod} \Lambda \), we denote by \( |M| \) the length of \( M \). The symbol \( \subset \) is used to denote proper inclusion. The Gabriel-Roiter (GR for short) measure \( \mu(M) \) for a \( \Lambda \)-module \( M \) was defined to be a rational number in [12] by induction as follows:

\[
\mu(M) = \begin{cases} 
0 & \text{if } M = 0; \\
\max_{N \subset M} \{\mu(N)\} & \text{if } M \text{ is decomposable; } \\
\max_{N \subset M} \{\mu(N)\} + \frac{1}{2^{|M|}} & \text{if } M \text{ is indecomposable.}
\end{cases}
\]

(In later discussion, we will use the original definition for our convenience, see [11] or Section 2.1 below.) The so-called Gabriel-Roiter submodules of an indecomposable module are defined to be the indecomposable proper submodules with maximal GR measure.

A rational number \( \mu \) is called a GR measure for an algebra \( \Lambda \) if there is an indecomposable \( \Lambda \)-module \( M \) such that \( \mu(M) = \mu \). Using the Gabriel-Roiter measure, Ringel obtained a partition of the module category for any Artin algebra of infinite representation type [11, 12]: there are infinitely many GR measures \( \mu_i \) and \( \mu_i \) with \( i \) natural numbers, such that

\[
\mu_1 < \mu_2 < \mu_3 < \ldots < \mu^3 < \mu^2 < \mu^1
\]

and such that any other GR measure \( \mu \) satisfies \( \mu_i < \mu < \mu^j \) for all \( i, j \). The GR measures \( \mu_i \) (resp. \( \mu^i \)) are called take-off (resp. landing) measures. Any other GR measure is called a central measure. An indecomposable module is called a take-off (resp. central, landing) module if its GR measure is a take-off (resp. central, landing) measure.

Key words and phrases. Tame quiver of type \( \tilde{A}_n \), string modules, Gabriel-Roiter measure, direct predecessor.

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To calculate the GR measure of a given indecomposable module, it is necessary to know the GR submodules. Thus it is interesting to know the number of the isomorphism classes of the GR submodules for a given indecomposable module. It was conjectured that for a representation-finite algebra (over an algebraically closed field), each indecomposable module has at most three GR submodules. In [5], we have proved the conjecture for representation-finite hereditary algebras. In this paper, we will start to study the GR submodules of string modules. In particular, we will show in Section 3 that each string module, which contains no band submodules, has at most two GR submodules, up to isomorphism. As an application, we show for a tame quiver \( Q \) of type \( \tilde{A}_n, n \geq 1 \), that if an indecomposable module on an exceptional regular component of the Auslander-Reiten quiver has, up to isomorphism, precisely two GR inclusions, then one of the two GR inclusions is an irreducible monomorphism. A description of the numbers of the GR submodules will also be presented there.

Let \( \mu, \mu' \) be two GR measures for \( \Lambda \). We call \( \mu' \) a direct successor of \( \mu \) if, first, \( \mu < \mu' \) and second, there does not exist a GR measure \( \mu'' \) with \( \mu < \mu'' < \mu' \). The so-called Successor Lemma in [12] states that any Gabriel-Roiter measure \( \mu \) different from \( \mu_1 \), the minimal one, over a representation-finite Artin algebra has a direct predecessor. In this paper, we start to study the GR measures admitting no direct predecessors for representation-infinite algebras. An ideal test class are the path algebras of tame quivers, as the representation theory of this category is very well understood and many first properties of the GR-measures for these algebras are already known [8]. Among them, the quivers of type \( \tilde{A}_n, n \geq 1 \), are of special interests because the GR submodules of an indecomposable module can be in some sense easily determined (see [7], or Proposition 2.2 below). In Section 4 it will be shown that for a quiver of type \( \tilde{A}_n \) there are only finitely many GR measures admitting no direct predecessors. This gives rise to the following question: does a representation-infinite (hereditary) algebra (over an algebraically closed field) being of tame type imply that there are only finitely many GR measures having no direct predecessors and vice versa? The proof of the above mentioned result for quivers of type \( \tilde{A}_n \) can be generalized to those of types \( \tilde{D}_n (n \geq 4), \tilde{E}_6, \tilde{E}_7 \) and \( \tilde{E}_8 \). Thus we partially answer the above question.

It was shown in [11] that all landing modules are preinjective modules in the sense of Auslander-Smalø [2]. However, not all preinjective modules are landing modules in general. It is interesting to study the preinjective modules, which are in the central part. In Section 5 We will show that for a tame quiver \( Q \) of type \( \tilde{A}_n \), if there is a preinjective central module, then there are actually infinitely many of them. However, it is possible that the central part does not contain any preinjective module. We characterize the tame quivers of type \( \tilde{A}_n \) with this property. In particular, we show that the quiver \( Q \) of type \( \tilde{A}_n \) is equipped with a sink-source orientation if and only if any indecomposable preinjective module is either a landing module or a take-off module.

Throughout, we fix an algebraically closed field \( k \) and by algebras, representations or modules we always mean finite dimensional ones over \( k \), unless stated otherwise.

2. Preliminaries and known results

2.1. The Gabriel-Roiter measure. We first recall the original definition of the Gabriel-Roiter measure [11]. Let \( \mathbb{N}=\{1,2,\ldots\} \) be the set of natural numbers and \( \mathcal{P}(\mathbb{N}) \) be the set of all subsets of \( \mathbb{N} \). A total order on \( \mathcal{P}(\mathbb{N}) \) can be defined as follows: if \( I,J \) are two different subsets of \( \mathbb{N} \), write \( I < J \)
if the smallest element in \((I \setminus J) \cup (J \setminus I)\) belongs to \(J\). Also we write \(I \ll J\) provided \(I \subset J\) and for all elements \(a \in I, b \in J \setminus I\), we have \(a < b\). We say that \(J\) starts with \(I\) if \(I = J\) or \(I \ll J\).

Let \(\Lambda\) be an Artin algebra and \(\text{mod}\ \Lambda\) be the category of finitely generated left \(\Lambda\)-modules. For each \(M \in \text{mod}\ \Lambda\), let \(\mu(M)\) be the maximum of the sets \(\{|M_1|, |M_2|, \ldots, |M_t|\}\), where \(M_1 \subset M_2 \subset \ldots \subset M_t\) is a chain of indecomposable submodules of \(M\). We call \(\mu(M)\) the Gabriel-Roiter measure of \(M\). If \(M\) is an indecomposable \(\Lambda\)-module, we call an inclusion \(\mu\) provided \(\mu(M) = \mu(T) \cup \{|M|\}\), thus if and only if every proper submodule of \(M\) has Gabriel-Roiter measure at most \(\mu(T)\). In this case, we call \(T\) a GR submodule of \(M\). A subset \(I\) of \(\mathbb{N}\) is called a GR measure of \(\Lambda\) if there is an indecomposable module \(M\) with \(\mu(M) = I\).

Remark Although in the introduction we define the Gabriel-Roiter measure in a different way, these two definitions (orders) coincide. In fact, for each \(I = \{a_i | i\} \in P(\mathbb{N})\), let \(\mu(I) = \sum_{i \geq 1} \frac{1}{2^{a_i}}\). Then \(I < J\) if and only if \(\mu(I) < \mu(J)\).

We denote by \(\mathcal{T}, \mathcal{C}\) and \(\mathcal{Z}\) the collection of indecomposable \(\Lambda\)-modules, which are in the take-off part, the central part and the landing part, respectively. Now we present one result concerning Gabriel-Roiter measures, which will be quite often used in our later discussion. The proofs can be found in \([7\text{, Section 3}]\).

**Proposition 2.1.** Let \(\Lambda\) an Artin algebra and \(X \subset M\) be a GR inclusion.

1. If \(\mu(X) < \mu(Y) < \mu(M)\), then \(|Y| > |M|\).
2. There is an irreducible monomorphism \(X \rightarrow Y\) with \(Y\) indecomposable and an epimorphism \(Y \rightarrow M\).

The first statement is a direct consequence of the definition. For a proof of the second statement, we refer to \([6\text{, Proposition 3.2}]\).

2.2. Let \(Q\) be a tame quiver of type \(\tilde{A}_n, n \geq 1, \tilde{B}_n, n \geq 4, \tilde{E}_6, \tilde{E}_7\) or \(\tilde{E}_8\), and \(\text{Rep}(Q)\) the category of finite dimensional representations over an algebraically closed field. We simply call the representations in \(\text{Rep}(Q)\) modules. We briefly recall some notations and refer to \([1, 10]\) for details. If \(X\) is a quasi-simple module, then there is a unique sequence \(X = X_1 \rightarrow X_2 \rightarrow \ldots \rightarrow X_r \rightarrow \ldots\) of irreducible monomorphisms. Any indecomposable regular module \(M\) is of the form \(M \cong X_i\) with \(X\) a quasi-simple module (quasi-socle of \(M\)) and \(i\) a natural number (quasi-length of \(M\)). The rank of an indecomposable regular module \(M\) is the minimal positive integer \(r\) such that \(\tau^r M = M\), where \(\tau = DT\tau\) denotes the Auslander-Reiten translation. A regular component (standard stable tube) of the Auslander-Reiten quiver of \(Q\) is called exceptional if its rank (the rank of any quasi-simple module on it) \(r > 1\). If \(X\) is quasi-simple of rank \(r\), then the dimension vector \(\dim X_i = \delta = \sum_{i=1}^r \tau^i X,\) where \(\delta\) is the minimal positive imaginary root. Let \(|\delta| = \sum_{\nu \in Q} \delta\). In particular, if \(Q\) is of type \(\tilde{A}_n\), then \(\delta_\nu = 1\) for each \(\nu \in Q\) and \(|\delta| = n + 1\). A quasi-simple module of rank 1 will be called a homogeneous quasi-simple module. We denote by \(H_i\) an indecomposable homogeneous module with quasi-length \(i\). (There are infinitely many homogeneous tubes. However, the GR measure \(\mu(H_i)\) does not depend on the choice of \(H_i\).) We denote by \(P, R\) and \(Z\) the collection of indecomposable preprojective, regular and preinjective modules, respectively.

We collect some known facts in the following proposition, which will be quite often used in our later discussion. The proofs can be found in \([7\text{, Section 3}]\).

**Proposition 2.2.** Let \(Q\) be a tame quiver of type \(\tilde{A}_n, n \geq 1\).

1. Let \(\iota: T \subset M\) be a GR inclusion.
a If $M \in \mathcal{P}$, then $\tau$ is an irreducible monomorphism.
b If $M \in \mathcal{R}$ is a quasi-simple module, then $T \in \mathcal{P}$.
c If $M = X_i \in \mathcal{R}$ with $X$ quasi-simple and $i > 1$, then $T \in \mathcal{P}$ or $T \cong X_{i-1}$.
d If $M \in \mathcal{I}$, then $T \in \mathcal{R}$.

(2) If $X \in \mathcal{P}$, then $X \in \mathcal{T}$ and $\mu(X) < \mu(H_1)$.

(3) Let $H_1$ be a homogeneous quasi-simple module. Then $\mu(H_1)$ is a central measure and $\mu(M) > \mu(H_1)$ if $M \in \mathcal{I}$ satisfies $\dim M > \delta$.

(4) Let $X$ be a quasi-simple module of rank $r$. Then
a If $\mu(X_r) < \mu(H_1)$, then $\mu(X_i) < \mu(H_j)$ for all $i \geq 1$ and $j \geq 1$.
b If $\mu(X_r) \geq \mu(H_1)$, then $X_i$ is the unique GR submodule of $X_{i+1}$ for every $i \geq r$. If in addition $r > 1$, then $\mu(X_i) > \mu(H_1)$ for all $i > r$ and $j \geq 1$.

(5) Let $\mathcal{T}$ be a stable tube of rank $r > 1$. Then there is a quasi-simple module $X$ on $\mathcal{T}$ such that $\mu(X_r) \geq \mu(H_1)$.

(6) Let $S$ be a quasi-simple module of rank $r$ which is also a simple module. Then $\mu(S_r) < \mu(H_1)$, and thus $\mu(S_j) < \mu(H_1)$ for all $j \geq 1$.

(7) Let $M \in \mathcal{I} \setminus \mathcal{T}$ and $N$ be a GR submodule of $M$. Thus $N \cong X_i$ for some quasi-simple module $X$ by (1)d. Then $\mu(M) > \mu(X_j)$ for all $j \geq 1$.

**Remark** Some statements in Proposition 2.2 hold in general. For example, the statements (2), (4) and (7) and the first argument of the statement (3) also hold for tame quivers of type $\tilde{E}_n$, $\tilde{E}_7$, and $\tilde{E}_8$. The statement (5) is known to be true for tame quivers which is not of the $\tilde{E}_8$ type.

**Lemma 2.3.** Let $Q$ be a tame quiver of type $\tilde{A}_n$. Then for every indecomposable preinjective module $M$, there is, in each regular component, precise one quasi-simple module $X$ such that $\Hom(X, M) \neq 0$. In particular, up to isomorphism, each indecomposable preinjective module contains in each regular component at most one GR submodule.

**Proof.** Let $M = \tau^s I_\nu$, where $I_\nu$ is an indecomposable injective module corresponding to a vertex $\nu \in Q$. It is obvious that there is a quasi-simple module $X$ on a given regular component such that $\Hom(X, I_\nu) \neq 0$. Thus $\Hom(\tau^sX, M) \neq 0$. Assume that $X$ and $Y$ are non-isomorphic quasi-simple modules on the same tube such that $\Hom(X, M) \neq 0 \neq \Hom(Y, M)$. Then $\Hom(\tau^sX, I_\nu) \neq 0 \neq \Hom(\tau^{-s}Y, I_\nu)$. Thus $(\dim \tau^{-s}X)_\nu \neq 0 \neq (\dim \tau^{-s}Y)_\nu$, which is impossible since $1 = \delta_\nu \geq (\dim \tau^{-s}X)_\nu + (\dim \tau^{-s}Y)_\nu$. \qed

3. THE NUMBER OF GR SUBMODULES

As we have mentioned in the introduction, the number of the GR submodules of a given indecomposable module over a representations-finite algebra is conjectured to be bounded by three. In this section, we will show for a string algebra that this number is always bounded by two for any indecomposable string module containing no band submodules. We will also describe the numbers of the GR submodules for different kinds of indecomposable modules over tame quivers $Q$ of type $\tilde{A}_n$.

3.1. String modules. We first recall what string modules are. For details, we refer to [3]. Let $\Gamma$ be a string algebra with underlying quiver $Q$. We denote by $s(C)$ and $e(C)$ the starting and the ending vertices of a given string $C$, respectively. Let $C = c_n c_{n-1} \cdots c_2 c_1$ be a string, the corresponding string module $M(C)$ is defined as follows: let $u_i = s(c_{i+1})$ for $0 \leq i \leq n - 1$ and $u_n = e(c_n)$. For
a vertex $\nu \in Q$, let $I_{\nu} = \{ i | u_i = \nu \} \subset \{0, 1, \ldots, n \}$. Then the vector space $M(C)_{\nu}$ associated to $\nu$ satisfies that $\dim M(C)_{\nu} = |I_{\nu}|$ and has $z_i, i \in I_{\nu}$ as basis. If for $1 \leq i \leq n$, the symbol $c_i$ is an arrow $\beta$, define $\beta(z_{i-1}) = z_i$. If for $1 \leq i \leq n$, the symbol $c_i$ is an inverse of an arrow $\beta$, define $\beta(z_i) = z_{i-1}$.

Note that the indecomposable string modules are uniquely determined by the underlying string, up to the equivalence given by $C \sim C^{-1}$.

If $C = E\beta F$ is a string with $\beta$ an arrow, then the string module $M(E)$ is a submodule of $M(C)$: let $E$ be of length $n$ and $F$ be of length $m$. Then $C$ has length $n + m + 1$. If $M(C)$ is given by $n + m + 2$ vectors $z_0, z_1, \ldots, z_{n+m+2}$, it is obvious that the space determined by the vectors $z_0, z_1, \ldots, z_n$ defines a submodule, which is $M(E)$. The corresponding factor module is $M(F)$. If $C = E\beta^{-1} F$ is a string with $\beta$ an arrow, we may obtain similarly an indecomposable submodule $M(F)$ of $M(C)$ with factor module $M(E)$.

3.2. A “covering” of a string module. Let $C = c_n c_{n-1} \cdots c_2 c_1$ be a string. We associate with $C$ a Dynkin quiver $\tilde{\alpha}_{n+1}$ as follows: the vertices are $u_i$, and there is an arrow $u_{i-1} \overset{\alpha}{\rightarrow} u_i$ if $c_i$ is an arrow, and an arrow $u_i \overset{\alpha}{\rightarrow} u_{i-1}$ in case $c_i$ is an inverse of an arrow. Let $M(C)$ be the string module and $M_{\tilde{\alpha}}(C)$ be the unique sincere indecomposable representation over $\tilde{\alpha}_{n+1}$.

Before going further, we introduce the follow lemma, which was proved in [12].

Lemma 3.1. Let $M$ and $N$ be indecomposable modules over an Artin algebra $\Lambda$ and $\text{Sing}(N, M)$ be the set of all non-injective homomorphisms. If $N$ is a GR submodule of $M$, then $\text{Sing}(N, M)$ is a subspace of $\text{Hom}_\Lambda(N, M)$.

From now on, we assume that $M(C)$ is a string module over $\Gamma$ determined by a string $C$ such that $M(C)$ contains no band submodules. Thus any submodule of $M$ is a string module.

Lemma 3.2. Let $N$ be a GR submodule of $M(C)$. Then there is a substring $C'$ of $C$, such that the submodule $X$ of $M(C)$ determined by $C'$ is isomorphic to $N$.

Proof. Let $f : N \subset M(C)$ be the inclusion. Then $f$ is a linear combination of a basis described in [4], say $f_1, \ldots, f_t$. Since by Lemma 3.1 $\text{Sing}(N, M)$ is a subspace, there is an $1 \leq i \leq t$ such that $f_i$ is a monomorphism. By the description of the basis, $X = \text{im} f_i$ is a submodule of $M$ determined by a substring $C'$ of $C$. It is obvious that $N \cong X$. \hfill $\square$

By this lemma, we may assume without loss of generality that a GR submodule $N$ of $M(C)$ is always given by a substring of $C$. Thus we may obtain in an obvious way an indecomposable submodule $G(N)$ of $M_{\tilde{\alpha}}(C)$ using the construction of the quiver $\tilde{\alpha}_{n+1}$.

Remark Note that different monomorphisms $f_i$ in the basis in the proof of Lemma 3.2 may give rise to different $G(N)$, which are indecomposable and pairwise non-isomorphic. They all have the same length $|N|$ by the construction.

More generally, let $N_1 \subset N_2 \subset \ldots \subset N_s = N \subset N_{s+1} = M(C)$ be a GR filtration of $M(C)$. By above discussion, we may assume that there is a sequence of substrings $C_1 \subset C_2 \subset \ldots \subset C_s \subset C$ such that $N_i$ is determined by the substring $C_i$. Thus we may define $G(N_i)$ using the construction of the quiver $\tilde{\alpha}_{n+1}$. Therefore, we get inclusions of indecomposable modules over the quiver $\tilde{\alpha}_{n+1}$: $G(N_1) \subset G(N_2) \subset \ldots \subset G(N_s) \subset M_{\tilde{\alpha}}(C)$. Conversely, if $T$ is a submodule of $M_{\tilde{\alpha}}(C)$ with inclusion map $f$, then there is a natural way to get a string submodule $F(T)$ of $M(C)$. We may also denote the inclusion by $f : F(T) \subset M$. Then under the inclusion, $G F(T) = T$. It is easily seen that
$\mathcal{F}(\mathcal{G}(N) \cong N$. Note that $\mathcal{F}$ preserves indecomposables, monomorphisms and lengths. In particular, $\mu(T) \leq \mu(\mathcal{F}(T))$.

**Lemma 3.3.** We keep the notations as above.

1. For each $i$, $\mathcal{G}(N_i)$ is a GR submodule of $\mathcal{G}(N_{i+1})$ and $\mu(\mathcal{G}(N_i)) = \mu(N_i)$.
2. If $T$ is a GR submodule of $M_\mu(C)$, then $\mathcal{F}(T)$ is a GR submodule of $M$.

**Proof.** (1) We use induction. Assume first that $i = 1$ and that $\mathcal{G}(N_1)$ is not a GR submodule of $\mathcal{G}(N_2)$. Let $X \subset \mathcal{G}(N_2)$ be a GR submodule and thus $\mathcal{F}(X)$ is isomorphic to a submodule of $N_2$. Since $\mathcal{F}$ preserves monomorphisms, $\mu(N_1) \geq \mu(\mathcal{F}(X)) \geq \mu(X) > \mu(\mathcal{G}(N_1))$. This is impossible since $N_1$ and $\mathcal{G}(N_1)$ are both simple modules. Thus $\mathcal{G}(N_1)$ is a GR submodule of $\mathcal{G}(N_2)$. It follows $\mu(\mathcal{G}(N_2)) = \mu(N_2)$ since $|\mathcal{G}(N_2)| = |N_2|$.

Now assume that $\mu(\mathcal{G}(N_i)) = \mu(N_i)$. Let $X$ be a GR submodule of $\mathcal{G}(N_{i+1})$. Then $\mu(\mathcal{G}(N_{i+1})) \leq \mu(X) \leq \mu(\mathcal{F}(X)) \leq \mu(N_i)$. Therefore, $\mu(\mathcal{G}(N_{i+1})) = \mu(N_{i+1})$ by induction and $\mathcal{G}(N_{i+1})$ is a GR submodule of $\mathcal{G}(N_{i+1})$. Hence $\mu(\mathcal{G}(N_{i+1})) = \mu(N_{i+1})$ since $|\mathcal{G}(N_{i+1})| = |N_{i+1}|$.

(2) Since $N$ is a GR submodule of $M(C)$, $\mu(T) = \mu(\mathcal{G}(N)) = \mu(N) \geq \mu(\mathcal{F}(T)) \geq \mu(T)$. Thus $\mu(N) = \mu(\mathcal{F}(T))$ and $\mathcal{F}(T)$ is a GR submodule of $M(C)$. 

For a Dynkin quiver of type $A$, we have shown in [5] the following result:

**Proposition 3.4.** Let $Q$ be a Dynkin quiver of type $A$. Then each indecomposable module has at most two GR submodules and each factor of a GR inclusion is a uniserial module.

As a consequence of this proposition and Lemma 3.3, we have

**Theorem 3.5.** Let $\Lambda$ be a string algebra and $M(C)$ be a string module containing no band submodules. Then $M(C)$ contains, up to isomorphism, at most two GR submodules and the factors of the GR inclusions are uniserial modules.

**Corollary 3.6.** If $\Lambda$ is a representation-finite string algebra, then each indecomposable module has, up to isomorphism, at most two GR submodules and the GR factors are uniserial.

3.3. Now we assume that $Q$ is a tame quiver of type $\tilde{A}_n$. Then every indecomposable regular module with rank $r > 1$ is a string module containing no band submodules, thus has at most two GR submodules up to isomorphism by above theorem.

**Proposition 3.7.** If an exceptional regular module has precisely two GR submodules, then one of the GR inclusions is an irreducible map. In particular, every exceptional regular module has at most one preprojective GR submodule, up to isomorphism.

Before proving this proposition, we briefly recall how the irreducible monomorphisms between string modules look like. We refer to [3] for details. Let $C = c_r \cdots c_2 c_1$ be a string such that $c_r$ is an arrow. If there is an arrow $\gamma$ with $e(\gamma) = e(c_r)$, let $D = d_1 \cdots d_2 d_1$ be a string with $s(D) = s(d_1) = s(\gamma)$ such that $d_i$ is an arrow for every $i$ and such that $t \geq 0$ is maximal. Then the natural inclusion $M(C) \to M(D) \gamma^{-1} C$ is an irreducible monomorphism. Similarly, Let $C = c_r \cdots c_2 c_1$ be a string such that $c_1$ is an arrow. If there is an arrow $\gamma$ with $e(\gamma) = s(c_1)$, let $D = d_i \cdots d_2 d_1$ be a string with $e(D) = e(d_i) = s(\gamma)$ such that $d_i$ is an inverse of an arrow for every $i$ and such that $t \geq 0$ is maximal. Then the natural inclusion $M(C) \to M(C \gamma D)$ is an irreducible monomorphism. Any irreducible monomorphism between string modules is of one of these forms.
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Proof. Let \( M(C) \) be an exceptional regular module with \( C = c_m \cdots c_2 c_1 \), which has precisely (up to isomorphism) two GR submodules. Then the module \( M_{\alpha}(C) \) also has two GR submodules, which are actually given by the irreducible monomorphism \( X \rightarrow M_{\alpha}(C) \) and \( Y \rightarrow M_{\alpha}(C) \). By definition of \( M_{\alpha}(C) \), we may identify the arrows \( \alpha_i \) or its inverse in \( \tilde{\mathbb{A}}_{m+1} \) with \( c_i \) in the string \( C \) of \( \mathbb{A}_n \). We may assume that \( X \) is determine by string \( E \) and \( M_{\alpha}(C) \) is determined by \( F \alpha^{-1} E \), where \( F \) is a composition of arrows or a trivial path and \( \alpha \) is an arrow. Then under the above identification, we have \( C = F \alpha^{-1} E \). Let \( M(C) \rightarrow M' \) be the unique irreducible monomorphism with \( M' \) determined by a string \( F' \beta^{-1} F \alpha^{-1} E \), where \( F' \) is a composition of arrows or a trivial path and \( \beta \) is an arrow. Thus either the ending vertex \( e(F) \) is a sink, or \( F \) is a trivial path. Again by the description of irreducible monomorphism, the inclusion \( F(X) \rightarrow M(C) \) is still an irreducible map. Note that \( F(X) \) is a GR submodule of \( M(C) \) by Lemma 3.3.

Remark Let \( Q \) be a tame quiver of type \( \tilde{\mathbb{A}}_n \) and \( M \) be a non-simple indecomposable module. Let \( gr(M) \) denote the number of the isomorphism classes of the GR submodules of \( M \).

1. If \( M \) is preprojective, each GR inclusion \( X \subset M \) is namely an irreducible map (Proposition 2.2(1)a). In particular, \( gr(M) \leq 2 \) since there are precisely two irreducible maps to \( M \), which are monomorphisms.

2. If \( M \) is a quasi-simple module of rank \( r > 1 \), then \( gr(M) = 1 \) since \( M \) is uniserial.

3. If \( M \) is a non-quasi-simple regular module of rank \( r > 1 \), then \( gr(M) \leq 2 \), and one of the GR inclusion is irreducible in case \( gr(M) = 2 \) (Proposition 3.7).

4. If \( M = X_i \) is a regular module with \( i > 1 \), where \( X \) is a quasi-simple module of rank \( r \geq 1 \) with \( \mu(X_r) \geq \mu(H_1) \), then \( gr(M) = 1 \) and the unique GR inclusion is an irreducible map (Proposition 2.2(4)b).

5. If \( M \) is preinjective, then \( M \) contains, up to isomorphism, at most one GR submodule in each regular component (Lemma 2.3). If we identify the homogeneous modules \( H_i \), then \( gr(M) \leq 3 \). Under the convention, if \( gr(M) = 3 \), then \( M \) contains a homogeneous module \( H_i \) and as well as an exceptional module \( X_j \) with rank \( r > 1 \) as GR submodules. Note that \( \mu(X_{mr}) \neq \mu(H_s) \) for any \( m > 1 \) and any \( s > 1 \). It follows that \( i = 1 \) and \( j = r \) and thus \( \delta = \dim X_j = \dim H_i \). Therefore, \( |M| < 2|\delta| \) (Proposition 2.1(2)).

6. A homogeneous simple module \( H_1 \) may contains more GR submodules. For example, if \( n \) is odd and \( Q \) is with a sink-source orientation (see [7] Example 3), then the GR measure of a homogeneous simple module is \( \mu(H_1) = \{1, 3, 5, \ldots, n, n+1\} \). There are up to isomorphism \( \frac{n+1}{2} \) indecomposable preprojective modules with length \( n \) and they are all non-isomorphic GR submodules of \( H_1 \). In general, \( gr(H_1) \) is bounded by the number of the indecomposable summands of the projective cover of \( H_1 \).

7. One may also define \( gr(M) \) as the number of the dimension vectors of the GR submodules of \( M \). Then it is easily seen that \( gr(M) \leq 2 \) for each indecomposable module \( M \), which is not a homogeneous quasi-simple module \( H_1 \).

4. Direct Predecessor

Let \( \Lambda \) be an Artin algebra and \( I \) and \( J \) be two different GR measures for \( \Lambda \). Then \( J \) is called a direct successor of \( I \) if, first, \( I < J \) and second, there does not exist a GR measure \( J' \) with \( I < J' < J \). It is easily seen that if \( J \) is the direct successor of \( I \), then \( J \) is a take-off (resp. central, landing) measure if and only if so is \( I \). Let \( I^1 \) be the largest GR measure, i.e. the GR measure of an
indecomposable injective module with maximal length. It was proved in [12] that any Gabriel-Roiter measure \( I \) different from \( I^1 \) has a direct successor. However, there are GR measures, which does not admit a direct predecessor. By the construction of the take-off measures and the landing measures [11], the GR measures having no direct predecessors are central measures.

4.1. From now on, we fix a tame quiver \( Q \) of type \( \widehat{k}_n \). The following proposition gives a GR measure possessing no direct predecessor.

**Proposition 4.1.** The GR measure \( \mu(H_1) \) of a homogeneous quasi-simple module \( H_1 \) has no direct predecessor.

**Proof.** Assume, to the contrary, that \( \mu(M) \) is the direct predecessor of \( \mu(H_1) \) for some indecomposable module \( M \). Since \( \mu(H_1) \) is a central measure, so is \( \mu(M) \). It follows that \( M \) is not preprojective. Let \( Y \) be a GR submodule of \( H_1 \). Since \( Y \) is preprojective, \( \mu(Y) < \mu(M) < \mu(H_1) \) and thus \( |M| > |H_1| \) (Proposition 2.11). If \( M \) is preinjective, then there is a monomorphism \( H_1 \to M \) because \( |M| > |H_1| \), and hence \( \mu(H_1) \nless \mu(M) \). This contradiction implies that \( M \) is a regular module.

Assume that \( M = X_t \) for some quasi-simple module \( X_t \) of rank \( r \) \( > 1 \). Because \( |X_t| = |M| > |H_1| \), we have \( i > r \). Therefore, \( \mu(X_r) < \mu(M) < \mu(H_1) \). It follows from that \( \mu(M) < \mu(X_1) < \mu(H_1) \) for all \( j \geq i \) (Proposition 2.14a). This is a contradiction. \( \Box \)

**Proposition 4.2.** Let \( M \in \mathcal{I} \setminus \mathcal{I} \). If \( \mu(N) \) is the direct predecessor of \( \mu(M) \) for some indecomposable module \( N \), then \( N \in \mathcal{I} \) and \( |N| > |M| \).

**Proof.** Since \( \mu(N) \) is not a take-off measure, \( N \) is not preprojective. Assume for a contradiction that \( N = Y_j \) is regular for some quasi-simple module \( Y_j \). Let \( X_t \) be a GR submodule of \( M \) for some quasi-simple module \( X_t \) and some \( t \geq 1 \) (Proposition 2.1d). Then \( \mu(M) > \mu(X_t) \) for all \( t \geq 1 \) by Proposition 2.7 and thus \( \mu(X_t) \neq \mu(Y_j) \) for any \( t \geq 1 \). Therefore \( \mu(X_t) < \mu(Y_j) < \mu(M) \).

It follows that \( |Y_j| > |M| \) and \( \mu(M) < \mu(Y_{j+1}) \) since \( \mu(N) = \mu(Y_j) \) is the direct predecessor of \( \mu(M) \). Notice that a GR submodule \( T \) of \( Y_{j+1} \) is either \( Y_j \) or a preprojective module. In particular \( \mu(T) < \mu(M) < \mu(Y_{j+1}) \). Thus \( |M| > |Y_{j+1}| \) by Proposition 2.1(1). This contradicts \( |N| = |Y_j| > |M| \). Therefore, \( N \) is preinjective. \( \Box \)

4.2. A regular module \( X_t \), \( i > r \), for some quasi-simple module \( X \) of rank \( r \) may contain a preprojective module as a GR submodule. However, this cannot happen if \( \mu(X_r) \geq \mu(H_1) \), in which case the irreducible monomorphisms \( X_r \to X_{r+1} \to X_{r+2} \to \ldots \) are GR inclusions (Proposition 2.6b). The GR measure \( \mu(X_r) \) for a quasi-simple module \( X \) of rank \( r \) is important when comparing the GR measures of regular modules \( X_t \) and those of homogeneous modules \( H_j \). Namely, there is a similar result that can be used to compare the GR measures of two non-homogeneous regular modules.

**Lemma 4.3.** Let \( X, Y \) be quasi-simple modules of rank \( r \) and \( s \), respectively. Assume that \( \mu(X_r) \geq \mu(H_1) \).

(1) If \( \mu(X_r) > \mu(Y_s) \), then \( \mu(X_i) > \mu(Y_j) \) for all \( i \geq r, j \geq 1 \).

(2) If \( \mu(X_i) = \mu(Y_j) \) for some \( i \geq 2r \), then \( r = s \) and \( \mu(X_i) = \mu(Y_i) \) for every \( t \geq r \).

(3) If \( \mu(X_{2r}) > \mu(Y_{s}) \), then \( \mu(X_i) > \mu(Y_j) \) for all \( i \geq 2r, j \geq 1 \).

**Proof.** (1) If \( \mu(Y_s) < \mu(H_1) \), then \( \mu(Y_j) < \mu(H_1) \) for all \( j \geq 1 \). Thus we may assume that \( \mu(Y_s) \nless \mu(H_1) \). Since for each \( j \geq s, \mu(Y_j) \) starts with \( \mu(Y_s) \) and \( |Y_s| = |X_r| = |\delta| \), we have \( \mu(X_r) > \mu(Y_j) \).
(2) By assumption, we have \( j \geq 2s \) because \( |Y_j| = |X_i| \geq 2|\delta| \). Since \( \mu(X_r) \geq \mu(H_1) \), we have \( \mu(Y_j) \geq \mu(H_1) \) and the irreducible monomorphisms in the sequences

\[
X_r \to X_{r+1} \to X_{r+2} \to \ldots, \quad \text{and} \quad Y_s \to Y_{s+1} \to Y_{s+2} \to \ldots
\]

are all GR inclusions (Proposition 2.2(4)). It follows that

\[
\mu(Y_j) = \mu(Y_s) \cup \{ |Y_{s+1}|, |Y_{s+2}|, \ldots, |Y_{2s-1}|, |Y_{2s}|, |Y_{2s+1}|, \ldots, |Y_j| \}
\]

\[
= \mu(X_r) \cup \{ |X_{r+1}|, |X_{r+2}|, \ldots, |X_{2r-1}|, |X_{2r}|, |X_{2r+1}|, \ldots, |X_i| \}
\]

Since \( |X_r| = |Y_s| = |\delta| \) and \( |X_{2r}| = |Y_{2s}| = 2|\delta| \), so \( \mu(X_r) = \mu(Y_s) \) and \( \mu(X_{2r}) = \mu(Y_{2s}) \). Note that

\[
|X_{r+l}| - |X_{r+l-1}| = |Y_{r+l}| - |Y_{r+l-1}|
\]

for all \( l \geq 1 \). It follows that \( r = s \) and \( \mu(X_t) = \mu(Y_t) \) for all \( t \geq r = s \).

(3) follows similarly. \( \square \)

**Corollary 4.4.** Let \( X \) be a quasi-simple module of rank \( r \) such that \( \mu(X_r) \geq \mu(H_1) \). If \( M \) is an indecomposable module such that \( \mu(M) = \mu(X_i) \) for some \( i \geq 2r \), then \( M \) is a regular module.

**Proof.** Assume, to the contrary, that \( M \) is preinjective. Let \( Y_j \) be a GR submodule of \( M \) for some quasi-simple module \( Y \) of rank \( s \). Then \( \mu(M) > \mu(Y_j) \) for all \( j \geq 1 \) by Proposition 2.2(7). Thus \( Y \not\cong X \) and \( t \geq 2s \) since \( |M| = |X_i| > 2|\delta| \) and since there is an epimorphism \( Y_{i+1} \to M \) by Proposition 2.1(2). Notice that \( \mu(Y_i) \geq \mu(H_1) \). Otherwise, we would have \( \mu(Y_i) < \mu(H_1) \). However, there is a monomorphism \( H_1 \to M \) since \( |M| > 2|\delta| \). We obtain a contradiction because \( Y_1 \) is a GR submodule of \( M \). Since \( Y_i \) is a GR submodule of \( N \) and \( \mu(M) = \mu(X_i) \), so \( \mu(Y_i) = \mu(X_{i-1}) \). Therefore, \( r = s \) and \( \mu(Y_{i+1}) = \mu(X_i) \) by above lemma. This contradicts \( |Y_{i+1}| > |M| = |X_i| \) (Proposition 2.1(1)). Thus \( M \) is regular. \( \square \)

4.3. We have seen in Proposition 2.2(4) that the irreducible maps \( H_1 \to H_2 \to H_3 \to \ldots \) are GR inclusions. One can show more: namely, in \( [7] \) (or \( [8] \) for general cases) we proved that \( \mu(H_{i+1}) \) is the direct successor of \( \mu(H_i) \) for each \( i \geq 1 \).

Let \( X \) be a quasi-simple module of rank \( r > 1 \). It is possible (for example, if \( \mu(X_r) \geq \mu(H_1) \)) that all irreducible maps \( X_r \to X_{r+1} \to X_{r+2} \to \ldots \) are GR inclusions. However, it is not true in general that \( \mu(X_{j+1}) \) is the direct successor of \( \mu(X_j) \) for all \( j \geq r \) (\( [7] \) Example 4(1)). The following proposition tells if \( \mu(X_r) \geq \mu(H_1) \) and if \( \mu(X_{j+1}) \) is not the direct successor of \( \mu(X_j) \), then \( j < 2r \).

**Proposition 4.5.** Let \( X \) be a quasi-simple module of rank \( r \) such that \( \mu(X_r) \geq \mu(H_1) \). Then \( \mu(X_{j+1}) \) is a direct successor of \( \mu(X_j) \) for each \( j \geq 2r \).

**Proof.** We may assume \( r > 1 \). We first show that there does not exist an indecomposable regular module \( M \) such that \( \mu(M) \) lies between \( \mu(X_j) \) and \( \mu(X_{j+1}) \) for any \( j \geq 2r \). Assume, to the contrary, that there exists a \( j \geq 2r \) and an indecomposable regular module \( M \) with \( \mu(X_j) < \mu(M) < \mu(X_{j+1}) \). We may assume that \( |M| \) is minimal. Then \( |M| > |X_{j+1}| > 2|\delta| \), since \( X_j \) is a GR submodule of \( X_{j+1} \). Let \( M = Y_i \) for some quasi-simple module \( Y \) of rank \( s > 1 \). It follows that \( \mu(Y_s) \geq \mu(H_1) \) and \( i > 2s \). Therefore, \( Y_{i-1} \) is a GR submodule of \( Y_i \) and

\[
\mu(Y_{i-1}) \leq \mu(X_j) < \mu(M) = \mu(Y_i) < \mu(X_{j+1})
\]

by minimality of \( |M| \). This implies \( \mu(Y_{i-1}) = \mu(X_j) \), since otherwise \( |X_j| > |M| > |X_{j+1}| \), which is a contradiction. Observe that \( i - 1 \geq 2s \) and \( j \geq 2r \). Then Lemma 4.3 implies \( \mu(X_t) = \mu(Y_i) \) for
all \( t \geq r = s \). This contradicts the assumption \( \mu(X_j) < \mu(M) = \mu(Y_t) < \mu(X_{j+1}) \). Therefore, there are no indecomposable regular modules \( M \) satisfying \( \mu(X_j) < \mu(M) < \mu(X_{j+1}) \) for any \( j \geq 2r \).

Now we assume that \( M \) is an indecomposable preinjective module such that \( \mu(X_j) < \mu(M) < \mu(X_{j+1}) \). Since \( \mu(X_t) \geq \mu(H_1) \), so \( X_j \) is a GR submodule of \( X_{j+1} \). It follows that \( |X_{j+1}| > |M| > |X_j| \) by Proposition 2.1(1). Let \( Y_t \) be a GR submodule of \( M \) for some quasi-simple module \( Y \) and \( i \geq 1 \). Then \( Y_t \not\cong X_t \) for all \( t \geq 0 \) by Proposition 2.2(7). Comparing the lengths, we have \( \mu(Y_t) \geq \mu(X_j) \). (Namely, if \( \mu(Y_t) < \mu(X_j) \), then \( |X_j| > |M| \) by Proposition 2.1(1) since \( Y_t \) is a GR submodule of \( M \). But on the other hand, \( |X_j| < |M| \) by previous discussion.) Thus Proposition 2.2(7) implies that \( \mu(X_j) < \mu(Y_{t+1}) < \mu(M) < \mu(X_{j+1}) \). Therefore, we get an indecomposable regular module \( Y_{t+1} \) with GR measure lying between \( \mu(X_j) \) and \( \mu(X_{j+1}) \), which is a contradiction.

The proof is completed. \( \square \)

4.4. Let \( X \) be a quasi-simple module of rank \( r \) such that \( \mu(X_r) > \mu(H_1) \). For a given \( i \geq 2r \), let \( \mu_{i,1} > \mu_{i,2} > \ldots > \mu_{i,t} \) be all different GR measures of the form \( \mu_{i,j} = \mu(X_i) \cup \{a_{i,j}\} \) and \( a_{i,j} \neq X_{i+1} \) for any \( 1 \leq j \leq t_i \). Notice that there are only finitely many such \( \mu_{i,j} \) for each given \( i \).

**Lemma 4.6.**

1. \( a_{i,j} < |X_{i+1}| \) for all \( 1 \leq j \leq t_i \) and \( a_{i,j} < a_{i,l} \) if \( 1 \leq j < l < t_i \).
2. \( \mu_{i,j} > \mu(X_i) \) for all \( 1 \leq j \leq t_i, t_i \geq 1 \).
3. \( \mu_{i,j} > \mu_{i,h} \) if \( i < l \).
4. If \( M \) is an indecomposable module such that \( \mu(M) = \mu_{i,j} \), then \( M \in \mathcal{I} \).

**Proof.**

1. If \( a_{i,j} > |X_{i+1}| \), then

\[
\mu_{i,j} = \mu(X_i) \cup \{a_{i,j}\} < \mu(X_i) \cup \{|X_{i+1}|\} = \mu(X_{i+1}).
\]

This contradicts \( \mu(X_{i+1}) \) is a direct successor of \( \mu(X_i) \) (Proposition 1.5). Thus \( a_{i,j} < |X_{i+1}| \).

2. follows from (1) and the fact that \( X_{2r} \subset X_{2r+1} \subset \ldots \subset X_t \subset X_{t+1} \subset \ldots \) is a sequence of GR inclusions.

3. If \( i < l \), then

\[
\mu_{i,h} = \mu(X_i) \cup \{a_{i,h}\} = \mu(X_i) \cup \{|X_{i+1}|, \ldots, |X_l|, a_{i,h}\} < \mu(X_i) \cup \{a_{i,j}\} = \mu_{i,j}.
\]

4. If \( M \) is not preinjective, then \( M \) is regular, say \( M = Y_t \) for some quasi-simple module \( Y \) of rank \( s \). Thus \( t > 2s \) since \( |M| > |X_s| \geq 2|\delta| \), and \( \mu(Y_s) \geq \mu(H_1) \). In particular, \( Y_{t-1} \) is a GR submodule of \( Y_t \) and \( \mu(Y_{t-1}) = \mu(X_s) < \mu(X_{t+1}) < \mu(M) = \mu(Y_t) \). This is a contradiction since \( \mu(Y_t) \) is also a direct successor of \( \mu(Y_{t-1}) \). \( \square \)

**Proposition 4.7.** The sequence of GR measures

\[
\ldots < \mu_{i+1,2} < \mu_{i+1,1} < \mu_{i,t_i} < \ldots < \mu_{i,j+1} < \mu_{i,j} < \ldots < \mu_{i,2} < \mu_{i,1}
\]

is a sequence of direct predecessors.

**Proof.** Let \( M \) be an indecomposable module such that

\[
\mu(X_i) \cup \{a_{i,j+1}\} = \mu_{i,j+1} < \mu(M) < \mu_{i,j} = \mu(X_i) \cup \{a_{i,j}\}.
\]
Then $\mu(M) = \mu(X_i) \cup \{b_1, b_2, \ldots, b_m\}$ with $a_{i,j} < b_1 \leq a_{i,j+1}$. By the choices of $\mu_{i,j}$, we have $m \geq 2$ and $b_1 = a_{i,j+1}$. This implies $M$ contains a submodule $N$ with $\mu(N) = \mu(X_i) \cup \{a_{i,j+1}\}$, which is thus a preinjective module by above lemma. However, an indecomposable preinjective module cannot be a submodule of any other indecomposable module. We therefore get a contradiction.

Now let $M$ be an indecomposable module such that

$\mu(X_i) < \mu(X_i) \cup \{b_{i+1,1}, a_{i+1,1}\} = \mu_{i+1,1} < \mu(M) = \mu(X_i) \cup \{a_{i,t}\}$.

It follows that $\mu(M) = \mu(X_i) \cup \{b_1, b_2, \ldots, b_m\}$. By definition of $\mu_{i,t}$, we have $b_1 = |X_{i+1}| < a_{i+1,1} < |X_{i+2}|$ and $m \geq 2$. From $b_2 \leq a_{i+1,1}$ and the definition of $\mu_{i+1,1}$, we obtain that $b_2 = a_{i+1,1}$ and $m \geq 3$. Therefore, $M$ contains an indecomposable preinjective module $N$ with GR measure $\mu(X_i) \cup \{|X_{i+1}|, a_{i+1,1}\}$ as a submodule, which is impossible.

The proof is completed. $\Box$

**Remark** We should note that some segments of the sequence of the GR measures in this proposition may not exist. In this case, we can still show as in the proof that for example, $\mu_{j,1}$ is a direct predecessor of $\mu_{i,t}$ for some $j \geq i + 2$.

**Remark** Assume that these $\mu_{i,j}$ constructed above are not landing measures. (For example, $X$ is a homogeneous simple module $H_1$. See Section 5). Since each GR measure different from $I^1$ has a direct successor, we may construct direct successors starting from $\mu_{i,t}$ for a fixed $i$. Let $\mu(M)$ be the direct successor of $\mu_{i,t}$. If $M$ is preinjective, then $|M| < |\mu_{i,t}| = a_{i,t}$ by Proposition 4.2. Thus after taking finitely many such direct successors, we obtain a regular measure (meaning that it is a GR measure of an indecomposable regular module) Proposition 4.2 tells that all direct successors starting with this regular measure are still regular ones. One the other direction, if there are infinitely many preinjective modules containing some $X_i$, $i \geq 2r$ as GR submodules, then the sequence $\mu_{i,j}$ is infinite (This does occur in some case. See Section 5). Thus we obtain a sequence of GR measures indexed by integers $\mathbb{Z}$.

4.5. We fix a tame quiver $Q$ of type $\tilde{A}_n$. There are always GR measures having no direct predecessors, for example, $\mu(H_1)$ (Proposition 4.1.1). We are going to show that the number of GR measures possessing no direct predecessors is always finite.

**Lemma 4.8.** Let $X$ be a quasi-simple module of rank $r > 1$. Assume that there is an $i \geq 1$ such that $X_i \in \mathcal{C}$ is a central module. Then there is an $i_0 \geq i$ such that $\mu(X_{j+1})$ is a direct successor of $\mu(X_j)$ for each $j \geq i_0$.

**Proof.** By Proposition 4.5.2 we may assume that $\mu(X_r) < \mu(H_1)$. Since $X_i$ is a central module, $X_j$ is the unique, up to isomorphism, GR submodule of $X_{j+1}$ for every $j \geq i$. We first show that there is a $j_0$ such that there does not exist a regular module with GR measure $\mu$ satisfying $\mu(X_j) < \mu < \mu(X_{j+1})$ for any $j \geq j_0$.

Let $Y$ be a quasi-simple module of rank $s$ such that $\mu(X_j) < \mu(Y_i) < \mu(X_{j+1})$ for some $j \geq i \geq r$ and $l \geq 1$. In this case, $Y_i$ is a GR submodule of $Y_{i+1}$ since $Y_i$ is a central module and thus $\mu(Y_i) > \mu(T)$ for all preprojective module $T$. Comparing the lengths, we have $\mu(Y_{i+l}) < \mu(X_{j+1})$, and similarly $\mu(Y_{h}) < \mu(X_{j+1})$ for all $h \geq 1$. Now replace $j$ by some $j' > j$ and repeat the above consideration. Since there are only finitely many quasi-simple modules $Z$ such that $\mu(Z_{r_Z}) \leq \mu(H_1)$, where $r_Z$ is the rank of $Z$, we may obtain an index $j_0$ such that a GR measure $\mu$ of an indecomposable regular module satisfies either $\mu < \mu(X_{j_0})$ or $\mu > \mu(X_j)$ for all $j \geq 1$. 


Fix the above chosen \( j_0 \). Assume that there is an indecomposable preinjective module \( M \) such that \( \mu(X_j) < \mu(M) < \mu(X_{j+1}) \) for some \( j \geq j_0 \). Then \( \mu(M) \) starts with \( \mu(X_j) \) and thus there is an indecomposable submodule \( N \) of \( M \) in a GR filtration of \( M \) such that \( \mu(N) = \mu(X_j) \). Note that \( N \) is a regular module and thus \( N = Y_i \) for some \( i \geq 1 \). If \( X_j \cong N \), then \( \mu(M) > \mu(X_j) \) for all \( j \geq 0 \), a contradiction. Therefore, \( X_j \not\cong N \). It follows that \( \mu(X_j) = \mu(N) < \mu(Y_{i+1}) < \mu(M) < \mu(X_{j+1}) \), which contradicts the choice of \( j_0 \). We can finish the proof by taking \( i_0 = j_0 \).

**Theorem 4.9.** Let \( Q \) be a tame quiver of type \( \tilde{A}_n \), \( n \geq 1 \). Then only finitely many GR measures have no direct predecessors.

**Proof.** We first show that only finitely many GR measures of regular modules have no direct predecessors. Let \( X \) be a quasi-simple module of rank \( r \geq 1 \). If \( \mu(X_r) \geq \mu(H_1) \), then for every \( i > 2r \), \( \mu(X_i) \) has a direct predecessor \( \mu(X_{i-1}) \) (Proposition 4.5). Thus we may assume that \( \mu(X_r) < \mu(H_1) \).

If every \( X_i \) is a take-off module, \( \mu(X_i) \) has direct predecessor by definition. If there is an index \( i \geq 1 \) such that \( X_j \) are central modules for all \( j \geq i \), then there is an index \( i_0 \geq i \) such that \( \mu(X_j) \) is a direct predecessor of \( \mu(X_{j+1}) \) for every \( j \geq i_0 \). Therefore, there are only finitely many GR measures of indecomposable regular modules having no direct predecessor.

Now it is sufficient to show that all but finitely many GR measures of preinjective modules have no direct predecessors. Let \( M \) be an indecomposable preinjective module. Since there are only finitely many isomorphism classes of indecomposable preinjective modules with length smaller than \( 2|\delta| \), we may assume that \( |M| > 2|\delta| \). Thus a GR submodule of \( M \) is \( X_i \) for some quasi-simple \( X \) of rank \( r \geq 1 \) and some \( i \geq 2r \). Notice that \( \mu(X_r) \geq \mu(H_1) \), since, otherwise, \( \mu(X_i) < \mu(H_1) < \mu(M) \) would imply \( |H_1| > |M| \), which is impossible. Without loss of generality, we may also assume that there are GR measures \( \mu \) starting with \( \mu(X_i) \) and \( \mu < \mu(M) \). (Namely, if such a \( \mu \) does not exist, we may replace \( M \) by an indecomposable preinjective module \( M' \) with \( |M'| > |M| + |\delta| \). Then the GR submodule of \( M' \) is \( Y_{i'} \) with \( Y \not\cong X \). By this way, we may finally find an integer \( d \) such that all indecomposable preinjective modules with length greater than \( d \) contain \( Z_i, l \geq 2r_Z \) as GR submodules for some fixed quasi-simple module \( Z \). Thus there are infinitely many indecomposable preinjective modules with GR measures starting with \( \mu(Z_l), l \geq 2r_Z \).) Then Proposition 4.7 ensures the existence of the direct predecessor of \( \mu(M) \).

4.6. Tame quivers. After showing that for a tame quiver of type \( \tilde{A}_n \), there are only finitely many GR measures having no direct predecessors, we realized that the fact is also true for any tame quiver, i.e. a quiver of type \( \tilde{D}_n, \tilde{E}_6, \tilde{E}_7 \) or \( \tilde{E}_8 \). We outline the proof of this fact in the following.

**Theorem 4.10.** Let \( \Lambda \) be a tame quiver. Then there are only finitely many GR measures having no direct predecessors.

The proof of this theorem is almost the same as that for the \( \tilde{A}_n \) case. As we have remarked after Proposition 2.2 that the statements (2), (4) and (7) in Proposition 2.2 hold in general \[8\]. Using these we can show Lemma 4.3 for all tame quivers. Proposition 4.5 remains true. But the proof should be changed a little bit because in general, a GR submodule of a preinjective module is not necessarily a regular module. The first part of the proof of Proposition 4.5 is valid in general cases. For the second part, we have to change as follows:

**Proof.** Assume that \( M \) is an indecomposable preinjective module such that \( \mu(X_i) < \mu(M) < \mu(X_{i+1}) \) with \( |M| \) minimal. Let \( N \) be a GR submodule of \( M \). Comparing the lengths, we have
\(\mu(X_i) \leq \mu(N)\). If \(N = Y_j\) is regular for some quasi-simple module \(Y\) of rank \(s\), then \(\mu(M) > \mu(Y_{j+1}) > \mu(Y_j) \geq \mu(X_i)\). This contradicts the first part of the proof. If \(N\) is preinjective, then \(\mu(N) = \mu(X_i)\) by the minimality of \(|M|\). Thus a GR filtration of \(N\) contains a regular module \(Z_{2t}\) for a quasi-simple \(Z\) of rank \(t\). It follows that \(\mu(X_{2t}) = \mu(Z_{2t})\). Thus by Lemma 4.3 and Proposition 2.2(7) we have \(\mu(M) > \mu(N) > \mu(Z_{i+1}) = \mu(X_{i+1})\), which is a contradiction. \(\square\)

Lemma 4.6 is true in general. However, Proposition 4.7 should be replaced by the following one:

**Proposition 4.11.**

1. There are only finitely many GR measures lying between \(\mu_{i,j}\) and \(\mu_{i,j+1}\).

2. There are only finitely many GR measures lying between \(\mu_{i,t}\) and \(\mu_{i+1,t}\). In particular, \(\mu_{i,j}\) has a direct predecessor.

**Proof.** Assume that \(M\) is an indecomposable module such that \(\mu(X_i) \cup \{a_{i,j+1}\} = \mu_{i,j+1} < \mu(M) < \mu_{i,j} = \mu(X_i) \cup \{a_{i,j}\}\). Then \(\mu(M) = \mu(X_i) \cup \{b_1, b_2, \ldots, b_m\}\). By definition of \(\mu_{i,j}\), we have \(b_1 = a_{i,j+1}\) and \(m \geq 2\). In particular, \(M\) has a GR filtration containing an indecomposable module \(N\) such that \(\mu(N) = \mu(X_i) \cup \{b_1\}\), which is thus preinjective. However, there are only finitely many indecomposable modules containing a given indecomposable preinjective module as a submodule. It follows that only finitely many GR measures starting with \(\mu(N) = \mu(X_i) \cup \{b_1\}\) exist. Therefore, the number of GR measures, which lies between \(\mu_{i,j+1}\) and \(\mu_{i,j}\) is finite for each \(i \geq 2r\).

2) follows similarly. Notice that the first remark after Proposition 4.7 still works for this case. \(\square\)

The remaining proof of Theorem 4.10 is similar.

5. Preinjective Central modules

In [11], it was proved that all landing modules are preinjective in the sense of Auslander and Smalø [2]. There may exist infinitely many preinjective central modules. In this section, we study the preinjective modules and the central part. Throughout this section, \(Q\) is a fixed tame quiver of type \(\tilde{A}_n\).

5.1. We first describe the landing modules.

**Proposition 5.1.** Let \(M\) be an indecomposable preinjective module. Then either \(M \in \mathcal{L}\) or \(\mu(M) < \mu(X)\) for some indecomposable regular module \(X\).

**Proof.** If \(\mu(M) = \mu(X_j)\) for some quasi-simple module \(X\), we thus have \(\mu(M) < \mu(X_j)\) for all \(j > i\). Thus we may assume that \(\mu(M) > \mu(X)\) for all regular modules \(X \in \mathcal{R}\). Let \(\mu_1\) be the direct successor of \(\mu(M)\) and \(\mathcal{A}(\mu_1)\) the collection of indecomposable modules with GR measure \(\mu_1\). It follows that \(\mathcal{A}(\mu_1)\) contains only preinjective modules. Let \(Y^1 \in \mathcal{A}(\mu_1)\) and \(X^1 \rightarrow Y^1\) be a GR inclusion. Since \(X^1 \in \mathcal{R}\), we have \(\mu(X^1) < \mu(M) < \mu(Y^1) = \mu_1\). Thus \(|M| > |Y^1|\). Let \(\mu_2\) be the direct successor of \(\mu_1\) and \(Y^2 \in \mathcal{A}(\mu_2)\). As above we have \(|Y^1| > |Y^2|\). Repeating this procedure, we get a sequence of indecomposable preinjective modules \(M = Y^0, Y^1, Y^2, \ldots, Y^n, \ldots\) such that \(\mu(Y^n)\) is the direct successor of \(\mu(Y^{i-1})\) and \(|Y^n| < |Y^{i-1}|\). Because the lengths decrease, there is some \(j < \infty\) such that \(\mu(Y^j)\) has no direct successor. It follows that \(\mu(Y^j) = I^1\) and \(\mu(M)\) is a landing measure. \(\square\)

**Corollary 5.2.** Let \(M\) be an indecomposable module. Then \(\mu(M) > \mu(X)\) for all regular modules \(X\) if and only if \(M\) is a landing module.
5.2. We partition the tame quivers of type (Proposition 2.2(5)). Thus preinjective central modules.
Assume that there is a stable tube of rank \( r > 1 \). Then all but finitely many landing modules contain only exceptional regular modules as GR submodules.

**Proposition 5.3.** If \( M, N \) are landing modules, then \( \mu(M) < \mu(N) \) if and only if \( |M| > |N| \).

**Proof.** Assume that \( \mu(M) < \mu(N) \). Let \( X \) be a GR submodule of \( N \). Since \( X \) is a regular module, we have \( \mu(X) < \mu(M) < \mu(N) \) and thus \( |M| > |N| \). \( \square \)

**Proposition 5.4.** Assume that there is a stable tube of rank \( r > 1 \). Then all but finitely many landing modules contain only exceptional regular modules as GR submodules.

**Proof.** Let \( M \) be a landing module which is thus preinjective. Thus the GR submodules of \( M \) are all regular modules. Assume that \( M \) contains homogeneous modules \( H_i \) as GR submodules. Let \( T \) be a stable tube of rank \( r > 1 \). Then there exists a quasi-simple module \( X \) on \( T \) such that \( \mu(X_i) \geq \mu(H_1) \) (Proposition 2.2(5)). Thus \( \mu(H_i) < \mu(X_{r+1}) < \mu(M) \) and therefore, \( |X_{r+1}| > |M| \). This implies \( i = 1 \) and \( |M| < 2\beta \). \( \square \)

**Corollary 5.5.** Assume that there is a stable tube of rank \( r > 1 \). If \( M \) is an indecomposable module containing homogeneous modules \( H_i \) as GR submodules for some \( i \geq 2 \), then \( M \) is a central module.

5.2. We partition the tame quivers of type \( \widetilde{k}_n \) into three classes; for each cases we study the possible preinjective central modules.

**Case 1** Assume that in the quiver \( Q \) there is a clockwise path of arrows \( \alpha_1\alpha_2 \) and a counter clockwise path \( \beta_1\beta_2 \) as follows:

\[
\begin{array}{c}
\bullet \\
\alpha_2 \\
\downarrow \\
\beta_2 \\
\end{array} \quad \begin{array}{c}
\bullet \\
\alpha_1 \\
\downarrow \\
\beta_1 \\
\end{array} \\
1 \\
\end{array}
\]

Let \( C \) be a string starting with \( \alpha_2^{-1} \) and ending with \( \beta_2 \). Thus \( s(C) = 1, e(C) = 2 \). It is obvious that the string modules \( M(C) \) contains both simple regular modules \( S(1) \) and \( S(2) \), which are in different regular components, as submodules. Thus \( M(C) \) is an indecomposable preinjective module. Fix such a string \( C \) such that the length of \( C \) is large enough, i.e. \( M(C) \) contains homogeneous modules \( H_i \) as submodules. The GR submodules of \( M(C) \) has one of the following forms \( S(1), S(2), H_i \) for some \( i, j, t \geq 1 \). However, \( \mu(S(1)), \mu(S(2)), \mu(H_t) \) for all \( i \geq 0 \) (Proposition 2.2(6)). Thus the GR submodules of \( M(C) \) are homogeneous modules. In particular, there are infinitely many indecomposable preinjective modules containing only homogeneous modules as GR submodules. Thus there are infinitely many preinjective central modules by Corollary 5.5.

As an example, we consider the following quiver \( Q = \widetilde{k}_{p,q} \), \( p + q = n + 1 \), with precisely one source and one sink:

\[
\begin{array}{c}
\bullet \\
\alpha_1 \\
\downarrow \\
\beta_1 \\
\end{array} \quad \begin{array}{c}
\bullet \\
\alpha_2 \\
\downarrow \\
\beta_2 \\
\end{array} \quad \begin{array}{c}
\bullet \\
\alpha_3 \\
\downarrow \\
\beta_3 \\
\end{array} \quad \cdots \quad \begin{array}{c}
\bullet \\
\alpha_{p-1} \\
\downarrow \\
\beta_{p-1} \\
\end{array} \quad \begin{array}{c}
\bullet \\
\alpha_p \\
\downarrow \\
\beta_p \\
\end{array} \\
1 \quad 2 \quad \cdots \quad n \quad n+1 \\
\end{array}
\]

There are two stable tubes \( T_X \) and \( T_Y \) consisting of string modules. The stable tube \( T_Y \) contains the string module \( Y \) determined by the string \( \beta_q\beta_{q-1}\cdots\beta_2\beta_1 \), and simple modules \( S \) corresponding to the vertices \( s(\alpha_i), 2 \leq i \leq p \) as quasi-simple modules. The rank of \( Y \) is \( p \). The other tube
we may easily deduce that the sequence of irreducible monomorphism above discussion.

There are infinitely many landing modules containing only exceptional modules of the form (compositions of arrows) are of length 1. In this case, all exceptional quasi-simple modules in one of:

$$Y_p = Y_{p+1} \subset \ldots \subset Y_j \subset \ldots$$ is a chain of GR inclusions and thus $\mu(X_q) = \mu(H_1)$.

Similarly, $Y_p \subset Y_{p+1} \subset \ldots \subset Y_j \subset \ldots$ is a chain of GR inclusions and $\mu(Y_p) = \mu(H_1)$.

Any non-sincere indecomposable module belongs to the take-off part. This is true because the GR submodule of $H_1$ is a uniserial module and has GR measure $\{1, 2, 3, \ldots, n\}$ and a non-sincere indecomposable module has length smaller than $|\delta|$. Let $M \in T$ be a sincere indecomposable preinjective module and $T \subset M$ a GR submodule. We claim that $T$ is isomorphic to some $H_1$, $X_{sq}$ or $Y_{tp}$ for some $i, s, t \geq 1$. First of all the $T \not\cong S_i$ for any simple regular module $S$ and any $i \geq 1$, since $\mu(S_i) < \mu(H_1)$ (Proposition 2.2(6)) and there is a monomorphism $H_1 \to M$ for each homogeneous module $H_1$. If, for example, $T \cong X_1$ for some $i \geq q$, then there is an epimorphism $X_{i+1} \to M$ (Proposition 2.1(2)). Thus $|X_i| < |M| < |X_{i+1}|$. However, $|X_{i+1}| - |X_i| = 1$ if $i$ is not divided by $q$.

Notice that if $p \geq 2$ and $q \geq 2$, then there are infinitely many preinjective central modules by above discussion.

**Case 2** $Q = \tilde{A}_p, 1$. Let’s keep the notations in the above example. By Proposition 5.4 we know that there are infinitely many landing modules containing only exceptional modules of the form $Y_i$ as GR submodules. Given an indecomposable preinjective module $M$ and its GR submodule $Y_i$, $i > p$. We claim that the GR submodules of $\tau M$ are homogeneous ones. Namely, if $\tau M$ contains an exceptional regular module $N$ as a GR submodule, then $N \cong Y_j$ for some $j \geq p$. In particular, both $M$ and $\tau M$ contains $Y$ as a submodule, i.e. $\mathrm{Hom}(Y, M) \neq 0 \neq \mathrm{Hom}(Y, \tau M)$. Therefore, we have $\mathrm{Hom}(\tau^{-1}Y, M) \neq 0 \neq \mathrm{Hom}(Y, M)$, which contradicts Lemma 2.3. Thus, there are infinitely many indecomposable preinjective modules containing only homogeneous modules as GR submodules and hence infinitely many preinjective central modules.

**Case 3** $Q \neq \tilde{A}_p, q$ is of the following form: all non-trivial clockwise (or counter clockwise) paths (compositions of arrows) are of length 1. In this case, all exceptional quasi-simple modules in one of the exceptional tubes are of length at least 2, and the quasi-simple modules on the other exceptional tube have length at most 2.

Let $p = \beta_1 \cdots \beta_2 \beta_1$ be a composition of arrows in $Q$ with maximal length. Thus there is an arrow $\alpha$ with ending vertex $e(\alpha) = e(p)$ and $s(\alpha)$ is a source. Let $X = M(p)$ be the string module, which is thus a quasi-simple module, say with rank $r$. By the maximality of $p$ and the description of irreducible maps between string modules, we may easily deduce that the sequence of irreducible monomorphism $X = X_1 \to X_2 \to \ldots \to X_r \to X_{r+1} \to \ldots$ is namely a sequence of GR inclusions. Therefore

$$\mu(X_{r+1}) = \{1, 2, 3, \ldots, t + 1 = |X_1|, |X_2|, |X_3|, \ldots, |X_r|, |X_{r+1}|\}$$

with $|X_i| - |X_{i-1}| \geq 2$ for $2 \leq i \leq r$ and $|X_{r+1}| = |X_r| + (t + 1)$.

Let $Y$ be the string module determined by the arrow $\alpha$. It is also a quasi-simple module, say with rank $s$. It is clear that $X$ and $Y$ are not in the same regular component. By the description of
Lemma 5.6. We keep the notations as above. If $n$ is not equipped with a sink-source orientation, then $\mu(Y_s) \geq \mu(X_r)$ and $\mu(Y_j) > \mu(X_i)$ for $i \geq 1$ and $j > s$.

5.3. Now we characterize the tame quivers $Q$ of type $\tilde{A}_n$ such that no indecomposable preinjective modules are central modules. We also show that there are always infinitely many preinjective central modules if any.

Theorem 5.7. Let $Q$ be a tame quiver of type $\tilde{A}_n$. Then $\mathcal{I} \cap \mathcal{C} = \emptyset$ if and only if $Q$ is equipped with a sink-source orientation.

Proof. If $n = 1$, then $Q$ is obvious of a sink-source orientation, and the central part contains precisely the regular modules (see, for example, [11]). Now we assume $\mathcal{I} \cap \mathcal{C} \neq \emptyset$, a sincere indecomposable preinjective is always a landing module. Then the proof of Proposition 5.4 implies that there is no indecomposable preinjective module containing a homogeneous module $H_i$, $i \geq 2$ as GR submodules. Therefore, by above partition of the tame quiver of type $\tilde{A}_n$, we need only to consider Case 3 and show that $\mathcal{I} \cap \mathcal{C} = \emptyset$ implies $t = 1$ (let’s keep the notations in case 3). Assume for a contradiction that $t > 1$. Let $S$ be the simple module corresponding to $\beta_t$. Thus $S$ is a quasi-simple of rank $s$ and $\tau S \cong Y$. Let $I$ be the (indecomposable) injective cover of $S$. It is obvious that Hom$(X, I) \neq 0$. Consider the indecomposable preinjective modules $\tau^{um} I$, where $u$ is a positive integer and $m = [r, s]$ is the lowest common multiple of $r$ and $s$. Since Hom$(S, \tau^{um} I) \neq 0 \neq$ Hom$(X, \tau^{um} I)$, a GR submodule of $\tau^{um} I$ is either $S_i$ or $X_j$. Notice that $\mu(H_i) > \mu(S_i)$ for all $i \geq 0$ since $S$ is simple. Therefore, for $u$ large enough, the unique GR submodule of $\tau^{um} I$ is $X_j$ for some $j \geq 1$ because no indecomposable preinjective modules containing $H_i$ as GR submodules for $i \geq 2$. In particular there are infinitely many preinjective modules containing GR submodules of the form $X_j, j \geq 1$. Thus we may select a GR inclusion $X_j \subset M$ with $M \in \mathcal{I}$ such $|X_j| > |Y_{s+1}|$. Because $\mu(X_j) < \mu(Y_{s+1}) < \mu(M)$, we have $|Y_{s+1}| > |M|$. This contradicts $|X_j| > |Y_{s+1}|$. Thus we have $t = 1$ and $Q$ is equipped with a sink-source orientation.

Conversely, if $Q$ is with a sink-source orientation, we may see directly that $\mathcal{I} \cap \mathcal{C} = \emptyset$ (for details, see [7, Example 3]).

Theorem 5.8. Let $Q$ be a tame quiver of type $\tilde{A}_n$. Then $\mathcal{I} \cap \mathcal{C} \neq \emptyset$ if and only if $|\mathcal{I} \cap \mathcal{C}| = \infty$.

Proof. We have seen in Corollary 5.5 that an indecomposable module containing homogeneous modules $H_i, i \geq 2$ as GR submodules is a central module. Thus we may assume that there are only finitely many indecomposable preinjective module containing homogeneous modules as GR submodules. Thus, we need only consider Case 3. Let’s keep the notations there. Then $\mathcal{I} \cap \mathcal{C} \neq \emptyset$ implies that $Q$ is not with a sink-source orientation. In particular, the length $t$ of the longest path of arrows $\beta_t \cdots \beta_1$ is greater than 1. Therefore, $\mu(Y_j) > \mu(X_i)$ for all $i \geq 1, j > s$. Again let $m = [r, s]$. By assumption, the GR submodules of $\tau^{um} I$ are of the form $X_i$ for almost all $u \geq 1$. To avoid a contradiction as in the proof of above theorem, $\mu(\tau^{um} I)$ is smaller than $\mu(Y_{s+1})$ for $u$ large enough and thus almost all $\tau^{um} I$ are central modules.
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