Exploiting Active Subspaces to Quantify Uncertainty in the Numerical Simulation of the HyShot II Scramjet

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Abstract

We present a computational analysis of the reactive flow in a hypersonic scramjet engine with emphasis on effects of uncertainties in the operating conditions. We employ a novel methodology based on \textit{active subspaces} to characterize the effects of the input uncertainty on the scramjet performance. The active subspace re-parameterizes the operating conditions from seven well characterized physical parameters to a single derived active variable. This dimension reduction enables otherwise intractable—given the cost of the simulation—computational studies to quantify uncertainty; bootstrapping provides confidence intervals on the studies’ results. In particular we (i) identify the parameters that contribute the most to the variation in the output quantity of interest, (ii) compute a global upper and lower bound on the quantity of interest, and (iii) classify sets of operating conditions as safe or unsafe corresponding to a threshold on the output quantity of interest. We repeat this analysis for two values of the fuel injection rate. These analyses provide a potential template for quantifying uncertainty in large-scale computational fluid dynamics simulations.

\textbf{Keywords:} uncertainty quantification, active subspace, hypersonic, scramjet

1. Introduction

Over the past decade there has been a renewed interest in numerical simulations of high-speed air-breathing propulsion systems for hypersonic vehicles, driven by fundamental advancements in computational tools which simulate
high Mach-number flight conditions. These tools are essential in the design process of such vehicles due to the challenges and costs associated with traditional physical prototyping. In light of the limited experience with sustained hypersonic flight, numerical simulations can be of critical value to understanding the behavior of this system, in particular characterizing the safe operability limits of the propulsion system.

Supersonic combustion engines (scramjets) are an economic alternative to rockets because they do not require on-board storage of the oxidizer. The HyShot II scramjet, shown in Figure 1, was designed to demonstrate supersonic combustion during flight in a simple configuration [17, 40], and it has since been the subject of multiple ground-based experimental campaigns in the High Enthalpy shock tunnel Göttinngen (HEG) of the German Aerospace Center (DLR) [11, 12, 15, 16, 25–28, 38]. The simplicity of the configuration and the availability of experimental data has made the HyShot II scramjet the subject of multiple computational investigations, based on either Reynolds-averaged Navier-Stokes (RANS) simulations [34] or large-eddy simulations (LES) [3, 10, 24].

Figure 1: Side view of the HyShot II scramjet outlining the various physics within the different parts of the geometry. Contours show numerical Schlieren results from a 2D simulation.

Although the geometry depicted in Figure 1 (and in detail in Figure 2) is simple and without moving parts, accurately predicting the internal flow structure is a very challenging problem. The difficulty results from the physical phenomena encountered within the engine when traveling at high velocity—namely turbulence, shocks, boundary layers, mixing, and combustion—all of which must be modeled. Required closures and disparate spatial and temporal scales of the various physical processes cause these simulations to be fairly expensive.

Scramjet operation requires a careful balance between maximizing thrust and maintaining stable and safe operation. At the conditions of interest here, the flow inside the HyShot II scramjet combustor has three different regimes [24, 26]. At low fuel-air equivalence ratios, the flow is supersonic throughout. At higher fuel-air equivalence ratios, a stable flow with a shock-train in the
combustor develops. Finally, at sufficiently high fuel-air equivalence ratios, the flow will unstart—a potentially catastrophic failure mode associated with a large loss in thrust. Scramjet design and operation is complicated by the fact that the maximum thrust occurs very close to the boundaries between these different regimes.

Previous efforts [30, 44, 45] studied various aspects of this problem both experimentally and computationally, but largely ignored the effect of the uncertainties on the estimation of the operability limit. These efforts enhance understanding of the physics of the problem, but do not provide sufficient confidence in the quantitative estimates of the maximum fuel flow rate compatible with safe operations. As a consequence safety margins based on experience are applied to the estimate. To produce more trustworthy uncertainty estimates, it is necessary to explicitly identify, estimate, and account for the full range of uncertainties that affect the operability limit. Many sources of uncertainty are present in the scenario of interest; both variability in the operating environment of the vehicle (e.g., fluctuations in temperature) and inadequacy of the constitutive models (e.g., the combustion model) introduce uncertainty in the simulation result. Our prior work studies the effects of uncertainties in the RANS turbulence models—and how to account for them in a physically meaningful manner [8]. In this paper, we focus on the operating environment in the HyShot II vehicle studied by DLR in the HEG shock tunnel [11, 12, 38]. In this context, the main source of variability corresponds to incomplete knowledge and controllability of the conditions within the HEG shock tunnel.

Monte Carlo and related random sampling methods are commonly used to quantify uncertainty in a simulation’s predictions—e.g., with moments, quantiles, or empirical density functions—given uncertainty in its inputs [32]. Estimates are unbiased, and one can develop confidence intervals for the uncertainty analysis from the Central Limit Theorem. However, when the simulation is expensive, sampling can be impractical for accurate statistics due to slow $n^{-1/2}$ convergence (where $n$ is the number of samples). An alternative approach for rapidly converging, biased estimation of the simulation’s predictions employs response surfaces—where one uses a few carefully selected runs of the expensive model to construct a cheaper response surface that is subsequently sampled. Popular response surface models in uncertainty quantification include polynomial chaos [14, 50], stochastic collocation [49], and Gaussian process regression [23, 36]. For polynomial variants, moments reduce to numerical quadrature on the integral formulation of averages. Response surfaces are most appropriate when the number of input parameters is sufficiently small.

In our case, we supplement statistical measures of the scramjet’s quantity of interest with its range; determining the range is posed as optimization. We employ active subspaces [5] to discover that the quantity of interest can be well represented by a univariate function of the active variable derived from the model’s inputs. We then exploit this low-dimensional model to (i) estimate the cumulative distribution function of the quantity of interest, (ii) determine the range of the quantity of interest, and (iii) identify the sets of parameters corresponding to safe operation of the scramjet. As a byproduct, the active subspace
reveals a subset of the model’s parameters whose perturbations produce the greatest change in the quantity of interest, i.e., a sensitivity analysis.

For this particular model, discovering the low-dimensional structure with the active subspace and quantifying uncertainty requires 16 simulation runs for each of two cases we consider. This number is remarkably small considering both the complexity of the simulation model and its dependence on seven independent input parameters. The exceptionally small computational cost is a result of the low-dimensional structure revealed by the active subspace.

The remainder of this paper is broadly structured in three sections: describing the HyShot II simulation in Section 2, characterizing the uncertainties in the operating conditions in Section 3, and quantifying the effects of the uncertainties in Section 4. We conclude with a brief summary.

2. Methodology for deterministic simulations

2.1. Geometry and computational grids

Figure 2 shows the HyShot II geometry including the intake ramp (forebody), isolator, combustion chamber, and nozzle details. The actual HEG system is much larger and includes the mounting structure and the instrumentation of the 1 : 1 scale model. The geometry in Figure 2 corresponds to one-half of the flight vehicle, which had symmetrically mounted fueled and un-fueled engines on top of a rocket. Both dual and single engine models have been investigated in the HEG [11, 12, 38].

![Figure 2: HyShot II scramjet geometry, taken from [34].](image)

The intake ramp is significantly wider than the combustion chamber, and thus the flow entering the isolator/combustor is very close to uniform in the spanwise direction. The gap between the intake ramp and body wall of the combustion chamber is a boundary-layer and isolator shock (emanating from the cowl wall leading edge) bleed channel. Within the combustor there are four
fuel injection ports spanning the width of the channel, shown in the slice view A–A. Additionally the model was mounted at a nominal angle-of-attack of 3.6°.

In order to reduce the computational cost of each simulation the HyShot II configuration is separated into two domains. The first is a 2D representation of the intake ramp and entrance to the combustion chamber, denoted as forebody and isolator in Figure 1, respectively. The second domain is a fully 3D representation of the combustion chamber and exit nozzle, including the fuel injection ports. This domain decomposition assumes that the combustion chamber inflow conditions, i.e., the flow dynamics upstream of fuel injection, are two-dimensional (no variation in the spanwise dimension). This assertion has been verified by DLR through comparison of 2D and 3D intake ramp simulations [13]. Leveraging these observations a similar domain decomposition has been applied by a variety of researchers numerically investigating the HyShot II scramjet [cf. 10, 21, 22].

The domain decomposition allows us to address the different modeling requirements in the 2D and 3D domains. Over the forebody and through the isolator, turbulence and transition phenomena dictate the flow structures. In contrast, the combustion chamber requires modeling of mixing and flow structures due to the injected fuel. The 2D simulations are dramatically less expensive computationally due to relaxed mesh resolution and modeling complexity. We next describe the meshes used in this analysis.

2.1.1. 2D Forebody

The grid for the 2D forebody/ramp is shown in Figure 3, where the downstream boundary is located at the fuel injection ports. In the wall-normal direction the grid is designed such that the first row of cells adjacent to the wall have $y_+ = 1$, resulting in just under 50k control volumes in the domain. In regions far from solid boundaries the cells are unstructured, which reduces numerical artifacts related to poor shock-grid alignment.
2.1.2. 3D Combustion Chamber

To reduce the cost of simulating the 3D domain we take advantage of symmetries and simulate only 1/8th of the total combustor. In the work of Pecnik et al. [33, 34] the results using a 1/2-span domain (including the sidewall and two injection ports) and a 1/8-span domain (including half an injection port and using symmetry planes in the spanwise directions) were compared. Those authors concluded that while shocks emanating from the sidewalls influence certain quantities of interest, in general the 1/8-span domain is acceptable for use in the analysis of the scramjet. The same conclusion has been reached by DLR and other analysts, all of whom perform HyShot II simulations using a 1/8-span domain [cf. 4, 10, 21, 22].

The grid is again designed such that the wall-normal spacing adjacent to the wall is $y_{+} = 1$, and the total number of control volumes in this domain is $1.2M$. To increase numerical stability, a corner radius of 0.06 mm is used at the fuel nozzle orifice; see Figure 4(a). The domain is essentially structured except near the injection port; see Figure 4(c).

2.1.3. Coupling the two domains

In the 3D domain the inflow condition for the oxidizer stream is taken from the 2D simulation. A wall-normal profile is extracted at $x = 352.68$ mm (absolute coordinates of the mesh), this profile is applied uniformly across the span of the combustion chamber inlet. The profile is extracted at a location where the oblique shock generated by the body wall leading edge is captured well above the boundary-layer resolving cells, see Figure 5.

2.2. Reynolds-averaged Navier-Stokes

Simulating the HyShot II scramjet with LES is tractable but costly [3, 24]. Wall-modeled LES uses approximately 100M cells, and wall-resolved LES would require 100B cells. It is not feasible to use LES for UQ, which may require many simulations to estimate statistics of the solution behavior.

RANS costs less than LES by modeling only the time-averaged flow. Also, the mesh can be much coarser than LES, which further reduces the cost. The trade off is that the physics of turbulence are now modeled; the accuracy of the result depends strongly on the accuracy of the turbulence model. RANS represents the most practical approach for performing UQ of the HyShot II system, and in general RANS is the most common approach used by hypersonic vehicle designers for simulating turbulence in their systems [21, 51].

2.2.1. Physics Modeling and Closures

Several RANS closures are needed in the multiphysics model. We next describe several specific choices.

Turbulence. The turbulence model used to determine the Reynolds stresses is based on the eddy-viscosity hypothesis and the $k–\omega$ SST formulation. The SST model is one of the most frequently used RANS models in industrial applications.
In many applications with compressible boundary layers and interactions of shock-waves and turbulence, SST gives reasonably accurate predictions [8, 31].

In the SST model, two transport equations are solved in addition to the RANS equations to describe the turbulent kinetic energy $k$ and the specific dissipation rate $\omega$ (having units of inverse time). The SST model blends the standard $k-\omega$ and $k-\epsilon$ models to accurately represent both the near wall regions and the response to high strain and pressure gradient.

Due to its popularity, several modified SST models have been proposed. We apply two specific limiter modifications [35]: (i) a limiter for the eddy-viscosity,

$$\mu_t = \frac{\rho a_1 k}{\max(a_1 \omega, \Omega F_2)},$$  \hspace{1cm} (1)

using the magnitude of vorticity $\Omega$ in the denominator, and (ii) a turbulence kinetic energy production limiter, which is meant to ensure realizable Reynolds
stresses,
\[ P_k = \min (P_k, 20 C_\mu \omega) . \] (2)

**Transition.** The laminar-turbulent boundary-layer transition process is not predicted, but rather modeled by manually specifying transition locations on the intake ramp, body wall, and cowl wall. These locations then affect the flow through inhibited production and destruction of turbulence kinetic energy in the boundary layers upstream of these points; this mimics the handling of transition in the \( \gamma - Re_\theta t \) transition model by Menter et al. [29].

**Combustion.** The combustion model is based on a flamelet/progress variable approach (FPVA), in which the chemistry is tabulated as a series of laminar flamelet solutions for a given set of boundary conditions and background pressure. In this manner the chemical composition is mapped *a priori* with respect to a small number of parameters used to search this table. The major assumption behind this approach is that chemistry is fast relative to the mixing time scales and can therefore be accurately represented by a small number of scalar quantities. This approach requires three additional transport equations for the mixture fraction, mixture fraction variance \( \tilde{Z}'^2 \), and progress variable \( \tilde{C} \). These are the values used with the table to provide species mass fractions and other properties that in turn influence the local temperature and pressure [33, 34, 41]. The 20 reaction \( H_2 - O_2 \) mechanism of [19] is used to generate the FPVA table. The FPVA framework is fairly common in subsonic application, as it has been developed based on a low Mach assumption. Formal descriptions of the development and extension to high speed flows are found in [37, 42].

### 2.2.2. Flow Solver

The present calculations solve the steady, compressible RANS equations (five PDEs), the \( k-\omega \) SST turbulence model (two PDEs), and the FPVA combustion...
model (three PDEs and a look-up table)—all with specified transition locations. The computations are carried out under the steady-state assumption although the unstart process is transient. This choice is motivated by the desire to detect the conditions that lead to the inception of the process rather than the need to model the entire unstart phenomenon.

The flow solver Joe, developed at Stanford’s Center for Turbulence Research, is used to perform these simulations. The code performs parallel calculations on a collocated unstructured mesh using a finite volume formulation. The discretization is second order in space; the gradients of flow quantities are calculated with a least-squares formulation. Comprehensive discussions of the numerical implementation as well as validation of Joe with respect to HyShot II simulations can be found in Peˇ cnik et al. [33, 34], Terrapon et al. [41].

2.3. Boundary conditions

In both the 2D and 3D domains solid walls are modeled as isothermal with \( T_w = 300 \text{K} \) due to the very short test time which prevents the steel walls from heating up [12]. The nozzle (combustor exit), bleed channel, and freestream (flow outside cowl side of vehicle) use a Neumann boundary condition. The value of \( P_{0,H_2} \) at the fuel inflow is specified based on the desired equivalence ratio, using the fact that \( P_{0,H_2} \) and \( \phi \) are linearly related in Table 1.

3. Characterizing the Sources of Uncertainty

The first step in uncertainty quantification is to identify and characterize the system’s uncertainties, i.e., describe them in mathematical terms. These uncertainties must be propagated through the solution to assess their impact on the quantity of interest. In this section we both introduce and characterize the sources of uncertainty in the HyShot II model.

Due to limited knowledge, all of the uncertainties are described by uniform distributions; we explain their ranges in this section. In general any available knowledge (e.g., from observations, theory, or expert opinion) is used to inform the uncertainty range specification. The parameters and associated ranges are summarized in Table 2.

3.1. Inflow conditions, mean quantities

The objective of this work is to reproduce conditions from the ground test experimental campaign carried out by researchers at DLR in the HEG. The conditions in the HEG represent the flight test from Boyce et al. [2] along with boundary conditions and comprehensive measurements in the combustor.

Stanford University and DLR collaborated to share experimental results, simulation results, and additional unpublished information [13]. When this study was initialized, a total of thirteen experimental runs in the HEG shock tunnel at the relevant conditions (i.e., mimicking the high altitude conditions encountered during the flight test) had been performed, four without fueling and nine with fueling. For each experimental run, the stagnation pressure \( P_0, \)
temperature $T_0$, and enthalpy $H_0$ of the nozzle supply region are measured (reported in Table 1). The fueled shots additionally measured the hydrogen fuel plenum pressure $P_{0,H2}$ and equivalence ratio $\phi$,

$$\phi = \frac{\dot{m}_{H2}}{\dot{m}_{O2}},$$

which is the ratio of the actual hydrogen mass flux to the theoretical value required for complete consumption of the oxidizer within the combustor—essentially a measure of the fuel flow rate. For all shots the hydrogen plenum temperature was 300K.

Although Table 1 reports three parameters to describe the gas state in the nozzle, the magnitudes of $T_0$ and $H_0$ are not independent. This can be seen in Figure 6, which shows these two quantities for all 13 experimental runs. The apparent linear relationship is physically justifiable: for a calorically perfect gas, the specific heat at constant pressure is the coefficient of proportionality. The least-squares fit relationship is

$$H_0 = \frac{T_0 - 508.1386}{6.8718 \times 10^{-4}}.$$

We thus take the $P_0$ and $H_0$ values from Table 1, and use Equation (4) to calculate the associated $T_0$. We transform these stagnation conditions into the static pressure, temperature, and velocity to use as inflow conditions in the

| Shot | $P_0$ [bar] | $T_0$ [K] | $H_0$ [MJ/kg] | $P_{H2}$ [bar] | $\phi$ |
|------|-------------|-----------|---------------|----------------|-------|
| fuel-off |           |           |               |                |       |
| 805  | 178.05     | 2742      | 3.25          |                |       |
| 807  | 181.19     | 2777      | 3.30          |                |       |
| 808  | 175.27     | 2716      | 3.21          |                |       |
| 814  | 176.64     | 2735      | 3.24          |                |       |
| fuel-on |           |           |               |                |       |
| 804  | 172.96     | 2652      | 3.22          | 5.41           | 0.341 |
| 809  | 177.80     | 2753      | 3.27          | 5.27           | 0.329 |
| 810  | 176.66     | 2705      | 3.20          | 5.73           | 0.351 |
| 811  | 178.81     | 2726      | 3.23          | 4.68           | 0.286 |
| 812  | 179.84     | 2729      | 3.23          | 5.32           | 0.325 |
| 816  | 173.06     | 2769      | 3.39          | 3.71           | 0.266 |
| 817  | 176.99     | 2796      | 3.13          | 5.28           | 0.315 |
| 827  | 170.23     | 2701      | 3.19          | 5.09           | 0.324 |
| 828  | 187.43     | 2796      | 3.28          | 4.84           | 0.288 |

Table 1: Experimental conditions from HyShot II ground tests performed in the HEG [13].
CFD. This is done using the following ratios provided by DLR

\[
\frac{P}{P_0} = 1.16 \times 10^{-4}, \quad (5a)
\]

\[
\frac{T}{T_0} = 0.0978, \quad (5b)
\]

\[
\frac{U_{\text{mag}}}{\sqrt{H_0}} = 1.332, \quad (5c)
\]

which are assumed valid for any given run-condition. Equation (5c) gives the velocity magnitude, which together with the angle-of-attack \( \alpha \) allows us to compute velocity components.

While the CFD solver takes the velocity components and the static pressure and temperature to specify the inflow condition, it is more consistent to characterize the uncertainty directly in the stagnation conditions \( (P_0 \text{ and } H_0) \) that are measured in the experiments. The small number of measurement data available (the 13 runs reported in Table 1) makes it difficult to characterize this uncertainty. Nevertheless, 13 samples in the table are used to compute sample means \( (\hat{P}_0 \text{ and } \hat{H}_0) \) and standard deviations \( (\hat{\sigma}_{P_0} \text{ and } \hat{\sigma}_{H_0}) \). The uncertainty range is conservatively defined as \( \hat{P}_0 \pm 3 \cdot \hat{\sigma}_{P_0} \) and similarly for \( H_0 \).

We also take the angle-of-attack \( \alpha \) as uncertain, due to both the physical mounting of the model in the tunnel and the flow-induced deviation from this angle during a run. The nominal angle-of-attack was \( 3.6^\circ \). Based on the expert opinion of an experienced experimentalist [M. Gamba, personal communication, July 5\textsuperscript{th}, 2012], we believe \( \alpha \) can only be specified within \( \pm 1^\circ \). Additional evidence suggests this value is uncertain based on analysis performed during the construction of the HyShot II model [11]. The weight of the model and
associated instrumentation caused static load deflection of the model up to 0.465\(^\circ\), a value which will most likely increase during dynamic loading. Further evidence that \(\alpha\) is uncertain is behavior of the HEG facility during operation, which causes the whole building to shake. One may reasonably assume that the model or test section is at an effective angle-of-attack relative to the flow (we ignore any possible uncertainty in the yaw angle of the vehicle). Thus the uncertainty range is specified as 3.6 \(\pm\) 1\(^\circ\). Table 2 summarizes the nominal value and endpoints of the uncertainty range for these parameters.

3.2. Inflow conditions, turbulence quantities

The \(k-\omega\) SST model requires specification of the turbulence kinetic energy \(k\) and the specific dissipation-rate \(\omega\) at the inflow. This is done by specifying a turbulence intensity \(I\) and turbulence dissipation length scale \(L_{t,\omega}\) (characteristic of the energy-carrying eddies). Thus,

\[
k = \frac{3}{2} (U_{mag} I)^2, \quad (6a)
\]

\[
\omega = \frac{\sqrt{k}}{C_{\mu}^{1/4} L_{t,\omega}}, \quad (6b)
\]

where \(U_{mag}\) is the mean velocity magnitude and \(C_{\mu} = 0.09\) [43].

There is no direct experimental data on either the nominal values of the turbulence intensity and length scale or on their deviation from these values. We therefore estimate both the nominal and the uncertainty.

Assumed constant stagnation pressure and pressure fluctuation data, specifically that \(P'_{rms}/P_0 = 1 \times 10^{-5}\), are used to estimate the nominal turbulence intensity as \(I = 1\%\). Combining this with expert opinion [M. Gamba, personal communication, July 5\(^{th}\), 2012], we arrive at a range of \(I = [0.1, 1.9]\).

To estimate the turbulence length scale, we employ expert opinion [M. Gamba and J. Larsson, personal communication, July 5\(^{th}\), 2012] that the largest turbulence length scales generated at the throat preceding the nozzle in the tunnel are roughly half the size of the throat. Furthermore, that 3D isotropic eddies should grow as the inverse of the density in the isentropic expansion in the nozzle, i.e., that

\[
\frac{L_t}{L_{t,throat}} \approx \left( \frac{P_0}{P} \frac{T}{T_0} \right)^{1/2}. \quad (7)
\]

The flow in the nozzle should approximately satisfy the 1D variable-area flow relation

\[
\left( \frac{A}{A^*} \right)^2 = \frac{1}{M^2} \left[ \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \right]^{(\gamma + 1)/(\gamma - 1)}, \quad (8)
\]

where \(\gamma\) is the ratio of specific heats. To achieve Mach 7.4 (the nominal HEG flow condition) the nozzle must have an area ratio of \(\frac{A}{A^*} = 133\). The diameter of the test section at the HyShot II model is approximately 610mm, thus the throat diameter is 53mm.
Substituting the pressure and temperature ratios from Equation (5) into Equation (7), we find a length scale ratio of 9.43, determining a nominal length scale as $L_{t,ω0} \approx 245\text{mm}$. A conservative uncertainty range is specified relative to this nominal value where $L_{t,ω} = L_{t,ω0} \pm 50\%$. Table 2 summarizes the nominal value and endpoints of the uncertainty range for these parameters.

### 3.3. Transition Locations

There are three transition locations within the HyShot II: along the intake ramp $x_{t,r}$, body wall $x_{t,b}$, and cowl wall $x_{t,c}$. Along the intake ramp face $x_{t,r}$ is determined from heat flux measurements along the surface. The resolution of thermocouples is very coarse, however, with only four thermocouples spanning 115mm [13]. The body and cowl transition locations are specified through expert opinion (based on the experience of the HEG facility operators) and there are no transition measurements available on these walls. Based on this information the three transition locations are assumed uncertain, though are still treated as fixed locations (ignoring uncertainty related to the assumption that these locations are stationary).

In this work we only consider uncertainty in $x_{t,r}$ and $x_{t,c}$, both of which reside in the 2D computational domain. The 2D simulations are significantly cheaper than their 3D counterpart, so the impact of considering additional uncertainties is relatively small. The 3D computations are more expensive due to both increased domain size and the additional modeling closures (e.g., combustion, mixing). Thus uncertainty in $x_{t,b}$ is ignored as a practical consideration to reduce computational cost of the UQ analysis.

DLR uses theoretical arguments to support the specified nominal value of $x_{t,r} = 145\text{mm}$ (relative to the intake ramp leading edge) in light of the poor thermocouple resolution. They state that this value matches well with the estimated location from transition theory

$$\frac{Re_θ}{M_e} \approx 200,$$

where $Re_θ$ is the Reynolds number based on momentum thickness and $M_e$ is the Mach number at the edge of the boundary-layer. As opposed to directly assuming the magnitude of $x_{t,r}$ is uncertain, we instead assume the critical value of $Re_θ/M_e = 200$ from transition theory is uncertain. We first characterize the uncertainty in this critical value,

$$\frac{Re_θ}{M_e} = 200 \cdot (1 \pm \varphi), \text{ where } \varphi = 20\% .$$

The value of $\varphi$ is selected based on the opinions of G. Iaccarino and J. Larsson, and is a conservative estimate. This ratio can be written as a linear function of the momentum thickness

$$\frac{Re_θ}{M_e} = \text{const} \cdot \theta(x).$$
Upstream of transition there is only laminar boundary-layer growth so the laminar momentum thickness equation can be substituted, where

$$\theta(x) = \text{const} \cdot \sqrt{x},$$

(12)

and the critical ratio can now be represented as a linear function of the square root of $x$

$$\frac{Re_\theta}{M_e} = \text{const} \cdot \sqrt{x}.$$

(13)

Linearizing about the nominal value $x_{t,r0}$ gives

$$\frac{Re_\theta}{M_e} = \text{const} \cdot \sqrt{x_{t,r0}} + \frac{\text{const}}{2\sqrt{x_{t,r0}}} (x - x_{t,r0}),$$

$$= \text{const} \cdot \sqrt{x_{t,r0}} \left( 1 \pm \frac{x - x_{t,r0}}{2x_{t,r0}} \right).$$

(14)

This is in the same form as Equation (10), thus the uncertainty $\varphi$ is

$$\varphi = \pm \frac{x - x_{t,r0}}{2x_{t,r0}}.$$

(15)

Rearranging we can represent the transition location uncertainty as a function of $\varphi$

$$x_{t,r} = x_{t,r0}(1 \pm 2\varphi),$$

(16)

thus the uncertainty range is $x_{t,r} = x_{t,r0} \cdot [0.6, 1.4]$ where $x_{t,r0} = 145$mm.

The same procedure is used to determine the uncertainty in $x_{t,c}$. The nominal transition location is specified as 50mm downstream of the cowl leading edge, so the range is defined as $x_{t,c} = x_{t,c0} \cdot [0.6, 1.4]$ where $x_{t,c0} = 50$mm. The uncertainty ranges and nominal values for these parameters are summarized in Table 2.

| Parameter                      | Min        | Nominal    | Max        | Units     |
|-------------------------------|------------|------------|------------|-----------|
| Stagnation Pressure           | 16.448     | 17.730     | 19.012     | MPa       |
| Stagnation Enthalpy           | 3.0551     | 3.2415     | 3.4280     | MJ/kg     |
| Angle of Attack               | 2.6        | 3.6        | 4.6        | deg.      |
| Turbulence Intensity          | 0.001      | 0.01       | 0.019      |           |
| Turbulence Length Scale       | 0.1325     | 0.245      | 0.3575     | m         |
| Ramp Transition Location      | 0.087      | 0.145      | 0.203      | m         |
| Cowl Transition Location      | 0.030      | 0.050      | 0.070      | m         |

Table 2: Summary of parameters and ranges used in the UQ analysis of the HyShot II scramjet within the 2D intake ramp simulation.
4. Uncertainty Quantification

The objective of this work is to determine the effect of the seven uncertain parameters on the scramjet performance. Specifically we evaluate the range of exit pressures compatible with the input variability. Mathematically, this requires two global optimizations (minimize and maximize) to reveal the exit pressure range. These optimizations are challenging for several reasons:

1. The exit pressure is a functional of the pressure field computed from the nonlinear multiphysics simulation, so we have no prior knowledge of exploitable structure like linearity, convexity, or concavity.
2. Each evaluation of the objective requires an expensive simulation; each run takes approximately two hours on available resources.
3. We cannot evaluate gradients or Hessians of the objective with respect to the inputs.
4. Function evaluations contain nonnegligible numerical noise due to (i) the fixed point iteration scheme that solves the compressible flow model and (ii) the fixed mesh constructed to capture shocks at the nominal value for the inputs; therefore, finite difference gradients cannot be trusted.
5. Given the above considerations, a seven-dimensional parameter space is extremely large for the global optimization.

These conditions are common in design problems with expensive computer simulations, and a practical approach is to optimize with the aid of response surfaces [1, 20, 48]. The trouble with response surfaces in our case is knowing where in the seven-dimensional space of inputs to evaluate the objective and construct the initial surrogate. Most methods involve an iterative construction that evaluates the objective at a new set of points to improve accuracy near the predicted optimum. Unfortunately, this is not feasible in our case, since each simulation must be carefully monitored for convergence; automating the selection of new simulation runs is impractical.

If we could detect some special structure in this complex model, then we could potentially exploit it to make the optimizations easier. In particular, if we could determine that the objective is roughly monotonic with respect to its most important inputs, then the optimization simplifies dramatically. For example, if an increase in angle of attack increases the exit pressure, then the model with the largest angle of attack should have a relatively large exit pressure; reason similarly for the other six inputs.

We employ a simple diagnostic visualization known as a sufficient summary plot [6] to determine if a one-dimensional active subspace exists in our particular model. In general, the active subspace is a set of directions along which the quantity of interest changes the most, on average. Dimension reduction is possible when the number of active directions is less than the number of input parameters. Active subspaces are based on the eigenvalues and eigenvectors of the uncentered covariance matrix of the gradient vector of the objective; details and mathematical formalism can be found in our recent work [5].
For HyShot II, we do not have access to the gradient of the quantity of interest (described in 4.1) with respect to the inflow parameters, so we cannot check for an effective dimension of the model, e.g., by examining the eigenvalues of the uncentered covariance of the gradient. However, we can use a simple check based on linear regression to reveal (if present) a dominant one-dimensional active subspace—that we can potentially exploit in the optimization. This check is closely related to regression graphics [6] for studying trends in regression surfaces through visualization.

4.1. Quantity of Interest

We treat the exit pressure as a critical measure of the scramjet performance. We repeat the analysis for two fuel injection rates to investigate the effects of uncertainties on the safe operating envelope of the engine, specifically with respect to the unstart phenomena. Numerical simulations that demonstrate unstart, i.e., the scenario in which the shock system has moved all the way to the upstream boundary, are computationally challenging—especially for large values of $\phi$. For this reason, we seek a proxy indicator for the unstart process.

Several possible unstart indicators have been proposed [9]. Presently, we use a function of the combustion chamber exit pressure. The motivation for this choice is related to the physics of compressible flow with heat addition (Rayleigh’s flow): a supersonic flow decelerates towards sonic conditions when heat is released through a combustion process. The corresponding increase in pressure achieves its maximum just before choking conditions are reached.

The unstart proxy is defined as the normalized integral of pressure over the last 10mm of the combustor

$$\frac{1}{\text{vol}(V)} \iiint_V P \, dx \, dy \, dz,$$

(17)

where $V$ is the volume between 0.64 $\leq x \leq$ 0.65m. This integral is essentially a function of the maximum exit pressure $P_{e,\text{max}}$, but is used instead because $P_{e,\text{max}}$ is a non-smooth quantity [46].

4.2. Identifying a one-dimensional active subspace

Next we generically describe a cheap procedure that serves as a quick check to reveal a one-dimensional active subspace in the quantity of interest as a function of its inputs; we then apply it to the HyShot II model. To carry out this quick check, the following ingredients must be available.

1. A scalar quantity of interest. The model should have a single output $f = f(x)$ that depends on the $m$ input parameters denoted $x = [x_1, \ldots, x_m]^T$.
2. A range for each input parameter. Determine a lower bound $x_l^i$ and an upper bound $x_u^i$ such that $x_l^i \leq x_i \leq x_u^i$ for each input parameter $x_i$.
3. Enough time and computing resources to fit a linear model. To fit the linear model below, we need to evaluate the model $f$ $O(m)$ times, e.g., $2m$ times.
Given these pieces, we run the following procedure.

1. Draw \( M \) samples \( \hat{x}_j \) uniformly at random from \([-1,1]^m\). We choose to sample uniformly in the range because in the case of the HyShot inputs there is no data defining a different measure on the space of the inputs.

2. Let
\[
x_j = \frac{1}{2} \left[(x^u - x^l) \cdot \hat{x}_j + (x^u + x^l)\right],
\]
where \( x^l \) is the \( m \)-vector of lower bounds and \( x^u \) is the \( m \)-vector of upper bounds; the dot operation (\( \cdot \)) is component-wise multiplication.

3. Compute \( f_j = f(x_j) \).

4. Compute the coefficients of the linear regression model
\[
f(x) \approx \hat{u}_0 + \hat{u}_1 \hat{x}_1 + \cdots + \hat{u}_m \hat{x}_m
\]
with least-squares
\[
\hat{u} = \arg\min_u \frac{1}{2} \|Au - f\|_2^2,
\]
where
\[
A = \begin{bmatrix}
1 & \hat{x}_1^T \\
& \vdots & \ddots & \vdots \\
& \vdots & \ddots & 1 \\
1 & \hat{x}_M^T
\end{bmatrix}, \quad \hat{u} = \begin{bmatrix}
\hat{u}_0 \\
\vdots \\
\hat{u}_m
\end{bmatrix}, \quad f = \begin{bmatrix}
f_1 \\
\vdots \\
f_M
\end{bmatrix}.
\]

5. Let \( w = u'/\|u'\| \) where \( u' = [\hat{u}_1, \ldots, \hat{u}_m]^T \) is the last \( m \) coefficients (i.e., the gradient) of the linear regression model.

6. Produce a scatter plot of \( w^T \hat{x}_j \) versus \( f_j \), which is the sufficient summary plot.

Here is a MATLAB implementation of this procedure, assuming there is a function \textit{mymodel.m} that computes the model output \( f \) given inputs and vectors \( x_l \) and \( x_u \) containing the lower and upper bounds of the inputs.

```matlab
% Produce a sufficient summary plot to check for a % dominant 1-d active subspace.
M = 4*m;
Xhat = 2*rand(m,M)-1;
X = 0.5*(repmat(xu-xl,1,M).*Xhat + repmat(xu+xl,1,M));
for j=1:M, f(j) = mymodel(X(:,j)); end
uhat = [ones(M,1) Xhat'] \ f';
w = uhat(2:m+1)/norm(uhat(2:m+1));
plot(Xhat'*w,f,'o');
```

When \( m = 2 \), this visualization is comparable to producing a three-dimensional plot of \( f \) versus \( x_1 \) and \( x_2 \), and then rotating the plot until a dominant, univariate trend emerges in the outputs \( f \). In effect, the normalized regression coefficients \( w \) provide the angles in \( m \) dimensions from which to view the outputs to reveal a trend.
We apply this procedure to the exit pressure from the HyShot II as a function of the $m = 7$ input parameters with ranges shown in Table 2. We repeat the study for two values of the fuel equivalence ratio $\phi$ defined in Equation (3), 0.30 and 0.35. For each $\phi$ we sample the inputs and run the model $M = 14$ times. Figure 7 shows the results of the diagnostic visualization. The $x$-axis of Figure 7 show values of the active variable $w^T \hat{x}$, which is a specific linear combination of the seven model inputs $x$ with normalized ranges. Figure 7 shows that a dominant monotonic—even linear—trend exists in the exit pressure as a function of the derived active variable $w^T \hat{x}$ for both $\phi = 0.30$ and $\phi = 0.35$. We can exploit this trend to easily determine the range of the exit pressure over all values of the seven input parameters.

![Figure 7](image)

(a) $\phi = 0.30$

(b) $\phi = 0.35$

Figure 7: The gray circles represent exit pressures at $\phi = 0.30$ and 0.35 computed from a set of 14 HyShot II simulations and plotted against the respective active variable $w^T \hat{x}$. A clear monotonic trend is present in both cases that we exploit to find the range of exit pressures. The overlapping gray circles distributed horizontally correspond to exit pressures plotted against 500 bootstrap replications of the active variable. The horizontal variability is a result of the relatively low oversampling (2x) when fitting the linear model. The gray squares show the exit pressures at the boundaries of the domain where the perceived trend suggests we find the upper and lower bounds of the exit pressure. The monotonic trend is validated by these two additional simulations for each value of $\phi$.

The values in the vector $w$ can be used to measure the global sensitivity of the exit pressure to each of the seven parameters. These values and their corresponding input parameters are shown in Table 3. They suggest that four of the seven parameters contribute the most to the global variability of the exit pressure. The angle of attack, the stagnation conditions (temperature and pressure) and the turbulence intensity dominate the transition location and the turbulence length scale. In general, it appears that the largest contributors to changes in the exit pressure are the inputs directly related to the intensity of the bow shock at the nose of the vehicle—more precisely to the thermodynamic post shock conditions and the flow entering the Hyshot combustor. The state
of the boundary layers (as determined by different transition location) plays a secondary role. It is more difficult to assess why the turbulent intensity plays such an important role. Caution is needed before interpreting this result, since RANS eddy-viscosity models (like the SST) are known to behave erratically across strong shock waves—like those present here [39]; a more detailed investigation is outside the scope of the present paper.

There is an important distinction between the equivalence ratios $\phi = 0.30$ and 0.35. The leading factors in determining changes in the exit pressure do not change, but the relative importance of stagnation pressure and enthalpy does. For the smaller equivalence ratio, the leading cause of variability in the exit pressure is attributed to changes in the bow shock. For the larger equivalence ratio, changes in stagnation enthalpy play a different role (the stagnation temperature is constant across a shock wave). In fact, the stagnation temperature directly impacts the combustion process within the engine. At high equivalence ratios the corresponding uncertainty becomes more important in determining the amount of heat released in the combustor and ultimately the exit pressure.

| Index | $\phi = 0.30$ | $\phi = 0.35$ | Parameter                     |
|-------|---------------|---------------|-------------------------------|
| 1     | 0.6933        | 0.6996        | Angle of Attack               |
| 2     | 0.5033        | 0.4823        | Turbulence Intensity          |
| 3     | 0.0067        | -0.0270       | Turbulence Length Scale       |
| 4     | 0.3468        | 0.1997        | Stagnation Pressure           |
| 5     | -0.3732       | -0.4738       | Stagnation Enthalpy           |
| 6     | -0.0524       | -0.0602       | Cowl Transition Location      |
| 7     | 0.0605        | 0.0957        | Ramp Transition Location      |

Table 3: The components of the vector $\mathbf{w}$ that define the active subspace computed independently for each value of $\phi$. Each component corresponds to one of the uncertainty sources in the HyShot II simulation as described in Section 3.

### 4.3. Bootstrap for $\mathbf{w}$

The elements of $\mathbf{w}$ depend on the samples $f_j$ used to fit the linear model. With finite samples, it is natural to ask if the gradient of the linear model has been sufficiently resolved to uncover the true one-dimensional active subspace. However, with a limited budget of function evaluations, checking convergence of $\mathbf{w}$ is not feasible. We use a bootstrap technique [7] to estimate the variability in the computed components of $\mathbf{w}$. Let

$$\pi^k = [\pi_1^k \ldots \pi_M^k], \quad k = 1, \ldots, N$$

be $M$-vectors of integers between 1 and $M$ sampled with replacement. (Recall that in our case, $M = 14$ is the number of HyShot II simulations we use to fit
the linear model.) Define the vectors $f_k$ by

$$f_k = \begin{bmatrix} f_{\pi_1}^k & \vdots & f_{\pi_M}^k \end{bmatrix} \quad (23)$$

and the matrices $A_k$ by

$$A_k = \begin{bmatrix} 1 & \hat{x}_{\pi_1}^T \\ \vdots \\ 1 & \hat{x}_{\pi_M}^T \end{bmatrix} \quad (24)$$

In words, $A_k$ is an $M \times (m + 1)$ matrix (where $m = 7$ is the number of HyShot II input parameters) whose rows are sampled with replacement from the rows of $A$ in (21). Similarly, $f_k$ is an $M$-vector whose elements are sampled with replacement from $f$. Compute $w_k$ for each pair $A_k$ and $f_k$, then build bootstrap histograms for the components of $w$.

We repeat this procedure for $k = 1, \ldots, 10^4$, which produces $10^4$ bootstrap replicates. The bootstrap histograms along with stem plots of the values of $w$ from Table 3 are shown for $\phi = 0.30$ in Figure 8 and for $\phi = 0.35$ in Figure 9. The sharp peak of the histograms around the stems suggests confidence in the computed directions. The large ranges in the histograms is a result of the low oversampling factor (2x) used to fit the linear model.

We could have included bootstrap confidence intervals for each component of $w$ in Table 3. But this might be misleading, since the components of $w$ are not independent. Instead, to get an idea of the variability in $w$ provided by the bootstrap, we choose a set of 500 bootstrap replicates of $w$ and plot the exit pressures $f_j$ against the corresponding active variables $w^T \hat{x}_j$. The result is a horizontal scatter of black dots around each point in Figure 7. The scatter provides a visual indication of the variability in the vectors $w$. The spread relative to the range of the active variable is again due to the low oversampling factor (2x) used to fit the linear model.

4.4. **Approximating the range of exit pressures**

The strong monotonic trend in the exit pressure as a function of the active variable $w^T \hat{x}$ suggests the following simple heuristic to find the maximum and minimum exit pressures. Define

$$\hat{x}_{\text{max}} = \underset{-1 \leq x \leq 1}{\text{argmax}} \ w^T \hat{x}, \quad f_{\text{max}} = f \left( \frac{1}{2} \left[ (x^u - x^l) \cdot \hat{x}_{\text{max}} + (x^u + x^l) \right] \right),$$

where the scale and shift of $\hat{x}_{\text{max}}$ returns it to the original parameter range. Similarly define

$$\hat{x}_{\text{min}} = \underset{-1 \leq x \leq 1}{\text{argmin}} \ w^T \hat{x}, \quad f_{\text{min}} = f \left( \frac{1}{2} \left[ (x^u - x^l) \cdot \hat{x}_{\text{min}} + (x^u + x^l) \right] \right).$$

(25)
Figure 8: Bootstrap histograms of the components of the active subspace vector $\mathbf{w}$ for the case where $\phi = 0.30$. The black stems are the computed components of $\mathbf{w}$ from Table 3. The caption of each subfigure name the specific inflow parameter. The sharp peaks around each of the stems provides confidence that the computed $\mathbf{w}$ are stable.

The range $[f_{\text{min}}, f_{\text{max}}]$ provides our estimate for the range of exit pressures from the HyShot II model. Computing these requires running the model four more times—two for each $\phi$—which is much cheaper than adaptively constructing a response surface. This is only successful because of the structure revealed in the active subspace. The maximum and minimum exit pressures are shown in Figure 7 as gray squares along with the runs used to determine the active subspace—all plotted against the active variable. At worst, these runs bound the initial set of 14 runs. At best, they provide estimates of the range of exit pressures over all values of the input parameters. Checking the necessary conditions for stationarity at these points is not feasible.

### 4.5. Constraining the exit pressure

To further demonstrate the utility of the active subspace, consider the following exercise in safety engineering. Suppose that the scramjet operates safely when the exit pressure is below 2.8 bar, but it nears unsafe operation above 2.8 bar. With the one-dimensional active subspace, we can quickly characterize the parameter regime that produces exit pressures below the threshold of 2.8 bars.

The first step is to build a response surface model of the exit pressure as a function of the active variable $\mathbf{w}^T \hat{\mathbf{x}}$. We could try to construct a response surface of the seven model input variables. But with only 16 model runs (the first 14 samples plus the two runs used to compute the bounds), our modeling choices would be very limited. The apparent low departure from a univariate trend in Figure 7 suggests that we can construct a useful response of just the active variable. In particular, we can use the 16 model evaluations for each value
of $\phi$ and construct a univariate quadratic regression surface that models exit pressure as a function of the active variable. This construction is equivalent to a single-index regression model for the exit pressure with a quadratic polynomial link function [18].

The coefficient of determination (i.e., $R^2$) is 0.997 for $\phi = 0.30$ and 0.998 for $\phi = 0.35$, which gives strong confidence in the quadratic model. We use the regression model to find the largest value of the active variable such that the upper 99% confidence limit of from the regression surface is less than the threshold of 2.8 bars. Figure 10 shows the regression surface and its upper 99% confidence limit for both values of $\phi$. The shaded region identifies the values of the active variable that produce exit pressures at or below the pressure threshold.

Let $y_{\text{max}}$ be the value of the active variable where the regression confidence bound crosses the pressure threshold. Then the safe region of the normalized input parameters is the set $S$ defined as

$$S = \{ \tilde{x} : \mathbf{w}^T \tilde{x} \leq y_{\text{max}}, -1 \leq \tilde{x} \leq 1 \}.$$  \hfill (27)

One can easily shift and scale this region to the space of the model’s true input variables $x$. The linear inequality constraint implies that the inputs are related with respect to the exit pressure. In other words, the complete range of safe angles of attack depends on the other input variables. The presence of the active subspace and the quality of the regression surface enable us to simply characterize these relationships.
Figure 10: A quadratic regression surface models the relationship between the active variable $w^T \hat{x}$ and the exit pressure. The solid line shows the mean prediction of the regression surface. The dotted line shows the upper 99% confidence limit for the prediction. We find the value of the active variable $y_{\text{max}}$ where the upper confidence limit crosses the safety threshold of 2.8 bars. All values of the active variable less than $y_{\text{max}}$ produce pressures within the safety limit. The set of safe input variables is shown in Equation (27).

The safe set $S$ defined in (27) is like a seven-dimensional box with the top chopped off by the hyperplane $\hat{x}^T w \leq y_{\text{max}}$. We can identify a new set of independent ranges for the input variables such that all inputs within those ranges produce exit pressures below the 2.8 bar safety threshold; this is like finding the largest seven-dimensional box that fits inside the set $S$. More precisely, we solve the following optimization problem,

\[
\text{maximum } \hat{x} \quad \prod_{i=1}^{m} |\hat{x}_i - \hat{x}_{i,\text{min}}|, \\
\text{subject to } \hat{x} \in S,
\]

where $\hat{x}_{i,\text{min}}$ are the components of the minimizer from (26). The maximizing components define the corner of the largest hyperrectangle opposite the corner $\hat{x}_{\text{min}}$.

We can interpret such analysis as backward uncertainty propagation that characterizes safe inputs given a characterization of a safe output under the constraint that the inputs be independent. For the present study, these spaces differ between the two fuel flow rates; see Figure 10. For $\phi = 0.30$, only angle of attack and turbulent intensity are affected by the safety constraint on the pressure; angle of attack must be less than 3.99 degrees, and turbulent intensity must be less than 0.018. For $\phi = 0.35$, the same parameters are constrained—angle of attack less than 3.31 and turbulent intensity less than 0.011—and a stricter minimum on stagnation enthalpy (greater than 2.23 MJ/Kg) appears. To explain this, we observe that low free-stream stagnation temperature leads to conditions that are further away from the adiabatic frame temperature leading
to increased heat deposition in the air-stream. This in turns leads to more deceleration and larger pressure increases in the chamber [47].

For completeness, we use the quadratic response surface of the active variable to estimate a cumulative distribution function of the exit pressure for both fuel injection rates. The procedure we follow is to sample uniformly from the scramjet’s seven-dimensional input space, and for each sample evaluate the active variable and the quadratic response of the active variable. Figure 11 shows estimates of the cumulative distribution function computed from these samples with a Gaussian kernel density estimator. The vertical lines show the estimated upper and lower bounds.

![Figure 11: Estimated cumulative distribution functions for exit pressure at both values of fuel equivalence ratio $\phi$. These are estimated with Gaussian kernel density estimates; the samples are drawn from the quadratic response surface of the active variable. Vertical lines show the estimated bounds for each case. Each of 500 gray lines is computed with a bootstrap replicates of the direction $w$.](image)

5. Conclusions

We present a numerical investigation of the reactive flow within a hydrogen fueled scramjet with the objective of studying the effect of uncertainties in operating on the overall performance. We carry out three-dimensional RANS simulations with a flamelet-based combustion model at two different equivalence ratios and record the pressure increase at the engine exit as a measure of the total combustion heat release. Seven uncertain factors are considered and their ranges are justified on the basis of experimental evidence or expert opinions. Uncertainty quantification is enabled by identifying a one-dimensional model that describes the relationship between the seven uncertain inputs and the quantity of interest. Such identification is based on active subspaces. This strategy proves to be extremely effective requiring only sixteen full simulations.
for each value of the fuel equivalence ratio. It also ranks the inputs to identify the most dominant ones in determining the output uncertainty. The study is complemented by a bootstrap estimate of the confidence. The results show that the overall exit pressure variability is higher at the high equivalence ratio (from 27% to 32%) and although the dominant inputs are the same (angle of attack, stagnation pressure and enthalpy and turbulence intensity) their relative importance changes. Specifically the stagnation enthalpy (temperature) becomes more important at high equivalence ratio. The active subspace is also exploited to study the impact of a constraint imposed on the maximum exit pressure. Once again the results indicate that the stagnation enthalpy uncertainty plays a critical role at high equivalence ratios.

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