UNIQUENESS OF POSITIVE SOLUTIONS WITH CONCENTRATION FOR THE SCHRÖDINGER-NEWTON PROBLEM

PENG LUO, SHUANGJIE PENG, AND CHUNHUA WANG

Abstract. We are concerned with the following Schrödinger-Newton problem
\[-\varepsilon^2 \Delta u + V(x)u = \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^2(\xi)\,d\xi}{|x-\xi|} \right) u, \quad x \in \mathbb{R}^3.\]

For \( \varepsilon \) small enough, we show the uniqueness of positive solutions concentrating at the nondegenerate critical points of \( V(x) \). The main tools are a local Pohozaev type of identity, blow-up analysis and the maximum principle. Our results also show that the asymptotic behavior of concentrated points to Schrödinger-Newton problem is quite different from those of Schrödinger equations.

1. Introduction and main results

The Schrödinger-Newton problem appeared in \cite{18} and can be used to describe the quantum mechanics of a polaron at rest. It was also used by Choquard to describe an electron trapped in its own hole in a certain approximating to Hartree-Fock theory of one component plasma in \cite{12}. Penrose in \cite{18} also derived it as a model of self-gravitating matter, in which quantum state reduction is understood as a gravitational phenomenon. Specifically, if \( m \) is the mass of the point, the interaction leads to the system in \( \mathbb{R}^3 \)
\[
\begin{aligned}
\frac{\varepsilon^2}{2m} \Delta u - V(x)u + \psi u &= 0, & x \in \mathbb{R}^3, \\
\Delta \psi + 4\pi \tau |u|^2 &= 0, & x \in \mathbb{R}^3,
\end{aligned}
\]

where \( u \) is the wave function, \( \psi \) is the gravitational potential energy, \( V(x) \) is a given Schrödinger potential, \( \varepsilon \) is the Planck constant, \( \tau = Gm^2 \) and \( G \) is the Newton’s constant of gravitation.

Let
\[ u(x) = \frac{\hat{u}}{4\varepsilon \sqrt{\pi\tau m}}, \quad V(x) = \frac{1}{2m} \hat{V}(x), \quad \psi(x) = \frac{1}{2m} \hat{\psi}(x). \]

Then system (1.1) can be written, maintaining the original notations, as
\[
\begin{aligned}
\varepsilon^2 \Delta u - V(x)u + \psi u &= 0, & x \in \mathbb{R}^3, \\
\varepsilon^2 \Delta \psi + |u|^2 &= 0, & x \in \mathbb{R}^3,
\end{aligned}
\]

The second equation in (1.2) can be explicitly solved with respect to \( \psi \), so that the system turns into the following single nonlocal equation
\[-\varepsilon^2 \Delta u + V(x)u = \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^2(\xi)\,d\xi}{|x-\xi|} \right) u, \quad x \in \mathbb{R}^3. \tag{1.3}\]

Also, (1.3) appears in the study of standing waves for the following nonlinear Hartree equations
\[ i\varepsilon \frac{\partial \varphi}{\partial t} = -\varepsilon^2 \Delta_x \varphi + (V(x) + E)\varphi - \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{\varphi^2(\xi)\,d\xi}{|x-\xi|} \right) \varphi, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+, \]

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with the form $\varphi(x,t) = e^{-iEt/\varepsilon}u(x)$, where $i$ is the imaginary unit and $\varepsilon$ is the Planck constant.

In recent decades, problem (1.3) has been extensively investigated. When $\varepsilon = 1$ and $V(x) = 1$, (1.3) changes into

$$-\Delta u + u = \frac{1}{8\pi \varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^2(\xi)}{|x - \xi|} d\xi \right) u, \quad x \in \mathbb{R}^3. \tag{1.4}$$

The existence and uniqueness of ground states for (1.4) was obtained with variational methods by Lieb [12], Lions [13] and Menzala [15]. Later, the nondegeneracy of the ground states for (1.4) was proved by Tod-Moroz [20] and Wei-Winter [22], which can be stated as follows:

**Theorem A.** (c.f [12][22]) There exists a unique radial solution $U_\alpha$ of the problem

$$
\begin{cases}
-\Delta u + V(\alpha)u = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{u^2(\xi)}{|x - \xi|} d\xi \right) u, \quad \text{in} \quad \mathbb{R}^3, \\
u(x) > 0, \quad \text{in} \quad \mathbb{R}^3, \quad u(0) = \max_{x \in \mathbb{R}^3} u(x).
\end{cases}
$$

The solution $U_\alpha$ is strictly decreasing and

$$\lim_{|x| \to \infty} U_\alpha(x)e^{|x||x|} = \lambda_0 > 0, \quad \lim_{|x| \to \infty} \frac{U'_\alpha(x)}{U_\alpha(x)} = -1,$$

for some constant $\lambda_0 > 0$. Moreover, if $\phi(x) \in H^1(\mathbb{R}^3)$ solves the linearized equation

$$-\Delta \phi(x) + V(\alpha)\phi(x) = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{U^2_\alpha(\xi)}{|x - \xi|} d\xi \right) \phi(x) + \frac{1}{4\pi} \left( \int_{\mathbb{R}^3} \frac{U_\alpha(\xi)\phi(\xi)}{|x - \xi|} d\xi \right) U_\alpha(x),$$

then $\phi(x)$ is a linear combination of $\partial U_\alpha/\partial x_j$, $j = 1, 2, 3$.

If $\varepsilon$ is small and $V(x)$ is not a constant, the existence of solutions with ground states for (1.3) under some conditions on $V(x)$ was proved by [14] since problem (1.3) has a variational structure. Moreover, the solution with ground states concentrates at certain point. Later, Wei-Winter [22] proved that (1.3) has a solution concentrating at $k$ points which are the local minimum points of $V(x)$. This also means the existence of multiple solutions. Concerning the existence of solutions with concentration in other cases, we can refer to [5][19] and the references therein.

On the other hand, the Schrödinger-Newton problem (1.3) is a special type of following Choquard equation:

$$-\varepsilon^2 \Delta u + V(x)u = \frac{1}{8\pi \varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^p(\xi)}{|x - \xi|^{N-\alpha}} d\xi \right) |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $\alpha \in (0, N)$ and $p > 1$. For the existence and concentration of positive solutions to the Choquard equation (1.5), one can refer to [16][17] and the references therein.

As far as we know, for nonlinear Schrödinger equations, the results on the uniqueness of solutions concentrating at some points are few. To obtain uniqueness of concentrating solutions, the classical moving plane method does not work. The main tools are the topological degree and local Pohozaev identity which can be found in [2][3][6][8]. However, for problem (1.3), whether the solution with concentration is unique is still open. In this paper, we intend to solve this type of problems partially by using local Pohozaev type of identity and blow-up analysis which was recently developed in [3][6][9]. However, we should point out that, compared with [3][6][9], to handle the nonlocal term in (1.3), there are many new difficulties, which will be discussed in more details later.

We assume that $V(x)$ is a bounded $C^1$ function satisfying $\inf_{x \in \mathbb{R}^3} V(x) > 0$. Define the following Sobolev space $H_\varepsilon$

$$H_\varepsilon := \left\{ u(x) \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} (\varepsilon^2|\nabla u(x)|^2 + V(x)u^2(x)) dx < \infty \right\},$$
and the corresponding norm
\[ \|u\|_\varepsilon = (u(x), u(x))^{1/2} = \left( \int_{\mathbb{R}^3} (\varepsilon^2|\nabla u(x)|^2 + V(x)u^2(x))dx \right)^{1/2}. \]

**Definition A.** (c.f. [2]) We call a family of nonnegative functions \( \{u_\varepsilon\}_{\varepsilon > 0} \) concentrate at a set of different points \( \{a_1, \cdots, a_k\} \subset \mathbb{R}^3 \) if there exist \( \{x_{i,\varepsilon}\}_{\varepsilon > 0} \subset \mathbb{R}^3, \|x_{i,\varepsilon} - a_i\| = o(1) \) for \( i = 1, \cdots, k \) and \( k \) nonnegative functions \( U_i \in H^1(\mathbb{R}^3) \) (1 \( \leq \) \( i \) \( \leq \) \( k \)) satisfying \( U_i(x) \neq 0 \) and \( U_i(0) = \max_{x \in \mathbb{R}^3} U_i(x) \) such that
\[ \|u_\varepsilon - \sum_{i=1}^k U_i(\frac{x - x_{i,\varepsilon}}{\varepsilon})\|_\varepsilon = o(\varepsilon^{3/2}). \] (1.6)

**Remark A.** Here the solutions in **Definition A** are consistent with those obtained by Secchi [19] and Wei-Winter [22].

Our main results are as follows.

**Theorem 1.1.** Let \( \{u_\varepsilon^{(1)}(x)\}_{\varepsilon > 0}, \{u_\varepsilon^{(2)}(x)\}_{\varepsilon > 0} \) be two families of positive solutions of (1.3) concentrating at a nondegenerate critical point \( a_1 \in \mathbb{R}^3 \) of \( V(x) \). Then for \( \varepsilon \) small enough, \( u_\varepsilon^{(1)}(x) \equiv u_\varepsilon^{(2)}(x) \)

must be of the form
\[ U_{a_1}(\frac{x - x_{1,\varepsilon}}{\varepsilon}) + w_\varepsilon(x), \] (1.7)

with \( x_{1,\varepsilon}, w_\varepsilon(x) \) satisfying, as \( \varepsilon \to 0, \)
\[ |x_{1,\varepsilon} - a_1| = o(\varepsilon), \text{ and } \|w_\varepsilon\|_\varepsilon = O(\varepsilon^{7/2}). \] (1.8)

**Theorem 1.2.** Let \( \{u_\varepsilon^{(1)}(x)\}_{\varepsilon > 0}, \{u_\varepsilon^{(2)}(x)\}_{\varepsilon > 0} \) be two families of positive solutions of (1.3) concentrating at \( k \) (\( k \geq 2 \)) different nondegenerate critical points \( \{a_1, \cdots, a_k\} \subset \mathbb{R}^3 \) of \( V(x) \). Then for \( \varepsilon \) small enough, \( u_\varepsilon^{(1)}(x) \equiv u_\varepsilon^{(2)}(x) \)

must be of the form
\[ \sum_{j=1}^k U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) + w_\varepsilon(x), \] (1.9)

with \( x_{j,\varepsilon}, w_\varepsilon(x) \) satisfying, for \( j = 1, \cdots, k, \) as \( \varepsilon \to 0, \)
\[ |x_{j,\varepsilon} - a_j| = O(\varepsilon) \text{ and } \|w_\varepsilon\|_\varepsilon = O(\varepsilon^{7/2}). \] (1.10)

Furthermore, there exist \( j_0 \in \{1, \cdots, k\}, C_1 > 0 \) and \( C_2 > 0 \) such that
\[ C_1\varepsilon \leq |x_{j_0,\varepsilon} - a_{j_0}| \leq C_2\varepsilon. \] (1.11)

**Remark 1.3.** For the existence of positive solutions to (1.3) concentrating at \( k \) different points, one can refer to Wei-Winter’s paper [22]. Also, we can prove that if the positive solutions to (1.3) concentrating at \( k \) different points, then these points must be the critical points of \( V(x) \) by Pohozaev identity. Theorem 1.2 shows the uniqueness of the solutions obtained by Wei-Winter in [22].

For Schrödinger equations, it is proved in [3] that \( |x_{j,\varepsilon} - a_j| = o(\varepsilon) \) for \( j = 1, \cdots, k. \) For the Schrödinger-Newton problem, we can also prove that \( |x_{1,\varepsilon} - a_1| = o(\varepsilon) \) in the single peak case. However, in the multi-bump case, we can only deduce that the order of \( |x_{j_0,\varepsilon} - a_{j_0}| \) is the same as \( \varepsilon \) for some \( j_0 \in \{1, \cdots, k\}. \) This means that the asymptotic behavior of concentrated points to Schrödinger-Newton problem is quite different from those of Schrödinger equations.
Remark 1.4. Recently, Xiang [24] proved the uniqueness and nondegeneracy of ground states to the above Choquard equation (1.5) with $V(x) = a > 0$ when $p \to 2$. However for general $p$, the uniqueness and nondegeneracy of ground states in Wei-Winter [22] is still an open problem. Correspondingly, our results cannot generalize to the Choquard equation for general $p$. However, our methods to handle the nonlocal term is useful to study the Schrödinger-Possion problem [10].

Our main idea is inspired by Cao-Li-Luo [3], Deng-Lin-Yan [6] and Guo-Peng-Yan [9]. Let $u_{\varepsilon}^{(1)}(x)$, $u_{\varepsilon}^{(2)}(x)$ be two different positive solutions concentrating at $\{a_{1}, \cdots, a_{k}\}$ for $k \geq 1$. Set

$$\eta_{\varepsilon}(x) = \frac{u_{\varepsilon}^{(1)}(x) - u_{\varepsilon}^{(2)}(x)}{\|u_{\varepsilon}^{(1)} - u_{\varepsilon}^{(2)}\|_{L^\infty(\mathbb{R}^3)}}. \quad (1.12)$$

Then we prove $\eta_{\varepsilon}(x) = o(1)$ for $x \in \mathbb{R}^3$, which is incompatible with the fact $\|\eta_{\varepsilon}\|_{L^\infty(\mathbb{R}^3)} = 1$. For the estimate near the nondegenerate critical points, we will use the blow-up analysis and local Pohozaev type of identity. But for the estimate away from the nondegenerate critical points, we will use the maximum principle.

Remark 1.5. Problem (1.3) is nonlocal, which will cause many differences compared with [3, 6].

First, to apply the blow-up analysis, we need to prove that

$$|x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}| = o(\varepsilon), \quad (1.13)$$

which cannot be obtained by $|x_{j,\varepsilon}^{(1)} - a_{j}| = O(\varepsilon)$ and $|x_{j,\varepsilon}^{(2)} - a_{j}| = O(\varepsilon)$ for $j = 1, 2, \cdots, k$ ($k \geq 2$). To obtain (1.13), we will apply local Pohozaev identity carefully in Proposition 4.7 below.

Next, after using the blow-up analysis, we will apply local Pohozaev identity again to obtain $\eta_{\varepsilon}(x) = o(1)$ near the nondegenerate critical points. To this aim, we need to estimate the error between the two solutions precisely in Proposition 4.7 below, where the classical Nash-Moser iteration will be used.

On the other hand, we would like to point out that the corresponding local Pohozaev identity will have two terms involving volume integral. Then to calculate the two integrals precisely, we need to use some symmetries skillfully by some observations. We will also use the maximum principle carefully due to the nonlocal term.

This paper will be organized as follows. In Section 2 and Section 3, we first establish some basic estimates of the solutions with concentration. Then we give the detailed proofs of Theorem 1.1 by using a local Pohozaev identity, blow-up analysis and the maximum principle. In Section 4, we obtain a precise estimate on the errors of the two solutions and the concentrated points. Combining these estimates and applying the methods in the proof of Theorem 1.1 we prove Theorem 1.2 in Section 5. To make the main clue clear, some important but tedious estimates and inequalities will be delayed to the Appendix. In the sequel, we will use $C$ to denote various generic positive constants. $O(t)$, $o(t)$ mean $|O(t)| \leq C|t|$, $o(t)/t \to 0$ as $t \to 0$. $o(1)$ denotes quantities that tend to 0 as $\varepsilon \to 0$.

2. The basic estimates

Proposition 2.1. Let $\{u_{\varepsilon}(x)\}_{\varepsilon > 0}$ be a family of positive solutions of (1.3) concentrating at different points $a_{1}, \cdots, a_{k}$ with $k \geq 1$. Then $u_{\varepsilon}(x)$ is of the form

$$u_{\varepsilon}(x) = \sum_{j=1}^{k} U_{a_{j}}(\frac{x - x_{j,\varepsilon}}{\varepsilon}) + w_{\varepsilon}(x), \quad (2.1)$$

with $x_{j,\varepsilon}$ and $w_{\varepsilon}(x)$ satisfying, for $j = 1, \cdots, k$, as $\varepsilon \to 0$,

$$|x_{j,\varepsilon} - a_{j}| = o(1) \text{ and } \|w_{\varepsilon}\|_{\varepsilon} = o(\varepsilon^{3/2}), \quad (2.2)$$
and

\[ (w_\varepsilon(x), U_{a_j}(\frac{x-x_i}{\varepsilon}))_\varepsilon = o(\varepsilon^3), \ (w_\varepsilon(x), \frac{U_{a_j}(x-x_i)}{\partial x^i})_\varepsilon = 0, \ i = 1, 2, 3. \] (2.3)

Proof. Let \( u_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_i, \varepsilon) \). Then

\[-\Delta u_{i,\varepsilon}(x) + V(\varepsilon x + x_i, \varepsilon)u_{i,\varepsilon}(x) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{u_{i,\varepsilon}^2(\xi)}{|x - \xi|} u_{i,\varepsilon}(x), \ x \in \mathbb{R}^3. \] (2.4)

Suppose that \( \varphi(x) \) is an arbitrarily fixed function in \( H^1(\mathbb{R}^3) \). By (2.4) we get

\[ \int_{\mathbb{R}^3} \nabla u_{i,\varepsilon}(x) \nabla \varphi(x) + V(\varepsilon x + x_i, \varepsilon)u_{i,\varepsilon}(x) \varphi(x)dx = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_{i,\varepsilon}^2(\xi)}{|x - \xi|} u_{i,\varepsilon}(x) \varphi(x)d\xi dx. \] (2.5)

By (1.6) and passing to a subsequence if necessary \( u_{i,\varepsilon}(x) \to U_i(x) \) weakly in \( H^1(\mathbb{R}^3) \) as \( \varepsilon \to 0 \). Then taking \( \varepsilon \to 0 \) in (2.5) we get

\[ \int_{\mathbb{R}^3} \nabla U_i(x) \nabla \varphi(x) + V(a_i)U_i(x) \varphi(x)dx = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U_i^2(\xi)}{|x - \xi|} U_i(x) \varphi(x)d\xi dx. \]

Therefore \( U_i(x) \) is a nonnegative weak solution of

\[
\begin{cases}
-\Delta u + V(a_i)u = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{u^2(\xi)}{|x - \xi|} d\xi \right) u, & \text{in } \mathbb{R}^3, \\
u(x) > 0, & \text{in } \mathbb{R}^3, \ u(0) = \max_{x \in \mathbb{R}^3} u(x). 
\end{cases}
\]

By regularity theory and maximum principle, we know \( U_i(x) > 0 \) in \( \mathbb{R}^3 \). Also from the uniqueness result in [22], we can show that \( U_i(x) = U_{a_i}(x) \) and \( U_i(x) \) decays exponentially at infinity. Then following the decomposition lemma in Lemma A.3 we can write \( u_\varepsilon(x) \) uniquely as (2.1) with \( x_j, \varepsilon \) and \( w_\varepsilon(x) \) satisfying (2.2) and (2.3) for \( j = 1, \ldots, k \), as \( \varepsilon \to 0 \). \( \square \)

Proposition 2.2. Suppose that \( u_\varepsilon(x) \) is a positive solution of (1.3) concentrating at different points \( a_1, \ldots, a_k \) with \( k \geq 1 \). Then for any fixed \( R \gg 1 \), there exist \( \theta > 0 \) and \( C > 0 \), such that

\[ u_\varepsilon(x) \leq Ce^{-\theta |x-x_i|/\varepsilon}, \text{ for } l = 1, \ldots, k \text{ and } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_j, \varepsilon). \] (2.6)

Proof. If \( u_\varepsilon(x) \) is a positive solution of (1.3), then we have

\[-\varepsilon^2 \Delta u_\varepsilon(x) + (V(x) - \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{u_\varepsilon^2(\xi)}{|x - \xi|} d\xi \right))u_\varepsilon(x) = 0, \ x \in \mathbb{R}^3. \] (2.7)

By (B.8) in the Appendix, we know that, for large fixed \( R \) and \( \varepsilon \) small enough,

\[ V(x) - \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{u_\varepsilon^2(\xi)}{|x - \xi|} d\xi \right) \geq m/2, \ \text{in } \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_j, \varepsilon), \] (2.8)

where \( m = \inf_{x \in \mathbb{R}^3} V(x) \). Then (2.7) and (2.8) imply

\[-\varepsilon^2 \Delta u_\varepsilon + \frac{m}{2} u_\varepsilon \leq 0, \ \text{in } \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R\varepsilon}(x_j, \varepsilon). \]

Define the operator \( L_\varepsilon \) as follows:

\[ L_\varepsilon v := -\varepsilon^2 \Delta v + \frac{m}{2} v, \text{ for all } v \in H^1(\mathbb{R}^3). \]
Then for $v_l(x) = e^{-\theta|x-x_l|/\varepsilon}$, where $0 < \theta < \sqrt{m/2}$ and $l \in \{1, 2, \ldots, k\}$, we have
\[
L_\varepsilon v_l(x) = -\varepsilon^2 \left( \frac{\theta^2}{\varepsilon^2} - \frac{2\theta}{|x-x_l|/\varepsilon} \right) e^{-\theta|x-x_l|/\varepsilon} + \frac{m}{2} e^{-\theta|x-x_l|/\varepsilon} = \left[ \frac{2\varepsilon\theta}{|x-x_l|} + \frac{m}{2} - \theta^2 \right] e^{-\theta|x-x_l|/\varepsilon} \geq 0.
\]

Next, we extend $u_\varepsilon(x)$ to $\mathbb{R}^3$ by 0 (still denoted as $u_\varepsilon(x)$) and let $\bar{v}_l(x) = cv_l(x) - u_\varepsilon(x)$, where $c > 0$, then
\[
L_\varepsilon \bar{v}_l = cL_\varepsilon v_l - L_\varepsilon u_\varepsilon \geq 0, \text{ in } \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R_\varepsilon}(x_j, \varepsilon).
\]

Also since $u_\varepsilon(x) \in C(\mathbb{R}^3)$, then there exists $C_0 > 0$ such that $|u_\varepsilon(x)| \leq C_0$, on $\partial \left( \bigcup_{j=1}^k B_{R_\varepsilon}(x_j, \varepsilon) \right)$. So taking $c = C_0 e^{R_\varepsilon}$, we have
\[
\bar{v}_l(x) = cv_l(x) - u_\varepsilon(x) \geq ce^{-R_\varepsilon} - C_0 \geq 0, \text{ on } \partial \left( \bigcup_{j=1}^k B_{R_\varepsilon}(x_j, \varepsilon) \right).
\]

Thus for the above fixed large $R$, we obtain
\[
\begin{cases}
L_\varepsilon \bar{v}_l(x) \geq 0, & \text{in } \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R_\varepsilon}(x_j, \varepsilon), \\
\bar{v}_l(x) \geq 0, & \text{on } \partial \left( \bigcup_{j=1}^k B_{R_\varepsilon}(x_j, \varepsilon) \right), \\
\bar{v}_l(x) \to 0, & \text{as } |x| \to +\infty.
\end{cases}
\]

Then by the maximum principle, we have $\bar{v}_l(x) \geq 0$, for $x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R_\varepsilon}(x_j, \varepsilon)$. This means \((2.4)\) \(\square\).

**Corollary 2.3.** Suppose that $u_\varepsilon(x)$ is a solution of \((1.3)\) as in Proposition 2.2.  
(1). Then for any fixed $R > 1$, there exists $\theta_1 > 0$ such that
\[
u_\varepsilon(x) = O(e^{-\theta_1 R}), \text{ for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{R_\varepsilon}(x_j, \varepsilon).
\]

(2). Then for any fixed $d > 0$, there exists $\theta_2 > 0$ such that
\[
u_\varepsilon(x) = O(e^{-\theta_2 |x|/\varepsilon}), \text{ for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^k B_{d}(x_j, \varepsilon).
\]

**Proof.** These are the direct results by Proposition 2.2 \(\square\)

By the regularity theory of elliptic equations in \([7]\), $u_\varepsilon(x)$ above is in fact a classical solution.

**Proposition 2.4.** Let $u(x)$ be the solution of \((1.3)\), then we have following local Pohozaev identity:
\[
\int_{\Omega} \frac{\partial V(x)}{\partial x^i} u^2(x) dx = \int_{\partial \Omega} \left( -\varepsilon^2 |\nabla u(x)|^2 + V(x) u^2(x) - \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \frac{u^2(\xi) u^2(x)}{|x-\xi|^3} d\xi \right) \nu_i(x) d\sigma \\
+ \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u^2(\xi) u^2(x) \frac{x^i-\xi^i}{|x-\xi|^3} d\xi dx, \text{ for } i = 1, 2, 3,
\]
where $\Omega$ is a bounded open domain of $\mathbb{R}^3$, $\nu(x) = (\nu_1(x), \nu_2(x), \nu_3(x))$ is the outward unit normal of $\partial \Omega$ and $x^i, \xi^i$ are the $i$-th components of $x, \xi$. 

Proof. \ref{eq:2.11} is obtained by multiplying $\frac{\partial u(x)}{\partial x^i}$ on both sides of \ref{eq:1.3} and integrating on $\Omega$. We omit the details. \hfill $\square$

3. PROOF OF THEOREM \ref{thm:1.1}

**Proposition 3.1.** Let $u_\varepsilon(x)$ be the solution of \ref{eq:1.3} concentrating at a nondegenerate critical point $a_1 \in \mathbb{R}^3$ of $V(x)$. Then it holds
\begin{equation}
|x_{1,\varepsilon} - a_1| = o(\varepsilon). \tag{3.1}
\end{equation}

Proof. First, for the small fixed constant $\bar{d} > 0$, taking $u(x) = u_\varepsilon(x)$ and $\Omega = B_d(x_{1,\varepsilon})$ in the Pohozaev identity \ref{eq:2.11} with any $d \in (\bar{d}, 2\bar{d})$, we have, for $i = 1, 2, 3$,
\begin{equation}
\int_{B_d(x_{1,\varepsilon})} \frac{\partial V(x)}{\partial x^i} u_\varepsilon^2(x) dx = \int_{\partial B_d(x_{1,\varepsilon})} A(x) d\sigma + \frac{1}{8\pi \varepsilon^2} \int_{B_d(x_{1,\varepsilon})} \int_{\mathbb{R}^3} u_\varepsilon^2(x) u_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \tag{3.2}
\end{equation}
where
\begin{equation}
A(x) = (-\varepsilon^2 |\nabla u_\varepsilon(x)|^2 + u_\varepsilon^2(x) V(x) - \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} u_\varepsilon^2(\xi) |x - \xi|^{-3} d\xi) v_1(x). \tag{3.3}
\end{equation}

Next for any $d \in (\bar{d}, 2\bar{d})$, \ref{eq:1.6} and (A.16) imply
\begin{equation}
\text{LHS of \ref{eq:3.2}} = \int_{B_d(x_{1,\varepsilon})} \frac{\partial V(x)}{\partial x^i} U_{a_1}^2 \left( \frac{x - x_{1,\varepsilon}}{\varepsilon} \right) dx + o(\varepsilon^4 + \varepsilon^3 |x_{1,\varepsilon} - a_1|) = \varepsilon^3 \left( \int_{\mathbb{R}^3} U_{a_1}^2 dx \right) + \sum_{l=1}^3 \frac{\partial^2 V(a_1)}{\partial x^i \partial x^l}(x_{1,\varepsilon}^l - a_1^l) + o(\varepsilon^4 + \varepsilon^3 |x_{1,\varepsilon} - a_1|), \tag{3.4}
\end{equation}
where $x_{1,\varepsilon}^l, a_1^l$ are the $l$-th components of $x_{1,\varepsilon}, a_1$.

On the other hand, using \ref{eq:2.10}, \ref{eq:A.2} and (C.13), there exists $d_\varepsilon \in (\bar{d}, 2\bar{d})$ such that
\begin{equation}
\int_{\partial B_{d_\varepsilon}(x_{1,\varepsilon})} A(x) d\sigma = O(e^{-\eta/\varepsilon} + \|w_\varepsilon\|_2) = O(\varepsilon^7). \tag{3.5}
\end{equation}

Also for any $d \in (\bar{d}, 2\bar{d})$, by symmetry and \ref{eq:2.10}, we deduce
\begin{equation}
\frac{1}{8\pi \varepsilon^2} \int_{B_d(x_{1,\varepsilon})} \int_{\mathbb{R}^3} u_\varepsilon^2(x) u_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx = \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_\varepsilon^2(x) u_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(e^{-\eta/\varepsilon}) = O(e^{-\eta/\varepsilon}). \tag{3.6}
\end{equation}

Let $d = d_\varepsilon$ in \ref{eq:3.2}, then \ref{eq:3.4}, \ref{eq:3.5} and \ref{eq:3.6} imply
\begin{equation}
\sum_{l=1}^3 \frac{\partial^2 V(a_1)}{\partial x^i \partial x^l}(x_{1,\varepsilon}^l - a_1^l) = o(|x_{1,\varepsilon} - a_1|) + o(\varepsilon). \tag{3.7}
\end{equation}

This means that \ref{eq:3.1} holds. \hfill $\square$

**Proposition 3.2.** Let $\eta_{1,\varepsilon}(x) = \eta_\varepsilon(\varepsilon x + x_{1,\varepsilon}^{(1)})$, then taking a subsequence necessarily, it holds
\begin{equation}
\eta_{1,\varepsilon}(x) \to b_{1,i} \frac{\partial U_{a_1}}{\partial x^i} \tag{3.7}
\end{equation}
uniformly in $C^1(B_R(0))$ for any $R > 0$, where $\eta_\varepsilon(x)$ is the function in \ref{eq:1.12} and $b_{1,i}, i = 1, 2, 3$ are some constants.
Proof. Since \( \| \eta_{1, \varepsilon} \|_{L^\infty(\mathbb{R}^3)} \leq 1 \), by the regularity theory, we know
\[
\eta_{1, \varepsilon}(x) \in C^{1, \alpha}_{\text{loc}}(\mathbb{R}^3) \quad \text{and} \quad \| \eta_{1, \varepsilon} \|_{C^{1, \alpha}_{\text{loc}}(\mathbb{R}^3)} \leq C, \quad \text{for some} \quad \alpha \in (0, 1).
\]
So we may assume that
\[
\eta_{1, \varepsilon}(x) \to \eta_1(x), \quad \text{in} \quad C_{\text{loc}}(\mathbb{R}^3).
\]
By direct calculations, we have
\[
-\Delta \eta_{1, \varepsilon}(x) = -V(\varepsilon x + x_{1, \varepsilon}^{(1)}) \eta_{1, \varepsilon}(x) + E_1(\varepsilon x + x_{1, \varepsilon}^{(1)}) \eta_{1, \varepsilon}(x) + E_2(\varepsilon x + x_{1, \varepsilon}^{(1)}),
\]
where \( E_1(\varepsilon x + x_{1, \varepsilon}^{(1)}) \) and \( E_2(\varepsilon x + x_{1, \varepsilon}^{(1)}) \) are in \([3.3]\). Then from \([3.14]\) and \([3.15]\), we get
\[
E_1(\varepsilon x + x_{1, \varepsilon}^{(1)}) = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{U_{a_1}(\xi)}{|x - \xi|} d\xi \right) + o(1), \quad x \in B_{d/\varepsilon}(0),
\]
and
\[
E_2(\varepsilon x + x_{1, \varepsilon}^{(1)}) = \frac{U_{a_1}(x)}{4\pi} \left( \int_{\mathbb{R}^3} \frac{U_{a_1}(\xi) \eta_{1, \varepsilon}(\xi)}{|x - \xi|} d\xi \right) + o(1), \quad x \in B_{d/\varepsilon}(0).
\]
Next, for any given \( \Phi(x) \in C_{0}^\infty(\mathbb{R}^3) \), we have
\[
\int_{\mathbb{R}^3} \left( -\Delta \eta_{1, \varepsilon}(x) + V(\varepsilon x + x_{1, \varepsilon}^{(1)}) \eta_{1, \varepsilon}(x) \right) \Phi(x) dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U_{a_1}^2(\xi)}{|x - \xi|} \eta_{1, \varepsilon}(x) \Phi(x) d\xi dx
\]
\[
- \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{U_{a_1}(\xi) \eta_{1, \varepsilon}(\xi) U_{a_1}(x) \Phi(x) d\xi dx = o(1) \| \Phi \|_{H^1(\mathbb{R}^3)}. \tag{3.9}
\]
Letting \( \varepsilon \to 0 \) in \([3.9]\) and using the elliptic regularity theory, we find that \( \eta_1(x) \) satisfies
\[
-\Delta \eta_1(x) + V(a_1) \eta_1(x) = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{U_{a_1}^2(\xi)}{|x - \xi|} d\xi \right) \eta_1(x) + \frac{1}{4\pi} \left( \int_{\mathbb{R}^3} \frac{U_{a_1}(\xi) \eta(x)}{|x - \xi|} d\xi \right) U_{a_1}(x), \quad \text{in} \quad \mathbb{R}^3,
\]
which gives \( \eta_1(x) = \sum_{i=1}^{3} b_{1,i} \frac{\partial U_{a_1}(x)}{\partial x^i} \). This means \([3.7]\). \(\square\)

**Proposition 3.3.** Let \( b_{1,i} \) be as in Proposition 3.2, then we have
\[
b_{1,i} = 0, \quad \text{for all} \quad i = 1, 2, 3.
\]

Proof. Since \( u_\varepsilon^{(1)}(x), \ u_\varepsilon^{(2)}(x) \) are the positive solutions of \([1.3]\), for the small fixed constant \( \tilde{d} > 0 \), using Pohozaev identity \([2.11]\) with any \( \delta \in (\tilde{d}, 2\tilde{d}) \), we deduce
\[
\int_{B_{\delta}(x_{1, \varepsilon}^{(1)})} \frac{\partial V(\varepsilon x)}{\partial x^i} (u_\varepsilon^{(1)}(x) + u_\varepsilon^{(2)}(x)) \cdot \eta_\varepsilon(x) dx
\]
\[
= \int_{\partial B_{\delta}(x_{1, \varepsilon}^{(1)})} B(x) dx \sigma + \frac{1}{8\pi \varepsilon^2} \int_{B_{\delta}(x_{1, \varepsilon}^{(1)})} \int_{\mathbb{R}^3} (u_\varepsilon^{(1)}(x))^2 (u_\varepsilon^{(1)}(\xi) + u_\varepsilon^{(2)}(\xi)) \eta_\varepsilon(\xi) \left| \frac{x^i - \xi^i}{|x - \xi|^3} \right| d\xi dx \tag{3.10}
\]
\[
+ \frac{1}{8\pi \varepsilon^2} \int_{B_{\delta}(x_{1, \varepsilon}^{(1)})} \int_{\mathbb{R}^3} (u_\varepsilon^{(2)}(\xi))^2 (u_\varepsilon^{(1)}(x) + u_\varepsilon^{(2)}(x)) \eta_\varepsilon(x) \left| \frac{x^i - \xi^i}{|x - \xi|^3} \right| d\xi dx,
\]
where

\[ B(x) = -\left( \varepsilon^2 \nabla (u^{(1)}_\varepsilon(x) + u^{(2)}_\varepsilon(x)) \cdot \nabla \eta_\varepsilon(x) + V(x)(u^{(1)}_\varepsilon(x) + u^{(2)}_\varepsilon(x))\eta_\varepsilon(x) \right) \nu_1(x) \]

\[ - \frac{(u^{(1)}_\varepsilon(x))^2}{8\pi\varepsilon^2} \int_{\mathbb{R}^3} \left( u^{(1)}_\varepsilon(\xi) + u^{(2)}_\varepsilon(\xi) \right) \eta_\varepsilon(\xi) |x - \xi|^{-1} d\xi - \frac{(u^{(1)}_\varepsilon(x) + u^{(2)}_\varepsilon(x)) \eta_\varepsilon(x) \nu_1(x)}{8\pi\varepsilon^2} \int_{\mathbb{R}^3} \left( u^{(2)}_\varepsilon(\xi) \right)^2 |x - \xi|^{-1} d\xi. \]

Now from (2.2), Proposition 3.1, Proposition 3.2 and (A.16), for any \( \delta \in (\bar{d}, 2\bar{d}) \), we have

\[ \text{LHS of } (3.10) = \int_{B_\delta(x^{(1)}_1, \varepsilon)} \frac{\partial V(x)}{\partial x^i} \left( U_{a_j} \left( \frac{x - x^{(1)}_1}{\varepsilon} \right) + U_{a_1} \left( \frac{x - x^{(2)}_1}{\varepsilon} \right) \right) \eta_\varepsilon(x) dx + O(\varepsilon^5) \]

\[ = \int_{B_\delta(x^{(1)}_1, \varepsilon)} \left( \sum_{l=1}^{3} \left( x^l - a^l_1 \right) \frac{\partial^2 V(a_1)}{\partial x^l \partial x^{l'}} \right) \left( U_{a_j} \left( \frac{x - x^{(1)}_1}{\varepsilon} \right) + U_{a_1} \left( \frac{x - x^{(2)}_1}{\varepsilon} \right) \right) \eta_\varepsilon(x) dx \]

\[ + \int_{B_\delta(x^{(1)}_1, \varepsilon)} o(|x - a_1|) \left( U_{a_j} \left( \frac{x - x^{(1)}_1}{\varepsilon} \right) + U_{a_1} \left( \frac{x - x^{(2)}_1}{\varepsilon} \right) \right) \eta_\varepsilon(x) dx + O(\varepsilon^5) \]

\[ = 2\varepsilon^4 \left( \sum_{l=1}^{3} \frac{\partial^2 V(a_1)}{\partial x^l \partial x^{l'}} \right) \int_{\mathbb{R}^3} x^l U_{a_1}(x) (b_{1,i} \frac{\partial U_{a_1}(x)}{\partial x^i}) dx + o(\varepsilon^4) \]

\[ = \varepsilon^4 \int_{\mathbb{R}^3} U_{a_1}(x) dx \left( \sum_{l=1}^{3} \frac{\partial^2 V(a_1)}{\partial x^l \partial x^{l'}} b_{1,i} \right) + o(\varepsilon^4), \]

where \( x^l, a^l_1 \) are the \( l \)-th components of \( x, a_1 \).

On the other hand, similar to (3.5), there exists \( \delta_\varepsilon \in (\bar{d}, 2\bar{d}) \), we obtain

\[ \int_{\partial B_{\delta_\varepsilon}(x^{(1)}_1, \varepsilon)} B(x) d\sigma = O(\varepsilon^{-\eta/\varepsilon} + \| w_\varepsilon \|_\varepsilon \| \eta_\varepsilon \|_\varepsilon) = O(\varepsilon^5). \]

Also, by (2.6), for any \( \delta \in (\bar{d}, 2\bar{d}) \), we have

\[ \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x^{(1)}_1, \varepsilon)} \int_{\mathbb{R}^3} \left( u^{(1)}_\varepsilon(x)^2 \left( u^{(1)}_\varepsilon(\xi) + u^{(2)}_\varepsilon(\xi) \right) \eta_\varepsilon(\xi) \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx \]

\[ = \frac{1}{8\pi\varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( u^{(2)}_\varepsilon(\xi)^2 \left( u^{(1)}_\varepsilon(x) + u^{(2)}_\varepsilon(x) \right) \eta_\varepsilon(x) \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(\varepsilon^{-\eta/\varepsilon}), \]

and

\[ \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x^{(1)}_1, \varepsilon)} \int_{\mathbb{R}^3} \left( u^{(2)}_\varepsilon(\xi)^2 \left( u^{(1)}_\varepsilon(x) + u^{(2)}_\varepsilon(x) \right) \eta_\varepsilon(x) \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx \]

\[ = \frac{1}{8\pi\varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( u^{(2)}_\varepsilon(\xi)^2 \left( u^{(1)}_\varepsilon(x) + u^{(2)}_\varepsilon(x) \right) \eta_\varepsilon(x) \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(\varepsilon^{-\eta/\varepsilon}). \]
Let $\delta = \delta_\varepsilon$ in (3.10), by symmetry, (3.13), (3.14) and (3.15), we get

\[ \text{RHS of (3.10)} \]
\[ = \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( (u_\varepsilon^{(1)}(x))^2 - (u_\varepsilon^{(2)}(x))^2 \right) (u_\varepsilon^{(1)}(\xi) + u_\varepsilon^{(2)}(\xi)) \eta_\varepsilon(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(e^{-\eta/\varepsilon}) \]
\[ = \frac{J_\varepsilon}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u_\varepsilon^{(1)}(x) + u_\varepsilon^{(2)}(x)) \eta_\varepsilon(x) (u_\varepsilon^{(1)}(\xi) + u_\varepsilon^{(2)}(\xi)) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(e^{-\eta/\varepsilon}) \]
\[ = O(e^{-\eta/\varepsilon}), \] (3.16)

where $J_\varepsilon := \| u_\varepsilon^{(1)}(\cdot) - u_\varepsilon^{(2)}(\cdot) \|_{L^\infty(\mathbb{R}^3)}$. Then (3.12) and (3.16) imply

\[ \sum_{l=1}^3 \frac{\partial^2 V(a_l)}{\partial x^i \partial x^i} b_{1,l} = o(1). \]

This means $b_{1,i} = 0$. Similarly, we can obtain $b_{1,i} = 0$, for all $i = 1, 2, 3$. \qed

**Proposition 3.4.** For any fixed $R > 0$, it holds

\[ \eta_\varepsilon(x) = o(1), \ x \in B_R(x_{1,\varepsilon}^{(1)}). \]

**Proof.** Proposition 3.2 and Proposition 3.3 show that for any fixed $R > 0$, $\eta_{1,\varepsilon}(x) = o(1)$ in $B_R(0)$. Also, we know $\eta_{1,\varepsilon}(x) = \eta_\varepsilon(x + x_{1,\varepsilon}^{(1)})$. Then $\eta_\varepsilon(x) = o(1), x \in B_R(x_{1,\varepsilon}^{(1)}).$ \qed

**Proposition 3.5.** For large $R > 0$ and fixed $\gamma \in (0, 1)$, there exists $\varepsilon_0$ such that

\[ |\eta_\varepsilon(x)| \leq \gamma, \ \text{for } x \in \mathbb{R}^3 \setminus B_{R_\varepsilon}(x_{1,\varepsilon}^{(1)}) \text{ and } \varepsilon \in (0, \varepsilon_0). \]

**Proof.** First, $\eta_\varepsilon(x)$ satisfies the following equation:

\[ -\varepsilon^2 \Delta \eta_\varepsilon(x) + V(x) \eta_\varepsilon(x) = E_1(x) \eta_\varepsilon(x) + E_2(x), \]

where $E_1(x)$ and $E_2(x)$ are the functions in (B.3). Let $\tilde{\eta}_\varepsilon(x) = \eta_\varepsilon(x) + \gamma$, then

\[ -\varepsilon^2 \Delta \tilde{\eta}_\varepsilon(x) + (V(x) - E_1(x)) \tilde{\eta}_\varepsilon(x) = E_2(x) + \gamma (V(x) - E_1(x)), \]

and

\[ \tilde{\eta}_\varepsilon(x) = \gamma + o_\varepsilon(1), \ \text{for } x \in \partial B_{R_\varepsilon}(x_{1,\varepsilon}^{(1)}) \text{ or } |x| \to \infty. \]

Also for large $R > 0$, (B.3) and (B.10) imply

\[ V(x) - E_1(x) \geq 0, \ \text{and} \ E_2(x) + \gamma (V(x) - E_1(x)) \geq 0, \ \text{for } x \in \mathbb{R}^3 \setminus B_{R_\varepsilon}(x_{1,\varepsilon}^{(1)}). \]

Then by the maximum principle, we have

\[ \min_{\mathbb{R}^3 \setminus B_{R_\varepsilon}(x_{1,\varepsilon}^{(1)})} \tilde{\eta}_\varepsilon(x) \geq - \max_{\partial (\mathbb{R}^3 \setminus B_{R_\varepsilon}(x_{1,\varepsilon}^{(1)}))} (\tilde{\eta}_\varepsilon(x))^- = 0. \]

This means

\[ \eta_\varepsilon(x) \geq -\gamma, \ \text{for } x \in \mathbb{R}^3 \setminus B_{R_\varepsilon}(x_{1,\varepsilon}^{(1)}). \]

Let $\tilde{\eta}_\varepsilon(x) = \eta_\varepsilon(x) - \gamma$, then

\[ -\varepsilon^2 \Delta \tilde{\eta}_\varepsilon(x) + (V(x) - E_1(x)) \tilde{\eta}_\varepsilon(x) = E_2(x) - \gamma (V(x) - E_1(x)). \]

Also

\[ \tilde{\eta}_\varepsilon(x) = -\gamma + o_\varepsilon(1), \ \text{for } x \in \partial B_{R_\varepsilon}(x_{1,\varepsilon}^{(1)}) \text{ or } |x| \to \infty, \]
and for large $R > 0$, (B.9) and (B.10) imply
\[
V(x) - E_1(x) \geq 0, \text{ and } E_2(x) - \gamma(V(x) - E_1(x)) \leq 0, \text{ for } x \in \mathbb{R}^3 \setminus B_{R\varepsilon}(x_{1,\varepsilon}^{(1)}).
\]
Then by the maximum principle, we have
\[
\max_{\mathbb{R}^3 \setminus B_{R\varepsilon}(x_{1,\varepsilon}^{(1)})} \tilde{\eta}_\varepsilon(x) \leq \max_{\partial(\mathbb{R}^3 \setminus B_{R\varepsilon}(x_{1,\varepsilon}^{(1)}) \setminus \{x_{1,\varepsilon}^{(1)}\})} (\tilde{\eta}_\varepsilon(x))^+ = 0.
\]
This means
\[
\eta_\varepsilon(x) \leq \gamma, \text{ for } x \in \mathbb{R}^3 \setminus B_{R\varepsilon}(x_{j,\varepsilon}^{(1)}).
\]

**Proof of Theorem 1.1**: Suppose that $u_\varepsilon^{(1)}(x)$, $u_\varepsilon^{(2)}(x)$ are two different positive solutions concentrating at the nondegenerate critical point $a_1$ of $V(x)$. From Proposition 3.4 and Proposition 3.5 for small $\varepsilon$ and fixed $\gamma \in (0, 1)$, we have $|\eta_\varepsilon(x)| \leq \gamma$, $x \in \mathbb{R}^3$, which contradicts with $\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$. So, $u_\varepsilon^{(1)}(x) \equiv u_\varepsilon^{(2)}(x)$ for small $\varepsilon$. Also (3.1) and (C.13) imply (1.8). \[\square\]

4. **More precise estimates**

**Proposition 4.1.** Let $u_\varepsilon(x)$ be the solution of (4.3) concentrating at $k$ ($k \geq 2$) different nondegenerate critical points $\{a_1, \ldots, a_k\} \subset \mathbb{R}^3$ of $V(x)$. Then it holds
\[
|x_{j,\varepsilon} - a_j| = O(\varepsilon), \text{ for } j = 1, 2, \ldots, k.
\]
Furthermore, there exist $j_0 \in \{1, \ldots, k\}$, $C_1 > 0$ and $C_2 > 0$ such that
\[
C_1 \varepsilon \leq |x_{j_0,\varepsilon} - a_{j_0}| \leq C_2 \varepsilon.
\]
**Proof.** First, for the small fixed constant $\tilde{d} > 0$, taking $u(x) = u_\varepsilon(x)$ and $\Omega = B_d(x_{j,\varepsilon})$ in the Pohozaev identity (2.11) with any $d \in (\tilde{d}, 2\tilde{d})$, we have
\[
\int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x^i} u_\varepsilon^2(x) dx = \int_{\partial B_d(x_{j,\varepsilon})} A(x) d\sigma + A_1, \text{ for } i = 1, 2, 3,
\]
where $A(x)$ is the function in (3.3) and
\[
A_1 = \frac{1}{8\pi \varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} u_\varepsilon^2(x) u_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx.
\]
By (A.10), (3.9) and Proposition 2.1, for any $d \in (\tilde{d}, 2\tilde{d})$, we obtain
\[
\text{LHS of (4.3) } = \int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x^i} U_{a_j}^2 \left(\frac{x - x_{j,\varepsilon}}{\varepsilon}\right) dx + O(\varepsilon^{-\eta/\varepsilon} + \|U_{a_j}(\cdot - x_{j,\varepsilon}/\varepsilon)\|_{\varepsilon} \|w_\varepsilon\|_{\varepsilon} + \|w_\varepsilon\|_{\varepsilon}^2)
\]
\[
= \int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x^i} U_{a_j}^2 \left(\frac{x - x_{j,\varepsilon}}{\varepsilon}\right) dx + O(\varepsilon^5 + \varepsilon^3 \max_{j=1, \ldots, k} |x_{j,\varepsilon} - a_j|^2).
\]
Next similar to (3.12), for any $d \in (\bar{d}, 2\bar{d})$, from (A.16), we have
\[
\int_{B_d(x_{j,\varepsilon})} \frac{\partial V(x)}{\partial x^i} U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) dx
\]
\[
= \sum_{l=1}^{3} \frac{\partial^2 V(a_j)}{\partial x^i \partial x^l} \int_{B_d(x_{j,\varepsilon})} (x^l - a_j^l) U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) dx + o\left( \int_{B_d(x_{j,\varepsilon})} |x - a_j| U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) dx \right) \tag{4.6}
\]
\[
= \varepsilon^3 \left( \int_{\mathbb{R}^3} U_{a_j}^2(x) dx \right) \sum_{l=1}^{3} \frac{\partial^2 V(a_j)}{\partial x^i \partial x^l} (x_{j,\varepsilon}^l - a_j^l) + o\left( \varepsilon^4 + \varepsilon^3 \max_{j=1,\ldots,k} |x_{j,\varepsilon} - a_j| \right),
\]
where $x_{j,\varepsilon}^l, a_j^l$ are the $l$-th components of $x_{j,\varepsilon}, a_j$. So (4.5) and (4.6) imply
\[
\text{LHS of (4.3)} = \varepsilon^3 \left( \int_{\mathbb{R}^3} U_{a_j}^2(x) dx \right) \sum_{l=1}^{3} \frac{\partial^2 V(a_j)}{\partial x^i \partial x^l} (x_{j,\varepsilon}^l - a_j^l) + o\left( \varepsilon^4 + \varepsilon^3 \max_{j=1,\ldots,k} |x_{j,\varepsilon} - a_j| \right). \tag{4.7}
\]
On the other hand, similar to (5.5), there exists $\delta_\varepsilon \in (\bar{d}, 2\bar{d})$, we get
\[
\int_{\partial B_{d_\varepsilon}(x_{j,\varepsilon})} A(x) d\sigma = O(\varepsilon^7). \tag{4.8}
\]
Let $d = d_\varepsilon$ in (4.3), then combining (4.8) and (4.10) below in Proposition 4.2 we deduce that
\[
\text{RHS of (4.3)} = O(\varepsilon^4 + \varepsilon^3 \max_{j=1,\ldots,k} |x_{j,\varepsilon} - a_j|^4). \tag{4.9}
\]
So (4.7) and (4.9) imply
\[
\sum_{l=1}^{3} \frac{\partial^2 V(a_j)}{\partial x^i \partial x^l} (x_{j,\varepsilon}^l - a_j^l) = o\left( \max_{j=1,\ldots,k} |x_{j,\varepsilon} - a_j| \right) + O(\varepsilon).
\]
This means that (4.11) holds.
Moreover, (4.1), (4.3), (4.7), (4.8) and (4.11) below in Proposition 4.2 imply (4.2). \hfill \Box

**Proposition 4.2.** For the small fixed constant $\bar{d} > 0$ and any $d \in (\bar{d}, 2\bar{d})$, it holds
\[
A_1 = O(\varepsilon^4). \tag{4.10}
\]
Furthermore, there exist $i_0 \in \{1, 2, 3\}$, $j_0 \in \{1, \ldots, k\}$, $C_3 > 0$ and $C_4 > 0$ such that
\[
C_3\varepsilon^4 \leq |A_1| \leq C_4\varepsilon^4. \tag{4.11}
\]
**Proof.** First, $A_1$ can be written as follows:
\[
A_1 = A_{1,1} + A_{1,2} + A_{1,3} + A_{1,4}, \tag{4.12}
\]
where
\[
A_{1,1} = \frac{1}{8\pi\varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) u_{2\varepsilon}^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
\[
A_{1,2} = \frac{1}{4\pi\varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) w_{\varepsilon}^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
\[
A_{1,3} = \frac{1}{8\pi\varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} w_{\varepsilon}^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
and
\[
A_{1,4} = \frac{1}{8\pi\varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} \frac{\partial V(x)}{\partial x^i} U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) dx.
\]
Then (4.10), (4.15), (4.17), (4.18) imply (4.11) for some \( i \).

Also, from (2.10) and (A.10), we obtain

Then (4.15) and (4.16) imply (4.10).

Then (4.12), (4.13), (4.14), (D.1), (D.9) and (D.17) imply

Also, from (2.10) and (A.10), we obtain

Then by (A.5) and (C.13), we get

and

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Also, since \( \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_i}^2(x) U_{a_i}^2(\xi + \frac{x_i - x_i}{\epsilon}) |x - \xi|^{-3} d\xi dx = O(\epsilon^6) \). (4.13)

Also, from (2.10) and (A.10), we obtain

Then (4.15) and (4.16) imply (4.10).

On the other hand, since \( a_i \) are different points, we can take \( i = i_0 \) and \( j = j_0 \) such that

Also, since \( B_{d/\epsilon}(-\frac{x_i - x_i}{\epsilon}) \subset \mathbb{R}^3 \backslash B_{2d/\epsilon}(0) \) for small fixed \( d > 0 \), we deduce

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Then (4.10), (4.15), (4.17), (4.18) imply (4.11) for some \( i_0, j_0, C_3 > 0 \) and \( C_4 > 0 \).
Proposition 4.3. Let \( u^{(1)}_\varepsilon(x) \) and \( u^{(2)}_\varepsilon(x) \) be the solutions of (1.3) concentrating at \( k (k \geq 2) \) different nondegenerate critical points \( \{a_1, \cdots, a_k\} \subset \mathbb{R}^3 \) of \( V(x) \), then
\[
|x^{(1)}_j - x^{(2)}_j| = o(\varepsilon), \text{ for } j = 1, 2, \cdots, k. \tag{4.19}
\]

Proof. First, for the small fixed constant \( \bar{d} > 0 \), taking \( u(x) = u^{(m)}_\varepsilon(x) = \sum_{l=1}^{k} U_{a_l}(x - x^{(m)}_l/\varepsilon) + w^{(m)}_\varepsilon(x) \) and \( \Omega = B_d(x^{(m)}_j, \varepsilon) \) in the Pohozaev identity (2.11) with any \( d \in (\bar{d}, 2\bar{d}) \) and \( m = 1, 2 \), then
\[
\int_{B_d(x^{(m)}_j, \varepsilon)} \frac{\partial V(x)}{\partial x^i} (u^{(m)}_\varepsilon(x))^2 \, dx = \int_{\partial B_d(x^{(m)}_j, \varepsilon)} A^{(m)}(x) \, d\sigma + A^{(m)}_1, \text{ for } i = 1, 2, 3, \tag{4.20}
\]
where
\[
A^{(m)}(x) = -\varepsilon^2 |\nabla u^{(m)}_\varepsilon(x)|^2 + (u^{(m)}_\varepsilon(x))^2 (V(x) - \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \frac{(u^{(m)}_\varepsilon(\xi))^2}{|x - \xi|} d\xi) \nu_i(x),
\]
and
\[
A^{(m)}_1 = \frac{1}{8\pi \varepsilon^2} \int_{B_d(x^{(m)}_j, \varepsilon)} \int_{\mathbb{R}^3} (u^{(m)}_\varepsilon(x))^2 (u^{(m)}_\varepsilon(\xi))^2 \frac{x^i - \xi^i}{|x - \xi|^3} d\xi \, dx.
\]

From (4.20), for any \( d \in (\bar{d}, 2\bar{d}) \), we have
\[
\int_{B_d(x^{(1)}_j, \varepsilon)} \frac{\partial V(x)}{\partial x^i} (u^{(1)}_\varepsilon(x))^2 \, dx - \int_{B_d(x^{(2)}_j, \varepsilon)} \frac{\partial V(x)}{\partial x^i} (u^{(1)}_\varepsilon(x))^2 \, dx = \int_{\partial B_d(x^{(1)}_j, \varepsilon)} A^{(1)}(x) \, d\sigma - \int_{\partial B_d(x^{(2)}_j, \varepsilon)} A^{(2)}(x) \, d\sigma + (A^{(1)}_1 - A^{(2)}_1). \tag{4.21}
\]

Then (4.1), (4.3) and (4.7) imply
\[
\text{LHS of (4.21)} = \varepsilon^3 \left( \int_{\mathbb{R}^3} U_{a_j}^2(x) dx \right) \sum_{l=1}^{3} \frac{\partial^2 V(a_j)}{\partial x^i \partial x^l} (x^{(1)}_l, x^{(2)}_l) + o(\varepsilon^4), \tag{4.22}
\]
where \( x^{(1)}_l, x^{(2)}_l \) are the \( l \)-th components of \( x^{(1)}_j, x^{(2)}_j \).

On the other hand, similar to (1.8), there exists \( d_\varepsilon \in (\bar{d}, 2\bar{d}) \) such that
\[
\int_{\partial B_{d_\varepsilon}(x^{(m)}_j, \varepsilon)} A^{(m)}(x) \, d\sigma = O(e^{-\eta/\varepsilon}), \text{ for } m = 1, 2. \tag{4.23}
\]

Also, from (D.17), we know
\[
A^{(m)}_1 = \sum_{l=1}^{k} \frac{(a^i_j - a^i_l)\varepsilon}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2(x)U_{a_l}^2(\xi) \frac{x^{(m)}_j - x^{(m)}_l}{\varepsilon} |x - \xi|^{-3} d\xi \, dx + o(\varepsilon^4), \tag{4.24}
\]
where \( a^i_j, a^i_l \) are the \( i \)-th components of \( a_j, a_l \).

Then (4.23), (4.24) and the exponential decay of \( U_{a_j}(x) \) imply
\[
\text{RHS of (4.21)} = \sum_{l=1}^{k} \frac{(a^i_j - a^i_l)\varepsilon}{8\pi} \int_{B_{d_\varepsilon}(0)} \int_{B_{d_\varepsilon}(0)} U_{a_j}^2(x)U_{a_l}^2(\xi) D(x, \xi) \, d\xi \, dx + o(\varepsilon^4), \tag{4.25}
\]
where

\[ D(x, \xi) = |x - \xi + \frac{x_{j,\xi}^{(1)} - x_{i,\xi}^{(1)}}{\varepsilon}|^{-3} - |x - \xi + \frac{x_{j,\xi}^{(2)} - x_{i,\xi}^{(2)}}{\varepsilon}|^{-3}. \]

Then using (2.2), we get

\[ D(x, \xi) = O(|x_{j,\xi}^{(1)} - x_{j,\xi}^{(2)} + x_{i,\xi}^{(1)} - x_{i,\xi}^{(2)}|\varepsilon^3) = o(\varepsilon^3), \text{ for } x \in B_{d/\varepsilon}(0), \xi \in B_{d/\varepsilon}(0). \]  

(4.26)

Then (4.25) and (4.26) imply

\[ \text{RHS of (4.21)} = o(\varepsilon^4). \]  

(4.27)

So from (4.22) and (4.27), we obtain (4.19).

\[ \square \]

**Proposition 4.4.** Let \( u^{(1)}_\varepsilon(x) \) and \( u^{(2)}_\varepsilon(x) \) be the solutions of (1.3) concentrating at \( k \) \((k \geq 2)\) different nondegenerate critical points \( \{a_1, \cdots, a_k\} \subset \mathbb{R}^3 \) of \( V(x) \), then

\[ \| w^{(1)}_\varepsilon - w^{(2)}_\varepsilon \|_\varepsilon = o(\varepsilon^{7/2}). \]  

(4.28)

**Proof.** First, from (C.3), we obtain

\[ M_\varepsilon(x, u^{(1)}_\varepsilon(x) - u^{(2)}_\varepsilon(x)) = M_\varepsilon(x, u^{(1)}_\varepsilon(x)) - M_\varepsilon(x, u^{(2)}_\varepsilon(x)) = \bar{N}_\varepsilon(x) + \bar{I}_\varepsilon(x), \]  

(4.29)

where

\[ \bar{N}_\varepsilon(x) = \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{(u^{(1)}_\varepsilon(\xi))^2}{|x - \xi|} (R^{(1)}_\varepsilon(x) + u^{(1)}_\varepsilon(x)) + \frac{w^{(1)}_\varepsilon(x)}{4\pi\varepsilon^2} \int_{\mathbb{R}^3} \frac{R^{(1)}_\varepsilon(\xi)w^{(1)}_\varepsilon(\xi)}{|x - \xi|} d\xi \right) - \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{(u^{(2)}_\varepsilon(\xi))^2}{|x - \xi|} (R^{(2)}_\varepsilon(x) + u^{(2)}_\varepsilon(x)) - \frac{w^{(2)}_\varepsilon(x)}{4\pi\varepsilon^2} \int_{\mathbb{R}^3} \frac{R^{(2)}_\varepsilon(\xi)w^{(2)}_\varepsilon(\xi)}{|x - \xi|} d\xi \right), \]  

(4.30)

and

\[ \bar{I}_\varepsilon(x) = \frac{W^{(1)}_{j,\varepsilon}(x)}{8\pi\varepsilon^2} \int_{\mathbb{R}^3} \frac{W^{(1)}_{j,\varepsilon}(\xi)U_{a_j}(\varepsilon - \xi/\varepsilon)}{|x - \xi|} d\xi + \sum_{j=1}^k (V(a_j) - V(x))U_j\left(\frac{x - x_{j,\xi}^{(1)}}{\varepsilon}\right) \]  

\[ - \frac{W^{(2)}_{j,\varepsilon}(x)}{8\pi\varepsilon^2} \int_{\mathbb{R}^3} \frac{W^{(2)}_{j,\varepsilon}(\xi)U_{a_j}(\varepsilon - \xi/\varepsilon)}{|x - \xi|} d\xi - \sum_{j=1}^k (V(a_j) - V(x))U_j\left(\frac{x - x_{j,\xi}^{(2)}}{\varepsilon}\right), \]  

(4.31)

where \( R^{(m)}_\varepsilon(x) \) and \( W^{(m)}_{j,\varepsilon}(x) \) are the functions in (C.1) and (C.2) for \( m = 1, 2 \).

Next, using (A.6) and (C.13), we obtain

\[ \int_{\mathbb{R}^3} \bar{N}_\varepsilon(x)(u^{(1)}_\varepsilon(x) - u^{(2)}_\varepsilon(x)) dx \]

\[ = O(\varepsilon^{-4} |x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}| \| R^{(1)}_\varepsilon \|_\varepsilon \sum_{m=1}^2 \| w^{(m)}_\varepsilon \|_\varepsilon^3) + O(\varepsilon^{-3} \sum_{m=1}^2 \| w^{(m)}_\varepsilon \|_\varepsilon^3 \| w^{(1)}_\varepsilon - w^{(2)}_\varepsilon \|_\varepsilon) \]

\[ + O(\varepsilon^{-3} \| R^{(1)}_\varepsilon \|_\varepsilon \sum_{m=1}^2 \| w^{(m)}_\varepsilon \|_\varepsilon^2 \| w^{(1)}_\varepsilon - w^{(2)}_\varepsilon \|_\varepsilon) \]

\[ = o(\varepsilon^9) + O(\varepsilon^{11/2}) \| w^{(1)}_\varepsilon - w^{(2)}_\varepsilon \|_\varepsilon. \]  

(4.32)

Also, similar to (C.12), we have

\[ \int_{\mathbb{R}^3} \bar{I}_\varepsilon(x)(u^{(1)}_\varepsilon(x) - u^{(2)}_\varepsilon(x)) dx = o(\varepsilon^{7/2}) \| w^{(1)}_\varepsilon - w^{(2)}_\varepsilon \|_\varepsilon. \]  

(4.33)
On the other hand, by Proposition [C.1] there exists $\rho'$ such that
\[
\int_{\mathbb{R}^3} M_\varepsilon(x, w_\varepsilon^{(1)}(x) - w_\varepsilon^{(2)}(x))(w_\varepsilon^{(1)}(x) - w_\varepsilon^{(2)}(x))dx \geq \rho'\|w_\varepsilon^{(1)} - w_\varepsilon^{(2)}\|^2. \tag{4.34}
\]
Then (4.29), (4.32), (4.33) and (4.34) imply (4.28). □

**Proposition 4.5.** It holds
\[
w_\varepsilon^{(1)}(x) - w_\varepsilon^{(2)}(x) = o(\varepsilon^2), \text{ in } \bigcup_{j=1}^{k} B_{R\varepsilon}(x_{j,\varepsilon}^{(1)}). \tag{4.35}
\]

**Proof.** Let $\bar{w}_{j,\varepsilon}(x) := w_\varepsilon^{(1)}(x) - w_\varepsilon^{(2)}(x) = w_\varepsilon^{(1)}(\varepsilon x + x_{j,\varepsilon}^{(1)}) - w_\varepsilon^{(2)}(\varepsilon x + x_{j,\varepsilon}^{(1)})$, then we have
\[
-\Delta \bar{w}_{j,\varepsilon}(x) = G_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)}) + N_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)}) + I_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)}),
\]
\[
G_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)}) = G(\varepsilon x + x_{j,\varepsilon}^{(1)}, w_{\varepsilon}^{(1)}(\varepsilon x + x_{j,\varepsilon}^{(1)})) - G(\varepsilon x + x_{j,\varepsilon}^{(1)}, w_{\varepsilon}^{(2)}(\varepsilon x + x_{j,\varepsilon}^{(1)})),
\]
where $G(\varepsilon x + x_{j,\varepsilon}^{(1)}, w_{\varepsilon}^{(1)}(\varepsilon x + x_{j,\varepsilon}^{(1)})), N_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)})$ and $I_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)})$ are the functions in (4.30) and (4.31) for $m = 1, 2$.

So by Nash-Moser iteration (Lemma A.2), we have, for any fixed $R > 0$, 
\[
\sup_{B_R(0)} \bar{w}_{j,\varepsilon}(x) \leq C\|\bar{w}_{j,\varepsilon}\|_{L^2(B_{2R}(0))} + C\|\bar{G}_\varepsilon(\varepsilon \cdot + x_{j,\varepsilon}^{(1)})\|_{L^2(\mathbb{R}^3)}
\]
\[
+ C\|\bar{N}_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)})\|_{L^2(\mathbb{R}^3)} + C\|\bar{I}_\varepsilon(\varepsilon \cdot + x_{j,\varepsilon}^{(1)})\|_{L^2(\mathbb{R}^3)}. \tag{4.36}
\]
Then using (4.19) and (4.23), we obtain
\[
\|\bar{w}_{j,\varepsilon}\|_{L^2(B_{2R}(0))} = O(\|\bar{w}_{j,\varepsilon}\|_{L^2(\mathbb{R}^3)}) = O(\varepsilon^{-3/2}\|w_\varepsilon^{(1)} - w_\varepsilon^{(2)}\|_\varepsilon) = o(\varepsilon^2). \tag{4.37}
\]

Also, from (4.28), (A.5), (A.10), (C.6) and (C.13), we deduce
\[
\|\bar{G}_\varepsilon(\varepsilon \cdot + x_{j,\varepsilon}^{(1)})\|_{L^2(\mathbb{R}^3)} = \varepsilon^{-\frac{5}{2}}\|\bar{G}_\varepsilon(\cdot)\|_{L^2(\mathbb{R}^3)}
\]
\[
= O(\varepsilon^{-\frac{5}{2}}\|w_\varepsilon^{(1)} - w_\varepsilon^{(2)}\|_\varepsilon) + O(\varepsilon^{-\frac{7}{2}}\|R_\varepsilon^{(1)}\|_\varepsilon) \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(R_\varepsilon^{(1)}(\xi))^2}{|x - \xi|^2} (w_\varepsilon^{(1)}(x) - w_\varepsilon^{(2)}(x))^2 dxd\xi \right)^{\frac{1}{2}}
\]
\[
+ O(\varepsilon^{-\frac{7}{2}}\|x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}\|_\varepsilon) \cdot \|R_\varepsilon^{(1)}\|_\varepsilon \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(R_\varepsilon^{(1)}(\xi))^2}{|x - \xi|^2} (w_\varepsilon^{(1)}(x))^2 dxd\xi \right)^{\frac{1}{2}} \tag{4.38}
\]
\[
= O(\varepsilon^{-\frac{5}{2}}\|w_\varepsilon^{(1)} - w_\varepsilon^{(2)}\|_\varepsilon) + O(\varepsilon^{-\frac{7}{2}}\|x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}\|_\varepsilon) \cdot \|w_\varepsilon^{(1)}\|_\varepsilon) = o(\varepsilon^2).
\]

Next, similar to (4.38), (C.11) and (C.12), we can obtain
\[
\|\bar{N}_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)})\|_{L^2(\mathbb{R}^3)} = \varepsilon^{-\frac{5}{2}}\|\bar{N}_\varepsilon(\cdot)\|_{L^2(\mathbb{R}^3)} = O(\varepsilon^{-\frac{5}{2}} \cdot \sum_{m=1}^{2} \|N(x, w_\varepsilon^{(m)}(x))\|_{L^2(\mathbb{R}^3)}) = O(\varepsilon^4), \tag{4.39}
\]
and
\[
\|\bar{I}_\varepsilon(\varepsilon x + x_{j,\varepsilon}^{(1)})\|_{L^2(\mathbb{R}^3)} = O(\varepsilon^{-\frac{1}{2}} \cdot \|x_{j,\varepsilon}^{(1)} - x_{j,\varepsilon}^{(2)}\|_\varepsilon) \cdot \|U_{a_j}(\cdot - x_{j,\varepsilon}^{(1)})\|_\varepsilon) + O(\varepsilon^{-7/6}) = o(\varepsilon^2). \tag{4.40}
\]
Then (4.36), (4.37), (4.38), (4.39) and (4.40) imply
\[
\sup_{B_R(0)} \bar{w}_{j,\varepsilon}(x) = o(\varepsilon^2), \text{ for } j = 1, 2, \ldots, k.
\]
This means (4.35). □
5. Proof of Theorem 1.2

Proposition 5.1. Let \( \eta_{j,\varepsilon}(x) = \eta_{\varepsilon}(\varepsilon x + x_{j,\varepsilon}^{(1)}) \) for \( j = 1, 2, \ldots, k \) and \( k \geq 2 \), then taking a subsequence necessarily, it holds

\[
\eta_{j,\varepsilon}(x) \to \sum_{i=1}^{3} d_{j,i} \frac{\partial U_{a_j}(x)}{\partial x^i}
\]
uniformly in \( C^1(B_R(0)) \) for any \( R > 0 \), where \( \eta_k(x) \) is the function in (1.12) and \( d_{j,i} \), \( i = 1, 2, 3 \) are some constants.

Proof. The proof is just as similar as that of Proposition 3.2. Here we want to point out that to obtain (3.8) for the case \( k \geq 2 \), the estimate (3.19) is crucial. \( \square \)

Proposition 5.2. Let \( d_{j,i} \) be as in Proposition 5.1, then we have

\[
d_{j,i} = 0, \quad \text{for all} \quad j = 1, \ldots, k, \quad i = 1, 2, 3.
\]  
(5.1)

Proof. Since \( u_{\varepsilon}^{(1)}(x), u_{\varepsilon}^{(2)}(x) \) are the positive solutions of (1.3), for the small fixed constant \( \bar{d} > 0 \) and any \( \varepsilon \in (d, 2d) \), similar to (3.10), we have

\[
\int_{B_{\varepsilon}(x_{j,\varepsilon}^{(1)})} \frac{\partial V(x)}{\partial x^i} \left( u_{\varepsilon}^{(1)}(x) + u_{\varepsilon}^{(2)}(x) \right) \cdot \eta_{\varepsilon}(x) dx = \int_{\partial B_{\varepsilon}(x_{j,\varepsilon}^{(1)})} B(x) d\sigma + F_1 + F_2,
\]  
(5.2)

where \( B(x) \) is the function in (3.11),

\[
F_1 = \frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x_{j,\varepsilon}^{(1)})} \int_{\mathbb{R}^3} \left( u_{\varepsilon}^{(1)}(x) \right)^2 \left( u_{\varepsilon}^{(1)}(\xi) + u_{\varepsilon}^{(2)}(\xi) \right) \eta_{\varepsilon}(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]

and

\[
F_2 = \frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x_{j,\varepsilon}^{(1)})} \int_{\mathbb{R}^3} \left( u_{\varepsilon}^{(2)}(\xi) \right)^2 \left( u_{\varepsilon}^{(1)}(x) + u_{\varepsilon}^{(2)}(x) \right) \eta_{\varepsilon}(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx.
\]

Then similar to (3.12) and (3.13), we know

\[
\text{LHS of (5.2)} = \varepsilon^4 \int_{\mathbb{R}^3} U_{a_j}^2(x) dx \left( \sum_{l=1}^{3} \frac{\partial^2 V(a_j)}{\partial x^l \partial x^l} d_{j,l} \right) + o(\varepsilon^4),
\]  
(5.3)

and

\[
\int_{\partial B_{\varepsilon}(x_{j,\varepsilon}^{(1)})} B(x) d\sigma = O(\varepsilon^5), \quad \text{for some} \quad \delta_\varepsilon \in (\bar{d}, 2\bar{d}).
\]  
(5.4)

Let \( \delta = \delta_\varepsilon \) in (5.2), then (5.3), (5.4) and (5.5) below in Proposition 5.3 imply

\[
\sum_{i=1}^{3} \frac{\partial^2 V(a_j)}{\partial x^i \partial x^i} d_{j,i} = o(1).
\]

This means \( d_{j,i} = 0, \) for \( i = 1, 2, 3 \). Similarly, we can obtain (5.1). \( \square \)

Proposition 5.3. For the small fixed constant \( \bar{d} > 0 \) and any \( \delta \in (\bar{d}, 2\bar{d}) \), it holds

\[
F_1 + F_2 = o(\varepsilon^4).
\]  
(5.5)

Proof. First, \( F_1 \) can be written as

\[
F_1 = F_{1,1} + F_{1,2} + F_{1,3} + F_{1,4},
\]  
(5.6)
where

\[ F_{1,1} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_j^{(1)}\varepsilon)} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_j^{(1)}\varepsilon}{\varepsilon} \right) (u_\varepsilon^{(1)}(\xi) + u_\varepsilon^{(2)}(\xi)) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ F_{1,2} = \frac{1}{4\pi\varepsilon^2} \int_{B_\delta(x_j^{(1)}\varepsilon)} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_j^{(1)}\varepsilon}{\varepsilon} \right) u_\varepsilon^{(1)}(x) \left( u_\varepsilon^{(1)}(\xi) + u_\varepsilon^{(2)}(\xi) \right) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ F_{1,3} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_j^{(1)}\varepsilon)} \int_{\mathbb{R}^3} \left( w_\varepsilon^{(1)}(x) \right)^2 (u_\varepsilon^{(1)}(\xi) + u_\varepsilon^{(2)}(\xi)) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ F_{1,4} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_j^{(1)}\varepsilon)} \int_{\mathbb{R}^3} W_{j,\varepsilon}(x) \left( 2u_\varepsilon^{(1)}(x) - W_{j,\varepsilon}(x) \right) \left( u_\varepsilon^{(1)}(\xi) + u_\varepsilon^{(2)}(\xi) \right) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

and \( W_{j,\varepsilon}(x) \) is the function in (C.2).

Next, \( F_2 \) can be written as follows:

\[ F_2 = F_{2,1} + F_{2,2} + F_{2,3}, \]  \hspace{1cm} (5.7)

where

\[ F_{2,1} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_j^{(1)}\varepsilon)} \int_{\mathbb{R}^3} (U_{a_j} \left( \frac{x - x_j^{(1)}\varepsilon}{\varepsilon} \right) + U_{a_j} \left( \frac{x - x_j^{(2)}\varepsilon}{\varepsilon} \right)) \eta_\varepsilon(x) \left( u_\varepsilon^{(1)}(\xi) \right)^2 \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ F_{2,2} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_j^{(1)}\varepsilon)} \int_{\mathbb{R}^3} \left( W_{j,\varepsilon}(x) + W_{j,\varepsilon}(x) \right) \eta_\varepsilon(x) \left( u_\varepsilon^{(2)}(\xi) \right)^2 \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ F_{2,3} = \frac{1}{8\pi\varepsilon^2} \int_{B_\delta(x_j^{(1)}\varepsilon)} \int_{\mathbb{R}^3} \left( w_\varepsilon^{(1)}(x) + w_\varepsilon^{(2)}(x) \right) \eta_\varepsilon(x) \left( u_\varepsilon^{(2)}(\xi) \right)^2 \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx. \]

Then (A.5), (A.10), (B.4), (C.13) imply

\[ F_{1,3} = O(\varepsilon^{-4}||u_\varepsilon^{(1)}||^2_2 ||u_\varepsilon^{(2)}||_2 + u_\varepsilon^{(2)}||\eta_\varepsilon||_2) = O(\varepsilon^6), \quad F_{1,4} = O(e^{-\eta/\varepsilon}) \quad \text{and} \quad F_{2,2} = O(e^{-\eta/\varepsilon}). \]  \hspace{1cm} (5.8)

Next from (5.8), (E.1), (E.9), (E.15) and (E.25), we know

\[ F_1 + F_2 = G + o(\varepsilon^4), \]  \hspace{1cm} (5.9)

where

\[ G = \frac{1}{8\pi\varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( U_{a_j} \left( \frac{x - x_j^{(1)}\varepsilon}{\varepsilon} \right) - U_{a_j} \left( \frac{x - x_j^{(2)}\varepsilon}{\varepsilon} \right) \right) \left( \sum_{m=1}^2 U_{a_j} \left( \frac{\xi - x_j^{(m)}\varepsilon}{\varepsilon} \right) \right) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx. \]

To estimate the term \( G \), we divide it into two cases.

**Case 1:**

\[ |x_j^{(1)}\varepsilon - x_j^{(2)}\varepsilon| = o(\varepsilon^3). \]  \hspace{1cm} (5.10)

Then from (5.10), (A.5) and (E.11), we obtain

\[ G = O(\varepsilon^{-4} \cdot \frac{|x_j^{(1)}\varepsilon - x_j^{(2)}\varepsilon|}{\varepsilon} \cdot \left| \sum_{m=1}^2 U_{a_j} \left( \frac{\xi - x_j^{(m)}\varepsilon}{\varepsilon} \right) \right| \cdot \eta_\varepsilon ||_\varepsilon) = o(\varepsilon^4). \]  \hspace{1cm} (5.11)

So in this case, (5.9) and (5.11) imply (5.5).
Case 2: For any fixed $C_0 > 0$, there exists $\{\varepsilon_i\}_{i=1}^{\infty}$ such that

$$\lim_{i \to +\infty} \varepsilon_i = 0 \quad \text{and} \quad |x_j^{(1)} - x_j^{(2)}| \geq C_0 \varepsilon_i^3, \quad \text{for all} \quad i = 1, \cdots, +\infty.$$ 

Then from (4.35) and (A.10), there exists some $C > 1$ such that

$$J_{\varepsilon} := \|u_\varepsilon^{(1)}(\cdot) - u_\varepsilon^{(2)}(\cdot)\|_{L^\infty(\mathbb{R}^3)} \geq \|u_\varepsilon^{(1)}(\cdot) - u_\varepsilon^{(2)}(\cdot)\|_{L^\infty(B_{\delta}(x_j^{(1)}))}^2 \geq C \varepsilon^{-1} |x_j^{(1)} - x_j^{(2)}| - C_0 \varepsilon^2 - e^{-\eta/\varepsilon} \geq \frac{C C_0}{4\varepsilon} |x_j^{(1)} - x_j^{(2)}|. \quad (5.12)$$

On the other hand, from (4.19) and (A.10), for small fixed $U$, then the exponential decay of $G_{\varepsilon}$ implies

$$\eta_\varepsilon(x) = J_{\varepsilon}^{-1} (U_{a_j}(\frac{x - x_j^{(1)}}{\varepsilon}) - U_{a_j}(\frac{x - x_j^{(2)}}{\varepsilon})) + J_{\varepsilon}^{-1} w_{j,\varepsilon}(x) + O(e^{-\eta/\varepsilon}), \quad \text{in} \quad B_d(x_j^{(1)}). \quad (5.13)$$

Then the exponential decay of $U_{a_j}(x)$ and (5.13) imply

$$U_{a_j}(\frac{x - x_j^{(1)}}{\varepsilon}) \eta_\varepsilon(x) = J_{\varepsilon}^{-1} U_{a_j}(\frac{x - x_j^{(1)}}{\varepsilon}) (U_{a_j}(\frac{x - x_j^{(1)}}{\varepsilon}) - U_{a_j}(\frac{x - x_j^{(2)}}{\varepsilon})) + J_{\varepsilon}^{-1} U_{a_j}(\frac{x - x_j^{(1)}}{\varepsilon}) w_{j,\varepsilon}(x) + O(e^{-\eta/\varepsilon}), \quad \text{in} \quad \mathbb{R}^3. \quad (5.14)$$

Then from the symmetry and (5.14), we have

$$G = \frac{1}{8\pi \varepsilon^2 J_{\varepsilon}} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( U_{a_j}(\frac{x - x_j^{(1)}}{\varepsilon}) - U_{a_j}(\frac{x - x_j^{(2)}}{\varepsilon}) \right) \left( U_{a_j}(\frac{\xi - x_j^{(1)}}{\varepsilon}) + U_{a_j}(\frac{\xi - x_j^{(2)}}{\varepsilon}) \right)$$

$$\cdot \left( w_\varepsilon^{(1)}(\xi) - w_\varepsilon^{(2)}(\xi) \right) \frac{x - \eta}{|x - \eta|} d\xi dx + O(e^{-\eta/\varepsilon}) \quad (5.15)$$

So in this case, (5.9) and (5.15) imply (5.5). \qed

**Proposition 5.4.** For any fixed $R > 0$, it holds

$$\eta_\varepsilon(x) = o(1), \quad x \in \bigcup_{j=1}^{k} B_{R \varepsilon}(x_j^{(1)}).$$

**Proof.** Similar to Proposition 3.4, this is the result of Proposition 5.1 and Proposition 5.2. \qed

**Proposition 5.5.** For large $R > 0$ and fixed $\gamma_1 \in (0, 1)$, there exists $\varepsilon_0$ such that

$$|\eta_\varepsilon(x)| \leq \gamma_1, \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R \varepsilon}(x_j^{(1)}) \quad \text{and} \quad \varepsilon \in (0, \varepsilon_0).$$

**Proof.** Similar to the proof of Proposition 5.5, we replace $B_{R \varepsilon}(x_j^{(1)})$ by $\bigcup_{j=1}^{k} B_{R \varepsilon}(x_j^{(1)})$. \qed

**Proof of Theorem 1.2.** Let $u^{(1)}_\varepsilon(x)$, $u^{(2)}_\varepsilon(x)$ be two different positive solutions concentrating at the nondegenerate critical points $\{a_1, \cdots, a_k\}$ of $V(x)$ for $k \geq 2$. Then Proposition 5.4 and Proposition 5.5 imply $|\eta_\varepsilon(x)| \leq \gamma_1$, for $x \in \mathbb{R}^3$, small $\varepsilon$ and fixed $\gamma_1 \in (0, 1)$, which contradicts with $\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$. So $u^{(1)}_\varepsilon(x) \equiv u^{(2)}_\varepsilon(x)$ for small $\varepsilon$. Also (4.11), (4.2) and (C.13) imply (1.10) and (1.11). \qed
Appendix

A. Some basic estimates

Lemma A.1 (Hardy-Littlewood-Sobolev inequality, c.f. [12]). Let \( p, r > 1, 0 < \lambda < 3, \frac{1}{p} + \frac{1}{r} + \frac{\lambda}{3} = 2, f \in L^p(\mathbb{R}^3), h \in L^r(\mathbb{R}^3), \) then there exists \( C(\lambda, p) > 0 \) such that

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(x) |x - y|^{-\lambda} h(y) dx dy \leq C(\lambda, p) \| f \|_{L^p(\mathbb{R}^3)} \| g \|_{L^r(\mathbb{R}^3)}.
\tag{A.1}
\]

Lemma A.2 (Nash-Moser iteration, c.f. Theorem 8.17 in [7]). If \( u \in H^1(\mathbb{R}^3) \) is the solution of

\(-\Delta u = f(x)\) in \( \mathbb{R}^3 \) and \( f \in L^{q/2}(\mathbb{R}^3) \) for some \( q > 3 \), then for any ball \( B_{2R}(y) \subset \mathbb{R}^3 \) and \( p > 1 \), there exists \( C = C(p, q) \) such that

\[
\sup_{B_{2R}(y)} u(x) \leq C \left(R^{-3/p} \| u \|_{L^2(B_{2R}(y))} + R^{2(1-3/q)} \| f \|_{L^{q/2}(\mathbb{R}^3)}\right).
\]

Lemma A.3 (Decomposition lemma, c.f. [1] or Lemma A.1 in [2]). For \( u(x) \in H_\varepsilon \), if there exist \( \delta_0 > 0, \varepsilon_0 > 0 \) such that

\[
\| u - \sum_{j=1}^k U_{a_j}(\frac{x - x_j \varepsilon}{\varepsilon}) \|_\varepsilon \leq \delta_0 \varepsilon^3, \text{ and } |x_j \varepsilon - a_j| \leq \delta, \text{ for all } \delta \in (0, \delta_0) \text{ and } \varepsilon \in (0, \varepsilon_0),
\]

then for all \( \delta \in (0, \delta_0) \) and \( \varepsilon \in (0, \varepsilon_0) \), the following minimization problem

\[
\inf \{ \varepsilon^{-3} \left\| u - \sum_{j=1}^k U_{a_j}(\frac{x - x_j \varepsilon}{\varepsilon}) \right\|_\varepsilon; x_j \varepsilon \in B_{4\delta}(a_j) \}
\]

has a unique solution which can be written as

\[
u = \sum_{j=1}^k \alpha_{j, \varepsilon} U_{a_j}(\frac{x - x_j \varepsilon}{\varepsilon}) + v_\varepsilon(x),\]

where \( |\alpha_{j, \varepsilon} - 1| \leq 2\delta, \ v_\varepsilon \in \bigcap_{j=1}^k E_{\varepsilon, a_j, x_j, \varepsilon} \) and

\[
E_{\varepsilon, a_j, x_j, \varepsilon} = \left\{ (u(x) \in H_\varepsilon : (u(x), U_{a_j}(\frac{x - x_j \varepsilon}{\varepsilon}))_\varepsilon = 0, (u(x), \frac{\partial U_{a_j}(\frac{x - x_j \varepsilon}{\varepsilon})}{\partial x})_\varepsilon = 0, i = 1, 2, 3 \right\}.
\]

Lemma A.4. Suppose \( f_\varepsilon \in L^1(\mathbb{R}^3) \cap C(\mathbb{R}^3) \), for any fixed small \( d > 0 \) independent of \( \varepsilon \) and \( x_\varepsilon \), there exists a small constant \( d_\varepsilon \in (d, 2d) \) such that

\[
\int_{\partial B_{d_\varepsilon}(x_\varepsilon)} |f_\varepsilon(x)| d\sigma \leq \frac{1}{d} \int_{\mathbb{R}^3} |f_\varepsilon(x)| dx.
\tag{A.2}
\]

Proof. First, for any fixed small \( d > 0 \) and \( x_\varepsilon \),

\[
\int_{d}^{2d} \int_{\partial B_{r}(x_\varepsilon)} |f_\varepsilon(x)| d\sigma dr = \int_{B_{2d}(x_\varepsilon) \setminus B_{d}(x_\varepsilon)} |f_\varepsilon(x)| dx \leq \int_{\mathbb{R}^3} |f_\varepsilon(x)| dx.
\tag{A.3}
\]

Also \( \int_{\partial B_r(x_\varepsilon)} |f_\varepsilon(x)| d\sigma \) is continuous with respect to \( r \). By mean value theorem of integrals, there exists \( d_\varepsilon \in (d, 2d) \) such that

\[
\int_{d}^{2d} \int_{\partial B_{r}(x_\varepsilon)} |f_\varepsilon(x)| d\sigma dr = d_\varepsilon \int_{\partial B_{d_\varepsilon}(x_\varepsilon)} |f_\varepsilon(x)| d\sigma.
\tag{A.4}
\]

Then (A.3) and (A.4) imply (A.2). \(\square\)
Lemma A.5. For any $u_1, u_2, u_3, u_4 \in H_\varepsilon$ and $0 < \lambda \leq 2$, then

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_1(\xi)u_2(\xi)u_3(x)u_4(x) \cdot |x - \xi|^{-\lambda}d\xi dx \leq C\varepsilon^{-\lambda} ||u_1||_\varepsilon ||u_2||_\varepsilon ||u_3||_\varepsilon ||u_4||_\varepsilon. \quad (A.5)$$

Proof. First, by Hardy-Littlewood-Sobolev inequality in Lemma A.1, we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_1(\xi)u_2(\xi)u_3(x)u_4(x) \cdot |x - \xi|^{-\lambda}d\xi dx \leq C||u_1 \cdot u_2||_{L^{\frac{6}{\lambda - 2}}(\mathbb{R}^3)} ||u_3 \cdot u_4||_{L^{\frac{6}{\lambda - 2}}(\mathbb{R}^3)}. \quad (A.6)$$

Next, for $0 < \lambda \leq 2$, by Hölder’s inequality and Sobolev embedding, we get

$$||u_1 \cdot u_2||_{L^{\frac{6}{\lambda - 2}}(\mathbb{R}^3)} \leq ||u_1||_{L^2(\mathbb{R}^3)} ||u_2||_{L^\frac{6}{\lambda - 2}(\mathbb{R}^3)} \leq ||u_1||_\varepsilon ||u_2||_\varepsilon^{\frac{3(2-\lambda)}{\lambda}} ||u_2||_{L^6(\mathbb{R}^3)} \leq \varepsilon^{-\frac{\lambda}{2}} ||u_1||_\varepsilon ||u_2||_\varepsilon. \quad (A.7)$$

Similarly, for $0 < \lambda \leq 2$, we have

$$||u_3 \cdot u_4||_{L^{\frac{6}{\lambda - 2}}(\mathbb{R}^3)} \leq \varepsilon^{-\lambda/2} ||u_3||_\varepsilon ||u_4||_\varepsilon. \quad (A.8)$$

Then (A.6), (A.7) and (A.8) imply (A.5). □

Lemma A.6. For any $u_1, u_2, u_3, u_4 \in H^1(\mathbb{R}^3)$, and $0 < \lambda \leq 2$, then

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} u_1(\xi)u_2(\xi)u_3(x)u_4(x) \cdot |x - \xi|^{\lambda}d\xi dx \leq C||u_1||_{H^1(\mathbb{R}^3)} ||u_2||_{H^1(\mathbb{R}^3)} ||u_3||_{H^1(\mathbb{R}^3)} ||u_4||_{H^1(\mathbb{R}^3)}. \quad (A.9)$$

Proof. Similar to the proof of Lemma A.5, we can obtain (A.9) by Hardy-Littlewood-Sobolev inequality, Hölder’s inequality and Sobolev embedding. □

Lemma A.7. (1) There exist two positive constants $d_1$ and $\eta$ such that, for $j = 1, 2, \cdots, k$,

$$U_{a_j}\left(\frac{x - x_{j,\varepsilon}}{\varepsilon}\right) = O(e^{-\eta/\varepsilon}), \text{ for } x \in \mathbb{R}^3 \setminus B_d(x_{j,\varepsilon}), \text{ and } 0 < d < d_1. \quad (A.10)$$

(2) Let $\{a_1, \cdots, a_k\} \subset \mathbb{R}^3$ be the different nondegenerate critical points of $V(x)$ with $k \geq 1$, then it holds

$$\int_{\mathbb{R}^3} (V(a_j) - V(x))U_{a_j}\left(\frac{x - x_{j,\varepsilon}}{\varepsilon}\right)u(x)dx = O(\varepsilon^{7/2} + \varepsilon^{3/2}|x_{j,\varepsilon} - a_j|^2) ||u||_\varepsilon, \quad (A.11)$$

and

$$\int_{\mathbb{R}^3} \frac{\partial V(x)}{\partial x^i}U_{a_j}\left(\frac{x - x_{j,\varepsilon}}{\varepsilon}\right)u(x)dx = O(\varepsilon^{5/2} + \varepsilon^{3/2}|x_{j,\varepsilon} - a_j|) ||u||_\varepsilon, \quad (A.12)$$

where $u(x) \in H_\varepsilon$ and $j = 1, 2, \cdots, k$.

Proof. First, the exponential decay of $U_{a_j}(x)$ implies (A.10). Next, since $a_j$ is a nondegenerate critical point of $V(x)$, we know

$$V(a_j) - V(x) = -\sum_{i=1}^3 \sum_{l=1}^3 (a_i^j - a_i^j)(x^l - a_l^j) \frac{\partial^2 V(a_j)}{\partial x^i \partial x^l} + o(|x - a_j|^2). \quad (A.13)$$
Then using (A.13) and Hölder’s inequality, for any small constant $d$, we have

$$\left| \int_{B_d(x_j,\varepsilon)} (V(a_j) - V(x)) U_{a_j} \left( \frac{x - x_j \varepsilon}{\varepsilon} \right) u(x) dx \right|$$

$$\leq C \int_{B_d(x_j,\varepsilon)} |x - a_j|^2 U_{a_j} \left( \frac{x - x_j \varepsilon}{\varepsilon} \right) |u(x)| dx$$

$$\leq C \left( \int_{B_d(x_j,\varepsilon)} |x - a_j|^4 U_{a_j}^2 \left( \frac{x - x_j \varepsilon}{\varepsilon} \right) dx \right)^{1/2} \left( \int_{B_d(x_j,\varepsilon)} u^2(x) dx \right)^{1/2}$$

(A.14)

$$\leq C \varepsilon \frac{1}{2} \left( \int_{B_d(0)} |x| + (x_j,\varepsilon - a_j)|^4 U_{a_j}^2(y) dy \right)^{1/2} \|u\|_\varepsilon$$

$$\leq C \varepsilon \frac{1}{2} \left( \varepsilon^2 + |x_j,\varepsilon - a_j|^2 \right) \|u\|_\varepsilon.$$ 

Also, by (A.10), we can deduce that

$$\left| \int_{\mathbb{R}^3 \setminus B_d(x_j,\varepsilon)} (V(a_j) - V(x)) U_{a_j} \left( \frac{x - x_j \varepsilon}{\varepsilon} \right) u(x) dx \right| \leq C e^{-\eta/\varepsilon} \|u\|_\varepsilon.$$  

(A.15)

Then from (A.14) and (A.15), we get (A.11).

Similarly, since $a_j$ is the nondegenerate critical point of $V(x)$, we know

$$\frac{\partial V(x)}{\partial x^i} = \frac{\partial V(x)}{\partial x^j} - \frac{\partial V(a_j)}{\partial x^i} = \sum_{l=1}^3 (x^l - a_j^l) \frac{\partial^2 V(a_j)}{\partial x^j \partial x^l} + o(|x - a_j|), \text{ for } i = 1, 2, 3.$$  

(A.16)

So similar to (A.14), from (A.16) and Hölder’s inequality, for any small fixed $d$, we have

$$\left| \int_{B_d(x_j,\varepsilon)} \frac{\partial V(x)}{\partial x^i} U_{a_j} \left( \frac{x - x_j \varepsilon}{\varepsilon} \right) u(x) dx \right| \leq C \int_{B_d(x_j,\varepsilon)} |x - a_j| U_{a_j} \left( \frac{x - x_j \varepsilon}{\varepsilon} \right) |u(x)| dx$$

(A.17)

$$\leq C \varepsilon \frac{1}{2} (\varepsilon + |x_j,\varepsilon - a_j|) \|u\|_\varepsilon.$$  

Also, by (A.10), we know

$$\left| \int_{\mathbb{R}^3 \setminus B_d(x_j,\varepsilon)} \frac{\partial V(x)}{\partial x^i} U_{a_j} \left( \frac{x - x_j \varepsilon}{\varepsilon} \right) u(x) dx \right| \leq C e^{-\eta/\varepsilon} \|u\|_\varepsilon.$$  

(A.18)

Then (A.17) and (A.18) imply (A.12).

\[\square\]

### B. Regularization and Some Calculations

Let $u_{\varepsilon}^{(1)}(x), u_{\varepsilon}^{(2)}(x)$ be two different positive solutions concentrating at $\{a_1, \cdots, a_k\}$. Set

$$\eta_\varepsilon(x) = \frac{u_{\varepsilon}^{(1)}(x) - u_{\varepsilon}^{(2)}(x)}{\|u_{\varepsilon}^{(1)} - u_{\varepsilon}^{(2)}\|_{L^\infty(\mathbb{R}^3)}}.$$  

(B.1)

Then we know $\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$ and

$$-\varepsilon^2 \Delta \eta_\varepsilon(x) + V(x) \eta_\varepsilon(x) = E_1(x) \eta_\varepsilon(x) + E_2(x), \quad x \in \mathbb{R}^3,$$  

(B.2)

where

$$E_1(x) = \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \left( \frac{u_{\varepsilon}^{(1)}(\xi)}{|x - \xi|} \right)^2 d\xi, \quad E_2(x) = \frac{u_{\varepsilon}^{(2)}(x)}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \frac{u_{\varepsilon}^{(1)}(\xi) + u_{\varepsilon}^{(2)}(\xi)}{|x - \xi|} \eta_\varepsilon(\xi) d\xi.$$  

(B.3)
Proposition B.1. For $\eta_\epsilon(x)$ defined by (B.1), we have
\[ \|\eta_\epsilon\|_\epsilon = O(\epsilon^{3/2}). \quad (B.4) \]

Proof. From (B.2) we have
\[ \|\eta_\epsilon\|_\epsilon^2 = \int_{\mathbb{R}^3} E_1(x)\eta_\epsilon^2(x)dx + \int_{\mathbb{R}^3} E_2(x)\eta_\epsilon(x)dx. \quad (B.5) \]

Next, by Hardy-Littlewood-Sobolev inequality, Hölder’s inequality and the fact $|\eta_\epsilon(x)| \leq 1$, we know
\[
\left| \int_{\mathbb{R}^3} E_1(x)\eta_\epsilon^2(x)dx \right| \leq C\epsilon^{-2} \left( \int_{\mathbb{R}^3} |u_\epsilon^{(1)}(\xi)|^{12/5}d\xi \right)^{5/6} \left( \int_{\mathbb{R}^3} |\eta_\epsilon(x)|^{12/5}dx \right)^{5/6} \\
\leq C\epsilon^{-2} \left( \int_{\mathbb{R}^3} |u_\epsilon^{(1)}(\xi)|^{12/5}d\xi \right)^{5/6} \left( \int_{\mathbb{R}^3} |\eta_\epsilon(x)|^{2}dx \right)^{5/6} \quad (B.6) \\
\leq C\epsilon^{1/2} \|\eta_\epsilon\|_{5/3}^2 \leq C\epsilon^3 + \frac{1}{2}\|\eta_\epsilon\|_\epsilon^2,
\]

and
\[
\left| \int_{\mathbb{R}^3} E_2(x)\eta_\epsilon(x)dx \right| \leq C\epsilon^{-2} \left( \int_{\mathbb{R}^3} |u_\epsilon^{(2)}(x)|^{6}dx \right)^{\frac{2}{3}} \cdot \left( \int_{\mathbb{R}^3} |(u_\epsilon^{(1)}(\xi) + u_\epsilon^{(2)}(\xi))|^{\frac{6}{5}}d\xi \right)^{\frac{2}{5}} \leq C\epsilon^3. \quad (B.7)
\]

Then (B.5), (B.6) and (B.7) imply (B.4). \hfill \Box

Lemma B.2. For any fixed $R > 0$, it holds
\[ E_1(x) = o(1) \cdot R + O\left(\frac{1}{R}\right), \text{ for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\epsilon}(x_j,\epsilon), \quad (B.8) \]

and
\[ E_2(x) = O(e^{-\theta'R}), \text{ for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\epsilon}(x_j,\epsilon) \text{ and some } \theta' > 0. \quad (B.9) \]

Proof. First, we know
\[
\{ \xi, |x - \xi| \leq R\epsilon/2 \} \subset \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\epsilon/2}(x_j,\epsilon), \text{ for } x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\epsilon}(x_j,\epsilon),
\]

and $\|u_\epsilon\|_\epsilon = O(\epsilon^{3/2})$. Then by (2.33), for $x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\epsilon}(x_j,\epsilon)$, it holds
\[
E_1(x) = \frac{1}{8\pi\epsilon^2} \int_{|x - \xi| \leq R\epsilon/2} (u_\epsilon^{(1)}(\xi))^2 |x - \xi|^{-1}d\xi + \frac{1}{8\pi\epsilon^2} \int_{|x - \xi| > R\epsilon/2} (u_\epsilon^{(1)}(\xi))^2 |x - \xi|^{-1}d\xi \\
= O(\epsilon^{-2} \int_{|x - \xi| \leq R\epsilon/2} (u_\epsilon^{(1)}(\xi))^2 |x - \xi|^{-1}d\xi) + O(e^{-2\theta'R^2}) + O\left(\frac{1}{R}\right). \quad (B.10)
\]

Also, by Hölder’s inequality, we have
\[
\int_{|x - \xi| \leq R\epsilon/2} (w_\epsilon^{(1)}(\xi))^2 |x - \xi|^{-1}d\xi \\
= O\left( \left( \int_{\mathbb{R}^3} (w_\epsilon^{(1)}(\xi))^6 d\xi \right)^{1/3} \left( \int_{|x - \xi| \leq R\epsilon/2} |x - \xi|^{-3/2}d\xi \right)^{2/3} \right) \\
= R \cdot O(e^{-1}\|w_\epsilon^{(1)}(\xi)\|_{6}^2) = o(\epsilon^3) \cdot R. \quad (B.11)
\]

Then (B.10) and (B.11) imply (B.8).
Next for \( x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{k} B_{R\varepsilon}(x_{j,\varepsilon}) \), we have

\[
E_2(x) = O(e^{-9R}) \cdot \varepsilon^{-2} \int_{\mathbb{R}^3} (u_{\varepsilon}^{(1)}(\xi) + u_{\varepsilon}^{(2)}(\xi)) |x - \xi|^{-1} \cdot |\eta_{\varepsilon}(\xi)|d\xi, \quad (B.12)
\]

and

\[
\begin{align*}
\int_{\mathbb{R}^3} (u_{\varepsilon}^{(1)}(\xi) + u_{\varepsilon}^{(2)}(\xi)) |x - \xi|^{-1} \cdot |\eta_{\varepsilon}(\xi)|d\xi \\
= \int_{|x - \xi| \leq R\varepsilon/2} (u_{\varepsilon}^{(1)}(\xi) + u_{\varepsilon}^{(2)}(\xi)) |x - \xi|^{-1} \cdot |\eta_{\varepsilon}(\xi)|d\xi + O\left( (R\varepsilon)^{-1} \|u_{\varepsilon}^{(1)} + u_{\varepsilon}^{(2)}\|_{\varepsilon}\|\eta_{\varepsilon}\|_{\varepsilon} \right) \quad (B.13)
\end{align*}
\]

\[
= O\left( \|u_{\varepsilon}^{(1)}(\cdot) + u_{\varepsilon}^{(2)}(\cdot)\|_{\varepsilon} \cdot \left( \int_{|x - \xi| \leq R\varepsilon/2} |x - \xi|^{-2}d\xi \right)^{1/2} + O(R^{-1}\varepsilon^2) = O((R^{3/2} + R^{-1})\varepsilon^2). \right)
\]

Then (B.12) and (B.13) imply (B.9). \hfill \Box

**Lemma B.3.** For any fixed small \( d > 0 \), it holds

\[
E_1(x) = \frac{1}{8\pi}\frac{1}{\varepsilon^2} \left( \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) |x - \xi|^{-1}d\xi \right) + o(1), \text{ in } B_d(x_{j,\varepsilon}^{(1)}), \quad (B.14)
\]

and

\[
E_2(x) = \frac{1}{4\pi} \cdot U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) \left( \int_{\mathbb{R}^3} U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) \eta_{\varepsilon} |x - \xi|^{-1}d\xi \right) + o(1), \text{ in } B_d(x_{j,\varepsilon}^{(1)}). \quad (B.15)
\]

**Proof.** For \( x \in B_d(x_{j,\varepsilon}^{(1)}) \), we have

\[
|E_1(x) - \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) |x - \xi|^{-1}d\xi \right)|
\]

\[
= O(\varepsilon^{-2} \int_{\mathbb{R}^3} |w_{\varepsilon}^{(1)}(\xi)| \cdot (u_{\varepsilon}^{(1)}(\xi) + U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right)) |x - \xi|^{-1}d\xi) + O(e^{-\eta/\varepsilon}) \quad (B.16)
\]

\[
= O(\varepsilon^{-2} \int_{|x - \xi| \leq C} |w_{\varepsilon}^{(1)}(\xi)| \cdot (u_{\varepsilon}^{(1)}(\xi) + U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right)) |x - \xi|^{-1}d\xi)
\]

\[
+ O(\varepsilon^{-2} \|u_{\varepsilon}^{(1)}(\cdot)\|_{\varepsilon} \cdot \|u_{\varepsilon}^{(1)}(\cdot) + U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right)\|_{\varepsilon}) + O(e^{-\eta/\varepsilon}),
\]

where \( C \) is a fixed constant.

On the other hand, by Hölder’s inequality, we know

\[
\int_{|x - \xi| \leq C} |w_{\varepsilon}^{(1)}(\xi)| \cdot (u_{\varepsilon}^{(1)}(\xi) + U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right)) |x - \xi|^{-1}d\xi
\]

\[
= O\left( \|w_{\varepsilon}^{(1)}(\cdot)\|_{L^6(\mathbb{R}^3)} \cdot \|u_{\varepsilon}^{(1)}(\cdot)\|_{L^6(\mathbb{R}^3)} \left( \int_{|x - \xi| \leq C} |x - \xi|^{-3}d\xi \right)^{1/3} \right)
\]

\[
= O(\varepsilon^{-1} \|u_{\varepsilon}^{(1)}(\cdot)\|_{\varepsilon} \cdot \|u_{\varepsilon}^{(1)}(\cdot) + U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right)\|_{\varepsilon}) = o(\varepsilon^2).
\]

Then (B.16) and (B.17) imply (B.14). Similar to the estimates of (B.14), combining Proposition B.1 and Proposition 4.3, we deduce (B.15). \hfill \Box
C. Estimates of the term $w_\varepsilon$

For convenience, we define the following notations:

$$R_\varepsilon(x) = \sum_{l=1}^{k} U_{a_l}(\frac{x - x_{l,\varepsilon}}{\varepsilon}), \quad R_{\varepsilon}^{(m)}(x) = \sum_{l=1}^{k} U_{a_l}(\frac{x - x_{l,\varepsilon}^{(m)}}{\varepsilon}), \quad \text{for } m = 1, 2,$$

(C.1)

and

$$W_{j,\varepsilon}(x) = \sum_{l=1,l\neq j}^{k} U_{a_l}(\frac{x - x_{l,\varepsilon}}{\varepsilon}), \quad W_{j,\varepsilon}^{(m)}(x) = \sum_{l=1,l\neq j}^{k} U_{a_l}(\frac{x - x_{l,\varepsilon}^{(m)}}{\varepsilon}), \quad \text{for } m = 1, 2.$$

(C.2)

Let $M_\varepsilon(x, w_\varepsilon(x))$ as follows:

$$M_\varepsilon(x, w_\varepsilon(x)) := -\varepsilon^2 \Delta w_\varepsilon(x) + G(x, w_\varepsilon(x)),$$

where

$$G(x, w_\varepsilon(x)) = V(x)w_\varepsilon(x) - \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \left( \frac{R_\varepsilon(\xi)}{|x - \xi|} \right)^2 d\xi \right) w_\varepsilon(x) + \frac{1}{4\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{R_\varepsilon(\xi)w_\varepsilon(\xi)}{|x - \xi|} d\xi \right) R_\varepsilon(x).$$

(C.4)

Let $u_\varepsilon(x) = R_\varepsilon(x) + w_\varepsilon(x)$ be the solution of (1.3), then

$$M_\varepsilon(x, w_\varepsilon(x)) = N(x, w_\varepsilon(x)) + l_\varepsilon(x),$$

(C.5)

where

$$N(x, w_\varepsilon(x)) = \frac{1}{8\pi\varepsilon^2} \left( \int_{\mathbb{R}^3} \frac{w_\varepsilon^2(\xi)}{|x - \xi|} d\xi \right) R_\varepsilon(x) + \frac{w_\varepsilon(x)}{4\pi\varepsilon^2} \int_{\mathbb{R}^3} \frac{R_\varepsilon(\xi)w_\varepsilon(\xi)}{|x - \xi|} d\xi,$$

and

$$l_\varepsilon(x) = \frac{W_{j,\varepsilon}(x)}{8\pi\varepsilon^2} \int_{\mathbb{R}^3} \frac{W_{j,\varepsilon}(\xi)U_{a_j}(\frac{\xi - x_{j,\varepsilon}}{\varepsilon})}{|x - \xi|} d\xi + \sum_{j=1}^{k} (V(a_j) - V(x))U_{a_j}(\frac{x - x_{j,\varepsilon}}{\varepsilon}).$$

(C.7)

**Proposition C.1.** Let $u_\varepsilon(x) = R_\varepsilon(x) + w_\varepsilon(x)$ be the solution of (1.3), then there exists a constant $\bar{\rho} > 0$ independent of $\varepsilon$ such that

$$\int_{\mathbb{R}^3} M_\varepsilon(x, w_\varepsilon(x))w_\varepsilon(x)dx \geq \bar{\rho} \|w_\varepsilon\|^2.$$

(C.8)

**Proof.** Similar to the proof of Proposition 3.1 in [4], we can prove (C.8) by the contradiction argument and blow-up analysis. For the more details, one can refer to [3, 4].

**Proposition C.2.** Suppose that $u_\varepsilon(x) = R_\varepsilon(x) + w_\varepsilon(x)$ is a positive solution of (1.3) and $\{a_1, \cdots, a_k\} \subset \mathbb{R}^3$ are the different nondegenerate critical points of $V(x)$ with $k \geq 1$, then it holds

$$\|w_\varepsilon\|_\varepsilon = O(\varepsilon^{7/2}) + O(\varepsilon^{3/2} \max_{j=1, \cdots, k} |x_{j,\varepsilon} - a_j|^2).$$

(C.9)

**Proof.** First, from Proposition C.1, we know

$$\|w_\varepsilon\|_\varepsilon \leq C \int_{\mathbb{R}^3} N(x, w_\varepsilon(x))w_\varepsilon(x)dx + C \int_{\mathbb{R}^3} l_\varepsilon(x)w_\varepsilon(x)dx.$$

(C.10)
Next, using \((2.2)\) and \((A.5)\), we deduce
\[
\int_{\mathbb{R}^3} N(x, w_\varepsilon(x)) w_\varepsilon(x) dx = \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_\varepsilon^2(\xi)}{|x - \xi|} (R_\varepsilon(x) + w_\varepsilon(x)) w_\varepsilon(x) dx d\xi
\]
\[
+ \frac{1}{4\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{R_\varepsilon(\xi) w_\varepsilon(\xi)}{|x - \xi|} w_\varepsilon^2(x) dx d\xi \tag{C.11}
\]
\[
= O(\varepsilon^{-3} \|w_\varepsilon\|_2^3 \cdot \|w_\varepsilon + R_\varepsilon\|_2) = o(1) \|w_\varepsilon\|_2^2.
\]

Also from \((A.10)\) and \((A.13)\), we have
\[
\int_{\mathbb{R}^3} l_\varepsilon(x) w_\varepsilon(x) dx = \sum_{j=1}^{k} \int_{\mathbb{R}^3} (V(a_j) - V(x)) U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(x) d\xi dx
\]
\[
- \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W_{j,\varepsilon}(\xi) U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(x) |x - \xi|^{-1} d\xi \tag{C.12}
\]
\[
= O(\varepsilon^{3/2} \|w_\varepsilon\|_2 (\varepsilon^2 + \max_{j=1,\ldots,k} |x_{j,\varepsilon} - a_j|^2 + \varepsilon^{-n/\varepsilon}).
\]

Then \((C.10)\), \((C.11)\) and \((C.12)\) imply \((C.9)\). \qed

**Proposition C.3.** Let \(u_\varepsilon(x)\) be a positive solution of \((1.3)\) as in Proposition \((C.2)\), then it holds
\[
\|w_\varepsilon\|_2 = O(\varepsilon^{7/2}). \tag{C.13}
\]

**Proof.** It follows from the results of Proposition \((B.1)\) and Proposition \((C.2)\) directly. \qed

**Lemma D.1.** It holds
\[
A_{1,1} = \frac{1}{4\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx
\]
\[
+ \frac{1}{8\pi \varepsilon^2} \sum_{l=1, l\neq j}^{k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_l} \left( \frac{\xi - x_{l,\varepsilon}}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(\varepsilon^6). \tag{D.1}
\]

**Proof.** First, \(A_{1,1}\) can be written as follows:
\[
A_{1,1} = A_{1,1,1} + A_{1,1,2} + A_{1,1,3} + A_{1,1,4} + A_{1,1,5}, \tag{D.2}
\]
where
\[
A_{1,1,1} = \frac{1}{8\pi \varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
\[
A_{1,1,2} = \frac{1}{4\pi \varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
\[
A_{1,1,3} = \frac{1}{8\pi \varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
\[
A_{1,1,4} = \frac{1}{4\pi \varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) W_{j,\varepsilon}(\xi) \left( U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) + w_\varepsilon(\xi) \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
\[
A_{1,1,5} = \frac{1}{4\pi \varepsilon^2} \int_{B_d(x_{j,\varepsilon})} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx.
\]
\[ A_{1,1,5} = \frac{1}{8\pi\varepsilon^2} \int_{B_d(x,j,\varepsilon)} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) (W_{j,\varepsilon}(\xi))^2 \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx. \]

Now by symmetry and (A.10), we have

\[ A_{1,1} = \frac{1}{8\pi\varepsilon^2} \int_{\mathbb{R}^3 \setminus B_d(x,j,\varepsilon)} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_j}^2 \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx = O(\varepsilon^{-\eta/\varepsilon}), \quad (D.3) \]

and

\[ A_{1,1,2} = \frac{1}{4\pi\varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(\varepsilon^{-\eta/\varepsilon}). \quad (D.4) \]

Next, by (A.5) and (C.13), we get

\[ A_{1,1,3} = O(\varepsilon^{-4} \left\| U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) \right\|^2 \left\| w_\varepsilon \right\|^2_\varepsilon) = O(\varepsilon^6). \quad (D.5) \]

Also, (2.10) and (A.10) imply

\[ W_{j,\varepsilon}(x) \left( U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) + w_\varepsilon(x) \right) = O(\varepsilon^{-\eta/\varepsilon}), \text{ for } x \in \mathbb{R}^3. \quad (D.6) \]

This means

\[ A_{1,1,4} = O(\varepsilon^{-\eta/\varepsilon}). \quad (D.7) \]

Also, from (A.10), we can deduce

\[ A_{1,1,5} = \frac{1}{8\pi\varepsilon^2} \sum_{l=1, l \neq j}^k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_l} \left( \frac{\xi - x_{l,\varepsilon}}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(\varepsilon^{-\eta/\varepsilon}). \quad (D.8) \]

Then (D.2), (D.3), (D.4), (D.5), (D.6) and (D.8) imply (D.1).

Lemma D.2. It holds

\[ A_{1,2} = -\frac{1}{4\pi\varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(\varepsilon^6). \quad (D.9) \]

Proof. First, \( A_{1,2} \) can be written as follows:

\[ A_{1,2} = A_{1,2,1} + A_{1,2,2} + A_{1,2,3} + A_{1,2,4} + A_{1,2,5}, \quad (D.10) \]

where

\[ A_{1,2,1} = \frac{1}{4\pi\varepsilon^2} \int_{B_d(x,j,\varepsilon)} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(x) U_{a_j}^2 \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ A_{1,2,2} = \frac{1}{2\pi\varepsilon^2} \int_{B_d(x,j,\varepsilon)} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(x) U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ A_{1,2,3} = \frac{1}{4\pi\varepsilon^2} \int_{B_d(x,j,\varepsilon)} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(x) w_\varepsilon^2(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ A_{1,2,4} = \frac{1}{2\pi\varepsilon^2} \int_{B_d(x,j,\varepsilon)} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(x) W_{j,\varepsilon}(\xi) \left( U_{a_j} \left( \frac{\xi - x_{j,\varepsilon}}{\varepsilon} \right) + w_\varepsilon(\xi) \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \]

\[ A_{1,2,5} = \frac{1}{4\pi\varepsilon^2} \int_{B_d(x,j,\varepsilon)} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) w_\varepsilon(x) (W_{j,\varepsilon}(\xi))^2 \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx. \]
Lemma D.3. Then (D.10), (D.11), (D.12), (D.13), (D.14) and (D.16) imply (D.9).

Next, by (A.5) and (C.13), we get

\[
A_{1,2,1} = -\frac{1}{4\pi\varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x-x_j^\varepsilon}{\varepsilon} \right) U_{a_i} \left( \frac{\xi - x_i^\varepsilon}{\varepsilon} \right) w_\varepsilon \left( \frac{x^i - \xi^i}{|x - \xi|^3} \right) d\xi dx + O(\varepsilon^{-n/\varepsilon}). \tag{D.11}
\]

Next, by (A.5) and (C.13), we get

\[
A_{1,2,2} = O(\varepsilon^{-4} \|U_{a_j} \left( \frac{\cdot - x_j^\varepsilon}{\varepsilon} \right)\|_2^2 \|w_\varepsilon\|_\varepsilon^2) = O(\varepsilon^6), \tag{D.12}
\]

and

\[
A_{1,2,3} = O(\varepsilon^{-4} \|U_{a_j} \left( \frac{\cdot - x_j^\varepsilon}{\varepsilon} \right)\|_\varepsilon \|w_\varepsilon\|_\varepsilon^3) = O(\varepsilon^8). \tag{D.13}
\]

Also, similar to (D.7), we have

\[
A_{1,2,4} = O(\varepsilon^{-n/\varepsilon}). \tag{D.14}
\]

On the other hand, for \( l \neq j \) and fixed small \( d \), from (A.10) and (C.13), we have

\[
\int_{B_d(x_j, \varepsilon)} \int_{\mathbb{R}^3} U_{a_j} \left( \frac{x-x_j^\varepsilon}{\varepsilon} \right) w_\varepsilon(x) U_{a_i}^2 \left( \frac{\xi - x_i^\varepsilon}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx
\]

\[
= \int_{B_d(x_j, \varepsilon)} \int_{\mathbb{R}^3 \setminus B_{2d}(x_j, \varepsilon)} U_{a_j} \left( \frac{x-x_j^\varepsilon}{\varepsilon} \right) w_\varepsilon(x) U_{a_i}^2 \left( \frac{\xi - x_i^\varepsilon}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(\varepsilon^{-n/\varepsilon}) \tag{D.15}
\]

\[
= O \left( \|U_{a_j} \left( \frac{\cdot - x_j^\varepsilon}{\varepsilon} \right)\| \|w_\varepsilon\| \|U_{a_i} \left( \frac{\cdot - x_i^\varepsilon}{\varepsilon} \right)\|^2 \right) = O(\varepsilon^8).
\]

Then (A.10) and (D.15) imply

\[
A_{1,2,5} = O(\varepsilon^6) + O(\varepsilon^{-n/\varepsilon}) = O(\varepsilon^6). \tag{D.16}
\]

Then (D.10), (D.11), (D.12), (D.13), (D.14) and (D.16) imply (D.9).

\[\square\]

Lemma D.3. For \( l \neq j \), it holds

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x-x_j^\varepsilon}{\varepsilon} \right) U_{a_i}^2 \left( \frac{\xi - x_i^\varepsilon}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx = \varepsilon^3 (a_j^i - a_i^j) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2(x) U_{a_i}^2 \left( \xi + \frac{x_j^\varepsilon - x_i^\varepsilon}{\varepsilon} \right) |x - \xi|^{-3} d\xi dx + o(\varepsilon^6). \tag{D.17}
\]

Proof. First, we have

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x-x_j^\varepsilon}{\varepsilon} \right) U_{a_i}^2 \left( \frac{\xi - x_i^\varepsilon}{\varepsilon} \right) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx = B_1 + B_2, \tag{D.18}
\]

where

\[
B_1 = \varepsilon^4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2(x) U_{a_i}^2 \left( \xi + \frac{x_j^\varepsilon - x_i^\varepsilon}{\varepsilon} \right) \frac{x^i}{|x - \xi|^3} d\xi dx,
\]

and

\[
B_2 = -\varepsilon^4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2(x) U_{a_i}^2 \left( \xi + \frac{x_j^\varepsilon - x_i^\varepsilon}{\varepsilon} \right) \frac{\xi^i}{|x - \xi|^3} d\xi dx.
\]
Then for small fixed $d > 0$, we know
\[
B_1 = \varepsilon^4 \int_{B_d/\varepsilon(0)} \int_{\mathbb{R}^3} \int_{B_d/\varepsilon(0)} U^2_{a_l}(x)U^2_{a_l}(\xi + \frac{x_{j,\varepsilon} - x_{l,\varepsilon}}{\varepsilon}) \frac{x^i}{|x - \xi|^3} d\xi dx + O(\varepsilon^{-n/\varepsilon})
\]
\[
= O(\varepsilon^7 \int_{\mathbb{R}^3} U^2_{a_j}(x)|x| dx \cdot (\int_{\mathbb{R}^3} U^2_{a_l}(\xi + \frac{x_{j,\varepsilon} - x_{l,\varepsilon}}{\varepsilon}) d\xi)) + O(\varepsilon^{-n/\varepsilon}) = O(\varepsilon^7).
\]
Also we have
\[
B_2 = -\varepsilon^4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U^2_{a_j}(x)U^2_{a_l}(\xi + \frac{x_{j,\varepsilon} - x_{l,\varepsilon}}{\varepsilon}) \frac{x^i}{|x - \xi|^3} d\xi dx 
+ (x^i_{j,\varepsilon} - x^i_{l,\varepsilon}) \varepsilon^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U^2_{a_j}(x)U^2_{a_l}(\xi + \frac{x_{j,\varepsilon} - x_{l,\varepsilon}}{\varepsilon}) |x - \xi|^{-3} d\xi dx.
\]
Next, similar to (D.19), we deduce
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U^2_{a_j}(x)U^2_{a_l}(\xi + \frac{x_{j,\varepsilon} - x_{l,\varepsilon}}{\varepsilon}) |x - \xi|^{-3} d\xi dx = O(\varepsilon^3),
\]
and
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U^2_{a_j}(x)U^2_{a_l}(\xi + \frac{x_{j,\varepsilon} - x_{l,\varepsilon}}{\varepsilon}) \frac{x^i}{|x - \xi|^3} d\xi dx = O(\varepsilon^3).
\]
Then using (2.2), (D.20), (D.21) and (D.22), we obtain
\[
B_2 = \varepsilon^3 (a^i_{j,\varepsilon} - a^i_{l,\varepsilon}) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U^2_{a_j}(x)U^2_{a_l}(\xi + \frac{x_{j,\varepsilon} - x_{l,\varepsilon}}{\varepsilon}) |x - \xi|^{-3} d\xi dx + o(\varepsilon^6).
\]
Then (D.18), (D.19) and (D.23) imply (D.17).

E. The estimates of $F_{1,1}$, $F_{1,2}$, $F_{2,1}$ and $F_{2,3}$ in (5.6) and (5.7)

Lemma E.1. It holds
\[
F_{1,1} = G_1 + \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U^2_{a_j}(x) \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) \left( w^{(1)}_{x}(\xi) + w^{(2)}_{x}(\xi) \right) \eta(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + o(\varepsilon^4),
\]
where
\[
G_1 = \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U^2_{a_j}(x) \left( \frac{x - x_{j,\varepsilon}}{\varepsilon} \right) \left( U_{a_j} \left( \frac{\xi - x^{(1)}_{j,\varepsilon}}{\varepsilon} \right) + U_{a_j} \left( \frac{\xi - x^{(2)}_{j,\varepsilon}}{\varepsilon} \right) \right) \eta(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx.
\]

Proof. $F_{1,1}$ can be written as
\[
F_{1,1} = F_{1,1,1} + F_{1,1,2} + F_{1,1,3},
\]
where
\[
F_{1,1,1} = \frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x^{(1)}_{j,\varepsilon})} \int_{\mathbb{R}^3} U^2_{a_j}(x) \left( \frac{x - x^{(1)}_{j,\varepsilon}}{\varepsilon} \right) \left( U_{a_j} \left( \frac{\xi - x^{(1)}_{j,\varepsilon}}{\varepsilon} \right) + U_{a_j} \left( \frac{\xi - x^{(2)}_{j,\varepsilon}}{\varepsilon} \right) \right) \eta(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
\[
F_{1,1,2} = \frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x^{(1)}_{j,\varepsilon})} \int_{\mathbb{R}^3} U^2_{a_j}(x) \left( \frac{x - x^{(1)}_{j,\varepsilon}}{\varepsilon} \right) \left( w^{(1)}_{x}(\xi) + w^{(2)}_{x}(\xi) \right) \eta(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,
\]
\[
F_{1,1,3} = \frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x^{(1)}_{j,\varepsilon})} \int_{\mathbb{R}^3} U^2_{a_j}(x) \left( \frac{x - x^{(1)}_{j,\varepsilon}}{\varepsilon} \right) \left( w^{(1)}_{x}(\xi) + w^{(2)}_{x}(\xi) \right) \eta(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx.
\]
Now, by (A.10), we get
\[ F_{1,1,1} = G_1 + O(e^{-\eta/\varepsilon}), \]
(E.3)
and
\[ F_{1,1,2} = \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) \left(w_{a,j}^{(1)}(x) + w_{i,j}^{(2)}(x)\right) \eta_{r}(x) \frac{x^i - \xi^i}{|x-\xi|^3} d\xi dx + O(e^{-\eta/\varepsilon}). \]
(E.4)
Next, using Proposition 5.1 we can calculate that, for \( l \neq j \),
\[ \frac{1}{8\pi \varepsilon^2} \int_{B_{\delta}(x_{j,\varepsilon}^{(1)})} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) U_{a_i}(\frac{\xi - x_{i,\varepsilon}^{(1)}}{\varepsilon}) \eta_{r}(x) \frac{x^i - \xi^i}{|x-\xi|^3} d\xi dx \]
\[ = \frac{\varepsilon^2}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) U_{a_i}(\xi) \eta_{r}(\xi) \frac{x^i - \xi^i}{|x-\xi|^3} d\xi dx + O(e^{-\eta/\varepsilon}) \]
\[ = \frac{\varepsilon^2}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) U_{a_i}(\xi) \left( \sum_{m=1}^{3} d_{m,l} \frac{\partial U_{a_l}(\xi)}{\partial \xi_m} \right) \frac{x^i - \xi^i}{|x-\xi|^3} d\xi dx \]
\[ + o(\varepsilon^2) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) U_{a_i}(\xi) \frac{x^i - \xi^i}{|x-\xi|^3} d\xi dx + O(e^{-\eta/\varepsilon}) \]
\[ = \frac{\varepsilon^2}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) U_{a_i}(\xi) \left( \sum_{m=1}^{3} d_{m,l} \frac{\partial x^i - \xi^i}{|x-\xi|^3} \right) d\xi dx + o(\varepsilon^4), \]
between (E.5) and (E.6), we have
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) U_{a_i}(\xi) (\xi-x_{i,\varepsilon}^{(1)}) U_{a_i}(\xi) |x-\xi|^{-3} d\xi dx = O(\varepsilon^3), \]
here we also use the following estimate, which can be found by (A.9),
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) U_{a_i}(\xi) |x-\xi|^\alpha d\xi dx = O(\varepsilon^\alpha), \]
for \( \alpha > 0 \), and \( l \neq j \).
Similar to (E.5) and (E.6), we have
\[ \frac{1}{8\pi \varepsilon^2} \int_{B_{\delta}(x_{j,\varepsilon}^{(1)})} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) U_{a_i}(\xi-x_{i,\varepsilon}^{(2)}) \eta_{r}(\xi) \frac{x^i - \xi^i}{|x-\xi|^3} d\xi dx = o(\varepsilon^4). \]
(E.7)
Then (E.5), (E.6) and (E.7) imply
\[ F_{1,1,3} = o(\varepsilon^4). \]
(E.8)
Then (E.1) can be deduced by (E.2), (E.3), (E.4) and (E.8).

Lemma E.2. It holds
\[ F_{1,2} = \frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x-x_{j,\varepsilon}^{(1)}) w_{e_{j,\varepsilon}^{(1)}}(x) U_{a_j}(\frac{\xi-x_{j,\varepsilon}^{(1)}}{\varepsilon}) \eta_{r}(\xi) \frac{x^i - \xi^i}{|x-\xi|^3} d\xi dx + o(\varepsilon^4). \]
(E.9)
Proof. First, we write $F_{1,2}$ as follows:

$$F_{1,2} = F_{1,2,1} + F_{1,2,2} + F_{1,2,3},$$  \(\text{(E.10)}\)

where

$$F_{1,2,1} = \frac{1}{4\pi \varepsilon^2} \int_{B_{\varepsilon}(x^{(1)})} \int_{\mathbb{R}^3} U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon}) w_{\varepsilon}^{(1)}(x) \left(U_{a_j}(\frac{\xi-x_j^{(1)}}{\varepsilon}) + U_{a_j}(\frac{\xi-x_j^{(2)}}{\varepsilon})\right) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,$$

$$F_{1,2,2} = \frac{1}{4\pi \varepsilon^2} \int_{B_{\varepsilon}(x^{(1)})} \int_{\mathbb{R}^3} U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon}) w_{\varepsilon}^{(1)}(x) \left(W_{j,\varepsilon}^{(1)}(\xi) + W_{j,\varepsilon}^{(2)}(\xi)\right) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,$$

$$F_{1,2,3} = \frac{1}{4\pi \varepsilon^2} \int_{B_{\varepsilon}(x^{(1)})} \int_{\mathbb{R}^3} U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon}) w_{\varepsilon}^{(1)}(x) \left(w_{\varepsilon}^{(1)}(\xi) + w_{\varepsilon}^{(2)}(\xi)\right) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx.$$

Now by direct calculation, we get

$$U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon}) - U_{a_j}(\frac{x-x_j^{(2)}}{\varepsilon}) = O\left(\frac{x_j^{(1)} - x_j^{(2)}}{\varepsilon} \cdot |\nabla U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon})|\right)$$

$$= O\left(\frac{|x_j^{(1)} - x_j^{(2)}|}{\varepsilon} U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon})\right).$$  \(\text{(E.11)}\)

Then by \(\text{(A.10)}, \text{(E.11)}\), we have

$$F_{1,2,1} = \frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon}) w_{\varepsilon}^{(1)}(x) U_{a_j}(\frac{\xi-x_j^{(2)}}{\varepsilon}) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + o(\varepsilon^4).$$  \(\text{(E.12)}\)

Also, similar to \(\text{(D.15)}\), we get

$$F_{1,2,2} = O(\varepsilon^{-2} ||U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon})||_\varepsilon \cdot ||w_{\varepsilon}^{(1)}||_\varepsilon \cdot ||W_{j,\varepsilon}(\cdot)| + W_{j,\varepsilon}(\cdot)||_\varepsilon \cdot ||\eta \varepsilon||_\varepsilon) + O(\varepsilon^{-\eta/\varepsilon}) = O(\varepsilon^6).$$  \(\text{(E.13)}\)

And by \(\text{(A.5)}\) and \(\text{(C.13)}\), we obtain

$$F_{1,2,3} = O(\varepsilon^{-4} ||U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon})||_\varepsilon \cdot ||w_{\varepsilon}^{(1)}||_\varepsilon \cdot ||w_{\varepsilon}^{(1)} + w_{\varepsilon}^{(2)}||_\varepsilon \cdot ||\eta \varepsilon||_\varepsilon) = O(\varepsilon^6).$$  \(\text{(E.14)}\)

Then \(\text{(E.9)}\) can be deduced by \(\text{(E.10)}, \text{(E.12)}, \text{(E.13)}\) and \(\text{(E.14)}\). \(\square\)

**Lemma E.3.** It holds

$$F_{2,1} = G_2 - \frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon}) w_{\varepsilon}^{(1)}(x) U_{a_j}(\frac{\xi-x_j^{(1)}}{\varepsilon}) \eta_\varepsilon(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + o(\varepsilon^4),$$  \(\text{(E.15)}\)

where

$$G_2 = -\frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(\frac{x-x_j^{(2)}}{\varepsilon}) \left(U_{a_j}(\frac{\xi-x_j^{(1)}}{\varepsilon}) + U_{a_j}(\frac{\xi-x_j^{(2)}}{\varepsilon})\right) \eta_\varepsilon(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx.$$

**Proof.** First, we write $F_{2,1}$ as follows:

$$F_{2,1} = F_{2,1,1} + F_{2,1,2} + F_{2,1,3} + F_{2,1,4} + F_{2,1,5} + F_{2,1,6},$$  \(\text{(E.16)}\)

where

$$F_{2,1,1} = \frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x^{(1)})} \int_{\mathbb{R}^3} \left(U_{a_j}(\frac{x-x_j^{(1)}}{\varepsilon}) + U_{a_j}(\frac{x-x_j^{(2)}}{\varepsilon})\right) \eta_\varepsilon(x) U_{a_j}(\frac{\xi-x_j^{(2)}}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx,$$
\[
\begin{align*}
F_{2,1,2} & = \frac{1}{4\pi \varepsilon^2} \int_{B_{\varepsilon}(x_j, \varepsilon)} \int_{\mathbb{R}^3} (U_{a_j}(\frac{x - x_j}{\varepsilon}) + U_{a_j}(\frac{\xi - x_j}{\varepsilon})) \eta_{x}(x) U_{a_j}(\frac{\xi - x_j}{\varepsilon}) w_{\varepsilon}^{(2)}(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \\
F_{2,1,3} & = \frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x_j, \varepsilon)} \int_{\mathbb{R}^3} (U_{a_j}(\frac{x - x_j}{\varepsilon}) + U_{a_j}(\frac{\xi - x_j}{\varepsilon})) \eta_{x}(x) (w_{\varepsilon}^{(2)}(\xi))^2 \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \\
F_{2,1,4} & = \frac{1}{4\pi \varepsilon^2} \int_{B_{\varepsilon}(x_j, \varepsilon)} \int_{\mathbb{R}^3} (U_{a_j}(\frac{x - x_j}{\varepsilon}) + U_{a_j}(\frac{\xi - x_j}{\varepsilon})) \eta_{x}(x) W_{j,\varepsilon}^{(2)}(\xi) U_{a_j}(\frac{\xi - x_j}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \\
F_{2,1,5} & = \frac{1}{4\pi \varepsilon^2} \int_{B_{\varepsilon}(x_j, \varepsilon)} \int_{\mathbb{R}^3} (U_{a_j}(\frac{x - x_j}{\varepsilon}) + U_{a_j}(\frac{\xi - x_j}{\varepsilon})) \eta_{x}(x) w_{\varepsilon}^{(2)}(\xi) U_{a_j}(\frac{\xi - x_j}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx, \\
F_{2,1,6} & = \frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x_j, \varepsilon)} \int_{\mathbb{R}^3} (U_{a_j}(\frac{x - x_j}{\varepsilon}) + U_{a_j}(\frac{\xi - x_j}{\varepsilon})) \eta_{x}(x) (W_{j,\varepsilon}^{(2)}(\xi))^2 \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx.
\end{align*}
\]

Now, by \([A.10]\) and symmetry, we have 
\[
F_{2,1,1} = G_2 + O(e^{-\eta/\varepsilon}).
\]  

Also similar to \([E.12]\), we know 
\[
\begin{align*}
F_{2,1,2} & = -\frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(\frac{x - x_j}{\varepsilon}) w_{\varepsilon}^{(2)}(x) U_{a_j}(\frac{\xi - x_j}{\varepsilon}) \eta_{x}(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + o(\varepsilon^4) \\
& = -\frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(\frac{x - x_j}{\varepsilon}) w_{\varepsilon}^{(1)}(x) U_{a_j}(\frac{\xi - x_j}{\varepsilon}) \eta_{x}(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O\left(||U_{a_j}(\frac{\cdot - x_j}{\varepsilon})||_2 ||w_{\varepsilon}^{(1)} - w_{\varepsilon}^{(2)}||_2 ||\eta_{x}||_2\right) + o(\varepsilon^4) \\
& = -\frac{1}{2\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(\frac{x - x_j}{\varepsilon}) w_{\varepsilon}^{(1)}(x) U_{a_j}(\frac{\xi - x_j}{\varepsilon}) \eta_{x}(x) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + o(\varepsilon^4).
\end{align*}
\]  

Next, by \([A.5]\) and \([C.13]\), we get 
\[
F_{2,1,3} = O(\varepsilon^{-4} ||U_{a_j}(\frac{\cdot - x_j}{\varepsilon}) + U_{a_j}(\frac{\cdot - x_j}{\varepsilon})||_2 ||\eta_{x}||_2 ||w_{\varepsilon}^{(2)}||_2^2) = O(\varepsilon^6).
\]  

And by \([D.6]\), we obtain 
\[
F_{2,1,4} = O(e^{-\eta/\varepsilon}) \text{ and } F_{2,1,5} = O(\varepsilon^6).
\]  

Also, \(l \neq j\), similar to \([E.5]\) and \([E.6]\), we have 
\[
\begin{align*}
\frac{1}{8\pi \varepsilon^2} \int_{B_{\varepsilon}(x_j, \varepsilon)} \int_{\mathbb{R}^3} U_{a_j}(\frac{x - x_j}{\varepsilon}) \eta_{x}(x) U_{a_1}(\frac{\xi - x_j}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx \\
= \frac{\varepsilon^2}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}(x) \eta_{x}(x) U_{a_1}(\xi + \frac{x_j - x_j}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} d\xi dx + O(e^{-\eta/\varepsilon}).
\end{align*}
\]
and
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{aj}(x) \eta_{j,\varepsilon}(x) U_{aj}^2(\xi + \frac{x_{j,\varepsilon}^{(1)} - x_{l,\varepsilon}^{(2)}}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx \\
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{aj}(x) \left( \sum_{m=1}^3 d_{m,j} \frac{\partial U_{aj}(x)}{\partial x_m} \right) U_{aj}^2(\xi + \frac{x_{j,\varepsilon}^{(1)} - x_{l,\varepsilon}^{(2)}}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx \\
+ o(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{aj}(x) U_{aj}^2(\xi + \frac{x_{j,\varepsilon}^{(1)} - x_{l,\varepsilon}^{(2)}}{\varepsilon}) \frac{1}{|x - \xi|^2} \, d\xi \, dx)
\]
\begin{equation}
= O\left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{aj}(x) U_{aj}^2(\xi + \frac{x_{j,\varepsilon}^{(1)} - x_{l,\varepsilon}^{(2)}}{\varepsilon}) |x - \xi|^{-3} \, d\xi \, dx \right) + o(\varepsilon^2)
\begin{equation}
= O(\varepsilon^3) + o(\varepsilon^2) = o(\varepsilon^2)
\end{equation}
\end{equation}

Similar to \((E.21)\) and \((E.22)\), we obtain
\begin{equation}
\frac{1}{8\pi \varepsilon^2} \int_{B_3(x, j, \varepsilon)} \int_{\mathbb{R}^3} U_{aj}(x - \frac{x_{j,\varepsilon}^{(2)}}{\varepsilon}) \eta_{j,\varepsilon}(x) U_{aj}^2(\xi - \frac{x_{j,\varepsilon}^{(2)}}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx = o(\varepsilon^4).
\end{equation}

Then \((E.21)\), \((E.22)\) and \((E.23)\) imply
\begin{equation}
F_{2,1,6} = o(\varepsilon^4).
\end{equation}

Then \((E.16)\), \((E.17)\), \((E.18)\), \((E.19)\), \((E.20)\) and \((E.21)\) imply \((E.15)\). \(\square\)

**Lemma E.4.** It holds
\begin{equation}
F_{2,3} = -\frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{aj}(x) \eta_{j,\varepsilon}(x) U_{aj}^2(\xi - \frac{x_{j,\varepsilon}^{(2)}}{\varepsilon}) (w_{\varepsilon}^{(1)}(\xi) + w_{\varepsilon}^{(2)}(\xi)) \eta_{j,\varepsilon}(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx + o(\varepsilon^4).
\end{equation}

**Proof.** First, we write \(F_{2,3}\) as follows:
\begin{equation}
F_{2,3} = F_{2,3,1} + F_{2,3,2} + F_{2,3,3} + F_{2,3,4} + F_{2,3,5} + F_{2,3,6},
\end{equation}
where
\begin{align*}
F_{2,3,1} & = \frac{1}{8\pi \varepsilon^2} \int_{B_3(x, j, \varepsilon)} \int_{\mathbb{R}^3} (w_{\varepsilon}^{(1)}(x) + w_{\varepsilon}^{(2)}(x)) \eta_{j,\varepsilon}(x) U_{aj}^2(\xi - \frac{x_{j,\varepsilon}^{(2)}}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx, \\
F_{2,3,2} & = \frac{1}{4\pi \varepsilon^2} \int_{B_3(x, j, \varepsilon)} \int_{\mathbb{R}^3} (w_{\varepsilon}^{(1)}(x) + w_{\varepsilon}^{(2)}(x)) \eta_{j,\varepsilon}(x) U_{aj}(\xi - \frac{x_{j,\varepsilon}^{(2)}}{\varepsilon}) w_{\varepsilon}^{(2)}(\xi) \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx, \\
F_{2,3,3} & = \frac{1}{8\pi \varepsilon^2} \int_{B_3(x, j, \varepsilon)} \int_{\mathbb{R}^3} (w_{\varepsilon}^{(1)}(x) + w_{\varepsilon}^{(2)}(x)) \eta_{j,\varepsilon}(x) (w_{\varepsilon}^{(2)}(\xi))^2 \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx, \\
F_{2,3,4} & = \frac{1}{4\pi \varepsilon^2} \int_{B_3(x, j, \varepsilon)} \int_{\mathbb{R}^3} (w_{\varepsilon}^{(1)}(x) + w_{\varepsilon}^{(2)}(x)) \eta_{j,\varepsilon}(x) W_{j,\varepsilon}^{(2)}(\xi) U_{aj}(\xi - \frac{x_{j,\varepsilon}^{(2)}}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx, \\
F_{2,3,5} & = \frac{1}{4\pi \varepsilon^2} \int_{B_3(x, j, \varepsilon)} \int_{\mathbb{R}^3} (w_{\varepsilon}^{(1)}(x) + w_{\varepsilon}^{(2)}(x)) \eta_{j,\varepsilon}(x) w_{\varepsilon}^{(2)}(\xi) U_{aj}(\xi - \frac{x_{j,\varepsilon}^{(2)}}{\varepsilon}) \frac{x^i - \xi^i}{|x - \xi|^3} \, d\xi \, dx,
\end{align*}
\[ F_{2,3,6} = \frac{1}{8\pi \varepsilon^2} \int_{B_1(\varepsilon \xi_j, \varepsilon)} \int_{\mathbb{R}^3} \left( w_\varepsilon^{(1)}(x) + w_\varepsilon^{(2)}(x) \right) \eta_\varepsilon(x) \left( W_{j,\varepsilon}^{(2)}(\xi) \right) x^i - \xi^i \| x - \xi \|^3 \, dx. \]

Now by (A.10), (E.11) and symmetry, we have

\[ F_{2,3,1} = \frac{1}{8\pi \varepsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} U_{a_j}^2 \left( \frac{x - x_{\varepsilon,j}}{\varepsilon} \right) \left( w_\varepsilon^{(1)}(\xi) + w_\varepsilon^{(2)}(\xi) \right) \eta_\varepsilon(\xi) x^i - \xi^i \| x - \xi \|^3 \, dx \]

Next, by (A.5) and (C.13), we get

\[ F_{2,3,2} = O(\varepsilon^{-4} \| w_\varepsilon^{(1)} \|_\varepsilon \| w_\varepsilon^{(2)} \|_\varepsilon \| \eta_\varepsilon \|_\varepsilon \| w_\varepsilon^{(2)} \|_\varepsilon^2) = O(\varepsilon^6), \]

and

\[ F_{2,3,3} = O(\varepsilon^{-4} \| w_\varepsilon^{(1)} \|_\varepsilon \| w_\varepsilon^{(2)} \|_\varepsilon \| \eta_\varepsilon \|_\varepsilon \| w_\varepsilon^{(2)} \|_\varepsilon^2) = O(\varepsilon^8). \]

Also by (D.6), we know

\[ F_{2,3,4} = O(\varepsilon^{-\eta/\varepsilon}) \text{ and } F_{2,3,5} = O(\varepsilon^6). \]

Next, similar to (D.15), we obtain

\[ F_{2,3,6} = O(\varepsilon^{-2} \| w_\varepsilon^{(1)} \|_\varepsilon \| w_\varepsilon^{(2)} \|_\varepsilon \| \eta_\varepsilon \|_\varepsilon ) + O(\varepsilon^{-\eta/\varepsilon}) = O(\varepsilon^6). \]

Then (E.26), (E.27), (E.28), (E.29), (E.30) and (E.31) imply (E.25). \( \square \)

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(Peng Luo) School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, China
E-mail address: pluo@mail.ccnu.edu.cn

(Shuangjie Peng) School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, China
E-mail address: sjpeng@mail.ccnu.edu.cn

(Chunhua Wang) School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, China
E-mail address: chunhuawang@mail.ccnu.edu.cn