O(N)–Universality Classes and the Mermin-Wagner Theorem

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We study how universality classes of O(N)–symmetric models depend continuously on the dimension $d$ and the number of field components $N$. We observe, from a renormalization group perspective, how the implications of the Mermin-Wagner-Hohenberg theorem set in as we gradually deform theory space towards $d = 2$. For dimension in the range $2 < d < 3$ we find, for any $N \geq 1$, a finite family of multi-critical effective potentials of increasing order. Apart for the $N = 1$ case, these disappear in $d = 2$ consistently with the Mermin-Wagner-Hohenberg theorem. Finally, we study $O(N = 0)$–universality classes and find an infinite family of these in two dimensions.

Introduction. Our modern understanding of quantum or statistical field theory is based on the ideas put forward by K. Wilson and formalized within the framework of the renormalization group (RG) \cite{1}. This approach considers all possible theories describing the quantum or statistical fluctuations of a given set of degrees of freedom, the fields, subject only to the constraints imposed by symmetry and dimensionality; this defines what we call theory space. The process of quantization on one side, or averaging on the other, is then seen as a trajectory connecting the bare action or Hamiltonian to the full quantum or statistical effective action. This trajectory can be of finite or infinite length (with respect to the RG time); in the first case one is performing an effective field theory calculation, while in the second case one needs an ending point for the trajectory: this usually is a fixed-point. RG fixed-points describe scale invariant theories, where fluctuations on all length scales are equally important; these theories, like lighthouses, shed light on the structure of theory space. They attract or repel surrounding theories giving rise to universality, a phenomenon that underlies both non-perturbative renormalization and the understanding of continuous phase transitions \cite{2}. Once all fixed-points are known we can reconstruct the general (topological) properties of the RG flow and acquire a deep understanding of a given class of models. A paradigmatic example of this is the $c$-theorem \cite{3}, which describes the RG flow between two dimensional theories.

Important information about two dimensional theories comes from exact results for particular lattice models; still, our ability to predict the universal features of two dimensional continuous phase transitions resides on our understanding of the structure of theory space. Three dimensional systems are much more difficult to treat exactly; here too, many analytical insights come from the RG study, otherwise one would have to resort to numerical methods. Deep insights, such as the role played by conformal symmetry in constraining statistical fluctuations, are also naturally embedded in the larger framework of RG analysis \cite{3}.

In this Letter we show how another fundamental and broad result like the Mermin-Wagner-Hohenberg theorem \cite{4,5}, which states that there cannot be continuous phase transitions in $d = 2$ systems characterized by continuous symmetries, fits in the RG picture. We will do this by studying scalar $O(N)$–models, a class of theories that has many applications to low dimensional systems; they can describe long polymer chains ($N = 0$), liquid-vapor ($N = 1$), superfluid helium ($N = 2$), ferromagnetic ($N = 3$) and QCD chiral ($N = 4$) phase transitions \cite{6,7}. Despite their relevance, there is no complete description of how universality classes of $O(N)$–models depend continuously on both $d$ and $N$. In this Letter we give such a description by studying scaling solutions of the effective average action \cite{6}. As a result we find many new $N \geq 2$ universality classes describing multi-critical models in fractal dimension $2 < d \leq 3$. In the $N = 0$ case we observe an infinite number of fixed-points in $d = 2$, analogue to the $N = 1$ minimal–models \cite{8}.

Flow equations. The effective average action (EAA) \[ \Gamma_k[\phi] \] is a functional that depends on the infrared scale $k$ and that interpolates smoothly between the bare action for $k \to \infty$ and the standard effective action for $k \to 0$ \cite{6}. The EAA satisfies an exact RG equation \cite{9} that describes its dependence upon changes of scale; this equation can be used to set up a framework where to concretely implement the RG ideas discussed above. It is generally quite difficult to follow exactly the flow of the EAA and to find the relative fixed-point functions: approximations are needed. One that retains important information about the structure of theory space is the one where all one-particle-irreducible (1PI) vertices of the EAA are evaluated at zero momenta. This defines the running effective potential \[ U_k(\rho) \] which is a function of the $O(N)$–invariant \[ \rho = \frac{1}{2} \phi^2 \]. In this approximation theory space is represented by the functional space of effective potentials. This space is still infinite dimensional and, at least at the qualitative level, $O(N)$–universality classes of the full theory can be found by determining the relative scaling solutions.

In terms of the running dimensionless effective potential \[ \bar{U}_k(\bar{\rho}) = k^{-2\eta/\nu} U_k(\rho) \], with \[ \bar{\rho} = k^{-(d-2\eta/\nu)} \rho \], a scaling solution \[ \partial_t \bar{U}_k(\bar{\rho}) = 0 \] satisfies the following ordinary differential equation \cite{6}:

\[
-(d - 2 + \eta) \bar{\rho} \frac{d \bar{U}_k}{d \bar{U}_k} + d \bar{U}_k = c_d (N - 1) \frac{1 - \frac{\eta}{d - 2}}{1 + \bar{U}_k^\nu} + c_d \frac{1 - \frac{\eta}{d - 2}}{1 + \bar{U}_k^\nu + 2\bar{\rho} \bar{U}_k^{\nu - 1}}.
\]

where \( c_d^{-1} = (4\pi)^{d/2} \Gamma(d/2 + 1) \). The anomalous dimension \( \eta \) fixes the scaling properties of the field at a particular fixed-point; to lowest order its value is related to the
proposed in [3]. For every $i$,$\eta_i$ (which we label by $\eta$) these it is possible to obtain the anomalous dimension $\eta$:

$$\eta = \frac{4\tilde{\rho}_0 \tilde{U}''(\tilde{\rho}_0)}{[1 + 2\tilde{\rho}_0 \tilde{U}''(\tilde{\rho}_0)]^2},$$

with $\tilde{\rho}_0$ the absolute minimum $\tilde{U}'(\tilde{\rho}_0) = 0$.

Every scaling solution, together with its domain of attraction, represents a different universality class; thus by finding the solutions of the system composed of (1) and (2) one can determine $O(N)$–universality classes. Differently from other implementations of the RG, all the analysis can be made leaving $d$ and $N$ as free parameters, permitting us to study how theory space depends on these.

Mermin-Wagner-Hohenberg theorem. We solve the fixed-point equations (1) and (2) by the iterative method proposed in [3]. For every $d$ and $N$ we find a discrete set of scaling solutions to these equations. These correspond to multi-critical potentials of increasing order with $i$ minima (which we label by $i$), which are potentials describing multi-critical transitions, in which one needs to tune multiple parameters to reach the critical point. For each of these it is possible to obtain the anomalous dimension $\eta_i$ as a function of $d$ and $N$. By studying the function $\eta_i(d, N)$ we can follow the evolution through theory space of the fixed-point representing the $i$-th multi-critical potential.

For $d > 4$ we find only the Gaussian fixed-point ($i = 1$); at $d = 4$, the upper critical dimension for $O(N)$–models, the Wilson-Fisher fixed-points ($i = 2$) start to branch away from the Gaussian fixed-point. In $d = 3$ these fixed-points describe the known universality classes of the Ising, XY, Heisenberg and other models; our estimates for the anomalous dimensions turn out to be in good agreement with estimates available in the literature [5, 10]. Approaching $d = 2$ one clearly observes that only the $N = 1$ anomalous dimension continues to grow [11]; for all other values of $N \geq 2$ the anomalous dimension bends downward to become zero exactly when $d = 2$. This is a non-trivial fact, not evident from the structure of equation (1), telling us that only the $O(N)$–model with discrete symmetry ($N = 1$) can have a second-order phase transition in two dimensions, while all the $O(N)$–models with continuous symmetry ($N \geq 2$) cannot. This result, that here emerges solely from the RG analysis, is commonly known as the Mermin-Wagner-Hohenberg (MWH) theorem [4]. In this respect Figure 1 shows the way in which the MWH theorem manifests itself in the RG framework; our analysis can be seen as a RG confirmation of this important theorem and can be the starting point for a new rigorous proof of it. Note also that, as expected from the exact solution [12], the anomalous dimension tends to zero for $N \to \infty$.

That the vanishing of the anomalous dimension implies that there are no continuous phase transitions for the $N \geq 2$ models in $d = 2$ can be confirmed by the analysis of the critical exponent $\nu_2(d, N)$, which indeed blows up for $d \to 2$ and $N \geq 2$ [13]. This allows us to distinguish the Spherical model, related to the $N \to \infty$ limit, from the Gaussian model, both having $\eta = 0$. Only the $N = 1$ model has a finite $\nu_2$ in two dimensions, in all other cases $\nu_2$ diverges upon approaching $d = 2$, as in the $N \to \infty$ limit where one knows exactly that $\nu_2(d, \infty) = \frac{1}{d-2}$.

The critical case $N = 2$ is known to have a distinguished behavior [14]. In this case one can observe all the distinctive properties of the Kosterlitz-Thouless phase transition by studying the properties of the RG flow [15].

Our functions $\eta_2(d, N)$ can be compared with large-$N$ expansion analogs [16] which fail to reproduce the small $N$ region, both qualitatively ($N = 1$) and quantitatively ($N < 10$). To our knowledge, our method is the only able to give accurate theoretical estimates valid for every $d$ and $N$.

To better discriminate between theories which can undergo a continuous phase transition in $d_* = 2$ and those which cannot, we extend the analysis of scaling solutions to non-integer $N$; in particular we want to see what happens around the critical value $N_*$ of $2$. The MWH theorem tells us that at $d = d_*$ the quantity $O(d, N) = \eta_2(d, N)/\eta_2(d, 1)$ can be seen as a sort of order parameter, meaning it is zero for $N > N_*$ and non-zero.
for $N < N_* $; but it tells us nothing about its continuity in $N$. Figure 2 shows that the RG analysis can say a lot more about this. First, we see that $O(d, N)$ evolves continuously with $N$ across $N_*$; second, we see that $O(d, N)$ can be written in a scaling form around the transition point $(d_*, N_*) = (2, 2)$; in particular we can write the following scaling relation:

$$O(d_*, N_*) \sim \begin{cases} 
(\frac{N_* - N}{N_* - N})^\Theta & N \rightarrow N_*^- \\
0 & N \rightarrow N_*^+ 
\end{cases},$$

(3)

where we introduced a new scaling exponent $\Theta$. A fit from the data displayed in Figure 2 gives the estimate $\Theta \approx 0.98$ which is quite close to one. Relation (3) tells us how theory space deforms as we vary the control parameter $N$. An interesting question is if relation (3) is universal, in the sense that the value of $\Theta$ is independent of the details of the implementation of the RG procedure but rather describes an inner property of the set of theory spaces parametrized by $N$. One can make a similar reasoning by keeping $N$ fixed at $N_*$ and varying $d$ around $d_* $:

$$O(d_*, N_*) \sim \begin{cases} 
0 & d \rightarrow d_*^- \\
(\frac{d - d_*}{d_*})^\Delta & d \rightarrow d_*^+ 
\end{cases};$$

(4)

where we introduced the new scaling exponent $\Delta$ and included the information, taken from ([17], that $\eta_2$ remains zero for $N \geq N_*$ and $d \leq d_*$. A fit from the data displayed in Figure 1 gives the approximate value $\Delta \approx 1.86$. Finally, we found that equation (4) has a discrete set of solutions only when the coefficient of the first term on the lhs is negative, thus our analysis applies when $\eta > 2 - d$. This fact prevents us from performing a complete analysis in the range $1 \leq d < 2$, where indeed studies of $O(N)$-models on fractals have shown that the MWH theorem is still valid ([17]).

**Multi-critical $O(N)$-models in fractal dimension.**

When new universality classes appear by branching from the Gaussian fixed-point it is easy to determine the relative critical dimensions, since the argument based on canonical dimensions is valid. In particular, the $i$-th multi-critical scaling solution appears at the upper critical dimension $d_{c,i} = 2 + \frac{1}{\eta_{ex,i}}$ ([3]). At these dimensions we see non-trivial fixed-points branching from the Gaussian for every $N \geq 2$, corresponding to potentials with $i$ minima when expressed in terms of the variable $2\sqrt{p}$.

The critical dimensions $d_{c,i}$ accumulate at $d = 2$ and thus one may naively expect to find, for any $N$, infinitely many universality classes in two dimension. Our analysis shows instead, see Figure 3 for the cases $i = 3, 4, 5$ and $N = 1, 2, 3, 4$, that this happens only in the $N = 1$ case, where the multi-critical fixed-points approach, in the limit $d \rightarrow 2$, the fixed-points representing minimal-models ([3]). For any other $N \geq 2$ we find that, consistently with the MWH theorem, the multi-critical scaling solutions, present in the range $2 < d < 3$, are instead absent in $d = 2$. This fact is a strong check of the general validity of the MWH theorem, which our analysis indicates is also applicable to multi-critical phase transitions. On the other side, we predict the existence of a whole family of $O(N)$-universality classes in fractal dimensions between two and three. To our knowledge these universality classes are new.

**The $N \rightarrow 0$ limit.** We now study the $N \rightarrow 0$ limit, that describes the universality class of self-avoiding random walks (SAW) ([18]). Figure 4 (Top) shows $\eta_2$ as function of $N$ in the interval between $-2 \leq N \leq 2.5$ for the cases $d = 2$ and $d = 3$. The anomalous dimension is continuous in the whole range; this is an indication that the $N \rightarrow 0$ limit is well defined. Figure 4 (Top) also shows, interestingly, that both the $d = 2$ and $d = 3$ curves tend to zero as $N \rightarrow -2$ where indeed the model is know to have Gaussian critical exponents in both dimensions ([19]).

We also find multi-critical scaling solutions for $N = 0$. The interesting thing here is that these solutions survive in infinite number when $d \rightarrow 2$. A plot of the first four anomalous dimensions is shown in Figure 4 (Bottom); these are numerically very similar to those of the $N = 1$ models (see Figure 1 and 3). This similarity is expected, as one may see by inspection of Figure 4 (Top). Even if the anomalous dimension is not a relevant physical parameter in the correspondence with SAW, we can use scaling relations to relate it to the physical critical exponents $\nu$ and $\gamma$. In $d = 2$ one finds the exact values $\nu_{ex} = \frac{4}{3}$ and $\gamma_{ex} = \frac{3}{3}$ ([20], and so $\nu_{ex} = 2 - \frac{\nu}{\gamma_{ex}} = \frac{4}{3} \nu \approx 0.208$; we find $\eta_2(2, 0) = 0.232$. In $d = 3$ one finds from Monte Carlo simulations the values $\nu_{MC} = 0.587$ and $\gamma_{MC} = 1.157$ ([7], and so $\nu_{MC} = 2 - \frac{\nu_{MC}}{\gamma_{MC}} \approx 0.029$; we find $\eta_2(3, 0) = 0.04$. As we said before, we cannot extend our method to $d < 2$ to compare with exact SAW critical exponents found on fractals ([21]). In any case, our analysis suggests that there...
is a countable family of $O(N = 0)$–universality classes in two dimensions. To our knowledge these are novel and may describe multi-critical phase transitions of some polymeric system.

**Discussion and outlook.** In this Letter we studied how universality classes of scalar theories with linearly realized $O(N)$–symmetry vary continuously with the dimension $d$ and with the number of field components $N$. As we varied these parameters, we followed the evolution of RG fixed-points by studying the scaling solutions of the RG equation \[^1\]. As in \[^3\], even if all our analysis was based on the study of a simple ODE, we were able to observe a very rich behavior.

Above four dimensions, as expected, we found only the Gaussian universality class; at $d = 4$ we observed the Wilson-Fisher universality classes appear. In fractal dimension between two and three we found non-trivial fixed-points for all $N$: these are novel universality classes that can, in principle, be observed in theoretical models on fractal lattices or in real physical systems.

Approaching two dimensions we observed the RG manifestation of the MWH theorem: only the $N = 1$ universality classes survived down to $d = 2$, while all the $N \geq 2$ ones disappeared. By considering $(d, N)$ as real parameters near $(d_c, N_c) = (2, 2)$ we found that the transition described by the MWH theorem, between theories that can undergo a continuous phase transition and theories that cannot, is continuous, and that the anomalous dimension, which can be seen as analogous to the order parameter, can be written in scaling form at the critical point $(2, 2)$. Our analysis revealed how different theory spaces parametrized by $N$ are related to each other; this information gives a deep RG understanding of the MWH theorem and could be used as the starting point for an extension of it.

Finally, we studied the $N \to 0$ limit; we found that it is continuous around $N = 0$ and we observed new $O(N = 0)$–universality classes in $d = 2$. These are analogous to the universality classes of $N = 1$ minimal-models and may describe particular multi-critical transitions of polymeric systems.

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