Schwinger terms in 1+1 dimensions

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July 13, 1997

Abstract

Two different approaches - Källen’s and Brandt’s methods - for calculation of the Schwinger terms in the 1+1 dimensional Abelian and non-Abelian free current algebras are discussed. These methods are applied to calculation of the single and double commutators. The validity of the Jacobi identities is examined in 1+1 and 3+1 dimensions and in this way is given natural restriction on the regularization. It is shown that the Jacobi identity cannot be broken in 1+1 dimensions even using the regularization which fails in the 3+1 dimensional case. A connection between the Schwinger term and anomaly is shown in the simplest case of the Schwinger model.

1 Introduction

Schwinger terms [1, 2, 3], i.e. terms proportional to the derivatives of the \(\delta\)-function in commutators of currents or gauge generators, play an important role in the investigations of various field theory models [4]-[28].

They are closely related to quantum anomalies and therefore to the completeness and the self-consistency of a given quantum field theory (QFT). This is the main reason they should be considered.

In particular, it is well known that the seagull terms which are introduced to covariantize the time ordered product of current operators do not cancel the Schwinger terms in the equal time commutator (ETC) and Feynman’s conjecture fails [36].

In this paper I will present a résumé of the results which can be obtained using two different approaches for the calculations of the Schwinger terms in the 1+1 dimensional Abelian and non-Abelian free current algebras.

Roughly speaking, if the algebra of the currents or the gauge generators [1]-[28] of the quantum theory is not closed we get an indication this theory
could be the anomalous and then the whole process of its quantization is highly non-trivial [29]-[33].

There are several methods for the calculation of the ETC. Here I use Källen’s [34] and Brandt’s [35] methods. From my point of view these two methods are more elegant and simple, especially in 1+1 dimensions, than e.g. the BJL-method [37, 38]. I think the main difference lies in the logic of the problem.

In Källen’s or Brandt’s methods we start with the canonical commutation relations, then we calculate the ETC and finally the anomaly (see Section 4).

In the case of the commutator of the two currents using BJL-technique we have to calculate the vacuum expectation value (VEV) of their T-product, i.e. all relevant Feynman graphs. Only after this we are able to find the anomaly and, subsequently, the ETC. This is obviously a rather indirect procedure.

Of course I am aware of the problems of Brandt’s and Källen’s method in 3+1 dimensions. Brandt’s method fails to give reliable results once we get to the spatial limit (e.g. [35]). Problem of Källen’s method lies in the possibility of changing the order of the integration (see (2.6)). Nevertheless I believe that it is possible to improve these methods.

Another open question, which is closely related to equal-time algebras, is a violation of the Jacobi identity. Again, it is possible to use BJL [40]-[46], Källen’s or Brandt’s (point-splitting) [39] methods.

The key point of the problem consists in correct definitions of current operators and their multiplication.

In general, it would be desirable that the Jacobi identity is fulfilled, since the multiplication of well-defined operators on the Hilbert space is necessarily associative. Regularization procedures defining the current commutators and respecting the associativity (i.e. the Jacobi identity) do exist (see Sect. VII); therefore they have to be considered as preferred (correct) ones.

For compact reviews which contain similar topics see [47]-[50].

This paper is organized as follows. In Sect. II resp. III the definitions of the ETC according to Källen [34] resp. Brandt [35] are given, the single commutators are calculated using both methods and the results are used in Sect. IV for the calculation of the anomaly.

In Sect. V resp. VI the double commutators are computed using the Källen’s and Brandt’s techniques and the results are used in Sect. VII for the discussion of the failure of the Jacobi identity in 1+1 and 3+1 dimensions.

2 Källen’s approach

The technique employed in this section was introduced by Källen [34]. Let us consider the function

\[ F^{ab}_{\mu\nu}(x - y) = \langle 0 | J^a_{\mu}(x) J^b_{\nu}(y) | 0 \rangle, \]

(2.1)
where \( |0\rangle \ldots |0\rangle \) means the VEV and

\[
J_\mu^a(x) \equiv \bar{\psi}(x) \gamma_\mu \tau^a \psi(x),
\]

where \( \gamma_\mu \) and \( \tau^a \) are the Dirac and Pauli matrices respectively.

Inserting the completeness relation between the two currents in (2.1) we obtain

\[
F_{\mu\nu}^{ab}(x-y) = \sum_n (0| J^a_\mu(x)|n\rangle \langle n| J^b_\nu(y)|0\rangle
\]

\[
= \sum_n (0| J^a_\mu(0)|n\rangle \langle n| J^b_\nu(0)|0\rangle e^{-ip^0(x-y)}
\]

\[
= \int d^2 p e^{-ip(x-y)} G_{\mu\nu}^{ab}(p) \theta(p^0),
\]

where the summation runs over the states with positive energy and

\[
G_{\mu\nu}^{ab}(p) = \sum_{\delta(p^n) - p} \delta(p^n - p)|0\rangle \langle 0| J^a_\mu(0)|n\rangle \langle n| J^b_\nu(0)|0\rangle.
\]

The many-particle state \( |n\rangle \) has total momentum \( p \). \( G_{\mu\nu}^{ab}(p) \) vanishes for \( p^2 < 0 \) or \( p^0 < 0 \).

We can find \( G_{\mu\nu}^{ab}(p) \) using several methods. The simplest one is to notice that \( G_{\mu\nu}^{ab}(p) \) must be a tensor of rank two with respect to Lorentz transformations and therefore its most general form is

\[
G_{\mu\nu}^{ab}(p) = A^{ab}(p^2) g_{\mu\nu} + B^{ab}(p^2) p_\mu p_\nu
\]

\[
= (p^2 g_{\mu\nu} - p_\mu p_\nu) G_1^{ab}(p^2) + g_{\mu\nu} G_2^{ab}(p^2).
\]

If the vector current \( J_\mu \) is conserved, i.e. \( \partial_\mu J^\mu = 0 \), then \( G_2^{ab}(p^2) \) vanishes and we can write the VEV of the commutator of the two currents as

\[
\langle 0| [J^a_\mu(x), J^b_\nu(y)] |0\rangle = F_{\mu\nu}^{ab}(x-y) - F_{\nu\mu}^{ab}(y-x)
\]

\[
= \int d^2 p e^{-ip(x-y)} (p^2 g_{\mu\nu} - p_\mu p_\nu) G_1^{ab}(p^2) \varepsilon(p^0) =
\]

\[
= \int_0^\infty dM^2 \int d^2 p e^{-ip(x-y)} (M^2 g_{\mu\nu} - p_\mu p_\nu) G_1^{ab}(M^2) \varepsilon(p^0) \delta(p^2 - M^2)
\]

\[
= 2\pi \int_0^\infty dM^2 G_1^{ab}(M^2) (M^2 g_{\mu\nu} + \partial_\mu \partial_\nu) \Delta_2(x-y,M^2),
\]

where

\[
\varepsilon(p^0) = \theta(p^0) - \theta(-p^0) = \frac{p^0}{|p^0|},
\]

\[\text{(2.7)}\]

\[\text{Here we assume (and later we show) that } G_1^{ab}(p^2) = G_2^{ab}(p^2).\]
and $\Delta_2(x - y, M^2)$ is the Pauli-Jordan function in two dimensions

$$\Delta_2(x - y, M^2) = \frac{1}{2\pi} \int d^2p \ e^{-ip(x-y)} \delta(p^2 - M^2) \varepsilon(p^0), \quad (2.8)$$

with the following familiar properties

$$\begin{align*}
\partial_0^x \Delta_2(x, M^2) |_{x^0=0} &= -i\delta(x^1), \\
\Delta_2(x, M^2) |_{x^0=0} &= 0.
\end{align*} \quad (2.9)$$

Using (2.8) and (2.9) we can immediately write

$$\langle 0 | [J_a^0(x), J_b^0(y)] | 0 \rangle_{E.T.} = \langle 0 | [J_a^1(x), J_b^1(y)] | 0 \rangle_{E.T.} = 0. \quad (2.10)$$

For the combination $\mu = 0$ and $\nu = 1$ we find

$$\langle 0 | [J_a^0, J_b^1] | 0 \rangle_{E.T.} = 2\pi \int_0^\infty dM^2 \ G_{1}^{ab}(M^2) \cdot \partial_1^x \delta(x^1 - y^1). \quad (2.11)$$

Interchanging the order of the integration in (2.6) is quite legal in our case because both integrals exist.

What remains is to calculate the function $G_{1}^{ab}(p^2)$. From (2.5) one finds

$$G_{\mu \nu}^{ab}(p) = (p^2 g_{\mu \nu} - p_\mu p_\nu) \ G_1^{ab}(p^2). \quad (2.12)$$

Contracting both sides with $g^{\mu \nu}$, using (2.4) and taking into account that only the state $|n\rangle$ containing one fermion-antifermion pair (as the currents are free) contributes we get

$$G_1^{ab}(p^2) = \frac{1}{p^2} \sum_{p_1+p_2} \langle 0 | J_a^0 | n \rangle \langle n | J_b^1 | 0 \rangle$$

$$= \frac{1}{(2\pi)^2 p^2} \int d^2p_1 \ d^2p_2 \ \delta(p-p_1-p_2) \delta(p_1^2 - m^2) \delta(p_2^2 - m^2) \theta(p_1^0) \theta(p_2^0) \times$$

$$\times \ Tr\{\gamma_\mu (\not{p}_1 + m) \gamma^\mu (\not{p}_2 - m)\} \cdot Tr\{x^a x^b\}$$

$$= -\frac{m^2}{\pi^2 p^2 \sqrt{p^2 (p^2 - 4m^2)}} \theta(p^2 - 4m^2) \ Tr\{x^a x^b\}, \quad (2.13)$$

where $m$ is the fermion mass. Because of

$$m^2 \int_4m^2 a \sqrt{a(a - 4m^2)} \ da = \frac{1}{2} \quad (2.14)$$

2For the 3+1 dimensional case this property does not hold.

3This integral has evidently a singularity for $m = 0$ which compensates the factor of $m^2$; obviously, such an "$m \cdot \frac{1}{m}$ effect" is completely analogous to that arising in the dispersive derivation of the axial anomaly through a relevant imaginary part (c.f. [47]).
and the normalization
\[ \text{Tr} \{ x^a x^b \} = \frac{1}{2} \delta^{ab}, \] (2.15)
we finally get
\[ \langle 0 | [J_a^0(x), J_b^1(y)] | 0 \rangle_{E.T.} = i \frac{\delta^{ab}}{2\pi} \partial_x^1 \delta(x^1 - y^1) \] (2.16)
and in the Abelian case
\[ \langle 0 | [J_0(x), J_1(y)] | 0 \rangle_{E.T.} = i \pi \partial_x^1 \delta(x^1 - y^1). \] (2.17)

3 Brandt’s approach

We define the ETC of the two local operators \( A(x) \) and \( B(y) \) as
\[ [A(x), B(y)]_{E.T.} \equiv \lim_{\xi \to 0, \xi' \to 0} [A(x, \xi), B(y, \xi')], \] (3.1)
where \( A(x, \xi) \) and \( B(y, \xi') \) are functions of the renormalized local operators \( \psi(x) \), \( \psi(x + \xi) \) evaluated at time \( x_0 \) and
\[ A(x) = \lim_{\xi \to 0} A(x, \xi); \quad \xi^0 = 0. \] (3.2)

We define the current \( J_\Gamma(x) \) as
\[ J_\Gamma(x) \equiv \lim_{\xi \to 0} J_\Gamma(x, \xi), \] (3.3)
where
\[ J_\Gamma(x, \xi) \equiv \frac{1}{2} [\bar{\psi}(x)\Gamma\psi(x + \xi) + \bar{\psi}(x + \xi)\Gamma\psi(x)] \] (3.4)
is the point-split current and
\[ \Gamma \in \{1, \gamma^\mu, \cdots; \gamma^\mu \tau^a, \cdots\}. \] (3.5)

Using the above definition and the relation
\[ \{\psi_{\alpha a}(x), \bar{\psi}_{\beta b}^+(y)\}_{E.T.} = \delta_{\alpha \beta} \delta^{ab} \delta(x^1 - y^1) \] (3.6)
it is easy to find the useful identities
\[ [\bar{\psi}(x)\Gamma\psi(y), \bar{\psi}(z)]_{E.T.} = \bar{\psi}(x)\Gamma \gamma^0 \delta(y^1 - z^1), \] \[ [\bar{\psi}(x)\Gamma\psi(y), \bar{\psi}(z)]_{E.T.} = -\gamma^0\Gamma \psi(y) \delta(x^1 - z^1), \] \[ [\bar{\psi}(x)\Gamma_A\psi(y), \bar{\psi}(z)\Gamma_B\psi(w)]_{E.T.} = \bar{\psi}(x)\Gamma_A \gamma^0 \Gamma_B \psi(w) \delta(y^1 - z^1) - \bar{\psi}(z)\Gamma_B \gamma^0 \Gamma_A \psi(y) \delta(x^1 - w^1). \] (3.7)
Now we can directly compute the single commutator of the two currents.

$$[J_{\Gamma_A}(x, \xi), J_{\Gamma_B}(y, \xi')]_{E.T.} =$$

$$= \frac{1}{2} \left[ \bar{\psi}(x) \Gamma_{A\bar{B}} \psi(x + \xi) \delta(x^1 + \xi^1 - y^1) - \bar{\psi}(x) \Gamma_{B^0 A} \psi(x + \xi) \delta(x^1 - y^1) + \right.$$

$$\left. + \bar{\psi}(x + \xi) \Gamma_{A\bar{B}} \psi(x) \delta(x^1 - y^1) - \bar{\psi}(x + \xi) \Gamma_{B^0 A} \psi(x) \delta(x^1 + \xi^1 - y^1) \right]$$

$$= \frac{1}{2} \left[ \bar{\psi}(x) \Gamma_{A\bar{B}} \psi(x + \xi) \sum_{i=0}^{\infty} (\xi_i^1)^i \partial_1^i \delta(x^1 - y^1) - \right.$$

$$\left. - \bar{\psi}(x) \Gamma_{B^0 A} \psi(x + \xi) \delta(x^1 - y^1) + \right.$$

$$\left. + \bar{\psi}(x + \xi) \Gamma_{A\bar{B}} \psi(x) \delta(x^1 - y^1) - \right.$$

$$\left. - \bar{\psi}(x + \xi) \Gamma_{B^0 A} \psi(x) \sum_{i=0}^{\infty} (\xi_i^1)^i \partial_1^i \delta(x^1 - y^1) \right]$$

$$= \frac{1}{2} \left[ \bar{\psi}(x) \Gamma_{\{A\bar{B}\}} \psi(x + \xi) + \bar{\psi}(x + \xi) \Gamma_{\{A\bar{B}\}} \psi(x) \right] \delta(x^1 - y^1) +$$

$$+ \frac{1}{2} \left[ \bar{\psi}(x) \Gamma_{A\bar{B}} \psi(x + \xi) - \bar{\psi}(x + \xi) \Gamma_{B^0 A} \psi(x) \right] \sum_{i=1}^{\infty} (\xi_i^1)^i \partial_1^i \delta(x^1 - y^1)$$

$$= J_{\Gamma_{\{A\bar{B}\}}}(x, \xi) \delta(x^1 - y^1) +$$

$$+ \frac{1}{2} \left[ \bar{\psi}(x) \Gamma_{A\bar{B}} \psi(x + \xi) - \bar{\psi}(x + \xi) \Gamma_{B^0 A} \psi(x) \right] \sum_{i=1}^{\infty} (\xi_i^1)^i \partial_1^i \delta(x^1 - y^1),$$

(3.8)

where the following notation was used

$$\Gamma_{A\bar{B}} \equiv \Gamma_A \gamma^0 \Gamma_B,$$

(3.9)

$$\Gamma_{\{A\bar{B}\}} \equiv \Gamma_{A\bar{B}} - \Gamma_{B^0 A}$$

(3.10)

and the limit $\xi' \to 0$ was already taken. It is easy to check that the result does not depend on the order in which we take the limit procedures.

The first term in (3.8) (after taking the limit $\xi \to 0$) is, as we could expect, the operator of the current.

Assuming about the Schwinger term that it is a c-number we find its form by calculating the VEV of the second term in (3.8).

Considering that

$$\lim_{\xi^0 \to 0^+} \langle 0 | \bar{\psi}(x) \Gamma_{A\bar{B}} \psi(x + \xi) | 0 \rangle = -\frac{i}{2\pi} \text{Tr} \left\{ \Gamma_{A\bar{B}} \gamma_1 \right\} \frac{\xi^1}{(\xi^1)^2 - i \varepsilon},$$

(3.11)
we find
\[
\lim_{\xi \to 0, \xi' \to 0} [J_{\Gamma_A}(x, \xi), J_{\Gamma_B}(y, \xi')] = \frac{i}{4\pi} \text{Tr} \left\{ [\Gamma_A^0, \Gamma_B] \gamma_1 \right\} \partial_1^x \delta(x^1 - y^1) - \frac{i}{4\pi} \text{Tr} \left\{ [\Gamma_A, \Gamma_B] \gamma_1 \right\} \partial_1^x \delta(x^1 - y^1).
\]

(3.12)

Finally we get the following Schwinger terms (S.T.) for the different combinations of the matrices
\[
\begin{align*}
\Gamma_A = \gamma_0, \quad \Gamma_B = \gamma_1 & \quad \text{S.T.} = \frac{i}{\pi} \partial_1^x \delta(x^1 - y^1), \\
\Gamma_A = \Gamma_B = \gamma_{0(1)} & \quad \text{S.T.} = 0, \\
\Gamma_A = \tau^a \gamma_0, \quad \Gamma_B = \tau^b \gamma_1 & \quad \text{S.T.} = \frac{i}{2\pi} \delta^{ab} \partial_1^x \delta(x^1 - y^1), \\
\Gamma_A = \tau^a \gamma_{0(1)}, \quad \Gamma_B = \tau^b \gamma_{0(1)} & \quad \text{S.T.} = 0.
\end{align*}
\]

(3.13)

Comparing this with the previous section we observe that both approaches - Källen’s and Brandt’s - give the same results.

4 Anomaly and Schwinger terms

In this section I follow essentially \[36\].

The massless Schwinger model has on the classical level the gauge respectively the chiral symmetries from which the conservation of the vector respectively the axial vector currents results
\[
\begin{align*}
\partial_\mu J^\mu(x) &= 0, \\
\partial_\nu J^5_\nu(y) &= 0,
\end{align*}
\]

(4.1)

where
\[
J^5_\nu = \bar{\psi}(x) \gamma^\nu \gamma^5 \psi(x),
\]

(4.2)

where \(\gamma^5 = \gamma^0 \gamma^1\).

Using in these equations and the canonical commutation relations (CCR) we may formally derive identities like
\[
\begin{align*}
\partial_\mu^x \langle 0 | T J^\mu(x) J^5_\nu(y) | 0 \rangle &= 0, \\
\partial_\nu^y \langle 0 | T J^\mu(x) J^5_\nu(y) | 0 \rangle &= 0.
\end{align*}
\]

(4.3)

On the quantum level these equations - the vector respectively axial vector Ward identities (VWI resp. AWI) - cannot be satisfied simultaneously \[36\] and then we speak about the anomaly respectively the anomalous Ward identities.
To show it we introduce a covariant two-point Green function

$$G^{\mu\nu} \equiv \langle 0 | T^* J^\mu(x) J^\nu(y) | 0 \rangle. \quad (4.4)$$

The $T^*$-product is defined for two Bose operators $A(x)$ and $B(y)$ as

$$T^*(A(x)B(y)) \equiv T(A(x)B(y); n) + C(x,y;n) + c(x,y), \quad (4.5)$$

where

$$T(A(x)B(y); n) \equiv \theta([x - y] \cdot n)A(x)B(y) + \theta([y - x] \cdot n)B(y)A(x) \quad (4.6)$$

is the generalized $T$-product, $n$ is a time-like vector, $n^2 = 1$, $C(x,y;n)$ and $c(x,y)$ are the so-called contact or seagull terms.\footnote{It is easy to see that for $n = (1,0)$ we get the ordinary $T$-product.}

To find $C(x,y;n)$ we use the property that $T^*$ is independent of $n$. We get (for details see [36])

$$C(x,y;n) = C(y;n)\delta^2(x-y), \quad (4.7)$$

where

$$C(y;n) = \int dn'_\alpha S^\alpha(y,n') \quad (4.8)$$

and $S^\alpha(y,n')$ is given by

$$\delta([x - y] \cdot n)[A(x),B(y)] = S(y;n)\delta^2(x-y) + S^\alpha(y;n)P_{\alpha\beta}\partial^\beta\delta^2(x-y). \quad (4.9)$$

$P_{\alpha\beta}$ is the spacelike projection operator

$$P_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta. \quad (4.10)$$

Note that $C(y;n)$ is determined up to an arbitrary function $C_0(y)!$

In the previous sections we have got the complete formulae for the commutators and now we can rewrite them in a more compact and useful form

$$\delta([x - y] \cdot n)[J^\mu(x),J^\nu(y)] = S^{\mu\nu;\alpha}(y;n)P_{\alpha\beta}\partial^\beta\delta^2(x-y), \quad (4.11)$$

with

$$S^{\mu\nu;\alpha}(y;n) = S(y)(n^\mu g^{\nu\alpha} + n^\nu g^{\mu\alpha}). \quad (4.12)$$

For 1+1 dimensions

$$J^\mu_5 = \varepsilon^{\mu\nu} J^\nu \quad (4.13)$$

and in case of (4.4) it is easy to find that \footnote{The seagull $C(x,y;n)$ in (4.3) carries two indices $\mu,\nu$ now.}

$$C^{\mu\nu}(x,n) = S(x)n^\mu n^\nu \delta^2(x-y) + C_0^{\mu\nu}(x) \quad (4.14)$$
and we get
\[ \partial_{\mu} T^* (J^\mu(x) J_5^\nu(y)) = (C_{0 \mu}^\nu(y) \partial_\mu + S(y) \partial_\nu) \delta^2(x - y). \] (4.15)

Now it is clear that if we want to preserve VWI we have to put
\[ C_{0 \mu \nu}(y) = -g_{\mu \nu} S(y) \] (4.16)
and then AWI is broken
\[ \partial_{\mu} T^* (J^\mu(x) J_5^\nu(y)) = S(y) \varepsilon^\mu_{\nu \gamma} \partial_\gamma \delta^2(x - y). \] (4.17)

On the contrary if we try to preserve AWI then VWI will be broken.

From the last equation the connection between the anomaly and the Schwinger term is clear.

In our simple case the Schwinger term is reduced to (see (2.17), (4.11))
\[ S(x) = \frac{i}{\pi} \]
and so AWI is given by
\[ \partial_{\mu} T^* (J^\mu(x) J_5^\nu(y)) = \frac{i}{\pi} \varepsilon^\mu_{\nu \gamma} \partial_\gamma \delta^2(x - y). \] (4.18)

5 Double commutators in Källen’s approach

In the calculation of the double commutators we proceed similarly as in Section 2.

Let us consider the following double commutator
\[ [J_A(x), J_B(y)], J_C(z)], \] (5.1)
later with the specific values of \( A, B \) and \( C \):
\[ A = \gamma_1 \tau^a, \quad B = \gamma_1 \tau^b, \quad C = \gamma_1 \tau^c. \] (5.2)

We introduce the function
\[ F_{\alpha \beta \gamma}^{abc} = \sum_{n, m} \langle 0 | J_\alpha^a(x) | n \rangle \langle n | J_\beta^b(y) | m \rangle \langle m | J_\gamma^c(z) | 0 \rangle \]
\[ = \sum_{n, m} \langle 0 | J_\alpha^a(n) | J_\beta^b| m \rangle \langle m | J_\gamma^c(0) e^{-i p^\gamma (x - y) e^{-i q^\gamma (y - z)}} \]
\[ = \int d^2 p d^2 q e^{-i p(x - y) e^{-i q(y - z)} \theta(p^0) \theta(q^0) G_{\alpha \beta \gamma}^{abc}(p, q)}, \] (5.3)

9
where
\[
G_{\alpha\beta\gamma}^{abc}(p, q) = \sum_{n, m} \delta^2(p^{(n)} - p)\delta^2(q^{(m)} - q)\langle 0|J^\alpha_n|n\rangle\langle n|J^\beta_m|m\rangle\langle m|J^\gamma_0|0 \rangle \tag{5.4}
\]

\[
\epsilon^{f} = \frac{i\pi f^{abc}}{8} \int \frac{d^2r d^2s d^2u}{\pi^4} \theta(r^0)\theta(s^0)\theta(u^0) \times
\]
\[
\times \delta(r^2 - m^2)\delta(s^2 - m^2)\delta(u^2 - m^2)\delta^2(p - r - s)\delta^2(q - s - u) \times
\]
\[
\times (r_1s_0u_0 + r_0s_1u_0 + r_0s_0u_1). \tag{5.5}
\]

The abbreviation \(\epsilon^{f}\) means the effective equality, i.e. we drop terms which do not contribute to the equal time limit. With this remark in mind

\[
F_{\alpha\beta\gamma}^{abc} = \frac{i\pi f^{abc}}{8} \int d^2r d^2s d^2u e^{-ip(x-y)}e^{-iq(y-z)}\theta(p^0)\theta(q^0) \times
\]
\[
\times \int \frac{d^2r d^2s d^2u}{\pi^4} \theta(r^0)\theta(s^0)\theta(u^0) \delta(r^2 - m^2)\delta(s^2 - m^2)\delta(u^2 - m^2) \times
\]
\[
\times \delta^2(p - r - s)\delta^2(q - s - u)(r_1s_0u_0 + r_0s_1u_0 + r_0s_0u_1). \tag{5.6}
\]

Using notation
\[
[t] = \theta(r^0)\theta(s^0)\theta(u^0)\delta(r^2 - m^2)\delta(s^2 - m^2)\delta(u^2 - m^2)(r_1s_0u_0 + r_0s_1u_0 + r_0s_0u_1), \tag{5.7}
\]

we can write
\[
[J_1^\alpha(x), J_1^\beta(y), J_1^\gamma(z)] = \frac{i\pi f^{abc}}{8} \int \frac{d^2r d^2s d^2u}{\pi^4} [t] \times
\]
\[
\times \left(e^{-ir(x-y)}e^{-is(x-z)}e^{-iu(y-z)} + e^{-ir(y-x)}e^{-is(y-z)}e^{-iu(x-z)} -
\]
\[
e^{-ir(x-z)}e^{-is(y-z)}e^{-iu(x-y)} - e^{-ir(z-x)}e^{-is(z-x)}e^{-iu(y-z)} \right) =
\]
\[
= \frac{i\pi f^{abc}}{8} \int \frac{d^2r d^2s d^2u}{\pi^4} [t] \times
\]
\[
\times \left(e^{-ir(x-y)}e^{-is(x-z)}e^{-iu(y-z)} \left(\theta(r^0)\theta(s^0)\theta(u^0)\theta(r^0 + s^0)\theta(s^0 + u^0) -
\]
\[
- \theta(-r^0)\theta(s^0)\theta(u^0)\theta(-r^0 + s^0)\theta(s^0 + u^0) \right) -
\]
\[
- e^{-ir(z-x)}e^{-is(y-z)}e^{-iu(x-z)} \theta(r^0)\theta(s^0)\theta(u^0)\theta(r^0 + s^0)\theta(s^0 + u^0) -
\]
\[
- \theta(r^0)\theta(s^0)\theta(-u^0)\theta(r^0 + s^0)\theta(s^0 - u^0) \right) \tag{5.8}
\]
Here we consider two ways how to calculate the double commutators using VEV of the double commutator of the free current using a simpler formalism. The procedures of the calculation of the remaining double commutators are practically the same and we can omit them.

In addition, in the following section we derive the general form (6.1.6) of the VEV of the double commutator of the free current using a simpler formalism.

\begin{equation}
\begin{split}
&= \frac{i\pi f^{abc}}{8} \int \frac{d^2r d^2s d^2u}{\pi^4} [ts] \times \\
&\quad \times e^{-ir(x-y)} e^{-is(x-z)} e^{-iu(y-z)} \varepsilon(r^0) \theta(s^0) \theta(u^0) \theta(t^0 + s^0) - \\
&- \frac{i\pi f^{abc}}{8} \int \frac{d^2r d^2s d^2u}{\pi^4} [ts] \times \\
&\quad \times e^{-ir(x-z)} e^{-is(x-y)} e^{-iu(x-y)} \varepsilon(u^0) \theta(r^0) \theta(s^0) \theta(r^0 + s^0)
\end{split}
\end{equation}

\begin{equation}
\begin{aligned}
&= \frac{i\pi f^{abc}}{4} \int \frac{d^2x}{2\pi} e^{-ir(x-y)} \varepsilon(r^0) r_0 \delta(r^2 - m^2) \int \frac{d^2s d^2u}{\pi^3} (s_0 u_1 + s_1 u_0) \times \\
&\quad \times \theta(s^0) \theta(u^0) \theta(u^0 + s^0) \delta(s^2 - m^2) \delta(u^2 - m^2) e^{-is(x-z)} e^{-iu(y-z)} - \\
&- \frac{i\pi f^{abc}}{4} \int \frac{d^2u}{2\pi} e^{-ir(x-y)} \varepsilon(u^0) u_0 \delta(u^2 - m^2) \int \frac{d^2r d^2s}{\pi^3} (r_0 s_1 + r_1 s_0) \times \\
&\quad \times \theta(r^0) \theta(s^0) \theta(r^0 + s^0) \delta(r^2 - m^2) \delta(s^2 - m^2) e^{-ir(x-z)} e^{-is(x-y)}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
&= \frac{i f^{abc}}{2} \delta(x - y) \int \frac{d^2t}{2} (e^{-it(z-x)} - e^{-it(x-z)}) \theta(t^0) \times \\
&\quad \times \int \frac{d^2r d^2s}{\pi^3} (r_0 s_1 + r_1 s_0) \theta(r^0) \theta(s^0) \delta(r^2 - m^2) \delta(s^2 - m^2) \delta(t - r - s)
\end{aligned}
\end{equation}

\begin{equation}
= \frac{i f^{abc}}{2\pi} \delta(x^1 - y^1) \delta(y - z),
\end{equation}

where in the last equality we used \([\mathcal{A}, \mathcal{L}]\) from the Appendix. The procedures of the calculation of the remaining double commutators are practically the same and we can omit them.

In addition, in the following section we derive the general form (6.1.6) of the VEV of the double commutator of the free current using a simpler formalism.

\section{Double commutators in Brandt’s approach}

Here we consider two ways how to calculate the double commutators using Brandt’s approach. One of them (I will call it the ”w-way”) could violate the Jacobi identity as was shown by Banerjee, Rothe & Rothe \[39\].

We start from the general form of the single commutator, which we got in the previous sections

$$[J_A(x), J_B(y)]_{E.T.} = \delta(x^1 - y^1)J_{[A^0B]}(y) + S_{AB}\partial_1^y \delta(x^1 - y^1).$$  \hfill (6.1)
We define the double commutator similarly as the single one (see (3.1))
\[
[[J_A(x),J_B(y)],J_C(z)]_{E.T.} \equiv \lim_{\xi \to 0,\xi' \to 0,\xi'' \to 0} [[J_A(x,\xi),J_B(y,\xi')],J_C(z,\xi'')]_{E.T.,}
\]
where we use $A$, resp. $B$, resp. $C$ as the shorthand notation for $\Gamma_A$, resp. $\Gamma_B$, resp. $\Gamma_C$.

6.1 W-way

The "w-way" consists in the assumption that we can put the limit procedures inside the external commutator i.e.
\[
\lim_{\xi'' \to 0} [\left[ \lim_{\xi \to 0,\xi' \to 0} \left[ [J_A(x,\xi),J_B(y,\xi')]_{E.T.},J_C(z,\xi'')]_{E.T.,} \right]_{E.T.},
\]
and then use (6.1). Employing this procedure we can write
\[
[[J_A(x),J_B(y)],J_C(z)]_{E.T.} = \left[ \delta(x^1 - y^1)J_{\{A;B\}}(y) + S_{AB}\partial^1 \delta(x^1 - y^1),J_C(z) \right]_{E.T.},
\]
Assuming that $S_{AB}$ is a c-number we find
\[
[[J_A(x),J_B(y)],J_C(z)]_{E.T.} = \delta(x^1 - y^1)\delta(y^1 - z^1)J_{\{A;B;C\}}(z) - \frac{i}{4\pi}S_{AB,C}\delta(x^1 - y^1)\partial^1 \delta(y^1 - z^1),
\]
where we introduced the notation
\[
S_{AB,C} = \text{Tr} \left\{ \{ [A \circ B] \circ C \} \gamma_1 \right\}.
\] (6.1.4)

For
\[
A = \tau^a \gamma^a, \ B = \tau^b \gamma^b, \ C = \tau^c \gamma^c
\] (6.1.5)
we obtain
\[
\langle 0 | [[J_A(x),J_B(y)],J_C(z)]_{E.T.} | 0 \rangle = -\frac{1}{2\pi}f^{abc} \times
\]
\[
\times \left( g^{a1}g^{b1}g^{c1} + g^{a0}g^{b0}g^{c1} + g^{a0}g^{b1}g^{c0} + g^{a1}g^{b0}g^{c0} \right) \delta(x^1 - y^1)\partial^1 \delta(y^1 - z^1),
\] (6.1.6)
where
\[
f^{abc} = -2i\text{Tr} \left\{ [\tau^a,\tau^b] \tau^c \right\}.
\] (6.1.7)
6.2 C-way

Using (6.7) one finds

\[ [[\bar{\psi}(x)A\psi(y), \bar{\psi}(z)B\psi(w)], \bar{\psi}(u)C\psi(v)]_{E.T.} = \]

\[ = \bar{\psi}(x)ABC\psi(v)\delta(w^1 - u^1)\delta(y^1 - z^1) - \]

\[ - \bar{\psi}(u)CAB\psi(w)\delta(x^1 - v^1)\delta(y^1 - z^1) - \]

\[ - \bar{\psi}(z)BAC\psi(v)\delta(y^1 - u^1)\delta(x^1 - w^1) + \]

\[ + \bar{\psi}(u)CBA\psi(y)\delta(z^1 - v^1)\delta(x^1 - w^1), \quad (6.2.1) \]

where for the case (6.1.5)

\[ ABC = \tau^a_\tau^b_\tau^c_\gamma_\alpha_0_\gamma_0_\gamma_0, \quad CAB = \tau^c_\tau^a_\tau^b_\gamma_\alpha_0_\gamma_0_\gamma_0, \quad \ldots. \quad (6.2.2) \]

What is really important is to take one of the two internal limits as the last one, i.e. \( \xi \) or \( \xi' \), otherwise we immediately go back to the ”w-way”. Concretely in (6.2.1), we can put

\[ x = x, \quad y = x + \xi, \quad z = w = y, \quad u = v = z. \quad (6.2.3) \]

It is now straightforward but tedious to show that the result is the same as that obtained by the ”w-way” (see (6.1.3), (6.1.6)).

Actually, in the case of 1+1 dimensions it is a little bit redundant to speak about two ways because both give the same result. But in the following section we will see their possible difference.

Concluding this section we can say that both methods, i.e. Brandt’s and Källen’s, give the same results for the form of the Schwinger terms in the double commutators of the free currents.

7 Failure of Jacobi identity?

In the previous section we have got the following general formula for the double commutator

\[ [[J_A(x), J_B(y)], J_C(z)]_{E.T.} = \delta(x^1 - y^1)\delta(y^1 - z^1)J_{[AB][BC]}(z) - \]

\[ - \frac{i}{4\pi}S_{AB,C}\delta(x^1 - y^1)\partial_1^y\delta(y^1 - z^1), \quad (7.1) \]

where the form of \( S_{AB,C} \) is given by (6.1.4).
Now using the identities

\[ \delta(x^1 - y^1) \delta(y - z) J_{[A^0, B]^e; c} (z) + \]
\[ + \delta(y^1 - z^1) \delta(z - x) J_{[B^0 C]^e; a} (x) + \]
\[ + \delta(z^1 - x^1) \delta(x - y) J_{[C^0 A]^e; b} (y) = 0 , \]  
\[ (7.2) \]

\[ \delta(x^1 - y^1) \partial^y \delta(y - z^1) + \]
\[ + \delta(y^1 - z^1) \partial^z \delta(z^1 - x^1) + \]
\[ + \delta(z^1 - x^1) \partial^x \delta(x^1 - y^1) = 0 \]  
\[ (7.3) \]

and

\[ S_{A B, C} = S_{B C, A} = S_{C A, B} , \]  
\[ (7.4) \]

it is easy to see that the Jacobi identity is fulfilled.

The Jacobi identity for the free currents cannot be broken in 1+1 dimensions because both ways - "w" and "c" - leads to the same results.

As an illustration that it does not have to be always true, I mention an example given in [39] in which these two procedures do give different results.

We consider the Jacobi identity for the operators \( J^i(x) \), \( J^j(y) \), \( J^0(z) \) in 3+1 dimensions.

Because of

\[ [J^0_3(x), J^j(y)]_{E.T.} = 0 , \]  
\[ (7.5) \]

\[ [J^i(y), J^j(z)]_{E.T.} = 2i \varepsilon^{ijk} J_5 k(y) \delta^3(y - z) , \]  
\[ (7.6) \]

\[ [J^0_3(x), J^i(y)]_{E.T.} = \frac{2i}{\pi^2} \left[ \frac{1}{\xi^2} C_2 + \frac{1}{8} C_4 \nabla^2 \right] \partial^i \delta^3(x - y) , \]  
\[ (7.7) \]

then using the "w-way" we find

\[ [J^0_3(x), [J^i(y), J^j(z)]]_{E.T.} = \frac{4}{\pi^2} \varepsilon^{ijk} \left[ \frac{1}{\xi^2} C_2 + \frac{1}{8} C_4 \nabla^2 \right] \partial_k \delta^3(x - y) \delta^3(y - z) , \]  
\[ (7.8) \]

\[ [V^i(y), [J^j(z), J^0_3(x)]]_{E.T.} = 0 , \]  
\[ (7.9) \]

\[ [V^j(z), [J^0_3(x), J^i(y)]]_{E.T.} = 0 , \]  
\[ (7.10) \]  

6It means that the formula (3.12) is not valid any more, but we can simply generalize the method for the 3+1 dimensional case [35].
and the Jacobi identity is evidently broken. The constants $C_2$ and $C_4$ are given by the equations

$$\left\langle \frac{\xi^i \xi^j}{\xi^2} \right\rangle = C_2 \delta^{ij}, \quad (7.11)$$

$$\left\langle \frac{\xi^i \xi^j \xi^k \xi^l}{(\xi^2)^2} \right\rangle = C_4 (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}), \quad (7.12)$$

where $\langle \rangle$ means the average over spatial directions.

When one takes the internal limit procedures at the end of the calculation, the results for the double commutators (7.9) and (7.10) are different [39].

$$[J^i(y), [J^j(z), J^0(x)]_{\text{E.T.}} = -\frac{2}{\pi^2} \varepsilon^{ijk} \left[ \frac{2C_2}{\varepsilon^2} \delta^3(x - y) \partial_k \delta^3(y - z) + C_4 \left( \frac{1}{4} \partial_k \delta^3(y - z) \nabla^2 \delta^3(x - y) + \frac{1}{2} \partial_l \delta^3(y - z) \cdot \partial_l \partial_k \delta^3(x - y) - \partial_l \partial_l \delta^3(y - z) \cdot \partial_k \delta^3(x - y) - \frac{1}{2} \nabla^2 \delta^3(y - z) \cdot \partial_k \delta^3(x - y) + \nabla^2 \partial_k \delta^3(y - z) \cdot \delta^3(x - y) \right) \right], \quad (7.13)$$

$$[J^0(x), [J^i(y), J^j(z)]_{\text{E.T.}} = \frac{2}{\pi^2} \varepsilon^{ijk} \left[ \frac{2C_2}{\varepsilon^2} \partial_k \delta^3(x - y) \delta^3(y - z) + \frac{C_4}{6} \left( 3 \nabla^2 \partial_k \delta^3(x - y) \delta^3(y - z) + \partial_k \delta^3(x - y) \nabla^2 \delta^3(y - z) + 2 \partial_l \partial_k \delta^3(y - z) \cdot \partial_k \delta^3(x - y) - \nabla^2 (x - y) \partial_k \delta^3(y - z) - 2 \partial_l \partial_k \delta^3(x - y) \cdot \partial_k \delta^3(y - z) \right) \right], \quad (7.14)$$

and the Jacobi identity is fulfilled.

**8 Conclusion**

I considered the single and double commutators of free currents in a 1+1 dimensional quantum field theory. In order to shed more light on their properties I used two methods to perform the calculation, Källen’s and Brandt’s, which are mathematically rigorous and elegant.

Using these results I determined the anomalous axial vector Ward identity and showed the correct way of the regularization of the double commutators which does not violate the Jacobi identity.
I hope that the statement from Introduction was clarified in this paper and I just work on improvements of point-splitting (Brandt’s) method in higher dimensions.

Acknowledgements

I would like to thank Prof. Bertlmann for giving me this interesting problem and to Prof. J. Horejší for useful discussions. My thanks belong to both of the two as well as Dr. Ch. Adam for their help with completion of this article. Also I am grateful to Dr. J. Novotný, Mgr. M. Schnabl for helpful discussions and to Dr. M. Stöhr for computer help.

Appendix

Here we derive the useful identity which is needed in the calculation of the double commutator.\footnote{For the notation see Section 2.}

Firstly we consider the single commutator of the two currents $J_{\mu}^a$ and $J_{\nu}^b$

$$\langle 0 | [J_{\mu}^a(x), J_{\nu}^b(y)] | 0 \rangle = \int d^2 p \ e^{-ip(x-y)} \theta(p^0) G_{\mu\nu}^{ab} (p^2), \quad (A.1)$$

where

$$G_{\mu\nu}^{ab} (p^2) = \frac{1}{4} \int \frac{d^2r \ d^2s}{\pi^2} \theta(r^0) \theta(s^0) \delta(r^2 - m^2) \delta(s^2 - m^2) \delta^2(t - r - s) \times \Tr \{ \gamma_\mu (r + m) \gamma_\nu (s - m) \} \Tr \{ \tau^a \tau^b \}. \quad (A.2)$$

For $\mu = 0$ and $\nu = 1$ we get

$$G_{01}^{ab} (p^2) = \frac{1}{2} \int \frac{d^2r \ d^2s}{\pi^2} \theta(r^0) \theta(s^0) \delta(r^2 - m^2) \delta(s^2 - m^2) \delta^2(t - r - s) \times (r_0 s_1 + r_1 s_0) \Tr \{ \tau^a \tau^b \}. \quad (A.3)$$

Finally

$$\langle 0 | [J_{\mu}^a(x), J_{\nu}^b(y)] | 0 \rangle = \frac{1}{2} \int \frac{d^2t}{2} (e^{-it(z-x)} - e^{-it(x-z)}) \theta(t^0) \times \int \frac{d^2r \ d^2s}{\pi^3} (r_0 s_1 + r_1 s_0) \theta(r^0) \theta(s^0) \times \delta(r^2 - m^2) \delta(s^2 - m^2) \delta^2(t - r - s) \Tr \{ \tau^a \tau^b \}. \quad (A.4)$$
For the other side we have from Section 2

\( (0 | [ J_0^a (x), J_1^b (y) ] | 0 )_{E.T.} = \frac{i}{\pi} \partial_1^a \delta (x^1 - y^1) \text{Tr} \{ e^{a \gamma^b} \} \).  \hspace{1cm} (A.5)

So we obtain the relation

\[
\frac{1}{2} \int \frac{d^2 t}{2} (e^{-it(z-x)} - e^{-it(x-z)}) \theta (t^0) \int \frac{d^2 r}{\pi^3} \frac{d^2 s}{\pi^3} (r_0 s_1 + r_1 s_0) \times \\
\times \theta (r^0) \theta (s^0) \delta (r^2 - m^2) \delta (s^2 - m^2) \delta^2 (t - r - s) = \frac{i}{\pi} \partial_1^a \delta (x^1 - y^1).
\]

\hspace{1cm} (A.6)

References

[1] J. Schwinger, Phys. Rev. 82 (1951) 664.
[2] J. Schwinger, Phys. Rev. Lett. 3 (1959) 269.
[3] T. Goto and I. Imamura, Prog. Theor. Phys. 14 (1955) 196.
[4] R. Stora, Algebraic structure and topological origin of anomalies, in: Recent progress in gauge theories, 1983 Cargèse Lectures, H. Lehmann (ed.), NATO ASI series, Plenum Press, New York 1984.
[5] B. Zumino, Chiral anomalies and differential geometry, in: Relativity, groups and topology II, 1983 Les Houches Lectures, B.S. DeWitt and R. Stora (eds.), North-Holland, Amsterdam 1984.
[6] R. Bertlmann, Anomalies in quantum field theory, International Series of Monographs on Physics 91, Clarendon – Oxford University Press, 1996.
[7] J. Mickelsson, Commun. Math. Phys. 97 (1985) 361.
[8] L. Faddeev, Phys. Lett. 145B (1984) 81.
[9] L. Faddeev and S. Shatashvili, Theor. Math. Phys. 60 (1984) 770.
[10] R. Jackiw, Field theoretic investigations in current algebra, Topological investigations of quantized gauge theories, in: Current algebra and anomalies, S.B. Treiman, R. Jackiw, B. Zumino and E. Witten (eds.), p. 81 and p. 211, World Scientific, Singapore 1985.
[11] R. Jackiw, Diverse topics in theoretical and mathematical physics, World Scientific, Singapore 1995.
[12] S.-G. Jo, Phys. Rev. D35 (1987) 3179.
[13] S.-G. Jo, Nucl. Phys. B259 (1985) 616.
[14] M. Kobayashi and A. Sugamoto, Phys. Lett. 159B (1985) 315.
[15] M. Kobayashi, K. Seo and A. Sugamoto, Nucl. Phys. B273 (1986) 607.
[16] A.J. Niemi and G.W. Semenoff, Phys. Rev. Lett. 55 (1985) 927.
[17] A.J. Niemi and G.W. Semenoff, Phys. Rev. Lett. 56 (1986) 1019.
[18] A.J. Niemi and G.W. Semenoff, Nucl. Phys. B276 (1986) 173.
[19] H. Sonoda, Phys. Lett. 156B (1985) 220.
[20] H. Sonoda, Nucl. Phys. B266 (1986) 410.
[21] S. Hosono, Nucl. Phys. B300 [FS 22] (1988) 238.
[22] R. Banerjee and S. Ghosh, Z. Phys C41 (1988) 121.
[23] R. Banerjee and S. Ghosh, Phys. Lett. 220B (1989) 581.
[24] R. Banerjee and S. Ghosh, Mod. Phys. Lett A5 (1989) 855.
[25] E. Langmann and J. Mickelsson, Phys. Lett. 338B (1994) 241.
[26] L. Faddeev and S. Shatashvili, Phys. Lett. 167B (1986) 225.
[27] K. Fujikawa, Phys. Lett. B171 (1986) 424.
[28] R. Bertlmann and T. Sykora, Phys. Rev. D56 (1997) 2236.
[29] R. Jackiw and R. Rajaraman, Phys. Rev. Lett., 54, 1219 (1985).
[30] R. Rajaraman, Phys. Lett. 154B (1985) 305.
[31] H. O. Girotti, H.J. Rothe and K.D. Rothe, Phys. Rev. D, vol. 33, no. 2, 514 (1986).
[32] H. J. Rothe and K.D. Rothe, Phys. Rev. D, vol. 40, no. 2, 545 (1989).
[33] N. P. Ilieva, Int. J. Mod. Phys. A4 (1989) 4567.
[34] G. K"allen, Gradient terms in commutators of currents and fields, Lectures given at winter schools in Karpacz and Schladming, February and March 1968.
[35] R. A. Brandt, Phys. Rev. 166 (1968) 1795.
[36] D. Gross and R. Jackiw, Nucl. Phys. B14 (1969) 269.
[37] J. Bjorken, Phys. Rev. 148 (1966) 1467.
[38] K. Johnson and F. E. Low, Prog. Theor. Phys. Suppl. 37-38 (1966) 74.

[39] R. Banerjee, H.J. Rothe and K.D. Rothe, Mod. Phys. Lett. A5 (1990) 1103.

[40] D. Levy, Nucl. Phys. B282 (1987) 367.

[41] R. Banerjee, H. J. Rothe and K. D. Rothe, Mod. Phys. Lett A6 (1990) 2147.

[42] R. Banerjee and H. J. Rothe, Mod. Phys. Lett. A6 (1991) 5287.

[43] J. Kubo, Nucl. Phys. B427 (1994) 398.

[44] J. M. Pawlowski, [hep-th/9610048](http://arxiv.org/abs/hep-th/9610048).

[45] J. M. Pawlowski, [hep-th/9701020](http://arxiv.org/abs/hep-th/9701020).

[46] J. P. Muniain and J. Wudka, Phys. Rev. D55 (1997) 5341.

[47] C. Adam, R. Bertlmann and P. Hofer, La Rivista del Nuovo Cimento, vol. 16, ser. 3, no. 8, 1993.

[48] G. Kelnhofer, Consistent and covariant Schwinger terms in anomalous gauge theories, Dissertation, University of Vienna, 1991.

[49] H. Grosse, E. Langmann, Int. J. Mod. Phys A, 7 (1992) 5045.

[50] J. M. Pawlowski, Anomalien und Verletzung der Jacobiidentität in chiralnen Eichtheorien, Dissertation, University of Jena, 1994.

[51] C. Itzykson, J.-B. Zuber, Quantum Field Theory, McGraw-Hill Inc., 1980.