1. Introduction

Symmetry-breaking phase transitions are ubiquitous in condensed matter systems and in quantum field theories. There is also good reason to believe that they feature in the very early history of the Universe. At many such transitions topological defects of one kind or another are formed. Because of their inherent stability, they can have important effects on the subsequent behaviour of the system.

In the first of these lectures I shall review a number of examples of spontaneous symmetry breaking, many of which will be discussed in more detail by other lecturers, and discuss their general features. The second lecture will be mainly devoted to the conditions under which topological defects can appear and their classification in terms of homotopy groups of the underlying vacuum manifold. In my final lecture, I will discuss the ‘cosmology in the laboratory’ experiments which have been done to try to test some of the ideas thrown up by discussions of defect formation in the early Universe by looking at analogous processes in condensed-matter systems.

2. Spontaneous symmetry breaking

Often a system has symmetries that are not shared by its ground state or vacuum state. This is the phenomenon of spontaneous symmetry breaking. It always implies a degeneracy of the ground state. In this lecture, I want to discuss a number of simple examples of this phenomenon.

The key signature of spontaneous symmetry breaking is the existence of some operator \( \phi \), the order parameter (for example the magnetization \( \mathbf{M} \) in a ferromagnet) whose ground-state expectation value is not invariant. Typically, the equilibrium state at high temperature is invariant —
for example, above the Curie temperature $T_c$ the expectation value of $\hat{M}$ vanishes. Symmetry breaking appears as the system is cooled below $T_c$. It spontaneously acquires magnetization in some random direction.

2.1. FERROMAGNET

As a first example, let us think about a Heisenberg ferromagnet [1], a system of spins $\hat{S}_r$ at lattice sites $r$, with Hamiltonian

$$
\hat{H} = -\frac{1}{2} \sum_r \sum_{r'} J(|r-r'|)\hat{S}_r \cdot \hat{S}_{r'} - H \cdot \sum_r \hat{S}_r, \tag{1}
$$

where $J$, assumed positive, is non-zero only for neighbouring spins, and $H$ represents a possible external magnetic field. For non-zero $H$, the ground state $|0\rangle$ of the system has all spins aligned in the direction of $H$:

$$
\langle 0 | \hat{S}_r | 0 \rangle = \frac{1}{2} h \cdot \sigma, \quad h = \frac{H}{|H|}. \tag{2}
$$

If we take the infinite-volume limit and then let $H \to 0$, this remains true.

Clearly, with $H = 0$, (1) is invariant under all rotations. The symmetry group is $SO(3)$ or, rather, its two-fold covering group $SU(2)$. However, $\langle 0 | \hat{S}_r | 0 \rangle$ is evidently not invariant. Applying the rotation operators to $|0\rangle$ yields an infinitely degenerate set of ground states $|0_h\rangle$, labelled by the directions of the unit vector $h$. Note that the scalar product of two distinct ground states tends to zero in the infinite-volume limit:

$$
|\langle 0_h | 0_{h'} \rangle| = \cos N \frac{\theta}{2} \to 0 \quad \text{as} \quad N \to \infty, \tag{3}
$$

where $N$ is the number of lattice sites and $\theta$ is the angle between $h$ and $h'$. In fact, these two states $|0_h\rangle$ and $|0_{h'}\rangle$ belong to separate, mutually orthogonal Hilbert spaces, with unitarily inequivalent representations of the commutation relations. No operator constructed from a finite set of spins has a nonvanishing matrix element between $|0_h\rangle$ and $|0_{h'}\rangle$. This is another characteristic of spontaneously broken symmetry.

Note that physically, unless we introduce a symmetry-breaking interaction with an external magnetic field, the different ground states are indistinguishable. Indeed it is possible (though not particularly helpful) to define an invariant ground state, a uniform superposition of all $|0_h\rangle$. 
2.2. FREE BOSE GAS

Another example of spontaneous symmetry breaking is Bose-Einstein condensation. Let us first consider a free Bose gas [2], with Hamiltonian

$$\hat{H} = \sum_k \epsilon_k \hat{a}_k \hat{a}^\dagger_k, \quad (4)$$

where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are the destruction and creation operators for a particle of momentum $\hbar \mathbf{k}$, satisfying the commutation relations

$$[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'}, \quad (5)$$

and

$$\epsilon_k = \frac{\hbar^2 k^2}{2m}. \quad (6)$$

The relevant symmetry here is the phase symmetry

$$\hat{a}_k \rightarrow e^{i\vartheta} \hat{a}_k, \quad \hat{a}_k^\dagger \rightarrow e^{-i\vartheta} \hat{a}_k^\dagger, \quad (7)$$

corresponding to the existence of a conserved particle number,

$$\hat{N} = \sum_k \hat{a}_k^\dagger \hat{a}_k. \quad (8)$$

In the grand canonical ensemble, the mean occupation number of mode $k$ is given by the Bose-Einstein distribution

$$\langle \hat{a}_k^\dagger \hat{a}_k \rangle = N_k \equiv \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}, \quad (9)$$

where $\beta = 1/k_B T$ is the inverse temperature and $\mu$ the chemical potential. If the mean number density $n = \langle \hat{N} \rangle / \Omega$ (where $\Omega$ is the volume) is specified, then the value of $\mu$ for any large temperature $T$ is given by

$$n = \frac{1}{\Omega} \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \approx \frac{1}{\Omega \beta \mu}, \quad (10)$$

As $T$ falls, $\mu$ increases. Clearly, to avoid a divergence, $\mu$ can never become positive, but eventually it approaches zero at some critical temperature $T_c$. At that point, the occupation number (9) diverges at $k = 0$, so the integral approximation in (10) is no longer adequate. We have to separate the $k = 0$ mode from the rest. For all the other values of $k$, we can still make the integral approximation, and set $\mu = 0$. Thus we get

$$n = n_0 + \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\epsilon_k} - 1}, \quad \text{with} \quad n_0 \approx \frac{1}{-\Omega \beta \mu}. \quad (11)$$
For $T < T_c$, $\mu$ becomes extremely small, of order $-k_B T/n \Omega$. Thus, a macroscopically significant fraction of all the particles is to be found in the single zero-momentum mode.

Note that from (11) we can calculate the critical temperature $T_c$ at which $n_0$ first becomes large. It is given by

$$n = \frac{1}{(2\pi)^2} \left( \frac{2mk_B T_c}{\hbar^2} \right)^{3/2} \int_0^\infty \frac{x^{1/2} dx}{e^x - 1},$$

which yields

$$T_c = \frac{2\pi}{\zeta(\frac{3}{2})^{2/3}} \frac{\hbar^2 n^{2/3}}{mk_B} = 3.31 \frac{\hbar^2 n^{2/3}}{mk_B}. \quad (13)$$

2.3. DEGENERATE GROUND STATE

To see how a degenerate ground state emerges in this case, it is more convenient to work in terms of the scalar field

$$\hat{\phi}(r) = \frac{1}{\Omega^{1/2}} \sum_k \hat{a}_k e^{i k \cdot r}. \quad (14)$$

Then we can write

$$\hat{H} = \int d^3 r \frac{\hbar^2}{2m} \nabla \hat{\phi}^\dagger(r) \cdot \nabla \hat{\phi}(r). \quad (15)$$

The phase symmetry (7) can now be expressed as

$$\hat{\phi}(r) \mapsto e^{i \theta} \hat{\phi}(r), \quad \hat{\phi}^\dagger(r) \mapsto e^{-i \theta} \hat{\phi}^\dagger(r). \quad (16)$$

Now let us introduce an explicit symmetry-breaking source term playing the same role as an external magnetic field for the ferromagnet. In this case, we take

$$\hat{H}_1 = -\int_\Omega d^3 r [j^* \hat{\phi}(r) + j \hat{\phi}^\dagger(r)] = -\Omega^{1/2} (j^* \hat{a}_0 + j \hat{a}_0^\dagger), \quad (17)$$

where $j$ is a complex number, and consider the limit $j \to 0$ only after letting $\Omega \to \infty$. Here $\hat{H}_1$ may be thought of as representing the possibility of particle exchange with the environment. We then find that the density operator $\rho_0$ for the zero-momentum mode is the same as with $j = 0$ but with a shifted field:

$$\rho_0 = (1 - e^{\beta \mu}) e^{\beta \mu \hat{b}_0^\dagger \hat{b}_0}, \quad (18)$$

where

$$\hat{b}_0 = \hat{a}_0 + \frac{\Omega^{1/2} j}{\mu}. \quad (19)$$
To find $\mu$ we must again use (10) and (11) but appropriately modified, namely

$$
\Omega n_0 = \langle \hat{a}_0^{\dagger} \hat{a}_0 \rangle = \langle \hat{b}_0^{\dagger} \hat{b}_0 \rangle + \frac{\Omega |j|^2}{\mu^2} = \frac{1}{e^{-\beta \mu} - 1} + \frac{\Omega |j|^2}{\mu^2}.
$$

(20)

Clearly, as $\Omega \to \infty$, the first term on the right becomes negligible, and so

$$
\mu \approx -\frac{|j|}{n_0^{1/2}}.
$$

(21)

(We assume for simplicity that $|j| \ll \sqrt{n_0}/\beta$, so that $\mu$ remains small enough to be negligible for the $k \neq 0$ modes.) Then we find that the ground state for the zero-momentum mode is a coherent state $|\phi\rangle$, such that $\hat{b}_0 |\phi\rangle = 0$, in which the expectation value of $\hat{\phi}$ is non-zero, with a phase constrained by the phase of $j$:

$$
\langle \hat{\phi}(r) \rangle = \phi \equiv \eta e^{i\vartheta} = n_0^{1/2} \frac{j}{|j|}.
$$

(22)

The coherent states may be labelled by the value of $\phi$, and are given explicitly by

$$
|\phi\rangle = \exp[\Omega^{1/2}(\phi \hat{a}_0^{\dagger} - \phi^{*} \hat{a}_0)]|0\rangle.
$$

(23)

Once again, we have a set of degenerate ground states labelled by a phase angle $\vartheta$. None of these states has a definite particle number. However, as $\Omega \to \infty$ the uncertainty in $n_0$ tends to zero. Moreover, as before, the scalar product between states with different values of $\vartheta$ tends to zero; one finds

$$
\langle \eta e^{i\vartheta} | \eta e^{i\chi} \rangle = \exp[-\Omega \eta^2 (1 - e^{-i(\vartheta - \chi)})] \to 0 \text{ as } \Omega \to \infty.
$$

(24)

Here too, these states belong to distinct orthogonal Hilbert spaces, carrying unitarily inequivalent representations of the canonical commutation relations.

It is important to note that degenerate ground states and symmetry breaking occur only in the infinite-volume limit. In a finite volume, there is always a unique ground state, a uniform superposition of ‘ground states’ with different phases. As in the case of the ferromagnet, the limits of infinite volume and zero external field do not commute. If we keep $j$ finite and let $\Omega \to \infty$, we arrive at a state with $\langle \hat{\phi}(r) \rangle = \eta e^{i\vartheta}$. On the other hand if we let $j \to 0$ first, the particle number remains definite, and we end up with a superposition of all the degenerate ground states. Physically (unless $j \neq 0$) the states are indistinguishable.
A Bose gas with interaction is described by the Hamiltonian
\[
\hat{H} = \int_{\Omega} d^3r \frac{\hbar^2}{2m} \nabla \hat{\phi}^\dagger(r) \cdot \nabla \hat{\phi}(r) + \frac{1}{2} \int_{\Omega} d^3r \int_{\Omega} d^3r' V(|r' - r|) \hat{\phi}^\dagger(r') \hat{\phi}(r) \hat{\phi}^\dagger(r) \hat{\phi}(r').
\] (25)

which is clearly still invariant under the phase transformations (16). So long as the interaction is weak, the qualitative picture is largely unchanged. Bose-Einstein condensation has been observed in alkali metal vapours, such as rubidium [3].

2.4. LIQUID HELIUM-4

The normal-to-superfluid ‘lambda transition’ at a temperature of about 2 K in liquid $^4$He is another example of a symmetry-breaking phase transitions in a bosonic system [4]. But unlike the case of alkali metal vapours, this is a system with strong interparticle interactions that substantially change the picture. Nevertheless, there are close similarities. The broken symmetry is still the phase symmetry associated with conservation of particle number; the transition may still be described in terms of a scalar field $\hat{\phi}$ which acquires a non-zero expectation value below the transition.

The most obvious difference caused by the interatomic interactions is in the nature of the excitations above the ground state. In $^4$He at low temperature, these are collective excitations, phonons, whose dispersion relation is linear near $k = 0$:

\[
\epsilon_k = c_s \hbar |k|,
\] (26)

where $c_s$ is the sound speed. At larger values of $|k|$ the graph curves downwards to a minimum; excitations near the minimum are called rotons.

A useful description of $^4$He is provided by the Ginzburg–Landau model [4], which may also be applied to other systems. The starting point is to consider the free energy, $F$ say, as a function of the order parameter $\phi$. At least in principle, $F[\phi]$ may be calculated in the usual way from the partition function $Z$, by restricting the sum over states to those with a given expectation value of the order parameter field, $\langle \hat{\phi}(r) \rangle = \phi(r)$. In the neighbourhood of the transition temperature, $F$ can be expanded in powers of $\phi$:

\[
F[\phi] = \int_{\Omega} d^3r \frac{\hbar^2}{2m} \nabla \phi^\dagger(r) \cdot \nabla \phi(r)
+ \int_{\Omega} d^3r \left[ \alpha(T) |\phi(r)|^2 + \frac{1}{2} \beta(T) |\phi(r)|^4 + \cdots \right],
\] (27)
where the higher terms are usually unimportant, at least for a qualitative description. The coefficient $\beta(T)$ is always positive and may usually be taken to be constant. At high temperature, $\alpha(T)$ is also positive so that the minimum of the free energy occurs at $\phi = 0$. At low temperature, however, $\alpha$ becomes negative, and the minimum occurs at

$$|\phi| = \left( \frac{-\alpha(T)}{\beta(T)} \right)^{1/2}.$$  

(28)

As usual, the phase of $\phi$ is arbitrary: we have a degenerate equilibrium state.

Note that the critical temperature $T_c$ is the temperature at which $\alpha(T) = 0$. Close to that point, we may take

$$\alpha(T) \approx \alpha_1 \cdot (T - T_c).$$  

(29)

A good qualitative picture of the behaviour of $^4$He is given by the two-fluid model, normal plus superfluid. The scalar field $\phi = \langle \hat{\phi} \rangle$ describes the superfluid component, defining both the superfluid density and velocity:

$$n_s = |\phi|^2, \quad v_s = \frac{\hbar}{m} \nabla \vartheta, \quad \text{where} \quad \phi = |\phi| e^{i\vartheta}.$$  

(30)

The normal component corresponds to single-particle (or, rather, single-quasiparticle) excitations above the ground state.

2.5. SUPERCONDUCTORS

The electrons in a solid constitute a Fermi gas rather than a Bose gas. It is not single electrons that condense but bound pairs of electrons, Cooper pairs [4]. There is an effective attractive force between electrons near the Fermi surface $k^2 = k_F^2$. At least for conventional superconductors, this force preferentially binds pairs with equal and opposite momenta and spins.

Below the critical temperature, we find that in the ground state

$$\langle \hat{a}_{k\uparrow} \hat{a}_{-k\downarrow} \rangle = F(k) \neq 0, \quad \text{for} \quad k = |k| \sim k_F.$$  

(31)

The order parameter $\hat{\phi}$ in this case can be taken to be an integral over such products of pairs of destruction operators, multiplied by the internal wave function of a Cooper pair.

There is an important difference between this and the examples discussed previously. The symmetry here is again the phase symmetry (16), but it is now a local, gauge symmetry: $\vartheta$ is allowed to be a function of space
and time, $\vartheta(t, r)$. This is possible because of the coupling to the electromagnetic field $A_\mu(x)$ which transforms as

$$A_\mu(x) = A_\mu(x) - \frac{\hbar}{2e} \partial_\mu \vartheta(x).$$

The factor of $2e$ in the denominator appears because this is the charge of a Cooper pair. It ensures that the covariant derivative

$$D_\mu \hat{\phi} \equiv \partial_\mu \hat{\phi} + 2i \frac{e}{\hbar} A_\mu \hat{\phi},$$

transforms in the same way as $\hat{\phi}$ itself. The Ginzburg–Landau model may be used for superconductors too, provided that the derivatives $\nabla \phi$ in (27) are replaced by covariant derivatives $D \phi$.

Symmetry breaking in gauge theories is a somewhat problematic concept. Indeed Elitzur's theorem [5, 6] says that spontaneous breaking of a local, gauge symmetry is impossible! — which might be thought to imply that what I have just told you is nonsense. More specifically, it says that, while for a global symmetry taking the infinite-volume limit and then letting $j \to 0$ may yield a state with $\langle \hat{\phi} \rangle \neq 0$, in a gauge theory we always have

$$\lim_{j \to 0} \lim_{\Omega \to \infty} \langle \hat{\phi} \rangle = 0.$$  

(34)

But one must be careful not to misinterpret this (entirely correct) theorem. It applies only in an explicitly gauge-invariant formalism. If, as is often done, we add a gauge-fixing term that explicitly breaks the local symmetry (e.g., by imposing the Coulomb gauge condition $\nabla \cdot A = 0$) then the remaining global symmetry can be broken spontaneously. We certainly can define and use gauge-non-invariant states with $\langle \hat{\phi} \rangle \neq 0$, though there must always be an alternative (but often inconvenient) gauge-invariant description.

A model widely used as an exemplar of symmetry breaking in particle physics is the Abelian Higgs model, the relativistic version of the Ginzburg–Landau model. It is described by the action integral

$$I = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^* D^\mu \phi - \frac{1}{2} \lambda (\phi^* \phi - \eta^2)^2 \right],$$

(35)

with

$$D_\mu \phi = \partial_\mu \phi + ie A_\mu \phi, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  

(36)

Here the coupling constant $\lambda$ plays the role of $\beta$ and $-\lambda \eta^2$ that of $\alpha$. (Note that here, and in general when dealing with relativistic models, I set $c = \hbar = 1$.) Symmetry breaking in this model is very similar to that in a superconductor.
2.6. LIQUID CRYSTALS

A very different example is provided by the isotropic to nematic transition in a liquid crystal [7]. A nematic liquid crystal is typically composed of rod-shaped molecules that like to line up parallel to one another. There is no long-range translational order: the molecules are free to flow past one another. But there is long-range orientational order. At any point in the liquid there is a preferred direction, characterized by a unit vector \( \mathbf{n} \), the director. Note that \( \mathbf{n} \) and \( -\mathbf{n} \) are completely equivalent.

The symmetry group here is the rotation group \( SO(3) \). Above the transition temperature \( T_c \), the system is completely isotropic, with the molecules randomly oriented, but below it, the rotational symmetry is broken.

A convenient choice for the order parameter in this case is the average mass quadrupole tensor \( Q \) of the molecules in a small region. When the directions of the molecules are isotropically distributed, \( Q = 0 \). But if they are aligned in the direction of \( \mathbf{n} \), it has the form

\[
Q = Q(3\mathbf{n}\mathbf{n} - 1).
\]

In particular, if \( \mathbf{n} \) is in the \( z \) direction, then

\[
Q = \begin{pmatrix}
-Q & 0 & 0 \\
0 & -Q & 0 \\
0 & 0 & 2Q
\end{pmatrix}.
\]

2.7. GENERIC CASE

Let us now examine the generic situation. (For more detail, see for example [8].) Suppose the system has a symmetry group \( G \). In other words, the Hamiltonian \( \hat{H} \) is invariant under every operation \( g \in G \):

\[
\hat{U}^{-1}(g)\hat{H}\hat{U}(g) = \hat{H} \quad \text{for all} \quad g \in G,
\]

where \( \hat{U}(g) \) is the unitary operator representing the operation \( g \) on the Hilbert space.

However, we assume also that there is an operator \( \hat{\phi} \) with a nonvanishing ground-state expectation value which transforms non-trivially under \( G \). Specifically, we consider a multiplet of operators \( \hat{\phi} = (\hat{\phi}_i)_{i=1...n} \) transforming according to some \( n \)-dimensional representation \( D \) of \( G \):

\[
\hat{U}^{-1}(g)\hat{\phi}_i\hat{U}(g) = \sum_j D_{ij}(g)\hat{\phi}_j,
\]

or more concisely

\[
\hat{U}^{-1}(g)\hat{\phi}\hat{U}(g) = D(g)\hat{\phi}.
\]
We suppose that the expectation value in the ground state $|0\rangle$, 

$$
\langle 0|\hat{\phi}|0\rangle = \phi_0,
$$

say, is not invariant:

$$
\langle 0|\hat{U}^{-1}(g)\hat{\phi}\hat{U}(g)|0\rangle = D(g)\phi_0 \neq \phi_0,
$$

for some $g \in G$. Obviously this implies that the ground state $|0\rangle$ is not invariant:

$$
\hat{U}(g)|0\rangle \neq |0\rangle.
$$

But by (39), $\hat{U}(g)|0\rangle$ is also an eigenstate of $\hat{H}$ with the same eigenvalue; the ground state is degenerate.

In general, not all elements of $G$ lead to distinct ground states. There may be some subgroup $H$ of elements such that

$$
D(h)\phi_0 = \phi_0 \quad \text{for all} \quad h \in H.
$$

The distinct degenerate ground states correspond to the distinct values of $\phi = D(g)\phi_0$. Hence they are in one-to-one correspondence with the left cosets of $H$ in $G$ (sets of elements of the form $gH$). These cosets are the elements of the quotient space

$$
M = G/H.
$$

This space may be regarded as the vacuum manifold or manifold of degenerate ground states.

For example, for a Heisenberg ferromagnet, $G = SU(2)$, and $H = U(1)$, the subgroup of rotations about the direction of the magnetization vector. Here $M = SU(2)/U(1) = S^2$, a two-sphere. For a Bose gas, $G = U(1)$, and $H$ comprises the identity element only, $H = 1 \equiv \{1\}$. Thus $M = S^1$, the circle.

A nematic is a slightly less trivial example. Here $G = SO(3)$; however, $H$ is not merely the subgroup $SO(2) \equiv C_\infty$ of rotations about $\mathbf{n}$. Rather, $H$ is the infinite dihedral group, $H = D_\infty$, which includes also rotations through $\pi$ about axes perpendicular to $\mathbf{n}$. Correspondingly $M$ is not the two-sphere but the real projective space $RP^2$, obtained from $S^2$ by identification of opposite points.

2.8. HELIUM-3

Finally let me turn to the particularly interesting, and relatively complicated, case of $^3$He [9, 10]. This lighter isotope also exhibits a phase transition, though at a much lower temperature than $^4$He, between 2 and 3 mK.
It is of course a Fermi liquid. So the mechanism of superfluidity is very different, similar to that of superconductivity. In this case we have Cooper pairs not of electrons but of $^3$He atoms. The order parameter can again be constructed from pairs of destruction operators.

There is however an important difference. In the original BCS model, the pairs were bound in an isotropic $^1S$ state; as indicated by the form of (31). But for a pair of $^3$He atoms close to the Fermi surface it turns out that the most attractive state is the $^3P$. The pairs have both unit orbital and unit spin angular momenta: $L = S = 1$. We need to consider a more general form of order parameter, related to the quantity

$$ F_{ab}(k) = \langle \hat{a}_{ka} \hat{a}_{-kb} \rangle, \quad a, b = \uparrow, \downarrow. $$

The fact that $S = 1$ tells us that $F$ should be symmetric in the spin indices $a$ and $b$, so it can be expanded in terms of the three independent symmetric $2 \times 2$ matrices, $\sigma_j \sigma_2$, where $\sigma_j$ are the Pauli matrices. The fact that $L = 1$ means that $F$ should be proportional to $k$ times a function of $k = |k|$ only. Thus we can write

$$ F_{ab}(k) = F(k) A_{ij}(\sigma_i \sigma_2)_{ab} k_j, $$

where the two-index tensor $A$ may be normalized by $\text{tr}(A^\dagger A) = 1$.

The order parameter is essentially $A$ times a scalar factor representing the density of Cooper pairs. Since it is now a $3 \times 3$ complex matrix rather than a scalar, the possible patterns of symmetry breaking are much more complex. There are in fact two distinct superfluid phases, $^3$He-A and $^3$He-B, which are stable in different regions of the phase diagram; the A phase is stable only at high pressure and at temperatures not far below the critical temperature. In the presence of a magnetic field, there is a third stable phase, the A1 phase.

The system exhibits a much larger symmetry than $^4$He. To a good approximation, it is symmetric under independent orbital and spin rotations, as well as under the phase rotations as before. Thus the symmetry group is

$$ G = SO(3)_S \times SO(3)_L \times U(1) $$

(There is also a weak spin-orbit coupling, whose effects I will discuss a little later.)

In the A phase, the order parameter takes the form

$$ A_{ij} = \frac{1}{\sqrt{2}} d_i (m_j + i n_j), $$

where $d, m$ and $n$ are unit vectors, with $m \perp n$. The vector $d$ defines an axis along which the component of $S$ vanishes. If we define $l = m \wedge n$, 

$$ A_{ij} = \frac{1}{\sqrt{2}} d_i (m_j + i n_j). $$

(There is also a weak spin-orbit coupling, whose effects I will discuss a little later.)
then \( l \) is an axis along which the component of \( L \) is +1. In this case, the subgroup \( H \) that leaves \( A \) invariant comprises spin rotations about the direction of \( d \), orbital rotations about \( l \) combined with compensating phase transformations, and, finally, the discrete transformation that reverses the signs of all three vectors. Hence

\[
H_A = U(1) \times U(1) \times \mathbb{Z}_2. \tag{51}
\]

Correspondingly, the vacuum manifold is

\[
M_A = G/H_A = S^2 \times SO(3)/\mathbb{Z}_2. \tag{52}
\]

Here, the elements of \( S^2 \) label the direction of \( d \), while \( SO(3) \) describes the orientation of the orthonormal triad \((l, m, n)\). The \( \mathbb{Z}_2 \) factor represents the identification \((d, m, n) \equiv (-d, -m, -n)\).

The \( B \) phase, by contrast, is characterized by an order parameter of the form

\[
A_{ij} = R_{ij} e^{i\theta}, \tag{53}
\]

where \( R \in SO(3) \) is a real, orthogonal matrix. In this case, the only elements of \( H \) are combined orbital and spin rotations, so

\[
H_B = SO(3) \quad \text{and} \quad M_B = G/H_B = SO(3) \times S^1. \tag{54}
\]

As I mentioned earlier, there is actually a weak spin-orbit coupling term in the Hamiltonian, which is only noticeable at long range, and which reduces the symmetry to

\[
G' = SO(3)J \times U(1) \quad \text{with} \quad J = L + S. \tag{55}
\]

Note that going from \( G \) to \( G' \) is not strictly speaking a case of spontaneous symmetry breaking. There are similarities: at short range, the symmetry appears to be the larger group \( G \); when we go to long range (or low energy), we see that the symmetry group is actually \( G' \). However, the true symmetry is always \( G' \); \( G \) is only approximate.

In the \( A \) phase the effect is to require that the vectors \( d \) and \( l \) be parallel or antiparallel, and in fact by selecting one of the two configurations related by inversion, we can ensure that \( d = l \). In this case, we find

\[
H'_A = U(1), \quad M'_A = G'/H'_A = SO(3). \tag{56}
\]

In the \( B \) phase, the restriction is that \( R \) in (53) is no longer an unconstrained orthogonal matrix, but a rotation matrix through a definite angle (the Leggett angle \( \theta_L = \arccos(-\frac{1}{4}) \)) about an arbitrary axis \( n \). Thus we find

\[
H'_B = SO(2), \quad M'_B = G'/H'_B = S^2 \times S^1. \tag{57}
\]
2.9. THE STANDARD MODEL OF PARTICLE PHYSICS

There are remarkable similarities between the symmetry breaking pattern of $^3$He and that found in the standard model of particle physics which incorporates quantum chromodynamics together with the unified electroweak theory of Weinberg and Salam. It is based on the symmetry group

$$ G = SU(3)_{\text{col}} \times SU(2)_I \times U(1)_Y, $$

where $I$ and $Y$ denote respectively the weak isospin and weak hypercharge. The symmetry breaking from this down to the observed low energy symmetry is described by the Higgs field, $\hat{\phi}$, which plays the role of the order parameter. It is a two-component complex scalar field invariant under the colour group $SU(3)_{\text{col}}$, belonging to the fundamental 2-dimensional representation of $SU(2)_I$, and with non-zero weak hypercharge $Y = 1$. It acquires a vacuum expectation value of the form

$$ \langle \hat{\phi} \rangle = \phi = \begin{pmatrix} 0 \\ v \end{pmatrix}, $$

thus reducing the symmetry to the subgroup

$$ H = SU(3)_{\text{col}} \times U(1)_{\text{em}}. $$

The generator of the remaining $U(1)$ symmetry is the electromagnetic charge

$$ Q = I_3 + \frac{1}{2}Y. $$

There may also be other stages of symmetry breaking at higher energies. The three independent coupling constants $g_3, g_2, g_1$ corresponding to the three factors in $G$ have a weak logarithmic energy dependence and appear to come to approximately the same value at an energy scale of about $10^{15}$ GeV, especially if supersymmetry is incorporated into the model [11, 12, 13]. This suggests that there may be a grand unified theory (GUT) uniting the strong, weak and electromagnetic interactions in a single theory with a symmetry group such as $SO(10)$. There would then be a phase transition (or a sequence of phase transitions) at that energy scale at which the GUT symmetry breaks to the symmetry group (58) of the standard model. If the model is supersymmetric, then there must also be a supersymmetry-breaking transition.

3. Defect formation

The appearance of topological defects is a common feature of symmetry-breaking phase transitions. In this lecture, I shall review the defects associated with the various transitions discussed earlier, and the general conditions for the existence of defects.
3.1. DISCRETE SYMMETRY BREAKING

The simplest possible field-theoretic model that exhibits symmetry breaking is a model of a real scalar field described by the action integral

\[ I = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{8} \lambda (\phi^2 - \eta^2)^2 \right]. \]  

(62)

Here the action is invariant under the reflection symmetry \( \phi \mapsto -\phi \). Thus the symmetry group is \( G = \mathbb{Z}_2 \), and the manifold of degenerate vacua reduces to a pair of points; the two vacuum states are characterized by

\[ \phi \equiv \langle \hat{\phi} \rangle = \pm \eta. \]  

(63)

At high temperature, the equilibrium state is symmetric, with \( \langle \hat{\phi} \rangle = 0 \). When the system cools through the critical temperature, \( \hat{\phi} \) acquires a non-zero expectation value, but the sign is chosen arbitrarily. So it may happen that in one region, it chooses \( \phi \approx \eta \) and in another \( \phi \approx -\eta \). When such regions meet, they must be separated by a planar defect, a domain wall, across which \( \phi \) goes smoothly from one value to the other. The minimum energy configuration is determined by a balance between gradient energy and potential energy. At zero temperature one finds for example that a domain wall in the \( xy \)-plane is described by

\[ \phi(z) = \eta \tanh \frac{z}{\xi}, \quad \text{with} \quad \xi = \frac{2}{\sqrt{\lambda \eta}}. \]  

(64)

As the system cools below \( T_c \), energy is trapped in the domain wall. In a sense the defect is a region of trapped old high-temperature phase, with the characteristic energy density that it had at \( T_c \). The wall is topologically stable. It can move, as one domain grows at the expense of the other, but it cannot simply break. A closed wall bounding a finite domain may of course shrink and eventually disappear. But this is a relatively slow process, so walls may have continuing effects.

3.2. ABELIAN VORTICES OR STRINGS

Now let us consider the case of an Abelian \( U(1) \) symmetry, such as that of superfluid \(^4\text{He}\). When the system is cooled through the transition temperature, the order parameter acquires a non-zero expectation value \( \phi = \eta e^{i\vartheta} \).

The magnitude \( \eta \) is determined by the minimization of the free energy, but the phase \( \vartheta \) is arbitrary. It is chosen randomly. However, in a large system there is no reason why the same choice should be made everywhere; \( \vartheta \) may vary from one part of the system to another. We should expect the choice to be made independently in widely separated regions, especially if we are
talking about a transition in the early Universe, where such regions may have had no prior causal contact.

When such a random choice is made, it may happen that around some large loop in space the value of $\vartheta$ varies through $2\pi$ or a multiple thereof. In such a case, $\phi$ must vanish somewhere inside the loop; indeed it must vanish all along a curve that threads through the loop. This is the core of a topological defect, a cosmic string or vortex.

If the string is along the $z$ axis, the order parameter around it typically takes the form

$$\phi(r, \varphi, z) = \eta f(r) e^{in\varphi},$$

(65)

where $r, \varphi, z$ are cylindrical polar coordinates, and $n$ is an integer, the winding number. The function $f$ has limiting values $f(0) = 0, f(\infty) = 1$. It may be determined by minimizing the Ginzburg–Landau free energy, (27).

For a superfluid, an important consequence of the expression (30) for the superfluid velocity is that the superfluid flow is irrotational: $\nabla \wedge \mathbf{v}_s = 0$. The vorticity vanishes everywhere, except in the core of the string, where the superfluid density vanishes. The string is a vortex. The form of (65) implies that there is a flow of superfluid around the string, with velocity

$$(v_s)_\varphi = \frac{\hbar}{m_4 r},$$

(66)

at large $r$, where $m_4$ is the mass of a $^4$He atom. Thus the circulation around the string is quantized:

$$\oint \mathbf{v}_s \cdot d\mathbf{r} = n \kappa_4,$$

(67)

where the circulation quantum is

$$\kappa_4 = \frac{2\pi \hbar}{m_4}.$$  

(68)

There is a similar vortex in $^3$He-$B$, but in that case $m_4$ is replaced by the mass of a Cooper pair, namely $2m_3$, so the circulation quantum is

$$\kappa_3 = \frac{\pi \hbar}{m_3}.$$  

(69)

An important feature of the string or vortex is its topological stability deriving from this quantization. It can move around, but cannot break. A vortex loop can disappear by shrinking to a point, but a long, straight vortex is stable.
3.3. VORTICES IN A GAUGE THEORY

If the symmetry is a gauge symmetry, with coupling to a gauge field $A_\mu$, then around the string $A$ has an azimuthal component,

$$A_\varphi(r, \varphi, z) = \frac{n\hbar}{er} g(r),$$  \hspace{1cm} (70)

where $e$ is the charge and $g$ has the same limiting values as $f$. By taking an integral round a large loop surrounding the string we find that it carries a quantized magnetic flux,

$$\Phi = \lim_{r \to \infty} \int_0^{2\pi} A_\varphi r d\varphi = n \frac{2\pi \hbar}{e}.$$  \hspace{1cm} (71)

The magnetic field is given by

$$B_z = \frac{n\hbar}{er} g'(r).$$  \hspace{1cm} (72)

(In the case of a superconductor $e$ in the above should be replaced by the charge 2$e$ of a Cooper pair, so the flux quantum is actually $\pi \hbar/e$.)

The functions $f$ and $g$ are determined by minimizing the free energy. For the Abelian Higgs model at zero temperature, with action integral (35) (setting $c = \hbar = 1$), they satisfy the equations

$$f'' + \frac{1}{r} f' - \frac{n^2}{r^2} (1 - g)^2 f + \lambda \eta^2 (1 - f^2) f = 0,$$

$$g'' - \frac{1}{r} g' + 2e^2 \eta^2 f^2 (1 - g) = 0.$$  \hspace{1cm} (73)

No analytic solution is known, but it is easy to find solutions numerically. Note that there are two length scales governing the large-$r$ behaviour of the functions, the inverse masses of the scalar and vector excitations (Higgs and gauge particles),

$$m_s^2 = 2 \lambda \eta^2, \quad \text{and} \quad m_\nu^2 = 2e^2 \eta^2.$$  \hspace{1cm} (74)

The asymptotic behaviour depends on the ratio of these two,

$$\kappa = \frac{m_s^2}{m_\nu^2} = \frac{\lambda}{e^2}.$$  \hspace{1cm} (75)

For $\kappa < 4$ one finds that at large $r$

$$1 - g \propto r^{1/2} e^{-m_\nu r}, \quad 1 - f \propto r^{-1/2} e^{-m_s r}.$$  \hspace{1cm} (76)
In this case, the string has a narrow core that constitutes a magnetic flux tube, while the order parameter reaches its vacuum value over a larger distance. On the other hand, when $\kappa > 4$, $m_v$ controls the behaviour of both $1 - f$ and $1 - g$, with

$$1 - g \propto r^{1/2} e^{-m_v r}, \quad 1 - f \propto r^{-1} e^{-2m_v r}. \quad (77)$$

In superconductors [4], the two length scales are known as the correlation length $\xi = \hbar/m_sc$ and the Landau penetration depth $\lambda = \hbar/m_vc$. Here large and small values of $\kappa$ distinguish so-called type-I from type-II superconductors. In a type-II superconductor, vortices with $|n| > 1$ are unstable; there is a repulsive force between parallel $n = 1$ vortices which can stabilize a lattice of vortices. Hence there is an intermediate range of magnetic field strength within which the field penetrates the superconductor but confined to a lattice of flux tubes.

### 3.4. DEFECTS IN NEMATICS

It is easy to construct a stable linear defect similar to a string in the case of a nematic, by allowing the director $n$ to rotate as one moves around the string through an angle $\pi$: e.g., we can take

$$n(r, \varphi, z) = \left(\cos \frac{\varphi}{2}, \sin \frac{\varphi}{2}, 0\right), \quad (78)$$

provided we include in the order parameter $Q$ a factor $f(r)$ that vanishes at $r = 0$. $Q$ then has no singularity because of the identification of $n$ and $-n$. This configuration is called a disclination [7].

Like the superfluid vortex, the disclination is topologically stable. If the rotation angle were $2\pi$ instead of $\pi$ it would not be. It could ‘escape into the third dimension’: at small $r$, we could rotate $n$ upwards until at $r = 0$ it points in the $z$ direction, thus allowing $f(0)$ to be nonvanishing.

In addition to this linear defect there can be a point defect in a nematic liquid. Away from the centre, $r = 0$ say, we can take

$$n(r) = \frac{\hat{r}}{r}, \quad (79)$$

again provided there is a factor in $Q$ that vanishes at the centre.

This is often called the hedgehog or monopole configuration. Like the vortex it is topologically stable, and cannot disappear spontaneously — though it may annihilate with an anti-hedgehog.

### 3.5. THE FUNDAMENTAL GROUP

The general conditions for the existence of defects can be expressed in terms of the topology of the vacuum manifold $M$, specifically its homotopy groups [14].
The existence of linear defects for example is related to the possibility of finding non-trivial closed loops in $M$. Let us recall that different points in $M$ correspond to different values of the order parameter labelling different degenerate vacua. A closed loop is a continuous map $\phi : I \to M$ from the unit interval of real numbers, $I = [0, 1] \subset \mathbb{R}$, to $M$ such that $\phi(0) = \phi(1)$ (or equivalently a map from the circle $S^1$ to $M$). Linear defects can exist if it is possible to find a closed loop that cannot be continuously shrunk to a point without leaving $M$, because then if the value of the order parameter around a loop in space follow this curve, it is not possible to fill in the values inside the loop continuously while remaining on $M$.

In general, two closed loops are homotopic if it is possible to deform one continuously into the other within $M$. This is an equivalence relation, so we may define homotopy classes of loops. For example, when $M$ is the circle $S^1$, the homotopy classes may be labelled by the winding number, the (algebraic) number of times that we traverse the circle while going from 0 to 1 in $I$.

The homotopy classes constitute the elements of a group, the fundamental group or first homotopy group of $M$, denoted by $\pi_1(M)$. To construct it, we introduce a base point $b \in M$, and consider loops starting and finishing at $b$, i.e. $\phi(0) = \phi(1) = b$. Then the product of two loops $\phi$ and $\psi$ is the loop constructed by following $\phi$ and then $\psi$: 

$$(\phi \cdot \psi)(t) = \begin{cases} 
\phi(2t) & \text{for } t \leq \frac{1}{2}, \\
\psi(2t - 1) & \text{for } t > \frac{1}{2}.
\end{cases}$$

It is easy to show that this is a relation between homotopy classes, and so defines a product on the set of classes, which thus becomes the group $\pi_1(M)$.

The condition for the existence of topologically stable linear defects, strings or vortices, is that the fundamental group be non-trivial: $\pi_1(M) \neq 1$.

For the Abelian case, where $M = S^1$, the fundamental group is simply the group of integers, $\pi_1(S^1) = \mathbb{Z}$. The distinct possible linear defects are labelled by the elements of this group, the winding numbers.

In a continuum version of the Heisenberg ferromagnet, we have $M = S^2$. On the sphere, all loops can be shrunk to a point, so $\pi_1(S^2) = 1$; there are no possible linear defects.

For the nematic, however, the situation is different. On the sphere $S^2$ all loops are homotopically trivial, but this is no longer true when we identify opposite points to form $RP^2$, because a curve that starts at one pole and ends at the opposite pole is closed in $RP^2$ but cannot be shrunk to a point. There is only one non-trivial homotopy class, because traversing the same loop twice gives a trivial loop; as we noted earlier, a disclination with a winding of $2\pi$ rather than $\pi$ is not stable. Hence the fundamental group in this case is $\pi_1(RP^2) = \mathbb{Z}_2 = \{0, 1\}$, the group of integers modulo 2.
3.6. THE SECOND HOMOTOPY GROUP

The conditions for the existence of other types of defects can also be expressed in terms of homotopy groups. For point defects such as the hedgehog the relevant question is whether there are non-shrinkable two-surfaces in $M$.

The second homotopy group [14] is defined in terms of closed two-surfaces, i.e. maps $\phi : I^2 \rightarrow M$ from the unit square to $M$, such that

$$\phi(0,t) = \phi(1,t) = b, \quad \phi(s,0) = \phi(s,1) = b \quad \text{for all } s \text{ and } t. \quad (81)$$

In other words, $\phi$ maps the whole boundary of $I^2$ to the chosen base point $b \in M$. In effect, it is a map from $S^2$ to $M$, in which one designated point is mapped to $b$.

Two closed surfaces are homotopic if one can be smoothly deformed into the other. This defines an equivalence relation, and hence a classification into homotopy classes of surfaces.

As before, we can introduce a product on the set of closed surfaces, by setting

$$(\phi \cdot \psi)(s,t) = \begin{cases} 
\phi(s,2t) & \text{for } t \leq \frac{1}{2}, \\
\psi(s,2t-1) & \text{for } t > \frac{1}{2}.
\end{cases} \quad (82)$$

This defines a product on the set of homotopy classes. (We could equally well have defined the product with the roles of $s$ and $t$ interchanged; it is easy to show that the results are homotopic to each other.) Thus we have defined the second homotopy group $\pi_2(M)$.

As a simple example, let us consider a continuum version of the Heisenberg ferromagnet, with $SU(2)$ symmetry and an order parameter $M$ transforming according to the 3-dimensional vector representation. Below the transition, the magnitude of $M$ is fixed but its direction is arbitrary. Thus the subgroup $H$ that leaves $M$ invariant is $H = U(1)$ and $M = SU(2)/U(1) = S^2$. In this case, the homotopy classes are labelled by an integer, the (algebraic) number of times the map wraps around the sphere. For example, a typical element of the homotopy class labelled by $n$ is the map of one sphere on another defined in terms of polar coordinates $\theta, \varphi$ by

$$\phi : S^2 \rightarrow S^2 : (\theta, \varphi) \mapsto (\theta, n\varphi). \quad (83)$$

For this case, therefore, $\pi_2(S^2) = \mathbb{Z}$, the group of integers.

For the nematic, we have to identify opposite points of $S^2$ and pass to $RP^2$ but this makes no difference to the classification of closed surfaces. We again have $\pi_2(RP^2) = \mathbb{Z}$, so the possible hedgehogs are labelled by an integer.
One thing this classification cannot tell us, however, is whether configurations with \(|n| > 1\) are actually stable. In some cases, it may be energetically favourable for a configuration with winding number \(n = 2\) for instance, to break up into two separate \(n = 1\) configurations. Whether this actually happens is a question of detailed dynamics.

### 3.7. DOMAIN WALLS

As we noted earlier, domain walls occur when a discrete symmetry is broken. More generally, the condition for the existence of domain walls is that the vacuum manifold \(M\) be disconnected. Domain walls are classified by the elements of what is often called the ‘zeroth homotopy group’, denoted by \(\pi_0(M)\), whose elements are in one-to-one correspondence with the connected components of \(M\). It is analogous to the higher homotopy groups: it may be regarded as classifying maps \(\phi : S^0 \to M\), where \(S^0\) is the 0-sphere (the boundary of the interval \([-1, 1] \subset \mathbb{R}\), namely the pair of points \(\{1, -1\}\)) in which the image of one chosen point is the base point of \(M\).

In a general case, the terminology is strictly speaking inaccurate, because \(\pi_0(M)\) is not a group. There is one special case in which it is so, namely when \(H = 1\), so that \(M\) is itself a group, \(M = G\). In this case, the connected component \(G_0\) of \(G\) containing the identity is an invariant subgroup (i.e., for any \(g \in G\), \(gG_0g^{-1} = G_0\)), and hence the quotient group \(G/G_0\) is defined; moreover

\[
\pi_0(G) = G/G_0. \tag{84}
\]

Another way of characterizing \(\pi_0(M)\) is as a quotient of two groups. If \(\phi \in M\) can be connected by a continuous path to \(\phi_0\), then one can always find a continuous path in \(G\) from the identity \(e\) to \(g\) such that \(D(g)\phi_0 = \phi\). Hence the connected component \(M_0 \subset M\) containing \(\phi_0\) may be identified with the set of elements \(\{g\phi_0 : g \in G_0\}\). The subgroup of \(G_0\) which leaves \(\phi_0\) unaltered is clearly \(H \cap G_0\). Hence,

\[
M_0 = G_0/(H \cap G_0). \tag{85}
\]

Now since \(G_0\) is an invariant subgroup in \(G\), it follows that \(H \cap G_0\) is also an invariant subgroup of \(H\). Thus \(H/(H \cap G_0)\) is a group, and moreover a subgroup of \(G/G_0\). One can then show that

\[
\pi_0(M) = (G/G_0)/(H/(H \cap G_0)). \tag{86}
\]

### 3.8. HELIUM-3

Finally, let us return to the interesting case of \(^3\text{He}\), beginning with the superfluid \(^3\text{He}-B\) phase, and initially ignoring the spin-orbit interaction.
We recall from (54) that \( M_B = SO(3) \times S^1 \). Here both the zeroth and second homotopy groups \( \pi_0(M_B) \) and \( \pi_2(M_B) \) are trivial, so there are no topologically stable domain walls or monopoles. However,

\[
\pi_1(M_B) = \mathbb{Z}_2 \times \mathbb{Z}.
\]  

(87)

so there are two different kinds of vortices. The factor \( \mathbb{Z} \) classifies vortices around which the phase changes by \( 2n\pi \), exactly as in the case of \( ^4\text{He} \). However, the \( \mathbb{Z}_2 \) factor arises because there are non-trivial loops in \( SO(3) = \mathbb{R}P^3 \). In a vortex corresponding to the non-trivial element of \( \mathbb{Z}_2 \) there is no actual circulation around the string, but rather a relative rotation of the orbital and spin angular momenta. These are called spin vortices as opposed to mass vortices.

Note that vortices may carry both types of quantum number simultaneously. Such a combination is a spin–mass vortex.

When we take account of the spin-orbit interaction the manifold is reduced, according to (57), to \( M'_B = S^2 \times S^1 \). In this case, we find

\[
\pi_2(M'_B) = \mathbb{Z}.
\]  

(88)

Viewed on a large scale there are monopole configurations. But since there are no short-range monopoles, these have no actual singularity. What happens is that the order parameter near the monopole is forced to leave the manifold \( M'_B \), but can remain everywhere on the larger manifold \( M_B \); the rotation angle in the order parameter, which is fixed to be the Leggett angle \( \theta_L \) at large distances, can tend smoothly to zero at the centre, but \( \phi \) itself remains non-zero.

We also find

\[
\pi_1(M'_B) = \mathbb{Z},
\]  

(89)

corresponding to the fact that the mass vortices are unaffected by the spin-orbit coupling, and survive to large distances. This is not the case, however, for the spin vortices, since there is no longer a \( \mathbb{Z}_2 \) factor. What happens is that these become attached to a long-range soliton or domain-wall feature. The order parameter at large distances around this vortex cannot lie everywhere on \( M'_B \), but to minimize the energy it does so except near one direction. Note that this is true in spite of the fact that \( \pi_0(M'_B) = 1 \), which means there are no truly stable domain walls: \( M'_B \) is connected, but the relevant point is that it is not possible in \( M_B \) to deform the relevant loop in such a way that it lies entirely in \( M'_B \). (Such cases may be classified by the relative homotopy groups of \( M_B \) and its subspace \( M'_B \), in this case the group \( \pi_1(M_B, M'_B) \).

Now let us turn to the \( A \) phase, for which according to (52) the ‘vacuum manifold’ is \( M_A = S^2 \times SO(3)/\mathbb{Z}_2 \). This space is again connected, so there
are no stable domain walls. However, we find
\[ \pi_1(M_A) = \mathbb{Z}_4, \quad \pi_2(M_A) = \mathbb{Z}. \] (90)

Thus there are monopoles, labelled by an integer winding number, and vortices labelled by an integer \( n \) modulo 4. On the other hand at long range the manifold, given by (56), is simply \( M'_A = SO(3) \), whence
\[ \pi_1(M'_A) = \mathbb{Z}_2, \quad \pi_2(M'_A) = 1. \] (91)

Hence there are no stable monopoles and only one class of stable vortices; the latter are to be identified with the \( n = 2 \) short-range vortices.

It is not hard to see what happens to the other short-range defects. For an \( n = \pm 1 \) short-range vortex, the corresponding loop in \( M_A \) cannot be deformed to lie entirely within \( M'_A \). In other words, we cannot make \( d \) parallel to \( l \) everywhere. The vortex becomes attached to a sheet or domain wall across which \( d \) rotates by \( \pi \) with a compensating rotation about \( l \).

Similarly, around a short-range monopole we cannot deform the order parameter so that \( d \) remains everywhere parallel to \( l \). The monopole becomes attached to a string in the centre of which \( d \) is in the opposite direction.

4. Cosmology in the Laboratory

Our present understanding of fundamental particle physics suggests that the Universe may have undergone a series of phase transitions very early in its history. One of the clearest signatures of these transitions would be the formation of stable topological defects with potentially significant cosmological effects. To predict these we need to estimate how many defects would have been formed and how they would have evolved during the subsequent cosmic expansion. Calculations of the behaviour of the system in the highly non-equilibrium context of a rapid phase transition are problematic, however, and it is hard to know whether they are reliable. There is no direct way of testing them, because we cannot do experiments on the early Universe.

But what we can do is to apply similar methods to analogous low-temperature transitions in condensed-matter systems, which often have a very similar mathematical description. Over the last few years, several experiments have been done in a variety of systems to test ideas drawn from cosmology. This has led to some extremely innovative and exciting condensed-matter physics, although the results are still somewhat confusing.
4.1. DEFECT FORMATION IN THE EARLY UNIVERSE

The electroweak transition, at about 100 GeV, where the $W$ and $Z$ particles acquire a mass through the Higgs mechanism, occurred when the age of the Universe was around $10^{-10}$ s. It is now believed that this is not in fact a genuine phase transition but rather a rapid but smooth crossover \cite{15}. (This is possible only because this is a gauge theory.) There was probably a later transition, the quark–hadron transition at which the soup of quarks and gluons separated into individual hadrons.

More interesting from a cosmological point of view, however, are the hypothetical transitions at even earlier times. If the idea of grand unification is correct, there would have been a phase transition of some kind at an energy scale of around $10^{15}$ GeV, corresponding to a time about $10^{-36}$ s after the Big Bang. In some models, we expect a sequence of phase transitions, as the symmetry is broken in several stages, for example

$$SO(10) \rightarrow SU(5) \rightarrow SU(3) \times SU(2) \times U(1), \quad (92)$$

or

$$SO(10) \rightarrow SU(4) \times SU(2) \times SU(2) \rightarrow SU(3) \times SU(2) \times U(1). \quad (93)$$

The most attractive GUTs are supersymmetric. Since supersymmetry is not manifest at low energies, it must have been broken at some intermediate time, possibly yielding another phase transition, perhaps at about 1 TeV.

Domain walls, strings and monopoles may all have been formed at early-Universe phase transitions, as indeed may more general composite objects of various kinds \cite{8}. Monopoles and domain walls are cosmologically problematic, for different reasons. Heavy domain walls, such as those that could have been formed in the early Universe, certainly do not exist in our Universe today, and monopoles could be present only in very small numbers. So if these defects were produced at all, there must have been a mechanism to remove them completely or almost completely at an early stage. Inflation has often been invoked to do this job.

Strings on the other hand could have survived in sufficient numbers to be cosmologically significant without violating any observational bounds. For a long time it was believed that they might serve to explain the initial inhomogeneities in the density of the Universe from which galaxies and clusters later evolved. The idea that strings alone could seed these density perturbations seems no longer viable, in the light of the data on the cosmic microwave background anisotropy. It is still perfectly possible to fit the data with models incorporating both strings and inflation \cite{16}, but a recent analysis concludes that strings probably do not make a significant
contribution [17]. They may, however, have had other important cosmological effects, for example in the generation of magnetic fields [18, 19, 20], high-energy cosmic rays [21, 22] and baryogenesis [23, 24].

For these reasons I shall restrict the discussion to the case of string formation. To be specific, let us consider the breaking of an Abelian $U(1)$ symmetry — though most of the discussion can easily be extended to non-Abelian symmetries.

**4.2. DEFECT FORMATION AT A FIRST-ORDER TRANSITION**

The nature of the early-Universe transitions is largely unknown, in particular the order of each transition. In some cases, as I mentioned, there may be no true transition at all. Defects may be formed in any event, but the mechanism depends strongly on the order. In relation to condensed-matter analogues, most interest attaches to second-order transitions, and that is the case I will spend most time on. But I shall begin with what is in some ways the simpler situation of a first-order transition.

In fact, the first ‘cosmology in the laboratory’ experiments were done with a first-order transition, namely the transition from normal isotropic liquid to nematic liquid crystal [25, 26].

Let us suppose, therefore, that there is a first-order transition, proceeding by bubble nucleation. Once the Universe reaches the relevant critical temperature, bubbles of the new low-temperature phase are born at random positions and start to grow until they eventually meet and merge. The nucleation rate $\gamma$ per unit space-time volume is given by an expression of the form

$$\gamma(T) = A(T)e^{-S_E(T)}$$  \hspace{1cm} (94)

where $S_E$ is the Euclidean action for a tunnelling solution, and the prefactor $A$ is typically of order $T^4$. The nucleation rate determines the characteristic distance $\xi$ between nucleation sites, such that the number of separate bubbles nucleating in a large volume $V$ is $V/\xi^3$. Typically the bubbles expand at relativistic speeds, and then $\xi$ is of order $\gamma^{-1/4}$.

In each new bubble the order parameter $\phi$ becomes non-zero, and must choose a random phase $\vartheta$. There is no reason why there should be any correlation between the phases in different bubbles (except conceivably in the case of very near neighbours). So it is reasonable to assume that each is an independent random variable, uniformly distributed between 0 and $2\pi$.

When two bubbles meet, an equilibration process will occur, leading to a phase $\vartheta$ smoothly interpolating between $\vartheta_1$ and $\vartheta_2$ across the boundary. It is reasonable to assume that it will do so by the shortest possible path, so that the total variation will always be less than $\pi$; this is called the geodesic rule. Numerical simulations have confirmed that it is usually though not
universally true — the rule may lead to a slight underestimate of the total number of defects formed [27, 28].

When these two bubbles encounter a third, it is possible that a string defect may be trapped along the line where they meet. This will happen if the net phase change from $\theta_1$ to $\theta_2$ to $\theta_3$ and back to $\theta_1$ is $\pm 2\pi$ rather than zero. If the geodesic rule applies and the three phases are strictly independent, the probability of this happening can easily be seen to be $\frac{1}{4}$.

Thus the total length of string formed in this process in a large volume $V$ will be of order $V/\xi^2$. The length of string per unit volume will be

$$L = \frac{k}{\xi^2}, \tag{95}$$

where $k$ is a numerical constant of order 1. (For example, if it is assumed that nucleation sites form a body-centred cubic lattice, one finds $k = 3/2^{7/6} = 1.34$. A random lattice would really be more appropriate; that might well give a somewhat smaller value.)

The first tests of this idea in condensed-matter systems were done in nematic liquid crystals, by studying the formation of disclination lines in the isotropic to nematic transition [25, 26]. The symmetry in that case is of course non-Abelian, but the principle is the same. We may assume that within each nucleating bubble of the nematic phase, the director $\mathbf{n}$ is an independent random variable, uniformly distributed over half the unit sphere (except near the walls where special effects come into play).

The analogue of the geodesic rule is then the assumption that across the interface between two bubbles, the director always turns by an angle less than $\pi$. In that case, the probability that a disclination will be trapped between three bubbles with independently oriented directors is $1/\pi$, so (95) should still hold.

The experiments did in fact show reasonably good agreement with the predictions. Further experiments have been done to check the correlations between defects and antidefects [29].

4.3. SECOND-ORDER TRANSITIONS

The argument is a little more complex in the case of a second-order phase transition. As the system cools through the critical temperature, the order parameter must acquire a non-zero value and choose a random phase. We may assume that the choice is made independently in widely separated regions. Thus there is a chance that defects will be trapped, and we should expect the formation of a random tangle of strings. What is less obvious is what the characteristic scale $\xi_{str}$ of this tangle should be. Here $\xi_{str}$ may be defined by the condition that in a randomly chosen volume $\xi_{str}^3$ there will
be on average a length $\xi_{\text{str}}$ of string. In other words, the length of string per unit volume is

$$L_{\text{str}} = \frac{1}{\xi_{\text{str}}}.$$  \hspace{1cm} (96)

What determines $\xi_{\text{str}}$? Obviously it is related to the correlation length $\xi_{\phi}$ of the order parameter, specifically of its phase. But this is not an answer. During a second-order phase transition, $\xi_{\phi}$ is varying rapidly. Indeed, it is characteristic of second-order transitions that the equilibrium correlation length $\xi_{\text{eq}}$ diverges at the critical temperature. So we must specify at what time or what temperature $\xi_{\text{str}}$ should be compared with $\xi_{\text{eq}}$.

An answer to this question has been given by Wojciech Zurek [30, 31, 32], following an earlier suggestion of mine [33]. It is clear that in a real system going through the transition at a finite rate, the true correlation length $\xi_{\phi}$ can never become infinite. In fact, for reasons of causality it can never increase faster than the speed of light. So, beyond the point where $\dot{\xi}_{\text{eq}} = c$, the adiabatic approximation, that $\xi_{\phi} \approx \xi_{\text{eq}}(T)$, ceases to be valid, and instead one may assume that $\xi_{\phi}$ will be more or less constant until after the transition, at least to the point where it again becomes equal to the decreasing $\xi_{\text{eq}}$. In a non-relativistic system, it is not the speed of light that is relevant, but some characteristic speed of the system.

Zurek has given an alternative argument leading to essentially the same conclusion, based on a comparison of the quench rate and relaxation rate of the system.

Let us assume that near the transition, the temperature varies linearly with time, so that

$$\epsilon \equiv 1 - \frac{T}{T_c} = \frac{t}{\tau_0}. \hspace{1cm} (97)$$

Here $\tau_0$ is the quench time. (We take $t = 0$ when $T = T_c$.) The equilibrium correlation length near $T_c$ has the form

$$\xi_{\text{eq}}(T) = \xi_0 |\epsilon|^{-\nu}, \hspace{1cm} (98)$$

where $\nu$ is a critical index. In mean field theory, $\nu = \frac{1}{2}$, and this is often an adequate approximation. For $^4$He, however, the renormalization group gives a more accurate value, $\nu = \frac{2}{3}$. Similarly the relaxation time $\tau$ diverges at $T_c$:

$$\tau(T) = \tau_0 |\epsilon|^{-\mu}, \hspace{1cm} (99)$$

where for $^4$He the critical index $\mu = 1$. This is the phenomenon of critical slowing down. The characteristic velocity is

$$c(T) = \frac{\xi_{\text{eq}}(T)}{\tau(T)} = \frac{\xi_0}{\tau_0} |\epsilon|^{\mu-\nu}. \hspace{1cm} (100)$$
Note that it vanishes at $T_c$. In $^4$He, this is the speed of second sound, a thermal wave in which the normal and superfluid components oscillate in antiphase.

Now information about the phase of the order parameter cannot propagate faster than the speed $c(T)$. Hence after the transition the distance over which phase information can propagate is the sonic horizon

$$h(t) = \int_0^t c(T(t'))dt' = \frac{1}{1 + \mu - \nu} \frac{\xi_0 \tau_q}{\tau_0} t^{1+\mu-\nu}. \quad (101)$$

This becomes equal to the equilibrium correlation length when

$$\epsilon = \epsilon_Z = \left[(1 + \mu - \nu) \frac{\tau_0}{\tau_q}\right]^{\frac{1}{1+\mu}}. \quad (102)$$

The time when this happens is the Zurek time

$$t_Z = \left[(1 + \mu - \nu) \tau_0 \tau_q \mu\right]^{\frac{1}{1+\mu}}. \quad (103)$$

It is reasonable to suppose, at least as a first crude approximation, that at the Zurek time the characteristic length scale $\xi_{str}$ of the tangle of strings or vortices should be equal to the correlation length:

$$\xi_{str}(t_Z) \sim \xi_Z = \xi_{eq}(t_Z) \sim \xi_0 \left(\frac{\tau_q}{\tau_0}\right)^{\frac{\mu}{1+\mu}}. \quad (104)$$

Equivalently, we expect the density of strings or vortices (i.e., the length per unit volume) to be approximately $1/\xi_{eq}^2$, i.e.,

$$L_{str}(t_Z) = \frac{\kappa}{\xi_0^2} \left(\frac{\tau_0}{\tau_q}\right)^{\frac{2\mu}{1+\mu}}, \quad (105)$$

where $\kappa$ is a numerical constant of order unity. Numerical simulations [34, 35] suggest that it should in fact be somewhat less than unity, perhaps of order 0.1. Note that in $^4$He, the exponent in (105) is $\frac{1}{2}$ in mean field theory, while using renormalization-group values it is $\frac{3}{5}$. This is the prediction that has to be tested.

4.4. EXPERIMENTS IN HELIUM-4

Zurek [30] initially suggested testing these predictions in superfluid $^4$He. Experiments designed to test his predictions have been performed by Peter McClintock's group at Lancaster using a rapid pressure quench. The experimental sample was contained in a small chamber that could be rapidly
expanded to lower the pressure, thereby sending it through the lambda transition into the superfluid phase. The number of vortices produced was found by measuring the attenuation of a second sound signal, generated by a small heater.

The first experiment [36] did in fact see evidence of vorticity generated during the quench, at roughly the predicted level. However, it was not conclusive for various reasons. Vorticity might have been produced by hydrodynamical effects at the walls. Also the capillary tube used to fill the chamber was closed at the outer end, so that during the expansion some helium was inevitably injected into the chamber, possibly again creating vorticity. Another problem was that it was not possible to measure the second-sound attenuation during the first 50 ms after the transition, so that later readings had to be extrapolated back to the relevant time.

To overcome these problems, the apparatus was redesigned to minimize the hydrodynamic effects, and the experiment repeated [37]. Somewhat disappointingly, the result was null: no vorticity was detected with the improved apparatus. One possible explanation for this is that the vortices produced may simply disappear too fast to be seen [38, 39]. The rate at which vorticity dissipates was measured in the rather different circumstances of vorticity generated by turbulent flow. It is not certain that the results can be carried over to the circumstances of this experiment.

A third version of the experiment, incorporating further improvements, is now being planned [40]. Results are eagerly awaited.

4.5. EXPERIMENTS IN HELIUM-3

There are a number of advantages in using $^3$He rather than $^4$He. One is that because the correlation length is much longer (40 to 100 nm, rather than less than 1 nm), a continuum (Ginzburg–Landau) description is much more accurate than in $^4$He. Moreover, the energy needed to generate a vortex is larger relative to the thermal energy, so it is easier to avoid extrinsic vortex formation. Another advantage is that since the nuclear spin is non-zero, one can use nuclear magnetic resonance to count the vortices.

Perhaps the greatest advantage, however, lies in the fact that one can induce a temperature- rather than pressure-driven transition. This is because of another characteristic of $^3$He, namely that it is a very efficient neutron absorber, via the reaction

$$n + ^3\text{He} \rightarrow p + ^3\text{H} + 764\ \text{keV}.$$  \hspace{1cm} (106)

Two experiments have been done with $^3$He, one in Grenoble [41] and one in Helsinki [42]. Both use $^3$He in the superfluid $B$ phase, and look for evidence of vortices similar to those in $^4$He. Both make use of the neutron
absorption reaction, by exposing the helium container to neutrons from a radioactive source. Each neutron absorbed releases 764 keV of energy, initially in the form of kinetic energy of the proton and triton. This serves to heat up a small region to above the transition temperature. It then rapidly cools, in a time of the order of 1 µs, and goes back through the transition into the superfluid phase. During this process we expect a random tangle of vortices to be generated.

In other respects the experiments are very different. The Grenoble experiment [41], using a sample of $^3$He-$B$ at a temperature much less than $T_c$ was essentially calorimetry. The total energy released, in the form of quasiparticles, following each neutron-absorption event was measured. Of the available 764 keV of energy about 50 keV is released in the form of ultraviolet radiation. However, the measured energy was in the range 600 to 650 keV, depending on the pressure, leaving a considerable shortfall. This is interpreted as being the energy lost to vortex formation. It is very hard to think of any other possible interpretation.

The main feature of the Helsinki experiment [42], using a sample of $^3$He-$B$ at a considerably higher temperature, not far below $T_c$, was the use of a rotating cryostat. If a container of helium is rotated rapidly, vortices are generated at the walls and migrate to form a central cluster parallel to the rotation axis. However, if the rotation is slower, no vortices can be formed. In $^3$He-$B$, it is possible to ensure that no vortices at all are present. We then have a remarkable situation. The normal fluid component is rotating with the container, but the superfluid component, which cannot support vorticity, is completely stationary. Thus there is a \textit{counterflow} velocity, a difference $v = v_s - v_n$ between the velocities of the two components. This introduces novel hydrodynamic effects; in particular a superfluid vortex moving relative to the normal fluid experiences the transverse \textit{Magnus force}.

In consequence vortices above a certain minimum size $r_0$ and correctly oriented are expanded until they reach the walls of the container, and then migrate to join a central cluster parallel to the axis. The number of vortices ‘captured’ in this way can be determined by nuclear magnetic resonance (NMR) measurements. It is possible to detect each individual vortex joining the cluster.

The number of vortices we expect to be captured can be predicted. It is essentially the number of vortices with sizes between the required minimum size $r_0$, which depends on the counterflow velocity $v$, and the maximum radius of the bubble. The size distribution of loops formed is expected to be scale invariant. This leads to a very simple prediction. There is a critical velocity $v_{cn}$ for neutron-induced vortex formation, which is substantially lower than the critical velocity $v_c$ for spontaneous vortex formation at the walls. If $v > v_{cn}$, the number of vortices captured after each neutron-absorption
event should have the form

\[ N = c \left( \frac{v}{v_{cn}} \right)^3 - 1 \]  \hspace{1cm} (107)

where \( c \) is a calculable constant. Remarkably enough, all the dependence on the bulk temperature, the pressure and the magnetic field is contained in the value of \( v_{cn} \). Hence if \( N \) is plotted against \( v^3 \) for various values of these parameters, one should see a set of straight lines with a common intercept at \(-c\) on the vertical axis. This simple prediction does fit the experimental results very well over a considerable parameter range, providing good evidence for the validity of the prediction.

It has also been possible to test the predicted dependence of \( v_{cn} \) on temperature, namely \( v_{cn} \propto \epsilon^{1/3} \). This again is a good fit to the data.

4.6. EXPERIMENTS IN SUPERCONDUCTORS

It is particularly interesting to test the predictions of defect formation in superconductors, because they provide an example of a gauge theory.

The first experiments [43] were done by a group at Technion, using a thin film of the high-temperature superconductor YBCO. The film was raised above the critical temperature by shining a light on it, and then allowed to cool. The object of the experiment was to determine the number of defects formed, in this case ‘fluxons’ each carrying one quantum of magnetic flux. What Carmi and Polturak measured, using a SQUID detector, was actually the net flux, i.e., the difference \( \Delta N = N_+ - N_- \) between the numbers of fluxons and antifluxons. In fact they saw no evidence for any fluxon formation, with an upper limit of \( |\Delta N| < 10 \).

This result has to be compared with predictions based on Zurek’s work. In this case the Zurek length \( \xi_Z \) is estimated to be about \( 10^{-7} \) m, so within the 1 cm² sample we should expect the total number of defects to be

\[ N = N_+ + N_- \approx 10^{10} \]  \hspace{1cm} (108)

The net flux may be estimated by assuming that the phase of the order parameter performs a random walk with a step length of \( \xi_Z \) and a typical angle \( \delta \sim \pi/2 \). This suggests that

\[ \Delta N \approx \frac{\delta}{2\pi} \sqrt{\frac{L}{\xi_Z}} = \frac{1}{4} \sqrt{\frac{L}{\xi_Z}}, \]  \hspace{1cm} (109)

where \( L \approx 20 \) mm is the perimeter of the sample. (Note that according to this argument \( \Delta N \) is of order \( N^{1/4} \).) This yields

\[ \Delta N \approx 100, \]  \hspace{1cm} (110)
in clear contradiction to the experimental results. It should be noted that this prediction is based on (105) with the constant $\kappa$ set equal to unity, so there may be scope for reducing it slightly, though probably not by enough to remove the discrepancy.

On the other hand, Carmi and Polturak in fact suggest that the disagreement is more serious, because in a gauge theory the mechanism of defect formation is different and the geodesic rule is unreliable [44, 45, 46], so one should perhaps expect $\Delta N$ to be of order $N^{1/2}$, leading to an estimate $\Delta N \approx 10^4$ which is obviously in very severe disagreement with the results. This is a point that needs further theoretical study.

However, the same group have also performed another experiment [47], with very different results. This involved a loop of superconducting wire laid down in a square-wave pattern across a grain boundary in the substrate so as to create a series of $N = 214$ Josephson junctions in series. As the wire cools it becomes superconducting before the Josephson junctions start to conduct, so in effect each segment of wire between neighbouring junctions is initially a separate system, so it is reasonable to assume that their phases are random and uncorrelated. Hence some flux will be trapped when the wire eventually becomes a single superconducting loop. The experiment revealed an r.m.s. flux of

$$\Delta N_{\text{exp}} = 7.4 \pm 0.7.$$  \hspace{1cm} (111)

The theoretical prediction in this case would be

$$\Delta N_{\text{th}} = \frac{1}{4}\sqrt{N} = 3.6.$$  \hspace{1cm} (112)

It is perhaps rather surprising that the experiment saw more flux than predicted. The authors suggest that this may again be due to a breakdown of the geodesic rule with an r.m.s. value of $\delta$ closer to $\pi$ than to $\pi/2$. (Arguably, if $\delta$ is uniformly distributed between $-\pi$ and $\pi$, we should use an r.m.s. value of $\pi/\sqrt{3}$ rather than $\pi/2$, but the difference is minimal, leading to $\Delta N_{\text{th}} = 4.3$.) There could also perhaps be a non-zero phase change along the section of the loop away from the Josephson junctions.

Recently experiments have been performed by a different group [48, 49] on annular Josephson tunnelling junctions, comprising two rings of superconducting material separated by a thin layer. When the system is cooled through the critical temperature and the rings become superconducting, one may expect that the random choice of phase will lead to trapping of fluxons. For the experiments done so far the predicted number trapped is less than one fluxon on average, which is not ideal. Nevertheless, they have detected flux trapping at roughly the predicted level. An important feature of this experiment is that it is possible to vary the quench rate and so test
4.7. DISCUSSION

Experiments with $^3\text{He}$, with liquid crystals and with superconducting loops have all confirmed the basic idea that defects are formed during rapid phase transitions. The best evidence so far that Zurek’s predictions of defect numbers are sound comes from the $^3\text{He}$ experiments, though the others are reasonably consistent.

On the other hand, neither the $^4\text{He}$ experiment nor that with a superconducting film have shown any evidence for defect formation.

At first sight, the discrepancy between the results with $^4\text{He}$ and $^3\text{He}$ may be surprising, but in fact the differences between the two systems are very great. Karra and Rivers [50] have argued that a very important factor is the great discrepancy between the widths of the ‘critical region’, below the critical temperature and above the Ginzburg temperature $T_G$ [51]. This is the temperature above which thermal fluctuations are large enough to create a significant transient population of thermally excited small vortex loops. It is given approximately by the condition that

$$\xi^3(T_G)\Delta F(T_G) = k_BT_G,$$

where $\Delta F$ is the difference in free energy between the ‘false-vacuum’ state with $\phi = 0$ and the broken-symmetry equilibrium state. Above this temperature, it appears, the formation of long vortices is suppressed. It happens that $^3\text{He}$ and $^4\text{He}$ are very different in regard to the width of the critical region between $T_G$ and $T_c$. In $^3\text{He}$ it is extremely narrow; $T_G$ is very close to $T_c$, at $\epsilon \approx 10^{-8}$. In $^4\text{He}$, on the other hand, $T_G$ is about half a degree below $T_c$. Karra and Rivers [50] used thermal field theory, with a Gaussian approximation, to show that the Zurek predictions should be approximately valid provided that

$$\epsilon(T_G)\frac{\tau_q}{\tau_0} \lesssim 100,$$

a condition that is very well satisfied for the $^3\text{He}$ experiments where the left hand side is about $10^{-5}$ and badly violated for those in $^4\text{He}$, where it is $10^{10}$.

Also puzzling is the discrepancy between the different experiments in superconductors. There is some doubt about how to compute the number of defects formed in a transition in a theory with a local gauge symmetry. There is another mechanism operating in a gauge theory [46, 52], but if anything this makes the discrepancy more puzzling because it tends to
suggest that the Zurek prediction of defect numbers would be an underestimate. On the other hand, it is worth noting that the inequality (114) is also seriously violated in the superconducting film experiment, though whether the argument leading to it is valid in the case of symmetry breaking in a gauge theory is not clear.

What is clear is that there is as yet no certainty about when the cosmology-based predictions of defect numbers are reliable. Only further experimental and theoretical work will resolve this question.

Acknowledgements

These lectures were presented at the Summer School on *Patterns of Symmetry Breaking*, supported by NATO as an Advanced Study Institute and by the European Science Foundation Programme *Cosmology in the Laboratory*. I am grateful to several participants for pointing out errors and suggesting improvements to the preliminary version.

References

1. Yeomans, J.M. (1992) *Statistical Mechanics of Phase Transitions*, Clarendon, Oxford.
2. Martin, P.A. and Rothen, F. (2002) *Many-body Problems and Quantum Field Theory: An Introduction*, Springer, Berlin.
3. Anderson, M.H., Ensher J.R., Matthews, M.R., Wieman, C.E. and Cornell, E.A. (1995) Observation of Bose–Einstein condensation in a dilute atomic vapor, *Science* 269, 198–201.
4. Tilley, D.R. and Tilley, J. (1990) *Superfluidity and Superconductivity*, 3rd ed., IoP Publishing, Bristol.
5. Elitzur, S. (1975) Impossibility of spontaneously breaking local symmetries, *Phys. Rev. D* 12, 3978–3982.
6. Itzykson, C. and Drouffe, J.-M. (1989) *Statistical Field Theory*, vol. 1, 3rd ed., p. 341, Cambridge University Press, Cambridge.
7. De Gennes, P.G. and Prost, J. (1993) *The Physics of Liquid Crystals*, Oxford University Press, Oxford.
8. Kibble, T.W.B. (2000) Classification of topological defects and their relevance to cosmology and elsewhere, in *Topological Defects and the Non-Equilibrium Dynamics of Symmetry Breaking Phase Transitions*, ed. Y.M. Bunkov and H. Godfrin, NATO Science Series C 549, 7–31, Kluwer Academic Publishers, Dordrecht.
9. Vollhardt, D. and Wölfle, P. (1990) *The Superfluid Phases of Helium–3*, Taylor and Francis, London.
10. Volovik, G.E. (1992) *Exotic Properties of Superfluid ^3^He*, World Scientific, Singapore.
11. Amaldi, U., de Boer, W. and Fürstenau, H. (1991) Comparison of grand unified theories with electroweak and strong coupling-constants measured at LEP, *Phys. Lett. B* 260, 447–455.
12. Amaldi, U., de Boer, W., Frampton, P.H., Fürstenau, H. and Liu, J.T. (1992) Consistency checks of grand unified theories, *Phys. Lett. B* 281, 374–382.
13. Haber, H.E. (1998) The status of the minimal supersymmetric standard model and beyond, *Nucl. Phys. Proc. Supp.* 62, 469–484.
14. Hu, S.-T. (1959) *Homotopy Theory*, Academic Press, New York.
15. Kajantie, K., Laine, M., Rummukainen, K. and Shaposhnikov, M. (1996) Is there a hot electroweak phase transition at $m(H)$ greater than or similar to $m(W)$?, *Phys. Rev. Lett.* **77**, 2887–2890.

16. Contaldi, C., Hindmarsh, M.B. and Magueijo, J. (1999) Cosmic microwave background and density fluctuations from strings plus inflation, *Phys. Rev. Lett.* **82**, 2034–2037.

17. Durrer, R., Kunz, M. and Melchiorri, A. (2002) Cosmic structure formation with topological defects, *Phys. Rep.* **364**, 1–81.

18. Vachaspati, T. and Vilenkin, A. (1991) Large-scale structure from wiggly cosmic strings, *Phys. Rev. Lett.* **67**, 2034–2037.

19. Durrer, R., Kunz, M. and Melchiorri, A. (2002) Cosmic structure formation with topological defects, *Phys. Rep.* **364**, 1–81.

20. Vachaspati, T. and Vilenkin, A. (1991) Large-scale structure from wiggly cosmic strings, *Phys. Rev. D* **51**, 5946–49.

21. Dimpoloulos, K. (1998) Primordial magnetic fields from superconducting string networks, *Phys. Rev. D* **57**, 4629–41.

22. Bonazzola, S. and Peter, P. (1997) Can high energy cosmic rays be vortons?, *Astropart. Phys.* **7**, 161–172.

23. Bhattacharjee, P. and Sigl, G. (2000) Origin and propagation of extremely high energy cosmic rays, *Phys. Rep.* **327**, 109–247.

24. Davis, A.C. and Perkins, W.B. (1997) Dissipating cosmic vortons and baryogenesis, *Phys. Lett.* **B392**, 46–50.

25. Dimpoloulos, K. and Davis, A.C. (1999) Cosmological consequences of superconducting string networks, *Phys. Lett.* **B446**, 238–246.

26. Chuang, I., Durrer, R., Turok, N. and Yurke, B. (1991) Cosmology in the laboratory — defect dynamics in liquid crystals, *Science* **251**, 1336–42.

27. Bowick, M.J., Chandar, L., Schiff, E.A. and Srivastava, A.M. (1994) The cosmological Kibble mechanism in the laboratory — string formation in liquid crystals, *Science* **263**, 943–5.

28. Srivastava, A.M. (1992) Numerical simulation of dynamical production of vortices by critical and subcritical bubbles, *Phys. Rev. D* **46**, 1353–67.

29. Pogosian, L. and Vachaspati, T. (1998) Relaxing the geodesic rule in defect formation algorithms, *Phys. Lett.* **B423B**, 45–48.

30. Digal, S., Ray, R. and Srivastava, A.M. (1999) Observing correlated production of defects–antidefects in liquid crystals, *Phys. Rev. Lett.* **83**, 5030–33.

31. Zurek, W.H. (1985) Cosmological experiments in superfluid helium, *Nature* **317**, 505–508.

32. Zurek, W.H. (1993) Cosmic strings in laboratory superfluids and topological remnants of other phase transitions, *Acta Phys. Polon.* **B24**, 1301–11.

33. Zurek, W.H. (1996) Cosmological experiments in condensed matter systems, *Phys. Rep.* **276**, 177–221.

34. Kibble, T.W.B. (1980) Some implications of a cosmological phase transition, *Phys. Rep.* **67C**, 183–199.

35. Lagina, P. and Zurek, W.H. (1997) Density of kinks after a quench: When symmetry breaks, how big are the pieces?, *Phys. Rev. Lett.* **78**, 2519–2522.

36. Rivers, R.J. (2000) Slow $^4$He quenches produce fuzzy, transient vortices, *Phys. Rev. Lett.* **84**, 1248–51.

37. Rivers, R.J. (2001) Zurek–Kibble causality bounds in time-dependent Ginzburg–Landau theory and quantum field theory, *J. Low Temp. Phys.* **124**, 41–83.
40. Hendry, P.C., Lawson, N.S. and McClintock, P.V.E. (2000) Does the Kibble mechanism operate in liquid He–4? J. Low Temp. Phys. 119, 249–256.
41. Bäuerle, C., Bunkov, Yu.M., Fisher, S.N., Godfrin, H. and Pickett, G.R. (1996) Laboratory simulation of cosmic string formation in the early Universe using superfluid He–3, Nature 382, 332–334.
42. Ruutu, V.M.H., Eltsov, V.B., Gill, A.J., Kibble, T.W.B., Krusius, M., Makhlin, Yu.G., Plaçais, B., Volovik, G.E. and Xu, W. (1996) Vortex formation in neutron-irradiated superfluid He–3 as an analogue of cosmological defect formation, Nature 382, 334–336.
43. Carmi, R. and Polturak, E. (1999) Search for spontaneous nucleation of magnetic flux during rapid cooling of YBa$_2$Cu$_3$O$_{7-δ}$ films through $T_c$, Phys. Rev. B 60, 7595–7600.
44. Rudaz, S. and Srivastava, A.M. (1993) On the production of flux vortices and magnetic monopoles in phase transitions, Mod. Phys. Lett. A 8, 1443–50.
45. Copeland, E.J. and Saffin, P. (1996) Bubble collisions in Abelian gauge theories and the geodesic rule, Phys. Rev. D 54, 6088–94.
46. Hindmarsh, M.B. and Rajantie, A. (2000) Defect formation and local gauge invariance, Phys. Rev. Lett. 85, 4660–63.
47. Carmi, R., Polturak, E. and Koren, G. (2000) Observation of spontaneous flux generation in a multi-Josephson-junction loop, Phys. Rev. Lett. 84, 4966–69.
48. Kavoussanaki, E., Monaco, R. and Rivers, R.J. (2000) Testing the Kibble–Zurek scenario with annular Josephson tunneling junctions, Phys. Rev. Lett. 85, 3452–5.
49. Monaco, R., Mygind, J. and Rivers, R.J. (2002) Zurek–Kibble domain structures: The dynamics of spontaneous vortex formation in annular Josephson tunneling junctions, Phys. Rev. Lett. 89, 080603.
50. Karra, G. and Rivers, R.J. (1997) Initial vortex densities after a temperature quench, Phys. Lett. 414B, 28–33.
51. See for instance ref. [4], p. 347; but see also Kleinert, H. and Schulte-Frohlinde, V. (2001), Critical properties of $\phi^4$-theories, World Scientific Publishing Co., Singapore, p. 18.
52. Hindmarsh, M.B. and Rajantie, A. (2001) Phase transition dynamics in the hot Abelian Higgs model, Phys. Rev. D 64, 065016.