A MULTISCALE-ANALYSIS OF STOCHASTIC BISTABLE REACTION-DIFFUSION EQUATIONS

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Abstract. A multiscale analysis of 1D stochastic bistable reaction-diffusion equations with additive noise is carried out w.r.t. travelling waves within the variational approach to stochastic partial differential equations. It is shown with explicit error estimates on appropriate function spaces that up to lower order w.r.t. the noise amplitude, the solution can be decomposed into the orthogonal sum of a travelling wave moving with random speed and into Gaussian fluctuations. A stochastic differential equation describing the speed of the travelling wave and a linear stochastic partial differential equation describing the fluctuations are derived in terms of the coefficients. Our results extend corresponding results obtained for stochastic neural field equations to the present class of stochastic dynamics.

1. Introduction

Consider the reaction-diffusion equation
\begin{equation}
\partial_t v(t, x) = \nu x^2 v(t, x) + b f(v(t, x)), \ t > 0, \ x \in \mathbb{R}
\end{equation}
for strictly positive constants \( \nu, b > 0 \) and bistable reaction term
\begin{align}
f(0) &= f(a) = f(1) = 0 \quad \text{for some } a \in (0, 1) \\
f(x) &< 0 \quad \text{for } x \in (0, a), f(x) > 0 \text{ for } x \in (a, 1) \\
f'(0) &< 0, f'(a) > 0, f'(1) < 0.
\end{align}

It is well-known that under the above assumptions (1.1) admits a travelling wave solution, i.e. a monotone increasing \( C^2 \)-function \( \hat{v} \), connecting the stable fixed points 0 and 1 of the reaction term, satisfying
\begin{equation}
c \hat{v}_x = \nu \hat{v}_{xx} + b f(\hat{v})
\end{equation}
for some wavespeed \( c \in \mathbb{R} \) and boundary conditions \( \hat{v}(-\infty) = 0, \hat{v}(+\infty) = 1 \), see, e.g. Theorem 12 in [6]. It is easy to see that \( \hat{v}(\cdot + ct) \) and all spatial translates \( \hat{v}(\cdot + x_0 + ct) \) are solutions of (1.1).

In the particular case of the cubic nonlinearity \( f(v) = v(1-v)(v-a) \), equation (1.1) reduces to the well-known Nagumo equation (cf. [14]), for which the travelling wave is explicitly given by
\[
\hat{v}(x) = \left(1 + e^{-\sqrt{\frac{b}{\nu}}x}\right)^{-1}
\]
propagating along the axis with speed \( c = \sqrt{2b(\frac{1}{2} - a)} \).

It can be observed in numerical simulations that the travelling wave solution persists up to apparently lower order fluctuations also under the impact of noisy perturbations (1.1). The term ”lower-order” here is to be understood w.r.t. the order of the noise amplitude and will be made precise in our subsequent analysis. Even more holds for small noise amplitude: similar to the deterministic case, the solution of (1.1) will converge quickly to some profile of the type
\begin{equation}
\gamma(t) = \hat{v}(x + \gamma(t)) + u(t, x),
\end{equation}
where \( \gamma(t) \) is the random position of the wave front and \( u \) denotes lower order fluctuations. For this reason we will call the solution \( v \) also a stochastic travelling wave. It is the main purpose of this
paper to rigorously derive a decomposition of type \([1.3]\) in the case of noisy perturbations induced by function-space valued additive Wiener noise (see e.g. \([2]\)), together with an identification of \(\gamma\) (resp. \(u\)) as solution of a stochastic differential equation (resp. linear stochastic partial differential equation) accompanied with explicit error estimates in terms of \(u\) in appropriate function spaces. More specifically, we consider the stochastic reaction-diffusion equation

\[ du(t, x) = \nu\partial_{xx}^2 v(t, x) dt + bf(v(t, x)) dt + \varepsilon\, dW(t, x), \quad t > 0, \quad x \in \mathbb{R} \]

where \(W(t), t \in [0, T]\), denotes a \(Q\)-Wiener process, taking values on a suitable Hilbert space \(H\) and \(\varepsilon > 0\) will be considered to be a small parameter.

The problem is obviously connected with the stability properties of the travelling wave solution \(\hat{v}\) and the latter problem has been intensively studied in the deterministic setting for a long time, mainly based on maximum principle and comparison techniques (cf. in particular \([3]\)), and on spectral considerations (cf. \([4, 7]\) and the more recent monograph \([3]\)). Both approaches however are not easy to carry over to the stochastic case, or even worse, require unnatural monotonicity conditions on the noise terms. Instead, we rather apply a pathwise stability analysis in the spirit of the classical Lyapunov approach to dynamical systems, first developed in \([17, 18]\).

Note that decomposition \([1.3]\) is not well-posed without further assumptions, since it involves two unknowns: the position \(\gamma(t)\) of the wave front and the remainder \(u(t, x)\). In addition, there are various possibilities to define the position of the wave front. Since pointwise criteria like, e.g. level sets of \(v(t, x)\) do not make much sense in the stochastic case, because of (spatially non-monotone) fluctuations, we take a minimiser

\[ \gamma(t) := \argmin_{\gamma \in \mathbb{R}} |v(t, \cdot) - \hat{v}(\cdot + \gamma)| \]

of the distance between \(v(t, \cdot)\) and the spatial translates of \(\hat{v}\) as the (in general non-unique) definition of the position of the wave-front.

Since global minima are difficult to characterise and difficult to handle in the context of dynamical equations, we will require only the necessary condition \(|v(t, \cdot) - \hat{v}(\cdot + \gamma(t)), \hat{v}_x(\cdot + \gamma(t))| = 0\) for local minima in the decomposition \([1.3]\). Here, \(|,\cdot|\) denotes the scalar product of the underlying Hilbert space \(H\). In \([8]\), the authors used this approach and obtained results on the corresponding decomposition up to the first random time \(\tau\), when the local minimum becomes a saddle point. Apart from the necessity of introducing the above mentioned stopping time, the position of the wave front will be a semi-martingale only, in particular non-differentiable. In addition, the decomposition obtained in this way will be non-explicit w.r.t. the small parameter \(\varepsilon\).

To avoid non-differentiability we will consider in this paper the following (differentiable) approximation \(\gamma^m(t)\), defined as the solution of the following pathwise ordinary differential equation

\[ \dot{\gamma}^m(t) = c + \langle v(t, \cdot) - \hat{v}(\cdot + \gamma^m(t)), \hat{v}_x(\cdot + \gamma^m(t)) \rangle \]

for suitable initial condition and a priori chosen relaxation parameter \(m > 0\). We will then show in Theorem \([3.3]\) as our first main result that \(\gamma^m(t)\) admits the decomposition

\[ \gamma^m(t) = ct + \varepsilon \, C_0^m(t) + o(\varepsilon) \]

where \(C_0^m(t) = \int_0^t c_0^m(s) \, ds\), and \(c_0^m\) is the unique strong solution of the stochastic ordinary differential equation

\[ dc_0^m(t) = -mc_0^m(t) \, dt + m \langle \hat{v}_x, dW(t) \rangle \]

for suitable initial condition. The fluctuations \(u(t) = v(t, \cdot) - \hat{v}(\cdot + ct + \varepsilon C_0^m(t))\) can be represented as

\[ u(t) = \varepsilon \, u_0^m(t) + o(\varepsilon) \]

where \(u_0^m\) is the unique variational solution of the (affine) linear stochastic partial differential equation

\[ du_0^m(t) = [\nu \Delta u_0^m(t) + bf(\cdot + ct)u_0^m(t) - c_0^m(t) \hat{v}_x(\cdot + ct)] \, dt + dW(t). \]

To compare our results to the analysis in \([8]\), based on the variational characterisation of local minima of \(|v(t, \cdot) - \hat{v}(\cdot + \gamma)|\), we also consider the asymptotics of our decomposition for \(m \to \infty\),
i.e. in the limit of immediate relaxation. We will show in Lemma 3.3 below that both processes \( \gamma^n(t) \) and \( u^n(t) \) converge as \( n \to \infty \) and we will identify the limiting processes again as solutions of stochastic (partial) differential equation together with explicit error estimates in the associated decomposition (see Theorem 3.3).}

### 2. Stochastic Reaction-Diffusion Equations in Weighted Sobolev Spaces

#### 2.1. Realisation as stochastic evolution equation

Thinking of a typical trajectory of a stochastically perturbed travelling wave one cannot expect the solution \( v \) of (1.4) to take values in \( L^2(\mathbb{R}) \), which raises the question in which sense and especially on which function space to model the above reaction-diffusion equation. If we assume the noise to be on stochastically perturbed travelling wave one cannot expect the solution to be a realisation as stochastic evolution equation.

This motivates us to derive a decomposition of \( v \) into \( v = u + \hat{v} \) and investigate properties of the difference \( u \) under a smallness condition on \( u^0 := v^0 - \hat{v} \). This approach to study the dynamics of stochastic travelling wave solutions has also been used in [1], [17], [10] and [11]. In the stability analysis below it turns out, that controlling \( u \) in \( L^2(\mathbb{R}) \) with respect to the Lebesgue measure is not sufficient, but that we need an additional control with respect to the measure

\[
\rho(x) = Z e^{-\frac{x^2}{2}}
\]

with a positive constant \( Z > 0 \). This measure is naturally associated to the equation and will be derived and motivated in Section 2.3. Assume \( (W(t))_{t \in [0,T]} \) to be a Q-Wiener process on the space \( L^2(1 + \rho) = L^2(\mathbb{R}, dx) \cap L^2(\mathbb{R}, \rho \, dx) \). First, we rewrite (1.4) in terms of \( u = v - \hat{v} \) and obtain

\[
du(t, x) = v \Delta u(t, x) \, dt + b \left( f(u(t, x) + \hat{v}(x + ct)) - f(\hat{v}(x + ct)) \right) \, dt + \epsilon dW(t, x), \quad t > 0, \quad x \in \mathbb{R}
\]

This SPDE can now be formulated as a stochastic evolution equation on \( L^2(1 + \rho) \). Following the approach of [15] we choose the Gelfand triple

\[
H^1(1 + \rho) \hookrightarrow L^2(1 + \rho) \hookrightarrow H^{1,*}(1 + \rho)
\]

with the norms

\[
\|u\|_{H^1(1 + \rho)}^2 = \int u^2(1 + \rho) \, dx \quad \text{on} \quad L^2(1 + \rho)
\]

and

\[
\|u\|_{H^{1,*}(1 + \rho)}^2 = \|u\|_{H^1(1 + \rho)}^2 + \|u_x\|_{H^1(1 + \rho)}^2 \quad \text{on} \quad H^1(1 + \rho).
\]

Define the operator \( \Delta : H^1(1 + \rho) \to H^{1,*}(1 + \rho) \) as

\[
H^{1,*}(\Delta v, u)_{H^1} = -\int \left( v_x(u(1 + \rho)) - \int v_x(u(1 + \rho)) \right) \, dx
\]

Let \( G : \mathbb{R} \times H^1(1 + \rho) \to H^{1,*}(1 + \rho) \) be given by

\[
H^1(v, G(t, u))_{H^{1,*}} = \int v(x) b \left( f(u(x) + \hat{v}(x + ct)) - f(\hat{v}(x + ct)) \right) (1 + \rho(x)) \, dx.
\]

In this terminology (2.1) corresponds to the stochastic evolution equation

\[
\begin{align*}
\frac{du}{dt} &= \nu \Delta u + G(t, u) dt + \epsilon dW \\
\end{align*}
\]

on \( L^2(1 + \rho) \). To prove well-posedness of this equation we impose the following additional assumptions on the global behaviour of the reaction term \( f \). We assume that the derivative \( f' \) is bounded from above by

\[
\alpha_1 := \sup_{x \in \mathbb{R}} f'(x) < \infty,
\]

that there exists a finite positive constant \( L \) such that

\[
|f(x_1) - f(x_2)| \leq L|x_1 - x_2| \left( 1 + x_1^2 + x_2^2 \right) \quad \forall x_1, x_2 \in \mathbb{R},
\]
which is typically satisfied for polynomials of third degree with leading negative coefficient and that there exists \( \eta_2 \) such that
\[
|f(u + v) - f(v) - f'(v)u| \leq \eta_2(1 + |u|)|u|^2 \quad \forall v \in [0, 1], u \in \mathbb{R}.
\]

Furthermore, to ensure even higher regularity of solutions of (2.2), namely weak spatial differentiability, we demand the following growth condition of the derivative \( f' \). There exists a constant \( \eta_3 \) > 0 with
\[
|f'(u + v)| \leq \eta_3(1 + |u|^2)
\]
and
\[
|f'(u + v) - f'(v)| \leq \eta_3(|u| + |u|^2) \quad \forall v \in [0, 1], u \in \mathbb{R}.
\]

For simplicity of the following analysis we also assume
\[
(A1) \quad \int_0^1 f(v) dv \geq 0
\]
This condition on the input parameter \( f \) or, in other words, on the associated potential \( F = \int f dv \) implies that the wave speed \( c \), as part of the travelling wave solution \( \hat{v} \), is nonnegative. To briefly illustrate this relation we multiply (1.2) by \( \hat{v} \) and integrate over the real line to obtain
\[
c \int \hat{v}^2 dx = \nu \int \hat{v}_x \hat{v}_x dx + b \int f(\hat{v}) \hat{v}_x dx = \nu \int \frac{d}{dx} \hat{v}^2 dx + b \int_0^1 f(\hat{v}) \hat{v} dx = b \int_0^1 f(\hat{v}) \hat{v} dx
\]
due to the boundary conditions of \( \hat{v} \) at \( \pm \infty \). This shows that the wave speed \( c \) and the integral on the right-hand side have the same sign. Assumption (A1) is required in our analysis only to ensure that \( q \) is monotone decreasing. Dropping this assumption would require to consider both cases, monotone decreasing (resp. increasing).

Under the above conditions the following existence and uniqueness result can be stated.

**Theorem 2.1.** Assume (B1) – (B3). For each \( T > 0, \varepsilon > 0 \) and each \( \eta \in L^2(1 + \rho) \) there exists a unique variational solution \( u \in L^2(\Omega; C([0, T]; L^2(1 + \rho))) \cap L^2(\Omega \times [0, T]; H^1(1 + \rho)) \), almost surely satisfying the integral equation
\[
u(t) = u^0 + \int_0^t \nu \Delta u(s) ds + \int_0^t G(s, u(s)) ds + \varepsilon W(t), \quad t \in [0, T]
\]
\[
 u^0 = \varepsilon \eta
\]
Here the Bochner integrals are defined in \( H^{1,*}(1 + \rho) \). Moreover, there exists a constant \( C(T, \omega, \|u^0\|^2_{1+\rho}) > 0 \) such that for all \( t \in [0, T] \)
\[
\sup_{s \leq t} \|u(s)\|^2_{1+\rho} + \int_0^t \|u(s)\|^2_{H^1(1+\rho)} ds \leq C(T, \omega, \|u^0\|^2_{1+\rho})
\]
and \( C(T, \omega, \|u^0\|^2_{1+\rho}) < \infty \) for almost all \( \omega \in \Omega \).

**Proof.** We show that the coefficients in (2.3) satisfy the conditions of Theorem 1.1 in [12]. Let \( u, w \in H^1(1 + \rho) \). \( G \) can be realised as a continuous mapping \( H^1(1 + \rho) \to H^{1,*}(1 + \rho) \): Using property (B2) and the elementary estimate \( \|u\|_{\infty} \leq \|u\|_{H^1(1+\rho)} \) one obtains
\[
H^{1,*} \langle G(t, u), w \rangle_{H^1} = b \int (f(u + \hat{v}(\cdot + ct)) - f(\hat{v}(\cdot + ct))) w(1 + \rho) dx
\]
\[
\leq bL \int \|u(3 + 2u^2)|w|(1 + \rho) dx \leq bL(3 + 2\|u\|^2_{H^1(1+\rho)}) \|w\|_{H^1(1+\rho)}
\]
The drift and diffusion operators are clearly hemicontinuous. For showing coercivity we estimate
\[
H^{1,*} \langle \nu \Delta u, w \rangle_{H^1} = -\nu \int u^2_x (1 + \rho) dx - \nu \int u_x u \rho_x dx \leq -\nu \|u\|^2_{H^1(1+\rho)} + \left( \nu + \frac{c^2}{2\rho} \right) \|u\|^2_{1+\rho}
\]
and

\[ H^1 \cdot \langle G(t, u), u \rangle_{H^1} = b \int \left( f(u + \dot{v}(\cdot + ct)) - f(\dot{v}(\cdot + ct)) \right) u(1 + \rho) \, dx \leq b \eta_1 \| u \|_{1 + \rho}^2 \]

since \( f \) is one-sided Lipschitz continuous, i.e. \( (f(x) - f(y))(x - y) \leq \eta_1 (x - y)^2 \) for all \( x, y \in \mathbb{R} \) using (B1). Weak monotonicity of the linear part of the drift operator is covered by the coercivity condition. For the nonlinear part with the one-sided Lipschitz estimate from above one obtains

\[ \| u \|_{H^1} \leq b \eta_1 \int (u_1 - u_2)^2(1 + \rho) \, dx = b \eta_1 \| u_1 - u_2 \|_{1 + \rho}^2 \]

Furthermore, \( G \) is of admissible growth: Let \( w \in H^1(1 + \rho) \) with \( \| w \|_{H^1(1 + \rho)} \leq 1 \). Applying condition (B2) yields

\[ H^1 \cdot \langle G(t, u), w \rangle_{H^1} \leq bL \int |u|(3 + 2u^2)w(1 + \rho) \, dx \leq 3bL\| u \|_{1 + \rho} + 2bL\| u \|_{H^1(1 + \rho)} \| u \|_{1 + \rho} \]

\[ \leq 3bL\| u \|_{H^1(1 + \rho)}(1 + \| u \|_{1 + \rho}) \]

For the second part of the statement an application of Itô’s formula as stated in [16, Theorem 4.2.5] together with the coercivity of the drift yields for all \( t \in [0, T] \)

\[ \| u(t) \|_{1 + \rho}^2 = \| u_0 \|_{1 + \rho}^2 + \int_0^t \left( 2H^1 \cdot \langle \nu \Delta u(s) + G(s, u(s)), u(s) \rangle_{H^1} + \| \sqrt{Q} \|_{L^2(L^2(1 + \rho))}^2 \right) ds \]

\[ + 2 \int_0^t \langle u(s), dW(s) \rangle \]

\[ \leq \| u_0 \|_{1 + \rho}^2 - 2\nu \int_0^t \| u(s) \|_{H^1(1 + \rho)}^2 \, ds + 2 \left( \nu + \frac{\rho^2}{2\nu} + b\eta_1 \right) \int_0^t \| u(s) \|_{1 + \rho}^2 \, ds \]

\[ + T \text{tr}(Q) + M(t) \]

where \( M(t), t \in [0, T], \) denotes the martingale part and \( \| \cdot \|_{L^2(L^2(1 + \rho))} \) the Hilbert-Schmidt norm on \( L^2(1 + \rho) \). In particular, setting \( \theta := 2 \left( \nu + \frac{\rho^2}{2\nu} + b\eta_1 \right) \) this implies

\[ \| u(t) \|_{1 + \rho}^2 \leq \| u_0 \|_{1 + \rho}^2 + \theta \int_0^t \| u(s) \|_{1 + \rho}^2 \, ds + T \text{tr}(Q) + M(t) \]

such that Gronwall’s lemma allows us to bound the expectation of the left-hand side by

\[ E \left[ \| u(t) \|_{1 + \rho}^2 \right] \leq e^{\theta t} \left( \| u_0 \|_{1 + \rho}^2 + T \text{tr}(Q) \right) \]

Taking first the supremum and then the expectation of all terms in [2.3] and inserting [2.4] we obtain the estimate

\[ E \left[ \sup_{t \in [0, T]} \| u(t) \|_{1 + \rho}^2 + 2\nu \int_0^T \| u(s) \|_{H^1(1 + \rho)}^2 \, ds \right] \]

\[ \leq \| u_0 \|_{1 + \rho}^2 + \theta \int_0^T E \left[ \| u(s) \|_{1 + \rho}^2 \right] \, ds + T \text{tr}(Q) + E \left[ \sup_{t \in [0, T]} |M(t)| \right] \]

\[ \leq \| u_0 \|_{1 + \rho}^2 + e^{\theta t} \left( \| u_0 \|_{1 + \rho}^2 + T \text{tr}(Q) \right) + T \text{tr}(Q) + E \left[ \sup_{t \in [0, T]} |M(t)| \right] \]

A control of the last expectation in terms of the martingale’s quadratic variation \( |M|, t \in [0, T] \), is provided by the Burkholder-Davis-Gundy inequality

\[ E \left[ \sup_{t \in [0, T]} |M(t)| \right] \leq CE \left[ |M|^{\frac{3}{2}} \right] = 2CE \left[ \left( \int_0^t \langle u(s), Q u(s) \rangle ds \right)^{\frac{3}{2}} \right] \]
Assumption 2.2. Let $\sqrt{Q}$ be a Hilbert-Schmidt operator from $L^2(1+\rho)$ to $H^1(1+\rho)$.

Remark 2.3. In the following analysis we will frequently consider translations of the measure $\rho$ w.r.t. deterministic as well as stochastic shifts, for which reason the following two properties of $\rho$ will become important:

(i) $\rho$ is decreasing, i.e. $\rho(x+y) \leq \rho(x)$ for all $y > 0$.
(ii) $\rho$ is of (at most) exponential growth, i.e., $\rho(x - \xi) \leq e^{M|\xi|}\rho(x)$ for $M = \frac{\rho}{\xi}$ and all $\xi \in \mathbb{R}$.

Lemma 2.4. Assume (B4). Let $p_t = e^{t\nu\Delta}$, $t \geq 0$, denote the semigroup generated by $\nu\Delta$. Then there exists a finite constant $C = C(T, ||\hat{\nu}||_{\infty})$ such that for all $u \in H^1(1+\rho)$ and for all $s \leq t$ the following bound holds:

$$||\partial_x p_t - s G(s, u)||_{1+\rho}^2 \leq C(1 + ||u||_{1+\rho}^2 ||u_x||_{2+\rho}^2)(||u||_{1+\rho}^2 + ||u_x||_{2+\rho}^2).$$

Proof. Let $u \in C^1(\mathbb{R})$. Then, (B4) leads to

$$|\partial_x p_t G(s, u)| = |p_t - s (\partial_x G(s, u))|
\leq p_t - s |f'(u + \hat{\nu}(\cdot + cs))u_x + (f'(u + \hat{\nu}(\cdot + cs)) - f'(\hat{\nu}(\cdot + cs)))\hat{\nu}_x (\cdot + cs)|
\leq p_t - s [\eta_3 (1 + u^2)|u_x| + \eta_3 (||u||^2 + u^2) \hat{\nu}_x (\cdot + cs)]$$

It is well-known that $\nu\Delta$ generates the Gaussian semigroup

$$p_t f(x) = \frac{1}{\sqrt{4\pi t}} \int f(x+y)e^{-\frac{x^2}{4t}} dy.$$  

Applying it to the measure $1 + \rho$ we obtain

$$p_t (1 + \rho)(x) = 1 + \frac{Ze^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \int e^{-\frac{y^2}{4t}} dy = 1 + \frac{Ze^{-\frac{x^2}{4t} + \frac{y^2}{4t}}}{\sqrt{4\pi t}} \int e^{-\frac{y^2 + 2xy}{4t}} dy = 1 + Ze^{-\frac{x^2 + y^2}{4t}}.$$
Hence, there exists a constant $C = C(\|\tilde{\nu}_x\|_{\infty})$ such that
\[
\int |\partial_x p_{t-s} G(s, u)|^2 (1 + \rho) \, dx \leq C \int p_{t-s} (u_x^2 + u^4 + u_x^2 + u^4) (1 + \rho) \, dx \\
= C \int (u_x^2 + u^4 + u_x^2 + u^4) (1 + p_{t-s} \rho) \, dx \\
\leq C e^{\frac{2}{1+\rho} t-s} \int (u_x^2 + u^4 + u_x^2 + u^4) (1 + \rho) \, dx \\
\leq C e^{\frac{2}{1+\rho} T} (1 + \|u\|_{L^\infty(dx)}^4) \int (u_x^2 + u^2) (1 + \rho) \, dx
\]
Now, with the elementary estimate $\|u\|_{L^\infty(dx)}^2 \leq 2 \|u\|_{1+\rho} \|u_x\|_{1+\rho}$ there exists a positive constant $C = C(T, \|\tilde{\nu}_x\|_{\infty})$ such that
\[
\|\partial_x p_{t-s} G(s, u)\|_{1+\rho}^2 \leq C (1 + \|u\|_{1+\rho} \|u_x\|_{1+\rho} (\|u\|_{1+\rho} + \|u_x\|_{1+\rho}^2).
\]

With this at hand we are able to formulate the following regularity statement:

**Proposition 2.5.** Under Assumption 2.2 for each $T > 0$ and each $\eta \in H^1(1 + \rho)$ there exists a unique variational solution $u \in C([0, T]; H^1(1 + \rho))$ of equation (2.3).

**Proof.** Let $u$ be the unique variational solution in $L^2(\Omega; C([0, T]; L^2(1 + \rho))) \cap L^2(\Omega \times (0, T); H^1(1 + \rho))$ from Theorem 2.1. Using the semigroup $p_t = e^{t\nu^2}$, $t \geq 0$, $u$ allows for the mild solution representation
\[
u(t) = p_t u^0 + \int_0^t p_{t-s} G(s, u(s)) \, ds + \int_0^t p_{t-s} dW(s)
\]
We will show that indeed $u$ is differentiable in space and the gradient takes values in $L^2(1 + \rho)$:
For every $t \in [0, T]$ with Lemma 2.1
\[
\|u_x(t)\|_{1+\rho}^2 \leq 2 \|p_t u_x^0\|_{1+\rho}^2 + 4 \|\partial_x p_{t-s} G(s, u(s))\|_{1+\rho}^2 \, ds + 4 \|\partial_x \int_0^t p_{t-s} dW(s)\|_{1+\rho}^2 \\
\leq 2 \|p_t u_x^0\|_{1+\rho}^2 + 4 C \int_0^t (1 + \|u(s)\|_{1+\rho}^2) \|u_x(s)\|_{1+\rho}^2 (\|u(s)\|_{1+\rho}^2 + \|u_x(s)\|_{1+\rho}^2) \, ds \\
+ 4 \|\partial_x \int_0^t p_{t-s} dW(s)\|_{1+\rho}^2 =: I + II + III, \text{ say.}
\]
Clearly,
\[
I \leq \sup_{t \in [0, T]} \|p_t\|^2 \|u_x^0\|_{1+\rho}^2 =: C_1(T, \|u_x^0\|_{1+\rho})
\]
For uniformly bounding $III$ note that for a $Q$-Wiener process $W$ on $H^1(1 + \rho)$ we know that
\[
\int_0^t p_{t-s} dW(s) \in L^2(\Omega, C([0, T]; H^1(1 + \rho)))
\]
and thus
\[
III \leq 4 \sup_{t \in [0, T]} \|\partial_x \int_0^t p_{t-s} dW(s)\|_{1+\rho}^2 =: C_3(T, \omega) < \infty \text{ for P-a.e. } \omega \in \Omega.
\]
The bound 2.4 is now applied to reduce the second term to
\[
II \leq C_{2.1} \int_0^t \|u_x(s)\|_{1+\rho}^4 \, ds + C_{2.2}
\]
for constants $C_{2.1} = C_{2.1}(T, \omega, \|u^0\|_{1+\rho})$ and $C_{2.2} = C_{2.2}(T, \omega, \|u^0\|_{1+\rho})$. Altogether, this leads to
\[
\|u_x(t)\|_{1+\rho} \leq C(T, \omega, \|u^0\|_{H^1(1+\rho)}) + C_{2.1} \int_0^t \|u_x(s)\|_{1+\rho}^4 \, ds.
\]
Set \( \alpha(t) := \|u_x(t)\|^2_{1+p} \). Then, Gronwall’s inequality together with \(2.3\) implies
\[
\|u_x(t)\|^2_{1+p} \leq C(T, \omega, \|u^0\|_{H^1(1+\rho)}) \exp \left( C_{2.1} \int_0^t \alpha(s) \, ds \right) \leq \tilde{C}(T, \omega, \|u^0\|_{H^1(1+\rho)})
\]
for a positive constant \( \tilde{C} \). The previous \( P \)-nullsets on which \( \tilde{C} \) is possibly infinite do not depend on \( t \) such that we even have
\[
\sup_{t \in [0,T]} \|u_x(t)\|^2_{1+p} \leq \tilde{C}(T, \omega, \|u^0\|_{H^1(1+\rho)})
\]
To prove continuity it remains to show that \( u_x \in C([0,T]; L^2(1+\rho)) \). Considering
\[
u_x(t) = pu_x^0 + \int_0^t \partial_x p_{t-s} G(s, u(s)) \, ds + \partial_x \int_0^t p_{t-s} dW(s) = I + II + III
\]
clearly \( I \) and \( III \) have the required regularity. For the second term note that with Lemma \(2.4\) for all \( s \leq t \) the integrand takes values in \( L^2(1+\rho) \) such that \( II \) is a continuous process on this space. \( \square \)

The objective of the following sections is to more thoroughly study the behaviour of \( u \), providing information about the deviation of the stochastic solution from the deterministic wave. We will identify a suitable stochastic phase of the chosen reference profile \( \hat{v} \) and subsequently quantify the remaining fluctuations around this profile in terms of the noise strength \( \varepsilon \). This first-order stability analysis with respect to \( \varepsilon \) follows closely the methods and ideas developed in \(11\) (and partially also in \(10\)) for the study of stochastic neural field equations. In the case of bistable reaction-diffusion equations the involved diffusion and reaction operators however are no longer bounded, such that an extended notion of solution as well as refined estimates have to be introduced. To have the flexibility of also working in the larger spaces \( L^2(\rho) \) and \( L^2(\mathbb{R}) \) we introduce a second Gelfand triple
\[
V \hookrightarrow H = H^* \hookrightarrow V^*
\]
with \( H = L^2(\mathbb{R}) \) and \( V = H^1(\mathbb{R}) \).

2.3. The frozen-wave setting. Since our aim is to study the dynamics of solutions whose initial profile is already close to a travelling wave solution, we linearise the nonlinearity \( f \) around the travelling wave \( \hat{v} \) and obtain the representation

\[
du(t) = [\mathcal{L}_t u(t) + R(t, u)] \, dt + \varepsilon \, dW(t)
\]
with
\[
(2.9) \quad \mathcal{L}_t : V \to V^*, \quad \mathcal{L}_t u = \nu \Delta u + b f'(\hat{v}(\cdot + ct)) \, u, \quad t \in [0,T]
\]
and the nonlinear remainder
\[
R : [0,T] \times V \to V^*, \quad R(t, u) = b \left( f(u + \hat{v}(\cdot + ct)) - f(\hat{v}(\cdot + ct)) - f'(\hat{v}(\cdot + ct))u \right)
\]

To identify the dynamical equations for the wave-speed for the stochastic travelling wave, as well as for the remaining fluctuations, it will be useful to introduce the frozen-wave operator \( \mathcal{L}^\#: V \to V^* \) given by
\[
\mathcal{L}^\# u := \nu \Delta u + b f'(\hat{v}) u - c \partial_x u.
\]
Note that by \(1.2\) we have \( \mathcal{L}^\# \hat{v}_x = 0 \). Likewise, one can easily check that an eigenvector \( \Psi \) of the adjoint operator corresponding to the eigenvalue 0 is given by \( \Psi(x) = e^{-\frac{c}{2} \varepsilon} \hat{v}_x(x) \). \( \Psi \) is strictly positive and we assume the following regularity property:

**Assumption 2.6.** The zero-eigenfunction \( \Psi \) of \( \mathcal{L}^\#_* \) satisfies \( \Psi \in V \).
Indeed, this regularity can be shown under the following additional assumption on the reaction function \( f \), namely
\[
\text{(A2)} \quad f \in C^2, f''(0) > 0 \quad \text{and} \quad f''(1) < 0
\]
saying that \( f \) is strictly convex in a small neighbourhood around 0 and strictly concave in a small neighbourhood around 1. The main implication of this additional assumption is then formulated in the next Lemma.

**Lemma 2.7.** Assume (A1) – (A2). Then \( \frac{f'(\hat{v})}{\hat{v}_x} \) is strictly increasing in \( x \) at \( \pm \infty \). In particular,
\[
\exists \gamma_- := \lim_{x \downarrow -\infty} \frac{b f'(\hat{v})}{\hat{v}_x} (x) = \frac{c}{2\nu} - \sqrt{\left( \frac{c}{2\nu} \right)^2 - \frac{b}{\nu} f'(0)} < 0
\]
and
\[
\exists \gamma_+ := \lim_{x \uparrow \infty} \frac{b f'(\hat{v})}{\hat{v}_x} (x) = \frac{c}{2\nu} + \sqrt{\left( \frac{c}{2\nu} \right)^2 - \frac{b}{\nu} f'(1)} > 0.
\]

**Proof.** We will first show that \( \lim_{|x| \to \infty} e^{-\frac{\nu x}{2}} \hat{v}_x^2 (x) = 0 \).
Indeed, \( \hat{v}_x \in L^1(\mathbb{R}) \), hence \( \lim_{n \to \infty} \hat{v}_x (x_n) = 0 \) for some sequence \( x_n \uparrow \infty \), implies that
\[
\hat{v}_x^2 (x) = \int_{x_n}^{x} \hat{v}_x \hat{v}_x \, dx
\]
and
\[
\hat{v}_x^2 (x) \leq \int_{x_n}^{x} \hat{v}_x \hat{v}_x \, dx + 2 \int_{x_n}^{x} \hat{v}_x \hat{v}_x \, dx + 2 \int_{x_n}^{x} \hat{v}_x \hat{v}_x \, dx
\]
Consequently,
\[
\hat{v}_x^2 (x) \leq \lim_{n \to \infty} \hat{v}_x^2 (x_n) + 2 \int_{x_n}^{x} \hat{v}_x \hat{v}_x \, dx + 2 \int_{x_n}^{x} \hat{v}_x \hat{v}_x \, dx
\]
In particular,
\[
\lim_{x \to \infty} \hat{v}_x^2 (x) \leq \limsup_{x \to \infty} \int_{x_n}^{x} \hat{v}_x \hat{v}_x \, dx = 0
\]
and thus, \( \lim_{x \to \infty} e^{-\frac{\nu x}{2}} \hat{v}_x^2 (x) = 0 \), too.

To see that \( \lim_{x \to -\infty} e^{-\frac{\nu x}{2}} \hat{v}_x^2 (x) = 0 \) note that for \( x \leq x_0 := \hat{v}^{-1}(a) \)
\[
d(x) = 2 \left( -\frac{c}{\nu} \hat{v}_x + \hat{v}_x \hat{v}_x \right) e^{-\frac{\nu x}{2}} \hat{v}_x (x) = -2 \frac{b}{\nu} e^{-\frac{\nu x}{2}} f' (\hat{v}) \hat{v}_x (x) > 0
\]
Consequently,
\[
\lim_{x \to -\infty} e^{-\frac{\nu x}{2}} \hat{v}_x^2 (x) = \inf_{x \leq \hat{v}^{-1}(a)} e^{-\frac{\nu x}{2}} \hat{v}_x^2 (x) =: \gamma < \infty
\]
and thus
\[
\lim_{x \to -\infty} e^{-\frac{\nu x}{2}} \hat{v}_x^2 (x) \leq \limsup_{x \to -\infty} e^{-\frac{\nu x}{2}} \hat{v}_x^2 (x) = 0
\]
Now define \( w := e^{-\frac{\nu x}{2}} \hat{v}_x \) and note that
\[
w_{xx} = \left( \frac{c}{2\nu} \right)^2 - \frac{b}{\nu} f'(\hat{v}) w,
\]
since differentiating \( c \hat{v}_x = \hat{v}_{xx} + b f'(\hat{v}) \) implies \( c \hat{v}_{xx} = \hat{v}_{xxx} + b f'(\hat{v}) \hat{v}_x \). Then Assumption (A2) implies that
\[
\frac{d}{dx} \left( w_x + \frac{b}{\nu} f'(\hat{v}) - \left( \frac{c}{2\nu} \right)^2 \right) w_x = \frac{b}{\nu} f''(\hat{v}) \hat{v}_x w_x^2
\]
is strictly positive (resp. negative) for \( x \downarrow -\infty \) (resp. \( x \uparrow +\infty \)). According to the previous part of the proof, \( \lim_{|x| \to \infty} w^2 (x) = 0 \), hence
\[
\lim_{|x| \to \infty} \left( w_x + \frac{b}{\nu} f'(\hat{v}) - \left( \frac{c}{2\nu} \right)^2 \right) w_x^2 \geq 0
\]
so that
\[ w_x^2 + \left( \frac{b}{\nu} f'(\hat{\nu}) - \left( \frac{c}{2\nu} \right)^2 \right) w^2 > 0 \]
for \( x \) at \( \pm \infty \). Using \( w_x = \left( \frac{b}{\nu} - \frac{b f'(\hat{\nu})}{\nu} \right) w \), we conclude that
\[ \left( \frac{c}{2\nu} - \frac{b f'(\hat{\nu})}{\nu} \right)^2 + \frac{b}{\nu} f'(\hat{\nu}) - \left( \frac{c}{2\nu} \right)^2 > 0 \]
or equivalently
\[ (2.11) \]
\[ \frac{b}{\nu} f'(\hat{\nu}) - \frac{b f'(\hat{\nu})}{\nu} \left( \frac{c}{\nu} - \frac{b f(\hat{\nu})}{\nu} \right) > 0 . \]
In particular,
\[ \frac{b}{\nu} \frac{d}{dx} f'(\hat{\nu}) - \frac{b f'(\hat{\nu})}{\nu} \frac{\hat{\nu}_x}{\hat{\nu}_x} > 0 \]
so that \( \frac{f(\hat{\nu})}{\hat{\nu}_x} \) is strictly increasing at \( \pm \infty \).

To prove the remaining identities for \( \gamma_\pm \) observe that by l’Hospital’s rule we obtain that
\[ \gamma_- = \lim_{x \to -\infty} \frac{b f'(\hat{\nu})}{\nu} (x) = \lim_{x \to -\infty} \frac{b f'(\hat{\nu})}{\nu} (x) \frac{\hat{\nu}_x}{\hat{\nu}_x} (x) = \frac{b f'(0)}{\frac{1}{\nu} - \gamma_-} \]
or equivalently, \( \gamma_- \left( \frac{1}{\nu} - \gamma_- \right) = \frac{b}{\nu} f'(0) \). Since \( \gamma_- < 0 \) we obtain the assertion. \( \gamma_+ \) can be computed similarly.

**Lemma 2.8.** Assume \((A1) - (A2)\). Then Assumption \((A0)\) is satisfied.

**Proof.** We have
\[ \| \Psi \|^2 = \| \Psi \|^2_H + \| \Psi_x \|^2 \leq \left( 1 + 2 \left( \frac{c^2}{\nu^2} \right) \right) \int e^{-2\frac{\hat{\nu}^2}{2\nu}} \hat{\nu}_x^2 dx + 2 \int e^{-2\frac{\hat{\nu}^2}{2\nu}} \hat{\nu}_x^2 dx \]
The previous Lemma implies that
\[ \frac{d}{dx} \left( e^{-\left( \frac{2\hat{\nu}^2}{\nu} - \gamma_+ \right)x} \hat{\nu}_x^2 \right) = - \left( 2 \frac{b f'(\hat{\nu})}{\nu} \hat{\nu}_x - \gamma_- \right) e^{-\left( \frac{2\hat{\nu}^2}{\nu} - \gamma_- \right)x} \hat{\nu}_x^2 \geq 0 \]
for \( x \downarrow -\infty \), hence \( M_- := \sup \int_x^\infty e^{-\gamma_- y} \hat{\nu}_x^2 dy < \infty \) which implies that
\[ (2.12) \]
\[ \int_{-\infty}^\infty e^{-2\frac{\hat{\nu}^2}{2\nu}} \hat{\nu}_x^2 dy \leq M_- \int_{-\infty}^\infty e^{-\gamma_- y} dy < \infty \] \( \forall x \).

Similarly,
\[ \frac{d}{dx} \left( e^{-\left( \frac{2\hat{\nu}^2}{\nu} - \gamma_- \right)x} \hat{\nu}_x^2 \right) = - \left( 2 \frac{b f'(\hat{\nu})}{\nu} \hat{\nu}_x - \gamma_+ \right) e^{-\left( \frac{2\hat{\nu}^2}{\nu} + \gamma_+ \right)x} \hat{\nu}_x^2 \leq 0 \]
for \( x \uparrow \infty \), hence \( M_+ := \sup \int_x^\infty e^{-\gamma_+ y} \hat{\nu}_x^2 dy < \infty \) which implies that
\[ (2.13) \]
\[ \int_{x}^\infty e^{-2\frac{\hat{\nu}^2}{2\nu}} \hat{\nu}_x^2 dy \leq M_+ \int_{x}^\infty e^{-\gamma_+ y} dy < \infty \] \( \forall x \).

Combining \((2.12)\) and \((2.13)\) we obtain that \( \int e^{-2\frac{\hat{\nu}^2}{2\nu}} \hat{\nu}_x^2 dx < \infty \).

Finally,
\[ \int e^{-2\frac{\hat{\nu}^2}{2\nu}} \hat{\nu}_x^2 dx \leq \left( \sup_{x \in \mathbb{R}} \left| \frac{\hat{\nu}_x}{\hat{\nu}_x} \right| \right)^2 \int e^{-2\frac{\hat{\nu}^2}{2\nu}} \hat{\nu}_x^2 dx \]
The above supremum can be bounded by
\[ \sup_{x \in \mathbb{R}} \left| \frac{\hat{\nu}_x}{\hat{\nu}_x} \right| \leq \frac{c}{\nu} + \sup_{x \in \mathbb{R}} \left| \frac{f(\hat{\nu})}{\hat{\nu}_x} \right| \]
which is finite again by the previous Lemma 2.7. \( \square \)
We normalise $\Psi$ such that $\langle \Psi, \hat{v}_x \rangle = 1$. As mentioned earlier, in the following stability analysis it will turn out to be of advantage to work in weighted measure spaces instead of the unweighted choices of $V$ and $H$. A natural choice for such a measure is

$$\rho(x) := \frac{\Psi(x)}{v_x(x)} = Ze^{-\hat{\varepsilon}x}$$

where $Z$ denotes the normalising constant. One reason for $\rho$ to be a natural choice is that in the space $L^2(\rho)$ the frozen-wave operator $\mathcal{L}^\#$ separates $\hat{v}_x$, which due to the identity $\partial_t \hat{v}(x + ct) = c\hat{v}_x(x + ct)$ is the direction of movement of the wave, from its orthogonal complement $\hat{v}_x^\perp$ in the following sense: For $u \in V$

$$0 = v(\langle u, \mathcal{L}^\#_* \Psi \rangle)_V = v(\langle \mathcal{L}^\# u, \Psi \rangle)_V$$

$$= -\nu \int u_x (\hat{v}_x \rho)_x dx + b \int f'(\hat{v})u \hat{v}_x \rho dx - c \int u_x \hat{v}_x \rho dx$$

$$= \nu \langle \nabla u, \hat{v}_x \rho \rangle_V + b \langle f'(\hat{v})u, \hat{v}_x \rangle_\rho - c \langle u_x, \hat{v}_x \rangle_\rho$$

$$= \nu \langle \mathcal{L}^\# u, \hat{v}_x \rho \rangle_V$$

Note that for $u \in H^2(\mathbb{R})$ this is the usual orthogonality in $L^2(\rho)$, i.e. $\mathcal{L}^\#(H^2(\mathbb{R})) \subset \hat{v}_x^\perp$ in $L^2(\rho)$.

Due to the fact that $\hat{v}_x$ is a zero-eigenvector of $\mathcal{L}^\#$ perturbations in that direction lead to a random phase shift in the dynamics. To also bound the spread of perturbations in the shape of orthogonal to $\hat{v}_x$ (see Theorem 3.3) we need to control the behaviour of the dynamics in directions orthogonal to $\hat{v}_x$, more precisely we assume the following contraction property

**Assumption 2.9.** There exist $\kappa > 0, C_* > 0$ such that for $u \in V$ with $\rho u \in V$

$$(C1) \quad \nu \langle \mathcal{L}^\# u, \rho \rangle_V \leq -\kappa \|u\|^2_\rho + C_* \langle \hat{v}_x, u \rangle_\rho^2$$

i.e. the flow generated by the frozen-wave operator is contracting on the orthogonal complement of $\hat{v}_x$ in $L^2(\rho)$.

Alternatively, this abstract assumption can again be replaced by the additional assumption (A2) on the reaction term $f$. We will prove the contraction property (C1) in analogy to [18] Theorem 1.5, where a similar spectral gap inequality for the (unfrozen) operator in the unweighted space $L^2(\mathbb{R})$ has been shown.

**Proposition 2.10.** Given (A1) - (A2) the frozen-wave operator $\mathcal{L}^\#$ satisfies Assumption 2.9.

**Proof.** For $u \in C^2(\mathbb{R})$ we write $u = h\hat{v}_x$. Due to the identity $\mathcal{L}^\# \hat{v}_x = 0$ we obtain

$$\mathcal{L}^\# u = vh_{xx} \hat{v}_x + 2v h_x \hat{v}_{xx} + v h \hat{v}_{xxx} + bf'(\hat{v}) \hat{v}_x h - ch_x \hat{v}_x - ch \hat{v}_{xx}$$

and the associated quadratic form

$$\mathcal{E}(h) := -\nu \langle \mathcal{L}^\# u, \rho \rangle_V = \nu \int h_x^2 \hat{v}_x^2 Ze^{-\hat{\varepsilon}x} dx .$$

Rewriting also (C1) in terms of $u = h\hat{v}_x$ and setting $w(x) = \hat{v}_x(x) e^{-\hat{\varepsilon}x}$ our aim is to prove an inequality of the type

$$-\nu \int h_x^2 w^2 dx \leq -\kappa \int h^2 w^2 dx + C_* \left( \int h w^2 dx \right)^2 .$$

Recall from the proof of Lemma 2.7 that

$$\frac{d}{dx} \left( e^{-\frac{2(\hat{\varepsilon}-\gamma_-)x}{\nu}} \hat{v}_x^2 \right) = - \left( \frac{b f'(\hat{v})}{\nu} \hat{v}_x - \gamma_- \right) e^{-\frac{2(\hat{\varepsilon}-\gamma_-)x}{\nu}} \hat{v}_x^2 \geq 0$$

for $x \downarrow -\infty$, so that

$$\int_{-\infty}^{x} e^{-\frac{2(\hat{\varepsilon}-\gamma_-)y}{\nu}} \hat{v}_x^2 dy \leq \int_{-\infty}^{x} e^{(\hat{\varepsilon}-\gamma_-)y} \hat{v}_x^2 dy e^{-\frac{2(\hat{\varepsilon}-\gamma_-)x}{\nu}} \hat{v}_x^2(x) = \frac{1}{\nu - \gamma_-} e^{-\hat{\varepsilon}x} \hat{v}_x^2(0)$$
for \( x \downarrow -\infty \). Since \( \int_{-\infty}^{0} e^{-\hat{\beta} y \hat{\nu}_x^2} dy < \infty \) and \( e^{-\hat{\beta} x \hat{\nu}_x^2} \) is locally bounded from below we can find a finite constant \( K_- \) such that

\[
\int_{-\infty}^{x} e^{-\hat{\beta} y \hat{\nu}_x^2} dy \leq K_- e^{-\hat{\beta} x \hat{\nu}_x^2} \quad \text{for all } x \leq 0.
\]

Similarly,

\[
d\frac{d}{dx} \left( e^{-\hat{\beta} x \hat{\nu}_x^2} \right) = - \left( \frac{b f(\hat{\nu})}{\hat{\nu}_x} - \gamma_+ \right) e^{-\hat{\beta} x \hat{\nu}_x^2} \leq 0
\]

for \( x \uparrow +\infty \), so that

\[
\int_{x}^{\infty} e^{-\hat{\beta} y \hat{\nu}_x^2} dy \leq \int_{x}^{\infty} e^{(\hat{\beta} - \gamma_+) y \hat{\nu}_x^2} dy e^{-\hat{\beta} x \hat{\nu}_x^2} \frac{1}{\gamma_+ - \frac{b f(\hat{\nu})}{\hat{\nu}_x}} e^{-\hat{\beta} x \hat{\nu}_x^2}
\]

for \( x \uparrow +\infty \), thereby using \( \gamma_+ - \frac{b}{\hat{v}} = \sqrt{(\frac{b}{\hat{v}})^2 - \frac{b}{\hat{v}} f'(1)} - \frac{b}{\hat{v}} > 0 \). Since \( \int_{0}^{\infty} e^{-\hat{\beta} y \hat{\nu}_x^2} dy < \infty \) and \( e^{-\hat{\beta} x \hat{\nu}_x^2} \) is locally bounded from below, we can also find a finite constant \( K_+ \) such that

\[
\int_{x}^{\infty} e^{-\hat{\beta} y \hat{\nu}_x^2} dy \leq K_+ e^{-\hat{\beta} x \hat{\nu}_x^2} \quad \text{for all } x \geq 0.
\]

Estimate (2.14) now implies for \( h \in C^1_b \) with \( h(0) = 0 \) that

\[
\int_{-\infty}^{0} h^2 w^2 \, dx = -2 \int_{-\infty}^{0} \int_{x}^{0} h_x y h w^2 \, dx \, dy = -2 \int_{-\infty}^{0} h_x(y) h(y) \int_{-\infty}^{y} w^2 \, dx \, dy
\]

\[
\leq 2K_- \int_{-\infty}^{0} |h_x(y) h(y)| w^2 \, dy \leq \frac{1}{2} \int_{-\infty}^{0} h^2 w^2 \, dx + 2K_- \int_{-\infty}^{0} h_x^2 \, w^2 \, dx
\]

hence

\[
\int_{-\infty}^{0} h^2 w^2 \, dx \leq 4K_- \int_{-\infty}^{0} h_x^2 \, w^2 \, dx
\]

and similarly, using (2.13),

\[
\int_{0}^{\infty} h^2 w^2 \, dx \leq 4K_+ \int_{0}^{\infty} h_x^2 \, w^2 \, dx.
\]

Combining both estimates we obtain the weighted Hardy type inequality

\[
\int h^2 w^2 \, dx \leq K_- \lor K_+ \int h_x^2 \, w^2 \, dx
\]

for any \( h \in C^1_b(\mathbb{R}) \) with \( h(0) = 0 \). For a general \( h \in C^1_b(\mathbb{R}) \) centering allows us to derive the inequality

\[
K_- \lor K_+ \int h_x^2 \, w^2 \, dx \geq \int (h - h(\hat{x}))^2 w^2 \, dx.
\]

We introduce the normalising constant \( W = \int w^2 \, dx \) and the normalised measure \( \tilde{w}^2 = W^{-1} w^2 \) to further estimate the above right hand side by

\[
\text{Var}_{\tilde{w}^2}(h) = \int \tilde{h}^2 \tilde{w}^2 \, dx - \left( \int \tilde{h} \tilde{w}^2 \, dx \right)^2.
\]

Altogether, this yields the desired Poincaré inequality

\[
\int h^2 w^2 \, dx \leq K_- \lor K_+ \int h_x^2 \, w^2 \, dx + W^{-1} \left( \int h w^2 \, dx \right)^2
\]

for any \( h \in C^1_b(\mathbb{R}) \).

\[\square\]

**Lemma 2.11.** Under Assumption 2.2, \( \mathcal{L}^{#} \) generates a contraction semigroup \( (P^#_t)_{t \geq 0} \) on \( \dot{v}^{1 \frac{1}{2}} \subset L^2(\rho) \) satisfying

\[
\|P^#_t u\|_\rho \leq e^{-\epsilon t}\|u\|_\rho
\]

(2.16)
Proof. Note that \((L^\#, C^2_\epsilon(\mathbb{R}))\) is symmetric on \(L^2(\rho)\) and the subspace \(\hat{v}_x^+\) is invariant under \(L^\#,\)

i.e. for \(u \in \hat{v}_x^+\) we have \(\langle L^\# u, \hat{v}_x \rho \rangle = \langle u, L^\# \rho \rangle = 0\) and thus \(L^\# u \in \hat{v}_x^+\). By Assumption 2.9 the operator \(L^\#\) is bounded from above by

\[
\langle L^\# u, u \rangle_\rho \leq -\kappa \|u\|^2_ho
\]
on the orthogonal complement \(\hat{v}_x^+\). It follows that \(L^\#\) is essentially self-adjoint and its Friedrichs extension \((L^\#, D(L^\#))\) generates a symmetric \(C_0\)-semigroup \((P^\# t)_t\) on \(\hat{v}_x^+\) (see [9]). It is easy to see that Assumption 2.9 extends to all \(u \in D(L^\#)\). In particular,

\[
\frac{1}{2} \frac{d}{dt} \|P^\# t u\|^2_\rho = \langle L^\# P^\# t u, P^\# t u \rangle_\rho \leq -\kappa \|P^\# t u\|^2_\rho
\]

which yields

\[
\|P^\# t u\|^2_\rho \leq e^{-2\kappa t} \|u\|^2_\rho
\]

for all \(u \in D(L^\#)\) and subsequently for all \(u \in \hat{v}_x^+\).

We would like to highlight that this contraction property is only needed for the asymptotic second moment estimate in Section 3.3 but not for the main multiscale decompositions in Theorem 3.3 and Theorem 3.5. A similar contraction property can in general not be expected to hold true for the non-autonomous linearisation \((L_t)\) given in (2.10). But yet we will show that this family of linear operators generates an evolution family on \(H^1(1 + \rho)\) facilitating a mild solution representation of \(u\) on this space. The family of linear operators \((L_t), t \in [0, T]\), can be seen as operators from \(H^3(1 + \rho) \rightarrow H^1(1 + \rho)\) satisfying

\[
\|L_t u\|_{H^1(1 + \rho)} \leq L_u \|u\|_{H^3(1 + \rho)}
\]

Lemma 2.12. \((L_t)\) generates an evolution family \((P_{t,s})_{0 \leq s \leq t \leq T}\) on \(H^1(1 + \rho)\) with

\[
\|P_{t,s} h\|_{H^1(1 + \rho)} \leq e^{L_s (t - s)} \|h\|_{H^1(1 + \rho)}
\]

for a constant \(L_s > 0\).

Proof. We first show that the Gaussian semigroup \(p_t = e^{t\Delta}, t \geq 0\), acts on \(H^1(1 + \rho)\): For \(u \in H^1(1 + \rho)\)

\[
\int (p_t u)^2(x) (1 + \rho(x)) \, dx \leq \int p_t(u^2)(x) (1 + \rho(x)) \, dx = \int u^2(p_t(1 + \rho))(x) \, dx
\]

using the symmetry of the semigroup. Together with (2.17) this enables us to obtain the bounds

\[
\|p_t u\|^2_{1 + \rho} \leq e^{t \Delta} \|u\|^2_{1 + \rho}
\]

as well as

\[
\|\partial_x p_t u\|^2_{1 + \rho} = \int (\partial_x p_t u)^2(x) (1 + \rho(x)) \, dx \leq \int p_t(u_x)^2(1 + \rho) \, dx \leq e^{t \Delta} \int (u_x)^2(1 + \rho) \, dx
\]

\[
= e^{t \Delta} \|u_x\|^2_{1 + \rho}.
\]

Set \(B(t) u := b f'(\hat{v}(\cdot + ct)) u, i.e. L_t = \nu \Delta + B(t)\). Defined like this, \((B(t))_{t \in [0, T]}\) is indeed a family of bounded perturbations of \(\nu \Delta\) with

\[
\sup_{\|u\| = 1} \|B(t) u\|_{H^1(1 + \rho)} \leq b^2 \|f'(\hat{v})\|_\infty^2 + \|f''(\hat{v})\|_\infty^2 \|\hat{v}_x\|_\infty^2
\]

uniformly in \(t \in [0, T]\). Hence, by [13], Ch. 5, Theorem 2.3 \(L_t = \nu \Delta + B(t)\) is a stable family of infinitesimal generators. In particular, there exists \(L_s > 0\) such that the associated evolution family satisfies (2.17).

Since \(u\) is a variational solution on \(H^1(1 + \rho)\) it can also be represented as a mild solution

\[
u(t,0) = \frac{\varepsilon}{\varepsilon} P_{t,s} R(s,u(s)) \, ds + \varepsilon \int_0^t P_{t,s} dW(s)
\]

Conditions for variational solutions to satisfy a mild solution representation are stated in [14].
3. Multiscale Analysis

3.1. Dynamical equation for phase-adaptation. To establish a description of the noise-induced phase shift of the wave profile arising due to the nonlinearity of the system the idea is to determine the stochastic phase by dynamically matching the deterministic profile \( \hat{v} \) with the stochastic solution \( v \). This matching is achieved by minimising the \( L^2 \)-distance

\[
(3.1) \quad C \mapsto \|v(\cdot, t) - \hat{v}(\cdot + ct + C)\|_\rho
\]

over all possible phases \( C \). Again, it turns out to be of advantage to work in the weighted measure space \( L^2(\rho) \), where the measure will now be moved with the wave. The following dynamics are designed to point in the direction of the negative gradient of \((3.1)\). Let \( m > 0 \) and consider the (pathwise) ODE

\[
(3.2) \quad \dot{C}^m(t) = m B(t, C^m(t)), \quad t \in [0, T] \\
C(0) = 0
\]

where

\[
B(t, C) = \langle v(t, \cdot) - \hat{v}(\cdot + ct + C), \hat{v}_x(\cdot + ct + C) \rangle_{\rho(\cdot + ct + C)} \\
= \langle v(t, \cdot) - \hat{v}(\cdot + ct + C), \hat{v}(\cdot + ct + C) \rangle
\]

Equation \((3.2)\) can be regarded as an alternative approach to the phase conditions specified by certain algebraic constraints in the classical stability analysis (refer to [7]) and has also been introduced in [10, 11, 13] and [18].

**Proposition 3.1** (Well-posedness). \textit{P-almost surely there exists a unique adapted solution} \( C \in C^1([0, T]) \) \textit{of the (pathwise) ODE} \((3.2)\).

**Proof.** With analogous arguments as in [18] and [10] \( B(t, C) \) is continuous in \((t, C) \in [0, T] \times \mathbb{R} \). It then suffices to show that \( B(t, C) \) is Lipschitz continuous in \( C \) with a Lipschitz constant independent of \( t \): For \( C_1, C_2 \in \mathbb{R}, t > 0 \)

\[
B(t, C_1) - B(t, C_2) = \langle \Psi(\cdot + ct + C_1) - \Psi(\cdot + ct + C_2), v(t) - \hat{v}(\cdot + ct) \rangle \\
+ \langle \Psi(\cdot + ct + C_1), \hat{v}(\cdot + ct) - \hat{v}(\cdot + ct + C_1) \rangle - \langle \Psi(\cdot + ct + C_2), \hat{v}(\cdot + ct - \hat{v}(\cdot + ct + C_2) \rangle \\
= I + II + III, \quad \text{say.}
\]

With

\[
|I| \leq \|\Psi(\cdot + ct + C_1) - \Psi(\cdot + ct + C_2)\|_H \|u(t)\|_H \leq \|\Psi\|_H |C_1 - C_2| \|u\|_{C([0,T];H)}
\]

and

\[
|II + III| = |\langle \Psi(\cdot + ct), \hat{v}(\cdot + ct - C_1) - \hat{v}(\cdot + ct) \rangle - \langle \Psi(\cdot + ct), \hat{v}(\cdot + ct - C_2) - \hat{v}(\cdot + ct) \rangle| \\
= |\langle \Psi(\cdot + ct), \hat{v}(\cdot + ct - C_1) - \hat{v}(\cdot + ct - C_2) \rangle| \\
\leq \|\Psi\|_H \|\hat{v}\|_H |C_1 - C_2|
\]

Therefore, since \( u \) is adapted and in \( L^p(\Omega; C([0, T]; H)) \), there exists a unique adapted process \( C \in L^p(\Omega; C^1([0, T])) \) that solves \((3.2)\). \( \square \)

Let \( \gamma^m(t) := ct + C^m(t) \). The initial representation \( v(t) = \hat{v}(\cdot + ct) + u(t) \) can now be replaced by

\[
(3.3) \quad v(t) = \hat{v}(\cdot + \gamma^m(t)) + u^m(t),
\]

where \( u^m : \Omega \times [0, T] \rightarrow H \) defined by

\[
u^m(t) = u(t) + \hat{v}(\cdot + ct) - \hat{v}(\cdot + \gamma^m(t)) = v(t) - \hat{v}(\cdot + \gamma^m(t))
\]

is an adapted process in \( L^2(\Omega; C([0, T]; H)) \cap L^2(\Omega \times (0, T); V) \). Moreover, \( u^m \) is the unique variational solution of the equation

\[
(3.4) \quad du^m(t) = \left[ \nu \Delta u^m(t) + G^m(t, u^m(t)) - (C^m)'(t) \hat{v}_x(\cdot + \gamma^m(t)) \right] dt + \varepsilon dW(t),
\]
\[
\begin{align*}
\dot{u}^m(t) &= \left[ \nu \Delta u^m(t) + b f'(\dot{v}(\cdot + \gamma^m(t))) u^m(t) - m(\Psi(\cdot + \gamma^m(t)), u^m(t)) \dot{v}_x(\cdot + \gamma^m(t)) \right] dt \\
&+ R^m(t, u^m(t)) dt + \varepsilon dW(t)
\end{align*}
\]

with initial profile \( u^m(0) = u^0 \) and
\[
\begin{align*}
G^m(t, u) &= b \left[ f(u + \dot{v}(\cdot + \gamma^m(t))) - f(\dot{v}(\cdot + \gamma^m(t))) \right] \\
R^m(t, u) &= G^m(t, u) - b f'(\dot{v}(\cdot + \gamma^m(t))) u.
\end{align*}
\]

For the Nagumo equation, for instance, the remainder \( R^m \) is explicitly given by
\[
R^m(t, u) = \frac{1}{2} f''(\dot{v}(\cdot + \gamma^m(t))) u^2 + \frac{1}{6} f'''(\dot{v}(\cdot + \gamma^m(t))) u^3 = \frac{1}{2} (1 + a - 3 \dot{v}(\cdot + \gamma^m(t))) u^2 - u^3.
\]

3.2. Multiscale decomposition of the fluctuations. For the subsequent analysis we demand higher regularity of the reaction function \( f \) by assuming that \( f \in C^3(\mathbb{R}) \). Let \( \rho_t(x) = \rho(x + ct) \) and for \( h \in C([0, T], H^1(1 + \rho)) \)
\[
\|h\|_T = \sup_{t \in [0, T]} \|h(t)\|_{H^1(1 + \rho)}.
\]

Likewise for \( f \in C[0, T] \) define
\[
|f|_T = \sup_{t \in [0, T]} |f(t)|.
\]

Additionally, we again assume (A1) - (A2) ensuring that \( \dot{v}_x \in H^1(1 + \rho) \).

We now derive an SDE for the stochastic perturbation \( c^m(t) := \dot{C}^m(t) \) of the wave speed. Note that while the phase adaptation \( C^m(t) \) is a process of bounded variation, the resulting adapted wave speed is of unbounded variation.

**Lemma 3.2.** The dynamically adapted wave speed \( c^m(t) = m \langle u^m(t), \Psi(\cdot + \gamma^m(t)) \rangle \) solves the SDE
\[
(3.5)
\]
\[
c^m(t) = c^m(0) + m \int_0^t \langle - c^m(s) + c^m(s) u^m(s), \Psi_x(\cdot + \gamma^m(s)) + \nu \langle R^m(s, u^m(s)), \Psi(\cdot + \gamma^m(s)) \rangle \rangle ds
\]
\[
+ \varepsilon m \int_0^t \langle \Psi(\cdot + \gamma^m(s)), dW(s) \rangle
\]
\[
c^m(0) = \varepsilon m \langle \eta, \Psi \rangle
\]

**Proof.** Applying Itô’s lemma we obtain
\[
c^m(t) - c^m(0) = m \int_0^t \nu \langle u^m(s), \Psi(\cdot + \gamma^m(s)) \rangle ds
\]
\[
- m \int_0^t \langle \langle \Psi(\cdot + \gamma^m(s), u^m(s)) \rangle, \dot{v}_x(\cdot + \gamma^m(s)), \Psi(\cdot + \gamma^m(t)) \rangle ds
\]
\[
+ m \int_0^t \langle \nu \langle R^m(s, u^m(s)), \Psi(\cdot + \gamma^m(s)) \rangle \rangle ds + \varepsilon m \int_0^t \langle dW(s), \Psi(\cdot + \gamma^m(s)) \rangle
\]
\[
+ m \int_0^t \langle u^m(s), \Psi_x(\cdot + \gamma^m(s))(c + c^m(s)) \rangle ds
\]
\[
= m \int_0^t \nu \langle u^m(s), \Psi(\cdot + \gamma^m(s)) \rangle ds
\]
\[
- m \int_0^t \langle c^m(s), \dot{v}(\cdot + \gamma^m(s)) \rangle ds + m \int_0^t \nu \langle R^m(s, u^m(s)), \Psi(\cdot + \gamma^m(s)) \rangle ds
\]
\[
+ m \int_0^t \langle c^m(s) \langle u^m(s), \Psi_x(\cdot + \gamma^m(s)) \rangle ds + \varepsilon m \int_0^t \langle dW(s), \Psi(\cdot + \gamma^m(s)) \rangle
\]
\[
= m \int_0^t \nu \langle u^m(s), \cdot - \gamma^m(s), \mathcal{L}^\# \Psi \rangle ds
\]
\[- m \int_0^t e^m(s) \, ds + m \int_0^t e^m(s) \langle u^m(s), \Psi_x(\cdot + \gamma^m(s)) \rangle \, ds \\
+ m \int_0^t \nu \cdot \langle R^m(s, u^m(s)), \Psi(\cdot + \gamma^m(s)) \rangle \, ds + \varepsilon m \int_0^t \langle dW(s), \Psi(\cdot + \gamma^m(s)) \rangle \]

\[\square\]

In order to investigate the dynamics on different scales of the noise strength we first formally identify the highest order terms in (3.5) as well as (3.4). Expecting that both \(C^m\) and \(u^m\) are of order \(\varepsilon\) leads us to define \(c_0^m\) to be the unique strong solution of

\begin{equation}
\begin{aligned}
dc_0^m(t) &= -m c_0^m(t) \, dt + m \langle \Psi(\cdot), dW(t) \rangle, \quad t \in [0, T] \\
c_0^m(0) &= m(\eta, \Psi)
\end{aligned}
\end{equation}

and \(u_0^m \in L^2(\Omega; C([0, T]; H)) \cap L^2(\Omega \times (0, T); V)\) to be the unique variational solution to

\begin{equation}
\begin{aligned}
du_0^m(t) &= [\mathcal{L}u_0^m(t) - c_0^m(t) \hat{v}_x(\cdot + ct)] \, dt + dW(t), \quad t \in [0, T] \\
u_0^m(0) &= \eta
\end{aligned}
\end{equation}

with \(u_0^m \in L^2(\Omega, C([0, T], H^1(1 + \rho)))\). This regularity holds true since \(u_0^m\) has the mild solution representation

\[u_0^m(t) = P_{t,0} \eta - \int_0^t P_{t,s} c_0^m(s) \hat{v}_x(\cdot + cs) \, ds + \int_0^t P_{t,s} dW(s)\]

and \(\hat{v}_x \in H^1(1 + \rho)\). We define the first order phase adaptation by \(C_0^m(t) = \int_0^t c_0^m(s) \, ds\) and the first order phase by \(\gamma_0^m(t) = ct + \varepsilon C_0^m(t)\). For \(\varepsilon > 0\) and \(q \in [0, 1]\) set

\begin{equation}
\tau_{q, \varepsilon} = \inf\{t \in [0, T] : \|u(t)\|_{H^1(1+\rho)} \geq \varepsilon^{1-q}\}
\end{equation}

where \(u\) is the solution from Proposition 2.5 and

\[\tau_{q, \varepsilon}^m = \inf\{t \in [0, T] : |C_0^m(t)| \geq \varepsilon^{-q}\}\]

Theorem 3.3. Let \(q < \frac{1}{2}\). On \(\{\tau_{q, \varepsilon} \wedge \tau_{q, \varepsilon}^m = T\}\) the stochastic travelling wave \(v\) can be decomposed into

\[v(t) = \hat{v}(\cdot + ct + \varepsilon C_0^m(t)) + \varepsilon u_0^m(t) + \varepsilon r^m(t)\]

with

\[\|r^m\|_T \leq \alpha(T) \varepsilon^{1-2q}\]

where the constant \(\alpha(T)\) is independent of \(m\) and \(\varepsilon\). Moreover,

\[\lim_{\varepsilon \to 0} P[\tau_{q, \varepsilon} \wedge \tau_{q, \varepsilon}^m = T] = 1\]

Proof. Let \(\tilde{u}_0^m(t) := v(t) - \hat{v}(\cdot + \gamma_0^m(t)) = u(t) + \hat{v}(\cdot + ct) - \hat{v}(\cdot + \gamma_0^m(t))\). By Taylor’s formula there exists \(\xi(t, x)\) with \(|\xi(t, x)| \leq \varepsilon |C_0^m(t)|\) uniformly in \(x\) such that

\[\tilde{u}_0^m(t) = u(t) + \hat{v}_x(\cdot + ct + \xi(t, \cdot)) \varepsilon C_0^m(t)\]

Hence, on \(\{\tau_{q, \varepsilon} \wedge \tau_{q, \varepsilon}^m = T\}\) using Remark 2.3

\[\|\tilde{u}_0^m(t)\|_{H^1(1+\rho)} \leq \|u(t)\|_{H^1(1+\rho)} + \varepsilon |C_0^m(t)| \|\hat{v}_x(\cdot + ct + \xi(t))\|_{H^1(1+\rho)} \leq \varepsilon^{1-q} \varepsilon^{1-q} \|\tilde{v}_x\|_{H^1(1+\rho(\xi(t)))} \leq \varepsilon^{1-q} \varepsilon^{1-q} (1 + \varepsilon^{M_\varepsilon(1-q)}) \leq C_1 \varepsilon^{1-q}\]

for a constant \(C_1 > 1\). The remainder process

\[r^m(t) = \frac{1}{\varepsilon} \left( v(t) - \hat{v}(\cdot + ct + \varepsilon C_0^m(t)) \right) - u_0^m(t)\]

is a variational solution of the following pathwise evolution equation:

\[dr^m(t) = \mathcal{L}r^m(t) \, dt + \frac{b}{\varepsilon} \left( f(\hat{v}(\cdot + \gamma_0^m(t)) + \tilde{u}_0^m(t)) - f(\hat{v}(\cdot + \gamma_0^m(t))) - f'(\hat{v}(\cdot + \gamma_0^m(t))) \tilde{u}_0^m(t) \right) dt\]
Using (2.17) and Remark 2.3 (i) we estimate the above expression using local bounds on

The second part can be controlled by

with $r^m(0) = 0$. Thus, it can be represented as a mild solution

Using (2.17) and Remark 2.3 (i) we estimate

The first part is bounded by applying condition (B3) as follows:

Therefore, on $\{\tau_{q, \epsilon} \wedge \tau_{r, \epsilon}^m = T\}$

Furthermore, using Taylor’s formula there exists an intermediate point $\xi(t, x)$ with $|\xi(t, x)| \leq |\tilde{u}_0^m(t, x)|$ such that

Note that even though $f''$ and $f'''$ are not assumed to be globally bounded, we can control the above expression using local bounds on $\{\tau_{q, \epsilon} \wedge \tau_{r, \epsilon}^m = T\}$. Since $\hat{\omega} \in [0, 1]$ and

for $\epsilon$ small enough, we know that $\hat{\omega}(x) + \tilde{u}_0^m(t, y) \in [-1, 2]$ for all $t > 0, x, y \in \mathbb{R}$, and therefore

The second part can be controlled by

and

and

with $r^m(0) = 0$. Thus, it can be represented as a mild solution

Using (2.17) and Remark 2.3 (i) we estimate the above expression using local bounds on
Thus, we obtain

\[ \frac{b}{\varepsilon} \left| f''(\dot{\gamma}(t)) \right| |\tilde{u}_0(t)| + \frac{b}{\varepsilon} \left| f''(\dot{\gamma}(t)) \right| |\tilde{u}_0(t)| \]

For the last part set

\[ b \left( |f''(\dot{\gamma}(t))| + |f''(\dot{\gamma}(t))| \right) \]

By Taylor’s theorem there exists

\[ C_0(t) \]

such that on \( \{ \tau_{q,\varepsilon} + \tau_{m} \} \)

\[ \| r_m(t) \| H^{1+\rho_1} \leq \frac{b^2}{\varepsilon} \left( |f''(\dot{\gamma}(t))| + |f''(\dot{\gamma}(t))| \right) \]

For the last part set

\[ R_m(t) := -\frac{1}{\varepsilon} (\dot{\gamma}(t) - \dot{\gamma}(t)) - \varepsilon C_0(t) \]

satisfying

\[ \left( \frac{d}{dt} - c\partial_x \right) R_m(t) = r_m(t). \]

Thus, we obtain

\[ \int_0^t P_{q,\varepsilon} r_m(s) ds = R_m(t) + \int_0^t P_{q,\varepsilon} \left( L_s - c\partial_x \right) R_m(s) ds \]

and

\[ \left\| \int_0^t P_{q,\varepsilon} r_m(s) ds \right\|_{H^{1+\rho_1}} \leq \left\| R_m(t) \right\|_{H^{1+\rho_1}} + \int_0^t \left\| P_{q,\varepsilon} \left( L_s - c\partial_x \right) R_m(s) \right\|_{H^{1+\rho_1}} ds \]

By Taylor’s theorem there exists \( \xi(t,x) \) with \( |\xi| \leq \varepsilon |C_0| \) such that the order of the first summand can be estimated by

\[ \left\| R_m(t) \right\|_{H^{1+\rho_1}} \leq \frac{\varepsilon}{2} |C_0(t)|^2 \left| |\tilde{u}_0(t)| + |\tilde{u}_0(t)| \right|_{H^{1+\rho_1}} \]

For estimating the order of the second summand one needs to control also higher derivatives of \( R_m \):

\[ \| R_m(t) \| L^{\varepsilon}(1+\rho_1) = \| R_m(t) \| H^{1+\rho_1} \]

with

\[ \varepsilon \partial_{xx} R_m(t) = \tilde{u}_{xx}(t) + \varepsilon C_0(t) - \varepsilon \tilde{u}_{xx}(t) - \varepsilon C_0(t) \tilde{u}_{xx}(t) \]
\[ \varepsilon \partial_{xxx} R_3^m (t) = \hat{v}_{xxx}(\cdot + ct + \varepsilon C_0^m (t)) - \hat{v}_{xxx}(\cdot + ct) - \varepsilon C_0^m (t) \hat{v}_{xxx}(\cdot + ct) \]

Note that differentiating (1.2) yields

\[ \nu \hat{v}_{xxxx} = c\hat{v}_{xxxx} - f'(\hat{v}) \hat{v}_{xxx} - 3f''(\hat{v}) \hat{v}_x - f'''(\hat{v}) \hat{v}_x^3 \]

implying that \( \hat{v} \in C^3 \) if \( f \in C^3 \). Again applying Taylor’s theorem there exist \( \xi_1(t) \), \( \xi_2(t) \) with \( |\xi_1,2(t)| \leq \varepsilon |C_0^m(t)| \) such that

\[ \| \partial_{xxx} R_3^m (t) \|_{L^2(1+\rho)} \leq \varepsilon \frac{|C_0^m(t)|^2}{2} (1 \vee e^{M \varepsilon^{-q}})^\frac{3}{2} \| \hat{v}_{xxx} \|_{1+\rho} \leq (1 \vee e^{M \varepsilon^{-q}})^\frac{3}{2} \| \hat{v}_{xxx} \|_{1+\rho} \leq C_4 \varepsilon^{1-2q} \]

for a constant \( C_4 > 0 \) and

\[ \| \partial_{xxx} R_3^m (t) \|_{L^2(1+\rho)} \leq \varepsilon \frac{|C_0^m(t)|^2}{2} (1 \vee e^{M \varepsilon^{-q}})^\frac{3}{2} \| \hat{v}_{xxx} \|_{L^2(1+\rho)} \leq C_5 \varepsilon^{1-2q} \]

with \( C_5 > 0 \). Altogether we obtain

\[ II \leq C \sup_{t \in [0, T]} \| \mathcal{L}_t - c \tilde{\partial}_x \|_{L(H^2(1+\rho_0),H^1(1+\rho_0))} \frac{e^{L_* t} - 1}{L_*} \varepsilon^{1-2q} \]

for a constant \( C > 0 \) independent of \( \varepsilon \). It remains to show that in the small-noise limit the above order estimate holds for \( P \)-almost all paths \( \omega \in \Omega \). If \( \tau_{q, \varepsilon} < T \) then, due to continuity, for \( t_0 := \tau_{q, \varepsilon} (\omega) \) we obtain

\[ \varepsilon^{1-q} \leq \| u(t_0) \|_{H^1(1+\rho_0)} \leq \varepsilon \| C_0^m(t_0) \bar{v}_x(\cdot + ct_0 + \xi(t_0)) + u_0^m(t_0) \|_{H^1(1+\rho_0)} + \varepsilon \| r^m(t_0) \|_{H^1(1+\rho_0)} \]

and therefore, using Markov’s inequality

\[ P[\tau_{q, \varepsilon} < T] \leq P[\| C_0^m(t_0) \bar{v}_x(\cdot + ct_0 + \xi(t_0)) + u_0^m(t_0) \|_{H^1(1+\rho_0)} \geq \varepsilon^{-q}(1 - \alpha(T)\varepsilon^{1-q})] \]

\[ \leq \frac{\varepsilon^{2q}}{2(1 - \alpha(T)\varepsilon^{1-q})^2} \left( E \left[ \| C_0^m \|_{T}^2 \right] (1 \vee e^{M \varepsilon^{-q}}) \| \bar{v}_x \|_{H^1(1+\rho)}^2 + E \left[ \| u_0^m \|_{T}^2 \right] \right) \]

\[ \longrightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \]

Likewise, the second stopping time converges as follows

\[ P[\tau_{q, \varepsilon}^m < T] \leq P[|C_0^m|_T \geq \varepsilon^{-q}] \leq \varepsilon^{2q} E \left[ |C_0^m|_T^2 \right] \rightarrow 0. \]

\[ \square \]

3.3. Immediate relaxation. From the definition of the stochastic phase adaptation process \( C^m \) it is clear that the initial goal of minimising the \( L^2(\rho) \)-distance between \( v \) and \( \hat{v} \) for every time \( t \in [0, T] \) can only approximately be achieved when choosing a finite relaxation rate \( m \). Although Theorem 3.3 shows that already for finite \( m \) a multiscale decomposition into processes of the expected order can be installed, we are interested in investigating the case of so-called immediate relaxation, i.e. the limit \( m \to \infty \), and show that in that case indeed a rigorous minimisation (on relevant orders of the noise strength) is achieved. As an alternative to this description of a stochastic phase one could try to adapt the analysis developed in \([8]\) for a generalised framework of neural field equations, where the dynamics of the local minimum of \( \ref{5.1} \) are explicitly described by an SDE up to a certain stopping time. This approach does not incorporate a gradient-descent procedure. As pointed out by the authors, studying instead the behaviour of the global minimum of \( \ref{5.1} \) would be much more complicated, since its dynamics will be highly discontinuous. To our knowledge, this has not been investigated for bistable reaction-diffusion equations. Below we will adapt the methods from \([11]\). It will turn out that the limit phase adaptation is a process of unbounded variation behaving almost like a Brownian motion, which is in accordance with the
Lemma 3.4. Define the processes $C_0$ and $u_0$ as

$$C_0(t) = \langle \eta, \Psi \rangle + \int_0^t \langle \Psi(\cdot + cs), dW(s) \rangle \quad \text{for } t > 0$$

and

$$u_0(t) = P_{t,0} \Pi_0 \eta + \int_0^t P_{t,s} \Pi s dW(s) \quad \text{for } t > 0$$

with $u_0(0) = \eta$. Then for any $\delta > 0$ almost surely

$$\sup_{\delta \leq t \leq T} |C_0^m(t) - C_0(t)| \longrightarrow 0$$

as well as

$$\sup_{\delta \leq t \leq T} \|u_0^m(t) - u_0(t)\|_{H^2(1+\rho_t)} \longrightarrow 0$$

Proof. Integrating (3.3) yields

$$C_0^m(t) = e^{-mt} \langle \eta, \Psi \rangle + \int_0^t e^{-m(t-s)} m \langle \Psi(\cdot + cs), dW(s) \rangle$$

and therefore

$$C_0^m(t) = (1 - e^{-mt}) \langle \eta, \Psi \rangle + \int_0^t (1 - e^{-m(t-s)}) \langle \Psi(\cdot + cs), dW(s) \rangle$$

With this, the difference between the approximate relaxation and the immediate relaxation process is given by

$$C_0(t) - C_0^m(t) = e^{-mt} \langle \eta, \Psi \rangle + \int_0^t e^{-m(t-s)} \langle \Psi(\cdot + cs), dW(s) \rangle =: e^{-mt} \langle \eta, \Psi \rangle + S_t$$

For the martingale term an integration by parts leads to

$$S_t = \langle \Psi(\cdot + ct), W(t) \rangle - \int_0^t me^{-m(t-s)} \langle \Psi(\cdot + cs), W(s) \rangle + c e^{-m(t-s)} \langle \Psi_x(\cdot + cs), W(s) \rangle ds$$

Now, the function $t \mapsto \langle \Psi(\cdot + ct), W(t) \rangle$ is Hölder continuous for any $\beta < \frac{1}{2}$ almost surely, i.e.

$$M_\beta(T, \omega) := \sup_{|t-s| \leq T} \left| \frac{\langle \Psi(\cdot + ct), W(t) \rangle - \langle \Psi(\cdot + cs), W(s) \rangle}{|t-s|^\beta} \right| < \infty \quad \text{a.s.}$$

Thus,

$$S_t \leq c \left| \int_0^t e^{-m(t-s)} \langle \Psi_x(\cdot + cs), W(s) \rangle ds \right| + M_\beta(T, \omega) \int_0^t me^{-m(t-s)} (t-s)^\beta ds$$

$$+ e^{-mt} \|\Psi(\cdot + ct, W(t))\|$$

$$\leq \frac{c}{m} \|\Psi_x\| \sup_{0 \leq t \leq T} \|W(t)\| + \frac{M_\beta(T, \omega)}{m^\beta} \Gamma(1+\beta) + e^{-mt} \|\Psi\| \sup_{0 \leq t \leq T} \|W(t)\|$$

where $\Gamma$ denotes the Gamma function

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx.$$
Hence,
\[
\sup_{\delta \leq t \leq T} |C_0(t) - C_0^m(t)| \\
\leq e^{-m\delta} \|\Psi\| \left( \|\eta\| + \sup_{0 \leq t \leq T} \|W(t)\| \right) + \frac{M_\beta(T, \omega)}{m^\beta} \Gamma(1 + \beta) + \frac{c}{m} \|\Psi_x\| \sup_{0 \leq t \leq T} \|W(t)\| \\
\xrightarrow{m \to \infty} 0 \quad \text{a.s.}
\]

Note that since \( L_t \hat{v}_x(\cdot + ct) = 0 \) we have \( P_{t,s} \hat{v}_x(\cdot + cs) = \hat{v}_x(\cdot + ct) \). Thus, the mild solution representation of the first-order fluctuations \( u_0^m \) is given by
\[
u_0^m(t) = P_{t,0} \eta - \int_0^t c_0^m(s) P_{t,s} \hat{v}_x(\cdot + cs) ds + \int_0^t P_{t,s} dW(s)
\]
\[
= P_{t,0} \Pi_0 \eta + \langle \eta, \hat{v}_x \rangle_\rho P_{t,0} \hat{v}_x - \int_0^t c_0^m(s) \hat{v}_x(\cdot + ct) ds + \int_0^t P_{t,s} dW(s)
\]
\[
= P_{t,0} \Pi_0 \eta + P_{t,0} \langle \eta, \Psi \rangle \hat{v}_x - \hat{v}_x(\cdot + ct) C_0^m(t) + \int_0^t P_{t,s} \Pi_s dW(s)
\]
\[
+ \int_0^t P_{t,s} \hat{v}_x(\cdot + cs) \langle \Psi(\cdot + cs), dW(s) \rangle
\]
\[
= u_0(t) + \hat{v}_x(\cdot + ct) \left( \langle \eta, \Psi \rangle + \int_0^t \langle \Psi(\cdot + cs), dW(s) \rangle - C_0^m(t) \right)
\]
\[
u_0(t) = u_0(t) + \hat{v}_x(\cdot + ct) (C_0(t) - C_0^m(t))
\]

Using this we obtain
\[
\sup_{\delta \leq t \leq T} \|u_0^m(t) - u_0(t)\|_{H^{1(1+\rho)}} \leq \sup_{\delta \leq t \leq T} |C_0^m(t) - C_0(t)| \|\hat{v}_x\|_{H^{1(1+\rho)}} \xrightarrow{m \to \infty} 0 \quad \text{a.s.}
\]

Indeed, in contrast to [8], where the existence of a phase-adaptation process has been shown up to a stopping time, Lemma 3.4 provides us with effective formulae for the first-order stochastic phase-adaptation and fluctuations for all times \( t \in [0, T] \). We now show that passing over to the limit from finite to immediate phase relaxation preserves the previous multiscale decomposition (Theorem 3.3).

**Theorem 3.5.** Let \( \tau_{q,\varepsilon} \) be defined as in (3.8) and set
\[
\tau_{q,\varepsilon}^\infty = \inf\{ t \in [0, T] : |C_0(t)| \geq \varepsilon^{-q} \} \wedge T
\]
Then on \( \{ \tau_{q,\varepsilon}^\infty \leq T \} \) the following multiscale decomposition of the stochastic travelling wave \( v \) holds:
\[
v(t) = \hat{v}(\cdot + ct + \varepsilon C_0(t)) + \varepsilon u_0(t) + \varepsilon r(t)
\]
with
\[
\|r\|_T \leq \alpha(T) \varepsilon^{1-2q}
\]
where \( \alpha(T) \) is a positive constant. Moreover, in the small-noise limit the above representation holds for almost every path \( \omega \in \Omega \), i.e.
\[
P[\tau_{q,\varepsilon}^\infty \leq T] \xrightarrow{\varepsilon \to 0} 1.
\]

**Proof.** Let \( t < \tau_{q,\varepsilon}^\infty \wedge \tau_{q,\varepsilon}^\infty \). Performing an integration by parts in (3.3), we obtain
\[
C_0^m(t) = (1 - e^{-m t}) \langle \eta, \Psi \rangle + \int_0^t m e^{-m(t-s)} \langle \Psi(\cdot + cs), W(s) \rangle ds
\]
\[
- c \int_0^t (1 - e^{-m(t-s)}) \langle \Psi_x(\cdot + cs), W(s) \rangle ds
\]
which yields for \( 0 < \delta < t \)
\[
|C_0^m(t)| \delta \leq \|\eta, \Psi\| + \|\Psi\| \|W\| + c \delta \|\Psi_x\| \|W\| \xrightarrow{\delta \to 0} \|\eta, \Psi\|
\]
Since $|C_0(t)| \approx |\langle \eta, \Psi \rangle|$ for $t$ close to 0, we know that $|C_0^m(t)|_\delta < \varepsilon^{-q}$. Furthermore, for any $\delta > 0$

$$\sup_{\delta \leq s \leq t} |C_0^m(s)| \rightarrow \sup_{\delta \leq s \leq t} |C_0(s)| < \varepsilon^{-q}.$$  

This implies $\{\tau_{q, \varepsilon} \wedge \tau_{q, \varepsilon}^\infty = T\} \subseteq \{\tau_{q, \varepsilon} \wedge \tau_{q, \varepsilon}^m = T\}$ for $m$ sufficiently large. Hence, applying Theorem 3.3 and Lemma 4.3 we obtain

$$\|e^t\|_{H^1(1+\rho_t)} = \|v(t) - \hat{v}(\cdot + ct + \varepsilon C_0(t)) - \varepsilon u_0(t)\|_{H^1(1+\rho_t)}$$

$$\leq \|v(t) - \hat{v}(\cdot + ct + \varepsilon C_0^m(t)) - \varepsilon u_0^m(t)\|_{H^1(1+\rho_t)} + \|\varepsilon u_0^m(t) - u_0(t)\|_{H^1(1+\rho_t)}$$

$$\leq \alpha(T) \varepsilon^{2-2q} + \varepsilon |C_0^m(t) - C_0(t)| \|\hat{v}_x\|_{H^1(1+\rho_t)}(1 + \varepsilon^{M_2^{1-q} - q} + \varepsilon |u_0^m(t) - u_0(t)|_{H^1(1+\rho_t)}$$

$$\rightarrow_{m \to \infty} \alpha(T) \varepsilon^{2-2q} \text{ a.s.}$$

In the limit $\varepsilon \to 0$ the above order estimate holds for almost every path $\omega \in \Omega$, i.e.

$$P[\tau_{q, \varepsilon} \wedge \tau_{q, \varepsilon}^\infty = T] \geq 1 - P[\tau_{q, \varepsilon} < T] - P[\tau_{q, \varepsilon}^\infty < T] \rightarrow_{\varepsilon \to 0} 1$$

with analogous arguments as in Theorem 3.3.

3.4. Statistical and geometrical properties of first-order phase adaptation and of fluctuations. We would like to compare our stochastic phase adaptation $C_0$ to the phase description obtained in [1], where the phenomenon of stochastic wave propagation has been (formally) investigated for (nonlocal) stochastic neural field equations. They concluded that to first order of the noise strength the stochastic perturbation of the phase is a Brownian motion. Taking a look at the variance of the immediate phase adaptation we see that for $t > 0$

$$\text{Var}(C_0(t)) = \text{Var} \left( \int_0^t \langle \Psi(\cdot + cs), dW(s) \rangle \right) = \int_0^t \langle \Psi(\cdot + cs), Q\Psi(\cdot + cs) \rangle \, ds \approx \langle \Psi, Q\Psi \rangle \, t$$

showing that $C_0$ is roughly diffusive if $Q$ is “almost” translation invariant. Note that strict translation invariance is excluded since $Q$ is of finite trace. Thus, the above statement can only be an heuristic approximative description. Our first-order fluctuations are indeed orthogonal to the direction of movement of the wave: For $t > 0$

$$\langle u_0(t), \hat{v}_x(\cdot + ct) \rangle_{\rho_t} = \langle P_{t, 0} \Pi_0 \eta, \Psi(\cdot + ct) \rangle + \langle \int_0^t P_{t, s} \Pi_s dW(s), \Psi(\cdot + ct) \rangle$$

$$= \langle \Pi_0 \eta, P_{t, 0}^* \Psi(\cdot + ct) \rangle + \int_0^t \langle P_{t, s}^* \Psi(\cdot + ct), \Pi_s dW(s) \rangle$$

$$= \langle \Pi_0 \eta, \Psi \rangle + \int_0^t \langle \Psi(\cdot + cs), \Pi_s dW(s) \rangle = 0.$$  

Likewise, in the frozen wave setting $u_0^\#$ is orthogonal to $\hat{v}_x$ in $L^2(\rho)$. As stated in Subsection 2.3 the frozen wave operator $\mathcal{L}_t^\#$ generates a contraction semigroup on $\hat{v}_x^t$, which allows for the mild solution representation

$$u_0^\#(t) = P_t^\# \Pi_0 \eta + \int_0^t P_{t, s}^\# \Pi_s dW(s) = P_t^\# \Pi_0 \eta + \int_0^t P_{t, s}^\# \Pi_0 \Phi_s dW(s).$$

Using the contraction property (2.19) yields

$$\|u_0(t)\|_{\rho_t} = \|u_0^\#(t)\|_{\rho} \leq e^{-\kappa t} \|\eta\|_{\rho} + \left\| \int_0^t P_{t, s}^\# \Pi_0 \Phi_s dW(s) \right\|_{\rho}.$$

This allows us to bound the expectation by

$$E \left[ \|u_0(t)\|^2_{\rho_t} \right] \leq 2e^{-2\kappa t} \|\eta\|^2_{\rho} + 2 \int_0^t \left\| P_{t, s}^\# \Pi_0 \Phi_s \sqrt{\varepsilon} \right\|^2_{L^2(L^2(1+\rho), L^2(\rho))} \, ds.$$
For a given orthonormal basis \((e_k)_{k \geq 1}\) of \(L^2(1 + \rho)\) we expand
\[
\|P_{t-s}^\# \Pi_0 \Phi_s \sqrt{Q} \|_{L^2(L^2(1 + \rho), L^2(\rho))}^2 = \sum_k \|P_{t-s}^\# \Pi_0 \Phi_s \sqrt{Q} e_k \|_{L^2(\rho)}^2 \leq e^{-2\kappa (t-s)} \sum_k \|\sqrt{Q} e_k \|_{\rho_s}^2
\]
Thus, we obtain the (asymptotic) second moment estimate
\[
E \left[ \|v_0(t) \|_{\rho_s}^2 \right] \leq 2e^{-2\kappa t} \|\eta\|_{\rho_s} + 2 \frac{\|\sqrt{Q}\|_{L^2(L^2(1 + \rho), L^2(\rho))}^2}{1 - e^{-2\kappa t}} \rightarrow \frac{\|\sqrt{Q}\|_{L^2(L^2(1 + \rho), L^2(\rho))}^2}{\kappa} \quad \text{as } t \to \infty.
\]

3.5. Minimisation. It is still open to verify that the above choice of \(C_0\) indeed realises the declared objective of minimising the distance between the stochastic wave \(v\) and all possible translations of the deterministic profile \(\hat{v}\), thus offering an apt description for a stochastic phase. Since all relevant dynamics have been considered on a scale of order \(\varepsilon\), it is natural to also investigate the minimisation property on this scale.

Proposition 3.6. For \(t < \tau_{\eta, \varepsilon} \wedge \tau_{\infty, \varepsilon}\) the function \(a \mapsto \|v - \hat{v}(\cdot + ct + \varepsilon a)\|_{\rho_s}^2\) is locally minimal to order \(\varepsilon\) at \(a = C_0(t)\).

Proof. Applying Theorem 3.5 we obtain
\[
\frac{1}{2} \frac{d}{da} \|v(t) - \hat{v}(\cdot + ct + \varepsilon a)\|_{\rho_s}^2 = -\varepsilon^2 (u_0(t) + r(t), \hat{v}_x(\cdot + \gamma_0(t)))_{\rho_s}
\]
\[
= -\varepsilon^2 \left( \langle \hat{v}_x(\cdot + ct), u_0(t) \rangle_{\rho_s} + \langle \hat{v}_x(\cdot + ct + \varepsilon C_0(t)) - \hat{v}_x(\cdot + ct), u_0(t) \rangle_{\rho_s} + \langle \hat{v}_x(\cdot + ct), r(t) \rangle_{\rho_s} + \langle \hat{v}_x(\cdot + ct + \varepsilon C_0(t)) - \hat{v}_x(\cdot + ct), r(t) \rangle_{\rho_s} \right) = o(\varepsilon^2).
\]
Here we used that \(t \hat{v}_x(\cdot + ct), u_0(t) \rangle_{\rho_s} = 0\) as shown in (3.10). For the second derivative we obtain
\[
\frac{1}{2} \frac{d^2}{da^2} \|v(t) - \hat{v}(\cdot + ct + \varepsilon a)\|_{\rho_s}^2 = \varepsilon^2 \langle \hat{v}_x(\cdot + ct) \rangle_{\rho_s}^2 + o(\varepsilon^2) > 0.
\]

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