Split extension classifiers in the category of cocommutative Hopf algebras

Marino Gran
*Université catholique de Louvain*

joint work with G. Kadjo, F. Sterck and J. Vercruysse

*Category Theory 2019*
University of Edinburgh
13 July 2019
Outline

“Abelian” versus “semi-abelian”

Cocommutative Hopf algebras

Split extension classifiers

A description in the case of Hopf algebras
Outline

“Abelian” versus “semi-abelian”

Cocommutative Hopf algebras

Split extension classifiers

A description in the case of Hopf algebras
“Abelian” versus “semi-abelian”

Definition
A category $\mathcal{C}$ is abelian if

- $\mathcal{C}$ has a 0-object
- $\mathcal{C}$ has finite products
- any arrow $f$ in $\mathcal{C}$ has a factorisation $f = i \circ p$

where $p$ is a normal epi and $i$ is a normal mono.
Ab is the typical example of abelian category:

- Ab has a 0-object: the trivial group \( \{0\} \)
- Ab has finite products
- any homomorphism \( f \) in Ab has a factorisation \( f = i \circ p \)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{p} & & \downarrow{i} \\
\rightarrow{f(X)} & & \\
\end{array}
\]

where \( p \) is a surjective homomorphism (= normal epi) and \( i \) is an inclusion as a normal subgroup (= normal mono).
Grp is not abelian:

- Grp has a 0-object: the trivial group
- Grp has finite products
- Problem: an arrow $f$ in Grp does not have a factorisation $f = i \circ p$

With $p$ a surjective homomorphism and $i$ an inclusion as a normal subgroup.
**Question**: is there a list of simple axioms to develop a unified treatment of the categories $\text{Grp}$, $\text{Rng}$, $\text{Lie}_K$, ...?
Question: is there a list of simple axioms to develop a unified treatment of the categories Grp, Rng, Lie_K, ... ?

S. Mac Lane, Duality for groups, Bull. Amer. Math. Soc. (1950)
Several proposals of “non-abelian contexts” for

**radical theory:**
S. A. Amitsur (1954), A.G. Kurosh (1959)

**non-abelian homological algebra:**
A. Frölich (1961), M. Gerstenhaber (1970), G. Orzech (1972)

**commutator theory:**
P. Higgins (1956), S.A. Huq (1968), etc.
Definition (G. Janelidze, L. Márki, W. Tholen, JPAA, 2002)

A finitely complete category $\mathcal{C}$ is semi-abelian if

1. $\mathcal{C}$ has a 0-object
2. $\mathcal{C}$ has $A + B$
3. $\mathcal{C}$ is (Barr)-exact
4. $\mathcal{C}$ is (Bourn)-protomodular:

\[ 0 \rightarrow K \xrightarrow{k} A \xleftarrow{f} B \]
\[ \downarrow u \quad \downarrow v \quad \downarrow w \]
\[ 0 \rightarrow K' \xrightarrow{k'} A' \xleftarrow{f'} B' \]

$u, w$ isomorphisms $\Rightarrow v$ isomorphism.
Examples
Grp, Rng, Lie$_K$, XMod (more generally, any variety of $\Omega$-groups)
Examples
Grp, Rng, Lie_K, XMod (more generally, any variety of Ω-groups)

Loop, Grp(Comp), Set^op, Heyt, etc.
Examples
Grp, Rng, Lie_k, XMod (more generally, any variety of \(\Omega\)-groups)

Loop, Grp(Comp), Set^op, Heyt, etc.

\[ \text{[ } C \text{ is abelian } \iff [ C \text{ and } C^\text{op} \text{ are semi-abelian} ]! \]
Examples
Grp, Rng, Lie$_K$, XMod (more generally, any variety of $\Omega$-groups)

Loop, Grp(Comp), Set$_{\ast}^{\text{op}}$, Heyt, etc.

[ $\mathcal{C}$ is abelian ] $\iff$ [ $\mathcal{C}$ and $\mathcal{C}^{\text{op}}$ are semi-abelian]!

Many new connections have been discovered between semi-abelian (co)homology and commutator theory in universal algebra.
Outline

“Abelian” versus “semi-abelian”

Cocommutative Hopf algebras

Split extension classifiers

A description in the case of Hopf algebras
Let $K$ be a field.

**Bialgebras**

A $K$-bialgebra $(A, m, u, \Delta, \epsilon)$ is both a $K$-algebra $(A, m, u)$ and a $K$-coalgebra $(A, \Delta, \epsilon)$, where $m, u, \Delta, \epsilon$ are linear maps such that

\[
\begin{align*}
A \otimes A \otimes A & \xrightarrow{1_A \otimes m} A \otimes A \\
A \otimes A & \xrightarrow{m} A
\end{align*}
\]

and

\[
\begin{align*}
A & \xrightarrow{\Delta} A \otimes A \\
A \otimes A & \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A
\end{align*}
\]

\[
\begin{align*}
A \otimes K & \xrightarrow{1_A \otimes \epsilon} A \otimes A \\
A & \xleftarrow{\epsilon \otimes 1_A} K \otimes A
\end{align*}
\]

and

\[
\begin{align*}
A \otimes A & \xrightarrow{1_A \otimes \epsilon} A \otimes A \\
A & \xleftarrow{\epsilon \otimes 1_A} K \otimes A
\end{align*}
\]

commute, and $m$ and $u$ are $K$-coalgebra morphisms.
A Hopf algebra \((A, m, u, \Delta, \epsilon, S)\) is a \(K\)-bialgebra with an antipode, a linear map \(S: A \to A\) making the following diagram commute:

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{1_A \otimes S} & A \otimes A \\
\Delta \downarrow & & \downarrow S \otimes 1_A \\
A & \xrightarrow{\epsilon} & K \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{m} & A \\
\downarrow \epsilon & & \downarrow u \\
K & & A
\end{array}
\]
A Hopf algebra \((A, m, u, \Delta, \epsilon, S)\) is a \(K\)-bialgebra with an antipode, a linear map \(S: A \to A\) making the following diagram commute:

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{1_A \otimes S} & A \otimes A \\
\Delta & & \Delta \\
A & \xrightarrow{\epsilon} & K & \xrightarrow{u} & A
\end{array}
\]

\((A, m, u, \Delta, \epsilon, S)\) is cocommutative if the following triangle commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \\
& \xleftarrow{\Delta} & \\
A \otimes A & \xrightarrow{tw} & A \otimes A
\end{array}
\]

In Sweedler’s notations : \(\Delta(a) = a_1 \otimes a_2 = a_2 \otimes a_1\), for any \(a \in A\).
Example
Any group $G$ gives the group-algebra

$$K[G] = \left\{ \sum_{g} \alpha_{g} g \mid g \in G, \right\},$$

which becomes a cocommutative Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}.$$
Example
Any group $G$ gives the group-algebra

$$K[G] = \{ \sum g \alpha_g \mid g \in G, \},$$

which becomes a cocommutative Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}.$$ 

In the category $\text{Hopf}_{K,\text{coc}}$ of cocommutative Hopf algebras there is the full subcategory

$$\text{GrpHopf}_K \subset \text{Hopf}_{K,\text{coc}}$$

of group Hopf algebras (= generated by grouplike elements).
Theorem (M. Gran, F. Sterck and J. Vercruysse, JPAA, 2019)
The category $\text{Hopf}_{K,coc}$ is semi-abelian.
**Theorem (M. Gran, F. Sterck and J. Vercruysse, JPAA, 2019)**

The category $\text{Hopf}_{K,coc}$ is semi-abelian.

**Remark**

The fact that $\text{Hopf}_{K,coc}$ is protomodular follows from

$$\text{Hopf}_{K,coc} \cong \text{Grp}(\text{Coalg}_{K,coc})$$
Theorem (M. Gran, F. Sterck and J. Vercruysse, JPAA, 2019)
The category $\text{Hopf}_{K,\text{coc}}$ is semi-abelian.

Remark
The fact that $\text{Hopf}_{K,\text{coc}}$ is protomodular follows from

$$\text{Hopf}_{K,\text{coc}} \cong \text{Grp}(\text{Coalg}_{K,\text{coc}})$$

The most difficult part is to prove that $\text{Hopf}_{K,\text{coc}}$ is a regular category (this was explained by F. Sterck in her talk).
In particular, this result implies

**Theorem (M. Takeuchi, Manuscr. Math., 1972)**

The category $\text{Hopf}_{K,\text{coc}}^{\text{comm}}$ is abelian.
In particular, this result implies

**Theorem (M. Takeuchi, Manuscr. Math., 1972)**
The category $\text{Hopf}_{K,coc}^{\text{comm}}$ is abelian.

Indeed:

$$\text{Hopf}_{K,coc}^{\text{comm}} = \text{Ab(Hopf}_{K,coc})$$.
In particular, this result implies

**Theorem (M. Takeuchi, Manuscr. Math., 1972)**
The category $\text{Hopf}_{K,coc}^{\text{comm}}$ is abelian.

Indeed:

$$\text{Hopf}_{K,coc}^{\text{comm}} = \text{Ab}(\text{Hopf}_{K,coc}).$$

$A \in \text{Hopf}_{K,coc}$ is abelian $\iff \Delta : A \to A \otimes A$ is a normal mono
In particular, this result implies

**Theorem (M. Takeuchi, Manuscr. Math., 1972)**

The category $\text{Hopf}_{K,coc}^{\text{comm}}$ is abelian.

Indeed:

$$\text{Hopf}_{K,coc}^{\text{comm}} = \text{Ab}(\text{Hopf}_{K,coc}).$$

$A \in \text{Hopf}_{K,coc}^{\text{comm}}$ is abelian $\iff \Delta: A \to A \otimes A$ is a normal mono $\iff A$ is commutative: $ab = ba$ $\iff A \in \text{Hopf}_{K,coc}^{\text{comm}}$
There is an adjunction

\[
\text{Hopf}^{\text{comm}}_{K, \text{coc}} = \text{Ab}(\text{Hopf}_{K, \text{coc}}) \rightleftharpoons \text{Hopf}_{K, \text{coc}} \]

In general, if \( C \) is semi-abelian, \( \text{Ab}(C) \) is abelian
There is an adjunction

\[
\text{Hopf}_{K, \text{coc}}^{\text{comm}} = \text{Ab}(\text{Hopf}_{K, \text{coc}}) \xleftarrow{\ab} \text{Hopf}_{K, \text{coc}}
\]

In general, if \( \mathcal{C} \) is semi-abelian, \( \text{Ab}(\mathcal{C}) \) is abelian

\[
\text{Ab}(\mathcal{C}) \xleftarrow{\ab} \mathcal{C}
\]

with unit of the adjunction

\[
A \xrightarrow{\eta_A} [A, A]
\]
Commutators
For general normal Hopf subalgebras $M, N$ of $A \in \text{Hopf}_{K,\text{coc}}$

\[
\begin{array}{c}
M \longrightarrow A \longleftarrow N \\
\end{array}
\]

one can compute the categorical commutator :

\[
[M, N]_{\text{Huq}} = \langle \{m_1 n_1 S(m_2) S(n_2) \mid m \in M, n \in N\} \rangle_A
\]

(where $\Delta(m) = m_1 \otimes m_2$ and $\Delta(n) = n_1 \otimes n_2$).
In $\text{Hopf}_{K,\text{coc}}$ the condition $[M, N]_{\text{Huq}} = 0$ is equivalent to the existence of a (unique) morphism $p: M \otimes N \to A$ making the diagram

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{(1_M, 0)} & M \\
\downarrow & & \downarrow & p & \downarrow \\
(0, 1_N) & \downarrow & \downarrow & & \downarrow \\
M & \xrightarrow{p} & \downarrow & N \\
\downarrow & & \downarrow & \downarrow & \downarrow \\
A & & A & & A
\end{array}
\]

commute, where $p(m \otimes n) = mn$, for any $m \otimes n \in M \otimes N$. 

This allows one to apply methods of commutator theory to $\text{Hopf}_{K,\text{coc}}$. 

In $\text{Hopf}_{K,\text{coc}}$ the condition $[M, N]_{\text{Huq}} = 0$ is equivalent to the existence of a (unique) morphism $p: M \otimes N \rightarrow A$ making the diagram

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{(1_M,0)} & (1_M,0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{p} & M & \xrightarrow{(0,1_N)} & (0,1_N) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\rho} & A & \xrightarrow{\rho} & A \\
\end{array}
\]

commute, where $p(m \otimes n) = mn$, for any $m \otimes n \in M \otimes N$.

This allows one to apply methods of commutator theory to $\text{Hopf}_{K,\text{coc}}$. 
Outline

“Abelian” versus “semi-abelian”

Cocommutative Hopf algebras

Split extension classifiers

A description in the case of Hopf algebras
Split extensions

In a semi-abelian category \( \mathcal{C} \) a **split extension** is a diagram

\[
0 \rightarrow X \xrightarrow{\kappa} A \xleftarrow{s} B \rightarrow 0 \tag{1}
\]

where \( \kappa = \text{Ker}(p) \) and \( p \circ s = 1_B \).
**Split extensions**

In a semi-abelian category $\mathbb{C}$ a **split extension** is a diagram

$$
0 \longrightarrow X \xrightarrow{\kappa} A \xleftarrow{s} B \xrightarrow{p} 0
$$

(1)

where $\kappa = \text{Ker}(p)$ and $p \circ s = 1_B$.

**Example**

In the category $\text{Grp}$ of groups each split extension (1) is determined by a morphism

$$
\chi : B \rightarrow \text{Aut}(X)
$$

where the action of $B$ on $X$ is given by

$$
\chi(b)(x) = s(b)x s(b)^{-1}
$$

for any $b \in B$ and $x \in X$. 
Given any \( X \in \text{Grp} \) there is a universal split extension

\[
0 \to X \xrightarrow{i_1} X \rtimes \text{Aut}(X) \xleftarrow{i_2} \text{Aut}(X) \to 0
\]

(with kernel \( X \)) with the following universal property:

\[
\exists ! \chi \downarrow \downarrow
\]
Given any $X \in \text{Grp}$ there is a universal split extension

$$
\begin{array}{c}
0 \rightarrow X \xrightarrow{i_1} X \rtimes \text{Aut}(X) \xleftarrow{i_2} \text{Aut}(X) \rightarrow 0
\end{array}
$$

(with kernel $X$) with the following universal property:

for any other split extension, there is a unique morphism

$$
\begin{array}{c}
0 \rightarrow X \xrightarrow{\kappa} A \xleftarrow{s} B \rightarrow 0
\end{array}
$$

$$
\begin{array}{c}
0 \rightarrow X \xrightarrow{i_1} X \rtimes \text{Aut}(X) \xleftarrow{i_2} \text{Aut}(X) \rightarrow 0.
\end{array}
$$
Given $X \in \text{Grp}$, the group $\text{Aut}(X)$ is the split extension classifier:

$$
0 \rightarrow X \xrightarrow{\kappa} A \leftarrow B \rightarrow 0
$$

$$
0 \rightarrow X \xrightarrow{i_1} X \rtimes \text{Aut}(X) \xleftarrow{i_2} \text{Aut}(X) \rightarrow 0.
$$

The category $\text{Grp}$ has representable actions in the sense of F. Borceux, G. Janelidze, G.M. Kelly, Comment. Math. Univ. Carolin. 2005.
The term “having representable actions” comes from the fact that

$$\text{SplExt}(−, X): \text{Grp}^{op} \to \text{Set}$$

is representable, with representing object $\text{Aut}(X)$:

$$\text{SplExt}(−, X) \cong \text{hom}(−, \text{Aut}(X)).$$
The term “having representable actions” comes from the fact that

$$\text{SplExt}(\_, X) : \text{Grp}^{op} \to \text{Set}$$

is representable, with representing object $\text{Aut}(X)$:

$$\text{SplExt}(\_, X) \cong \text{hom}(\_, \text{Aut}(X)).$$

**Split extensions in Grp** correspond to actions:

$$\text{Act}(\_, X) \cong \text{SplExt}(\_, X) \cong \text{hom}(\_, \text{Aut}(X))$$
Split extensions in the category of Lie algebras

Similarly, for any $L \in \text{Lie}_K$ the Lie algebra $\text{Der}(L)$ of derivations is a split extension classifier

\[ 0 \rightarrow L \xrightarrow{\kappa} A \xleftarrow{s} B \rightarrow 0 \]
\[ 0 \rightarrow L \xrightarrow{i_1} L \times \text{Der}(L) \xleftarrow{i_2} \text{Der}(L) \rightarrow 0 \]

where the Lie algebra action is

\[ \rho(b)(l) = [s(b), l] \]
Split extensions in the category of Lie algebras

Similarly, for any $L \in \text{Lie}_K$ the Lie algebra $\text{Der}(L)$ of derivations is a split extension classifier

$$
0 \rightarrow L \xrightarrow{\kappa} A \xleftarrow{s} B \rightarrow 0
$$

$$
0 \rightarrow L \xrightarrow{i_1} L \times \text{Der}(L) \xleftarrow{i_2} \text{Der}(L) \rightarrow 0
$$

where the Lie algebra action is

$$
\rho(b)(l) = [s(b), l]
$$

$$
\text{Act}(\_ , L) \cong \text{SplExt}(\_ , L) \cong \text{hom}(\_ , \text{Der}(L))
$$
In general, a semi-abelian category \( \mathbb{C} \) has representable actions if any object \( X \in \mathbb{C} \) has a split extension classifier, denoted by \([X]\), with

\[
0 \rightarrow X \xrightarrow{\kappa} \overline{X} \xleftarrow{s} [X] \rightarrow 0
\]

a universal split extension (with kernel \( X \)).
Outline

“Abelian” versus “semi-abelian”

Cocommutative Hopf algebras

Split extension classifiers

A description in the case of Hopf algebras
Split extensions in cocommutative Hopf algebras

In $\text{Hopf}_{K,coc}$ any split extension

$$
0 \longrightarrow X \xrightarrow{\kappa} A \xleftarrow{s} B \longrightarrow 0
$$

is canonically isomorphic to the semidirect product exact sequence

$$
0 \longrightarrow X \xrightarrow{\kappa} A \xleftarrow{s} B \longrightarrow 0
$$
Semidirect product

In the split exact sequence

\[ 0 \rightarrow X \xrightarrow{i_1} X \rtimes B \xrightarrow{i_2} B \xleftarrow{p_2} 0 \]  

the semidirect product \( X \rtimes B \) is the vector space \( X \otimes B \) equipped with the cocommutative Hopf algebra structure:

- \( M_{X \rtimes B}(x \otimes b, x' \otimes b') = x(b_1 \cdot x') \otimes b_2 b' \)
- \( \Delta_{X \rtimes B} = (1_X \otimes \text{tw} \otimes 1_B)(\Delta_X \otimes \Delta_B) \)
- \( u_{X \rtimes B} = u_X \otimes u_B \) and \( \epsilon_{X \rtimes B} = \epsilon_X \otimes \epsilon_B \)
- \( S(x \otimes b) = (S_B(b_1)) \cdot S_X(x) \otimes S_B(b_2) \)

(here \( b \cdot x \) denotes the action of \( b \) on \( x \) corresponding to \( 0 \rightarrow X \overset{\kappa}{\rightarrow} A \overset{s}{\leftarrow} B \rightarrow 0 \))
When $K$ is an algebraically closed field of characteristic 0:

**Theorem (Milnor-Moore, Ann. Math. 1965)**

For any cocommutative Hopf $K$-algebra $H$ there is a split extension

$$
0 \longrightarrow \mathcal{U}(L_H) \xrightarrow{i_1} H \cong \mathcal{U}(L_H) \rtimes K[G_H] \xleftarrow{i_2} K[G_H] \longrightarrow 0
$$

\[ \mathcal{U}(L_H) \] is the universal enveloping algebra of the Lie algebra $L_H = \{ x \in H | \Delta(x) = 1 \otimes x + x \otimes 1 \}$ of primitive elements of $H$; $K[G_H]$ is the group Hopf algebra generated by the grouplike elements $G_H = \{ x \in H | \Delta(x) = x \otimes x, \epsilon(x) = 1 \}$ of $H$. 
When $K$ is an algebraically closed field of characteristic 0:

**Theorem (Milnor-Moore, Ann. Math. 1965)**

For any cocommutative Hopf $K$-algebra $H$ there is a split extension

$$0 \longrightarrow \mathcal{U}(L_H) \xrightarrow{i_1} H \cong \mathcal{U}(L_H) \rtimes K[G_H] \xrightarrow{i_2} K[G_H] \longrightarrow 0$$

- $\mathcal{U}(L_H)$ is the universal enveloping algebra of the Lie algebra
  
  $$L_H = \{ x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1 \}$$

  of **primitive elements** of $H$;
When $K$ is an algebraically closed field of characteristic 0:

**Theorem (Milnor-Moore, Ann. Math. 1965)**

For any cocommutative Hopf $K$-algebra $H$ there is a split extension

$$0 \rightarrow \mathcal{U}(L_H) \overset{i_1}{\rightarrow} H \cong \mathcal{U}(L_H) \rtimes K[G_H] \overset{i_2}{\underset{p_2}{\leftarrow}} K[G_H] \rightarrow 0$$

- $\mathcal{U}(L_H)$ is the universal enveloping algebra of the Lie algebra

$$L_H = \{ x \in H \mid \Delta(x) = 1 \otimes x + x \otimes 1 \}$$

of primitive elements of $H$;

- $K[G_H]$ is the group Hopf algebra generated by the grouplike elements

$$G_H = \{ x \in H \mid \Delta(x) = x \otimes x, \epsilon(x) = 1 \}$$

of $H$. 
This result can be used to prove

**Proposition (M.G., G. Kadjo and J. Vercruysse (APCS, 2016))**

When $K$ is an algebraically closed field with characteristic 0, the pair $(\text{PrimHopf}_K, \text{GrpHopf}_K)$ of full subcategories of $\text{Hopf}_K,\text{coc}$ is a hereditary torsion theory.

Moreover, the category of groups is a localization of $\text{Hopf}_K,\text{coc}$, i.e. the reflector $F: \text{Hopf}_K,\text{coc} \to \text{Grp}$ preserves finite limits.
This result can be used to prove

**Proposition (M.G., G. Kadjo and J. Vercruysse (APCS, 2016))**

When $K$ is an algebraically closed field with characteristic 0, the pair

$$(\text{PrimHopf}_K, \text{GrpHopf}_K)$$

of full subcategories of $\text{Hopf}_{K, \text{coc}}$ is a hereditary torsion theory.

Moreover, the category of groups is a localization of $\text{Hopf}_{K, \text{coc}}$

Grp $\xleftarrow{F} \perp \xrightarrow{\bot} \text{Hopf}_{K, \text{coc}}$

i.e. the reflector $F: \text{Hopf}_{K, \text{coc}} \to \text{Grp}$ preserves finite limits.
Split extension classifier in $\text{Hopf}_{K,\text{coc}}$

The category $\text{Hopf}_{K,\text{coc}}$ has representable actions in the sense of Borceux, Janelidze, Kelly (2005).
Split extension classifier in $\text{Hopf}_{K,coc}$

The category $\text{Hopf}_{K,coc}$ has representable actions in the sense of Borceux, Janelidze, Kelly (2005).

It is natural to look for an explicit description of the split extension classifier $[H]$ of any cocommutative Hopf algebra $H$. 
The “group Hopf algebra part” of $[H]$ is

$$K[\text{Aut}_{\text{Hopf}}(H)]$$

where $\text{Aut}_{\text{Hopf}}(H)$ is the group of Hopf automorphisms of $H$. 
The “group Hopf algebra part” of $[H]$ is

$$K[\text{Aut}_{\text{Hopf}}(H)]$$

where $\text{Aut}_{\text{Hopf}}(H)$ is the group of Hopf automorphisms of $H$.

To define the “primitive part” of $[H]$ one needs the following

**Definition**

A Hopf derivation of a Hopf algebra $(H, m, u, \Delta, \epsilon, S)$ is a linear endomorphism $\psi : H \to H$ that is a derivation

$$\psi \circ m = m \circ (\psi \otimes \text{id} + \text{id} \otimes \psi)$$

and a coderivation

$$\Delta \circ \psi = (\psi \otimes \text{id} + \text{id} \otimes \psi) \circ \Delta.$$
One writes $\text{Der}_{\text{Hopf}}(H)$ for the Lie algebra of Hopf derivations, where

$$[\psi_1, \psi_2] = \psi_1 \circ \psi_2 - \psi_2 \circ \psi_1, \quad \forall \psi_1, \psi_2 \in \text{Der}_{\text{Hopf}}(H).$$
One writes $\text{Der}_{\text{Hopf}}(H)$ for the Lie algebra of Hopf derivations, where

$$[\psi_1, \psi_2] = \psi_1 \circ \psi_2 - \psi_2 \circ \psi_1, \quad \forall \psi_1, \psi_2 \in \text{Der}_{\text{Hopf}}(H).$$

By applying the universal enveloping algebra functor $\mathcal{U}: \text{Lie}_K \rightarrow \text{Hopf}_{K,\text{coc}}$ one gets the primitive Hopf algebra

$$\mathcal{U}(\text{Der}_{\text{Hopf}}(H)).$$
One writes $\text{Der}_{\text{Hopf}}(H)$ for the Lie algebra of Hopf derivations, where

$$[\psi_1, \psi_2] = \psi_1 \circ \psi_2 - \psi_2 \circ \psi_1, \quad \forall \psi_1, \psi_2 \in \text{Der}_{\text{Hopf}}(H).$$

By applying the universal enveloping algebra functor $\mathcal{U}: \text{Lie}_K \to \text{Hopf}_K,\text{coc}$ one gets the primitive Hopf algebra

$$\mathcal{U}(\text{Der}_{\text{Hopf}}(H))$$

One defines

$$[H] = \mathcal{U}(\text{Der}_{\text{Hopf}}(H)) \rtimes \bar{\rho} \ K[\text{Aut}_{\text{Hopf}}(H)]$$

where the action

$$\bar{\rho}: K[\text{Aut}_{\text{Hopf}}(H)] \otimes \mathcal{U}(\text{Der}_{\text{Hopf}}(H)) \to \mathcal{U}(\text{Der}_{\text{Hopf}}(H))$$

is determined by $\bar{\rho}(\phi \otimes \psi) = \phi \circ \psi \circ \phi^{-1}$. 
Theorem (M.G., G. Kadjo and J. Vercruysse, BBMS 2018)
Let $K$ be an algebraically closed field of characteristic zero. Then

$$[H] = \mathcal{U}(\text{Der}_{\text{Hopf}}(H)) \rtimes_{\rho} K[\text{Aut}_{\text{Hopf}}(H)]$$

is the split extension classifier of $H$ in $\text{Hopf}_{K,coc}$.
Theorem (M.G., G. Kadjo and J. Vercruysse, BBMS 2018)
Let $K$ be an algebraically closed field of characteristic zero. Then

$$[H] = \mathcal{U}(\text{Der}_{\text{Hopf}}(H)) \rtimes K[\text{Aut}_{\text{Hopf}}(H)]$$

is the split extension classifier of $H$ in $\text{Hopf}_{K,coc}$

There is a universal split extension

$$0 \longrightarrow H \longrightarrow H \rtimes_* [H] \longrightarrow [H] \longrightarrow 0$$

where the action $*: [H] \otimes H \rightarrow H$ is defined by

$$(\phi \otimes \psi) \ast h = \psi(\phi(h))$$

for any $\phi \otimes \psi \in [H] = \mathcal{U}(\text{Der}_{\text{Hopf}}(H)) \rtimes K[\text{Aut}_{\text{Hopf}}(H)]$, and $h \in H$. 
Center
When a semi-abelian category $\mathcal{C}$ is action representable, the categorical center $Z(X)$ of an object $X$ can be obtained as the kernel of the canonical arrow $\chi$ in

\[
\begin{array}{ccccccc}
0 & \rightarrow & X & \rightarrow & X \times X & \leftarrow & X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & X & \rightarrow & X \times X & \leftarrow & X \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & X & \rightarrow & X \times [X] & \leftarrow & [X] & \rightarrow & 0 \\
\end{array}
\]

(see A. Cigoli and S. Mantovani, JPAA, 2012).
**Example**

In the case of groups, this corresponds to the fact that the center $Z(G)$ of a group $G$ is the kernel of the conjugation map $\chi$ in

\[
0 \rightarrow G \rightarrow G \times G \xleftarrow{\bar{\chi}} G \rightarrow G \rightarrow 0
\]

\[
0 \rightarrow G \rightarrow G \times Aut(G) \xleftarrow{\bar{\chi}} Aut(G) \rightarrow 0
\]

where $\chi(g)(h) = ghg^{-1}$, for any $g, h \in G$. 

\[
\begin{array}{cccccc}
0 & \rightarrow & G & \rightarrow & G \times G & \xleftarrow{\bar{\chi}} & G & \rightarrow & 0 \\
& & \downarrow & & \bar{\chi} & & \downarrow & & \\
0 & \rightarrow & G & \rightarrow & G \times Aut(G) & \xleftarrow{\bar{\chi}} & Aut(G) & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \rightarrow & G & \rightarrow & G \times Aut(G) & \xleftarrow{\bar{\chi}} & Aut(G) & \rightarrow & 0 \\
& & \downarrow & & \bar{\chi} & & \downarrow & & \\
0 & \rightarrow & G & \rightarrow & G \times Aut(G) & \xleftarrow{\bar{\chi}} & Aut(G) & \rightarrow & 0
\end{array}
\]
**Definition (N. Andruskiewitsch, Canad. J. Math. 1996)**

Given a Hopf algebra $A$, the Hopf center $HZ(A)$ is the largest Hopf subalgebra of $A$ contained in the algebraic center $Z_{alg}(A)$ of $A$, where

$$Z_{alg}(A) = \{ a \in A \mid ab = ba, \forall b \in A \}.$$
Definition (N. Andruskiewitsch, Canad. J. Math. 1996)
Given a Hopf algebra $A$, the Hopf center $HZ(A)$ is the largest Hopf subalgebra of $A$ contained in the algebraic center $Z_{alg}(A)$ of $A$, where

$$Z_{alg}(A) = \{ a \in A \mid ab = ba, \forall b \in A \}.$$

Proposition (M.G., G. Kadjo and J. Vercruysse, 2018)
When $A$ is cocommutative, the categorical center $Z(A)$ of $A$ coincides with the Hopf center $HZ(A)$:

$$Z(A) = HZ(A) = \{ a \in A \mid \Delta(a) \in A \otimes Z_{alg}(A) \}.$$
Final remarks
It is interesting to adopt the approach based on semi-abelian categories in the study of (cocommutative) Hopf algebras.
Final remarks
It is interesting to adopt the approach based on semi-abelian categories in the study of (cocommutative) Hopf algebras.

The case of general Hopf algebras is more subtle, since limits in $\text{Hopf}_K$ are difficult to compute.
Final remarks

It is interesting to adopt the approach based on semi-abelian categories in the study of (cocommutative) Hopf algebras.

The case of general Hopf algebras is more subtle, since limits in $\text{Hopf}_K$ are difficult to compute.

The approach based on *Schreier split extensions* (due to Sobral, Martins-Ferreira, Montoli, Bourn) could be useful to study some exactness properties of $\text{Hopf}_K$. 
References

• G. Janelidze, L. Márki and W. Tholen, Semi-abelian categories, J. Pure Appl. Algebra (2002)
• F. Borceux, G. Janelidze and G.M. Kelly, Internal object actions, Comment. Math. Univ. Carolin. (2005)
• M. Takeuchi, A correspondence between Hopf ideals and sub-Hopf algebras, Manuscr. Mathematica (1972)
• J. Milnor and J. Moore, On the structure of Hopf algebras, Ann. Math. (1965)
• M. Gran, G. Kadjo and J. Vercruysse, Split extension classifiers in the category of cocommutative Hopf algebras, Bull. Belgian Math. Society (2018)
• M. Gran, F. Sterck and J. Vercruysse, A semi-abelian extension of a theorem by Takeuchi, J. Pure Appl. Algebra (2019)
• N. Andruskiewitsch, Notes on extensions of Hopf algebras, Canad. J. Math. (1996)