Limited memory predictors based on polynomial approximation of periodic exponents

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Abstract

The paper presents transfer functions for limited memory time-invariant linear integral predictors for continuous time processes such that the corresponding predicting kernels have bounded support. It is shown that processes with exponentially decaying Fourier transforms are predictable with these predictors in some weak sense, meaning that convolution integrals over the future times can be approximated by causal convolutions over past times. For a given predicting horizon, the predictors are based on polynomial approximation of a periodic exponent (complex sinusoid) in a weighted $L_2$-space.

Key words: forecasting, transfer functions, weak predicability

1 Introduction

We study pathwise predictability and predictors of continuous time processes in deterministic setting and in the framework of the frequency analysis. It is well known that certain restrictions on frequency distribution can ensure additional opportunities for prediction and interpolation of the processes; see, e.g., Slepian (1978), Knab (1979), Papoulis (1985), Marvasti (1986), Vaidyanathan (1987), Lyman et al (2000, 2001) and the bibliography therein. These works considered predictability of band-limited processes; the predictors were non-robust with respect to small noise in high frequencies; see, e.g., the discussion in Higgins (1996), Chapter 17.

We consider some special linear weak predictability: instead of predictability of the original processes, we study predictability of sets of anticausal convolution integrals based on linear time-invariant integral predictors. This version of predictability was introduced in Dokuchaev (2008) for band-limited and high-frequency processes. In Dokuchaev (2021), the problem was
considered for processes single point spectrum degeneracy. In Dokuchaev (2021), the problem was considered for processes with exponentially decaying Fourier transforms. In these works, integral predictors with kernels featuring unlimited support were derived. Respectively, the predicting algorithms based on these predictors would require unlimited history of observations for the underlying processes.

Following the setting from Dokuchaev (2010), the present paper considers processes with exponentially decaying Fourier transforms. It is known that these processes are analytic and therefore allow a unique extension from any open interval. In particular, an arbitrarily accurate prediction can be achieved via Taylor series expansions with sufficiently small steps and sufficiently high order of the Taylor polynomials. However, this would require calculations of large number of derivatives for the underlying processes which is impractical. The result of Dokuchaev (2010) allowed to use "universal" integral type predictors instead of calculating derivatives of the underlying process. The goal of the present paper is to develop integral predictors with limited memory, i.e. such that the corresponding convolution kernels have bounded support, and the corresponding predicting algorithm requires history of observations on some finite time interval. The setting with limited memory predictors was first suggested in Lyman et al (2000) and Lyman and Edmonson (2001) for stochastic stationary band-limited processes. In Lyman et al (2000), an existence result for the predictors but the predictors are not derived. In Lyman and Edmonson (2001) the predictors are obtained for the case of a known preselected spectral density.

In the present paper, some principally new predictors are obtained via polynomials approximating a periodic exponent $e^{i\omega T}$ in exponentially weighted $L_2$-spaces, where $\omega \in \mathbb{R}$, and where $T > 0$ is a preselected prediction horizon. These predictors allow a compact explicit representation in the time domain given by equation (3.2) below. In addition, these predictors allow explicit representation in the frequency domain via their transfer functions given by equation (3.3) below. The choice of the predictors is independent on the spectral characteristics of input processes.

The paper is organized in the following manner. In Section 2 we formulate the definitions and background facts related to the linear weak predictability. In Section 3 we formulate the main theorems on predictability and predictors (Theorem 1 and Theorem 2). In Section 4 we discuss possible choices of approximating polynomials. In Section 5 we discuss the robustness of the predictors. Section 7 contains the proofs. Finally, in Section 8 we discuss our results.
2 Problem setting and definitions

Let $x(t)$ be a currently observable continuous time process, $t \in \mathbb{R}$. The goal is to estimate, at current times $t$, the values $y(t) = \int_{t}^{t+T} h(t-s)x(s)ds$, using historical values of the observable process $x(s)|_{s \leq t}$. Here $h(\cdot)$ is a given kernel, and $T > 0$ is a given prediction horizon.

In this case, $x$ has to be found, and where $\tau > 0$. We call $\hat{h}$ a predictor or predicting kernel.

To describe admissible classes of $h$ and $\hat{h}$, we need some notations and definitions. Let $R^+ \triangleq [0, +\infty)$, $C^+ \triangleq \{z \in C : \text{ Re } z > 0\}$, $C^- \triangleq \{z \in C : \text{ Re } z < 0\}$, $i = \sqrt{-1}$.

For $p \in [1, +\infty]$, we denote by $L_p(R, R)$ and $L_p(R, C)$ the usual $L_p$-spaces of functions $x : R \to R$ and $x : R \to C$ respectively.

For $x \in L_p(R, C)$, $p = 1, 2$, we denote by $X = \mathcal{F}x$ the function defined on $iR$ as the Fourier transform of $x$;

$$X(i\omega) = (\mathcal{F}x)(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}x(t)dt, \quad \omega \in R.$$ 

If $x \in L_2(R, C)$, then $X$ is defined as an element of $L_2(iR, C)$, i.e., $X(i\cdot) \in L_2(R, C)$.

For $x \in L_p(R, C)$, $p = 1, 2$, such that $x(t) = 0$ for $t < 0$, we denote by $Lx$ the Laplace transform

$$X(z) = (Lx)(z) \triangleq \int_{0}^{\infty} e^{-zt}x(t)dt, \quad z \in C^+. \quad (2.1)$$

In this case, $X|_{iR} = \mathcal{F}x$.

Let $H^r$ be the Hardy space of holomorphic functions $U(z)$ on $C^+$ with finite norm $\|U\|_{H^r} = \sup_{s > 0} \|U(s+i)\|_{L_r(R, C)}$, $r \in [1, +\infty]$; see, e.g., Duren (1970)).

Let $H_r^\infty$ be the Hardy space of holomorphic functions $U(z)$ on $C^-$ with finite norm $\|h\|_{H_r^\infty} = \sup_{s > 0} \|U(-s+i)\|_{L_r(R, C)}$, $r \in [1, +\infty]$.

**Definition 1** For $\theta \in [0, +\infty)$ and $T \in (0, +\infty)$, we denote by $\mathcal{H}(-T, \theta)$ the set of functions $h : R \to R$ such that $h(t) = 0$ for $t \notin [-T, \theta]$ and such that $h \in C^\infty(R)$.

It can be shown that the traces of functions from $\mathcal{H}(-T, \theta)$ are everywhere dense in $L_2(-T, \theta)$: let $\kappa_\varepsilon(t)$ be defined as $\kappa_\varepsilon(t) = \varepsilon^{-1}\kappa_1(t/\varepsilon)$, where $\kappa_1(t)$ is the so-called Sobolev kernel defined as $\kappa_1(t) = \kappa^{-1}\exp((t^2(t^2-1)^{-1})1_{|t|<1})$, where $\kappa = \int_{-1}^{1} \exp(t^2(t^2-1)^{-1})dt$.

Let $\bar{h} \in L_2(R)$ be a function vanishing outside $(-T, \theta)$. Then, for any $\varepsilon > 0$, functions $h_\varepsilon$, defined as the convolutions

$$h_\varepsilon(t) = \int_{-\infty}^{\infty} \kappa_\varepsilon(t-s)1_{[-T+\varepsilon, \theta-\varepsilon]}(s)\bar{h}(s)ds,$$

belong to $\mathcal{H}(-T, \theta)$ and approximate $\bar{h}$ in $L_2(R)$.
Remark 1

Let \( \hat{h}(0, \tau) \) be the class of functions \( \hat{h} : \mathbb{R} \to \mathbb{C} \) such that \( \hat{h} \in L_2(\mathbb{R}; \mathbb{C}) \), \( \hat{h}(t) = 0 \) for \( t \notin [0, \tau] \) and such that \( \hat{H}() = L\hat{h} \in \mathbb{H}^2 \cap \mathbb{H}^\infty \).

Definition 2

For \( \tau > 0 \), let \( \hat{H}(0, \tau) \) be the class of functions \( \hat{h} : \mathbb{R} \to \mathbb{C} \) such that \( \hat{h} \in L_2(\mathbb{R}; \mathbb{C}) \), \( \hat{h}(t) = 0 \) for \( t \notin [0, \tau] \) and such that \( \hat{H}() = L\hat{h} \in \mathbb{H}^2 \cap \mathbb{H}^\infty \).

Definition 3

Let \( \bar{X} \) be a class of processes \( x \) from \( L_2(\mathbb{R}; \mathbb{C}) \cup L_1(\mathbb{R}; \mathbb{C}) \). Let \( T > 0, \theta \geq 0 \), and \( \tau > 0 \), be given.

(i) We say that the class \( \bar{X} \) is linearly weakly \((\tau, \theta)\)-predictable with the prediction horizon \( T \) if, for any \( h(\cdot) \in \mathcal{H}(-\theta, T) \), there exists a sequence \( \{\hat{h}_{d}(\cdot)\}_{d=1}^{+\infty} = \{\hat{h}_{d}(\cdot, \bar{X}, k)\}_{d=1}^{+\infty} \subset \hat{H}(0, \tau) \) such that

\[
\sup_{t \in \mathbb{R}} |y(t) - \hat{y}_{d}(t)| \to 0 \quad \text{as} \quad d \to +\infty \quad \forall x \in \mathcal{X},
\]

where

\[
y(t) \overset{\Delta}{=} \int_{t}^{t+T} h(t - s)x(s)ds, \quad \hat{y}_{d}(t) \overset{\Delta}{=} \int_{t-\tau}^{t} \hat{h}_{d}(t - s)x(s)ds.
\]

The process \( \hat{y}_{d}(t) \) is the prediction of the process \( y(t) \).

(ii) Let the set \( \mathcal{F}(\bar{X}) \overset{\Delta}{=} \{X = \mathcal{F}x, \ x \in \bar{X}\} \) be provided with a norm \( \| \cdot \| \). We say that the class \( \bar{X} \) is linearly weakly \((\tau, \theta)\)-predictable with the prediction horizon \( T \) uniformly with respect to these norm \( \| \cdot \| \), if, for any \( h(\cdot) \in \mathcal{H}(-\theta, T) \), there exists a sequence \( \{\hat{h}_{d}(\cdot)\} = \{\hat{h}_{d}(\cdot, \bar{X}, h, \| \cdot \|)\} \subset \hat{H}(0, \tau) \) such that

\[
\sup_{t \in \mathbb{R}} |y(t) - \hat{y}_{d}(t)| \to 0 \quad \text{uniformly in} \quad \{x \in \bar{X} : \|X\| \leq 1, \ X = \mathcal{F}x\}.
\]

Here \( y(\cdot) \) and \( \hat{y}_{d}(\cdot) \) are defined in part (i) of this definition.

Remark 1

We include the case where \( \theta > 0 \), because the choice of \( \theta = 0 \) would allow only \( h \) vanishing at zero. For these kernels, the values of \( x \) at the nearest future times are not covered by the prediction.

3 The main result

For \( r > 0 \), let \( L_{2,r}(\mathbb{R}; \mathbb{C}) \) be the Hilbert space of processes \( u : \mathbb{R} \to \mathbb{C} \) with the norm

\[
\|u\|_{L_{2,r}(\mathbb{R}, \mathbb{C})} = \left( \int_{-\infty}^{+\infty} e^{r|\omega|} |u(\omega)|^2d\omega \right)^{1/2}.
\]

Let \( \mathcal{X}(r) \) be the set of processes \( x \in L_2(\mathbb{R}; \mathbb{C}) \) such that \( \|X(i)\|_{L_{2,r}(\mathbb{R}, \mathbb{C})} < +\infty \) for \( X = \mathcal{F}x \).

Let \( \mathcal{U}(r) \) be a class of processes \( x(\cdot) \in \mathcal{X}(r) \) such that \( \|X(i)\|_{L_{2,r}(\mathbb{R}, \mathbb{C})} \leq 1 \).
Theorem 1 For any $r > 0$, $T > 0$, $\theta \geq 0$, and $\tau > 0$, the following holds.

(i) The class $X(r)$ is $(T + \theta, \theta)$-predictable in the weak sense with the prediction horizon $T$.

(ii) The class $U(r)$ is linearly weakly $(T + \theta, \theta)$-predictable with the prediction horizon $T$ uniformly with respect to the norm $\| \cdot \|_{L_2,r(\mathbb{R})}$.

Remark 2 It can be seen that if $h \in H(-\theta, T)$ and $x \in X(r)$, then $y \in X(r)$, where $y$ is such as described in Definition 3. It follows from the fact that $\sup_{\omega \in \mathbb{R}} |H(i\omega)| < +\infty$ for $H = \mathcal{F}h$. In addition, if $x \in U(r)$, then $y/\sup_{\omega \in \mathbb{R}} |H(i\omega)| \in U(r)$. In general, the process $y$ represents certain smoothing of $x$, since $\sup_{\omega \in \mathbb{R}} |\omega^k H(i\omega)| < +\infty$ for all $k \in \mathbb{R}$; however, $h \notin X(r)$.

Remark 3 In Dokuchaev (2010), a related linear weak predictability was established for processes from $X(r)$. However, the predictability therein was established for a predictor requiring infinite history of observations. Theorem 1 above establishes predictability with predictors requiring a finite period of historical observations.

3.1 A family of predictors

The question arises how to find the predicting kernels. We suggest a possible choice of the kernels; they are given explicitly in the frequency domain.

Let $h \in H(-\theta, T)$ and $H = \mathcal{F}h$.

By the choice of $h$, we have that $H(i\omega) = Q(i\omega) e^{i\omega T}$, where $Q \in \mathbb{H}^2 \cap \mathbb{H}^\infty$ is such that $Q = \mathcal{L}q$, where $q(t) \equiv h(t - T)$.

Let $H(i\omega)|_{\omega \in \mathbb{R}}$ be extended on $\mathbb{C}$ as $H(z) = Q(z) e^{z T}$, $z \in \mathbb{C}$.

Theorem 2 The following holds.

(i) There exists a sequence of polynomials $\{\psi_d(z)\}_{d=1}^\infty$ of order $d$ such that

$$\|e^{iT \cdot} - \psi_d(i \cdot)\|^2_{L_2,-r(\mathbb{R},\mathbb{C})} = \int_{-\infty}^{\infty} |e^{iT \omega} - \psi_d(i \omega)|^2 e^{-r|\omega|} d\omega \rightarrow 0 \quad \text{as} \quad d \rightarrow +\infty. \quad (3.1)$$

(ii) For $d = 1, 2, \ldots, z \in \mathbb{C}$, set

$$\widehat{h}_d(t) \triangleq \sum_{k=0}^{d} a_{dk} \frac{d^k h}{dt^k}(t + T),$$
where \( a_{dk} \) are the coefficients of the polynomials \( \psi_d(z) = \sum_{k=0}^{d} a_{dk} z^k \). Then \( \hat{h}_d(\cdot) \in \hat{H}(0, T + \theta) \) for all \( d \), and the predictability of the processes considered in Theorem 1(i)-(ii) can be ensured with the sequence of these predicting kernels, i.e., with

\[
\hat{y}_d(t) = \int_{t-T-\theta}^{t} \hat{h}_d(t-s) x(s) ds. \tag{3.2}
\]

Theorem 2 describes predictor kernels in the time domain. These predictors can be represented explicitly in the frequency domain via their transfer functions

\[
\hat{H}_d(z) = e^{-Tz} \psi_d(z) H(z), \quad \hat{h}_d = F^{-1} \hat{H}_d|_{R}. \tag{3.3}
\]

Clearly, if the coefficients of a polynomial \( \psi_d \) are real, then the predicting kernel \( h_d(t) \) is real valued. In any case, if the underlying process \( x \) is real valued, then one should replace \( \hat{h}_d(t) \) by its real part.

4 On selection of polynomials \( \psi_d \)

Polynomials \( \{\psi_k\}_{k=0}^{\infty} \) required for the predictors can be constructed from projections of the function \( e^{i\omega T} \) on the truncated orthonormal basis in the Hilbert space \( L_{2,-r}(R, +\infty) \) using the Gram–Schmidt procedure as the following.

Let \( u_k(\omega) \doteq \omega^k, \ k = 0, 1, 2, \ldots, \) and let

\[
w_0 = v_0 = u_0/\|u_0\|_{L_{2,-r}(R, C)},\quad v_k = u_k - \sum_{p=0}^{k-1} \langle u_k, v_p \rangle_{L_{2,-r}(R, C)} v_p, \quad w_k = v_k/\|v_k\|_{L_{2,-r}(R, C)}, \quad k = 1, 2, \ldots.
\]

In this case,

\[
\|w_k\|_{L_{2,-r}(R, C)} = 1, \quad k = 0, 1, \ldots, \quad \langle w_k, w_l \rangle_{L_{2,-r}(R, C)} = 0, \quad k, l = 0, 1, \ldots, \ k \neq l.
\]

**Theorem 3** Let \( c_k \doteq \langle e^{iT \cdot}, w_k \rangle_{L_{2,-r}(R, C)} \), \( k = 0, 1, 2, \ldots \). Then the polynomials

\[
\psi^GS_d(\omega) \doteq \sum_{k=0}^{d} c_k w_k(\omega)
\]

are such as required in Theorem 2(i). Moreover, they are optimal in the sense that

\[
\|e^{iT \cdot} - \psi^GS_d(\cdot)\|_{L_{2,-r}(R, C)} \leq \|e^{iT \cdot} - \psi_d(\cdot)\|_{L_{2,-r}(R, C)}
\]

for any \( d \) and any polynomial \( \psi_d \) of order \( d \).
For the case of small prediction horizon \( T < r \), the polynomials \( \psi_d \) can be constructed explicitly (although optimality in the sense of Theorem 3(iii) will not be preserved).

**Theorem 4** For the case where \( T < r \), the polynomials \( \psi_d(z) \) satisfying the assumptions of Theorem 2(i) can be constructed as\footnote{5} \( \psi_d(z) = \sum_{k=0}^{d} \frac{T^k z^k}{k!} \), i.e., as truncated Taylor expansions of \( e^{Tz} \).

It can be noted that the choice of polynomials in Theorems 3, 4 depends only on \( T \) and \( r \) only.

## 5 On robustness with respect to noise contamination

It is shown below that the predictors introduced in Theorem 2 and designed for processes from \( \mathcal{X}(r) \) feature some robustness with respect to noise contamination.

Suppose that \( r > 0 \) and either \( p = 1 \) or \( p = 2 \) is given.

Assume that the predictors are applied to a process \( x \in L_2(\mathbb{R}, C) \) such that \( x = x_0 + \eta \), where \( x_0 \in \mathcal{X}(r) \), and where \( \eta \in L_p(\mathbb{R}, C) \cap L_2(\mathbb{R}, C) \) represents the noise. We assume that either \( p = 1 \) or \( p = 2 \).

Let \( X = \mathcal{F}x \), \( X_0 = \mathcal{F}x_0 \), and \( N = \mathcal{F}\eta \).

We assume that \( X_0(i) \in L_2(\mathbb{R}, C) \) and \( \|N(i)\|_{L_p(\mathbb{R}, C)} = \nu \). The parameter \( \nu \geq 0 \) represents the intensity of the noise.

By the assumptions, the predictors are constructed as in Theorem 2 under the hypothesis that \( \nu = 0 \), i.e. that \( x = x_0 \in \mathcal{X} \). By Theorems 1, 2 for an arbitrarily small \( \varepsilon > 0 \), there exists \( d \) such that, if the hypothesis that \( \nu = 0 \) is correct, then

\[
E_d \triangleq \|\hat{y}_{d,0} - y\|_{L_\infty(\mathbb{R}, C)} \leq \varepsilon,
\]

where \( \hat{y}_{d,0} = \hat{h}_d * x \) be defined via convolutions as in Theorem 2 with \( \nu = 0 \).

Let us estimate the prediction error for the case where \( \nu > 0 \). Let \( \hat{y}_{d,\eta} \) be defined by (3.2) with \( x = x_0 + \eta \). We have that

\[
\|\hat{y}_{d,\eta} - y\|_{L_\infty(\mathbb{R}, C)} \leq E_d + E_{\eta,d},
\]

where

\[
E_{\eta,d} = \|\hat{h}_d * \eta - h * \eta\|_{L_\infty(\mathbb{R}, C)}
\]
represents the additional error caused by the presence of a high-frequency noise \( \eta \notin \mathcal{X}(r) \) (when \( \nu > 0 \)). We have that
\[
E_{\nu,d} \leq \frac{1}{2\pi} \| (\tilde{H}_d(i\cdot) - H(i\cdot))N(i\cdot) \|_{L_1(\mathbb{R},\mathbb{C})}.
\]
It follows that
\[
\| \tilde{y} - y \|_{L_{\infty}(\mathbb{R},\mathbb{C})} \leq \varepsilon + \frac{\nu}{2\pi} \| \tilde{H}_d(i\cdot) \|_{L_q(\mathbb{R},\mathbb{C})} + \| H(i\cdot) \|_{L_q(\mathbb{R},\mathbb{C})},
\]
where \( q = +\infty \) for \( p = 1 \) and \( q = 2 \) for \( p = 2 \).

Therefore, it can be concluded that the prediction is robust with respect to noise contamination for any given \( \varepsilon \). On the other hand, if \( \varepsilon \to 0 \) then \( \gamma \to +\infty \) and \( \kappa \to +\infty \). In this case, the right hand part of (5.1) is increasing for any given \( \nu > 0 \). Therefore, the error in the presence of noise will be large for a predictor targeting too small a size of the error for the noiseless processes from \( \mathcal{X}(r) \).

The equations describing the dependence of \( (\varepsilon, \tilde{\kappa}_d) \) on \( d \) could be derived similarly to estimates in Dokuchaev (2012), Section 6, where discrete time setting was considered. We leave it for future research.

6 On numerical implementation

6.1 An algorithm based on time discretisation

The predictor described above requires to calculate higher order derivative of the kernel \( h \). The property of the admissible kernel \( h \) make it difficult to find its derivatives even in rare cases where these derivatives can be found explicitly. For example, this is the case for the kernel
\[
h(t) = \frac{2}{T} \kappa_1 \left( \frac{2(t - T/2)}{T} \right).
\]
that belongs to the class \( \mathcal{H}(-T,0) \).

We suggest to streamline calculations via replacing the derivatives from \( h \) by the corresponding finite differences.

Let us consider the problem of forecasting of the integral
\[
A = \int_0^T h(T - s)x(s)ds
\]
based on observations of \( x(t) \) for \( t \in [-T,0] \), where \( h \) is defined by (6.1). This case is covered by Theorem 1 with \( \theta = 0 \).
Let us select an integer $n > 0$; this will be the number of sampling points $t \in [-T, 0]$ for the observable process $x(t)$. Let $\vec{t} = \{ t_k \}_{k=1}^n \in \mathbb{R}^n$ be such that $0 = t_1 < t_2 < \cdots < t_n = T$. We will use observations $\{ x(t_k - T) \}_{k=1}^n$.

For a function $f : \mathbb{R}^n \to \mathbb{R}$ be a function, we define a vector $D(\vec{t}, f) \in \mathbb{R}^n$ such that its $k$th component is $(f(t_{k+1}) - f(t_k))/(t_{k+1} - t_k)$ for $k = 1, \ldots, n-1$, and that its $n$th component is zero.

It can be noted that the first $n-1$ components of the vector $D(\vec{t}, f)$ are finite differences approximating the derivative $df/dt$ as $n \to +\infty$ for differentiable functions $f$. We select the last component to be zero since we will need to calculate this vector only for functions $f$ vanishing at the endpoints together with all derivatives; in that case, zero is the best approximation.

Let an integer $d > 0$ be given, and let $a_{dk}$ be defined as in Theorem 4.

For $h \in \mathcal{H}(-T, 0)$, let $q(t) = h(t + T)$, and let

$$D_0 = \{q_j\}_{j=1}^n, \quad D_1 = D(\vec{t}, D_0), \ldots, D_k = D(\vec{t}, D_{k-1}), \ldots.$$ 

Let $\hat{h} \in \mathbb{R}^n$ be defined as

$$\hat{h} = \sum_{k=0}^d a_{kd} D_k.$$ 

Then the estimate $\hat{A}_{n,d}$ of $A$ can be calculated as

$$\hat{A}_{n,d} = \sum_{k=1}^{n-1} \hat{h}(n - k)x(t_k - T)(t_{k+1} - t_k). \quad (6.3)$$

This is an approximation of estimate (3.2) after discretization in time.

### 6.2 Some numerical experiments

We made some numerical experiments.

For selected varying $T > 0$ and $d > 0$,

We considered input processes $x \in \mathcal{X}(r)$ with $r > T$ obtained via the Monte-Carlo simulation as the following.

1. We calculated vectors $\vec{h}$ using coefficients $a_{dk} = T^k/k!$ such as described in Theorem 4 by the choice of $y$, we have

2. We simulated random input $x(t)$. At each Monte-Carlo simulation, be selected independent random numbers $a, b, c, d, e, f$ uniformly distributed over intervals $[-500, 500]$. 

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independent random numbers $p, q$ uniformly distributed over intervals $[1, 20]$ and independent random numbers $\alpha, \beta$ uniformly distributed over intervals $[0.05, 500]$. For each set of number, we defined a process

$$\bar{x}(t) = \frac{a}{p+T+iT} + \frac{b}{q+T+iT} + c \text{sinc}(t/\alpha + e) + d \text{sinc}(t/\beta + f)$$

It can be noted that the first two terms in this sum represent processes from $X(p + T)$ and $X(q + T)$ respectively, and the second two terms represent band limited processes that belong to $X(r)$ for any $r > 0$. This means that $\bar{x} \in X(r)$ for some $r > T$.

3. For some given $\sigma \geq 0$, random noise processes $\eta$ were simulated as Gaussian process $\{\eta(t)\}_{t=-n}^{n}$ with independent values such that $E\eta_r(t) = 0$ and $\text{Var} \eta_r(t) = \sigma^2$. A noise contaminated process $x = \bar{x} + \eta_r$ was created to replace $\bar{x}$ in the simulation.

4. For each $x(\cdot)$, we calculated the value $A$ defined by (6.2).

5. For each $x(\cdot)$, we calculated the value $\hat{A}_{n,d}$ defined by (6.3).

6. For each $x(\cdot)$, we calculated the forecast error $\hat{A}_{n,d} - A$ and the relative forecast error

$$E(n, d, x(\cdot)) = \frac{\hat{A}_{n,d} - A}{|A|}$$

7. We calculated the mean relative error

$$E(n, d, \sigma) = \mathbb{E}E(n, d, x(\cdot)).$$

Here $\mathbb{E}$ means the average over the Monte-Carlo simulations.

We have used R software. We have used $T = 0.2$, and we have used 1000 Monte-Carlo simulations for each set of parameters.

For the case where $\sigma = 0$, i.e., without the additional noise, we obtained that

$$E(500, 5, 0) = 0.026, \quad E(1000, 5, 0) = 3 \cdot 10^{-6},$$

$$E(3000, 4, 0) = 7 \cdot 10^{-6}, \quad E(3000, 5, 0) = 1 \cdot 10^{-6}.$$
It follows from (7.1)-(7.2) that

\[ E(20000, 4, 0.05) = 0.121, \quad E(40000, 4, 0.05) = 0.069, \]
\[ E(4000, 4, 0.02) = 0.106, \quad E(3000, 4, 0.02) = 0.1265, \]
\[ E(2500, 4, 0.01) = 0.066, \quad E(3000, 4, 0.01) = 0.060, \]
\[ E(4000, 4, 0.01) = 0.051, \quad E(10000, 4, 0.01) = 0.026. \]

The results of these experiments confirm the effectiveness of the method and some robustness with respect to noise contamination.

7 Proofs

Theorem 1 follows immediately from Theorem 2.

Proof of Theorem 2. By the Completeness Theorem for polynomials (Higgins (1977), p.31), it follows that there exists sequence of polynomials \( \{ \tilde{\psi}_d(\omega) \}_{d=1}^{\infty} \) in \( \mathbb{R} \) of order \( d \) such that

\[
\| e^{iT} - \tilde{\psi}_d(\cdot) \|_{L_2(-r, R, C)}^2 = \int_{-\infty}^{\infty} |e^{iT\omega} - \tilde{\psi}_d(\omega)|^2 e^{-r|\omega|} d\omega \to 0 \quad \text{as} \quad d \to +\infty.
\]

The coefficients \( a_k \) of desired polynomials \( \psi_d(z) = \sum_{k=0}^{d} a_k z^k \) can be constructed by adjustment the signs of the coefficients for the polynomials \( \tilde{\psi}_d(\omega) = \sum_{k=0}^{d} \tilde{a}_k \omega^k \) such that \( \psi_d(i\omega) \equiv \tilde{\psi}_d(\omega) \), i.e., \( \tilde{a}_k = a_k i^k \) and \( a_k = \tilde{a}_k i^{-k} \). This proves statement (i).

Let us prove statement (ii). Clearly, \( q(t) = 0 \) for \( t < 0 \) and \( q \in C^{\infty}(\mathbb{R}) \). Hence \( z^n Q(z) \in \mathbb{H}_2 \cap \mathbb{H}_\infty \) for any integer \( n \geq 0 \). It follows that

\[
\tilde{H}(z) = \psi_d(z) Q(z) \in \mathbb{H}_2 \cap \mathbb{H}_\infty. \tag{7.1}
\]

Further, \( Q(z) e^{(T+\theta)z} \in \mathbb{H}_2 \). Hence

\[
\tilde{H}(z) e^{(T+\theta)z} = e^{-Tz} \psi_d(z) e^{Tz} Q(z) e^{(T+\theta)z} = \psi_d(z) Q(z) e^{(T+\theta)z} \in \mathbb{H}_2. \tag{7.2}
\]

It follows from (7.1)-(7.2) that \( \tilde{H} \in \mathcal{H}(-\theta, T) \).

For \( x \in \mathcal{X}(r) \), let \( X(i\omega) = \mathcal{F} x, \ Y(i\omega) = \mathcal{F} y = H(i\omega) X(i\omega), \ \tilde{Y}_d(i\omega) = \tilde{H}_d(i\omega) X(i\omega), \) and \( \tilde{y} = \mathcal{F}^{-1} \tilde{Y}_d \).

We have that

\[
\| \tilde{y}_d - y \|_{L_\infty(\mathbb{R})} \leq \frac{1}{2\pi} \| (\tilde{H}_d(i\cdot) - H(i\cdot)) X(i\cdot) \|_{L_1(\mathbb{R})}. \tag{7.3}
\]
Furthermore,
\[
\| (\tilde{H}_d(i\cdot) - H(i\cdot))X(i\cdot) \|_{L_1(\mathbb{R})} = \int_{-\infty}^{\infty} \left| (e^{-i\omega T} \psi_d(i\omega) - 1)e^{i\omega T} Q(i\omega)X(i\omega) \right| d\omega \\
= \int_{-\infty}^{\infty} e^{-r|\omega|/2} \left| (e^{-i\omega T} \psi_d(i\omega) - 1)e^{i\omega T} e^{r|\omega|/2} Q(i\omega)X(i\omega) \right| d\omega \\
= \int_{-\infty}^{\infty} e^{-r|\omega|/2} \left| (\psi_d(i\omega) - e^{i\omega T}) e^{r|\omega|/2} e^{i\omega T} Q(i\omega)X(i\omega) \right| d\omega \leq \alpha_d^{1/2} \beta^{1/2}.
\]
(7.4)

Here
\[
\alpha_d = \int_{-\infty}^{\infty} e^{-r|\omega|}|\psi_d(i\omega) - e^{i\omega T}|^2 d\omega, \quad \beta = \int_{-\infty}^{\infty} e^{r|\omega|} |e^{i\omega T} Q(i\omega)X(i\omega)|^2 d\omega.
\]

By the choice of \( \psi_d \), it follows that
\[
\alpha_d \to 0 \quad \text{as} \quad d \to +\infty. \tag{7.5}
\]

Since \( q \in C^\infty(\mathbb{R}; \mathbb{R}) \) and has a bounded support, it follows that \( \sup_{\omega} |Q(i\omega)| < +\infty \). Hence
\[
|\beta| \leq \sup_{\omega} |Q(i\omega)| \int_{-\infty}^{\infty} e^{r|\omega|}|X(i\omega)|^2 d\omega. \tag{7.6}
\]

We have that \( \beta \) is finite for each \( X \) under the assumptions of of Theorem 2(i)-(ii), and that \( \beta \) is bounded over \( x \in U(r) \) under the assumptions of of Theorem 2(ii). Then estimates (7.3)-(7.4) imply the proof of Theorem 2. \( \square \)

**Proof of Theorem 3**: follows from the completeness of polynomials in \( L_{2,-r}(\mathbb{R}, \mathbb{C}) \) (Higgins (1996), p.31), and from the orthonormality of the sequence \( \{w_k\} \) implied by the properties of the Gram-Schmidt orthogonalization process.

**Proof of Theorem 4**: We have that
\[
\psi_d(i\omega) = \sum_{k=0}^{d} \frac{(Ti\omega)^k}{k!} = C_d(\omega) + iS_d(\omega),
\]
where
\[
C_d(\omega) = \sum_{m=0}^{\infty} \frac{(Ti\omega)^{2m}}{(2m)!} = (-1)^m \sum_{m=0}^{\infty} \frac{T^{2m} \omega^{2m}}{(2m)!}
\]
and
\[
S_d(\omega) = \frac{1}{i} \sum_{m=0}^{\infty} \frac{(Ti\omega)^{2m+1}}{(2m)!} = (-1)^m \sum_{m=0}^{\infty} \frac{T^{2m+1} \omega^{2m}}{(2m)!}.
\]
We have that \( C_d(i\omega) \) and \( S_d(i\omega) \) are truncated Taylor expansions for \( \cos(T\omega) \) and \( \sin(T\omega) \) respectively. Hence
\[
\int_{-\infty}^{\infty} |\cos(T\omega) - C_{d-1}(i\omega)||e^{-r|\omega|}|d\omega \leq \int_{-\infty}^{\infty} \frac{T^d}{d!} |\omega|^d e^{-r|\omega|} d\omega = \frac{2T^d}{d!} \frac{d!}{r^{d-1}} = \frac{2T^d}{r^{d-1}}.
\]
By the assumptions, we have that $T < r$. Hence
\[
\int_{-\infty}^{\infty} |\cos(T\omega) - C_d(i\omega)|^2 e^{-r|\omega|} d\omega \to 0 \quad \text{as} \quad d \to +\infty.
\]
Similarly, we obtain that
\[
\int_{-\infty}^{\infty} |\sin(T\omega) - S_d(i\omega)|^2 e^{-r|\omega|} d\omega \to 0 \quad \text{as} \quad d \to +\infty.
\]
This completes the proof of Theorem 4.

8 Discussion and future research

The present paper studies prediction of continuous time processes in pathwise deterministic setting. The paper suggests linear integral predictors with limited memory for prediction of anti-causal convolutions with finite horizon.

The predictors are described explicitly via polynomials approximating periodic exponents (complex sinusoids) defined by the predicting horizon.

1. The predictors do not depend on the shape of the spectrum of the underlying process.
2. The predictors are not error-free; however, the error can be made arbitrarily small with a choice of larger degrees of polynomials.
3. Some predictors for the same class of underlying processes were obtained earlier in Dokuchaev (2010). However, the predictors in were quite different: they required unlimited history of observations.
4. The method leads to a relatively simple numerical algorithm based on time discretization.
5. Predictors feature some robustness with respect to noise contamination. This means that the predictors can be applied for processes that are not necessarily in the class $\mathcal{X}(r)$, provided that a certain forecasting error is tolerable.

Since $\sup_{\omega} |\hat{h}_d(i\omega)|$ could increase fast as $d \to +\infty$, the method would require calculations with large numbers to achieve high predicting accuracy. We leave this for the future research.

References
Dokuchaev, N.G. (2008). The predictability of band-limited, high-frequency, and mixed processes in the presence of ideal low-pass filters. *Journal of Physics A: Mathematical and Theoretical* **41** No 38, 382002 (7pp)

Dokuchaev, N. (2010). Predictability on finite horizon for processes with exponential decrease of energy on higher frequencies. *Signal Processing* **90** Iss. 2, 696–701.

Dokuchaev, N. (2012). Predictors for discrete time processes with energy decay on higher frequencies. *IEEE Transactions on Signal Processing* **60**, No. 11, 6027-6030.

Dokuchaev, N. (2021). Pathwise continuous time weak predictability and single point spectrum degeneracy. *Applied and Computational Harmonic Analysis*, (53) 116–131.

Duren P. (1970) *Theory of $H^p$-Spaces*. Academic Press, New York.

Higgins, J.R. (1977). Completeness and Basis Properties of Sets of Special Functions. Cambridge University Press. 1977.

Higgins, J.R. (1996). Sampling Theory in Fourier and Signal Analysis. Oxford University Press, New York.

Knab J.J. (1979). Interpolation of band-limited functions using the approximate prolate series. *IEEE Transactions on Information Theory* **25**(6), 717–720.

Lyman R.J, Edmonson W.W., McCullough S., and Rao M. (2000). The predictability of continuous-time, bandlimited processes. *IEEE Transactions on Signal Processing* **48**, Iss. 2, 311–316.

Lyman R.J and Edmonson W.W. (2001). Linear prediction of bandlimited processes with flat spectral densities. *IEEE Transactions on Signal Processing* **49**, Iss. 7, 1564–1569.

Marvasti F. (1986). Comments on ”A note on the predictability of band-limited processes.” *Proceedings of the IEEE*, **74**(11), 1596.

Papoulis A. (1985). A note on the predictability of band-limited processes. *Proceedings of the IEEE*, **73**(8), 1332–1333.

Slepian D. (1978). Prolate spheroidal wave functions, Fourier analysis, and uncertainty–V: The discrete case. *Bell System Technical Journal*, **57**(5), 1371–1430.

Vaidyanathan P.P. (1987). On predicting a band-limited signal based on past sample values. *Proceedings of the IEEE*, **75**(8), 1125–1127.