QUANTUM $SU(2, 2)$-HARMONIC OSCILLATOR

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Abstract

The $SU(2, 2)$-harmonic oscillator on the phase space $\mathcal{A}(2, 2) = SU(2, 2)/S(U(2) \times U(2))$ is quantized using the coherent states. The quantum Hamiltonian is the Toeplitz operator corresponding to the square of the distance with respect to the $SU(2, 2)$-invariant Kähler metric on the phase space. Its spectrum, depending on the choice of representation of $SU(2, 2)$, is computed.

1 Introduction

The $SU(2, 2)$-harmonic oscillator is the generalization of the model harmonic oscillator with the flat phase space. In our case the phase space $\mathcal{A}(2, 2) = SU(2, 2)/S(U(2) \times U(2)) \simeq SO(4, 2)/SO(4) \times SO(2)$ is the eight dimensional conformal domain, on which the canonical coordinates $(x^\mu, p^\mu), \mu = 0, \ldots, 3$ can be globally introduced.

The spaces of this type are well known as Cartan classical domains. They appear in physics and mathematics considered by many authors. The complex geometry of these spaces and, in particular, its applications in conformal theories has been investigated in work of Coquereaux and Jadczyk (see [1] and references there). The geometry of $\mathcal{A}(2, 2)$ is related to the space–time geometry. The Shilov boundary of $\mathcal{A}(2, 2)$ is the compactified Minkowski space–
time, endowed with the conformal structure of the signature $(+, -, -, -)$. The compactification is obtained by addition a light cone at infinity to the usual Minkowski space–time.

As it is suggested in [2] the conformal domain can be considered as the replacement of the space-time on the micro scale. This interpretation is based on the Born’s reciprocity idea of the symmetry between the space-time and the energy-momentum space. The reciprocity symmetry can be reformulated as the symmetry of the conformal domain. In the consequence these spaces are not distinguished on micro scale. The Minkowski space is interpreted as the very-high-mass, or very-high-energy-momentum-transfer limit of the conformal domain.

The $SU(2,2)$-harmonic oscillator is the one-body system. It is obtained from the two-body interacting system by introducing the "center of the mass" coordinates. The interaction is $SU(2,2)$-invariant. The covariant harmonic oscillators are used in quark models. In these models the interaction between quarks are given by the harmonic oscillator potential. The model of the relativistic hadron consisting of two quarks interacting in that way can be found in [3]. These models have been considered by many authors (see references in [3]). It is tempting to interpret our model along the similar lines.

The quantization by using the Berezin–Weyl calculus [4] provides the quantum Hamiltonian as the Toeplitz operator [1]. This scheme of quantization involves the system of Perelomov’s generalized coherent states for $SU(2,2)$ (see [5], [6]). The representation spaces of the quantization are the Hilbert spaces of the holomorphic functions on the domain, corresponding to the members of the discrete series of unitary irreducible representations of $SU(2,2)$. These representations spaces have their counterparts in Minkowski space–time as spaces of distributional boundary values (see [7]).

By the quantization procedure for different representations we obtain different spectrum of the quantum Hamiltonian. In contrast to the geometric quantization, this quantization does not contain the prequantization stage. For all representations the quantum Hamiltonian has discrete and degenerate spectrum. The $SU(1,1)$-harmonic oscillator has been considered in [8].

We use the $S$–parametrization of $\mathcal{A}(2,2)$ introduced in [1]. This parametrization provides the description of the geometry of $\mathcal{A}(2,2)$ in terms of its symmetries.
2. SU(2,2) – HARMONIC OSCILLATOR

The classical Hamiltonian of the SU(2,2) – harmonic oscillator in the S-parametrization [1] of A(2,2) is given by the function:

\[ H = \frac{1}{4} \text{Tr}(\ln^2(S_0S)) \]  
(2.1)

This function, up to multiplication constant, is the square of the distance from the origin \( S_0 \) of \( A(2,2) \), with respect to the SU(2,2) – invariant Kähler metric on \( A(2,2) \). The function (2.1) is the generalization of the Hamiltonian of the harmonic oscillator with the flat phase space. In the flat case the Hamiltonian of the harmonic oscillator can be obtained in this way from the Kähler metric on the phase space \( \Gamma = C^N \).

\( A(2,2) \) can be realized as the complex bounded domain

\[ 1 - ZZ^+ > 0, \]  
(2.2)

where the points of \( A(2,2) \) are parametrized by \( Z \in M_2(C) \). Let us introduce the following coordinates on \( A(2,2) \) given by [7]:

\[ Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_2 \\ 0 \end{bmatrix} \]  
(2.3)

\[ u_1 = e^{i\phi_1\sigma_3} e^{i\theta_1\sigma_1}, \quad u_2 = e^{i\phi_2\sigma_3} e^{i\theta_2\sigma_1}, \]

\[ \lambda_1 = r_+ e^{i\alpha}, \quad \lambda_2 = r_- e^{i\beta}, \]

where the Pauli matrices are: \( \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), \( \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \). The conditions for the coordinates are:

\[ 0 \leq r_+, \, r_- < 1 \]  
(2.4)

\[ 0 \leq \theta_1, \, \theta_2 \leq \frac{\pi}{2} \]

\[ 0 \leq \alpha, \, \beta, \, \phi_1, \, \phi_2 \leq 2\pi \]

Let us express the function (2.1) by the variables (2.4). Using the relation between parametrizations we can write:

\[ Y \equiv \frac{S_0S - 1}{S_0S + 1} = \frac{1}{2} \begin{bmatrix} Z + Z^+ & i(Z - Z^+) \\ i(Z - Z^+) & -(Z + Z^+) \end{bmatrix} \]  
(2.5)
Then we have:

\[ H = \frac{1}{4} \text{Tr} \left( \ln^2 (S_0 S) \right) = \frac{1}{4} \text{Tr} \left( \ln^2 \frac{1 + Y}{1 - Y} \right) \]  

(2.6)

The Hamiltonian (2.1) is invariant under the action of the isotropy group of the origin \( S_0 \), then it is invariant under the transformation \( Z \rightarrow UZV^+ = \begin{vmatrix} r_+ & 0 \\ 0 & r_- \end{vmatrix} \), where \( U \) and \( V \) are the unitary matrices given by the singular value decomposition for \( Z \). Using this invariance we obtain:

\[ H = \frac{1}{2} \left( \ln^2 \frac{1 + r_+}{1 - r_+} + \ln^2 \frac{1 + r_-}{1 - r_-} \right). \]  

(2.7)

The canonical variables can be introduced by:

\[ Z = \left( x^n + i\frac{p^n}{p^2} \right) \sigma, \]  

(2.8)

where the unbounded parametrization of \( \mathcal{A}(2,2) \) by \( Z \in M_2(\mathbb{C}) \) is used. In this parametrization the condition (2.2) reads:

\[ -i(Z - Z^+) > 0. \]  

(2.9)

The relation between (2.2) and (2.9) parametrizations is given by the Cayley transformation:

\[ Z = \frac{1 + iZ}{1 - iZ} \]  

(2.10)

3. COHERENT STATES FOR SU(2,2)

Let us consider the discrete series of unitary irreducible representations of \( SU(2,2) \), which are realized in the spaces of holomorphic functions on \( \mathcal{A}(2,2) \), namely the representation of the series \( d_0 \) in Graev’s classification [7]. The members of the series \( d_0 \) are labeled by the integer number \( n = 4, 5, \ldots \) and two spin labels \( j_1, j_2 \). In our case \( j_1 = j_2 = 0 \).

Let \( |dZ| \) denotes the Euclidean measure on \( \mathcal{A}(2,2) \). Let \( d\mu_n \) denotes the normalized measure given by:

\[ d\mu_n(Z) = N_n \left| \det(1 - ZZ^+) \right|^{n-4} |dZ|, \quad n = 4, 5, \ldots \]  

(3.1)
where \( N_n = \frac{(n-3)(n-2)^2(n-1)}{\pi^4} \) is the normalization constant so that \( f \, d\mu_n = 1 \).

The space of functions on \( \mathcal{A}(2, 2) \):

\[
\mathcal{F}_n = \left\{ f \text{ holomorphic} : \|f\|_n^2 = \int |f(Z)|^2 d\mu_n(Z) < \infty \right\}
\]

is the Hilbert space with the scalar product:

\[
\langle f | g \rangle = \int \overline{f(Z)} g(Z) d\mu_n(Z), \; f, g \in \mathcal{F}_n.
\]

The transformation:

\[
T^n (g) f(Z) = [\det(CZ + D)]^{-n} f((AZ + B)(CZ + D)^{-1}),
\]

\[ f \in \mathcal{F}_n, \; g^{-1} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \in SU(2, 2), \]

defines the unitary irreducible representation of \( SU(2, 2) \) in \( \mathcal{F}_n \). The system of coherent states of type \((T^n | \Psi_0 = 1 \rangle \)

\[
(T^n | \Psi_0 = 1 \rangle = [\det(CZ + D)]^{-n} |1 = | \Psi_0 > \in \mathcal{F}_n. \]

The states obtained by (3.3) can be parametrized by points of \( \mathcal{A}(2, 2) \):

\[
|\zeta > = \frac{[\det(1 - \zeta^+ \zeta)]^{n/2}}{[\det(1 - \zeta^+ Z)]^{n}}, \; 1 - \zeta^+ > 0.
\]

The family \(|\zeta > : 1 - \zeta^+ > 0\} \) forms the system of generalized coherent states for \( SU(2, 2) \) \[6\]. It has the property of the resolution of the unity:

\[
N_n \int |\zeta > < \zeta | d\mu(\zeta) = 1_{\mathcal{F}_n},
\]

where \( d\mu(\zeta) = [\det(1 - \zeta^+ \zeta)]^{-4}|d\zeta| \) is the \( SU(2, 2) \)-invariant measure on \( \mathcal{A}(2, 2) \). Every state \( |\Psi > \in \mathcal{F}_n \) has the continuous representation:

\[
|\Psi > \rightarrow < \zeta | \Psi > = C_{\Psi}(\zeta) = [\det(1 - \zeta^+ \zeta)]^{n/2} \Psi(\zeta).
\]
The representation (3.8) has the property:

\[ C_\Psi(\zeta') = \int K(\zeta', \zeta)C_\Psi(\zeta) d\mu(\zeta), \quad (3.9) \]

where

\[ K(\zeta', \zeta) = N_n < \zeta'|\zeta> = N_n \frac{[\det(1 - \zeta^+\zeta')]^{n/2} [\det(1 - \zeta^+\zeta)]^{n/2}}{[\det(1 - \zeta^+\zeta')]^n} \quad (3.10) \]

is the reproducing kernel:

\[ K(\zeta, \zeta'') = \int K(\zeta, \zeta')K(\zeta', \zeta'') d\mu(\zeta'). \quad (3.11) \]

4. QUANTUM SU(2,2)-HARMONIC OSCILLATOR

The classical system on \(A(2,2)\) can be quantized by using the Berezin-Weyl calculus. This scheme of the quantization involves the system of coherent states.

For every representation \(T^{(n)}\) of \(SU(2,2)\), \(n = 4, 5, \ldots\) we obtain different quantizations in the representation spaces \(F_n\). The operator corresponding to the classical observable is its Toeplitz operator constructed by using the generalized Bergman projection.

Let \(L^2(d\mu_n)\) denotes the Hilbert space of the measurable and square integrable functions on \(A(2,2)\) with respect to the measure \(d\mu_n\). The generalized Bergman projection \([4]\):

\[ P_B : L^2(d\mu_n) \longrightarrow F_n, P_B^+ = P_B = P_B^2 \quad (4.1) \]

is given by:

\[ (P_B f)(Z) = \int L_n(Z, \zeta)f(\zeta, \zeta^+)d\mu_n(\zeta), \quad f \in L^2(d\mu_n), \quad (4.2) \]

where \(L_n(\zeta', \zeta) = [\det(1 - \zeta^+\zeta')]^{-n}\) is the generalized Bergman kernel. The quantization associates to each function \(f \in L^2(d\mu_n)\) an operator \(\hat{f}\) in \(F_n\) \([1]\):

\[ f \longrightarrow \hat{f} = N_n \int f(\zeta, \zeta^+) |\zeta> <\zeta| d\mu(\zeta). \quad (4.3) \]
Acting by \( \hat{f} \) on \( |\Psi\rangle \in \mathcal{F}_n \) we have:

\[
\hat{f} |\Psi\rangle = P_B(f \cdot \Psi).
\]

Then the operator (4.3):

\[
\hat{f} = P_B \circ f \circ P_B
\]

is the Toeplitz operator corresponding to the function \( f \).

Let us describe the orthonormal base in \( \mathcal{F}_n \). The base consists of the functions [7]:

\[
\triangle_{jm_{q_1q_2}}(Z) = (N^{jm})^{-1}(\det Z)^m D_{q_1q_2}^j(Z),
\]

\[
m = 0, 1, 2, \ldots \quad 2j = 0, 1, 2, \ldots \quad -j \leq q_1, q_2 \leq j
\]

where the function \( D_{q_1q_2}^j \) is the extension of the polynomial well known from the \( SU(2) \) representation theory:

\[
D_{q_1q_2}^j(Z) = \left[ (j + q_1)! (j - q_1)! \right]^{1/2} \sum_{s=\max(0,q_1+q_2)}^{\min(j-q_1,j+q_2)} \binom{j+q_2}{s} \times
\]

\[
\times \left( \frac{j-q_2}{s-q_1-q_2} \right) z_{11}^{s-j-q_1-s} z_{12}^{j+q_1-s} z_{21}^{j+q_2-s} z_{22}^{s-q_1-q_2}
\]

and the normalization constant is given by:

\[
(N^{jm})^2 = (n-1)(n-2)^2(n-3) \frac{(n-3)!(n-4)!(m+2j+1)!m!}{(2j+1)(m+n-2)!(m+2j+n-1)!}
\]

(4.8)

The orthonormality of (4.6) reads:

\[
\langle \triangle_{jm_{q_1q_2}} \mid \triangle_{j'm'_{q'_1q'_2}} \rangle = \delta_{j',j} \delta_{m',m} \delta_{q'_1,q_1} \delta_{q'_2,q_2}
\]

(4.9)

By the quantization (4.3) of the \( SU(2,2) \)-harmonic oscillator we obtain the quantum Hamiltonian:

\[
\hat{H} = P_B \circ H \circ P_B, \quad H \in L^2(d\mu_n)
\]

(4.10)

In order to find the spectrum of the operator (4.10) let us compute the matrix element:

\[
\langle \triangle_{j'm'_{q'_1q'_2}} \mid \hat{H} \triangle_{jm_{q_1q_2}} \rangle = \langle \triangle_{j'm'_{q'_1q'_2}} \mid H \triangle_{jm_{q_1q_2}} \rangle
\]

(4.11)
In this order we use the coordinates \((2.3)\). After some calculations we obtain:

\[
\langle \Delta^{'j,m'}_{q_1q_2'} | \hat{H} \Delta^{'j,m}_{q_1q_2} \rangle = \delta^{'j,j} \delta^{'m,m} \delta^{'q_1,q_1} \delta^{'q_2,q_2} E^{(n)}_{q_1q_2} \tag{4.12}
\]

\[
E^{(n)}_{q_1q_2} = \langle \Delta^{'j,m}_{q_1q_2} | \hat{H} \Delta^{'j,m}_{q_1q_2} \rangle
\]

Then the operator \(\hat{H}\) is diagonal in the base \((4.6)\), while its eigenvalues are \(\langle \Delta^{'j,m}_{q_1q_2} \rangle\):

\[
\hat{H} \Delta^{'j,m}_{q_1q_2} = E^{(n)}_{q_1q_2} \Delta^{'j,m}_{q_1q_2} \tag{4.13}
\]

The eigenvalues are given by the integral:

\[
E^{(n)}_{q_1q_2} = \alpha_{j,m,n} \sum_{q=-j}^{q=n} \sum_{i=0}^{n-4} \sum_{l=0}^{-4} (-1)^{i+l} \binom{n-4}{i} \binom{n-4}{l} \times \tag{4.14}
\]

\[
\times \int_0^1 dr_+ \int_0^1 dr_- \left( \ln^2 \frac{1+r_+}{1-r_+} + \ln^2 \frac{1+r_-}{1-r_-} \right) r_+^{2(j+m+q+i)+1} r_-^{2(j+m-q+l)+1} \left( r_+^2 - r_-^2 \right)^2,
\]

where

\[
\alpha_{j,m,n} = \frac{(m+n-2)!(m+2j+n-1)!}{(2j+1)(n-3)!(n-4)!m!(m+2j+1)!}
\]

Let us denote for \(N = 0, 1, 2, \ldots\)

\[
S_1(N) \equiv \frac{1}{N+1} \sum_{a=0}^{N} \frac{1}{2a+1} \tag{4.15}
\]

\[
S_2(N) \equiv \begin{cases} 
\frac{1}{N+1} \sum_{b=1}^{N} \sum_{a=b}^{N} \frac{1}{2a+1} \cdot \frac{1}{2b} , & N = 1, 2, \ldots \\
0 , & N = 0 
\end{cases}
\]

\[
S(N) \equiv S_1(N) \ln 2 + S_2(N)
\]

The integral \((4.14)\) can be computed using the formula:

\[
\int_0^1 \ln^2 \frac{1+r}{1-r} \cdot r^{2N+1} dr = 4S(N) \tag{4.16}
\]

The eigenvalues are given by the formula:
\[
(n)
E_{jm}^{nq_2} = \frac{4}{(2j + 1)(n - 3)!} \sum_{i=0}^{n-4} (-1)^i \binom{n - 4}{i} \times
\]
\[
\times \left\{ \frac{(m + 2j + n - 1)!}{(m + 2j + 1)!} \left[ (m + n - 2)S(2j + m + 2 + i) 
- (m + 1)S(2j + m + 1 + i) \right] + 
\frac{(m + n - 2)!}{m!} [(2j + m + 2)S(m + i) - (2j + m + n - 1)S(m + 1 + i)] \right\}
\]

We observe that the eigenvalue does not depend on \(q_1, q_2\) indices. Then the eigenvalue \((n) E_{jm}^{nq_2}\) is \((2j + 1)^2\) degenerate.

5. Remarks

The result of the quantization depends on the choice of representation of \(SU(2, 2)\). The question arises how to interpret this choice. According to the Berezin’s interpretation [4] the number of representation depends on parameter \(h\), which plays the role of the Planck constant. By taking the limit \(h \to 0\) the correspondence principle is obtained. From this point of view the relation between this parameter and the Planck constant in (2.8) is not clear.

The Hamiltonian of the \(SU(2, 2)\)-harmonic oscillator may also be interpreted as the generalization of the Born’s quantum metric operator, which plays the crucial role in the reciprocity theory. This fact may encouraged us to interpret the spectrum of the quantum Hamiltonian in the spirit of this theory.

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