Reductions of the Volterra lattice

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We exhibit three classes of algebraic constraints which are shown compatible with Volterra lattice.

1. Introduction

In this Letter we discuss classical discrete system — Volterra lattice \[ \frac{\partial r(i)}{\partial t} = r(i) (r(i+1) - r(i-1)). \] (1)

Physical applications of this differential-difference system are well-known (see, for example, Refs. [2], [3]). In particular, the system (1) can be interpreted as kinetic equation describing stimulated scattering of plasma oscillations by ions. This system has been thoroughly studied for a number of initial conditions [4], [5], [6] by inverse scattering transform method. A number of works is concerned with the question: what a boundary conditions are consistent with higher flows in Volterra and Toda lattice hierarchy (see, for example [7], [8], [9], [10], [11], [12]). In particular some of their results show that imposing some special boundary conditions for the lattices yields finite-dimensional systems corresponding to finite growth Lie algebras.

Our principal goal in the Letter is to show three denumerable classes of invariant submanifolds of the Volterra lattice which are defined by some algebraic constraints. We show that each of these constraints are compatible with the Volterra equation itself and do not analyse their compatibility with the higher flows. As a result we are forced to consider finite-dimensional systems of ordinary differential equations with rational dependence on unknown functions supplemented by discrete symmetry transformation.

To make the matter more clear, let us remind firstly the notion of differential constraints compatible with a given system of differential equations [13]. For our aims, we may restrict ourselves by consideration of scalar evolutionary equation \( E \) in the following form:

\[ \frac{\partial r}{\partial t} = F(r, r', ..., r^{(m)}), \quad ' \equiv \frac{\partial}{\partial x}. \] (2)

Let us denote by [\( E \)] the union this equation and its differential consequences with respect to \( x \in \mathbb{R}^1 \). Let the equation (2) be supplemented by differential constraint \( H \)

\[ h(r, r', ..., r^{(n)}) = 0, \] (3)

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where \( h \), as well as \( F \), is supposed to be some locally analytic function of its arguments. One says that differential constraint (3) is compatible with (2) or, in other words, define invariant submanifold for (2), if
\[
D_t(h)|_{E \cap H} = 0,
\]
(4)
where \( D_t \) stands for total derivative with respect to \( t \in \mathbb{R}^1 \). The equation (4), whose solutions are some differential functions, is referred to as determining one [13].

The situation is considerably simplified when one can resolve (3) with respect to higher-order derivative as
\[
r^{(n)} = S(r, r', ..., r^{(n-1)}).
\]
Then practical recipe to solve determining equation (4) consists of successively replacing \( r^{(n)} \to S \).

The method of differential constraints allows us to select some classes of partial solutions of given equation (system of equations) by solving (3) and further analysis. Observe that this method can be applied to both integrable and nonintegrable equations. For integrable equations one can expect that differential constraint being considered as ordinary differential equation turns out to be integrable in some sense.

Let us return now to the Volterra lattice which is evolutionary equation of the form
\[
\frac{\partial r(i)}{\partial t} = F(r(i + 1), r(i), r(i - 1)).
\]
Here the discrete variable \( i \in \mathbb{Z} \) plays the role of “space” variable \( x \) in (2). Additional constraints in this case are not differential but algebraic and solutions of determining equation are some locally analytic functions
\[
h = h(r(i), ..., r(i + n)).
\]
We believe that determining equation (4) can be successfully applied in discrete case and show this on example of Volterra and Toda lattice.

In Ref. [14] we found an infinite class of algebraic constraints for the Volterra lattice (see, below (8)) but there we do not used determining equation and showed compatibility in the following equivalent way. Suppose that one can resolve the equation \( h = 0 \) as
\[
r(i + n) = S_1(r(i), ..., r(i + n - 1))
\]
and as
\[
r(i - 1) = S_2(r(i), ..., r(i + n - 1)).
\]
The latter equation can be derived only after shifting \( i \to i - 1 \). Identifying \( y_1 = r(i), ..., y_n = r(i + n - 1) \) for some value \( i = i_0 \), one is led to some first-order finite dimensional system of ordinary differential equations on unknown functions \( y_k(t) \). Note that provided that \( i \) is some fixed integer, (5) and (6) become some boundary conditions for the Volterra lattice. Then one defines “new” functions \( \tilde{y}_1 = r(i + 1), ..., \tilde{y}_n = r(i + n) \). These functions are related with “old” ones by invertible relations
\[
\tilde{y}_1 = y_2, ..., \tilde{y}_{n-1} = y_n, \quad \tilde{y}_n = S_1(y_1, ..., y_n).
\]
Then one requires that the collection \( \{ \tilde{y}_k(t) \} \) also represent the solution of the finite-dimensional system. It is checked by straightforward computations. If so, then one can conclude that algebraic constraint under consideration is compatible with the Volterra lattice, while the mapping (7) can be recognized as symmetry transformation for attached finite-dimensional system.

The Letter is organized as follows. In Section 2, we formulate our main theorem which generalize the result of [14]. In Section 3, we write down finite dimensional systems to which the Volterra lattice is reduced under corresponding constraints. Finally, in Section 4, we present relevant results for the Toda lattice. Most part of material of this Section can be found in [14].

2. Constraints compatible with Volterra lattice

Our main result is in the following theorem

Theorem 2.1. Each one of the following constraints

\[ \sum_{s=1}^{N} r(i + s - 1) = \prod_{s=0}^{N+1} r(i + s - 1), \quad N \geq 1, \]  

(8)

\[ \sum_{s=1}^{2M+1} r(i + s - 1) = \prod_{s=1}^{M+1} r(i + 2s - 2), \quad M \geq 1 \]  

(9)

and

\[ \sum_{s=1}^{2M} r(i + s - 1) \cdot \sum_{s=1}^{2M} r(i + s) = \prod_{s=1}^{2M+1} r(i + s - 1), \quad M \geq 1 \]  

(10)

is consistent with Volterra lattice (1).

To prove the theorem, one need in the following lemma:

Lemma 2.1. The quantity

\[ I_N(i) = \sum_{k=1}^{N} r(i + k - 1) \left( \sum_{s=k}^{N} r(i + s + 1) \right) \]  

(11)

\[ = \sum_{k=1}^{N} r(i + k + 1) \left( \sum_{s=1}^{k} r(i + s - 1) \right) \]  

(12)

is integral for difference system (8) (with corresponding \( N \)). The quantities \( I_{2M}(i) \) and \( I_{2M-1}(i) \) are integrals for the difference system (9) and (10), respectively.

Proofs of the above lemma and theorem are quite technical and we find that it is suitable to put them in Appendix.

3. Finite-dimensional systems

Let us present in this Section attached finite-dimensional systems for all three classes of constraints.
Identify \( y_1 = r(i), \ldots, y_{N+1} = r(i + N) \) for some fixed value \( i = i_0 \). The constraint (8) force this set of functions to be a solution of the system

\[
\begin{align*}
\dot{y}_1 &= y_1 y_2 - \frac{y_1 + \ldots + y_N}{y_2 \ldots y_{N+1}}, \\
\dot{y}_k &= y_k (y_{k+1} - y_{k-1}), \quad k = 2, \ldots, N, \\
\dot{y}_{N+1} &= \frac{y_2 + \ldots + y_{N+1}}{y_1 \ldots y_N} - y_N y_{N+1}.
\end{align*}
\] (13)

From the above theorem we already know that the constraint (8) (for any \( N \)) is compatible with the Volterra lattice. On the level of the system (13) this means that \( r(i) \)'s for all \( i \in \mathbb{Z} \) being expressed via \( y_k \)'s must solve the Volterra lattice. Consider “new” variables \( \{\tilde{y}_1, \ldots, \tilde{y}_{N+1}\} \) defined by shifting \( i \rightarrow i + 1 \), i.e. \( \tilde{y}_1 = r(i + 1), \ldots, \tilde{y}_{N+1} = r(i + N + 1) \). Thanks to the Theorem 2.1 these “new” functions also represent a solution of (13) being expressed, taking into account, (8) as

\[
\begin{align*}
\tilde{y}_1 &= y_2, \ldots, \\
\tilde{y}_N &= y_{N+1}, \\
\tilde{y}_{N+1} &= \frac{y_2 + \ldots + y_{N+1}}{y_1 \ldots y_N}.
\end{align*}
\] (14)

From what we already know, we can conclude that any solution of the system (13) supplemented by the mapping (14) gives suitable solution of the Volterra lattice. Observe that equations yielding time-evolution (13) and (14) have common integral

\[
I_N = \sum_{k=1}^{N-1} y_{k+2} \left( \sum_{s=1}^{k} y_s \right) + \frac{(y_1 + \ldots + y_N)(y_2 + \ldots + y_{N+1})}{y_1 \ldots y_{N+1}}.
\]

Remark that \( I_1 \equiv 1 \).

Similar arguments are relevant for constrains of the second class (9). Corresponding finite-dimensional system together with compatible mapping (symmetry transformation) read

\[
\begin{align*}
\dot{z}_1 &= z_1 \left( z_2 + \frac{z_1 + \ldots + z_{2M}}{1 - z_2 z_4 \ldots z_{2M}} \right), \\
\dot{z}_k &= z_k (z_{k+1} - z_{k-1}), \quad k = 2, \ldots, 2M - 1, \\
\dot{z}_{2M} &= -z_{2M} \left( \frac{z_1 + \ldots + z_{2M}}{1 - z_1 z_3 \ldots z_{2M-1}} + z_{2M-1} \right), \\
\bar{z}_1 &= z_2, \ldots, \bar{z}_{2M-1} = z_{2M}, \\
\bar{z}_{2M} &= \frac{z_1 + \ldots + z_{2M}}{z_1 z_3 \ldots z_{2M-1} - 1}.
\end{align*}
\] (15)

Equations (15) and (16) have an integral

\[
T_{2M} = \sum_{k=1}^{2M-2} z_{k+2} \left( \sum_{s=1}^{k} z_s \right)
\]
\[ + \sum_{k=1}^{2M} z_k \cdot \frac{z_1 z_3 \ldots z_{2M-1} + \sum_{k=2}^{2M} z_k + z_2 z_4 \ldots z_{2M} \cdot \sum_{k=1}^{2M-1} z_k - \sum_{k=2}^{2M-1} z_k}{(z_1 z_3 \ldots z_{2M-1} - 1)(z_2 z_4 \ldots z_{2M} - 1)} \]

As for constraints (10), making use similar calculations, we obtain the following system:

\[ \dot{w}_1 = w_1 \left( w_2 + \frac{(w_1 + \ldots + w_{2M-1})(w_1 + \ldots + w_{2M})}{w_1 + \ldots + w_{2M} - w_1 w_2 \ldots w_{2M}} \right), \]

\[ \dot{w}_k = w_k(w_{k+1} - w_{k-1}), \quad k = 2, \ldots, 2M - 1, \quad (17) \]

\[ \dot{w}_{2M} = -w_{2M} \left( \frac{(w_2 + \ldots + w_{2M})(w_1 + \ldots + w_{2M})}{w_1 w_2 \ldots w_{2M} - w_1 - \ldots - w_{2M}} + w_{2M-1} \right) \]

with corresponding symmetry transformation

\[ \tilde{w}_1 = w_2, \ldots, \tilde{w}_{2M-1} = w_{2M}, \]

\[ \tilde{w}_{2M} = \frac{(w_2 + \ldots + w_{2M})(w_1 + \ldots + w_{2M})}{w_1 w_2 \ldots w_{2M} - w_1 - \ldots - w_{2M}}. \quad (18) \]

The system (17) with compatible mapping (18) has the integral

\[ T_{2M-1} = \sum_{k=1}^{2M-2} \sum_{s=1}^{k} w_{k+2} \left( \sum_{s=1}^{k} w_s \right) \]

\[ + \sum_{k=1}^{2M} w_k \cdot \frac{\sum_{k=1}^{2M-1} w_k \cdot \sum_{k=2}^{2M} w_{k}}{w_1 w_2 \ldots w_{2M} - w_1 - \ldots - w_{2M}}. \]

4. Toda lattice

Constraints compatible with Toda lattice

\[ \dot{q}_1(i) = q_2(i+1) - q_2(i), \]

\[ \dot{q}_2(i) = q_2(i)(q_1(i) - q_1(i-1)) \quad (19) \]

can be obtained by using well known lattice Miura transformation

\[ q_1(i) = r(2i) + r(2i+1), \]

\[ q_2(i) = r(2i-1)r(2i). \quad (20) \]

For even \( N = 2P \) from (18) we immediately derive (14)

\[ \sum_{s=1}^{P} q_1(i + s - 1) = \prod_{s=1}^{P+1} q_2(i + s - 1) \quad (21) \]

By analogy with the case of Volterra lattice one can prove
Lemma 4.1. The quantity

\[ J_P(i) = \sum_{k=1}^{P} q_1(i + k - 1) \left( \sum_{s=k}^{P} q_1(i + s) \right) - \sum_{k=1}^{P} q_2(i + k) \]

\[ = \sum_{k=1}^{P} q_1(i + k) \left( \sum_{s=1}^{k} q_1(i + s - 1) \right) - \sum_{k=1}^{P} q_2(i + k) \]

is integral for difference system (21).

The integral \( J_P(i) \) is calculated as \( J_P(i) \mid_{(20)} = I_2 P(i) \). Along the lines as in Section 1 we are led to

Theorem 4.1. The constraint (21) is compatible with Toda lattice (19).

Instead of proving of this Theorem we observe that consistency condition reads \( J_P(i - 1) = J_P(i) \) which is valid by virtue of the Lemma 4.1.

To describe the reductions of the Toda lattice in terms of finite-dimensional systems it is convenient to pass from polynomial to exponential form of the latter with the help of ansatz

\[ q_1(i) = \dot{u}_i, \quad q_2(i) = e^{u_i - u_{i-1}}. \]

In variables \( u_i \) Toda lattice becomes [15]

\[ \ddot{u}_i = e^{u_{i+1} - u_i} - e^{u_i - u_{i-1}} \tag{22} \]

while the constraint (21) turns into

\[ \sum_{s=1}^{P} \dot{u}_{i+s-1} = e^{u_i + P - u_{i-1}} \tag{23} \]

Define a finite collection of variables attached to (23) as \( v_1 = u_i, \ldots, v_{P+1} = u_{i+P} \). Then as can be checked the constraint (23) leads to the system

\[ \ddot{v}_1 = e^{v_{2-v_1} - (\ddot{v}_1 + \ldots + \ddot{v}_{P})e^{v_1 - v_{P+1}}, \]

\[ \ddot{v}_k = e^{v_{k+1-v_k} - v_{k+1-v_k-1}, \quad k = 2, \ldots, P, \tag{24} \]

\[ \ddot{v}_{P+1} = (\ddot{v}_2 + \ldots + \ddot{v}_{P+1})e^{v_1 - v_{P+1}} - e^{v_{P+1-v_P}} \]

with corresponding symmetry transformation

\[ \ddot{v}_1 = v_2, \ldots, \ddot{v}_P = v_{P+1}, \]

\[ \ddot{v}_{P+1} = v_1 + \log(\ddot{v}_2 + \ldots + \ddot{v}_{P+1}). \]

Using Miura transformation (20) one can easy prove [14]

Proposition 4.1. The relations

\[ y_{2k-1} + y_{2k} = \ddot{v}_k, \quad k = 1, \ldots, P \]

\[ y_{2P+1} + \frac{y_2 + \ldots + y_{2P+1}}{y_1 y_2 \ldots y_{2P+1}} = \ddot{v}_{P+1}, \tag{25} \]
\[ y_{2k} y_{2k+1} = e^{v_{k+1} - v_k}, \quad k = 1, ..., P \]

realize the correspondence between the systems (24) and (13) with \( N = 2P \).

As was noticed in [14], the system (24), for any \( P \), admits Lagrangian and consequently Hamiltonian representation. Lagrangian is given by

\[
L = \sum_{k<l} \dot{v}_k \dot{v}_l + \sum_{k=1}^{P} e^{v_{k+1}-v_k} + \left( \frac{1}{2} \dot{v}_1 + \sum_{k=2}^{P} \dot{v}_k + \frac{1}{2} \dot{v}_{P+1} \right) e^{v_1-v_{P+1}}.
\]

It is natural to suppose that systems (13), (15), (17) and (24) may be integrable in the sense of Liouville-Arnold theorem. We are going to present the relevant material concerned with first integrals, Lax pairs, Painlevé analysis of the finite-dimensional systems under consideration in subsequent publications.

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Appendix

A. Proof of the Lemma 2.1.

First, notice that it is far from obvious that (11) \( \equiv \) (12). To prove this, we need of use induction by \( N \). To this aim, we observe that the recurrence relation

\[
I_{N+1}(i) = I_N(i) + r(i + N + 2) \cdot \sum_{s=1}^{N+1} r(i + s - 1)
\]

is valid both for (11) and for (12) with \( I_1(i) = r(i) r(i + 2) \). For \( N = 1 \) the identity (11) \( \equiv \) (12) is obvious. Suppose now that this is true for some positive integer \( N \), then

\[
I_{N+1}(i)|_{11} = I_N(i)|_{11} + r(i + N + 2) \cdot \sum_{s=1}^{N+1} r(i + s - 1) = I_N(i)|_{12}
\]

\[
+ r(i + N + 2) \cdot \sum_{s=1}^{N+1} r(i + s - 1) = I_{N+1}(i)|_{12}
\]

Therefore the identity (11) \( \equiv \) (12) is proved.

Now let us to show that by virtue of (8) the relation \( I_N(i+1) = I_N(i) \) is valid. We have

\[
I_N(i+1)|_{12} = \sum_{k=1}^{N-1} r(i + k + 2) \left( \sum_{s=1}^{k} r(i + s) \right) + r(i + N + 2) \cdot \sum_{s=1}^{N} r(i + s).
\]

(26)

Shifting in (8) \( i \to i + 2 \) one can rewrite it as

\[
r(i + N + 2) = \frac{\sum_{s=1}^{N} r(i + s + 1)}{\prod_{s=0}^{N} r(i + s + 1)}
\]
Substituting the latter in (26) we have 

\[ I_N(i + 1) = \sum_{k=1}^{N-1} r(i + k + 2) \left( \sum_{s=1}^{k} r(i + s) \right) + \frac{\sum_{s=1}^{N} r(i + s) \cdot \sum_{s=1}^{N} r(i + s + 1)}{\prod_{s=0}^{N-1} r(i + s + 1)} \]  

(27)

Make of use again the constraint (8) in the form

\[ r(i) = \frac{\sum_{s=1}^{N} r(i + s)}{\prod_{s=0}^{N-1} r(i + s + 1)}. \]

Substituting that in (26) we obtain

\[ I_N(i + 1) = \sum_{k=1}^{N-1} r(i + k + 2) \left( \sum_{s=1}^{k} r(i + s) \right) + r(i) \cdot \sum_{s=1}^{N} r(i + s - 1) = I_N(i). \]

The similar reasonings are used to prove that \( I_{2M}(i) \) is integral for (11) while \( I_{2M-1}(i) \) is that for (10).

**B. Proof of the Theorem 2.1.**

Consider the constraint (8). It can be written as

\[ h = \sum_{s=1}^{N} r(i + s - 1) - \prod_{s=0}^{N+1} r(i + s - 1) = 0 \]

By virtue of the Volterra lattice equations (11) we have the following

\[ D_t \left( \sum_{s=1}^{N} r(i + s - 1) \right) = r(i + N - 1)r(i + N) - r(i - 1)r(i). \]  

(28)

On the other hand, taking into account (5), we obtain

\[ D_t \left( \prod_{s=0}^{N+1} r(i + s - 1) \right) = \sum_{s=1}^{N} r(i + s - 1) \cdot (r(i + N) + r(i + N + 1) - r(i - 1) - r(i - 2)). \]  

(29)

To show that this \( h \) solves determining equation, one needs to rewrite the relation (28) = (29) in terms of \( I_N(i) \) to use then Lemma 2.1.

Observe that the relation (28) = (29) can be rewritten as

\[ r(i - 2) \cdot \sum_{s=1}^{N} r(i + s - 1) + r(i - 1) \cdot \sum_{s=2}^{N} r(i + s - 1) \]

\[ = r(i + N) \cdot \sum_{s=1}^{N-1} r(i + s - 1) + r(i + N - 1) \cdot \sum_{s=2}^{N} r(i + s - 1) \]  

(30)

Adding \( I_{N-2}(i) \) to l.h.s. of (30) and \( I_{N-2}(i) \) to r.h.s. of that we obtain

\[ I_N(i - 2) \]  

(11) = \[ I_N(i) \]  

(12).

By virtue of the Lemma 2.1 the latter is identity. Therefore the part of the Theorem concerning class of constraints (8) is proved. Similar arguments are applied for (9) and (10). We only remark that in these cases we obtain consistency conditions in the form

\[ I_{2M}(i - 1) = I_{2M}(i) \]  

for (9) and

\[ \sum_{s=1}^{2M} r(i + s) \cdot (I_{2M-1}(i) - I_{2M-1}(i - 1)) + \sum_{s=1}^{2M} r(i + s - 1) \cdot (I_{2M-1}(i + 1) - I_{2M-1}(i)) = 0. \]  

for (10), respectively.
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