The Abrikosov Flux Lattice in Planar Crystals of YBa$_2$Cu$_3$O$_{7-\delta}$

M. Harrison and W. Barford

Department of Physics, The University of Sheffield,
Sheffield, S3 7RH, United Kingdom.

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Abstract

Anisotropic London theory is used to predict the Abrikosov flux lattice for arbitrary field orientations in crystals of YBa$_2$Cu$_3$O$_{7-\delta}$ by minimising the Gibbs free energy for samples of a planar geometry. At low fields the “vortex-chain” state exists, i.e. the inter-chain vortex separation scales as the inverse flux density, $1/B$, whereas the intra-chain vortex separation is constant. At higher fields there is a cross-over to the uniaxially distorted hexagonal lattice where the inter-vortex separation scales as $1/B^{1/2}$. At low fields the vortex lattice is inclined towards the $\hat{c}$ axis, rotating towards the applied field at higher fields. The results of this calculation are in close agreement to the Bitter pattern experiments of Gammel et al. assuming an anisotropy ratio $\gamma$ of 5 and an in-plane penetration depth $\lambda_{a}$ of 1.413 Å, indicating that the “vortex-chain” state has been observed in these experiments.

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* To whom correspondence should be addressed.
I. INTRODUCTION

An interesting aspect of high temperature superconductors is the nature of the mixed state in which magnetic flux lines (or vortices) penetrate the superconducting sample when the applied field exceeds $H_{c1}$. In isotropic materials the vortices are arranged in a regular hexagonal lattice. In the anisotropic high temperature superconductors, however, the vortex lattice is distorted for an applied field not parallel to the crystalline axis. The detailed structure and arrangement of vortices in these materials has been the subject of much experimental and theoretical attention in the last few years.

For high magnetic fields the distortion of the equilibrium flux line lattice depends on the anisotropy ratio and orientation of the lattice with respect to the crystalline axis. At low magnetic fields, however, it has been predicted that the flux line lattice becomes distorted into a series of equally spaced vortex chains, running parallel to the plane containing the applied field and the $\hat{c}$ axis: the $\theta$ plane. This “vortex-chain” structure arises from the vortex-vortex interaction becoming attractive for magnetic fields not parallel to one of the principle crystallographic axes. The minima of the interaction are situated at a distance $x_{\text{min}}$ on either side of the vortex core in the $\theta$ plane. The lattice is no longer expected to scale uniformly with field; the intra-chain spacing, the distance between vortices within a chain, is determined by $x_{\text{min}}$. For a fixed magnetic flux density the intra-vortex spacing remains constant, and hence the inter-vortex spacing, the distance between chains, is inversely proportional to the flux density.

A number of experiments have investigated the flux lattice directly; both on the surface of the sample, and within the bulk. Small angle neutron scattering experiments have been performed to probe the magnetic field distribution within single crystals at high fields. Yethriaj et al. observed a distorted hexagonal lattice in YBa$_2$Cu$_3$O$_{7-\delta}$, in agreement with the predictions of London theory. Keimer et al., however, observed vortex-chains for an applied field of $B = 0.5T$ inclined at $80^\circ$ to the $\hat{c}$ axis. The presence of vortex-chains at these large fields is attributed to an exponential softening of the vortex lattice shear modulus, as
predicted by Ivlev and Kopnin\textsuperscript{8}.

High-resolution Bitter decoration experiments have also been performed to directly observe the flux line lattice emerging from the surface of single, twin free crystals of YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{7−δ} in low fields by Gammel and \textit{et al}\textsuperscript{1}. Their results show the variation of the intra- and inter- chain distance as a function of the normal component of the applied field, \(H_Z\). These results indicate that both the intra- and inter- chain distances scale as roughly \(1/\sqrt{H_Z}\), and hence do not appear to obey the vortex-chain state predictions. However, since the experiments are performed on flat, platelet crystals, demagnetisation effects play an important rôle in determining the density and orientation of the flux lattice with respect to the crystalline axis. It is the purpose of this paper to show that once the demagnetisation effects have been considered the experimental results fit the vortex-chain predictions very well. This implies that, although the Bitter decoration experiments are surface probes, they do provide useful information on the bulk properties of the lattice. It also demonstrates that three dimensional anisotropic London theory provides a good description of the flux lattice in YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{7−δ}.

The plan of this paper is as follows: in \textsection II we discuss the thermodynamics of planar superconductors in a magnetic field. This is the situation relevant to most of the recent decoration experiments on high temperature superconductors. \textsection III discusses the magnetic field distributions obtained from anisotropic London theory, which enables the pairwise interaction between vortices to be calculated. In \textsection IV we explain the details of the lattice summation, for both the low and high field limits, while \textsection V describes the results and a comparison with the experimental observations. We summarise in \textsection VI.

II. THE THERMODYNAMICS OF PLANAR SUPERCONDUCTORS

The high resolution Bitter pattern experiments of Gammel \textit{et al}\textsuperscript{1} were performed on flat platelets approximately 0.5mm x 0.5mm square, and between 5 to 40 \(\mu\)m in thickness. Thus, demagnetisation effects must be considered in the determination of the flux line lattice. In
figure 1 we illustrate the coordinate system used in this paper. The \((\hat{X}, \hat{Y}, \hat{Z})\) axes define the crystallographic axes with \(\hat{Z}\) being parallel to \(\hat{c}\) and \((\hat{X} - \hat{Y})\) lying in the \((\hat{a} - \hat{b})\) plane.

In general an applied field, \(H_a\), will be oriented at an angle \(\phi\) with respect to the \(\hat{c}\) axis. Since we are only concerned with uniaxial anisotropy, the plane enclosed by \(\hat{c}\) and \(H_a\) is a symmetry plane. Hence, the magnetic flux density, \(B\), defined as the average flux density per unit cell, also lies in this symmetry plane, at an angle \(\theta\) with respect to the \(\hat{c}\) axis. We therefore define a coordinate axes \((\hat{x}, \hat{y}, \hat{z})\) by a rotation of \(\theta\) about the \(\hat{Y}\) axis, such that \(\hat{z}\) is parallel to \(B\).

To simplify matters we will assume that the platelets are infinite in the \(\hat{a} - \hat{b}\) \((\hat{X} - \hat{Y})\) plane. The boundary conditions are then that the normal component of \(B\) and the parallel component of \(H\) to the surfaces are continuous, \(i.e.,\)

\[
B_z = B_a \cos \phi = B \cos \theta 
\]

and

\[
H_x = H_a \sin \phi, \quad (2)
\]

where \(B_a = H_a\). Equation (1) is generally satisfied in Bitter pattern experiments on flat platelets; see for example, figure (1) in reference [1]. The flux density is determined by,

\[
B = \frac{B_a \cos \phi}{\cos \theta}. \quad (3)
\]

In the presence of fixed external currents and temperature the equilibrium flux line lattice is obtained by minimising the Gibbs free energy,

\[
G(H) = U - \frac{B \cdot H}{4\pi}, \quad (4)
\]

where \(U\) contains the interaction between the flux lines and their self-energies, and we have defined \(G\) in CGS units.

The Gibbs energy is stationary with respect to the normal component of \(B\), and hence the correct thermodynamic potential to be minimised is,
\[ G = U - \frac{B_X H_X}{4\pi} \]  

(5)

where \( H_X \) is given by equation (2) and \( B_X \) by

\[ B_X = B \sin \theta. \]  

(6)

As shown by Kogan\(^9\), the general expression for the pairwise interaction per unit length, \( U_{12} \), between two parallel vortices is

\[ U_{12} = \frac{\phi_0}{4\pi} h_z(\mathbf{r}_{12}) \]  

(7)

where \( h_z(\mathbf{r}_{12}) \) is the longitudinal component of the magnetic field due to vortex 2 at the position of vortex 1 (and \textit{vice versa}). In a periodic lattice all points are equivalent. Hence, the lattice sum is obtained by summing the pairwise interactions between a vortex at the origin and another at the lattice position \( \mathbf{R}_i \), multiplied by the vortex number density, \( n \). Then the interaction energy per unit volume is,

\[ U = \frac{n\phi_0}{8\pi} \sum_{\mathbf{R}_i} h_z(\mathbf{R}_i). \]  

(8)

This sum includes the self-energy of the vortices, taken as the limit \( \mathbf{R}_i \to 0 \).

Using the Poisson sum formula equation (8) can also be expressed as the reciprocal space sum,

\[ U = \frac{n^2\phi_0}{8\pi} \sum_{\mathbf{G}_j} \tilde{h}_z(\mathbf{G}_j), \]  

(9)

where \( \tilde{h}_z(\mathbf{G}) \) is the Fourier transform of \( h_z(\mathbf{R}) \) and \( \mathbf{G}_j \) are the set of reciprocal lattice vectors.

The prescription for obtaining the equilibrium flux line lattice is now as follows. First \( U \) is minimised with respect to the vortex positions for a given density and orientation of the flux lattice with respect to the \( \hat{c} \) axis. Finally, \( U \) is substituted into the Gibbs free energy which is minimised as a function of the orientation of the flux lattice. Before \( U \) can be calculated, however, the longitudinal component of the field distribution must be known. This is derived from the London theory in the next section.
III. ANISOTROPIC LONDON THEORY

The high temperature superconductors may be viewed as a stack of superconducting layers coupled via Josephson tunnelling. This is conveniently described by the Lawrence-Doniach model\textsuperscript{10}. The Lawrence-Doniach model introduces four length scales: the in-plane coherence length ($\xi_{ab}$) and penetration depth ($\lambda_{ab}$), the distance between planes ($s$) and the Josephson length scale, $\lambda_J = \xi_{ab} \sqrt{2/\rho}$, where $\rho$ is the dimensionless Josephson coupling constant. $\lambda_J$ has the physical significance that for distances from the vortex core of less than $\lambda_J$ the phase differences between planes are large. Conversely, for distances from the core of greater than $\lambda_J$ the phase differences between planes are small. In the latter case the Josephson currents are small and in the extreme type II limit the Lawrence-Doniach model becomes equivalent to the three dimensional London theory with uniaxial anisotropy.

In YBa$_2$Cu$_3$O$_{7-\delta}$ $\lambda_J \approx 60\text{Å}$, which for most typical field strengths is much smaller than the inter-vortex spacing. Hence, the vortex-vortex interactions are accurately described by London theory. Within the London approximation the basic equation describing the magnetic field distribution for an isolated vortex is

$$\mathbf{h} + (\nabla \times \mathbf{A}) \cdot \nabla \times \mathbf{h} = \phi_o \delta (\mathbf{r}) \hat{z},$$

where $\mathbf{A} = m c^2 / 4 \pi n_s e^2$, and $m$ is the effective mass tensor. For uniaxial anisotropy $\Lambda$ has two degenerate eigenvalues: $\Lambda_a$ associated with screening currents flowing in the plane, and $\Lambda_c$ associated with screening currents flowing along the $\hat{c}$-axis. The anisotropy ratio,

$$\gamma = \sqrt{\frac{\Lambda_c}{\Lambda_a}} = \frac{\lambda_c}{\lambda_a}$$

is defined by $\lambda_J/s$. Fourier transforming and inverting equation (10) we obtain\textsuperscript{11}

$$\tilde{\mathbf{h}}(\mathbf{k}) = \frac{1}{(1 + \Lambda_a k^2)} \left( \hat{z} - \frac{(\Lambda_c - \Lambda_a) Q \cdot Q}{1 + \Lambda_a k^2 + (\Lambda_c - \Lambda_a) Q^2} \right) \phi_o,$$

where $Q = k \times \hat{c}$.

The field component parallel to the vortex is
\[ \tilde{h}_z(k) = \frac{\phi_o (1 + \lambda_{zz}^2 k^2)}{BC} \] (13)

where \( B = 1 + \lambda_{zz}^2 k_x^2 + \lambda_{zy}^2 k_y^2 \), \( C = 1 + \lambda_{zz}^2 k^2 \), \( k^2 = k_x^2 + k_y^2 \) and \( \lambda_{zz}^2 = \lambda_a^2 \sin^2 \theta + \lambda_c^2 \cos^2 \theta \).

Equation (13) is conveniently expressed as

\[ \tilde{h}_z(k) = \tilde{h}_1(k) - \tilde{h}_2(k) = \frac{\phi_o}{\lambda_a^2} \left( \frac{\lambda_a^2}{B} - \frac{\lambda_{zz}^2 - \lambda_a^2}{BC} \right). \] (14)

The expression for \( h_z(r) \) is obtained from the inverse Fourier transform of \( \tilde{h}_z(k) \) to produce

\[ h_z(r) = h_1(r) - h_2(r), \] (15)

where

\[ h_1(r) = \frac{\phi_o}{2\pi} \frac{\lambda_{zz}}{\lambda_a^2 \lambda_c} K_0(\rho_o), \]

\[ \rho_o^2 = \frac{x^2}{\lambda_{zz}^2} + \frac{y^2}{\lambda_c^2} \] (16)

and

\[ h_2(r) = \frac{\phi_o}{2\pi} \left( \frac{\lambda_{zz}^2 - \lambda_a^2}{2\lambda_a^2} \right) \int_0^1 du \frac{\rho}{a(u)b(u)} K_1(\rho), \] (17)

with

\[ \rho^2 = \frac{x^2}{a^2(u)} + \frac{y^2}{b^2(u)}, \] (18)

\[ a^2(u) = \lambda_{zz}^2 - (\lambda_{zz}^2 - \lambda_a^2) u \] (19)

and

\[ b^2(u) = \lambda_c^2 - (\lambda_c^2 - \lambda_a^2) u. \] (20)

\( K_0 \) and \( K_1 \) are the zeroth and first order Bessel functions, respectively.

In isotropic superconductors the magnetic field due to a vortex is parallel to the vortex core and positive. Hence, the Lorentz forces between a pair of vortices is centrally directed and repulsive. In anisotropic superconductors, however, there are both longitudinal and
transverse components to the vortex field. Furthermore, the longitudinal component becomes negative with minima situated at a distance $x_{\text{min}}$ either side of the core in the $\hat{x}$ direction (i.e. in the plane enclosing the $\hat{c}$ axis and $\mathbf{B}$). The Lorentz force between a pair of vortices therefore becomes attractive for certain relative orientations. It is this attractive interaction which has led to the suggestion of the vortex-chain regime for flux densities where the average vortex spacing becomes comparable to $x_{\text{min}}$. In figure 2 we plot $x_{\text{min}}$ as a function of the orientation of the vortex with respect to the $\hat{c}$ axis for anisotropy ratios of 5.

IV. THE LATTICE SUMMATION

The interaction energy of a lattice of vortices is given in equations (8) and (9). In this section we discuss the most efficient way of performing these summations.

A. Low Field Limit

In the low field limit, defined by $L \gg \lambda$, where $L$ is the average inter-vortex spacing, the real space sum converges rapidly. The area of the unit cell, $s$, and hence $n$, is defined by,

$$s = \frac{1}{n} = \frac{\phi_0}{B} = \frac{L^2 \sqrt{3}}{2}.$$  \hspace{1cm} (21)

The effect of the uniaxial anisotropy is to both lift the orientational degeneracy of the hexagonal lattice and to cause a uniaxial distortion of the equilateral triangles. A deformation of $\beta_x$ in the $x$ direction and by $\beta_y$ in the $y$ direction will produce a family of isosceles triangles. For a fixed flux density $s$ is constant, so that $\beta_x \beta_y = 1$, and hence we denote $\beta_x = 1/\beta_y = \beta$. There are two orientations of the lattice which are compatible with the uniaxial symmetry. However, as shown in ref. [3] the orientation of the lattice which minimises the energy for high flux densities is that illustrated in figure 3. This is also the orientation which minimises the energy in the low density limit.

A general lattice vector corresponding to the unit cell shown in figure 3 is
\[ \mathbf{R}_{mn} = m \mathbf{a}_1 + n \mathbf{a}_2, \]

where \( \mathbf{a}_1 = L \beta \hat{\mathbf{x}}, \)
\[ \mathbf{a}_2 = L \left( \frac{\beta}{2} \hat{\mathbf{x}} + \frac{\sqrt{3}}{2\beta} \hat{\mathbf{y}} \right), \quad (22) \]

and \( m \) and \( n \) are integers. The summations of \( h_1(\mathbf{r}) \) and \( h_2(\mathbf{r}) \) are performed over ellipses defined by the contours of \( h_1(\mathbf{r}) \) and \( h_2(\mathbf{r}) \), respectively.

In the London approximation the vortex self-energy is divergent due to the inability of the theory to describe the core adequately. The theory assumes a constant local magnetic field across the core, its value being given at \( r = \xi \). However, in anisotropic superconductors the vortex core is not circular, but elliptical for fields oriented away from \( \hat{\mathbf{c}} \). Hence, an elliptical cut-off is required to remove the divergence at small \( r \), namely \( \xi_x = \xi_{zz} \) and \( \xi_y = \xi_a \), where \( \xi_{zz}^2 = \xi_a^2 \cos^2 \theta + \xi_c^2 \sin^2 \theta \). (\( \xi_a \) and \( \xi_c \) are the in-plane and out-off-plane coherence lengths).

The self-energy is therefore given by
\[ U_{\text{self-energy}} = \frac{\phi_o}{2\pi} \frac{\lambda_{zz}}{\lambda_a \lambda_c} K_o \left( \frac{1}{\kappa} \right) \quad (23) \]

where \( \kappa = \lambda_{ab}/\xi_c = \lambda_c/\xi_a \).

**B. High Field Limit**

For high magnetic fields, \( L \ll \lambda \), the real space sums converge slowly. The summation of equation (8) can be more easily evaluated in reciprocal space to obtain the lattice energy
\[ U = \frac{B n}{8\pi} \sum_{pq} \bar{h}_z(\mathbf{G}_{pq}), \quad (24) \]

where \( \bar{h}_z(\mathbf{G}_{pq}) = \bar{h}_1(\mathbf{G}_{pq}) - \bar{h}_2(\mathbf{G}_{pq}) \) and
\[ \bar{h}_1(\mathbf{G}_{pq}) = \frac{\phi_o \lambda_{zz}^2}{\lambda_{zz}^2 \lambda_a^2 + \lambda_c^2 \lambda_{zz} G_x^2}, \quad (25) \]

\[ \bar{h}_2(\mathbf{G}_{pq}) = \phi_o \left( \frac{\lambda_{zz}^2 - \lambda_a^2}{\lambda_a^2} \right) \int_0^1 \frac{du}{(1 + a(u)^2 G_x^2 + b(u)^2 G_y^2)^2}. \quad (26) \]
The coefficients \( a(u) \) and \( b(u) \) are given in equations (19) and (20). \( \mathbf{G}_{pq} \) describes the set of reciprocal lattice vectors,

\[
\mathbf{G}_{pq} = pb_1 + qb_2,
\]

where

\[
b_1 = G_o \left( \frac{\sqrt{3}}{2\beta} \hat{x} - \frac{\beta}{2} \hat{y} \right),
\]

and

\[
b_2 = G_o \beta \hat{y},
\]

with \( G_o = \frac{2\pi L}{s} \), and \( p \) and \( q \) are integers.

The sum of \( \tilde{h}_2(\mathbf{G}_{pq}) \) converges rapidly. However, the sum of \( \tilde{h}_1(\mathbf{G}_{pq}) \) is logarithmically diverging because of the \( 1/G^2 \) behaviour. Physically this corresponds to the self-energy of the vortex core. There are various ways to deal with this divergence. One method is to use a Gaussian cut-off in the reciprocal space sum. Another is to use the Ewald summation method, originally used by Fetter for type II superconductors. The advantage of the latter method is that it explicitly removes the formally diverging self energy term, so we adopt it in this paper.

The details of the extension of Fetter’s method to anisotropic superconductors are shown in the appendix. Here we simply quote the result for the total energy per unit volume, excluding the self-energy, as

\[
U = \frac{Bn\phi_o \lambda_z^2}{8\pi \lambda_a^2} \left( 1 - \alpha^2 + \sum_{\mathbf{G}\neq0} \frac{\exp\left( -\alpha^2 \mathbf{G}^2 \right)}{\mathbf{G}^2 \left( 1 + \mathbf{G}^2 \right)} \right) + \frac{1}{4\pi n\lambda_z \lambda_c} \sum_{\mathbf{R}_i \neq 0} E_1 \left( \frac{\mathbf{R}_i^2}{4\alpha^2} \right)
\]

\[
- \frac{Bn\phi_o}{8\pi} \left( \frac{\lambda_z^2 - \lambda_a^2}{\lambda_a^2} \right) \sum_{\mathbf{G}} \int_0^1 du \left( \frac{1}{1 + a^2 G_x^2 + b^2 G_y^2} \right)^2
\]

where \( \alpha^2 = 1/4\pi n, \mathbf{G}^2 = \lambda_z^2 G_x^2 + \lambda_c^2 G_y^2, \mathbf{R}_i^2 = (\mathbf{R}_i/\lambda_z)^2 + (\mathbf{R}_i/\lambda_c)^2 \) and \( E_1 \) is the exponential integral.

The vortex self-energy (obtained in the limit \( \mathbf{R}_i \to 0 \)) is not included in this summation, but can be evaluated as before by employing an elliptical cut-off for the vortex core, namely \( x \to \xi_x = \xi_z \) and \( y \to \xi_y = \xi_c \). Hence, with the substitution \( \mathbf{R}_i = 1/\kappa \), the self-energy term becomes

\[
U_{self-energy} = \frac{B\phi_o \lambda_z}{32\pi^2 \lambda_a^2 \lambda_c} E_1 \left( \frac{\pi n}{\kappa^2} \right)
\]
V. RESULTS AND DISCUSSION

We now turn to a discussion of our predictions for the lattice parameters, and their comparison with experiment. Figure 4a shows our predictions and the experimental values from [1] of the intra-chain (D) and inter-chain (C) distances \textit{when projected onto the a-b plane}, versus the normal component of the applied field. The applied field is at an angle of $40^\circ$ to the $\hat{c}$ axis. We have taken the anisotropy parameter $\gamma$ as 5, the in-plane penetration depth $\lambda_a$ as 1 413Å, and the in-plane coherence length $\xi_a$ as 16Å. For this choice of parameters there is very good agreement with experiment. The results for the same parameters are shown in figure 4b for an applied field at $70^\circ$ to the $\hat{c}$ axis, where the fit is less good. A possible explanation for this might be due to a misalignment of the crystal. The inset of figure 4b shows the flux line lattice orientation with respect to the $\hat{c}$-axis as a function of the applied magnetic field. For low magnetic fields the flux lattice is oriented nearly parallel to the $\hat{c}$ axis, slowly aligning with the applied field direction with increasing magnetic intensity.

To show that the vortex-chain structure has indeed been observed in reference [1] we must take into account the effects of the rotation of the flux lattice as a function of applied field. This has two consequences: first, the position of the minimum in the vortex-vortex interaction, $x_{min}$, will change, as it is angular dependent. This angular dependence is shown in figure 2. Second, the magnetic flux density in the plane normal to the flux lattice is not linearly related to the applied field, but given by equation (3). We therefore scale the results in the following way. The intra-chain distance \textit{in the plane normal to the flux lattice}, $d = D \cos \theta$, is divided by $x_{min}$ and plotted against the inverse flux density, $1/B$. In the vortex-chain regime this quantity should be a constant. To preserve areas the inter-chain distance is multiplied by $x_{min}$, and should be inversely proportional to the flux density. Figures 5a and 5b shows the log-log plot of $d/x_{min}$ and $c \times x_{min}$ against $1/B$ for the applied field orientations of $40^\circ$ and $70^\circ$, respectively. We have also plotted the experimental points scaled in the same way. Evidently, for low fields, both the theoretical and experimental results are in close agreement with the vortex-chain predictions, with $d/x_{min}$ roughly unity.
and almost independent of B.

At higher fields, scaling of the unit cell parameters vary with the magnetic flux density as $\sim 1/B^{1/2}$, as expected. The anisotropic distortion of the hexagonal lattice agrees with the perturbation expansion of [3] with,

$$\beta = \left( \frac{\sin^2 \theta + \gamma^2 \cos^2 \theta}{\gamma^2} \right)^{1/4}. \quad (30)$$

At high fields the flux lattice is parallel to the applied field, so $\theta = \phi$.

As a final comparison to experiment we calculate the value of the intra-chain distance on the surface of the sample as a function of the orientation of the applied field for a fixed normal component of 12 Oe. This is shown in figure 6 with the experimental values of [1] for comparison. Again, there is reasonable agreement for this choice of parameters.

VI. CONCLUSIONS

In summary, we have calculated the equilibrium flux line lattice in planar crystals of YBa$_2$Cu$_3$O$_{7-\delta}$ using three dimensional anisotropic London theory. By taking into account demagnetisation effects, which cause the flux line lattice to orient away from the applied field towards the crystalline axis, we are able to show that the low field Bitter pattern experiments of Gammel et al. demonstrate the existence of the “vortex-chain” state. In this state the intra-chain distance is independent of flux density, whereas the inter-chain distance scales as the inverse flux density, $1/B$. We also demonstrate that as the field strength is increased there is a smooth cross-over to the distorted hexagonal lattice in which the inter-vortex spacings scale uniformly as $1/B^{1/2}$.

The agreements between theory and experiment are found for an anisotropy ratio $\gamma$ of 5 and the in-plane penetration depth $\lambda_a$ of 1 413Å. This value of the penetration depth is the zero temperature penetration depth, and not the value of the penetration depth at the irreversibility line quoted for twinned samples. Since the samples used in ref [1] are relatively twin-free, consisting of untwinned regions of at least 100µm square, we deduce that locally the lattice is not frozen but has assumed its zero temperature equilibrium configurations.
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In this appendix we use the Ewald summation method to derive the interaction energy of a lattice of vortices in the high density limit.

The reciprocal space sum

\[ S = \sum_{\text{all } \mathbf{G}} \frac{1}{1 + \mathbf{G}^2} \quad (A1) \]

diverges logarithmically for large \( \mathbf{G} \), where \( \mathbf{G}^2 = \lambda^2 \mathbf{G}_x^2 + \lambda^2 \mathbf{G}_y^2 \).

(A1) may be written as

\[ S = 1 + \sum_{\mathbf{G} \neq 0} \frac{1}{\mathbf{G}^2} - \sum_{\mathbf{G} \neq 0} \frac{1}{\mathbf{G}^2 \left( 1 + \mathbf{G}^2 \right)}, \quad (A2) \]

where the third term converges, leaving the second term to consider. Using the identity

\[ \frac{1}{\mathbf{G}^2} = 2 \int_0^\infty \xi \exp \left( -\xi^2 \mathbf{G}^2 \right) \, d\xi, \quad (A3) \]

and splitting the integral into two parts, we obtain

\[ \sum_{\mathbf{G} \neq 0} \frac{1}{\mathbf{G}^2} = \sum_{\mathbf{G} \neq 0} \left[ 2 \int_0^\alpha \xi \exp \left( -\xi^2 \mathbf{G}^2 \right) \, d\xi + \int_\alpha^\infty \xi \exp \left( -\xi^2 \mathbf{G}^2 \right) \, d\xi \right] \]

\[ = \sum_{\mathbf{G} \neq 0} \left[ 2 \int_0^\alpha \xi \exp \left( -\xi^2 \mathbf{G}^2 \right) \, d\xi \right] + \exp \left( -\alpha^2 \mathbf{G}^2 / \mathbf{G}^2 \right) \quad (A4) \]

where the choice of \( \alpha = (4\pi n)^{-1/2} \) maximises the rate of convergence and \( n \) is the real space density of lattice points. The second term of (A4) now has a Gaussian cut-off and converges easily. However, the logarithmic divergence has to be extracted from the first term. Writing the first term of (A4) as a sum over all \( \mathbf{G} \),

\[ T = \sum_{\text{all } \mathbf{G}} \sum_{\mathbf{G} \neq 0} 2 \int_0^\alpha \xi \exp \left( -\xi^2 \mathbf{G}^2 \right) \, d\xi \]

\[ - 2 \int_0^\alpha \xi \, d\xi \]

and using the Poisson sum formula, \( n \sum_{\mathbf{G}} \mathbf{F}(\mathbf{G}) = \sum_{\mathbf{R}} \mathbf{F}(\mathbf{R}) \), we obtain

\[ T = \frac{1}{4\pi n \lambda z \lambda c} \sum_{\text{all } \mathbf{R}_c} \sum_{\alpha} 2 \int_0^\alpha \exp \left( -\frac{\mathbf{R}_c^2}{4\xi^2} \right) \frac{d\xi}{\xi} - \alpha^2, \quad (A5) \]
where \( \tilde{R}^2 = (R_x/\lambda_{zz})^2 + (R_y/\lambda_c)^2 \).

Finally, collecting terms,

\[
S = 1 + \sum_{G \neq 0} \left[ \frac{\exp\left(-\alpha^2 \tilde{G}^2\right)}{G^2} - \frac{1}{G^2 (1 + G^2)} \right]
+ \frac{1}{4\pi n \lambda_{zz} \lambda_c} \sum_{R_i} E_1 \left( \frac{\tilde{R}_i^2}{4\alpha^2} \right) - \alpha^2,
\]

(A6)

where the exponential integral is

\[
E_1 (x) = \int_x^{\infty} \frac{\exp(-t)}{t} dt.
\]

The logarithmically diverging contribution comes from the term \( \tilde{R}_i \to 0 \) in (A6).
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Figure Captions

Figure 1. The slab geometry used in the decoration experiments, with the \( \hat{c} \)-axis perpendicular to the surface of the slab. The \( (\hat{X}, \hat{Y}, \hat{Z}) \) axes define the crystallographic axes with \( \hat{Z} \) being parallel to \( \hat{c} \) and \( (\hat{X} - \hat{Y}) \) lying in the \( (\hat{a} - \hat{b}) \) plane. The external field, \( H_a \), is applied at an angle \( \phi \) to \( \hat{c} \), producing a vortex lattice of magnetic flux density, \( B \), inclined at an angle \( \theta \) to \( \hat{c} \). This vortex lattice is described by a general coordinate axes \( (\hat{x}, \hat{y}, \hat{z}) \) by a rotation of \( \theta \) about the \( \hat{Y} \)-axis such that \( \hat{c} \), \( \hat{x} \) and \( \hat{z} \) are coplanar and \( B \) is parallel to \( \hat{z} \).

Figure 2. The position of minimum of the inter-vortex potential, \( x_{min} \), in units of \( \lambda = \sqrt{\lambda_a \lambda_c} \) versus the orientation of the vortex for \( \gamma = 5 \).

Figure 3. (a) The isotropic flux line lattice is described by equilateral triangles of side \( L \). (b) A deformation of \( \beta \) in the \( x \) direction and by \( 1/\beta \) in the \( y \) direction will produce a family of isosceles triangles that describe the anisotropic flux line lattice.

Figure 4(a). Intra-chain distance (D): theory, solid line; experiment, open diamonds, and inter-chain distance (C): theory, dashed line; experiment, filled diamonds, projected onto the \( a-b \) plane. Applied field orientation of 40°, in-plane penetration depth \( \lambda_a \) of 1 413Å, in-plane coherence length \( \xi_a \) of 16Å and anisotropy ratio \( \gamma \) of 5. The inset shows the vortex chain.

Figure 4(b). The same as fig 4(a) with an applied field orientation of 70°. The inset shows the orientation of the flux line lattice as a function of the applied field: solid line - \( \phi = 40^\circ \) and dotted line - \( \phi = 70^\circ \).

Figure 5(a). Log-log plot of intra-chain distance divided by \( x_{min} \): theory, solid line; experiment, open diamonds and inter-chain distance multiplied by \( x_{min} \): theory, dashed line; experiment, filled diamonds, in the plane normal to the flux lattice. Applied field at 40°.

Figure 5(b). The same as fig 5(a) with the applied field at 70°.

Figure 6. The intra-chain distance on the surface of the sample as a function of on the applied field orientation for a fixed normal component of 12 Oe, open squares - experiment.