Article

Fekete-Szegö Type Problems and Their Applications for a Subclass of \( q \)-Starlike Functions with Respect to Symmetrical Points

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Abstract: In this article, by using the concept of the quantum (or \( q \)-) calculus and a general conic domain \( \Omega_{k,q} \), we study a new subclass of normalized analytic functions with respect to symmetrical points in an open unit disk. We solve the Fekete-Szegö type problems for this newly-defined subclass of analytic functions. We also discuss some applications of the main results by using a \( q \)-Bernardi integral operator.

Keywords: analytic functions; quantum (or \( q \)-) calculus; conic domain; \( q \)-derivative operator; Hankel determinant; Toeplitz matrices; Fekete-Szegö problem; \( q \)-Bernardi integral operator

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1. Introduction and Definitions

Let \( \mathcal{A} \) denote the class of all functions \( f \) which are analytic in the open unit disk

\[
\mathbb{E} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \]

and has the normalized Taylor-Maclaurin series expansion of the following form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}
\]
Let $S$ be the subclass of all functions in $A$ that are univalent in $E$ (see [1]): If $f$ and $g \in A$, the function $f$ is said to be subordinate to the function $g$, written as $f \prec g$, if there exists an analytic function $w$ in $E$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in E),$$

such that $f(z) = g(w(z))$. Furthermore, the following equivalence will hold true (see [2]), if $g$ is univalent in $E$.

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(E) \subset g(E).$$

Let $P$ denote the well-known Carathéodory class of functions $p$, which are analytic in the open unit disk $E$ with

$$\Re(p(z)) > 0 \quad \text{and} \quad p(0) = 1.$$

If $p \in P$, then it has the form given by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (2)$$

where $|c_n| \leq 2 \ (n \in \mathbb{N}).$

If $f$ is univalent in $E$ and $f(E)$ is a star-shaped domain with respect to the origin, then $f$ is called starlike in $E$ with respect to the origin. The analytical condition of a starlike function in $E$ is given as follows:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in E).$$

The class of all such functions is denoted by $S^*$. A function $f \in A$ is said to be starlike with respect to symmetrical points (see [3]) if it satisfies the inequality:

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \quad (z \in E).$$

The class of all functions in $S$ which are starlike with respect to symmetrical points is denoted by $S^*_s$. Furthermore, we denote two interesting subclasses of $S$ by $k$-UCV and $k$-ST ($0 \leq k < \infty$) of functions which are, respectively, $k$-uniformly convex and $k$-starlike in $E$ defined for $0 \leq k < \infty$ by

$$k-UCV = \left\{ f : f \in S \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left|\frac{zf''(z)}{f'(z)}\right| (z \in E) \right\}$$

and

$$k-ST = \left\{ f : f \in S \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > k \left|\frac{zf'(z)}{f(z)} - 1\right| (z \in E) \right\}.$$

Kanas et al. (see [4,5]; see also [6]) defined and studied classes of $k$-starlike functions and $k$-uniformly convex functions subject to the conic domain $\Omega_k (k \geq 0)$, where

$$\Omega_k = \left\{ u + iv : u^2 > k^2 (u-1)^2 + v^2 \right\}.$$
For this conic domain, the following functions play the role of extremal functions:

\[
p_k(z) = \begin{cases} 
A_1(z) & (k = 0) \\
A_1(z) & (k = 1) \\
A_3(z) & (0 < k < 1) \\
A_4(z) & (k > 1), 
\end{cases}
\]  

where

\[
A_1(z) = \frac{1 + z}{1 - z},
\]
\[
A_2(z) = 1 + \frac{2}{\pi^2} \log \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2,
\]
\[
A_3(z) = 1 + \frac{2}{1 - k^2} \sinh^2 \left\{ \left( \frac{\pi}{2\pi} \arccos k \right) \right\},
\]
\[
A_4(z) = 1 + \frac{1}{k^2 - 1} \sin \left( \frac{\pi}{2K(i)} \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2 - (ix)^2}} dx \right) + \frac{1}{1 - k^2},
\]

\(i \in (0, 1),\) and

\[k = \cosh \left( \frac{\pi K(i)}{K(i)} \right),\]

\(K(i)\) is the first kind of Legendre’s complete elliptic integral (see, for details [4,5]). Indeed, from (4), we have

\[
p_k(z) = 1 + P_1z + P_2z^2 + P_3z^3 + \ldots \quad \text{(5)}
\]
We first give some basic definitions of the quantum (or \(q\)-) calculus that will help us in the upcoming sections. We also provide some notations and concepts used in this investigation.

**Definition 1.** Let \(q \in (0, 1)\) and the \(q\)-factorial \([n]_q!\) be defined as follows:

\[
[n]_q! = \begin{cases} 
1 & (n = 0) \\
\prod_{k=1}^{n-1} [k]_q & (n \in \mathbb{N}).
\end{cases}
\]  
\(6\)

**Definition 2.** The generalized \(q\)-Pochhammer symbol \([t]_{n,q} (t \in \mathbb{C})\) is defined as follows:

\[
[t]_{n,q} = \frac{(q^n q^n)}{(1 - q^n)} = \begin{cases} 
1 & (n = 0) \\
[t]_q[t+1]_q[t+2]_q \ldots [t+n-1]_q & (n \in \mathbb{N}).
\end{cases}
\]

**Definition 3.** The \(q\)-Gamma function is defined as follows:

\[
\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1 \quad (t > 0).
\]

**Definition 4.** (see [26]) For \(f \in A\), the \(q\)-derivative operator or \(q\)-difference operator are defined as follows:

\[
D_q f(z) = \frac{f(z) - f(qz)}{(1 - q)z} \quad (z \in \mathbb{E}).
\]  
\(7\)

From (1) and (7), we have

\[
D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.
\]  
\(8\)

Moreover, for \(n \in \mathbb{N}\) and \(z \in \mathbb{E}\), we get

\[
D_q z^n = [n]_q z^{n-1}, \quad D_q \left( \sum_{n=1}^{\infty} a_n z^n \right) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.
\]

When \(q \to 1\), the \(q\)-difference operator \(D_q\) approaches the ordinary differential operator:

\[
\lim_{q \to 1^-} (D_q f)(z) = f'(z).
\]

**Definition 5.** (see [8]) We say that a function \(f \in A\) belongs to the class \(S^*_q\) if

\[
f(0) = 1 = f'(0)
\]  
\(9\)

and

\[
\left| \frac{z (D_q f)(z)}{f(z)} - \frac{1}{1 - q} \right| \leq \frac{1}{1 - q}.
\]  
\(10\)

By applying the principle of subordination, the conditions (9) and (10) can be written as follows (see [27]):

\[
\frac{z (D_q f)(z)}{f(z)} \leq 1 + \frac{z}{1 - qz}.
\]

Now, making use of the quantum (or \(q\)-) calculus and the principle of subordination, we define \(q\)-starlike and \(q\)-convex functions with respect to symmetrical points as follows.
Definition 6. An analytic function $f$ is said to be in the class $S^*_k(q)$ if
\[
\left| \frac{2z \, (D_q f)\,(z)}{f(z) - f(-z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.
\] (11)

By applying the principle of subordination, the condition (11) can be written as follows:
\[
\frac{2z \, (D_q f)\,(z)}{f(z) - f(-z)} < 1 + z \frac{1}{1-q}.
\]

Definition 7. (see [9]) Let $k \in [0, \infty)$ and $q \in (0,1)$. A function $p$ is said to be in the class $k\mathcal{P}_q$ if and only if
\[
p(z) < p_{k,q}(z),
\] (12)

where
\[
p_{k,q}(z) = 2p_k(z) \{(1+q) + (1-q) \, p_k(z)\}^{-1}
\] (13)
and $p_k(z)$ is given by (5).

Geometrically, a function $p \in k\mathcal{P}_q$ takes on all values from the domain $\Omega_{k,q}$, which is defined as follows:
\[
\Omega_{k,q} = \left\{ w : \Re \left( \frac{(1+q) \, w}{(q-1) \, w + 2} \right) > k \left| \frac{(1+q) \, w - 1}{(q-1) \, w + 2} \right| \right\}.
\]

Remark 1. If $q \to 1^-$, then $\Omega_{k,q} = \Omega_k$ is given by (3).

Remark 2. For $q \to 1^-$, then $k\mathcal{P}_q = \mathcal{P}(p_k)$, where $\mathcal{P}(p_k)$ is defined in [4].

In the present investigation, by using the quantum (or $q$-) calculus and the general conic domain $\Omega_{k,q}$, we focus on the Hankel determinant, the Toeplitz matrices and the Fekete-Szegő problems for the function class $S^*_k(q)$.

Definition 8. An analytic function $f$ is said to be in the class $k\mathcal{S}^*_k(q)$ if
\[
\frac{2z \, (D_q f)\,(z)}{f(z) - f(-z)} \in k\mathcal{P}_q
\]
or, equivalently,
\[
\Re \left( \frac{(1+q) \, \frac{2z(D_q f)(z)}{f(z) - f(-z)}}{(q-1) \, \frac{2z(D_q f)(z)}{f(z) - f(-z)} + 2} \right) > k \left| \frac{(1+q) \, \frac{2z(D_q f)(z)}{f(z) - f(-z)} - 1}{(q-1) \, \frac{2z(D_q f)(z)}{f(z) - f(-z)} + 2} \right|.
\] (14)

Special Case:

For $k = 0$ and $q \to 1^-$, then the class $k\mathcal{S}^*_k(q)$ reduces to $\mathcal{S}^*_k$ (see [3]).

Let $n \in \mathbb{N}_0$ and $j \in \mathbb{N}$. The $j$th Hankel determinant was introduced and studied in [28]:
\[
H_j(n) = \begin{vmatrix}
    a_n & a_{n+1} & \cdots & a_{n+j-1} \\
    a_{n+1} & a_{n+2} & \cdots & a_{n+j-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+j-1} & a_{n+j-2} & \cdots & a_{n+2j-2} \\
\end{vmatrix}
\]
where \( a_1 = 1 \). Several authors have studied \( H_j(n) \). In particular, sharp upper bounds on \( H_j(2) \) were obtained in [29–31] for several classes. The Hankel determinant \( H_2(1) \) represents a Fekete-Szegő functional \( |a_3 - a_2^2| \). This functional has been further generalized as \( |a_3 - \mu a_2^2| \) for some real or complex \( \mu \) and also the functional \( |a_2 a_4 - a_3^2| \) is equivalent to \( H_2(2) \) [30]. Babalola [32] studied the Hankel determinant \( H_3(1) \).

The symmetric Toeplitz determinant \( T_j(n) \) is defined as follows:

\[
T_j(n) = \begin{vmatrix}
 a_n & a_{n+1} & \ldots & a_{n+j-1} \\
 a_{n+1} & \ddots & \ddots & \vdots \\
 \vdots & \ddots & \ddots & \ddots \\
 a_{n+j-1} & \ldots & \ldots & a_n \\
\end{vmatrix}
\]

so that

\[
T_2(2) = \begin{vmatrix}
 a_2 & a_3 \\
 a_3 & a_2 \\
\end{vmatrix}, \quad T_2(3) = \begin{vmatrix}
 a_3 & a_4 \\
 a_4 & a_3 \\
\end{vmatrix}, \quad T_3(2) = \begin{vmatrix}
 a_2 & a_3 & a_4 \\
 a_3 & a_2 & a_3 \\
 a_4 & a_3 & a_2 \\
\end{vmatrix}
\]

and so on. The problem of finding the best possible bounds for \( ||a_{n+1} - a_n|| \) has a long history (see [33]). It is known from [33] that

\[
||a_{n+1} - a_n|| < c,
\]

for a constant \( c \).

**Lemma 1.** (see [31]) If \( p \) is analytic in \( E \) and of the form (2), then

\[
2c_2 = c_1^2 + x(4 - c_1^2)
\]

and

\[
4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - (4 - c_1^2)c_1 x^2 + 2(4 - c_1^2)(1 - \left| x^2 \right|)z
\]

and, for some \( x, z \in \mathbb{C} \), with \( |x| \leq 1 \), and \( |z| \leq 1 \).

**Lemma 2.** (see also [34]) If \( p \) is analytic in \( E \) and of the form (2), and if \( \mu \in \mathbb{C} \) (\( 1 \leq k \leq n - 1 \)), then

\[
|c_n - \mu c_{n-k}| \leq 2 \max(1, |2\mu - 1|)
\]

**Lemma 3.** (see [35]; see also [33]) If the function \( p \) given by (2) is analytic in \( E \), then

\[
|c_n| \leq 2 \quad (n \in \mathbb{N}).
\]

The above inequality is sharp for the function \( f \) given by

\[
f(z) = \frac{1+z}{1-z}.
\]

**Lemma 4.** (see [35]) If \( p \) is analytic in \( E \) and of the form (2), then

\[
|c_2 - vc_1^2| \leq \begin{cases} 
-4v + 2 & (v < 0) \\
2 & (0 \leq v \leq 1) \\
4v - 2 & (v > 1)
\end{cases}
\]

The equality holds true for the function \( p \) given by

\[
p(z) = \frac{1+z}{1-z}
\]
or by one of its rotations, when \( v < 0 \) or \( v > 1 \). In addition, the equality holds true for the function \( p \) given by

\[
p(z) = \frac{1 + z^2}{1 - z^2}
\]

or by one of its rotations, when \( 0 < v < 1 \). If \( v = 0 \), the equality holds true if and only if

\[
p(z) = \left( \frac{1}{2} + \frac{1}{2} \lambda \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \frac{1}{2} \lambda \right) \frac{1 - z}{1 + z}, \quad (0 \leq \lambda \leq 1)
\]

or one of its rotations. If \( v = 1 \), the equality holds true if and only if \( p(z) \) is the reciprocal of one of the functions such that the equality holds true in the case when \( v = 0 \). In addition, the above upper bound is sharp and it can be improved as follows when:

\[
|c_2 - vc_1^2| + |c_1|^2 \leq 2 \quad (0 < v \leq \frac{1}{2})
\]

and

\[
|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2 \quad \left( \frac{1}{2} < v \leq 1 \right).
\]

2. Main Results

**Theorem 1.** Let the function \( f \) given by (1) belong to the class \( k-S^{*}_s(q) \). Then

\[
|a_2| \leq \frac{1}{2} P_1,
\]

\[
|a_3| \leq \frac{1}{2q} \left\{ P_1 + \left| P_2 - P_1 + \left( \frac{q - 1}{4q} P_2^3 \right) \right| \right\}
\]

and

\[
|a_4| \leq \frac{1}{2(1 + q^2)} \left\{ P_1 + \left| 2 (P_2 - P_1) + \left( \frac{2q^2 - 2q + 1}{2q} P_2^3 \right) + \left| P_3 + P_1 - 2P_2 \right| \right| \right\}.
\]

**Proof.** For \( f \in k-S^{*}_s(q) \), we have

\[
2z \left( D_q f \right)(z) = h(z) \prec H_k(z),
\]

where

\[
H_k(z) = 2p_k(z) [(1 + q) + (1 - q) p_k(z)]^{-1}
\]

and \( p_k(z) \) is given by (5).

The function \( p(z) \) with \( p(0) = 1 \) is given as follows:

\[
p(z) = \frac{1 + \frac{1}{1 - H_k^{-1}(h(z))}}{1 - \frac{1}{1 - H_k^{-1}(h(z))}} = 1 + c_1 z + c_2 z^2 + \ldots.
\]

After some computation involving (16), we have

\[
h(z) = H_k \left( \frac{p(z) + 1}{p(z) - 1} \right).
\]
Therefore, we find that

\[
H_k \left( \frac{p(z) + 1}{p(z) - 1} \right) = 1 + \left( \frac{q + 1}{2} \right) \left[ \frac{P_1 c_1}{2} z + \left( \frac{P_1 c_2}{2} + \frac{1}{4} \left( P_2 - P_1 + \frac{(q - 1) P_1^2}{2} \right) \right) c_1 \right] z^2 \\
+ \left( \frac{P_1 c_3}{2} + \left( \frac{P_2}{2} - \frac{P_1}{2} + \frac{(q - 1) P_1^2}{4} \right) c_1 c_2 \right) \left( \frac{P_1}{8} - \frac{P_2}{4} \right) z^3 + \left( \frac{P_3}{8} - \frac{1}{8} (q - 1) P_1 P_2 - \frac{1}{32} (q - 1)^2 P_1^3 \right) c_1^3 \right] .
\]

We also have

\[
2z \left( D_q f \right) (z) = 1 + 2q a_2 z + q \left[ 2 \right]_q a_3 z^2 + \left\{ [4]_q a_4 - [2]_q a_2 a_3 \right\} z^3 + \cdots.
\]

Comparing the corresponding coefficients in (17) and (18) along with Lemma 3, we obtain the required result. \( \square \)

**Theorem 2.** Let the analytic function \( f \in A \) be in the class \( k-S^*_k(q) \). Then

\[
T_3(2) \leq \left[ \frac{P_1}{8} + \frac{1}{2 (1 + q^2)} \left( \Omega_1 + \Omega_2 \right) \right] \left( \frac{P_1^2}{4} + 16 | \Omega_3 | + \frac{P_1^2}{2q^2} + 2 P_1^2 \Omega_5 \right) 2 - \frac{\Omega_4}{\Omega_5 P_1^2} \right] ,
\]

where

\[
\Omega_1 = P_1 + \left\{ 2 P_2 - 2 P_1 + \Omega_8 P_1^2 \right\} ,
\]

and

\[
\Omega_2 = \left| P_3 + P_1 - 2 P_2 - \Omega_6 P_1^2 + \Omega_7 P_1 P_2 + (q - 1) \Omega_8 P_1^3 \right| .
\]

Furthermore, we have

\[
\Omega_3 = 2 P_1 \Omega_5 \left( \frac{P_1}{8} + \frac{P_1}{8} - \frac{P_1}{4} - \frac{1}{8} \Omega_6 P_1^2 + \frac{1}{8} \Omega_7 P_1 P_2 + \frac{1}{8} (q - 1) \Omega_8 P_1^3 \right) \\
- \frac{1}{2q^2} \left( \frac{P_2}{4} - \frac{P_1}{4} + (q - 1) \frac{P_1^2}{8} \right)^2 ,
\]

\[
\Omega_4 = \frac{P_1}{2q^2} \left( \frac{P_2}{4} - \frac{P_1}{4} + (q - 1) \frac{P_1^2}{8} \right) - 2 P_1 \Omega_5 \left( \frac{P_2}{2} - \frac{P_1}{2} - \frac{1}{4} \Omega_6 P_1^2 \right) ,
\]

\[
\Omega_5 = \frac{1}{16 (1 + q^2)} ,
\]

\[
\Omega_6 = \left( \frac{q^2 - 2q + 1}{2q} \right) ,
\]

\[
\Omega_7 = \left( \frac{1 - 2q^2 + 2q}{2q} \right) .
\]
and
\[ \Omega_8 = \left( \frac{q^2 - q + 1}{4q} \right). \]

Here \( P_1 \) and \( P_2 \) are given in (5).

**Proof.** By comparison of coefficients in (17) and (18), we can obtain

\[ a_2 = \frac{1}{4} P_1 c_1 \]  
\[ a_3 = \frac{1}{2q} \left\{ \frac{1}{2} P_1 c_2 + \left( \frac{P_2}{4} - \frac{P_1}{4} + \left( \frac{q - 1}{8} \right) \right) c_1 \right\} \]  
\[ a_4 = \frac{1}{2(1 + q^2)} \left[ \frac{1}{2} P_1 c_3 + \left( \frac{P_2 - P_1}{2} + \left( \frac{2q^2 - 2q + 1}{8q} \right) \right) c_1 \right] c_2 \]
\[ + \left\{ \frac{1}{8} \left( P_3 + P_1 - 2P_2 - \frac{1}{2q} \left( \frac{2q^2 - 2q + 1}{P_1^2} \right) \right) \right\} \frac{1}{16q} P_1 P_2 \]
\[ + \left\{ \frac{1}{8} \frac{2q^2 - 2q}{(q - 1) \left( \frac{q^2 - q + 1}{32q} \right) \right\} c_1 c_2 \]  
\[ + \left\{ \frac{1}{8} \left( \frac{2q^2 - 2q + 1}{P_1^2} \right) \right\} \frac{1}{16q} P_1 P_2 \]
\[ + \left\{ \frac{1}{8} \frac{2q^2 - 2q}{(q - 1) \left( \frac{q^2 - q + 1}{32q} \right) \right\} c_1 c_2 \]  
\[ + \left\{ \frac{1}{8} \frac{2q^2 - 2q + 1}{P_1^2} \right\} \frac{1}{16q} P_1 P_2 \]
(21)

A detailed calculation for \( T_3(2) \) yields
\[ T_3(2) = (a_2 - a_4) \left( a_2^2 - 2a_3^2 + a_2a_4 \right). \]

Now, if \( f \in k-S_+^+ (q) \), then we have
\[ |a_2 - a_4| \leq |a_2| + |a_4|, \]
\[ |a_2 - a_4| \leq \frac{1}{2} P_1 + \frac{1}{16 (1 + q^2)} (\Omega_1 + \Omega_2). \]  
(22)

We need to maximize \( |a_2^2 - 2a_3^2 + a_2a_4| \) for \( f \in k-S_+^+ (q) \). Thus, by writing \( a_2, a_3, a_4 \) in terms of \( c_1, c_2, c_3 \), with the help of (19) and (21), we get
\[ |a_2^2 - 2a_3^2 + a_2a_4| \leq \frac{p_2^2 c_2^2}{4} + \Omega_3 c_4^2 - \Omega_4 c_7 c_2 - \frac{p_2^2 c_2^2}{8q} + \Omega_5 P_1^2 c_1 c_3. \]  
(23)

Finally, applying the triangle inequality, Lemma 2 and Lemma 3 along with (22) and (23), we obtained the required result. \( \square \)

**Theorem 3.** If an analytic function \( f \in A \) is in the class \( k-S_+^+ (q) \), then
\[ |a_2 a_4 - a_3^2| \leq \frac{p_1^2}{4q^2}. \]

**Proof.** Making use of (19), (20) and (21), we have
\[ a_2 a_4 - a_3^2 = \lambda_1 c_1 c_3 + \lambda_2 c_1^2 c_2 - \lambda_3 c_2^2 + \lambda_4 c_3, \]  
(24)

where
\[ \lambda_1 = \frac{p_1^2}{16 (1 + q^2)}. \]
\[
\lambda_2 = \left( \frac{1}{16q^2 (1 + q^2)} \right) p_1^2 - \left( \frac{1}{16q^2 (1 + q^2)} \right) p_1 p_2 + \left( \frac{2 - q}{64q^2 (1 + q^2)} \right) p_3^2.
\]

\[
\lambda_3 = \frac{p_1^2}{16q^2}
\]

and
\[
\lambda_4 = \left( \frac{1}{16 (1 + q^2)} \right) p_1 p_3 - \left( \frac{1}{64q^2 (1 + q^2)} \right) p_1^2 + \left( \frac{1}{32q^2 (1 + q^2)} \right) p_1 p_2 + \left( \frac{q - 2}{128q^2 (1 + q^2)} \right) p_3^2
\]
\[
+ \left( \frac{4q^2 - 4q^3 - q + 2}{128q^2 (1 + q^2)} \right) p_1 p_2 + \left( \frac{q - 1}{256q^2 (1 + q^2)} \right) p_1 - \left( \frac{1}{64q^2} \right) p_2^2.
\]

By using Lemma 1, we take
\[
Y = 4 - c_1^2 \quad \text{and} \quad Z = \left( 1 - |x|^2 \right) z.
\]

Without loss of generality, we assume that \( c = c_1 \) \((0 \leq c \leq 2)\), so that
\[
a_2 a_4 - a_3^2 = \frac{1}{4} \left( \lambda_1 + 2 \lambda_2 - \lambda_3 + 4 \lambda_4 \right) c^4 + \frac{1}{2} \left( \lambda_1 + \lambda_2 - \lambda_3 \right) Y c^2 x
\]
\[
- \frac{1}{4} \lambda_1 Y c^2 x^2 - \frac{1}{4} \lambda_3 Y^2 x^2 + \frac{1}{2} \lambda_1 c Y Z.
\]

Taking the moduli on both sides of (25) and using the triangle inequality, we find that
\[
|a_2 a_4 - a_3^2| \leq \left| \frac{1}{4} \left( \lambda_1 + 2 \lambda_2 - \lambda_3 + 4 \lambda_4 \right) \right| c^4 + \left| \frac{1}{2} \left( \lambda_1 + \lambda_2 - \lambda_3 \right) \right| Y c^2 |x|
\]
\[
+ \left| \frac{1}{4} \lambda_1 \right| Y c^2 |x|^2 + \left| \frac{1}{4} \lambda_3 \right| Y^2 |x|^2 + \left| \frac{1}{2} \lambda_1 \right| \left( 1 - |x|^2 \right) c Y.
\]

This can be written as follows:
\[
|a_2 a_4 - a_3^2| \leq |A_\lambda| c^4 + |B_\lambda| |x| Y c^2 + \left| \frac{1}{4} \lambda_1 \right| |x|^2 Y c^2 + \left| \frac{1}{4} \lambda_3 \right| |x|^2 Y^2 + \left| \frac{1}{2} \lambda_1 \right| \left( 1 - |x|^2 \right) Y c
\]
\[
= G(|x|),
\]
where
\[
A_\lambda = \frac{1}{4} \left( \lambda_1 + 2\lambda_2 - \lambda_3 + 4\lambda_4 \right)
\]
\[
B_\lambda = \frac{1}{2} \left( \lambda_1 + \lambda_2 - \lambda_3 \right).
\]

Now, trivially, we have
\[
G'(c, |x|) > 0
\]
on the closed interval \([0, 1]\), which shows that \(G(c, |x|)\) is an increasing function in the interval \([0, 1]\). Therefore, the maximum value occurs at \(x = 1\) and we have
\[
\max \{ G(c, |1|) \} = G(c),
\]
\[
G(c, |1|) = |A_\lambda| c + |B_\lambda| Y c^2 + \left| \frac{\lambda_1}{4} \right| Y c^2 + \left| \frac{\lambda_3}{4} \right| Y^2
\]
and
\[
G(c) = |A_\lambda| c^4 + |B_\lambda| Y c^2 + \left| \frac{\lambda_1}{4} \right| Y c^2 + \left| \frac{\lambda_3}{4} \right| Y^2.
\]
Hence, by putting \(Y = 4 - c^2\) and after some simplification, we have
\[
G(c) = \left( |A_\lambda| - |B_\lambda| - \left| \frac{\lambda_1}{4} \right| + \left| \frac{\lambda_3}{4} \right| \right) c^4 + 4 \left( |B_\lambda| + \left| \frac{\lambda_1}{4} \right| - \left| \frac{\lambda_3}{2} \right| \right) c^2 + 4 |\lambda_3|.
\]
We consider \(G'(c) = 0\), for the optimum value of \(G(c)\), which implies that \(c = 0\). Thus, \(G(c)\) has a maximum value at \(c = 0\). Hence, the maximum value of \(G(c)\) is given by
\[
\max \{ G(c) \} = 4 |\lambda_3|,
\]
which occurs at \(c = 0\) or
\[
c^2 = \frac{4 \left( |B_\lambda| + \left| \frac{\lambda_1}{4} \right| - \left| \frac{\lambda_3}{2} \right| \right)}{|A_\lambda| - |B_\lambda| - \left| \frac{\lambda_1}{4} \right| + \left| \frac{\lambda_3}{4} \right|}.
\]
Hence, by putting
\[
\lambda_3 = \frac{p_1^2}{16q^2}
\]
in (27) and after some simplification, we obtain the desired result. \(\square\)

For \(q \to 1\), \(k = 0\), and \(p_1 = 2\) in Theorem 3, we have the following known result for the class \(S^*_s\).

**Corollary 1.** (see [36]) If an analytic function \(f \in A\) that belongs to the class \(S^*_s\), then
\[
\left| a_2 a_4 - a_3^2 \right| \leq 1.
\]

**2.1. The Fekete-Szegö Problem**

**Theorem 4.** Let the function \(f \in A\) given by (1) belong to the class \(k-S^*_s(q)\). Then...
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{4q} (2P_2 + (q(1 - \mu) - 1)P_1^2) & (\mu \leq \delta_1) \\ \frac{P_1}{4q} & (\delta_1 \leq \mu \leq \delta_2) \\ -\frac{1}{4q} (2P_2 + (q(1 - \mu) - 1)P_1^2) & (\mu \leq \delta_1), \end{cases} \]

where

\[ \delta_1 = \frac{2P_2 + P_1 [(q - 1)P_1 - 2]}{qP_1^2}, \]
\[ \delta_2 = \frac{2P_2 + P_1 [(q - 1)P_1 + 2]}{qP_1^2}. \]

**Proof.** From (19) and (20), we have

\[ a_3 - \mu a_2^2 = \frac{P_1}{4q} \left( c_2 - v c_1^2 \right), \]

where

\[ v = \frac{1}{2} \left( 1 - \frac{P_2}{P_1} - \frac{(q - 1)P_1 - 2}{2} + \frac{\mu P_1}{2} \right). \]  

(28)

By applying the triangle inequality and Lemma 4, we obtain Theorem 4. \( \square \)

If we set \( k = 0 \) and \( q \to 1^- \) in Theorem 4, we thus obtain the following known result.

**Corollary 2.** (see [37]) If an analytic function \( f \in S^*_s(\phi) \), then

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2} (P_2 - \frac{\mu}{2} P_1^2) & (\mu \leq \delta_1) \\ \frac{P}{2} & (\delta_1 \leq \mu \leq \delta_2) \\ -\frac{1}{2} (P_2 - \frac{\mu}{2} P_1^2) & (\mu \leq \delta_1), \end{cases} \]

where

\[ \delta_1 = \frac{2(P_2 - P_1)}{P_1^2}, \]
\[ \delta_2 = \frac{2(P_2 + P_1)}{P_1^2}. \]

Let \( \delta_1 \leq \mu \leq \delta_2 \). Then, in view of Lemma 4, Theorem 4 can be improved as follows.

**Theorem 5.** If the function \( f \) given by (1) belongs to the class \( S^*_s(q) \) and if

\[ \delta_1 \leq \mu \leq \delta_3 = \frac{2P_2}{qP_1^2}, \]

then

\[ |a_3 - \mu a_2^2| + \frac{1}{qP_1^2} \left( 2(P_2 - P_1) - (q - 1)P_1^2 + \mu q P_1^2 \right) |a_2|^2 \leq \frac{P_1}{2q}. \]
Furthermore, if $\delta_3 \leq \mu \leq \delta_2$, then
\[
|a_3 - \mu a_2^2| + \frac{1}{qP_1^2} (2(P_2 + P_1) - (q - 1)P_2^2 - \mu qP_1^2) |a_2|^2 \leq \frac{P_1}{2q}.
\]

If we set $k = 0$ and $q \to 1^-$, we obtain the following known result.

**Corollary 3.** (see [37]) If an analytic function $f \in S^*_s(\phi)$ and if

\[
\delta_1 \leq \mu \leq \delta_3 \leq \frac{2P_2}{P_1^2},
\]

then
\[
|a_3 - \mu a_2^2| + \frac{1}{qP_1^2} (2(P_2 + P_1) + \mu P_1^2) |a_2|^2 \leq \frac{P_1}{2}.
\]

Moreover, if

\[
\delta_3 \leq \mu \leq \delta_2,
\]

then
\[
|a_3 - \mu a_2^2| + \frac{1}{P_1^2} (2(P_2 + P_1) - \mu P_1^2) |a_2|^2 \leq \frac{P_1}{2}.
\]

2.2. Applications of the Main Results

In this section, firstly we recall that the Bernardi integral operator $F_\beta$ given in [38] as follows:

\[
F_\beta(f(z)) = \frac{1 + \beta}{z^\beta} \int_0^z t^{\beta-1} f(t)dt \quad (f \in A, \beta > -1).
\]

The $q$-integral of the function $f$ on $[0, z]$ is defined as follows (see, for example [39]):

\[
\int_0^z f(t) dq_t = (1 - q) z \sum_{k=0}^\infty q^k f(q^k z),
\]

and $q$-integral of the function $z^n$ is given by

\[
\int_0^z z^n dq_t = \frac{z^{n+1}}{[n + 1]_q}, \quad (29)
\]

where $n \neq -1$ and for $q \to 1^-$, Equation (29) becomes

\[
\int_0^z h_1(t) dt = \frac{z^{n+1}}{n + 1}.
\]

Noor [39] introduced the $q$-Bernardi integral operator $B_q(z)$ as follows:

\[
B_q(z) = F_\beta(f(z)) = \frac{[1 + \beta]_q}{z^\beta} \int_0^z t^{\beta-1} f(t) dq_t \quad (\beta > -1).
\]

Let $f \in A$. Then, by using Equations (29) and (8), we obtain the following power series for the function $B_q(z)$ in the open unit disk $E$ as follows:

\[
B_q(z) = z + \sum_{n=2}^\infty \frac{[1 + \beta]_q}{[n + \beta]_q} a_n z^n.
\]

Clearly, $B_q(z)$ is analytic in the open unit disk $E$. 

Let

$$B_n = \frac{[1 + \beta]_q}{[n + \beta]_q} \quad (n \geq 1).$$  \hspace{1cm} (32)

Applying Theorem 1 on Equation (31), we obtain the following result.

**Theorem 6.** If the function $B_q(z)$ given by (31) belongs to the class $k$-$S^*_q (q)$, where $k \in [0, 1]$, then

$$|a_2| \leq \frac{1}{2B_2} P_1,$$

$$|a_3| \leq \frac{(1 + q)}{2 ([3]_q B_3 - 1)} \left[ P_1 + \left| P_2 - P_1 + \left( \frac{(q - 1)P_1^2}{2} \right) \right| \right]$$

and

$$|a_4| \leq \frac{(1 + q)}{2 ([4]_q B_4)} \left[ P_1 + \left| P_2 - P_1 + \frac{2 ([3]_q B_3 - 1) (q - 1) + (1 + q)}{2 ([3]_q B_3 - 1)} P_1^2 \right| \right] + \left| P_3 + P_1 - 2P_2 - \frac{2 ([3]_q B_3 - 1) (q - 1) + (1 + q)}{2 ([3]_q B_3 - 1)} \left( P_1^2 + P_1 P_2 \right) \right| + \frac{(q - 1)}{4} \left( \frac{(q - 1) ([3]_q B_3 - 1) + (1 + q)}{([3]_q B_3 - 1)} P_1^3 \right),$$

where $B_2, B_3$ and $B_4$ are given in (32).

Applying Theorem 2 to Equation (31), we obtain the following result.

**Theorem 7.** If the function $B_q(z)$ given by (31) belongs to the class $k$-$S^*_q (q)$, then

$$T_3(2) \leq \left\{ \frac{P_1}{2B_2} + \frac{1 + q}{2 ([4]_q B_4)} (\Omega_{10} + \Omega_{11}) \right\} \left\{ \frac{P_1^2}{4B_2^2} + 16 |\Omega_{12}| + \frac{(1 + q)^2 P_1^2}{2 ([3]_q B_3 - 1)^2} \right\} + 2P_1^2 \Omega_{14} \left| 2 - \frac{\Omega_{13}}{P_1^2 \Omega_{14}} \right|,
where
\[
\begin{align*}
\Omega_{10} &= P_1 + \left| 2P_2 - 2P_1 + \Omega_{15}P_1^2 \right|, \\
\Omega_{11} &= \left| P_3 + P_1 - 2P_2 + \Omega_{15}P_1^2 + \Omega_{16}P_1P_2 + (q - 1) \Omega_{17}P_1^3 \right|, \\
\Omega_{12} &= 2P_1\Omega_{14} \left( \frac{P_3}{8} + \frac{P_1}{8} - \frac{P_2}{4} - \frac{1}{8} \Omega_{15}P_1^2 + \frac{1}{8} \Omega_{16}P_1P_2 + \frac{1}{8} (q - 1) \Omega_{17}P_1^3 \right) \\
&\quad - \frac{(1 + q)^2}{2 \left( [3]_q B_3 - 1 \right)^2} \left( \frac{P_2}{4} - \frac{P_1}{4} + (q - 1) \frac{P_1^2}{8} \right)^2, \\
\Omega_{13} &= \frac{(1 + q)^2 P_1}{2 \left( [3]_q B_3 - 1 \right)^2} \left( \frac{P_2}{4} - \frac{P_1}{4} + (q - 1) \frac{P_1^2}{8} \right) + 2P_1\Omega_{14} \left( \frac{P_2}{2} - \frac{P_1}{2} + \frac{1}{4} \Omega_{15}P_1^2 \right), \\
\Omega_{14} &= \frac{(1 + q)}{16B_2B_4 [4]_q}, \\
\Omega_{15} &= \frac{2 \left( [3]_q B_3 - 1 \right) (q - 1) + (1 + q)}{2 \left( [3]_q B_3 - 1 \right)}, \\
\Omega_{16} &= \left( \frac{(1 + q) - 2 \left( [3]_q B_3 - 1 \right) (q - 1)}{2 \left( [3]_q B_3 - 1 \right)} \right), \\
\Omega_{17} &= \left( \frac{\left( [3]_q B_3 - 1 \right) (q - 1) + (1 + q)}{4 \left( [3]_q B_3 - 1 \right)} \right).
\end{align*}
\]

where \( P_1 \) and \( P_2 \) are given in (5).

Applying Theorem 3 to Equation (31), we obtain the following result.

**Theorem 8.** If the function \( B_q(z) \) given by (31) belongs to the class \( k-S^*_s(q) \), then

\[
\left| a_2a_4 - a_3^2 \right| \leq \frac{(1 + q)^2 P_1^2}{4 \left( [3]_q B_3 - 1 \right)^2}.
\]

For \( q \to 1^- \), \( k = 0 \), \( \beta = 0 \) and \( p_1 = 2 \) in Theorem (8), we have the following known result for the class \( S^*_s \).

**Corollary 4.** (see [36]) Let \( f \in S^*_s \) be of the form (1). Then

\[
\left| a_2a_4 - a_3^2 \right| \leq 1.
\]
Theorem 9. If the function $B_q(z)$ given by (31) belongs to the class $k-S^*_q(q)$, then

$$\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} \frac{(1+q)B_2^2}{4[3]_q B_3 - 1} \left( 2P_2 + \left( (q - 1) - \mu \frac{[3]_q B_2^2 - 1}{(1+q)B_2^2} \right) P_1^2 \right) & (\mu \leq \delta_1) \\
\frac{(1+q)P_1}{4[3]_q B_3 - 1}, & (\delta_1 \leq \mu \leq \delta_2) \\
- \frac{(1+q)P_1}{4[3]_q B_3 - 1} \left( 2P_2 + \left( (q - 1) - \mu \frac{[3]_q B_2^2 - 1}{(1+q)B_2^2} \right) P_1^2 \right) & (\mu \leq \delta_1), \end{cases}$$

where

$$\delta_1 = \frac{(1+q)B_2^2}{[3]_q B_3 - 1} \left( 2P_2 + P_1 ((q - 1) P_1 - 2) \right) \left( \frac{\mu}{P_1^2} \right)$$

and

$$\delta_2 = \frac{(1+q)B_2^2}{[3]_q B_3 - 1} \left( 2P_2 + P_1 ((q - 1) P_1 + 2) \right) \left( \frac{\mu}{P_1^2} \right).$$

If we set $k = 0$, $\beta = 0$ and $q \to 1-$ in Theorem 9, we obtain the following known result.

Corollary 5. (see [37]) If an analytic function $f \in A$ belongs to the class $S^*_q(\phi)$, then

$$\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} \frac{1}{2} (P_2 - \mu P_1^2) & (\mu \leq \delta_1) \\
\frac{P_1}{2}, & (\delta_1 \leq \mu \leq \delta_2) \\
- \frac{1}{2} (P_2 - \mu P_1^2) & (\mu \geq \delta_2), \end{cases}$$

where

$$\delta_1 = \frac{2}{P_1^2} (P_2 - P_1)$$

and

$$\delta_2 = \frac{2}{P_1^2} (P_2 + P_1).$$

3. Conclusions

We have made use of the general conic domain $\Omega_{k,q}$ and the quantum (or $q$-) calculus to introduce and investigate several new subclasses of $q$-starlike functions with respect to symmetrical points in open unit disk $\mathbb{E}$. We have studied some interesting results such as the Hankel determinant, the Toeplitz matrices, and the Fekete-Szegö inequalities. We have also discussed some applications of our main results by using a $q$-Bernardi integral operator.

For further investigation, we can easily follow a known relationship between the $q$-analysis and $(p,q)$-analysis (see [25] (p. 340, Equations (9.1), (9.2) and (9.3))) and the results for the $q$-analogues, which we have included in this paper for $0 < q < 1$, can then be easily transformed into the related results for the $(p,q)$-analogues with $0 < q < p \leq 1$ by adding a rather redundant (or superfluous) parameter $p$ (see, for details [25] (p. 340)).
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