Characteristics of Cosmic String Scaling Configurations

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Abstract

Using the formalism developed in a previous paper we analyze the cosmological implications of our conclusions concerning the scaling behaviour of a network of cosmic strings, in particular the idea that once gravitational backreaction becomes important the transient scaling regime so far explored by numerical simulations will be replaced by a new ‘full scaling’ regime, with slightly different parameters. This has consequences for the normalization of the string tension on all scales. We show how the new scaling solutions affect the gravitational wave bound from nucleosynthesis and the milli-second pulsar, and also describe the consequences from large scale structure up to COBE scales.

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Cosmic strings formed at an early phase transition may have interesting cosmological effects. To test this idea, we need to know how a network of strings evolves. In a recent publication [1] we developed a detailed analytic formalism aimed at improving our understanding of the evolution of such a network in a $k = 0$ Friedmann-Roberston-Walker (FRW) universe. The approach is based on calculating the evolution of the probability, $p[r(l)]$, of a random segment of left-moving string of length $l$ having an extension $r$ at time $t$. One of our most important conclusions was that the regime so far studied in numerical simulations is a ‘transient scaling’ regime, which will be succeeded once gravitational back-reaction becomes important by ‘full scaling’, characterized by rather different parameters. In this letter we address some of the cosmological consequences arising from the new results.

In [1] we introduced three length scales $\xi(t)$, $\bar{\xi}(t)$ and $\zeta(t)$ related to the long string density, the persistence length along the long strings and the small-scale structure on the string network respectively. In terms of the mean square extension, $K(l) \equiv r^2$, $\bar{\xi}$ and $\zeta$ are defined by

$$K(l, t) \approx 2\bar{\xi}(t)l, \quad l \gg t,$$

$$\approx l^2 - \frac{l^3}{3\zeta(t)}, \quad l \ll t,$$

whereas $\xi(t) = (V/L)^{1/2}$ is the interstring distance for a total length, $L$, of string in a large volume $V$. We introduced the dimensionless distance $\gamma, \bar{\gamma}$ and $\epsilon$ defined by $\gamma = 1/H\xi$, $\bar{\gamma} = 1/H\bar{\xi}$, $\epsilon = 1/H\zeta$, where $H = \dot{R}/R$ is the Hubble parameter; $R \propto t^{1/p}$, with $p = 2$ in the radiation era and $p = \frac{3}{2}$ in the matter era. With this definition, in the radiation era, $\gamma$ and $\bar{\gamma}$ are identical to the variables used in [2] and [3], but in the matter era they are half as large. In terms of these variables the evolution equations become [1]

$$- pt\frac{\dot{\gamma}}{\gamma} = -p + \left(\frac{3}{2} - \bar{\alpha}_{nl}\right) - \frac{F}{4} + \frac{c}{2}\bar{\gamma} + \frac{G\mu}{2}\epsilon,$$

$$- pt\frac{\dot{\bar{\gamma}}}{\bar{\gamma}} = -p + \left(\bar{\alpha}_{nl} + \bar{G} - \frac{F}{2}\right) - \frac{\chi\gamma^2}{w\bar{\gamma}} + \frac{l^2}{2}\bar{\gamma} + G\mu\epsilon,$$

$$- pt\frac{\dot{\epsilon}}{\epsilon} = -p + \left(\bar{\alpha}_{nl} + \frac{3 - 12C}{2}F\right) - \frac{\chi\gamma^2}{\epsilon} + kc\bar{\gamma} + G\mu\dot{C}\epsilon.$$

The coefficient parameters and functions in (3)–(5) were fully defined in [1]; here we recall the definitions briefly. The parameter $\bar{\alpha} = 1 - 2\langle x^2 \rangle$ used in [2] and [3] was written as $\bar{\alpha} = \bar{\alpha}_{nl} + \frac{1}{2} F$, where $\bar{\alpha}_{nl}$ relates to the effect of loop formation and is probably small compared to $F$, which is defined in [1], Eq. (7.17), and is the contribution of stretching. From the simulations of Bennett & Bouchet [4], we have, in the radiation- and matter-dominated eras respectively,

$$\bar{\alpha}_{RD} = 0.14 \pm 0.04 \quad \bar{\alpha}_{MD} = 0.26 \pm 0.04,$$

so $F$ is likely to be almost twice these values. The function $\bar{G}$, defined in [1], Eq. (7.23), also represents a stretching effect; it is larger than $F$ but of the same order, around unity. The rate of loop formation is determined by $c$, which probably lies in the range 0.1–0.5. The
effect of gravitational radiation is represented by the parameter $\Gamma G \mu$, where $G$ is Newton’s constant, $\mu$ the string tension and $\Gamma$ a numerical constant of order $10^{-100}$. The probability of intercommuting is governed by the constant $\chi$, of order 0.1, and $w$, defined by the rate at which $K(l)$ approaches its asymptotic form for large $l$, lies between 0 and 1. The function $I$ describes the effect of loop formation. It is certainly less than $2c$. In [1] it was argued that $I/2c$ is likely to be only slightly less than unity, but this argument is not correct; in fact it could in principle lie anywhere between zero and unity. $C$ is a constant relating to the stretching of strings on very small scales, which is probably also small. The two most critical parameters are $k$, which describes the excess small-scale kinkiness on loops as compared to long strings, and $\hat{C}$ which determines the rate at which gravitational back-reaction smoothes the small-scale kinkiness.

Initially, soon after strings are formed, the gravitational radiation terms in (3)–(5) are negligible. Then, as we argued in [1], $\gamma$ and $\bar{\gamma}$ will reach scaling values, while (if $k$ is not too large — see below) $\epsilon$ starts to grow. This is the transient scaling regime. The corresponding scaling values $\gamma_{\text{transient}}$ and $\bar{\gamma}_{\text{transient}}$ are found by setting to zero the right hand sides of (3) and (4).

This is the only scaling regime accessible to the numerical simulations [4–7]. From (3) and (4) above, assuming that $\bar{\alpha}_n l$ is negligible, we have (the error bars are our estimates):

$$c\bar{\gamma}_{\text{transient, RD}} = 1.14 \pm 0.04,$$
$$c\bar{\gamma}_{\text{transient, MD}} = 0.26 \pm 0.04.$$ (7)

The second of these is really a purely notional figure, since the transient scaling regime will not last into the matter-dominated era. The simulations of [4] also give

$$\gamma_{\text{transient, RD}} = 7.2 \pm 1.4,$$
$$\gamma_{\text{transient, MD}} = 2.8 \pm 0.7.$$ (8)

With reasonable estimates for $c$, this shows that $\gamma_{\text{transient}}$ and $\bar{\gamma}_{\text{transient}}$ are of roughly similar magnitude.

In using Equations (3) and (4), which determine the transient scaling values, we have assumed that $\bar{\alpha}_n l$ and the gravitational radiation terms are negligible. There are then four remaining parameters: $\chi$, $w$, $c$ and $I$. The latter two are themselves in principle functions of $\gamma$ and $\bar{\gamma}$, given by [1], Eqs. (6.17) and (6.27), but there are considerable uncertainties in these formulae relating to the effects of the small-scale structure, and it seems better therefore at this stage to regard them as independent parameters; $c$ at least can be estimated directly from the simulations.

We have explored numerically the solutions of Eqs. (3) and (4) for the full ranges of the four parameters $\chi$, $w$, $c$ and $\delta = 1 - I/2c$. Both $w$ and $\delta$ are restricted to lying within the range 0 to 1. In general, we find that the value of $c$ is fairly tightly constrained; the other parameters less so. In the radiation era, for $\chi = 0.1$, the ranges within which we find solutions consistent with Eqs. (3)–(8) are

$$0.10 < c < 0.18,$$
$$0.62 < w < 1,$$ (RD)
$$0 < \delta < 0.21.$$ (9)
Generally speaking, the smaller values of $w$ or the larger values of $c$ require small values of $\delta$. In the matter era, conditions are less restrictive; in fact $\delta$ is unconstrained:

$$0.08 < c < 0.14,$$

$$0.26 < w < 1, \quad \text{(MD)}$$

$$0 < \delta < 1.$$  \hfill (10)

In both cases, lowering the value of $\chi$ tends to relax the constraints; with $\chi = 0.06$, much wider ranges of $w$ and $\delta$ are possible — for example, in the radiation era $\delta$ can be as large as 0.6.

The predicted values of $\gamma$ are generally towards the lower end of the range allowed by (8):

$$5.8 < \gamma_{\text{transient,RD}} < 6.5, \quad 6.5 < \tilde{\gamma}_{\text{transient,RD}} < 11,$$

$$2.1 < \gamma_{\text{transient,MD}} < 2.8, \quad 1.7 < \tilde{\gamma}_{\text{transient,MD}} < 3.0.$$  \hfill (11)

It is interesting that $\tilde{\gamma}$ tends to be bigger than $\gamma$ in the radiation era, but about the same size in the matter era. We also note that $\chi$ and $\gamma$ always appear in (3)–(5) in the combination $\chi \gamma^2$; hence lowering $\chi$ will always increase the scaling value of $\gamma$.

During the transient scaling regime, $\epsilon$ grows until the point where the gravitational radiation terms become important. The important question is whether then a full scaling regime is reached, in which all three of $\gamma$, $\tilde{\gamma}$, $\epsilon$ are constant. If there is such a solution, it is in the region of parameter space where $\epsilon \gg \gamma_{\text{full}} \sim \tilde{\gamma}_{\text{full}}$, which means that the intercommuting term in the third equation, $\chi \gamma^2 / \epsilon$, is negligible. If we neglect it, then, assuming we know the values of the various constants, it is straightforward to solve for the scaling parameters. As we showed in [4], the existence of a full scaling solution depends primarily on the magnitudes of $\dot{C}$ and $k$; we require $\dot{C} > \dot{C}_{\text{cr}} > k$, where

$$\dot{C}_{\text{cr}} = \frac{p - \bar{\alpha}_{nl} - \frac{3}{2}(1 - 4C)F}{2p - 3 + \bar{\alpha}_{nl} + \frac{1}{2}F}.$$  \hfill (12)

and is typically of order one. Unfortunately, we do not know the values of $\dot{C}$ and $k$, but we shall assume, as seems likely, that these inequalities are satisfied.

It is important to realize that the simulations performed so far, which neglect gravitational radiation, may have provided misleading information about the true values of the scaling parameters. From the evolution equations, (3)–(5), it is clear that when the gravitational radiation terms become significant, then $\tilde{\gamma}$ decreases. Without knowing the values of $\dot{C}$ and $k$ it is not possible to predict the extent of the decrease, but what we can do is to examine the relationships between the changing values of $\gamma$, $\tilde{\gamma}$ and $\epsilon$.

From Eq. (3), we might expect that when full scaling is reached the values of $c \tilde{\gamma}$ and of $\bar{\epsilon} = \Gamma G \mu \epsilon$ should be comparable; as $\bar{\epsilon}$ increases, $c \tilde{\gamma}$ must decrease. The ratio between the two is determined by the values of $\dot{C}$ and $k$; by [4], Eq. (9.22),

$$\frac{\bar{\epsilon}}{c \tilde{\gamma}} = \frac{\Gamma G \mu \epsilon}{c \tilde{\gamma}} = \frac{\dot{C}_{\text{cr}} - k}{\dot{C} - \dot{C}_{\text{cr}}}.$$  \hfill (13)
We have studied the effect of changing $\bar{\varepsilon}$ on the values of $\gamma$ and $\bar{\gamma}$ predicted by Eqs. (3) and (4). As we noted, as $\bar{\varepsilon}$ increases, $c\bar{\gamma}$ decreases. It is not so obvious from the equations what happens to $\gamma$. However, we find that in all cases it too decreases, though by a smaller factor.

In the radiation era, the dependence of $\gamma$ and $\bar{\gamma}$ on $\bar{\varepsilon}$ is nearly linear; less so in the matter era. For example, if we choose the parameter values $\chi = 0.1, c = 0.1, \delta = 0.1, w = 0.8$, we find the dependence shown in Figure 1. Other parameter values within the allowed range give very similar curves. When $\bar{\varepsilon} = c\bar{\gamma}$, $\bar{\gamma}$ has decreased by nearly a half in the radiation era, and in the matter era by about a third. At the same point, $\gamma$ has decreased in the radiation era by about 25%; in the matter era by 10%.

From (11), reasonable estimates seem to be $\gamma_{\text{full, RD}} = 3$ to $6$, $c\gamma_{\text{full, RD}} = 0.2$ to $1$, $\gamma_{\text{full, MD}} = 1.4$ to $2.6$, $c\gamma_{\text{full, MD}} = 0.05$ to $0.27$, (14)

For observational tests, we need to know how far out we should expect to have to look to find a cosmic string. What is most important here is the scaling value of $\gamma$ in the recent, matter-dominated epoch. With $H_0 = 2/3t = 100h$ km s$^{-1}$Mpc$^{-1}$ $(0.5 < h < 1)$, the mean interstring distance today would be $(\gamma_{\text{full, MD}}H_0)^{-1} = (1200$ to $2000)h^{-1}$ Mpc, with the nearest string to us being a distance say half that, i.e. in the range $z = 0.2$ to $0.5$.

We are also of course interested in the nearest loops, so we turn next to them. The loops that are born off the network are usually produced with quite large centre of mass velocities, and lose energy both through red-shifting and, at a constant rate, to gravitational radiation, $\dot{E}_{\text{gr}} = -\Gamma_{\text{loops}}G\mu^2$, until their rest-mass is reduced to zero. Here $\Gamma_{\text{loops}}$ is a constant of order 50–100. We assume that the fraction of the total energy which will eventually be radiated is $f$, i.e., $(1-f)E_b$ is red-shifted away, where $E_b$ is the average energy once it has reached a stable non-self-intercommuting form.

As a first approximation, we imagine that the loop immediately loses the fraction $(1-f)$ of its total energy, leaving a static loop of length $fE_b/\mu$, which loses its energy by gravitational radiation, a fairly good picture as long as the lifetime is long compared to $t_b$. Given this, we take the effective length of the loop at birth to be $l_b = fE_b/\mu \equiv (\kappa - 1)\Gamma_{\text{loops}}G\mu t_b$, where we have introduced the dimensionless parameter $\kappa$. Note that $\kappa$ is identical to the $1/\beta$ of Allen and Caldwell [8] and Allen and Shellard [9]. At a later time the loop’s length is $l = \Gamma_{\text{loops}}G\mu(\kappa t_b - t)$; it disappears at $t_d = \kappa t_b$. Numerical simulations indicate $\kappa$ to be somewhere between 2 and 10, the latter value certainly satisfying the assumption we have made on the role of the initial red shifting. (Even for $\kappa \approx 2$, the corrections are not large.)

In a volume $V$, the rate at which loops are born from long strings is given by,

$$\dot{N} = \frac{\nu V}{(\kappa - 1)\Gamma_{\text{loops}}G\mu t^4_b},$$

where

$$\nu = \frac{fc\bar{\gamma}\gamma^2}{p^3}.\quad (16)$$

Taking $f$ to be around 0.7 while $\gamma$ and $c\bar{\gamma}$ are given by the estimates above, we obtain

$$\nu_{\text{RD}} = 0.2 \text{ to } 3, \quad \nu_{\text{MD}} = 0.02 \text{ to } 0.4.\quad (17)$$
The parameter $\nu$ is not very well constrained. However, we predict a somewhat lower rate of loop formation than for example ref. [8].

Any possible direct observational test of the cosmic string scenario depends on knowing how many long strings or loops we may expect to see within a reasonable distance, and how large they would be. We therefore turn to estimates of these quantities.

The total number density of loops at time $t$ is obtained from $\dot{N}$, allowing for the expansion factor:

$$\dot{n} = \int_{t/\kappa}^{t} \frac{\dot{N}(t_b)}{V} dt_b \left( \frac{R_b}{R} \right)^3,$$

where $R(t)$ is the FRW scale factor. Except for the transition region between RD and MD, we obtain

$$\dot{n} = A(\kappa) \frac{\nu}{\Gamma_{\text{loops}} G \mu t^3},$$

with

$$A_{\text{RD}}(\kappa) = \frac{2 \kappa^{3/2} - 3 \kappa + 1}{3 \kappa - 1}, \quad A_{\text{MD}}(\kappa) = 1.$$ (20)

This then gives the mean distance between loops, the present distance being

$$\dot{n}^{-1/3} = \left( \frac{\Gamma_{\text{loops}} G \mu}{\nu} \right)^{1/3} t = \left( \frac{\Gamma_2 \mu_6}{\nu} \right)^{1/3} 90 h^{-1} \text{ Mpc}$$

where $\Gamma_2 = \Gamma_{\text{loops}}/100$ and $\mu_6 = 10^6 G \mu$. Taking $\nu_{\text{MD}} = 0.1$, $\Gamma_{\text{loops}} = 100$, $G \mu = 10^{-6}$, we obtain $\dot{n}^{-1/3} \simeq 200 h^{-1} \text{ Mpc}$. The nearest loop to us is probably at about half this distance. If this is the case there are probably a few hundred loops with $z < 0.3$.

The total length of all loops in a volume $V$ at time $t$ is given by

$$L_{\text{loops}} = \int_{t/\kappa}^{t} \frac{\dot{N}(t_b)}{V} dt_b \left( \frac{R_b}{R} \right)^3 \Gamma_{\text{loops}} G \mu (\kappa t_b - t),$$

from which we can write the energy density in loops,

$$\rho_{\text{loops}} = B(\kappa) \frac{\mu \nu}{t^2},$$

where

$$B_{\text{RD}}(\kappa) = \frac{2(2\kappa^{3/2} - 3\kappa + 1)}{3(\kappa - 1)};$$

$$B_{\text{MD}}(\kappa) = \frac{\kappa - 1}{\kappa - 1} \ln \kappa - 1.$$ (24)

The mean length of a loop is

$$\bar{l} = \frac{B(\kappa)}{A(\kappa)} \Gamma_{\text{loops}} G \mu t.$$ (25)

Thus at the present time, we have
\[ \bar{l} = B_{\text{MD}}(\kappa) \Gamma_2 \mu_6 200 h^{-1} \text{kpc}, \]  

with a corresponding mass

\[ M_{\text{loop}} = \mu \bar{l} = B_{\text{MD}}(\kappa) \Gamma_2 \mu_6^2 4 \times 10^{12} M_\odot. \]

The median loop size is given by the same expression but with \( B_{\text{MD}} \) replaced by \( \frac{\kappa - 1}{\kappa + 1} \). Thus we see that the size of a typical loop is larger than a typical galaxy while its mass is that of a very large galaxy. In other words, surviving loops are significant objects gravitationally.

The total gravitational wave energy generated by cosmic strings is the sum of that generated by loops and by long strings. In the RD era, we have

\[ \rho_{gr}(t) = \mu [A_{\text{RD}}(\kappa) \nu + \nu_{ls}] \frac{1}{t^2} \int_t^1 \frac{dt_r}{t_r} \left( \frac{t}{t_r} \right)^3 \left( \frac{R_t}{R_r} \right)^4, \]

where

\[ \nu_{ls} = \frac{\Gamma G \mu \epsilon \gamma^2}{p^3}. \]

Here the lower limit is the time of formation of cosmic strings, or rather the time at which they start to move freely and so lose a significant amount of energy to gravitational radiation. It is interesting to note that the ratio between the parameters \( \nu_{ls} \) and \( \nu \) which govern the contributions of long strings and loops is given by

\[ \frac{\nu_{ls}}{\nu} = \frac{\Gamma G \mu \epsilon}{f c^2 \gamma}, \]

i.e., essentially the ratio (13). Since \( \gamma \) decreases as \( \epsilon \) increases, if \( \nu_{ls} \) is near its upper limit, then \( \nu \) must be relatively small, and vice versa — if more energy is dissipated as gravitational radiation less must be available for loop formation.

The spectrum of gravitational waves emitted by cosmic strings is constrained by the nucleosynthesis limit on relativistic particle species, as well as the pulsar timing on a stochastic gravitational wave background. At the time of nucleosynthesis we obtain

\[ \Omega_{gr}(t_{\text{nuc}}) = \frac{32 \pi G \mu}{3} [A_{\text{RD}}(\kappa) \nu + \nu_{ls}] \left( \frac{g_{\ast, \text{nuc}}}{g_{\ast, r}} \right)^{1/3} \ln \frac{t_{\text{nuc}}}{t_f}. \]

Typically, we have \( T_{\text{nuc}}/T_f \simeq 10^{-16}, \ g_{\ast, \text{nuc}}/g_{\ast, r} = 10.75/106.75 \simeq 0.1 \). With these values, (31) becomes

\[ \Omega_{gr}(t_{\text{nuc}}) = 1.2 \times 10^{-3} \mu_6 [A_{\text{RD}}(\kappa) \nu + \nu_{ls}]. \]

The nucleosynthesis limit places a limit on the energy density in neutrinos plus any relativistic particle species beyond the photons and electrons present at \( t_{\text{nuc}} \). It is bounded by the equivalent energy density of \( N_\nu \) neutrino species. In terms of the energy density in gravitational radiation at nucleosynthesis we obtain

\[ \Omega_{gr} \leq 0.163 \times (N_\nu - 3) \Omega_{\text{rad}}, \]
where three light neutrino species have been assumed. According to [10], the current observational limit on light (≤ 10 MeV) neutrino species is $N_\nu \leq 3.1$, which yields the bound

$$\Omega_{gr}(t_{\text{nuc}}) \leq 0.02. \quad (34)$$

Combining the two expressions we obtain

$$\mu_6[A_{RD}(\kappa)\nu + \nu_h] < 15. \quad (35)$$

which is not at all restrictive for any values of $\nu$ and $\kappa$ in the expected range. A more stringent limit, $N_\nu \leq 3.04$, has been given recently [11]. Using this limit would change the bounds in (34) and (35) to 0.007 and 5, respectively. This bound is still not hard to satisfy, but may be somewhat restrictive if either of the terms within the square brackets in (35) is close to its upper limit.

This though is probably not the tightest gravitational radiation bound. The pulsar timing limit requires that the energy density in gravitational radiation, in a logarithmic interval at $f = (8.2 \text{ years})^{-1}$ satisfy the inequality [12]

$$\frac{d\Omega_{gr}}{d\ln f} \bigg|_{f=(8.2 \text{ years})^{-1}} \leq 1.0 \times 10^{-7} h^{-2}. \quad (36)$$

This gives

$$\mu_6 < 0.2 \sqrt{\frac{\kappa - 1}{\kappa}} \left(\frac{10}{\nu}\right) \left(\frac{0.1}{\nu}\right)^{3/2} \frac{\Gamma_2^{1/2}}{h^{7/2}}. \quad (37)$$

(The details of this derivation will be presented elsewhere.) Note the strong dependence on $\nu$, $\kappa$ and $h$. For $h = 0.5$ the condition is not particularly restrictive, though it puts bounds on $\nu$ and $\kappa$, but for $h = 1.0$ it is a very restrictive condition, which could only be satisfied by making $\kappa$ and $\nu$ very small, say $\kappa = 2$, $\nu \leq 0.05$.

The growth of primordial density perturbations generated by strings provides a separate constraint on $G\mu$. In the work of Albrecht and Stebbins [13] with strings and hot dark matter, the power spectrum of the overdensity fluctuations is approximated by an integral in conformal time over the product of the transfer function describing how the initial perturbation spectrum is evolved to the present, and a “form factor”, $F$, which describes the input from the string. It is in this term that the string scaling behaviour is hidden. In particular $F$ is given in terms of the wavenumber $k$ as

$$F(k\xi/R) = \frac{2}{\pi^2} \frac{\beta^2 \Sigma_\chi^2}{\xi^2} \left(\frac{1}{1 + 2(k\chi/R)^2}\right). \quad (38)$$

Briefly, $\xi$ and $\chi$ correspond roughly to $\xi$ and $\bar{\xi}$ in our notation, $\beta$ gives the r.m.s. velocity of the strings, and $\Sigma$ the increase in the surface density of the wakes due to the wiggliness. For simplicity, we will assume that the only parameters that are affected are $\chi$ and $\xi$. Albrecht and Stebbins incorporate the small scale-structure on the network into a renormalized string tension $\mu$, and let this value vary depending on the type of string they are attempting to mimic. We would like to know in what direction the power spectrum goes if the scaling solutions are changed, by increasing $\chi$ and $\xi$. If $\chi/\xi$ is doubled (which is roughly the upper
limit of what we might expect), then the form factor $F$ is multiplied by 4 and so therefore is the power spectrum. Hence in order to fit the large scale structure data, we would naively expect that the normalization of $\mu$ must go down by a factor of two. For the case where they attempt to match the simulations of ref. [4], they use $\beta \Sigma \sim 1.2$ and (for $h = 0.5$) $G\mu_{\text{ren}} = 4.0 \times 10^{-6}$. We would obtain $G\mu_{\text{ren}} \approx 2 \times 10^{-6}$, a value well within current allowed limits on $G\mu$.

The third observational test which constrains the string parameters arises from anisotropy observations on the microwave background sky on both large and small angular scales. There are few cosmic string calculations of this effect to date, unlike say the situation in inflation or global textures. In a recent paper, Perivolaropoulos [14] obtains an analytic expression for $\Delta T/T$ in terms of the interstring distance $\xi$. There are many assumptions in the calculation, including the use of straight strings and the neglect of loops, which make direct comparison somewhat dubious. However, his final result, in our notation, is that $\Delta T/T \propto G\mu \gamma$. Perivolaropoulos quotes a result of $G\mu = (2.0 \pm 0.5) \times 10^{-6}$. If as we believe, $\gamma$ is somewhat less than the prediction of ref. [4], this should be increased, but only very slightly.

We conclude that the modifications suggested by our analytic study slightly weaken the observational constraints on $G\mu$. For $h = 0.5$, they are fully compatible with current observational limits on this parameter, but for $h = 1.0$, it is very difficult to accommodate the gravitational-wave bound. However, there are still many uncertainties that need to be resolved before we can make really definitive predictions. The biggest problem is to find a way of estimating the parameters associated with the small-scale structure on the strings, in particular $\hat{C}$ and $k$ which determine the effect on the small scale length of the gravitational back-reaction and loop formation, respectively.

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FIGURES

FIG. 1.
Variation of $\gamma$ and $\bar{\gamma}$ with $\bar{\epsilon}$ in the full scaling regime (a) in the radiation-dominated and (b) in the matter-dominated era.