REMARKS ON AFFINE SPRINGER FIBRES

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Let $G$ be a simply connected almost simple algebraic group over $\mathbb{C}$ and let $\mathfrak{g}$ be the Lie algebra of $G$. Let $B$ be a Borel subgroup of $G$, let $T$ be a maximal torus of $B$ and let $t, b$ be the Lie algebras of $T, B$. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. For any nilpotent element $N \in \mathfrak{g}$ we set $\mathcal{B}_N = \{ b \in \mathcal{B}; N \in b \}$ (a Springer fibre). In [KL] an affine analogue of $\mathcal{B}_N$ ("affine Springer fibre") was introduced. Let $F = \mathbb{C}((\epsilon)), A = \mathbb{C}[[\epsilon]]$, where $\epsilon$ is an indeterminate and let $\mathfrak{g}(F) = F \otimes \mathfrak{g}$ (a Lie algebra over $F$), $L = A \otimes \mathfrak{g}$ (a Lie algebra over $A$). An element $\xi \in \mathfrak{g}(F)$ is said to be topologically nilpotent if $\lim_{n \to \infty} \text{ad}(\xi)^n = 0$ in $\text{End}_F(\mathfrak{g}(F))$. Let $\tilde{X}$ be the set of all Iwahori subalgebras of $\mathfrak{g}(F)$; this is an increasing union of projective varieties over $\mathbb{C}$. According to [KL], for any regular semisimple, topologically nilpotent element $\xi \in \mathfrak{g}(F)$, the set $\tilde{X}_\xi = \{ I \in \tilde{X}; \xi \in I \}$ is a nonempty, locally finite union of projective varieties all of the same dimension, say $b_\xi$. Let $[\tilde{X}_\xi]$ be the set of irreducible components of $\tilde{X}_\xi$, a finite or countable set.

In the remainder of this paper, $h$ denotes a fixed regular element in $t$. Then $\epsilon h \in \mathfrak{g}(F)$ is regular semisimple, topologically nilpotent so that the affine Springer fibre $\tilde{X}_{\epsilon h} = \{ I \in \tilde{X}; \epsilon h \in I \}$ is defined. From [KL, §5] we see that $b_{\epsilon h} = \nu$ where $\nu = \dim \mathcal{B}$. As in [KL, §3], there is a free abelian group $\Lambda$ (see Sec.2) of rank equal to the rank of $\mathfrak{g}$ which acts freely on $\tilde{X}_{\epsilon h}$ in such a way that the induced $\Lambda$-action on $[\tilde{X}_{\epsilon h}]$ is also free and has only finitely many orbits. In this paper we will describe a fundamental domain for the $\Lambda$-action on $\tilde{X}_{\epsilon h}$. Namely, let $\mathcal{S}'$ be the Steinberg variety of triples $(E, b_1, b_2)$ where $b_1 \in \mathcal{B}, b_2 \in \mathcal{B}$ and $E \in b_1 \cap b_2$ is nilpotent. Let $\mathcal{S}$ be the fibre at $b$ of the projection $\mathcal{S}' \to \mathcal{B}, (E, b_1, b_2) \mapsto b_2$. We can identify $\mathcal{S}$ with $\{(E, b_1); b_1 \in \mathcal{B}, E \in n \cap b_1\}$. We state the following result.

**Theorem.** There is a locally closed subvariety of $\tilde{X}_{\epsilon h}$ which is a fundamental domain for the $\Lambda$-action on $\tilde{X}_{\epsilon h}$ such that $\tilde{X}_{\epsilon h}$ is isomorphic to $\mathcal{S}$.

From the theorem one can deduce some information on the representation of the affine Weyl group on the vector space $\mathbb{C}[\tilde{X}_{\epsilon h}]$ with basis $[\tilde{X}_{\epsilon h}]$ defined in [L2], see Section 6.

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2. Let $U$ be the unipotent radical of $B$. Let $n$ be the Lie algebra of $U$. Let $G(F), U(F), T(F)$ be the group of $F$-points of $G, U, F$ respectively. Let $G(F)$ be the group of $F$-points of $G$. Note that $G(F)$ acts naturally on $g(F)$ by the adjoint representation $g : x \mapsto \text{Ad}(g)(x)$. Let $\Lambda$ be the subgroup of $T(F)$ consisting of the elements $\chi(\varepsilon)$ where $\chi$ runs over the one parameter subgroups $C^* \rightarrow T$ (viewed as homomorphisms $F^* \rightarrow T(F)$). Let $X$ be the set of $A$-Lie subalgebras of $g(F)$ of the form $\text{Ad}(g)(L)$ for some $g \in G(F)$. We shall regard $X$ as an increasing union of projective algebraic varieties over $C$ as in [L1, §11]. For each $L' \in X$, $L'/\varepsilon L'$ inherits from $L'$ a bracket operation and becomes a simple Lie algebra over $C$. Let $\pi_{L'} : L' \rightarrow L'/\varepsilon L'$ be the obvious map. Let $B_{L'}$ be the set of Borel subalgebras of $L'/\varepsilon L'$. Now $\tilde{X}$ consists of all $C$-Lie subalgebra of $g(F)$ of the form $\pi_{L'}^{-1}(b')$ for some $L' \in X$ and some $b' \in B_{L'}$. We define $\pi : \tilde{X} \rightarrow X$ by $I \mapsto L'$ where $I \subset L'$. Note that $g : I \mapsto \text{Ad}(g)I$ is a well defined action of $G(F)$ on $\tilde{X}$ which is transitive. According to [KL], $t : I \mapsto \text{Ad}(t)I$ defines a free action of $\Lambda$ on $\tilde{X}_{\varepsilon h} = \{I \in \tilde{X} ; \varepsilon h \in I\}$ inducing a free action of $\Lambda$ with finitely many orbits on $[\tilde{X}_{\varepsilon h}]$. Let $X_{\varepsilon h} = \{L' \in X ; \varepsilon h \in L'\}$.

If $\xi \in n(F) := F \otimes n$ then $\exp(\xi) \in U(F)$ is well defined. Let $n(F)' = \bigoplus_{i \in \mathbb{Z}; i < 0} \varepsilon^i n \subset n(F)$. Let $U(F)' = \{\exp(\xi) ; \xi \in n(F)\} \subset U(F)$. It is well known that any $L' \in X$ can be written in the form $\text{Ad}(t)\text{Ad}(u)L$ where $t \in \Lambda, u \in U(F)'$ are uniquely determined. Hence we have a partition $X_{\varepsilon h} = \sqcup_{t \in A} X_{\varepsilon h, t}$ where $X_{\varepsilon h, t} = \{\text{Ad}(t)\text{Ad}(u)L ; u \in U(F)', \varepsilon h \in \text{Ad}(u)L\}$ is a locally closed subset of $X_{\varepsilon h}$. Let $\tilde{X}_{\varepsilon h, t} = \pi^{-1}(X_{\varepsilon h, t})$. This is a locally closed subset of $\tilde{X}_{\varepsilon h}$. Let $\Omega = X_{\varepsilon h, 1}, \tilde{\Omega} = \tilde{X}_{\varepsilon h, 1} = \pi^{-1}(\Omega)$. Note that

(a) $\tilde{X}_{\varepsilon h} = \sqcup_{t \in A} \tilde{X}_{\varepsilon h, t}$

as a set. Thus, $\tilde{\Omega}$ is a fundamental domain for the $\Lambda$-action on $\tilde{X}_{\varepsilon h}$. Let $\omega = \{E \in n(F)'; \text{Ad}(\exp(E))(\varepsilon h) \in L\}$. In preparation for the proof of the theorem we will prove the following result.

**Lemma 3.** The map $E = \varepsilon^{-1}E_1 + \varepsilon^{-2}E_2 + \varepsilon^{-3}E_3 + \ldots \mapsto E_1$ is a bijection $\phi : \omega \rightarrow n$. (Here $E_1, E_2, E_3, \ldots$ is a sequence of elements of $n$ with $E_i = 0$ for large $i$.)

The equation defining $\omega$ is $\exp(\text{ad}(E))(\varepsilon h) \in L$ that is
\[
\varepsilon h + \sum_{i \geq 1} \varepsilon^{-i+1}[E_i, h] + \left(1/2\right) \sum_{i,j \geq 1} \varepsilon^{-i-j+1}[E_i, [E_j, h]] \\
+ \left(1/6\right) \sum_{i,j,k \geq 1} \varepsilon^{-i-j-k+1}[E_i, [E_j, [E_k, h]]] + \cdots \in L,
\]
that is
\[
\sum_{i \geq 2} \varepsilon^{-i+1}[E_i, h] + \left(1/2\right) \sum_{i,j \geq 1} \varepsilon^{-i-j+1}[E_i, [E_j, h]] \\
+ \left(1/6\right) \sum_{i,j,k \geq 1} \varepsilon^{-i-j-k+1}[E_i, [E_j, [E_k, h]]] + \cdots \in L,
\]
that is

\[ [E_r, h] = -(1/2) \sum_{i,j \geq 1,i+j=r} [E_i, [E_j, h]] \]

\[ - (1/6) \sum_{i,j,k \geq 1,i+j+k=r} [E_i, [E_j, [E_k, h]]] + \ldots \]

for \( r = 2, 3, \ldots \). In the right hand side we have \( i < r, j < r, k < r, \) etc. Hence if \( E_r' \) is known for \( r' < r \) then \([E_r, h]\) is a well defined element of \( \mathfrak{n} \). Hence \( E_r \) is a well defined element of \( \mathfrak{n} \). (Note that \( E \mapsto [E, h] \) is a vector space isomorphism \( \mathfrak{n} \xrightarrow{\sim} \mathfrak{n} \).

It remains to show that \( E_r = 0 \) for large \( r \). For \( r \geq 1 \) let \( \mathfrak{n}^r \) be the subspace of \( \mathfrak{n} \) spanned by all iterated brackets of \( r \) elements of \( \mathfrak{n} \). (Thus, \( \mathfrak{n}^1 = \mathfrak{n}, \mathfrak{n}^2 \) is spanned by \([a, b]\) with \( a, b \) in \( \mathfrak{n} \), \( \mathfrak{n}^3 \) is spanned by \([a, b], c\) with \( a, b, c \) in \( \mathfrak{n} \), etc.) Note that

(b) \( E \mapsto [E, h] \) is an isomorphism \( \mathfrak{n}^r \xrightarrow{\sim} \mathfrak{n}^r \) for any \( r \geq 1 \).

We show by induction on \( r \) that

(c) \( E_r \in \mathfrak{n}^r \) for \( r = 1, 2, \ldots \)

For \( r = 1 \) this is clear. Assume now that \( r \geq 2 \). From (a) and the induction hypothesis we deduce that \([E_r, h] \in \mathfrak{n}^r \). Using (b) we see that for some \( E' \in \mathfrak{n}^r \) we have \([E_r, h] = [E', h] \), hence \([E_r - E', h] = 0 \), hence \( E_r = E' \). Thus \( E_r \in \mathfrak{n}^r \), proving (c). Since \( \mathfrak{n}^r = 0 \) for large \( r \) we see that \( E_r = 0 \) for large \( r \). This completes the proof of the lemma.

4. For \( E \in \mathfrak{n} \) we set \( u_E = \exp(E) \in U(F)' \) where \( E = \phi^{-1}(E) \) (see Lemma 3). Note that \( \text{Ad}(u_E)(ch) \in L \). Now \( \mathbb{E} \mapsto \text{Ad}(\exp(-E))L \) is a bijection \( \psi : \omega \xrightarrow{\sim} \forall \). Hence \( \psi' := \psi\phi^{-1} : \mathfrak{n} \rightarrow \Omega \) is a bijection. We have \( \psi'(E) = \text{Ad}(u_E^{-1})L \). We show:

(a) Let \( E \in \mathfrak{n} \) and let \( L_E = \text{Ad}(u_E^{-1})L \in X \). Note that \( ch \in L_E \). Then \( \pi_{L_E}(ch) \in L_E/\epsilon L_E \) and \( \pi_L([-E, h]) \in L/\epsilon L \) correspond to each other under the Lie algebra isomorphism \( \tau_E : L/\epsilon L \xrightarrow{\sim} L_E/\epsilon L_E \) induced by \( \text{Ad}(u_E^{-1}) : L \xrightarrow{\sim} L_E \).

We must show that \( \text{Ad}(u_E)(ch) = -[E, h] \mod \epsilon L \) or that \( \text{Ad}(\exp(E))(ch) = -[E, h] \mod \epsilon L \) where \( E = \epsilon^{-1}E_1 + \epsilon^{-2}E_2 + \epsilon^{-3}E_3 + \ldots \) corresponds to \( E = E_1 \) as in Lemma 3. Thus we must show that

\[
ch + \sum_{i \geq 1} \epsilon^{-i+1}[E_i, h] + (1/2) \sum_{i,j \geq 1} \epsilon^{-i-j+1}[E_i, [E_j, h]]
+ (1/6) \sum_{i,j,k \geq 1} \epsilon^{-i-j-k+1}[E_i, [E_j, [E_k, h]]] + \cdots = -[E_1, h] \mod \epsilon L,
\]

or that

\[
\sum_{i \geq 2} \epsilon^{-i+1}[E_i, h] + (1/2) \sum_{i,j \geq 1} \epsilon^{-i-j+1}[E_i, [E_j, h]]
+ (1/6) \sum_{i,j,k \geq 1} \epsilon^{-i-j-k+1}[E_i, [E_j, [E_k, h]]] + \cdots \in \epsilon L.
\]

But the left hand side is actually zero, by the proof of Lemma 3. This proves (a).
From (a) we deduce:
(b) the map \( \beta \mapsto \tau_E(\beta) \) is a bijection \( \{ \beta \in B_L; \pi_L([-E, h]) \in \beta \} \to \{ \beta' \in B_{LE}; \pi_{LE}(eh) \in \beta' \} \).
Taking union over all \( E \in \mathfrak{n} \) and using the bijection \( \psi': \mathfrak{n} \to \Omega \) we deduce
(c) the map \( (E, \beta) \mapsto \pi_{LE}^{-1}(\tau_E(\beta)) \) is a bijection \( \{(E, \beta) \in \mathfrak{n} \times B_L; \pi_L([-E, h]) \in \beta \} \xrightarrow{\sim} \check{\Omega} \).

We consider the bijection
(d) \( \{(E, \beta) \in \mathfrak{n} \times B_L; \pi_L([-E, h]) \in \beta \} \to \mathcal{G} \)
given by \( (E, \beta) \mapsto (-[E, h], b_1) \) where \( b_1 \in B \) is defined by \( \pi_L(b_1) = \beta \). The composition of the inverse of (d) with the bijection (c) is a bijection
(e) \( \mathcal{G} \xrightarrow{\sim} \check{\Omega} \).

From the proof we see that the bijection (e) is an isomorphism of algebraic varieties. This proves the theorem.

5. Let \( NT \) be the normalizer of \( T \) in \( G \) and let \( W = NT/T \) be the Weyl group. For any \( w \in W \) let \( B_w \) be the variety consisting of all \( b_1 \in B \) such that \( (b, b_1) \) are in relative position \( w \). Note that \( B_w \) is isomorphic to \( C^{|w|} \) where \( |w| \in \mathbb{N} \) is the length of \( w \). Let \( \mathcal{G}_w = \{(E, b_1) \in \mathcal{G}; b_1 \in B_w\} \). The second projection \( \mathcal{G}_w \to B_w \) makes \( \mathcal{G}_w \) into a vector bundle with fibres of dimension \( \nu - |w| \). Hence \( \mathcal{G}_w \) is isomorphic to \( C^\nu \) as an algebraic variety. We have a partition \( \mathcal{G} = \bigsqcup_{w \in W} \mathcal{G}_w \) (as a set) with \( \mathcal{G}_w \) locally closed in \( \mathcal{G} \) (the closure of \( \mathcal{G}_w \) in \( \mathcal{G} \) is denoted by \( \overline{\mathcal{G}_w} \)). Hence we have a partition \( \check{\Omega} = \bigsqcup_{w \in W} \check{\Omega}_w \) (as a set) where \( \check{\Omega}_w \) corresponds to \( \mathcal{G}_w \) under 4(e). Note that \( \check{\Omega}_w \) is isomorphic to \( C^\nu \) as an algebraic variety and that \( \check{\Omega}_w \) is locally closed in \( \check{\Omega} \). For \( w \in W, t \in \Lambda \) we set \( \check{\Omega}_{w,t} = \text{Ad}(t)\check{\Omega}_w \). Using 2(a) we see that
(a) \( \check{X}_{ch} = \bigsqcup_{(w,t) \in W \times \Lambda} \check{\Omega}_{w,t} \)
as a set, where \( \check{\Omega}_{w,t} \) is locally closed in \( \check{X}_{ch} \) and is isomorphic to \( C^\nu \). Let \( \check{\Omega}_{w,t} \) be the closure of \( \check{\Omega}_{w,t} \) in \( \check{X}_{ch} \). Note that \( \check{\Omega}_{w,t} \) is open dense in \( \check{\Omega}_{w,t} \). Since \( \check{X}_{ch} \) is of pure dimension \( \nu \), we see that
(b) \( (w, t) \mapsto \check{\Omega}_{w,t} \) is a bijection \( W \times \Lambda \xrightarrow{\sim} [\check{X}_{ch}] \).
In particular,
(c) the number of \( \Lambda \)-orbits on \( [\check{X}_{ch}] \) is equal to the order of \( W \).
A result closely related to (c) (but not (c) itself) appears in [TS].

6. Let \( [\mathcal{G}] \) be the set of irreducible components of \( \mathcal{G} \) (a finite set naturally indexed by \( W \) by \( w \mapsto [\mathcal{G}_w] \)). The bijection 5(b) gives rise to an imbedding \( [\mathcal{G}] \to [\check{X}_{ch}] \), \( [\mathcal{G}_w] \mapsto \check{\Omega}_{w,1} \) hence to an imbedding of vector spaces
(a) \( C[\mathcal{G}] \to C[\check{X}_{ch}] \)
with bases \( [\mathcal{G}], [\check{X}_{ch}] \). Springer has shown that \( W \) acts naturally on \( C[\mathcal{G}] \) (this is known to be the regular representation of \( W \) in a nonstandard basis). In [L2] it is shown that the affine Weyl group of \( G \) acts naturally on \( C[\check{X}_{ch}] \). Hence, by restriction, \( W \) acts on \( C[\check{X}_{ch}] \). From the definitions we see that the imbedding (a) is compatible with the \( W \)-actions.
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