Manin matrices of type $C$: multi-parametric deformation

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Abstract

We constructed a multi-parametric deformation of the Brauer algebra representation related with the symplectic Lie algebras. The notion of Manin matrix of type $C$ was generalised to the case of the multi-parametric deformation by using this representation and corresponding quadratic algebras. We derived pairing operators for these quadratic algebras and minors for the considered Manin matrices. The rank of pairing operators and dimensions of components of quadratic algebras were calculated.

Keywords: Manin matrices; quadratic algebras; Brauer algebra; multi-parametric deformation.

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1 Introduction

In [Man87, Man88] Yuri Manin considered some ‘non-commutative’ transformations between arbitrary quadratic algebras. He interpreted the quadratic algebras and these transformations as a quantum version of vector spaces and representations on them (see [S21] for details).

The notion ‘Manin matrix’ was introduced in [CF] for a matrix over a non-commutative ring satisfying some commutation relations. Such a matrix describes a non-commutative transformation between polynomial algebras. The Manin matrices and their determinants have a lot of interesting properties and applications [CF, CM, CFR]. In [CFRS] they were generalised for the \( q \)-case, where \( q \) is a parameter of the deformation of the polynomial algebras. The MacMahon Master Theorem and Cayley–Hamilton–Newton identities for the \( q \)-Manin matrices were obtained in [GLZ, FH07, FH08] and [IO] respectively. Some properties and applications of the super-version of Manin matrices were derived in [MR].

The Manin matrices were described for arbitrary quadratic algebras in terms of idempotent operators in [S20]. In this general case we cannot generalise the notion of determinant, but we introduced two types of minors for Manin matrices by using non-degenerate pairings between quadratic algebras.

Multi-parametric deformation of the polynomial algebras and corresponding matrices were considered by Manin for the super-case in [Man89]. In [S20, § 3.3, § 6.1] the (non-super) multi-parametric case was considered: we obtained idempotents describing the corresponding quadratic algebras and calculated the minors of Manin matrices by using representations of the symmetric groups. In terms of the classification of simple Lie algebras one can refer to this case as ‘multi-parametric Manin matrices of type \( A \)’.

Manin matrices of types \( B \), \( C \) and \( D \) were introduced by A. Molev in [Molev] for the non-deformed case. They are related with the representations of Brauer algebras. The corresponding quadratic algebras and minors of Manin matrices of these types were described in [S20, § 7]. Since the quadratic algebras over an algebraically closed field of characteristic 0 for different orthogonal/symplectic forms are isomorphic to each other, it is enough to take a canonical form in this case.

Here we generalise the Manin matrices of type \( C \) for the multi-parametric case. We do not suppose that the basic field is algebraically closed, but any symplectic form is reduced to the canonical form over an arbitrary field of characteristic 0. For the orthogonal case (types \( B \) and \( D \)) one needs to consider a general symmetric non-degenerate form, we leave this case for further research.

The structure of the paper is following. In Section 2 we recall necessary notions and facts on quadratic algebras and Manin matrices. In Section 3 we recall the multi-parametric type \( A \) case. We also recall some definitions and statements on Brauer algebra in Section 4. In Section 5 we construct a deformed representation of the Brauer algebra. In Section 6 we use this representation to define a multi-parametric version of Manin matrices of type \( C \) and write some of their minors. In Section 7 we calculate dimensions of the graded components of the quadratic algebras (which equal to the ranks of pairing operators) and prove that the
other minors vanish.

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2 Manin matrices for quadratic algebras

We fix a field $\mathbb{K}$ of characteristic $\text{char } \mathbb{K} = 0$ and an integer $k \geq 2$. Recall the general notions introduced in [S20].

Let $V$ be a finite-dimensional vector space (over $\mathbb{K}$) with a basis $(e_i)$ labelled by some set of indices. Denote by $E^i_j \in \text{End}(V)$ the operators acting by the formula $E^i_j e_l = \delta^i_l e_j$. They form a basis of $\text{End}(V)$. We use the following tensor notations. Let $T$ be an operator acting on the vector space $V \otimes V = \mathbb{K} \otimes V$. It has the form $T = \sum_{i,j,l,m} T_{ij}^{lm} E^i_j \otimes E^m_l$, where $T_{ij}^{lm} \in \mathbb{K}$ are its entries in the basis $(e_i \otimes e_j)$. For $a, b \in \{1, \ldots, k\}$ such that $a \neq b$ denote by $T^{(ab)}$ or $T^{(a,b)}$ the following operator acting on the vector space $V^\otimes k = V \otimes \cdots \otimes V$:

$$T^{(ab)} = \sum_{i,j,l,m} T_{ij}^{lm} (1 \otimes \cdots \otimes E^i_j \otimes \cdots \otimes E^m_l \otimes \cdots \otimes 1),$$

(2.1)

where $E^i_j$ and $E^m_l$ are on the $a$-th and $b$-th sites respectively.

Let $V$ be a finite-dimensional vector space with a basis $(e_i)$. The vector space $\text{Hom}(\tilde{V}, V)$ has a basis consisting of the operators $E^i_a$ acting as $E^i_a e_{\beta} = \delta^i_\beta e_i$. Let $\mathfrak{R}$ be an algebra (i.e. associative unital algebra over $\mathbb{K}$) and let $M = \sum_{i} M^i_a \otimes E^i_a \in \mathfrak{R} \otimes \text{End}(V)$. We identify the elements $r \in \mathfrak{R}$ and $H \in \text{Hom}(\tilde{V}, V)$ with the elements $r \otimes 1$ and $1 \otimes H$ of $\mathfrak{R} \otimes \text{Hom}(\tilde{V}, V)$ respectively, so that $M = \sum_{i,a} M^i_a E^i_a$. Denote by $M^{(a)}$ an element of $\mathfrak{R} \otimes \text{Hom}(\tilde{V}^\otimes k, V^\otimes k) = \mathfrak{R} \otimes \text{Hom}(\tilde{V}, V) \otimes \cdots \otimes \text{Hom}(\tilde{V}, V)$ of the form

$$M^{(a)} = \sum_{i,a} M^i_a (1 \otimes \cdots \otimes E^i_a \otimes \cdots \otimes 1),$$

(2.2)

where $E^i_a$ is in the $a$-th tensor factor $\text{Hom}(\tilde{V}, V)$.

Let $(e^i)$ be a basis of the vector space $V^*$ dual to the basis $(e_i)$. Let $A \in \text{End}(V \otimes V)$ be an idempotent operator, i.e. $A^2 = A$. It acts on the basis vectors $e_i \otimes e_j \in V \otimes V$ and $e^i \otimes e^j \in V^* \otimes V^*$ from the left and from the right respectively: $A(e_i \otimes e_j) = \sum_{l,m} A_{lm}^i e_l \otimes e_m$, $(e^i \otimes e^j) A = \sum_{l,m} A_{lm}^i e^l \otimes e^m$. Denote by $\mathfrak{X}_A(\mathbb{K})$, $\mathfrak{X}^*_A(\mathbb{K})$, $\Xi_A(\mathbb{K})$ and $\Xi^*_A(\mathbb{K})$ the quadratic algebras generated by $x^i$, $x_i$, $\psi^i$, $\psi_i$ respectively with the relations

$$A(X \otimes X) = 0, \quad (X^* \otimes X^*) A = 0, \quad (\Psi \otimes \Psi)(1 - A) = 0, \quad (1 - A)(\Psi^* \otimes \Psi^*) = 0,$$

(2.3)

where $X = \sum_i x^i e_i$, $X^* = \sum_i x_i e^i$, $\Psi = \sum_i \psi_i e_i$, $\Psi^* = \sum_i \psi^i e_i$.

In [S20] § 5.3 we introduce the notion of pairing operators $S_{(k)}$, $A_{(k)} \in \text{End}(V^\otimes k)$ for an idempotent $A$. These operators define non-degenerate pairings $\mathfrak{X}_A(\mathbb{K}) \times \mathfrak{X}^*_A(\mathbb{K}) \to \mathbb{K}$ and
\[ \Xi_A(\mathbb{K}) \times \Xi_A(\mathbb{K}) \to \mathbb{K} \] by the matrix formulae
\[ \langle X \otimes \cdots \otimes X, X^* \otimes \cdots \otimes X^* \rangle = S(k), \quad \langle \Psi^* \otimes \cdots \otimes \Psi^*, \Psi \otimes \cdots \otimes \Psi \rangle = A(k), \quad (2.4) \]
where the dots ‘\( \cdots \)’ mean that there are \( k \) tensor factors in the products. Recall also that \( A(1) = S(1) = 1, A(2) = A, S(2) = 1 - A \).

Consider two idempotents \( A \in \text{End}(V \otimes V) \) and \( \tilde{A} \in \text{End}(\tilde{V} \otimes \tilde{V}) \) with entries \( A_{lm}^{ij}, \tilde{A}_{\gamma\delta}^{ab} \). Recall that Manin matrix over an algebra \( \mathcal{R} \) for a pair of idempotents \((A, \tilde{A})\) or simply \((A, \tilde{A})\)-Manin matrix is an operator \( M \in \mathcal{R} \otimes \text{Hom}(\tilde{V}, V) \) with entries \( M_{ab} \in \mathcal{R} \) satisfying the relation
\[ AM^{(1)}M^{(2)}(1 - \tilde{A}) = 0. \quad (2.5) \]
Entry-wise it is written as \( \sum_{l,m} A_{lm}^{ij} M_{\gamma \delta}^{lm} = \sum_{l,m,a,b} A_{lm}^{ij} M_{a \beta}^{lm} \tilde{A}_{\gamma \delta}^{ab} \). The Manin matrices are related with the quadratic algebras as follows. There is a bijection between \((A, \tilde{A})\)-Manin matrices \( M \) over \( \mathcal{R} \), graded homomorphisms \( f_M : \mathcal{X}_A(\mathbb{K}) \to \mathcal{R} \otimes \mathcal{X}_{\tilde{A}}(\mathbb{K}) \) and graded homomorphisms \( f^M : \Xi_A(\mathbb{K}) \to \mathcal{R} \otimes \Xi_{\tilde{A}}(\mathbb{K}) \) (see [S20, § 2.5] for details).

In [S20, § 5.4] we defined two generalisations of minors: \( S \)-minors and \( A \)-minors. For an \((A, \tilde{A})\)-Manin matrix \( M \) they are entries of the operators
\[ \text{Min}_{\tilde{S}(k)} M = M^{(1)} \cdots M^{(k)} \tilde{S}(k) \langle f_M(X \otimes \cdots \otimes X), \tilde{X}^* \otimes \cdots \otimes \tilde{X}^* \rangle, \quad (2.6) \]
\[ \text{Min}^{A(k)} M = A(k) M^{(1)} \cdots M^{(k)} \langle \Psi^* \otimes \cdots \otimes \Psi^*, f^M(\tilde{\Psi} \otimes \cdots \otimes \tilde{\Psi}) \rangle, \quad (2.7) \]
where \( \tilde{S}(k) \) and \( A(k) \) are pairing \( S \)- and \( A \)-operators for the idempotents \( \tilde{A} \) and \( A \) respectively (see [S20, § 5.3]). The \( S \)- and \( A \)-minors of \( M \) describe the \( k \)-th graded component of the homomorphism \( f_M \) and \( f^M \) respectively.

## 3 Manin matrices of type A

Here we recall multi-parametric case described in [Man89] and [S20, § 3.3, 6.1].

Recall that the permutation operator \( P = \sum_{i,j} E_i^j \otimes E_j^i \) acting on the space \( V \otimes V \) defines two representations of the permutation group \( S_k \) on the space \( V^\otimes k \); these are \( \sigma_a \mapsto \pm P^{(a, a+1)} \), where \( \sigma_a = \sigma_{a, a+1} \in S_k \) are adjacent transpositions. The polynomial algebra and Grassmann algebra are quadratic algebras \( \mathcal{X}_A(\mathbb{K}) \) and \( \Xi_A(\mathbb{K}) \) for the idempotent \( A = \frac{1 - E}{2} \).

Let \( \tilde{q} \) be a matrix with non-zero entries \( q_{ij} \in \mathbb{K} \setminus \{0\} \). We call it a parameter matrix iff
\[ q_{ij} = q_{ji}^{-1}, \quad q_{ii} = 1. \quad (3.1) \]
Define \( \tilde{q} \)-permutation operator
\[ P_{\tilde{q}} = \sum_{i,j} q_{ij} E_j^i \otimes E_i^j. \quad (3.2) \]
This is an operator \( P_\tilde{q} \in \text{End}(V \otimes V) \) with the entries \((P_\tilde{q})^{j\ell}_{ij} = q_{ij} \delta^\ell_i \delta^j_\ell\). It is involutive and satisfies the braid relation:

\[
P_\tilde{q}^2 = 1, \quad P_\tilde{q}^{(12)} P_\tilde{q}^{(23)} P_\tilde{q}^{(12)} = P_\tilde{q}^{(23)} P_\tilde{q}^{(12)} P_\tilde{q}^{(23)}. \quad (3.3)
\]

Due to (3.3) the formulae \( \sigma_{\alpha} \mapsto \pm P_\tilde{q}^{(\alpha, \alpha+1)} \) gives representations \( \rho^\pm_\tilde{q} : S_k \to \text{End}(V^{\otimes k}) \) of the permutation group \( S_k \).

The idempotent \( A_\tilde{q} = \frac{1-\rho^+_\tilde{q}}{2} \) defines multi-parametric deformations of the polynomial and Grassmann algebras. These are the quadratic algebras \( \mathfrak{X}_{A_\tilde{q}}(\mathbb{K}) \) and \( \Xi_{A_\tilde{q}}(\mathbb{K}) \) defined by the commutation relations \( x^i x^j = q_{ij}^{-1} x^j x^i \) and \( \psi^i \psi^j = -q_{ij} \psi^j \psi^i \). Since the relations for \( \mathfrak{X}_{A_\tilde{q}}(\mathbb{K}) \) and \( \Xi_{A_\tilde{q}}(\mathbb{K}) \) are \( x^i x^j = q_{ij}^{-1} x^j x^i \) and \( \psi^i \psi^j = -q_{ij}^{-1} \psi^j \psi^i \), we have isomorphisms \( \mathfrak{X}_{A_\tilde{q}}(\mathbb{K}) \cong \mathfrak{X}_{A_\tilde{q}'}(\mathbb{K}) \) and \( \Xi_{A_\tilde{q}}(\mathbb{K}) \cong \Xi_{A_\tilde{q}'}(\mathbb{K}) \), where \( \tilde{q}' = (q_{ij}^{-1}) \). The pairing \( S \)- and \( A \)-operators for \( A_\tilde{q} \) are

\[
S_{A_\tilde{q},(k)} = \rho^+_\tilde{q}(h(k)), \quad A_{A_\tilde{q},(k)} = \rho^-_\tilde{q}(h(k)), \quad (3.4)
\]

where \( h(k) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \) and \( \rho^\pm_\tilde{q} : \mathbb{K}[S_k] \to \text{End}(V^{\otimes k}) \) are the corresponding representations of the group algebra \( \mathbb{K}[S_k] \).

Consider an \( \tilde{n} \times \tilde{n} \) parameter matrix \( \tilde{p} \), where \( \tilde{n} = \dim \tilde{V} \). An \( (A_\tilde{q}, A_\tilde{p}) \)-Manin matrix is called briefly \( (\tilde{q}, \tilde{p}) \)-Manin matrix. Its \( S \)- and \( A \)-minors are defined by the formulae (2.6) and (2.7) with \( \tilde{S}_{(k)} = S_{\tilde{p},(k)} \) and \( A_{(k)} = A_{\tilde{q},(k)} \).

Finally recall some explicit formulae needed below. We have the following relations in the algebras \( \Xi_{A_\tilde{q}}(\mathbb{K}) \) and \( \mathfrak{X}_{A_\tilde{q}}(\mathbb{K}) \): for any \( k \)-tuple of indices \( I = (i_1, \ldots, i_k) \) and permutation \( \sigma \in S_k \), we have

\[
\psi_{i_{\sigma(1)}} \cdots \psi_{i_{\sigma(k)}} = \frac{(-1)^{\sigma}}{\mu_I(\tilde{q}, \sigma)} \psi_{i_1} \cdots \psi_{i_k}, \quad \psi^{i_{\sigma(1)}} \cdots \psi^{i_{\sigma(k)}} = (-1)^{\sigma} \mu_I(\tilde{q}, \sigma) \psi^{i_1} \cdots \psi^{i_k}, \quad (3.5)
\]

where \(( -1)^{\sigma} \) is the sign of \( \sigma \) and

\[
\mu_I(\tilde{q}, \sigma) = \prod_{\sigma^{-1}(t) > \sigma^{-1}(s) > t} q_{i_s i_t}. \quad (3.6)
\]

The non-vanishing entries of the \( A \)-operator \( A_{\tilde{q},(k)} \) are

\[
(A_{\tilde{q}})^{i_{\sigma(1)}, \ldots, i_{\sigma(k)}}_{i_1, \ldots, i_k} = \langle \psi_{i_{\sigma(1)}} \cdots \psi_{i_{\sigma(k)}} : \psi_{i_1} \cdots \psi_{i_k} \rangle = \frac{1}{k!} (-1)^{\sigma} \mu_I(\tilde{q}, \sigma), \quad (3.7)
\]

where \( \sigma \in S_k \) and \( I = (i_1, \ldots, i_k) \) is a \( k \)-tuple of pairwise different indices\(^1\).

\(^1\)In [S20, eq. (6.5)] we proved the formula (3.7) for the case \( i_1 < \ldots < i_k \), but it still holds for any pairwise different indices since it does not depend on the order in the set of indices. Explicitly one can check (3.7) by using the formula \( \mu_I(\tilde{q}, \sigma) = \mu_J(\tilde{q}, \tau) \mu_I(\tilde{q}, \sigma) \), where \( J = (i_{\sigma(1)}, \ldots, i_{\sigma(k)}) \), which in turn follows from (3.5).
4 Idempotents in Brauer algebra

We introduce the Brauer algebra $\mathcal{B}_k$ by following [N, BW] (see also [IM, MO, Molev]). Let $\omega \in \mathbb{K}$ be a parameter. The Brauer algebra $\mathcal{B}_k(\omega)$ is an algebra generated by the elements $\sigma_1, \ldots, \sigma_{k-1}$ and $\epsilon_1, \ldots, \epsilon_{k-1}$ with the commutation relations

$$\sigma_a^2 = 1, \quad \epsilon_a^2 = \omega \epsilon_a, \quad \sigma_a \epsilon_a = \epsilon_a \sigma_a = \epsilon_a, \quad a = 1, \ldots, k - 1,$$

$$\sigma_a \sigma_b = \sigma_b \sigma_a, \quad \epsilon_a \epsilon_b = \epsilon_b \epsilon_a, \quad \sigma_a \epsilon_b = \epsilon_b \sigma_a, \quad |a - b| > 1,$$

$$\sigma_a \sigma_{a+1} \sigma_a = \sigma_{a+1} \sigma_a \sigma_{a+1}, \quad \epsilon_a \epsilon_{a+1} \epsilon_a = \epsilon_a, \quad \epsilon_{a+1} \epsilon_a \epsilon_{a+1} = \epsilon_{a+1},$$

$$\sigma_a \epsilon_{a+1} \epsilon_a = \sigma_{a+1} \epsilon_a, \quad \epsilon_{a+1} \epsilon_a \sigma_{a+1} = \epsilon_a \sigma_{a+1}, \quad a = 1, \ldots, k - 2.$$ 

The subalgebra of $\mathcal{B}_k(\omega)$ generated by the elements $\sigma_1, \ldots, \sigma_{k-1}$ is naturally identified with the group algebra $\mathbb{K}[S_k]$.

Suppose $\omega \neq 0$. The relations (4.1) imply that the elements

$$v_a = \frac{1 + \sigma_a}{2} - \frac{\epsilon_a}{\omega}, \quad a = 1, \ldots, k - 1,$$

are idempotents: $v_a^2 = v_a$.

In order to consider ‘higher’ idempotents we need the elements $\sigma_{ab} = \sigma_{ba} = \tau_{a,b-1} \sigma_{b-1} \tau_{a,b-1}^{-1}$, $\epsilon_{ab} = \epsilon_{ba} = \tau_{a,b-1} \tau_{a,b-1}^{-1} \epsilon_{b-1} a < b$, where $\tau_{ab} = \sigma_a \sigma_{a+1} \cdots \sigma_{b-1}$ is the cycle of the length $b - a$ in the group $S_k$. In particular, $\tau_{aa} = 1$, so we have $\sigma_{a,a+1} = \sigma_{a+1,a} = \sigma_a$ and $\epsilon_{a,a+1} = \epsilon_{a+1,a} = \epsilon_a$. Note that the elements $\sigma_{ab} \in S_k$ are the standard transpositions and that $\sigma \epsilon_{ab} \sigma^{-1} = \epsilon_{\sigma(a) \sigma(b)}$ for any $\sigma \in S_k$.

The idempotents were obtained in [HS, I] for the more general Birman–Murakami–Wenzl algebra. In the particular case of the Brauer algebra the expression found in [I] reduces as follows (see [IR, Molev, S20]):

$$s_{(k)} = \frac{1}{k!} \prod_{b=2}^{k} \frac{(y_b + 1)(y_b + \omega + b - 3)}{2b + \omega - 4},$$

(4.5)

where $y_b = \sum_{a=1}^{b-1} (\sigma_{ab} - \epsilon_{ab})$ and we suppose that the denominators do not vanish:

$$\omega \notin \{0, -2, -4, \ldots, 4 - 2k\}.$$ 

(4.6)

For $k = 2$ the idempotent (4.3) coincides with (4.4), i.e. $s_{(2)} = v_1$. An ‘$R$-matrix’ form of the idempotents $s_{(k)}$ was found in [IM, MO].

It was proved in [I] § 4.3 (see also [HS, Molev, eq. (1.31)], [S20, Prop.7.10]) that (4.5) satisfies

$$\sigma_a s_{(k)} = s_{(k)} \sigma_a = s_{(k)}, \quad \epsilon_a s_{(k)} = s_{(k)} \epsilon_a = 0, \quad a = 1, \ldots, k - 1.$$ 

(4.7)
The relations (4.2) are obvious. Then, it is straightforward to check the last two relations (4.1) follow from Proof.

Consider a deformation of $q_i$ of the form $q_i = \lambda_i q_i + \mu_i q_i^2$, where $\lambda_i$ and $\mu_i$ are independent parameters satisfying (3.1). It is defined by $q_i \mapsto -q_i$ (the substitution $q_i \mapsto -q_i$ corresponds to the renumbering of the basis $e_i \mapsto e_{-i}$). Let $q = (q_i)$ be a family of the parameters $q_{ij}$, $q_i$ satisfying (3.1), (5.3). It is defined by $r(r+1)/2$ independent parameters: $q_1, \ldots, q_r, q_{ij}, 1 \leq i < j \leq r$. Note that the simultaneous change $q_i \mapsto -q_i$ preserves the operator (5.2). Sometimes we use the notations $P_q = P_{\bar{q}}$ and $Q_q = \bar{Q}_{\bar{q}}$.

Theorem 5.1. A family $q = (q_{ij}, q_i)$ satisfying (3.1) and (5.3) defines a representation $\rho_q: B_k(-2r) \to \text{End}(V^{\otimes k})$ by the formulae

$$\sigma_a \mapsto -P_{q(a,a+1)}^{(a,a+1)}, \quad \epsilon_a \mapsto -Q_{q(a,a+1)}^{(a,a+1)}$$

(5.4)

Proof. We already know that the commutation relations between $\rho_q(\sigma_a)$ are satisfied. The last two relations (3.1) follow from

$$Q_{q}^2 = 2rQ_q, \quad P_qQ_q = Q_qP_q = -Q_q.$$  

(5.5)

The relations (1.2) are obvious. Then, it is straightforward to check

$$Q_{q}^{(12)}Q_{q}^{(23)}Q_{q}^{(12)} = Q_{q}^{(12)}, \quad P_{q}^{(12)}Q_{q}^{(23)}Q_{q}^{(12)} = -P_{q}^{(23)}Q_{q}^{(12)}.$$  

(5.6)
These imply the second and forth relations (4.3). Let $q'$ be the family of the inverted parameters: $q' = (q_{ij}^{-1}, q_i^{-1})$. We have

$$P_{q'} = P_q^{(21)}, \quad Q_{q'} = Q_q^{(21)}.$$  \hfill (5.7)

By applying (5.7) to the first relation (5.6) for $q'$ we derive the third relation (4.3). The matrix transposition of the second relation (5.6), application of $P_q^\top = P_q$, $Q_q^\top = Q_q$ (5.8) and (5.7) gives the fifth relation (4.3).

\section{Manin matrices of type $C$ and their minors}

Consider the idempotent

$$C_q = \rho_q(v_1) = \frac{1 - P_q}{2} - \frac{Q_q}{2r}.$$  \hfill (6.1)

Let us write the commutation relations for the quadratic algebras $\Xi_{C_q}(K)$ and $\Xi_{C_q^*}(K)$. Note that $Q_q C_q = C_q Q_q = 0$. By multiplying the matrix relation $(\Psi \otimes \Psi)(1 - C_q) = 0$ by $Q_q$ from the right we obtain $(\Psi \otimes \Psi)Q_q = 0$, so this relation is equivalent to the system

$$(\Psi \otimes \Psi)(1 + P_q) = 0, \quad (\Psi \otimes \Psi)Q_q = 0.$$  \hfill (6.2)

Analogously, the relations for $\Xi_{C_q^*}(K)$ can be written in the form

$$(1 + P_q)(\Psi^* \otimes \Psi^*) = 0, \quad Q_q(\Psi^* \otimes \Psi^*) = 0.$$  \hfill (6.3)

Due to the identities (5.8) the matrix transposition of the relations (6.3) gives the relations (6.2) for $q' = (q_{ij}^{-1}, q_i^{-1})$, so the formula $\psi^i \mapsto \psi_i$ gives the isomorphism of quadratic algebras $\Xi_{C_q^*}(K) \cong \Xi_{C_q}(K)$.

By writing the commutation relations (6.2) explicitly we see that $\Xi_{C_q}(K)$ is a quadratic algebra with the generators $\psi_{-r}, \ldots, \psi_{-1}, \psi_1, \ldots, \psi_r$ and relations

$$\psi_i \psi_j = -q_{ij} \psi_j \psi_i, \quad \sum_{i=1}^r q_i \psi_i \psi_{-i} = 0.$$  \hfill (6.4)

The first relations define the $\tilde{q}$-Grassmann algebra $\Xi_{A_q}(K)$ (see [Man89], [S20], § 3.3]), so $\Xi_{C_q}(K)$ is the quotient of $\Xi_{A_q}(K)$ by one additional commutation relation. The quadratic algebra $\Xi_{C_q^*}(K)$ is generated by $\psi^{-r}, \ldots, \psi^{-1}, \psi^1, \ldots, \psi^r$ with the relations

$$\psi^i \psi^j = -q_{ij}^{-1} \psi^j \psi^i, \quad \sum_{i=1}^r q_i^{-1} \psi^i \psi^{-i} = 0.$$  \hfill (6.5)

\footnote{In the notations of [S20] the idempotent (6.1) is a deformation of $\tilde{B}_{2r}$.}
By substituting $A = C_q$ to the first of (6.2) we see that $\mathfrak{X}_{C_q}(\mathbb{K})$ is the algebra generated by $x^r, \ldots, x^{-1}, x^1, \ldots, x^\lambda$ with the commutation relations $x^i x^j - q_j^{-1} x^j x^i = \varepsilon q_i \delta_{i, j} - \lambda$, where $\lambda = \frac{1}{r} \sum_{i=1}^{r} (q_i^{-1} x^i x^{-i} - q_i x^{-i} x^i)$. Again we have the isomorphism $\mathfrak{X}_{C_q}(\mathbb{K}) \cong \mathfrak{X}_{C_q}(\mathbb{K})$.

The Molev’s definition of Manin matrix of type $C$ (see [Molev § 5.6], [S20 § 7.1]) is generalised to the case of multi-parametric deformation. We propose the term $(q, \hat{p})$-Manin matrix of type $C$ for a $(C_q, A_{\hat{p}})$-Manin matrix. Let us discuss minors of these matrices.

In [S20 § 6.1] we calculated $S$-operators for the idempotent $A_{\hat{p}}$ and $S$-minors for an $(A, A_{\hat{p}})$-Manin matrix, where $A \in \text{End}(V \otimes V)$ is an arbitrary idempotent (in particular, we can take $A = C_q$).

The $A$-minors of a $(q, \hat{p})$-Manin matrix of type $C$ (or, more generally, of a $(C_q, \hat{A})$-Manin matrix, where $\hat{A} \in \text{End}(V \otimes \hat{V})$ is an arbitrary idempotent) are expressed through the pairing $A$-operators for the idempotent $C_q$ by the formula (2.7). To find these $A$-operators we remind the following result [S20 Th. 5.26].

**Lemma 6.1.** Consider an idempotent $A \in \text{End}(V \otimes V)$. Let $\rho: \mathfrak{U}_k \to \text{End}(V^{\otimes k})$ and $\varepsilon: \mathfrak{U}_k \to \mathbb{K}$ be representation and augmentation of an algebra $\mathfrak{U}_k$. Let $s(\varepsilon(k)) \in \mathfrak{U}_k$ be such that

$$us(\varepsilon(k)) = s(\varepsilon(k))u = \varepsilon(u)s(\varepsilon(k)) \quad \forall u \in \mathfrak{U}_k, \quad \varepsilon(s(\varepsilon(k))) = 1. \quad (6.6)$$

Suppose there exist elements $u_1, \ldots, u_{k-1} \in \mathfrak{U}_k$ such that $\rho(u_a) = 1 - A^{(a, a+1)}$ and $\varepsilon(u_a) = 0$ for all $a = 1, \ldots, k - 1$. If $\rho$ and $\varepsilon$ satisfy

$$\rho(u)(\Psi \otimes \cdots \otimes \Psi) = \varepsilon(u)(\Psi \otimes \cdots \otimes \Psi), \quad (6.7)$$

$$\rho(u)(\Psi^* \otimes \cdots \otimes \Psi^*) = \varepsilon(u)(\Psi^* \otimes \cdots \otimes \Psi^*) \quad (6.8)$$

for all $u \in \mathfrak{U}_k$, then $A(k) = \rho(s(\varepsilon(k))) \in \text{End}(V^{\otimes k})$ is the $k$-th $A$-operator for the idempotent $A$.

The pairing $A$-operators for the idempotent $A = C_q$ are obtained by application of this fact to the Brauer algebra $\mathfrak{U}_k = B_k(-2r)$, its representation (5.4) and the augmentation

$$\varepsilon: B_k(\omega) \to \mathbb{K}, \quad \varepsilon(\sigma_a) = 1, \quad \varepsilon(\epsilon_a) = 0. \quad (6.9)$$

**Theorem 6.2.** The operators

$$C_{q, (k)}(a) = \rho_q(s(\varepsilon(k))) \in \text{End}(V^{\otimes k}), \quad k = 2, 3, \ldots, r + 1, \quad (6.10)$$

are the pairing $A$-operators for the idempotent $C_q$.

**Proof.** First of all we note that the restriction on $k$ in (6.10) for a fixed $r$ corresponds exactly to the condition (4.6) for $\omega = -2r$. The formula (4.7) means exactly that the conditions (6.6) are valid for the generators $u = \sigma_a$ and $u = \epsilon_a$, so they hold for any $u \in \mathfrak{U}_k$ (the normalisation $\varepsilon(s(\varepsilon(k))) = 1$ is obvious). Since the elements (1.4) satisfy $\rho_q(v_a) = C_q^{(a, a+1)}$ and $\varepsilon(v_a) = 1$, $a = 1, \ldots, k - 1$, we have $\rho_q(u_a) = 1 - C_q^{(a, a+1)}$ and $\varepsilon(u_a) = 0$ for $u_a = 1 - v_a$. The relations (6.2), (6.3) imply that the conditions (6.7), (6.8) (where $\rho = \rho_q$) are valid for
the generators $u = \sigma_a$, $u = \epsilon_a$ and hence for any element $u \in \fr{U}_k$. Thus Lemma 6.1 can be applied.

Let $M$ be a $(C_q, \tilde{A})$-Manin matrix, where $\tilde{A} \in \End(\tilde{V} \otimes \tilde{V})$ is an arbitrary idempotent. Then its $A$-minors are entries of the operator $C_{q(k)} M^{(1)} \cdots M^{(k)} \in R \otimes \Hom(\tilde{V}^{\otimes k}, V^{\otimes k})$, where $k \leq r + 1$. Below we prove that the $k$-th $A$-minors of $M$ vanish for $k \geq r + 1$.

## 7 Dimensions of components of the quadratic algebra

Here we calculate the dimensions of the graded components $\Xi_{C_q}(R)_k$ of the quadratic algebra $\Xi_{C_q}(R)$. For the non-deformed case $q_{ij} = q = 1$ we found these dimensions in [S20 § 7.2] by using the abstract (full) trace map $\tr_{1,...,k} : B_k(\omega) \rightarrow R$ and the formulae [Molev eq. (1.52)]

$$\tr_{1,...,k} s(k) = \frac{\omega + 2k - 2}{\omega + k - 2} \binom{\omega + 2k - 2}{k}.$$  

(7.1)

**Theorem 7.1.** The graded components of the quadratic algebra $\Xi_{C_q}(R)$ have the dimensions

$$\dim \Xi_{C_q}(R)_k = \frac{2r - 2k + 2}{k} \binom{2r + 1}{k - 1},$$  

(7.2)

$$\dim \Xi_{C_q}(R)_k = 0,$$  

(7.3)

**Proof.** Recall that the dimension of the vector space $\Xi_{C_q}(R)_k$ coincides with the rank of the $k$-th $A$-operator $C_{q(k)}$ (if it exists) [S20 eq. (5.31)]. Since the pairing operator is an idempotent, its rank is equal to its trace: $\dim \Xi_{C_q}(R)_k = \rk C_{q(k)} = \tr_{V^{\otimes k}} C_{q(k)}$. Remind the calculation for the non-deformed case. Due to the commutative diagram [Molev eq. (1.77)] we have $\tr_{V^{\otimes k}} (\rho(u)) = (-1)^k \tr_{1,...,k} u \forall u \in B_k(-2r)$, where $\rho = \rho_q$ for $q_{ij} = q = 1$. Hence by substituting $\omega = -2r$ to (7.1) we obtain the formula (7.2) for this case. To prove it for general $q$ we show that $\tr_{V^{\otimes k}} C_{q(k)}$ does not depend on $q$, if $k = 2, \ldots, r + 1$ (for $k = 1$ it is obvious: $\dim \Xi_{C_q}(R)_1 = \dim V = 2r$). The idempotent $\idemp$ can be written in the following form [Molev Prop. 1.2.5]:

$$s(k) = h(k) \sum_{t=0}^{[k/2]} (-1)^t \binom{\omega/2 + k - 2}{t}^{-1} \sum_{1 \leq a_t < b_t \leq k} \epsilon_{a_1 b_1} \cdots \epsilon_{a_t b_t},$$  

(7.4)

where the internal sum is over the non-intersecting pairs of indices $(a_1, b_1), \ldots, (a_t, b_t)$. This expression is well-defined since $\omega/2 + k - 2 = -r + k - 2 < 0$. By using the formulae $\sigma h(k) = h(k) \sigma = h(k) \forall \sigma \in S_k$ and $\sigma \epsilon_{a b} \sigma^{-1} = \epsilon_{\sigma(a) \sigma(b)}$ we see that the trace $\tr_{V^{\otimes k}} C_{q(k)} = \tr_{V^{\otimes k}} \rho_q(s(k))$ reduces to a linear combination of the terms $\tr_{V^{\otimes k}} \rho_q(h(k)\epsilon_{12} \cdots \epsilon_{2t-1,2t})$ with coefficients independent of $q$. Let us check that these terms does not depend on $q$ as well.

Note that $\rho_q(h(k))$ equals to the operator $A_{q(k)}$ with the entries (3.7), hence we have

$$\tr_{V^{\otimes k}} \rho_q(h(k)\epsilon_{12} \cdots \epsilon_{2t-1,2t}) = \sum_{j_1 \cdots j_{k-1}, j_1 \cdots j_2t} (A_{q})^{j_1 \cdots j_{k-1}, j_1 \cdots j_2t}_{i_1 \cdots i_{k-1}, i_1 \cdots i_2t} \prod_{p=1}^{t} (Q_q)^{i_{2p-1}, i_{2p}}.$$  

(7.5)
A term in this sum does not vanish only if the indices \( i_1, \ldots, i_k \) are pairwise different, the indices \( j_1, \ldots, j_{2t} \) have the form \( j_1 = i_{\sigma(1)}, \ldots, j_{2t} = i_{\sigma(2t)} \) for some \( \sigma \in S_{2t} \subset S_k \) and \( i_{2p-1} = -i_{2p}, \quad i_{\sigma(2p-1)} = -i_{\sigma(2p)}, \quad p = 1, \ldots, t. \) (7.6)

The conditions (7.6) imply that \( \sigma = \sigma' \tau, \) where \( \sigma' = \sigma a_1 \cdots \sigma a_m \) for some odd \( a_1, \ldots, a_m \) such that \( 1 \leq a_1 < \cdots < a_m < 2t \) and \( \tau \in S_{2t} \) permutes the pairs \((1, 2), (3, 4), \ldots, (2t - 1, 2t)\) in some way. Hence up to a \( q \)-independent factor the term in the sum (7.5) equals

\[
\mu_I(\hat{q}, \sigma) \prod_{p=1}^t (q_{i_{2p-1}} q_{i_{\sigma(2p-1)}}^{-1}) = \mu_I(\hat{q}, \sigma) \prod_{p=1}^t (q_{i_{2p-1}} q_{i_{\sigma(2p-1)}}^{-1}) = \mu_I(\hat{q}, \sigma) \prod_{s=1}^m q_{i_{a_s}}^2. \quad (7.7)
\]

Let \( \psi_{-r}, \ldots, \psi_{-1}, \psi_1, \ldots, \psi_r \) be the generators of the \( \hat{q} \)-Grassmann algebra \( \Xi_{A_k}(\mathbb{K}) \). Due to (5.3) they satisfy \( \psi_i \psi_{-i} = \psi_j \psi_{-j} = 0 \), so by taking into account (7.6) we obtain

\[
\psi_{i_{\sigma(1)}} \cdots \psi_{i_{\sigma(k)}} = \psi_{i_{\sigma'(1)}} \cdots \psi_{i_{\sigma'(k)}}. \quad (7.8)
\]

By virtue of (5.3) this implies \( \mu_I(\hat{q}, \sigma) = \mu_I(\hat{q}, \sigma') \). The direct calculation of \( \mu_I(\hat{q}, \sigma') = \mu_I(\hat{q}, \sigma a_1 \cdots \sigma a_m) \) by the formula (3.6) gives the product \( \prod_{s=1}^m q_{i_{a_s}, i_{a_s+1}} \). Hence by using (7.6) and (5.3) again we derive

\[
\mu_I(\hat{q}, \sigma) = \mu_I(\hat{q}, \sigma') = \prod_{s=1}^m q_{i_{a_s}, i_{a_s+1}} = \prod_{s=1}^m q_{i_{a_s}, -i_{a_s}} = \prod_{s=1}^m q_{i_{a_s}}^{-2}. \quad (7.9)
\]

By substituting (7.9) into (7.7) we see that the traces (7.5) do not depend on \( q \). This proves the formula (7.2) for general \( q \). For \( k = r + 1 \) it gives \( \dim \Xi_{C_k}(\mathbb{K})_{r+1} = 0 \) and hence all the higher components \( \Xi_{C_k}(\mathbb{K})_k \) are also zero. □

In particular, the formula (7.3) implies that the \( k \)-th pairing \( A \)-operator and the corresponding minors vanish for \( k \geq r + 1 \).

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