Research Article

On Local Weak Solutions for Fractional in Time SOBOLEV-Type Inequalities

Mohamed Jleli and Bessem Samet

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Bessem Samet; bsamet@ksu.edu.sa

Received 8 July 2020; Revised 24 August 2020; Accepted 10 September 2020; Published 23 September 2020

Academic Editor: Zoran Mitrovic

Copyright © 2020 Mohamed Jleli and Bessem Samet. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider two fractional in time nonlinear Sobolev-type inequalities involving potential terms, where the fractional derivatives are defined in the sense of Caputo. For both problems, we study the existence and nonexistence of nontrivial local weak solutions. Namely, we show that there exists a critical exponent according to which we have existence or nonexistence.

1. Introduction and Main Results

We consider the fractional in time Sobolev-type inequalities

\[
\begin{align*}
\partial_t^\sigma \eta(t,z) + \Delta_\lambda \eta(t,z)(1 + |z|^2)^{\rho/2} |\eta(t,z)|^{p-2} \eta(t,z) & \leq 0, \\
\eta(0,z) = \eta_0(z), & \quad z \in \mathbb{R}^N,
\end{align*}
\]

where \(\sigma \in (0,1)\) and \(\partial_t^\lambda = \lambda \in \{\sigma, \sigma + 1\}\) is the derivative of fractional order \(\lambda\) in the sense of Caputo. Namely, we are concerned with the existence and nonexistence of nontrivial local weak solutions to problems (1) and (2). We shall establish that there exists a critical exponent \(p_c\) that depends on \(N\) and \(\rho\) such that if \(p < p_c\), then problems (1) and (2) admit no nontrivial local weak solutions (i.e., we have an instantaneous blow-up), while if \(p \geq p_c\), then the considered problems admit local solutions for some initial values. In the proofs of our results, we use the test function method with some integral estimates. For more details about the test function method and its applications to partial differential equations, we refer to [1–3] and the references therein.

The absence of solutions (complete blow-up phenomenon) was observed in [4] for the following elliptic inequality with a singular potential term

\[
-\Delta_\eta \geq \frac{\eta^2}{|z|^2} \text{in } \mathbb{R}^N \setminus \{0\}, \quad \eta \geq 0 \text{ a.e.,}
\]

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\) containing 0. In the same reference, an instantaneous blow-up result was obtained for the parabolic analogue of (3), namely

\[
\partial_t \eta - \Delta_\eta \geq \frac{\eta^2}{|z|^2} \text{in } \mathbb{R}^N \setminus \{0\}, \quad \eta \geq 0 \text{ a.e.,}
\]

Notice that the method in [4] is based on comparison principles. In [5], using the test function method and avoiding the maximum principle, instantaneous blow-up results were obtained for certain classes of elliptic and parabolic inequalities including as special cases (3) and (4). For more...
results on instantaneous blow-up for nonlinear evolution equations, we refer to [6–9] and the references therein. The investigation of instantaneous blow-up for linear Sobolev-type equations was first considered in [10]. Namely, the following problem was studied

\[ \partial_t (\eta_{xx} + \eta) = \eta_{xx}, \eta(0, z) = \eta_0(z), \eta(t, 0) = \eta(t, L), L > 0. \]  

(5)

In the limit case \( \sigma = 1 \) and \( \rho = 0 \), (1) and (2) (with equalities instead of inequalities), reduce, respectively, to

\[ \begin{cases} 
\partial_t + \Delta_\eta(t, z) + \eta(t, z)\partial_t \eta(t, z) = 0, (t, z) \in (0, \infty) \times \mathbb{R}^N, \\
\eta(0, z) = \eta_0(z), z \in \mathbb{R}^N
\end{cases} \]  

(6)

\[ \begin{cases} 
\partial_t + \Delta_\eta(t, z) + \eta(t, z)\partial_t \eta(t, z) = 0, (t, z) \in (0, \infty) \times \mathbb{R}^N, \\
\eta(0, z) = \eta_0(z), \partial_n \eta(0, z) = \eta_1(z), z \in \mathbb{R}^N.
\end{cases} \]  

(7)

The nonexistence of local weak solutions to (6) and (7) was considered in [11] by the use of the test function method. When \( N \in \{1, 2\} \), it was proved that for all \( \rho > 1 \), (6) and (7) admit no nontrivial local weak solutions. If \( N \geq 3 \), it was shown that if \( 1 < \rho \leq N/N - 2 \), then (6) and (7) admit no nontrivial local weak solutions, while if \( \rho > N/N - 2 \), then local solutions exist. Our aim in this paper is to study the instantaneous blow-up for the fractional in time versions of (6) and (7) with the potential term \( V(z) = (1 + |z|^2)^{\rho/2} \).

For existence results for stationary problems involving potential terms, see for example [12] and the references therein.

Notice that the study of fractional in time Sobolev-type equations was first considered in [13], where the nonexistence of global weak solutions was investigated.

Before mentioning our main results, let us define the meaning of solutions to (1) and (2).

Let \( T \in (0, \infty) \). Given \( \kappa > 0 \) and \( \omega \in C([0, T]) \), \( T > 0 \), we define the fractional integral operators

\[ (\mathcal{J}_\sigma^\omega)(t) = \frac{1}{T(\kappa)} \int_0^t (t - \zeta)^{\kappa - 1} \omega(\zeta) d\zeta, 0 < t \leq T \]  

and

\[ (\mathcal{J}_\tau^\omega)(t) = \frac{1}{T(\kappa)} \int_t^\tau (\zeta - t)^{\kappa - 1} \omega(\zeta) d\zeta, 0 < t \leq T. \]  

(8)

(9)

Given \( \omega_i \in C([0, T]) \), \( i = 1, 2 \), one has (see e.g., [14, 15])

\[ \int_0^T (\mathcal{J}_0^\omega_1)(t) \omega_2(t) dt = \int_0^T (\mathcal{J}_0^\omega_2)(t) \omega_1(t) dt. \]  

(10)

For \( \sigma \in (0, 1) \), the derivatives of fractional orders \( \sigma \) and \( \sigma + 1 \) in the Caputo sense are defined, respectively, by

\[ (\partial_t^\sigma \omega)(t) = \left( \mathcal{J}_0^{1-\sigma} \omega' \right)(t), 0 < t \leq T, \omega \in C^1([0, T]) \]

\[ (\partial_t^{\sigma+1} \omega)(t) = \left( \mathcal{J}_0^{-\sigma} \omega' \right)(t), 0 < t \leq T, \omega \in C^2([0, T]). \]  

(11)

Using the above notions and property (10), we define local weak solutions to problem (1) as follows.

**Definition 1.** Let \( \eta_0 \in L^1_{\text{loc}}(\mathbb{R}^N) \). We say that \( \eta \) is a local weak solution to (1), if there exists \( \eta \in L^p_{\text{loc}}(\Lambda_{T,N}) \) and satisfies

\[ \int_{\Lambda_{T,N}} (1 + |\xi|^2)^{\rho/2} |\eta|^p \xi dz dt - \int_{\mathbb{R}^N} \eta_0(z) \Delta (\mathcal{J}_{T}^{1-\sigma} \xi)(0, z) dz \leq - \int_{\Lambda_{T,N}} \eta \Delta \xi dz dt + \int_{\Lambda_{T,N}} \eta \Delta (\mathcal{J}_{T}^{1-\sigma} \xi) dz dt, \]  

(12)

for all \( \xi \in \text{CoO}(\Lambda_{T,N}), \xi \geq 0 \), with \( \text{supp}_z(\xi) \subset cR^N \). Here, \( \Lambda_{T,N} \) is the product set \([0, T] \times R^N \). Local weak solutions to problem (2) are defined as follows.

**Definition 2.** Let \( \eta_0, \eta_1 \in L^1_{\text{loc}}(\mathbb{R}^N) \). We say that \( \eta \) is a local weak solution to (2), if there exists \( \eta \in L^p_{\text{loc}}(\Lambda_{T,N}) \) and satisfies

\[ \int_{\Lambda_{T,N}} (1 + |\xi|^2)^{\rho/2} |\eta|^p \xi dz dt + \int_{\mathbb{R}^N} \eta_0(z) \Delta (\mathcal{J}_{T}^{1-\sigma} \xi)(0, z) dz \]

\[ - \int_{\mathbb{R}^N} \eta_1(z) \Delta (\mathcal{J}_{T}^{1-\sigma} \xi)(0, z) dz \leq - \int_{\Lambda_{T,N}} \eta \Delta \xi dz dt \]

\[ - \int_{\Lambda_{T,N}} \eta \Delta (\mathcal{J}_{T}^{1-\sigma} \xi) dz dt. \]  

(13)

for all \( \xi \in \text{CoO}(\Lambda_{T,N}), \xi \geq 0 \), with \( \text{supp}_z(\xi) \subset cR^N \) and \( dT \langle JT^1 - \sigma \xi \rangle(T, \cdot) = 0 \).

Now, we state our results. We first define the (critical) exponent

\[ p_c = \begin{cases} 
\infty & \text{if } N \in \{1, 2\}, \\
\frac{N + \rho}{N - 2} & \text{if } N \geq 3.
\end{cases} \]  

(14)

**Theorem 3.** Let \( N \geq 1, \rho > -2, \) and \( \sigma \in (0, 1) \).

(i) If \( \eta_0 \in L^p(\mathbb{R}^N) \) and \( 1 < p < p_c \), then problem (1) admits no nontrivial local weak solution

(ii) If \( N \geq 3 \) and \( p > p_c \), then problem (1) admits local solutions for some \( \eta_0 > 0 \).
Theorem 4. Let $N \geq 1$, $p > -2$, and $\sigma \in (0, 1)$.

(i) If $\eta_0, \eta_1 \in L^1(\mathbb{R}^N)$ and $1 < p < p_\sigma$, then problem (2) admits no nontrivial local weak solution.

(ii) If $N \geq 3$ and $p > p_\sigma$, then problem (2) admits local solutions for some $\eta_0 > 0$ and $\eta_1 = 0$.

The next section contains some preliminary estimates that will be useful in the proofs of our results. Section 3 is devoted to the proofs of Theorems 3 and 4.

2. Preliminaries

For $S, T \in (0, \infty)$, let

$$\alpha(t) = (1 - T^{-1}t)^m, t \in [0, T]$$

and

$$\beta(z) = F\left(\frac{|z|^2}{S^2}\right)^{m}, z \in \mathbb{R}^N,$$

where $m \gg 1$ (i.e., sufficiently large) is a natural number and $F \in C^{\infty}(\mathbb{R}_+)$ satisfies

$$0 \leq F \leq 1, F_{|[0,1]} \equiv 1, F_{|[2,\infty)} \equiv 0.$$  \hfill (17)

Let us introduce the function

$$\xi(t, z) = \alpha(t)\beta(z), \quad (t, z) \in A_{T,N}.$$  \hfill (18)

Clearly, one has $\xi \in C^\infty(A_{T,N}), \xi \geq 0$ and $\sup_{S} p_\sigma(\xi) < \infty$.

Lemma 5. The function $\alpha$ defined by (15) satisfies the following properties:

$$\left(\mathcal{F}_t^k\alpha\right)(t) = \frac{m!}{\Gamma(k + m + 1)} T^{-m}(T - t)^{k+m}, t \in [0, T],$$

Taking $\tau = T - \tau/T - t$, the above integral reduces to

$$\left(\mathcal{F}_t^k\alpha\right)(t) = \frac{1}{\Gamma(k)} T^{-m}(T - t)^{k+m} \int_0^t (1 - r)^{k-1} r^m dr$$

which proves (19). Next, (20) and (21) follow by differentiating (19).

Lemma 6. For sufficiently large $S$, one has

$$\int_{\mathbb{R}^n} \beta(z)^{-1/p-1} \left(1 + |z|^2\right)^{-p/2(p-1)} |\Delta \beta(z)|^{p-1} dz \leq CS^{N-2p+p/p-1}.$$  \hfill (24)

Here, $C > 0$ is a constant (independent on $S$).

Proof. Using (16) and (17), one obtains

$$\int_{\mathbb{R}^n} \beta(z)^{-1/p-1} \left(1 + |z|^2\right)^{-p/2(p-1)} |\Delta \beta(z)|^{p-1} dz$$

On the other hand, an elementary calculation shows that

$$|\Delta \beta(z)| \leq CS^{-2} F\left(\frac{|z|^2}{S^2}\right)^{m-2}, S < |z| < \sqrt{2S}.$$  \hfill (26)

Here and below, $C > 0$ is a constant independent on $S$, whose value may change from line to line. Hence, one deduces that

$$\int_{S < |z| < \sqrt{2S}} \beta(z)^{-1/p-1} \left(1 + |z|^2\right)^{-p/2(p-1)} |\Delta \beta(z)|^{p-1} dz$$

which proves the desired result.

The following result is obvious.
Lemma 7. For $T > 0$, one has
\[ \int_0^T a(t) dt = \frac{T}{m+1}. \]  
(28)

Using (20), one obtains easily the following result.

Lemma 8. For $T > 0$, one has
\[ \int_0^T a(t)^{-1/p-1} |\partial_n(\mathcal{F}^\kappa_\alpha)(t)| dt = CT^{1+(\kappa-1)/p-1}, \]  
(29)

where $\kappa \in (0, 1)$.

Using (21), the following result follows.

Lemma 9. For $T > 0$, one has
\[ \int_0^T a(t)^{-1/p-1} |\partial_n(\mathcal{F}^\kappa_\alpha)(t)| dt = CT^{1+(\kappa-2)/p-1}, \]  
(30)

where $\kappa \in (0, 1)$.

3. Proofs of the Main Results

Proof of Theorem 10.

(i) Suppose that $\eta \in L^p_{loc}(\Lambda_{T,N})$ is a nontrivial local wessak solution to (1) for some fixed $T \in (0, \infty)$. Then, using (12) with $\xi$ is the function defined by (18), one obtains
\[ \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt - \int_{\mathbb{R}^N} \eta_0(z) \Delta(\mathcal{J}^{1-\sigma}_T \xi(0,z)) dz \leq \int_{\mathcal{A}_{T,N}} |\eta||\Delta \xi| dz dt + \int_{\mathcal{A}_{T,N}} |\eta||\Delta \xi| dz dt. \]  
(31)

Next, using $\varepsilon$-Young inequality with $0 < \varepsilon < 1/2$, one obtains
\[ \int_{\mathcal{A}_{T,N}} |\eta||\Delta \xi| dz dt \leq \varepsilon \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt + C \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt + C \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt. \]  
(32)

Similarly, one has
\[ \int_{\mathcal{A}_{T,N}} |\eta||\Delta \xi| dz dt \leq \varepsilon \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt + \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt. \]  
(33)

It follows from (31), (32), and (33) that
\[ (1-2\varepsilon) \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt + \int_{\mathbb{R}^N} \eta_0(z) \Delta(\mathcal{J}^{1-\sigma}_T \xi(0,z)) dz \leq C \left( \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt + \int_{\mathbb{R}^N} \beta(z)^{-1/p-1} (1 + |z|^2)^{p/2}(p-1) |\Delta \xi|^{p-1} dz dt \right). \]  
(34)

On the other hand, by (18), one has
\[ \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt = \left( \int_0^T a(t) dt \right) \left( \int_{\mathbb{R}^N} \beta(z)^{-1/p-1} (1 + |z|^2)^{p/2}(p-1) |\Delta \xi|^{p-1} dz dt \right). \]  
(35)

Hence, using Lemmas 6 and 7, for sufficiently large $S$, one obtains
\[ \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt \leq CT^{\sigma-2p/p-1}. \]  
(36)

Again, by (18), one has
\[ \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt = \left( \int_0^T a(t) dt \right) \left( \int_{\mathbb{R}^N} \beta(z)^{-1/p-1} (1 + |z|^2)^{p/2}(p-1) |\Delta \xi|^{p-1} dz dt \right). \]  
(37)

Therefore, using Lemma 6 and Lemma 8 with $\kappa = 1 - \sigma$, for sufficiently large $S$, one obtains
\[ \int_{\mathcal{A}_{T,N}} (1 + |z|^2)^{p/2} |\eta|^p \xi dz dt \leq CT^{1-\sigma/p-1} S^{\sigma-2p/p-1}. \]  
(38)

On the other hand,
\[ \int_{\mathbb{R}^N} |\eta_0(z)| |\Delta(\mathcal{J}^{1-\sigma}_T \xi(0,z))| dz = \left| \left( \mathcal{J}^{1-\sigma}_T \alpha \right)(0) \right| \]  
(39)

Hence, using (19) (with $\kappa = 1 - \sigma$) and (26), one deduces that
Hence, passing to the infimum limit as $S \to \infty$ in (42), using (17), (41), and Fatou’s lemma, one deduces that

$$
\lim_{S \to \infty} \int_{\mathbb{R}^N} |\eta_0(x)||\Delta (f^{1-\sigma}_T \xi(0, z))| dz = 0.
$$

Next, it follows from (34), (36), (38), and (40) that

$$
\int_0^T \int_{\mathbb{R}^N} (1 + |z|^2)^{p/2} |\eta|^p a(t) F \left( \frac{|z|^2}{S^2} \right)^m dz dt \\
\leq C (TS^{N-2p} + S^{N-2p-1}) + T^{1-\sigma} S^{N-2p-1} (42)
$$

which contradicts the fact that $\eta$ is nontrivial.

(ii) Let

$$
\frac{\rho + 2}{p - 1} \leq a < N - 2
$$

$$
\kappa = [a(N - a - 2)]^{1/p - 1}.
$$

Notice that since $p > p_\sigma$, the set of values of $a$ satisfying (44) is nonempty. Consider the function

$$
w(z) = \kappa (1 + |z|^2)^{-\alpha/2}, z \in \mathbb{R}^N.
$$

An elementary calculation shows that

$$
\Delta w(z) = \kappa a (1 + |z|^2)^{-\alpha/2} \left[ N + (N - a - 2)|z|^2 \right].
$$

Hence, using (44), (45), and (46), one obtains

$$
-\Delta w(z) = (1 + |z|^2)^{\rho/2} |w(z)|^p = \kappa a (1 + |z|^2)^{-\alpha/2 - 2} \\
\cdot \left[ N + N - a - 2|z|^2 \right] - \eta^p (1 + |z|^2)^{\rho/2 - \alpha/2 + \frac{1}{2}} \\
> \rho a(N - a - 2) (1 + |z|^2)^{-\alpha/2 - 1} - \eta^p (1 + |z|^2)^{\rho/2 - \alpha/2 + \frac{1}{2}} \\
= \rho a(N - a - 2) (1 + |z|^2)^{-\alpha/2 - 1} \\
\cdot \left[ 1 - \eta^p \left( \frac{1 + |z|^2}{a(N - a - 2)} \right)^{\rho/2 - \alpha/2 + \frac{1}{2}} \right] \\
= \rho a(N - a - 2) (1 + |z|^2)^{-\alpha/2 - 2} \\
\cdot \left[ 1 - (1 + |z|^2)^{\rho/2 - \alpha/2 + 1/2} \right] > 0.
$$

This shows that

$$
\eta(t, z) = w(z), t \geq 0, z \in \mathbb{R}^N,
$$

is a global solution (so local solution) to (1) with $\eta_0 = w > 0$.

**Proof of Theorem 11.**

(i) Suppose that $\eta \in L^1_{loc}(A_{T, N})$ is a nontrivial local weak solution to (2) for some fixed $T \in (0, \infty)$. Then, using (13) with $\xi$ is the function defined by (18), and one obtains

$$
\int_{A_{T, N}} (1 + |z|^2)^{\rho/2} |\eta|^p \xi dz dt \\
\leq \int_{A_{T, N}} |\eta||\Delta \xi|dz dt \\
+ \int_{A_{T, N}} |\eta||\Delta \bar{\eta}_T (f^{1-\sigma}_T \xi(0, z))| dz dt \\
+ \int_{\mathbb{R}^N} |\eta_0(z)||\Delta \bar{\eta}_T (f^{1-\sigma}_T \xi(0, z))| dz \\
+ \int_{\mathbb{R}^N} |\eta_1(z)||\Delta \bar{\eta}_T (f^{1-\sigma}_T \xi(0, z))| dz.
$$

Notice that by (20) (with $\kappa = 1 - \sigma$), one has $\bar{\eta}_T (f^{1-\sigma}_T \xi(T, \cdot)) \equiv 0$. Hence, by Definition 2, the choice of the test function $\xi$ defined by (18) is allowed. Next, following the same arguments used in the proof of part (i) of Theorem 3, by the use of $\varepsilon$-Young inequality, one obtains

$$
\int_{A_{T, N}} (1 + |z|^2)^{\rho/2} |\eta|^p \xi dz dt \\
\leq C \sum_{j=1}^5 A_j(S).
$$
where
\[ A_1(S) = \int_{\mathbb{R}^N} \xi^{-1/p-1} (1 + |z|^2)^{-p/2(p-1)} |\Delta \xi|^{p-1} dzdt, \]
\[ A_2(S) = \int_{\mathbb{R}^N} \xi^{-1/p-1} (1 + |z|^2)^{-p/2(p-1)} |\partial_t (\mathcal{F}_T^{-1}\sigma \xi)|^{p-1} dzdt, \]
\[ A_3(S) = \int_{\mathbb{R}^N} \xi^{-1/p-1} (1 + |z|^2)^{-p/2(p-1)} |\partial_t (\mathcal{F}_T^{-1}\sigma \xi)|^{p-1} dzdt, \]
\[ A_4(S) = \int_{\mathbb{R}^N} |\eta_0(z)||\Delta (\mathcal{F}_T^{-1}\sigma \xi)(0,z)| dz, \]
\[ A_5(S) = \int_{\mathbb{R}^N} |\eta_1(z)||\Delta (\mathcal{F}_T^{-1}\sigma \xi)(0,z)| dz. \]

Notice that by (18), one has
\[ A_3(S) = (\int_0^T \alpha(t)^{-1/p-1} |\partial_t (\mathcal{F}_T^{-1}\sigma \xi)(t) dt \right)^+ \right) \cdot (\int_{\mathbb{R}^N} \beta(z)^{-1/p-1} (1 + |z|^2)^{-p/2(p-1)} |\Delta \beta(z)|^{p-1} dz). \]

Hence, using Lemma 6 and Lemma 9 with \( \kappa = 1 - \sigma \), for sufficiently large \( S \), one deduces that
\[ A_3(S) \leq CT^{-\alpha(p+1)/p-1} \sigma^{N-2+p/p-1}. \]

Furthermore, by (19) and (20) (with \( \kappa = 1 - \sigma \)), one has
\[ A_4(S) = |\partial_t (\mathcal{F}_T^{-1}\sigma \xi)(0)| \int_{\mathbb{R}^N} |\eta_0(z)||\Delta \beta(z)| dz \]
\[ = CT^{-\sigma} \int_{\mathbb{R}^N} |\eta_0(z)||\Delta \beta(z)| dz \]
\[ A_5(S) = |(\mathcal{F}_T^{-1}\sigma \xi)(0)| \int_{\mathbb{R}^N} |\eta_1(z)||\Delta \beta(z)| dz \]
\[ = CT^{1-\sigma} \int_{\mathbb{R}^N} |\eta_1(z)||\Delta \beta(z)| dz. \]

Next, using (36), (38), (51), (54), (55), and (56), for sufficiently large \( S \), one obtains
\[ \int_0^T \int_{\mathbb{R}^N} (1 + |z|^2)^{p/2} |\eta|^p \alpha(t) F \left( \frac{|z|^2}{S^2} \right) m dzdt \]
\[ \leq C \left[ (T + T^{1-\sigma p/p-1} + T^{-1-\sigma(p+1)/p-1}) \sigma^{N-2+p/p-1} \right] \]
\[ + \int_{\mathbb{R}^N} \left( T^{-\sigma} |\eta_0(z)| + T^{1-\sigma} |\eta_1(z)| \right) |\Delta \beta(z)| dz \]

Notice that by (26), since \( \eta_0, \eta_1 \in L^1(\mathbb{R}^N) \), one has
\[ \lim_{S \to \infty} \int_{\mathbb{R}^N} (T^{-\sigma} |\eta_0(z)| + T^{1-\sigma} |\eta_1(z)|) |\Delta \beta(z)| dz = 0. \]

Therefore, passing to the infimum limit as \( S \to \infty \) in (57) and using Fatou’s lemma, since \( 1 < p < p_c \), it holds that
\[ \int_0^T \int_{\mathbb{R}^N} (1 + |z|^2)^{p/2} |\eta|^p \alpha(t) m dzdt = 0, \]
which contradicts the fact that \( \eta \) is nontrivial.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

Authors’ Contributions

All authors contributed equally to this work.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Saud University for funding this work through research group No. RGP-237.

References

[1] V. A. Galaktionov, E. Mitidieri, and S. I. Pohozaev, “Blow-up for higher-order parabolic, hyperbolic, dispersion and Schrödinger equations,” in Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015.
[2] E. Mitidieri and S. I. Pohozaev, “Towards a unified approach to nonexistence of solutions for a class of differential inequalities,” Milan Journal of Mathematics, vol. 72, no. 1, pp. 129–162, 2004.
[3] S. I. Pohozaev, “Essentially nonlinear capacities induced by differential operators,” Doklady Akademii Nauk, vol. 357, no. 5, pp. 592–594, 1997.
[4] H. Brezis and X. Cabré, “Some simple nonlinear PDE’s without solutions,” Bollettino della Unione Matematica Italiana, vol. 8, no. 2, pp. 223–262, 1998.
[5] S. I. Pohozaev and A. Tesei, “Nonexistence of local solutions to semilinear partial differential inequalities,” Annales de l’Institut Henri Poincare/Analyse non lineaire, vol. 21, no. 4, pp. 487–502, 2004.
[6] X. Cabré, “Extremal solutions and instantaneous complete blow-up for elliptic and parabolic problems,” Contemporary Mathematics, vol. 446, pp. 159–174, 2007.
[7] E. I. Galakhov, “On the instantaneous blow-up of solutions of some quasilinear evolution problems,” Differential Equations, vol. 46, no. 3, pp. 329–338, 2010.
[8] J. A. Goldstein and I. Kombe, “Instantaneous blow up,” Contemporary Mathematics, vol. 327, pp. 141–149, 2003.
[9] S. I. Pohozaev and A. Tesei, “Instantaneous blow-up of solutions to a class of hyperbolic inequalities,” Electronic Journal of Differential Equations, vol. 8, pp. 155–165, 2002.

[10] B. D. Coleman, R. J. Duffin, and V. J. Mizel, “Instability, uniqueness and nonexistence theorems for the equation \( u_t = u_{xx} - u_{xtx} \) on a strip,” Archive for Rational Mechanics and Analysis, vol. 19, no. 2, pp. 100–116, 1965.

[11] M. O. Korpusov, “Critical exponents of instantaneous blow-up or local solubility of non-linear equations of Sobolev type,” Izvestiya: Mathematics, vol. 79, no. 5, pp. 955–1012, 2015.

[12] C. Vetro, “A model of capillary phenomena in \( \mathbb{R}^N \) with subcritical growth,” Rendiconti Lincei-Matematica e Applicazioni, vol. 31, no. 2, pp. 335–347, 2020.

[13] A. Alsaeedi, M. S. Alhothuali, B. Ahmad, S. Kerbal, and M. Kirane, “Nonlinear fractional differential equations of Sobolev type,” Mathematical Methods in the Applied Sciences, vol. 37, no. 13, pp. 2009–2016, 2014.

[14] A. Kilbas, M. H. Srivastava, and J. J. Trujillo, Theory and Application of Fractional Differential Equations, North Holland Mathematics Studies, vol. 204, Elsevier, 2006.

[15] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.