FEEDBACK CONTROL OF NOISE IN A 2-D NONLINEAR STRUCTURAL ACOUSTICS MODEL

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Abstract. A time domain feedback control methodology for reducing sound pressure levels in a nonlinear 2-D structural acoustics application is presented. The interior noise in this problem is generated through vibrations of one wall of the cavity (in this case a beam), and control is implemented through the excitation of piezoceramic patches which are bonded to the beam. These patches are mounted in pairs and are wired so as to create pure bending moments which directly affect the manner in which the structure vibrates. This application of control in this manner leads to an unbounded control input term and the implications of this are discussed. The coupling between the beam vibrations and the interior acoustic response is inherently nonlinear, and this is addressed when developing a control scheme for the problem. Gains for the problem are calculated using a periodic LQR theory and are then fed back into the nonlinear system with results being demonstrated by a set of numerical examples. In particular, these examples demonstrate the viability of the method in cases involving excitation involving a large number of frequencies through both spatially uniform and nonuniform exterior forces.

1. Introduction. The use of piezoceramic patches as actuators in structural acoustic systems has proven very successful in recent years, and their applicability in a variety of settings involving structural vibrations is likely to burgeon in the near future. Their role as actuators stems from the piezoelectric property that a strain in the material is produced when a voltage is applied. By bonding the patches to the underlying structure either individually or in pairs, forces and/or bending moments can be created which then affect the manner in which the underlying structure vibrates. The effectiveness of the patches as actuators for controlling structural vibrations is further augmented by the fact that they are lightweight (and hence, even though they do significantly alter the passive structural dynamics [11], they do so without significantly mass loading the structure), space efficient, durable and relatively inexpensive. Furthermore, they can be molded in a variety of shapes so as to fit various underlying structures.
In many structural acoustics applications, the problem of controlling interior noise is directly correlated with the problem of reducing structural vibrations since the interior noise fields are often generated by the vibrations of an enclosing structure. An important example of this is the problem of controlling interior aircraft cabin noise which is caused by fuselage vibrations that are induced by the low frequency high magnitude exterior noise fields generated by the engines. In this example, the problem is exacerbated by the use of lighter structural materials and the development of a new class of turboprop and turbofan engines which are very fuel efficient but also very loud.

As a result of the coupling between the structural vibrations and the impinging noise fields in examples such as this, the use of piezoceramic patches as actuators in structural acoustic systems has been studied experimentally, as well as analytically and numerically with both frequency domain [12, 14, 15, 16, 17, 18, 21, 22] and state space [1, 5] control techniques being considered.

When modeling structural acoustic problems involving the reduction of interior sound pressure levels through the activation of piezoceramic patches, four components are usually considered. They consist of the interior acoustic field, the structural vibrations, the coupling between the vibrating structure and the acoustic field, and the interactions between the piezoceramic patch and the underlying structure to which it is bonded. Due to the small magnitudes of the structural vibrations and the observation that the interior acoustic fields have sound pressure levels which are usually less than 150 dB, the acoustic and structural components are often considered to be linear, and this is the assumption which is made here. Because the careful modeling of the patch/structure interactions is integral in model-based control schemes, this has been carefully studied on its own right and models for the forces and moments resulting from the excitation of patches which are bonded to a variety of substructures and excited either individually or in pairs can be found in [10]. Finally, the coupling between the acoustic field and the vibrating structure manifests itself through pressure and momentum conditions which are ultimately nonlinear since they occur at the surface of the vibrating structure. In previous works (see [1, 5]), these nonlinear coupling terms have been approximated by their linear components and control schemes have been developed for the approximating linear systems. In this paper, the fully nonlinear coupling terms are retained and a scheme for controlling the resulting nonlinear system is presented.

The structural acoustic system under consideration in this work is motivated by an experimental apparatus that has been designed and constructed in the Acoustics Division, NASA LaRC. The apparatus consists of a concrete cylinder with a thin, circular, flexible plate at one end. An exterior noise source causes vibrations in the plate which in turn cause unwanted interior noise. Due to the materials used in constructing the apparatus, the chamber walls other than the plate are considered to be completely rigid which leads to hard wall boundary conditions in the resulting models.

In our presentation below, a 2-D model representing a slice from the experimental setup is discussed. Here the structural acoustic system consists of a 2-D cavity with an Euler-Bernoulli beam at one end. The strong and weak forms of the system equations are presented in the next section, and the advantages of the latter in this context are discussed. Because the incorporation of the feedback control is through actuators covering only sections of the boundary, the resulting system contains an
unbounded control input term, and an abstract formulation for the problem in this context is given which proves to be useful when developing an infinite dimensional state space optimal control strategy for the system. In this control scheme, the gains and tracking contributions for the linearized system are determined using an LQR theory for problems having periodic forcing terms and are then applied to the nonlinear system of interest. The problem of approximating the system dynamics is addressed in Section 3, and the finite dimensional control problem is discussed in the fourth section. In particular, numerical techniques are discussed which improve the efficiency of the scheme through the off-line calculation of the gains and tracking terms. Numerical examples demonstrating the effectiveness of the control strategy are given in Section 5. In the examples, a variety of forcing functions are chosen to illustrate the control method in cases involving multiple frequency excitation (up to twenty frequencies are considered), nonsymmetric excitation, and excitation of the system when the temporal coefficients are also periodic. Through these examples, the effectiveness of this time domain control method is demonstrated for structural acoustic applications which are difficult if not impossible to treat using frequency domain control techniques.

2. The System Equations. Consider the 2-D structural acoustic system consisting of a variable domain \( \Omega(t) \) with a vibrating wall \( \Gamma_0(t) \) at one end as depicted in Figure 1. The periodic forcing function \( f \) models an exterior noise field which through acoustic/structure interactions results in unwanted interior noise. As motivated by the conditions in the experimental apparatus described above, hard wall conditions are assumed on the fixed boundary \( \Gamma \). The strong and weak forms of the equations of motion describing the dynamics of the system are now presented with emphasis placed on developing a formulation which is conducive to the development of a feedback control methodology for the problem.

![Diagram of 2D domain](image)

**Fig. 1.** The 2-D domain.

2.1. Strong Form of the System Equations. For the range of magnitudes involved in these simulations, the acoustic response inside the cavity can be described either in terms of a velocity potential \( \phi \) or the acoustic pressure \( p \) (the two are related through the relationship \( p = \rho_f \phi_t \) where \( \rho_f \) is the equilibrium density of the
atmosphere). Motivated by control theoretic considerations and to simplify the presentation which follows, we choose the velocity potential as the second-order state variable. The acoustic dynamics inside the hardwalled cavity are then modeled by the undamped linear wave equation

\[ \phi_{tt} = c^2 \Delta \phi , \quad (x, y) \in \Omega(t), t > 0, \]

\[ \nabla \phi \cdot \hat{n} = 0 , \quad (x, y) \in \Gamma, t > 0 \]

where \( c \) is the speed of sound in the cavity and \( \hat{n} \) is the outward unit normal to the cavity. We note that air damping inside the cavity is omitted due to the relatively small dimensions of the experimental cavity being simulated, and can readily be incorporated for significantly larger cavities.

It is assumed that the fourth boundary \( \Gamma_0(t) \) consists of an impenetrable fixed-end Euler-Bernoulli beam with Kelvin-Voigt damping. Letting \( w \) denote the transverse displacement of a beam having width \( b \) and thickness \( h \), the equation of motion for the transverse vibrations of the beam are given by

\[
\rho w_{tt} + \frac{\partial^2 M}{\partial x^2}(t, x) = -\rho_f \phi_t(t, x, w(t, x)) + f(t, x) , \quad 0 < x < a , t > 0 ,
\]

\[ w(t, 0) = \frac{\partial w}{\partial x}(t, 0) = w(t, a) = \frac{\partial w}{\partial x}(t, a) = 0 , \quad t > 0 , \]

where \( f \) is the applied force due to pressure from the exterior noise field, \( \rho_f \phi_t(t, x, w(t, x)) \) is the backpressure due to the ensuing acoustic waves inside the cavity (this latter term is in general nonlinear since its effect occurs on the surface of the vibrating beam) and \( M \) is the total beam moment. For pairs of patches having thickness \( T \), width \( b \) and edges at \( x_1 \) and \( x_2 \), the density of the structure is given by (e.g., see [6, 9, 10])

\[ \rho(x) = \rho_b h b + 2 b \left( \rho_b T_b + \rho_{pe} T \right) \chi_{pe}(x) \tag{2.1} \]

where the characteristic function is given by

\[ \chi_{pe}(x) = \begin{cases} 1 & , \quad x_1 \leq x \leq x_2 \\ 0 & , \quad \text{otherwise} \end{cases} \]

Here \( T_b \) denotes the thickness of the bonding layer which results from gluing the patches to the beam, and \( \rho_b, \rho_{bl} \) and \( \rho_{pe} \) are the densities of the beam, glue and patch, respectively.

The general beam moment

\[ M(t, x) = M(t, x) - M_{pe}(t, x) \]

consists of an internal component \( M \), depending on material and geometric properties of the beam and patches, and an external component \( M_{pe} \) (the control term) which results from the activation of the patches through an applied voltage.

For a beam undergoing pure bending motion with out-of-phase excitation of the patches, the internal and external moments are given by

\[ M(t, x) = EI(x) \frac{\partial^2 w}{\partial x^2} + c_D I(x) \frac{\partial^3 w}{\partial x^2 \partial t} \]

\[ M_{pe}(t, x) = K^B V \chi_{pe}(x) \]
where $V$ is the voltage into the patches. As shown in [9], the stiffness, damping and control constants for the combined structure are

$$
EI(x) = E_b \frac{h^3 b}{12} + \frac{2b}{3} [E_{bt} a_{3bt} + E_{pe} a_{3pe}] \chi_{pe}(x)
$$

$$
c_D I(x) = c_{Db} \frac{h^3 b}{12} + \frac{2b}{3} [c_{Dbt} a_{3bt} + c_{Dpe} a_{3pe}] \chi_{pe}(x)
$$

$$
\kappa^B = E_{pe} b d_{31} (h + 2T_{bt} + T).
$$

Here $E_b, E_{bt}, E_{pe}$ are the Young's moduli for the beam, bonding layer and patch, respectively. The coefficients $c_{Db}$ and $c_{Dbt}$ are the Kelvin-Voigt damping coefficients for the beam and bonding layer while the coefficient $c_{Dpe}$ is taken to be a combination of the Kelvin-Voigt damping coefficient for the patch and the damping (i.e., energy dissipation) related to the production of current in the patches when the structure vibrates. The constants $a_{3bt}$ and $a_{3pe}$ given by $a_{3bt} = (h/2 + T_{bt})^3 - (h/2)^3$ and $a_{3pe} = (h/2 + T_{bt} + T)^3 - (h/2 + T_{bt})^3$ result from the integration of stresses through the bonding layer and patch. Finally, $d_{31}$ is a piezoceramic constant which relates the amount of strain produced in the patch to the level of voltage being applied.

**FIG. 2. Acoustic cavity with piezoceramic patches creating pure bending moments.**

In the model discussed thus far, the structural and cavity acoustic responses are coupled through the backpressure $\rho j \phi(t, x, w(t, x))$ on the surface of the beam. A second coupling equation is the continuity of velocity (or momentum) condition (here $j$ is the unit normal in the $y$-direction)

$$
w_i(t, x) = \nabla \phi(t, x, w(t, x)) \cdot j, \quad 0 < x < a, t > 0
$$

which results from the assumption that the beam is impenetrable to air.

The model thus developed is nonlinear in the terms:

(i): variable domain $\Omega(t)$,

(ii): pressure $-\rho j \phi(t, x, w(t, x))$,

(iii): velocity $\nabla \phi(t, x, w(t, x)) \cdot j$. 

Under an assumption of small displacements \((w(t,x) = \hat{w}(t,x) + \delta \) where \(\hat{w} \equiv 0\)) which is inherent in the Euler-Bernoulli formulation, the variable domain \(\Omega(t)\) can be approximated by the fixed domain \(\Omega \equiv [0,a] \times [0,\ell]\) as shown in Figure 2. Note that with this assumption, the velocity term \(\nabla \phi(t, x, w(t,x)) \cdot \hat{n}\) can be approximated by the normal term \(\nabla \phi(t, x, w(t,x)) \cdot \hat{n}\) which arises when developing the weak form of the equation. The fully nonlinear form of the pressure coupling term will be retained throughout the following discussion.

For a coupled system in which \(s\) pairs of patches are bonded to the beam and excited out-of-phase, the acoustic, structural and coupling components just discussed can be combined to yield the approximate nonlinear model

\[
\begin{align*}
\phi_{tt} &= c^2 \Delta \phi \quad (x, y) \in \Omega, t > 0, \\
\nabla \phi \cdot \hat{n} &= 0 \quad (x, y) \in \Gamma, t > 0, \\
\nabla \phi(t, x, w(t,x)) \cdot \hat{n} &= w_i(t, x) \quad 0 < x < a, t > 0, \\
\rho w_{tt} + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} + c_D I \frac{\partial^3 w}{\partial x^2 \partial t} \right) &= -\rho_f \phi_t(t, x, w(t,x)) + f(t, x) + \frac{\partial^2}{\partial x^2} \sum_{i=1}^{s} K_B^i u_i(t) \chi_{pe_i}(x), \quad 0 < x < a, \quad t > 0, \\
w(t, 0) &= \frac{\partial w}{\partial x}(t, 0) = w(t, a) = \frac{\partial w}{\partial x}(t, a) = 0 \quad t > 0, \\
\phi(0, x, y) &= \phi_0(x, y) \quad , \quad w(0, x) = w_0(x) \\
\phi_t(0, x, y) &= \phi_1(x, y) \quad , \quad w_t(0, x) = w_1(x).
\end{align*}
\]

Here \(u_i(t)\) is the voltage being applied to the \(i^{th}\) patch and \(\chi_{pe_i}\) denotes the characteristic function over the \(i^{th}\) patch. We point out that the piezoceramic material parameters \(K_B^i, i = 1, \cdots, s\) as well as the beam parameters \(\rho, EI\) and \(c_D I\) are considered to be unknown and must be estimated using data fitting techniques as discussed in [7].

We also reiterate that the parameters \(K_B^i, \rho, EI\) and \(c_D I\) are piecewise constant in nature due to the presence and differing material properties of the bonding layer and patches (see (2.2) as well as the results in [11]). This leads to difficulties with the strong form of the system equations since it involves the second derivatives of the Heaviside function (equivalently, derivatives of the Dirac delta) thus yielding unbounded structural coefficients and control input operator. To avoid these problems, it is advantageous to formulate the problem in weak or variational form (the use of the variational form also permits the use of basis functions having less smoothness than those used when approximating the solution to the strong form of the equations).

### 2.2. Weak Form of the System Equations

To formulate the weak of variational form of the problem, we introduce the Hilbert spaces \(H = L^2(\Omega) \times L^2(\Gamma_0)\) and \(V = H_0^1(\Omega) \times H_0^1(\Gamma_0)\) with the inner products

\[
\left\langle \left( \begin{array}{l} \phi \\ w \end{array} \right), \left( \begin{array}{l} \xi \\ \eta \end{array} \right) \right\rangle_H = \int_{\Omega} \frac{\rho_f}{c^2} \phi \xi d\omega + \int_{\Gamma_0} \rho_f \eta d\gamma
\]
and
\[
\left\langle \begin{pmatrix} \phi \\ w \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle_V = \int_{\Omega} \rho_f \nabla \phi \cdot \nabla \xi \, d\omega + \int_{\Gamma_0} E ID^2 w D^2 \eta \, d\gamma,
\]
respectively. Here \( L^2(\Omega) \) and \( H^1(\Omega) \) denote the quotient spaces of \( L^2 \) and \( H^1 \) over the constant functions and \( H_0^2(\Gamma_0) \) is given by \( H_0^2(\Gamma_0) = \{ \psi \in H^2(\Gamma_0) : \psi(x) = \psi'(x) = 0 \text{ at } x = 0, a \} \) (the use of the quotient space results from the fact that the velocity potential \( \phi \) is determined only up to a constant).

The multiplication of the wave equation by an arbitrary function \( \xi \in \bar{H}^1(\Omega) \), integration over the domain, and the application of Green’s theorem results in
\[
\frac{\partial f}{c^2} \int_{\Omega(t)} \phi_{tt} \xi d\omega = \rho_f \int_{\Gamma_0(t)} (\nabla \phi(t, x, w(t, x)) \cdot \hat{n}) \xi d\gamma - \rho_f \int_{\Omega(t)} \nabla \phi \cdot \nabla \xi d\omega.
\]

The approximation of the variable domain \( \Omega(t) \) by the fixed domain \( \Omega \equiv [0, a] \times [0, \ell] \) (which implies that the velocity term \( \nabla \phi(t, x, w(t, x)) \cdot \hat{j} \) can be approximated by the normal term \( \nabla \phi(t, x, w(t, x)) \cdot \hat{n} \)) then yields
\[
\frac{\partial f}{c^2} \int_{\Omega} \phi_{tt} \xi d\omega = \rho_f \int_{\Gamma_0} w_t \xi d\gamma - \rho_f \int_{\Omega} \nabla \phi \cdot \nabla \xi d\omega
\]
for all \( \xi \in \bar{H}^1(\Omega) \) as an approximating weak form of the wave equation. Likewise, integration of the beam equation yields
\[
\int_{\Gamma_0} \rho_b w_{tt} \eta d\gamma + \int_{\Gamma_0} E ID^2 w D^2 \eta \, d\gamma + \int_{\Gamma_0} c_D ID^2 w_t D^2 \eta \, d\gamma
\]
\[= - \int_{\Gamma_0} \rho_f \phi_t(w) \eta d\gamma + \int_{\Gamma_0} \sum_{i=1}^k \mathcal{K}_i u_i(t) \chi_{pe_i} D^2 \eta \, d\gamma + \int_{\Gamma_0} f \eta d\gamma
\]
for all \( \eta \in H_0^2(\Gamma_0) \) (again, \( \chi_{pe_i}(x) \) denotes the characteristic function over the \( i^{th} \) patch).

The weak forms of the wave and beam equation just described are in second order form, and it is desirable at this point to write them as a first order system in order to place them in a form which is amenable to the application of LQR optimal control results. This is accomplished by defining the product spaces \( H \times V \) and \( V = V \times V \) with the norms
\[
\left| (\Phi, \Psi) \right|^2_H = \left| \Phi \right|^2_V + \left| \Psi \right|^2_H,
\]
\[
\left| (\Phi, \Psi) \right|^2_V = \left| \Phi \right|^2_V + \left| \Psi \right|^2_V
\]
and taking the state to be \( Z = (\phi, w, \dot{\phi}, \dot{w}) \). Note that the state now contains a multiple of the pressure since \( p = \rho_f \dot{\phi} \). The nonlinear first-order variational form
is then given by
\[\int_{\Omega} \frac{\partial f}{\partial x} (\phi)_{t} \xi d\omega + \int_{\Gamma_0} \rho (\dot{w})_{t} \eta d\gamma + \int_{\Omega} \rho f \nabla \phi \cdot \nabla \xi d\omega + \int_{\Gamma_0} EID^2 w D^2 \eta d\gamma + \int_{\Gamma_0} \left\{ c_D ID^2 \omega D^2 \eta + \rho f \left[ \phi(\omega) \eta - \dot{w} \xi \right] \right\} d\gamma \]
\[= \int_{\Gamma_0} \sum_{i=1}^{s} K_i^B u_i(t) \chi_{\text{pec}} D^2 \eta d\gamma + \int_{\Gamma_0} f \eta d\gamma \] (2.4)

for all \((\xi, \eta)\) in \(V\).

We point out that in this variational form, the derivatives have been transferred from the beam and patch moments onto the test functions. This eliminates the problem of having to approximate the derivatives of the characteristic function and the Dirac delta as is necessary with the strong form of the equations.

The system (2.4) can be formally approximated by replacing the state variables by their finite dimensional approximations and constructing the resulting matrix system, and it is in this form that we will consider approximation strategies in the next section. In order to discuss the formulation and well-posedness of the infinite dimensional control problem, however, it is advantageous to pose the problem in an abstract Cauchy format, and this is the subject of the next subsection.

2.3. Abstract First Order Formulation. In order to pose the system (2.4) in abstract Cauchy format, we will first formulate the system in terms of sesquilinear forms and the bounded operators which they define. This follows the theoretical work in [2, 4] and further details concerning the abstract formulation of structural acoustic systems of this type can be found in [1, 7, 8].

We first point out that the Hilbert spaces \(H\) and \(V\) form a Gelfand triple \(V \hookrightarrow H \simeq H^* \hookrightarrow V^*\) with pivot space \(H\) (further details concerning the basic definitions and fundamental functional analysis theory here can be found in [23]). For \(\Phi = (\phi, w)\) and \(\Psi = (\xi, \eta)\) in \(V\), we then form the sesquilinear forms

\[\sigma_1(\Phi, \Psi) = \int_{\Omega} \rho f \nabla \phi \cdot \nabla \xi d\omega + \int_{\Gamma_0} EID^2 w D^2 \eta d\gamma,\]
\[\sigma_2(\Phi, \Psi) = \int_{\Gamma_0} \left\{ c_D ID^2 \omega D^2 \eta + \rho f (\phi \eta - w \xi) \right\} d\gamma\]

where \(\sigma_i : V \times V \to \mathbb{C}\), \(i = 1, 2\) are bounded and satisfy various coercivity conditions (see [8] for details for a similar system). As a result of the boundedness, we can define operators \(A_1, A_2 \in \mathcal{L}(V, V^*)\) by

\[\langle A_i \Phi, \Psi \rangle_{V^*, V} = \sigma_i(\Phi, \Psi)\]

for \(i = 1, 2\). To account for the control contributions, we let \(U\) denote the Hilbert space containing the control inputs and we define the control operator \(B \in \mathcal{L}(U, V^*)\) by

\[\langle Bu, \Psi \rangle_{V^*, V} = \int_{\Gamma_0} \sum_{i=1}^{s} K_i^B u_i(t) \chi_{\text{pec}} D^2 \eta d\gamma\]
for $\Psi \in V$, where $\langle \cdot , \cdot \rangle_{V^*, V}$ is the usual duality pairing. This then yields the product space term $B u(t) = (0, B u(t)) \in V^* \times V^*$ where again, $V = V \times V$.

In second order form, the external forcing and nonlinear perturbation terms are given by $F = (0, f/\rho b)$ and $G(z, z_t) = (0, -\rho_f \tilde{\phi}_t(w))$ (here, $\tilde{\phi}_t(w) = \phi_t(t, x, w(t, x)) - \phi(t, x, 0)$ denotes the nonlinear perturbation to the linear coupling term) which yields the product space terms $F(t) = (0, F(t))$ and $G(Z(t)) = (0, G(z(t), z_t(t)))$.

The weak form (2.4) is formally equivalent to the system

$$\mathcal{Z}_t(t) = A \mathcal{Z}(t) + B u(t) + F(t) + G(Z(t))$$

in $V^*$ where

$$\text{dom } A = \{ \Theta = (\Upsilon, \Lambda) \in H : \Lambda \in V, A_1 \Upsilon + A_2 \Lambda \in H \}$$

$$A = \begin{bmatrix} 0 & I \\ -A_1 & -A_2 \end{bmatrix}.$$  

As discussed in [7, 8], the abstract Cauchy equation (2.5) is in a form which is amenable to the development of well-posedness results for the open loop system. This is accomplished by showing that the operator $A$ generates a $C_0$-semigroup $T(t)$ on $H$ which can then be extended to a space containing the control input terms. Under suitable hypotheses concerning the Lipschitz continuity of the nonhomogeneous terms $B u$ and $F$ as well as the nonlinear coupling term, the well-posedness of the open loop system can be demonstrated. Moreover, this abstract formulation of the problem provides a framework within which an effective infinite dimensional control strategy can be developed and well-posedness results for the resulting closed loop system can be discussed.

2.4. The Infinite Dimensional Control Problem. The goal in the infinite dimensional control problem is to determine an optimal voltage which, when applied out-of-phase to the patches, causes a reduction in the magnitude of the state variables. Since the first-order state includes $\phi_t$ which is a multiple of the pressure, this then leads to a reduction in interior noise levels. In the linearized case (as discussed in [1, 5]) the feedback gains were calculated from an LQR theory for problems with periodic forcing functions (this is a reasonable assumption here since the forcing function models the exterior noise which in this problem is generated by the revolution of turboprop or turbofan blades).

This strategy cannot be directly applied here since the nonlinear coupling term has been retained. Instead, the following strategy was adopted. The infinite dimensional system was linearized by replacing the nonlinear coupling term $\phi_t(t, x, w(t, x))$ by its linear component $\phi_t(t, x, 0)$. This linearization is motivated by the assumption of small beam displacements which is inherent in the Euler-Bernoulli theory (for physically reasonable input forces, the beam displacements are of the order $10^{-5}$m for the geometries of interest). The feedback gains for this approximate linearized system were calculated from a periodic LQR theory (see [1]) and were then incorporated into the nonlinear problem to create a stable nonlinear closed loop control system.
Discussing first the linearized problem, the periodic LQR problem consists of finding \( u \in L^2(0, \tau; U) \) which minimizes a quadratic cost functional of the form

\[
J(u) = \frac{1}{2} \int_0^\tau \{ \langle QZ(t), Z(t) \rangle_H + \langle Ru(t), u(t) \rangle_U \} \, dt
\]

subject to \( Z = (\phi, w, \phi_t, w_t) \) satisfying the linear boundary value problem \( Z(t) = AZ(t) + Bu(t) + F(t) \) with \( Z(0) = Z(\tau) \). The operator \( Q \) can be chosen so as to emphasize the minimization of particular state variables as well as to create windows that can be used to decrease state variations of certain frequencies. The control space \( U \) is taken to be \( \mathbb{R}^s \) if \( s \) patches are used in the model, and it is assumed that the operator \( R \in \mathcal{L}(U) \) is an \( s \times s \) diagonal matrix where \( r_{ii} > 0, i = 1, \cdots, s \), is the weight on the controlling voltage into the \( i^{th} \) patch. In the case that \( B \) is bounded on \( H \), a complete feedback theory for this periodic problem can be given as discussed in [13]. This theory can be extended to also include the case of unbounded \( B \), i.e., \( B \in \mathcal{L}(U, V^*) \), of interest here (see [3]). Under usual stabilizability and detectability assumptions on the system as well as standard assumptions on \( Q \), the optimal control is given by

\[
u(t) = -R^{-1}B^* [\Pi Z(t) - r(t)] \tag{2.6}
\]

where \( \Pi \in \mathcal{L}(V^*, V) \) is the unique nonnegative self-adjoint solution of the algebraic Riccati equation

\[
A^* \Pi + \Pi A - \Pi BR^{-1}B^* \Pi + Q = 0. \tag{2.7}
\]

Here \( r \) is the unique \( \tau \)-periodic solution of

\[
\dot{r}(t) = -(A^* - \Pi BR^{-1}B^*) r(t) + \Pi F(t) \tag{2.8}
\]

and the optimal trajectory \( Z \) is the solution of

\[
\dot{Z}(t) = (A - BR^{-1}B^* \Pi) Z(t) + BR^{-1}B^* r(t) + F(t). \tag{2.9}
\]

As discussed in greater detail in the next section where the corresponding finite dimensional control problem is considered, an effective strategy for controlling the original nonlinear system is to determine the gains and tracking solution for the linearized model and feed these back into the nonlinear system thus yielding the nonlinear closed loop system

\[
\dot{Z}(t) = (A - BR^{-1}B^* \Pi) Z(t) + BR^{-1}B^* r(t) + F(t) + G(Z(t))
\]

where \( \Pi \) and \( r \) are determined from (2.7) and (2.8), respectively.

Using the results in [3], one finds that under appropriate assumptions, the operator \( A - BR^{-1}B^* \Pi \) also generates an exponentially stable \( C_0 \)-semigroup \( S(t) \) on the state space \( H \). As in the open loop case, this semigroup can then be extended through extrapolation space techniques to a larger space containing the control input and forcing terms. With the assumption that the input term \( F \) is sufficiently smooth so as to assure the necessary continuity in nonhomogeneous terms, the well-posedness of the closed loop nonlinear system can also be obtained.
3. Finite Dimensional Approximation. The discussion thus far has centered around the infinite dimensional model for the nonlinear structural acoustic system. In this section we present a Galerkin scheme for obtaining a finite dimensional approximation to the model which can be used when simulating the nonlinear system dynamics, estimating physical parameters and determining control gains. This was accomplished by discretizing the potential and beam in terms of spline and spectral expansions, respectively.

As shown in [1] where the corresponding linear problem was considered, a suitable Galerkin expansion for the potential is

\[ \phi^N(t, x, y) = \sum_{j=0}^{m_x} \sum_{i=0}^{m_y} \phi_{ij}(t)P_i^m(x)P_j^p(y) \]

where \( P_i^m(x) \) and \( P_j^p(y) \) denote the standard Legendre polynomials that have been scaled by transformation to the intervals \([0, a]\) and \([0, \ell]\), respectively. The condition \( i+j \neq 0 \) eliminates the constant function thus guaranteeing that the set of functions is suitable as a basis for the quotient spaces \( \bar{L}_2(\Omega) \) and \( \bar{H}^1(\Omega) \).

The beam displacement was discretized in terms of cubic splines since they satisfy the smoothness requirement as well as being easily implemented when adapting to the fixed-end boundary conditions and patch discretizations. Specifically, the approximate beam displacement was taken to be the linear combination

\[ w^N(t, x) = \sum_{i=1}^{n-1} w_i(t)B^m_i(x) \]

where \( B^m_i \) is the \( i^{th} \) cubic spline which has been modified to satisfy the boundary conditions (see [1] for details).

The \( m \) and \( n-1 \) dimensional approximating beam and cavity subspaces were taken to be \( H^m = \text{span} \{ B^m_i \}_{i=1}^m \) and \( H^p = \text{span} \{ B^p_i \}_{i=1}^{n-1} \), respectively, where \( B^m_i \) and \( B^p_i \) are the \( i^{th} \) beam and cavity bases described above. Defining \( N = m+n-1 \), the approximating state space was then taken to be \( H^N = H^m \times H^p \) and the product space for the first order system was \( \mathcal{H}^N = H^N \times H^N \).

By restricting the infinite dimensional system (2.4) to \( \mathcal{H}^N \times \mathcal{H}^N \) and choosing \( (\xi, \eta) \) in \( H^N \), one obtains the nonlinear finite dimensional system

\[
\begin{align*}
\int_\Omega \frac{p_f}{c_T} \phi^N_{tt} B^m_i \, d\omega &+ \int_{\Gamma_0} \rho_b w^N_{tt} B^m_p \, d\gamma \\
&+ \int_\Omega \rho_f \nabla \phi^N \cdot \nabla B^m_i \, d\omega + \int_{\Gamma_0} EID^2 w^N D^2 B^m_p \, d\gamma \\
&+ \int_{\Gamma_0} \left\{ c_DID^2 w^N D^2 B^m_p + \rho_f \left[ \phi^N_i (w^N) B^m_p - w^N_i B^m_p \right] \right\} \, d\gamma \\
&= \int_{\Gamma_0} \sum_{i=1}^s K^B_i w_i(t) \chi_{pe} D^2 B^m_p \, d\gamma \\
&+ \int_{\Gamma_0} f B^m_p \, d\gamma
\end{align*}
\]
for $\ell = 1, \ldots, m$ and $p = 1, \ldots, n - 1$. This yields the nonlinear system

$$
M^N \dot{y}^N(t) = \tilde{A}^N (y^N(t)) + \tilde{B}^N u(t) + \tilde{F}^N(t)
$$

where

$$
y^N(t) = \left( \begin{array}{c} \partial^N(t) \\ \dot{\partial}^N(t) \end{array} \right)
$$

and $\tilde{A}^N (y^N(t))$ is a nonlinear operator which is a function of $y^N(t)$. The vector $\dot{\partial}^N(t) = (\phi_{m1}^N(t), \phi_{m2}^N(t), \ldots, \phi_{mn}^N(t), w_1^N(t), w_2^N(t), \ldots, w_{m-1}^N(t))^T$ contains the $N \times 1 = (m + n - 1) \times 1$ approximate state coefficients while $u(t) = (u_1(t), \ldots, u_s(t))^T$ contains the $s$ control variables. The full system has the form

$$
\begin{bmatrix}
M_1^N & 0 \\
0 & M_2^N
\end{bmatrix}
\begin{bmatrix}
\dot{\partial}^N(t) \\
\dot{\partial}^N(t)
\end{bmatrix}
= 
\begin{bmatrix}
0 & M_1^N \\
-A_1^N & -A_2^N (w^N(t))
\end{bmatrix}
\begin{bmatrix}
\partial^N(t) \\
\dot{\partial}^N(t)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
\tilde{B}^N
\end{bmatrix}
u(t) + 
\begin{bmatrix}
0 \\
\tilde{F}^N(t)
\end{bmatrix}
$$

with

$$
M_1^N = \text{diag}[M_{11}^N, M_{12}^N], \quad M_2^N = \text{diag}[M_{21}^N, M_{22}^N],
$$

$$
A_2^N (w^N(t)) = 
\begin{bmatrix}
0 & A_{21}^N \\
A_{22}^N (w^N(t)) & A_{22}^N
\end{bmatrix}
, \quad A_1^N = \text{diag}[A_{11}^N, A_{12}^N]
$$

and

$$
\tilde{B}^N = \begin{bmatrix} 0, \tilde{B}_2^N \end{bmatrix}^T, \quad \tilde{F}^N(t) = \begin{bmatrix} 0, \tilde{F}_2^N(t) \end{bmatrix}^2.
$$

The matrices $M_{21}^N$ and $A_{11}^N$ are the mass and stiffness matrices which arise when solving the uncoupled wave equation with Neumann boundary conditions while $M_{22}^N$, $A_{12}^N$ and $A_{22}^N$ are the mass, stiffness and damping matrices which arise when solving the damped beam equation with fixed boundary conditions. The matrices $M_1^N$ and $M_2^N$ result from the choice of $V$ inner product (see [1]). The contributions from the coupling terms are contained in the matrix $A_{21}^N$ and operator $A_{22}^N (w^N(t))$ while the control, forcing and initial terms are contained in $\tilde{B}^N$, $\tilde{F}_2^N(t)$ and $\tilde{y}_0^N$, respectively. A more detailed description of the various component matrices can be found in [1].

4. The Finite Dimensional Control Problem. Because the underlying equations of motion and hence their finite dimensional approximations are nonlinear, LQR optimal control results cannot be directly applied to the problem. Instead, as mentioned in the discussion of the infinite dimensional control problem, the following strategy was adopted. The system was linearized and the gains and tracking solutions for the corresponding periodic LQR problem were calculated. These operators were then incorporated into the nonlinear finite dimensional approximating system to yield a suboptimal feedback control system. As demonstrated by the results in the next section, the strategy is very effective for reducing interior sound pressure levels.
4.1. Linear Periodic Control Problem. As discussed in [1], the approximation of the nonlinear coupling term \( \phi(t, x, w(t, x)) \) by its linear component, and the projection of the resulting system into the finite dimensional subspace \( H^N \times H^N \) yields the linear finite dimensional Cauchy equation

\[
\dot{y}^N(t) = A^Ny^N(t) + B^Nu(t) + F^N(t)
\]

\[
y^N(0) = y^N_0
\]

with the components of the linear stiffness matrix found in [1].

The periodic finite dimensional control problem is then to find \( u \in L^2(0, \tau) \) which minimizes

\[
J^N(u) = \frac{1}{2} \int_0^\tau \left\{ \langle Q^Ny^N(t), y^N(t) \rangle_{R^N} + \langle Ru(t), u(t) \rangle_{R^s} \right\} dt , \quad N = m + n - 1
\]

where \( y^N \) solves (4.1), \( \tau \) is the period, \( R \) is an \( s \times s \) diagonal matrix and \( r_{ii} > 0, i = 1, \ldots, s \) is the weight or penalty on the controlling voltage into the \( i^{th} \) patch.

The nonnegative definite matrix \( Q^N \) is chosen in a manner so as to emphasize the minimization of particular state variables. From energy considerations as discussed in [1], an appropriate choice for \( Q^N \) in this case is

\[
Q^N = M^N D
\]

where \( M^N \) is the mass matrix, and the diagonal matrix \( D \) is given by

\[
D = \text{diag} [d_1 I^m, d_2 I^{n-1}, d_3 I^m, d_4 I^{n-1}] .
\]

Here \( I^k, k = m, n - 1, \) denotes a \( k \times k \) identity matrix and the parameters \( d_i \) are chosen to enhance stability and performance of the feedback. Note that with this choice, the first inner product in the definition of \( J^N(u) \) simplifies to

\[
\langle Q^Ny^N(t), y^N(t) \rangle_{R^N} = d_1 \int_\Omega \rho_f |\nabla \phi^N|^2 d\omega + d_2 \int_{\Gamma_0} EI (D^2 w^N)^2 d\gamma
\]

\[
+ d_3 \int_\Omega \frac{\rho_f}{c^2} (\dot{\phi}^N)^2 d\omega + d_4 \int_{\Gamma_0} \rho_b (\dot{w}^N)^2 d\gamma.
\]

Hence the control \( u \) which minimizes \( J^N(u) \) subject to (4.2) also minimizes the energy in the sense that

\[
T^N_b = \frac{1}{2} \int_{\Gamma_0} \rho_b (\dot{w}^N)^2 d\gamma
\]

\[
P^N_b = \frac{1}{2} \int_{\Gamma_0} EI (D^2 w^N)^2 d\gamma
\]

\[
T^N_w = \frac{1}{2} \int_\Omega \rho_f |\nabla \phi^N|^2 d\omega
\]

\[
P^N_w = \frac{1}{2} \int_\Omega \frac{\rho_f}{c^2} (\dot{\phi}^N)^2 d\omega
\]

are the finite dimensional approximations to the kinetic and potential energies of the undamped beam and wave equation, respectively.

The optimal control is then given by

\[
u^N(t) = R^{-1}(B^N)^T \left[ r^N(t) - \Pi^N y^N(t) \right]
\]
where $\Pi^N$ is the solution to the algebraic Riccati equation
\begin{equation}
(A^N)^T \Pi^N + \Pi^N A^N - \Pi^N B^N R^{-1}(B^N)^T \Pi^N + Q^N = 0. \tag{4.3}
\end{equation}
For the regulator problem with periodic forcing function $F^N(t)$, $r^N(t)$ must satisfy the linear differential equation
\begin{equation}
\begin{aligned}
\dot{r}^N(t) &= - [A^N - B^N R^{-1}(B^N)^T \Pi^N]^T r^N(t) + \Pi^N F^N(t) \\
r^N(0) &= r^N(\tau)
\end{aligned} \tag{4.4}
\end{equation}
while the optimal trajectory is the solution to the linear differential equation
\begin{equation}
\begin{aligned}
\dot{y}^N(t) &= [A^N - B^N R^{-1}(B^N)^T \Pi^N] y^N(t) + B^N R^{-1}(B^N)^T r^N(t) + F^N(t) \\
y^N(0) &= y^N(\tau).
\end{aligned}
\end{equation}

The finite dimensional optimal control, Riccati solution, tracking equation and closed loop system can be compared with the original infinite dimensional relations given in (2.6), (2.7), (2.8) and (2.9), respectively. In order to guarantee the convergence $\Pi^N \rightarrow \Pi$, $r^N \rightarrow r$, and hence the convergence of $u^N \rightarrow u$, it is sufficient to impose various conditions on the original and approximate systems. These hypotheses include convergence requirements for the uncontrolled problem as well as the requirement that the approximation systems preserve stabilizability and detectability margins uniformly. A fully developed theory (see [3]) is available for the case when $\mathcal{F} \equiv 0$ (in this case the tracking variable $r$ does not appear in the solution) even when $\mathcal{B}$ is unbounded. However, the theory in [3] requires strong damping in the second-order system whereas the only damping in our system is the strong Kelvin-Voigt damping in the beam (damping in the cavity was omitted due to the relatively small dimensions involved). Although the convergence theory of [3] does not directly apply here, numerical tests indicate that convergence is obtained even though this system contains only weak or boundary damping.

4.2. Nonlinear Control Problem. To extend these results to the nonlinear system of interest, the linear gains were calculated and incorporated into the nonlinear system (3.1), thus yielding the suboptimal control
\begin{equation}
\begin{aligned}
\dot{y}^N(t) &= A^N (y^N(t)) - B^N R^{-1}(B^N)^T \Pi^N y^N(t) + B^N R^{-1}(B^N)^T r^N(t) + F^N(t) \\
y^N(0) &= y^N(\tau),
\end{aligned}
\end{equation}
where $A^N = (M^N)^{-1} A^N$. The Riccati matrix $\Pi^N$ and tracking vector $r^N(t)$ are solutions to (4.3) and (4.4) which arise when formulating the corresponding LQR problem.

We point out that the Riccati matrix and tracking solution can be computed off-line and then treated as filters when applying the feedback methodology to the nonlinear problem of interest. The Riccati matrix was calculated first using Potter’s method. This was then used in solving the tracking equation. Because numerical evidence indicated that the tracking solution was quite nearly periodic on temporal intervals $[0, \tau]$ where $\tau$ was chosen so as to be commensurate with the various frequencies of the forcing function, the tracking problem was solved as an initial value problem with $r(\tau) = 0$. The interval $[0, \tau]$ was divided into $ndpts$ subintervals and the tracking solution at times $t_i = i * \tau / ndpts$, $i = 1, ndpts$ was calculated and
stored in the $m \cdot (n - 1) \times ndpts$ matrix $R^N$ with the $i^{th}$ column containing the solution at time $t_i$. We again note that this was performed off-line and hence did not produce delays in the feedback process in the system to be controlled.

When controlling the actual system, the matrices $\Pi^N$ and $R^N$ were loaded and the vector $\eta^N(t)$, $t \in [t_i, t_{i+1}] \subset [0, \tau]$ was obtained by interpolating between the values in the $i^{th}$ and $(i + 1)^{st}$ columns of $R^N$. In this manner the feeding back of information into the nonlinear system could be efficiently accomplished.

5. Numerical Examples. To demonstrate the previously described control methodology, a .6 m by 1 m 2-D cavity with a flexible beam at one end was considered (see Figure 3). The beam was assumed to have width and thickness .1 m and .005 m, respectively, and the Young’s modulus and beam density were taken to be $E = 7.1 \times 10^{10} \text{ N/m}^2$ and $\rho_b = 2700 \text{ kg/m}^3$. This yielded the stiffness parameter $EI = 73.96 \text{ Nm}^2$ and linear mass density $\rho = 1.35 \text{ kg/m}$ in those regions of the beam not covered with patches. The damping parameter for the beam was chosen to be $c_D = .001 \text{ kg m/s/sec}$. The speed of sound and atmospheric density inside the cavity were taken to be $c = 343 \text{ m/sec}$ and $\rho_f = 1.21 \text{ kg/m}^3$, respectively.

![Graphical representation of the acoustic cavity with one controlling patch.](image)

**FIG. 3. The acoustic cavity with one controlling patch.**

Control was implemented via patches having thickness $T = .000508 \text{ m}$ and width $b = .1 \text{ m}$ (we point out that the chosen thickness value corresponds to 20 mil which is a commercially available thickness for piezoceramic patches). The Young’s modulus and density were taken to be $E_{pe} = 6.3 \times 10^{10} \text{ N/m}^2$ and $\rho_{pe} = 7650 \text{ kg/m}^3$ which are reasonable for a patch made from G-1195 piezoceramic material [20].

From (2.1) and (2.2) we see that the density and stiffness coefficient in the region of the combined beam and patch (Region 2) will be greater than that of the beam (Region 1) (see Figure 3). We also assume that the damping coefficient will be slightly larger in Region 2 than Region 1. Finally, we assumed a bonding layer of thickness $T_{bl} = .0001 \text{ m}$.

For testing purposes, the structural parameter values were chosen as specified in Table 1. Finally, the constant $K^B = E_{pe}bd_{31}(h + 2T_{bl} + T)$, which arises when modeling the actuation due to the patch, was taken to be $K^B = .0067$ (this latter
value was obtained by taking $d_{31} = 1.9 \times 10^{-10} \text{ m/V}$ which is the value specified for G-1195 piezoceramic material).

|        | $\rho$ ($\text{kg/m}$) | $EI$ ($\text{Nm}^2$) | $c_D I$ ($\text{kg m}^2/\text{sec}$) |
|--------|------------------------|-----------------------|-------------------------------------|
| Region 1 | 1.35                   | 73.96                 | .001                                |
| Region 2 | 2.115                  | 125.4                 | .00125                              |

Table 1. Values of the structural parameters. Region 1 is that part of the structure consisting solely of beam while Region 2 consists of the combined beam and patch.

We reiterate that in actual applications, the parameters $\rho, EI, c_D I$ and $K^B$ are considered to be unknown and are estimated using data fitting techniques (see [7]). While the expressions given (2.1) and (2.2) can be used as starting values in the parameter estimation routines, experimental evidence (see [11]) has indicated that the final parameter values can vary quite significantly from the analytic values due to the contributions from the bonding layer, variation in the measurement of physical constants, and nonuniformities in the various materials. This combined with the lack of analytic expressions for the damping constant necessitates the use of parameter estimation techniques with actual physical structures before model-based control strategies can be implemented.

In order to compare the system frequencies observed in the following examples with the analytic frequencies of an undamped beam containing no patches and an uncoupled acoustic cavity having the above dimensions, it is useful to list the individual beam and cavity frequencies. We emphasize however, that while these analytic frequencies will be close to those observed for the fully coupled system, differences will occur due to the damping in the beam, the presence of the patches on the beam, and the coupling between the dynamics of the beam and the acoustic response inside the cavity.

|        | 73.2 | 201.8 | 395.6 |
|--------|------|-------|-------|
| cavity | 171.5| 285.8 | 333.4 | 343.3 | 571.7 | 596.8 | 666.7 |

Table 2. Natural frequencies for the uncoupled beam and cavity (in hertz).

For a fixed-fixed beam of length $a$, the natural frequencies (in hertz) are given by

$$f_i = \frac{\lambda_i^2}{2\pi a^2} \sqrt{\frac{EI}{\rho}}$$

where again, $a = 6 \text{ m}$, $EI = 73.96 \text{ Nm}^2$, $\rho = 1.35 \text{ kg/m}$ and the first three values of $\lambda_i$ are $\lambda_1 = 4.7300, \lambda_2 = 7.8532$ and $\lambda_3 = 10.9956$. Also, for pressure oscillations in a cavity having dimensions $a \times \ell$, the natural frequencies are given by

$$f_{mn} = \frac{c}{2} \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{\ell}\right)^2}, \quad m = 1, \ldots, n = 0, \ldots$$

The individual frequencies for the uncoupled beam and cavity are summarized in Table 2. We point out that the beam frequency $f_2 = 201.8$ Hz and the cavity
frequencies $f_{01} = 285.8$ Hz and $f_{11} = 333.4$ Hz are asymmetric in the $x$-direction and hence the related coupled system frequencies will be observed only when the system is excited by a nonsymmetric force.

In the following examples, a variety of forcing functions modeling exterior noise fields of various forms are considered. In each case, the uncontrolled and controlled systems are considered and the effects of feeding the gains and tracking component from the linearized system into the nonlinear problem of interest are demonstrated. The various forcing functions were chosen to illustrate the effectiveness of this strategy in cases involving multiple frequency excitation, nonsymmetric excitation, and excitation of the system when the temporal coefficients are also periodic. In such cases, time domain state space control methods offer distinct and significant advantages over frequency domain control techniques.

![Diagram](Image)

**Fig. 4.** The acoustic cavity with observation points $p_1 = (0.3, 0.1), p_2 = (0.45, 0.1), p_3 = (0.3, 0.8)$ and $b_1 = 0.3$.

### 5.1. Example 1

The forcing function in this example was taken to be

$$f(t, x) = \sin(150\pi t) + \sin(170\pi t) + \sin(215\pi t) + \sin(240\pi t) + \sin(272\pi t) + \sin(302\pi t) + \sin(320\pi t) + \sin(365\pi t) + \sin(390\pi t) + \sin(420\pi t) + \sin(454\pi t) + \sin(480\pi t) + \sin(500\pi t) + \sin(536\pi t) + \sin(570\pi t) + \sin(603\pi t) + \sin(615\pi t) + \sin(650\pi t) + \sin(670\pi t) + \sin(720\pi t)$$

which models a plane wave (uniform in space) having frequencies ranging from 75 Hz to 360 Hz. The exterior pressure wave exhibits an root mean square (rms) sound pressure level of 124 dB.

|            | $b_1 = 0.3$ | $p_1 = (0.3, 0.1)$ | $p_2 = (0.45, 0.1)$ | $p_3 = (0.3, 0.8)$ |
|------------|-------------|--------------------|--------------------|--------------------|
| Uncontrolled System | 3.0e-05 m   | 104.5 dB           | 104.6 dB           | 103.5 dB           |
| Controlled System   | 6.6e-06 m   | 83.8 dB            | 79.4 dB            | 78.0 dB            |

**Table 3.** RMS decibel and displacement levels in the uncontrolled and controlled cases.
In approximating the open loop dynamics of the system, the limits \( m_x = m_y = 10 \) and \( n = 16 \) were used for a total of 120 cavity and 15 beam basis functions. Time histories of the beam displacement at \( X = .3 \) and acoustic pressure at \((X, Y) = (.3, .1)\) are plotted in Figure 5 with corresponding frequency plots in Figure 6.

![Uncontrolled Beam Displacement at X=.3](image1)

![Controlled Beam Displacement at X=.3](image2)

![Uncontrolled Acoustic Pressure at (X,Y) = (.3, .1)](image3)

![Controlled Acoustic Pressure at (X,Y) = (.3, .1)](image4)

**FIG. 5.** Uncontrolled and controlled beam displacements and pressures at the points \( X = .3 \) and \((X, Y) = (.3, .1)\) throughout the time interval \([0, .2]\). Control is started at time \( T = 0 \).

As noted in the frequency plots of the uncontrolled dynamics, the primary responses correspond with the natural frequencies of the coupled system occurring at 63.5, 180.7 and 395.5 Hz (note that these natural frequencies differ slightly from the analytic uncoupled values 73.2, 171.5 and 395.6 summarized in Table 2 due to the presence of the patches, the damping in the beam, and the coupling between the beam and cavity dynamics). While the remaining driving frequencies can be found if the plots are magnified, the energy in these responses is much less than that found in frequencies which are directly exciting natural frequencies of the system. We point out that the strong responses at system frequencies are transient, however, and when run far enough out in time, the system dynamics eventually reflect only the driving frequencies.

The uncontrolled rms sound pressure levels at the cavity points \( p1 = (.3, .1), p2 = (.45, .1) \) and \( p3 = (.3, .8) \) (see Figure 4) as well as the rms beam displacement at the central point \( b1 = .3 \) are summarized in Table 3. As expected, the strongest excitation of the interior field is recorded at the observation points closest to the
Fig. 6. Uncontrolled and controlled beam and cavity frequencies.

Fig. 7. The controlling voltage $u(t)$ when control is started at time $T = 0$.

vibrating beam. At those points, the sound pressure level is approximately 20 dB less than that of the driving exterior field.
Control Started at Time $T = 0$. Control was then implemented in the model through the excitation of a centered patch covering 1/6 of the beam length (hence $x_1$ and $x_2$ in Figure 3 have the values .25 and .35, respectively). The quadratic cost functional parameters were taken to be $d_1 = d_2 = d_4 = 1$, $d_3 = 10^4$ and $R = 10^{-6}$ with $d_3$ of much larger magnitude than $d_1, d_2$ or $d_4$ to emphasize the penalization of large pressure variations. To control the system, the Riccati matrix $\Pi^N$ and tracking trajectory on the interval $[0, 1]$ were calculated for the linearized system and then incorporated into the nonlinear model in the manner discussed in the last section.

![Uncontrolled and Controlled Beam Displacements and Pressures](image)

**Fig. 8.** Uncontrolled and controlled beam displacements and pressures at the points $X = .3$ and $(X,Y) = (.3, .1)$ throughout the time interval $[0, .2]$. Control is started at time $T = .1$.

Plots of the time history and frequencies of the controlled beam displacement and acoustic pressure levels are reported in Figures 5 and 6 opposite those of the uncontrolled dynamics. It is seen that after a brief transient period (which is partially due to an initial pressure surge in the exciting exterior pressure field), the controlled displacement and pressures are substantially reduced and maintained at a very low level as compared with the uncontrolled quantities. To quantify the reduction, the rms sound pressure levels of the controlled pressure levels at the points $p_1, p_2$ and $p_3$ were calculated and found to be 83.8, 79.4 and 78.0 dB, respectively (see Table 2). When compared with the uncontrolled pressures, this yields a reduction of 20.7, 25.2 and 25.5 dB at the three points.
The controlling voltage $u(t)$ is plotted in Figure 7. It has a maximum magnitude of 83.2 volts and an rms level of 24.5 volts. Since it is very reasonable to apply $10 - 15$ rms $V/mil$ to the patches [19] and these patches have 20 mil thicknesses, this is a physically reasonable voltage to apply to the system.

**Control Started at Time $T = .1$.** To test the control strategy in the case when pressure is allowed to first build up in the cavity before control is implemented, the system was run uncontrolled for .1 second at which time the controlling voltage was started. By starting the control at this point, the control scheme is starting with an initial beam displacement of $3.09 \times 10^{-5}$ m and an initial pressure of $-3.93$ $N/m^2$.

As demonstrated by the uncontrolled and controlled trajectories in Figure 8, the system undergoes a brief transient period after which, both the beam displacement and acoustic pressure are reduced to and maintained at the levels obtained when control was started at time $T = 0$. The controlling voltage in this case is plotted in Figure 9. After an initial transient period, it too settles into the levels observed when control was implemented at $T = 0$. This demonstrates the feasibility of the method for substantially reducing sound pressure levels when noise has been allowed to build up in the cavity before control is implemented.

![The Controlling Voltage](image)

**FIG. 9.** The controlling voltage $u(t)$ when control is started at time $T = .1$.

### 5.2. Example 2.

The forcing function in this example has the same frequency distribution as that in the first example except that here the wave is nonsymmetric in space. Specifically, the exterior force is taken to be

$$f(t, x) = [\sin(150\pi t) + \sin(170\pi t) + \sin(215\pi t) + \sin(240\pi t) + \sin(272\pi t) + \sin(302\pi t) + \sin(320\pi t) + \sin(365\pi t) + \sin(390\pi t) + \sin(420\pi t) + \sin(454\pi t) + \sin(480\pi t) + \sin(500\pi t) + \sin(536\pi t) + \sin(570\pi t) + \sin(603\pi t) + \sin(615\pi t) + \sin(650\pi t) + \sin(670\pi t) + \sin(720\pi t)] - [6x(.6 - x^2)\sin(4\pi x)]$$
which has an rms sound pressure level of 124.3 dB at the point \( x = .378 \) (it is at this point that the function \( \tilde{f} = 6x(.6 - x^2)\sin(4\pi x) \), plotted in Figure 10, reaches its maximum magnitude value).

\[ 
\begin{align*}
\text{(a) External Spatial Force and Patch Placement} \\
\text{(b) The acoustic cavity with observation points } b1 = .15 \text{ and } p1 = (.15,.1), p2 = (.375,.1), p3 = (.15,.8).
\end{align*}
\]

As was the case in the last example, numerical tests indicated that the open and closed loop dynamics were resolved with the limits \( m_x = m_y = 10, N = 16 \) (120 cavity and 15 beam basis functions), and the following results were obtained with these choices. Time histories of the uncontrolled beam displacement and acoustic pressure at the points \( b1 = .15 \) and \( p1 = (.15,.1) \) are plotted in Figure 11 with corresponding frequency plots in Figure 12. The rms displacement and decibel levels at the points \( b1, p1, p2 \) and \( p3 \) are reported in Table 4.

|                  | \( b1 = .15 \) | \( p1 = (.15,.1) \) | \( p2 = (.375,.1) \) | \( p3 = (.15,.8) \) |
|------------------|----------------|-------------------|-------------------|-------------------|
| Uncontrolled System | 6.5e-06 m      | 99.0 dB           | 99.3 dB           | 99.1 dB           |
| Controlled System  | 8.4e-07 m      | 69.9 dB           | 72.6 dB           | 70.6 dB           |

**Table 4. RMS decibel and displacement levels in the uncontrolled and controlled cases.**

In addition to the driving frequencies, the plots of the uncontrolled beam and acoustic pressure frequencies in Figure 12 exhibit very strong system responses at 62.3, 180.7, 286.9 and 336.9 hertz. The system responses at 62.3 and 180.7 hertz correspond to the first symmetric frequencies of the uncoupled beam and cavity (see Table 2). The system value 62.3 differs slightly from the value of 63.5 obtained in the last example due to the fact that two nonsymmetrically placed patches are
**Fig. 11.** Uncontrolled and controlled beam displacements and pressures at the points $X = .15$ and $(X, Y) = (.15, 1)$ throughout the time interval $[0, 3]$.

**Fig. 12.** Uncontrolled and controlled beam and cavity frequencies.
 bonded to the structure as opposed to the single centered patch used in Example 5.1. The system responses at 286.9 and 336.9 hertz correspond to the analytic values of 285.8 and 333.4 hertz which are the natural frequencies of the first two asymmetric modes in an isolated cavity of this size. These latter responses are due to the nonsymmetric nature of the forcing function in this example and lead to greater energy in higher frequency oscillations than was observed in the last example.
Due to the nonsymmetric nature of the forcing function and hence the system response, control was implemented through the excitation of two piezoceramic patch pairs covering the regions $[.1, .2]$ and $[.35, .45]$ as shown in Figure 10. The penalty terms in the weighting matrix $Q^N$ of (4.2) were taken to be $d_1 = d_2 = d_4 = 1$ and $d_3 = 10^4$ which again emphasizes the penalization of large pressure values. The control operator $R$ of (4.2) in this case was a $2 \times 2$ diagonal matrix with the elements taken to be $r_{11} = r_{22} = 10^{-6}$. The resulting voltages for the left and right patches are plotted in Figure 14. The voltage into the left patch attains a maximum magnitude of 41.6 volts and has an rms value of 8.4 volts while the controlling voltage into the right patch has an rms value of 10.9 volts and a maximum magnitude of 39.8 volts. The slightly larger rms value of the voltage into the right patch reflects the larger magnitude of the exterior forcing term in that region of the structure.

|                | b1 = .3 | p1 = (.3,.1) | p2 = (.45,.1) | p3 = (3,.8) |
|----------------|---------|--------------|--------------|------------|
| Uncontrolled System | 3.0e-05 m | 99.7 dB | 98.0 dB | 93.4 dB |
| Controlled System   | 5.9e-06 m | 87.4 dB | 77.4 dB | 76.6 dB |

**Table 5.** RMS decibel and displacement levels in the uncontrolled and controlled cases.

From the time histories of the controlled pressure and displacement in Figure 11 as well as the values in Table 4, it can be seen that the acoustic pressure and beam displacement are significantly reduced through the feedback mechanism described in Section 4.2. This is further emphasized by the surface plots in Figure 13 which show the uncontrolled and controlled interior pressure fields at the times $T = .1786$ and $T = .2426$ (these temporal values were chosen since the minimum and maximum pressure values at the point $p1$ occur at these times). In addition to illustrating the asymmetries in the interior pressure field in the uncontrolled case, these plots demonstrate that the acoustic sound pressure levels are uniformly reduced throughout the cavity by the application of the controlling voltages. Moreover, as demonstrated by the time histories in Figure 11, these low levels are maintained throughout the time interval of interest.

We point out that in addition to the choice of patches covering the regions $[.1, .2]$ and $[.35, .45]$, several other locations and patch lengths were also tried with good control results being obtained for a variety of configurations. Hence, while the control results can be optimized by tailoring the patch configuration according to the shape of the exterior force, the determination of precise patch locations is not crucial to the success of the scheme due to the robustness of the method.

### 5.3. Example 3.

This example differs from Example 1 in that here the magnitudes of the various components of the forcing function are allowed to also be time dependent and periodic. Specifically, $f$ was taken to be

$$f(t, x) = 2 \cos(10\pi t) \sin(150\pi t) + 4 \cos(16\pi t) \sin(302\pi t) + 6 \cos(22\pi t) \sin(454\pi t) + 8 \cos(28\pi t) \sin(720\pi t)$$

which models a plane wave with underlying frequencies ranging from 75 Hz to 360 Hz with time dependent coefficients having frequencies ranging from 5 Hz to 14 Hz. The rms sound pressure level of the exterior force is 128.7 dB.
From numerical tests, it was determined that the open loop dynamics were resolved with the limits \( m_x = m_y = 10, N = 16 \) (120 cavity and 15 beam basis functions), and the following uncontrolled approximate solutions were obtained with these choices. Time histories of the uncontrolled beam displacement and acoustic pressure at the points \( b1 = .3 \) and \( p1 = (.3, .1) \) (see Figure 4) are plotted in Figure 15 with corresponding frequency plots in Figure 16. The rms decibel and magnitude levels of the pressure and beam displacement at the points \( b1 = .3, p1 = (.3, .1), p2 = (.45, .1) \) and \( p3 = (.3, .8) \) are given in Table 5. As expected, the largest sound pressure levels are found at the centered point \( p1 \) which lies closest to the vibrating beam.

**Fig. 15.** Uncontrolled and controlled beam displacements and pressures at the points \( X = .3 \) and \( (X,Y) = (.3,.1) \) throughout the time interval \([0, 2]\).

Control was implemented through the excitation of a centered patch pair covering 1/3 of the beam length with the control parameters taken to be \( d_1 = d_2 = d_4 = 1, d_3 = 10^4 \) and \( R = 10^{-6} \). Although several other combinations were tried, these reported values provided a good balance between the amount of pressure reduction obtained and the amount of controlling voltage being applied to the patches.

Due to the higher frequency closed loop dynamics which resulted when the controlling voltage was applied to the patches (see Figure 16), a larger number of cavity and beam basis functions were needed in order to resolve the system dynamics. The time histories of the approximate controlled beam displacement and interior cavity pressure plotted in Figure 15 as well as the corresponding rms values in Table 5 were
obtained with 168 cavity and 19 beam basis functions \((m_x = m_y = 12, N = 20)\). By comparing the open and closed loop rms values in Table 5, it can be seen that the reduction in acoustic pressure levels ranges from 12.3 dB at \(p1\) in the front of the cavity to 16.8 dB at \(p3\) in the back of the chamber.

The time history of the controlled pressure field also shows a very strong beat phenomenon which results from the closely grouped response frequencies as can be seen in Figure 16. This beat phenomenon is transient, however, and will diminish as the system reaches steady state.

In order to further demonstrate the reduction in sound pressure levels throughout the cavity, surface plots of the uncontrolled and controlled pressure levels at the times \(T = .1076\) and \(T = .1814\) are given in Figure 17 (the maximum and minimum values of the pressure at the point \(p1\) occur at these times). As demonstrated by these figures, the sound pressure levels are uniformly reduced throughout the cavity with the largest levels in both cases occurring at the front of the cavity which is adjacent to the vibrating beam.

**Fig. 16. Uncontrolled and controlled beam displacements and cavity frequencies.**

Finally, the controlling voltage for this example is plotted in Figure 18. This voltage has an rms level of 48.4 volts and a maximum magnitude of 137.5 volts which are physically reasonable levels to apply to patches of this thickness. The beat phenomenon observed in the time history of the controlled pressure is also very apparent here and again results from the close proximity of the driving frequencies (see Figure 16).
Fig. 17. Uncontrolled and controlled pressure fields at $T = .1076$ and $T = .1814$.

Fig. 18. The controlling voltage $u(t)$ when control is started at time $T = 0$. 
6. Conclusions. In this paper, a time domain feedback control scheme for a class of nonlinear structural acoustics applications has been presented. The specific problem described here consists of a 2-D enclosed cavity with a thin beam at one end. A periodic exterior pressure field causes vibrations in the beam which through acoustic/structure interactions leads to unwanted interior sound pressure levels. Control is implemented via piezoceramic patches on the beam which produce bending moments when an out-of-phase voltage is applied.

Because the coupling between the structural vibrations and the acoustic pressure response is inherently nonlinear, the formulation of the problem leads to a nonlinear coupled system and hence LQR feedback control results cannot be applied directly. Instead, the following strategy was adopted. The feedback gains and tracking solutions for the linearized problem were calculated using a periodic LQR theory and the results were stored off-line. These gains and tracking components were then treated as filters and incorporated into the nonlinear problem of interest with the results being documented in a set of examples (other examples directly comparing the control method for the linear and nonlinear systems can be found in [6]).

The treatment of the problem in this manner and the ensuing set of examples extends the linear results of [1, 5] in two important aspects. First, the retention of the nonlinear coupling terms throughout the treatment of the problem provides a model which more accurately describes the dynamics of the actual physical system than does the linear model considered previously. The success of the control scheme in significantly reducing the sound pressure levels in this nonlinear model also indicates that the physical system with its inherent nonlinear coupling mechanisms can be successfully controlled using a methodology of this type (in practice, this will involve the development of a state estimator since the full state cannot be measured).

The second extension of the previous work is through the choice of examples. In creating examples to demonstrate the control of interior sound pressure levels in this nonlinear system, emphasis was placed on choosing physically realistic exterior pressure fields which excite system responses that would be difficult to control using frequency domain techniques. In the first two examples, the exterior field contained twenty exciting frequencies and the control techniques were demonstrated for both uniform and nonuniform (in space) pressure fronts. In these examples, the time domain control methodology was applied in two ways. In the first, control was applied immediately so that a low level of acoustic pressure was maintained throughout the entire time interval of interest. The scheme was also demonstrated for the case in which the excited system was allowed to run uncontrolled for an initial time period before feedback control was started, and it was showed that after a brief transient interval, the pressure and displacement levels were reduced to the levels attained when control was started at the beginning of the excitation interval. The exterior field in the third example was doubly periodic in the sense that the pressure oscillations as well as the time dependent coefficients were taken to be periodic. This models a multiply periodic exterior noise field that has a slowly varying underlying set of frequencies. In all cases, it was shown that the interior sound pressure levels could be significantly reduced and maintained at a low level of magnitude through the application of the controlling voltages (determined via the time domain control methodology) to the patches.
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