REMARKS ON QUASI-ISOMETRIC NON-EMBEDDABILITY INTO UNIFORMLY CONVEX BANACH SPACES

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Abstract. We construct a locally finite graph and a bounded geometry metric space which do not admit a quasi-isometric embedding into any uniformly convex Banach space. Connections with the geometry of $c_0$ and superreflexivity are discussed.

The question of coarse embeddability into uniformly convex Banach spaces became interesting after the recent work of G. Kasparov and G. Yu, who showed the coarse Novikov Conjecture, i.e. that the coarse assembly map in K-theory is injective, for bounded geometry metric spaces coarsely embeddable into uniformly convex Banach spaces [KY]. So far there is no example of a bounded geometry metric space which wouldn’t admit a coarse embedding into any uniformly convex Banach space - such spaces are hard to find even if we restrict the target space to be Hilbert.

In this note we consider quasi-isometric embeddings, a special case of coarse embedding that may be described as large scale biLipschitz. We construct a locally finite graph which does not admit a quasi-isometric embedding into any uniformly convex Banach space. In contrast to this note that every separable metric space admits a biLipschitz embedding into a strictly convex Banach space, namely into the space $c_0$ with an equivalent strictly convex norm, by a classic result of I. Aharoni [Ah].

It also turns out that our methods can be applied to a $c_0$-type of geometry. As a result we find an explicit geometric obstruction to uniform quasi-isometric embeddability (i.e. with embedding constants independent of $n$) of the $\ell^n_\infty$'s into any uniformly convex Banach space. In the proof one can directly see how the Lipschitz constants of the embedding are being pushed away to infinity as $n$ grows larger.

In the last section we also comment our observations in view of Bourgain’s paper on superreflexivity [Bo], from which quasi-isometric non-embeddability of trees into uniformly convex Banach spaces can be deduced. Bourgain’s proof is of different nature, using classic characterization of superreflexivity due to R.C. James and G. Pisier. We show that our graph has essentially the opposite of the hyperbolic geometry on the tree and that the two results are independent of each other.

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The inspiration for the construction of the graph $\Gamma$ comes from the work of T. Laakso on $A_\infty$-weighted metrics on the plane [La], similar spaces were also considered in [NR] in the context of metric embeddings of finite graphs into Hilbert spaces. I would like to greatly thank Piotr Hajłasz from whom I learned about Laakso’s work and my advisor Guoliang Yu for his support and guidance. I am also very grateful to William Johnson for illuminating discussions in which the connection with Banach space geometry (among other things) became apparent.

1. Construction of the graph $\Gamma$

The main idea is to build a graph with a fractal-like structure in such way that the deformed geometry is reflected on the large scale. We will first inductively construct a sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$ of finite graphs which will be the building blocks of $\Gamma$ (compare [NR]).

Define $\Gamma_0$ to be a single edge of length 1. To construct $\Gamma_1$ take four copies of $\Gamma_0$ and denote by $p_i$ and $q_i$ the vertices of $i$-th copy, $i = 0, 1, 2, 3$. The graph $\Gamma_1$ is constructed by identifying $q_i$ with $p_{i+1}$ with $i \mod 4$. Equip $\Gamma_1$ with the path metric.

Similarly to construct $\Gamma_{n+1}$ take four copies of $\Gamma_n$, denote two vertices of valence $2^n$ that are distance $2^n$ away from each other in the $i$-th copy by $p_i$ and $q_i$, $i = 0, 1, 2, 3$ (there are two pairs of such vertices in $\Gamma_n$), and identify $q_i$ with $p_{i+1}$ with $i \mod 4$. Equip each $\Gamma_n$ with a path metric. For each $\Gamma_n$ a pair of vertices of valence $2^n$ and distance $2^n$ from each other will be called a pair of primary vertices. In every $\Gamma_n$ there are exactly two pairs of primary vertices. Note that $\text{diam} \, \Gamma_n = 2^n$.

For each $n \in \mathbb{N}$ denote by $p_n$ and $q_n$ a pair of primary vertices in $\Gamma_n$. Construct the graph $\Gamma$ by identifying $q_n$ with $p_{n+1}$ for every $n \in \mathbb{N}$ (see Fig. 2) and extending the metric in the obvious way. Note that one of the pairs of primary vertices in $\Gamma_n$ changes valence under the isometric embeddings $\Gamma_n \subseteq \Gamma_{n+1}$ and $\Gamma_n \subseteq \Gamma$, however we will refer to them without change as primary vertices of an isometric copy of $\Gamma_n$.\[\text{Figure 1. The graph } \Gamma_3\]
Actually, for our purposes it would be enough to consider just the set of all vertices of $\Gamma$, however we found the quote from the Introduction to [Gr]: 'Given a discrete metric space $\Gamma$, one can make it more palatable by adding some meat to $\Gamma$ in the form of edges and higher dimensional simplices with vertices in $\Gamma$, without changing the quasi-isometry type', quite applicable in this situation.

Recall that a metric space $X$ is called locally finite if there exists a discrete subset $\mathcal{N} \subseteq X$ and a constant $C > 0$ such that for every $x \in X$ there is a $y \in \mathcal{N}$ satisfying $d(x, y) \leq C$ and for every $y \in \mathcal{N}$ the number of elements in every ball around $y$ in $\mathcal{N}$ is finite. The graph $\Gamma$ is a locally finite metric space.

2. Round ball spaces and nonexistence of the embedding

In this section we prove that the graph $\Gamma$ constructed above does not admit a quasi-isometric embedding into any round ball metric space, which we define below.

Definition 1. Let $X, Y$ be metric spaces. We say that $f : X \to Y$ is a quasi-isometry if there are constants $L > 0$, $C \geq 0$ such that
\[
L^{-1}d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq Ld_X(x, y) + C
\]
for all $x, y \in X$.

Definition 2 ([La]). A metric space $X$ is called a round ball space if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that
\[
\text{diam} \left( B \left( x, \frac{1 + \delta_\varepsilon}{2}d(x, y) \right) \cap B \left( y, \frac{1 + \delta_\varepsilon}{2}d(x, y) \right) \right) < \varepsilon d(x, y)
\]
for all $x, y \in X$.

The round ball condition generalizes to metric spaces the classical notion of uniform convexity for Banach spaces. This is made precise in the following.
Proposition 3 ([La]). A Banach space is a round ball space if and only if it is uniformly convex.

Given a map \( f : X \to Y \) between metric spaces and points \( x, y \in X \), by \( L_{x,y} \) we denote the Lipschitz constant of \( f|_{\{x,y\}} \), i.e.

\[
L_{x,y} = \frac{d_Y(f(x), f(y))}{d_X(x, y)}.
\]

The following lemma can be extracted from [La].

Lemma 4. Let \( x_1, x_2, x_3, x_4 \in X \) satisfy

1. \( d_X(x_1, x_3) = d_X(x_2, x_4) = C \)
2. \( d_X(x_1, x_2) = d_X(x_1, x_3) = d_X(x_2, x_4) = d_X(x_3, x_4) = C/2 \)

for some constant \( C > 0 \). If \( f : X \to Y \) is a map to a round ball metric space \( Y \) and the restriction \( f|_{\{x_1,x_2,x_3,x_4\}} \) is biLipschitz with constant \( L \) then

\[
\max \{ L_{x_1,x_2}, L_{x_1,x_4}, L_{x_3,x_2}, L_{x_3,x_4} \} \geq (1 + \delta_{L-2})L_{x_1,x_3}.
\]

Proof. Condition (1) implies that

\[
d_Y(f(x_2), f(x_4)) \geq L^{-2}d_Y(f(x_1), f(x_3)).
\]

If \( B(f(x_1), R) \cap B(f(x_3), R) \) contains \( f(x_2) \) and \( f(x_4) \) then we must have \( R \geq \frac{1}{2}(1 + \delta_{L-2})d_Y(f(x_1), f(x_3)) \), by the round ball condition. Since this is the case when we take

\[
R = \max \{ d_Y(f(x_1), f(x_2)), d_Y(f(x_1), f(x_4)), d_Y(f(x_3), f(x_2)), d_Y(f(x_3), f(x_4)) \},
\]

the assertion follows. \( \square \)

Theorem 5. The space \( \Gamma \) does not admit a quasi-isometric embedding into any round ball space.

Proof. Assume that there exists a quasi-isometric embedding of the metric space \( \Gamma \) into a round ball space \( Y \). Observe that such an embedding is biLipschitz for large distances, i.e. there are constants \( L > 0 \) and \( S > 0 \) such that

\[
L^{-1}d_Y(x, y) \leq d_Y(f(x), f(y)) \leq Ld_Y(x, y)
\]

for all \( x, y \in \Gamma \) satisfying \( d_Y(x, y) \geq S \). Choose \( n \in \mathbb{N} \) such that \((1 + \delta_{L-2})^n L^{-1} > L \). In \( \Gamma \) choose points \( x_1^{(0)} \) and \( x_3^{(0)} \) that are images of primary vertices in an isometric copy of \( \Gamma_k \) for some \( k \in \mathbb{N} \) and satisfy \( d_Y(x_1^{(0)}, x_3^{(0)}) > 2^{n+1}S \). Denote by \( x_2^{(0)} \) and \( x_4^{(0)} \) the second pair of primary vertices in this copy of \( \Gamma_k \). The quadruple of points \( x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)} \) satisfies the hypothesis of lemma 4 and we get

\[
\max \{ L_{x_1^{(0)}, x_2^{(0)}}, L_{x_1^{(0)}, x_4^{(0)}}, L_{x_3^{(0)}, x_2^{(0)}}, L_{x_3^{(0)}, x_4^{(0)}} \} > (1 + \delta_{L-2})L_{x_1^{(0)}, x_3^{(0)}}
\]

Rename the pair for which the max on the left side is attained to \( x_1^{(1)} \) and \( x_3^{(1)} \). These two points constitute a pair of primary vertices of an isometric copy of \( \Gamma_{k-1} \). Denote by \( x_2^{(1)} \) and \( x_4^{(1)} \) the remaining pair of primary vertices in \( \Gamma_{k-1} \). Applying lemma 4 to the quadrupule \( x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)} \) and the renaming procedure together with equation (1) we get a pair of points \( x_1^{(2)} \) and \( x_3^{(2)} \) satisfying

\[
L_{x_1^{(2)}, x_3^{(2)}} \geq (1 + \delta_{L-2})L_{x_1^{(0)}, x_3^{(0)}}.
\]
The points $x_1^{(2)}$ and $x_3^{(2)}$ are again a pair of primary vertices of an isometric copy of $\Gamma_{k-2}$.

Continuing in this way after $n$ steps we will get a pair of points $x_1^{(n)}$, $x_3^{(n)}$ satisfying $d_\Gamma(x_1^{(n)}, x_3^{(n)}) \geq S$ and

$$L_{x_1^{(n)}, x_3^{(n)}} \geq (1 + \delta L^{-2}) n L_{x_1^{(0)}, x_3^{(0)}}.$$ 

By the choice of $n$ we get a contradiction to the Lipschitz condition for large distances. □

We remark that what is essential for the proof of Theorem 5 is the existence of an isometric copy of $\Gamma_n$ in $\Gamma$ for arbitrarily large $n$. Given the sequence $\{\Gamma_n\}_{n \in \mathbb{N}}$ one can thus create similar examples using constructions like e.g. disjoint union with an appropriate metric.

**Bounded geometry example.** We also want to indicate that it is now easy to modify the construction of $\Gamma$ to get a bounded geometry metric space with the property of non-embeddability. Simply consider for each $n \in \mathbb{N}$ the space $V_n$, the set of vertices of $\Gamma_n$ with the metric multiplied by $n$ and glue them together as in the construction of $\Gamma$. The resulting space however is not quasi-geodesic.

### 3. The geometry of $c_0$ and quasi-isometric embeddings

The fact that a 1-net in $c_0$ does not admit a biLipschitz for large distances embedding into any uniformly convex Banach space is well known in Banach space geometry, one can prove it using e.g. ultrapowers. We will show however how the methods from the previous section can be implemented in the $\ell_\infty^n$’s and $c_0$ and we will give an explicit geometric obstruction to uniform quasi-isometric embeddability (i.e. with constants $L$ and $C$ independent of $n$)¹ of $\ell_\infty^n$’s into uniformly convex Banach spaces.

Consider the embedding of $\ell_\infty^n$ into $\ell_\infty^{n+1}$ given by adding $(n+1)$-st coordinate 0 to each vector in $\ell_\infty^n$. For vectors $v, w \in \ell_\infty^n$ take their images under this embedding $\tilde{v}, \tilde{w} \in \ell_\infty^{n+1}$ and the vectors $x = \frac{\tilde{v} + \tilde{w}}{2} + (0, 0, ..., 0, -\frac{\|\tilde{v} - \tilde{w}\|}{2})$, $y = \frac{\tilde{v} + \tilde{w}}{2} + (0, 0, ..., 0, \frac{\|\tilde{v} - \tilde{w}\|}{2})$. It is easy to check that the quadruple of points $\tilde{v}, \tilde{w}, x, y$ satisfies the hypothesis of lemma 4, thus we can apply with no change the procedure from the proof of Theorem 5 and recover

**Proposition 6.** Let $X$ be a Banach space containing $\ell_\infty^n$’s quasi-isometrically uniformly. Then $X$ does not admit a quasi-isometric embedding into any uniformly convex Banach space.

The same argument gives a purely metric proof of the fact that the unit ball in $c_0$ does not admit a biLipschitz embedding into any uniformly convex Banach space, which is again obvious once we appeal to the linear structure of $c_0$.

¹We do not use here the standard notion of uniform containment of a sequence $X_n$ of finite dimensional spaces in a Banach space, namely that for every $\varepsilon > 0$ there is an isomorphic embedding of $X_n$ with distortion less than $1 + \varepsilon$ for every $n$, since it emphasizes the infinitesimal aspect of uniformity. Our definition is suitable for the purposes of large scale geometric behavior.
4. Some remarks on a paper of Bourgain

For the purposes of the coarse Novikov Conjecture [KY] one can consider coarse embeddings into superreflexive Banach spaces, since these are exactly the ones that admit an equivalent uniformly convex norm, due to a theorem of Enflo [En]. It might be thus interesting to confront the above observations with a paper of J. Bourgain in which a metric characterization of superreflexivity was given [Bo]. The necessary and sufficient condition for a Banach space to be superreflexive was shown to be biLipschitz uniform non-embeddability of a sequence of trees $T_j$ defined below (see [Bo] for a precise formulation).

Denote $\Omega_n = \{-1,1\}^n$, $T_n = \bigcup_{i \leq n} \Omega_i$, $T = \bigcup_{j=1}^{\infty} T_j$ and again add "some meat" to $T$ in the form of edges in the obvious way and denote the resulting space by $\mathcal{T}$. First note that quasi-isometric embeddability of $T$ (or equivalently $\mathcal{T}$) into a superreflexive Banach space implies the existence of a biLipschitz embedding of $T$ into such a space.

Lemma 7. Let $X$ be a discrete metric space and assume that $X$ embeds quasi-isometrically into a superreflexive Banach space. Then $X$ admits a biLipschitz embedding into a superreflexive Banach space.

Proof. Let $f : X \to E$ be the quasi-isometric embedding. Define $\tilde{f} : X \to E \oplus \ell_2(X)$ by the formula $\tilde{f}(x) = f(x) \oplus \delta_x$. $E \oplus \ell_2(X)$ is super-reflexive and it is easy to verify that $\tilde{f}$ is a biLipschitz embedding.

Thus $\mathcal{T}$ also does not admit a quasi-isometric embedding into any superreflexive Banach space, the argument in [Bo] is however of probabilistic nature, as mentioned earlier.

The geometry of the graph $\Gamma$ is intuitively the very opposite of the hyperbolic geometry on a tree. This is indicated already by topological invariants, but also the geometries of these spaces are very different on the large scale. The next proposition shows that Theorem 5 and Bourgain’s result cannot be deduced from each other.

Proposition 8. (1) $\mathcal{T}$ does not admit a coarse embedding into $\Gamma$

(2) $\Gamma$ does not admit a coarse embedding into $\mathcal{T}$.

Proof. To see (1) note the obvious fact that since the graph is infinite in just one direction, a coarse embedding from $\mathbb{R}$ to $\Gamma$ must map both infinite ends of $\mathbb{R}$ in the same direction in $\Gamma$ and distant points on the line have to cross close to the joints of $\Gamma$, i.e. the points in which $\Gamma_n$ is glued with $\Gamma_{n+1}$. Since $\mathbb{R}$ is isometrically embedded in the tree $\mathcal{T}$, the assertion follows.

Similarly for (2) take two infinite geodesic rays in $\Gamma$ and observe that they have to pass through the joints so that arbitrarily far on these geodesic rays some points are identified. In an isometric copy of $\Gamma_n \subseteq \Gamma$ the pair of primary vertices that are not joints is at distance $2^n$ apart. Thus a coarse embedding of $\Gamma$ into $\mathcal{T}$ from some point on cannot map these vertices into the same branch in $\mathcal{T}$. But that means that the joints as points on two geodesic rays will be mapped to points whose distance grows to infinity, which gives a contradiction.

Note that to get quasi-isometric non-embeddability of $\Gamma$ into $\mathcal{T}$ it is enough to use Theorem 5, since any $\mathbb{R}$-tree is a round ball space.

\[\text{2}\] Although it is not of great importance, it is convenient to take the direct sum with the $\ell_1$-norm
The intuition behind the facts presented above might be the following: the tree $T$ carries an $\ell_1$-geometry, while the geometry of $\Gamma$ resembles the one of $c_0$. Recall that $\ell_1$ and $c_0$ are the standard examples of non-uniformly convex Banach spaces, thus the presence of any of these geometries should be an obstruction to quasi-isometric embeddability into uniformly convex Banach spaces.

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