Secular dynamics of a coplanar, non-resonant planetary system under the general relativity and quadrupole moment perturbations

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ABSTRACT
We construct a secular theory of a coplanar system of \( N \) planets not involved in strong mean motion resonances, and which are far from collision zones. Besides the point-to-point Newtonian mutual interactions, we consider the general relativistic corrections to the gravitational potential of the star and the innermost planet, and also a modification of this potential by the quadrupole moment and tidal distortion of the star. We focus on hierarchical planetary systems. After averaging the model Hamiltonian with a simple algorithm making use of very basic properties of the Keplerian motion, we obtain analytical secular theory of high order in the semimajor axes ratio. A great precision of the analytic approximation is demonstrated with the numerical integrations of the equations of motion. A survey regarding model parameters (the masses, semimajor axes, spin rate of the star) reveals a rich and non-trivial dynamics of the secular system. Our study is focused on its equilibria. Such solutions predicted by the classic secular theory, which correspond to aligned (mode I) or anti-aligned (mode II) apsides, may be strongly affected by the gravitational corrections. The so-called true secular resonance, which is a new feature of the classic two-planet problem discovered by Michtchenko & Malhotra, may appear in other, different regions of the phase space of the generalized model. We found bifurcations of mode II from which emerge new, yet unknown in the literature, secularly unstable equilibria and a complex structure of the phase space. These equilibria may imply secularly unstable orbital configurations even for initially moderate eccentricities. The point mass gravity corrections can affect the long-term stability in the secular time-scale, which may directly depend on the age of the host star through its spin rate. We also analyse the secular dynamics of the \( \upsilon \) Andromedae system in the realm of the generalized model. Also in this case of the three-planet system, new secular equilibria may appear.

Key words: methods: analytical – methods: \( N \)-body simulations – celestial mechanics – planetary systems.

1 INTRODUCTION
The currently known sample of extrasolar planets\(^1\) comprises many multiple-planet configurations. Their architectures are diverse and, usually, very different from the Solar system configuration. Some of them consist of so-called hot Jupiters or hot Neptunes, with semimajor axes \( \sim 0.05 \) au and the orbital periods of a few days. The multiple-planet systems may also contain companions in relatively distant orbits. Such systems are non-resonant (or hierarchical).

The discovery of multiplyplanet systems and their unexpected orbital diversity initiated interest in their secular evolution. Generally, the analysis of secular interactions, including tidal effects, general relativity (GR) and quadrupole moment (QM) corrections to the Newtonian gravitation (NG) of point masses, follows the theory of compact multiple stellar systems (Mardling & Lin 2002; Nagasawa & Lin 2005). The secular evolution of planetary systems has been already intensively studied in this context by many authors. For instance, Adams & Laughlin (2006b) consider the GR corrections in the sample of detected extrasolar systems. They have shown that the GR interactions can be particularly important for the long-term dynamics of short-period planets. Also Adams & Laughlin (2006a) and Mardling (2007) study the evolution of such objects taking into account the tidal circularization of their orbits, and secular excitation of the eccentricity of the innermost planet by more distant companions. These secular effects imply constraints on the orbital elements of the innermost orbits, including yet undetected Earthlike planets, which are expected to be found in such systems. The quadrupole moment and relativistic effects may detectably affect the light curves of stars hosting transiting planets on a time-scale

\(^1\)See http://exoplanet.eu, http://exoplanets.org, http://exoplanets.eu.
of a few years (Miralda-Escudé 2002; Agol et al. 2005; Winn et al. 2005). In general, the interplay of apparently subtle perturbations with the point mass Newtonian gravity depends on many physical and orbital parameters, and it may lead to non-trivial and interesting dynamical phenomena. Still, the problem is of particular importance for studies of the long-term stability of the Solar system (see very recent works by Benítez & Gallardo 2008; Laskar 2008).

Because a fully general study of such effects would be a very difficult task, we introduce some simplification to the planetary model. We skip tidal effects caused by dissipative interactions of extended star and planetary bodies, in particular the tidal friction (TF) damping the eccentricity and the tidal distortion (TD) that modifies planetary figures, and could influence their gravitational interaction with the parent star. The tidal effects are typically much smaller than the leading Newtonian, relativistic and quadrupole moment contributions (Mardling 2007). Also their time-scale is usually much longer than of the most significant conservative effects. In this paper, we focus on the secular dynamics of planetary systems over an intermediate time-scale relevant to the conservative perturbations. Our goal is to obtain a qualitative picture of the secular system that may be useful as the first order approximation to a more general model of the long-term planetary dynamics.

Moreover, we show in this paper that even if we skip the non-conservative tidal effects then the GR and/or QM corrections to the point mass Newtonian gravity can be much more significant for the secular dynamics than the mutual NG interactions alone. These effects may change qualitatively the view of the dynamics of planetary systems as predicted by the Laplace–Lagrange theory (Murray & Dermott 2000) or its recent versions (e.g. Lee & Peale 2003; Michtchenko & Malhotra 2004; Libert & Henrard 2005; Michtchenko, Ferraz-Mello & Beaué 2006; Veras & Armitage 2007; Migaszewski & Goździewski 2008a).

The model without dissipative effects may be investigated with the help of conservative Hamiltonian theory. Assuming that planetary orbits are well separated and the system is far from mean motion resonances and collision zones, we can apply the averaging proposition (Arnold, Kozlov & Neishtadt 1993) to derive the long-term evolution of their orbital elements. This approach can be classified among secular theories having a long history in the context of the Solar system (Brouwer & Clemence 1961; Murray & Dermott 2000).

This paper is a step towards a generalization of the secular model of a coplanar two-planet system by Michtchenko & Malhotra (2004) and its analytical version applied to an N-planet system in (Migaszewski & Goździewski 2008a). To explore the phase space globally without restrictions on eccentricities, we simplify the equations of motion through the averaging of perturbations to the Keplerian motion, with the help of the seminumerical method proposed by Michtchenko & Malhotra (2004). To obtain analytical results, we also use a very simple averaging algorithm that can be applied to perturbations dependent on the mutual distance of interacting bodies (Migaszewski & Goździewski 2008a). These works demonstrate that the secular evolution can be precisely described in wide ranges of the orbital parameters, including eccentricities up to 0.8–0.9, for well separated (hierarchical) orbits with a small ratio of the semimajor axes, \( \alpha \sim 0.1 \). Here, the secular NG theory is very helpful to derive the more general model including the GR and QM effects. Basically, without the averaging, the only possibility of investigating the long-term secular dynamics relies on numerical solutions of the equations of motion (Mardling & Lin 2002). However, due to extremely different time-scales which are related as days (the orbital periods of inner planets) to \( 10^3\sim10^6 \) yr of the secular orbital evolution, the numerical integrations are of very limited use when we want to investigate large volumes of initial conditions rather than isolated orbits. In such a case, the analytical theory can be very helpful to get a deep insight into the secular dynamics. Moreover, when necessary, particular solutions can be studied in detail with the help of the direct numerical integrations.

The plan of this paper is the following. In Section 2 we formulate the generalized model of a coplanar, N-planet system, accounting for the relativistic and quadrupole moment corrections to the NG-perturbed motion of the innermost planet. To make the paper self-consistent, we average out the perturbations to the Keplerian motion with the help of the averaging algorithm described in Migaszewski & Goździewski (2008a). In Section 3 we apply the analytical and numerical tools to study the influence of the GR and QM corrections on the secular evolution. In particular, we recall the concept of the so-called representative plane of initial conditions (Michtchenko & Malhotra 2004). We focus on the search for stationary solutions in the averaged and reduced systems (periodic orbits in the full systems), and we investigate their stability and bifurcations with the help of phase diagrams. In this section we also derive interesting conclusions on the stability of the unaveraged systems. In Section 4, we study the phase space of two-planet systems in wide ranges of parameters governing their orbital configurations (masses, semimajor axes ratios, eccentricities) and physical parameters (flattening of the star). To illustrate the application of the secular theory to multi-planet configurations, we consider the three-planet \( \nu \) And system, and we investigate the QM (stellar rotation) influence on its secular orbital evolution.

## 2 Generalized Model of N-Planet System

The dynamics of the planetary system can be modelled by the Hamiltonian function written with respect to canonical Poincaré variables (see e.g. Poincaré 1897; Laskar & Robutel 1995), and expressed by a sum of two terms, \( \mathcal{H} = \mathcal{H}_{\text{kepl}} + \mathcal{H}_{\text{pert}} \), where

\[
\mathcal{H}_{\text{kepl}} = \sum_{i=1}^{N} \left( \frac{p_{i}^2}{2\mu_i} - \frac{\mu_i \beta_i}{\tau_i} \right)
\]

stands for the integrable part comprising the direct sum of the relative, Keplerian motions of \( N \) planets and the host star. Here, the dominant point mass of the star is \( m_0 \), and \( m_i \ll m_0, i = 1, \ldots, N \) are the point masses of the \( N \) planets. For each planet–star pair we define the mass parameter \( \mu_i = k^2 (m_0 + m_i) \) where \( k \) is the Gauss gravitational constant, and \( \beta_i = (1/m_i + 1/m_0)^{-1} \) are the so-called reduced masses. We consider the perturbing Hamiltonian \( \mathcal{H}_{\text{pert}} \) as a sum of three terms,

\[
\mathcal{H}_{\text{pert}} = \mathcal{H}_{\text{NG}} + \mathcal{H}_{\text{GR}} + \mathcal{H}_{\text{QM}},
\]

where \( \mathcal{H}_{\text{NG}} \) is for the mutual point mass interactions between planets, \( \mathcal{H}_{\text{GR}} \) is for the general (post-Newtonian) relativistic corrections to the Newtonian gravity, and \( \mathcal{H}_{\text{QM}} \) takes into account the dynamical flattening and tidal distortion of the parent star.

We consider the secular effects of \( \mathcal{H}_{\text{NG}} \), which can be expressed as follows:

\[
\mathcal{H}_{\text{NG}} = \sum_{i=1}^{N} \sum_{j>i}^{N} \left( -\frac{k^2 m_i m_j}{\Delta x_{ij}} + \frac{p_i \cdot p_j}{m_0} \right).
\]

where \( \Delta x_{ij} \) are the position vectors of planets relative to the star, \( p_i \) are for their conjugate momenta relative to the barycentre of the
whole \((N + 1)\)-body system, \(\Delta_{ij} = \|r_i - r_j\|\) denote the relative distance between planets \(i\) and \(j\).

For hierarchical planetary systems (weakly interacting binaries), the secular time-scale of Newtonian point mass interactions may be as short as \(10^7\) yr, up to Myr. In general, these interactions cause slow circulation of the apsidal lines. The GR correction to the NG potential of the star–planet system also leads to the circulation of pericentres. Moreover, the time-scale of this effect may be comparable to that forced by mutual Newtonian interactions between planets. In such a situation, one should necessarily include the relativistic corrections to the model of motion.

2.1 General relativistic corrections

The GR correction will be applied only to the innermost planet, so we skip the direct GR perturbations on the motion of the more distant companions, as well as the mutual GR interactions caused by planetary masses. Usually, the GR corrections to the Newtonian potential are expressed in terms of the PPN formalism (e.g. Kidder 1995). Alternatively, we found a very clear paper by Richardson & Kelly (1988) conveniently providing the explicit Hamiltonian of the two-body problem with the GR term. Following these authors, \(\mathcal{H}_{GR} \equiv \mathcal{H}_{GR}/\beta\) (i.e. \(\mathcal{H}_{GR}\) rescaled by the reduced mass) may be written as follows:

\[
\mathcal{H}_{GR} = \gamma_{1} P_{r}^{4} + \gamma_{2} \frac{P_{r}^{2}}{r} + \gamma_{3} \frac{(P_{r})^2}{r^2} + \gamma_{4} \frac{1}{r^2},
\]

where \(\gamma_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}\) are coefficients defined through

\[
\gamma_{1} = \frac{(1 - 3\nu)}{8c^2}, \quad \gamma_{2} = -\frac{\mu(3 + \nu)}{2c^2}, \quad \gamma_{3} = \frac{\mu^2}{2c^2}, \quad \gamma_{4} = -\frac{\mu\nu}{2c^2},
\]

and \(c\) is the velocity of light in vacuum, \(\mu = k^2(m_0 + m_1), v \equiv m_0/m_1/(m_0 + m_1)^2\), \(r\) is the astrocentric distance and \(P\) is the astrocentric momentum of the innermost planet (normalized through the reduced mass):

\[
P = v + \frac{1}{c^2} \left[ 4\gamma_{1}(v \cdot v)v + 2\gamma_{2} v + 2\gamma_{4} (r \cdot v) r \right],
\]

where \(v \equiv \dot{r}\) stands for the astrocentric velocity of the innermost planet (the relativistic corrections from other planetary bodies in the system are skipped). Hence, in the relativistic Hamiltonian, we put \(P = v\) with the accuracy of \(O(c^{-2})\) and then the Hamiltonian is conserved up to the order of \(O(c^{-4})\).

2.2 Rotational and tidal distortions of the star

Fast rotating stars are significantly flattened and that in turn may lead to important deviations of the NG potential. In the absence of a close planet and non-radial pulsations, the star has rotational symmetry, regarding its mass density and shape. The gravitational potential of such a body can be expanded in harmonic series expressed in terms of the Stokes coefficients. The well-known property of this expansion is that the gravitational potential of rotationally symmetric bodies retains only terms related to the so-called zonal harmonics:

\[
\mathcal{H}_{\text{spin}} = \mu \frac{R_{\odot}}{r} \sum_{l=2}^{\infty} J_{l} \left( \frac{R_{\odot}}{r} \right)^{l} P_{l}(\sin \phi),
\]

where \(R_{\odot}\) stands for the characteristic radius of a sphere encompassing the body, and it can be fixed as the equatorial radius of the star, \(P_{l}(\sin \phi)\) is the Legendre polynomial of the \(l\)th order, \(\phi\) is the astrocentric latitude and \(J_{l}\) are non-dimensional Stokes coefficients, \(l \geq 1\). It can be shown that for rotationally symmetric objects, \(J_{l} = 0\) for \(l\) odd. Basically, \(J_{l}\) and the leading term with \(J_{2}\), in particular, can be determined numerically with the help of the theory of stellar interiors combined with a model of rotation and helioseismic data (Pijpers 1998; Godier & Rozelot 1999). To calculate these coefficients, one must know the mass density as well as the hydrostatic figure of the star. There are attempts to estimate \(J_{2}\) for stars hosting planets. For instance, Iorio (2006) determined the quadrupole moment of HD 209458 (Charbonneau et al. 2000) as \(J_{2} \approx 3.5 \times 10^{-3}\), however with a large error of \(\sim 10^{-3}\), making the result not very credible. In general, the estimates of \(J_{2}\) are very uncertain even for the well explored Sun. For this slowly rotating dwarf (with rotational period of \(\sim 30\) d), \(J_{2} \sim 10^{-3}\) with an upper bound of \(10^{-4}\). Moreover, in the literature we found approximations of the current \(J_{2}\) of the Sun in the range up to \(10^{-3}\).

Due to indefiniteness of higher order zonal harmonics, we consider the dynamical effects of the leading \(J_{2}\) term only. For a coplanar system, in which planets move around the star in its equatorial plane, the first non-zero term in the harmonic series, equation (6), has the form

\[
\mathcal{H}_{\text{spin}} \approx -\frac{1}{2} \mu \beta J_{2} R_{\odot}^{2} \frac{1}{r^{3}},
\]

The above formulae may be conveniently expressed (Mardling & Lin 2002) in terms of the stellar spin frequency:

\[
\mathcal{H}_{\text{spin}} \approx -\frac{1}{6} \beta R_{\odot}^{5} (1 + m_{1}/m_{0}) k_{L} \Omega^{2} \frac{1}{r^{3}},
\]

where \(k_{L}\) is the tidal Love number (Murray & Dermott 2000) which can be related to \(J_{2}\) through

\[
k_{L} = \frac{3 J_{2}}{q}, \quad q = \Omega^{2} R_{\odot}^{6} k_{L}^{2} m_{0},
\]

where \(q\) is the ratio of centrifugal acceleration and gravitation at the surface of the star and \(\Omega\) is the spin rate. Hence, we have

\[
J_{2} = \frac{1}{3} k_{L} \frac{m_{2} \Omega^{2}}{k_{L}^{2} m_{0}}.
\]

In this paper we adopt the standard value of \(k_{L} = 0.02\) (Nagasawa & Lin 2005). Obviously, the dynamical flattening is more significant for fast rotating stars. Before the zero-age main-sequence (ZAMS) stage, the rotational periods may be as low as a few days, down to 1 d for young (~100 Myr) Sun-like stars. The rotational periods of ~40 d are typical for 8-Gyr-old, evolved objects. According to the scaling rule of \(J_{2} \sim \Omega^{2}\), the zonal harmonics of young objects may be as large as \(10^{-4}\). Because the rotational period may change by two orders of magnitude during the lifetime of the star, also the quadrupole moment may change by a few orders of magnitude. Hence, flattening caused by fast rotation not only can affect significantly the orbital acceleration which can compete with the GR and mutual NG contributions, but it may also introduce a dependence of the system dynamics on the age of the parent star (Nagasawa & Lin 2005, see also Sections 5 and 6).

We also consider a contribution of the tidal bulge (TB) caused by the presence of a point mass innermost planet, assuming that only the extended star is distorted due to the tidal interactions. Then the Hamiltonian is corrected by the following term (Mardling & Lin 2002):

\[
\mathcal{H}_{TB} = -\beta R_{\odot}^{5} k_{L}^{2} m_{1} (1 + m_{1}/m_{0}) k_{L} \frac{1}{r^{3}}.
\]

Both gravitational corrections are then \(\mathcal{H}_{QM} = \mathcal{H}_{\text{spin}} + \mathcal{H}_{TB}\).

We account for direct dynamical effects related to \(\mathcal{H}_{QM}\) only in the motion of the innermost planet although these perturbations can
be also quite easily included for the remaining star–planet pairs. In the realm of the secular theory of non-resonant, hierarchical systems, we assume that other companions are much more distant from the star. For the semimajor axes ratio $\sim 0.1$, the GR + QM perturbations acting on such outer companions are by orders of magnitude smaller than for the innermost planet. However, the dynamics of the innermost body influences indirectly the secular dynamics of the whole system. This will be demonstrated in this paper in the cases of two- and three-planet configurations. We also underline that with the simplified model of interactions, we can study the dynamics of quite compact star–planet configurations because they can be parametrized, in general, by individual planetary masses, semimajor axes and physical parameters of the star.

### 2.3 The secular model of a non-resonant planetary system

To apply the canonical perturbation theory, we first transform $\mathcal{H}$ to the following form:

$$\mathcal{H}(I, \phi) = \mathcal{H}_{\text{kepl}}(I) + \mathcal{H}_{\text{pert}}(I, \phi),$$

where $(I, \phi)$ stand for the action–angle variables, and $\mathcal{H}_{\text{pert}}(I, \phi) \sim \epsilon \mathcal{H}_{\text{kepl}}(I)$, where $\epsilon \ll 1$ is a small parameter. In this paper, we apply the well-known approach to analyse the equations of motion induced by Hamiltonian, equation (12), that relies on the averaging proposition (see e.g. Arnold et al. 1993). By averaging the perturbations with respect to the fast angles (the mean longitudes or the mean anomalies) over their periods, we obtain the secular Hamiltonian which does not depend on these fast angles. Simultaneously, the conjugate momenta to the fast angles become integrals of the secular problem. In the planetary system with a dominant stellar mass, the orbits (apsidal lines) also slowly rotate due to mutual interactions, hence the longitudes of periastron and the longitudes of node become slow angles. Assuming that no strong mean motion resonances are present, and the system is far enough from collisions, the averaging makes it possible to reduce the number of degrees of freedom, and to obtain qualitative information on the long-term changes of the slowly varying orbital elements (i.e. on the slow angles and their conjugate momenta).

The transformation of the Hamiltonian to the required form (equation 12) may be accomplished by expressing this Hamiltonian with respect to the (modified) Delaunay canonical elements. These variables can be related to the Keplerian canonical elements (Murray & Dermott 2000). Actually, we use the following set of canonical action–angle variables which can be obtained after an appropriate canonical transformation of the Delaunay elements:

$$I_i \equiv M_i, \quad L_i = \beta_i \sqrt{|\mu_i|}, \quad g_i = \sigma_i, \quad G_i = L_i \sqrt{1 - e_i^2}, \quad h_i = \Omega_i, \quad H_i = G_i (\cos I_i - 1),$$

where $M_i$ are the mean anomalies, $\sigma_i$ stand for canonical semimajor axes, $e_i$ are the eccentricities, $I_i$ denote inclinations, $\sigma_i$ are the longitudes of pericentre and $\Omega_i$ denote the longitudes of ascending node. The choice of $\sigma_i$ instead of $\omega_i$ is important for further applications, because $\Omega_i$ are undefined (and irrelevant) for the dynamics of the coplanar system. We note that the geometrical, canonical elements $(a_i, e_i, I_i, \sigma_i, \Omega_i)$ may be derived through the formal transformation between classic (astrocentric) Keplerian elements and the relative Cartesian coordinates (e.g. Ferraz-Mello, Michtchenko & Beaugé 2006; Morbidelli 2002), with appropriate rescaling of the astrocentric velocities. In the settings adopted here, the Cartesian coordinates are understood as Poincaré coordinates, i.e. astrocentric positions of planets, and canonical momenta taken relative to the barycentre of the system.

The $N$-planet Hamiltonian expressed in terms of the modified Delaunay variables (13) has the form of

$$\mathcal{H} = - \sum_{i=1}^{N} \frac{\mu_i^2 \beta_i^4}{2 L_i^2} + \mathcal{H}_{\text{pert}} (L_i, I_i, G_i, g_i, H_i, h_i).$$

In this Hamiltonian, $I_i$ play the role of the fast angles. In the absence of strong mean motion resonances, these angles can be eliminated by the following averaging formulæ:

$$\mathcal{H}_{\text{sec}} = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{H}_{\text{sec}} dM_1 \cdots dM_N + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_{\text{GR}} dM_1 + \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_{\text{QM}} dM_1.$$  

Hence, we can rewrite the secular Hamiltonian in the symbolic form of

$$\mathcal{H}_{\text{sec}} = \langle \mathcal{H}_{\text{sec}} \rangle + \langle \mathcal{H}_{\text{GR}} \rangle + \langle \mathcal{H}_{\text{QM}} \rangle.$$  

Now, we try to average out each component of this Hamiltonian expressed with respect to the fast angles (the mean anomalies).

### 2.4 Averaging Newtonian point-to-point interactions

A simple averaging of the Hamiltonian of the classic planetary $(N + 1)$-body model is described in our previous paper (Migaszewski & Goździewski 2008a). The algorithm makes use of the very basic properties of the Keplerian motion and it relies on appropriate change of integration variables in equation (14). It may be also applied to perturbations expressed through powers of the relative distance.

To recall the main result, the secular Hamiltonian of the $N$-planet system can be described as a sum of two-body Hamiltonians evaluated over all pairs of planets:

$$\langle \mathcal{H}_{\text{sec}} \rangle = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \langle \mathcal{H}_{\text{sec}}^{i,j} \rangle,$$  

The direct part of the disturbing Hamiltonian reads as follows:

$$\langle \mathcal{H}_{\text{sec}}^{i,j} \rangle = - \frac{k^2 m_i m_j}{a_{i,j}} \times \left[ 1 + \sqrt{1 - e_{i,j}^2} \sum_{t=2}^{\infty} \left( \frac{t}{1 - e_{i,j}} \right)^{t} \mathcal{R}_{i,j}^{t}(e_i, e_j, \Delta \sigma_{i,j}) \right].$$  

The explicit formulæ for functions $\mathcal{R}_{i,j}^{t}(e_i, e_j, \Delta \sigma_{i,j})$ are given in Migaszewski & Goździewski (2008a). It is well known that the indirect part averages out to a constant and it does not contribute to the secular dynamics (Brouwer & Clemence 1961).

### 2.5 Averaging the PPN relativistic potential

Making use of the formulæ for the relativistic PPN Hamiltonian by Richardson & Kelly (1988), we write down the mean relativistic potential as follows:

$$\langle \mathcal{H}_{\text{GR}} \rangle = \gamma_i (v^i) +\gamma_r \left( \frac{v^2}{r} \right) + \gamma_r \left( \frac{(r \cdot r)^2}{r^4} \right) + \gamma_r \left( \frac{1}{r^2} \right).$$  

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with the accuracy of $O(c^{-2})$. We should average out each component of this Hamiltonian over the mean anomalies. It appears to be possible with the algorithm in Migaszewski & Goździewski (2008a). To average out the whole PPN Hamiltonian, we must calculate a few integrals written in the general form of
\[ \langle \mathcal{X} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{X} \, dM = \frac{1}{2\pi} \int_0^{2\pi} [\mathcal{X} \, J] \, df, \]
where $J$ is a scaling function defined by
\[ J \equiv \frac{dM}{df} = \frac{(1 - e^2)^{1/2}}{(1 + e \cos f)^3} \]
and $f$ is the true anomaly of the innermost planet (note that we skip index ‘1’ of that planet). Components of the mean Hamiltonian can be written explicitly as follows:
\[ \langle v^4 \rangle = \frac{n^2 a^4}{\sqrt{1 - e^2}} \left[ 1 + 3 \frac{2}{e^2} + e^2 F_1 \right], \]
\[ \langle \frac{v^2}{r} \rangle = \frac{n^2 a}{\sqrt{1 - e^2}} \left[ 1 + e^2 F_2 \right], \]
\[ \langle \frac{(r \cdot v)^2}{r^3} \rangle = \frac{n^2 a}{\sqrt{1 - e^2}} e^2 F_2, \]
\[ \langle \frac{1}{r^2} \rangle = \frac{1}{a^2 \sqrt{1 - e^2}}. \]

Functions $F_1$ and $F_2$ are given through
\[ F_1 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2 f}{(1 + e \cos f)^3} \, df, \]
\[ F_2 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2 f}{1 + e \cos f} \, df. \]

These integrals can be calculated in the closed form
\[ F_1 = \frac{3(2 - e^2 - 2\sqrt{1 - e^2})}{2e^2}, \quad F_2 = \frac{1 - \sqrt{1 - e^2}}{e^{-2}}. \]

Finally, the secular relativistic potential is the following:
\[ \langle H_{GR} \rangle_R = -\frac{3\mu^2}{c^2 a^2 \sqrt{1 - e^2}} + \frac{\mu^2 (15 - \nu)}{8a^2 c^2}. \]

Because all secular Hamiltonian corrections we account for in this paper (see below) do not depend on the mean anomalies, $L_{1,2}$ are constants of motion. It also means that the second term in equation (28) reduces to the constant and does not contribute to the long-term dynamics. Writing down the secular Hamiltonian with respect to the action–angle variables, we have
\[ \langle H_{GR} \rangle_R = -\frac{3\mu^2 \beta^2}{c^2 L^3 G} + \text{const.} \]

Here, the action variables $(L, G)$ are defined with usual formulae known in the Keplerian motion theory. We recall that in the secular PPN Hamiltonian, only the star–innermost planet interactions are considered. The perturbation has been also scaled by the reduced mass. To properly add the GR star–inner planet interactions to the Hamiltonian of $N$ planets, we should account for the mass factor. Because the canonical Delaunay elements of the $N$-planet system are defined through equation (13), hence, to obtain the proper scaling, we should multiply the two-body Hamiltonian, equation (29), through $\beta \equiv 1/(1/m_0 + 1/m_1)$. Finally, we obtain the following secular Hamiltonian related to the GR correction:
\[ \langle H_{GR} \rangle_R = -\frac{3\mu^2 \beta^2}{c^2 L^3 G} + \text{const.} \]

In this Hamiltonian all slow angles are cyclic; hence $L$ and $G$ would be integrals of motion in the absence of mutual planetary inter-actions. However, the element $G$ is no longer constant when we consider the NG contributions. The evolution of canonical angle $\sigma$ reads as follows:
\[ \sigma_{GR} = \frac{d}{d\tau} \langle H_{GR} \rangle_R = \frac{3\mu^2 \beta^2}{c^2 L^3 G^2} = \frac{3\mu^2}{c^2 a^2 \sqrt{1 - e^2}}. \]

This is the well-known formula describing the relativistic advance of pericentre. We derive it here to keep the paper self-consistent. In the above formulae, we skip the symbol $\langle \cdots \rangle$ of the mean, which basically should encompass the canonical angle $\sigma$ and the conjugate actions $(L, G)$ as well as the symbols of orbital elements $(a, e)$. However, we should keep in mind that after calculating the average over the fast angles, these actions/elements have the sense of the mean actions/elements.

### 2.6 The effect of quadrupole moment of the star

To average out the quadrupole moment of the star, we have to calculate integrals from the astrocentric distance of the planet taken in the power of ‘$-3$’ for the spin distortion, and the power of ‘$-6$’ for the tidal distortion. To calculate these averages, we express the astrocentric distance $r$ of the inner planet with respect to the true anomaly and we replace the integration variables $dM = J \, df$. For $l > 1$, we find
\[ \langle \frac{1}{r^l} \rangle = \frac{(1 - e^2)^{l/2}}{a^l (1 - e^2)^{l-1}} \sum_{i=0}^{l-2} \binom{l-2}{i} e^i \langle \cos^i f \rangle, \]
where, for $s$ even, the average of $\langle \cos^s f \rangle$ over the mean anomaly reads as follows:
\[ \langle \cos^s f \rangle = \frac{2^{s/2} \pi}{\Gamma(1/2 - s/2) \Gamma(s + 1)} \]
while for $s$ odd, the average over the mean anomaly $\langle \cos f \rangle = 0$. The average of the leading term in $\langle H_{spin} \rangle$ has the form
\[ \langle H_{spin} \rangle \approx -\frac{\beta k_1 (1 + m_1/m_0) R_0^2 \Omega^2}{6a^3 (1 - e^2)^{3/2}}. \]

Having this secular Hamiltonian, we can calculate the apsidal frequency forced by the spin-induced QM of the star:
\[ \sigma_{spin} = \frac{\partial \langle H_{spin} \rangle}{\partial G} = \frac{\beta k_1 (1 + m_1/m_0) \mu R_0^2 \Omega^2}{2G^4 L^3}. \]

Similarly, we calculate the leading term in $\langle H_{TB} \rangle$:
\[ \langle H_{TB} \rangle \approx -\frac{\beta k_1 R_0 f^2 (3e^2 + 3e + 1) k^2 m_1 (1 + m_1/m_0)}{a^4 (1 - e^2)^{3/2}} \]
and the apsidal frequency due to this correction:
\[ \sigma_{TB} = \frac{\partial \langle H_{TB} \rangle}{\partial G} \approx -\frac{15\beta^{13} k_1 R_0^4 (G^4 - 14L^2 G^2 + 21L^4) m_1 \mu}{8G^8 L^4 m_0}. \]

The tidal bulge induced by a close companion may be very important in some configurations. In fact, the TB effect may be even dominant over the spin-induced quadrupole moment perturbations. For instance, for $\nu$ And b it is $\sim 20$ times as big (see Mandling & Lin 2002; Nagasawa & Lin 2005, for details).
Figure 1. The long-term, secular evolution of eccentricities of the innermost planets of known non-resonant extrasolar systems. Red curves are for the point mass planets, interacting mutually through Newtonian forces. The blue curves are for generalized model of motion, including additional effects (general relativity and the quadrupole moment of the star). For all tested planetary systems, we fix $T_{\text{rot}} = 30$ d, $k_L = 0.02$. Each panel is labelled by the names of the parent stars together with the number of planets written in brackets.

3 TOOLS TO ANALYSE THE SECULAR DYNAMICS

To obtain an overview of the dynamical influence of the GR and QM corrections on the point mass NG dynamics, we compare the secular evolution of the eccentricity of the innermost planet in the sample of detected multiple extrasolar systems. In this experiment, we basically follow Adams & Laughlin (2006b), however, solving the averaged equations of motion by means of the numerical integrator. Moreover, we carry out this test to illustrate the different and rich behaviours of multiplanet systems in the detected sample; a systematic analysis is described in subsequent sections. The results are shown in Fig. 1. Each panel in this figure is labelled with the parent star name. The orbital elements of the selected systems, and the parameters of their parent stars come from the Jean Schneider Encyclopedia of Extrasolar Planets.\(^2\) In this experiment, we fixed rotational period of the star, $T_{\text{rot}} = 30$ d, $k_L = 0.02$ (hence $J_2 \sim 10^{-7}$ to $10^{-6}$).

In some cases, the differences between predictions of the generalized theory and by the classical model are significant, in some other cases both theories give practically the same outcome. Looking at the elements of the examined systems, we may conclude that the corrections become very important for systems with the innermost planet close to the star, and with other bodies relatively distant.

In fact, it is already well known (e.g. Adams & Laughlin 2006b) that the additional effects are important when the strength and characteristic time-scale of perturbations stemming from the GR and QM become comparable with the secular time-scale of mutual NG interactions. Then the GR and QM regarded as perturbing effects may induce a significant contribution to the secular evolution and the interplay of these effects with the NG interactions may lead to very complex and rich dynamics. As we will see below, this condition is particularly well satisfied for strictly hierarchical systems, e.g. HD 217107, HD 38529 (Fischer et al. 2001), HD 190360 (Vogt et al. 2005), HD 11964 (Butler et al. 2006), \(\nu\) Andromedae (Butler et al. 1999). The secular time-scales can be also comparable when the planetary masses are relatively small because the NG-induced apsidal frequency decreases proportionally to the products of these

\(^2\)http://exoplanet.eu.
masse while, in the first approximation, the GR and spin-induced apsidal motion of the innermost orbit does not depend directly on the planetary mass (see equations 31 and 35).

Still, the results illustrated in Fig. 1 have limited significance for the study of the global dynamics. Drawing one-dimensional time–orbital element plots, we can analyse the dynamical evolution only for a few isolated initial conditions. This can be a serious drawback, if we recall that the initial conditions of the discovered systems are still known with large uncertainties. Some critically important parameters governing the secular evolution, like the masses, nodal lines and inclinations of orbits are poorly constrained by the observations or undetermined at all. There is also a technical problem: the direct integrations over the secular time-scale are CPU intensive. Yet looking at isolated configurations, we obtain only a local view of the dynamics. Instead, following the methodology of Poincaré, we should try to understand globally the perturbing effects and the resulting dynamics. We should investigate the whole families of solutions rather than a few isolated phase-space trajectories. In such a case, the application of secular analytical theories becomes critically important.

3.1 Representative plane of initial conditions and equilibria

The simplest class of solutions, which can be studied most effectively, are the equilibria (or stationary solutions). In the multi-parameter dynamical systems, their positions (coordinates), stability and bifurcations provide information on the general structure of the phase space. Hence, we focus on the stationary solutions emerging in the secular model of a two-planet system with the GR and QM corrections.

After the averaging of \( \mathcal{H} \) over the mean anomalies (i.e. over the orbital periods), the secular Hamiltonian (\( \mathcal{H}_{\text{sec}} \)) does not depend on \( I_i = M_i \) anymore. Therefore, as we mentioned already, the conjugate actions \( L_i \) become constants of motion (hence, the mean semimajor axes are also constant). Moreover, in the coplanar problem the longitudes of nodes are undefined and irrelevant for the dynamics. The secular energy of the two-planet system does not depend on individual longitudes of pericentres \( \sigma_i \), but only on their difference \( \Delta \sigma = \sigma_1 - \sigma_2 \) (Brouwer & Clemence 1961). These basic facts can be expressed with the help of an appropriate canonical transformation (Michtchenko & Malhotra 2004):

\[
\Delta \sigma = \sigma_1 - \sigma_2, \quad G_1, \tag{38}
\]

\[
\sigma_2, \quad C = G_1 + G_2. \tag{39}
\]

Because \( \sigma_2 \) is the cyclic angle, the conjugate momentum equal to the total angular momentum \( C \) is conserved. For a fixed angular momentum as a constant parameter, the phase space of the reduced system becomes two dimensional, and the system is integrable. We can now choose \( \Delta \sigma \) as the canonical angle and \( G_1 \) as the conjugate momentum. Alternatively, the role of the momentum may be attributed to \( e_1 \). Obviously, \( G_2 \) (or \( e_2 \)) becomes a dependent variable (through the \( C \) integral).

To study the dynamics of the reduced secular system in a global manner, we follow a concept of the so-called representative plane of initial conditions introduced by Michtchenko & Malhotra (2004). The representative plane (\( S \)) comprises points in the phase space which lie on a specific plane crossing all phase trajectories of the system. In the cited paper, it has been shown that a good choice of the \( S \) plane flows from the condition of vanishing derivatives of the secular Hamiltonian over \( \Delta \sigma \). It is equivalent to the symmetry of secular interactions with respect to fixed apsidal line of a selected orbit and it means that time derivatives of the conjugate momenta (which can be expressed by eccentricities) must vanish. Indeed, in accord with the law of conservation of the total angular momentum, the eccentricities must reach at the same time maximal and minimal values along the phase trajectories, hence \( e_1 = 0 \), \( i = 1, 2 \). Regarding the classic problem with Newtonian point-to-point interactions, this condition is satisfied when \( \Delta \sigma = 0 \) or \( \Delta \sigma = \pi \) (Michtchenko & Malhotra 2004) (see also Migaszewski & Goździewski 2008a).

If we consider the generalized problem of a two-planet system then the concept of the representative plane is still valid. The small perturbations which we introduce do not change the dimension of the phase space and the motion remains on perturbed Keplerian orbits, only the form of the secular Hamiltonian is modified. Moreover, we may expect that qualitative properties of the secular system may be changed because we introduce a few new parameters describing the physical set-up of the studied system and interactions governing its dynamical evolution. To encompass both the specific values of angle \( \Delta \sigma = 0, \pi \), the representative plane can be defined by the following set of points:

\[
S = \{ e_1 \cos \Delta \sigma \times e_2; e_1, e_2 \in [0, 1] \},
\]

where \( \Delta \sigma = \pi, e_1 \cos \Delta \sigma < 0 \) for the left-hand half-plane (called the \( S_L \) plane, hereafter), and \( \Delta \sigma = 0, e_1 \cos \Delta \sigma > 0 \) are for the right-hand half-plane (the \( S_R \) plane). Hence, the sign of \( e_1 \cos \Delta \sigma \) on the \( x \)-axis tells us the value of angle \( \Delta \sigma \).

3.2 A general view of the representative plane

To illustrate the influence of additional perturbations on the secular dynamics of the classic model, we compute a set of the \( S \) planes for wide ranges of orbital and physical parameters of the studied two-planet configurations. Before we discuss these results, first we explain how the \( S \) plane illustrates the dynamical structure of the phase space. Conveniently, Fig. 2, which is derived for specific orbital parameters (given in the caption), reveals all relevant features in a condensed form and its description can be regarded as a guide useful for the further analysis.

We start from the left-hand panel of Fig. 2. Smooth half-ellipse like curves marked with thin lines are for the levels of the secular energy of the generalized problem. The straight lines are for the collision line of orbits defined through \( e_i(1 \pm e_i) = e_j(1 - e_j) \). The relative magnitude of the corrections to the secular Hamiltonian is represented by contour levels of the following coefficient:

\[
\kappa(e_1 \cos \Delta \sigma, e_2) = \left| \frac{\partial \langle \mathcal{H}_{\text{GR}} \rangle / \partial G_1 + \partial \langle \mathcal{H}_{\text{QM}} \rangle / \partial G_1}{\partial \langle \mathcal{H}_{\text{sec}} \rangle / \partial G_1} \right|. \tag{40}
\]

i.e. the ratio of the apsidal frequency of the inner pericentre induced by \( \langle \mathcal{H}_{\text{GR}} \rangle + \langle \mathcal{H}_{\text{QM}} \rangle \) relative to the ‘natural’ apsidal frequency in the point mass Newtonian model. Regions of the \( S \) plane where \( \kappa > 0.1 \) are colour coded, according to the levels of constant \( \kappa \). Yellow colour encodes \( \kappa \geq 1 \), meaning that the apsidal frequency of the innermost pericentre forced by perturbing GR + QM corrections is larger than the relative pericentre frequency \( \Delta \sigma \) caused by the secular NG interactions between planets.

The thick curves defined through

\[
\frac{\partial \mathcal{H}_{\text{sec}}}{\partial G_1} = 0 \tag{41}
\]

can be attributed to the stationary solutions of the reduced, two-dimensional system with \( (G_1, \Delta \sigma) \) variables, because coordinates of all points of these curves must also satisfy

\[
\frac{\partial \mathcal{H}_{\text{sec}}}{\partial \Delta \sigma} = 0, \tag{42}
\]
according to the definition of the $S$ plane. Such solutions are periodic orbits of the full secular system (equations 38–39). We examine the Lyapunov stability of these equilibria with the help of the Lyapunov theorem, adopting $H_{\text{sec}}$ as the Lyapunov function in the cases when they correspond to its maximum (or a minimum) in the reduced two-dimensional phase space. Because the investigated dynamical system has one degree of freedom, the stable or unstable equilibria can be easily identified as extrema or saddles of the secular Hamiltonian in the representative plane, respectively.

The positions of equilibria [or stationary modes (Michtchenko & Malhotra 2004)] help us to distinguish between different types (families) of orbits characterized by librations of $\Delta \varpi$ around a particular value (libration centre). The red, thick curves drawn in both panels of Fig. 2 are for stationary modes in the classic model (Michtchenko & Malhotra 2004), while the equilibria in the generalized model are drawn with thick, blue curves. In the $S_0$ half-plane, these stationary solutions are classified by Michtchenko & Malhotra (2004) as mode I equilibria, and they are characterized by librations of $\Delta \varpi$ around 0 in the neighbouring trajectories. In the $S_0$ half-plane, we can find also Lyapunov stable mode II solutions related to librations of $\Delta \varpi$ around $\pi$ in close trajectories. Some parts of the equilibrium curves are marked with violet colour (for the generalized model). These points denote unstable equilibria (UE) accompanied by the true secular resonance (the TSR hereafter). Such unstable equilibria in the $S_0$ plane are discovered by Michtchenko & Malhotra (2004) in the secular classic coplanar model of two planets. Obviously, mode I and mode II solutions, known as generic features of the classic model, exist also in the generalized problem. However, their positions in the phase space may be heavily affected by apparently subtle GR and QM perturbations.

Note that along the red thick curves representing equilibria in the classic model, $\kappa$ is undefined.

Now, let us examine the right-hand panel of Fig. 2. In this plot, besides levels of the secular energy of the classic model (red, thin curves), we also plot such levels for the generalized problem (green, thin curves). We can observe a significant discrepancy between the shapes of contour levels of both Hamiltonians.

In this plot we mark also a few specific levels of $H_{\text{sec}}$: $\mathcal{E}_a = -2.247$, $\mathcal{E}_b = -2.24955$, $\mathcal{E}_c = -2.25$, $\mathcal{E}_d = -2.2525$, $\mathcal{E}_e = -2.256$, $\mathcal{E}_f = -2.26$, respectively; here, we skipped all constant terms (the Keplerian part, indirect term, constant relativistic term) in the full secular Hamiltonian, and the energy values are given in terms of $10^{-5} \text{M}_{\odot} \text{au}^2 \text{yr}^{-2}$ when $1 \text{M}_{\odot}$. 1 au and 1 sideral year are taken as units of mass, distance and time, respectively (then the Gauss constant $k = 2\pi\epsilon$). Because the motion of the secular system must be confined to a fixed energy level, points at which a particular energy level crosses the curve representing stationary solutions tell us distinct equilibria and their bifurcations. For instance, following the energy level labelled with $a$ in the right-hand panel of Fig. 2, we see that it crosses the equilibrium curves at two points. The first one, found in the $S_0$ half-plane, corresponds to the mode I solution. The second cross-point is found in the $S_0$ half-plane, and it marks the mode II equilibrium.

To estimate the precision of the analytical theory, the energy levels in Fig. 2 are calculated with exact semi-analytical averaging (Michtchenko & Malhotra 2004) that makes use of precise adaptive Gauss–Legendre quadratures ( Migaszewski & Goździewski 2008b). For a comparison, the stationary modes are computed with both methods: we recall that thick curves represent equilibrium calculated with the semi-analytic averaging while the thin curves (over-plotted on them) are for stationary solutions calculated with the help of the analytical theory outlined in Section 2 (the right-hand panel of Fig. 2). The results of these methods are in excellent accord. The most significant discrepancies between the results appear in the $S_0$ half-plane (for $\Delta \varpi = 0$), in the regime of large eccentricity $e_2$. In such a case, the secular series representing $H_{\text{sec}}$ diverge over the anticollision line (marked with the red, straight line). This problem is discussed in detail in Migaszewski & Goździewski (2008a). Obviously, over this anticollision line, $r_2 > r_1$ at some parts of orbits, breaking the underlying assumption that we require to expand the inverse of the mutual distance in convergent series. Moreover, we demonstrate that the analytical theory reproduces the dynamics of hierarchical configurations up to very large $e_2$. Another, direct test of the precision of the analytic theory is given in Section 3.4.
Figure 3. Phase diagrams computed for a two-planet secular system described with the same parameters as used for the construction of Fig. 2. Each phase diagram is accompanied by a smaller plot of the fundamental frequency $g$ of the secular solutions calculated for initial conditions lying on the $y \equiv e_1 \cos \Delta \sigma = 0$-axis. Panels in the top row are for the secular energy $E_a = -2.247 \times 10^{-5}$, panels in the bottom row are for $E_c = -2.25 \times 10^{-5}$. These levels are labelled in Fig. 2 with (a) and (c), respectively. The left-hand panels are for the $(e_1 \cos \Delta \sigma, e_1 \sin \Delta \sigma)$ plane, the right-hand panels are for the $(e_2 \cos \Delta \sigma, e_2 \sin \Delta \sigma)$ plane. Shaded regions mark different libration zones around mode I ($\Delta \sigma = 0$) and mode II ($\Delta \sigma = \pi$), respectively. UE marks unstable equilibria accompanied by libration zones around $\Delta \sigma = \pi$ and centred at the TSR solutions. The true separatrices are indicated by $g \to 0$. See the text for more details.

3.3 Phase diagrams and the non-linear secular resonance

To investigate more closely the secular dynamics related to the new mode II and UE solutions, and to understand the structure of the phase space in more detail, we compute a number of phase plots (or phase diagrams). The secular energy is kept constant and we draw the phase trajectories in the $(e_1 \cos \Delta \sigma, e_1 \sin \Delta \sigma)$, as well as $(e_2 \cos \Delta \sigma, e_2 \sin \Delta \sigma)$ planes, choosing the initial conditions along the fixed energy level. The phase plots are computed with the help of the analytic theory. For a reference, we recall the right-hand panel of Fig. 2 which illustrates the $S$ plane of $(e_1 \cos \Delta \sigma, e_2)$. We recall that the UE of the generalized model are marked with violet curves.

The phase diagrams are illustrated in Fig. 3. The top row is for the energy level of $E_a = -2.247 \times 10^{-5}$ which is labelled with a in the right-hand panel of Fig. 2. This energy level crosses the equilibrium curves in two points which can be recognized in the phase diagrams as libration centres of $\Delta \sigma = 0$ (labelled with mode I), and of $\Delta \sigma = \pi$ (labelled with mode II), respectively. Overall, the phase diagrams look like qualitatively the same as in the classic model. Both libration modes are separated from the circulation zone of $\Delta \sigma$ by false separatrices (Pauwels 1983; Michtchenko & Malhotra 2004), i.e. the transition between each mode and the circulation of $\Delta \sigma$ does not involve solutions with infinite period. This is illustrated in smaller, bottom plots accompanying each phase diagram, which show the secular, fundamental frequency $g$ of the mean system for initial conditions lying on the $x$-axis of the respective phase diagram. This frequency has been determined through solutions of the secular equations of motion. Clearly, when the true separatrices cross the $e_{1,2} \cos \Delta \sigma$-axis, $g$ decreases to 0. Crossings of false separatrices do not lead to any discontinuity of smooth plots of $g$.

The phase diagrams are much more complicated for energy levels labelled with b-f, which cross the equilibrium curves at more than two points. We analyse in detail the energy level (c) which...
intersects the curve of stationary solutions in *four* points. These points can be recognized in the phase diagrams shown in the bottom panels of Fig. 3. We can identify them easily in the *x*-axis of these diagrams (because such points of the *S* plane have *y* ≡ *e*₂*ₙ* sin Δ*σ* = 0). Starting at the *S₀* plane and following the energy level counterclockwise, we have a stable mode I equilibrium surrounded by a large zone of librations of Δ*σ* around 0 which corresponds to a single crossing point of the energy level and the equilibrium curve in the *S₀* plane. In the *S₁* plane, we have *three* such points, two of them are Lyapunov stable, and one point in the middle is an unstable equilibrium. This part of the phase space, as seen in Fig. 3, encompass a figure-eight shaded area, involving two islands of librations (TSRs, or elliptic points) around Δ*σ* = π, and the hyperbolic UE point lying in the middle between them. Both libration centres are characterized by Δ*σ* = π. Because they are related to stable equilibria separated by the hyperbolic structure of the UE, two parts of the phase curve surrounding the islands of the TSRs and that meet in the UE must form a real separatrix. The whole structure may be still surrounded by a zone of librations of Δ*σ* around π. It is shaded in light grey. Let us note that this mode II libration area is confined to the *S₁* plane (hence, in this particular case, angle Δ*σ* does not pass through 0).

A sequence of (*e*₂ cos Δ*σ*, *e*₂ sin Δ*σ*) diagrams for the secular energy levels *a*–*f* are shown in Fig. 4. Looking at these levels plotted in the *S* plane, we can now follow a development of dynamical structures related to the different modes of motion. In particular, phase diagrams Figs 4(b) and (d) reveal bifurcations of mode II from which emerge the UE and TSR solutions. Clearly, the bifurcations may be identified with points at which the given energy level is tangent to the equilibrium curves in the *S* plane.

The phase diagrams assure us that the secular dynamics of the generalized model can be much more complex and rich due to the GR and QM corrections to the NG Hamiltonian than the secular dynamics in the classic model.

### 3.4 Numerical test of the analytic secular theory

Finally, we illustrate limitations of the secular theory and we compare its results with the outcome of the direct numerical integrations. This comparison is also directly related to the dynamical stability of the planetary system.

First, we constructed the numerical model of the generalized system independently of the analytical model. We wrote the equations of motion with respect to the Jacobi reference frame, using the formulation of Mardling & Lin (2002) who call it the direct code. In this code, the GR acceleration is modelled with the PPN formulae given in Kidder (1995). In that way we have the possibility to check the analytic theory in a completely independent way, which also prevents copying logical errors which could be done during the averaging. After some experiments, we also found that the choice of the reference frame (e.g. related to Jacobi, Poincaré or classic astrocentric coordinates) is in fact irrelevant for the results of this test. We also do not account for the difference between the osculating and the mean elements.

Using the direct code, we integrated numerically a few phase diagrams for nominally *the same initial conditions and parameters* used to draw plots in Figs 4(a)–(f) with the help of the analytic, secular model. We computed osculating elements related to the Jacobi reference frame over a few secular cycles. The numerically derived phase curves are drawn with black, filled circles in

![Figure 4](https://academic.oup.com/mnras/article-abstract/392/1/2/1069953/1069953)
the respective panels of Fig. 4. The solutions obtained with the analytic theory are overplotted on these numerical solutions with thinner, green curves. Subsequent five panels of Figs 4(a)–(e) reveal that the agreement of both sets of solutions is excellent. The analytic theory reproduces qualitative features seen in the phase plots, and their structure with great accuracy. We find that both solutions coincide even in the regime of large eccentricities. Remarkably, the direct code integrations require a CPU time that is by a few orders of magnitude longer than the calculations carried out with the help of the analytical theory.

However, in the last panel of Fig. 4(f) we can observe significant deviations of the analytic solutions from the numerical theory, particularly in the outer parts of the phase diagram. In fact, in this case $e_2$ is so large that the assumptions of the secular theory are broken. After examining the $S$ plane (Fig. 2), we can see that the energy level corresponding to the last panel passes close to the collision line. To illustrate the real border of the dynamical stability, we examined the dynamical character of solutions in the $S$ plane with the help of the spectral number technique (Michchenko & Ferraz-Mello 2001). This simple FFT-based algorithm makes it possible to distinguish between chaotic and regular solutions. The dynamical maps shown in Fig. 5 are constructed by counting the number of frequencies in the FFT spectrum of the time series, $\sigma(t) = a(t) \exp(i \lambda(t))$, where $a(t)$ and $\lambda(t)$ are the temporal canonical semimajor axes and mean longitude of each planet. The number of peaks (the spectral number, SN hereafter) in the spectrum over some noise level tells us the character of orbit. Orbits with large SN (greater than 1000) are very chaotic, while SN $\sim 1$ means a small number of frequencies and a regular, quasi-periodic phase trajectory. Each point in the dynamical maps represents a phase trajectory that was integrated over $\sim 10^3 P_2$. Although such a time span is relevant for the short-term dynamics only, the calculations took a very long CPU time (a few days on 24 AMD-CPU cores). The results are shown in two panels of Fig. 5. The left-hand panel is for the classic model (only NG interactions are included), while the right-hand panel is constructed for the generalized model.

Let us analyse the left-hand panel of Fig. 5 for the classic model. Solutions, which appear strongly chaotic, are marked with colours (darker point means larger SN and a more chaotic system). Clearly, the border of stable motions is irregular and is shifted towards small $e_2$ by 0.1–0.2 with respect to the formal, geometrical collision line bordering the triangular region (it is drawn in both panels of Fig. 5). The thick red curves mark the equilibrium of mode I and mode II, respectively. Shaded regions are for the initial conditions in the $S$ plane corresponding to orbital configurations with librating $\Delta \sigma$. In the right-hand panel of Fig. 5, we show the equilibrium curves of the generalized model and libration zones of $\Delta \sigma$ associated with these equilibria. We mark again the SN signature of the short-term dynamics. We note that the border of stability is quite different from those of the classic model (compare with the left-hand panel of Fig. 5). This is a very clear example showing that apparently subtle GR + QM effects may affect the short-term stability of the system in a significant way.

The dynamical map for the generalized model (Fig. 5, the right-hand panel) helps us to identify the source of unstable behaviour seen in Fig. 4(f), revealing that some initial conditions lead to erratic and irregular behaviour. In the dynamical map, we mark two levels of $H_{\infty}$ corresponding to phase diagrams drawn in Figs 4(e) and (f), respectively. The energy level (e) lies entirely in the regular region, in spite of the fact that $e_2$ may reach values as large as 0.8. The neighbouring level (f) may touch the unstable zone, and that is why the orbital evolution at this energy level may become very unstable. Because the motion must be confined to the fixed energy level, due to the secular evolution, some initial conditions may be transported to the chaotic zone (by excitation of the eccentricity) during roughly a half of the secular period. Then the short-term, strong chaos can destabilize the system immediately. Following the fixed energy curve, we can also identify three islands of stable motions. The first one lies in the right-hand half-plane of the phase diagram and is associated with librations of $\Delta \sigma$ around 0, see the neighbourhood of the corresponding cross point in the $S_1$ plane, Fig. 2. In the $S_2$ plane, we can find corresponding unstable equilibrium and two stable solutions with associated libration islands shown in Fig. 4(f) (see the left-hand half-plane of the phase diagram) around $0, -0.1$ and $(0, -0.78)$, respectively.

Finally, to illustrate the development of the secular instability, we solved the equations of motion of the full system, starting very close to the UE lying on the energy level between levels $e$ and $f$, as

Figure 5. The $S$ planes for the classic model (the left-hand panel) and for the generalized model (the right-hand panel) for the two-planet system analysed in Fig. 2. The thick, red curves mark stationary modes in the classic model; the thick blue curves are for the equilibria in the generalized model. Shaded areas indicate zones of $\Delta \sigma$ librations. The thick red lines are for the collision line of orbits. Colours code log SN of the outer orbit, characterizing solutions derived numerically with the help of the direct code. Black points are for strongly chaotic solutions with log SN $\sim 3$, white colour is for regular solutions with log SN $\sim 0$, intermediate values are marked with the colour scale above the panels, accordingly (see the text for more details).
marked in Fig. 2, and Figs 4(e) and (f). The solution is illustrated in the phase diagram in the left-hand panel of Fig. 6. For reference, the numerical solution is overplotted on the analytically derived separatrices of the UE. The corresponding time evolution of the orbital elements is illustrated in the right-hand panels of Fig. 6. Clearly, during quite a long time the full system stays close to the UE, but after \( \sim 10^7 \) yr it follows a trajectory close to the inner separatrix, and finally begins to move close to the outer separatrix, approaching large \( e_2 \). During this evolution, we observe not only very irregular behaviour of \( a_2 \) and both eccentricities, but also \( \Delta \sigma \) changing from large amplitude librations around 0 to circulations. Although the configuration seems bounded during many secular periods, such behaviour may be classified as strongly chaotic.

4 PARAMETRIC SURVEY OF TWO-PLANET SYSTEMS

The characterization of the phase space with the help of the representative plane can be very useful to conduct a survey of the basic features of the secular dynamics. In particular, we want to understand how it depends on the physical and orbital parameters governing the magnitude of the GR and QM interactions. In subsequent diagrams of the \( S \) plane, we will always mark the collision and anticollision lines. In this way, we can determine the border of validity of the analytic approach. Yet to derive the stationary modes possibly exactly, in the whole permitted range of eccentricity, we compute their locations with the help of the semi-analytical averaging algorithm.

4.1 Dependence of the secular dynamics on the masses

The results of the survey of the secular dynamics of two-planet systems, including GR and QM interactions, for varied planetary masses, are illustrated in Fig. 7. We fix the system parameters as follows: the mass of the parent star is \( m_0 = 1 \, M_\odot \), the equatorial radius of the star \( R_0 = 1 \, M_\odot \), \( T_{\text{rot}} = 30 \, \text{d} \), \( k_i = 0.02 \) (then \( J_2 \sim 10^{-7} \)), the semimajor axes of the planets are \( a_1 = 0.1 \, \text{au} \) and \( a_2 = 1.0 \, \text{au} \), respectively. Hence, we consider a typical hierarchical configuration with the orbital periods ratio \( \sim 30 \). In this experiment, the planetary masses are varied, but their ratio is kept constant, \( m_1/m_2 = 3 \).

Reference, the mass of the inner planet is labelled in the top left-hand corner of each respective panel in Fig. 7. Basically, in this test we can also analyse the effect of unknown inclination of the coplanar system on the long-term dynamics and stability. However, as we show in a recent work regarding the 14 Herculis planetary system (Goździewski, Migaszewski & Konacki 2008), planetary masses derived from observations do not necessarily always scale according to the mass factor \( 1/\sin i \). If the minimal masses are large then the mutual interactions in low-inclination configurations can strongly modify the RV signal and even the mass hierarchy may be reversed in the orbital fits.

Fig. 7 reveals that curves representing stationary modes, which are known in the classic problem, are usually significantly shifted and/or distorted. Also new features of the \( S \) plane appear and can be seen in the top left-hand panel of Fig. 7. In general, the distortions of equilibrium curves are stronger when the masses are smaller. It is quite straightforward to explain this effect. When the planetary masses decrease, also their mutual interactions (scaled by \( m_1 m_2 \)) are decreasing. Yet the pericentre frequency induced by \( \bar{\mathcal{H}}_{\text{sec}} \) is scaled by the mass product. Simultaneously, the GR and the spin-induced apsidal frequencies do not depend on the planetary masses directly (see equations 31 and 35, respectively), and they can be regarded as approximately constant in the given mass range. Therefore \( \kappa \) increases with decreasing \( m_1 \) and \( m_2 \). Then, also the assumptions of the secular theory are better fulfilled.

Some parts of the stationary curves in the \( S_\text{c} \) half-plane (for \( \Delta \sigma = \pi \)) comprise unstable equilibria (they are marked with violet colour). As we mentioned already, to the best of our knowledge, such solutions are yet unknown in the literature. Similarly to the non-classic equilibria discovered by Michtchenko & Malhotra (2004), these solutions are accompanied by the TSR solutions and correspond to saddles of the secular Hamiltonian. The behaviour of neighbouring solutions tells us that they are Lyapunov unstable. This has been analysed in Section 3.

Actually, the sequence of panels in Fig. 7 illustrates a characteristic development of curves representing the equilibria, including the UE solutions. When the masses are relatively large (see the top left-hand panel of Fig. 7), the equilibrium curves are distorted and the unstable equilibria appear at the very edge of the \( S_\text{c} \) plane, in the range of moderate and large values of eccentricity. In contrast,
in the classic model, the UE solutions can appear (in fact, they were found) only for $\Delta \sigma = 0$ (Michtchenko & Malhotra 2004). Seemingly, the new UE branch located in the $S_e$ plane is specific only for this model. When the masses decrease then $k$ grows (so the GR + QM effects become comparable in magnitude to the NG interactions). This leads to further distortion of mode II curves and to expanding the UE part towards moderate $e_1$. At some point (between $m_1 \approx 1.5 m_J$ and $m_1 \approx 1.2 m_J$) both stationary curves meet in a bifurcation point. Here, we can explain the particular choice of parameters used to construct Fig. 2. When the masses become smaller, the equilibrium curves separate along $e_2$. We note that already for $m_1 \approx 1.2 m_J$, the $S$ plane is dominated by the GR + QM corrections. We recall that in the classic case, the qualitative features of the $S$ plane do not depend on the masses individually (Michtchenko & Malhotra 2004), only on their ratio in the approximation of small values (see also the sequence of plots in Fig. 7). This conclusion is not true anymore in the realm of the generalized model.

### 4.2 Dependence of the secular dynamics on semimajor axes

In the next experiment, we investigate the dependence of the secular dynamics of the generalized model on individual semimajor axes; note that the dynamics of the classic model depend only on their ratio, $\alpha$. The results are illustrated in Fig. 8. We proceed in the same manner as to draw Figs 2 and 7. We seek stationary solutions, and we overplot the found equilibria on colour coded contour levels of coefficient $k$. The primary parameters of the tested configurations are the following: $m_J = 1 M_J$, $R = 1 M_J$, $T_{\text{rot}} = 30$ d, $k_1 = 0.02$, $m_1 = 0.4 m_J$, $m_2 = 0.2 m_J$. The ratio of semimajor axes is kept constant, $\alpha = a_1/a_2 = 0.1$, while the individual $a_1, a_2$ are varied. For reference, the nominal value of $a_1$ is labelled in the top-left-hand corner in each respective panel. For decreasing $a_1$, the derivatives of $H_{\text{NG}}, H_{\text{GR}}, H_{\text{QM}}$ over $G_1$ increase; hence the magnitude of the respective correction to the apsidal frequency grows. Moreover, the GR and QM induced apsidal frequency increase faster than the rate of the pericentre advance forced by the NG interactions. In the region of $S$ plane painted in yellow, the GR and QM perturbations dominate over the NG interactions. This region expands quickly with decreasing semimajor axis of the inner planet, $a_1$. Already for $a_1 \sim 0.4$ au (which is similar to the semimajor axis of Mercury in the Solar system), the apparent corrections to the classic model may contribute much more to the pericentre frequency of the innermost planet than the point mass Newtonian interactions.

The top left-hand panel in Fig. 8 is for $a_1 = 2$ au and $a_2 = 20$ au, respectively. For these parameters, a shift of the curve of stationary solutions, when compared to the ones in the classic model, is already significant. In the next panel ($a_1 = 1$ au, $a_2 = 10$ au) the distortion of curves representing stationary modes is even stronger. Moreover, the UE mode present in the classic model in the $S_e$ plane cannot be found in that half-plane anymore. Simultaneously, new solutions appear at the very edge of the $S_e$ plane, in the range of large $e_1$. For $a_1 = 0.5$ au, this branch of stationary modes is even more extended. Starting with this value of $a_1$, the structure of the $S$ plane with respect to the generalized model is very different from those in the classic case. For smaller $a_1$, the curves of stationary modes are still more distorted. Clearly, these distortions cannot be regarded as small. This result is quite unexpected, recalling that the semimajor axes and the planetary masses by no means are ‘extreme’. In spite

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**Figure 7.** The representative energy planes ($e_1 \cos \Delta \sigma, e_2$) for $\Delta \sigma = 0$ (the right-hand half-planes, $S_0$) and $\Delta \sigma = \pi$ (the left-hand half-planes, $S_\pi$). Colour contours are for the ratio of the apsidal frequency induced by the general relativity and quadrupole moment to the apsidal frequency caused by mutual interactions between planets. Thick curves mark positions of stationary solutions. Red lines are for stable equilibria in the classic model. Blue and violet curves mark the positions of stable and unstable equilibria of the generalized model. The thin lines are for the secular energy levels of the generalized model. The thick, skewed lines are for the collision lines defined with $a_1 (1 \pm e_1) = a_2 (1 \pm e_2)$. Parameters of these systems are as follows: $m_0 = 1 M_\odot, a_1 = 0.1$ au, $a_2 = 1$ au, $R_0 = 1 R_\odot, T_{\text{rot}} = 30$ d, $k_1 = 0.02$. Each panel was calculated for varied planetary masses, under the condition of constant mass ratio $m_1/m_2 = 3$. The mass of the inner planet is written in the top left-hand corner of each panel.
of these ‘typical’ parameters, the secular theories of the classic and generalized models lead to a qualitatively different view of the phase space. We stress again that the notion of the GR and QM effects as corrections (or small perturbations) to the secular Hamiltonian should be understood in quite a new light.

4.3 Dependence of the secular dynamics on the stellar spin

In the last parametric survey, we study the dependence of the secular dynamics in the realm of the generalized model on the stellar spin (or, effectively, on the second zonal harmonic $J_2$). Fig. 9 illustrates the $S$ plane computed for following parameters: $m_0 = 1 M_{\odot}$, $m_1 = 1.25 m_1, m_2 = 0.25 m_1, a_1 = 0.1$ au, $a_2 = 1$ au, $R_0 = 1 M_{\odot}, k_{l} = 0.02$. The top left-hand panel in Fig. 9 is for the GR correction only and subsequent plots are for decreasing rotation period (generalized model) $T_{rot}$ of the star (its particular values label the respective plots). This sequence corresponds to increasing $J_2$.

The top left-hand panel of Fig. 9 is for a spherical, non-rotating star ($J_2 = 0$, no tidal bulge), the bottom right-hand panel is for very fast rotation characteristic for a young object (then $J_2 = 1 \times 10^{-4}$). Curiously, changes of the spin encompassing that range are enough to induce new families and stationary solutions which we described above. They are represented, as before, by thick violet curves drawn in the $S_n$ plane. Actually, the sequence of plots simulates variations of flattening during the lifetime of the star. Hence, after skipping dissipative tidal perturbations, we may conclude that the structure of the phase space of the secular system (and, in general, also its dynamical stability) may depend not only on the observed or measured orbital parameters but also on the age and the spin rate of the host star.

The structure of the $S$ plane and stationary modes which are illustrated in Fig. 9 remind us closely of Figs 7 and 8. In fact, as we already mentioned, the dependence of the dynamics on the model parameters may be described uniformly through $\kappa$, see equation (40). Indeed, the increase of the stellar spin leads to an increase of the pericentre frequency and the numerator of equation (40) (note that other terms are constant). The same behaviour of $\kappa$ is caused by decreasing masses (see the sequence of diagrams in Fig. 7 and also discussion in Section 4.1). In that case, the two terms of the numerator of $\kappa$ do not change, but the denominator decreases. Finally, if the semimajor axes decrease in constant ratio, $\kappa$ also grows because the GR- and QM-induced correction to the apsidal frequency grows faster than the NG-forced apsidal frequency. In that sense, all point mass gravity corrections governed by the parameter changes described above modify similarly the structure of the $S$ plane.

Considering our simplified secular model, the existence of the branch of stationary solutions in the regime of large $e_1$ and small $e_2$ may seem questionable in the real planetary configurations. In that regime, the tidal perturbations may become very significant for the secular evolution. However, the position of the bifurcation point in the $S_n$ plane, where the two branches of equilibrium curves meet, and which marks the extent of the branch, is shifted towards moderate $e_1 \sim 0.6$–0.7 when the mass ratio $m_1/m_2$ grows. This effect can be observed in the geometric evolution of these branches in Figs 8, 7 and 9 which form a sequence of $m_1/m_2 = 2, 3, 5$, respectively. Hence, planetary configurations in that regime may really be found in nature.
5 The Secular Dynamics of the \( \nu \) and System

Finally, we consider the generalized coplanar problem of three planets. This model can be still described by the Hamiltonian written in the general form of equations (1)–(4) and (7)–(11) with \( N = 3 \). The averaged perturbing Hamiltonian has the form of equations (15), (16), (17), (30), (34) and (36). To study the properties of the secular system, we introduce the following set of angle–action variables related to the Poincaré canonical elements (Migaszewski & Gożdziewski 2008a):

\[
\sigma_1 \equiv \sigma_3 - \sigma_1, \quad G_1', \\
\sigma_2 \equiv \sigma_3 - \sigma_2, \quad G_2', \\
\sigma_3 \equiv -\sigma_3, \quad \text{AMD} = G_1' + G_2' + G_3',
\]

where \( G_i' \equiv L_i - G \) (see equation 13). Because \( \sigma_1 \) is the cyclic angle, the angular momentum deficit (AMD) is conserved, and the reduced system has two degrees of freedom. These variables make it possible to construct the representative energy plane in a similar manner as for the two-planet system. Moreover, the choice of such a plane is not unique. One of the possible definitions can be derived through a direct analogy to the two-planet system case. The symmetric representative plane can be defined as the set of phase-space points fulfilling the following relation:

\[
\frac{\partial H_{sec}}{\partial \sigma_i} = 0, \quad i = 1, 2, 3,
\]

with the simultaneous conditions that \( \sigma_i = 0, \pi \). For details, see Migaszewski & Gożdziewski (2008a).

A sequence of symmetric representative planes shown in Fig. 10 illustrates the qualitative properties of the generalized secular model of the \( \nu \) Andromedae planetary system (Butler et al. 1999). This system comprises three planets with masses and semimajor axes derived through the radial velocity observations: \( m_0 = 1.27 \, M_\odot \), \( m_1 = 0.69 \, m_0 \), \( m_2 = 1.98 \, m_0 \), \( m_3 = 3.95 \, m_0 \), \( a_1 = 0.25 \, \text{au} \), \( a_2 = 0.83 \, \text{au} \), \( a_3 = 2.51 \, \text{au} \). The constant of the AMD integral (effectively, the integral of the total angular momentum) was obtained for the following eccentricities of the nominal system: \( e_1 = 0.029, e_2 = 0.254, e_3 = 0.242 \).

We consider the representative plane for varying age of the parent star, starting with approximately 30 Myr before entering the ZAMS. Subsequent plots are labelled by the lifetime \( \tau \) relative to this moment taken as the zero-point of the time-scale, rotation period \( T_{rot} \) and stellar radius \( R_0 \) expressed in units of the Sun’s radius. For reference, the classic model and with the GR correction are illustrated in the first two top left-hand panels of Fig. 10. These panels are derived for a non-rotating, non-distorted spherical star. When the star is spinning, its second zonal harmonic may be as large \( J_2 \sim 10^{-3} \). The current equatorial radius of \( \nu \) And is approximately \( R_0 = 1.26 \, M_\odot \). We note that the stellar radius and the spin period of the star at \( \tau = -30 \) Myr are taken from Nagasawa & Lin (2005), and were linearly interpolated over \( \tau \in [-30, 0] \) Myr.

The results are again quite surprising. After adding the GR corrections to the Hamiltonian of the classic model, the overall view of the phase space changes significantly. The saddle of the secular Hamiltonian which is present in the classic model now vanishes. In its place, a new maximum of the secular Hamiltonian appears. Moreover, at the bottom half-plane of the representative plane, close to

\[ \text{Figure 9.} \] The representative energy planes \( (e_1 \cos \sigma, e_2) \) for \( \Delta \sigma = 0 \) (the right-hand half-planes, \( \mathcal{S}_0 \)) and \( \Delta \sigma = \pi \) (the left-hand half-planes, \( \mathcal{S}_\pi \)). Colours are for the ratio of the apsidal frequency induced by the general relativity and quadrupole moment to the apsidal frequency caused by mutual interactions between planets. The thick lines mark the stationary solutions. The thick red curves are for stable equilibria in the classic model. The blue and violet curves mark positions of stable and unstable equilibria in the generalized model, respectively. Thin lines are for the energy levels of a generalized model. The thick skew lines are for collision lines defined with \( \sigma_i (1 \pm e_i) = \sigma_j (1 - e_j) \). Parameters of the system are the following: \( m_0 = 1.0 \, M_\odot, m_1 = 1.25 \, m_0, m_2 = 0.25 \, m_0, a_1 = 0.1 \, \text{au}, a_2 = 1.0 \, \text{au}, R_0 = 1.27 \, \text{M}_\odot, k_L = 0.02 \). Subsequent panels are for different rotational periods of the star, \( T_{rot} \), which is labelled in each respective plot. For reference, the top left-hand panel is for the GR correction only.
the border of the permitted region of motion, two new saddle points appear.

The next diagrams illustrate changes of the structure of the $S$ plane and the development of equilibria in a sequence simulating time evolution of the stellar spin. At the beginning, before the star enters the ZAMS, the characteristic plane reveals a sharp maximum and a saddle in the very edge of the region of permitted motions. The thin curve surrounding the maximum marks the energy level of the nominal system. When the rotation period increases up to $\sim 8 \text{d}$, the secondary extremum (the minimum) emerges in place of the saddle and it persists shortly before the ZAMS stage and for longer rotational periods.

Curiously, the only feature seen in the energy diagrams, which survives the spin variations during the whole lifetime of the star, and persists in the generalized model (with the GR and QM corrections), is the stable equilibrium point related to the maximum of the secular Hamiltonian, which is found in the range of small $e_1$ and moderate $e_2$. The nominal system appears in the energy level drawn with the grey (green), thick line surrounding this maximum of $\mathcal{H}_{sec}$. We also may notice that close to this equilibrium of the generalized model, a saddle of the classic model appears which is linearly stable.

6 CONCLUSIONS

In this paper, we consider a generalized secular theory of a coplanar, $N$-planet system. Extending the model analysed in recent works devoted to secular planetary dynamics with mutual Newtonian point-to-point interactions (e.g. Michtchenko & Malhotra 2004; Libert & Henrard 2005; Rodríguez & Gallardo 2005; Migaszewski & Goździewski 2008a), we consider the influence of the general relativity and quadrupole moment of the parent star on the secular dynamics of the innermost planet and stability of the whole planetary system. In general, these corrections to the classic model still do not cover all physics governing the dynamics of such systems. In some cases (e.g. of the short-period hot Jupiters), the tidal, dissipative torques acting between the inner planet and the star may be significant for the orbital evolution. However, our main goal is rather to extend the classic model with perturbations that are conservative and may be well modelled in the realm of the Hamiltonian mechanics than to build a complete, general secular theory. Still, this approach is useful for a wide class of systems, when the tidal interactions may be regarded as secondary effects, or are acting during a much longer characteristic time-scale than the GR and QM perturbations. As a reward, for paying the price of a less general model, we may investigate the secular dynamics in a global manner.

Our analytic model follows assumptions required by the averaging theorem. Technically, the averaging has been done with the help of a very simple method. This algorithm relies on an appropriate change of integration variables. It does not incorporate any classic Fourier expansion of the perturbing function. We obtain a very precise analytic model of the coplanar, $N$-planet system in terms of the semimajor axes ratio. It can be regarded as a generalization of the recent analytic secular theories of the classic model investigated in many recent papers (e.g. Ford, Kozinsky & Rasio 2000; Lee & Peale 2003; Michtchenko & Malhotra 2004; Libert & Henrard 2005). On the other hand, our work also covers, to some extent, the global dynamics of the generalized model studied in Mardling & Lin (2002) and Nagasawa & Lin (2005) with the help of the Gauss/Lagrange planetary equations of motion. We stress, however, that our investigations are devoted to a more narrow class of systems (regarding the conservative perturbations).

A general conclusion which can be derived on the basis of the generalized theory is quite unexpected. Even in a case when the orbital parameters cannot be regarded as extreme, the corrections to the classic Hamiltonian stemming from the general relativity and...
the quadrupole moment of the star may affect the secular dynamics dramatically. Not only the structure of the phase space of the secular model changes, and new branches of stationary solutions appear. These solutions may bifurcate within small relative ranges of the parameters (e.g. when the spin of the parent star is changing). We show that there is no simple and general recipe to predict the behaviour of the secular system, when the perturbations are ‘switched on’. The secular dynamics of the generalized model becomes extremely complex and rich. For some combinations of the system parameters, the notion of the GR and QM effects as corrections to the point mass NG interactions does not seem proper anymore. In some cases, these effects may be more important for the secular dynamics than the mutual, point mass Newtonian interactions between the planets.

We also show that these effects may be significant for the dynamical stability of planetary systems both in the short term and on the secular time-scales. For instance, the QM generated perturbations may directly depend on the star age and its physical parameters \((k_1, \varrho_0)\). In turn, these effects may strongly influence the structure of the phase space and can imply short-term, strong chaotic orbital evolution during a few secular periods.

The direct tests of the analytic theory are very encouraging. The results justify its great accuracy. The precision of the analytic calculations is very important for studying the global dynamics of hierarchical systems. The alternative numerical approach would require huge CPU time because the hierarchical planetary systems evolve during very different time-scales. Then the CPU requirements of the direct numerical integrations are by orders of magnitude larger than those needed by the analytic formulae.

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