ASYMPTOTIC EXPANSION OF THE ANNEALED GREEN’S FUNCTION AND ITS DERIVATIVES

MATTHIAS KELLER AND MARIUS LEMM

Abstract. We consider random elliptic equations in dimension \( d \geq 3 \) at small ellipticity contrast. We derive the large-distance asymptotic expansion of the annealed Green’s function up to order 4 in \( d = 3 \) and up to order \( d+2 \) for \( d \geq 4 \). We also derive asymptotic expansions of its derivatives. The obtained precision lies far beyond what is established in prior results in stochastic homogenization theory. Our proof builds on a recent breakthrough in perturbative stochastic homogenization by Bourgain in a refined version shown by Kim and the second author, and on Fourier-analytic techniques of Uchiyama.

1. Introduction

The regularity of Green’s functions and their derivatives forms the backbone of classical elliptic regularity theory for divergence-form operators \([5, 17, 21, 22]\). Here we consider divergence-form discrete elliptic operators of the form

\[
\nabla^* A_\omega \nabla, \quad \text{on} \ell^2(\mathbb{Z}^d), \quad d \geq 3,
\]

with a random, elliptic coefficient matrix \( A_\omega(x) \in \mathbb{R}^{d \times d} \). The central goal of the very lively field of stochastic homogenization theory is to understand the large-distance behavior of solutions to \( \nabla^* A_\omega(x) \nabla u_\omega(x) = f(x) \). Naturally, Green’s functions play a central role in this endeavour; see \([1, 2, 3, 11]\) and references therein.

In 2018, J. Bourgain \([4]\) introduced a completely novel approach to studying such equations in the regime of small ellipticity contrast. He takes the coefficients to be

\[
A_\omega(x) = (1 + \delta \sigma_\omega(x)) I_d
\]

with \( \{\sigma_\omega(x)\}_{x \in \mathbb{Z}^d} \) a family of independent and identically distributed bounded random variables, \( I_d \) the \( d \times d \) identity matrix, and \( \delta > 0 \) is a small parameter. A key point is that inspired by an earlier unpublished note of I.M. Sigal, from the outset Bourgain’s focus lies not with deriving an effective large-distance description of (random) solutions \( u_\omega(x) \), but only of their average \( \langle u_\omega(x) \rangle \). This is equivalent to studying the annealed (i.e., averaged) Green’s function

\[
G(x) = \langle G_\omega(x, 0) \rangle = \left\langle \frac{1}{\nabla^* A_\omega \nabla} \delta_0(x, 0) \right\rangle,
\]

where \( \frac{1}{\nabla^* A_\omega \nabla} \) is the operator inverse of \( \nabla^* A_\omega \nabla \), see \([10]\). The main result of \([4]\), which was subsequently refined by Kim and the second author \([16]\), establishes that \( \langle G_\omega(x, 0) \rangle \) can be represented as a convergent perturbation series in \( \delta > 0 \) with explicit large-distance decay bounds. (See Theorem 2.3 below for the precise statement.) This fact has several non-trivial consequences. For instance, \([9]\) showed that it allows to define higher-order correctors beyond what was previously believed.
possible. A related, intriguing possibility is to extend the main result of [4, 16] (see Theorem 2.3 below) to arbitrary ellipticity contrast. This is known as the Bourgain-Spencer conjecture which remains open. See [8, 10] for recent results in this direction.

For our purposes here, we stay within the small-ellipticity contrast regime and instead focus on a different consequence of the main results in [4, 16]. Namely, [16, Corollary 3.1] proves that for all multi-indices \( \alpha \) with \( |\alpha| \leq d + 1 \),

\[
|\nabla^\alpha G(x)| \leq C_\alpha (1 + |x|)^{2-d-|\alpha|}, \quad x \in \mathbb{Z}^d.
\]

That is, the first \( d + 1 \) derivatives match the expected power-law scaling that is familiar from the free Laplacian. (Before these works, (1.3) was only known for \( |\alpha| \leq 2 \), in any dimension and for any ellipticity contrast [2, 6, 18, 19].)

In the present work, we build further on the results of [4, 16] to prove a previously unforeseen strong refinement of (1.3), a precise asymptotic expansion of \( \nabla^\alpha G(x) \) as \( |x| \to \infty \).

Our main results can be summarized as follows.

- Theorem 2.6 provides the asymptotic expansion of \( G(x) \) as \( |x| \to \infty \) up to order 4 in \( d = 3 \) and up to order \( d + 2 \) for \( d \geq 4 \).
- Corollary 2.8 contains analogous asymptotic expansions for the derivatives \( \nabla^\alpha G \) with \( |\alpha| \leq 3 \) in \( d = 3 \) and \( |\alpha| \leq d + 1 \) in \( d \geq 4 \). For every derivative taken, one loses one order in the asymptotic expansion of \( G(x) \), so when one reaches the last derivative, we only identify the leading-order asymptotics.

In general, these main results go far beyond what can be achieved by the powerful methods of homogenization theory in the regime of small ellipticity contrast. The reason is partly that those methods first describe the random Green’s function \( G_\omega(x,0) \) which is harder to understand due to probabilistic fluctuations, see e.g. [2, Theorem 8.20 & Section 9.2], [4, Corollary 3], [13, Proposition 4.2] and [20, Theorem 5.1]. These results can be averaged post-hoc to obtain information on \( G(x) = \langle G(x,0) \rangle \) and its derivatives. For instance, it is well-known that the leading term in the expansion is the homogenized Green’s function; see [2, 18] and the other references above. However, the resulting bounds will be much less precise than the expansion we show here in Theorem 2.6 or Corollary 2.8. In particular, there does not appear to be any result on derivatives of order \( \geq 3 \) in the literature.

The paper is organized as follows.

- In Section 2 we define the setting and state the main results.
- In Section 3 we prove Theorem 2.6.
- In Section 4 we prove Corollary 2.8.
- In the Appendix, we include a self-contained derivation of the leading term in Theorem 2.6 based on [16, Theorem 1.1].

An open question related to diffusion processes is discussed in Subsection 2.5.

We mention that a version of the main results of this paper originally appeared in the preprint [15, Version 1] about optimal Hardy weights on \( \mathbb{Z}^d \). The present paper has been split off from that work.
2. Setup and main results

We begin by reviewing the setup and the main results of [4, 16] which form the backbone of our asymptotic expansion. Afterwards, we state our main results, Theorem 2.6 and Corollary 2.8.

2.1. Basic setting. Recall the definition of the discrete derivative denoted by $\nabla = (\nabla_1, \nabla_2, \ldots, \nabla_d)^T$. For a function $u : \mathbb{Z}^d \to \mathbb{R}$ or $\mathbb{C}$,

$$\nabla_j u(x) = u(x + e_j) - u(x)$$

with $e_j$ the $j$th canonical basis vector. Its $\ell^2(\mathbb{Z}^d)$-adjoint is denoted by $\nabla^* = (\nabla^*_1, \ldots, \nabla^*_d)$ and acts as

$$\nabla^*_j u(x) := u(x - e_j) - u(x).$$

Then $\nabla^* \nabla = -\Delta$ is the usual discrete Laplacian, a positive operator.

Assumption 2.1. Let $\{\sigma_\omega(x)\}_{x \in \mathbb{Z}^d}$ be a family of independent and identically random variables bounded by 1. We shall consider the random divergence-form operator

$$L_\omega = -\Delta + \delta \nabla^* \sigma_\omega I_d \nabla,$$

on $\ell^2(\mathbb{Z}^d)$. We will suppress the identity matrix $I_d$ from the notation. Note that $L_\omega$ is of the form (1.1) with the coefficients chosen by (1.2). We assume that $\delta \in (0, 1)$ so that $L_\omega$ is uniformly elliptic. We denote the Green’s function of $L_\omega$ by $G_\omega(x, 0)$, the unique solution to $L_\omega G_\omega(x, 0) = \delta_0(x)$ which is well-defined for $d \geq 3$.

Our main object of interest is the annealed Green’s function $G(x) = \langle G(x, 0) \rangle$.

The relevance of the annealed Green’s function is that it governs the behavior of averaged solutions. For instance, take [16, Corollary 1.6]. It says that for any $f \in \ell^{p_d}(\mathbb{Z}^d)$ with $p_d^{-1} = \frac{1}{2} + \frac{1}{d}$ (the critical Sobolev index), there exists a unique random solution $u_\omega \in \ell^{q_d}(\mathbb{Z}^d)$, with the Hölder dual $q_d$ of $p_d$ so that $u_\omega$ solves the equation $L_\omega u_\omega = f$ and the averaged solution is given by the formula

$$\langle u_\omega \rangle = G * f.$$

Thus we see that the decay properties of $G$ determine the decay properties of the averaged solution $\langle u_\omega \rangle$. The same is true for derivatives of all orders.

The correspondence is clearest when $f$ is compactly supported. Note that decay rates of a function and its derivatives are the most natural way to measure regularity on $\mathbb{Z}^d$.

Remark 2.2. The coefficients $\sigma_\omega$ are taken to be a multiple of the identity matrix $I_d$ only for simplicity. The same techniques apply if the i.i.d. perturbation is any symmetric matrix [16, Remark 1.4].

2.2. Background on the annealed Green’s function. In [4], Bourgain shows that the annealed Green’s function arises itself as a Green’s function of a matrix-valued convolution operator, called $\mathcal{L}$ below, which arises as the harmonic mean of the original random operator. This “parent operator” for $G$ can be realized as a bounded operator

$$\mathcal{L} : H^1(\mathbb{Z}^d) \to H^{-1}(\mathbb{Z}^d).$$
defined via the discrete Sobolev spaces
\[ H^1(\mathbb{Z}^d) = \Lambda^{-1}(\ell^2(\mathbb{Z}^d)), \quad H^{-1}(\mathbb{Z}^d) = \Lambda(\ell^2(\mathbb{Z}^d)), \quad \Lambda = (-\Delta)^{1/2}. \]

We refer to [16, Section 2.1] for the details and to [9, Lemma 1.1] for an alternative definition of \( L \) via the Lax-Milgram theorem.

The breakthrough result of [4] gives a precise description of \( L \) of the following form
\[
(2.2) \quad L = \Delta + \nabla^* K^\delta \nabla,
\]
where \( \Delta \) is the free Laplacian and \( K^\delta \) is a \( d \times d \) matrix-valued convolution operator whose components satisfy a decay estimate. This decay estimate was subsequently improved to the (conjecturally nearly optimal) rate \(-3d + \varepsilon\) in [16] which we use here.

We now summarize these results. We notationally identify the convolution operator \( K^\delta \) with its matrix-valued kernel \( K^\delta_j(x - y) \in \mathbb{R}^{d \times d} \).

**Theorem 2.3** ([4, 16]). Let \( d \geq 3 \) and \( \varepsilon \in (0, 1) \). There exists \( c_d > 0 \) so that for all \( \delta \in (0, c_d) \), the representation \( (2.2) \) holds with the following decay estimate on the convolution kernel
\[
|K^\delta_{j,k}(x - y)| \leq C_d \delta^2 (1 + |x - y|)^{-3d + \varepsilon}, \quad j, k \in \{1, \ldots, d\}.
\]

For the purposes of this paper, we can choose \( \varepsilon = \frac{1}{2} \) for definiteness so that, for \( \delta \in (0, c_d) \),
\[
(2.3) \quad |K^\delta_{j,k}(x - y)| \leq C_d \delta^2 (1 + |x - y|)^{-3d + \varepsilon}, \quad j, k \in \{1, \ldots, d\}.
\]

The usefulness of \( (2.3) \) lies in the fact that it guarantees the existence of moments of \( K^\delta \) up to order \( 2d - 1 \). The existence of higher moments has meaning in homogenization theory, where it can be shown to be equivalent to the existence of a previously unforeseen higher-order corrector theory up to order \( 2d \), [9].

Of particular importance for the leading-order behavior is the \( d \times d \) matrix
\[
(2.4) \quad Q = I_d + \sum_{x \in \mathbb{Z}^d} K^\delta(x).
\]

In the language of homogenization theory,
\[
(2.5) \quad Q = \frac{\mathbf{I} + (\mathbf{I})^T}{2}
\]
corresponds to the symmetrized lowest-order homogenized coefficients [9, Eq. (2.5)].

**Proposition 2.4.** The \( d \times d \) matrix \( Q \) is symmetric and there exist constants \( c_d, C_d > 0 \) so that for all \( \delta \in (0, c_d) \),
\[
(2.6) \quad 1 - C_d \delta^2 \leq Q \leq 1 + C_d \delta^2.
\]

**Proof.** For the symmetry of \( Q \), we use the power series representation of \( K^\delta \), cf. [4, Eq. (2.5)] and [16, Eq. (1.14)]. To write this down, we require some basic objects and notation from [4, 16]. Let \( \Omega \) denote the underlying probability space to the \( \{\sigma_\omega(x)\}_{x \in \mathbb{Z}^d} \).

First, we express the expectation as a projection operator on the extended space
\[
P : L^2(\mathbb{Z}^d \times \Omega) \to \ell^2(\mathbb{Z}^d) \subset L^2(\mathbb{Z}^d \times \Omega)
\]
\[
u(x, \omega) \mapsto \langle u \rangle(x)
\]
Here $L^2(\mathbb{Z}^d \times \Omega)$ is defined with respect to the counting measure on $\mathbb{Z}^d$ and the probability measure on $\Omega$. We write $P^\perp = I_{L^2(\mathbb{Z}^d \times \Omega)} - P$ for the projection onto the orthogonal complement.

Second, we write $\sigma$ for the multiplication operator
\[
\sigma : L^2(\mathbb{Z}^d \times \Omega) \to L^2(\mathbb{Z}^d \times \Omega), \quad (\sigma u)(x, \omega) = \sigma_\omega(x)u(x, \omega).
\]

Third, we introduce the operator-valued $d \times d$ matrix $K$ whose components are the operators $K_{j,k} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ which are defined as Fourier multiplication by the functions
\[
F(\theta) = \frac{(e^{i\theta_j} - 1)(e^{-i\theta_k} - 1)}{2 \sum_{j=1}^d (\cos \theta_j - 1)}
\]
using the following convention for the Fourier transform
\[
\hat{f}(\theta) = \sum_{x \in \mathbb{Z}^d} e^{-ix \cdot \theta} f(x), \quad \theta \in [-\pi, \pi]^d.
\]
(The operator $K$ can be formally written as $\nabla \nabla^* \Delta$ and is also known as the discrete Helmholtz projection.) Equivalently, the operator $K_{j,k}$ is a convolution operator with the convolution kernel
\[
K_{j,k}(x - y) = \int_{[-\pi, \pi]^d} e^{i(x-y) \cdot \theta} F(\theta) \frac{d^d \theta}{(2\pi)^d}.
\]

Then we lift $K$ to the extended space $L^2(\mathbb{Z}^d \times \Omega)$ by acting trivially on the random component and we abuse notation by calling the resulting operator $K$ as well.

With these preparations complete, we can write down the following power series representation of $K^\delta$, cf. [16, Eq. (1.14)]
\[
K^\delta = \delta \sum_{n=1}^\infty (-\delta)^n P\sigma(K P^\perp \sigma)^n.
\]

We remark that the convergence of this series is proved in [4, 16].

We claim that (2.8) implies that the convolution kernel satisfies
\[
(K^\delta)^T(x) = K^\delta(-x).
\]
Let us prove this. By (2.8), we have
\[
(K^\delta)^T(x)
= \delta \sum_{n=1}^\infty (-\delta)^n P\sigma(K^T P^\perp \sigma)^n(x, 0)
= \delta \sum_{n=1}^\infty (-\delta)^n P\sigma(x) \sum_{x_1, \ldots, x_{n-1} \in \mathbb{Z}^d} K^T(x - x_1) P^\perp \sigma(x_1) \ldots K^T(x_{n-1}) P^\perp \sigma(0).
\]
By (2.7), we have $K^T(x) = K(-x)$. Introducing the reflected configuration $\tilde{\sigma}_\omega(x) = \sigma_\omega(-x)$ and reflecting the summation variables $x_\alpha$ to $-x_\alpha$, we obtain
\[
(K^\delta)^T(x) = \delta \sum_{n=1}^\infty (-\delta)^n P\sigma(K P^\perp \tilde{\sigma})^n(-x, 0).
\]
Thanks to Assumption 2.1, $\tilde{\sigma}$ has the same distribution as $\sigma$. Due to the presence of the first $P$ projection in (2.10), $\tilde{\sigma}$ only appears in an averaged sense in that equation and so (2.9) is proved.
Combining (2.4) and (2.9) with the change of variables $x \to -x$, we conclude
\[
Q^T = I_d + \sum_{x \in \mathbb{Z}^d} (K^\delta(x))^T = I_d + \sum_{x \in \mathbb{Z}^d} K^\delta(x) = Q
\]
as desired.

Finally, the decay estimate (2.3) implies $\|\hat{K}^\delta(0)\| \leq C_d\delta^2$ and so (2.6) follows from the spectral theorem. \hfill \Box

From now on we assume that $\delta$ is sufficiently small such that (2.3) holds and $Q$ is positive definite.

**Remark 2.5.** A curious but rather unexplored property of $K^\delta(x)$ is that, despite being an averaged object, it holds enough information to fully characterize the law of the probability measure of the random coefficients $\{w_x\}_{x \in \mathbb{Z}^d}$ [16, Proposition 1.8].

**2.3. Main result 1: Green’s function asymptotics.** The asymptotic expansion involves the modified spatial variable
\[
\tilde{x} = \sigma Q^{-1/2}x, \quad \text{with } \sigma = (\det Q)^{1/(2d)},
\]
and the universal constant
\[
\kappa_d = \frac{1}{2} \pi^{-d/2} \Gamma(d/2 - 1).
\]

For $d \geq 3$, we denote
\[
m_d = \begin{cases} 
3, & \text{if } d = 3, \\
  d + 1, & \text{if } d \geq 4.
\end{cases}
\]

We now state our first main result, a large-distance asymptotic expansion of $G(x) = \langle G(x, 0) \rangle$ of order $m_d + 1$.

**Theorem 2.6 (Asymptotic expansion of the annealed Green’s function).** Let $d \geq 3$. There exists $c_d > 0$ so that for all $\delta \in (0, c_d)$ the following holds. There are polynomials $U_1, \ldots, U_{m_d}$ with $U_k$ having degree at most $3k$ so that
\[
\langle G(x) \rangle = \frac{\kappa_d}{\sigma^d} |x|^{2-d} + \sum_{k=1}^{m_d} U_k \left( \frac{\tilde{x}}{|\tilde{x}|} \right) |\tilde{x}|^{2-d-k} + o(|\tilde{x}|^{2-d-m_d}), \quad \text{as } |x| \to \infty.
\]

The proof of Theorem 2.6 is given in Section 3.

The polynomials $U_k$ are defined in an explicit manner following [23]. They are given as Fourier transforms of fractions of the form $P_{2d-2+2k}(\xi)$ where $P_{2d-2+2k}$ is a homogeneous polynomial of degree $2d - 2 + k$. These polynomials can be explicitly computed as moments of the function $T : \mathbb{Z}^d \to \mathbb{R}$ defined by
\[
T(x) = \frac{1}{2} \delta_{x=0} + \frac{1}{4d} \delta_{|x|=1} + \frac{1}{4d} \sum_{j,k=1}^{d} \left( -K_{j,k}^\delta(x) + K_{j,k}^\delta(x - e_j) \right)
\]
\[
+ K_{j,k}^\delta(x - e_k) - K_{j,k}^\delta(x - e_j - e_k) \big).
\]

For instance, we have
\[
U_1(\omega) = \int_{\mathbb{R}^d} P_{2d-1}(\xi) \xi^2 e^{-i\omega \cdot \xi} d\xi,
\]
\[
P_{2d-1}(\xi) = -\frac{2i}{3\sigma^4(2\pi)^d} |\xi|^{2d-4} \sum_{x \in \mathbb{Z}^d} T(x)(\xi \cdot x)^3.
\]
Here the Fourier transform giving $U_1(\omega)$ is defined in the sense of tempered distributions on $\mathbb{R}^d \setminus \{0\}$ and the integral can be computed via \cite[Lemma 2.1]{13}.

**Remark 2.7.**

(i) In the context of homogenization theory, the leading term in the expansion (2.14),

$$\frac{K_d}{\sigma^2} |\tilde{x}|^{2-d} = G_{\text{hom}}(x),$$

is well-known as the **homogenized Green’s function** $G_{\text{hom}}(x)$.

(ii) By (2.11) and Proposition 2.4, we have the comparability

$$C_d,\delta^{-1}|x| \leq |\tilde{x}| \leq C_d,\delta |x|$$

for an appropriate constant $C_d,\delta > 1$. In particular, $|x| \to \infty$ and $|\tilde{x}| \to \infty$ are equivalent and $o(|x|^{-k}) = o(|\tilde{x}|^{-k})$.

(iii) As described above, the polynomials $U_k$ are computable from moments of the operator $K^\delta$ alone. In view of the results in \cite{8}, it is possible to rephrase the asymptotic expansion in terms of higher-order correctors. This reformulation could pave the way for extending Theorem 2.6 beyond the small-ellipticity regime, i.e., to all $\delta \in (0, 1)$, as further progress is made on the Bourgain-Spencer conjecture, cf. \cite{8}.

(iv) In Appendix A we give a short self-contained argument for readers interested in seeing how the matrix $Q$ from Theorem 2.3 arises in $G(x)$ by Taylor expansion around the origin in Fourier space. This yields the leading asymptotic order in Theorem 2.6. The idea for this is to reduce $G(x)$ to the Green’s function of the free Laplacian, using dyadic pigeonholing to control the error terms.

2.4. **Main result 2: Asymptotics of Green’s function derivatives.** In the discrete setting, pointwise asymptotics up to order $N$ of a function yield pointwise asymptotics of its first derivatives up to order $N-1$. This procedure can be iterated for higher derivatives, with a loss of one asymptotic order per derivative. Since our expansion (2.14) has $m_d + 1$ terms, we can describe asymptotics of the derivatives $\langle \nabla^\alpha G \rangle$ with $|\alpha| \leq m_d$ up to order $m_d - |\alpha|$ with $m_d$ defined in (2.13). To this end, we recall the notation of the discrete derivative $\nabla_j$

$$\nabla_j u(x) = u(x + e_j) - u(x).$$

For a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$, $\alpha_j \geq 0$, we write $|\alpha| = \sum_{j=1}^d \alpha_j$ and

$$\nabla^\alpha = \nabla_1^{\alpha_1} \cdots \nabla_d^{\alpha_d}.$$

By linearity, $\nabla^\alpha G(x) = \langle \nabla^\alpha G(x, 0) \rangle$.

**Corollary 2.8 (Asymptotic expansion of derivatives of $G$).** Under the same assumptions as in Theorem 2.6, let $\alpha \in \mathbb{N}_0^d$ be a multi-index with $|\alpha| \leq m_d$.

Then, as $|x| \to \infty$,

$$\nabla^\alpha G(x) = \nabla^\alpha \left( \frac{K_d}{\sigma^2} |\tilde{x}|^{2-d} + \sum_{k=1}^{m_d - |\alpha|} U_k \left( \frac{\tilde{x}}{|\tilde{x}|} \right) |\tilde{x}|^{2-d-k} \right) + o(|\tilde{x}|^{2-d-m_d}).$$

**Remark 2.9.**
(i) Since each $U_k$ is a polynomial and thus smooth, we can apply the mean value theorem to bound the discrete derivative by the corresponding continuum derivative, cf. [7, Lemma, p. 6], taking into account the linear change of variables $\tilde{x} = \sigma Q^{-1/2}x$. Together with Proposition 2.4, this readily implies that the decay rates of each term is controlled by

$$\nabla^\alpha \left( U_k \left( \frac{\tilde{x}}{|\tilde{x}|} \right) |\tilde{x}|^{2-d-k} \right) = O(|\tilde{x}|^{2-d-k-|\alpha|}).$$

In summary, (2.17) indeed gives an asymptotic expansion comprising $m_d - |\alpha|$ orders.

(ii) The leading term does not involve $U_k$ and is therefore particularly easy to compute. For example, we have the following leading-order gradient asymptotic

$$\langle \nabla G([x,x+se_j]) \rangle = \frac{s}{2} + \frac{d}{2} \kappa_d |\tilde{x}|^{2-d} + O(|\tilde{x}|^{-d}), \quad \text{as } |x| \to \infty.$$  

We give a proof of this fact in Section 4.

(iii) For the maximal value $|\alpha| = m_d$, (2.17) reduces to

$$\langle \nabla^\alpha G([x,x+se_j]) \rangle = \frac{\kappa_d}{\sigma^2} \nabla^\alpha |\tilde{x}|^{2-d} + o(|\tilde{x}|^{2-d-m_d}), \quad \text{as } |x| \to \infty,$$

so it just manages to capture the leading order asymptotic of the $m_d$-th derivative.

2.5. An open question: Does $\mathcal{L}$ generate a random walk? We close the presentation of the main results by describing an interesting open problem.

In the setting of [4, 16] described above, one may ask whether the operator $\mathcal{L}$ from (2.2) is again the generator of a random walk on $\mathbb{Z}^d$. More precisely, one can write $m(\theta) = 4d(1 - \hat{T}(\theta))$ with the function $T$ defined in (2.15).

An interesting simple-to-state question is then the following: Is $T(x) \geq 0$ for all $x \in \mathbb{Z}^d$? If so, $T(x)$ can be interpreted as the transition function of a random walk with generator $\mathcal{L}$, at least up to a multiplicative factor 4d. This would mean that probabilistic averages of solutions are themselves governed by bona fide diffusion process, whose dynamics may in turn hold non-trivial information about the non-averaged processes.

Such a direct dynamical meaning of the operator $\mathcal{L}$ is not at all obvious and would be remarkable. We encountered this question when noting that Uchiyama’s analysis [23] would apply more directly if $T(x) \geq 0$. Our initial investigations indicate that identifying the conditions under which $T(x) \geq 0$ is true is connected to subtle questions concerning componentwise positivity of matrix inverses [14].

3. Proof of Theorem 2.6

The proof relies on Theorem 2.3, specifically the decay estimate (2.3) which is the main result of [16], and the generalization of a delicate Fourier analysis developed by Uchiyama [23] in the probabilistic setting of random walks.

3.1. Fourier-space representation. Recall that the averaged Green’s function $\langle G \rangle$ is the Green’s function of the operator $\mathcal{L}$ which can be described via Theorem 2.3 as (2.2) and the decay estimate (2.3).
Equivalently, the operator \( L \) is a Fourier multiplier with the symbol \( m : \mathbb{T}^d \to \mathbb{C} \), on the torus \( \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d \), given by

\[
m(\theta) = 2 \sum_{j=1}^{d} (1 - \cos \theta_j) + \sum_{1 \leq j, k \leq d} (e^{-i\theta_j} - 1) \widehat{K}_{j,k}^{\delta}(\theta)(e^{i\theta_k} - 1).
\]

By integration by parts, the decay bound (2.3) then implies the regularity

\[
\widehat{K}_{j,k}^{\delta} \in C^{2d-1}(\mathbb{T}^d), \quad j, k \in \{1, \ldots, d\}
\]

which will be used many times in the following argument.

By Taylor expansion of the Fourier multiplier (3.1) at the origin, we find that the lowest order is quadratic and given by (2.4), i.e.,

\[
\mathcal{Q} = \text{Hess}(m)(0) = \mathbf{I}_d + \widehat{K}^{\delta}(0).
\]

This also means we can express the averaged Green’s function as a Fourier multiplier.

\[
\mathcal{G}(x) = \int_{\mathbb{T}^d} e^{ix\cdot\theta} \frac{1}{m(\theta)} \frac{d^d\theta}{(2\pi)^d}
\]

To see that the integral is well-defined for \( d \geq 3 \), observe that for \( \delta \) small enough \( m \) vanishes only at the origin \( \theta = 0 \) by (2.3). Thus, by Taylor expansion, \( \frac{1}{m(\theta)} \) only has a quadratic singularity at the origin.

The goal is to perform asymptotic analysis of (3.2) as \( |x| \to \infty \). This is a delicate stationary phase argument which has to take special care of the singularity at the origin in Fourier space. A hands-on approach to obtain the leading term which is based on the harmonic analysis ideas in [16, Appendix A] is explored in the Appendix. To derive the full asymptotic expansion, we draw on the techniques of Uchiyama [23] who elegantly accounts for cancelations of naively non-integrable terms. While Uchiyama assumes he is in a probabilistic setting which may not pertain to the averaged Green’s function, cf. Section 2.5, we show now that his argument extends to our case.

To make contact with the probabilistic perspective, we denote

\[
m(\theta) = 4d \left(1 - \hat{T}(\theta)\right)
\]

with \( T : \mathbb{Z}^d \to \mathbb{R} \) given as in Section 2.5 i.e.,

\[
\hat{T}(x) = \frac{1}{2} \delta_{x=0} + \frac{1}{4d} \delta_{|x|=1} + \frac{1}{4d} \sum_{j,k=1}^{d} \left( - K_{j,k}^{\delta}(x) + K_{j,k}^{\delta}(x-e_j) + K_{j,k}^{\delta}(x-e_k) - K_{j,k}^{\delta}(x-e_j-e_k) \right).
\]

Note that we produced the term \( \frac{1}{2} \delta_{x=0} \) by adding and subtracting a constant in (3.3). This is a common technical trick in the context of discrete random walks to remove periodicity, cf. (3.6) below.
3.2. Properties of $T$. As mentioned above, our goal is to extend [23, Theorem 2] to our situation. In a first step, we verify the assumptions of that theorem with the exception of $T(x) \geq 0$. The function $T$ satisfies the following properties assumed in [23] for small $\delta$. For all these properties the decay bound (2.3), which is $|K_{j,k}^\delta(x)| \leq C_d \delta^2(1 + |x|^{-3d+1/2})$, from [16, Theorem 1.1] is of the essence.

(i) $T$ has zero mean. Indeed, by (2.3), we can use Fubini and a change of variables to see

\[
\sum_{x \in \mathbb{Z}^d} xT(x) = \frac{1}{4d} \sum_{x \in \mathbb{Z}^d} x \delta_{|x|=1} + \frac{1}{4d} \sum_{j,k=1}^d \sum_{x \in \mathbb{Z}^d} K_{j,k}^\delta(x) (-x + (x + e_j) + (x + e_k) - (x + e_j + e_k)) = 0.
\]

(ii) The smallest subgroup of $\mathbb{Z}^d$ generated by

\[
\{ x \in \mathbb{Z}^d : T(x) > 0 \}
\]

is equal to $\mathbb{Z}^d$. This is an aperiodicity property. To see it is true, note that we can use the decay bound (2.3), to conclude that for all sufficiently small $\delta > 0$, we have $T(0) > 0$ and $T(\pm e_j) > 0$ for $j = 1, \ldots, d$.

(iii) The decay bound (2.3) also implies the summability of

\[
\sum_{x \in \mathbb{Z}^d} |T(x)||x|^{2+m_d} < \infty, \quad d = 3 \text{ or } d \geq 5,
\]

\[
\sum_{x \in \mathbb{Z}^d} |T(x)||x|^{2+m_d} \ln |x| < \infty, \quad d = 4,
\]

were $m_d = 2d - 3$, $d = 3, 4$ and $m_d = d + 1$, $d \geq 5$, was defined in (2.13).

Together (i)-(iii) verify the assumptions of Theorem 2 in [23] with $m$ equal to $m_d$ and $T$ called $p$ there, with the exception of non-negativity.

3.3. Verification of non-vanishing condition. We confirm that the fact that $T$ may be negative does not pose any problems in the proof. This step uses Proposition 2.4 and the extension is applicable as long as $Q$ is strictly positive semidefinite.

The proof of [23, Theorem 2] is contained in Section 4 of that paper. The proof makes use of general estimates on Fourier integrals taken from Sections 2 and 3 of [23] which do not depend on the non-negativity of $p$. This concerns Lemma 2.1, Lemma 3.1 and Corollary 3.1 from [23]. These are used in Section 4 together with the absolute summability (3.7) to control the error terms. The only step where the loss of non-negativity requires a short argument is the proof of the non-vanishing condition $c(\theta)^2 + s(\theta)^2 > 0$ which is obtained on page 226 of [23] from positivity and aperiodicity. We now verify this condition to our context.

For $\theta \in \mathbb{T}^d$, we set

\[
c(\theta) = \sum_{x \in \mathbb{Z}^d} T(x)(1 - \cos(\theta \cdot x)), \quad s(\theta) = \sum_{x \in \mathbb{Z}^d} T(x) \sin(\theta \cdot x).
\]

Lemma 3.1. For sufficiently small $\delta > 0$, we have

\[
c(\theta)^2 + s(\theta)^2 > 0
\]
for \( \theta \in \mathbb{T}^d \setminus \{0\} \).

**Proof.** By Taylor expansion around \( \theta = 0 \), we obtain

\[
c(\theta) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} T(x)(\theta \cdot x)^2 + \mathcal{O}(\theta^4),
\]

\[
s(\theta) = \sum_{x \in \mathbb{Z}^d} T(x)(\theta \cdot x) + \mathcal{O}(\theta^5),
\]

where the error terms are controlled by the decay bound of \( K^\delta \), (2.3) which enters via the definition of \( T \) given in (2.15). By the definitions of \( Q \), (2.4), and \( T \), (2.15), we have

\[
Q = \text{Hess}(m)(0) = -4d \text{Hess}(\tilde{T})(0).
\]

Using this for \( c \) and the zero mean property (3.5) of \( T \), for \( s \) we obtain

\[
c(\theta) = \frac{1}{2d} \langle \theta, Q\theta \rangle + \mathcal{O}(\theta^4), \quad s(\theta) = \mathcal{O}(\theta^5).
\]

Now Proposition 2.4 says that for sufficiently small \( \delta > c \)

Thus, choosing \( \delta \) small enough that \( \delta^2 C_d \leq \mu/2 \), we conclude that \( c(\theta)^2 + s(\theta)^2 > 0 \) holds on \( K \) as well. This proves Lemma 3.1. \( \square \)

### 3.4. Conclusion

We are now ready to complete the proof of Theorem 2.6. Thanks to (i)-(iii), Lemma 3.1 and the paragraph preceding it, the proof of Theorem 2 from [23] extends to our situation and yields an asymptotic expansion similar to (2.14).

Namely, taking account of the rescaling by \( 4d \) that we introduced in (3.3), we have the asymptotic expansion

\[
\mathcal{G}(x) = \frac{1}{4d (\sigma')^2} |x'|^{2-d} + \sum_{k=1}^{m_d} U_k \left( \frac{x'}{|x'|} \right) |x'|^{2-d-k} + o(|x'|^{2-d-m_d}), \quad \text{as } |x| \to \infty,
\]

where \( x' = \sigma'(Q')^{-1/2}x \) and \( Q' \) is the \( d \times d \) matrix generating the second-moment functional

\[
\langle \theta, Q' \theta \rangle = \sum_{x \in \mathbb{Z}^d} T(x)(x \cdot \theta)^2, \quad \sigma' = (\det Q')^{1/(2d)}.
\]

When we compare this with our claim (2.14), we see that the latter features \( \tilde{x} = \sigma Q^{-1/2}x \) instead, with the matrix \( Q \) defined in (2.4). These are related via

\[
Q' = \frac{1}{4d} Q.
\]
To see this, we use the Fourier representation and recall (3.3) and (3.1) to find
\[ Q_{i,j}' = \sum_{x \in \mathbb{Z}^d} T(x) x_i x_j = -\left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \hat{T} \right)(0) = \frac{1}{4d} \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} m \right)(0) = \frac{1}{4d} Q_{i,j} \]
for all \( i, j \in \{1, \ldots, d \} \).

Finally, we employ the identity (3.3) and its consequence \( \sigma' = (4d)^{-1/2} \sigma \) in (3.8). For the leading term, we note that \( |x'| = |x| \) and \( \frac{1}{4d \sigma'} = \frac{1}{\sigma} \). For the subleading term, we note that \( \frac{x'}{|x'|} = \frac{\tilde{x}}{|\tilde{x}|} \). Absorbing the factors of \( (4d)^{d+k-2} \) into the \( U_1, \ldots, U_m \) then yields (2.14). This proves Theorem 2.6. \( \square \)

4. Proof of Corollary 2.8 and Formula (2.19)

Proof of Corollary 2.8. Let \( 1 \leq j \leq d, s \in \{ \pm 1 \} \) and let \( \alpha \in \mathbb{N}_0^d \) be a multi-index with \( |\alpha| \leq m_d \). We apply Theorem 2.6. Regarding the \( U_k \) terms, (2.18) shows that
\[ \nabla^\alpha \left( \sum_{k=m_d-|\alpha|+1}^{m_d} U_k \left( \frac{\tilde{x}}{|\tilde{x}|} \right) |\tilde{x}|^{2-d-k} \right) \in O(|\tilde{x}|^{1-d-m_d}) \subseteq o(|\tilde{x}|^{2-d-m_d}). \]

Regarding the \( o(|\tilde{x}|^{2-d-m_d}) \) error term appearing in Theorem 2.6, we note that for \( f \in o(|\tilde{x}|^{2-d-m_d}) \), the triangle inequality implies \( \nabla^\alpha f(|x, x + se_j|) \in o(|\tilde{x}|^{2-d-m_d}) \), so the error does not get worse under discrete differentiation. This proves Corollary 2.8. \( \square \)

Proof of Formula (2.19). We recall the notation (2.11), i.e., \( \tilde{x} = \sigma Q^{-1/2} x \). We first use Corollary 2.8 and the bound (2.18) for all \( k \geq 1 \) to find
\[ \nabla_j \mathcal{G}(x) = \frac{K_d}{\sigma^2} (|\tilde{x} + \tilde{e}_j|^2 - |\tilde{x}|^2) + O(|\tilde{x}|^{-d}) \]
To compute the leading term, we expand \( (1 + y)^q = 1 + qy + O(y^2) \) to obtain
\[ |\tilde{x} + \tilde{e}_j|^2 - |\tilde{x}|^2 = |\tilde{x}|^2 \left( 1 + \left( \frac{\tilde{x}}{|\tilde{x}|} \right) \tilde{e}_j + \left| \frac{\tilde{e}_j}{|\tilde{x}|} \right|^2 \frac{2}{|\tilde{x}|^2} - 1 \right) \]
\[ = |\tilde{x}|^{2-d} \frac{2-d}{2} \left( \frac{\tilde{x}}{|\tilde{x}|} \tilde{e}_j \right) + O(|\tilde{x}|^{-d}), \quad \text{as } |x| \rightarrow \infty, \]
where we also made use of the equivalence of the norms \( |x| \) and \( |\tilde{x}| \), cf. (2.16). \( \square \)

Acknowledgments

The authors are grateful to Scott Armstrong and Mitia Duerinckx for useful remarks. MK acknowledges the financial support of the German Science Foundation.

Appendix A. Direct Argument for the Leading Order in Theorem 2.6

In this appendix, we give a self-contained proof of the lowest order asymptotic in Theorem 2.6 i.e.,
\[ \mathcal{G}(x) = \frac{K_d}{\sigma^2} |\tilde{x}|^{2-d} + O(|x|^{1-d}), \quad \text{as } |x| \rightarrow \infty, \]
where again \( \tilde{x} = \sigma Q^{-1/2} x \). The approach is to use a Taylor expansion around the origin in Fourier space which is justified by Theorem 2.3 and controlled by adapting the dyadic pigeonholing from [16, Appendix A].
Proof of (A.1). We recall Definition (A.1) of $m(\theta)$ and the fact that $\hat{R}_{j,k}^\delta \in C^{2d-1}(\mathbb{T}^d)$ for all $j, k \in \{1, \ldots, d\}$. We first isolate the lowest, quadratic order of $m(\theta)$ by setting

$$m(\theta) = m_0(\theta) + \tilde{m}(\theta), \quad \text{where } m_0(\theta) = \langle \theta, \mathcal{Q} \theta \rangle.$$ 

Observe that $\tilde{m} \in C^{2d-1}(\mathbb{T}^d)$ satisfies

| $\alpha$ | $1$ | $3 - |\alpha|$ |
|----------|-----|---------------|

(A.2) $|D^\alpha \tilde{m}(\theta)| \leq C_\alpha |\theta|^{3-|\alpha|}$, $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq 2d - 1$.

We can decompose

| $\frac{1}{m(\theta)}$ | $\frac{1}{m_0(\theta)}$ | $\frac{\tilde{m}(\theta)}{m_0(\theta)m(\theta)}$ |

(A.3)

The next lemma then implies (A.1).

Lemma A.1. For all $\delta \geq 0$ sufficiently small, we have

(A.4) $\int_{\mathbb{T}^d} e^{ix \cdot \theta} \frac{\tilde{m}(\theta)}{m_0(\theta)m(\theta)} \frac{d^d\theta}{(2\pi)^d} = O(|x|^{1-d})$, as $|x| \to \infty$.

(A.5) $\int_{\mathbb{T}^d} e^{ix \cdot \theta} \frac{1}{m_0(\theta)} \frac{d^d\theta}{(2\pi)^d} = \frac{\kappa_\delta}{\sigma^2} |\hat{x}|^{2-d} + O(|x|^{1-d})$, as $|x| \to \infty$.

Proof. We loosely follow Appendix A in [16] where a similar problem is treated and start with the proof of (A.4). We define the function $F$ on $\mathbb{T}^d$ by

$$F(\theta) = \frac{\tilde{m}(\theta)}{m_0(\theta)m(\theta)}.$$ 

and note that, due to (A.2) and the quadratic vanishing order of $m(\theta)$ and $m_0(\theta)$ at the origin, we have

(A.6) $|F(\theta)| \leq C_\delta |\theta|^{-1}$.

Let $\varphi : [0, \infty) \to \mathbb{R}$ be a smooth cutoff function with $\varphi = 1$ on $[0, 2\pi]$ which is supported on $[0, 4\pi]$. Define $\psi_l(r) := \varphi(r) - \varphi(2r)$ and $\bar{\psi}_l(r) = \psi(2r)$ for all $l \geq 1$ and $r \geq 0$. Note that this defines a partition of unity $\sum_{l \geq 1} \psi_l(r) = 1$ for all $r \neq 0$. We decompose

(A.7) $\int_{\mathbb{T}^d} e^{ix \cdot \theta} F(\theta) \frac{d^d\theta}{(2\pi)^d} = \sum_{l \geq 0} f_l(x)$,

where we rescaled and introduced

$$f_l(x) = 2^{-ld} \int_{\mathbb{T}^d} e^{ix \cdot \theta} F_l(\theta) \frac{d^d\theta}{(2\pi)^d}, \quad F_l(\theta) = \bar{\psi}(|\theta|) F(2^{-l}\theta).$$

Note that (A.6) implies $|F_l(\theta)| \leq C_\delta 2^l |\theta|^{-1}$, so we can use the triangle inequality to obtain

(A.8) $|f_l(x)| \leq C_\delta 2^{-l(d-1)}$.

This is useful for for $|x| 2^{-l} \leq 1$, while for $|x| 2^{-l} \geq 1$ it can be improved to

(A.9) $|f_l(x)| \leq C_\delta \frac{2^{-l(d-1)}}{(|x| 2^{-l})^d}$. 


To prove (A.9), we assume without loss of generality that $|x_1| = \max_{1 \leq j \leq d} |x_j|$ and use $d$-fold integration by parts to write

$$f_i(x) = i^d \frac{2^{-ld}}{|x_1|^{2-l}d} \int_{\mathbb{T}^d} e^{ix_\theta \cdot \theta} \frac{d^d \theta}{(2\pi)^d}$$

Next, we observe that (A.2) and the quadratic behavior of both $m_0$ and $m$ at the origin yield that $\|\partial_\theta^d F_i\|_\infty \leq C_d 2^{-l(d-1)}$; compare Lemma A.1 in [10]. Applying this bound to (A.10) yields (A.9).

We use (A.8) and (A.9) and bound the resulting geometric series to find

$$\sum_{l \geq 0} |f_l(x)| \leq C_d \sum_{l \geq \log_2 |x|} 2^{-l(d-1)} + C_d \sum_{0 \leq l \leq \log_2 |x|} \frac{2^{-l(d-1)}}{(|x|2^{-l})^d} \leq C_d |x|^{1-d}.$$}

In view of (A.7), this proves (A.4).

Next, we turn to the proof of (A.5). By Proposition 2.3, the matrix $Q$ is symmetric and positive definite. Hence, by a change of variables,

$$\int_{\mathbb{T}^d} e^{ix_\theta \cdot \theta} \frac{1}{m_0(\theta)^2} \frac{d^d \theta}{(2\pi)^d} = \sigma^{-d} \int_{\mathbb{T}^d} e^{i(Q^{-1/2}x)_\theta \cdot \theta} \frac{1}{|\theta|^2} \frac{d^d \theta}{(2\pi)^d}.$$}

For the volume element, we used that $\det(Q^{1/2}) = (\det Q)^{1/2} = \sigma^d$ since $Q$ is symmetric. By applying (A.4) with $\delta = 0$, we obtain

$$\int_{\mathbb{T}^d} e^{iQ^{-1/2}x_\theta \cdot \theta} \frac{1}{|\theta|^2} \frac{d^d \theta}{(2\pi)^d} = \int_{\mathbb{T}^d} e^{i(Q^{-1/2}x_\theta \cdot \theta)} \frac{1}{2 \sum_{j=1}^d (1 - \cos \theta_j)} \frac{d^d \theta}{(2\pi)^d} + O(|x|^{1-d}).$$}

We recognize the first integral as the Green’s function of the free Laplacian on $\mathbb{Z}^d$ evaluated at the point $Q^{-1/2}x$. The standard asymptotic formula for the free Laplacian gives

$$\int_{\mathbb{T}^d} e^{iQ^{-1/2}x_\theta \cdot \theta} \frac{d^d \theta}{m_0(\theta)^2} = \frac{\kappa_d}{\sigma_d^d} Q^{-1/2}x|^{2-d} + O(|x|^{1-d}) = \frac{\kappa_d}{\sigma_d^d} x|^{2-d} + O(|x|^{1-d}).$$}

This proves (A.5) and thus Lemma A.1.

References

1. S. Armstrong, T. Kuusi, and J.-C. Mourrat, The additive structure of elliptic homogenization Invent. Math. 208 (2017), no. 3, 999-1154
2. S. Armstrong, T. Kuusi, and J.-C. Mourrat, Quantitative stochastic homogenization and large-scale regularity, 352 (2019), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer
3. P. Bella, A. Giunti, F. Otto, Quantitative stochastic homogenization : local control of homogenization error through corrector, In: Mathematics and materials / Mark J. Bowick... (eds.) Providence, RI : American mathematical society, 2017, 301-327
4. J. Bourgain, On a homogenization problem, J. Stat. Phys. 172 (2018), no. 2, 314–320
5. E. De Giorgi, Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 3 (1957), 25–43
6. T. Delmotte and J.-D. Deuschel, On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to $\nabla \phi$ interface model, Probab. Theory Relat. Fields, 133 (2005), no. 3, 358–390
7. W.F. Donoghue, Jr., Monotone Matrix Functions and Analytic Continuation, Springer (1974)
8. M. Duerinckx, Non-perturbative approach to the Bourgain- Spencer conjecture in stochastic homogenization, arXiv:2102.06319 [math.AP]
9. M. Duerinckx, A. Gloria, and M. Lemm, A remark on a surprising result by Bourgain in homogenization, Comm. Part. Diff. Eq. 44 (2019), no. 12, 1345-1357
10. M. Duerinckx, M. Lemm, F. Pagano, in preparation.
11. A. Gloria, S. Neukamm, and F. Otto, *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, Invent. Math. **199** (2015), no. 2, 455–515.

12. A. Gloria, and F. Otto, *An optimal variance estimate in stochastic homogenization of discrete elliptic equations*, Ann. Probab., **39** (2011), no. 3, 779–856.

13. Y. Gu and J.-C. Mourrat, *Scaling limit of fluctuations in stochastic homogenization*, Multiscale Model. Simul., **14** (2016), no. 1, 452–81.

14. C.R. Johnson, R.L. Smith, *Inverse M-matrices, II*, Linear Algebra Appl. **435** (2011), 953–983.

15. M. Keller and M. Lemm, *On optimal Hardy weights for the Euclidean lattice*, version 1, [arXiv:2103.17019][math.AP]

16. J. Kim and M. Lemm, *On the averaged Green’s function of an elliptic equation with random coefficients*, Arch. Ration. Mech. Anal. **234** (2019), no. 3, 1121-1166.

17. Littman, W., Stampacchia, G., Weinberger, H. F., *Regular points for elliptic equations with discontinuous coefficients*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) 17 (1963), 43–77.

18. D. Marahrens and F. Otto, *Annealed estimates on the Green function*, Probab. Theory Related Fields **163** (2015), no. 3-4, 527–573.

19. D. Marahrens and F. Otto, *On annealed elliptic Green’s function estimates*, Mathematica bohemica **140** (2015), no. 4, 489–506.

20. J.C. Mourrat and F. Otto, *Correlation structure of the corrector in stochastic homogenization*, Ann. Probab., **44** (2016), no. 5, 3207-3233.

21. J. Moser, *A new proof of De Giorgi’s theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468.

22. J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958) 931–954.

23. K. Uchiyama. *Green’s functions for random walks on \( \mathbb{Z}^N \).* Proc. London Math. Soc. (3), 77(1):215–240, 1998.

**Matthias Keller, Universität Potsdam, Institut für Mathematik, 14476 Potsdam, Germany**

*Email address: matthias.keller@uni-potsdam.de*

**Marius Lemm, Institute of Mathematics, EPFL, 1015 Lausanne, Switzerland**

*Email address: marius.lemm@epfl.ch*