ON THE CAUCHY PROBLEM FOR THE MUSKAT EQUATION.
II: CRITICAL INITIAL DATA

THOMAS ALAZARD AND QUOC-HUNG NGUYEN

Abstract. We prove that the Cauchy problem for the Muskat equation is well-posed locally in time for any initial data in the critical space of Lipschitz functions with three-half derivative in $L^2$. Moreover, we prove that the solution exists globally in time under a smallness assumption.

1. Introduction

Consider the Muskat equation
\begin{equation}
\partial_t f = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \, d\alpha,
\end{equation}
where $\Delta_\alpha f$ is the slope, defined by
\begin{equation}
\Delta_\alpha f(x, t) = \frac{f(x, t) - f(x - \alpha, t)}{\alpha}.
\end{equation}

The Cauchy problem for the latter equation has attracted a lot of attention in recent years. In particular, the well-posedness of the Cauchy problem has been established in many sub-critical spaces, see [14, 19, 22, 1, 24]. We also refer the reader to our companion paper [2] where we initiate the study of the Cauchy problem for non-Lipschitz initial data, following earlier works in [19, 7, 18, 20] which allowed arbitrary large slopes. The main result of this paper states that the Cauchy problem is in fact well-posed for arbitrary initial data in a critical space.

**Theorem 1.1.** i) For any initial data $f_0$ in $\dot{W}^{1, \infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})$, there exists a time $T > 0$ such that the Cauchy problem for the Muskat equation has a unique solution
\begin{equation}
f \in C([0, T]; \dot{W}^{1, \infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})) \cap L^2(0, T; \dot{H}^2(\mathbb{R})).
\end{equation}

ii) Moreover, there exists a positive constant $\delta$ such that, for any initial data $f_0$ in $\dot{W}^{1, \infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})$ satisfying
\begin{equation}
(1 + \|f_0\|_{\dot{W}^{1, \infty}}^4) \|f_0\|_{\dot{H}^{3/2}}^2 \leq \delta,
\end{equation}
the Cauchy problem for the Muskat equation has a unique global solution
\begin{equation}
f \in C([0, +\infty); \dot{W}^{1, \infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})) \cap L^2(0, +\infty; \dot{H}^2(\mathbb{R})).
\end{equation}
Some remarks are in order.

- The first clarification to be made is that we denoted by $\dot{W}^{1,\infty}(\mathbb{R})$ the space of Lipschitz functions, and by $\dot{H}^{s}(\mathbb{R})$ (resp. $\dot{H}^{s}(\mathbb{R})$) the classical Sobolev (resp. homogeneous Sobolev) space of order $s$, with $s = 3/2$ or $s = 2$ in the statement. They are equipped with the norm defined by

$$\|u\|_{\dot{W}^{1,\infty}} := \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|},$$

and

$$\|u\|_{\dot{H}^{s}} := \left( \int_{\mathbb{R}} |\xi|^{2s} |\hat{f}(\xi)|^{2} \, d\xi \right)^{1/2}, \quad \|u\|_{\dot{H}^{s}}^{2} = \|u\|_{\dot{H}^{s}}^{2} + \|u\|_{L^{2}}^{2}.$$  

Now, recall that the Muskat equation is invariant by the change of unknowns:

$$f(t, x) \mapsto f_{\lambda}(t, x) := \frac{1}{\lambda} f(\lambda t, \lambda x).$$

Now, by a direct calculation, one verifies that

$$\|f_{\lambda}\|_{t=0} = \|f_{0}\|_{\dot{W}^{1,\infty} \cap \dot{H}^{3/2}}.$$  

This means that the space $\dot{W}^{1,\infty}(\mathbb{R}) \cap \dot{H}^{3/2}(\mathbb{R})$ is a critical space for the study of the Cauchy problem.

- Let us discuss now statement $ii)$, that is the global well-posedness component of this result. This is a 2D analogous to the recent result by Gancedo and Lazar [20] for the 3D problem; it improves on a previous result by Córdoba and Lazar [18] which proves a similar global existence result for the 2D-problem with a similar smallness assumption, but under the extra assumption that the initial data belongs to $H^{5/2}(\mathbb{R})$.

- We now come to statement $i)$ about the local well-posedness result for arbitrary initial data. This is, in our opinion, the main new result in this paper. Since we are working in a critical space, this result is optimal in several directions.

Firstly, it is known that the Cauchy problem is not well-posed globally in time: there are blow-up results for some large enough data by Castro, Córdoba, Fefferman, Gancedo and López-Fernández ([8, 9, 10]).

A striking consequence of these blow-up results is that one cannot solve the Cauchy problem for a time $T$ which depends only on the norm of $f_{0}$ in $\dot{W}^{1,\infty}(\mathbb{R}) \cap \dot{H}^{3/2}(\mathbb{R})$. Otherwise, one would obtain a global existence result for any initial data by an immediate scaling argument using (6). Notice that this argument does not contradict our main result: it means instead that the time of existence must depend on the initial data itself, and not only on its norm. This shows that one cannot prove statement $i)$ by using classical Sobolev energy estimates. This in turn poses new challenging questions since on the other hand the Muskat equation is a quasi-linear equation. To overcome this problem, we will estimate the solution for a norm whose definition depends on the initial data.
We will also prove a result which elaborates on the previous discussion, stating that whenever one controls a bigger norm than the critical one, the time of existence is bounded from below on a neighborhood of the initial data.

To introduce this result, let us fix some notations.

**Definition 1.2.** Given a real number $s \geq 0$ and a function $\phi : [0, \infty) \to (0, \infty)$ satisfying the following assumptions:

(H1) $\phi$ is increasing and $\lim_{r \to +\infty} \phi(r) = \infty$ when $r$ goes to $+\infty$;
(H2) there is a positive constant $c_0$ such that $\phi(2r) \leq c_0 \phi(r)$ for any $r \geq 0$;
(H3) the function $r \mapsto \phi(r)/\log(4 + r)$ is decreasing on $[0, \infty)$.

Then $|D|^{s, \phi}$ denotes the Fourier multiplier with symbol $|\xi|^{s}\phi(|\xi|)$, so that

$$\mathcal{F}(|D|^{s, \phi}f)(\xi) = |\xi|^{s}\phi(|\xi|)\mathcal{F}(f)(\xi).$$

Moreover, we define the space

$$\mathcal{X}^{s, \phi}(\mathbb{R}) = \{ f \in W^{1, \infty}(\mathbb{R}) \cap L^2(\mathbb{R}) : |D|^{s, \phi}(|D_x|)f \in L^2(\mathbb{R}) \},$$

equipped with the norm

$$\|f\|_{\mathcal{X}^{s, \phi}} := \|f\|_{W^{1, \infty}} + \|f\|_{L^2} + \left( \int_{\mathbb{R}} |\xi|^{2s} (\phi(|\xi|))^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

**Remark 1.3.** The Fourier multiplier $|D|^{s, \phi}$ with $\phi(r) = \log(2 + r)^a$ was introduced and studied in [5, 4, 3] for $s \in [0, 1)$ (also see [23]).

**Theorem 1.4.** Consider a real number $M_0 > 0$ and a function $\phi$ satisfying assumptions (H1)–(H3) in Definition 1.2. Then there exists a time $T_0 > 0$ such that, for any initial data $f_0$ in $\mathcal{X}^{\frac{3}{2}, \phi}(\mathbb{R})$ satisfying

$$\|f_0\|_{\mathcal{X}^{\frac{3}{2}, \phi}} \leq M_0,$$

the Cauchy problem for the Muskat equation has a unique solution

$$f \in C([0, T_0]; W^{1, \infty}(\mathbb{R}) \cap H^\frac{3}{2}(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})).$$

**Remark 1.5.** Statement i) in Theorem 1.1 is a consequence of Theorem 1.4. Indeed, it is easily seen that (cf [2, Lemma 3.8]), for any $f_0$ in the critical space $W^{1, \infty}(\mathbb{R}) \cap H^\frac{3}{2}(\mathbb{R})$, one may find a function $\phi$ such that $f_0$ belongs to $\mathcal{X}^{\frac{3}{2}, \phi}(\mathbb{R})$ (and satisfying assumptions (H1)–(H3) in Definition 1.2).

Theorem 1.1 and Theorem 1.4 are proved in the next section.

**Acknowledgments.** T.A. acknowledges the SingFlows project (grant ANR-18-CE40-0027) of the French National Research Agency (ANR). Q-H.N. is supported by the Shanghai Tech University startup fund.
2. Proof

2.1. Regularization. In order to rigorously justify the computations, we want to handle smooth functions (hereafter, a ‘smooth function’ is by definition a function which belong to $C^1([0,T]; H^\mu(\mathbb{R}))$ for any $\mu \in [0, +\infty)$ and some $T > 0$). To do so, we must regularize the initial data and also consider an approximation of the Muskat equation. For our purposes, we further need to consider a regularization of the Muskat equation which will be compatible with the Sobolev and Lipschitz estimates. It turns out that this is a delicate technical problem.

Our strategy will consists in smoothing the equation in two different ways: i) by introducing a cut-off function in the singular integral, removing wave-length shorter than some parameter $\varepsilon$ and ii) by adding a parabolic term of order 2 with a small viscosity of size $|\log(\varepsilon)|^{-1}$.

More precisely, we introduce the following Cauchy problem depending on the parameter $\varepsilon \in (0, 1]$:

\begin{equation}
\begin{aligned}
\partial_t f - |\log(\varepsilon)|^{-1} \partial_x^2 f &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \left(1 - \chi_\varepsilon \left(\frac{\alpha}{\varepsilon}\right)\right) d\alpha, \\
\left|f\right|_{t=0} &= f_0 * \chi_\varepsilon,
\end{aligned}
\end{equation}

where $\chi_\varepsilon(x) = \varepsilon^{-1} \chi(x/\varepsilon)$ where $\chi$ is a smooth bump function satisfying $0 \leq \chi \leq 1$ and

$\chi(y) = \chi(-y), \quad \chi(y) = 1 \text{ for } |y| \leq \frac{1}{4}, \quad \chi(y) = 0 \text{ for } |y| \geq 2, \quad \int_{\mathbb{R}} \chi \, dy = 1.$

The equation (8) does not belong to a general class of parabolic equations. However, we will see that it can be studied by standard tools in functional analysis together with two estimates for the nonlinearity in the Muskat equation which plays a central role in our analysis.

**Proposition 2.1.** For any $\varepsilon$ in $(0, 1]$ and any initial data $f_0$ in $H^{3/2}(\mathbb{R})$, there exists a time $T_\varepsilon > 0$ such that the Cauchy problem (8) has a unique solution $f_\varepsilon$ satisfying

$$f_\varepsilon \in C^1([0, T_\varepsilon]; H^\infty(\mathbb{R})).$$

Moreover, either $T_\varepsilon = +\infty$ or

$$\limsup_{t \to T_\varepsilon} \|f_\varepsilon(t)\|_{H^{3/2}} = +\infty.$$

We postpone the proof of this proposition to §2.9.

2.2. An estimate of the Lipschitz norm.

**Lemma 2.2.** For any real number $\beta_0$ in $(0, 1/2)$, there exists a positive constant $C_0 > 0$ such that, for any $\varepsilon \in (0, 1]$ and any smooth solution $f \in C^1([0, T]; H^\infty(\mathbb{R}))$ of the Muskat equation (8),

\begin{equation}
\frac{d}{dt} \|f(t)\|_{W^{1, \infty}} \leq C_0 \|f(t)\|_{H^{2}}^2 + C_0 \varepsilon^{\beta_0} \|f(t)\|_{C^{2, \beta_0}}^4,
\end{equation}
where \( \|u\|_{C^{2,\beta}_0} = \|\partial_{xx} u\|_{C^{0,\beta}_0} = \sup_{x,y \in \mathbb{R}} \frac{|(\partial_{xx} u)(x) - (\partial_{xx} u)(y)|}{|x - y|^\beta} \).

**Proof.** The proof is partially based on arguments from [17, 7, 20]. Firstly, it follows from the proof of [17, Lemma 5.1] that
\[
\partial_x \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f(x)}{1 + (\Delta_\alpha f(x))^2} \, d\alpha = \frac{\partial^2_x f(t, x)}{2\pi} \int \frac{1}{1 + (\Delta_\alpha f(t, x))^2} - \frac{1}{1 + (\Delta_{-\alpha} f(t, x))^2} \, d\alpha - \frac{2}{\pi} \int \frac{\partial_x f(t, x) - \Delta_\alpha f(t, x)}{\alpha^2} \frac{1 + \partial_x f(t, x) \Delta_\alpha f(t, x)}{1 + (\Delta_\alpha f(t, x))^2} \, d\alpha.
\]

Moreover,
\[
\left| \partial_x \left( \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \left( 1 - \chi \left( \frac{|\alpha|}{\varepsilon} \right) \right) \, d\alpha \right| \leq \int_{|\alpha| \leq 2\varepsilon} (|\Delta_\alpha f_{xx}| + |\Delta_\alpha f_x|) \, d\alpha
\]
\[
\leq \int_{\mathbb{R}} |\Delta_\alpha f_x|^2 \, d\alpha + C\varepsilon^{\beta_0} \|f_{xx}(t)\|_{C^{0,\beta}_0},
\]
where we used the notations \( f_x = \partial_x f \) and \( f_{xx} = \partial_{xx} f \). Thus, for any \( t \) and any \( x \), we have
\[
(\partial_x \partial_t f)(t, x) - |\log(\varepsilon)|^{-1} \partial^2_x f_x(t, x)
\leq \frac{\partial^2_x f(t, x)}{2\pi} \int \frac{1}{1 + (\Delta_\alpha f(t, x))^2} - \frac{1}{1 + (\Delta_{-\alpha} f(t, x))^2} \, d\alpha - \frac{2}{\pi} \int \frac{\partial_x f(t, x) - \Delta_\alpha f(t, x)}{\alpha^2} \frac{1 + \partial_x f(t, x) \Delta_\alpha f(t, x)}{1 + (\Delta_\alpha f(t, x))^2} \, d\alpha
\]
\[
+ C \int |\Delta_\alpha f_x(t, x)|^2 \, d\alpha + C\varepsilon^{\beta_0} \|f_{xx}(t)\|_{C^{0,\beta}_0}.
\]

Consider the function \( \varphi(t) = \|\partial_x f(t)\|_{L^\infty} \) and a function \( t \mapsto x_t \) such that \( \|\partial_x f(t)\|_{L^\infty} = (\partial_x f)(t, x_t) \).

Then \((\partial^2_x f)(t, x_t) = 0 \) and \(-(\partial_{xx} f)(t, x_t) \geq 0 \). So, it follows from (11) that
\[
\dot{\varphi}(t) \leq -\frac{2}{\pi} \int \frac{\partial_x f(t, x_t) - \Delta_\alpha f(t, x_t)}{\alpha^2} \, d\alpha
\]
\[
- \frac{2}{\pi} \int \frac{(\partial_x f(t, x_t) - \Delta_\alpha f(t, x_t))^2}{\alpha^2} \frac{\Delta_\alpha f(t, x_t)}{1 + (\Delta_\alpha f(t, x_t))^2} \, d\alpha
\]
\[
+ C \int |\Delta_\alpha f_x(t, x_t)|^2 \, d\alpha + C\varepsilon^{\beta_0} \|f_{xx}(t)\|_{C^{0,\beta}_0}.
\]

As already observed in [17] (see also [7, 20]), the first term in the right-hand side has a sign since \( \partial_x f(t, x_t) \geq \Delta_\alpha f(t, x_t) \) for any \( \alpha \). It follows that
\[
\dot{\varphi}(t) \leq \frac{1}{\pi} \int \frac{(\partial_x f(t, x_t) - \Delta_\alpha f(t, x_t))^2}{\alpha^2} \, d\alpha + C \int |\Delta_\alpha f_x(t, x_t)|^2 \, d\alpha
\]
\[
+ C\varepsilon^{\beta_0} \|f_{xx}(t)\|_{C^{0,\beta}_0}.
\]
We now apply Hardy’s inequality to infer that
\[
\int \frac{(\partial_x f(t,x_t) - \Delta_x f(t,x_t))^2}{\alpha^2} \, da \lesssim \int |\Delta_a f_x(t,x_t)|^2 \, da.
\]
Consequently, we end up with
\[
\dot{\varphi}(t) \lesssim \int \|\Delta_a f_x(t)\|^2_{L^\infty} \, da + \varepsilon^{\beta_0} \|f_{xx}(t)\|_{C^{\beta_0,\beta_0}}.
\]
Introducing the difference operator \(\delta_a g(x) = g(x) - g(x - \alpha)\), the previous inequality is better formulated as follows:
\[
\dot{\varphi}(t) \lesssim \int \|\delta_a (\partial_x f)(t)\|^2_{L^\infty} \, da + \varepsilon^{\beta_0} \|f_{xx}(t)\|_{C^{\beta_0,\beta_0}}.
\]
Now the right-hand side is equivalent to the following homogeneous Besov norm:
\[
\|\partial_x f(t)\|_{B^{1/2}_{2,\infty}}^2 \quad \text{(see [27, 28] or Section 2 in [2])}.
\]
Then it follows from Sobolev embeddings that
\[
\dot{\varphi}(t) \lesssim \|f(t)\|^2_{H^2} + \varepsilon^{\beta_0} \|f_{xx}(t)\|_{C^{\beta_0,\beta_0}}
\]
which is the wanted result. \(\square\)

2.3. Sobolev estimates. In this paragraph we recall a generalized Sobolev energy estimate proved in our companion paper [2]. By generalized Sobolev energy estimate, we mean that, instead of estimating the \(L^\infty_t(L^2_x)\)-norm of \((-\Delta)^s f\), we shall estimate the \(L^\infty_t(L^2_x)\)-norm of \(|D|^s f\) for some function \(\phi\) satisfying the assumptions in Definition 1.2.

There two technical results that we will borrow from [2]. The first result, which is Lemma 3.4 in [2], gives an energy estimate.

**Lemma 2.3.** There exists a positive constant \(C\) such that, for any \(T > 0\) and any smooth solution \(f \in C^1([0,T]; H^\infty(\mathbb{R}))\) to (1), there holds
\[
\frac{d}{dt} \|D|^s f\|^2_{L^2} + \int_\mathbb{R} \frac{|\|D|^s f|^2|}{1 + |(\partial_x f)^2|} \, dx + |\log(\varepsilon)|^{-1} \int_\mathbb{R} |D|^s f|^2 \, dx \leq CQ(f) \|D|^s f\|^2_{L^2},
\]
where
\[
Q(f) = \left( \|f\|^2_{H^2} + \|f\|^2_{H^{1/2}} \right) \|D|^s f\|^2_{L^2} + \|D|^s f\|^2_{L^2} \|f\|^2_{H^{1/2}}
\]
\[
+ \left( \|f\|^2_{H^{3/2}} + \|f\|^2_{H^{1/2}} \right) \|D|^s f\|^2_{L^2} \|f\|^2_{H^{1/2}}.
\]

**Remark 2.4.** Some explanations are in order since the reader may notice several modifications compared to our paper [2]. Firstly, in [2] we considered a function \(\phi\) whose definition depends on an extra function \(\kappa\). Here we ignore this point since it is irrelevant for the present analysis. Indeed, the functions \(\phi\) and \(\kappa\) are shown in [2] to be equivalent (such that \(\kappa(\lambda) \leq \phi(\lambda) \leq C \kappa(\lambda)\)), and the distinction between them served only to organize the proof. Secondly, in [2] we also assume that \(\phi(r)\) is bounded from below by \((\log(4 + r))^a\) for some \(a \geq 0\). Here we will use that this property holds with \(a = 0\). Once the previous clarifications have been done,
it remains to explain that in [2] we consider the equation (1) while here we work with (8). The elliptic term \((-\partial_{xx}^2)\) is trivial to handle since in [2] we only applied an \(L^2\)-energy estimate and since the latter operator is positive. Eventually, the cut-off function \((1 - \chi(\alpha/\varepsilon))\) is also harmless in the various computations used to prove Lemma 3.4 in [2].

Secondly, we recall two interpolation inequalities from [2, Lemma 3.5]. Hereafter, we use the notations
\[
\begin{align*}
A_{\phi}(t) &= \| |D|^{\frac{3}{2}} \phi f(t) \|_{L^2}^2, \\
B_{\phi}(t) &= \| |D|^{2} \phi f(t) \|_{L^2}^2, \\
P_{\phi}(t) &= \| |D|^{\frac{5}{2}} \phi f(t) \|_{L^2}^2, \\
\end{align*}
\]
and
\[
\mu_{\phi}(t) = \left( \phi \left( \frac{B(t)}{A(t)} \right) \right)^{-1}.
\]

**Lemma 2.5.** Consider a real number \(7/4 \leq s \leq 2\). Then, there exists a positive constant \(C\) such that, for any \(T > 0\), any smooth solution \(f \in C^1([0, T]; H^\infty(\mathbb{R}))\) to (8) and any \(t \in [0, T]\),
\[
\begin{align*}
\|f(t)\|_{H^s} &\leq C \mu_{\phi}(t) A_{\phi}(t)^{2-s} B_{\phi}(t)^{s-{\frac{3}{2}}}, \\
\| |D|^{\frac{3}{2}} \phi^2 f(t) \|_{L^2} &\leq C \mu_{\phi}(t) A_{\phi}(t) B_{\phi}(t)^{\frac{3}{4}}.
\end{align*}
\]

From these two lemmas, we get at once the following

**Proposition 2.6.** There exist two positive constants \(C_1\) and \(C_2\) such that, for any \(T > 0\) and any smooth solution \(f \in C^1([0, T]; H^\infty(\mathbb{R}))\) of the Muskat equation (8),
\[
\begin{align*}
\frac{d}{dt} A_{\phi}(t) + C_1 \frac{B_{\phi}(t)}{1 + \|f_x(t)\|_{L^\infty}^2} + |\log(\varepsilon)|^{-1} P_{\phi}(t) \\
&\quad \leq C_2 \left( \sqrt{A_{\phi}(t) + A_{\phi}(t)} \right) \mu_{\phi}(t) B_{\phi}(t).
\end{align*}
\]

We will also an estimate for the \(L^2\)-norm.

**Lemma 2.7.** There holds
\[
\frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 \leq C \varepsilon \frac{1}{2} \|f\|_{H^\frac{3}{2}} \|f\|_{L^2}.
\]

In particular,
\[
\|f(t)\|_{L^2} \leq \|f_0\|_{L^2} + C \varepsilon \frac{1}{2} \int_0^t \|f(\tau)\|_{H^\frac{3}{2}} d\tau.
\]

**Proof.** Set
\[
R_{\varepsilon}(f) = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_{\alpha} f}{1 + (\Delta_{\alpha} f)^2} \chi\left( \frac{\alpha}{\varepsilon} \right) d\alpha.
\]
We multiply the equation by \(f\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \| f(t) \|_{L^2}^2 \leq \frac{1}{\pi} \left( \int_{\mathbb{R}} \frac{\partial_x \Delta f}{1 + (\Delta f)^2} \, d\alpha, f \right) + \langle R_\varepsilon(f), f \rangle.
\]
Now, by [13, Section 2], the first term in the right-hand side has a sign. Indeed:
\[
\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \frac{\partial_x \Delta f}{1 + (\Delta f)^2} \, d\alpha \right] f(x) \, dx
= - \int_{\mathbb{R}^2} \log \left[ \sqrt{1 + \left( \frac{f(t, x) - f(t, x - \alpha)}{\alpha^2} \right)^2} \right] \, dx \, d\alpha.
\]
It remains to estimate \( R_\varepsilon(f) \). To do so, we use the estimate (22) to get
\[
\| R_\varepsilon(f) \|_{L^2} \lesssim \int_{|\alpha| \leq 2\varepsilon} \| \Delta f_x \|_{L^2} \, d\alpha
\lesssim \varepsilon^\frac{1}{2} \left( \int_{\mathbb{R}} \| \Delta f_x \|_{L^2}^2 \, d\alpha \right)^\frac{1}{2} \lesssim \varepsilon^\frac{1}{2} \| f \|_{H^{\frac{3}{2}}},
\]
which completes the proof. \( \square \)

2.4. Estimate of the Hölder norm. To exploit the Sobolev energy estimate (16), the main difficulty is to estimate from above the factor \( 1 + \| f_x(t) \|_{L^\infty}^2 \). This is where we will apply Lemma 2.2. This in turn requires to estimate the Hölder norm \( \| \cdot \|_{C^{2,\beta}} \) of \( f \). This is the purpose of the following result.

We will prove an estimate valid on arbitrary large time scale, which will be used later to prove a global existence result.

**Proposition 2.8.** For any \( 0 < \beta < 1/2 \), there exist two positive constant \( \varepsilon_0 \) and \( c_0 \) such that, for any \( \varepsilon \in (0, \varepsilon_0] \), any smooth solution \( f \in C^1([0, T]; H^\infty(\mathbb{R})) \) of the Muskat equation (8), and any time \( t \leq \min \{ \varepsilon^{-c_0}, T \} \), there holds
\[
\varepsilon^\beta \int_0^t \| f(\tau) \|_{C^{2,\beta}} \, d\tau \leq \varepsilon^\frac{\beta}{2} \| f_0 \|_{H^{\frac{3}{2}}}
+ \varepsilon^\frac{\beta}{2} \left( 2 + \sup_{s \in [0, t]} \| f(s) \|_{H^{\frac{3}{2}}} \right)^2 \log \left( 2 + \int_0^t \| f(s) \|_{H^{\frac{3}{2}}}^2 \, ds \right) \left( \int_0^t \| f(s) \|_{H^{\frac{3}{2}}}^2 \, ds \right)^\frac{1}{2}.
\]

**Proof.** The classical Sobolev embeddings implies that
\[
\| f(t) \|_{C^{2,\beta}} \lesssim \| f(t) \|_{H^{\frac{3}{2} + \beta}}.
\]
To estimate the latter Sobolev norm, the key point will be to apply the following interpolation inequality.

**Lemma 2.9.** Consider three real numbers
\[
\gamma > 0, \quad \beta_1 > 0 \quad \text{and} \quad 0 < \beta_2 < 2.
\]
Then, there exists a constant $C$ such that, for any function $g = g(t, x)$,
\[
\|g(t)\|_{\dot{H}^\gamma} \lesssim \frac{1}{(\nu t)^{\frac{2\gamma}{7}}} \|g(0)\|_{\dot{H}^{\gamma-\beta_1}} + \int_0^t \frac{1}{(\nu (t-s))^{\frac{2\gamma}{7}}} \|\partial_t g(s) - \nu \partial_{xx} g(s)\|_{\dot{H}^{\gamma-\beta_2}} \, ds.
\]
(20)

**Proof.** Set $G := \partial_t g - \nu \partial_{xx} g$. Then, one has,
\[
\dot{g}(t, \xi) = e^{-\nu|\xi|^2} \dot{g}(0, \xi) + \int_0^t e^{-\nu(t-s)|\xi|^2} \dot{G}(s, \xi) \, ds.
\]
The desired results then follows from Minkowski’s inequality. $\square$

Now, apply (20) with
\[
\gamma = \frac{5}{2} + \beta, \quad \beta_1 = 1 + \beta, \quad \beta_2 = \frac{3}{2} + \beta, \quad \nu = |\log(\varepsilon)|^{-1},
\]
to get
\[
\|f(t)\|_{C^{2,\beta}} \lesssim \|f(t)\|_{\dot{H}^\frac{3}{2} + \beta} \lesssim |\log(\varepsilon)|^\frac{1+\beta}{2} t^{-\frac{1+\beta}{2}} \|f_0\|_{\dot{H}^\frac{1}{2}} + \int_0^t |\log(\varepsilon)|^\frac{3+\beta}{2} (t-s)^{-\frac{3+\beta}{2}} \|\partial_t f - |\log(\varepsilon)|^{-1} \partial_x^3 f\|_{\dot{H}^1} \, ds.
\]
(21)

It remains to estimate the $\dot{H}^1$-norm of $\partial_t f - |\log(\varepsilon)|^{-1} \partial_{xx} f$. In view of the equation (8), this is equivalent to bound the $\dot{H}^1$-norm of
\[
\frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \left(1 - \chi\left(\frac{\alpha}{\varepsilon}\right)\right) \, d\alpha.
\]
We will split the latter term into two pieces and estimate them separately.

Firstly, directly from (10) and Minkowski’s inequality, we obtain that
\[
\left\| \frac{1}{\pi} \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \chi\left(\frac{\alpha}{\varepsilon}\right) \, d\alpha \right\|_{\dot{H}^1} \lesssim \int_{|\alpha| \leq 2\varepsilon} \left(\|\Delta_\alpha f_{xx}\|_{L^2} + \|\Delta_\alpha f_x\|_{L^4}^2\right) \, d\alpha \lesssim \varepsilon^{\frac{3}{2} + \beta} \left(\int_{\mathbb{R}} \|\Delta_\alpha f_{xx}\|_{L^2} \|\alpha|^{-2\beta} \, d\alpha\right)^{\frac{1}{2}} + \int_{\mathbb{R}} \|\Delta_\alpha f_x\|_{L^4}^2 \, d\alpha.
\]
(22)

Now we use the following inequality:
\[
\int_{\mathbb{R}^2} |\Delta_\alpha \tilde{f}|^2 |\alpha|^{-2\beta} \, d\alpha \, dx \sim \|\tilde{f}\|_{H^{\frac{3}{2} + \beta}}^2.
\]
Indeed,
\[
\int_{\mathbb{R}^2} |\Delta_\alpha \tilde{f}|^2 |\alpha|^{-2\beta} \, d\alpha \, dx = \int_{\mathbb{R}^2} \left[\frac{|\tilde{f}(x) - \tilde{f}(x - \alpha)|}{|\alpha|^{1/2 + \beta}}\right]^2 \, d\alpha \, dx \sim \|\tilde{f}\|_{H^{\frac{3}{2} + \beta}}^2.
\]

Similarly, using Sobolev embedding in Besov’s spaces, we get
\[
\int_{\mathbb{R}} \|\Delta_\alpha f_x\|_{L^4}^2 \, d\alpha \lesssim \|f\|_{H^{\frac{3}{2}}}^2.
\]
It follows that
\[
\left\| \frac{1}{\pi} \int_R \frac{\partial_x \Delta f}{1 + (\Delta f)^2} \chi \left( \frac{\alpha}{\varepsilon} \right) \, d\alpha \right\|_{\dot{H}^1} \lesssim \varepsilon^{\frac{1}{2} + \beta} \| f \|_{\dot{H}^{\frac{7}{2} + \beta}} + \| f \|_{\dot{H}^2} \| f \|_{\dot{H}^{\frac{3}{2}}},
\]
where we used an interpolation inequality in Sobolev spaces. On the other hand, it follows from the estimate (44) below that,
\[
\left\| \int_R \frac{\partial_x \Delta f}{1 + (\Delta f)^2} \, d\alpha \right\|_{\dot{H}^1} \lesssim \| \mathcal{T}(f) f \|_{\dot{H}^1}
\]
\[
\lesssim \left( 1 + \| f \|_{\dot{H}^{\frac{7}{2}}} \right)^2 \log \left( 2 + \| f \|_{\dot{H}^2}^2 \right)^{\frac{1}{2}} \| f \|_{\dot{H}^2}.
\]
By gathering the two previous estimates, we conclude that
\[
\| \partial_t f - |\log(\varepsilon)|^{-1} \partial_x^2 f \|_{\dot{H}^1} \lesssim \varepsilon^{\frac{1}{2} + \beta} \| f \|_{\dot{H}^{\frac{7}{2} + \beta}} + \left( 1 + \| f \|_{\dot{H}^{\frac{7}{2}}} \right)^2 \log \left( 2 + \| f \|_{\dot{H}^2}^2 \right)^{\frac{1}{2}} \| f \|_{\dot{H}^2}.
\]
Set
\[
b = \frac{\frac{3}{2} + \beta}{2}.
\]
By reporting this bound in (21), we find that
\[
\| f(t) \|_{\dot{H}^{\frac{7}{2} + \beta}} \lesssim |\log(\varepsilon)|^{\frac{1 + \beta}{2}} t^{-\frac{1 + \beta}{2}} \| f_0 \|_{\dot{H}^{\frac{3}{2}}} + \varepsilon^{\frac{1}{2} + \beta} \log(\varepsilon)^b \int_0^t (t - s)^{-b} \| f(s) \|_{\dot{H}^{\frac{7}{2} + \beta}} \, ds
\]
\[
+ |\log(\varepsilon)|^b \int_0^t (t - s)^{-b} \left( 1 + \| f(s) \|_{\dot{H}^{\frac{7}{2}}} \right)^2 \log \left( 2 + \| f(s) \|_{\dot{H}^2}^2 \right)^{\frac{1}{2}} \| f(s) \|_{\dot{H}^2} \, ds.
\]
So,
\[
\int_0^t \| f(\tau) \|_{\dot{H}^{\frac{7}{2} + \beta}} \, d\tau \lesssim |\log(\varepsilon)|^{\frac{1 + \beta}{2}} t^{1 - \frac{1 + \beta}{2}} \| f_0 \|_{\dot{H}^{\frac{3}{2}}} + \varepsilon^{\frac{1}{2} + \beta} |\log(\varepsilon)|^b t^{1 - b} \int_0^t \| f(s) \|_{\dot{H}^{\frac{7}{2} + \beta}} \, ds
\]
\[
+ |\log(\varepsilon)|^b t^{1 - b} \int_0^t \left( 1 + \| f(s) \|_{\dot{H}^{\frac{7}{2}}} \right)^2 \log \left( 2 + \| f(s) \|_{\dot{H}^2}^2 \right)^{\frac{1}{2}} \| f(s) \|_{\dot{H}^2} \, ds.
\]
As a result, there exists $c_0 > 0$ and $\varepsilon_0 \leq 1$ such that, if $t \leq \varepsilon^{-c_0}$ and $\varepsilon \leq \varepsilon_0$,
\[
\int_0^t \| f(\tau) \|_{\dot{H}^{\frac{7}{2} + \beta}} \, d\tau \leq \varepsilon^{-\beta} \| f_0 \|_{\dot{H}^{\frac{3}{2}}}
\]
\[
+ |\log(\varepsilon)|^b t^{1 - b} \mathcal{K}(t) \int_0^t \log \left( 2 + \| f(s) \|_{\dot{H}^2}^2 \right)^{\frac{1}{2}} \| f(s) \|_{\dot{H}^2} \, ds,
\]
where
\[
\mathcal{K}(t) = \sup_{s \in [0,t]} \left( 1 + \| f(s) \|_{\dot{H}^{\frac{7}{2}}} \right)^2.
\]
Now observe that
\[
\int_0^t \log \left( 2 + \| f(s) \|^2_{H^2} \right)^{\frac{1}{2}} \| f(s) \|_{H^2} \, ds \\
\leq (t + 1)^{\frac{3}{4}} \log \left( 2 + \int_0^t \| f(s) \|^2_{H^2} \, ds \right)^{\frac{1}{2}} \left( \int_0^t \| f(s) \|^2_{H^2} \, ds \right)^{\frac{1}{2}}.
\]
Therefore, up to modifying the values of \( c_0 > 0 \) and \( \varepsilon_0 \), we see that, for \( t \leq \varepsilon^{-c_0} \) and \( \varepsilon \leq \varepsilon_0 \), we have
\[
\varepsilon^\beta \int_0^t \| f(\tau) \|_{C^{2,\beta}} \, d\tau \lesssim \varepsilon \| f_0 \|_{H^\frac{3}{2}} \\
+ \varepsilon \| f_0 \|_{H^\frac{3}{2}} \log \left( 2 + \int_0^t \| f(s) \|^2_{H^2} \, ds \right)^{\frac{1}{2}} \left( \int_0^t \| f(s) \|^2_{H^2} \, ds \right)^{\frac{1}{2}}.
\]
This completes the proof. \( \square \)

2.5. Global in time estimates, under a smallness assumption.

**Proposition 2.10.** Let \( T > 0 \) and consider a smooth solution \( f \in C^1([0, T], H^\infty(\mathbb{R})) \) of the Muskat equation (8). Set
\[
K = 1 + 16 \left( \frac{C_2}{C_1} \right)^2
\]
and assume that
\[
2 \left( K + \frac{C_0}{C_1} \right)^{\frac{3}{2}} \left( 2 + \| \partial_x f_0 \|_{L^\infty} \right)^2 \| f_0 \|_{H^\frac{3}{2}} \leq 1,
\]
where the constants \( C_0, C_1, C_2 \) are as defined in the statements of Lemma 2.2 and Proposition 2.6. Then there exists \( \varepsilon_0 \) depending only on \( C_0, C_1, C_2 \) and \( \| f_0 \|_{L^2} \) such that, if \( \varepsilon \leq \varepsilon_0 \), then
\[
\sup_{0 \leq \tau \leq T} \| f(\tau) \|_{H^\frac{3}{2}} \leq \frac{1}{\sqrt{K \left( 2 + \| \partial_x f_0 \|_{L^\infty} \right)^2}} \quad \text{and} \quad \int_0^T \| f(\tau) \|^2_{H^\frac{3}{2}} \, d\tau \leq \frac{1}{C_0}.
\]

**Proof.** We apply the previous a priori estimate (16) in the simplest case where \( \phi = 1 \). With this choice, the quantities \( A_\phi \) and \( B_\phi \) defined by (13) simplify to
\[
A(t) = \| D^{\frac{3}{2}} f(t) \|^2_{L^2}, \\
B(t) = \| D^2 f(t) \|^2_{L^2} = \| f(t) \|^2_{H^2}.
\]
Introduce the set
\[
I = \left\{ t \in [0, T] : \int_0^t B(\tau) \, d\tau \leq \frac{2}{3C_0} \quad \text{and} \quad \sup_{0 \leq \tau \leq t} A(\tau) \leq \frac{1}{K \left( 2 + \| \partial_x f_0 \|_{L^\infty} \right)^2} \right\}.
\]
We want to prove that \( I = [0, T] \). Since 0 belongs to \( I \) by assumption on the initial data, and since \( I \) is closed, it suffices to prove that \( I \) is open. To do so, we consider
a time \( t^* \in [0, T) \) which belongs to \( I \). Our goal is to prove that

\[
\int_0^t B(\tau) \, d\tau \leq \frac{1}{2C_0} \quad \text{and} \quad \sup_{0 \leq \tau \leq t^*} A(\tau) \leq \frac{1}{4K(2 + \|\partial_x f_0\|_{L^\infty})^2}.
\]

This will imply at once that \( t^* \) belongs to the interior of \( I \).

Since \( \mu(t) = 1 \) for \( \phi = 1 \), the estimate (16) implies that there are two positives constants \( C_1, C_2 \) such that

\[
\frac{d}{dt} A(t) + C_1 \frac{B(t)}{1 + \|\partial_x f(t)\|_{L^\infty}^2} \leq C_2 \left( A(t) + \sqrt{A(t)} \right) B(t).
\]

By combining Proposition 2.8 with Lemma 2.2, we get, for any \( t \),

\[
\|\partial_x f(t)\|_{L^\infty} - \|\partial_x f_0\|_{L^\infty} \leq C_0 \int_0^t B(\tau) \, d\tau + C_0 \varepsilon^\frac{d}{2} \|f_0\|_{H^\frac{d}{2}} + C_0 \varepsilon^\frac{d}{2} \left[ \sup_{s \in [0, t]} \left( 1 + \|f(s)\|_{H^\frac{d}{2}} \right) \log \left( 2 + \int_0^t B(\tau) \, d\tau \right) \right] \left( \int_0^t B(\tau) \, d\tau \right)^\frac{1}{2}.
\]

By (17),

\[
\sup_{s \in [0, t]} \|f(s)\|_{H^\frac{d}{2}} \leq \|f_0\|_{L^2} + (1 + C_1 \varepsilon^\frac{d}{2}) \sup_{s \in [0, t]} \|f(s)\|_{H^\frac{d}{2}}.
\]

If \( t \leq t^* \) and \( \varepsilon \ll 1 \), then the bound on the integral of \( B \) implies that

\[
\|\partial_x f(t)\|_{L^\infty} - \|\partial_x f_0\|_{L^\infty} \leq \frac{2}{3}.
\]

So the right-hand side above can be bounded by some absolute constant depending only on \( C_0 \) and \( \varepsilon \). For \( \varepsilon \) small enough, we conclude that

\[
1 + \|\partial_x f(t)\|_{L^\infty} \leq 2 + \|\partial_x f_0\|_{L^\infty}.
\]

On the other hand, if \( t^* \in I \), then for any \( t \leq t^* \) we have

\[
A(t) + \sqrt{A(t)} \leq 2\sqrt{A(t)} \leq \frac{2}{\sqrt{K(2 + \|\partial_x f_0\|_{L^\infty})^2}}.
\]

Consequently, for any \( t \leq t^* \), we have

\[
\frac{d}{dt} A(t) + C_1 \frac{B(t)}{(2 + \|\partial_x f_0\|_{L^\infty})^2} \leq \frac{2C_2}{\sqrt{K(2 + \|\partial_x f_0\|_{L^\infty})^2}} B(t).
\]

By definition of \( K \), we have

\[
K \geq \frac{16C_2^2}{C_1^2},
\]

so, for any \( t \leq t^* \),

\[
\frac{d}{dt} A(t) + C_1 \frac{2}{2} \left( \frac{B(t)}{2 + \|\partial_x f_0\|_{L^\infty}} \right) \leq 0.
\]

Integrate this on the time interval \( [0, t^*] \), to infer that

\[
\sup_{t \in [0, t^*]} A(t) + \frac{C_1}{2} \int_0^{t^*} B(t) \, dt \leq A(0).
\]
Using the smallness assumption (25), the previous inequality (30) implies at once that

\[
\sup_{t \in [0,T^*]} A(t) \leq A(0) \leq \frac{1}{4K(2 + \|\partial_x f_0\|_{L^\infty})^2},
\]

\[
\int_0^{T^*} B(t) \, dt \leq \frac{2(2 + \|\partial_x f_0\|_{L^\infty})^2}{C_1} A(0) \leq \frac{1}{2C_0}.
\]

These are the wanted bootstrap inequalities. As explained above, by connexity, this proves that \( I = [0, T] \), which implies the desired results in (26). \( \square \)

2.6. A priori estimates locally in time, for arbitrary initial data.

**Proposition 2.11.** Consider \( \phi \) satisfying assumptions (H1)–(H3) in Definition 1.2. Let \( T > 0 \) and consider a smooth solution \( f \in C^1([0, T], H^\infty(\mathbb{R})) \) of the Muskat equation (8). For any \( M_0 > 0 \) there exists \( \varepsilon_0 > 0 \) and \( T_0 > 0 \) such that the following properties holds. If \( \varepsilon \in (0, \varepsilon_0) \) and

\[
\| |D|^{2/3} f(0)|^2_{L^2} \leq M_0,
\]

then, with \( T^* = \min\{T, T_0\} \), there holds

\[
\sup_{t \in [0,T^*]} A_\phi(t) \leq 5M_0, \quad \int_0^{T^*} \mu_\phi(t)^2 B_\phi(t) \, dt \leq \frac{1}{C_0},
\]

where \( A_\phi, B_\phi, \mu_\phi \) are defined in (13) while \( C_0 \) is given by Lemma 2.2.

**Proof.** For this proof we skip the index \( \phi \) and write simply \( A, B, \mu \).

Since (see (14)),

\[
\|f(t)\|_{H^2} \leq C_\mu(t)B(t)^{1/2}.
\]

We then apply Proposition 2.8 for some fixed parameter \( \beta > 0 \). Then, it follows from (16) that

\[
\frac{d}{dt} A(t) + C_1 \frac{B(t)}{\nu(t)^2} \leq C_2 \left( \sqrt{A(t)} + A(t) \right) \mu(t)B(t),
\]

where

\[
\nu(t) = 1 + \|\partial_x f_0\|_{L^\infty} + C_0 \int_0^t \mu(\tau)^2 B(\tau) \, d\tau + C_0 \varepsilon_0^\frac{1}{2} \|f_0\|_{H^\frac{7}{2}} + C_0 \varepsilon_0^{\frac{5}{2}} \left[ \sup_{\tau \in [0, t]} \left( 1 + \|f(\tau)\|_{H^\frac{7}{2}} \right)^2 \right] \log \left( 2 + \int_0^t \mu(\tau)^2 B(\tau) \, d\tau \right)^{\frac{1}{2}} \left( \int_0^t \mu(\tau)^2 B(\tau) \, d\tau \right)^{\frac{1}{2}}.
\]

Given a positive number \( T_0 \) to be determined, introduce the set

\[
I(T_0) = \left\{ t \in [0, \min\{T, T_0\}] ; \int_0^t \mu(\tau)^2 B(\tau) \, d\tau \leq \frac{2}{3C_0} \text{ and } \sup_{0 \leq \tau \leq t} A(\tau) \leq 5M_0 \right\}.
\]

We want to prove that \( I(T_0) = [0, \min\{T, T_0\}] \). Since 0 belongs to \( I(T_0) \) by assumption on the initial data, and since \( I(T_0) \) is closed, it suffices to prove that \( I(T_0) \) is
open. To do so, we consider a time \( t^* \in [0, \min\{T, T_0\} \} \) which belongs to \( I(T_0) \). Our goal is to prove that
\[
\int_0^{t^*} \mu(\tau)^2 B(\tau) \, d\tau \leq \frac{1}{2C_0} \quad \text{and} \quad \sup_{0 \leq \tau \leq t^*} A(\tau) \leq 4M_0.
\]
This will imply at once that \( t^* \) belongs to the interior of \( I(T_0) \).

As in the previous proof, we use (10) to write
\[
(32) \quad \sup_{s \in [0, t]} \| f(s) \|_{\dot{H}^{\frac{3}{2}}} \leq \| f_0 \|_{L^2} + (1 + C \varepsilon^2 t) \sup_{s \in [0, t]} \| f(s) \|_{\dot{H}^{\frac{3}{2}}}.
\]

It \( t \leq t^* \) with \( t^* \in I(T_0) \), then
\[
\nu(t) \leq 1 + \| \partial_x f_0 \|_{L^\infty} + \frac{2}{3} + C_0 \varepsilon^2 M_0
\]
\[
+ C_0 \varepsilon^2 (1 + \| f_0 \|_{L^2} + 6M_0)^2 \log \left( 2 + \frac{2}{3C_0} \right) \frac{1}{2} \left( \frac{2}{3C_0} \right)^{\frac{1}{2}}.
\]
Hence, one can define \( \varepsilon_0 \) small enough, depending only on \( M_0, \| f_0 \|_{L^2} \) and the fixed parameter \( \beta \), such that if \( \varepsilon \leq \varepsilon_0 \) and if \( t^* \in I(T_0) \), then for any \( t \in [0, t^*] \), we have
\[
\nu(t) \leq 2 + \| \partial_x f_0 \|_{L^\infty}.
\]

Consequently
\[
\frac{d}{dt} A(t) + C_1 \frac{B(t)}{(2 + \| \partial_x f_0 \|_{L^\infty})^2} \leq C_2 \left( A(t) + \sqrt{A(t)} \right) \mu(t) B(t).
\]

Introduce the function
\[
\mathcal{E}(r, m) := \sup_{\rho \geq 0} \left\{ C_2 \left( \sqrt{\rho} + r \right) \left( \frac{\rho}{\sqrt{r}} \right)^{-1} \rho - \frac{C_1}{2} \frac{\rho}{m} \right\}.
\]

Then, for any \( t \in [0, t^*] \), we have
\[
\frac{d}{dt} A(t) + \frac{C_1}{2} \frac{B(t)}{(2 + \| \partial_x f_0 \|_{L^\infty})^2} \leq \mathcal{E} \left( A(t), \| \partial_x f_0 \|_{L^\infty} \right).
\]

Assume that the number \( T_0 \) satisfies
\[
T_0 \leq \frac{A(0)}{4\mathcal{E}(4A(0), \| \partial_x f_0 \|_{L^\infty})}.
\]

Then, for any \( t \leq t^* \), we get that
\[
\sup_{\tau \leq t} A(\tau) + \frac{C_1}{2} \frac{1}{(2 + \| \partial_x f_0 \|_{L^\infty})^2} \int_0^t B(\tau) \, d\tau \leq 4A(0).
\]

In particular, for \( t = t^* \), this gives
\[
(33) \quad \sup_{t \leq t^*} A(t) \leq 4A(0), \quad \int_0^{t^*} B(t) \, dt \leq \frac{8A(0)}{C_1} (2 + \| \partial_x f_0 \|_{L^\infty})^2.
\]

To get the result, we must show that
\[
(34) \quad C_0 \int_0^T \mu(t)^2 B(t) \, dt \leq \frac{1}{2}.
\]
Recall that
\[ \mu(t) = \left( \phi \left( \frac{B(t)}{A(t)} \right) \right)^{-1}. \]
Since \( \phi \) is increasing and since \( A(t) \leq 4A(0) \), we have
\[ \mu(t) \leq \left( \phi \left( \frac{B(t)}{4A(0)} \right) \right)^{-1}. \]
Now, we claim that the function \( F : [0, +\infty) \to [0, +\infty) \), defined by
\[ F(r) = \left( \phi \left( \frac{r}{4A(0)} \right) \right)^{-1} r, \]
is increasing. To see this decompose \( F(r) \) under the form
\[ F(r) = F_1(r) F_2(r) \]
with
\[ F_1(r) = \frac{r}{(\log(\lambda_0 + r))^2} \quad F_2(r) = \frac{\log(\lambda_0 + r)}{\phi(r/4A(0))}. \]
Then
\[
\int_0^{t^*} \mu(t)B(t) \, dt \leq \int_0^{t^*} \left( \phi \left( \frac{B(t)}{4A(0)} \right) \right)^{-2} B(t) \, dt \\
\leq \int_0^{t^*} \left( \phi \left( \frac{r}{4A(0)} \right) \right)^{-2} r \, dt + \int_0^{t^*} \left( \phi \left( \frac{r}{4A(0)} \right) \right)^{-2} B(t) \, dt \\
\leq \left( \phi \left( \frac{r}{4A(0)} \right) \right)^{-2} 8A(0) \left( 2 + \|\partial_x f_0\|_{L^\infty} \right)^2,
\]
for any \( r \geq 1 \). Now we successively determine two numbers \( r_0 > 1 \) and \( T_0 > 0 \) such that
\[ C_0 \left( \phi \left( \frac{r_0}{4A(0)} \right) \right)^{-2} 8A(0) \left( 2 + \|\partial_x f_0\|_{L^\infty} \right)^2 = \frac{1}{4}, \]
and
\[ T_0 \left( \phi \left( \frac{r_0}{4A(0)} \right) \right)^{-2} r_0 = \frac{1}{4}. \]
With this choice we get (34) and we obtain that \( I(T_0) = [0, \min\{T,T_0\}] \), which is equivalent to the statement of the proposition. \( \square \)

2.7. Transfer of compactness. Previously, we have prove a priori estimates for the spatial derivatives. In this paragraph, we gather results from which we will infer estimates for the time derivatives as well as for the nonlinearity in the Muskat equation. These estimates serve to pass to the limit the equation (which is needed to regularize the solutions).

The Muskat equation (1) can be written under the form
\[ \partial_t f + |D| f = T(f)f, \]
where \( T(f) \) is the operator defined by
\[ T(f)g = -\frac{1}{\pi} \int_\mathbb{R} (\partial_\alpha \Delta_\alpha g) \frac{(\Delta_\alpha f)^2}{1 + (\Delta_\alpha f)^2} \, d\alpha. \]
We recall the following result from Proposition 2.3 in [1] and from Remark 2.9 and Propositions 2.10 and 2.13 in [2].

**Proposition 2.12.** i) For all $\delta \in [0, 1/2)$, there exists a constant $C > 0$ such that, for all functions $f_1, f_2 \in H^{1-\delta}(\mathbb{R}) \cap H^{\frac{3}{2}+\delta}(\mathbb{R})$,
\[
\| (T(f_1) - T(f_2)) f_2 \|_{L^2} \leq C \| f_1 - f_2 \|_{H^{1-\delta}} \| f_2 \|_{H^{\frac{3}{2}+\delta}}.
\]

ii) One can decompose the nonlinearity under the form
\[
T(f)g = \frac{(\partial_x f)^2}{1 + (\partial_x f)^2} |D| g + V(f) \partial_x g + R(f, g),
\]
where the coefficient $V(f)$ and the remainder term $R(f, g)$ satisfy the following estimates:
\[
\| V(f) \|_{L^\infty} \leq C \int_{\mathbb{R}} |\xi| |\hat{f}(\xi)| \, d\xi,
\]
\[
\| R(f, g) \|_{L^2} \leq C \| g \|_{H^{\frac{3}{2}}} \| f \|_{H^\frac{3}{2}},
\]
for some absolute constant $C$. Moreover,
\[
\| T(f) f \|_{H^1} \leq C \left( \| f \|_{H^\frac{3}{2}} + \| f \|_{H^\frac{3}{2}}^2 + 1 + \| V(f) \|_{L^\infty} \right) \| f \|_{H^2},
\]
and,
\[
| (V(f_1) \partial_x g, |D| g) | \leq C \left( \| f \|_{H^2} + \| f \|_{H^\frac{3}{2}}^2 \right) \| g \|_{H^{\frac{3}{2}}} \| g \|_{H^1}.
\]

For later purpose, we need a refinement of (42).

**Proposition 2.13.** There exists a positive constant $C > 0$ such that, for all function $f \in H^2(\mathbb{R})$,
\[
\| T(f) f \|_{H^1} \leq C \left( 1 + \| f \|_{H^\frac{3}{2}} \right)^2 \log \left( 2 + \| f \|_{H^2}^2 \right)^{\frac{1}{2}} \| f \|_{H^2}.
\]

**Proof.** In view of (42) and (40), it is sufficient to estimate the $L^1$-norm of $|\xi| \hat{f}$. Write,
\[
\int_{\mathbb{R}} |\xi| |\hat{f}| \, d\xi = \int_{|\xi| > \lambda} |\xi|^{-1} |\xi|^2 |\hat{f}| \, d\xi + \int_{|\xi| \leq \lambda} (|\xi| + 1)^{-\frac{1}{2}} |\xi| (1 + |\xi|)^{\frac{1}{2}} |\hat{f}| \, d\xi
\]
\[
\lesssim \left( \int_{|\xi| > \lambda} \frac{1}{|\xi|^2} \, d\xi \right)^{\frac{1}{2}} \| f \|_{H^2} + \left( \int_{|\xi| \leq \lambda} \frac{1}{|\xi|} \, d\xi \right)^{\frac{1}{2}} \left( \| f \|_{H^\frac{3}{2}} + \| f \|_{L^2} \right)
\]
\[
\lesssim \lambda^{-\frac{1}{2}} \| f \|_{H^2} + \log(1 + \lambda)^{\frac{1}{2}} \left( \| f \|_{H^\frac{3}{2}} + \| f \|_{L^2} \right).
\]

Choosing $\lambda = \| f \|_{H^2}^2$, we obtain
\[
\int_{\mathbb{R}} |\xi| |\hat{f}| \, d\xi \lesssim 1 + \log(1 + \| f \|_{H^2}^2)^{\frac{1}{2}} \left( \| f \|_{H^\frac{3}{2}} + \| f \|_{L^2} \right).
\]

By reporting this (40) and then using (42), we get the desired result (44). □

By using the equation (37), we deduce at once the following bound.
Corollary 2.14. There exists a non-decreasing function $\mathcal{F}: \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any $T > 0$, any $\varepsilon$ and any smooth solution $f$ in $C^1([0, T]; H^\infty(\mathbb{R}))$ of the Muskat equation (8), if one sets

$$M_\varepsilon(T) = \sup_{t \in [0, T]} \left( \|f(t)\|^2_{H^2} + \|f(t)\|^2_{L^2} \right) + \int_0^T \|f(t)\|^2_{H^1} dt + \log(\varepsilon)^{-1} \int_0^T \|f(t)\|^2_{H^2} dt$$

then,

$$(45) \quad \int_0^T \frac{\|T(f)\|^2_{H^1}}{\log(2 + \|T(f)\|^2_{H^1})} dt \leq \mathcal{F}(M_\varepsilon(T)),$$

and

$$(46) \quad \int_0^T \frac{\|\partial_t f\|^2_{H^1}}{\log(2 + \|\partial_t f\|^2_{H^1})} dt \leq \mathcal{F}(M_\varepsilon(T)).$$

Proof. Let $C$ be the constant given by Proposition 2.13 and set $\tilde{C} = \max\{C, 1\}$. We claim that

$$\frac{\|T(f)\|^2_{H^1}}{\log(2 + \|T(f)\|^2_{H^1})} \leq \tilde{C}^2 \left( \|f\|^2_{H^2} + \|f\|^2_{H^1} + \|f\|^2_{H^2} + 1 \right)^2 \|f\|^2_{H^2}.$$

If $\|T(f)\|^2_{H^1} \leq \|f\|^2_{H^2}$, then this is obvious. Otherwise, this follows at once from (44). This implies (45).

The proof of (46) follows from similar argument, using the equation to estimate $\partial_t f$ in terms of $T(f) f$.

\[ \square \]

It follows from the previous results that one can extract from the solutions of the approximate Cauchy problems (8) a sub-sequence converging to a solution of the Muskat equation (1). Since it is rather classical, we do not include the details and refer for instance to [16, 18].

2.8. Uniqueness. To prove the uniqueness of the solution to the Cauchy problem for rough initial data, we shall prove an estimate for the difference of two solutions.

Proposition 2.15. Let $T > 0$ and consider two solutions $f_1, f_2$ of the Muskat equation, with initial data $f_{1,0}, f_{2,0}$ respectively, satisfying

$$f_k \in C^0([0, T]; \dot{W}^{1,\infty}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})) \cap C^1([0, T]; \dot{H}^{1/2}(\mathbb{R})) \cap L^2(0, T; \dot{H}^{3/2}(\mathbb{R})), \quad k = 1, 2.$$ 

Assume that

$$\sup_{t \in [0, T]} \left( \|f_k(t)\|^2_{H^2} + \|f_k(t)\|^2_{W^{1,\infty}} \right) + \int_0^T \|f_k\|^2_{H^2} dt \leq M < \infty, \quad k = 1, 2.$$ 

Then the difference $g = f_1 - f_2$ is estimated by

$$(48) \quad \sup_{t \in [0, T]} \|g(t)\|^2_{H^2} \leq \|g(0)\|^2_{H^2} \exp \left( C(M + 1)^5 \int_0^T \left( \|f_1\|^2_{H^2} + \|f_2\|^2_{H^2} \right) dt \right).$$
Proof. Since \( \partial_t f_k + |D| f_k = T(f_k) f_k \), it follows from the decomposition (39) of \( T(f_k) f_k \) that the difference \( g = f_1 - f_2 \) satisfies

\[ \partial_t g + \frac{|D| g}{1 + (\partial_x f_1)^2} = V(f_1) \partial_x g + R(f_1, g) + (T(f_2 + g) - T(f_2)) f_2. \]

Since \( g \) belongs to \( C^1([0, T]; \dot{H}^{\frac{3}{2}}(\mathbb{R})) \), we may take the \( L^2 \)-scalar product of this equation with \( |D| g \) to get

\[ \frac{1}{2} \frac{d}{dt} \|g\|_{\dot{H}^{\frac{3}{2}}}^2 + \int \frac{(|D| g)^2}{1 + (\partial_x f_1)^2} \, dx \leq \| V(f_1) \partial_x g, |D| g \| + \| R(f_1, g) \|_{L^2} \| g \|_{H^1} \]

\[ + \| (T(f_2 + g) - T(f_2)) f_2 \|_{L^2} \| g \|_{H^1}. \]

It follows from Proposition 2.12 that

\[ \frac{d}{dt} \|g\|_{\dot{H}^{\frac{3}{2}}}^2 + M^{-1} \|g\|_{H^1}^2 \lesssim \left( \| f_1 \|_{H^2} + \| f_1 \|_{\dot{H}^{\frac{3}{2}}}^2 \right) \| g \|_{\dot{H}^{\frac{3}{2}}} \| g \|_{H^1} \]

\[ + \| f_2 \|_{H^2} \| f_2 \|_{\dot{H}^{\frac{3}{2}}} \| g \|_{\dot{H}^{\frac{3}{2}}} \| g \|_{H^1} \]

\[ \lesssim \| f_1 \|_{H^2} (1 + M) \| g \|_{\dot{H}^{\frac{3}{2}}} \| g \|_{H^1} + M \| f_2 \|_{H^2} \| f_2 \|_{\dot{H}^{\frac{3}{2}}} \| g \|_{\dot{H}^{\frac{3}{2}}} \| g \|_{H^1}. \]

Thus, thanks to Holder’s inequality, one gets

\[ \frac{d}{dt} \|g\|_{\dot{H}^{\frac{3}{2}}}^2 + \frac{1}{2M} \|g\|_{H^1}^2 \leq C(M + 1)^5 \left( \| f_1 \|_{H^2}^2 + \| f_2 \|_{H^2}^2 \right) \| g \|_{\dot{H}^{\frac{3}{2}}}^2 \]

which in turn implies (48). \( \square \)

2.9. The Cauchy problem for the approximate equations. It remains to prove Proposition 2.1.

Rewrite the equation (8) under the form

\[ \partial_t f - |\log(\varepsilon)|^{-1} \partial_x^2 f = N_\varepsilon(f), \]

with

\[ N_\varepsilon(f) = \frac{1}{\pi} \int_\mathbb{R} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \left( 1 - \chi \left( \frac{|\alpha|}{\varepsilon} \right) \right) \, d\alpha. \]

The next proposition shows that Equation (49) can be seen as a sub-critical parabolic equation.

Lemma 2.16. There holds

\[ \| N_\varepsilon(f) \|_{H^1} \lesssim \varepsilon^\frac{1}{2} \| f \|_{\dot{H}^{\frac{3}{2}}} + \left( 1 + \| f \|_{H^2} \right)^2 \log \left( 2 + \| f \|_{H^2}^2 \right)^{\frac{1}{2}} \| f \|_{H^2}, \]

and

\[ \| N_\varepsilon(f) \|_{L^2} \leq C \left( 1 + \| f \|_{H^2} \right)^2. \]
Proof. The estimate (50) follows at once from (23) and (24). To prove (51), we decompose $N_\varepsilon(f) = -|D|f + T(f)f + R_\varepsilon(f)$ where $T(f)$ is the operator already introduced in §2.7 and the remainder $R_\varepsilon(f)$ is as defined by (18). Recall from Proposition 2.3 in [1] that
\[
\|T(f)f\|_{L^2} \lesssim \|f\|_{H^1} \|f\|_{H^{\frac{5}{2}}},
\]
So the wanted conclusion follows from the estimate (19) for $R_\varepsilon(f)$. □

Multiply the latter equation by $(I - \Delta)^{3/2}f$ and integrate in time, to obtain
\[
\frac{1}{2} \frac{d}{dt} \|f\|_{H^{\frac{5}{2}}}^2 + |\log(\varepsilon)|^{-1} \|||D|f\|_{H^{\frac{5}{2}}}^2 \leq \|N_\varepsilon(f)\|_{H^1} \|f\|_{H^2}.
\]
Recall that
\[
\|N_\varepsilon(f)\|_{H^1} \lesssim \varepsilon^\frac{1}{2} \|f\|_{H^{\frac{5}{2}}} + \left(1 + \|f\|_{H^{\frac{5}{2}}}^2\right)^2 \log \left(2 + \|f\|_{H^2}^2\right)^{\frac{1}{2}} \|f\|_{H^2},
\]
Since $\varepsilon^\frac{1}{2} \ll |\log(\varepsilon)|^{-1}$ for $\varepsilon \ll 1$, we can absorb the contribution of $\varepsilon^\frac{1}{2} \|f\|_{H^{\frac{5}{2}}}$ in the right-hand side of (53) by the left-hand side of (52). On the other hand, since $5/2 > 2$, one can absorb the contribution of the other terms by using the Hölder’s inequality. This proves an a priori estimate for (49). We also get easily a contraction estimate similar to (but much simpler) the one given by Proposition 2.15. Then by using classical tools for semi-linear equations, we conclude that the Cauchy problem for (49) can be solved by standard iterative scheme.

References

[1] Thomas Alazard and Omar Lazar. Paralinearization of the Muskat equation and application to the Cauchy problem. Arch. Ration. Mech. Anal., 237(2):545–583, 2020.
[2] Thomas Alazard and Quoc-Hung Nguyen. On the Cauchy problem for the Muskat equation with non-Lipschitz initial data. arXiv preprint arXiv:2009.04343, 2020.
[3] Elia Brué and Quoc-Hung Nguyen. Advection diffusion equations with sobolev velocity field. arXiv:2003.08198v1.
[4] Elia Brué and Quoc-Hung Nguyen. On the Sobolev space of functions with derivative of logarithmic order. Adv. Nonlinear Anal., 9(1):836–849, 2020.
[5] Elia Brué and Quoc-Hung Nguyen. Sharp regularity estimates for solutions to the continuity equation drifted by sobolev vector fields. arXiv:1806.03466v2, 2020.
[6] Elia Brué and Quoc-Hung Nguyen. Sobolev estimates for solutions of the transport equation and ode flows associated to non-lipschitz drifts. Mathematische Annalen, 2020.
[7] Stephen Cameron. Global well-posedness for the two-dimensional Muskat problem with slope less than 1. Anal. PDE, 12(4):997–1022, 2019.
[8] Ángel Castro, Diego Córdoba, Charles Fefferman, and Francisco Gancedo. Breakdown of smoothness for the Muskat problem. Arch. Ration. Mech. Anal., 208(3):805–909, 2013.
[9] Ángel Castro, Diego Córdoba, Charles Fefferman, and Francisco Gancedo. Splash singularities for the one-phase Muskat problem in stable regimes. Arch. Ration. Mech. Anal., 222(1):213–243, 2016.
[10] Ángel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, and María López-Fernández. Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves. Ann. of Math. (2), 175(2):909–948, 2012.
[11] C. H. Arthur Cheng, Rafael Granero-Belinchón, and Steve Shkoller. Well-posedness of the Muskat problem with $H^2$ initial data. Adv. Math., 286:32–104, 2016.
[12] Peter Constantin, Diego Córdoba, Francisco Gancedo, Luis Rodríguez-Piazza, and Robert M. Strain. On the Muskat problem: global in time results in 2D and 3D. *Amer. J. Math.*, 138(6):1455–1494, 2016.

[13] Peter Constantin, Diego Córdoba, Francisco Gancedo, and Robert M. Strain. On the global existence for the Muskat problem. *J. Eur. Math. Soc. (JEMS)*, 15(1):201–227, 2013.

[14] Peter Constantin, Francisco Gancedo, Roman Shvydkoy, and Vlad Vicol. Global regularity for 2D Muskat equations with finite slope. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 34(4):1041–1074, 2017.

[15] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo. Interface evolution: the Hele-Shaw and Muskat problems. *Ann. of Math. (2)*, 173(1):477–542, 2011.

[16] Diego Córdoba and Francisco Gancedo. Contour dynamics of incompressible 3-D fluids in a porous medium with different densities. *Comm. Math. Phys.*, 273(2):445–471, 2007.

[17] Diego Córdoba and Francisco Gancedo. A maximum principle for the Muskat problem for fluids with different densities. *Comm. Math. Phys.*, 286(2):681-696, 2009.

[18] Diego Córdoba and Omar Lazar. Global well-posedness for the 2d stable muskat problem in $H^4$. *arXiv:1803.07528*.

[19] Fan Deng, Zhen Lei, and Fanghua Lin. On the two-dimensional Muskat problem with monotone large initial data. *Comm. Pure Appl. Math.*, 70(6):1115–1145, 2017.

[20] Francisco Gancedo and Omar Lazar. Global well-posedness for the 3d muskat problem in the critical sobolev space. *arXiv:2006.01787*.

[21] Rafael Granero-Belinchón and Stefano Scrobogna. On an asymptotic model for free boundary darcy flow in porous media. *arXiv:1810.11798*.

[22] Bogdan-Vasile Matioc. The Muskat problem in two dimensions: equivalence of formulations, well-posedness, and regularity results. *Anal. PDE*, 12(2):281–332, 2019.

[23] Quoc-Hung Nguyen. Quantitative estimates for regular lagrangian flows with bv vector fields. *arXiv:1805.01182*, 2018.

[24] Huy Q. Nguyen and Benoit Pausader. A paradifferential approach for well-posedness of the Muskat problem. *Arch. Ration. Mech. Anal.*, 237(1):35–100, 2020.

[25] Neel Patel and Robert M. Strain. Large time decay estimates for the Muskat equation. *Comm. Partial Differential Equations*, 42(6):977–999, 2017.

[26] Michael Siegel, Russel E. Caflisch, and Sam Howison. Global existence, singular solutions, and ill-posedness for the Muskat problem. *Comm. Pure Appl. Math.*, 57(10):1374–1411, 2004.

[27] Hans Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[28] Hans Triebel. Characterizations of Besov-Hardy-Sobolev spaces: a unified approach. *J. Approx. Theory*, 52(2):162–203, 1988.

[29] Fahuai Yi. Global classical solution of Muskat free boundary problem. *J. Math. Anal. Appl.*, 288(2):442–461, 2003.

**Thomas Alazard**
Université Paris-Saclay, ENS Paris-Saclay, CNRS,
Centre Borelli UMR9010, avenue des Sciences, F-91190 Gif-sur-Yvette France.

**Quoc-Hung Nguyen**
ShanghaiTech University,
393 Middle Huaxia Road, Pudong, Shanghai, 201210,
China