Smith-Purcell radiation from surface waves

A R Mkrtchyan, L Sh Grigoryan and A A Saharian
Institute of Applied Problems in Physics, 25 Nersessian Street, 0014 Yerevan, Armenia
E-mail: malpic@sci.am, levonshg@mail.ru, saharian@ysu.am

Abstract. We consider the radiation from an electron in flight over a surface wave of an arbitrary profile excited in a plane interface. For an electron bunch the conditions are specified under which the overall radiation essentially exceeds the incoherent part. In particular, it is shown that the radiation from the bunch with asymmetric density distribution of electrons in the longitudinal direction is partially coherent for waves with wavelengths much shorter than the characteristic longitudinal size of the bunch. The quantum radiation due to the dynamical Casimir effect induced by surface waves excited on a plane boundary is discussed.

1. Introduction
Surface waves have wide applications in various fields of science and technology. In the present talk, based on [1]-[4], we discuss the radiation from a charged particle flying over surface acoustic wave. The physics of this phenomenon is similar to that for the radiation of particle flying over a diffraction grating (Smith-Purcell radiation, for a review see [5]). The latter is used for the generation of the radiation in the range of millimeter and submillimeter waves.

2. Radiation from a single electron
Let a charge \( q \) move with constant velocity \( v \) parallel to a surface wave excited in the plane boundary of a homogeneous medium with permittivity \( \epsilon \) (see figure 1). If the axis \( z \) is aligned with the particle trajectory, then the equation of the interface has the form

\[
x = x_0(z,t) = -d + f(k_0 z \mp \omega_0 t),
\]

(1)

where \( k_0 \) and \( \omega_0 \) are the wave number and cyclic frequency, \( d \) is the distance from the non-excited interface, and \( f(u) \) is the function describing the wave profile, \( f(u + 2\pi) = f(u) \). We assume that the particle moves in the vacuum, \( x > x_0(z,t) \), and will consider the radiation in this region. From the symmetry properties of the problem it follows that the emission angle \( \theta \) (with respect to \( v \)) and the frequency \( \omega \) of the emitted photon are related by (\( m \) is an integer)

\[
\omega = m(k_0v \mp \omega_0)/(1 - \beta \cos \theta), \quad \beta = v/c.
\]

(2)

The dependence of the radiation intensity on \( d \) is given by the factor \( e^{-2\omega d \sigma/v} \), where

\[
\sigma = \sqrt{(1 - \beta^2)\omega_1^2/\omega^2 + \beta^2 \sin^2 \theta \sin^2 \varphi}, \quad \omega_1 = \omega \pm m\omega_0,
\]

(3)

with \( \varphi \) being the polar angle with respect to the \( x \) axis in the plane perpendicular to the particle trajectory. This factor does not depend on the wave profile in (1) and is determined by the dependence on distance \( d \) of the field spectral components for a uniformly moving particle.
In order to determine the radiation intensity we assume that the amplitude of the surface wave is small. Under the condition $β√ε - 1 > 1$, the spectral-angular distribution of the radiation intensity (per unit path length) in the region $x > 0$ is given by the expression

$$
\frac{dW}{dωdΩ} = \frac{2q^2(ε - 1)}{πc^2v^2} \sum_m ω^3 sin²θ cos²ϕ |f_m|^2 e^{-2ωσd/v} \times \frac{A_1σ^2 + A_2sin²θ sin²ϕ + A_3β^2sin^4θ sin^4ϕ}{[β^2 + (ε + 1) σ^2] (σ_2 + ε sin θ cos ϕ)^2} δ(cos θ - 1/β + mk_0/v/ω), \tag{4}
$$

where $dΩ = sin ϑdϑdϕ$, and we have assumed that $v ≫ ω_0/k_0$. In (4), $f_m = \frac{1}{2π} \int_{-π}^{+π} du f(u)e^{-imυ}$ is the Fourier transform of the profile function,

$$
σ_1 = \sqrt{β^2(ε - sin^2θ sin^2ϕ)} - 1, \quad σ_2 = \sqrt{ε - 1 + sin^2θ sin^2ϕ}, \tag{5}
$$

$$
A_1 = [σ_1σ_2 cos θ - ε(1 - sin^2θ sin^2ϕ)]^2 + sin^2θ(σ_1σ_2 cos φ + sin θ sin^2φ) - ε cos θ sin ϕ]^2, \tag{6}
A_2 = σ^2(σ_1 cos θ + ε sin θ sin ϕ)^2 + β^2(σ_2 sin θ cos ϕ + β cos θ sin^2θ sin^2ϕ + cos θ^2)^2, \tag{5}
A_3 = σ_2^2(1 - β cos ϕ)^2 + [cos θ + β sin ϕ(σ_2 cos ϕ + sin ϕ sin^2φ)]^2. \tag{6}
$$

The radiation intensity for the case of a perfectly reflecting surface is obtained from (4) taking the limit $ε → 0$. The Smith-Purcell radiation from this type of static surfaces has been widely discussed in the literature.

For a sinusoidal surface wave, $f(u) = a sin u$, one has $f_m = ±(a/2i)δ_{m±1}$. In this case the only contribution to the radiation intensity comes from the harmonic $m = 1$. The parameters of the radiation may be effectively controlled by tuning the characteristics of the surface wave. For an illustration we plot in figure 2 the spectral distribution of the radiation intensity, $dW/dω = \int dΩ dW/dωdΩ$, in the case of a sinusoidal wave excited on the surface of a quartz plate ($ε = 3.75$) as a function of the frequency of the radiated photon for various values of the ratio $d/λ_0$ (numbers near the curves), where $λ_0 = 2π/k_0$ is the wavelength of the surface wave. The full (dashed) curves correspond to the electron energy $E_e = 10$ MeV ($E_e = 50$ MeV). Note that $ω/k_0c = λ_0/λ$ with $λ$ being the wavelength of the radiated photon. The corresponding radiation angle is related to the frequency by the relation $cos θ = 1/β - k_0c/ω$. In accordance with this relation, for relativistic electrons, large values of $ω/(k_0c)$ correspond to small angles $θ$. For $θ ≫ γ^{-1}$ the radiation intensity is relatively insensitive to the particle energy, whereas for $θ ≪ γ^{-1}$ the intensity strongly increases with increasing energy. From figure 2 the suppression of the radiation intensity with increasing $d$ is well seen.
3. Radiation from an electron bunch

As a source of the radiation, in this section we consider a cold bunch consisting of $N$ electrons and moving with constant velocity $v$ along the $z$ axis (for coherence effects in the Smith-Purcell radiation see also [6]). The density of the current in the bunch can be written in the form $j = qv \sum_{j=1}^{N} \delta(r - R_j - vt)$, with $R_j = (X_j, Y_j, Z_j)$ being the position of the $j$-th particle at the initial moment $t = 0$. The spectral density of the radiation energy flux for a given $m$, $P_m^{(N)}(\omega)$, defined by the relation

$$\int_0^\infty d\omega \sum_m P_m^{(N)}(\omega) = \frac{c}{4\pi} \int_{-\infty}^{+\infty} dt \text{[EH]},$$

with $\text{E}$ and $\text{H}$ being the electric and magnetic fields, can be written as $P_m^{(N)}(\omega) = S_N P_m^{(1)}(\omega)$. Here, $P_m^{(1)}(\omega)$ is the corresponding function for the radiation from a single electron and $S_N = |\sum_{j=1}^{N} \exp(-\omega_1 \sigma X_j/v - ik_2 Y_j - i\omega_1 Z_j/v)|^2$. Assuming that the coordinates of the $j$-th particle are independent random variables and averaging over the position of a particle in the bunch, we obtain

$$\langle P_m^{(N)}(\omega) \rangle = \langle S_N \rangle P_m^{(1)}(\omega), \quad \langle S_N \rangle = Nh + N(N - 1)|h_xh_yh_z|^2,$$

where we have introduced the notations

$$h = g(-2\omega \sigma/v), \quad h_x = g(\omega \sigma/v), \quad h_y = \langle \exp(iK_yY) \rangle, \quad h_z = \langle \exp(iK_zZ) \rangle,$$

with $g(u) = \langle \exp(uX) \rangle$, $K_y = k_y = (\omega/v) \sin \theta \sin \varphi$, $K_z = \omega_1/v$. In (8), the term proportional to $N^2$ determines the contribution of coherent effects. Conventionally it is assumed that the coherent radiation is produced at wavelengths longer than the electron bunch length. However, as we shall see below, this conclusion depends on the distribution of electrons in the bunch.

As it is seen from (9), the form factors in $y$ and $z$ directions are determined by the Fourier transforms of the corresponding bunch distributions. For Gaussian distributions:

$$f_y(Y) = (\sqrt{2\pi}b_y)^{-1} \exp\left(-Y^2/(2b_y^2)\right), \quad f_z(Z) = (\sqrt{2\pi}b_z)^{-1} \exp\left(-Z^2/(2b_z^2)\right),$$

one finds $|h_x|^2 = e^{-k_2^2b_y^2}$, $|h_z|^2 = e^{-\omega_1^2b_z^2/v^2}$. Assuming that for all particles in the bunch we have $X_j > -d + a$, along the $x$-direction we take the distribution function

$$f_x(X) = C \exp\left(-X^2/(2b_x^2)\right), \quad x > -b, \quad C^{-1} = \sqrt{\pi/2b_x} \text{erfc}(-b/\sqrt{2b_x}),$$
and \( f_x(X) = 0 \) for \( X < -b \), where \( b < d - a \) and \( \text{erfc}(x) \) is the error function. For function (11) one gets
\[
g(u) = e^{u^2b_z^2/2} \frac{\text{erfc}(-b/(\sqrt{2}b_z) - bu/\sqrt{2})}{\text{erfc}(-b/(\sqrt{2}b_z))}.
\] (12)

In the limit \( b_z \ll b \) and \( b_z^2u \ll b \), we obtain a simple result \( g(u) \approx e^{u^2b_z^2/2} \). In this limiting case the form factor is reduced to
\[
\langle S_N \rangle = N \exp \left(2\omega^2\sigma^2b_z^2/v^2\right) \left[1 + (N - 1) \exp \left(-\omega^2\sigma^2b_z^2/v^2 - k_0^2b_z^2 - \omega^2b_z^2/v^2\right)\right].
\] (13)

As we see, for a Gaussian distribution the relative contribution of coherent effects is exponentially suppressed in the case \( b_1 > \lambda \), \( l = x, y, z \). This result is a consequence of the mathematical fact that for a function \( f(x) \in C^\infty(R) \) one has the estimate \( F(u) \equiv \int_{-\infty}^{\infty} f(x)e^{iuu}dx = O(u^{-\infty}), \ u \rightarrow +\infty \), where \( u = 2\pi b_z/\lambda \) for the longitudinal form factor of the relativistic bunch.

It should be noted that due to various beam manipulations the bunch shape can be highly non-Gaussian. In the estimate given above the continuity condition for the function \( f(x) \) and for infinite number of its derivatives is essential. It can be seen that when \( f(x) \in C^{n-1}(R) \), and the derivative \( f^{(n)}(x) \) is discontinuous at the point \( x_1 \), we have the asymptotic estimate
\[
F(u) = (-iu)^{-n-1} \left[f^{(n)}(x_1+) - f^{(n)}(x_1-)\right], \quad u \rightarrow +\infty.
\] (14)

Unlike the previous case, now the form factor for the short wavelengths decreases slower, as power-law, \( (2\pi b_1/\lambda)^{-n-1} \). In the coherent part of the radiation this form factor is multiplied by a large number of particles per bunch and the coherent effects dominate under the condition \( 2\pi b_1/\lambda < N^{1/2(n+1)} \). The radiation intensity is enhanced by the factor \( N(\lambda/2\pi b_1)^{2(n+1)} \). For instance, in the case of \( n = 1 \), \( N \sim 10^{10} \), the coherent radiation dominates for \( 100\lambda > b_1 \).

As an example, let us consider the case when the distribution function in the \( z \) direction has an asymmetric Gaussian form (see also [7])
\[
f(z) = \frac{2}{\sqrt{2}\pi(1 + p)b_z} \left[\exp\left(-\frac{z^2}{2p^2b_z^2}\right)\theta(-z) + \exp\left(-\frac{z^2}{2b_z^2}\right)\theta(z)\right],
\] (15)

where \( \theta(z) \) is the unit step function, \( z_0 = (1 + p)b_z \) is the characteristic bunch length, parameter \( p \) determines the degree of bunch asymmetry. When the transverse size of the bunch is shorter than the radiation wavelength, then the relative contribution of the coherent effects is estimated as \( N|F(\omega_1/v)|^2 \), where
\[
F(u) = (p + 1)^{-1} \left\{e^{-u^2} + pe^{-p^2u^2} - 2i[W(t) - pW(pt)]/\sqrt{\pi}\right\},
\] (16)

with the notations \( W(t) = \int_0^t \exp(p^2 - t^2)dt, \ t = ub_z/\sqrt{2} \). The expression \( |F(u)|^2 \) is invariant with respect to the replacement \( p \rightarrow 1/p, b_z \rightarrow b_zp \) that corresponds to the mirror reversal of the bunch. When the electron distribution is symmetric (\( p = 1 \)), the second summand in the figure braces of (16) vanishes and, as was mentioned earlier, the form factor exponentially decreases for short wavelengths \( \lambda < 2\pi b_z/\beta \). For \( p ub_z \gg 1 \) the asymptotic behavior has the form \( F(u) \sim i\sqrt{2/\pi(1 - p)/(u^3b_z^2p^2)} \) and for an asymmetrical bunch the contribution of the coherent effects can be dominant even in the case when the bunch length is greater than the radiation wavelength. Note that for the distribution function (15) one has \( n = 3 \) and this estimate is in agreement with (14).

Let \( f(z, a) \) be a continuous distribution function depending on the parameter \( a \), and \( \lim_{a \rightarrow 0} f(z, a) = f(z) \). The integral \( F(u, a) \) for \( f(z, a) \) uniformly converges and hence
lim_{a \to 0} F(u, a) = F(u). It follows from here that the estimate presented above is valid for continuous functions as well if they are sufficiently close to the corresponding discontinuous function (the corresponding derivative is sufficiently large). The final conclusion can be formulated as follows. If for the distribution function one has

\[ \lambda \frac{d^n f}{dz^n} \ll 2\pi, \quad i = 1, ..., n - 1; \quad \lambda \frac{d^n f}{dz^n} \gg 2\pi, \quad (17) \]

with \( z_0 \) being the bunch length, then the relative contribution of coherent effects into the radiation intensity is proportional to \( N(\lambda/2\pi z_0)^{2n} \) and the radiation is partially coherent in the case \( \lambda < z_0 \) but \( \lambda > 2\pi z_0 N^{-1/2n} \). In this case the main contribution into the radiation intensity comes from the parts of the bunch with large derivatives of the distribution function in the sense of the second condition in (17). For example, in the case of asymmetric distribution (15), when \( ub_z \gg 1 \) and \( pub_z \lesssim 1 \), with \( u = \omega/v \), the main contribution comes from the left Gaussian tail with \( z < 0 \). For this tail \( df/d(z/b_z) \sim u/(pub_z \gg u, \) at \( z \sim p b_z \) and therefore \( F(u) \sim 1/(ub_z) \). This can be seen directly from the exact relation (16) as well.

4. Quantum radiation from surface waves

The radiation discussed in the previous sections is a classical effect. In quantum field theory the radiation can be present in the absence of charged particles. This radiation arises as a result of the interaction of a surface wave with quantum fluctuations of the electromagnetic vacuum. This phenomenon is a good example of the dynamical Casimir effect (for reviews see [8, 9]). The Casimir effect is one of the most interesting manifestations of nontrivial properties of the quantum vacuum and arises due to the imposition of boundary conditions on the field operator.

For simplicity we consider a quantum scalar field \( \varphi(x) \) in a spacetime region with the boundary \( S \) on which the field operator obeys Dirichlet boundary condition. The field equation and the boundary condition have the form (here and in what follows we use the units \( c = 1, \hbar = 1) \)

\[ \left( \nabla \cdot \nabla + m^2 \right) \varphi(x) = 0, \quad \varphi(x)|_{x \in S} = 0. \quad (18) \]

In the case of a moving boundary, the interaction with quantum fluctuations of the field can lead to the creation of real quanta out of vacuum. Let \( \xi(x) = n^l \xi(x) \) describes the displacement of the hypersurface \( S \) from the static hypersurface \( S_0 \): \( x^l + \xi(x) \in S \) if \( x^l \in S_0 \). Here \( n^l \) is the unit normal to the boundary \( S_0 \). Assuming that the displacement is small, in the first approximation, for the number of quanta radiated in the interval of quantum numbers \( (\nu, \nu + d\nu) \) one has the formula [10, 11]

\[ n(\nu) d\nu = \int d\nu' \int_{S_0} d\Sigma n^l \xi(x) \left| \left[ \partial_l \varphi_{0\nu'}(x) \right| |\partial_l \varphi_{0\nu}(x) | \right|^2 d\nu, \quad (19) \]

where \( \{ \varphi_{0\nu}(x), \varphi_{0\nu}'(x) \} \) is a complete set of eigenfunctions with the set of quantum numbers \( \nu \) which are solutions of the boundary-value problem (18) with \( S = S_0 \). Below we consider applications of the general formula (19) to surface waves excited on a plane boundary, assuming that the excitation vanishes in the asymptotic regions \( t \to \pm \infty \).

As the hypersurface \( S_0 \) we take the plane \( x = 0 \). For the density of the number of quanta in the momentum space from (19) we find the expression

\[ n(k) = \frac{1}{16\pi^3} \int d\nu \frac{k^2 k^2}{\omega} |\xi(\omega + \nu', k, k')|^2, \quad k = (k_1, k_\perp), \quad \omega = \sqrt{k^2 + m^2}, \quad (20) \]

where \( k_\perp = (k_2, k_3) \), \( \xi(\omega, k_\perp) = \int_{-\infty}^{t=\infty} dt' dy dz \xi(t, y, z) e^{ik_\perp k_\perp - i\omega t} \). In the case of a running surface wave excited on the surface of a plane mirror the corresponding displacement function has the
form $\xi(t,y,z) = f_s(z - v_0 t)$ and, hence, $\xi(\omega, k) = (2\pi)^2 \delta(k_2) \delta(\omega - k_3 v_0) \int_{-\infty}^{\infty} dz f_s(z) e^{i k_3 z}$. One has $k_3 \leq \omega$ and for $v_0 < 1$ this expression vanishes. In this case the radiation is absent. This result becomes obvious if we pass to the reference frame moving with the surface wave. In the case of harmonic oscillations, $\xi(t,y,z) = \xi_0 \cos(k_0 z - \omega_0 t)$, with $\omega_0 > k_0$, the quanta are radiated satisfying the condition $\omega_0 - \omega \geq \sqrt{k_2^2 + (k_0 - k_3)^2}$. Introducing spherical coordinates in the momentum space, $k_1 = \omega \cos \vartheta$, $k_2 = \omega \sin \vartheta \sin \phi$, $k_3 = \omega \sin \vartheta \cos \phi$, $0 \leq \vartheta \leq \pi/2$, $0 \leq \phi \leq 2\pi$, for the spectral-angular density of the number of the radiated quanta we have

$$
\frac{dN}{d\omega d\Omega} = \frac{\xi_0^2 \omega^3}{8 \pi^3} \cos^2 \vartheta (\omega_0^2 - 2 \omega_0 \omega - k_0^2 + 2 k_0 \omega \sin \vartheta \cos \phi + \omega^2 \cos^2 \vartheta)^{1/2},
$$

(21)

where $\vartheta$ is the angle between the normal to the mirror and the direction of the radiation. For a given $\omega$, the allowed angular region for the radiation is determined by the non-negativity condition for the expression under the square root in (21). The total radiated energy is obtained from (21) as $E = \int \omega dN$.

The limit $k_0 \to 0$ of formula (21) corresponds to the mirror oscillating as a whole. In this case, the spectral density of the radiation is obtained after the integration over the solid angle:

$$
\frac{dN}{d\omega} = \frac{\xi_0^2 \omega^4}{32 \pi^2} \left[ u(1-u)^3 + u^3(1-u) + (1/2)(1-2u)^2 \ln |1-2u| \right], \quad u = \omega/\omega_0.
$$

(22)

The expression on the right-hand side of (22) takes its maximum value at $u = 1/2$.

Now we turn to the case of a standing surface wave excited on the surface of a plane mirror in the strip $0 \leq z \leq l$, $-\infty < y < +\infty$:

$$
\xi(t,y,z) = \xi_0 \cos(\omega_0 t) \sin(k_0 z), \quad k_0 = \pi n/l, \quad n = 1, 2, 3, \ldots,
$$

(23)

for $0 \leq z \leq l$ and $\xi(t,y,z) = 0$ otherwise. For the number of the radiated quanta per unit time and per unit length along $y$ we find

$$
n(k) = \frac{k_0^2 \xi_0^2}{8 \pi^4 \omega} \int_{w_1}^{w_2} \frac{dw}{(1-w^2)^2} \left( \frac{\sqrt{(w-w_1)(w_2-w)}}{1-(-1)^n \cos(\pi n w)} \right),
$$

(24)

where $w_{2,1} = [k_3 \pm \sqrt{(\omega_0 - \omega)^2 - k_3^2}] / k_0$. The conditions $\omega = \sqrt{k_1^2 + k_2^2 + k_3^2} \leq \omega_0$ and $|k_2| \leq \omega_0 - \omega$ should be satisfied for the presence of the radiation. In particular, the second inequality imposes a constraint on the angular region for the radiation.

References

[1] Mkrtchyan A R Grigorian L Sh Saharian A A Mkrtchyan A H and Didenko A N 1989 Izvestia AN Arm. SSR, Fizika 24 62
[2] Mkrtchyan A R Grigorian L Sh Saharian A A and Didenko A N 1991 Zhurnal Teh. Fiz. 61 21
[3] Mkrtchyan A R Grigorian L Sh Saharian A A and Didenko A N 1991 Acustica 75, 184
[4] Saharian A A Mkrtchyan A R Gevorgian L A Grigorian L Sh and Khachatryan B V 2001 Nucl. Instr. and Meth. B 173 211
[5] Potylitsyn A P 2011 Electromagnetic Radiation of Electrons in Periodic Structures (Berlin: Springer)
[6] Mkrtchyan A R Gevorgian L A Grigorian L Sh Khachatryan B V and Saharian A A 1998 Nucl. Instr. and Meth. B 145 67
[7] Korkhmassian N A Gevorgian L A and Petrosyan M L 1977 Zhurnal Teh. Fiz. 47 1583
[8] Bordag M Klimchitskaya G L Mohideen U and Mostepanenko V M 2009 Advances in the Casimir Effect (Oxford: University Oxford Press)
[9] Dalvit D Milonni P Roberts D and da Rosa F (Editors) 2011 Lecture Notes in Physics: Casimir Physics Vol. 834 (Berlin: Springer)
[10] Grigoryan L Sh and Saharian A A 1997 J. Contemp. Phys. 32 223
[11] Grigoryan L Sh and Saharian A A 1997 J. Contemp. Phys. 32 275