ON STOCHASTIC İTO PROCESSES WITH DRIFT IN $L_d$

N.V. KRYLOV

Abstract. For İto stochastic processes in $\mathbb{R}^d$ with drift in $L^d$ Aleksandrov’s type estimates are established in the elliptic and parabolic settings. They are applied to estimating the resolvent operators of the corresponding elliptic and parabolic operators in $L_p$ and $L_{p+1}$, respectively, where $p \geq d$.

1. Introduction

Let $d_1$ be an integer $> 1$, $(\Omega, F, P)$ be a complete probability space, and let $(w_t, F_t)$ be a $d_1$-dimensional Wiener process on this space with complete, relative to $F, P, \sigma$-fields $F_t$. Let $\sigma_t, t \geq 0$, be a progressively measurable process with values in the set of $d \times d_1$-matrices and let $b_t, t \geq 0$, be an $\mathbb{R}^d$-valued progressively measurable process. Assume that for any $T \in [0, \infty)$ and $\omega$

$$\int_0^T (|\sigma_t|^2 + |b_t|) \, dt < \infty. \quad (1.1)$$

Under this condition the stochastic process

$$x_t = \int_0^t \sigma_s \, dw_s + \int_0^t b_s \, ds \quad (1.2)$$

is well defined. Fix a nonnegative Borel $b$ on $\mathbb{R}^d$. 

Assumption 1.1. We have $\|b\| := \|b\|_{L_d(\mathbb{R}^d)} < \infty$ and

$$|b_t| \leq b(x_t) \sqrt{\det \sigma_t} \quad (1.3)$$

for all $(\omega, t)$, where $a_t = (1/2)\sigma_t \sigma_t^*$.

This assumption is supposed to hold throughout the whole article.

Our goal is to establish Aleksandrov’s type estimates in the elliptic and parabolic settings and then apply them to estimating the resolvent operators of the corresponding elliptic and parabolic operators in $L_p$ and $L_{p+1}$, respectively, where $p \geq d$.

Before stating our starting result proved in Section 2 introduce

$$B_R := \{x \in \mathbb{R}^d : |x| < R\},$$

and let $\tau_R(x)$ be the first exit time of $x + x_t$ from $B_R$ (equal to infinity if $x + x_t$ never exits from $B_R$). Also let $\tau_R = \tau_R(0)$.

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**Theorem 1.1.** There is a constant $N_{d,\|b\|}$ (depending only on $d$ and $\|b\|$) such that for any $R \in (0, \infty)$, $x \in B_R$, and nonnegative Borel $f$ given on $\mathbb{R}^d$ we have
\[
E \int_0^{\tau_R(x)} f(x + x_t) \sqrt{\det a_t} \, dt \leq N_{d,\|b\|} R \|f\|_{L^d(B_R)}.
\]
(1.4)

As a result of future development of this theorem we have the following proved in Section 2.

**Corollary 1.2.** Suppose that for all $\omega \in \Omega$ and $t \geq 0$ we have $\text{tr} a_t \leq K$, where $K \in (0, \infty)$ is a fixed number. Then for $0 \leq s < t < \infty$ and $R > 0$ we have
\[
P(\max_{r \in [s,t]} |x_r - x_s| \geq R \sqrt{t - s}) \leq 2e^{-\beta R^2/K},
\]
where $\beta = \beta(d,\|b\|) > 0$. In particular, for any $n \geq 0$
\[
E \max_{r \in [s,t]} |x_r - x_s|^{2n} \leq N(t - s)^n,
\]
(1.6)

where $N = N(n,d,K,\|b\|)$.

**Remark 1.1.** The reader might have noticed that estimate (1.6) has the same form for large $t - s$ and for small ones. This is not the case if $b$ is just bounded.

The literature on the stochastic equations with singular drift is quite impressive. We only mention one of the inspiring ones [13] and some of the most recent articles [4], [14], [18], [16], [3], [17]. Also see numerous references therein. The goals in these articles are much more ambitious than here, where we mostly concentrate around the results like Theorem 1.1 and Corollary 1.2. In all of the above mentioned sources apart from [14] the coefficient $b_t$ is allowed to depend on $x_t$ and $t$: $b_t = b(t,x_t)$ and it is supposed that
\[
\left( \int \left( \int |b(t,x)|^p \, dx \right)^{q/p} \, dt \right)^{1/q} < \infty,
\]
where
\[
\frac{d}{p} + \frac{1}{q} < 1.
\]
(1.7)

This excludes $p = d$ even if $b(t,x)$ is independent of $t$. In [14] the setting is closer to ours and again it is assumed that $p > d$. Observe in passing that the parabolic PDE counterparts of Theorem 1.1 or of our Theorem 4.1 are obtained in spaces with mixed norms in [12] with $\leq$ in place of $<$ in (1.7).

These results are not applicable in our situation, in particular, because $x_t$ is not supposed to solve any equation.

One may ask what happens if we only have $b \in L^p$ with $p < d$. It turns out that then estimate (1.4) may break down. For instance, let $dx_t = dw_t + b_t(x_t) \, dt$, $x_0 = 0$, where $b_t(x) = -(d/2)x|x|^{-2}I_{1>|x|>\epsilon}$, and in (1.4) take $x = 0$, $f \equiv 1$, $R = 1$. Then simple computations in polar coordinates
show that the left-hand side of (1.4) tends to infinity as \( \varepsilon \downarrow 0 \). At the same time, for any \( p < d \) the \( L_p \)-norms of \( b_\varepsilon \) stay bounded.

On the basis of the results in this article, the author intends to show in a subsequent article that stochastic equations with measurable diffusion and drift in \( L_d \) admit solutions, that \( L_d \) in (1.4) can be replaced with \( L_p \) for some \( p < d \) if the process \( x_t \) is uniformly nondegenerate, that Itô’s formula is applicable to \( u \in W^2_p \) for some \( p < d \) and so on. In particular, we will present some results concerning estimates of the time spent in sets of small measure, the probability to reach such sets, Fanghua Lin’s estimates playing a major role in the Sobolev space theory of fully nonlinear elliptic equations, boundary behavior of solutions of the corresponding elliptic equations with first order coefficients in \( L_d \), and the probability to pass through narrow tubes (see [10]). It is worth saying that the main driving force behind our results is an idea of Safonov from [15].

Many results of this article could be obtained by using the theory of fully nonlinear equations presented, for instance, in [9]. This theory is ultimately based on Theorems 6.1 and 4.3 and it turns out that we only need these two theorem rather than rather sophisticated theory from [9] to derive our results.

The article is organized as follows. In Section 2 we prove various extensions of Aleksandrov’s type estimates for the elliptic case (when \( f \) in estimates like (1.4) is independent of \( t \)) in bounded domains and in the whole space. The latter allows us to also prove Corollary 1.2.

In Section 3 we give an application of the results from Section 2 to showing that the resolvents of elliptic operators with drift in \( L_d \) decay in a normal way. This is an improvement over some results in [8].

Section 4 is devoted to the parabolic case and estimates like (1.4) but with \( f(t, x + x_t) \) in place of \( f(x + x_t) \). Its results are applied in Section 5 to estimating the resolvent of parabolic operators.

The final Section 6 is an Appendix, where we prove an auxiliary fact related to the theory of concave functions.

We finish the introduction with some notation and a stipulation about constants. In the proofs of various results we use the symbol \( N \) to denote finite nonnegative constants which may change from one occurrence to another and, if in the statement of a result there are constants called \( N \) which are claimed to depend only on certain parameters, then in the proof of the result the constants \( N \) also depend only on the same parameters unless specifically stated otherwise. Of course, if we write \( N = N(...) \), this means that \( N \) depends only on what is inside the parentheses.

For \( \delta \in (0, 1) \) by \( \mathbb{S}_\delta \) we mean the set of \( d \times d \)-symmetric matrices whose eigenvalues are between \( \delta \) and \( \delta^{-1} \). Introduce

\[ B_R(x) = \{ y : |y - x| < R \}, \quad a_\pm = a^\pm = (1/2)(|a| \pm a), \]
\[ D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j. \]
We use the notation $u^{(\varepsilon)} = u \ast \zeta_\varepsilon$, where $\zeta_\varepsilon(x) = \varepsilon^{-d} \zeta(x/\varepsilon)$, $\varepsilon > 0$, and $\zeta$ is a nonnegative $C^\infty$-function with support in $B_1$ whose integral is equal to one.

2. Elliptic case

Recall that Assumption 1.1 is supposed to be satisfied.

Our first result is Theorem 1.1, which is actually a particular case of Theorem 5.2 of [7], proved there by using a heavy artillery from the theory of fully nonlinear elliptic partial differential equations. We provide its proof based on the very initial knowledge of the solvability of the Monge-Ampère equations and estimates of its solutions, which, actually, after a long development became also one of the cornerstones of the theory of fully nonlinear elliptic partial differential equations. The proof will be given after some preparations. By setting $f = 1$ in (1.4) we obtain the following.

**Corollary 2.1.** We have

$$E \int_0^{\tau_2(x)} \sqrt{\det a_t} \, dt \leq NR^2,$$

where $N$ depends only on $d$ and $\|b\|$.

**Remark 2.1.** The proof of Theorem 1.1 consists of several steps. First we let to the reader to check that the constant $N_d,\|b\|$ is independent of $R$ by replacing $x + x_t$ with $(x + x_t)/R$. In the new situation condition (1.3) will change, but the new function $b$ will have the same $L_d(\mathbb{R}^d)$-norm as the original one. Therefore, it suffices to prove the theorem for $R = 1$.

Then the left-hand side of (1.4) will only increase if we replace $\tau_1(x)$ with the first exit time from $B_2(x)$. It follows that we may assume that $x = 0$ and $R = 2$.

Now it becomes clear that to prove Theorem 1.1 it suffices to prove the following.

**Theorem 2.2.** There is a constant $N_{d,\|b\|}$ such that for any nonnegative Borel $f$ given on $B_2$ we have

$$E \int_0^\tau f(x_t) \sqrt{\det a_t} \, dt \leq N_{d,\|b\|} \|f\|_{L_d(B_2)},$$

where $\tau = \tau_2$.

To prove this theorem we need two lemmas in which the assumptions of the theorem are supposed to hold and we also need the following result proved in the Appendix.

**Theorem 2.3.** Let $\theta$ be 0 or 1 and let $f$ be a nonnegative function on $\mathbb{R}^d$ such that $f^d$ has finite integral over $B_2$ and $f = 0$ outside $B_2$. Then there exists a convex function $z$ on $B_4$ which is nonpositive and such that
(a) we have in $B_4$
\[ |z| \leq 8\Phi \left( \int_{B_2} f^d \, dx \right), \tag{2.2} \]
where $\Phi$ is the inverse function of
\[ \rho \to \int_{|p| \leq \rho} \frac{1}{(1 + \theta|p|)^d} \, dp, \]
(b) for any $\varepsilon \in (0, 2)$ and nonnegative symmetric matrix $a$, in $B_2$ we have
\[ a^{ij} D_{ij} z^{(\varepsilon)} \geq d \sqrt[4]{\det a} \left( f(1 + \theta|Dz|) \right)^{(\varepsilon)}. \tag{2.3} \]

Remark 2.2. As is easy to see there is a constant $N_d$ depending only on $d$ such that
\[ 8\Phi(t) \leq N_d t^{1/d} \text{ if } \theta = 0, \quad 8\Phi(t) \leq N_d t^{1/d} \exp(Ndt) \text{ if } \theta = 1. \tag{2.4} \]

Lemma 2.4. Let $\gamma$ be a stopping time such that $\gamma \leq \tau$ and there is a constant $N$ such that (2.1) holds with that $N$ and $\gamma$ in place of $\tau$ for any nonnegative Borel $f$ given on $B_2$. Then it also holds with ($\gamma$ in place of $\tau$ and) an $N$ that depends only on $d$ and $\|b\|$. Proof. It suffices to concentrate on the $f$’s that are continuous in $\mathbb{R}^d$ and vanish outside $B_2$. Fix such an $f$ and define $z_1$ as a function from Theorem 2.3 when $\theta = 0$. By Remark 2.2
\[ |z_1| \leq N_d F, \quad F := \|f\|_{L_d(B_2)}. \]
Since, $z_1$ is convex and nonpositive, in $B_3$ we have
\[ |Dz_1(x)| \leq |z_1/(4 - |x|)| \leq |z_1| \leq N_d F. \tag{2.5} \]
Then define $z_2$ as a function from Theorem 2.3 when $\theta = 1$ and $f = b/d$. Finally, set $z = z_1 + N_d F z_2$.

In light of (2.3) in $B_2$, for any nonnegative symmetric matrix $a$, we have
\[ a^{ij} D_{ij} z^{(\varepsilon)} - b \sqrt[4]{\det a} |Dz^{(\varepsilon)}| \geq d \sqrt[4]{\det a} f^{(\varepsilon)} - b \sqrt[4]{\det a} |Dz^{(\varepsilon)}| \]
\[ + N_d F \sqrt[4]{\det a} (b(1 + |Dz_2|))^{(\varepsilon)} - N_d F b \sqrt[4]{\det a} |Dz^{(\varepsilon)}|. \]
Owing to (2.5), the last expression in $B_2$, for $\varepsilon \in (0, 1)$, is greater than
\[ d \sqrt[4]{\det a} f^{(\varepsilon)} - N_d F b \sqrt[4]{\det a} I_\varepsilon, \]
where
\[ I_\varepsilon := (b(1 + |Dz_2|))^{(\varepsilon)} - b - b |Dz_2^{(\varepsilon)}|. \]
Similarly to (2.5), $I_\varepsilon \leq N(b + b^{(\varepsilon)})$, by the Lebesgue theorem $I_\varepsilon \to 0$ as $\varepsilon \downarrow 0$ almost everywhere, and hence as $\varepsilon \downarrow 0$ we have
\[ \|I_\varepsilon\|_{L_d(B_2)} \to 0. \tag{2.6} \]
By observing that thanks to (1.3), for $t \leq \tau$,
\[ a^{ij} D_{ij} z^{(\varepsilon)}(x_t) + b_i Dz^{(\varepsilon)}(x_t) \geq a^{ij} D_{ij} z^{(\varepsilon)}(x_t) - b(x_t) \sqrt[4]{\det a_t} |Dz^{(\varepsilon)}(x_t)| \tag{2.7} \]
and using Itô’s formula, we get that for small \( \varepsilon \) and and \( T \in [0, \infty) \)
\[
Ez^{(\varepsilon)}(x_{\gamma \wedge T}) = z^{(\varepsilon)}(0) + E \int_0^{\gamma \wedge T} \left( a_i^j D_{ij} z^{(\varepsilon)}(x_t) + b_i^j D_{ij} z^{(\varepsilon)}(x_t) \right) dt \]
\[
\geq z^{(\varepsilon)}(0) + dE \int_0^{\gamma \wedge T} \sqrt{\det a_t} f^{(\varepsilon)}(x_t) dt
\]
\[
- N_d F E \int_0^{\gamma} \sqrt{\det a_t} |I_{B_2}(z_t)| dt. \tag{2.8}
\]

The last term in (2.8), by absolute value, is less than a constant independent of \( \varepsilon \) times the norm in (2.6) by assumption. Therefore, it tends to zero as \( \varepsilon \downarrow 0 \). The first term in (2.8) is nonpositive. Therefore, by passing to the limit in (2.8) first as \( \varepsilon \downarrow 0 \) and then as \( T \to \infty \) and also using that \( f \) is continuous we find
\[
E \int_0^{\gamma} f(x_t) \sqrt{\det a_t} dt \leq d^{-1} |z(0)|
\]
\[
\leq d^{-1} N_d F + d^{-1} N_d F N_d(\|b\|/d)^{1/d} \exp(N_d(\|b\|/d)^d).
\]
Hence (2.1) holds with \( \gamma \) in place of \( \tau \) and \( N = N_{d,\|b\|} \), where
\[
N_{d,\|b\|} := d^{-1} N_d + d^{-1} N_d N_d(\|b\|/d)^{1/d} \exp(N_d(\|b\|/d)^d). \tag{2.9}
\]

The lemma is proved. \( \square \)

**Lemma 2.5.** Let \( \gamma \) be a stopping time such that \( \gamma \leq \tau \) and (2.1) holds with \( (N_{d,\|b\|}) \) from (2.9) and \( \gamma \) in place of \( \tau \) for any nonnegative Borel \( f \) given on \( B_2 \). Introduce
\[
\gamma' = \tau \wedge \inf\{t \geq \gamma : \int_\gamma^t |b_s| ds \geq 1\}.
\]
Then there is a constant \( N \) such that (2.1) holds with that \( N \) and \( \gamma' \) in place of \( N_{d,\|b\|} \) and \( \tau \), respectively, for any nonnegative Borel \( f \) given on \( B_2 \).

Proof. We take the same \( f \), \( z_1 \), and \( z_2 \) as in the proof of Lemma 2.4 but use (2.7) only for \( t \leq \gamma' \). On the interval \( [\gamma, \gamma'] \) we use that \( |x_t| < 2 \), so that
\[
|Dz^{(\varepsilon)}(x_t)| \leq dN_{d,\|b\|} F \text{ implying that on } [\gamma, \gamma']
\]
\[
a_i^j D_{ij} z^{(\varepsilon)}(x_t) + b_i^j D_{ij} z^{(\varepsilon)}(x_t) \geq d \sqrt{\det a_t} f^{(\varepsilon)}(x_t) - dN_{d,\|b\|} F |b_t|.
\]
We also use that
\[
E \int_\gamma^{\gamma'} |b_t| dt \leq 1.
\]
Hence, similarly to (2.8) we get
\[
Ez^{(\varepsilon)}(x_{\gamma' \wedge T}) = z^{(\varepsilon)}(0) + E \int_0^{\gamma' \wedge T} \left( a_i^j D_{ij} z^{(\varepsilon)}(x_t) + b_i^j D_{ij} z^{(\varepsilon)}(x_t) \right) dt
\]
\[
\geq z^{(\varepsilon)}(0) + dE \int_0^{\gamma' \wedge T} \sqrt{\det a_t} f^{(\varepsilon)}(x_t) dt
\]
STOCHASTIC PROCESSES WITH DRIFT IN $L_d$

\[-N_d F \int_0^\tau \sqrt{\det a_t} |I_\varepsilon(x_t)| \, dt - dN_{d,\|b\|} F. \quad (2.10)\]

As $\varepsilon \downarrow$ the term with $I_\varepsilon$ disappears in light of our assumption and of what is said in the proof of Lemma 2.4. After that as in that proof we come to (2.1) with $N = 2N_{d,\|b\|}$ and $\gamma'$ in place of $N_{d,\|b\|}$ and $\tau$, respectively. The lemma is proved. □

Proof of Theorem 2.2. Set $\gamma_0 = 0$ and define recursively $\gamma_n$, $n = 1, 2, \ldots$, by

$$
\gamma_n = \tau \wedge \inf \{ t \geq \gamma_{n-1} : \int_{\gamma_{n-1}}^t |b_s| \, ds \geq 1 \}.
$$

In light of (1.1), $\gamma_n \to \tau$ as $n \to \infty$. Furthermore, since estimate (2.1) is obviously true for $\gamma_0 (= 0)$ in place of $\tau$, Lemma 2.5 implies that (2.1) is true with a constant, perhaps different from $N_{d,\|b\|}$ and with $\gamma_1$ in place of $\gamma$. An obvious induction proves that, for any $n$, (2.1) is true for $\gamma_n$ in place of $\tau$. Letting $n \to \infty$ we get the desired result. The theorem is proved. □

Remark 2.3. Theorem 1.1 as many results below admits a natural generalization with conditional expectations. This generalization is obtained by tedious and not informative repeating the proof with obvious changes. We mean the following which we call the conditional version of Theorem 1.1.

Let $\gamma$ be a stopping time. Then $x_t$ on the set $\{ \gamma < \infty \}$ is given for $t \geq \gamma$ as

$$
x_t = x_\gamma + \int_\gamma^t \sigma_t \, dw_t + \int_\gamma^t b_t \, dt.
$$

Also on the set $\{ \gamma < \infty \}$ introduce $\tau$ as the first time after $\gamma$ when $x_t$ exits from $B_R(x_\gamma)$. Then for any nonnegative Borel $f$ given on $\mathbb{R}^d$ on the set $\{ \gamma < \infty \}$ (a.s) we have

$$
E\left[ \int_{\gamma}^\tau f(x_t) \sqrt{\det a_t} \, dt \mid F_\gamma \right] \leq N_{d,\|b\|} R \| f \|_{L_d(B_R(x_\gamma))}. \quad (2.11)
$$

Then standard measure-theoretic arguments show that on the set $\{ \gamma < \infty \} \times \mathbb{R}^d$ there exists a function $G_{\gamma,\tau}(x) = G_{\gamma,\tau}(\omega, x)$ which is $F_{\gamma} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and such that, for any nonnegative Borel $f$ given on $\mathbb{R}^d$, on the set $\{ \gamma < \infty \}$ (a.s) we have

$$
E\left[ \int_{\gamma}^\tau f(x_t) \sqrt{\det a_t} \, dt \mid F_\gamma \right] = \int_{\mathbb{R}^d} f(x) G_{\gamma,\tau}(x) \, dx.
$$

In addition (2.11) shows that on the set $\{ \gamma < \infty \}$ (a.s)

$$
\| G_{\gamma,\tau} \|_{L_d/(d-1)(B_R(x_\gamma))} \leq N_{d,\|b\|} R. \quad (2.12)
$$

Let us show briefly how to obtain (2.11). First by repeating the above proof one shows that (a.s.)

$$
E\left[ \int_{\gamma}^\tau f(x_t - x_\gamma) \sqrt{\det a_t} \, dt \mid F_\gamma \right] \leq N_{d,\|b\|} R \| f \|_{L_d(B_R)}.
$$
Then by considering first continuous $f(x, y)$ and then arbitrary nonnegative Borel, one gets that (a.s.)
\[
E\left[ \int_{\gamma} f(x_t - x_{\gamma}, x_{\gamma}) \sqrt{\det a_t} \, dt \mid \mathcal{F}_\gamma \right] \leq N_d \|b\| R \|f(\cdot, x_{\gamma})\|_{L_d(B_R)}.
\]

Finally, the substitution $f(x, y) = f(x + y)$ leads to (2.11).

This remark allows us to generalize Corollary 2.1. For $0 \leq s \leq t$ set
\[
\psi_{s,t} = \int_s^t \sqrt{\det a_r} \, dr, \quad \psi_t = \psi_{0,t}.
\]

Lemma 2.6. For $n = 1, 2, \ldots$ and $x \in B_R$ we have
\[
I_n := E\psi_{\tau_R(x)}^n \leq N^n n! R^{2n},
\]
where $N$ depends only on $d$ and $\|b\|$.

Proof. Observe that
\[
I_{n+1} = (n + 1)E \int_0^{\tau_R(x)} \sqrt{\det a_t} \, E\left( \psi_{t,\tau_R(x)}^n \mid \mathcal{F}_t \right) \, dt.
\]
If our assertion is true for a given $n$, then by its conditional version
\[
I_{n+1} \leq (n + 1)N^n n! R^{2n} E \int_0^{\tau_R(x)} \sqrt{\det a_t} \, dt.
\]
After that Corollary 2.1 and the induction on $n$ finish the proof. □

Corollary 2.7. There are constants $N, \nu > 0$ depending only on $d$ and $\|b\|$ such that $E \exp(\nu \psi_{\tau_R(x)}/R^2) \leq N$ for any $R \in (0, \infty)$ and $x \in B_R$. In particular, (by Chebyshev’s inequality) for any $t > 0$,
\[
P(\psi_{\tau_R(x)} \geq t) \leq Ne^{-\nu t/R^2}.
\]

Corollary 2.7 basically says that $\tau_R(x)$ is smaller than a constant times $R^2$. We want to show that in a sense the converse is also true. For that, it is convenient to introduce
\[
\phi_t = \phi_{0,t}, \quad \phi_{s,t} = \int_s^t \text{tr} a_r \, dr.
\]

Here is a fact of immense importance for the future development.

Lemma 2.8. There is a constant $R = R_d, \|b\| \in [2, \infty)$ depending only on $d$ and $\|b\|$ such that
\[
E I_{\tau < \infty} \exp\left( - \phi_\tau \right) \leq 1/2,
\]
where $\tau = \tau_R$.

Proof. We use a Safonov’s idea from [15] based on the fact that the integrals of $\mathcal{b}^d$ should be small an a variety of sets. Fix an integer $k \geq 1$ to be specified later and define $\Gamma^i = B_{2(i+1)} \setminus B_{2i}$. Then for any integer $n \geq k$ there are at least $m = n - k$ sets $\Gamma^i$ with $i \leq n$ such that
\[
\|b\|_{L_d(\Gamma^i)} \leq \|b\|/k^{1/d}.
\]
Let us call such sets “good” and let $i_1 < \ldots < i_m$ be such that each $\Gamma^{i_j}$ is good. Introduce
\[
\tau^j = \inf\{t \geq 0 : |x_t| = 2i_j + 1\} \quad (\inf\emptyset := \infty),
\]
which (if finite) is the first time $x_t$ touches the middle sphere in $\Gamma^{i_j}$, and define $\gamma^j$ as the first exit time of $x_t$ from $B_1(x_{\tau^j})$ after $\tau^j$.

Observe that simple manipulations show that the function $p(x) := \cosh |x|$ satisfies
\[
a^s D_{rs} p - p tr a \leq 0
\]
in $B_1$ for any symmetric nonnegative $d \times d$-matrix $a$. By using Itô’s formula and observing that $|Dp| \leq p \leq \cosh 1$ in $B_1$ and $|b^t D_r p(x_t - x_{\tau^j})| \leq b(x_t) \sqrt{\det a_t} \cosh 1$ for $t \in [\tau^j, \gamma^j)$ we see that for any $\varepsilon > 0$ on the set $\{\tau^j < \infty\}$ (a.s.)
\[
\cosh 1 E \left[ \exp \left( - \int_{\tau^j}^{\gamma^j} (\varepsilon + tr a_t) dt \right) \mid F_{\tau^j} \right] 
\]
\[
= E \left[ p(x_{\gamma^j} - x_{\tau^j}) \exp \left( - \int_{\tau^j}^{\gamma^j} (\varepsilon + tr a_t) dt \right) \mid F_{\tau^j} \right] 
\]
\[
\leq 1 + \cosh 1 E \left[ \int_{\tau^j}^{\gamma^j} b(x_t) \sqrt{\det a_t} dt \mid F_{\tau^j} \right] 
\]
\[
= 1 + \cosh 1 E \left[ \int_{\tau^j}^{\gamma^j} I_{\Gamma^{i_j}} b(x_t) \sqrt{\det a_t} dt \mid F_{\tau^j} \right].
\]

By the way, the role of $\varepsilon$ is that the second term in the above sequence makes sense even if $\gamma^j = \infty$.

By the choice of $\Gamma^{i_j}$ and Remark 2.3 we can estimate the last term and conclude that on the set $\{\tau^j < \infty\}$ (a.s.)
\[
\cosh 1 E \left[ \exp \left( - \int_{\tau^j}^{\gamma^j} (\varepsilon + tr a_t) dt \right) \mid F_{\tau^j} \right] \leq 1 + (\cosh 1) N_{d,\|b\|/\|b\|/k^{1/d}},
\]
\[
E \left[ \exp \left( - \int_{\tau^j}^{\gamma^j} (\varepsilon + tr a_t) dt \right) \mid F_{\tau^j} \right] \leq (\cosh 1)^{-1} + N_{d,\|b\|/\|b\|/k^{1/d}}. \quad (2.14)
\]

We now choose $k$ and $n$, depending only on $d$ and $\|b\|$, so that
\[
((\cosh 1)^{-1} + N_{d,\|b\|/\|b\|/k^{1/d}})^{n-k} \leq 1/2
\]
and set
\[
R = 2(n + 1) \quad (\geq 2(i_m + 1)).
\]

After that it only remains to observe that, obviously,
\[
E \exp \left( - \int_{0}^{\tau} (\varepsilon + tr a_t) dt \right)
\]
is less than the expectation of the product over $j = 1, \ldots, m$ of the left-hand sides of (2.14), and thus, less than $1/2$ for any $\varepsilon > 0$, and then send $\varepsilon \downarrow 0$.

The lemma is proved. \qed
**Corollary 2.9.** For any \( R \in (0, \infty) \)

\[
EI_{\tau_R < \infty} \exp \left( - (R_{d,||b||}/R)^2 \phi_{\tau_R} \right) \leq 1/2.
\]  
(2.15)

In particular, for \( t_0 := R_{d,||b||}^{-2} \ln 3/2 \) and any \( R \in (0, \infty) \) we have

\[
P(\phi_{\tau_R} \leq t_0 R^2) \leq 3/4,
\]  
(2.16)

provided that (a.s.)

\[
\int_0^\infty \text{tr} \, a_t \, dt = \infty.
\]  
(2.17)

Proof. Replacing \( x_t \) with \( (R_{d,||b||}/R)x_t \) shows that it suffices to concentrate on \( R = R_{d,||b||} \) (see Remark 2.1). In that case (2.15) is identical with (2.13). As far as (2.16) is concerned it suffices to observe that for \( R \) the left-hand side equals

\[
P(\phi_{\tau_R} \leq \ln 3/2) = P(\tau_R < \infty, \phi_{\tau_R} \leq \ln 3/2) = P(\tau_R < \infty, e^{-\phi_{\tau_R}} \geq 2/3)
\]

\[
\leq (3/2)EI_{\tau_R < \infty} e^{-\phi_{\tau_R}} \leq 3/4.
\]

The following gives a more general form to Corollary 2.9.

**Theorem 2.10.** For any \( \kappa \in (0, 1) \), \( R \in (0, \infty) \), \( x \in B_{\kappa R} \), and \( \lambda \geq 0 \),

\[
EI_{\tau_R(x) < \infty} \exp \left( - \lambda \phi_{\tau_R(x)} \right) \leq 2e^{-\sqrt{\lambda(1-\kappa)}R/N},
\]  
(2.18)

where \( N = R_{d,||b||}/\ln 2 \). In particular, for any \( R, t \in (0, \infty) \) we have

\[
P(\phi_{\tau_R(x)} \leq tR^2) \leq 2 \exp \left( - \frac{\beta(1-\kappa)^2}{t} \right),
\]  
(2.19)

where \( \beta := 4^{-1}R_{d,||b||}^{-2} \ln^2 2 \left( < 1 \right) \), provided that (2.17) holds (a.s.).

Proof. Observe that \( \phi_{\tau_R(x)} \geq \phi_{\tau'_{R}(x)} \), where \( \tau'_{R}(x) \) is the first exit time of \( x + x_t \) from \( B_{1(1-\kappa)R}(x) \). It follows that in the proof of (2.18) we may assume that \( \kappa = 0 \) and \( x = 0 \). Then, as usual (see Remark 2.1) we may assume that \( R = 1 \). In that case take \( n \geq 1 \), to be specified later, and introduce \( \tau^k, k = 1, ..., n \), as the first exit time of \( x_t \) from \( B_{1/n}(x_{\tau^k}) \). Also let \( \gamma^k, k = 1, ..., n \), be the first exit times of \( x_t \) from \( B_{1/n}(x_{\tau^k}) \) after \( \tau^{k-1} \) \((\tau^0 := 0)\). Observe that, obviously \( \phi_{\tau_1} \geq \phi_{\tau_{1},\gamma^1} + ... + \phi_{\tau_{n-1},\gamma^n} \) and by the conditional version of Corollary 2.9 on the set \( \{ \tau^{k-1} < \infty \} \) (a.s.)

\[
E\left( I_{\tau^k \infty} \exp \left( - R_{d,||b||}^2 n^2 \phi_{\tau^{k-1},\gamma^k} \right) \mid \mathcal{F}_{\tau^{k-1}} \right) \leq 1/2.
\]

After that we get

\[
EI_{\tau_1 < \infty} \exp \left( - R_{d,||b||}^2 n^2 \phi_{\tau_1} \right) \leq 1/2^n,
\]

which for \( n = \lfloor \sqrt{\lambda}/R_{d,||b||} \rfloor \) yields (2.18).

We prove (2.19) again assuming that \( R = 1 \). We have

\[
P(\phi_{\tau_1(x)} \leq t) = P(\tau_1(x) < \infty, \exp(-\lambda \phi_{\tau_1(x)} \geq \exp(-\lambda t))
\]

\[
\leq 2 \exp \left( - \frac{\beta(1-\kappa)^2}{t} \right)
\]

where \( \beta := 4^{-1} \ln^2 2 \left( < 1 \right) \), provided that (2.17) holds (a.s.).
For $\sqrt{\lambda} = (1 - \kappa)/(2Nt)$ we get (2.19). The theorem is proved.

**Proof of Corollary 1.2.** First observe that (1.6) follows from (1.5). Next, having in mind the conditional versions of our results we convince ourselves that while proving (1.5) we may assume $s = 0$. Finally, using scaling $(x_t \to cx_{t/c^2})$ reduces the case of general $t > 0$ to that of $t = 1$. In that case, it only remains to observe that

$$P(\sup_{t \leq 1} |x_t| \geq R) = P(\tau_R \leq 1) \leq P(\phi_{\tau_R} \leq K) \leq 2e^{-\beta R^2/K}.$$  

It is instructive to compare the following with Corollary 2.1.

**Corollary 2.11.** Suppose that (2.17) holds (a.s.). Then there is a constant $N = N(d, ||b||)$ such that for any $R \in (0, \infty)$

$$NE \int_0^{\tau_R} \text{tr} \ a_t \ dt \geq R^2.$$  

(2.20)

Indeed

$$E\phi_{\tau_R} \geq t_0 R^2 P(\phi_{\tau_R} > t_0 R^2) \geq (1/4)t_0 R^2.$$  

Here is a technically convenient form of Corollary 2.11 to be used in a subsequent article (see [10]), while estimating the time spend by $x_t$ in sets of small measure.

**Corollary 2.12.** Let $\delta \in (0, 1)$ and assume that $\text{tr} \ a_t \geq \delta$ for all $(\omega, t)$. Then for any $\kappa \in (0, 1)$ there exists $\nu = \nu(\kappa, d, \delta, ||b||) > 0$ such that for any $R \in (0, \infty)$ and $x \in B_{\kappa R}$

$$E\tau \geq \nu R^2,$$  

(2.21)

where $\tau = \tau_R(x)$.

Indeed, $\tau \geq \gamma$, where $\gamma$ is the first exit time of $x + x_t$ from $B_{(1-\kappa)R}(x)$ for which by Corollary 2.11

$$E\gamma \geq \delta^{-1} E \int_0^{\gamma} \text{tr} \ a_t \ dt \geq \delta^{-1}(1 - \kappa)^2 R^2 \mu(d, ||b||)$$

with $\mu = \mu(d, ||b||) > 0$.

The following modestly looking version of Corollary 2.11 will play a major role in a subsequent paper while showing that $d$ in Theorems 1.1 can be replaced with a smaller number if $a_t \in S_\delta$ for all $\omega, t$ (see [10]).

**Lemma 2.13.** Let $a_t \in S_\delta$ for all $\omega, t$ and $\varepsilon \in (0, 1]$. Then there is a constant $N = N(d, \delta, ||b||, \varepsilon)$ such that for any $R \in (0, \infty)$

$$NE \int_0^{\tau_R \wedge (\varepsilon R^2)} e^{-t} \ dt \geq R^2 \wedge 1.$$  

(2.22)
Proof. Observe that, for $t \geq 0$, $e^{-t} \geq 1 - t$, and
\[
I := E \int_0^{\tau_R \wedge (\varepsilon R^2)} e^{-t} \, dt \geq E(\tau_R \wedge (\varepsilon R^2)) - (1/2)E(\tau_R \wedge (\varepsilon R^2))^2 \\
\geq E(\tau_R \wedge (\varepsilon R^2)) - (1/2)R^4.
\]

Here by Theorem 2.10, for $t \in (0, \varepsilon)$,
\[
E(\tau_R \wedge (\varepsilon R^2)) \geq tR^2 P(\tau_R > tR^2) \geq R^2 t \left(1 - 2 \exp \left(-\frac{\beta}{t}\right)\right),
\]
where $\beta = \beta(d, \delta, \|b\|) > 0$, so that for an appropriate $t = t(\beta, \varepsilon) > 0$
\[
I \geq N^{-1}R^2 - NR^4,
\]
where the last expression is greater than $N^{-1}R^2$ for $R \leq R_0 = R_0(d, \delta, \|b\|, \varepsilon)$
and (2.22) holds for such $R$. However, for $R \geq R_0$ by Corollary 2.9 for an appropriate $s_0 = s_0(d, \delta, \|b\|) > 0$
\[
I \geq EI_{R > s_0}R^2 \int_0^{(s_0 \wedge \varepsilon)R^2} e^{-t} \, dt \geq (1/4) \int_0^{(s_0 \wedge \varepsilon)R^2} e^{-t} \, dt.
\]

The lemma is proved. \hfill \Box

Theorem 2.14 which is a simple corollary of Theorems 1.1 and 2.10 can be used to prove Harnack’s inequality for diffusion processes with drift in $L_d$.

**Theorem 2.14.** Assume that $a_t \in S_\delta$ for all $(\omega, t)$. Then for any $\kappa \in (0, 1)$ there is a function $q(\gamma)$, $\gamma \in (0, 1)$, depending only on $d, \delta, \|b\|, \kappa$, and, naturally, on $\gamma$, such that for any $R \in (0, \infty)$, $x \in B_R$, and closed $\Gamma \subset B_R$ satisfying $|\Gamma| \geq \gamma |B_R|$ we have
\[
P(\tau_\Gamma(x) \leq \tau_R(x)) \geq q(\gamma),
\]
where $\tau_\Gamma(x)$ is the first time the process $x + a_t$ hits $\Gamma$. Furthermore, $q(\gamma) \to 1$ as $\gamma \uparrow 1$.

Proof. By using scalings we reduce the general case to the one in which $R = 1$. In that case for any $\varepsilon > 0$ we have
\[
P(\tau_\Gamma(x) > \tau_1(x)) \leq P(\tau_1(x) = \int_0^{\tau_1(x)} I_{B_1 \setminus \Gamma}(x + a_t) \, dt)
\leq P(\tau_1(x) \leq \varepsilon) + \varepsilon^{-1}E \int_0^{\tau_1(x)} I_{B_1 \setminus \Gamma}(x + a_t) \, dt.
\]

In light of Theorems 2.10 and 1.1 we can estimate the right-hand side and then obtain
\[
P(\tau_\Gamma(x) > \tau_R(x)) \leq 2e^{-N/\varepsilon} + N\varepsilon^{-1}|B_1 \setminus \Gamma|^{1/d}
\leq 2e^{-N/\varepsilon} + N\varepsilon^{-1}(1 - \gamma)^{1/d}
\]
where the constants $N$ depend only on $d, \delta, \kappa$, and $\|b\|$. By denoting
\[
q(\gamma) = 1 - \inf_{\varepsilon > 0} \left(2e^{-N/\varepsilon} + N\varepsilon^{-1}(1 - \gamma)^{1/d}\right),
\]
...
we get what we claimed. □

Remark 2.4. In a subsequent article (see [10]) we will see that \( Nq(\gamma) \geq \gamma^\beta \) for any \( \gamma \in (0, 1) \), where \( N \) and \( \beta > 0 \) depend only on \( d, \delta, \|b\|, \kappa \).

Next, we turn our attention to estimates like in Theorem 1.1 but on the infinite time interval.

**Lemma 2.15.** For any Borel nonnegative \( f \) vanishing outside \( B_1 \) and any \( x \in \mathbb{R}^d \) we have

\[
E \int_0^\infty e^{-\phi_t} f(x + x_t) \sqrt{\det a_t} \, dt \leq N \|f\|_{L_d(B_1)},
\]

(2.23)

where \( N \) depends only on \( d \) and \( \|b\| \).

Proof. We may assume that \( f \) is bounded. In that case introduce \( \mathfrak{M} \) as the collection of all stopping times and for \( \tau \in \mathfrak{M} \) set

\[
\bar{u}_\tau := \text{esssup} \Omega u_\tau, \quad \bar{u} = \sup_{\tau \in \mathfrak{M}} u_\tau,
\]

where for a fixed \( x \in \mathbb{R}^d \)

\[
u_\tau := E \left[ \int_\tau^\infty e^{-\phi_{\tau,t}} f(x + x_t) \sqrt{\det a_t} \, dt \mid \mathcal{F}_\tau \right].
\]

Observe that, since \( \sqrt{\det a_t} \leq \text{tr} a_t \) and \( f \) is bounded, \( \bar{u} < \infty \).

Take \( R_d,\|b\| \) from Lemma 2.8, take a \( \tau \in \mathfrak{M} \), and define \( \gamma \) as the first time after \( \tau \) when \( |x_t - x_\tau| \geq R_d,\|b\| \). On the set where \( \tau < \infty \) and \( |x + x_\tau| \geq 1 + R_d,\|b\| \) we have (recall that \( f \) is zero outside \( B_1 \))

\[
u_\tau = E \left[ e^{-\phi_{\tau,\gamma}} E \left[ \int_\gamma^\infty e^{-\phi_{t,t}} f(x + x_t) \sqrt{\det a_t} \, dt \mid \mathcal{F}_\gamma \right] \mid \mathcal{F}_\tau \right]
\]

\[= E \left[ e^{-\phi_{\tau,\gamma}} u_\gamma \mid \mathcal{F}_\tau \right] \leq \bar{u} E \left[ e^{-\phi_{\tau,\gamma}} \mid \mathcal{F}_\tau \right],
\]

where by the conditional version of Lemma 2.8 the last conditional expectation is less than \( 1/2 \) (a.s.).

On the set where \( \tau < \infty \) and \( |x + x_\tau| \leq 1 + R_d,\|b\| \) we have \( B_1 \subset B_\rho(x + x_\tau) \), where \( \rho = 3 + 2R_d,\|b\| \). The choice of \( \rho \) is dictated by the fact that the outside of \( B_\rho(x + x_\tau) \) is also outside of \( B_{1+R_d,\|b\|} \). Therefore, if we define \( \theta \) as the first time after \( \tau \) when \( |x_t - x_\tau| \geq \rho \), then by the above \( u_\theta \leq \bar{u}/2 \) (a.s.). After that we observe that

\[
u_\tau = E \left[ \int_\tau^\theta e^{-\phi_{\tau,t}} f(x + x_t) \sqrt{\det a_t} \, dt + e^{-\phi_{\tau,\theta}} u_\theta \mid \mathcal{F}_\tau \right] \leq E \left[ \int_\tau^\theta e^{-\phi_{\tau,t}} f(x + x_t) \sqrt{\det a_t} \, dt \mid \mathcal{F}_\tau \right] + \bar{u}/2,
\]

where by the conditional version of Theorem 1.1 the last conditional expectation is less than a constant \( N \) depending only on \( d \) and \( \|b\| \) times the \( L_d \)-norm of \( f \).
Thus in both cases of the location of $x + x_\tau$ we have (a.s.)
\[ u_\tau \leq N \| f \|_{L_d(B_1)} + (1/2) \bar{u}. \]
The arbitrariness of $\tau$ leads to the conclusion that
\[ \bar{u} \leq N \| f \|_{L_d(B_1)} + (1/2) \bar{u} \]
and the result follows. \qed

**Corollary 2.16.** For any Borel nonnegative $f$ vanishing outside $B_1$ and $x \in \mathbb{R}^d$ we have
\[ I := E \int_0^\infty e^{-\phi_t} f(x + x_t) \sqrt[4]{\det a_t} \, dt \leq N e^{-\mu |x|} \| f \|_{L_d(B_1)}, \]
where $N$ and $\mu > 0$ depend only on $d$ and $\| b \|$.

Indeed, we only need to consider the case that $|x| \geq 2$ in which we let $\tau$ be the first exit time of $x_t$ from $B_{|x| - 1}$ and observe that in the notation from Lemma 2.15
\[ I = EI_{\tau<\infty} e^{-\phi_\tau} u_\tau \leq N \| f \|_{L_d(B_1)} EI_{\tau<\infty} e^{-\phi_\tau}. \]
After that it only remains to observe that $EI_{\tau<\infty} e^{-\phi_\tau} \leq N e^{-|x|/N}$ by Theorem 2.10.

**Theorem 2.17.** Let $p \geq d$. Then there exists constants $N$ and $\mu > 0$, depending only on $d$, $p$, and $\| b \|$, such that for any $\lambda > 0$ and Borel nonnegative $f$ given on $\mathbb{R}^d$ we have
\[ u(y) = E \int_0^\infty e^{-\phi_t} f(x_t) \sqrt[4]{\det a_t} \, dt \leq N e^{-\mu |y|} \| f \|_{L_p(\mathbb{R}^d)} \]
(2.24)
where $\Psi(x) = \exp(\sqrt{\lambda} |x|)$.

Proof. Since the case of arbitrary $\lambda > 0$ is reduced to the case $\lambda = 1$ by simply replacing $x_t$ with $x_t \sqrt{\lambda}$, we only concentrate on $\lambda = 1$. Take a nonnegative $\zeta \in C_0^\infty(B_1)$ which integrates to one, for $y \in \mathbb{R}^d$ set $f_y(x) = \zeta(x-y)f(x)$, and introduce
\[ u(y) = E \int_0^\infty e^{-\phi_t} f_y(x_t) \sqrt[4]{\det a_t} \, dt. \]
By Corollary 2.16, for any $y \in \mathbb{R}^d$,
\[ u(y) \leq N e^{-2\mu |y|} \| f_y \|_{L_d(\mathbb{R}^d)} \]
\[ \leq N e^{-\mu |y|} \| \Psi^{-1} f_y \|_{L_d(\mathbb{R}^d)} \leq N e^{-\mu |y|} \| \Psi^{-1} f_y \|_{L_p(\mathbb{R}^d)}. \]
After that it only remains to note that
\[ \int_{\mathbb{R}^d} u(y) \, dy = u, \int_{\mathbb{R}^d} \| \Psi^{-1} f_y \|_{L_p(\mathbb{R}^d)}^p \, dy = \| \zeta \|_{L_p(\mathbb{R}^d)} \| \Psi^{-1} f \|_{L_p(\mathbb{R}^d)} \]
and use Hölder’s inequality ($q = p/(p-1)$)
\[ \int_{\mathbb{R}^d} e^{-\mu |y|} \| \Psi^{-1} f_y \|_{L_p(\mathbb{R}^d)} \, dy \leq \left[ \int_{\mathbb{R}^d} e^{-\mu |y|} \, dy \right]^{1/q} \left[ \int_{\mathbb{R}^d} \| \Psi^{-1} f_y \|_{L_p(\mathbb{R}^d)}^p \, dy \right]^{1/p}. \]
The theorem is proved. □

Here is somewhat unexpected consequence of Theorem 2.17.

The second proof of (1.6) in Corollary 1.2. As in the proof of Corollary 1.2 given after Theorem 2.10 we may assume \( s = 0 \) and \( t = 1 \). Using Hölder’s inequality allows us to concentrate on \( n \geq 1 \). In this case observe that using Itô’s formula easily yields that

\[
|x_n|^2 n \leq N \int_0^1 (|x_s|^{2n-1} b(x_s) + |x_s|^{2(n-1)}) \, ds + 2n \int_0^1 |x_s|^{2(n-1)} x_s \sigma_s \, dw_s,
\]

where in light of (2.24) with \( p = d \) the last term is a square integrable martingale, so that the expectation of its supremum over \( r \leq 1 \) is dominated by a constant times the square root of

\[
I := E \int_0^1 |x_s|^{4n-2} \, ds \leq e E \int_0^1 e^{-s} |x_s|^{4n-2} \, ds.
\]

We use (2.24) with an appropriate \( \lambda = \lambda(d, \delta) \) to get that

\[
I \leq N \| \Psi^{-1}_\lambda \| L_d(\mathbb{R}^d) = N.
\]

Similarly,

\[
E \int_0^1 |x_s|^{2(n-1)} \, ds \leq N \| \Psi^{-1}_\lambda \| L_p(\mathbb{R}^d) = N.
\]

Finally,

\[
E \int_0^1 |x_s|^{2n-1} b(x_s) \, ds \leq N \| \Psi^{-1}_\lambda \| L_d(\mathbb{R}^d) \leq N,
\]

where the last inequality holds because \( \Psi^{-1}_\lambda |x|^{2n-1} \) is bounded. This proves (1.6).

We finish the section with one more result which will be extended in a subsequent article to wider range of \( p \) but more narrow set of processes (see [10]).

Theorem 2.18. Let \( p \geq d \). Then there exists constants \( N \) and \( \mu > 0 \), depending only on \( d, p, \) and \( \| b \| \), and \( R_0 = R_0(d, \| b \|) \), such that for any \( \lambda > 0, R \in [0, \infty) \), and Borel nonnegative \( f \) given on \( \mathbb{R}^d \) we have

\[
E \int_0^\infty e^{-\lambda_\mu t (B_{R}^c)} f(x_t) \sqrt{\det \Sigma_t} \, dt \leq N(R\sqrt{\lambda} + R_0)^{2-d/p} \lambda^{d/(2p)-1} \| \Psi^{-1}_{R,\lambda} f \| L_p(\mathbb{R}^d),
\]

where \( \Psi_{R,\lambda}(x) = \exp \left( \sqrt{\lambda} \mu \text{dist} (x, B_{R+R_0}/\sqrt{\lambda}) \right) \) and

\[
\phi_t(B_{R}^c) = \int_0^t \text{tr} \Sigma_s \chi_{|x_s| \geq R} \, ds.
\]
Proof. The change of coordinates \( x_t = \sqrt{\lambda} y_t \) shows that we may assume that \( \lambda = 1 \). For \( R = 0 \) the result follows from Theorem 2.17. In the general case, take \( R_0 = R(d, \|b\|) = R_d,\|b\| \) from Lemma 2.8 and define recursively,

\[ \gamma_0 = 0 \]

\[ \tau^n = \inf \{ t \geq \gamma^n : x_t \notin B_{R+R_0} \}, \quad \gamma^{n+1} = \inf \{ t \geq \tau^n : x_t \in B_R \}. \]

Also for simplicity write \( \phi_t^R \) in place of \( \phi_t(B_R^c) \).

By the conditional version of Theorem 2.17 on the set \( \tau^n < \infty \) (a.s.)

\[
E \left( \int_{\tau^n}^{\gamma^{n+1}} e^{-\phi_t^R} f(x_t) \sqrt{\det a_t} \, dt \mid \mathcal{F}_{\tau^n} \right) = e^{-\phi_{\tau^n}} E \left( \int_{\tau^n}^{\gamma^{n+1}} e^{-\phi_t^R - \phi_{\tau^n}} f(x_t) \sqrt{\det a_t} \, dt \mid \mathcal{F}_{\tau^n} \right) 
\leq e^{-\phi_{\tau^n}} N_{\Psi R_1} \| f \|_{L_p(\mathbb{R}^d)}.
\]

Furthermore, by the choice of \( R_0 \) and the conditional version of Lemma 2.8 for \( n \geq 1 \)

\[
EI_{\tau^n < \infty} e^{-\phi_{\tau^n}} \leq EI_{\tau^{n-1} < \infty} e^{-\phi_{\tau^{n-1}}} E \left( I_{\gamma^n < \infty} e^{-\phi_{\gamma^n}} \mid \mathcal{F}_{\tau^{n-1}} \right) \leq (1/2) EI_{\tau^n < \infty} e^{-\phi_{\tau^n}}. 
\]

It follows that for \( n \geq 0 \)

\[
EI_{\tau^n < \infty} e^{-\phi_{\tau^n}} \leq (1/2)^n, \quad (2.26)
\]

\[
E \int_{\tau^n}^{\gamma^{n+1}} e^{-\phi_t^R} f(x_t) \sqrt{\det a_t} \, dt \leq N (1/2)^n \| \Psi_{R_1}^{-1} f \|_{L_p(\mathbb{R}^d)}. \quad (2.27)
\]

On the other hand, by Theorem 1.1 and H"older’s inequality on the set \( \{ \gamma^n < \infty \} \) (a.s.)

\[
e^{\phi_{\gamma^n}} E \left( \int_{\gamma^n}^{\tau^n} e^{-\phi_t^R} f(x_t) \sqrt{\det a_t} \, dt \mid \mathcal{F}_{\gamma^n} \right) 
\leq E \left( \int_{\gamma^n}^{\tau^n} f(x_t) \sqrt{\det a_t} \, dt \mid \mathcal{F}_{\gamma^n} \right)
\leq N(R + R_0) \| f \|_{L_d(B_{R+R_0})} \leq N(R + R_0)^{2-d/p} \| f \|_{L_p(B_{R+R_0})}
\]

\[
= N(R + R_0)^{2-d/p} \| \Psi_{R_1}^{-1} f \|_{L_p(B_{R+R_0})},
\]

where the last equality holds because \( \Psi_{R_\lambda} = 1 \) on \( B_{R+R_0} \). Furthermore, for \( n \geq 1 \),

\[
EI_{\gamma^n < \infty} e^{-\phi_{\gamma^n}} \leq EI_{\tau^{n-1} < \infty} e^{-\phi_{\tau^{n-1}}},
\]

so that owing to (2.26)

\[
EI_{\gamma^n < \infty} e^{-\phi_{\gamma^n}} \leq (1/2)^{n-1},
\]
which is also true for $n = 0$. Hence for $n \geq 0$

$$E \int_{\gamma_n} e^{-\phi R} f(x_t) \sqrt{\det a_t} dt \leq N(1/2)^{n-1}(R + R_0)^{2-d/p}\|\Psi^{-1}_{R,t} f\|_{L_p(B_{R+R_0})}.$$ 

After that it only remains to take into account (2.27) and the fact that ($R_0 \geq 2$ and)

$$\int_0^\infty = \infty \sum_{n=0}^\infty \left( \int_{\gamma_n} \gamma_n + \int_{\gamma_n+1} \tau_n \right)$$

The theorem is proved. \qed

3. An application to elliptic equations

Let $a(x)$ be a Borel measurable function on $\mathbb{R}^d$ with values in $\mathbb{S}_d$, where $\delta \in (0, 1)$ is a fixed constant. Let $b(x)$ be a Borel measurable function on $\mathbb{R}^d$ with values in $\mathbb{R}^d$ such that

$$\|b\|_{L^d(\mathbb{R}^d)} \leq \|b\|,$$

where $\|b\| < \infty$ is a fixed number. Define

$$L = (1/2) a^{ij} D_{ij} + b^i D_i \quad (D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j).$$

Here is the result of this section. Such results play a crucial role in the proof that one can pass to the limit under the sign of fully nonlinear elliptic operators when the arguments of these operators converge only in a very weak sense (see, for instance, Section 4.2 in [9]).

Theorem 3.1 for $R = \infty$ is obtained in [8] however with $\lambda$ in (3.1) restricted from below by a constant depending on how fast $\|(b - \mu)_+\|_{L_d(\mathbb{R}^d)} \to 0$ as $\mu \to \infty$. This is caused, in part, by the fact that $b = b_1 + b_2$, where $b_1$ is bounded and $b_2 \in L_d$, in [8].

**Theorem 3.1.** Let $p \geq d$ and $R \in (0, \infty]$. Then there exists a constant $N = N(d, \delta, \|b\|) \geq 0$ such that for any $\lambda > 0$ and $u \in W^{2, \infty}_{p, \text{loc}}(B_R) \cap C(B_R)$ ($B_\infty = \mathbb{R}^d$, $C(\mathbb{R}^d)$ is the set of bounded continuous functions on $\mathbb{R}^d$) we have

$$\lambda \|u_+\|_{L^p(B_{R/2})} \leq N \|\lambda u - Lu_+\|_{L^p(B_R)} + N \lambda R^{d/p} e^{-R^\delta/N} \sup_{\partial B_R} u_+, \quad (3.1)$$

where the last term should be dropped if $R = \infty$.

**Proof.** First we note that scaling the coordinates allows us to assume that $\lambda = 1$. By having in mind the possibility to approximate $B_R$ from inside by similar domains, we see that we may assume that $u \in W^2_p(B_R)$ and $R < \infty$. We may also assume that the norm on the right in (3.1) is finite. Then we can replace $L$ with $L_n := I_{|b| \geq n}\Delta + I_{|b| < n} L$ and if (3.1) is true for $L_n$ in place of $L$ we can pass to the limit by the dominated convergence theorem. Thus, we may assume that $b$ is bounded. In that case it suffices to prove (3.1) for
$u \in C^2(\bar{B}_R)$. Finally, by using mollifications we reduce this case further to the case of smooth bounded $a, b$.

In that case take $x_0 \in B_{R/2}$, and denote by $x_t$ a solution of

$$x_t = x_0 + \int_0^t \sqrt{a(x_s)}\,dw_s + \int_0^t b(x_s)\,ds$$

existing on a probability space carrying a $d$-dimensional Wiener process $w_t$.

By Itô’s formula we conclude that for any $T \in (0, \infty)$

$$u(x_0) = E \int_0^{T \wedge \tau_R} e^{-t} f(x_t)\,dt + E e^{-T \wedge \tau_R} u(x_{T \wedge \tau_R}),$$

where $\tau_R$ is the first exit time of $x_t$ from $B_R$. By sending $T \to \infty$ we obtain

$$u(x_0) = E \int_0^{\tau_R} e^{-t} f(x_t)\,dt + E e^{-\tau_R} u(x_{\tau_R}) =: I(x_0) + J(x_0). \quad (3.2)$$

Note that, owing to (2.18), $E e^{-\tau_R} \leq 2 e^{-R/N}$, so that

$$J_+(x_0) \leq 2e^{-R/N} \sup_{\partial B_R} u_+, \quad \|J_+\|_{L^p(B_{R/2})} \leq N R^{d/p} e^{-R/N} \sup_{\partial B_R} u_+.$$

To estimate $I(x_0)$, we use the same method as in the proof of Theorem 2.17, first setting $f = 0$ outside $B_R$ and observing that

$$I(x_0) \leq E \int_0^\infty e^{-t} f(x_t)\,dt =: \bar{I}(x_0).$$

Take a nonnegative $\zeta \in C_0^\infty(B_1)$ which integrates to one, for $y \in \mathbb{R}^d$ set $f_y(x) = \zeta(x - y) f(x)$, introduce

$$I(x_0, y) = E \int_0^\infty e^{-t} (f_y(x_t))_+\,dt$$

and observe that by Corollary 2.16

$$I(x_0, y) \leq N e^{-|x_0 - y|} \|f_y\|_+ \|L^p(\mathbb{R}^d).$$

Also obviously

$$\bar{I}(x_0) \leq \int_{\mathbb{R}^d} I(x_0, y)\,dy,$$

so that

$$\bar{I}_+(x_0) \leq N \int_{\mathbb{R}^d} e^{-|x_0 - y|} \|f_y\|_+ \|L^p(\mathbb{R}^d)\,dy,$$

where the integral is the convolution of two functions. Hence

$$\|\bar{I}_+\|_{L^p(\mathbb{R}^d)} \leq N \int_{\mathbb{R}^d} e^{-|y|}\,dy \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \zeta^p(x - y) |f_+(x)|^p \,dx \right) dy \right)^{1/p}$$

$$= N \|f_+\|_{L^p(\mathbb{R}^d)} = N \|f_+\|_{L^p(R_R)}$$

and the theorem is proved. \qed
4. Parabolic case

Recall that Assumption 1.1 is supposed to be satisfied. Define

\[ C_{T,R}(t_0, x_0) = [t_0, t_0 + T] \times B_R(x_0), \quad C_R(x_0) = [0, \infty) \times B_R(x_0), \]

\[ C_{T,R} = C_{T,R}(0,0), \quad C_R = C_R(0), \quad \partial_t = \frac{\partial}{\partial t}. \]

Here is our first result.

**Theorem 4.1.** There is a constant \( N_{d,||b||} \) (depending only on \( d \) and \( ||b|| \)) such that for any \( R \in (0, \infty) \), \( x \in B_R \), and nonnegative Borel \( f \) given on \( C_R \) we have

\[ E \int_{\tau_R(x)}^{T} f(t, x + x_t) \sqrt{\det a_t} \, dt \leq N_{d,||b||} R^{d/(d+1)} ||f||_{L_{d+1}(C_R)}. \]

(4.1)

As in the case of Theorem 1.1 to prove Theorem 4.1 it suffices to prove the following.

**Theorem 4.2.** There is a constant \( N_{d,||b||} \) such that for any nonnegative Borel \( f \) given on \( C_2 \) we have

\[ E \int_{0}^{\tau} f(t, x_t) \sqrt{\det a_t} \, dt \leq N_{d,||b||} ||f||_{L_{d+1}(C_2)}. \]

(4.2)

where \( \tau = \tau_2 \).

We need the following analytic result that is a particular case of Theorem 2 in [5] in which we use the notation \( f^{(\varepsilon)} = f * \xi_\varepsilon \), where \( \xi_\varepsilon(t, x) = \varepsilon^{-d-1} \xi(t/\varepsilon, x/\varepsilon) \) and \( \xi \) is a nonnegative \( C^\infty \) function with unit integral and support in \((-1, 0) \times B_1\).

**Theorem 4.3.** Let a nonnegative function \( f \in L_{d+1}(C_4) \) be such that \( f = 0 \) outside \( C_2 \). Then on \( C_4 \) there exists a bounded nonpositive function \( z(t, x) \) such that

a) \( z(t, x) \) is convex in \( x \) for \( x \in B_4 \), \( t \geq 0 \), and increasing in \( t \) for \( x \in B_4 \);

b) for all nonnegative symmetric \( d \times d \) matrices \( a = (a^{ij}) \) and \( \varepsilon \in (0,2) \) we have on \( C_2 \) that

\[ \partial_t z^{(\varepsilon)} + a^{ij} D_{ij} z^{(\varepsilon)} \geq \alpha \sqrt{\det a f^{(\varepsilon)}}, \]

(4.3)

where \( \alpha = \alpha(d) > 0 \) is a constant;

c) there exists a constant \( N_d = N(d) \) such that

\[ |z| \leq N_d ||f||_{L_{d+1}(C_2)}. \]

(4.4)

**Proof of Theorem 4.2.** It suffices to concentrate on continuous \( f \)'s as in Theorem 4.3. Take such an \( f \) and take \( z \) from Theorem 4.3. Set \( z_1 = z \), denote

\[ F = ||f||_{L_{d+1}(C_2)}, \]

and with \( z_2 \) as in the proof of Lemma 2.4 and \( N_d \) from (4.4) introduce \( z = z_1 + N_d F z_2 \).
In light of (4.3) for all small \( \varepsilon > 0 \) and nonnegative symmetric \( d \times d \) matrix \( a \) in \( C_2 \) we have

\[
\partial_t z^{(\varepsilon)} + a^{ij} D_{ij} z^{(\varepsilon)} - b \sqrt{\det a} |Dz^{(\varepsilon)}| \geq \alpha^{d+1/2} \sqrt{\det a} f^{(\varepsilon)} - b \sqrt{\det a} |Dz^{(\varepsilon)}| + N_d F \sqrt{\det a} \left( b(1 + |Dz_2|) \right)^{\varepsilon} - N_d F b \sqrt{\det a} |Dz_2^{(\varepsilon)}|.
\]

Owing to (2.5), the last expression on \( C_2 \) is greater than

\[
\alpha^{d+1/2} \sqrt{\det a} f^{(\varepsilon)} - N_d F \sqrt{\det a} I_\varepsilon,
\]

where

\[
I_\varepsilon := \left( b(1 + |Dz_2|) \right)^{\varepsilon} - b |Dz_2^{(\varepsilon)}|
\]

is a function of \( x \) alone, for which (2.6) holds.

By observing that thanks to (1.3), for \( t \leq \tau \),

\[
b^i Dz^{(\varepsilon)}(t, x_t) \geq -b(\varepsilon t) \sqrt{\det a_t} |Dz^{(\varepsilon)}(t, x_t)|
\]

and using Itô’s formula, we get that for small \( \varepsilon \) and \( T \in (0, \infty) \)

\[
0 \geq E z^{(\varepsilon)}(\tau \wedge T, x_{\tau \wedge T}) = z^{(\varepsilon)}(0) + E \int_0^{\tau \wedge T} \left( \partial_t z^{(\varepsilon)}(t, x_t) + a^{ij} D_{ij} z^{(\varepsilon)}(t, x_t) + b^i Dz^{(\varepsilon)}(t, x_t) \right) dt
\]

\[
\geq z^{(\varepsilon)}(0) + \alpha E \int_0^{\tau \wedge T} \sqrt{\det a_t} f^{(\varepsilon)}(t, x_t) dt
\]

\[
- N_d F E \int_0^T \sqrt{\det a_t} |I_\varepsilon(x_t)| dt.
\]

The last term in (4.5) tends to zero as \( \varepsilon \downarrow 0 \) in light of (2.6) and Theorem 1.1. By passing to the limit in (4.5) as \( \varepsilon \downarrow 0 \) and \( T \to \infty \) and also using that \( f \) is continuous we find

\[
E \int_0^\tau f(t, x_t) \sqrt{\det a_t} dt \leq \alpha^{-1} |z(0)|
\]

\[
\leq \alpha^{-1} N_d F + \alpha^{-1} N_d F N_d(|b|/d)^{1/d} \exp(N_d(|b|/d)^{d/4}.
\]

The theorem is proved. \( \square \)

Below, for \( \Gamma \subset \mathbb{R}^{d+1} \), by \( |\Gamma| \) we mean its Lebesgue measure. The next theorem will be used in a subsequent article to prove the Harnack inequalities for harmonic and caloric functions associated with diffusion processes with drift in \( L_d \) (see [11]).

**Theorem 4.4.** Assume that \( a_t \in S_\delta \) for all \( (\omega, t) \). Then for any \( \kappa \in (0, 1) \) there is a function \( q(\gamma) \), \( \gamma \in (0, 1) \), depending only on \( d, \delta, |b|, \kappa \), and, naturally, on \( \gamma \), such that for any \( R \in (0, \infty) \), \( x \in B_{\kappa R} \), and closed \( \Gamma \subset C_{R^2, R} \) satisfying \( |\Gamma| \geq \gamma |C_{R^2, R}| \) we have

\[
P(\tau_\Gamma(x) \leq \tau_{R^2, R}(x)) \geq q(\gamma),
\]

where \( \tau_\Gamma(x) \) is the first time the process \( (t, x + x_t) \) hits \( \Gamma \) and \( \tau_{R^2, R}(x) \) is its first exit time from \( C_{R^2, R} \). Furthermore, \( q(\gamma) \to 1 \) as \( \gamma \uparrow 1 \).
Proof. By using scalings we reduce the general case to the one in which $R = 1$. In that case for any $\varepsilon \in (0, 1)$ we have

$$P(\tau_1(x) > \tau_{1,1}(x)) \leq P(\tau_{1,1}(x) = \int_0^{\tau_{1,1}(x)} I_{C_{1,1}}(t, x + x_t) \, dt) \leq P(\tau_1(x) \leq \varepsilon) + \varepsilon^{-1} E \int_0^{\tau_{1,1}(x)} I_{C_{1,1}}(t, x + x_t) \, dt.$$ 

In light of Theorems 2.10 and 4.1 we can estimate the right-hand side and then obtain

$$P(\tau_1(x) \leq \varepsilon) + \varepsilon^{-1} E \int_0^{\tau_{1,1}(x)} I_{C_{1,1}}(t, x + x_t) \, dt \leq 2e^{-N/\varepsilon} + N\varepsilon^{-1}(1 - \gamma)^{1/(d+1)}$$

where the constants $N$ depend only on $d, \delta, \kappa$, and $\|b\|$. By denoting

$$q(\gamma) = 1 - \inf_{\varepsilon \in (0, 1)} \left(2e^{-N/\varepsilon} + N\varepsilon^{-1}(1 - \gamma)^{1/(d+1)}\right),$$

we get what we claimed. □

Next, we turn our attention to estimates like in Theorem 4.1 but on the infinite time interval.

**Lemma 4.5.** For any $x \in \mathbb{R}^d$, and Borel nonnegative $f$ on $\mathbb{R}^{d+1}$ vanishing outside $C_1$ we have

$$E \int_0^\infty e^{-\phi_t} f(t, x + x_t) \frac{d^{1/2}}{\sqrt{\det a_t}} \, dt \leq N\|f\|_{L^{d+1}(C_1)},$$

where $N$ depends only on $d$ and $\|b\|$.

Proof. Following the proof of Lemma 2.15 we may assume that $f$ is bounded. In that case introduce $\mathcal{M}$ as the collection of all stopping times and for $\tau \in \mathcal{M}$ set

$$\bar{u}_\tau := \text{esssup}_{\Omega} u_\tau, \quad \bar{u} = \sup_{\tau \in \mathcal{M}} u_\tau,$$

where, for a fixed $\varepsilon > 0$, $u_\tau := E \left[ \int_\tau^\infty e^{-\varepsilon(t-\tau) - \phi_{\tau,t}} f(t, x + x_t) \frac{d^{1/2}}{\sqrt{\det a_t}} \, dt \mid \mathcal{F}_\tau \right]$.

Observe that since $d^{1/2}\sqrt{\det a_t} \leq \varepsilon + \text{tr} a_t$ and $f$ is bounded, $\bar{u} < \infty$. This allows us to repeat literally the proof of Lemma 2.15 with only one change that instead of Theorem 1.1 one should refer to Theorem 4.1. Then we get (4.6) with $\varepsilon t + \phi_t$ in place of $\phi_t$. After that it only remains to set $\varepsilon \downarrow 0$. □

In the same way as Corollary 2.16 is obtained we arrive at the following.

**Corollary 4.6.** For any Borel nonnegative $f$ vanishing outside $C_1$ and $x \in \mathbb{R}^d$ we have

$$E \int_0^\infty e^{-\phi_t} f(t, x + x_t) \frac{d^{1/2}}{\sqrt{\det a_t}} \, dt \leq Ne^{-\mu|x|}\|f\|_{L^{d+1}(C_1)},$$

where $N$ and $\mu > 0$ depend only on $d$ and $\|b\|$.
Next, by repeating almost literally the proof of Theorem 2.17 in case $p = d$ we obtain the following.

**Theorem 4.7.** There exists constants $N$ and $\mu > 0$, depending only on $d$ and $\|b\|$, such that for any $\lambda > 0$ and Borel nonnegative $f$ given on $\mathbb{R}^{d+1}$ we have

$$E \int_0^\infty e^{-\lambda \phi_t} f(t, x_t) \, dt \leq N \lambda^{-d/(2d+2)} \|\Psi^{-1}_\lambda f\|_{L^{d+1}(\mathbb{R}^{d+1})},$$

(4.7)

where $\Psi_\lambda(x) = \exp(\sqrt{\lambda \mu} |x|)$.

By plugging $e^{-\mu t} f$ in place of $f$ in (4.7) and then using Hölder’s inequality on its right-hand side we get the following.

**Corollary 4.8.** Let $p \geq d + 1$. There exists constants $N$ and $\mu > 0$, depending only on $d, p$, and $\|b\|$, such that for any $\lambda > 0$ and Borel nonnegative $f$ given on $\mathbb{R}^{d+1}$ we have

$$E \int_0^\infty e^{-\lambda t - \lambda \phi_t} f(t, x_t) \, dt \leq N \lambda^{(d+2)/(2p)-1} \|\Phi_\lambda f\|_{L^p(\mathbb{R}^{d+1})},$$

(4.8)

where $\Phi_\lambda(x) = \exp(-\sqrt{\lambda \mu} |x| - \lambda t/2)$.

5. **An application to parabolic equations**

Let $a(t, x)$ be a Borel measurable function on $\mathbb{R}^{d+1}$ with values in $\mathbb{S}_d$, where $\delta \in (0, 1)$ is a fixed constant. Let $b(t, x)$ be a Borel measurable function on $\mathbb{R}^{d+1}$ with values in $\mathbb{R}^d$ such that there exists a function $b \in L_d(\mathbb{R}^d)$ for which

$$|b(t, x)| \leq b(x)$$

for all $t$ and $x$. Set

$$\|b\| = \|b\|_{L_d(\mathbb{R}^d)},$$

and define

$$L = \partial_t + (1/2) a^{ij} D_{ij} + b^i D_i.$$  

Also for $t \in [0, R^2)$ set

$$C_{R^2, R}(t) = [t, R^2) \times B_R, \quad \partial C_{R^2, R}(t) = \partial C_{R^2, R}(t) \setminus \{t\} \times B_R,$$

and for $t = 0$ drop the argument $t$ in the above notation.

Here is the result of this section. Such results play a crucial role in the proof that one can pass to the limit under the sign of fully nonlinear elliptic operators when the arguments of these operators converge only in a very weak sense (see, for instance, Section 4.2 in [9]).

Theorem 5.1 for $R = \infty$ is obtained in [8] however with $\lambda$ in (5.2) restricted from below by a constant depending on how fast $\|(|b| - \mu) + \|L_{d+1}(\mathbb{R}^{d+1}) \to 0$ as $\mu \to \infty$. This is caused, in part, by the fact that $b = b_1 + b_2$, where $b_1$ is bounded and $b_2 \in L_{d+1}$, in [8].
Theorem 5.1. Let $p \geq d+1$ and $R \in (0, \infty)$. Then there exists a constant $N = N(d, \delta, \|b\|) \geq 0$ such that for any $\lambda > 0$ and $u \in W_{p, \text{loc}}^{1,2}(C'_{R^2,R}) \cap C'_{R^2,R}$, $C_{\lambda, \infty} = \{(t, x) : t \geq 0\}$, $C(C_{\lambda, \infty})$ is the set of bounded continuous functions on $C_{\lambda, \infty}$, we have

$$
\|u(t_0, \cdot)\|_{L_p(B_{R/2})} \leq N\lambda^{-(p-1)/p} \|\lambda u - Lu\|_{L_p(C'_{R^2,R})} + NR^{d/p}e^{-R\sqrt{\lambda}/N} \sup_{\partial C_{R^2,R}(t_0)} u_+,
$$

(5.1)

for any $t_0 \in [0, R^2/4]$ and

$$
\lambda \|u_+\|_{L_p(C'_{R^2,R/2})} \leq N\|\lambda u - Lu\|_{L_p(C'_{R^2,R})} + N\lambda R^{d+2/p}e^{-R\sqrt{\lambda}/N} \sup_{\partial C_{R^2,R}} u_+,
$$

(5.2)

where in both estimates the last terms should be dropped if $R = \infty$.

Proof. We follow the proof of Theorem 3.1 and convince ourselves that it suffices to concentrate on smooth $a, b,$ and $u$. Also it suffices to prove (5.1) for $\lambda = 1$ and (5.2) for $\lambda = 2$.

We start with (5.1), take $(t_0, x_0) \in C'_{R^2,R/2}$, and denote by $x_t, t \geq t_0$, a solution of

$$
x_t = x_0 + \int_{t_0}^t \sqrt{a(s, x_s)} \, dw_s + \int_{t_0}^t b(s, x_s) \, ds
$$

existing on a probability space carrying a $d$-dimensional Wiener process $w_t$. Denote by $\tau$ the first time $t$ when $(t, x_t)$ exits from $C_{R^2,R}$ after time $t_0$ that is the minimum of $R^2 - t_0 \geq (3/4)R^2$ and the first time $\gamma$ when $x_t$ exits from $B_R$ after time $t_0$.

Define $f = u - Lu$, $\tilde{f} = fI_{C_{R^2,R}}$.

By Itô's formula we conclude that

$$
u(t_0, x_0) = E\int_{t_0}^\tau e^{-(t-t_0)}\tilde{f}(t, x_t) \, dt + Eu(\tau, x_\tau)e^{-(\tau-t_0)} =: I + J.
$$

(5.3)

Note that, for any $\varepsilon \in (0, 1)$

$$
Ee^{-(\tau-t_0)} \leq P(\tau - t_0 < \varepsilon(R^2 - t_0)) + e^{-(3/4)\varepsilon R^2},
$$

where the first term on the right is $P(\gamma - t_0 < \varepsilon(R^2 - t_0))$ which owing to (2.19) is dominated by twice $\exp(-\beta/\varepsilon)$, where $\beta = \beta(d, \delta, \|b\|) > 0$. It follows that, if $R > 1$, by taking $\varepsilon = 1/R$, we get

$$
Ee^{-(\tau-t_0)} \leq 2e^{-\beta R}
$$

with another $\beta = \beta(d, \delta, \|b\|) > 0$, for which we may safely assume that $\beta \leq 1$. However, if $R < 1$,

$$
Ee^{-(\tau-t_0)} \leq e^\beta e^{-\beta R}.
$$

It follows that

$$
J \leq ee^{-\beta R} \sup_{\partial C_{R^2,R}(t_0)} u_+.
$$
To estimate $I$, we use the same method as in the proof of Theorem 2.17. Take a nonnegative $\zeta \in C^\infty_0(B_1)$ which integrates to one, for $y \in \mathbb{R}^d$ set $\tilde{f}_y(t, x) = \zeta(x - y)f(t, x)$, introduce

$$I(y) = E \int_{t_0}^\infty e^{-(t-t_0)}(\tilde{f}_y(t, x_t))_+ \, dt$$

and observe that by Corollary 4.6

$$I(y) \leq Ne^{-\mu|x-y|}\|(\tilde{f}_y)_+\|_{L_p(\mathbb{R}^{d+1})}.$$

As a result,

$$u_+(t_0, x_0) \leq N \int_{\mathbb{R}^d} e^{-\mu|x-y|}\|(\tilde{f}_y)_+\|_{L_p(\mathbb{R}^{d+1})} \, dy + e e^{-\beta R} \sup_{\partial^rC_{R^2, R}(t_0)} u_+. \quad (5.4)$$

Here the first term on the right is a convolution whose $L_p(\mathbb{R}^d)$-norm is less than the $L_1$-norm of $e^{-\mu|x|}$ times

$$\left( \int_{\mathbb{R}^d} \|(\tilde{f}_y)_+\|_{L_p(\mathbb{R}^{d+1})} \, dy \right)^{1/p} = N\|\tilde{f}_+\|_{L_p(\mathbb{R}^{d+1})} = N\|f_+\|_{L_p(C_{R^2, R}(t_0))}.$$ 

After that (5.1) for $\lambda = 1$ follows.

To prove (5.2) for $\lambda = 2$, observe that by taking $e^{-(t-t_0)}u(t, x)$ in place of $u$ in (5.1) with $\lambda = 1$ we obtain

$$\|u(t_0, \cdot)\|_{L_p(B_{R/2})} \leq N \int_{0}^{R^2} I_{t_0 \leq t} e^{-p(t-t_0)}\|((2u - Lu)_+)\|_{L_p(B_R)} \, dt + Ne^{-pR/2} \sup_{\partial^rC_{R^2, R}} (u_+)^p.$$

By integrating through this estimate with respect to $t_0 \in [0, R^2/4]$ we immediately obtain (5.2) with $\lambda = 2$. The theorem is proved.

Remark 5.1. In light of Theorem 4.1, estimates (5.1) and (5.1) are of little use when $R$ is bounded and $\lambda$ is small. The real strength of these estimates reveals when $\lambda$ is large or $R = \infty$.

6. Appendix: Proof of Theorem 2.3

Let $z(x)$ be a finite convex function on $B_4$. For each point $x_0 \in B_4$ define $\phi(z, x_0)$ as the collection of $p \in \mathbb{R}^d$ for each of which there exists $b \in \mathbb{R}$ such that the plane $z = (p, x) + b$ is a supporting plane for the graph of $z(x)$ at point $(z(x_0), x_0)$. For any Borel set $\Gamma \subset B_4$ Aleksandrov [2] sets

$$\phi(z, \Gamma) = \bigcup_{x \in \Gamma} \phi(z, x), \quad \nu(z, \Gamma) = \text{Vol} \phi(z, \Gamma)$$

and proves that $\nu(z, \Gamma)$ is a measure finite on any compact subset of $B_4$. He calls $\phi(z, \Gamma)$ the normal image of $\Gamma$ relative to $z$.

We are going to consider the equation

$$\int_{\phi(z, \Gamma)} \frac{1}{(1 + \theta|p|)^d} \, dp = \int_{\Gamma} f^d(x) \, dx, \quad (6.1)$$
which is an equation about the unknown convex \( z \) which should be satisfied for any Borel \( \Gamma \subset B_4 \), where \( \theta \) is a parameter and \( f \) is a given function.

The following is a particular case of Theorem 4 of [2], with (2.2) being a particular case of estimate (2.8) of [2].

**Theorem 6.1.** Let \( \theta = 0 \) or \( 1 \) and let \( f \) be a nonnegative function on \( B_4 \) such that \( f^d \) has finite integral over \( B_4 \). Then equation (6.1) has a solution \( z \) which is nonpositive in \( B_4 \) and satisfies estimate (2.2).

According to [2] the left-hand side of (6.1) is a measure for any convex \( z \). Recall (see, for instance, [6]) that for any convex \( z \) on \( B_4 \) its gradient \( Dz \) is well defined almost everywhere and is a monotone function. It admits an extension to the whole of \( B_4 \) and the extension denoted by \( (z_1, \ldots, z_d) \) is still monotone. According to Theorem 2.1 of [6] the first generalized derivatives of the extension are (signed) measures, finite on any compact subset of \( B_4 \).

Denote by \( z_{ij}^{(0)}(x) \) the density of the measure \( D_jz_i(dx) \).

By Theorem 3.2 of [6] the density of the absolutely continuous part of the left-hand side of (6.1) for any convex \( z \) equals

\[
\frac{1}{(1 + \theta |Dz(x)|^d)} \det (z_{ij}^{(0)}(x))
\]

Next, recall that \( \zeta \in C_0^\infty(B_1) \) is a nonnegative radially symmetric function which integrates to one, for \( \varepsilon > 0 \) we define \( \zeta_\varepsilon(x) = \varepsilon^{-d} \zeta(x/\varepsilon) \), and for any distribution \( u \) we use the notation \( u(\varepsilon) = u * \zeta_\varepsilon \). One knows that if \( u \) is a measure, then \( u(\varepsilon) \to u^{(0)} \) as \( \varepsilon \downarrow 0 \) almost everywhere, where \( u^{(0)} \) is the density of the absolutely continuous part of the measure \( u \). Owing to this and the fact that the \( D_j z_i \)’s are the generalized derivatives of \( z \) coinciding with \( z_i \)’s (as generalized functions), we get (a.e.)

\[
z_{ij}^{(0)} = \lim_{\varepsilon \downarrow 0} (D_jz_i)^{(\varepsilon)} = \lim_{\varepsilon \downarrow 0} D_j(z_i)^{(\varepsilon)} = \lim_{\varepsilon \downarrow 0} D_j(D_i z)^{(\varepsilon)} = \lim_{\varepsilon \downarrow 0} (D_{ij}z)^{(\varepsilon)} = z_{ij}^{(0)},
\]

where \( z_{ij}^{(0)} \) is the density of the absolutely continuous part of the measure \( D_{ij} z \) which is the generalized derivative of order \( D_{ij} \) of \( z \). Just in case, recall that these generalized derivatives are signed measures existing for any convex \( z \). In this way we arrive at the first assertion of the following corollary, which is stated without proof in a slightly more general situation by Aleksandrov in [2] and claimed to be a simple consequence of his arguments in [1].

**Corollary 6.2.** The solution \( z \) from Theorem 6.1 satisfies (a.e. \( B_4 \))

\[
\det(z_{ij}^{(0)}) = f^d(1 + \theta |Dz(x)|^d).
\]

Furthermore, for any \( \varepsilon \in (0, 2) \) and nonnegative symmetric matrix \( a \), (2.3) holds in \( B_2 \).
To prove the second assertion of the corollary, observe that, since the matrix of generalized derivatives \((D_{ij}z)\) is nonnegative, owing to the inequality between the arithmetic and the geometric means, we have that in the sense of generalized functions
\[
\sqrt[4]{\det(z_{ij}^{(0)})} \sqrt[4]{\det a} \, dx \leq (1/d)a^{ij}z_{ij}^{(0)} \, dx \leq (1/d)a^{ij}D_{ij}z(dx).
\]
Hence,
\[
(1/d)a^{ij}D_{ij}z(dx) \geq \sqrt[4]{\det a(f(1 + \theta|Dz|)} \, dx
\]
and it only remains to take convolutions of both sides with \(\zeta_\varepsilon\). □

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E-mail address: nkrylov@umn.edu

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455