Duality Symmetries in Non-Linear
Gauge Theories

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Abstract

Duality symmetries are discussed for non-linear gauge theories of
\((n-1)\)-th rank antisymmetric tensor fields in general even dimensions
\(d = 2n\). When there are \(M\) field strengths and no scalar fields, the
duality symmetry groups should be compact. We find conditions on
the Lagrangian required by compact duality symmetries and show an
example of duality invariant non-linear theories. We also discuss how to
enlarge the duality symmetries to non-compact groups by coupling scalar
fields described by non-linear sigma models.

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1. Introduction

The free Maxwell’s equations are invariant under SO(2) rotations of the electric field and the magnetic field into each other. The invariance of this kind is called the duality invariance. In the relativistic notation duality transformations in the Maxwell’s theory are rotations of the electro-magnetic field strength \( F_{\mu\nu} \) into its dual \( \tilde{F}_{\mu\nu} = \frac{i}{2} \epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} \). The duality invariance has generalizations in higher even dimensions. In \( d = 2n \) dimensions one can consider duality rotations of \( n \)-th rank field strengths of antisymmetric tensor fields and their duals. These duality symmetries naturally appear in supergravities \cite{1}, \cite{2}. Recently, duality symmetries play an important role in non-perturbative analysis of string theories \cite{4}.

In ref. \cite{4} the general discussion of duality symmetries in \( d = 4 \) was given. It was shown that possible duality groups are \( \text{Sp}(2M, \mathbb{R}) \) or its subgroup when there are \( M \) field strengths, and that the Lagrangian is not invariant under duality transformations but should transforms in a definite way. Non-compact duality symmetries are possible when the theories contain scalar fields described by a G/H non-linear sigma model, where G is a non-compact group and H is a maximal compact subgroup. These results were generalized to theories in higher dimensions in ref. \cite{5}. It was found that possible duality groups are \( \text{Sp}(2M, \mathbb{R}) \) or its subgroup in \( d = 4k \) and \( \text{SO}(M, M) \) or its subgroup in \( d = 4k + 2 \).

More recently, duality symmetries were studied for non-linear gauge theories, whose Lagrangians are not quadratic in the field strengths of the gauge fields. Such Lagrangians appear as low energy effective theories in string theories. In refs. \cite{6}, \cite{7} conditions on the Lagrangians required by a compact duality symmetry U(1) were obtained when there is a single field strength in \( d = 4 \). One of the theories satisfying the conditions is the Born-Infeld theory

\[
\mathcal{L}_{BI} = \frac{1}{g^2} \left[ 1 - \sqrt{-\det (\eta_{\mu\nu} + gF_{\mu\nu})} \right],
\]

which appears as a low energy effective theory in open string theories. The compact duality symmetry U(1) can be enlarged to a non-compact symmetry \( \text{Sp}(2, \mathbb{R}) \sim \text{SL}(2, \mathbb{R}) \) by coupling scalar fields (dilaton and axion fields).

The purpose of the present paper is to generalize these results on duality symmetries in non-linear gauge theories to theories which contain more than one field
strengths, and also to theories in arbitrary even dimensions. (Theories with a single field strength in $d = 4k$ dimensions were already studied in ref. [6].) Such studies will be useful in discussing string dualities. When there are $M$ field strengths and no scalar fields, the duality symmetry group should be a compact group. We find conditions on the Lagrangian required by compact duality symmetries and show an example of duality invariant non-linear theories up to sixth order in the field strengths. We also discuss how to couple non-linear sigma models to enlarge the duality symmetry groups to non-compact ones.

In the next section we review the general structure of duality symmetries in general even dimensions. In sect. 3 we work out the details of $G/H$ non-linear sigma models for groups $G$, $H$ relevant to duality symmetries. In sect. 4 conditions of duality invariance on the Lagrangian are obtained and an example of non-linear theories satisfying them is given. Finally, couplings to non-linear sigma models are discussed in sect. 5.

2. Duality symmetries

In this section we review the general structure of duality symmetries in even dimensions $d = 2n$ [4], [5]. We consider theories of $(n - 1)$-th rank antisymmetric tensor gauge fields $B^a_{\mu_1 \cdots \mu_n-1}(x)$ ($a = 1, \cdots, M$) interacting with other fields $\phi_i(x)$. Field strengths of the tensor gauge fields and their duals are defined as

$$F^a_{\mu_1 \cdots \mu_n} = n \partial_{[\mu_1} B^a_{\mu_2 \cdots \mu_n]},$$

$$\tilde{F}^a_{\mu_1 \cdots \mu_n} = \frac{1}{n!} \epsilon^{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} F^a_{\nu_1 \cdots \nu_n},$$

(2.1)

where the indices in $[\mu_1 \cdots \mu_n]$ are totally antisymmetrized with unit strength and $\epsilon^{\mu_1 \cdots \mu_d}$ is the Levi-Civita symbol. As the inclusion of gravity is straightforward, we consider theories in flat space-time. In $d$ dimensions the duality operation satisfies

$$\tilde{F} = \epsilon F, \quad \epsilon = \begin{cases} +1 & \text{for } d = 4k + 2, \\ -1 & \text{for } d = 4k. \end{cases}$$

(2.2)

We consider a Lagrangian which is a function of $M$ field strengths and of other fields $\phi_i(x)$ and their derivatives

$$\mathcal{L} = \mathcal{L}(F^a_{\mu_1 \cdots \mu_n}, \phi_i, \partial_{\mu} \phi_i).$$

(2.3)
The gauge fields $B^{\mu_1 \cdots \mu_{n-1}}$ appear only through their field strengths $F^{\mu_1 \cdots \mu_n}$.

The equations of motion for $B^{\mu_1 \cdots \mu_{n-1}}$ and the Bianchi identities are

$$\partial_{\mu_1} \left( \tilde{G}^{\mu_1 \cdots \mu_n} \right) = 0, \quad \partial_{\mu_1} \left( \tilde{F}^{\mu_1 \cdots \mu_n} \right) = 0,$$

(2.4)

where the duals of antisymmetric tensors $G^{\mu_1 \cdots \mu_n}$ are defined by

$$\tilde{G}^{\mu_1 \cdots \mu_n} = n! \frac{\partial L}{\partial F^{\mu_1 \cdots \mu_n}}.$$  

(2.5)

For the free Maxwell theory in four dimensions, we obtain $G^{\mu \nu} = \tilde{F}^{\mu \nu}$. Eq. (2.4) is invariant under transformations

$$\delta \tilde{G}^{a \mu_1 \cdots \mu_n} = n! \frac{\partial L}{\partial F^a} - \tilde{G}^{b \mu_1 \cdots \mu_n} D^a_{\mu_1 \cdots \mu_n} - \tilde{G}^{b \mu_1 \cdots \mu_n} C^b_{\mu_1 \cdots \mu_n} \partial G^c_{\mu_1 \cdots \mu_n}.$$  

(2.7)

This should be consistent with the transformation of $G$ given in eq. (2.6). By equating these two transformation laws we obtain

$$\frac{\partial}{\partial F^a} \left( n! \delta L - \frac{1}{2} \tilde{F}^b B^{b \mu_1 \cdots \mu_n} - \frac{1}{2} \tilde{C}^{b \mu_1 \cdots \mu_n} \partial G^c_{\mu_1 \cdots \mu_n} \right) - \left( A^{ab} + D^{ba} \right) n! \frac{\partial L}{\partial F^b}$$

$$= \frac{1}{2} \left( B^{ab} + \epsilon B^{ba} \right) \tilde{F}^b + \frac{1}{2} \tilde{G}^b C^{b \mu_1 \cdots \mu_n} \partial G^c_{\mu_1 \cdots \mu_n}.$$  

(2.8)

When there exist non-trivial interactions, this equation gives conditions on the transformation parameters

$$A^{ab} + D^{ba} = \alpha \delta^{ab}, \quad B^{ab} = -\epsilon B^{ba}, \quad C^{ab} = -\epsilon C^{ba},$$  

(2.9)
where $\alpha$ is an arbitrary constant, as well as a condition on the variation of the Lagrangian
\[
\frac{\partial}{\partial F^a} \left( \delta \mathcal{L} - \frac{1}{2n!} F^a B^{ab} \tilde{F}^b - \frac{1}{2n!} \tilde{G}^a C^{ab} G^b - \alpha \mathcal{L} \right) = 0. \]
\tag{2.10}

The equations of motion for $\phi^i$ are
\[
E_i \equiv \left( \frac{\partial}{\partial \phi^i} - \partial_{\mu} \frac{\partial}{\partial (\partial_{\mu} \phi^i)} \right) \mathcal{L} = 0. \]
\tag{2.11}

The covariance of these equations under the duality transformation (2.4)
\[
\delta E_i = -\frac{\partial \xi^j}{\partial \phi^i} E_j \]
\tag{2.12}
requires another condition on the variation of the Lagrangian
\[
\left( \frac{\partial}{\partial \phi^i} - \partial_{\mu} \frac{\partial}{\partial (\partial_{\mu} \phi^i)} \right) \left( \delta \mathcal{L} - \frac{1}{2n!} \tilde{G}^a C^{ab} G^b \right) = 0. \]
\tag{2.13}

We can now find out possible duality groups from eqs. (2.9), (2.10) and (2.13). Comparing eqs. (2.10) and (2.13) we find $\alpha = 0$. Then, the conditions on the parameters (2.9) can be written as
\[
X^T \Omega + \Omega X = 0, \]
\tag{2.14}
where
\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \epsilon \\ 1 & 0 \end{pmatrix}. \]
\tag{2.15}

For $d = 4k$ ($\epsilon = -1$) $\Omega$ is an antisymmetric matrix and the above condition implies that the group of duality transformations is $\text{Sp}(2M, \mathbb{R})$ or its subgroup. On the other hand, for $d = 4k + 2$ ($\epsilon = +1$) $\Omega$ is a symmetric matrix, which can be diagonalized to $\text{diag}(1, -1)$. The group of duality transformations in this case is $\text{SO}(M, M)$ or
its subgroup. Eqs. (2.9), (2.10) and (2.13) also require that the Lagrangian must transform as

$$\delta \mathcal{L} = \frac{1}{2n!} F^a B^{ab} \tilde{F}^b + \frac{1}{2n!} \tilde{G}^a C^{ab} G^b$$

$$= \delta \left( \frac{1}{2n!} F^a \tilde{G}^a \right). \quad (2.16)$$

Thus, although the Lagrangian is not invariant under the duality transformations, it transforms in a definite way. It can be shown that a derivative of the Lagrangian with respect to an invariant parameter is invariant under the duality transformations. The invariant parameter can be an invariant external field such as the metric. Thus, the energy-momentum tensor, which is given by a functional derivative of the Lagrangian with respect to the metric, is invariant under the duality transformations.

In field theories with non-compact symmetries one has to check the absence of ghosts, i.e., negative norm states. When no scalar fields are present, only compact subgroups of $\text{Sp}(2M, \mathbb{R})$ or $\text{SO}(M, M)$ are possible as duality symmetries without introducing ghosts. The maximal compact subgroup of $\text{Sp}(2M, \mathbb{R})$ and $\text{SO}(M, M)$ are $\text{U}(M)$ and $\text{SO}(M) \times \text{SO}(M)$ respectively. Elements of these compact groups are given by $X$ in eq. (2.15) with

$$D = A, \quad C = \epsilon B, \quad A^T = -A, \quad B^T = -\epsilon B. \quad (2.17)$$

Non-compact duality symmetries are possible when scalar fields described by a $G/H$ non-linear sigma model are present. Here, $G$ is a non-compact duality group and $H$ is a maximal compact subgroup of $G$. In the next section we work out the details of $G/H$ non-linear sigma models for $G = \text{Sp}(2M, \mathbb{R})$, $H = \text{U}(M)$ in $d = 4k$ and $G = \text{SO}(M, M)$, $H = \text{SO}(M) \times \text{SO}(M)$ in $d = 4k + 2$.

3. Non-linear sigma models

The $G/H$ non-linear sigma model [8], [4] is a theory of $G/H$-valued scalar fields, which is invariant under rigid $G$ transformations. For our purpose $G$ is a non-compact Lie group and $H$ is a maximal compact subgroup of $G$. We represent the scalar fields by a $G$-valued scalar field $V(x)$ and require local $H$ invariance. Since we do not introduce independent $H$ gauge fields, the $H$ part of $V(x)$ can be gauged away
and physical degrees of freedom are on a coset space $G/H$. Under the rigid $G$ transformation $g$ and the local $H$ transformation $h(x)$ the scalar field $V(x)$ transforms as

$$V(x) \to gV(x)h^{-1}(x). \quad (3.1)$$

To construct the action we decompose the Lie algebra $G$ of $G$ as $G = H + N$, where $H$ is the Lie algebra of $H$ and $N$ is its orthogonal complement in $G$. The orthogonality is defined with respective to the trace in a certain representation: $\text{tr}(HN) = 0$. The $G$-valued field $V^{-1}\partial_\mu V$ is decomposed as

$$V^{-1}\partial_\mu V = Q_\mu + P_\mu, \quad Q_\mu \in H, \quad P_\mu \in N. \quad (3.2)$$

The transformation laws of $Q_\mu$ and $P_\mu$ under the local $H$ transformation in (3.1) are found to be

$$Q_\mu \to hQ_\mu h^{-1} + h\partial_\mu h^{-1}, \quad P_\mu \to hP_\mu h^{-1}, \quad (3.3)$$

while they are invariant under the rigid $G$ transformations. We see that $Q_\mu$ transforms as an $H$ gauge field, while $P_\mu$ is covariant under the $H$ transformations. By using these quantities we can construct actions which are invariant under the rigid $G$ and the local $H$ transformations. The kinetic term of the scalar fields is

$$\mathcal{L}_{\text{scalar}} = \frac{1}{2} \text{tr}(P_\mu P^\mu), \quad (3.4)$$

which is quadratic in derivatives of $V$ and is manifestly invariant under the rigid $G$ and the local $H$ transformations. The $H$ gauge field $Q_\mu$ can be used to define the covariant derivatives on other fields transforming under the local $H$. The covariant quantity $P_\mu$ also can be used to construct invariant terms in the action.

Let us work out the details of the theories relevant to duality symmetries. For $G = \text{Sp}(2M, \mathbb{R})$, $H = \text{U}(M)$ in $d = 4k$ the $G$ transformations in the complex basis are

$$\begin{pmatrix} F + iG \\ F - iG \end{pmatrix} \to \begin{pmatrix} a & b^* \\ b & a^* \end{pmatrix} \begin{pmatrix} F + iG \\ F - iG \end{pmatrix}, \quad a^\dagger a - b^\dagger b = 1, \quad a^T b - b^T a = 0, \quad (3.5)$$
where \(a, b\) are \(M \times M\) complex matrices. The subgroup \(H = U(M)\) corresponds to the case \(b = 0\). In this basis the \(G\)-valued scalar field is expressed as

\[
V(x) = \begin{pmatrix}
\phi_0(x) & \phi_1^*(x) \\
\phi_1(x) & \phi_0^*(x)
\end{pmatrix}, \quad \phi_0^* \phi_0 - \phi_1^* \phi_1 = 1, \quad \phi_0^T \phi_1 - \phi_1^T \phi_0 = 0.
\] (3.6)

The physical degrees of freedom are represented by an \(H\)-invariant variable

\[
z = \phi_1^*(\phi_0^*)^{-1} = z^T,
\] (3.7)

which transforms under \(G\) as

\[
z \rightarrow (az + b^*)(bz + a)^{-1}.
\] (3.8)

Alternatively, we can use a variable

\[
S = S_1 + iS_2 = i \frac{1 - z}{1 + z}, \quad S_1, S_2 \in \mathbb{R}
\] (3.9)

to represent physical degrees of freedom. Its infinitesimal \(G\) transformation is

\[
\delta S = AS + B - SCS - SD.
\] (3.10)

The kinetic term for the scalar fields (3.4) becomes

\[
L_{\text{scalar}} = \text{tr} \left[ \left(\phi_0^T \partial_\mu \phi_1 - \phi_1^T \partial_\mu \phi_0 \right)^* \left(\phi_0^T \partial^\mu \phi_1 - \phi_1^T \partial^\mu \phi_0 \right) \right]
= \text{tr} \left[ \frac{1}{1 - z^* z} \partial_\mu z^* \frac{1}{1 - z z^*} \partial^\mu z \right]
= - \text{tr} \left[ \frac{1}{S - S^*} \partial_\mu S^* \frac{1}{S - S^*} \partial^\mu S \right].
\] (3.11)

We now turn to the case \(G = SO(M, M), H = SO(M) \times SO(M)\) in \(d = 4k + 2\). In the \(\Omega\)-diagonal basis the \(G\) transformations are written as

\[
\begin{pmatrix}
F + G \\
F - G
\end{pmatrix} \rightarrow \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
F + G \\
F - G
\end{pmatrix},
\]

\[a^T a - c^T c = 1, \quad d^T d - b^T b = 1, \quad a^T b - c^T d = 0.
\] (3.12)
where \( a, b, c, d \) are \( M \times M \) real matrices. The subgroup \( H = \text{SO}(M) \times \text{SO}(M) \) corresponds to the case \( b = c = 0 \). The \( G \)-valued scalar field is parametrized as
\[
V(x) = \begin{pmatrix} \phi_1(x) & \psi_2(x) \\ \psi_1(x) & \phi_2(x) \end{pmatrix},
\quad \phi_1 \phi_1 - \psi_1 \psi_1 = 1, \quad \phi_2 \phi_2 - \psi_2 \psi_2 = 1, \quad \phi_1 \psi_2 - \psi_1 \phi_2 = 0. \tag{3.13}
\]
We define an \( H \)-invariant variable
\[
z = (\psi_1 (\phi_1)^{-1})^T = \psi_2 (\phi_2)^{-1}, \tag{3.14}
\]
which transforms under the \( G \) transformation as
\[
z \rightarrow (az + b)(cz + d)^{-1}. \tag{3.15}
\]
Alternatively, we can use a variable
\[
S = -S_1 + S_2 = \frac{1 - z}{1 + z}, \quad S_1^T = -S_1, \quad S_2^T = S_2, \tag{3.16}
\]
whose infinitesimal \( G \) transformation is
\[
\delta S = AS - B + SCS - SD. \tag{3.17}
\]
The kinetic term (3.4) then becomes
\[
\mathcal{L}_{\text{scalar}} = \text{tr} \left[ \left( \phi_1^T \partial_\mu \psi_2 - \psi_1^T \partial_\mu \phi_2 \right) \left( \phi_2^T \partial^\mu \psi_1 - \psi_2^T \partial^\mu \phi_1 \right) \right]
\]
\[
= \text{tr} \left[ \frac{1}{1 - z^T z} \partial_\mu z^T \frac{1}{1 - z^T z} \partial^\mu z \right]
\]
\[
= \text{tr} \left[ \frac{1}{S + S^T} \partial_\mu S^T \frac{1}{S + S^T} \partial^\mu S \right]. \tag{3.18}
\]
4. Non-linear gauge theories

Let us consider duality invariant gauge theories, which are not quadratic in the field strengths. In this section we shall first consider theories which contain only gauge fields and no other fields. The duality symmetry group should be a compact group. We consider the maximal symmetric cases: $\text{U}(M)$ for $d = 4k$ and $\text{SO}(M) \times \text{SO}(M)$ for $d = 4k + 2$. The transformation parameters satisfy eq. (2.17).

According to the general discussion in sect. 2 the Lagrangian must satisfies eq. (2.16). On the other hand, assuming that the Lagrangian is a function of only the field strengths $F^a$ we obtain

$$
\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta F^a} = \frac{1\!}{n!} \left( C^{ab} G^b + D^{ab} F^b \right) \tilde{G}^a. \tag{4.1}
$$

By equating these two expressions and by using eq. (2.17) we obtain the conditions

$$
F^{a\mu_1\cdots\mu_n} \tilde{F}_{\mu_1\cdots\mu_n} + G^{a\mu_1\cdots\mu_n} \tilde{G}_{\mu_1\cdots\mu_n} = 0, \tag{4.2}
$$

$$
F^{a\mu_1\cdots\mu_n} \tilde{G}_{\mu_1\cdots\mu_n} - F^{b\mu_1\cdots\mu_n} \tilde{G}_{\mu_1\cdots\mu_n} = 0. \tag{4.3}
$$

The first condition (4.2) is a generalization of the condition for a single gauge field obtained in refs. [6], [7]. From eq. (2.5) the second condition (4.3) can be rewritten as

$$
F^a \frac{\partial \mathcal{L}}{\partial F^b} - F^b \frac{\partial \mathcal{L}}{\partial F^a} = 0. \tag{4.4}
$$

We see that it requires the $\text{SO}(M)$ invariance of the Lagrangian when $F^a$ transform as an $M$-dimensional vector of $\text{SO}(M)$.

Let us obtain an example of the Lagrangians which satisfy these conditions. We expand the Lagrangian in $F$ and consider terms up to and including of order $O(F^6)$

$$
\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + O(F^8), \tag{4.5}
$$

where $\mathcal{L}_m$ is of order $O(F^m)$. We make the following ansatz

$$
\begin{align*}
\mathcal{L}_2 &= \alpha_1 (F^a F^a) + \alpha_2 (F^a \tilde{F}^a), \\
\mathcal{L}_4 &= \beta_1 (F^a F^a)^2 + \beta_2 (F^a F^b)^2 + \beta_3 (F^a \tilde{F}^a)^2 + \beta_4 (F^a \tilde{F}^b)^2, \\
\mathcal{L}_6 &= \gamma_1 (F^a F^a)^3 + \gamma_2 (F^a F^b)^2 (F^c F^c) + \gamma_3 (F^a F^b)(F^a F^c)(F^b F^c) \\
&\quad + \gamma_4 (F^a \tilde{F}^a)^2 (F^c F^c) + \gamma_5 (F^a \tilde{F}^b)^2 (F^c F^c) \\
&\quad + \gamma_6 (F^a \tilde{F}^b)(F^a F^b)(F^c \tilde{F}^c) + \gamma_7 (F^a \tilde{F}^b)(F^a \tilde{F}^c)(F^b \tilde{F}^c), \tag{4.6}
\end{align*}
$$
where \((F^a F^b) = F^a_{\mu_1 \cdots \mu_n} F^{b\mu_1 \cdots \mu_n}\) etc. We note that the terms proportional to \((F^a \tilde{F}^a)\) identically vanish in \(d = 4k + 2\) since \((F^a \tilde{F}^a) = -(\tilde{F}^a F^a) = 0\). Since the indices \(a, b, \cdots\) are appropriately contracted, the condition (1.3) is automatically satisfied. Substituting eqs. (1.5), (1.6) into eq. (1.2) and using the identities

\[(F_1 \tilde{F}_2) = -\epsilon(\tilde{F}_1 F_2), \quad (\tilde{F}_1 \tilde{F}_2) = -(F_1 F_2),\]

we find that the condition (1.2) is satisfied at this order when the non-vanishing parameters are

\[\alpha_1 = \pm \frac{1}{2n!}, \quad \beta_2 = \beta_4 = \alpha_1 \beta, \quad \gamma_3 = \gamma_7 = 2\alpha_1 \beta^2.\] (4.8)

Thus the Lagrangian up to sixth order in \(F\) is

\[
\mathcal{L} = \pm \frac{1}{2n!} \left[ (F^a F^a) + \beta \left\{ (F^a F^b)(F^a F^b) + (F^a \tilde{F}^b)(F^a \tilde{F}^b) \right\} 
+ 2\beta^2 \left\{ (F^a F^b)(F^a F^c)(F^b F^c) + (F^a \tilde{F}^b)(F^a \tilde{F}^c)(F^b F^c) \right\} \right] + \mathcal{O}(F^8).\] (4.9)

For \(M=1\) in four dimensions we see that this Lagrangian coincides with the Born-Infeld Lagrangian (1.1) up to this order.

5. Coupling to scalar fields

The compact duality symmetries in the previous section can be enlarged to non-compact symmetries by introducing scalar fields. In this section we shall discuss couplings of scalar fields described by G/H non-linear sigma models to non-linear gauge theories. We require that the theories are invariant under the maximal non-compact duality symmetries \(\text{Sp}(2M, \mathbb{R})\) in \(d = 4k\) or \(\text{SO}(M, M)\) in \(4k + 2\).

We use the fields \(S_1, S_2\) introduced in eqs. (3.9), (3.16) to represent the G/H-valued scalar fields. They have symmetry properties \(S_1^T = -\epsilon S_1, S_2^T = S_2\). It is convenient to introduce the “vielbein” \(R^{ab}\) for the “metric” \(S_2^{ab}\) by

\[S_2 = R^T R.\] (5.1)
For a given $S_2$, $R$ is determined only up to local $\text{SO}(M)$ transformations

$$
\delta R(x) = \Lambda(x) R(x), \quad \Lambda^T(x) = -\Lambda(x). \tag{5.2}
$$

When the Lagrangian is expressed by using $R$, it should be invariant under these local $\text{SO}(M)$ transformations. The duality transformations of $S_1$, $S_2$ can be derived from eqs. (3.10), (3.17)

$$
\delta S_1 = A S_1 + B - S_1 C S_1 - \epsilon R^T R C R^T R - S_1 D,
\delta R = -R C S_1 - R D, \tag{5.3}
$$

where the transformation parameters satisfy eq. (2.14).

Let us obtain conditions for the duality invariance of the theory. Since the scalar kinetic term (3.11) or (3.18) is invariant under the duality transformations (5.3) by itself, we consider the Lagrangian $\mathcal{L}$ excluding it. Then, we can assume that $\mathcal{L}$ does not depend on derivatives of the scalar fields. The Lagrangian must satisfies the transformation property (2.16). On the other hand, the variation can be expressed also as

$$
\delta \mathcal{L} = \delta F^a \frac{\partial \mathcal{L}}{\partial F^b} + \delta S_1^{ab} \frac{\partial \mathcal{L}}{\partial S_1^{ab}} + \delta R^{ab} \frac{\partial \mathcal{L}}{\partial R^{ab}}, \tag{5.4}
$$

where $\delta F^a$ and $\delta S_1$, $\delta R$ are given in eqs. (2.6) and (5.3) respectively. By equating these two expressions we obtain the conditions on the Lagrangian. From the coefficients of $B^{ab}$ we obtain

$$
\frac{\partial \mathcal{L}}{\partial S_1^{ab}} = \frac{1}{2n!} F^a \tilde{F}^b. \tag{5.5}
$$

This condition can be solved by

$$
\mathcal{L} = \frac{1}{2n!} F^a S_1^{ab} \tilde{F}^b + \tilde{\mathcal{L}}(F, R), \tag{5.6}
$$

where $\tilde{\mathcal{L}}(F, R)$ is independent of $S_1$. Next, from the coefficients of $A^{ab} = -D^{ba}$ we obtain

$$
F^a \frac{\partial \mathcal{L}}{\partial F^b} - R^{ab} \frac{\partial \mathcal{L}}{\partial R^{ca}} = 0. \tag{5.7}
$$
To solve this condition we introduce

\[ \mathcal{F}^a = R^{ab} F^b, \]  

(5.8)

and regard \( \mathcal{L} \) as a function of \( \mathcal{F} \) and \( R \). By using identities

\[
\begin{align*}
\frac{\partial \mathcal{L}(F, R)}{\partial F^b} &= R^{cb} \frac{\partial \mathcal{L}(F, R)}{\partial F^c}, \\
\frac{\partial \mathcal{L}(F, R)}{\partial R^{ca}} &= \frac{\partial \mathcal{L}(F, R)}{\partial R^{ca}} + F^a \frac{\partial \mathcal{L}(F, R)}{\partial F^c},
\end{align*}
\]

(5.9)

eq. (5.7) becomes

\[ \frac{\partial \mathcal{L}(F, R)}{\partial R^{ab}} = 0. \]

(5.10)

Thus, \( \mathcal{L}(F, R) = \hat{\mathcal{L}}(F) \) is independent of \( R \). Finally, from the coefficients of \( C^{ab} \) we obtain a condition

\[ \mathcal{F}^a \tilde{F}^b + G^a \tilde{G}^b = 0, \]

(5.11)

where we have defined

\[ \tilde{G}^a = n! \frac{\partial \hat{\mathcal{L}}(F)}{\partial F^a}. \]

(5.12)

We must also require the invariance of the Lagrangian under the local \( \text{SO}(M) \) transformations (5.2). It gives a condition

\[ \mathcal{F}^a \tilde{G}^b - \mathcal{F}^b \tilde{G}^a = 0. \]

(5.13)

To summarize the duality invariance requires that the Lagrangian takes the form

\[
\mathcal{L} = \mathcal{L}_{\text{scalar}}(S, \partial_\mu S) + \frac{1}{2n!} F^a S^{ab} \tilde{F}^b + \hat{\mathcal{L}}(\mathcal{F}),
\]

(5.14)

where the scalar kinetic term \( \mathcal{L}_{\text{scalar}} \) given in eq. (3.11) or (3.18) is included and \( \hat{\mathcal{L}}(\mathcal{F}) \) should satisfy eqs. (5.11), (5.13). The conditions (5.11), (5.13) take the same form as eqs. (4.1), (4.3). Therefore, once we find a theory containing only gauge fields which has a compact duality symmetry, we can construct a theory which has
non-compact duality symmetry by coupling it to scalar fields as in eq. (5.14). We note that the transformations of \( F \) and \( G \) have a simple form

\[
\delta F^a = (RCR^T)^{ab} G^b + \Lambda^{ab} F^b,
\]

\[
\delta G^a = \epsilon (RCR^T)^{ab} F^b + \Lambda^{ab} G^b.
\]

These have the same form as the compact U(M) or SO(M) \( \times \) SO(M) transformations in eq. (2.17) but with field dependent and local transformation parameters.

For \( M = 1 \) in four dimensions these results reduce to those in refs. [6], [7]. In particular, the scalar fields have only two components \( S_1 = a \) and \( R = e^{-\frac{1}{2}\phi} \), where \( a(x) \) and \( \phi(x) \) are the axion field and the dilaton field respectively. The Lagrangian (5.14) then becomes

\[
\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{1}{2} e^{2\phi} (\partial_{\mu} a)^2 + \frac{1}{4} a F_{\mu\nu} \tilde{F}^{\mu\nu} + \bar{\mathcal{L}} \left( e^{-\frac{1}{2}\phi} F_{\mu\nu} \right),
\]

where \( \bar{\mathcal{L}} \) must satisfies eq. (5.11). The condition (5.13) is automatically satisfied for \( M = 1 \).

Note added

After completing this paper we noticed ref. [9], in which duality symmetries of supergravities in general space-time dimensions were discussed in considerable details. We would like to thank B. Julia for informing us about these works.
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