Some desultory remarks concerning algebraic cycles and Calabi–Yau threefolds

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Abstract We study some conjectures about Chow groups of varieties of geometric genus one. Some examples are given of Calabi–Yau threefolds where these conjectures can be verified, using the theory of finite–dimensional motives.

Keywords Algebraic cycles · Chow groups · motives · finite–dimensional motives · Calabi–Yau threefolds

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1 Introduction

This note is about some specific questions concerning Chow groups $A^*X$ of complex varieties. In this field, the following relative version of Bloch’s conjecture occupies a central position:

Conjecture 1 (Bloch [6]) Let $X$ be a smooth projective variety over $\mathbb{C}$ of dimension $n$. Let $\Gamma \in A^n(X \times X)$ be a correspondence such that

$$\Gamma_* = \text{id}: H^{j,0}(X) \to H^{j,0}(X) \text{ for all } j = 2, \ldots, n.$$ 

Then

$$\Gamma_* = \text{id}: A^n_{\alpha,\beta}(X) \to A^n_{\alpha,\beta}(X).$$

(Here $A^n_{\alpha,\beta}(X)$ denotes the subgroup of 0–cycles in the kernel of the Albanese map.)

This conjecture is open in most interesting cases.

A second conjecture addressed in this note is specific to varieties with $p_g = 1$:

Conjecture 2 Let $X$ be a smooth projective variety over $\mathbb{C}$ of dimension $n$, and $p_g(X) = 1$. Then there exists a “transcendental motive” $t(X) \in \mathcal{M}_{\text{rat}}$, responsible for $H^{n,0}(X)$, which is indecomposable: any submotive of $t(X)$ is either 0 or $t(X)$.

This is motivated by results of Voisin [45] and Pedrini [30], who prove that for certain $K3$ surfaces, the transcendental motive $t_2(X)$ (in the sense of [22]) is indecomposable.

A third conjecture addressed in this note is the following conjecture made by Voisin concerning self–products of varieties of geometric genus one. (For simplicity, we only state the conjecture in the case of odd dimension.)
Remark 7 The threefold studied in great detail, the rigidity of which is Calabi–Yau. The variety desingularization

Conjecture 3 (Voisin [42]) Let X be a smooth projective variety of odd dimension n, with \( p_g(X) = 1 \) and \( h^{3,0}(X) = 0 \) for \( 0 < j < n \). For any \( k \geq 2 \), let the symmetric group \( S_k \) act on \( X^k \) by permutation of the factors. Let \( pr_k : X^k \rightarrow X^{k-1} \) denote the projection obtained by omitting one of the factors. Then the induced map

\[
(pr_k)_* : A_j(X^k)^{S_k} \rightarrow A_j(X^{k-1})
\]
is injective for \( j \leq k - 2 \).

In this note, using elementary arguments, some examples are given of Calabi–Yau threefolds where these conjectures are verified. The following is a sample of this (slightly more general statements can be found below):

**Theorem (=Theorems 28, 33 and 21)** Let X be a Calabi–Yau threefold which is rationally dominated by a product of elliptic curves. Then conjecture 2 and a weak form of conjecture 1 are true for X. If in addition \( h^{2,1}(X) = 0 \), then conjecture 3 is true for X.

One example where this applies is Beauville’s threefold [4]; other examples are given below. The main tool used in this note is the theory of finite–dimensional motives of Kimura and O’Sullivan [23].

**Conventions** All varieties will be projective irreducible varieties over \( \mathbb{C} \).

For smooth X, we will denote by \( A^j(X) \) the Chow group \( CH^j(X) \otimes \mathbb{Q} \) of codimension j cycles under rational equivalence. The notations \( A^j_{hom}(X) \) and \( A^j_{AJ}(X) \) will denote the subgroup of homologically trivial and Abel–Jacobi trivial cycles respectively. \( M_{rat} \) will denote the (contravariant) category of Chow motives with \( \mathbb{Q} \)-coefficients over \( \mathbb{C} \). For a smooth projective variety over \( \mathbb{C} \), \( h(X) = (X, \Delta_X, 0) \) will denote its motive in \( M_{rat} \). \( H^*(X) \) will denote singular cohomology with \( \mathbb{Q} \)-coefficients.

2 Some Calabi–Yau threefolds

This section presents some examples of Calabi–Yau threefolds to which our arguments apply.

**Definition 4 (Calabi–Yau)** In this note, a smooth projective variety X of dimension 3 is called a Calabi–Yau threefold if \( h^{3,0}(X) = 1 \) and \( h^{1,0}(X) = h^{2,0}(X) = 0 \).

**Remark 5** Definition 4 is non–standard; usually, one requires that the canonical bundle is trivial. For the purposes of the present note, however, definition 4 suffices.

2.1 Rigid examples

**Example 6 (Beauville [4], Strominger–Witten [36])** Let E be the Fermat elliptic curve, and let \( \varphi : E \rightarrow E \) be the automorphism given by \((x, y, z) \rightarrow (x, y, \zeta z)\), where \( \zeta \) is a primitive third root of unity. Let

\[
\varphi_3 = \varphi \times \varphi \times \varphi : E^3 \rightarrow E^3
\]

be the automorphism acting as \( \varphi \) on each factor. Let \( \tilde{E}^3 \rightarrow E^3 \) denote the blow–up of the 27 fixed points of \( \varphi^3 \), and let

\[
\tilde{\varphi_3} : \tilde{E}^3 \rightarrow \tilde{E}^3
\]
denote the automorphism induced by \( \varphi^3 \). The quotient

\[
Z := \tilde{E}^3 / \tilde{\varphi_3}
\]
is a smooth Calabi–Yau threefold, which is rigid (i.e. \( h^{2,1}(Z) = 0 \)).

**Remark 7** The threefold Z is relevant in string theory. Indeed, as explained in the nice article [13] (where Z is studied in great detail), the rigidity of Z posed a conundrum to physicists: the mirror of Z cannot be a projective threefold! This is discussed in [11], and led to the subsequent development of a theory of generalized mirror symmetry [9].

**Example 8 ([13], [33])** The group \( G = (\mathbb{Z}_3)^2 = \langle \zeta \times \zeta \times \zeta, \zeta \times \zeta^2 \times 1 \rangle \) acts on \( E^3 \), and there exists a desingularization

\[
Z_2 \rightarrow E^3 / G
\]
which is Calabi–Yau. The variety \( Z_2 \) is rigid [33].
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2.2 More (not necessarily rigid) examples

Example 9 (Oguiso–Sakurai [28]) The varieties $X_{3,1}$ and $X_{3,2}$ constructed in [28, Theorem 3.4] are Calabi–Yau threefolds, obtained as crepant resolutions of quotients $E^3/G$, where $E$ is an elliptic curve and $G \subset \text{Aut}(E^3)$ a certain group.

Example 10 (Borcea–Voisin) Let $S$ be a K3 surface admitting a non–symplectic involution $\alpha$ which fixes $k = 10$ rational curves. Let $E$ be an elliptic curve, and let $\iota: E \rightarrow E$ be the involution $z \mapsto -z$. There exists a desingularization $X \rightarrow (S \times E)/(\alpha \times \iota)$ which is Calabi–Yau; it has $h^{2,1}(X) = 11 - k$ [43, 8]. To be sure, the Borcea–Voisin construction exists more generally for any $k \leq 10$ [43, 8]; in this note, however, we only consider the extremal case $k = 10$. In this easy case of the Borcea–Voisin construction, the K3 surface $S$ is rationally dominated by a product of elliptic curves. Also (as explained in [14, 2.4]), $X$ is birational to a double cover of $\mathbb{P}^3$ branched along 8 planes.

Remark 11 In [10], the Borcea–Voisin construction is generalized, to include quotients of higher–order automorphisms of $S \times E$. In some cases, e.g. [10, Table 2 lines 18 and 19], the resulting Calabi–Yau threefold is rationally dominated by curves, and is rigid (cf. [10, Remarks 6.3 and 6.5]).

3 Preliminaries

3.1 Standard conjecture $B(X)$

Let $X$ be a smooth projective variety of dimension $n$, and $h \in H^2(X, \mathbb{Q})$ the class of an ample line bundle. The hard Lefschetz theorem asserts that the map

$$L^{n-i}: H^i(X, \mathbb{Q}) \rightarrow H^{2n-i}(X, \mathbb{Q})$$

obtained by cupping with $h^{n-i}$ is an isomorphism, for any $i < n$. One of the standard conjectures asserts that the inverse isomorphism is algebraic:

**Definition 12** Given a variety $X$, we say that $B(X)$ holds if for all ample $h$, and all $i < n$ the isomorphism

$$(L^{n-i})^{-1}: H^{2n-i}(X, \mathbb{Q}) \cong H^i(X, \mathbb{Q})$$

is induced by a correspondence.

**Remark 13** It is known that $B(X)$ holds for the following varieties: curves, surfaces, abelian varieties [24, 25], threefolds not of general type [17], hyperkähler varieties of $K3^{[n]}$-type [12], $n$–dimensional varieties $X$ which have $A_i(X)$ supported on a subvariety of dimension $i + 2$ for all $i \leq \frac{n-2}{2}$ [38, Theorem 7.1], $n$–dimensional varieties $X$ which have $H_i(X) = N^{n-2}H_i(X)$ for all $i > n$ [39, Theorem 4.2], products and hyperplane sections of any of these [24, 25].

For smooth projective varieties over $\mathbb{C}$, the standard conjecture $B(X)$ implies the standard conjecture $D(X)$, i.e homological and numerical equivalence coincide on $X$ and $X \times X$ [24, 25].

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1 The definition of Calabi–Yau variety in [28] is different from ours, as it is not required that $h^{2,0} = 0$; however (as noted in [13, Section 4.1]), the varieties $X_{3,1}$ and $X_{3,2}$ do have $h^{2,0} = 0$. 

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3.2 Coniveau and niveau filtration

**Definition 14 (Coniveau filtration)** Let $X$ be a quasi–projective variety. The coniveau filtration on cohomology and on homology is defined as

\[ N^c H^j(X, \mathbb{Q}) = \sum \text{Im}(H^j_Y(X, \mathbb{Q}) \to H^j(X, \mathbb{Q})) ; \]
\[ N^c H_i(X, \mathbb{Q}) = \sum \text{Im}(H_i(Z, \mathbb{Q}) \to H_i(X, \mathbb{Q})) , \]

where $Y$ runs over codimension $\geq c$ subvarieties of $X$, and $Z$ over dimension $\leq i - c$ subvarieties.

Vial introduced the following variant of the coniveau filtration:

**Definition 15 (Niveau filtration)** Let $X$ be a smooth projective variety. The niveau filtration on homology is defined as

\[ \tilde{N}^j H_i(X) = \sum \text{Im}(H_{i-2j}(Z) \to H_i(X)) , \]

where the union runs over all smooth projective varieties $Z$ of dimension $i - 2j$, and all correspondences $\Gamma \in A_{i-2j}(Z \times X)$. The niveau filtration on cohomology is defined as

\[ \tilde{N}^c H^j(X) := \tilde{N}^{c+1-n} H_{2n-i}X . \]

**Remark 16** The niveau filtration is included in the coniveau filtration:

\[ \tilde{N}^j H_i(X) \subset N^j H_i(X) . \]

These two filtrations are expected to coincide; indeed, Vial shows this is true if and only if the Lefschetz standard conjecture is true for all varieties [41, Proposition 1.1].

Using the truth of the Lefschetz standard conjecture in degree $\leq 1$, it can be checked [41, page 6 "Properties"] that the two filtrations coincide in a certain range:

\[ \tilde{N}^j H^i(X) = N^j H^i(X) \text{ for all } j \geq i - \frac{1}{2} . \]

3.3 Finite–dimensional motives

We refer to [23], [2], [21], [27] for basics on finite–dimensional motives. A crucial property is the nilpotence theorem, which allows to lift relations between cycles from homological to rational equivalence:

**Theorem 17 (Kimura [23])** Let $X$ be a smooth projective variety of dimension $n$ with finite–dimensional motive. Let $\Gamma \in A^n(X \times X)$ be a correspondence which is numerically trivial. Then there is $N \in \mathbb{N}$ such that

\[ \Gamma^N = 0 \in A^n(X \times X) . \]

Conjecturally, any variety has finite–dimensional motive [23]. We are still far from knowing this, but at least there are quite a few non–trivial examples:

**Remark 18** The following varieties have finite–dimensional motive: abelian varieties, varieties dominated by products of curves [23], K3 surfaces with Picard number 19 or 20 [29], surfaces not of general type with $p_g = 0$ [17, Theorem 2.11], many examples of surfaces of general type with $p_g = 0$ [31, 45], generalized Kummer varieties [47, Remark 2.9(ii)], 3–folds and 4–folds with nef tangent bundle [18, 19], varieties of dimension $\leq 3$ rationally dominated by products of curves [40, Example 3.15], varieties $X$ with $A_{i,1}X = 0$ for all $i$ [39, Theorem 4] (in particular, Fano 3–folds [16]), products of varieties with finite–dimensional motive [23].

**Remark 19** It is worth pointing out that up till now, all examples of finite-dimensional motives happen to be in the tensor subcategory generated by Chow motives of curves. On the other hand, “many” motives are known to lie outside this subcategory, e.g. the motive of a general hypersurface in $\mathbb{P}^3$ [3, Remark 2.34].
4 Bloch conjecture for some Calabi–Yau threefolds

Definition 20 Let $X$ be a Calabi–Yau threefold. A correspondence $\Gamma \in A^3(X \times X)$ is called symplectic if

$$\Gamma_* = \text{id}: \quad H^{0,3}(X) \to H^{0,3}(X).$$

Theorem 21 Let $X$ be a Calabi–Yau threefold. Assume moreover
(i) $X$ has finite-dimensional motive;
(ii) $B(X)$ is true;
(iii) the generalized Hodge conjecture is true for $H^3(X)$.

Let $\Gamma \in A^3(X \times X)$ be a symplectic correspondence. Then

$$\Gamma_* : \quad A^3_{\text{hom}}X \to A^3_{\text{hom}}X$$

is an isomorphism.

Remark 22 In case $X$ is not of general type (i.e., if we adhere to the usual definition of Calabi–Yau varieties), hypothesis (ii) is always fulfilled [37].

Proof Hypotheses (i) and (ii) ensure the existence of a refined Chow–K"unneth decomposition $\Pi_{i,j}$ as in [41]. There is a splitting

$$H^3(X) = H^3_{\text{tr}}(X) \oplus \tilde{N}^1 H^3(X),$$

where the “transcendental cohomology” $H^3_{\text{tr}}(X)$ is defined as

$$H^3_{\text{tr}}(X) := (H^3_{3,0}, H^3_{3,3})(X) \subset H^3(X).$$

Hypothesis (iii) implies that

$$((\Gamma - \Delta) \circ \Pi_{3,0})_* H^3(X) = 0,$$

in view of lemma 23 below. This means that

$$\Gamma - \Delta = (\Gamma - \Delta) \circ (\sum_{(i,j) \neq (0,3)} \Pi_{i,j}) \quad \text{in} \quad H^6(X \times X).$$

By construction of the $\Pi_{i,j}$, this implies

$$\Gamma - \Delta = R_0 + R_1 + R_2 \quad \text{in} \quad H^6(X \times X),$$

where $R_0, R_1, R_2$ are cycles supported on (point) $\times X$, resp. on (divisor) $\times$ (divisor), resp. on $X \times$ (point). That is, the cycle

$$\Gamma - \Delta - R_0 - R_1 - R_2 \in A^3(X \times X)$$

is homologically trivial. Applying the nilpotence theorem, and noting that the $R_\ell$ do not act on $A^3_{\text{hom}}X$, it follows that there exists $N \in \mathbb{N}$ such that

$$(\Gamma^{\circ N})_* = \text{id}: \quad A^3_{\text{hom}}X \to A^3_{\text{hom}}X.$$

In particular,

$$\Gamma_* : \quad A^3_{\text{hom}}X \to A^3_{\text{hom}}X$$

is injective and surjective.

Lemma 23 Let $X$ be a Calabi–Yau threefold, and assume the generalized Hodge conjecture is true for $H^3(X)$. Let $\Gamma \in A^3(X \times X)$ be a symplectic correspondence. Then

$$\Gamma_* = \text{id}: \quad H^3_{\text{tr}}(X) \to H^3_{\text{tr}}(X).$$
Proof The intersection pairing on $H^3(X)$ respects the decomposition
\[ H^3(X) = H^3_{tr}(X) \oplus \tilde{N}^1 H^3(X), \]
i.e. restriction induces a non-degenerate pairing
\[ H^3_{tr}(X) \otimes H^3_{tr}(X) \to H^6(X), \]
and hence $H^3_{tr}(X)$ and $\tilde{N}^1 H^3(X)$ are orthogonal with respect to the intersection pairing.

Let $\omega \in H^{0,3}(X)$ be a generator. By the truth of the generalized Hodge conjecture and remark 16, we have
\[ \tilde{N}^1 H^3(X) = N^1 H^3(X) = \{ a \in H^3(X) | a_C \cdot \omega = 0 \}. \]

Let $K \subset H^3(X)$ denote the kernel
\[ K := \ker((\Gamma - \Delta)_* : H^3(X) \to H^3(X)). \]
Since the correspondence $\Gamma$ is symplectic, we have (by definition)
\[ H^{0,3}(X) \subset K_C := \ker((\Gamma - \Delta)_* : H^3(X, \mathbb{C}) \to H^3(X, \mathbb{C})). \]
But then,
\[ K^\perp \subset \{ a \in H^3(X) | a_C \cdot \omega = 0 \} = \tilde{N}^1 H^3(X) \]
(here $^\perp$ denotes the orthogonal complement with respect to the intersection pairing on $H^3(X)$). This implies
\[ K \supset \tilde{N}^1 H^3(X)^\perp = H^3_{tr}(X). \]

Remark 24 Lemma 23 is inspired by the analogous result for $K3$ surfaces, which can be found in [45, Proof of Corollary 3.11] or [30, Lemma 2.5].

Remark 25 As for examples which satisfy the hypotheses of theorem 21 all the examples of section 2 will do. Indeed, all examples in section 2 are rationally dominated by products of elliptic curves. As such, they have finite-dimensional motive and $B(X)$ is true. The generalized Hodge conjecture is true for products of elliptic curves [1, Theorem 6.1] (NB: for products of Fermat curves, it suffices to refer to [35]); any blow-up of $E_1 \times E_2 \times E_3$ still satisfies the generalized Hodge conjecture in degree 3, hence so do the Calabi–Yau varieties of section 2 as they are dominated by such a blow-up.

5 Indecomposability

Definition 26 Let $X$ be a smooth projective variety of dimension $n \leq 5$. Assume $B(X)$ holds and $X$ has finite-dimensional motive. Then we define the “transcendental motive” $t(X)$ as
\[ t(X) := (X, \Pi_{n,0}, 0) \in M_{rat}, \]
where $\Pi_{n,0}$ is the refined Chow–Künneth projector constructed by Vial [41, Theorem 2].

Remark 27 The fact that $t(X)$ is well-defined up to isomorphism follows from [22, Theorem 7.7.3] and [41, Proposition 1.8]. In case $n = 2$, $t(X)$ coincides with the “transcendental part” $t_2(X)$ constructed for any surface in [22].

Theorem 28 Let $X$ be a Calabi–Yau threefold. Assume moreover
(i) $X$ has finite-dimensional motive;
(ii) $B(X)$ is true;
(iii) the generalized Hodge conjecture is true for $H^3(X)$.

Then $t(X)$ is indecomposable: any non-zero submotive of $t(X)$ coincides with $t(X)$. 

Proof Suppose $V = (X, v, 0) \subset t(X)$ is a submotive which is not the whole motive $t(X)$. Then in particular,

$$H^3(V) \subsetneq H^3(t(X)) = H^3_{\text{rat}}(X).$$

(Indeed, suppose we have equality. Then $V = t(X)$ in $\mathcal{M}_{\text{hom}}$, and using finite-dimensionality this implies $V = t(X)$ in $\mathcal{M}_{\text{rat}}$, contradiction.) But $H^3_{\text{rat}}(X)$ does not have non-trivial sub-Hodge structures: indeed, suppose $H^3(V, \mathbb{C})$ contains $H^{3,0}(X)$. Then

$$(v - c\Delta)_*, H^{3,0}(X) = 0,$$

for some non-zero $c \in \mathbb{Q}$. But then, as in the proof of lemma \[23\]

$$\left(\ker((v - c\Delta)|_{H^3})\right)^\perp \subset \tilde{N}^1 H^3(X),$$

whence

$$\ker((v - c\Delta)|_{H^3}) \supset \tilde{N}^1 H^3(X) = H^3_{\text{rat}}(X);$$

this is absurd as it contradicts the fact that $H^3(V) \neq H^3_{\text{rat}}(X)$. Suppose next that $H^3(V, \mathbb{C})$ does not contain $H^{3,0}(X)$, i.e. $v_* H^{3,0}(X) = 0$. Then, again as in the proof of lemma \[23\] we find that

$$\left(\ker(v|_{H^3})\right)^\perp \subset \tilde{N}^1 H^3(X),$$

whence

$$\ker(v|_{H^3}) \supset \tilde{N}^1 H^3(X) = H^3_{\text{rat}}(X);$$

it follows that $H^*(V) = 0$ and so (using finite-dimensionality) $V = 0$ in $\mathcal{M}_{\text{rat}}$.

**Corollary 29** Let $X$ be as in theorem \[25\] Let $G \subset \text{Aut}(X)$ be a finite group of finite order automorphisms.

(i) If $g \in G$ is symplectic, then

$$\text{A}^3_{\text{hom}}(X) = \text{A}^3_{\text{hom}}(Y),$$

where $Y$ denotes a resolution of singularities of the quotient $X/G$.

(ii) If $g \in G$ is not symplectic, then

$$\text{A}^3_{\text{hom}}(Y) = 0.$$

**Proof** (i) After blowing up $X$ (which doesn’t change $\text{A}^3$), we may assume the rational map $p: X \rightarrow Y$ is a morphism, i.e. $Y = X/G$. The morphism $p$ induces a map of motives

$$p: t(X) \rightarrow t(Y) \text{ in } \mathcal{M}_{\text{rat}}.$$

Since

$$p_* p^* = s \cdot \text{id}: \text{A}^3_{\text{hom}}(Y) \rightarrow \text{A}^3_{\text{hom}}(Y)$$

(where $s$ is the number of elements of $G$), this map of motives has a right-inverse (given by $1/s$ times the transpose of the graph of $p$). By general properties of pseudo-abelian categories, this means \[13\] Remark 1.7] that $t(Y)$ is (non-canonically) a direct summand of $t(X)$, i.e. we can write

$$t(X) = T_0 \oplus T_1 \text{ in } \mathcal{M}_{\text{rat}},$$

such that $p$ induces an isomorphism $T_0 \cong t(Y)$. The motive $T_0$ cannot be 0 (if it were 0, then a fortiori $t(Y) \in \mathcal{M}_{\text{hom}}$ would be 0 and hence $H^{3,0}(X) = H^{3,0}(Y) = 0$, which is absurd). Applying theorem \[25\] it follows that $T_0 = t(X)$ and so

$$p: t(X) \xrightarrow{\cong} t(Y) \text{ in } \mathcal{M}_{\text{rat}}.$$

(ii) As in the proof of (i), we have a splitting

$$t(X) = T_0 \oplus T_1 \text{ in } \mathcal{M}_{\text{rat}},$$

such that $p$ restricts to an isomorphism $T_0 \cong t(Y)$. The motive $T_0$ cannot be all of $t(X)$ (if it were, then also $p: t(X) \cong t(Y)$ in $\mathcal{M}_{\text{hom}}$ and hence $H^{3,0}(X) \cong H^{3,0}(Y)$. But this is absurd, for the projector $\frac{1}{s} \sum_{g \in G} \Gamma_g$ acts as 0 on $H^{3,0}(X)$). It follows that $T_0 = 0$ and so

$$t(Y) = 0 \text{ in } \mathcal{M}_{\text{rat}}.$$
Voisin’s conjecture

Conjecture 30 (Voisin [22]) Let \( X \) be a Calabi–Yau threefold. For any \( k \geq 2 \), let the symmetric group \( S_k \) act on \( X^k \) by permutation of the factors. Let \( pr_k : X^k \to X^{k-1} \) denote the projection obtained by omitting one of the factors. The induced map

\[
(pr_k)_* : \ A_j(X^k)^{S_k} \to A_j(X^{k-1})
\]

is injective for \( j \leq k - 2 \).

Remark 31 Suppose \( X \) has a Chow–Künneth decomposition \( h(X) = \sum h^i(X) \) in \( \mathcal{M}_{\text{rat}} \). Then conjecture 30 is equivalent to the following: for any \( k \geq 2 \), the Chow motive \( \text{Sym}^k h^3(X) \) satisfies

\[
A_j(\text{Sym}^k h^3(X)) = 0 \quad \text{for all } j \leq k - 2.
\]

In case \( k = 2 \), conjecture 30 predicts the following concrete statement about 0–cycles: let \( a, a' \in A^3_{\text{rat}}(X) \) be two 0–cycles of degree 0. Then

\[
a \times a' = -a' \times a \quad \text{in } A^6(X \times X).
\]

Remark 32 A conjecture similar to conjecture 30 can be formulated for varieties of geometric genus 1 in any dimension. We refer to [42] and [46, Conjecture 4.37 and Example 4.40] for precise statements, and verifications in certain cases.

Theorem 33 Let \( X \) be a Calabi–Yau threefold. Assume moreover

(i) \( X \) has finite–dimensional motive;
(ii) \( B(X) \) is true;
(iii) \( X \) is rigid, i.e. \( h^{2,1}(X) = 0 \).

Then conjecture 30 is true for \( X \).

Proof Hypotheses (i) and (ii) ensure the existence of a Chow–Künneth decomposition \( \Pi_i \), i.e.

\[
h(X) = h^0(X) \oplus \cdots \oplus h^6(X) \quad \text{in } \mathcal{M}_{\text{rat}}.
\]

where \( h^i(X) = (X, \Pi_i, 0) \). Let

\[
A_k := \frac{1}{k!} \left( \sum_{\sigma \in S_k} \Gamma_{\sigma} \right) \circ (\Pi_3^k) \in (A_{3k}(X^k \times X^k).
\]

On the level of cohomology, the correspondence \( A_k \) is a projector on \( \text{Sym}^k H^3(X) \subset H^{3k}(X^k) \); on Chow–theoretical level \( A_k \) is idempotent and defines the Chow motive \( \text{Sym}^k h^3(X) \) in the language of [23].

Hypothesis (iii) implies that \( \dim H^3(X) = 2 \), hence for \( k \geq 3 \) one has

\[
A_k = 0 \quad \text{in } H^{6k}(X^k \times X^k).
\]

Using the nilpotence theorem, it follows that

\[
A_k = 0 \quad \text{in } A^{3k}(X^k \times X^k).
\]

It only remains to check the case \( k = 2 \). Note that \( \text{Sym}^2 H^3(X) \) has dimension 1, and

\[
\text{Sym}^2 H^3(X) \subset H^6(X^2) \cap F^3.
\]

What’s more, the Hodge conjecture is true for this subspace, since

\[
\text{Sym}^2 H^3(X) = \mathbb{Q} \cdot \Pi_3 \subset H^6(X^2).
\]

It follows that

\[
A_2 = \Pi_3 \times \Pi_3 \quad \text{in } H^{12}(X^2 \times X^2),
\]

and hence (using the nilpotence theorem)

\[
A_2 = \Pi_3 \times \Pi_3 \quad \text{in } A^6(X^2 \times X^2).
\]

It follows that

\[
(A_2)_*(A_j(X^2)) = 0 \quad \text{for all } j \leq 2,
\]

i.e. a strong form of Voisin’s conjecture is true.
Remark 34  Theorem 33 applies to the examples in subsection 2.1 and also to the two examples of remark 11.

Remark 35  In the proof of theorem 33, we have used the condition \( \dim H^3(X) = 2 \), which is a consequence of hypothesis (iii). By replacing in the proof the correspondence \( H_3 \) by \( H_{3,0} \) (i.e., replacing the motive \( h^3(X) \) by \( t(X) \)), it is enough to assume

\[
\dim H^3_{tr}(X) = 2,
\]

a condition a priori weaker than (iii).

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