Generalized Lie Algebroids - Examples by Distinguished Lie Algebroids with Applications to Optimal Control

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Abstract

We will prove that the generalized Lie algebroid is a distinguished example by Lie algebroid. The generality of it with respect to the Lie algebroid is similar with the generality of the pull-back vector bundle with respect to the vector bundle. Next, we will prove that the proof of Theorem 3.1 from [15] is a misconception and the mentioned Theorem has no validity. Finally, we anatomize an optimal control problem solvable in the generalized Lie algebroid framework whereas Lie algebroid instrumentation can not solve it.

Keywords: (Pre)algebra, generalized (almost) Lie algebra, generalized (almost) Lie algebroid, skew-algebroid, optimal control problem.

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1 Introduction

Important applications of Lie algebras in physics and mechanics (see [32]) inspired many authors to study these spaces and to generalize them to other spaces such as Lie superalgebras [21], affine (Kac-Moody) Lie algebras [22], quasisimple Lie algebras [17], (locally) extended affine Lie algebras [1, 26, 27] and invariant affine reflection algebras [28].

A natural generalization of the usual Lie algebra over the field \( \mathbb{K} \) were introduced by Palais [29] and Rinehart [31]. It is called Lie d-ring or Lie-Rinehart algebra (see [18, 19, 20] for more details). For an easy access, see the following definition of Lie-Rinehart algebra.

Definition 1.1. A Lie-Rinehart algebra over a field \( \mathbb{K} \) is a triple \((A, [ \cdot, \cdot ]_A, (\mathcal{F}, \cdot), \rho)\) such that

\[ LR1. \ (A, [ \cdot, \cdot ]_A) \text{ is a Lie algebra over } \mathbb{K}; \]
\[ LR2. \ (\mathcal{F}, \cdot) \text{ is a commutative and unitary algebra over } \mathbb{K}; \]
\[ LR3. \ A \text{ is a module over } \mathcal{F}; \]
\[ LR4. \ The \ modules \ morphism \ \rho \text{ from } A \text{ to } Der (\mathcal{F}) \text{ (see Definition 2.4), called anchor map, is an algebras morphism and satisfy the compatibility condition} \]

\[ [u, f \cdot v]_A = f \cdot [u, v]_A + \rho(u)(f) \cdot v, \]

for any \( u, v \in A \) and \( f \in \mathcal{F} \).

The study of Lie algebroids were considerably improved by J. Pradines in [30]. He noticed that the Lie algebroids are infinitesimal versions of Lie groupoids in a functorial manner.

Remark 1.2. Using the general framework of Lie-Rinehart algebras, a Lie algebroid can be regarded as a triple \(((F, \nu, N), [ \cdot, \cdot ]_F, (\rho, Id_N))\), where \((F, \nu, N)\) is a vector bundle, \([ \cdot, \cdot ]_F\) is an operation on the module of sections \( \Gamma (F, \nu, N) \) and \((\rho, Id_N)\) is a vector bundles morphism from \((F, \nu, N)\) to \((TN, \tau_N, N)\) such that the triple

\[ ((\Gamma (F, \nu, N), [ \cdot, \cdot ]_F), (\mathcal{F} (N), \cdot), \Gamma (\rho, Id_N)), \]

is a Lie-Rinehart algebra over \( \mathbb{R} \), where \( \Gamma (\rho, Id_N)\) is the modules morphism associated to the vector bundles morphism \((\rho, Id_N)\) (for a detailed illustration, see Definition 2.5).

A first generalization of the Lie algebroid is the skew-algebroid. It was introduced by J. Grabowski and P. Urbański in [13, 14] and it was used in analytical mechanics [10, 11]. Also, we remark that the
interests for skew algebroids is given by the natural geometric framework offered for the nonholonomic mechanics (see [12]). For more applications of skew-algebroids, see e.g. [9].

A possible generalization of the Lie algebroid was introduced recently in literature by Arcuș in [2] and is called generalized Lie algebroid by the following

**Definition 1.3.** A generalized Lie algebroid is a triple \((F,\nu,N),[\cdot,\cdot]_F, (\rho,\eta))\) given by the diagrams

\[
\begin{array}{ccc}
(F,[\cdot,\cdot]_F) & \xrightarrow{\rho} & (TM,[\cdot,\cdot]_TM) \\
\downarrow \nu & & \downarrow \tau_M \\
N & \xrightarrow{\eta} & M \\
\end{array}
\quad
\begin{array}{ccc}
(TM,[\cdot,\cdot]_TM) & \xrightarrow{Th} & (TN,[\cdot,\cdot]_TN) \\
\downarrow \tau_M & & \downarrow \tau_N \\
N & \xrightarrow{h} & N \\
\end{array}
\]

where \(h\) and \(\eta\) are arbitrary diffeomorphisms, \((\rho,\eta)\) is a vector bundles morphism from \((F,\nu,N)\) to \((TM,\tau_M,M)\) and the operation

\[
\Gamma(F,\nu,N) \times \Gamma(F,\nu,N) \xrightarrow{[\cdot,\cdot]_F} \Gamma(F,\nu,N)
\]

fulfills

\[
[u,f \cdot v]_F = f \cdot [u,v]_F + (\Gamma(Th \circ \rho, h \circ \eta))(u) (f) \cdot v,
\]

such that the couple \((\Gamma(F,\nu,N),[\cdot,\cdot]_F)\) is a Lie algebra over \(F(N)\). The anchor map \(\Gamma(Th \circ \rho, h \circ \eta)\) is given by the equality

\[
\Gamma(Th \circ \rho, h \circ \eta)(u^\alpha t^\alpha) (f) = u^\alpha \rho^i \cdot \left(\frac{\partial (f \circ h)}{\partial x^i} \circ h^{-1}\right),
\]

for any \(u^\alpha t^\alpha \in \Gamma(F,\nu,N)\) and \(f \in F(N)\).

First of all, studying the definition of generalized Lie algebroid, we remark that the operation \([\cdot,\cdot]_F\) is not \(F(N)\)-bilinear and so, the couple \((\Gamma(F,\nu,N),[\cdot,\cdot]_F)\) can not be regarded as a Lie algebra over \(F(N)\) in the usual sense. In [7] a new term, prealgebra, was introduced in order to extend the notion of algebra. Basically information about prealgebras are presented in the second part of Section 2 of this paper. Using the notion of (almost) Lie (pre)algebra, we obtain a new extension of the usual Lie algebra, called generalized (almost) Lie algebra.

**Definition 1.4.** A generalized (almost) Lie algebra over unitary and commutative ring \(F\) is a triple \((A,[\cdot,\cdot]_A, (\rho,\rho_0))\) satisfying

1. \((A,[\cdot,\cdot]_A)\) is a (almost) Lie prealgebra over \(F\);
2. \((\rho,\rho_0)\) is a modules morphism from \((A,+,.\cdot)\) to \((Der(F),+,.\cdot)\) (called anchor map) such that \(\rho_0\) is invertible and satisfies the compatibility condition

\[
[u,f \cdot v]_A = f \cdot [u,v]_A + \rho_0^{-1} (\rho(u)(\rho_0(f))) \cdot v,
\]

for any \(u,v \in A\) and \(f \in F\).

The generalized (almost) Lie algebra \((A,[\cdot,\cdot]_A, (\rho,Id_{x}))\) will be denoted \((A,[\cdot,\cdot]_A,\rho)\).

More information about generalized (almost) Lie algebras is presented in Section 3 (see also [7, 8]). In Example 3.8 we prove that the class of generalized (almost) Lie algebras is larger than the class of Lie-Rinehart algebras. So, the affirmation ”...... the author’s “generalized Lie algebra” is actually a particular case of a Lie pseudoalgebra (Lie-Rinehart algebra) mentioned in the introduction.” from [15], pag. 5 is completely false.

**Remark 1.5.** A skew-algebroid/Lie algebroid can be regarded as a triple \(((F,\nu,N),[\cdot,\cdot]_F, (\rho,Id_N))\) given by the diagrams

\[
\begin{array}{ccc}
(F,[\cdot,\cdot]_F) & \xrightarrow{\rho} & (TN,[\cdot,\cdot]_TN) \\
\downarrow \nu & & \downarrow \tau_N \\
N & \xrightarrow{Id_N} & N \\
\end{array}
\]

where \((\rho,Id_N)\) is a vector bundles morphism from \((F,\nu,N)\) to \((TN,\tau_N,N)\) and \([\cdot,\cdot]_F\) is an operation on \(\Gamma(F,\nu,N)\) such that the triple

\[
(\Gamma(F,\nu,N),[\cdot,\cdot]_F,\Gamma(\rho,Id_N))
\]

is a generalized almost Lie algebra/generalized Lie algebra over \(F(N)\).
Secondly, studying the definition of generalized Lie algebroid, we remark that the anchor map 
\( \Gamma ( Th \circ \rho, h \circ \eta ) \) is only a notation, because we can not discuss about the vector bundles morphism 
\( ( Th \circ \rho, h \circ \eta ) \). We can discuss only about the composition vector bundles morphism 
\( ( Th, h ) \circ ( \rho, \eta ) \).

In the Theorem 3.11, we prove that the generalized Lie algebroid is an example by distinguished Lie
algebroid. The generality of it with respect to the Lie algebroid is similar with the generality of the
pull-back vector bundle with respect to the vector bundle. Every vector bundle can be regarded as the
pull-back of it through identity. Similar, every Lie algebroid can be regarded as a particular generalized
Lie algebroid such that \( \eta = Id_N = h. \) This was the motivation for the name of generalized Lie algebroid.

Section 4, is devoted to show that the proof of Theorem 3.1 (as the only result) from [15] is a
misconception and so why the authors of [15] have wrong ratiocination.

Using the Euclidean 3-dimensional manifold \( \Sigma \) with the differentiable structure given by the differentiable atlass \( \{(\Sigma, \varphi_\Sigma)\} \), where
\[
\varphi_\Sigma: \Sigma \to \mathbb{R}^3, \quad x \mapsto (x^1, x^2, x^3),
\]
in Section 5, we put the optimal control problem of finding the curve \([0, T] \to \Sigma \) given by
\[
(\varphi_\Sigma \circ c)(t) = (x^1(t), x^2(t), x^3(t))
\]
and the sections \( u = y^1 \frac{\partial}{\partial x^1} \in \Gamma ( T\Sigma_{\text{Imc}}, \tau_{\Sigma}, \text{Imc} ) \) which verify the control system
\[
\begin{align*}
\frac{dx^1}{dt} &= -x^2 y^2 + y^3, \\
\frac{dx^2}{dt} &= -x^1 y^1 - x^2 y^2 + y^3, \\
\frac{dx^3}{dt} &= y^1,
\end{align*}
\]
and which are solutions of an ODE by Lagrange type, where \( L \) is the Lagrange fundamental function
given by
\[
L ( x, y ) = \frac{1}{2} \left[ (y^1)^2 + (y^2)^2 + (y^3)^2 \right].
\]

We know that, using the dual of a Lie algebroid and the Legendre transformation defined by a regular
Lagrangian, Alan Weinstein gave a theory of Lagrangian systems on Lie algebroids and obtained the
Euler-Lagrange equations. Similar to Klein’s formalism for ordinary Lagrangian Mechanics [23], Alan
Weinstein proposed in [33] a developement of a Lagrangian formalism directly on a Lie algebroid.

P. Liberman showed in [24] that it is not possible to develop this formalism, if one considers the
tangent bundle of a Lie algebroid as a space for developing the theory. E. Martínez in [25] gave a full
description of a Lagrangian formalism using the prolongation of a Lie algebroid presented by K. Mackenzie
and P. J. Higgins in [16].

In the paper [3], C. M. Arcuţ presented a Lagrangian formalism using the commutative diagrams
\[
\begin{array}{c}
F \xrightarrow{g} (F, [\cdot]_{F,h}) \xrightarrow{\rho} TN \xrightarrow{Th} TN \\
\downarrow \nu \quad \downarrow \nu \quad \downarrow \tau_N \quad \downarrow \tau_N \quad \downarrow \tau_N,
\end{array}
\]

where \((g, h)\) is a locally invertible vector bundles morphism.

After some calculus, in the end of Section 5, we pass the diagrams
\[
\begin{array}{c}
\dot{c} \xrightarrow{c} T\Sigma \xrightarrow{g} (F, [\cdot]_{F,sO}) \xrightarrow{\rho} T\Sigma \xrightarrow{T sO} T\Sigma \\
\downarrow \tau_{\Sigma} \quad \downarrow \tau_{\Sigma} \quad \downarrow \tau_{\Sigma} \quad \downarrow \tau_{\Sigma} \quad \downarrow \tau_{\Sigma},
\end{array}
\]

where the vector bundle \((T\Sigma, \tau_{\Sigma}, \Sigma)\) is anchored by the generalized Lie algebroid
\(( (F, \nu, N), [\cdot]_{F,sO}, (\rho, Id_{\Sigma}) ) \) with the help of a left invertible vector bundles morphism \(( g, sO ) \).

It is important to remark that our optimal control problem can not be solve with the help of the
previous theories of Lagrangian formalism for (generalized) Lie algebroids, because \( \dim \Gamma ( T\Sigma, \tau_{\Sigma}, \Sigma ) = 3 \neq 2 = \dim \Gamma ( F, \tau_{\Sigma}, \Sigma ) \). So, the introduction of generalized Lie algebroids is motivated by the usual
problems from optimal control theory.
2 Preliminaries

In this section, we present basic notions about modules. All the examples are from the geometry of vector bundles and all the vector bundles that we used have paracompact basis. Also, we remark that if \((A,\) is a commutative group, then \((\text{End}(A),+,\circ)\) is an unitary ring.

**Definition 2.1.** If \((\mathcal{F},\ast,\cdot)\) is an unitary ring and there exists an unitary rings morphism

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \text{End}(A) \\
f & \longmapsto & \phi_f,
\end{array}
\]

then we will say that \(\mathcal{F}\) acts on \(A\) with the help of representations \(\phi_f, f \in \mathcal{F}\) and the triple \((A,+,\phi)\) will be called module over \(\mathcal{F}\). In addition, if the application \(\phi\) is injective, then \((A,+,\phi)\) will be called faithful module over \(\mathcal{F}\). In particular, if \(\phi_f(u) = f \cdot u\), for any \(f \in \mathcal{F}\) and \(u \in A\), then we will say that \((A,+,\cdot)\) or \(A\) is a module over \(\mathcal{F}\).

**Example 2.2.** If \((F,\nu,N)\) is a vector bundle, then the set of sections \(\Gamma (F,\nu,N)\) can be regarded as a faithful module over \(\mathcal{F}(N)\) with respect to the usual action ".".

**Example 2.3.** If \((F,\nu,N)\) and \((E,\pi,M)\) are two vector bundles and \(\varphi_0 \in \text{Man}(N,M)\), then \((\Gamma (F,\nu,N),+\),\(\circ)\) is a module over \(\mathcal{F}(M)\), where \(\circ\) is the action given by

\[g \circ z = \varphi_0^* (g) \cdot z,\]

for any \(g \in \mathcal{F}(M)\) and \(z \in \Gamma (F,\nu,N)\). We denoted by \(\text{Man}\) the category of manifolds.

**Definition 2.4.** If \((A,+,\phi)\) is a module over unitary ring \(\mathcal{F}\) and \((B,\boxplus,\psi)\) is a module over unitary ring \(\mathcal{G}\), then we define the morphisms set with the source \((A,+,\phi)\) and the target \((B,\boxplus,\psi)\) by

\[
\left\{ (\alpha,\alpha_0) \in \text{Hom}(A,B) \times \text{Hom}(\mathcal{G},\mathcal{G}) : \alpha (\phi_f (u)) = \psi_{\alpha_0(f)} (\alpha (u)), \forall (f \in \mathcal{F}, u \in A) \right\}.
\]

Sometime, the modules morphism \((\alpha,\text{Id}_{\mathcal{F}})\) will be denoted by \(\alpha\). The category of modules will be denoted by \(\text{Mod}\).

**Definition 2.5.** Let \((F,\nu,N)\) and \((E,\pi,M)\) be two vector bundles. The pair \((\varphi,\varphi_0)\) given by the commutative diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & E \\
\nu \downarrow & & \pi \downarrow \\
N & \xrightarrow{\varphi_0} & M \\
(t_a) & & (s_a)
\end{array}
\]

is a vector bundles morphism from \((F,\nu,N)\) to \((E,\pi,M)\), if there exists \(\Phi_{\varphi}^a \in \mathcal{F}(M)\) such that the application \(\Gamma (\varphi,\varphi_0)\) from \((\Gamma (F,\nu,N),+\),\(\circ)\) over \(\mathcal{F}(M)\) to \((\Gamma (E,\pi,M),+\),\(\circ)\) over \(\mathcal{F}(M)\) given by

\[\Gamma (\varphi,\varphi_0) (z^a \circ t_a) = z^a \cdot \Gamma (\varphi,\varphi_0) (t_a) = (z^a \cdot \Phi_{\varphi}^a) \cdot s_a,\]

is a modules morphism. In literature, \(\varphi\) is a vector bundles morphism covering \(\varphi_0\). The modules morphism \(\Gamma (\varphi,\varphi_0)\) will be called the modules morphism associated to the vector bundles morphism \((\varphi,\varphi_0)\).

Note that \(\varphi|_{F_x}\) is a real vector spaces morphism from \(F_x\) to \(E_{\varphi_0(x)}\) satisfying

\[\varphi ((z^a \circ t_a) (x)) = (\Gamma (\varphi,\varphi_0) (z^a \circ t_a)) (\varphi_0 (x)).\]

The following is a locally representation of a vector bundle morphism.

**Proposition 2.6.** Let \((F,\nu,N)\) and \((E,\pi,M)\) be two vector bundles. The pair \((\varphi,\varphi_0)\) given by the commutative diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & E \\
\nu \downarrow & & \pi \downarrow \\
N & \xrightarrow{\varphi_0} & M \\
(t_a) & & (s_a)
\end{array}
\]
where $\varphi_0 \in \text{Diff}(N,M)$, is a vector bundles morphism if and only if there exists $\varphi_\alpha^0 \in \mathcal{F}(N)$ such that
\[
\Gamma(\varphi,\varphi_0)(z^\alpha \cdot t_\alpha) = (z^\alpha \cdot \varphi_\alpha^0) \circ \varphi_0^{-1} \cdot s_a.
\]

Proof. Because $(\varphi,\varphi_0)$ is a vector bundles morphism from $(F,\nu,N)$ to $(E,\pi,M)$ if and only if
\[
\Gamma(\varphi,\varphi_0)(z^\alpha \cdot t_\alpha) = \Gamma(\varphi,\varphi_0)(\varphi_0^{-1} \circ t_\alpha) = (z^\alpha \cdot \varphi_\alpha^0) \circ \varphi_0^{-1} \cdot s_a
\]
we obtain the conclusion of the proposition. \qed

**Corollary 2.7.** Let $(F,\nu,N)$ and $(E,\pi,N)$ be two vector bundles. The pair $(\varphi,\text{Id}_N)$ given by the commutative diagram
\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & E \\
\nu \downarrow & & \pi \downarrow \\
N & \xrightarrow{\text{Id}_N} & N \\
(t_\alpha) & \xrightarrow{} & (s_a)
\end{array}
\]
is a vector bundles morphism if and only if there exists $\varphi_\alpha^0 \in \mathcal{F}(N)$ such that
\[
\Gamma(\varphi,\text{Id}_N)(z^\alpha \cdot t_\alpha) = (z^\alpha \cdot \varphi_\alpha^0) \cdot s_a.
\]

**Definition 2.8.** A prealgebra over $\mathcal{F}$ is a pair $(A,\cdot|\cdot)_A$, where $A$ is a module over $\mathcal{F}$ and the operation $[\cdot,\cdot]_A$ is biadditive. In particular, if the operation $[\cdot,\cdot]_A$ is hihomogenous, then the pair $(A,\cdot|\cdot)_A$ is called an algebra over $\mathcal{F}$.

**Example 2.9.** The set $\text{Der}(\mathcal{F})$ of derivations of the unitary ring $\mathcal{F}$ is a module over $\mathcal{F}$ and the operation $[,]_{\text{Der}(\mathcal{F})}$ given by
\[
[X,Y]_{\text{Der}(\mathcal{F})}(f) = X \circ Y(f) - Y \circ X(f), \quad \forall f \in \mathcal{F},
\]
satisfy the compatibility condition
\[
[X,f \cdot Y]_{\text{Der}(\mathcal{F})} = f \cdot [X,Y]_{\text{Der}(\mathcal{F})} + X(f) \cdot Y,
\]
for any $X,Y \in \text{Der}(\mathcal{F})$ and $f \in \mathcal{F}$. So, $\big(\text{Der}(\mathcal{F}),[,]_{\text{Der}(\mathcal{F})}\big)$ is a prealgebra over $\mathcal{F}$, but it is not an algebra over $\mathcal{F}$. Therefore, the class of prealgebras is larger than the class of algebras.

**Definition 2.10.** If $(A,\cdot|\cdot)_A$ and $(A',\cdot|\cdot)_{A'}$ are (pre)algebras over over $\mathcal{F}$ and $\mathcal{F}'$ respectively, then the set
\[
\{ (\alpha,\alpha_0) \in \text{Mod}(A,A') : \alpha([u,v]_A) = [\alpha(u),\alpha(v)]_{A'} \},
\]
will be called the set of morphisms from $(A,\cdot|\cdot)_A$ to $(A',\cdot|\cdot)_{A'}$.

We note by $\text{(Pre)Alg}$ the category of (pre)algebras.

**Definition 2.11.** If $(A,\cdot|\cdot)_A$ is a (pre)algebra over $\mathcal{F}$ such that $[u,u]_A = 0$, for any $u \in A$, then we will say that $(A,\cdot|\cdot)_A$ is an almost Lie (pre)algebra over $\mathcal{F}$. In addition, if
\[
[u,v,z]_A + [z,[u,v]_A]_A + [v,[z,u]_A]_A = 0,
\]
for any $u,v,z \in A$, then we say that the triple $(A,\cdot|\cdot)_A$ is a Lie (pre)algebra over $\mathcal{F}$.

We denote by $\text{Lie(Pre)Alg}$ the category of Lie (pre)algebras.

**Example 2.12.** $\big(\text{Der}(\mathcal{F}),[,]_{\text{Der}(\mathcal{F})}\big)$ is a Lie prealgebra over $\mathcal{F}$.
3 Generalized (Almost) Lie Algebras/Algebroids

We begin this section by a new extension of the usual notion of Lie algebra as the following.

**Definition 3.1.** A generalized (almost) Lie algebra over unitary and commutative ring $\mathcal{F}$ is a triple $(A,[,]_A, (\rho, \rho_0))$ satisfying

1. $(A,[,]_A)$ is a (almost) Lie prealgebra over $\mathcal{F}$;
2. $(\rho, \rho_0)$ is a modules morphism from $(A, +, \cdot)$ to $(\text{Der}(\mathcal{F}), +, \cdot)$ (called anchor map) such that $\rho_0$ is invertible and satisfies the compatibility condition

$$[u, f \cdot v]_A = f \cdot [u, v]_A + \rho_0^{-1}(\rho(u)(\rho_0(f))) \cdot v,$$

for any $u, v \in A$ and $f \in \mathcal{F}$.

The generalized Lie algebra $(A,[,]_A, (\rho, Id_{\mathcal{F}}))$ will be denoted $(A,[,]_A, \rho)$.

**Remark 3.2.** The triple $(A,[,]_A, (\rho, \rho_0))$ is a generalized Lie algebra over unitary and commutative ring $\mathcal{F}$ if and only if $(A,[,]_A, (\rho, \rho_0))$ is a generalized almost Lie algebra over unitary and commutative ring $\mathcal{F}$ (see [7],[8]) such that

$$[u, [v, z]_A]_A + [z, [u, v]_A]_A + [v, [z, u]_A]_A = 0,$$

for any $u, v, z \in A$.

**Theorem 3.3.** If $(A,[,]_A, \rho)$ is a generalized Lie $\mathcal{F}$-algebra such that $A$ is a faithful module, then $\rho$ is a prealgebras morphism from $(A,[,]_A)$ to $(\text{Der}(\mathcal{F}), [,]_{\text{Der}(\mathcal{F})})$.

**Proof.** Let $u, v, w \in A$ and $f \in \mathcal{F}$. Since $(A,[,]_A)$ is a Lie $\mathcal{F}$-algebra, we have the Jacobi identity:

$$[[u, v]_A, fw]_A = [u, [v, fw]_A]_A - [v, [u, fw]_A]_A. \tag{3.2}$$

Using the definition of generalized Lie $\mathcal{F}$-algebra, we obtain

$$[[u, v]_A, fw]_A = [u, f[v, w]_A]_A + \rho[u, v]_A(f) \cdot w.$$

Similarly, we get

$$[u, [v, w]_A]_A = [u, f[v, w]_A]_A + \rho(u)(f) \cdot [v, w]_A + \rho(u)(\rho(v)(f)) \cdot w,$$

and

$$[v, [u, w]_A]_A = [v, f[u, w]_A]_A + \rho(u)(f) \cdot [v, w]_A + \rho(u)(\rho(v)(f)) \cdot w.$$

Setting three above equations in (3.2) and using Jacobi identity, we obtain

$$\rho[u, v]_A(f) \cdot w = \rho(u)(\rho(v)(f)) \cdot w - \rho(v)(\rho(u)(f)) \cdot w$$

and consequently

$$\rho[u, v]_A = [\rho(u), \rho(v)]_{\text{Der}(\mathcal{F})}.$$

Thus, the modules morphism $\rho$ is a prealgebras morphism from $(A,[,]_A)$ to $(\text{Der}(\mathcal{F}), [,]_{\text{Der}(\mathcal{F})})$.

**Proposition 3.4.** If $((A,[,]_A), (\mathcal{F}, \cdot), \rho)$ is a Lie-Rinehart algebra over the field $\mathbb{K}$, then the triple $(A,[,]_A, \rho)$ is a generalized Lie algebra over $\mathcal{F}$ such that its anchor map $\rho$ is a prealgebras morphism from $(A,[,]_A)$ to $(\text{Der}(\mathcal{F}), [,]_{\text{Der}(\mathcal{F})})$. \qed


Example 3.6. If $(A,[\cdot,\cdot]_A,\rho)$ is a generalized Lie algebra over $\mathbb{K}$, then the triple $(\text{Der}(F),[\cdot,\cdot]_{\text{Der}(F)},\rho)$ can not be regarded as a Lie-Rinehart algebra over a field $\mathbb{K}$. So, if $(A,[\cdot,\cdot]_A,\rho)$ is a generalized Lie algebra such that $A$ is not a faithful module over $\mathcal{F}$, then the triple $((A,[\cdot,\cdot]_A),(\mathcal{F},\cdot),\rho)$ can not be regarded as a Lie-Rinehart algebra over a field $\mathbb{K}$. 

Corollary 3.5. If $(A,[\cdot,\cdot]_A,\rho)$ is a generalized Lie algebra over $\mathcal{F}$ such that its anchor map $\rho$ is not a prealgebras morphism from $(A,[\cdot,\cdot]_A)$ to $(\text{Der}(\mathcal{F}),[\cdot,\cdot]_{\text{Der}(\mathcal{F})})$, then the triple $((A,[\cdot,\cdot]_A),(\mathcal{F},\cdot),\rho)$ can not be regarded as a Lie-Rinehart algebra over a field $\mathbb{K}$.

Example 3.6. If $(L,[\cdot,\cdot]_L)$ is a Lie algebra over $\mathcal{F}$, then considering the null anchor map 0 from $L$ to $\text{Der}(\mathcal{F})$, it results that $(L,[\cdot,\cdot]_L,0)$ is a generalized Lie algebra over $\mathcal{F}$.

Example 3.7. If $\mathcal{F}$ is an unitary and commutative ring, then considering the usual Lie bracket $[\cdot,\cdot]_{\text{Der}(\mathcal{F})}$ and the identity anchor map $\text{Id}_{\text{Der}(\mathcal{F})}$, we will obtain that the triple $$(\text{Der}(\mathcal{F}),[\cdot,\cdot]_{\text{Der}(\mathcal{F})},\text{Id}_{\text{Der}(\mathcal{F})})$$ is a generalized Lie algebra over $\mathcal{F}$.

In the following we present a generalized Lie algebra which is not Lie-Rinehart algebra.

Example 3.8. Let $\mathcal{F}$ be an unitary and commutative ring. If $\text{Der}(\mathcal{F})$ is a free module with basis $\{\partial_i\}_{i\in\mathbb{N}}$ and $\rho$ is a modules endomorphism of $\text{Der}(\mathcal{F})$, then we define the application $\text{Der}(\mathcal{F}) \times \text{Der}(\mathcal{F}) \rightarrow \text{End}(\mathcal{F})$ given by the equality

$$X \cdot Y = Y^i(X \circ \partial_i) + \rho(X)(Y^i)\partial_i,$$

where $Y = Y^i\partial_i$ and "$\circ$" is the usual composition application. Now, we define

$$[X,Y]_{\text{Der}(\mathcal{F})} = X \cdot Y - Y \cdot X, \quad \forall X,Y \in \text{Der}(\mathcal{F}).$$

Using the Leibniz property of $\partial_i$ and $\partial_j$ we get

$$[X,Y]_{\text{Der}(\mathcal{F})}(f \cdot g) = [X,Y]_{\text{Der}(\mathcal{F})}(f) \cdot g + f \cdot [X,Y]_{\text{Der}(\mathcal{F})}(g).$$

Thus, $[X,Y]_{\text{Der}(\mathcal{F})} \in \text{Der}(\mathcal{F})$. Moreover, if $f \in \mathcal{F}$, then we have

$$[X,fY]_{\text{Der}(\mathcal{F})} = f[X,Y]_{\text{Der}(\mathcal{F})} + \rho(X)(f)Y.$$

Easily, we obtain that

$$[f \cdot X,f \cdot X]_{\text{Der}(\mathcal{F})} = f \cdot [f \cdot X,f]_{\text{Der}(\mathcal{F})} + (f \cdot \rho(X)(f)) \cdot X.$$

Also, the Jacobi identity holds for this bracket. We remark that if $\rho(\partial_i \circ \partial_j) = \rho(\partial_i) \circ \rho(\partial_j)$, then $\rho$ is a prealgebras morphism from $((\text{Der}(\mathcal{F})),[\cdot,\cdot]_{\text{Der}(\mathcal{F})})$ to $((\text{Der}(\mathcal{F})),[\cdot,\cdot]_{\text{Der}(\mathcal{F})})$. As the equality $\rho(\partial_i \circ \partial_j) = \rho(\partial_i) \circ \rho(\partial_j)$ can not be check, because $\partial_i \circ \partial_j \notin \text{Der}(\mathcal{F})$, then $\rho$ can not be a prealgebras morphism from $((\text{Der}(\mathcal{F})),[\cdot,\cdot]_{\text{Der}(\mathcal{F})})$ to $((\text{Der}(\mathcal{F})),[\cdot,\cdot]_{\text{Der}(\mathcal{F})})$. Using Corollary 3.5, it result that the triple $(\text{Der}(\mathcal{F}),(\mathcal{F},\cdot),\rho)$ can not be regarded as a Lie-Rinehart algebra over a field $\mathbb{K}$.
**Definition 3.9.** A generalized (almost) Lie algebras morphism from \((A, [\cdot, \cdot]_A, (\rho, \rho_0))\) over \(\mathcal{F}\) to \((A', [\cdot, \cdot]_{A'}, (\rho', \rho'_0))\) over \(\mathcal{F}'\) is a couple \(((a, a_0), (b, b_0))\), where

\[(a, a_0) \in \text{PreAlg}((A, [\cdot, \cdot]_A), (A', [\cdot, \cdot]_{A'}))\]

and

\[(b, b_0) \in \text{PreAlg}\left(\left([\text{Der} (\mathcal{F}), [\cdot, \cdot]_{\text{Der} (\mathcal{F})}], ([\text{Der} (\mathcal{F}'), [\cdot, \cdot]_{\text{Der} (\mathcal{F}')}])\right)\right)\]

such that the following diagrams are commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{a} & A' \\
\rho & \downarrow & \rho' \\
\text{Der} (\mathcal{F}) & \xrightarrow{b} & \text{Der} (\mathcal{F}')
\end{array}
\]

\[
\begin{array}{cccc}
\mathcal{F} & \xrightarrow{a_0} & \mathcal{F}' \\
\rho_0 & \downarrow & \rho'_0 \\
\mathcal{F} & \xrightarrow{b_0} & \mathcal{F}'
\end{array}
\]

\[
\begin{array}{cccc}
\mathcal{F} & \xrightarrow{a} & \mathcal{F}' \\
\rho & \downarrow & \rho' \\
\mathcal{F} & \xrightarrow{b} & \mathcal{F}'
\end{array}
\]

and

\[
(a \circ a) (u) \circ (b \circ \rho) (v) - (a \circ a) (u) \circ (b \circ \rho) (v) \in \text{Der} (\mathcal{F}'),
\]

for any \(u, v \in A\). Every endomorphism \(((a, a_0), (\text{id}_{\text{Der} (\mathcal{F}}), \text{id}_{\mathcal{F}}))\) will be note simply \((a, a_0)\).

We will note by \text{gala} the category of generalized almost Lie algebras/generalized Lie algebras.

**Remark 3.10.** Using the general framework of generalized Lie algebras, a generalized Lie algebroid can be regarded as a triple \(((F, \nu, N), [\cdot, \cdot]_{F, h}, (\rho, \eta))\) given by the diagrams

\[
\begin{array}{ccc}
(F, [\cdot, \cdot]_F) & \xrightarrow{\rho} & (TM, [\cdot]_{TM}) & \xrightarrow{\tau} & (TN, [\cdot]_{TN}) \\
\nu & \downarrow & \tau_M & \downarrow & \tau_N \\
N & \xrightarrow{\eta} & M & \xrightarrow{h} & N
\end{array}
\]

where \(h\) and \(\eta\) are arbitrary diffeomorphisms, \((\rho, \eta)\) is a vector bundles morphism from \((F, \nu, N)\) to \((TM, \tau_M, M)\) and \([\cdot]_{F, h}\) is an operation on \(\Gamma (F, \nu, N)\) such that the triple

\[
\left(\Gamma (F, \nu, N), [\cdot]_{F, h}, \Gamma (Th \circ \rho, h \circ \eta)\right)
\]

is a generalized Lie algebra over \(\mathcal{F}(N)\), where the anchor map \(\Gamma (Th \circ \rho, h \circ \eta)\) is given by the equality

\[
(\Gamma (Th \circ \rho, h \circ \eta)(u^\alpha t_\alpha)) (f) = u^\alpha \rho_\alpha \cdot \left(\frac{\partial (f \circ h)}{\partial x^\alpha} \circ h^{-1}\right),
\]

for any \(u^\alpha t_\alpha \in \Gamma (F, \nu, N)\) and \(f \in \mathcal{F}(N)\).

**Proposition 3.11.** A generalized Lie algebroid is a distinguished example by Lie algebroid.

**Proof.** Let \(((F, \nu, N), [\cdot]_{F, h}, (\rho, \eta))\) be a generalized Lie algebroid. Because

\[
\frac{\partial (f \circ h)}{\partial x^i} \circ h^{-1} = \frac{\partial h^i}{\partial x^t} \circ h^{-1} \cdot \frac{\partial f}{\partial x^t},
\]

then

\[
(\Gamma (Th \circ \rho, h \circ \eta)(u^\alpha t_\alpha)) (f) = u^\alpha \rho_\alpha \cdot \left(\frac{\partial h^i}{\partial x^t} \circ h^{-1}\right) \frac{\partial f}{\partial x^t}
\]

\[
\xrightarrow{\text{put}} \left(\frac{\partial u^\alpha}{\partial x^t} \frac{\partial}{\partial x^t}\right) (f) = \Gamma (\theta, \text{id}_{\mathcal{N}})(u^\alpha t_\alpha) (f)
\]

where \((\theta, \text{id}_{\mathcal{N}})\) is a vector bundles morphism from \((F, \nu, N)\) to \((TN, \tau_N, N)\). Therefore, we obtain that the anchor map \(\Gamma (Th \circ \rho, h \circ \eta)\) is a modules morphism. As \(\Gamma (F, \nu, N)\) is a faithful module, then the anchor map of the generalized Lie algebra

\[
\left(\Gamma (F, \nu, N), [\cdot]_{F, h}, \Gamma (\theta, \text{id}_{\mathcal{N}})\right)
\]

is a prealgebra morphism. Therefore, the triple

\[
\left(\left(\Gamma (F, \nu, N), [\cdot]_{F, h}\right), \left(F(\mathcal{N}), \cdot\right), \Gamma (\theta, \text{id}_{\mathcal{N}})\right)
\]

is a Lie-Rinehart algebra over \(\mathbb{R}\) and so, the triple

\[
\left((F, \nu, N), [\cdot]_{F, h}, (\theta, \text{id}_{\mathcal{N}})\right)
\]

is a Lie algebroid. \(\square\)
4 Why the Proof of Theorem 3.1 in [15] Does Not Work?

This section is devoted to detail that Theorem 3.1 in [15] is based on a completely false assumption and so is not valid. Indeed, component-wise composition of vector bundle morphism has no sense as a new vector bundle morphism; and this is their false assumption.

In the theory of vector bundles, it is very famous that if \((\varphi, \varphi_0)\) and \((\psi, \psi_0)\) are two vector bundles morphisms given by the commutative diagrams

\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & E \\
\nu \downarrow & & \pi \downarrow \\
N & \xrightarrow{\varphi_0} & M \\
(\tau) & \xrightarrow{\psi_0} & (\xi) \\
\end{array}
\]

such that \(\varphi_0 \in \text{Diff} (N, M)\) and \(\psi_0 \in \text{Diff} (M, P)\), then we can discuss only about the composition vector bundles morphism \((\psi, \psi_0) \circ (\varphi, \varphi_0)\) such that

\[
\Gamma ((\psi, \psi_0) \circ (\varphi, \varphi_0)) (z^\alpha \cdot t_\alpha) = \Gamma (\psi, \psi_0) \circ \Gamma (\varphi, \varphi_0) (z^\alpha \cdot t_\alpha)
\]

\[
= \Gamma (\psi, \psi_0) ((z^\alpha \cdot \varphi_0) \circ \varphi_0^{-1} \cdot s_\alpha)
\]

\[
= (((z^\alpha \cdot \varphi_0) \circ \varphi_0^{-1}) \cdot \psi_0) \circ \psi_0^{-1} \cdot t_\alpha.
\]

Indeed, it is trivial that the pair \((\psi \circ \varphi, \psi_0 \circ \varphi_0)\) can not be regarded as a vector bundles morphism.

So, we can establish the following for generalized Lie algebroids as special kinds of data structures dealing with the vector bundles.

**Corollary 4.1.** If \(((F, \nu, N), [], {}_{F, h}, (\rho, \eta))\) is a generalized Lie algebroid, then we can not discuss about the vector bundles morphism \(\Phi \circ \text{put} = Th \circ \rho \text{ covering } \phi \circ \text{put} = h \circ \eta\).

**Remark 4.2.** If \((\varphi, \varphi_0)\) and \((\psi, \varphi_0^{-1})\) are two vector bundles morphisms given by the commutative diagrams:

\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & E \\
\nu \downarrow & & \pi \downarrow \\
N & \xrightarrow{\varphi_0} & M \\
(\tau) & \xrightarrow{\varphi_0^{-1}} & (\xi) \\
\end{array}
\]

then the pair \((\psi \circ \varphi, Id_N) \circ \text{put} = (\psi \circ \varphi, \varphi_0^{-1} \circ \varphi_0)\) can not be regarded as a vector bundles morphism from \((F, \nu, N)\) to \((G, \tau, N)\). We can discuss only about the composition vector bundles morphism \(\psi, \varphi_0^{-1} \circ (\varphi, \varphi_0)\).

Finally, by notations used in [15] and gathering above discussions, we derive the following statement.

**Corollary 4.3.** If \(((F, \nu, N), [], {}_{F, h}, (\rho, \eta))\) is a generalized Lie algebroid, then we can not discuss about the vector bundles morphism \((T\phi)^{-1} \circ \Phi \text{ covering } Id_N = \phi^{-1} \circ \phi\).

So, the affirmation "... then \([,]_\Phi\) is a Lie algebroid bracket on \(F\) with respect to a "traditional" anchor map \(\Psi : F \rightarrow TN\) covering the identity, \(\Psi = (T\phi)^{-1} \circ \Phi\)" is false.

Therefore, the proof of Theorem 3.1 from [15] is based on a misconception and is not legal. It breaks Theorem 1.3 down as the only statement in [15].

5 Optimal control problem

Here, we detailed the optimal control problem presented in the introduction. The result shall appoint the importance of generalized Lie algebroids in real occasions.

If, for any \(x \in \Sigma\), we consider \((\dot{x}^1, \dot{x}^2, \dot{x}^3) = \varphi_\Sigma \circ s_0 (x)\), then the equations system (1.1) is equivalent to the following

\[
\frac{d\dot{x}_i^1}{dt} = -\ddot{x}^2 y^2 - y^1,
\]

\[
\frac{d\dot{x}_i^2}{dt} = -\ddot{x}^1 y^1 - \ddot{x}^2 y^2 - y^3,
\]

\[
\frac{d\dot{x}_i^3}{dt} = -y^1.
\]
As
\[
\begin{pmatrix}
\frac{d\bar{x}^1}{dt} \\
\frac{d\bar{x}^2}{dt}
\end{pmatrix}
= \begin{pmatrix}
0 & -\bar{x}^2 & -1 \\
-\bar{x}^1 & -\bar{x}^2 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
y^1 \\
y^2 \\
y^3
\end{pmatrix},
\]
and
\[
\begin{pmatrix}
0 & -\bar{x}^2 & -1 \\
-\bar{x}^1 & -\bar{x}^2 & -1
\end{pmatrix} = \begin{pmatrix}
\bar{x}^1 & -1 \\
0 & -1
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 & 0 \\
-\bar{x}^1 & 1 & 0
\end{pmatrix},
\]
it results that we can consider the vector subbundle \((F, \tau_\Sigma, \Sigma)\) of the vector bundle \((T\Sigma, \tau_\Sigma, \Sigma)\) such that \(\Gamma(F, \tau_\Sigma, \Sigma) = \text{Span}(t_1, t_2)\), where
\[
t_1 = \frac{\partial}{\partial \bar{x}^1}, \quad t_2 = \bar{x}^1 \frac{\partial}{\partial \bar{x}^1} + \bar{x}^2 \frac{\partial}{\partial \bar{x}^2} + \frac{\partial}{\partial \bar{x}^3}.
\]
Denoting by \((\rho, \text{Id}_\Sigma)\) the vector bundles morphism from \((F, \tau_\Sigma, \Sigma)\) to \((T\Sigma, \tau_\Sigma, \Sigma)\) given by
\[
\Gamma(\rho, \text{Id}_\Sigma) \begin{pmatrix}
t_1 \\
t_2
\end{pmatrix} = \begin{pmatrix}
1 \\
\bar{x}^1
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial \bar{x}^1} \\
\frac{\partial}{\partial \bar{x}^2}
\end{pmatrix}
\end{pmatrix},
\]
we obtain the Lie algebroid \(((F, \tau_\Sigma, \Sigma), [,], \tau_\Sigma, (\rho, \text{Id}_\Sigma))\). Now, we denote by \((g, s_0)\) the vector bundles morphism from \((T\Sigma, \tau_\Sigma, \Sigma)\) to \((F, \tau_\Sigma, \Sigma)\) given by
\[
\Gamma(g, s_0) \begin{pmatrix}
\frac{\partial}{\partial \bar{x}^1} \\
\frac{\partial}{\partial \bar{x}^2}
\end{pmatrix} = \begin{pmatrix}
\bar{x}^1 & -1 \\
0 & -1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
t_1 \\
t_2
\end{pmatrix},
\]
and we consider the vector bundles morphism \((T s_0, s_0)\) from \((T\Sigma, \tau_\Sigma, \Sigma)\) to \((T\Sigma, \tau_\Sigma, \Sigma)\) given by
\[
\Gamma(T s_0, s_0) \begin{pmatrix}
\frac{\partial}{\partial \bar{x}^1} \\
\frac{\partial}{\partial \bar{x}^2}
\end{pmatrix} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial \bar{x}^1} \\
\frac{\partial}{\partial \bar{x}^2} \\
\frac{\partial}{\partial \bar{x}^3}
\end{pmatrix},
\]
where \((\frac{\partial}{\partial \bar{x}^1}, \frac{\partial}{\partial \bar{x}^2}, \frac{\partial}{\partial \bar{x}^3})\) is the natural base. Since
\[
\begin{pmatrix}
\bar{x}^1 & -1 \\
0 & -1 \\
-1 & 0
\end{pmatrix} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
-\bar{x}^1 & 1 \\
0 & 1 \\
1 & 0
\end{pmatrix},
\]
then we deduce
\[
\Gamma(g, s_0) = \Gamma(R, \text{Id}_\Sigma) \circ \Gamma(T s_0, s_0),
\]
where \((R, \text{Id}_\Sigma)\) is a vector bundles morphism from \((T\Sigma, \tau_\Sigma, \Sigma)\) to \((F, \tau_\Sigma, \Sigma)\) given by
\[
\Gamma(R, \text{Id}_\Sigma) \begin{pmatrix}
\frac{\partial}{\partial \bar{x}^1} \\
\frac{\partial}{\partial \bar{x}^2}
\end{pmatrix} = \begin{pmatrix}
-\bar{x}^1 & 1 \\
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
t_1 \\
t_2
\end{pmatrix}.
\]
If we denote by \(R\) the matrix
\[
\begin{pmatrix}
-\bar{x}^1 & 1 \\
0 & 1 \\
1 & 0
\end{pmatrix},
\]
then
\[
R^t = \begin{pmatrix}
-\bar{x}^1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}, \quad R^t \circ R = \begin{pmatrix}
1 + (\bar{x}^1)^2 & -\bar{x}^1 \\
-\bar{x}^1 & 2
\end{pmatrix}
\text{ and } \det(R^t \circ R) = 2 + (\bar{x}^1)^2 \neq 0.
\]
Easily, we obtain
\[
(R^t \circ R)^{-1} = \frac{1}{2 + (\bar{x}^1)^2} \begin{pmatrix}
2 & \bar{x}^1 \\
\bar{x}^1 & 1 + (\bar{x}^1)^2
\end{pmatrix},
\]
and so
\[ R_{\text{left}}^{-1} \circ \pi = (\pi' \circ R)^{-1} \circ \pi' = \frac{1}{2 + (\tilde{x}^1)^2} \begin{pmatrix} \frac{-\tilde{x}^1}{2 + (\tilde{x}^1)^2} & \frac{\tilde{x}^1}{2 + (\tilde{x}^1)^2} & \frac{2}{2 + (\tilde{x}^1)^2} \\ 1 & 1 + (\tilde{x}^1)^2 & \tilde{x}^1 \end{pmatrix}, \]
is the left inverse of the matrix \( R \), because \( R_{\text{left}}^{-1} \cdot R = I_2 \). Thus we obtain the left inverse vector bundle morphism \((R_{\text{left}}^{-1} \cdot I_d_{\Sigma})\) from \((F, \tau_{\Sigma}, \Sigma)\) to \((T \Sigma, \tau_{\Sigma}, \Sigma)\) given by
\[ \Gamma((R_{\text{left}}^{-1} \cdot I_d_{\Sigma})(t_1 \ t_2)) = \frac{1}{2 + (\tilde{x}^1)^2} \begin{pmatrix} \frac{-\tilde{x}^1}{2 + (\tilde{x}^1)^2} & \frac{\tilde{x}^1}{2 + (\tilde{x}^1)^2} & \frac{2}{2 + (\tilde{x}^1)^2} \\ 1 & 1 + (\tilde{x}^1)^2 & \tilde{x}^1 \end{pmatrix}, \]

If \( g = Ts_{\Sigma}^{-1} \cdot R_{\text{left}}^{-1} \), then \((\tilde{g}, s_{\Sigma}^{-1})\) is a vector bundles morphism from \((F, \tau_{\Sigma}, \Sigma)\) to \((T \Sigma, \tau_{\Sigma}, \Sigma)\) and
\[ \Gamma(\tilde{g}, s_{\Sigma}^{-1})(t_1 \ t_2) = \begin{pmatrix} \frac{g}{2 + (x^1)^2} & \frac{-x^1}{2 + (x^1)^2} & \frac{1}{2 + (x^1)^2} \\ \frac{2}{2 + (x^1)^2} & \frac{2 + (x^1)^2}{2 + (x^1)^2} & \frac{2}{2 + (x^1)^2} \end{pmatrix}, \]

As \( g_{\tilde{g}} \cdot g_{s_{\Sigma}} = \delta_{\tilde{g}}, \) it results that the vector bundles morphism \((g, s_{\Sigma})\) is left invertible and \((\tilde{g}, s_{\Sigma}^{-1})\) is its left inverse. So, we pass the diagram
\[
\begin{array}{ccccccc}
\dot{c} \mapsto & T \Sigma & \xrightarrow{g} & (F, [\cdot]_{F,s_{\Sigma}}) & \xrightarrow{\rho} & T \Sigma & \xrightarrow{Ts_{\Sigma}} & T \Sigma \\
\Sigma & \xrightarrow{s_{\Sigma}} & \Sigma & \xrightarrow{I_{\Sigma}} & \Sigma & \xrightarrow{s_{\Sigma}} & \Sigma \\
\end{array}
\]

where the vector bundle \((T \Sigma, \tau_{\Sigma}, \Sigma)\) is anchored by the generalized Lie algebroid \(((F, \nu, N), [\cdot]_{F,s_{\Sigma}}, (\rho, I_{d_{\Sigma}}))\) with the help of a left invertible vector bundles morphism \((g, s_{\Sigma})\).

In the end of this paper, we ask:

- Can we develop a Lagrangian formalism directly on a vector bundle \((E, \pi, M)\) anchored by a (generalized) Lie algebroid \(((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))\) with the help of a left invertible vector bundles morphism \((g, h)\) similar to Klein’s formalism for ordinary Lagrangian Mechanics?

We suppose that it is possible, but it is necessary to extend the notion of pullback vector bundle and, using it, we can obtain a new version of the Lie algebroid generalized tangent bundle or of the prolongation Lie algebroid. Also, we suppose that, in particular situations, this space was used in all our papers including theory of connections [2], mechanics and optimal control [3], Kaluza-Klein G-spaces [4], Weil’s theory [5], vertical and complete lifts [6].

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