0. Introduction

One of the basic problems in arithmetic mirror symmetry is to compare the number of rational points on a mirror pair of Calabi-Yau varieties. At present, no general algebraic geometric definition is known for a mirror pair. But an important class of mirror pairs comes from certain quotient construction. In this paper, we study the congruence relation for the number of rational points on a quotient mirror pair of varieties over finite fields. Our main result is the following theorem:

**Theorem 0.1.** Let $X_0$ be a smooth projective variety over the finite field $\mathbb{F}_q$ with $q$ elements of characteristic $p$. Suppose $X_0$ has a smooth projective lifting $X$ over the Witt ring $W = W(\mathbb{F}_q)$ such that the $W$-modules $H^i(X, \Omega^s_{X/W})$ are free. Let $G$ be a finite group of $W$-automorphisms acting on the right of $X$. Suppose $G$ acts trivially on $H^i(X, \mathcal{O}_X)$ for all $i$. Then for any natural number $k$, we have the congruence

$$\#X_0(\mathbb{F}_q^k) \equiv \#(X_0/G)(\mathbb{F}_q^k) \pmod{q^k},$$
where \( \#X_0(F_{q^k}) \) (resp. \( \#(X_0/G)(F_{q^k}) \)) denotes the number of elements of the sets of \( F_{q^k} \)-rational points of \( X_0 \) (resp. \( X_0/G \)).

The main application of the above theorem is to Calabi-Yau varieties. This gives the following theorem announced in [W], which was the main motivation of the present paper.

**Theorem 0.2.** Let \( X_0 \) be a geometrically connected smooth projective Calabi-Yau variety of dimension \( n \) over the finite field \( F_q \) with \( q \) elements of characteristic \( p \). Suppose \( X_0 \) has a smooth projective lifting \( X \) over the Witt ring \( W = W(F_q) \) such that the \( W \)-modules \( H^r(X, \Omega^n_{X/W}) \) are free. Let \( G \) be a finite group of \( W \)-automorphisms acting on the right of \( X \). Suppose \( G \) fixes a non-zero \( n \)-form on \( X \). Then for any natural number \( k \), we have the congruence

\[
\#X_0(F_{q^k}) \equiv \#(X_0/G)(F_{q^k}) \pmod{q^k}.
\]

**Proof.** If \( X \) is a Calabi-Yau scheme over \( W \) of dimension \( n \), then \( H^i(X, \mathcal{O}_X) = 0 \) for \( i \neq 0, n \) and \( G \) acts trivially on them. If the generic fiber of \( X \) is geometrically connected, then \( G \) acts trivially on \( H^0(X, \mathcal{O}_X) \). By Serre duality, \( H^n(X, \mathcal{O}_X) \) is dual to \( H^0(X, \Omega^n_{X/W}) \). Since \( X \) is Calabi-Yau, \( \Omega^n_{X/W} \) is a trivial invertible sheaf. In order for \( G \) to act trivially on \( H^n(X, \mathcal{O}_X/W) \), it suffices for \( G \) to fix a nonzero \( n \)-form. Theorem 0.2 thus follows from Theorem 0.1.

In particular, we have the following corollary:

**Corollary 0.3.** Let \( X_0 \) be the smooth \((n - 1)\)-dimensional hypersurface

\[
x_0^{n+1} + \cdots + x_n^{n+1} + \lambda x_0 \cdots x_n = 0
\]

in \( \mathbb{P}^n_{\mathbb{F}_q} \), where \( \lambda \in \mathbb{F}_q \). Let

\[
G = \{ (\zeta_0, \ldots, \zeta_n) | \zeta_i \in \mathbb{F}_q, \zeta_i^{n+1} = 1, \prod_{i=0}^{n} \zeta_i = 1 \}.
\]
Consider the action $G \times X_0 \to X_0$ defined by

$$(\zeta_0, \ldots, \zeta_n) \times [x_0 : \ldots : x_n] \mapsto [\zeta_0 x_0 : \ldots : \zeta_n x_n].$$

We have $\#X_0(\mathbb{F}_{q^k}) \equiv \#(X_0/G)(\mathbb{F}_{q^k}) \pmod{q^k}$ for any natural number $k$.

It is well known that the above hypersurface is Calabi-Yau. A $G$-equivariant nonzero $(n-1)$-form is

$$(-1)^i dx_0 \wedge \ldots \wedge \hat{dx}_i \wedge \ldots \wedge dx_n \over 1 + \sum_{j \neq i} x_j^{n+1} - \lambda \prod_{j \neq i} x_j$$
on the affine space $x_i = 1$ of $\mathbb{P}^n$.

It is known that for the above hypersurface $X_0$, $X_0/G$ is a strong singular mirror of $X_0$ if $(n+1)|(q-1)$. It is conjectured in [W] that for a strong mirror pair of Calabi-Yau varieties $\{X_0, X'_0\}$ over the finite field $\mathbb{F}_q$, we have $\#X_0(\mathbb{F}_{q^k}) \equiv \#X'_0(\mathbb{F}_{q^k}) \pmod{q^k}$ for any integer $k$. See [W] for a fuller discussion on this and other arithmetic mirror conjectures. In the situation of Theorem 0.2, if $X/G$ is a singular mirror of $X$ and if $Y$ is a smooth crepant resolution of $X/G$, then the pair $(X, Y)$ forms a strong mirror pair of smooth projective Calabi-Yau varieties. The congruence mirror conjecture in this case then reduces to showing the congruence

$$\#(X/G)(\mathbb{F}_{q^k}) \equiv \#Y(\mathbb{F}_{q^k}) \pmod{q^k}.$$ 

Another application of the theorem is to geometrically connected varieties with the property $H^i(X, \mathcal{O}_X) = 0$ for all $i \neq 0$. Again in this case, $G$ acts trivially on $H^i(X, \mathcal{O}_X)$ for all $i$. Let $\overline{K}$ be the algebraic closure of the fraction field of $W = W(\mathbb{F}_q)$. By [E], if the $l$-adic cohomology group $H^i(X \otimes_W \overline{K}, \mathbb{Q}_l)$ satisfies the coniveau 1 condition for each $i \neq 0$, that is, if any cohomology class in $H^i(X \otimes_W \overline{K}, \mathbb{Q}_l)$ vanishes in $H^i(U, \mathbb{Q}_l)$ when restricted to some nonempty open $U \subset X \otimes_W \overline{K}$, then we have $H^i(X, \mathcal{O}_X) = 0$ for all $i \neq 0$. The converse is true if we assume the generalized Hodge conjecture. It turns out that in this case, we can prove a theorem stronger than Theorem 0.1. We don’t need to assume $X_0$ can be lifted to $W$.

**Theorem 0.4.** Let $X_0$ be a smooth geometrically connected projective variety over the finite field $\mathbb{F}_q$. Suppose $H^i(X_0, \mathcal{O}_{X_0}) = 0$ for all $i \neq 0$. Then for any natural number $k$, we have

$$\#X_0(\mathbb{F}_{q^k}) \equiv 1 \pmod{q^k}.$$
Let $G$ be a finite group of $\mathbf{F}_q$-automorphisms acting on the right of $X_0$. We have
\[
\#(X_0/G)(\mathbf{F}_{q^k}) \equiv \#X_0(\mathbf{F}_{q^k}) \equiv 1 \pmod{q^k}.
\]

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1. Proof of the Theorems

First we introduce some notations. For any smooth proper scheme $X_0$ over $\mathbf{F}_q$, let $H^i(X_0/W)$ be the crystalline cohomology group of $X_0$. It is a finitely generated module over the Witt ring $W = W(\mathbf{F}_q)$. Denote by $F : X_0 \rightarrow X_0$ the Frobenius correspondence, that is, it is the identity map on the underlying topological space of $X_0$, and it maps a section of $\mathcal{O}_{X_0}$ to its $q$-th power.

Let $\kappa$ be a field and let $Z$ be a scheme over $\kappa$. Denote by $|Z|$ the set of Zariski closed points in $Z$. For any $z \in |Z|$, define $\deg(z) = [k(z) : \kappa]$, where $k(z)$ is the residue field at $z$. Let $f : Z \rightarrow Z$ be a $\kappa$-endomorphism with isolated fixed points. Set
\[
Z^f = \{z \in |Z| | f(z) = z \text{ and } f \text{ induces identity on } k(z)\},
\]
and define
\[
\Lambda(f) = \sum_{z \in Z^f} \deg(z).
\]
Let $\kappa'$ be a field extending $\kappa$ and let $f' : Z \otimes_\kappa \kappa' \rightarrow Z \otimes_\kappa \kappa'$ be the base change of $f$. Then we have $\Lambda(f) = \Lambda(f')$. 

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Lemma 1.1. Let $X_0$ be a smooth projective variety over the finite field $\mathbf{F}_q$, let $g : X_0 \to X_0$ be an $\mathbf{F}_q$-automorphism of finite order, and let $K = \text{Frac} W$ be the fraction field of $W = W(\mathbf{F}_q)$. Then $\text{Tr}(F^k, H^i(X_0/W) \otimes_W K)$ and $\text{Tr}(gF^k, H^i(X_0/W) \otimes_W K)$ are algebraic integers for any positive integer $k$ and any $i$, and

$$
\Lambda(F^k) = \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(F^k, H^i(X_0/W) \otimes_W K),
$$

$$
\Lambda(gF^k) = \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(gF^k, H^i(X_0/W) \otimes_W K).
$$

Proof. Let $l$ be a prime number distinct from $p$. By Deligne’s theorem ([D] 3.3.9), $\text{Tr}(F^k, H^i(X_0 \otimes_{\mathbf{F}_q} \overline{\mathbf{F}_q}, \overline{\mathbf{Q}_l}))$ are algebraic integers. By the comparison theorem of Katz-Messing ([KM]), we have

$$
\text{Tr}(F^k, H^i(X_0/W) \otimes_W K) = \text{Tr}(F^k, H^i(X_0 \otimes_{\mathbf{F}_q} \overline{\mathbf{F}_q}, \overline{\mathbf{Q}_l})).
$$

So $\text{Tr}(F^k, H^i(X_0/W) \otimes_W K)$ are algebraic integers. The formula for $\Lambda(F^k)$ follows from the Lefschetz fixed point formula in crystalline cohomology theory ([B] Théorème VII 3.1.9).

We will reduce the statements about $gF^k$ to the corresponding statements for $F^k$. Suppose $g : X_0 \to X_0$ has finite order $m$. Let $X_1 = X_0 \times_{\text{Spec} \mathbf{F}_q} \text{Spec} \mathbf{F}_{q^m}$, and let $\varphi \in \text{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_q)$ be the Frobenius substitution. For any $\sigma \in \text{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_q)$, we have $\sigma = \varphi^k$ for some integer $k$ uniquely determined modulo $m$. Define

$$
f_\sigma : X_1 \to X_1
$$

to be the isomorphism of schemes

$$
f_\sigma = (\text{id}_{X_0} \times \sigma^*) \circ (g^{-k} \times \text{id}_{\text{Spec} \mathbf{F}_{q^m}}) : X_0 \times_{\text{Spec} \mathbf{F}_q} \text{Spec} \mathbf{F}_{q^m} \to X_0 \times_{\text{Spec} \mathbf{F}_q} \text{Spec} \mathbf{F}_{q^m}.
$$

Note that $f_\sigma$ is independent of the choice of $k$ since $g$ has order $m$. Since $g^{-k} \times \text{id}_{\text{Spec} \mathbf{F}_{q^m}}$ is an $\mathbf{F}_{q^m}$-morphism of $X_1$, the following diagram commutes:

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_\sigma} & X_1 \\
\downarrow & & \downarrow \\
\text{Spec} \mathbf{F}_{q^m} & \xrightarrow{\sigma^*} & \text{Spec} \mathbf{F}_{q^m}.
\end{array}
$$
Moreover we have

\[ f_\tau f_\sigma = f_{\sigma \tau} \]

for any \( \sigma, \tau \in \text{Gal}(F_{q^m}/F_q) \). By the theory of galois descent, ([S] Chapter V, No. 20, or Corollarie 7.7 in [SGA 1] Exposé VIII), there exists a scheme \( X_0' \) over \( \text{Spec} F_q \) such that we have an \( F_{q^m} \)-isomorphism

\[ X_1 \cong X_0' \times_{\text{Spec} F_q} \text{Spec} F_{q^m} \]

and the following diagrams commute:

\[
\begin{array}{c}
X_1 \xrightarrow{f_\sigma} X_1 \\
\cong \downarrow \cong \\
X_0' \times_{\text{Spec} F_q} \text{Spec} F_{q^m} \xrightarrow{\text{id}_{X_0'} \times \sigma^*} X_0' \times_{\text{Spec} F_q} \text{Spec} F_{q^m}.
\end{array}
\]

For any scheme \( Z \) of characteristic \( p \), let \( F_Z : Z \to Z \) be the Frobenius correspondence, that is, \( F_Z \) is identity on the underlying topological space and the morphism of sheaves \( F_Z^* : \mathcal{O}_Z \to F_Z \mathcal{O}_Z \) maps each section to its \( q \)-th power. On \( X_1 = X_0 \times_{\text{Spec} F_q} \text{Spec} F_{q^m} \), we have

\[
F_{X_1} = (\text{id}_{X_0} \times \varphi^*) \circ (F_{X_0} \times \text{id}_{\text{Spec} F_{q^m}}) = f_\varphi \circ (g \times \text{id}_{\text{Spec} F_{q^m}}) \circ (F_{X_0} \times \text{id}_{\text{Spec} F_{q^m}})
= f_\varphi \circ (gF_{X_0} \times \text{id}_{\text{Spec} F_{q^m}}).
\]

Through the isomorphism \( X_1 \cong X_0' \times_{\text{Spec} F_q} \text{Spec} F_{q^m}, \) \( F_{X_1} \) is identified with \((\text{id}_{X_0'} \times \varphi^*) \circ (F_{X_0'} \times \text{id}_{\text{Spec} F_{q^m}})\). Moreover, the commutative diagram above shows that \( f_\varphi \) is identified with \( \text{id}_{X_0'} \times \varphi^* \). So the morphism \( gF_{X_0} \times \text{id}_{\text{Spec} F_{q^m}} \) on \( X_0 \times_{F_q} F_{q^m} \) is identified with the morphism \( F_{X_0'} \times \text{id}_{\text{Spec} F_{q^m}} \) on \( X_0' \times_{\text{Spec} F_q} F_{q^m} \). So we have

\[
\text{Tr} \left( gF_{X_0} \times \text{id}_{F_{q^m}}, H^i \left( X_0 \times_{F_q} F_{q^m}/W(F_{q^m}) \right) \otimes_{W(F_{q^m})} \text{Frac}(W(F_{q^m})) \right) \\
= \text{Tr} \left( F_{X_0'} \times \text{id}_{F_{q^m}}, H^i \left( X_0' \times_{F_q} F_{q^m}/W(F_{q^m}) \right) \otimes_{W(F_{q^m})} \text{Frac}(W(F_{q^m})) \right).
\]

By the base change theorem in crystalline cohomology theory ([B] Corollaire V 3.5.7), we have

\[
\text{Tr}(gF_{X_0}, H^i(X_0/W) \otimes_W K)
\]
\[
\begin{align*}
\text{Tr} & \left( gF_{X_0} \times \text{id}_{F_{q^m}}, H^i \left( X_0 \times_{F_q} F_{q^m} / W(F_{q^m}) \right) \otimes_{W(F_{q^m})} \text{Frac}(W(F_{q^m})) \right), \\
\text{Tr} & \left( F_{X_0'}, H^i(X_0'/W) \otimes_W K \right) \\
\text{Tr} & \left( F_{X_0'} \times \text{id}_{F_{q^m}}, H^i \left( X_0' \times_{F_q} F_{q^m} / W(F_{q^m}) \right) \otimes_{W(F_{q^m})} \text{Frac}(W(F_{q^m})) \right).
\end{align*}
\]

So we have
\[
\begin{align*}
\text{Tr} (gF_{X_0}, H^i(X_0/W) \otimes_W K) &= \text{Tr} (F_{X_0'}, H^i(X_0'/W) \otimes_W K).
\end{align*}
\]

In particular, \( \text{Tr} (gF_{X_0}, H^i(X_0/W) \otimes_W K) \) are algebraic integers for all \( i \). Moreover, we have
\[
\begin{align*}
\Lambda (gF_{X_0}) &= \Lambda (gF_{X_0} \times \text{id}_{\text{Spec} F_{q^m}}) \\
&= \Lambda (F_{X_0'} \times \text{id}_{\text{Spec} F_{q^m}}) \\
&= \Lambda (F_{X_0'}) \\
&= \sum_{i=0}^{2 \dim X_0} (-1)^i \text{Tr} (F_{X_0'}, H^i(X_0'/W) \otimes_W K) \\
&= \sum_{i=0}^{2 \dim X_0} (-1)^i \text{Tr} (gF_{X_0}, H^i(X_0/W) \otimes_W K).
\end{align*}
\]

This proves the statements for \( gF \). To prove the statements for \( gF_k \), we use the base change from \( F_q \) to \( F_{q^k} \).

**Lemma 1.2.** Under the condition of Theorem 0.1, we have
\[
\begin{align*}
\text{Tr} (gF^k, H^i(X_0/W) \otimes_W K) &\equiv \text{Tr} (F^k, H^i(X_0/W) \otimes_W K) \pmod{q^k}
\end{align*}
\]
for all \( i \).

**Proof.** Let \( H^i = H^i(X_0/W) \). Recall that \( H^i \) can be identified with the de Rham cohomology of the lifting \( X \) of \( X_0 \) to \( W = W(F_q) \). (Confer [B] Théorème V 2.3.2). On \( H^i \), we have the Hodge filtration
\[
H^i = F^0 H^i \supset F^1 H^i \supset \cdots
\]
and this filtration is $G$ stable. By a result of Mazur (the property (8.2) on page 65 of [M]), we have

$$F(F^1H^i) ⊂ qH^i.$$  

We have

$$H^i/F^1H^i = F^0H^i/F^1H^i ≅ H^i(X, O_X).$$

Choose a basis $\{e_1, \ldots, e_s\}$ of $F^1H^i$ and extend it to a basis $\{e_1, \ldots, e_s, e_{s+1}, \ldots, e_{s+t}\}$ of $H^i$. Since $F^k(F^1H^i) ⊂ q^kH^i$, the matrix of $F^k$ on $H^i$ with respect to the above basis is of the form

$$\begin{pmatrix} q^kA & q^kB \\ C & D \end{pmatrix},$$

where $A$ is an $s \times s$ matrix, $B$ is an $s \times t$ matrix, $C$ is a $t \times s$ matrix, and $D$ is a $t \times t$ matrix. Since $G$ acts trivially on $H^i/F^1H^i ≅ H^i(X, O_X)$ and $G$ preserves the Hodge filtration, the matrix of $g \in G$ on $H^i$ with respect to the above basis is of the form

$$\begin{pmatrix} P & O \\ Q & I \end{pmatrix},$$

where $P$ is an $s \times s$ matrix, $O$ is the $s \times t$ zero matrix, $Q$ is a $t \times s$ matrix, and $I$ is the $t \times t$ identity matrix. So the matrix of $gF^k$ is

$$\begin{pmatrix} q^kA & q^kB \\ C & D \end{pmatrix} \begin{pmatrix} P & O \\ Q & I \end{pmatrix} = \begin{pmatrix} q^kAP + q^kBQ & q^kB \\ CP + DQ & D \end{pmatrix}.$$  

We have

$$\text{Tr}(gF^k, H^i) = \text{Tr}(q^kAP + q^kBQ) + \text{Tr}(D).$$

On the other hand, we have

$$\text{Tr}(F^k, H^i) = \text{Tr}(q^kA) + \text{Tr}(D).$$

So we have

$$\text{Tr}(gF^k, H^i) \equiv \text{Tr}(F^k, H^i) \pmod{q^k}.$$  

This finishes the proof of Lemma 1.2.
Lemma 1.3. Let $X_0$ be a quasi-projective scheme over $\mathbb{F}_q$, let $G$ be a finite group acting on the right of $X_0$. Then for any natural number $k$, we have

$$\#(X_0/G)(\mathbb{F}_{q^k}) = \frac{1}{\#G} \sum_{g \in G} \Lambda(gF^k).$$

Proof. This result is well known. We include a proof here for completeness. Let $Y_0 = X_0/G$, and let $|X_0|$ (resp. $|Y_0|$) be the set of Zariski closed point in $X_0$ (resp. $Y_0$). For any $x \in |X_0|$, define the decomposition subgroup at $x$ by

$$G_d(x) = \{ g \in G \mid gx = x \}$$

and the inertia subgroup at $x$ by

$$G_i(x) = \{ g \in G_d(x) \mid g \text{ induces identity on the residue field } k(x) \text{ at } x \}.$$

Let $y$ be the image of $x$ in $Y_0$. By Proposition 1.1 in Exposé V of [SGA 1], we have an isomorphism

$$G_d(x)/G_i(x) \cong \text{Gal}(k(x)/k(y)),$$

and for any $y \in |Y_0|$, there are exactly $\frac{\#G}{\#G_d(x)}$ Zariski closed points in $X_0$ above $y$ and each of these closed points has degree $\deg(y) \frac{\#G_d(x)}{\#G_i(x)}$. We have

$$\#Y_0(\mathbb{F}_{q^k}) = \sum_{y \in |Y_0|, \deg(y)|k} \deg(y)$$

$$= \frac{1}{\#G} \sum_{y \in |Y_0|, \deg(y)|k} \frac{\#G}{\#G_d(x)} \frac{\#G_d(x)}{\#G_i(x)} \deg(y)$$

$$= \frac{1}{\#G} \sum_{y \in |Y_0|, \deg(y)|k} \sum_{x \in |X_0|, x \mapsto y} \deg(x) \#G_i(x).$$

Let $y \in |Y_0|$ be a Zariski closed point with $\deg(y)|k$, let $x \in |X_0|$ be a point above $y$, and let $\phi_y \in \text{Gal}(k(x)/k(y))$ be the Frobenius substitution. Suppose $g \in G_d(x)$ and $g^{-1} \mapsto \phi_y^{-\deg(y)}$ under the canonical homomorphism $G_d(x) \to \text{Gal}(k(x)/k(y))$. Then $gF^k(x) = x$ and $gF^k$ induces identity on $k(x)$. Conversely, if $x$ is a Zariski closed point in $X_0$ such that $gF^k(x) = x$ and $gF^k$ induces identity on $k(x)$, then $g \in G_d(x)$,
deg(y)|k, and \( g^{-1} \mapsto \phi_y^{deg(y)} \), where \( y \) is the image of \( x \) in \( Y_0 \). On the other hand, there are exactly \( \#G_i(x) \) elements \( g \) in \( G_d(x) \) such that \( g^{-1} \mapsto \phi_y^{deg(y)} \). So we finally get
\[
\#Y_0(F_{q^k}) = \frac{1}{\#G} \sum_{g \in G} \sum_{x \in [X_0], x \mapsto y} \deg(x) \#G_i(x)
\]
\[
= \frac{1}{\#G} \sum_{g \in G} \Lambda(gF^k).
\]
This proves Lemma 1.3.

Now we are ready to prove Theorem 0.1. By Lemmas 1.3 and 1.1, we have
\[
\#(X_0/G)(F_{q^k}) = \frac{1}{\#G} \sum_{g \in G} \Lambda(gF^k)
\]
\[
= \frac{1}{\#G} \sum_{g \in G} \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(gF^k, H^i(X_0/W) \otimes W K).
\]
By Lemmas 1.1 and 1.2, \( \text{Tr}(gF^k, H^i(X_0/W) \otimes W K) \) and \( \text{Tr}(F^k, H^i(X_0/W) \otimes W K) \) are algebraic integers, and
\[
\text{Tr}(gF^k, H^i(X_0/W) \otimes W K) \equiv \text{Tr}(F^k, H^i(X_0/W) \otimes W K) \pmod{q^k}.
\]
From now on, we work over the integral closure of \( p \)-adic integers. Let \( \text{ord}_q(\#G) = c \), a non-negative rational number. For each \( k \geq c \), we have
\[
\#(X_0/G)(F_{q^k}) = \frac{1}{\#G} \sum_{g \in G} \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(gF^k, H^i(X_0/W) \otimes W K)
\]
\[
\equiv \frac{1}{\#G} \sum_{g \in G} \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(F^k, H^i(X_0/W) \otimes W K) \pmod{q^{k-c}}
\]
\[
\equiv \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(F^k, H^i(X_0/W) \otimes W K) \pmod{q^{k-c}}
\]
\[
\equiv \#X_0(F_{q^k}) \pmod{q^{k-c}}.
\]
Let $Z(X_0, T)$ and $Z(X_0, T)$ be the zeta-functions of $X_0$ and $X_0/G$, respectively. They are rational functions. Recall that we have

\[
\frac{d}{dT} \log Z(X_0, T) = \sum_{k=1}^{\infty} \#X_0(F_{q^k})T^{k-1},
\]

\[
\frac{d}{dT} \log Z(X_0/G, T) = \sum_{k=1}^{\infty} \#(X_0/G)(F_{q^k})T^{k-1}.
\]

Take a factorization

\[
Z(X_0, T) = \prod_{i=1}^{m} (1 - \alpha_i T)^{-n_i}, \quad \alpha_i \neq 0
\]

where the $\alpha_i$’s are distinct and the $n_i$’s are non-zero integers. Taking logarithmic derivative on both sides, we get

\[
\sum_{k=1}^{\infty} (\#X_0(F_{q^k}) - \#(X_0/G)(F_{q^k}))T^{k-1} = \sum_{i=1}^{m} \frac{n_i \alpha_i}{1 - \alpha_i T}.
\]

Using the congruence

\[
\#(X_0/G)(F_{q^k}) \equiv \#X_0(F_{q^k}) \pmod{q^{k-c}}
\]

for all $k \geq c$, one deduces that the above power series is $p$-adic analytic in the open disk $\text{ord}_q(T) > -1$. This implies that each $\alpha_i$ satisfies $\text{ord}_q(\alpha_i) \geq 1$, that is, each $\alpha_i$ is divisible by $q$. We conclude that

\[
\#X_0(F_{q^k}) - \#(X_0/G)(F_{q^k}) = \sum_{i=1}^{m} n_i \alpha_i^k \equiv 0 \pmod{q^k}.
\]

This finishes the proof of Theorem 0.1.

Let’s prove Theorem 0.4. By Ogus’ generalization of Mazur’s theorem ([BO] Theorem 8.39), the Newton polygon of the Frobenius correspondence $F$ on $H^i(X_0/W) \otimes W K$ lies on or above the Hodge polygon of $X_0$. For any $i \neq 0$, we have $H^i(X_0, O_{X_0}) = 0$. So the slope of each line segment on the Newton polygon is at least 1, that is, all the eigenvalues of $F^k$ on $H^i(X_0/W) \otimes W K$ are divisible by $q^k$ (as $p$-adic integers). So we have

\[
\text{Tr}(F^k, H^i(X_0/W) \otimes W K) \equiv 0 \pmod{q^k}
\]
for all \( i \neq 0 \). Since \( X_0 \) is geometrically connected, we have

\[
\text{Tr}(F^k, H^0(X_0/W) \otimes W K) = 1.
\]

So by Lemma 1.1, we have

\[
\#X_0(F_{q^k}) = \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(F^k, H^i(X_0/W) \otimes W K) \equiv 1 \pmod{q^k}.
\]

Now let \( G \) be a finite group acting on the right of \( X_0 \). For any \( g \in G \), since \( g \) has finite order, the action of \( g \) on \( H^i(X_0/W) \otimes W K \) is diagonalizable and all its eigenvalues are roots of unity. Combining with the fact that \( F \) commutes with \( g \), we see that all the eigenvalues of \( gF^k \) on \( H^i(X_0/W) \otimes W K \) are also divisible by \( q^k \) for any \( i \neq 0 \). So we have

\[
\text{Tr}(gF^k, H^i(X_0/W) \otimes W K) \equiv \text{Tr}(F^k, H^i(X_0/W) \otimes W K) \equiv 0 \pmod{q^k}
\]

for all \( i \neq 0 \). Since \( X_0 \) is geometrically connected, we have

\[
\text{Tr}(gF^k, H^0(X_0/W) \otimes W K) = \text{Tr}(F^k, H^0(X_0/W) \otimes W K) = 1.
\]

Again let \( \text{ord}_q(#G) = c \). For each \( k \geq c \), by Lemmas 1.1, 1.3, and the above discussion, we have

\[
(X_0/G)(F_{q^k}) = \frac{1}{\#G} \sum_{g \in G} \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(gF^k, H^i(X_0/W) \otimes W K) \equiv \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(F^k, H^i(X_0/W) \otimes W K) \pmod{q^{k-c}}
\]

\[
\equiv \sum_{i=0}^{2\dim X_0} (-1)^i \text{Tr}(F^k, H^i(X_0/W) \otimes W K) \pmod{q^{k-c}}
\]

\[
\equiv \#X_0(F_{q^k}) \pmod{q^{k-c}}.
\]

As in the proof of Theorem 0.1, this implies that

\[
(X_0/G)(F_{q^k}) \equiv #X_0(F_{q^k}) \pmod{q^k}.
\]
This finishes the proof of Theorem 0.4.

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