Global classical solutions to the spherically symmetric Nordstr"om-Vlasov system

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Abstract
Classical solutions of the spherically symmetric Nordstr"om-Vlasov system are shown to exist globally in time. The main motivation for investigating the mathematical properties of the Nordstr"om-Vlasov system is its relation to the Einstein-Vlasov system. The former is not a physically correct model, but it is expected to capture some of the typical features of the latter, which constitutes a physically satisfactory, relativistic model but is mathematically much more complex. We show that classical solutions of the spherically symmetric Nordstr"om-Vlasov system exist globally in time for compactly supported initial data under the additional condition that there is a lower bound on the modulus of the angular momentum of the initial particle system. We emphasize that this is not a smallness condition and that our result holds for arbitrary large initial data satisfying this hypothesis.

1 Introduction
In astrophysics, systems such as galaxies or globular clusters are often modeled as a large ensemble of particles (stars) which interact only by the gravitational field which they create collectively. In such systems collisions among the particles are sufficiently rare to be neglected. Let \( f = f(t,x,p) \geq 0 \) denote the density of the particles in phase-space, where \( t \in \mathbb{R} \) denotes time, \( x \in \mathbb{R}^3 \) position, and \( p \in \mathbb{R}^3 \) momentum. This density function satisfies the Vlasov equation—a continuity equation on phase space—coupled to the field equation(s) for the gravitational field. In the non-relativistic case, i.e., for Newtonian gravity, the field is governed by Poisson’s equation and the resulting system is called the Vlasov-Poisson system. The initial value problem for this system is by now well understood, and satisfactory global existence results have been obtained, cf. [8, 10, 11, 17]. In general relativity, the Poisson equation is substituted by
Einstein’s equations, which coupled to the Vlasov equation yield the Einstein-
Vlasov system. In contrast to the Vlasov-Poisson system this system is highly
non-trivial even when matter is left out, since the Einstein equations by them-
selves constitute a nonlinear system of PDE’s. Little is known in general about
the global structure of vacuum solutions, and even less is known for matter
spacetimes. However, for small initial data there is a global existence result
for general, asymptotically flat vacuum spacetimes [6]. By imposing symmetry
conditions on spacetime global existence results are known also for initial data
unrestricted in size. Some of these results also hold for matter spacetimes when
matter is described by the Vlasov equation, i.e., for the Einstein-Vlasov system.
It should be pointed out that similar global results for other phenomenological
matter models are not known, indicating that in general relativity a kinetic de-
scription of matter is mathematically convenient. For a review on global results
for the Einstein equations and for the Einstein-Vlasov system, cf. [15, 1]. The
global existence results that hold for unrestricted data for the Einstein-Vlasov
system all concern cosmological spacetimes which are spatially compact. In the
asymptotically flat case, where for example gravitational collapse of an isolated
body is studied, the global structure of solutions to the Einstein-Vlasov system
is not known for large initial data, even in the spherically symmetric case. Some
qualitative information is obtained in [13]. On the other hand, for small spheri-
cally symmetric initial data, spacetime is known to be geodesically complete and
thus to contain no singularities [12]. For large data singularities will form—cf.
[15, 14]—and the central problem of weak cosmic censorship in general relativity
can be studied, i.e., to show that formation of singularities will always result in
black holes. The significance of the latter problem motivates a study of a less
complex but related model, the Nordström-Vlasov system.

2 The Nordström-Vlasov system

In the present paper we investigate a relativistic model which is obtained by
coupling the Vlasov equation to the Nordström scalar gravitation theory [9]. In
this theory, the gravitational effects are mediated by a scalar field \( \phi \), and the
system reads

\[
\partial_t^2 \phi - \Delta_x \phi = -e^{4\phi} \int f \frac{dp}{\sqrt{1+p^2}}, \tag{2.1}
\]

\[
\partial_t f + \hat{p} \cdot \nabla_x f - \left[ (\partial_t \phi + \hat{p} \cdot \nabla_x \phi) p + (1 + p^2)^{-1/2} \nabla_x \phi \right] \cdot \nabla_p f = 0. \tag{2.2}
\]

Here

\[
\hat{p} := \frac{p}{\sqrt{1+p^2}}
\]
denotes the relativistic velocity of a particle with momentum \( p \), \( p^2 = |p|^2 \), and
units are chosen such that the mass of each particle, the gravitational constant,
and the speed of light are all equal to unity. A solution \((f, \phi)\) of this system is in-
terpreted as follows: The spacetime is a Lorentzian manifold with a conformally
flat metric which, in the coordinates \((t,x)\), takes the form

\[ g_{\mu\nu} = e^{2\phi} \text{diag}(-1,1,1,1). \]

The particle distribution defined on the mass shell in this metric is given by

\[ f_{\text{ph}}(t,x,p) = f(t,x,e^{\phi}p). \]  

(2.3)

More details on the derivation of this model are given in [2]. It should be emphasized that although the system does not constitute a physically correct model it still captures some of the essential features of the Einstein-Vlasov system which are not present in the Vlasov-Poisson system. In particular, the Nordström-Vlasov model does allow for propagation of gravitational waves, even in the spherically symmetric case. The hope is that the analysis of this model will lead to a better mathematical understanding of a whole class of nonlinear partial differential equations and eventually to a better understanding of the Einstein-Vlasov system. For previous studies of the Nordström-Vlasov system we refer to [4, 5], where existence of local classical and global weak solutions is established, and to [3] where the non-relativistic limit is considered.

It turns out to be convenient to rewrite the system in terms of the new unknowns \((f,\phi)\), where \(f\) is given by

\[ f(t,x,p) = e^{4\phi(t,x)} f(t,x,p). \]  

(2.4)

The system then takes the form

\[ \partial_t^2 \phi - \Delta_x \phi = -\mu, \]  

(2.5)

\[ \mu(t,x) = \int f(t,x,p) \frac{dp}{\sqrt{1+p^2}}. \]  

(2.6)

\[ Sf - \left[ (S\phi)t + (1+p^2)^{-1/2} \nabla_x \phi \right] \cdot \nabla_p f = 4fS\phi, \]  

(2.7)

where

\[ S := \partial_t + \hat{\mathbf{p}} \cdot \nabla_x \]

is the free-transport operator. The function \(\mu\) is the trace of the energy-momentum tensor. We supply the system with the initial conditions

\[ f(0,x,p) = f^{\text{in}}(x,p), \quad \phi(0,x) = \phi_0^{\text{in}}(x), \quad \partial_t \phi(0,x) = \phi_1^{\text{in}}(x), \quad x, p \in \mathbb{R}^3, \]  

(2.8)

and we assume that the initial data have the regularity

\[ f^{\text{in}} \in C^1_c(\mathbb{R}^6), \quad \phi_0^{\text{in}} \in C^3_b(\mathbb{R}^3), \quad \phi_1^{\text{in}} \in C^2_b(\mathbb{R}^3). \]  

(2.9)

Here the subscript \(c\) indicates that the functions under consideration have compact support, \(b\) indicates that they are bounded together with their derivatives up to the indicated order.
We impose spherical symmetry on the initial data. By uniqueness, spherical symmetry propagates from the initial data so that the solution \((f, \phi)\) is also spherically symmetric, i.e.,

\[ f(t,Ax,Ap) = f(t,x,p), \quad \phi(t,Ax) = \phi(t,x), \quad x,p \in \mathbb{R}^3, \quad t \geq 0, \quad A \in \text{SO}(3). \]

By abuse of notation we can then write

\[ f(t,x,p) = f(t,r,u,\alpha), \quad \phi(t,x) = \phi(t,r) \]

where

\[ r := |x|, \quad u := |p|, \quad \alpha := \angle(x,p), \]

the latter denoting the angle between \(x\) and \(p\). In particular,

\[ \nabla_x \phi(t,x) = \partial_r \phi(t,r) \frac{x}{r}. \]

Consider the characteristic system

\[ \dot{x} = \hat{p}, \quad \dot{p} = -(S\phi)p - (1+p^2)^{-1/2} \nabla_x \phi \]

of the Vlasov equation \((2.7)\). Due to spherical symmetry the angular momentum is conserved along solutions of the characteristic system. Indeed, defining

\[ L := r^2 u^2 \sin^2 \alpha = |x \times p|^2, \]

we find that along characteristics

\[
\frac{d}{ds} \left( e^{2\phi(s,x(s))} L(x(s),p(s)) \right) = 2e^{2\phi} L(\partial_t \phi + \nabla_x \phi \cdot \hat{p}) \\
+ 2e^{2\phi} (x \times \hat{p}) \cdot \left[ \hat{p} \times p - x \times (S\phi)p + (1+p^2)^{-1/2} \partial_r \phi \frac{x}{r} \right] \\
= 2e^{2\phi} LS\phi - 2e^{2\phi} LS\phi = 0. \quad (2.10)
\]

The initial datum \(f^\text{in}\) is assumed to satisfy the additional condition that there is a positive number \(L_0\) such that

\[ L(x,p) = r^2 u^2 \sin^2 \alpha \geq L_0 > 0, \quad (x,p) \in \text{supp} f^\text{in}. \quad (2.11) \]

In \([4, 5]\) a continuation criterion has been proved for the Nordström-Vlasov system which says that a classical solution can be extended beyond \(t = T\) if the support of the momentum can be controlled on the time interval \([0, T]\). More precisely, if the quantity

\[ P(t) = \sup\{|p| : 0 \leq s < t, \quad (x,p) \in \text{supp} f(s)\}, \quad (2.12) \]

is bounded on the time interval \([0, T]\) then the solution can be extended to a larger time interval \([0, T + \delta]\) for some \(\delta > 0\). Hence, if \([0, T]\) is assumed to be the maximal life span of a solution, a bound of \(P(t)\) on \([0, T]\) implies that \(T = \infty\).
In this work we show that the quantity $P(t)$ can be controlled if the initial data satisfy the condition (2.11). Our main result can now be formulated.

**Theorem** For spherically symmetric initial data $(f^0,\phi^0)$ which satisfy the conditions (2.9) and (2.11), there exists a unique global classical solution $(f,\phi) \in C^1([0,\infty[ \times \mathbb{R}^6) \times C^2([0,\infty[ \times \mathbb{R}^3)$ of the Nordström-Vlasov system (2.5), (2.6), (2.7), which satisfies the initial conditions (2.8).

For the proof it will be essential to look at the behaviour of another quantity along characteristics (besides the angular momentum): Along characteristics,

$$ \frac{d}{ds} \left( e^\phi \sqrt{1+p^2} \right) = \partial_t \phi e^\phi \frac{1}{\sqrt{1+p^2}}. \quad (2.13) $$

It is of interest to note that this relation is independent of the symmetry assumption; the quantity under consideration is the particle energy which would be conserved for a static solution.

### 3 Proof of the theorem

We assume that the length of the maximal existence interval $T$ is finite and show in three steps that the quantity $P$ from (2.12) is bounded on $[0,T]$ which then contradicts the continuation criterion [5, Prop. 2].

**Step 1:** An estimate for $\partial_t \phi$

We split $\phi$ as follows:

$$ \phi = \phi_{\text{hom}} + \psi. \quad (3.1) $$

Here $\phi_{\text{hom}}$ is the solution of the homogeneous wave equation with the given data, and $\psi$ is the solution of the inhomogeneous wave equation with zero data. Clearly,

$$ ||\phi_{\text{hom}}(t)||_{\infty} + ||\partial_t \phi_{\text{hom}}(t)||_{\infty} \leq C, \quad 0 \leq t < T; \quad (3.2) $$

recall that we assume $T < \infty$. By [7, Lemma 7],

$$ \psi(t,x) = -\int_{|x-y|\leq t} \mu(t-|x-y|,y) \frac{dy}{|x-y|} = -\frac{1}{2r} \int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \mu(\tau,\lambda) \lambda d\lambda d\tau. $$

Note that $\psi \leq 0$ so that $\phi \leq \phi_{\text{hom}} < C$. Distinguishing the cases $0 < r < t$ and $r \geq t$ we get

$$ \partial_r \psi(t,r) \leq \frac{1}{2r} \int_0^{(t-r)_+} \mu(\tau,t-r-\tau)(t-r-\tau) d\tau, \quad (3.3) $$

where $(t-r)_+$ denotes the positive part of $t-r$. Define

$$ \widetilde{P}(t) := \sup \{e^{\phi(s,x)}|p|| (s,p) \in \text{supp} f(s), \quad 0 \leq s \leq t \}. $$

Then by conservation of angular momentum,

$$ r \mu(t,r) = 2\pi \int_0^{\infty} \int_0^{\pi} f(t,r,u,\alpha) r \sin \alpha u d\alpha \frac{u}{\sqrt{1+u^2}} du. $$
\begin{align*}
&\leq C e^{-\phi(t,r)} \int_0^\infty \int_0^\pi f(t,r,u,\alpha) \, du \, d\alpha \\
&= C e^{3\phi(t,r)} \int_0^{e^{-\phi(t,r)}e(t)} \int_0^\pi e^{-4\phi(t,r)} f(t,r,u,\alpha) \, du \, d\alpha \\
&\leq C e^{2\phi(t,r)} \tilde{P}(t) \leq C\bar{P}(t); \quad (3.4)
\end{align*}

recall that \( \phi \leq \phi_{\text{hom}} \leq C \) and

\[ f(t,x,p) = f^\text{in}(X(0),P(0)) \exp \left[ 4\phi(t,x) - 4\phi^\text{in}_0(X(0)) \right] \leq C, \quad (3.5) \]

where \((X(s), P(s))\) are the characteristics of the Vlasov equation with \((X,P)(t) = (x,p)\). If we substitute (3.4) into (3.3) we get the estimate

\[ \partial_t \psi(t,r) \leq \frac{C_2}{r} \tilde{P}(t), \quad r > 0, \quad 0 \leq t < T. \quad (3.6) \]

**Step 2: A bound on \( \tilde{P} \)**

With (3.2) and (3.6) equation (2.13) implies that

\[ \frac{d}{ds} \left( e^\phi \sqrt{1 + p^2} \right) \leq C + C\tilde{P}(s) \frac{1}{ru} \leq C + C\tilde{P}(s) e^{\phi(s,r)}r \sin \alpha. \]

Here we used the fact that \( e^\phi \leq C \). At this stage we recall the assumption (2.11) on the data, namely

\[ L(x,p) = r^2 u^2 \sin^2 \alpha \geq L_0 > 0, \quad (x,p) \in \text{supp} f^\text{in}. \quad (3.7) \]

Using the fact that \( e^{2\phi} L \) is constant along characteristics we get

\[ \frac{d}{ds} \left( e^\phi \sqrt{1 + p^2} \right) \leq C + C\tilde{P}(s), \]

and by Gronwall, \( \tilde{P} \) is bounded.

**Step 3: Completion of the proof**

Using the bound on \( \tilde{P} \) and (3.6) we can bound \( \mu \),

\[ \mu(t,x) = \int_{e^{\phi(t,x)} \leq C} f(t,x,p) \frac{1}{\sqrt{1 + p^2}} \, dp \leq C e^{4\phi(t,x)} \int_{e^{\phi(t,x)} \leq C} dp \leq C, \]

and hence also \( \phi \),

\[ |\phi(t,x)| \leq C + C \int_{|x-y| \leq 1} \frac{dy}{|x-y|} \leq C. \]

Hence the bound on \( \tilde{P} \) implies a bound on \( |p| \) over the support of \( f \), by the continuation criterion [5, Prop. 2] global existence follows, and the proof of the theorem is complete.

**Acknowledgment:** S. C. acknowledges support by the European HYKE network (contract HPRN-CT-2002-00282).
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