Superfield Extended BRST Quantization in General Coordinates

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Abstract

We propose a superfield formalism of Lagrangian BRST–antiBRST quantization of arbitrary gauge theories in general coordinates with the base manifold of fields and antifields described in terms of both bosonic and fermionic variables.

1 Introduction

The principle of extended BRST symmetry provides the basis of several Lagrangian quantization schemes for general gauge theories, including the well-known $Sp(2)$-covariant approach \[1\] and its different modifications, e.g., the superfield formalism \[2\] and the two versions of triplectic quantization \[3, 4\]. In order to reveal the geometric content of extended BRST symmetry, it is important to study these quantization methods in general coordinates (see, e.g., \[5, 6\] and references therein).

In the recent paper \[6\], it was shown that the geometry of the $Sp(2)$-covariant and triplectic schemes is the geometry of an even symplectic supermanifold equipped with a scalar density function and a flat symmetric connection (covariant derivative), while the geometry of the modified triplectic quantization also includes a symmetric structure (analogous to a metric tensor). The study of \[6\] generalizes the concept of triplectic supermanifolds, introduced in \[5\], to the case of base manifolds \[5, 6\] containing not only bosonic but also fermionic variables.

In this paper, we propose a superfield version of the quantization scheme developed in \[5, 6\]. The superfield description naturally involves an extension of supermanifolds used in \[5, 6\]. Namely, the triplectic supermanifold is extended to the complete supermanifold of variables used in the original $Sp(2)$-covariant approach. Note that in Darboux coordinates a similar extension takes place in the superfield formulation \[2\] of the $Sp(2)$-covariant scheme.

The paper is organized as follows. In Section 2, we propose a superfield extension of triplectic supermanifolds and introduce an operation of covariant differentiation on such supermanifolds, following the approach of our previous works \[5, 6\]. In Section 3, we propose a manifest realization of the (modified) triplectic algebra \[3, 4\] and outline a suitable quantization procedure along the lines of \[5, 6\]. In Section 4, we summarize the results and make concluding remarks.

We use DeWitt’s condensed notation \[7\] and apply tensor analysis on supermanifolds \[8\]. Left-hand derivatives with respect to some variables $x^i$ are denoted as $\partial_x A = \partial A/\partial x^i$. Right-hand derivatives with respect to $x^i$ are labelled by the subscript ”r”, and the notation $A_r = \partial_r A/\partial x^i$ is used. The covariant derivative $\nabla$ (and other operators acting on tensor fields) is assumed to act from the right: $A \nabla$; if necessary, the action of an operator from the right is indicated by an arrow, e.g., $\nabla \rightarrow$. Raising the $Sp(2)$-group indices is performed with the help of the antisymmetric second rank tensor $\varepsilon^{ab}$ $(a, b = 1, 2)$: $\theta^a = \varepsilon^{ab} \theta_b$, $\varepsilon^{ac} \varepsilon_{cb} = \delta_a^b$. The Grassmann parity of a quantity $A$ is denoted by $\epsilon(A).$
2 Superfield Extension of Triplectic Supermanifolds

The supervariables used in various realizations of extended BRST symmetry can be naturally combined into a set \((x^i, \theta^i_a, y^i, \bar{y}^i), i = 1, 2, \ldots, N = 2n\). Thus, the supermanifolds of the triplectic \([4]\) and modified triplectic \([4]\) quantization schemes consist of the variables \(x^i = (\phi^A, \bar{\phi}_A)\) and \(\theta^i_a = (\pi^A_a, \bar{\phi}^*_Aa)\), where \(\phi^A\) are the fields of the configuration space of a general gauge theory; the antifields \(\bar{\phi}_A\) are the sources of the combined BRST–antiBRST symmetry; the antifields \(\bar{\phi}^*_Aa\) are the sources of the BRST and antiBRST transformations; while \(\pi^Aa\) are auxiliary (gauge-fixing) fields. A superfield description \([2]\) of extended BRST symmetry requires an extension of triplectic supermanifolds \([3, 4]\) by the additional (external) variables \(y^i = (\lambda^A, J_A)\) arising in the original \(Sp(2)\)-covariant scheme \([1]\), where \(\lambda^A\) are auxiliary (gauge-fixing) fields, and \(J_A\) are the sources to the fields \(\phi^A\). The realization of extended BRST symmetry in general coordinates \([3]\) is based on a tensor analysis on supermanifolds with coordinates \((x^i, \theta^i_a)\). In this section, we propose a superfield formulation of the analysis \([6]\).

2.1 Superfields, Component Transformations

Let us consider a superspace spanned by space-time coordinates and an \(Sp(2)\)-doublet of anticommuting coordinates \(\eta^a\). Any function \(f(\eta)\) has a component representation,

\[
f(\eta) = f_0 + \eta^a f_a + \eta^2 f_3, \quad \eta^2 \equiv \frac{1}{2} \epsilon^{ab} \eta_a \eta^b,
\]

and an integral representation,

\[
f(\eta) = \int d^2 \eta' \delta(\eta' - \eta) f(\eta'), \quad \delta(\eta' - \eta) = (\eta' - \eta)^2,
\]

where integration over \(\eta^a\) is given by

\[
\int d^2 \eta = 0, \quad \int d^2 \eta^a = 0, \quad \int d^2 \eta^a \eta^b = \epsilon^{ab}.
\]

In particular, for any superfield \(f(\eta)\) we have

\[
\int d^2 \eta \frac{\partial f(\eta)}{\partial \eta^a} = 0,
\]

which implies the property of integration by parts

\[
\int d^2 \eta \frac{\partial f(\eta)}{\partial \eta^a} g(\eta) = -\int d^2 \eta (-1)^{\epsilon(f)} f(\eta) \frac{\partial g(\eta)}{\partial \eta^a},
\]

where derivatives with respect to \(\eta^a\) are taken from the left.

Let us now introduce a set of superfields \(z^i(\eta), \epsilon(z^i) = \epsilon_i, i = 1, \ldots, N\), with the component notation

\[
z^i(\eta) = x^i + \eta^a \theta^i_a + \eta^2 y^i,
\]

and the following distribution of Grassmann parity:

\[
\epsilon(x^i) = \epsilon(y^i) = \epsilon_i, \quad \epsilon(\theta^i_a) = \epsilon_i + 1.
\]

We shall identify the components \((x^i, \theta^i_a, y^i)\) with local coordinates of a supermanifold \(\mathcal{N}\), \(\dim \mathcal{N} = 4N\), where the submanifold \(\mathcal{M}\), \(\dim \mathcal{M} = 3N\), with coordinates \((x^i, \theta^i_a)\) is chosen as a triplectic supermanifold \([5, 6]\). We accordingly define the following transformations of the local coordinates:

\[
\bar{x}^i = \bar{x}^i(x), \quad \bar{\theta}^i_a = \theta^i_a \frac{\partial \bar{x}^i}{\partial x^j}, \quad \bar{y}^i = y^i,
\]

where \(\bar{x}^i = \bar{x}^i(x)\) are transformations on the submanifold \(\mathcal{M}\), \(\dim \mathcal{M} = N\), with coordinates \((x^i)\), called the base supermanifold \([8]\). The transformations of the coordinates \((x^i, \theta^i_a)\) are identical with
the transformations which define a triplectic supermanifold [5,6]. The superfield derivative \( \frac{\partial}{\partial z^i(\eta)} \) with respect to variations \( \delta z^i(\eta) = \delta x^i + \eta^a \delta \theta_a \) induced by the component transformations [1],

\[
\frac{\partial}{\partial z^i(\eta)} = \frac{\partial}{\partial \theta_a} \eta_a + \frac{\partial}{\partial x^i} \eta^2,
\]

is trivial on the external variables \( y^i \). Using the derivative (2) and the transformations (1), we can introduce a superfield extension of covariant differentiation [5,6] on triplectic supermanifolds \( \mathcal{M} \).

### 2.2 Superfield Extension of Triplectic Covariant Derivative

As a preliminary step, we shall discuss some elements of tensor analysis on the base supermanifold \( M \), referring for a detailed treatment of supermanifolds to the monograph [8]. To this end, let us consider a local coordinate system \((x) = (x^1, \ldots, x^N)\) on the base supermanifold \( M \) in the vicinity of a point \( P \). Let the sets \( \{e_i\} \) and \( \{e^i\} \) be coordinate bases in the tangent space \( T_PM \) and the cotangent space \( T^*_PM \), respectively. Under a change of coordinates \((x) \to (\bar{x})\), the basis vectors in \( T_PM \) and \( T^*_PM \) transform according to

\[
\bar{e}^i = e^j \frac{\partial \bar{x}^j}{\partial x^i}, \quad \bar{e}_i = e^j \frac{\partial x^i}{\partial \bar{x}^j}.
\]

The transformation matrices obey the following relations:

\[
\frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{\partial x^j} = \delta^i_j, \quad \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j} = \delta^i_j, \quad \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{\partial x^j} = \delta^i_j, \quad \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^j} = \delta^j_k.
\]

A tensor field of type \((n, m)\) with rank \( n + m \) is given by a set of functions \( T^{i_1 \ldots i_n}_{j_1 \ldots j_m}(x) \), with Grassmann parity \( \epsilon(T^{i_1 \ldots i_n}_{j_1 \ldots j_m}) = \epsilon(T) + \epsilon_{i_1} + \cdots + \epsilon_{i_n} + \epsilon_{j_1} + \cdots + \epsilon_{j_m} \), which transform under a change of coordinates \((x) \to (\bar{x})\), according to

\[
\bar{T}^{i_1 \ldots i_n}_{j_1 \ldots j_m} = T^{i_1 \ldots i_n}_{j_1 \ldots j_m} + \sum_{r=1}^{n} T^{i_1 \ldots \hat{i}_r \ldots i_n}_{j_1 \ldots j_m} \Gamma^{i_r}_{i_k} (\epsilon_{i_k} + \epsilon_{j_k}) + \sum_{r=1}^{m} \epsilon_{j_k} (\epsilon_{j_k} + \epsilon_{i_k}) + \sum_{r=1}^{n-1} \sum_{p=r+1}^{m} \epsilon_{j_k} \epsilon_{i_p} (\epsilon_{i_p} + \epsilon_{i_k})
\]

\[
\times (\bar{T}^{i_1 \ldots i_{n-1} i_n}_{j_1 \ldots j_{m-1} j_m} - \sum_{s=1}^{m} T^{i_1 \ldots i_n}_{j_1 \ldots \hat{j}_s \ldots j_m} \Gamma^{j_s}_{j_k} (\epsilon_{j_s} + \epsilon_{i_k}) \sum_{p=s+1}^{m} \epsilon_{j_p} (\epsilon_{i_p} + \epsilon_{j_p})).
\]

In particular, it is easy to see that the unit matrix \( \delta^i_j \) is a tensor field of type \((1, 1)\).

By analogy with tensor analysis on manifolds, on supermanifolds one introduces an operation \( \nabla \equiv \nabla_i \) (\( i \)) of covariant differentiation of tensor fields, by the requirement that this operation should map a tensor field of type \((n, m)\) into a tensor field of type \((n, m + 1)\), and that, in case one can introduce local Cartesian coordinates, it should reduce to the usual differentiation. On an arbitrary supermanifold \( M \), a covariant derivative is given by a variety of differentiations with respect to separate coordinates, \( \nabla = (\nabla_i) \). These differentiations are local operations, acting on a tensor field of type \((n, m)\) by the rule

\[
T^{i_1 \ldots i_n}_{j_1 \ldots j_m} \nabla_k = T^{i_1 \ldots i_n}_{j_1 \ldots j_m,k} + \sum_{r=1}^{n} T^{i_1 \ldots \hat{i}_r \ldots i_n}_{j_1 \ldots j_m} \Gamma^{i_r}_{i_k} (\epsilon_{i_k} + \epsilon_{j_k}) + \sum_{s=1}^{m} \epsilon_{j_k} (\epsilon_{j_k} + \epsilon_{i_k}) + \sum_{r=1}^{n-1} \sum_{p=r+1}^{m} \epsilon_{j_k} \epsilon_{i_p} (\epsilon_{i_p} + \epsilon_{j_p})
\]

\[
- \sum_{s=1}^{m} T^{i_1 \ldots i_n}_{j_1 \ldots \hat{j}_s \ldots j_m} \Gamma^{j_s}_{j_k} (\epsilon_{j_s} + \epsilon_{i_k}) \sum_{p=s+1}^{m} \epsilon_{j_p} (\epsilon_{i_p} + \epsilon_{j_p}),
\]

where \( \Gamma^{k}_{ij}(x) \) are generalized Christoffel symbols (connection coefficients), subject to the transformation law

\[
\Gamma^{M}_{ij,k} = \sum_{s=1}^{m} \epsilon_{j_k} (\epsilon_{j_k} + \epsilon_{i_k}) \frac{\partial x^k}{\partial x^i} \frac{\partial x^m}{\partial x^j} \frac{\partial x^n}{\partial x^k} \Gamma^{i}_{mn} \frac{\partial x^n}{\partial x^j} + \frac{\partial \bar{x}^k}{\partial \bar{x}^j} \frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j}.
\]
In this paper, we restrict the consideration to symmetric connections, i.e., those possessing the property

\[ M^{i} k_{ij} = (-1)^{\varepsilon_i \varepsilon_j} M^{i} k_{ji}. \]

Note that this property is fulfilled automatically in case a local Cartesian system can be introduced on the supermanifold \( M \).

The curvature tensor \( M^{i} m_{jk}(x) \) is defined by the action of the (generalized) commutator of covariant derivatives \( \left[ M^{i} \nabla_{j}, M^{i} \nabla_{k} \right] = M^{i} \nabla_{k} M^{i} \nabla_{j} - (-1)^{\varepsilon_{ij}} M^{i} \nabla_{j} M^{i} \nabla_{k} \) on a vector field \( T^{i} \) by the rule

\[ T^{i} \left[ M^{j} \nabla_{j}, M^{k} \nabla_{k} \right] = (-1)^{\varepsilon_m (\varepsilon_i + 1)} T^{m} M^{i} R^{m} m_{jk}. \]

A straightforward calculation yields the following result:

\[ M^{i} M^{i} m_{jk} = - M^{i} m_{jk} + M^{i} m_{jk} (-1)^{\varepsilon_{i} \varepsilon_{k}} + M^{i} j M^{i} l m (-1)^{\varepsilon_{j} \varepsilon_{m}} - M^{i} k M^{i} l m (-1)^{\varepsilon_{k} (\varepsilon_{m} + \varepsilon_{j})}. \]  

(5)

The curvature tensor \( M^{i} ) \) possesses the property of generalized symmetry

\[ M^{i} m_{jk} = - (-1)^{\varepsilon_{j} \varepsilon_{k}} M^{i} m_{kj}. \]

and obeys the Jacobi identity

\[ (-1)^{\varepsilon_{j} \varepsilon_{k}} M^{i} m_{jkl} + \text{cycle } (j, k, l) \equiv 0. \]

On triplectic supermanifolds \( M \), one defines \( \mathcal{D} \) covariant differentiation of tensor fields transforming as tensors on the base supermanifold \( M \). In a similar way, we introduce a superfield extension of the triplectic covariant derivative. Given in mind the coordinate transformations \( \Gamma \) on the supermanifold \( \mathcal{N} \), we define a tensor field of type \( (n, m) \) and rank \( n + m \) as a geometric object which in any local coordinate system \( (x, \theta, y) \) is given by a set of functions \( T^{i_{1} \ldots i_{n}} j_{1} \ldots j_{m} (z) \) transforming by the tensor law \( \mathcal{D} \). Let us define the superfield covariant derivative \( \mathcal{D} \equiv \mathcal{D} \) in a Cartesian coordinate system to coincide with \( \frac{\partial}{\partial z \left( \eta \right)} \), given by \( \mathcal{D} \). Then, in general coordinates, \( \mathcal{D} = \left( \mathcal{D}_{i} (\eta) \right) \) becomes

\[ \mathcal{D}_{i} (\eta) = \frac{\partial}{\partial \theta_{a}^{i}} \eta_{a} + \mathcal{M} i \nabla_{i} \eta^{2}. \]

Here, each term of the \( \eta \)-expansion acts as a covariant differentiation of tensor fields \( T^{i_{1} \ldots i_{n}} j_{1} \ldots j_{m} (z) \).

The component \( \mathcal{M} i \nabla_{i} \) is an extension of the covariant derivative \( \nabla_{i} \), given by \( \mathcal{D} \) on the base supermanifold \( M \), namely,

\[ \mathcal{M} i \nabla_{i} = \mathcal{M} i \nabla_{i} - \frac{\partial}{\partial \theta_{a}^{i}} \theta_{a}^{i} \mathcal{M} k m \left( -1 \right)^{\varepsilon_{m (\varepsilon_{k} + 1)}}. \]  

(6)

The operation \( \mathcal{M} i \nabla_{i} \) coincides\(^{1}\) with the triplectic covariant derivative \( \mathcal{D} \).

Since by definition \( (x^{i}, \theta_{a}^{i}) \) are independent coordinates, \( \mathcal{D} \) imply that the vectors \( \theta_{a}^{i} \) are covariantly constant with respect to \( \mathcal{M} i \nabla_{i} \), namely,

\[ \theta_{a}^{i} \mathcal{M} j = 0. \]  

(7)

By virtue of \( \mathcal{D} \), \( \mathcal{D} \), the commutator of two superfield covariant derivatives \( \mathcal{D}_{i} (\eta) \) has the form

\[ \left[ \mathcal{D}_{i} (\eta), \mathcal{D}_{j} (\eta') \right] = \left[ \mathcal{M} i \nabla_{i}, \mathcal{M} j \nabla_{j} \right] \eta^{2} (\eta')^{2}. \]

\(^{1}\)To observe the coincidence of \( \mathcal{D} \) with the triplectic covariant derivative \( \mathcal{D} \), one should go over to the parameterization \( (x^{i}, \theta_{a}^{i}) \), where \( \theta_{a}^{i} \) transform as vectors of the tangent space \( T_{p} M \) (for details, see Section 3.3).
From (6), (7), it follows that the action of this commutator on a scalar field $T = T(z)$ is given by

$$T \left[ \mathcal{D}_i(\eta), \mathcal{D}_j(\eta') \right] = (-1)^{\epsilon_m(\epsilon_n+1)} \eta^2 (\eta')^2 \frac{\partial_x T}{\partial \theta^a_m} g^m_n R^m_{nij},$$

where $R^m_{nij}$ is the curvature tensor on the base supermanifold.

### 3 Superfield Realization of (Modified) Triplectic Algebra

The extended BRST quantization in general coordinates is based on a realization of the so-called triplectic and modified triplectic operator algebras. The operators obeying these algebras are originally defined on triplectic supermanifolds $\mathcal{M}$. In this section, we propose a superfield formulation of the triplectic algebra, which permits us to formulate a superfield realization of extended BRST quantization in general coordinates, along the lines of [6].

#### 3.1 Triplectic and Modified Triplectic Algebras

The triplectic algebra includes two sets of second- and first-order operators, $\hat{\Delta}^a$ and $\hat{V}^a$, respectively, having the Grassmann parity $\epsilon(\Delta^a) = \epsilon(V^a) = 1$, and obeying the following relations:

$$\Delta \langle a \Delta^b \rangle = 0, \quad V \langle a V^b \rangle = 0, \quad V^a \Delta^b + \Delta^b V^a = 0. \quad (8)$$

The modified triplectic quantization, in comparison with the $Sp(2)$-covariant approach and the triplectic scheme, involves an additional $Sp(2)$-doublet of first-order operators $\hat{U}^a$, $\epsilon(U^a) = 1$, with the modified triplectic algebra given by the relations

$$\Delta \langle a \Delta^b \rangle = 0, \quad V \langle a V^b \rangle = 0, \quad U \langle a U^b \rangle = 0, \quad V^a \Delta^b + \Delta^b V^a = 0, \quad \Delta \langle a U^b \rangle + U \langle a \Delta^b \rangle = 0, \quad U^a V^b + V^a U^b = 0. \quad (9)$$

In [8], the curly brackets denote symmetrization with respect to the enclosed indices $a$ and $b$.

Using the odd second-order differential operators $\Delta^a$, one can introduce a pair of bilinear operations $(\ ,\ )^a$, by the rule

$$(F, G)^a = (-1)^{\epsilon(G)}(FG)\Delta^a - (-1)^{\epsilon(G)}(F\Delta^a)G - F(G\Delta^a). \quad (10)$$

The operations possess the Grassmann parity $\epsilon((F, G)^a) = \epsilon(F) + \epsilon(G) + 1$ and obey the following symmetry property:

$$(F, G)^a = -(-1)^{(\epsilon(G)+1)(\epsilon(F)+1)}(G, F)^a.$$  

The operations are linear with respect to both arguments,

$$(F + G, H)^a = (F, H)^a + (G, H)^a, \quad (F, G + H)^a = (F, G)^a + (F, H)^a,$$

and obey the Leibniz rule

$$(F, GH)^a = (F, G)^a H + (F, H)^a G(-1)^{\epsilon(G)\epsilon(H)}.$$  

Due to the properties of the operators $\Delta^a$, the odd bracket operations satisfy the generalized Jacobi identity

$$(F, (G, H))^{(a)b}(-1)^{\epsilon(F)+1)(\epsilon(H)+1)} + \text{cycle}(F, G, H) \equiv 0.$$

In view of their properties, the operations $(\ ,\ )^a$ form a set of antibrackets, such as those introduced for the first time in [1]. Therefore, having an explicit realization of operators $\Delta^a$ with the properties, one can generate the extended antibrackets explicitly, using [10]. Explicit realizations of $\Delta^a$ are known in two cases: in Darboux coordinates [1 3 4], and in general coordinates on triplectic supermanifolds $\mathcal{M}$, where the base supermanifold $M$ is a flat Fedosov supermanifold [6 9].
3.2 Realization of Triplectic Algebra

To find an explicit superfield realization of the triplectic algebra \( \mathfrak{S} \) in general coordinates, we shall use the assumptions of [6] concerning the properties of the base supermanifold \( M \). Thus, we equip \( M \) with a Poisson structure, namely, with a nondegenerate \( \text{even} \) second-rank tensor field \( \omega^{ij}(x) \), and its inverse \( \omega_{ij}(x) \), \( \epsilon(\omega^{ij}) = \epsilon(\omega_{ij}) = \epsilon_i + \epsilon_j \),

\[
\omega^{ij} \omega_{kj}(-1)^{\epsilon_k} = \delta^i_j, \quad \omega_{ik} \omega^{kj}(-1)^{\epsilon_i} = \delta^j_i,
\]

obeying the properties of generalized antisymmetry

\[
\omega^{ij} = -(-1)^{\epsilon_i \epsilon_j} \omega^{ji} \Leftrightarrow \omega_{ij} = -(-1)^{\epsilon_i \epsilon_j} \omega_{ji},
\]

and satisfying the following Jacobi identities:

\[
\omega^{ij} \partial_i \omega_{jk}(-1)^{\epsilon_k} + \text{cycle}(i,j,k) \equiv 0 \iff \omega_{ij,k}(-1)^{\epsilon_k} + \text{cycle}(i,j,k) \equiv 0.
\]

The tensor field \( \omega^{ij} \) defines a Poisson bracket [6], and, due to its nondegeneracy, also a corresponding \( \text{even} \) symplectic structure [6] on the base supermanifold. In view of this fact, the supermanifold \( M \) can be regarded as an \( \text{even} \) Poisson supermanifold, as well as an \( \text{even} \) symplectic supermanifold. Following [6], we demand that the covariant derivative \( \nabla_i \) should respect the Poisson structure \( \omega^{ij} \),

\[
\omega^{ij} \nabla_k = 0 \iff \omega_{ij,k} \nabla_k = 0,
\]

which provides the covariant constancy of the differential two-form \( \omega = \omega_{ij} dx^i \wedge dx^j \). Thus, the base supermanifold \( M \) can be regarded as an even Poisson supermanifold, as well as an \( \text{even} \) symplectic supermanifold. Following [6], we can formally identify \( \omega^{ij} \) and \( \omega_{ij} \) with some functions of the supervariables \( z^i(\eta) \), i.e., \( \Omega^{ij}(z) = \omega^{ij}(x) \) and \( \Omega_{ij}(z) = \omega_{ij}(x) \). It is obvious that the tensor fields \( \Omega^{ij} \) and \( \Omega_{ij} \) are covariantly constant:

\[
\Omega^{ij} \nabla_k(\eta) = \Omega_{ij,k} \nabla_k(\eta) = 0.
\]

The introduced structures allow one to equip the supermanifold \( \mathcal{N} \) with a superfield \( Sp(2) \)-irreducible second-rank tensor \( S_{ab} \),

\[
S_{ab} = \frac{1}{6} \int d^2 \eta \eta^2 \frac{\partial z^i}{\partial \eta^a} \Omega_{ij} \frac{\partial z^j}{\partial \eta^b}, \quad \epsilon(S_{ab}) = 0,
\]

invariant under changes of local coordinates on \( \mathcal{N} \), i.e., \( \tilde{S}_{ab} = S_{ba} \), and symmetric with respect to the \( Sp(2) \)-indices, \( S_{ab} = S_{ba} \).

Following [6], we also equip the base supermanifold \( M \) with a scalar density \( \rho(x) \), \( \epsilon(\rho) = 0 \). Using the covariant derivative \( \nabla_i(\eta) \), we can construct a superfield \( Sp(2) \)-doublet of odd second-order differential operators \( \Delta^a \), acting as scalars on the supermanifold \( \mathcal{N} \),

\[
\Delta^a = \int d^2 \eta \eta^2 \left( \nabla_i \frac{\partial}{\partial \eta^a} \right) \Omega^{ij} \left[ \left( \bar{\nabla}_j + \frac{1}{2}(\mathcal{R} \nabla)_{,j} \right) \frac{\partial_r}{\partial \eta^2} \right] (1)^{\epsilon_i + \epsilon_j},
\]

where \( \mathcal{R}(z) \equiv \rho(x) \).

The operators \( \Omega^{ij} \) generate a superfield \( Sp(2) \)-doublet of antibracket operations,

\[
(F,G)^a = -\int d^2 \eta \eta^2 \left( \Omega^{ij} \frac{\partial}{\partial \eta^a} \right) \Omega^{ij} \frac{\partial}{\partial \eta^a} (GD)_{,j} (1)^{\epsilon_i + \epsilon(j)} - (1)^{\epsilon(F)} + 1(1)^{\epsilon(G)} + 1(F \leftrightarrow G).
\]

These operations possess all the properties of extended antibrackets [1], except the Jacobi identity, which is closely related to the properties [6] of anticommutativity and nilpotency of \( \Delta^a \).

Using the operations [14] and the irreducible second-rank \( Sp(2) \)-tensor \( S_{ab} \) in [12], we define the following \( Sp(2) \)-doublet of odd first-order differential operators \( V_a \):

\[
\nabla_a = (\cdot, S_{ab})^b = -\frac{1}{2} \int d^2 \eta \eta^2 \left( \bar{\nabla}_i \frac{\partial}{\partial \eta^2} \right) \frac{\partial z^i}{\partial \eta^a}.
\]
Straightforward calculations, analogous to \( [6] \), with allowance for the manifest form of the operators \( \Delta^a, V^a, [13], [15] \), show that there exists such a choice of the density function \( \mathcal{R} \),

\[
\mathcal{R} = -\log \text{sdet} \left( \Omega^{ij} \right),
\]

that the triplectic algebra \( [5] \) is fulfilled on \( \mathcal{N} \) in case the base supermanifold \( M \) is a flat Fedosov supermanifold:

\[
\mathring{M}^{i}_{mj} = 0,
\]

with the curvature tensor \( \mathring{M}^{i}_{mj} \) given by \( [5] \). Thus, we have explicitly realized the extended antibrackets \( [14] \) and the triplectic algebra \( [5] \) of the generating operators \( \Delta^a, V^a \).

### 3.3 Realization of Modified Triplectic Algebra

In view of \( [5] \), to complete the explicit superfield realization of the modified triplectic algebra \( [9] \) in general coordinates, it remains to construct the operators \( U^a \). To this end, following \( [6] \), we introduce another geometrical structure on the base supermanifold \( M \). Namely, we consider a symmetric second-rank tensor \( g_{ij}(x) = (-1)^{\epsilon_i \epsilon_j} g_{ji}(x) \), which we identify with a tensor field \( G_{ij}(z) \). The introduced tensor field can be used to construct on \( \mathcal{N} \) an \( Sp(2) \) scalar function \( S_0 \), the so-called anti-Hamiltonian,

\[
S_0 = \frac{1}{2} \varepsilon^{ab} \int d^2 \eta \eta^2 \frac{\partial_r z^i}{\partial \eta^a} G_{ij} \frac{\partial_r z^j}{\partial \eta^b}, \quad \epsilon(S_0) = 0.
\]

(16)

The anti-Hamiltonian \( S_0 \) generates vector fields \( U^a \),

\[
\mathring{U}^a = (\cdot, S_0)^a = \int d^2 \eta \eta^2 \left[ \left( \mathring{\mathcal{D}}_i \frac{\partial_r}{\partial \eta^a} \right) \Omega^{im} G_{mn} \frac{\partial_z n}{\partial \eta^a} (-1)^m \right.
\]

\[
+ \frac{1}{2} \left( \mathring{\mathcal{D}}_i \frac{\partial_r}{\partial \eta^a} \right) \Omega^{ij} \frac{\partial_z m}{\partial \eta^c} \left( G_{mn} \mathring{\mathcal{D}}_j \frac{\partial_r}{\partial \eta^c} \right) \frac{\partial_z n}{\partial \eta^a} (-1)^{\epsilon_i + \epsilon_j + \epsilon_m} \right].
\]

The algebraic conditions \( [9] \) yield the following equations for \( S_0 \):

\[
(S_0, S_0)^a = 0, \quad S_0 V^a = 0, \quad S_0 \Delta^a = 0.
\]

(17)

Solutions of these equations always exist. An example of such solutions can be found in the class of covariantly constant tensor fields \( G_{ij}, G_{ij} \mathcal{D}_k = 0 \). We do not restrict ourselves to this special case, and simply assume that equations \( [17] \) are fulfilled. Thus, we obtain a realization of the modified triplectic algebra \( [9] \), and have at our disposal all the ingredients for the quantization of general gauge theories within the modified triplectic scheme.

### 3.4 Quantization

The quantization procedure repeats all the essential steps taken for the first time in \( [5] \), and leads to the vacuum functional

\[
Z = \int dz \mathcal{D}_0 \exp \{ (i/\hbar)[W + X + \alpha S_0] \},
\]

(18)

where \( \alpha \) is an arbitrary constant; the function \( S_0 \) is given by \( [16] \), while the quantum action \( W = W(z) \) and the gauge-fixing functional \( X = X(z) \) satisfy the following quantum master equations:

\[
\frac{1}{2}(W, W)^a + W V^a = i\hbar W \Delta^a,
\]

(19)

\[
\frac{1}{2}(X, X)^a + X U^a = i\hbar X \Delta^a.
\]

(20)

\[ ^2 \text{In the class of covariantly constant tensors } G_{ij}, \text{ solutions of } [17] \text{ can be selected by imposing the condition } G_{ij} (\mathcal{R} \mathcal{D}_k) \Omega^{jk} = 0. \text{ The simplest solution of this kind is given by a covariantly constant scalar density } \mathcal{R}. \]
In (18), integration over the supervariables is understood as integration over their components, 
\[ dz = dx \, d\theta \, dy, \]
with the integration measure \( \mathcal{D}_0 \) given by 
\[ \mathcal{D}_0 = [s\text{det}\,(\Omega^{ij})]^{-3/2}. \]

In (19) and (20), we have introduced operators \( \mathcal{V}^a, \mathcal{U}^a \), according to 
\[ \mathcal{V}^a = \frac{1}{2} (\alpha U^a + \beta V^a + \gamma U^a), \quad \mathcal{U}^a = \frac{1}{2} (\alpha U^a - \beta V^a - \gamma U^a). \]
It is obvious that for arbitrary constants \( \alpha, \beta, \gamma \) the operators \( \mathcal{V}^a, \mathcal{U}^a \) obey the properties  
\[ \mathcal{V}^{(a} \mathcal{V}^{b)} = 0, \quad \mathcal{U}^{(a} \mathcal{U}^{b)} = 0, \quad \mathcal{V}^{(a} \mathcal{U}^{b)} + \mathcal{U}^{(a} \mathcal{V}^{b)} = 0. \]

Therefore, the operators \( \Delta^a, \mathcal{V}^a, \mathcal{U}^a \) also realize the modified triplectic algebra.

The integrand of the vacuum functional (18) is invariant under extended BRST transformations defined by the generators 
\[ \delta \alpha = (\cdot, W - X)^a + \mathcal{V}^a - \mathcal{U}^a. \]
In the usual manner, this allows one to prove that, for every given set of the parameters \( \alpha, \beta, \gamma \), the vacuum functional (18) does not depend on a choice of the gauge-fixing function \( X \).

Let us analyze the component structure of the proposed quantization scheme in order to establish its relation with the modified triplectic quantization in general coordinates [6]. To this end, note that the integration measure \( \mathcal{D}_0 \) and the function \( S_0 \), 
\[ S_0 = \frac{1}{2} \varepsilon^{ab} \theta_i^\alpha g_{ij} \theta_j^\beta (-1)^{\epsilon_i + \epsilon_j}, \]
coincide with the corresponding objects of [6]. The operators \( \Delta^a, \mathcal{V}^a, \mathcal{U}^a \) and antibrackets \( (\cdot, \cdot)^a \) have the form 
\[ \Delta^a = (-1)^{\epsilon_i} \partial \partial \partial_{\theta^a}, \quad \mathcal{V}^a = \frac{1}{2} \varepsilon^{ab} M_{i}^{\alpha\beta} \frac{1}{2} \rho_{i}, \quad \mathcal{U}^a = -\varepsilon^{ab} \omega^{ij} \theta_j^b, \]
\[ \mathcal{U}^a = -\varepsilon^{ab} \omega^{ij} \theta_j^b, \quad \mathcal{U}^a = -\varepsilon^{ab} \omega^{ij} \theta_j^b, \quad \mathcal{U}^a = -\varepsilon^{ab} \omega^{ij} \theta_j^b, \]
\[ (F, G)^a = \left( F \mathcal{V}^p \right) \frac{\partial G}{\partial \theta_{ia}} - (-1)^{(\epsilon(F) + 1)(\epsilon(G) + 1)} \left( G \mathcal{V}^p \right) \frac{\partial F}{\partial \theta_{ia}}, \]
where \( \theta_{ia} \), defined by \( \theta_i^a = \omega^{ij} \theta_j^a (-1)^{\epsilon_i} \), are covariantly constant covectors, \( \theta_{ia} \mathcal{M} \mathcal{N} \mathcal{J} = 0 \), while \( \omega_{ia} \mathcal{M} \mathcal{J} \mathcal{N} \mathcal{P} \mathcal{Q} \mathcal{R} \mathcal{S} \) transform as vectors. The above component expressions imply that the operators \( \Delta^a, \mathcal{V}^a, \mathcal{U}^a \) and antibrackets coincide with the corresponding objects of [6], which follows from the coincidence of \( \mathcal{V}^a \) with the triplectic covariant derivative \( \mathcal{M} \mathcal{N} \mathcal{J} \mathcal{K} \mathcal{L} \mathcal{M} \mathcal{K} \mathcal{A} \mathcal{B} \mathcal{C} \mathcal{D} \mathcal{E} \mathcal{F} \mathcal{G} \mathcal{H} \mathcal{I} \mathcal{J} \mathcal{K} \mathcal{L} \mathcal{M} \mathcal{N} \mathcal{O} \mathcal{P} \mathcal{Q} \mathcal{R} \mathcal{S} \mathcal{T} \mathcal{U} \mathcal{V} \mathcal{W} \mathcal{X} \mathcal{Y} \mathcal{Z}, \]

In case local Cartesian coordinates can be introduced on \( M \), the coincidence of derivatives is automatic, while in the case of arbitrary connection coefficients, the coincidence takes place since \( M \) is a Fedosov supermanifold, namely, due to (11). Equations (14), (20) formally coincide with the master equations of [6], because the external variables \( y^i \) enter only as arguments of \( W(z) \) and \( X(z) \). In Darboux coordinates \( (\tilde{z}^\mu, y^i) \), \( y^i = (\lambda^\alpha, J^a) \), one can choose solutions of (19), (20) as solutions of the master equations of [6], namely, \( W = W(\tilde{z}), X = X(\tilde{z}, \lambda) \). Since in the coordinates \( (\tilde{z}^\mu, y^i) \) the tensor \( \omega^{ij} \) can be chosen [5] such that \( \mathcal{D}_0 = \text{const} \), the vacuum functional (18) reduces to 
\[ Z = \int d\tilde{z} \, d\lambda \exp \{ (i/\hbar)[W(\tilde{z}) + X(\tilde{z}, \lambda) + \alpha S_0(\tilde{z})] \}, \]
which is identical with the vacuum functional [6], written in Darboux coordinates.
4 Conclusion

In this paper, we have proposed a superfield realization of extended BRST symmetry in general coordinates, along the lines of our recent works \cite{5, 6} on modified triplectic quantization. We have found an explicit superfield realization of the modified triplectic algebra of generating operators $\Delta^a$, $V^a$, $U^a$ on an extended supermanifold $\mathcal{N}$, obtained from the triplectic supermanifold $\mathcal{M}$ by adding external supervariables, which, in Darboux coordinates, can be interpreted as sources $J_A$ to the fields and as auxiliary gauge-fixing variables $\lambda^A$. The present study applies the essential ingredients of \cite{5, 6}, and has the same general features. Thus, the base supermanifold $\mathcal{M}$ of fields and antifields is a flat Fedosov supermanifold equipped with a symmetric structure. As in \cite{5, 6}, the formalism is characterized by free parameters, $(\alpha, \beta, \gamma)$, whose specific choice in Darboux coordinates reproduces all the known schemes of covariant quantization based on extended BRST symmetry (for details, see \cite{5}). Every specific choice of the free parameters $(\alpha, \beta, \gamma)$ yields a gauge-independent vacuum functional and, therefore, a gauge independent $S$-matrix (see \cite{12}).

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