Infrared effects in inflationary correlation functions

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Abstract
In this paper, I briefly review the status of infrared effects which occur when using inflationary models to calculate initial conditions for a subsequent hot, dense plasma phase. Three types of divergences have been identified in the literature: secular, ‘time-dependent’ logarithms, which grow with time spent outside the horizon; ‘box-cutoff’ logarithms, which encode a dependence on the infrared cutoff when calculating in a finite-size box; and ‘quantum’ logarithms, which depend on the ratio of a scale characterizing new physics to the scale of whatever process is under consideration, and whose interpretation is the same as the conventional field theory. I review the calculations in which these divergences appear, and discuss the methods which have been developed to deal with them.

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1. Introduction
Looking out at the universe from the vantage point of the Earth, we see a small fluctuation \( \delta T \) in the temperature of the cosmic microwave background radiation (CMB). Over the last decade, advances in observational astronomy have allowed us to commence a detailed study of \( \delta T \)’s statistical properties. We aim to compare these properties with those of small fluctuations which, according to our present ideas, are expected to have existed in the very early universe.

These fluctuations evolve in a hot, dense plasma of tightly coupled baryons and photons. If we suppose the plasma era was preceded by an epoch of primordial inflation, then initial conditions for these fluctuations can be calculated from the parameters of the inflationary model. The variance predicted by this approach was computed in a now-classic series of early papers [1–6], and later extended to the skewness [7–12] and kurtosis [13–16]. With sufficiently precise experimental data in hand, we can work backwards from observations of these statistical properties to determine what the initial conditions must have been. The result is a realistic prospect of constraining parameters in certain inflationary models.

This ambitious programme was begun in earnest following the arrival of high-quality CMB maps made by the Wilkinson Microwave Anisotropy Probe (WMAP) [17].
precise data are expected from the Planck Surveyor satellite. Once it had become clear that the statistical properties of $\delta T$ carried important information about whatever physics was operative during inflation, theorists were soon tempted to refine their calculations in the hope that additional details could be extracted. A similar programme in electroweak physics had provided important clues about the unknown details of electroweak symmetry breaking\(^1\), and it was reasonable to explore the possibility of similar gains in cosmology. The observable statistical properties of the inflationary density perturbation are encoded in its $n$-point correlation functions, with the leading contribution to each function typically suppressed by $(n - 1)$ powers of the small quantity $(H/M_P)^2$ \[19\], where $H$ is the Hubble rate during inflation and $M_P = (8\pi G)^{-1/2}$ is the Planck mass. Two types of refinement were possible: either to calculate to the lowest order in $(H/M_P)^2$ at progressively larger $n$, or to calculate higher order corrections with $n$ fixed.

It had been known for a long time that perturbation theory beyond leading order in $(H/M_P)^2$ was complicated by troublesome infrared behaviour. Sasaki, Suzuki, Yamamoto and Yokoyama considered a $\lambda \phi^n$ theory with $n \geq 3$, coupled non-minimally to gravity, and evaluated each $n$-point function perturbatively \[20, 21\]. Sasaki \textit{et al.} noticed that the $n$-point functions evaluated in de Sitter space had a potential divergence in the far future, which they called ‘superexpansionary’. We will rephrase the analysis of Sasaki \textit{et al.} in modern notation and discuss the meaning of these divergences in sections 2.1 and 3.

Later, Mukhanov, Abramo and Brandenberger studied a different sort of infrared effect, namely the accumulation of long-wavelength fluctuations in an expanding universe \[22, 23\]. (See also Abramo and Woodard \[24\].) This is different to the analysis of Sasaki \textit{et al.}, which had been based on a purely geometrical divergence present even in a spacetime without perturbations. Whether such divergences are physical is complicated by the question of gauge, which was studied by Unruh \[25\]. Later, Losic and Unruh were able to give a gauge-invariant argument that infrared terms generically become large in nearly de Sitter spacetimes \[26–29\]. We will discuss corrections of this type in sections 2.2 and 4. Divergences of a third kind had long been studied by Prokopec, Tsamis, Woodard and their collaborators. These authors took the quantum nature of correlation functions seriously, and calculated loop corrections just as one would when studying scattering experiments. Unfortunately, the complexity of both the rules for calculating these loops and the computations themselves are somewhat greater than that in Minkowski space. These calculations have been reported in a large literature. Together with quantum effects studied by others, including Boyanovsky, de Vega and Sanchez, a recent sample can be found in \[30–41\]. We will consider effects of this type in section 2.3.

Perturbative calculations of correlation functions from inflation had been studied in the 1980s by Allen, Grinstein and Wise \[7\] and later by Falk, Rangarajan and Srednicki \[8\]. The sophistication of such calculations increased dramatically in the years following Maldacena’s calculation of the 3-point function in single-field slow-roll inflation \[10\]. At least in part, this increase was a consequence of the more complicated models of inflation which had been developed during the 1990s and early 2000s, and which theorists hoped would yield a distinctive pattern of correlations \[12, 16, 42, 43\]. These developments are reviewed pedagogically in a recent article by Chen \[44\].

Since 2002, significant theoretical effort has been expended in refining our understanding of both inflationary correlation functions themselves, and the models which can generate significantly non-Gaussian statistics. The calculations reviewed in this paper were a consequence of the reinvigoration of interest in infrared issues which followed this effort.

\(^1\) More details of this ‘precision electroweak programme’ can be found in the review \textit{Electroweak model and constraints on new physics}, prepared by the Particle Data Group \[18\].

\(^2\)
Logarithms exhibiting secular growth with time were discovered in calculations of $n$-point correlation functions with $n \geq 3$ [8, 45–48]. Other calculations, if taken at face value, gave divergent answers in the infinite-volume limit. Divergences could be avoided by carrying out the calculation in a finite box, but any such procedure left behind residual logarithms involving the box cutoff [46, 47, 49–53]. Further logarithms were encountered while studying the influence of unknown physics at energy scales even larger than the inflationary scale [54–61].

Infrared effects are not a phenomenon unique to de Sitter calculations but are present in many applications of quantum field theory. Their appearance typically signals the presence of a non-trivial background, which cannot be described by asymptotic in- and out-states containing a definite number of particles. In scattering calculations, infrared divergences allow putative fixed-particle states to develop a slowly varying field, made out of an accumulation of arbitrarily many soft particles radiated on approach to or recoil from the scattering event. This ‘Bloch–Nordsieck’ or ‘initial state radiation’ phenomenon is studied in many textbooks on quantum field theory [62]. A similar effect can occur in the confined phase of quantum chromodynamics, where soft particles can be radiated by partons moving inside nuclei. These particles are trapped by the strong QCD force, and change the background colour field. This effect may dramatically alter the mix of partons observed by an impinging probe, such as a photon, and cannot be neglected in an accurate comparison with experiment [63–68]. The effective $W$ approximation is a related example, in which the same infrared effects control the $W^\pm$ content of colliding nuclei. The results are very practical. For example, this method can be used to predict the rate of Higgs production from $W^+W^-$ fusion [69].

Throughout this paper, Planck’s constant $\hbar$ and the speed of light $c$ are set to unity. Many expressions are written in terms of the reduced Planck mass, $M_P^{-2} = 8\pi G$. For brevity, a scalar field which contributes to the energy–momentum tensor $T_{ab}$ is referred to as ‘active’, whereas a field making no contribution to $T_{ab}$ is referred to as a ‘spectator’. In a cosmological context, such scalars are isocurvature modes.

2. A zoology of logarithms

In this section, I briefly review the sources of logarithmic divergences in inflationary correlation functions.

2.1. Time-dependent logarithms

A decade after calculations of the 2-point function had determined the variance in $\delta T$ expected from an inflationary initial condition, the corresponding skewness was calculated by Falk, Rangarajan and Srednicki [8] in an approximation where gravitational interactions were neglected$^2$. At that time, data from the COBE satellite [71] had established the existence of temperature fluctuations in the microwave background at the level $\delta T \sim 10^{-5}T$, where $T \approx 2.75$ K was the average microwave temperature over the sky. In inflationary models $\delta T/T$ is closely related to the parameter $(H/M_P)^2$. Since the skewness is proportional to $(H/M_P)^2$, it was already evident that any non-Gaussianity would be very small.

Let us suppose inflation is driven by a single scalar field with potential $V$. We define the slow-roll parameters by

$^2$ It was later argued by Gangui et al [9] and in more detail by Maldacena [10] that gravitational interactions in fact provide a dominant contribution to the skewness, but the contribution calculated by Falk et al is still present, if subdominant. See [44] or the review of inflationary non-Gaussianities by Koyama in this issue [70].
and use a conformal time variable, \( \tau \), related to cosmic time \( t \) by \( \tau = \int_0^t \sqrt{\frac{a}{a'}} dt' \). When measured in this variable, horizon crossing occurs for a mode of wavelength \( k \) when \( k \tau = -1 \), and after \( N \) e-folds outside the horizon we have \( k \tau = -e^{-N} \). The infinite future of de Sitter space corresponds to \( \vert k \tau \vert \to 0 \). In this notation the result of Falk, Rangarajan and Srednicki can be written as

\[
\langle \delta \phi(k_1) \delta \phi(k_2) \delta \phi(k_3) \rangle = (2\pi)^3 \delta(k_1 + k_2 + k_3) \frac{H^4_0}{4 \sqrt{2} M_P} \frac{\xi^2}{1 + \ln \left| k \tau_e \right|} \sum_{j,k,l} \ln \left| \frac{k}{k_j} \right| k_j^3 \ln \left| k \tau_e \right| + \cdots,
\]

where a subscript ‘\( * \)’ denotes evaluation at the time \( \tau_e \) and ‘\( \cdots \)’ denotes terms of lower order in the slow-roll expansion, together with other terms which do not grow as \( \left| k \tau_e \right| \to 0 \) [45, 48]. Higher powers of \( \ln \left| k \tau_e \right| \) may be generated at higher order in the slow-roll expansion, either by retaining such terms explicitly in the tree-level calculation or by including loop corrections, as we shall shortly explain. A similar effect can be observed in the massless limit of the calculation by Chen and Wang [73].

The result is a power series in \( \ln \left| k \tau_e \right| \) with an apparent divergence in the limit \( \tau_e \to 0 \), as Sasaki et al had predicted [20]. Series representations of this type were later applied to inflationary correlation functions by Gong and Stewart [74, 75], who obtained them by systematically solving Mukhanov’s equation [5] using a Green’s function approach. Gong and Stewart remarked that the correction linear in logarithms was typically proportional to \( \sim \ln \left| k \tau_e \right| \), whereas the correction quadratic in logarithms was typically proportional to \( \sim (\ln \left| k \tau_e \right|)^2 \) and so on. One could therefore expect the series to break down as a predictive instrument when \( \ln \left| k \tau_e \right| \sim e^{-1} \). This occurs when the mode \( k \) is of order \( e^{-1} \) e-folds outside the horizon.

Taking into account the construction of Gong and Stewart, and bearing in mind that \( \tau_e \) is the time at which we wish to evaluate the correlation functions, it seems clear that there is nothing mysterious about the appearance of powers of \( \ln \left| k \tau_e \right| \). Such terms merely express that correlations evolve outside the horizon. This evolution is mainly driven by the evolving classical background field [45], but we shall see in section 3.3 that intrinsically quantum contributions can also be present. We conclude that, as in other applications of quantum field theory, properties of the background play a role in interpreting the physics of infrared divergences. In the present case, correlation functions can be evaluated at any time of interest provided the series of logarithms can be summed or ignored. Therefore, for observational purposes the key question is the value of \( \epsilon \) when modes of interest leave the horizon. We would typically wish to evaluate correlation functions at the end of inflation, where initial

\[ \epsilon = \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2, \quad \eta = M_P^2 \frac{V''}{V}, \quad \xi^2 = M_P^2 \frac{V'V'''}{V^2}, \]

3 If our purpose were only to calculate the density fluctuation produced by a single-field model of inflation, then the logarithmic divergence in equation (2) would be entirely spurious [25]. The density fluctuation is determined by the comoving curvature perturbation, \( \xi \), which is conserved on superhorizon scales, \( \xi = 0 \), in the absence of entropy modes. Occasionally it has been argued that an estimate of the non-Gaussian yield at the end of inflation can be obtained by setting \( \ln \left| k \tau_e \right| \sim 60 \), but this example shows clearly that unless care is taken to account for possible cancellations, any such procedure can be quite misleading. The possibility of such cancellations when reasoning with cutoff-dependent terms in a general effective field theory was pointed out by Burgess and London [72]. In the inflationary context, the pitfalls of this procedure were noted in [48]. I would like to thank Xingang Chen for discussions on this point.

4 There is no general proof that the power series in \( \ln \left| k \tau_e \right| \) always orders itself in this form, and according to a general theorem due to Weinberg (to be discussed in section 3.1) there may be cases where it does not. In many practical examples where the slow-roll approximation applies, however, this structure appears. In these expressions, the symbol \( \sim \) is used to mean that the coefficient of \( \ln \left| k \tau_e \right| \) is a quantity of order \( \epsilon \) (but not necessarily \( \epsilon \) itself), whereas the coefficient of \( (\ln \left| k \tau_e \right|)^2 \) is of order \( \epsilon^2 \), and so on.
conditions for the subsequent evolution must be set. If $\epsilon \gtrsim 10^{-2}$, the end of inflation may be uncomfortably close to the era when $|\epsilon \ln|k\tau_*|| \sim 1$.

We will return to this question in section 3, where we will argue that a reasonable prescription exists for handling time-dependent logarithms which does not impair our ability to extract a predictive initial condition even if $|\epsilon \ln|k\tau_*|| \sim 1$ at the time of interest. We can already note one obvious strategy. If only logarithmic divergences are present, then better control over the power series may be achieved by ‘resumming’—that is, accounting for the contribution of—all terms at the same order as the leading logarithms, which are terms of the form $(\epsilon \ln|k\tau_*|)^n$ for all $n \geq 0$ [53]. For this purpose the principal tool is the renormalization group equation. Such an analysis has been carried out by Burgess, Holman, Leblond and Shandera [76] whose method will be outlined in section 3. Unfortunately, this method leaves open the question of whether all divergences in the $|k\tau_*| \to 0$ limit are comparatively tame logarithms, or if more aggressive behaviour can occur.

This question was taken up by Weinberg in 2005 [54], and later by Chaicherdsakul [56]. Weinberg was able to prove that in many inflationary models, but not all, the worst divergences would be logarithmic. In a later publication, this theorem was extended to fields of higher spin [55]. What of the possibility of faster divergences, which are apparently allowed by the analysis of Sasaki et al [20]? These would grow like powers of the scale factor, $a(t)$, corresponding to powers rather than logarithms of $|k\tau_*|$. In many cases such aggressive growth would induce a failure of predictivity long before the end of inflation. Also, whereas the coefficient which accompanies a logarithmic divergence is physically meaningful, and can be used as an input to renormalization group calculations, power-law divergences are by contrast mostly meaningless. The late-time physics whose presence they signal must be found elsewhere: it cannot usually be distilled from the properties of the divergences themselves. Although Weinberg’s theorem identifies a class of models containing interactions which may give rise to such fast divergences, it does not appear that any previously proposed inflationary model makes essential use of such interactions. For this reason they have not yet received much attention.

2.2. Box-cutoff logarithms

Until nonlinear questions became pressing, inflationary perturbation theory was dominated by the traditional Lifshitz approach [77, 78], in which one separates each field into a background $\phi$ and perturbation $\delta \phi$. At second order and above, such calculations become more arduous, although they have now been carried to an impressive degree of refinement [79–81].

The separate universe principle is an alternative to the Lifshitz approach. According to this principle, a volume of spacetime containing a background field $\phi$ and perturbation $\delta \phi$ behaves (on scales sufficiently large that gradients may be neglected) just like an unperturbed universe containing the homogeneous field value $\phi + \delta \phi$. If we solve for an arbitrary quantity $U$ with initial conditions set by the value of a background field at time $\tau_*$ in the unperturbed universe, we can determine the values taken by this quantity on superhorizon scales in the perturbed universe using the trivial identity

$$\delta U(\tau) = U(\tau, \phi_* + \delta \phi) - U(\tau, \phi_*),$$

where $\tau$ is the time at which we wish to evaluate $\delta U$. This might typically be at the end of inflation or later. Whatever time we choose, equation (3) allows correlations of $\delta U$ to be computed provided we know the correlation functions of $\delta \phi$ at time $\tau_*$. For example, the first contribution to the 2-point function of $\delta U$ is $(\delta U(\tau)\delta U(\tau)) = U_{\phi_*}^2(\tau)\langle \delta \phi^2 \rangle_\tau$, where $U_{\phi_*}$ denotes the partial derivative of $U$ with respect to $\phi_*$. 


Higher-order terms present in equation (3) imply that other contributions must exist. Let us restore spatial arguments for clarity. Even if \( \delta \phi \) has Gaussian statistics at time \( \tau \), there will be contributions of the form
\[
\langle \delta U(\tau, x_1) \delta U(\tau, x_2) \rangle \supseteq \frac{1}{6} U_{\phi, \phi, \phi} \langle \delta \phi(x_1) \delta \phi(x_3) \delta \phi(x_2) \delta \phi(x_4) \rangle + \frac{1}{6} U_{\phi, \phi, \phi, \phi} \langle \delta \phi(x_1) \delta \phi(x_2) \delta \phi(x_3) \delta \phi(x_4) \rangle + \langle \delta \phi(x_1) \delta \phi(x_2) \rangle,
\]
where \((x_1 \leftrightarrow x_2)\) denotes the preceding term with \(x_1\) and \(x_2\) exchanged and the symbol \('\supseteq'\) means that the correlation function contains this contribution among others. Indeed, these 4-point correlations will be accompanied by six-, eight- and higher 2-point correlation functions for all \(n\). If \(\delta \phi\) has non-Gaussian statistics, then even more contributions will be generated. Zaballa, Rodríguez and Lyth introduced a set of diagrammatic (‘Feynman-like’) rules designed to keep track of these terms \([82]\). Generalized rules were discussed by Byrnes, Koyama, Sasaki and Wands \([51]\) and in \([53]\), but practical calculations are rarely of sufficient complexity to require them.

These contributions depend on the correlations among \(\delta \phi\), when \(|x_2 - x_1|\) becomes large, which are identified most simply in Fourier space. We find that \(\langle \delta \phi^2(x_1) \delta \phi^2(x_2) \rangle\), receives contributions of the form
\[
\langle \delta \phi^2(x_1) \delta \phi^2(x_2) \rangle \supseteq \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (x_1 - x_2)} \int \frac{d^3q}{(2\pi)^3} P(|k - q|) P(q),
\]
in which \(P(q)\) is the power spectrum of \(\delta \phi\), defined by
\[
\langle \delta \phi(k_1) \delta \phi(k_2) \rangle = (2\pi)^3 \delta(k_1 + k_2) P(k). \tag{6}
\]
In the limit \(|x_1 - x_2| \to \infty\), significant contributions to the \(k\)-integral in equation (5) arise only from the region \(k \ll |x_1 - x_2|^{-1}\). The behaviour of the integrand in the small-\(k\) limit depends crucially on \(P(q)\). For a scale-invariant power spectrum generated during inflation \(P(q) \sim H^2/(2q^3)\), which gives logarithmic singularities in the \(q\)-integral. Boubekeur and Lyth \([49]\), and later Lyth and Rodríguez \([50]\), remarked that because the logarithmic behaviour is softened in the limit \(q \gg k\), the log-divergent part only receives contributions for \(q \lesssim k\). Therefore, to a reasonable approximation,
\[
\langle \delta \phi^2(x_1) \delta \phi^2(x_2) \rangle \approx \left( \frac{H^2}{2\pi^2} \right)^2 \int \frac{d^3k}{2k^3} (\ln kL) e^{ik \cdot (x_1 - x_2)}, \tag{7}
\]
where \(L \gg |x_1 - x_2|\) is an infrared cutoff. This expression diverges in the limit \(L \to \infty\).

It is clear at once that this divergence is not a physical prediction. It depends on what is assumed about correlations among the perturbations in the infinite volume limit, where \(|x_2 - x_1| \to \infty\). Even within the framework we are using, it is easy to see that if \(P(q)\) has constant red tilt then this logarithmic divergence is exchanged for a power law. If \(P(q)\) has constant blue tilt the integral is convergent. An exactly logarithmic divergence occurs only for a scale-invariant \(P(q)\). For example, if \(\delta \phi\) has even a very small mass this will give rise to a finite correlation length, beyond which correlations decay exponentially. The divergence in equation (7) would then be absent. We will see later that there are good physical reasons to believe this is what should happen in practice, at least in certain theories, and some evidence from concrete calculations that it does.

In any case we should also recognize that the framework we have been discussing is inadequate for the description of correlations on very large scales. Why is this? To calculate the power spectrum from a model of inflation we must typically assume that the background field \(\phi\) is spatially homogeneous and depends only on time, whereas the spatially dependent fluctuation \(\delta \phi\) satisfies \(|\delta \phi| \ll |\phi|\) everywhere. Although this is reasonable on scales not too much larger than the de Sitter horizon, it need not happen that every field admits such a
decomposition as we pass to the infinite volume limit. If this is the case we should set the scale $L$ to be conservatively smaller than the largest scale for which a spatially homogeneous background $\phi$ can be found. The calculation makes sense within this ‘box’. If we are forced to discuss correlations on larger scales, they will have to be determined by patching together a mosaic of boxes in which the homogeneous background field may take different values. This point of view was advocated by Boubekeur and Lyth [49], and later refined by Byrnes, Koyama, Sasaki and Wands [51], Lyth [52, 83] and other authors [53, 84–86].

This cannot be the whole story, because equation (7) shows that the mosaicking procedure leaves behind terms involving $\ln kL$. These logarithms depend on the arbitrary scale $L$. Assuming the physics of the scalar field zero-mode to be quasi-classical on very large scales, the missing element is an accurate map of the average scalar field value within each box of the mosaic. This map naturally depends on the scale $L$, but the $L$ dependence of physical quantities cancels when we study correlations within the mosaic as a whole. These issues will be discussed in more detail in section 4. We see again that the appearance of infrared divergences is connected with the existence of non-trivial background configurations. In the present case the map could consist of an inventory of boxes, pairing each box with the average field value within. Alternatively, in a theory such as inflation where predictions are essentially statistical, it could simply consist of a probability distribution for the average field value, taken over the ensemble of boxes.

2.3. Logarithms from new physics

In the previous two sections we have emphasized the role of the background field configuration in generating infrared divergences. When determining correlations among small fluctuations we take the background to be homogeneous and approximately time independent. In practice it may be neither, leading to the emergence of compensating logarithms.

There is another source of logarithmic corrections which is unconnected with the background field configuration. Suppose we define some quantum field theory at a scale $\mu$. In using this field theory to compute correlations in vacuum—where there is no background at all—we are familiar with the appearance of ‘large logarithms’ of the form $\ln E/\mu$, where $E$ is an energy scale characteristic of the correlation in question. Corrections of this sort occur in any quantum field theory, and the field theories we use to compute inflationary correlations are no exception. Such logarithms were studied by Weinberg [54, 55] who computed the correction induced by loops of $N$ different spectator fields in a model where inflation is driven by the vacuum energy associated with a single scalar field.

(See figure 1. In contrast to Minkowski space, each external leg is evaluated at the same late time $\tau_*$, and therefore these diagrams should be read from the middle to the outside. The interior of each diagram functions rather like an instanton\(^5\). Breaking the left-hand diagram down the middle, four quanta of the spectator field nucleate below the de Sitter horizon, and propagate until one pair annihilates at time $\tau'$ and a second pair at time $\tau''$. The product of each annihilation is a quantum of the active field, which propagates freely until the surface $\tau = \tau_*$ (represented by the outgoing arrows). The shared history of these two particles generates a correlation. In an alternative and equally acceptable interpretation, a pair of active quanta nucleate. One particle propagates directly to the surface $\tau = \tau_*$, whereas the second spontaneously fluctuates into a pair of spectator quanta at time $\tau'$, and at a later time $\tau''$ coalesce to form another quantum of the active field before propagating to the final surface. The right-hand diagram can be interpreted similarly, with the self-loop either providing a

\(^5\) The conventional prescription for calculating correlation functions in an interacting vacuum can be related to the Hartle–Hawking state [87].
Figure 1. One-loop corrections to the power spectrum of an active scalar field. In the left-hand diagram, a loop of spectator scalar fields, represented by dashed lines, corrects the 2-point function of an active field, represented by solid lines. This loop was computed by Weinberg [54, 55]. An error in the numerical coefficient was corrected by Adshead, Easther and Lim [58]. In the right-hand diagram, a self-loop corrects the same 2-point function of an active scalar, first computed in [57] and again in [58]. In Minkowski space this diagram would factorize, leaving a scale-free integral over the loop momentum \( q \). The non-trivial time dependence of de Sitter endows the loop with a scale of the form of equation (8). This diagram is the leading correction when self-loops are included. It would be accompanied by self-loops of the same form as the left-hand diagram, which are suppressed by powers of the slow-roll parameter \( \epsilon \). For spectator fields, there is no contribution from the right-hand diagram, so the left-hand loop is the leading term.

As in Minkowski space, these logarithms arise from ultraviolet divergent momentum integrals which we can study by cutting off the integral for \( q > \Lambda \). For a correlation dominated by wavenumbers of order \( k \), the typical integrals with which we are confronted take a form similar to

\[
\int_{|q|<\Lambda} \frac{d^3q}{(2\pi)^3} \frac{k}{q^3} \left| \mathbf{q} + \mathbf{k} \right|^2 = \frac{k^3}{6\pi^2} \ln \frac{\Lambda}{k} + \text{finite},
\]

where \( \mathbf{q} \) is the comoving three-momentum which circulates in the loop, and ‘finite’ denotes terms which do not diverge in the limit \( \Lambda \to \infty \). It is the appearance of \( k \) in the combination \( |\mathbf{q} + \mathbf{k}| \) which gives the logarithm a non-trivial \( k \) dependence, and by the usual arguments we can be assured that the coefficient of the logarithm is independent of the precise ultraviolet regulator we choose. This does not make equation (8) entirely unambiguous because its interpretation depends on the meaning we assign to \( \Lambda \), as we shall discuss shortly. Whatever its meaning, we see the emergence of the structure \( E/\mu \), with \( E = k \) characterizing the scale of the correlation, and \( \Lambda = \mu \) representing the point at which we define the theory.

These logarithms are a different species to the infrared logarithms of sections 2.1–2.2, which diverged for fixed \( k \) when the calculation was taken to occur in a box of infinite spatial or temporal extent. Equation (8) is finite in this limit. It is true that a divergence occurs when \( k \to 0 \), but a divergence in this limit occurs even at the tree-level\(^6\). We nevertheless include these ultraviolet logarithms in this discussion of infrared effects because they alert us

\(^6\) This divergence arises from our convention of normalizing the field modes as Minkowski space oscillators deep inside the horizon, but it is not clear that there is any physical effect. In practical models, the divergence will be cut off if inflation began at a finite point in the past. Parker and collaborators have argued that an extra prescription is required to define appropriately renormalized composite operators in the quantum field theory, which entails subtraction of these modes [88, 89]. A consensus on this issue does not yet appear to exist.

Following Weinberg’s calculation of loop corrections of the form (8), and others, [53] combined the effect of secular and box-cutoff logarithms with that of an ultraviolet logarithm at the same order of perturbation theory. In this work, the scale \( A \) which accompanies the ultraviolet logarithm was erroneously identified with the scale \( L \) which accompanies the box-cutoff logarithm. This error was corrected from v3 onwards of the arXiv version of [53]. I would like to thank E Dimastrogiovanni for drawing this to my attention.
to the fact that the large-scale (that is, infrared) structure in the background fields depends on physics at large values of the Hubble rate, $H$ (that is, in the ultraviolet). This feature of gravity, which swaps ultraviolet and infrared physics, is known from previous applications of the holographic renormalization group to domain wall spacetimes which are asymptotically anti de Sitter [90–92] or de Sitter [93–95]. We will return to this question when we revisit the mosaicking prescription outlined in section 2.2, because it is clear that the large-scale structure of the mosaic will depend on what is assumed about the matter field theory in the ultraviolet.

We must still decide what meaning should be attached to $\Lambda_1$. In calculating loop corrections, Weinberg [54, 55] made use of dimensional regularization. This entails an analytic continuation to $3 + \epsilon$ spatial dimensions, after which a limit must be extracted by dropping singular terms as $\epsilon \to 0$. Later, Chaicherdsakul [56] and Adshead, Easther and Lim [58, 96] employed the same technique. Because $\Lambda$ does not appear explicitly in this method, being hidden in subtractions associated with the pole at $\epsilon = 0$, these papers quoted the loop correction as a multiple of $\ln k$ but left the scale $\Lambda$ implicit. Equation (8) used an alternative procedure, imposing a sharp limit on the comoving momenta which contribute to each integral. This method was employed in [57], and later by Dimastrogiovanni and Bartolo [59], and gives results in agreement with dimensional regularization.

In either approach, $\Lambda$ must be interpreted as a comoving scale. van der Meulen and Smit remarked that because comoving scales are not physical, logarithms of the form $\ln \Lambda/k$ were not easy to interpret [97]. It had previously been observed in [57] that if one takes $\Lambda$ to be associated with a fixed physical scale $\Lambda_P$, then at an arbitrary time the corresponding comoving scale is $\Lambda = \Lambda_P a$ where $a$ is the scale factor. But at which time should the cutoff be evaluated? Senatore and Zaldarriaga argued that an appropriate choice would be the time of horizon crossing [61], for which $a = k/H_k$ if $H_k$ is the value of the Hubble parameter as the wavenumber $k$ crosses the horizon, at time $\tau = \tau_k$. If so we could conclude that equation (8), and other similar integrals, should be replaced by

$$\int \frac{d^3q}{(2\pi)^3} \frac{k^3q^2}{|q+k|^2} = \frac{k^3}{6\pi^2} \ln \frac{\Lambda_P}{H_k} + \text{finite}. \quad (9)$$

A similar conclusion had been reached earlier, by a different method, in [60]. This would give a result in precise analogy with the logarithm $\ln E/\mu$ encountered in scattering calculations, but seems incompatible with the use of dimensional regularization. However, Senatore and Zaldarriaga went on to argue that the free-field propagators which appear in the Feynman rules should also be calculated by analytic continuation of the Mukhanov equation to $3 + \epsilon$ spatial dimensions. This was not attempted in [54–56, 58]. Analytically continuing in this way generates extra terms which enable dimensional regularization to reproduce (9). At the present time it seems unclear whether the horizon-crossing cutoff can be obtained from a more fundamental principle, or whether one must simply adopt it as a prescription giving reasonable results consistent with the approximate de Sitter symmetry. It is also unclear how this form of dimensional regularization should be extended to momentum integrals arising from field redefinitions, which are to be discussed below. At one loop, these can be dealt with using a fixed momentum cutoff, so this ambiguity is not yet pressing.

**Corrections to the power spectrum.** For convenience, some results on loop corrections will be collected in this section.

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7 Senatore and Zaldarriaga observed that the choice $a = k/H_k$ in the comoving cutoff was equivalent to including only quanta beneath the cutoff at the earliest of the times $\tau'$ and $\tau''$ in the left-hand diagram of figure 1. In a more complicated diagram, one would choose the earliest such time. A similar argument was given in [60], which discussed the influence of quanta which redshift under the horizon at times subsequent to $\tau'$. 

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Weinberg gave his calculation in the comoving gauge, where the role of active scalar field was taken by the comoving curvature perturbation $\xi$. Adshead, Easther and Lim worked in the uniform curvature gauge, where the action describing a 3-point contact interaction between an active field $\phi$ and a collection of $N$ spectator fields $s^a$ can be written as [11]

$$S_N \equiv \int d^4x \frac{\alpha^2}{2M_p^2 H^2} \left[ \frac{1}{2} \frac{\delta L}{\delta \phi} \left[ \partial^{-2}(s_a \partial^2 s_a) - \frac{1}{2} s_a \partial s_a \right] - \frac{1}{2} \delta \phi s_a \partial s_a - \partial^{-2} \delta \phi \partial^2 s_a \right],$$

(10)

where $d^4x = d^4x \, d\tau$, summation of repeated $a$ indices is implied, a prime denotes a derivative with respect to conformal time and an overdot denotes a derivative with respect to cosmic time. In this equation, following the notation of Maldacena [10], $\delta L/\delta \phi$ represents the first-order equation of motion for $\phi$. This term can be removed by a field redefinition\(^8\)

$$\delta \phi = \sigma - \frac{\phi}{4M_p^2 H} \left( \partial^{-2}(s_a \partial^2 s_a) - \frac{1}{2} s_a \partial s_a \right).$$

(11)

The rules for evaluating correlation functions such as $\langle \sigma \sigma \rangle$ are discussed in the review paper by Koyama elsewhere in this issue [70]. One finds the power spectrum of $\sigma$ to be

$$P^{(\sigma)}(k) = \frac{H^2}{2k^3} \left( 1 - N \epsilon_\Lambda \frac{H^2}{M_p^2} \ln \frac{\Lambda_p}{H_*} + \cdots \right),$$

(12)

where now the time of observation $\tau_* \approx \tau_{h(k)}$ is chosen to be almost immediately after the time of horizon crossing $\tau_{h(k)} \approx \tau_{h(k)}$.

To obtain the 2-point correlation function of $\delta \phi$, the effect of the field redefinition must be reversed. One finds

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle_{\sigma} \equiv \langle \sigma(k_1) \sigma(k_2) \rangle_{\sigma},$$

$$- \frac{\phi_*}{4M_p^2 H_*^2} \left( \sigma(k_1) \left( \partial^{-2}(s_a \partial^2 s_a) - \frac{1}{2} s_a \partial s_a \right) \right)_{k_1} + (k_1 \leftrightarrow k_2)$$

$$+ \frac{\phi^2_*}{16M_p^2 H_*^2} \left( \partial^{-2}(s_a \partial^2 s_a) - \frac{1}{2} s_a \partial s_a \right)_{k_1} \left( \partial^{-2}(s_\beta \partial^2 s_\beta) - \frac{1}{2} s_\beta \partial s_\beta \right)_{k_2},$$

(13)

where $(k_1 \leftrightarrow k_2)$ denotes the previous term with $k_1$ and $k_2$ exchanged, and $(AB)_k$ is the convolution of $A$ and $B$ with argument $k$. In evaluating this expression we encounter 3-point correlations of the form $\langle \sigma s_a s_a \rangle$ and 4-point correlations of the form $\langle s_a s_\beta s_\beta s_\beta \rangle$, but the 4-point terms give no contribution to $\ln(\Lambda_p/H_*)$. The 3-point correlations can be evaluated using the general formula for a 3-point function of scalar fields given in [11], yielding

$$\langle \sigma(p_1) s_a(p_2) \sigma(p_3) \rangle_{\sigma} = (2\pi)^3 \delta(p_1 + p_2 + p_3) \frac{H_*^4}{4 \prod_i p_i^3 M_p^2} \phi_*^4 \delta s_a s_\beta$$

$$\times \left( -8 \frac{p_2^3 p_3^3}{p_1^3} + p_1 \left( p_2^2 - p_2^3 - p_3^3 \right) \right),$$

(14)

where $p_i = p_1 + p_2 + p_3$. After some calculation, one finds that the second term in brackets ($\cdots$) does not contribute, and the power spectrum of $\phi$ can be written

$$P_\sigma(k) = \frac{H_*^2}{2k^3} \left( 1 - \frac{3N \epsilon_\Lambda H_*^2}{40\pi^2 M_p^2} \ln \frac{\Lambda_p}{H_*} + \cdots \right).$$

(15)

---

\(^8\) This procedure is notationally misleading, because $\delta L/\delta \phi$ is zero by construction (at least to leading order) when evaluated on a propagator and therefore gives no contribution whether we subtract it or not. The $\delta L/\delta \phi$ term is a proxy for boundary terms, which following the notation of Maldacena [10] have not been written explicitly, which do not vanish on solutions to the equations of motion [11]. It was observed in [98] that the boundary terms and $\delta L/\delta \phi$ terms are both removed by the field definition, allowing this notational trick to make sense.
Figure 2. Graviton loop corrections to the power spectrum of a scalar field, calculated by Dimastrogiovanni and Bartolo [59]. Unlike the case of scalar loops, the right-hand diagram is not slow-roll suppressed compared to the left-hand diagram. As before, the interior of these diagrams can be considered as a sort of instanton for the nucleation of gravitational (wavy lines) and scalar quanta (straight lines), which propagate to the time of observation $\tau_e$ on the external legs of the diagram.

From this calculation we see that field redefinitions reshuffle the coefficient of $\ln(\Lambda_P/H_\epsilon)$. Since gauge transformations are a form of field redefinition, we must expect the numerical coefficient $-3/40$ in equation (15) to shift under a change of gauge. In particular, this will happen if we change to comoving gauge and obtain the correlation function of the curvature perturbation $\zeta$. Therefore, equation (15) cannot be compared directly to Weinberg’s result in [54], but is an equally good indicator of the magnitude of the loop correction9.

Some time after Weinberg’s $\zeta$-gauge calculation of a spectator loop [54], the self-loop correction to the power spectrum of an active scalar field was obtained [57]10. This corresponds to the right-hand diagram of figure1 and was found to give a correction to the power spectrum of the active field immediately after horizon crossing which amounted to

$$P_e(k) = \frac{H_e^2}{2k^3} \left( 1 + \frac{1}{3\pi^2} \frac{H_e^2}{M_P^2} \ln \frac{\Lambda_P}{H_e} + \cdots \right).$$

(16)

Together, equations (15) and (16) give the largest part of the loop correction for a single scalar field accompanied by $N$ isocurvature fields. The terms which are neglected by this combination are higher order in $\epsilon$ but do not include compensating factors of $N$. Adshead, Easther and Lim also considered loops similar to the left-hand diagram of figure1, with active fields circulating in the interior of the diagram [58]. In principle, these allow the species of active scalars to fluctuate into each other, but have not yet been calculated.

**Corrections from graviton loops.** Dimastrogiovanni and Bartolo obtained corrections from graviton loops [59], shown in figure 2.

Before discussing these, consider the self-loop correction for an active scalar field. Wick contraction of an external field operator with an operator in the interior of the loop endows the right-hand diagram of figure 1 with a dependence on the external wavenumber. This happens because operators like $\partial^{-2}$ applied to the interior of the loop generate integrals similar to those in equation (8), containing $|k+q|$ in the denominator. In the right-hand diagram of figure 2 the

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9 A calculation of this loop in the uniform curvature gauge was given in [58], but the contribution from the field redefinition was not included. In Weinberg’s calculation, given in the comoving gauge, a similar redefinition was dropped, which changes the answer. (A numerical error in Weinberg’s calculation was corrected by Adshead, Easther and Lim [58], but these authors did not include the contribution from the field redefinition either.) I would like to thank Peter Adshead for communicating the outcome of his calculations concerning this issue.

10 This calculation depends on a set of Feynman rules for the active field, which were obtained in the same reference. Unfortunately a sign error was present in one of these rules, which led to an incorrect numerical coefficient in the final answer. This sign error was detected and corrected in [58], but owing to some typographical errors the final numerical coefficient determined by these authors was again given incorrectly. The correct coefficient was first given in v3 of [57].
graviton pair interior to the loop must contract among themselves, and likewise the external fields. Therefore no dependence on the external wavenumber is generated, and this diagram factorizes into an overall renormalization of the tree-level two-point function,

\[ P^\prime(k) = \frac{H_s^2}{2k^3} \left( 1 - \frac{1}{2\pi^2 M_p^2} \ln(\Lambda_p \alpha_s L) + \cdots \right), \]  

where \( L \) is an infrared cutoff of the type which appears in box-cutoff logarithms, as discussed in section 2.2. For the left-hand diagram, one finds\(^{11}\)

\[ P^\prime(k) = \frac{H_s^2}{2k^3} \left( 1 + \frac{1}{6\pi^2 M_p^2} \ln \frac{\Lambda_p}{H_s} + \frac{1}{2\pi^2 M_p^2} \ln k L + \cdots \right), \]  

where ‘\( \cdots \)’ denotes terms which vanish at late times together with any subleading contributions. These examples are interesting because they show that quantum loops can generate ‘box-cutoff’ effects in addition to ‘new physics’ logarithms, which was suggested but not demonstrated in \[83\]. It is not yet clear how to interpret the association of a \( \ln k L \) term with a loop containing spin-2 gravitational quanta, which lead to traceless deformations of the metric. However, these terms all cancel in the total one-loop correction, as we now explain.

Together, equations (15)–(18) give the largest part of the loop correction for each active scalar field in a theory containing \( N_s \) spectator fields.\(^{12}\) The terms which are neglected by this combination are higher-order in \( \epsilon^\ast \) but do not include compensating factors of \( N_s \). First, the \( k \)-dependent logarithms in equations (16) and (18) cancel between themselves, to leave a logarithm depending only on the ultraviolet and infrared regulators. Second, this logarithm cancels with that in equation (17); only the contribution of equation (15) is left. Therefore, to this accuracy, the total loop correction is entirely free of the infrared regulator,

\[ P^\text{total}(k) = \frac{H_s^2}{2k^3} \left( 1 - \frac{3N_s \epsilon^\ast}{40\pi^2 M_p^2} \ln \frac{\Lambda_p}{H_s} + \cdots \right). \]  

Remarkably, this receives contributions from the isocurvature fields alone. It is not clear whether this cancellation is accidental, or enforced by some deeper principle. For example, in the case of a single minimally coupled scalar field in de Sitter space (obtained by passage to the limit \( \epsilon \to 0 \)), this absence of one-loop corrections becomes exact and is consistent with de Sitter invariance. If the cancellation is accidental there seems no reason why \( L \)-dependent logarithms should not appear at subleading order in \( \epsilon^\ast \), or at two loops and higher. It would be of considerable interest to determine whether this is the case.

Burgess \textit{et al} \[76\] emphasized that, if infrared-sensitive contributions occur, one should obtain a time independent answer when working with physical ultraviolet and infrared cutoffs, \( \Lambda_p \) and \( L_p \). In their analysis, this was achieved because the ultraviolet and infrared terms combined to give \( \ln \Lambda_p L_p \). If one chooses the infrared cutoff to be comoving—as we are doing in this discussion—then de Sitter invariance is broken and one cannot avoid a time dependent logarithm of the form rejected by Senatore and Zaldarriaga as incompatible with eternal inflation. In the present framework, whether a similar problem exists depends on the time-dependent distribution of background field values which removes \( L \) from equation (19) and similar equations.

\(^{11}\) Dimastrogiovanni and Bartolo quoted their answer in a different form, but equation (17) gives the only relevant terms in the limit \( |k\tau_s| \to 0 \). Dimastrogiovanni and Bartolo found a slightly different coefficient for each logarithm. As this paper was being completed, a recalculation of the graviton loop appeared in \[145\], which agrees with the result given \[146\].

\(^{12}\) Adshead, Easther and Lim argued that for active fields only \textit{self}-loops of the leading-order interaction give physical logarithmic corrections \[58\], for the reasons discussed above equation (18). Loop corrections to the power spectrum of one active species from other active species are suppressed by slow-roll factors.
\[ k - q \]

Figure 3. Loop in pure $\phi^3$ theory. A factor of the coupling $g$ is present at each vertex.

$\phi^3$ loop. A somewhat less physical example is the loop in pure $\phi^3$ theory, which will play an interesting role in the discussion of time dependence in section 3. Taking the interaction to be

\[ S_3 \supseteq \int d^4 x \, a^2 \frac{g}{3} \delta \phi^3, \]

one arrives at the loop of figure 3. van der Meulen and Smit calculated this loop, accounting for both ultraviolet and infrared effects. It was later recalculated by various groups, since it provides a useful test case [60, 76, 84]. One finds

\[ P_\ast (k) = \frac{H_1^2}{2k^3} \left( 1 + \frac{g^2}{18\pi^2} \right) \ln kL - \frac{4g^2}{27\pi^2} \ln \frac{\Lambda_p}{H_*} + \cdots. \]  

(21)

$\phi^4$ loop. A loop in $\phi^4$ theory was calculated by Tsamis and Woodard [99, 100] and Petri [101], and later by Burgess, Holman, Leblond and Shandera [76]. The interaction is

\[ S_4 \supseteq \int d^4 x \, a^4 \frac{\lambda}{4!} \delta \phi^4, \]

and the diagram is the same as the right-hand loop of figure 1, although with a different vertex. The result is

\[ P_\ast (k) = \frac{H_1^2}{2k^3} \left( 1 + \frac{\lambda}{12\pi^2} \ln kL \ln |k\tau_\ast| + \cdots \right). \]

(23)

Other examples of loops have been considered in the literature. In [60] a set of loop corrections to the 2- and 3-point functions of an active scalar field were calculated in a theory where the scalars coupled to the field strength of an Abelian gauge field. Although their structure is rather complicated, these loops gave results which are not qualitatively different to those we have already studied.

3. Time evolution

As in any quantum field theory, the diagrams in figures 1–3 are calculated by identifying quasi-free propagating modes which travel along the arcs of the diagram and interact at vertices. In each calculation discussed above, the quasi-free modes behave like massless scalar fields in eternal de Sitter space which have 2-point correlations of the form

\[
\langle \delta \phi (k_1, \tau_1) \delta \phi (k_2, \tau_2) \rangle = (2\pi)^3 \delta (k_1 + k_2) \frac{H_1 H_2}{2k_1} \times \left\{ \begin{array}{ll}
(1 - ik_1 \tau_1)(1 + ik_2 \tau_2) e^{i(k_1 \tau_1 - k_2 \tau_2)} & \text{if } \tau_1 < \tau_2 \\
(1 + ik_1 \tau_1)(1 - ik_2 \tau_2) e^{i(k_2 \tau_2 - k_1 \tau_1)} & \text{otherwise},
\end{array} \right.
\]

(24)
where \( H_i = H(\tau_i) \) and \( k_i = |k_i| \). The magnitudes are equal on-shell, where conservation of momentum is satisfied, so \( k_1 = k_2 \). The distinction has been maintained in equation (24) because in practical calculations one must often calculate off-shell where \( k_1 \neq k_2 \), only taking the on-shell limit at the end of the calculation [15, 44, 60, 96]; the same is true for higher \( n \)-point functions where one must keep the \( k_i \) all distinct. Equation (24) is built out of elementary wavefunctions proportional to \( (1 - i k \tau) e^{i k \tau} \), which asymptotes to unity in the infinite future of de Sitter space where \( \tau \rightarrow 0^- \). Therefore, once correlations have been generated using equation (24) they persist into the indefinite future. Had we instead taken the quasi-free modes to have a mass of order \( m \), we would have obtained elementary wavefunctions of the form \( (-k \tau)^{1/2} a^{-1}(\tau) H^{(2)}_\nu(-k \tau) \), where \( H^{(1,2)}_\nu \) are Hankel functions of the first and second kind, respectively, and the order of these functions, \( \nu \), is related to the mass \( m \) by

\[
\nu^2 = \frac{9}{4} - \frac{m^2}{H^2}.
\]

For \( m \gg H \) this gives the Hankel functions an imaginary order. In the limit \( \tau \rightarrow 0 \) the massive de Sitter wavefunction would have an asymptotic expansion proportional to

\[
\sim (-k \tau)^{3/2-\nu} \left( \frac{-i^\nu \Gamma(\nu)}{\pi} + \frac{(-k \tau)^{2\nu} (1 + i \cot \pi \nu)}{\Gamma(1+\nu)} \right) + \cdots,
\]

where ‘\( \cdots \)’ denotes terms which converge to zero more quickly than those which have been written. Clearly, correlations are suppressed at late times whenever \( \nu < 3/2 \) and vanish entirely as \( \tau \rightarrow 0^- \). Having obtained expressions of this form, Riotto and Sloth [102] observed that equation (26) expresses nothing more profound than that a field of mass \( m \) has a proper correlation length of order \( 1/m \), and in the infinite future of de Sitter space the cosmological expansion has carried any two spatially distinct points far beyond a proper separation of order \( 1/m \). For this reason, correlations between spatially separated points must decay.

Weinberg remarked that this mismatch between equation (24) and the decay of correlations in the real universe compels us to account for mass terms non-perturbatively [55]. In principle we could deal with a term in the Lagrangian of the form \( m^2 \delta \phi^2 \) by including its contribution in the elementary wavefunctions of the quasi-free modes, or by counting it among the interactions. If we include it among the interactions, then we compute correlations using equation (24) and are obliged to make the implicit assumption that they do not subsequently decay. This may yield reasonable results, supposing the correlations evolve only slowly, provided we do not ask for their properties too long after horizon crossing. Nevertheless, where the correlations truly decay this approximation must lead to trouble at late times. Since we expect most fields to acquire a non-trivial potential except where protected by an exact symmetry, we also expect the fluctuations in these fields to develop small masses. As was explained in section 2.1, it is our failure to account for this time evolution in the background fields, and the subsequent decay of correlations, which is responsible for the appearance of the secular logarithms \( \ln |k \tau_*| \).

What if a field has a potential containing no mass term, but other higher-order interactions? This situation was analysed by Senatore and Zaldarriaga [61], who argued that on any potential of this kind the background field would still roll down and evolve. For this reason, we would expect a mass term to be generated for the fluctuations after resumming a sufficient number of insertions in the propagator. This would subsequently cause correlations to decay. Therefore this situation is qualitatively the same as the case of a mass term in the potential. We will see an explicit example in section 3.3.
### 3.1. Weinberg’s theorem and related results

How fast can we expect correlations to evolve? In section 2.1 we discussed a theorem due to Weinberg [54] which guarantees that only logarithms of \(|k\tau_*|\) appear in inflationary correlation functions, and not power laws of the form \(|k\tau_*|^{-n}\) for \(n \geq 1\) which in principle are allowed [20]. (Contributions proportional to positive powers of \(|k\tau_*|\) are generically present, but contribute nothing at late times.) The precise statement of this theorem applies to scalar and tensor excitations. In three dimensions, we have

**Theorem.** In three space dimensions, power law divergences are absent at late times in correlations among scalar and tensor quanta, provided all interactions in the Lagrangian, when written in cosmic time, fall into one of two classes:

(a) interactions containing strictly less than one factor of \(a(t)\); or

(b) interactions which may grow as fast as \(a(t)\) (but no faster), and which contain only fields rather than derivatives of fields.

Weinberg proved this theorem by studying commutators among the fields and their derivatives at late times, and then counting how these commutators could appear in the possible elementary interactions. This method of proof does not make use of the explicit slow-roll, massless solution (24). However, Senatore and Zaldarriaga later remarked that a refinement of this theorem might be possible [61], because the asymptotic estimates obtained by Weinberg did assume that the fields were massless. In addition, since each elementary interaction was considered in isolation, the possibility of cancellations among groups of interactions was not considered. At the time of writing, these possible refinements have not been studied.

In a later publication, Weinberg extended this theorem to Abelian and non-Abelian vector fields, and Dirac fermions [55] (see also [56]). These fields have rather better behaviour than the scalar fields considered in the original theorem, and correlation functions among them typically converge at late times.

### 3.2. Resummation by the \(\delta N\) formula

For the remainder of this discussion, we will restrict attention to theories satisfying the conditions of Weinberg’s theorem. Therefore, only secular logarithms will appear as \(\tau_* \to 0\), and not the faster power-law divergences.

Up to this point we have not made our choice of gauge explicit, but we have been discussing fluctuations in scalar fields. This will usually correspond to the uniform curvature gauge. The quantity whose correlations are accessible to observation in the CMB is often taken to be the comoving curvature perturbation \(\zeta\). (Indeed, Weinberg’s theorem and the spectator loop discussed in section 2.3 were given directly in this gauge.) The statistical properties of \(\zeta\) are determined from a superposition of the active fields in the theory. Many prescriptions have been employed to obtain this superposition [103, 104], but a simple method is to observe that \(\zeta\) is a local fluctuation in the aggregate cosmological expansion, \(\zeta = \delta \ln a\), where \(a\) is the scale factor. We can therefore use the separate universe formula, equation (3). One usually defines \(\ln a/a_0 = N\), where \(N\) is the number of e-folds from a reference time where \(a = a_0\). We take the initial slice to be a uniform curvature hypersurface, and the final slice to be a surface of uniform energy density, after which we arrive at the \(\delta N\) formula [50, 105, 106], reviewed elsewhere in this issue by Tanaka, Suyama and Yokoyama [107] and Wands [108]:

\[
\zeta = \frac{\partial N}{\partial \phi_*} \delta \phi_* + \frac{1}{2} \frac{\partial^2 N}{\partial \phi_*^2} \delta \phi_*^2 + \cdots.
\]  

(27)
Among other things, this is a gauge transformation. The $\delta N$ formula reproduces those terms from the nonlinear transformation between the uniform curvature gauge and comoving gauge [10] which do not vanish on superhorizon scales.

Equation (27) enables the correlation functions of $\zeta$ to be determined from those of $\delta \phi_*$, which in turn are computed using the massless propagator (24). In the foregoing discussion we argued that equation (24) prevents correlations from decaying after they have been generated, and that this corresponds to neglecting evolution of the background fields. The $\delta N$ formula ameliorates this difficulty. Suppose we wish to compute correlations on some scale $k$. If we take the initial time, $\tau_*$, to be shortly after the time of horizon exit associated with $k$, then we do not commit a gross error in failing to account for their subsequent decay. Moreover, the time dependence summarized by $N$ accounts for evolution of the background fields as they roll down their potentials. Therefore, we expect equation (27) to incorporate the infrared physics whose neglect gave rise to secular logarithms. This argument was given in [53] and is supported by the analysis of van der Meulen and Smit [97], although a formal proof is still lacking.

The same conclusion can be reached by considering the logarithms themselves. If $\tau_*$ is chosen not long after horizon crossing, then the terms $\ln|k \tau_*|$ are of order unity. If the leading term containing $n$ powers of $\ln|k \tau_*|$ occurs in the combination $\sim (\epsilon \ln|k \tau_*|)^n$, then, not long after horizon crossing, all terms containing logarithms will be small compared to the tree level whenever $\epsilon \ll 1$. Therefore, we can neglect all logarithms in using equation (27) to compute correlations of $\zeta$. The effect has been to ‘resum’ their effect into the background evolution of the scalar fields, which is described by the accumulating number of e-folds, $N$. For example, in a single-field model of inflation this argument reproduces the expected conclusion that $\zeta$ does not evolve. But the virtue of the $\delta N$ method is that it is not restricted to the single-field case. In a multiple-field scenario where $\zeta$ has non-trivial evolution, the $\delta N$ formula will correctly describe the growth and decay of fluctuations.

There may be problems when using equation (27) to obtain the correlation functions of $\zeta$ if one wishes to include the ‘quantum’ logarithms of section 2.3. This procedure was applied in [53, 57–60]. However, as we have already observed, the $\delta N$ formula is a gauge transformation, and it was argued in section 2.3 that such transformations will typically modify the coefficient of $\ln \Lambda_P / M_P$. Although this possibility has yet to be investigated in detail, the large number of ultraviolet terms discarded by the $\delta N$ formula (which should be compared, for example, with equation (A.8) of [10]) would apparently make a mismatch likely. The $\delta N$ formula retains all relevant infrared physics, so we would not expect similar difficulties with secular and box-cutoff logarithms.

### 3.3. Resummation by the dynamical renormalization group

For the purpose of comparison with observation, the $\delta N$ method is presumably sufficient and allows us to extract initial conditions from typical inflationary models without any loss of predictivity. In many models of inflation we wish to obtain the properties of correlations only 50–60 e-folds after they are synthesized. In a conventional model, the only source of appreciable time evolution during these 50–60 e-folds comes from the classical variation of the background. Nevertheless, as a point of principle, one might prefer a more satisfactory resolution. Some time after the resummation interpretation of $\delta N$ had been proposed, Burgess, Holman, Leblond and Shandera remarked that it did not account for the possibility of secular time evolution from quantum effects [76], such as those of equations (17), (21) and (23). These terms typically appear in combination with the small parameter $(H / M_P)^2$, and are therefore small unless many e-folds have elapsed since horizon crossing.
Burgess et al suggested that these logarithms could be resummed using the method of the renormalization group, and applied this technique to the quartic loop of equation (23). We briefly review their method. The starting point is to introduce an arbitrary constant, c, which is unity in the tree-level power spectrum:

$$P_*(c, k) = \frac{c H_*^2}{2k^3}.$$  

Our strategy is to resum the large logarithms into c. To do so, rewrite equation (23) in terms of an arbitrary intermediate time scale \( \vartheta \):

$$P_*(k) = \frac{c H_*^2}{2k^3} \left( 1 + \frac{\lambda}{12 \pi^2} \ln k \ln \vartheta \right) \left( 1 + \frac{\lambda}{12 \pi^2} \ln k [\ln |k \tau_*| - \ln \vartheta] \right).$$  

Clearly, the combination of overall prefactor and first bracket is of the same form as \( P_*(c, k) \), with \( c = c(\vartheta) \). We can therefore replace one by the other. Whatever the result, it must be independent of the arbitrary time \( \vartheta \). Applying the familiar method of Gell-Mann and Low [109], we find

$$\frac{dc}{d\vartheta} = \frac{\lambda}{12 \pi^2} c \ln k \ln \vartheta.$$  

(30)

Solving this differential equation and setting \( \vartheta = |k \tau_*| \), we conclude

$$P_*(k) = \frac{H_*^2}{2k^3} |k \tau_*|^\delta (1 + O(\epsilon^2 \ln |k \tau_*|)),$$  

(31)

where \( \delta \) satisfies

$$\delta = \frac{\lambda}{12 \pi^2} \ln k L.$$  

(32)

The error in this procedure is of order \( \epsilon^2 \ln |k \tau_*| \). Therefore, equations (31)–(32) are trustworthy even when \( |\epsilon \ln |k \tau_*|| \sim 1 \), giving rise to a resummation of the leading logarithms. Subleading logarithms could be included by accounting for terms of order \( \lambda^2 \) or higher in equation (30).

Equation (31) describes decaying correlations. On the basis of the foregoing discussion, we expect this behaviour to represent the correct infrared physics in a large class of theories. Indeed, comparison with equation (26) shows that the effect of \( c(|k \tau_*|) \) is to introduce a dependence on \( |k \tau_*| \) comparable to the case of a massive scalar field. Making use of equation (25) and recalling that the square of equation (26) yields the power spectrum, we see that equation (31) is equivalent to a dynamically generated mass \( M_{\text{eff}} \) [76]:

$$M_{\text{eff}}^2 = \frac{\lambda}{8 \pi^2} H_*^2 \ln k L.$$  

(33)

If \( 1/k \) and \( L \) do not generate an exponential hierarchy and \( \lambda \sim O(1) \), then \( M_{\text{eff}} \) is of the order of the Hubble rate. We can regard this as a concrete example of the argument of Senatore and Zaldarriaga [61] discussed just before section 3.1.

This conclusion does not apply to every theory. Burgess, Holman, Leblond and Shandera observed that the same argument applied to pure \( \phi^3 \) theory does not yield a dynamically generated mass. The obstruction comes from the power of \( \ln |k \tau_*| \) in the leading term of equation (21). Ignoring the ultraviolet logarithm proportional to \( \ln \Lambda_P/M_P \), which plays no role here, resumming the secular logarithms would give

$$P_*(k) = \frac{H_*^2}{2k^3} \exp \left( \frac{\epsilon^2}{18 \pi^2} \ln k [\ln |k \tau_*|] \right) (1 + O(\epsilon^2 \ln |k \tau_*|)),$$  

(34)

Since this does not suppress the power spectrum by a positive power of \( |k \tau_*| \) at late times we cannot interpret it in terms of an effective value of \( \nu \) in equation (26), and thus an effective mass
\( M_{\text{eff}} \). Indeed, (34) has rather poor behaviour in the limit \(|k \tau_*| \to 0\), because the exponential diverges there. One should not take this behaviour literally. Burgess et al included logarithms of cubic order which have been neglected here, and which would become negative in the limit \( \tau_* \to 0^- \). Thus, in pure \( \phi^3 \) theory the behaviour of the exponential in the far future becomes a delicate question.

This is not difficult to interpret. In pure \( \phi^3 \) theory the Hamiltonian is unbounded below, and there is nothing to prevent fluctuations (and their correlations) growing indefinitely. Burgess et al remarked that this pathological behaviour made studying the late-time behaviour of this theory problematic. Following the argument of Senatore and Zaldarriaga [61], we may expect that in realistic theories behaviour comparable to (31) is more generic.

4. Spatial evolution

Less is known about the resolution of what we have called box-cutoff logarithms, of the form \( \ln(kL) \), which were left behind in section 2.2 after tiling the spatial region of interest into a mosaic of boxes.

4.1. Mosaicking prescriptions

Our discussion in section 2.2 was based on an assumption that the role of physics at scales larger than \( L \) was to determine a quasi-classical background field configuration in each box. A proof of the validity of this assumption is not known. What would it entail? Consider an expectation value of local operators \( O_1, \ldots, O_n \) evaluated at spatial positions \( x_1, \ldots, x_n \) with roughly common separations \( |x_i - x_j| \sim 1/k \) which are small on the characteristic scale of the mosaic, so that \( kL \gg 1 \). We wish to evaluate this correlation function at time \( \tau_* \), just after horizon exit of the wavenumber \( k \), which can be achieved by averaging against some wavefunctional \( \Psi_1 \):
In typical applications, the expectation value (35) does not depend on the prior history of the universe up to the point of horizon crossing. It is therefore independent of almost all boundary data represented by $O_\partial L$, retaining only a possible dependence on the boundary configuration near the surface $\tau = \tau_e$. If the path integral is dominated by semiclassical field configurations which vary only slowly on the scale $L$, then to a good approximation this remaining dependence will involve only the roughly homogeneous boundary value at $\tau_e$, and not its spatial gradients. Equation (37) suggests that expectation values are to be calculated by fixing a homogeneous background field configuration at the surface of horizon crossing and summing over unrestricted field histories for fluctuations in the interior. Finally, one averages over the fixed background configurations with an unknown probability measure, $P$, which is derived from the integral over exterior field configurations. At this point we have arrived at the prescription of section 2.2.

This construction makes $P$ a distribution over field configurations at a fixed spatial position in an ensemble of realizations, but if ergodicity applies we can equally interpret it as a probability distribution over different boxes in the same realization. The above argument is merely heuristic, and the existence of such a decoupling limit has not yet been demonstrated. Nevertheless, equations (35)–(37) show a strong similarity with path integral approaches to the operator product expansion (‘OPE’) in flat space quantum field theory. A manifestly local and covariant formulation of this expansion in curved spacetime has been given by Hollands and Wald, which in principle could be applied to the study of infrared effects in nearly de Sitter spacetimes. However, this has not yet been done.

A similar conclusion was reached by Allen and Folacci [117] and later Kumar, Leblond, and Rajaraman [118], who studied ambiguities in defining the zero-mode for the Bunch–Davies propagator associated with a massless, minimally coupled scalar field in de Sitter space. Kumar et al. noted that similar ambiguities were removed in Minkowski space owing to requirements imposed by the principle of cluster decomposition. Exploiting the extra freedom available in de Sitter space, they succeeded in subtracting infrared divergences by performing an appropriate redefinition. In a field theory this redefinition would naturally become spatially dependent on scales which are large compared with those appearing in expectation values. The result is to reproduce the conclusion that infrared divergences derive from a supposition that calculations are to be carried out with a definite, homogeneous expectation value for the background field configuration. From this point of view, equation (37) can be thought of as an analogue of cluster decomposition in de Sitter.

These ideas have a large literature of their own. Kirsten and Garriga [119] used related reasoning to represent the quantized massless scalar field as the product of a Hilbert space, associated with the zero-mode, and a Fock space corresponding to excitations of finite wavenumber. Related ideas were explored by Moncrief [120], Higuchi and collaborators [121–124], and later Giddings and Marolf [125]. Urakawa and Maeda investigated the same physics from a different perspective.

4.2. The $L$-dependence of physical predictions

While discussing logarithms generated by ultraviolet effects, Senatore and Zaldarriaga remarked that combinations such as $\ln \Lambda/\kappa$ in equation (8) were unphysical [61], because there is no physical comoving scale which could accompany $\kappa$. However, this argument does not entirely preclude the appearance of $\kappa$ in any logarithm. For example, the
combination $\ln|k\tau^*|$ measures by how many e-folds the wavenumber $k$ is outside the horizon at time $\tau^*$ and is a sensible physical quantity. Nevertheless, this example shows that one must be careful in dealing with the combination $\ln kL$, in which $L$ is also a comoving scale.

Let us first idealize to an inflating volume which is spatially infinite. As in section 2.2, we divide this volume into boxes of size $L$, in each of which the field is spatially homogeneous with small perturbations. We have seen that correlations computed within these boxes depend on the arbitrary scale $L$, but (assuming only zero-modes need be retained from the long wavelength physics) an answer independent of $L$ is obtained if we average the correlations over boxes [128]. This requires a knowledge of the homogeneous background field within each box of the mosaic. If merely probabilistic answers are acceptable, it may only be necessary to use information about the statistical distribution of field values over the ensemble of boxes. Cosmological perturbation theory has been discussed from this point of view by many authors; see, for example, the recent textbook by Mukhanov [129].

If we assume that only zero-modes need be retained, as in section 4.1, it follows that this argument does not depend on a careful discussion of how the total $L$ dependence can somehow cancel between the $L$-dependent box-sized correlations and the distribution of field values among boxes. The schematic argument of equations (35)–(37) shows that in a full treatment, where the entire ultraviolet physics and initial conditions are known, we have done nothing more than dividing the calculation into two parts. When we assemble these two halves, the final answer must be independent of $L$. This happens automatically, no matter how remarkably fine-tuned the necessary cancellations appear, and is a standard argument in the application of effective field theories [72]. The scale $L$ therefore disappears from the answer and we need not worry about its physical significance. In practice, the distribution of field values among boxes of the mosaic will evolve in time. After convolving with this distribution, $L$-dependent terms would be traded for a time dependence which could itself be represented by secular logarithms.

However, it may not be the case that this $L$-independent answer is the quantity we wish to compare with observation. In discussing similar questions almost 20 years ago, Salopek and Bond remarked that we only wish to calculate what we can observe [130, 131]. What can we observe? Only the density fluctuation within our presently observable patch of the universe [52, 83–85, 132]. Observationally it is a matter of perfect indifference to us (although, of course, of surpassing theoretical interest) whether the distant, unobserved universe contains many regions similar to our own, or whether all distant regions are quite dissimilar. Whichever is correct, the answer will not change the result of satellite observations made today.

If we subscribe to the programme outlined in section 1, then we must suppose that our observable region of the universe passed through a phase of inflation which eventually came to an end. We assume that the classical background associated with whatever scalar fields supplied the necessary vacuum energy became roughly homogeneous within a box somewhat larger than our present horizon. This box should be large enough to contain many regions of comparable size to the presently observable universe, but need not be exponentially larger [52, 83]. We wish to compute correlations in the density fluctuation which would be recorded by a typical observer whose local patch experienced the same gross cosmological history. For this reason it may be incorrect to compare the $L$-independent answer discussed above with experiment, because in this answer the correlations specific to our patch are commingled with correlations recorded by observers experiencing a gross cosmological history quite different to our own.
Therefore, let us choose $L$ so that $\ln kL \sim 1$, making $L$ a little larger than our presently observable universe\footnote{This prescription seems to have been rediscovered several times, beginning with Salopek and Bond\cite{130,131}. In our present context, the choice $\ln kL \sim 1$ was suggested by Boubekeur and Lyth\cite{49}, who referred to it as a ‘minimal box’. Its properties were later studied in a series of papers by Lyth and collaborators\cite{50,52,83,86}. The same prescription, in various forms, was given by Bartolo et al\cite{84} and later by Enqvist et al\cite{85} and Kumar et al\cite{133}.}. The problem we face in this scenario is to determine the correct homogeneous background in which we should carry out our calculation. In general the answer depends on the distribution of field values within the ensemble of boxes, but there is a simple class of models in which this question is trivial. In any single-field model with a unique reheating minimum, there is a unique value of the background field, $\phi = \phi_{60}$, when inflation is of the order of 60 e-folds from ending. The prior history of the large-scale universe is irrelevant \cite{25,84}. To compute what would be recorded by a typical observer in this model, we are entitled to carry out our calculation within a box carrying an approximately homogeneous background field with value $\phi \sim \phi_{60}$. In this picture, the effective field theories we use to study slow-roll inflation are to be understood only as a description of the process by which an inflationary trajectory arrives in its final reheating minimum, causing inflation to end.

In the distant future, our observable region of the universe will expand to include modes which are presently unobservable. Enqvist, Nurmi, Podolsky and Rigopoulos remarked that an inflationary theorist, calculating millions of years in the future, would be obliged to work in a box of slightly larger size $L' > L$\cite{85}. The argument of Boubekeur and Lyth\cite{49}, which was extended to two loops by Bartolo et al\cite{84}, shows that physical observables compose correctly under the process of averaging $L$-sized boxes to make an $L'$-sized box.

4.3. Stochastic inflation

In a scenario which is more general than single-field inflation, there may be many terminal vacua in which the universe can reheat, rather than a unique choice. In these models there is no preferred background configuration—analogue to the single-field configuration $\phi \sim \phi_{60}$—in which we should carry out our computations. Even where a terminal vacuum can be selected in advance, it was shown by Byrnes, Choi and Hall that the final fluctuations may depend sensitively on the path by which the fields arrive at the minimum \cite{134,135} (reviewed in \cite{136}).

To determine what would be recorded by a typical observer we must know at least the frequency with which these minima are populated by boxes in which inflation terminates. This can be ascertained only if we know the distribution of field values within boxes of the mosaic, and (as we have discussed above) this distribution is an ultraviolet-dependent quantity. Without knowledge of the complete ultraviolet physics, including any relevant initial conditions, we cannot calculate it from first principles.

Apart from our interest in observational predictions, there may be other reasons to enquire about the large-scale disposition of the scalar fields. Senatore and Zaldarriaga\cite{61} emphasized that we would like to know the circumstances under which eternal inflation can occur \cite{137,138}. It is possible that this depends on effects arising from loops at high order in perturbation theory. For this reason we may wish to go beyond the narrow view advanced above, according to which our theories of slow-roll inflation should be restricted to a description of how inflation ends. Using a model of slow-roll inflation to describe evolution within a large spatial volume over very long times would again require us to know the distribution of field values over the entire mosaic of boxes.

Clearly we cannot hope to calculate this distribution \textit{ab initio}. Suppose, however, that by some means we know the distribution of field values over the mosaic of boxes at time $t_i$,
at which point we suppose that the Hubble scale took the value $H_i$. If we have an effective description of the processes by which correlations are established at energies lower than $H_i$, then we can calculate the subsequent evolution of the mosaic.

Similar techniques are used in many applications where the long-range physics likewise cannot be determined from first principles. A simple example is the distribution of partons within colliding nuclei [128, 139]. Since this distribution depends on the unknown details of confinement in QCD it cannot be calculated from first principles. However, its evolution with energy can be calculated at energies large enough that the perturbative regime of QCD is a good description. In the inflationary case, it is the regime of small $H/M_P$ which is accessible to perturbative calculations. A simple method of calculating this evolution was suggested by Starobinsky [140] and later applied to the calculation of correlation functions in the deep infrared by Starobinsky and Yokoyama [141].

In Starobinsky’s method, we suppose that the background field value within a given box of the mosaic is known. Our discussion of time evolution in sections 2.1 and 3 shows that we will obtain reasonable answers only if we account for the slow roll-down of this field. In a single-field scenario, the evolution of the background field value is given by

$$\dot{\phi} = -\frac{V'}{3H} + \frac{H^{3/2}}{2\pi} \theta(t),$$

(38)

where $V(\phi)$ is the potential and $\theta(t)$ is a Gaussian noise term satisfying

$$\langle \theta(t) \theta(t') \rangle = \delta(t-t').$$

(39)

In a careful treatment, we would find that this noise term depends on what we assume about the physics responsible for generating correlations within this box. Equation (38) is a Langevin equation, which can be equivalently expressed as an evolution equation for the distribution function of field values over the mosaic, $f(\phi)$,

$$\frac{\partial f}{\partial t} = \frac{1}{3H} \frac{\partial (fV')}{\partial \phi} + \frac{H^3}{8\pi^2} \frac{\partial^2 f}{\partial \phi^2}$$

(40)

Various refinements of this equation are possible, which were discussed by Salopek and Bond [130].

It has been suggested that stochastic inflation may represent one possible method of determining the distribution of field values over the entire mosaic of boxes [84, 85, 128]. Although this idea is attractive, there are several drawbacks. First, equations (38)–(40) depend on an initial condition for the distribution $f(\phi)$. This initial condition is sensitive to the details of ultraviolet physics and is no more calculable from first principles than is $f(\phi)$ itself. This difficulty is usually circumvented by seeking stationary solutions of the Fokker–Planck equation equation (40). However, it is not clear under which circumstances this is a reasonable choice, or whether the result genuinely approximates what would be obtained if we were to retain the complete ultraviolet physics.

Second, the evolution equations (38) and (40) depend on what we assume about physics at high values of the Hubble parameter, up to $H \sim M_P$. In particular, these equations assume that slow-roll inflation is a good description throughout this regime. This is analogous to the supposition that the standard model is a good description of physics at energies from the electroweak frontier at $\sim 1$ TeV up to the GUT scale at $\sim 10^{16}$ GeV. This assumption allows renormalization group evolution of the $SU(2)$, $U(1)_{Y}$ and strong coupling constants to predict a unification near the GUT scale in the same way that equation (40) allows us to predict the behaviour of $\phi$ at values of $H$ far above those which gave rise to observable density fluctuations. In either case, it is perfectly reasonable that unguessed physics intervenes to spoil the conclusion. If so, the true evolution of the background fields might follow laws
very different to equations (38) and (40). Alternatively, at sufficiently high values of $H$, the concept of a homogeneous background field itself may lose meaning. In that case the present discussion would be invalidated, becoming relevant only at lower energies.

It was emphasized above that, assuming only zero modes need be retained from the quasi-classical physics on large scales, cancellation of $L$ was automatic after averaging over the distribution of field values within the mosaic. Since there is no recipe to construct this distribution, it must be checked individually for each candidate. Unfortunately, it is non-trivial to verify that the $L$-dependence carried by a solution to equation (40) would precisely cancel that $L$-dependence of correlations computed within a box $L$. At the time of writing, this does not appear to have been verified. A complete cancellation would depend on retaining sufficient information about the relevant long-range physics in equations (38)–(40), and it is far from obvious that this is the case. However, since the same physics which gives rise to the box-cutoff logarithms is used to compute the evolution of $\phi$ in equation (38), it is at least plausible that the $L$ dependence of $f$ may have the requisite properties. A similar property is exhibited by the parton distribution functions which describe the physics internal to colliding nuclei [139]. A first step in this direction was given by Kuhnel and Schwarz [142] who argued that large-scale correlations were strongly suppressed by stochastic effects, leading to infrared finite correlation functions.

4.4. Dynamically generated masses

In sections 2.1 and 3 we discussed the role of an evolving background field in suppressing correlations at late times, and in section 3.3 we reviewed the argument of Burgess et al [76], according to which the secular logarithms could be resummed into a dynamically generated mass. This mass drives any correlations to zero in the far future, as the field settles in its minimum and loses memory of its past history.

Equation (38) and its associated Fokker–Planck equation, equation (40), account for both fluctuations and time evolution in the background field. A fluctuation $\theta$ in equation (38) can push the field up the potential. At later times, the term $-V'/3H$ causes it to roll towards the minimum. Therefore, we would not expect statistical properties computed using the stochastic framework to suffer from the pathological failure to decay associated with correlations established using equation (24). This expectation is borne out in explicit calculations.

Starobinsky and Yokoyama worked with a model of $\lambda \phi^4$ inflation, defined by the interaction of equation (22), and calculated the evolution of correlations in this theory using the stochastic method [141]. Working in the massless case for simplicity, they were able to use the Fokker–Planck equation (40) to derive an evolution equation for the 2-point function $\langle \phi^2 \rangle$:

$$\frac{\partial}{\partial t} \langle \phi^2 \rangle = \frac{H^3}{4\pi^2} - \frac{\lambda}{9H} \langle \phi^2 \rangle^2.$$  (41)

Starobinsky and Yokoyama observed that the solution of this equation gave a constant

$$\langle \phi^2 \rangle = \frac{3}{2\pi} \frac{H^2}{\sqrt{\lambda}},$$  (42)

at late times, and remarked that this was equivalent to the dynamical generation of a mass of order $H$. We briefly review this conclusion using the analysis of Riotto and Sloth [102], who derived a ‘gap’ equation for the $\phi$ propagator $G$:

$$\left( \square + \frac{\lambda}{6} \langle \phi^2 \rangle + G(x, x') \right) G(x, x') = \frac{i}{\sqrt{-g}} \delta(x - x').$$  (43)
where the differential operator $\Box = g^{ab}\nabla_a \nabla_b$ operates on the variable $x$, and $\nabla_a$ is a covariant derivative compatible with the metric $g_{ab}$. Equation (43) is obtained using the method of the two-particle irreducible action. For details the reader may consult the original article by Riotto and Sloth [102], or the textbook by Calzetta and Hu [143]. After translating to Fourier space and substituting Starobinsky and Yokoyama’s late-time solution for $\langle \phi^2 \rangle$, given by equation (42), in equation (43) one finds that the Green’s function $G(x, x')$ should be built out of solutions to the homogeneous equation

$$\ddot{G} + 3H \dot{G} + \left( \frac{k^2}{a^2} + \frac{\sqrt{\lambda}}{4\pi} H^2 \right) G = 0.$$  

(44)

This is the Klein–Gordon equation for a field of mass

$$M_{\text{eff}}^2 = \frac{\sqrt{\lambda}}{4\pi} H^2.$$  

(45)

Later, Burgess et al [76] argued that this mass could be reproduced from the dynamical mass generated by the renormalization group, equation (33), at least in the large-$N$ limit of an $O(N)$-symmetric theory similar to that studied by Riotto and Sloth [102] and Petri [101].

### 4.5. Tensor-to-scalar ratio

If an exact shift symmetry protects the potential of some field, then it experiences no classical evolution. An example might be a Goldstone boson or axion, or more generally any isocurvature field which can be interpreted as an angle. Once perturbations have been generated in such a field, they do not decay\(^{15}\). In this case we would not expect resummation to generate a mass, and there is no prediction for the value taken by such a field within our presently observable patch, although it may be possible to estimate a distribution [83, 86]. Hertzberg, Tegmark and Wilczek [144] argued that such scenarios represented a test case of anthropic reasoning.

What would happen if a microwave background observable depended on the value of such a field? We would not be able to predict which measurements should be recorded by CMB survey satellites, such as the WMAP and Planck satellite missions. A similar argument was given by Sloth, who pointed out that an observable such as the tensor fraction, $r$, may depend on a ratio of two quantities which receive loop corrections scaling differently as the box size increases [46, 47]. The expected value of $r$ will depend on the box in which correlations are computed. However, as has been stressed throughout, to obtain observables one must compute a conditional expectation value, contingent on inflation ending in our own vacuum (almost surely) everywhere within the box. For this reason, the observational relevance of infrared corrections in $r$ is not yet clear.

### 5. Discussion

Infrared effects have attracted considerable interest, but at the time of writing it has not been possible to extract a definitive prediction of a novel effect from these calculations—that is,\(^{15}\) during inflation, quantum fluctuations will completely randomize the value of such a field. Nevertheless, the existence of a shift symmetry means the field value is unobservable and there seem no unacceptable consequences associated with this behaviour. If the field is massless but interacting—such as at a critical point—then the field value is in principle observable. (This example was suggested by Clifford Burgess and Louis Leblond.) For sufficiently strong interactions the field presumably rolls down the resulting potential at late times, as described by Senatore and Zaldarriaga [61], preventing an unphysical divergence to arbitrarily large field values. However, it is not known what conditions must be satisfied by the interactions for divergences to be absent.
an effect which is not already implicit in calculations carried out locally around background fields set at a point on the scalar potential.

The best studied examples of infrared effects are the time-dependent secular logarithms introduced in section 2.1. Over short timescales these logarithms represent evolution of correlations, as the background fields slowly roll down their potentials. At later times quantum effects can become important, but their role remains somewhat unclear. The calculation of Burgess et al., supported by the arguments of Senatore and Zaldarriaga, suggests that in stable theories they will suppress correlations at late times.

Box-cutoff logarithms represent a different problem: the possibility that, after many e-folds of inflation, spatial gradients develop in the background scalar fields. If we only wish to use our theories of inflation to study observable correlations generated on approach to reheating, then this may not be a significant problem. On the other hand, if we wish to study the evolution of a large spatial volume undergoing inflation for a long period of time, we expose ourselves to the possibility of unknown new physics at large values of $H/M_P$.

The gauge-invariant arguments of Losic and Unruh appear to cast doubt on the idea that perturbation theory remains globally well defined for an indefinite period in de Sitter or nearly de Sitter spacetimes.

Except in special cases, such as single-field inflation, the study of correlations generated near exit from inflation is still ultraviolet-sensitive (in the sense of physics at large values of $H/M_P$) and infrared-sensitive (in the sense of physics on very large length scales), because we do not know in which reheating minimum inflation is likely to terminate. This need not be a problem if observation can accurately determine the properties of our local minimum. In that case, we would be able to understand all the properties of our vacuum, but we would not be able to understand how to embed this minimum in the larger landscape of the inflationary potential. However, this problem is not new and has been with us since the early days of the inflationary paradigm.

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