Analysis and geometry
on $\mathbb{R}_+$-marked configuration space

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Abstract

We carry out analysis and geometry on a marked configuration space $\Omega_{\mathbb{R}^+}$ over a Riemannian manifold $X$ with marks from the space $\mathbb{R}^+$ as a natural generalization of the work [J. Func. Anal. 154 (1998), 444–500]. As a transformation group $\mathfrak{G}$ on $\Omega_{\mathbb{R}^+}$ we take the “lifting” to $\Omega_{\mathbb{R}^+}$ of the action on $X \times \mathbb{R}^+$ of the semidirect product of the group $\text{Diff}_0(X)$ of diffeomorphisms on $X$ with compact support and the group $\mathbb{R}^X_+$ of smooth currents, i.e., all $C^\infty$ mappings of $X$ into $\mathbb{R}^+$ which are equal to one outside a compact set. The marked Poisson measure $\pi_{\tilde{\sigma}}$ on $\Omega_{\mathbb{R}^+}$ with Lévy measure $\tilde{\sigma}$ on $X \times \mathbb{R}^+$ is proven to be quasiinvariant under the action of $\mathfrak{G}$. Then, we derive a geometry on $\Omega_{\mathbb{R}^+}$ by a natural “lifting” of the corresponding geometry on $X \times \mathbb{R}^+$. In particular, we construct a gradient $\nabla^\Omega$ and divergence $\text{div}^\Omega$. The associated volume elements, i.e., all probability measures $\mu$ on $\Omega_{\mathbb{R}^+}$ with respect to which $\nabla^\Omega$ and $\text{div}^\Omega$ become dual operators on $L^2(\Omega_{\mathbb{R}^+}; \mu)$ are identified as the mixed Poisson measures with mean measure equal to a multiple of $\tilde{\sigma}$. As a direct consequence of our results, we obtain marked Poisson space representations of the group $\mathfrak{G}$ and its Lie algebra $\mathfrak{g}$. We investigate also Dirichlet forms and Dirichlet operators connected with (mixed) marked Poisson measures. In particular, we obtain conditions of ergodicity of the semigroups generated by the Dirichlet operators. A possible generalization of the results of the paper to the case where the marks belong to a homogeneous space of a Lie group is noted.
## Contents

0 Introduction .................................................. 3

1 Marked Poisson measures ..................... 6
   1.1 Marked configuration space .................. 6
   1.2 Marked Poisson measure .................... 8

2 Transformations of the marked Poisson measure .... 11
   2.1 Group of transformations of the marked configuration space .... 11
   2.2 $\mathcal{G}$-quasiinvariance of the marked Poisson measure ........ 12
   2.3 Transformations of compound Poisson space and quasiinvariance of compound Poisson measures .................. 14

3 The differential geometry of marked configuration space .... 15
   3.1 The tangent bundle of $\Omega_{X}^{\mathbb{R}^{+}}$ .................. 15
   3.2 Integration by parts and divergence on the marked Poisson space .... 19
   3.3 Integration by parts characterization .................... 23
   3.4 A lifting of the geometry ......................... 25
   3.5 Representations of the Lie algebra of the group $\mathcal{G}$ ........ 26

4 Intrinsic Dirichlet forms on marked Poisson spaces ...... 29
   4.1 Definition of the intrinsic Dirichlet form .................. 29
   4.2 Intrinsic Dirichlet operators .................... 31
   4.3 The heat semigroup and ergodicity .................... 35

5 Intrinsic Dirichlet operator and second quantization ...... 36
   5.1 Marked Poisson gradient and chaos decomposition ............ 36
   5.2 Second quantization on the marked Poisson space ............ 39
   5.3 The intrinsic Dirichlet operator as a second quantization .... 40

6 Appendix .................................................. 41
0 Introduction

In recent few years, stochastic analysis and differential geometry on configuration spaces have been considerably developed in a series of papers by S. Albeverio et al. [2–5], see also [26]. It has been shown that the geometry of the configuration space \( \Gamma_X \) over a Riemannian manifold \( X \) can be constructed via a simple “lifting procedure” and is completely determined by the Riemannian structure of \( X \). The mixed Poisson measures appeared to be exactly “volume elements” corresponding to the differential geometry introduced on \( \Gamma_X \). Intrinsic Dirichlet forms and operators, their canonical processes, Gibbs measures on configuration spaces, integration by parts characterization of canonical Gibbs measures, stochastic dynamics corresponding to Gibbs measures as well as many other problems were treated in the above framework.

A starting point for this analysis, more exactly, for the definition of differentiation on the configuration space, were the representation of the group of diffeomorphisms \( \text{Diff}_0(X) \) on \( X \) with compact support that was constructed by A. M. Vershik et al. [31] (see also [25, 27, 12]) and the fact, following from the Skorokhod theorem, that the Poisson measure is quasiinvariant with respect to the group \( \text{Diff}_0(X) \).

On the other hand, starting with the same work [31], many researchers consider representations also on marked (or compound) Poisson spaces. At the same time, in statistical mechanics of continuous systems, marked Poisson measures and their Gibbsian perturbations are used for the description of many concrete models, see e.g. [1]. Hence, it is natural to ask about geometry and analysis on these spaces. The first work in this direction was the paper [19], in which, just as in the case of the usual Poisson measure, the action of the group \( \text{Diff}_0(X) \) was used for the definition of the differentiation. However, this group proved to be too small for reconstructing mixed marked Poisson measures as “volume elements,” which means that \( \text{Diff}_0(X) \) is to be extended in a proper way.

Let us recall that the configuration space \( \Gamma_X \) is defined as

\[
\Gamma_X := \{ \gamma \subset X \mid \#(\gamma \cap K) < \infty \text{ for each compact } K \subset X \},
\]

where \( \#(\cdot) \) denotes the cardinality of a set. Then, the marked configuration space \( \Omega_X^M \) over \( X \) with marks from, generally speaking, a manifold \( M \) is defined as

\[
\Omega_X^M := \{ (\gamma, m) \mid \gamma \in \Gamma_X, m \in M^\gamma \},
\]

where \( M^\gamma \) stands for the set of all maps \( \gamma \ni x \mapsto m_x \in M \). Let \( \bar{\sigma} \) be a Radon measure on \( X \times M \) such that \( \bar{\sigma}(K \times M) < \infty \) for each compact \( K \subset X \) and \( \bar{\sigma} \) is nonatomic in \( X \), i.e., \( \bar{\sigma}(\{x\} \times M) = 0 \) for each \( x \in X \). Then, one can define on \( \Omega_X^M \) a marked Poisson measure \( \pi_{\bar{\sigma}} \) with Lévy measure \( \bar{\sigma} \).

Of course, one could consider \( \pi_{\bar{\sigma}} \) as a usual Poisson measure on the configuration space \( \Gamma_{X \times M} \) over the Cartesian product of the underlying manifold \( X \) and the space
of marks $M$, and study the properties of this measure using the results of [2–5]. However, such an approach does not distinguish between the two different natures of $X$ and $M$ and the different roles that these play in physics. Thus, our aim is to introduce and study such transformations of the marked configuration space which do “feel” this difference and lead to an appropriate stochastic analysis and differential geometry.

In this paper, we will be concerned with the model case where the space of marks $M$ is just $\mathbb{R}_+$. In our forthcoming papers [16, 17], we will generalize our results to the case where $M$ is a homogeneous space of a Lie group and the marked Poisson measures are replaced with Gibbs measures of Ruelle-type on $\Omega_{\mathbb{R}_+}^X$ (compare with [5, 21]).

Let $\mathbb{R}_+^X$ denote the group of smooth currents, i.e., all $C^\infty$ mappings $X \ni x \mapsto \theta(x) \in \mathbb{R}_+$ which are equal to one outside a compact set (depending on $\theta$). We define the group $\mathfrak{G}$ as the semidirect product of the groups $\text{Diff}_0(X)$ and $\mathbb{R}_+^X$: for $g_1 = (\psi_1, \theta_1)$ and $g_2 = (\psi_2, \theta_2)$, where $\psi_1, \psi_2 \in \text{Diff}_0(X)$ and $\theta_1, \theta_2 \in \mathbb{R}_+^X$, the multiplication of $g_1$ and $g_2$ is given by

$$g_1 g_2 = (\psi_1 \circ \psi_2, \theta_1(\theta_2 \circ \psi_1^{-1})).$$

The group $\mathfrak{G}$ acts in $X \times \mathbb{R}_+$ as follows: for any $g = (\psi, \theta) \in \mathfrak{G}$

$$X \times \mathbb{R}_+ \ni (x, s) \mapsto g(x, s) = (\psi(x), \theta(\psi(x))s) \in X \times \mathbb{R}_+.$$ 

Since each $\omega \in \Omega_{\mathbb{R}_+}^X$ can be interpreted as a subset of $X \times \mathbb{R}_+$, the action of $\mathfrak{G}$ can be lifted to an action on $\Omega_{\mathbb{R}_+}^X$. The marked Poisson measure $\pi_\sigma$ is proven to be quasiinvariant under it. Thus, we can easily construct, in particular, a representation of $\mathfrak{G}$ in $L^2(\pi_\sigma)$.

We note that the groups of smooth (as well as measurable and continuous) currents are classical objects in representation theory, see e.g. [30, 8, 9, 32, 12] and references therein for different representations of these groups. It should be stressed, however, that our representation of $\mathfrak{G}$ is reducible, because so is the regular representation of $\mathfrak{G}$ in $L^2(\pi_\sigma)$ (see subsec. 3.5 for details).

Thus, having introduced the action of the group $\mathfrak{G}$ on $\Omega_{\mathbb{R}_+}^X$, we proceed to derive analysis and geometry on $\Omega_{\mathbb{R}_+}^X$ in a way parallel to the work [4], dealing with the usual configuration space $\Gamma_X$. In particular, we note that the Lie algebra $\mathfrak{g}$ of the group $\mathfrak{G}$ is given by $\mathfrak{g} = V_0(X) \times C_0^\infty(X)$, where $V_0(X)$ is the algebra of $C^\infty$ vector fields on $X$ having compact support and $C_0^\infty(X)$ is the algebra of $C^\infty$ functions from $X$ into $\mathbb{R}$ with compact support. For each $(v, a) \in \mathfrak{g}$, we define the notion of a directional derivative of a function $F: \Omega_{\mathbb{R}_+}^X \rightarrow \mathbb{R}$ along $(v, a)$, which is denoted by $\nabla_{(v,a)}^\Omega F$. We obtain an explicit form of this derivative on a special set $\mathcal{FC}_0^\infty(\mathfrak{D}, \Omega_{\mathbb{R}_+}^X)$ of smooth cylinder functions on $\Omega_{\mathbb{R}_+}^X$, which, in turn, motivates our definition of a tangent
bundle $T(\Omega^R_X)$ of $\Omega^R_X$, and of a gradient $\nabla^\Omega F$. We note only that the tangent space $T_\omega(\Omega^R_X)$ to the marked configuration space $\Omega^R_X$ at a point $\omega = (\gamma, s) \in \Omega^R_X$ is given by

$$T_\omega(\Omega^R_X) := L^2(X \to T(X) + \mathbb{R}; \gamma).$$

Next, we derive an integration by parts formula on $\Omega^R_X$, that is, we get an explicit formula for the dual operator $\text{div}^\Omega$ of the gradient $\nabla^\Omega$ on $\Omega^R_X$. We prove that the probability measures on $\Omega^R_X$ for which $\nabla^\Omega$ and $\text{div}^\Omega$ become dual operators (with respect to $\langle \cdot, \cdot \rangle_{T(\Omega^R_X)}$) are exactly the mixed marked Poisson measures

$$\mu_{\nu, \tilde{\sigma}} = \int_{\mathbb{R}^+} \pi_{z\tilde{\sigma}} \nu(dz),$$

where $\nu$ is a probability measure on $\mathbb{R}^+$ (with finite first moment) and $\pi_{z\tilde{\sigma}}$ is the marked Poisson measure on $\Omega^R_X$ with Lévy measure $z\tilde{\sigma}$, $z \geq 0$. This means that the mixed marked Poisson measures are exactly the “volume elements” corresponding to our differential geometry on $\Omega^R_X$.

Thus, having identified the right volume elements on $\Omega^R_X$, we introduce for each measure $\mu_{\nu, \tilde{\sigma}}$ the first order Sobolev space $H^{1,2}_0(\Omega^R_X, \mu_{\nu, \tilde{\sigma}})$ by closing the corresponding Dirichlet form

$$\mathcal{E}^{\Omega}_{\mu_{\nu, \tilde{\sigma}}}(F, G) = \int_{\Omega^R_X} \langle \nabla^\Omega F, \nabla^\Omega G \rangle_{T(\Omega^R_X)} d\pi_{\nu, \tilde{\sigma}}, \quad F, G \in FC^\infty(\mathcal{D}, \Omega^R_X),$$

on $L^2(\Omega^R_X, \mu_{\nu, \tilde{\sigma}})$. Just as in the analysis on the usual configuration space, this is the step where we really start doing generic infinite dimensional analysis. The corresponding Dirichlet operator is denoted by $H^{\Omega}_{\mu_{\nu, \tilde{\sigma}}}$; it is a positive definite self-adjoint operator on $L^2(\Omega^R_X, \mu_{\nu, \tilde{\sigma}})$. The heat semigroup $(\exp(-tH^{\Omega}_{\mu_{\nu, \tilde{\sigma}}}))_{t \geq 0}$ generated by it is calculated explicitly. The results on the ergodicity of this semigroup are absolutely analogous to the corresponding results of [4]. Particularly, we have ergodicity if and only if $\mu_{\nu, \tilde{\sigma}} = \pi_{z\tilde{\sigma}}$ for some $z > 0$, i.e., $\mu_{\nu, \tilde{\sigma}}$ is a (pure) marked Poisson measure.

We also clarify the relation between the intrinsic geometry on $\Omega^R_X$ we have constructed with another kind of extrinsic geometry on $\Omega^R_X$ which is based on fixing the marked Poisson measure $\pi_{\tilde{\sigma}}$ and considering the unitary isomorphism between $L^2(\Omega^R_X, \pi_{\tilde{\sigma}})$ and the corresponding Fock space

$$\mathcal{F}(L^2(X \times \mathbb{R}^+_\sigma; \tilde{\sigma})) = \bigoplus_{n=0}^{\infty} \hat{L}^2((X \times \mathbb{R}^+)^n, n! \tilde{\sigma}^\otimes n),$$

where $\hat{L}^2((X \times \mathbb{R}^+)^n, n! \tilde{\sigma}^\otimes n)$ is the subspace of symmetric functions from $L^2((X \times \mathbb{R}^+)^n, n! \tilde{\sigma}^\otimes n)$. Our main result here is to prove that $H^{\Omega}_{\pi_{\tilde{\sigma}}}$ is unitarily equivalent (under the above isomorphism) to the second quantization operator of the Dirichlet operator $H^{X \times \mathbb{R}^+}_{\tilde{\sigma}}$ on the $L^2(X \times \mathbb{R}^+; \tilde{\sigma})$ space.
As a consequence of the results of this paper, we obtain a representation on the marked Poisson space \( L^2(\pi_\tilde{\sigma}) \) not only of the group \( \mathfrak{g} \), but also of its Lie algebra \( \mathfrak{g} \).

Finally, we note that one can construct in a natural way a bijection between the marked configuration space \( \Omega^\mathbb{R}_+ X \) and the “compound configuration” space \( \Omega_X \) over \( X \) defined as
\[
\Omega_X = \{ v(\cdot) = \sum_{x \in \gamma} s_x \varepsilon_x(\cdot) \mid \gamma \in \Gamma_X, s_x \in \mathbb{R}_+ \}.
\]
The image of the marked Poisson measure under this bijection is the compound Poisson measure on \( \Omega_X \). So, all results of this paper can be easily reformulated in terms of compound Poisson measures.

1 Marked Poisson measures

1.1 Marked configuration space

Let \( X \) be a connected, oriented \( C^\infty \) (non-compact) Riemannian manifold. The configuration space \( \Gamma_X \) over \( X \) is defined as the set of all locally finite subsets in \( X \):
\[
\Gamma_X := \{ \gamma \subset X \mid \#(\gamma \cap K) < \infty \text{ for each compact } K \subset X \},
\]
where \( \#(\cdot) \) denotes the cardinality of a set. One can identify any \( \gamma \in \Gamma_X \) with the positive integer-valued Radon measure
\[
\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}_p(X) \subset \mathcal{M}(X),
\]
where \( \sum_{x \in \emptyset} \varepsilon_x := \text{zero measure} \) and \( \mathcal{M}(X) \) (resp. \( \mathcal{M}_p(X) \)) denotes the set of all positive (resp. positive integer-valued) Radon measures on \( \mathcal{B}(X) \).

Let \( \mathbb{R}_+ := (0, +\infty) \). The marked configuration space \( \Omega^\mathbb{R}_+ X \) over \( X \) with marks from \( \mathbb{R}_+ \) is defined as
\[
\Omega^\mathbb{R}_+ X := \{ \omega = (\gamma, s) \mid \gamma \in \Gamma_X, s \in \mathbb{R}_+^\gamma \},
\]
where \( \mathbb{R}_+^\gamma \) stands for the set of all maps \( \gamma \ni x \mapsto s_x \in \mathbb{R}_+ \). Equivalently, we can define \( \Omega^\mathbb{R}_+ X \) as the collection of locally finite subsets in \( X \times \mathbb{R}_+ \) having the following properties:
\[
\Omega^\mathbb{R}_+ X = \left\{ \omega \subset X \times \mathbb{R}_+ \bigg| \begin{array}{l}
\text{a) } \forall (x, s), (x', s') \in \omega : (x, s) \neq (x', s') \Rightarrow x \neq x' \\
\text{b) } \Pr_X \omega \in \Gamma_X
\end{array} \right\},
\]
where \( \Pr_X \) denotes the projection of the Cartesian product of \( X \) and \( \mathbb{R}_+ \) onto \( X \). Again, each \( \omega \in \Omega^\mathbb{R}_+ X \) can be identified with the measure
\[
\sum_{(x, s) \in \omega} \varepsilon_{(x, s)} \in \mathcal{M}_p(X \times \mathbb{R}_+) \subset \mathcal{M}(X \times \mathbb{R}_+).
\]
It is worth noting that, for any bijection \( \phi: X \times \mathbb{R}_+ \to X \times \mathbb{R}_+ \), the image of the measure \( \omega(\cdot) \) under the mapping \( \phi \), \( (\phi^* \omega)(\cdot) \), coincides with \( (\phi(\omega))(\cdot) \), i.e.,
\[
(\phi^* \omega)(\cdot) = (\phi(\omega))(\cdot), \quad \omega \in \Omega_X^{\mathbb{R}_+},
\]
where \( \phi(\omega) = \{ \phi(x, s) \mid (x, s) \in \omega \} \) is the image of \( \omega \) as a subset of \( X \times \mathbb{R}_+ \).

Let \( \mathcal{B}_c(X) \) and \( \mathcal{O}_c(X) \) denote the families of all Borel, resp. open subsets of \( X \) that have compact closure. Let also \( \mathcal{B}_c(X \times \mathbb{R}_+) \) denote the family of all Borel subsets of \( X \times \mathbb{R}_+ \) whose projection on \( X \) belongs to \( \mathcal{B}_c(X) \).

Denote by \( C_{0,b}(X \times \mathbb{R}_+) \) the set of real-valued bounded continuous functions \( f \) on \( X \times \mathbb{R}_+ \) such that \( \text{supp} f \in \mathcal{B}_c(X \times \mathbb{R}_+) \). As usually, we set for any \( f \in C_{0,b}(X \times \mathbb{R}_+) \) and \( \omega \in \Omega_X^{\mathbb{R}_+} \)
\[
\langle f, \omega \rangle = \int_{X \times \mathbb{R}_+} f(x, s) \omega(dx, ds) = \sum_{(x,s) \in \omega} f(x,s).
\]
Notice that, because of the definition of \( \Omega_X^{\mathbb{R}_+} \), there are only a finite number of addends in the latter series.

Now, we are going to discuss the measurable structure of the space \( \Omega_X^{\mathbb{R}_+} \). We will use a “localized” description of the Borel \( \sigma \)-algebra \( \mathcal{B}(\Omega_X^{\mathbb{R}_+}) \) over \( \Omega_X^{\mathbb{R}_+} \).

For \( \Lambda \in \mathcal{O}_c(X) \), define
\[
\Omega^{\mathbb{R}_+}_\Lambda := \{ \omega \in \Omega_X^{\mathbb{R}_+} \mid \text{Pr}_X \omega \subset \Lambda \}
\]
and for \( n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \)
\[
\Omega^{\mathbb{R}_+}_\Lambda(n) := \{ \omega \in \Omega^{\mathbb{R}_+}_\Lambda \mid \#(\omega) = n \}.
\]
It is obvious that
\[
\Omega^{\mathbb{R}_+}_\Lambda = \bigsqcup_{n=0}^{\infty} \Omega^{\mathbb{R}_+}_\Lambda(n).
\]
Let
\[
\Lambda_{mk} := \Lambda \times \mathbb{R}_+
\]
(i.e., \( \Lambda_{mk} \) is the set of all “marked” elements of \( \Lambda \)) and let
\[
\tilde{\Lambda}^{n}_{mk} := \{(x_1, s_1), \ldots, (x_n, s_n) \} \in \Lambda_{mk}^n \mid x_j \neq x_k \text{ if } j \neq k \}.
\]
There is a bijection
\[
L^{(n)}_\Lambda: \tilde{\Lambda}^{n}_{mk}/\mathcal{S}_n \mapsto \Omega^{\mathbb{R}_+}_\Lambda(n)
\]
given by
\[
L^{(n)}_\Lambda: ((x_1, s_1), \ldots, (x_n, s_n)) \mapsto \{(x_1, s_1), \ldots, (x_n, s_n)\} \in \Omega^{\mathbb{R}_+}_\Lambda(n),
\]
where $\mathcal{S}_n$ is the permutation group over $\{1, \ldots, n\}$. On $\Lambda_{mk}^n / \mathcal{S}_n$ one introduces the related metric

$$\delta[(x_1, s_1), \ldots, (x_n, s_n); (x'_1, s'_1), \ldots, (x'_n, s'_n)] = \inf_{\pi \in \mathcal{S}_n} d^n[((x_1, s_1), \ldots, (x_n, s_n)); ((x'_{\pi(1)}, s'_{\pi(1)}), \ldots, (x'_{\pi(n)}, s'_{\pi(n)}))],$$

where $d^n$ is the metric on $\Lambda_{mk}^n$ driven from the original metrics on $X$ and $\mathbb{R}_+$. Then, $\tilde{\Lambda}_{mk}^n / \mathcal{S}_n$ becomes an open set in $\Lambda_{mk}^n / \mathcal{S}_n$ and let $\tilde{B}(\Lambda_{mk}^n / \mathcal{S}_n)$ be the trace $\sigma$-algebra on $\tilde{\Lambda}_{mk}^n / \mathcal{S}_n$ generated by $\mathcal{B}(\Lambda_{mk}^n / \mathcal{S}_n)$. Let then $\mathcal{B}(\Omega_{X^+}^n (n))$ be the image $\sigma$-algebra of $\mathcal{B}(\Lambda_{mk}^n / \mathcal{S}_n)$ under the bijection $L_{\Lambda}^{(n)}$ and let $\mathcal{B}(\Omega_{X^+}^n)$ be the $\sigma$-algebra on $\Omega_{X^+}^n$ generated by the usual topology of (disjoint) union of topological spaces.

For any $\Lambda \in \mathcal{O}_c(X)$, there is a natural restriction map

$$p_{\Lambda} : \Omega_{X^+}^n \mapsto \Omega_{\Lambda}^n,$$

defined by

$$\Omega_{X^+}^n \ni \omega \mapsto p_{\Lambda}(\omega) := \omega \cap \Lambda_{mk} \in \Omega_{\Lambda}^n.$$

The topology on $\Omega_{X^+}^n$ is defined as the weakest topology making all the mappings $p_{\Lambda}$ continuous. The associated $\sigma$-algebra is denoted by $\mathcal{B}(\Omega_{X^+}^n)$.

For each $B \in \mathcal{B}_c(X \times \mathbb{R}_+)$, we introduce a function $N_B : \Omega_{X^+}^n \rightarrow \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ such that

$$N_B(\omega) := #(\omega \cap B), \quad \omega \in \Omega_{X^+}^n. \quad (1.2)$$

Then, it is not hard to see that $\mathcal{B}(\Omega_{X^+}^n)$ is the smallest $\sigma$-algebra on $\Omega_{X^+}^n$ such that all the functions $N_B$ are measurable.

### 1.2 Marked Poisson measure

In order to construct a marked Poisson measure, we fix:

(i) an intensity measure $\sigma$ on the underlying manifold $X$, which is supposed to be a nonatomic Radon one,

(ii) a non-negative function

$$X \times \mathcal{B}(\mathbb{R}_+) \ni (x, \Delta) \mapsto p(x, \Delta) \in \mathbb{R}_+$$

such that, for $\sigma$-a.a. $x \in X$, $p(x, \cdot)$ is a finite Radon measure on $\mathbb{R}_+$.

Now, we define a measure $\tilde{\sigma}$ on $(X \times \mathbb{R}_+, \mathcal{B}(X \times \mathbb{R}_+))$ as follows:

$$\tilde{\sigma}(A) = \int_A p(x, ds) \sigma(dx), \quad A \in \mathcal{B}(X \times \mathbb{R}_+). \quad (1.3)$$
We will suppose that the measure \( \tilde{\sigma} \) is infinite and for any \( \Lambda \in B_c(X) \)

\[
\tilde{\sigma}(\Lambda_{mk}) = \int_X 1_\Lambda(x)p(x, \mathbb{R}_+) \sigma(\text{d}x) < \infty,
\]

(1.4)
i.e., \( p(x, \mathbb{R}_+) \in L^1_{\text{loc}}(\sigma) \).

Now, we wish to introduce a marked Poisson measure on \( \Omega^{R_+}_X \) (cf. e.g. [15, 14]). To this end, we take first the measure \( \tilde{\sigma}^{\otimes n} \) on \( (X \times \mathbb{R}_+)^n \), and for any \( \Lambda \in O_c(X) \), \( \tilde{\sigma}^{\otimes n} \) can be considered as a finite measure on \( \Lambda_{mk}^n \). Since \( \sigma \) is nonatomic, we get \( \tilde{\sigma}^{\otimes 2}(D_\Lambda) = 0 \), where

\[
D_\Lambda = \left\{ ((x_1, s_1), (x_2, s_2)) \in \Lambda_{mk}^2 \mid x_1 = x_2 \right\} = \left\{ (x_1, x_2) \in X^2 \mid x_1 = x_2 \right\} \times \mathbb{R}_+^2.
\]

Therefore,

\[
\tilde{\sigma}^{\otimes n}(\Lambda_{mk}^n \setminus \tilde{\Lambda}_{mk}^n) = 0
\]

and we can consider \( \tilde{\sigma}^{\otimes n} \) as a measure on \( (\tilde{\Lambda}_{mk}^n / \mathcal{G}_n, \mathcal{B}(\tilde{\Lambda}_{mk}^n / \mathcal{G}_n)) \) such that

\[
\tilde{\sigma}^{\otimes n}(\tilde{\Lambda}_{mk}^n / \mathcal{G}_n) = \tilde{\sigma}(\Lambda_{mk})^n.
\]

Denote by \( \tilde{\sigma}_{\Lambda,n} := \tilde{\sigma}^{\otimes n} \circ (\mathcal{L}_\Lambda^{(n)})^{-1} \) the image measure on \( \Omega^{R_+}_X(n) \) under the bijection (1.1). Then, we can define a measure \( \lambda^\Lambda_\tilde{\sigma} \) on \( \Omega^{R_+}_X \) by

\[
\lambda^\Lambda_\tilde{\sigma} := \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\sigma}_{\Lambda,n},
\]

where \( \tilde{\sigma}_{\Lambda,0} := \varepsilon_\emptyset \) on \( \Omega^{R_+}_X(0) = \{ \emptyset \} \). The measure \( \lambda^\Lambda_\tilde{\sigma} \) is finite and \( \lambda^\Lambda_\tilde{\sigma}(\Omega^{R_+}_X) = e^{\tilde{\sigma}(\Lambda_{mk})} \).

Hence, the measure

\[
\pi^\Lambda_{\tilde{\sigma}} := e^{-\tilde{\sigma}(\Lambda_{mk})} \lambda^\Lambda_\tilde{\sigma}
\]

is a probability measure on \( \mathcal{B}(\Omega^{R_+}_X) \). It is not hard to check the consistency property of the family \( \{ \pi^\Lambda_{\tilde{\sigma}} \mid \Lambda \in O_c(X) \} \) and thus to obtain a unique probability measure \( \pi_{\tilde{\sigma}} \) on \( \mathcal{B}(\Omega^{R_+}_X) \) such that

\[
\pi^\Lambda_{\tilde{\sigma}} = p^\Lambda_{\tilde{\sigma}} \pi_{\tilde{\sigma}}, \quad \Lambda \in O_c(X).
\]

This measure \( \pi_{\tilde{\sigma}} \) will be called a marked Poisson measure with Lévy measure \( \tilde{\sigma} \).

For any function \( \varphi \in C_{0,b}(X \times \mathbb{R}_+) \), it is easy to calculate the Laplace transform of the measure \( \pi_{\tilde{\sigma}} \)

\[
\ell_{\pi_{\tilde{\sigma}}} (\varphi) := \int_{\Omega^{R_+}_X} e^{\langle \varphi, \omega \rangle} \pi_{\tilde{\sigma}} (\text{d}\omega) = \exp \left( \int_{X \times \mathbb{R}_+} (e^{\varphi(x,s)} - 1) \tilde{\sigma}(\text{d}x, \text{d}s) \right),
\]

(1.5)
Particularly, if $C_0(X)$ denotes the space of continuous functions on $X$ with compact support, we obtain, for each $u \in C_0(X)$ such that $u(x) \leq 0$ for $\sigma$-a.a. $x \in X$,

$$L_{\tilde{\sigma}}(u) := \int_{\Omega_X} \exp \left[ \langle su, \omega \rangle \right] \pi_{\tilde{\sigma}}(d\omega)$$

$$= \int_{\Omega_X} \exp \left[ \sum_{x \in \gamma} s_x u(x) \right] \pi_{\tilde{\sigma}}(d(\gamma, s))$$

$$= \exp \left( \int_{X \times \mathbb{R}^+} (e^{su(x)} - 1) \tilde{\sigma}(dx, ds) \right).$$

This formula is also sufficient to define $\pi_{\tilde{\sigma}}$.

**Example 1.** Let $p(x, \cdot) \equiv \varepsilon_1(\cdot)$, $x \in X$. Then, $\tilde{\sigma} = \sigma \otimes \varepsilon_1$ and $\pi_{\tilde{\sigma}}$ is just the Poisson measure on $(\Gamma_X, \mathcal{B}(\Gamma_X))$ with intensity $\sigma$.

**Example 2.** Let $p(x, \cdot) \equiv \tau(\cdot)$, $x \in X$, where $\tau$ is a finite measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Now, $\tilde{\sigma} = \hat{\sigma} = \sigma \otimes \tau$ and $\pi_{\tilde{\sigma}}$ coincides with the marked Poisson measure under consideration in [19]. Notice that the choice of $\tilde{\sigma} = \hat{\sigma}$ as a product measure means a position-independent marking, while the choice of a general $\tilde{\sigma}$ of the form (1.3) leads to a positive-depending marking.

In what follows, we will suppose that the measure $\sigma$ is equivalent to the Riemannian volume $m$ on $X$: $\sigma(dx) = \rho(x) m(dx)$ with $\rho > 0$ $m$-a.s., and that for $m$-a.a. $x \in X$ $p(x, \cdot)$ is equivalent to the restriction of the Lebesgue measure to $\mathbb{R}_+$, denoted by $\lambda$:

$$p(x, ds) = p(x, s) \lambda(ds) \quad \text{with} \quad p(x, s) > 0 \lambda\text{-a.a.} \ s \in \mathbb{R}_+.$$  

Thus, the measure $\tilde{\sigma}$ can be written in the form

$$\tilde{\sigma}(dx, ds) = \rho(x)p(x, s) \lambda(ds)m(dx).$$

The condition $\tilde{\sigma}(|\Lambda_{mk}|) < \infty$, $\Lambda \in \mathcal{B}_c(X)$, implies that the function

$$q(x, s) := \rho(x)p(x, s)$$

satisfies

$$q^{1/2} \in L^2_{\text{loc}}(X; m) \otimes L^2(\mathbb{R}_+; \lambda).$$

(1.6)

We will suppose additionally that the following stronger condition is fulfilled:

$$\left( \max\{1, s\} q(x, s) \right)^{1/2} \in L^2_{\text{loc}}(X; m) \otimes L^2(\mathbb{R}_+; \lambda).$$

(1.7)
2 Transformations of the marked Poisson measure

2.1 Group of transformations of the marked configuration space

We are looking for a natural group $G$ of transformations of $\Omega^R_+$ such that

(i) $\tilde{\pi}_\sigma$ is $G$-quasiinvariant;

(ii) $G$ is big enough to reconstruct $\tilde{\pi}_\sigma$ by the Radon–Nikodym density $\frac{dg^*\pi_{\tilde{\sigma}}}{d\pi_{\tilde{\sigma}}}$, where $g$ runs through $G$.

Let us recall that in the work [19] the group $\text{Diff}_0(X)$ was taken as $G$, just in the same way as in the case of the usual Poisson measure [4]. Here, $\text{Diff}_0(X)$ stands for the group of diffeomorphisms of $X$ with compact support, i.e., each $\psi \in \text{Diff}_0(X)$ is a diffeomorphism of $X$ that is equal to the identity outside a compact set (depending on $\psi$). The group $\text{Diff}_0(X)$ satisfies (i). However, unlike the case of the Poisson measure, the condition (ii) is not now satisfied, because, for example, in the case where $\tilde{\sigma} = \sigma \otimes \tau$, there is no information about the measure $\tau$ that is contained in $\frac{d\psi^*\pi_{\tilde{\sigma}}}{d\pi_{\tilde{\sigma}}}$, see [19].

This is why we need a proper extension of the group $\text{Diff}_0(X)$. Let us consider the group of smooth currents, i.e., all $C^\infty$ mappings $X \ni x \mapsto \theta(x) \in \mathbb{R}_+$, which are equal to one outside a compact set (depending on $\theta$). A multiplication $\theta_1 \theta_2$ in this group is defined as the pointwise multiplication of mappings $\theta_1$ and $\theta_2$. In representation theory this group is denoted by $R^X_+$, or $C^\infty_0(X; \mathbb{R}_+)$. The group $\text{Diff}_0(X)$ acts in $R^X_+$ by automorphisms: for each $\psi \in \text{Diff}_0(X)$,

$$R^X_+ \ni \theta \mapsto \alpha(\psi) \theta := \theta \circ \psi^{-1} \in R^X_+.$$ 

Thus, we can endow the Cartesian product of $\text{Diff}_0(X)$ and $R^X_+$ with the following multiplication: for $g_1 = (\psi_1, \theta_1), g_2 = (\psi_2, \theta_2) \in \text{Diff}_0(X) \times R^X_+$

$$g_1 g_2 = (\psi_1 \circ \psi_2, \theta_1 (\theta_2 \circ \psi^{-1}_1))$$

and obtain a semidirect product

$$\text{Diff}_0(X) \times \alpha R^X_+ =: \mathcal{G}$$

of the groups $\text{Diff}_0(X)$ and $R^X_+$. 
The group $G$ acts in $X \times \mathbb{R}_+$ in the following way: for any $g = (\psi, \theta) \in G$

$$X \times \mathbb{R}_+ \ni (x, s) \mapsto g(x, s) = (\psi(x), \theta(\psi(x))s) \in X \times \mathbb{R}_+. \quad (2.1)$$

If $\text{id}$ denotes the identity diffeomorphism of $X$ and $\mathbf{1}$ is the function identically equal to one on $X$, then we will just identify $\psi$ with $(\psi, \mathbf{1})$ and $\theta$ with $(\text{id}, \theta)$. The action (2.1) of an arbitrary $g = (\psi, \theta)$ can be represented now as

$$(x, s) \mapsto g(x, s) = \theta \psi(x, s),$$

where

$$\psi(x, s) = (\psi(x), s), \quad \theta(x, s) = (x, \theta(x)s).$$

For any $g = (\psi, \theta) \in G$, denote $K_g := K_\psi \cup K_\theta$, where $K_\psi$ and $K_\theta$ are the minimal closed sets in $X$ outside of which $\psi = \text{id}$ and $\theta = \mathbf{1}$, respectively. Evidently, $K_g \in \mathcal{B}_c(X)$,

$$g(K_g)_{mk} = (K_g)_{mk},$$

and $g$ is the identity transformation outside $(K_g)_{mk}$.

Noting that

$$g^{-1}(x, s) = (\psi, \theta)^{-1}(x, s) = (\psi^{-1}, \theta^{-1} \circ \psi)(x, s) = (\psi^{-1}(x), \theta^{-1}(x)s),$$

we easily deduce the following

**Proposition 2.1** The measure $\tilde{\sigma}$ is $G$-quasiinvariant and for any $g = (\psi, \theta) \in G$ the Radon–Nikodym density is given by

$$p_{\tilde{\sigma}}(x, s) := \frac{d(g^*\tilde{\sigma})}{d\tilde{\sigma}}(x, s) = \frac{q(\psi^{-1}(x), \theta^{-1}(x)s)}{q(x, s)\theta(x)}J^{\psi}_m(x),$$

if $(x, s) \in \{0 < q(x, s) < \infty\} \cap \{0 < q(\psi^{-1}(x), \theta^{-1}(x)s) < \infty\}$,

$$p_{\tilde{\sigma}}(x, s) = 1, \quad \text{otherwise},$$

where $J^{\psi}_m$ is the Jacobian determinant of $\psi$ (w.r.t. the Riemannian volume $m$).

### 2.2 $G$-quasiinvariance of the marked Poisson measure

Any $g \in G$ defines by (2.1) a transformation of $X \times \mathbb{R}_+$, and, consequently, $g$ has the following “lifting” from $X \times \mathbb{R}_+$ to $\Omega^R_X$:

$$\Omega^R_X \ni \omega \mapsto g(\omega) = \{ g(x, s) \mid (x, s) \in \omega \} \in \Omega^R_X. \quad (2.2)$$

(Note that, for a given $\omega \in \Omega^R_X$, $g(\omega)$ indeed belongs to $\Omega^R_X$ and coincides with $\omega$ for all but a finite number of points.) The mapping (2.2) is obviously measurable and we can define the image $g^*\pi_{\tilde{\sigma}}$ as usually. The following proposition is an analog of a corresponding fact about Poisson measures.
Proposition 2.2 For any $g \in \mathcal{G}$, we have

$$g^*\pi_{\tilde{\sigma}} = \pi_{g^*\tilde{\sigma}}.$$

Proof. The proof is the same as for the usual Poisson measure $\pi_{\sigma}$ with intensity $\sigma$ and $\phi \in \text{Diff}_0(\mathbb{X})$ (e.g., [4]), one has just to calculate the Laplace transform of the measure $g^*\pi_{\tilde{\sigma}}$ for any $f \in \mathcal{C}_{0,b}(\mathbb{X} \times \mathbb{R}_+)$ and to use the formula (1.5). □

Proposition 2.3 The marked Poisson measure $\pi_{\tilde{\sigma}}$ is quasiinvariant w.r.t. the group $\mathcal{G}$, and for any $g \in \mathcal{G}$ we have

$$\frac{d(g^*\pi_{\tilde{\sigma}})}{d\pi_{\tilde{\sigma}}}(\omega) = \prod_{(x,s) \in \omega} p_g^\tilde{\sigma}(x,s). \quad (2.3)$$

Proof. By Proposition 2.1, the measures $g^*\tilde{\sigma}$ and $\tilde{\sigma}$ are equivalent for each $g \in \mathcal{G}$. To apply the Skorokhod theorem on absolute continuity of Poisson measures, we have to check the integrability condition

$$\sqrt{p_g^\tilde{\sigma}} - 1 \in L^2(\mathbb{X} \times \mathbb{R}_+; \tilde{\sigma}),$$

or the more restrictive one

$$p_g^\tilde{\sigma} - 1 \in L^1(\mathbb{X} \times \mathbb{R}_+; \tilde{\sigma})$$

(see, e.g., [28, 29]). But the change of variables formula gives

$$\int_{\mathbb{X} \times \mathbb{R}_+} |p_g^\tilde{\sigma}(x, s) - 1| \tilde{\sigma}(dx, ds) = \int_{(K_g)_{mk}} |p^\tilde{\sigma}_g(x, s) - 1| \tilde{\sigma}(dx, ds) \leq 2\tilde{\sigma}((K_g)_{mk}) < \infty.$$

Hence, $\frac{d(g^*\pi_{\tilde{\sigma}})}{d\pi_{\tilde{\sigma}}}$ is equal to the right hand side of (2.3) multiplied by the factor

$$\exp \left[ \int_{\mathbb{X} \times \mathbb{R}_+} (1 - p_g^\tilde{\sigma}(x, s)) \tilde{\sigma}(dx, ds) \right].$$

Finally, noting that

$$\int_{\mathbb{X} \times \mathbb{R}_+} (1 - p_g^\tilde{\sigma}(x, s)) \tilde{\sigma}(dx, ds) = \int_{(K_g)_{mk}} (1 - p^\tilde{\sigma}_g(x, s)) \tilde{\sigma}(dx, ds) = \tilde{\sigma}((K_g)_{mk}) - \tilde{\sigma}((K_g)_{mk}) = 0,$$

we conclude the proposition. □

Remark 2.1 Notice that only a finite (depending on $\omega$) number of factors in the product on the right hand side of (2.3) are not equal to one.
2.3 Transformations of compound Poisson space and quasi-invariance of compound Poisson measures

Let us recall that the “compound configuration” space $\Omega_X$ over $X$ is defined as follows (see e.g. [20, 19]):

$$\Omega_X := \{ \nu(\cdot) = \sum_{x \in \gamma} s_x \varepsilon_x(\cdot) \mid \gamma \in \Gamma_X, s_x \in \mathbb{R}_+ \}.$$  \hspace{1cm} (2.4)

Notice that, for any $\nu \in \Omega_X$, \{ \begin{align*} \{(x, s_x) \mid x \in \gamma\} &= (\gamma, s) = \omega \in \Omega_X^{\mathbb{R}_+}, \end{align*} \hspace{1cm} (2.5) \end{align*} \}

where the function $s \in \mathbb{R}_+^\gamma$ is defined by $s(x) = s_x$, $x \in \gamma$.

Any $\nu \in \Omega_X$ is a Radon measure on $(X, \mathcal{B}(X))$. Hence, $\Omega_X$ can be equipped with the Borel measurable structure as a subset of $\mathcal{M}(X)$ with the vague topology. Moreover, for any $u \in C_0(X)$, we have

$$\langle u, \nu \rangle = \sum_{x \in \gamma} s_x u(x) = \langle su, \omega \rangle$$

with $\omega$ of the form (2.5). Thus, the measurable space $(\Omega_X, \mathcal{B}(\Omega_X))$ is nothing but the image of the marked configuration space $(\Omega_X^{\mathbb{R}_+}, \mathcal{B}(\Omega_X^{\mathbb{R}_+}))$ under the transformation

$$\Omega_X^{\mathbb{R}_+} \ni \omega \mapsto (I \omega)(\cdot) = \sum_{(x, s_x) \in \omega} s_x \varepsilon_x(\cdot) \in \Omega_X,$$  \hspace{1cm} (2.6)

which determines a one-to-one correspondence between $\Omega_X^{\mathbb{R}_+}$ and $\Omega_X$. This is why the group $\mathfrak{S}$ generates the group $\widehat{\mathfrak{S}} = \{ \hat{g} = IgI^{-1} \mid g \in \mathfrak{S} \}$ of measurable transformations of the “compound configuration” space $\Omega_X$.

For our fixed measure $\tilde{\sigma}$ on $X \times \mathbb{R}_+$, we define a compound Poisson measure $\mu^{\text{CP}}_\tilde{\sigma}$ as the image of the marked Poisson measure $\pi_{\tilde{\sigma}}$ under the transformation (2.6), i.e.,

$$\mu^{\text{CP}}_\tilde{\sigma} := I^* \pi_{\tilde{\sigma}}.$$  \hspace{1cm} (2.7)

This definition generalizes evidently the corresponding one from [20, 19] (compare with Example 2; in particular, $\mu^{\text{CP}}_{\sigma \times \tau}$ coincides with the $\pi_{\tilde{\sigma}}$ from [19]).

Thus, by using the bijection (2.6) and the definition (2.7), all results of this paper can be easily reformulated in terms of the compound Poisson space $(\Omega_X, \mathcal{B}(\Omega_X), \mu^{\text{CP}}_{\tilde{\sigma}})$. For example, we deduce from Proposition 2.3 the $\widehat{\mathfrak{S}}$-quasiinvariance of the compound Poisson measure. Namely, the following statement holds.
Proposition 2.4 The compound Poisson measure $\mu_{\tilde{\sigma}}^{\text{CP}}$ is quasiinvariant with respect to the group $\tilde{\mathcal{G}}$, and for any $\hat{g} \in \tilde{\mathcal{G}}$

$$
\frac{d(\hat{g}^* \mu_{\tilde{\sigma}}^{\text{CP}})}{d\mu_{\tilde{\sigma}}^{\text{CP}}}(v) = \prod_{(x,s) \in \mathcal{I}^{-1}v} p_{\tilde{\sigma}}^\varphi(x, s). \tag{2.8}
$$

Proof. By Proposition 2.3, for any $g \in \mathcal{G}$, $g^* \pi_{\tilde{\sigma}} \sim \pi_{\tilde{\sigma}}$ and the Radon–Nikodym derivative has the form (2.3). Since $\mathcal{I}$ is a bijection, the equality (2.7) yields that, for any $\hat{g} \in \hat{\mathcal{G}}$, $\hat{g}^* \mu_{\tilde{\sigma}}^{\text{CP}} \sim \mu_{\tilde{\sigma}}^{\text{CP}}$, and the value of $\frac{d\hat{g}^* \mu_{\tilde{\sigma}}^{\text{CP}}}{d\mu_{\tilde{\sigma}}^{\text{CP}}}$ at a point $v \in \Omega_X$ coincides with the value of $\frac{dg^* \pi_{\tilde{\sigma}}}{d\pi_{\tilde{\sigma}}}$ at the point $\omega = \mathcal{I}^{-1}v$. ■

Remark 2.2 Setting in (2.8) $\tilde{\sigma} = \sigma \times \tau$ and $\hat{g}^* = \mathcal{I}(\psi, 1)\mathcal{I}^{-1}$, one obtains as a consequence of Proposition 2.4 the result of Proposition 2.8 from [19].

3 The differential geometry of marked configuration space

3.1 The tangent bundle of $\Omega^{\mathbb{R}^+}_X$

Let us denote by $V_0(X)$ the set of $C^\infty$ vector fields on $X$ (i.e., smooth sections of $T(X)$) that have compact support. Let also $C^\infty_0(X)$ stand for the set of all $C^\infty$ functions from $X$ into $\mathbb{R}$ that have compact support. Then, $\mathfrak{g} := V_0(X) \times C^\infty_0(X)$ can be thought of as a Lie algebra that corresponds to the Lie group $\mathcal{G}$. More precisely, for any fixed $v \in V_0(X)$ and for any $x \in X$, the curve

$$
\mathbb{R} \ni t \mapsto \psi_t^v(x) \in X
$$

is defined as the solution of the following Cauchy problem

$$
\begin{aligned}
\begin{cases}
\frac{d}{dt} \psi_t^v(x) = v(\psi_t^v(x)), \\
\psi_0^v(x) = x.
\end{cases}
\end{aligned} \tag{3.1}
$$

Then, the mappings $\{\psi_t^v, t \in \mathbb{R}\}$ form a one-parameter subgroup of diffeomorphisms in $\text{Diff}_0(X)$ (see, e.g., [7]):

1) $\forall t \in \mathbb{R}$ \hspace{1em} $\psi_t^v \in \text{Diff}_0(X)$,

2) $\forall t_1, t_2 \in \mathbb{R}$ \hspace{1em} $\psi_{t_1}^v \circ \psi_{t_2}^v = \psi_{t_1+t_2}^v$. 

15
Next, for each function \( a \in C_0^\infty(X) \), we define

\[
\theta^a_t = \theta^a_t(x) := e^{ta(x)} \in \mathbb{R}_X,
\]

and \( \{\theta^a_t, \ t \in \mathbb{R}\} \) form a one parameter subgroup of \( \mathbb{R}_X \).

Thus, we can consider, for an arbitrary \((v, a) \in \mathfrak{g}\), the curve \( \{\psi^v_t, \theta^a_t\}, t \in \mathbb{R}\) in \( \mathfrak{g} \). (Notice that this curve does not form a subgroup of \( \mathfrak{g} \).) Hence, to any \( \omega \in \Omega^+_{X} \) there corresponds the following curve in \( \Omega^+_{X} \):

\[
\mathbb{R} \ni t \mapsto (\psi^v_t, \theta^a_t) \omega \in \Omega^+_{X}.
\]

Define now, for a function \( F: \Omega^+_{X} \rightarrow \mathbb{R} \), the directional derivative of \( F \) along \((v, a)\) as

\[
(\nabla_{(v,a)} F)(\omega) := \left. \frac{d}{dt} F((\psi^v_t, \theta^a_t) \omega) \right|_{t=0},
\]

provided the right hand side exists. We will also denote by \( \nabla^v_{\psi} \) and \( \nabla^a_{\theta} \) the directional derivatives along \((v, 0)\) and \((0, a)\), respectively.

Absolutely analogously, one defines for a function \( \varphi: X \times \mathbb{R}_+ \rightarrow \mathbb{R} \) the directional derivative of \( \varphi \) along \((v, a)\):

\[
(\nabla^{X \times \mathbb{R}_+}_{(v,a)} \varphi)(x, s) = \left. \frac{d}{dt} \varphi((\psi^v_t, \theta^a_t)(x, s)) \right|_{t=0}. \tag{3.2}
\]

Then, for a continuously differentiable function \( \varphi \), we have from (2.1), (3.1), and (3.2)

\[
(\nabla^{X \times \mathbb{R}_+}_{(v,a)} \varphi)(x, s) = (\nabla^X \varphi(x, s), v(x))_{T_x(X)} + s \left. \frac{\partial}{\partial s} \varphi(\psi^v_t(x), \theta^a_t(\psi^v_t(x))s)e^{ta(\psi^v_t(x))} \right|_{t=0}
\]

\[
	imes (a(\psi^v_t(x)) + t(\nabla^X a(\psi^v_t(x), v(\psi^v_t(x)))_{T_{\psi^v_t(x)}(X)})\left|_{t=0}
\]

\[
= (\nabla^X \varphi(x, s), v(x))_{T_x(X)} + s \left. \frac{\partial}{\partial s} \varphi(x, s)a(x) \right|_{s=0}
\]

\[
= (\nabla^{X \times \mathbb{R}_+} \varphi(x, s), (v(x), a(x)))_{T_{(x,s)(X \times \mathbb{R}_+)}.} \tag{3.3}
\]

Here,

\[
T_{(x,s)(X \times \mathbb{R}_+)} := \nabla^X(x, T_x(X) + \mathbb{R})
\]

and

\[
\nabla^{X \times \mathbb{R}_+} := (\nabla^X, \nabla^{\mathbb{R}_+}),
\]
where $\nabla^X$ denotes the gradient on $X$ and

$$\nabla^{R_+} = s \frac{\partial}{\partial s}.$$  

Let us introduce a special class of “nice functions” on $\Omega^R_+$. We will say that a function $\varphi : X \times R_+ \to R$ is a $C^\infty$-function on $X \times R_+$ if it is $C^\infty$ with respect to the gradient $\nabla^X$ and the usual derivative in $s$. Denote by $\mathcal{D}$ the set of all $C^\infty$-functions $\varphi$ on $X \times R_+$ with support from $B_c(X \times R_+)$ such that $\varphi$ and all its derivatives are bounded and moreover the following estimate holds for all $n \in \mathbb{N}$

$$\left| \frac{\partial^n}{\partial s^n} \varphi(x, s) \right| \leq C_{n, \varphi} (\max \{1, s\})^{-n}. \quad (3.4)$$

Next, let $C_b^\infty(R^N)$ stand for the space of all $C^\infty$-functions on $R^N$ which together with all their derivatives are bounded. Then, we can introduce $FC_b^\infty(\mathcal{D}, \Omega^R_X)$ as the set of all functions $F : \Omega^R_X \to R$ of the form

$$F(\omega) = g_F(\langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_N, \omega \rangle), \quad \omega \in \Omega^R_X, \quad (3.5)$$

where $\varphi_1, \ldots, \varphi_N \in \mathcal{D}$ and $g_F \in C_b^\infty(R^N)$ (compare with [4]). $FC_b^\infty(\mathcal{D}, \Omega^R_X)$ will be called the set of smooth cylinder functions on $\Omega^R_X$.

For any $F \in FC_b^\infty(\mathcal{D}, \Omega^R_X)$ of the form (3.5) and a given $(v, a) \in g$, we have, just as in [4],

$$F((\psi_i^v, \theta_i^a)\omega) = g_F((\langle \varphi_1, (\psi_i^v, \theta_i^a)\omega \rangle, \ldots, \langle \varphi_N, (\psi_i^v, \theta_i^a)\omega \rangle)$$

$$= g_F((\langle \varphi_1 \circ (\psi_i^v, \theta_i^a), \omega \rangle, \ldots, \langle \varphi_N \circ (\psi_i^v, \theta_i^a), \omega \rangle),$$

and therefore

$$(\nabla^\Omega_{(v,a)} F)(\omega) = \sum_{j=1}^N \frac{\partial g_F}{\partial r_j}(\langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_N, \omega \rangle) \langle \nabla^{X \times R_+}_{(v,a)} \varphi_j, \omega \rangle. \quad (3.6)$$

In particular, we conclude from (3.6) that

$$\nabla^\Omega_{(v,a)} = \nabla^\Omega_v + \nabla^\Omega_a. \quad (3.7)$$

The expression of $\nabla^\Omega_{(v,a)}$ on smooth cylinder functions motivates the following definition.

**Definition 3.1** The tangent space $T_\omega(\Omega^R_X)$ to the marked configuration space $\Omega^R_X$ at a point $\omega = (\gamma, s) \in \Omega^R_X$ is defined as the Hilbert space

$$T_\omega(\Omega^R_X) : = L^2(X \to T(X) + \mathbb{R}; \gamma)$$

$$= L^2(X \to T(X); \gamma) \oplus L^2(X \to \mathbb{R}; \gamma)$$
with scalar product
\[
\langle V_1^\omega, V_2^\omega \rangle_{T_\omega(\Omega^\mathbb{R}^+)} = \int_X \langle (V_1^\omega(x)T_\omega(x), V_2^\omega(x)T_\omega(x))T_\omega(x) + V_1^\omega(x)R V_2^\omega(x)R \rangle \gamma(dx),
\] (3.8)
where \( V_1^\omega, V_2^\omega \in T_\omega(\Omega^\mathbb{R}^+) \) and \( V_\omega(x)T_\omega(x) \) and \( V_\omega(x)R \) denote the projection of \( V_\omega(x) \in T_x(X) + \mathbb{R} \) onto \( T_x(X) \) and \( \mathbb{R} \), respectively. The corresponding tangent bundle is
\[
T(\Omega^\mathbb{R}^+) = \bigcup_{\omega \in \Omega^\mathbb{R}^+} T_\omega(\Omega^\mathbb{R}^+).
\]
As usual in Riemannian geometry, having directional derivatives and a Hilbert space as a tangent space, we can introduce a gradient.

**Definition 3.2** We define an intrinsic gradient of a function \( F: \Omega^\mathbb{R}^+ \to \mathbb{R} \) as a mapping
\[
\Omega^\mathbb{R}^+ \ni \omega \mapsto (\nabla^\Omega F)(\omega) \in T_\omega(\Omega^\mathbb{R}^+)
\]
such that, for any \((v,a) \in g\),
\[
(\nabla^\Omega F)(\omega) = \langle (\nabla^\Omega F)(\omega), (v,a) \rangle_{T_\omega(\Omega^\mathbb{R}^+)}. \tag{3.9}
\]
By (3.6) and (3.3) we have, for an arbitrary \( F \in \mathcal{F}C_0^\infty(\mathfrak{D}, \Omega^\mathbb{R}^+) \) of the form (3.5) and each \( \omega = (\gamma, s) \in \Omega^\mathbb{R}^+ \),
\[
(\nabla^\Omega F)(\omega; x) = \sum_{j=1}^N \frac{\partial g_F}{\partial r_j}(\langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_N, \omega \rangle) \nabla^{X \times \mathbb{R}^+} \varphi_j(x, s_x), \quad x \in \gamma. \tag{3.9}
\]

**Remark 3.1** The operator \( s \frac{\partial}{\partial s} \) relates to a representation of the group of dilations in \( L^2(\mathbb{R}^+) \) (cf. e.g. [11]). Namely, for any \( \theta > 0 \), we put
\[
L^2(\mathbb{R}^+) \ni f(\cdot) \mapsto f(\theta \cdot) \in L^2(\mathbb{R}^+).
\]
Setting here \( \theta_t^a := e^{at}, \ a \in \mathbb{R}, \ t \in \mathbb{R} \), we obtain
\[
\frac{d}{dt} f(e^{at}s) \big|_{t=0} = s f'(s)a =: (\nabla_a^\mathbb{R} f)(s) = (\nabla^\mathbb{R} f, a)_{\mathbb{R}}. \tag{3.10}
\]
It is easy to verify that the following operator equality holds:
\[
\nabla^\mathbb{R} = S_\mathcal{L} \nabla^\mathbb{R} S_\mathcal{L}^{-1},
\]
where the mapping
\[
C^1(\mathbb{R}) \ni f \mapsto S_\mathcal{L} f := f \circ \mathcal{L} \in C^1(\mathbb{R}^+) \]
is the isomorphism generated by the bijection \( \mathbb{R}_+ \ni s \mapsto Ls = \log s \in \mathbb{R} \). Notice that, if \( U_L \) is the unitary between the spaces \( L^2(\mathbb{R}) \) and \( L^2(\mathbb{R}_+) \) generated by \( L \), i.e., if 
\[
(U_L f)(s) := \frac{f(Ls)}{\sqrt{s}}, \quad f \in L^2(\mathbb{R}), \ s \in \mathbb{R}_+, 
\]
then \( U_L \nabla^R U_L^{-1} \) coincides with \( \nabla^{\mathbb{R}_+} + \frac{1}{2} I \).

Finally, we note that the condition (3.4) means, in fact, that an \( \mathbb{R}_+ \)-derivative of an arbitrary order of a function from \( \mathcal{D} \) belongs again to \( \mathcal{D} \).

### 3.2 Integration by parts and divergence on the marked Poisson space

Let the marked configuration space \( \Omega^{\mathbb{R}_+}_X \) be equipped with the marked Poisson measure \( \tilde{\sigma} \), and in addition to the condition (1.7) we will suppose that

\[
\sqrt{q} \in H^{1,2}(X \times \mathbb{R}_+). \tag{3.11}
\]

Here, \( H^{1,2}(X \times \mathbb{R}_+) \) denotes the local Sobolev space of order 1 constructed with respect to the gradient \( \nabla^{X \times \mathbb{R}_+} \) in the space \( L^2_{loc}(X; m) \otimes L^2(\mathbb{R}_+; \lambda) \), i.e., \( H^{1,2}(X \times \mathbb{R}_+) \) consists of functions \( f \) defined on \( X \times \mathbb{R}_+ \) such that, for any set \( A \in B_c(X \times \mathbb{R}_+) \), the restriction of \( f \) to \( A \) coincides with the restriction to \( A \) of some function \( \varphi \) from the Sobolev space \( H^{1,2}(X \times \mathbb{R}_+) \) constructed as the closure of \( \mathcal{D} \) with respect to the norm

\[
\| \varphi \|^2_{1,2} := \int_{X \times \mathbb{R}_+} \left( |\nabla^X \varphi(x,s)|^2 + s^2 |\partial_s \varphi(x,s)|^2 + |\varphi(x,s)|^2 \right) m(dx) \lambda(ds).
\]

The set \( \mathcal{F}C^\infty_b(\mathcal{D}, \Omega^{\mathbb{R}_+}_X) \) is a dense subset in the space

\[
L^2(\Omega^{\mathbb{R}_+}_X, \mathcal{B}(\Omega^{\mathbb{R}_+}_X), \pi_{\tilde{\sigma}}) =: L^2(\pi_{\tilde{\sigma}}).
\]

For any \( (v, a) \in \mathfrak{g} \), we have a differential operator in \( L^2(\pi_{\tilde{\sigma}}) \) on the domain \( \mathcal{F}C^\infty_b(\mathcal{D}, \Omega^{\mathbb{R}_+}_X) \) given by

\[
\mathcal{F}C^\infty_b(\mathcal{D}, \Omega^{\mathbb{R}_+}_X) \ni F \mapsto \nabla_{(v,a)} F \in L^2(\pi_{\tilde{\sigma}}).
\]

Our aim now is to compute the adjoint operator \( \nabla_{(v,a)}^\ast \) in \( L^2(\pi_{\tilde{\sigma}}) \). It corresponds, of course, to an integration by parts formula with respect to the measure \( \pi_{\tilde{\sigma}} \).

But first we present the corresponding formula on \( X \times \mathbb{R}_+ \).

**Definition 3.3** For any \( (v, a) \in \mathfrak{g} \), the logarithmic derivative of the measure \( \tilde{\sigma} \) along \( (v, a) \) is defined as the following function on \( X \times \mathbb{R}_+ \):

\[
\beta_{(v,a)}^{\tilde{\sigma}} := \beta^\sigma_v + \beta^\sigma_a
\]
with
\[ \beta_v^a(x, s) : = \frac{\nabla^X q(x, s)}{q(x, s)} + \text{div}^X v(x) \]
\[ = \left( \frac{\nabla^X q(x, s)}{q(x, s)}, v(x) \right)_{T_x(X)} + \text{div}^X v(x), \]
\[ \text{div}^X = \text{div}^X_m \text{ being the divergence on } X \text{ w.r.t. } m, \text{ and} \]
\[ \beta_a^\sigma(x, s) = s \frac{\partial}{\partial s} q(x, s) q(x, s) a(x) + a(x). \]

Upon (3.11), we conclude that, for each \((v, a) \in g\), the function \(\nabla^X \log q\) is quadratically integrable with respect to the measure \(\tilde{\sigma}\) on any \(\Lambda_{mk} \in B_0(X \times \mathbb{R}_+)\). Therefore, taking to notice that \(\tilde{\sigma}(\Lambda_{mk}) < \infty\), we conclude that each function \(\beta^\sigma_{(v, a)}\), \((v, a) \in g\), is absolutely integrable on every \(\Lambda_{mk}\) with respect to \(\tilde{\sigma}\).

**Lemma 3.1 (Integration by parts formula on \(X \times \mathbb{R}_+)\)** For all \(\varphi_1, \varphi_2 \in \mathcal{D}\), we have
\[
\int_{X \times \mathbb{R}_+} (\nabla^X_{(v, a)} \varphi_1)(x, s) \varphi_2(x, s) \tilde{\sigma}(dx, ds) = \]
\[= - \int_{X \times \mathbb{R}_+} \varphi_1(x, s) (\nabla^X_{(v, a)} \varphi_2)(x, s) \tilde{\sigma}(dx, ds) \]
\[ - \int_{X \times \mathbb{R}_+} \varphi_1(x, s) \varphi_2(x, s) \beta^\sigma_{(v, a)}(x, s) \tilde{\sigma}(dx, ds). \]

**Proof.** Since \(\beta^\sigma_v\) is exactly the logarithmic derivative of the measure \(\tilde{\sigma}\) along the vector field \(v\), to prove the lemma it suffices to see that, for arbitrary \(f_1, f_2 \in C_0^\infty(\mathbb{R}_+), a \in \mathbb{R}\), and for \(m\)-a.a. \(x \in X\), we have by virtue of (1.7)
\[
\int_{\mathbb{R}_+} (\nabla^R_{a} f_1)(s) f_2(s) p(x, ds) = \int_{\mathbb{R}_+} as f_1'(s) f_2(s) p(x, s) \lambda(ds) \]
\[= a f_1(s) f_2(s) sp(x, s) \bigg|_{s=\infty}^{s=0} - \int_{\mathbb{R}_+} f_1(s) a \frac{d}{ds} (f_2(s) sp(x, s)) \lambda(ds) \]
\[= - \int_{\mathbb{R}_+} f_1(s) (\nabla^R_{a} f_2)(s) p(x, ds) - \int_{\mathbb{R}_+} f_1(s) f_2(s) \beta^p_{(x, \cdot)}(s) p(x, ds). \]

Here
\[ \beta^p_{a}(x, s) = \frac{s \frac{d}{ds} p(x, s)}{p(x, s)} a + a \]
and we have used the formulas (3.10) and (1.7). □
Definition 3.4 For any $(v,a) \in \mathfrak{g}$, the logarithmic derivative of the marked Poisson measure $\pi_{\tilde{\sigma}}$ along $(v,a)$ is defined as the following function on $\Omega^{\mathbb{R}^+_X}$:

$$\Omega^{\mathbb{R}^+_X} \ni \omega \mapsto B_{(v,a)}^{\pi_{\tilde{\sigma}}}(\omega) := \langle \beta_{(v,a)}^{\pi_{\tilde{\sigma}}}, \omega \rangle.$$  \hspace{1cm} (3.12)

A motivation for this definition is given by the following theorem.

Theorem 3.1 (Integration by parts formula) For all $F, G \in FC_b^\infty(\mathcal{D}, \Omega^{\mathbb{R}^+_X})$ and each $(v,a) \in \mathfrak{g}$, we have

$$\int_{\Omega^{\mathbb{R}^+_X}} (\nabla_{(v,a)}^{\Omega} F)(\omega) G(\omega) \pi_{\tilde{\sigma}}(d\omega) = -\int_{\Omega^{\mathbb{R}^+_X}} F(\omega) (\nabla_{(v,a)}^{\Omega} G)(\omega) \pi_{\tilde{\sigma}}(d\omega) - \int_{\Omega^{\mathbb{R}^+_X}} F(\omega) G(\omega) B_{(v,a)}^{\pi_{\tilde{\sigma}}}(\omega) \pi_{\tilde{\sigma}}(d\omega),$$ \hspace{1cm} (3.13)

or

$$\nabla_{(v,a)}^{\Omega} = -\nabla_{(v,a)}^{\Omega} - B_{(v,a)}^{\pi_{\tilde{\sigma}}}(\omega)$$ \hspace{1cm} (3.14)

as an operator equality on the domain $FC_b^\infty(\mathcal{D}, \Omega^{\mathbb{R}^+_X})$ in $L^2(\pi_{\tilde{\sigma}})$.

Proof. Because of (3.7), the formula (3.14) will be proved if we prove it first for the operator $\nabla_{a}^{\Omega}$, i.e., when $a(x) \equiv 0$, and then for the operator $\nabla_{v}^{\Omega}$, i.e., when $v(x) = 0 \in T_{x}(X)$ for all $x \in X$. We present below only the proof for $\nabla_{a}^{\Omega}$, since the proof for $\nabla_{v}^{\Omega}$ is basically the same as that of the integration by parts formula in case of Poisson measures [4].

By Proposition 2.2, we have for all $a \in C_0^\infty(X)$

$$\int_{\Omega^{\mathbb{R}^+_X}} F(\theta_{t}^{a}(\omega)) G(\omega) \pi_{\tilde{\sigma}}(d\omega) = \int_{\Omega^{\mathbb{R}^+_X}} F(\omega) G(\theta_{t}^{a}(\omega)) \pi_{\theta_t^a \tilde{\sigma}}(d\omega).$$

Differentiating this equation with respect to $t$, interchanging $d/dt$ with the integrals and setting $t = 0$, the l.h.s. becomes the l.h.s. of (3.13). To see that the r.h.s. then also coincides with the r.h.s. of (3.13), we note that

$$\frac{d}{dt} G(\theta_{-t}^{a}(\omega))|_{t=0} = - (\nabla_{a}^{\Omega} G)(\omega),$$

and by Proposition 2.3

$$\frac{d}{dt} \left[ \frac{d\pi_{\theta_{-t}^{a} \tilde{\sigma}}}{d\pi_{\tilde{\sigma}}}(\omega) \right]|_{t=0} = \sum_{(x,s) \in \omega} \left. \frac{d}{dt} p_{\theta_{-t}^{a}}(x,s) \right|_{t=0}$$

$$= \sum_{(x,s) \in \omega} \left. \frac{d}{dt} \left[ \frac{p(x, e^{-ta}(x)s)}{p(x,s)e^{ta}(x)} \right] \right|_{t=0} = - \langle \beta_{a}^{\pi_{\tilde{\sigma}}}, \omega \rangle = -B_{a}^{\pi_{\tilde{\sigma}}}(\omega).$$
Definition 3.5 For a vector field

\[ V : \Omega^R_X \ni \omega \mapsto V_\omega \in T_\omega(\Omega^R_X), \]
the divergence \( \text{div}_\pi^\Omega V \) is defined via the duality relation

\[ \int_{\Omega^R_X} \langle V_\omega, \nabla^\Omega F(\omega) \rangle_{T_\omega(\Omega^R_X)} \pi_\sigma(d\omega) = -\int_{\Omega^R_X} F(\omega)(\text{div}_\pi^\Omega V)(\omega) \pi_\sigma(d\omega) \]
for all \( F \in FC^\infty_b(\mathcal{D}, \Omega^R_X) \), provided it exists (i.e., provided

\[ F \mapsto \int_{\Omega^R_X} \langle V_\omega, \nabla^\Omega F(\omega) \rangle_{T_\omega(\Omega^R_X)} \pi_\sigma(d\omega) \]
is continuous on \( L^2(\pi_\sigma) \)).

A class of smooth vector fields on \( \Omega^R_X \) for which the divergence can be computed in an explicit form is described in the following proposition.

Proposition 3.1 For any vector field

\[ V_\omega(x) = \sum_{j=1}^N G_j(\omega)(v_j(x), a_j(x)), \quad \omega \in \Omega^R_X, \ x \in X, \]
with \( G_j \in FC^\infty_b(\mathcal{D}, \Omega^R_X) \), \( (v_j, a_j) \in \mathfrak{g}, \ j = 1, \ldots, N \), we have

\[ (\text{div}_\pi^\Omega V)(\omega) = \sum_{j=1}^N (\nabla_{(v_j, a_j)}^\Omega G_j)(\omega) + \sum_{j=1}^N B_{(v_j, a_j)}^\sigma(\omega) G_j(\omega) \]

\[ = \sum_{j=1}^N \langle \nabla^\Omega G_j(\omega), (v_j, a_j) \rangle_{T_\omega(\Omega^R_X)} + \sum_{j=1}^N \langle \beta_{(v_j, a_j)}^\sigma, \omega \rangle G_j(\omega). \]

Proof. Due to the linearity of \( \nabla^\Omega \), it is sufficient to consider the case \( N = 1 \), i.e.,
\( V_\omega(x) = G(\omega)(v(x), a(x)) \). By Theorem 3.1, we have for all \( F \in FC^\infty_b(\mathcal{D}, \Omega^R_X) \)

\[ -\int_{\Omega^R_X} \langle V_\omega, \nabla^\Omega F(\omega) \rangle_{T_\omega(\Omega^R_X)} \pi_\sigma(d\omega) = -\int_{\Omega^R_X} G(\omega)\nabla^\Omega_{(v, a)} F(\omega) \pi_\sigma(d\omega) \]

\[ = \int_{\Omega^R_X} (\nabla^\Omega_{(v, a)} G)(\omega) F(\omega) \pi_\sigma(d\omega) + \int_{\Omega^R_X} G(\omega) F(\omega) B_{(v, a)}^\sigma(\omega) \pi_\sigma(d\omega), \]
which yields

\[ (\text{div}_\pi^\Omega V)(\omega) = \nabla^\Omega_{(v, a)} G(\omega) + B_{(v, a)}^\sigma(\omega) G(\omega) \]

\[ = \langle \nabla^\Omega G(\omega), (v, a) \rangle_{T_\omega(\Omega^R_X)} + \langle \beta_{(v, a)}^\sigma, \omega \rangle G(\omega). \]
Remark 3.2 Extending the definition of \( B_{\pi}^{\tilde{\sigma}} \) in (3.12) to the class of vector fields \( V = \sum_{j=1}^{N} G_j \otimes (v_j, a_j) \) by
\[
B_{\pi}^{\tilde{\sigma}} V(\omega) := \sum_{j=1}^{N} \langle \beta_{(v_j, a_j)}(v_j, a_j), \omega \rangle G_j(\omega) + \sum_{j=1}^{N} \langle \nabla_{(v_j, a_j)} G_j(\omega) \rangle + \sum_{j=1}^{N} \nabla \Omega(v_j, a_j) G_j(\omega),
\]
we obtain that
\[
\text{div} \Omega_{\pi}^{\tilde{\sigma}} \cdot = B_{\pi}^{\tilde{\sigma}}.
\]
In particular, if \((v, a) \in \mathfrak{g}\), it follows, for the “constant” vector field \( V_\omega \equiv (v, a) \) on \( \Omega_{X_+}^{\mathbb{R}} \), that
\[
\text{div} \Omega_{\pi}^{\tilde{\sigma}}(v,a) = \langle \text{div}_{X}^{\mathbb{R}}(v,a), \omega \rangle,
\]
where \( \text{div}_{X}^{\mathbb{R}}(v,a) = \beta_{(v,a)}(v,a) \) is the divergence of \( \tilde{\sigma} \) on \( X \times \mathbb{R} \) w.r.t. \((v,a)\):
\[
\int_{X \times \mathbb{R}^+} \langle \nabla_{X \times \mathbb{R}^+} \varphi(x,s), (v(x),a(x)) \rangle_{T_{(v,a)}(X \times \mathbb{R}^+)} \tilde{\sigma}(dx,ds) = -\int_{X \times \mathbb{R}^+} \varphi(x,s) \langle \text{div}_{X}^{\mathbb{R}}(v,a), (x,s) \rangle \tilde{\sigma}(dx,ds), \quad \varphi \in \mathcal{D}.
\]

3.3 Integration by parts characterization

In the works [4, 5] it was shown that the mixed Poisson measures are exactly the “volume elements” corresponding to the differential geometry on the configuration space \( \Gamma_{X_+} \). Now, we wish to prove that an analogous statement holds true in our case of \( \Omega_{X_+}^{\mathbb{R}} \) for mixed marked Poisson measures.

We start with a lemma that describes \( \tilde{\sigma} \) as a unique (up to a constant) measure \( \xi \) on \( X \) such that \( \text{div}_{X}^{\mathbb{R}} \) is the dual operator \( \nabla_{X}^{\mathbb{R}} \) on \( L^2(\xi) \) of \( \nabla_{X}^{\mathbb{R}} \) when considered with domains \( V_0(\Lambda) \times C_0^{\infty}(\Lambda) \), resp. \( C_0^{\infty}(\Lambda) \) (i.e., the set of all \((v,a) \in \mathfrak{g}\), resp. \( \varphi \in \mathcal{D} \) with support in \( \Lambda \), resp. \( \Lambda_{mk} \)).

**Lemma 3.2** Let
\[
\begin{align*}
\frac{\nabla^{X}_{v} q(x,s)}{q(x,s)} & \in L^1_{\text{loc}}(X \times \mathbb{R}^{+}; m \otimes \lambda), \quad v \in V_0(X), \\
\frac{\partial}{\partial x} p(x,s) & \in L^1_{\text{loc}}(X \times \mathbb{R}^{+}; m \otimes \lambda).
\end{align*}
\]

Then, for every \( \Lambda \in \mathcal{O}_c(X) \) the measures \( z \tilde{\sigma} \), \( z \geq 0 \), are the only positive Radon measures \( \xi \) on \( \Lambda_{mk} \) such that \( \text{div}_{X}^{\mathbb{R}} \) is the dual operator on \( L^2(\xi) \) of \( \nabla_{X}^{\mathbb{R}} \) when considered with domains \( V_0(\Lambda) \times C_0^{\infty}(\Lambda) \), resp. \( C_0^{\infty}(\Lambda_{mk}) \) (i.e., the set of all \((v,a) \in \mathfrak{g}\), resp. \( \varphi \in \mathcal{D} \) with support in \( \Lambda \), resp. \( \Lambda_{mk} \)).
Proof. By using the condition (3.15), the lemma is obtained in complete analogy with Remark 4.1 (iii) in [5]. Indeed, let \( q_1(x,s) \) and \( q_2(x,s) \) be two densities w.r.t. \( m \otimes \lambda \) for which the logarithmic derivatives coincide. Then, we get

\[
\nabla_v^X \log q_1(x,s) = \nabla_v^X \log q_2(x,s), \quad v \in V_0(X),
\]

\[
\frac{\partial}{\partial s} \log q_1(x,s) = \frac{\partial}{\partial s} \log q_2(x,s) \quad m \otimes \lambda \text{-a.s.},
\]

which yields respectively

\[
q_1(x,s) = q_2(x,s)c(s),
\]

\[
q_1(x,s) = q_2(x,s)\tilde{c}(x) \quad m \otimes \lambda \text{-a.s.}
\]

Therefore, \( q_1(x,s) = \text{const} \) \( q_2(x,s) \) \( m \otimes \lambda \)-a.s. \( \blacksquare \)

Let \( \nu \) be a probability measure on \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\). Then, we define a mixed Poisson measure as follows:

\[
\mu_{\nu,\tilde{\sigma}} = \int_{\mathbb{R}_+} \pi_{x,\tilde{\sigma}} \nu(dz) \quad (3.16)
\]

Here, \( \pi_{0\tilde{\sigma}} \) denotes the Dirac measure on \( \Omega_{X}^{\mathbb{R}_+} \) with mass in \( \omega = \emptyset \). Let \( \mathcal{M}_l(\Omega_{X}^{\mathbb{R}_+}) \), \( l \in [1, \infty) \), denote the set of all probability measures on \((\Omega_{X}^{\mathbb{R}_+}, \mathcal{B}(\Omega_{X}^{\mathbb{R}_+}))\) such that

\[
\int_{\Omega_{X}^{\mathbb{R}_+}} |\langle f, \omega \rangle|^l \mu(d\omega) < \infty \quad \text{for all } f \in C_{0,b}(X \times \mathbb{R}_+), \quad f \geq 0.
\]

Clearly, \( \mu_{\nu,\tilde{\sigma}} \in \mathcal{M}_1(\Omega_{X}^{\mathbb{R}_+}) \) if and only if

\[
\int_{\mathbb{R}_+} z^l \nu(dz) < \infty \quad (3.17)
\]

We define \((\text{IbP})^{\tilde{\sigma}}\) to be the set of all \( \mu \in \mathcal{M}_1(\Omega_{X}^{\mathbb{R}_+}) \) with the property that \( \omega \mapsto \langle \beta_{(v,a)}^{\tilde{\sigma}}, \omega \rangle \) is \( \mu \)-integrable for all \( (v,a) \in \mathfrak{g} \) and which satisfy (3.13) with \( \mu \) replacing \( \pi_{\tilde{\sigma}} \) for all \( F, G \in \mathcal{F}C^\infty_b(\mathfrak{D}, \Omega_{X}^{\mathbb{R}_+}) \), \( (v,a) \in \mathfrak{g} \). We note that (3.13) makes sense only for such measures and that \( B_{(v,a)}^{\pi_{\tilde{\sigma}}} \) depends only on \( \tilde{\sigma} \) not on \( \pi_{\tilde{\sigma}} \). Obviously, since \( \nabla_{(v,a)}^{X \times \mathbb{R}_+} \) obeys the product rule for all \( (v,a) \in \mathfrak{g} \), we can always take \( G \equiv 1 \). Furthermore, \((\text{IbP})^{\tilde{\sigma}}\) is convex.

**Theorem 3.2** Let the condition (3.15) be satisfied. Then, the following conditions are equivalent:

(i) \( \mu \in (\text{IbP})^{\tilde{\sigma}} \);

(ii) \( \mu = \mu_{\nu,\tilde{\sigma}} \) for some probability measure \( \nu \) on \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\) satisfying (3.17) with \( l = 1 \).
Proof. The part (ii)⇒(i) is trivial. In order to prove (i)⇒(ii), we have to modify the corresponding part of the proof of Theorem 4.1 in [5]. We present this modification in Appendix. ■

As a direct consequence of Theorem 3.2, we obtain

Corollary 3.1 Suppose that the condition (3.15) is satisfied. Then the extreme points of (IbP) are exactly \( \pi_{z,\bar{\sigma}}, \ z \geq 0 \).

3.4 A lifting of the geometry

Just as in the case of the geometry on the configuration space, we can present an interpretation of the formulas obtained in subsections 3.1–3.3 via a simple “lifting rule.”

Suppose that \( f \in C^0_{\text{c}}(X \times \mathbb{R}_+), \) or more generally \( f \) is an arbitrary measurable function on \( X \times \mathbb{R}_+ \) for which there exists (depending on \( f \)) \( \Lambda \in B_c(X) \) such that \( \text{supp} \ f \subset \Lambda_{mk} \). Then, \( f \) generates a (cylinder) function on \( \Omega_{X}^{\mathbb{R}_+} \) by the formula

\[
L_f(\omega) := \langle f, \omega \rangle, \quad \omega \in \Omega_{X}^{\mathbb{R}_+}.
\]

We will call \( L_f \) the lifting of \( f \).

As before, any vector field \( (v, a) \in \mathfrak{g}, \)

\[
(v, a): \ X \ni x \mapsto (v(x), a(x)) \in T_{(x,s)}(X \times \mathbb{R}_+) = T_{x}(X) \oplus \mathbb{R},
\]

can be considered as a vector field on \( \Omega_{X}^{\mathbb{R}_+} \) (the lifting of \( (v, a) \)), which we denote by \( L_{(v, a)} \):

\[
L_{(v, a)}: \Omega_{X}^{\mathbb{R}_+} \ni \omega = \{ \gamma, s \} \mapsto \{ x \mapsto (v(x), a(x)) \} \in T_{\omega}(\Omega_{X}^{\mathbb{R}_+}) = L^2(\gamma \mapsto T(\gamma) \oplus \mathbb{R}; \gamma).
\]

For \((v, a), (w, b) \in \mathfrak{g}, \) the formula (3.8) can be written as follows:

\[
\langle L_{(v, a)}, L_{(w, b)} \rangle_{T_{\omega}(\Omega_{X}^{\mathbb{R}_+})} = L_{((v, a), (w, b)))_{T_{(x,s)}(X \times \mathbb{R}_+)}(\omega),
\]

i.e., the scalar product of lifted vector fields is computed as the lifting of the scalar product

\[
\langle (v(x), a(x)), (w(x), b(x)) \rangle_{T_{(x,s)}(X \times \mathbb{R}_+)} = f(x).
\]

This rule can be used as a definition of the tangent space \( T_{\omega}(\Omega_{X}^{\mathbb{R}_+}) \).

The formula (3.6) has now the following interpretation:

\[
\left( \nabla_{(v, a)}^{\Omega} L_{\varphi} \right)(\omega) = L_{\nabla_{(v, a)}^{\Omega} \varphi}(\omega), \quad \varphi \in \mathcal{D}, \ \omega \in \Omega_{X}^{\mathbb{R}_+}, \tag{3.18}
\]
and the “lifting rule” for the gradient is given by

\[
(\nabla^\Omega L_\varphi)(\gamma, s) : \gamma \ni x \mapsto \nabla^{X \times \mathbb{R}^+} \varphi(x, s_x).
\] (3.19)

As follows from (3.12), the logarithmic derivative \(B_{\pi_{\bar{\sigma}}}^{\pi_{\bar{\sigma}}} : \Omega_{X}^+ \rightarrow \mathbb{R}\) is obtained via the lifting procedure of the corresponding logarithmic derivative \(\beta_{(v,a)}^{\bar{\sigma}} : X \times \mathbb{R}_+ \rightarrow \mathbb{R}\), namely,

\[
B_{(v,a)}^{\pi_{\bar{\sigma}}}(\omega) = L_{\beta_{(v,a)}^{\bar{\sigma}}}(\omega),
\]
or equivalently, one has for the divergence of a lifted vector field:

\[
\text{div}^\Omega L_{(v,a)} = L_{\text{div}_{\bar{\sigma}}^{X \times \mathbb{R}_+}}(v, a).
\] (3.20)

We underline that by (3.18) and (3.19) one recovers the action of \(\nabla^\Omega_{(v,a)}\) and \(\nabla^\Omega\) on all functions from \(FC_{b}^{\infty}(\mathcal{D}, \Omega_{X}^+\mathbb{R}_+)\) algebraically from requiring the product or the chain rule to hold. Also, the action of \(\text{div}^\Omega_{\pi_{\bar{\sigma}}}\) on more general cylindrical vector fields follows as in Remark 3.2 if one assumes the usual product rule for \(\text{div}^\Omega_{\bar{\sigma}}\) to hold.

### 3.5 Representations of the Lie algebra of the group \(\mathfrak{G}\)

Using the \(\mathfrak{G}\)-quasiinvariance of \(\pi_{\bar{\sigma}}\), we can define a unitary representation of the group \(\mathfrak{G} = \text{Diff}_0(X) \times \mathbb{R}_+^X\) in the space \(L^2(\pi_{\bar{\sigma}})\). Namely, for \(g \in \mathfrak{G}\), we define a unitary operator

\[
(V_{\pi_{\bar{\sigma}}}(g)F)(\omega) := F(g(\omega)) \sqrt{\frac{dg^{-1*\pi_{\bar{\sigma}}}}{d\pi_{\bar{\sigma}}}}(\omega), \quad F \in L^2(\pi_{\bar{\sigma}}).
\]

Then, we have

\[
V_{\pi_{\bar{\sigma}}}(g_1)V_{\pi_{\bar{\sigma}}}(g_2) = V_{\pi_{\bar{\sigma}}}(g_1g_2), \quad g_1, g_2 \in \mathfrak{G}.
\]

As has been noted in the introduction, we have the following proposition.

**Proposition 3.2** The representation \(V_{\pi_{\bar{\sigma}}}\) of \(\mathfrak{G}\) is reducible.

**Proof.** Let us suppose for simplicity of notations that \(\bar{\sigma}\) is a product measure: \(\bar{\sigma} = \tilde{\sigma} = \sigma \otimes \tau\), \(\tau(ds) = p(s) \lambda(ds)\) (it will be seen that our proof can be generalized to the case of a general \(\bar{\sigma}\) without difficulties). Consider the regular representation of the group of dilations on \(L^2(\tau)\):

\[
(U_\tau(\theta)\varphi)(s) = \varphi(\theta s) \sqrt{\frac{p(\theta s)}{p(s)}}, \quad \theta \in \mathbb{R}_+, \varphi \in L^2(\tau).
\]
This representation is reducible (one can also easily see that it is unitarily equivalent to the regular representation of the additive group $\mathbb{R}$ on $L^2(\mathbb{R})$, see e.g. [33]). Let $\mathcal{H}$ be an arbitrary subspace of $L^2(\tau)$ that is invariant with respect to this representation and such that $\mathcal{H} \neq \{0\}$ and $\mathcal{H} \neq L^2(\tau)$. Then, the subspace $L^2(\sigma) \otimes \mathcal{H}$ of $L^2(\tilde{\sigma})$ is invariant with respect to the regular representation of the group $\mathfrak{G}$ on $L^2(\tilde{\sigma})$:

$$(V_\sigma(g)f)(x, s) = f(g(x, s))\sqrt{\frac{dg^{-1}\tilde{\sigma}}{d\tilde{\sigma}}}(x, s), \quad g \in \mathfrak{G}, f \in L^2(\tilde{\sigma}).$$

For a set $\Upsilon \in \mathcal{B}(X \times \mathbb{R}_+)$ denote by $B_\Upsilon(\Omega^R_X)$ the $\sigma$-algebra on $\Omega^R_X$ generated by the mappings $N_B$ (defined by (1.2)), where $B \in \mathcal{B}(X \times \mathbb{R}_+)$ and $B \subset \Upsilon$. By definition, each $B_\Upsilon(\Omega^R_X)$ is a sub-$\sigma$-algebra of $B(\Omega^R_X)$. It is easy to see that, for any $\Lambda \in \mathcal{O}_c(X)$, there is a natural isomorphism between the $\sigma$-algebras $B_{\Lambda_{mk}}(\Omega^R_X)$ and $B(\Omega^R_X)$. Then, by the definition of a marked Poisson measure, we get

$$L^2_{\Lambda}(\sigma) := L^2(\Omega^R_X, B_{\Lambda_{mk}}(\Omega^R_X), \sigma) \simeq L^2(\Omega^R_X, B(\Omega^R_X), \sigma_{\Lambda})$$

$$= \bigoplus_{n=0}^\infty L^2(\Omega^R_X(n), B(\Omega^R_X(n)), \sigma_{\Lambda} | \Omega^R_X(n))$$

$$= e^{-\tilde{\sigma}(\Lambda_{mk})} \bigoplus_{n=0}^\infty \frac{1}{n!} \dot{L}^2(\Lambda_{mk}, \sigma_{\Lambda}) \otimes \dot{H}^n,$$

where $\dot{L}^2(\Lambda_{mk}, \sigma_{\Lambda}) = (L^2(\Lambda_{mk}, \sigma)) \hat{\otimes}^n$, $\hat{\otimes}$ denoting symmetric tensor product.

We note that $L^2_{\Lambda}(\sigma)$ is a subspace of $L^2_{\Lambda}(\sigma)$ provided $\Lambda \subset \Lambda'$, and the whole space $L^2(\sigma)$ is the “limit” of the spaces $L^2_{\Lambda}(\sigma)$ as $\Lambda \nearrow X$, i.e., each function $F \in L^2(\sigma)$ can be represented as the $L^2(\sigma)$ limit of a sequence of functions $F_{\Lambda} \in L^2_{\Lambda}(\sigma)$ as $\Lambda \nearrow X$.

Now, for each $\Lambda \in \mathcal{O}_c(X)$, we define a subspace $R_{\Lambda}$ of $L^2_{\Lambda}(\sigma)$ as follows:

$$R_{\Lambda} := e^{-\tilde{\sigma}(\Lambda_{mk})} \bigoplus_{n=0}^\infty \frac{1}{n!} (L^2(\Lambda; \sigma) \otimes \mathcal{H}) \hat{\otimes}^n$$

$$= e^{-\tilde{\sigma}(\Lambda_{mk})} \bigoplus_{n=0}^\infty \frac{1}{n!} \dot{L}^2(\Lambda; \sigma_{\Lambda}) \otimes \dot{H}^n.$$

Again, for arbitrary $\Lambda, \Lambda' \in \mathcal{O}_c(X)$, $\Lambda \subset \Lambda'$, we obtain the inclusion $R_{\Lambda} \subset R_{\Lambda'}$, and let $R$ be the “limit” of the $R_{\Lambda}$ spaces as $\Lambda \nearrow X$. Evidently, $R$ does not coincide with the whole $L^2(\sigma)$ (an arbitrary function from the space}

$$e^{-\tilde{\sigma}(\Lambda_{mk})} \bigoplus_{n=0}^\infty \frac{1}{n!} \dot{L}^2(\Lambda; \sigma_{\Lambda}) \otimes \dot{H}^n,$$
where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$ in $L^2(\tau)$, will be orthogonal to all space $R_\Lambda$ with $\Lambda \subset \Lambda'$, and therefore to $R$.

It is easy to see that, for each fixed $g \in \mathfrak{S}$, all spaces $R_\Lambda$ such that $K_g \subset \Lambda$ are invariant with respect to the operator $V_{\pi_s}(g)$, and hence so is the space $R$, which concludes the proof. ■

As in subsec. 3.1, to any vector field $v \in V_0(X)$ there corresponds a one-parameter subgroup of diffeomorphisms $\psi_t^v$, $t \in \mathbb{R}$. It generates a one-parameter unitary group

$$V_{\pi_s}(\psi_t^v) := \exp[itJ_{\pi_s}(v)], \quad t \in \mathbb{R},$$

where $J_{\pi_s}(v)$ denotes the self adjoint generator of this group. Analogously, to a one-parameter unitary group $\theta_t^a$, $a \in C^\infty_0(X)$, there corresponds a one-parameter unitary group

$$V_{\pi_s}(\theta^a_t) := \exp[itI_{\pi_s}(a)]$$

with a generator $I_{\pi_s}(a)$.

**Proposition 3.3** For any $v \in V_0(X)$ and $a \in C^\infty_0(X)$, the following operator equalities on the domain $FC^{\infty}_b(\mathcal{D}, \Omega^{\mathbb{R}^+})$ hold:

$$J_{\pi_s}(v) = \frac{1}{i} \nabla^\Omega_v + \frac{1}{2i} B_{\pi_s}^v,$$

$$I_{\pi_s}(a) = \frac{1}{i} \nabla^\Omega_a + \frac{1}{2i} B_{\pi_s}^a.$$

**Proof.** These equalities follow immediately from the definition of the directional derivatives $\nabla^\Omega_v$ and $\nabla^\Omega_a$, Theorem 3.1, and the form of the operators $V_{\pi_s}(\psi_t^v)$ and $V_{\pi_s}(\theta^a_t)$. ■

For any $(v, a) \in \mathfrak{g}$, define an operator

$$R_{\pi_s}(v, a) := J_{\pi_s}(v) + I_{\pi_s}(a).$$

By Proposition 3.3,

$$R_{\pi_s}(v, a) = \frac{1}{i} \nabla^\Omega_{(v, a)} + \frac{1}{2i} B^\pi_{(v, a)}.$$

We wish to derive now a commutation relation between these operators.

**Lemma 3.3** The Lie-bracket $[(v_1, a_1), (v_2, a_2)]$ of the vector fields $(v_1, a_1), (v_2, a_2) \in \mathfrak{g}$, i.e., a vector field from $\mathfrak{g}$ such that

$$\nabla_{[(v_1, a_1), (v_2, a_2)]} = \nabla_{(v_1, a_1)} \nabla_{(v_2, a_2)} \nabla_{(v_2, a_2)} - \nabla_{(v_2, a_2)} \nabla_{(v_1, a_1)}$$

on $\mathcal{D}$, is given by

$$[(v_1, a_1), (v_2, a_2)] = ([v_1, v_2], \nabla_{v_1} a_2 - \nabla_{v_2} a_1),$$

where $[v_1, v_2]$ is the Lie-bracket of the vector fields $v_1, v_2$ on $X$. 28
Proof. The lemma is obtained by a direct computation if one uses the following evident relations which hold on $\mathcal{D}$:

$$\begin{align*}
\nabla_{v_1} X \nabla_{v_2} X - \nabla_{v_2} X \nabla_{v_1} X &= \nabla_{[v_1,v_2]} X, \quad v_1, v_2 \in V_0(X), \\
\nabla_{a_1}^\mathbb{R} + \nabla_{a_2}^\mathbb{R} - \nabla_{a_2}^\mathbb{R} \nabla_{a_1}^\mathbb{R} &= 0, \quad a_1, a_2 \in C_0^\infty(X), \\
\nabla_{v} X \nabla_{a}^\mathbb{R} + \nabla_{a}^\mathbb{R} \nabla_{v} X &= \nabla_{a}^\mathbb{R}, \quad v \in V_0(X), \ a \in C_0^\infty(X). \ 
\end{align*}$$

Thus, the operators $\left(\pi_a^\mathbb{R}, \nabla_{\Omega} \right)$ immediately imply

$\left(\pi_a^\mathbb{K}, \nabla_{\Omega} \right) = 0$.

Therefore, by using the chain rule, we conclude that the lemma will be proved if we show that

$$\nabla_{[v_1,a_1]} \nabla_{[v_2,a_2]} - \nabla_{[v_2,a_2]} \nabla_{[v_1,a_1]} = \nabla_{[(v_1,a_1),(v_2,a_2)]}$$

on $\mathcal{F}_b^\infty(\mathfrak{D}, \Omega^\mathbb{R}_X)$.

In particular,

$$\begin{align*}
\left[J_{\pi_a}(v_1), J_{\pi_a}(v_2)\right] &= -iJ_{\pi_a}([v_1,v_2]), \quad v_1, v_2 \in V_0(X), \\
\left[I_{\pi_a}(a_1), I_{\pi_a}(a_2)\right] &= 0, \quad a_1, a_2 \in C_0^\infty(X), \\
\left[J_{\pi_a}(v), I_{\pi_a}(a)\right] &= -iI_{\pi_a}(\nabla_{v} a), \quad v \in V_0(X), \ a \in C_0^\infty(X). 
\end{align*}$$

Proof. First we note that Lemma 3.3 and (3.6) immediately imply

$$\nabla_{(v_1,a_1)} \nabla_{(v_2,a_2)} - \nabla_{(v_2,a_2)} \nabla_{(v_1,a_1)} = \nabla_{[(v_1,a_1),(v_2,a_2)]}$$

on $\mathcal{F}_b^\infty(\mathfrak{D}, \Omega^\mathbb{R}_X)$.

But upon the representation

$$B_{(v,a)}^\pi(\omega) = \langle \nabla_{(v,a)}^\mathbb{K} \log q + \text{div}^\mathbb{K} v + a, \omega \rangle,$$

we easily derive (3.21) again from Lemma 3.3.

Thus, the operators $\mathcal{R}_{\pi_a}(v,a), (v,a) \in \mathfrak{g}$, give a marked Poisson space representation of the Lie algebra $\mathfrak{g}$ of the group $\mathfrak{G}$.



4 Intrinsic Dirichlet forms

on marked Poisson spaces

4.1 Definition of the intrinsic Dirichlet form

We start with introducing some useful spaces of smooth cylinder functions on $\Omega^\mathbb{R}_X$ in addition to $\mathcal{F}^\infty(\mathfrak{D}, \Omega^\mathbb{R}_X)$. By $\mathcal{F}^\mathbb{P}(\mathfrak{D}, \Omega^\mathbb{R}_X)$ we denote the set of all cylinder
functions of the form (3.5) in which the generating function \( g_F \) is a polynomial on \( \mathbb{R}^N \), i.e., \( g_F \in \mathcal{P}(\mathbb{R}^N) \). Analogously, we define \( \mathcal{F}_p^\infty(\mathcal{D}, \Omega_{X}^{\mathbb{R}^+}) \) where \( f_F \in C_p^\infty(\mathbb{R}^N) \) (=the set of all \( C^\infty \)-functions \( f \) on \( \mathbb{R}^N \) such that \( f \) and its partial derivatives of any order are polynomially bounded).

We have obviously

\[
\mathcal{F}_p^\infty(\mathcal{D}, \Omega_{X}^{\mathbb{R}^+}) \subset \mathcal{F}_b^\infty(\mathcal{D}, \Omega_{X}^{\mathbb{R}^+}),
\]

and these are algebras with respect to the usual operations. The existence of the Laplace transform \( \ell_{\pi_\sigma}(f) \) for each \( f \in C_{0,b}(X \times \mathbb{R}^+) \) implies, in particular, that \( \mathcal{F}_p^\infty(\mathcal{D}, \Omega_{X}^{\mathbb{R}^+}) \subset L^2(\pi_\sigma). \)

**Definition 4.1** For \( F, G \in \mathcal{F}_p^\infty(\mathcal{D}, \Omega_{X}^{\mathbb{R}^+}) \), we introduce a pre-Dirichlet form as

\[
\mathcal{E}_{\pi_\sigma}(F, G) = \int_{\Omega_{X}^{\mathbb{R}^+}} \langle \nabla^\Omega F(\omega), \nabla^\Omega G(\omega) \rangle_{T_{\omega}(\Omega_{X}^{\mathbb{R}^+})} \pi_\sigma(d\omega). \tag{4.1}
\]

Note that, for all \( F \in \mathcal{F}_p^\infty(\mathcal{D}, \Omega_{X}^{\mathbb{R}^+}) \), the formula (3.9) is still valid and therefore, for \( F = g_F(\langle \varphi_1, \cdot \rangle, \ldots, \langle \varphi_N, \cdot \rangle) \) and \( G = g_G(\langle \xi_1, \cdot \rangle, \ldots, \langle \xi_M, \cdot \rangle) \) from \( \mathcal{F}_p^\infty(\mathcal{D}, \Omega_{X}^{\mathbb{R}^+}) \), we have

\[
\langle \nabla^\Omega F(\omega), \nabla^\Omega G(\omega) \rangle_{T_{\omega}(\Omega_{X}^{\mathbb{R}^+})} = \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{\partial g_F}{\partial r_j}(\langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_N, \omega \rangle) \frac{\partial g_G}{\partial r_k}(\langle \xi_1, \omega \rangle, \ldots, \langle \xi_M, \omega \rangle) \times
\]

\[
\times \int_X \langle \nabla X, \varphi_j(x, s_x), \nabla X, \xi_k(x, s_x) \rangle_{T_{(x, s_x)}(X \times \mathbb{R}^+)} \gamma(dx)
\]

\[
= \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{\partial g_F}{\partial r_j}(\langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_N, \omega \rangle) \frac{\partial g_G}{\partial r_k}(\langle \xi_1, \omega \rangle, \ldots, \langle \xi_M, \omega \rangle) \times
\]

\[
\times \langle \nabla X, \varphi_j, \nabla X, \xi_k \rangle_{T(X \times \mathbb{R}^+), \omega}. \tag{4.2}
\]

Since for \( \varphi, \xi \in \mathcal{D} \), the function

\[
\langle \nabla X, \varphi(x, s), \nabla X, \xi(x, s) \rangle_{T_{(x, s)}(X \times \mathbb{R}^+)} = \langle \nabla X, \varphi(x, s), \nabla X, \xi(x, s) \rangle_{T_{x}(X)} + \nabla_{\mathbb{R}^+} \varphi(x, s) \nabla_{\mathbb{R}^+} \xi(x, s)
\]

belongs to \( C_{0,b}(X \times \mathbb{R}^+) \) (see (3.4)), we conclude that

\[
\langle \nabla^\Omega F(\cdot), \nabla^\Omega (\cdot) \rangle_{T(\Omega_{X}^{\mathbb{R}^+})} \in L^1(\pi_\sigma), \quad F, G \in \mathcal{F}_p^\infty(\mathcal{D}, \Omega_{X}^{\mathbb{R}^+}).
\]
and so (4.1) is well defined.

We will call $\mathcal{E}_{π_Ω}$ the intrinsic pre-Dirichlet form corresponding to the marked Poisson measure $π_Ω$ on $Ω^R_+$. In the next subsection we will prove the closability of $\mathcal{E}_{π_Ω}$.

### 4.2 Intrinsic Dirichlet operators

Let us introduce a differential operator $H_{Ω}^{Ω} π_Ω$ on the domain $\mathcal{F}C^∞_b (D, Ω^R_+ X)$ which is given on any $F ∈ \mathcal{F}C^∞_b (D, Ω^R_+ X)$ of the form (3.5) by the formula

$$\langle H_{Ω}^{Ω} π_Ω F, ω \rangle := -\sum_{j=1}^{N} \frac{∂gF}{∂r_j}(⟨φ_1, ω⟩, \ldots, ⟨φ_N, ω⟩) \times$$

$$\times \int_{X×R_+} [\Delta^X φ_j(x, s) + s^2 \frac{∂^2}{∂s^2}φ_j(x, s)$$

$$+ ⟨∇^X φ_j(x, s), ∇^X ξ(x, s)⟩_{T(x, s)}(X×R_+) + 2s \frac{∂}{∂s}φ_j(x, s)] ω(dx, ds),$$

where $\Delta^X$ denotes the Laplace–Beltrami operator corresponding to $∇^X$. Since

$$⟨∇^X φ_j(x, s), ∇^X ξ(x, s)⟩_{T(x, s)}(X×R_+) ∈ L^2(π_Ω) \cap L^1(π_Ω)$$

(see subsec. 3.2, in particular, the condition (3.11)), the r.h.s. of (4.3) is well defined as an element of $L^2(π_Ω)$. To show that the operator $H_{π_Ω}^{Ω}$ is well defined, we still have to show that its definition does not depend on the representation of $F$ in (3.5), which will be done below.

Let us consider also the pre-Dirichlet operator corresponding to the measure $π_Ω$ on $X × R_+$ and to the gradient $∇^{X×R_+}$:

$$\mathcal{E}_{π_Ω}^{X×R_+}(φ, ξ) := \int_{X×R_+} ⟨∇^{X×R_+} φ(x, s), ∇^{X×R_+} ξ(x, s)⟩_{T(x, s)}(X×R_+) π_Ω(dx, ds)$$

$$= \int_{X×R_+} [⟨∇^X φ(x, s), ∇^X ξ(x, s)⟩_{T(x)}(X) + ∇^{R_+} φ(x, s)∇^{R_+} ξ(x, s)] π_Ω(dx, ds),$$

where $φ, ξ ∈ \mathcal{D}$. This form is associated with the Dirichlet operator

$$H_{π_Ω}^{X×R_+} := H_{π_Ω}^{X} + H_{π_Ω}^{R_+}$$

(4.5)
on $\mathcal{D}$ which satisfies
\begin{equation}
\mathcal{E}_\sigma^{X\times\mathbb{R}^+}(\varphi, \xi) = (H_\sigma^{X\times\mathbb{R}^+}\varphi, \xi)_{L^2(\sigma)}, \quad \varphi, \xi \in \mathcal{D}.
\end{equation}

Here, $H_\sigma^X$ and $H_\sigma^{R+}$ are the Dirichlet operators of $\nabla^X$ and $\nabla^{R+}$, respectively. They are given by the formulas
\begin{align}
H_\sigma^X\varphi(x, s) &:= -\Delta^X \varphi(x, s) - \langle \nabla^X \log q(x, s), \nabla^X \varphi(x, s) \rangle_{T_x(\mathcal{X})}, \\
H_\sigma^{R+}\varphi(x, s) &:= -\Delta^{R+} \varphi(x, s) - \nabla^{R+} \log q(x, s)\nabla^{R+} \varphi(x, s),
\end{align}

where
\begin{align}
\Delta^{R+} = \text{div}^{R+} \nabla^{R+} &= -(\nabla^{R+})^* \nabla^{R+} \\
&= s^2 \frac{\partial^2}{\partial s^2} + 2s \frac{\partial}{\partial s} = s^2 \frac{\partial^2}{\partial s^2} + 2\nabla^{R+}.
\end{align}

The closure of the form $\mathcal{E}_\sigma^{X\times\mathbb{R}^+}$ on $L^2(\pi_\sigma)$ is denoted by $(\mathcal{E}_\sigma^{X\times\mathbb{R}^+}, D(\mathcal{E}_\sigma^{X\times\mathbb{R}^+}))$. This form generates a positive selfadjoint operator in $L^2(\pi_\sigma)$ (the so-called Friedrichs extension of $H_\sigma^{X\times\mathbb{R}^+}$, see e.g. [6]). For this extension we preserve the notation $H_\sigma^{X\times\mathbb{R}^+}$ and denote the domain by $D(H_\sigma^{X\times\mathbb{R}^+})$.

Using the underlying Dirichlet operator, we obtain the representation
\begin{align}
(H_\pi^\Omega F)(\omega) &:= -\sum_{j,k=1}^N \frac{\partial^2 F}{\partial r_j \partial r_k}(\langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_n, \omega \rangle)\langle \nabla^{X\times\mathbb{R}^+} \varphi_j, \nabla^{X\times\mathbb{R}^+} \varphi_k \rangle_{T(X\times\mathbb{R}^+), \omega} \\
&\quad + \sum_{j=1}^N \frac{\partial F}{\partial r_j}(\langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_n, \omega \rangle)\langle H_\sigma^{X\times\mathbb{R}^+} \varphi_j, \omega \rangle.
\end{align}

The following theorem implies that $H_\pi^\Omega$ is well defined as a linear operator on $FC_b^\infty(\mathcal{D}, \Omega_X^{\mathbb{R}^+})$, i.e., independently of the representation of $F$ as in (3.5).

**Theorem 4.1** The operator $H_\pi^\Omega$ is associated with the intrinsic Dirichlet form $\mathcal{E}_\pi^\Omega$, i.e., for all $F, G \in FC_b^\infty(\mathcal{D}, \Omega_X^{\mathbb{R}^+})$
\begin{equation}
\mathcal{E}_\pi^\Omega(F, G) = (H_\pi^\Omega F, G)_{L^2(\pi_\sigma)};
\end{equation}
or
\begin{equation}
H_\pi^\Omega = -\text{div}^\Omega \nabla^\Omega \quad \text{on} \quad FC_b^\infty(\mathcal{D}, \Omega_X^{\mathbb{R}^+}).
\end{equation}

We call $H_\pi^\Omega$ the intrinsic Dirichlet operator of the measure $\pi_\sigma$. 

32
Proof. For the shortness of notations we will prove the formula (4.9) in the case where \( F, G \in \mathcal{F}C_b^\infty(\mathfrak{D}, \Omega_X^{R^+}) \) are of the form

\[
F = g_F(\langle \varphi, \omega \rangle), \quad G = g_G(\langle \xi, \omega \rangle).
\]

However, it is a trivial step to generalize the proof for general \( F, G \).

Let \( \Lambda \in \mathcal{O}_c(X) \) be chosen so that the supports of the functions \( \varphi \) and \( \xi \) are in \( \Lambda_{nk} \). Then, by (4.1) and (4.2)

\[
\mathcal{E}_{\pi^\Lambda}(\varphi, \omega) = \int_{\Omega_X^+} \left( \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda_{nk}} \nabla X^{R^+}_i \varphi(x_i, s_i) \nabla X^{R^+}_i \xi(x_i, s_i) \right) \cdot \sigma(dx, ds) + \pi_\sigma(d\omega)
\]

where \( \nabla X^{R^+}_i \) denotes the \( \nabla X^{R^+} \) gradient in the \( (x_i, s_i) \) variables. We note that the vector field

\[
\nabla X^{R^+}_i g_F(\varphi(x_1, s_1) + \cdots + \varphi(x_n, s_n)) = g'_F(\varphi(x_1, s_1) + \cdots + \varphi(x_n, s_n)) \nabla X^{R^+}_i \varphi(x_i, s_i)
\]

has support in the \( x_i \) variable in \( \Lambda \) and the vector field (function)

\[
\nabla X^{R^+}_i g_F(\varphi(x_1, s_1) + \cdots + \varphi(x_n, s_n)) = g'_F(\varphi(x_1, s_1) + \cdots + \varphi(x_n, s_n)) s_i \frac{\partial}{\partial s_i} \varphi(x_i, s_i)
\]

is bounded, while \( s_i q(x_i, s_i) \) is equal to zero as \( s_i = 0 \) and \( s_i = \infty \) for \( m \)-a.a. \( x \in X \).

Therefore,

\[
\mathcal{E}_{\pi^\Lambda}(F, G) = e^{-\tilde{\sigma}(\Lambda_{nk})} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda_{nk}} \left( \sum_{i=1}^{n} H^{(X^{R^+})}_{\tilde{\sigma}} g_F(\varphi(x_1, s_1) + \cdots + \varphi(x_n, s_n)) \right) \times \sigma(dx, ds) + \pi_\sigma(d\omega)
\]

\[
= -e^{-\tilde{\sigma}(\Lambda_{nk})} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda_{nk}} \left( \sum_{i=1}^{n} g'_F(\varphi(x_1, s_1) + \cdots + \varphi(x_n, s_n)) \right) \times \sigma(dx, ds) + \pi_\sigma(d\omega)
\]

\[
+ g'_F(\varphi(x_1, s_1) + \cdots + \varphi(x_n, s_n)) \left( \nabla X^{R^+} \log q(x_i, s_i), \nabla X^{R^+} \varphi(x_i, s_i) \right)_{T(x_i, s_i)(X^{R^+})}
\]

\[
+ g'_F(\varphi(x_1, s_1) + \cdots + \varphi(x_n, s_n)) \left( \nabla X^{R^+} \varphi(x_i, s_i), \nabla X^{R^+} \varphi(x_i, s_i) \right)_{T(x_i, s_i)(X^{R^+})}
\]

33
\[ + \Delta^X \varphi(x_i, s_i) + \Delta^{\mathbb{R}^+} \varphi(x_i, s_i) \right] \times \times g_G(\xi(x_1, s_1) + \cdots + \xi(x_n, s_n)) \tilde{\sigma}(dx_1, ds_1) \cdots \tilde{\sigma}(dx_n, ds_n) \]

\[ = \int_{\Omega^X} F(\omega) G(\omega) \pi_{\tilde{\sigma}}(d\omega). \]

**Remark 4.1** The operator \( H_{\tilde{\sigma}}^\Omega \) can be naturally extended to cylinder functions of the form

\[ F(\omega) := e^{\langle \omega, \omega \rangle}, \quad \varphi \in \mathcal{D}, \omega \in \Omega^X_{\tilde{\sigma}}; \]

since such \( F \) belong to \( L^2(\pi_{\tilde{\sigma}}) \). We then have

\[ H_{\tilde{\sigma}}^\Omega e^{\langle \varphi, \omega \rangle} = \langle H_{\tilde{\sigma}}^{X \times \mathbb{R}^+} \varphi - |\nabla^{X \times \mathbb{R}^+} \varphi|_{\tilde{T}(X \times \mathbb{R}^+)}^2, \omega \rangle e^{\langle \varphi, \omega \rangle}. \] (4.10)

As an immediate consequence of Theorem 4.1 we obtain

**Corollary 4.1** \((\mathcal{E}_{\tilde{\sigma}}^\Omega, \mathcal{F}C_b^\infty(\mathcal{D}, \Omega_{X \tilde{T}}^{\mathbb{R}^+}))\) is closable on \( L^2(\pi_{\tilde{\sigma}}) \). Its closure \((\mathcal{E}_{\tilde{\sigma}}^\Omega, D(\mathcal{E}_{\tilde{\sigma}}^\Omega))\) is associated with a positive definite selfadjoint operator, the Friedrichs extension of \( H_{\tilde{\sigma}}^\Omega \), which we also denote by \( H_{\tilde{\sigma}}^\Omega \) (and its domain by \( D(H_{\tilde{\sigma}}^\Omega) \)).

Clearly, \( \nabla^\Omega \) also extends to \( D(\mathcal{E}_{\tilde{\sigma}}^\Omega) \). We denote this extension by \( \nabla^\Omega \).

**Corollary 4.2** Let

\[ F(\omega) := g_F(\langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_N, \omega \rangle), \quad \omega \in \Omega^X_{\tilde{\sigma}}; \]

\[ \varphi_1, \ldots, \varphi_N \in D(\mathcal{E}_{\tilde{\sigma}}^{X \times \mathbb{R}^+}), \quad g_F \in C_b^\infty(\mathbb{R}^N). \] (4.11)

Then \( F \in D(\mathcal{E}_{\tilde{\sigma}}^\Omega) \) and

\[ (\nabla^\Omega F)(\omega; x) = \sum_{j=1}^N \frac{\partial g_F}{\partial r_j}( \langle \varphi_1, \omega \rangle, \ldots, \langle \varphi_N, \omega \rangle ) \nabla^{X \times \mathbb{R}^+} \varphi_j(x, s_x). \]

**Proof.** By approximation this is an immediate consequence of (3.9) and the fact that, for all \( 1 \leq i \leq N \),

\[ \int \langle |\nabla^{X \times \mathbb{R}^+} \varphi_i|_{\tilde{T}(X \times \mathbb{R}^+)}^2, \omega \rangle \pi_{\tilde{\sigma}}(d\omega) = \mathcal{E}_{\tilde{\sigma}}^{X \times \mathbb{R}^+}(\varphi_i, \varphi_i). \] (4.12)

**Remark 4.2** Let \( \mu_{\nu, \tilde{\sigma}} \in \mathcal{M}_2(\Omega^{\mathbb{R}^+}_X) \) be given as in (3.16). Then, by Theorem 3.2, \( (ii) \Rightarrow (i) \), all results above are valid with \( \mu_{\nu, \tilde{\sigma}} \) replacing \( \pi_{\tilde{\sigma}} \). By (4.8) we have

\[ H_{\pi_{\tilde{\sigma}}}^\Omega = H_{\mu_{\nu, \tilde{\sigma}}}^\Omega \quad \text{on} \quad \mathcal{F}C_b^\infty(\mathcal{D}, \Omega_{X \tilde{T}}^{\mathbb{R}^+}). \]

We note that the r.h.s. of (4.8) only depends on \( \tilde{\sigma} \) and the Riemannian structure of \( X \times \mathbb{R}^+ \). The respective Friedrichs extensions on \( L^2(\mu_{\nu, \tilde{\sigma}}) \) again denoted by \( H_{\mu_{\nu, \tilde{\sigma}}}^\Omega \), however do not coincide.
4.3 The heat semigroup and ergodicity

The results of this subsection are obtained absolutely analogously to the corresponding results of the paper [4], so we omit the proofs.

For $\mu_{\nu,\tilde{\sigma}} \in \mathcal{M}_2(\Omega_{X}^{\mathbb{R}^+})$ let $T_{\mu_{\nu,\tilde{\sigma}}}^{\Omega}(t) := \exp(-tH_{\mu_{\nu,\tilde{\sigma}}})$, $t > 0$. Define

$$E(\mathcal{D}_1, \Omega_{X}^{\mathbb{R}^+}) = \text{l.h.} \left\{ \exp(\langle \log(1 + \varphi), \cdot \rangle) \mid \varphi \in \mathcal{D}_1 \right\},$$

where l.h. means the linear hull and

$$\mathcal{D}_1 := \left\{ \varphi \in D(H_{\sigma}^{X \times \mathbb{R}^+}) \cap L^1(\tilde{\sigma}) \mid H_{\sigma}^{X \times \mathbb{R}^+} \varphi \in L^1(\tilde{\sigma}) \right\},$$

and $-\delta \leq \varphi \leq 0$ for some $\delta \in (0, 1)$.

**Proposition 4.1** Let $\mu_{\nu,\tilde{\sigma}}$ be as in (3.16). Assume that $H_{\sigma}^{X \times \mathbb{R}^+}$ is conservative, i.e.,

$$\int_{X \times \mathbb{R}^+} (H_{\sigma}^{X \times \mathbb{R}^+} \varphi)(x, s) \tilde{\sigma}(dx, ds) = 0$$

for all $\varphi \in D(H_{\sigma}^{X \times \mathbb{R}^+}) \cap L^1(\tilde{\sigma})$ such that $H_{\sigma}^{X \times \mathbb{R}^+} \varphi \in L^1(\tilde{\sigma})$, and suppose that $(H_{\sigma}^{X \times \mathbb{R}^+}, \mathcal{D})$ is essentially selfadjoint on $L^2(\tilde{\sigma})$. Then

$$T_{\mu_{\nu,\tilde{\sigma}}}^{\Omega}(t) \exp(\langle \log(1 + \varphi), \cdot \rangle) = \exp(\langle \log(1 + e^{-tH_{\sigma}^{X \times \mathbb{R}^+}} \varphi), \cdot \rangle), \quad \varphi \in \mathcal{D}_1, \quad (4.13)$$

$$E(\mathcal{D}_1, \Omega_{X}^{\mathbb{R}^+}) \subset D(H_{\mu_{\nu,\tilde{\sigma}}}),$$

and

$$H_{\mu_{\nu,\tilde{\sigma}}}^{\Omega} \exp(\langle \log(1 + \varphi), \cdot \rangle) = \langle (1 + \varphi)^{-1}H_{\sigma}^{X \times \mathbb{R}^+} \varphi, \cdot \rangle \exp(\langle \log(1 + \varphi), \cdot \rangle), \quad \varphi \in \mathcal{D}_1.$$}

**Remark 4.3** (i) The condition of selfadjointness of $H_{\sigma}^{X \times \mathbb{R}^+}$ on $\mathcal{D}$ is fulfilled if $X$ is complete and $|\beta_{\tilde{\sigma}}|_{T(X \times \mathbb{R}^+)} \in L^p_{\text{loc}}(X; m) \otimes L^p(\mathbb{R}^+_\lambda)$ for some $p \geq \dim(X) + 1$.

(ii) Since $(\exp(-tH_{\sigma}^{X \times \mathbb{R}^+}))_{t>0}$ is sub-Markovian (i.e., $0 \leq \exp(-tH_{\sigma}^{X \times \mathbb{R}^+}) \varphi \leq 1$ for all $t > 0$ and $\varphi \in L^2(\tilde{\sigma})$, $0 \leq \varphi \leq 1$), because $(\mathcal{E}_{\sigma}^{X \times \mathbb{R}^+}, D(\mathcal{E}_{\sigma}^{X \times \mathbb{R}^+}))$ is a Dirichlet form, by a simple approximation argument Proposition 4.1 implies that the equality (4.13) holds for $t > 0$ and all $\varphi \in L^1(\tilde{\sigma})$, $-1 < \varphi \leq 0$.

**Theorem 4.2** Let the conditions of Proposition 4.1 hold. Then $E(\mathcal{D}_1, \Omega_{X}^{\mathbb{R}^+})$ is an operator core for the Friedrichs extension $H_{\mu_{\nu,\tilde{\sigma}}}^{\Omega}$ on $L^2(\mu_{\nu,\tilde{\sigma}})$. (In other words: $(H_{\mu_{\nu,\tilde{\sigma}}}^{\Omega}, E(\mathcal{D}_1, \Omega_{X}^{\mathbb{R}^+}))$ is essentially selfadjoint on $L^2(\mu_{\nu,\tilde{\sigma}})$.)
Theorem 4.3 Suppose that the conditions of Theorem 3.2 and Proposition 4.1 hold. Then the following assertions are equivalent:

(i) \( \mu_{\nu, \tilde{\sigma}} = \pi_{z\tilde{\sigma}} \) for some \( z > 0 \).

(ii) \( (E_{\mu_{\nu, \tilde{\sigma}}}, D(E_{\mu_{\nu, \tilde{\sigma}}})) \) is irreducible (i.e., for \( F \in D(E_{\mu_{\nu, \tilde{\sigma}}}) \), \( E_{\mu_{\nu, \tilde{\sigma}}}(F, F) = 0 \) implies that \( F = \text{const} \)).

(iii) \( (T_{\mu_{\nu, \tilde{\sigma}}}^{\Omega}(t))_{t>0} \) is irreducible (i.e., if \( G \in L^2(\mu_{\nu, \tilde{\sigma}}) \) such that \( T_{\mu_{\nu, \tilde{\sigma}}}^{\Omega}(t)(GF) = GT_{\mu_{\nu, \tilde{\sigma}}}^{\Omega}(t)F \) for all \( F \in L^\infty(\mu_{\nu, \tilde{\sigma}}), t > 0 \), then \( G = \text{const} \)).

(iv) If \( F \in L^2(\mu_{\nu, \tilde{\sigma}}) \) such that \( T_{\mu_{\nu, \tilde{\sigma}}}^{\Omega}(t)F = F \) for all \( T > 0 \), then \( F = \text{const} \).

(v) \( T_{\mu_{\nu, \tilde{\sigma}}}^{\Omega}(t) \not\equiv 1 \) and ergodic (i.e.,

\[
\int \left( T_{\mu_{\nu, \tilde{\sigma}}}^{\Omega}(t)F - \int F \, d\mu_{\nu, \tilde{\sigma}} \right)^2 \, d\mu_{\nu, \tilde{\sigma}} \to 0 \quad \text{as } t \to 0
\]

for all \( F \in L^2(\mu_{\nu, \tilde{\sigma}}) \).

(vi) If \( F \in D(H_{\mu_{\nu, \tilde{\sigma}}}^{\Omega}) \) with \( H_{\mu_{\nu, \tilde{\sigma}}}^{\Omega} = 0 \), then \( F = \text{const} \).

Remark 4.4 Let us consider the diffusion process \( M \) on \( X \times \mathbb{R}_+ \) associated to the Dirichlet form \( (E_{\tilde{\sigma}^X \times \mathbb{R}_+}, D(E_{\tilde{\sigma}^X \times \mathbb{R}_+})) \). This process can be interpreted as distorted Brownian motion on the manifold \( X \times \mathbb{R}_+ \). More precisely, the diffusion of points \( x \in X \) is associated to the Dirichlet form of the measure \( \sigma \), so that it is distorted Brownian motion on \( X \). The diffusion of marks \( s_x, x \in X \), can be obtained by the exponential change of time \( \mathbb{R} \ni t \mapsto L^{-1}t = e^t = s \in \mathbb{R}_+ \) in the distorted Brownian motion on \( \mathbb{R} \) associated to the Dirichlet form of the measure \( \mathcal{L}^*p(x, \cdot) \). This follows from the representation (4.4)–(4.7) of the Dirichlet form \( E_{\tilde{\sigma}^X \times \mathbb{R}_+} \) and Remark 3.1.

The existence of a diffusion process \( M \) corresponding to the Dirichlet form \( (E_{\mu_{\nu, \tilde{\sigma}}}^{\Omega}, D(E_{\mu_{\nu, \tilde{\sigma}}}^{\Omega})) \) and its identification with the independent infinite particle process (on \( X \times \mathbb{R}_+ \)) may be proved by the same arguments as in [4]. By analogy with the case of the process \( M \) on \( X \times \mathbb{R}_+ \), one can call \( M \) distorted Brownian motion on \( \Omega_{X, \mu_{\nu, \tilde{\sigma}}}^{\mathbb{R}_+} \).

5 Intrinsic Dirichlet operator and second quantization

In this section, we want to describe the Fock space realization of the marked Poisson spaces and show that \( H_{\pi_{z\tilde{\sigma}}}^{\Omega} \) is the second quantization of the operator \( H_{\tilde{\sigma}^X \times \mathbb{R}_+}^{\Omega} \).

5.1 Marked Poisson gradient and chaos decomposition

Let us define another “gradient” on functions \( F: \Omega_{X, \mu_{\nu, \tilde{\sigma}}}^{\mathbb{R}_+} \to \mathbb{R} \), which has specific useful properties on the marked Poisson space.
Definition 5.1 For any \( F \in \mathcal{FC}^\infty_p(\mathcal{D}, \Omega^R_X) \) we define the marked Poisson gradient \( \nabla^{MP} \) as

\[
(\nabla^{MP} F)(\omega, (x, s)) := F(\omega + \varepsilon_{(x, s)}) - F(\omega), \quad \omega \in \Omega^R_X, \ (x, s) \in X \times \mathbb{R}_+.
\]

Let us mention that the operation

\[
\Omega^R_X \ni \omega \mapsto \omega + \varepsilon_{(x, s)} \in \Omega^R_X
\]

is a \( \pi_{\tilde{\sigma}} \)-a.e. well-defined map because of the property

\[
\pi_{\tilde{\sigma}}\{(\omega = (\gamma, s) \in \Omega^R_X \mid x \in \gamma)\} = 0
\]

for an arbitrary \( x \in X \) (which easily follows from the construction of \( \pi_{\tilde{\sigma}} \)). We consider \( \nabla^{MP} \) as a mapping

\[
\nabla^{MP}: \mathcal{FC}^\infty_p(\mathcal{D}, \Omega^R_X) \ni F \mapsto \nabla^{MP} F \in L^2(\tilde{\sigma}) \otimes L^2(\pi_{\tilde{\sigma}})
\]

that corresponds to using the Hilbert space \( L^2(\tilde{\sigma}) \) as a tangent space at any point \( \omega \in \Omega^R_X \). Thus, for any \( \varphi \in \mathcal{D} \), we can introduce the directional derivative

\[
(\nabla^{MP}_\varphi F)(\omega) = \langle \nabla^{MP} F(\omega), \varphi \rangle_{L^2(\tilde{\sigma})}
\]

\[
= \int_{X \times \mathbb{R}_+} (F(\omega + \varepsilon_{(x, s)}) - F(\omega)) \varphi(x, s) \tilde{\sigma}(dx, ds).
\]

The most important feature of the marked Poisson gradient is that it produces (via a corresponding “integration by parts formula”) the orthogonal system of Charlier polynomials on \( (\Omega^R_X, \mathcal{B}(\Omega^R_X), \pi_{\tilde{\sigma}}) \). Below, we describe this construction in detail using the isomorphism between \( L^2(\pi_{\tilde{\sigma}}) \) and the Fock space (see [13, 18, 23])

Let \( \mathcal{F}(L^2(\tilde{\sigma})) \) denote the symmetric Fock space over \( L^2(\tilde{\sigma}) \):

\[
\mathcal{F}(L^2(\tilde{\sigma})) := \bigoplus_{n=0}^{\infty} \mathcal{F}_n(L^2(\tilde{\sigma})) n!,
\]

where

\[
\mathcal{F}_n(L^2(\tilde{\sigma})) := (L^2(\tilde{\sigma}))^{\otimes n} = \hat{L}^2((X \times \mathbb{R}_+)^n, \tilde{\sigma}_{\otimes n})
\]

and \( \mathcal{F}_0(L^2(\tilde{\sigma})) := \mathbb{R} \). Thus, for each \( F = (f^{(n)})_{n=0}^\infty \in \mathcal{F}(L^2(\tilde{\sigma})) \)

\[
\|F\|_{\mathcal{F}(L^2(\tilde{\sigma}))}^2 = \sum_{n=0}^{\infty} |f^{(n)}|_{\hat{L}^2(\tilde{\sigma}_{\otimes n})} n!.
\]

By \( \mathcal{F}_{\text{fin}}(\mathcal{D}) \) we denote the dense subset of \( \mathcal{F}(L^2(\tilde{\sigma})) \) consisting of finite sequences \( (f^{(n)})_{n=0}^N, \ n \in \mathbb{Z}_+ \), such that each \( f^{(n)} \) belongs to \( \mathcal{F}_n(\mathcal{D}) := a.\mathcal{D}^{\otimes n} \), the \( n \)-th symmetric algebraic tensor power of \( \mathcal{D} \):

\[
a.\mathcal{D}^{\otimes n} := \{ \varphi_1 \otimes \cdots \otimes \varphi_n \mid \varphi_i \in \mathcal{D} \}.
\]
In virtue of the polarization identity, the latter set is spanned just by the vectors of
the form $\varphi^{\otimes n}$ with $\varphi \in \mathcal{D}$.

Now, we define a linear mapping

$$F_{\text{fin}}(\mathcal{D}) \ni F = (f^{(n)})_{n=0} \mapsto IF = (IF)(\omega) = \sum_{n=0}^{N} Q_n(f^{(n)}; \omega) \in \mathcal{FP}(\mathcal{D}, \Omega^{x}_{\mathbb{R}^+}) \quad (5.1)$$

by using the following recursion relation:

$$Q_{n+1}(\varphi^{\otimes (n+1)}; \omega) = Q_n(\varphi^{\otimes n}; \omega)(\langle \omega, \varphi \rangle - \langle \varphi \rangle_{\tilde{\sigma}})$$

$$- nQ_n(\varphi^{\otimes (n-1)} \tilde{\otimes} (\varphi^2) ; \omega) - nQ_{n-1}(\varphi^{\otimes (n-1)} ; \omega) \langle \varphi^2 \rangle_{\tilde{\sigma}},$$

$$Q_0(1; \omega) = 1, \quad \varphi \in \mathcal{D}. \quad (5.2)$$

Here, we denoted by $\langle \varphi \rangle_{\tilde{\sigma}} := \int \varphi \, d\tilde{\sigma}$. Notice that, since $\mathcal{D}$ is an algebra under
pointwise multiplication of functions, the latter definition is correct.

It is not hard to see that the mapping (5.1) is one-to-one. Moreover, the following
proposition holds:

**Proposition 5.1** The mapping (5.1) can be extended by continuity to a unitary
isomorphism between the spaces $\mathcal{F}(L^2(\tilde{\sigma}))$ and $L^2(\pi_{\tilde{\sigma}})$.

For each $\varphi \in \mathcal{D}$, let us define the creation and annihilation operators in $\mathcal{F}(L^2(\tilde{\sigma}))$
by

$$a^+(\varphi)\psi^{\otimes n} = \varphi \otimes \psi^{\otimes n}, \quad a^-(\varphi)\psi^{\otimes n} = n(\langle \varphi, \psi \rangle_{L^2(\tilde{\sigma})}) \psi^{\otimes (n-1)}, \quad \psi \in \mathcal{D}.$$

We will denote by the same letters the images of these operators under the
unitary $I$.

**Proposition 5.2** We have, for each $\varphi \in \mathcal{D}$,

$$a^- (\varphi) = \nabla^{\text{MP}}_\varphi, \quad a^+ (\varphi) = \nabla^{\text{MP}^*}_\varphi.$$

In particular,

$$Q_n(\varphi_1 \otimes \cdots \otimes \varphi_n; \omega) = (\nabla^{\text{MP}^*}_\varphi_1 \cdots \nabla^{\text{MP}^*}_\varphi_n 1)(\omega), \quad \omega \in \Omega^{x}_{\mathbb{R}^+}.$$

Finally, for each $\varphi \in \mathcal{D}$ we introduce the Poisson exponential

$$e(\varphi; \cdot) := \sum_{n=0}^{\infty} \frac{1}{n!} Q_n(\varphi^{\otimes n}; \cdot) = I(\text{Exp} \varphi),$$

where

$$\text{Exp} \varphi = \left( \frac{1}{n!} \varphi^{\otimes n} \right)_{n=0}^{\infty}.$$

Then, one can show that, for $\varphi > -1$,

$$e(\varphi; \omega) = \exp \left[ \langle \log (1 + \varphi), \omega \rangle - \langle \varphi \rangle_{\tilde{\sigma}} \right], \quad \omega \in \Omega^{x}_{\mathbb{R}^+}. \quad (5.3)$$
5.2 Second quantization on the marked Poisson space

Let $B$ be a contraction on $L^2(\tilde{\sigma})$, i.e., $B \in \mathcal{L}(L^2(\tilde{\sigma}), L^2(\tilde{\sigma}))$, $\|B\| \leq 1$. Then, we can define the operator $\text{Exp} B$ as the contraction on $\mathcal{F}(L^2(\tilde{\sigma}))$ given by

$$\text{Exp} B \upharpoonright \mathcal{F}_n(L^2(\tilde{\sigma})) := B \otimes \cdots \otimes B \quad (n \text{ times}), \ n \in \mathbb{N},$$

$$\text{Exp} B \upharpoonright \mathcal{F}_0(L^2(\tilde{\sigma})) := 1.$$

For any selfadjoint positive operator $A$ in $L^2(\tilde{\sigma})$, we have a contraction semigroup $e^{-tA}, \ t \geq 0$, and it is possible to introduce a positive selfadjoint operator $d \text{Exp} A$ as the generator of the semigroup $\text{Exp}(e^{-tA}), \ t \geq 0$:

$$\text{Exp}(e^{-tA}) = \exp(-td \text{Exp} A). \quad (5.4)$$

The operator $d \text{Exp} A$ is called the second quantization of $A$. We denote by $H^\text{MP}_A$ the image of the operator $d \text{Exp} A$ in the marked Poisson space $L^2(\pi_{\tilde{\sigma}})$.

**Theorem 5.1** Let $\mathfrak{D} \subset \text{Dom} A$. Then, the symmetric bilinear form corresponding to the operator $H^\text{MP}_A$ has the following representation:

$$(H^\text{MP}_A F, G)_{L^2(\pi_{\tilde{\sigma}})} = \int_{\Omega^+_X} (\nabla^\text{MP} F, A \nabla^\text{MP} G)_{L^2(\tilde{\sigma})} \pi_{\tilde{\sigma}}(d\omega) \quad (5.5)$$

for all $F, G \in \mathcal{F}(\mathfrak{D}, \Omega^+_X)$.

**Remark 5.1** The bilinear form (5.5) uses the marked Poisson gradient $\nabla^\text{MP}$ and a coefficient operator $A > 0$. We will call

$$\mathcal{E}^\text{MP}_{\pi_{\tilde{\sigma}}, A}(F, G) = \int_{\Omega^+_X} (\nabla^\text{MP} F, A \nabla^\text{MP} G)_{L^2(\tilde{\sigma})} \pi_{\tilde{\sigma}}(d\omega)$$

the marked Poisson pre-Dirichlet form with coefficient $A$.

**Proof of Theorem 5.1.** The proof is analogous to that of Theorem 5.1 in [4]. Using again the fact that $\mathfrak{D}$ is an algebra under pointwise multiplication, one easily concludes that, for any $F \in \mathcal{F}(\mathfrak{D}, \Omega^+_X)$ and any $\omega \in \Omega^+_X$, the gradient $\nabla^\text{MP} F(\omega, (x, s))$ is a function in $\mathfrak{D}$ and hence

$$(\nabla^\text{MP} F, A \nabla^\text{MP} G)_{L^2(\tilde{\sigma})} \in \mathcal{F}(\mathfrak{D}, \Omega^+_X),$$

so that the form (5.5) is well-defined. Then, one verifies the formula (5.5) by using Propositions 5.1, 5.2 and the explicit formula for $d \text{Exp} A$ on $\mathcal{F}_n(\mathfrak{D})$:

$$d \text{Exp} A \varphi^\otimes n = n(A \varphi) \widehat{\otimes} \varphi^\otimes (n-1), \quad \varphi \in \mathfrak{D}. \quad \blacksquare$$
5.3 The intrinsic Dirichlet operator as a second quantization

The following two theorems are again analogous to corresponding results (Theorems 5.2 and 5.3) in [4], so we omit their proofs.

Let us consider the special case of the second quantization operator \( \exp A \) where the operator \( A \) coincides with the Dirichlet operator \( H_{\sigma}^{X \times \mathbb{R}^+} \).

**Theorem 5.2** We have the equality

\[
H_{H_{\sigma}^{X \times \mathbb{R}^+}}^{\Omega} = H_{\pi_{\sigma}}^{\Omega}
\]

on the dense domain \( \mathcal{F}C_\mathcal{D}_\mathcal{F}(\Omega_{X,\mathbb{R}^+}) \). In particular, for all \( F, G \in \mathcal{F}C_\mathcal{D}_\mathcal{E}(\Omega_{X,\mathbb{R}^+}) \)

\[
\int_{\Omega_{X,\mathbb{R}^+}} \langle \nabla^\Omega F(\omega), \nabla^\Omega G(\omega) \rangle_{\Omega_{\mathbb{R}^+},\pi_{\sigma}}(d\omega)
= \int_{\Omega_{X,\mathbb{R}^+}} \langle \nabla^{\Omega_{\mathbb{R}^+}} F(\omega), H_{\sigma}^{X \times \mathbb{R}^+} \nabla^{\Omega_{\mathbb{R}^+}} G(\omega) \rangle_{L^2(\sigma)}(d\omega),
\]

or

\[
\nabla^\Omega \nabla = \nabla^{\Omega_{\mathbb{R}^+}} H_{\sigma}^{X \times \mathbb{R}^+} \nabla^{\Omega_{\mathbb{R}^+}}
\]

as an equality on \( \mathcal{F}C_\mathcal{D}_\mathcal{E}(\Omega_{X,\mathbb{R}^+}) \).

**Theorem 5.3** Suppose that the operator \( H_{\sigma}^{X \times \mathbb{R}^+} \) is essentially selfadjoint on the domain \( \mathcal{D} \subset \text{Dom}(H_{\sigma}^{X \times \mathbb{R}^+}) \). Then, the intrinsic Dirichlet operator \( H_{\pi_{\sigma}}^{\Omega} \) is essentially selfadjoint on the domain \( \mathcal{F}C_\mathcal{D}_\mathcal{F}(\Omega_{X,\mathbb{R}^+}) \).

**Remark 5.2** Notice that in Theorem 5.3 we do not suppose the operator \( H_{\sigma}^{X \times \mathbb{R}^+} \) to be conservative. So, this theorem is a generalization of Theorem 4.2 in the special case where \( \mu_{\nu,\tilde{\sigma}} = \pi_{\tilde{\sigma}} \).

**Corollary 5.1** Suppose that the condition of Theorem 5.3 is satisfied and let \( T_{\pi_{\sigma}}(t) = \exp(-tH_{\pi_{\sigma}}^{\Omega}), t > 0 \). Then, for each \( \varphi \in \mathcal{D}, \varphi > -1 \), we have

\[
T_{\pi_{\sigma}}(t) \exp(\langle \log(1+\varphi), \cdot \rangle) = \exp \left[ \langle \log(1+e^{-tH_{\sigma}^{X \times \mathbb{R}^+}} \varphi), \cdot \rangle - \langle (e^{-tH_{\sigma}^{X \times \mathbb{R}^+}} - 1) \varphi \rangle_{\sigma} \right]. \quad (5.6)
\]

**Proof.** The formula (5.6) follows from Proposition 5.1, (5.3), (5.4) and Theorems 5.2 and 5.3. ■

**Remark 5.3** If \( H_{\sigma}^{X \times \mathbb{R}^+} \) is conservative, then

\[
\int (e^{-tH_{\sigma}^{X \times \mathbb{R}^+}} - 1) \varphi \, d\sigma = 0 \quad \text{for all } t \geq 0,
\]

and so in this case (5.6) coincides with (4.13) for \( \varphi \in \mathcal{D}, \varphi > -1 \).
6 Appendix

We will prove now the part (i)$\Rightarrow$(ii) of Theorem 3.2. To this end, we present first some definitions and reformulation of known statements.

For any $\Upsilon \in \mathcal{B}(X)$ and $\omega \in \Omega_X^+$, denote $\omega \cap \Upsilon := \omega \cap \Upsilon$.

**Definition 6.1** For $\Lambda \in \mathcal{O}_c(X)$, we define for $\omega \in \Omega_X^+$ and $\Delta \in \mathcal{B}(\Omega_X^+)$

$$\tilde{\Pi}_{\Lambda mk}^\sigma(\omega, \Delta) := \int_{\Omega_X^+} 1_\Delta(\omega(\chi \setminus \Lambda)_{mk} + \omega'_{\Lambda mk}) \pi_{\sigma'}(d\omega' \mid N_{\Lambda mk}(\cdot) = \omega(\Lambda_{mk})). \quad (6.1)$$

A probability measure $\mu$ on $(\Omega_X^+, \mathcal{B}(\Omega_X^+))$ is called a canonical Gibbs measure for the free case if

$$\mu(\Delta) = \int_{\Omega_X^+} \tilde{\Pi}_{\Lambda mk}^\sigma(\omega, \Delta) \mu(d\omega).$$

Let $\mathcal{G}_c(\tilde{\sigma})$ denote the set of all such probability measures.

For $\Lambda \in \mathcal{O}_c(X)$ denote by $\tilde{\mathcal{B}}_{(\chi \setminus \Lambda)_{mk}}(\Omega_X^+)$ the $\sigma$-algebra generated by the $\sigma$-algebra $\mathcal{B}_{(\chi \setminus \Lambda)_{mk}}(\Omega_X^+)$ and the mappings $N_{\Lambda mk}$. Then, one easily deduces the following proposition.

**Proposition 6.1** A probability measure $\mu$ on $(\Omega_X^+, \mathcal{B}(\Omega_X^+))$ belongs to $\mathcal{G}_c(\tilde{\sigma})$ if and only if for each bounded measurable function $G$ on $\Omega_X^+$

$$E_\mu[G \mid \tilde{\mathcal{B}}_{(\chi \setminus \Lambda)_{mk}}(\Omega_X^+)] = \tilde{\Pi}_{\Lambda mk}^\sigma G \quad \mu\text{-a.e.}$$

Here, for a sub-$\sigma$-algebra $\Sigma \subset \mathcal{B}(\Omega_X^+)$, $E_\mu[\cdot \mid \Sigma]$ denotes the conditional expectation w.r.t. $\mu$ and $\Sigma$ and

$$\tilde{\Pi}_{\Lambda mk}^\sigma G = (\tilde{\Pi}_{\Lambda mk}^\sigma G)(\omega) := \int_{\Omega_X^+} G(\omega') \tilde{\Pi}_{\Lambda mk}^\sigma(\omega, d\omega').$$

The following theorem, which is in fact due to [24], see also [10] and [22], can be obtained from the original one by a simple modification of the proof.

**Theorem 6.1** Let $\mu$ be a probability measure on $(\Omega_X^+, \mathcal{B}(\Omega_X^+))$. Then, $\mu \in \mathcal{G}_c(\tilde{\sigma})$ if and only if there exists a probability measure $\nu$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$\mu = \int_{\mathbb{R}_+} \pi_{z\tilde{\sigma}} \nu(dz).$$
Hence, by virtue of Proposition 6.1 and Theorem 6.1, it suffices to show that if 
\( \mu \in (\text{IbP})^\sigma \) then for \( \mu \)-a.e. \( \omega \in \Omega^{\mathbb{R}^+}_X \)
\[
\mathbb{E}_\mu[F \mid \hat{B}_{(X \setminus \Lambda)_{mk}}](\omega) = \hat{\sigma}(\Lambda_{mk})^{-\omega(\Lambda_{mk})} \times 
\int_{\Lambda_{mk}} \cdots \int_{\Lambda_{mk}} F(\varepsilon_{(x_1,s_1)} + \cdots + \varepsilon_{(x_\omega(\Lambda_{mk}),s_\omega(\Lambda_{mk}))}) \hat{\sigma}(dx_1,ds_1) \cdots \hat{\sigma}(dx_\omega(\Lambda_{mk}),ds_\omega(\Lambda_{mk}))
\]
for all bounded \( B_{\Lambda_{mk}}(\Omega^{\mathbb{R}^+}_X) \) measurable functions \( F : \Omega^{\mathbb{R}^+}_X \to \mathbb{R}_+ \) and all \( \Lambda \in \mathcal{O}_c(X) \). Here, the r.h.s. of (6.2) is understood to be equal to \( F(\emptyset) \) if \( \omega(\Lambda_{mk}) = 0 \). So, fix \( \Lambda \in \mathcal{O}_c(X) \).

Claim. Let \( g = (v,a) \in V_0(\Lambda) \times C_0(\Lambda) \), let \( K_1, \ldots, K_M \) be arbitrary subsets of \( (X \setminus \Lambda)_{mk} \) from \( B_c(X \times \mathbb{R}_+) \), and let \( \varphi_1, \ldots, \varphi_N \in C_0^\infty(\Lambda_{mk}) \). Let \( G := g_G(N_{K_1}, \ldots, N_{K_M}) \), \( F := g_F(\langle \varphi_1, \cdot \rangle, \ldots, \langle \varphi_N, \cdot \rangle) \) with \( g_G \in C_b^\infty(\mathbb{R}^N) \), \( g_F \in C_b^\infty(\mathbb{R}^M) \) and \( n \in \mathbb{N} \). Then
\[
\int G(\chi)G(\chi)F \, d\mu = - \int G(\chi)B_{\chi} \, d\mu.
\]
To prove the claim, for \( l \in \mathbb{N}, 1 \leq m \leq M \), we choose functions \( \chi_l, \chi_{ml} \in C_0^\infty(X \times \mathbb{R}_+) \) taking values in \([0,1]\) such that
a) \( \chi_l = 1 \) on a neighborhood of \( K_g \) (see Section 2), \( \chi_l \leq 1_{\Lambda_{mk}} \), and \( \chi_l \to 1_{\Lambda_{mk}} \) as \( l \to \infty \).

b) \( \chi_{ml} = 1 \) on \( K_m \), \( \chi_{ml} = 0 \) in a neighborhood of \( K_g \), and \( \chi_{ml} \to 1_{K_m} \) as \( l \to \infty \). Furthermore, let \( g \in C_b^\infty(\mathbb{R}) \) be such that \( 1_{(n)} \leq g \leq 1_{[n-\frac{1}{2},n+\frac{1}{2}]} \). Then, for every \( \omega \in \Omega^{\mathbb{R}^+}_X \)
\[
G_l(\omega) := g_G(\langle \chi_l, \omega \rangle, \ldots, \langle \chi_M, \omega \rangle) = G(\omega)
\]
and
\[
g_l(\omega) := g(\langle \chi_l, \omega \rangle) = 1_{\{\Lambda_{mk}=n\}}(\omega)
\]
for all \( l > l(\omega) \). Moreover, for all \( l \in \mathbb{N}, 1 \leq m \leq M \), we have that \( \nabla_{(v,a)}^{X \times \mathbb{R}_+} \chi_l \equiv 0 = \nabla_{(v,a)}^{X \times \mathbb{R}_+} \chi_{ml} \). Hence,
\[
\nabla_{(v,a)}^\Omega g_l = 0 = \nabla_{(v,a)}^\Omega G_l.
\]
Consequently, since \( \mu \in (\text{IbP})^\sigma \),
\[
\int G(\chi)G(\chi)F \, d\mu = \lim_{l \to \infty} \int G_l g_l \nabla_{(v,a)}^\Omega F \, d\mu
\]
\[
= - \lim_{l \to \infty} \left[ \int (g_l \nabla_{(v,a)}^\Omega G_l + G_l \nabla_{(v,a)}^\Omega g_l) F \, d\mu + \int G_l g_l F B_{(v,a)}^{\pi_\sigma} \, d\nu \right]
\]
\[
= - \int G(\chi)B_{(v,a)}^{\pi_\sigma} \, d\nu.
\]
and the claim is proven.

The claim immediately implies that for $F$ and $(v, a)$ as in the claim

$$\mathbb{E}_\mu[\nabla_{(v,a)}^\Omega F \mid \tilde{B}(X \setminus \Lambda)_{in\cap} (\Omega^R_X)] = -\mathbb{E}_\mu[B_{(v,a)}^\Omega F \mid \tilde{B}(X \setminus \Lambda)_{in\cap} (\Omega^R_X)] \quad \mu\text{-a.e.} \quad (6.3)$$

Now, set $A_1 := \Lambda_{mk}$ and pick $A_i \in \mathcal{B}_c((X \setminus \Lambda)_{mk})$, $i \in \mathbb{N}$, $i \geq 2$, closed under finite intersections such that

$$T := (N_{A_i})_{i \in \mathbb{N}} : \Omega^R_X \to \mathbb{Z}_+\text{ finite}$$

generates $\tilde{B}(X \setminus \Lambda)_{in\cap} (\Omega^R_X)$. Disintegrating $\mu$ w.r.t. $T$, there exists a probability kernel

$$\tilde{\mu} : \mathbb{Z}_+ \times \mathcal{B}(\Omega^R_X) \to [0, 1]$$

such that

$$\mu = \tilde{\mu}(n, d\omega)(\mu \circ T^{-1})(d\omega), \quad (6.4)$$

i.e., $\tilde{\mu}(n, d\omega)$ is a regular conditional probability corresponding to $\mathbb{E}_\mu[\cdot \mid T = n]$, $n \in \mathbb{Z}_+$.

From now on all statements depending on $n$ below are meant as statements which are true $(\mu \circ T^{-1})(d\omega)$-a.e. It follows from (6.3) that

$$\int \nabla_{(v,a)}^\Omega F(\omega) \tilde{\mu}(n, d\omega) = -\int \langle \beta_{(v,a)}^\Omega, \omega \rangle F(\omega) \tilde{\mu}(n, d\omega) \quad (6.5)$$

for all $(v, a)$ and $F$ as in the claim. Furthermore, since clearly $\tilde{\mu}(n, d\omega)$ is supported by $\{T = n\}$ and since for $n \in \mathbb{N}$ $\{\omega \in \Lambda_{mk} \mid \omega(\Lambda_{mk}) = n\}$ is isomorphic (as a measurable space) to $\tilde{\Lambda}_{mk}/\mathfrak{S}_n$, there exists a probability measure $\mu_\omega$ on $\tilde{\Lambda}_{mk}$ invariant under $\mathfrak{S}_n$ such that for all positive $\mathcal{B}(\Omega^R_X)$-measurable functions $G$ on $\Omega^R_X$

$$\int G(\omega) \tilde{\mu}(n, d\omega) = \int_{\Lambda_{mk}} G(\sum_{k=1}^n \varepsilon_{(x_k, s_k)} + \tilde{\omega}(X \setminus \Lambda)_{in\cap}) \mu_\omega(dx_1, ds_1, \ldots, dx_n, ds_n), \quad (6.6)$$

where $n = (n_1, n_2, n_3, \ldots)$ and $\tilde{\omega}(X \setminus \Lambda)_{in\cap}$ is the unique element in $\Omega^R_X$ such that $\tilde{\omega}(X \setminus \Lambda)_{in\cap}(A_i) = n_i$ for all $i \geq 2$. By the invariance under $\mathfrak{S}_n$, it suffices to determine $\mu_\omega$ on just one of the $n$! connected components of $\tilde{\Lambda}_{mk}$. Let $\tilde{\Lambda}_{mk}$ denote this component. Hence, it suffices to determine $\mu_\omega$ on $M := \Lambda_{mk} \times \cdots \times \Lambda_{mk} \subset o\tilde{\Lambda}_{mk}$, $i\Lambda$, $1 \leq i \leq n$, pairwise disjoint open subsets of $\Lambda$. Therefore, by Lemma 3.2 and a simple disintegration argument, it is enough to show that, for all $\varphi_i \in C_{0, b}(i\Lambda_{mk})$ and all $(v_i, a_i) \in V_0(i\Lambda) \times C_0(i\Lambda)$, $1 \leq i \leq n$,

$$\int \varphi_1(x_1, s_1) \cdots \langle \nabla_{X \times \mathbb{R}^+} \varphi_1(v_i, a_i) \rangle_{X \times \mathbb{R}^+} (x_i, s_i) \cdots (x_n, s_n) \mu_\omega(dx_1, ds_1, \ldots, dx_n, ds_n)$$

$$= -\int \beta_{(v_i, a_i)}^\Omega \varphi_1(x_1, s_1) \cdots \varphi_n(x_n, s_n) \mu_\omega(dx_1, ds_1, \ldots, dx_n, ds_n). \quad (6.7)$$
Because then by Lemma 3.2 we see that $\mu_n$ is up to a constant equal to $\tilde{\sigma}^n$ on $\Lambda^{mk}_1 \times \cdots \times \Lambda^{mk}_n$, hence since $\mu_n$ is a probability measure on $\tilde{\Lambda}^n_{mk}$, it follows that

$$\mu_n = \tilde{\sigma}(\Lambda^{mk})^{-n}\tilde{\sigma}^n,$$

and (6.2) follows then from (6.4) since the case $n = 0$ is trivial. To show (6.7) we fix $i \in \{1, \ldots, n\}$. Choosing $F$ in (6.5) so that $F := \prod_{j=1}^n \langle \varphi_j, \cdot \rangle$ on $\{\omega \subset \Lambda^{mk} | \omega(\Lambda^{mk}) = n\}$, we conclude that

$$\int \langle \varphi_1, \omega \rangle \cdots \langle \nabla_{(v_i,a_i)}^X \varphi_i, \omega \rangle \cdots \langle \varphi_n, \omega \rangle \tilde{\mu}(n, d\omega) = -\int \langle \beta_{(v_i,a_i)}, \omega \rangle \langle \varphi_1, \omega \rangle \cdots \langle \varphi_n, \omega \rangle \tilde{\mu}(n, d\omega).$$

Hence, by (6.6)

$$\int \tilde{\Lambda}^n_{mk} \prod_{j \neq i}^{n} (\varphi_j(x_1, s_1) + \cdots + \varphi_j(x_n, s_n)) \nabla_{(v_i,a_i)}^X \varphi_i(x_k, s_k) \mu_{\underline{\omega}}(dx_1, ds_1, \ldots, dx_n, ds_n)$$

$$= -\int \tilde{\Lambda}^n_{mk} \sum_{k=1}^{n} \beta_{(v_i,a_i)}^\underline{\omega}(x_k, s_k) \times$$

$$\times \prod_{j=1}^{n} (\varphi_j(x_1, s_1) + \cdots + \varphi_j(x_n, s_n)) \mu_{\underline{\omega}}(dx_1, ds_1, \ldots, dx_n, ds_n).$$

(6.8)

Since both integrals are invariant under $\mathfrak{S}_n$, (6.8) also holds if we only take the integrals over $\tilde{\Lambda}^n_{mk}$. But then (6.8) directly turns into (6.7), since $\varphi_i, a_i$ have support in $\Lambda_i, 1 \leq i \leq n$, which in turn are pairwise disjoint. ■

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