Derivation of Diagonalized Identical Equations for 3 by 3 Circulant and Quasi-Circulant Matrices and Their Application to
Winograd 7-Point FFT

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Abstract The derivation of the 7-point Winograd fast Fourier transform (FFT) requires complex steps such as using the Rader prime algorithm to turn an N-point discrete Fourier transform (DFT) into an (N − 1)-point convolution and then using the Chinese remainder theorem for polynomials to find the set of remainders. In this paper, we describe a simpler method for deriving the 7-point Winograd FFT using the diagonalized identical equations of 3 by 3 circulant and quasi-circulant matrices. These diagonalized identical equations of 3 by 3 matrices are not found in the literature and are newly derived.

Keywords: 3 by 3 circulant matrix, 3 by 3 quasi-circulant matrix, Winograd 7-point FFT

1. Introduction

Given an odd prime number \( N \), the traditional method of deriving the Winograd \( N \)-point fast Fourier transform (FFT) requires the following three steps:

\textbf{Step 1:} Using the Rader prime algorithm, an \( N \)-point discrete Fourier transform (DFT) is turned into an \( (N - 1) \)-point convolution \( Y_c(z) = \langle H(z)X(z) \rangle_{z^{-N-1}} \), where \( \langle \ast \rangle_{d(z)} \) represents the polynomial remainder of \( \ast \) when divided by nonzero polynomial \( d(z) \). \( Y_c(z) \), \( H(z) \), and \( X(z) \) are the \( Z \)-transforms of the output of the \( (N - 1) \)-point convolution, the impulse response corresponding to the \( (N - 1) \times (N - 1) \) circulant matrix, and the input data, respectively \cite{[1],[2]}. However, the \( Z \)-transform is the inverse of variable \( z \) for the conventional \( Z \)-transform.

\textbf{Step 2:} To reduce the number of multiplications and additions, the Winograd convolution algorithm is used to factorize \( z^N - 1 \) into a product of \( K \) relatively prime factors: \( \prod_{k=1}^{K} D_k(z) \). Then, the set of remainders of \( H(z) \) when divided by \( D_k(z) \) and the set of remainders of \( X(z) \) when divided by \( D_k(z) \) are found. \( H(z) \) and \( X(z) \) can be obtained using the Chinese remainder theorem for polynomials, and \( Y(z) = \langle H(z)X(z) \rangle_{z^{-N-1}} \) is computed using \( H(z) \) and \( X(z) \) \cite{[1],[3]}. 

\textbf{Step 3:} From \( Y(z) \), we obtain the relation

\[ y = A_{N\times L(N)}D_{L(N)}B_{L(N)\times N}x \]  

where \( y = (y(0) \ y(1) \ldots y(N - 1))^T \) and \( x = (x(0) \ x(1) \ldots x(N - 1))^T \) are the output and input vectors of degree \( N \), respectively, and the superscript \( T \) denotes the transpose of the matrix \cite{[4]}. \( A_{N\times L(N)} \) is an \( N \times L(N) \) matrix with elements of \( 0, \pm 1, \pm j \), and is referred to here as the post-addition matrix. \( D_{L(N)} \) is an \( L(N) \times L(N) \) diagonal matrix with elements of real numbers. \( B_{L(N)\times N} \) is an \( L(N) \times N \) matrix with elements of \( 0, \pm 1 \), and is referred to here as the pre-addition matrix. For example, in the case of \( N = 7 \), because \( D_{L(N)} \) becomes a \( 9 \times 9 \) matrix, \( L(7) \) equals 9.

Let the \( i \times i \) circulant and \( i \times i \) quasi-circulant matrices be denoted as \( R_i \) and \( S_i \), respectively. \( R_6 \) is obtained from the 7-point DFT using the Rader prime algorithm. When \( R_6 \) is divided into four \( 3 \times 3 \) block matrices, there are two types of matrix. The sum of the two types of matrix produces \( R_3 \), and their difference produces \( S_3 \). Therefore, if diagonalized identical equations for \( 3 \times 3 \) circulant and quasi-circulant matrices exist, the Winograd 7-point FFT can be obtained by only matrix operations. However, the diagonalized identical equations for \( 3 \times 3 \) circulant and quasi-circulant matrices are not found in the
literature. Only the diagonalized identical equations of $2 \times 2$ circulant matrix $R_2$ and quasi-circulant matrix $S_2$ have been published as shown in the following equations [3, 5]:

$$
R_2 = \begin{pmatrix}
p_1 & p_2 \\
p_2 & p_1
\end{pmatrix} = \begin{pmatrix}1 & 1 \\
1 & -1
\end{pmatrix}
$$

From Eq. (5),

$$
S_2 = \begin{pmatrix}
q_1 & q_2 \\
-q_2 & q_1
\end{pmatrix} = \begin{pmatrix}0 & 1 \\
1 & -1
\end{pmatrix}
$$

Then, $r_1, r_2, r_3$ are also represented by

$$
\begin{align*}
r_1 &= a + c + d \\
r_2 &= a - b + d \\
r_3 &= a - b + c
\end{align*}
$$

From Eqs. (6) and (8), the $3 \times 3$ circulant matrix $R_3$ can be represented as

$$
R_3 = \begin{pmatrix}1 & 1 & 0 \\
1 & -1 & 1 \\
1 & 0 & -1
\end{pmatrix} \begin{pmatrix}a & a & a \\
b & -b & 0 \\
c & 0 & -c
\end{pmatrix}
$$

The second factor on the right side of Eq. (9) can be written as

$$
\begin{pmatrix}a \\
b \\
c
\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 \\
1 & -1 & 0 \\
1 & 0 & -1
\end{pmatrix}
$$

then Eq. (9) can be written as

$$
R_3 = A_{3 \times 4,1}D_{4,1}B_{4 \times 3,1}
$$

\section{Derivation of diagonalized identical equation for $S_3$}

The $3 \times 3$ quasi-circulant matrix $S_3$ is defined as

$$
S_3 = \begin{pmatrix}s_1 & s_2 & s_3 \\
-s_3 & s_1 & s_2 \\
-s_2 & -s_3 & s_1
\end{pmatrix}
$$

We define $\alpha, \beta, \gamma, \delta$ as follows:

$$
\begin{align*}
\alpha &= \frac{1}{3}(s_1 - s_2 + s_3) \\
\beta &= \frac{1}{3}(2s_1 + s_2 - s_3) \\
\gamma &= \frac{1}{3}(s_1 + 2s_2 - s_3) \\
\delta &= \frac{1}{3}(s_1 - s_2 - 2s_3)
\end{align*}
$$

Then, $s_1, s_2, s_3$ are represented as

$$
\begin{align*}
s_1 &= \alpha + \beta \\
s_2 &= -\alpha + \gamma \\
s_3 &= \alpha - \delta
\end{align*}
$$

Furthermore, from Eq. (13), $\beta, \gamma, \delta$ are represented as

$$
\begin{align*}
\beta &= \gamma + \delta \\
\gamma &= \beta - \delta \\
\delta &= \beta - \gamma
\end{align*}
$$

Thus, $s_1, s_2, s_3$ are also represented by

$$
\begin{align*}
s_1 &= \alpha + \gamma + \delta \\
s_2 &= -\alpha + \beta - \delta \\
s_3 &= \alpha - \beta + \gamma
\end{align*}
$$
From Eqs. (14) and (16), the $3 \times 3$ quasi-circulant matrix $S_3$ can be represented as

$$S_3 = \begin{pmatrix}
1 & 1 & 0 & 1 \\
-1 & 1 & -1 & 0 \\
1 & 0 & -1 & -1
\end{pmatrix} \begin{pmatrix}
\alpha & -\alpha & \alpha \\
\beta & \beta & 0 \\
\gamma & 0 & -\gamma \\
0 & -\delta & -\delta
\end{pmatrix}.$$  

(17)

The second factor on the right side of Eq. (17) can be written as

$$\begin{pmatrix}
\alpha & 0 & \beta \\
0 & \gamma & -\delta \\
\gamma & \delta & \gamma
\end{pmatrix} \begin{pmatrix}
1 & -1 & 1 \\
1 & 1 & 0 \\
1 & 0 & -1 \\
0 & -1 & -1
\end{pmatrix}$$  

(18)

then Eq. (17) can be written as

$$S_3 = A_{3 \times 4,2}D_{4,2}B_{4 \times 3,2}$$  

(19)

3. Derivation of Winograd 7-Point FFT

3.1 Transformation of 7-point DFT matrix using Rader prime algorithm

The 7-point DFT is defined as

$$y(k) = \sum_{n=0}^{7-1} W_7^{nk} x(n), \quad 0 \leq k \leq 7 - 1$$  

(20)

where $(\ast)_d$ represents the remainder of $\ast$ when divided by positive integer $d$. The input and output vectors of degree 7 are defined as $x = (x(0) \ldots x(6))^T$ and $y = (y(0) \ldots y(6))^T$, respectively. From Eq. (20), the 7-point DFT can be represented in matrix notation as

$$y = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & W_7 & W_7^2 & W_7^3 & W_7^4 & W_7^5 & W_7^6 \\
1 & W_7^2 & W_7 & W_7^3 & W_7^4 & W_7^5 & W_7^6 \\
1 & W_7^3 & W_7^2 & W_7 & W_7^4 & W_7^5 & W_7^6 \\
1 & W_7^4 & W_7^3 & W_7^2 & W_7 & W_7^4 & W_7^5 \\
1 & W_7^5 & W_7^4 & W_7^3 & W_7^2 & W_7 & W_7^4 \\
1 & W_7^6 & W_7^5 & W_7^4 & W_7^3 & W_7^2 & W_7^1
\end{pmatrix} x$$  

(21)

where

$$W_7^m = e^{-j2\pi m/7} = \cos\left(-\frac{2\pi m}{7}\right) + j \sin\left(-\frac{2\pi m}{7}\right)$$

$$W_7 = (W_7)^0 = R_{1,7} - jI_{1,7}$$

$$W_7^2 = (W_7)^2 = -R_{2,7} + jI_{2,7}$$

$$W_7^3 = (W_7)^3 = -R_{3,7} - jI_{3,7}$$

$$W_7^4 = (W_7)^4 = R_{1,7} + jI_{1,7}$$

$$W_7^5 = (W_7)^5 = -R_{2,7} - jI_{2,7}$$

$$W_7^6 = (W_7)^6 = R_{3,7} + jI_{3,7}$$

(22)

The superscript $\ast$ denotes the complex conjugate, and $R_{n,7}$ and $I_{n,7}$ are the absolute values of the real and imaginary parts of $W_7^r$, respectively.

The Rader prime algorithm can be used to compute a Fourier transform with a block length equal to a prime integer. Because $n$ and $k$ in Eq. (20) both take the value zero, and zero is not a power of the primitive root, the zero time component (with $n = 0$) and the zero frequency component (with $k = 0$) must be treated particularly. To this end, we rewrite Eq. (20) as

$$y(0) = \sum_{n=0}^{7-1} x(n)$$  

(23)

$$y(k) = x(0) + \sum_{n=1}^{7-1} W_7^{nk} x(n), \quad 1 \leq k \leq 7 - 1$$  

(24)

The indexes $n$ and $k$ in Eq. (24) can be permuted by equations using the primitive root $3$ of $N = 7$ as

$$n = (3^m)_7, \quad 0 \leq m \leq 7 - 2$$  

(25)

$$k = (3^{-\ell})_7 = (3^{7-\ell})_7, \quad 0 \leq \ell \leq 7 - 2$$  

(26)

From Eqs. (25) and (26),

$$\langle nk \rangle_7 = (3^{m-\ell})_7, \quad 0 \leq m - \ell \leq 7 - 2$$  

(27)

is obtained. Table 1 shows the relationships between $m$ and $n$, $\ell$ and $k$, and $m - \ell$ and $\langle nk \rangle_7$.

From Eqs. (25)-(27), Eq. (24) can be represented as

$$y((3^{-\ell})_7) = y(0) + \sum_{m=1}^{7-1} W_7^{(3^m-\ell)_7} x((3^m)_7),$$

$$0 \leq \ell \leq 7 - 2$$  

(28)

Subtracting Eq. (23) from Eq. (28), we obtain

$$y((3^{-\ell})_7) - y(0) = \sum_{m=0}^{7-2} (W_7^{(3^m-\ell)_7} - 1) x((3^m)_7),$$

$$0 \leq \ell \leq 7 - 2$$  

(29)

Defining $W_7^{(3^m-\ell)_7} - 1$ as

$$w_{(m-\ell)_7} = W_7^{(3^m-\ell)_7} - 1$$  

(30)

Eq. (29) becomes

$$y((3^{-\ell})_7) - y(0) = \sum_{m=0}^{7-2} w_{(m-\ell)_7} x((3^m)_7),$$

$$0 \leq \ell \leq 7 - 2$$  

(31)

Table 1 Relationships between $m$ and $n$, $\ell$ and $k$, and $m - \ell$ and $\langle nk \rangle_7$

| $m$ | $n$ | $\ell$ | $k$ | $m - \ell$ | $\langle nk \rangle_7$ |
|-----|-----|-------|-----|-----------|---------------------|
| 0   | 1   | 0     | 1   | 0         | 1                   |
| 1   | 3   | 1     | 5   | 1         | -5                  |
| 2   | 2   | 2     | 4   | 2         | -4                  |
| 3   | 6   | 3     | 6   | 3         | -3                  |
| 4   | 4   | 4     | 2   | 4         | -2                  |
| 5   | 5   | 5     | 3   | 5         | -1                  |

Table 1 Relationships between $m$ and $n$, $\ell$ and $k$, and $m - \ell$ and $\langle nk \rangle_7$
From Eq. (31) and Table 1,

$$\mathbf{y}_c = \begin{pmatrix} w_1 & w_3 & w_5 & w_6 & w_4 & w_2 \\ w_5 & w_1 & w_2 & w_6 & w_4 & w_3 \\ w_4 & w_5 & w_1 & w_2 & w_6 & w_2 \\ w_2 & w_6 & w_5 & w_4 & w_1 & w_3 \\ w_3 & w_5 & w_4 & w_6 & w_1 & w_2 \end{pmatrix} \mathbf{x}_c$$  \hspace{1cm} (32)

can be obtained, where $\mathbf{y}_c$ and $\mathbf{x}_c$ are, respectively, the output and input vectors with data sequences scrambled as

$$\mathbf{y}_c = \begin{pmatrix} y(1) - y(0) \\ y(5) - y(0) \\ y(4) - y(0) \\ y(6) - y(0) \\ y(2) - y(0) \\ y(3) - y(0) \end{pmatrix}, \quad \mathbf{x}_c = \begin{pmatrix} x(1) \\ x(3) \\ x(2) \\ x(6) \\ x(4) \\ x(5) \end{pmatrix}$$  \hspace{1cm} (33)

Because the matrix on the right side of Eq. (32) is the $6 \times 6$ circulant matrix $\mathbf{R}_6$, Eq. (32) can be written as

$$\mathbf{y}_c = \mathbf{R}_6 \mathbf{x}_c$$  \hspace{1cm} (34)

From Eqs. (22) and (30),

$$\begin{align*}
w_1 &= w_6^* = R_{1,7} - 1 - jI_{1,7} \\
w_2 &= w_5^* = R_{2,7} - 1 - jI_{2,7} \\
w_3 &= w_4^* = R_{3,7} - 1 - jI_{3,7}
\end{align*}$$  \hspace{1cm} (35)

is obtained. The vectors $\mathbf{y}_g$ and $\mathbf{x}_g$ are formed by unscrambling the elements of $\mathbf{y}_c$ and $\mathbf{x}_c$, respectively. Then, the stride permutation matrices $\mathbf{P}_{6,(A)}$ from $\mathbf{y}_g$ to $\mathbf{y}_c$ and $\mathbf{P}_{6,(B)}$ from $\mathbf{x}_g$ to $\mathbf{x}_c$ are obtained as

$$\mathbf{P}_{6,(A)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (36)

$$\mathbf{P}_{6,(B)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (37)

With Eqs. (36) and (37), Eq. (34) becomes

$$\mathbf{y}_g = \mathbf{P}_{6,(A)}^{-1} \mathbf{R}_6 \mathbf{P}_{6,(B)} \mathbf{x}_g$$  \hspace{1cm} (38)

where $\mathbf{P}_{6,(A)}^{-1}$ is the inverse matrix of $\mathbf{P}_{6,(A)}$ and $\mathbf{P}_{6,(A)}^{-1} = \mathbf{P}_{6,(A)}^T$.

### 3.2 Derivation of Winograd 7-point FFT using diagonalized identical equations of $\mathbf{R}_3$ and $\mathbf{S}_3$

When $\mathbf{R}_6$ is divided into four $3 \times 3$ matrices as Eq. (32), two types of $3 \times 3$ block matrix appear as

$$\mathbf{W}_{3,1} = \begin{pmatrix} w_1 & w_3 & w_2 \\ w_5 & w_1 & w_3 \\ w_4 & w_5 & w_1 \end{pmatrix}$$  \hspace{1cm} (39)

$$\mathbf{W}_{3,2} = \begin{pmatrix} w_6 & w_4 & w_5 \\ w_2 & w_6 & w_4 \\ w_3 & w_2 & w_6 \end{pmatrix}$$  \hspace{1cm} (40)

Then, Eq. (34) can be represented as

$$\mathbf{y}_c = \mathbf{R}_6 \mathbf{x}_c = \begin{pmatrix} \mathbf{W}_{3,1} & \mathbf{W}_{3,2} \end{pmatrix} \begin{pmatrix} \mathbf{W}_{3,1} & \mathbf{W}_{3,2} \end{pmatrix} \mathbf{x}_c$$  \hspace{1cm} (41)

Denoting the $i \times i$ identity matrix and the $i \times i$ zero matrix as $\mathbf{I}_i$ and $\mathbf{O}_i$, respectively, and extending the elements in Eq. (2) for the $3 \times 3$ matrix, the matrix on the right side of Eq. (41) is expressed as

$$\mathbf{R}_6 = \begin{pmatrix} \mathbf{W}_{3,1} & \mathbf{W}_{3,2} \\ \mathbf{W}_{3,2} & \mathbf{W}_{3,1} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & -\mathbf{I}_3 \end{pmatrix} \begin{pmatrix} \mathbf{O}_3 & \frac{1}{2}(\mathbf{W}_{3,1} + \mathbf{W}_{3,2}) \\ \frac{1}{2}(\mathbf{W}_{3,1} + \mathbf{W}_{3,2}) & \mathbf{O}_3 \end{pmatrix}$$  \hspace{1cm} (42)

From Eqs. (35), (39), and (40), the two diagonal elements in the second matrix on the right side of Eq. (42) can be expressed as

$$\frac{1}{2}(\mathbf{W}_{3,1} + \mathbf{W}_{3,2}) = \frac{1}{2} \begin{pmatrix} w_1 + w_1^* & w_3 + w_3^* & w_2 + w_2^* \\ w_2^* + w_2 & w_1 + w_1^* & w_3 + w_3^* \\ w_3^* + w_3 & w_2^* + w_2 & w_1 + w_1^* \end{pmatrix}$$

$$= \begin{pmatrix} R_{1,7} - 1 & -R_{3,7} - 1 & -R_{2,7} - 1 \\ -R_{2,7} - 1 & R_{1,7} - 1 & -R_{3,7} - 1 \\ -R_{3,7} - 1 & -R_{2,7} - 1 & R_{1,7} - 1 \end{pmatrix} = \mathbf{R}_3$$  \hspace{1cm} (43)

$$\frac{1}{2}(\mathbf{W}_{3,1} - \mathbf{W}_{3,2}) = \frac{1}{2} \begin{pmatrix} w_1 - w_1^* & w_3 - w_3^* & w_2 - w_2^* \\ w_2^* - w_2 & w_1 - w_1^* & w_3 - w_3^* \\ w_3^* - w_3 & w_2^* - w_2 & w_1 - w_1^* \end{pmatrix}$$

$$= \begin{pmatrix} -jI_{1,7} & -jI_{3,7} & -jI_{2,7} \\ jI_{2,7} & -jI_{1,7} & -jI_{3,7} \\ jI_{3,7} & jI_{2,7} & -jI_{1,7} \end{pmatrix} = \mathbf{S}_3$$  \hspace{1cm} (44)

With Eqs. (43) and (44), Eq. (42) becomes

$$\mathbf{R}_6 = \begin{pmatrix} \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & -\mathbf{I}_3 \end{pmatrix} \begin{pmatrix} \mathbf{R}_3 & \mathbf{O}_3 & \mathbf{S}_3 \\ \mathbf{O}_3 & \mathbf{S}_3 & \mathbf{I}_3 \end{pmatrix} \begin{pmatrix} \mathbf{I}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & -\mathbf{I}_3 \end{pmatrix}$$  \hspace{1cm} (45)
Then, substituting Eqs. (11) and (19) into Eq. (45), we obtain

$$R_6 = \begin{pmatrix} I_3 & I_3 \\ I_3 & -I_3 \end{pmatrix} \cdot \begin{pmatrix} A_{3\times4}D_{4\times3}B_{4\times3} & O_3 \\ O_3 & A_{3\times4}D_{4\times3}B_{4\times3} \end{pmatrix} = \begin{pmatrix} I_3 & I_3 \\ I_3 & -I_3 \end{pmatrix} \cdot \begin{pmatrix} A_{3\times4} & O_{3\times4} \\ O_{3\times4} & A_{3\times4} \end{pmatrix} \cdot \begin{pmatrix} D_{4\times4} & O_4 \\ O_4 & D_{4\times4} \end{pmatrix} = \begin{pmatrix} B_{4\times4} & O_{4\times4} \\ O_{4\times4} & B_{4\times4} \end{pmatrix}$$

where $O_{i\times\ell}$ is the $i \times \ell$ zero matrix. Also, substituting Eq. (46) into Eq. (38),

$$y_9 = P_{9,\ell}^{-1} \begin{pmatrix} A_{3\times4} & A_{3\times4} \\ A_{3\times4} & -A_{3\times4} \end{pmatrix} \begin{pmatrix} D_{4\times4} & O_4 \\ O_4 & D_{4\times4} \end{pmatrix} = \begin{pmatrix} B_{4\times4} & B_{4\times4} \\ B_{4\times4} & -B_{4\times4} \end{pmatrix} P_{9,\ell}x_g$$

is obtained. With Eqs. (9), (10), (17) and (18), Eq. (47) becomes

$$y_9 = P_{9,\ell}^{-1} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To remove the stride permutation matrices in Eq. (48), exchanging the rows in the matrix of post-addition, and exchanging the columns in the matrix of pre-addition, we obtain

$$y_g = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ a & b & c & d & \alpha_j & \beta_j & \gamma_j & \delta_j \end{pmatrix}$$

From Eqs. (5), (13), (43), and (44), we obtain

$$a = \frac{1}{4}(R_{1,7} - R_{2,7} - R_{3,7}) - 1$$

$$b = \frac{1}{4}(R_{1,7} + R_{2,7} + R_{3,7})$$

$$c = \frac{1}{4}(R_{1,7} - R_{2,7} + 2R_{3,7})$$

$$d = \frac{1}{4}(R_{1,7} + R_{2,7} - R_{3,7})$$

$$\alpha_j = -j \frac{1}{4}(I_{1,7} + I_{2,7} - I_{3,7})$$

$$\beta_j = -j \frac{1}{4}(2I_{1,7} - I_{2,7} + I_{3,7})$$

$$\gamma_j = -j \frac{1}{4}(I_{1,7} + I_{2,7} + 2I_{3,7})$$

$$\delta_j = -j \frac{1}{4}(I_{1,7} - 2I_{2,7} + I_{3,7})$$

In Eq. (49), moving $-j$ from the diagonal matrix and absorbing it into the matrix of post-additions, Eq. (49) becomes

$$y_g = \begin{pmatrix} 1 & 1 & 0 & 1 & -j & -j & 0 & -j \\ 1 & 1 & 1 & 1 & 0 & -j & j & -j \\ 1 & 0 & 1 & -1 & -j & j & j & 0 \\ 1 & 0 & 1 & -1 & j & j & j & 0 \\ 1 & 1 & 0 & 1 & j & j & j & 0 \end{pmatrix}$$

$$y_g = \begin{pmatrix} a & b & c & d & \alpha_j & \beta_j & \gamma_j & \delta_j \\ \alpha_j & \beta_j & \gamma_j & \delta_j \end{pmatrix}$$
Adding Eq. (23) to each element of the three matrices in Eq. (51), the Winograd 7-point FFT is obtained, where

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & -1 & -1 & 0 & 1 \\
1 & -1 & 0 & 0 & -1 & 1 \\
0 & -1 & 1 & 1 & -1 & 0 \\
1 & 1 & -1 & 1 & -1 & -1 \\
0 & 1 & 0 & 1 & 0 & -1 \\
1 & -1 & 0 & 0 & 1 & -1 \\
0 & -1 & -1 & 1 & 1 & 0
\end{pmatrix}
\]

\(x_a \) (51)

\[
y = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & -j & 0 & -j \\
1 & 1 & -1 & 1 & 0 & -j & j & -j \\
1 & 1 & 0 & -1 & -1 & j & 0 & j & -j \\
1 & 1 & -1 & 0 & -1 & -j & 0 & j & j \\
1 & 1 & -1 & 1 & 0 & j & j & j & 0 \\
1 & 1 & 1 & 0 & 1 & j & j & j & 0
\end{pmatrix}
\]

\(y \) (52)

is obtained, where

\[
\begin{align*}
\theta &= \frac{2\pi}{7} \\
a &= \frac{1}{3}(\cos \theta + \cos 2\theta + \cos 3\theta) - 1 \\
b &= \frac{1}{3}(2\cos \theta - \cos 2\theta - \cos 3\theta) \\
c &= \frac{1}{3}(\cos \theta + \cos 2\theta - 2\cos 3\theta) \\
d &= \frac{1}{3}(-\cos \theta + 2\cos 2\theta + 3\cos 3\theta) \\
\alpha_j &= \frac{1}{3}(\sin \theta + \sin 2\theta - \sin 3\theta) \\
\beta_j &= \frac{1}{3}(2\sin \theta + 2\sin 2\theta - \sin 3\theta) \\
\gamma_j &= \frac{1}{3}(\sin \theta + \sin 2\theta + 2\sin 3\theta) \\
\delta_j &= \frac{1}{3}(\sin \theta - 2\sin 2\theta - \sin 3\theta)
\end{align*}
\]

With Eq. (53), the output of Eq. (52) becomes equal to that of the 7-point DFT.

4. Conclusion

The conventional derivation procedure for the Winograd 7-point FFT from the 7-point DFT obtains \(Y(z)\) from a set of remainders for polynomials using the Winograd short convolution algorithms based on the Chinese remainder theorem for polynomials, then Eq. (1) can be derived from \(Y(z)\). In contrast, in this paper, we obtain \(R_7\) and \(S_7\) from \(R_6\), and we obtain the Winograd 7-point FFT using Eqs. (11) and (20).

Diagonalized identical equations for \(3 \times 3\) circulant and quasi-circulant matrices are applicable to the derivation of the Winograd 13-, 97-, and 193-point FFTs. Therefore, these diagonalized identical equations can be applied to a broad range of problems. Also, as a special case, these diagonalized identical equations have been applied to deduce the Winograd 9-point FFT, where 9 is not an odd prime number [6].

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