Initial Data Set For Cosmology: Application to Matching Condition

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Abstract

In Einstein theory of gravity the initial configuration of metric field and its time derivative are related to matter configuration by four equations called constraints. We use the method of conformal metrics (York Method) to solve constraints and find an analytic set of consistent initial data for linearized Einstein field equations in a perturbed constant curvature space-time. They are explicitly covariant and more compact than decomposition of quantities to scalar, vector and tensor. This method is independent of type and physics of matter fields and is extendable to higher-order perturbative calculations. As an application example, we apply this method to two commonly used matching conditions during a phase transition and compare and interpret the results.

Keywords:
cosmology – relativity: initial conditions – cosmology: perturbations – relativity: conformal metric

1 Introduction

By a set of initial conditions, we mean a configuration for matter, radiation, and metric fields, and their time derivatives on a given initial 3-space. This information is necessary for solving Einstein equations. The task of defining this configuration is not trivial [Lichnerowicz 1944], [Arnowitt et al. 1962]. In contrast to Newtonian theory, due to diffeomorphism gauge symmetry, the initial value of the metric and its time derivative can not be specified in an arbitrary way. The configuration of matter fields partially determines them.

To define initial conditions in general relativity properly, one must define a time slicing of the space-time i.e. a diffeomorphism between the space-time and a 3+1 manifold [York 1972], [O Murchadha & York1974], [Choquet-Bruhat & York 1980], [Durrer 1994]. From this operation, in addition to evolution equations for geometrical quantities, one obtains the restriction of the Einstein equations on the space-like 3-space component of the 3+1 manifold. These equations don’t evolve with time and for this reason they are called constraints [Arnowitt et al. 1962]. The initial condition for geometric and matter fields is defined as their configuration on a space-like 3-space. As constraints are not dynamical equations, it is necessary and sufficient that field configuration on the initial 3-space satisfy them.

In cosmology, specially for studying the evolution of small perturbations, it is customary to decompose Einstein equations, as well as energy-momentum tensor, to scalar, vector, and tensor components. Evolution equations and constraints for each type of fluctuations are solved separately. This procedure drops some of components from constraints
and makes them easier to solve, specially when only one of components, usually scalar component, is studied. Nevertheless, evolution equations and constraints remain coupled and one has to solve them together. Moreover, the resulting equations are not usually explicitly covariant (although it is possible to perform the decomposition in a covariant manner [Stewart 1990]).

Having an exact solution of constraints is also important in numerical solution of evolution equations. Usually, it is very difficult to keep the conservation of energy-momentum in a numerical calculation. The exact solution of constraints assures that the initial data satisfies the conservation laws. It also can be used at each step of calculation for checking/correction of conservation violation.

The mathematical aspects of the initial conditions for the Einstein field equations have been studied and clarified in an outstanding work by J.W. York on the initial condition problem and its relation to conformal gravity [York 1972], [ `O Murchadha & York1974]. This formalism separates initially constrained components of metric and extrinsic curvature, and allows a more detailed insight to the physical nature of unconstrained components without knowing anything about the matter content of the theory.

Here we apply this method to linearized Einstein equations, solve constraints analytically, and find a consistent set of initial conditions for perturbed constant curvature cosmologies. The results are independent of details of the physical model under consideration. As an application example, we use the results of this method to discuss matching during a phase transition.

We first briefly remind the mathematical formulation of the problem. Then, we explain analytical solution of constraints for flat and constant curvature perturbed space-times and finally we apply the solution of constraints to matching conditions on a phase transition surface and we obtain the initial values allowed for unconstrained components.

## 2 3+1 Gravity

We assume that by a diffeomorphism transformation, the space-time is divisible to a space-like 3-manifold and a one-dimensional time like space (curve). The general form of the metric is:

$$ds^2 = -\alpha^2 d\eta^2 + g_{ij}(dx^i + \beta^i d\eta)(dx^j + \beta^j d\eta)$$

$$dt^2 \equiv \alpha^2 d\eta^2$$

(1)

\(\eta\) is the conformal time. \(\alpha(\eta, x)\) and \(\beta(\eta, x)\) are called lapse function and shift vector (in 3-space). The 3-tensor \(g_{ij}\) is the induced metric of the 3-space.

Einstein field equations can be expressed by a set of first-order differential equations which depend on the field set \(\{\alpha, \beta^i, g_{ij}, K_{ij}\}\). The extrinsic curvature \(K_{ij}\) is defined as the covariant derivative of the normal vector to the 3-space \(K_{ij} = -n_{i;j} = -\alpha^{(4)} \Gamma_{ij}^0\) [Misner et al. 1973]. The vector \(n^\mu\) is the normal unit vector of the 3-space. With respect to
to these fields, Einstein field equations take the following form [Piran 1980]:

\[
\frac{\partial g_{ij}}{\partial \eta} = -2\alpha g_{ik}K^k_j + \beta^k g_{ij,k} + g_{ik}\beta^k_j + g_{jk}\beta^k_i. 
\] (2)

\[
\frac{\partial K^j_i}{\partial \eta} = \beta K^j_{i,k} + K^j_k \beta^k_i - K^k_i \beta^k_j - \alpha^j_i + \alpha(KK^j_i + (3)R^j_i + 8\pi G g^{jk}T_{ik}). 
\] (3)

\[
(3)R - K^j_i K^i_j + K^2 = 16\pi G T_*^*, \quad (4)
\]

\[
K^j_{i,j} - K^j_i = -8\pi G T^*_i, \quad (5)
\]

The quantities \( T_*^i = T_{\mu\nu}^i n^\mu B^\nu_i \) and \( T^*_i = T_{\mu\nu}^i n^\mu n^\nu \) can be interpreted respectively as projection of energy-momentum flux on the 3-space and on its normal. \( B^\mu_i \) is the projection operator on the 3-space, \( B^\mu_i n^\mu = 0 \). For metric (1):

\[
n^\mu = \alpha^{-1}(1, \beta^i). \quad (6)
\]

\[
T_*^i = \alpha^{-1}(T_{0i} - T_{ik}\beta^k). \quad (7)
\]

\[
T^*_i = \alpha^{-2}(T_{00} - 2T_{0i}\beta^i + T_{ij}\beta^i\beta^j). \quad (8)
\]

Equations (2) and (3) are dynamical equations for the evolution of the 3-space. Equations (4) and (5) don’t have explicit time dependence. They are constraints. If they are satisfied by the metric and extrinsic curvature at the initial time \( t_0 \) on the 3-space, they stay valid for ever.

Gauge symmetry allows to fix arbitrarily the value of \( \alpha(t, x) \) and \( \beta(t, x) \). In synchronous gauge that we will use in this letter, \( \alpha \) depends only on \( t \) and \( \beta^i(t, x) = 0 \). The gauge symmetry assures that these relations remain valid after the evolution of the dynamical system. In fact, 4 of 16 equations (2) - (5) are automatically satisfied due to Bianacci identities. This results to an equal number of equations and fields. The physical reason behind existence of constraint equations is the gauge symmetry. A system with gravitational interaction is a constrained dynamical system and the initial value of \( g_{ij} \) and \( K_{ij} \) can not be arbitrarily chosen. The initial \( g_{ij} \) and \( K_{ij} \) fields configuration on the initial 3-space depends on the matter configuration and must satisfy constraints (4) and (5).

### 3 Conformal Metric Method

In [York 1971] and [O’Murchadha & York1974] it has been proved that unconstrained degrees of freedom of a gravitational system on a space-like 3-space are conformally equivalent. This allows to separate constrained and unconstrained components of \( g_{ij} \) and \( K_{ij} \). The extrinsic curvature tensor of the 3-space can be decomposed to traceless-transverse, longitudinal (traceless), and trace components:

\[
K_{ij} = S_{ij} + (LW)_{ij} + \frac{1}{3}Kg_{ij}. \quad (9)
\]

\[
(LW)_{ij} = W_{ij} + W_{ji} - \frac{2}{3}g_{ij}W^c_c. \quad (10)
\]
The scalar $K$ is the trace of $K_{ij}$ and $S_{ij}$ is its traceless-transverse component ($S_{ij}^{ij} = 0$). $W^i$ is a 3-vector field that generates the longitudinal (traceless) part of $K_{ij}$.

The conformal transformation of a 3-metric $g_{ij}$ is defined as $\bar{g}_{ij} = \phi^4 g_{ij}$. We call $g_{ij}$ the base metric and $\bar{g}_{ij}$ the physical metric which satisfies Einstein equations. Under this transformation, (9) and (10) keep their form. The only difference between base or physical extrinsic curvature decomposition is the use of base or physical metric in the contraction of indexes. Therefore:

$$\bar{K}_{ij} = \bar{S}_{ij} + (\bar{L}W)_{ij} + \frac{1}{3}K\bar{g}_{ij}. \quad (11)$$

$$\bar{W}^i = \bar{S}_{ij} + (\bar{L}W)^i_j = W_{ij} + W_{j|i} - \frac{2}{3}g_{ij}W^c_c. \quad (12)$$

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + 2\phi^{-1}(\delta^i_j\phi_k + \delta^i_k\phi_j - g_{jk}\phi^i). \quad (13)$$

The trace $K$ and the vector potential $W^i$ are considered as invariant under this transformation. The traceless-transverse component $S_{ij}$ is chosen to transform as:

$$\bar{S}_{ij} \equiv \phi^{-2}S_{ij}. \quad (14)$$

(When only one initial 3-space is considered, the conformal transformation of $S_{ij}$ is arbitrary. However, it has been recently shown that by assuming two infinitesimally close 3-spaces and a mapping between them, this transformation rule is imposed by Einstein equations [York 1998]. Even though the choice for transformation of $T^*i$ and $T^*_s$ is not unique, following choices guarantee the existence of a non-spacelike energy flow from the 3-space [O Murchadha & York 1974]:

$$\bar{T}^*i = \phi^{-10}T^*i. \quad (15)$$

$$\bar{T}^*_s = \phi^{-8}T^*_s. \quad (16)$$

With these relations, the longitudinal component must transform as:

$$(\bar{L}W)^i_j = \phi^{-4}(LW)^i_j. \quad (17)$$

$\bar{g}_{ij}$ and $\bar{K}_{ij}$ satisfy the constraint equations (9) and (10) if $W^i$ and $\phi$ satisfy following equations:

$$[\phi^6(\bar{L}W)^i_j]_i = \frac{1}{3}\phi^6K_{ij} - 8\pi T^*i. \quad (18)$$

$$-8\Delta\phi = -R\phi + M_{TT}\phi^{-7} + 2M_{TL}\phi^{-1} + (M_L - \frac{2}{3}K)\phi^5 + 16\pi GT^*_s\phi^{-3}. \quad (19)$$

$$M_{TT} = g_{ac}g_{bd}S^{ab}S^{cd}. \quad (20)$$

$$M_{TL} = g_{ac}g_{bd}(LW)^{cd}. \quad (21)$$

$$M_L = g_{ac}g_{bd}(LW)^{ab}(LW)^{cd}. \quad (22)$$

All indexes and derivatives are contracted with the base metric $g_{ij}$. Equations (18) and (19) are obtained from constraint equations (9) and (10). These equations show that $\phi$ and $W^i$ are the real constrained degrees of freedom of the initial configuration of fields. The
unconstrained degrees of freedom are $g_{ij}$, $S_{ij}$ and $K$ and they are all conformally related to their physical counterpart. Note also that this formulation of the initial value problem is gauge independent.

The real difficulty in solving equations (18) and (19) is that they are highly nonlinear and in general completely coupled. This fact has encouraged the use of other methods, specially for nonlinear numerical calculations [Piran 1980].

4 Initial Data for Linearized Einstein Equations in Flat Cosmology

We consider an initial equal-time 3-space $\Sigma$ in a flat universe with small perturbations (it is always possible to redefine coordinates such that the 3-space become equal-time). A priori $\Sigma$ can be any hypersurface, but interesting cases are those with constant $K$ or $K = K_{homo} + \delta K$, such that $K_{ij} = (\delta K)_{ij}$ and $K_{homo}$ is regarded as the trace of $K_{ij}$ for a 3-space obtained from mapping $\Sigma$ to the homogeneous background manifold. In fact, $K$ is an appropriate quantity to time label a spacelike 3-space because it has a non-negative derivative in the direction of timelike orthogonal vector to the 3-space [York 1972]. Other physically interesting 3-spaces e.g. one with constant energy in a perturbative theory are close to a constant $K$ 3-space. Here we only consider these types of 3-spaces for imposing initial conditions.

We take the base metric to be the metric in a flat homogeneous Freedman-Lemaître cosmology, i.e.:

$$g_{ij} = \alpha^2(t)\delta_{ij}. \quad (23)$$

This choice is physically motivated, because for a homogeneous universe, this metric satisfies Einstein equations. In addition, it simplifies all calculations.

With this choice, in a flat universe with small perturbations, the conformal factor will be close to 1:

$$\phi(x, t) = 1 + \delta\phi(x, t) \quad (24)$$

For linearized Einstein equations in synchronous gauge, the physical metric $\bar{g}_{ij}$ is:

$$\bar{g}_{ij} = \alpha^2(t)(\delta_{ij} + h_{ij})$$

$$= \alpha^2(t)\phi^4(x, t)\delta_{ij} = \alpha^2(t)\delta_{ij}(1 + 4\delta\phi(x, t)). \quad (25)$$

$$h_{ij} = 4\delta_{ij}\delta\phi. \quad (26)$$

This special form of metric fluctuation is not a restriction of perturbations to scalars and is not equivalent to Newtonian gauge because its application is uniquely on the initial 3-space. As we have seen in the previous section, the transverse-traceless component of the extrinsic curvature is an unconstrained quantity. If on the initial 3-space it is not zero, this pure tensorial curvature perturbation will contribute to $g_{ij}$ evolution (see (2)) and induces a purely tensorial metric perturbation.

Another important point is that it is always possible to choose a coordinate system on the 3-space with a metric like (25). It is well known that synchronous gauge does not
define the gauge completely and allows a redefinition of the coordinates which preserves the gauge. In a gauge preserving coordinate transformation like the following:

\[ t' = t \quad \text{and} \quad x'^i = x^i + \partial^i \psi + \varepsilon^{ij} \omega_j \]

(27)

the arbitrary fields \( \psi \) (local translation) and \( \omega_i \) (local rotation) can be chosen such that \( \tilde{g}_{ij} \) gets the form given in (25). After calculating the initial condition, one can return to the previous coordinates or to any other gauge.

From definition of projected energy-momentum tensor Equation (8), in synchronous gauge \( \tilde{T}^{	ext{si}} = \tilde{T}_0^i \). In an approximately homogeneous and flat universe with small perturbations, \( \tilde{T}_0^i = \mathcal{O}(1) \) in the comoving frame. Therefore, \( \tilde{T}^{	ext{si}} = 0 + \delta \tilde{T}^{	ext{si}} \). Regarding (13), at first-order of approximation, \( T^{	ext{si}} = \tilde{T}^{	ext{si}} \). From (8), one can conclude that at zero-order \( \hat{K}_{ij} = 0 \). This means that at zero-order, the physical extrinsic curvature is transverse, and therefore, in spaces with small perturbations, the value of its longitudinal part must be small. Equation (17) shows that the longitudinal part of the base extrinsic curvature also must be small.

For small perturbations, equation (18) becomes:

\[
[(1 + 6\delta \phi)(LW)^{ij}]_{ij} = -8\pi G T^{	ext{si}} + \frac{1}{3}(1 + 6\delta \phi)K^{|i|}.
\]

(28)

The term \( \delta \phi(LW)^{ij} \) is of second order and one expects that for smooth fluctuations, its derivative must be negligible with respect to \( (LW)^{ij}_{ij} \). As mentioned earlier, we assume that \( K^{|i|} = \mathcal{O}(1) \). Therefore, if:

\[ (\delta \phi(LW)^{ij})_{ij} \ll (LW)^{ij}_{ij}, \]

(29)

(28) reduces to:

\[
(LW)^{ij}_{ij} = -8\pi G T^{	ext{si}} + \frac{K^{|i|}}{3} \approx -8\pi G \tilde{T}^{	ext{si}} + \frac{K^{|i|}}{3}.
\]

(30)

The solution of this equation in the case of a flat base metric is trivial. With a variable change \( x'^i = \alpha(t) x^i \), (30) becomes:

\[
\partial'^j \partial'^i W^i + \frac{1}{3} \partial'^i \partial'^j W^j = -8\pi G T^{	ext{si}}(x', t) + \frac{\partial'^i K}{3}.
\]

\[ \partial' \] means partial derivative with respect to \( x' \). The solution in real space is:

\[
W^i(x, t) = \frac{1}{(2\pi)^3} \int d^3 k e^{-i\alpha(t)\delta_{ij} k^i x^j} \frac{2\pi G}{k^4} \left[ k^2 (4T^{	ext{si}}(k, t) - \frac{k^i \delta K}{8\pi G}) - k^i k^j T^{	ext{sj}}(k, t) \right].
\]

(32)

For solving equation (19), one needs also \( S_{ij} \) the transverse part of the base extrinsic curvature. It is an unconstrained component and must be chosen according to the physics of the system. To understand better the physical rôle of \( S_{ij} \), note that its definition:

\[ S_{ij} = 0 \]

(33)
can have non-trivial solutions. It has been proven that this equation allows a solution with a singularity called monopole solution (a black hole, a point like object of finite mass, or a geometrical singularity at origin) [Bowen & York 1980]:

\[ S^{ij} = \frac{3}{2r^2}(P^i u^j + P^j u^i - (g^{ij} - u^i u^j)P^c u_c) + \]

\[ \frac{3}{r^3}(\epsilon^{imn} S_m u_n u^j + \epsilon^{imn} S_m u_n u^i). \]  

(34)

The vector \( \vec{r} \) is the radial vector through origin and \( u^i \) is the radial unit vector, \( P^i \) and \( S^i \) are respectively the linear and angular 3-velocity of the singular point. This solution can be extended to a solution with \( N \) singularity at \( \vec{r}^{(i)} \), \( i = 1, \ldots, N \) [Bowen & York 1980, Brandt & Bruegmann 1997]. In fact, this is a solution for (30) in vacuum. However, in this case, \( (LW)^{ij} \) is transverse and it is possible to add (34) to any solution of (33) or in general to \( K_{ij} \). The existence of this solution reflects the effect of a velocity field on the extrinsic curvature and on the evolution of the metric (equation (3)). Note also that \( T^{\mu\nu} \) does not fix the velocity field. This makes a solution of type (34) independent of (32). In decomposition method, the linear and angular velocity fields appear in the decomposition of \( T^{\mu\nu} \) and their geometrical role is not as evident as here.

The solution (34) can be generalized to a non-singular matter distribution:

\[ S^{ij}_{vel} = \int_{V \setminus \{x\}} \sqrt{g} d^3 x' \frac{3}{2|\vec{r} - \vec{r}'|^2} \]

\[ (p^i(\vec{r}') u^j(\vec{r}') + p^j(\vec{r}') u^i(\vec{r}') - (g^{ij} - u^i u^j)p^c(\vec{r}') u_c) + \]

\[ \int_{V \setminus \{x\}} \sqrt{g} d^3 x' \frac{3}{|\vec{r} - \vec{r}'|^3} \]

\[ (\epsilon^{imn} S_m(\vec{r}') u_n u^j + \epsilon^{imn} S_m(\vec{r}') u_n u^i). \]

(35)

The integration is performed in the horizon of point \( x \) except the point itself. Vector \( u^i \) is in the direction of the source at \( \vec{r}' \). The vectors \( p^i(\vec{r}) \) and \( s^i(\vec{r}) \) are linear and angular velocity field densities with following definitions:

\[ \epsilon^{ijk} p_{ijk} = 0, \quad s^i_{ij} = 0. \]  

(36)

In general, these vectors can be obtained from matter distribution. Boltzmann equation relates them to non-gravitational interactions [Ehlers 1971].

Finally, \( S^{ij} \) can have a purely transverse-traceless component independent of matter distribution and related to relic gravitational waves. Therefore \( S^{ij} = S^{ij}_{vel} + S^{ij}_{relic} \).

Having \( S^{ij} \) for a given distribution of matter and relic perturbations, we can now use equations (20) to (22) to determine the coefficients of (19). Using approximation (24) and the fact that for the chosen base metric \( (3) R = 0 \), equation (19) changes to:

\[ \Delta \delta \phi = -\frac{1}{8} \left[ (M_{TT} + 2M_{TL} + M_L - \right. \]

\[ \frac{2}{3} K_{homo}^2 + 16\pi G T^*_homo) - (7M_{TT} + 2M_{TL} - 5M_L + \]

\[ \frac{10}{3} K^2 + 48\pi G T^*_shomo) \delta \phi + 16\pi \delta T^*_h - 4 \frac{1}{3} K_{homo} \delta K. \]  

(37)
In a homogeneous universe:
\[ \bar{g}_{ij} = \alpha^2 \delta_{ij} \quad \text{and} \quad \bar{K}_{ij} = -\frac{d\alpha}{d\eta} \delta_{ij}. \] (38)

From (4), one can see that:
\[ 16\pi G \bar{T}_{homo}^* - \frac{2}{3} K_{homo}^2 = 16\pi G \bar{T}_{homo}^* - \frac{2}{3} K_{homo}^2 = 0. \] (39)

To keep \( \delta \phi \) small according to our small perturbation assumption, the matter has to have a small average momentum. This makes all coefficients \( M_{TT}, M_{TL}, \) and \( M_L \) of second-order and negligible. It is also another demonstration of power of this method. [Veeraraghavan & Stebbins 1990] also arrives at the same conclusion for the vorticity of a perfect fluid. Here this result is obtained without any assumption about type and state equation of matter. At first-order, the velocity perturbations contribute only to the evolution of the metric (equation (2)) and not to constraints.

With these perturbative simplifications, (37) reduces to:
\[ \triangle \delta \phi = -\frac{2}{6} K_{homo} \delta K - \frac{1}{6} K_{homo} \delta K = \left( \frac{5}{12} K^2 + 6\pi G \bar{T}_{homo}^* \right) \delta \phi - 2\pi G \delta T_{homo}^* + \frac{1}{6} K_{homo} \delta K. \] (40)

and can be solved as:
\[ \delta \phi(x, t) = \frac{1}{\alpha(t)^2} \int d^3x' \frac{2\pi G \delta \bar{T}_{homo}^*(x', t) - \frac{1}{6} K_{homo} \delta \bar{K}(x', t)}{|\vec{r}' - \vec{r}|} = \frac{1}{(2\pi)^3} \int d^3k e^{-i\alpha(t) \delta_{ij} k^i x^j} \frac{2\pi G \delta \bar{T}_{homo}^*(k, t) - \frac{1}{6} K_{homo} \delta \bar{K}(k, t)}{k^2 + \frac{5}{12} K_{homo}^2 + 6\pi G \bar{T}_{homo}^*}. \] (41)

(\( \vec{r}' \) and \( \vec{r} \) are as in (35)). From the form of the base metric, \( T_{homo}^* = T_{homo}^* \).

The first expression for \( \delta \phi \) shows that physical metric \( \bar{g}_{ij} \) is similar to the metric for scalar perturbations alone (see e.g. [Bertschinger 1996]). It is the effect of special coordinate choice on the 3-space.

We now have all quantities necessary for definition of a set of initial data and solution of evolution equations (2) and (3). The essential characteristic of linearized constraints is that equations for \( W^i \) and \( \phi \) are decoupled. In fact, for all order of perturbations, \( \delta \phi \) decouples from (28) and therefore, this method is easily applicable to higher-order perturbative calculations.

### 5 Non-Flat Cosmogonies

For the general case of a space-time with constant curvature, we can use a flat base metric as before. The perturbative expansion of \( \phi \) will take the following form:
\[ \phi(x, t) = \phi_0(1 + \delta \phi(x, t)). \] (42)
\[ \phi_0 = \frac{1}{(1 + \frac{K_{homo}^2}{4r^2})^\frac{1}{2}}. \] (43)
\[ r^2 = \delta_{ij} x^i x^j \]  
\[ \bar{g}_{ij} = \alpha^2(t)(\gamma_{ij} + h_{ij}) = \alpha^2(t)\phi^3(x, t)\delta_{ij} = \alpha^2(t)\gamma_{ij}(1 + 4\delta\phi(x, t)) \]  
\[ \gamma_{ij} = \frac{\delta_{ij}}{(1 + \dot{K}'r^2)^{\frac{1}{2}}}. \]  

\[ \dot{\bar{g}}_{ij} = \alpha^2(t)(\gamma_{ij} + \delta_{ij}) \]  
\[ = \alpha^2(t)\phi^3(x, t)(\gamma_{ij} + \delta_{ij}) = \alpha^2(t)\phi^3(x, t)(1 + \dot{K}'r^2) \]  
\[ \phi^4(x, t) \]  
\[ \gamma_{ij} = \delta_{ij}(1 + \hat{K} \frac{4}{r^2}). \]

After performing a variable change from \( x \) to \( x' \) as before, the equation for \( W^i \) will become:

\[ \partial'_j\left[ \frac{\partial'^i W^j + \partial'^j W^i - \frac{2}{3}\delta_{ij}'\partial' c V^c}{(1 + \dot{K}'r^2)^{3}} \right] = -8\pi GT^* r^i + \frac{\delta^i K}{3}. \]  

(47)

where \( \dot{K}' = \frac{\dot{K}}{a(t)^2} \). To solve this equation, we assume, without loss of generality, that \( W^i \) can be decomposed to:

\[ W^i = \psi V^i - \partial'_i U. \]  

(48)

\[ \psi = (1 + \hat{K}' \frac{4}{r^2}). \]  

(49)

Putting (48) into (47) gives a system of equations for \( V^i \) and \( U \):

\[ \partial'^i \partial'_j V^j + \partial'^j \partial'_j V^i = -8\pi GT^*(x', t) + \frac{K^i}{3}. \]  

(50)

\[ \partial'^4 U = \frac{3}{4} \partial'_i \partial'_j F^{ij}. \]  

(51)

\[ F^{ij} \equiv V^j \partial'^i \psi + V^i \partial'^j \psi + \frac{2}{3} \delta^{ij} V^c \partial'_c \psi. \]  

(52)

Equation (50) is exactly the same as equation (31) and therefore its solution is the same as (32). In (51), the right hand side is known once (50) is solved and the solution of (51) is:

\[ U(x, t) = -\frac{1}{(2\pi)^3} \int d^3ke^{-i\theta k x} \sum_{l} 3k_i k_j F^{ij}(k) \frac{1}{4k^4}. \]  

(53)

Equation (17) changes to:

\[ \triangle \delta\phi = \left( \frac{5}{12} K_{homo}^2 + 6\pi GT_{s homo}^* \right) \phi_0^5 \phi_0^3 \delta\phi - 2\pi G\phi_0^{-3} \delta T_*^* - \frac{4}{3} \phi_0^5 K_{homo} \delta K. \]  

(54)

To solve this equation, one can expand \( \delta\phi \) and \( \delta T_*^* \) with respect to Spherical Harmonic functions:

\[ \delta\phi = \sum_m a_m(r)Y_m(\theta, \varphi) \]  

\[ \delta T_*^* = \sum_m T_m(r)Y_m(\theta, \varphi) \]  

\[ \delta K = \sum_m K_m(r)Y_m(\theta, \varphi) \]  

(55)

This results to the following equation for \( a_m \):

\[ r^2 \frac{d^2 a_m}{dr^2} + 2r \frac{da_m}{dr} + (l(l + 1) - r^2 A(r))a_m = \]  

\[ \frac{d}{dr}(r^2 \frac{da_m}{dr}) + (l(l + 1) - r^2 A(r))a_m = r^2(B(r)T_m + D(r)K_m). \]  

(56)
\[ A(r) \equiv \alpha^2(t)\left(\frac{5}{12}K_{\text{homo}}^2 + 6\pi GT_{\text{homo}}^*\phi_0^4\right) \] 
\[ B(r) \equiv -2\pi G\alpha^2(t)\phi_0^{-3} \] 
\[ D(r) \equiv -\frac{4}{3}\phi_0^5K_{\text{homo}}. \]

This equation has a polynomial homogeneous solution. For simplifying our notation, in the following we ignore \( m \) and \( l \) indexes. We assume:

\[ a(r) = \sum_n d_n r^n. \]  

for the homogeneous solution. Replacing \( a_{ml} \) in (56) with this expansion, one obtains the following recurrent expression for the coefficients \( d_n \):

\[ d_n = \sum_{i=0}^{8} \frac{C_{2i}}{C_1}[(n - 2i + 2)(n - 2i + 3) + l(l + 1)]d_{n-2i+2}. \]  

One can construct two independent solutions for two boundary conditions:

1) \( |a| < \infty \) for \( r \to 0 \implies d_n = 0, \ n < 0. \)

2) \( |a| < \infty \) for \( r \to \infty \implies d_n = 0, \ n > -16. \)

As the derivative terms of (56) are complete, a Green function can be found for this equation ([Zwillinger 1989]). We call the above homogeneous solutions \( a^{(1)}(r) \) and \( a^{(2)}(r) \). The Green function and the complete solution of \( a_{ml} \) are determined as:

\[ G(r; r') = \begin{cases} 
\frac{a^{(1)}(r)a^{(2)}(r')}{r^2\omega(r')} & 0 \leq r \leq r', \\
\frac{a^{(1)}(r')a^{(2)}(r)}{r^2\omega(r)} & r' \leq r < \infty 
\end{cases} \] 

\[ \omega(r) = a^{(1)}(r)\frac{da^{(2)}(r)}{dr} - a^{(2)}(r)\frac{da^{(1)}(r)}{dr}. \] 

\[ a_{ml}(r) = \int dr' r'^2(B(r')T_{ml} + D(r')K_{ml})G(r; r'). \]

This completes the solution of constraints for perturbed constant curvature spaces.

### 6 Matching Condition

The results obtained in previous sections can be applied to any perturbative cosmological context. An straightforward and interesting application is the determination of initial metric and extrinsic curvature perturbations in a universe filled with species which have
a distribution $f(x,t,p)$ in their classical phase space. If they have mutual interactions, one usually has to solve the Einstein-Boltzmann equations numerically. Division of these equations to scalar, vector and tensor components increases significantly the amount of equations to be solved and consequently the calculation time. This issue and the detail of calculation will be discussed elsewhere [Ziaeepour 1997], [Ziaeepour 1999].

Here we use constraint solutions to find a general expression for matching condition on a surface of phase transition. This issue has been already discussed in [Deruelle & Mukhanov 1995], [Deruelle et al. 1997], [Uzan et al. 1998]. We show that on the light of results obtained above, the matching becomes trivial and its physical interpretation more transparent. For simplicity, in this section we consider only the case of a flat cosmology. Using formalism presented in the previous section, the extension to a curved space-time is straightforward.

The matter in the early universe has gone through a few number of phase transitions. One of the consequences of these transitions is the formation of topological defects which could have been the source of initial perturbations (if not completely, at least partially [Contaldi et al. 1998]). To study the evolution of fluctuations after their formation, it is usually assumed that transition was very fast i.e. its duration was much shorter than evolution time until matter-radiation equilibrium. In this case, a phase transition approximately defines an equal time 3-space in the space-time.

The perturbation of the energy-momentum tensor after transition is written as:

$$\delta \bar{T}_{\mu\nu} = \delta \bar{T}^{\text{rad}}_{\mu\nu} + \Theta_{\mu\nu}$$  \hspace{1cm} (69)$$

where $\delta \bar{T}^{\text{rad}}_{\mu\nu}$ is the perturbation in the ordinary matter (assumed to be relativistic), and $\Theta_{\mu\nu}$ is the energy-momentum tensor of defects. In stiff approximation, it is assumed that $\Theta_{\mu\nu}$ evolves separately and is treated as an external source.

The power spectrum of defects after their formation can only be determined by numerical simulations [Stephen et al. 1998]. It needs a complete simulation of quantum processes during phase transition and their decoherence which leads to macroscopic classical behavior of defects. Such a simulation is not yet available and consequently, it is necessary to use phenomenological arguments to fix the initial conditions in simulations of perturbation evolution in presence of defects [Veeraraghavan & Stebbins 1990], [Pen et al. 1994], [Durrer & Sakellariadou 1997].

It is usually assumed that the effect of defect formation on matter and radiation is tiny and consequently perturbative. For studying the evolution of large scale (wavelength) perturbations which are important for the formation of large structures today, the lowest terms in the expansion of the power spectrum of defects is enough and give an analytical expression for initial perturbations [Deruelle et al. 1997]. Some of amplitudes of the expansion terms can be fixed by using the conservation of $\Theta_{\mu\nu}$, i.e $\Theta_{\mu\nu} = 0$. Others are usually fixed by some physical arguments, like causality, and/or a matching condition that relates a physical quantity in two parts of the space-time separated by the phase transition surface. Matching along with other choices define a model for initial perturbations. These conditions must be at least consistent and treated together.

For physical reasons like conservation of energy-momentum tensor, the phase transition surface is assumed to be a surface of constant density. However, this definition is ambiguous. In a flat space-time before phase transition, a constant density 3-space is flat. After transition, in general, the constant density surface is not equal-time and flat (keeping the
same coordinate definition). To remove this ambiguity, we assume that the isomorphism between homogeneous and perturbed space-time manifolds $M_{\text{homo}}$ and $M$, maps $\Sigma$, the transition surface, to a flat equal-time (thus constant density) 3-space. This means that at $O(0)$ the density is constant but not at $O(1)$ (It is the general case. Evidently, it is always possible to redefine time parameter such that $\Sigma$ becomes an equal time and constant density surface).

By definition, the space time must be differentiable everywhere including on the initial 3-space. This means that $a(t)$, $\bar{g}_{ij}$ and $\bar{K}_{ij}$ must be continuous [Deruelle & Mukhanov 1995]:

$$[\bar{g}_{ij}]_{\pm} = 0 \quad \text{and} \quad [\bar{K}_{ij}]_{\pm} = 0. \quad (70)$$

where for any quantity $Q$, $[Q]_{\pm} \equiv \lim_{\epsilon \to 0} Q(t_{PT} + \epsilon) - Q(t_{PT} - \epsilon)$. Variable $t_{PT}$ is the phase transition comoving time. Note that if these conditions are satisfied in one gauge, a continuous gauge transformation fulfills them too. Therefore in this sense, they are gauge invariant.

The relation (70) is used as the matching condition on the phase transition surface. Here we apply the solution of constraints (4) and (5) to the matching conditions, and determine the unconstrained quantities $S_{ij}$ and $K$ on the initial 3-space.

From (10) and (32):

$$\left(\frac{LW}{k^4}\right)_{ij}(k,t) = \frac{2\pi G}{k^4} [4k^2(k^i T^{*j} + k^j T^{*i}) - 2k_c T^{*c}(k^i k^j + k^2 g^{ij}) -$$

$$\frac{k^2 \delta K}{4\pi G} (k^i k^j - \frac{k^2 g^{ij}}{3})]. \quad (71)$$

and from (14), (17), and (11):

$$\bar{K}_{ij} = (S_{ij}^{\text{homo}} + (LW)_{ij}^{\text{homo}} + K_{ij} g^{ij} + \delta S_{ij} + \delta (LW)_{ij} -$$

$$10S_{ij}^{\text{homo}} + 4(LW)_{ij}^{\text{homo}} + \frac{4}{3} K_{ij} g^{ij}) \delta \phi + g^{ij} \delta K. \quad (72)$$

Equation (10) gives $\delta \phi$ as a function of $\delta \bar{T}^*_i$:

$$\delta \phi = \frac{2\pi G \delta \bar{T}^* + \frac{H}{2} \delta K(k,t)}{k^2} \quad H \equiv \frac{\dot{\alpha}}{\alpha}. \quad (73)$$

Dots denote derivative with respect to comoving time $t$. Before phase transition, Universe is homogeneous and the physical metric and extrinsic curvature are equal to their bare counterparts. The matching condition for induced metric (70) and definition of physical and bare metric (25) and (23) leads to $\delta \phi = 0$ and thereafter $\bar{g}_{ij} = g_{ij} = \alpha^2(t) \delta_{ij}$ and:

$$\delta K = -\frac{4\pi G}{H} \delta \bar{T}^*. \quad (74)$$

The matching condition for extrinsic curvature $\bar{K}_{ij}$ is $\delta \bar{K} = 0$. In consequence, its three independent components i.e. $\delta \bar{K}$, $(LW)^{ij}$ (it is $O(1)$), and $S^{ij}$ must separately be zero. From (74) and (71):

$$\delta \bar{T}^* = T^{*i} = 0. \quad (75)$$
\((\mathcal{F})\) means that there is no total density or velocity perturbation on the initial 3-space.
If matter is a perfect fluid or a scalar field, i.e. without viscosity, \(\delta T_{ij} = 0\).
The above results show that the matching condition \((\mathcal{F})\) is sufficient for determination
of the initial value of metric and extrinsic curvature and leads to an isocurvature initial
condition for large wavelengths perturbations for matter without viscosity (e.g. a mixture
of perfect fluid and a scalar field). No degree of freedom rests to be chosen or fixed by
other physical arguments.
Some authors (e.g. \([\text{Pen et al. 1994}]\) and \([\text{Turok N. 1996}]\)) use an ordinarily (in contrast
to covariantly) conserved pseudo-energy-momentum tensor to define a matching condition.
In terms of induced metric and extrinsic curvature of the initial 3-space, this tensor
has the following form:
\[
\tau_{\mu\nu} = \left[ \delta T_{00} + \frac{H}{4\pi G} \delta \bar{K} \right. \delta T_{0i} - \frac{a^2 H}{4\pi G} (H \delta h_{ij} - \delta K_{ij} + \delta_{ij} \delta K) \left. \delta T_{0i} \right] (76)
\]
If we impose the continuity of \(h_{ij}\) and \(\bar{K}_{ij}\) as before, from \((\mathcal{F})\) and \((\mathcal{F})\) one can triv-
ially conclude that \(\tau_{\mu\nu}\) conservation condition is fulfilled. In a flat space-time the \(k \neq 0\)
modes of \(\tau_{\mu\nu}\) are zero. \([\text{Pen et al. 1994}]\) and \([\text{Turok N. 1996}]\) use this property as matching
condition and set \(\tau_{00} = 0\) and \(\tau_{0i} = 0\). In contrast to the first prescription, these
ditions don’t fix all the independent degrees of freedom. For fixing the rest, Pen et al.
choose a relation between density perturbation of radiation and dark matter (based on
the assumption of a white noise power spectrum at superhorizon scales). \([\text{Turok N. 1996}]\)
chooses these quantities such that at superhorizon scale there is not any perturbation in
the ratio of different components of the plasma. It is equivalent to an unperturbed initial
condition.
In \([\text{Uzan et al. 1998}]\), first \((\mathcal{F})\) is used as matching condition and a series of relations
between components of metric, extrinsic curvature and energy-momentum tensor of de-
fects is obtained. Then it is shown that to first order, these conditions is equivalent to
\(\tau_{00} = 0\) i.e. the condition used by \([\text{Pen et al. 1994}]\) and \([\text{Turok N. 1996}]\) for matching.
From \((\mathcal{F}), (\mathcal{F})\) and \((\mathcal{F})\) one can immediately and without lengthy demonstration of
\([\text{Uzan et al. 1998}]\) conclude that continuity of \(\bar{g}_{ij}\) and \(\bar{K}_{ij}\) imposes \(\tau_{00} = 0\). But the in-
verse is not true, i.e. if \(\tau_{00} = 0\) and \(\tau_{0i} = 0\), it does not necessarily mean that there is no
total initial perturbation even for a matter without viscosity. Therefore, equivalence of
two matching prescriptions is one directional and model dependent. Specially, for some
models like perfect fluid and scalar field, the purely geometric matching of Deruelle et
al. completely fixes the initial value of the spectrum \((\mathcal{F})\) and \((\mathcal{F})\), but not the other
prescription. The advantage of the method presented here is that it is compact and in-
dependent of the details of the model. This makes the interpretation of the results easier
and more transparent.
Finally, we rise the following question: How physically meaningful is matching? A match-
ing prescription is used in the circumstances that not enough physical information about
the model is available to fix the initial value of geometry and spectrum. However, even
a purely geometrical and physically well motivated prescription of Deruelle et al. leads
automatically to an isocurvature perturbations in a mixture of perfect fluid and scalar
field. This can be a too much simplification of the reality. One interpretation of this
result is that any physical process needs a finite time to happen and a complicate process
like defect formation can not be replaced by a geometric matching on a 3-space. In this situation, it is probably more reasonable to use some phenomenological arguments for choosing the initial conditions.

In conclusion, York method for separation of dependent and independent degrees of freedom in Einstein equations is used to solve analytically the constraint equations for small perturbations in space-times with constant curvature. The solution is independent of details of the matter model. The results is applied to two commonly used matching prescriptions and it is shown that their equivalence is partial and model dependent.

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