A Classical WR Model with $q$ Particle Types

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Abstract A version of the Widom–Rowlinson model is considered, where particles of $q$ types coexist, subject to pairwise hard-core exclusions. For $q \leq 4$, in the case of large equal fugacities, we give a complete description of the pure phase picture based on the theory of dominant ground states.

Keywords $q$-Type Widom–Rowlinson model · Hard-core exclusion diameters · Large equal fugacities · Stable and unstable types · DLR measures · Pure phases · Contours · Polymer expansions

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1 Introduction

1.1 Preliminaries

The Widom–Rowlinson (WR) model in Euclidean space $\mathbb{R}^d$ with two types of particles was invented in [24]. It was considered in numerous papers among which we mention [1, 4, 6, 10, 20] as most closely related to the current study. In this paper we analyze a $q$-type version of the WR model with equal fugacities and arbitrary hard-core exclusion diameters between particles of different types. Particles of the same type do not interact with each other. Formally, particles of types $i, j \in \{1, \ldots, q\}$ located at points $x, x' \in \mathbb{R}^d$ interact via a hard-core exclusion potential

$$\Phi_{ij}(x, x') = \begin{cases} 0, & |x - x'| > D(i, j), \\ +\infty, & |x - x'| \leq D(i, j), \end{cases}$$

where $D(i, j) = D(j, i) \in (0, \infty)$ is the hard-core diameter. As was said, we assume identical fugacities:

$$z_1 = \cdots = z_q = z > 0.$$  \hfill (1.2)

In addition, we require a triangular-type inequality

$$D(i, j) < D(i, l) + D(j, l)$$

for any pairwise distinct triple $i, j, l \in \{1, \ldots, q\}$. Property (1.2) guarantees that two particles of types $i$ and $j$ do not interact with each other as soon as they are separated by a layer of particles of type $l$, provided that this layer has a finite thickness and high enough density of particles.

Under assumptions (1.1), (1.2) and (1.3), for $q \leq 4$ and for large values of $z$, we specify the set of ergodic infinite-volume Gibbs/Dobrushin–Lanford–Ruelle (DLR) measures (pure phases) for each collection of hard-core diameters

$$\mathbb{D} = \{D(i, j), 1 \leq i < j \leq q\}. \hfill (1.4)$$

See Theorem 1.1.

The ergodic DLR measures are constructed in the infinite-volume (thermodynamic) limit $\Lambda \nearrow \mathbb{R}^d$ from Gibbs distributions in a bounded ‘box’ $\Lambda$ with boundary conditions (possibly randomized) outside $\Lambda$ induced by particles of a given type $i \in \{1, \ldots, q\}$. Particle configurations specifying boundary conditions are taken to be dense enough in the following sense: given a partition of $\mathbb{R}^d$ into a grid of cubic cells of a fixed size smaller than $a = \min [D(i, j) : 1 \leq i < j \leq q]$, the boundary condition configuration is required to have at least one particle of type $i$ inside each cell located outside $\Lambda$. Depending on collection $\mathbb{D}$, some particle types turn out to be ‘stable’ meaning that the corresponding boundary conditions generate a pure phase. The remaining particle types are ‘unstable’ as they generate only convex combinations of the pure phases produced by stable types. See Theorem 1.2 where these convex combinations are specified explicitly.

Methodologically, the present work is related to the Pirogov–Sinai theory (PST), [18, 19, 23, 25]. In Ref. [1] an extension of the PST for the WR model has been proposed where, for a given collection $\mathbb{D}$, the vector $\mathbf{z} = (z_1, \ldots, z_q)$ of fugacities $z_1, \ldots, z_q > 0$ is varied to

A short article [17] discusses a lattice version of the model and explains ideas behind the proofs, and mentions possible developments stemming from our approach.
achieve a pure phase coexistence between different subsets of particle types. Given a subset of particle types, \( Q \subset \{1, \ldots, q\} \), a hypersurface \( H = H(Q) \) of dimension \( q + 1 - \sharp Q \) in the fugacity orthant \( \mathbb{R}^{q+1}_+ \) has been constructed such that for \( z \) lying in this hyper-surface the only pure phases are the ones generated by the particle types \( i \in Q \). (Here and below, \( \sharp \) stands for the cardinality of a given (finite) set.) As is customary in the PST, these hypersurfaces are obtained by equalizing sums of absolutely convergent series for the free energies of the coexisting pure phases. The series are constructed using a rather involved perturbation theory. Consequently, given a point \( z \in \mathbb{R}^{q+1}_+ \) (e.g., \( z = (z, \ldots, z) \) as in Eq. (1.2)), it is hard to conclude in which hypersurface \( H \) it lies. Solving such an ‘inverse problem’ requires an additional effort which frequently employs a so-called dominant ground states analysis; see, e.g., [2,3,15,16].

The current paper presents a solution to the aforementioned inverse problem for the case of equal (large) fugacities. Contrary to [1] we consider fugacities fixed and, in a sense, vary hard-core diameter collection \( D \). In this set-up, for \( q \leq 4 \) and the fugacity \( z \) large enough it is possible to specify phase coexistence regions explicitly in terms of linear inequalities between hard-core diameters. Our analysis is still based upon a perturbation theory and carried out in the spirit of [2,3,15,16]. It turns out that for \( q \leq 4 \), only few leading terms of the perturbation series are need to be analyzed in detail; the results can be expressed via linear inequalities between hard-core diameters. For \( q > 4 \) one needs to analyze higher order terms which eventually leads to rather cumbersome equations. We consider this challenge as quite interesting and promising.\(^2\) Cf. Remark 1.2 below.

It is instructive to compare our results with those obtained for a symmetric (continuous) version of the WR model where the fugacities \( z_i = z, 1 \leq i \leq q \), and the diameters \( D(i, j) \equiv D > 0, 1 \leq i < j \leq q \). Historically, a symmetric version of the WR model (with \( q = 2 \)) was the first one to be analyzed in [20] through the so-called Peierls argument. A further progress for symmetric models was achieved in [10] and later on in [6], for an arbitrary \( q \). In particular, in Ref. [6] some interaction potentials were added, within and between particle types, which obey the symmetry of the model and have a more generic form than a hard-core exclusion. The existence of at least \( q \) distinct pure phases for \( z \) large enough has been proven in [6], based on a clever stochastic dominance type of an argument. Other applications of stochastic dominance and the FKG-inequality type of arguments to the WR model can be found in [4,7].

In contrast to [1,6], our set-up is partially symmetric (as the fugacities \( z_i \) are equal) and partially without a symmetry (as the hard-core exclusion diameter collection \( D \) is arbitrary). Given a collection \( D \), our Theorems 1.1 and 1.2 calculate the set of pure phases which appear to be the same for all \( z > z_0(D) \). Such independence of \( z \) is a consequence of a symmetry because several pure phases are observed only if there exists a permutation of the set \( \{1, \ldots, q\} \) mapping \( D \) into itself and the corresponding stable particle types into each other (unstable particle types are not necessarily symmetric under this map). To establish Theorems 1.1 and 1.2 we employ the machinery of the PST which is designed to treat cases without any symmetry as in [1]. We do not know any ‘simpler’ inequalities or stochastic dominance-based arguments which can produce the same results. For some specific cases of \( D \) one can establish that for any bounded box \( \Lambda \) and any value of fugacity \( z \) the partition function with stable boundary conditions is not less than the corresponding partition function with unstable boundary conditions. This could slightly simplify the proof of Theorem 1.1.

\(^2\) Our hope is that the methodology presented here in a self-contained manner could be used as the base for further developments of the WR model and the PST in a wider context.
We do not use such arguments as they cannot be applied to an arbitrary $\mathbb{D}$. We again refer the reader to Remark 1.2 where some particular models are discussed.

1.2 A Formal Description of the Model. The Results

The definition of the phase space and the standard sigma-algebras of its subsets for a $q$-type WR model can be found in [6]. The set-up adopted in [6] provides a way to describe the DLR measures for this model.

In this paper we use an alternative approach, starting with $q$ independent Poisson processes of rate $z > 0$ in $\mathbb{R}^d$, with samples $X_1, \ldots, X_q$. We write $X = (X_1, \ldots, X_q)$, where $X_i$ identifies a type $i$ particle configuration and $X$ is a multi-type particle configuration. Given a bounded open subset $\Lambda \subset \mathbb{R}^d$, we set $X_i^\Lambda = X_i \cap \Lambda$ and $X^\Lambda = (X_1^\Lambda, \ldots, X_q^\Lambda)$. Then $X_1^\Lambda, \ldots, X_q^\Lambda$ are finite subsets in $\Lambda$ such that

(i) $X_1^\Lambda, \ldots, X_q^\Lambda$ are independent,

(ii) $\forall 1 \leq i \leq q$, 

$$P(\sharp X_i^\Lambda = s) = \frac{e^{-\nu(\Lambda)z^s} \nu(\Lambda)^s}{s!}, \quad s = 0, 1, \ldots,$$

(1.5)

(iii) $\forall 1 \leq i \leq q$, the distribution of points $X$ from $X^\Lambda$, conditional on $\sharp X_i^\Lambda = s$, is obtained by symmetrizing a sample of $s$ independent uniform random points in $\Lambda$.

Here and below, $\nu(\cdot)$ stands for the Lebesgue volume and $P$ for the underlying (joint) Poisson processes distribution.

In a number of forthcoming definitions, we refer a cubic box

$$\Lambda = [-L, L]^d, \quad \text{where} \quad \frac{L}{R} \text{ is a positive integer.}$$

(1.6)

Here $R \leq a = \min \{D(i, j) : 1 \leq i < j \leq q\}$ is a ‘coarse-graining’ constant which will be chosen later. Such a box is naturally divided into cubic cells of size $R$; such a cell will be denoted by $\Upsilon$. The choice of $R$ will not affect the forthcoming constructions, and we will omit this symbol from the notations. Let $\mu_\Lambda = \mu_{\Lambda, \mathbb{D}, z}$ denote the conditional probability distribution

$$\mu_\Lambda(\cdot) := P\left( \cdot \mid A(\Lambda) \right) = \frac{P\left( \cdot \cap A(\Lambda) \right)}{P(A(\Lambda))},$$

(1.7)

where $A(\Lambda) = A(\mathbb{D}, \Lambda)$ stands for the event

$$A(\Lambda) := \left\{ X = (X_1, \ldots, X_q) : |x - x'| > D(i, j), \quad \forall x \in X_i^\Lambda, x' \in X_j^\Lambda, \quad 1 \leq i < j \leq q \right\}.$$  

(1.8)

Then $\mu_\Lambda$ is called the Gibbs distribution in $\Lambda$ corresponding to the potential (1.1), fugacity $z$ and an empty boundary condition outside $\Lambda$. Multi-type particle configurations $X$ belonging to $A(\Lambda)$ are called admissible in $\Lambda$. (The adjective multi-type will be omitted.)

Next, consider an event $A$ defined by

$$A := \left\{ X = (X_1, \ldots, X_q) : |x - x'| > D(i, j), \quad \forall x \in X_i \text{ and } x' \in X_j, \quad 1 \leq i < j \leq q \right\}.$$  

(1.9)
A configuration $\mathbf{X} = \{X_1, \ldots, X_q\} \in \mathcal{A}$ is called admissible. Further, given $\mathbf{Y} \in \mathcal{A}$ and denoting $\Lambda^c = \mathbb{R}^d \setminus \Lambda$, set:

$$\mathcal{A} \left( \Lambda \mid Y^{\Lambda^c} \right) := \left\{ \mathbf{X} = (X_1, \ldots, X_q) : \mathbf{X}^\Lambda \vee Y^{\Lambda^c} \in \mathcal{A} \right\}. \quad (1.10)$$

Here, as before, $\mathbf{X}^\Lambda = (X_1^\Lambda, \ldots, X_q^\Lambda)$, $X_j^\Lambda = X_j \cap \Lambda$, and $\mathbf{X}^\Lambda \vee Y^{\Lambda^c}$ denotes the concatenated configuration:

$$\mathbf{X}^\Lambda \vee Y^{\Lambda^c} = (X_1^\Lambda \cup Y_1^{\Lambda^c}, \ldots, X_q^\Lambda \cup Y_q^{\Lambda^c}). \quad (1.11)$$

The Gibbs distribution $\mu_{\Lambda} \left( \cdot \mid Y^{\Lambda^c} \right)$ in $\Lambda$ with the boundary condition $Y^{\Lambda^c}$ is determined by

$$\mu_{\Lambda} \left( \cdot \mid Y^{\Lambda^c} \right) := \mathbb{P} \left( \cdot \mid \mathcal{A} \left( \Lambda \mid Y^{\Lambda^c} \right) \right) = \frac{\mathbb{P} \left( \cdot \cap \mathcal{A} \left( \Lambda \mid Y^{\Lambda^c} \right) \right)}{\mathbb{P} \left( \mathcal{A} \left( \Lambda \mid Y^{\Lambda^c} \right) \right)}. \quad (1.12)$$

Next, it is convenient to introduce a Gibbs distribution $\mu_{\Lambda} \left( \cdot \mid i \right)$ with a randomized boundary condition of type $i$ outside $\Lambda$. To this end, we pass from $\Lambda$ to the ‘twice-extended’ box $2^R \Lambda$:

$$2^R \Lambda = [-2R - L, L + 2R]^d. \quad (1.13)$$

Cube $2^R \Lambda$ is obtained by adding a layer of width $2R$ to $\Lambda$; equivalently, $2^R \Lambda$ is the union of cells belonging to cubes of linear size $5R$ centered at $Y \subset \Lambda$. Given $i \in \{1, \ldots, q\}$, set:

$$\mathcal{A}(\Lambda \mid i) := \left\{ \mathbf{X} = (X_1, \ldots, X_q) \in \mathcal{A} \left( 2^R \Lambda \right) : X_j \cap Y \neq \emptyset \text{ for each cell } Y \subset 2^R \Lambda \setminus \Lambda \right\}. \quad (1.14)$$

Further, set:

$$\mu_{\Lambda} \left( \cdot \mid i \right) := \mathbb{P} \left( \cdot \mid \mathcal{A}(\Lambda \mid i) \right) = \frac{\mathbb{P} \left( \cdot \cap \mathcal{A}(\Lambda \mid i) \right)}{\mathbb{P} \left( \mathcal{A}(\Lambda \mid i) \right)}. \quad (1.15)$$

We note that the choice of width $2R$ is not occasional: it reflects a convenient screening property discussed after Definition 2.1 in the next section. Furthermore, our choice of $R$ guarantees that for $\mathbf{X} \in \mathcal{A}(\Lambda \mid i)$ the intersection $X_j \cap Y = \emptyset$ for all $Y \subset 2^R \Lambda \setminus \Lambda$ and $j \neq i$.

As was indicated above, we study so-called infinite-volume Gibbs/DLR probability measures, corresponding with hard-core exclusion potentials $\Phi_{ij}$ as in (1.1), (1.3) and with symmetric fugacities equal to $\varepsilon$ as in (1.2). The definition of such a measure $\mu = \mu_{\varepsilon, 2}$ follows the Dobrushin–Lanford–Ruelle (DLR) equation. Given $\Lambda$ as in Eq. (1.6), let $\mathcal{X}(\Lambda)$ and $\mathcal{X}(\Lambda^c)$ denote the sigma-algebras generated by restrictions $\mathbf{X}^\Lambda$ and $\mathbf{X}^{\Lambda^c}$, respectively. Then the restriction of the conditional distribution $\mu(\cdot \mid \mathcal{X}(\Lambda^c))$ to $\mathcal{X}^\Lambda$ coincides with $\mu_{\Lambda} \left( \cdot \mid Y^{\Lambda^c} \right)$ for $\mu$-a.a. $\mathbf{Y}$. In other words, for any $B \in \mathcal{X}^\Lambda$

$$\left[ \mu \left( B \mid \mathcal{X}(\Lambda^c) \right) \right] (\mathbf{Y}) = \mu_{\Lambda} \left( B \mid Y^{\Lambda^c} \right). \quad (1.16)$$

Examples of DLR-measures are (weak) limiting points for measures $\mu_{\Lambda} \left( \cdot \mid i \right)$, $\mu_{\Lambda} \left( \cdot \mid Y^{\Lambda^c} \right)$ or $\mu_{\Lambda} \left( \cdot \mid i \right)$ when $\Lambda$ is taken as in Eq. (1.6) and $\Lambda \nearrow \mathbb{R}^d$ (we will also use the notation...
$L \to \infty$). In fact, the set of DLR measures is the closure of the convex hull of limit points of probability distributions $\mu(A) \left( \frac{Y^{\Lambda_{k}}}{\Lambda^{k}} \right)$ with arbitrary (admissible) boundary conditions $Y^{\Lambda_{k}}$.

A shift-invariant and ergodic DLR measure is interpreted as a pure phase. In this paper, for $q = 2, 3, 4$, we propose the following algorithm of identifying the pure phases. Let us list the distinct values of the hard-core diameters in an increasing order:

$$a = a(1) < \cdots < a(k) := b$$

where $1 \leq k \leq q(q - 1)/2$.

For each $j = 1, \ldots, q$ we form a $k$-dimensional incidence-occupation number vector $n(j)$ with non-negative integer entries:

$$n(j) = (n(j, 1), \ldots, n(j, k)).$$

(1.18)

Here $n(j, l)$ stands for the number of types $i \in \{1, \ldots, q\} \setminus \{j\}$ such that $D(j, i) = a(l), 1 \leq l \leq k$. Then consider the collection $S$ of types $j$ for which the vector $n(j)$ is lexicographically maximal:

$$S = S(\mathbb{D}) := \left\{ j : n(j') \leq n(j) \forall 1 \leq j' \leq q \right\}. \quad (1.19)$$

Mnemonically, to form set $S$ we first select the types $j \in \{1, \ldots, q\}$ which have the maximal number $n(j, 1)$ of incident ‘edges’ $(j, j')$ with $D(j, j') = a(1)$ (i.e., the maximal number of types $j'$ that repel type $j$ with the shortest hard-core diameter $a(1)$). From these we select those types $j$ which have the maximal number $n(j, 2)$ (representing edges $(j, j')$ with $D(j, j') = a(2)$), and so on. Pictorially speaking, types $i \in S$ are ‘most tolerant’ to other types $j$ in terms of a ‘direct’ repulsion. Cf. Remark 1.3 below.

Geometrically, for $q = 4$, particle types $j$ can be placed at the vertices of a tetrahedron, and edges $j \leftrightarrow l$ can be ‘painted’ in different colors: e.g., green when $D(j, l) = a(1)$, blue when $D(j, l) = a(2)$, etc. (We do not mean here that $D(i, j)$ is equal to the length of an edge.) In this context, vector $n(j)$ takes into account the multiplicities of single-edge paths of different colors beginning at $j$.

Values $i \in S$ are called stable particle types or stable types for short.

**Theorem 1.1** Assume that collection $\mathbb{D}$ satisfies the triangular property (1.3). Then, for $d \geq 2$ and $q = 2, 3, 4$, there exists $z_0 \in (0, \infty)$ (depending on the collection $\mathbb{D}$) such that $\forall z \geq z_0$:

(I) $\forall i \in S$ there exists a pure phase $\mu(\cdot \| i)$ (a shift-invariant, ergodic DLR-measure, with exponential mixing) obtained as the limit

$$\mu_{\Lambda^{\mathbb{R}^d}}(\cdot \| i) = \lim_{\Lambda \to \mathbb{R}^d} \mu_{\Lambda}(\cdot \| i).$$

(1.20)

Measures $\mu(\cdot \| i)$ are distinct for different $i \in S$.

(II) Every shift-periodic DLR-measure is a convex linear combination of measures $\mu(\cdot \| i), i \in S$.

**Remark 1.1** Under an additional assumption that $k = 1$ (i.e., when $D(i, j) \equiv a$), assertion (I) has been proved for any value of $q \geq 2$ in [21] and generalized to a wider class if symmetric interactions in [6].

The only place using the triangular property is Eq. (2.33). ▲
Theorem 1.2 Take \( q = 4 \) and adopt the same assumptions as in Theorem 1.1. Then, for \( d \geq 2 \) and \( \forall z \geq z_0 \), the following assertions hold true.

(I) Suppose that \( \sharp S = 1 \) and, without loss of generality, \( S = \{1\} \) (i.e., 1 is a unique stable particle type). Then

\[
\lim_{\Lambda \nearrow \mathbb{R}^d} \mu_\Lambda (\cdot \parallel j) = \mu (\cdot \parallel 1) \quad \forall j \in \{1, 2, 3, 4\}.
\]

Moreover, any family of Gibbs distributions \( \mu_\Lambda (\cdot \mid \mathbf{Y}^{\Lambda \mathbb{C}}) \) with boundary conditions \( \mathbf{Y}^{\Lambda \mathbb{C}} \) induced by \( \mathbf{Y} \in A \) converges to \( \mu (\cdot \parallel i) \) as \( \Lambda \nearrow \mathbb{R}^d \).

(IIa) Suppose that \( \sharp S = 2 \) and \( S = \{1, 2\} \) (i.e., there are two stable types, 1 and 2). Suppose \( D(1, 3) = D(2, 3) \). Then \( D(1, 4) = D(2, 4) \), and vice versa. In this case,

\[
\lim_{\Lambda \nearrow \mathbb{R}^d} \mu_\Lambda (\cdot \parallel 3) = \lim_{\Lambda \nearrow \mathbb{R}^d} \mu_\Lambda (\cdot \parallel 4) = \frac{1}{2} \mu (\cdot \parallel 1) + \frac{1}{2} \mu (\cdot \parallel 2).
\]

(IIb) Suppose again that \( \sharp S = 2 \) and \( S = \{1, 2\} \). Suppose \( D(1, 3) \neq D(2, 3) \) then \( D(1, 4) \neq D(2, 4) \), and vice versa. In this case \( D(1, 4) = D(2, 3) \) and \( D(1, 3) = D(2, 4) \). Furthermore,

\[
\lim_{\Lambda \nearrow \mathbb{R}^d} \mu_\Lambda (\cdot \parallel 3) = \begin{cases} \mu (\cdot \parallel 1), & \text{if } D(1, 3) < D(2, 3), \\ \mu (\cdot \parallel 2), & \text{if } D(2, 3) < D(1, 3), \end{cases}
\]

and, similarly,

\[
\lim_{\Lambda \nearrow \mathbb{R}^d} \mu_\Lambda (\cdot \parallel 4) = \begin{cases} \mu (\cdot \parallel 1), & \text{if } D(1, 4) < D(2, 4), \\ \mu (\cdot \parallel 2), & \text{if } D(2, 4) < D(1, 4). \end{cases}
\]

(III) Suppose \( \sharp S = 3 \) and \( S = \{1, 2, 3\} \) (i.e., there are three stable types, 1, 2 and 3). Then

\[
\lim_{\Lambda \nearrow \mathbb{R}^d} \mu_\Lambda (\cdot \parallel 4) = \frac{1}{3} \mu (\cdot \parallel 1) + \frac{1}{3} \mu (\cdot \parallel 2) + \frac{1}{3} \mu (\cdot \parallel 3).
\]

(IV) In all cases \( \sharp S = 1, 2, 3, 4 \),

\[
\lim_{\Lambda \nearrow \mathbb{R}^d} \mu_\Lambda (\cdot) = \frac{1}{\sharp S} \sum_{i \in S} \mu (\cdot \parallel i).
\]

Remark 1.2 Assertions (IIa), (IIb), (III) and (IV) have a transparent geometrical meaning. Viz., conditions specifying case (IIa) mean that unstable types 3 and 4 are positioned symmetrically relative to types 1 and 2 forming set \( S \). Next, cases (III) and (IV) possess a maximum geometric symmetry which leads to uniform coefficients. In contrast, in case (IIb) inequalities in (1.23) and (1.24) indicate that the unstable type generates a phase corresponding to a ‘closer’ type from \( S \).

Remark 1.3 For \( q = 5 \), the particle types can be placed at the vertices of a simplex in \( \mathbb{R}^4 \) (a pentagram). Here, the algorithm of specifying the set \( S \) is based on the analysis of both single-edge and two-edge paths issued from a given vertex \( i \). The number of edges in a path needed to be taken into account increases with \( q \) (in general, non-monotonically). Formally, for \( q \geq 5 \) it is still true that any type \( i \) which creates a pure phase lies in \( S \). However, not every \( i \in S \) yields a pure phase; the specification of the subset \( S^{\text{st}} \subset S \) corresponding to pure phases remains an open question.
The subsequent sections are dedicated to the proofs. Formally, we assume that \( q = 4 \); for \( q = 2, 3 \) the proofs simplify. The proof of assertion (I) of Theorem 1.1 is completed in Sect. 4.4 and that of assertion (II) in Sect. 5.5. The proof of Theorem 1.2 is completed in Sects. 5.1, 5.2, 5.3, and 5.4.

2 Contours: Definitions and Main Facts

The aim of this section is to introduce a representation of the WR model under consideration in terms of `contours`. To this end, we first perform a coarse graining procedure reducing the original model to a model on a cubic lattice \( \mathbb{Z}^d \). We then proceed with definitions of `rarefied' and `crystalline' partition functions that are standard concepts employed in the PST; see Eqs. (2.34) and (2.35). The crystalline partition function plays a role of (and introduced as) a statistical weight of an external contour. As usual, key properties are factorizations into products; cf. Eqs. (2.33), (2.34) and (2.36).

2.1 Hard-Core Exclusion Diameters

Given set \( D \) of hard-core exclusion diameters we select a constant \( R > 0 \) which is used as a discretization scale. This constant is taken to be small enough such that

\[
\min \left[ a, \min_{1 \leq l_1 < l_2 \leq k} |a(l_1) - a(l_2)| \right] > 10Rd; \quad (2.1)
\]

the number 10 is selected for definiteness only and is not optimal. Rescaling of \( \mathbb{R}^d \) with magnification \( 1/R \) maps our original model into a similar model with rescaled values of \( D(i, j) \) and \( z \). Accordingly, the previous requirement takes the form

\[
\min \left[ a, \min_{1 \leq l_1 < l_2 \leq k} |a(l_1) - a(l_2)| \right] > 10d. \quad (2.2)
\]

Without loss of generality we assume now that \( R = 1 \) and (2.2) is satisfied.

2.2 The Contour Definition

Our next aim is to translate the original model given in \( \mathbb{R}^d \) into a lattice model living in \( \mathbb{Z}^d \) (cf. [1,20]). A peculiarity of the WR model is that it is convenient to introduce contours without discussing (formal) Hamiltonians.

For this purpose we set:

\[
\mathbb{B}(y, S) := \left\{ x = (x^1, \ldots, x^d) \in \mathbb{R}^d : \sum_{i=1}^d (x^i - y^i)^2 \leq S^2 \right\},
\]

\[
\mathbb{B}(S) := \mathbb{B}(0, S),
\]

\[
\Upsilon(y) := \left\{ x = (x^1, \ldots, x^d) \in \mathbb{R}^d : |x^i - y^i| < \frac{1}{2} \quad i = 1, \ldots, d \right\}. \quad (2.3)
\]

In other words, \( \mathbb{B}(y, S) \) and \( \mathbb{B}(S) \) in (2.3) are closed balls of radius \( S \), and \( \Upsilon(y) \) is a unit cube (cell) centered at a point \( y = (y^1, \ldots, y^d) \in \mathbb{Z}^d \) with integer coordinates. For most of the time, the reference to the center \( y \) will be not important; accordingly, the argument \( y \) will be omitted. From now on, \( \Lambda \) will be the closure of a union of a finitely many unit cells \( \Upsilon(y) \) (an example of which is the cube (1.6)); we will sometimes refer to such \( \Lambda \) as a box. All definitions given in Sect. 1 remain valid for non-cubic boxes.
Depending on the context we will sometimes identify a cell $\Upsilon$ with its center $\Upsilon$ and write $\Upsilon \in \mathbb{Z}^d$; similarly, a box $\Lambda$ is identified with the collection of centers of cells $\Upsilon \subset \Lambda$ and treated as a subset in $\mathbb{Z}^d$. The number of lattice sites in $\Lambda \subset \mathbb{Z}^d$ will coincide with the Euclidean volume $\nu(\Lambda)$ of $\Lambda \subset \mathbb{R}^d$.

We will need some notation for ‘extended’ and ‘reduced’ boxes produced from a given box, $\Lambda$. To this end, for a given unit cell $\Upsilon$ denote by $Q_m(\Upsilon)$ a cube of integer linear size $m$ concentric with $\Upsilon$. Then $\Upsilon = Q_1(\Upsilon)$ and we set:

$$m\Lambda = \bigcup_{\Upsilon \subset \Lambda} Q_{2m+1}(\Upsilon) \text{ and } m\Lambda = \bigcup_{\Upsilon : Q_{2m+1}(\Upsilon) \subset \Lambda} \Upsilon.$$ (2.4)

Let $\mathbf{X} \in A(\Lambda)$ be a particle configuration admissible in $\Lambda \subset \mathbb{Z}^d$. Then each unit cell $\Upsilon \subset \Lambda$ cannot have particles of more than one type in it. This gives rise to a map

$$\phi_{\Lambda} : \Upsilon \subset \Lambda \mapsto \phi_{\Lambda}(\Upsilon, \mathbf{X}^\Lambda) \in \{0, 1, \ldots, q\}$$ (2.5)

attaching types to unit cells in $\Lambda$. (Here $\phi_{\Lambda}(\Upsilon, \mathbf{X}^\Lambda) = 0$ means cell $\Upsilon$ has no particles in $\mathbf{X}$.) The collection of pairs $(\Upsilon, \phi_{\Lambda}(\Upsilon, \mathbf{X}^\Lambda))$, where $\Upsilon \subset \Lambda$, is called an admissible cell configuration in $\Lambda$ generated by $\mathbf{X}$. Moreover, a map

$$\phi_{\Lambda} : \Upsilon \subset \Lambda \mapsto \{0, 1, \ldots, q\}$$ (2.6)

is called an admissible cell configuration in $\Lambda$ with empty boundary conditions if there exists a particle configuration $\mathbf{X} \in A(\Lambda)$ such that $\phi_{\Lambda}(\Upsilon) = \phi_{\Lambda}(\Upsilon, \mathbf{X}^\Lambda) \forall \Upsilon \subset \Lambda$. We refer to $\mathbf{X}$ as a generating particle configuration for $\phi_{\Lambda}$. Furthermore, the value $\phi_{\Lambda}(\Upsilon)$ is called the type of cell $\Upsilon$ in $\phi_{\Lambda}$. Memonically, the difference between $\phi_{\Lambda}$ and $\phi_{\underline{\L}}$ is that the former is always used with a reference to a generating particle configuration $\mathbf{X}$ whereas the latter can be considered without bearing in mind a particular generating particle configuration (which is guaranteed to exist and transfers a number of properties to the generated cell configuration $\phi_{\underline{\L}}$). The same is true about $\phi$ and $\phi$ introduced in Eqs. (2.7) and (2.8) below.

The set of admissible cell configurations in $\Lambda$ with empty boundary conditions is denoted by $C(\Lambda)$. The map $\phi_{\underline{\L}}$ and probability distribution $\mu_{\underline{\L}}(\cdot)$ defined in (1.7) induces a probability distribution on $C(\Lambda)$ which we again denote by $\mu_{\underline{\L}}(\cdot)$.

Similarly, we denote by $C(\Lambda \parallel i)$ the set of admissible cell configurations $\phi_{2\Lambda} \in C(2\Lambda) \parallel i)$ generated by $\mathbf{X} \in A(\Lambda \parallel i)$. Without loss of generality we denote these configurations $\phi_{i\Lambda}$ rather than $\phi_{i\widetilde{\Lambda}}$ as their value is equal to $i$ for each $\Upsilon \subset 2\Lambda \setminus \Lambda$. The probability distribution on $C(\Lambda \parallel i)$ induced by $\mu_{\Lambda}(\cdot \parallel i)$ defined in (1.15) is also denoted by $\mu_{\Lambda}(\cdot \parallel i)$.

Now let $\mathbf{X} \in A$. In a similar manner we define a map

$$\phi : \Upsilon \in \mathbb{Z}^d \mapsto \phi(\Upsilon, \mathbf{X}) \in \{0, 0, \ldots, q\}.$$ (2.7)

The collection of pairs $(\Upsilon, \phi(\Upsilon, \mathbf{X}))$ is called an admissible cell configuration in $\mathbb{Z}^d$ generated by $\mathbf{X}$. Further, a map

$$\phi : \Upsilon \in \mathbb{Z}^d \mapsto \{0, 1, \ldots, q\}$$ (2.8)

is called an admissible cell configuration in $\mathbb{Z}^d$ if there exists a particle configuration $\mathbf{X} \in A$ such that $\phi(\Upsilon) = \phi(\Upsilon, \mathbf{X})$. Here we again say that $\phi(\Upsilon)$ is the type of cell $\Upsilon$ in $\phi$. We denote by $C$ the set of admissible cell configurations in $\mathbb{Z}^d$. If $\phi_{\Lambda} \in C(\Lambda \parallel i)$ then its extension $\phi_{\Lambda,i}^* \in C$ is defined as

\[ \sqrt{\text{ Springer}} \]
For the rest of the paper we are working mainly in terms of admissible cell configurations (i.e. in terms of discretized lattice model) passing back to particle configurations when necessary. The exact procedures of recalculating statements given in terms of cell configurations to the corresponding statements in terms of particle configurations are self-evident except for few cases detailed later in this section.

**Definition 2.1** Given an admissible cell configuration \( \phi \in \mathcal{C} \), we say that a cell \( \Upsilon \) of non-0 type \( i \) is in phase \( i \) in \( \phi \) (or in a generating particle configuration \( \mathbf{X} \)) if all of 5\( d \) cells in the cube \( 2^\Upsilon \) have attached type \( i \).

Next, given an admissible cell configuration \( \phi_\Lambda \in \mathcal{C}(\Lambda||i) \), a cell \( \Upsilon \subset \Lambda \) of non-0 type \( j \) is said to be in phase \( j \) in \( \phi_\Lambda \) (or in a generating particle configuration \( \mathbf{X}^\Lambda \)) if each cell of \( 2^\Upsilon \cap \Lambda \) has attached type \( j \).

The property behind Definition 2.1 is that inside a cell in phase \( i \) any position of a particle of type \( i \) in a generating particle configuration \( \mathbf{X} \) is allowed, regardless of particle positions in any other cell. In probabilistic terms, restrictions \( \mathbf{X}^\Upsilon \) and \( \mathbf{X}^\Lambda \cap \Upsilon \) are conditionally independent, given that cell \( \Upsilon \) is in phase \( i \). Here we refer to any measure \( \mu_\Lambda \left( \cdot \mid [\Upsilon] \right) \), \( \mu_\Lambda \left( \cdot \mid [i] \right) \) or \( \mu_\Lambda (\cdot) \), with \( \Lambda \) containing \( 2^\Upsilon \), provided that the probability of the condition under such a measure is positive.

Indeed, given \( i, j \in \{1, \ldots, q\} \) with \( i \neq j \), consider the union \( U = \bigcup_{x \in \Upsilon} \mathbb{B}(x, D(i, j)) \).

Then, for any choice of points \( y \) in each of the \( 5^d - 1 \) cells in \( 2^\Upsilon \setminus \Upsilon \) (at least one point in a cell), the union of Euclidean balls of radius \( D(i, j) \) with centers at points \( y \) contains \( U \).

(Note that the above statement is not true if \( 2^\Upsilon \) is replaced with \( 1^\Upsilon \)). This is the screening phenomenon mentioned in the previous section.

Because of the screening phenomenon, Definition 2.1 does not refer to the interaction radius \( b \) (or to collection \( \mathbb{D} \) specifying the potentials \( \Phi_{ij} \)). In the literature the cube surrounding cell \( \Upsilon \) has traditionally the size larger than the radius of interaction. See, e.g., [1, 8, 23, 25].

**Definition 2.2** Given an admissible cell configuration \( \phi \in \mathcal{C} \), consider the union \( P(\phi) \) of all cells \( \Upsilon \in \mathbb{Z}^d \) which are in some (non-0) phase in \( \phi \). The complement \( \mathbb{Z}^d \setminus P(\phi) \) is denoted by \( C(\phi) \). Similarly for \( \phi_\Lambda \in \mathcal{C}(\Lambda||i) \) consider the sets \( P(\phi_\Lambda) \) and \( C(\phi_\Lambda) = \mathbb{Z}^d \setminus P(\phi_\Lambda) \), where \( \phi_\Lambda \) is defined in (2.9).

A contour \( \Gamma = \Gamma(\phi) \) or \( \Gamma = \Gamma(\phi_\Lambda) \) generated by an admissible cell configuration \( \phi \in \mathcal{C} \) or \( \phi_\Lambda \in \mathcal{C}(\Lambda||i) \) is a connected component of \( \mathcal{C}(\phi) \) or a connected component of \( \mathcal{C}(\phi_\Lambda) \) considered with the types of cells that lie in this connected component. The set of unit cells (with discarded types) forming contour \( \Gamma \) is denoted by \( B(\Gamma) \) and called the base of contour \( \Gamma \) (as before, \( B(\Gamma) \) can be identified with a subset of \( \mathbb{R}^d \)). The set \( B(\Gamma) \) contains a subset \( O(\Gamma) \) formed by unit cells with type 0.

Note that for a contour \( \Gamma(\phi_\Lambda) \) the set \( B(\Gamma) \) is always finite and may contain unit cells \( \Upsilon \not\subset \Lambda \). However, \( B(\Gamma) \subset 2\Lambda \) and \( O(\Gamma) \subset \Lambda \). The simplest example is where \( \phi_\Lambda \) attaches type 0 to all unit cells in \( \Lambda \).

It is clear that the Euclidean volumes satisfy:

\[
0 < 5^{-d} \nu(B(\Gamma)) \leq \nu(O(\Gamma)) < \nu(B(\Gamma)), \tag{2.10}
\]
as any cube of size 5 centered at any cell of B(Γ) contains at least one cell of type 0 (empty cell).

**Definition 2.3** Given an admissible cell configuration \( \phi \in C \) or \( \phi^\Lambda \in C(\Lambda \| i) \) for some \( i = 1, \ldots, q \), let Γ be a contour in \( \phi \) or \( \phi^\Lambda \). If B(Γ) is finite then the complement \( B(\Gamma)^c = \mathbb{Z}^d \setminus B(\Gamma) \) has a single connected component which is called the exterior of Γ and denoted by E(Γ). The region \( \mathbb{Z}^d \setminus (B(\Gamma) \cup E(\Gamma)) \) is denoted by \( I(\Gamma) \) and called the interior of Γ (it may be empty); its connected components are denoted by \( I_s(\Gamma) \) and labeled by \( s = 1, 2, \ldots \) in an arbitrary order. ▲

**Definition 2.4** For \( \phi \in C \) or \( \phi^\Lambda \in C(\Lambda \| i) \), let Γ be a contour in \( \phi \) or \( \phi^\Lambda \) with finite B(Γ). It is not hard to see that under \( \phi \) all unit cells \( \Upsilon \subset B(\Gamma) \) adjacent to \( E(\Gamma) \) have the same type \( i^E(\Gamma) \in \{1, \ldots, q\} \) called the external type for Γ. The same is true under \( \phi^\Lambda \) (cf. (2.9)); we will employ the same term external type and the same notation \( i^E(\Gamma) \). Further, in both situations (with \( \phi \) and \( \phi^\Lambda \)) the unit cells \( \Upsilon \subset B(\Gamma) \) adjacent to \( I_s(\Gamma) \) also have the same type \( i_s(\Gamma) = i(I_s(\Gamma)) \in \{1, \ldots, q\} \). The value \( i_s(\Gamma) \) is called the \( s \)-th internal type for Γ. ▲

**Definition 2.5** Given \( \phi \in C \), the set of contours in \( \phi \) is called the contour collection in \( \mathbb{Z}^d \) generated by \( \phi \) and denoted by \( \Gamma(\phi) \). (The collection \( \Gamma(\phi) \) can be finite or countable.) Likewise, for \( \phi^\Lambda \in C(\Lambda \| i) \), the collection of contours in \( \phi^\Lambda \) is denoted by \( \Gamma(\phi^\Lambda) \) and called the contour collection in \( \Lambda \) generated by \( \phi^\Lambda \). (The collection \( \Gamma(\phi^\Lambda) \) is always finite.)

It is convenient to introduce the maps

\[
\begin{align*}
\gamma & : \phi \in C \mapsto \Gamma(\phi), \\
\gamma \circ \phi & : \Xi \in A \mapsto \phi \mapsto \Gamma(\phi), \\
\gamma^\Lambda & : \phi^\Lambda \in C(\Lambda \| i) \mapsto \Gamma(\phi^\Lambda), \\
\gamma^\Lambda \circ \phi^\Lambda & : \Xi \in A(\Lambda \| i) \mapsto \phi^\Lambda \mapsto \Gamma(\phi^\Lambda).
\end{align*}
\]

For brevity we will say that Γ is a contour in/from an admissible cell configuration \( \phi \in C \) or \( \phi \in C(\Lambda \| i) \) if Γ belongs to the contour collection \( \Gamma(\phi) \) or \( \Gamma(\phi^\Lambda) \).

Suppose Γ is a contour in \( \phi \in C \) with finite B(Γ). Define \( \phi^\ast_i \in C \) as

\[
\phi^\ast_i(\Upsilon) := \begin{cases} 
\phi(\Upsilon) & \text{if } \Upsilon \subset B(\Gamma) \\
i^E(\Gamma) & \text{if } \Upsilon \subset E(\Gamma) \\
i_s(\Gamma) & \text{if } \Upsilon \subset I_s(\Gamma).
\end{cases}
\] (2.12)

Then, clearly, the contour collection \( \Gamma(\phi^\ast_i) \) consists of a single contour Γ. A similar construction can be performed for any contour Γ from \( \phi^\Lambda \in C(\Lambda \| i) \) with \( i^E(\Gamma) = i \). The corresponding admissible cell configuration from \( C(\Lambda \| i) \) is denoted by \( \phi^\ast_{i,\Gamma} \). Evidently, \( \phi^\ast_{i,\Gamma} \) is just a restriction of \( \phi^\ast_i \) to \( \Lambda \). Also, the cell configuration \( \phi^\ast_i \) defined in (2.12) should not be confused with the cell configuration \( \phi^\ast_{i,\Gamma} \) defined in (2.9).

For any cell configuration \( \phi \) and any box \( \Lambda \) let \( \phi^\Lambda \) be the restriction of \( \phi \) to \( \Lambda \). Note that \( \phi^\ast_i \) is uniquely determined by its restriction \( \phi^\ast_i \mid B(\Gamma) \) to B(Γ). In fact, it will be convenient to identify a contour Γ with a pair

\[
\Gamma = (B, \varphi_B),
\] (2.13)

where \( B = B(\Gamma) \) is a connected union of cubes of size 5 and \( \varphi_B = \varphi(\Gamma) \) is a restriction \( \phi \mid_B \) of a cell configuration \( \phi \in C \) having a single contour \( \Gamma(\phi) \) with \( B(\Gamma(\phi)) = B \).
Definition 2.6 A contour $\Gamma = (B, \varphi_B)$ is called non-separating if $\Gamma$ is drawn within a single type, i.e. for each $\Upsilon \subset B$ the value of $\varphi_B(\Upsilon)$ is either 0 or some $i \in \{1, \ldots, q\}$. Otherwise, contour $\Gamma$ is called separating. (See Fig. 1).

In Fig. 1 it is assumed that the picture is being drawn on a small size square lattice as is shown in the magnified insert in Fig. 1(i). The white color indicates empty cells which form the set $O(\Gamma)$. The width of $O(\Gamma)$ respects the hard-core diameters $D(\cdot, \cdot)$. Different shades of grey colors represent unit cells containing different non-0 particle types. The set $B(\Gamma)$ is slightly wider than $O(\Gamma)$ and includes additional strips of width 2 formed by non-0 cells, as can be seen in the magnified insert in Fig. 1(i). In this insert a small piece of $B(\Gamma)$ is drawn in not black colors. Large connected areas colored some shade of grey represent either $E(\Gamma)$ or connected components of $I(\Gamma)$ (there are 3 of them). Figure 1(i) shows non-separating contour. The small rectangular area is magnified to show the details. Figure 1(ii) shows separating contour. Small grey colored areas surrounded by white represent unit cells not in any phase. They are drawn out of proportion (thicker than 5 unit cells) to make them visible.

Definition 2.7 Given a cell configuration $\phi \in \mathcal{C}$, let $\Gamma$ be a contour from $\phi$. The contour $\Gamma$ is called external (in contour collection $\Gamma = \Gamma(\phi)$) if there exist a path on $\mathbb{Z}^d$ beginning in $B(\Gamma)$ and going to infinity (i.e., reaching outside any cube with center at the origin) while passing through unit cells in phase $\iota E(\Gamma)$. An external contour will be denoted by $\Gamma^E$. The intersection $\cap E(\Gamma^E)$ taken over all external contours $\Gamma^E \in \Gamma$ is denoted by $E(\Gamma)$ and called the exterior of contour collection $\Gamma$. If $E(\Gamma)$ is a non-empty connected set then the value $\iota E(\Gamma^E)$ is the same for all external contours $\Gamma^E \in \Gamma$, and $\phi$ assigns this value to all $\Upsilon \subset E(\Gamma)$. We denote this value by $\iota E(\Gamma)$.

Similarly, for $\underline{\phi_A} \in \mathcal{C}(\Lambda || i)$ a contour $\Gamma$ from $\Gamma = \Gamma(\underline{\phi_A}) = \Gamma(\underline{\phi_A}, i)$ is called external if $\Gamma$ is external in the cell configuration $\underline{\phi_A}, i \in \mathcal{C}$ introduced in Eq. (2.9). In that case $\iota E(\Gamma) = i$. We set $E_A(\Gamma) = \Lambda \cap E(\Gamma(\underline{\phi_A}, i))$. Clearly, $\iota E(\Gamma) = i$. ▲
Denote
\[
C(\mathbb{Z}^d \| i) := \left\{ \phi \in C : E(\Gamma(\phi)) \neq \emptyset \text{ is connected,} \right. \\
\left. \quad \text{and } \phi(\Upsilon) = i \text{ for all } \Upsilon \subset E(\Gamma(\phi)) \right\}. 
\tag{2.14}
\]

Pictorially, a cell configuration $\phi \in C(\mathbb{Z}^d \| i)$ consists of an ‘external sea’ of cells $\Upsilon$ in phase $i$ and finitely or countably many ‘islands’ of a finite size represented by external contours $\Gamma^E$, all of them having $i^E(\Gamma^E) = i$.

Next,
\[
G(\mathbb{Z}^d \| i) := \left\{ \Gamma : \Gamma = \Gamma(\phi), \ \phi \in C(\mathbb{Z}^d \| i) \right\}. 
\tag{2.15}
\]

Equivalently, $G(\mathbb{Z}^d \| i)$ consists of finite or countable contour collections $\Gamma$ such that any unit cell $\Upsilon \in \mathbb{Z}^d$ is either in phase $i$ or belongs to the union $I(\Gamma^E) \cup B(\Gamma^E)$ where $\Gamma^E$ is an external contour in $\Gamma$ with $i^E(\Gamma^E) = i$.

Figure 2 above shows a contour collection including 13 contours, of which 7 are external. We should stress that all figures in the paper should be viewed as drawn on a small-size square lattice.

2.3 Compatible Contour Collections

In this section we complete the discretization of the model. We define compatible collections of contours and statistical weights of contours. Then we rewrite the partition functions originally defined in terms of particle configurations as partition functions over compatible collections of contours (or equivalently cell configurations). Finally, we define Gibbs/DLR measures on compatible collections of contours (or equivalently cell configurations) and explain how the original measures on particle configurations can be reconstructed from them.

**Definition 2.8** Let $\Gamma$ be a contour collection generated by $\phi \in C$ or $\phi^* \in C(\Lambda \| i)$. Let $\Gamma_1, \Gamma_2 \in \Gamma$ be two different contours. We say that $\Gamma_1$ and $\Gamma_2$ are separated (in $\Gamma$) if there is a contour $\Gamma_3 \in \Gamma$ different from $\Gamma_1$ and $\Gamma_2$ such that there is no path on $\mathbb{Z}^d$ joining $B(\Gamma_1)$ and $B(\Gamma_2)$ and avoiding $B(\Gamma_3)$. In this case we say that $\Gamma_3$ separates $\Gamma_1$ and $\Gamma_2$. 

\[\mathbb{S}\text{ Springer}\]
In the opposite case (when $\Gamma_1$ and $\Gamma_2$ are not separated) there are two disjoint possibilities:

(i) $\Gamma_1$ and $\Gamma_2$ are nested, i.e. either $B(\Gamma_1) \subset I(\Gamma_2)$ or $B(\Gamma_2) \subset I(\Gamma_1)$.
(ii) $\Gamma_1$ and $\Gamma_2$ are mutually external, i.e. $B(\Gamma_1) \cap I(\Gamma_2) = I(\Gamma_1) \cap B(\Gamma_2) = \emptyset$.

It is not hard to see that in a contour collection $\Gamma$ generated by $\phi$ or $\phi^*_\Lambda,i$ the following compatibility conditions are always fulfilled:

(i) For any two different contours $\Gamma_1, \Gamma_2 \in \Gamma$,
\[ B(\Gamma_1) \cap B(\Gamma_2) = \emptyset. \] (2.16)
(ii) For any two mutually external contours $\Gamma_1, \Gamma_2 \in \Gamma$,
\[ t^E(\Gamma_1) = t^E(\Gamma_2). \] (2.17)
(iii) For any two nested contours $\Gamma_1, \Gamma_2 \in \Gamma$ such that $B(\Gamma_1) \subset I(\Gamma_2)$,
\[ t^E(\Gamma_1) = t_s(\Gamma_2). \] (2.18)

Observe that conditions (2.16)–(2.18) are expressed in terms of bases $B(\Gamma)$ and types $\varphi(\Gamma)$ only. Therefore, one can think of compatible collections $\Gamma$ of pairs $\Gamma = (B, \varphi)$. In fact, for any such finite or countable collection of pairs $\Gamma$ there exists a unique admissible cell configuration
\[ \phi^*_\Gamma = \varphi^{-1}(\Gamma). \] (2.19)

Cf. Eq. (2.12). Thus, we can speak about contours and compatible collection of contours with or without specifying explicitly the generating cell configuration $\phi$. If this configuration is not specified then we always assume that it is equal to $\phi^*_\Gamma$.

Now, given a box $\Lambda$, we say that a (finite) compatible collection $\Gamma$ is associated with $C(\Lambda \parallel i)$ if $\phi^*_\Gamma$ attaches type $i$ to each cell $\gamma \not\subseteq \Lambda$, i.e. $t^E(\Gamma) = i$. The set of such collections is denoted by $G(\Lambda \parallel i)$. It is not hard to see that
\[ G(\Lambda \parallel i) = \varphi^*_{\Lambda} \circ \phi_{\Lambda} (A(\Lambda \parallel i)), \quad 1 \leq i \leq q. \] (2.20)

Observe that for $\Gamma \in G(\Lambda \parallel i)$ and $\Gamma \in \Gamma$ we have $O(\Gamma) \subseteq \Lambda$. Moreover, for $\Gamma \in G(\Lambda \parallel i)$, and $\Gamma \in \Gamma$, base $Br(\Gamma) \subseteq 2\Lambda$.

We use Eq. (2.20) for making a passage from measures $\mu_{\Lambda}(\cdot \parallel i)$ to measures on compatible collections $\Gamma \in G(\Lambda \parallel i)$. Referring to (1.14), consider the event
\[ A(\Lambda, = i) = \left\{ X : X^\Lambda \in A(\Lambda \parallel i), \phi_{\Lambda}(X^\Lambda) = i \right\}. \] (2.21)
(The condition $\phi_{\Lambda}(X^\Lambda) = i$ is equivalent to $\varphi^*_{\Lambda} \circ \phi_{\Lambda}(X^\Lambda) = \emptyset$.) Then
\[ \varrho = e^{-z(q-1)}u(\Lambda)(1 - e^{-z})^{-u(\Lambda)}. \] (2.22)

Note that the RHS in (2.22) does not depend on $i$. Throughout the paper (beginning with Eq. (2.31)), the value
\[ \varrho = e^{-z} \] (2.23)
plays the role of a small parameter. With this notation the probability $\varrho(\Lambda, = i) = (\varrho^{q-1}(1 - \varrho))^{u(\Lambda)}$.\[ Springer\]
Further, for $\Gamma \in \mathcal{G}(\Lambda \| i)$ we set:

$$\mathcal{A}(\Gamma, \Lambda \| i) = \left\{ \mathbf{X} \in \mathcal{A}(\Lambda \| i) : \gamma_\Lambda \circ \phi_\Lambda (\mathbf{X}^\Lambda) = \Gamma \right\}, \quad (2.24)$$

and for any $\Gamma$ with $O(\Gamma) \subseteq \Lambda$

$$\mathcal{A}(\Gamma, \Lambda) = \left\{ \mathbf{X} \in \mathcal{A}(\Lambda \| i^\varnothing(\Gamma)) : \gamma_\Lambda \circ \phi_\Lambda (\mathbf{X}^\Lambda) = \Gamma \right\}. \quad (2.25)$$

In other words, (2.25) means that the whole contour configuration $\gamma_\Lambda \circ \phi_\Lambda (\mathbf{X}^\Lambda)$ is reduced to a single contour $\Gamma$.

Probability distributions on admissible particle configurations and probability distributions on compatible contour collections are closely related. The measure $\mu_\Lambda(\cdot \| i)$ defined on events $B \in \mathcal{A}(\Lambda \| i)$ (see (1.15)) induces the measure $\mu_\Lambda(\Gamma \| i)$ on compatible contour collections $\Gamma \in \mathcal{G}(\Lambda \| i)$:

$$\mu_\Lambda(\Gamma \| i) = \frac{\mu_\Lambda(\mathcal{A}(\Gamma, \Lambda \| i) \| i)}{\mu_\Lambda(\mathcal{A}(\Lambda \| i) \| i)}. \quad (2.26)$$

Recall that according to (1.15)

$$\mu_\Lambda(\mathcal{A}(\Gamma, \Lambda \| i) \| i) = \frac{\mathbb{P}(\mathcal{A}(\Gamma, \Lambda \| i))}{\mathbb{P}(\mathcal{A}(\Lambda \| i))}. \quad (2.27)$$

Going back from contours to particles, given contour collection $\Gamma \in \mathcal{G}(\Lambda \| i)$, the conditional distribution of the generating particle configuration $\mathbf{X}$ such that $\gamma_\Lambda \circ \phi_\Lambda (\mathbf{X}^\Lambda) = \Gamma$ under measure $\mu_\Lambda(\cdot \| i)$ is determined by

$$\mu_\Lambda(\cdot \| i; \Gamma) = \frac{\mathbb{P}(\cdot \cap \mathcal{A}(\Gamma, \Lambda \| i))}{\mathbb{P}(\mathcal{A}(\Lambda, \Lambda \| i))}. \quad (2.28)$$

For an events $B$ localized inside $E_\Lambda(\Gamma)$ we have that

$$\mu_\Lambda(B \| i; \Gamma) = \frac{\mathbb{P}(B \cap \mathcal{A}(\Lambda, \Lambda = i))}{\mathbb{P}(\mathcal{A}(\Lambda, \Lambda = i))}, \quad (2.29)$$

which is a reformulation of the screening phenomenon: conditional on cell $\Upsilon \subset \Lambda$ being in phase $i \neq 0$, the positions of (type $i$) particles inside $\Upsilon$ are independent of particle positions inside any other cell.

A more convenient expression for $\mu_\Lambda(\Gamma \| i)$ can be derived in terms of statistical weights of contours. A statistical weight of a contour $\Gamma$ is defined as

$$w(\Gamma) := \frac{\mathbb{P}(\mathcal{A}(\Gamma, \Lambda))}{\mathbb{P}(\mathcal{A}(\Lambda, \Lambda = i^\varnothing(\Gamma)))}. \quad (2.30)$$

Referring to Eq. (2.22) and the screening phenomenon, we conclude that the ratio in RHS of (2.30) does not depend on $\Lambda \supseteq O(\Gamma)$ (or equivalently $\varnothing \Lambda \supseteq B(\Gamma)$). That is why $\Lambda$ is absent in the LHS of (2.30).

For $z > 0$ large enough, $w(\Gamma)$ is exponentially small in volume of $B(\Gamma)$ as it satisfies straightforward bounds

$$\left( \frac{v}{1 - v} \right)^{\nu(B(\Gamma))} < w(\Gamma) < \left( \frac{v}{1 - v} \right)^{\nu(O(\Gamma))}, \quad (2.31)$$

cf. (2.10).
A statistical weight of a compatible contour collection $\Gamma \in \mathcal{G}(\Lambda \parallel i)$ is defined as

$$w(\Gamma) = \frac{\mathbb{P}(A(\Gamma, \Lambda \parallel i))}{\mathbb{P}(A(\Lambda, = i))}.$$  \tag{2.32}

The ratio in the RHS of (2.32) also does not depend on $\Lambda$ if $\Lambda \supseteq \text{O}(\Gamma)$ for all $\Gamma \in \mathcal{G}$. The normalizing factor $\mathbb{P}(A(\Lambda, = i))$ used in (2.32), (2.30) represents the statistical weight of a ‘ground state’ cell configuration $\phi_{\Lambda} \equiv i$ such that the statistical weight (2.32) of empty contour collection is 1.

The following factorization property is crucial for the forthcoming analysis

$$w(\Gamma) = \prod_{\Gamma \in \mathcal{G}(\Lambda \parallel i)} w(\Gamma).$$  \tag{2.33}

This factorization holds because of the triangle property (1.3) and the screening phenomenon. In a sense, the ‘contour language’ owes its convenience to (2.33).

The set $\mathcal{G}(\Lambda \parallel i)$ of contour collections with the statistical weights $w(\Gamma)$ yields the contour ensemble in $\Lambda$ with the external type $i$. For this ensemble, following [23] (see also [1]), we define the ‘rarefied’ partition function in $\Lambda$ with the external type $i \in \{1, \ldots, q\}$:

$$\Xi(\Lambda \parallel i) := \frac{\mathbb{P}(A(\Lambda \parallel i))}{\mathbb{P}(A(\Lambda, = i))} = \sum_{\Gamma \in \mathcal{G}(\Lambda \parallel i)} \prod_{\Gamma \in \mathcal{G}(\Lambda \parallel i)} w(\Gamma).$$  \tag{2.34}

An empty collection $\Gamma$ enters the RHS of (2.34) with the statistical weight 1. In case $\Lambda = \emptyset$, we set $\Xi(\Lambda \parallel i) = 1$. Representation (2.34) (and similar representation (2.40) below) makes possible the use of established methods of cluster or polymer expansions.

A specific feature of the model under consideration (with $q \leq 4$) is that $\Xi(\Lambda \parallel i)$ gives the same value $\forall i \in S$ (for all values of $z$). This fact is established in Lemma 3.3 and repeatedly used in Sects. 4 and 5. It would be possible to expect that $\Xi(\Lambda \parallel i)$ is maximized when $i \in S$. This property would have simplified our proofs, but we couldn’t verify it. (It turns out to be false for $z$ small and some $D$.)

It is also convenient to use the external contour ensemble in box $\Lambda$ with the external type $i$. To this end, for any contour $\Gamma = (B, \varphi_B)$ with finite $B$ we introduce the external statistical weight, otherwise known as the ‘crystalline’ partition function

$$w(\Gamma) := \frac{\mathbb{P}\left\{A(\Gamma, B(\Gamma) \cup I(\Gamma) \parallel i(\Gamma))\right\}}{\mathbb{P}\left\{A(\Gamma, B(\Gamma) \cup I(\Gamma), = i(\Gamma))\right\}}.$$  \tag{2.35}

A useful consequence of (2.34) is that

$$w(\Gamma) = w(\Gamma) \prod_s \Xi(3I_s(\Gamma) \parallel i_s(\Gamma))$$  \tag{2.36}

where $3I_s(\Gamma)$ is defined in accordance with (2.4).

**Definition 2.9** We say that a contour collection from $\mathcal{G}(\Lambda \parallel i)$ is external if it consists of mutually external contours. In this case we use the notation $\Gamma^E$. An equivalent definition is that the following properties hold:

$$(B(\Gamma_1^E) \cup I(\Gamma_1^E)) \cap (B(\Gamma_2^E) \cup I(\Gamma_2^E)) = \emptyset,$$  \tag{2.37}

and

$$i^E(\Gamma_1^E) = i^E(\Gamma_2^E) = i.$$  \tag{2.38}
The set of external contour collections from $\mathcal{G}(\Lambda \parallel i)$ is denoted by $\mathcal{G}(\Lambda \parallel i; E)$. ▲

Given an external contour collection $\Gamma^E \in \mathcal{G}(\Lambda \parallel i; E)$, we set:

$$w(\Gamma^E) := \prod_{\Gamma^E \in \Gamma^E} w(\Gamma^E).$$  \tag{2.39}

Together with statistical weight (2.39), set $\mathcal{G}(\Lambda \parallel i; E)$ forms an external contour ensemble in $\Lambda$ with the external type $i$.

With this definitions at hand, we obtain the following representation for the partition function $\Xi(\Lambda \parallel i)$ in (2.34):

$$\Xi(\Lambda \parallel i) = \sum_{\Gamma^E \in \mathcal{G}(\Lambda \parallel i; E)} w(\Gamma^E) \prod_{\Gamma^E \in \Gamma^E} w(\Gamma^E).$$  \tag{2.40}

Substituting of (2.36) into the RHS of Eq. (2.40) reveals the following recursive property of rarefied partition functions:

$$\Xi(\Lambda \parallel i) = \sum_{\Gamma^E \in \mathcal{G}(\Lambda \parallel i; E)} \prod_{\Gamma^E \in \Gamma^E} w(\Gamma^E) \prod_s \Xi(3I_s(\Gamma^E)) \overline{l}_s(\Gamma^E)).$$  \tag{2.41}

As in (2.36), the reduced box $3I_s(\Gamma^E)$ is obtained from the interior component $I_s(\Gamma^E)$ in accordance with (2.4). The probability $\mu(\Gamma^E \parallel i)$ of having $\Gamma^E$ as a collection of external contours has the form

$$\mu(\Gamma^E \parallel i) = \frac{w(\Gamma^E)}{\Xi(\Lambda \parallel i)}.  \tag{2.42}$$

Denote by

$$\mathcal{G}(\Gamma^E) = \{\Gamma^\prime \in \mathcal{G} \parallel i; E) \ : \ \Gamma^E \in \Gamma^\prime \} \quad \forall \Gamma^E \in \Gamma^E \}$$  \tag{2.43}

the subset of $\mathcal{G}(\Lambda \parallel i; E)$ consisting of external contour collections having $\Gamma^E$ as a subcollection. Then it follows from (2.41) and (2.42) that

$$\mu_\Lambda(\mathcal{G}(\Gamma^E) \parallel i) \leq \prod_{\Gamma^E \in \Gamma^E} w(\Gamma^E) \prod_s \Xi(3I_s(\Gamma^E)) \overline{l}_s(\Gamma^E)) \overline{l}(\Gamma^E).$$  \tag{2.44}

This inequality is known as the Peierls bound. We will reformulate it in Sect. 4 (cf. (4.23)) and use repeatedly in Sect. 5; see, e.g., Eq. (5.24).

Concluding this section, we make the following remark. The machinery developed in Sects. 3 and 4 will allow us to construct, for each $i \in \mathbb{S}$, limiting DLR measures concentrated on the space of compatible contour collections $\mathcal{G}(\mathbb{Z}^d \parallel i)$ (see Eq. (2.15)), or equivalently, on the space of cell configurations $\mathcal{C}(\mathbb{Z}^d \parallel i)$ (see Eq. (2.14)). We can denote these measures by $\mu(d\Gamma \parallel i)$ and $\mu(d\phi \parallel i)$, respectively. In turn, measures $\mu(d\Gamma \parallel i)$ and $\mu(d\phi \parallel i)$ will generate according to (2.28) and (2.29) the measure $\mu(dX \parallel i)$ sitting on the set

$$\mathcal{A}(\mathbb{R}^d \parallel i) = \{X \in \mathcal{A} : \phi \circ \phi(X) \in \mathcal{G}(\mathbb{Z}^d \parallel i)\}.  \tag{2.45}$$

(Recall, map $\phi$ has been introduced in (2.7).) Measure $\mu(dX \parallel i)$ is given by the limit (1.20) and satisfies the DLR equation (1.16). The key fact here is that, with $\mu(d\phi \parallel i)$-probability 1, for any bounded box $\Lambda$ there exists a random bounded box $\Lambda^*(\phi, \Lambda) \supset \Lambda$ such that in
cell configuration $\phi$ all unit cells of annulus $^2\Lambda^*(\phi, \Lambda) \setminus ^2\Lambda^*(\phi, \Lambda)$ are in phase $i$. Therefore, given function $f : \mathbf{X} \to \mathbb{R}$ depending on $\mathbf{X}^\Lambda$,}

$$\int_{\mathcal{A}(\mathbb{R}^d \| i)} f(\mathbf{X}) \mu(\mathbf{dX} \| i) = \int_{\mathcal{C}(\mathbb{Z}^d \| i)} \mu(\mathbf{d\phi} \| i) \int_{\mathcal{A}(\Lambda^*(\phi, \Lambda) \| i)} f(\mathbf{X}) \mu_{\Lambda^*(\phi, \Lambda)}(\mathbf{dX} \| i; \Gamma(\phi)),$$

(2.46)

which implies DLR property for $\mu(\mathbf{dX} \| i)$.

### 3 The Ensemble of Small Contours

In Sect. 3 we use concepts and terminology defined in Appendix at the end of the paper. More precisely, the appendix collects results about abstract polymer models. (In fact, it is an abridged version of Sect. 3 from [16].) The point is that if a specific ensemble of contours satisfies the generic conditions of Theorem 6.1 (mainly (6.3)) then it provides a ‘full control’ over the corresponding finite-volume Gibbs distributions and their infinite-volume limits. We note that in this paper the polymer expansion analysis follows [9], with (technical) modifications proposed in [16] (cf. also Ref. [12]). There exist several improvements of this approach; see [5] and the references therein; these improvements may help in a more accurate assessment of the threshold value $z_0$ in Theorems 1.1 and 1.2.

#### 3.1 The Definition of a Small Contour

In this subsection we define an ensemble of ‘small’ contours. Then in Sect. 3.2 we compare free energies of these ensembles for different external particle types. It appears that the free energy is maximal for the stable external types. An unstable particle type has a deficiency in the free energy, which gives a quantitative characterization of the stable type dominance. See Lemma 3.4 for more details.

**Definition 3.1** A contour $\Gamma$ is called **small** if $\text{diam} I(\Gamma)$, the diameter of its interior $I(\Gamma)$, is less than $b^{2d}$ (see (1.17) for the definition of $b$). We employ the notation $\Gamma^S$ for small contours; the index or argument $S$ is used for the same purpose in a number of places below. In particular, $\Gamma^S$ stands for a collection containing only (mutually external) small contours. Contours that are not small are referred to as **large**.

The threshold $b^{2d}$ is selected (rather arbitrarily) to be larger than $v(\mathbb{B}(b))$; cf. Eq. (3.11). This is important for the analysis in Sects. 4 and 5. Viz., see the proof of Lemma 4.1 (specifically, Eq. (4.14). Any independent on $z$ number will work for the analysis in the current section.

Let $\mathcal{G}(\Lambda \| i; S) \subset \mathcal{G}(\Lambda \| i; E)$ be the set of contour collections $\Gamma^S$ in $\Lambda$ with the external type $i$. (Recall, the set $\mathcal{G}(\Lambda \| i)$ was defined in Eq. (2.20) and its subset $\mathcal{G}(\Lambda \| i; E)$ was introduced in Definition 2.9.) Then, for any small contour $\Gamma^S \in \Gamma^S \in \mathcal{G}(\Lambda \| i; S)$, the external type $\iota^E(\Gamma^S) = i$. Accordingly, with $\phi$ as in (2.7), set:

$$\mathcal{A}(\Lambda \| i; S) = \{ \mathbf{X} \in \mathcal{A}(\Lambda \| i) : \gamma \circ \phi(\mathbf{X}) \in \mathcal{G}(\Lambda \| i; S) \}.$$  

(3.1)
The compatibility condition for small contours is given by (2.37)–(2.38) and the corresponding partition function is defined by

\[ \Xi(\Lambda \parallel i; S) := \sum_{\Gamma^S \in \mathcal{G}(\Lambda \parallel i; S)} \prod_{\Gamma^S \in \Gamma^S} w(\Gamma^S). \tag{3.2} \]

Here \( w(\Gamma) \) is the external statistical weight determined by (2.35) or equivalently (2.36). We call \( \Xi(\Lambda \parallel i; S) \) the partition function in the small contour ensemble in \( \Lambda \).

**Lemma 3.1** Suppose \( z \) is large enough. Then, for any finite box \( \Lambda \) and \( v \) as in (2.23),

\[ \Xi(\Lambda \parallel i; S) \leq (1 + v^A \frac{1}{1 - v})^{\nu(\Lambda)}. \tag{3.3} \]

Here and below,

\[ A = \nu(\mathbb{B}(a - 2)), \tag{3.4} \]

where \( a \) is defined in (2.9).

**Proof of Lemma 3.1.** To estimate the partition function \( \Xi(\Lambda \parallel i) \) from above we use the representation (2.34) and the upper estimate in (2.31):

\[ 0 < w(\Gamma) < \left( \frac{v}{1 - v} \right)^{\nu(O(\Gamma))}. \tag{3.5} \]

Relaxing a compatibility condition can only increase the partition function. To that end, we introduce a partition function \( \tilde{\Xi}(\Lambda \parallel i) \) with relaxed compatibility condition for contributing contours. Namely, in \( \tilde{\Xi}(\Lambda \parallel i) \) we allow the separating contours to overlap with each other and with non-separating contours. Also we allow inside \( \Lambda \) separating contours \( \Gamma \) with \( i^E(\Gamma) \neq i \), i.e. we remove both restrictions (2.37) and (2.38) for separating contours, making those contours completely independent from each other and from non-separating contours. The relaxed partition function \( \tilde{\Xi}(\Lambda \parallel i) \) can be calculated exactly:

\[ \tilde{\Xi}(\Lambda \parallel i) = (1 - v)^{-\nu(\Lambda)} \prod_{\Gamma: \text{B}(\Gamma) \subseteq \Lambda} (1 + w(\Gamma)), \tag{3.6} \]

where the product is taken over all separating contours \( \Gamma \) inside \( \Lambda \).

The trademark of the contour techniques is a summation over weighted connected lattice subsets. Typically, the weights are exponentially small compared to the number of lattice sites in the subset. On the other hand, the number of different connected lattice subsets of cardinality \( N \) containing the origin is at most \( c^N \), where \( c \) depends on dimension \( d \) and the radius of connectivity. Indeed, each connected subset has a spanning tree rooted at the origin. The spanning tree can be traversed by a path starting at the root and passing through every link of the tree only twice. Therefore, the length of the path is at most \( 2N \). Obviously, the number of \( r \)-connected lattice paths of length \( 2N \) is smaller than \( c(d, r)^{2N} \), where \( c(d, r) \) is a number of \( r \)-neighbors for a given lattice site.

We apply these considerations to the sum \( \sum_{\Gamma: O(\Gamma) \supseteq \mathcal{T}(0)} w(\Gamma) \), where all \( \Gamma \) are separating. Each contour \( \Gamma \) contributing to this sum has \( \nu(O(\Gamma)) \geq A + 1 \). In addition, \( O(\Gamma) \) is 5-connected. Finally, \( \text{B}(\Gamma) \subseteq 2O(\Gamma) \) and therefore there is at most \( q^{sd} c^{A+1/2} \) possibilities to reconstruct \( \text{B}(\Gamma) \) from \( O(\Gamma) \) and specify a configuration \( \varphi_{\text{B}(\Gamma)} \). Thus,

\[ \sum_{\Gamma: O(\Gamma) \supseteq \mathcal{T}(0)} w(\Gamma) \leq \sum_{n=A+1}^{\infty} \left( q^{sd} c \frac{1}{1 - v} \right)^n \leq v^{A+1/2}, \tag{3.7} \]
where the last inequality is true for $z$ large enough. Moreover,

$$\prod_{\Gamma: B(\Gamma) \subseteq \Lambda} (1 + w(\Gamma)) < \left(1 + 2 \sum_{\Gamma: O(\Gamma) \supseteq \Omega(0)} w(\Gamma)\right)^{v(\Lambda)} < \left(1 + v^A\right)^{v(\Lambda)}, \quad (3.8)$$

where both inequalities are true for $z$ large enough. (Variations of (3.7) and (3.8) could be seen in a number of forthcoming arguments.)

Lemma 3.1 is instrumental in estimating statistical weights $w(\Gamma^S)$ as it provides the control over $\prod_s \Xi \left(3 I_s(\Gamma^S) \parallel I_s(\Gamma)\right)$ in (2.36).

**Lemma 3.2** For any small contour $\Gamma^S$, the external statistical weight $w(\Gamma^S)$ (see (2.35)) obeys

$$w(\Gamma^S) > \left(\frac{v}{1-v}\right)^{v(B(\Gamma^S))} (1-v)^{-v(I(\Gamma^S))} \quad (3.9)$$

and

$$w(\Gamma^S) < \left(\frac{v}{1-v}\right)^{v(O(\Gamma^S))} \left(\frac{1 + v^A}{1 - v}\right)^{v(I(\Gamma^S))} \quad (3.10)$$

where $v = e^{-z}$ and $A$ is defined in Eq. (3.4).

**Proof of Lemma 3.2.** The lemma is a direct consequence of (2.31) and (3.3).

The first factor in both (3.9) and (3.10) is the main part of the estimate as the second factor is close to 1. Indeed, for $z$ large enough,

$$1 < \left(\frac{1 + v^A}{1 - v}\right)^{v(I(\Gamma^S))} < \left(\frac{1 + v^A}{1 - v}\right)^{b 2^d} < 1 + v^{0.9} \quad (3.11)$$

as $v(I(\Gamma^S)) < b 2^d$. (Among all positive numbers less than 1 the value 0.9 has been selected only for definiteness.)

### 3.2 Free Energies of Small Contour Ensembles

In this subsection we apply polymer expansion technique (cf. Sect. 1) to calculate free energies of small contour ensembles. This analysis culminates in Lemma 3.4 and its Corollary 3.5 which demonstrate that the stable, in a sense of (1.19), particle types have maximal free energies of small contour ensembles.

Representation (3.2) and upper bound (3.10) imply that $\Xi(\Lambda \parallel i; S)$ can be calculated using Theorem 6.1. Indeed, the symmetric compatibility relation $\Gamma^S_1 \sim \Gamma^S_2$ for small contours $\Gamma^S_1$ and $\Gamma^S_2$ is given by (2.37) and (2.38). The function $a(\cdot)$ in Theorem 6.1 can be selected as

$$a(\Gamma^S) := v(B(\Gamma^S)) \log (1 + v^{0.9}). \quad (3.12)$$

The bound (6.3) follows from the inequality

$$\sum_{\Gamma^S: B(\Gamma^S) \nsubseteq B(b 2^d)} w(\Gamma^S) (1 + v^{0.8}) v(B(\Gamma^S)) \leq \log (1 + v^{0.9}), \quad (3.13)$$
which can be verified for $z$ large enough (with respect to $b$) using the same enumerating arguments as in the proof of Lemma 3.1, together with estimates (3.10) and (3.11).

For a polymer $\Pi^S = \{ \Gamma^S \}$ formed by small contours $\Gamma^S$ (see Appendix) we define the base $B(\Pi^S) = \bigcup_{\Gamma^S \in \Pi^S} B(\Gamma^S)$ and the statistical weight $w(\Pi^S)$ as in (6.5). Then, due to shift-invariance of the statistical weights $w(\Gamma)$, the representation (6.4) can be rewritten as

$$
\log \mathbb{E}(\Lambda \parallel i; S) = v(\Lambda) f(i; S) + r(\Lambda \parallel i; S). \tag{3.14}
$$

Here the principal term $f(i; S)$ represents the free energy of the (infinite-volume) small contour ensemble with the external type $i$:

$$
f(i; S) = \sum_{\Pi^S: B(\Pi^S) \supseteq \Upsilon(0)} \frac{w(\Pi^S)}{v(B(\Pi^S))} \tag{3.15}
$$

and $r(\Lambda \parallel i; S)$ is the remainder:

$$
r(\Lambda \parallel i; S) = -\sum_{\Pi^S: B(\Pi^S) \cap \Lambda \neq \emptyset, B(\Pi^S) \cap \Lambda^0 \neq \emptyset} w(\Pi^S) \frac{v(B(\Pi^S) \cap \Lambda)}{v(B(\Pi^S))}. \tag{3.16}
$$

To assess $r(\Lambda \parallel i; S)$, we apply the bound from Eq. (6.6) and plug into it the definition (3.12) and bound (3.10). Then one can see that for $z$ large enough

$$
|r(\Lambda \parallel i; S)| < v(\partial \Lambda) v^{0.9} \tag{3.17}
$$

where $\partial \Lambda = \Lambda \setminus \Lambda^0$.

With representation (3.15) at hand, we are ready to compare the free energies $f(i; S)$ for different types $i$. Recall, that collection $S$ is defined in (1.19) using vectors $n(i)$ specified in (1.18). For $q = 4$, a given type $i$ is also characterized by an ordered ‘incident’ collection of 3 hard-core exclusion diameters. We assume that it is an increasing order.

**Lemma 3.3** Suppose that for two types, 1 and 2, we have coinciding collections of ordered incident hard-core exclusion diameters:

$$
D(1, j_1) = D(2, k_1) \leq D(1, j_2) = D(2, k_2) \leq D(1, j_3) = D(2, k_3), \tag{3.18}
$$

where

$$
\{j_1, j_2, j_3\} = \{2, 3, 4\} \text{ and } \{k_1, k_2, k_3\} = \{1, 3, 4\}. \tag{3.19}
$$

Then, for any box $\Lambda$, the permutation $\{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ with

$$
1 \mapsto 2, \; j_1 \mapsto k_1, \; j_2 \mapsto k_2, \; j_3 \mapsto k_3 \tag{3.20}
$$

defines a 1-1 map between events $\mathcal{A}(\Lambda \parallel 1; S)$ and $\mathcal{A}(\Lambda \parallel 2; S)$ as well as a 1-1 map between events $\mathcal{A}(\Lambda \parallel 1)$ and $\mathcal{A}(\Lambda \parallel 2)$ (modulo subsets of $\mathbb{P}$-probability zero). These maps preserve the probability distribution $\mathbb{P}$, which implies that

$$
\mathbb{E}(\Lambda \parallel 1; S) = \mathbb{E}(\Lambda \parallel 2; S) \tag{3.21}
$$

and

$$
\mathbb{E}(\Lambda \parallel 1) = \mathbb{E}(\Lambda \parallel 2). \tag{3.22}
$$
Proof of Lemma 3.3. It is enough to consider only two possibilities for the map (3.20):

\[ 1 \leftrightarrow 2, \ 3 \leftrightarrow 4 \quad \text{or} \quad 1 \leftrightarrow 2, \ 3 \leftrightarrow 3, \ 4 \leftrightarrow 4. \tag{3.23} \]

After that observation the claim is verified directly, by inspecting all possible relations between values \( D(i, j) \) listed in (3.18).

Remark 3.1 A direct analog of Lemma 3.3 for \( q = 5 \) fails: the fact that types 1 and 2 have \( n(1) = n(2) \) does not imply that there exists a permutation with the above properties.

For \( 1 \leq i < j \leq 4 \) introduce the quantities

\[ A(i, j) = \nu(B(D(i, j) - 2)), \quad B(i, j) = \nu(B(D(i, j) + 2)). \tag{3.24} \]

Lemma 3.4 Suppose that for two types, 1 and 2, we have different ordered collections of incident hard-core exclusion diameters:

\[ D(1, j_1) \leq D(1, j_2) \leq D(1, j_3) \text{ and } D(2, k_1) \leq D(2, k_2) \leq D(2, k_3). \]

If \( D(1, j_1) < D(2, k_1) \) then

\[ f(1; \mathcal{S}) - f(2; \mathcal{S}) > 0.5v^{B(1, j_1)}. \tag{3.25} \]

If \( D(1, j_1) = D(2, k_1) \) but \( D(1, j_2) < D(2, k_2) \) then

\[ f(1; \mathcal{S}) - f(2; \mathcal{S}) > 0.5v^{B(1, j_2)}. \tag{3.26} \]

Finally, if \( D(1, j_1) = D(2, k_1), \ D(1, j_2) = D(2, k_2) \) but \( D(1, j_3) < D(2, k_3) \) then

\[ f(1; \mathcal{S}) - f(2; \mathcal{S}) > 0.5v^{B(1, j_3)}. \tag{3.27} \]

Proof of Lemma 3.4. The statement of the lemma is a straightforward (although tedious) consequence of representation (3.15) and bounds (3.9)–(3.10).

In fact, suppose that \( D(1, j_1) < D(2, k_1) \). Then the summands in the RHS of (3.15) for \( f(1; \mathcal{S}) \) and \( f(2; \mathcal{S}) \) generated exclusively by non-separating small contours (i.e. the polymers consisting only of non-separating small contours) are identical. Hence, these summands denoted by \( f_{ns}(1; \mathcal{S}) \) and \( f_{ns}(2; \mathcal{S}) \) respectively, cancel each other when one takes the difference \( f(1; \mathcal{S}) - f(2; \mathcal{S}) \).

Further, according to (6.6), for \( z \) large enough, the contribution \( f_{s}(2; \mathcal{S}) \) to \( f(2; \mathcal{S}) \) provided by polymers containing at least one separating small contour does not exceed

\[
\sum_{\Gamma_{s}^{\mathcal{S}}: B(\Gamma_{s}^{\mathcal{S}}) \supseteq \Gamma_{0}, \ \Gamma_{s}^{\mathcal{S}} \text{ is separating}} w(\Gamma_{s}^{\mathcal{S}}) e^{a(\Gamma_{s}^{\mathcal{S}})} \\
\leq \sum_{\Gamma_{s}^{\mathcal{S}}: B(\Gamma_{s}^{\mathcal{S}}) \supseteq \Gamma_{0}, \ \Gamma_{s}^{\mathcal{S}} \text{ is separating}} \left( \frac{v}{1 - v} \right)^{\nu(O(\Gamma_{s}^{\mathcal{S}}))} \left( \frac{1 + v^{A} + v^{0.9}}{1 - v} \right)^{\nu(I(\Gamma_{s}^{\mathcal{S}}))} (1 + v^{0.9})^{\nu(B(\Gamma_{s}^{\mathcal{S}}))}, \tag{3.28} \]

where the inequality is due to (3.12) and (3.10), and \( A \) is as in (3.4). According to (3.11), for \( z \) large enough

\[ \left( \frac{1 + v^{A}}{1 - v} \right)^{\nu(I(\Gamma_{s}^{\mathcal{S}}))} < (1 + v^{0.9}) < (1 + v^{0.8})^{0.5}. \tag{3.29} \]
Next, \(A(2, k_1)\) yields a lower bound for the amount of empty cells in a separating small contour \(\Gamma^S\) contributing to \(f(i_2; S)\) and, similarly to (3.7), for \(\bar{z}\) large enough

\[
\sum_{\Gamma^S; B(\Gamma^S) \supseteq \Upsilon(0), \Gamma^S \text{ is separating}} \left( \frac{v}{1 - v} \right)^{\nu(B(\Gamma^S))} (1 + v^{0.9}) v(B(\Gamma^S)) \leq \sum_{n=A(2,k_1)+1}^{\infty} (q_1^{sd} c (1 + v^{0.9})^{sd} \frac{v}{1 - v})^n < v^{A(2,k_1)} (1 + v^{0.8})^{0.5}. \tag{3.30}
\]

Combining (3.29) and (3.30), we conclude

\[
f_S(2; S) < \sum_{\Gamma^S; B(\Gamma^S) \supseteq \Upsilon(0), \Gamma^S \text{ is separating}} w(\Gamma^S) e^{a(\Gamma^S)} < v^{A(2,k_1)} (1 + v^{0.8}). \tag{3.31}
\]

On the other hand, \(f(1; S)\) incorporates the contribution from small contours which are contained in a ball of radius \(D(1, j_1) + 2\) with a single cell in the middle occupied by type \(j_1\). We call them dominating small contours. To estimate \(f(1; S)\) from below we can consider an ensemble containing only dominating small contours and all non-separating small contours. Denote the corresponding free energy by \(\tilde{f}(1; S)\). Clearly, \(\tilde{f}(1; S) < f(1; S)\). The contribution \(f_{ns}(1; S)\) provided by polymers consisting only from non-separating small contours is identical for both \(f(1; S)\) and \(\tilde{f}(1; S)\). The contribution \(\tilde{f}_{sd}(1; S)\) to \(\tilde{f}(1; S)\) from the polymers consisting of a single dominating small contour \(\Gamma^S\) satisfies the lower bound

\[
\tilde{f}_{sd}(1; S) = w(\Gamma^S) > v^{B(1,j_1)} \tag{3.32}
\]

because of Eq. (3.9).

The remaining part \(\tilde{f}_{nsd}(1; S)\) of \(\tilde{f}(1; S)\) collects the contribution of polymers containing at least two small contours with at least one of them being a dominating small contour. By virtue of Eq. (6.6), the sum of absolute values of statistical weights of all polymers containing a dominating small contour \(\Gamma^S\) and at least one other small contour does not exceed

\[
\tilde{f}_{nsd}(1; S) < w(\Gamma^S) (e^{a(\Gamma^S)} - 1) < 2w(\Gamma^S) a(\Gamma^S) < 2w(\Gamma^S) B(1, j_1) v^{0.9} < 0.1 w(\Gamma^S). \tag{3.33}
\]

where \(a(\Gamma^S)\) is defined in (3.12) ans \(\bar{z}\) is so large that \(2B(1, j_1)v^{0.9} < 0.1\).

Thus,

\[
f(1; S) > \tilde{f}(1; S) > f_{ns}(1; S) + \tilde{f}_{sd}(1; S) - \tilde{f}_{nsd}(1; S) \tag{3.34}
\]

Comparing (3.34) to (3.31) yields (3.25), provided that \(\bar{z}\) is large enough and the difference \(D(1, j_1) - D(2, k_1) > 4\) (which is guaranteed by (2.1)).

The proof of (3.26) and (3.27) is similar: we first identify common parts in partition functions \(f(1; S)\) and \(f(2; S)\) and then compare remaining parts. For instance, in the case of (3.26) the contribution to \(f(1; S)\) from all polymers containing only non-separating small contours and small contours separating 1 and \(j_1\) is the same as the contribution to \(f(2; S)\) from all polymers containing only non-separating small contours and small contours separating 2 and \(k_1\). On the other hand, in this case \(f(1; S)\) incorporates the contribution from dominating small contours which are contained in a ball of radius \(D(1, j_2) + 2\) with a single cell in the middle occupied by type \(j_2\). As in (3.34), this contribution is bounded from below by
0.9 \nu^{B(1, f_2)}. As to the remaining part in \( f(2; S) \), it incorporates polymers containing at least one small contour separating 2 and \( k_2 \) or at least one small contour separating 2 and \( k_3 \). This contribution is upper-bounded by an expression similar to (3.31), with \( A(2, k_1) \) replaced by \( A(2, k_2) \).

\[ \] 

**Corollary 3.5** The types \( i \in \{1, 2, 3, 4 \} \) generating the maximal value of \( f(\cdot; S) \) are precisely those forming set \( S \) defined in (1.19).

**Proof of Corollary 3.5.** Comparing assumption on collection \( \mathbb{D} \) used in Lemma 3.4 with the definition of \( S \) implies the corollary.

### 4 Unrestricted Partition Functions \( \Xi(A \parallel i) \)

The purpose of this section is to analyze complete ensembles of contours. In particular, we derive a polymer expansion for \( \log \Xi(A \parallel i) \), \( i \in S \), similar to that in Eqs. (3.14)–(3.17) for \( \log \Xi(A \parallel i; S) \). Recall, the rarefied partition function \( \Xi(A \parallel i) \) was defined in Eqs. (2.34) and (2.40) and its restricted version \( \Xi(A \parallel i; S) \) in (3.2). We have to distinguish between stable partition functions \( \Xi(A \parallel i) \), where \( i \in S \), and unstable partition functions \( \Xi(A \parallel i) \), with \( i \not\in S \). For unstable partition function we need upper bounds. In these bounds we employ new objects which we call ‘boundary layers’: these are groups of large contours (see Definition 3.1) organized in a certain manner; cf. Definition 4.1. For stable partition functions we construct the aforementioned expansion in terms of contours with renormalized statistical weights (4.20). Upper bounds for unstable partition functions are needed to control these renormalized statistical weights. The desired representation of \( \Xi(A \parallel i), i \in S \), suitable for applying polymer expansion Theorem 6.1 is achieved in Eq. (4.22).

#### 4.1 Upper Bounds for \( \Xi(A \parallel i), i \not\in S \). Boundary Layers

We start with the definition of a boundary layer. Let \( G(A \parallel i) \) be as in Eq. (2.20). Given a compatible contour collection \( \Gamma \in G(A \parallel i) \), we use a procedure of ‘erasing’ the small contours \( \Gamma^S \in \Gamma \). Namely, erasing an small contour \( \Gamma^S \) means that \( B(\Gamma^S) \cup I(\Gamma^S) \) is filled with the type \( i^E(\Gamma^S) \) in the corresponding cell configuration. Equivalently erasing means removing \( \Gamma^S \) from the collection \( \Gamma \).

**Definition 4.1** Consider a compatible contour collection \( \Gamma \in G(A \parallel i), i \not\in S \) and erase all small contours \( \Gamma^S \in \Gamma \). The remaining large contours from \( \Gamma \) form a collection \( \Gamma^L \) which determines a cell configuration, \( \phi^*_\Gamma \in C(\mathbb{Z}^d \parallel i) \), cf. (2.19). By construction, \( \phi^*_\Gamma \mid _{\Lambda^B} = i \).

We take the union \( U = U \left( \phi^*_\Gamma \right) \) of all cells \( \Upsilon \subset \Lambda \) that are not in a phase \( l \in S \) in \( \phi^*_\Gamma \).

A connected component of set \( U \) which is adjacent to \( \Lambda^B \) is called the base of the boundary layer in \( \Gamma \). (It is not hard to see that among connected components of \( U \) there is only one adjacent to \( \Lambda^B \).) This connected component, considered together with the types of unit cells belonging to it, is called a boundary layer in \( \Gamma \). A boundary layer is denoted by \( \Gamma^U(\Gamma) \) and its base by \( B(\Gamma^U(\Gamma)) \); index \( U \) means unstable. We use a shorter notation \( \Gamma^U \) when it does not create a confusion.

The boundary layer can also be considered for a particle configuration \( \Xi \in \mathcal{A}(A \parallel i) \) (by referring to maps \( \gamma_{\Lambda} \) and \( \phi_{\Lambda} \) from (2.5) and (2.11)); this point of view will be particularly convenient in Sect. 5 (see Definition 5.2).
An alternative understanding of a boundary layer $\Gamma^U$ is that it is a compatible collection of large contours $\Gamma^L_m (\Gamma^U)$ such that:

(i) $\Gamma^U$ contains 0 or more large contours $\Gamma^L_{k,E} (\Gamma^U)$ such that at least one of internal types $\iota_S (\Gamma^L_{k,E} (\Gamma^U)) \in S$, i.e. is stable. We denote the corresponding interiors $I_s (\Gamma^L_{k,E} (\Gamma^U))$ of $\Gamma^L_m (\Gamma^U)$ by $I_{k,s} (\Gamma^U)$ and the types $\iota_S (\Gamma^L_{k,E} (\Gamma^U))$ by $\iota_{k,s} (\Gamma^U)$.

(ii) $\Gamma^U$ contains 0 or more large contours $\Gamma^L_{n,I} (\Gamma^U)$ which are different from large contours $\Gamma^L_{k,E} (\Gamma^U)$ described in item (i) above.

(iii) Each connected component $V_j (\Gamma^U)$ of the set $B (\Gamma^U) \setminus \bigcup_m B_L (\Gamma^U)$ has the same type $\iota (V_j (\Gamma^U))$ for all constituting unit cells, and this type is unstable.

Here the index $L$ means large, $E$ external and $I$ internal. Item (iii) is actually a consequence of items (i), (ii) and the compatibility of contours $\Gamma^L_m (\Gamma^U)$ with each other. The meaning of notations

\begin{equation}
B^L (\Gamma^U) = \bigcup_m B_L (\Gamma^U), \quad O^L (\Gamma^U) = \bigcup_m O_L (\Gamma^U),
V (\Gamma^U) = \bigcup_j V_j (\Gamma^U) = B (\Gamma^U) \setminus B^L (\Gamma^U)
\end{equation}

is straightforward.

Figure 3 above illustrates (for $d = 2$) properties of a boundary layer $\Gamma^U$ obtained after erasing the small contours.

There are four large contours $\Gamma^L_{k,E} = \Gamma^L_{k,E} (\Gamma^U)$: each of them has at least one stable internal type. One large contour does not have a stable internal type: this is $\Gamma^{L,I}_1 (\Gamma^U)$. Together, the large contours $\Gamma^L_{k,E}, 1 \leq k \leq 4$, and $\Gamma^{L,I}_1$ form the contour collection $\{ \Gamma^L_m (\Gamma^U), 1 \leq m \leq 5 \}$. Here set $S$ consists of a single type represented by a light gray color. The
connected areas $V_j = V_j(\Gamma^U)$ are filled with unstable types which are represented by three different shades of darker gray color. The unstable type of $V_1$ is the one specified in the boundary condition. Anything drawn or written in black represents a notation and is not a part of the boundary layer. Specifically the thick black line is the boundary of $\Lambda$ which is also the external boundary of $\Lambda(\Gamma^U)$. The thin black lines point to the internal boundary of $\Lambda(\Gamma^U)$. The order of numeration for the large contours $\Gamma_k^m, \Gamma_n^m, \Gamma_n^{\Lambda}$ or $\Gamma_l^m$ plays no role; the same is true about the numeration for areas $V_j$.

Upon returning the erased small contours we define

$$w(\Gamma^U) := \prod_m w(\Gamma_m^U) \prod_j \mathbb{E}(V_j(\Gamma^U) \parallel I(V_j(\Gamma^U)) ; S).$$  

(4.2)

Here the product over $m$ collects the contribution of large contours constituting $\Gamma^U$ while the product over $j$ collects the contribution of small contours that we put back in their places. Then the definition of $\Gamma^U$ implies that for $i \notin S$

$$\mathbb{E}(\Lambda \parallel i) = \sum_{\Gamma^U \in \mathcal{G}(\Lambda \parallel i)} w(\Gamma^U) \prod_{k,s} \mathbb{E}(3I_{k,s}(\Gamma^U) \parallel I_{k,s}(\Gamma^U)).$$

(4.3)

cf. (2.41). Define the renormalized statistical weight of boundary layer $\Gamma^U$:

$$W(\Gamma^U) := \prod_m w(\Gamma_m^U) \prod_j \mathbb{E}(V_j(\Gamma^U) \parallel I(V_j(\Gamma^U)) ; S) \prod_j \mathbb{E}(V_j(\Gamma^U) \parallel I ; S),$$

(4.4)

where type $l \in S$. Note that, according to Lemma 3.3, partition functions $\mathbb{E}(V_j(\Gamma^U) \parallel I ; S)$ and $\mathbb{E}(V_j(\Gamma^U) \parallel I)$ do not depend on $l \in S$ as long as $q \leq 4$. On the other hand, for any $\Gamma^U \in \mathcal{G}(\Lambda \parallel i)$,

$$\mathbb{E}(\Lambda \parallel l) \geq \prod_j \mathbb{E}(V_j(\Gamma^U) \parallel I ; S) \prod_{k,s} \mathbb{E}(3I_{k,s}(\Gamma^U) \parallel I).$$

(4.5)

Dividing (4.3) by (4.5) and using definition (4.4), we estimate

$$\mathbb{E}(\Lambda \parallel l)^{-1} w(\Gamma^U) \prod_{k,s} \mathbb{E}(3I_{k,s}(\Gamma^U) \parallel I_{k,s}(\Gamma^U)) \leq W(\Gamma^U)$$

(4.6)

and

$$\frac{\mathbb{E}(\Lambda \parallel i)}{\mathbb{E}(\Lambda \parallel l)} \leq \sum_{\Gamma^U \in \mathcal{G}(\Lambda \parallel i)} W(\Gamma^U),$$

(4.7)

where $i \notin S$ and $l \in S$.

**Lemma 4.1** Let

$$a(\Gamma^U) = \nu(\mathcal{B}^L(\Gamma^U)) \log (1 + v^{0.8}) + \nu(\Lambda(\Gamma^U)) \log (1 + v^{b2d}).$$

(4.8)

Then, assuming $z$ large enough, we have that for any box $\Lambda$,

$$\sum_{\Gamma^U \in \mathcal{G}(\Lambda \parallel i)} W(\Gamma^U) e^{a(\Gamma^U)} < (1 + v^{0.5b2d})^{\nu(\bar{\Lambda})}.$$  

(4.9)
Proof of Lemma 4.1. We prove Lemma 4.1 by organizing the large contours $\Gamma_m^L (\Gamma^U)$ of a given boundary layer $\Gamma^U$ in a tree-like structure $T = T (\Gamma^U)$.

The root of the tree is represented by the boundary $\partial \Lambda$. Next, the first-level vertices are identified with those large contours $\Gamma_m^L (\Gamma^U)$ whose base $B (\Gamma_m^L)$ can be connected to $\partial \Lambda$ by a sequence of unit cells from $V (\Gamma^U)$ which is parallel to the first coordinate axis and not intersecting $B^L (\Gamma^U)$, cf (4.1). For every first-level large contour $\Gamma_m^L$ we choose a particular sequence of unit cells satisfying the above non-intersection condition and treat it as an edge $e (\partial \Lambda, \Gamma_m^L)$ joining the root $\partial \Lambda$ and the vertex $\Gamma_m^L$.

The second-level vertices are those among the remaining large contours $\Gamma_m^L$ whose base can be connected to the base of a first-level large contour with a horizontal sequence of unit cells from $V (\Gamma^U)$ still not intersecting $B^L (\Gamma^U)$. Again, for every such large contour we choose a sequence of unit cells with these properties and treat it as an edge. This process can be continued further, until all large contours of $\Gamma^U$ are included in $T$. By construction, the sequences of units cells representing edges of the tree are pair-wise disjoint. See Fig. 4.

Next, we take the definition (4.4) of renormalized statistical weight $W (\Gamma^U)$ and plug in it bound (2.31), representation (3.14), bound (3.17), and bounds (3.25)–(3.27). The result is a series of inequalities in Eqs. (4.10)–(4.18) involving $W (\Gamma^U)$, in one form or another. First,

$$W (\Gamma^U) < \left( \frac{v}{1 - v} \right)^{\nu (O^2 (\Gamma^U))} e^{\nu (B^L (\Gamma^U)) - \nu (2b)^d \nu (V (\Gamma^U))}$$

$$\leq \left( v (1 + 5^d 2^{v0.9}) \right)^{\nu (O^2 (\Gamma^U))} e^{-\nu (2b)^d \nu (V (\Gamma^U))},$$

(4.10)

because $B (1, j)$ in (3.25)–(3.27) is smaller than $(2b)^d$. Consequently, from (4.10):

$$W (\Gamma^U) e^{a (\Gamma^U)} \leq \left( v (1 + 5^d 2^{v0.8}) \right)^{\nu (O^2 (\Gamma^U))} e^{-0.5 \nu (2b)^d \nu (V (\Gamma^U))}.$$ 

(4.11)

By construction, if $E (\Gamma^U)$ denotes the collection of all edges of $T$ and $\nu (E (\Gamma^U))$ stands for the volume of $E (\Gamma^U)$ then

$$\nu (E (\Gamma^U)) \leq \nu (V (\Gamma^U)).$$

(4.12)
Bounds (4.11) and (4.12) imply that
\[ W(\Gamma^u) e^{d(\Gamma^u)} \leq \left( v(1 + 5^d 2 v^{0.8}) \right)^{\nu(O^u(\Gamma^u))} e^{-0.5 v^{(2b)d} \nu(\mathcal{E}(\Gamma^u))} \leq b^{2d} + 1. \tag{4.13} \]

We finish the proof of Lemma 4.1 by induction in the number of levels in \( T \). The root node of a tree is different from the remaining nodes as it is given by \( \partial \Lambda \) and does not have an associated large contour. A subtree rooted at a first-level node has only nodes which all are large contours. From now on and until it is said otherwise we consider such ‘homogeneous’ trees. Later we perform the final estimate including the root \( \partial \Lambda \).

A single-level homogeneous tree consists of a single large contour \( \Gamma^L \). Any large contour \( \Gamma^L \) has
\[ \nu(O(\Gamma^L)) > b^{2d} + 1. \tag{4.14} \]

Therefore for \( z \) large enough, the following bound can be verified similarly to (3.30)
\[ \sum_{\Gamma^L: E(\Gamma^L) \supseteq \Upsilon(0)} \left( v(1 + 5^d 2 v^{0.8}) \right)^{\nu(O(\Gamma^L))} \leq v^{b^{2d}+1}. \tag{4.15} \]

Denote by \( \Gamma^L_{\ast}(T) \) the root large contour of a homogeneous tree and suppose that the estimate
\[ \sum_{T: E(\Gamma^L_{\ast}) \supseteq \Upsilon(0)} \left( v(1 + 5^d 2 v^{0.8}) \right)^{\nu(O(\Gamma^L_{\ast}))} \leq v^{b^{2d}}. \tag{4.16} \]

has been verified for homogeneous trees with at most \( N \) levels. A tree with at most \( N + 1 \) levels can be decomposed into:

(i) the root large contour \( \Gamma^L_{\ast} \),
(ii) at most \( \nu(B(\Gamma^L_{\ast})) \) edges issued from the root large contour \( \Gamma^L_{\ast} \) and leading to the first-level large contours \( \Gamma^L_r \), and
(iii) a collection of sub-trees with maximum \( N \) levels rooted at each of \( \Gamma^L_r \).

The volume \( \nu(e(\Gamma^L_{\ast}, \Gamma^L_r)) \) occupied by the edge \( e(\Gamma^L_{\ast}, \Gamma^L_r) \) joining LBCs \( \Gamma^L_{\ast} \) and \( \Gamma^L_r \) can be any positive integer \( t \). Thus, the sum of the weights taken from the RHS of (4.13), over all trees with at most \( N + 1 \) levels, does not exceed
\[ \leq \sum_{\Gamma^L_{\ast}: E(\Gamma^L_{\ast}) \supseteq \Upsilon(0)} \left( v(1 + 5^d 2 v^{0.8}) \right)^{\nu(O(\Gamma^L_{\ast}))} \left( 1 + v^{b^{2d}} \sum_{t=1}^{\infty} e^{-0.5 v^{(2b)d} t} \right)^{\nu(B(\Gamma^L_{\ast}))} \leq b^{2d}, \tag{4.17} \]

which reproduces (4.16). Thus, the estimate (4.16) is true for any homogeneous tree.

Finally, we return to our inhomogeneous tree (where the root is represented by the boundary \( \partial \Lambda \)). By using (4.13), we estimate the contribution of all uniform trees growing from the root \( \partial \Lambda \):
\[ \sum_{\Gamma^u \in \partial(\Lambda \parallel i)} W(\Gamma^u) e^{d(\Gamma^u)} \leq \left( 1 + v^{b^{2d}} \sum_{r=1}^{\infty} e^{-0.5 v^{(2b)d} r} \right)^{\nu(\partial \Lambda)} \leq \left( 1 + v^{0.5 b^{2d}} \right)^{\nu(\partial \Lambda)}. \tag{4.18} \]

This finishes the proof of Lemma 4.1.
Corollary 4.2 For any \( i \notin S \) and any \( l \in S \)
\[
\frac{\Xi(\Lambda \| i)}{\Xi(\Lambda \| l)} < \left(1 + v^{0.5 b^2 d}\right)^{v(\partial \Lambda)}
\] (4.19)
for \( z \) large enough.

Proof of Corollary 4.2. Substitute (4.9) into RHS of (4.7). □

4.2 Polymer Expansions for \( \Xi(\Lambda \| i), i \in S \), and Theorem 1.1(I)

The purpose of this section is to prove assertion(I) of Theorem 1.1 by deriving for \( \Xi(\Lambda \| i), i \in S \), the representation (4.22) of the type (6.1) and then applying Theorem 6.1.

In analogy with (4.3)–(4.5), define a renormalized statistical weight \( W(\Gamma) \) of contour \( \Gamma \):
\[
W(\Gamma) = w(\Gamma) \prod_s \frac{\Xi(3I_s(\Gamma) \| \iota_l(\Gamma))}{\Xi(3I_s(\Gamma) \| \iota_E(\Gamma))}
\] (4.20)
and rewrite (2.40) as
\[
\Xi(\Lambda \| l) = \sum_{\Gamma^E \in \mathcal{G}(\Lambda \| i; \varepsilon)} \prod_{\Gamma^E \in \mathcal{G}^E} W(\Gamma^E) \prod_s \Xi(3I_s(\Gamma^E) \| i).
\] (4.21)
Here we used the fact that \( \Xi(3I_s(\Gamma^E) \| \iota^E(\Gamma^E)) = \Xi(3I_s(\Gamma^E) \| i) \) for all external contours \( \Gamma^E \in \mathcal{G}^E \). Iterating (4.21) we obtain the desired representation
\[
\Xi(\Lambda \| i) = \sum_{\Gamma} \prod_{\Gamma^E \in \mathcal{G}^E} W(\Gamma^E),
\] (4.22)
where the sum is taken over the collections of contours \( \Gamma \) such that for any contour \( \Gamma \) from this collection \( B(\Gamma) \subseteq 2\Lambda \), \( \iota^E(\Gamma) = i \) and all \( B(\Gamma) \) are mutually disjoint.

Representation (4.22) gives rise to a host of facts and constructions. Viz., the Peierls bound (2.44) takes a simple form
\[
\mu_\Lambda (\mathcal{G}(\Gamma^E) \| i) \leq \prod_{\Gamma^E \in \mathcal{G}^E} W(\Gamma^E).
\] (4.23)
As was said, we will use (4.23) in Sect. 5, beginning with Eq. (5.24).

Proof of Theorem 1.1(I) Assertion(I) of Theorem 1.1 follows in a standard way from the convergence of polymer expansion for the partition functions \( \Xi(\Lambda \| i) \) in (4.22). In turn, this convergence is implied by Theorem 6.1 as soon as condition (6.3) is verified.

Extending the definition in (3.12), we set
\[
a(\Gamma) = v(B(\Gamma)) \log (1 + v^{0.9})
\] (4.24)
for any contour \( \Gamma \) (large or small). The inequality
\[
\sum_{\Gamma : B(\Gamma) \supset \mathcal{Y}(0)} W(\Gamma)(1 + v^{0.8})^{\mu(B(\Gamma))} \leq \log (1 + v^{0.9}),
\] (4.25)
can be verified, after plugging in the definition (4.20) and estimate (4.19), by using the same enumerating arguments as in the proof of (3.13). The desired bound (6.3) follows.

The polymer expansion Theorem 6.1 implies existence, for \( i \in S \), of a pure phase \( \mu(\cdot \| i) \) (cf. (1.20)) which is shift-invariant, ergodic and has an exponential decay of correlations.
For measure $\mu(\cdot \parallel i)$, $i \in S$, the probability that the type of $\Upsilon(0)$ is $i$ tends to 1 as $z \to \infty$. Consequently, all pure phases named in Theorem 1.1(I) are different (and, due to the Lemma 3.3, symmetric). This completes the proof of assertion (I) in Theorem 1.1.

5 Unstable Boundary Condition

In Sect. 5 we prove Theorem 1.2 and assertion (II) of Theorem 1.1. Our approach is based on papers [11,25]. (We also use constructions from [13,14].) In our opinion, constructions performed here yield simplified versions of those used in [11,25]. Although the results of this section are claimed for $q = 4$, we keep using a general $q$ as it would clarify the nature of the bounds used. See, e.g., (5.6), (5.7), (5.9) and so on.

5.1 Proof of Theorem 1.2 (I)

Recall, we assume that a single stable type is 1. We will verify that

$$\mu(\cdot \parallel \Lambda_1) = \lim_{L \to \infty} \mu_{\Lambda_L}(\cdot \parallel \Lambda_1)$$

is the only limit Gibbs/DLR state of the model. In the course of the proof we develop an argument that is later used in the proof of the remaining parts in Theorem 1.2 and of assertion (II) of Theorem 1.1.

Given $L > 0$, we work with a cubic box $\Lambda_L$ of size $L$; cf. (1.6). Next, given an admissible particle configuration $Y = (Y_1, \ldots, Y_q) \in A$, we consider the distribution $\mu_{\Lambda_L}(\cdot \mid Y_{\Lambda_L}^{\Lambda_L^c})$ defined in Eq. (1.12). It is instructive to consider a boundary layer $\Gamma^U$ for the boundary condition $Y_{\Lambda_L}^{\Lambda_L^c}$: this is a generalization of the similar concept introduced in Sect. 4 for the boundary condition containing only particles of a given unstable type. Namely, $\Gamma^U$ is the connected component (or connected components) of the unit cells not in a stable phase adjacent to $\Lambda_L^c$. As before, the meaning of the boundary layer is to describe a transition from particles of unstable types present in $Y_{\Lambda_L}^{\Lambda_L^c}$ to particles of the stable type occurring in box $\Lambda_L$. Details are given below.

Our aim is to show that, with probability tending to 1 as $L \to \infty$, the boundary layer does not penetrate into the smaller concentric cubic box $\Lambda_{L/2}$. If this is the case for a particle configuration $X \in A(\Lambda_{L/2} \mid Y_{\Lambda_L}^{\Lambda_L^c})$ then there exists a box $\Lambda' = \Lambda'(X^{\Lambda_L})$ such that $\Lambda_{L/2} \subseteq \Lambda' \subseteq \Lambda_L$ and $X \in A(\Lambda' \parallel \Lambda_1)$. From the polymer expansions constructed for $\mu(\cdot \parallel \Lambda_1)$ in Sect. 4 we know that the measure $\mu_{\Lambda_L}(\cdot \parallel \Lambda_1)$ forgets the boundary conditions exponentially fast. Thus,

$$\lim_{L \to \infty} \mu_{\Lambda_L}(\cdot \mid Y_{\Lambda_L}^{\Lambda_L^c}) = \lim_{L \to \infty} \mu_{\Lambda'}(\cdot \parallel \Lambda_1)$$

which yields assertion (II) of Theorem 1.2.

We will now give a semi-formal review of the ideas, and we take some liberties during this stage of presentation. In particular, saying that a quantity behaves like $e^{-L}$ we mean that $e^{-L}$ is an upper bound for the quantity. Also we omit various positive constants, e.g. writing $e^{-L}$ instead of $e^{-cL}$, as constants are not essential for the arguments. We also use the notation $\Delta_L$ for the annulus $\Lambda_{L+b} \setminus \Lambda_L$ of thickness $b$ where $b$ is the largest hard-core diameter; cf. Eq. (1.17).

Graphically our ideas are presented by Fig. 5. Here the base of the boundary layer is shown as the complement to the area occupied by the light grey color. The frames (i) and (ii) demonstrate two possibilities for the boundary layer to penetrate into $\Lambda_{L/2}$; we will check that each of these possibilities occurs with probability tending to 0 as $L \to \infty$. 
In frame (i) there is an annulus $\Delta L'$, with $3L/4 < L' < L$, with a relatively low number of unstable unit cells inside it. When we estimate the probability of such a boundary layer, the unstable unit cells inside $\Delta L'$ serve as pinpoints for connected components of empty cells (depicted, as always, in a white color). At least one of these components is stretched over a long distance, at least $3L/4 - L/2 = L/4$, which carries a small probability. Furthermore, a relatively small amount of pinpoints guarantees that the entropy-type contribution to the probability coming from relatively short components is beaten by this small probability. This makes the overall probability of such a boundary layer negligible as $L \to \infty$. Cf. Eq. (5.15).

In frame (ii) no such annulus $\Delta L'$ can be found, and therefore a considerable part of $\Lambda L \setminus L_{3L/4}$ is covered by the boundary layer, although the amount of empty unit cells is not that large. This again makes probability of this type of boundary layers small. Cf. Eq. (5.33).

More precisely, if the boundary condition $\textbf{Y}^{\Delta L'}$, where $3L/4 < L' < L$, contains inside $\Delta L'$ less than $\sqrt{L}$ unit cells which are not in phase 1 then the probability for the boundary layer to penetrate inside $\Lambda L/2$ decays like $e^{-L}$. This property can be verified based on the following observations (1)--(5):

1. Only $\textbf{Y}^{\Delta L'}$ affects $\mu_{\Lambda L'} \left( \cdot \cdot \cdot \textbf{Y}^{\Delta L'} \right)$.
2. Each unstable unit cell inside $\Delta L'$ induces a large contour belonging to the boundary layer. (These induced large contours are not necessarily different for different unstable unit cells).
3. To penetrate into $\Lambda L/2$ the boundary layer should have at least $L/2$ unit cells in the base of at least one of the above large contours.
4. The $\mu_{\Lambda L'} \left( \cdot \cdot \cdot \textbf{Y}^{\Delta L'} \right)$-probability of the collection of large contours induced by unstable unit cells inside $\Delta L'$ is not larger than the $\mu_{\Lambda L'+b} \left( \cdot \cdot \cdot \| 1 \right)$-probability of the same collection multiplied by the factor of order $e^{\sqrt{L}}$. This statement is true because there are at most $\sqrt{L}$ unstable unit cells inside $\Delta L'$.
5. Due to the presence of a long large contour in the collection the corresponding $\mu_{\Lambda L'+b} \left( \cdot \cdot \cdot \| 1 \right)$-probability of the collection decays as $2^{\sqrt{L}} e^{-L}$. Here the first factor esti-
mates from above the contribution of all large contours in the collection which do not reach $\Lambda_{L/2}$. The second factor is the contribution of the large contour which reaches $\Lambda_{L/2}$. For large $L$ the factor of order $e^{-L}$ dominates the factors of order $2\sqrt{L}$ which gives us the desired estimate.

This argument covers the picture in frame (i) of Fig. 5.

To handle boundary conditions which contain more than $\sqrt{L}$ unstable unit cells inside $\Delta_L$ we split $\Lambda_L \setminus \Lambda_{3L/4}$ into annuli of thickness $b$: $\Lambda_{3L/4}$, $\Lambda_{3L/4+b}$, $\ldots$, $\Lambda_{L-b}$. First, we apply the preceding argument to $\Lambda_{L'} = \Lambda_{3L/4}$ assuming that the boundary condition $Z^{\Lambda_{3L/4}}$ contains at most $\sqrt{L}$ unstable unit cells. Then we proceed by induction, allowing the next annulus to have more unstable unit cells than the preceding one, for the price of doubling the estimate available for the preceding annulus. We select $e^{n/\sqrt{L}}$ as the maximal allowed amount of unstable unit cells in the $n$-th annulus. Note that this quantity remains smaller than $\sqrt{L}$ as soon as $n < \frac{1}{2}\sqrt{L} \log L$, and it becomes larger than the volume of the corresponding annulus for $n > 2d\sqrt{L} \log L$. Therefore, we need at most $2d\sqrt{L} \log L$ induction steps to reproduce the desired estimate for an arbitrary boundary condition. As each induction step worsens the estimate by the factor 2, at the end of the induction we accumulate the factor of $2^{2d\sqrt{L} \log L}$. This factor is still multiplied by the initial estimate $e^{-L}$. Thus, for large $L$ we have the desired resulting estimate $2^{2d\sqrt{L} \log L} e^{-L} \leq e^{-L/2}$.

The induction step itself is performed in the following way. Fix the configuration in the annulus number $n$ such that this configuration contains at most $e^{n/\sqrt{L}}$ unstable unit cells. This configuration serves as a boundary condition for $\Lambda_{3L/4+nb}$. For any configuration inside $\Lambda_{3L/4+nb}$ there are only two possibilities (1) and (2):

1. One of the layers indexed by $n' < n$ has less than $e^{n'/\sqrt{L}}$ unstable unit cells. In this case by induction we already know that the probability for the boundary layer to penetrate inside $\Lambda_{L/2}$ is small.

2. The annuli indexed by $n' < n$ have in total at least $\sum_{n'=1}^{n-1} e^{n'/\sqrt{L}} > \sqrt{L} e^{n/\sqrt{L}} / 2$ unstable unit cells. The $\mu_{\Lambda_{3L/4+nb}} (\cdot \| 1)$-probability of this event behaves like $\exp(-\sqrt{L} e^{n/\sqrt{L}} / 2)$, as follows from Lemma 4.1. The corresponding $\mu_{\Lambda_{3L/4+nb}} (\cdot | Z^{\Lambda_{3L/4+nb}})$-probability of the same event is at most $\exp(e^{n/\sqrt{L}})$ times larger. For large enough $L$ we obtain:

$$\exp\left(-\sqrt{L} e^{n/\sqrt{L}} / 2 + e^{n/\sqrt{L}}\right) < \exp\left(-\sqrt{L} e^{n/\sqrt{L}} / 4\right).$$

Adding the last estimate to the estimates obtained in the previous induction steps we at most double them. This completes the induction step and the argument covering the picture in frame (ii) of Fig. 5.

We turn now to formal definitions and proofs. The event $A\left(\Lambda_L | Y^{A_L}\right)$ defined in Eq. (1.10) is written in terms of the restriction $X^{A_L}$ of the configuration $X$. This restriction is a collection of finite sets $X_j^{A_L} = X_j \cap \Lambda_L$, $1 \leq j \leq q$ belonging to $\Lambda_L$. With a certain abuse of notation, we write $X^{A_L} \in A\left(\Lambda_L | Y^{A_L}\right)$ meaning that $X \in A\left(\Lambda_L | Y^{A_L}\right)$. In the argument below we need to embed the sets $X_j^{A_L}$ in a larger box, $A' \subseteq \Lambda_L$. For this purpose, the notation $X^{A_L} \vee \emptyset^{A_L \setminus A'}$ will be employed. A similar meaning is assigned to symbol $X^{A'} \vee \emptyset^{A_L \setminus A'}$ when box $A' \subseteq \Lambda_L$.

Further, the events $A(A)$ and $A(\Lambda\| i)$ in Eqs. (1.8) and (1.14) are also written in terms of the restriction $X^{A}$. Consequently, with the same degree of the notational abuse we write
Fig. 6 Sets $\Lambda^L$ and $\Lambda^L_b$

$X^L \cup \Lambda^L \subset A(\Lambda \parallel 1)$ when $\Lambda^L \supset \Lambda$, and $X^L \cup \Lambda^L \subset A(\Lambda \parallel 1)$ when $\Lambda^L \subset \Lambda$, by following a similar meaning. Under these agreements, it becomes obvious that

$$X^L \in A(\Lambda^L \mid Y^L) \implies X^L \cup \Lambda^L \in A(\Lambda + 1 \parallel 1)$$

(5.2)

and, for $L > b$,

$$X^{L-b} \in A(\Lambda_{L-b} \parallel 1) \implies X^{L-b} \cup \Lambda^{L-b} \in A(\Lambda^L \mid Y^L_0).$$

(5.3)

Extending these manipulations to events, we obtain that

$$B \subseteq A(\Lambda \mid Y^L_0) \implies B \cup \Lambda^L \subset A(\Lambda + 1 \parallel 1)$$

(5.4)

and

$$B \subseteq A(\Lambda_{L-b} \parallel 1) \implies B \cup \Lambda^{L-b} \subset A(\Lambda^L \mid Y^L_0).$$

(5.5)

Consequently, for an event $B \subseteq A(\Lambda \mid Y^L_0)$ we have

$$\mu(\Lambda \mid Y^L_0) \leq \mu(\Lambda + 1 \parallel 1) \cdot \frac{P(A(\Lambda_{L-b} \parallel 1))}{P(A(\Lambda + 1 \parallel 1))}$$

(5.6)

In fact, we will need a more precise version of (5.6).

**Definition 5.1** We construct the set $\Lambda^L = \Lambda^L(\Lambda^L; 1)$ by removing from $\Lambda^L$ the cubes of size $2b + 1$ centered at unit cells that are not in phase 1 in $Y^L$. Similarly, consider the set $\Lambda^L_0 = \Lambda(\Lambda^L_0; 1)$ obtained by adding to $\Lambda^L$ the cubes of size $2b + 1$ centered at unit cells that are not in phase 1 in $Y^L$. Fig. 6.
Assume that \( Y^{\Lambda_L} \) contains \( \leq n \) unit cells that are not in phase 1. Then, for an event \( B \subseteq A \left( \Lambda | Y^{\Lambda_L} \right) \),

\[
\mu_{\Lambda_L} \left( B \mid Y^{\Lambda_L} \right) < \mu_{\star \Lambda_L} \left( B \lor \square \Lambda_L \mid \Lambda \right) \frac{\mathbb{P}(A(\star \Lambda_L \mid 1))}{\mathbb{P}(A(\star \Lambda_L \mid 1))} \leq \frac{\left( \frac{q}{v} \right)^n}{(2b+1)^d} \tag{5.7}
\]

Using inequalities (5.6) and (5.7), it is possible to obtain an upper bound for \( \mu_{\Lambda_L} \left( B \mid Y^{\Lambda_L} \right) \) from a similar bound for \( \mu_{\star \Lambda_L} \left( B \lor \square \Lambda_L \mid \Lambda \right) \). Since type 1 is stable we can estimate probabilities of events in \( A(\cdot \mid 1) \) by using the machinery developed in Sects. 3 and 4.

Our next step is a formal definition of the boundary layer in \( \Lambda_L \) for the case of an arbitrary boundary condition \( Y^{\Lambda_L} \), cf. Definition 4.1.

**Definition 5.2** Consider a cubic box \( \Lambda_L \) and a particle configurations \( Y \in A \). Given a particle configuration \( X \in A \left( \Lambda | Y^{\Lambda_L} \right) \), consider the set \( U \) of the unit cells \( \Upsilon \subseteq \Lambda_L \) which are not in phase 1 in \( X^{\Lambda_L} \). A base of the boundary layer (in \( X^{\Lambda_L} \)) under boundary condition \( Y^{\Lambda_L} \) is the union of those connected components of the set \( U \) which are adjacent to unstable unit cells in \( \Lambda_L \). As in Definition 4.1, we refer to a boundary layer \( \Gamma^U = \Gamma^U(X^{\Lambda_L}) \) under boundary condition \( Y^{\Lambda_L} \) by considering its base, \( B(\Gamma^U) \), together with the types attached to unit cells \( \Upsilon \in B(\Gamma^U) \).

Equivalently, the boundary layer can be defined in the following way. First, we construct the set \( *\Lambda_L = *\Lambda_L(Y^{\Lambda_L}; 1) \) as described in Definition 5.1 and pass from a configuration \( X \in A \left( \Lambda | Y^{\Lambda_L} \right) \) to its restriction \( *\Lambda_L \subseteq A (\star \Lambda_L \mid 1) \). Due to the definition of \( *\Lambda_L \), all unit cells of \( *\Lambda_L \setminus \Lambda_L \) are empty and therefore belong to the bases of some external contours \( \Gamma_k^E \). Among these external contours there are large contours which we denote by \( \Gamma_{l,E} \). Some of the external large contours \( \Gamma_{l,E} \) have interior components \( I_{l,E} \) with unstable types \( t_{l,E} \). Any such component gives rise to a boundary layer \( \Gamma_{l,t}^U \) with base \( B(\Gamma_{l,t}^U) \) as described by Definition 4.1. The full collection of boundary layers \( \Gamma_{l,t}^U \), for varying \( l \) and \( t \), together with all external contours \( \Gamma_k^E \), forms the boundary layer \( \Gamma^U \) under boundary condition \( Y^{\Lambda_L} \).

Consequently, the base \( B(\Gamma^U) = \left( \bigcup_{l,t} B(\Gamma_{l,t}^U) \right) \cup \left( \bigcup_k B(\Gamma_k^E) \right) \) as in Definition 4.1 \( B(\Gamma^U) \subseteq \Lambda_L \).

In Fig. 7, the contours have been drawn in configuration \( X^{\Lambda_L} \lor \bigcup_{l,t} B(\Gamma_{l,t}^U) \). The base \( B(\Gamma^U) \) of the boundary layer is the part of \( \Lambda_L \) which is not colored light grey. There are four contours in the collection \( \{ \Gamma_k^E \} \). Their bases are shown as connected components of white color inside \( \Lambda_L \). Among these contours one is small and three are large. These large contours constitute the collection \( \{ \Gamma_{l,E} \} \). Five connected components of dark colors constitute the collection \( \{ B(\Gamma_{l,t}^U) \} \). The remaining contours of the configuration \( X^{\Lambda_L} \) are not shown. They are situated inside light grey areas in \( \Lambda_L \).

We are interested in estimating of \( \mu_{\Lambda_L} \left( \cdot | Y^{\Lambda_L} \right) \)-probability of the event

\[
B_0 = \left\{ X^{\Lambda_L} \in A \left( \Lambda_L | Y^{\Lambda_L} \right) : B(\Gamma^U) \cap \Lambda_{L/2} \neq \emptyset \right\}. \tag{5.8}
\]

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Note that event $\mathcal{B}_0$ occurs only if among $\Gamma^L_{s,E}$, for which either $\Lambda_{L/2} \subset \text{I}(\Gamma^L_{s,E})$ or $\text{B}(\Gamma^L_{s,E}) \cap \Lambda_{L/2} \neq \emptyset$. In both cases $\nu(\text{B}(\Gamma^L_{s,E})) > L/2$ (regardless of dimension $d$).

Assume for now that $\mathcal{Y}^\Delta_L$ has at most $\sqrt{L}$ unit cells not in the phase 1. Then, according to (5.7),

$$\mu_{\Lambda_L} \left( \mathcal{B}_0 | \mathcal{Y}^\Delta_L \right) < \mu^{\ast}_{\Lambda_L} \left( \mathcal{B}_0 \lor \varnothing^{\ast\Lambda_L \setminus \Lambda_L} \parallel 1 \right) \left( \frac{q}{v} \right) \sqrt{L} (2b+1)^d. \tag{5.9}$$

The upper bound for $\mu^{\ast}_{\Lambda_L} \left( \mathcal{B}_0 \lor \varnothing^{\ast\Lambda_L \setminus \Lambda_L} \parallel 1 \right)$ can be obtained from the following considerations:

1. As follows from Peierls estimate (4.23), the probability to have all large contours of $\Gamma^L_{l,E}$ among entire collection of external contours in $\ast \Lambda_L$ does not exceed the product of renormalized statistical weights $\prod_l W(\Gamma^L_{l,E})$.

2. For any large contour $\Gamma^L_{l,E}$ there exists a unit cell $\Upsilon \subset \ast \Lambda_L \setminus \Lambda$ such that $\Upsilon \subset \text{B}(\Gamma^L_{l,E})$. This unit cell $\Upsilon$ can be used as a pinpoint in summation over bases $\text{B}(\Gamma^L_{s,E}) \supset \Upsilon$ as in (4.25).

3. The product $\prod_l W(\Gamma^L_{l,E})$ includes a large contour $\Gamma^s_{l,E}$ with $\nu(\text{B}(\Gamma^s_{l,E})) > L/2$. Therefore $W(\Gamma^s_{l,E}) \leq 2 (v/(1-v))^{L/5d^2}$ as follows from (4.19) and (2.31). There are at most $\sqrt{L}$ possibilities to select, among unstable unit cells in $\Delta_L$, a unit cell $\Upsilon$ giving rise to $\text{B}(\Gamma^s_{l,E})$.

4. The remaining part of the product $\prod_l W(\Gamma^L_{l,E})$ corresponds to large contours ‘originated’ from remaining at most $\sqrt{L} - 1$ unstable unit cells in $\Delta_L$. The sum of factors

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**Fig. 7** Boundary layer under a generic boundary condition
Combining (1)–(4), we conclude that
\[ \mu_{\Lambda_L} \left( \mathcal{B}_0 | Y^{\Lambda_L}_E \right) \leq \left( \frac{q}{v} \right)^{\sqrt{L}} \left( \frac{v}{1 - v} \right)^{L/5d} \frac{\sqrt{L}}{L^{(5d/3)}} \] (5.10)
provided \( L \) is large enough.

Next, set \( \Lambda_{(n)} = \Lambda_{3L/4 + nb} \). Consider the annuli \( \Delta_{(n)} = \Lambda_{3L/4 + nb} \setminus \Lambda_{3L/4 + nb} \) indexed by an integer \( n \), \( 1 \leq n \leq 2d \sqrt{T} \log L \), and introduce a threshold value
\[ G(n) = \max(1, e^{n/\sqrt{T}}) \] (5.11)
as an upper bound for the amount of unstable unit cells in the corresponding annulus. For \( n \geq 2d \sqrt{T} \log L \) we have
\[ G(n) > (3L/4 + nb + b)^d - (3L/4 + nb)^d \] (5.12)
so that the above threshold is not really a limitation.

Consider the part of the boundary layer \( \Gamma^U \) which lies outside \( \Lambda_{(n)} \), i.e. consider the restriction of \( \Gamma^U \) to \( B_{(\Lambda^U)} \setminus \Lambda_{(n)} \). We denote by \( B_{(\Lambda^U)} \setminus \Lambda_{(n)} \) the part of \( B_{(\Lambda^U)} \) adjacent to \( \Lambda_{(n)} \). Further, define the event
\[ B_n = \left\{ X^{\Lambda_L} \in A \left( \Lambda_L | Y^{\Lambda_L}_E \right) : \nu \left( B_{(\Lambda^U)}(\Gamma^U) \cap \Delta_{(n)} \right) \leq G(n) \right\}. \] (5.13)

In other words, event \( B_n \) occurs when \( \Delta_{(n)} \) contains at most \( G(n) \) unstable unit cells from \( B_{(\Lambda^U)}(\Gamma^U) \).

For \( n < \frac{1}{2} \sqrt{T} \log L \) we have
\[ G(n) < \sqrt{L}. \] (5.14)

Thus, adjusting the argument leading to (5.10) we obtain
\[ \mu_{\Lambda_{(n)}} \left( \mathcal{B}_0 | Z^{\Lambda_{(n)}}_E \right) \leq \nu((3L/4 + bn)/(5d/3)) \] (5.15)
provided \( L \) is large enough and \( Z^{\Lambda_{(n)}}_E \) contains at most \( G(n) \) unstable unit cells inside the annulus \( \Delta_{(n)} \). The difference between (5.15) and (5.10) is that the large contour \( \Gamma^{L, E} \) has \( \nu(B(\Gamma^{L, E})) > (3L/4 + bn)/2 \). Set:
\[ \mu_{\Lambda L} \left( \mathcal{B}_0 | \mathcal{B}_n, Y^{\Lambda_L}_E \right) := \frac{\mu_{\Lambda_L} \left( \mathcal{B}_0 \cap \mathcal{B}_n | Y^{\Lambda_L}_E \right)}{\mu_{\Lambda_L} \left( \mathcal{B}_n | Y^{\Lambda_L}_E \right)} \] (5.16)

By using the DLR property together with (5.15), we obtain that for \( n < \frac{1}{2} \sqrt{T} \log L \),
\[ \mu_{\Lambda L} \left( \mathcal{B}_0 | \mathcal{B}_n, Y^{\Lambda_L}_E \right) = \frac{\int \mu_{\Lambda_{(n)}} \left( \mathcal{B}_0 | Z \right) d\mu_{\Lambda_L} \left( Z | Y^{\Lambda_L}_E \right)}{\int \mathcal{B}_n d\mu_{\Lambda_L} \left( Z | Y^{\Lambda_L}_E \right)} \leq \nu((3L/4 + bn)/(5d/3)) \] (5.17)
where \( Z = Z^\Lambda_c(n) \). Writing, for brevity, \( \mu_{\Lambda_L} (B_0 | B_n) \) instead of \( \mu_{\Lambda_L} \left( B_0 \mid B_n, \mathcal{Y}^{\Lambda_c}_L \right) \), we can estimate:

\[
\frac{1}{2} \sqrt{L} \log L \sum_{n' = 1}^{\frac{1}{2} \sqrt{L} \log L} \mu_{\Lambda_L} (B_0 | B_{n'}) \leq v^{3L/(5d24)} \tag{5.18}
\]

provided that \( L \) is large enough.

On the other hand, for \( n \geq \frac{1}{2} \sqrt{L} \log L \), with \( B_n^g := \bigcap_{n' = 1}^{n-1} B_{n'}^c \), we have:

\[
\mu_{\Lambda_L} (B_0 | B_n) \leq \sum_{n' = 1}^{n-1} \mu_{\Lambda_L} (B_0 | B_{n'}) \mu_{\Lambda_L} (B_{n'} | B_n) + \mu_{\Lambda_L} (B_0 | B_n^g) \mu_{\Lambda_L} (B_n^g | B_n) \tag{5.19}
\]

\[
\leq \sum_{n' = 1}^{n-1} \mu_{\Lambda_L} (B_0 | B_{n'}) + \mu_{\Lambda_L} (B_n^g | B_n) .
\]

Owing to (5.7) and a calculation similar to (5.17),

\[
\mu_{\Lambda_L} (B_n^g | B_n) \leq \sup \left[ \mu_{\Lambda_{(a)}} (B_n^g | Z) : Z \in B_n \right] \tag{5.20}
\]

where again \( Z = Z^\Lambda_c(n) \). The subsequent inequalities (5.22)–(5.32) aim at estimating the probability \( \mu_{\Lambda_{(a)}} (B_n^g | Z) \) uniformly in \( Z \in B_n \).

The event \( B_n^g \) is that inside \( \Lambda_{(a)} \setminus \Lambda_{3L/4} \) the boundary layer contains at least

\[
M = M(n) := \sum_{n' = 1}^{n-1} G(n') \tag{5.21}
\]

unstable unit cells. The probability of this event can be estimated in the following way:

\[
\mu_{\Lambda_{(a)}} (B_n^g | Z^{\Lambda_{(a)}}) \leq \left( \frac{q}{v} \right)^{G(n)} \mu^{* \Lambda_{(a)}} (B_n^g \parallel 1) ; \tag{5.22}
\]

cf. Eq. (5.9). Recall, box \( ^* \Lambda_{(a)} \) obtained from cube \( \Lambda_{(a)} \) is \( Z^{\Lambda_{(a)}} \)-dependent (see Definition 5.1). Further, we write

\[
\mu^{* \Lambda_{(a)}} (B_n^g \parallel 1) = \sum_{\Gamma^{L,E}} z_{\Lambda_{(a)}} \left( G(\Gamma^{L,E}) \parallel 1 \right) \prod_{\Gamma^{L,E} \in \Gamma^{L,E}} \prod_{I} \Xi \left( 3I_{I} (\Gamma^{L,E}) \parallel \ell_{I} (\Gamma^{L,E}) \right)
\times \sum_{\Gamma^{L,E}} w(\Gamma^{U}) \prod_{k,s} \Xi \left( 3I_{k,s} (\Gamma^{U}) \parallel \ell_{k,s} (\Gamma^{U}) \right) \tag{5.23}
\]

Here the sum \( \sum_{\Gamma^{L,E}} z_{\Lambda_{(a)}} \) is taken over collections \( \Gamma^{L,E} \) of external large contours \( \Gamma^{L,E} \) with base \( B(\Gamma^{L,E}) = \bigcup_{\Gamma^{L,E} \in \Gamma^{L,E}} B(\Gamma^{L,E}) \) such that each \( B(\Gamma^{L,E}) \) contains a unit cell unstable in \( Z^{\Lambda_{(a)}} \). The event \( G(\Gamma^{L,E}) \) is defined in (2.43). Next, the sum \( \sum_{\Gamma^{L,E}} w(\Gamma^{U}) \) is taken over boundary layers \( \Gamma^{U} \) in \( \bigcup_{\Gamma^{L,E}} 3I_{I} (\Gamma^{L,E}) \) such that \( v(B(\Gamma^{U})) \geq M \). Note that for each \( \Gamma^{L,E} \) only components \( 3I_{I} (\Gamma^{L,E}) \) having unstable \( \ell_{I} (\Gamma^{L,E}) \) contribute to the double union above (and correspondingly to the double product in (5.23)). The statistical weight \( w(\Gamma^{U}) \) is defined.

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by (4.2) and the sum over $\Gamma^U$ in (5.23) originates from (4.3). To be more precise, Eq. (4.2) defines $w(\Gamma^U(3I_\ell(\Gamma^{L,E}))$ for the boundary layer $\Gamma^U(3I_\ell(\Gamma^{L,E}))$ inside $3I_\ell(\Gamma^{L,E}))$ and $w(\Gamma^U)$ is obtained as the double product $\prod_{\Gamma^{L,E} \in \Gamma^{L,E}} \prod_{\ell} w(\Gamma^{U}(3I_\ell(\Gamma^{L,E})))$. Finally, in the product $\prod_{k,s}$ the boundary layer $\Gamma^U$ is treated as a single object, i.e. indexes $k$ and $s$ are defined according to item (i) below Definition 4.1 and are not partitioned by index $t$.

By virtue of the Peierls bound (4.23), the probability of event $\mathcal{G}(\Gamma^{L,E})$ can be estimated by the product of renormalized statistical weights:

$$\mu^{*}\Lambda_{(n)}(\mathcal{G}(\Gamma^{L,E}) \| 1) \leq \prod_{\Gamma^{L,E} \in \Gamma^{L,E}} W(\Gamma^{L,E}).$$

(5.24)

Hence, with the help of (4.20) and (4.6), the RHS in (5.23) is less than or equal to

$$\sum_{\Gamma^{L,E}} z^{*,*}\Lambda_{(n)} \prod_{\Gamma^{L,E} \in \Gamma^{L,E}} w(\Gamma^{L,E}) \sum_{\Gamma^U}^{\Gamma^{L,E},M} M W(\Gamma^U),$$

(5.25)

which can be seen by multiplying the fraction in (5.23) by the double product $\prod_{\Gamma^{L,E} \in \Gamma^{L,E}} \prod_{\ell} \Xi(3I_\ell(\Gamma^{L,E}) \| 1)$ and dividing the sum $\sum_{\Gamma^U}^{\Gamma^{L,E},M}$ by the same quantity; cf. (4.4).

Now, referring to (4.8), we conclude that the expression (5.25) does not exceed

$$\exp\left(-Mv b^{2d}\right) \sum_{\Gamma^{L,E}} z^{*,*}\Lambda_{(n)} \prod_{\Gamma^{L,E} \in \Gamma^{L,E}} w(\Gamma^{L,E}) \sum_{\Gamma^U}^{\Gamma^{L,E},M} M W(\Gamma^U) e^{a(\Gamma^U)}$$

(5.26)

since $a(\Gamma^U) \geq v(B(\Gamma^U)) v b^{2d}$.

Next, we apply the bound (4.9):

$$\sum_{\Gamma^U}^{\Gamma^{L,E},M} W(\Gamma^U) e^{a(\Gamma^U)} \leq \prod_{\Gamma^{L,E} \in \Gamma^{L,E}} \left(1 + v 0.5 b^{2d}\right)^{v(B(\Gamma^{L,E}))}.$$ 

(5.27)

Therefore, expression (5.26) is less than or equal to

$$\exp\left(-Mv b^{2d}\right) \sum_{\Gamma^{L,E}} z^{*,*}\Lambda_{(n)} \prod_{\Gamma^{L,E} \in \Gamma^{L,E}} w(\Gamma^{L,E}) \left(1 + v 0.5 b^{2d}\right)^{v(B(\Gamma^{L,E}))}.$$ 

(5.28)

Further, as follows from (4.19), statistical weights $w(\Gamma)$ and $W(\Gamma)$ obey

$$w(\Gamma^{L,E}) \leq W(\Gamma^{L,E}) \left(1 + v 0.5 b^{2d}\right)^{v(\Gamma^{L,E})}.$$ 

(5.29)

because $\sum_{\ell} v(\partial(3I_\ell(\Gamma))) \leq v(\Gamma)$. This yields that (5.28) does not exceed

$$\exp\left(-Mv b^{2d}\right) \sum_{\Gamma^{L,E}} z^{*,*}\Lambda_{(n)} \prod_{\Gamma^{L,E} \in \Gamma^{L,E}} W(\Gamma^{L,E}) \left(1 + v 0.5 b^{2d}\right)^{2v(B(\Gamma^{L,E}))}.$$ 

(5.30)

which in turn is not bigger than

$$\exp\left(-Mv b^{2d}\right) (1 + v 0.9)^{G(n)}.$$ 

(5.31)

Here we used (4.25) and the fact that we have at most $G(n)$ reference unit cells inside $*\Lambda_{(n)} \setminus \Lambda_{(n)}$, and each base $B(\Gamma^{L,E})$ contains at least one such reference unit cell. Note that the quantities (5.31) do not depend on particle configuration $Z$. 

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All in all, for the probability in the LHS of (5.20) is estimated as follows:

\[
\mu_{\Lambda_L}(B^*_n | B_n) \leq \left( \frac{q}{v} \right)^{G(n)} \exp \left( -\sum_{n'=1}^{n-1} G(n') \nu^{b_{2d}} \right) (1 + \nu^{0.9})^{G(n)}. \tag{5.32}
\]

Observe that \(\forall \ n \geq 1\), as follows from the definition (5.11),

\[
\sum_{n'=1}^{n-1} G(n') \geq \frac{1}{2} \sqrt{L} G(n).
\]

For \(L\) large enough this yields

\[
\mu_{\Lambda_L}(B^*_n | B_n) \leq \exp \left( -\frac{1}{4} G(n) \sqrt{L} \nu^{b_{2d}} \right)
\]

which, for \(n \geq \frac{1}{2} \sqrt{L} \ln L\), leads to

\[
\mu_{\Lambda_L}(B^*_n | B_n) \leq \exp \left( -\frac{1}{4} L \nu^{b_{2d}} \right). \tag{5.33}
\]

Therefore, with the help of (5.18), we obtain a recursive bound for the probability in the LHS of (5.19): for \(n \geq \frac{1}{2} \sqrt{L} \ln L\),

\[
\mu_{\Lambda_L}(B_0 | B_n) \leq v^{3L/(5d24)} + \sum_{n'=\frac{1}{2} \sqrt{L} \log L}^{n-1} \mu_{\Lambda_L}(B_0 | B_{n'}) + \exp \left( -\frac{1}{4} L \nu^{b_{2d}} \right). \tag{5.34}
\]

This yields that for \(n \geq \frac{1}{2} \sqrt{L} \ln L\),

\[
\mu_{\Lambda_L}(B_0 | B_n) \leq 2^n \left[ v^{3L/(5d24)} + \exp \left( -\frac{1}{4} L \nu^{b_{2d}} \right) \right]. \tag{5.35}
\]

Finally, observe that for \(n = 2d \sqrt{L} \log L\) and \(L\) large enough, by virtue of (5.12), \(B_n\) is the full event. Therefore, for \(L\) large enough,

\[
\mu_{\Lambda_L}(B_0 | Y^{\Lambda_L^G}) = \mu_{\Lambda_L}(B_0 | B_n, Y^{\Lambda_L^G})
\]

\[
\leq 2^{2d \sqrt{L} \ln L} \left[ v^{3L/(5d24)} + \exp \left( -\frac{1}{4} L \nu^{b_{2d}} \right) \right] \tag{5.36}
\]

Denote by \(B_0^C\) the complement of \(B_0\) and consider a cylindrical event \(B\) localized in \(\Lambda_L/4\). Then, for \(L\) large enough, a standard consequence of polymer expansion is

\[
\left| \frac{\mu_{\Lambda_L}(B \cap B_0^C | Y^{\Lambda_L^G}) - \mu(B \| 1)}{\mu_{\Lambda_L}(B_0^C | Y^{\Lambda_L^G})} \right| \leq e^{-cL}. \tag{5.37}
\]
where \( c \in (0, \infty) \) is a constant. Finally,

\[
\left| \mu_L (B|Y^A_L) - \mu(\parallel 1) \right| \leq \left| \frac{\mu_L (B \cap B_0|Y^A_L)}{\mu_L (B_0|Y^A_L)} - \mu(\parallel 1) \right| \mu_L (B_0|Y^A_L) \\
+ \left| \frac{\mu_L (B \cap B_0^0|Y^A_L)}{\mu_L (B_0^0|Y^A_L)} - \mu(\parallel 1) \right| \mu_L (B_0^0|Y^A_L)
\]

\( \leq e^{-cL} + \exp \left( -\frac{1}{8} L v b^{2d} \right) \). (5.38)

The RHS in (5.38) tends to 0 as \( L \to \infty \). This completes the proof of Theorem 1 (I).

5.2 Proof of Assertions (IIa), (III) and (IV) of Theorem 1.2

Here we suppose that \( S = \{ 1, 2 \} \) (and hence, unstable types are 3 and 4). A common property in cases (IIa) and (IV) is a complete symmetry between stable types 1 and 2. Namely, for any particle configuration \( X \) with restrictions \( X^A_L \) and \( X^A_L \) we define the corresponding factorized particle configurations \( X^F, X^A_L, F \) and \( X^A_L, F \) by replacing each particle of a stable type with a particle of a factorised type \( S \). (Here \( S \) stands for stable.) In general, more than one original particle configuration is mapped to the same factorized particle configuration. To be more specific, if in the factorized particle configuration the number of \( D(1, 2) \)-connected components of \( S \)-type particles equals \( k \) then the number of original particle configurations mapped to this factorized configuration is \( 2^k \). (Recall, in this sub-section \( \#S = 2 \).) Assigning to the factorized particle configurations additional statistical weights \( 2^k \), we obtain an equivalent model which is a kind of the FK representation for the original model (cf. [4,7]).

Figure 8 is a factorized version of Fig. 2. We assume that two lighter grey colors in Fig. 2 correspond to stable types 1 and 2. In Fig. 8 both of them are drawn as a single light grey color which corresponds to the type \( S \).

All constructions from the previous sections can be repeated for the factorized model with minor modifications which we list below.
In the original model the measure $\mathbb{P}$ was defined for particles of the type 1, 2, 3 and 4. In the factorized model this measure $\mathbb{P}$ is defined for particles of the type 3, 4 and $\mathbb{S}$.

The original objects carrying the notation $(\cdot \parallel i)$ with $i = 1, 2$ have a direct analogues $(\cdot \parallel S)$ in the factorized model. To define in the factorized model the statistical weight of a contour $\Gamma^F$ we need to integrate over the generating configurations $X^F \in \mathcal{A}(\Gamma^F, \Lambda)$ the statistical weight $2^k(X^F)^{-1}$. Here $k(X^F)$ is the number of $D(1, 2)$-connected component of $S$-type particles in $X^F$; cf. (2.30). We use the power $k - 1$ rather than $k$ because the external $D(1, 2)$-connected component (i.e. the one intersecting $E(\Gamma^F)$) inherits its original type from the boundary condition or from the enclosing contour. Consequently, the analogue of the upper bound in (2.31) has an additional factor $2^{\nu(B(\Gamma^F))}$.

Similarly, in analogues of bounds (5.6) and (5.7) we replace $\left(\frac{q}{v}\right)$ with $\left(\frac{2q}{v}\right)$ estimating from above the maximal number of $D(1, 2)$-connected components of $S$-type particles in the boundary condition $Y^{A_L}_{1^F, F}$. (Recall, we work with $q = 4$.)

The definition of the event $B_0$ (see (5.8)) remains valid for the factorized model. Moreover, repeating the argument (5.15)–(5.36) from Sect. 5.1 we conclude that the probabilities of the complement event $B_0^F$ under factorized measures $\mu_{A_L} (\cdot \parallel 3)$, $\mu_{A_L} (\cdot \parallel 4)$ and $\mu_{A_L} (\cdot)$ (the latter is with the empty boundary condition) tend to $1$ as $L \to \infty$. Note that the probability laws for random configurations $X^L_3$ and $X^L_4$ under measures $\mu_{A_L} (\cdot \parallel j)$, $j = 3, 4$, and $\mu_{A} (\cdot)$ are the same in both factorized and original models. If in the factorized model $X^L_{A_L, F} \in B_0^F$ then this factorized configuration contains an annulus enclosing $A_{L/2}$ and formed by unit cells of $S$. Due to the symmetry between stable types 1 and 2, the corresponding original configuration $X^L_{A_L} \in B_0^F$ contains, inside this annulus particles of specific type $i \in S$ with the probability which is $1/2$ of the probability of the factorized configuration. Moving back from probabilities of the factorized configurations $X^L_{A_L, F}$ to probabilities of the original configurations $X^L_{A_L}$, we see that, for unstable types $j = 3, 4$,

$$\lim_{L \to \infty} \mu_{A_L} (\cdot \parallel j) = \lim_{L \to \infty} \mu_{A} (\cdot) = \frac{1}{2} \left[ \mu (\cdot \parallel 1) + \mu (\cdot \parallel 2) \right]. \quad (5.39)$$

This yields Assertion (IIa) of Theorem 1.2 and Assertion (IV) in the case where $S = \{1, 2\}$.

Note that the technique from Sect. 5.1 which we applied to the factorized model implies the uniqueness of the limit Gibbs state in the factorized model. This uniqueness does not imply, of course, the uniqueness in the original model. In particular, the not translation periodic limit Gibbs states which can exist in the original model can’t be detected by the factorized model as the sigma-algebra of the factorized model is not fine enough.

Modifications of the above argument covering Assertion (III) (with $\not\exists S = 3$) are straightforward. The same is true of Assertion (IV) in case $\not\exists S = 3$. For the proof in case $\not\exists S = 4$, see Sect. 5.4.

5.3 Proof of Assertions (IIb) of Theorem 1.2

For the definiteness we again assume that the stable types are 1 and 2, now with $D(4, 2) = D(3, 1) < D(3, 2) = D(4, 1)$. In this case types 1 and 2 are not symmetric with respect to a given unstable type, e.g. type 3. Nevertheless, the pair of types (1, 3) is symmetric relative to the pair (2, 4), in the sense that if we simultaneously replace type 1 with 2 and type 3 with 4 then an admissible configuration remains admissible. Based on this symmetry, we use the ideas of Sect. 5.2 and work with factorized configurations.

The factorized model now contains two factorized particle types: $S$ and $u$. The factorized particle type $S$ replaces stable types 1 and 2 while the factorized particle type $u$ replaces...
unstable types 3 and 4. We say that two factorized particles are connected if the distance between them is less than or equal to: \(D(1, 2)\) for two \(S\)-type particles, \(D(3, 4)\) for two \(u\)-type particles, and \(D(1, 4)\) for two particles of different factorized type. Each connected component of factorized particles has two implementations by original particles: \((S = 1, u = 3)\) and \((S = 2, u = 4)\). Therefore, if a factorized particle configuration \(\mathbf{X}^F \in \mathcal{A}\) contains \(k\) connected components of the above type then the corresponding statistical weight is \(2^k(\mathbf{X}^F)\).

Repeating the argument from Sect. 5.2, we conclude that \(\mu_{\Lambda_L}(E^C_0 \| u)\) tends to 1 as \(L \to \infty\). Here the meaning of the notation \((\cdot \| u)\) is straightforward.

The factorized particle configuration belonging to the event \(E^C_0\) can be implemented by the original configurations in 4 different ways. The difference between implementations is in the choice of the unstable particle type which is present in the boundary condition for \(\Lambda_L\) and the choice of the stable particle type placed in the annulus enclosing \(\Lambda_L/2\). Each specific choice is a pair of unstable and stable particle types, e.g. \((3, 1)\).

It turns out that the \(\mu_{\Lambda_L}(\cdot \| u)\)-probability of pairs \((3, 2)\) and \((4, 1)\) tends to 0 as \(L \to \infty\). Indeed, any boundary layer configuration implementing the pair \((3, 2)\) can be mapped into a boundary layer configuration implementing the pair \((3, 1)\) such that the integral of the statistical weights of all particle configurations implementing the pair \((3, 2)\) which are mapped to the same configuration implementing the pair \((3, 1)\) is at least \(\psi^{-L/2d}\) times smaller than the statistical weight of the configuration they are mapped to. The \(1 \leftrightarrow 2, 3 \leftrightarrow 4\) symmetry defines a similar map for pairs \((4, 1)\) and \((4, 2)\).

Recall that the boundary layer is a collection of large contours. Then the aforementioned map is constructed by erasing in this collection of the most external large contour satisfying the following conditions: it encloses \(\Lambda_L/2\) and implements a transition from the type 3 outside to the type 4 inside. If such large contour is not present in the boundary layer then the map is constructed by replacing the most internal large contour satisfying the following conditions: it encloses \(\Lambda_L/2\) and implements a transition from the type 3 outside to the type 2 inside. The replacing large contour is chosen to be the smallest large contour implementing the transition from the type 3 outside to the type 1 inside and having the same interior as the original large contour. Here the smallest large contour \(\Gamma\) means the large contour having minimal possible \(\nu(O(\Gamma))\).

Figure 9 shows two types of the original boundary layers (frames (i) and (ii)) and the resulting boundary layer (frame (iii)).

The desired estimate of the statistical weights is a consequence of the facts that: the map decreases the number of empty unit cells in the boundary layer by at least \(2d(L/2)^{d-1}\) unit cells, all removed empty unit cells are 5-connected and this connected set encloses \(\Lambda_L/2\).
Due to the symmetry, the surviving pairs (3, 1) and (4, 2) have the same $\mu_{\Lambda L}(-\|u)-\nabla\mu_{\Lambda L}$-probability. This yields Assertion (IIb).

5.4 Proof of Assertion (IV) of Theorem 1.2

In Sects. 5.2 and 5.3 we showed how to construct factorized models for all cases except for the case $\varpi = 4$. Assume for the definiteness that $D(1, 2) = D(3, 4) = a$ and $D(1, 4) = b$ as in the previous section. Then the model is symmetric under the same transformation $1 \leftrightarrow 2, 3 \leftrightarrow 4$ and the factorized model is constructed as in the previous section by using two factorized types $S'$ and $S''$. The factorized type $S'$ combines particles of the types 1 and 2 while the factorized type $S''$ combines particles of the types 3 and 4. The types $S'$ and $S''$ are symmetric again such we can perform yet another factorization and define a model in terms of a single stable phase $S$. As in previous sections, a factorized configuration $\mathbf{x}^{A_L, B_0} \in B_0^0$ contains and annulus enclosing $A_{L/2}$ and formed by unit cells of type $S$. Due to the symmetry, the corresponding original configuration $\mathbf{x}^{A_L} \in B_0^0$ contains, inside this annulus particles of specific type $i$ with the probability which is $1/4$ of the probability of the factorized configuration. Moving back from probabilities of the factorized configurations $\mathbf{x}^{A_L, B_0}$ to probabilities of the original configurations $\mathbf{x}^{A_L}$, we arrive at Assertion (IV) in case where $\varpi = 4$.

Note that in the model consisting of a single particle type $S$ the probability of $B_0$ can easily be estimated without the machinery from previous sections. In this case the event $B_0$ simply means that there exists a contour $\Gamma^\varnothing$ adjacent to $\Lambda_L$ and intersecting $A_{L/2}$. Due to the Peierls bound, the probability of such contour is less than $e^{u(O(\Gamma^\varnothing))}$. The amount of contours originating from a given unit cell $\Upsilon \in \partial \Lambda_L$ and having given value of $u(O(\Gamma^\varnothing))$ is less than $e^{u(O(\Gamma^\varnothing))}$. There is at most $2dL^{d-1}$ possibilities to chose the originating unit cell $\Upsilon$. Thus, the probability of $B_0$ does not exceed $2dL^{d-1}(cV)u(O(\Gamma^\varnothing))$.

5.5 Proof of Assertion (II) of Theorem 1.1

In this section we follow ideas from [25] (cf. Lemma in Sect. 3.2 of [25]).

Given a unit cell $\Upsilon$ denote by $B_\Upsilon$ the event formed by particle configurations $\mathbf{x} \in A$ for which there is no bounded box $\Lambda' \supset \Upsilon$ and a particle type $i$ (stable or unstable) such that $\mathbf{x}^{A'} \in A(\Lambda' \parallel i)$. It is not hard to see that

$$B_\Upsilon = \{ \mathbf{x} \in A : \Upsilon \subset B(\Gamma) \text{ for some contour } \Gamma \in \Gamma(\mathbf{x}) \text{ with unbounded base } B(\Gamma) \}. \quad (5.40)$$

Similarly, consider a box $\Lambda$ and an admissible particle configuration $\mathbf{y} \in A$. Then for a given unit cell $\Upsilon \subset \Lambda$, we define the event $B_{\Upsilon, \Lambda} \mathbf{y} \in A$ consisting of particle configurations $\mathbf{x}$ such that $\mathbf{x}^{A_0} = \mathbf{y}^{A_0}$ and there is no bounded box $\Lambda' : \Upsilon \subset \Lambda' \subset \Lambda$ and a particle type $i$ (stable or unstable) such that $\mathbf{x}^{A'} \in A(\Lambda' \parallel i)$. Equivalently,

$$B_{\Upsilon, \Lambda} \mathbf{y} = \{ \mathbf{x} \in A : \mathbf{x}^{A_0} = \mathbf{y}^{A_0}, \Upsilon \subset B(\Gamma) \text{ for some contour } \Gamma \in \Gamma(\mathbf{x}) \text{ such that } B(\Gamma) \cap \Lambda_0 \neq \emptyset \}. \quad (5.41)$$

Obviously, for $\mathbf{y} \notin B_\Upsilon$,

$$\bigcap_{\Lambda' : \Lambda' \supset \Upsilon} B_{\Upsilon, \Lambda} \mathbf{y} = \emptyset. \quad (5.42)$$
Furthermore,\[ B_T = \bigcap_{\Lambda: \Lambda \supseteq \Upsilon} B_{T, \Lambda} \text{ where } B_{T, \Lambda} = \bigcup_{\Upsilon \in \mathcal{A}} B_{T, \Lambda, \Upsilon}. \tag{5.43} \]

Let $\mu(\cdot)$ be a translation periodic DLR measure. Suppose that\[ \mu(B_T) > 0 \tag{5.44} \]
for some $\Upsilon$. Due to translation periodicity of $\mu$ and owing to (5.43),\[ \sum_{\Upsilon \subset \Lambda_L} \mu(B_{T, \Lambda_L}) > \sum_{\Upsilon \subset \Lambda_L} \mu(B_T) > \varepsilon \nu(\Lambda_L) = \varepsilon L^d \tag{5.45} \]
where $\varepsilon > 0$.

The sum in the LHS of (5.45) gives an expectation of the amount of unit cells inside $\Lambda_L$ which belong to bases of so-called interface (or open) contours in $\Lambda$. An interface contour in $\Lambda$ is a contour $\Gamma$ that has more than one connected component of the intersection $E(\Gamma) \cap \Lambda$. (Recall, $E(\Gamma)$ stands for the exterior of $\Gamma$; see Definition 2.3.) Note that for an interface contour $\Gamma$, base $B(\Gamma)$ is always adjacent to $\Lambda_L$. Cf. Fig. 10.

Consider the event\[ B_{M, \Lambda_L} = \left\{ \mathbf{X} \in \mathcal{A} : \sum_{\Delta \subset \Lambda} \nu(B(\Gamma)) = M \right\} \tag{5.46} \]
where the sum $\sum_{\Delta \subset \Lambda} \nu(B)$ is taken over all interface contours $\Gamma \in \Gamma_2(\mathbf{X})$ in $\Lambda$. By virtue of the DLR property, the expectation\[ \sum_{M \geq 0} M \mu(B_{M, \Lambda_L}) = \int_{\mathcal{A}} \mu(d\mathbf{Y}) \sum_{M \geq 0} M \mu(\nu(B_{M, \Lambda_L} | Y^{\Lambda_L})). \tag{5.47} \]
According to (5.6),\[ \mu(\nu(B_{M, \Lambda_L} | Y^{\Lambda_L}) < \mu_{\Lambda_L+b}(B_{M, \Lambda_L} \vee \nabla_{\Lambda} \uparrow 1) \left( \frac{q}{\nu} \right)^{\nu(\Lambda_{L+b} \setminus \Lambda_{L-b})}; \tag{5.48} \]
here we assume, without loss of generality, that type 1 is stable. The probability 
\( \mu_{A_{L+b}} \left( \mathcal{B}_{M,\Lambda_L} \cup \mathcal{A}_{L} \parallel 1 \right) \) can be upper-bounded by using (2.31) and polymer expansions from Sect. 4. The resulting inequality is

\[
\mu_{A_{L+b}} \left( \mathcal{B}_{M,\Lambda_L} \cup \mathcal{A}_{L} \parallel 1 \right) \leq \left( \frac{v}{1 - v} \right)^{0.5M/5^d} \tag{5.49}
\]

as, together with \( \mathcal{A}_{L} \), the interface contours form a single contour inside \( \Lambda_{L+b} \) containing at least \( M \) unit cells in its base. Partitioning the sum \( \sum_{M \geq 0} \) in the RHS of (5.47) into \( \sum_{0 \leq M \leq L^d - 1/2} \) and \( \sum_{M > L^d - 1/2} \) and plugging into the later sum estimates (5.48) and (5.49) we conclude that for \( L \) large enough,

\[
\sum_{\mathcal{T} \subseteq \Lambda_L} \mu(\mathcal{B}_{\mathcal{T},\Lambda_L}) = \sum_{M = 0}^{\infty} M \mu(\mathcal{B}_{M,\Lambda_L}) \leq 2L^{d-1/2}. \tag{5.50}
\]

But (5.50) contradicts (5.45), and so the assumption (5.44) is false.

The negation of (5.44) means that with \( \mu \)-probability 1 a particle configuration \( X \in \mathcal{A} \) does not have unbounded contours in \( \Gamma(X) \). Thus, for any given \( L > 1 \), with \( \mu \)-probability 1 there exists a box \( \Lambda \supset \Lambda_L \) such that the restriction \( X^{\gamma \Lambda \setminus \Lambda} \) contains at least one particle of the same type in each unit cell \( \gamma \subseteq 2\Lambda \setminus \Lambda \). Therefore, the \( \mu \)-probability of the event that \( \Lambda_{L/2} \) is enclosed by an annulus of unit cells in a stable phase tends to 1 as \( L \to \infty \) as was shown in Sects. 5.1, 5.2, and 5.3. This implies Assertion (II) of Theorem 1.1.

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Appendix: The Polymer Expansion Theorem

Consider a finite or countable set \( \Theta \) the elements of which are called (abstract) contours and denoted \( \theta, \theta' \), etc. Fix some anti-reflexive and symmetric relation \( \sim \) on \( \Theta \times \Theta \) (with \( \theta \not\sim \theta \) and \( \theta \sim \theta' \) equivalent to \( \theta' \sim \theta \)). A pair \( \theta, \theta' \in \Theta \times \Theta \) is called incompatible \( (\theta \not\sim \theta') \) if it does not belong to the relation and compatible \( (\theta \sim \theta') \) in the opposite case. (In our context, two contours are compatible when they are mutually external and have the same external type.) A collection \( \{\theta_j\} \) is called a compatible collection of contours if any two its elements are compatible. Every contour \( \theta \) is assigned a (generally speaking) complex-valued statistical weight denoted by \( w(\theta) \), and for any finite \( \Lambda \subseteq \Theta \) an (abstract) partition function is defined as

\[
Z(\Lambda) = \sum_{\{\theta_j\} \subseteq \Lambda} \prod_j w(\theta_j), \tag{6.1}
\]

where the sum is extended to all compatible collections of contours \( \theta_i \in \Lambda \). The empty collection is compatible by definition, and it is included in \( Z(\Lambda) \) with statistical weight 1.

A polymer \( \Pi = [\theta_j^{a_j}] \) is an (unordered) finite collection of different contours \( \theta_i \in \Theta \) taken with positive integer multiplicities \( a_i \), such that for every pair \( \theta', \theta'' \in \Pi \) there exists a sequence \( \theta'' = \theta_1, \theta_2, \ldots, \theta_s = \theta'' \in \Pi \) with \( \theta_{i_j} \not\sim \theta_{i_{j+1}}, \ j = 1, 2, \ldots, s - 1 \). The notation \( \Pi \subseteq \Lambda \) means that \( \theta_i \in \Lambda \) for every \( \theta_i \in \Pi \).
With every polymer $\Pi$ we associate an (abstract) graph $G(\Pi)$ which consists of $\sum_i \alpha_i$ vertices labeled by the contours from $\Pi$ and edges joining every two vertices labeled by incompatible contours. As follows from the definition of $\Pi$, graph $G(\Pi)$ is connected. We denote by $r(\Pi)$ the quantity

$$r(\Pi) = \prod_i (\alpha_i !)^{-1} \sum_{G' \subseteq G(\Pi)} (-1)^{\sharp \mathcal{E}(G')}$$

(6.2)

(the Moebius-type inversion coefficient). Here the sum is taken over all connected subgraphs $G'$ of $G(\Pi)$ containing all $\sum_i \alpha_i$ vertices, and $\sharp \mathcal{E}(G')$ denotes the number of edges in $G'$. For any $\theta \in \Pi$ we denote by $\alpha(\theta, \Pi)$ the multiplicity of contour $\theta$ in polymer $\Pi$.

The Polymer expansion theorem (Theorem 6.1 below) is a modification of assertions from [9] (see also [22]) which has been established in [16].

**Theorem 6.1** Suppose that there exists a function $a(\theta) : \Theta \mapsto \mathbb{R}^+$ such that for any contour $\theta$

$$\sum_{\theta' : \theta' \neq \theta} \left| w(\theta') e^{a(\theta')} \right| \leq a(\theta).$$

(6.3)

Then, for any finite $\Lambda$,

$$\log Z(\Lambda) = \sum_{\Pi \subseteq \Lambda} w(\Pi),$$

(6.4)

where the statistical weight of a polymer $\Pi = [\theta_i^{\alpha_i}]$ equals

$$w(\Pi) = r(\Pi) \prod_i w(\theta_i)^{\alpha_i}.$$  

(6.5)

Moreover, the series (6.4) for $\log Z(\Lambda)$ absolutely converges in view of the bound

$$\sum_{\Pi : \Theta \supseteq \theta} \alpha(\theta, \Pi) |w(\Pi)| \leq |w(\theta)| e^{a(\theta)},$$

(6.6)

which holds true for any contour $\theta$.

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