Identical quantum subsystems can possess a property which does not have any classical counterpart: indistinguishability. As a long-debated phenomenon, identical particles’ indistinguishability has been shown to be at the heart of various fundamental physical results. When concerned with the spatial degree of freedom, identical constituents can be made indistinguishable by overlapping their spatial wave functions via appropriately defined spatial deformations. By the laws of quantum mechanics, any measurement designed to resolve a quantity which depends on the spatial degree of freedom only and performed on the regions of overlap is not able to assign the measured outcome to one specific particle within the system. The result is an entangled state where the measured property is shared between the identical constituents. In this work, we present a coherent formalization of the concept of deformation in a general N-particle scenario, together with a suitable measure of the degree of indistinguishability. We highlight the basic differences with non-identical
particles scenarios and discuss the inherent role of spatial deformations as entanglement activators within the spatially localized operations and classical communication operational framework.

This article is part of the theme issue ‘Identity, individuality and indistinguishability in physics and mathematics’.

1. Introduction: identity and indistinguishability

In physics, particles are said to be identical if their intrinsic physical properties, such as mass, electric charge and (total) spin, are the same [1,2]. This is the case, for example, of subatomic particles such as electrons, photons, quarks, of atomic nuclei and of atoms and molecules themselves. Particles identity is a cornerstone of both classical and quantum physics which provides the core of the inductive approach to the investigation of Nature’s fundamental laws: the assumption that all the electrons in the universe possess the same electric charge, mass, spin, etc., allows one to conclude that some fundamental properties extrapolated from the behaviour of a sample of electrons observed in a laboratory also hold for all the other electrons in the universe.

Despite being frequently used as synonyms, particles identity is not the same as particles indistinguishability. Being a purely quantum phenomenon, the latter is more strictly related to the concept of individual addressability [3,4]. Identical particles can indeed still be distinguished one from the other when their extrinsic properties, such as their position or the projection of their angular momentum along an axis, are different. This is clear in the classical world where two physical systems, even when microscopic and identical, always occupy distinct positions in space at a fixed time, thus always being potentially individually addressed by following their trajectory [2]. On the contrary, this is not always true in quantum mechanics, where the wave-like and probabilistic description of physical systems allows different particles wave functions to be spatially overlapped, thus having a non-zero probability of simultaneously occupying the same region of space. When this situation occurs, any measurement of quantities depending only on the particles position performed on the region of overlap does not allow the observer to understand to which specific particle the measured outcome belongs to. This is the case, for example, of two synchronized photon sources $A$ and $B$ emitting single-photons impinging, with a certain probability, on a restricted detecting spatial region. If a single-photon detector in that region clicks, we now have no way of knowing from which source the detected photon is coming from: in this situation, we say that there is no which-way information and the particles of interest are said to be indistinguishable [1,4].

The difference between identity and indistinguishability is particularly evident in the everyday experience. It is indeed this difference which allows one to relate observed results to specific samples in an experiment: for example, we can talk about the characterization of a specific laser source carried out in a laboratory in Buenos Aires only because the photons emitted by such a source are very well distinguishable (not spatially overlapped, in this case) from the ones emitted by a neon sign in Tokyo, despite all the photons being identical [3,5–7]. Still, the laser must be very well isolated from other light sources to be sure that the characterized device is the laser and not a street lamp nearby. Thus, differently from particle identity, particle indistinguishability depends on the variable degrees of freedom involved. As a crucial consequence, indistinguishability is a meaningful concept only when related to the discrimination capability of the measurement device employed to probe those degrees of freedom.

To better clarify this point, let us recover the above mentioned example of two synchronized single-photon emitters and let us now assume that source $A$ is known to emit photons with horizontal polarization, while source $B$ produces only vertically polarized ones. Furthermore, let us suppose that the polarization is not changed by the dynamics. If the single-photon detector placed on the region of spatial overlap is designed to discriminate also the photon polarization, we now have a way to understand whether the origin of the particle causing the click is
source $A$ or $B$. In other words, the two photons can now be individually addressed and are not indistinguishable anymore despite being identical and spatially overlapped. Similarly, if we now further assume the polarization of the two photons to be the same, we could employ a measurement device capable of detecting their energy to discriminate among them. Even the emission time can be used to discriminate between the two particles if we know one source to emit before the other. Finally, the number of detectors can be set to distinguish the two particles, too. For example, let us consider that a photon emitted from source $A$ can only reach regions $L$ and $C$ while a photon emitted from source $B$ can impinge only on $C$ and $R$, with $L$, $C$ and $R$ distinct: a single-photon detector placed in region $C$ would be unable to distinguish the two particles, while the addition of a second detector on $L$ would be enough to reconstruct the origin of every click.

Summing up, particles are always assumed, often implicitly, to be (or not to be) indistinguishable to the eyes of the employed measurement devices, while they are universally identical or non-identical. From the experimental point of view, the actual generation of indistinguishable photons is actually a hard operation of fine tuning and synchronization. From now on, we will always implicitly refer to spatial indistinguishability when not otherwise specified, i.e. to the indistinguishability of particles spatially overlapped in relation to detectors for which no which-way information exists.

In this article, we characterize the degree of indistinguishability in a general $N$-particle quantum system. This is achieved by formalizing and extending the idea of deformation operations. Firstly introduced in [8] and later exploited in [9,10] in the particular scenario of bipartite systems, deformations provide a mathematical framework suitable to describe the manipulation of identical constituents when particles’ indistinguishability is involved. They account for processes where indistinguishability is generated starting from identical, yet distinguishable particles, and vice versa. Remarkably, they play a fundamental role in devising a coherent extension of the traditional local operations and classical communication (LOCC) framework to systems of indistinguishable constituents, whereas the latter fails due to resorting on particles’ individuality. After a short summary of the no-label approach to identical particles [4,11] in §2, we introduce, formalize and generalize deformations in §3. In §4, we retrieve the definition of an entropic measure of spatial indistinguishability firstly introduced in [12], extending it to the multipartite scenario and to a general amount of degrees of freedom. Finally, in §5, we review and employ the spatially localized operation and classical communication (sLOCC) operational framework, which highlights the importance of spatial deformations as a fundamental tool for the manipulation of identical constituents in many practical applications, as confirmed by recent experiments. To help the reader to grasp the main aspects of the manuscript, we conclude each section with a brief summary of the discussed arguments.

2. The no-label formalism

As is well known, particles living in a three-dimensional space can be divided into two macro groups: bosons, with integer spin, and fermions, with semi-integer spin. According to the symmetrization postulate, the global state describing an ensemble of identical bosons must remain the same when the role of any pair of particles is exchanged: bosonic states are symmetric under particles swapping. On the contrary, fermionic states are ruled to be anti-symmetric under analogous particles exchange [7]. The existence of such a postulate is at the heart of the Pauli exclusion principle and sets the ground for fundamental results in modern physics, from models to analyse Bose–Einstein condensates to the description of the behaviour of neutron stars.

To deal with these conditions, the standard approach to identical particles assigns unphysical (unobservable) labels to each constituent, ensuring that the global state exhibits the correct symmetry when any two labels are switched [1]. For example, let us consider two non-entangled particles with spatial wave functions $\psi_1, \psi_2$. If the two particles are non-identical, their global state is simply given by the tensor product $|\Psi^{(2)}\rangle = |\psi_1\rangle_A \otimes |\psi_2\rangle_B$, where the labels $A$ and $B$ encompass all the other physical degrees of freedom as well as the properties which makes the two constituents different. Differently, if the two particles are identical and indistinguishable,
labels $A$ and $B$ becomes simply fictitious names without any physical meaning and the global state must be written as [7]

$$|\psi^{(2)}\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle_A \otimes |\psi_2\rangle_B + \eta |\psi_2\rangle_A \otimes |\psi_1\rangle_B),$$

(2.1)

in order to satisfy the symmetrization postulate, where $\eta = 1$ for bosons and $\eta = -1$ for fermions.

The approach leading to equation (2.1), despite being the most frequently used even in didactic textbooks, is known to be affected by some formal problems [6,13]. For example, the necessity to symmetrize/antisymmetrize states by hand as in equation (2.1) leads to the emergence of fictitious entanglement when this is evaluated using standard tools such as the von Neumann entropy of the reduced density matrix. This is tackled by adopting ad hoc treatments to probe the existence of quantum correlations among identical particles systems. In addition, such methods require to treat bosons and fermions differently. In order to overcome these problems, a plethora of alternative approaches to deal with identical particles has been proposed over time [3,4,11,13–27].

Among these methods, the no-label approach recognizes the origin of the problem in the unphysical labels $A$ and $B$ appearing in equation (2.1), removing them from the formalism [4,11]. In this way, global states are simply given by a list of the single particle states: considering once again the example of two constituents with single spatial wave functions $\psi_1$ and $\psi_2$, the global state is written as $|\psi^{(2)}\rangle := |\psi_1\rangle \otimes |\psi_2\rangle$. If the two particles are distinguishable, e.g. not spatially overlapped, the global state is still a product state. Nonetheless, when they are not perfectly distinguishable, the global state cannot be written as a tensor product anymore: $|\psi^{(2)}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$. Similarly, the global Hilbert space $\mathcal{H}^{(2)}$ is generally not the tensor product of the single particle Hilbert spaces $\mathcal{H}^{(1)}_1$ and $\mathcal{H}^{(1)}_2$: $\mathcal{H}^{(2)} \neq \mathcal{H}^{(1)}_1 \otimes \mathcal{H}^{(1)}_2$. The formalism easily accounts for other single particle degrees of freedom. For example, let us consider a bipartite state of two identical qubits, one with spatial wave function $\psi_1$ and pseudospin $\uparrow$, and the other one analogously characterized by $\psi_2$ and $\downarrow$. Within the standard approach, such a state is given by $|\Psi^{(2)}\rangle = \frac{1}{\sqrt{2}}(|\psi_1\uparrow\rangle_A \otimes |\psi_2\downarrow\rangle_B + \eta |\psi_2\downarrow\rangle_A \otimes |\psi_1\uparrow\rangle_B)$, analogously to equation (2.1). In the no-label approach, instead, the same situation is simply described by the state $|\psi^{(2)}\rangle = |\psi_1\uparrow\rangle \otimes |\psi_2\downarrow\rangle$, with $|\psi^{(2)}\rangle = |\psi_1\uparrow\rangle \otimes |\psi_2\downarrow\rangle$ if $\psi_1$ and $\psi_2$ do not overlap (distinguishable particles) and $|\psi^{(2)}\rangle \neq |\psi_1\uparrow\rangle \otimes |\psi_2\downarrow\rangle$ otherwise (indistinguishable particles). The advantages of the no-label formalism emerge even more clearly when dealing with more complicated scenarios: consider, for example, the previously mentioned situation, where this time we do not know whether the particle with spatial wave function $\psi_1$ is characterized by the pseudospin $\uparrow$ and the one with $\psi_2$ by $\downarrow$ or vice versa. When such an uncertainty is maximum, two possible states for the bipartite system are the Bell triplet ($+$) and singlet ($-$) maximally entangled states, which the symmetrization postulate in the standard formalism rules to be written as

$$|\psi^{(2)}\rangle = \frac{1}{2}(|\psi_1\uparrow\rangle_A \otimes |\psi_2\downarrow\rangle_B \pm |\psi_1\downarrow\rangle_A \otimes |\psi_2\uparrow\rangle_B + \eta (|\psi_2\downarrow\rangle_A \otimes |\psi_1\uparrow\rangle_B \pm |\psi_2\uparrow\rangle_A \otimes |\psi_1\downarrow\rangle_B).$$

(2.2)

The no-label approach noticeably simplifies the notation, allowing to rewrite the same state as $|\psi^{(2)}\rangle = |\psi_1\uparrow\rangle \pm |\psi_2\downarrow\rangle \pm |\psi_1\downarrow\rangle \pm |\psi_2\uparrow\rangle$, or equivalently [11] as $|\psi^{(2)}\rangle = |\psi_1,\psi_2\rangle_{\pm\pm} \otimes |\uparrow,\downarrow\rangle$, where the subscript indicates the symmetry of the state: $|\alpha,\beta\rangle_{\pm\pm} = \pm |\beta,\alpha\rangle$. Finally, the generalization to the $N$-particle scenario is straightforward, with the global state $|\psi^{(N)}\rangle := |\psi_1,\psi_2,\ldots,\psi_N\rangle$ generally satisfying $|\psi^{(N)}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle$. For a more exhaustive review of how the most common two-particle states are written within the no-label approach, compared to their expressions in the standard approach with fictitious labels, see figure 1.

Given the two-particle state $|\psi_1,\psi_2\rangle$, the cornerstone of the no-label approach is provided by the definition of the probability amplitude related to finding the system in the state $|\psi_1,\psi_2\rangle$, which takes into account the eventual indistinguishability of the constituents. According to the meaning of indistinguishability discussed in §1, the impossibility to discriminate between the two particles
should reasonably lead to both of them contributing with their probability amplitude of being found in $\varphi_1$ and $\varphi_2$. Thus, we define

$$\langle \varphi_1, \varphi_2 | \psi_1, \psi_2 \rangle := \langle \varphi_1 | \psi_1 \rangle \langle \varphi_2 | \psi_2 \rangle + \eta \langle \varphi_1 | \psi_2 \rangle \langle \varphi_2 | \psi_1 \rangle.$$  \hspace{1cm} (2.3)

Remarkably, this definition directly encodes the statistical exchange phase $\eta$: within the no-label approach, the statistical information about the identical particles nature is encoded in the transition amplitudes, rather than in the symmetrization of the quantum state. Some important characteristics of the formalism can be directly derived from (2.3): comparing $\langle \varphi_1, \varphi_2 | \psi_1, \psi_2 \rangle$ with $\langle \varphi_1, \varphi_2 | \psi_1, \psi_2 \rangle$, it follows that

$$|\psi_2, \varphi_1\rangle = \eta |\psi_1, \varphi_2\rangle.$$  \hspace{1cm} (2.4)

(see the note at the bottom of figure 1). Furthermore, state $|\psi_1, \psi_2\rangle$ is not, in general, normalized: indeed, it can be easily checked that (assuming the single particle wave functions $\psi_1, \psi_2$ to be properly normalized)

$$\langle \psi_1, \psi_2 | \psi_1, \psi_2 \rangle = 1 + \eta |\langle \psi_1 | \psi_2 \rangle|^2 := C_\perp^2,$$  \hspace{1cm} (2.5)

implying that the correctly normalized two particle state is

$$|\Psi^2\rangle_N = \frac{|\psi_1, \psi_2\rangle}{C_\perp}.$$  \hspace{1cm} (2.6)

Note that, when the spatial overlap is null (i.e. distinguishable particles), $\langle \psi_1 | \psi_2 \rangle = 0$ and the normalized two particle state simply reduces to $|\Psi^2\rangle_N = |\psi_1, \psi_2\rangle$. Equations (2.3), (2.4) and (2.6) should reasonably lead to both of them contributing with their probability amplitude of being found in $\varphi_1$ and $\varphi_2$. Thus, we define

$$\langle \varphi_1, \varphi_2 | \psi_1, \psi_2 \rangle := \langle \varphi_1 | \psi_1 \rangle \langle \varphi_2 | \psi_2 \rangle + \eta \langle \varphi_1 | \psi_2 \rangle \langle \varphi_2 | \psi_1 \rangle.$$  \hspace{1cm} (2.3)

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Within this scenario, the action of a single particle operator $O$ on the permutation. Under particle swapping, the state behaves as

$$|\psi_{\alpha_1}, \psi_{\alpha_2}, \ldots, \psi_{\alpha_N}\rangle = \eta^{P_{\alpha}} |\psi_1, \psi_2, \ldots, \psi_N\rangle,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is any arbitrary permutation of $(1, 2, \ldots, N)$, while $P_{\alpha}$ is the parity of the permutation. Under particle swapping, the $N$-particle state behaves as

$$|\psi_{\alpha_1}, \psi_{\alpha_2}, \ldots, \psi_{\alpha_N}\rangle = \eta^{P_{\alpha}} |\psi_1, \psi_2, \ldots, \psi_N\rangle,$$

while the properly normalized state is simply given by

$$|\Psi^{(N)}\rangle_N = \frac{|\psi_1, \ldots, \psi_N\rangle}{\sqrt{\langle\psi_1, \ldots, \psi_N | \psi_1, \ldots, \psi_N\rangle}}.$$

Note that, if all the single particle wave functions are non-overlapping and individually modified in the same way, deformations consist in transformations acting differently, but still localized operations and classical communication in space on operations which are localized particles, where no individual constituent can be defined. In such a situation, one can instead rely on operations which are localized in space, rather than on single elements, leading to the spatially localized operations and classical communication (sLOCC) framework discussed further in §5 [27,29].

Within this scenario, the action of a single particle operator $O^{(1)}_X$ localized on the spatial region $X$ on the multipartite state $|\Psi^{(N)}\rangle$ is defined, according to the no-label approach, as

$$O^{(1)}_X |\Psi^{(N)}\rangle := \sum_i |(X|\psi_i\rangle| \psi_1, \ldots, O^{(1)}_X \psi_i, \ldots, \psi_N\rangle,$$

where $|X\rangle$ denotes a generic state of a particle spatially localized on $X$, and the presence of at least one such constituent is assumed [8]. Remarkably, the operational necessity of focusing on a specific region of space rather than on individual particles is reflected, in equation (2.11), by the sum being weighted by the probability amplitudes associated with each particle being in the region $X$. Note that, when the region $X$ is wide enough to enclose the whole spatial distribution of $|\Psi^{(N)}\rangle$, equation (2.11) reduces to

$$O^{(1)}_X |\Psi^{(N)}\rangle := \sum_i |\psi_1, \ldots, O^{(1)}_X \psi_i, \ldots, \psi_N\rangle,$$

which is the usual single particle operator acting on a state of $N$ identical particles.

In this section, we have briefly reviewed the no-label approach, an alternative formalism to deal with identical particles which avoid resorting on unphysical labels. We have shown its main features, with particular attention to the main advantages it provides over the standard, label-based, approach. Within this formalism, we have introduced the sLOCC framework as an operational method to deal with indistinguishable particles, postponing a more detailed discussion to §5. Finally, we remark the spreading which the no-label approach is undergoing to, having been used in different works such as, but not limited to, references [8,26,30–34].

### 3. Deformations

In this section, we discuss and formalize the concept of deformation, a tool of particular importance when applied to systems of identical particles.

In contrast to global unitary transformations where all the elements of a multipartite state are modified in the same way, deformations consist in transformations acting differently, but still...
unitarily, on each particle, thus changing the relative relations among the constituents. Given an $N$-partite state $|\Psi^{(N)}\rangle = |\psi_1, \psi_2, \ldots, \psi_N\rangle$ of either distinguishable or indistinguishable particles, the action of the deformation $D_{a,X}^{(N)}|\Psi^{(N)}\rangle$ is defined, within the no-label approach, as

$$D_{a,X}^{(N)}|\Psi^{(N)}\rangle := \left( U_{a_1,X_1}^{(1)} \otimes U_{a_2,X_2}^{(1)} \otimes \cdots \otimes U_{a_N,X_N}^{(1)} \right) |\Psi^{(N)}\rangle = \sum_\alpha \langle X_1 | \psi_{\alpha_1} (X_2 | \psi_{\alpha_2} \cdots (X_N | \psi_{\alpha_N}) | \times \eta^{P_\alpha} | U_{a_1,X_1}^{(1)} \psi_{\alpha_1}, U_{a_2,X_2}^{(1)} \psi_{\alpha_2}, \ldots, U_{a_N,X_N}^{(1)} \psi_{\alpha_N} \rangle. $$

(3.1)

Here, the elements $a_j$ in $a = (a_1, a_2, \ldots, a_N)$ identify the type of transformation represented by the single particle unitary operator $U_{a_j,X_j}^{(1)}$, and encode the set of parameters required to determine it, while $X_j \in X = (X_1, X_2, \ldots, X_N)$ denotes its region of action. $\alpha$ and $P_\alpha$ are as in equation (2.7). In general, for a deformation $a_j \neq a_i$ for $j \neq i$, equation (3.1) holds when each operator acts on at least one particle, i.e. $\exists \alpha : \forall i \exists j : (X_i | \psi_{\alpha_j}) \neq 0$.

The probability amplitudes weighting the sum in equation (3.1) account, as in (2.11), for the spatially localized approach required when the constituents are indistinguishable. When they are distinguishable, either being identical or non-identical, we can individually address each of them within the traditional LOCC framework and drop the subscript $X$, so that equation (3.1) becomes

$$D_{a}^{(N)} |\Psi^{(N)}\rangle = |U_{a_1}^{(1)} \psi_1, U_{a_2}^{(1)} \psi_2, \ldots, U_{a_N}^{(1)} \psi_N\rangle. $$

(3.2)

Moreover, deformations are unitary when dealing with non-identical particles. Indeed, in this case, we are sure that the constituents are left distinguishable by the deformation. Thus, the right-hand side of equation (3.2) reduces in this case to a tensor product, namely

$$D_{a}^{(N)} |\psi^{(N)}\rangle = |U_{a_1}^{(1)} \psi_1 \otimes |U_{a_2}^{(1)} \psi_2 \otimes \cdots \otimes |U_{a_N}^{(1)} \psi_N\rangle. $$

(3.3)

Hence, one has

$$\langle D_{a}^{(N)}|\psi^{(N)}\rangle|D_{a}^{(N)}|\psi^{(N)}\rangle = \langle U_{a_1}^{(1)} \psi_1 | U_{a_1}^{(1)} \psi_1 \rangle \langle U_{a_2}^{(1)} \psi_2 | U_{a_2}^{(1)} \psi_2 \rangle \cdots \langle U_{a_N}^{(1)} \psi_N | U_{a_N}^{(1)} \psi_N \rangle

= \langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle \cdots \langle \psi_N | \psi_N \rangle,$$

which implies $\langle \Psi^{(N)} | [D_{a}^{(N)}]^\dagger D_{a}^{(N)} | \Psi^{(N)} \rangle = \langle \Psi^{(N)} | |\Psi^{(N)}\rangle$, finally leading to

$$[D_{a}^{(N)}] \dagger D_{a}^{(N)} = I. $$

(3.4)

Remarkably, this is in general not true any more for identical constituents, not even when initially distinguishable. From the physical point of view, this is so because the deformation can change the relative spatial overlap of particles, thus leading to the emergence of indistinguishability manifested in the cross-inner products appearing in the right-hand side of equation (2.7). In order to explicitly show this, let us consider the scenario of $N = 2$ distinguishable but identical particles. Before applying the deformation, from equation (2.10), we have

$$\langle \Psi^{(2)} | \Psi^{(2)} \rangle = \langle \psi_1, \psi_2 | \psi_1, \psi_2 \rangle = \langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle.$$

After the deformation, instead, from equation (2.7), it holds that

$$\langle D_{a}^{(2)}|\Psi^{(2)}\rangle|D_{a}^{(2)}|\Psi^{(2)}\rangle = \langle U_{a_1}^{(1)} \psi_1, U_{a_2}^{(1)} \psi_2 | U_{a_1}^{(1)} \psi_1, U_{a_2}^{(1)} \psi_2 \rangle

= \langle U_{a_1}^{(1)} \psi_1 | U_{a_1}^{(1)} \psi_1 \rangle \langle U_{a_2}^{(1)} \psi_2 | U_{a_2}^{(1)} \psi_2 \rangle + \eta \langle U_{a_1}^{(1)} \psi_1 | U_{a_2}^{(1)} \psi_2 \rangle^2

= \langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle + \eta \langle [U_{a_1}^{(1)}]^\dagger U_{a_2}^{(1)} | \psi_2 \rangle^2.$$

Since, in general, $[U_{a_1}^{(1)}]^\dagger U_{a_2}^{(1)} \neq 0$, it follows that

$$\langle \Psi^{(2)} | [D_{a}^{(2)}] \dagger D_{a}^{(2)} | \Psi^{(2)} \rangle \neq \langle \Psi^{(2)} | \Psi^{(2)} \rangle \Rightarrow [D_{a}^{(2)}] \dagger D_{a}^{(2)} \neq I. $$

(3.5)
Figure 2. (a,b) Example of non-unitary deformation of three identical and initially distinguishable particles. The particle localized in region $X_1$ undergoes a spin rotation, while the ones in regions $X_2$ and $X_3$ get spatially overlapped over region $X_4$, where spatial indistinguishability is generated. (c,d) Example of unitary deformation of three identical and distinguishable particles. The particle localized in region $X_1$ undergoes a spin rotation, the one in region $X_2$ sees a unitary restriction of its wave function support to a region $X_2' \subset X_2$, while the particle in region $X_3$ gets spatially translated to $X_4$. No indistinguishability is generated by the process.

We thus conclude that deformations are unitary when applied to non-identical particles and, in general, non-unitary for identical ones. The latter situation is schematically represented in figure 2a,b, where we depict an example of deformation acting on three identical, nonetheless distinguishable, particles leading to the generation of spatial indistinguishability, thus being non-unitary. In figure 2c,d, instead, we report a pictorial representation of the particular scenario where three identical, distinguishable particles are manipulated via a deformation which does not generate indistinguishability, thus retaining unitarity.

Clearly, when normalization is important, states of indistinguishable particles obtained by a deformation can be straightforwardly normalized: given a system of $N$ identical particles in a general mixed state $\rho$, the normalized state after the deformation is

$$\rho_N = \frac{D\rho D^\dagger}{\text{Tr}[D^\dagger D\rho]}, \quad (3.6)$$

where we omit superscripts and subscripts of the deformation operator for simplicity.

In this section, we have introduced and mathematically formalized deformations, namely transformations consisting in a set of unitary operations of particular relevance when dealing with identical particles. We have highlighted their crucial role as operational tools to generate indistinguishability among initially distinguishable, identical constituents, and their suitability to deal with them afterwards. Particular attention has been paid to their general non-unitarity, linking it to their physical interpretation and to the scenario of application.

4. Entropic measure of indistinguishability

As we shall discuss in the next section, spatial indistinguishability provides an important quantum resource which can be accessed within the sLOCC operational framework for
different goals. Within this picture, deformations generating indistinguishability from previously distinguishable constituents provide the key tool to activate such resources. In order to quantitatively demonstrate this, we need a way to quantify indistinguishability. To this aim, we now introduce an entropic measure of generalized indistinguishability, of which spatial indistinguishability is derived as a particular case.

Let us consider an elementary N-particle state |ψ(N)⟩ = |ψ₁, ψ₂, . . . , ψ_N⟩; here, each ψ_j encodes both the single particle spatial wave function χ_j and all the other relevant degrees of freedom given by the eigenvalues of a complete set of commuting observables and gathered in a vector σ_j, so that |ψ_j⟩ = |χ_jσ_j⟩. We now identify N regions of space S₁, S₂, . . . , S_N where we set N single particle detectors, corresponding to the spatial modes |S₁⟩, |S₂⟩, . . . , |S_N⟩. Since they represent detection regions, we require such spatial modes to be non-overlapping. Nonetheless, we allow for two or more detectors to be same, that is |S_i⟩ = |S_j⟩ for any 1 ≤ i, j ≤ N. Typically, the detectors will be sensible to the spatial position of particles (by construction) and to a subset α of the degrees of freedom encoded in σ, while being unable to detect the remaining β (where σ_j = α_j ∪ β_j ∀ j = 1, . . . , N). For example, each detector could be capable of detecting the energy of a particle impinging on the spatial region where it is set, without having access to its spin. A single particle detection performed in the region S_k giving as outcomes the set of values α_k is thus described by the projection operator P_k^{(1)} = | ⟨S_k⟩ α_k β_k⟩ ⟨S_k⟩ α_k β_k |, while the probability of such an outcome when detecting a particle whose state is |ψ_j⟩ = |χ_jα_j β_j⟩ is given by

\[ P_{k,j} = ⟨ψ_j| P_k^{(1)} |ψ_j⟩ = \sum_β |⟨S_k⟩ α_k β_j⟩|^2 = |⟨S_k⟩ α_j | β_j⟩|^2. \] (4.1)

A global simultaneous detection of the multipartite state giving as outcomes α₁ for the particle in the region S₁, α₂ for the one in S₂, and so on, is described by the action of the N-particle projection operator

\[ P_{\{S_k, α_k\}}^{(N)} = \bigotimes_{k=1}^N P_k^{(1)}. \] (4.2)

We now introduce the joint probability related to the projective measurement in equation (4.2) of detecting in the region S₁ with degrees of freedom α₁ the particle whose state is |ψ_{j₁}⟩, in the region S₂ with degrees of freedom α₂ the one whose state is |ψ_{j₂}⟩, and so on, that is

\[ P_{\{S_k, α_k\}}^{j_1, . . . , j_N} = \prod_{k=1}^N P_{j_k, k}. \] (4.3)

With respect to the projective measurement in equation (4.2), we define the degree of indistinguishability of the N-particle state as

\[ I_{\{S_k, α_k\}} = - \sum_{j_1, . . . , j_N=1}^{N \atop \forall i \neq j} \frac{P_{\{S_k, α_k\}}^{j_1, . . . , j_N}}{Z} \log_2 \frac{P_{\{S_k, α_k\}}^{j_1, . . . , j_N}}{Z}, \] (4.4)

where we have indicated the partition function

\[ Z = \sum_{j_1, . . . , j_N=1}^{N \atop \forall i \neq j} P_{\{S_k, α_k\}}^{j_1, . . . , j_N}. \] (4.5)

When all the particles are spatially separated, there is at most only one non-null joint probability contributing to equation (4.4). In particular, if they are perfectly localized on one region each and the values of their accessible degrees of freedom are \{α_k\}_{k=1}^N, such a probability is equal to 1 and I reaches its minimum I_{\{S_k, α_k\}} = 0: particles are perfectly distinguishable with respect to the measurement given by equation (4.2). On the contrary, if all the constituents are equally distributed over all the N spatial regions and possess the same values α₁ = α₂ = . . . = α_N, then all
the joint probabilities contribute equally to equation (4.4): we have maximally indistinguishable particles and \( I \) takes its maximum value \( I_{\{S_\alpha \alpha \}} = \log_2 N! \).

In what follows, we shall be interested in the scenario where the detectors are only sensible to the spatial degree of freedom. This situation is derived from the above described picture by setting \( \alpha = \emptyset \), so that equation (4.4) reduces to a measure of the degree of spatial indistinguishability, as described in the next section.

In this section, we have generalized the notion of spatial indistinguishability introduced in [12]. As we will discuss in the next section, such a quantity allows to probe and further disclose the role of identical particles indistinguishability within quantum technology protocols involving the sLOCC operational framework.

5. Accessing quantum indistinguishability resources: the sLOCC operational framework

As discussed in §2, indistinguishable particles cannot be addressed with the traditional LOCC framework, since this relies on the possibility to individually manipulate and thus distinguish the single constituents. From an operational point of view, we thus resort to the sLOCC framework to access the quantum properties of an indistinguishable particles state [27,29].

(a) Presentation of the operational framework

For simplicity, we present the sLOCC framework within the simple scenario of two identical qubits with opposite pseudospin, initially distinguishable and localized in the distinct spatial regions \( A \) and \( B \). Following the original formulation [27], we take the bipartite system to be in the initial elementary state \( |\psi\rangle_{AB} = |A \uparrow, B \downarrow \rangle \). Note that \( |\psi\rangle_{AB} \) is normalized, since \( \langle A|B \rangle = 0 \). Applying the notions introduced in §3, we proceed by deforming such a state to make the two single particle wave functions spatially overlap over two distinct regions \( L \) and \( R \) corresponding to the normalized spatial modes \( |L\rangle, |R\rangle \). This amount to performing the transformation

\[
|\psi\rangle_{AB} \rightarrow |\psi\rangle_{D} = |\psi_{1} \uparrow, \psi_{2} \downarrow \rangle , \tag{5.1}
\]

where \( |\psi_{1}\rangle = l |L\rangle + r |R\rangle \) and \( |\psi_{2}\rangle = l' |L\rangle + r' |R\rangle \). Here, the complex coefficients \( l, l', r, r' \) determine the different probabilities of finding each particle in each region and satisfy the relation \( |l|^2 + |r|^2 = |l'|^2 + |r'|^2 = 1 \). Following what discussed in §1, we highlight that, despite being spatially indistinguishable, the two qubits in state \( |\psi\rangle_{D} \) can still be discriminated by a device capable of detecting their spin direction, which has been left unchanged by the deformation. Finally, the deformation has left the state normalized: indeed, it holds that

\[
D(\langle \psi | \psi \rangle_{D}) = \left( \langle \psi_{1} | \psi_{1} \rangle \langle \uparrow | \uparrow \rangle \right) \left( \langle \psi_{2} | \psi_{2} \rangle \langle \downarrow | \downarrow \rangle \right) + \eta \langle \psi_{1} | \psi_{2} \rangle \langle \uparrow | \downarrow \rangle |^2 = 1 . \tag{5.2}
\]

We now set two single particle detectors on \( L \) and \( R \), respectively, and perform a coincidence measurement, preserving the state if both of them detect a particle and discarding it otherwise. Crucially, the detectors are unable to access the spin direction, so that the two qubits are effectively indistinguishable to their eyes. Thus, this part of the process amounts to a postselected measurement where state \( |\psi\rangle_{D} \) is projected on the subspace spanned by the basis

\[
B_{LR} = \{ |L \uparrow, R \uparrow \rangle, |L \uparrow, R \downarrow \rangle, |L \downarrow, R \uparrow \rangle, |L \downarrow, R \downarrow \rangle \} \tag{5.2}
\]

via the corresponding projection operator

\[
\Pi_{LR} = \sum_{\sigma, \tau = \uparrow, \downarrow} |\sigma \rangle \langle R \tau | \langle L \sigma | \tau \rangle . \tag{5.3}
\]

After the proper normalization, the resulting state is given by

\[
|\psi\rangle_{LR} = \frac{\Pi_{LR} |\psi\rangle_{D}}{\sqrt{D(\langle \psi | \Pi_{LR} | \psi \rangle_{D})}} = \frac{l' |L \uparrow, R \downarrow \rangle + \eta l r |L \downarrow, R \uparrow \rangle}{\sqrt{|l'|^2 + |l r|^2}} , \tag{5.4}
\]
postselected with probability

\[ P_{LR} = D(\Psi | \Pi_{LR} | \Psi) = |l'|^2 + |l|^2. \]  

(5.5)

Note that the two qubits in the final state \( |\Psi_{LR}\rangle \) of equation (5.4) are distinguishable, since one of them is now localized in region \( L \) while the other in region \( R \).

(b) Analysis and possible applications

The first aspect that emerges from equation (5.4) is that the final state \( |\Psi_{LR}\rangle \) is an entangled state, provided \( l, l', r, r' \neq 0 \). Since the initial state was non-entangled, we thus conclude that the sLOCC protocol can be used to generate entanglement [27,29]. Remarkably, the superposition of states \( |L \uparrow, R \downarrow\rangle \) and \( |L \downarrow, R \uparrow\rangle \) is a direct consequence of the impossibility for the two detectors to understand which one of the two qubits they have detected, namely if the one with spin \( \uparrow \) generated in \( A \) or the one with spin \( \downarrow \) generated in \( B \). In other words, the origin of the quantum correlations in the sLOCC-generated state of equation (5.4) is the no-which-way information discussed in §1 deriving from the achieved spatial indistinguishability. For this reason, we say that deformations leading to indistinguishability activate entanglement, while the sLOCC measurement allows to access it. To further stress this point, we remark that \( |\Psi_{LR}\rangle \) is non-entangled whenever at least one among \( l, l', r, r' \) is null; indeed, this amounts to the scenario where (at least) one of the qubits is perfectly localized either on \( L \) or on \( R \), so that the coincidence click required by the sLOCC measurement allows to precisely track the origin of both the particles. This is the situation occurring, e.g. when no deformation is performed, so that \( l = l' = 1 \) and \( r = r' = 0 \): particles remain distinguishable and no entanglement is generated.

From equation (4.4) with \( \alpha = \{\theta\} \), \( N = 2 \), and \( S_1 = L, S_2 = R \), the amount of spatial indistinguishability obtained with the deformation can be properly quantified by the entropic measure introduced in §4 [12]

\[ I_{LR} = -\frac{|l|^2 |r|^2}{Z} \log_2 \frac{|l|^2 |r|^2}{Z} - \frac{|l'|^2 |r'|^2}{Z} \log_2 \frac{|l'|^2 |r'|^2}{Z}, \]  

(5.6)

where \( Z = |l|^2 |r|^2 + |l'|^2 |r'|^2 \). Such a quantity takes into account the no-which-way information, taking the minimum value \( I = 0 \) when no overlap is present \( (l = 1, r = 1 \) or \( l' = 1, r = 1 \): distinguishable particles) and the maximum one \( I = 1 \) when the overlap is maximum \( (l = l' = r = r' = 1/\sqrt{2} \) maximally indistinguishable particles).

The role of indistinguishability as a resource for quantum technologies within the sLOCC framework has been investigated by several recent experiments. Remarkably, in [29], the authors have experimentally implemented the deformation+sLOCC protocol with two photons initially prepared in the state \( |\Psi_{AB}\rangle \). They have performed quantum teleportation with the final state of equation (5.4), thus showing that the achieved entanglement is physical. Furthermore, by directly accessing the value of \( l, l', r, r' \) they fixed \( l = r = 1/\sqrt{2} \) to make \( I \) a function of just one parameter and showed that the amount of quantum correlations present in the state produced by the sLOCC protocol, as quantified by the entanglement of formation [35], is proportional to the degree of spatial indistinguishability achieved. In particular, when \( Z = 1 \) we see from equation (5.4) that the sLOCC process generates the maximally entangled state \( |\Psi_{LR}^{\text{max}}\rangle = (|L \uparrow, R \downarrow\rangle + \eta |L \downarrow, R \uparrow\rangle)/\sqrt{2} \).

In [8–10,12,36], the authors considered the more realistic scenario where the deformation+sLOCC protocol is applied to two qubits in different scenarios involving local noisy environments. In the analysed situations, which involve non-elementary states, the authors have shown that the process can be employed to prepare entangled states. Thus, the sLOCC operational framework can be used to achieve quantum correlations in a way which is robust under the detrimental action of local noise.

Another relevant element emerging from the sLOCC-prepared state, as can be noticed from equation (5.4), is the factor \( \eta := e^{i\theta} \) encoding the exchange phase \( \theta \), with \( \theta = 0 \) for bosons and \( \theta = \pi \) for fermions being at the core of the symmetrization postulate discussed in §2. Although many decades have passed after the first formulation of the postulate, a first direct experimental measurement of the bosonic exchange phase has been only recently achieved with two photons.
in an all-optical set up [37,38]. This is mainly due to the difficulty in designing a set up manually generating a superposition between a reference state and its physically permuted one, from which later extrapolating the relative exchange phase via interferometry. Thanks to its reliance on spatial indistinguishability, the sLOCC process allows to avoid such a difficulty by letting \( \theta \) naturally emerge. Exploiting this effect, in [39], the authors designed and experimentally implemented a photonic set up capable of directly measuring the exchange phase of two real bosons and of simulated fermions and anyons by applying interferometry to the sLOCC-produced state of equation (5.4). Remarkably, the introduced theoretical set up is general and could be suitably adapted to directly measure the exchange phase of even real fermions and anyons.

Finally, spatial indistinguishability of identical particles undergoing the sLOCC measurement has been shown to provide a useful resource of quantum coherence yielding an advantage in quantum metrology [40,41], whereas the endurance of quantum coherence within systems of indistinguishable particles in non-dissipative noisy quantum networks was demonstrated in [42].

It is interesting to highlight the connection between the deformation + sLOCC operational framework and the entanglement extraction protocol [25]. In the latter, a single-mode state of indistinguishable particles is split over distinct modes. The resulting particle number distribution is then measured along such modes, postselecting only those states which respect a desired partition. Being the resulting modes distinct, this allows to access the entanglement between groups of identical particles whose accessibility was previously ruled out by their single-mode indistinguishability. In relation to this framework, the mode splitting operation is a particular case of deformation acting on already indistinguishable particles. Furthermore, deformations such as mode merging operations can be seen themselves as the preparation step required to achieve the entanglement extraction single-mode starting point. Furthermore, the particles distribution postselected measurement and the sLOCC projection are clearly related, since they both make quantum correlations accessible by making an indistinguishable state distinguishable. Nonetheless, while entanglement extraction focuses on the splitting of an already indistinguishable state to show that quantum correlations inaccessible within identical systems are actually physically meaningful and constitute useful resources in their own right [18,25], the sLOCC process presents itself as an alternative operational framework where indistinguishability is generated over previously arranged detection regions with the goal of generating, restoring, and/or manipulating entanglement in actual practical applications.

In this section, we have discussed the sLOCC operational framework suitable to deal with indistinguishable particles. In contrast to the localized operations and classical communication approach traditionally employed when dealing with distinguishable constituents, in the sLOCC framework different operations are localized in space rather than on specific particles, whose indistinguishability makes individually unaddressable. We have briefly commented on the main works reporting possible practical applications of deformations and the sLOCC protocol, where spatial indistinguishability of identical constituents is shown to provide an important resource to achieve quantum information tasks such as entanglement generation, entanglement restoration, coherence generation and the direct measurement of particles’ exchange phase. Finally, we have briefly compared the sLOCC operational framework with the entanglement extraction protocol.

6. Conclusion

In conclusion, we have discussed and elucidated the distinction between the concepts of particle identity and particle indistinguishability in quantum mechanics. We have presented a concise review of the no-label approach as a suitable tool to deal with indistinguishable constituents, as introduced in [4] and further deepened in [8,11]. We have introduced a coherent formalization of deformations acting on either distinguishable or indistinguishable multipartite states, providing an extension of the indistinguishability entropic measure introduced in [8] to the general \( N \)-partite scenario. We have highlighted the relevance of deformations as operations exploitable to activate quantum correlations to be later accessed within the sLOCC operational framework. Finally, we
have briefly discussed the relations between the sLOCC protocol and the entanglement extraction one as operational frameworks.

Given the results presented in this work, we believe that deformations, together with the sLOCC operational framework, have the potential to become a useful technique for many real-world applications exploiting quantum technologies. Indeed, identical particles constitute the main building blocks of platforms such as quantum networks, quantum computers and quantum measurement systems. For instance, spatial indistinguishability of identical constituents generated by properly tuned deformations could be exploited to shield from noise the fundamental quantum correlation properties required for quantum cryptographic protocols, or the coherence of qubits used to run quantum algorithms. Furthermore, the entanglement-restoration characteristics of the presented techniques could be further investigated to preserve the super-sensitivity of states carrying information in quantum sensing and metrology protocols.

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References

1. Cohen-Tannoudji C, Diu B, Laloe F, Dui B. 2006 Quantum mechanics (2 vol. set). New York, NY: Wiley-Interscience.
2. Ghirardi G, Marinatto L, Weber T. 2002 Entanglement and properties of composite quantum systems: a conceptual and mathematical analysis. J. Stat. Phys. 108, 49–122. (doi:10.1023/A:1015439502289)
3. Tichy MC, de Melo F, Kus M, Mintert F, Buchleitner A. 2013 Entanglement of identical particles and the detection process. Fortschr. Phys. 61, 225–237. (doi:10.1002/prop.201200079)
4. Lo Franco R, Compagno G. 2016 Quantum entanglement of identical particles by standard information-theoretic notions. Sci. Rep. 6, 20603. (doi:10.1038/srep20603)
5. Herbut F, Vujicic M. 1987 Irrelevance of the Pauli principle in distant correlations between identical fermions. J. Phys. A. Math. Gen. 20, 5555–5563. (doi:10.1088/0305-4470/20/16/030)
6. Tichy MC, Mintert F, Buchleitner A. 2011 Essential entanglement for atomic and molecular physics. J. Phys. B: At. Mol. Opt. Phys. 44, 192001. (doi:10.1088/0953-4075/44/19/192001)
7. Peres A. 2002 Quantum theory: concepts and methods. New York, NY: Springer.
8. Nosrati F, Castellini A, Compagno G, Lo Franco R. 2020 Dynamics of spatially indistinguishable particles and quantum entanglement protection. Phys. Rev. A 102, 062429. (doi:10.1103/PhysRevA.102.062429)
9. Piccolini M, Nosrati F, Compagno G, Livreri P, Morandotti R, Lo Franco R. 2021 Entanglement robustness via spatial deformation of identical particle wave functions. Entropy 23, 708. (doi:10.3390/e23060708)
10. Piccolini M, Nosrati F, Morandotti R, Lo Franco R. 2022 Indistinguishability-enhanced entanglement recovery by spatially localized operations and classical communication. Open Sys. Info. Dyn. 28, 2150020. (doi:10.1142/S1230161221500207)
11. Compagno G, Castellini A, Lo Franco R. 2018 Dealing with indistinguishable particles and their entanglement. Phil. Trans. R. Soc. A 376, 20170317. (doi:10.1098/rsta.2017.0317)
12. Nosrati F, Castellini A, Compagno G, Lo Franco R. 2020 Robust entanglement preparation against noise by controlling spatial indistinguishability. *npj Quant. Inf.* 6, 1–7. (doi:10.1038/s41534-020-0271-7)

13. Ghirardi G, Marinatto L. 2004 General criterion for the entanglement of two indistinguishable particles. *Phys. Rev. A* 70, 012109. (doi:10.1103/PhysRevA.70.012109)

14. Li YS, Zeng B, Liu XS, Long GL. 2001 Entanglement in a two-identical-particle system. *Phys. Rev. A* 64, 054302. (doi:10.1103/PhysRevA.64.054302)

15. Paskauskas R, You L. 2001 Quantum correlations in two-boson wave functions. *Phys. Rev. A* 64, 042310. (doi:10.1103/PhysRevA.64.042310)

16. Schliemann J, Cirac JI, Kus M, Lewenstein M, Loss D. 2001 Quantum correlations in two-fermion systems. *Phys. Rev. A* 64, 022303. (doi:10.1103/PhysRevA.64.022303)

17. Zanardi P. 2002 Quantum entanglement in fermionic lattices. *Phys. Rev. A* 65, 042101. (doi:10.1103/PhysRevA.65.042101)

18. Morris B, Yadin B, Fadel M, Zibold T, Treutlein P, Adesso G. 2020 Entanglement between identical particles is a useful and consistent resource. *Phys. Rev. X* 10, 041012. (doi:10.1103/PhysRevX.10.041012)

19. Eckert K, Schliemann J, Bruss D, Lewenstein M. 2002 Quantum correlations in systems of indistinguishable particles. *Ann. Phys.* 299, 88–127. (doi:10.1006/aphy.2002.6268)

20. Balachandran AP, Govindarajan TR, de Queiroz AR, Reyes-Lega AF. 2013 Entanglement and particle identity: a unifying approach. *Phys. Rev. Lett.* 110, 080503. (doi:10.1103/PhysRevLett.110.080503)

21. Cunden FD, Di Martino S, Facchi P, Florio G. 2014 Spatial separation and entanglement of identical particles. *Int. J. Quantum Inform.* 12, 1461001. (doi:10.1142/S0219749914610012)

22. Sasaki T, Ichikawa T, Tsutsui I. 2011 Entanglement of indistinguishable particles. *Phys. Rev. A* 83, 012113. (doi:10.1103/PhysRevA.83.012113)

23. Bose S, Home D. 2002 Generic entanglement generation, quantum statistics, and complementarity. *Phys. Rev. Lett.* 88, 050401. (doi:10.1103/PhysRevLett.88.050401)

24. Bose S, Home D. 2013 Duality in entanglement enabling a test of quantum indistinguishability unaffected by interactions. *Phys. Rev. Lett.* 110, 140404. (doi:10.1103/PhysRevLett.110.140404)

25. Killoran N, Cramer M, Plenio MB. 2014 Extracting entanglement from identical particles. *Phys. Rev. Lett.* 112, 150501. (doi:10.1103/PhysRevLett.112.150501)

26. Sciara S, Lo Franco R, Compagno G. 2017 Universality of Schmidt decomposition and particle identity. *Sci. Rep.* 7, 44675. (doi:10.1038/srep44675)

27. Lo Franco R, Compagno G. 2018 Indistinguishability of elementary systems as a resource for quantum information processing. *Phys. Rev. Lett.* 120, 240403. (doi:10.1103/PhysRevLett.120.240403)

28. Horodecki R, Horodecki P, Horodecki M, Horodecki K. 2009 Quantum entanglement. *Rev. Mod. Phys.* 81, 865–942. (doi:10.1103/RevModPhys.81.865)

29. Sun K *et al.* 2020 Experimental quantum entanglement and teleportation by tuning remote spatial indistinguishability of independent photons. *Opt. Lett.* 45, 6410–6413. (doi:10.1364/OL.401735)

30. Lourenço AC, Debarba T, Duzzioni EL. 2019 Entanglement of indistinguishable particles: a comparative study. *Phys. Rev. A* 99, 012341. (doi:10.1103/PhysRevA.99.012341)

31. Chin S, Huh J. 2019 Entanglement of identical particles and coherence in the first quantization language. *Phys. Rev. A* 99, 052345. (doi:10.1103/PhysRevA.99.052345)

32. Chin S, Huh J. 2019 Reduced density matrix of identical particles from three aspects: the first quantization, exterior products, and GNS representation. Preprint. (https://arxiv.org/abs/1906.00542)

33. Qureshi T, Rizwan U. 2017 Hanbury Brown–Twiss effect with wave packets. *Quanta* 6, 61–69. (doi:10.12743/quanta.v6i6.66)

34. Mani H, Ramadas N, Sreedhar V. 2020 Quantum entanglement in one-dimensional anyons. *Phys. Rev. A* 101, 022314. (doi:10.1103/PhysRevA.101.022314)

35. Wootters WK. 1998 Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.* 80, 2245–2248. (doi:10.1103/PhysRevLett.80.2245)

36. Piccolini M, Giovannetti V, Lo Franco R. 2023 Asymptotically-deterministic robust preparation of maximally entangled bosonic states. Preprint. (https://arxiv.org/abs/2303.11484)
37. Tschernig K, Müller C, Smoor M, Kroh T, Wolters J, Benson O, Busch K, Pérez-Leija A. 2021 Direct observation of the particle exchange phase of photons. *Nat. Photonics* **15**, 671–675. (doi:10.1038/s41566-021-00818-7)

38. Lo Franco R. 2021 Directly proving the Bosonic nature of photons. *Nat. Photonics* **15**, 638–639. (doi:10.1038/s41566-021-00867-y)

39. Wang Y et al. 2022 Proof-of-principle direct measurement of particle statistical phase. *Phys. Rev. Appl.* **18**, 064024. (doi:10.1103/PhysRevApplied.18.064024)

40. Castellini A, Lo Franco R, Lami L, Winter A, Adesso G, Compagno G. 2019 Indistinguishability-enabled coherence for quantum metrology. *Phys. Rev. A* **100**, 012308. (doi:10.1103/PhysRevA.100.012308)

41. Sun K et al. 2022 Activation of indistinguishability-based quantum coherence for enhanced metrological applications with particle statistics imprint. *Proc. Natl Acad. Sci. USA* **119**, e2119765119. (doi:10.1073/pnas.2119765119)

42. Perez-Leija A, Guzmán-Silva D, León-Montiel RDJ, Gräfe M, Heinrich M, Moya-Cessa H, Busch K, Szameit A. 2018 Endurance of quantum coherence due to particle indistinguishability in noisy quantum networks. *npj Quant. Inf.* **4**, 45. (doi:10.1038/s41534-018-0094-y)