Abstract

We show that any covariant scattering amplitude of the $W_3$ string will contain, as part of its integrand, a factor that obeys the differential equations satisfied by an Ising model correlation function. This factor can thus be identified with such a correlation function, in agreement with a previous result of the authors. The $W_3$ string is also shown to contain an $N = 2$ parafermion theory, and hence to contain in addition the non-linear infinite-dimensional $W$-algebra corresponding to this parafermion theory. The physical states form a representation of this algebra.
Introduction

One of the most remarkable features to emerge from the development of $W_N$ strings is their relation to the unitary minimal models. The first suggestion that such a relation existed arose from the observation of a numerical coincidence [1,2], namely that the central charge of a subset of the ghost and matter fields of a $W_N$ string was the same as that of a unitary minimal model. The simplest example of this is given by the $W_3$ string, which is related to the Ising model. This string can be constructed from scalar fields $\phi$ and $X^\mu$, with $\mu = 0, 1, \ldots, D - 1$, together with reparametrization ghosts $b, c$ and ghosts $d, e$ corresponding to $W_3$-transformations. It can then be seen that the fields $\phi, d$ and $e$ have a total central charge of $1/2$, which is precisely that of the Ising model. Furthermore, the intercepts associated with the $X^\mu$-sector of physical states of the $W_N$ string are related to the weights of primary fields of the corresponding minimal model [1,2].

More recently, direct connections between $W_N$ strings and the unitary minimal models have been discovered. The partition function of the $W_3$ string, which gives the count of states at each level, has been shown to involve the characters of the Ising model [3]. In addition, the tree-level scattering amplitudes for the $W_3$ string contain Ising model correlation functions, and satisfy the factorization and duality properties expected of a string theory [4].

In order to explain these connections in more detail, let us review some facts about $W_3$ strings. The physical states of the $W_3$ string theory are given by the cohomology of the BRST charge $Q$. It has been found [5] that, with the exception of certain discrete states, this cohomology can be obtained by the action of a picture-changing operator $P$ and a screening charge $S$ on three distinct states $|\psi_a\rangle$, with $a = 1, 15/16$ and $1/2$. For $a = 1$ and $15/16$, the states $|\psi_a\rangle$ have the form

$$|\psi_a\rangle = c e^{i\beta(a,0)\phi} V X(a) |0\rangle$$

$$\equiv V(a,0)|0\rangle,$$

(1)

while $|\psi_{1/2}\rangle$ has the form

$$|\psi_{1/2}\rangle = (\sqrt{522}e^{-i/\sqrt{522}}\partial e)^{i\beta(1/2,0)\phi} V X(1/2)|0\rangle$$

$$\equiv V(1/2,0)|0\rangle;$$

(2)
here $|0\rangle$ is the invariant vacuum state, $V^X(a)$ is a conformal operator of weight $a$ constructed from $X^\mu$ alone, and the momenta $\beta$ are given by

$$\beta(1, 0) = 8iQ/7, \quad \beta(15/16, 0) = iQ, \quad \beta(1/2, 0) = 4iQ/7.$$  (3)

The screening operator $S$, which is used to generate further states in the cohomology, is given by [5]

$$S = \oint dz \left( d - \frac{5i}{3\sqrt{58}} \partial b - \frac{2}{261} \partial b b e - \frac{41}{3\sqrt{58}} d b e \right) e^{i\beta_s \phi}$$  (4)

with $\beta_s = -2iQ/7$.

We have seen that, with the exception of the discrete states, the physical states are encoded in $V^X(a)$. The corresponding count of states has been shown [3] to be given by the character

$$\left\{ \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{D-2}} \right\} \chi_h(x),$$  (5)

where $\chi_h$ is the Ising model character for the states of weight $h = 1 - a$.

Following the general tree-level calculations of $W_3$ string scattering amplitudes given in [4], covariant approaches to $W_3$ string scattering have been considered. The first such attempt [6,7] involved calculations of correlation functions of only vertex operators and picture-changing operators, and led to results that disagreed with those of reference [4]. These amplitudes did not share the connection with Ising model correlation functions found in [4], and in addition they violated general principles of S-matrix theory and string theory. The use of vertex and picture-changing operators was a straightforward generalization of previous covariant string scattering calculations; in view of the expected role of $W$-moduli it is perhaps not surprising that this simple generalization is not adequate for $W$-strings.

In reference [5] a general covariant procedure for calculating $W_3$-scattering was given. This involved using not only the vertices $V(a, 0)$ and the picture changing operator $P$ but also the screening charge $S$ in the computation of scattering amplitudes, and led to amplitudes in agreement with those found earlier [4] and which satisfied the general principles expected. A number of specific amplitudes
were evaluated explicitly using this formalism in reference [5]. It is the purpose of this paper to show quite generally that all amplitudes calculated in this way contain correlation functions of the Ising model.

In reference [7] a field redefinition that leads to a considerable simplification of the physical states was given for the $W_3$ string. The redefined fields are given by

$$
\begin{align*}
\tilde{c} &= c + \frac{7\sqrt{58}i}{174} \partial e - \frac{8}{261} b \partial e e - \frac{4\sqrt{29}i}{87} \partial \phi e \\
\tilde{b} &= b \\
\tilde{e} &= e \\
\tilde{d} &= d + \frac{7\sqrt{58}i}{174} \partial b - \frac{8}{261} \partial b e e + \frac{4\sqrt{29}i}{87} \partial \phi e \\
\tilde{\phi} &= \phi - \frac{4\sqrt{29}i}{87} b e,
\end{align*}
$$

whereupon the BRST charge $Q$ [8] takes the form $Q = Q_0 + Q_1$, with

$$
Q_0 = \oint j_0, \quad Q_1 = \oint j_1,
$$

where

$$
\begin{align*}
j_0 &= \tilde{c} \left( T_X + T_{\tilde{\phi}} + T_{\tilde{d} \tilde{e}} + \frac{1}{2} T_{\tilde{b} \tilde{e}} \right) \\
j_1 &= -\frac{4\sqrt{29}i}{261} \tilde{c} \left( 2(\partial \tilde{\phi})^3 + 6Q \partial^2 \tilde{\phi} \partial \tilde{\phi} + \frac{19}{4} \partial^3 \tilde{\phi} + 9\partial \tilde{\phi} \partial \tilde{d} \partial \tilde{e} + 3Q \partial \tilde{d} \partial \tilde{e} \right).
\end{align*}
$$

In $j_0$ the stress tensors for the redefined fields are of precisely the same form as those for the original fields. $Q_0$ and $Q_1$ then satisfy the relations

$$
Q_0^2 = Q_1^2 = 0, \quad \{Q_0, Q_1\} = 0.
$$

In the remainder of this paper we shall work with the redefined fields, but for simplicity of notation we drop the tilde.

It is useful to note that there are two different types of ghost number, $N_{b,c}$ and $N_{d,e}$, corresponding to the ghosts $b, c$ and $d, e$, so that operators can be characterized by the pair $(N_{b,c}, N_{d,e})$. Thus the fields $b, c, d$ and $e$ have ghost numbers $(-1, 0)$, $(1, 0)$, $(0, -1)$ and $(0, 1)$ respectively. Then $Q_0$ and $Q_1$ have ghost numbers $(1, 0)$
and \((0, 1)\), so that if \(X\) is some operator that commutes with \(Q\) and which is an eigenstate of both \(N_{b,c}\) and \(N_{d,e}\), then \(X\) will commute with both \(Q_0\) and \(Q_1\).

In terms of the redefined fields, the basic vertices \(V(a, 0)\) are

\[
V(a, 0) = c \partial e e^{i\beta(a,0)\phi} V^X(a)
\]  

(11)

for \(a = 0, 15/16\), and

\[
V(1/2, 0) = c e^{i\beta(1/2,0)\phi} V^X(1/2).
\]  

(12)

Further, the screening charge takes the particularly simple form

\[
S = \oint dz d e^{i\beta_s\phi}.
\]  

(13)

Hence both \(Q_0\) and \(Q_1\) commute separately with the vertices \(V(a, 0)\) and with \(S\).

In general, however, the picture changing operator \(P\) does not commute separately with \(Q_0\) and \(Q_1\). The most general form of the picture changing operator is some linear combination of \([Q, \phi]\) and \([Q, X^\mu]\). An explicit calculation shows that while \(Q_0\) and \(Q_1\) each commute with \([Q, X^\mu]\), the same is not true of \([Q, \phi]\). For our purposes, however, the important point to note is that the only contribution of \(P\) to a scattering amplitude comes from the term \([Q_1, \phi]\). This can be seen by considering the ghost numbers of operators contributing to a scattering amplitude.

An \(N\) string amplitude will contain 3 unintegrated vertex operators \(V(a, 0)(z)\) together with \(N - 3\) integrated vertex operators \(\int dz \oint dw b(w)V(a, 0)(w)\). The total \(b, c\) ghost number of these operators is 3, which is precisely what is needed to get a non-zero correlation function. The scattering amplitude is obtained by including insertions of \(S\) and \(P\), in addition to the physical state vertex operators.

Since \(S\) has ghost number \((0, -1)\), and \(P\) consists of terms having ghost number either \((1, 0)\) or \((0, 1)\), it follows that the only terms in \(P\) that can contribute are those of ghost number \((0, 1)\). These terms are given precisely by \([Q_1, \phi]\), as stated above. It is then evident that \(Q_1\) commutes with this part of \(P\), and so \(Q_1\) commutes with all of the operators used in constructing a scattering amplitude for the \(W_3\) string. Furthermore, it can easily be seen that all correlation functions that arise in this way will factorize into two separate correlation functions, one involving the spacetime fields \(X^\mu\) and the ghosts \(b, c\), and the other involving \(\phi\).
and the ghosts $d, e$. Our aim is now to show that the correlation function of $\phi$ and $d, e$ is, in fact, equal to a correlation function of the Ising model.

At this point it is convenient to define some fields $\phi_h$ as follows:

$$
\phi_h = \begin{cases} 
\partial e e^{i\beta(1-h,0)}\phi, & \text{for } h = 0, 1/16 \\
e e^{i\beta(1/2,0)\phi} & \text{for } h = 1/2.
\end{cases}
$$

The vertices $V(a,0)$ are then given in terms of $\phi_{1-a}$ by

$$
V(a,0) = cV^X(a)\phi_{1-a}.
$$

The fields $\phi_h$ are constructed only from the scalar field $\phi$ and the ghost $e$. It is straightforward to show that, with respect to the stress tensor $T_\phi + T_{d,e}$ for these fields, $\phi_h$ is a primary field with conformal weight $h$. We have therefore constructed primary fields of weights $0$, $1/16$ and $1/2$ with respect to a stress tensor having $c = 1/2$, and we would like to show that the correlation functions of these fields coincide with those of the Ising model. The fact that the weights of these fields are the same as those of the Ising model is not in itself sufficient to show this; what is needed is to consider the representations of the Virasoro algebra constructed from these fields.

It is well-known that the minimal conformal field theory models are solvable as a consequence of the fact that there exist descendants of highest weight states that are themselves highest weight states. Such descendant states can consistently be set to zero, and doing this leads to differential equations that must be satisfied by correlation functions involving these primary fields. For the minimal models, these differential equations enable all correlation functions of primary fields to be calculated.

Let us consider in more detail the $(p,q)$ minimal model, which has a central charge $c = 1 - 6(p-q)^2/pq$ and a closed operator product algebra for a set of primary fields $\phi_{r,s}$ of weight $h_{r,s} = [(rp - sq)^2 - (p-q)^2]/4pq$, with $1 \leq r \leq q-1$ and $1 \leq s \leq p-1$. The primary field $\phi_{r,s}$ has a descendant that is a highest weight field at level $rs$. Since it is possible to write $\phi_{r,s} = \phi_{q-r,p-s}$, this field also has a highest weight descendant at level $(q-r)(p-s)$. This field therefore possesses two highest weight descendants, and these can be shown to be distinct in the sense that neither one can be expressed as a descendant of the other. Furthermore, all
other highest weight descendants in the Virasoro representation constructed from \( \phi_{r,s} \) can be written as descendants of these two fields.

The Ising model is obtained by taking \( p = 4, q = 3 \). The three primary fields of this model are \( \phi_{1,1} = \phi_{2,3} \), with weight 0, \( \phi_{2,1} = \phi_{1,3} \), with weight 1/2, and \( \phi_{1,2} = \phi_{2,2} \) with weight 1/16. Setting the highest weight descendants of these fields to zero then gives the conditions

\[
\hat{L}_{-1}\phi_{1,1} = 0
\]
\[
(\hat{L}_{-6} + \frac{22}{9}\hat{L}_{-4}\hat{L}_{-2} - \frac{31}{36}\hat{L}_{-3}^2 - \frac{16}{27}\hat{L}_{-2}^3)\phi_{1,1} = 0
\]
\[
(\hat{L}_{-2} - \frac{3}{4}\hat{L}_{-1})\phi_{2,1} = 0
\]
\[
(\hat{L}_{-3} - \frac{4}{5}\hat{L}_{-1}\hat{L}_{-2} + \frac{4}{15}\hat{L}_{-1}^3)\phi_{2,1} = 0
\]
\[
(\hat{L}_{-2} - \frac{4}{3}\hat{L}_{-1})\phi_{1,2} = 0
\]
\[
(\hat{L}_{-4} - \frac{122}{147}\hat{L}_{-1}\hat{L}_{-3} + \frac{50}{147}\hat{L}_{-1}^2\hat{L}_{-2} - \frac{4}{49}\hat{L}_{-1}^4 - \frac{1}{36}\hat{L}_{-2}^2)\phi_{1,2} = 0
\]

Here we have used the notation \( \hat{L}_n \phi(w) = \oint_w \frac{dz}{2\pi i} (z-w)^{n+1} T(z) \phi(w) \).

Given a particular realization of the fields of a \( c = 1/2 \) conformal field theory, it may or may not be the case that the above highest weight descendants actually vanish. Our method for showing the occurrence of Ising model correlation functions in \( W_3 \) string scattering amplitudes will be to consider such highest weight descendants that can be constructed from the vertex operators of the \( W_3 \) strings. These descendants do not vanish identically, but we will show that nevertheless they do give zero in correlation functions. This is sufficient to enable us to deduce the presence of Ising model correlation functions in \( W_3 \) string scattering.

Let us therefore consider the highest weight descendants of the fields \( \phi_h \) given earlier. The simplest possible example is \( \hat{L}_{-1}\phi_0 \), which is equal to \( \partial \phi_0 / \partial z \). Evidently this is not zero, but it is simple to show that it can be written as the commutator of \( Q_1 \) with some other operator. The obvious candidate for this operator is \( ee^{i\beta(1,0)} \phi \) and indeed we find

\[
\frac{\partial \phi_0}{\partial z} = \frac{3\sqrt{29}}{5\sqrt{2}} i \left\{ Q_1, ee^{i\beta(1,0)} \phi \right\}.
\]

We have checked, using Mathematica [9] and the OPEdefs package of Thielemans [10], that a similar picture holds for each of the other highest weight descendants
of the fields \( \phi_n \) given in eqn (16); each such highest weight descendant can be written as the commutator of \( Q_1 \) with some primary field. Thus \( \phi_{1/2} \) satisfies the conditions

\[
(\hat{L}_{-2} - \frac{3}{4} \hat{L}_{-1}^2) \phi_{1/2} = -\frac{3\sqrt{29}}{4\sqrt{2}} [Q_1, e^{-4Q/7\phi}]
\]

\[
(\hat{L}_{-3} - \frac{4}{5} \hat{L}_{-1} \hat{L}_{-2} + \frac{4}{15} \hat{L}_{-1}^3) \phi_{1/2} = -\frac{3\sqrt{29}}{10} [Q_1, \partial \phi e^{-4Q/7\phi} + \frac{1}{\sqrt{2}} de e^{-4Q/7\phi}]
\]

(18)

and \( \phi_{1/16} \) satisfies the conditions

\[
(\hat{L}_{-2} - \frac{4}{3} \hat{L}_{-1}^2) \phi_{1/16} = -\frac{\sqrt{29}}{5} \left\{ Q_1, \frac{7}{\sqrt{2}} \partial e e^{-Q\phi} - 8e \partial \phi e^{-Q\phi} \right\}
\]

(19)

and

\[
(\hat{L}_{-4} - \frac{122}{147} \hat{L}_{-1} \hat{L}_{-3} + \frac{50}{147} \hat{L}_{-1}^2 \hat{L}_{-2} - \frac{4}{49} \hat{L}_{-1}^4 - \frac{1}{36} \hat{L}_{-2}^2) \phi_{1,2} = \frac{9\sqrt{29}}{4} \left\{ Q_1, \frac{745Q}{14112} de \partial^2 e e^{-Q\phi} + \frac{25Q}{378} \partial de \partial e e^{-Q\phi} - \frac{431Q}{15876} \partial^3 e e^{-Q\phi} \right.
\]

\[
+ \frac{565}{7056} de \partial \phi \partial e e^{-Q\phi} + \frac{18071}{42336} \partial^2 e \partial \phi e^{-Q\phi} - \frac{103Q}{882} \partial e (\partial \phi)^2 e^{-Q\phi}
\]

\[
+ \frac{1}{441} e(\partial \phi)^3 e^{-Q\phi} - \frac{29Q}{126} e \partial \phi \partial^2 \phi e^{-Q\phi} + \frac{23}{63} \partial e \partial \phi e^{-Q\phi} + \frac{17}{2016} e \partial^3 \phi e^{-Q\phi} \}
\]

(20)

There is an analogous condition for the level-6 descendant of the field \( \phi_0 \), but since this is lengthy we omit it here.

Since we have argued previously that \( Q_1 \) commutes with all operators used to construct \( W_3 \) string scattering amplitudes, it follows that the (anti-)commutator of \( Q_1 \) with any well-defined operator will vanish in all correlation functions of interest to us. Hence all of the above highest weight descendants can be set to zero, which implies that the correlation functions of \( \phi_0, \phi_{1/16} \) and \( \phi_{1/2} \) satisfy certain differential equations. These differential equations, which are sufficient to determine completely the correlation functions, are precisely those satisfied by correlation functions of the Ising model. We therefore conclude that the integrand of any \( W_3 \) string scattering amplitude will contain a factor that is an Ising model correlation function.

It was first observed [1,2] that, for certain low-level physical states of the two-scalar \( W_3 \) string, one can regard the remaining scalar outside the Ising sector as
providing a Liouville dressing to the Ising vertices. It is clear from reference [5] that this interpretation holds for all vertices in the theory, so that the Ising vertices \( \phi_h \) are dressed in this way. Since, as we have now shown, the Ising sector vertices do indeed lead to Ising correlation functions as part of the scattering amplitudes, it would be interesting to carry out a detailed comparison between the full scattering amplitudes for the \( W_3 \) string and those found for the Ising model coupled to 2-dimensional gravity.

The \( W_3 \) string in fact contains an infinite-dimensional algebra that elucidates the structure of the null states found above. In order to see this we consider the BRST charge \( Q_1 \) associated with the Ising sector of the \( W_3 \) string. This charge can be used to define a new spin-3 generator

\[
U^{(3)}(z) \equiv \frac{9\sqrt{29}i}{4}\{Q_1, d(z)\}
\]

\[
= 2(\partial \phi)^3 + 6Q \partial^2 \phi \partial \phi + \frac{19}{4} \partial^3 \phi + 18\partial \phi \partial d \partial e
\]

\[
+ 6Q \partial d \partial e + 9\partial \phi \partial d \partial e + 9\partial^2 \phi \partial d \partial e + 3Q \partial^2 d \partial e.
\]

Both \( U^{(3)} \) and the stress-tensor \( U^{(2)} \equiv T_\phi + T_{de} \) commute with the BRST charge \( Q_1 \), but these fields do not form a closed algebra amongst themselves. The operator product of \( U^{(3)} \) with itself can be easily found from

\[
d(z)U^{(3)}(w) = -6 \frac{R^{(3)}(w)}{(z-w)^2} - 3 \frac{\partial R^{(3)}(w)}{z-w} + \ldots,
\]

where \( R^{(3)} = 3\phi d + Q \partial d \). Taking the commutator of this operator product with \( Q_1 \) then defines a new spin-4 generator \( U^{(4)} \) occurring in the operator product of \( U^{(3)} \) with itself:

\[
U^{(3)}(z)U^{(3)}(w) = -6 \frac{U^{(4)}(w)}{(z-w)^2} - 3 \frac{\partial U^{(4)}(w)}{z-w} + \ldots.
\]

The operator product of \( U^{(3)} \) with \( U^{(4)} \) can be obtained in a similar way by considering \( R^{(3)}(z)U^{(4)}(w) \) and then taking the anticommutator with \( Q_1 \). The resulting OPE is found to be non-linear, and contains a new spin-5 field \( U^{(5)} \) in addition to \( U^{(3)} \) and \( U^{(2)} \). The existence of such an algebraic structure was first noticed in reference [11] at the classical level, when considering under what conditions a \( W_N \) algebra would have a BRST charge with the decomposition \( Q = Q_0 + Q_1 \) with \( Q_0 \) and \( Q_1 \) both squaring to zero.
It thus emerges that there is a non-linear infinite-dimensional algebra associated with the Ising sector. This algebra is generated by fields $U^{(s)}$ of spin $s$, with $s \geq 2$, and $U^{(s)}$ is of the form $U^{(s)} = \{Q_1, R^{(s)}\}$ for $s \geq 3$. It follows that the only central term that arises in this algebra is in the OPE of $U^{(2)}$ with itself.

This infinite-dimensional algebra can be used to rewrite equations (17-20) in terms of null vectors. Let us consider $\phi_0$, for example. This is a highest weight state of the algebra, in the sense that

$$\hat{U}^{(s)}_n \phi_0 = 0, \quad n \geq 1; \quad (24)$$

where $\hat{U}^{(s)}_n$ is defined by the operator product expansion $U^{(s)}(z)\phi(w) = \sum(z - w)^{-n-s}(\hat{U}^{(s)}_n \phi)(w)$. Using the fact that $e^{e i \beta (1, 0)} = \hat{d}_{-1} \phi_0$, we can rewrite eqn (17) as the vanishing of a descendant field within this infinite-dimensional algebra,

$$\left(\hat{U}^{(3)}_{-1} + \frac{15Q}{7} \hat{L}_{-1}\right) \phi_0 = 0. \quad (25)$$

Similarly we see that $\phi_{1/16}$ is a highest weight field of the infinite-dimensional non-linear algebra and that eqn (19) can be written as

$$\left\{4Q \hat{L}_{-2} - \frac{4}{3} \hat{L}^2_{-1} + 17\hat{W}_{-2} - 32\hat{L}_{-1}\hat{W}_{-1}\right\} \phi_{1/16} = 0. \quad (26)$$

A similar reformulation can be given for all of equations (17-20).

We now wish to investigate in more detail the structure of the non-linear infinite-dimensional algebra discussed above. To this end, we shall show that it is a parafermion-generated $W$-algebra and identify the parafermion currents in the Ising sector. A parafermion theory [12] contains currents $\psi_k, \bar{\psi}_{-k} \equiv \psi_k^\dagger$, for $k = 0, \ldots, N - 1$, with both $\psi_k$ and $\psi_k^\dagger$ having conformal dimensions $\tilde{h}_k = k(N - k)/N, \tilde{\bar{h}}_k = 0$. $\psi_0$ is taken to be the identity operator. The central charge of this theory is $c = 2(N - 1)/(N + 2)$, and so to make contact with the Ising sector with $c = 1/2$ we choose $N = 2$. $\psi_1$ and $\psi_{-1}$ then have conformal weight $h = 1/2$. There are two operators in the cohomology of $Q_1$ that suggest themselves for the roles of the parafermionic currents, namely

$$\psi_1 = \phi_{1/2} = e^{e i \beta (1/2, 0)} \phi \quad (27)$$

and

$$\psi_{-1} = \phi(1/2, 1) = \{6 \partial d \partial e + \sqrt{2} \partial \phi \partial d - 3 \sqrt{2} \partial^2 \phi d - 3/2 \partial^2 d\} e^{e i \beta (1/2, 1)} \phi, \quad (28)$$
where \( \beta(1/2, n) = (4 - 8n)iQ/7 \) and \( \phi(h, n) \) is defined by \( V(1 - h, n) \equiv cV^X(1 - h)\phi(h, n), \phi(h, 0) = \phi_h. \) If we bosonize the ghosts by writing \( d = e^{-i\rho} \) and \( e = e^{i\rho}, \) in which case their energy-momentum tensor becomes \( T_{de} = -1/2(\partial \rho)^2 + 5i/2\partial^2 \rho, \) we obtain

\[
\psi_1 = e^{i\rho + i\beta(1/2, 0)\phi}
\]  

(29)

and

\[
\psi_{-1} = \{-i\sqrt{2}\partial \phi \partial \rho - 3\sqrt{2}\partial^2 \phi + 9i/2\partial^2 \rho - 3/2(\partial \rho)^2\}e^{-i\rho + i\beta(1/2, 1)\phi}.
\]  

(30)

A straightforward calculation gives

\[
\psi_1(z)\psi_{-1}(w) = \frac{1}{z - w} \sum_{n=0}^{\infty} (z - w)^n S_n,
\]  

(31)

where

\[
S_n = \frac{(n - 1)(n - 2)}{2} P_n - (n - 2)\partial P_{n-1} + \frac{1}{2} \partial^2 P_{n-2} + 2U^{(2)} P_{n-2}
\]  

(31)

and

\[
P_n = e^{-i\rho - i\beta(1/2, 0)\phi} \frac{\partial^n}{n!} e^{i\rho + i\beta(1/2, 0)\phi}.
\]  

(33)

Evaluating explicitly the right hand side, we find

\[
\psi_1(z)\psi_{-1}(w) = \frac{1}{z - w} \times
\]

\[
\left\{ 1 + (z - w)^2 2U^{(2)} + (z - w)^3 \partial U^{(2)} + \sqrt{2} (z - w)^3 U^{(3)} + O(z - w)^4 \right\},
\]  

(34)

where \( U^{(2)} \) is the total stress tensor for \( \phi \) and \( \rho \) and

\[
U^{(3)} = 2(\partial \phi)^3 + 6Q\partial^2 \phi \partial \phi + 19/4\partial^3 \phi
\]

\[
+ 9/2\partial \phi ((\partial \rho)^2 - 3i\partial^2 \rho) - 9i\partial^2 \phi \partial \rho
\]

\[
- Q \left( 3\partial \rho \partial^2 \rho + 2i\partial^3 \rho + i(\partial \rho)^3 \right)
\]  

(35)

is the bosonized form for \( U^{(3)} \).

The first two terms in equation (34) confirm that \( \psi_1 \) and \( \psi_{-1} \) are indeed parafermions. It has been explained in references [13] how a parafermionic operator product expansion contains in its regular part an infinite number of terms
which define the generators of an infinite-dimensional non-linear \( W \)-algebra denoted \( W_\infty(N) \) [13]. Eqns (31) and (35) show that the algebra \( W_\infty(-2) \) generated by the parafermionic currents \( \psi_1 \) and \( \psi_{-1} \) is precisely the algebra discussed above.

It is known that the algebra \( W_\infty(-2) \) constructed from the parafermions \( \psi_1 \) and \( \psi_1^\dagger \) has a realisation in terms of two free fields [14]. It appears, however, that the realisation that we have constructed here is inequivalent to that of reference [13]. This can be seen by redefining the fields \( \phi \) and \( \rho \) so that only one of them has a background charge; although the stress tensor obtained in this way agrees with that of reference [13], the spin-3 primary field does not.

In addition to containing parafermionic currents, the Ising sector also carries a representation of the parafermionic algebra. In the notation of reference [12], the \( N = 2 \) parafermion algebra can be represented on the fields \( \varphi_{[m]} \), where \( l = 0, 1 \) and \( m = -l, -l+2, \ldots, 4-l-2 \). The fields \( \varphi^l \equiv \varphi_{[l]} \) are parafermionic primary fields, in the sense that they satisfy

\[
A_{l/2+p}\varphi^l = A_{-l/2+p+1}\varphi^l = 0, \quad p \geq 0,
\]

where \( A_\nu \) and \( A_\nu^\dagger \) are the modes of \( \psi_1 \) and \( \psi_{-1} \) respectively acting on \( \varphi^l \). The fields \( \varphi^0 \) and \( \varphi^1 \) have dimensions 0 and 1/16 respectively. The field \( \varphi^0 \) is the identity operator and can be identified with \( \phi(1,1) \). The other primary field is \( \varphi^1 \), which we identify with \( e^{i\rho+i\beta(15/16,1)}\phi \). It is straightforward to verify that this is a parafermionic highest weight state.

The parafermions, and all of the above fields, commute with \( Q_1 \) since they are defined in terms of \( \phi(h,n) \) which itself commutes with \( Q_1 \). It is also straightforward to show that \( \psi_1 \) and \( \psi_{-1} \) commute with the screening charge \( S \), and hence all the generators of \( W_\infty(-2) \) commute with both \( S \) and \( Q_1 \).

The \( W_\infty(-2) \) algebra is associated, through the underlying parafermions, with a level 2 \( SU(2) \) WZWN model, and one can use one of the other scalar fields of the \( W_3 \) string to realize the full \( SU(2) \) currents rather than just the \( SU(2)/U(1) \) of the parafermions. One can also incorporate \( N = 2 \) supersymmetry into the model by a similar procedure. Further results, and details of the role of parafermions, will be given elsewhere [15].

The properties of the \( W_N \) string theories for \( N \geq 4 \) are largely unknown, although the numerical phenomenology of references [1,2] extends to these theories. It
has also been found [3] that if these theories obey a no-ghost theorem then their partition functions must involve the characters of the unitary minimal models with $c = 1 - 6/N(N + 1)$. There is evidence, however, that the $W_N$ string involves an $N - 1$ parafermion theory, thus generalizing the above results for the $W_3$ string. A $W_N$ string theory involves scalar fields $\phi_j$, for $j = 2, \ldots, N$, and ghosts $c_j, b_j$ for the same values of $j$. The field $\phi_2$ can be replaced by any number of spacetime fields $X^\mu$ provided they have the same central charge as $\phi_2$. The final scalar $\phi_N$ has a background charge such that its central charge is $1 + 3(N - 1)(2N + 1)^2/(N + 1)$, while the highest spin ghosts $c_N, b_N$ have a central charge $-2(6N^2 - 6N + 1)$. Hence the $\phi_N, b_N, c_N$ system has a total central charge of $2(N - 2)/(N + 1)$, which is just that of an $N - 1$ parafermionic theory. It is thus natural to suppose that a $W_N$ string theory can be written as a $W_{N-1}$ theory coupled to an $N - 1$ parafermion theory. We recall that such a parafermion theory generates the nonlinear infinite dimensional algebra $W_\infty(-(N - 1))$, which contains the usual $W_{N-1}$ algebra together with higher spin generators with no central term, which lends support to the above conjecture. Clearly, the states of the $W_N$ string will provide a realisation of these symmetries. These conjectures extend and generalize those of reference [11].

It is also natural to suppose that the BRST charge for the $W_N$ string, denoted $Q(W_N)$, can be written as

$$Q(W_N) = Q_0(W_{N-1}) + Q_1^{(N-1)},$$

where $Q_1^{(N-1)}$ is a function of the fields $\phi^N, c^N$ and $b^N$ and is such that the spin-$N$ field in the $W_\infty(-(N - 1))$ algebra is given by the anticommutator of $Q_1$ with $b^N$. Assigning a separate ghost number to $c^N, b^N$ we would conclude that $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. Given the two boson realization of the $N - 1$ parafermions, we can use it to construct the BRST charge $Q(W_N)$ from $Q(W_{N-1})$ [15].

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While this paper was being prepared for publication we received a paper [16] by C. Hull which commented on the connection with the Ising model and which, following reference [11], mentioned the presence of a $W_\infty$ algebra in the $W_3$ string. The nature of this algebra was not clarified.
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