RESEARCH ARTICLE

The second moment of $GL(n) \times GL(n)$ Rankin–Selberg $L$-functions

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Abstract
We prove an asymptotic expansion of the second moment of the central values of the $GL(n) \times GL(n)$ Rankin–Selberg $L$-functions $L(1/2, \pi \otimes \pi_0)$ for a fixed cuspidal automorphic representation $\pi_0$ over the family of $\pi$ with analytic conductors bounded by a quantity that is tending to infinity. Our proof uses the integral representations of the $L$-functions, period with regularised Eisenstein series and the invariance properties of the analytic newvectors.

Contents
1 Introduction ........................................ 2
  1.1 Sketch of the proof ................................ 4
  1.2 What’s next? .................................... 6
2 Basic notations and preliminaries .......... 7
  2.1 Basic notations .................................. 7
  2.2 Domains and measures .......................... 7
  2.3 Automorphic representations ................... 9
  2.4 Maximal Eisenstein series ..................... 10
  2.5 Genericity and Kirillov model ................. 10
  2.6 Zeta integrals ................................... 11
  2.7 Plancherel formula ............................. 12
  2.8 Spectral weights ............................... 12
  2.9 Analytic newvectors ............................ 13
  2.10 Main theorem .................................. 14
3 The Fourier expansion of maximal Eisenstein series 14
4 Proof of the main theorem ................. 18
  4.1 Choices of the local components .............. 18
  4.2 Computation of the spectral side .......... 18
  4.3 Computation of the period side .......... 20
5 Analysis of the degenerate terms in the period side 21
6 Analysis of the regularised term in the period side 29
7 Analysis of the spectral side .......... 33
1. Introduction

The asymptotic evaluation of higher moments of the central $L$-values carries important arithmetic information: for example, subconvex bounds or nonvanishing results for the central $L$-values. This evaluation becomes more and more difficult as the moment, the degrees of the $L$-functions or the rank of the underlying group increase.

Obtaining subconvex bounds – that is, proving bounds of the form

$$L(1/2, \pi) \ll C(\pi)^{1/4 - \delta}, \quad \delta > 0,$$

where $C(\pi)$ is the analytic conductor of an automorphic representation $\pi$, is an extremely difficult problem with respect to the current technology. A narrow, but important for applications, class of automorphic representations suffers from yet another major technical difficulty named conductor-drop. These representations are usually functorially lifted from smaller groups and have unusually small analytic conductors.

For example, if $\pi$ varies over automorphic representations for $\text{PGL}_2(\mathbb{Q})$ with $C(\pi)$ being of size $T$, then the size of the analytic conductor of the Rankin–Selberg convolution $\pi \otimes \bar{\pi}$ is roughly $T^2$, where $\bar{\pi}$ is the contragredient of $\pi$, whereas $C(\pi \otimes \pi')$ has size $T^4$ if $C(\pi')$ is of size $T^2$ but $\pi'$ is away from $\bar{\pi}$. That is, the $\text{PGL}(4)$-subfamily of $\pi \otimes \bar{\pi}$ shows the conductor-drop phenomena. Another example of a family that sees conductor-dropping is the $\text{PGL}(3)$-family of $\text{Sym}^2\pi$, where $\pi$ varies over a $\text{PGL}(2)$ family (the subconvexity problem for this family is directly related to the arithmetic quantum unique ergodicity problem for $\text{SL}_2(\mathbb{R})$). This happens due to one of the Langlands parameters of $\text{Sym}^2\pi$ being extremely small compared to the others. The families defined by the Plancherel balls with a large radius (for example, dilated) or high centre often exclude these narrow classes. Thus moment estimates over these families do not usually become fruitful to yield a subconvex bound of an $L$-function that has conductor drop; see, for example, [2, 31, 30].

One naturally interesting and important family of automorphic representations can be given by representations with growing conductors: for example,

$$\mathcal{F}_X := \{ \pi \text{ automorphic representation for } \text{PGL}_n(\mathbb{Z}) \mid C(\pi) < X \},$$

with $X \to \infty$. The family $\mathcal{F}_X$, unlike the families defined by the Plancherel balls, is indifferent to the conductor-drop issue. So a Lindelöf-consistent estimate for a high enough moment over the family $\mathcal{F}_X$ will likely produce a subconvex estimate even for the $L$-functions suffering from conductor-drop. Here, by Lindelöf-consistent (also called Lindelöf on average) estimate for the $2k$th moment, we mean the estimate

$$\mathbb{E}_{\mathcal{F}_X} |L(1/2, \pi)|^{2k} \ll \epsilon X^\epsilon,$$

where $\mathbb{E}$ denotes the average. On the other hand, a more interesting and difficult question would be to find an asymptotic formula of (a suitably weighted and smoothened version of) the above average whose leading term is believed to be a polynomial in $\log X$.

However, the family $\mathcal{F}_X$ becomes quite large as $X$ tends to infinity. One informally has $|\mathcal{F}_X| \asymp X^{n-1}$; see [6] for the corresponding nonarchimedean analogue. This is why, to obtain a subconvex bound of an $L$-function attached to an element in $\mathcal{F}_X$, one needs to evaluate quite a high moment asymptotically or at least estimate in the Lindelöf-consistent manner. For example, we need to estimate an amplified $4(n - 1)$th moment over $\mathcal{F}_X$ even to break the convexity barrier. Unfortunately, the current technology is not advanced enough to tackle such a high moment of these $L$-functions due to the large size of the conductors. Hence a natural, informal question arises regarding the race between the sizes of the conductors of the $L$-functions and the families: as a function of $n$, how high of a moment can be asymptotically evaluated (or estimated in a Lindelöf-consistent manner) over the family $\mathcal{F}_X$?

---

1. The generalised Lindelöf hypothesis predicts that any $\delta < 1/4$ is achievable.
Such a question has been addressed in the literature for low-rank groups. We may try to guess an answer to our proposed informal question by looking at the small number of examples in low ranks. For $n = 2$ in [22], the authors obtained an asymptotic formula of the 4th moment over a family in the nonarchimedean conductor aspect and restricted only to the holomorphic forms. In [3], the authors proved a Lindelöf-consistent upper bound of the 6th moment in the nonarchimedean conductor aspect for $n = 3$. These are the best possible estimates so far for small $n$, which allows us to wonder whether the 2nd moment can be asymptotically evaluated over the family $\mathcal{F}_X$. However, if we work on the GL($n$) rather than on the PGL($n$) family – that is, if we do an extra central average – we expect that an asymptotic formula of the $2n + 2$th moment is achievable.

Our primary motivation is to prove an asymptotic formula for the 2th moment of the central $L$-values for PGL($n$) with $n \geq 3$, over the family $\mathcal{F}_X$, using the integral representations of the $L$-functions and spectral theory. If $\pi$ is an automorphic representation for PGL($n$), then

$$L(1/2, \pi)^n = L(1/2, \pi \otimes E_0),$$

where $E_0$ is the minimal Eisenstein series for PGL($n$) with trivial Langlands parameters and $\otimes$ denotes the Rankin–Selberg convolution. Thus evaluating the 2th moment of $L(1/2, \pi)$ is the same as evaluating the second moment of $L(1/2, \pi \otimes E_0)$. However, the approach of the integral representations and the spectral decomposition encounters severe analytic difficulties due to the growth of $E_0$ near the cusp: for example, $E_0$ fails to be square integrable in the fundamental domain. To avoid this particular technical difficulty, we may replace $E_0$ with a fixed cusp form and try to evaluate their second moment asymptotically.

Let $n \geq 3$. In this article, we evaluate the second moment of the central Rankin–Selberg $L$-values $L(1/2, \pi \otimes \pi_0)$, where $\pi$ varies over a family of automorphic representations for PGL($n$) that are unramified at all the finite places and the archimedean conductors are growing to infinity. Here $\pi_0$ is a fixed cuspidal representation for PGL($n$) $(\text{i.e., unramified at the finite places})$, which is again unramified at all the finite places. Below we informally describe our main theorem.

**Theorem 1.1** (Informal version). Let $n \geq 3$ and $\pi_0$ be a cuspidal automorphic representation for PGL($n$) $(\text{i.e., unramified at the finite places})$, which is tempered at $\infty$. Let $\pi$ vary over the generic automorphic representations in $\mathcal{F}_X$. Then we have an asymptotic formula of the following (weighted) average

$$\mathbb{E}_{\pi \in \mathcal{F}_X} \text{generic} \left[ \frac{|L(1/2, \pi \otimes \pi_0)|^2}{L(1, \pi, \text{Ad})} \right] \text{+ continuous} = n \frac{\zeta(n/2)^2}{\zeta(n)} L(1, \pi_0, \text{Ad}) \log X + O_{\pi_0}(1),$$

as $X$ tends to infinity.

For the actual formal statement, we refer to Theorem 2.1.

**Remark 1.2.** In Theorem 1.1, by ‘continuous’, we mean the corresponding terms from the generic noncuspidal spectrum. In the actual statement – that is, Theorem 2.1 – we do a specific weighted average over the full generic automorphic spectrum such that the weights are uniformly bounded away from zero on the cuspidal spectrum with analytic conductors bounded by $X$. Consequently, we also need to change the harmonic weight $L(1, \pi, \text{Ad})$ by an equivalent arithmetic factor for the noncuspidal spectrum.

This is the first instance of an asymptotic evaluation of the second moment of a family of $L$-functions with arbitrary high degree. In general, for a pair of groups $H \leq G$ and their representations $\pi$ and $\Pi$, respectively, it is an interesting question to asymptotically evaluate moments of the central $L$-values of the Rankin–Selberg product $\Pi \otimes \pi$ (if defined). Previously, in [31], Nelson–Venkatesh asymptotically evaluated the first moment keeping $\Pi$ fixed and letting $\pi$ vary over a dilated Plancherel ball when $(G, H)$ are Gan–Gross–Prasad pairs and, more interestingly, allowing arbitrary weights in the spectral side. More recently, in [30], Nelson proved a Lindelöf-consistent upper bound of the first moment for the groups $(G, H) = (U(n + 1), U(n))$ in the nonsplit case, keeping $\pi$ fixed and letting $\Pi$ vary over a
Plancherel ball with high centre. Both [31, 30] assume that the family of $L(s, \Pi \otimes \pi)$ does not show any conductor-dropping. The method in [30] also yields an asymptotic formula with power savings of a specific weighted first moment over this family. Blomer in [1] obtained a Lindelöf-consistent upper bound of the second moment for $G = H = \text{GL}(n)$, keeping $\Pi$ a fixed cuspidal representation and letting $\pi$ vary in a Plancherel ball. On the contrary to [31, 30], he proves a Lindelöf-consistent upper bound when the family of $L(s, \Pi \otimes \pi)$ shows conductor-dropping. However, his method does not yield an asymptotic formula.

There have been quite a few results for asymptotic formulas and upper bounds on rank $\leq 2$ and degree $\leq 4$. In particular, we refer to [4]: the authors prove an asymptotic formula for $\text{GL}(2) \times \text{GL}(2)$ Rankin–Selberg $L$-functions, fixing one of the representations but with an extra average over the centre of $\text{GL}(2)$. In [10], an asymptotic formula for the sixth moment of the $L$-values attached to holomorphic cusp forms for $\text{GL}(2)$ is achieved, but again with an extra average over the centre of $\text{GL}(2)$.

### 1.1. Sketch of the proof

Our point of departure is similar to [1] and [32]. We use the spectral decomposition of $\text{PGL}(n)$ and integral representations of the $L$-functions. We start by choosing $\phi_0 \in \pi_0$ such that the Whittaker function $W_0$ of $\phi_0$ is an analytic newvector; see Section 2.9 for a brief description of the analytic newvectors. Such $W_0$ in the Kirillov model of $\pi_0$ can be described by a fixed bump function. Let $Eis(f_x)$ be the maximal Eisenstein series $\text{PGL}_n(\mathbb{Z})$ attached to a generalised principal series vector $f_x$. Also let $X$ be a large real number and $x$ be the diagonal element in $\text{PGL}_n(\mathbb{R})$ given by $\text{diag}(X, \ldots, X, 1)$. We translate the Eisenstein series by $x$ to obtain $Eis(f_x)(.x)$.

For this subsection, let $\mathcal{X} := \text{PGL}_n(\mathbb{Z}) \setminus \text{PGL}_n(\mathbb{R})$ and $N$ be the maximal unipotent of the upper triangular matrices in $\text{PGL}_n(\mathbb{R})$. We start by writing the inner product

$$\langle \phi_0 \text{Eis}(f_{1/2})(.x), \phi_0 \text{Eis}(f_{1/2})(.x) \rangle = \langle |\phi_0|^2, |\text{Eis}(f_{1/2})|^2(.x) \rangle,$$

where all the inner products above are the usual $L^2$-inner product on the fundamental domain $\mathcal{X}$. Note that both of the sides of equation (1.1) are absolutely convergent as $\phi_0$ decays rapidly at the cusps.

We use Parseval’s identity on the left-hand side over $\text{PGL}(n)$. A typical term corresponding to an automorphic representation $\pi$ in the spectral sum would look like

$$\left| \int_{\mathcal{X}} \phi_0(g) \overline{\phi(g)} \text{Eis}(f_{1/2})(gx)dg \right|^2 = \frac{|L(1/2, \tilde{\pi} \otimes \pi_0)|^2}{L(1, \pi, \text{Ad})} |Z_\pi(f_{1/2}, W, W_0)|^2,$$

and

$$Z_\pi(f_x, W, W_0) = \int_{N \setminus \text{PGL}_n(\mathbb{R})} W_0(g) \overline{W(g)} f_x(gx)dg$$

is the local zeta integral.

We choose $f_x$ such that $f_{1/2} \left[ \begin{array}{cc} I & 0 \\ c & 1 \end{array} \right]$ is supported on $|c| < \tau$ for some $\tau > 0$ sufficiently small, so that $W_0 f_{1/2}(.x)$ would mimic a smoothened characteristic function of the archimedean congruence subgroup $K_0(X, \tau)$ (see equation (2.8)). If $W$ is an analytic newvector (see Section 2.9), then the invariance property of $W$ will yield that $Z_\pi(f_{1/2}, W, W_0) \gg 1$ if $C(\pi) < X$. We use

$$\sum_{W} |Z_\pi(f_{1/2}, W, W_0)|^2$$

as the spectral weights where in the above sum $W$ traverses some orthonormal basis of $\pi$. We point out on the naive similarities between the spectral weight here and the one used in, for example, [18, Theorem 1]. However, the invariance property that is needed here is a bit stronger than the invariance used in
[18, Theorem 1]: we only needed invariance at points near the identity in $GL_n(\mathbb{R})$ in [18], whereas here, we have to gain an invariance that is uniform for all elements in $GL_{n-1}(\mathbb{R})$. The method of using the approximate invariance of the newvectors is similar to [32] for $GL(2)$, where in the nonarchimedean aspect, the exact invariance is used. This analysis is done in Section 7.

We now explain how we proceed to give an asymptotic expansion of the right-hand side of equation (1.1). The heuristic idea, at least to obtain an upper bound, is to make the change of variables in the period of the right-hand side of equation (1.1) to write it as

$$\int_X |\phi_0(gx^{-1})|^2 |Eis(f_{1/2})(g)|^2 dg$$

and then bound this period by

$$\leq \|\phi_0\|^2_{L^\infty(X)} \int_X |Eis(f_{1/2})(g)|^2 dg.$$  

But unfortunately, $Eis(f_{1/2})$ (barely!) fails to be square integrable on $X$. That is why we have to regularise the period. We adopt the regularising techniques of Zagier [34]; see also [27, 29]. First we deform $|Eis(f_{1/2})|^2$ as $Eis(f_{1/2+s})Eis(f_{1/2})$ for $s$ lying in some generic position with very small $\Re(s)$. From the Fourier expansions of the Eisenstein series, we can pick off the nonintegrable terms in the product $Eis(f_{1/2+s})Eis(f_{1/2})$ and call their sum $F_s$. Then we construct a regularised Eisenstein series by

$$\tilde{E}(s, .) := Eis(f_{1/2+s})Eis(f_{1/2}) - Eis(F_s).$$

We will check that $\tilde{E}(s, .)$ lies in $L^2(X)$. Consequently, we regularise the period as

$$\langle |\phi_0(x^{-1})|^2, |Eis(f_{1/2})|^2 \rangle = \lim_{s \to 0} \langle |\phi_0(x^{-1})|^2, Eis(f_{1/2})Eis(f_{1/2+s}) \rangle$$

$$= \lim_{s \to 0} \langle |\phi_0(x^{-1})|^2, \tilde{E}(s, .) \rangle + \lim_{s \to 0} \langle |\phi_0(x^{-1})|^2, Eis(F_s) \rangle.$$  

We call the first summand the regularised term, which, upon rigorous application of the heuristic above, can be proved to be of bounded size. The second summand is called the degenerate term and yields the main term.

Up to some nonarchimedean factors involving $L(1, \pi_0, Ad)$, the degenerate term is of the form

$$\partial_{s=0} Z_x(f_{1/2}f_{1/2+s}, W_0, \overline{W_0}) - \partial_{s=0} Z_x(Mf_{1/2}Mf_{1/2+s}, W_0, \overline{W_0}),$$

where $M$ is certain intertwining operator that arises in the constant term of a maximal parabolic Eisenstein series. One main difficulty of the paper is asymptotically evaluating the above two derivatives. The first one is comparatively easy to understand as one can apply the support condition of $f_{1/2}f_{1/2+s}\left[\begin{array}{c} I \\ c \\ 1 \end{array}\right]x$, which is concentrated on $c = O(1/X)$ and the approximate invariance of $W_0$. The second one is more technical to analyse. The intertwined vector $Mf_{1/2+s}\left[\begin{array}{c} I \\ c \\ 1 \end{array}\right]x$, which on the matrices $\left[\begin{array}{c} I \\ c \\ 1 \end{array}\right]$, essentially mimics a Fourier transform of $f_{1/2+s}$, has support of size $c = O(X)$. So we cannot get away with just the invariance properties of $W_0$. In this case, we understand a more detailed shape of the intertwined vectors via the Iwasawa decomposition on the matrices of the form $\left[\begin{array}{c} I \\ cX \\ 1 \end{array}\right]$. This analysis is done in Section 5.

On the other hand, to analyse the regularised term, we understand the growth of the (degenerate) Fourier terms of $Eis(f_s)$ for $s$ being close to 0, 1/2 or 1. This analysis relies on the analytic properties
of the intertwining operators attached to various Weyl elements and functional analytic properties of the Eisenstein series. This analysis is done in Section 6.

Remark 1.3. We remark that our method of proof, which is uniform for \( n \geq 3 \), can also be made to work for \( n = 2 \) with a slight modification with a modified main term (the statement of our theorem does not in any way make sense for \( n = 2 \)). The main terms in the asymptotic expansion are the artefacts of the nonintegrable terms among the product of the constant terms in the Fourier expansion of \( E \) and \( \text{Eis}(f_{1/2}) \) and \( \text{Eis}(f_{1/2+s}) \). The constant term of \( \text{Eis}(f_s) \) looks like \( \sum_w M_w f_s \), where \( M_w \) are certain intertwining operators and \( w \) runs over a set of Weyl elements attached to the underlying parabolic subgroup (see Section 3). If \( n \geq 3 \), then the nonintegrable terms in the above-mentioned product are of the form \( f_{1/2} f_{1/2+s} \) and \( M f_{1/2} f_{1/2+s} \), where \( M \) is the intertwiner attached to the relative long Weyl element. In particular, the off-diagonal terms of the from \( f_{1/2} M f_{1/2+s} \) are integrable. Such a phenomenon does not happen for \( n = 2 \). In this case (where the maximal Eisenstein series is also a minimal Eisenstein series), the off-diagonal terms are also nonintegrable.

As described in the sketch of the proof, eventually we need to deform the principal series vector to regularise the Eisenstein series. The number of deformations needed in the Langlands parameters of the associated principal series vector depends on the number of nonintegrable terms in the product of the constant terms. For \( n \geq 3 \), we need to deform only one of the parameters of the principal series vector to regularise the corresponding maximal parabolic Eisenstein series. However, for \( n = 2 \), for the reasons stated above, to regularise the Eisenstein series, we need to deform two (i.e., both) of the parameters. This modification will produce more degenerate terms, and consequently, a different main term with a different constant will appear; see [4].

1.2. What’s next?

As we have described above, the motivating question for us is to find an asymptotic expansion of the 2\( n \)-th moment of the central \( L \)-values for \( \text{PGL}(n) \), and to do that, we need to replace \( \phi_0 \) with a minimal Eisenstein series \( E_0 \) with trivial Langlands parameters. As, in particular, \( E_0 \) is not in \( L^\infty \), our current proof obviously fails (see the sketch of the proof), and that is why we need to regularise \( E_0 \) as well. However, this regularisation increases the analytic difficulties manifold. We need to employ a regularised version of the spectral decomposition (and Parseval), as in, for example, [29, 27], to follow the same strategy as in the sketch of the proof of the main theorem. On the other hand, regularising both the Eisenstein series involved in the period \( \langle |E_0|^2, |\text{Eis}(f_{1/2})(x)|^2 \rangle \) will introduce many more degenerate terms, which will typically have higher-order poles at the critical point. This will likely yield a higher power of \( \log X \) in the main term. It will be interesting to see if the constant appearing in the main term is the same as predicted by the random matrix models; see [9]. However, we leave this to future work.

It is natural to speculate what happens for the second moment of the Rankin–Selberg \( L \)-functions for other \( (\text{GL}(n), \text{GL}(m)) \) pairs with \( m \neq n \) and the \( \text{GL}(m) \) form being fixed (cuspidal or Eisenstein). If \( m < n \), we believe that the problems become simpler than the \( m = n \) case as the degrees, and hence conductors, become lower. Similarly, for \( m > n \), we expect the problems to be much more difficult for high degree and conductor size. In particular, it will be very interesting to see if we can push the method in this paper at least to the case \( m = n + 1 \) case. More interestingly, if \( n = 3 \) and the fixed form is a minimal Eisenstein series, then we will have a Lindelöf-consistent eighth moment (the convexity barrier) of \( L \)-functions of \( \text{PGL}(3) \) over the family \( \mathcal{F}_X \).

Remark 1.4. We briefly remark that one may try to explicite the constant contribution of the asymptotic expansion in the main theorem and obtain a power-saving error term as in [4]. One possible way to obtain finer asymptotics in the regularised part is to spectrally expand the period \( \langle |\phi_0|^2, \tilde{E}_s(.x) \rangle \) over the \( \text{PGL}(n) \) automorphic spectrum. Then one may use the existence of a spectral gap and explicit decay of the matrix coefficient for \( n \geq 3 \) to obtain that \( \langle |\phi_0|^2, \phi \rangle \langle \phi, \tilde{E}_s(.x) \rangle \), at least for a tempered \( \phi \), will decay polynomially in \( X \). However, it is not yet clear to us how to explicite the constant term and get an error term with polynomial saving in the degenerate part; see Remark 5.4.
2. Basic notations and preliminaries

2.1. Basic notations

We use adèlic language. Let \( r \geq 3 \). For any ring \( R \) by \( G(R) \), we denote the set of points \( \text{GL}_r(R)/R^\times \). In this paper, \( R \) will denote the adèles \( \mathbb{A} \) over \( \mathbb{Q} \) or the local fields \( \mathbb{R}, \mathbb{Q}_p \) or rational numbers \( \mathbb{Q} \) or the local ring \( \mathbb{Z}_p \). We drop the ring \( R \) from the notation \( G(R) \) if the ring is clear from the context.

Let \( N \) be the maximal unipotent subgroup of \( G \) consisting of upper triangular matrices. For \( q \in \mathbb{A}^{r-1} \), we define a character of \( N(\mathbb{A}) \) by

\[
\psi_q(n(x)) = \psi_0 \left( \sum_{i=1}^{r-1} q_i x_{i,i+1} \right), \quad n(x) := (x_{i,j})_{i,j},
\]

where \( \psi_0 \) is an additive character of \( \mathbb{Q}\backslash \mathbb{A} \). We abbreviate \( \psi_{(1,...,1)} \) by \( \psi \). We call \( \psi_q \) nondegenerate if \( q_i \neq 0 \) for \( 1 \leq i \leq r-1 \); otherwise, we call \( \psi_q \) degenerate.

Let \( A \) be the set of diagonal matrices in \( G \), which we identify with \( \begin{pmatrix} A_{r-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \), where \( A_{r-1} \) is the set of diagonal matrices in \( \text{GL}(r-1) \). We parametrise elements of \( A_{r-1} \) as \( a(y) := \text{diag}(y_1, \ldots, y_{r-1}) \).

Let \( K := \prod_{p < \infty} K_p \) be the standard maximal compact in \( G(\mathbb{A}) \), where \( K_p := G(\mathbb{Z}_p) \) for \( p < \infty \) and \( K_\infty := \text{PO}_d(\mathbb{R}) \).

For any factorisable function \( f \) on \( G(\mathbb{A}) \) by \( f_p \), we denote the \( p \)th component of \( f \), which is a function on \( G(\mathbb{Q}_p) \).

2.2. Domains and measures

We fix Haar measures on \( G \) and its subgroups, and a \( G \)-invariant measure of \( N \backslash G \). If the subgroup is compact, then we normalise the Haar measure to be a probability measure. Let \( \delta \) denote the modular character on \( A \). It is defined by

\[
\delta(a(y)) := \prod_{j=1}^{r-1} |y_j|^{1/(r-1-j)}
\]

and is trivially extended to \( NA \).

To integrate over \( N(\mathbb{R}) \backslash G(\mathbb{R}) \), we use two different types of coordinates according to efficiency. The first one is Bruhat (with respect to the standard maximal parabolic) coordinates. First, note that the set of elements of the form \( \begin{pmatrix} h & b' \\ c & 0 \end{pmatrix} \) with \( h \in \text{GL}_{r-1}(\mathbb{R}) \) and row vectors \( b, c \in \mathbb{R}^{r-1} \) has zero measure in \( \text{GL}_r(\mathbb{R}) \) with respect to its Haar measure. Thus while integrating over \( G(\mathbb{R}) \), we integrate over the points of the form

\[
\begin{pmatrix} h & b' \\ c & 0 \end{pmatrix} \begin{pmatrix} I_{r-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \quad h \in \text{GL}_{r-1}(\mathbb{R}) \text{ and } b, c \in \mathbb{R}^{r-1} \text{ row vectors}.
\]

Similarly, the set of points of the form

\[
g = \begin{pmatrix} h \\ 1 \end{pmatrix} \begin{pmatrix} I_{r-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \quad h \in N_{r-1}(\mathbb{R}) \backslash \text{GL}_{r-1}(\mathbb{R}) \text{ and } c \in \mathbb{R}^{r-1} \text{ row vector}
\]

has full measure in \( N(\mathbb{R}) \backslash G(\mathbb{R}) \). We use these coordinates to integrate over \( N(\mathbb{R}) \backslash G(\mathbb{R}) \) using the invariant measure

\[
dg = \frac{dh}{|\det(h)|} dc,
\]
where $dc$ denotes the Lebesgue measure and $dh$ is the $\text{GL}_{r-1}(\mathbb{R})$-invariant Haar measure on $N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})$. Description of the above invariant measure follows from [23, eq. (5.14)] and discussion above that. However, there is a more direct way to see this. Let $\phi \in C_c(G(\mathbb{R}))$ be measurable. Then it follows from [14, Proposition 1.4.3] that

$$
\int_{G(\mathbb{R})} \phi(g) dg = \int_{\mathbb{R}^{n-1}} \phi \left[ \left( \begin{array}{c} h \\ b' \\
 c \\ 1 \end{array} \right) \right] \prod_{i,j} d_L h_{ij} \prod_i d_L b_i \prod_i d_L c_i \frac{\det \left( \begin{array}{c} h \\ b' \\
 c \\ 1 \end{array} \right) \right]^n, 
$$

where $d_L x$ denotes the Lebesgue measure on $\mathbb{R}$. Noting that

$$
\left( \begin{array}{c} h \\ b' \\
 c \\ 1 \end{array} \right) = \left( \begin{array}{c} I_{r-1} \\ b' \\
 1 \\ 1 \end{array} \right) \left( \begin{array}{c} h - b' c \\
 1 \\
 c \\ 1 \end{array} \right) \left( \begin{array}{c} I_{r-1} \\
 1 \\
 1 \\
 c \\ 1 \end{array} \right),
$$

we can write

$$
\int_{G(\mathbb{R})} \phi(g) dg = \int_{\mathbb{R}^{n-1}} \phi \left[ \left( \begin{array}{c} I_{r-1} \\ b' \\
 1 \\ 1 \end{array} \right) \left( \begin{array}{c} h - b' c \\
 1 \\
 c \\ 1 \end{array} \right) \left( \begin{array}{c} I_{r-1} \\
 1 \\
 1 \\
 c \\ 1 \end{array} \right) \right] \prod_{i,j} d_L h_{ij} \prod_i d_L b_i \prod_i d_L c_i \frac{\det(h - b' c)^n}{\det(h)^n}.
$$

Fixing $b, c$ and changing variables $h_{ij} \mapsto h_{ij} + (b' c)_{ij}$, we can write the above as

$$
\int_{G(\mathbb{R})} \phi(g) dg = \int_{\mathbb{R}^{n-1}} \phi \left[ \left( \begin{array}{c} I_{r-1} \\ b' \\
 1 \\ 1 \end{array} \right) \left( \begin{array}{c} h \\
 1 \\
 c \\ 1 \end{array} \right) \left( \begin{array}{c} I_{r-1} \\
 1 \\
 1 \\
 c \\ 1 \end{array} \right) \right] \prod_{i,j} d_L h_{ij} \prod_i d_L b_i \prod_i d_L c_i \frac{\det(h)^n}{\det(h)^n}.
$$

Noting that $\prod_{i,j} d_L h_{ij} \frac{\det(h)^n}{\det(h)^n} = dh$ and taking the $N(\mathbb{R})$-quotient on the left, we deduce the invariant measure on $N(\mathbb{R}) \setminus G(\mathbb{R})$.

On the other hand, when we integrate on $N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})$, we use Iwasawa coordinates. We write

$$N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R}) \ni h = a(y)k, \quad a(y) \in A_{r-1}, k \in K_{r-1},
$$

where $K_{r-1}$ is the standard maximal compact in $\text{GL}_{r-1}$, with the measure

$$dh = \frac{\prod_i d^x y_i}{\delta(a(y))} dk,
$$

where $dk$ is the probability Haar measure on $K_{r-1}$.

Let $X := G(\mathbb{Q}) \setminus G(\mathbb{A})$. We fix a fundamental domain $X$ in $G(\mathbb{A})$ of the form

$$D \times K_f, \quad D \subseteq G(\mathbb{R}), K_f := \prod_{\rho < \infty} K_{\rho},
$$

that is contained in a Siegel domain of the form $S \times K_f$, where

$$S := \{G(\mathbb{R}) \ni g = n(x) \left( \begin{array}{c} a(y) \\
 1 \end{array} \right) k \mid |x_{i,j}| < 1, |y_i| > y_0, k \in K_\infty \},
$$

(2.1)

where $y_0 > 0$ is an explicit constant dependent only on the group. The above follows from strong approximation for $\text{GL}(n)$ and [14, §1.3].

We equip $X$ with the $G(\mathbb{A})$-invariant probability measure that in Iwasawa coordinates is given by

$$X \ni g = n(x) \left( \begin{array}{c} a(y) \\
 1 \end{array} \right) k, \quad dg = \prod_{j,k} dx_{j,k} \frac{\prod_i d^x y_i}{\delta(a(y))|\det(a(y))|} dk,$$
where \( n(x) \in N(\mathbb{R}) \) and \( dx_{i,j} \) is the usual Lebesgue measure. Note that \( \delta \left( \frac{\partial(y)}{1} \right) = \delta(a(y)) \det(a(y)) \).

### 2.3. Automorphic representations

We briefly describe the classes of local and global representations that are relevant in this paper. We refer to [28], [12, §5] for details.

Let \( \mathbb{X} \) be the isomorphism class of irreducible unitary automorphic representations that are unramified at all finite places and appear in the spectral decomposition of \( L^2(\mathbb{X}) \). Similarly, by \( \mathbb{X}_{\text{gen}} \), we denote the subclass of generic representations in \( \mathbb{X} \): that is, the class of representations that have (unique) Whittaker models.

We first mention the Langlands description for \( \mathbb{X}_{\text{gen}} \). We take a partition \( r = r_1 + \cdots + r_k \). Let \( \pi_i \) be a unitary cuspidal automorphic representation for GL\(_r_i(\mathbb{Q})\) (if \( r_j = 1 \), we take \( \pi_j \) to be a unitary Hecke character). Consider the normalised parabolic induction \( \Pi(r_1) \times \cdots \times \Pi(r_k) \) to \( G \) of the tensor product \( \pi_1 \otimes \cdots \otimes \pi_k \). There exists a unique irreducible constituent of \( \Pi \), which we denote by the isobaric sum \( \pi_1 \boxplus \cdots \boxplus \pi_k \). Then the Langlands classification says that every element in \( \mathbb{X}_{\text{gen}} \) is isomorphic to an isobaric sum \( \boxplus_j \pi_j' \) for some partition \( r = \sum_j r_j' \) and some cuspidal representation \( \pi_j' \) of GL\(_{r_j'}\).

We recall from [28] that we call \( \pi' \in \mathbb{X} \) a discrete series if \( \pi' \) appears discretely in the spectral decomposition of \( L^2(\mathbb{X}) \). The elements of \( \pi' \) are square-integrable automorphic forms for \( G(\mathbb{Q}) \). Mœglin– Waldspurger classified the discrete series for \( G(\mathbb{Q}) \) via the iterated residues of generic automorphic forms; see [28]. The Langlands description of \( \mathbb{X} \) says that every element in \( \mathbb{X} \) is isomorphic to an isobaric sum \( \boxplus_j \pi_j' \) for some partition \( r = \sum_j r_j' \) and some discrete series representation \( \pi_j' \) of GL\(_{r_j'}\).

We fix an automorphic Plancherel measure \( d\mu_{\text{aut}} \) on \( \mathbb{X} \) compatible with the invariant probability measure on \( \mathbb{X} \). If \( \pi \) is a discrete series, then \( d\mu_{\text{aut}}(\pi) \) is absolutely continuous to the counting measure at \( \pi \). On the other hand, if \( \pi \) is an Eisenstein series induced from a twisted discrete series \( \pi'|.|^s \), where \( \pi' \) is a discrete series on a Levi subgroup \( M \) and \( \lambda \) lies in the purely imaginary dual of the Cartan subalgebra of \( M \), then \( d\mu_{\text{aut}}(\pi) \) is absolute continuous to the product of the counting measure at \( \pi' \) and the Lebesgue measure at \( \lambda \).

For any \( \pi \in \mathbb{X} \), we denote the \( p \)th component of \( \pi \) by \( \pi_p \) for \( p \leq \infty \). The generalised Ramanujan conjecture predicts that if \( \pi \) is cuspidal, then \( \pi_p \) is tempered for all \( p \leq \infty \). In this paper, we assume that certain cuspidal representations are \( \vartheta \)-tempered at the archimedean place, whose definition we recall below.

First we describe the Langlands description of the generic representations of \( G(\mathbb{R}) \). Let \( r' \in \{1, 2\} \) and \( \sigma \) be an essentially square integrable (square integrable mod centre) representation of GL\(_{r'}(\mathbb{R}) \). That is, if \( r' = 2 \), then \( \sigma \) is a discrete series of GL\(_2(\mathbb{R}) \); and if \( r' = 1 \), then \( \sigma \) is a unitary character of GL\(_1(\mathbb{R}) \). By the Langlands classification, we know that any generic unitary irreducible representation \( \xi \) of \( G(\mathbb{R}) \) is isomorphic to a normalised parabolic induction of

\[
\sigma_1 | \det|^s_1 \otimes \cdots \otimes \sigma_k | \det|^s_k ,
\]

from a Levi subgroup attached to a partition of \( r = \sum_{j=1}^k r_j \) with \( r_j \in \{1, 2\} \), where \( \sigma_j \) is an essentially square integrable representation of GL\(_{r_j}(\mathbb{C})\) and \( s_j \in \mathbb{C} \) with \( \sum_{j=1}^k r_j s_j = 0 \) and \( \Re(s_1) \geq \cdots \geq \Re(s_k) \).

Let \( \vartheta \geq 0 \). We say that \( \xi \) is \( \vartheta \)-tempered if all such \( s_j \) have real parts in \([-\vartheta, \vartheta] \). By [26], if \( \pi \) is cuspidal, then \( \pi_\infty \) is \( \vartheta \)-tempered with \( \vartheta = 1/2 - 1/(1+r^2) \).

We denote the analytic conductor of \( \pi \) by \( C(\pi) \). Note that as \( \pi \in \mathbb{X} \) is unramified at all the finite places, we have \( C(\pi) = C(\pi_\infty) \). If \( \{\mu_i\} \in \mathbb{C}^r \) are the Langlands parameters of \( \pi_\infty \), then we define (see [15]) \( C(\pi_\infty) := \prod_{i=1}^r (1 + |\mu_i|) \).
2.4. Maximal Eisenstein series

Let $P$ be the standard parabolic subgroup in $G$ attached to the $r = (r - 1) + 1$ partition. We choose a generalised principal series vector

$$f_s \in \mathcal{I}_{r-1,1}(s) := \text{Ind}_{P(\mathbb{A})}^G(\delta_{(\mathbb{A})}) \mid \det \mid^s \Pi \mid^{-s(r-1)}, \quad s \in \mathbb{C},$$

by

$$f_s(g) = f_s,\phi(g) := \int_{\mathbb{H}^r} \Phi(te_r g) |\det(tg)|^s d^x t,$$  \hspace{1cm} (2.2)

where $\Phi \in \mathcal{S}(\mathbb{A}^r)$ is a Schwartz–Bruhat, factorisable function and $e_r = (0, \ldots, 0, 1) \in \mathbb{A}^r$. The integral in equation (2.2) converges for $\Re(s) > 1/r$ and then can be extended meromorphically to the whole complex plane. By $\hat{\Phi}$, we denote Fourier transform of $\Phi$ that is defined by

$$\hat{\Phi}(x) := \int_{\mathbb{A}^r} \Phi(u) \psi_0(x_1 u_1 + \cdots + x_r u_r) du.$$  

We abbreviate $f_s,\phi$ as $\hat{f}_s$. We record the following transformation property of $f_s$, which can be seen from equation (2.2):

$$f_s \left[ \begin{array}{c} h \\ 1 \end{array} \right] g = |\det(h)|^s f_s(g), \quad h \in \text{GL}_{r-1}(\mathbb{A}), g \in G(\mathbb{A}).$$  \hspace{1cm} (2.3)

Finally, we define the maximal Eisenstein series associated to $f_s$ by

$$\text{Eis}(f_s)(g) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} f_s(\gamma g).$$

The above definition is valid for $s$ in a right half plane and then can be extended to all of $\mathbb{C}$ by meromorphic continuation. From [21, §4], [11, §2.3.1], we know that for $f_s,\phi \in \mathcal{I}_{r-1,1}(s)$ with some $\Phi \in \mathcal{S}(\mathbb{A}^r)$, the maximal parabolic Eisenstein series $\text{Eis}(f_s,\phi)$ has at most simple poles at $s = 0$ and $s = 1$. The residues at these poles are independent of $g$.

Let $\tilde{P}$ be the maximal parabolic subgroup in $G$ attached to the partition $r = 1 + (r - 1)$ (the associate parabolic to $P$). We can similarly construct an associated Eisenstein series from a vector $\tilde{f}_s \in \mathcal{I}_{1,r-1}(s)$ defined analogously. All of the properties of an Eisenstein series associated to $P$ hold analogously for the same associated to $\tilde{P}$.

2.5. Genericity and Kirillov model

We briefly review the Whittaker and Kirillov models of a generic representation of $G$ over a local field; see [16] for details. In this subsection, we only work locally, without mentioning the underlying local field. Fix a nondegenerate additive character $\psi$ of $N < G$. Consider the space of Whittaker functions on $G$ by

$$\mathcal{W}(G) := \left\{ W \in C^\infty(G) \mid \begin{array}{l} W(ng) = \psi(n)W(g), n \in N, g \in G; \\ W \text{ grows at most polynomially in } g \end{array} \right\},$$

on which $G$ acts by right translation.

We call an irreducible representation $\pi$ of $G$ generic if there exists a $G$-equivariant embedding $\pi \hookrightarrow \mathcal{W}(G)$. For generic $\pi$, we identify $\pi$ with its image in $\mathcal{W}(G)$, which we call the Whittaker model of $\pi$ under this embedding.
It is known (for example, see [16] for the case of an archimedean local field) from the theory of the Kirillov model that if \( \pi \) is an irreducible generic representation of \( \text{PGL}(r) \), then

\[
\pi \ni W \mapsto \left\{ \text{GL}(r - 1) \ni g \mapsto W \left( \begin{pmatrix} g & \hbar \\ 1 & 1 \end{pmatrix} \right) \right\}
\]
is injective and the space of the restricted Whittaker functions in the right-hand side, which is called the Kirillov model, is isomorphic to \( \pi \) as well. It is also known that the space \( C^\infty_c(N_{r-1}(\mathbb{R}) \backslash \text{GL}_{r-1}(\mathbb{R}), \psi) \) is contained in \( \pi \) under this realisation; see [16, Proposition 5].

If \( \pi \) is also unitary, then we can give a unitary structure on its Whittaker model by the inner product

\[
\langle W_1, W_2 \rangle := \int_{N_{r-1}(\mathbb{R}) \backslash \text{GL}_{r-1}(\mathbb{R})} W_1 \left( \begin{pmatrix} h & 0 \\ 1 & 1 \end{pmatrix} \right) \overline{W_2} \left( \begin{pmatrix} h & 0 \\ 1 & 1 \end{pmatrix} \right) dh;
\]

that is, we have \( \langle W_1(.g), W_2(.g) \rangle = \langle W_1, W_2 \rangle \) for \( g \in G \).

### 2.6. Zeta integrals

We review the theory of the \( \text{GL}(r) \times \text{GL}(r) \) Rankin–Selberg integral. We refer to [11] for details. We choose \( \phi_0 \in \pi_0 \) with a factorisable Whittaker function \( W_0 \) and a maximal Eisenstein series \( \text{Eis}(f_s) \) attached to some vector \( f_s \) in the generalised Principal series \( \mathcal{I}_{r-1}(s) \), as defined in Section 2.4. Let \( \pi \in \hat{\mathbb{H}} \) and \( \phi \in \pi \) be any automorphic form. One defines the global Rankin–Selberg integral [11, §2.3.2] of \( \phi_0 \) and \( \phi \) by

\[
\Psi(f_s, \phi_0, \tilde{\phi}) := \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})} \phi_0(g) \overline{\phi(g)} \text{Eis}(f_s)(g) dg,
\]

where \( s \in \mathbb{C} \) is such that \( \text{Eis}(f_s) \) is regular. As \( \phi_0 \) is cuspidal, the above integral converges absolutely. For \( s \) in a right half plane performing a standard unfolding-folding, one gets

\[
\Psi(f_s, \phi_0, \tilde{\phi}) = \int_{P(\mathbb{Q}) \backslash G(\mathbb{A})} \phi_0(g) \overline{\phi(g)} f_s(g) dg.
\]

The above integral representation of \( \Psi(f_s, \phi_0, \tilde{\phi}) \) has a meromorphic continuation to all \( s \in \mathbb{C} \). It is known that if \( \pi \) and \( \pi_0 \) are cuspidal, then the only possible poles of \( \Psi \) are simple and can occur at \( \Re(s) = 0, 1 \).

We may choose \( f_s \) to be factorisable, which can be done by choosing \( \Phi \in S(\mathbb{A}^r) \), as in Section 2.4, to be factorisable. Furthermore, if we assume that \( \pi \) is generic and \( \phi \in \pi \) has a factorisable Whittaker function \( W_\phi \), then for all \( s \in \mathbb{C} \), the global zeta integral is Eulerian – that is, factors in local zeta integrals

\[
\Psi(f_s, \phi, \tilde{\phi}) = \Psi_{\infty}(f_s, \omega, W_0, \overline{W_\phi}) \prod_{p < \infty} \Psi_p(f_s, \omega, W_0, \overline{W_\phi}),
\]

where the local zeta integral \( \Psi_{\infty} \) is defined by

\[
\Psi_{\infty}(f_s, \omega, W_0, \overline{W_\phi}) := \int_{N(\mathbb{R}) \backslash G(\mathbb{R})} W_0, \overline{W_\phi} \Phi_{\infty}(e_r g) | \det(g)|^s dg,
\]

for \( s \) being in some right half plane and then can be meromorphically continued to the whole complex plane. Similarly, the nonarchimedean zeta integral \( \Psi_p \) is defined by replacing \( \infty \) with \( p \) and \( \mathbb{R} \) with \( \mathbb{Q}_p \).
It is known that if \( \pi_{0,p} \) and \( \pi_p \) are unitary and \( \vartheta_0 \) and \( \vartheta \) tempered, respectively, then the above integral representation of \( \Psi_p \) is valid for \( \Re(s) \geq 1/2 \) if \( \vartheta + \vartheta_0 < 1/2 \) and \( p \leq \infty \) (this can be seen in the archimedean case from the bounds of the Whittaker functions in Lemma 7.2).

We record the local functional equation satisfied by \( \Psi_\infty \). From \([11, \text{Theorem 3.2}]\), we have

\[
\int_{N(\mathbb{R}) \setminus \GL_r(\mathbb{R})} W_{0,\infty}(g) \overline{W_{\phi,\infty}(g) \Phi_\infty(e_r g)} |\det(g)|^{1-s} dg = \gamma_\infty(s, \pi_{0,\infty} \otimes \pi_\infty, \psi) \int_{N(\mathbb{R}) \setminus \GL_r(\mathbb{R})} W_{0,\infty}(g) W_{\phi,\infty}(g) \Phi_\infty(e_r g) |\det(g)|^s dg.
\]

Here \( \tilde{W} \) denotes the contragredient Whittaker function of \( W \) defined by \( \tilde{W}(g) := W(wg^{-1}) \), where \( w \) is the long Weyl element in \( G(\mathbb{R}) \) and \( \gamma_\infty(.,.,\psi) \) denotes the local archimedean \( \gamma \)-factor. As the additive character \( \psi \) is fixed throughout the paper, we drop \( \psi \) from the notation of \( \gamma_\infty \). Folding the above integrals over \( \mathbb{R}^\times \), we can also rewrite the local functional equation as

\[
\Psi_\infty(\hat{f}_{1-s,\infty}, \tilde{W}_{0,\infty}, \overline{W_\infty}) = \gamma_\infty(s, \pi_{0,\infty} \otimes \pi_\infty) \Psi_\infty(f_{s,\infty}, W_{0,\infty}, \overline{W_\infty}), \tag{2.4}
\]

for any \( W_\infty \in \pi_\infty \), and \( f \) is related to \( \Phi \) according to equation (2.2). From the definition of the \( \gamma \)-factors (see \([11, \text{p. 120}]\)), one can check that \(|\gamma_\infty(1/2, \Pi)| = 1\) if \( \Pi \) is unitary.

### 2.7. Plancherel formula

We refer to \([27, \text{§2.2}]\) for a more detailed discussion of the Plancherel formula.

Recall the automorphic Plancherel measure \( d\mu_{\text{aut}} \) on \( \hat{\mathbb{X}} \) from Section 2.3. Let \( \phi_1, \phi_2 \in \mathcal{C}^\infty(\mathbb{X}) \) with rapid decay at all cusps. We record a Plancherel formula (i.e., a spectral decomposition) of the inner product between \( \phi_1 \) and \( \phi_2 \),

\[
\langle \phi_1, \phi_2 \rangle = \int_{\mathbb{R}} \sum_{\phi \in \mathcal{B}(\pi)} \langle \phi_1, \phi \rangle \langle \phi, \phi_2 \rangle d\mu_{\text{aut}}(\pi), \tag{2.5}
\]

where \( \mathcal{B}(\pi) \) is an orthonormal basis of \( \pi \) and

\[
\langle f_1, f_2 \rangle := \int_{\mathbb{X}} f_1(g) \overline{f_2(g)} dg.
\]

The identity equation (2.5) is independent of choice of \( \mathcal{B}(\pi) \).

Rapid decay properties of \( \phi_i \) imply that all the inner products on the right-hand side of equation (2.5) converge. One can show by the trace class property in \( L^2(\mathbb{X}) \) of some inverse Laplacian that the right-hand side of equation (2.5) converges absolutely.

### 2.8. Spectral weights

Let \( \pi, \pi_0 \in \hat{\mathbb{X}}_{\text{gen}} \) with \( W_{0,\infty} \in \pi_{0,\infty} \) and \( f_s \in \mathcal{I}_{r-1,1}(s) \). We define the spectral weight

\[
J(f_{s,\infty} W_{0,\infty}, \pi_\infty) := \sum_{W_\infty \in \mathcal{B}(\pi_\infty)} |\Psi_\infty(f_{s,\infty}, W_{0,\infty}, \overline{W_\infty})|^2. \tag{2.6}
\]

Here \( \mathcal{B}(\pi_\infty) \) is an orthonormal basis of \( \pi_\infty \). The sum in the right-hand side of equation (2.6) is absolutely convergent and is independent of choice of \( \mathcal{B}(\pi_\infty) \); see \([5, \text{Appendix 4}]\).
The definition of $J$ involves only the archimedean components of the representations and functions. In fact, one can define $J$ for any irreducible generic unitary representation $\sigma$ of $G(\mathbb{R})$ and $\beta \in C^\infty(N(\mathbb{R}) \backslash G(\mathbb{R}), \psi_\infty)$ with sufficient decay at infinity, by

$$J(\beta, \sigma) := \sum_{W \in \mathcal{B}(\sigma)} \left| \int_{N(\mathbb{R}) \backslash G(\mathbb{R})} \beta(g) \overline{W(g)} dg \right|^2.$$ 

Then, using the Whittaker–Plancherel formula (see [33, Chapter 15]), one can obtain that

$$\int_{G(\mathbb{R})} J(\beta, \sigma) d\mu_{\text{loc}}(\sigma) = \int_{N(\mathbb{R}) \backslash G(\mathbb{R})} |\beta(g)|^2 dg =: \|\beta\|^2_{L^2(N(\mathbb{R}) \backslash G(\mathbb{R}))}.$$ (2.7)

Here $\overline{G(\mathbb{R})}$ is the tempered unitary dual of $G(\mathbb{R})$ equipped with the local Plancherel measure $d\mu_{\text{loc}}$ compatible with the chosen Haar measure on $G(\mathbb{R})$ (see [6, §4.13.2]).

2.9. Analytic newvectors

Analytic newvectors are certain approximate archimedean analogues of the classical nonarchimedean newvectors pioneered by Casselman [8] and Jacquet–Piatetski-Shapiro–Shalika [20]. Let $K_0(p^N) \subset \text{PGL}_r(\mathbb{Z}_p)$ be the subgroup of matrices whose last rows are congruent to $(0, \ldots, 0, *) \mod p^N$. Let $\sigma$ be a generic irreducible representation of $\text{PGL}_r(\mathbb{Q}_p)$, and let $N_0$ be the minimal nonnegative integer such that $\sigma$ contains a nonzero vector $v$ that is invariant by $K_0(p^{N_0})$. Let $C(\sigma)$ be the conductor of $\sigma$, which can be defined in terms of the local gamma factor attached to $\sigma$. Then the main theorem of [8, 20] states that the real number $p^{N_0}$ is equal to $C(\sigma)$. One calls such a $v$ a newvector of $\sigma$. In [20], the authors call newvectors essential vectors; and in some literature, the authors call them newforms.

In [19] the authors produce an approximate analogue of this theorem at the archimedean place. Let $X > 1$ be tending to infinity and $\tau > 0$ be sufficiently small but fixed. We define an approximate congruence subgroup $K_0(X, \tau) \subseteq \text{PGL}_r(\mathbb{R})$, which is an archimedean analogue of the subgroup $K_0(p^N)$, in the following way: it is the image in $\text{PGL}_r(\mathbb{R})$ of

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_r(\mathbb{R}) \right| \begin{array}{l} a \in \text{GL}_{r-1}(\mathbb{R}), \quad |a - 1_{r-1}| < \tau, \quad |b| < \tau, \\ d \in \text{GL}_1(\mathbb{R}), \quad |c| < \frac{\tau}{X}, \quad |d - 1| < \tau \end{array} \right\}. $$ (2.8)

Here, $|.|$ denotes an arbitrary fixed norm on the corresponding spaces of matrices. Fix $0 \leq \theta < 1/2$. Then in [19, Theorem 1], the authors show that for all $\epsilon > 0$, there is a $\tau > 0$ such that for all generic irreducible unitary $\theta$-tempered representation $\pi$ of $\text{PGL}_r(\mathbb{R})$, there is a unit vector $v \in \pi$ such that

$$\|\pi(g)v - v\|_\pi < \epsilon \quad \text{for all } g \in K_0(C(\pi), \tau),$$

where $C(\pi)$ is the analytic conductor of $\pi$. We call such a vector $v$ an analytic newvector of $\pi$.

The authors also prove that [19, Theorem 7] any unit vector $v$ in the Kirillov model of $\pi$ that can be given by a function in $C_c^\infty(N_{r-1}(\mathbb{R}) \backslash \text{GL}_{r-1}(\mathbb{R}), \psi_\infty)^{0_{r-1}(\mathbb{R})}$ is a newvector. Moreover, $v$ can be chosen in a way such that if $W$ is the image of $v$ in the corresponding Whittaker model, then also

$$|W(g) - W(1)| < \epsilon$$

for all $g \in K_0(C(\pi), \tau)$ and $W(1) \asymp 1$. 

2.10. Main theorem

Theorem 2.1. Let \( r \geq 3 \) and \( X \) be tending to infinity. Let \( \pi_0 \) be a fixed cuspidal representation in \( \hat{\mathcal{X}} \) such that \( \pi_{0,\infty} \) is \( \theta_0 \)-tempered for some \( 0 \leq \theta_0 < 1/(r^2 + 1) \). We define a weight function
\[
J_X : \hat{\mathcal{X}}_{\text{gen}} \to \mathbb{R}_{\geq 0},
\]
as in equation (4.5), which satisfies the following properties:

- \( J_X(\pi) \) only depends on the archimedean component of \( \pi \) (with an abuse of notation, we write \( J_X(\pi) = J_X(\pi_\infty) \)).
- If \( \pi_\infty \) is \( \theta \)-tempered such that \( \theta + \theta_0 < 1/2 \) and \( C(\pi_\infty) < X \), then \( J_X(\pi_\infty) \gg \pi_0 \).
- \( \int_{G(\mathbb{R})} J_X(\pi_\infty) d\mu_{\text{loc}}(\pi_\infty) = X^{r-1} \).

And finally, we have
\[
\int_{\hat{\mathcal{X}}_{\text{gen}}} \frac{|L(1/2, \tilde{\pi} \otimes \pi_0)|^2}{\ell(\pi)} J_X(\pi) d\mu_{\text{aut}}(\pi) = X^{r-1} \left( r \frac{\zeta(r/2)^2}{\zeta(r)} L(1, \pi_0, \text{Ad}) \log X + O_{\pi_0}(1) \right),
\]
where \( \tilde{\pi} \) is the contragredient of \( \pi \). Here \( \ell(\pi) \) is defined as in equation (4.3) and only depends on the nonarchimedean data of \( \pi \).

If \( \pi \) is cuspidal, then \( \ell(\pi) \asymp L(1, \pi, \text{Ad}) \) with an absolute implied constant and thus \( \ell(\pi) \ll_{\epsilon} C(\pi)^{\epsilon} \), which follows from [25].

We note that if \( \pi \in \hat{\mathcal{X}}_{\text{gen}} \), then \( \pi_\infty \) is \( \theta \)-tempered for \( \theta < 1/2 \), which is a result in [26]. Thus the \( \theta_0 \)-temperedness assumption of \( \pi_{0,\infty} \) in Theorem 2.1 implies that \( J_X(\pi) \gg 1 \) for all \( \pi \in \hat{\mathcal{X}}_{\text{gen}} \) with \( C(\pi) < X \). Moreover, the family
\[
\mathcal{F}^\text{gen}_X := \{ \text{generic automorphic representations} \pi \text{ of } \text{PGL}_r(\mathbb{Z}) \text{ with } C(\pi) < X \}
\]
has \( \ell(\pi)^{-1} \)-weighted cardinality \( \asymp X^{r-1} \). This is essentially contained in the proof of [19, Theorem 9]. Hence, \( J_X \) can be realised as a smoothened characteristic function of the \( \ell(\pi)^{-1} \)-weighted family \( \mathcal{F}^\text{gen}_X \).

Consequently, we have an immediate corollary of Theorem 2.1.

Corollary 2.2. Let \( \pi_0 \) be as in Theorem 2.1. Then
\[
\sum_{\hat{\mathcal{X}} \ni \pi \text{ cuspidal} \atop C(\pi) < X} \frac{|L(1/2, \pi \otimes \pi_0)|^2}{L(1, \pi, \text{Ad})} \ll_{\pi_0} X^{r-1} \log X
\]
as \( X \) tends to infinity.

This is the sharpest possible (Lindelöf on average) second-moment estimate of the cuspidal Rankin–Selberg central \( L \)-values.

3. The Fourier expansion of maximal Eisenstein series

We recall some useful information about the Fourier expansion of maximal Eisenstein series. The computation is essentially done in [24], but we extract the relevant computation for completeness.

Let \( f_s \) be a holomorphic section in the generalised principal series \( I_{r-1,1}(s) \) such that \( f_s \) is constructed from a Schwartz–Bruhat function \( \Phi \in S(\mathbb{A}^r) \), as described in Section 2.4. Let \( \text{Eis}(f_s) \) be the Eisenstein series attached to \( f_s \).
We want to understand the Fourier expansion of $E(s)$. It is a straightforward calculation using the Bruhat decomposition. We sketch out the essential details for completeness. Let $R(s)$ be sufficiently large. We temporarily allow $\psi$ to be a possibly degenerate character of $N$. Then

$$\int_{N(G)\backslash N(\mathbb{A})} E(s)(ng)\overline{\psi(n)}dn = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \int_{N(G)\backslash N(\mathbb{A})} f_s(\gamma ng)\overline{\psi(n)}dn. \tag{3.1}$$

We start with the Bruhat decomposition of $G(\mathbb{Q})$ with respect to $P(\mathbb{Q})$. The Bruhat cells are indexed by a subset of the Weyl group, namely, the subset of Weyl elements $w$ such that $w\alpha > 0$ for all simple roots $\alpha$ other than $\alpha_0$ that determines $P$.

**Lemma 3.1.** We define the Weyl elements

$$w_i := \begin{pmatrix} I_{i-1} & I_{r-i} \\ 1 & \end{pmatrix}, \quad 1 \leq i \leq r.$$ 

Also let $N_i$ be the subgroup of $N$ of the form

$$N_i := \left\{ n := \begin{pmatrix} I_{i-1} & x \\ 1 & I_{r-i} \end{pmatrix} \mid x := (x_1, \ldots, x_{r-i}) \right\}.$$ 

Then

$$G(\mathbb{Q}) = \bigcup_{i=1}^r P(\mathbb{Q})w_iN_i(\mathbb{Q}),$$

where the union is disjoint.

**Proof.** Any $\gamma \in G(\mathbb{Q})$ has the bottom row of the form $(0, \ldots, 0, d, \ast, \ldots, \ast)$, where $d \neq 0$ and occurs at the $i$th position for some $1 \leq i \leq r$. There exists an element $x \in N_i$ such that $\gamma = dy'x$ with $y'$ having bottom row of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the $i$th position. It can readily be checked that $y'w_i^{-1} \in P(\mathbb{Q})$. Clearly, the union is disjoint. \hfill $\square$

Using Lemma 3.1 and the left-$P(\mathbb{Q})$ invariance of $f_s$, we can rewrite the right-hand side of equation (3.1) as

$$\sum_{i=1}^r \sum_{\gamma \in N_i(\mathbb{Q})} \int_{N(G)\backslash N(\mathbb{A})} f_s(\gamma ng)\overline{\psi(n)}dn.$$ 

Note that $N_i := N \cap w_i^{-1}N'w_i$. Hence, $N = N_i\tilde{N}_i$, where $\tilde{N}_i := N \cap w_i^{-1}Nw_i$. It can be checked that

$$\tilde{N}_i = \{ n \in N \mid e_in = e_i \},$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the $i$th place.

We write an element $n \in N$ as $n_1n_2$ with $n_1 \in N_i$ and $n_2 \in \tilde{N}_i$. Unfolding the $N_i(\mathbb{Q})$ sum, we obtain that the right-hand side of equation (3.1) is equal to

$$\sum_{i=1}^r \int_{\tilde{N}_i(\mathbb{Q})\backslash \tilde{N}_i(\mathbb{A})} \overline{\psi(n_2)} \int_{N_i(\mathbb{A})} f_s(w_in_1n_2g)\overline{\psi(n_1)}dn_1dn_2.$$
There exist \( n'_1 \in \tilde{N}_i \) and \( n'_2 \in N_i \) such that \( n'_1 n'_2 = n_1 n_2 \) and and \( n'_2 \in N \) such that \( n'_2 w_i = w_i n'_2 \). Appealing to the left-\( N(\mathbb{A}) \) invariance of \( f_s \), we conclude that the above expression is

\[
\sum_{i=1}^{r} \int_{N_i(\mathbb{A})} \psi(n_2) f_s(w_i n'_2 g) \psi(n_1) dn_1 dn_2.
\]

We check that if \( e_i n_1 = (0, 1, x) \) for some \( x \in \mathbb{A}^{r-1} \), then \( n'_1 = (0, 1, xu) \) for some upper triangular unipotent matrix \( u \) in \( GL_{r-1}(\mathbb{A}) \). Also, \( \psi(n_1) \) is equal to \( \psi(n'_1) \). Thus, making the change of variables \( xu \mapsto x \), we obtain that

\[
\int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} Eis(f_s)(ng) \psi(n) dn = \sum_{i=1}^{r} \int_{N_i(\mathbb{Q}) \setminus N_i(\mathbb{A})} \psi(n') dn' \int_{N_i(\mathbb{A})} f_s(w_i ng) \psi(n) dn. \tag{3.2}
\]

Clearly, if \( \psi \) is nondegenerate, the above is zero. In particular, if \( \psi \) is of the form \( \theta_{\tilde{q}} \) for some \( \tilde{q} := (q_j)_j \in \mathbb{Q}^{r-1} \), then the \( i \)th summand, for \( i < r \), on the right-hand side of equation (3.2) does not identically vanish only if \( q_j = 0 \) for all \( j \neq i \). For \( i = r \), the same happens only if \( \tilde{q} = 0 \), in which case the summand is equal to \( f_s(g) \). For \( q \in \mathbb{Q}^r \), we denote \((0, \ldots, 0, q, 0, \ldots, 0)\), where \( q \) is at the \( i \)th place, by \( i(q) \).

We define (again on a right half plane, and extend by meromorphic continuation) twisted intertwining operators on \( \mathcal{I}_{r-1,1}(s) \ni f_s \) attached to the Weyl element \( w_i \) by

\[
M^q_i f_s(g) := \int_{N_i(\mathbb{A})} f_s(w_i g) \psi_{i(q)}(n) dn. \tag{3.3}
\]

Thus we obtain the following Fourier expansion of \( Eis(f_s)(g) \).

**Lemma 3.2.** Let \( f_s \) and \( Eis(f_s)(g) \) be as above. Then

\[
Eis(f_s)(g) = f_s(g) + \sum_{q \in \mathbb{Q}} \sum_{i=1}^{r-1} M^q_i f_s(g).
\]

The terms \( f_s \) and \( M^0_i f_s \) are the constant terms of \( Eis(f_s)(g) \).

Let us write \( g \in G(\mathbb{A}) \) in its Iwasawa coordinates \( g = nak \), where \( a := \begin{pmatrix} a(y) \\ 1 \end{pmatrix} \). Then for \( i < r \), we have

\[
M^q_i f_s(g) = \int_{N_i(\mathbb{A})} f_s(w_i n' nak) \psi_{i(q)}(n) dn',
\]

which is defined for \( \Re(s) \) large enough and can be meromorphically continued.

We work exactly as before to compute the above integral. We write \( n = n_1 n_2 \) with \( n_1 \in N_i \) and \( n_2 \in \tilde{N}_i \) and make the change of variables \( n' \mapsto n'n_{i}^{-1} \). Then we write \( w_i n' n_2 = n'_2 w_i n'' \) for some \( n'_2 \in N \) and \( n'' \) that is related to \( n' \) as before, and we make the change of variables \( n'' \mapsto n' \). We use the left \( N(\mathbb{A}) \)-invariance of \( f_s \) and the fact that \( \psi(n') = \psi(n'') \).

Finally, we make the change of variables \( n' \mapsto an'a^{-1} \) and use the transformation property of \( f_s \) as in equation (2.3) to obtain

\[
M^q_i f_s(g) = \psi_{i(q)}(n_1) \prod_{j=1}^{i-1} |y_j|^{(1-s)(r-j)} \int_{N_i(\mathbb{A})} f_s(w_i nk) \psi_{i(q)}(ana^{-1}) dn. \tag{3.4}
\]
We first study the integral on the right-hand side of equation (3.4) for \( q = 0 \). We use the construction of \( f_s \) using \( \Phi \in S(\mathbb{A}^r) \) as in Section 2.4. We also parametrise \( n \) so that \( e_i n = (0, 1, x) \) with \( x \in \mathbb{A}^{r-i} \) and make the change of variables \( x \mapsto x/t \) to write the integral as

\[
\int_{\mathbb{A}^{r-i}} \int_{\mathbb{A}^{r}} (k, \Phi)(0, t, x)|t|^{r-s-r+i} d^x d t.
\]

Here \( (k, \Phi)(x) := \Phi(xk) \). Using Tate’s functional equation (see [7, Proposition 3.1.6]), we can rewrite the above as

\[
\int_{\mathbb{A}^{r}} \frac{1}{(k, \Phi)^i} \frac{(te_i)|t|^{r-i+1-r} d^t}{d x},
\]

where the partial Fourier transform \( \hat{\Phi}^i \) is defined by

\[
\hat{\Phi}^i(x_1, \ldots, x_r) := \int_{\mathbb{A}^{r-i}} \Phi(x_1, \ldots, x_{i-1}, u_1, \ldots, u_{r-i+1})(x_i u_1 + \cdots + x_r u_{r-i+1}) d u.
\]

In particular, it can be seen that

\[
M_{1f_s}(g) = f_{1-s, \Phi(wg^{-i})} =: \tilde{f}_s(g), \quad \text{(3.5)}
\]

where \( w \) is the long Weyl element. It can be checked that \( \tilde{f} \) lies in the principal series \( \mathcal{I}_{1, r-1}(1 - s) \) arising from the associate parabolic \( \tilde{P} \).

Now, for \( q \neq 0 \), the integral on the right-hand side of equation (3.4) gives rise to a degenerate Whittaker function. Parametrising \( n \in N_1(\mathbb{A}) \) as in Lemma 3.1, one can see that \( \psi_{i(q)}(ana^{-1}) = \psi_0(q y_i x_1) \); that is, the value \( \psi_{i(q)}(ana^{-1}) \) depends on \( a \) only through \( y_i \). We define

\[
W_{f_s}^i(q y_i, k) := \int_{N_1(\mathbb{A})} f_i(w, nk) \psi_{i(q)}(ana^{-1}) d n.
\]

Again, the above is defined for \( \Re(s) \) sufficiently large and can be extended analytically to all of \( \mathbb{C} \), and it can be shown that \( W_{f_s}^i(t, k) \) decays rapidly as \( t \to \infty \). We prove these claims in Lemma 6.3 (although these results are implicitly done in [17]). In particular, we have

\[
M_{qf_s}^i(g) = W_{f_s}^i(q y_i, k) \psi_0(q x_{i, i+1})
\]

for \( q \neq 0 \).

We summarise the above results and rewrite Lemma 3.2 in the following proposition to record the Fourier expansion of a maximal Eisenstein series.

**Proposition 3.3.** Let \( f_s \in \mathcal{I}_{r-1, 1}(s) \) be a holomorphic section and \( \text{Eis}(f_s)(g) \) be the corresponding maximal Eisenstein series. Let \( g = n(x) \left( \begin{array}{c} a(y) \\ 1 \end{array} \right) \) be its Iwasawa decomposition. Then

\[
\text{Eis}(f_s)(g) = f_s(g) + \sum_{i=1}^{r-1} \prod_{j=1}^{i-1} |y_j|^{s j} \prod_{j=i}^{r-1} |y_j|^{(1-s)(r-j)} \left[ M_{qf_s}^i(k) + \sum_{q \in \mathbb{Q}^*} W_{f_s}^i(q y_i, k) \psi_0(q x_{i, i+1}) \right].
\]

The terms containing \( M_{qf_s}^i \) are the constant terms of \( \text{Eis}(f_s)(g) \), and the terms containing \( W_{f_s}^i \) are holomorphic in \( s \). The above sum converges absolutely and uniformly on compacta.
4. Proof of the main theorem

4.1. Choices of the local components

We start by choosing various vectors and auxiliary test functions. Let $\pi_0 \in \hat{X}_{\text{gen}}$ be the fixed cuspidal representation as in Theorem 2.1. Let $\phi_0 \in \pi_0$ with Whittaker function $W_0 = \bigotimes_{p \leq \infty} W_{0,p}$, such that $W_{0,p}$ are unramified for $p < \infty$ with $W_{0,p}(1) = 1$.

Here and elsewhere in the paper, the index set $\{p \leq \infty\}$ denotes the set of places $\{\infty\} \cup \{p \text{ prime in } \mathbb{Z}\}$.

We choose $W_{0,\infty} \in \pi_{0,\infty}$ so that

$$\|W_{0,\infty}\|_{\pi_{0,\infty}} = 1,$$

and $W_{0,\infty} \begin{bmatrix} \cdot \cr 1 \end{bmatrix} \in C_c^\infty(N_{f-1}(\mathbb{R}) \setminus GL_{f-1}(\mathbb{R}), \psi_{\infty})^{O_{f-1}(\mathbb{R})},$

whose existence is guaranteed by the theory of Kirillov model. We choose $S(\mathbb{A}^r) \ni \Phi = \bigotimes_{p \leq \infty} \Phi_p$ with

$$\Phi_p := \text{char}(\mathbb{Z}_p^r) = \hat{\Phi}_p \text{ for } p < \infty,$$

and for $\tau > 0$ sufficiently small but fixed

$$\Phi_\infty \in C_c^\infty(B_\tau(0, \ldots, 0, 1)),$$

such that $\Phi_\infty$ is nonnegative and has values sufficiently concentrated near 1. Here $B_\tau$ denotes the ball of radius $\tau$. Thus $\Phi_\infty$ can be thought as a smoothened characteristic function of $B_\tau(0, \ldots, 0, 1)$.

Let $f_\sigma := f_{\sigma,\Phi} \in I_{r-1,1}(s)$ be associated to $\Phi$ according to equation (2.2). The choice of $\Phi_\infty$ ensures that there exist sufficiently small $\tau_1, \tau_2 > 0$ (depending on $\tau$ and $\Phi_\infty$) such that $f_{1/2,\infty}\begin{bmatrix} I_{r-1} \\
- c \\
1 \end{bmatrix}$ is supported on $|c| \leq \tau_1$ and has values $\asymp 1$ for $|c| \leq \tau_2$. This implies that

$$\int_{\mathbb{R}^{r-1}} |f_{1/2,\infty}|^2 \begin{bmatrix} I_{r-1} \\
- c \\
1 \end{bmatrix} dc \asymp 1,$$

with an absolute implied constant. We renormalise $\Phi_\infty$ by an absolute constant so that the above integral is 1.

4.2. Computation of the spectral side

Let $\text{Eis}(f_\sigma) := \text{Eis}(f_{\sigma,\Phi})$ be the maximal Eisenstein series attached to $f_{\sigma,\Phi} \in I_{r-1,1}(s)$, which is defined in Section 4.1. Let $X > 1$ be a large number tending to infinity and

$$A(\mathbb{A}) \ni x := (x_p)_p, \quad x_{\infty} := \text{diag}(X, \ldots, X, 1) \in A(\mathbb{R}) \text{ and } x_p = 1 \text{ for all } p < \infty.$$

Our point of departure is the following period, which we write in two different ways:

$$\int_{\mathbb{X}} |\phi_0(g)|^2 |\text{Eis}(f_\sigma)(gx)|^2 dg = \langle \phi_0 \text{Eis}(f_\sigma)(.x), \phi_0 \text{Eis}(f_\sigma)(.x) \rangle. \quad (4.1)$$

We use the Parseval relation in equation (2.5) and the notations in Section 2.6 to write the right-hand side of equation (4.1) as

$$\int_{\mathbb{X}} \sum_{\phi \in \mathcal{B}(\pi)} |\Psi(f_\sigma(.x), \phi_0, \phi)|^2 d\mu_{\text{aut}}(\pi), \quad (4.2)$$

where $\mathcal{B}(\pi)$ is an orthonormal basis of $\pi$. 
Lemma 4.1. Let $\pi \in \hat{\mathcal{H}} \setminus \hat{\mathcal{H}}_{\text{gen}}$ be a nongeneric representation. Then $\Psi(f_s, \phi_0, \tilde{\phi}) = 0$ for all $\phi \in \pi$ and $s \in \mathbb{C}$.

Proof. For $\Re(s)$ sufficiently large, we have (see Section 2.6)

$$
\Psi(f_s, \phi_0, \tilde{\phi}) = \int_{P(\mathbb{Q}) \setminus G(\mathbb{A})} \phi_0(g)\overline{\phi(g)}f_s(g)dg.
$$

We follow the computation of [11, p.104–105]. We use the Fourier expansion

$$
\phi_0(g) = \sum_{\gamma \in N(\mathbb{Q}) \setminus P(\mathbb{Q})} W_0(\gamma g)
$$

and the left $P(\mathbb{Q})$-invariance of $f_s$ and unfold over $P(\mathbb{Q})$ to get

$$
\Psi(f_s, \phi_0, \tilde{\phi}) = \int_{N(\mathbb{Q}) \setminus G(\mathbb{A})} W_0(g)\overline{\phi(g)}f_s(g)dg.
$$

We fold the last integral over $N(\mathbb{A})$ and use the left $N$-equivariance of $W_0$ and the left $N$-invariance of $f_s$ to obtain

$$
\Psi(f_s, \phi_0, \tilde{\phi}) = \int_{N(\mathbb{Q}) \setminus G(\mathbb{A})} W_0(g)f_s(g)\int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} \overline{\phi(ng)}\psi(n)dndg.
$$

By definition, the inner integral vanishes as $\phi$ is nongeneric. Finally, by analytic continuation of $\Psi$, we extend the result for all $s \in \mathbb{C}$. \qed

Thus Lemma 4.1 allows us to reduce the integral in equation (4.2) only over $\hat{\mathcal{H}}_{\text{gen}}$. Once we restrict to $\pi \in \hat{\mathcal{H}}_{\text{gen}}$, we can use the Eulerian property of the zeta integral $\Psi$ as in Section 2.6. If $\phi \in \pi$ with $\|\phi\|_\pi = 1$ and Whittaker function $W_\phi = \bigotimes_{p \leq \infty} W_p$ such that $W_p$ is unramified and $W_p(1) = 1$ for $p < \infty$, then by Schur’s lemma, we have

$$
\|\phi\|^2_\pi = \ell(\pi)\|W_\infty\|^2_{\pi, \infty},
$$

where $\ell(\pi)$ only depends on the nonarchimedean data of $\pi$. A standard Rankin–Selberg computation\(^2\) yields that $\ell(\pi) = L(1, \pi, \text{Ad})$ for a cuspidal $\pi$. In fact, in our case, $\ell(\pi)$ is equal to $L(1, \pi, \text{Ad})$ up to a positive constant dependent only on $n$.

Another standard computation [11, Theorem 3.3] shows that

$$
\Psi_p(f_s, p, W_{0,p}, \overline{W_p}) = L_p(s, \pi_0 \otimes \overline{\pi}), \quad p < \infty,
$$

where $f_s$ is as chosen in Section 4.1 and $L_p(s, \cdot)$ denotes the unramified $p$-adic Euler factor of $L(s, \cdot)$. Thus by meromorphic continuation, we have

$$
\Psi(f_s, \phi_0, \tilde{\phi}) = L(s, \pi_0 \otimes \overline{\pi})\Psi_\infty(f_{s, \infty}, W_{0, \infty}, \overline{W_\infty})
$$

for all $s \in \mathbb{C}$ whenever both sides of the above are defined. Using the equation above, and recalling the harmonic weight in equation (4.3) and the spectral weight in equation (2.6), we obtain that equation (4.2) is equal to

$$
\int_{\hat{\mathcal{H}}_{\text{gen}}} \frac{|L(s, \pi_0 \otimes \overline{\pi})|^2}{\ell(\pi)} J(f_{s, \infty}(x_\infty)W_{0, \infty}, \pi_\infty)d\mu_{\text{aut}}(\pi).
$$

\(^2\)See [1, eq. (3.11)] and the computation there for the spherical case; the general case follows similarly.
We appeal to the holomorphicity of the zeta integrals to specify $s = 1/2$ and define a normalised spectral weight $J_X(\pi, \infty)$ by

$$J_X(\pi) := J_X(\pi, \infty) := X^{r-1} J(f_{1/2, \infty}(x, \infty) W_{0, \infty}, \pi). \quad (4.5)$$

Thus we write the main equation of our proof:

$$\int_{\mathbb{Z}_{\text{gen}}} \frac{|L(1/2, \pi_0 \otimes \tilde{\pi})|^2}{\ell(\pi)} J_X(\pi) d\mu_{\text{aut}}(\pi) = X^{r-1} \langle |\phi_0|^2, |\text{Eis}(f_{1/2})(x)|^2 \rangle. \quad (4.6)$$

### 4.3. Computation of the period side

Again recall the choices of the local factors in Section 4.1. We write $f_s = \bigotimes_{p \leq \infty} f_{s, p}$; then for $k_p \in K_p$,

$$f_{s, p}(k_p) = \int_{\mathbb{Q}_p^\times} \Phi_p(te_r k_p) |\det(tk_p)|^s d^r t.$$  

Here we fix Haar measures $d^r t$ on $Q^\times_p$ and (respectively, $d t$ on $Q_p$) such that $\text{vol}(\mathbb{Z}_p^\times) = 1$ (respectively, $\text{vol}(\mathbb{Z}_p) = 1$).

First, we record that $f_{s, p}$ is an unramified vector in $I_{r-1, 1}(s)_p$ and $\tilde{f}_{s, p}$ is an unramified vector in $I_{1, r-1}(1-s)_p$, which is a generalised principal series attached to the opposite parabolic of $P$. Note that $te_r k_p \in \mathbb{Z}_p^\times$ if and only if $t \in \mathbb{Z}_p$. Thus

$$f_{s, p}(k_p) = \sum_{m=0}^{\infty} p^{-mr s} = (1 - p^{-rs})^{-1},$$

for $\Re(s) > 0$. Similarly, using equation (3.5), we have

$$\tilde{f}_{s, p}(k_p) = f_{1-s, \Phi_p}(w k_p^{-t}) = (1 - p^{-r(1-s)})^{-1}$$

for $\Re(s) < 1$. Thus for $g$ in the fundamental domain of $\mathbb{X}$, we can write

$$f_s(g) = \zeta(rs) f_{s, \infty}(g_\infty) \quad (4.7)$$

and

$$\tilde{f}_s(g) = \zeta(r - rs) \tilde{f}_{s, \infty}(g_\infty), \quad (4.8)$$

for all $s \in \mathbb{C}$, which can be achieved by meromorphic continuation.

Let $\Re(s)$ be sufficiently small. From Proposition 3.3, we can see that among the constant terms of $\text{Eis}(f_{1/2+s})(g)$, the terms that do not lie in $L^2(\mathbb{X})$ are $f_{1/2+s}(g)$ and $\tilde{f}_{1/2+s}(g)$. Similarly, one checks that the constant terms of

$$\overline{\text{Eis}(f_{1/2})\text{Eis}(f_{1/2+s})} - \overline{f_{1/2}f_{1/2+s}} - \overline{\tilde{f}_{1/2}\tilde{f}_{1/2+s}}$$

are integrable in $L^2(\mathbb{X})$. Inspired by this, we define a regularised Eisenstein series of the form

$$\overline{E}_s := \overline{\text{Eis}(f_{1/2})\text{Eis}(f_{1/2+s})} - \text{Eis}(\overline{f_{1/2}f_{1/2+s}}) - \text{Eis}(\overline{\tilde{f}_{1/2}\tilde{f}_{1/2+s}}). \quad (4.9)$$
Proof of Theorem 2.1. Recall equations (4.6) and (4.9). We write the inner product on the right-hand side of equation (4.6) as
\[
\lim_{s \to 0} \langle |\phi_0|^2, E_s(x) \rangle + \lim_{s \to 0} \left[ \langle |\phi_0|^2, \text{Eis}(\overline{f_{1/2}^s f_{1/2+}^s})(x) \rangle + \langle |\phi_0|^2, \text{Eis}(\overline{\tilde{f}_{1/2}^s \tilde{f}_{1/2+}^s})(x) \rangle \right].
\]
The second term is the degenerate term as in equation (5.1). From equation (5.4), Proposition 5.2 and Lemma 5.1, we obtain that the second term above is
\[
rL(1, \pi_0, \text{Ad}) \frac{\xi(r/2)^2}{\xi(r)} \log X + O_{\pi_0}(1).
\]
On the other hand, we write the first term above, which is the regularised term, as
\[
\lim_{s \to 0} \int_X |\phi_0|^2 (g^{-1})E_s(g) dg,
\]
and bound this by
\[
\|\phi_0\|_{L^\infty(X)}^2 \int_X |E_s(g)| dg.
\]
From Proposition 6.1, we know that the last integral is convergent for \(s\) being sufficiently small, and \(E_s\) is holomorphic in a sufficiently small neighbourhood of \(s = 0\). Thus, using Cauchy’s residue theorem, we can write the above limit as
\[
\int_{|s| = \epsilon} \frac{1}{s} \int_X |\phi_0|^2 (g^{-1})E_s(g) dg \frac{ds}{2\pi i}
\]
for some arbitrary small but fixed \(\epsilon > 0\). Applying Proposition 6.1 once again, we confirm the above integral is \(O_{\phi_0, \epsilon}(1)\).

Now nonnegativity and the first property of the spectral weight \(J_X(\pi)\) follow from the definition in equation (4.5). The second property follows from Proposition 7.1. Finally, the third property follows from equation (2.7) and Lemma 5.1. \(\Box\)

5. Analysis of the degenerate terms in the period side

In this section, we analyse the degenerate terms
\[
\lim_{s \to 0} \left[ \langle |\phi_0|^2, \text{Eis}(\overline{f_{1/2}^s f_{1/2+}^s})(x) \rangle + \langle |\phi_0|^2, \text{Eis}(\overline{\tilde{f}_{1/2}^s \tilde{f}_{1/2+}^s})(x) \rangle \right].
\]
Note that \(\overline{f_{1/2}^s f_{1/2+}^s} \in \mathcal{H}_{r-1,1}(1 + s)\) is such that its local component \(\overline{f_{1/2, p}^s f_{1/2+}^s, p}\) is unramified for \(p < \infty\). Thus by the uniqueness of the spherical vector, \(\overline{f_{1/2, p}^s f_{1/2+}^s, p} \in \mathcal{H}_{r-1,1}(1 + s)\) is a multiple of the unramified vector
\[
g \mapsto \int_{\mathbb{Q}_p^\times} \Phi_p(ter g) |\det(tg)|^{1+s} dt.
\]
Comparing the values of the functions at the identity as before, we check that the multiple is
\[
\frac{(1 - p^{-r/2})^{-1}(1 - p^{-r/2 - rs})^{-1}}{(1 - p^{-r rs})^{-1}} \text{ for } p < \infty.
\]
We compute the first term inside the limit in equation (5.1) for \( \Re(s) \) large. Doing a similar computation as in Section 2.6 and using equation (4.4), we obtain

\[
\langle |\phi_0|^2, \text{Eis}(f_{1/2}f_{1/2+s})(.x) \rangle = \frac{\zeta(r/2)\zeta(r/2 + rs)}{\zeta(r + rs)} L(1 + s, \pi_0 \otimes \pi_0) \\
\Psi_\infty(f_{1/2,\infty}f_{1/2+s,\infty}(.x_\infty), W_{0,\infty}, \overline{W_{0,\infty}}). \tag{5.2}
\]

Finally, we meromorphically continue the above to the whole complex plane.

Similarly, we compute the second term inside the limit in equation (5.1). Note that in this case, \( \tilde{f} \) lies in \( \mathcal{I}_{1,r-1} \) associated to the parabolic \( \tilde{P} \). Working as in Section 2.6, we obtain that

\[
\langle |\phi_0|^2, \text{Eis}(\tilde{f}_{1/2}\tilde{f}_{1/2+s})(.x) \rangle = \int_{P(\mathbb{Q})\backslash G(\mathbb{A})} |\tilde{\phi}_0|^2(g)\tilde{f}_{1/2}\tilde{f}_{1/2+s}(gx)dg.
\]

We recall the definition of \( \tilde{f} \) in equation (3.5) and make the change of variables \( g \mapsto pg^{-1} \) to obtain that the above is equal to

\[
\int_{P(\mathbb{Q})\backslash G(\mathbb{A})} |\tilde{\phi}_0|^2(g)\tilde{f}_{1/2}\tilde{f}_{1/2-s}(gx^{-1})dg,
\]

where \( \tilde{\phi}_0(g) := \phi_0(pg^{-1}) \), which lies in the contragredient representation \( \pi_0 \). Note that \( \tilde{\Phi}_p = \Phi_p \) for \( p < \infty \). Thus, doing a calculation similar to that preceding equation (5.2), we obtain

\[
\langle |\phi_0|^2, \text{Eis}(\tilde{f}_{1/2}\tilde{f}_{1/2+s})(.x) \rangle = \frac{\zeta(r/2)\zeta(r/2 - rs)}{\zeta(r - rs)} L(1 - s, \pi_0 \otimes \pi_0) \\
\Psi_\infty(\tilde{f}_{1/2,\infty}\tilde{f}_{1/2-s,\infty}(.x_\infty^{-1}), \tilde{W}_{0,\infty}, \overline{\tilde{W}_{0,\infty}}). \tag{5.3}
\]

Recalling the definition of the contragredient \( \tilde{W}_0 \) and making the change of variables \( g_\infty \mapsto wg_\infty^{-1} \) in the definition of the zeta integral \( \Psi_\infty \), we also have

\[
\Psi_\infty(\tilde{f}_{1/2,\infty}\tilde{f}_{1/2-s,\infty}(.x_\infty^{-1}), \tilde{W}_{0,\infty}, \overline{\tilde{W}_{0,\infty}}) = \Psi_\infty(\tilde{f}_{1/2,\infty}\tilde{f}_{1/2+s,\infty}(.x_\infty), W_{0,\infty}, \overline{W_{0,\infty}}).
\]

In the following Lemma 5.1, we prove the archimedean factors \( \Psi_\infty \) on the right-hand side of equations (5.2) and (5.3) are equal for \( s = 0 \). We first record that

\[
\Psi_\infty(|h|^2, W_{0,\infty}, \overline{W_{0,\infty}}) = \|hW_{0,\infty}\|^2_{L^2(N(\mathbb{R})\backslash G(\mathbb{R}))},
\]

where \( h \) is either \( f_{1/2,\infty}(.x_\infty) \) or \( \tilde{f}_{1/2,\infty}(.x_\infty) \).

**Lemma 5.1.** Recall the choices of the local components in Section 4.1. We have

\[
\|W_{0,\infty}f_{1/2,\infty}(.x_\infty)\|^2 = \|W_{0,\infty}\tilde{f}_{1/2,\infty}(.x_\infty)\|^2 = \|W_{0,\infty}f_{1/2,\infty}\|^2 = 1,
\]

where all the norms are taken in \( L^2(N(\mathbb{R})\backslash G(\mathbb{R})) \).

**Proof.** To ease the notations, we drop \( \infty \) from the subscripts in this proof.

First recall that

\[
\tilde{f}_{1/2}(g) = \tilde{f}_{1/2}(wg^{-1}),
\]

...
which implies, by a change of variable $g \mapsto wg^{-t}$, that

$$\|W_0\tilde{f}_{1/2}(.x)\|^2 = \int_{N(\mathbb{R}) \backslash G(\mathbb{R})} |W_0(g)|^2 |\tilde{f}_{1/2}(wg^{-t}x^{-1})|^2 dg = \int_{N(\mathbb{R}) \backslash G(\mathbb{R})} |\tilde{W}_0(g)|^2 |\tilde{f}_{1/2}(gx^{-1})|^2 dg.$$  

We make the change of variables $g \mapsto gx$ and then employ the Whittaker–Plancherel formula as in equation (2.7) to write the above as

$$\|\tilde{\pi}_0(x)\tilde{W}_0\tilde{f}_{1/2}\|^2 = \int_{G(\mathbb{R})} \sum_{W \in B(\sigma)} |\Psi(\tilde{f}_{1/2}, \tilde{\pi}_0(x)\tilde{W}_0, W)|^2 d\mu_{\text{loc}}(\sigma).$$

We use the $GL(r) \times GL(r)$ local functional equation as in equation (2.4) and the unitarity of the gamma factor at $1/2$ to obtain that

$$|\Psi(\tilde{f}_{1/2}, \tilde{\pi}_0(x)\tilde{W}_0, W)|^2 = |\Psi(f_{1/2}, \pi_0(x^{-1})W_0, \tilde{W})|^2.$$  

Consequently, applying the Whittaker–Plancherel again with the orthonormal basis $B(\sigma) := \{\sigma(x)\tilde{W}\}$, we obtain

$$\|W_0\tilde{f}_{1/2}(.x)\|^2 = \|\tilde{\pi}_0(x)\tilde{W}_0\tilde{f}_{1/2}\|^2 = \|\pi_0(x^{-1})W_0f_{1/2}\|^2 = \|W_0f_{1/2}(.x)\|^2,$$

which proves the first equality.

Thus now it is enough to prove that

$$\|W_0f_{1/2}\|^2 = \|W_0f_{1/2}(.x)\|^2.$$  

We use Bruhat coordinates to write

$$\int_{N(\mathbb{R}) \backslash G(\mathbb{R})} |W_0|^2 |f_{1/2}|^2 \left| \frac{X}{c} \right| d\mu_{\text{loc}}(X)$$

$$= \int_{N_{r-1}(\mathbb{R}) \backslash G L_{r-1}(\mathbb{R})} \int_{\mathbb{R}^{r-1}} |W_0|^2 \left| \begin{pmatrix} h & 0 \\ c & 1 \end{pmatrix} \right| |f_{1/2}|^2 \left| \begin{pmatrix} hX & 0 \\ cX & 1 \end{pmatrix} \right| dc \; dh \left| \det(h) \right|.$$  

Using the transformation property of $f_{1/2}$ as in equation (2.3) and making the change of variables $c \mapsto c/X$, we obtain that the above is equal to

$$\int_{\mathbb{R}^{r-1}} |f_{1/2}|^2 \left| \begin{pmatrix} I_{r-1} & 0 \\ c & 1 \end{pmatrix} \right| \int_{N_{r-1}(\mathbb{R}) \backslash G L_{r-1}(\mathbb{R})} |W_0|^2 \left| \begin{pmatrix} h & 0 \\ c/X & 1 \end{pmatrix} \right| dh \; dc.$$  

Using the $G$-invariance of the inner product in the Whittaker model as in Section 2.5, we conclude that the inner integral above is equal to

$$\int_{N_{r-1}(\mathbb{R}) \backslash G L_{r-1}(\mathbb{R})} |W_0|^2 \left| \begin{pmatrix} h & 0 \\ 1 & 1 \end{pmatrix} \right| dh = \int_{N_{r-1}(\mathbb{R}) \backslash G L_{r-1}(\mathbb{R})} |W_0|^2 \left| \begin{pmatrix} h & 0 \\ 1 & c \end{pmatrix} \right| dh.$$  

Thus, reverse engineering the above manipulation with Bruhat coordinates (that is, taking $X = 1$), we conclude the proof of the first two equalities.

From the above proof, we also obtain that

$$\|W_0f_{1/2}\|^2 = \|W_0\|^2 \pi_0 \int_{\mathbb{R}^{r-1}} |f_{1/2}|^2 \left| \begin{pmatrix} I_{r-1} & 0 \\ c & 1 \end{pmatrix} \right| dc.$$  

We deduce the last equality recalling the normalisations of $W_0$ and $f_{1/2}. \square$
It is known that (see [11, Theorem 4.2]) if \( \pi_0 \) is cuspidal, then \( L(s, \pi_0 \otimes \pi_0) \) has a simple pole at \( s = 1 \) with residue \( L(1, \pi_0, \text{Ad}) \). Let us write

\[
L(1 + s, \pi_0 \otimes \pi_0) = \frac{L(1, \pi_0, \text{Ad})}{s} + O(\pi_0(1),
\]

as \( s \to 0 \). Thus, using equations (5.2) and (5.3) and Lemma 5.1, we can evaluate the limit in equation (5.1) as

\[
\lim_{s \to 0} \left[ \langle |\phi_0|^2, \text{Eis}(\mathcal{f}_{1/2}f_{1+2s})(.x) \rangle + \langle |\phi_0|^2, \text{Eis}(\mathcal{f}_{1/2}\mathcal{f}_{1+2s})(.x) \rangle \right] = \]

\[
L(1, \pi_0, \text{Ad}) \frac{\zeta(r/2)^2}{\zeta(r)} \Psi'(f_{1/2,\infty}, W_{0,\infty}) + O_{\pi_0, \Phi}(1),
\]

(5.4)

where \( \Psi'(f_{1/2,\infty}, W_{0,\infty}) \) is defined as

\[
\partial_{s=0} \left( \Psi_{\infty}(f_{1/2,\infty}f_{1+2s,\infty}(x), W_{0,\infty}, W_{0,\infty}) - \Psi_{\infty}(f_{1/2,\infty}f_{1+2s,\infty}(x), W_{0,\infty}, W_{0,\infty}) \right).
\]

Here and elsewhere in the paper, we write \( \partial_{s=0} \) as an abbreviation of \( \frac{\partial}{\partial s} \big|_{s=0} \).

**Proposition 5.2.** We have

\[
\Psi'(f_{1/2,\infty}, W_{0,\infty}) = r \log x + O_{W_{0,\infty}, \Phi}(1)
\]

as \( X \) tends to infinity.

Proposition 5.2 follows immediately from the following Lemma 5.3, Lemma 5.5 and Lemma 5.1. Again, to ease the notations, we drop \( \infty \) subscripts from the proofs of the next two lemmata.

**Lemma 5.3.** We have

\[
\partial_{s=0} \Psi_{\infty}(f_{1/2,\infty}f_{1+2s,\infty}(x), W_{0,\infty}, W_{0,\infty}) = -\log x \|W_{0,\infty}f_{1/2,\infty}\|^2 + O_{W_{0,\infty}, \Phi}(1)
\]

as \( X \) tends to infinity.

**Proof.** We start by make the change of variables \( g \mapsto w g^{-t} \) in the zeta integral to write

\[
\Psi(f_{1/2,\infty}f_{1+2s}(x), W_0, W_0) = \Psi(f_{1/2,\infty}f_{1+2s}^{-1}(x^{-1}), W_0, W_0).
\]

We use Bruhat coordinates as in the proof of Lemma 5.1 to write the above zeta integral as

\[
\int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} \int_{\mathbb{R}^{r-1}} |\tilde{W}_0|^2 \left[ \begin{pmatrix} h & c \\ c & 1 \end{pmatrix} \right] \tilde{f}_{1/2} \tilde{f}_{1-2s} \left[ \begin{pmatrix} h/X & c/X \\ c & 1 \end{pmatrix} \right] dc \frac{dh}{|\det(h)|}.
\]

Again, as in the proof of Lemma 5.1, we use the transformation property of \( \tilde{f}_{1/2} \tilde{f}_{1-2s} \) as in equation (2.3) and make the change of variables \( c \mapsto cX \) to obtain that the above is equal to

\[
X^{(r-1)s} \int_{\mathbb{R}^{r-1}} \tilde{f}_{1/2} \tilde{f}_{1-2s} \left[ \begin{pmatrix} I_{r-1} & c \\ c & 1 \end{pmatrix} \right] \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} |\tilde{W}_0|^2 \left[ \begin{pmatrix} h/cX & 1 \\ 1 & 1 \end{pmatrix} \right] |\det(h)|^{-s} dh dc.
\]
Differentiating at \( s = 0 \), we obtain that the above is equal to

\[
(r - 1) \log X \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[ \begin{pmatrix} I_{r-1} & c \\ 0 & 1 \end{pmatrix} \right] \int_{N_{r-1}(\mathbb{R})/GL_{r-1}(\mathbb{R})} |\tilde{W}_0|^2 \left[ \begin{pmatrix} h \\ cX \mathbf{1} \end{pmatrix} \right] dhdcdc
\]

\[
+ \int_{\mathbb{R}^{r-1}} \frac{\partial}{\partial s} f_{1/2} \left. \right|_{s=0} f_{1/2-s} \left[ \begin{pmatrix} I_{r-1} & c \\ 0 & 1 \end{pmatrix} \right] \int_{N_{r-1}(\mathbb{R})/GL_{r-1}(\mathbb{R})} |\tilde{W}_0|^2 \left[ \begin{pmatrix} h \\ cX \mathbf{1} \end{pmatrix} \right] dhdcdc
\]

\[
- \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[ \begin{pmatrix} I_{r-1} & c \\ 0 & 1 \end{pmatrix} \right] \int_{N_{r-1}(\mathbb{R})/GL_{r-1}(\mathbb{R})} |\tilde{W}_0|^2 \left[ \begin{pmatrix} h \\ cX \mathbf{1} \end{pmatrix} \right] \log |\det(h)| dhdcdc. \quad (5.5)
\]

The first summand in equation (5.5) is easy to understand. Using the invariance of the unitary product exactly as in the proof of Lemma 5.1, we can yield that the first summand is equal to

\[
(r - 1) \log X \|\tilde{W}_0\|_{\tilde{\sigma}_0} \|\hat{f}_{1/2}\|^2.
\]

From the Whittaker–Plancherel expansion in equation (2.7), the \( GL(r) \times GL(r-1) \) local functional equation (see [11, Proposition 3.2]), and the unitarity of the \( \gamma \)-factor, as in the proof of Lemma 5.1, one also gets that \( \|\tilde{W}_0\|_{\tilde{\sigma}_0} = \|W_0\|_{\tilde{\sigma}_0} \).

We claim that the second summand in equation (5.5) is of bounded size. Note that again the invariance of the unitary inner product implies that the inner integral is equal to \( \|\tilde{W}_0\|^2_{\tilde{\sigma}_0} \). Thus, using Cauchy’s integral formula, we can write the second summand as

\[
\|\tilde{W}_0\|_{\tilde{\sigma}_0}^2 \int_{|s| = \epsilon} \frac{1}{s^2} \int_{\mathbb{R}^{r-1}} \overline{f_{1/2}} \hat{f}_{1/2-s} \left[ \begin{pmatrix} I_{r-1} & c \\ 0 & 1 \end{pmatrix} \right] dcds \frac{ds}{\pi i}
\]

for some sufficiently small \( \epsilon > 0 \). To show that the above integrals converge, we start with the Iwasawa decomposition of \( \left[ \begin{pmatrix} I_{r-1} & c \\ 0 & 1 \end{pmatrix} \right] \). One can check by induction or otherwise that there exists a \( \tilde{z}(c) \in \mathbb{R}^\times \) so that

\[
\left[ \begin{pmatrix} I_{r-1} & c \\ 0 & 1 \end{pmatrix} \right] = \tilde{z}(c) \hat{\eta}(c) \left[ \begin{pmatrix} \tilde{a}(c) & 0 \\ c \tilde{b}(c) & 1 \end{pmatrix} \right] \hat{k}(c); \quad \hat{\eta}(c) \in N(\mathbb{R}), \hat{k}(c) \in K_\infty,
\]

\[
\tilde{a}(c) := \text{diag}(a_1(c), \ldots, a_{r-1}(c)); \quad a_i(c) := \frac{\sqrt{1 + c_i^2 + \cdots + c_{i-1}^2}}{\sqrt{1 + c_i^2 + \cdots + c_{r-1}^2} \sqrt{1 + |c|^2}}. \quad (5.6)
\]

Thus, using the transformation property in equation (2.3), we get

\[
\|\tilde{W}_0\|_{\tilde{\sigma}_0}^2 \int_{\mathbb{R}^{r-1}} \overline{f_{1/2}} \hat{f}_{1/2-s} \left[ \begin{pmatrix} I_{r-1} & c \\ 0 & 1 \end{pmatrix} \right] \|\tilde{W}_0\|_{\tilde{\sigma}_0} \|\hat{f}_{1/2} \hat{f}_{1/2-s}\|_{L^\infty(K_\infty)}.
\]

Thus the second summand of equation (5.5) is bounded by

\[
\ll_{\epsilon,f,W_0} \int_{\mathbb{R}^{r-1}} (1 + |c|^2)^{-r/2(1-\epsilon)} dc.
\]

The above integral is convergent for sufficiently small \( \epsilon \).
We now focus on the third summand in equation \((5.5)\). In the inner integral, we use Iwasawa coordinates for \(h = ak\), move the \(K\)-integral outside and make the change of variables \(c \mapsto ck\) to rewrite it as

\[
-\int_{O_{r-1} \setminus \{0\}} \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[ \begin{pmatrix} I_{r-1} & \ 0 \\ \ 0 & 1 \end{pmatrix} \right] \log |\det(a)| \frac{da}{\delta(a)} dcdk.
\]

We use the Iwasawa decomposition of \(\begin{pmatrix} I_{r-1} & \ 0 \\ \ 0 & 1 \end{pmatrix} \) as in equation \((5.6)\) to write it as \(\bar{n}(cX) \begin{pmatrix} \tilde{a}(cX) \\ 1 \end{pmatrix} \widetilde{k}(cX)\). Then, using the left \(N(\mathbb{R})\)-invariance of \(|\bar{W}_0|^2\) and changing variable \(a \mapsto a \times \tilde{a}(cX)^{-1}\), we obtain that the above quantity is equal to

\[
\int_{O_{r-1} \times \mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[ \begin{pmatrix} I_{r-1} & \ 0 \\ \ 0 & 1 \end{pmatrix} \right] \log |\det(\tilde{a}(cX))| \frac{da}{\delta(a)} dcdk - \int_{O_{r-1} \setminus \{0\}} \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[ \begin{pmatrix} I_{r-1} & \ 0 \\ \ 0 & 1 \end{pmatrix} \right] \delta(\tilde{a}(cX)) \log |\det(a)| \frac{da}{\delta(a)} dcdk.
\]

We write the above as \(A - B\), where \(A\) denotes the first term and \(B\) denotes the second term above. To analyse \(A\), we reverse-engineer the above process: make the change of variables \(a \mapsto a \times \tilde{a}(cX)\), use the left \(N(\mathbb{R})\)-invariance of \(|\bar{W}_0|^2\) and make the change of variables \(c \mapsto ck^{-1}\) to obtain

\[
A = \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[ \begin{pmatrix} I_{r-1} & \ 0 \\ \ 0 & 1 \end{pmatrix} \right] \int_{O_{r-1} \setminus \{0\}} \log |\det(a(ck^{-1}X))| \frac{da}{\delta(a)} dcdk.
\]

But \(\det(\tilde{a}(cX)) = (1 + X^2 |c|^2)^{-r/2} = \det(a(ck^{-1}X))\) for all \(k \in O_{r-1}(\mathbb{R})\). Using that, we can move the integral over \(O_{r-1}(\mathbb{R})\) to couple with the integral over \(A_{r-1}(\mathbb{R})\) to obtain an integral over \(N_{r-1}(\mathbb{R}) \setminus GL_{r-1}(\mathbb{R})\). Then, once again appealing to the invariance of the unitary product, we obtain

\[
A = \|\bar{W}_0\|_\infty^2 \int_{\mathbb{R}^{r-1}} |\hat{f}_{1/2}|^2 \left[ \begin{pmatrix} I_{r-1} & \ 0 \\ \ 0 & 1 \end{pmatrix} \right] \log |\det(\tilde{a}(cX))| dc.
\]

Note that

\[
\log |\det(\tilde{a}(cX))| = -\frac{r}{2} \log(1 + X^2 |c|^2)
\]

\[
= -\frac{r}{2} \log(1 + X^2) + O(\log(1 + |c|^2)) = -r \log X + O_{\epsilon}(1 + |c|^2).
\]

Using the Iwasawa decomposition and transformation property of \(\hat{f}_{1/2}\) as in equation \((2.3)\), similar to the second case, we obtain
\[ A + r \log X \| \tilde{W}_0 \|_{H^0}^2 \int_{\mathbb{R}^{r-1}} |\tilde{f}_{1/2}|^2 \left( \begin{array}{c} I^{r-1} \\ c \\ 1 \end{array} \right) \] 
\[ \ll \tilde{W}_0, \varepsilon \| \tilde{f}_{1/2} \|_{L^2(K_\infty)}^2 \int_{\mathbb{R}^{r-1}} (1 + |c|^2)^{-r/2 + \varepsilon} dc \ll \Phi, \tilde{W}_0. \]

Working as in the proof of Lemma 5.1, we check that
\[ \| W_0 \|_{H^0}^2 \int_{\mathbb{R}^{r-1}} |\tilde{f}_{1/2}|^2 \left( \begin{array}{c} I^{r-1} \\ c \\ 1 \end{array} \right) \] 
\[ \ll \tilde{W}_0, \varepsilon \| \tilde{f}_{1/2} \|_{L^2(K_\infty)}^2 \int_{\mathbb{R}^{r-1}} (1 + |c|^2)^{-r/2 + \varepsilon} dc \ll \Phi, \tilde{W}_0. \]

Thus we obtain
\[ A = -r \log X \| W_0f_{1/2} \|^2 + O_{W_0, f}(1). \quad (5.7) \]

Now we prove that \( B \) is of bounded size. To prove that, we first claim that
\[ \int_{A_{r-1}(\mathbb{R})} |W_0| \left( \begin{array}{c} a \\ 1 \\ k(cX) \\ k \\ 1 \end{array} \right) \left( \begin{array}{c} k(cX) \\ k \\ 1 \end{array} \right) \log |\det(a)| \frac{da}{\delta(a)} \ll \tilde{W}_0 \]
uniformly in \( c \). We assume the claim. Now note that
\[ \delta(\bar{a}(cX)) = \frac{(1 + X^2|c|^2)^{r/2 - 1}}{\prod_{i=1}^{r-2} (1 + c_i^2 X^2 + \cdots + c_i^2 X^2)} \ll \frac{X^{r-2}(1 + |c|^2)^{r/2 - 1}}{\prod_{i=1}^{r-2} (1 + c_i^2 X^2)}. \]

We use the Iwasawa decomposition and work as before. Using transformation of \( \tilde{f}_{1/2} \) as in equation (2.3), we thus get
\[ B \ll \tilde{W}_0, f \int_{O_{r-1}(\mathbb{R})} \int_{\mathbb{R}^{r-1}} (1 + |ck|^2)^{-r/2} X^{r-2}(1 + |c|^2)^{r/2 - 1} \frac{dcdk}{\prod_{i=1}^{r-2} (1 + c_i^2 X^2)} \ll \int_{\mathbb{R}^{r-1}} \prod_{i=1}^{r-1} (1 + c_i^2)^{-1} dc, \]

which we obtain by noting that \( |ck| = |c| \) for \( k \in O_{r-1}(\mathbb{R}) \) and making the change of variables \( c_i \mapsto c_i/X \) for \( i \leq r - 2 \). It is easy to see that the above integral is convergent, which yields that
\[ B = O_{W_0, f}(1). \]

Now, to prove the claim above, let \( \omega := \tilde{k}(cX) \left( \begin{array}{c} k \\ 1 \end{array} \right) \in K_\infty \) implicitly depending on \( cX \). Note that from Lemma 7.2, we get that
\[ \bar{\pi}_0(\omega) \tilde{W}_0 \left( \begin{array}{c} a(y) \\ 1 \end{array} \right) \ll \epsilon, M, \tilde{W}_0 \delta^{1/2 - \varepsilon} (a(y)) |\det(a(y))|^{1/2 - \varepsilon} \prod_{i=1}^{r-1} \min(1, |y_i|^{-M}). \]

Thus we obtain
\[ \int_{A_{r-1}(\mathbb{R})} |\bar{\pi}_0(\omega) \tilde{W}_0| \left( \begin{array}{c} a \\ 1 \end{array} \right) \log |\det(a)| \frac{da}{\delta(a)} \ll \tilde{W}_0, \eta, M \int_{\mathbb{R}^{r-1}} \prod_{i=1}^{r-1} \min(1, |y_i|^{-M}) |\det(a(y))|^{1-2\theta_0 - \varepsilon} (|\det(a(y))|^\varepsilon + |\det(a(y))|^{-\varepsilon}) \prod_i d^r y_i. \]
Employing the bound of $\partial_0$ from the statement of Theorem 2.1, we check that the above integral is convergent for large enough $M$ and sufficiently small $\epsilon > 0$, which yields the claim.

\[ \square \]

**Remark 5.4.** In the very last estimate of the proof of Lemma 5.3, we can only prove that the integral of the Whittaker function is of bounded size. It is not clear to us if or how one can improve the estimate to be a constant plus a power-saving error term. This would potentially explicate the constant term of the asymptotic expansion in Theorem 2.1 with a power-saving error term; see Remark 1.4.

**Lemma 5.5.** We have

\[ \partial_{s=0} \Psi_{\infty} \left( \int_{0}^{\infty} f_{1/2+s, \infty} \right), W_{0, \infty}, \overline{W_{0, \infty}} \right) = (r - 1) \log X \left\| W_{0, \infty} f_{1/2, \infty} \right\|^2 + O_{W_{0, \infty}, \Phi_{\infty}} (1) \]

as $X$ tends to infinity.

**Proof.** The proof of this lemma is very similar to (and easier than) the proof of Lemma 5.3. We first write $\Psi_{\int_{1/2} f_{1/2+s} \,(x), W_{0}, \overline{W_{0}}}$ as

\[ X^{(r-1)s} \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} \int_{\mathbb{R}^{n-1}} |W_{0}|^2 \left[ \left( \frac{h}{c/X} \right) \right] f_{1/2} f_{1/2+s} \left[ \left[ \begin{array}{c} I_{r-1} \\ c \\ 1 \end{array} \right] \right] dc |\det(h)|^s dh. \]

Note that the $s = 0$ derivative in the the statement of this lemma can be computed exactly the same as we did in the calculation of equation (5.5) of Lemma 5.3 and can be seen equal to

\[ (r - 1) \log X \int_{\mathbb{R}^{n-1}} |f_{1/2}|^2 \left[ \left( \begin{array}{c} I_{r-1} \\ c \\ 1 \end{array} \right) \right] \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} |W_{0}|^2 \left[ \left( \begin{array}{c} h \\ c/X \end{array} \right) \right] dh dc \]

\[ + \int_{\mathbb{R}^{n-1}} \overline{f_{1/2}} \partial_{s=0} f_{1/2+s} \left[ \left[ \begin{array}{c} I_{r-1} \\ c \\ 1 \end{array} \right] \right] \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} |W_{0}|^2 \left[ \left( \begin{array}{c} h \\ c/X \end{array} \right) \right] dh dc \]

\[ + \int_{\mathbb{R}^{n-1}} |f_{1/2}|^2 \left[ \left( \begin{array}{c} I_{r-1} \\ c \\ 1 \end{array} \right) \right] \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} |W_{0}|^2 \left[ \left( \begin{array}{c} h \\ c/X \end{array} \right) \right] \log |\det(h)| dh dc. \]

Exactly as in the proof of Lemma 5.3, we can check (for example, changing $\hat{f}_{1/2-s}$ to $f_{1/2+s}$ and $\overline{W_{0}}$ to $W_{0}$) that the first and second summands in equation (5.8) are

\[ (r - 1) \log X \left\| W_{0} f_{1/2} \right\|^2 \]

and $O_{W_{0}, f} (1)$, respectively. We claim that the third summand in equation (5.8) is also $O_{W_{0}, f} (1)$, which yields the lemma.

From the relation between $f$ and $\Phi$ from equation (2.2), we write

\[ f_{1/2} \left[ \left( \begin{array}{c} I_{r-1} \\ c \\ 1 \end{array} \right) \right] = \int_{\mathbb{R}^n} \Phi(t(c, 1)) |t|^{r/2} \, dt. \]

Recall the choice of $\Phi$ in Section 4.1. Support of $\Phi$ being on $B T(0, \ldots, 0, 1)$ implies that in the above integral, $t \approx 1$ and hence $c \ll 1$. Below, we show that

\[ \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} |W_{0}|^2 \left[ \left( \begin{array}{c} h \\ c/X \end{array} \right) \right] \log |\det(h)| dh \ll W_{0} 1, \]

which clearly implies our claim above.
We write $h = ak$ in Iwasawa coordinates and let $\omega := \begin{pmatrix} k & c/X \\ 0 & 1 \end{pmatrix}$. Note that as $k \in O_{r-1}(\mathbb{R})$ and $c/X \ll 1$, there exists a fixed compact set $\Omega \in G(\mathbb{R})$ such that $\omega \in \Omega$ for all relevant $c$ and $k$. Thus it is enough to show that

$$\int_{A_{r-1}(\mathbb{R})} |\pi_0(\omega)w_0|^2 \left| \begin{pmatrix} a \\ 1 \end{pmatrix} \right| \log |\det(a)| \frac{da}{\delta(a)} \ll w_0, \Omega.$$ 

This can be done similarly as we did at the end of the proof of Lemma 5.3.

6. Analysis of the regularised term in the period side

Let $s \in \mathbb{C}$ with sufficiently small $\Re(s)$. Recall the regularised Eisenstein series $\tilde{E}_s$ from equation (4.9). The main proposition of this section is the following.

**Proposition 6.1.** $\tilde{E}_s$ is holomorphic in a sufficiently small neighbourhood of $s = 0$ and is integrable on $\mathbb{X}$.

Note from the definition in equation (4.9) that $\tilde{E}_s$ is holomorphic in a punctured neighbourhood of $s = 0$. Thus it is enough to prove that $\tilde{E}_s$ is holomorphic at $s = 0$. Recall the description of the poles of the maximal Eisenstein series in Section 2.4. We know, in particular, $Eis(f_{1/2})Eis(f_{1/2+s})$ is holomorphic at $s = 0$, and we thus only need to show the following.

**Lemma 6.2.** For fixed $g \in \mathbb{X}$,

$$Eis(f_{1/2}f_{1/2+s})(g) + Eis(\overline{f_{1/2}f_{1/2+s}})(g)$$

is holomorphic at $s = 0$.

**Proof.** Our argument is to show that the residues $R$ and $\tilde{R}$ (which are independent of $g$) at the simple poles at $s = 0$ of $Eis(f_{1/2}f_{1/2+s})$ and $Eis(\overline{f_{1/2}f_{1/2+s}})$, respectively, cancel each other.

Let $\phi_0$ be the cusp form as we have chosen in Section 4.1. From equation (5.2), we get

$$R \Vert \phi_0 \Vert_2^2 = \text{Res}_{s=0}(\langle |\phi_0|^2, Eis(f_{1/2}f_{1/2+s}) \rangle) = \frac{\zeta(r/2)^2}{\zeta(r)} L(1, \pi_0, \text{Ad}) \Psi_\infty(\langle |f_{1/2,\infty}|^2, W_{0,\infty}, \overline{W_{0,\infty}} \rangle).$$

Similarly, from equation (5.3), we get

$$\tilde{R} \Vert \phi_0 \Vert_2^2 = \text{Res}_{s=0}(\langle |\phi_0|^2, Eis(\overline{f_{1/2}f_{1/2+s}}) \rangle) = -\frac{\zeta(r/2)^2}{\zeta(r)} L(1, \pi_0, \text{Ad}) \Psi_\infty(\langle |\overline{f_{1/2,\infty}}|^2, W_{0,\infty}, \overline{W_{0,\infty}} \rangle).$$

From Lemma 5.1 with $x_\infty = 1$ (and the equation preceding Lemma 5.1), we conclude that the $\Psi_\infty$ factors in the above expressions of $R$ and $\tilde{R}$ are equal.

Now we prove some preparatory lemmata to prove the integrability of $\tilde{E}_s$ on $\mathbb{X}$. We actually show that $\tilde{E}_s$ is integrable in the Siegel domain $\mathbb{S}$ as in equation (2.1), which contains $\mathbb{X}$. Let $g \in \mathbb{S}$ with $g = (g_\infty, k_f)$, where $g_\infty = n_{\infty} \begin{pmatrix} a(y_{\infty}) \\ 1 \end{pmatrix} k_\infty \in G(\mathbb{R})$ in Iwasawa coordinates and $k_f \in K_f := \prod_{p<\infty} K_p$. As $g \in \mathbb{S}$, we have $y_{j,\infty} \gg 1$. We recall the quantities in Proposition 3.3 from Section 3.
Lemma 6.3. Suppose that $i < r$. Let $s \in \mathbb{C}$ be away from a pole of $M^0_{i}f_s$ with $|\Re(s)| < 2$. Then

$$\|M^0_{i}f_s\|_{L^\infty(K)} \ll 1.$$  

Further, let $\mathbb{R}^x \times K_f \ni (y_\infty, 1) =: y$. Then for all $k \in K$ and $s$ with $|\Re(s)| < 2$,

$$\sum_{q \in \mathbb{Q}^x} W^i_{f_s}(qy, k) \ll_N |y_\infty|^{-N},$$

where the sum in the left-hand side converges absolutely.

Proof. In this proof, we assume that $\Phi \in S(\mathbb{A}^r)$ is an arbitrary Schwartz function. We get that for $k \in K$,

$$M^0_{i}f_s(k) = \int_{\mathbb{A}^r-i} \int_{\mathbb{A}^x} (k, \Phi)(0, t, x)|t|^{r_s-r+i}d^x t \ll \int_{|x|,|t| < 1} |t|^{r_s-r+i}d^x t \ll_K \Re(s) 1,$$

if $\Re(s)$ is sufficiently large. On the other hand, using the Tate functional equation and working similarly, we obtain

$$M^0_{i}f_s(k) = \int_{\mathbb{A}^r-i} \int_{\mathbb{A}^x} (\tilde{k}, \Phi)^i (te_i)|t|^{r_i-1-r_s} \ll_K \Re(s) 1,$$

if $\Re(s)$ is sufficiently negative. Using the Phragmén–Lindelöf convexity principle and the compactness of $K$, we deduce the first claim.

Let $z \in \mathbb{A}$ and $k \in K$. Following a similar computation after equation (3.4) in Section 3, we get that

$$W^i_{f_s}(z, k) = \int_{\mathbb{A}^r-i} \int_{\mathbb{A}^x} (k, \Phi)(0, t, x)\overline{\psi_0(zx_1/t)}|t|^{r_s-r+i}d^x t.$$

This converges absolutely if $\Re(s)$ is sufficiently large.

We first concentrate on the $x_1$ integral. In the archimedean component of this integral, we integrate by parts with respect to the $x_1, \infty$ variables. This yields that the archimedean integral is bounded by $\ll_N |t_\infty|^{-N}$ for all large $N$.

In the $p$-adic component, we note that compact support of $\Phi_p$ forces $x_{1, p}$ to vary over a compact space. This implies that the $p$-adic integral vanishes unless $|z_p/t_p| \ll 1$. However, the support condition of $\Phi_p$ ensures that $|t_p| \ll 1$, which in turn restricts $z_p$ to be of bounded size.

Thus we can analytically continue the integral representation of $W^i_{f_s}$ to $\Re(s)$ sufficiently negative, but fixed. Altogether, estimating the integrals as before, we obtain if $\Re(s) \geq -2$; then for sufficiently large $N$, we have

$$W^i_{f_s}(z, k) \ll_K \Re(s) |z_\infty|^{-N} \prod_{p < \infty} \text{char}_{|z_p| \ll 1}.$$

Thus for $q \in \mathbb{Q}^x$ and $y$ as in the statement of this lemma, we have

$$W^i_{f_s}(qy, k) \ll_K |y_\infty q_\infty|^{-N},$$

if the denominator of $q$ is bounded; otherwise, the above is zero. Thus the sum over $q \in \mathbb{Q}^x$ is absolutely convergent for sufficiently large $N$. We conclude using the compactness of $K$. $\square$

Lemma 6.4. Let $g \in S$ and $s \in \mathbb{C}$ with sufficiently small $\Re(s)$. Then

$$\overline{\text{Eis}(f_{1/2})(g)} \text{Eis}(f_{1/2+s})(g) - \overline{f_{1/2}(g)} f_{1/2+s}(g) - \overline{f_{1/2}(g)} \overline{f_{1/2+s}(g)} \ll \delta^{1-\eta} \left( a(y_\infty) \right)$$

for some $\eta > 0$. 

\[\delta \left( \left[ \begin{array}{c} a(y_{\infty}) \\ 1 \end{array} \right] \right) = \prod_{j=1}^{r-1} |y_{j,\infty}|^{j(r-j)}.\]

Note that \(y_{j,\infty} \gg 1\) as \(g \in \mathbb{S}\). Thus it is enough to show that the exponents of \(|y_{j,\infty}|\) arising in the left-hand side in the expression in the lemma are less than \(j(r-j)\).

We recall equation (3.4) and for \(1 < i < r\) write
\[H^i_s(g) = \sum_{q \in \mathbb{Q}} M^q_i f_{1/2+s}(g)\]
\[= M^0_i f_{1/2+s}(g) + \sum_{q \in \mathbb{Q}^n} \psi_i(q)(\eta_1) \prod_{j=1}^{i-1} |y_j|^{(1/2+s)_j} \prod_{j=i}^{r-1} |y_j|^{(1/2-s)(r-j)} W^{i}_f_{1/2+s}(qy_i, k),\]
where \(n_1\) is a unipotent element as in Section 2.4. Using Lemma 6.3, we obtain that
\[H^i_s(g) \ll \prod_{j=1}^{i-1} |y_{j,\infty}|^{(1/2+\Re(s))j} \prod_{j=i}^{r-1} |y_{j,\infty}|^{(1/2-\Re(s))(r-j)}\]

On the other hand, we similarly obtain
\[H^1_s(g) := \sum_{q \in \mathbb{Q}^n} M^q_1 f_{1/2+s}(g) \ll_{K,N} \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1/2-\Re(s))(r-j)-N \delta_{j-1}}.\]

We also record that
\[f_{1/2+s}(g) \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1/2+\Re(s))j}\]
and
\[\tilde{f}_{1/2+s}(g) \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1/2-\Re(s))(r-j)}\]

We use Lemma 3.2 to rewrite
\[\Eis(f_{1/2+s})(g) = f_{1/2+s}(g) + \tilde{f}_{1/2+s}(g) + H^1_s(g) + \sum_{1 \leq i < r} H^i_s(g).\]

After multiplying \(\Eis(f_{1/2})\) and \(\Eis(f_{1/2+s})\) using the above expression and subtracting the terms \(\overline{f_{1/2}f_{1/2+s}}\) and \(\overline{f_{1/2}\tilde{f}_{1/2+s}}\), we are left with the following type of terms whose bounds are given below:
\[\overline{f_{1/2}f_{1/2+s}}(g) \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{r/2-(r-j)\Re(s)}\]

If we replace the left-hand side above with \(\overline{f_{1/2}f_{1/2+s}}(g)\), then a similar inequality holds, with the exponent in the right-hand side being \(r/2 + j\Re(s)\). In any case, for sufficiently small \(\Re(s)\) and \(r \geq 3\), we have
\[r/2 + r|\Re(s)| < j(r-j), \quad 1 \leq j < r.\]
A similar estimate can be done for $\overline{f_{1/2}H^1_s(g)}$. Next we check that

$$H^1_0(g) \overline{f_{1/2+s}} \ll \prod_{j=1}^{r-1} |y_{j,0}|^{(1-\Re(s))(r-j)-N\delta_{j=1}}.$$ 

A similar estimate can be obtained if we replace $\overline{f_{1/2+s}}$ by $H^1_s$ on the left-hand side above. Clearly, for sufficiently small $\Re(s)$, we have

$$(1 - \Re(s))(r-j) - N\delta_{j=1} < (r-j).$$

Finally, for $1 < i < r$, the exponent of $y_{i,0}$ of $\overline{H^1_0}$ is $\leq (r-2)/2$. On the other hand, the same of $G_s$ is $\leq (1/2 + |\Re(s)|)(r-1)$ for $G_s$ being one of $f_{1/2+s}$, $\overline{f_{1/2+s}}$, or $H^1_s$ with $i < r$. So the exponent of $y_{i,0}$ of the product $\overline{H^1_0G_s}$ for $1 < i < r$ is

$$\leq (r-2 + r-1)/2 + (r-1)|\Re(s)| < j(r-j)$$

for sufficiently small $\Re(s)$.

Similarly, one estimates remaining terms of the form $\overline{H^1_0f_{1/2+s}}$, $\overline{G_0H^1_s}$ and $\overline{f_{1/2}H^1_s}$, which we leave for the reader. Hence we conclude the proof. 

\begin{lemma}
Let $g \in \mathbb{S}$ and $s \in \mathbb{C}$ with sufficiently small $\Re(s)$. Then

$$\text{Eis}(\overline{f_{1/2}f_{1/2+s}})(g) - \overline{f_{1/2}(g)f_{1/2+s}(g)} \ll \delta^{1-\eta} \left[ |a(y_{\infty})| \right]$$

and also

$$\text{Eis}(\overline{f_{1/2}f_{1/2+s}})(g) - \overline{f_{1/2}(g)f_{1/2+s}(g)} \ll \delta^{1-\eta} \left[ |a(y_{\infty})| \right]$$

for some $\eta > 0$.
\end{lemma}

\begin{proof}
We take a very similar path as in the proof of Lemma 6.4. Let $s \in \mathbb{C}$ be away from the poles of the relevant Eisenstein series and $\Re(s)$ be sufficiently small.

First note that $\overline{f_{1/2}f_{1/2+s}} \in \mathcal{I}_{r-1}(1-s)$. We use the functional equation of the Eisenstein series [11, Proposition 2.1]: there exists $\tilde{F}_s \in \mathcal{I}_{r-1,1}(s)$ such that

$$\text{Eis}(\overline{f_{1/2}f_{1/2+s}}) = \text{Eis}(\tilde{F}_s).$$

In fact, $\tilde{F}_s$ is the preimage of $\overline{f_{1/2}f_{1/2+s}}$ under the standard intertwiner from $\mathcal{I}_{r-1}(1-s)$ to $\mathcal{I}_{r-1,1}(s)$: that is,

$$\overline{f_{1/2}f_{1/2+s}} = M^0_1 \tilde{F}_s,$$

and in particular, $\tilde{F}_s$ is holomorphic in a sufficiently small neighbourhood of $s = 0$.

From Lemma 3.2, we get that

$$\text{Eis}(\overline{f_{1/2}f_{1/2+s}}) - \overline{f_{1/2}(g)f_{1/2+s}(g)} = \text{Eis}(\tilde{F}_s)(g) - M^0_1 \tilde{F}_s(g)$$

$$= \tilde{F}_s(g) + \sum_{q \in \mathbb{Z}} M^q_1 \tilde{F}_s(g) + \sum_{1 < i < r} \sum_{q \in \mathbb{Z}} M^q_1 \tilde{F}_s(g).$$

\end{proof}
We now bound each summand above similarly to the proof of Lemma 6.4. As $\tilde{F}_s \in \mathcal{I}_{r-1,1}(s)$, we obtain that

$$\tilde{F}_s(g) \ll \left| \tilde{F}_s \left( \begin{array}{c} a(y_\infty) \\ 1 \end{array} \right) \right| \ll \prod_{j=1}^{r-1} |y_{j,\infty}|^{\Re(s)}.$$ 

Here we applied $\|\tilde{F}_s\|_{L^\infty(K)} \ll 1$, which can be deduced similarly to the proof of Lemma 6.3 and applying holomorphicity of $\tilde{F}_s$ for small $s$.

On the other hand, once again recalling equation (3.4), we obtain for $1 < i < r$ that

$$\sum_{q \in \mathbb{Q}} M^q_i \tilde{F}_s(g) \ll M^q_i \tilde{F}_s \left( \begin{array}{c} a(y_\infty) \\ 1 \end{array} \right)$$

$$+ \prod_{j=1}^{i-1} |y_{j,\infty}|^{j\Re(s)} \prod_{j=i}^{r-1} |y_{j,\infty}|^{(1-\Re(s))(r-j)} \sum_{q \in \mathbb{Q}^\times} \|W^i_{F_s}(qy_i, \cdot)\|_{L^\infty(K)}$$

$$\ll \prod_{j=1}^{i-1} |y_{j,\infty}|^{j\Re(s)} \prod_{j=i}^{r-1} |y_{j,\infty}|^{(1-\Re(s))(r-j)}.$$ 

In the last estimate above, we used that

$$\sum_{q \in \mathbb{Q}^\times} \|W^i_{F_s}(qy_i, \cdot)\|_{L^\infty(K)} \ll_N |y_{i,\infty}|^{-N},$$

which can be deduced similarly to the proof of Lemma 6.3. Similarly, we deduce that

$$\sum_{q \in \mathbb{Q}^\times} M^q_i \tilde{F}_s(g) \ll N \prod_{j=1}^{r-1} |y_{j,\infty}|^{(1-\Re(s))(r-j)-N} \delta_{j+1},$$

In each case, the exponent of $y_{j,\infty}$ is strictly smaller than $j(r-j)$, which concludes the proof for the second assertion for sufficiently small $\Re(s)$. The first assertion can be proved similarly (and more easily), which we leave for the reader. \qed

**Proof of Proposition 6.1.** In Lemma 6.2, we have already proved the holomorphicity of $E_s$ at $s = 0$. From Lemma 6.4 and Lemma 6.5, we conclude by the triangle inequality that for sufficiently small $\Re(s)$ and $g \in \mathcal{S}$,

$$\tilde{E}_s(g) \ll \delta^{1-\eta} \left[ \begin{array}{c} a(y_\infty) \\ 1 \end{array} \right]$$

for some $\eta > 0$. Thus

$$\int_X |\tilde{E}_s(g)| dg \ll \int_{y_{j,\infty} > 1} \delta^{-\eta} \left[ \begin{array}{c} a(y_\infty) \\ 1 \end{array} \right] \prod_j d^\times y_{j,\infty}.$$ 

The last integral is convergent, and we conclude. \qed

**7. Analysis of the spectral side**

Recall the spectral weight $J_X(\pi_\infty)$ from equation (4.5), the choices of the local components from Section 4.1, and the $\vartheta_1$-temperedness assumption on $\pi_{0,\infty}$ from the statement of Theorem 2.1. In this section, we prove the remaining second property of the spectral weight as described in Theorem 2.1. That
is, we show that $J_X(\pi_\infty)$ is uniformly bounded away from zero if $\pi_\infty$ is $\theta$-tempered with $\theta + \theta_0 < 1/2$ and $C(\pi_\infty) < X$.

**Proposition 7.1.** Let $\pi \in \mathcal{X}_{\mathfrak{g}_{\text{gen}}}$ be such that $\pi_\infty$ is $\theta$-tempered with $\theta + \theta_0 < 1/2$. Let $\pi_0$ be the cuspidal automorphic representation as in Theorem 2.1. Then

$$J_X(\pi_\infty) \gg 1, \quad \text{if } C(\pi_\infty) < X,$$

where the implied constant possibly depends on $W_0, \infty, \Phi_\infty$.

For the rest of this section, to ease notations, we drop the $\infty$-subscript everywhere.

We recall the notations and definition of the Sobolev norm $S_d$ as in [27, §2.3.2], [19, §3.9]. Let $\{H\}$ be a basis of $\text{Lie}(G(\mathbb{R}))$. We define a Laplacian on $G(\mathbb{R})$ by

$$\mathcal{D} := 1 - \sum_H H^2,$$

which is positive definite and self-adjoint on any unitary representation $\xi$ of $G(\mathbb{R})$. For any $v \in \xi$, we define the $d$th Sobolev norm of $v$ by

$$S_d(v) := \|\mathcal{D}^d v\|_{\xi}.$$

We refer to [27, §2.4] for a collection of useful properties of the Sobolev norm.

Let $W \in \pi$ be a unit vector such that in the Kirillov model $W$ is given by

$$W[\left(\begin{array}{c}g \\ 1 \end{array}\right)] = W_0[\left(\begin{array}{c}g \\ 1 \end{array}\right)].$$

We take a very similar path as in the proof of [19, Lemma 5.2] for $\pi_0$ being a tempered representation. Here we modify the proof to accommodate the $\theta_0$-tempered case.

**Lemma 7.2.** Let $W_0$ be as in Section 4.1. Let $A_{r-1}(\mathbb{R})O_{r-1}(\mathbb{R}) \ni h = ak$ as before. If $c \ll 1$, then

$$W_0[\left(\begin{array}{c}h \\ c/X 1 \end{array}\right)] \ll \eta, \pi_0 |\det(a)|^{-\theta_0}\delta^{1/2-\eta}\left(\begin{array}{c}a \\ 1 \end{array}\right)\min(1, a^{-M}_{r-1}) \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M})$$

for any $\eta > 0$.

This lemma is proved in [19, Lemma 5.2] for $\pi_0$ being a tempered representation. Here we modify the proof to accommodate the $\theta_0$-tempered case.

**Proof.** Let $W'_0 := \pi_0[\left(\begin{array}{c}k \\ c/X 1 \end{array}\right)]W_0$. Note that $k \in O_{r-1}(\mathbb{R})$ and $|c| \ll 1$ vary over compact sets. Hence it is enough to show that

$$W'_0[\left(\begin{array}{c}a \\ 1 \end{array}\right)] \ll \eta, M, \pi_0 |\det(a)|^{-\theta_0}\delta^{1/2-\eta}\left(\begin{array}{c}a \\ 1 \end{array}\right)\min(1, a^{-M}_{r-1}) \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}).$$

We take a very similar path as in the proof of [19, Lemma 5.2].

We define $W_1 := d\pi_0(Y^M)(W'_0)$, where $Y$ is a Lie algebra element such that

$$d\pi_0(Y)W'_0\left(\begin{array}{c}a \\ 1 \end{array}\right) = a_{r-1}W'_0\left(\begin{array}{c}a \\ 1 \end{array}\right).$$
Thus it is enough to prove that
\begin{equation}
W_1 \left[ \begin{pmatrix} a \\ 1 \end{pmatrix} \right] \ll_{\eta,M,\pi_0} |\det(a)|^{-\theta_0 \delta^{1/2-\eta}} \left[ \begin{pmatrix} a \\ 1 \end{pmatrix} \right] \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}). \tag{7.3}
\end{equation}

We use the Dixmier–Malliavin Lemma (see [13]) to find finitely many \( \alpha_i \in C_c^\infty(G(\mathbb{R})) \) and \( W_i \in \pi_0^\infty \) such that
\[ W_1 = \sum_i \pi_0(\alpha_i) W_i. \]

Thus to prove equation (7.3), it is enough to show equation (7.3) with \( W_1 \) replaced by \( \pi_0(\alpha_i) W_i =: W_2 \) for each \( i \).

Let \( \sigma \in \mathbb{R} \). We use the Whittaker–Plancherel formula to expand
\[ |\det(a)|^{-\sigma} W_2 \left[ \begin{pmatrix} a \\ 1 \end{pmatrix} \right] = \int_{\text{GL}_0^{-1}(\mathbb{R})} \sum_{W' \in B(\pi')} W'(a) Z_{W_2,\sigma}(W') d\mu_{\text{loc}}(\pi'), \tag{7.4} \]
which is valid for \( \sigma \) in some left half plane. Here
\[
Z_{W_2,\sigma}(W') := \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} W_2 \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \bar{W}(h) |\det(h)|^{-\sigma} dh
= \gamma(1/2 - \sigma, \pi_0 \otimes \bar{\pi})^{-1} \omega_{\pi'}(-1)^{r-1} \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} \bar{W} \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \bar{W}(h) |\det(h)|^\sigma dh.
\]

In the last line, we have used the \( \text{GL}(r) \times \text{GL}(r-1) \) local functional equation. Here \( \gamma(.) \) denotes the local gamma factor and \( \omega_{\pi'} \) denotes the central character of \( \pi' \). Finally, \( \bar{W} \) denotes the contragredient of \( W \) defined by \( \bar{W}(g) := W(w g^{-1}) \), where \( w \) is the long Weyl element of the respective group.

Let \( \tilde{a}_i(g) := \alpha_i(g^{-1}) \). Let \( N^* \) be the unipotent radical of upper triangular matrices attached to the partition \( r = (r-1) + 1 \). Recalling that \( W_2 = \pi_0(\alpha_i) W_i \), we can write
\[
\bar{W} \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] = \int_{G(\mathbb{R})} \tilde{a}_i(g) \bar{W}_i \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} g \right] dg
= \int_{N^*} \tilde{a}_i(n^* g) \psi_{e_{r-1} h(n^*)} dn^* dg,
\]
where \( e_{r-1} \) is the row vector \((0, \ldots, 0, 1)\). Then we have that
\[
Z_{W_2,\sigma}(W') = \gamma(1/2 - \sigma, \pi_0 \otimes \bar{\pi})^{-1} \omega_{\pi'}(-1)^{r-1}
\int_{N^*} \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} \bar{W}_i \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} g \right] \int_{N^*} \tilde{a}_i(n^* g) \psi_{e_{r-1} h(n^*)} dn^* \bar{W}'(h) |\det(h)|^\sigma dh dg.
\]

We choose an orthonormal basis \( B(\pi') \) consisting of eigenfunctions of the Laplacian \( D' \) on \( \text{GL}_{r-1}(\mathbb{R}) \), as defined in equation (7.1), and integrate by parts the \( h \)-integral \( L \) times with respect to \( D' \). We note that \( W' \otimes |\det|^{\sigma} \) is also an eigenfunction of \( D' \). We recall a bound of the gamma factor from [19, Lemma 3.1]:
\[
\gamma(1/2 - \sigma, \pi_0 \otimes \bar{\pi}) \ll_{\sigma, \pi_0} C(\pi')^r |\sigma|.
\]
We apply the Cauchy–Schwarz on the above $h$ integral. Then we use the above bound of the gamma factor and unitarity of $\pi_0$ to obtain that

$$Z_{W_2, \sigma}(W') \ll C(\pi')^{r \sigma} \lambda_{W'}^{-L} \int_{N^* \setminus G(\mathbb{R})} \left( \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} \left| D' L \left( \int_{N^*} \alpha_i(n^* g) \overline{\psi}_{e_{r-1} h}(n^*) dn^* \right) \right| \right)^{2dh} \frac{\det(h)}{|\sigma|^2} \frac{dh}{dg},$$

where $\lambda_{W'}$ is the $D'$-eigenvalue of $\tilde{W}'$. The above $N^*$-integral gives rise to a Schwartz function in $e_{r-1} h$, which can be seen integrating by parts several times in the $N^*$-integral. Thus

$$D' L \left( \int_{N^*} \alpha_i(n^* g) \overline{\psi}_{e_{r-1} h}(n^*) dn^* \right) \ll \min(1, |e_{r-1} h|^{-N}).$$

Noting that $g$ varies over a compact set in $G(\mathbb{R})$ modulo $N^*$, we obtain that

$$Z_{W_2, \sigma}(W') \ll C(\pi')^{r \sigma} \lambda_{W'}^{-L} \left( \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} \min(1, |e_{r-1} h|^{-N}) |\tilde{W}'(h)|^2 \det(h)^2 dh \right)^{1/2}.$$

We use [19, Lemma 5.2] on $\tilde{W}'$ (which is in the tempered representation $\tilde{\pi}'$) to check that the above integral is absolutely convergent for any $\sigma > 0$. In particular, from the location of the first pole of $\gamma(1/2 - \sigma, \pi_0 \otimes \pi^-)$, we may conclude that one can choose $\sigma$ in $(0, 1/2 - \theta_0)$ in the definition of $Z_{W_2, \sigma}(W')$.

Again, we use [19, Lemma 5.2] to estimate $W'(a)$ in equation (7.4) by

$$\ll \delta^{1/2 - \eta}(a) \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}) \lambda_{W'}^d,$$

where $d$ only depends on $M$. We choose $\sigma = 1/2 - \theta_0 - \eta$ to obtain that

$$W_2 \left[ \begin{pmatrix} a \\ 1 \end{pmatrix} \right] \ll_{\eta, M, \pi_0} |\det(a)|^{-\theta_0} \delta^{1/2 - \eta} \left[ \begin{pmatrix} a \\ 1 \end{pmatrix} \right] \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}) \int_{\text{GL}_{r-1}(\mathbb{R})} C(\pi')^{r \sigma} \sum_{W' \in B(\pi')} \lambda_{W'}^{d-L} d\mu_0(\pi').$$

We make $L$ sufficiently large and invoke [19, Lemma 3.3] to conclude that the above sum and integral are absolutely convergent. 

**Lemma 7.3.** Let $W$ be as in equation (7.2) and $W_0$ be as in Section 4.1. Let $V$ be $W$ or $W_0$ and $\xi$ be $\pi$ or $\pi_0$, respectively. Also let $A_{r-1}(\mathbb{R}) \Omega_{r-1}(\mathbb{R}) \ni h = ak$, where $a = \text{diag}(a_1, \ldots, a_{r-1})$ and $|c| \ll 1$. Then for any sufficiently small $\eta > 0$,

$$V \left[ \begin{pmatrix} h \\ c/X \end{pmatrix} \right] - V \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \ll_{\eta} |\det(a)|^{-\theta} \delta^{1/2 - \eta} \left[ \begin{pmatrix} a \\ 1 \end{pmatrix} \right] C(\xi)|c|/X.$$

Here $\theta$ is $\theta_0$ or $\theta_0$ depending on whether $\xi$ is $\pi$ or $\pi_0$, respectively.

This is essentially the main result of analytic newvectors, proved in [19, Proposition 4.1], but in a more quantitative form. We need to only modify the proof of [19, Proposition 4.1], and we describe that here.
Proof. Let \( \sigma \in \mathbb{R} \) be in some left half plane. As in the proof of [19, Proposition 4.1], we write the difference in the lemma as

\[
\int_{\text{GL}_{r-1}(\mathbb{R})} \omega_{\pi'} ((-1)^{r-1} C(\xi)^{-1}) C(\xi)^{\tau, -1} \gamma(1/2 - \sigma, \xi \otimes \pi')^{-1} \sum_{W' \in B(\pi')} W'(h)|\det(h)|^{-\sigma} dh
\]

where \( w' \) is the long Weyl element of \( \text{GL}(r - 1) \). Note that \( \pi' \) is tempered. We now use [19, Lemma 5.2] for tempered representations to estimate

\[ W'(h) \ll \delta^{1/2 - \eta}(a)S_d(W') \]

for any \( \eta > 0 \) and some \( d > 0 \). We choose \( \sigma = 1/2 - \theta - \eta \) (which is admissible) and proceed as in the proof of [19, Proposition 4.1] to conclude. \( \square \)

Proof of Proposition 7.1. Recall the definition of \( J_X \) from equations (4.5) and (2.6). In the expression of equation (2.6), we choose a basis \( B(\pi) \) containing an analytic newvector \( W \) as in equation (7.2). To show the required lower bound of \( J_X \), it is enough to drop all but the term containing \( W \) from the sum in equation (2.6) by positivity and show that

\[ X^{r-1}|\Psi(f_{1/2}(x), W_0, \overline{W})|^2 \gg 1 \]

if \( C(\pi) < X \).

First, using equation (2.2) and the choices of the local components as in Section 4.1, we get

\[ f_{1/2}\left[ \left( \begin{smallmatrix} |r-1 & \hfill c \\
-1 & 1 \end{smallmatrix} \right) \right] = \int_{\mathbb{R}^x} \Phi(t(c, 1))|t|^{r/2}dt \geq 0. \]

The support condition of \( \Phi \) in Section 4.1 implies that the above vanishes unless \( |c| < \tau \). We use Bruhat coordinates and make the change of variables to write \( X^{r-1/2} \Psi(f_{1/2}(x), W_0, \overline{W}) \) as

\[
\int_{\mathbb{R}^r} W_0 \left[ \left( \frac{h}{c/X} \right) \right] W \left[ \left( \frac{h}{c/X} \right) \right] f_{1/2}\left[ \left( \begin{smallmatrix} |r-1 & \hfill c \\
-1 & 1 \end{smallmatrix} \right) \right] dc \frac{dh}{|\det(h)|^{1/2}}.
\]

We use Lemma 7.3 for \( W_0 \), noting that \( |c| < \tau \) and \( C(\pi) < X \), to obtain that the above integral is

\[
\int_{\mathbb{R}^r} \left[ \frac{h}{1} \right] W_0 \left[ \left( \frac{h}{c/X} \right) \right] f_{1/2}\left[ \left( \begin{smallmatrix} |r-1 & \hfill c \\
-1 & 1 \end{smallmatrix} \right) \right] dc \frac{dh}{|\det(h)|^{1/2}} + O(\eta) \left( \tau \int_{\mathbb{R}^r} \left[ \frac{h}{1} \right] dc \frac{dh}{|\det(h)|^{1/2}} \right).
\]

We use Lemma 7.3 for \( W_0 \) and the definition of \( W \) in the Kirillov model as in equation (7.2) to obtain that the main term of equation (7.5) is equal to
\[
\int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} |W_0|^2 \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \frac{dh}{|\det(h)|^{1/2}} \int_{\mathbb{R}^{r-1}} f_{1/2} \left[ \begin{pmatrix} l_{r-1} \\ c \\ 1 \end{pmatrix} \right] dc \\
+ O_{\pi_0} \left( \frac{1}{X} \int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} |W_0|^2 \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \delta^{1/2-\eta} \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \frac{dh}{|\det(h)|^{1/2+\theta_0}} \int_{\mathbb{R}^{r-1}} f_{1/2} \left[ \begin{pmatrix} l_{r-1} \\ c \\ 1 \end{pmatrix} \right] dc \right). 
\] (7.6)

From the choice of \( \Phi \) in Section 4.1, we obtain that

\[
0 \leq \int_{\mathbb{R}^{r-1}} f_{1/2} \left[ \begin{pmatrix} l_{r-1} \\ c \\ 1 \end{pmatrix} \right] dc \simeq 1.
\]

Also, the choice of \( W_0 \) in Section 4.1 ensures that

\[
\int_{N_{r-1}(\mathbb{R}) \setminus \text{GL}_{r-1}(\mathbb{R})} |W_0|^2 \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \frac{dh}{|\det(h)|^{1/2}} \simeq \pi_0 1.
\]

So the main term of equation (7.6) is

\[
\simeq \pi_0 \int_{\mathbb{R}^{r-1}} f_{1/2} \left[ \begin{pmatrix} l_{r-1} \\ c \\ 1 \end{pmatrix} \right] dc.
\]

On the other hand, the error term in equation (7.6) is trivially \( \ll \pi_0, \tau X^{-1} \), which follows from the support condition of \( W_0 \) as in Section 4.1. In total, we obtain that equation (7.6), which is the main term of equation (7.5), is

\[
\simeq \pi_0 \int_{\mathbb{R}^{r-1}} f_{1/2} \left[ \begin{pmatrix} l_{r-1} \\ c \\ 1 \end{pmatrix} \right] dc + O_{\pi_0, \tau} (1/X).
\]

Now we focus on the error term of equation (7.5). We use Iwasawa coordinates in the integral and use Lemma 7.2 to estimate the error term by

\[
\ll M^2 \tau \int_{\mathbb{R}^{r-1}} f_{1/2} \left[ \begin{pmatrix} l_{r-1} \\ c \\ 1 \end{pmatrix} \right] dc \int_{A_{r-1}(\mathbb{R})} |\det(a)|^{1/2-\phi-\theta_0} \delta^{-\eta} \left[ \begin{pmatrix} h \\ 1 \end{pmatrix} \right] \\
\times \min(1, a_{r-1}^{-M}) \prod_{i=1}^{r-2} \min(1, (a_i/a_{i+1})^{-M}) d^\infty a.
\]

We recall the assumption that \( \phi + \theta_0 < 1/2 \). Hence, the inner integral is convergent for sufficiently small \( \eta \) and large enough \( M \). Thus we obtain that equation (7.5) is

\[
\simeq \pi_0 \left( 1 + \tau O_{\pi_0} (1) \right) \int_{\mathbb{R}^{r-1}} f_{1/2} \left[ \begin{pmatrix} l_{r-1} \\ c \\ 1 \end{pmatrix} \right] dc + O_{\pi_0, \tau} (1/X).
\]

We conclude that the above is \( \gg 1 \) by making \( \tau \) sufficiently small but fixed. \( \square \)

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