Gauge freedom in path integrals in Abelian gauge theory

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We extend the gauge symmetry of an Abelian gauge field to incorporate quantum gauge degrees of freedom. We twice apply the Harada–Tsutsui gauge recovery procedure to gauge-fixed theories. First, starting from the Faddeev–Popov path integral in the Landau gauge, we recover the gauge symmetry by introducing an additional field as an extended gauge degree of freedom. Fixing the extended gauge symmetry by the usual Faddeev–Popov procedure, we obtain the theory of Type I gaugeon formalism. Next, applying the same procedure to the resulting gauge-fixed theory, we obtain a theory equivalent to the extended Type I gaugeon formalism.

Subject Index B05

1. Introduction

The standard formalism of canonically quantized gauge theories [1–5] does not consider quantum-level gauge transformations. There is no quantum gauge freedom, since the quantum theory is defined only after the gauge fixing. Within the broader framework of Yokoyama’s gaugeon formalism [6–15], we can consider quantum gauge transformations as $q$-number gauge transformations among a family of Lorentz covariant linear gauges. In this formalism, quantum gauge freedom is provided by a set of extra fields, called gaugeon fields. The gaugeon formalism has been studied not only in Abelian fields [6,7,10,16–18] and Yang–Mills fields [11–15,19–21] but also in the Higgs model [8,21,22], chiral gauge theory [9], Schwinger’s model [23], the Rarita–Schwinger field [24], string theory [25,26], and gravity [27,28].

Yokoyama and Kubo [7] proposed two types of gaugeon theories for Abelian gauge fields, which they referred to as Type I and Type II theories. The Lagrangian of each theory has a gauge-fixing parameter $\alpha$ that can be shifted from $\alpha$ to $\alpha + \tau$ by a $q$-number gauge transformation. The tree-level photon propagator can be expressed as

$$\langle A_\mu A_\nu \rangle \sim \frac{1}{k^2} \left( g_{\mu\nu} + (a - 1) \frac{k_\mu k_\nu}{k^2} \right),$$

where the parameter $a$ is defined as

$$a = \epsilon \alpha^2 \quad (\epsilon = \pm 1) \quad \text{for Type I},$$

$$a = \alpha \quad \text{for Type II}.$$  

(1.2a)

(1.2b)

In Type I theory, the $q$-number gauge transformation can change the absolute value, but not the sign, of the parameter $a$; in Type II theory the parameter $a$ can be arbitrarily altered.
The Lagrangian of the Abelian gauge field $A_\mu$ in Type I theory \cite{6,17} is given by
\begin{equation}
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial_\mu B A^\mu - \partial_\mu Y_0 \partial^\mu Y + \frac{\epsilon}{2} (Y_0 + \alpha B)^2 - i \partial_\mu c_4 \partial^\mu c - i \partial_\mu K_4 \partial^\mu K, \tag{1.3}
\end{equation}
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $B$ is the Nakanishi–Lautrup field, $c_4$ and $c$ are the usual Faddeev–Popov (FP) ghosts, $\alpha$ is the gauge-fixing parameter, $Y_0$ and $Y_0$ are gaugeon fields, and $K$ and $K_4$ are FP ghosts for the gaugeon fields, which are introduced to ensure the BRST (Becchi–Rouet–Stora–Tyutin) symmetry \cite{17}. This Lagrangian permits the $q$-number gauge transformation where we vary the gauge-fixing parameter $\alpha$. The transformation is defined by
\begin{align*}
A_\mu &\to \hat{A}_\mu = A_\mu + \tau \partial_\mu Y, \\
Y_0 &\to \hat{Y}_0 = Y_0 - \tau B, \\
B &\to \hat{B} = B, \quad Y \to \hat{Y} = Y, \quad (1.4) \\
c &\to \hat{c} = c + \tau K, \quad c_4 \to \hat{c}_4 = c_4, \\
K &\to \hat{K} = K, \quad K_4 \to \hat{K}_4 = K_4 - \tau c_4,
\end{align*}
with $\tau$ being a parameter of the transformation. Under this transformation the Lagrangian (1.3) becomes
\begin{equation}
\mathcal{L} (\phi_A; \alpha) = \mathcal{L} \left( \hat{\phi}_A; \hat{\alpha} \right), \tag{1.5}
\end{equation}
where $\phi_A$ collectively represents all fields and $\hat{\alpha}$ is defined by
\begin{equation}
\hat{\alpha} = \alpha + \tau. \tag{1.6}
\end{equation}

The Lagrangian and $q$-number gauge transformation in Type II theory are described in \cite{7}. BRST symmetric Type II theory is given by \cite{16}.

The Lagrangian of the extended Type I theory, investigated by Endo \cite{18}, is given by
\begin{equation}
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial_\mu B A^\mu + (Y_{1s} + \alpha_1 B)(Y_{2s} + \alpha_2 B) - \partial_\mu Y_{1s} \partial^\mu Y_1 - \partial_\mu Y_{2s} \partial^\mu Y_2 \\
- i \partial_\mu c_4 \partial^\mu c - i \partial_\mu K_{1s} \partial^\mu K_1 - i \partial_\mu K_{2s} \partial^\mu K_2, \tag{1.7}
\end{equation}
where $Y_i$ and $Y_{is}$ ($i = 1, 2$) are two sets of gaugeon fields, $K_i$ and $K_{is}$ are two sets of FP ghosts for the gaugeon fields, and the constants $\alpha_i$ are the gauge-fixing parameters. The corresponding parameter $a$ of the tree-level photon propagator (1.1) is given by $a = 2\alpha_1 \alpha_2$. Thus, this theory extends the Type I gaugeon formalism by setting $\alpha$ as quadratic in the gauge-fixing parameters [cf. (1.2a)]. Because the parameter $a$ (and its sign) can be changed into arbitrary values by the $q$-number gauge transformation ($\alpha_1 \to \alpha_1 + \tau_1$, $\alpha_2 \to \alpha_2 + \tau_2$), this theory possesses some characteristics of Type II theory. The Lagrangian can also be written as
\begin{equation}
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial_\mu B A^\mu + \frac{1}{2} (Y_{+s} + \alpha_+ B)^2 - \frac{1}{2} (Y_{-s} + \alpha_- B)^2 - \partial_\mu Y_{+s} \partial^\mu Y_+ \\
- \partial_\mu Y_{-s} \partial^\mu Y_- - i \partial_\mu c_4 \partial^\mu c - i \partial_\mu K_{+s} \partial^\mu K_+ - i \partial_\mu K_{-s} \partial^\mu K_- \tag{1.8}
\end{equation}
where $Y_{\pm}$ are defined by
\begin{equation}
Y_{\pm} = \frac{1}{\sqrt{2}} (Y_1 \pm Y_2) \tag{1.9}
\end{equation}
and $Y_{\pm s}, K_{\pm}, K_{\pm s},$ and $\alpha_{\pm}$ are defined similarly.

Gaugeon theories for the Yang–Mills fields have been proposed by Yokoyama \cite{11} and Yokoyama, Takeda, and Monda \cite{15}. The BRST symmetric theories have been obtained by Abe \cite{19} and Koseki,
Sato, and Endo [20]. Although these theories are easily shown to be equivalent to the standard formalism in the Landau gauge \((a = 0)\), their equivalence to the standard formalism in non-Landau gauges \((a \neq 0)\) cannot be demonstrated. Therefore, these theories should be compared with the Abelian gaugeon theory, which is equivalent to non-Landau gauge theory \((a \neq 0)\) as well as to the Landau gauge \((a = 0)\) [17].

Sakoda [29] extended the gauge freedom of Yang–Mills fields using the gauge recovery procedure for gauge non-invariant functionals proposed by Babelon, Schaposnik, and Viallet [30] and Harada and Tsutsui [31,32]. Sakoda’s theory includes the two gauges of the standard formalism: the Landau gauge and a non-Landau \(a\)-gauge. Sakoda’s theory considers the total Fock space, which embeds the Fock spaces of both gauges of the standard formalism. In this theory, the \(q\)-number gauge transformation connects the Landau gauge and non-Landau \(a\)-gauge. Different from the gaugeon formalism, the \(q\)-number transformation of Sakoda’s theory cannot arbitrarily change the gauge parameter, but allows only \(\alpha = 0\) and \(\alpha = a\).

In this paper, we further extend the gauge freedom to allow more flexibility in the gauge parameter than in Sakoda’s theory. As a first step, we consider the Abelian gauge field. Starting with the Faddeev–Popov path integral in the Landau gauge, we extend the gauge freedom by twice applying the Harada–Tsutsui gauge recovery procedure [32]. In contrast, Sakoda [29] applied this procedure once to the Yang–Mills field.

The remainder of this paper is organized as follows. Section 2 reviews the Harada–Tsutsui gauge recovery procedure for gauge non-invariant functionals [32] and Sakoda’s path integral [29] of Yang–Mills fields. In Sect. 3, we extend the gauge symmetry of Abelian gauge fields twice using the Harada–Tsutsui gauge recovery procedure. In Sect. 4, we relate our theory to the gaugeon formalism and show that our theory is equivalent to the extended Type I gaugeon formalism.

2. Path integral of the gauge non-invariant functional

2.1. Harada–Tsutsui gauge recovery procedure

Harada and Tsutsui’s procedure extends the gauge degrees of freedom of the gauge non-invariant functional [32]. We illustrate their procedure on a system of gauge non-invariant Yang–Mills fields \(A_\mu\). Such a system might comprise massive Yang–Mills fields.

The action \(S_0[A]\) of the system is not invariant,

\[
S_0[A^g] \neq S_0[A],
\]

under the gauge transformation

\[
A_\mu \to A^g_\mu = g A_\mu g^{-1} + ig \partial_\mu g^{-1},
\]

where \(g\) is a group-valued function. The usual path integral of the system is given by

\[
Z_0 = \int D A_\mu \ e^{i S_0[A]},
\]

which leads to non-renormalizable propagators in the massive Yang–Mills case.

Now, we promote the group-valued function \(g(x)\) to a dynamical variable, and define an extended action by

\[
S[A, g] \equiv S_0[A^g],
\]

which is now invariant under the extended gauge transformation,

\[
A \to A^h, \quad g \to g^h = g h^{-1},
\]

in the extended gauge parameter space \((a \neq 0)\).
where $h(x)$ is a group-valued function. The formal path integral for $S[A, g]$,

$$Z_{\text{div}} = \int \mathcal{D}A \mathcal{D}g \, e^{iS[A, g]},$$  \hspace{1cm} (2.6)

is divergent since $S[A, g]$ is gauge invariant. To factor out the divergent gauge volume, we require gauge fixing (if $g = 1$, $Z_{\text{div}}$ reduces to $Z_0$). Expressing the gauge-fixing condition as

$$f[g, A] = 0,$$  \hspace{1cm} (2.7)

the corresponding FP determinant $\Delta_{\text{FP}}[A, g]$ is given by

$$1 = \Delta_{\text{FP}}[A, g] \int \mathcal{D}h \delta \left( f[gh, A^{h^{-1}}] \right).$$  \hspace{1cm} (2.8)

Inserting (2.8) into (2.6) and factoring out the gauge volume, we obtain

$$Z = \int \mathcal{D}A \mathcal{D}g \Delta_{\text{FP}}[A, g] \delta(f[g, A]) e^{iS[A, g]}.$$  \hspace{1cm} (2.9)

We can also consider 't Hooft averaging. Instead of the gauge-fixing condition (2.7), we use

$$f[g, A] = C(x),$$  \hspace{1cm} (2.10)

where $C(x)$ is an arbitrary $c$-number function. Averaging the path integral over $C(x)$ with the Gaussian weight

$$\exp \left[ -\frac{i}{2a} \int d^4x C(x)^2 \right],$$  \hspace{1cm} (2.11)

we obtain

$$Z = \int \mathcal{D}A \mathcal{D}g \mathcal{D}\Phi \mathcal{D}C \Delta_{\text{FP}}[A, g] e^{iS[A, g]+i \int d^4x (\Phi f[g, A] - C) - C^2/2a}$$

$$= \int \mathcal{D}A \mathcal{D}g \mathcal{D}\Phi \Delta_{\text{FP}}[A, g] e^{iS[A, g]+i \int d^4x (\Phi f[g, A] + a\Phi^2/2)}.$$  \hspace{1cm} (2.12)

The first line expresses the delta functional as a Fourier integral with respect to a field $\Phi$. Equation (2.12) yields renormalizable propagators in the massive Yang–Mills case.

### 2.2. Sakoda’s method

Sakoda [29] extended the gauge freedom of the gauge-fixed Yang–Mills fields in the Landau gauge by applying the Harada–Tsutsui procedure. This method is briefly explained below.

The Landau-gauge Lagrangian of a Yang–Mills field $A_\mu$ is given by

$$\mathcal{L}_L = 2 \text{tr} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu A_\mu + i \bar{c} \partial^\mu D_\mu c \right],$$  \hspace{1cm} (2.13)

where $F^{\mu\nu}$ is the field strength, $B$ is the Nakanishi–Lautrup field, and $c$ and $\bar{c}$ are the FP ghosts. We express the path integral as

$$Z_0 = \int \mathcal{D}A \mathcal{D}B \mathcal{D}\bar{c} \mathcal{D}c \, e^{i \int \mathcal{L}_L d^4x}$$

$$= \int \mathcal{D}A \mathcal{D}B \, I_0[A, B],$$  \hspace{1cm} (2.14)

where

$$I_0[A, B] = \Delta[A] e^{i \int d^4x 2 \text{tr}( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu A_\mu )},$$  \hspace{1cm} (2.15)

$$\Delta[A] = \det \partial^\mu D_\mu.$$  \hspace{1cm} (2.16)
Since we consider a gauge-fixed system, the functional \( I_0[A, B] \) is not gauge invariant under the gauge transformation

\[
A_{\mu} \rightarrow A_{\mu}' = gA_{\mu}g^{-1} + ig\partial_{\mu}g^{-1},
\]
\[
B \rightarrow B' = B.
\]

Now, we promote the group-valued function \( g(x) \) to a dynamical variable and define

\[
\tilde{I}_0[A, B, g] \equiv I_0[A^g, B^g] = \Delta[A^g] \exp \left\{ i \int d^4x \ 2\mathrm{tr} \left( -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + B \partial^{\mu} A_{\mu}^g \right) \right\}.
\]

The functional \( \tilde{I}_0[A, B, g] \) is invariant under the extended gauge transformation,

\[
A \rightarrow A^h,
\]
\[
g \rightarrow g^h = gh^{-1},
\]
\[
B \rightarrow B^h = B,
\]

where \( h(x) \) is a group-valued function. The formal path integral for \( \tilde{I}_0[A, B, g] \),

\[
Z_{\text{div}} = \int \mathcal{D} A \mathcal{D} B \mathcal{D} g \tilde{I}_0[A, B, g],
\]

is divergent since \( \tilde{I}_0[A, B, g] \) is now gauge invariant. To factor out the divergent gauge volume, we require gauge fixing. For this purpose, we consider the gauge-fixing condition

\[
f[g, A] \equiv \bar{\alpha}^\mu A_{\mu} - \partial^\mu A_{\mu}^g = C,
\]

where \( C \) is an arbitrary \( c \)-number function. The corresponding FP determinant \( \Delta_{\text{FP}}[A, g, C] \) is then given by

\[
1 = \Delta_{\text{FP}}[A, g, C] \int \mathcal{D} h \delta \left( f \left[ gh, A^h \right] - C \right).
\]

Inserting \( (2.22) \) into \( (2.20) \) and factoring out the gauge volume, we obtain

\[
Z = \int \mathcal{D} A \mathcal{D} B \mathcal{D} g \tilde{I}_0[A, B, g] \Delta[A] \delta(f[g, A] - C).
\]

In \( (2.23) \), \( \Delta_{\text{FP}}[A, g, C] \) was evaluated as

\[
\Delta_{\text{FP}}[A, g, C] \delta(f[g, A] - C) = \det(\partial^\mu D_\mu) \delta(f[g, A] - C)
\]
\[
= \Delta[A] \delta(f[g, A] - C).
\]

Expressing the delta functional as a Fourier integral with respect to \( \Phi \) and applying ’t Hooft averaging with a Gaussian weight, we obtain

\[
Z = \int \mathcal{D} A \mathcal{D} B \mathcal{D} g \mathcal{D} \Phi \tilde{I}_0[A, B, g] \Delta[A] \exp \left\{ i \int d^4x \ 2\mathrm{tr} \left( \Phi f[g, A] + \frac{a}{2} \Phi^2 \right) \right\},
\]

where \( a \) is the gauge-fixing parameter. The corresponding Lagrangian is given by

\[
\mathcal{L} = 2\mathrm{tr} \left[ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + B \partial^\mu A_{\mu}^g + i \bar{\eta} \partial^\mu D_{\mu} \eta^g + \Phi f[g, A] + \frac{a}{2} \Phi^2 + i \bar{\eta} \partial^\mu D_{\mu} \eta \right].
\]

Here, the determinant \( \Delta[A^g] \) in \( \tilde{I}_0[A, B, g] \) has been expressed in terms of the FP ghosts \( \eta \) and \( \bar{\eta} \):

\[
\Delta[A^g] = \int \mathcal{D} \bar{\eta} \mathcal{D} \eta \exp \left\{ i \int d^4x \ 2\mathrm{tr} \left( i \bar{\eta} \partial^\mu D_{\mu} \eta \right) \right\},
\]
where

\[ D^g_{\mu} \eta^g = \partial^\mu \eta^g - i \left[ A^g_{\mu}, \eta^g \right], \quad \eta^g = g \eta g^{-1}. \] (2.28)

The Lagrangian (2.26) is invariant under the BRST transformations \( \delta, \tilde{\delta}, \) and \( \delta_B = \delta + \tilde{\delta} : \)

\[
\begin{align*}
\delta A_{\mu} &= D_{\mu} c, \quad \delta g = -i g c, \\
\delta c &= i c^2, \quad \delta \eta = i \{ c, \eta \}, \\
\delta \tilde{c} &= i \Phi, \quad \delta \Phi = \delta B = \delta \tilde{\eta} = 0,
\end{align*}
\] (2.29)

and

\[
\begin{align*}
\tilde{\delta} A_{\mu} &= 0, \quad \tilde{\delta} g = -i g \eta, \\
\tilde{\delta} \eta &= i \eta^2, \quad \tilde{\delta} c = 0, \quad \tilde{\delta} \tilde{c} = 0, \\
\tilde{\delta} \tilde{\eta} &= i (\Phi - B), \quad \tilde{\delta} \Phi = \tilde{\delta} B = 0.
\end{align*}
\] (2.30)

These transformations satisfy the nilpotency condition, \( \delta^2 = \tilde{\delta}^2 = \delta_B^2 = \{ \delta, \tilde{\delta} \} = 0. \) We denote the corresponding BRST charges by \( Q, \tilde{Q}, \) and \( Q_B. \)

In Sakoda’s theory, we can consider the two subspaces of the total Fock space using the BRST charges. One is the subspace \( \ker \tilde{Q} = \{|\phi\rangle; \tilde{Q}|\phi\rangle = 0 \}; \) the other is \( \ker Q. \) The subspace \( \ker \tilde{Q} \) corresponds to the Fock space of the standard formalism of the \( a \)-gauge. To see this, we express the Lagrangian (2.26) as

\[
\mathcal{L} = 2 \text{tr} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu A_{\mu} + \frac{a}{2} B^2 + i \tilde{c} \partial^\mu D_{\mu} c \right] \\
- i \tilde{\delta} \left( 2 \text{tr} \left[ \tilde{\eta} \left( (\partial^\mu A_{\mu} - \partial^\mu A^g_{\mu}) + \frac{a}{2} (\Phi + B) \right) \right] \right). \] (2.31)

The term in the second line is \( \tilde{Q} \)-exact, and thus ignorable in the subspace \( \ker \tilde{Q}; \) the remaining term is nothing but the \( a \)-gauge Lagrangian. To show that the subspace \( \ker Q \) corresponds to the Fock space of the standard formalism of the Landau gauge, we denote the Landau-gauge fields \( A'_{\mu} \) by \( A'_{\mu} = A^g_{\mu} \) and express the Lagrangian as

\[
\mathcal{L} = 2 \text{tr} \left[ -\frac{1}{4} F^{\prime \mu\nu} F'_{\mu\nu} + B \partial^\mu A'_{\mu} + i \tilde{c} \partial^\mu D'_{\mu} \eta' \right] \\
- i \tilde{\delta} \left( 2 \text{tr} \left[ \tilde{c} \left( (\partial^\mu A'_{\mu} - \partial^\mu A_{\mu}^g) + \frac{a}{2} \Phi \right) \right] \right), \] (2.32)

where \( F'_{\mu\nu} \) is the field strength of \( A'_{\mu}, D'_{\mu} \) is the covariant derivative corresponding to \( A'_{\mu}, \) and \( \eta' = \eta^g. \) The term in the second line is ignorable in the subspace \( \ker Q; \) the remaining term is the Landau-gauge Lagrangian. Thus, in Sakoda’s theory, the subspaces of the total Fock space identify the Fock spaces of the standard theory of the \( a \)-gauge and Landau gauge. The \( a \)-gauge field \( A_{\mu} \) and Landau-gauge field \( A'_{\mu} \) are connected through the \( q \)-number gauge transformation \( g(x). \) The

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1 We note here that another expression,

\[
\mathcal{L} = 2 \text{tr} \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \Phi \partial^\mu A_{\mu} + \frac{a}{2} \Phi^2 + i \tilde{c} \partial^\mu D_{\mu} c \right] + i \tilde{\delta} \left( 2 \text{tr} \left[ \tilde{\eta} \partial^\mu A^g_{\mu} \right] \right),
\]

would be helpful to analyze the subspace \( \ker \tilde{Q} \) by using the BRST charge \( Q. \) The field \( \Phi \) (rather than \( B \)) plays the role of the Nakanishi–Lautrup field in this \( a \)-gauge theory.
q-number transformation of Sakoda’s theory limits the gauge parameter to only two values, \( \alpha = 0 \) and \( \alpha = a \). Considering this, Sakoda’s theory differs from the gaugeon formulation of the Yang–Mills field \([15,19,20]\). (Strictly speaking, using Sakoda’s q-number transformation \( g(x) \) we can define another q-number gauge transformation \( \{g(x)\}^\tau \) with an arbitrary real number \( \tau \). This transformation changes the gauge parameter \( a \) into \( a(1 - \tau)^2 \) in the tree-level propagator of \( A_\mu \). We do not, however, consider this transformation at present, since the transformed Lagrangian would have complicated terms and it would not be easy to analyze the theory in this gauge.\(^2\)

3. Successive extension of the gauge freedom of the Abelian field

3.1. Sakoda’s extension

To ensure that this section remains self-contained, we repeat here Sakoda’s arguments in the Abelian case.

We start with the Landau-gauge Lagrangian of the Abelian gauge field given by

\[
L_L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu A_\mu + i \bar{c} \partial^\mu \partial_\mu c. \tag{3.1}
\]

The path integral is expressed as

\[
Z_0 = \int DADBD\bar{c}Dc \, e^{i \int d^4x L_L} = \int DADB \, I_0[A, B], \tag{3.2}
\]

where

\[
I_0[A, B] = \Delta \, e^{i \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu A_\mu \right)}, \tag{3.3}
\]

\[
\Delta = \det \partial^\mu \partial_\mu. \tag{3.4}
\]

Since we consider a gauge-fixed system, the functional \( I_0[A, B] \) is not gauge invariant under the gauge transformation

\[
A_\mu \to A_\mu^\theta = A_\mu + \partial_\mu \theta, \quad B \to B^\theta = B, \tag{3.5}
\]

where \( \theta \) is an arbitrary scalar function. Now, we promote the function \( \theta \) to a dynamical variable and define

\[
\tilde{I}_0[A, B, \theta] \equiv I_0[A^\theta, B^\theta] = \Delta \, e^{i \int d^4x \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu (A_\mu + \partial_\mu \theta) \right\}}. \tag{3.6}
\]

The functional \( \tilde{I}_0[A, B, \theta] \) is invariant under the extended gauge transformation,

\[
A_\mu \to A_\mu + \partial_\mu \lambda, \quad \theta \to \theta - \lambda, \quad B \to B, \tag{3.7}
\]

\(^2\) In the Abelian limit, the situation becomes simple. The Abelian limit of Sakoda’s theory is equivalent to the Abelian gaugeon formalism (see Sects. 3.1 and 4.1); the transformation \( \{g(x)\}^\tau \) becomes a usual q-number gauge transformation of the Abelian gaugeon formalism.
where λ is an arbitrary scalar function. The formal path integral for \( \tilde{I}_0[A, B, \theta] \),
\[
Z_{\text{div}} = \int DADBD\theta \tilde{I}_0[A, B, \theta],
\]
(3.8)
is divergent since \( \tilde{I}_0[A, B, \theta] \) is now gauge invariant. To factor out the divergent gauge volume, we require gauge fixing. For this purpose, we consider the gauge-fixing condition
\[
f[\theta, A] \equiv \partial^\mu A_\mu - \partial^\mu A^B_\mu = C.
\]
(3.9)
The corresponding FP determinant is then given by
\[
1 = \Delta_{\text{FP}}[A, \theta, C] \int D\lambda \delta \left( -\partial^\mu \partial_\mu \theta \lambda - C \right).
\]
(3.10)
Inserting (3.10) into (3.8) and factoring out the gauge volume, we obtain
\[
Z_1 = \int DADBD\theta \tilde{I}_0[A, B, \theta] \Delta \delta \left( -\partial^\mu \partial_\mu \theta - C \right),
\]
(3.11)
where we have evaluated \( \Delta_{\text{FP}}[A, \theta, C] \) as
\[
\Delta_{\text{FP}}[A, \theta, C] \delta \left( -\partial^\mu \partial_\mu \theta - C \right) = \det \left( \partial^\mu \partial_\mu \right) \delta \left( -\partial^\mu \partial_\mu \theta - C \right) = \Delta \delta \left( -\partial^\mu \partial_\mu \theta - C \right).
\]
(3.12)
Expressing the delta functional as a Fourier integral with respect to \( \Phi_1 \) and applying 't Hooft averaging with a Gaussian weight, we obtain
\[
Z_1 = \int DADBD\theta D\Phi_1 I_1[A, B, \theta, \Phi_1],
\]
(3.13)
with
\[
I_1[A, B, \theta, \Phi] = \Delta \Delta \exp \left[ i \int d^4x \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu (A_\mu + \partial_\mu \theta) - \Phi \partial^\mu \partial_\mu \theta + \frac{a}{2} \Phi^2 \right\} \right].
\]
(3.14)
where \( a \) is the gauge-fixing parameter. The two FP determinants can be expressed in terms of two pairs of ghost fields:
\[
\Delta \Delta = \int D\bar{c} Dc D\bar{\eta} D\eta \exp \left[ i \int d^4x \left\{ i\bar{c} \partial^\mu \partial_\mu c + i\bar{\eta} \partial^\mu \partial_\mu \eta \right\} \right].
\]
(3.15)
Summarizing these results, we obtain the Lagrangian of the first extension of the gauge freedom as
\[
\mathcal{L}_{\text{1st}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu (A_\mu + \partial_\mu \theta) - \Phi \partial^\mu \partial_\mu \theta + \frac{1}{2} a \Phi^2 + i\bar{c} \partial^\mu \partial_\mu c + i\bar{\eta} \partial^\mu \partial_\mu \eta.
\]
(3.16)
Because we extended the gauge freedom by the method of Sakoda [29], the above Lagrangian is the Abelian limit of Sakoda’s Yang–Mills Lagrangian (2.26). As implied by Sakoda [29], (3.16) is equivalent to the Lagrangian of the gaugeon formalism. (We will confirm this in Sect. 4.1.) The Lagrangian is invariant under the following Abelian version of Sakoda’s BRST transformation:
\[
\delta_B A_\mu = \partial_\mu c, \quad \delta_B \theta = -(c + \eta),
\]
\[
\delta_B \bar{c} = i\Phi, \quad \delta_B \bar{\eta} = i(\Phi - B),
\]
\[
\delta_B B = \delta_B \Phi = \delta_B c = \delta_B \eta = 0.
\]
(3.17)
This transformation satisfies the nilpotency \( \delta_B^2 = 0 \). We can also find \( \delta \) and \( \tilde{\delta} \) transformations satisfying \( \delta_B = \delta + \tilde{\delta} \), as in the Yang–Mills case (2.29) and (2.30).
3.2. The successive extension

Starting from the path integral (3.13), we again extend the gauge freedom of the Lagrangian $L_{1st}$. The functional $I_1[A, B, \theta, \Phi]$ is not gauge invariant under the gauge transformation

$$A_\mu \rightarrow A_\mu^X = A_\mu + \partial_\mu \chi,$$

$$\theta \rightarrow \theta^X = \theta - \chi,$$

$$B \rightarrow B^X = B,$$

$$\Phi \rightarrow \Phi^X = \Phi,$$

(3.18)

where $\chi$ is an arbitrary scalar function. Now, we promote the function $\chi$ to a dynamical variable and define

$$\tilde{I}_1[A, B, \theta, \Phi, \chi]$$

$$\equiv I_1[A^X, B^X, \theta^X, \Phi^X]$$

$$= \Delta \Delta \exp \left[ i \int d^4x \left\{ -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + B \partial^\mu (A_\mu + \partial_\mu \theta) - \Phi \partial^\mu \partial_\mu (\theta - \chi) + \frac{1}{2} a \Phi^2 \right\} \right],$$

(3.19)

where we have used $A_\mu^X + \partial_\mu \theta^X = A_\mu + \partial_\mu \theta$. The functional $\tilde{I}_1[A, B, \theta, \Phi, \chi]$ is gauge invariant under the following extended gauge transformation:

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda,$$

$$\theta \rightarrow \theta - \lambda,$$

$$\chi \rightarrow \chi - \lambda,$$

$$B \rightarrow B,$$

$$\Phi \rightarrow \Phi,$$

(3.20)

where $\lambda$ is an arbitrary scalar function. The formal path integral for $\tilde{I}_1[A, B, \theta, \Phi, \chi]$, $Z_{\text{div}}$, is divergent since $\tilde{I}_1[A, B, \theta, \Phi, \chi]$ is now gauge invariant. To factor out the divergent gauge volume, we require gauge fixing. We consider the gauge-fixing condition

$$f[\chi, A] \equiv \partial^\mu A_\mu - \partial^\mu A_\mu^X = C,$$

(3.22)

where $C$ is an arbitrary $c$-number function. The corresponding FP determinant $\Delta_{\text{FP}}[A, \chi, C]$ is then defined as

$$1 = \Delta_{\text{FP}}[A, \chi, C] \int \mathcal{D}\delta \left(-\partial^\mu \partial_\mu \chi^X - C\right).$$

(3.23)

Inserting (3.23) into (3.21) and factoring out the gauge volume, we obtain

$$Z_2 = \int \mathcal{D}A \mathcal{D}B \mathcal{D}\theta \mathcal{D}\Phi \mathcal{D}\chi \mathcal{D}\lambda \Delta_{\text{FP}}(A, \chi, C) \Delta \delta(-\partial^\mu \partial_\mu \chi - C),$$

(3.24)

where we have evaluated $\Delta_{\text{FP}}[A, \chi, C]$ as

$$\Delta_{\text{FP}}[A, \chi, C] \delta(-\partial^\mu \partial_\mu - C) = \det(\partial^\mu \partial_\mu) \delta(-\partial^\mu \partial_\mu \chi - C) = \Delta \delta(-\partial^\mu \partial_\mu \chi - C).$$

(3.25)
Expressing the delta functional as a Fourier integral with respect to $\phi$ and applying ’t Hooft averaging with a Gaussian weight, we obtain

$$Z_2 = \int \mathcal{D}A \mathcal{D}B \mathcal{D}\Phi \mathcal{D}\chi \mathcal{D}\phi I_2[A, B, \theta, \Phi, \chi, \phi],$$

(3.26)

with

$$I_2[A, B, \theta, \Phi, \chi, \phi] = \Delta \Delta \Delta \exp \left[ i \int d^4x \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu (A_\mu + \partial_\mu \theta) - \Phi \partial^\mu \partial_\mu (\theta - \chi) + \frac{a}{2} \Phi^2 - \phi \partial^\mu \partial_\mu \chi + \frac{a'}{2} \phi^2 \right\} \right],$$

(3.27)

where $a'$ is another gauge-fixing parameter. The Lagrangian of the successive extension of the gauge degree of freedom is expressed by

$$\mathcal{L}_{2nd} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu A_\mu + (B - \Phi) \partial^\mu \partial_\mu \theta + (\Phi - \phi) \partial^\mu \partial_\mu \chi + \frac{a}{2} \Phi^2 + \frac{a'}{2} \phi^2 + i \bar{c} \partial^\mu \partial_\mu c + i \bar{\eta} \partial^\mu \partial_\mu \eta + i \bar{\xi} \partial^\mu \partial_\mu \xi,$$

(3.28)

where the third $\Delta$ has been expressed in terms of the ghost fields $\xi$ and $\bar{\xi}$:

$$\det(\partial^\mu \partial_\mu) = \int \mathcal{D}\bar{\xi} \mathcal{D}\xi \exp \left( i \int d^4x i \bar{\xi} \partial^\mu \partial_\mu \xi \right).$$

(3.29)

This Lagrangian is invariant under the following BRST transformation:

$$\delta_B A_\mu = \partial_\mu c, \quad \delta_B \theta = \eta,$$

$$\delta_B \chi = \xi, \quad \delta_B \bar{c} = iB,$$

$$\delta_B \bar{\eta} = i(B - \Phi), \quad \delta_B \bar{\xi} = i(\Phi - \phi),$$

$$\delta_B B = \delta_B \Phi = \delta_B \phi = \delta_B c = \delta_B \eta = \delta_B \xi = 0,$$

(3.30)

which satisfies the nilpotency $\delta_B^2 = 0$.

4. Relation to the gaugeon formalism

4.1. Equivalence of $\mathcal{L}_{1st}$ and the Type I gaugeon Lagrangian

We first confirm that the Lagrangian (3.16) of Sakoda’s extension is equivalent to the Lagrangian of Type I gaugeon theory (1.3). Redefining the fields as

$$\theta = -\alpha Y,$$

$$\Phi = \frac{1}{\alpha} Y_\ast + B,$$

$$\eta = K, \quad \bar{\eta} = K_\ast,$$

(4.1)

where $\alpha$ is a numerical parameter satisfying $a = \varepsilon \alpha^2$ ($\varepsilon = a/|a|$), we can rewrite the Lagrangian (3.16) as

$$\mathcal{L}_{1st} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + B \partial^\mu A_\mu + Y_\ast \partial^\mu \partial_\mu Y + \frac{1}{2} \varepsilon (Y_\ast + \alpha B)^2 + i c_\ast \partial^\mu \partial_\mu c + i K_\ast \partial^\mu \partial_\mu K,$$

(4.2)

which is exactly (1.3). It should be noted that the field $\theta$ introduced as an extended gauge freedom plays the role of a gaugeon field.
4.2. Equivalence of $L_{2nd}$ and the extended Type I Lagrangian

Next, we show that the Lagrangian (3.28) of the successive extension is equivalent to the extended Type I Lagrangian (1.8). Redefining the fields as

$$\theta = -\alpha_+ Y_+ - \alpha_- Y_-,$$

$$\Phi = \frac{1}{\alpha_+} Y_{++} + B,$$

$$\chi = -\alpha_- Y_-,$$

$$\phi = \frac{1}{\alpha_-} Y_{--} + B,$$

$$\eta = K_+, \quad \bar{\eta} = K_{++},$$

$$\xi = K_-, \quad \bar{\xi} = K_{--},$$

(4.3)

where $\alpha_+$ and $\alpha_-$ are numerical parameters satisfying

$$a = \varepsilon \alpha_+^2 \quad (\varepsilon = a/|a|), \quad a' = \varepsilon' \alpha_-^2 \quad (\varepsilon' = a'/|a'|),$$

(4.4)

we can rewrite the Lagrangian (3.28) as

$$L_{2nd} = -\frac{1}{4} F_{\mu \nu}^A F_{\mu \nu}^A + B \partial^\mu A_\mu + Y_{++} \partial^\mu \partial_\mu Y_+ + Y_{--} \partial^\mu \partial_\mu Y_- + \frac{1}{2} \varepsilon (Y_{++} + \alpha_+ B)^2$$

$$+ \frac{1}{2} \varepsilon' (Y_{--} + \alpha_- B)^2 + i c_s \partial^\mu \partial_\mu c + i K_{++} \partial^\mu \partial_\mu K_+ + i K_{--} \partial^\mu \partial_\mu K_-.$$  

(4.5)

Provided that the sign factors $\varepsilon$ and $\varepsilon'$ differ, this Lagrangian is equivalent to the extended Type I gaugeon Lagrangian (1.8). In the successive extension, the real scalar fields $\theta$ and $\chi$ introduced as extended gauge degrees of freedom play the roles of gaugeon fields $Y_+$ and $Y_-$. 

5. Summary and discussion

Starting from the Faddeev–Popov path integral in the Landau gauge of Abelian gauge theory, we successively extended the gauge symmetry to incorporate quantum gauge degrees of freedom. Following Sakoda’s treatment of Yang–Mills fields, we applied (in each extension) the Harada–Tsutsui gauge recovery procedure to the gauge non-invariant functional. The Lagrangian resulting from the first extension agrees with the Abelian limit of Sakoda’s Yang–Mills Lagrangian, and is equivalent to that of Type I gaugeon theory. The scalar field $\theta$ introduced as an extended gauge degree of freedom plays the role of a gaugeon field $Y$. The theory obtained by the second extension is equivalent to extended Type I theory if the signs ($\varepsilon$ and $\varepsilon'$) differ. The scalar fields $\theta$ and $\chi$ introduced as extended gauge degrees of freedom play the roles of gaugeon fields $Y_+$ and $Y_-$. 

One might consider what happens when we further repeat Sakoda’s extensions of gauge freedom. In the Abelian case, extending the gauge freedom three times or more would not yield any new features, at least from the viewpoint of the photon propagator (1.1). For example, when we apply Sakoda’s extension to the Lagrangian $L_{2nd}$ (4.5), we obtain the third pair of gaugeon fields $(Y_{3^+}, Y_3)$, corresponding FP ghosts $(K_{3^+}, K_3)$, and the third extended Lagrangian

$$L_{3rd} = L_{2nd} + \frac{1}{2} \varepsilon_3 (Y_{3^+} + \alpha_3 B)^2 + Y_{3^+} \partial^\mu \partial_\mu Y_3 + i K_{3^+} \partial^\mu \partial_\mu K_3.$$  

(5.1)

where $\varepsilon_3$ is a sign factor and $\alpha_3$ a numerical parameter. The photon propagator following from (5.1) is again expressed as (1.1) where the parameter $a$ is now given by

$$a = \varepsilon (\alpha_+)^2 + \varepsilon' (\alpha_-)^2 + \varepsilon_3 (\alpha_3)^2.$$  

(5.2)
Thus the third extension does not expand the region of the values of the parameter $a$; the second extension is enough to give the parameter $a$ an arbitrary value. In the non-Abelian case, non-trivial features may appear when we apply Sakoda's extensions multiple times. Exploring this possibility is our next task.

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