A Lagrangian model for the evolution of turbulent magnetic and passive scalar fields

T. Hater,1 H. Homann,1,2 and R. Grauer1
1Theoretische Physik I, Ruhr-Universität Bochum,
Universitätsstr. 150, D-44780 Bochum (Germany)
2Université de Nice-Sophia Antipolis, CNRS, Observatoire de la Côte d’Azur,
Laboratoire Cassiopée, Bd. de l’Observatoire, 06300 Nice, France

In this paper we present an extension of the Recent Fluid Deformation (RFD) closure introduced by Chevillard and Meneveau [1] which was developed for modeling the time evolution of Lagrangian fluctuations in incompressible Navier-Stokes turbulence. We apply the RFD closure to study the evolution of magnetic and passive scalar fluctuations. This comparison is especially interesting since the stretching term for the magnetic field and for the gradient of the passive scalar are similar but differ by a sign such that the effect of stretching and compression by the turbulent velocity field is reversed. Probability density functions (PDFs) of magnetic fluctuations and fluctuations of the gradient of the passive scalar obtained from the RFD closure are compared against PDFs obtained from direct numerical simulations.

PACS numbers: 47.27.eb, 47.10.-g, 02.50.Ey, 52.30Cv, 52.35.Ra, 52.65.Kj

I. INTRODUCTION

In recent years several Lagrangian type closures have been developed to model fluctuations of turbulent incompressible fields [1–4]. The starting point for all these models goes back to the so-called Restricted Euler closure [5, 6] which models the time evolution of the gradient tensor $A_{ij} := \partial_i u_j$

$$\frac{\partial}{\partial t} A_{ij} = - \left( A_{ik} A_{kj} - \frac{\delta_{ij}}{3} A_{mk} A_{km} \right). \quad (1)$$

This model is derived using the incompressibility constraint and neglects the viscous term and the anisotropic part of the pressure Hessian. The Restricted Euler model is already able to capture geometric features of vortex stretching and alignment but fails as a robust model to obtain stationary statistics due to the appearance of finite time singularities.

Several strategies have been applied to regularize the restricted Euler model where the most prominent ones are the Tetrad Model [2] which models the anisotropic part of the pressure Hessian and the Recent Fluid Deformation approach which models both the viscous term and the anisotropic part of the pressure Hessian [1].

In this paper we extend the Recent Fluid Deformation approach to the case of the time evolution of the gradient of passive scalar and to kinematic magnetohydrodynamic turbulence. Comparing these two examples is especially interesting since both examples differ mainly by a sign in the stretching term. Thus structures which are expanded in one equation (e.g. gradient of passive scalar) are compressed in the other (e.g. MHD). Comparison with direct numerical simulations are performed to test the range of applicability of the extended Recent Fluid Deformation models.

II. VELOCITY GRADIENTS

Before we apply the Recent Fluid Deformation approach to the gradient of a passive scalar and to kinematic MHD turbulence, we first recall the main steps in the derivation of this model. Detailed information can be found in [1, 2]. Starting with the Navier-Stokes equation

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u \quad , \quad \nabla \cdot u = 0 \quad (2)$$

one takes the gradient

$$\partial_t \partial_k u_i + \partial_k (u_j \partial_j u_i) + \partial_k p = \nu \partial_k u_i \quad (3)$$

and denotes the velocity gradient $\partial_k u_j$ as $A_{ij}$. Using the Lagrangian material derivative $\frac{d}{dt} = (\partial_t + u_i \partial_i)$ eqn. (2) reduces to

$$\frac{dA_{ij}}{dt} + A_{ik} A_{kj} = - \partial_{ij} p + \nu \partial_k u_i \quad , \quad (4)$$

Assuming the pressure Hessian to be isotropic $\partial_{ij} p = \frac{\partial^2 p}{\partial x_i \partial x_j}$ and neglecting the viscous term would result in the Restricted Euler model given by eqn. (1). In order to improve the model for the pressure Hessian, Chevillard and Meneveau [1] consider the pressure Hessian $P_{nm} = \frac{\partial^2 p}{\partial x_n \partial x_m}$ expressed in the Lagrangian frame and postulate that the Lagrangian pressure is isotropic at some given time $t_0$. Applying the change between Eulerian and Lagrangian coordinates $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial x_i}$ twice - while neglecting higher orders - one obtains

$$\frac{\partial^2 p}{\partial x_i \partial x_j} \simeq \frac{\partial X_n}{\partial x_i} \frac{\partial X_m}{\partial x_j} \frac{\partial^2 p}{\partial X_n \partial X_m} = - \frac{A_{ik} A_{kj}}{C_{kk}^2} \quad (5)$$

where $C = \exp(\tau A) \exp(\tau A^T)$ is the short-time Cauchy-Green tensor resulting from the integration of the flow map. The time $\tau$ is chosen of the order of the Kolmogorov time so that it is possible to linearize the integration from $\partial_t u_j$ to $\partial_t X_j$. 

\[ \text{arXiv:1010.1598v2 [physics.plasm-ph]} \]
In order to study the dynamics of the gradient of the passive scalar $\nabla \theta$ using the Recent Fluid Deformation approach (top) and obtained from direct numerical simulations (bottom).

Applying a similar reasoning to the viscous term results in the approximation $\nu A_{ij} \simeq \frac{C_{ij}^{-1}}{C_{kk}} A_{ij}$, $T$ being a characteristic friction time such that the Reynolds number $R$ is proportional to $(\frac{L}{\tau})^2$. In all simulations described below, the parameters $T = 1$ and $\tau = 0.05$ were chosen which result in a Taylor-Reynolds number of $R = 77$. This value is in the range of the performed DNS simulations (see Table I).

Finally, one obtains a stochastic ordinary differential equation (SDE)

$$\frac{dA_{ij}}{dt} = - \left( A_{ik} A_{kj} - \frac{A_{ik} A_{kj}}{C_{kk}} C_{ij}^{-1} + \frac{C_{kk}^{-1}}{3T} A_{ij} \right) dt + dW_{ij}$$

where the stochastic forcing term $W_{ij}$ is added to represent large scale forcing. The SDE (6) can be integrated with standard methods for stochastic equations [7, 8].

### III. EXTENSION TO PASSIVE ADMIXTURES

In this section we apply the Recent Fluid Deformation theory to the case of an admixture $\theta$ which is passively advected by the fluid according to

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta.$$  \hspace{1cm} (7)

In order to study the dynamics of the gradient of the passive field $\theta$ we take the derivative of eqn. (7) and obtain

$$\partial_t \partial_i (\partial_i \theta) + \partial_i (u_k \partial_i \theta) = \kappa \partial_{kk} (\partial_i \theta).$$  \hspace{1cm} (8)

Using again the Lagrangian material derivative $\frac{d}{dt} = (\partial_t + u_j \partial_j)$ eqn. (8) reduces to

$$\frac{d}{dt} (\nabla \theta)_i = - A_{ik} (\nabla \theta)_k + \kappa \partial_{kk} (\nabla \theta)_i.$$  \hspace{1cm} (9)

In this equation, the gradient tensor $A_{ij}$ is obtained from integrating eqn. (6). Modeling the viscous term as for eqn. (8) and adding a stochastic forcing $V$, we obtain the stochastic differential equation (SDE)

$$d(\nabla \theta)_i = - \left[ A_{ik} (\nabla \theta)_k + \frac{C_{kk}^{-1}}{3T} (\nabla \theta)_i \right] dt + dV_i$$  \hspace{1cm} (10)

This equation is integrated together with the equation (6) for the velocity gradient tensor. The forcing is chosen to be vectorial gaussian noise, delta-correlated in time, as for the velocity gradient. The forcing amplitude is $\sqrt{2d}$ as for a standard random walk. The parameters for the result shown here are $T_\theta = 1$ and $\tau = 0.05$.

In order to test the applicability of the Recent Fluid Deformation approximation, we performed direct numerical simulations for passive scalar turbulence. For this, we utilized the pseudo-spectral simulation framework LATU [9]. Table I summarizes the relevant parameters of the simulation.

This model extension has been analyzed in [10] in great detail, considering important phenomena like amplification of the gradient’s norm and alignment to strain principal axes. But in contrast to this publication, we focus on the comparison of different turbulent systems.

The top of Figure 1 shows the probability distribution function (PDF) of fluctuations of $\nabla \theta$ obtained from integrating eqns. (6) and (10). In the bottom of Figure 1 the PDF obtained from the direct numerical spectral simulation is shown. The qualitative agreement is excellent and the tails of the PDFs of $\nabla \theta$ approximately follow an exponential decay behavior as predicted by Shraiman and Siggia [11] for a Gaussian velocity field.
We start with the incompressible MHD equations in the kinematic regime neglecting the back-reaction of magnetic fluctuations on the velocity field. The equation for the time evolution of the magnetic field $B$ reads
\[ \partial_t B + \mathbf{u} \cdot \nabla B = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \Delta \mathbf{B} \, , \] (11)
where the velocity field obeys the Navier-Stokes equations (2). In the Lagrangian frame this equation takes the form
\[ \frac{d}{dt} B_i = B_k A_{ki} + \eta \Delta B_i \, . \] (12)

Again, using the same Recent Fluid Deformation approximation for the resistive term and adding a stochastic forcing, which is as above gaussian and white-in-time, we obtain
\[ dB_i = - \left( -B_k A_{ki} + \frac{C_{kk}^{-1}}{3T_B} B_i \right) dt + dU_i \, . \] (13)

Note that this equation differs from (10) by the transpose in $A$ and - more important - the sign of the first term on the right hand side. Thus directions, where the magnetic field is stretched, are directions, where the gradient of the passive scalar is compressed and vice versa. This very different influence of the stretching term results in a qualitatively different PDF for the magnetic field fluctuations compared to the $\nabla \theta$ fluctuations. In order to test whether the Recent Fluid Deformation approach is able to reproduce the different role of the stretching term, we performed direct numerical simulations of kinematic MHD turbulence. Here, we again performed simulations using the framework LATU and the relevant parameters are given in Table I. The simulation was started with random initial condition for the magnetic field $B$ and an already fully turbulent velocity field. External forcing was only applied to the velocity field by keeping the large scale Fourier-modes $k \leq 2$ constant. The top of Figure 2 shows the PDF of magnetic field fluctuations calculated using equations (4) and (13). This PDF has to be compared to the PDF obtained from direct numerical simulation shown in the bottom of Figure 2. It is again remarkable that the simple stochastic ODE model (13) is able to capture correctly the shape of the PDF of magnetic field fluctuations.

Since we considered the kinematic MHD equations in the Dynamo regime, the PDF of the stochastic model could not be obtained from a time series as in the case of the passive scalar since the magnetic field is growing exponentially.

To obtain meaningful statistics ensemble averaging was used instead of time sampling. Numerically, a random initial state was generated and integrated a sufficiently large number of time steps, to retain no statistical influence of the initial state. The final state enters the statistics and a new realization is generated via another random initial condition. The PDF was obtained by sampling over $10^6$ initial conditions.

For illustration of the kinematic dynamo effect and a consistency check with direct numerical simulations of the kinematic MHD equations we tracked the local magnetic energy $B^2(X(t), t)$ in Figure 3 for $T_B = 6$, a value well in the dynamo regime of the model. For this simulation only the velocity gradient itself was driven by stochastic forcing, as outlined above, while the magnetic field was undriven. The forcing amplitude was scaled with $\sqrt{2 dt}$, analog to the scaling for a standard random walk.

The parameter $\tau$ of the model was chosen to be $\tau = 0.05$. The magnetic diffusion time $T_B$ scale is varied to explore the reaction of the dynamo effect. Below the critical parameter of approximately $T_B \approx 4$ we observed no growing magnetic energy.

The corresponding growth rate, estimated from an exponential growth $\approx \exp(\gamma t/\tau)$, is about $\gamma \approx 0.032$. The growth rate compares well with the growth rate of our DNS simulation, where we obtain $\gamma \approx 0.03$. Although the identification of the correct Kolmogorov timescale $\tau$ is not without uncertainties, it is remarkable that this value of the growth rate is in the range of data obtained by kinematic dynamo simulations (see also Table I in and references therein).

### V. CONCLUSION & OUTLOOK

In this paper we have shown that the natural extension of the Recent Fluid Deformation model for Navier-Stokes turbulence 1 to the case of fluctuations of the gradient of a passive scalar and to magnetic field fluctuations is able to produce probability distribution functions that agree well with PDFs obtained from direct numerical simulations. The PDFs from the stochastic ODEs 10.
are obtained with a fraction of the computing resources necessary for direct numerical simulations. The next step is to include the back-reaction of the magnetic field on the fluid flow. If this can be managed, then generating magnetic field fluctuations using the Recent Fluid Deformation model is a tempting alternative for e.g. the problem of cosmic ray propagation where magnetic field fluctuations are generated by other means which are not able to capture intermittency effects (see [13, 14]).

Acknowledgments

R.G. acknowledges stimulating discussions with L. Chevillard during the workshop “Euler Equations: 250 Years On”. Access to the JUGENE BlueGene/P computer at the FZ Jülich was made available through project HBO22. This work benefited from support through DFG-FOR1048.

[1] L. Chevillard and C. Meneveau, Phys. Rev. Lett. 97, 174501 (2006).
[2] M. Chertkov, A. Pumir, and B. Shraiman, Phys. Fluids 11, 2394 (1999).
[3] A. Naso and A. Pumir, Phys. Rev. E 72, 056318 (2005).
[4] L. Chevillard and C. Meneveau, C. R. Méc. 335, 187 (2007).
[5] P. Vieillefosse, Physica A 125, 150 (1984).
[6] B. J. Cantwell, Phys. Fluids A 4, 782 (1992).
[7] E. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations (Springer, Berlin, 1999).
[8] D. J. Higham, SIAM Review 43 (2001).
[9] H. Homann, O. Kamps, R. Friedrich, and R. Grauer, New Journal of Physics 11, 73020 (2009).
[10] M. Gonzalez, Phys. Fluids 21, 055104 (2009).
[11] B. I. Shraimann and E. D. Siggia, Phys. Rev. E 49, 2912 (1994).
[12] G. L. Eyink, arXiv:1008.4959 (2010).
[13] J. Giacalone and J. R. Jokipii, The Astrophysical Journal 520, 204 (1999).
[14] G. Zimbardo, P. Pommois, and P. Veltri, The Astrophysical Journal 639, L91 (2006).