New fermion discretizations and their applications

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Why New Fermions?

(1) Further understanding on lattice field theory

(2) Improvement of lattice QCD simulations
Lattice fermion improvement

Wilson: $O(a)$ errors & bad chiral properties

◆ Smeared-link clover (UV filtered, $O(a)$ improved)
  DeGrand, Hasenfratz, Kovacs, MILC(98), BMW’s intensive works
◆ Twisted-mass (unphysical zero-mode removed, $O(a)$ improved)
  Frezzotti, et.al. ALPHA(00), ETM’s intensive works

Staggered: taste breaking at $O(a^2)$

◆ HISQ (fat-link & $O(a^2)$ Symanzik)  Follana, et.al.(06), MILC’s intensive works
◆ HYP  Hasenfratz, Knechtli(01)  ◆ Fat7  Orginos, Toussaint et.al.(99)  ◆ Asqtad  Lepage (98)

Domain-wall, Overlap: Numerical cost

◆ Fixed topology (kernel zero-mode removed, locality)  Fukaya, et.al. (06), JLQCD intensive works
◆ Reweighting (enlarge 5th size, chiral properties)  Hasenfratz et.al.(08), Ishikawa, et.al. (10)
◆ Hyercube overlap (perfect kernel→locality, scaling)  Bietenholtz, et.al. (99)(12)  etc....
New setups can contribute?

1. Flavored mass
   Adams (09) cf.) Golterman, Smit (84)
   Staggered overlap $\rightarrow$ CPU time reduction (overlap)?
   Taste symmetry improved (staggered)?

2. Central branch
   Kimura, Komatsu,TM, Noumi, Torii, Aoki (11)
   Wilson w/o additive renorm. $\rightarrow$ Chiral symmetry (No fine-tuning?)
   $O(a)$ improved?

3. Minimal-doubling
   Karsten(81) Wilczek(87) Creutz(07) Borici(07)
   Chiral two-flavor w/ ultra locality $\rightarrow$ Better chiral property?
1. Flavored mass
**Naive**

\[ \sum_{\mu} C_{\mu} \]

\[ C_{\mu} = (T_{+\mu} + T_{-\mu})/2 \]

\[ T_{\pm\mu} \psi_n = U_{n,\pm\mu} \psi_n \pm \mu \]

**Wilson**

\[ U(4) \times U(4) \]
the kinetic term of this action has larger symmetry than the action of the continuum theory:

\[ C_\mu = (T_{+\mu} + T_{-\mu})/2 \]

\[ T_{\pm\mu} \psi_n = U_{n,\pm\mu} \psi_{n\pm\mu} \]

In this section, we first review the \( U(4) \) symmetries of the naive fermion in \([7, 42]\) using the spin-flavor representation. Then we apply the same method to the Wilson fermion action, which is invariant under only \( \text{NG bosons} \) when the symmetry is spontaneously broken. The existence of these sixteen NG indices is due to the spontaneous breakdown of symmetry.

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In section 3, we discuss the symmetries of the naive fermion and the minimally doubled fermions.

Section 4 is devoted to a summary of the lattice fermion via the spin-flavor representation. In section 3, we discuss the symmetries of the naive fermion and the minimally doubled fermions.

We show, however, that an additional \( U(1) \) vector symmetry is realized by tuning the flavor parameter of the naive fermion and the Wilson fermion. This investigation helps us understand the case of staggered fermions.

The free energy of the Wilson fermion is expressed as a function of the mass parameter and the lattice spacing. In this section, we study how the free energy of the Wilson fermion is related to the symmetry enhancement and its spontaneous breakdown.
The kinetic term of this action has the following flavor and chiral symmetry:

\[
\sum_{\mu} C_{\mu}
\]

where \( C_{\mu} = (T_{+\mu} + T_{-\mu})/2 \)

\( T_{\pm \mu} \psi_n = U_{n,\pm \mu} \psi_{n,\pm \mu} \)

\[
\Gamma_X^{(+)}, \Gamma_X^{(-)} \in \left\{ 1_4, \right\}
\]

\[
\Gamma_X^{(+)} \in \left\{ 1_4, \right\}
\]

\[
\Gamma_X^{(-)} \in \left\{ \right\}
\]
Drawback: additive mass renormalization, i.e., fine-tuning $f$

- Include taste-dependent mass term:

\[
\sum_{sym.} C_1 C_2 C_3 C_4
\]

\[
C_\mu = (T_{+\mu} + T_{-\mu})/2
\]

\[
T_{\pm\mu} \psi_n = U_{n,\pm\mu} \psi_n \pm\mu
\]

**U(4) x U(4) → U(2) x U(2)**

- For $\Gamma_X^{(+)}$
  \[
  \Gamma_X^{(+)} \in \left\{ 1_4, (-1)^{n_1 + \cdots + n_4} \gamma_5 \right\}
  \]

- For $\Gamma_X^{(-)}$
  \[
  \Gamma_X^{(-)} \in \left\{ (-1)^{n_{\mu,\nu}} \frac{[\gamma_\mu, \gamma_\nu]}{2} \right\}
  \]
**Naive flavored mass**

Creutz, Kimura, TM (10)

\[
M_V = \sum_\mu C_\mu, \quad \text{Vector (1-link)}
\]

\[
M_T = \sum_{\text{perm. sym.}} \sum C_\mu C_\nu, \quad \text{Tensor (2-link)}
\]

\[
M_A = \sum_{\text{perm. sym.}} \sum_\nu \prod C_\nu, \quad \text{Axial-V (3-link)}
\]

\[
M_P = \sum_{\text{sym. } \mu=1} 4 \prod C_\mu, \quad \text{Pseudo-S (4-link)}
\]

- **gamma-5 hermiticity**

- **2nd derivative terms** \( \sum_n \bar{\psi}_n (M_P - 1) \psi_n \rightarrow -a \int d^4 x \bar{\psi}(x) D_\mu^2 \psi(x) + O(a^2) \)

- **Cousins of Wilson fermion**
Naive $\rightarrow$ Wilson$^{(9)}$ $\rightarrow$ Domain-wall Overlap$^{(9)}$

Flavored-mass

Spin diag.

Staggered
Naive → Staggered → Wilson → Domain-wall Overlap

Wilson → St.Wilson → St.Dm-wall St.Overlap

Faster domain-wall & overlap !?
**Flavored mass**  Golterman, Smit (1984)  Adams(2009)

### Staggered

\[
\epsilon_x \sum_{sym.} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4 \sim (1 \otimes \xi_5) + O(a)
\]

\{C_0, \Xi_\mu, I_s, R_{\mu\nu}\} \times \{U^\epsilon(1)\}_{m=0}

### St. Wilson  Adams (09)

\[\xi_5 = -1 \rightarrow \text{physical sector : } \ell\]

\[\xi_5 = +1 \rightarrow \text{decoupled sector : } \hbar\]

- Practical form

\[\eta_\mu D_\mu + r(1 + M_A) + m \quad M_A = \epsilon_x \sum_{sym.} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4\]

With this mass shift:

\[\xi_5 = -1 \rightarrow \text{physical sector : } \ell\]

\[\xi_5 = +1 \rightarrow \text{decoupled sector : } \hbar\]
Flavored mass  

Golterman, Smit (1984)  
Adams (2009)

Staggered

\[
\epsilon_x \sum_{\text{sym.}} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4 \sim (1 \otimes \xi_5) + O(a)
\]

{\(C_0, \Xi_\mu, I_s, R_{\mu \nu}\} \times \{U^e(1)\}_{m=0} \rightarrow \}

St. Wilson  
Adams (09)

\[\xi_5 = -1 \rightarrow \text{physical sector: } \ell\]
\[\xi_5 = +1 \rightarrow \text{decoupled sector: } \bar{\ell}\]

No parameter tuning for Lorentz symmetry!

Practical form

\[
\eta_\mu D_\mu + r (1 + M_A) + m \quad M_A = \epsilon_x \sum_{\text{sym.}} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4
\]

Wilson-like term  
mass parameter

With this mass shift →
\[\xi_5 = -1 \rightarrow \text{physical sector: } \ell\]
\[\xi_5 = +1 \rightarrow \text{decoupled sector: } \bar{\ell}\]
◆ **Staggered-Wilson (Domain-wall, Overlap)**

How to apply:
- As Wilson → Mass parameter tuning required
- As Domain-wall → 5th dimension introduced
- As Overlap → Overlap formula with StWil kernel

- Spin diagonalization  Creutz, Kimura, TM (10)

\[ \bar{\psi}_x C_1 C_2 C_3 C_4 \psi_x \quad \rightarrow \quad \pm \chi_x (\epsilon \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4) \chi_x \]

- Index theorem  Adams (09)  Follana, Azcoiti, Di Carlo, Vaquero (11)

\[ H_{SW}(m) = \epsilon (D_{SW} - m) \]

\[ \lambda(m) \]

Index\((D_{sw}) = -\text{Spectral flow}(H_{sw})\)

\[ H_W(m) = \gamma_5(D_W - m) \]
• **Aoki phase** Creutz, Kimura, TM(11) TM, Nakano, Kimura, Ohnishi(12)

  Strong-coupling lattice QCD & 2d model
  → **Implies parity-flavor broken phase**

  *ChPT analysis required → 1st or 2nd order?*

• **Another type (Hoelbling type)** Hoelblin (10), de Forcrand, Kurkela, Panero (10)

  \[
  \sum_{\mu \nu = 12, 34} i \epsilon_{\mu \nu \mu \nu} (C_{\mu} C_{\nu} + C_{\nu} C_{\mu}) \\
  \sim (\mathbf{1} \otimes i(\sigma_{12} + \sigma_{34})) + O(a)
  \]

  \[
  \{C_{0}, \Xi_{\mu}, I_{s}, R_{\mu \nu}\} \times \{U^{e}(1)\}_{m=0} \rightarrow \{C_{T}, \Xi'_{\mu}, R_{12}, R_{34}, R_{13} R_{42}\}
  \]

  → Requires fine-tuning of parameters for Lorentz sym. continuum Sharpe (12)

  Let’s focus only on Adams type.
§ Potential problems of $\eta_{\mu} D_{\mu} + r(1 + M_A)$

1. Lorentz symmetry restored?

Euclidian Lorentz symmetry, C, P, T
→ likely to be restored from $\{C_0, \Xi'_\mu, R_{\mu\nu}\}$

No parameter tuning!

2. Multi-link terms require numerical costs?

(i) 24 terms for symmetric sum, (ii) 4 transporters

VS

One component fermion (small matrix size)
§ Potential advantages of $\eta_\mu D_\mu + r(1 + M_A)$

1. could reduce numerical costs in 2-flavor overlap
   
   One-component action $\rightarrow$ Small matrix size of propagator

2. could reduce influence of taste-breaking for 2-flavor
   
   Staggered sym. vs 4 tastes

   Halved staggered sym. vs 2 tastes

   The situation should be different, but better or worse?
1. Numerical costs reduced? de Forcrand, Kurkela, Panero (2011)

**Staggered-Overlap Dirac propagator**

- **Small matrix size**
  Requires fewer Matrix-Vector multiplications for sign function!

- **4-link hopping terms**
  Gauge fluctuation is raised to 4th power!
  → splitting of two branches reduced

**Staggered-Wilson is better as an overlap kernel, but not much better.**
2. How about taste breaking? Sharpe (12)

**Pion spectrum**

§ Staggered \( \{C_0, \Xi_\mu, I_s, R_{\mu\nu}\} \times \{U^c(1)\}_{m=0} \rightarrow \{C_0, \Xi_j, I_s, R_{ij}\} \) Transfer-matrix sym.

classify 15 pseudoscalar operators Golterman (1986)

1 : \( \xi_4, \xi_{45}, \xi_5 \)
3 : \( \xi_i, \xi_{i5}, \xi_i \xi_j \xi_i4 \)

7 irreps

§ Staggered-Wilson \[ \{C_0, \Xi'_\mu, R_{\mu\nu}\} \rightarrow \{C_0, \Xi'_j, R_{ij}\} \]

Irreps mix in \( \xi_5 \) pairs

\[ 1 \ \& \ \xi_5 \rightarrow \bar{\ell}\ell, \bar{h}h \]

Physical sector \[ \bar{\ell}(\gamma_5 \otimes 1)\ell \]

\( \eta' \)

\[ \xi_4 \ \& \ \xi_{45} \rightarrow \bar{\ell}h, \bar{h}\ell \]

\[ \bar{\ell}\gamma_j \ell, \bar{h}\gamma_j h \]

\[ \bar{\ell}(\gamma_5 \otimes \sigma_i)\ell \]

\( \pi_0, \pi^\pm \)

States in 3d irrep

cf.) \( \text{SU}(2) \) in ChPT potential upto \( O(a^4), O(a^2m) \)

Discrete symmetries are sufficient for degenerate pion triplet!
◆ Short summary

• Adams fermion will work as 2-flavor Wilson.

• Taste-breaking exists, but small enough to have degenerate pion triplet.

• Further study is needed to reveal numerical merit or demerit.

How about other mesons and baryons?

Usual improvement works? (Fixed topology, smearing)
Naive \rightarrow Wilson^{(0)} \rightarrow Domain-wall^{(0)}

Naive \rightarrow Flavored-mass \rightarrow Staggered

Staggered \rightarrow Wilson^{(0)} \rightarrow Domain-wall^{(0)}

Staggered \rightarrow Flavored-mass \rightarrow St.Wilson

St.Wilson \rightarrow St.Dm-wall^{(0)}

St.Wilson \rightarrow St.Overlap

St.Overlap
2. Central-branch
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- Wilson w/o onsite term  \( M_W = m + 4r = 0 \)

\[
S = \frac{1}{2} \sum_{x, \mu} \bar{\psi}_x [\gamma_\mu (U_{x, \mu} \psi_{x+\mu} - U_{x, -\mu} \psi_{x-\mu}) - (U_{x, \mu} \psi_{x+\mu} + U_{x, -\mu} \psi_{x-\mu})]
\]

\[\psi_x \rightarrow e^{i\theta(-1)^{x_1+x_2+x_3+x_4}} \psi_x, \quad \bar{\psi}_x \rightarrow \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}\]

\[
\Gamma_X^{(+)} = \left\{ 1_4, (-1)^{n_1+\ldots+n_4} \gamma_5, (-1)^{\bar{n}_\mu} \gamma_\mu, (-1)^{n_\mu} i \gamma_\mu \gamma_5, (-1)^{n_\mu, \nu} \frac{[\gamma_\mu, \gamma_\nu]}{2} \right\}
\]

\[
\Gamma_X^{(-)} = \left\{ (-1)^{n_1+\ldots+n_4} 1_4, \gamma_5, (-1)^{n_\mu} \gamma_\mu, (-1)^{\bar{n}_\mu} \gamma_\mu \gamma_5, (-1)^{n_\mu, \nu} \frac{[\gamma_\mu, \gamma_\nu]}{2} \right\}
\]
2. Central-branch

- Wilson w/o onsite term  $M_W \equiv m + 4r = 0$

$$S = \frac{1}{2} \sum_{x,\mu} \bar{\psi}_x [\gamma_\mu (U_{x,\mu} \psi_{x+\mu} - U_{x,-\mu} \psi_{x-\mu}) - (U_{x,\mu} \psi_{x+\mu} + U_{x,-\mu} \psi_{x-\mu})]$$

$\Rightarrow$ Another $U(1)!$

$$\psi_x \rightarrow e^{i\theta(-1)^x} \psi_x, \quad \bar{\psi}_x \rightarrow \bar{\psi}_x e^{i\theta(-1)^x}$$

$$\Gamma_X^{(+) \in \left\{ 1_4, \right\}}$$

$$\Gamma_X^{(-) \in \left\{ (-1)^{n_1+\cdots+n_4} 1_4, \right\}}$$

$\gamma_5 \otimes \xi_5$ Prohibits additive mass renormalization! 
SSB gives NG boson!
**Strong-coupling QCD**  
Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

\[
cosh(m_{SPA}) = 1 + \frac{2M_W^2 (16 + M_W^2)}{16 - 15M_W^2}
\]

NG boson associated with SSB of U(1)

- Pion (eta) condensate \( \langle \psi \gamma_5 \psi \rangle \neq 0 \)
- No chiral condensate \( \langle \bar{\psi} \psi \rangle = 0 \)

§ Advantages

- No additive mass renormalization (no fine-tuning)
- SSB of U(1) and massless NG boson
- No O(a) errors

§ Potential drawbacks

- sign problem
- U(1) problem
- Quark mass

Twisted-mass works?

\( \bar{\psi} \psi \leftrightarrow \bar{\psi} \gamma_5 \psi \) change of mass base

6-flavor massless QCD

6th-rooting works?

→ 12-flavor QCD

Could be a new possibility of 12-flavor lattice QCD
Central points for other flavored masses

- For other naive flavored mass terms

  \( M_A : U(1) \) restored  
  \( M_T : U(2) \) restored  
  \( M_P : U(4) \) restored

- For staggered flavored mass terms

  \( M_{\tilde{A}} : C_T' \bar{\Xi}, C_T'I \) restored  
  \( M_{\tilde{H}} : C_T' \) restored  

\[ C_T' : \chi_x \to \bar{\chi}_x^T, \quad \bar{\chi}_x \to \chi_x^T, \quad U_{x,\mu} \to U_{x,\mu}^* \]

|       | \( C_T' \) | \( \Xi_\mu \) | \( I_\mu \) | \( C_T' \bar{\Xi}_\mu \) | \( C_T' I_\mu \) | \( \Xi_\mu I_\mu \) |
|-------|------------|---------------|-------------|-----------------|-----------------|--------------------|
| \( S_{st} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( S_A \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( S_{\tilde{H}} \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |
| \( S_m \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) | \( \times \) |

Table 1: Invariance (\( \circ \)) or non-invariance (\( \times \)) of the staggered kinetic term \( S_{st} \), Adams-type term \( S_A \), Hoelbling-type term \( S_{\tilde{H}} \) and the usual mass term \( S_m \) under \( C_T', \Xi_\mu, I_\mu \) and their combinations.

With \( A \mu = 0 \) or 1 and \( \sum_\mu A_\mu \neq 0 \). Ref. [6] shows by classifying operators by timeslice group that these pions fall into 7 irreducible representations of symmetry group of the corresponding transfer matrix at finite lattice spacing:

1. \( \xi_4, \xi_{45}, \xi_5 \)

2. \( \xi_i, \xi_{i5}, \xi_{ij} \)

Here we take the 4th direction as time. Moreover, it is shown from staggered chiral perturbation theory in Ref. [7] that \( SO(4) \) flavor and Lorentz symmetries hold in the \( O(a^2) \) chiral perturbation (pion) potential. Thus 15-plet falls into 4 irreducible representations up to \( O(a^4), O(a^2m) \) and \( O(a^2p^2) \) as

1. \( \xi_5 \)

2. \( \xi_{\mu}, \xi_{\mu 5} \)

3. \( \xi_{\mu \nu} \)

It means that there are three degeneracies in lattice-pseudo pion spectrum in the leading discretization errors.

Staggered fermions with flavored mass

In this section we investigate symmetries of staggered-Wilson fermions and the spectrum of pseudo-scalar states.
3. Minimal-doubling
3. Minimal-doubling

Flavored imaginary chemical potential term lifts species degeneracy.

cf.) Flavored mass in Wilson

\[
\begin{align*}
\text{Wilson} & : & \sum_{\mu} (1 - \cos p_{\mu}) \\
\text{Flavored chemical-pot.} & : & (i) \gamma_4 \sum_{j=1}^{3} (1 - \cos p_j) \quad \rightarrow \quad \text{keeping one chiral sym.}
\end{align*}
\]

Finite-mass system(Wil) \Leftrightarrow \text{Finite-density system}(FCP)

◆ Advantage
  • U(1) chiral symmetry
  • Ultra-local
  • 2 flavor possible

◆ Drawbacks
  • Hypercubic symmetry breaking
  • Tuning parameters for a correct continuum limit

\begin{align*}
\text{Bedaque, Buchoff, Tiburzi, Walker-Loud(08)} \\
\text{Capitani, Creutz, Weber, Wittig (09)(10)}
\end{align*}
◆ Symmetries  Bedaque, Buchoff, Tiburzi, Walker-Loud (08)

(1) $U(1)$ chiral symmetry
(2) P
(3) CT
(4) Cubic symmetry.

◆ Counterterms  Capitani, Creutz, Weber, Wittig (09)(10)

$\text{dim} 3 \quad \bar{\psi}_n i \gamma_4 \psi_n \quad \text{dim} 4 \quad \bar{\psi}_n \gamma_4 D\psi_n \quad F_{i4} F_{i4}$

*Fine-tuning of three parameters are required for Lorentz sym.*

◆ Chiral phase structure  TM (12)

*Nontrivial phase diagram in the parameter space*
**Finite \((T, \mu)\) QCD with FCP**  Misumi, Kimura, Ohnishi (2012)

Still fine-tuning for \(O(1/a)\) chemical potential renorm..... cf.)additive mass in Wilson
But the discrete symmetries suit this case.

\[
S_{md} = \sum_x \left[ \frac{1}{2} \sum_{j=1}^{3} \bar{\psi}_x \gamma_j (U_{x,x+j} \psi_{x+j} - U_{x,x-j} \psi_{x-j}) + \frac{1}{2} \bar{\psi}_x \gamma_4 \left( e^{\mu} U_{x,x+4} \psi_{x+4} - e^{-\mu} U_{x,x-4} \psi_{x-4} \right) + \frac{i}{2} \sum_{j=1}^{3} \bar{\psi}_x \gamma_4 (2 \psi_x - U_{x,x+j} \psi_{x+j} - U_{x,x-j} \psi_{x-j}) + i d_3 \bar{\psi}_x \gamma_4 \psi_x \right]
\]

§ Strong-coupling study

*Effective potential of \(\sigma\) as a function of \(T, \mu\) and \(d_3\)*

Chiral phase structure
- 1st and 2nd phase transition \((m=0)\)
- 1st, critical point and crossover \((m\neq0)\)

*New possibility of \((T,\mu)\) lattice QCD!*
4. **Summary**

1. **Flavored-mass terms** give us new types of Wilson and overlap fermions.

2. **Staggered-Wilson** can be an alternative Wilson and overlap for 2-flavor QCD (3 degenerate pion spectrum)

3. **Central-branch fermion** is a new possibility of use of Wilson for many-flavor QCD without fine-tuning of parameters.

4. **Flavored-chemical-potential fermion** would be useful for finite-temperature & density lattice QCD.
Related talks

Tuesday 15:30 Room 8       Taro KIMURA
“QCD Phase diagram with 2-flavor discretization”

Wednesday 9:30 Room 5    Takashi NAKANO
“Strong coupling analysis of Aoki phase in St-Wil fermions”
Back-up slides
◆ Spectral flow

(i) Hermitian operator

$$H(m) = \gamma_5 (D - m) \quad (H^2 = D^\dagger D + m^2 \geq 0)$$

(ii) Eigenvalue flow \( \lambda_i(m) \)

$$\lambda_0(m) = \mp m \quad \text{only for zero modes}$$

zero mode : low-lying crossing

chirality : minus the sign of slope

* lattice theory (Wilson fermion)

(i) Hermitian operator

$$H_W(m) = \gamma_5 (D_W - m)$$

(ii) Eigenvalue flow

would-be zero modes : low-lying real crossing

approximate chirality : \( \lambda'(m) = -\psi(m)^\dagger \gamma_5 \psi(m) \)

Index\((D_W) = -\text{Spectral flow}(H_W)\)

\[ \text{Index}(D_W) = (-1)^{d/2} Q \]

※ Spectral flow :

Crossings counted with \(\pm\) slopes

2d Wilson fermion

\( H_W(m) = \gamma_5 (D_W - m) \)

R.Edwards, U.Heller, R.Narayanan (1998)
For generalized Wilson fermions

\[ H_{gw} = \gamma_5(D_{nf} - M_P) \]

\[ \text{Index}(D_{gw}) = -\text{Spectral flow}(H_{gw}) \]

\[ \text{Index}(D_{gw}) = 2^d(-1)^{d/2} Q \]

※ gauge configuration:

\[ U_{x,1} = e^{i\omega x_2}, \quad U_{x,2} = \begin{cases} 1 & (x_2 = 1, 2, \cdots, L - 1) \\ e^{i\omega Lx_1} & (x_2 = L) \end{cases}, \quad \omega = 2\pi Q. \]

For staggered-Wilson fermions

\[ H_{sw} = \epsilon(D_{st} - M^{(A)}_f) = \Gamma_{55}(D_{st} - M^{(A)}_f) \]

\[ \text{Index}(D_{sw}) = -\text{Spectral flow}(H_{sw}) \]

\[ \text{Index}(D_{sw}) = 2^{d/2}(-1)^{d/2} Q \]

Index theorem holds for them.
**Overlap formulation**

negative-mass mode in $D_w \rightarrow$ massless mode in $D_{ov}$

*Low-lying crossings are far from high-lying ones*

- **Generalized overlap**
  
  $$D_{go} = 1 + \gamma_5 \frac{H_{gw}(m)}{\sqrt{H_{gw}^2(m)}}$$

  *Any-flavor (1~15) overlap is possible!*

  cf.) 2 or 3-flavor overlap $\rightarrow$ lattice QCD

  12-flavor overlap $\rightarrow$ conformal window

- **Staggered-overlap**

  $$D_{so} = 1 + \Gamma_{55} \frac{H_{sw}(m)}{\sqrt{H_{sw}^2(m)}}$$

  *Less expensive overlap!*

  cf.) 1/4 matrix size $\rightarrow$ less CPU cost for Lanczos process
• **Shift symmetry** → broken to 2-link shift for $S_A$
  broken to 4-link shift for $S_H$

  $$S_\rho : \chi_x \rightarrow \zeta_\rho(x)\chi_{x+\hat{\rho}}, \quad \bar{\chi}_x \rightarrow \zeta_\rho(x)\bar{\chi}_{x+\hat{\rho}}, \quad U_{\mu,x} \rightarrow U_{\mu,x+\hat{\rho}}$$

  $$S_\mu : \phi(p) \rightarrow \exp(ip_\mu)\Xi_\mu \phi(p)$$

• **Axis reversal** → broken to shifted axis reversal

  $$T_\rho : \chi_x \rightarrow (-1)^{x_\rho}\chi_{I_\rho x}, \quad \bar{\chi}_x \rightarrow (-1)^{x_\rho}\bar{\chi}_{I_\rho x}, \quad U_{\mu,x} \rightarrow U_{\mu,I_\rho x}$$

  $$T_\rho : \phi(p) \rightarrow \Gamma_\rho \Xi_\rho \Xi_\rho \phi(Ip)$$

• **Rotation** → remain in $S_A$
  broken to subgroup in $S_H$

  $$R_{\rho\sigma} : \chi_x \rightarrow S_R(R^{-1}_x)\chi_{R^{-1}_x}, \quad \bar{\chi}_x \rightarrow S_R(R^{-1}_x)\bar{\chi}_{R^{-1}_x}, \quad U_{\mu,x} \rightarrow U_{\mu,Rx}$$

  $$R_{\rho\sigma} : \phi(p) \rightarrow \exp(-\frac{i\pi}{4}\Gamma_\rho \Gamma_\sigma)\exp(-\frac{i\pi}{4}\Xi_\rho \Xi_\sigma) \phi(R^{-1}p)$$

• **Conjugation** → remain in $S_A$
  broken in $S_H$

  $$C : \chi_x \rightarrow \epsilon_x \bar{\chi}_x^T, \quad \bar{\chi}_x \rightarrow -\epsilon_x \bar{\chi}_x^T, \quad U_{\mu,x} \rightarrow U^*_{\mu,x}$$

  $$C : \phi(p) \rightarrow \bar{\phi}(-p)^T$$

**Axis and Rotation** → ($\Gamma_4 \times SW_{4,\text{diag}}$)
Details of StWil symmetries

\[ \{ \Xi_\mu, I_s, R_{\mu\nu} \} \rightarrow \Gamma_4 \ltimes SW_4 \]

\[ \{ \Xi'_\mu, R_{\mu\nu} \} \rightarrow \Gamma_3 \ltimes SW_4 \]

Physical-sector symmetry

\[ \Xi'_j \Xi'_4 R^2_{4j} = \Xi_j \Xi_4 \sim (1 \otimes \sigma_j) \]

\[ \Xi'_4 R^2_{34} R^2_{12} = \Xi_4 I_s \sim (\gamma_4 \otimes 1) \]

\[ C_0 \Xi'_2 \Xi'_4 R^2_{24} \sim C \]
Details of timeslice symmetries

Enlarged staggered sym : \( \{ C_0, \Xi_\mu, I_s, R_{\mu\nu}, T^{1/2}_\mu \} \) \( \Xi_\mu^2 = 1 \)

\[ \rightarrow T^{1/2}_\mu \times [\{ C_0, \Xi_\mu \} \times \{ R_{\mu\nu}, I_s \}] = (\otimes_j Z_{N_\mu}) \times [\Gamma_{4,1} \times W_4] \]

Timeslice sym : \( T^{1/2}_\mu \times [\{ C_0, \Xi_\mu \} \times \{ R_{ij}, I_s \}] = (\otimes_j Z_{N_j}) \times [\Gamma_{4,1} \times W_3] \)

Relevant group at rest
\[ \Gamma_{4,1} \times W_3 \sim [\{ R_{ij}, \Xi_{ij} \} \times \{ C_0, \Xi_4, \Xi_{123}, I_s \}] / Z_2 \]
\[ = [\{ R_{ij}, \tilde{R}_{4i} \equiv \epsilon_{ijk} R_{jk} \Xi_{kj} \} \times \{ C_0, \Xi_4, \Xi_{123}, C_0 \Xi_4 I_s \}] / Z_2 \]
\[ = [SW_4 \times \Gamma_{2,2}] / Z_2 \]

Staggered-Wilson
\[ \{ C_0, \Xi'_\mu, R_{\mu\nu}, T'_{1/2}_\mu \} \sim [\{ R_{ij}, \Xi'_{ij} \} \times \{ C_0, \Xi'_4, \Xi'_{123}, I_s \}] / Z_2 \]
\[ = [\{ R_{ij}, \tilde{R}_{4i} \equiv \epsilon_{ijk} R_{jk} \Xi'_{kj} \} \times \{ C_0, \Xi'_4, \Xi'_ {123} \} / Z_2 \]
\[ = [SW_4 \times \Gamma_{1,2}] / Z_2 \]
Dim3, 4 : \( Q(1 \otimes \xi_F)Q \quad \bar{Q}(\gamma_\mu \otimes \xi_F)D_\mu Q \) for \( \xi_F = 1 \) or \( \xi_5 \) \( \rightarrow \bar{\ell}\gamma_\mu D_\mu \ell, \quad \bar{\ell}\ell \)

Dim5 \( O(a) \) : \( \bar{Q}(i\sigma_{\mu\nu}F_{\mu\nu} \otimes \xi_F)Q \) for \( \xi_F = 1 \) or \( \xi_5 \) \( \rightarrow \bar{\ell}i\sigma_{\mu\nu}F_{\mu\nu}\ell \)

*No unphysical term nor taste-breaking term up to \( O(a) \)*

Dim6 \( O(a^2) \) : 2 types of four-fermi operators \( \mathcal{L}_{6}^{FF(A)} \) and \( \mathcal{L}_{6}^{FF(B)} \)

In \( \mathcal{L}_{6}^{FF(A)} \) the spin and flavor independently forms scalar

25 operators with \( \xi_5 \) pair \( \rightarrow \) 50 operators

\[ \text{SA, SV, AS, VS, PV, PA, VP, AP, TV, TA, VT, AT, AA, PP, SP, PS, ST, PT, TS, TP, VV, AA, VA, AV, TT} \]

\( \rightarrow \) No taste-breaking. No derivative terms. Contributes to potential \( \mathcal{V}_{6}^{FF(A)} \)

In \( \mathcal{L}_{6}^{FF(B)} \) the spin and flavor are not independent

10 operators with \( \xi_5 \) pair \( \rightarrow \) 20 operators

\[ \text{TV, TA, VT, AT, VV, AA, VA, AV, TT} \]

\( \rightarrow \) Taste-breaking. Derivative terms. No contribution to potential \( \mathcal{V}_{6}^{FF(B)} \)

*No taste-breaking in ChPT potential upto \( O(a^2) \): SU(2)*