Demagnetizing Factors for Cylinders

Du-Xing Chen, James A. Brug, Member, IEEE, and Ronald B. Goldfarb, Senior Member, IEEE

Abstract—Fluxmetric (ballistic) and magnetometric demagnetizing factors $N_r$ and $N_r'$ for cylinders as functions of susceptibility $\chi$ and the ratio $y$ of length to diameter have been evaluated. Using a one-dimensional model when $\gamma \geq 10$, $N_r$ was calculated for $-1 \leq \chi < \infty$ and $N_r'$ was calculated for $\chi = \infty$. Using a two-dimensional model when $0.01 \leq \gamma \leq 50$, an important range for magnetometer measurements, $N_r$ and $N_r'$ were calculated for $-1 \leq \chi < \infty$. Demagnetizing factors for $\chi < 0$ are applicable to superconductors. For $\chi = 0$, suitable for weakly magnetic or saturated ferromagnetic materials, $N_r$ and $N_r'$ were computed exactly using inductance formulas.

I. INTRODUCTION

The study of demagnetizing factors for ellipsoids and their degenerate forms (spheres, infinite plates, infinite cylinders) dates from the work of Poisson [1] and was elaborated upon by Thomson [2], Evans and Smith [3], and Maxwell [4]. Experimental investigations on finite right circular cylinders began in the 1870's, when ballistic galvanometers were first used in magnetic measurements of iron [5], [6]. The literature distinguishes between "magnetometric" and "fluxmetric" (or "ballistic") demagnetizing factors $N_r$ and $N_r'$ [7]. $N_r$ refers to an average of magnetization over the entire specimen and is appropriate for magnetometer measurements of small samples. $N_r'$ refers to an average of magnetization at the midplane of the sample and is appropriate for measurements made with short search coils. Values of the demagnetizing factor were deduced from the shearing of the magnetic hysteresis loop [8], a procedure originally developed by Lord Rayleigh for ellipsoids [9], or by measuring the magnetization and the field at the cylinder's side [10]. Factors for cylinders of several aspect ratios were published [7], [8], [11], [12]. By the 1900's, it was apparent that the values of the factors depended on the susceptibility $\chi$ of the material [13]–[15].

Although it has been criticized from a pedagogical point of view [16], [17], the use of fictitious magnetic poles to calculate demagnetizing fields has been universal. The first theoretical treatment of magnetic pole distributions in finite cylinders was by Green [18]. An early model to attempt to explain experimental data considered point magnetic poles at each end of a cylinder [19]. This simplistic model could be used only for long uniformly magnetized cylinders, and the results deviated significantly from experimental data on ferromagnetic samples. During the 1920's and 1930's, there were several theoretical papers on $N_r$ for material with constant susceptibility $\chi$. The results were given as functions of $\chi$ and the length-to-diameter ratio $y$. These used one-dimensional models with approximations as needed to suit the computational techniques of the time. The first systematic theoretical calculation of $N_r$ for the high susceptibility case was done by Würschmidt [20], [21]. He calculated $N_r$ of cylinders using a one-dimensional model in which the cylinder had side surface poles and point end poles. He used Taylor expansions for the magnetization and the demagnetizing field at the midplane. The calculation was complicated, and he completed it only for the case $y = 50$ and $\chi \rightarrow \infty$. For the $\gamma$ and $\chi$ dependence of $N_r$, he gave qualitative results using the first few terms of the expansion. A similar approach with simpler expressions was used by Neumann and Warmuth [22], who calculated $N_r$ for $\chi \rightarrow \infty$ and $y \geq 10$.

To obtain the susceptibility dependence of $N_r$, Stäblein and Schlechtweg [23] used a quadratic approximation and two linear differential equations. The model was improved by substituting uniform end-surface poles for point end poles. Their results included 30 values of $N_r$ for $10 \leq \gamma \leq 500$ and $12.56 \leq \chi < \infty$. An extension of the $\gamma$ region to 0 was achieved by Warmuth [24]–[26], who fitted existing data and extrapolated graphically using the demagnetizing factor $N$ of ellipsoids as a reference. The values of $N_r$ calculated from the one-dimensional models were consistent with the data of ballistic measurements on soft magnetic materials. Bozorth and Chapin [27] compiled the results, which were later plotted in Bozorth's book [28].

To obtain axial demagnetizing factors more accurately, especially for short cylinders, two-dimensional calculations are needed. The simplest case is $\chi = 0$, where $N_r$ and $N_r'$ as functions of $\gamma$ can be derived analytically. The approximation $\chi = 0$ applies to diamagnets, paramagnets, and saturated ferromagnets. $N_r$ for 25 values of $\gamma$ from 0.2 to 1000 were obtained accurately to four significant figures by Brown [29] from a calculation of self-inductance [30] and listed as a table in Brown's book [31]. Crabtree [32] obtained the same values for the average demagnetizing factor by integration of the local field over the cylindrical volume. Moskowitz et al. extended
Brown’s method to cylinders of polygonal cross section [33], and Kaczér and Klem extended it to hollow cylinders [34]. \( N_f \) for \( \gamma = 0 \) was calculated exactly by Joseph [35]. Approximate values for \( N_n \) and \( N_f \) for \( \gamma = 0 \), accurate for large \( \gamma \), were calculated by Vallabh Sharma using uniformly magnetized volume elements [36]. Sato and Ishii [37] obtained a simple expression to approximate \( N_n \) for \( \gamma = 0 \). Chen and Li [38], [39] obtained \( N_f \) for \( \gamma = 0 \) using magnetostatic potential calculations.

The susceptibilities \( \chi = -1 \) and \( \chi \rightarrow \infty \) correspond to perfectly diamagnetic and ideally soft ferromagnetic materials, respectively. \( N_n \) and \( N_f \) for these susceptibilities were first treated by Taylor for perfectly conducting cylinders [40], [41]. He developed a method introduced by Smythe that expressed charge densities on the side and ends in terms of a set of orthogonal polynomials, and expanded the electrostatic potential at the cylinder center [42]-[44]. Taylor calculated electric and magnetic polarizabilities for conducting cylinders for 0.25 \( \leq \gamma \leq 4 \) in both the longitudinal and transverse directions. \( N_n(\infty) \) can be deduced from his electric polarizability results because of the analogy between electrostatics and magnetostatics. Because his calculation for magnetic polarizability was for a uniform quasi-static but nonpenetrating applied field, \( N_n(-1) \) can also be deduced from his results. According to Taylor, his convergence error was less than 0.1% for the longitudinal direction.

Using a similar approach with simpler base functions, Templeton et al. calculated axial \( N_f \) for \( \gamma \rightarrow \infty \) for 0.05 \( \leq \gamma \leq 250 \) [45], [46]. The fact that the side and end-pole densities have basically a \( \delta^{-1/3} \) dependence, where \( \delta \) is the distance from the corner, was used to construct the set of polynomials. To estimate their error, Templeton and Arrott calculated the root-mean-square deviation of the normalized potential from 0 and found it to be less than 0.3% [45]. Compared to an approximate formula with 8 adjustable parameters, the deviations of their 12 computed \( N_f(\infty) \) values were less than 0.25%. The work was based on their earlier magnetostatic analysis of the magnetization process in soft ferromagnetic cylinders with constant end-pole densities [47], [48]. The details of the calculation were published by Aharoni, who also calculated the self-energy of cylinders [49], and more generally, cylinders with nonuniform magnetization [50].

For susceptibilities other than 0, \(-1, \) and \( \infty \), different techniques have been used. Archer and Guancial [51] and Fawzi et al. [52] calculated the distribution of magnetization and magnetic field in long cylinders with large susceptibilities using volume and boundary integral equations. Using experimental resistance network analogs, Okoshi [53] obtained \( N_f \) for \( \gamma \rightarrow \infty \), and Yamamoto and Yamada [54] obtained \( N_f \) and \( N_n \) for large \( \gamma \).

Several papers have treated demagnetizing factors at points. Joseph and Schlömann [55] solved for local demagnetizing factors in uniformly magnetized cylinders and used a series expansion to account for nonuniform magnetization. Kraus [56] determined the complete local demagnetizing tensor for uniformly magnetized cylinders. Brug and Wolf [57] calculated the magnetization distribution in disks and obtained the local demagnetizing factor for materials that undergo phase transitions.

In Zijlstra’s book [58], \( N_f \) and \( N_n \) are plotted. These types of graphs and tables appear in other books on magnetism and magnetic materials, and they are widely used, sometimes inappropriately, in magnetic measurements of ferromagnetic, ferrimagnetic, weakly magnetic, and superconducting materials. However, there remain some problems. For \( \chi = 0 \), the most accurate case, the number of \( \gamma \) values for \( N_n \) and \( N_f \) is insufficient for accurate interpolation. For \( \chi \neq 0 \), almost all books give results obtained before 1950, and there are no data for \( \chi < 0 \). For long cylinders (\( \gamma > 10 \)), there is a lack of data on the \( \chi \) dependence of \( N_f \), and there are no data on \( N_n \). For short cylinders (\( \gamma < 10 \)), there are even less data, and those that exist have large errors because they were obtained by extrapolation. In summary, there is no complete picture for the \( \chi \) and \( \chi \) dependence of \( N_f \) and \( N_n \).

In this paper, we calculate \( N_f \) and \( N_n \) for a complete range of \( \gamma \) and \( \chi \). Susceptibility \( \chi \) is traditionally assumed to be constant in the material and is therefore defined as \( M/H \), where \( M \) is the magnetic moment per unit volume and \( H \) is the internal magnetic field. For the case \( \chi = 0 \), in which the magnetization is uniform, we give 61 exact inductance calculations of \( N_n \) and \( N_f \) for \( 10^{-5} \leq \gamma \leq 10^5 \). For \( \chi \neq 0 \), more elaborate methods are used. For \( \gamma > 10 \), the variation of magnetization across the radius of the cylinder is negligible at the midplane, and we calculate \( N_f \) as a function of \( \gamma \) and \( \chi \left( -1 \leq \chi < \infty \right) \) based on the one-dimensional model of Stäblein and Schlechtweg [23]. Unlike them, we use Taylor expansions for \( M(\zeta) \), calculate the demagnetizing field \( H_d(\zeta) \) directly at 25 points along the axis, and obtain more accurate results. The model is also applicable to \( N_n \) for \( \gamma \rightarrow \infty \). For \( 0.01 \leq \gamma \leq 50 \), a two-dimensional finite element method is used that takes into account the variation of magnetic pole density along the side and ends of the cylinder. Values of \( N_n \) and \( N_f \) are given for \(-1 \leq \chi < \infty \).

II. FLUXMETRIC AND MAGNETOMETRIC DEMAGNETIZING FACTORS

The demagnetizing correction is nontrivial for samples in open magnetic circuits. An exact correction can be obtained only for ellipsoids [4], [59], [60], where both the magnetization \( M \) and the demagnetizing field \( H_d \) are uniform under a uniform applied field \( H_a \). If the three principal ellipsoid axes coincide with the \( x, y, \) and \( z \) axes, the internal field is

\[
H = H_a + H_d = H_a - \vec{N}M, \tag{1}
\]

where \( \vec{N} \) is the demagnetizing tensor,

\[
\vec{N} = \begin{bmatrix}
N_x & 0 & 0 \\
0 & N_y & 0 \\
0 & 0 & N_z
\end{bmatrix}. \tag{2a}
\]
with
\[ N_f + N_x + N_y = 1. \] (2b)

If the applied field is along one of the principal axes, we have
\[
H = H_L + H_d = H_d - NM, \tag{3}
\]
where \( N \) is called the demagnetizing factor. In SI units, \( 0 \leq N \leq 1 \). In cylindrical samples, which are commonly used in magnetic measurements, the demagnetizing field is not uniform, and two kinds of susceptibility-dependent demagnetizing factors are defined.

If the sample is located in a uniform applied field \( H_d \) along its axis, the fluxmetric (or ballistic) demagnetizing factor \( N_f \) is defined as the ratio of the average demagnetizing field to the average magnetization at the midplane perpendicular to the axis. The magnetometric demagnetizing factor \( N_m \) is defined as the ratio of the average demagnetizing field to the average magnetization of the entire sample [58]:
\[
\int H_d ds = -N_f \int M ds, \tag{4}
\]
\[
\int H_d dv = -N_m \int M dv. \tag{5}
\]

\( N_f \) and \( N_m \) are functions of the ratio \( \gamma \) of cylinder length to diameter and the susceptibility \( \chi \) of the material. For ferromagnetic or ferrimagnetic materials, this \( \chi \) should be regarded as an effective \( \chi \), similar to the differential susceptibility \( dM/dH \) at the corresponding magnetic state. In [58], the definition of \( N_m \) is limited to \( \chi = 0 \).

### III. \( N_m \) AND \( N_f \) FOR \( \chi = 0 \) DETERMINED BY INDUCTANCE CALCULATIONS

Brown [29] showed how \( N_m \) could be determined using a self-inductance calculation in which a uniformly magnetized cylinder was modeled as a solenoid. In fact, both \( N_m \) and \( N_f \) may be calculated using the mutual inductance of two model solenoids of the same diameter. \( N_m \) is obtained when the solenoids have the same length, and the problem reduces to the self-inductance calculation. \( N_f \) is obtained when the length of one of the model solenoids approaches 0, and the problem is that of the mutual inductance of a solenoid and a single-turn loop located at its midplane. In this section we calculate exact values of \( N_m \) and \( N_f \) for \( \chi = 0 \) and a wide range of \( \gamma \).

### A. Formulas for Inductance

An exact formula for the self-inductance \( L_s \) of a thin solenoid of length \( 2l \), radius \( a \), and number of turns \( n \) is [61]
\[
L_s = 2\mu_0 n^2 / (3l^2) \left\{ \left( a^2 + l^2 \right)^{1/2} - l \right\} \cdot [F(k_x) + (a^2 - l^2)E(k_x)] - a^3, \tag{6}
\]
where \( F(k) \) and \( E(k) \) are the complete elliptic integrals of the first and second kind of modulus \( k \), which is defined by
\[
k_x^2 = a^2 / (a^2 + l^2), \tag{7}
\]
and \( \mu_0 \) is the permeability of vacuum.

An exact formula for the mutual inductance \( L_m \) of the same thin solenoid and a coaxial single-turn loop of the same radius at its midplane is [62]
\[
L_m = \mu_0 \pi a / k_m \left\{ F(k_m) - E(k_m) \right\}, \tag{8}
\]
where the modulus \( k_m \) is defined by
\[
k_m^2 = 4a^2 / (4a^2 + l^2). \tag{9}
\]

Cohen [63] derived an exact general formula for the mutual inductance of two concentric coaxial thin solenoids (denoted by subscripts 1 and 2). We have used it successfully for these calculations, as an alternative to (6) and (8), with \( a_1 = a_2 \), in the limits \( l_2 = l_1 \) (for \( N_m \)) and \( l_2 \to 0 \) (for \( N_f \)).

### B. Relationship between \( N_m \) and \( L_s \), \( L_m \) and \( L_s \)

The flux density \( B \) in a magnetic material is related to the internal field \( H \) and the magnetization \( M \): \( B = \mu_0 H + M \), where \( H \) is, in general, related to the applied and demagnetizing fields as defined for ellipsoids in (1). Thus \( B = \mu_0 (H_x + H_d + M) \). Following Brown [29], we define \( B' \) as the Amperian flux density:
\[
B' = B - \mu_0 H_d = \mu_0 (H_d + M). \tag{10}
\]
When \( \chi = 0 \), a cylinder in an axial field has a uniform magnetization \( M \). An ideal thin solenoid carrying current \( I \) through \( n \) turns over a length \( 2l \) is equivalent, with respect to the \( B' \) field, to a longitudinally magnetized cylinder coincident with it [29]. Thus the cylinder can be modeled as a solenoid with the same \( M \), and its average \( H_d \) can be obtained from \( M \) and average \( B' \) using (10). We take the solenoid as having one turn \( (n = 1) \), so
\[
M = I / (2l). \tag{11}
\]

For the entire volume, we can obtain the average \( B' \) from the average flux \( \Phi \) in the solenoid as
\[
\langle B' \rangle = \Phi / (\pi a^2). \tag{12a}
\]

Thus the average demagnetizing field can be obtained from (10) and (12a) as
\[
\langle H_d \rangle = \Phi / (\mu_0 \pi a^2) - M. \tag{13a}
\]

The definition of self-inductance is
\[
L_s = \Phi / I. \tag{14a}
\]

From (11), (13a), and (14a), we obtain the final expression for the magnetometric demagnetizing factor:
\[
N_m = -\langle H_d \rangle / M = 1 - 2lL_s / (\mu_0 \pi a^2). \tag{15}
\]

For \( N_f \), we obtain the average \( B' \) at the midplane from the flux \( \Phi_0 \) in the one-turn secondary loop of radius \( a \):
\[
\langle B' \rangle = \Phi_0 / (\pi a^2). \tag{12b}
\]
The average demagnetizing field is
\[ \langle H_z \rangle = \Phi_0 / \left( \mu_0 \pi a^2 \right) - M. \tag{13b} \]

The definition of mutual inductance is
\[ L_m = \Phi_0 / I. \tag{14b} \]

The final expression for the fluxmetric demagnetizing factor is
\[ N_f = -\langle H_2 \rangle / M = 1 - 2L_m / (\mu_0 \pi a^2). \tag{16} \]

Equations (15) and (16) have been derived, by direct integration rather than inductance formulas, by Joseph [35].

C. Results

Values of \( N_m(\chi = 0) \) and \( N_f(\chi = 0) \) as functions of \( \gamma (= l/a) \) computed using (6), (15), (8), and (16) are given in Table I. For \( N_m(\chi = 0) \), the data agree with those given by Brown [29], [31]. For \( N_f \) the data agree with those obtained by Joseph [35] and by Chen and Li [38]. In Table I we also give \( N \) for ellipsoids of revolution with longitudinal axes 2l and transverse axes 2a calculated from well-known formulas [4], [59], [60].

IV. ONE-DIMENSIONAL MODEL FOR LONG CYLINDERS

A. Calculation of \( M \)

Assume that a cylinder of length 2l and diameter 2a is located in a uniform applied field \( H_t \) along the \( z \) axis, as shown in Fig. 1. The material has constant susceptibility \( \chi \), which leads to
\[ B = \mu_0 (M + H) = \mu_0 M \left( 1 + 1 / \chi \right) \tag{17} \]

at any point inside the cylinder. Since \( \nabla \cdot B = 0 \), the volume magnetic pole density, proportional to \( \nabla \cdot M \), equals 0 inside the cylinder; that is, all poles are on the surface.

We further assume for this one-dimensional model that \( M_z \), the \( z \) component of \( M \), is uniform in each cross section of the cylinder, and can be expressed by a scalar quantity as
\[ M(z) = M_z(z) = \sum_{i=0}^{n} M_{2i}(z/l)^{2i}, \tag{18} \]

where \( M_{2i} \) (i = 0, 1, \ldots, \( n \)) are constants. \( B, H_t \), and \( H_r \) can also be written as scalar quantities.

For a section of cylinder of length \( dz \) at \( z \),
\[ \int \mathbf{M} \cdot ds = \pi a^2 dM(z) + 2\pi a M(z)dz = 0, \tag{19a} \]

because \( \int \mathbf{M} \cdot ds = \int \nabla \cdot M \, dv \) and \( \nabla \cdot M = 0. \) \( M(z) \) is the radial component of \( M \) at the side surface. Substituting \( \sigma(z) = \mu_0 M_z(z) \) gives, on the side surface, the surface magnetic pole density
\[ \sigma(z) = -\frac{1}{2} \mu_0 a \, dM(z) / dz = -(\mu_0 a / l) \sum_{i=0}^{n} i M_{2i}(z/l)^{2i-1}. \tag{19b} \]

On the end planes of the cylinder, we have uniform surface magnetic pole densities:
\[ \sigma(\pm l) = \pm \mu_0 M(l) = \pm \mu_0 \sum_{i=0}^{n} M_{2i}. \tag{20} \]
Although the side surface magnetic pole density is a function of \( z \) only, our assumptions of both uniform \( M \), and uniform \( a(z) \) are, in fact, contradictory for constant \( x; a(z) \) and the side pole density would produce a nonuniform \( H_{dz} \), the \( z \) component of the demagnetizing field, which would lead to a nonuniform \( M \).

However, if \( y (= l/a) \) is large (\( \geq 10 \)), and if we consider only \( N_f, \) for which the middle part of the cylinder is more important, this one-dimensional model is a good approximation.

**B. Calculation of \( N_f \) and \( N_m \) for Long Cylinders \((l \gg 10)\)**

In this model, in a given \( H_a \), the demagnetizing field \( H_d(z) = H_{d1}(z) + H_{d2}(z) \),

\[
H_{d1}(z) = -a(2\mu_0)^{-1} \int_{z-l}^{z+l} a(z) (z - \xi) \cdot \left[ (z - \xi)^2 + a^2 \right]^{-3/2} d\xi
= \sum_{i=0}^{n} F_i(z) M_{2i}, \tag{22a}
\]

where

\[
F_i(z) = i a^2 (2l^2)^{-1} \int_{z-l}^{z+l} \zeta^{2i-1}(z - \xi) \cdot \left[ (z - \xi)^2 + a^2 \right]^{-3/2} d\xi. \tag{22b}
\]

The fields produced by the end poles at \( z = \pm l \) are

\[
H_{d2}(z) = -2(2\mu_0)^{-1} \int_{0}^{a} \sigma (\pm l) (\pm l - \xi) \cdot r[\pm l - \xi]^2 + r^2]^{-3/2} dr
= \frac{1}{2} M(l) \left\{ (l + \xi)(l + \xi)^2 + a^2 \right\}^{-1/2} - 1, \tag{23}
\]

from which

\[
H_{d2}(z) = H_{d1}(z) + H_{d2}(z) = \sum_{i=0}^{n} F_i(z) M_{2i}, \tag{24a}
\]

where

\[
F_i(z) = \frac{1}{2} (l + \xi)[(l + \xi)^2 + a^2]^{-1/2}
+ \frac{1}{2} (l - \xi)[(l - \xi)^2 + a^2]^{-1/2} - 1. \tag{24b}
\]

From (3), and considering the \( z \) dependence of \( H \) and \( H_d \), we have

\[
H_d(z) - M(z)/\chi = -H_a \tag{25}
\]

Rewriting the variable \( z \) in \( H_{d1}, F_i, F_2, \) and \( H_{d2} \), as \( z, \) and substituting (22a) and (24a) into (21), and (18) and (21) into (25), we obtain

\[
\sum_{i=0}^{n} \left\{ F_i(z) + F_2(z) - (z/l)^2/\chi \right\} M_{2i} = -H_a. \tag{26}
\]

This is a general equation relating the expansion coefficients \( M_{2i} \), of magnetization, the applied field \( H_a \), the susceptibility \( \chi \), and the position \( z \) for a cylinder of length \( 2l \) and diameter \( 2a \). In our problem, \( H_a, l, a, \) and \( \chi \) are given, and the \( n + 1 \) coefficients \( M_{2i} \) are unknown. We can choose \( n + 1 \) positions, \( z = z_0, z_1, \cdots, z_n, \) and get a set of \( n + 1 \) linear equations. \( M_{2i} (i = 0, 1, \cdots, n) \) are then obtained by solving these equations simultaneously. \( N_f \) and \( N_m \) can be obtained according to (4), (5), and (25) as

\[
N_f = -H_d(0)/M_0 = H_a/M_0 - 1/\chi, \tag{27}
\]

\[
N_m = -\left\langle H_d \right\rangle / \left\langle M \right\rangle = H_a/\left\langle M \right\rangle - 1/\chi, \tag{28}
\]

where the \( \langle \cdot \rangle \) brackets denote the volume average, and

\[
\left\langle M \right\rangle = l^{-1} \int_{0}^{l} M(z) dz = \sum_{i=0}^{n} M_{2i}/(2l + 1). \tag{29}
\]

In principle, the larger the number of terms in the expansion equation (26), the more accurate are the results. However, if \( n \) is too large, the computed function \( M(z) \) oscillates. For computation, \( n = 12 \) is a practical choice, and

\[
z_i = il/12, \quad (i = 0, 1, \cdots, 12), \tag{30}
\]

\[
H_a = 1, \tag{31a}
\]

\[
a = 1 \tag{31b}
\]

Table II and Fig. 2(a) and 2(b) give the calculated \( N_f \) as functions of \( \gamma \) and \( \chi \) for \(-1 \leq \chi \leq 10^6 \). When \( \chi \to \infty \), \( H_d \) is uniform in the cylinder (\( H_d = -H_a \)), and \( M \) close to the ends is very small. As a result, values of \( N_f(\chi \to \infty) \) obtained from this model are expected to be fairly accurate due to rather small end effects. Similar data were presented in [64]. Fig. 3 gives \( N_m \) and \( N_f \) as functions of \( \gamma \) when \( \chi = 0.001 \) and \( 10^6 \). The demagnetizing factors \( N \) for ellipsoids (dashed curve) are located between \( N_m \) and \( N_f \) for \( \chi \to \infty \).
### Table II

| $\gamma$ | 10  | 20  | 50  | 100 | 200 | 500 | 1000 | 2000 |
|----------|-----|-----|-----|-----|-----|-----|------|------|
| $x$      | ($10^{-3}$) | ($10^{-2}$) | ($10^{-1}$) | ($10^{-1}$) | ($10^{-2}$) | ($10^{-3}$) | ($10^{-4}$) | ($10^{-5}$) |
| $N_f$    |      |      |      |      |      |      |      |      |
| $-1$     | 3.965 | 1.130 | 1.900 | 4.776 | 1.196 | 1.917 | 4.796 | 1.199 |
| $-0.8$   | 4.239 | 1.162 | 1.931 | 4.852 | 1.215 | 1.945 | 4.865 | 1.216 |
| $-0.4$   | 4.642 | 1.210 | 1.972 | 4.944 | 1.237 | 1.980 | 4.950 | 1.237 |
| $0$      | 4.963 | 1.248 | 1.999 | 5.000 | 1.250 | 2.000 | 5.000 | 1.250 |
| $0.1$    | 5.037 | 1.256 | 2.005 | 5.011 | 1.252 | 2.004 | 5.010 | 1.252 |
| $0.2$    | 5.108 | 1.265 | 2.011 | 5.021 | 1.255 | 2.007 | 5.018 | 1.255 |
| $0.5$    | 5.315 | 1.289 | 2.026 | 5.047 | 1.260 | 2.016 | 5.040 | 1.260 |
| $1$      | 5.639 | 1.328 | 2.049 | 5.082 | 1.268 | 2.027 | 5.067 | 1.267 |
| $2$      | 6.231 | 1.401 | 2.089 | 5.135 | 1.277 | 2.040 | 5.100 | 1.275 |
| $3$      | 7.695 | 1.616 | 2.191 | 5.247 | 1.293 | 2.058 | 5.143 | 1.286 |
| $5$      | 9.377 | 1.967 | 2.352 | 5.394 | 1.308 | 2.071 | 5.168 | 1.292 |
| $10$     | 11.23 | 2.565 | 2.688 | 5.654 | 1.332 | 2.083 | 5.187 | 1.296 |
| $50$     | 13.21 | 3.600 | 3.803 | 6.494 | 1.395 | 2.106 | 5.211 | 1.299 |
| $100$    | 14.14 | 4.294 | 5.448 | 8.132 | 1.500 | 2.141 | 5.238 | 1.302 |
| $200$    | 14.68 | 4.776 | 7.497 | 11.77 | 1.734 | 2.210 | 5.288 | 1.305 |
| $500$    | 15.03 | 5.126 | 9.800 | 19.91 | 2.661 | 2.808 | 5.438 | 1.316 |
| $10^3$   | 15.15 | 5.255 | 10.90 | 26.32 | 4.221 | 2.808 | 5.684 | 1.333 |
| $2 \times 10^3$ | 15.21 | 5.322 | 11.53 | 31.12 | 6.275 | 3.889 | 6.186 | 1.369 |
| $5 \times 10^3$ | 15.25 | 5.363 | 11.95 | 34.78 | 8.642 | 5.70 | 8.223 | 1.474 |
| $10^4$   | 15.26 | 5.377 | 12.09 | 36.16 | 9.779 | 11.68 | 13.12 | 1.669 |
| $2 \times 10^4$ | 15.27 | 5.384 | 12.16 | 36.88 | 10.44 | 15.47 | 22.71 | 2.237 |
| $5 \times 10^4$ | 15.28 | 5.391 | 12.21 | 37.33 | 10.87 | 18.77 | 38.76 | 4.620 |
| $10^5$   | 15.27 | 5.390 | 12.22 | 37.48 | 11.02 | 20.11 | 47.73 | 7.811 |
| $2 \times 10^5$ | 15.28 | 5.390 | 12.23 | 37.55 | 11.10 | 20.83 | 53.77 | 11.10 |
| $5 \times 10^5$ | 15.28 | 5.391 | 12.24 | 37.60 | 11.14 | 21.28 | 57.94 | 14.19 |
| $10^6$   | 15.28 | 5.391 | 12.24 | 37.63 | 11.17 | 21.58 | 60.82 | 16.83 |
| $2 \times 10^6$ | 15.28 | 5.391 | 12.24 | 37.63 | 11.17 | 21.59 | 60.99 | 16.99 |
| $5 \times 10^6$ | 15.28 | 5.391 | 12.24 | 37.63 | 11.17 | 21.59 | 60.99 | 16.99 |

*The row for $x = 0$ is comparable to data for $N_f(0)$ in Table I. The last row gives $N_f(\infty)$ calculated by Templeton and Arrott [45].

---

*Fig. 2. Calculated $N_f$ from the one-dimensional model. (a) For $10 \leq \gamma < 200$, the curves from top to bottom are for $x = \infty$, $1000$, $300$, $100$, $30$, $10$, $3$, $1$, $0$, and $-1$. (b) For $10 \leq \gamma < 1000$, the curves from top to bottom are for $x = \infty$, $10^6$, $3 \times 10^6$, $10^5$, $3 \times 10^5$, $10^4$, $300$, $100$, $0$, and $-1*.}
Earlier results for \( \gamma > 10 \) exist for \( N_f \) but not for \( N_m \). Würschmidt's result for \( \gamma = 50 \) and \( \chi \to \infty \) is 3.4% smaller than our result [20], [21]. Neumann and War- 
muth's results deviate from our results by 1.5%, 0.0%, -3.4%, -5.3%, and -8.6% for \( \gamma = 10, 20, 50, 100, \) and 1000 [22]. For \( \chi \to \infty \), the maximum deviation of the 
results of Stäbelin and Schlachweg from our results is 3%; but for their smallest \( \chi \) (12.56), the maximum 
difference is above 30% [23]. Templeton and Arrott's [45] 
results for \( \chi \to \infty \) (Table II) are the most accurate, and 
our data agree within 1.1%. Compared with the exact 
results for \( \chi = 0 \) in Table I, the data in Table II have errors 
of 0.7%, 0.2%, and 0.0% for \( \gamma = 10, 20, \) and \( \gamma \geq 50 \).

V. TWO-DIMENSIONAL MODEL FOR SHORT CYLINDERS

A. Calculation of Surface Pole Density

The magnetization, in general, varies throughout 
the cylinder in both the radial and axial directions. A calculation 
of demagnetizing factors for a short cylinder (\( \gamma < 10 \)) 
must take this variation into account, especially near 
the corners, where the magnetization sharply diverges for 
susceptibilities far from 0. As in the one-dimensional 
model, we assume that the cylinder consists of material 
with constant \( \chi \), so the demagnetizing fields are 
completely specified by the surface pole density.

To obtain the distribution of poles on the surface for a 
specified susceptibility, we divide the surface of the 
cylinder into a set of nonoverlapping elements of uniform 
pole density. With the cylindrical symmetry there is no 
azimuthal dependence of the pole distribution, and we are 
therefore able to use elements in the shape of rings about 
the central axis. We solve the set of equations that de- 
scribes the interaction between these rings of surface poles 
and use the resulting distribution to obtain the magnetometric 
and fluxmetric demagnetizing factors.

The method of dividing the surface of a magnetized 
body into a set of interacting elements of uniform pole 
density has been used before for the calculation of the 
magnetic fields of rectangular bodies. Ruehli and Ellis 
[65] assumed a constant susceptibility, and Normann and 
Mende [66] used a field dependent magnetization with 
the assumption that the volume pole distribution is negligible. 
Both of these studies were interested primarily in the field 
and magnetization distributions rather than the demagne-
tizing factors. A method that involves dividing the vol-
ume into uniformly magnetized elements was used by 
Brug and Wolf [57] for the case of thin disks that undergo 
magnetic phase transitions. They used a demagnetizing 
matrix that was derived by Hegedus, Kadar, and Della 
Torre [67], [68] for interacting volume elements in cylin-
drical geometries. Volume elements have also been used 
by Soinski [69] for rectangular and ring-shaped samples. 
A method of obtaining demagnetizing fields in bodies of 
arbitrary shape was presented by Vallabh Sharma [36] 
using rectangular volume elements. Templeton and Arrott 
[45] used the principle that the magnetic potential is 0 at 
each point on a grid inside a body with infinite suscepti-
bility. They calculated the demagnetizing factors of cy-
linders and bars and later extended this to the case of a 
material that saturates [70].

The following method solves for the pole distribution 
at the surface of a cylinder for an arbitrary value of the 
susceptibility. The surface is divided into rings of area 
\( \pi (r^2 - r_i^2) \) on the end planes of the cylinder and into rings 
of area \( 2 \pi a z_2 - z_1 \) on the side surface of the cylinder. 
The cylinder diameter is \( 2a, r_1, \) and \( r_2 \) are the inner 
and outer radii of the end-surface ring, and \( z_1 \) and \( z_2 \) are 
the side-surface ring limits. The \( z \) and \( r \) components of 
the demagnetizing field at a point \( i \) that results from the 
surface poles \( \sigma_j \) at each ring are given by

\[
\begin{align*}
H_{lc}^i &= -\sum_j N_j \sigma_j / \mu_0, \\
H_{lr}^i &= -\sum_j N_j \sigma_j / \mu_0,
\end{align*}
\]

where, for example, \( N_j \) is a scalar factor that relates the 
surface pole on the \( j \)th ring to the \( r \) component of the demagnetizing 
field that it produces at the \( i \)th ring. A method of calculating these factors is given in the Appendix.

The interactions between the surface poles at each ring 
are specified using the equation \( \bar{M} = \chi (H_a + H_b) \). 
The demagnetizing field \( H_j \), given in (32), is written in terms 
of the magnetization at each ring using \( \sigma_j / \mu_0 = n \cdot M'_j = M'_j \), where \( n \) is the unit vector 
outward normal to the surface of the ring, and \( M' \) is the magnetization at the \( j \)th ring. The resulting equations are for the magnetization at 
each point,

\[
\begin{align*}
M'_i &= \chi \left( H_a - \sum_j N_j \sigma_j M'_j \right), \\
M_i &= -\chi \sum_j N_j \sigma_j M'_j.
\end{align*}
\]
where we have constrained the applied field to be in the
axial direction. Because only the normal component of \( M \)
is needed at each point, the equations can be rewritten as
\[
H_a = \chi^{-1}M_a + \sum_j N_j^a M_j^a
\]
(rings on the top end-plane, \(34a\)).
\[
H_a = -\chi^{-1}M_a + \sum_j N_j^a M_j^a
\]
(rings on the bottom end-plane, \(34b\)).
\[
0 = \chi^{-1}M_a + \sum_j N_j^a M_j^a
\]
(rings on the side surface, \(34c\)).

With the surface of the cylinder divided into \( n \) rings, there will be a set of \( n \) simultaneous linear equations. These equations are solved for \( M_a \) at each ring using matrix inversion [71].

The number of rings required to adequately specify the surface pole density depends on how rapidly the pole distribution varies. For both large susceptibilities and susceptibilities near \(-1\), the pole distribution diverges sharply near the corner of the cylinder. To reduce the total number of rings, the density of the rings is made roughly proportional to the pole density. For the case of infinite susceptibility, the pole density near the corner at the end-plane is approximated as [45]
\[
a_0(r, \pm 1) \propto \pm [(1 - r/a)^{-1/3} - (1 + r/a)^{-1/3}]. \tag{35}\]

We let the width of the rings \( \Delta r \) be inversely proportional to \( a_0 \) near the corner of the cylinder, which gives
\[
\Delta r = \{a_0 + a[[(1 - r/a)^{-1/3} - (1 + r/a)^{-1/3}]\}^{-1}. \tag{36}\]

A similar equation determines the width of the rings on the side surface of the cylinder. The adjustable parameters \( a_0 \) and \( a \) are determined by the total number of rings used in the calculation. A total of 148 rings was used, 37 on each end-plane and 74 on the side of the cylinder. For \( \gamma = 10, 20, \) and \( 50, \) the calculations were repeated with 8 end-plane rings and 132 side rings.

B. Calculation of \( N_{f,m} \) and \( N_f \)

After obtaining the surface pole distribution, there are two ways to calculate the demagnetizing factors, based on the two equations
\[
N_{f,m} = -\langle H_a \rangle / \langle M_a \rangle = H_a / \langle M_a \rangle - \chi^{-1}, \tag{37a}\]
\[
N_f = -\langle H_a \rangle / \langle M_a \rangle = -\langle H_a \rangle / [\chi(H_a + \langle H_a \rangle)], \tag{37b}\]

where \( H_a \) and \( M_a \) are averaged over the midplane for \( N_f \) and over the volume for \( N_{f,m} \).

For the first method, using (37a), \( N_{f,m} \) is calculated from \( \langle M_a \rangle \). We have \( \int M \cdot \text{d}s = 0 \) for a material with constant susceptibility. Therefore, for each transverse cross section of the cylinder, the average \( M_a \) multiplied by the cross-sectional area \( S \) equals the surface integration of \( M_a \) over the cylinder surface above the cross section. In the case of \( N_f \), the cross section is taken to be the midplane. We have, from (37a),
\[
N_f = H_a S \int_{S_1} M_a \text{d}s - \chi^{-1} = H_a S \sum_j M_a s_j - \chi^{-1}. \tag{38}\]

The second method, using (37b), requires a calculation of \( \langle H_a \rangle \). We use (32a), with \( z \) denoting an interior point of the cylinder. Again, the average is taken over the midplane for \( N_f \) and over the volume for \( N_{f,m} \).

Since the first method involves surface flux calculations, while the second involves interior field calculations, we refer to them as the surface and the volume methods, respectively. Table III gives some examples of the results obtained from both methods. We see from this table that the results do not agree, especially for large \( \gamma \) or large \( \chi \). The differences are even larger for \( N_{f,m} \). The source of the disagreement is a systematic error that is due almost entirely to the finite number of elements with which the surface pole density is calculated.

The field produced by a magnetic pole is very sensitive to the distance \( r \) between the pole and the point at which the field is considered, with an \( r^{-2} \) dependence. In the two-dimensional model, the division of the surface of the cylinder into rings will produce a discretization error because \( H_a \) is calculated in the center of a region of uniform pole density. This error is mainly due to the division of the side surface; at the center of each ring on this surface, the normal component of \( H_a \) is produced by poles on the ring itself with significant contribution from poles on adjacent rings. For the end planes, the poles on the same
surface do not contribute to the normal component of \( H_d \), except for the pole at the point being considered. A region of uniform pole density adjacent to the point does not produce a discretization error.

Fig. 4 gives the surface pole distribution of a cylinder with \( \gamma = 1 \) for \( \chi = -1, 0, 0.0001, 1, \text{ and } 10000 \), calculated with the two-dimensional model. The normalized pole density is expressed as \( M_s / M_s(r = 0) \). Therefore, all the curves have the actual sign of the pole density when \( H_d \) is positive. The curves shown are for the top half of the cylinder with \( z > 0 \); they are odd functions of \( z \). As can be seen from the figure, when \( \chi \to \infty \), the side poles are broadly distributed with a larger density, and this should give a relatively large error. The side poles are narrowly distributed for \( \chi < 0 \), giving a small error. For the cases of small or zero side-pole density (low or zero \( \chi \)), the error will be small or zero. The error becomes larger as \( \gamma \) increases because the size of the rings on the side surface also increases. The relative effect of the end poles increases and the error decreases as \( \gamma \) decreases.

This systematic error is correctable if it is not too large. For the surface and volume methods, we write the demagnetizing factors as \( N_s \) and \( N_v \), and write \( \langle M_s \rangle \) and \( \langle H_d \rangle \) as \( M_s^z \) and \( H_d^z \), respectively. For simplicity, the corrected quantities are written as \( N_{s, c} \) and \( N_{v, c} \) when \( \gamma \) and \( \chi \) are arbitrarily set equal to 1. From (37), we have

\[
N_{s, c}^{-1} = -\chi (H_d^z)^{-1} + 1, \tag{39a}
\]

\[
N_{v, c}^{-1} = -\chi (H_d^z)^{-1} + 1, \tag{39b}
\]

We assume that the error in the surface pole density calculation is not too large so that

\[
H_d^z / H_d \approx M_{s, c} / M_c. \tag{39c}
\]

Thus we have from (39)

\[
N_v (1 + N_v \chi) / [N (1 + N_s \chi)] = (1 + N_v \chi) / (1 + N_s \chi). \tag{40a}
\]

Solving (40a) for \( N \) gives the interpolation/extrapolation equation

\[
N = N_v (1 + N_v \chi) / (1 + N_s \chi). \tag{40b}
\]

We use (40b) and the results (up to 5 significant digits) from the two methods to obtain the final \( N_f \) and \( N_v \). They are presented in Tables IV and V and Figs. 5, 6, and 7. The entries in Table III provide data for illustrative computations. Values for \( \gamma \leq 10 \) were obtained with the 74-side-ring calculation. The values for \( \gamma = 20 \) and 50 are from the 132-side-ring calculation, which gives a smaller discretization error for these cases. As an example of the worst case, \( N_s \) for \( \chi = 10000 \) and \( \gamma = 20 \) is 55.46 \( \times 10^{-4} \) when computed with 74 side rings, compared to 53.85 \( \times 10^{-4} \) for 132 side rings.

With the corrected \( N_{s, c} \), we can correct \( M_s \) and \( H_d \) using (39b) and (39d) and compare them with \( M_{s, c} \) and \( H_{d, c} \) calculated from (39a) and (39c). We find that, for all \( N_s \) and \( N_v \), the discretization error is less than 10%, giving a demagnetizing factor error of \( < 1 \% \). Exceptions are when \( \gamma = 20 \) and \( \chi \geq 100 \) for the 74-side-ring calculation, and when \( \gamma = 50 \) and \( \chi \geq 100 \) or \( \gamma = 10 \) and \( \chi = 10000 \) for the 132-side-ring calculation. Thus (39e) is rather accurate in most cases and the interpolation/extrapolation approach is valid.

A comparison can be made with less-general published results obtained for specific values of \( \chi \). The results agree with the exact self-inductance calculations in Table I to four significant figures except for \( N_v \) for \( \gamma > 1 \) when, in the worst case, the error is 0.12%. The two-dimensional calculation for \( N_s \) can be compared with \( N_s \), data deduced from the work of Taylor [40], [41] as shown in Section VI-A. From Taylor’s data, we calculate five values each of \( N_s (\gamma = 1) \) and \( N_v (\gamma = \infty) \) for \( 0.25 \leq \gamma \leq 4 \) (Table VI). The greatest difference between the two-dimensional results and Taylor’s method is 0.22% for \( N_v (\gamma = 1) \) and 0.17% for \( N_s (\gamma = \infty) \). The results are compared with the values calculated by Templeton and Arrott [45]. The largest deviation is 0.98% at \( \gamma = 10 \).

The results of \( N_f \) for \( \gamma \geq 10 \) can be compared with our one-dimensional calculation (Table II). The deviations of the one-dimensional from the two-dimensional results are \(-1.2% \), \(-1.2% \), \(0.6% \), \(1.8% \), \(0.7% \), and \(-4.2% \) for \( \gamma = 10 \) and \( \chi = 10000, 100, 10, 1, 0, \text{ and } -1 \), respectively. For \( \gamma = 20 \) and 50 the maximum deviation reduces to 1.2% and 1.9%, respectively. Such deviations are due mainly to the approximations made in the one-dimensional model.
TABLE IV

\( N_{\varphi} \) as Function of \( \chi \) and \( \gamma \) Calculated Using the Two-Dimensional Model.

| \( \gamma \) | \( N_{\varphi} (10^{-6}) \) |
|---|---|
| 0.01 | 9781 |
| 0.02 | 9576 |
| 0.03 | 9383 |
| 0.04 | 9201 |
| 0.05 | 9028 |
| 0.06 | 8864 |
| 0.07 | 8704 |
| 0.08 | 8558 |
| 0.09 | 8424 |
| 0.1 | 9265 |
| 0.2 | 9064 |
| 0.3 | 8866 |
| 0.4 | 8685 |
| 0.5 | 8454 |
| 0.6 | 8238 |
| 0.7 | 8035 |
| 0.8 | 7846 |
| 0.9 | 7669 |
| 1.0 | 7503 |
| 1.5 | 1199 |
| 2.0 | 6896 |
| 2.5 | 4497 |
| 3.0 | 322.7 |
| 3.5 | 2491 |
| 4.0 | 1991 |
| 5.0 | 1387 |
| 7.0 | 77.75 |
| 10.0 | 41.38 |
| 20.0 | 11.44 |
| 50.0 | 1.937 |

TABLE V

\( N_{\varphi} \) as Function of \( \chi \) and \( \gamma \) Calculated Using the Two-Dimensional Model.

| \( \gamma \) | \( N_{\varphi} (10^{-6}) \) |
|---|---|
| 0.01 | 9780 |
| 0.02 | 9576 |
| 0.03 | 9383 |
| 0.04 | 9201 |
| 0.05 | 9028 |
| 0.06 | 8864 |
| 0.07 | 8704 |
| 0.08 | 8558 |
| 0.09 | 8424 |
| 0.1 | 9265 |
| 0.2 | 9064 |
| 0.3 | 8866 |
| 0.4 | 8685 |
| 0.5 | 8454 |
| 0.6 | 8238 |
| 0.7 | 8035 |
| 0.8 | 7846 |
| 0.9 | 7669 |
| 1.0 | 7503 |
| 1.5 | 1199 |
| 2.0 | 6896 |
| 2.5 | 4497 |
| 3.0 | 322.7 |
| 3.5 | 2491 |
| 4.0 | 1991 |
| 5.0 | 1387 |
| 7.0 | 77.75 |
| 10.0 | 41.38 |
| 20.0 | 11.44 |
| 50.0 | 1.937 |

VI. DISCUSSION

A. \( N_{m} \) for \( \chi = 0, \infty, \) and \(-1\)

In ellipsoids, a uniform applied field produces a magnetization and a demagnetizing field that are both uniform. In cylinders, the magnetization and the demagnetizing field are both nonuniform except in two cases. For \( \chi = 0 \), the magnetization is uniform but the demagnetizing field is not. For \( \chi = \infty \), the demagnetizing field is uniform (and exactly opposite to the applied field) but the magnetization is not. For \( \chi = -1 \), the nonuniform magnetization and nonuniform demagnetizing field in the cylinder combine to exactly cancel the applied field so that the flux density \( B = 0 \). In these three cases, demagnetizing factors can be calculated more accurately by introducing electromagnetic scalar potentials. Moreover, there are some simple relations among \( N_{mx} \), \( N_{my} \), and \( N_{mz} \), the magnetometric demagnetizing factors along the three principal orthogonal axes.

For \( \chi = 0 \), a cylinder has \( N_{mx} \), \( N_{my} \), and \( N_{mz} \) that are
I. Introduction

CHEN er al. DEMAGNETIZING FACTORS FOR CYLINDERS

II. Theory

According to the more general theorem of Brown and Morris [72]-[74], as examples from Table I, cylinders with $\gamma = 0.22$, 0.80, and 1.6 are equivalent to ellipsoids with $\gamma = 0.30$, 0.90, and 1.6. A cylinder with $\gamma = 0.9065$ is equivalent to a sphere ($N = 0.3333$) [36]. A cube is also equivalent to a sphere according to the theorem [75]-[77]. Experimentally, for weakly magnetic and saturated ferromagnetic materials, these two shape-isotropic geometries can be used as alternatives to spheres. However, the theorem of Brown and Morris on the equivalence of a body of arbitrary geometry (including a cylinder) and a possible ellipsoid cannot lead to an \textit{a priori} estimate of the value of $N_m(\chi = 0)$ except for a body with center symmetry, such as a cube.

For cylinders, the transverse magnetometric demagnetizing factors are


equivalent to those of a possible ellipsoid of revolution, with

$$N_{mr} + N_{my} + N_{mz} = 1, \quad (41)$$

Equations (41) and (42) are valid for $N_m$ only when $\chi = 0$. For $\chi > 0$, the sum is less than 1, while for $\chi < 0$, it is greater than 1. If $N_f$ is considered, rather than $N_{mz}$, the sum is always less than 1.

For a given $\gamma$, values of $N_f(\chi \to \infty)$ and $N_m(\chi \to \infty)$ are rather close to $N$ for ellipsoids. We can fit the $N_f(\infty)$ and $N_m(\infty)$ versus $\gamma$ data in Table II and Figs. 2 and 3 by simple equations:

$$N_f(\chi \to \infty, \gamma) = N(\gamma)[1 + 0.12(\log \gamma - 3)], \quad (43a)$$

$$N_m(\chi \to \infty, \gamma) = N(\gamma)[1 + 0.25(\log \gamma - 1)], \quad (43b)$$

where the demagnetizing factor $N(\gamma)$ for ellipsoids can be found in Table I. In the range $10 \leq \gamma \leq 100$, (43a) and (43b) fit the model results with maximum deviations of 1% and 1.5%, respectively. Including the results for $\gamma < 10$ in Tables IV and V, the lower boundaries of the $\gamma$ range with the same indicated errors extend to 2 and 7 for $N_f$ and $N_m$, respectively.

Taylor calculated the anisotropic electric ($\alpha$) and magnetic ($\beta$) polarizabilities of conducting cylinders [40], [41]. In Taylor's terminology, "conducting" means "without field penetration," so electrically $E = 0$ and magnetically $B = 0$. Therefore we can relate the longitudinal $\alpha_l$ and transverse $\alpha_t$ electric polarizabilities to $N_m(\infty)$, and the longitudinal $\beta_l$ and transverse $\beta_t$ magnetic polarizabilities to $N_m(1)$. For a conducting solid of revolution with volume $V_o$,

$$\alpha_l/(e_0V_o) = N_m(1)^{-1}, \quad (44a)$$

$$\alpha_t/(e_0V_o) = N_m(\infty)^{-1}, \quad (44b)$$

$$\mu_0\beta_l/V_o = [N_m(-1) - 1]^{-1}, \quad (44c)$$

$$\mu_0\beta_t/V_o = [N_m(-1) - 1]^{-1}, \quad (44d)$$

where $e_0$ is the permittivity of vacuum. Equations (44a) and (44b) are in accordance with the analogy between
TABLE VI

| γ  | Nₘ(∞) | Nₘ(∞) | Nₘ(∞) | Nₘ(∞) | Nₘ(∞) | Nₘ(∞) |
|----|-------|-------|-------|-------|-------|-------|
| 0  | 0.5712| 0.6764| 0.2136| 1.036 | 1.114 |
| 0.25| 0.6718| 0.8948| 0.2371| 1.103 | 1.114 |
| 0.5 | 0.2371| 0.8853| 0.2928| 1.114 |
| 1   | 0.3154| 0.3692| 0.3669| 1.030 |
| 2   | 0.3829| 0.2341| 0.4237| 1.0815|
| 4   | 0.4319| 0.3161| 0.4506| 1.0553|
| ∞   | 0.5   | 1     | 0.5   | 1     |

The data are obtained from the electric and magnetic polarizabilities of conducting cylinders calculated by Taylor [40], [41]. Σ(∞) and Σ(-1) are the sums Nₘ + Nₙ + Nₙ for x = ∞ and x = -1.

electrostatic shielding in a conductor and magnetostatic shielding in the same body but with x → ∞.

There is a relation between βₗ and αₗ [41],

\[ \mu_0 \beta_{li} = -\alpha_{li} / (2e_0), \]

which leads to an exact relation between Nₘ(∞) and Nₘ(-1),

\[ Nₘ(∞) = Nₘ(-1) = \frac{1}{2} [1 - Nₘ(-1)]. \]

Table VI lists Nₘ(∞), Nₘ(∞), Nₘ(∞), Nₘ(∞), Nₘ(∞), and Nₘ(∞) for short cylinders based on the α and β data given by Taylor. From this table we see that the sum of the Nₘ's deviate from 1 by about 11%. For other values of x, the deviation should be less, and this could help one estimate Nₘ for different values of x and γ.

B. General Rules for Nₗ and Nₘ as Functions of x and γ

We give some general rules for the variation of Nₗ and Nₘ with γ and x based on the tables and figures. (a) For a given x, both Nₗ and Nₘ decrease with increasing γ. This is because the demagnetizing effect is an "end effect," although the demagnetizing fields depend on both end and side surface poles. When γ increases, the effect of cylinder ends on the midplane and the entire volume decreases. (b) When both γ and x are fixed, Nₘ(γ, x) > Nₗ(γ, x). That is, the end effect is weaker at the midplane. (c) Nₘ decreases with increasing γ at any value of x. (d) With increasing γ, Nₗ increases when x > 1 and decreases when x < 1; there is a region around x = 1 where Nₗ is insensitive to x. Rules (b), (c), and (d) give the following relation for Nₗ(γ) and Nₘ(γ) when x > 1:

\[ Nₘ(∞) > Nₘ(n) > Nₘ(0) > Nₗ(∞) > Nₗ(n) > Nₗ(0) > Nₗ(-1). \]

(e) For γ > 1, the ratio Nₘ/Nₗ increases with increasing γ and decreasing x. When γ increases from 1 to 100, Nₘ(0)/Nₗ(0) increases from 1.37 to 8 to 800; Nₘ(∞)/Nₗ(∞) is smaller and increases from 1.14 to 1.32 to 1.46. (f) The minimum γ for Nₗ(γ) > 0.99Nₗ(∞) increases with increasing γ. In fluxmetric ferromagnetic measurements, this rule tells us that Nₗ(∞) can be used for dM/dH larger than a minimum value that depends on γ. For γ = 10, 100, and 1000, the minimum values are 200, 5 x 10⁶, and 10⁷.
C. Minimum $\chi$ for $N_f(\chi) > 0.99N_f(\infty)$

For $\chi > 0$, the demagnetizing effect resembles the response of an amplifier with negative feedback. The input, output, and gain of the amplifier are $H_a, M$, and $\chi$, respectively. After operation, $M$ becomes $H_a (H_a = -NM)$, where $N$ is the local demagnetizing factor, with sign opposite to that of $H_a$, which feeds back to the input. The result is that $M$ becomes smaller and nonuniform, while $H$ becomes smaller than $H_a$. There is perfect feedback when $\chi \to \infty$, and $H = 0$ everywhere. Since $H / H_a = (1 + N\chi)^{-1}$, where $N$ is the local demagnetizing factor, in order to have $N_f(\chi)$ almost equal to $N_f(\infty)$, $(1 + N\chi)$ must be sufficiently large for $H / H_a$ to be small everywhere. Actually $N_f$ is the smallest local $N$, and the origin of rule (f) may be understood as follows.

$N_f(\infty)$ decreases with increasing $\gamma$. To have $N_f(\chi)$ almost equal to $N_f(\infty)$, a smaller $N_f(\infty)$ must be balanced by a larger $\chi$ to fulfill the condition of a sufficiently large $(1 + N\chi)$. Therefore, with increasing $\gamma$, the minimum $\chi$ to satisfy $N_f(\chi) > 0.99N_f(\infty)$ increases. From Table II we can deduce this minimum $\chi$ to be $k / N_f(\infty)$ with $15 < k < 17$.

D. Position Dependence of $M, H_d$, and $\sigma$

We have explained rules (a), (b), and (f). To understand the other rules, we have to know the details of the position dependence of $M, H_d$, and $\sigma$. Fig. 8(a), 8(b), and 8(c) gives the curves of $M(z)/M(0), H_d(z)/H_a$, and $\sigma(z)/\sigma(1)$ as functions of $z$ for $\gamma = 20$. The curves in graphs (a), (b), and (c) are for $\chi = -1, -0.5, 0.001, 30$, and $10^5$, respectively. Graphs (d), (e), and (f) are for $\chi = 0.001$ on an expanded scale.

Fig. 8. $M(z)/M(0)$ (graphs (a) and (d)), $H_d(z)/H_a$ (graphs (b) and (c)), and $\sigma(z)/\sigma(1)$ (graphs (c) and (f)) as functions of $z/l$ for a cylinder with $\gamma = 20$. The curves in graphs (a), (b), and (c) are for $\chi = -1, -0.5, 0.001, 30$, and $10^5$, respectively. Graphs (d), (e), and (f) are for $\chi = 0.001$ on an expanded scale.

E. $\chi$ Dependence of $N_m/N_f$

For $\chi = 10^5$ (that is, $\chi \to \infty$) we can see in Fig. 8(b) that $H_d(z)$ is a constant equal to $-H_a$ in the entire cylinder, which makes $H$ very small (0 for $\chi \to \infty$) and $M(z)$
finite. $M(z)$ is approximately parabolic as shown in Fig. 8(a) and as already pointed out by other authors [21], [27]. The average $H_d$ is equal to $-H_o$. For the midplane and the entire volume, respectively, the average $M$ is $M(0)$ and approximately $0.67M(0)$. Therefore, $N_m/N_f$ should be close to 1.5.

When $\chi$ decreases to 30, the variation of $M(z)$ is gradual for small $z$ and sudden in the end regions. This increases the volume-averaged $M$ to more than $0.67M(0)$. But because $H_d(z)$ becomes rather $z$ dependent and its absolute value at $z = l$ is much larger than at $z = 0$ [Fig. 8(b)], $N_m/N_f$ is larger.

As $\chi$ decreases to 0.001, both $M(z)$ and $H_d(z)$ remain nearly constant in Fig. 8(a) and 8(b). The reason is that $M(z)$ is very small compared to $H_d(z)$, so $H_d(z)$ is even smaller than $H_d$. The variation of the extremely small $H_d(z)$ cannot be seen in Fig. 8(b), and the modification of the field by such a small $H_d(z)$ causes an invisible change in $M$ in Fig. 8(a). However, if we expand the scales [Fig. 8(c) and 8(f)], we can see that the variation of $M(z)$ and $H_d(z)$ continues the trend of decreasing $\chi$ from $10^5$ to 30. This makes $N_m/N_f$ even larger.

When $\chi$ is small but negative, both $M(z)$ and $H_d(z)$ change their signs. At this point, $N_f$ and $N_m$ continue the trend without sudden change. The situation for $\chi < 0$ can be seen from the curves for $\chi = -0.5$ and $-1$ in Fig. 8(a) and 8(b). For these two cases, $H_d(z)$ and the absolute value of $M(z)$ remain constant until $z > 0.8l$, and then they suddenly increase. This makes $N_m/N_f$ largest when $\chi = -1$. Further discussion of this in terms of pole distribution is given in the next section.

F. $\chi$ Dependence of $N_f$

To understand the susceptibility dependence of $N_f$ for $\gamma > 1$, we focus on the magnetic pole distribution shown in Fig. 8(c) and 8(f). For the largest $\chi$, $\sigma(z)$ varies with $z$ almost linearly on the cylinder surface except for the regions close to the ends. This means that the magnetic poles are the most uniformly distributed on the cylinder, and $H_d(0)$, which has a greater contribution from the poles in the central region than from the ends, is the largest. Thus $N_f$ is its largest for the highest $\chi$.

When $\chi$ is decreasing, the variation of $\sigma(z)$ is progressively greater in the end regions, while the magnetic pole density in the central region becomes gradually lower. Therefore, $N_f$ decreases with decreasing $\chi$. When $\chi = 0$, all the poles are at the ends, $\sigma(z)$ is 0, and $N_f$ should be its smallest. In fact, $N_f$ continues to decrease when $\chi$ becomes negative. The reason is that, although $\sigma(z)$ remains 0 in a large central region for $\chi < 0$, $\sigma(z)$ for $z$ close to $l$ increases with decreasing $\chi$ and its sign is opposite to that of $\sigma(z+l)$ [Fig. 8(c)]. The side poles produce a field at the center directed opposite to the field produced by the end poles, so that the demagnetizing effect of the end poles is partially compensated by the effect of the side poles. This makes $N_f$ a little lower than for $\chi = 0$. The side poles close to the ends, with signs opposite to those of the end poles, have a different effect on $N_m$. They greatly increase the value of $H_d$ in the end regions, so the volume-averaged $N_m$ increases with decreasing $\chi$ even for negative $\chi$. Finally, we have the largest value of $N_m/N_f$ at $\chi = -1$.

When $\gamma < 1$, the susceptibility dependence of $N_f$ is the opposite. We can explain this as follows. For oblate cylinders, the end poles are the main contributors to the demagnetizing field $H_d$ at the midplane. For a given $M(z)$ at the midplane, the end-pole density is smallest when $\gamma \to \infty$ since, in this case, the poles are the most uniformly distributed on the entire surface. Therefore $N_f$ is the smallest. A smaller positive $\chi$ repels the poles to the end regions, which gives rise to a larger end-pole density and a larger $N_f$. Although all the poles are distributed on the ends when $\chi = 0$, the pole density on the ends is not the largest. This is because, when $\chi < 0$, the end-pole density has a sign opposite to that of the side-pole density nearby, as can be seen from Fig. 4; thus the end poles are further enhanced. As a consequence, the end pole density increases continuously with decreasing $\chi$ regardless of its sign, and $N_f$ takes its largest value when $\chi = -1$.

G. Error Transmission in Susceptibility Measurements

From the above analysis we see that, at present, the accuracy of $N_m$ and $N_f$ can reach 1% in general. To know if this accuracy is good enough for the purpose of magnetic measurements, we examine the influence of the error in $N_m$ or $N_f$ on susceptibility measurements. We consider a cylinder consisting of material with constant $\chi$ in a longitudinal applied field $H_o$. An external susceptibility $\chi_e$ can be defined as $\langle M \rangle/H_o$, where the average is taken over the whole volume or the midplane, depending on whether $N_m$ or $N_f$ is considered. From (3), replacing $N$ by $N_{f,m}$ and $M$ by $\langle M \rangle$, we obtain

$$\chi = \chi_e(1 - N_{f,m}\chi_e)^{-1}. \quad (47)$$

This equation is accurate under the above assumptions and definitions. From (47), the relative error in $\chi$ caused by the calculation error of $N_{f,m}$ and the measurement error of $\chi_e$ can be derived as

$$|\Delta \chi/\chi| = |N_{f,m}| |\Delta N_{f,m}/N_{f,m}| + (\chi/\chi_e) |\Delta \chi_e/\chi_e|. \quad (48)$$

On the right-hand side of (48), the first term is the transmission error from the erroneous $N_{f,m}$ calculation to the $\chi$ determination. This error equals the error in $N_{f,m}$ multiplied by a factor $\alpha_l = |N_{f,m}\chi_e|$, which can be obtained from the results of this work. We examine three typical cases. In the first case, $|\chi|$ and $\alpha_l$ are small; the error in $N_{f,m}$ can be large and still result in a small transmission error. For example, when $|\chi| = 0.1$, we have $\alpha_l < 0.1$ since $N_{f,m} < 1$, and less than 10% of the $N_{f,m}$ error is reflected in the final $\chi$ result. The second case is a magnetometric measurement for $\chi = -1$. $\alpha_l$, obtained from Table V, is 0.37, 0.17, and 0.06 for $\gamma = 1$, 3, and 10; only a small part of the error in $N_{f,m}$ is transmitted to the final $\chi$ result. In the third case, high-$\chi$ materials are considered. To reduce the error due to the large $\alpha_l$, fluxmetric measurements should be made with long samples, since $N_f < N_m$ and $N_f$ decreases with increasing $\gamma$. To
ensure that \( a_1 < 1 \) based on Table II and Fig. 2, \( \gamma \) should be 12, 58, 200, and 700, for \( \chi = 10^3, 10^5, 10^6, \) and \( 10^7 \), respectively.

The second term in (48) is the transmission error due to the measurement error of \( x_e \). The corresponding factor is \( a_2 = \chi / x_e \). For the first case where \( a_1 \) is very small, \( a_2 \) is very close to 1 since \( \chi = x_e \), so the error in \( \chi \) is almost the same as the \( x_e \) measurement error. For our second case, from (47), we have \( a_1 + a_2 = 1 \). This is interesting because, when \( \chi \) is small so that \( N_{\chi} \to 1 \) and \( a_1 \to 1 \), the \( x \) error is mainly due to the error in \( N_{\chi} \), no matter how large the \( x_e \) measurement error is. For the high-\( \chi \) case, if \( a_2 = 2 \) from (47), leading to a double transmission error to \( x \) from the \( x_e \) measuring error. As a consequence, to obtain accurate results of \( x \) for high-\( \chi \) materials, both accurate \( N_f \) and accurate \( x Chad \) are strongly required if \( \gamma \) cannot be made very large.

In summary, to determine \( \chi \) accurately, the higher the \(|\chi|\) of the material, the higher is the required calculation accuracy of \( N_{\chi e} \). From a magnetic measurement point of view, our calculation accuracy for \( N_{\chi e} \) at \( \chi = 0 \) is more than adequate, and the number of calculated points is sufficient for accurate interpolation over a wide range of values. For other \( \chi \) values, the requirement for calculation accuracy will be determined by the particular purpose of the magnetic measurement; a 1% accuracy may be sufficient for accurate interpolation over a wide range of values. For other \( \chi \) values, the requirement for calculation accuracy will be determined by the particular purpose of the magnetic measurement; a 1% accuracy may be sufficient for many uses but is inadequate for others. For example, our 0.2% accuracy for \( \chi = -1 \) is required for superconductor calibration standards.

H. Application to Materials with \( \chi > 0 \)

Most materials have \( \chi > 0 \), and our results can be used for demagnetization corrections of their magnetic measurements. For materials with linear or nearly linear magnetization curves, our \( N_{\chi e} \) values are satisfactory. These include paramagnets, spin glasses, weakly magnetic materials, and iron-powder cores and ordinary ferromagnets in the initial and saturation states. However, even in these cases, some caution is required. We give an example below.

For magnetometric measurements of weakly magnetic materials (\( \chi < 0.01 \)) only very small demagnetizing corrections are needed. However, such materials can also be measured by fluxmetric methods, as recommended by at least one measurement standard [79]. A source of error in fluxmetric measurements is if the sample diameter is less than that of the measurement coil. A large demagnetizing effect would occur in the measurement of \( M \) because the measured flux linkage is contributed not only by the \( M \) of the sample but also by the \( H_d \) within the coil volume produced by the sample’s poles. Furthermore, fluxmetric measurements on weakly magnetic materials require many coil turns, which ensures that this error will arise. The error in \( \chi \) due to this effect can be as large as 30%, even if the requirements of [79] are followed [38], [64].

The magnetization curves of ferromagnetic materials are nonlinear, and it is difficult to assign to them specific \( \chi \) and \( N_{\chi e} \) values except in the initial state and when approaching saturation, as mentioned above. However, our results can still be used satisfactorily for long magnetically soft materials over a wide field range. We can regard \( \chi \) as the normal susceptibility \( \chi_n = M_n / H_n \), where the subscript \( m \) denotes the maximum value at the endpoints of a symmetric magnetization loop, and use \( N_f(\chi) \).

To extract an unknown \( \chi \) from fluxmetric measurements of \( x_e \) on samples with known \( \gamma \), one uses \( N_f \). But a knowledge of \( \chi \) is required to select the appropriate \( N_f \) value. The known \( \chi_n \) and \( \gamma \) and the unknown \( \chi \) and \( N_f \) are related by (47) and Table II or Fig. 2, so the unknowns can be calculated simultaneously. An iterative process may be used. \( N_f \) is estimated based on the measured \( x_e \) using Table II or Fig. 2, and \( \chi \) is calculated from \( x_e \) and \( N_f \) using (47). Then a better estimate of \( N_f \) is made. Since the differential susceptibility is field dependent in ferromagnets, this treatment involves some error. The resultant \( \chi \) is an effective susceptibility \( \chi_{ef} \). Its value is between \( \chi \) and an averaged differential susceptibility in the sample. We have \( \chi_{ef} = \chi_n \) when \( H_n \to 0 \) or \( H_n \to \infty \), or when the sample has very large \( \gamma \) and \( \chi_n \), and \( H_n \) is close to \( H_n(\chi_{e max}) \). In other cases, including the use of \( N_{\chi e} \) and Tables IV and V, the resultant \( \chi_{ef} \) may be larger or smaller than \( \chi_n \), depending on the measurement conditions.

For measurements of semihard ferromagnets with intrinsic coercivity \( H_e \), Zijlstra [58] has suggested a simple method to obtain a reasonably accurate loop in which the demagnetizing corrections for \( M \) and \( H \) are performed using \( N_f(0) \) and \( N_f(\infty) \), respectively. With our results, nonzero finite effective \( \chi \) (corresponding to the differential susceptibilities at \( H = H_e \) and \( H = H_n \)) can be used more properly.

I. Remarks Concerning \( \chi < 0 \) and Nonuniform \( \chi \)

For normal diamagnetic materials with uniform \( \chi \), values of \( N_{\chi e}(\chi = 0) \) are more than adequate for experimental work. However, large negative values of \( \chi \) arise in ac magnetic measurements of normal conductors and both ac and dc measurements of superconductors, where bulk magnetic moments have their source in eddy currents and shielding supercurrents, respectively. These magnetic moments allow us to ascribe values of \( M \) or \( H \) and \( \chi \) to these materials.

In an ideal type-I superconductor, \( \chi = -1 \) because \( B \) and the permeability \( \mu(1 + \chi) \) both equal 0 at every point in the material. The same applies to a normal conductor in an ac field when the skin depth is negligible compared to its dimensions. Thus there is an equivalence between these cylinders and a normal perfectly diamagnetic cylinder. For these cases, our values of \( N_{\chi e}(\chi = -1) \) and \( N_f(\chi = -1) \) can be used. We have verified this experimentally with a magnetometric low-field ac susceptibility measurement of a niobium cylinder (\( \gamma = 1.033 \)) below the critical temperature. The susceptometer was calibrated using the known demagnetizing factor and dipole field of a superconducting niobium sphere [80]. We accounted for a 0.4% volume decrease of both the standard sphere and the sample cylinder upon cooling to 4 K, and deduced a value of \( N_{\chi e} \) equal to 0.361 ± 0.001 from (47), with the uncertainty based on the measured scatter in \( x_e \).
Our two-dimensional calculations give 0.3622. Thus we see that cylindrical superconducting standards for magnetic measurements for use at low fields and temperatures can be made using accurate values of \( N_{\text{m}}(1) \).

The results of this work have to be used more carefully for materials that do not have constant susceptibility. In these cases, an effective susceptibility should be found. For example, the \( M(H) \) curve of a type-II superconductor in fields below the lower critical field \( H_c \) is linear, with \( \chi = 1 \). In the mixed state, \( M(H) \) increases when \( H > H_c \) and the effective \( \chi \) should be close to the differential susceptibility at each point, which is positive. This causes a discontinuity in the value of \( N_{\text{m}} \) above \( H_c \), and a proper demagnetizing correction should take account of this effect. Similar caveats apply to the intermediate state of superconductors.

Normal conducting cylinders in ac fields have \( M(H) \) loops, and a complex susceptibility with a negative real part can be defined [81]. However, this susceptibility is due to eddy currents constrained by the skin effect, different from our model assumption of uniform susceptibility. \( N_{\text{f},m} \) for \( \chi = -1 \) may be used in the limit of small skin depth. Otherwise, to obtain good results, \( \gamma \) must be large enough so that only a small correction is needed; our \( N_{\text{f},m} \) values for an effective \( \chi < 0 \) can be used. A similar case arises in hard superconductors, where a portion of the magnetization comes from penetrated supercurrents that follow the critical-state model [82].

Since most cases of magnetic measurements involve nonlinear magnetization curves, the demagnetizing correction using the factors calculated for constant \( \chi \) should be made cautiously. For this, a deep understanding of the magnetization process and the demagnetizing effect is most important.

**APPENDIX**

The demagnetizing field produced by a ring-shaped distribution of magnetic poles is calculated below. Results are presented both for the case where a ring has a finite width, as on the end-plane of a cylinder, or a finite height, as on the side surface of a cylinder. The method follows that of Gray [83], who calculated the field produced by a disk of charge. The pole density is taken to be uniform over the width or height of the ring.

The magnetic scalar potential at a point \( (r_2, z_2) \) produced by a pole density \( \sigma \) on the \( j \)th ring is given by

\[
\Phi_{\sigma} = \frac{(a/4\pi \mu_0)}{2} \sum_{j=1}^{n} \int_{0}^{r_1} r_j \left[ (z_j - z_2)^2 + r_j^2 \right] \cos \phi \, dr_j \, d\phi,
\]

The effective susceptibility is

\[
\Phi_{\text{e}} = \frac{(a/4\pi \mu_0)}{2} \sum_{j=1}^{n} \int_{0}^{r_1} \left[ (z_j - z_2)^2 + a^2 + r_j^2 \right] \cos \phi \, dr_j \, d\phi,
\]

where \( \Phi_{\sigma} \) and \( \Phi_{\text{e}} \) represent the potential due to a ring on the end-plane and side surface, respectively. The parameters \( r_1 \) and \( r_2 \) are the inner and outer radii of the ring of poles on the end surface, and \( z_1 \) and \( z_2 \) are the limits in the \( z \) direction of the ring on the side surface. The limits of integration are changed to give

\[
\Phi_{\sigma} = \int_{0}^{r_1} F(r, \phi) \, dr \, d\phi - \int_{0}^{r_1} F(r, \phi) \, dr \, d\phi,
\]

\[
\Phi_{\text{e}} = \int_{0}^{r_1} G(z, \phi) \, dz \, d\phi - \int_{0}^{r_1} G(z, \phi) \, dz \, d\phi,
\]

where \( F(r, \phi) \) and \( G(z, \phi) \) represent the integrands in (A1) and (A2). The definite integral \( \int F(r, \phi) \, dr \, d\phi \) has been evaluated by Gray [83]. Using a similar method, we evaluate the other integrals with the results,

\[
\int_{0}^{r_1} F(r, \phi) \, dr \, d\phi = (ar_2/2\mu_0) \int_{0}^{\infty} \exp (-\lambda |z_j - z_2|) \lambda^{-1} J_1(\lambda r_2) J_0(\lambda r_1) \, d\lambda,
\]

\[
\int_{0}^{r_1} G(z, \phi) \, dz \, d\phi = (a/2\mu_0) \int_{0}^{\infty} \exp (\lambda |z_j|) \lambda^{-1} J_0(\lambda r_2) J_1(\lambda r_1) \, d\lambda,
\]

\[
\left[ z_j/(2|z_j|) \right] \int_{0}^{a} \exp (-\lambda |z_j|) J_1(\lambda r_2) J_0(\lambda r_1) \, d\lambda - r_1 \int_{0}^{\infty} \exp (-\lambda |z_j|) J_1(\lambda r_2) J_0(\lambda r_1) \, d\lambda,
\]

\[
\left[ 1/2 \right] r_2 \int_{0}^{\infty} \exp (-\lambda |z_j|) J_1(\lambda r_2) J_0(\lambda r_1) \, d\lambda - r_1 \int_{0}^{\infty} \exp (-\lambda |z_j|) J_0(\lambda r_2) J_1(\lambda r_1) \, d\lambda,
\]

and, for rings on the side surface, as

\[
\int_{0}^{a} \exp (-\lambda |z_j|) J_0(\lambda r_2) J_1(\lambda r_1) \, d\lambda = -\frac{1}{2} \int_{0}^{\infty} \exp (-\lambda |z_j|) J_0(\lambda r_2) J_1(\lambda r_1) \, d\lambda + a \int_{0}^{\infty} \exp (-\lambda |z_j|) J_1(\lambda r_2) J_0(\lambda r_1) \, d\lambda.
\]
\[ N^+ = (\mu_0 / a) \theta_{2r} \theta_{2r} - f^+_r, \]

\[ = \left[ z_1 / (2 |z_1|) \right] a \int_0^\infty \exp \left( -|z_1| \right) J_0(\lambda r) d\lambda \]

\[ - \left[ z_2 / (2 |z_2|) \right] a \int_0^\infty \exp \left( -|z_2| \right) J_0(\lambda r) d\lambda. \]

\[ (A11) \]

We have arbitrarily set \( z_j = 0 \) for ease of notation and have included the factor \( z_j / |z_j| \) when needed to account for the sign reversal that occurs when \( z_j < z_i \). The integrals in the \( f \) factors are given in [84] and reduce to

\[ f^+_z = \left\{ -k_j \right\} |z_j| \left[ 4 \pi (a_2 b_j)^{-1} \right]^{-1} F(k_j) \]

\[ - (2 \pi)^{-1} \Lambda(\alpha_2, \beta_2) + \frac{1}{2} z_j / |z_j| \] \( (a_2 > b_2) \).

\[ (A12a) \]

\[ f^+_z = \left\{ -k_j \right\} |z_j| \left[ 4 \pi a_j (a_2 b_j)^{-1} \right]^{-1} F(k_j) \]

\[ + (2 \pi)^{-1} \Lambda(\alpha_2, \beta_2) z_j / |z_j| \] \( (a_2 < b_2) \).

\[ (A12b) \]

\[ f^+_z = - (a_2 / b_j)^{1 / 2} (\pi k_j)^{-1} \left[ (1 - k_j^2 / 2) F(k_j) - E(k_j) \right], \]

\[ (A13) \]

\[ f^+_z = -k_j (a_2 / b_j)^{1 / 2} (2 \pi)^{-1} F(k_j), \]

where \( a_j = r_j \) and \( b_j = r_j \), and

\[ f^+_z = \left\{ -k_j \right\} |z_j| b_2 (4 \pi a_j)^{-1} F(k_j) \]

\[ + b_2 (4 a_2)^{-1} z_j / |z_j| \] \( (a_2 = b_2) \).

\[ (A15a) \]

\[ f^+_z = \left\{ -k_j \right\} |z_j| b_2 (4 \pi a_2 a_j)^{-1} F(k_j) \]

\[ + b_2 (2 \pi a_j)^{-1} \Lambda(\alpha_2, \beta_2) z_j / |z_j| \] \( (a_2 < b_2) \).

\[ (A15b) \]

where \( a_j = r_j \) and \( b_j \) is the radius of the cylinder. The factors \( f^+_r, f^+_\theta, f^+_\phi \), and \( f^+_\zeta \) are given by the same expressions except \( r_j \) is replaced by \( r_j \) and \( z_j \) is replaced by \( z_j \).

For each case \( k = 4 ab / [(a + b)^2 + z^2] \), and \( F(k) \) and \( E(k) \) are the complete elliptic integrals of the first and second kind. \( \Lambda(\alpha, \beta) \) is related to the Heun lambda function and is expressed in terms of the elliptic integrals of the third kind. \( \Lambda(\alpha, \beta) = (1 - p)^{1 / 2} (1 - k^2 / p)^{1 / 2} \Pi(\alpha, p, \pi / 2) \). The parameters \( p, \alpha, \) and \( \beta \) are specified by \( p = k^2 / (1 - (1 - k^2) \sin^2 \beta), k = \sin \alpha, \) and \( \sin^2 \beta = z^2 / [(a - b)^2 + z^2] \). The elliptic integrals are evaluated numerically using the procedure given in [85].

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Du-Xing Chen was born in Shanghai, China, in 1941. He studied physics at Beijing University from 1958 to 1964, received the M.S. degree from the Institute of Physics, Academia Sinica, in 1981, and received the Ph.D. degree from the Central Iron and Steel Research Institute, Beijing, in 1985. He is a visiting Professor in the Electromagnetism Group at the Universitat Autònoma de Barcelona, Spain. His areas of research are magnetic measurements, magnetic materials, and superconductor theory and measurements. He was a visiting scientist at the Royal Institute of Technology, Stockholm, from 1985 to 1989, and at the National Institute of Standards and Technology, Boulder, Colorado, during 1988 and 1990. He has published two books entitled Physical Basis of Magnetic Measurements (China Mechanical Industry, Beijing, 1985) and Ballistic and Bridge Methods of Magnetic Measurements of Materials (China Metrology, Beijing, 1990).

James A. Brug (M'87) was born in Cody, WY, in 1954. He received the B.S. degree in physics from Montana State University and the Ph.D. degree in applied physics from Yale University. He is currently on leave from Hewlett-Packard Laboratories in Palo Alto, CA, as an ASEE postdoctoral fellow at the National Institute of Standards and Technology in Boulder, Colorado. His research has focused on noise mechanisms in magnetic thin films, measurement of magnetic properties at small length scales, and the magnetics of magnetoresistive recording heads. Dr. Brug is a member of the American Physical Society.

Ronald B. Goldfarb (M'79-SM'86) was born in Mexico City, Mexico, in 1951. He received the B.A. degree in electrical engineering and the M.S. degree in materials science from Rice University and the M.S. and Ph.D. degrees in physics from Colorado State University. He is a member of the Superconductor and Magnetic Measurements Group at the National Institute of Standards and Technology (NIST) in Boulder, CO. His areas of research are ac losses and coupling effects in low- and high-temperature superconductors, cryogenic magnetic properties of concentrated spin glasses and weakly magnetic alloys, and magnetic instrument development. From 1979 to 1981 he was a National Research Council postdoctoral research associate at NIST. Dr. Goldfarb is a member of the American Physical Society and the American Society for Testing and Materials.