A THEORY OF BRANCHES FOR ALGEBRAIC CURVES

TRISTRAM DE PIRO

Abstract. This paper develops some of the methods of the "Italian School" of algebraic geometry in the context of infinitesimals. The results of this paper have no claim to originality, they can be found in [10], we have only made the arguments acceptable by modern standards. However, as the question of rigor was the main criticism of their approach, this is still a useful project. The results are limited to algebraic curves. As well as being interesting in their own right, it is hoped that these may also help the reader to appreciate their sophisticated approach to algebraic surfaces and an understanding of singularities. The constructions are also relevant to current research in Zariski structures, which have played a major role both in model theoretic applications to diophantine geometry and in recent work on non-commutative geometry.

1. Introduction, Preliminary Definitions, Lemmas and Notation

We begin this section with the preliminary reminder to the reader that the following results are concerned with algebraic curves. However, the constructions involved are geometric and rely heavily on the techniques of Zariski structures, originally developed in [12] and [3]. One might, therefore, speculate that the results could, in themselves, be used to develop further the general theory of such structures. Our starting point is the main Theorem 17.1 of [12], also formulated for Zariski geometries in [3];

Theorem 1.1. Main Theorem 17.1 of [12]

Let $M$ be a Zariski structure and $C$ a presmooth Zariski curve in $M$. If $C$ is non-linear, then there exists a nonconstant continuous map:

$$f : C \to P^1(K)$$

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Moreover, $f$ is a finite map ($f^{-1}(x)$ is finite for every $x \in C$), and for any $n$, for any definable $S \subseteq C^n$, the image $f(S)$ is a constructible subset (in the sense of algebraic geometry) of $[P^1(K)]^n$.

Remarks 1.2. Here, $K$ denotes an algebraically closed field, we refer the reader to the original paper for the remaining terminology of the Theorem.

Using this theorem, one can already see that there exists a close connection between a geometric theory of algebraic curves and Zariski curves. We begin this section by pointing out some of the main obstacles to developing the results of this paper in the context of Zariski curves, leaving the resolution of the main technical problems for another occasion. This discussion continues up to $(††)$, when we introduce the main notation and preliminary lemmas of the paper.

In Section 2, the first major obstacle that we encounter is a suitable generalisation of the notion of a linear system. Using Theorem 1.1, any linear system $\Sigma$ of algebraic hypersurfaces on $[P^1(K)]^n$ will define a linear system of Zariski hypersurfaces on $C^n$, by composing with the finite cover $f(*)$. One would then expect to be able to develop much of the theory of $g^n_\alpha$ given in Section 2 for such systems, applied to a Zariski curve $S \subseteq C^n$. This follows from the following observations. First, there exists a generalised Bezout’s theorem, holding in Zariski structures, see the paper [12], hence one might hope to obtain an analogous result to Theorem 2.3. Secondly, one would expect that the local calculations, by algebraic power series, which we used in Lemma 2.10, would transfer to intersections on $S$. This uses the fact that, at a point where $f$ is unramified, $S$ and the algebraic curve $f(S)$ are locally isomorphic, in the sense of infinitesimal neighborhoods, and, at a point $p$ where $f$ is ramified, with multiplicity $r$, we have the straightforward relation;

$$I_{\text{italian}}^{(\Sigma)}(p, S, f^*(\phi_\lambda)) = rI_{\text{italian}}^{(\Sigma)}(f(p), f(S), \phi_\lambda)$$

where I have used the notation of Theorem 2.3, $\phi_\lambda$ denotes a hypersurface in $[P^1(K)]^n$ and $f^*(\phi_\lambda)$ denotes its inverse image in $S^n$.

In Section 3, the notion of a multiple point is introduced. As this is defined locally, one would expect this definition to be generalisable to Zariski curves. Also, the geometric notion of 2 algebraic curves
being biunivocal has a natural generalisation to Zariski curves. However, the main result of Section 3, that any algebraic curve is birational (biunivocal) to an algebraic curve without multiple points, is not so easily transferred. This follows from the simple observation that there exist Zariski curves $S$, which cannot be embedded in a projective space $P^n(K)$, see Section 10 of [3]. This failure of Theorem 3.3 could be explained alternatively, by noting that the combinatorics involved must fail for multiplicity calculations using Zariski curves. The solution to this problem seems to be quite difficult and one must presumably attempt to resolve it by introducing a larger class of hypersurfaces than that defined by a linear system $\Sigma$, as in (*) above. See also the remarks below.

In Section 4, the method of Conic projections is not immediately transferable to Zariski structures, as one needs to define what is meant by a line in this context. However, we note the following geometric property of a line in relation to other algebraic curves;

**Theorem 1.3.** (Luroth) Let $f : l \rightarrow C$ be a finite morphism of a line $l$ onto a projective algebraic curve $C$. Then $f$ is biunivocal.

*Proof.* See [2].

This theorem in fact characterises a line $l$ up to birationality. Its proof requires a global topological property of the line, namely that the genus of $l$ is zero. Although the above property may be formulated for Zariski curves, it does not guarantee the existence of such a curve $S \subseteq (C^n)^{eq}$, which is not trivially biunivocal to an algebraic line. One would expect the solution of this problem to require more advanced techniques, such as a geometric definition of the genus of a Zariski curve. Severi, in fact, gives such a definition for algebraic curves in [10] and one might hope that his definition would generalise to Zariski curves. One could then hope to extend the methods of Conic projections in this context.

The results of Section 5 rely centrally on the main result of Section 3, hence their generalisation to Zariski curves require a resolution of the problems noted above. One should observe that the Italian geometers definition of a branch is very different to a local definition using algebraic power series, see also the remarks later in this section, hence cannot be straightforwardly generalised using Theorem 1.1. Given a Zariski curve $S \subseteq C^n$ and $p \in S$, let $\{\gamma_1^p, \ldots, \gamma_n^p\}$ enumerate
the branches of \( f(p) \in f(S) \). We can then define;

\[
\gamma^j_p = \{ x \in S \cap V_p : f(x) \in \gamma^j_p \}
\]

In the case when \( f \) is unramified (in the sense of Zariski structures) at \( p \), this would give an adequate definition of a branch \( \gamma^j_p \) for the Zariski curve \( C \). However, the definition is clearly inadequate when \( p \) is a point of ramification and requires a generalisation of the methods of Section 5.

The results of Section 6 depend mainly on Cayley’s classification of singularities. As the main technical tool used in the proof of this result is the method of algebraic power series, by using Theorem 1.1, one would expect the results given in this Section to generalise more easily to Zariski curves.

\((\uparrow\uparrow)\) We will work in the language \( \mathcal{L}_{\text{spec}} \), as defined in [7]. \( P(L) \) will denote \( \bigcup_{n \geq 1} P^n(L) \) where \( L \) is an algebraically closed field. Unless otherwise stated, we will assume that the field has characteristic 0, this is to avoid problems concerning Frobenius. The results that we prove in this paper hold in arbitrary characteristic, if we avoid exceptional cases, however we need to make certain modifications to the proof. We will discuss these modifications in the final section of the paper.

We assume the existence of a universal specialisation \( P(K) \to \pi P(L) \) where \( K \) is algebraically closed and \( L \subset K \). Given \( l \in L \), we will denote its infinitesimal neighborhood by \( V_l \), that is \( \pi^{-1}(l) \). As we noted in the paper [7], it is not strictly necessary to consider a universal specialisation when defining the non-standard intersection multiplicity of curves, one need only consider a prime model of the theory \( T_{\text{spec}} \). However, some of our proofs will require more refined infinitesimal arguments which are not first order in the language \( \mathcal{L}_{\text{spec}} \) and therefore cannot immediately be transferred to a prime model. We will only refer to the non-standard model when using infinitesimal arguments. We assume the reader is familiar with the arguments employed in the papers [5], [6] and [12]. Of particular importance are the following notions:

(i). The technique of Zariski structures. We assume that \( P^n(L) \) may be considered as a Zariski structure in the topology induced by algebraically closed subvarieties. When referring to the dimension of an algebraically closed subvariety \( V \), we will use the model theoretic
definition as given in the paper [5]. We will assume the reader is acquainted with the notions surrounding the meaning of "generic" in this context.

(ii). The method of algebraic power series and their relation to infinitesimals. This technique was explored extensively in the papers [5] and [6]. In the following paper, we will use power series methods to parametrise the branches of algebraic curves without being overly rigorous. By a branch in this context, we refer to the "Newtonian definition" rather than the one used by the Italian school of algebraic geometry, which is the subject of this paper.

(iii). The non-standard statement and proof of Bezout’s theorem for projective algebraic curves in \( P^2(L) \). This was given in the paper [6].

We will assume that \( L \) has infinite transcendence degree and therefore has the property;

Given any subfield \( L_0 \prec L \) of finite transcendence degree and an integer \( n \geq 1 \), we can find \( \tilde{a}_n \in P^n(L) \) which is generic over \( L_0 \). (*)

We refer the reader to (i) above for the relevant definition of generic. In general, when referring to a generic point, we will mean generic with respect to some algebraically closed field of finite transcendence degree. This field will be the algebraic closure of the parameters defining any algebraic object given in the specific context.

By a projective algebraic curve, we will mean a closed irreducible algebraic subvariety \( C \) of \( P^n(L) \) for some \( n \geq 1 \) having dimension 1. Occasionally, we will consider the case when \( C \) has distinct irreducible components \( \{C_1, \ldots, C_r\} \), which will be made clear in a given situation. In both cases, we need only consider the usual Zariski topology on \( P^n(L) \) as given in (i) above. By a plane projective algebraic curve, we will mean a projective algebraic curve contained in \( P^2(L) \). By a projective line \( l \) in \( P^n \), we will mean a projective algebraic curve isomorphic to \( P^1(L) \). Any distinct points \( \{p_1, p_2\} \) in \( P^n(L) \) determine a unique projective line denoted by \( l_{p_1p_2} \). We will call the line generic if there exists a generic pair \( \{p_1, p_2\} \) determining it. We occasionally assume the existence of the closed algebraic variety \( I \subset (P^n \times P^n) \setminus \Delta \times P^n \), which parametrises the family of lines in \( P^n \), defined by;
\[ I(a, b, y) \equiv y \in I_{ab} \]

In order to see that this does define a closed algebraic variety, take the standard open cover \( \{ U_i := (X_i \neq 0) : 0 \leq i \leq n \} \) of \( P^n \). Then observe that \( I \cap ((U_i \times U_j) \setminus \Delta \times P^n) \) is locally trivialisable by the maps;

\[ \Theta_{ij} : (U_i \times U_j) \setminus \Delta \times P^1 \to I \cap (U_i \times U_j \times P^n) \]

\[ \Theta_{ij}(a, b, [t_0 : t_1]) = [t_0^{a_1} + t_1^{b_1} : \ldots : t_0^{a_i} + t_1^{b_i} : \ldots : t_0^{a_n} + t_1^{b_n}] \]

and the transition functions \( \Theta_{ijkl} = \Theta_{ij} \circ \Theta_{kl}^{-1} \) are algebraic. In general, we will leave the reader to check in the course of the paper that certain naturally defined algebraic varieties are in fact algebraic. By a projective plane \( P \) in \( P^n \), we will mean a closed irreducible projective subvariety of \( P^n \), isomorphic to \( P^r(L) \), for some \( 0 \leq r \leq n \). We did, however, use a special notation for a plane projective curve, as defined above. Any sequence of points \( \bar{a} \) determines a unique projective plane \( P_{\bar{a}} \), defined as the intersection of all planes containing \( \bar{a} \). We will call a sequence \( \{ p_0, \ldots, p_r \} \) linearly independent if \( P_{p_0, \ldots, p_r} \) is isomorphic to \( P^r(L) \). As before, if \( U \subset (P^n)^{r+1} \) defines the open subset of linearly independent elements, we can define the cover \( I \subset U \times P^n \), which parametrises the family of \( r \)-dimensional planes in \( P^n \);

\[ I(p_0, \ldots, p_r, y) \equiv y \in P_{p_1, \ldots, p_r} \]

As before, one can see that this is a closed projective algebraic subvariety of \( U \times P^n \).

By a non-singular projective algebraic curve, we mean a projective algebraic curve \( C \subset P^n \) which is non-singular in the sense of [2] (p32). Given any point \( p \in C \), we then define its tangent line \( l_p \) as follows;

By Theorem 8.17 of [2], there exist homogeneous polynomials \( \{ G_1, \ldots, G_{n-1} \} \) such that \( C \) is defined in an affine open neighborhood \( U \) of \( p \) in \( P^n \) by the homogenous ideal \( J = \langle G_1, \ldots, G_{n-1} \rangle \) (\( \ast \)). Let \( dG_j = \frac{\partial G_j}{\partial X_0} X_0 + \ldots + \frac{\partial G_j}{\partial X_n} X_n \) be the differential of \( G_j \). Then \( dG_j \) defines a family of hyperplanes, parametrised by an open neighborhood of \( p \) in \( C \). If \( x_j = \frac{X_j}{X_0} \) is a choice of affine coordinate system containing \( p \) and
are given another choice of \( \{2\} \), we can find a matrix (calculation, using the fact that;)

\[
G^\text{res}_j(x_1, \ldots, x_n) = \frac{G_i}{X_0^{\deg(G_i)}},
\]

and \( J^\text{res} = \langle G^\text{res}_1, \ldots, G^\text{res}_{n-1} \rangle \). Then \( dG^\text{res}_j(p) \) defines a family of affine hyperplanes, parametrised by an open neighborhood of \( p \) in \( C \). We claim that \( dG^\text{res}_j(p) = dG_j(p)^\text{res} \), this follows by an easy algebraic calculation, using the fact that;

\[
\frac{\partial G_i}{\partial x_0}(p_0) + \ldots + \frac{\partial G_i}{\partial x_n}(p_0) = 0
\]

The differentials \( \{dG^\text{res}_1, \ldots, dG^\text{res}_{n-1}\} \) are independent at \( p \), by the same Theorem 8.17 of \([2]\), hence \( \bigcap_{1 \leq j \leq n-1} dG_j(p) \), defines a line \( l_p \subset P^n \), which we call the tangent line. We need to show that the definition is independent of the choice of \( \{G_1, \ldots, G_{n-1}\} \). Suppose that we are given another choice \( \{H_1, \ldots, H_{n-1}\} \). Again, using Theorem 8.17 of \([2]\), we can find a matrix \( (f_{ij})_{1 \leq i, j \leq n-1} \) of polynomials in \( R(U) \), for some affine open neighborhood \( U \) of \( p \) in \( P^n \) (**), such that the matrix of values \( (f_{ij}(p))_{1 \leq i, j \leq n-1} \) is invertible and;

\[
H^\text{res}_i = \sum_{1 \leq j \leq n-1} f_{ij} G^\text{res}_j \pmod{(J^\text{res}(U))^2}
\]

Then, by properties of differentials, and the fact that, for \( g \in (J^\text{res}(U))^2 \), \( dg(p) \equiv 0 \), we have;

\[
dH^\text{res}_i(p) = \sum_{1 \leq j \leq n-1} f_{ij}(p) dG^\text{res}_j(p)
\]

This implies that \( \bigcap_{1 \leq j \leq n-1} dG_j(p) = \bigcap_{1 \leq j \leq n-1} dH_j(p) \) as required.

We define the tangent variety \( Tang(C) \) to be \( \bigcup_{x \in C} l_x \). We claim that this is a closed projective subvariety of \( P^n \). In order to see this, using the notation of the above argument, let \( \{U_i\} \) be an affine cover of \( P^n \) and let \( \{G_{i1}, \ldots, G_{ij}, \ldots, G_{in-1}\} \) be homogeneous polynomials with the properties (*) and (**) given above. Let \( W \subset P^n \times P^n \) be the closed projective variety, defined on \( P^n \times U_i \) by;

\[
W_i(\tilde{X}, \tilde{Y}) \equiv U_i(\tilde{Y}) \wedge \bigwedge_{1 \leq j \leq n-1} G_{ij}(\tilde{Y}) = 0 \wedge \bigwedge_{1 \leq j \leq n-1} dG_{ij}(\tilde{Y}), \tilde{X} = 0
\]

Then, by completeness, the variety \( V(\tilde{X}) \equiv (\exists \tilde{Y}) W(\tilde{X}, \tilde{Y}) \) is a closed projective variety of \( P^n \). The above argument shows that \( \tilde{V} = Tang(C) \).

We will require a more sophisticated notation when considering hypersurfaces \( H \) of \( P^n(L) \) for \( n \geq 1 \). Namely, by a hypersurface of degree
we will mean a homogenous polynomial \( F(X_0, X_1, \ldots, X_n) \) of degree \( m \) in the variables \([X_0 : X_1 : \ldots : X_n]\). In the case when \( F \) is irreducible or has distinct irreducible factors \( \{F_1, \ldots, F_r\} \), such a hypersurface may be considered as the union of \( r \) distinct irreducible closed algebraic subvarieties of codimension 1 in \( P^n(L) \). We will then refer to the hypersurface as having distinct irreducible components. Again, we can understand such hypersurfaces using the usual Zariski topology as given in (i) above. By a hyperplane, we will mean an irreducible hypersurface isomorphic to \( P^{n-1}(L) \). Equivalently, a hyperplane is defined by a homogeneous polynomial of degree 1. In general, let \( F = F_1^{n_1} \cdots F_r^{n_r} \) be the factorisation of \( F \) into irreducibles for \( 1 \leq j \leq r \). We will want to take into account the "non-reduced" character of \( F \) if some \( n_j \geq 2 \). Therefore, given an irreducible homogeneous polynomial \( G \) of degree \( m \) and an integer \( s \geq 1 \), we will refer to \( G^s \) as an \( s \)-fold component of degree \( ms \). Geometrically, we interpret \( G^s \) as follows;

Consider the space of all homogenous polynomials of degree \( ms \) parametrised by \( P^N(L) \). Let \( W \subset P^N \times P^n \) be the irreducible projective variety defined by \( W(\bar{x}, \bar{y}) \) iff \( y \in \text{Zero}(F_{\bar{x}}) \), where \( F_{\bar{x}} \) is the homogenous polynomial (defined uniquely up to scalars) by the parameter \( \bar{x} \). The coefficients of the homogeneous polynomial \( G^s \) determine uniquely an element \( \bar{a} \) in \( P^N \) and we can consider \( G^s \) as the fibre \( W(\bar{a}) \subset P^n \). In the Zariski topology, this consists of the variety defined by the irreducible polynomial \( G \). However, in the language \( L_{\text{spec}} \), we can realise its non-reduced nature using the following lemmas;

**Lemma 1.4.** Let \( S \subset P^n \) be an irreducible hypersurface and \( C \) a projective algebraic curve, then, if \( C \) is not contained in \( S \), \( S \cap C \) consists of a finite non-empty set of points.

*Proof.* The proof follows immediately from the Projective Dimension Theorem, see for example [2]. \( \square \)

**Lemma 1.5.** Let \( G = 0 \) define an irreducible hypersurface of degree \( m \). Let \( l \) be a generic line, then \( l \) intersects \( G = 0 \) in precisely \( m \) points.

*Proof.* Let \( Par_l = P^n \times P^n \setminus \Delta \) be the parameter space for lines in \( P^n \) as defined above. Let \( Par_m \) be the projective parameter space for all homogeneous forms of degree \( m \). Then we can form the variety \( W \subset Par_l \times Par_m \times P^n \) given by;
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$W(l, x, y) \iff y \in l \cap \text{Zero}(G_x),$

where $G_x$ is the homogenous polynomial corresponding to the parameter $x$. If $l$ is chosen to be generic over the parameters defining $G_x$, then $l \cap \text{Zero}(G_x)$ is finite. It follows that there exists an open subset $U \subset \text{Par}_l \times \text{Par}_m$ consisting of parameters $\{l, x\}$ such that $l \cap G_x$ has finite intersection. As $U$ is smooth, the finite cover $W$ restricted to $U$ is equidimensional and we may apply Zariski structure arguments, as was done in [6]. Considering $W$ as a finite cover of $U$, if $y \in l \cap \text{Zero}(G_x)$, we define $\text{Mult}_y(l, G_x)$ to be $\text{Mult}(l, x, y)(W/U)$ in the sense of Zariski structures. Using the notation in [6], we can also define $\text{LeftMult}_y(l, G_x)$ and $\text{RightMult}_y(l, G_x)$. In more geometric language;

$\text{LeftMult}_y(l, G_x) = \text{Card}(V_y \cap l' \cap \text{Zero}(G_x))$

where $l'$ is a generic infinitesimal variation of $l$ in the nonstandard model $P(K)$.

$\text{RightMult}_y(l, G_x) = \text{Card}(V_y \cap l \cap \text{Zero}(G_{x'}))$

where $x' \in V_x$ is generic in $\text{Par}_m$, considered as a variety in the non-standard model $P(K)$.

We now claim that there exists a line $l$ with exactly $m$ points of intersection with $G_{x_0} = G$. If $G_{x_0}$ defines a projective algebraic curve in $P^2(L)$, the result follows immediately from an application of Bezout’s theorem and the fact that there exists a line $l$ having only (algebraically) transverse intersections with $G_{x_0}$. This result was shown in [6]. Otherwise, we obtain the case that $G_{x_0}$ defines a plane projective curve by repeated application of Bertini’s Theorem for Hyperplane Sections, see [4]. Namely, we can find a 2-dimensional plane $P$ such that $P \cap G_{x_0}$ defines a plane projective algebraic curve $C_{x_0}$. We may assume that such a plane $P$ is determined by a generic triple $\{a, b, c\}$. We have an isomorphism $\theta : P^2(L) \to P$ given by;

$\theta([Y_0, Y_1, Y_2]) = [\frac{a_0}{a_i}Y_0 + \frac{b_0}{b_i}Y_1 + \frac{c_0}{c_i}Y_2 : \ldots : \frac{a_0}{a_i}Y_0 + \frac{b_0}{b_i}Y_1 + \frac{c_0}{c_i}Y_2]$

If the form $G_{x_0}$ is given by $\sum_{i_0 + \ldots + i_n = m} x_0^{i_0} \ldots x_n^{i_n} X_0^{i_0} \ldots X_n^{i_n}$, then the corresponding curve $C_{x_0}$ is given in coordinates $\{Y_0, Y_1, Y_2\}$ by;
where $F_{j_0j_1j_2}$ is an algebraic function of $\{a, b, c, \bar{x}\}$ and, in particular, linear in the variables $\{\bar{x}\}$. By the assumption that $C_{x_0}$ does not vanish identically on $P$, we must have that $F_{j_0j_1j_2}$ is not identically zero for some $(j_0j_1j_2)$ with $j_0 + j_1 + j_2 = m$. It follows that $C_{x_0}$ defines a plane projective curve of degree $m$. Now, by the above argument, we may find a line $l_0 \subset P$ having exactly $m$ intersections with $C_{x_0}$. Therefore, $l_0$ has exactly $m$ intersections with $G_{x_0}$ as well.

Now consider the fibre $U(x_0) = \{l : l \cap G_{x_0} \text{ is finite} \}$ and restrict the cover $W$ to $U(x_0)$. By the above calculation, we have that:

$$\sum_{y \in l_0 \cap G_{x_0}} \text{Mult}_{l_0,y}(W/U(x_0)) = \sum_{y \in l_0 \cap G_{x_0}} \text{LeftMult}_{y}(l_0, G_{x_0}) \geq m \ (*)$$

By elementary properties of Zariski structures, if $l$ is chosen to be generic over the parameters defining $G_{x_0}$, then $(\ast)$ holds, with $l$ replacing $l_0$, and, moreover, $\text{LeftMult}_{y}(l, G_{x_0}) = 1$ for each intersection $y \in l \cap G_{x_0}$. This implies that $l$ has at least $m$ points of intersection with $G_{x_0} = G$. In order to obtain equality, we now consider the fibre $U(l) = \{x : l \cap G_x \text{ is finite} \}$ and restrict the cover $W$ to $U(l)$. It will follow from the result given in the next section, the Hyperspatial Bezout Theorem, that, for any $x \in U(l)$:

$$\sum_{y \in l_0 \cap G_{x}} \text{Mult}_{l_0,y}(W/U(l)) = \sum_{y \in l \cap G_{x}} \text{RightMult}_{y}(l, G_{x}) = \text{deg}(l)\text{deg}(G_{x})$$

where, for a projective algebraic curve, degree is given by Definition 1.12. We clearly have that $\text{deg}(l) = 1$ and, by assumption, that $\text{deg}(G_{x_0}) = m$. Hence, we must have that $l$ intersects $G_{x_0}$ in exactly $m$ points and, moreover, $\text{RightMult}_{y}(l, G_{x_0}) = 1$ for each intersection $y \in l \cap G_{x_0}$ as well.

The lemma is now proved but we can give an algebraic formulation of the result. Namely, if $l$ is determined by the generic pair $\{a, b\}$, then we have an isomorphism $\theta : P^1(L) \rightarrow l$ given by:

$$\theta([Y_0, Y_1]) = \left[ \frac{a_0}{a_1} Y_0 + \frac{b_0}{b_1} Y_1 : \ldots : \frac{a_m}{a_i} Y_0 + \frac{b_m}{b_i} Y_1 \right]$$

If the form $G_{x_0}$ is given by $\sum_{i_0+\ldots+i_n=m} x_{i_0} \ldots x_{i_n} X_{i_0} \ldots X_{i_n}$, then the equation of $G_{x_0}$ on $l$ is given in coordinates $\{Y_0, Y_1\}$ by the homogenous
polynomial;

\[ P_{x_0}(Y_0, Y_1) = \sum_{j_0+j_1=m} F_{j_0 j_1}(a, b, \bar{x}_0) Y_0^{j_0} Y_1^{j_1} \]

We clearly have, by the same reasoning as above, that \( P_{x_0} \) has degree \( m \). The result of the lemma gives that \( P_{x_0} \) has exactly \( m \) roots in \( P^1(L) \). Hence, these roots are all distinct. It follows that the scheme theoretic intersection of \( l \) and \( G = 0 \) consists of \( m \) distinct reduced points. That is \( l \) intersects \( G = 0 \) algebraically transversely.

\[ \square \]

We now extend the lemma to include reducible varieties. As before \( G \) is an irreducible algebraic form of degree \( m \) and we consider the power \( G^s \) for some \( s \geq 1 \). We will say that \( y \in l \cap G^s = 0 \) is counted \( r \)-times if \( \text{RightMult}_y(l, G^s) = r \) in the sense defined above.

**Lemma 1.6.** Let \( G = 0 \) define an irreducible hypersurface of degree \( m \) and let \( s \geq 1 \). Let \( l \) be a generic line, then \( l \) intersects \( G^s = 0 \) in \( m \) points each counted \( s \)-times.

**Proof.** We first show that \( l \) is generic with respect to \( G = 0 \). Let \( \lambda = \{\lambda_i\} \) be the parameters defining \( G = 0 \) and let \( \mu = \{\mu_j\} \) be the parameters defining \( G^s = 0 \). We claim that \( \lambda \) and \( \mu \) are interdefinable, considered as elements of the projective spaces \( \text{Par}_m \) and \( \text{Par}_{ms} \), in the structure \( P(L) \), which was considered in \([5]\). The fact that \( \mu \in \text{dcl}(\lambda) \) is clear. Conversely, let \( \alpha \) be an automorphism fixing \( \mu \) and let \( G_{\alpha(\lambda)} \) be the algebraic form of degree \( m \) obtained from \( G = G_\lambda \). As \( (G_\lambda)^s = G_\mu \), we have that \( (G_{\alpha(\lambda)})^s = G_\mu \). Hence, \( \text{Zero}(G_{\alpha(\lambda)}) = \text{Zero}(G_\lambda) \). By the projective Nullstellensatz, and the fact that both algebraic forms are irreducible, we must have that \( \langle G_\lambda \rangle \leq \langle G_{\alpha(\lambda)} \rangle \). Hence, there must exist a unit \( U \) in the ring \( L[X_0, \ldots, X_n] \) such that \( G_\lambda = U G_{\alpha(\lambda)} \). As the only such units are scalars, we obtain immediately that \( \alpha(\lambda) = \lambda \) in \( \text{Par}_{ms} \). As \( P(L) \) is sufficiently saturated, we obtain that \( \lambda \in \text{dcl}(\mu) \).

By the previous lemma, we obtain that \( l \) intersects \( G = 0 \) in exactly \( m \) points, hence \( l \) intersects \( G^s = 0 \) in exactly \( m \) points as well. Therefore, the first part of the lemma is shown. We now apply the Hyperspatial Bezout Theorem, see below, to obtain that:

\[ \sum_{y \in l \cap G^s = 0} \text{RightMult}_y(l, G^s) = \deg(l) \deg(G^s) = ms \]
Hence, the lemma is shown by proving that for any \( y \in l \cap G^s = 0 \), \( \text{RightMult}_y(l, G^s) \geq s \). As before, we choose a generic plane \( P \) containing \( l \), such that \( C_{x_0} = P \cap G_{x_0} \) defines a projective algebraic curve of degree \( m \). Using an explicit presentation of an isomorphism \( \theta : \mathbb{P}^2(L) \to L \), as was done above, we clearly have that the intersection \( (P \cap (G_{x_0})^s = 0) \) consists of the non-reduced curve \( C_{x_0}^s = 0 \), which has degree \( ms \). We now apply a result from the paper \([6]\),(Lemma 4.16), which gives that, for \( y \in l \cap G_{x_0} = 0 \);

\[
\text{RightMult}_y(l, C_{x_0}^s, \text{Par}_{Q_{ms}}) = s \cdot \text{RightMult}_y(l, C_{x_0}, \text{Par}_{Q_{ms}}) \geq s
\]

where we have use the fact that the parameter defining \( C_{x_0}^s \) moves in the projective parameter space \( Q_{ms} \) defining plane projective curves of degree \( ms \) in \( \mathbb{P}^2(L) \). We now claim that, for \( y \in l \cap G_{x_0} = 0 \);

\[
\text{RightMult}_y(l, G_{x_0}^s) \geq \text{RightMult}_y(l, C_{x_0}^s, \text{Par}_{Q_{ms}}) \quad (*)
\]

Recall that, given \( G_x \) an algebraic form of degree \( m \), the restriction of \( G_x \) to the plane \( P_{abc} \) is given by the formula;

\[
C_x(Y_0, Y_1, Y_2) = \sum_{j_0+j_1+j_2=m} F_{j_0j_1j_2}(a, b, c, \bar{x})Y_0^{j_0}Y_1^{j_1}Y_2^{j_2}
\]

We first claim that each \( F_{j_0j_1j_2} \) is not identically zero \((**\)) Let \( H_x \) be a hyperplane given in coordinates by;

\[
H_x = \sum_{r=0}^n x_iX_i = 0
\]

Then the restriction of \( H_x \) to \( P_{abc} \) is given by the plane;

\[
P_x = (\sum_{r=0}^n x_0^r a_r)Y_0 + (\sum_{r=0}^n x_1^r b_r)Y_1 + (\sum_{r=0}^n x_2^r c_r)Y_2
\]

By elementary linear algebra, we can find hyperplanes \( \{H_0, H_1, H_2\} \) whose restriction to \( P_{abc} \) define the planes \( \{Y_0 = 0, Y_1 = 0, Y_2 = 0 \} \). It follows by direct calculation that the algebraic form of degree \( m \) defined by \( H_0^0 H_1^1 H_2^2 \) restricts to the curve of degree \( m \) defined by \( Y_0^{j_0} Y_1^{j_1} Y_2^{j_2} = 0 \). Hence, \((**\)) is shown. Now consider the function;

\[
\mathcal{F} = \{F_{j_0j_1j_2} : \text{Par}_m \to \text{Par}_{Q_m}\}
\]

By earlier remarks, the algebraic function \( \mathcal{F} \) is linear in the variables \( \{x_i\} \) and defined over \( \{a, b, c\} \). Hence, its image defines a plane
P \subset Par_{Q_m}. By the above calculation, this plane P contains the linearly independent set \( \{ p_{j_0j_1j_2} : j_0 + j_1 + j_2 = m \} \) where \( C_{p_{j_0j_1j_2}} \) defines the curve \( Y_0^j Y_1^i Y_2^j = 0 \). Hence, \( F \) is surjective. Moreover, by elementary facts about linear maps, the fibres of \( F \) are equidimensional. We can then show (\( \ast \)). Suppose that;

\[
RightMult_y(l, C_{x_0}^s, Par_{Q_m}) = k
\]

Then one can find \( x' \in V_{x_1} \cap Par_{Q_m}, \) generic over the parameter \( x_1 \) defining \( C_{x_0}^s \), such that \( C_{x'} \) intersects \( l \) in the distinct points \( \{ y_1, \ldots, y_k \} \subset V_y \). As \( x_1 \) is regular for the cover \( \overline{F} \); if \( x_2 \) defines \( G_{x_0}^s \), we can find \( x'' \in V_{x_2} \cap Par_{Q_m} \), such that \( \overline{F}(x'') = x' \). The algebraic form defined by \( G_{x''} \) then intersects the plane \( P_{abc} \) in the curve \( C_{x'} \), hence it must intersect the line \( l \) in the distinct points \( \{ y_1, \ldots, y_k \} \subset V_y \) as well. This implies that;

\[
RightMult_y(l, G_{x_0}^s) \geq k
\]

Hence (\( \ast \)) is shown. The lemma is then proved. In this lemma we have not shown anything interesting algebraically. Namely, if one considers the restriction of \( G_{x_0}^s \) to \( l \), we obtain the homogeneous polynomial \( P_{x_0}^s \). By the previous lemma, \( P_{x_0} \) has \( m \) distinct roots in \( P^1(L) \), hence \( P_{x_0}^s \) has \( m \) distinct roots with multiplicity \( s \). Therefore, the scheme theoretic intersection of \( l \) with \( G_{x_0}^s \) consists of \( m \) distinct copies of the non-reduced scheme \( L[t]/(t)^s \). The usefulness of the result will be shown in the following lemmas.

\[\square\]

\textbf{Remarks 1.7.} Note that the latter part of the argument in fact shows that, for any line intersecting \( G_{x_0}^s \) in finitely many points, we must have that each point of intersection \( y \) is counted at least \( s \)-times.

\textbf{Lemma 1.8.} Let \( F = 0 \) define a hypersurface of degree \( k \). Let \( F = F_1^{m_1} \cdot \ldots \cdot F_j^{m_j} \cdot \ldots \cdot F_r^{m_r} \) be its factorisation into irreducibles, with degree(\( F_j \)) = \( m_j \). Then there exists a line \( l \), intersecting each component \( F_j \) in exactly \( m_j \) points, each counted \( n_j \) times, with the property that the sets \( \{ (F_j \cap l) : 1 \leq j \leq r \} \) are pairwise disjoint. Moreover, the set of lines having this property form a Zariski open subset of \( Par_1 \), defined over the parameters of \( F = 0 \).

\textbf{Proof.} Let \( (x_1, \ldots, x_r) \in Par_{m_1} \times \ldots \times Par_{m_r} \) be the tuple defining each reduced irreducible component of \( F = 0 \). By an elementary argument, extending the proof in Lemma 1.6, the tuple is interalgebraic with the tuple \( x \in Par_k \) defining the hypersurface \( F = 0 \). Let
\(\theta_{(x_j,m_j,n_j)}(y) \subset Par_l\) be the statement that a line \(l_y\) intersects \(F_j\) in exactly \(m_j\) points, each counted \(n_j\) times;

\[
\exists z_1 \neq ... \neq z_{m_j} \left[ \bigwedge_{1 \leq i \leq m_j} z_i \in l_y \cap F_{j,x_j} \land \text{RightMult}_{z_i}(l_y,F_{j,x_j}) = n_j \land \forall w(w \in l_y \cap F_{j,x_j} \rightarrow \bigvee_{1 \leq i \leq m_j} z_i = w) \right]
\]

By definability of multiplicity in Zariski structures, Lemmas 1.5 and Lemmas 1.6, each \(\theta_{(x_j,m_j,n_j)}(y)\) is definable over \(x_j\) and is a Zariski dense algebraic subset of \(Par_l\). By the previous remark, the complement of \(\theta_{(x_j,m_j,n_j)}(y)\) in the set of lines having finite intersection with \(F_j\) is given by;

\[
\exists w\left[ w \in l_y \cap F_{j,x_j} \land \text{RightMult}_w(l_y,F_{j,x_j}) \geq n_j + 1 \right]
\]

It follows that \(\theta_{(x_j,m_j,n_j)}(y)\) defines a Zariski open subset of \(Par_l\). Now let;

\[
\theta(y) = \bigwedge_{1 \leq j \leq r} \theta_{(x_j,m_j,n_j)}(y)
\]

Then \(\theta(y)\) defines a Zariski open subset of \(Par_l\). Finally, let \(W\) be the union of the pairwise intersections of the irreducible components \(F_j\). Then, by elementary dimension theory, \(W\) is Zariski closed of dimension at most \(n - 2\). Hence, the condition on \(Par_l\) that a line passes through \(W\) defines a proper closed set over the parameters \((x_1, \ldots, x_r)\). Let \(U(y)\) be the Zariski open complement of this set in \(Par_l\). Then any line \(l\) satisfying \(\theta(y) \land U(y)\) has the properties required of the lemma.

**Definition 1.9.** We will call a line \(l\) satisfying the conclusion of the lemma transverse to \(F\).

We will now give an alternative characterisation of transversality.

**Lemma 1.10.** Let \(F = 0\) define a hypersurface of degree \(k\). Then a line \(l\) is transverse to \(F\) iff \(l\) intersects \(F\) in finitely many points and, for each \(y \in l \cap (F = 0)\), \(\text{LeftMult}_y(l,F) = 1\). Moreover, the notion of transversality may be formulated by a predicate in the language \(L_{\text{spec}}\), \(\text{Transverse}_k \subset Par_k \times Par_l;\)

\(\text{Transverse}_k(\lambda,l)\) iff \(l\) is transverse to \(F_\lambda\)

**Proof.** The first part of the proof is clear using previous results of this section. For the second part, use the results in Section 3 of [7]. □
We now have;

**Lemma 1.11. A Nullstellensatz for Non-Reduced Hypersurfaces**

Let \( F = 0 \) define a hypersurface of degree \( k \) and let \( \text{Zero}(F) = \text{Zero}(F_{\lambda_1}) \cup \ldots \cup \text{Zero}(F_{\lambda_r}) \) be its geometric factorisation into irreducibles, using the Zariski topology. Let \( \sigma(\lambda_j, m_j, n_j) \subset \text{Par}_{m_j} \), for \( 1 \leq j \leq r \), be the predicates defined in \( \mathcal{L}_{\text{spec}} \) by:

A transverse line to \( F_{\lambda_j} \) intersects \( F_{\lambda_j} \) in exactly \( m_j \) points, each counted \( n_j \) times.

Then the original homogeneous polynomial \( F \) is characterised, up to scalars, by the sequence;

\[
(\text{Zero}(F_{\lambda_1}), \ldots, \text{Zero}(F_{\lambda_r}), \sigma(\lambda_1, m_1, n_1), \ldots, \sigma(\lambda_r, m_r, n_r))
\]

**Proof.** By the proof of previous results from this section, the formulae \( \sigma(\lambda_j, m_j, n_j) \), for \( 1 \leq j \leq r \), determine the multiplicity of each component \( F_{\lambda_j} \). The result then follows by uniqueness of factorisation. \( \square \)

We will refer to a hypersurface as generic if the parameter defining it is generic in the parameter space of all hypersurfaces of the same degree.

**Definition 1.12.** The degree of a projective algebraic curve \( C \) is the number of intersections with a generic hyperplane.

We need to check this is a good definition. Let \((P^n)^*\) be the dual space of \( P^n \). \((P^n)^*\) is the parameter space for all hyperplanes \( H \) in \( P^n \). We have that;

\[
\{ a \in (P^n)^* : \text{dim}(C \cap H_a) \geq 1 \}
\]

is closed in \((P^n)^*\), hence, for generic \( a \in (P^n)^* \), \( C \cap H_a \) is finite (and non-empty). Choosing some generic \( a \) in \((P^n)^*\), let \( m \) be the number of intersections of \( H_a \) with \( C \). Let \( \theta(x) \subset (P^n)^* \) be the statement;

\[
\exists x_1 \neq \ldots \neq x_m (\bigwedge_{1 \leq i \leq m} x_i \in C \cap H_x \land \forall y (y \in C \cap H_x \rightarrow \bigvee_{1 \leq i \leq m} x_i = y))
\]

\( \theta(x) \) is algebraic and defined over \( \emptyset \), hence, as it contains \( a \), must be Zariski dense in \((P^n)^*\). In particular, it contains any generic \( a \) in \((P^n)^*\).
Definition 1.13. The degree of a hypersurface $F$ is the degree of the homogenous polynomial defining it. (See the above remarks)

In the following paper, the notion of birationality between projective algebraic curves, will be central.

Definition 1.14. We define a linear system $\Sigma$ on $P^r$ to be the collection of algebraic forms of degree $k$, for some $k \geq 1$, corresponding to a plane, which we will denote by $\text{Par}_\Sigma$, contained in $\text{Par}_k$, the parameter space of homogeneous polynomials of degree $k$. If $\text{Par}_\Sigma$ has dimension $n$, we define a basis of $\Sigma$ to be an ordered set of $n+1$ forms corresponding to a maximally independent set of parameters in $\text{Par}_\Sigma$. Equivalently, a basis of $\Sigma$ is an ordered system;

\[
\{\phi_0(X_0, \ldots, X_r), \ldots, \phi_n(X_0, \ldots, X_r)\}
\]

of homogeneous polynomials of degree $k$ belonging to $\Sigma$ which are independent, that is there do not exist parameters $\{\lambda_0, \ldots, \lambda_n\}$ such that;

\[
\lambda_0\phi_0 + \ldots + \lambda_n\phi_n \equiv 0
\]

Definition 1.15. Given a linear system $\Sigma$ of dimension $n$ on $P^r$, we define the base locus of the system $\Sigma$ by;

\[
\text{Base}(\Sigma) = \{\bar{x} \in P^r : \phi_0(\bar{x}) = \ldots = \phi_n(\bar{x}) = 0\}
\]

for any basis of $\Sigma$. Given any 2 bases $\{\phi_0, \ldots, \phi_j, \ldots, \phi_n\}$ and $\{\psi_0, \ldots, \psi_i, \ldots, \psi_n\}$ of $\Sigma$, we can find an invertible matrix of scalars $(\lambda_{ij})_{0 \leq i, j \leq n}$ such that;

\[
\psi_i = \sum_{j=0}^{n} \lambda_{ij}\phi_j
\]

Hence, the base locus of $\Sigma$ is well defined. As a basis corresponds to a maximally independent sequence in $\text{Par}_\Sigma$, we could equivalently define;

\[
\text{Base}(\Sigma) = \{\bar{x} \in P^r : \phi(\bar{x}) = 0\}
\]

for every algebraic form $\phi$ belonging to $\Sigma$.

Lemma 1.16. Let $\Sigma$ be a linear system of dimension $n$ and degree $k$ on $P^r$. Then, a choice of basis $B$ for $\Sigma$ defines a morphism $\Phi_{\Sigma,B} :
$P^r \setminus \text{Base}(\Sigma) \to P^n$ with the property that Image$(\phi_{\Sigma,B})$ is not contained in any hyperplane section of $P^n$. Moreover, given any 2 bases \{B, B'\} of \Sigma, there exists a homography $\theta_{B,B'} : P^n \to P^n$ such that;

$$\Phi_{\Sigma,B'} = \theta_{B,B'} \circ \Phi_{\Sigma,B}$$

**Proof.** Given a choice of basis $B = \{\phi_0, \ldots, \phi_n\}$ for $\Sigma$, one checks that the map defined by;

$$\Phi_{\Sigma,B}([X_0 : \ldots : X_r]) = [\phi_0(X_0, \ldots, X_r) : \ldots : \phi_n(X_0, \ldots, X_r)]$$

is a morphism with the required properties. Now suppose that \{\phi_0, \ldots, \phi_j, \ldots, \phi_n\} and \{\psi_0, \ldots, \psi_i, \ldots, \psi_n\} are 2 bases $B$ and $B'$ for $\Sigma$. Let $(\lambda_{ij})_{0 \leq i,j \leq n}$ be the matrix of scalars as given in Definition 1.15. Then one can define a homography $\theta_{B,B'}$ by;

$$\theta_{B,B'}([Y_0 : \ldots : Y_n]) = [\sum_{j=0}^n \lambda_{0j} Y_j : \ldots : \sum_{j=0}^n \lambda_{ij} Y_j : \ldots : \sum_{j=0}^n \lambda_{nj} Y_j]$$

It is clear that this homography has the required property of the lemma.

□

**Definition 1.17.** We define a rational map from $P^r$ to $P^n$ to be a morphism defined by a choice of basis for a linear system $\Sigma$.

**Remarks 1.18.** Given a linear system $\Sigma$, we will generally refer to a morphism given by Lemma 1.16 as simply $\Phi_{\Sigma}$, leaving the reader to remember that a choice of basis is involved. As any 2 such choices differ by a homography, any properties of one morphism transfer directly to the other, so one hopes that this terminology will not cause confusion. More geometrically, observe that, if $x \in P^r \setminus \text{Base}(\Sigma)$, then the set of algebraic forms in $\Sigma$, vanishing at $x$, defines a hyperplane $H_x \subset \Sigma$. A choice of basis $\{\phi_0, \ldots, \phi_n\}$ for $\Sigma$ identifies this hyperplane $H_x$ with a point $[\phi_0(x), \ldots, \phi_n(x)]$ of the dual space $P^{n*}$.

**Definition 1.19.** We say that two projective algebraic curves $C_1$ and $C_2$ are birational if there exists $U \subset C_1$ and $V \subset C_2$, with $U$ and $V$ open in $C_1$ and $C_2$ respectively, such that $U$ and $V$ are isomorphic as algebraic varieties. We will use the notation $\Phi : C_1 \rightsquigarrow C_2$ for a birational map.

We will require the following presentation of birational maps;
Lemma 1.20. Let $C_1 \subset P^r$ and $C_2 \subset P^n$ be birational projective algebraic curves, as in Definition 1.16, with the property that no hyperplane section of $P^r$ or $P^n$ contains $C_1$ or $C_2$ respectively. Then we can find linear systems $\{\Sigma, \Sigma'\}$, rational maps;

$$\phi_\Sigma : P^r \setminus \text{Base}(\Sigma) \to P^n \quad \phi_\Sigma' : P^n \setminus \text{Base}(\Sigma') \to P^r$$

and open subsets $\{U', V'\}$ of $\{C_1, C_2\}$, which are disjoint from $\{\text{Base}(\Sigma), \text{Base}(\Sigma')\}$, such that the restrictions $\phi_\Sigma : U' \to V'$ and $\phi_\Sigma' : V' \to U'$ are (inverse) isomorphisms.

Proof. As usual, let $[X_0 : \ldots : X_r]$ and $[Y_0 : \ldots : Y_n]$ be homogeneous coordinates for $P^r$ and $P^n$ respectively. Taking the hyperplanes $X_0 = 0$ and $Y_0 = 0$, we can find affine presentations;

$$\frac{L[x_1, \ldots, x_r]}{J_1} = R(C_1 \setminus C_1 \cap (X_0 = 0)) = R(U'')$$

$$\frac{L[y_1, \ldots, y_n]}{J_2} = R(C_2 \setminus C_2 \cap (Y_0 = 0)) = R(V'')$$

where $\{U'', V''\}$ are open subsets of $\{C_1, C_2\}$, $\{J_1, J_2\}$ are prime ideals.

We can then find $U' \subset U'' \cap U$ and $V' \subset V'' \cap V$ such that $U'$ and $V'$ are isomorphic as algebraic subvarieties of $C_1$ and $C_2$ (consider the elements of $U$ which are mapped to $V''$ by the original isomorphism). Now choose polynomials $F(\bar{x})$ and $G(\bar{y})$ such that;

$$\frac{L[x_1, \ldots, x_r]}{J_1^{\phi}} = R(U') \quad \frac{L[y_1, \ldots, y_n]}{J_2^{\psi}} = R(V')$$

As $R(U') \cong R(V')$, we can find rational functions $\{\phi_1(\bar{x}), \ldots, \phi_n(\bar{x})\}$ and $\{\psi_1(\bar{y}), \ldots, \psi_r(\bar{y})\}$ (with denominators powers of $F$ and $G$ respectively) defining morphisms;

$$\Phi : A^r \setminus \{F = 0\} \to A^n \quad \Psi : A^n \setminus \{G = 0\} \to A^r$$

and representing the isomorphism $U' \cong V'$. We now show how to convert $\Phi$ into $\Phi_\Sigma$. By equating denominators, we are able to write $\{\phi_1(\bar{x}), \ldots, \phi_n(\bar{x})\}$ as $\{\frac{p_1(\bar{x})}{q(\bar{x})}, \ldots, \frac{p_n(\bar{x})}{q(\bar{x})}\}$. Now make the substitutions $x_i = \frac{x_i}{Q(x)}$ in $\{p_1(\bar{x}), \ldots, p_n(\bar{x}), q(\bar{x})\}$ and multiply through by the highest power of $X_0$ to obtain homogeneous polynomials of the same degree $\{P_1(\bar{X}), \ldots, P_n(\bar{X}), Q(\bar{X})\}$. Let $\Sigma$ be the linear system defined by the
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plane spanned by these homogeneous polynomials and define $\Phi_\Sigma$ by;

$$\rho Y_0 = Q(X_0, \ldots, X_r), \rho Y_1 = P_1(X_0, \ldots, X_r), \ldots, \rho Y_n = P_n(X_0, \ldots, X_r)$$

where $\rho$ is a constant of proportionality. (This is an alternative notation for a map between projective spaces, used frequently in papers by the Italian geometers Castelnouvo, Enriques and Severi). By the assumption on $\{C_1, C_2\}$ concerning hyperplane sections, the homogeneous polynomials $\{Q, P_1, \ldots, P_n\}$ form a basis for $\Sigma$, hence this defines a rational map. Similarly, one can find a linear system $\Sigma'$ and convert $\Psi$ into a rational map $\phi_{\Sigma'}$. The rest of the properties of the lemma follow immediately from the construction.

We should also note the following equivalent criteria for birationality of projective algebraic curves;

**Lemma 1.21.** Let $C_1$ and $C_2$ be projective algebraic curves. Then $C_1$ and $C_2$ are birational iff;

(i). There is an isomorphism of function fields $L(C_1) \cong L(C_2)$.

(ii). (In characteristic 0) There exist $a_1$ generic in $C_1$, $a_2$ generic in $C_2$ and an algebraic relation $\text{Rational}(x, y)$ such that $a_1 \in \text{dcl}_{\text{Rational}}(a_2)$ and $a_2 \in \text{dcl}_{\text{Rational}}(a_1)$.

(In characteristic $p$) One can use the same criteria but must pay attention to the presence of the Frobenius map, see the paper [5] for details on how to resolve this issue.

**Definition 1.22.** Let $\Phi : C_1 \leftrightarrow C_2$ be a birational map, as in Definition 1.19. We define the correspondence $\Gamma_{\Phi} \subseteq C_1 \times C_2$ associated to $\Phi$ to be;

$$(\text{Graph}(\Phi) \subseteq U \times V)$$

where, for $W$ an algebraic subset of $C_1 \times C_2$, we let $\bar{W}$ denote its Zariski closure.

**Definition 1.23.** Let $C_1$ and $C_2$ be projective algebraic curves. We will say that 2 birational maps $\Phi_1 : C_1 \leftrightarrow C_2$ and $\Phi_2 : C_1 \leftrightarrow C_2$ are equivalent if there exists $U \subseteq C_1$ such that $\Phi_1$ and $\Phi_2$ are both defined and agree on $U$. Clearly, equivalence of birational maps is an equivalence relation.
Lemma 1.24. Let $\Phi_1 : C_1 \to C_2$ and $\Phi_2 : C_1 \to C_2$ be equivalent birational maps, then $\Gamma_{\Phi_1} = \Gamma_{\Phi_2}$.

Proof. Immediate from the definitions.

Definition 1.25. We will denote the equivalence class of a birational map $\Phi$ by $[\Phi]$. By the above lemma, we can associate a correspondence $\Gamma_{[\Phi]}$ to an equivalence class of birational maps.

Lemma 1.26. Obstruction to Birationality at Singular Points

Let $\Gamma_{[\Phi]}$ be a birational correspondence between $C_1$ and $C_2$. If $x$ is a non-singular point of $C_1$, there exists a unique corresponding point $y$ of $C_2$ and vice-versa.

Proof. Let $U \subset C_1$ be the set of non-singular points of $C_1$. We can consider $\Gamma_{\Phi}$ as a cover of $U$. As $U$ is smooth, we may apply the technique of Zariski structures for this cover. By birationality, if $x \in U$ is generic, there exists a unique $(xy) \in \Gamma_{\Phi}$ and, moreover, $\text{Mult}(y/x) = 1$. This last fact was shown, for example, in the paper [5] and given originally in [12]. By further properties of Zariski structures, again see either of the above, the total multiplicity of points in the cover over $U$ is preserved, in particular, for any $x \in U$, there exists a unique corresponding $(xy) \in \Gamma_{\Phi}$.

Remarks 1.27. Note that non-singularity is not necessarily preserved when associating $y$ to $x$ in the above lemma. This motivates the following definition.

Definition 1.28. Given a birational correspondence $\Gamma_{[\Phi]}$, we define the canonical sets $U_{[\Phi]} \subset \Gamma_{[\Phi]}$, $V_{[\Phi]} \subset C_1$ and $W_{[\Phi]} \subset C_2$ to be the sets:

$U_{[\Phi]} = \{(x, y) \in \Gamma_{[\Phi]} : \text{NonSing}(x), \text{NonSing}(y)\}$

$V_{[\Phi]} = \pi_1(U_{[\Phi]})$

$W_{[\Phi]} = \pi_2(U_{[\Phi]})$

Lemma 1.29. Given a birational correspondence $\Gamma_{[\Phi]}$, there exists an isomorphism $\Phi_1 : V_{[\Phi]} \to W_{[\Phi]}$ such that $\Gamma_{[\Phi]} = \Gamma_{[\Phi_1]}$

Proof. By an elementary result in algebraic geometry, see for example [2], a morphism $\Phi : U \subset C_1 \to P^n$, where $U$ is an open subset of $C_1$, extends uniquely to the non-singular points of $C_1$. Combining this with Lemma 1.26, we obtain immediately the result.
We now need to relate the canonical sets of a birational correspondence $\Gamma_{[\Phi]}$ with a particular presentation of $[\Phi]$ given by Lemma 1.20;

**Definition 1.30.** Let $\phi_\Sigma$ be as in Lemma 1.20, we define the canonical open sets associated to $\phi_\Sigma$ to be;

$$V_{\phi_\Sigma} = V_{[\Phi]} \setminus \text{Base}(\Sigma)$$

$$W_{\phi_\Sigma} = \phi_\Sigma(V_{\phi_\Sigma})$$

**Lemma 1.31.** We have the following relations between canonical sets;

$$V_{\phi_\Sigma} \subset V_{[\Phi]} \subset \text{NonSing}(C_1)$$

$$W_{\phi_\Sigma} \subset W_{[\Phi]} \subset \text{NonSing}(C_2)$$

$\phi_\Sigma : V_{\phi_\Sigma} \to W_{\phi_\Sigma}$ is an isomorphism.

$V_{\phi_\Sigma}$ and $\text{Base}(\Sigma)$ are disjoint.

**Proof.** The proof is an easy exercise. \qed

**Remarks 1.32.** It would be desirable to find a particular presentation $\Phi_\Sigma$ of a birational class $[\Phi]$, for which $\text{Base}(\Sigma)$ is disjoint from the canonical set $V_{[\Phi]}$. In general, one can easily prove the weaker result;

There exist 2 presentations $\Phi_{\Sigma_1}$ and $\Phi_{\Sigma_2}$ of a birational class $[\Phi]$, such that;

$$V_{[\Phi]} = V_{\Phi_{\Sigma_1}} \cup V_{\Phi_{\Sigma_2}}$$

$$W_{[\Phi]} = W_{\Phi_{\Sigma_1}} \cup W_{\Phi_{\Sigma_2}}$$

We also note that the choice of $\Sigma$ presenting a birational class $[\Phi]$ is far from unique. For example, let $\text{Id} : P^1 \to P^1$ be the identity map. This isomorphism can be represented by any of the birational maps;

$$\phi_n[X_0 : X_1] = [X_0^n : X_0^{n-1}X_1] \ (n \geq 1)$$

Let $\Sigma_n$ be the linear system of dimension 1 and degree $n$ defined by the pair of homogeneous polynomials $\{X_0^n, X_0^{n-1}X_1\}$. Then $\phi_n = \Phi_{\Sigma_n}$ (with this choice of basis).
We finally note the following well known theorem, see for example [2];

**Theorem 1.33.** Let $C$ be a projective algebraic curve, then $C$ is birational to a plane projective algebraic curve.

2. **A Basic Theory of $g^r_n$**

We begin this section with the definition of intersection multiplicity used by the original Italian school of algebraic geometry.

**Definition 2.1.** Let $C \subset P^w$ be a projective algebraic curve. Let $\text{Par}_F$ be the projective parameter space for all hypersurfaces of a given degree $e$ and let $U = \{ \lambda \in \text{Par}_F : |C \cap F_\lambda| < \infty \}$ be the open subvariety of $\text{Par}_F$ corresponding to hypersurfaces of degree $e$ having finite intersection with $C$. For $\lambda \in U$, $p \in C \cap F_\lambda$, we define;

$$I_{\text{italian}}(p, C, F_\lambda) = \text{Card}(C \cap F'_\lambda \cap V_p)$$

for $\lambda' \in V_\lambda$ generic in $U$.

**Remarks 2.2.** That this is a rigorous definition follows from general properties of Zariski structures. The definition is the same as $\text{RightMult}_p(C, F_\lambda)$ which we considered in the previous section. We will often abbreviate the notation $I_{\text{italian}}(p, C, F_\lambda) = s$ by saying that $p$ is counted $s$ times for the intersection of $C$ with $F_\lambda$.

The basic theory of $g^r_n$ relies principally on the following result;

**Theorem 2.3.** Hyperspatial Bezout

Let $C \subset P^w$ be a projective algebraic curve of degree $d$ and $F_\lambda$ a hypersurface of degree $e$ having finite intersection with $C$. Then;

$$\sum_{p \in C \cap F_\lambda} I_{\text{italian}}(p, C, F_\lambda) = de$$

We first require the following lemma, preserving the notation from Definition 2.1;

**Lemma 2.4.** Let $H_\lambda$ be a generic hyperplane, then;

$$I_{\text{italian}}(p, C, H_\lambda) = 1 \text{ for all } p \in C \cap H_\lambda$$
and each \( p \in C \cap H_\lambda \) is non-singular.

In the Italian terminology, each point \( p \) of intersection is counted once or the intersection is transverse. Using the methods developed in Section 1, it is not difficult to prove that each point of intersection is transverse (using the scheme theoretic definition).

**Proof.** Suppose, for contradiction, that \( I_{\text{italian}}(p_1, C, H_\lambda) \geq 2 \) for some \( p_1 \in C \cap H_\lambda \). Let \( \{p_1, \ldots, p_d\} \) be the total set of intersections, where \( d \) is the degree of \( C \), see Definition 1.12. Then we can find \( \lambda' \in \mathcal{V}_\lambda \) generic in \( U \) and a distinct pair \( \{p_1^1, p_1^2\} \) in \( \mathcal{V}_{p_1} \cap C \cap H_{\lambda'} \). By properties of Zariski structures, see [5] or [12], we can also find \( \{p_1^1, p_1^2, \ldots, p_1^d\} \) such that \( p_1^j \in C \cap H_{\lambda'} \cap \mathcal{V}_{p_j} \) for \( 2 \leq j \leq d \). It follows from the definition of an infinitesimal neighborhood that, for \( p \neq q \) with \( \{p, q\} \subset P^w \), \( \mathcal{V}_p \) and \( \mathcal{V}_q \) are disjoint. Hence, \( \{p_1^1, p_1^2, p_1^3, \ldots, p_1^d\} \) defines a distinct set of intersections of \( C \) with \( H_{\lambda'} \). It follows that \( C \) and \( H_{\lambda'} \) have at least \( d + 1 \) intersections, contradicting the facts that \( \lambda' \) is generic in \( U \) and the degree of \( C \) is equal to \( d \). For the second part of the lemma, observe that the set of nonsingular points \( \text{NonSing}(C) \) is a dense open subset of \( C \), defined over the field of definition of \( C \). The condition that a hyperplane passes through at least one point of \( (C \setminus \text{NonSing}(C)) \) is therefore a union of finitely many proper hyperplane conditions on \( \text{Par}_H \), also defined over the parameters of \( C \) and the tuple \( \{\lambda_1, \ldots, \lambda_j\} \), for \( 1 \leq j \leq e - 1 \). As \( H_\lambda \) was chosen to be generic, its parameter \( \lambda \) cannot lie inside \( P_1 \cup \ldots \cup P_m \). Hence, the result is shown.

We now complete the proof of Theorem 2.3;

**Proof.** Choose \( \{\lambda_1, \ldots, \lambda_e\} \) independent generic tuples in \( P^{w*} = \text{Par}_H \), the parameter space for hyperplanes on \( P^w \). Let \( F_e \) be the form of degree \( e \) defined by;

\[
F_e = H_{\lambda_1} \cdot \ldots \cdot H_{\lambda_e} = \sum i_0 + \ldots + i_w = e \lambda_{i_0 \ldots i_w} Y_0^{i_0} \ldots Y_w^{i_w} = 0
\]

We first claim that the intersections \( C \cap H_{\lambda_j} \) are pairwise disjoint sets for \( 1 \leq j \leq e \). The condition that a hyperplane \( H_{\mu} \) passes through at least one point of the intersection \( \{(C \cap H_{\lambda_1}) \cup \ldots \cup (C \cap H_{\lambda_j})\} \) is a union of finitely many proper closed hyperplane conditions on \( \text{Par}_H \), defined over the parameters of \( C \) and the tuple \( \{\lambda_1, \ldots, \lambda_j\} \), for \( 1 \leq j \leq e - 1 \). As the tuples \( \{\lambda_1, \ldots, \lambda_e\} \) were chosen to be generically independent in \( \text{Par}_H \), the result follows. Now, by Lemma 2.4 and the definition of
we obtain a total number $de$ of intersections between $C$ and $F_e$. We claim that for each point $p$ of intersection;

$$I_{\text{italian}}(p, C, F_e) = 1 \ (*)$$

This does not follow immediately from Lemma 2.4 as the parameter 

$$\{\lambda_{i_0...i_n}\}$$

defining $F_e$ is allowed to vary in the parameter space $\text{Par}_e$ of all forms of degree $e$. We prove the claim by reducing the problem to one about plane projective curves.

We use Lemma 1.20 and Theorem 1.33 to find a plane projective algebraic curve $C_1 \subset P^2$ and a linear system $\Sigma$ such that $\Phi_\Sigma : C_1 \sim C$. Let $\{\phi_0, \ldots, \phi_w\}$ be a basis for $\Sigma$, defining the birational map $\Phi_\Sigma$. We may suppose that each $\phi_i$ is homogenous of degree $k$ in the variables $\{X_0, X_1, X_2\}$ for $P^2$. Let $\{V[\phi], V[\Phi], W[\phi], W[\Phi]\}$ be the canonical sets associated to $\Gamma_\Phi$, see Definitions 1.28 and 1.30. Note that the canonical sets are all definable over the data of $\Phi_\Sigma$. Hence, we may, without loss of generality, assume that the point $p$ given in $(\ast)$ above lies in $W[\Phi]$ and its corresponding $p' \in C_1$ lies in $V[\Phi]$. In particular, $p'$ defines a non-singular point of the curve $C_1$. Now, given an algebraic form $F_\mu$ of degree $e$;

$$F_\mu = \Sigma_{i_0+...+i_w=e} \mu_{i_0...i_n} Y_{i_0}^0 \ldots Y_{i_w}^w = 0$$

we obtain a corresponding algebraic curve $\psi_\mu$ of degree $ke$ on $P^2$ given by the equation;

$$\psi_\mu = \Sigma_{i_0+...+i_w=e} \phi_{i_0}^{i_0} \ldots \phi_{i_w}^{i_w} = 0 \ (\dagger)$$

We claim that;

$$I_{\text{italian}}(p, C, F_\mu) \leq I_{\text{italian}}(p', C_1, \psi_\mu) \ (**)$$

For suppose that $p$ is counted $s$-times for the intersection of $C$ with $F_\mu$, then we can find $\mu' \in V_\mu$ generic in $U$ such that $C \cap F_{\mu'} \cap V_p$ consists of the distinct points $\{p_1, \ldots, p_s\}$. By elementary properties of infinitesimals, $\{p_1, \ldots, p_s\} \subset W[\Phi]$, hence we can find a corresponding distinct set $\{p'_1, \ldots, p'_s\}$ in $V[\Phi]$. By the defining property of a specialisation, the fact that the correspondence $\Gamma[\Phi]$ is closed and the definition of $\psi_{\mu'}$ we must have that $\{p'_1, \ldots, p'_s\} \subset C_1 \cap V_p \cap \psi_{\mu'}$. As the map $\theta : \text{Par}_e \to \text{Par}_{ke}$, defined by $(\dagger)$, is algebraic, it follows that $\psi_{\mu'}$ defines an infinitesimal variation of $\psi_\mu$ in the space of all algebraic
curves of degree $ke$ on $P^2$. Hence, it follows that $p'$ is counted at least $s$-times for the intersection of $C_1$ with $\psi_\mu$. Therefore, (***) is shown.

Now, given a hyperplane $H_\mu$;

$$H_\mu = \mu_0 Y_0 + \ldots + \mu_w Y_w = 0$$

we obtain a corresponding algebraic curve $\phi_\mu$ of degree $k$ on $P^2$, defined by the equation;

$$\phi_\mu = \mu_0 \phi_0 + \ldots + \mu_w \phi_w = 0$$

Corresponding to the factorisation $F_\lambda = F_e = H_{\lambda_1} \ldots H_{\lambda_e}$, we obtain the factorisation $\psi_\lambda = \phi_{\lambda_1} \ldots \phi_{\lambda_e}$. Therefore, in order to show (\(\star\)), it will be sufficient to prove that;

$$I_{\text{italian}}(p', C, \phi_{\lambda_1} \ldots \phi_{\lambda_e}) = 1$$

Let $p$ belong (uniquely) to the intersection $C \cap H_{\lambda_j}$. We claim first that;

$$I_{\text{italian}}(p, C, \phi_{\lambda_j}) = 1$$

We clearly have that the linear system $\Sigma$ consists of $\{\phi_\lambda : \lambda \in P^w\}$. Hence, as by construction $p'$ does not belong to Base($\Sigma$) and is non-singular, the result in fact follows from Lemma 2.4 and a local result given later in this section, Lemma 2.10, which is independent of this theorem. By results of [6] on plane projective curves, (***) follows. Hence, (\(\star\)) is shown as well.

We have now proved that there exists a form $F_e$ of degree $e$ which intersects $C$ in exactly $de$ points with multiplicity. The theorem now follows immediately from the corresponding result in Zariski structures that, for a finite equidimensional cover $G \subset Par_e \times P^w$;

$$\Sigma_{x \in G(\lambda)} \text{Mult}_{(\lambda, x)}(G/Par_F)$$

is preserved.

\[
\]

Using this theorem, we develop a basic theory of $g^n_\mu$ on $C$, a projective algebraic curve in $P^w$. Suppose that we are given a linear system $\Sigma$ of dimension $r$, consisting of algebraic forms $\phi_\lambda$, parametrised by $Par_\Sigma$. We will assume that $C \cap \phi_\lambda$ has finite intersection for each $\lambda \in Par_\Sigma$,
which we will abbreviate by saying that $\Sigma$ has finite intersection with $C$. Then, for $\lambda \in \text{Par}_\Sigma$, we obtain the weighted set of points:

$$W_\lambda = \{n_{p_1}, \ldots, n_{p_m}\}$$

where

$$\{p_1, \ldots, p_m\} = C \cap \phi_\lambda$$

and

$$I_{\text{italian}}(p_j, C, \phi_\lambda) = n_{p_j} \text{ for } 1 \leq j \leq m.$$ 

By Theorem 2.3, the total weight $n_{p_1} + \ldots + n_{p_m}$ of these points is always equal to $de$.

It follows that, as $\lambda$ varies in $\text{Par}_\Sigma$, we obtain a series of weighted sets $\text{Series}(\Sigma) = \{W_\lambda : \lambda \in \text{Par}(\Sigma)\}$. We now make the following definition;

**Definition 2.5.** We define $\text{order}(\text{Series}(\Sigma))$ to be the total weight of any of the sets in $\text{Series}(\Sigma)$. We define $\text{dim}(\text{Series}(\Sigma))$ to be $\text{dim}(\Sigma)$. We define $g^e_n(\Sigma)$ to be the series of weighted sets parametrised by $\text{Par}_\Sigma$.

where:

$$n = \text{order}(\text{Series}(\Sigma)) \text{ and } r = \text{dim}(\text{Series}(\Sigma)).$$

We now make the following local analysis of $g^e_n(\Sigma)$.

**Definition 2.6.** Let $\Sigma$ be a linear system having finite intersection with $C \subset P^w$. If $\phi_\lambda$ belongs to $\Sigma$ and $p \in C \cap \phi_\lambda$, we define:

$$I^\Sigma_{\text{italian}}(p, C, \phi_\lambda) = \text{Card}(C \cap \phi_\lambda \cap \mathcal{V}_p) \text{ for } \lambda' \in \mathcal{V}_\lambda \text{ generic in } \text{Par}_\Sigma.$$

**Remarks 2.7.** This is a good definition as $\text{Par}_\Sigma$ is smooth. The difference between $I_{\text{italian}}$ and $I^\Sigma_{\text{italian}}$ is that in the first case we can vary the parameter $\lambda$ over all forms of degree $e$, while, in the second case, we restrict the parameter to forms of the linear system defined by $\Sigma$.

**Definition 2.8.** We will refer to $C \setminus \text{Base}(\Sigma)$ as the set of mobile points for the system $\Sigma$. 

We now make the following preliminary definition:

**Definition 2.9.** We will define a coincident mobile point for $\Sigma$ to be a point $p \in C \cap \phi_\lambda$, for some $\lambda \in \text{Par}_\Sigma$, such that $p$ lies outside $\text{Base}(\Sigma)$ and with the further property that;

$$I^\Sigma_{\text{italian}}(p, C, \phi_\lambda) \leq I_{\text{italian}}(p, C, \phi_\lambda)$$

It is an important property of linear systems in characteristic 0 that there do not exist coincident mobile points. (See the final section for the corresponding result in arbitrary characteristic.) We will prove this in the following lemmas, the notation of Definition 2.6 will be maintained until Lemma 2.17.

**Lemma 2.10. Non-Existence of Coincident Mobile Points**

Let $p \in (C \setminus \text{Base}(\Sigma)) \cap \phi_\lambda$ be a non-singular point. Then;

$$I_{\text{italian}}(p, C, \phi_\lambda) = I^\Sigma_{\text{italian}}(p, C, \phi_\lambda)$$

**Proof.** We prove this by induction on $m = I_{\text{italian}}(p, C, \phi_\lambda)$. The case $m = 1$ is clear as we always have that;

$$I^\Sigma_{\text{italian}}(p, C, \phi_\lambda) \leq I_{\text{italian}}(p, C, \phi_\lambda)$$

Suppose that $I_{\text{italian}}(p, C, \phi_\lambda) = m + 1$. Let $\lambda' \in \text{Par}_\Sigma \cap \mathcal{V}_\lambda$ be generic and let $\{p_1, \ldots, p_r\}$ enumerate $\mathcal{V}_p \cap C \cap \phi_{\lambda'}$. Suppose $r \geq 2$, then, by results of [6], (summability of specialisation), we have that;

$$I_{\text{italian}}(p_j, C, \phi_{\lambda'}) \leq m \text{ for each } 1 \leq j \leq r.$$  

By properties of infinitesimal neighborhoods, each $p_j$ is non-singular and lies in $C \setminus \text{Base}(\Sigma)$. Hence, by the induction hypothesis, it follows that;

$$I^\Sigma_{\text{italian}}(p_j, C, \phi_{\lambda'}) = I_{\text{italian}}(p_j, C, \phi_{\lambda'}) \text{ for each } 1 \leq j \leq r$$

Again, using the same result from [6], (summability of specialisation), we have that;
$$I_{\text{italian}}^{(\Sigma)}(p, C, \phi_\lambda) = \sum_{1 \leq j \leq r} I_{\text{italian}}^{(\Sigma)}(p_j, C, \phi_{\lambda'})$$

where the $$(\Sigma)$$ notation is used to show that the result holds for either of the above defined multiplicities. Hence;

$$I_{\text{italian}}(p, C, \phi_\lambda) = I_{\text{italian}}^{\Sigma}(p, C, \phi_\lambda)$$

We may, therefore, assume that;

For any generic $$\lambda' \in \mathcal{V}_\lambda \cap \text{Par}_\Sigma$$, there exists a unique $$p' \in \mathcal{V}_p \cap C \cap \phi_{\lambda'}$$, with $$I_{\text{italian}}(p', C, \phi_{\lambda'}) = I_{\text{italian}}(p, C, \phi_\lambda)$$ (\ast)\]

Now, as $$p \notin \text{Base}(\Sigma)$$, we can find $$\phi_\mu$$ with $$p \notin C \cap \phi_\mu$$. We consider the pencil of algebraic forms defined by $$\{\phi_\lambda, \phi_\mu\}$$. We may assume $$p$$ is in finite position on the curve $$C$$. For ease of notation, we will assume that $$w = 2$$ and make the generalisation to arbitrary dimension $$w$$ at the end of the lemma. Let $$f(X, Y) = 0$$ define the curve $$C$$ in affine coordinates such that $$p$$ corresponds to the point $$(0, 0)$$. We rewrite the pencil of curves $$(\phi_\lambda, \phi_\mu)$$ in affine coordinates, which gives the 1-parameter family;

$$g(X, Y; t) = \sum_{i+j \leq \deg(\Sigma)} (\lambda_{ij} + t\mu_{ij})X^iY^j = 0$$

The following calculation is somewhat informal, see remark (ii) of Section 1;

As $$p \notin \text{Base}(\Sigma)$$, the function $$\frac{\phi_\lambda}{\phi_\mu}$$ is defined at $$p = (0, 0)$$ and gives an algebraic morphism;

$$\frac{\phi_\lambda}{\phi_\mu} : W \subset C \to \text{Par}_t$$

on some open $$W \subset C$$, with $$\frac{\phi_\lambda}{\phi_\mu}(0, 0) = 0$$.

The cover graph $$(\frac{\phi_\lambda}{\phi_\mu}) \subset W \times \text{Par}_t$$ of $$\text{Par}_t$$ is Zariski unramified at $$(0, 0, 0)$$ as, given generic $$t \in \mathcal{V}_0$$, there exists a unique $$(x, y)$$ such that $$(x, y) \in \mathcal{V}_{(0, 0)} \cap C \cap (\phi_\lambda + t\phi_\mu)$$, by (\ast) and the fact that $$\text{Par}_t$$ is smooth. By results of [5], (Theorem 6.10), if we assume the ground field $$L$$ has characteristic 0, the cover is etale at $$(0, 0, 0)$$. We will make the modification for non-zero characteristic in the final section of this paper. (1)
Now, as \( f(X,Y) = 0 \) is non-singular at \( p = (0, 0) \), we can apply the implicit function theorem to obtain a parametrisation in algebraic power series \((x(t), y(t))\) of the branch at \((0, 0)\) such that \( f(x(t), y(t)) = 0 \). (See the remark (ii) in Section 1 and [6] for the correct interpretation of these power series on appropriate etale covers and the corresponding definition of a branch.) We then obtain a map defined by algebraic power series \( \theta : A^{1, \text{et}} \to \text{Par}_t \) given by:

\[
\theta(t) = \frac{\partial \lambda}{\partial \mu}(x(t), y(t))
\]

where \( A^{1, \text{et}} \) is an etale cover of \( A^1 \) over the distinguished point \((0, 0)\). We have that \( \theta \) is etale at \((0, 0)\), as the composition of etale maps is etale. By the inverse function theorem, we may find an etale isomorphism \( \rho : A^{1, \text{et}} \to A^{1, \text{et}} \) such that \( \theta(\rho(t)) = t \) and \((x_1(t), y_1(t)) = (x(\rho(t)), y(\rho(t)))\) also parametrises the branch at \((0, 0)\).

We therefore have that:

\[
g(x_1(t), y_1(t); t) = 0 \quad (**) \quad (2)
\]

Now, by assumption, \( I_{\text{italian}}(p, C, \phi_\lambda) \geq 2 \). Hence, by results of [6]:

\[
g(X, Y; 0) \text{ is algebraically tangent to } f(X, Y) = 0 \text{ at } (0, 0) \quad (3).
\]

By the chain rule and (**) we have that:

\[
\frac{\partial g}{\partial X}(x_1(t), y_1(t)) x_1'(t) + \frac{\partial g}{\partial Y}(x_1(t), y_1(t)) y_1'(t) + \frac{\partial g}{\partial t}(x_1(t), y_1(t)) = 0 \quad (***) \quad (4)
\]

Hence, at \( t = 0 \), we have that \( \frac{\partial \rho}{\partial t}(0, 0) = 0 \), that is \( p = (0, 0) \) belongs to \( \phi_\mu = 0 \), which is a contradiction. The calculation (***) holds for formal power series in \( L[[t]] \). In particular, it holds for algebraic power series. We now give a brief justification for this calculation;

Let \( v = \text{ord}_t \) be the standard valuation on the power series ring \( L[[t]] \). Given a power series \( f \in L[[t]] \) and a sequence of power series \( \{f_n : n \in \mathbb{Z}_{\geq 0}\} \), we will say that \( \{f_n\} \) converges to \( f \), abbreviated by \( \{f_n\} \to f \), if:

\[
(\forall m \in \mathbb{Z}_{\geq 0})(\exists n(m) \in \mathbb{Z}_{\geq 0})(\forall k \geq n(m))[v(f - f_k) \geq m]
\]
Now choose sequences \( \{x^n(t), y^n(t)\} \) of polynomials in \( L[t] \) such that \( \{x^n(t)\} \to x_1(t) \) and \( \{y^n(t)\} \to y_1(t) \). We claim that:

\[
\{g_n = g(x^n(t), y^n(t); t)\} \to g(x_1(t), y_1(t); t)
\]

This follows by standard continuity arguments for polynomials in the non-archimedean topology induced on \( L[[t]] \) by \( v \). We then have that:

\[
g'_{n}(t) = \frac{\partial g}{\partial X}(x^n(t), y^n(t), t)x'_{1}(t) + \frac{\partial g}{\partial Y}(x^n(t), y^n(t), t)y'_{1}(t) + \frac{\partial g}{\partial t}(x^n(t), y^n(t), t)
\]

This follows from the fact that the chain rule and product rule hold in the polynomial ring \( L[[t]] \), even in non-zero characteristic. We now claim that \( \{x^n(t)'\} \to x_1(t)' \) and \( \{y^n(t)'\} \to y_1(t)' \). This holds by the definition of convergence and the fact that, for a power series \( f \in L[[t]] \), if \( \text{ord}_t(f) = r \), then \( \text{ord}_t(f') \geq r - 1 \). Using standard continuity arguments and uniqueness of limits, one obtains the result (4). One can also give a geometric interpretation of the calculation (4) using duality arguments. We will discuss this problem on another occasion.

In order to finish the argument, we claim that:

\[
\frac{\partial g_{0}}{\partial X}(0,0)x'_{1}(0) + \frac{\partial g_{0}}{\partial Y}(0,0)x'_{2}(0) = 0 \tag{5}
\]

This follows from (3) and the fact that algebraic tangency can be characterised by the property that \( Dg_{0} \) at \( (0,0) \) contains the tangent line \( l_{p} \) of \( C \). This is clear if \( g_{0} \) is non-singular at \( p \), in particular if \( g_{0} \) has a non-reduced component at \( p \). Otherwise, it follows easily from [6] or [2]. Hence, at \( t = 0 \), we have that \( \frac{\partial g}{\partial t}(0,0) = 0 \), that is \( p = (0,0) \) belongs to \( \phi_{\mu} = 0 \), which is a contradiction.

We now consider the case for arbitrary dimension \( w \). We will use Theorem 1.33 to find a plane projective curve \( C_1 \subset P^2 \) birational to \( C \). Using Lemma 1.20, we can find a linearly independent system \( \Sigma' \) and a birational presentation \( \Psi_{\Sigma'} : C_1 \mapsto C \). We will assume that the point \( p \) under consideration lies inside the canonical set \( W_{\Psi_{\Sigma'}} \), with corresponding \( p' \in V_{\Psi_{\Sigma'}} \). This can in fact always be arranged, see the section on Conic Projections. However, for the moment, we can, if necessary, replace the set \( \text{NonSing}(C) \) by \( W_{\Psi_{\Sigma'}} \). Now, we follow through the calculation given above for \( w = 2 \). The argument up to (1) is unaffected. We first justify the calculation (2). Let \( f(X,Y) = 0 \) be an affine representation of \( C_1 \), such that the point \( p' \) corresponds
to $(0,0)$. Then, we may obtain a local power series representation
$(x(t),y(t))$ of $f(X,Y)$ at $(0,0)$ and, applying $\Psi_{\Sigma'}$, a
local power series representation $\Psi_{\Sigma'}(x(t),y(t)) = (x_1(t),\ldots,x_w(t))$ of the corresponding $p \in C$. We may then, applying
the same argument, obtain the relation
\[ g(x_1(t),\ldots,x_w(t);t) = 0, \]
where
\[ g(X_1,\ldots,X_w;X) = \sum_{i_0+\ldots+i_w=\deg(\Sigma)} \lambda_{i_0\ldots i_w} X_1^{i_0} \ldots X_n^{i_w} + t^{\mu_{i_0\ldots i_w}} X_1^{i_0} \ldots X_n^{i_w} \]
is an affine representation of the pencil $\phi_{\lambda} + t\phi_{\mu}$.

We now need to justify the calculation in (3). Write $\phi_{\lambda}$ in the form;
\[ \phi_{\lambda} = \sum_{i_0+\ldots+i_w=\deg(\Sigma)} \lambda_{i_0\ldots i_w} X_1^{i_0} \ldots X_n^{i_w} = 0 \]
Let $\Sigma' = \{\psi_0,\ldots,\psi_w\}$, then the assumption that $\phi_{\lambda}$ passes through $p$ implies that the curve;
\[ D = \sum_{i_0+\ldots+i_w=\deg(\Sigma)} \lambda_{i_0\ldots i_w} \psi_0^{i_0} \ldots \psi_n^{i_w} = 0 \]
passes through the corresponding $p'$ of $C_1$. By the fact that
$I_{italian}(p,C,\phi_{\lambda}) \geq 2$, we can vary the coefficients $\{\lambda_{i_0\ldots i_w}\}$ of $\phi_{\lambda}$ to obtain distinct intersections $\{x'',x'''\}$ in $C \cap \phi_{\lambda'} \cap V_p$. By properties of infinitesimals, these intersections lie in the fundamental set $W_{\Psi_{\Sigma'}}$. Hence, we can find corresponding intersections $\{x''',x''''\}$ in $V_{p'} \cap V_{\psi_{\Sigma'}}$ with the corresponding variation of $D$. This implies that;
\[ I_{italian}(p',C_1,D) \geq 2 \]
By results of the paper [6], $D$ must be algebraically tangent to the curve $C_1$ at $p'$. Hence, by the chain rule, and the characterisation of algebraic tangency given above, $\phi_{\lambda}$ is algebraically tangent to the curve $C$ at $p$, in the sense that its differential $D\phi_{\lambda}$ at $p$, contains the tangent line $l_p$ of $C$ ($\ast$). The reader should also look at the proof of Theorem 2.3, where a similar calculation was carried out. Finally, we need to justify (4). This is clear from the calculation done above. The final step (5) is also clear from the corresponding calculation and ($\ast$).

\[ \square \]

**Remarks 2.11.** The lemma fails for non-linear systems. Let $C$ be defined in affine coordinates $(x,y)$ by $y = 0$ and let $\{\phi_t\}$ be the pencil of curves, defined in characteristic 0, by $y = (x-t)^2 = x^2 - 2tx + t^2$. 
By construction, each $\phi_t$ is tangent to $y = 0$ at $(t,0)$. It follows that each $(t,0) \in C$ is a coincident mobile point for $\phi_t$.

**Lemma 2.12.** Suppose that $p \in C \setminus \text{Base}(\Sigma) \cap \phi_\lambda$ is a singular point, then;

$$I_{\text{italian}}(p, C, \phi_\lambda) = I_{\text{italian}}^\Sigma(p, C, \phi_\lambda).$$

**Proof.** Suppose that $I_{\text{italian}}^\Sigma(p, C, \phi_\lambda) = m$. As $p \notin \text{Base}(\Sigma)$, the condition that $\phi_\lambda$ passes through $p$ defines a proper closed subset of the parameter space $\text{Par}_\Sigma$. Hence, we can find $\lambda' \in V_\lambda$ generic in $\text{Par}_\Sigma$ and $\{p_1, \ldots, p_m\} = C \cap \phi_{\lambda'} \cap V_p$, distinct from $p$, witnessing this multiplicity. As both $\text{NonSing}(C)$ and $C \setminus \text{Base}(\Sigma)$ are open and defined over $L$, we have that $\{p_1, \ldots, p_m\}$ must lie in the intersection of these sets. Applying the result of the previous lemma, we must have that;

$$I_{\text{italian}}(p_j, C, \phi_{\lambda'}) = 1 \text{ for } 1 \leq j \leq m.$$

Hence, by summability of specialisation, again see the paper [6], we must have that $I_{\text{italian}}(p, C, \phi_\lambda) = m$ as required. $\square$

In the following Lemma 2.13, Lemma 2.16 and Lemma 2.17, by a canonical set on $C$, we will mean either a set of the form $V_{\phi_\Sigma_1}$, for the domain of a birational map $\phi_{\Sigma_1}$, or a set of the form $W_{\psi_\Sigma_2}$, for the image of a birational map $\psi_{\Sigma_2}$, see also Definition 1.30. For ease of notation, we will abbreviate either of these sets by $W$. In particular, $W$ may include the canonical set $V_{\phi_\Sigma}$ defined by (any) choice of basis for the linear system $\Sigma$.

**Lemma 2.13.** Multiplicity at non-base points witnessed by transverse intersections in the canonical sets.

Let $p \in C \setminus \text{Base}(\Sigma)$, then, if $m = I_{\text{italian}}(p, C, \phi_\lambda)$, we can find $\lambda' \in V_\lambda$, generic in $\text{Par}_\Sigma$, and distinct $\{p_1, \ldots, p_m\} = C \cap \phi_{\lambda'} \cap V_p$ such that $\{p_1, \ldots, p_m\}$ lies in the canonical set $W$ with the point $p$ removed, $W \setminus \{p\}$, and the intersection of $C$ with $\phi_{\lambda'}$ at each $p_j$ is transverse for $1 \leq j \leq m$.

**Proof.** As $p \notin \text{Base}(\Sigma)$, the condition that $\phi_\lambda$ does not pass through $p$ defines an open subset of $\text{Par}_\Sigma$. By the previous lemma, taking generic (over $L$), $\lambda' \in V_\lambda$, we can find $\{p_1, \ldots, p_m\} = C \cap \phi_{\lambda'} \cap V_p$, distinct from $p$. Finally, $C \setminus W$ defines a finite subset of $C$ (over $L$). Clearly, $\{p_1, \ldots, p_m\}$ avoid this set, otherwise, by properties of specialisations, some $p_j$ would equal $p$ for $1 \leq j \leq m$. Finally, the transversality result
follows from the fact that $I_{\text{italian}}(p_j, C, \phi_{\lambda'}) = 1$ for $1 \leq j \leq m$, (using Lemmas 2.10 and 2.12 again).

We have analogous results to Lemmas 2.10, 2.12 and 2.13 for points in $\text{Base}(\Sigma)$;

We first require the following;

**Lemma 2.14.** Let $p \in C \cap \text{Base}(\Sigma)$, then there exists an open subset $U_p \subset \text{Par}(\Sigma)$ and an integer $I_p \geq 1$ such that;

$$I_{\text{italian}}(p, C, \phi_{\lambda}) = I_p \text{ for } \lambda \in U_p.$$  

and

$$I_{\text{italian}}(p, C, \phi_{\lambda}) \geq I_p \text{ for } \lambda \in \text{Par}_\Sigma.$$ 

***Proof.*** By properties of Zariski structures, we have that;

$$W_k = \{ \lambda \in \text{Par}(\Sigma) : I_{\text{italian}}(p, C, \phi_{\lambda}) \geq k \}$$ 

are definable and Zariski closed in $\text{Par}_\Sigma$. The result then follows by taking $I_p = \min_{\lambda \in \text{Par}_\Sigma} I_{\text{italian}}(p, C, \phi_{\lambda})$ and the fact that $\text{Par}_\Sigma$ is irreducible.

We can now formulate the corresponding version of Lemmas 2.12 and 2.13 for base points;

**Lemma 2.15.** Let $p \in C \cap \text{Base}(\Sigma) \cap \phi_{\lambda}$. Then;

$$I_{\text{italian}}(p, C, \phi_{\lambda}) = I_p + I_{\Sigma_{\text{italian}}}(p, C, \phi_{\lambda}) - 1$$

***Proof.*** Let $m = I_{\Sigma_{\text{italian}}}(p, C, \phi_{\lambda})$. Choosing $\lambda' \in \mathcal{V}_{\lambda}$ generic in $\text{Par}_\Sigma$, we can find $\{p_1, \ldots, p_{m-1}\} = C \cap \mathcal{V}_p \cap \phi_{\lambda'}$, distinct from $p$, witnessing this multiplicity. Therefore, for $1 \leq j \leq m - 1$, $p_j \notin \text{Base}(\Sigma)$, by properties of specialisations and the fact that $\text{Base}(\Sigma)$ is finite and defined over $L$. Hence, we can apply the results of Lemmas 2.10 and 2.12 to conclude that $I_{\text{italian}}(p_j, C, \phi_{\lambda'}) = 1$ for $1 \leq j \leq m - 1$. As $\lambda'$ was generic in $\text{Par}_\Sigma$, using the previous Lemma 2.14, we have that $I_{\text{italian}}(p, C, \phi_{\lambda'}) = I_p$. Now, it follows easily, using summability of
specialisation (see the paper [6]), that $I_{\text{italian}}(p, C, \phi_\lambda) = I_p + (m - 1)$. The lemma is proved. □

**Lemma 2.16.** Let $p \in C \cap \text{Base}(\Sigma) \cap \phi_\lambda$, then, if $m = I_{\text{italian}}^\Sigma(p, C, \phi_\lambda)$, we can find $\lambda' \in \mathcal{V}_\lambda$, generic in $\text{Par}_{\Sigma}$, and $\{p, p_1, \ldots, p_{m-1}\} = C \cap \phi_{\lambda'} \cap \mathcal{V}_p$ witnessing this multiplicity such that $\{p_1, \ldots, p_{m-1}\}$ lie in the canonical set $W$, see the explanation before Lemma 2.13, and the intersections $C \cap \phi_{\lambda'}$ at $p_j$ are transverse for $1 \leq j \leq m - 1$.

**Proof.** Use the proof of Lemma 2.13, basic properties of infinitesimals and the fact that $W$ is open in $C$ and definable over $L$. □

**Lemma 2.17.** Generic Intersections

Fix a canonical set $W$ and let $\phi_\lambda$ be generic in $\Sigma$, then each point of intersection of $C$ with $\phi_\lambda$ outside $\text{Base}(\Sigma)$ lies inside the canonical set $W$ and is transverse.

**Proof.** The finitely many points of $(C \setminus W)$ are defined over the data of $\{W, C\}$. Hence, the condition on $\text{Par}_{\Sigma}$ that $\phi_\lambda$ intersects a point of $(C \setminus W)$ outside $\text{Base}(\Sigma)$ consists of a finite union of proper hyperplanes defined over the data of $\{W, C\}$. Therefore, for generic $\phi_\lambda$, each point of intersection of $\phi_\lambda$ with $C$, outside $\text{Base}(\Sigma)$, lies inside $W$. Now observe that the condition of transversality between $C$ and $\phi_\lambda$, inside $W$, defines a constructible condition on $\text{Par}_{\Sigma}$, over the data of $\{C, W\}$. Namely:

$$\theta(\lambda) \equiv \forall y[(y \in \phi_\lambda \cap W) \rightarrow \text{NonSing}(y) \land \text{RightMult}_y(C, \phi_\lambda) = 1]$$

By Lemmas 2.13 and 2.16, the condition is Zariski dense in $\text{Par}_{\Sigma}$. Hence, the result follows. □

We now make the following definitions;

**Definition 2.18.** Let $\Sigma$ be a linear system defining a $g^r_n(\Sigma)$ on a projective algebraic curve $C \subset P^w$. Let $\{W_\lambda\} = \text{Series}(\Sigma)$. If $p \in W_\lambda$, we say that:

- $p$ is $s$-fold ($s$-plo in Italian) for the $g^r_n(\Sigma)$ if $I_{\text{italian}}(p, C, \phi_\lambda) \geq s$.

- $p$ is counted (contato) $s$-times for the $g^r_n(\Sigma)$ if $p$ has multiplicity $s$ in $W_\lambda$, equivalently $I_{\text{italian}}(p, C, \phi_\lambda) = s$. 

p has $s$-fold contact (contatto) with $\phi_\lambda$ if $I_{\text{italian}}^\Sigma(p, C, \phi_\lambda) = s$.

**Remarks 2.19.** The Italian terminology is generally quite confusing. The above lemmas show that the discrepancy between contatto and contatto occurs only at fixed points of the system $\Sigma$. The philosophy behind their approach is that algebraic calculations may be reduced to visual arguments using the ideas that a $s$-fold contact at $p$ is a limit of $s$-points converging along the curve from intersections with forms in the system $\Sigma$ and that these points are preserved by birationality. In the case of a fixed point, $p$ may be counted more times than its actual contact with $\phi_\lambda$ and this excess intersection is never actually visually manifested by a variation. The Italian approach is to ignore this excess intersection. This motivates the following definition;

**Definition 2.20.** Let a $g^r_n(\Sigma)$ be given on $C$. For $p \in C \cap \phi_\lambda$, we define;

$$I_{\text{mobile italian}}(p, C, \phi_\lambda) = \text{Card}(C \cap \phi_\lambda \cap \{V_p \setminus p\}), \lambda' \in V_\lambda \text{ generic in } \text{Par}_\Sigma.$$

**Remarks 2.21.** One needs to check, as usual, that this is a good definition. This follows, for example, by Lemma 2.14, Remarks 2.7 and Lemma 2.15.

**Lemma 2.22.** For $p \notin \text{Base}(\Sigma)$, $I_{\text{mobile italian}}(p, C, \phi_\lambda) = I_{\text{italian}}(p, C, \phi_\lambda)$

For $p \in \text{Base}(\Sigma)$, we have that;

$$I_p + I_{\text{mobile italian}}(p, C, \phi_\lambda) = I_{\text{italian}}(p, C, \phi_\lambda)$$

**Proof.** The proof follows immediately from the same results cited in the previous remark. $\square$

We now make the following definition;

**Definition 2.23.** By a $g^r_n$, we mean the series obtained from a given $g^r_{n'}(\Sigma)$ by removing some (possibly all) of the fixed point contributions $I_x$ in $\text{Base}(\Sigma)$. That is, we subtract some part of $I_x$ from each weighted set $W_\lambda$. We define $n$ to be the total multiplicity of each $W_\lambda$ after subtracting some of the fixed point contribution, so $n \leq n'$. We say that the $g^r_n$ has no fixed points if all the fixed point contributions are removed. We refine the Italian terminology for a $g^r_n$ by saying that $x$ is $s$-fold for $W_\lambda$ if it appears in the weighted set with multiplicity at least $s$ and $x$
is counted $s$-times for $W_\lambda$ if it appears with multiplicity exactly $s$. We define \emph{Base($g^n_r$)} to be \{$x \in C : \forall \lambda(x \in W_\lambda)$\}.

We now have the following;

\textbf{Lemma 2.24.} For a given $g^n_r$, we always have that $r \leq n$.

\textit{Proof.} Let the $g^n_r$ be defined by $\Sigma$, a linear system of dimension $r$, having finite intersection with $C$. Pick $\{p_1, \ldots, p_r\}$ independent generic points of $C$, not contained in \emph{Base($\Sigma$)}. The condition that $\phi_\lambda$ passes through $p_j$ defines a proper hyperplane condition $H_{p_j}$ on $\text{Par}_\Sigma$. The base points of the subsystem defined by $H_{p_1} \cap \cdots \cap H_{p_j}$ are defined over $p_1, \ldots, p_j$ and finite, for $1 \leq j \leq r-1$, as, by assumption, no form in $\Sigma$ contains $C$. We must, therefore, have that $\dim(H_{p_1} \cap \cdots \cap H_{p_r}) = 0$. That is there exists a unique $\phi_\lambda$ in $\Sigma$ passing through $\{p_1, \ldots, p_r\}$. We must have that the total intersection multiplicity of $C$ with $\phi_\lambda$ outside the fixed contribution is at least $r$, by construction, hence $r \leq n$ as required.

\textbf{Definition 2.25.} \textit{Subordinate Systems} 

We say that 

$g^{r_1}_{n_1} \subseteq g^{r_2}_{n_2}$  

if there exist linear systems $\Sigma' \subseteq \Sigma$ (having finite intersection with $C$) of dimension $r_1$ and dimension $r_2$ respectively such that $g^{r_1}_{n_1}$ is obtained from $\Sigma'$, $g^{r_2}_{n_2}$ is obtained from $\Sigma$ and, for each $\lambda \in \text{Par}_{\Sigma'}$, $W_\lambda \subseteq V_\lambda$. Here, by $\{W_\lambda, V_\lambda\}$, we mean the weighted sets parametrised by $\{g^{r_1}_{n_1}, g^{r_2}_{n_2}\}$ and, by $W_\lambda \subseteq V_\lambda$, we mean that $n_{p_i} \leq m_{p_i}$ where $W_\lambda = \{n_{p_1}, \ldots, n_{p_r}\}$ and $V_\lambda = \{m_{p_1}, \ldots, m_{p_r}\}$.

\textbf{Remarks 2.26.} Note that the relationship of subordination is clearly transitive. That is, if $g^{r_1}_{n_1} \subseteq g^{r_2}_{n_2} \subseteq g^{r_3}_{n_3}$, then $g^{r_1}_{n_1} \subseteq g^{r_3}_{n_3}$.

\textbf{Definition 2.27.} \textit{Composite Systems}

We say that a $g^n_r$, defined by $\Sigma$, is composite, if, for generic $p \in C$, every weighted set $W_\lambda$ containing $p$ also contains a distinct $p'(p)$ with $p' \notin \text{Base}(\Sigma)$.

\textbf{Remarks 2.28.} Note that the definition of composite is well defined, for, if the given $g^n_r$ is defined by $\Sigma$, the statement;
θ(x) ≡ \exists x' \notin \text{Base}(\Sigma) \forall \lambda \in \text{Par}_\Sigma(x \in C \cap \phi_\lambda \rightarrow x' \in C \cap \phi_\lambda)

defines a constructible subset of C. Hence, if it holds for some generic p, it holds for any generic p in C. In modern terminology, we would say that the given $g^*_n$ separates points (generically), see Proposition 7.3 of [2].

**Definition 2.29. Simple Systems**

We say that a $g^*_n$ is simple if it is not composite.

The importance of simple $g^*_n$ is due to the following:

**Lemma 2.30. Construction of a Birational Model**

A simple $g^*_n$ on C defines a projective image $C' \subset P^r$, birational to C.

*Proof.* Let the $g^*_n$ be defined by a linear system $\Sigma$, with a choice of basis $\{\phi_0, \ldots, \phi_r\}$, (having finite intersection with C), possibly after removing some fixed point contribution. Let $\Phi_\Sigma$ be the morphism defined as in Lemma 1.16. This morphism is defined on an open subset $U = C \setminus \text{Base}(\Sigma)$ of C. By continuity, the image of $\Phi_\Sigma$ on U is irreducible, hence either defines a constructible $V \subset P^r$ of dimension 1 or the image is a point. We can clearly exclude the second case, otherwise we can find $\{\phi_\lambda, \phi_\mu\}$ differing by a constant of proportionality $\rho$ on U, therefore $\phi_\lambda - \rho \phi_\mu$ contains C. Let $C' = \overline{V}$, then $C'$ is an irreducible projective algebraic curve. We claim that $C'$ is birational to C. If not, then, using Lemma 1.21 or just the definition of birationality, in characteristic 0, for generic $x \in C$, we can find a distinct $x' \in U$ such that $\Phi_\Sigma(x) = \Phi_\Sigma(x')$. The choice of basis for $\Sigma$ determines an isomorphism of $\text{Par}_\Sigma$ with $P^r$. Using the parametrisation of $\text{Par}_\Sigma$ given by this isomorphism, if $\phi_\lambda$ passes through x, the corresponding hyperplane $H_\lambda \subset P^r$ would pass through $\Phi_\Sigma(x)$ and, therefore, $\phi_\lambda$ would pass through $x'$ as well. This contradicts simplicity. The lemma may, of course, fail in non-zero characteristic. We refer the reader to the final section for the problems associated to Frobenius. □

We also have the following transfer results:

**Lemma 2.31.** Let a simple $g^*_n$ on C be given, as in the previous lemma, defining a birational projective image $C' \subset P^r$. Let $V_{\phi_\Sigma}$ and $W_{\phi_\Sigma}$ denote the canonical sets associated to this birational map. Then, given
a $g^d_m$ on $C'$, without fixed points, there exists a corresponding $g^d_m$ on $C$, without fixed points, and, for any corresponding pair $\{x, x'\}$ in $\{V_{\phi_{\Sigma}}, W_{\phi_{\Sigma}}\}$, $x$ is counted $s$ times in $V_{\lambda}$ iff $x'$ is counted $s$ times in $W_{\lambda}$, where $\{V_{\lambda}, W_{\lambda}\}$ are the weighted sets parametrised by the $g^d_m$, on $\{C, C'\}$ respectively.

Proof. Let the $g^d_m$ on $C'$ be defined by a linear system $\Sigma'$ of dimension $d$, with basis $\{\psi_0, \ldots, \psi_d\}$, after removing all the fixed point contribution. We then obtain a corresponding linear system $\Sigma''$ defined by:

$$\{\theta_\lambda = \lambda_0 \psi_0(\phi_0, \ldots, \phi_r) + \ldots + \lambda_d \psi_d(\phi_0, \ldots, \phi_r) = 0\}$$

Clearly, $\Sigma''$ has finite intersection with $C'$, hence it defines a $g^d_m'$ after removing all fixed point contributions. Now, let $\{x, x'\}$ be a corresponding pair in $\{V_{\phi_{\Sigma}}, W_{\phi_{\Sigma}}\}$. Suppose that $x'$ is counted $s$ times in $W_{\lambda}$, then, as the $g^d_m$ on $C'$ has no fixed points, we have that:

$$I_{\text{mobile}}(x', C', \psi_\lambda) = s$$

Therefore, we can find $\lambda' \in V_{\lambda}$ generic in $Par_{\Sigma'}$ and

$$\{x'_1, \ldots, x'_s\} = C' \cap \psi_{\lambda'} \cap V_{x'} \setminus \{x'\}$$

By properties of infinitesimals, $\{x'_1, \ldots, x'_s\} \subset W_{\phi_{\Sigma}}$, hence, we can find corresponding $\{x_1, \ldots, x_s\} \subset V_{\phi_{\Sigma}} \cap \theta_{\lambda'} \cap V_x \setminus \{x\}$. It follows that $I_{\text{mobile}}(x, C, \theta_{\lambda}) \geq s$, and equality follows from the converse argument. Therefore, as the corresponding $g^d_m$ has no fixed points, $x$ is counted $s$ times in $V_{\lambda}$. The converse uses the same argument. It remains to show that $m = m'$. By Lemmas 2.16 and 2.17 and the fact that the $g^d_m$ has no fixed points, a generic $V_{\lambda}$ consists of $m'$ points each counted once inside $V_{\phi_{\Sigma}}$. Therefore, by the above argument, these points are each counted once inside the corresponding $W_{\lambda}$. Hence, $m' \leq m$. We obtain $m \leq m'$ by the reversal of this argument.

\[\square\]

Lemma 2.32. Let a simple $g^n_r$ on $C$ be given, defining a birational map $\Phi_{\Sigma} : C \hookrightarrow C' \subset P^r$. Then, if $d$ is the degree of $C'$, we have that $d' = d + I$ for the $g^n_r(\Sigma)$ defined by $\Sigma$, where $I$ is the total fixed point contribution from $\text{Base}(\Sigma)$.

Proof. Let $k$ be the degree of $\phi_{\Sigma}$ and $v$ the degree of $C$, then we claim that:
By Lemma 2.4, if $H_\lambda$ is a generic hyperplane, it cuts $C'$ transversely in $d$ distinct points. We may also assume that these points lie inside the canonical set $W_{\Phi_S}$ as this is defined over the data of $\Phi_\Sigma$. Let $\{p_1, \ldots, p_d\}$ be the corresponding points of $V_{\Phi_S}$ and $\phi_\lambda$ the corresponding form in $\Sigma$. By, for example, Lemma 2.17;

$$I_{\text{italian}}(p_j, C, \phi_\lambda) = 1 \text{ for } 1 \leq j \leq d$$

There can be no more intersections of $\phi_\lambda$ with $C$ outside $\text{Base}(\Sigma)$, otherwise one would obtain a corresponding intersection of $H_\lambda$ with $C'$ outside $W_{\Phi_S}$. Hence, the total multiplicity of intersection $I_{\text{italian}}(C, \phi_\lambda)$ between $C$ and $\phi_\lambda$ is exactly $I + d$, using Lemma 2.14. By Theorem 2.3 and the fact that $\phi_\lambda$ is a form of degree $k$, we also have that the total multiplicity $I_{\text{italian}}(C, \phi_\lambda)$ is $kv$. As, by definition, $d' = kv$, The result follows.

$$\square$$

3. The Construction of a Birational Model of a Plane Projective Algebraic Curve without Multiple Points

We first make the following definition.

**Definition 3.1.** Let $C \subset P^w$ be a projective algebraic curve, not contained in any hyperplane of $P^w$. We define a point $p \in C$ to be $s$-fold on $C$ if, for every hyperplane $H$ passing through $p$;

$$I_{\text{italian}}(p, C, H) \geq s.$$ 

and equality is attained. We define $p$ to be a multiple point if it is $s$-fold for some $s \geq 2$. We define $p$ to be simple if it is not multiple.

We have the following lemma;

**Lemma 3.2.** Let $C \subset P^2$ be a projective algebraic curve and $p$ a point on $C$. Let $F(X, Y) = 0$ be an affine representation of $C$ such that the point $p$ corresponds to $(0, 0)$. Then $p$ is $s$-fold on $C$ iff we can write $F$ in the form;

$$F(X, Y) = \Sigma_{(i+j) \geq s} a_{ij} X^i Y^j$$
In particular, \( p \) is non-singular iff it is not multiple.

Proof. Suppose that \( p \) is \( s \)-fold on \( C \), then, for a generic line \( l \) defined by \( aX - bY = 0 \), we have that;

\[
I_{\text{italian}}(p, C, l) \geq s
\]

It follows, by the results of [6], that we also have;

\[
I_{\text{italian}}(p, l, C) = I_{\text{italian}}(p, C, l) \geq s\quad(\ast)
\]

Now, suppose that \( F(X, Y) \) has the expansion;

\[
F(X, Y) = \sum_{i+j \leq \deg(F)} a_{ij}X^iY^j.
\]

and that \( a_{ij} \neq 0 \) for some \((i, j)\) with \( i + j < s\) (**). We can parametrise the branch at \((0, 0)\) of \( l \) algebraically by;

\[
(x(t), y(t)) = (bt, at)
\]

We make the substitution of this parametrisation into \( F(X, Y) \) and obtain;

\[
F(x(t), y(t)) = \sum a_{ij} x(t)^i y(t)^j = \sum a_{ij} a^i b^j t^{i+j}
\]

By (**), and generic choice of \( \{a, b\} \), this expansion has order \( s_1 < s \). In characteristic 0, it then follows, by the method of [6], see also Theorem 6.1 of this paper, that, for the pencil \( \Sigma_1 \) defined by \( \{F_t \equiv F(X, Y) + t = 0\} \);

\[
I_{\text{italian}}^{\Sigma_1}(p, l, F) = s_1.
\]

In particular, as \( p \notin \text{Base}(\Sigma_1) \), we have, by Lemma 2.10, that;

\[
I_{\text{italian}}(p, l, F) = s_1
\]

and therefore, by \((\ast)\), that;

\[
I_{\text{italian}}(p, C, l) = s_1
\]
contradicting the fact that \( p \) was \( s \)-fold on \( C \). It follows that (**\#) doesn’t hold, as required. See, however, the final section for the problem in non-zero characteristic. For the converse direction, suppose that \( F(X,Y) \) has the expansion;

\[
F(X,Y) = \Sigma_{i+j\geq s} a_{ij} X^i Y^j
\]

Then, by a direct calculation and using the argument above, one has that, for any line;

\[
I_{italian}(p, l, F) \geq s.
\]

Hence, by (\#) again, \( p \) must be \( s \)-fold on \( C \).

Using this result, it follows immediately, by considering the vector \((\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y})\) evaluated at \((0,0)\), that \( p \) is non-singular on \( C \) iff \( p \) is not multiple.

\[\square\]

Our main result in this section will be the following;

**Theorem 3.3.** Let \( C \subset P^2 \) be a projective algebraic curve, then there exists \( C_1 \subset P^w \) such that \( C \) and \( C_1 \) are birational and \( C_1 \) has no multiple points.

The proof will proceed using the basic theory of \( g_n^r \) that we developed in the previous section. We require the following definition;

**Definition 3.4.** Let a \( g_n^r \) without fixed points be given on \( C \). We will call the \( g_n^r \) transverse if the following property holds;

There does not exist a subordinate \( g_{n'}^{r-1} \subset g_n^r \) such that \( n' \leq n - 2 \).

We then have;

**Lemma 3.5.** Let a \( g_n^r \) on \( C \) be given without fixed points. Then, if the \( g_n^r \) is transverse, it must be simple.

**Proof.** Suppose that the \( g_n^r \), defined by \( \Sigma \), is not simple, then it must be composite. Therefore, for generic \( x \in C \), there exists an \( x'(x) \notin \)
Base(Σ) such that every weighted set \( W_{\lambda} \) containing \( x \) also contains \( x'(x) \). As \( x \) is generic, \( x \notin Base(\Sigma) \), hence the subsystem \( \Sigma_x \subset \Sigma \), consisting of forms in \( \Sigma \) passing through \( x \), has dimension \( r - 1 \). Define a \( g_{n'}^{-1} \) from \( \Sigma_x \) by removing the fixed point contribution of \( g_{n'}^{-1}(\Sigma_x) \).

We claim that \( g_{n'}^{-1} \subset g_n^r \) (\(*\)). We clearly have that \( \Sigma_x \subset \Sigma \). As \( x \) is generic, \( x \notin \Sigma_x \). Suppose that \( p \) appears in a \( W_{\lambda} \) defined by \( g_{n'}^{-1} \) with multiplicity \( s \). Then, by definition, \( I_{\text{mobile}}(p, C, \phi_\lambda) = s \), calculated with respect to \( \Sigma_x \). It follows easily from the above lemmas that \( I_{\text{mobile}}(p, C, \phi_\lambda) = s \), calculated with respect to \( \Sigma \), as well. Hence, \( p \) also appears in the corresponding \( V_{\lambda} \) with multiplicity \( s \). This gives the claim (\(*\)). We show that \( n' \leq n - 2 \). Let \( \{p_1, \ldots, p_r\} \) be the base points of \( \Sigma \). Then \( \Sigma_x \) has base points containing the set \( \{p_1, \ldots, p_r, x, x'(x)\} \). Clearly, the fixed point contribution of \( \Sigma_x \) at \( \{p_1, \ldots, p_r\} \) can only increase the fixed point contribution of \( \Sigma \) and, moreover, the fixed point contribution of \( \Sigma_x \) at \( \{x, x'\} \) must be at least 2. Hence, using the Hyperspatial Bezout Theorem and the definition of \( \{g_n^r, g_{n'}^{-1}\} \), we have that \( n' \leq n - 2 \). This contradicts the fact that the original \( g_n^r \) was transverse. Hence, the \( g_n^r \) must be simple.

Following from this, we have the following;

**Lemma 3.6.** Let a \( g_n^r \) be given on \( C \) without fixed points such that \( g_n^r \) is transverse. Then the \( g_n^r \) defines a birational map \( \Phi : C \dashrightarrow C_1 \subset P^r \) and, moreover, \( C_1 \) has no multiple points.

**Proof.** The first part of the lemma follows from the previous Lemma 3.5 and Lemma 2.30. See the final section, though, for the problem in non-zero characteristic. It remains to prove that \( C_1 \) has no multiple points. Consider the \( g_n^r \) without fixed points on \( C_1 \), defined by the linear system \( \Sigma \) of hyperplane sections. By Lemma 2.31, there exists a corresponding \( g_{n'}^r \) on \( C \) without fixed points. By its construction in Lemma 2.31, it equals the original \( g_n^r \). Now suppose that there exists a multiple point \( p \) of \( C_1 \). We consider the subsystem \( \Sigma_p \) of hyperplane sections passing through \( p \). This defines a \( g_{n'}^{-1} \subset g_n^r \) and, as \( p \) is multiple, after removing the fixed point contribution, we obtain a \( g_{n'}^{-1} \subset g_n^r \) with \( n' \leq n - 2 \). Using Lemma 2.31 again, we obtain a corresponding \( g_{n'}^{-1} \) on \( C \). We claim that \( g_{n'}^{-1} \subset g_n^r \). This follows easily from the fact that \( \Sigma_p \subset \Sigma \), the \( g_{n'}^{-1} \) has no fixed points and the argument of the previous lemma. This contradicts the fact that the original \( g_n^r \) on \( C \) was transverse. Hence, \( C_1 \) has no multiple points. 

\( \square \)
We now give the proof of Theorem 3.3. We find a transverse $g_n^r$ on $C$ using a combinatorial argument;

Proof. (Theorem 3.3)

Suppose that $C$ has order $n \geq 2$ (the case when $C$ has order $n = 1$ is obvious.) Consider the independent system $\Sigma$ consisting of all plane algebraic curves defined by homogeneous forms of order $n - 1$. Clearly, no form in $\Sigma$ may contain $C$, and the system has no fixed points on $C$, hence the system defines a $g_{n_1}^{r_1}$, without fixed points, of dimension $r_1 = \frac{(n-1)(n+2)}{2}$ and order $n_1 = n(n-1)$. If this $g_{n_1}^{r_1}$ is transverse, the proof is complete. Hence, we may suppose that there exists a $g_{n_2}^{r_2} \subset g_{n_1}^{r_1}$, without fixed points, such that $r_2 = r_1 - 1$ and $n_2 \leq n_1 - 2$. Again, if this $g_{n_2}^{r_2}$ is transverse, the proof is complete. Hence, we may suppose that there exists a sequence;

$$g_{n_i}^{r_i} \subset g_{n_{i-1}}^{r_{i-1}} \subset \ldots \subset g_{n_1}^{r_1}$$

with $r_i = r_1 - (i - 1)$ and $n_i \leq n_1 - 2(i - 1)$. ($\ast$)

We need to show that this sequence terminates. By Lemma 2.24, we always have that $r_i \leq n_i$. Combining this with ($\ast$), we have that;

$$r_1 - (i - 1) \leq n_1 - 2(i - 1)$$

Therefore;

$$i \leq n_1 - r_1 + 1 = n(n - 1) - \frac{(n-1)(n+2)}{2} + 1 = \frac{(n-1)(n-2)}{2} + 1 \quad (\ast \ast)$$

Now suppose that equality is attained in ($\ast \ast$), then we would have that;

$$r_i = r_1 - (i - 1) = \frac{(n-1)(n+2)}{2} - \frac{(n-1)(n-2)}{2} = 2(n - 1) \geq 2$$

$$n_i \leq n_1 - 2(i - 1) = n(n - 1) - (n - 1)(n - 2) = 2(n - 1) \quad (\ast \ast \ast)$$

This implies that $n_i \leq r_i$ and hence $n_i = r_i$. Therefore, we must have that there exists a $g_{n_i}^{m_i} \subset g_{n_i}^{r_i}$ for each $i$ with $m \geq 2$ and the sequence terminates in fewer than $\frac{(n-1)(n-2)}{2} + 1$ steps. The final $g_{n_i}^{r_i}$ in the sequence then defines a birational map from $C$ to $C_1 \subset P^w$ without multiple points ($w \geq 2$), as required.
Remarks 3.7. The terminology of transverse $g^r_C$ is partly motivated by
the following fact. Suppose that a transverse $g^r_C$ on $C$ is defined by a
linear system $\Sigma$, possibly after removing some fixed point contribution,
then, if $x \in C \setminus \text{Base}(\Sigma)$ is non-singular, there exists an algebraic form
$\phi_{x}$ from $\Sigma$ such that $\phi_{x}(x) = 0$ but $\phi_{x}$ is not algebraically tangent to
$C$ at $x$. The proof of this follows straightforwardly from the definition of
transversality and the fact that, if $\phi_{x}$ is algebraically tangent to $C$ at
$x$, then $I_{\text{italian}}(x, C, \phi_{x}) \geq 2$ (see the proof of Lemma 4.2). In modern
terminology, one calls this property separating tangent vectors. See, for
example Proposition 7.3 of [2]. The full motivation, however, comes
from Theorem 6.10.

4. The Method of Conic Projections

The purpose of this section is to explore the Italian technique of
projecting a curve onto a plane. The method of conic projections is
extremely old and can be found in [1], where projective notions are
explicitly incorporated in the discussion of perspective. Severi himself
also wrote an article on the subject of [1] in [11]. We assume that we
are given a projective algebraic curve $C \subset P^w$ for some $w > 2$ and that
$C$ is not contained in any hyperplane section (otherwise reduce to this
lower dimension).

The construction;

Let $\Omega \subset P^w$ be a plane of dimension $w - k - 1$ and $\omega \subset P^w$ a plane
of dimension $2 \leq k < w$ such that $\Omega \cap \omega = \emptyset$. We define the projection
of $C$ from $\Omega$ to $\omega$ as follows;

Let $P \in C$. We may assume that $P$ does not lie on $\Omega$. Let $< \Omega, P >$
be the intersection of all hyperplanes containing $\Omega$ and $P$. It is a plane
of dimension $w - k$. Now, by elementary dimension theory, we must
have that;

$$\dim(< \Omega, P > \cap \omega) \geq k + (w - k) - w = 0$$

We may exclude the case that $< \Omega, P >$ and $\omega$ intersect in a line
$l$, as then $\Omega$ and $l$ would intersect in a point $Q$, contradicting the fact
that $\Omega \cap \omega = \emptyset$. Hence, $< \Omega, P > \cap \omega$ defines a point $pr(P)$. We may
repeat this construction for the cofinitely many points $U \subset C$ which
do not lie on $\Omega$. Now, consider the statement;

$$\phi(y) \equiv [y \in \omega \land \exists x \exists w (x \in \Omega \land w \in U \land y \in l_{xw})]$$

By elimination of quantifiers for algebraically closed fields, this clearly defines an algebraic set consisting of $\{pr(w) : w \in U\}$. We call this the projection $pr(U)$ of $U$. As $U$ is irreducible, it follows that $pr(U)$ is irreducible. Moreover, $pr(U)$ has dimension 1, otherwise $pr(U)$ would consist of a single point $Q$, in which case $U$ and therefore $C$ would be contained in the plane $\langle \Omega, Q \rangle$, contradicting the assumption. We define the projection $proj(C)$ of $C$ from $\Omega$ to $\omega$ to be the closure $pr(U)$ of $pr(U)$ in $\omega$. We can define a correspondence $\Gamma \subset U \times pr(U)$ by;

$$\Gamma(w, y) \equiv [y \in pr(U) \land w \in U \land \exists x (x \in \Omega \land y \in l_{xw})]$$

We define the associated correspondence $\Gamma_{pr} \subset C \times proj(C)$ to be the Zariski closure $\Gamma$. Note that, in the case when $\Omega \cap C = \emptyset$, the correspondence $\Gamma$ defines an algebraic function (in the sense of model theory) $pr$ from $C$ to $pr(C)$. By the model theoretic description of definable closure in algebraically closed fields of characteristic 0, $pr$ defines a morphism from $C$ to $pr(C)$. See the final section for the problem in non-zero characteristic.

**Remarks 4.1.** Note that when $\Omega$ is a point $P$ and $\omega$ is a hyperplane, the construction is equivalent to forming the cone;

$$Cone(C) = \bigcup_{x \in C} l_{xP}$$

and taking the intersection $\omega \cap Cone(C)$. This is the reason for the terminology of "conic projections". The case when $w = 3$ is explicitly discussed in [1], $P$ represents the eye of the observer wishing to obtain a representation of a curve on a plane.

We now prove the following general lemma on conic projections;

**Lemma 4.2.** Let $w \geq 3$ and suppose that $\{A, B\}$ are independent generic points of $C$. Then the line $l_{AB}$ does not otherwise meet the curve.

**Proof.** Suppose, for contradiction, that we can find independent generic points $\{A, B\}$ on $C$ such that $l_{AB}$ intersects $C$ in a new point $P$. As $A$ and $B$ are generic, they are non-singular points on the curve $C$, hence we can define the tangent lines $l_A$ and $l_B$ at these points. We claim the
For any hyperplane $H_\lambda$ containing $l_A$ and passing through $A$;

$$I_{italian}(A, C, H_\lambda) \geq 2 \quad (\ast)$$

This is the converse to a result already proved in Lemma 2.10 above. It can be proved in a similar way, namely fix a birational map $\Phi$ between $C$ and $C_1 \subset P^2$ such that $A$ and its corresponding non-singular point $A'$ on $C_1$ lie in the fundamental sets $V_{\Phi}$ and $W_{\Phi}$ (use the fact that $A$ is generic). By the chain rule, the corresponding $\phi_\lambda$ passes through $A'$ and is tangent to the curve $C_1$. Hence, by results of [6], we have that;

$$I_{italian}(A', C_1, \phi_\lambda) \geq 2$$

Using the technique of Section 2, it follows easily that $(\ast)$ holds as required.

Now choose a generic hyperplane $H$ in $P^w$. Let $pr$ be the projection of $C$ from $P$ to $H$. Let $D = pr(A) = pr(B)$. We have that $pr(C)$ is defined over $P$ and, moreover, that $D$ is generic in $pr(C)$. This follows as, otherwise, $dim(D/P) = 0$, therefore, as $\{A, B\} \subset l_{PD}$, $dim(A, B/P) = 0$, which implies $dim(A, B) = 1$, contradicting the fact that $\{A, B\}$ were independent generic. Hence, $D$ defines a non-singular point on the curve $pr(C)$. Now, let $l_1$ and $l_2$ be the projections of the lines $l_A$ and $l_B$. (We will deal with the degenerate case when one of these projections is a point below $(\dagger)$). Clearly $l_1$ and $l_2$ pass through $D$. We claim that they have the property $(\ast)$. Let $H_\lambda$ be a hyperplane of $H$ passing through $l_1$. Then the hyperplane $<H_\lambda, P>$ of $P^w$ passes through $l_A$. Let $\lambda' \in V_{\lambda}$ be generic in $Par_H$, then, we may assume that $D \notin pr(C) \cap H_{\lambda'}$. Therefore, the corresponding hyperplane $<H_{\lambda'}, P>$ does not pass through $A$. Now, using the property $(\ast)$ of $l_A$, Lemma 2.10 and the fact that $<H_{\lambda'}, P>$ is an infinitesimal variation of $<H_\lambda, P>$, we can find distinct $\{A_1, A_2\} \subset V_A \cap C \cap <H_{\lambda'}, P>$. It follows that $\{pr(A_1), pr(A_2)\} \subset V_D \cap pr(C) \cap H_{\lambda'}$. We claim that $pr(A_1)$ and $pr(A_2)$ are distinct. Suppose not, then $pr(A_1) = pr(A_2) = D' \in V_D \cap pr(C)$. Consider the following finite cover $F \subset pr(C) \times P^w$;

$$F(y, x) \equiv y \in pr(C) \land x \in C \cap l_y$$
As we have shown, \( (DA) \) is generic for this cover, hence, by properties of Zariski structures, \( \text{mult}_{(DA)}(F/pr(C)) = 1 \). However, this contradicts the fact that we can find \( D' \in \mathcal{V}_D \cap \text{pr}(C) \) and distinct \( \{A_1, A_2\} \subset \mathcal{V}_A \) such that \( F(D'A_1) \) and \( F(D'A_2) \). Therefore, we have shown that \( I_{\text{Italian}}(D, \text{pr}(C), H_\lambda) \geq 2 \). Hence, \( l_1 \) and \( l_2 \) have the property \((*)\). We now claim that \( l_1 = l_2 \) \((**)\). Suppose not, then, if \( l_D \) is the tangent line to \( \text{pr}(C) \) at \( D \), it must be distinct from say \( l_1 \). Choose a hyperplane \( H_\lambda \) of \( \omega \) passing through \( l_1 \) but not through \( l_D \). We have that \( I_{\text{Italian}}(D, \text{pr}(C), H_\lambda) \geq 2 \), but, using part of the proof of Lemma 2.10, \( H_\lambda \) must then pass through \( l_D \). Hence, \((**)\) is shown.

Now, by \((**)\), we have that the tangent lines \( l_A \) and \( l_B \) both lie on the plane spanned by \( l_B \) and \( l_{AB} \). Consider the statement;

\[ x \in \text{NonSing}(C) \text{ and the tangent line } l_x \text{ lies on the plane spanned by } l_B \text{ and } l_{xB}. \]

It defines an algebraic subset of \( \text{NonSing}(C) \) over \( B \), and, moreover, as it holds for \( A \), which was assumed to be generic in \( C \) and independent from \( B \), it defines an open subset \( V \) of \( C \). \((***)\)

Now choose a generic hyperplane \( H \) and let \( \text{pr}_B(C) \) be the projection of \( C \) from \( B \) to \( H \). By the proof (below) that the degenerate case \((\dagger)\) cannot occur, and the argument above, we can find an open \( W \subset V \), such that, for each \( x \in W \), \( l_x \) is projected to \( l_{\text{pr}_B(x)} \). We claim that;

For \( y \in \text{pr}_B(W) \), the \( l_y \) intersect in a point \( Q \). \((****)\)

Let \( y_1 \) and \( y_2 \) be independent generic points in \( \text{pr}_B(W) \), then we can find corresponding \( x_1 \) and \( x_2 \) in \( W \) such that \( < l_{B_{y_1}}, l_{y_1} > = < l_{B_{x_1}}, l_{x_1} > \) and \( < l_{B_{y_2}}, l_{y_2} > = < l_{B_{x_2}}, l_{x_2} > \), \((*****)\). As \( x_1 \) and \( x_2 \) lie in \( V \), by \((***)\), we have that;

\[ < l_{B_{x_1}}, l_{x_1} > \cap < l_{B_{x_2}}, l_{x_2} > = l_B \text{ \((******)\)} \]

We can assume that \( l_{y_1} \neq l_{y_2} \), \((\dagger\dagger)\), as if \( l_{y_1} = l_{y_2} \), then, considering \( \{y \in \text{pr}_B(C) : l_y = l_{y_1}\} \), we have all the tangent lines \( l_y \) of \( C \) equal \( l_{y_1} \). Now, if \( \text{pr}_B(C) \) is not a line \( l \), by Lemma 2.17, we can find a hyperplane \( H \), with \( I_{\text{Italian}}(p, \text{pr}_B(C), H) = 1 \), for \( p \in (\text{pr}_B(C) \cap H) \), and \( \text{Card} (\text{pr}_B(C) \cap H) \geq 2 \). Picking \( p, q \in (\text{pr}_B(C) \cap H) \), we have that \( l_p = l_q \), therefore, \( l_{pq} = l_p \subset H \). Then, by \((*)\), we have that \( I_{\text{Italian}}(p, C, H) \geq 2 \), a contradiction. Then \( \text{pr}_B(C) \) is a line, which,
we may assume, is not the case. Now, by \((\ast \ast \ast \ast \ast)\), we have that \(\langle l_{y_1}, l_{B_{y_1}} \rangle \cap \langle l_{y_2}, l_{B_{y_2}} \rangle = l_B, (\ast \ast \ast \ast \ast \ast)\). Then, using \((\dagger \dagger)\), and applying \(pr_B\) to both sides of \((\ast \ast \ast \ast \ast \ast \ast)\), we have that \(l_{y_1} \cap l_{y_2} = pr_B(l_B) = Q\), as required. Now, to obtain the final contradiction, we need to show that \((\ast \ast \ast)\) cannot occur. This also covers the degenerate case \((\dagger)\). We show;

If \(C\) is a projective algebraic curve in \(P^w\), with the property that there exists an open \(W \subset \text{Nonsing}(C)\) such that each \(l_x\) for \(x \in W\) intersects in a point \(Q\), then \(C\) is a line \(l\). \((\dagger)\)

Choose a generic hyperplane \(H\) and consider the projection \(pr_Q(C)\) of \(C\) from \(Q\) to \(H\). If this projection is a point, then \(C\) is a line \(l\). Hence, we may assume that \(pr_Q(C)\) is a projective algebraic curve in \(H\). Suppose that \(x\) is generic in \(pr_Q(C)\), with \(pr_Q(y) = x\) and \(y\) generic in \(C\). Then \(x\) and \(y\) are nonsingular and, moreover, if \(H_\lambda \subset H\) is any hyperplane passing through \(x\), the corresponding hyperplane \(\langle H_\lambda, Q \rangle\) passes through \(y\). By the assumption, \(\langle H_\lambda, Q \rangle\) contains the tangent line \(l_y\). Hence, by the proof given above,

\[
\min\{I_{\text{italian}}(y, C, \langle H_\lambda, Q \rangle), I_{\text{italian}}(x, C, H_\lambda)\} \geq 2.
\]

Now, as \(x\) is nonsingular, we can find a hyperplane \(H_\lambda\) passing through \(x\), not containing \(l_x\). By the usual argument, given above, we obtain a contradiction. Therefore, \((\dagger)\) is shown.

As we assumed that our original \(C\) was not contained in a \(P^2\), the proof of the lemma is shown. The lemma also holds in non-zero characteristic even though, surprisingly, the result \((\dagger)\) turns out to be false. We will deal with these problems in the final section.

\[\square\]

**Remarks 4.3.** The proof of the above lemma is attributed to Castelnuovo in [10], but I have been unable to find a convenient reference.

**Definition 4.4.** We will define a correspondence \(\Gamma \subset C_1 \times C_2\), where \(C_1\) and \(C_2\) are projective algebraic curves, to be generally biunivocal, if, for generic \(x \in C_1\) there exists a unique generic \(y \in C_2\) such that \(\Gamma(x, y)\) and \(\Gamma(x', y)\) implies that \(x = x'\) \((\ast)\). We will say that \(\Gamma\) is biunivocal at \(x\) if \((\ast)\) holds.

We then have as an immediate consequence of the above that;
**Lemma 4.5.** Let $C \subset P^w$, $(w \geq 3)$, be a projective algebraic curve, not contained in any hyperplane section. Fix some hyperplane $H$. Then, if $P$ is a generic point of the curve, the projection $pr_P$ from $P$ to $H$ is generally biunivocal on $C$.

We now note the following;

**Lemma 4.6.** Let $C \subset P^w$ be as in the above lemma. Fix a plane $\omega$ of dimension $w - k \geq 2$. Then, if $\{P_1, \ldots, P_k\}$ are independent generic points of $C$, the projection from $\Omega = \langle P_1, \ldots, P_k \rangle$ to $\omega$ is generally biunivocal.

**Proof.** Choose a sequence of planes $\omega = \omega_0 \subset \omega_1 \subset \ldots \subset \omega_{k-1}$ such that $\dim(\omega_i) = \dim(\omega) + i$ for $i \leq k - 1$, with field of definition equal to that of $\omega$. The projection $pr_{P_i}$ from $P_i$ to $\omega_{k-1}$ is generally biunivocal, and $\{pr_{P_1}(P_2), \ldots, pr_{P_1}(P_k)\}$ define independent generic points on $pr_{P_1}(C)$. Repeating the argument $k$ times, we obtain that the composition $pr_{P_k} \circ \ldots \circ pr_{P_1}$ is biunivocal as a projection to $\omega$. We claim that $pr_{<P_1,\ldots,P_k>} = pr_{P_k} \circ \ldots \circ pr_{P_1}$. This can be checked for a generic point $x$ of $C$. Suppose, inductively, that $pr_{<P_1,\ldots,P_i>} (x) = pr_{P_i} \circ \ldots \circ pr_{P_1} (x) = z$. Let $u = pr_{P_{i+1}} (z)$, then there exists $z' = <P_1, \ldots, P_{i+1}> \cap \omega_{k-i}$ such that $l_{z,z'} \cap \omega_{k-i-1} = u$. We have that $<P_1, \ldots, P_i, x > \cap \omega_{k-i} = z$, hence $<P_1, \ldots, P_i, P_{i+1}, x > \cap \omega_{k-i-1} = u$. This shows that $pr_{<P_1,\ldots,P_{i+1}>} (x) = u$ and, therefore, by induction, that $pr_{<P_1,\ldots,P_k>} (x) = pr_{P_k} \circ \ldots \circ pr_{P_1} (x)$. Hence (*) is shown. It follows immediately that the projection from $\Omega = \langle P_1, \ldots, P_k \rangle$ to $\omega$ is biunivocal on $C$ as required. \hfill \Box

As a consequence, we note the following, which is of independent interest;

**Lemma 4.7.** Let $C \subset P^w$ be as in the previous lemma. Let $\{P_1, \ldots, P_k\}$ be independent generic points on $C$ for $k \leq w - 1$. Then the hyperplane $<P_1, \ldots, P_k>$ does not otherwise encounter $C$.

**Proof.** By the previous lemma, the projection $pr_{<P_1,\ldots,P_{k-1}>}$ is generally biunivocal. Suppose there existed another intersection $Q$ of $<P_1, \ldots, P_k>$ with $C$. Then $pr_{<P_1,\ldots,P_{k-1}>}(Q) = pr_{<P_1,\ldots,P_{k-1}>}(P_k)$, contradicting the definition of generally biunivocal. \hfill \Box

**Theorem 4.8.** Let $C \subset P^w$ be a projective algebraic curve, then $C$ is birational to a plane projective curve.
Proof. By Lemma 4.6, we can find a generally biunivocal correspondence between \( C \) and \( C_1 \subset P^2 \). In characteristic 0, we can therefore find \( U \subset C \) and \( V \subset C_1 \) such that \( U \cong V \). This defines a birational map \( \Phi_C : C \leftarrow \rightarrow C_1 \). \( \square \)

We now use the techniques of this section to prove some further results which will be required later. We have first;

**Lemma 4.9.** Let \( C \subset P^w \) be a projective algebraic curve, not contained in any hyperplane section, and \( x \in C \). Then there exists a plane \( \Omega \) of dimension \( w - 3 \) and a plane \( \omega \) of dimension 2 such that the projection \( \text{pr} \) from \( \Omega \) to \( \omega \) is generally biunivocal and, moreover, biunivocal at \( x \).

*Proof.* Let \( H \) be a hyperplane containing \( x \), then, by the assumption on \( C \), it intersects \( C \) in a finite number \( r \) of points. It follows that we can find \( P \) generic in \( H \) such that \( l_{Px} \) does not otherwise encounter the curve \( C \). Now choose a further hyperplane \( H' \) not containing \( P \). Let \( Q \) be generic on \( C \) with \( Q \) independent from \( P \). Suppose that \( l_{PQ} \) intersects \( C \) in a new point \( R \), \((*)\). Then \( \{Q,R\} \) must form a generic independent pair. Otherwise, as \( P \in l_{QR} \), we would have that \( \text{dim}(P/Q) = 0 \) and therefore \( \text{dim}(P) = 0 \) which is a contradiction. Now, we can imitate the proof of Lemma 4.2 for the independent pair \( \{Q,R\} \) and the projection \( \text{pr}_P \) to obtain a contradiction. It follows that \((*)\) cannot occur, hence the projection \( \text{pr}_P \) is generally biunivocal and, moreover, by construction, biunivocal at \( x \). Now, repeat this argument \( w - 2 \) times, to find a sequence of points \( \{P_1,\ldots,P_{w-2}\} \) and planes \( \{\omega = \omega_1 \subset \omega_2 \subset \ldots \subset \omega_{w-1} = P^w\} \) such that the projection \( \text{pr}_{P_j} \) from \( \omega_{j+1} \) to \( \omega_j \) is generally biunivocal on \( C \) and, moreover, biunivocal at \( x \), for \( 1 \leq j \leq w - 2 \). It follows that the projection \( \text{pr}_{<P_1,\ldots,P_{w-2}>} \) from \( <P_1,\ldots,P_{w-2}> \) to \( \omega \) has the required property. \( \square \)

**Definition 4.10.** Let \( C \subset P^w \) be a projective algebraic curve. Let \( \text{pr} \) be a projection from \( \Omega \) to \( \omega \) such that \( \text{pr} \) is biunivocal at \( x \). We call \( <\Omega,x> \) the axis of the projection at \( x \).

We now refine Lemma 4.9 to ensure that the axis of projection is not in "special position" with respect to \( x \).

**Definition 4.11.** Special Position

Let \( C \subset P^w \) be a projective algebraic curve and \( x \) an \( s \)-fold point on \( C \), (see Definition 3.1). Let \( \Omega \) be a plane passing through \( x \). We
say that $\Omega$ is in special position if for every hyperplane $H$ containing $\Omega$;

\[ I_{\text{italian}}(x, C, H) \geq s + 1. \]

**Lemma 4.12.** Let hypotheses be as in Lemma 4.9. Let $x$ be an $s$-fold point of $C$. Then we can obtain the conclusion of Lemma 4.9, with the extra requirement that the axis of projection $<\Omega, x>$ is not in special position.

**Proof.** As $x$ is $s$-fold on $C$, we can find a hyperplane $H$ passing through $x$ such that $I_{\text{italian}}(x, C, H) = s$. Now imitate the proof of Lemma 4.9 by choosing a point $P_1$, generic on $H$ such that $l_{P_1x}$ does not otherwise encounter the curve $C$. Repeating the argument for the projected hyperplane $pr_{P_1}(H)$ in $P^{w-1}$, we obtain a series $\{P_1, \ldots, P_{w-2}\}$ as in the conclusion of Lemma 4.9 and, by construction, the axis of projection $<P_1, \ldots, P_{w-2}, x> \subset H$. Hence, $<\Omega, x>$ is not in special position. 

\[\square\]

We can now prove;

**Lemma 4.13.** Let $C \subset P^w$ be a projective algebraic curve. Suppose that $x \in C$ and $pr$ is a projection from $\Omega$ to $\omega = P^2$ satisfying the conclusion of Lemma 4.12. Then $x$ is $s$-fold on $C$ iff $pr(x)$ is $s$-fold on $pr(C)$.

**Proof.** Let $V = \{x \in C : pr \text{ is biunivocal at } x\}$ and $W = pr(V)$. $V$ and $W$ are open subsets of $C$ and $pr(C)$ respectively and are in bijective correspondence. Let $\{H_\lambda : \lambda \in \text{Par}_H\}$ be the independent system $\Sigma$ of lines in $P^2$. We then obtain a corresponding independent system $\Sigma'$ on $P^w$ defined by $\{<\Omega, H_\lambda > : \lambda \in \text{Par}_H\}$. Clearly, $\Sigma$ has no base points on $pr(C)$. We claim that $\Sigma'$ has no base points on $C$. Suppose that there existed a base point $y \in C$ for $\Sigma'$, then clearly $pr(y)$ would be a base point for $\Sigma$ on $pr(C)$, which is a contradiction. We now claim that for $y \in V$ and corresponding $pr(y) \in W$, that;

\[ I_{\text{italian}}(y, C, <\Omega, H_\lambda>) = I_{\text{italian}}(pr(y), pr(C), H_\lambda) \] (*)

By the fact that $\Sigma'$ has no base points, and results of Section 2, we have that;

\[ I_{\text{italian}}(y, C, <\Omega, H_\lambda>) = I_{\text{italian}}^{\Sigma'}(y, C, <\Omega, H_\lambda>) \]
Therefore, the result follows immediately from the definition of $I_{\text{Italian}}^\Sigma$ and the fact that $V$ and $W$ are in bijective correspondence.

We now claim that $deg(C) = deg(pr(C))$. Suppose that $deg(pr(C)) = d$, then, by the proof of Lemma 2.4, we can find a generic plane $H_\lambda$ intersecting $pr(C)$ transversely at $d$ distinct points inside $W$. By $(\ast)$, the corresponding hyperplane $<\Omega, H_\lambda>$ intersects $C$ transversely at $d$ distinct points inside $V$. Hence, by general properties of infinitesimals, $deg(C) = d$ as well.

Now, let $\Sigma''$ be the independent system defined by the lines in $P^2$ passing through $pr(x)$. By Bezout, it defines a $g_d^r(\Sigma'')$ on $pr(C)$. Suppose that $pr(x)$ is $s_1$-fold on $pr(C)$, then, after removing the fixed point contribution at $pr(x)$, we obtain a $g_{d-s_1}^r$ on $pr(C)$ without fixed points. Let $\Sigma'''$ be the independent system defined by $<\Omega,\Sigma''>$. It consists exactly of the hyperplanes passing through the axis of projection $<\Omega, x>$. Again, by hyperspatial Bezout, it defines a $g_d^r(\Sigma''')$ on $C$. As $pr$ was assumed to be biunivocal at $x$, this $g_d^r(\Sigma''')$ has a unique fixed point at $x$. Moreover, by the assumption on $pr$ that the axis of projection is not in special position, if $x$ is $s_2$-fold on $C$, then its fixed point contribution at $x$ is $s_2$. Hence, after removing this contribution, we obtain a $g_{d-s_2}^r$ on $C$ without fixed points. Now, using Lemma 2.17, for generic $\lambda$, the weighted set $W_\lambda$ for $g_{d-s_2}^r$ consists of $d - s_2$ points, each counted once, lying inside $V$. Using $(\ast)$, the corresponding $V_\lambda$ for $g_{d-s_1}^r$ consists of $d - s_2$ points, each counted once, lying inside $W$. By Lemma 2.17 again, we must have that $d - s_1 = d - s_2$, hence $s_1 = s_2$ as required. □

As an easy consequence of the above lemma, we have;

**Lemma 4.14.** Let $C \subset P^w$ be a projective algebraic curve and suppose that $x \in C$, then $x$ is non-singular iff $x$ is not multiple.

**Proof.** Choose a projection $pr$ as in Lemma 4.13. Then, $x$ is not multiple iff $pr(x)$ is not multiple. By Lemma 3.2, $pr(x)$ is not multiple iff $pr(x)$ is non-singular. In characteristic 0, using the fact that $pr$ is generally biunivocal and an elementary model theoretic argument, we can find an inverse morphism $\phi : W' \to V'$, where $\{W', V'\}$ are open subsets of $\{W, V\}$ given in the previous lemma. It follows that $V' \cong W'$. As $pr(x)$ is non-singular, we can extend the morphism $\phi$ to include $pr(x)$ in its domain, by biunivocity of $pr$ at $x$, we must have that $\phi(pr(x)) = x$. Hence, we may assume that $x$ lies in $V'$ and $pr(x)$
lies in $W'$. Therefore, $pr(x)$ is non-singular iff $x$ is non-singular. Hence, the result is shown.

Combining this result with Theorem 3.3, we then have the following;

**Theorem 4.15.** Let $C \subset P^w$ be a projective algebraic curve, then $C$ is birational to a non-singular projective algebraic curve $C_1 \subset P^{w'}$.

We finish this section with the following application of the method of Conic Projections, the result will not be required later in the paper. Some part of the proof will require methods developed in Sections 5 and 6, we refer the reader to Definition 6.3 for the terminology ”node”. I have not seen a reasonable algebraic proof of this result.

**Theorem 4.16.** Let $C \subset P^w$ be a projective algebraic curve, then $C$ is birational to a plane projective algebraic with at most nodes as singularities.

**Proof.** We may, by the previous theorem, assume that $C$ is non-singular. We first reduce to the case $w = 3$. That is, we claim that $C$ is birational to a non-singular projective algebraic curve $C' \subset P^3$. Let $V_3$ be the variety of chords on $C$. That is;

$$V_3 = \overline{\bigcup ab : (a, b) \in C^2 \setminus \Delta}$$

We may assume that $C$ is not contained in any plane of dimension 2 $\dagger$ and that $w \geq 4$. We then claim that $V_3$ has dimension 3. Let $a \in C$ and define;

$$\text{Cone}_a = \overline{\{ \bigcup l_{ab} : b \in (C \setminus a) \}}$$

As $C$ is irreducible, so is $\text{Cone}_a$. (use the fact that for any component $W$ of $\text{Cone}_a$, $\{b \in (C \setminus a) : l_{ab} \subset W\}$ is a closed subset of $C \setminus a$). Suppose that $\text{Cone}_a$ has dimension 1. Then there exists $b$ such that;

$$\text{Cone}_a = l_{ab}$$

It follows immediately that $C$ is contained in $l_{ab}$, contradicting the hypothesis (†). Hence, $\text{Cone}_a$ has dimension 2. Now, suppose that $V_3$ has dimension 2. Then there exist finitely many points $\{a_1, \ldots, a_m\}$ such that;
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\[ V_3 = \text{Cone}_{a_1} \cup \ldots \cup \text{Cone}_{a_m} \]

In particular, we can find distinct \( \{a, b\} \subset C \) such that \( \text{Cone}_a = \text{Cone}_b \). Choose \( a' \in C \) distinct from \( \{a, b\} \). Then we may assume that \( l_{ax} \) has infinite intersection with \( \text{Cone}_b \setminus \delta(\text{Cone}_a) \) and therefore \( C \subset H_{aa'b} \), contradicting the hypothesis (†). Hence, \( V_3 \) has dimension 3 as required. Now define the tangent variety on \( C \):

\[ \text{Tang}(C) = \{ \bigcup l_a : a \in C, l_a \text{ is the tangent line to } C \text{ at } a \} \]

We claim that \( \text{Tang}(C) \) is a closed subvariety of \( V_3 \) of dimension 2. In order to see that \( \text{Tang}(C) \subset V_3 \), it is sufficient to prove that \( l_a \subset \text{Cone}_a \). Consider the following covers \( F, F^* \subset (C \setminus \{a\}) \times P^w \);

\[ F(x, y) \equiv x \in (C \setminus \{a\}) \land y \in l_{ax} \]
\[ F^*(x, \lambda) \equiv x \in (C \setminus \{a\}) \land H_\lambda \text{ contains } l_{ax} \]

Let \( \bar{F} \) and \( \bar{F}^* \) be the closures of these covers inside \( C \times P^w \). We have the incidence relation \( I \subset P^w \times P^w \) given by;

\[ I(x, \lambda) \equiv x \in H_\lambda \]

By definition, for \( x \in (C \setminus \{a\}) \), we have that;

\[ I(\bar{F}(x), \bar{F}^*(x)) \]

Hence, this relation holds at \( \{a\} \) as well (⋆). Now, by the proof of Theorem 6.7, the fibre \( \bar{F}^*(a) \) consists exactly of the hyperplanes \( H_\lambda \) containing \( l_a \). Hence, by (⋆), we must have that the fibre \( F(a) \) defines \( l_a \). In order to complete the proof, observe that the projection \( pr : C \times P^w \to P^w \) defines a morphism from \( F \) to \( \text{Cone}_a \). Let;

\[ \Gamma_{pr}(x, y, z) \subset \bar{F} \times \text{Cone}_a \subset C \times P^w \times P^w \]

be the graph of this projection. Then, for \( x \in (C \setminus \{a\}) \), the fibre \( \Gamma_{pr}(x) = l_{ax} \times l_{ax} \). Hence, the fibre \( \Gamma_{pr}(a) = l_a \times l_a \). This proves that \( l_a \subset \text{Cone}_a \) as required. We clearly have that \( \text{Tang}(C) \) has dimension 2 if \( C \) is not contained in a line \( l \), contradicting the hypothesis (†). The fact that \( \text{Tang}(C) \) is a closed subvariety of \( V_3 \) follows immediately from the fact that \( \text{Tang}(C) \) is a closed subvariety of \( P^w \).
Now choose a plane $\Omega$ of dimension $w - 4$ such that $\Omega \cap V_3 = \emptyset$. Let $pr$ be the projection from $\Omega$ to $\omega$ where $\omega$ is a plane of dimension 3 and $\Omega \cap \omega = \emptyset$. Let $pr(C) \subset \omega$ be the projection of $C$. We claim that $pr(C)$ is birational to $C$ and non-singular. First, observe that $pr$ is defined everywhere on $C$ as $C \subset V_3$ and $\Omega \cap V_3 = \emptyset$. Secondly, we have that $pr$ is biunivocal everywhere on $C$. For suppose that $pr(x) = pr(x')$, then $<\Omega, x> = <\Omega, x'> = <\Omega, pr(x)>$, hence $l_{xx'}$ intersects $\Omega$, contradicting the fact that $\Omega \cap V_3 = \emptyset$. Finally, we have that for every $x \in C$, the axis of projection $<\Omega, x>$ is not in special position, see Definition 4.11. This follows from the fact that, as $x$ is non-singular, $<\Omega, x>$ would only be in special position if it contained the tangent line $l_x$. As $Tang(C) \subset V_3$, this would contradict the fact that $\Omega \cap V_3 = \emptyset$. Now, by Lemmas 4.13 and 4.14, we must have that $pr(x)$ is non-singular for every $x \in C$, that is $pr(C)$ is non-singular. We may, therefore, using the argument of Lemma 4.14 invert the morphism $pr$ to obtain an isomorphism between $C$ and $pr(C)$. In particular, $C$ and $pr(C)$ are birational.

We now consider the curve $C' = pr(C) \subset P^3$. In order to prove the theorem, it will be sufficient to show that $C'$ is birational to a plane projective curve with at most nodes as singularities. We now define the following 5 varieties, we use the notation $l$ to denote a line and $P$ to define a 2-dimensional plane;

(i). $Tangent(C') = \{ \bigcup l_a : a \in C, l_a \text{ the tangent line at } C \}$

(ii). $Trisecant(C') = \{ \bigcup l_{abc} : (a, b, c) \in C^3 \text{ distinct}, l_{abc} \supset \{a, b, c\} \}$

By a bitangent plane $P_{ab}$, we mean a hyperplane passing through the tangent lines $l_a$ and $l_b$ for distinct $\{a, b\}$ in $C$. By an osculatory plane $P_a$, we refer the reader to Definition 6.3.

We define $Supp(P_{ab}) = \{a, b\}$ and $Supp(P_a) = C' \cap P_a$. We then consider;

(iii). $Bitangent Chord(C') = \{ \bigcup l_{ab} : (a, b) \in (C^2 \setminus \Delta), l_{ab} \supset Supp(P_{ab}) \}$

(iv). $Osculatory Chord(C') = \{ \bigcup_{a \in C} l_{ab} : (a, b) \in (Supp(P_a)^2 \setminus \Delta) \}$

By Remark 6.6, there exists a finite set $W \subset C'$, consisting of points which are the origins of non-ordinary branches. We let;

(v). $Singular Cone(C') = \bigcup_{y \in W} Cone_y$
As we have already seen, \( \text{Tangent}(C') \) defines a 2-dimensional algebraic variety, unless \( C' \) is contained in a line, which we may assume is not the case. By Lemma 4.2, there are no trisecant lines passing through an independent generic pair \((a,b)\) of \( C'^2 \), unless \( C' \) is contained in a plane, which again we may exclude. The statement \( D(x,y) \subset C'^2 \setminus \Delta \), given by:

\[
D(x,y) \equiv \{(x,y) \in C'^2 : (x \neq y) \land \forall w (w \neq x \land w \neq y \land w \in C' \cap l_{xy})\}
\]

is clearly constructible and has the property that \( \overline{D} = C'^2 \). This clearly implies that \( \dim(\text{Trisecant}(C')) \leq 2 \). We now claim that, if \((a,b)\) is an independent generic pair in \( C'^2 \), there is no bitangent plane \( P_{a,b} \). Let \( \Sigma = \{ H_\lambda : \lambda \in P^3 \} \) be the system of hyperplanes in \( P^3 \). \( \Sigma \) has no fixed points on \( C' \), hence the linear condition that \( H_\lambda \) passes through \( l_a \) has codimension 2 and defines a 1-dimension family \( \Sigma_1 \subset \Sigma \). As we may assume that \( C' \) is not contained in any hyperplane section, \( \Sigma_1 \) has finite intersection with \( C' \). As \( b \) is independent generic from \( a \), it cannot be a base point for the new system \( \Sigma_1 \). Hence, using the results of Theorems 6.2 and 6.5, the condition that \( H_\lambda \in \Sigma_1 \) passes through \( l_b \) is also a codimension 2 linear condition. As \( \Sigma_1 \) is 1-dimensional, this implies the claim. Now consider the statement \( D'(x,y) \subset C'^2 \setminus \Delta \), given by:

\[
D'(x,y) \equiv \{(x,y) \in C'^2 : (x \neq y) \land \forall \lambda (H_\lambda \supset l_a \cup l_b)\}
\]

Again, this is constructible and has the property that \( \overline{D'} = C'^2 \). This clearly implies that \( \dim(\text{Bitangent Chord}(C')) \leq 2 \). For any given \( a \in C \), there exists a unique osculatory plane \( P_a \) passing through \( a \). Hence, there exist finitely many lines of the form \( l_{ab} \) for \( b \in (C^2 \setminus \Delta) \cap \text{Supp}(P_a) \). This clearly implies that \( \dim(\text{Osculatory Chord}(C')) \leq 2 \). Finally, by the above consideration, we have that \( \dim(\text{Singular Cone}(C')) = 2 \).

We now choose a point \( P \) in \( P^3 \) such that \( P \) lies outside the above defined varieties. This is clearly possible as they all have dimension at most 2. It is easily shown that a generic \( P \), lies on finitely many bisecant lines of \( C' \), (†), see [2]. Let \( \omega \) be a plane of dimension 2 not containing \( P \) and let \( \text{pr}_P(C') \) be the projection of \( C' \) from \( P \) to \( \omega \). We claim that \( \text{pr}_P(C') \) is birational to \( C' \) with at most nodes as singularities. Clearly, the birational property holds by (†). Now, first, suppose that \( y \in \text{pr}_P(C') \) is singular, then, by Lemma 4.14, it must be multiple. By Lemma 4.13 and the fact that \( C' \) is non-singular, this can only occur if
either $|pr_P^{-1}(y)| \geq 2$ or there exists a unique $x$ such that $pr_P(x) = y$ and the axis of projection $< P, x >$ is in special position. We may exclude the second possibility on the grounds that $P$ is disjoint from $Tang(C')$. We may therefore assume that $|pr_P^{-1}(y)| \geq 2$. Now, we may exclude the possibility that $|pr_P^{-1}(y)| \geq 3$, on the grounds that $P$ is disjoint from $Trisecant(C')$. Hence, we may assume that $|pr_P^{-1}(y)| = 2$. Now, as $C'$ is non-singular, by Definition 5.2, there exist exactly 2 branches centred at $y$ on $pr_P(C')$. Moreover, we may assume that both elements of $pr_P^{-1}(y) = \{x_1, x_2\}$ are the origins of ordinary branches, otherwise $P$ would lie inside $Singular Cone(C')$. If $P$ is situated on the osculatory plane $Px_1$ of say $x_1$, then $(x_1x_2) \subset Supp(P_{x_1})$ and, hence, as $P \in l_{x_1x_2}$, we would have that $P$ lies in $Osculatory Chord(C')$, which is not the case. Hence, we may apply Theorem 6.4, to obtain that both branches have character $(1, 1)$. Finally, let $l_a$ and $l_b$ be the tangent lines to the 2 branches $\{\gamma_1^y, \gamma_2^y\}$ at $y$. We claim that $l_a$ and $l_b$ are distinct ($\ast$). By definition, we have that;

$$I_{italian}(y, \gamma_1^y, pr_P(C'), l_a) = 2$$

Hence, using Definition 5.15 and Lemma 5.17, we can find an infinitesimal variation $l'_a$ of $l_a$ intersecting $\gamma_1^y$ in distinct points $\{y', y''\}$. By Definition 5.15, we can find distinct points $\{x'_1, x''_1\}$ in $C' \cap V_{x_1}$ such that $pr_P(x'_1) = y'$ and $pr_P(x''_1) = y''$. We clearly have that that $< l'_a, P >$ is an infinitesimal variation of $< l_a, P >$ which intersects $C'$ in $\{x'_1, x''_1\}$. Therefore;

$$I_{italian}(x_1, C', < l_a, P >) \geq 2$$

and hence, by previous arguments, $< l_a, P >$ contains the tangent line $l_{x_1}$. As the axis of projection $l_{P_{x_1}}$ was not in special position, we must have that $pr_P(l_{x_1}) = l_a$. Similarly, $pr_P(l_{x_2}) = l_b$. Now, if $l_a = l_b$, we would have that $< l_a, P >$ is a bitangent plane to $C'$, $l_{x_1x_2}$ is a bitangent chord and $P$ would belong to the variety $Bitangent Chord(C')$. As this is not the case, the claim ($\ast$) is proved. This completes the theorem.

\[\square\]

5. A Theory of Branches for Algebraic Curves

We now develop a theory of branches for algebraic curves, using the techniques of the previous sections.
Let $C \subset P^w$ be a projective algebraic curve. Then, by Theorem 4.15, we can find $C^{ns} \subset P^w$ which is non-singular and birational to $C$. Let $\Phi^{ns} : C^{ns} \leftrightarrow C$ be the birational map between $C^{ns}$ and $C$. As $C^{ns}$ is non-singular, $\Phi^{ns}$ extends to a totally defined morphism from $C^{ns}$ to $C$. As usual, we let $\Gamma_{\Phi^{ns}}$ denote the graph of the correspondence between $C^{ns}$ and $C$. It has the property that, given any $x \in C^{ns}$, there exists a unique $y \in C$ such that $\Gamma_{\Phi^{ns}}(x, y)$. We claim the following;

**Lemma 5.1.** Let $C_1 \subset P^{w_1}$ and $C_2 \subset P^{w_2}$ be any two non-singular birational models of $C$, with corresponding morphisms $\Phi_1^{ns}$ and $\Phi_2^{ns}$. Then there exists a unique isomorphism $\Phi : C_1 \leftrightarrow C_2$ with the property that $\Phi_2^{ns} \circ \Phi = \Phi_1^{ns}$ and $\Phi_1^{ns} \circ \Phi^{-1} = \Phi_2^{ns}$.

**Proof.** As $\Phi_2^{ns}$ is a birational map, we can invert it to give a birational map $\Phi_2^{-1} \circ \Phi_1^{ns} : C_1 \leftrightarrow C_2$. Then the composition $\Phi_2^{-1} \circ \Phi_1^{ns} : C_1 \leftrightarrow C_2$ is birational as well. As $C_1$ is non-singular, we can extend this birational map to a totally defined morphism $\Phi : C_1 \rightarrow C_2$. By the same argument, we can find a totally defined morphism $\Psi : C_2 \rightarrow C_1$ with the property that $\Psi$ inverts $\Phi$ as a birational map. We claim that $\Psi$ inverts $\Phi$ as a morphism. We have that $\Psi \circ \Phi : C_1 \rightarrow C_1$ is a morphism with the property that there exists an open $U \subset C_1$ such that $\Psi \circ \Phi | U = Id_U$. Then $Graph(\Psi \circ \Phi) \subset C_1 \times C_1$ is closed irreducible and intersects the diagonal $\Delta$ in an open dense subset. Therefore, $Graph(\Psi \circ \Phi) = \Delta$, hence $\Psi \circ \Phi = Id_{C_1}$. Similarly, one shows that $\Phi \circ \Psi = Id_{C_2}$. Therefore, $\Phi$ is an isomorphism. By construction, $\Phi_1^{ns}$ and $\Phi_2^{ns} \circ \Phi$ agree as birational maps and are totally defined, hence, by a similar argument, they agree as morphisms. Similarly, $\Phi_2^{ns}$ and $\Phi_1^{ns} \circ \Phi^{-1}$ agree as morphisms. For the uniqueness statement, use the fact that any 2 isomorphisms $\Phi_1$ and $\Phi_2$, satisfying the properties of $\Phi$, would agree on an open subset $U$ of $C_1$, hence must be identical. \hfill $\square$

Now fix a point $O$ of $C$. Let $\Gamma_{\Phi^{ns}} \subset C^{ns} \times C$ be the graph of the correspondence defined above. We denote by $\{O_1, \ldots, O_t\}$ the fibre $\Gamma_{\Phi^{ns}}(x, O)$. Note that, by the previous lemma, if we are given 2 non-singular models with correspondences $\Gamma_{\Phi_1^{ns}}$ and $\Gamma_{\Phi_2^{ns}}$, then we have a correspondence $O_1 \sim O_1', O_2 \sim O_2', \ldots, O_t \sim O_t'$ between the fibres $\Gamma_{\Phi_1^{ns}}(x, O)$ and $\Gamma_{\Phi_2^{ns}}(x, O)$, given by $O_j \sim O_j'$ iff $\Phi(O_j) = O_j'$, where $\Phi$ is the isomorphism given by the previous lemma. Moreover, as $\Phi$ is an isomorphism, we also have a correspondence of infinitesimal neighborhoods $V_{O_1} \sim V_{O_1}', \ldots, V_{O_t} \sim V_{O_t}'$, given by $V_{O_j} \sim V_{O_j}'$ iff $\Phi : V_{O_j} \cong V_{O_j}'$, here we mean that $\Phi$ is a bijection of sets. We now make the following
Definition 5.2. Let \( C \subset P^w \) be a projective algebraic curve. Suppose that \( O \in C \). For \( 1 \leq j \leq t \), we define the branches \( \gamma^j_O \) at \( O \) to be the equivalence classes \([V_O^j]\) of the infinitesimal neighborhoods of \( O_j \) in the fibre \( \Gamma^\text{ns}(x, O) \) of any non-singular model \( C^\text{ns} \) of \( C \). We define \( \gamma_O \) to be the union of the branches at \( O \).

Remarks 5.3. Note that the definition does not depend on the choice of a non-singular model \( C^\text{ns} \), however, for computational purposes, it is convenient to think of a branch \( \gamma \) as the infinitesimal neighborhood of some \( O_j \) in a fixed non-singular model \( C^\text{ns} \).

We now have the following lemmas concerning branches;

Lemma 5.4. Let \( C \subset P^w \) and \( O \in C \) be an \( s \)-fold point. Then \( O \) is the origin of at most \( s \) branches. In particular, a non-singular point is the origin of a single branch.

Proof. Suppose that \( O \) is the origin of \( t \) branches. Then there exists a non-singular model \( C^\text{ns} \) and a birational map \( \Phi : C^\text{ns} \to C \) such that \( \Phi^{-1}(O) = \{O_1, \ldots, O_t\} \). By a slight extension to Remarks 1.32, and the fact that \( \{O_1, \ldots, O_t\} \) are non-singular, we may assume that \( \text{Base}(\Sigma) \) is disjoint from this set (*). Let \( \Sigma \) define the system of hyperplanes in \( P^w \). It defines a \( g(\Sigma) \) without fixed points on \( C \), where \( d = \text{deg}(C) \).

By Lemma 2.32, there is a corresponding \( g(\Sigma) \) on \( C^\text{ns} \), where \( I \) is the total fixed point contribution at \( \text{Base}(\Sigma) \). Now let \( \Sigma' \subset \Sigma \) define the system of hyperplanes passing through \( O \). It defines a \( g(\Sigma') \) on \( C \) with corresponding \( g(\Sigma') \) on \( C^\text{ns} \). As \( O \) is \( s \)-fold on \( C \), we can find \( g(\Sigma') \subset g(\Sigma') \) on \( C \) without fixed points. By Lemma 2.31, we can find a corresponding \( g(\Sigma') \subset g(\Sigma') \) on \( C^\text{ns} \), without fixed points (**). Now each \( \phi_\lambda \) in \( \Sigma' \) must pass through \( \{O_1, \ldots, O_t\} \). Moreover, the fixed point contribution from \( \Sigma' \) at \( \text{Base}(\Sigma) \) must be at least \( I \).

Hence, by the assumption (*), the total fixed point contribution from \( \text{Base}(\Sigma') \) must be at least \( t + I \). Hence, by (**), \( d - s \leq (d + I) - (t + I) \), therefore \( t \leq s \). The lemma is proved.

\( \square \)

We now make the following definition;

Definition 5.5. Let \( C_1 \subset P^w_1, C_2 \subset P^w_2, C_3 \subset P^w_3 \) be birational projective algebraic curve with correspondences \( \Gamma_{\Phi_1} \subset C_1 \times C_2 \) and
Γ_Φ_2 ⊂ C_2 × C_3. We define the composition Γ_Φ_2 ∘ Γ_Φ_1 ⊂ C_1 × C_3 to be;

Γ_Φ_2 ∘ Γ_Φ_1(xz) ≡ ∃ y ∈ C_2 ∧ Γ_Φ_1(xy) ∧ Γ_Φ_2(yz))

**Lemma 5.6.** Let hypotheses be as in the previous lemma, then if Φ_1 : C_1 → C_2 and Φ_2 : C_2 → C_3 are birational maps;

Γ_Φ_2 ∘ Γ_Φ_1 = Γ_Φ_2 ∘ Φ_1

**Proof.** The proof is a straightforward consequence of the fact that both correspondences obviously agree on a Zariski open subset. □

**Lemma 5.7.** Birational Invariance of Branches

Let C_1 ⊂ P^{w_1} and C_2 ⊂ P^{w_2} be birational projective algebraic curves with correspondence Γ[Φ] ⊂ C_1 × C_2. Fix O ∈ C_2 and let \{P_1, \ldots, P_s\} = Γ[Φ](x, O). Then, [Φ] induces an injective map;

[Φ] : \gamma_O → U_{1≤k≤s} γ_{P_k}

and, moreover;

[Φ] : U_{O∈C_2} γ_O → U_{O∈C_1} γ_O

is a bijection, with inverse given by [Φ^-1]*.

**Proof.** We first claim that there exists a non-singular model C^{ns}_1 of C_1 and C_2 with morphisms Φ_1 : C^{ns}_1 → C_1 and Φ_2 : C^{ns}_2 → C_2 such that Φ_2 = Φ ∘ Φ_1 and Φ_1 = Φ^-1 ∘ Φ_2 as birational maps (*). Choose a non-singular model C^{ns}_1 of C_1 with morphism Φ_1 : C^{ns}_1 → C_1. Then Φ ∘ Φ_1 defines a birational map from C^{ns}_1 to C_2. As C^{ns}_1 is non-singular, it extends (uniquely) to a birational morphism Φ_2 : C^{ns}_2 → C_2. Clearly, \{C^{ns}_1, Φ_1, Φ_2\} have the property (*). In order to define the map [Φ] *, let O ∈ C_2 and suppose that γ^j_O is a branch corresponding to the equivalence class [V_O], where O_j lies in the fibre Φ_2^-1(O). We claim that Φ_1(O_j) lies in the fibre Γ[Φ](x, O) (**). As Φ ∘ Φ_1 = Φ_2 as birational maps, it follows, by Lemma 1.21, that Γ_Φ ∘ Φ_1 = Γ_Φ_2. However, by the previous lemma, Γ_Φ ∘ Φ_1 = Γ_Φ ∘ Φ_1, which gives (**). Then, if Φ_1(O_j) = P_k, and Φ_1^-1(P_k) = \{P'_k, \ldots, P'_kr, \ldots, P'_kl\}, for 1 ≤ r ≤ l, we must have that [V_O] = [V_{P'_r}], for some r. Hence, we can set [Φ]^*(γ^j_O) = γ^r_{P_k}. One needs to check that this definition does not depend
either on the choice of the non-singular model $C^{ns}$ or the choice of the birational map $\Phi$ representing the correspondence $\Gamma_{[\Phi]}$. For the first claim, after fixing a choice of $\Phi$, note that by the requirement $(*)$, $\Phi_1$ is uniquely determined by the choice of $\{C^{ns}, \Phi_2\}$. By Lemma 5.1, for any 2 choices $\{C^{ns}_1, \Phi_1\}$ and $\{C^{ns}_2, \Phi_2\}$, there exists a connecting isomorphism $\Theta : (C^{ns}_1, \Phi_1) \leftrightarrow (C^{ns}_2, \Phi_2)$, which must therefore be a connecting isomorphism $\Theta : (C^{ns}_1, \Phi_1) \leftrightarrow (C^{ns}_2, \Phi_1)$. The result then follows immediately by definition of the branch as an equivalence class of infinitesimal neighborhoods. For the second claim, after fixing a choice of $\{C^{ns}, \Phi_1, \Phi_2\}$ and replacing $\Phi$ by an equivalent $\Phi'$, one still obtains $(*)$ for the original $\{C^{ns}, \Phi_1, \Phi_2\}$ and the result follows trivially. The rest of the lemma now follows by proving that $[\Phi^{-1}]^*\text{ inverts } [\Phi]^*$. This is trivial to check using the particular choice $\{C^{ns}, \Phi_1, \Phi_2, \Phi\}$. □

Remarks 5.8. Note that points are not necessarily preserved by birational maps, but, by the above, branches are always preserved.

We now refine the Italian definition of intersection multiplicity, in order to take into account the above construction of a branch. Suppose that $C \subset P^w$ is a projective algebraic curve. We let $Par_F$ be the projective parameter space for all hypersurfaces of a given degree $e$. Now, fix a particular form $F_\lambda$ of degree $e$, such that $F_\lambda$ has finite intersection with $C$. The condition that a hypersurface of degree $e$ contains $C$ is clearly linear on $Par_F$, (use Theorem 2.3 to show that the condition is equivalent to the plane condition $P$ on $Par_F$ of passing through $de + 1$ distinct points on the curve $C$.) Now fix a maximal linear system $\Sigma$ containing $F_\lambda$, such that $\Sigma$ has empty intersection with $P$. It is trivial to check that the corresponding $g^*_n(\Sigma)$ on $C$ has no fixed points. Now fix a non-singular model $C^{ns}$ of $C$, with corresponding birational morphism $\Phi_\Sigma$. By the transfer result Lemma 2.31, we obtain a corresponding $g^*_n$ without fixed points on $C^{ns}$. Let $Card(O, V_\lambda, g^*_n)$ denote the number of times $O \in C^{ns}$ is counted for this $g^*_n$ in the weighted set $V_\lambda$. We then define;

Definition 5.9. $I_{\text{italian}}(p, \gamma^j_p, C, F_\lambda) = Card(p_j, V_\lambda, g^*_n) \ (\ast)$

where the branch $\gamma^j_p$ corresponds to $[V_{p_j}]$ in the fibre $\Gamma_{[\Phi_\Sigma]}(x, p)$.

Remarks 5.10. One may, without loss of generality, assume that $Base(\Sigma')$ is disjoint from the fibre $\Gamma_{[\Phi_\Sigma]}(x, p)$. In which case, the definition becomes the more familiar;
\[ I_{\text{italian}}(p, \gamma_p^j, C, F_\lambda) = \text{Card}(C^{ns} \cap F_\lambda \cap V_{p_j}), \lambda' \in V_\lambda \text{ generic in } Par_\Sigma. \]

Here, we have used the notation \( \{ F_\lambda : \lambda \in Par_\Sigma \} \) to denote the family of "lifted" forms on \( C^{ns} \) corresponding to the family \( \{ F_\lambda : \lambda \in Par_\Sigma \} \) on \( C \), by way of the birational morphism \( \Phi_{\Sigma'} \).

**Lemma 5.11.** Given \( \Sigma \) containing \( F_\lambda \). The definition \((\ast)\) does not depend on the choice of the non-singular model \( C^{ns} \) and birational morphism \( \Phi_{\Sigma'} \).

**Proof.** We divide the proof into the following 2 cases;

**Case 1.** \((C^{ns}, \Phi)\) is fixed and we have 2 presentations \( \Phi_{\Sigma_1} \) and \( \Phi_{\Sigma_2} \) of the birational morphism \( \Phi \), as given by Lemma 1.20, such that \( \text{Base}(\Sigma_1) \) possibly includes some of the points in \( \Gamma_{[\Phi]}(x, p) \), while \( \text{Base}(\Sigma_2) \) is disjoint from this set.

Let \( V = V_{\Phi_{\Sigma_1}} \cap V_{\Phi_{\Sigma_2}} \), with corresponding \( W \subset C \). Denote the weighted sets of the 2 given \( g_n^\nu \) on \( C^{ns} \), in the definition \((\ast)\), corresponding to the presentations \( \Phi_{\Sigma_1} \) and \( \Phi_{\Sigma_2} \), by \( \{ V_1^\lambda \} \) and \( \{ V_2^\lambda \} \). By the proof of Lemma 2.31, if \( x \in V \), then \( x \) is counted \( s \)-times in \( V_1^\lambda \) if \( x \) is counted \( s \)-times in \( V_2^\lambda \) \((\dagger)\). Now, we claim that \( \text{Card}(p_j, V_1^\lambda, g_n^\nu) = \text{Card}(p_j, V_2^\lambda, g_n^\nu) \). This follows by \((\dagger)\) and an easy application of, say Lemma 2.16, to witness both these cardinalities in the canonical set \( V \).

**Case 2.** We have 2 non-singular models \((C_1^{ns}, \Phi_1)\) and \((C_2^{ns}, \Phi_2)\), with presentations \( \Phi_{\Sigma_1} \) and \( \Phi_{\Sigma_2} \), such that \( \text{Base}(\Sigma_1) \) is disjoint from \( \Gamma_{[\Phi_1]}(x, p) \) and \( \text{Base}(\Sigma_2) \) is disjoint from \( \Gamma_{[\Phi_2]}(x, p) \).

Using Lemma 5.1, we can find an isomorphism \( \Phi : C_1 \to C_2 \), such that \( \Phi_2 \circ \Phi = \Phi_1 \). Let \( \{ G_\lambda : \lambda \in Par_\Sigma \} \) denote the lifted forms on \( C_1^{ns} \) corresponding to the morphism \( \Phi_{\Sigma_1} \) and \( \{ H_\lambda : \lambda \in Par_\Sigma \} \) denote the lifted forms on \( C_2^{ns} \) corresponding to the morphism \( \Phi_{\Sigma_2} \). We need to check that:

\[ \text{Card}(C_1^{ns} \cap G_\lambda \cap V_{p_j}) = \text{Card}(C_2^{ns} \cap H_\lambda \cap V_{q_j}), \lambda' \in V_\lambda \cap Par_\Sigma \text{ generic.} \]

where \( \Phi(p_j) = q_j \) and \( \{ p_j, q_j \} \) in \( \{ C_1^{ns}, C_2^{ns} \} \) respectively, correspond to the branch \( \gamma_p^j \).

Suppose that \( \text{Card}(C_2^{ns} \cap H_\lambda \cap V_{q_j}) = n \). We may, without loss of generality, assume that \( \text{Base}(\Phi_{\Sigma_2}) \) is disjoint from \( \Gamma_{[\Phi_1]}(x, p) \) in a
particular presentation $\Phi_{\Sigma_3}$ of the morphism $\Phi$ (†). Hence, $\text{Base}(\Phi_{\Sigma_2} \circ \Phi_{\Sigma_3})$ is disjoint from $\Gamma_{[\Phi_1]}(x, p)$ as well (††). Let $\{\Phi_{\Sigma_3}^\ast H_\lambda : \lambda \in \text{Par}_\Sigma\}$ denote the lifted forms on $C^{ns}_1$ corresponding to this presentation. By the fact that $\Phi$ is an isomorphism and (†), we have $\text{Card}(C^{ns}_1 \cap \Phi^\ast H_\lambda \cap \mathcal{V}_p) = n$. Now let $V = V_{\Phi_{\Sigma_2}} \cap V_{\Phi_{\Sigma_3}}$. By (††) and Lemma 2.13, we may witness $\text{Card}(C^{ns}_1 \cap \Phi^\ast H_\lambda \cap \mathcal{V}_p) = n$ inside the canonical set $V$.

Now, using the fact that $\Phi_2 \circ \Phi = \Phi_1$ as birational maps, and Lemma 2.31, it follows that $\text{Card}(C^{ns}_1 \cap \mathcal{G}_\lambda \cap \mathcal{V}_p) = n$.

\[\square\]

**Lemma 5.12.** The definition (†) does not depend on the choice of a maximal independent system $\Sigma$ containing $F_\lambda$, having finite intersection with $C$.

**Proof.** Let $\Sigma$ be a maximal independent system containing $F_\lambda$. We claim first that $\Sigma$ has no base points on $C$, (†). For suppose that $w \in \text{Base}(\Sigma) \cap C$. Let $F_\mu$ be any form of degree $e$ having finite intersection with $C$. Then $< F_\mu, \Sigma >$ defines a system of higher dimension, hence must intersect $H$ in a point. That is, we can find parameters $\{\alpha, \beta\}$ and a fixed $F_\lambda$ in $\Sigma$ such that $\alpha F_\mu + \beta F_\lambda$ contains $C$. It follows immediately that $w$ must also be a base point for $F_\mu$. Clearly, we can find a form of degree $e$, having finite intersection with $C$, which doesn’t contain $w$. This gives a contradiction and proves (†). Now, given a choice of $\Sigma$ containing $F_\lambda$, by the proof of the previous lemma, we may assume that:

\[I_{\text{Italian}}(p, \gamma_j, C, F_\lambda) = \text{Card}(C^{ns} \cap F_\lambda \cap \mathcal{V}_p), \lambda' \in \mathcal{V}_\lambda \text{ generic in Par}_\Sigma.\]

As $p$ is not a base point for $\Sigma$ on $C$, it follows immediately that $p_j$ is not a base point for $\Sigma$ on $C^{ns}$. Hence, by Lemma 2.12, we have that:

\[I_{\text{Italian}}(p, \gamma_j, C, F_\lambda) = I_{\text{Italian}}(p_j, C^{ns}, F_\lambda)\]

Clearly, this equality does not depend on the particular choice of $\Sigma$ containing $F_\lambda$ but only on the presentation of the morphism $\Phi : C^{ns} \rightarrow C$. The result follows.

\[\square\]

Following from the definition of the Italian intersection multiplicity at a branch, we obtain a more refined version of Bezout’s theorem;

**Theorem 5.13.** Hyperspatial Bezout, Branched Version
Let $C \subset \mathbb{P}^w$ be a projective algebraic curve of degree $d$ and $F_\lambda$ a hypersurface of degree $e$, intersecting $C$ in finitely many points. Let $p$ be a point of intersection with branches given by $\{\gamma^1_\lambda, \ldots, \gamma^n_\lambda\}$. Then,

$$I_{\text{italian}}(p, C, F_\lambda) = \sum_{1 \leq j \leq n} I_{\text{italian}}(p, \gamma^j_\lambda, C, F_\lambda)$$

In particular,

$$\sum_{p \in C \cap F_\lambda} \sum_{1 \leq j \leq n} I_{\text{italian}}(p, \gamma^j_\lambda, C, F_\lambda) = de$$

Proof. Fix a non-singular model $(C^{ns}, \Phi)$ of $C$. Let $\Phi_{\Sigma}$ be a particular presentation of the morphism $\Phi$ such that $\text{Base}(\Phi_{\Sigma})$ is disjoint from the fibre $\Gamma_{\Phi}(x, p) = \{p_1, \ldots, p_n\}$ (*). Let $\Sigma$ be a linear system containing $F_\lambda$ such that $\Sigma$ has finite intersection with $C$ and;

$$I_{\text{italian}}(p, C, F_\lambda) = I_{\Sigma}^{\text{italian}}(p, C, F_\lambda)$$

Let $V_{\Phi_{\Sigma}} \subset C^{ns}$ and $W_{\Phi_{\Sigma}} \subset C$ be the canonical sets associated to the morphism $\Phi_{\Sigma}$. By Lemma 2.13, we can witness $m = I_{\text{italian}}(p, C, F_\lambda)$ by transverse intersections $\{x_1, \ldots, x_m\} = C \cap F_{\lambda} \cap V_\lambda$ inside the canonical set $W_{\Phi_{\Sigma}}$, for $\lambda$ generic in $\text{Par}_{\Sigma}$. Again, by Lemma 2.13 and (*), we can find $\lambda'$ such that for each $p_j \in \Gamma_{[\Phi]}(x, p)$, $m_j = I_{\text{italian}}(p, \gamma^j_\lambda, C, F_\lambda)$ is witnessed by transverse intersections $\{y^j_{1}, \ldots, y^j_{m_j}\} = C^{ns} \cap F_{\lambda'} \cap V_{\lambda'}$ inside the canonical set $V_{\Phi_{\Sigma}}$. We need to show that $m_1 + \ldots + m_j + \ldots + m_n = m$. By properties of infinitesimals and the fact that $\{p_1, \ldots, p_n\}$ are distinct, the sets $\{y^1_{1}, \ldots, y^1_{m_1}\}, \ldots, \{y^n_{1}, \ldots, y^n_{m_n}\}$ are disjoint. If $y^j_i$ belongs to one of these sets, then $y^j_i \in V_{p_i}$, hence $\Phi_{\Sigma}(y^j_i) \in V_p$. Moreover, $\Phi_{\Sigma}(y^j_i) \subset C \cap F_{\lambda'}$. It follows that $y^j_i$ corresponds to a unique $x_k$ in $\{x_1, \ldots, x_m\}$. As each $y^j_i$ lies in $V_{\Phi_{\Sigma}}$, this clearly gives an injection from $\{y^1_{1}, \ldots, y^1_{m_1}, \ldots, y^n_{1}, \ldots, y^n_{m_n}\}$ to $\{x_1, \ldots, x_m\}$. Hence, $m_1 + \ldots + m_n \leq m$. To prove equality, suppose that $x_{k}$ lies in $\{x_1, \ldots, x_m\}$. Then there exists a unique $y_k$ with $\Phi_{\Sigma}(y_k) = x_k$. By a similar argument to the above, $y_k$ must appear in one of the sets $\{y^1_{1}, \ldots, y^1_{m_1}\}, \ldots, \{y^n_{1}, \ldots, y^n_{m_n}\}$. This gives the result.

\[\Box\]

**Remarks 5.14.** One may use this version of Bézout’s theorem in order to develop a more refined theory of $g'_n$.

We also simplify the branch terminology for later applications;
Definition 5.15. Let $C \subset P^w$ be a projective algebraic curve and $C^{ns} \subset P^{w_1}$ some non-singular birational model with birational morphism $\Phi^{ns} : C^{ns} \to C$. Let $\gamma^j_p$ correspond to the infinitesimal neighborhood $\mathcal{V}_{p_j}$ of $p_j$ in $\Gamma_{\Phi^{ns}}(x, p)$. Then we will also define the branch $\gamma^j_p$ by the formula:

$$\gamma^j_p := \{ x \in C : \exists y(\Gamma_{\Phi^{ns}}(y, x) \land y \in \mathcal{V}_{p_j}) \}$$

Remarks 5.16. Note that the definition uses the language $\mathcal{L}_{spec}$, and, in particular, is not algebraic.

We have the following;

Lemma 5.17. Definition 5.15 does not depend on the choice of non-singular model $C^{ns}$ and morphism $\Phi^{ns}$. Lemma 5.7 may be reformulated replacing the old definition of a branch with Definition 5.15. Finally, we have, with hypotheses as for the old definition of intersection multiplicity at a branch, that;

$$I_{italian}(p, \gamma^j_p, C, F_\lambda) = Card(C \cap F_\lambda') \cap \mathcal{V}_{p_j}, \lambda' \in \mathcal{V}_{\lambda} \text{ generic in } Par_{\Sigma} (\ast)$$

Proof. The first part follows immediately from Lemma 5.1 and the fact that all the data of the lemma may be taken inside a standard model. The second part is similar, follow through the proof of Lemma 5.7. The final part may be checked by following carefully through the proofs of Lemmas 5.11 and 5.12. □

Remarks 5.18. Note that we could not have simplified the above presentation by taking (\ast) as our original definition of intersection multiplicity at a branch. The main reason being that the arguments on Zariski multiplicities require us to count intersections inside $C \cap \mathcal{V}_{p}$, rather than the smaller $\mathcal{L}_{spec}$ definable $C \cap \gamma^j_p$.

We now reformulate the preliminary definitions of Section 2 in terms of branches.

Let $C \subset P^w$ be a projective algebraic curve and let $\Sigma$ be a linear system, having finite intersection with $C$. Let $g^n_\nu(\Sigma)$ be defined by this linear system $\Sigma$ and let $g^n_\nu' \subset g^n_\nu$ be obtained by removing its fixed point contribution. Fix a non-singular model $(C^{ns}, \Phi)$ of $C$, with corresponding presentation $\Phi_{\Sigma}$. By the transfer result Lemma 2.31, we obtain a
corresponding $g_n^r$ without fixed points on $C^{ns}$. Let $\text{Card}(O, V_\lambda, g_n^r)$ denote the number of times $O \in C^{ns}$ is counted for this $g_n^r$ in the weighted set $V_\lambda$. We then make the following definition;

**Definition 5.19.** $I_{\Sigma,\text{mobile}}(p, \gamma_p^j, C, \phi_\lambda) = \text{Card}(p_j, V_\lambda, g_n^r) \ (*)$

where the branch $\gamma_p^j$ corresponds to $[V_{p_j}]$ in the fibre $\Gamma_{[\Phi_{\Sigma'}]}(x, p)$.

As before, one needs the following lemma;

**Lemma 5.20.** The definition $(*)$ does not depend on the choice of non-singular model $C^{ns}$ and birational morphism $\Phi_{\Sigma'}$.

**Proof.** The proof is the same as Lemma 5.11, we leave the details to the reader.  \square

We now make the following definition;

**Definition 5.21.** Let hypotheses be as in Definition 5.19, then we define:

$I_{\Sigma,\text{italian}}(p, \gamma_p^j, C, \phi_\lambda) = I_{\Sigma,\text{mobile}}(p, \gamma_p^j, C, \phi_\lambda) + 1$ if $p \in \text{Base}(\Sigma)$

$I_{\Sigma,\text{italian}}(p, \gamma_p^j, C, \phi_\lambda) = I_{\Sigma,\text{mobile}}(p, \gamma_p^j, C, \phi_\lambda)$ if $p \notin \text{Base}(\Sigma)$

**Lemma 5.22.** Let notation be as in the previous definition. As in Remarks 5.10, if $(C^{ns}, \Phi)$ is a non-singular model of $C$, with presentation $\Phi_{\Sigma'}$, such that $\text{Base}(\Sigma')$ is disjoint from the fibre $\Gamma_{[\Phi_{\Sigma'}]}(x, p)$, then we have that;

$I_{\Sigma,\text{italian}}(p, \gamma_p^j, C, \phi_\lambda) = \text{Card}(C^{ns} \cap \overline{\phi_{\lambda'}} \cap V_{p_j}), \lambda' \in V_\lambda$ generic in $\text{Par}_\Sigma$

where again we have used the notation $\{\overline{\phi_\lambda} : \lambda \in \text{Par}_\Sigma\}$ to denote the family of "lifted" forms, as in Remarks 5.10.

**Proof.** We divide the proof into the following cases;

Case 1. $p \notin \text{Base}(\Sigma)$.

Then, by Definition 5.21, we have that;
\[ I^\Sigma_{\text{italian}}(p, C, \gamma_p^j, \phi_\lambda) = I^\Sigma_{\text{mobile}}_{\text{italian}}(p, C, \gamma_p^j, \phi_\lambda) \quad (1) \]

By the assumption on \( \Sigma' \), we have that \( p_j \notin \text{Base}(\Sigma) \) for the "lifted" family of forms on \( C_{\text{ns}} \), corresponding to \( \Sigma \), (†). By the transfer result, Lemma 2.31, the \( g^r_p \) on \( C_{\text{ns}} \), used to define \( I^\Sigma_{\text{mobile}}_{\text{italian}}(p, C, \gamma_p^j, \phi_\lambda) \), is obtained from the "lifted" family of form on \( C_{\text{ns}} \) after removing all fixed point contributions. Therefore, as by (†), this lifted family has no fixed point contribution at \( p_j \), we must have that;

\[ I_{\text{italian}}(p_j, C_{\text{ns}}, \phi_\lambda) = I^\Sigma_{\text{italian}}_{\text{mobile}}(p, C, \gamma_p^j, \phi_\lambda) \quad (2) \]

Combining (1), (2) and using Lemma 2.10, we have that;

\[ I^\Sigma_{\text{italian}}(p_j, C_{\text{ns}}, \phi_\lambda) = I^\Sigma_{\text{italian}}(p, C, \gamma_p^j, \phi_\lambda) \]

The result for this case now follows immediately from the definition of \( I^\Sigma_{\text{italian}}(p, C_{\text{ns}}, \phi_\lambda) \).

Case 2. \( p \in \text{Base}(\Sigma) \).

Then, by Definition 5.21, we have that;

\[ I^\Sigma_{\text{italian}}(p, C, \gamma_p^j, \phi_\lambda) = I^\Sigma_{\text{italian}}_{\text{mobile}}(p, C, \gamma_p^j, \phi_\lambda) + 1 \quad (1) \]

In this case, we have that \( p_j \in \text{Base}(\Sigma) \) for the "lifted" family of forms on \( C_{\text{ns}} \), corresponding to \( \Sigma \), (†). Let \( I_{p_j} \) be the fixed point contribution for this family, as defined in Lemma 2.14. Then, by a similar argument to the above, we have that;

\[ I_{\text{italian}}(p_j, C_{\text{ns}}, \phi_\lambda) = I_{p_j} + I^\Sigma_{\text{italian}}_{\text{mobile}}(p, C, \gamma_p^j, \phi_\lambda) \quad (2) \]

Using Lemma 2.15, we have that;

\[ I_{\text{italian}}(p_j, C_{\text{ns}}, \phi_\lambda) = I_{p_j} + I^\Sigma_{\text{italian}}(p_j, C_{\text{ns}}, \phi_\lambda) - 1 \quad (3) \]

Combining (1), (2), (3) gives that;

\[ I^\Sigma_{\text{italian}}(p_j, C_{\text{ns}}, \phi_\lambda) = I^\Sigma_{\text{italian}}(p, C, \gamma_p^j, \phi_\lambda) \]
Again, the result for this case follows immediately from the definition of $I_{\text{italian}}^\Sigma(p_j, C^{\text{ns}}, \overline{\phi}_\lambda)$.

□

As an easy consequence of the previous lemma, we have that;

**Lemma 5.23.** Let $C \subset P^w$ be a projective algebraic curve. Let $\Sigma$ be a linear system, having finite intersection with $C$. Then, if $\gamma^j_p$ is a branch centred at $p$ and $\phi_\lambda$ belongs to $\Sigma$;

$$I_{\text{italian}}^\Sigma(p, C, \gamma^j_p, \phi_\lambda) = \text{Card}(C \cap \gamma^j_p \cap \phi_\lambda') \text{ for } \lambda' \in \mathcal{V}_\lambda \text{ generic in } \text{Par}_\Sigma.$$

$$I_{\text{italian}}^{\Sigma, \text{mobile}}(p, C, \gamma^j_p, \phi_\lambda) = \text{Card}(C \cap (\gamma^j_p \setminus p) \cap \phi_\lambda') \text{ for } \lambda' \in \mathcal{V}_\lambda \text{ generic in } \text{Par}_\Sigma.$$

where $\gamma^j_p$ was given in Definition 5.15.

**Proof.** The first part of the lemma follows immediately from Lemma 5.22 and the Definition 5.15 of a branch. The second part follows from Definition 5.21 and the first part. □

We then reformulate the remaining results of Section 2 in terms of branches. The notation of Lemma 5.23 will be use for the remainder of this section.

**Lemma 5.24.** Non-Existence of Coincident Mobile Points along a Branch

Let $C \subset P^w$ be a projective algebraic curve. Let $\Sigma$ be a linear system, having finite intersection with $C$, such that $p \in C \setminus \text{Base}(\Sigma)$. Then, if $\gamma^j_p$ is a branch centred at $p$ and $\phi_\lambda$ belongs to $\Sigma$;

$$I_{\text{italian}}(p, C, \gamma^j_p, \phi_\lambda) = I_{\text{italian}}^\Sigma(p, C, \gamma^j_p, \phi_\lambda) = I_{\text{italian}}^{\Sigma, \text{mobile}}(p, C, \gamma^j_p, \phi_\lambda)$$

**Proof.** Let $(C^{\text{ns}}, \Phi)$ be a non-singular model of $C$, with presentation $\Phi_\Sigma'$, such that $\text{Base}(\Sigma')$ is disjoint from $\Gamma_{[\Phi]}(x, p)$. Let $\{\overline{\phi}_\lambda\}$ be the ”lifted” family of algebraic forms on $C^{\text{ns}}$ defined by $\Sigma$. By Lemma 5.22, we have that;

$$I_{\text{italian}}^\Sigma(p, C, \gamma^j_p, \phi_\lambda) = I_{\text{italian}}^\Sigma(p_j, C^{\text{ns}}, \overline{\phi}_\lambda) \ (1)$$
By Remarks 5.10, we have that;

\[ I_{\text{italian}}(p, C, \gamma; \phi_\lambda) = I_{\text{italian}}(p_j, C^{ns}, \bar{\phi}_\lambda) \]  \hspace{1em} (2)

Using the fact that \( p \notin \text{Base}(\Sigma) \) and the hypotheses on \( \Phi_\Sigma' \), it follows that \( p_j \notin \text{Base}(\Sigma) \), for the lifted system defined by \( \Sigma \). Hence, we may apply Lemma 2.10, to obtain that;

\[ I^\Sigma_{\text{italian}}(p_j, C^{ns}, \bar{\phi}_\lambda) = I_{\text{italian}}(p_j, C^{ns}, \bar{\phi}_\lambda) \]  \hspace{1em} (3)

The result follows by combining (1), (2) and (3) and using Definition 5.21.

\[ \square \]

**Lemma 5.25.** Branch Multiplicity at non-base points witnessed by transverse intersections along the branch

Let \( p \in C \setminus \text{Base}(\Sigma) \) and let \( \gamma^j_p \) be a branch centred at \( p \). Then, if \( m = I_{\text{italian}}(p, C, \gamma^j_p, \phi_\lambda) \), we can find \( \lambda' \in V_\lambda \), generic in \( \text{Par}_\Sigma \), and distinct \( \{p_1, \ldots, p_m\} \) such that the intersection of \( C \) with \( \phi_{\lambda'} \cap \gamma^j_p \setminus p \) at each \( p_i \) is transverse for \( 1 \leq i \leq m \).

**Proof.** By Lemmas 5.23 and 5.24, for \( \lambda' \in V_\lambda \), generic in \( \text{Par}_\Sigma \), the intersection \( C \cap \phi_{\lambda'} \cap \gamma^j_p \) consists of \( m \) distinct points \( \{p_1, \ldots, p_m\} \). The condition on \( \text{Par}_\Sigma \) that \( \phi_\lambda \) passes through \( p \) defines a proper closed subset, hence we may assume these points are all distinct from \( p \). Finally, the transversality result follows from, say Lemma 2.17, using the fact that \( \{p_1, \ldots, p_m\} \) cannot lie inside \( \text{Base}(\Sigma) \).

\[ \square \]

Again, we have analogous results to Lemmas 5.24 and 5.25 for points in \( \text{Base}(\Sigma) \);

We first require the following;

**Lemma 5.26.** Let \( p \in C \cap \text{Base}(\Sigma) \) and \( \gamma^j_p \) a branch centred at \( p \). Then there exists an open subset \( U_{\gamma^j_p} \subset \text{Par}_\Sigma \) and an integer \( I_{\gamma^j_p} \geq 1 \) such that;

\[ I_{\text{italian}}(p, C, \gamma^j_p, \phi_\lambda) = I_{\gamma^j_p} \text{ for } \lambda \in U_{\gamma^j_p} \]

and
\[ I_{\text{italian}}(p, C, \gamma^j, \phi_\lambda) \geq I_{\gamma^j_p} \text{ for } \lambda \in \text{Par}_\Sigma \]

**Proof.** Let \((C^{ns}, \Phi)\) be a non-singular model with presentation \(\Phi_{\Sigma'}\) such that \(\text{Base}(\Sigma')\) is disjoint from \(\Gamma_{[\Phi]}(x, p)\). Then, by the proof of Lemma 5.12, we have that, for \(\lambda \in \text{Par}_\Sigma\):

\[ I_{\text{italian}}(p, \gamma^j_p, C, \phi_\lambda) = I_{\text{italian}}(p, C^{ns}, \phi_\lambda) \]

By properties of Zariski structures;

\[ W_k = \{ \lambda \in \text{Par}_\Sigma : I_{\text{italian}}(p, C^{ns}, \phi_\lambda) \geq k \} \]

are definable and Zariski closed in \(\text{Par}_\Sigma\). The result then follows by taking \(I_{\gamma^j_p} = \min_{\lambda \in \text{Par}_\Sigma} I_{\text{italian}}(p, C^{ns}, \phi_\lambda)\) and using the fact that \(\text{Par}_\Sigma\) is irreducible.

\[ \square \]

We can now formulate analogous results to Lemmas 5.24 and 5.25 for base points;

**Lemma 5.27.** Let \(p \in C \cap \text{Base}(\Sigma) \cap \phi_\lambda \) and \(\gamma^j_p\) a branch centred at \(p\). Then;

\[ I_{\text{italian}}(p, \gamma^j_p, C, \phi_\lambda) = I_{\gamma^j_p} + I_{\text{italian}}^{\Sigma}(p, \gamma^j_p, C, \phi_\lambda) - 1 \]

and

\[ I_{\text{italian}}(p, \gamma^j_p, C, \phi_\lambda) = I_{\gamma^j_p} + I_{\text{italian}}^{\Sigma, \text{mobile}}(p, \gamma^j_p, C, \phi_\lambda) \]

**Proof.** In order to prove the first part of the lemma, suppose that \(m = I_{\text{italian}}^{\Sigma}(p, \gamma^j_p, C, \phi_\lambda)\). By Lemma 5.23, choosing \(\lambda' \in \mathcal{V}_\lambda\) generic in \(\text{Par}_\Sigma\), we can find \(\{p_1, \ldots, p_{m-1}\} = C \cap \phi_{\lambda'} \cap (\mathcal{V}_p \setminus p)\), distinct from \(p\), witnessing this multiplicity. Using the fact that \(\{p_1, \ldots, p_{m-1}\}\) lie outside \(\text{Base}(\Sigma)\), we may apply Lemma 2.17 to obtain that the intersections at these points are transverse. By the previous Lemma 5.26, we have that \(I_{\text{italian}}(p, \gamma^j_p, C, \phi_{\lambda'}) = I_{\gamma^j_p}\). Now choose a nonsingular model \((C^{ns}, \Phi)\), with presentation \(\Phi_{\Sigma'}\), such that \(\text{Base}(\Sigma')\) is disjoint from \(\Gamma_{[\Phi]}(x, p)\). By definition 5.15 of a branch, we can find \(\{p'_1, \ldots, p'_m\} = C^{ns} \cap \phi_{\lambda'} \cap (\mathcal{V}_p \setminus p)\). By properties of specialisations, \(\text{Base}(\Sigma')\) is also disjoint from this set. We then have that the intersections between \(C^{ns}\) and \(\overline{\phi_{\lambda'}}\) are also transverse at these points and that
$I_{\text{italian}}(p_j, C^{ns}, \phi) = I_{\gamma^p}$. It then follows by summability of specialisation, see [6], that:

$I_{\text{italian}}(p_j, C^{ns}, \phi) = I_{\gamma^p} + (m - 1)$.

Again, using the presentation of $(C^{ns}, \Phi)$, we obtain that:

$I_{\text{italian}}(p, \gamma_j^p, C, \phi) = I_{\gamma^p} + (m - 1)$.

Hence, the result follows. The second part of the lemma follows immediately from the Definition 5.21 of $I^{\Sigma}_{\text{italian}}(p, C, \gamma_j^p, \phi)$ at a base point. □

**Lemma 5.28.** Let $p \in C \cap \text{Base}(\Sigma)$ and let $\gamma_j^p$ be a branch centred at $p$. Then, if $m = I^{\Sigma}_{\text{italian}}(p, C, \gamma_j^p, \phi)$, we can find $\lambda' \in V_\lambda$, generic in $\text{Par}_\Sigma$, and distinct $\{p_1, \ldots, p_{m-1}\} = C \cap \phi \lambda' \cap (\gamma_j^p \setminus p)$ such that the intersection of $C$ with $\phi \lambda'$ at each $p_i$ is transverse for $1 \leq i \leq m - 1$. If $m = I^{\Sigma, \text{mobile}}_{\text{italian}}(p, C, \gamma_j^p, \phi)$, then the same results for distinct $\{p_1, \ldots, p_m\}$ with the same properties.

**Proof.** The first part of the lemma is a straightforward consequence of Lemma 5.23, properties of infinitesimals, (to show that $\{p_1, \ldots, p_{m-1}\}$ lie outside $\text{Base}(\Sigma)$) and Lemma 2.17 (to obtain transversality). The second part of the lemma also follows from Lemma 5.23 and Lemma 2.17 (to obtain transversality). □

We now note the following, connecting the original definition of $I_{\text{italian}}$ with its branched version at non-singular points.

**Lemma 5.29.** Let $p$ be a nonsingular point of the curve $C$, then there exists a unique branch $\gamma_p$ centred at $p$ and;

$I_{\text{italian}}(p, C, \phi) = I_{\text{italian}}(p, C, \gamma_p, \phi)$ (1)

$I^{\Sigma, \text{mobile}}_{\text{italian}}(p, C, \phi) = I^{\Sigma, \text{mobile}}_{\text{italian}}(p, C, \gamma_p, \phi)$ (2)

**Proof.** The fact that there exists a unique branch $\gamma_p$, centred at $p$, follows from Lemma 5.4. By definition 5.15 of a branch, we then have that:

$\gamma_p = C \cap V_p$
Now (1) follows from Lemma 5.17, Lemma 2.10 and Definition 2.6. While (2) follows from Definition 2.20 and Lemma 5.23.

6. CAYLEY’S CLASSIFICATION OF SINGULARITIES

The purpose of this section is to develop a theory of singularities for algebraic curves based on the work of Cayley. In order to make this theory rigorous, one first needs to find a method of parametrising the branches of an algebraic curve. This is the content of the following theorem;

Theorem 6.1. Analytic Representation of a Branch

Let \( C \subset P^w \) be a projective algebraic curve. Suppose that \( C \) is defined by equations \( \{ F_1(x_1, \ldots, x_w), \ldots, F_m(x_1, \ldots, x_w) \} \) in affine coordinates \( x_i = \frac{x_i}{X_0} \). Let \( p \in C \) correspond to the point \( \bar{0} \) in this coordinate system. Then there exist algebraic power series \( \{ x_1(t), \ldots, x_w(t) \} \) such that \( x_1(t) = \ldots = x_w(t) = 0, F_j(x_1(t), \ldots, x_w(t)) = 0 \) for \( 1 \leq j \leq m \) and with the property that, for any algebraic function \( F_\lambda(x_1, \ldots, x_w) \);

\[
F_\lambda(x_1(t), \ldots, x_w(t)) \equiv 0 \text{ iff } F_\lambda \text{ vanishes on } C.
\]

Otherwise, \( F_\lambda \) has finite intersection with \( C \) and

\[
\text{ord}_t F_\lambda(x_1(t), \ldots, x_w(t)) = I_{\text{italian}}(p, \gamma^j_p, C, F_\lambda) \quad (*)
\]

We refer to the power series as parametrising the branch \( \gamma^j_p \).

Proof. We first prove the theorem in the case when \( C \subset P^w \) is a non-singular projective algebraic curve. By Lemma 4.13, we can find a plane projective algebraic curve \( C_1 \subset P^2 \) such that \( \{ C, C_1 \} \) are birational and there exists a corresponding point \( p_1 \in C_1 \) such that \( p_1 \) is non-singular. Let \( \Phi_\Sigma \) and \( \Psi_\Sigma' \) be presentations such that \( \Psi_\Sigma' = \Phi_\Sigma^{-1} \) as a birational map. Without loss of generality, we may assume that \( V_{\Phi_\Sigma} = W_{\Psi_\Sigma'} \subset C \) and \( V_{\Psi_\Sigma'} = W_{\Phi_\Sigma} \subset C_1 \). Moreover, we may assume that \( \{ p, p_1 \} \) lie in \( \{ \Phi_{\Sigma}, \Psi_{\Sigma'} \} \) and correspond to the origins of the affine coordinate systems \( (x_1, \ldots, x_w) \) and \( (y_1, y_2) \). Let \( \Sigma' = \{ \psi_0, \psi_1, \ldots, \psi_w \} \) and let \( (y_1(t), y_2(t)) \) be an analytic representation of \( p_1 \in C_1 \), given by the inverse function theorem. We obtain an analytic representation of \( p \in C \) by the formula;
We can homogenise the power series representation of \( p(G) \). Hence, we can invert the power series \( H \). This clearly proves that \( x_j(t) \) is a formal power series in \( L[[t]] \). That \( x_j(0) = 0 \) for \( 1 \leq j \leq w \) follows from the corresponding property for \((y_1(t), y_2(t))\) and the fact that \( p \) is situated at the origin of the coordinate system \((x_1, \ldots, x_w)\). Finally, we need to check that \( x_j(t) \) define algebraic power series. This follows obviously from the fact that \( \psi_j \) and \( \psi_0 \) define algebraic functions. Now, suppose that \( \{F_1, \ldots, F_m\} \) are defining equations for \( C \). Let \( \{F'_1, \ldots, F'_m\} \) be the corresponding equations written in homogeneous form for the variables \( \{X_0, \ldots, X_w\} \), where \( x_j = \frac{X_j}{X_0} \). Let \( G(Y_0, Y_1, Y_2) \) be the defining equation for \( C_1 \). We can homogenise the power series representation of \( p_1 \in C_1 \) by \((Y_0(t) : Y_1(t) : Y_2(t)) = (1 : y_1(t) : y_2(t))\). Then we must have that \( G(1 : y_1(t) : y_2(t)) = 0 \). Now \( F'_k(\psi_0, \ldots, \psi_w) \) vanishes identically on \( C_1 \), hence, by the projective Nullstellensatz, there exists a homogeneous \( H_k(Y_0, Y_1, Y_2) \) such that:

\[
F'_k(\psi_0, \ldots, \psi_w) = H_k G
\]

It follows that:

\[
F'_k(\psi_0(1 : y_1(t) : y_2(t)), \ldots, \psi_w(1 : y_1(t) : y_2(t))) \equiv 0
\]

therefore:

\[
F'_k(1 : \frac{\psi_1(y_1(t), y_2(t))}{\psi_0(y_1(t), y_2(t))}, \ldots, \frac{\psi_w(y_1(t), y_2(t))}{\psi_0(y_1(t), y_2(t))}) \equiv 0
\]

which gives:

\[
F_k(x_1(t), \ldots, x_w(t)) \equiv 0
\]

as required. The property that an algebraic function \( F_\lambda \) vanishes on \((x_1(t), \ldots, x_w(t))\) iff it vanishes on \( C \) can be proved in a similar way to the above argument, invoking Theorem 2.10 of the paper [8]. Alternatively, it can be proved directly, using the fact that, as \((x_1(t), \ldots, x_w(t))\) define algebraic power series, \((y_1 - x_1(t), \ldots, y_w - x_w(t))\) defines the equation of a curve \( C' \) on some etale cover \( i : (A^w, (0))^{et} \to (A^w, (0)) \) such that \( i(C') \subset C \). If \( F_\lambda \) vanishes on \( C' \), then it must vanish on an open subset \( U \) of \( C \), hence as \( F_\lambda \) is closed, must vanish on all of
\[ C \] as required. Finally, we need to check the property \((*)\). Suppose that \(F_\lambda\) is an algebraic function with \(m = \ord_t F_\lambda(x_1(t), \ldots, x_w(t))\), passing through \(p\). Choose \(\Sigma_1\) containing \(F_\lambda\) such that \(\Sigma_1\) has finite intersection with \(C\) and \(p \notin \Base(\Sigma_1)\). It follows, using Lemma 2.12, that \(I_{\italian}(p, C, F_\lambda) = I_{\italian}^{\Sigma_1}(p, C, F_\lambda)\), \((\dagger)\). Using Lemma 2.31, we can transfer the system \(\Sigma_1\) to a system on \(C_1\). Let \(G_\lambda\) be the corresponding algebraic curve to the algebraic form \(F_\lambda\). We must have that \(p_1 \notin \Base(\Sigma_1)\), otherwise, as \(p_1\) belongs to the canonical set \(V_{\psi_{\Sigma'}}\), we would have that \(p\) belongs to \(\Base(\Sigma_1)\) as well. Hence, using Lemma 2.12 again, we must have that \(I_{\italian}(p_1, C_1, G_\lambda) = I_{\italian}^{\Sigma_1}(p_1, C_1, G_\lambda)\), \((\dagger\dagger)\). By direct calculation, we have that;

\[
G_\lambda(y_1(t), y_2(t)) = \psi_0^r(y_1(t), y_2(t))F_\lambda(x_1(t), \ldots, x_w(t))
\]

for some \(r \leq 0\). As \(\ord_t \psi_0(y_1(t), y_2(t)) = 0\), we have that;

\[
\ord_t G_\lambda(y_1(t), y_2(t)) = \ord_t F_\lambda(x_1(t), \ldots, x_w(t)) = m
\]

Now, by Theorem 5.1 of the paper [6] and \((\dagger\dagger)\), it follows that;

\[
I_{\italian}(p_1, C_1, G_\lambda) = I_{\italian}^{\Sigma_1}(p_1, C_1, G_\lambda) = m
\]

Now, using Lemma 2.31 and the fact that \(\{p, p_1\}\) lie in the canonical sets \(V_{\Sigma'}\) and \(W_{\Sigma'}\), we must have that \(I_{\italian}^{\Sigma_1}(p, C, F_\lambda) = m\) as well. Hence, by \((\dagger)\), it follows that \(I_{\italian}(p, C, F_\lambda) = m\). As \(C\) is a non-singular model of itself, this proves the claim \((*)\) in this special case.

We now assume that \(C \subset P^w\) is any projective algebraic curve. Suppose that \(C^{ns} \subset P^w\) is a non-singular model of \(C\) with birational morphism \(\Phi_{\Sigma'} : C^{ns} \to C\) such that the branch \(\gamma_j\) corresponds to \(V_{\psi_j}\) in the fibre \(\Gamma_{[\psi]}(x, p)\), disjoint from \(\Base(\Sigma')\). As before, we may assume that \(\{p, p_j\}\) correspond to the origins of the coordinate systems \((x_1, \ldots, x_w)\) and \((y_1, \ldots, y_w)\). Let \(\Sigma' = \{\phi_0, \ldots, \phi_w\}\). By the previous argument, we can find an analytic representation \((y_1(t), \ldots, y_w(t))\) of \(p_j\) in \(C^{ns}\), with the properties given in the statement of the theorem. As before, we obtain an analytic representation of the corresponding \(p \in C\), by the formula;

\[
(x_1(t), \ldots, x_w(t)) = \left(\frac{\phi_0}{\phi_0}(y_1(t), \ldots, y_w(t)), \ldots, \frac{\phi_w}{\phi_0}(y_1(t), \ldots, y_w(t))\right)
\]
One checks that this has the required properties up to the property (\(\ast\)) by a direct imitation of the proof above, with the minor modification that the projective Nullstellensatz for \(C^{ns}\) gives that, if \(\{G_1, \ldots, G_k\}\) are defining equations for \(C^{ns}\), then, if \(F\) vanishes on \(C^{ns}\), there must exist homogeneous \(\{H_1, \ldots, H_k\}\) such that \(F = H_1G_1 + \ldots + H_kG_k\). Alternatively, one can refer the parametrisation to a non-singular point of a plane projective curve, in which case the argument up to (\(\ast\)) is identical.

We now verify the property (\(\ast\)). Suppose that \(F_\lambda\) is an algebraic function with \(\text{ord}_tF_\lambda(x_1(t), \ldots, x_w(t)) = m\). Let \(\overline{F}_\lambda\) be the corresponding function on \(C^{ns}\), obtained from the presentation \(\Phi_\Sigma\). By Lemmas 5.11 and 5.12;

\[
I_{\text{italian}}(p, \gamma_j^p, C, F_\lambda) = I_{\text{italian}}(p_j, C^{ns}, \overline{F}_\lambda) \quad (\ast\ast)
\]

We claim that \(\text{ord}_t\overline{F}_\lambda(y_1(t), \ldots, y_w'(t)) = m\). This follows by repeating the argument given above. By the properties of \((y_1(t), \ldots, y_w'(t))\) and the result verified in the case of a non-singular curve, we obtain immediately that \(I_{\text{italian}}(p_j, C^{ns}, \overline{F}_\lambda) = m\). Combined with (\(\ast\ast\)), this gives the required result.

\[\square\]

Using the analytic representation, we obtain the following classification of singularities due to Cayley;

**Theorem 6.2. Cayley’s Classification of Singularities**

Let \(C \subset P^w\) be a projective algebraic curve which is not contained in any hyperplane section. Let \(\gamma_j^p\) be a branch of the algebraic curve centred at \(p\). Then we can assign a sequence of non-negative integers \((\alpha_0, \alpha_1, \ldots, \alpha_{w-1})\), called the character of the branch, which has the following property;

Let \(\Sigma\) be the system of hyperplanes passing through \(p\). Then there exists a filtration of \(\Sigma\) into subsystems of hyperplanes;

\[
\Sigma_{w-1} \subset \Sigma_{w-2} \subset \ldots \subset \Sigma_1 \subset \Sigma_0 = \Sigma
\]
with \( \dim(\Sigma_i) = (w - 1) - i \), such that, for any hyperplane \( H_\lambda \) passing through \( p \), we have that:

\[
H_\lambda \in \Sigma_i \text{ iff } I_{\text{italian}}(p, \gamma^j_p, C, H_\lambda) \geq \alpha_0 + \alpha_1 + \ldots + \alpha_i. \tag{\dagger}
\]

Moreover, these are the only multiplicities which occur.

**Proof.** Without loss of generality, we may assume that \( p \) is situated at the origin of the coordinate system \((x_1, \ldots, x_w)\). By the previous theorem, we can find an analytic parameterisation of the branch \( \gamma^j_p \) of the form:

\[
x_k(t) = a_{k,1}t + a_{k,2}t^2 + \ldots + a_{k,n}t^n + \ldots, \text{ for } 1 \leq k \leq w
\]

Let \( \sum_{k=1}^{w} \lambda_k x_k = 0 \) be the equation of a hyperplane \( H_\lambda \) passing through \( p \). Then, we can write \( H_\lambda(x_1(t), \ldots, x_k(t)) \) in the form:

\[
(\sum_{k=1}^{w} \lambda_k a_{k,1})t + \ldots + (\sum_{k=1}^{w} \lambda_k a_{k,n})t^n + O(t^n) \tag{*}
\]

Let \( \bar{a}_n = (a_{1,n}, \ldots, a_{k,n}, \ldots, a_{w,n}) \). We claim that we can find a sequence \((\bar{a}_{m_1}, \ldots, \bar{a}_{m_w})\), for \( m_1 < \ldots < m_i < \ldots < m_w \), such that:

(i). \( \{\bar{a}_{m_1}, \ldots, \bar{a}_{m_w}\} \) is linearly independent

(ii). \( V_{m_i} = \langle \bar{a}_{n_1}, \ldots, \bar{a}_{n_i} \rangle = \langle \bar{a}_{m_1}, \ldots, \bar{a}_{m_i} \rangle \), for \( 1 \leq i \leq w \)

(iii). If there exist \( \{n_1, \ldots, n_i\} \) for \( n_1 < \ldots < n_i \) with \( n_1 \leq m_1, \ldots, n_i \leq m_i \) such that

\[
V_{m_i} = \langle \bar{a}_{n_1}, \ldots, \bar{a}_{n_i} \rangle,
\]

then \( n_1 = m_1, \ldots, n_i = m_i \).

(iv). \( V_k = V_{m_i} \) for \( m_i \leq k < m_{i+1} \).

The first three properties may be proved by induction on \( i \). For \( i = 1 \), choose the first non-zero vector \( \bar{a}_{m_1} \). For the inductive step, assume we have found \( \{\bar{a}_{m_1}, \ldots, \bar{a}_{m_i}\} \) with the required properties \((i) - (iii)\). We claim that there exists \( \bar{a}_{m_{i+1}} \), with \( m_{i+1} > m_i \), such that \( \bar{a}_{m_{i+1}} \notin V_{m_i} \), \((**))\). Suppose not. As \( i < w \), the condition;
\( \bigwedge_{s=1}^{i} (\sum_{k=1}^{w} \lambda_k a_{k,m_s} = 0) \)

defines a non-empty plane \( P \) in the dimension \( w - 1 \) parameter space \( \text{Par}_H \) of hyperplanes passing through \( p \). Choosing \( \lambda \in P \) and using \((*)\), it follows that \( \text{ord}_t H_\lambda(x_1(t), \ldots, x_w(t)) \geq n \) for all \( n > 0 \). Therefore, \( H_\lambda(x_1(t), \ldots, x_w(t)) \equiv 0 \) and, by Theorem 6.1, \( H_\lambda \) must contain \( C \). This contradicts the assumption that \( C \) is not contained in any hyperplane section. Using \((***)\), choose \( m_{i+1} \) minimal such that \( \bar{a}_{m_{i+1}} \notin V_{m_i} \). Properties \((i)\) and \((ii)\) are trivial to verify. For \((iii)\), assume that \( \{\bar{a}_{n_1}, \ldots, \bar{a}_{n_{i+1}}\} \) are as given in the hypotheses. Then, the sequence must form a linearly independent set. Hence, we must have that \( V_{m_i} = \langle \bar{a}_{n_1}, \ldots, \bar{a}_{n_i} \rangle \). By the induction hypothesis, \( n_1 = m_1, \ldots, n_i = m_i \). Then, \( \bar{a}_{m_{i+1}} \notin V_{m_i} \). Now, by minimality of \( m_{i+1} \), we also have that \( \bar{a}_{m_{i+1}} = \bar{a}_{n_{i+1}} \) as required.

Property \((iv)\) follows easily from properties \((i)-(iii)\). We clearly have that \( V_{m_i} \subset V_k \subset V_{m_{i+1}} \), for \( m_i \leq k < m_{i+1} \). If \( V_k \neq V_{m_i} \), then, by \((i),(ii)\), \( V_k = V_{m_{i+1}} \). This clearly contradicts \((iii)\).

Now, define;

\[ \Sigma_i = \{ \lambda \in \text{Par}_{H,p} : \bigwedge_{s=1}^{i} (\sum_{k=1}^{w} \lambda_k a_{k,m_s} = 0) \} \], for, \( 1 \leq i \leq w-1 \),

Then, we obtain a filtration;

\[ \Sigma_{w-1} \subset \ldots \subset \Sigma_i \subset \ldots \subset \Sigma_0 = \Sigma \]

with \( \text{dim}(\Sigma_i) = (w-1) - i \) as in the statement of the theorem. Define;

\[ \alpha_0 = m_1 \text{ and } \alpha_i = m_{i+1} - m_i \text{ for } 1 \leq i \leq w-1. \]

We need to verify the property \((\dagger)\). Suppose that \( H_\lambda \in \Sigma_i \), for \( i \geq 1 \). Then \( H_\lambda \) contains the plane \( V_{m_i} \) spanned by \( \{\bar{a}_{m_1}, \ldots, \bar{a}_{m_i}\} \). Hence, by \((iv)\), it contains the plane \( V_k \) for \( k < m_{i+1} \). Then, by \((*)\), \( \text{ord}_t(H_\lambda(x_1(t), \ldots, x_w(t))) \geq m_{i+1} \) and, by Theorem 6.1,

\[ I_{\text{italian}}(p, \gamma^i_p, C, H_\lambda) \geq m_{i+1} = \alpha_0 + \ldots + \alpha_i. \]

Conversely, suppose that
\[ \text{ord}_t H_\lambda(x_1(t), \ldots, x_w(t)) \geq \alpha_0 + \ldots + \alpha_i = m_i + 1, \text{ for some } i \geq 1. \]

Then \( H_\lambda \) contains the plane \( V_k \) for \( k < m_i + 1 \). In particular, it contains the plane \( V_{m_i} \). Hence \( H_\lambda \in \Sigma_i \). The remaining case amounts to showing that \( I_{\text{Italian}}(p, \gamma_0^j, C, H_\lambda) \geq \alpha_0 \) for any hyperplane \( H_\lambda \) passing through \( p \). This follows immediately from \( (*) \), Theorem 6.1 and the fact that \( \bar{a}_m \) was the first non-zero vector in the sequence \( \{ \bar{a}_n : n < \omega \} \).

The remark made after the property \( (*) \) follows immediately from the property \( (iv) \) of the sequence \( \{ \bar{a}_m, \ldots, \bar{a}_w \} \).

\[ \square \]

Definition 6.3. In accordance with the Italian terminology, we refer to \( \alpha_0 \) as the order of the branch, \( \alpha_j \) as the \( j \)'th range of the branch, for \( 1 \leq j \leq w - 2 \), and \( \alpha_{w-1} \) as the final range or class of the branch. We define \( \langle \bar{a}_{m_1}, \ldots, \bar{a}_{m_k} \rangle \) to be the \( k \)'th osculatory plane at \( p \) for \( 1 \leq k \leq w - 1 \). We also define the \( w - 1 \)'th osculatory plane to be the osculatory plane. We define the tuple \( (\alpha_0, \ldots, \alpha_{w-1}) \) to be the character of the branch. Cayley referred to branches of order 1 as linear and superlinear otherwise. He referred to branches having a character of the form \( (1, 1, \ldots, 1) \) as ordinary. The Italian geometers refer to a simple point, which is the origin of an ordinary branch, as an ordinary simple point. Note that for a simple (equivalently non-singular) point, the \( 1 \)'st osculatory plane is the same as the tangent line. We will now also use the terminology, tangent line to a branch, to describe the \( 1 \)'st osculatory plane at any branch. We define a node of a plane curve to be the origin of at most 2 ordinary branches with distinct tangent lines, this definition was used in Theorem 4.16. We also used, in Theorem 4.16, the fact that a generic point of an algebraic curve is an ordinary simple point, (this is not true when the field has non-zero characteristic, see the final section) a rigorous proof of this result requires duality arguments, we postpone this proof for another occasion.

We have the following important results on the projection of a branch, see Section 4 for the relevant definitions.

Theorem 6.4. Let \( C \subset P^w \) be a projective algebraic curve, as defined in the previous theorems of this section, and \( \gamma_0 \) a branch centred at \( O \) with character \( (\alpha_0, \ldots, \alpha_{w-1}) \). Let \( P \) be chosen generically in \( P^w \), then the projection \( \text{pr}_P(\gamma_0) \) has character:

\[ (\alpha_0, \ldots, \alpha_{w-2}) \]
If $P$ is situated generically on the osculatory plane, then $pr_P(\gamma_O)$ has character;

$$(\alpha_0, \ldots, \alpha_{w-2} + \alpha_{w-1})$$

More generally, if $P$ is situated on the $k$'th osculatory plane for $1 \leq k \leq w - 2$, and not on an osculatory plane of lower order, then the projection $pr_P(\gamma_O)$ has character;

$$(\alpha_0, \ldots, \alpha_{k-2}, \alpha_{k-1} + \alpha_k, \alpha_{k+1}, \ldots, \alpha_{w-1}).$$

Proof. First note that, by Lemma 4.9, if $P$ is situated generically on the $k$'th osculatory plane for any $k \geq 1$, and $H'$ is any hyperplane not containing $P$, the projection $pr_P$ is generally biunivocal. Hence, by Lemma 5.7, the projection $pr_P(\gamma_O)$ is well defined. We first claim that for any hyperplane $H_\lambda$ in $P^{w-1}$;

$$I_{\text{italian}}(O, \gamma_O, C, pr_P^{-1}(H_\lambda)) = I_{\text{italian}}(pr_P(O), pr_P(\gamma_O), pr_P(C), H_\lambda) \ (*)$$

This follows by using the proof of Lemma 4.13 and the fact that the multiplicity is calculated at a branch, to replace the use of biunivocity. Now, if $P$ is situated in generic position in $P^w$, then \{pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_{w-1}})\} forms a linearly independent sequence passing through $pr_P(O) \in P^{w-1}$ and, for any hyperplane $H_\lambda \subset P^{w-1}$, we have that, for $i \leq w - 2$;

$$< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_i}) > \subset H_\lambda \text{ iff } < \bar{a}_{m_1}, \ldots, \bar{a}_{m_i} > \subset < P, H_\lambda >$$

It follows by $(*)$ and Theorem 6.2 that;

$$I_{\text{italian}}(pr_P(O), pr_P(\gamma_O), pr_P(C), H_\lambda) \geq \alpha_0 + \ldots \alpha_i$$

iff

$$< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_i}) > \subset H_\lambda \quad (i \leq w - 2)$$

Hence, by Theorem 6.2 again, we have that the branch $pr_P(\gamma_O)$ has character $(\alpha_0, \ldots, \alpha_{w-2})$.

If $P$ is situated generically on the $k$'th osculatory plane, but not on an osculatory plane of lower order, then, the sequence
\{pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_{k-1}}), pr_P(\bar{a}_{m_k+1}), \ldots, pr_P(\bar{a}_{m_{w-1}})\} \text{ forms a linearly}
\text{ independent set, and, for } 1 \leq i \leq k - 2;

< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_i}) > \subset H_\lambda \text{ iff } < \bar{a}_{m_1}, \ldots, \bar{a}_{m_i} > \subset < P, H_\lambda >

whereas;

< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_{k-1}}) > \subset H_\lambda \text{ iff } < \bar{a}_{m_1}, \ldots, \bar{a}_{m_k} > \subset < P, H_\lambda >

and, for } 1 \leq j \leq (w - 2 - k);

< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_{k-1}}), pr_P(\bar{a}_{m_{k+1}}), \ldots, pr_P(\bar{a}_{m_{k+j}}) > \subset H_\lambda

iff

< \bar{a}_{m_1}, \ldots, \bar{a}_{m_{k+j}} > \subset < P, H_\lambda >

The rest of the theorem then follows immediately by the same argument as given above.

\[\square\]

We also have the following important consequence of Theorem 6.1.

**Theorem 6.5. Linearity of Multiplicity at a Branch**

Let } C \subset P^w \text{ be a projective algebraic curve and } \gamma^j_p \text{ a branch of } C,\text{ centred at } p. \text{ Let } \Sigma \text{ be an independent system having finite intersection with } C. \text{ Then, for any } k \geq 1, \text{ the condition;}

\{\lambda \in \text{Par}_\Sigma : I_{\text{italian}}(p, \gamma^j_p, C, F_\lambda) \geq k\} \text{ (*)}

is linear and definable.

**Proof.** That (*) is definable follows from the definition of multiplicity at a branch and elementary facts about Zariski structures. Moreover, (* ) is definable over the parameters of } C \text{ and the point } p. \text{ In order to prove linearity, suppose that } F_\lambda \text{ and } F_\mu \text{ belong to } \Sigma \text{ and satisfy (*)}. \text{ Let } (x_1(t), \ldots, x_w(t)) \text{ be a parameterisation of the branch } \gamma^j_p \text{ as given by Theorem 6.1. Then;}

\[k \leq \min\{\text{ord}_t F_\lambda(x_1(t), \ldots, x_w(t)), \text{ord}_t F_\mu(x_1(t), \ldots, x_w(t))\}\]
Then, for any constant $c$;

$$(F_\lambda + cF_\mu)(x_1(t), \ldots, x_w(t)) = F_\lambda(x_1(t), \ldots, x_w(t)) + cF_\mu(x_1(t), \ldots, x_w(t))$$

Hence,

$$ord_t(F_\lambda + cF_\mu)(x_1(t), \ldots, x_w(t)) \geq k$$

as well. This shows that the pencil of curves generated by $\{F_\lambda, F_\mu\}$ satisfies $(\ast)$. Let $W$ be the closed projective subvariety of $\text{Par}_\Sigma$ defined by the condition $(\ast)$. Then, $W$ has the property that for any $\{a, b\} \subset W$, $l_{ab} \subset W$. It follows easily that $W$ defines a plane $H$ in $\text{Par}_\Sigma$ as required. (Use the fact that for any tuple $\{a_1, \ldots, a_n\}$ in $W$, the plane $H_{a_1, \ldots, a_n} \subset W$ and a dimension argument)

□

Remarks 6.6. Note that, given $\Sigma$ of dimension $n$ as in the statement of the theorem, we can, using the above theorem, find a sequence $\{\beta_0, \ldots, \beta_{n-1}\}$ and a filtration;

$$\Sigma_{n-1} \subset \ldots \subset \Sigma_i \subset \ldots \subset \Sigma_0 \subset \Sigma$$

with $\dim(\Sigma_i) = (n - 1) - i$ such that;

$$I_{\text{Italian}}(p, \gamma, C, F_\lambda) \geq \beta_0 + \ldots + \beta_i \text{ iff } F_\lambda \in \Sigma_i$$

and these are the only multiplicities which can occur. Note also that, as an easy consequence of the theorem, given any tuple $(\beta_0, \ldots, \beta_{n-1})$ and $\Sigma$ as above;

$$\{x \in C : x \text{ has character } (\beta_0, \ldots, \beta_{n-1}) \text{ with respect to } \Sigma\}$$

is constructible and defined over the field of definition of $C$ and $\Sigma$. In particular, it follows from the previous Definition 6.3, in characteristic 0, that there exist only finitely many points on $C$ which are the origins of non-ordinary branches.

We have the following important characterisation of multiplicity at a branch;

Theorem 6.7. Multiplicity at a Branch as a Specialised Condition
Let $C$ and $\Sigma$ be as in the statement of Theorem 6.5 and the following remark. Fix independent generic points (over $L$) $\{p_{0j}, \ldots, p_{i,j}\}$ in $\gamma_p^j$. Then the system $\Sigma_i$ may be obtained by specialising the condition:

$$\{\lambda \in \text{Par}_\Sigma : F_\lambda = 0 \text{ passes through } \{p_{0j}, \ldots, p_{i,j}\}\}.$$

That is, if $F_\lambda \in \Sigma_i$, there exists $\lambda' \in \mathcal{V}_\lambda$ such that $F_{\lambda'}$ passes through $\{p_{0j}, \ldots, p_{i,j}\}$, while, if $F_{\lambda'}$ passes through $\{p_{0j}, \ldots, p_{i,j}\}$, then its specialisation $F_\lambda$ belongs to $\Sigma_i$.

Proof. We will first assume that the curve $C$ is non-singular, (hence, by Lemma 5.4, there exists a single branch at $p$). Consider the cover $F_i \subset (C \setminus p) \times \text{Par}_\Sigma$ given by:

$$F_{i+1}(x, \lambda) \equiv (x \in C \cap F_\lambda) \land (F_\lambda \in \Sigma_i) \; (i \geq 0)$$

For generic $q \in C$, the fibre $F_0(q,y)$ has dimension $n - 1 - (i + 1)$. Hence, we obtain an open subset $U \subset C$ such that $F_{i+1} \subset (U \setminus p) \times \text{Par}_\Sigma$ is regular with fibre dimension $n - 1 - (i + 1)$. Let $\bar{F}_{i+1}$ be the closure of $F_{i+1}$ in $U \times \text{Par}_\Sigma$. We claim that $\Sigma_{i+1}$ is defined by the fibre $\bar{F}_{i+1}(p,y)$. First observe that, as $p$ has codimension 1 in $U$, $\text{dim}(\bar{F}_{i+1}(p,y)) \leq n - 1 - (i + 1)$. As $p$ is non-singular, each component of the fibre $\bar{F}_{i+1}(p,y)$ has dimension at least $n - 1 - (i + 1)$. Suppose that $\bar{F}_{i+1}(p,\lambda)$ holds, then, by regularity of $p$ for the cover $\bar{F}_{i+1}$, given $p' \in U \cap V_p$ generic, we can find $\lambda' \in \mathcal{V}_\lambda$ such that $F_{i+1}(p',\lambda')$, that is $F_{\lambda'}$ passes through $p'$ and $F_{\lambda'}$ belongs to $\Sigma_i$. By definition of $\Sigma_i$, we have that $I_{\text{Italian}}(p, C, F_{\lambda'}) \geq \beta_0 + \ldots + \beta_i$. As $p'$ is distinct from $p$, by summability of specialisation, see the paper [6], we must have that $I_{\text{Italian}}(p, C, F_\lambda) \geq \beta_0 + \ldots + \beta_i + 1$. Therefore, in fact, by the above theorem, $I_{\text{Italian}}(p, C, F_\lambda) \geq \beta_0 + \ldots + \beta_{i+1}$ and $F_\lambda$ belongs to $\Sigma_{i+1}$. By dimension considerations, it follows that $\bar{F}_{i+1}(p,y)$ defines the system $\Sigma_{i+1}$ as required.

We now prove one direction of the theorem by induction on $i \geq 0$. Suppose that $\{p_{0j}, \ldots, p_{i+1,j}\}$ are given independent generic points in $C \cap \mathcal{V}_p$. By the above, if $F_\lambda$ belongs to $\Sigma_{i+1}$, then there exists $\lambda' \in \mathcal{V}_\lambda$ such that $F_{\lambda'}$ belongs to $\Sigma_i$ and passes through $p_{i+1,j}$. Moreover, as all the covers $F_i$ are defined over the field of definition $L$ of $C$, we may take $\lambda'$ to lie in the field $L_1 = L(p_{i+1,j})^{alg}$. Hence, $F_{\lambda'}$ does not pass through any of the other independent generic points $\{p_{0j}, \ldots, p_{i,j}\}$. Let $L_2 = L(p_{0j}, \ldots, p_{i,j})^{alg}$. As $\text{dim}(p_{kj}/L) = 1$, for $1 \leq k \leq i$, we may,
without loss of generality, assume that \( L_2 \) is linearly disjoint from \( L_1 \) over \( L \). Hence, by the amalgamation property for the universal specialisation, see the paper [7], we have that \( \{p_0, \ldots, p_i\} \) still belong to \( \mathcal{V}_p \cap C \) when taking \( P(L_1) \) as the standard model. Now, we consider the subsystem \( \Sigma' \subset \Sigma \) defined by;

\[
\Sigma' = \{ F_\lambda : F_\lambda \text{ passes through } p_{i+1,j} \}
\]

As \( p_{i+1,j} \) was chosen to be generic, it cannot be a base point for any of the subsystems;

\[
\Sigma_{n-1} \subset \ldots \subset \Sigma_i \subset \ldots \subset \Sigma_0 \subset \Sigma
\]

Hence, we obtain a corresponding filtration;

\[
\Sigma_{n-2} \cap \Sigma' \subset \ldots \subset \Sigma_i \cap \Sigma' \subset \ldots \subset \Sigma_0 \cap \Sigma' \subset \Sigma'
\]

with the properties in Remarks 6.6. We now apply the induction hypothesis to \( F_{\lambda'} \in \Sigma_i \cap \Sigma' \). We can find \( \lambda'' \in \mathcal{V}_{\lambda'} \) such that \( F_{\lambda''} \) passes through \( \{p_0, \ldots, p_i\} \) and \( F_{\lambda''} \) belongs to \( \Sigma' \). Hence \( F_{\lambda''} \) passes through \( \{p_0, \ldots, p_{i+1,j}\} \). Finally, note that \( \lambda'' \in \mathcal{V}_{\lambda} \), if one considers \( P(L) \) rather than \( P(L_1) \) as the standard model. Hence, one direction of the theorem is proved.

The converse direction may also be proved by induction on \( i \geq 0 \). Suppose that \( F_{\lambda''} \) passes through independent generic points \( \{p_0, \ldots, p_{i+1,j}\} \) in \( \gamma_p^j \). As before, we may consider \( p_{i+1,j} \) as belonging to the standard model \( P(L_1) \) and \( \{p_0, \ldots, p_i\} \) as belonging to \( \mathcal{V}_p \cap C \), relative to \( P(L_1) \). We again consider the subsystem \( \Sigma' \subset \Sigma \) as defined above. Let \( \lambda' \) be the specialisation of \( \lambda'' \) relative to \( P(L_1) \). Then, \( F_{\lambda'} \) belongs to \( \Sigma' \), as \( \Sigma' \) is defined over \( p_{i+1,j} \). Moreover, by the inductive hypothesis applied to \( \Sigma', \) \( F_{\lambda'} \) also belongs to \( \Sigma_i \). We now apply the argument at the beginning of this proof, with \( P(L) \) as the standard model, to obtain that \( F_{\lambda} \) belongs to \( \Sigma_{i+1} \), where \( \lambda \) is the specialisation of \( \lambda' \), relative to \( P(L) \). Clearly \( \lambda'' \) specialises to \( \lambda \), hence the converse direction is proved.

We still need to consider the case for arbitrary \( C \subset P^w \). Let \( C^{ns} \subset P^w \) be a non-singular model of \( C \) and suppose that the presentation \( \Phi_{\Sigma'} \) of \( (C^{ns}, \Phi^{ns}) \) has \( Base(\Sigma') \) disjoint from the fibre \( \Gamma_{[\Phi^{ns}]}(y, p) \). Let \( \gamma_p^j \) correspond to the infinitesimal neighborhood \( \mathcal{V}_{p_j} \) of \( p_j \) in \( \Gamma_{[\Phi^{ns}]}(y, p) \)
and let $\{F_\lambda\}$ be the system $\Sigma$ of lifted forms on $C^{ns}$ corresponding to the space of forms $\{F_\lambda\}$ in $\Sigma$. By Lemma 5.12, it follows that, for $\lambda \in \text{Par}_\Sigma$, $I_{\text{italian}}(p, \gamma^j_p, C, F_\lambda) = I_{\text{italian}}(p_j, C^{ns}, F_\lambda)$, hence the character $(\beta_0, \ldots, \beta_n)$ of the branch $\gamma^j_p$ with respect to the system $\Sigma$ is the same as the character of the branch $\gamma^j_{p_j}$ with respect to the lifted system $\Sigma$. Moreover, we obtain the same filtration of $\text{Par}_\Sigma$, as given in Remark 6.6, for both systems with respect to the branches $\{\gamma^j_p, \gamma^j_{p_j}\}$.

Now, suppose that we are given independent generic points $\{p_{0j}, \ldots, p_{ij}\}$ in $\gamma^j_p$. Then, by definition of $\gamma^j_p$, we can find corresponding independent generic points $\{p'_{0j}, \ldots, p'_{ij}\}$ in $\gamma^j_{p_j}$. Now suppose that $F_\lambda \in \Sigma_i$, then the corresponding $F_\lambda$ belongs to $\Sigma_i$. By the proof of the above theorem for non-singular curves, we can find $\lambda' \in \mathcal{V}_\lambda$ such that $F_{\lambda'}$ passes through $\{p'_{0j}, \ldots, p'_{ij}\}$. Then, by definition, the corresponding $F_{\lambda'}$ passes through $\{p_{0j}, \ldots, p_{ij}\}$. Conversely, suppose that we can find $\lambda' \in \mathcal{V}_\lambda$ such that $F_{\lambda'}$ passes through $\{p'_{0j}, \ldots, p'_{ij}\}$. Then the corresponding $F_{\lambda'}$ passes through $\{p_{0j}, \ldots, p_{ij}\}$. By the proof for non-singular curves, the specialisation $F_\lambda$ belongs to $\Sigma_i$. Hence, by the observation (†) above, the corresponding $F_{\lambda}$ belongs to $\Sigma_i$ as well. The theorem is then proved. □

Remarks 6.8. One can give a slightly more geometric interpretation of the preceding theorem as follows:

Consider the cover $F \subset C^{i+1} \times \text{Par}_\Sigma$ given by:

$$F(p_1, \ldots, p_{i+1}, \lambda) \equiv \{p_1, \ldots, p_{i+1}\} \subset C \cap F_\lambda = 0$$

Generically, the cover $F$ over $C^{i+1}$ has fibre dimension $n - (i + 1)$. For a tuple $(p, \ldots, p) \in \Delta^{i+1}$, the dimension of the fibre $F(p, \ldots, p)$ is $n - 1$. By the above, $\Sigma_i \subset F(p, \ldots, p)$, which has dimension $n - (i + 1)$, is regular for the cover, in the sense of the above theorem.

The theorem may be construed as a generalisation of an intuitive notion of tangency. $i + 1$ independent generic points on the branch $\gamma^j_p$ determine a projective plane $H_i$ of dimension $i$. As these $i + 1$ points converge independently to $p$, the plane $H_i$ converges to the $i$'th osculatory plane at $p$. As with the proofs we have given of many of the original arguments in [10], the method using infinitesimals in fact reverses this type of thinking in favour of a more visual approach. In this case, we have shown that, by moving the $i$'th osculatory plane away from $p$, we can cut the branch $\gamma^j_p$ in $i + 1$ independent generic points.
The theorem also provides an effective method of computing osculatory planes at a branch $\gamma_p^j$ for $p \in C$.

We now require the following lemma;

**Lemma 6.9.** Let $F_{\lambda'}$ have finite intersection with $C$, where the parameter $\lambda'$ is taken inside the non-standard model $P(K)$. Then there exists a maximally independent set $\{p_{0j}, \ldots, p_{ij}\}$ of generic intersections (over $L$) inside $\gamma_p^j$.

**Proof.** Let $W$ be the finite set of intersections inside $\gamma_p^j$ of $F_{\lambda'}$ with $C$. As $C$ is strongly minimal, if $\dim(W/L) = i + 1$, then there exists a basis $\{p_{0j}, \ldots, p_{ij}\}$ of $W$ over $L$. In particular, we have that $\{p_{0j}, \ldots, p_{ij}\}$ are generically independent points of $C$ and are maximally independent in $W$. 

We then have the following theorem;

**Theorem 6.10.** Intersections along a Branch

Let hypotheses be as in the previous theorem. Let $i$ be maximal such that $F_{\lambda'}$ belongs to $\Sigma_i$ and suppose that $F_{\lambda'}$ intersects $\gamma_p^j$ in the maximally independent set of generic points $\{p_{i+1,j}, \ldots, p_{i+r,j}\}$ over $L$, where $r \leq (n-1) - i$. Then, if the branch $\gamma_p^j$ has character $(\beta_0, \ldots, \beta_n)$ with respect to the system $\Sigma$, $F_{\lambda'}$ intersects $(\gamma_p^j \setminus p)$ in at least $\beta_{i+1} + \ldots + \beta_{i+r}$ points, counted with multiplicity.

**Proof.** We assume first that $C$ is non-singular. Let $F_{\lambda}$ be the specialisation of $F_{\lambda'}$ relative to the standard model $P(L)$. Then by Theorem 6.7, $F_{\lambda}$ belongs to $\Sigma_{i+r}$ (replace the system $\Sigma$ in Theorem 6.7 by the system $\Sigma_i$.) Hence, $I_{\text{italian}}(p, C, F_{\lambda}) \geq \beta_0 + \ldots + \beta_{i+r}$, whereas $I_{\text{italian}}(p, C, F_{\lambda'}) = \beta_0 + \ldots + \beta_i$. It follows immediately, by summability of specialisation, see the paper [6], that the total multiplicity of intersections of $F_{\lambda'}$ with $C$ inside the branch $(\gamma_p^j \setminus p)$ is at least;

$$(\beta_0 + \ldots + \beta_{i+r}) - (\beta_0 + \ldots + \beta_i) = \beta_{i+1} + \ldots + \beta_{i+r}$$

as required. If $C$ is singular, let $(C^{ns}, \Phi^{ns})$ be a non-singular model, with a presentation $\Phi_{\Sigma'}$ such that $\text{Base}(\Sigma')$ is disjoint from $\Gamma_{[\Phi]}(x, p)$. Then, given the maximally independent set of generic points $\{p_{i+1,j}, \ldots, p_{i+r,j}\}$, in $\gamma_p^j$, for the intersection of $C$ with $F_{\lambda'}$, we obtain a maximally independent set for the intersection $F_{\lambda'} \cap C \cap V_p$. By the above, and the fact that the character of the branch $\gamma_p^j$ with respect
to \( \{F_\lambda\} \) equals the character of the branch \( \gamma_{p_j} \) with respect to \( \overline{F}_\lambda \), we obtain that \( \overline{F}_\lambda \) intersects the branch \((\gamma_{p_j} \setminus p_j)\) in at least \( \beta_{i+1} + \ldots + \beta_{i+r} \) points with multiplicity. Hence, using for example Lemma 5.12, and the fact that \( \text{Base}(\Sigma') \) is disjoint from \( \gamma_{p_j} \), we obtain that \( F_\lambda \) intersects \((\gamma_j \setminus p_j)\) in at least \( \beta_i + 1 + \ldots + \beta_i + r \) points with multiplicity as well. □

Remarks 6.11. Note that, in the statement of the theorem, one cannot obtain that \( F_\lambda \) intersects the branch \((\gamma_j \setminus p_j)\) in exactly \( \beta_i + 1 + \ldots + \beta_i + r \) points, with multiplicity. For example, consider the algebraic curve \( C \) given in affine coordinates by \( x^3 - y^2 = 0 \). At \((0,0)\), this has a cusp singularity with character \( (2,1) \). The tangent line or \( 1 \)’st osculatory plane, is given by \( y = 0 \). The line \( y - \epsilon = 0 \), where \( \epsilon \) is an infinitesimal, cuts the branch of \( C \) at \((0,0)\) in exactly \( 3 = 2 + 1 \) points. However, the total transcendence degree (over \( L \)) of these points is clearly \( 1 \). Neither can one obtain that \( F_\lambda \) intersects the branch \((\gamma_j \setminus p_j)\) transversely. For example, consider the algebraic curve \( C \) given in affine coordinates by \( y - x^2 = 0 \). Let \( \Sigma \) be the \( 2 \)-dimensional system consisting of (projective) lines. As \((0,0)\) is an ordinary simple point of \( C \), it has character \( (1,1) \), which is also the character of \((0,0)\) with respect to the system \( \Sigma \). Again, the tangent line or \( 1 \)’st osculatory plane, is given by \( y = 0 \). The line \( y = (2\epsilon)x - \epsilon^2 \) cuts the branch of \( C \) at \((0,0)\) in exactly one point \((\epsilon, \epsilon^2)\), with multiplicity \( 2 \), and specialises to \( y = 0 \).

Remarks 6.12. The above theorem is critical in calculations relating to the transformation of branches by duality. We save this point of view for another occasion.

We finish this section by applying the above ideas to examine the behaviour of hyperplane systems on an arbitrary projective algebraic curve;

**Lemma 6.13.** Let \( C \subset P^w \) be any projective algebraic curve and let;

\[ V^* \subset \text{NonSing}(C) \times P^{w*} \text{ be } \{ (x, \lambda) : x \in \text{NonSing}(C) \land H_\lambda \supset l_x \} \]

Then \( V^* \) defines an irreducible algebraic variety and, if \( \bar{V}^* \subset C \times P^{w*} \) defines its Zariski closure, then, for a singular point \( p \), which is the origin of branches \( \{ \gamma^1_p, \ldots, \gamma^m_p \} \), the fibre \( \bar{V}^*(p) \) consists exactly of the parameters for hyperplanes \( H_\lambda \), which contain at least one of the tangent lines \( l^1_p, (1 \leq j \leq m) \).

**Proof.** The fact that \( V^* \) defines an irreducible algebraic variety follows easily from arguments given in Section 1. We let \( \Sigma \) be the linear system defined by the set of hyperplanes \( \{ H_\lambda : \lambda \in P^{w*} \} \) and let \( C^{na} \) be a
nonsingular model of $C$, with presentation $\Phi_\Sigma$, such that $Base(\Sigma')$ is disjoint from the fibres \(\{\Gamma_{[\phi]}(x, p_1), \ldots, \Gamma_{[\phi]}(x, p_n)\}\), where \(\{p_1, \ldots, p_n\}\) denote the singular points of $C$, (\(*\)). Let \(\{V_{\Phi_\Sigma}, W_{\Phi_\Sigma}\}\) be the canonical sets associated to this presentation and let $\Sigma$ also denote the lifted system of forms on $C^{ns}$. We now lift the cover $V^*$ to $C^{ns}$ by defining $V^{*,\text{ift}} \subset V_{\Phi_\Sigma} \times P^{ws}$:

\[
V^{*,\text{ift}} = \{(x, \lambda) : V^*(\Phi_\Sigma(x), \lambda)\}
\]

Using the fact that $V^*$ defines an algebraic variety, it is clear that $V^{*,\text{ift}}$ defines an algebraic variety as well. Let $\bar{V}^{*,\text{ift}}$ be the Zariski closure of this cover inside $C^{ns} \times P^w$. Now let $p$ be a singular point of $C$, let $\gamma_p$ be a branch centred at $p$ and let $p_j$ be the corresponding point of $C^{ns}$. By the assumption (\(*\)), the character of the branch $\gamma_p$, with respect to the system $\Sigma$, is the same as the character of the branch $\gamma_{p_j}$, with respect to the lifted system $\Sigma$. In particular, the filtration given by Remarks 6.6 is the same. As in Remarks 6.6, we let $\Sigma_1 \subset \Sigma$ consist of the set of hyperplanes \(\{H_{\lambda} : H_{\lambda} \supset l_{p_j}\}\). We now claim that the fibre $\bar{V}^{*,\text{ift}}(p_j, z)$ defines $\Sigma_1$, (\(**\)). The proof is similar to the beginning of Theorem 6.7. Using the fact that $p_j$ is non-singular, we have that $\dim(\bar{V}^{*,\text{ift}}(p_j, z)) = w - 2$, (\(\dagger\)), and is regular for the cover $\bar{V}^{*,\text{ift}}/C^{ns}$. Now, suppose that $\bar{V}^{*,\text{ift}}(p_j, \lambda)$ holds, then, given generic $p_j^\prime, p_j^\prime\prime \in (V_{p_j} \cap V_{\Phi_\Sigma})$, we can find $\lambda' \in V_{\lambda}$ such that $\bar{V}^{*,\text{ift}}(p_j^\prime, \lambda')$. In particular, using the assumption (\(*\)) again, the corresponding form $\overline{H}_{\lambda'}$ must contain the tangent line $l_{p_j^\prime}$. Now, applying the result of Theorem 6.7 to the linear system $\Sigma_{p_j^\prime} \subset \Sigma$ of forms passing through $p_j^\prime$, taking $\{p_j, p_j^\prime\}$ as belonging to the standard model, we can find $p_j'' \in V_{p_j^\prime}$, generically independent from $\{p_j, p_j^\prime\}$, and $\lambda'' \in V_{\lambda}$ such that $\overline{H}_{\lambda''}$ passes through $\{p_j^\prime, p_j^\prime\}$. As the pair $\{p_j^\prime, p_j^\prime\}$ are generically independent over $p_j$ and belong to $V_{p_j}$, taking only $p_j$ as belonging to the standard model, again applying Theorem 6.7, we obtain that the specialisation $\overline{H}_{\lambda}$ of $\overline{H}_{\lambda''}$ belongs to $\Sigma_1$. Hence, $\Sigma_1 \subset \bar{V}^{*,\text{ift}}(p_j, y)$ and the result (\(**\)) then follows from this and the dimension consideration (\(\dagger\)). We now project the cover $\bar{V}^{*,\text{ift}}$ to a cover $W^* \subset C \times P^w$, by defining:

\[
W^* = \{(y, \lambda) : y \in C \land \exists x[\Gamma_{[\phi]}(x, y) \wedge \bar{V}^{*,\text{ift}}(x, \lambda)]\}
\]

By construction, we have that $W^*$ is an irreducible closed projective variety, such that its restriction $W^*|_{W_{\Phi_\Sigma}}$ agrees with $V^*$. Hence,
Moreover, for a singular point $p$ of $C$, the fibre $W^*(p)$ consists exactly of the parameters for hyperplanes $H_\lambda$, which contain at least one of the tangent lines $l_{\gamma_j^p}$, where $\{\gamma_1^p, \ldots, \gamma_j^p, \ldots, \gamma_m^p\}$ are the branches centred at $p$. This completes the proof.

$$\square$$

**Lemma 6.14.** Let $C \subset P^w$ be any projective algebraic curve and let $O$ be a fixed point of $C$, possibly singular, let:

$$V^* \subset (C \setminus \{O\}) \times P^{w*} \subset \{(x, \lambda) : x \in C \setminus \{O\} \wedge H_\lambda \ni l_{Ox}\}$$

Then $V^*$ defines an irreducible algebraic variety and, if $\bar{V}^* \subset C \times P^{w*}$ defines its Zariski closure, then, if $O$ is the origin of branches $\{\gamma_1^O, \ldots, \gamma_m^O\}$, the fibre $\bar{V}^*(O)$ consists exactly of the parameters for hyperplanes $H_\lambda$, which contain at least one of the tangent lines $l_{\gamma_j^O}$, $(1 \leq j \leq m)$.

**Proof.** The proof is similar to Lemma 6.13. Using the same notation there, we choose a presentation $\Phi_{\Sigma'}$ of a nonsingular model $C^{ns}$. We define $\Sigma_1 \subset \Sigma$ to consist of the set of hyperplanes $\{H_\lambda : H_\lambda \ni l_{\gamma_j^O}\}$, where $\gamma_j^O$ is one of the branches centred at $O$. We define $V^*, lift$ and $\bar{V}^*, lift$ as in the previous Lemma 6.13, using $\Phi_{\Sigma'}$, (with the corresponding modification of $V^*$). The aim is then to show the corresponding statement ($**$) of Lemma 6.13, that the fibre $\bar{V}^*(O_j, z)$ defines $\Sigma_1$, where $O_j$ corresponds to $\gamma_j^O$ in $C^{ns}$. As before, we have that $\dim(\bar{V}^*(O_j, z)) = w - 2$, ($\dagger$), and that $O_j$ is regular for the cover $(V^{*, lift}/C^{ns})$. If $V^{*, lift}(O_j, \lambda)$ holds, then, given generic $O'_j \in (V_{O_j} \cap V_{\Phi_{\Sigma'}})$, we can find $\lambda' \in \Lambda'$ such that $V^{*, lift}(O'_j, \lambda')$. The corresponding form $H_{\lambda'}$ then contains $l_{O'O'}$ for some $O'$, generically independent from $O$, lying on $\gamma_j^O$. Applying Theorem 6.7 to the set of hyperplanes $\Sigma_0$ passing through $O$, we obtain that the specialisation $H_\lambda$ contains the tangent line $l_{\gamma_j^O}$. Combining this result with the dimension consideration ($\dagger$), we obtain ($**$) as required. The result then follows by the same argument as Lemma 6.13. $\square$

### 7. Some Remarks on Frobenius

When the field has non-zero characteristic, many of the above arguments are complicated by the Frobenius morphism. However, we take the point of view that this is an exception rather than a general rule, hence the results are true if we exclude unusual cases. We will consider each of the previous sections separately and point out where
to make these modifications. We briefly remind the reader that given, algebraic curves $C_1$ and $C_2$, by a generally biunivocal map, denoted for this section only using the repetition of notation, $\phi : C_1 \leftrightarrow C_2$, we mean a morphism $\phi$, defined on an open subset $U \subset C_1$, such that $\phi$ defines a bijective correspondence between $U$ and an open subset $V$ of $C_2$. In characteristic 0, a generally biunivocal map is birational, in the sense of Definition 1.19. However, this is not true when the field has non-zero characteristic, Frobenius being a counterexample.

Section 1. The results of this section hold in arbitrary characteristic.

Section 2. We encounter the first problem in Theorem 2.3, the proof of which depends on Lemma 2.10. Unfortunately, Lemma 2.10 is not true in arbitrary characteristic. However, as we will explain below, Lemma 2.10 is true for a linear system $\Sigma$ which defines a birational morphism $\Phi_\Sigma$ on $C$. As this was assumed in Theorem 2.3, its proof does hold in non-zero characteristic.

Lemma 2.10 does not hold in arbitrary characteristic. Let $C$ be the algebraic curve defined by $y = 0$ in affine coordinates $(x, y)$. Consider the linear system $\Sigma$ of dimension 1 defined by $\phi_t(x, y) := (y = x^2 + t)$ in characteristic 2. Then $\phi_t$ is tangent to $C$ at $(t^{1/2}, 0)$ for all $t$. In particular, $(0, 0)$ is a coincident mobile point for the linear system. The reason for the failure of the lemma is that the function $F(x, y) = y - x^2$ defines a Zariski unramified morphism on $y = 0$ at $(0, 0)$, which is not etale, it is just the Frobenius map in characteristic 2. One can avoid such cases by insisting that the linear system $\Sigma$ under consideration defines a separable morphism on $C$ (\#). With this extra requirement and a result from $[3]$ (Theorem 6.11) that any locally Zariski unramified separable morphism between curves is locally etale, the proof of Lemma 2.10 holds.

The remaining results of the section are unaffected, with the restriction (\#) on $\Sigma$ in non-zero characteristic. In particular, Lemma 2.30 holds with this restriction. The proof of the Lemma gives the existence of a generally biunivocal morphism $\phi$. By separability, this induces an isomorphism of the function fields of the respective curves. By an elementary algebraic and model theoretic argument, see for example $[2]$, (Theorem 4.4 p25), one can then invert the morphism $\phi$ in the sense of Definition 1.19. Therefore, $\phi$ will define a birational map.
Section 3. The proof of Lemma 3.2 requires results from Section 2 which may not hold in certain exceptional cases. However, the Lemma is still true in arbitrary characteristic. One should replace the use of Lemma 2.30 by invoking general results for plane curves in \[6\]. Lemma 3.6 and Theorem 3.3 also holds, if we replace birational with biunivocal. In order to obtain the full statement of Theorem 3.3 in arbitrary characteristic, one can use the following argument:

We obtain from the argument of Theorem 3.3, in arbitrary characteristic, a generally biunivocal morphism \(\phi\) from \(C \subset \mathbb{P}^2\) to \(C_1 \subset \mathbb{P}^w\). This induces an inclusion of function fields \(\phi^* : L(C_1) \to L(C)\). We may factor this extension as \(L(C_1) \subset L(D) \subset L(C)\), with \(D\) an algebraic curve, \(L(D) \subset L(C)\) a purely inseparable extension and \(L(C_1) \subset L(D)\) a separable extension. We, therefore, obtain rational maps \(\phi_1 : C \rightsquigarrow D\) and \(\phi_2 : D \rightsquigarrow C_1\), such that \(\phi_2 \circ \phi_1\) is equivalent to \(\phi\) as a biunivocal map between \(C\) and \(C_1\). Now, using the method of \[5\] (Remarks 6.5), we may find an algebraic curve \(C' \subset \mathbb{P}^2\) (apply some power of Frobenius to the coefficients defining \(C\)) and a morphism \(\text{Frob}^n : C \to C'\), together with a birational map \(\phi_3 : D \rightsquigarrow C'\) such that \(\text{Frob}^n\) and \(\phi_3 \circ \phi_1\) are equivalent as biunivocal maps between \(C\) and \(C'\). We now obtain a separable rational map \(\phi_4 = \phi_2 \circ \phi_3^{-1} : C' \rightsquigarrow C_1\), such that \(\phi_4 \circ \text{Frob}^n\) and \(\phi\) are equivalent as biunivocal morphisms. Now let \(U \subset \text{NonSing}(C)\) be an open set on which \(\phi\) and \(\phi_4 \circ \text{Frob}^n\) are defined and agree as morphisms. By an elementary application of the chain rule and the fact that the differential of the Frobenius morphism is identically zero, one obtains that, for any \(x \in U\), \((D\phi)_x\) contains the tangent line \(l_x\) of \(C\). By the methods in the introduction of Section 1, this is in fact a closed condition on \((D\phi)\), hence, in fact \((D\phi)_x\) contains the tangent line \(l_x\) of \(C\), for \(x \in \text{NonSing}(C)\), at any point where \(\phi\) is defined. We can summarise this more generally in the following lemma:

**Lemma 7.1.** Let \(\phi : C \rightsquigarrow \mathbb{P}^w\) be an inseparable rational map, then, for any nonsingular point \(x\) of \(C\) at which \(\phi\) is defined, \((D\phi)_x\) contains the tangent line \(l_x\) of \(C\).

By Remark 3.7, this property is excluded for a transverse \(g_n^*\) as used in Theorem 3.3.

Section 4. The projection construction defined at the beginning of the section may fail to define a separable morphism in non-zero characteristic. However, using Lemma 7.1 and methods from Section 1, one can easily show that this only occurs for projective curves \(C\) with the property that, for every \(x \in \text{NonSing}(C)\), the tangent line \(l_x\) passes
through a given point $P$. In this case, the projection of $C$ from $P$ onto any hyperplane will be inseparable. In [2] such curves are called strange. Non-singular strange curves were completely classified by Samuel in [9];

**Theorem 7.2.** The only strange non-singular projective algebraic curves are the line and the conic in characteristic 2.

However, there are examples of other singular strange projective algebraic curves in $P^w$, for $w \geq 2$, not contained in any hyperplane section. For example, the curves $Fr_w \subset P^w$ obtained by iterating Frobenius, given parametrically by:

$$(t, t^p, t^{p^2}, \ldots, t^{p^{w-1}})$$

in characteristic $p$.

For these examples, Lemma 4.2 fails. In order to see this, pick independent points $\{T, S\}$ on $Fr_w$ given by $(t, t^p, \ldots, t^{p^{w-1}})$ and $(s, s^p, \ldots, s^{p^{w-1}})$. Then, the equation of the chord $l_{TS}$ is given parametrically by:

$$(t + \lambda s, t^p + \lambda s^p, \ldots, t^{p^{w-1}} + \lambda s^{p^{w-1}})$$

If $t + \lambda s = v$, and $V$ is given by $(v, v^p, \ldots, v^{p^{w-1}})$, then we have that the chord $l_{TS}$ meets $V$, distinct from $\{T, S\}$, iff we can solve $\lambda^{p-1} = 1, \ldots, \lambda^{p^{w-1}-1} = 1$ for $\lambda \neq 1$. This is clearly possible if $p \geq 3$. In this case, we would have that the chord $l_{TS}$ intersects $Fr_w$ in at least $p$ points.

Lemma 4.2 holds in arbitrary characteristic, if we exclude singular strange projective curves, however the proof should be modified as it involves Lemma 2.10 applied to a projection. If $C$ is a non-singular strange curve, using the classification given above, the theorem has no content as we assumed that $C$ was not contained in any hyperplane section.

**Lemma 7.3.** Lemma 4.2 in arbitrary characteristic, excluding singular strange projective curves

Let $C \subset P^w$, for $w \geq 3$, not contained in any hyperplane section and such that $C$ is not a singular strange projective curve. Suppose that $\{A, B\}$ are independent generic points of $C$, then the line $l_{AB}$ does not otherwise meet the curve $C$. 
Proof. We use the same notation as in Lemma 4.2. Let $pr_P$ be the projection defined in this lemma. Suppose that $pr_P$ is inseparable, then, by the above remarks $C$ is a strange projective algebraic curve, such that all its tangent lines $l_x$, for $x \in \text{NonSing}(C)$, pass through $P$. Hence, we can assume that $pr_P$ is separable. We now show that the degenerate case ($\dagger$) cannot occur. Suppose that $pr_P(l_A)$ is a point. We have that $\dim_P(A) = 1$, hence we can find an open $W \subset \text{NonSing}(C)$, defined over $P$, such that, for $x \in W$, $l_x$ passes through $P$. In particular, as $\dim_P(B) = 1$, $l_B$ passes through $P$, hence we must have that $l_A = l_B = AB$. As $A$ and $B$ were independent, it follows easily that $C$ must be a line $l$, which is a contradiction. We can now follow through the proof of Lemma 4.2 to obtain that; 

There exists an open $W \subset \text{NonSing}(C)$, defined over the field of definition of $C$, such that, for $y \in pr_B(W)$, the $l_y$ intersect in a point $Q$. ($***$)

It follows that, for $x \in W$, the $l_x$ intersect $l_B$. In particular, $l_A$ intersects $l_B$. If $l_A = l_B$, we obtain that $C$ is a line, hence we may assume that $l_A \cap l_B = Q$. As $B$ was generic, we can find an open subset $W' \subset \text{NonSing}(C)$, defined over $AB$, such that, for $x \in W'$, the $l_x$ intersect $l_A$ and $l_B$. Then, either, for such $x \in W'$, the $l_x$ all pass through $Q$ or the $l_x$ all lie in the plane $P_{AB}$ defined by $l_A$ and $l_B$. In the first case, we have that $C$ is a strange curve, contradicting the hypotheses. In the second case, we use the fact that the plane $Pl = P_{AB}$ must be defined over the field of definition of $C$ and then, by the fact that the generic chord $l_{AB}$ lies in $Pl$, that $C$ must be contained in $Pl$ as well, contradicting the hypotheses. $\square$

Lemma 4.5 is true if we exclude singular strange projective curves. In order to obtain the corresponding result for a singular strange projective curve $C$, pick a generic point point $P \in P^w$. Let $x \in C$ be generic and independent from $P$. We claim that $l_{P_x}$ does not otherwise meet the curve (*). If not, we can find $y \in C$, distinct from $x$, such that $P \in l_{xy}$. Hence, $\dim(P/xy) \leq 1$ and $\dim(P/x) \geq 3$. Now calculate $\dim(Pxy)$ in two different ways;

(i). $\dim(Pxy) = \dim(P/xy) + \dim(xy) \leq 1 + 2 = 3$

(ii).$\dim(Pxy) = \dim(y/Px) + \dim(Px) \geq 0 + 4 = 4$
This clearly gives a contradiction. It follows, using (*), by an elementary model theoretic argument, that the projection \( pr_P \) onto any hyperplane \( H \) will be generally biunivocal on \( C \). Lemma 4.6 may also be easily modified to include the case of singular strange curves. Theorem 4.8 holds in arbitrary characteristic by the modifications of Lemma 4.5 and Lemma 4.6 and by ensuring that the projections \( pr_P \) always define separable morphisms. In the case of strange curves, we can always ensure this by picking the centre of projection \( P \) to be disjoint from the \textit{bad} point \( Q \), defined as the intersection of the tangent lines. Lemma 4.9 is still true in arbitrary characteristic but the proof needs to be modified in order to take into account singular strange projective curves, (we implicitly used Lemma 4.2 in the proof). Using the same notation as in the lemma, given \( x \in C \), using the same argument, we can find \( P \in P^w \) generic, such that \( l_x \) does not otherwise meet the curve. Now using the modification of Lemma 4.5, the projection from \( P \) will be generally biunivocal and, by construction, biunivocal at \( x \). We can then obtain the lemma by repeating this argument. Lemma 4.12 holds in arbitrary characteristic provided the projection \( pr \) is separable. As we have already remarked, this can always be arranged in non-zero characteristic. Lemma 4.14 holds in arbitrary characteristic, using the previous modified lemmas, and the fact that a separable biunivocal map, between \( C \) and \( pr(C) \), may be inverted to give a birational map including the nonsingular points of \( pr(C) \). It follows that Theorem 4.15 holds in arbitrary characteristic as well, by the modifications from Section 3. Finally, Theorem 4.16 holds by checking the result for certain further unusual curves, depending on generalisations of results in later sections, (see \((†)\) below). We should note that, without these generalisations, Theorem 4.16 still holds if one accepts the weaker definition of a node as the origin of 2 linear branches (see Definition 6.3).

Section 5. The results of this section hold in arbitrary characteristic up to Lemma 5.24. We only make the remark that it is always possible to choose a \textit{maximal} linear system such that it defines a separable morphism on a projective curve \( C \). The proof of Lemma 5.24 has the same complications as Lemma 2.10. Again, we can avoid these complications and recover the remaining results of the section in arbitrary characteristic, by the assumption on the linear system \( \Sigma \) that it defines a separable morphism on \( C \).

Section 6. The results up to Definition 6.3 hold, by appropriate choices of linear systems \( \Sigma \). In Definition 6.3, the claim that a generic point of an algebraic curve is an ordinary simple point does not hold in
arbitrary characteristic, (†). An example is given by the plane quartic curve \( F(x, y, z) = x^3y + y^3z + z^3x = 0 \) over a field of characteristic 3. Every point of this curve is an inflection point, that is a point with character (1, 2). In this case, the natural duality map;

\[
DF : C \to C^*
\]

\[
[x : y : z] \mapsto [Fx : Fy : Fz] = [z^3 : x^3 : y^3]
\]

is purely inseparable. In order to prove Theorem 4.16 in arbitrary characteristic, one needs to classify such exceptional curves. This can be done, using work of Plücker on the transformation of branches by duality, we save this point of view for another occasion.

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MATHEMATICS DEPARTMENT, THE UNIVERSITY OF CAMERINO, CAMERINO, ITALY
E-mail address: tristam.depiro@unicam.it