Prediction of Two-Node Tandem Queue with Feedback Having State and Time Dependent Service Rates

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Abstract: Queuing networks are current area of great research and application interest, in view of their increased applicability in - modelling manufacturing facilities and computer/communication networks, production and assembly lines, hospitals, transport systems, banks and so forth. The present paper develops and analyses a tandem queuing model having two nodes with feedback. Such type of model finds its application in many fields, e.g. telecommunication, inventory, hospitals, traffic control and so on. For example, in hospitals, a patient first visits the doctor and goes through a series of examinations, after which he revisits the doctor for medication. The inter-arrival time is exponentially distributed, whereas the service rate for each node depends on the aggregate of customers in the respective queues and thus follows non-homogeneous Poisson process. After constructing the difference –differential equations, and using PGF technique, we have obtained the joint probability distribution for queue size. Some key measures such as the “average number of customers in queue”, “utilization time of each queue”, “average waiting time in each queue” are computed.

Key Words: Tandem Queues, Feedback, PGF

1. Introduction:
There are situations where the departures from one service station (node) form the arrivals to another service station, such queuing systems are called queuing networks. Tandem queues are queuing networks where, service facilities are in series. Taylor and Jackson were credited as the pioneer in the study of sequence of queues in series [1]. Burke proved that the distribution of the output of a queue with Poisson arrival, exponential service and infinite capacity was also Poisson with same mean value as the arrival rate, thus each queue could be treated independently [2]. Shukla and Bhadoriya, Balsama et al. gave an extensive review on queuing network model with finite capacity, work and development on tandem queuing system [3] [4]. Most of the earlier works on tandem were leveraged upon time independent Poisson process. However, it has been found that in many real life situations, the parameters vary with the time. Newell was the first to consider queue with time dependent arrival rate [5]. Maggu studied tandem queuing model having two nodes having phase type service [6]. Massey et al. studied queuing...
networks with non-stationary Poisson input [7]. Worthington et al. evaluated time dependent behaviour of queuing system by DTM approach [8]. Some of the authors with significant contributions are SrinivasaRao et al. [9], Suhasini et al. [10], Srinivasa et al. [11], Aparajitha and Kumar [12]. They studied queuing models with non-homogeneous bulk arrivals, state and time dependent service rates, Binomial bulk arrival, load dependent service rates, non-homogeneous and state dependent service rates to follow Poisson process.

Queues with feedback generally represents situation where customer after service revisits the server once again. Such types of queues with feedback are prominent in health sector, telecom sector, production process etc.; where the chance of rework is high. Finch introduced the concept of feedback in cyclic queues [13]. Takas analysed a single server queue with feedback [14]. Mei et al. evaluated response times in a two-node queuing network with feedback [15].

In this paper, we have considered a two-node tandem queue having state dependent non-homogeneous service rates with feedback. Customers arrive at the first service counter and join the second service counter after availing services from the first. If the customer finds the service satisfactory he leaves the system or revisits the first service station if unsatisfied. Transient probabilities and some performance measures are obtained for the given system.

2. Assumptions for the queuing system considered in the paper are given as follows:

i. The arrivals follow a “Poisson process with mean arrival rate $\lambda$”.

ii. The service rate at each node is “dependent on the number of customers in the queue and follows non-homogeneous Poisson process with mean service rate $\mu_1(t)$ and $\mu_2(t)$ respectively both as function of time; $t$”. This time “t” is written like this for special consideration. We define $\mu_1(t)$ and $\mu_2(t)$ as:

\[
\mu_1(t) = \alpha_1 + \beta_1 t
\]

\[
\mu_2(t) = \alpha_2 + \beta_2 t
\]

iii. On service completion the customer can either leave the system with probability q or can revisit the first server with probability p, so we have $p+q=1$. So, system has a Bernoulli feedback.

3. The difference-differential equations governing the system are:

\[
\frac{\partial P_{n_1,n_2}(t)}{\partial t} = -[\lambda + n_1\mu_1(t) + qn_2\mu_2(t)]P_{n_1,n_2}(t) + (n_1 + 1)\mu_1(t)P_{n_1+1,n_2-1}(t) + q(n_2 + 1)\mu_2(t)P_{n_1,n_2+1}(t) + \lambda P_{n_1-1,n_2}(t)
\]

\[
\begin{align*}
\frac{\partial P_{0,n_2}(t)}{\partial t} & = -[\lambda + qn_2\mu_2(t)]P_{0,n_2}(t) + \mu_1(t)P_{1,n_2-1}(t) + q(n_2 + 1)\mu_2(t)P_{0,n_2+1}(t) \\
\frac{\partial P_{n_1,0}(t)}{\partial t} & = -[\lambda + n_1\mu_1(t)]P_{n_1,0}(t) + \mu_2(t)P_{n_1,1}(t) + \lambda P_{n_1-1,0}(t)
\end{align*}
\]

\[
\frac{\partial P_{n_1,n_2}(t)}{\partial t} = -\lambda P_{0,0}(t) + q\mu_2(t)P_{0,1}(t) + \lambda P_{n_1-1,0}(t)
\]

\[
\frac{\partial P_{n_1,n_2}(t)}{\partial t} = -\lambda P_{0,0}(t) + q\mu_2(t)P_{0,1}(t)
\] (1)
We define PGF of $P_{n_1,n_2}(t)$ as

$$F(X,Y,t) = \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} P_{n_1,n_2}(t)x^{n_1}y^{n_2}$$  \hspace{1cm} (2)

After some algebraic operation to equation (1) using equation (2) we get

$$\frac{\partial F(X,Y,t)}{\partial t} = \lambda(x-1)F(X,Y,t) + q\mu_2(t)(1-y)\frac{\partial F(X,Y,t)}{\partial y} + \mu_1(t)(y-x)\frac{\partial F(X,Y,t)}{\partial x}$$ \hspace{1cm} (3)

Thus, linear partial differential equation of first order in three independent variables $X$, $Y$ & $t$ and the Lagrange’s auxiliary equation is:

$$\frac{dt}{1} = \frac{dx}{\mu_1(t)(y-x)} = \frac{dy}{-q\mu_2(t)(1-y)} = \frac{dF}{-\lambda(1-x)F(X,Y,t)}$$ \hspace{1cm} (4)

From first and third terms of equation (4), we have

$$c_1 = (y-1)e^{-\int \mu_1(t)dt}$$

From first and second terms of equation (4), we have

$$c_2 = xe^{-\int \mu_1(t)dt} + (y-1)e^{-\int q\mu_2(t)dt} \left( \int \mu_1(t)e^{\int [q\mu_2(t)-\mu_1(t)]dt} dt \right) + \int \mu_1(t)e^{-\int \mu_1(t)dt} dt$$

From first and fourth terms of equation (4), we have

$$c_3 = F(X,Y,t)exp \left[ -\lambda \left( xe^{-\int \mu_1(t)dt} + (y-1) e^{-\int q\mu_2(t)dt} \left( \int \mu_1(t)e^{\int [q\mu_2(t)-\mu_1(t)]dt} dt \right) \right. \right. $$

$$+ \int \mu_1(t)e^{-\int \mu_1(t)dt} dt \int e^{\int \mu_1(t)dt} dt$$

$$\left. \left. - (y-1) e^{-\int \mu_2(t)dt} \left( \int e^{\int \mu_1(t)dt} \left( \int \mu_3(t)e^{\int [q\mu_2(t)-\mu_1(t)]dt} dt \right) dt \right) \right]$$

where $c_1$, $c_2$, and $c_3$ are arbitrarily defined constants.

The general solution can be obtained using boundary conditions,

$p_{0,0}(0) = 1, p_{0,0}(t) = 0, \text{for all } t > 0$

The PGF of the system size for both the queues by time $t$ is:
4. Performance Measures for the Queuing System

The probability that the queue is empty is given by

\[
P_{0,0}(t) = \exp \left[ -\lambda \left( e^{-\frac{(\alpha_1 t + \beta_1 t^2)}{2}} \left( \int_0^t e^{\alpha_1 v + \beta_1 v^2} \frac{dv}{\alpha_1} - \frac{1}{\alpha_1} \right) \right) + e^{-q \left( \frac{(\alpha_2 t + \beta_1 t^2)}{2} \right)} \left( \frac{1}{\alpha_1} \right) - \right]
\]

\[
\int_0^t (\alpha_1 + \beta_1 v) e^{(q\alpha_2 - \alpha_1) v + (q\beta_2 - \beta_1) v^2} \frac{dv}{\alpha_1} - \right]
\]

\[
\left. \int_0^t (\alpha_1 + \beta_1 v) e^{(q\alpha_2 - \alpha_1) v + (q\beta_2 - \beta_1) v^2} \frac{dv}{\alpha_1} \right] - \int_0^t e^{\alpha_1 v + \beta_1 v^2} \left( \int_0^t (\alpha_1 + \beta_1 v) e^{(q\alpha_2 - \alpha_1) v + (q\beta_2 - \beta_1) v^2} \frac{dv}{\alpha_1} \right) \left( \int_0^t \frac{1}{q\alpha_2} \right)
\]

(6)

Taking y=1 in \( F(X,Y,t) \), we obtain the PGF of the system size for first queue as

\[
F(X, t) = \exp \left[ \lambda (x - 1) e^{-\frac{(\alpha_1 t + \beta_1 t^2)}{2}} \left( \int_0^t e^{(\alpha_1 v + \beta_1 v^2)} \frac{dv}{\alpha_1} - \frac{1}{\alpha_1} \right) \right] \quad ; \quad \lambda < \alpha_1, \beta_1
\]

(7)

By expanding \( F(X, t) \), we obtain “the probability that first queue is empty”:

\[
P_{0,0}(t) = \exp \left[ -\lambda e^{-\frac{(\alpha_1 t + \beta_1 t^2)}{2}} \left( \int_0^t e^{(\alpha_1 v + \beta_1 v^2)} \frac{dv}{\alpha_1} - \frac{1}{\alpha_1} \right) \right]
\]

(8)
Measures for first queue:

1. Mean number of customers:

\[ L_1(t) = \lambda e^{-\left(\frac{a_1 t + \beta_1 t^2}{2}\right)} \left( \int_0^t e^{\left(\frac{a_1 v + \beta_1 v^2}{2}\right)} dv - \frac{1}{\alpha_1} \right) \]  

(9)

2. Utilization time:

\[ U_1(t) = 1 - \exp \left[ -\lambda e^{-\left(\frac{a_1 t + \beta_1 t^2}{2}\right)} \left( \int_0^t e^{\left(\frac{a_1 v + \beta_1 v^2}{2}\right)} dv - \frac{1}{\alpha_1} \right) \right] \]  

(10)

3. Throughput of the service station is:

\[ P_1(t) = (\alpha_1 + \beta_1 t) \left[ 1 - \exp \left[ -\lambda e^{-\left(\frac{a_1 t + \beta_1 t^2}{2}\right)} \left( \int_0^t e^{\left(\frac{a_1 v + \beta_1 v^2}{2}\right)} dv - \frac{1}{\alpha_1} \right) \right] \right] \]  

(11)

4. The average waiting time:

\[ W_1(t) = \frac{L_1(t)}{P_1'(t)} \]

\[ = \frac{\lambda e^{-\left(\frac{a_1 t + \beta_1 t^2}{2}\right)} \left( \int_0^t e^{\left(\frac{a_1 v + \beta_1 v^2}{2}\right)} dv - \frac{1}{\alpha_1} \right)}{(\alpha_1 + \beta_1 t) \left[ 1 - \exp \left[ -\lambda e^{-\left(\frac{a_1 t + \beta_1 t^2}{2}\right)} \left( \int_0^t e^{\left(\frac{a_1 v + \beta_1 v^2}{2}\right)} dv - \frac{1}{\alpha_1} \right) \right] \right]} \]  

(12)

Taking \( x=1 \) in \( F(X,Y,t) \), we obtain the PGF of the second queue size as

\[ F(Y, t) = \exp \left[ \lambda \left( (y-1)e^{-q(\beta_2 t^2/2)} \left( \frac{1}{q \alpha_2 - \alpha_1} - \int_0^t (\alpha_1 + \beta_1 v) e^{(q\alpha_2 - \alpha_1)v + (q\beta_2 - \beta_1)v^2/2} dv \right) \right) \right. \]

\[ + (y-1)e^{-q(\beta_2 t^2/2)} \left[ \int_0^t e^{\alpha_1 v + \beta_1 v^2/2} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(q\alpha_2 - \alpha_1)v + (q\beta_2 - \beta_1)v^2/2} dv \right] \]

\[ - \left. \int_0^t e^{\alpha_1 v + \beta_1 v^2/2} \left( \int_0^t (\alpha_1 + \beta_1 v) e^{(q\alpha_2 - \alpha_1)v + (q\beta_2 - \beta_1)v^2/2} dv - \frac{1}{q \alpha_2} \right) dv \right] \]

\[ \lambda < \min (\alpha_1 + \beta_1 t, q(\alpha_2 + \beta_2 t)) \]  

(13)
By expanding $F(Y, t)$, we obtain “the probability that second queue is empty”:

$$
P_0(t) = \exp \left[ \lambda \left( e^{-q(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left( \frac{1}{q \alpha_2 - \alpha_1} \int_0^t (\alpha_1 + \beta_1 v) e^{(q \alpha_2 - \alpha_1)v + (q \beta_2 - \beta_1)\frac{v^2}{2}} dv \right) \right. \\
+ e^{-q(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left. \int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(q \alpha_2 - \alpha_1)v + (q \beta_2 - \beta_1)\frac{v^2}{2}} dv \right) \\
- \int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} \left( \int_0^t (\alpha_1 + \beta_1 v) e^{(q \alpha_2 - \alpha_1)v + (q \beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv - \frac{1}{q \alpha_2} \right] \right] 
$$

(14)

Measures for second queue:

1. Mean number of customers:

$$
L_2(t) = \exp \left[ \lambda \left( e^{-q(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left( \frac{1}{q \alpha_2 - \alpha_1} \int_0^t (\alpha_1 + \beta_1 v) e^{(q \alpha_2 - \alpha_1)v + (q \beta_2 - \beta_1)\frac{v^2}{2}} dv \right) \right. \\
+ e^{-q(\alpha_2 t + \beta_2 \frac{t^2}{2})} \left. \int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} dv \int_0^t (\alpha_1 + \beta_1 v) e^{(q \alpha_2 - \alpha_1)v + (q \beta_2 - \beta_1)\frac{v^2}{2}} dv \right) \\
- \int_0^t e^{\alpha_1 v + \beta_1 \frac{v^2}{2}} \left( \int_0^t (\alpha_1 + \beta_1 v) e^{(q \alpha_2 - \alpha_1)v + (q \beta_2 - \beta_1)\frac{v^2}{2}} dv \right) dv - \frac{1}{q \alpha_2} \right] \right] 
$$

(15)

2. Utilization time:
Throughput of the service station is:

$$U_2(t) = 1 - \exp \left[ -q \left( \frac{\alpha_1 + \beta_1}{\alpha_2} \right) \left( \int_0^t (\alpha_1 + \beta_1) e^{(q\alpha_2 - \alpha_1)u + (q\beta_2 - \beta_1)v} \frac{du}{\alpha_1} \right) \right. \\
+ \int_0^t e^{(q\alpha_2 - \alpha_1)u + (q\beta_2 - \beta_1)v} \left( \int_0^t (\alpha_1 + \beta_1) e^{(q\alpha_2 - \alpha_1)u + (q\beta_2 - \beta_1)v} \frac{du}{\alpha_1} \right) \right] $$

(16)

3. Throughput of the service station is:

$$P'_2(t) = (\alpha_1 + \beta_1) \left( 1 - \exp \left[ -q \left( \frac{\alpha_1 + \beta_1}{\alpha_2} \right) \left( \int_0^t (\alpha_1 + \beta_1) e^{(q\alpha_2 - \alpha_1)u + (q\beta_2 - \beta_1)v} \frac{du}{\alpha_1} \right) \right. \\
+ \int_0^t e^{(q\alpha_2 - \alpha_1)u + (q\beta_2 - \beta_1)v} \left( \int_0^t (\alpha_1 + \beta_1) e^{(q\alpha_2 - \alpha_1)u + (q\beta_2 - \beta_1)v} \frac{du}{\alpha_1} \right) \right] $$

(17)

4. Average waiting time:

$$W_2(t) = \frac{L_2(t)}{P'_2(t)}$$

(18)

Where $L_2(t) & P'_2(t)$ have already been obtained above.

5. The expected number of customers in the queuing system by time t is:

$$L(t) = L_1(t) + L_2(t) =$$

$$\lambda e^{-\left( \frac{\alpha_1 + \beta_1}{\alpha_2} \right)} \left( \int_0^t e^{(\alpha_1 + \beta_1)u} \frac{du}{\alpha_1} \right)$$


Conclusion:
The present work develops and analyses a two-node tandem queue with feedback having state and time dependent service rates. The concept of feedback has made the model to deal with many more pragmatic situations. The model can be used for evaluation of performance measures of some production system, hospital managements, and communication networks where service rate depends on the aggregate of customers in the system and on time as well.

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