HOMOLOGICAL REGULARITIES AND CONVEXITIES

E. KIRKMAN, R. WON AND J.J. ZHANG

Abstract. This paper concerns homological notions of regularity for noncommutative algebras. Properties of an algebra $A$ are reflected in the regularities of certain (complexes of) $A$-modules. We study the classical Tor-regularity and Castelnuovo–Mumford regularity, which were generalized from the commutative setting to the noncommutative setting by Jørgensen, Dong, and Wu. We also introduce two new numerical homological invariants: concavity and Artin–Schelter regularity. Artin–Schelter regular algebras occupy a central position in noncommutative algebra and noncommutative algebraic geometry, and we use these invariants to establish criteria which can be used to determine whether a noetherian connected graded algebra is Artin–Schelter regular.

0. Introduction

Let $k$ be a base field. An $\mathbb{N}$-graded algebra $A := \bigoplus_{i=0}^{\infty} A_i$ is called connected graded if $A_0 = k$. Throughout let $A$ denote a connected $\mathbb{N}$-graded $k$-algebra unless otherwise stated. The Hilbert series $h_A(t)$ of $A$ will be recalled in Definition 1.1.

A seminal result due to Stanley [St2, Theorem 4.4] states

\begin{equation}
\tag{E0.1.1}
h_A(t^{-1}) = \pm t^\ell h_A(t)
\end{equation}

for some integer $\ell$.

The above theorem provides a surprising and effective criterion which is equivalent to the Gorenstein property. For example, if $G$ is a finite group acting linearly on the polynomial ring $B := k[x_1, \ldots, x_n]$, then the Hilbert series of the fixed subring $B^G$ can be calculated using Molien’s Theorem [Ki, Proposition 1.6]. One can then use (E0.1.1) to determine whether or not $B^G$ is Gorenstein. In general, verifying (E0.1.1) is much easier than verifying the Gorenstein property directly. A noncommutative version of Stanley’s criterion was proved in [JZ, Theorems 6.1 and 6.2].

Note that a commutative connected graded algebra $A$ is regular (i.e., has finite global dimension) if and only if $A$ is isomorphic to a polynomial ring $k[x_1, \cdots, x_n]$ with $\deg x_i > 0$ for all $i$. Hence, in the commutative case, checking whether or not $A$ is regular is relatively easy. However, there are many more noncommutative connected graded regular algebras, namely, the so-called Artin–Schelter regular algebras, which we now define. For a connected graded algebra $A$, let $m = A_{\geq 1}$. The trivial graded $A$-bimodule $A/m$ is also denoted by $k$.

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Definition 0.2. [AS, p.171] A connected graded algebra $T$ is called Artin–Schelter Gorenstein (or $AS$ Gorenstein, for short) if the following conditions hold:

(a) $T$ has injective dimension $d < \infty$ on the left and on the right,
(b) $\Ext^i_T(Tk, TT) = \Ext^i_T(kT, TT) = 0$ for all $i \neq d$, and
(c) $\Ext^d_T(Tk, TT) \cong \Ext^d_T(kT, TT) \cong k(l)$ for some integer $l$. Here $l$ is called the $AS$ index of $T$.

In this case, we say $T$ is of type $(d, l)$. If in addition,

(d) $T$ has finite global dimension, and
(e) $T$ has finite Gelfand–Kirillov dimension,

then $T$ is called Artin–Schelter regular (or $AS$ regular, for short) of dimension $d$.

In this paper we generally reserve the letters $S$ and $T$ for $AS$ regular algebras.

Artin–Schelter regular algebras play an important role in noncommutative algebraic geometry and many other subjects [AS, ATV1, ATV2, RRZ]. Therefore, it would be very useful to have reasonably verifiable criteria for the $AS$ regular property.

Recall that for a graded left $A$-module $M$, we say that $M$ has a linear resolution (or simply say that $M$ is linear) if $M$ has a minimal free $A$-resolution of the form

\[ \cdots \rightarrow A(-i)^{\beta_i} \rightarrow A(-i + 1)^{\beta_{i-1}} \rightarrow \cdots \rightarrow A(-1)^{\beta_1} \rightarrow A^{\beta_0} \rightarrow M \rightarrow 0, \]

for some integers $\beta_i$, or equivalently, $\Tor^A_j(k, M) = 0$ for all $j \neq i$. In this paper it is necessary to deal with (cochain) complexes of graded $A$-modules instead of graded $A$-modules. A complex $X$ of graded left $A$-modules is naturally graded by homological and internal degrees. By examining certain complexes, properties of $A$ can be reflected in the relationships between these degrees. For example, $A$ is Koszul if the trivial graded $A$-module $k$ has a linear resolution, or, equivalently if the Tor-regularity [Definition 0.4] of $k$ is 0. Here is one of our main results.

Theorem 0.3 (Theorem 4.6). Let $A$ be a noetherian connected graded $s$-Cohen–Macaulay algebra. Suppose that there is a graded algebra map $f : T \rightarrow A$, where $T$ is a Koszul Artin–Schelter regular algebra, such that the induced modules $AT$ and $TA$ are both finitely generated. Then the following are equivalent:

(a) $A$ is Artin–Schelter regular,
(b) the Hilbert series $h_A(t)$ of $A$ satisfies $\deg h_A(t) = -s$.

In the commutative case, the existence of the desired map $f : T \rightarrow A$ is automatic (assuming $A$ is generated in degree 1). Similar to Theorem 0.1, the combinatorial property in Theorem 0.3(b) is much easier to check than the definition of AS regularity and so Theorem 0.3 provides an important criterion which can be used to determine whether or not $A$ is AS regular. In the rest of paper, we will prove other criteria for determining AS regularity by using more sophisticated numerical homological invariants.

We now recall or introduce the following homological invariants that will be studied throughout the paper:

- Tor-regularity,
- Castelnuovo–Mumford regularity,
- (numerical) Artin–Schelter regularity,
- concavities associated to the Tor- and Castelnuovo–Mumford regularities.
Definition 0.4. [Jo2, Jo3, DW] The Tor-regularity of a nonzero complex \( X \) of graded left \( A \)-modules is defined to be

\[
\text{Torreg}(X) = \sup_{i,j \in \mathbb{Z}} \{ j - i \mid \text{Tor}_{i}^{A}(\mathbb{k}, X)_{j} \neq 0 \}.
\]

It is clear that Torreg\((X)\) provides a measure of the growth of the degrees of the generators of the free modules in a minimal free resolution of \( X \).

Since Torreg\((A_{k}) \geq 0\) and the equality holds if and only if \( A \) is Koszul, Torreg\((A_{k})\) can be regarded as an invariant that measures how far \( A \) is from being Koszul, at least in the noncommutative setting. When \( A \) is a commutative algebra generated in degree one, Torreg\((A_{k})\) takes on only the values of 0 or \( \infty \) [AP], so it indicates only whether or not \( A \) is Koszul. However, in the noncommutative case, Torreg\((A_{k})\) can be any value in \( \mathbb{N} \cup \{ +\infty \} \) (see Example 2.4(4) and Lemma 5.6).

Next we recall the definition of Castelnuovo–Mumford regularity, which was first introduced and studied in the noncommutative setting by Jørgensen, Dong, and Wu [Jo2, Jo3, DW]. Recall that the \( i \)-th local cohomology of a complex \( X \) of graded \( A \)-modules is defined to be

\[
H_{i}^{m}(X) = \lim_{n \to \infty} \text{Ext}_{A}^{i}(A/m^{n}, X).
\]

See [Jo1, Jo2] and Section 1 for more details.

Definition 0.5. [Jo2, Jo3, DW] Let \( A \) be a noetherian connected graded algebra and let \( X \) be a nonzero complex of graded left \( A \)-modules. The Castelnuovo–Mumford regularity (or CM regularity, for short) of \( X \) is defined to be

\[
\text{CMreg}(X) = \sup_{i,j \in \mathbb{Z}} \{ j + i \mid H_{i}^{m}(X)_{j} \neq 0 \}.
\]

The notions of regularity defined in Definitions 0.4 and 0.5 are natural generalizations of the classical Tor- and Castelnuovo–Mumford regularities in the commutative case. For a finitely generated graded \( A \)-module \( M \), the relationship between the regularities Torreg\((M)\) and CMreg\((M)\) has been studied in the literature. When \( A \) is a polynomial ring generated in degree 1, Torreg\((M)\) = CMreg\((M)\) for any finitely generated graded \( A \)-module \( M \) [EG]. Other relationships between these invariants were established in the commutative case [Röm] and were extended to the noncommutative case in [Jo2, Jo3, DW]. In this paper we provide further relationships between these invariants in the noncommutative case.

Now we introduce a new numerical homological invariant associated to every noetherian connected graded algebra \( A \).

Definition 0.6. Let \( A \) be a noetherian connected graded algebra. The Artin–Schelter regularity (or AS regularity) of \( A \) is defined to be

\[
\text{ASreg}(A) = \text{Torreg}(k) + \text{CMreg}(A).
\]

We remark that in the literature, the phrase “AS regularity of \( A \)” often refers to \( A \) possessing the Artin–Schelter regular property given in Definition 0.2. Here we use use the phrase “AS regularity of \( A \)” to refer to the numerical invariant of \( A \) defined in Definition 0.6, which is related to the AS regular property. Indeed, by [Jo2, Theorems 2.5 and 2.6] or Corollary 2.6(2), ASreg\((A)\) \( \geq 0 \); and by Theorem 0.8, equality holds if and only if \( A \) is AS regular. Hence ASreg\((A)\) can be considered as an invariant that measures how far \( A \) is from being AS regular.
By Example 2.4(3), if $T$ is AS regular of type $(d, l)$, then

(E0.6.1) \[ \text{CMreg}(T) = d - l = -\text{Torreg}(k) \]

which will appear in several places in this paper. As a consequence, $\text{ASreg}(T) = 0$. Some further computations of regularities in the non-Koszul case are provided in Example 2.4(3).

The next theorem is another main result of the paper which was announced in [DW, Proposition 5.6] without proof. It is also an extension of [Röm, Theorem 4.2]. See Corollary 3.4 for a related result.

**Theorem 0.7** (Theorem 2.8). Let $A$ be a noetherian connected graded algebra with balanced dualizing complex. Let $X$ be a nonzero object in $\text{D}^b_{\text{fg}}(A\text{-Gr})$ with finite projective dimension. Then

\[ \text{CMreg}(X) = \text{Torreg}(X) + \text{CMreg}(A). \]

As a companion to Theorem 0.7, we generalize two very nice results of Dong and Wu [DW, Theorems 4.10 and 5.4] to the not-necessarily Koszul setting.

**Theorem 0.8.** Let $A$ be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent:

(a) $A$ is AS regular,

(b) $\text{ASreg}(A) = 0$.

Another main goal of this paper is to introduce a new notion of regularity, which we call the **concavity** of a graded algebra. If $f : A \to B$ is a graded algebra homomorphism between two $\mathbb{N}$-graded algebras. We say that $f$ is a **finite map** if the modules $A_B$ and $B_A$ are finitely generated.

**Definition 0.9.** Let $P$ be a numerical invariant that is defined on all AS regular algebras. Let $A$ be a locally finite $\mathbb{N}$-graded algebra (that is not necessarily connected graded).

1. The $P$-**concavity** of $A$ is defined to be

\[ c_P(A) := \inf \{ P(T) \mid T \text{ is AS regular and there is a finite map } f : T \to A \}. \]

If no such $T$ exists, we define $c_P(A) = \infty$. A similar convention is used in the other parts of this definition.

If $P$ is a numerical invariant that is defined on locally finite $\mathbb{N}$-graded algebras and if $P(A)$ is finite, then we define the **normalized** $P$-**concavity** of $A$ to be

\[ c_{P,-}(A) := c_P(A) - P(A). \]

Below are some special cases.

2. If we take $P = \text{Torreg}$, then the Torreg-**concavity** of $A$ is defined to be

\[ c_{\text{Torreg}}(A) := \inf \{ \text{Torreg}(T) \mid T \text{ is AS regular and there is a finite map } f : T \to A \}. \]

3. If we take $P = -\text{CMreg}$, then the **concavity** of $A$ is defined to be

\[ c(A) := \inf \{ -\text{CMreg}(T) \mid T \text{ is AS regular and there is a finite map } f : T \to A \}. \]

The **normalized concavity** of $A$ is defined to be

\[ c_{-}(A) := c(A) + \text{CMreg}(A). \]
In this paper we focus primarily on the concavity and the normalized concavity of the invariant \( P = -\text{CMreg} \). By (E0.6.1), for any AS regular algebra \( T \), \( \text{CMreg}(T) \leq 0 \) and so by definition, for any locally finite \( \mathbb{N} \)-graded algebra \( A \), we have, \( c(A) \geq 0 \). If \( S \) is a noetherian connected graded AS regular algebra, then \( c(S) = 0 \) if and only if \( S \) is a Koszul [Theorem 0.10(2)]. Let us explain the motivation behind the terminology “concavity”. By analogy with the commutative case, for noncommutative \( T \), we can imagine \( \text{Spec} \ T \) as a noncommutative space associated to \( T \).

By definition, the concavity of \( A \) depends on all finite maps \( f \) from AS regular algebras \( T \) to \( A \). However, when \( A \) itself is AS regular, this invariant can be calculated by taking \( f \) to be the identity map, as shown in the following theorem, which can be considered as an extension of Theorem 0.3 in the case that \( A = S \) is AS regular.

**Theorem 0.10.** Let \( T \) and \( S \) be noetherian AS regular algebras.

1. \( c(S) = -\text{CMreg}(S) \).
2. \( S \) is Koszul if and only if \( c(S) = 0 \).
3. Suppose \( f : T \to S \) is a finite map. Then \( c(T) \geq c(S) \). If further \( c(T) = c(S) \), then \( f \) is surjective and \( T \)-modules and \( S \)-modules are linear Cohen–Macaulay modules over \( T \).

As a corollary to the above results, we see that if there is a finite map between AS regular algebras \( f : T \to S \), then the Koszul property of \( S \) is controlled by the Koszul property of \( T \).

**Corollary 0.11.** (Corollary 4.4) Let \( T \) and \( S \) be noetherian AS regular algebras and let \( f : T \to S \) be a finite map.

1. If \( T \) is Koszul (or equivalently, \( \text{Torreg}(T-k) = 0 \)), then \( S \) is Koszul. Further, \( f \) is surjective and \( T \)-modules and \( S \)-modules are linear Cohen–Macaulay modules over \( T \).
2. Suppose \( \text{Torreg}(T-k) \leq \deg(k \otimes_T S) \). Then \( S \) is Koszul.
3. Suppose \( \text{Torreg}(T-k) = 1 \). If \( f \) is not surjective, then \( S \) is Koszul. As a consequence if \( T \) is a proper subalgebra of \( S \), then \( S \) is Koszul.

We also show that (normalized) concavity is related to the AS regular property.

**Theorem 0.12.** Let \( A \) be a noetherian connected graded algebra with balanced dualizing complex. Let \( c_-(A) \) be the normalized concavity defined in Definition 0.9(4). Then the following hold.

1. \( c_-(A) \geq 0 \).
2. \( c_-(A) = 0 \) if and only if \( A \) is AS regular.

By Theorem 0.12, \( c_-(A) \) can be viewed as an invariant, similar to \( \text{ASreg}(A) \), that measures how far away \( A \) is from being AS regular. Here is a list of numerical invariants that qualify as indicators of the AS regular property for different special classes of algebras:
(i) $\deg h_A(t)$ [Theorem 0.3],
(ii) $\text{ASreg}(A)$ [Theorem 0.8],
(iii) $c_-(A)$ [Theorem 0.12],
(iv) $\text{CMreg}(A)$ [Theorem 4.5], and
(v) $\text{Torreg}(T_A)$ [Remark 5.10].

**Remark 0.13.** As noted in [St1, p.506] (or [KKZ2, p.256]) we have the following hierarchy of homological properties for commutative graded algebras

\[
\text{regular} \implies \text{hypersurface} \implies \text{complete intersection} \implies \text{Gorenstein} \implies \text{Cohen–Macaulay}.
\]

In this paper, we mainly study the “regular” property. It would be very interesting if new numerical (and hopefully computable) invariants could be found to detect or characterize the other properties in the above diagram. Of course Stanley’s Theorem 0.1 for the Gorenstein property is our model. Some necessary conditions to be a “noncommutative complete intersection” are given in [KKZ2].

To conclude the introduction, we mention one application of concavity to noncommutative invariant theory. In [KWZ], the authors used CMreg to bound the degrees of generators of invariant subrings $T^H$ when a semisimple Hopf algebra $H$ acts on an AS regular algebra $T$ homogeneously. The following proposition is an easy consequence of [KWZ, Theorem 0.8]. For any connected graded algebra $A$, let

\[
\beta_1(A) = \deg \text{Tor}^A_1(k,k).
\]

Then $\beta_1(A)$ is the largest degree of an element in a minimal generating set of $A$ and $\beta_2(A)$ is the largest degree of an element in a minimal relation set of $A$. We have the following lower bounds for $c(T^H)$ in terms of the degrees of the generators and the relations of $T^H$.

**Proposition 0.14.** Let $H$ be a semisimple Hopf algebra acting on a noetherian AS regular algebra $T$ homogeneously. Let $R = T^H$ denote the invariant subring of this action. Then the following hold:

1. $c(R) \geq \beta_1(R) - 1$, and
2. $c(R) \geq \min \left\{ \frac{1}{2} \beta_2(R) - \text{CMreg}(T), \frac{1}{2} (\beta_2(R) - \text{CMreg}(T) - 1), \beta_2(R) - 2 \right\}$. 

By [KKZ1, Lemma 3.2(b)], $T^H$ is Cohen–Macaulay with balanced dualizing complex. Now if we are given a noetherian Cohen–Macaulay domain $A$ with balanced dualizing complex (or even a noetherian AS Gorenstein domain), it is generally very difficult to determine if there exist $(T,H)$ such that $A$ is isomorphic to $T^H$. The simple inequality in Proposition 0.14(1) provides an easy way of showing that a noetherian connected graded algebra $A$ cannot be isomorphic to any invariant subring $T^H$ of an AS regular algebra $T$ under a semisimple Hopf algebra $H$ action (if $c(A) < \beta_1(A) - 1$). For example, let

\[
A = k[x_1, \cdots, x_n][t]/(t^2 = f(x_1, \cdots, x_n))
\]

where $\deg x_i = 1$, $\deg t \geq 2$, and $f$ is an irreducible homogeneous polynomial in $x_i$ of degree $(2\deg t)$. It follows from the definition that

\[
0 = c(A) < 1 \leq \deg t - 1 = \beta_1(A) - 1.
\]

Therefore $A$ cannot be isomorphic to $T^H$ by Proposition 0.14(1). See Example 4.2 for further examples.
The paper is organized as follows. Section 1 recalls some basic definitions and properties from homological algebra (including local cohomology). Section 2 gives the definitions of invariants such as Castelnuovo–Mumford regularity, Ext- or Tor-regularity and basic inequalities and equalities relating these regularities. Theorem 0.7 is proved in Section 2. In Section 3 we define AS regularity and prove Theorem 0.8. Section 4 concerns concavity associated to the Castelnuovo–Mumford regularity. We prove Theorems 0.10, 0.12 and Proposition 0.14 on concavity in Section 4. Finally Section 5 contains some examples, questions and remarks.

1. Preliminaries

For an \( \mathbb{N} \)-graded \( k \)-algebra \( A \), we let \( A \text{-Gr} \) denote the category of \( \mathbb{Z} \)-graded left \( A \)-modules. When convenient, we identify the graded right \( A \)-modules with graded left \( A^{op} \)-modules and denote the category \( A^{op} \text{-Gr} \). The derived category of complexes of graded \( A \)-modules is denoted \( D(\mathbb{A} \text{-Gr}) \). We use the standard notation \( D^+ (\mathbb{A} \text{-Gr}) \), \( D^- (\mathbb{A} \text{-Gr}) \), and \( D^b (\mathbb{A} \text{-Gr}) \) for the full subcategories of complexes which are bounded below, bounded above, and bounded, respectively. We use the subscript \( fg \) to denote the full subcategories consisting of complexes with finitely generated cohomology, e.g., \( D^b_{fg} (\mathbb{A} \text{-Gr}) \). We use the standard convention that a left \( A \)-module \( M \) can be viewed as complex concentrated in position 0.

Let \( \ell \) be an integer. For a graded \( A \)-module \( M \), the shifted \( A \)-module \( M(\ell) \) is defined by \( M(\ell)_m = M_{m+\ell} \) for all \( m \in \mathbb{Z} \). For a cochain complex \( X = (X^i, d^i : X^i \to X^{i+1}) \), we define two notions of shifting: \( X(\ell) \) shifts the degrees of each graded vector space \( X^i(\ell)_m = X^i_{m+\ell} \) for all \( i, m \in \mathbb{Z} \) and \( X[\ell] \) shifts the complex \( X[\ell]_i = X^i_{i+\ell} \) for all \( i \in \mathbb{Z} \).

**Definition 1.1.** Let \( A := \bigoplus_{i \geq 0} A_i \) be an \( \mathbb{N} \)-graded locally finite algebra. The *Hilbert series* of \( A \) is defined to be

\[
h_A(t) = \sum_{i \in \mathbb{N}} (\dim_k A_i) t^i.
\]

Similarly, if \( M = \bigoplus_{i \in \mathbb{Z}} M_i \) is a \( \mathbb{Z} \)-graded \( A \)-module (or \( \mathbb{Z} \)-graded vector space), the *Hilbert series* of \( M \) is defined to be

\[
h_M(t) = \sum_{i \in \mathbb{Z}} (\dim_k M_i) t^i.
\]

We say that \( M \) is *locally finite* if \( \dim_k M_d < \infty \) for all \( d \in \mathbb{Z} \). Define the *degree* of \( M \) to be the maximal degree of the nonzero homogeneous elements in \( M \), namely,

\[
(E1.0.1) \quad \deg(M) = \inf \{ d \mid (M)_d = 0 \} - 1 = \sup \{ d \mid (M)_d \neq 0 \} \quad \in \quad \mathbb{Z} \cup \{ \pm \infty \}.
\]

By convention, we define \( \deg(0) = -\infty \). Similarly, we define

\[
(E1.0.2) \quad \ged(M) = \sup \{ d \mid (M)_d = 0 \} + 1 = \inf \{ d \mid (M)_d \neq 0 \} \quad \in \quad \mathbb{Z} \cup \{ \pm \infty \}.
\]

By convention, we define \( \ged(0) = \infty \).

For a nonzero cochain complex \( X \) in \( D(A \text{-Gr}) \), the *degree* of \( X \) is defined to be

\[
(E1.0.3) \quad \deg(X) = \sup_{m, n \in \mathbb{Z}} \{ m + n \mid H^n(X)_m \neq 0 \} = \sup_{n \in \mathbb{Z}} \{ \deg H^n(X) + n \}
\]
where $H^d(X)$ is the $d$th homology of the complex $X$. Similarly, the ged of $X$ is defined to be
\begin{equation}
\text{ged}(X) = \inf_{m,n \in \mathbb{Z}} \{ m + n \mid H^n(X)_m \neq 0 \} = \inf_{n \in \mathbb{Z}} \{ \text{ged} H^n(X) + n \}.
\end{equation}
We also define
\begin{align*}
\sup(X) &= \sup \{ d \mid H^d(X) \neq 0 \} \\
\inf(X) &= \inf \{ d \mid H^d(X) \neq 0 \}.
\end{align*}

Let $A$ be a connected graded algebra. Recall that $\mathfrak{m}$ denotes the graded Jacobson radical (or maximal graded ideal) $A_{\geq 1}$ and that $\mathfrak{k}$ denotes the graded $A$-bimodule $A/\mathfrak{m}$. For a graded left $A$-module $M$, let
\begin{equation}
t^A_i(AM) = \deg \text{Tor}^A_i(\mathfrak{k}, M).
\end{equation}
If $M$ is a graded right $A$-module, let
\begin{equation}
t^A_i(M_A) = \deg \text{Tor}^A_i(M, \mathfrak{k}).
\end{equation}
It is clear that $t^A_i(AM) = t^A_i(M_A)$. If the context is clear, we will use $t^A_i(M)$ instead of $t^A_i(AM)$ (or $t^A_i(M_A)$).

For each graded left $A$-module $M$, we define
\begin{equation}
\Gamma_m(M) = \{ x \in M \mid A_{\geq n}x = 0 \text{ for some } n \geq 1 \} = \lim_{n \to \infty} \text{Hom}_A(A/A_{\geq n}, M)
\end{equation}
and call this the $\mathfrak{m}$-torsion submodule of $M$. It is standard that the functor $\Gamma_m(-)$ is a left exact functor $A-\text{Gr} \to A-\text{Gr}$. Since this category has enough injectives, the $i$th right derived functors, denoted by $H^i_m$ or $R^i \Gamma_m$, are defined and called the local cohomology functors, see [AZ, VdB, Jo1, Jo2] for more details. For a complex $X$ of graded left $A$-modules, the $i$th local cohomology functor of $X$ is given in (E0.4.2). For example, if $M$ is a graded left $A$-module, then
\begin{equation}
H^i_m(M) = R^i \Gamma_m(M) := \lim_{n \to \infty} \text{Ext}^i_A(A/A_{\geq n}, M).
\end{equation}

**Definition 1.2.** Let $A$ be a connected graded noetherian graded algebra. Let $M$ be a finitely generated graded left $A$-module. We call $M$ $s$-Cohen–Macaulay or simply Cohen–Macaulay if $H^i_m(M) = 0$ for all $i \neq s$ and $H^s_m(M) \neq 0$. We say $A$ is Cohen–Macaulay if $A_A$ is Cohen–Macaulay.

Throughout the rest of this paper, when we need a dualizing complex we assume the following hypothesis; we refer the reader to [Ye] for the definitions of a dualizing complex and a balanced dualizing complex.

**Hypothesis 1.3.** Let $A$ be a noetherian connected graded algebra with balanced dualizing complex. In this case by [VdB, Theorem 6.3] the balanced dualizing complex will be given by $R \Gamma_m(A)'$, where $'$ denotes the graded vector space dual.

The local cohomological dimension of a graded $A$-module $M$ is defined to be
\begin{equation}
\text{lcd}(M) := \sup \{ i \in \mathbb{Z} \mid H^i_m(M) \neq 0 \}
\end{equation}
and the cohomological dimension of $\Gamma_m$ is defined to be
\begin{equation}
\text{cd}(\Gamma_m) = \sup_{M \in A-\text{Gr}} \{ \text{lcd}(M) \}.
\end{equation}

We will use the following *Local Duality Theorem* of Van den Bergh several times.
**Theorem 1.4.** [VdB, Theorem 5.1] Let $A$ be a noetherian connected graded $\k$-algebra with $\text{cd}(\Gamma_m) < \infty$ and let $C$ be a connected graded $\k$-algebra. Then for any $X \in D((A \otimes C^\op)\text{-Gr})$ there is an isomorphism

$$R\Gamma_m(X)' \cong R\text{Hom}_A(X, R\Gamma_m(A)')$$

in $D((C \otimes A^\op)\text{-Gr})$.

2. Equalities and inequalities

In this section we study the relationships between the regularities defined in the previous sections, recalling and generalizing results of Jørgensen, Dong, and Wu [Jo2, Jo3, DW]. Throughout this section, we assume Hypothesis 1.3.

Recall, from Definition 0.5, that the Castelnuovo–Mumford regularity of cochain complex $X$ of left $A$-modules is defined to be

$$\text{CMreg}(X) = \deg(R\Gamma_m(X)) = \sup_{i,j \in \mathbb{Z}} \{j + i \mid H^i_m(X)_j \neq 0\}$$

As noted in [Jo2, Observation 2.3] if $X \in D^b_{fg}(A\text{-Gr})$ then $R\Gamma_m(X)' \in D^b_{fg}(A^\op\text{-Gr})$ and $R\Gamma_m(X)' \neq 0$. It follows that $\text{CMreg}(X)$ is finite. In particular, if $A$ is a finitely generated commutative algebra, then $\text{CMreg}(X)$ is finite for all $X \in D^b_{fg}(A\text{-Gr})$.

In Example 5.1, we show that there exists a noetherian connected domain which does not satisfy Hypothesis 1.3 with $\text{GKdim} A = 2$ with $\text{CMreg}(A) = \infty$.

**Example 2.1.** Assume Hypothesis 1.3.

1. If $M$ is a finite-dimensional nonzero graded left $A$-module, then $H^i_m(M) = 0$ for all $i \neq 0$ and $H^0_m(M) = M$ so

$$\text{CMreg}(M) = \deg(M).$$

(E2.1.1) A more general case is considered in part (4), as a finite-dimensional module is 0-Cohen–Macaulay.

2. Let $A$ be an AS Gorenstein algebra of type $(d, l)$. Then $\text{CMreg}(A) = d - l$. This is a well-known fact, which is a consequence of [AZ, Theorem 8.1(3)].

3. Let $A$ be an AS regular algebra of type $(d, l)$. Recall that when regarded as a rational function, $\deg_t h_A(t) = -l$ [StZ, Proposition 3.1(4)]. Hence,

$$\text{CMreg}(A) = d - l = \text{gldim} A + \deg_t h_A(t).$$

By the second statement in [StZ, Proposition 3.1(4)], $d \leq l$. As a consequence,

$$\text{CMreg}(A) = d - l \leq 0.$$  

(E2.1.2)

4. If $M$ is $s$-Cohen–Macaulay, then, by definition,

$$\text{CMreg}(M) = s + \deg(H^s_m(M)).$$

(E2.1.3)

Recall that the Tor-regularity of a nonzero complex $X$ of graded left $A$-modules was defined in Definition 0.4, namely

$$\text{Torreg}(X) = \deg(\k \otimes^L_A X) = \sup_{i,j \in \mathbb{Z}} \{j - i \mid \text{Tor}^i_j(\k, X)_j \neq 0\}.$$
**Definition 2.2.** Let $X$ be a nonzero object in $\mathcal{D}^b_{\text{fg}}(A\text{-Gr})$. The Ext-regularity of $X$ is defined to be
\[
\text{Ext}^{\text{reg}}(X) = -\text{ged}(R\text{Hom}_A(X, k)) = -\inf_{i \in \mathbb{Z}} \{\text{ged}(\text{Ext}^i_A(X, k)) + i\}.
\]

By [DW, Remark 4.5], if $X$ has a finitely generated minimal free resolution over $A$, then $\text{Ext}^{\text{reg}}(X) = \text{Tor}^{\text{reg}}(X)$, and we will not distinguish between $\text{Ext}^{\text{reg}}(X)$ and $\text{Tor}^{\text{reg}}(X)$ in this case. The following easy lemma is useful for computing $\text{Tor}^{\text{reg}}$ of modules over tensor products.

Throughout this paper, $\otimes$ means $\otimes k$.

**Lemma 2.3.** Let $A$ and $B$ be connected graded algebras. Let $P$ (resp. $Q$) be a nonzero object in $\mathcal{D}^b_{\text{fg}}(A\text{-Gr})$ (resp. $\mathcal{D}^b_{\text{fg}}(B\text{-Gr})$). Then $\text{Tor}^{\text{reg}}((A \otimes B)P \otimes Q) = \text{Tor}^{\text{reg}}(A P) + \text{Tor}^{\text{reg}}(B Q)$.

**Proof.** Replacing $P$ by its (minimal) free resolution, we can assume that each term $P^i$ is a finitely generated free $A$-module. The same applies to $Q$. Let $X$ and $Y$ denote the complexes given by tensoring $P$ and $Q$ with $k_A$ and $k_B$ respectively on the left. Then $\text{Tor}^A_i(k, P)$ and $\text{Tor}^B_i(k, Q)$ can be computed by taking homology of $X$ and $Y$, respectively. Further, $\text{Tor}^{A \otimes B}_i(k, P \otimes Q)$ can be computed by taking homology of the complex $X \otimes Y$. By the K"unneth formula (see, e.g. [Ro, Theorem 10.8.1]), we have
\[
\bigoplus_{p+q=n} \text{Tor}^A_p(k, P) \otimes \text{Tor}^B_q(k, Q) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \\
\cong H_n(X \otimes Y) \\
\cong \text{Tor}^{A \otimes B}_n(k, P \otimes Q).
\]
Therefore, using the convention that $\text{deg}(0) = -\infty$,
\[
\text{Tor}^{\text{reg}}((A \otimes B)P \otimes Q) = \sup_{n \in \mathbb{Z}} \{\text{deg}((\text{Tor}^{A \otimes B}_n(k, P \otimes Q)) - n\}
\]
\[
= \sup_{p,q \in \mathbb{Z}} \{\text{deg}((\text{Tor}^A_p(k, P)) + \text{deg}(\text{Tor}^B_q(k, Q)) - (p + q))
\]
\[
= \sup_{p \in \mathbb{Z}} \{\text{deg}((\text{Tor}^A_p(k, P)) - p\} + \sup_{q \in \mathbb{Z}} \{\text{deg}(\text{Tor}^B_q(k, Q)) - q\}
\]
\[
= \text{Tor}^{\text{reg}}(A P) + \text{Tor}^{\text{reg}}(B Q),
\]
as desired. \(\square\)

**Example 2.4.** Assume Hypothesis 1.3.

(1) If $M \in A\text{-Gr}$ and $r = \text{Tor}^{\text{reg}}(M)$, then
\[(E2.4.1) t^A_i(M) := \text{deg}(\text{Tor}^A_i(k, M)) \leq (r + i)
\]
for all $i$.

(2) $\text{Ext}^{\text{reg}}(A) = \text{Tor}^{\text{reg}}(A) = 0$.

(3) Let $T$ be any noetherian AS regular algebra. It is well-known that
\[(E2.4.2) \text{Tor}^{\text{reg}}(T k) = -t + d(= - \text{CM}^{\text{reg}}(T)).
\]
This assertion follows from [StZ, Proposition 3.1(3) and eq. (3.2)].

Next we give an explicit example. Let $T$ be a non-Koszul AS regular algebra of global dimension 3 that is generated in degree 1. By [AS], $T$ is
generated by two elements that satisfy cubic relations. So the minimal free resolution of $T_k$ has the form
\[ 0 \to T(-4) \to T(-3)^{\oplus 2} \to T(-1)^{\oplus 2} \to T \to k \to 0. \]
Therefore, $T$ is of type $(3,4)$ and
\[ t_i^T(k) = \begin{cases} 
0, & i = 0, \\
1, & i = 1, \\
3, & i = 2, \\
4, & i = 3, \\
-\infty, & i > 3.
\end{cases} \]
By Example 2.1(3), $\text{CMreg}(T) = -4 + 3 = -1$ and it is easy to check that
\[ \text{Torreg}(k) = \max\{0, 1-1, 3-2, 4-3, -\infty\} = 1. \]
As a consequence, $\text{Torreg}(k) = -\text{CMreg}(T)$, or equivalently,
\[ \text{ASreg}(T) = 0. \]
Of course this equation always holds for all AS regular algebras by (E2.4.2).

(4) Let $n$ be a fixed positive integer. Let $A$ be any finitely generated commutative Koszul algebra (but not regular) and let $T$ be the algebra in part (3).
Let $B = A \otimes T^\otimes n$. By a similar argument to [KWZ2, Lemma 2.7], using the Künneth formula, one can easily check that
\[ \text{Torreg}(B_k) = n \] [Lemma 2.3] and that $B$ is neither AS regular nor Koszul.

The following result of Jørgensen plays an important role in this paper.

**Theorem 2.5.** [Jo2, Theorems 2.5 and 2.6] Let $A$ be a noetherian connected graded algebra with a balanced dualizing complex, and let $X$ be a nonzero object in $D^b_{fg}(A\text{-Gr})$.

1. $\text{Torreg}(X) \leq \text{CMreg}(X) + \text{Torreg}(k)$.
2. $\text{CMreg}(X) \leq \text{Torreg}(X) + \text{CMreg}(A)$.

We have the following immediate consequences.

**Corollary 2.6.** Let $A$ be a noetherian connected graded algebra with a balanced dualizing complex and let $X$ be a nonzero object in $D^b_{fg}(A\text{-Gr})$.

1. If $\text{Torreg}(k)$ is finite, then so is $\text{Torreg}(X)$.
2. $\text{ASreg}(A) \geq 0$.
3. If $\text{ASreg}(A) = 0$, then equality holds in both parts of Theorem 2.5. Namely,
\[ \text{Torreg}(X) = \text{CMreg}(X) + \text{Torreg}(k). \]

**Proof.** (1) By Theorem 2.5(1),
\[ \text{Torreg}(X) \leq \text{CMreg}(X) + \text{Torreg}(k), \]
and since $\text{CMreg}(X)$ is finite, the result follows.

(2) The statement follows by taking the sum of two inequalities in Theorem 2.5.

(3) When $\text{ASreg}(A) = 0$ then $\text{Torreg}(k) = -\text{CMreg}(A)$. Hence, using both inequalities in Theorem 2.5, we have
\[ \text{Torreg}(X) \leq \text{CMreg}(X) + \text{Torreg}(k) = \text{CMreg}(X) - \text{CMreg}(A) \leq \text{Torreg}(X) \]
and so equality holds throughout. 

We will show that if $X$ has finite projective dimension, then Theorem 2.5(2) becomes an equality. We will need the following straightforward lemma. Recall that for a cochain complex $X$ and $\ell \in \mathbb{Z}$, the complexes $X(\ell)$ and $X[\ell]$ were defined at the beginning of Section 1.

**Lemma 2.7.** Let $A$ be a noetherian connected graded algebra and $X$ be a nonzero complex of graded left $A$-modules. Suppose $\deg(X)$ is finite. Then

$$\deg(X[1]) = \deg(X) - 1 \quad \text{and} \quad \deg(X(1)) = \deg(X) - 1.$$  

Similar equations hold for $\text{ged}(X)$, $\text{CMreg}(X)$, $\text{Extreg}(X)$, and $\text{Torreg}(X)$.

For a cochain complex

$$X = \cdots \rightarrow X_{s-1} \rightarrow X_s \rightarrow X_{s+1} \rightarrow \cdots,$$

we denote the **brutal truncations** of $X$ by

$$X^\geq_s := \cdots \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow X_s \rightarrow X_{s+1} \rightarrow \cdots$$

and

$$X^\leq_s := \cdots \rightarrow X_{s-1} \rightarrow X_s \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \cdots.$$  

(We remark that the notation $X^\geq_s$ and $X^\leq_s$ may mean different truncations in other papers.)

**Theorem 2.8.** Let $A$ be a noetherian connected graded algebra with a balanced dualizing complex. Let $X$ be a nonzero object in $\text{D}^b_{\text{fg}}(A\text{-Gr})$ with finite projective dimension. Then

$$\text{CMreg}(X) = \text{Torreg}(X) + \text{CMreg}(A).$$

**Proof.** By Lemma 2.7, we may assume that $X^n = 0$ for all $n \geq 1$. Let $F$ be a minimal free resolution of $X$, which we write as

$$F : \cdots \rightarrow 0 \rightarrow F^{-s} \xrightarrow{d^{-s}} \cdots \rightarrow F^{-1} \xrightarrow{d^{-1}} F^0 \rightarrow 0 \rightarrow \cdots$$

for some $s \geq 0$. We will prove the assertion by induction on $s$, which is (an upper bound on) the projective dimension of $X$.

For the initial step, we assume that $s = 0$, or $X = F^0 = \bigoplus_i A(-a_i)$ for some integers $a_i$. In this case, it is clear that

$$\text{Torreg}(X) = \text{Torreg}(F^0) = \max_i \{a_i\} =: a.$$  

By Lemma 2.7,

$$\text{CMreg}(X) = \text{CMreg} \left( \bigoplus_i A(-a_i) \right) = \max_i \{\text{CMreg}(A(-a_i))\}$$

$$= \text{CMreg}(A) + \max_i \{a_i\} = \text{CMreg}(A) + \text{Torreg}(X),$$

so the assertion holds for $X = F^0$ as required.

For the inductive step, assume that $s > 0$. Let $F^\leq_{s-1}$ be the brutal truncation of the complex $F$

$$F^\leq_{s-1} : \cdots \rightarrow 0 \rightarrow F^{-s} \rightarrow \cdots \rightarrow F^{-1} \rightarrow 0 \rightarrow 0 \rightarrow \cdots,$$

which is obtained by replacing $F^0$ by 0. We have a distinguished triangle in $\text{D}^b_{\text{fg}}(A\text{-Gr})$

$$F^0 \xrightarrow{f} F \rightarrow F^\leq_{s-1} \rightarrow F^0[1]$$

(E2.8.1)
where $F^0$ is viewed as a complex concentrated at position 0 and $f$ is the inclusion. Let $G$ be the complex $F^{\leq -1}[1]$, which is a minimal free complex concentrated in position $\{-s - 1, \cdots, 0\}$. Then we have a distinguished triangle in $\mathbb{D}^b_{\text{lg}}(A\text{-Gr})$

(E2.8.2) \[ G \xrightarrow{\phi_2} F^0 \xrightarrow{f} F \to G[1] \]

obtained by rotating (E2.8.1). By the induction hypothesis, the assertion holds for both $G$ and $F^0$. We need to show that the assertion holds for $X$, or equivalently, for $F$, as $F \cong X$ in $\mathbb{D}^b_{\text{lg}}(A\text{-Gr})$. By Theorem 2.5(2), the assertion is equivalent to

(E2.8.3) \[ \text{CMreg}(F) \geq \text{Torreg}(F) + \text{CMreg}(A). \]

We fix the following temporary notation:

\[ a = \text{Torreg}(F^0), \quad b = \text{Torreg}(G), \quad c = \text{Torreg}(F) = \text{Torreg}(X), \]

and

\[ a = \text{CMreg}(F^0), \quad \beta = \text{CMreg}(G), \quad \gamma = \text{CMreg}(F) = \text{CMreg}(X). \]

By definition and the minimality of $F$, we have

\[ c = \max \{\text{Torreg}(F^0), \text{Torreg}(F^{\leq -1})\} = \max \{a, b - 1\}. \]

By the above equation, we have,

(E2.8.4) \[ a \leq c, \quad \text{and} \quad b - 1 \leq c, \]

and $c$ must equal to either $a$ or $b - 1$. There are two cases:

**Case 1.** $c = a$ and $a \geq b$.

**Case 2.** $c = b - 1$,

**Case 1:** Suppose that $c = a$ and $a \geq b$. By the definition of $a$, we have $F^0 = A(-a) \oplus C^0$ where $C^0$ is a graded free left $A$-module. Let $\phi_1 : F^0 \to A(-a)$ be the corresponding projection. By the definition of $\alpha := \text{CMreg}(F^0)$, there is an integer $j \in \mathbb{Z}$ such that $H^{j}_m(F^0)_{a-j} \neq 0$ and the induced projection

\[ \tau_1 := H^j_m(\phi_1)_{a-j} : H^j_m(F^0)_{a-j} \to H^j_m(A(-a))_{a-j} \]

is nonzero. The triangle (E2.8.2) gives rise to a long exact sequence

(E2.8.5) \[ \cdots \to H^j_m(G)_{a-j} \to H^j_m(F^0)_{a-j} \to H^j_m(F)_{a-j} \to H^{j+1}_m(F^0)_{a-j} \to \cdots. \]

If

\[ \tau_2 := H^j_m(\phi_2)_{a-j} : H^j_m(G)_{a-j} \to H^j_m(F^0)_{a-j} \]

is not surjective, then (E2.8.5) implies that $H^j_m(F)_{a-j} \neq 0$. By definition, the assumption that $a = c$, and the induction hypothesis, we have

\[ \text{CMreg}(F) \geq a - j + j = a \]

(induction hypothesis)

\[ = a + \text{CMreg}(A) \]

(assumption in Case 1)

\[ = \text{Torreg}(A) + \text{CMreg}(A) \]

as desired.

It remains to show the claim that $\tau_2$ is not surjective. Assume to the contrary that $\tau_2$ is surjective. Then so is the composed map

\[ \tau_3 := \tau_1 \circ \tau_2 : H^j_m(G)_{a-j} \to H^j_m(A(-a))_{a-j}. \]
In particular, $\tau_3$ is not a zero map. Note that
\[ \tau_3 = \tau_1 \circ \tau_2 = H^j_m(\phi_1)_{\alpha-j} \circ H^j_m(\phi_2)_{\alpha-j} = H^j_m(\phi_1 \circ \phi_2)_{\alpha-j}, \]
which implies that $\phi_3 := \phi_1 \circ \phi_2$ is nonzero in $D^b_{\text{reg}}(A, \text{Gr})$. Consider $F$ as the cone of the map $\phi_2 : G \to F^0$; it is clear that $\phi_2$ is the map from the top row $G$ to the middle row $F^0$ in the following diagram
\[
\begin{array}{cccccc}
F^{-s} & \longrightarrow & \cdots & \longrightarrow & F^{-2} & \longrightarrow & F^{-1} & \longrightarrow & 0 \\
0 & \downarrow & & \downarrow & d^{-1}=\phi_2 & & 0 \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & F^0 & \longrightarrow & 0 \\
0 & \downarrow & & \downarrow & \phi_1 & & 0 \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & A(-a) & \longrightarrow & 0.
\end{array}
\]

Since $b \leq a$, $F^{-1}$, which is the zeroth term in the minimal free resolution of $G$, is generated in degree $\leq a$. Since $F$ is a minimal free resolution im $\phi_2 = \text{im} d^{-1} \subseteq mF^0$, and consequently, im $\phi_3 \subseteq mA(-a)$. For every generator $x$ in $F^{-1}$, which has degree $\leq a$, the image $\phi_3(x)$ lies in $mA(-a)$, which has degree at least $a + 1$. Therefore $\phi_3(x) = 0$. This implies that $\phi_3(F^{-1}) = 0$, yielding a contradiction. So we have proved the claim and finished the proof in Case 1.

**Case 2:** Suppose $c = b - 1$. By the definition of $\beta := \text{CMreg}(G)$, there is an integer $j \in \mathbb{Z}$ such that $H^j_m(G)_{\beta-j} \neq 0$. The triangle (E2.8.2) gives rise to a long exact sequence
\[
\cdots \to H^{j-1}_m(F)_{\beta-j} \to H^j_m(G)_{\beta-j} \to H^j_m(F^0)_{\beta-j} \to \cdots.
\]

By the induction hypothesis, the assumption that $c < b$, (E2.8.4), and the definitions of $\alpha, \beta, a$ and $b$, we have
\[
\begin{align*}
\beta &= \text{CMreg}(G) \\
&= \text{Torreg}(G) + \text{CMreg}(A) \quad \text{(induction hypothesis)} \\
&= b + \text{CMreg}(A) > c + \text{CMreg}(A) \\
&\geq a + \text{CMreg}(A) = \text{CMreg}(F^0) \quad \text{(induction hypothesis)} \\
&= \alpha,
\end{align*}
\]
which implies that $H^j_m(F^0)_{\beta-j} = 0$. Since $H^j_m(G)_{\beta-j} \neq 0$ by definition, (E2.8.6) implies that $H^{j-1}_m(F)_{\beta-j} \neq 0$. By definition, $\text{CMreg}(F) \geq \beta - j + (j - 1) = \beta - 1$. This inequality implies that
\[
\text{CMreg}(F) \geq \beta - 1 = \text{CMreg}(G) - 1
\]
\[
= \text{Torreg}(G) + \text{CMreg}(A) - 1 \quad \text{(induction hypothesis)} \\
= b + \text{CMreg}(A) - 1 = (b - 1) + \text{CMreg}(A) \\
= c + \text{CMreg}(A) \\
= \text{Torreg}(F) + \text{CMreg}(A),
\]
as desired, see (E2.8.3).

Combining these two cases completes the proof. \qed

We conclude this section with the following remark.
Remark 2.9. Let $A$ and $B$ be noetherian connected graded algebras with balanced dualizing complexes. Let $P$ (resp. $Q$) be a nonzero object in $D^b_{fg}(A\text{-Gr})$ (resp. $D^b_{fg}(B\text{-Gr})$). Then

$$\text{CMreg}((A\otimes B)P \otimes Q) = \text{CMreg}(AP) + \text{CMreg}(BQ).$$

The proof of the above equality is similar to the proof of Lemma 2.3 using a version of the K"unneth formula for local cohomology, so it is omitted.

3. Artin–Schelter Regularities

In this section we prove results that are related to AS regular algebras. Recall from Definition 0.6 that the AS regularity of $A$ is defined to be

$$\text{ASreg}(A) = \text{Torreg}(k) + \text{CMreg}(A).$$

In general, $\text{ASreg}(A)$ can be any positive integer, as the next example shows.

Example 3.1. Let $d \geq 2$ be an integer and let $B = k[x]/(x^d)$ with $\deg x = 1$. By Example 2.1(2), $\text{CMreg}(B) = d - 1$. By an easy computation,

$$\deg \text{Tor}_n^B(k,k) = \begin{cases} 0 & n = 0, \\ \left\lfloor \frac{n}{2} \right\rfloor d & n > 0 \text{ is even}, \\ 1 + \left\lfloor \frac{n}{2} \right\rfloor d & n > 0 \text{ is odd}. \end{cases}$$

As a consequence, if $d > 2$, then $\text{Torreg}(Bk) = \infty$ and $\text{ASreg}(B) = \infty$. If $d = 2$, then $\text{Torreg}(Bk) = 0$ and $\text{ASreg}(B) = d - 1 = 1$.

Now let $d = 2$ so $B = k[x]/(x^2)$ and let $C$ be the algebra $B^\otimes m$ for a positive integer $m$. Then by Example 2.1(2), $\text{CMreg}(C) = m$. Since $C$ is Koszul, $\text{Torreg}(Ck) = 0$. Therefore $\text{ASreg}(C) = m$.

The goal of this section is to prove Theorem 0.8. We begin with a generalization of a nice result of Dong and Wu [DW, Theorem 4.10] that provides the first step towards to the proof of Theorem 0.8.

Theorem 3.2. Let $A$ be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent:

(i) $A$ is AS regular.

(ii) $A$ is Cohen–Macaulay and $\text{ASreg}(A) = 0$.

When $A$ is Koszul, then [DW, Theorem 4.10] can be recovered from Theorem 3.2 since standard AS Gorenstein algebras satisfy (ii) in the above theorem.

Proof of Theorem 3.2. We first prove that (i) implies (ii). Suppose that $A$ is AS regular of type $(d,1)$. It is well-known that $A$ is Cohen–Macaulay. By (E2.4.2), $\text{ASreg}(A) = 0$.

We now show that (ii) implies (i). Let $A$ be noetherian connected graded with balanced dualizing complex. If $\text{projdim}_A k < \infty$, then $A$ has finite global dimension. Since $A$ is noetherian, if it has finite global dimension, then it has finite GK dimension. By [Zh, Theorem 0.3], $A$ is AS Gorenstein and so $A$ is AS regular by definition. Hence, it suffices to show that $\text{projdim}_k k < \infty$.

Let

$$F : \cdots \to F_i \to \cdots \to F_0 \to k \to 0$$

be a resolution of $k$.
be a minimal free resolution of the trivial left $A$-module $k$. Since $A$ is Cohen–Macaulay, by [VdB, Theorem 6.3], the balanced dualizing complex over $A$ is

$$ R := R \Gamma_m(A)' \cong \omega[d] $$

where $\omega$ is a dualizing $A$-bimodule and $d := \text{lcm}(A)$. By the Local Duality Theorem (Theorem 1.4), for every complex $X$ of graded left $A$-modules,

$$ \text{R} \Gamma_m(X)' \cong \text{R Hom}_A(X, R) \cong \text{R Hom}_A(X, \omega[d]). $$

Since the dualizing complex has finite injective dimension, we obtain that

$$ \text{injdim}(\omega) = d < \infty. $$

By the consequence of the Local Duality Theorem (E3.2.1), $\Gamma_m$ has cohomological dimension $d$.

For each $j \geq 0$, let $Z_j(F)$ denote the $j$th syzygy of the complex $F$. We will show that $Z_j(F) = 0$ for $j \gg 0$, which implies that $\text{projdim} k < \infty$ as desired. Assume to the contrary that $Z_j(F) \neq 0$ for all $j \geq 0$. Note that

$$ \cdots \rightarrow F_{j+2} \rightarrow F_{j+1} \rightarrow Z_j(F) \rightarrow 0 $$

is a minimal free resolution of $Z_j(F)$.

**Claim.** For all $j \geq d$, $t^{A}_{j+1}(k) \leq t^{A}_j(k)$.

**Proof of the claim.** By the balanced condition, $\text{Ext}^i_A(k, \omega) = 0$ for all $i \neq d = \text{injdim} \omega$. By induction on syzygies, we have $\text{Ext}^i_A(Z_{j-1}(F), \omega) = 0$ for all $i \neq 0$. Further, by induction, one sees that $\text{Ext}^i_A(Z_{j-1}(F), \omega) = 0$ for all $i \neq 0$ and all $j \geq d$. From now on, we fix $j \geq d$. By local duality (E3.2.1), we obtain that $H^i_m(Z_{j-1}(F)) = 0$ for all $i \neq d$. Since $A$ is Cohen–Macaulay, $H^i_m(F_j) = 0$ for all $i \neq d$. Applying $\text{R} \Gamma_m(-)$ to the short exact sequence

$$ 0 \rightarrow Z_j(F) \rightarrow F_j \rightarrow Z_{j-1}(F) \rightarrow 0, $$

we obtain a long exact sequence, which has only three nonzero terms yielding a short exact sequence

$$ 0 \rightarrow H^0_m(Z_j(F)) \rightarrow H^d_m(F_j) \rightarrow H^d_m(Z_{j-1}(F)) \rightarrow 0. $$

The above short exact sequence implies that $\deg H^d_m(Z_j(F)) \leq \deg H^d_m(F_j)$. By definition, $\text{CMreg}(Z_j(F)) \leq \text{CMreg}(F_j)$. Since $\text{ASreg}(A) = 0$, Theorem 2.6(3), for any $X \in D^b_{fg}(A-\text{Gr})$,

$$ \text{Torreg}(X) = \text{CMreg}(X) + c, $$

where $c = \text{Torreg}_A(k) = -\text{CMreg}(A)$. Then

$$ t^{A}_{j+1}(k) = t^{A}_{0}(F_{j+1}) = t^{A}_0(Z_j(F)) $$

$$ = t^{A}_0(Z_j(F)) - 0 $$

$$ \leq \text{Torreg}(Z_j(F)) = \text{CMreg}(Z_j(F)) + c $$

$$ \leq \text{CMreg}(F_j) + c = \text{Torreg}(F_j) $$

$$ = \sup\{t^{A}_i(F_j) - i \mid i \in \mathbb{Z}\} $$

$$ = t^{A}_0(F_j) - 0 = t^{A}_0(F_j) $$

$$ = t^{A}_j(k) $$

as desired. This finishes the proof of the claim.
Since \( t^A_j(k) \geq j \) when \( F_j \neq 0 \), then for \( j \gg 0 \), the claim contradicts the fact that \( t^A_j(k) \geq j \). Therefore we obtain a contradiction, and hence \( \text{projdim} k < \infty \) as required. \( \square \)

Next we work to complete the proof of Theorem 0.8, which is a generalization of [DW, Theorem 5.4]. We need the following lemma [DW, Lemma 5.3]. Let \( R \) be a nonzero object in \( D^b(A-\text{Gr}) \) that will be a balanced dualizing complex over \( A \) in the proof of Theorem 0.8. By \([VdB, \text{Theorem 6.3}]\), \( R \) is a finitely generated free graded left \( A \)-module. Set \( s = \inf R \). Then \( f \) naturally induces a morphism from the truncated complex \( F^\geq s \to R \), denoted by \( \overline{f} \).

**Lemma 3.3.** Retain the above notation.

(1) [DW, Lemma 5.3] There is a quasi-isomorphism \( g : F^{\leq s-1} \to \text{cone}(\overline{f}) \).

(2) \( H^i(\text{cone}(\overline{f})) = \begin{cases} F^{s-1}/\text{im}(d^{s-2}_X), & \text{if } i = s-1, \\ 0, & \text{if } i \neq s-1. \end{cases} \)

(3) \( F^{\leq s-1} \) is a minimal free resolution of \( \text{cone}(\overline{f}) \).

**Proof.** Part (1) is a consequence of [DW, Lemma 5.3] as \( f \) is a quasi-isomorphism.

Parts (2) and (3) follow immediately from part (1). \( \square \)

Now we are ready to prove Theorem 0.8.

**Proof of Theorem 0.8.** By (E2.4.2), (i) implies (ii).

We now show that (ii) implies (i). Let \( c = -\text{CMreg}(A) = \text{Torreg}(k) \). By Corollary 2.6(3),

\[
\text{Extreg}(X) = \text{Torreg}(X) = \text{CMreg}(X) + c
\]

for any nonzero object \( X \) in \( D^b_{fg}(A-\text{Gr}) \).

Let \( R \) be a balanced dualizing complex over \( A \). Then by [VdB, Theorem 6.3], \( R \cong R \Gamma_m(A)' \). By assumption, \( \text{CMreg}(A) = -c \), so for all \( i \in \mathbb{Z} \), we have \( H^i_m(A)_{> -(i+c)} = 0 \). Hence, for all integers \( i \), \( H^i(R)_{< -i+c} = 0 \), or \( H^i(R(c))_{< -i} = 0 \), and so by [DW, Lemma 5.2], \( \text{Ext}^i_A(R(c), k)_{> -i} = 0 \). By a degree shift, we obtain that \( \text{Ext}^i_A(R, k)_{> -i-c} = 0 \) for all \( i \in \mathbb{Z} \).

Since \( R \) is a balanced dualizing complex over \( A \), by definition, \( R \Gamma_m(R) \cong A' \), which implies that \( \text{CMreg}(R) = 0 \). Hence, by the first paragraph, \( \text{Extreg}(R) = \text{Torreg}(R) = \text{CMreg}(R) + c = c \). By definition of \( \text{Extreg}(R) \), for all integers \( i \), we have \( \text{Ext}^i_A(R, k)_{< -i-c} = 0 \). Combining this with the last paragraph, we obtain that

(E3.3.1) \[
\text{Ext}^i_A(R, k)_j \neq 0 \quad \text{if and only if } j = -i - c.
\]

Since \( R \in D^b_{fg}(A-\text{Gr}) \), \( R \) has a minimal free resolution \( F \xrightarrow{\sim} R \) such that each term in \( F \) is a finitely generated free graded left \( A \)-module. Set

\[
s = \inf R = \inf \{ j \mid H^j(R) \neq 0 \}.
\]

By [VdB, Theorem 6.3], \( R \cong R \Gamma_m(A)' \), which implies that

\[
s = -\sup \{ j \mid H^j_m(A) \neq 0 \}.
\]
Since both brutal truncations $F^{≥s}$ and $F^{≤s−1}$ of $F$ are minimal free complexes, (E3.3.1) implies that

\[
\text{Ext}_{A}^{s}(F^{≥s}) = \begin{cases} 
  c, & \text{if } F^{≥s} \text{ is not acyclic} \\
  -\infty, & \text{if } F^{≥s} \text{ is acyclic}
\end{cases}
\]

and

\[
\text{Ext}_{A}^{s}(F^{≤s−1}) = \begin{cases} 
  c, & \text{if } F^{≤s−1} \text{ is not acyclic} \\
  -\infty, & \text{if } F^{≤s−1} \text{ is acyclic}
\end{cases}
\]

By the choice of $s$, $F^{s} \neq 0$ and $\text{Ext}_{A}^{s}(F^{≥s}, k) = \text{Hom}_{A}(F^{s}, k) \neq 0$. Hence $F^{≥s}$ is not acyclic, and it follows that $\text{Ext}_{A}^{s}(F^{≥s}) = c$ by (E3.3.2). Let $f$ be the quasi-isomorphism from $F \to R$. Then $f$ naturally induces a morphism from $F^{≥s}$ to $R$, denoted by $\overline{f}$. Hence we are in the setting of Lemma 3.3.

We claim that $\overline{f}$ is a quasi-isomorphism. Suppose on the contrary that $\overline{f}$ is not a quasi-isomorphism. Then $Y := \text{cone}(\overline{f}) \neq 0$ in $D_{\mathfrak{R}}(A\text{-Gr})$. Consider the following distinguished triangle

\[
F^{≥s} \xrightarrow{\overline{f}} R \to Y := \text{cone}(\overline{f}) \to F^{≥s}[1].
\]

Since $R \Gamma_{m}(R) \cong A'$ in $D(A^{e}\text{-Gr})$, we have the exact sequences

\[
0 \to H_{m}^{1}(Y) \to H_{m}^{0}(F^{≥s}) \to H_{m}^{0}(R) \to H_{m}^{0}(Y) \to H_{m}^{1}(F^{≥s}) \to 0,
\]

and the isomorphism

\[
H_{m}^{j−1}(Y) \cong H_{m}^{j}(F^{≥s}), \quad j \neq 0, 1.
\]

By Lemma 3.3(2), $Y \cong H^{s−1}(Y)[1−s]$ in $D_{\mathfrak{R}}(A\text{-Gr})$. By the Local Duality Theorem 1.4, we have isomorphisms

\[
R \Gamma_{m}(Y)' \cong \text{RHom}_{A}(Y, R) \cong \text{RHom}_{A}(H^{s−1}(Y)[1−s], R)
\]

\[
\cong \text{RHom}_{A}(H^{s−1}(Y), R)[s−1].
\]

By taking the 0th cohomology, we obtain the following isomorphism

\[
H_{m}^{0}(Y)' \cong \text{Ext}^{s−1}(H^{s−1}(Y), R).
\]

Since $H^{j}(R) = 0$ for all $j < s$, one sees that

\[
H_{m}^{0}(Y) \cong \text{Ext}^{s−1}(H^{s−1}(Y), R)' = 0.
\]

Since $\text{CMReg}(F^{≥s}) = \text{ExtReg}(F^{≥s}) − c = c − c = 0$, $H_{m}^{j−1}(Y)_{≥j} = 0$ for $j \neq 1$ by (E3.3.4) and (E3.3.5). By definition, $\text{CMReg}(Y) ≤ −1$. However, Lemma 3.3(3) says that $F^{≤s−1}$ is a minimal free resolution of $Y$. Hence (E3.3.3) implies that $\text{ExtReg}(Y) = c$, and consequently, by the first paragraph, $\text{CMReg}(Y) = c − c = 0$. This yields a contradiction. Therefore $\overline{f}$ is a quasi-isomorphism and $Y \cong 0$.

Since $F^{≥s−1}$ is a minimal free resolution of $Y$, $F^{j} = 0$ for all $j ≤ s − 1$, which means that the projective dimension of $R$ is finite. By [DW, Theorem 3.6], $A$ is AS Gorenstein. As a consequence, $A$ is Cohen–Macaulay. It follows from Theorem 3.2 that $A$ is AS regular.

**Corollary 3.4.** Let $A$ be a noetherian connected graded algebra with balanced dualizing complex. Then the following are equivalent:

(a) $A$ is AS regular.

(b) $\text{Torreg}(X) = \text{CMreg}(X) + \text{Torreg}(k)$ for every nonzero $X \in D_{\mathfrak{R}}(A\text{-Gr})$. 

\[\□\]
(c) \(\text{CMreg}(X) = \text{Torreg}(X) + \text{CMreg}(A)\) for every nonzero \(X \in D_{\text{reg}}^b(A-\text{Gr})\).

(d) There is a constant \(c\) such that \(\text{CMreg}(X) = \text{Torreg}(X) + c\) for every nonzero \(X \in D_{\text{reg}}^b(A-\text{Gr})\).

(e) There is a constant \(c\) such that \(\text{CMreg}(M) = \text{Torreg}(M) + c\) for every nonzero finitely generated graded left \(A\)-module \(M\).

**Proof.** By Theorem 0.8 and Corollary 2.6(3), we have that (a) implies both (b) and (c). It is clear that either (b) or (c) implies (d), and also that (d) implies (e). Hence, we need only show that (e) implies (a).

Suppose there is a constant \(c\) such that \(\text{CMreg}(M) = \text{Torreg}(M) + c\) for every nonzero finitely generated graded left \(A\)-module \(M\). Since \(\text{Torreg}(A) = 0\) [Example 2.4(2)], then by setting \(M = A\), we have that \(c = \text{CMreg}(A)\). Since \(\text{CMreg}(k) = 0\) [Example 2.1(1)], setting \(M = k\) implies that \(c = -\text{Torreg}(k)\). Therefore \(\text{ASreg}(A) = \text{CMreg}(A) + \text{Torreg}(k) = c - c = 0\).

Hence, by Theorem 0.8, \(A\) is AS regular. \(\Box\)

We remark that Corollary 3.4 is a generalization of [Röm, Theorem 4.1].

4. Concavity

In this section we use the letters \(A\) and \(B\) for connected graded noetherian algebras, \(S\) and \(T\) for connected graded noetherian AS regular algebras, and \(F\) and \(G\) for general locally finite \(\mathbb{N}\)-graded noetherian algebras.

For a locally finite \(\mathbb{N}\)-graded noetherian algebra \(F\), let

\[(E4.0.1) \quad \Phi(F) := \{T \mid \text{there is a finite map } \phi : T \to F\},\]

where \(T\) ranges over all connected graded noetherian AS regular algebras.

Now we recall Definition 0.9. Let \(P\) be any numerical invariant that is defined for locally finite \(\mathbb{N}\)-graded noetherian rings (or a subclass of such algebras).

(1) The \(P\)-concavity of \(F\) is defined to be

\[c_P(F) := \inf_{T \in \Phi(F)} \{P(T)\} .\]

If no such \(T\) exists, we write \(c_P(F) = \infty\). The normalized \(P\)-concavity of \(F\) is defined to be

\[c_{P,-}(F) := c_P(F) - P(F) .\]

(2) The concavity of \(F\) is defined to

\[c(F) := \inf_{T \in \Phi(F)} \{-\text{CMreg}(T)\} \]

and the normalized concavity of \(F\) is defined to be

\[c_{-}(F) := c(F) + \text{CMreg}(F) .\]

**Proposition 4.1.** Let \(f : F \to G\) be a finite map of locally finite graded noetherian algebras. Then \(c_P(F) \geq c_P(G)\). As a consequence, the following hold.

(1) If \(F\) is a subalgebra of \(G\) such that \(FG\) and \(GF\) are finitely generated, then \(c_P(F) \geq c_P(G)\).

(2) If \(H\) is a finite dimensional semisimple Hopf algebra acting on \(F\) homogeneously, then \(c_P(F^H) \geq c_P(F) \geq c_P(F^H)\).

(3) \(c_P(F) \geq c_P(M_n(F))\).
(4) Let $t$ be a commutative indeterminate of degree 1. Then $c(F[t]) = c(F)$.

Proof. By definition, $c_P(F) = \inf_{T \in \Phi(F)}\{P(T)\}$. If $T \in \Phi(F)$, then by definition there is a finite map $\phi : T \to F$. But since $f : F \to G$ is a finite map, the composition $f \circ \phi : T \to F$ is a finite map. Hence $T \in \Phi(G)$, so $\Phi(F) \subseteq \Phi(G)$, which implies that $c_P(F) \geq c_P(G)$.

Parts (1)–(3) are immediate consequences of the main assertion.

For part (4), note that there is a finite map $F[t] \to F$ given by sending $t$ to 0. Hence, taking $P = \text{CMreg}$ in the main assertion, we have $c(F[t]) \geq c(F)$. Fix a real number $\epsilon > 0$. By definition of $c(F)$, there is a noetherian AS regular algebra $T$ of type $(d, l)$ and a finite map $\phi : T \to F$ such that $-\text{CMreg}(T) \leq c(F) + \epsilon$. Then $T[t] \to F[t]$ is a finite map. Hence,

\[
c(F[t]) \leq -\text{CMreg}(T[t]) = -(d + 1) - (l + \deg t) = -(d - l) - (1 - \deg t) = -(d - l) \leq -\text{CMreg}(T) \leq c(F) + \epsilon.
\]

Since $\epsilon$ was arbitrary, we obtain that $c(F[t]) \leq c(F)$. Combined with the previous paragraph, we conclude that $c(F[t]) = c(F)$.

We do not have any examples with strict inequality $c_P(F) > c_P(F \# H)$ and $c_P(F) > c_P(M_n(F))$. Proposition 0.14 can be used to provide many examples with $c_P(F \# H) > c_P(F)$. We now prove Proposition 0.14. For the rest of the paper we consider only connected graded noetherian algebras.

Proof of Proposition 0.14. Let $H$ be a semisimple Hopf algebra acting on a noetherian AS regular algebra $T$ homogeneously and let $R = T^H$ denote the invariant subring.

(1) By [KWZ, Theorem 0.8(1)], if $S \in \Phi(R)$, then

\[
\beta_1(R) \leq \max\{\beta_1(S), \text{CMreg}(T) - \text{CMreg}(S)\}.
\]

It is clear that $\text{CMreg}(T) - \text{CMreg}(S) \leq -\text{CMreg}(S) \leq -\text{CMreg}(S) + 1$. By definition and Theorem 0.7,

\[
\beta_1(S) = t^S_k(k) \leq \text{Torreg}(sk) + 1 = \text{CMreg}(sk) - \text{CMreg}(S) + 1 = -\text{CMreg}(S) + 1
\]

which implies that $\beta_1(R) \leq -\text{CMreg}(S) + 1$, or equivalently, $-\text{CMreg}(S) \geq \beta_1(R) - 1$. Since $S \in \Phi(R)$ was arbitrary, we obtain that $c(R) \geq \beta_1(R) - 1$.

(2) By [KWZ, Theorem 0.8(2)], if $S \in \Phi(R)$, then

\[
\beta_2(R) \leq \max\{2(\text{CMreg}(T) - \text{CMreg}(S)), \text{CMreg}(T) - \text{CMreg}(S) + \beta_1(S), \beta_2(S)\}.
\]

If $\beta_2(R) \leq 2(\text{CMreg}(T) - \text{CMreg}(S))$, then $-\text{CMreg}(S) \geq \frac{1}{2}\beta_2(R) - \text{CMreg}(T)$. Since $S \in \Phi(R)$ was arbitrary, we obtain that

\[
c(R) \geq \frac{1}{2}\beta_2(R) - \text{CMreg}(T).
\]

If $\beta_2(R) \leq \text{CMreg}(T) - \text{CMreg}(S) + \beta_1(S)$, then $\beta_2(R) \leq \text{CMreg}(T) - \text{CMreg}(S) + (-\text{CMreg}(S) + 1)$, which implies that

\[
c(R) \geq \frac{1}{2}(\beta_2(R) - \text{CMreg}(T) - 1).
\]
Finally, if \( \beta_2(R) \leq \beta_2(S) \), then
\[
\beta_2(R) \leq \beta_2(S) \leq \text{Torreg}(k) + 2 \leq \text{CMreg}(k) - \text{CMreg}(S) + 2 = -\text{CMreg}(S) + 2,
\]
which implies that \( c(R) \geq \beta_2(R) - 2 \).

Combining these three cases, we obtain that
\[
c(R) \geq \min \left\{ \frac{1}{2} \beta_2(R) - \text{CMreg}(T), \frac{1}{2} (\beta_2(R) - 1), \beta_2(R) - 2 \right\},
\]
as desired. \( \square \)

The inequalities in Proposition 0.14 can be used to test if an algebra is isomorphic to the invariant subring of a semisimple Hopf action on an AS regular algebra.

**Example 4.2.** Let \( A \) be a connected graded algebra that is not generated in degree 1 (or equivalently \( \beta_1(A) \geq 2 \)). Suppose there is a noetherian Koszul AS regular algebra \( S \in \Phi(A) \). Then \( c(A) = 0 \) by definition and so by Proposition 0.14(1), \( A \) cannot be isomorphic to an invariant subring \( T^H \).

A specific example is the graded algebra
\[
A = k[x_1, \ldots, x_n, t_1, \ldots, t_m]/(t_j^2 - f_j(x_1, \ldots, x_n) \mid j = 1, \ldots, m)
\]
where \( \deg x_i = 1 \) for all \( i = 1, \ldots, n \), \( \deg t_{j_0} > 1 \) for some \( j_0 \), and each \( f_j \) is a homogeneous polynomial in the \( x_i \)'s of degree equal to \( 2 \deg t_j \). Then \( c(A) = 0 \), as the polynomial ring \( k[x_1, \ldots, x_n] \) is in \( \Phi(A) \). It is clear that \( \beta_1(A) = \max_{1 \leq j \leq m} \{ \deg t_j \} > 1 \).

Our next aim is to prove Theorems 0.10 and 0.12. We begin by proving the following lemma. As usual, let \( T \) denote a noetherian AS regular algebra.

**Lemma 4.3.** Let \( A \) be a connected graded noetherian algebra, let \( \phi : T \to A \) be a finite map, and let \( M \) be a finitely generated graded left \( A \)-module. For parts (2)–(5), we further assume that \( TA \) is linear.

1. \( \text{Torreg}(TA) \leq \text{Torreg}(T) + \text{Torreg}(AM) \).
2. For all \( s \geq 0 \), we have \( t^T_s(k) - s \leq \max_{0 \leq i \leq s} \{ t^A_i(k) - i \} \).
3. For all \( 0 \leq s \leq \gldim T \), we have \( \max_{0 \leq i \leq s} \{ t^A_i(k) - i \} = \max_{0 \leq i \leq s} \{ t^T_i(k) - i \} \).
4. For all \( j > \gldim T \), we have
   \[
t^A_j(k) - j \leq \text{Torreg}(T) = \max_{0 \leq i \leq \gldim T} \{ t^T_i(k) - i \}.
   \]
5. \( \text{Torreg}(A) = \text{Torreg}(T) \). As a consequence, then \( A \) is Koszul if and only if \( T \) is Koszul.

**Proof.** We will use the change of rings spectral sequence given in [Ro, Theorem 10.60], namely:

\[
E^2_{p,q} := \text{Tor}_p^A(T, k_T, A) \Rightarrow \text{Tor}_p^T(k_T, T, M).
\]

The \( E^2 \)-page of the spectral sequence is similar to the one given after [KWZ, Lemma 5.1].
(1) By (E4.3.1), we have
\[ t_n^T(TM) = \deg \Tor^T_n(k_T, TM) \]
\[ \leq \max_{p+q=n} \{ \deg E_{p,q}^2 \} = \max_{p+q=n} \{ \deg (\Tor^A_p(\Tor^T_q(k_T, A), AM)) \} \]
\[ \leq \max_{p+q=n} \{ \deg \Tor^A_p(k, A) + \deg \Tor^T_q(k, A) \} \]
\[ = \max_{p+q=n} \{ t^A_p(AM) + t^T_q(TA) \}. \]

Then
\[ t_n^T(TM) - n \leq \max_{p+q=n} \{ t^A_p(AM) - p + t^T_q(TA) - q \} \]
\[ \leq \Torreg(AM) + \Torreg(TA). \]

Now the assertion follows from the definition.

(2) For the remainder of the proof, we assume that \( TA \) is linear, that is, we assume for all \( q \) that there exists some \( n_q \geq 0 \) such that \( \Tor^T_q(k_T, A) = k(-q)^{n_q} \).

We claim that the map \( f : T \rightarrow A \) is surjective. Since \( TA \) is linear, \( k \otimes_T A = k^{\oplus n} \) for some positive integer \( n \). So \( TA \) is generated by elements of degree 0. But \( A \) has only one element in degree 0 (up to a scalar). So \( TA \) is generated by a single element of degree 0. This implies that the map \( \phi : T \rightarrow A \) is surjective. Note that \( n_q = 0 \) for all \( q > \projdim_0 TA \). By taking \( M = k \) in (E4.3.1), we have
\[ t_n^T(k) = \deg \Tor^T_n(k_T, TA) \]
(E4.3.2)
\[ \leq \max_{p+q=n} \{ \deg E_{p,q}^2 \} = \max_{p+q=n} \{ \deg \Tor^A_p(\Tor^T_q(k_T, A), Ak) \} \]
\[ \leq \max_{p+q=n} \{ \deg \Tor^A_p(k, Ak) + q \} = \max_{p+q=n} \{ t^A_p(k) + q \}. \]

Therefore the assertion in (2) follows.

(3) We use induction on \( s \) from 0 to \( g := \gl(T) \). Since \( t_0^T(k) = t_0^A(k) = 0 \), the assertion holds for \( s = 0 \). Now assume that \( s > 0 \) and assume the assertion holds for \( s - 1 \). Part (2) above implies that \( \max_{0 \leq i \leq s} \{ t^A_i(k) - i \} \geq \max_{0 \leq i \leq s} \{ t^T_i(k) - i \} \). Hence we need only show that \( \max_{0 \leq i \leq s} \{ t^A_i(k) - i \} \leq \max_{0 \leq i \leq s} \{ t^T_i(k) - i \} \). By the induction hypothesis, it suffices to show that
\[ t^A_s(k) - s \leq \max_{0 \leq i \leq s} \{ t^T_i(k) - i \}. \]

By [StZ, (3-4), p.1600], \( \{ t^T_0(k), \ldots, t^T_g(k) \} \) is strictly increasing, or equivalently,
\[ \{ t^T_0(k), t^T_1(k) - 1, \ldots, t^T_{g-1}(k) - (g-1), t^T_g(k) - g \} \]
(E4.3.4)

is non-decreasing. So (E4.3.3) is equivalent to
\[ t^A_s(k) - s \leq t^T_s(k) - s, \quad \text{or} \quad t^A_s(k) \leq t^T_s(k). \]

Let \( \alpha = t^T_s(k) \) and \( \beta = t^A_s(k) \). Using the spectral sequence (E4.3.1) (and an inequality in (E4.3.2)), we see that for each \( p < s \),
\[ \deg E_{p,s-p}^\infty \leq \deg E_{p,s-p}^2 \leq \max_{p+q=s-p, p < s} \{ t^A_p(k) + q \} \]
\[ = \max_{0 \leq p \leq s-1} \{ t^A_p(k) - p \} + s = \max_{0 \leq i \leq s-1} \{ t^T_i(k) - i \} + s \]
\[ \leq t^T_s(k) - s + s = t^A_s(k) = \alpha. \]
Hence, if \( \beta > \alpha \), then \( (E_{p,q}^{\infty})_S = 0 \) for all \( p < s \). Similarly, for all \( r \geq 2 \) and for all \( p+q < s \), we can show \( \deg E_{p,q}^r \leq t^A_p(k) \leq \alpha \). Observe that if \( r \geq 2 \), the incoming differentials to \( E^r_{s,0} \) are all zero, and the outgoing differentials map to \( E^r_{s-r,r-1} \) with \( \deg E^r_{s-r,r-1} \leq \alpha \). Hence, \( 0 \neq (E_{s,0}^2)_S \) is in the kernel of all outgoing differentials and so \( (E_{s,0}^2)_S \) survives on the \( \infty \)-page. Thus, by (E4.3.1), \( \text{Tor}^S_T(k,k) \neq 0 \), which contradicts that \( \deg \text{Tor}^T_S(k,k) = \alpha < \beta \). Therefore \( \beta \leq \alpha \), which is (E4.3.5). This completes the inductive step and the proof.

(4) We prove this by induction on \( j > \text{gldim} \, T \). The initial step and the inductive step are similar, so we treat them together. Let \( \gamma = \text{Tor}^r_T(k) \). Similar to the proof of part (3), one sees that for every \( r \geq 2 \)

\[
\deg E^r_{j-r,r-1} \leq \max_{p \leq j-1} \{ t^A_p(k) - p + (j-1) \} \leq \gamma + (j-1)
\]

where the second inequality is the inductive step when \( j > \text{gldim} \, T + 1 \), and is part (3) when \( j = \text{gldim} \, T + 1 \). Since \( E_{j,0}^\infty = 0 \) as \( j > \text{gldim} \, T \), the “outgoing differential” argument in the proof of part (3) shows that

\[
\deg E^r_{j,0} \leq \max_{r \geq 2} \{ \deg E^r_{j-r,r-1} \} \leq \gamma + j - 1.
\]

This is equivalent to

\[
t^A_j(k) \leq \gamma + j - 1 < \gamma + j,
\]

and therefore the assertion holds.

(5) The equation follows from parts (3) and (4). The consequence follows from the fact that \( A \) is Koszul if and only if \( \text{Tor}^r_S(A) = 0 \). \( \square \)

Now we are ready to prove Theorems 0.10 and 0.12.

**Proof of Theorem 0.10.** Fix a noetherian AS regular algebra \( S \). We first remark that by [AZ, Theorem 8.3(3)], if there is a finite map \( T \rightarrow S \), then for any finitely generated graded \( S \)-module \( M \), we have \( \text{CMreg}(TM) = \text{CMreg}(SM) \). In particular, \( \text{CMreg}(TS) = \text{CMreg}(SM) \).

(1) Suppose \( T \) is any noetherian AS regular algebra and suppose \( T \rightarrow S \) is a finite map. By Theorem 0.7 and the fact that \( \text{Tor}^r_T(S) \geq 0 \), we obtain that

\[
\text{CMreg}(T) = \text{CMreg}(TS) - \text{Tor}^r_T(S) \leq \text{CMreg}(S)
\]

and hence, \( - \text{CMreg}(T) \geq - \text{CMreg}(S) \). Therefore, \( c(S) \geq - \text{CMreg}(S) \). By definition, it is clear that \( c(S) \leq - \text{CMreg}(S) \), and so we have equality, as desired.

(2) We use part (1) in the next proof. If \( S \) is Koszul, then \( c(S) = - \text{CMreg}(S) = \text{Tor}^r_S(k) = 0 \). Conversely, if \( c(S) = 0 \), then \( \text{Tor}^r_S(k) = 0 \), which implies that \( S \) is Koszul.

(3) Suppose \( T \) is a noetherian AS regular algebra and \( f : T \rightarrow S \) is a finite map. By Theorem 0.7, we have

\[
\text{CMreg}(S)(= \text{CMreg}(SM)) = \text{CMreg}(TS) = \text{Tor}^r_T(S) + \text{CMreg}(T).
\]

This implies that \( - \text{CMreg}(T) \geq - \text{CMreg}(S) \), as \( \text{Tor}^r_T(S) \geq 0 \) by definition. The first assertion now follows from part (1).

If \( c(T) = c(S) \), then by part (1), we have \( \text{CMreg}(T) = \text{CMreg}(S) = \text{CMreg}(TS) \). Hence, by Theorem 0.7, \( \text{Tor}^r_T(S) = 0 \), which means that \( TS \) is linear. Similarly, \( S_T \) is linear. By the proof of Lemma 4.3(2), the map \( f : T \rightarrow S \) is surjective.

Since the local cohomology of \( S \) can be computed as a left \( S \)-module or as a left \( T \)-module, \( H^i_S(TS) \) is zero unless \( i = \text{gldim} \, S \). Hence \( TS \) is Cohen–Macaulay. \( \square \)
Proof of Theorem 0.12. (1) Let $T$ be a noetherian AS regular algebra, $A$ be a noetherian connected graded algebra, and suppose $\phi : T \to A$ is a finite map. By Theorem 0.7,
\[
\CMreg(TA) = \Torreg(TA) + \CMreg(T).
\]
Since $\CMreg(TA) = \CMreg(A) A =: \CMreg(A)$, we have
\[
- \CMreg(T) = \Torreg(TA) - \CMreg(A) \geq - \CMreg(A).
\]
Then $c(A) \geq - \CMreg(A)$. By definition of $c_-(A)$, we obtain that $c_-(A) = c(A) + \CMreg(A) \geq 0$.

(2) If $A$ is AS regular, by letting $T = A$, we obtain that $c(A) \leq - \CMreg(A)$.
As a consequence, $c_-(A) \leq 0$. By part (1), $c_-(A) = 0$.

Conversely, suppose $c_-(A) = 0$, or equivalently, $c(A) = - \CMreg(A)$. By definition and the fact that $c(A)$ is an integer, there is a noetherian AS regular algebra $T$ with a finite map $\phi : T \to A$ such that $- \CMreg(T) = c(A) = - \CMreg(A)$. By Theorem 0.7,
\[
\Torreg(TA) = \CMreg(A) - \CMreg(T) = 0.
\]
Then $TA$ is a linear left $T$-module (i.e., $t_i^T(TA) = 0$ if $\Tor_i^T(k, TA) \neq 0$).

By Lemma 4.3(5), $\Torreg(Ak) = \Torreg(Tk)$. We know that $\CMreg(A) = \CMreg(T)$ and since $T$ is AS regular, we have $\CMreg(T) = - \Torreg(Tk)$. Combining these equations, we obtain that
\[
\ASreg(A) = \CMreg(A) + \Torreg(Ak) = \CMreg(T) + \Torreg(Tk) = \ASreg(T) = 0.
\]
Finally the assertion follows from Theorem 0.8. \hfill $\square$

We collect some criteria for Koszulness of AS regular algebras that follow from our previous results.

Corollary 4.4. Let $T$ and $S$ be noetherian AS regular algebras and $f : T \to S$ be a finite map.

(1) If $T$ is Koszul (or equivalently, $\Torreg(Tk) = 0$), then so is $S$. Further, $f$ is surjective and $T$ and $S_T$ are linear Cohen–Macaulay modules over $T$.

(2) Suppose $\Torreg(Tk) \leq \deg(k \otimes_T S)$. Then $S$ is Koszul.

(3) Suppose $\Torreg(Tk) = 1$. If $f$ is not surjective, then $S$ is Koszul. As a consequence if $T$ is a proper subalgebra of $S$, then $S$ is Koszul.

Proof. (1) Suppose $T$ is a Koszul noetherian AS regular algebra and suppose $f : T \to S$ is a finite map. By Theorem 0.10(3) and the fact that $T$ is Koszul, we have
\[
0 = - \CMreg(T) \geq - \CMreg(S) \geq 0.
\]
Hence $\CMreg(S) = \CMreg(T) = 0$. The result follows from parts (2) and (3) of Theorem 0.10 (the Koszulness of $S$ also follows from Lemma 4.3(5)).

(2) Suppose $S$ and $T$ are AS regular. By Theorem 0.7, we have
\[
\CMreg(S) = \CMreg(TS) = \Torreg(TS) + \CMreg(T).
\]
By Theorem 0.8, $\CMreg(S) = - \Torreg(sk)$ and $\CMreg(T) = - \Torreg(tk)$, so
\[
0 \leq \Torreg(sk) = - \CMreg(S) = - \CMreg(T) - \Torreg(TS) = \Torreg(tk) - \Torreg(TS) \leq \deg(k \otimes_T S).
\]
Hence, if $\Torreg(tk) \leq \deg(k \otimes_T S)$, then $\Torreg(sk) = 0$, whence $S$ is Koszul.

(3) If $f$ is not surjective, then $\deg(k \otimes_T S) \geq 1$ and so by part (2), $S$ is Koszul. \hfill $\square$
Theorem 4.5. Let $A$ be a noetherian connected graded algebra. Suppose that there is a finite map $f : T \to A$, where $T$ is a noetherian Koszul AS regular algebra. Then the following hold.

1. $\text{CMreg}(A) \geq 0$ and $\text{CMreg}(A) = 0$ if and only if $A$ is AS regular (and Koszul).

2. If $A$ is commutative and generated in degree 1, then $\text{CMreg}(A) \geq 0$, and $A$ is a polynomial ring if and only if $\text{CMreg}(A) = 0$.

Part (2) is an improvement of [Röm, Theorem 4.1(iv) ⇔ (v)].

Proof of Theorem 4.5. (1) Since the map $\phi : T \to A$ is finite and $T$ is Koszul, by Theorem 0.7,

$$\text{CMreg}(A) = \text{CMreg}(T) = \text{Torreg}(T,A) + \text{CMreg}(T) = \text{Torreg}(T,A) \geq 0.$$ 

Since $T \in \Phi(A)$ and $T$ is Koszul, by definition,

$$0 \leq c(A) \leq -\text{CMreg}(T) = 0,$$

which implies that $c(A) = 0$. Consequently, by the definition of $c_-(A)$, we have $c_-(A) = \text{CMreg}(A)$. Now by Theorem 0.12(2), we have that $A$ is AS regular if and only if $\text{CMreg}(A) = 0$. If this happens, $\text{CMreg}(A) = 0$ implies that $\text{Torreg}(A,k) = 0$ by Theorem 0.8, and hence $A$ is Koszul.

(2) Let $T = k[A_1]$. Then $A = T/J$ for some ideal $J$. Since $T$ is noetherian, Koszul and AS regular, then the assertion follows from part (1). □

Note that part (2) of the above theorem fails when $A$ is not commutative [Example 5.4(ii)]. Now we are ready to prove Theorem 0.3. Recall that the Hilbert series of a graded $A$-module $M$, denoted by $h_M(t)$, is given in Definition 1.1. If $h_M(t)$ is a rational function, then the $a$-invariant of $M$, denoted by $a(M)$, is defined to be the $t$-degree of the rational function $h_M(t)$.

Theorem 4.6. Let $A$ be a noetherian connected graded $s$-Cohen–Macaulay algebra. Suppose that there is a finite map $f : T \to A$, where $T$ is a noetherian Koszul AS regular algebra. Then the following are equivalent.

(a) $A$ is AS regular.

(b) $A$ is AS regular and Koszul.

(c) $\deg h_A(t) = -s$.

Proof. Since there is a finite map to $A$ from an AS regular algebra $T$, by taking the minimal free resolution of the graded $T$-module $A$, it follows that the Hilbert series $h_A(t)$ is rational. Then by the proof of [KWZ, Theorem 4.7(2)], we have

$$\text{CMreg}(A) = s + a(A) = s + \deg h_A(t).$$

Now the assertions follows from Theorem 4.5(1). □

The following simple corollary is useful in practice.

Corollary 4.7. Let $A$ and $B$ be algebras satisfying Hypothesis 1.3. Assume that there is a finite map $f : A \to B$ such that $A B$ is linear of finite projective dimension. Then $A$ is AS regular if and only if $B$ is AS regular.

Proof. Suppose $B$ is AS regular. By the change of rings spectral sequence given in [Ro, Theorem 10.60] (or see (E4.3.1)), we have

$$\text{gldim } A = \text{projdim } A k \leq \text{projdim } B k + \text{projdim } A B < \infty.$$
By [Zh, Theorem 0.3], $A$ is AS regular.

Now we assume $A$ is AS regular. By Lemma 4.3(5), $\text{Torreg}(Ak) = \text{Torreg}(Bk)$. By Theorem 0.7,

$$\text{CMreg}(B) = \text{CMreg}(AB) = \text{Torreg}(AB) + \text{CMreg}(A) = \text{CMreg}(A).$$

Then

$$\text{ASreg}(B) = \text{Torreg}(Bk) + \text{CMreg}(B) = \text{Torreg}(Ak) + \text{CMreg}(A) = \text{ASreg}(A) = 0.$$

The assertion follows from Theorem 0.8. □

5. Examples and Remarks

This section contains some examples and remarks. To save space we will omit some non-essential details. Our first example shows that it is necessary to assume Hypothesis 1.3 for most of the results in this paper.

**Example 5.1.** We refer to [AZ, SZ] for the definition of the $\chi$-condition. Let $A$ be the noetherian connected graded domain of GK-dimension 2 given in [SZ, Theorem 2.3] which does not satisfy the $\chi$-condition. By [VdB, Theorem 6.3], $A$ does not admit a balanced dualizing complex. Further, by [SZ, Theorem 2.3], $\text{Ext}_A^1(k, A)$ is not bounded above. Since $\text{Hom}_A(k, A) = 0$, $H_m^1(A)$ contains $\text{Ext}_A^1(k, A)$ as a graded $k$-subspace. Therefore $\text{deg } H_m^1(A) = +\infty$, and consequently, $\text{CMreg}(A) = +\infty$.

If there is a finite map $T \to A$ for some noetherian AS regular algebra $T$, then [AZ, Theorem 8.1(1) and Lemma 8.2(4)] implies that $A$ satisfies the $\chi$-condition. Since $A$ does not satisfy the $\chi$-condition, we conclude that there does not exist a finite map from a noetherian AS regular algebra to $A$. As a consequence, $c(A) = +\infty$ by definition.

In [RS], Rogalski and Sierra provide a family of examples of noetherian Koszul algebras of global dimension 4 for which the $\chi$-condition fails. By similar arguments, these algebras have infinite CM regularity and concavity.

Before we give more examples, we prove some well-known lemmas.

**Lemma 5.2.** Let $A$ be a noetherian connected graded algebra and let $\Omega$ be a regular normal element of degree 1 or 2. Let $B = A/(\Omega)$. Then $\text{Torreg}(Bk) \leq \text{Torreg}(Ak)$.

Consequently, the following hold.

1. If $A$ has finite global dimension, then $\text{Torreg}(Bk) < \infty$.
2. [P, Theorem 1.2] If $A$ is Koszul, then so is $B$.
3. Suppose $\text{deg } \Omega = 1$. If $A$ has finite global dimension, then so does $B$.

*Proof.* The first part of the proof is copied from the proof of [KKZ2, Theorem 1.11]. Since $B$ is a factor ring of $A$, there is a graded version of the change of rings spectral sequence given in [Ro, Theorem 10.71]

$$E^2_{p,q} := \text{Tor}_p^B(k, \text{Tor}_q^A(B, k)) \implies \text{Tor}_r^1(k, k).$$

Let $a = \text{deg } \Omega$. Since $B = A/(\Omega)$ and $\Omega$ is a regular normal element, we have $\text{Tor}_{a}^1(B, k) = k$, $\text{Tor}_1^1(B, k) = k(-a)$ and $\text{Tor}_i^1(B, k) = 0$ for $i > 1$. Hence the $E^2$-page of the spectral sequence (E5.2.1) has only two possibly nonzero rows; namely

$q = 0 : \text{Tor}_p^B(k, k)$ for $p = 0, 1, 2, \ldots$, and

$q = 1 : \text{Tor}_p^B(k, k(-a))$ for $p = 0, 1, 2, \ldots$. 
Since (E5.2.1) converges, we have $\text{Tor}_B^B(k, k) = \text{Tor}_A^A(k, k) = k$ and a long exact sequence
\[
\cdots \to \text{Tor}_4^B(k, k) \to \text{Tor}_2^B(k, k)(-a) \to \\
\to \text{Tor}_3^A(k, k) \to \text{Tor}_1^B(k, k)(-a) \to \\
\to \text{Tor}_2^B(k, k) \to \text{Tor}_0^B(k, k)(-a) \to \\
\to \text{Tor}_1^A(k, k) \to 0.
\]
By the $k$-th row from the bottom in the above exact sequence, we have
\[
\deg(\text{Tor}_k^B(k, k)) - k \leq \max\{\deg(\text{Tor}_{k-2}^A(k, k)) - k, \deg(\text{Tor}_{k-2}^B(k, k) - (a - 2))\}
\]
where we use $\text{Tor}_{k-2}^B(k, k) - (a - 2) \leq \text{Tor}(A|A)$ by the induction hypothesis and $(2 - a) \geq 0$ as $a = 1$ or $2$. The assertion follows from the definition.

Parts (1) and (2) of this lemma follow immediately from the main assertion.

(3) For each $k > \text{gldim} A + 1$, we have an exact sequence
\[
0(= \text{Tor}_k^A(k, k)) \to \text{Tor}_k^B(k, k) \to \text{Tor}_{k-2}^B(k, k)(-1) \to 0(= \text{Tor}_{k-1}^A(k, k)).
\]
So $\deg \text{Tor}_k^B(k, k) = \deg \text{Tor}_{k-2}^B(k, k) + 1$.

If $\text{Tor}_k^B(k, k) \neq 0$, $B$ being connected graded implies that $\deg \text{Tor}_k^B(k, k) \geq \deg \text{Tor}_{k-2}^B(k, k) + 2$ by using the minimal free resolution of $k$. This yields a contradiction, and thus $\text{Tor}_k^B(k, k) = 0$, or $\text{gldim} B < k < \infty$. \hfill $\square$

See Lemma 4.3 and Corollary 4.7 for related results. Note that the inequality $\text{Tor}^{AB}(k, k) \leq \text{Tor}^{AB}(A|A)$ in Lemma 5.2 can be strict, see Example 5.4(iv). Part (3) of Lemma 5.2 is a special case of Corollary 4.7 if $A$ satisfies Hypothesis 1.3.

**Lemma 5.3.** Let $A$ be a noetherian connected graded AS Gorenstein algebra of type $(d, l)$. Let $\Omega$ be a regular normal element of degree $a$.

1. If $a \geq l - d + 2$, then $A/\Omega$ is not AS regular.

2. Suppose $A$ is generated in degree 1 and is not Koszul. If $a \geq \max\{3, l - d + 1\}$, then $A/\Omega$ is not AS regular.

**Proof.** By the Rees Lemma, $B := A/\Omega$ is also AS Gorenstein (of injective dimension $d - 1$). Applying $H_m^A(-)$ to the short exact sequence of left $A$-modules,
\[
0 \to A(-a) \to A \to B \to 0,
\]
we obtain that $H_m^A(B) = 0$ for all $i \neq d - 1$ and
\[
0 \to H_m^{d-1}(B) \to H_m^d(A(-a)) \to H_m^d(A) \to 0
\]
is an exact sequence. Since $H_m^d(A)$ is nonzero and bounded above,
\[
\deg H_m^{d-1}(B) = \deg H_m^d(A(-a)) = \deg H_m^d(A) + a = -l + a.
\]
This implies that
\[
(E5.3.1) \quad \text{CMreg}(B) = (d - 1) + (-l + a) = (d - l) + (a - 1).
\]
As a consequence, $B$ is of type $(d - 1, l - a)$. 

Let $B$ be a factor ring $T/\langle \Omega \rangle$ where $\Omega$ is a regular normal element of degree $a$. The algebra $B$ is $s$-Cohen–Macaulay where $s = \text{gldim } T - 1 = d - 1$ and by (E5.3.1),
\[(E5.4.1) \quad \text{CMreg}(B) = \text{CMreg}(T) + (a - 1).\]

If $a$ is 1 or 2, by Lemma 5.2(1), $\text{Torreg}(B/k) < \infty$.

Now we give some explicit examples.

(i) Let $A = k[x]/(x^a)$ where $a \geq 2$. If $a \geq 3$, $\text{Torreg}(A/k) = \infty$ by Example 3.1. This means that the assertion in Lemma 5.2(1) fails if $\deg \Omega > 2$.

(ii) Let $T$ be the algebra $k\langle x, y \rangle/(x^2y - yx^2, xy^2 - y^2x)$, which is a noetherian AS regular algebra of type $(3, 4)$ (a special case of the algebra given in Example 2.4(3)). By Example 2.4(3), $\text{Torreg}(T/k) = 1$.

It is easy to see that $\Omega := x^2$ is a normal regular element of $T$. Let $B = T/\langle \Omega \rangle$. Then, by Lemma 5.2(1), $\text{Torreg}(B/k) \leq \text{Torreg}(T/k) = 1$. Since $B$ is isomorphic to $k\langle x, y \rangle/(x^2, xy^2 - y^2x)$ which is not Koszul, $\text{Torreg}(B/k) \geq 1$. Therefore $\text{Torreg}(B/k) = 1$. By (E5.4.1),
\[
\text{CMreg}(B) = \text{CMreg}(T) + (2 - 1) = -\text{Torreg}(T/k) + 1 = -1 + 1 = 0.
\]
Hence $\text{ASreg}(B) = 1$ and $B$ is not AS regular. This example shows that Theorem 4.5(2) fails without the commutativity assumption.

Let $A = B^\otimes n$ for any $n \geq 1$. Then it is easy to check that $\text{CMreg}(A) = 0$ and $\text{Torreg}(A/k) = \text{ASreg}(A) = n$. As a consequence, $A$ is not AS regular. By Theorem 4.5(1), there does not exist a finite map from a Koszul AS regular algebra to $A$ (for any $n \geq 1$).

(iii) Let $B$ be the algebra in part (ii). Since $T \to B$ is a surjective map, $c(B) \leq -\text{CMreg}(T) = \text{Torreg}(T/k) = 1$. We claim that $c(B) = 1$. First, by definition, $c(B) \geq 0$, and hence $c(B)$ is either 0 or 1. Now, since $\text{CMreg}(B) = 0$, we have $c_-(B) = c(B) + \text{CMreg}(B) = c(B)$, which implies that $c_-(B)$ is either 0 or 1. Since $B$ is not AS regular by part (ii), by Theorem 0.12, $c_-(B) = 1$. Therefore $c(B) = c_-(B) = 1$ and we have proved the claim.

It is easy to see that
\[
1 \leq c(B^\otimes n) = c_-(B^\otimes n) \leq n.
\]

It would be interesting to work out the exact value of $c(B^\otimes n)$.

(iv) Let $T$ be as in part (ii) and let $\Omega$ be $xy - yx$. Then $\Omega$ is a regular normal element of $T$ such that $B := T/\langle \Omega \rangle$ is the commutative polynomial ring
proof. (1) First assume Lemma 5.6.  

One can calculate a nonnegative integer by the hypothesis. Let $E$ there are nonnegative integers $p, q$ such that $p + q = 1$.  

By definition [Definitions 0.6 and 0.9(3)], the last two invariants are dependent on the first three. We now seek to understand what values are possible for the first three invariants.

**Definition 5.5.** Let $A$ be a noetherian connected graded algebra with balanced dualizing complex.

1. The $ta$-pair associated to $A$ is defined to be
   $$ta(A) := (\text{Torreg}(A), \text{ASreg}(A)) \in (\mathbb{N} \times \mathbb{N}) \cup \{(+\infty, +\infty)\}.$$  

2. The $tc$-pair associated to $A$ is defined to be
   $$tc(A) := (\text{Torreg}(A), \text{CMreg}(A)) \in (\mathbb{N} \cup \{+\infty\}) \times \mathbb{Z}.$$  

Note that $\text{CMreg}(A) \geq -\text{Torreg}(A)$ for every $A$.

In the next lemma, let $D$ denote a noetherian connected graded algebra with balanced dualizing complex.

**Lemma 5.6.** (1) For every $t \in \mathbb{N} \cup \{+\infty\}$ and $c \in \mathbb{Z}$ with $c \geq -t$, there is an algebra $D$ such that $ta(D) = (t, c)$.

(2) For every $t \in \mathbb{N}$ and $c \in \mathbb{Z}$ with $c \geq -t$, there is an algebra $D$ such that $ta(D) = (t, c)$ and $c(D) = t$.

(3) For every pair $(t, a) \in (\mathbb{N} \times \mathbb{N}) \cup \{(+\infty, +\infty)\}$, there is an algebra $D$ such that $ta(D) = (t, a)$.

(4) For every pair $(t, b) \in \mathbb{N} \times \mathbb{N}$, there is an algebra $D$ such that
   $$\text{Torreg}(A(k), c_-(D)) = (c(D), c_-(D)) = (t, b).$$

**Proof.** (1) First assume $t = \infty$. Let $T$ be the algebra in Example 5.4(ii). Then $\text{CMreg}(T) = -1$. Let $A = k[x]/(x^3)$. Then $\text{CMreg}(A) = 2$. For every integer $c$ there are nonnegative integers $p, q$ such that $c = (-1)p + 2q$. Let $D = T \otimes p \otimes A \otimes q$. One can calculate
   $$\text{CMreg}(D) = p \text{CMreg}(T) + q \text{CMreg}(A) = p(-1) + q(2) = c$$

as desired.

Now we assume that $t \in \mathbb{N}$. Let $T$ be the algebra in Example 5.4(ii). Then $\text{Torreg}(T) = 1$ and $\text{Torreg}(T \otimes k) = t$ by Lemma 2.3. Let $a = t + c$ which is a nonnegative integer by the hypothesis. Let $E$ be the algebra $k[x]/(x^2)$ which has $\text{CMreg}(E) = 1$ and $\text{Torreg}(E) = 0$. Let $D = T \otimes t \otimes E \otimes E$. Then
   $$\text{CMreg}(D) = t \text{CMreg}(T) + a \text{CMreg}(E) = -t + a = c$$

and
   $$\text{Torreg}(D) = t \text{Torreg}(T) + a \text{Torreg}(E) = t + a \cdot 0 = t$$
as desired.

(2) We use the second half of the proof of (1) and note that
\[ c(D) = c(T^\otimes t \otimes E^\otimes a) = c(T^\otimes t) = t \]
as desired.

(3) Since \( \text{ASreg}(A) = \text{Torreg}(A) + \text{CMreg}(A) \), the assertion follows easily from part (1).

(4) Since \( c_-(A) = c(A) + \text{CMreg}(A) \), the assertion follows easily from part (2). \( \square \)

Note that the algebras \( A \) in the above proof are generated in degree 1 and are module-finite over their centers. We conclude the paper with the following remarks.

**Remark 5.7.** Assume Hypothesis 1.3.

(1) It would be interesting to work out the range of \( c(A) \) (resp. \( c_-(A) \)) for every fixed \( tc \)-pair \( (t, c) \). When \( c = -t \), by Theorems 0.8 and 0.12(1), \( c(A) \) must be \(-c\). However, computing concavity \( c(A) \) and normalized concavity \( c_-(A) \) is generally much harder than computing \( \text{Torreg}(A) \) and \( \text{CMreg}(A) \).

(2) Further it would be interesting to study algebras with \( ta \) near \((0, 0)\), for example, \( ta(A) = (1, 0) \) or \( ta(A) = (0, 1) \). It is not clear to us whether an algebra \( A \) with \( ta(A) = (0, 1) \) is \( AS \) Gorenstein, although we do not have any counterexample to this.

**Remark 5.8.** It is an open question whether the tensor product of two noetherian \( AS \) regular algebras (or two algebras satisfying Hypothesis 1.3) is always noetherian. If we ignore this issue, we can consider the behavior of regularities with respect to the tensor product.

(1) [Lemma 2.3] \( \text{Torreg}(A) \) is additive in the following sense
\[ \text{Torreg}(A \otimes B) = \text{Torreg}(A) + \text{Torreg}(B) \].

(2) [Remark 2.9] \( \text{CMreg}(A) \) is additive in the following sense
\[ \text{CMreg}(A \otimes B) = \text{CMreg}(A) + \text{CMreg}(B) \].

(3) It follows from parts (1) and (2), \( \text{ASreg}(A) \) is additive in the following sense
\[ \text{ASreg}(A \otimes B) = \text{ASreg}(A) + \text{ASreg}(B) \].

(4) We do not know if \( c(A) \) (resp. \( c_-(A) \)) is additive. It follows from the definition that
\[ c(A \otimes B) \leq c(A) + c(B) \].
The same is true for \( c_-(A) \).

**Remark 5.9.** It follows immediately from Theorem 0.10(2) that if \( S \) and \( T \) are noetherian \( AS \) regular algebras and that \( f : T \to S \) and \( g : S \to T \) are finite maps, then both \( f \) and \( g \) are isomorphisms of graded algebras.

The above statement fails if we assume that \( S \) and \( T \) are only \( AS \) Gorenstein. Let \( S = T = k[x]/(x^2) \), which is noetherian \( AS \) Gorenstein. Let \( f(= g) : T \to T \) be defined by sending \( a + bx \to a \) for all \( a, b \in k \). Then \( f \) is a finite map that is not an isomorphism of graded algebras.

**Remark 5.10.** Let \( T \) be a noetherian \( AS \) regular Koszul algebra and let \( A \) be a connected graded algebra. Suppose \( f : T \to A \) is a finite map. It is trivial that

(1) \( \text{Torreg}(T_A) \geq 0 \).
By Theorem 0.7,
\[
\text{Torreg}(TA) = \text{Torreg}(TA) + \text{CMreg}(T) = \text{CMreg}(TA) = \text{CMreg}(A).
\]
Now it follows from Theorem 4.5(1) that
\[
(2) \quad \text{Torreg}(TA) = 0, \text{ or equivalently, } TA \text{ is linear, if and only if } A \text{ is AS regular (and Koszul)}.
\]
Therefore Torreg(TA) qualifies as an indicator of the AS regular property in this case.

We can say more in terms of the Hilbert series of the algebras.

**Remark 5.11.** We continue Remark 5.10 and consider a special case when \(h_T(t) = (1 - t)^{-d}\) where \(d = \text{gldim } T\). Let \(f : T \to A\) be a finite map.

1. Then \(A\) is AS regular if and only if there is an integer \(0 \leq d' \leq d\) such that the \(i\)-th term in the minimal free resolution of the \(T\)-module \(TA\) is isomorphic to \(T(-i)^{d'}\) for all \(i\). If this happens, \(A\) is Koszul and \(h_A(t) = (1 - t)^{-d+d'}\).
2. If \(A\) is known to be Cohen–Macaulay, then \(A\) is AS regular if and only if \(h_A(t) = (1 - t)^{-d+d'}\) for an integer \(0 \leq d' \leq d\). If this happens, all properties in part (1) hold.

**Remark 5.12.** Let \(A\) and \(B\) be two connected graded algebras. We say that \(A\) and \(B\) are finite-equivalent if there are finite graded algebra homomorphisms \(f : A \to B\) and \(g : B \to A\). It is easy to that being finite-equivalent is an equivalence relation. For example, \(k\) and \(k[x]/(x^a)\) with \(a \geq 2\) are finite-equivalent. It follows from Proposition 4.1 that

1. if \(A\) and \(B\) are finite equivalent, then \(c(A) = c(B)\).

The concavity \(c(A)\) also has the following properties:

1. If \(A\) is commutative and generated in degree 1, then \(c(A) = 0\).
2. Proposition 4.1(4) If \(x\) is a commutative indeterminate of degree 1, then \(c(A[x]) = c(A)\).

By Remark 5.9, if \(S\) and \(T\) are finite-equivalent noetherian AS regular algebras, then \(S \cong T\). Properties (1) and (2) fails for other invariants that we have defined in this paper. See Remark 5.8 for property (3).

**Remark 5.13.** The relationship between \(\text{Torreg}(A_k)\) and \(\text{CMreg}(A)\) (resp. \(c(A)\) and \(\text{CMreg}(A)\)) is given in Lemma 5.6(1,2). It seems that there is a mysterious (maybe closer) connection between \(\text{Torreg}(A_k)\) and \(c(A)\). By Jørgensen’s Theorem 2.5,
\[
\text{Torreg}(A_k) \geq -\text{CMreg}(A)
\]
and by Theorem 0.12(1) and the definition,
\[
c(A) = c_-(A) - \text{CMreg}(A) \geq -\text{CMreg}(A).
\]
By Lemma 5.6(2), for each \(t \geq 0\), there is an algebra \(D\) such that \(\text{Torreg}(A_k) = c(A) = t\).

Based on this very limited evidence, we ask the following two questions.

**Question 5.14.** Suppose both \(c(A)\) and \(\text{Torreg}(A_k)\) are finite. Then is \(\text{Torreg}(A_k)\) uniformly bounded by a function of \(c(A)\) (or vice versa)?
A special case of the above question is related to a nice result in [AP] (also see [Rörm, Theorem 2.3]). Is there a noncommutative version of this result?

**Question 5.15.** Suppose \(c(A) = 0\) and \(\text{Torreg}(A, k) < \infty\). Is then \(\text{Torreg}(A, k) = 0\) (or equivalently, is \(A\) Koszul)?

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Kirkman: Department of Mathematics & Statistics, P. O. Box 7388, Wake Forest University, Winston-Salem, NC 27109, USA
Email address: kirkman@wfu.edu

Won: Department of Mathematics, The George Washington University, Washington, DC 20052, USA
Email address: robertwon@gwu.edu

Zhang: Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA
Email address: zhang@math.washington.edu