Locally Self-adjoint Extensions of Nonlinear Smooth Operators and Abstract Boundary Conditions

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Abstract. In a real Hilbert spaces $\mathcal{H}$ a smooth operator $F$ is studied, whose derivative $F'(x)$ at each point $x \in \text{Dom}(F)$ is a symmetric operator. In terms of abstract boundary conditions locally self-adjoint extensions of this operator are described. We use some concepts and facts from symplectic differential geometry.

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1. Introduction

The role played by the theory of extensions of linear operators in Hilbert space in the analysis of linear boundary value problems is well known [D-Sh], [Mr]. Self-adjoint boundary value problems are very important from the physical point of view. They describe conservative systems, i.e. systems which are subordinated to some conservation law. For example, the boundary value problems for the Euler equations generated by a wide class of variation problems are self-adjoint in the above sense. The problem of description of linear self-adjoint boundary conditions for linear differential equations (ordinary or partial) is reduced to the problem of description of self-adjoint extensions of the corresponding linear symmetric operators in a Hilbert space. At first the theory of such extensions was constructed by John von Neumann [11] and afterwards it was developed in papers of K. Friedrichs, H. Freudenthal, J. Calkin, M. Krein and other authors ([4], [5], [2], [8]). Some generalizations of Friedrichs extension to the nonlinear case were constructed in [13] and [17].

The self-adjoint extensions mentioned above can be described by means of so called ”abstract boundary conditions”. Conditions of such kind are given by the Calkin Theorem [2]. Let us formulate it.

**Calkin Theorem.** Let $F_0$ be a closed linear symmetric operator in a Hilbert space $\mathcal{H}$ with a dense domain $\text{Dom}(F_0)$ and with finite defects $\{m, m\}$. Denote by $\langle \cdot, \cdot \rangle$ the following bilinear form defined on $\text{Dom}(F_0^*)$:

$$\langle v_1, v_2 \rangle = (F_0^* v_1, v_2) - (v_1, F_0^* v_2).$$

A linear set $D \subseteq \mathcal{H}$ is the domain of a self-adjoint extension $F$ of the operator $F_0$ if and only if there exists such a system of elements $v_1, v_2, \ldots, v_m \in \mathcal{H}$, that

(a) $v_1, v_2, \ldots, v_m$ are linearly independent modulo $\text{Dom}(F_0)$;
(b) $\langle v_i, v_j \rangle = 0 \ \forall i, j \in \{1, 2, \ldots, m\}$;
(c) $D = \{f \in \text{Dom}(F_0^*) : \langle f, v_i \rangle = 0 \ \forall i \in \{1, 2, \ldots, m\}\}$

The complete description of self-adjoint extensions of linear symmetric ordinary differential operators (regular and singular) on the base of the Calkin Theorem was carried out in the book of M. A. Naimark [Nm] (see also [3]).

Consider the orthogonal sum:

$$\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}. \quad (1.1)$$

We denote the inner product in $\mathcal{H}^2$ by $\langle \cdot, \cdot \rangle_2$ and by $\text{pr}_1, \text{pr}_2$ we denote the projections on the first and the second summands in the orthogonal sum $\mathcal{H}^2$, defined by (1.1). In the coordinate form we will write for any $h \in \mathcal{H}^2$: $h = \{\text{pr}_1 h, \text{pr}_2 h\}$. A nonlinear generalization of the theory of self-adjoint extensions of linear symmetric operators was carried out in the papers of the author [17]-[21]. Before to expound this generalization, we introduce a definition [18]:

**Definition 1.1.** Let $F$ be an operator acting in a real Hilbert space $\mathcal{H}$. We call it **graphically smooth**, if its graph $\text{graph}(F)$ is such a $C^1$-submanifold of $\mathcal{H}^2$, that for any $x \in \text{Dom}(F)$ the tangent space $T_\zeta(\text{graph}(F)) \ (\zeta = \ldots$
\{x, F(x)\} is the graph of some operator \(F'(x)\). We call this operator \textit{graphic derivative} of \(F\) at the point \(x\). If for any \(x \in \text{Dom}(F)\) the operator \(F'(x)\) is symmetric (self-adjoint), then we call the operator \(F\) \textit{locally symmetric} (locally self-adjoint).

\textbf{Remark 1.2.} We mean a smooth submanifold \(X\) of some smooth manifold \(Y\) as a range of an injective immersion of a smooth manifold \(\hat{X}\) into \(Y\) ([18], Definition 1.1). It is a submanifold in a generalized sense. When the immersion is a homeomorphism on its range \(X\) in the topology induced on the latter by the \(Y\)-topology, then we have the submanifold \(X\) in the commonly accepted sense. According to our terminology, in this case it is called \textit{the regular submanifold}.

We considered in [18], [19] a nonlinear operator \(F_0\) defined in a real Hilbert space \(H\), whose domain is a translated linear set: \(\text{Dom}(F_0) = D_0 + \{x_0\}\), where \(D_0\) is a dense linear set in \(H\), \(x_0\) is a distinguished point in \(H\). In applications the space \(L_2(\Omega)\) plays usually the role of the space \(H\), where \(\Omega\) is a domain in the space \(\mathbb{R}^m\), as the linear set \(D_0\) participates often the set \(C_0^\infty(\Omega)\) of smooth functions with compact supports in \(\Omega\); \(F_0\) is a nonlinear differential operator. We suppose that at each point \(x \in \text{Dom}(F_0)\) the operator \(F_0\) has the Gateaux derivative \(F_0'(x)\) along \(\text{Dom}(F_0)\) and the latter is a linear symmetric operator in \(H\) with \(\text{Dom}(F_0) = D_0\). We considered the following operator fields:

\[\{\hat{F}'_0(x)\}_{x \in D_0},\ \{F^*_0(x)\}_{x \in D_0},\]

where \(\hat{F}'_0(x)\) is the closure of the operator \(F_0'(x)\). It turns out, that under some conditions these fields are integrable in some sense [18]. Their antiderivatives \(\hat{F}, \tilde{F}\), which satisfy the conditions \(\hat{F}(x_0) = \tilde{F}(x_0) = F_0(x_0)\) are extensions of the operator \(F_0\) and \(F_0 \subseteq \hat{F} \subseteq \tilde{F}\). We have called the operators \(\hat{F}, \tilde{F}\) the \textit{minimal} and \textit{maximal} extensions of the operator \(F_0\) respectively. They may be considered as nonlinear generalizations of the closure \(\hat{F}_0\) for a linear symmetric operator \(F_0\) and of the adjoin operator \(F_0^*\) to the latter. If \(F_0\) is a nonlinear differential operator (ordinary or partial elliptic), then its maximal extension \(\hat{F}\) is the maximal differential operator with the generalized derivatives (by Sobolev) [1], defined by the given differential operation in the space \(L_2(\Omega)\). Notice that under some conditions the operator \(\hat{F}\) is graphically smooth and the operator \(\tilde{F}\) is locally symmetric.

In the papers [20], [21] the following problem was posed and solved: to describe the all locally self-adjoint extensions \(F\) of the operator \(\tilde{F}\), which are restrictions of the operator \(\hat{F} : \hat{F} \subseteq F \subseteq \tilde{F}\). This description was carried out in terms of symplectic differential geometry ([14], Chapt. VII). Let us illustrate this for a simple linear ordinary differential operator \(L_0\) in the space \(\mathcal{H} = L_2[0,1]\), defined on the set

\[\text{Dom}(L_0) = C_0^\infty(0,1) = \{u \in C^\infty(0,1) | \text{supp}(u) \subset (0,1)\}\]
by the operation $lu = -\frac{d^2 u}{dx^2}$. The operator $L_0$ is symmetric. The problem of
a description of the all self-adjoint boundary value problems for the differential
equation $l(u) = f$ is equivalent to a description of the all self-adjoint
extensions of the operator $L_0$. Let $\tilde{L}_0$ be the closure of the operator $L_0$. It is
clear that $L_0 \subseteq L_0^*$ and that every self-adjoint extension $L$ of the operator $L_0$
has the property: $\tilde{L}_0 \subseteq L \subseteq L_0^*$. It is known [10] that the operator $L_0^*$ is the
maximal differential operator defined in the space $\mathcal{H}$ by the operation $l$, i.e.

$$\text{Dom}(L_0^*) = \{ u \in \mathcal{H} \mid u, u' \in \text{Abc}(0, 1), lu \in \mathcal{H} \},$$

(1.2)

where Abc$(0, 1)$ is the set of the all absolutely continuous on $(0, 1)$ functions.
Furthermore, $L_0^*u = lu$ for any $u \in \text{Dom}(L_0^*)$. On the other hand, $\text{Dom}(\tilde{L}_0)$ is
described in the following manner:

$$\text{Dom}(\tilde{L}_0) = \{ u \in \text{Dom}(L_0^*) \mid u_0 = u'_0 = u_1 = u'_1 = 0 \},$$

(1.3)

where

$$u_0 = u(0+), \quad u'_0 = u'(0+), \quad u_1 = u(1-), \quad u'_1 = u'(1-)$$

(1.4)

(these limits exist for any $u \in \text{Dom}(L_0^*)$ [10]). Let us light up the above prob-
lem of description of self-adjoint extensions with the point of view of the sym-
plectic geometry. Consider on $\mathcal{H}^2$ the bilinear form: $j(u, v) = (\text{pr}_1 u, \text{pr}_2 v) -
(\text{pr}_2 u, \text{pr}_1 v)$. It can be considered as a closed differential 2-form on $\mathcal{H}^2$, since
$j = da$, where $\alpha$ is the following 1-form on $\mathcal{H}^2$:

$$\alpha(\zeta)u := -(\text{pr}_2 \zeta, \text{pr}_1 u) \quad (\zeta, u \in \mathcal{H}^2).$$

Since the form $j$ is non-degenerate, it is a symplectic form ([14], Chapt.
VII) on the space $\mathcal{H}^2$. It is known that the graphs $\Gamma_0 = \text{graph}(\tilde{L}_0)$, $\Gamma^* =
\text{graph}(L_0^*)$ are respectively an isotropic and a coisotropic subspace of the symplectic space
$(\mathcal{H}^2, j)$ and a linear operator $L$ acting in $\mathcal{H}$ is self-adjoint
if and if graph($L$) is a Lagrangian subspace of $(\mathcal{H}^2, j)$ [20]. Therefore our
problem is equivalent to the following one: to describe all the Lagrangian
subspaces $\Gamma$ of $(\mathcal{H}^2, j)$, which contain the isotropic subspace $\Gamma_0$ and simul-
taneously are contained in the coisotropic subspace $\Gamma^*$. These subspaces are
parameterized by the Lagrangian subspaces $\tilde{\Gamma}$ of the symplectic space $(\mathcal{C}, \tilde{j})$,
where $\mathcal{C} = \Gamma^* \ominus \Gamma_0$ and $\tilde{j}$ is the lifting of the form $j$ from $\mathcal{H}^2$ into $\mathcal{C}$ by means
of the natural embedding: $i_\mathcal{C} : \mathcal{C} \to \mathcal{H}^2$. In other words, the operator $L$
is a desired self-adjoint extension of $L_0$ if and only if graph($L$) = $\Gamma_0 \ominus \tilde{\Gamma}$ for
some Lagrangian subspace $\tilde{\Gamma}$ of $(\mathcal{C}, \tilde{j})$. Let us show that $\tilde{\Gamma}$ plays a role of a
boundary condition. Using the descriptions (1.2), (1.3), (1.4) of the domains
$\text{Dom}(L_0^*)$, $\text{Dom}(\tilde{L}_0)$ and the Green formula:

$$(lu, v) - (u, lv) = (u'v - uv')_1 - (u'v - uv')_0 \quad \forall u, v \in \text{Dom}(L_0^*)$$

we can show that the symplectic space $(\mathcal{C}, \tilde{j})$ is symplectically isometric to
the symplectic space $(\tilde{\mathcal{C}}, \tilde{j})$, where

$$\tilde{\mathcal{C}} = \{ \eta_u \in \mathbb{R}^4 : \eta_u = (u_0, u'_0, u_1, u'_1), \ u \in \text{Dom}(L_0^*) \}$$
and \( \hat{j}(\eta_u, \eta_v) = (u'v - uv')_1 - (u'v - uv')_0 \). Then the choice of a Lagrangian subspace \( \hat{\Gamma} \) of \( (\hat{C}, \hat{j}) \) is equivalent to the choice of self-adjoint boundary conditions.

In the present paper we carry out further development of the theory worked out in the papers [20], [21] with the purpose to obtain a description of locally self-adjoint extensions of the above mentioned nonlinear locally symmetric operators in terms of some abstract boundary conditions.

We shall use terminology and notations from the book [9] for linear, multilinear and smooth structures and vector bundles.

Remark 1.3. The following question appears: why we restrict ourselves to the case of a real Hilbert space \( \mathcal{H} \)? It turns out that in a complex Hilbert space the class of graphically smooth locally self-adjoint operators is reduced to the rather small class of ones, whose graphs are piecewise-affine submanifolds in \( \mathcal{H}^2 \). This fact is based on the following claim, which follows from Theorem 2.0 of [6]: an analytic function \( \Phi(z) \) defined in a domain \( G \subseteq \mathbb{C} \), whose values are unitary operators, must be constant. Indeed, the graph of a graphically smooth operator \( F \) in a complex Hilbert space \( \mathcal{H} \) should be an analytic submanifold of \( \mathcal{H}^2 \). For any \( \zeta \in \text{graph}(F) \) consider the Cayley transform of the derivative \( F'(pr_1\zeta) \):

\[
V(\zeta) = \left( F'(pr_1\zeta) - iI \right) \left( F'(pr_1\zeta) + iI \right)^{-1},
\]

which is a unitary operator ([10], Chapt. IV, Sect. 14, no 3). It is possible to show that the function \( V : \text{graph}(F) \to L(\mathcal{H}) \) is analytic in the uniform operator topology. Then using the claim, formulated above, it is possible to show that for any point \( \zeta_0 \in \text{graph}(F) \) there is its neighborhood \( U(\zeta_0) \subset \text{graph}(F) \) such that \( F'(pr_1\zeta) \equiv \text{const} \) in \( U(\zeta_0) \).

2. Preliminaries

2.1. Maximal and minimal extensions

As in [18], we consider an operator \( F_0 \), acting in a real Hilbert space \( \mathcal{H} \), whose domain of definition \( \text{Dom}(F_0) \) is a translated lineal:

\[
\text{Dom}(F_0) = D_0 + \{ x_0 \},
\]

where \( D_0 \) is a dense lineal in \( \mathcal{H} \), \( x_0 \) is a distinguished point in \( \mathcal{H} \). We will suppose that the operator \( F_0 \) satisfies the conditions A)-F) of [18]. Briefly speaking it means, that the operator \( F_0 \) has at each point \( x \in \text{Dom}(F_0) \) the Gateaux derivative \( F_0'(x) \) along \( \text{Dom}(F_0) \) and the latter is a linear symmetric operator in \( \mathcal{H} \) with \( \text{Dom}(F_0'(x)) = D_0 \). Denote briefly by \( D(x) \) the closure \( \overline{F_0'(x)} \) of the operator \( F_0'(x) \). Furthermore, we imposed the following conditions on the operator field \( \{ D(x) \}_{x \in \mathcal{H}} \):

(a) this field admits an extension on the whole \( \mathcal{H} \); for the operators of the extended field the previous notation is preserved: \( (D(x))_{x \in \mathcal{H}} \);
(b) there exists a norm \( \| \cdot \|_+ \), defined on \( D_0 \), such that the completion \( \mathcal{H}_+ \) of the lineal \( D_0 \) with respect of this norm is continuously imbedded in \( \mathcal{H} \) and for any \( x \in \mathcal{H} \) the norm
\[
\| v \|_x = (\| \mathbf{D}(x)v \|^2 + \| v \|^2)^{\frac{1}{2}}
\]
is equivalent to the norm \( \| v \|_+ \); the last fact implies the relation:
\[
\forall x \in \mathcal{H} \quad \operatorname{Dom}(\mathbf{D}(x)) = \mathcal{H}_+;
\]
(c) the following relation holds:
\[
\mathbf{D}(\cdot) \in C(\mathcal{H}, L(\mathcal{H}_+; \mathcal{H})).
\]

We have constructed in [18] a so called maximal extension of the operator \( F_0 \), which is the conjugate operator \( F_0^* \) to \( F_0 \) in the linear case. Let us remind briefly such construction. For each fixed \( x \in \operatorname{Dom}(F_0) \) we consider the following operator \( \Phi'(x) \), which is a member of the class \( L(\mathcal{H}; \mathcal{H}_+^*) \):
\[
\forall h \in \mathcal{H}, v \in \mathcal{H}_+ : \quad \Phi'(x)h(v) = (\mathbf{D}(x)v, h). \tag{2.1}
\]
The condition (c) imply, that the operator field \( \Phi'(x)_{x \in \mathcal{H}} \) is continuous with respect to the operator norm. It turns out that this field is integrable ([18], Lemma 2.1). Let \( \Phi \) be its antiderivative, defined by the relation:
\[
\Phi(x) = f + \int_{x_0}^x \Phi'(u)du, \tag{2.2}
\]
where \( f \) is a functional of the form:
\[
f(v) = (v, F_0(x_0)), \tag{2.3}
\]
which belongs to \( \mathcal{H}^* \subset \mathcal{H}_+^* \). It is clear that:
\[
\Phi \in C^1(\mathcal{H}, \mathcal{H}_+^*). \tag{2.4}
\]
We consider the set \( \tilde{D} = \Phi^{-1}(\mathcal{H}^* \cap \text{Im}(\Phi)) \), i.e.
\[
\tilde{D} = \{ x \in \mathcal{H} : \exists y \in \mathcal{H}, \forall v \in \mathcal{H}_+ : \quad (\Phi(x))(v) = (v, y) \}. \tag{2.5}
\]
Since \( \mathcal{H}_+ \) is dense in \( \mathcal{H} \), then in definition (2.5) an unique \( x \in \tilde{D} \) corresponds to each \( y \in \mathcal{H} \). Thus we have an operator \( \tilde{F} \) acting in \( \mathcal{H} \) with \( \operatorname{Dom}(\tilde{F}) = \tilde{D} \), defined by the condition:
\[
\forall v \in \mathcal{H}_+ : \quad (\Phi(x))(v) = (v, \tilde{F}(x)). \tag{2.6}
\]
It turns out that the operator \( \tilde{F} \) is an extension of the operator \( F_0 \) ([18], Lemma 2.2). We have called this extension the maximal extension of the operator \( F_0 \). According to the definition (2.6) of the operator \( \tilde{F} \), its graph
\[
\mathcal{M} = \text{graph}(\tilde{F}) \tag{2.7}
\]
is the set of zeros of an operator \( \Theta \), which is a member of the class \( C^1(\mathcal{H}^2, \mathcal{H}_+^*) \) and is defined by the relation:
\[
(\Theta(\zeta))(v) = (\Phi(x))(v) - (v, y), \tag{2.8}
\]
where \( v \in \mathcal{H}_+ \), \( \zeta = x, y \in \mathcal{H}^2 \). By Theorem 2.1 of [18] the set \( \mathcal{M} \) is a regular \( C^1 \)-submanifold of \( \mathcal{H}^2 \). So the operator \( \tilde{F} \) is closed and graphically smooth.
(18), Definition 1.2) and at each point \(x \in \tilde{D}\) its graphic derivative \(\tilde{F}'(x)\) is of the form:

\[
\tilde{F}'(x) = (D(x))^* \tag{2.9}
\]

(17), Theorem 2.1).

By Theorem 4.1 of [18] the domain of definition \(\tilde{D}\) (2.5) of the operator \(\tilde{F}\) is \(\mathcal{H}_+\)-saturated, i.e.

\[
\tilde{D} = \tilde{D} + \mathcal{H}_+. \tag{2.10}
\]

Then we can define the following extension of the operator \(F_0\):

\[
\tilde{F} = \tilde{F}|_{\{x_0\} + \mathcal{H}_+}. \tag{2.11}
\]

We have called it the minimal extension of the operator \(F_0\). It may be considered as a nonlinear generalization of the closure of the operator \(F_0\). Indeed, by Lemma 4.1 of [18] the operator \(\tilde{F}\) is graphically smooth and at each point \(x \in \text{Dom}(\tilde{F})\) its graphic derivative \(\tilde{F}'(x)\) coincides with \(D(x)\). Let us remind that the last operator is the closure of \(F_0'(x)\). Since for each \(x \in H\) the operator \(D(x)\) is symmetric, then the operator \(\tilde{F}\) is locally symmetric (19), Definition 6.1).

In [18] we have defined on the manifold \(M\) (2.9) the following Abelian group of mappings:

\[
\mathcal{G} = \{G(\cdot, v)\}_{v \in \mathcal{H}_+}, \tag{2.12}
\]

where

\[
\forall \zeta \in M, \forall v \in \mathcal{H}_+ \quad G(\zeta, v) = \{\text{pr}_1 \zeta + v, \tilde{F}(\text{pr}_1 \zeta + v)\}. \tag{2.13}
\]

In view of the property (2.10), the right part of (2.13) is well defined. This group can be considered as a representation of the additive group \(\mathcal{H}_+\), i.e.

\[
\forall \zeta \in M, \forall v_1, v_2 \in \mathcal{H}_+ \quad G(G(\zeta, v_1), v_2) = G(\zeta, v_1 + v_2). \tag{2.14}
\]

It turns out, that the manifold \(M\) is a regular \(C^2\)– submanifold of \(\mathcal{H}^2\) and

\[
G(\cdot, \cdot) \in C^1(M \times \mathcal{H}_+, M), \tag{2.15}
\]

if the operator field \(\{D(x)\}_{x \in \mathcal{H}}\) satisfies the following additional conditions ([18], Lemma 4.2, [19], Theorem 5.1):

(d) if the mapping \(D(\text{pr}_1 \zeta)\) is considered as acting from \(M\) into \(L(\mathcal{H}_+; \mathcal{H})\), then at each \(\zeta \in M\) it has the linear Gateaux derivative \(D(\text{pr}_1 \zeta)\)'s:

\[
(e) \quad \text{for any fixed } \zeta \in M, v \in \mathcal{H}_+ \text{ the operator } (D(\text{pr}_1 (\zeta)v)' , \text{ which is defined on } \text{pr}_1(T(\zeta)(M))\text{, admits an extension on all of } \mathcal{H}, \text{ which we denote by } D'(\text{pr}_1 (\zeta)(v, \cdot));
\]

(f) \(D'(\text{pr}_1 \zeta)(\cdot, \cdot)\) is a locally uniformly continuous mapping from \(M\) into \(L(\mathcal{H}_+, \mathcal{H}; \mathcal{H})\).
2.2. Locally self-adjoint extension and the foliation generated by the group \( \mathcal{G} \)

We have called a graphically smooth operator \( F \) locally self-adjoint, if its graphic derivative \( F'(x) \) at any point \( x \in \text{Dom}(f) \) is a self-adjoint operator ([19], Definition 6.1). In the papers [20], [21] the following problem was raised and solved: to describe all locally self-adjoint extensions \( F \) of the operator \( F_0 \), which satisfy the condition:

\[
\widehat{F}_0 \subset F \subset \tilde{F}.
\] (2.16)

The description mentioned above was carried out in terms of the symplectic differential geometry ([14], Chapt VII). We consider the space \( \mathcal{H}^2 \) as a symplectic manifold \( (\mathcal{H}^2, j) \) with the symplectic form \( j \), described in Introduction, i.e.,

\[
\forall \zeta \in \mathcal{H}^2, \; \forall h_1, h_2 \in T_\zeta(\mathcal{H}^2)(\equiv \mathcal{H}^2) \; j(\zeta)(h_1, h_2) = (Jh_1, h_2)_2,
\]

where the operator \( J \) is defined by the relation:

\[
\forall h \in \mathcal{H}^2 \; Jh = \{-\text{pr}_2 h_2, \text{pr}_1 h\}.
\]

This operator is skew-self-adjoint and unitary:

\[
J^* = -J, \quad J^2 = I.
\]

Notice that we identify the tangent bundle \( T(\mathcal{H}^2) \) with \( \mathcal{H}^2 \times \mathcal{H}^2 \).

A graphically smooth operator \( F \) is locally symmetric (locally self-adjoint), iff \( \text{graph}(F) \) is an isotropic (Lagrangian) submanifold of the symplectic manifold \( (\mathcal{H}^2, j) \) ([20], Proposition 1.1).

The equality (2.9) implies, that \( \mathcal{M} \) is a coisotropic submanifold of \( (\mathcal{H}^2, j) \) [20].

The above raised problem on locally self-adjoint extensions may be reformulated in the following manner: to describe all Lagrangian submanifolds \( L \) of \( (\mathcal{H}^2, j) \), such that

\[
\Gamma_0 \subset L \subset \mathcal{M},
\] (2.17)

where

\[
\Gamma_0 = \text{graph}(\widehat{F}).
\] (2.18)

Such manifolds \( L \) are the graphs of desired locally self-adjoint extensions \( F \) of the operator \( F_0 \), satisfying the condition (2.16). This problem was solved in [21] also in a more restricted sense. We have described so called regularly locally self-adjoint extensions, i.e. such ones, whose graphs are regular submanifolds of \( \mathcal{H}^2 \) ([19], Remark 2.1).

Let us remind the main statements of [20] concerning above mentioned extensions. In their construction an important role play the orbits \( \Gamma_\zeta \) of the group \( \mathcal{G} \) ([2.12], (2.13)), which have the form:

\[
\forall \zeta \in \mathcal{M} \quad \Gamma_\zeta = \text{graph}(\widehat{F}_\zeta),
\]

\[
\widehat{F}_\zeta = \widehat{F}|_{\{\text{pr}_1 \zeta\} + \mathcal{H}_+}.
\] (2.19)

In particular

\[
\widehat{F} = \widehat{F}_\zeta_0,
\] (2.20)
where \( \zeta_0 = \{ x_0, F_0(x_0) \} \)
(see (2.12)). Furthermore:
\[ \Gamma_0 = \Gamma_{\zeta_0} \]
(see (2.18)).

The orbits \( \Gamma_{\zeta} \) have the property:
\[ \forall \zeta_1, \zeta_2 \in M \text{ either } \Gamma_{\zeta_1} \cap \Gamma_{\zeta_2} = \emptyset \text{ or } \Gamma_{\zeta_1} = \Gamma_{\zeta_2}. \]

This means that the family of these orbits
\[ F = \{ \Gamma_{\zeta} \}_{\zeta \in M}. \]  (2.21)
forms a partition of the manifold \( M \). It turns out that under the conditions (a)-(f) this partition forms a \( C^1 \)-foliation on \( M \) in the sense of R.S. Palais, i.e. it is a local \( C^1 \)-bundle ([12]; [20], Definition 2.1). A fibered chart, corresponding to above foliation near any point \( \zeta_* \in M \), may be chosen in the form:
\[ (U, \psi^{-1}, \mathcal{H}_+ \times B), \]  (2.22)
where
\[ U = \psi(V \times W), \]  (2.23)
\( V \) is a neighborhood of 0 in \( \mathcal{H}_+ \),
\[ W = \phi(\hat{W}) \]  (2.24)
and
\[ (\hat{W}, \phi, B), \quad \zeta_* \in W \]  (2.25)
is a chart at the point \( \zeta_* \) for a regular \( C^1 \)-submanifold \( \mathcal{C} \) of \( M \) with a model Banach space \( B \). This submanifold complements each leaf \( \Gamma_{\zeta} \) in \( M \). This means that if \( \mathcal{C} \cap \Gamma_{\zeta} \neq \emptyset \), then
\[ \forall \xi \in \mathcal{C} \cap \Gamma_{\zeta} \text{ } T_{\xi}(\Gamma_{\zeta}) \cap T_{\xi}(\mathcal{C}) = \{ 0 \} \text{ and } T_{\xi}(\Gamma_{\zeta}) + T_{\xi}(\mathcal{C}) = T_{\xi}(M). \]
The mapping \( \psi \), which participates in the chart (2.22) has the form:
\[ \forall \eta \in W, v \in V \text{ } \psi(v, \eta) = G(\phi^{-1}(\eta), v). \]
Hence the leaves \( \Gamma_{\zeta} \) are \( C^1 \)-submanifolds of \( M \) with the model space \( H_+ \). We called linear connected components of sets \( \psi(V \times W) \cup \Gamma_{\zeta} \) sections of the fibered chart (2.22). In our case such sections have the form:
\[ \sigma_{\zeta} = \psi(V \times \eta), \eta \in W. \]

We dealt with so called regular foliation ([12]; [20], Definition 2.3). This means that near each point \( \zeta \in M \) there exists a so called regular fibered chart of the form (2.22), such that each leaf \( \Gamma_{\zeta} \) of the foliation intersects \( U \) in at most one section \( \sigma_{\zeta} \). The following conditions ensure the regularity of the foliation \( F \), defined by (2.21) ([20], Theorem 2.1):

(g) the coercive estimate holds for the maximal extension \( \tilde{F} \) of the operator \( F_0 \):
\[ \forall x \in \text{Dom}(\tilde{F}), v \in \mathcal{H}_+ \text{ } \| \tilde{F}(x + v) - \tilde{F}(x) \|^2 + \| v \|^2 \geq \gamma(\| x \|, \| v \|_+), \]
where $\gamma(y, z)$ is a continuous function in the domain $y \geq 0$, $z \geq 0$ and satisfies the conditions:

1) $\gamma(y, z) > 0$, when $z \neq 0$;
2) $\gamma(y, 0) = 0$;
3) for any fixed $y \geq 0$ the function $\gamma(y, z)$ increases with respect to $z$.

If the foliation $\mathcal{F}$ is regular, then it may be endowed with the structure of a $C^1$-manifold, whose topology coincides with the quotient topology, generated by the partition of $\mathcal{M}$ into leafs of $\mathcal{F}$. This structure is defined in the following manner. Consider near any point $\zeta \in \mathcal{M}$ a regular fibered chart of the form (2.22)–(2.23). Let $\Pi_{\mathcal{F}}$ be the natural projection of $\mathcal{M}$ on the partition $\mathcal{F}$, i.e

$$\Pi_{\mathcal{F}}(\zeta) = \Gamma_\zeta.$$  \hspace{1cm} (2.26)

We set

$$\hat{U} = \Pi_{\mathcal{F}}(U).$$  \hspace{1cm} (2.27)

Then the submanifold

$$\hat{W} = \psi(0 \times W)$$  \hspace{1cm} (2.28)

intersects each leaf $\Gamma \in \hat{U}$ at an unique point

$$\xi = \pi(\Gamma).$$  \hspace{1cm} (2.29)

Consider the mapping

$$\tilde{\phi} = \phi \cdot \pi,$$  \hspace{1cm} (2.30)

where $\phi$ is the mapping, participating in the chart (2.25). It turns out that the charts of the form

$$(\hat{U}, \hat{\phi}, B)$$  \hspace{1cm} (2.31)

(see (2.26), (2.27), (2.29), (2.30)) constitute a $C^1$-atlas on $\mathcal{F}$ [20].

Since in view of (2.9), (2.19),

$$\forall \zeta, \xi \in \mathcal{M} \quad (\hat{F}_\zeta)'(pr_1\xi) = D(pr_1\xi),$$  \hspace{1cm} (2.32)

then the leafs $\Gamma_\zeta$ are isotropic submanifolds of the symplectic manifold $(\mathcal{H}^2, j)$. Furthermore, the pair

$$(\hat{W}, \omega),$$  \hspace{1cm} (2.33)

where

$$\omega = i^*_{\hat{W}}(j),$$  \hspace{1cm} (2.34)

is a symplectic manifold, since at each $\zeta \in \hat{W}$ the leaf $\Gamma_\zeta$ complements the submanifold $\hat{W}$, defined by (2.28), in the coisotropic submanifold $\mathcal{M}$. Hence the pair

$$(\hat{U}, \hat{\omega}),$$  \hspace{1cm} (2.35)

where

$$\hat{\omega} = \pi^*(\omega)$$  \hspace{1cm} (2.36)

is a symplectic manifold (see (2.27), (2.29)). Thus, the foliation $\mathcal{F}$ may be considered as a symplectic manifold in the local sense, described above.

Let us formulate the main result of the paper [20] (Theorem 3.1):
Proposition 2.1. Assume the foliation $\mathcal{F}$ is regular and the conditions (a)-(f) are satisfied. A graphically smooth operator $F$ is regularly locally self-adjoint and and satisfies the condition (2.16), if the following conditions are satisfied:

1) $\text{graph}(F)$ is a $\mathcal{F}$-saturated set, i.e. for any $\zeta \in \text{graph}(F)$ the leaf $\Gamma_\zeta \subset \text{graph}(F)$;

2) for any leaf $\Gamma \in \Pi_\mathcal{F}(\text{graph}(F))$ (see (2.26)) there exists its neighborhood $\hat{U} \subset \mathcal{F}$ of the form (2.27), such that the set

$$\mathcal{L} = \hat{U} \cap \Pi_\mathcal{F}(\text{graph}(F))$$

is a regular Lagrangian submanifold of the symplectic manifold $(\hat{U}, \hat{\omega})$.

If $\text{graph}(F)$ is a closed $C^2$-submanifold in $\mathcal{M}$, then above conditions are necessary.

We will describe a specific class of regularly locally self-adjoint extensions of the operator $F_0$. For this aim we need the following definition:

Definition 2.2. Let $N$ be a subset of $\mathcal{M}$. We call the set

$$\Pi_\mathcal{F}^{-1}(\Pi_\mathcal{F}(N)) = \bigcup_{\zeta \in N} \Gamma_\zeta$$

the $\mathcal{F}$-saturation of the set $N$ and denote it by $N_s$.

We call $N$ $\mathcal{F}$-saturated, if $N_s = N$.

The following statement is valid ([20], Theorem 3.4):

Proposition 2.3. Assume that the foliation $\mathcal{F}$ is regular and the conditions (a)-(f) are satisfied. Let $\hat{W}$ be a regular $C^1$-submanifold of $\mathcal{M}$ such that each leaf $\Gamma$ of the foliation $\mathcal{F}$ intersects the latter in at most one point and complements it in $\mathcal{M}$ at each intersection point (if it exists). Then in order an operator $F$ would be a regularly self-adjoint extension of $F_0$ it is sufficient that

$$\text{graph}(F) = L_s,$$

where $L$ is a regular Lagrangian submanifold of the symplectic manifold $(\hat{W}, \omega)$, defined by (2.33), (2.34).

Definition 2.4. We call the operator $F$, described in Proposition 2.3, the locally self-adjoint extension, defined by the Lagrangian submanifold $L$ of the symplectic manifold $(\hat{W}, \omega)$.

3. Abstract boundary conditions

The problem of a description of above mentioned locally self-adjoint extensions of the operator $F_0$ in terms of so called abstract boundary conditions can be formulated explicitly in the following manner: what properties the mapping $\Theta : \mathcal{M} \to E$ ($E$ is a Banach space) must have in order the set of its zeros

$$\mathcal{L} = \{ \zeta \in \mathcal{M} : \Theta(\zeta) = 0 \}$$

would be the graph of the desired extension?
3.1. Symplectomorphic property of the group $G$

Before to solve above problem, we will establish a symplectomorphic property of the mappings $G(\cdot, v)$ of the group $G$.

**Proposition 3.1.** Under the conditions (a)-(f) each mapping $G(\cdot, v)$ ($v \in H_+$) realizes a $C^1$-symplectomorphism of the coisotropic $C^2$-submanifold of the symlectic manifold $(H^2, J)$ into itself, i.e.

$$\forall v \in H_+ \forall \zeta \in M, \forall h_1, h_2 \in T_\zeta(M)$$

$$ (JG_\zeta(\zeta, v)h_1, G_\zeta(\zeta, v)h_2) = (Jh_1, h_2). \quad (3.1) $$

**Proof.** For any fixed $v \in H_+$ consider the vector field on $H^2$:

$$ \xi_v(\zeta) = \{v, D(\text{pr}_1 \zeta)\}. \quad (3.2) $$

Definition (2.13) of the mappings $G(\cdot, v)$ and the relation$
\forall \zeta \in M \ D(\text{pr}_1 \zeta) \subset \tilde{\mathcal{F}}'(\text{pr}_1 \zeta)$

imply, that the restriction

$$ \hat{\xi}_v = \xi_v |_M \quad (3.3) $$

forms a vector field on $M$ and the restriction of the flow $U(t, \zeta, v)$ of the field $\xi_v$ on $M$ coincides with the flow $\hat{U}(t, \zeta, v)$ of the field $\hat{\xi}_v$, which has the property:

$$ \forall \zeta \in M \quad \hat{U}(t, \zeta, v) = U(t, \zeta, v) = G(\zeta, tv). \quad (3.4) $$

In view of the conditions d), e), f), the field $\hat{\xi}_v$ belongs to the class $C^1$.

For the proof of our proposition it is sufficient to show, that the vector field $\xi_v$ is Hamiltonian, i.e. there exists a scalar function $f \in C^2(H^2, \mathbb{R})$, such that

$$ \forall \zeta \in H^2, h \in H^2 \quad (J\xi_v(\zeta), h) = df(\zeta)h. \quad (3.5) $$

Indeed, assuming that the relation (3.5) holds, consider the Lie derivative $L_{\xi_v}(\hat{j})$ of the 2-form $\hat{j} = i^*_M j$ along the field $\hat{\xi}_v$ defined by (3.3) ([14], Chapt. VII, Sect 2). Putting $\hat{f} = f|_M$, we obtain according to the definition of the Lie derivative:

$$ \forall \zeta \in M \quad \frac{d}{dt}(\hat{U}(t, \zeta, v) \ast \hat{j})|_{t=0} = L_{\xi_v}(\hat{j}) = $$

$$ d(\hat{j} \vee \hat{\xi}_v) + D(\hat{j}) \vee \hat{\xi}_v = d((J\xi_v, \cdot)_2) = d(\hat{f}(\cdot)) = 0, $$

hence the flow $\hat{U}(t, \cdot, v)$ is a symplectomorphism for each fixed $t \in \mathbb{R}$. Then desired property (3.1) follows from (3.4).

We now turn to the proof of the relation (3.5). The latter is equivalent to the fact that the vector field

$$ \eta_v(\zeta) = J\xi_v(\zeta) \quad (3.6) $$

is integrable, i.e. for any closed contour $C \subset H^2$

$$ \int_C (\eta_v(\zeta), d\zeta)_2 = 0. \quad (3.7) $$
Let $\Lambda$ be the following lineal in $\mathcal{H}^2$:  
\[ \Lambda = \text{Dom}(F_0) \oplus \mathcal{H}, \]  
which is dense in $\mathcal{H}^2$:  
\[ \text{cl}(\Lambda) = \mathcal{H}^2. \]  
Suppose $\zeta_0, \zeta_1 \in \Lambda$ and $L_{0,1}$ is the segment of the straight, connecting the points $\zeta_0, \zeta_1$:  
\[ L_{0,1} = \{ \zeta \in \mathcal{H}^2 : \zeta = \zeta(t) = (1 - t)\zeta_0 + t\zeta_1, \ 0 \leq t \leq 1 \}. \]  
Consider the integral:  
\[ I_{0,1} = \int_{L_{0,1}} (\eta_v(\zeta), d\zeta)_2 \]  (see (3.2), (3.6)). We have:  
\[ I_{0,1} = \int_0^1 \left( (D(\text{pr}_1(\zeta(t)))v, \text{pr}_1(\zeta_1 - \zeta_0))dt - (v, \text{pr}_2(\zeta_1 - \zeta_0)) \right) = \\
(v, \int_0^1 F_0'(\text{pr}_1(\zeta_1 - \zeta_0))dt) - (v, \text{pr}_2(\zeta_1 - \zeta_0)) = \Psi(\zeta_1, v) - \Psi(\zeta_0, v), \]  where  
\[ \Psi(\zeta, v) = (v, F_0(\text{pr}_1\zeta) - \text{pr}_2\zeta). \]  The last representation of the integral $I_{0,1}$ implies:  
\[ \int_\Delta (\eta_v(\zeta), d\zeta)_2 = 0 \]  for any triangular contour $\Delta$ with vertices in $\Lambda$ (3.8). From (3.9) and the continuity of the field $\eta_v$ we conclude, that the same equality holds for any triangular contour $\Delta \subset \mathcal{H}^2$. Then by the Gavurin theorem [15] the relation (3.7) is valid, i.e. the field $\eta_v$ is integrable. 

3.2. Defect of the operator $F_0$

Consider a linear closed symmetric operator $A_0$ acting in a real Hilbert space $\mathcal{H}$ with a dense domain $\text{Dom}(A_0)$, the complexification $\mathcal{H}_c = \mathcal{H} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathcal{H}$ and the natural extension $A_{0,c}$ of $A_0$ to $\mathcal{H}_c$, defined by $A_{0,c}(x \otimes z) = (A_0x) \otimes z$ for any $x \in \text{Dom}(A_0)$ and $z \in \mathbb{C}$. It is known that $A_{0,c}$ is a symmetric operator acting in $\mathcal{H}_c$ with the dense domain $\text{Dom}(A_{0,c}) = \text{Dom}((A_0) \otimes \mathbb{C})$. We shall consider deficiency index of $A_{0,c}$ ([10], Chapt IV, Sect.14, no 7). The following claim is valid:

**Proposition 3.2.** (i) Defect numbers of the operator $A_{0,c}$ are equal each to other, i.e., its deficiency index has the form $(m, m)$;

(ii) The equality  
\[ \dim(\text{graph}(A_{0,c}^*)) \oplus \text{graph}(A_0)) = 2m. \]  

(3.10)
is valid.
Proof. (i) Let \((n_i, n_{-i})\) be the deficiency index of \(A_{0,c}\). This means that \(n_i = \dim_c(\mathcal{N}_i)\) and \(n_{-i} = \dim_c(\mathcal{N}_{-i})\), where \(\mathcal{N}_i = \ker(A_{0,c}^*-iI)\) and \(\mathcal{N}_{-i} = \ker(A_{0,c}^*+iI)\). Here \(\dim_c\) denotes the complex dimension of subspaces in \(\mathcal{H}_c\).

It is not difficult to show that \(JA_{0,c}^* = A_{0,c}J\), where \(J\) is the conjugation operator in \(\mathcal{H}_c\). Hence \(\mathcal{N}_i = J\mathcal{N}_{-i}\), i.e., \(n_i = n_{-i}\). Claim (i) is proven.

(ii) By the Neumann formula for \(\text{Dom}(A_{0,c}^*)\) ([10], Chapt IV, Sect.14, Theorem 4): \(\text{Dom}(A_{0,c}^*) = \text{Dom}(A_{0,c}^*) + \mathcal{N}_i + \mathcal{N}_{-i}\), where the summands are linearly independent. Hence, in view of claim (i),

\[
\dim_c(\text{Dom}(A_{0,c}^*)) = 2m (\text{mod}(\text{Dom}(A_{0,c})))
\]

therefore \(\dim(\text{Dom}(A_{0,c}^*)) = 2m (\text{mod}(\text{Dom}(A_{0})))\). The last equality implies easily (3.10). Claim (ii) is proven.

\[\square\]

Definition 3.3. We shall call the number \(m\), defined by (3.10), defect of the operator \(A_0\) and denote it by \(\text{def}(A_0)\).

We shall deal with a nonlinear operator \(F_0\), for which the closure of \(D(x)\) of its graphic derivative \((F_0'(x))\) has at all points of \(\text{Dom}(F_0)\) the same finite defect:

\[
\forall x \in \text{Dom}(F_0) \quad \text{def}(D(x)) = m < \infty.
\]

(3.11)

It turns out, that the relation (3.11) holds for any \(x \in \text{Dom}(F_0)\), if it is true for at least one \(x_0 \in \text{Dom}(F_0)\). More explicitly, the following statement is true:

**Proposition 3.4.** If the conditions (a)-(d) are fulfilled and

\[
\exists x_0 \in \text{Dom}(F_0) \quad \text{def}(D(x)) = m < \infty,
\]

(3.12)

then

\[
\forall x \in \mathcal{H} \quad \text{def}(D(x)) = m.
\]

(3.13)

**Proof.** In view of the condition (c), the subspaces \(\text{graph}(D(x))\) depend continuously on \(x\) with respect to the gap metric ([7], Chapt. IV, Theorem 2.14). The subspaces \(\text{graph}((D(x))^*)\) have the same property, since

\[
\text{graph}((D(x))^*) = J(\text{graph}(D(x)))^\perp.
\]

Therefore the subspaces

\[
\mathcal{C}_x = \text{graph}((D(x))^*) \odot \text{graph}(D(x))
\]

depend continuously on \(x\) with respect to the gap metric too, hence their dimensions are equal to each other. Thus, in view of Proposition 3.2, (3.13) is valid.

\[\square\]

The proved proposition brings us to the following definition:

**Definition 3.5.** If for the operator \(F_0\) the condition (3.12) is fulfilled, then we say, that \(F_0\) has a finite defect \(m\) and write \(m = \text{def}(F_0)\).
3.3. Description of abstract boundary conditions in the linear case

We now turn to description of abstract boundary conditions. Before we will establish corresponding statement for the linear case. Let \( A_0 \) be a closed symmetric operator in \( \mathcal{H} \) with a finite defect \( m \) and \( \Theta \) be a linear operator such that

\[
\Theta \in L(M, R^m),
\]

where

\[
M = \text{graph}(A_0^*).
\]

Since \( M, R^m \) are Hilbert spaces, then

\[
\Theta^* \in L(R^m, M).
\]

The following statement describes self-adjoint extensions of \( A_0 \) in terms of abstract boundary conditions:

**Lemma 3.6.** The subspace

\[
\mathcal{L} = \ker(\Theta)
\]

is the graph of a self-adjoint extension \( A \) of the operator \( A_0 \) iff the following conditions are fulfilled:

\[
\text{Im}(\Theta) = R^m; \quad \text{Im}(\Theta^*) \subset C,
\]

where

\[
C = M \cap J(M),
\]

and

\[
\Theta J \Theta^* = 0.
\]

**Proof.** Assume that

\[
\mathcal{L} = \text{graph}(A),
\]

\[
A_0 \subset A,
\]

\[
A = A^*.
\]

Consider the subspace

\[
\Gamma = \text{graph}(A_0).
\]

Since \( A_0 \) is a symmetric operator, then it is a restriction of the operator \( A^* \). This means that \( \Gamma \) is an isotropic subspace of the symplectic space \( (\mathcal{H}^2, j) \), which is contained in the coisotropic subspace \( M \), defined by (3.15):

\[
\Gamma \subset M.
\]

Furthermore, in view of (3.15), we have:

\[
M = (J\Gamma)^\perp.
\]

According to (3.14), (3.22), we can rewrite (3.23) in the form:

\[
\Gamma \subset \ker(\Theta).
\]

From (3.20), (3.27) we conclude:

\[
C = M \cap \Gamma^\perp.
\]
Then the inclusion (3.28) is equivalent to the following one:

\[
\text{Im}(\Theta^*) = \mathcal{M} \ominus \ker(\Theta) \subset \mathcal{C}.
\]

So (3.23) is equivalent to (3.19)-(3.20). From (3.20) we conclude:

\[
J(\mathcal{C}) = \mathcal{C}.
\] (3.30)

Then the 2-form \( \hat{j}(u, v) = (\hat{J}u, v)_2 \), where

\[
\hat{J} = J|_{\mathcal{C}},
\] (3.31)

is a symplectic form in the space \( \mathcal{C} \). So the pair \( (\mathcal{C}, \hat{j}) \) form a symplectic space. In view of (3.27) and the equality (3.29),

\[
\dim(\mathcal{C}) = 2m.
\] (3.32)

Consider the subspace:

\[
L = \mathcal{L} \cap \mathcal{C}.
\] (3.33)

Then by (3.29) we have:

\[
\mathcal{L} = L \oplus \Gamma.
\] (3.34)

Since \( \Gamma \) is an isotropic subspace of \( (\mathcal{H}^2, j) \), it is the \( J \)-orthogonal complement to the coisotropic subspace \( \mathcal{M} \) (see (3.27) and (3.26)) is valid. Then the representation (3.34) implies that the subspace \( \mathcal{L} \) is Lagrangian in \( (\mathcal{H}^2, j) \), iff the subspace \( L \) (3.33) is Lagrangian in \( (\mathcal{C}, \hat{j}) \).

We have by (3.17), (3.33) that

\[
L = \ker(\hat{\Theta}),
\] (3.35)

where

\[
\hat{\Theta} = \Theta|_{\mathcal{C}}.
\] (3.36)

From (3.28), (3.29) we conclude, that the equality (3.18) is equivalent to the following one:

\[
\text{Im}(\hat{\Theta}) = \mathbb{R}^m.
\] (3.37)

Furthermore, by (3.19), (3.36) \( \hat{\Theta}^* = \Theta^* \). Then taking into account (3.30), (3.31), we obtain that the relation (3.21) is equivalent to

\[
\hat{\Theta}\hat{J}\hat{\Theta}^* = 0.
\]

Since by (3.30) \( \hat{J}^{-1} = -\hat{J} \), the last equality can be rewritten in the form:

\[
\hat{\Theta}\hat{J}^{-1}\hat{\Theta}^* = 0.
\] (3.38)

By (3.32) and Proposition 4.1 the relations (3.36), (3.37), (3.38) are equivalent to the fact, that \( L \) is a Lagrangian subspace of the symplectic space \( (\mathcal{C}, \hat{j}) \), i.e. as it was mentioned above, \( L \) is a Lagrangian subspace of \( (\mathcal{H}^2, j) \). The last fact is equivalent to (3.24). So the relations (3.22)-(3.24) are equivalent to (3.18)-(3.21).
3.4. Main results

We now turn to a description of abstract locally self-adjoint boundary conditions of our nonlinear operator $F_0$, which we suppose to have a finite defect $m$ (Definition 3.5).

**Theorem 3.7.** Assume that the conditions (a)-(f) are fulfilled and the foliation $F$ is regular. A non-empty subset $\mathcal{L} \subset \mathcal{M}$ is the graph of a locally self-adjoint extension of the operator $F_0$, if the following conditions are fulfilled:

1) $\mathcal{L} \cap \text{graph}(F_0) \neq \emptyset$;
2) there exists a neighborhood $U(\mathcal{L})$ of $\mathcal{L}$ in $\mathcal{M}$ and a mapping $\Theta \in C^2(U(\mathcal{L}), \mathbb{R}^m)$, (3.39)

such that $\mathcal{L}$ satisfies the condition

$$\mathcal{L} = \{\zeta \in \mathcal{M} : \Theta(\zeta) = 0\}$$

and at each point $\zeta \in U(\mathcal{L})$ the following conditions are fulfilled:

$$\text{Im}(\Theta'(\zeta)) = \mathbb{R}^m,$$

$$\text{Im}((\Theta'(\zeta))^*) \subset \mathcal{C}_\zeta,$$

where

$$\mathcal{C}_\zeta = T_\zeta(\mathcal{M}) \cap J(T_\zeta(\mathcal{M}))$$

and

$$\Theta'(\zeta)J(\Theta'(\zeta))^* = 0.$$ (3.44)

**Proof.** By conditions (3.39)-(3.41) the set $\mathcal{L}$ is a $C^2$-submanifold of $\mathcal{M}$ and

$$\forall \zeta \in \mathcal{L} \quad T_\zeta(\mathcal{L}) = \ker(\Theta'(\zeta)).$$

This fact, conditions (3.41)-(3.44) and Lemma 3.6 imply that for any $\zeta \in \mathcal{L}$ the subspace $T_\zeta(\mathcal{L})$ is the graph of a self-adjoint operator. This means that $\mathcal{L}$ is the graph of a locally self-adjoint operator $F$. By the condition 1) there exists a point $y_0 \in \text{pr}_1(\mathcal{L}) \cap \text{Dom}(F_0)$. On the other hand by the Lemma 3.1 of [20] the submanifold $\mathcal{L}$ is $F$-saturated, hence

$$\Gamma_{\xi_0} = \text{graph}(\hat{F}) \subset \mathcal{L}, \quad \xi_0 = \{y_0, F_0(y_0)\}.$$  

Recall that $\hat{F}$ is the minimal extension of the operator $F_0$ (definition (2.11)). Thus, we have:

$$F_0 \subset \hat{F} \subset F,$$

i.e. the operator $F$ is an extension of $F_0$. □

A converse statement to Theorem 3.7 is valid. We use there the notion of a locally self-adjoint extension, defined by a Lagrangian submanifold of the symplectic manifold $(\hat{W}, \omega)$ (see (2.33), (2.34), Definition 2.3 and Proposition 2.3).
Theorem 3.8. Assume that the conditions (a)-(f) are fulfilled, the foliation \( F \) is regular and a locally self-adjoint extension \( F \) of the operator \( F_0 \) is defined by a Lagrangian \( C^3 \)-submanifold \( \mathcal{L}_1 \) of a symplectic \( C^3 \)-submanifold \( (\widehat{W}, \omega) \) of \( (\mathcal{H}^2, j) \) such that \( \widehat{W} \subset \mathcal{M} \). Furthermore, assume that \( \widehat{W} \) is \( C^3 \)-diffeomorphic to an open domain \( \mathcal{C}_1 \) of the space \( \mathbb{R}^{2m} \) and \( \mathcal{L}_1 \) is \( C^3 \)-diffeomorphic to a lineal. Then there exists a neighborhood \( U(\mathcal{L}) \) of the submanifold \( \mathcal{L} = \text{graph}(F) \) in \( \mathcal{M} \) and a smooth mapping \( \Theta \) from \( U(\mathcal{L}) \) into \( \mathbb{R}^m \), which satisfy the conditions (3.40)-(3.44) of Theorem 3.7.

Proof. By Theorem A.10, there exists a neighborhood \( U(\mathcal{L}_1) \) of the manifold \( \mathcal{L}_1 \) in \( \widehat{W} \) and a mapping

\[
\hat{\Theta} \in C^1(U(\mathcal{L}_1), \mathbb{R}^m), \tag{3.45}
\]

\[
\mathcal{L}_1 = \left\{ \zeta \in \widehat{W} : \hat{\Theta}(\zeta) = 0 \right\} \tag{3.46}
\]

and for any point \( \zeta \in U(\mathcal{L}) \) the derivative \( \hat{\Theta}'(\zeta) \) satisfies the conditions:

\[
\text{Im}(\hat{\Theta}'(\zeta)) = \mathbb{R}^m, \tag{3.47}
\]

\[
\hat{\Theta}'(\zeta)(\Omega(\zeta))^{-1}(\hat{\Theta}'(\zeta))^* = 0, \tag{3.48}
\]

where \( \Omega(\zeta) \) is the skew-self-adjoint operator in \( T_{\zeta}(\widehat{W}) \), defined by the 2-form \( \omega(2.32) \), i.e.

\[
\Omega(\zeta) = P_{\zeta}JP_{\zeta}
\]

and \( P_{\zeta} \) is the orthogonal projection on \( T_{\zeta}(\widehat{W}) \). By the property of the manifold \( \widehat{W} \) (Proposition 2.3) for any \( \zeta \in \widehat{W}_c \) (Definition 2.2) there exists an unique \( \hat{\zeta} \in \widehat{W} \), such that

\[
\exists v \in \mathcal{H}_+ \quad \zeta = G(\hat{\zeta}, v). \tag{3.49}
\]

Then we can extend the mapping \( \hat{\Theta} \) on the neighborhood of \( \mathcal{L} = \text{graph}(F) \)

\[
U(\mathcal{L}) = (U(\mathcal{L}_1))s \tag{3.50}
\]

in the following manner:

\[
\Theta(\zeta) = \hat{\Theta}(\hat{\zeta}). \tag{3.51}
\]

In view of (3.45) and of \( C^1 \)-smoothness of the mappings \( G(\cdot, v) \) (19, Theorem 5.1), the mapping \( \Theta \) belongs to the class \( C^1(U(\mathcal{L}), \mathbb{R}_m) \). Let us take a point \( \hat{\zeta} \in U(\mathcal{L}_1) \) and consider the following subspace of the tangent space \( T_{\zeta}(\widehat{W}) \):

\[
L_1(\hat{\zeta}) = \text{ker}(\hat{\Theta}'(\hat{\zeta})).
\]

The relations (3.47), (3.48) and Proposition 4.1 imply that \( L_1(\hat{\zeta}) \) is a Lagrangian subspace of the symplectic space \( (T_{\zeta}(\widehat{W}), \omega(\hat{\zeta})) \). Consider the foliation \( \mathcal{F} \) defined by \( \hat{\Theta}(\hat{\zeta}) \), and the tangent spaces \( T_{\zeta}(\Gamma_{\zeta}) \) to its leaves \( \Gamma_{\zeta} \). Let us remind that for each \( \zeta \in \mathcal{M} \) the isotropic subspace \( T_{\zeta}(\mathcal{L}) \) of \( (\mathcal{H}^2, j) \) is contained in the coisotropic subspace \( T_{\zeta}(\mathcal{M}) \) and it is the \( j \)-orthogonal complement to \( T_{\zeta}(\mathcal{M}) \) in \( \mathcal{H}^2 \). Furthermore, we have by (2.19):

\[
T_{\zeta}(\Gamma_{\zeta}) = \text{graph}(D(\text{pr}_1(\zeta))). \tag{3.52}
\]
Then for any \( \hat{\zeta} \in U(L_1) \) the subspace

\[
L(\hat{\zeta}) = L_1(\hat{\zeta}) + T_{\hat{\zeta}}(\Gamma_{\hat{\zeta}})
\]  

(3.53)
is Lagrangian in \((H^2, j)\). By the definition of the mapping \( \Theta \) (see (3.49), (3.51)), it is constant along the leafs \( \Gamma_{\zeta} \) of the foliation \( F \). Since \( L = (L_1)_{s} \), then (3.40) is valid. Furthermore we have:

\[
\forall \zeta \in U(L) \quad T_{\zeta}(\Gamma_{\zeta}) \subset \ker(\Theta'(\zeta))
\]

(3.54)
(see (3.50)), hence we conclude from (3.53), that for any \( \hat{\zeta} \in U(L_1) \)

\[
\ker(\Theta'(\hat{\zeta})) = L(\hat{\zeta}).
\]

Consequently for any point \( \zeta \in U(L) \), represented by the formula (3.49), we have:

\[
\ker(\Theta'(\zeta)) = L(\zeta),
\]

(3.55)
where

\[
L(\zeta) = G_{\zeta}(L(\hat{\zeta}), v).
\]
The last equality and symplectomorphic property of the mappings \( G(\cdot, v) \) (Proposition 3.1) imply, that for each \( \zeta \in U(L) \) the subspace \( L(\zeta) \) is Lagrangian in \((H^2, j)\), i.e. it is the graph of a self-adjoint operator \( F'(\text{pr}_1(\zeta)) \). From the equalities (3.52), (3.55) and the inclusion (3.54) we conclude, that \( F'(\text{pr}_1(\zeta)) \) is an extension of of the symmetric operator \( D(\text{pr}_1(\zeta)) \), which have the defect \( m \) (Proposition 3.4 and Definition 3.5). The last fact, the equality (3.55) and Lemma 3.6 imply the conditions (3.41)-(3.44) of Theorem 3.7. \( \square \)

Appendix A. Some statements from symplectic differential geometry.

We use in our paper a characterization of a Lagrangian submanifold \( L \) of a symplectic manifold \((\mathcal{C}, \omega) \) \((\text{dim}(\mathcal{C}) = 2m)\) in terms of a smooth mapping \( \Theta : \mathcal{C} \to \mathbb{R}^m \), whose set of zeros coincides with \( L \). In order to establish this characterization, we need some statements on extensions of symplectomorphisms.

A.1. Linear case

A.1°. In this subsection we consider the problem mentioned above for the linear case. Let \( H^{2m} \) be \( 2m \)-dimensional real Hilbert space with an inner product \((\cdot, \cdot) \) \((m < \infty)\) and \( \Omega \) be a linear invertible skew-self-adjoint operator in \( H : \quad \Omega^* = -\Omega \). We set for any \( u,v \in H^{2m} \):

\[
\omega(u, v) = (\Omega u, v).
\]

Then the pair \((H, \omega)\) form a symplectic space.

We will use some well known elementary notions and facts from the symplectic linear algebra: isotropic, coisotropic and Lagrangian subspaces and existence of them; symplectic basis and existence of it and other ones ([}
By \( \mathcal{L}^\perp \) we denote the \( \Omega \)-orthogonal complement to a subspace \( \mathcal{L} \in \mathcal{H}^{2m} \) in \( \mathcal{H} \),
\[
\mathcal{L}^\perp = (\Omega(\mathcal{L}))^\perp.
\]
Assume that a subspace \( \mathcal{L} \subseteq \mathcal{H} \) is defined in the following manner:
\[
\mathcal{L} = \ker(\Theta), \quad (A.1)
\]
where
\[
\Theta \in L(\mathcal{H}^{2m}, \mathbb{R}^m). \quad (A.2)
\]
Our purpose to obtain a criterion ensuring that the subspace \( \mathcal{L} \) is Lagrangian.

Before we state some auxiliary statements.

**Lemma A.1.** The following relation is valid:
\[
\mathcal{L}^\perp = \text{Im}(\Omega^{-1}\Theta^*). \quad (A.3)
\]

*Proof.* We have:
\[
\Omega(\mathcal{L}) = \ker(\Theta\Omega^{-1}).
\]
Then
\[
\mathcal{L}^\perp = \text{Im}((\Theta\Omega^{-1})^*) = \text{Im}(\Omega^{-1}\Theta^*). \quad \square
\]

The following statement is the straightforward consequence of the previous lemma:

**Lemma A.2.** The subspace \( \mathcal{L} \) (see (A.1), (A.2)) is coisotropic if and only if the following identity holds:
\[
\Theta\Omega^{-1}\Theta^* = 0. \quad (A.4)
\]

We turn now to a consideration of Lagrangian subspaces. The following statement is valid:

**Lemma A.3.** A subspace \( \mathcal{L} \subset \mathcal{H}^{2m} \) is Lagrangian in \( (\mathcal{H}^{2m}, \omega) \), if and only if it is coisotropic (isotropic) there and
\[
\dim(\mathcal{L}) = m. \quad (A.5)
\]

*Proof.* Let \( P \) be the orthogonal projector on \( \mathcal{L} \). Setting in (A.3) \( \Theta = I - P \), we obtain:
\[
\mathcal{L}^\perp = \text{Im}(\Omega^{-1}(I - P)).
\]
Then \( \mathcal{L} \) is coisotropic (isotropic) if and only if the inclusion is valid:
\[
\text{Im}(\Omega^{-1}(I - P)) \subseteq \text{Im}(P) \quad (\text{Im}(P) \subseteq \text{Im}(\Omega^{-1}(I - P))). \quad (A.6)
\]
On the other hand, (A.5) is equivalent to:
\[
\dim(\text{Im}(\Omega^{-1}(I - P))) = \dim(\text{Im}(P)) = m. \quad (A.7)
\]
The inclusion (A.6) together with the equality (A.7) are equivalent to the equality
\[
\text{Im}(\Omega^{-1}(I - P)) = \text{Im}(P),
\]
which means that \( \mathcal{L}^\perp = \mathcal{L} \), i.e. the subspace \( \mathcal{L} \) is Lagrangian. \quad \square
As a consequence of Lemma A.2 and Lemma A.3 we obtain the main statement of this subsection:

**Proposition A.4.** The subspace $L \subseteq H^{2m}$ (see (A.1), (A.2)) is Lagrangian in $(H^{2m}, \omega)$ if and only if the operator $\Theta$ satisfies the condition (A.4) and the relation:

$$\text{Im}(\Theta) = \mathbb{R}^m$$  \hspace{1cm} (A.8)

**Proof.** The condition (A.8) is equivalent to (A.5). Then by Lemmas A.2, A.3 we obtain our statement. \hfill $\square$

### A.2. Symplectic connection

We need a so called symplectic connection, which permit to carry out ”parallel translations” of vectors on symplectic manifolds in similar manner, as on Riemannian manifolds. Notice that symplectic connections were studied in [16].

It will be convenient for us to consider a representation of a symplectic $C^r$-smooth manifold ($r \geq 2$) with the help of charts. Let $\mathcal{D}$ be an open subset of $\mathbb{R}^m$, which can be considered to be a symplectic manifold, if a closed non-degenerate differential 2-form is defined on it:

$$\forall x \in \mathcal{D}, \forall u, v \in T(D) \equiv \mathbb{R}^m \quad \omega(x)(u, v) = (\Omega(x)u, v),$$  \hspace{1cm} (A.9)

where

$$\Omega(x) \in C^{r-1}(\mathcal{D}, L(\mathbb{R}^{2m})).$$

and for each $x \in \mathcal{D}$ $\Omega(x)$ is an invertible skew-self-adjoint operator. For any fixed $x \in \mathcal{D}$ we consider the operator $\Omega(x)$, which is conjugate to the identity operator $I$ with respect to the form $\omega(x)$. In other words $\Omega(x)$ acts from $\mathbb{R}^{2m}$ into $(\mathbb{R}^{2m})^*$ and

$$\forall u, v \in \mathbb{R}^{2m} \quad (\tilde{\Omega}(x)u, v) = \omega(x)(v, u).$$  \hspace{1cm} (A.10)

It is easily to see that each $\tilde{\Omega}(x)$ is invertible and

$$\tilde{\Omega}(x) \in C^{r-1}(\mathcal{D}, L(\mathbb{R}^{2m}, (\mathbb{R}^{2m})^*)).$$

If we identify $(\mathbb{R}^{2m})^*$ with $\mathbb{R}^{2m}$, then $\Omega(x) = (\Omega(x))^* = -\Omega(x)$, Consider the following $C^{r-2}$-mapping from $\mathcal{D}$ into the space $L(\mathbb{R}^{2m}, \mathbb{R}^{2m})$ of antisymmetric 2-forms, taking their values in $\mathbb{R}^{2m}$:

$$\forall x \in \mathcal{D}, \forall h, \xi \in \mathbb{R}^{2m} \quad \Gamma(x)(h, \xi) = -((\tilde{\Omega}(x))^{-1}(dx(\Omega(x)))(\xi, h))).$$  \hspace{1cm} (A.11)

It turns out that this mapping has the property of a connection, defined by the 2-form $\omega$, if the last one is considered to generate a pseudo-Riemannian metric on $\mathcal{D}$. Consider on $\mathcal{D}$ a $C^r$-smooth curve $\gamma: x = x(t), \ t \in [a, b]$ and the following linear differential equation of a ”parallel translation” of vectors along $\gamma$:

$$\frac{d\xi}{dt} = \Gamma(x(t))(\frac{dx}{dt}, \xi),$$  \hspace{1cm} (A.12)

in which $\xi(t) \in T_x(t)(\mathcal{D}) \equiv \mathbb{R}^{2m}$ for each $t \in [a, b]$.
We have:

**Proof.** Let us take two arbitrary vectors \( v_1, v_2 \in T_x(t_0)(\mathcal{D}) \) and the corresponding solutions of (A.12):

\[
\xi_k(t) = U_\gamma(t, t_0)v_k \quad (k = 1, 2).
\]

In order to prove our statement, we ought to show that

\[
\frac{d}{dt}[\omega(x(t))(\xi_1(t), \xi_2(t))] \equiv 0. \tag{A.13}
\]

We have:

\[
\frac{d}{dt}[\omega(x(t))(\xi_1(t), \xi_2(t))] = \omega(x(t))\left(\frac{d\xi_1(t)}{dt}, \xi_2(t)\right) + \omega(x(t))(\xi_1(t), \frac{d\xi_2(t)}{dt}) + \omega_x(x(t))(\xi_1(t), \xi_2(t)) \cdot \frac{dx(t)}{dt}. \tag{A.14}
\]

Taking into account (A.10), (A.11) and definition (A.9) of the operator \( \tilde{\Omega} \), we have:

\[
\omega(x(t))(\frac{d\xi_1(t)}{dt}, \xi_2(t)) = -\omega_x(x(t))(\xi_1(t), dx(t)dt) \cdot \xi_2(t), \tag{A.15}
\]

\[
\omega(x(t))(\xi_1(t), \frac{d\xi_2(t)}{dt}) = \omega_x(x(t))(\xi_2(t), \frac{dx(t)}{dt}) \cdot \xi_1(t). \tag{A.16}
\]

From the identities (A.14) - (A.16) we obtain, taking into account that \( \omega \) is a closed 2-form:

\[
\frac{d}{dt}[\omega(x(t))(\xi(t), \xi(t))] = \omega_x(x(t))(\xi_2(t), \frac{dx(t)}{dt}) \cdot \xi_1(t) - \omega_x(x(t))(\xi_1(t), \frac{dx(t)}{dt}) \cdot \xi_2(t) + \omega_x(x(t))(\xi_1(t), \xi_2(t)) \cdot \frac{dx(t)}{dt} =
\]

\[
d\omega(x(t))(\xi_1(t), \xi_2(t), \frac{dx(t)}{dt}) = 0,
\]

i.e. the identity (A.13) is valid. \( \Box \)

### A.3. Extension of a diffeomorphism to a tubular neighborhood

We need some statements on an extension of a diffeomorphism between two manifolds to a diffeomorphism between their tubular neighborhoods. Before we state a statement on a trivialization of some class of vector bundles.

**Lemma A.6.** Let \( \{N_n\}_{n \in \mathbb{E}} \) be a family of subspaces of a Hilbert space \( \mathcal{H} \) (\( E \) is a Banach space), such that the orthogonal projectors \( P(l) \) on them have the property:

\[
P(\cdot) \in C^r(E, L(\mathcal{H})).
\]

Then there exists a family \( \{U(l)\}_{l \in \mathbb{E}} \) of topological linear isomorphisms from \( N_0 \) onto \( N_1 \), such that

\[
U(\cdot) \in C^{r-1}(E, L(N_0, \mathcal{H})).
\]
In other words, the mapping
\[ U(l, v) = \{ l, U(l)v \} \quad (l \in E, v \in N_0) \] (A.17)
realizes a $C^{r-1}$-smooth BP-morphism between the trivial vector bundle $E \times N_0 \to E$ and the vector bundle, defined by the family \{ $N_l$ \}_{l \in E}.

Proof. We set $Q(l) = I - P(l)$. For any fixed $l \in E$ consider the following evolution equation in the Hilbert space $H$:
\[
\frac{d\xi(t)}{dt} = \left[ \frac{dP(tl)}{dt} P(tl) + \frac{dQ(tl)}{dt} Q(tl) \right] \xi.
\]
It is known that the evolution operator $V_l(t, t_0)$ of this equation is unitary and it "turns" the subspaces $N_l$ and $N^\perp_l$, i.e.,
\[ P(tl)V_l(t, t_0) = V_l(t, t_0)P(t_0l), \quad Q(tl)V_l(t, t_0) = V_l(t, t_0)Q(t_0l) \]
(see [ ], Chapt. ). It is easily to see that the mapping
\[ U(l) = V_l(1, 0)|_{N_0} \]
has desired properties. □

The following statement on extension of diffeomorphisms is valid:

**Proposition A.7.** Let $\mathcal{L}$ be a $C^r$-smooth ($r \geq 3$) regular submanifold of a Hilbert space $\mathcal{H}$. Assume that a mapping $U_{\mathcal{L}}$ realizes a $C^r$-diffeomorphism between a subspace $\hat{\mathcal{L}}$ of $\mathcal{H}$ and the submanifold $\mathcal{L}$. Then it can be extended to a $C^{r-2}$-diffeomorphism $U_T$ between a tubular neighborhood $T(\mathcal{L})$ of $\mathcal{L}$ and a tubular neighborhood $T(\hat{\mathcal{L}})$ of $\hat{\mathcal{L}}$.

Proof. Consider the normal bundle \{ $N_x$ \}_{x \in \mathcal{L}} on the submanifold $\mathcal{L}$:
\[ \forall x \in \mathcal{L} : \quad N_x = (T_x(\mathcal{L}))^\perp. \]
Furthermore, consider on $\mathcal{H}$ the geodesic flow, generated by $\mathcal{H}$-metric and the exponential mapping $Exp$ defined by this flow [9]. Let $T(\mathcal{L})$ be the tubular neighborhood defined by the normal bundle $N(\mathcal{L})$ and the mapping $Exp$, i.e.,
\[ T(\mathcal{L}) = f_N(Z_N), \]
where
\[ f_N = Exp|_{N(\mathcal{L})}, \]
$Z_N$ is a neighborhood of the zero section $\zeta_\mathcal{L}$ of $N(\mathcal{L})$ and $f_N$ realizes a $C^{r-2}$-diffeomorphism between $Z_N$ and $T(\mathcal{L})$. Let us transfer the bundle $N(\mathcal{L})$ from the base $\mathcal{L}$ to the base $\hat{\mathcal{L}}$ with the help of mapping $U_{\mathcal{L}}$, i.e. consider the following vector bundle:
\[ \hat{N}(\hat{\mathcal{L}}) = (U_{\mathcal{L}}^{-1})^*(N(\mathcal{L}))) \] (A.18)
Let us notice that $U_{\mathcal{L}}^*$ is a BP-isomorphism and $(U_{\mathcal{L}}^{-1})^*^{-1} = U_{\mathcal{L}}^*$. By Lemma A.6 there exists a $C^{r-1}$-smooth BP-isomorphism $\mathcal{U}$ between the trivial vector

\[ \text{1} \text{Here and in the sequel } f^* \text{ denotes the lifting of a vector bundle } E \to X \text{ from the base } X \text{ to the base } \hat{X} \text{ with the help of a mapping } f : \ X \to X \ [9]. \text{ We hope that this notation will not cause a confusion with the notation of adjoint linear homomorphisms.} \]
bundle \( \hat{L} \times N_0 \to \hat{L} \) and the vector bundle \( N(\mathcal{L}) \). Since \( \text{codim}(\hat{L}) = \text{codim}(\mathcal{L}) \), then there exists a topological linear isomorphism \( S \) between \( \hat{N} = (\hat{L})^\perp \) and \( N_0 \). Let \( P_{\hat{N}} \) be the orthogonal projection on \( \hat{N} \). Then the mapping
\[
\hat{U}(x) = \{(I - P_{\hat{N}})x, S(P_{\hat{N}}(x))\}
\]
realizes a topological linear isomorphism between \( \mathcal{H} \) and \( \hat{L} \times N_0 \).

Consider the mapping \( \Phi = \hat{U}^{-1}U^{-1}U_\mathcal{L}^*f_N^{-1} \), which realizes a \( C^{r-2} \)-diffeomorphism between \( \mathcal{T}(\mathcal{L}) \) and a tubular neighborhood \( \mathcal{T}(\hat{L}) \) of \( \hat{L} \). Then the mapping \( U_\mathcal{T} = \Phi^{-1} \) is the desired extension of \( U_\mathcal{L} \).

We need also the following statement on a diffeomorphism of neighborhoods of a subspace \( \mathcal{L} \) of a Hilbert space \( \mathcal{H} \), generated by a BP-automorphism of the restriction of the tangent bundle \( T(\mathcal{H}) \) on \( \mathcal{L} \). Let \( \mathcal{L}' \) be a complement of \( \mathcal{L} \) in \( \mathcal{H} \), \( P_\mathcal{L} \) be the projection on \( \mathcal{L} \) along to \( \mathcal{L}' \), \( Q_\mathcal{L} = I - P_{\mathcal{L}'} \).

**Proposition A.8.** Let \( W \) be a \( C^r \)-smooth BP-automorphism of the vector bundle \( T(\mathcal{H})|_\mathcal{L} \), such that
\[
W|_{T(\mathcal{L})} = \text{id}_{T(\mathcal{L})}.
\]
Then there exists a \( C^r \)-diffeomorphism \( V \) of a neighborhood \( \mathcal{T}_1(\mathcal{L}) \) of \( \mathcal{L} \) onto a neighborhood \( \mathcal{T}_2(\mathcal{L}) \) of \( \mathcal{L} \), such that
\[
V|_\mathcal{L} = \text{id}_\mathcal{L} \quad \text{(A.19)}
\]
and
\[
T(V)|_{T(\mathcal{H})|_\mathcal{L}} = W. \quad \text{(A.20)}
\]

**Proof.** For any \( x \in \mathcal{L} \) the BP-automorphism \( W \) has the following form on the fiber \( T_x(\mathcal{H}) \) (\( \equiv \mathcal{H} \)):
\[
W(x) = \text{Id}_\mathcal{L} + U(x),
\]
where
\[
U(x) = W(x)|_{\mathcal{L}'}.
\]
Furthermore, we have:
\[
W(x) \in \text{Laut}(\mathcal{H}). \quad \text{(A.21)}
\]
Consider the following mapping of \( \mathcal{H} \) into itself:
\[
\tilde{V}(x) = P_{\mathcal{L}}x + U(P_{\mathcal{L}}x)Q_{\mathcal{L}}x.
\]
It is obviously that \( \text{(A.20)} \) is valid. Let us calculate the derivative of \( V(x) \) along any direction \( \xi \in \mathcal{H} \):
\[
V'(x)\xi = P_{\mathcal{L}}\xi + U_x(P_{\mathcal{L}}x)Q_{\mathcal{L}}x \cdot P_{\mathcal{L}}\xi + U(P_{\mathcal{L}}x)Q_{\mathcal{L}}\xi.
\]
In particular, for \( x \in \mathcal{L} \):
\[
V'(x)\xi = P_{\mathcal{L}}\xi + U(P_{\mathcal{L}}x)Q_{\mathcal{L}}\xi = W(x)\xi,
\]
hence \( \text{(A.21)} \) holds. Furthermore, in view of \( \text{(A.22)} \), the mapping \( \tilde{V} \) is a local \( C^r \)-diffeomorphism at each point \( x \in \mathcal{L} \). There is only to construct a tubular neighborhood \( \mathcal{T}_1(\mathcal{L}) \) of \( \mathcal{L} \), such that the restriction
\[
V = V|_{\mathcal{T}_1(\mathcal{L})}
\]
realizes a $C^r$-diffeomorphism on its range $\mathcal{T}_2(\mathcal{L})$. □

A.4. Main statement

In this subsection we shall state the main statement of this section, which give a description of a Lagrangian manifold in terms of a mapping, whose set of zeros coincides with it. We establish before a statement on an extension of a symplectomorphism from a Lagrangian manifold to its tubular neighborhood. This statement can be considered to be a global variant of Givental Theorem for the case of Lagrangian manifolds ([14], Chapt VII, Sect. 3).

**Proposition A.9.** Let $(\mathcal{C}, \omega)$ be a symplectic $C^r$-manifold ($r \geq 3$) of a finite dimension $2m$ and $\mathcal{L}$ be its Lagrangian submanifold. Assume that $\mathcal{C}$ is $C^r$-diffeomorphic to an open domain $\mathcal{C}$ of the space $\mathbb{R}^{2m}$ with the canonical symplectic form $j$ and $\mathcal{L}$ is $C^r$-diffeomorphic to a Lagrangian subspace $\hat{\mathcal{L}}$ of $(\mathbb{R}^{2m}, j)$. Then there exists a $C^{r-2}$-diffeomorphism $\Psi$ from a tubular neighborhood $\mathcal{T}(\hat{\mathcal{L}})$ of $\hat{\mathcal{L}}$ onto a tubular neighborhood $\mathcal{T}(\mathcal{L})$ of $\mathcal{L}$, such that:

$$\Psi(\hat{\mathcal{L}}) = \mathcal{L} \quad \text{(A.22)}$$

and

$$\Psi^* (\omega|_{\mathcal{T}(\mathcal{L})}) = j|_{\mathcal{T}(\hat{\mathcal{L}})} \quad \text{(A.23)}$$

**Proof.** Assume that we have constructed a $C^{r-1}$-diffeomorphism $S$ from a tubular neighborhood $\mathcal{T}_0(\hat{\mathcal{L}})$ of $\hat{\mathcal{L}}$ onto a tubular neighborhood $\mathcal{T}_0(\mathcal{L})$ of $\mathcal{L}$, such that

$$S(\hat{\mathcal{L}}) = \mathcal{L} \quad \text{(A.24)}$$

and

$$S^* (\omega|_{\mathcal{T}(\mathcal{L})}|_{\hat{\mathcal{L}}}) = j|_{\mathcal{T}(\hat{\mathcal{L}})}|_{\hat{\mathcal{L}}} \quad \text{(A.25)}$$

Then setting

$$\tilde{\omega} = S^* (\omega|_{\mathcal{T}(\mathcal{L})}),$$

we have by (A.25) that

$$\tilde{\omega}|_{\mathcal{T}(\mathcal{L})}|_{\hat{\mathcal{L}}} = j|_{\mathcal{T}(\hat{\mathcal{L}})}|_{\hat{\mathcal{L}}}.$$

Then by the Veinstein theorem ([14], Chapt. ) there exist tubular neighborhoods $\mathcal{T}(\hat{\mathcal{L}}) \subseteq \mathcal{T}_0(\mathcal{L})$, $\mathcal{T}_1(\hat{\mathcal{L}}) \subseteq \mathcal{T}_0(\hat{\mathcal{L}})$ of $\hat{\mathcal{L}}$ and a $C^{r-1}$-diffeomorphism $\Phi$ from $\mathcal{T}(\hat{\mathcal{L}})$ onto $\mathcal{T}_1(\mathcal{L})$, such that

$$\Phi|_{\hat{\mathcal{L}}} = id_{\hat{\mathcal{L}}}, \quad \Phi^* (\omega|_{\mathcal{T}_1(\mathcal{L})}) = j|_{\mathcal{T}(\hat{\mathcal{L}})}.$$

Then the mapping $\Psi = S \circ \Phi$ satisfies the conditions (A.22), (A.23) with $\mathcal{T}(\mathcal{L}) = \Psi(\mathcal{T}(\hat{\mathcal{L}}))$.

We turn now to the construction of the mapping $S$ satisfying the conditions (A.24), (A.25). Before let us map the manifold $\mathcal{C}$ onto the open domain $\hat{\mathcal{C}}$ of $\mathbb{R}^{2m}$ by means of a $C^r$-diffeomorphism $U_\mathcal{C}$. We set $\hat{\mathcal{L}} = U_\mathcal{C}(\mathcal{L})$. Let $U_{\mathcal{L}}$ be a $C^r$-diffeomorphism from $\hat{\mathcal{L}}$ onto the Lagrangian subspace $\hat{\mathcal{L}} \subseteq \mathbb{R}^{2m}$. Then by Proposition A.7 there exists a $C^{r-2}$-diffeomorphic extension $U_T$
of \((U_L')^{-1}\), which maps a tubular neighborhood \(\mathcal{T}_2(\hat{L})\) of \(\hat{L}\) onto a tubular neighborhood \(\mathcal{T}_3(\hat{L})\) of \(\hat{L}\). We set
\[
\mathcal{T}_3(L) = (U_C)^{-1}(\mathcal{T}_3(\hat{L})), \hat{\omega} = (U_C^{-1} \circ U_T) \ast (\omega|_{T(\mathcal{T}_3(L))}).
\]
Since \(U_C^{-1} \circ U_T(\hat{L}) = L\) and \(L\) is Lagrangian in \((\mathcal{T}_3(L), \omega)\), then \(\hat{L}\) is Lagrangian in \((\mathcal{T}_3(L), \hat{\omega})\).

Consider on the symplectic manifold \((\mathcal{T}_2(\hat{L}), \hat{\omega})\) the symplectic connection \(\Gamma(x)(\cdot, \cdot)\) (see (A.11)). We shall translate vectors with the help of this connection along the rays
\[
\gamma_l : \ x = tl, \ t \geq 0, \ l \in L.
\]
In other words, we consider the family of differential equations:
\[
\frac{d\xi}{dt} = \Gamma(tl)(l, \xi).
\] (A.26)
For a fixed \(l \in L\) we denote by \(U_l(t, t_0)\) the evolution operator of the equation (A.26). Let us show that \(\hat{L}\) is invariant with respect of this operator. This is equivalent to the fact that for each fixed \(l \in L, \ t \geq 0\) the restriction of the vector field \(\{\Gamma(tl)(l, \xi)\}_{\xi \in T_2(\hat{L})}\) on \(\hat{L}\) is a vector field on \(\hat{L}\). This means that
\[
\forall \xi \in \hat{L} : \ \Gamma(tl)(l, \xi) \in T(\hat{L}) (\equiv \hat{L}).
\] (A.27)
Let us prove the last property. Since \(\hat{L}\) is Lagrangian, then for any \(l \in \hat{L}\) and \(\xi, h \in T_2(\hat{L})\)
\[
\omega(tl)(h, \xi) = 0,
\] (A.28)
in particular,
\[
\omega(tl)(l, \xi) = 0.
\]
The last identity implies that
\[
\forall h \in T_2(\hat{L}) : \ d_\xi(\omega(tl)(l, \xi)) \cdot h = -td_\xi\omega(tl)(l, \xi) \cdot h + \omega(tl)(h, \xi) = 0.
\] (A.29)
In view of (A.28), the identity (A.29) is equivalent to the following one:
\[
\forall h \in T_2(\hat{L}) : \ d_x\omega(tl)(l, \xi) \cdot h = 0.
\] (A.29)
According to the definition of the operator \(\hat{\Omega}(x)\) (A.10) and of \(\Gamma(x)(\cdot, \cdot)\) (A.11) the relation \(y = \Gamma(tl)(l, \xi)\) is equivalent to
\[
\forall h \in T_2(\hat{L}) : \ \omega(tl)(y, h) = -d_x\omega(tl)(l, \xi) \cdot h.
\]
Then (A.29), (A.30) and the fact that \(\hat{L}\) is a Lagrangian submanifold of the symplectic manifold \((\mathcal{T}_2(\hat{L}), \hat{\omega})\), imply the inclusion: \(y \in T_2(\hat{L})\). So, the invariance of \(\hat{L}\) with respect to the evolution operator of the equation (A.26) is proved.

We now turn to a symplectomorphic trivialization of the vector bundle \(T(\mathcal{T}_2(\hat{L}))|_{\hat{L}}\) by means of the above symplectic connection. Let us set
\[
\mathcal{W}(\xi_0, l) = \{l, W_l\xi_0\},
\]
where
\[
W_l = U_l(1, 0).
\]
It is easy to see that the mapping $\mathcal{W}$ realizes a $C^{r-2}$-smooth BP-morphism of the trivial vector bundle $\hat{\mathcal{L}} \times T_0(\mathbb{R}^{2m}) \to \hat{\mathcal{L}}$ onto the vector bundle $T(T_2(\mathcal{L}))|_{\hat{\mathcal{L}}}$. The invariance of $\hat{\mathcal{L}}$, which have been proved above, implies the following property of $\mathcal{W}$:

$$\mathcal{W}(\hat{\mathcal{L}} \times T_0(\mathcal{L})) = T(\hat{\mathcal{L}}). \quad \text{(A.30)}$$

Furthermore, by the property of the symplectic connection (Proposition [A.5]) on each fibre $\{l\} \times T_0(\mathbb{R}^{2m})$ the mapping $\mathcal{W}$ realizes a linear symplectic isomorphism between the symplectic spaces $(\{l\} \times T_0(\mathbb{R}^{2m}), \hat{\omega}(0))$ and $(T_l(T_2(\hat{\mathcal{L}})), \hat{\omega}(l))$. For a convenience we shall identify the spaces $T_l(\mathbb{R}^{2m})$ with $\mathbb{R}^{2m}$. We choose a symplectic basis in the symplectic space $(\mathbb{R}^{2m}, j)$ (respectively, in $(\mathbb{R}^{2m}, \hat{\omega}(0))$) in the following manner:

$$e_1^{(0)}, e_2^{(0)}, \ldots, e_m^{(0)}, f_1^{(0)}, f_2^{(0)}, \ldots, f_m^{(0)}$$

and, respectively,

$$e_1^{(0)}, e_2^{(0)}, \ldots, e_m^{(0)}, \hat{f}_1^{(0)}, \hat{f}_2^{(0)}, \ldots, \hat{f}_m^{(0)},$$

where $e_k \in \hat{\mathcal{L}}$ ($k = 1, 2, \ldots, m$). Let us remind that the subspace $\hat{\mathcal{L}}$ is Lagrangian in the both symplectic spaces. Consider the symplectic linear diffeomorphism $S$ from $(\mathbb{R}^{2m}, j)$ onto $(\mathbb{R}^{2m}, \hat{\omega}(0))$, defined by the correspondence of the above bases. It is clear that

$$S|_{\hat{\mathcal{L}}} = id_{\hat{\mathcal{L}}} \quad \text{(A.31)}$$

Then for each $l \in \hat{\mathcal{L}}$ the mapping $W_l \circ S$ realizes a symplectomorphism between $(\mathbb{R}^{2m}, j)$ and $(T_l(T_2(\hat{\mathcal{L}})), \hat{\omega}(l))$.

Our aim is to construct for any $l \in \mathcal{L}$ a symplectomorphism $\hat{W}_l$ between the above symplectic spaces, such that the mapping

$$\hat{W}(l, v) = l, \hat{W}_l v \quad \text{(A.32)}$$

realizes a $C^{r-2}$-smooth BP-morphism between the vector bundles

$$T(\mathbb{R}^{2m})|_{\hat{\mathcal{L}}}, \quad T(T_2(\hat{\mathcal{L}}))|_{\hat{\mathcal{L}}}$$

and, moreover, the relation holds

$$\mathcal{W}|_{T(\hat{\mathcal{L}})} = id_{T(\hat{\mathcal{L}})}, \quad \text{(A.33)}$$

if we identify on the both manifolds $\mathbb{R}^{2m}, T_2(\hat{\mathcal{L}})$ the vector bundles $T(\hat{\mathcal{L}})$ and $\hat{\mathcal{L}} \times \hat{\mathcal{L}}$. Taking into account the last remark, we can consider $e_1^{(0)}, e_2^{(0)}, \ldots, e_m^{(0)}$ to be a basis in each tangent space $T_l(\hat{\mathcal{L}})$. In view of (A.30), (A.31),

$$\forall l \in \hat{\mathcal{L}} : \quad W_l \circ S(T_l(\hat{\mathcal{L}})) = T_l(\mathcal{L}).$$

Then the vector functions

$$e_k(l) = (W_l \circ S)^{-1}(e_k^{(0)}) \quad (k = 1, 2, \ldots, m)$$
are $C^{r-2}$-smooth and for each fixed $l \in \hat{\mathcal{L}}$ they form a basis in the tangent space $T_l(\hat{\mathcal{L}})$ identified with $\hat{\mathcal{L}}$. Consider in $\mathbb{R}^{2m}$, $j$) the following Lagrangian subspace $\hat{\mathcal{L}}'$, which is complementary to $\hat{\mathcal{L}}$:

$$\mathcal{L} = \text{span}(f_1^{(0)}, f_2^{(0)}, \ldots, f_m^{(0)}).$$

Let us construct a basis in $\hat{\mathcal{L}}'$ for each fixed $l \in \hat{\mathcal{L}}$

$$\tilde{f}_1(l), \tilde{f}_2(l), \ldots, \tilde{f}_m(l),$$

such that the vectors

$$e_1(l), e_2(l), \ldots, e_m(l), \tilde{f}_1(l), \tilde{f}_2(l), \ldots, \tilde{f}_m(l)$$

form a symplectic basis in $(\mathbb{R}^{2m}, j)$. The vectors $\tilde{f}_i(l)$ have the form:

$$\sum_{k=1}^{m} a_{ik} f_k^{(0)},$$

where for each fixed $i$ the numbers $a_{ik}$ ($k = 1, 2, \ldots, m$) form the solution of the following linear system:

$$\sum_{k=1}^{m} a_{ik} j(f_k^{0}, e_s(l)) = \delta_{is} (s = 1, 2, \ldots, m).$$

It is easy to see that this system has the unique solution and the vector functions $\tilde{f}_i(l)$ are $C^{r-2}$-smooth.

Consider the sequence of vectors:

$$e_1^{(0)}, e_2^{(0)}, \ldots, e_m^{(0)}, f_1(l), f_2(l), \ldots, f_m(l),$$

where

$$f_i(l) = W_l \circ S(\tilde{f}_i(l)).$$

Since $e_i^{(0)} = W_l \circ S(e_i(l))$ and $W_l \circ S$ realizes a linear symplectic isomorphism between $(\mathbb{R}^{2m}, j)$ and $(T_l(\mathbb{R}^{2m}), \hat{\omega}(l))$, then the system (A.34) form a symplectic basis in the last symplectic space. Let us construct the linear symplectic isomorphism $\tilde{W}_l$ between $(\mathbb{R}^{2m}, j)$ and $(T_l(\mathbb{R}^{2m}), \hat{\omega}(l))$ by means of the correspondence of the symplectic bases

$$\tilde{W}_l(e_i^{(0)}) = e_i^{(0)}, \quad \tilde{W}_l(f_i^{(0)}) = f_i(l) \ (i = 1, 2, \ldots, m).$$

Then the mapping of the form (A.32) defines a $C^{r-2}$-smooth BP-morphism $\tilde{W}$ between the vector bundles $T|_{\mathcal{L}}(\mathbb{R}^{2m})|_{\hat{\mathcal{L}}}$ and $T(T_2(\hat{\mathcal{L}}))|_{\hat{\mathcal{L}}}$ having the property (A.33). Furthermore, this isomorphism is symplectic, i.e.,

$$\tilde{W}^*|_{T(T_2(\hat{\mathcal{L}}))|_{\hat{\mathcal{L}}}} = j|_{T(\mathbb{R}^{2m})|_{\hat{\mathcal{L}}}}.$$ (A.35)

By Proposition A.8 we can construct a diffeomorphism $\mathcal{V}$ from a tubular neighborhood $\mathcal{T}(\mathcal{L})$ of $\hat{\mathcal{L}}$ onto a tubular neighborhood $T_1(\hat{\mathcal{L}}) \subseteq T_3(\hat{\mathcal{L}})$ of $\hat{\mathcal{L}}$, such that

$$\mathcal{V}|_{\hat{\mathcal{L}}} = \text{id}_{\hat{\mathcal{L}}}$$

and

$$T(\mathcal{V})|_{T(T_1(\hat{\mathcal{L})))|_{\hat{\mathcal{L}}}} = \tilde{W}.$$
The last relations and the equalities \( \text{(A.26)} \), \( \text{(A.35)} \) imply that the mapping
\[
S = U_C \circ U_T \circ V
\]
has the properties \( \text{(A.24)} \), \( \text{(A.25)} \).

Let us turn now to the main statement of this section.

**Theorem A.10.** Let \((\mathcal{C}, \omega)\) be a symplectic \( C^r \) -manifold of a finite dimension \( 2m \) endowed by a Riemannian metric \( \{(\cdot, \cdot)_x\}_{x \in \mathcal{C}} \). A non-empty subset \( \mathcal{L} \subseteq \mathcal{C} \) is a Lagrangian submanifold of \((\mathcal{C}, \omega)\), if there exists a tubular neighborhood \( T(\mathcal{L}) \) of \( \mathcal{L} \) and a mapping \( \Theta \) from \( T(\mathcal{L}) \) into the space \( \mathbb{R}^m \) satisfying the conditions:
\[
\Theta \in C^r(T(\mathcal{L}), \mathbb{R}^m), \tag{A.36}
\]
\[
\mathcal{L} = \{ x \in T(\mathcal{L}) : \Theta(x) = 0 \}, \tag{A.37}
\]
\[
\forall x \in T(\mathcal{L}) : \text{Im}(\Theta'(x)) = \mathbb{R}^m, \tag{A.38}
\]
\[
\forall x \in T(\mathcal{L}) : \Theta'(x)(\Omega(x))^{-1}(\Theta'(x))^* = 0, \tag{A.39}
\]
where the operator \((\Theta'(x))^*\) is conjugate to the operator \(\Theta'(x)\) (if the latter is considered to be acting from the Hilbert space \( (T_x(\mathcal{C}), (\cdot, \cdot)_x) \) into the Hilbert space \( \mathbb{R}^m \)) and the operator \(\Omega(x)\) is a skew-self-adjoint invertible operator generating in the first space the 2-form \(\omega(x)(\cdot, \cdot)\), i.e.,
\[
\forall u, v \in T_x(\mathcal{C}) : \omega(x)(u, v) = (\Omega(x)u, v)_{\mathcal{C}}.
\]

If \( r \geq 3 \), the manifold \( \mathcal{C} \) is \( C^r \) -diffeomorphic to an open domain \( \hat{\mathcal{C}} \) of the space \( \mathbb{R}^{2m} \) and \( \mathcal{L} \) is \( C^{r-2} \)-diffeomorphic to a linear space, then the necessary condition for \( \mathcal{L} \) to be a Lagrangian submanifold of \((\mathcal{C}, \omega)\) is the existence of a mapping \( \Theta \), such that the conditions \( \text{(A.36)} - \text{(A.39)} \) are satisfied for it with \( r - 2 \) instead of \( r \).

**Proof.** The first part of our theorem follows from Proposition \( \text{[A.4]} \) applied to the tangent space at each point of \( T(\mathcal{L}) \). Let us prove the second part.

According to the condition imposed on the Lagrangian submanifold \( \mathcal{L} \), there exists a \( C^r \)-diffeomorphism between it and a Lagrangian subspace \( \hat{\mathcal{L}} \) of the symplectic space \( (\mathbb{R}^{2m}, j) \). Then by Proposition \( \text{[A.8]} \) there exists a \( C^{r-2} \)-diffeomorphism \( \Psi \) from a tubular neighborhood \( T(\hat{\mathcal{L}}) \) of \( \hat{\mathcal{L}} \) onto a tubular neighborhood \( T(\mathcal{L}) \) of \( \mathcal{L} \) satisfying the conditions \( \text{(A.22)} \), \( \text{(A.23)} \). Let \( P \) be the orthogonal projection on the subspace \( \hat{\mathcal{L}} \subseteq \mathbb{R}^{2m} \), i.e.,
\[
\hat{\mathcal{L}} = \ker(I - P). \tag{A.40}
\]

Since \( \hat{\mathcal{L}} \) is Lagrangian, then by Proposition \( \text{[A.4]} \) in which \( \mathbb{R}^m = \hat{\mathcal{L}}^\perp \), the conditions are fulfilled:
\[
\text{rank}(P) = m, \tag{A.41}
\]
\[
(I - P)J(I - P) = 0. \tag{A.42}
\]

We set \( \Theta = (I - P)\Psi^{-1} \). Taking into account \( \text{(A.40)} - \text{(A.42)} \), we obtain, that this mapping \( \Theta \) satisfies the desired conditions. \[\square\]
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