Kwok Sau Fa

Nonlocal description of a falling body through the air

Abstract In this present work we consider a falling body through the air under the influence of gravity. In particular, we consider the experimental data based upon the free fall of six men in the atmosphere of the earth. In order to describe this process we employ a nonlocal dissipative force. We show that our description, by using an exponential memory kernel, can fit the experimental data as well as that obtained by a local dissipative force.

Keywords free fall system · nonlocal dissipative force · gravity · exponential memory kernel

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1 Introduction

Address(es) of author(s) should be given
Recently, the idea of employing a nonlocal dissipative force has captured so much interest in the scientific community. For instance, in diffusion processes, generalized Langevin equations have been employed to describe anomalous diffusion regimes, including both subdiffusion and superdiffusion. According to Kubo [1] the frictional memory kernel is related to the form of the correlation function for random force when the system is in equilibrium state. As can be noted, anomalous diffusion transport is present in diverse physical systems, e.g., it can be found in porous media [2,3], polymers [4,5], amorphous semiconductors [6,7] and composite heterogeneous films [8].

Approaches based on the fractional derivatives have also been employed to describe, for instance, anomalous diffusion transport [9,10] and viscoelastic systems [11,12,13]. The fractional derivatives are also nonlocal and they can be related to memory effects. Recent works on the theme, have revealed that a fractional derivative can behave as a damping term [9,10]. In this respect, a simple fractional oscillator has also been employed to investigate the physical behavior of a fractional derivative, and it supports this point of view [14,15].

In this work we consider a simple process involving a falling body through the air. We also consider an observed process based upon the free fall of six men in the atmosphere of the earth. As it is well-known the usual description employs the Newton second law with a dissipative term in function of the velocity of the falling body. For this observed process the usual model, with the dissipative force proportional to \( v^2 \), is employed. This phenomenological model is nonlinear. However, other descriptions can also be used to describe the process and they can fit the observed data equally well. In our description,
we work on the assumption that the dissipative force is nonlocal which is related to the memory effect. We analyze our model by employing short and long memory kernels. Our results are compared with the result of the usual approach and experimental data.

2 Local and nonlocal models

When an object falls from rest through the air surrounding the earth, it experiences a resisting force that opposes to the relative motion in which the object moves relatively to the air. Experimentally, this resisting force is related to the relative speed \( v \). For slow speeds the resisting force is in magnitude proportional to the speed. This last assumption is usually employed to describe diffusion processes such as in the Langevin equation \[16\]. However, in other cases it may be proportional to the square (or some other power) of the speed. Applying the Newton second law \[17,18\] we have:

\[
m \frac{dv}{dt} = mg - mF(v),
\]

where \( g \) is the free-fall acceleration and \( F(v) \) is a function of the velocity. We note that for \( F(v) \sim v^2 \), Eq. (1) is nonlinear and it corresponds to the well-known Riccati equation \[19\]. In particular, for \( F(v) = K_1 v/m \), we have the following solutions for \( z \) and \( v \):

\[
v(t) = \frac{mg}{K_1} - c_1 \exp\left[-\frac{K_1}{m} t\right]
\]

and
\begin{align}
  z(t) &= z_0 - \frac{c_1 m}{K_1} + \frac{m}{K_1} \left( gt + c_1 \exp\left[ -\frac{K_1}{m} t \right] \right), \\

  \text{where } c_1 &= \frac{mg}{K_1 - v_0}, \text{ and } z_0 \text{ is the initial position and } v_0 \text{ is the initial velocity.}
\end{align}

For \( F(v) = K_2 v^2 / m \), we obtain

\begin{align}
  v(t) &= b \frac{1 + c_2 \exp[-pt]}{1 - c_2 \exp[-pt]}, \\

  \text{and}
\end{align}

\begin{align}
  z(t) &= z_0 + b \frac{p}{p} \ln \left( \frac{(1 - c_2 \exp[-pt])^2}{(1 - c_2)^2 \exp[-pt]} \right), \\

  \text{where } b^2 &= \frac{mg}{K_2}, \text{ and } c_2 = \frac{(v_0 - b)}{(v_0 + b)}.
\end{align}

For these cases, Eq. (1) shows that the resisting force of an object increases with the increase of the relative speed \( v \). Consequently, when the body falls enough, it reaches a constant terminal velocity. For solutions (2) and (4) the terminal velocities are obtained by taking \( t \to \infty \), and the results are:

\begin{align}
  V_1 &= \frac{mg}{K_1}, \\

  V_2 &= \sqrt{\frac{mg}{K_2}}.
\end{align}

Now one considers the above system by assuming a nonlocal dissipative force given by

\begin{align}
  F(v) &= \int_0^t \gamma(t - \tau) v(\tau) d\tau,
\end{align}
where $\gamma(t)$ is the memory kernel. We note that Eq. (1) is linear with $F(v)$ given by (8), and its solutions can be obtained by using the Laplace transform. For $\gamma(t) \sim \delta(t)$ one has a local dissipative force with $F(v) \sim v$. For a nonlocal memory kernel we choose the exponential and power-law functions. We see that these functions have short and long tails, respectively. Our idea, with these functions, is to analyze the influence of memory function on the system described by Eq. (1) for short and long memory effects.

For exponential memory kernel $\gamma(t) = \gamma_0 e^{-\lambda t}$ we obtain the following solution:

$$z = z_0 + A + \frac{g \lambda}{\gamma_0} t - \left[ A - \left( v_0 - \frac{g \lambda}{\gamma_0} \frac{\lambda}{2} A \right) t \right] e^{-\frac{\lambda}{2} t}, \quad \gamma_0 = \frac{\lambda^2}{4}, \quad (9)$$

$$v(t) = \frac{g \lambda}{\gamma_0} + \left[ v_0 - \frac{g \lambda}{\gamma_0} \frac{\lambda}{2} \left( v_0 - \frac{g \lambda}{\gamma_0} \frac{\lambda}{2} A \right) t \right] e^{-\frac{\lambda}{2} t}, \quad \gamma_0 = \frac{\lambda^2}{4}, \quad (10)$$

$$z = z_0 + A + \frac{g \lambda}{\gamma_0} t + \left[ \left( v_0 - \frac{g \lambda}{\gamma_0} \frac{\lambda}{2} A \right) \frac{\sin(B_1 t)}{B_1} - A \cos(B_1 t) \right] e^{-\frac{\lambda}{2} t}, \quad \gamma_0 > \frac{\lambda^2}{4}, \quad (11)$$

$$v(t) = \frac{g \lambda}{\gamma_0} + \left( v_0 - \frac{g \lambda}{\gamma_0} \frac{\lambda}{2} A \right) \cos(B_1 t) e^{-\frac{\lambda}{2} t} + \left[ A B_1 - \frac{\lambda}{2 B_1} \left( v_0 - \frac{g \lambda}{\gamma_0} \frac{\lambda}{2} A \right) \sin(B_1 t) e^{-\frac{\lambda}{2} t}, \quad \gamma_0 > \frac{\lambda^2}{4}, \quad (12)$$

$$z = z_0 + A + \frac{g \lambda}{\gamma_0} t + \left[ \left( v_0 - \frac{g \lambda}{\gamma_0} \frac{\lambda}{2} A \right) \frac{\sinh(B_2 t)}{B_2} - A \cosh(B_2 t) \right] e^{-\frac{\lambda}{2} t}, \quad \gamma_0 < \frac{\lambda^2}{4}, \quad (13)$$

and
\[ v(t) = \frac{g\lambda}{\gamma_0} + \left( v_0 - \frac{g\lambda}{\gamma_0} \right) \cosh(B_2 t) e^{-\frac{\lambda t}{2}} - \left[ AB_2 + \frac{\lambda}{2B_2} \left( v_0 - \frac{g\lambda}{\gamma_0} - \frac{\lambda}{2} A \right) \right] \sinh(B_2 t) e^{-\frac{\lambda t}{2}}, \quad \gamma_0 < \frac{\lambda^2}{4}, \quad (14) \]

where \( A = (g + \gamma v_0 - \gamma^2 \lambda^2 / \gamma_0) / \gamma_0, B_1 = \sqrt{\gamma_0 - \lambda^2 / 4} \) and \( B_2 = \sqrt{\lambda^2 / 4 - \gamma_0} \).

For the three cases we can verify that \( v(t = 0) = v_0 \) and they give the same expression for the terminal velocity which is given by

\[ V = \frac{g\lambda}{\gamma_0}. \quad (15) \]

Now, it is more convenient to write Eqs. (9), (11) and (13) in terms of the terminal velocity

\[ z = z_0 + \frac{v_0 V}{g} - \frac{3 V^2}{4} t + V t + \left( \frac{3 V^2}{4} + \left( v_0 + \frac{V}{2} \right) t \right) e^{-\frac{g\lambda t}{2}}, \quad \gamma_0 = \frac{\lambda^2}{4}, \quad (16) \]

\[ z = z_0 + A + V t - \left[ \left( v_0 - V - \frac{\lambda}{2} A \right) \frac{\sin(B_1 t)}{B_1} - A \cos(B_1 t) \right] e^{-\frac{\lambda t}{2}}, \quad \gamma_0 > \frac{\lambda^2}{4}, \quad (17) \]

and

\[ z = z_0 + A + V t + \left[ \left( v_0 - V - \frac{\lambda}{2} A \right) \frac{\sinh(B_2 t)}{B_2} - A \cosh(B_2 t) \right] e^{-\frac{\lambda t}{2}}, \quad \gamma_0 < \frac{\lambda^2}{4}, \quad (18) \]

where \( A = V / \lambda + v_0 V / g - V^2 / g, B_1 = \sqrt{g\lambda / V - \lambda^2 / 4} \) and \( B_2 = \sqrt{\lambda^2 / 4 - g\lambda / V} \).

We note that \( \gamma_0 \) does not appear explicitly in these last expressions. In particular, the solution for \( \gamma_0 = \frac{\lambda^2}{4} \) only contains one unknown parameter such as in the usual approach. For two other cases the free fall can be described by one unknown parameter \( \lambda \) which can be determined by the experimental data.

In order to obtain some insights on the solutions of the nonlocal approach (9)-(14) and the solution of the usual approach given by Eq. (5) let us fit the
Experimental data from the free fall of six men in the atmosphere of the earth, from a maximum altitude of 31,400 to 2,100 feet which was accomplished in 116 seconds [19]. The average weight of the men and their equipment was 261.2 pounds and the average velocity of the lowest point corresponding to a drop of 29,300 feet in 116 seconds is \( V = 251.4 \text{ feet per second} \). The experimental data are shown in Table 1 with the values of \( H - z, H = 31,400 \).

To find the velocity of the lowest point \( V \) we set \( v_0 = 0 \) and \( z_0 = 0 \). We obtain from Eq. (5) the following approximate expression

\[
z = V_2 t - \left( \ln 2 \right) \frac{V_2^2}{g}. \tag{19}
\]

Substituting \( z = 29,300 \), \( t = 116 \) and \( g = 32.2 \) we find \( V_2 = 265.7 \text{ ft/s} \). This value is somewhat larger than the experimental result. In the case of \( F(v) \sim v \), the terminal velocity is larger than \( V_2 \), thus this model should be put aside. For the solution (16) we find the following approximate expression

\[
z = V t - \frac{3 V^2}{4 g}. \tag{20}
\]

Note that the expressions (19) and (20) are similar and the values \( \ln 2 \) and 3/4 are close. From Eq. (20) we find \( V = 266.89 \text{ ft/s} \). The parameter \( \lambda \) can be estimated from (15) and we find \( \lambda = 0.483 \text{ s}^{-1} \).

In Fig. 1 we show the comparison of the observed values with those obtained by the solutions (5) and (16). It is clear that the solution (16) gives the adjustment as well as that of the usual approach. In Fig. 2 we show the behavior of the solutions (5), (16), (17) and (18). For the solution (5) the terminal velocity is \( V_2 = 265.7 \text{ ft/s} \), while for other solutions the terminal velocity is \( V = 266.89 \text{ ft/s} \). The plots show that the curves are close to each
other. Thus, all of them can be used to describe the free fall system equally well.

Next, we consider a long tail memory kernel given by $\gamma(t) = \gamma_1 t^{-\alpha}$ for $0 < \alpha < 1$. The solutions for $z(t)$ and $v(t)$ are given by

$$z(t) = z_0 + gt^2 E_{2-\alpha,3}(-\gamma^* t^{2-\alpha}) + v_0 t E_{2-\alpha,2}(-\gamma^* t^{2-\alpha})$$  \hspace{1cm} (21)

and

$$v(t) = gt E_{2-\alpha,2}(-\gamma^* t^{2-\alpha}) + v_0 E_{2-\alpha,1}(-\gamma^* t^{2-\alpha})$$ \hspace{1cm} (22)

where $\gamma^* = \gamma_1 \Gamma(1-\alpha)$, $\Gamma(x)$ is the gamma function and $E_{\mu,\nu}$ is the generalized Mittag-Leffler function defined by $E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(n\mu + \nu)}$ \hspace{1cm} [20, 21]. For $t = 0$ we verify that $v(t = 0) = v_0$. In order to verify the terminal velocity we should analyze the behavior of Eq. (22) for large time. To do so, we use the following asymptotic approximation:

$$E_{\mu,\nu}(z) \sim -\frac{1}{z \Gamma(\nu - \mu)}.$$ \hspace{1cm} (23)

Then, we obtain from (22), for $v_0 = 0$, the following approximation:

$$v(t) \sim \frac{gt^{\alpha-1}}{\gamma^* \Gamma(\alpha)}.$$ \hspace{1cm} (24)

This last result shows that Eq. (22) can not give a terminal velocity. Then, a long time memory such as given by the power-law memory kernel, based on Eqs. (1) and (8), is not appropriate to describe the free fall system through the air.
3 Conclusion

In summary, we have investigated the falling body problem by using the local and nonlocal dissipative forces. We note that nonlocal dissipative forces are frequently used to describe diffusion processes by a generalized Langevin equation. Then, this kind of approach represents a well-known generalization of the usual approach. We have shown that the results of a nonlocal dissipative force given by the exponential memory kernel can fit the experimental data as well as the result obtained by the local dissipative force $F(v) \sim v^2$. In contrast with the nonlinear force $F(v) \sim v^2$, the nonlocal dissipative force (8) gives a linear differential equation. On the other hand, the power-law memory kernel does not give a terminal velocity and consequently it alone can not be used to describe the free fall system. These results suggest that a short memory effect in the nonlocal approach may be appropriate to describe the free fall system. The presence of memory effect in the system may be related to the fact that when the body moves through the air, it displaces the particles in its immediate vicinity. Then, the surrounding flow field is altered and acts back on the body, resulting in an action which depends on the past motion of the body. This feedback process which leads to a memory effect can be reflected to the motion of a particle in a fluid [22]. Further, we have also verified Eq. (1) with an additional term for the dissipative force given by $F(v) = K_1v/m + \gamma_0 \int_0^t \exp\left(-\lambda(t - \tau)\right)v(\tau)d\tau$. For this case we can improve the adjustment, i.e., we can fit the experimental data as well as the result of the usual approach with a lower terminal velocity ($V = 265$ ft/s). In Fig. 3 we show the comparison of the result with that described by the
usual approach. In the case of a fractional derivative, it can also be employed to replace the classical Newtonian force, but it does not lead to improve the result obtained by the usual approach significantly [23].

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Figure Captions

Fig. 1 - Adjustment of the experimental data with the expressions (5) (dotted line) and (16) (solid line). The terminal velocity for the expression (5) is given by $V_2 = 265.7 \text{ ft/s}$, whereas the terminal velocity for the expression (16) is given by $V = 266.89 \text{ ft/s}$.

Fig. 2 - Plots of the expressions (5), (16), (17) and (18). The terminal velocity for the expression (5) is given by $V_2 = 265.7 \text{ ft/s}$, whereas the terminal velocity for the three other expressions is given by $V = 266.89 \text{ ft/s}$.

Fig. 3 - Adjustment of the experimental data with the expression (5) (dotted line) and the result obtained by Eq. (1) with the dissipative force given by $F(v) = K_1 v/m + \gamma_0 \int_0^t \exp(-\lambda(t-\tau))v(\tau)d\tau$ (solid line). The terminal velocity for the expression (5) is given by $V_2 = 265.7 \text{ ft/s}$, whereas the terminal velocity for the second case is given by $V = 265 \text{ ft/s}$, where $\lambda_1 = \lambda + K_1/m$ and $\gamma_0 + \lambda K_1/m = \lambda_1^2/4$. 
| t   | H-z   | t   | H-z   | t   | H-z   |
|-----|-------|-----|-------|-----|-------|
| 0   | 31,400| 46.7| 20,550| 88.1| 9,400 |
| 9.9 | 30,780| 51.3| 19,400| 92.7| 8,140 |
| 14.5| 30,200| 55.9| 18,100| 97.3| 7,080 |
| 19.1| 28,850| 60.5| 16,600| 101.9| 5,700|
| 23.7| 27,700| 65.1| 15,150| 106.5| 4,450|
| 28.3| 26,150| 69.7| 14,070| 111.1| 3,170|
| 32.9| 24,600| 74.3| 12,800| 116.0| 2,100|
| 37.5| 23,200| 78.9| 11,650|     |       |
| 42.1| 21,750| 83.5| 10,400|     |       |

**Table 1** Estimated values of $H - z$ for the fall from an altitude of 31,400 to 2,100 feet, with $z = 0$ when $t = 0$. 