Integrability of Liénard systems with a weak saddle

Armengol Gasull and Jaume Giné

Abstract. We characterize the local analytic integrability of weak saddles for complex Liénard systems, \( \dot{x} = y - F(x) \), \( \dot{y} = ax \), \( 0 \neq a \in \mathbb{C} \), with \( F \) analytic at 0 and \( F(0) = F'(0) = 0 \). We prove that they are locally integrable at the origin if and only if \( F(x) \) is an even function. This result implies the well-known characterization of the centers for real Liénard systems. Our proof is based on finding the obstructions for the existence of a formal integral at the complex saddle, by computing the so-called resonant saddle quantities.

Mathematics Subject Classification. Primary 37C15, Secondary 34A34.

Keywords. Center problem, Analytic integrability, Weak saddle, Liénard equation.

1. Introduction and main results

Since the pioneering works of Élie and Henri Cartan [1] and Balthasar van der Pol [25,26] where Liénard-type differential equations [15],

\[
\frac{d^2 x}{dt^2} + f(x) \frac{dx}{dt} + g(x) = \ddot{x} + f(x) \dot{x} + g(x) = 0,
\]

appear in electrical problems, many other situations have been modeled by differential equations that can be transformed into them. For instance, Liénard equations appear in mechanical problems, in predator–prey systems [9] and chemical or biochemical reactions [17,19]. Recall that Hilbert 16th problem for polynomial differential equations of degree \( d \) asks for a bound for their number of limit cycles that depends only on \( d \). It is remarkable that this question restricted to planar polynomial quadratic differential equations (\( d = 2 \)) can be reduced to the study of some Liénard-type differential equations, see [27].

Two of the main questions about them are to know whether they are integrable or not and to give criteria for controlling their number of limit cycles. This paper deals with the first one.

Poincaré proved that if a real planar analytic differential system has a weak focus at a critical point, then this point is a center if and only if the equation has an analytic first integral defined in a neighborhood of this point, see, for instance, [14,18,20,23]. Then, the characterization of the centers for non-degenerate critical points is equivalent to the characterization of the local analytic integrable cases.

Let us recall the simple and well-understood characterization of the centers, and so the local analytic integrable cases, for classical real analytic Liénard systems, \( g(x) \equiv x \). If we write these equations in \( \mathbb{R}^2 \) as

\[
\dot{x} = \frac{dx}{dt} = y + F(x), \quad \dot{y} = -x,
\]

with \( F(x) = - \int_0^x f(s) \, ds \) analytic at zero and \( F(0) = F'(0) = 0 \), then the origin is a center if and only if \( F \) is an even function, see, for instance, [4,7,29]. We include for completeness a very simple proof, which essentially is the one appearing in [4,29]. Write \( F = F^o + F^e \), where \( F^e(x) = (F(x) + F(-x))/2 \) is the even part of \( F \) and \( F^o \) its odd part. If \( F = F^o \), then the origin is a center by the classical criterion of
Consider the scalar product 

\[(y + F(x), -x) \cdot (y + F^e(x), -x) = x(F(x) - F^e(x)) = xF^o(x) = ax^{2k+2} + O_{2k+3}(x),\]

where as usual \((u, v)⊥ = (-v, u)\) is a perpendicular vector to \((u, v)\). Then, on a neighborhood of the origin, it does not change sign and vanishes only on \(x = 0\), which is not an invariant curve of the Liénard system. Therefore, the level sets of the Liénard system corresponding to \(F = F^e\), which are closed curves surrounding the origin, are curves without contact for the flow of the original Liénard system. This implies that the origin is not a center, as we wanted to prove. In other words, we have used the first integral of the integrable case \(F = F^e\) as a Lyapunov function for the non-integrable one \(F \neq F^e\). Another proof of this result is given in [6]. It is based on the computation of the so-called Lyapunov constants, which are the obstructions for finding an analytic first integral of the Liénard system.

One of the motivations of this paper is to prove a similar result, but for classical Liénard systems with a weak saddle at the origin,

\[
\dot{x} = y + F(x), \quad \dot{y} = x,
\]

with \(F\), also analytic and \(F(0) = F'(0) = 0\). Notice that its eigenvalues are ±1. Recall that a saddle point is called weak if its eigenvalues are ±\(\lambda\) with \(0 \neq \lambda \in \mathbb{R}\). That is, weak saddles are hyperbolic saddles such that the divergence of the vector field vanishes on them.

In this situation, we have not the clear geometric interpretation of integrable system as the one for which the orbits are closed and a more analytical approach is needed. In fact, the method that we will use is the computation of the so-called (resonant) saddle quantities that we will recall below, which as we will see, is similar to the computation of the Lyapunov quantities used in [24]. It is also worth mentioning that the duality between centers and saddles, given by the complex change of variables and time \((x_1, y_1, t_1) = (ix, y, it)\), that transforms one case into the other one when considering the system in \(\mathbb{C}^2\), does not allow to use the simple geometric interpretation that we have on \(\mathbb{R}^2\). Other Liénard-like complex systems with a weak saddle are studied in [11].

Before stating our main result, we introduce some notation and motivate the extension of our problem to \(\mathbb{C}^2\).

The general center problem for analytic vector fields in \(\mathbb{R}^2\) with an elementary singular point of the form

\[
\dot{x} = -y + O_2(x, y), \quad \dot{y} = x + O_2(x, y),
\]

is sometimes embedded by the change of variable \(u = x + iy\) and the corresponding conjugate variable \(v = x - iy\) into the complex vector fields of the form

\[
\dot{u} = u + O_2(u, v), \quad \dot{v} = -v + O_2(u, v).
\]

The next extension of the above system is to consider analytic vector fields in \(\mathbb{C}^2\) of the form

\[
\dot{u} = \lambda u + O_2(u, v), \quad \dot{v} = -\mu v + O_2(u, v), \quad (1)
\]

where \(\lambda, \mu \in \mathbb{C}\setminus\{0\}\).

It is already proved by Poincaré [13, 21, 22] that if \(\lambda/\mu \notin \mathbb{Q}^+\), then the differential equation (1) has no local analytic first integral in a neighborhood of zero. The case \(\lambda/\mu = p/q \in \mathbb{Q}^+\), with gcd\((p, q) = 1\), is called \([p : -q]\) resonant case, and in this situation, adding some more necessary conditions, the local analytic integrability is sometimes possible. Let us briefly recall how to obtain these conditions.

Changing the time, if necessary, the \([p : -q]\) resonant case can be written as

\[
\dot{u} = pu + O_2(u, v), \quad \dot{v} = -qv + O_2(u, v), \quad (2)
\]
with \( p, q \in \mathbb{Z}^+ \). For this system, the linear part has the analytic first integral \( H_0(u, v) = u^q v^p \), and we can seek the conditions for the existence of an analytic first integral \( H(u, v) = H_0(u, v) + O_{p+q+1}(u, v) \) for system (2). Hence, we get the equation \( \dot{H}(u, v) = \dot{v_1}H_0^2(u, v) + v_3H_0^3(u, v) + \cdots \), and the so-called \([p : -q]\) resonant saddle quantities \( v_i \) are polynomials of the coefficients of system (2). If all the \( v_i \) are zero we say that we have a formal analytic resonant saddle, see [12,28] and references therein. From this result, we obtain also the existence of a local analytic first integral, see [23].

In this work, we aim to give a simple and self-contained proof of the characterization of the integrable complex analytic differential systems in \( \mathbb{C}^2 \) of the form

\[
\dot{x} = y + F(x), \quad \dot{y} = g(x),
\]

where \( F(x) \) is an analytic function of \( x \) without linear and constant terms. We prove:

**Theorem 1.** System (3) is locally integrable at the origin if and only if \( F(x) \) is an even function of \( x \).

Notice that if we restrict our attention to \((x, y) \in \mathbb{R}^2\), our result covers the classical Liénard case with a weak focus at the origin, \( a = -1 \), and the weak saddle case, \( a = 1 \), which has a \([1 : -1]\) resonance. Our result also extends to \( \mathbb{C}^2 \) some of the results of [16].

As a corollary of Theorem 1 and using the ideas of Cherkas [4] and Christopher, Lloyd and Pearson [7], we get the characterization of the real analytic integrable weak saddles for general Liénard systems, see also [2,3,8,10]. This corollary is also recently proved in [16] and extends to the weak saddle case the known results for the weak focus case.

Consider the differential systems in \( \mathbb{R}^2 \) of the form

\[
\dot{x} = y + F(x), \quad \dot{y} = g(x),
\]

where \( F \) and \( g \) are analytic functions of \( x \), \( F(x) \) without linear and constant terms and with \( g(0) = 0 \) and \( g'(0) \neq 0 \) and set \( G(x) = \int_0^x g(\xi) d\xi \).

**Corollary 2.** System (4) has an integrable resonant weak saddle (resp. weak focus) at the origin if and only if \( g'(0) > 0 \) (resp. \( g'(0) < 0 \)) and \( F(x) = \phi(G(x)) \), for some analytic function \( \phi \) with \( \phi(0) = 0 \).

### 2. Proof of the results

**Proof of Theorem 1.** System (3) can be transformed into system

\[
\dot{x} = y + F(x), \quad \dot{y} = x,
\]

doing the change of variables \( x = x, Y = by \) and the change of time \( dt = bds \) with \( b \) any of the roots \( 1/\sqrt{a} \). Observe that for simplicity we write again \( y \) instead of \( Y \).

Consider \( F(x) = \sum_{i=2}^{\infty} a_i x^i \). As explained in the introduction, to find the saddle quantities we propose a formal first integral of the form

\[
H(x, y) = x^2 - y^2 + \sum_{k=3}^{\infty} H_k(x, y),
\]

where \( H_k(x, y) \) are homogeneous polynomials that can be written

\[
H_k(x, y) = \sum_{i+j=k} c_{i,j} x^i y^j.
\]
Now we compute the derivative of $H$ along the vector field associated with system (5), and we obtain a linear system for each function $H_k$. For instance, for $H_3$ we have the linear system

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
3 & 0 & 2 & 0 \\
0 & 2 & 0 & 3 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
c_{3,0} \\
c_{2,1} \\
c_{1,2} \\
c_{0,3}
\end{pmatrix}
= -
\begin{pmatrix}
2a_2 \\
0 \\
0 \\
0
\end{pmatrix}.
$$

From the last equation, we obtain $c_{1,2} = 0$. Now we go to the second equation and we have $c_{3,0} = 0$. These two conditions are the only ones that we will use for the next steps of the proof. The linear system for $H_4$ is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 2 & 0 & 0 & 0 \\
0 & 3 & 0 & 3 & 0 & 0 \\
0 & 0 & 2 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
c_{4,0} \\
c_{3,1} \\
c_{2,2} \\
c_{1,3} \\
c_{0,4}
\end{pmatrix}
= -
\begin{pmatrix}
2a_3 + 3a_2c_{3,0} \\
2a_2c_{2,1} \\
a_2c_{1,2} \\
0 \\
0
\end{pmatrix}.
$$

From the last equation, we obtain $c_{1,3} = 0$. Next, we consider the third equation, and taking into account that $c_{1,2} = 0$, we deduce $c_{3,1} = 0$. Finally, the first equation gives us that $a_3$ must be zero, taking into account that $c_{5,0} = 0$, obtaining the first condition to have formal integrability for system (5). The linear system for $H_5$ is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
c_{5,0} \\
c_{4,1} \\
c_{3,2} \\
c_{2,3} \\
c_{1,4} \\
c_{0,5}
\end{pmatrix}
= -
\begin{pmatrix}
2a_4 + 3a_3c_{3,0} + 4a_2c_{4,0} \\
2a_3c_{2,1} + 3a_2c_{3,1} + 4a_2c_{4,1} \\
a_3c_{1,2} + 2a_2c_{2,2} \\
a_2c_{1,3} \\
0 \\
0
\end{pmatrix}.
$$

From the last equation, we have $c_{1,4} = 0$, now going to the fourth one, and taking into account that $c_{1,3} = 0$, we arrive to $c_{3,2} = 0$. Finally, from the second equation, and taking into account that $a_3 = 0$ and $c_{3,1} = 0$, we have $c_{5,0} = 0$. Similarly, the linear system for $H_6$ is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
c_{6,0} \\
c_{5,1} \\
c_{4,2} \\
c_{3,3} \\
c_{2,4} \\
c_{1,5} \\
c_{0,6}
\end{pmatrix}
= -
\begin{pmatrix}
2a_5 + 3a_4c_{3,0} + 4a_3c_{4,0} + 5a_2c_{5,0} \\
2a_4c_{2,1} + 3a_3c_{3,1} + 4a_2c_{4,1} \\
a_4c_{1,2} + 2a_3c_{2,2} + 3a_2c_{3,2} \\
a_3c_{1,3} + 2a_2c_{2,3} \\
a_2c_{1,4} \\
0 \\
0
\end{pmatrix}.
$$

Now, taking into account that $c_{1,2} = c_{3,0} = 0$, $c_{1,4} = c_{3,2} = c_{5,0} = 0$, and $a_3 = 0$ we obtain that $c_{1,5} = c_{3,3} = c_{5,1} = 0$ and that $a_5$ must be zero, which is the second condition to have integrability for system (5).

Let us prove the theorem by induction. Our induction hypothesis is that for each $n$ odd, the necessary conditions for system (5) to have a formal first integral $H$ of the form (6) are:

$$
c_{n,0} = c_{n-2,2} = \cdots = c_{3,n-3} = c_{1,n-1} = 0, \quad c_{n,1} = c_{n-2,3} = \cdots = c_{3,n-2} = c_{1,n} = 0,
$$

and $a_3 = a_5 = \cdots = a_n = 0$.

In fact, until now, we have prove the induction hypothesis for $n = 3$ and $n = 5$. Clearly, the proof for $n = 5$ is not needed to use the induction method, but we have included it for the sake of clarity.
For any \( m \), we have that the linear system obtained that (6) is a formal first integral is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 3 & 0 & \cdots & m-1 \\
0 & \cdots & 0 & 0 & 2 & 0 & m \\
0 & \cdots & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
c_{m,0} \\
c_{m-1,1} \\
c_{m-2,2} \\
\vdots \\
c_{2,m-2} \\
c_{1,m-1} \\
c_{0,m}
\end{pmatrix}
+ \begin{pmatrix}
2a_{m-1} + 3a_{m-2}c_{3,0} + \cdots + (m-2)a_{3}c_{m-2,0} + (m-1)a_{2}c_{m-1,0} \\
a_{m-2}c_{21} + 3a_{m-3}c_{31} + \cdots + (m-3)a_{3}c_{m-3,1} + (m-2)a_{2}c_{m-2,1} \\
a_{m-3}c_{12} + 2a_{m-4}c_{22} + \cdots + (m-4)a_{3}c_{m-4,2} + (m-3)a_{2}c_{m-3,2} \\
\vdots \\
a_{2}c_{1,m-2} \\
0 \\
0
\end{pmatrix}.
\]

Let us prove our induction hypothesis for any \( n \) (odd) assuming that it is true for \( n - 2 \).

When \( m = n \) is odd, the determinant of the above linear system is different from zero, and then, the system is compatible and determined. Moreover, using the induction hypothesis and starting from the last equation and going up in the other equations, we arrive to \( c_{n,0} = c_{n-2,2} = \cdots = c_{3,n-3} = c_{1,n-1} = 0 \), as we wanted to prove.

However, for \( m = n + 1 \) even we have that the determinant of the corresponding linear system is zero. Anyhow, using again our induction hypothesis, from the last equation we have \( c_{1,n} = 0 \), and going up in the equations we get that \( c_{n,1} = c_{n-2,3} = \cdots = c_{1,n} = 0 \) and the first equation reduces to

\[
0 = 2a_n + 3a_{n-1}c_{3,0} + \cdots + (n-1)a_{3}c_{n-1,0} + na_{2}c_{n,0}.
\]

Then, we know that \( c_{i,0} = 0 \) if \( i \) is odd and if \( i \) is even the \( c_{i,0} \) is multiplied by \( a_i \) with \( i \) odd. Since, moreover again by induction hypothesis, we know that \( a_3 = a_5 = \cdots = a_{n-2} = 0 \) we obtain \( 0 = 2a_n \) which implies \( a_n = 0 \), as we wanted to show.

This proves that system (5) has only a formal first integral when \( F \) is even. Then, applying [23, Cor. 3.2.6] we have that under this hypothesis system (5) has an analytic first integral around the origin. \( \square \)

Proof of Corollary 2. Following [4,7], let us see that system (4) can be locally transformed into a classical Liénard system in \( \mathbb{R}^2 \) of the form

\[
\dot{x} = y + \tilde{F}(x), \quad \dot{y} = \sigma x,
\]

where \( \sigma = \text{sign}(g'(0)) \) and \( \tilde{F} \) is an analytic function. Let \( u \) be the root of \( 2\sigma G(x) \) that has the same sign that \( x \),

\[
u = \phi(x) = \text{sign}(x) \sqrt{2\sigma G(x)} = x \sqrt{|g'(0)|(1 + O_1(x))},
\]

which is well defined and analytic in a neighborhood of \( x = 0 \). Moreover, it has an analytic inverse \( x = \xi(u) = u(1 + O_1(u))/\sqrt{|g'(0)|} \). The transformation \( u = \phi(x) \) converts system (4) into the new system

\[
\dot{u} = \frac{\sigma g(\xi(u))}{u}(y + F(\xi(u))), \quad \dot{x} = g(\xi(u)).
\]
Since \( g(\xi(u))/u = \sigma \sqrt{|g'(0)|} + O_1(u) \) is analytic and nonzero in a neighborhood of the origin, we can do the change of time \( dt/ds = \sigma u/g(\xi(u)) \) and it becomes the classical Liénard system,
\[
u' = y + F(\xi(u)), \quad \dot{x} = \sigma u.
\]
By Theorem 1, the condition to have integrability for such system is that \( F(\xi(u)) \) is an analytic even function of \( u \). The chain of equalities
\[
F(\xi(u)) = \hat{\phi}(u^2) = \hat{\phi}(2\sigma G(x)) = \phi(G(x))
\]
proves the corollary.

By using Lüroth theorem as in [5], we get the following effective characterization of real integrable polynomial Liénard systems with a weak saddle at the origin:

**Corollary 3.** Consider system (4) with \( F \) and \( G \) polynomials. Then, it has an integrable resonant weak saddle at the origin if and only if there exist polynomials \( \hat{F}, \hat{G} \) and \( u, w \) with \( u(0) = u'(0) = 0, u''(0) \neq 0 \), such that
\[
F(x) = \hat{F}(u(x)), \quad G(x) = \hat{G}(u(x)).
\]

**The strong saddles case**

The problem of the local integrability for strong (that is, non-weak) saddles of Liénard systems is not considered in our work. Notice that if instead of system (4) we take in \( \mathbb{R}^2 \) the system
\[
\dot{x} = y + F(x), \quad \dot{y} = x, \quad (7)
\]
with \( F(0) = 0 \) but \( F'(0) \neq 0 \), it has a strong saddle at the origin. It is convenient to write \( F'(0) = 1/c - c \), with \( c \in \mathbb{R}^+ \backslash \{1\} \). We remove the case \( c = 1 \) to avoid the weak saddle case. Then, its eigenvalues are \( -c < 0 < 1/c \), and the Poincaré necessary condition for having an analytic first integral at the origin is \( c^2 = q/p \in \mathbb{Q}^+ \). In fact, if \( F(x) \equiv (\sqrt{p/q} - \sqrt{q/p}) x \) it is not difficult to prove that
\[
H(x, y) = \left(\sqrt{p} x + \sqrt{q} y\right)^q \left(\sqrt{p} y - \sqrt{q} x\right)^p
\]
is a first integral of the linear Liénard system. We do not know if there are nonlinear integrable cases in system (7).

**Acknowledgements**

The Armengol Gasull was supported by a MINECO Grant Number MTM2013-40998-P and by a CIRIT Grant Number 2014SGR568. The Jaume Giné was partially supported by a MINECO/ FEDER Grant Number MTM2014-53703-P and an AGAUR (Generalitat de Catalunya) Grant Number 2014SGR 1204.

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(Received: April 26, 2016; revised: November 30, 2016)