REMARKS ON TOPOLOGY OF STABLE TRANSLATING SOLITONS

KEITA KUNIKAWA∗ AND SHUNSUKE SAITO†

ABSTRACT. We show that any complete \( f \)-stable translating soliton \( M \) admits no codimension one cycle which does not disconnect \( M \).

1. Introduction

A translating soliton (translator for short) is an oriented, connected smoothly immersed hypersurface \( x : M^n \rightarrow \mathbb{R}^{n+1} \) on which its mean curvature vector \( H \) satisfies

\[
H = T^\perp,
\]

where \( T \in \mathbb{R}^{n+1} \) is a fixed unit length constant vector and \( T^\perp \) denotes its normal projection onto the normal bundle \( T^\perp M \). Translators are known as type II singularity model of the mean curvature flow and hence classification of translators gives us a better understanding of singularities. In [7], Martin, Savas-Halilaj and Smoczyk studied topological aspects of translators, and they asked whether there exists a complete translator which has finitely many genus when \( \dim M = 2 \). Later, Smith gave an answer of this question. Adopting desingularisation technique, he constructed new complete embedded translators which have arbitrary finite number of genus and three ends. Note that complete embedded translators of infinitely many genus were already known by Nguyen in [9], [10] before the works [7], [13].

On the other hand, translators are critical points of the weighted volume functional \( \mathcal{A}_T(M) \) with respect to the weighted measure \( e^{-\langle x, T \rangle} \, d\text{vol}_M \). In the following, we set \( f(x) := -\langle x, T \rangle \in C^\infty(M) \) and \( d\text{vol}_f := e^{-f} \, d\text{vol}_M \). A translator \( x : M^n \rightarrow \mathbb{R}^{n+1} \) is called \( f \)-stable if the second derivative of \( \mathcal{A}_f \) is nonnegative for all compactly supported normal variations of \( x \). Shahriyari [12] studied the \( f \)-stability of translators and he showed that any translating graph \( M^2 \subset \mathbb{R}^3 \) must be \( f \)-stable. Later, this fact was generalized into higher dimensions by Xin [15]. Clearly, every complete graph has no genus (when \( \dim M = 2 \)) and only one end. Hence, roughly speaking, simple topology implies \( f \)-stability. Our interest is the converse, that is, topological properties of translators under \( f \)-stability. In this direction, Impera and Rimoldi showed the following.

Proposition 1.1 (Impera-Rimoldi [5]). Suppose that \( n \geq 2 \), then any complete \( f \)-stable translator \( x : M^n \rightarrow \mathbb{R}^{n+1} \) has at most only one end.

Their result indicates that \( f \)-stable translators must be topologically simple (at least in the sense of ends). As an immediate consequence, all examples constructed by Smith [13] can not be \( f \)-stable since they have three ends.

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In the present paper we also confirm topological simplicity of $f$-stable translators from a view point of a relation between weighted $L^2$ harmonic 1-forms and codimension one cycles (the details are explained in a later section).

**Theorem 1.2.** Any complete $f$-stable translator $x : M^n \to \mathbb{R}^{n+1}$ admits no codimension one cycle which does not disconnect $M$.

For a surface case $x : M^2 \to \mathbb{R}^3$, our theorem implies that any complete $f$-stable translator has no genus.

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2. Preliminaries

Throughout the paper, we assume that $x : M^n \to \mathbb{R}^{n+1}$ is a codimension one, smooth complete translator with the induced metric $g := x^* \langle \cdot, \cdot \rangle$ and the second fundamental form $A$. Also we assume that $M$ is oriented and connected.

2.1. Stability of translators. As mentioned in the Introduction, a translator is defined by the equation $H = T^\perp$ for some unit length constant vector $T \in \mathbb{R}^{n+1}$. By the usual first variation formula, it is not difficult to see that for a hypersurface being a translator is equivalent to being a critical point of the weighted volume functional $A_T(M)$. Then the $f$-stability (or it is also called $L$-stability) of a translator is defined by

$$\left. \frac{d^2}{dt^2} \right|_{t=0} A_T(x_t(M)) \geq 0$$

for any compactly supported normal variation $x_t : M \to \mathbb{R}^{n+1}$ with $x_0 = x$. Let $V \in T^\perp M$ be a variation vector field of such a compactly supported normal variation $x_t$. Then $f$-stability for translators can be written as

$$\int_M |\nabla^\perp V|^2 - |A|^2 |V|^2 \, dvol_f \geq 0,$$

where $\nabla^\perp$ denotes the normal connection on $T^\perp M$. See [5], [6] or [12] for more details.

2.2. Weighted Laplacian. Let $A^p(M)$ be the space of smooth $p$-forms on $M$ and $A^p_c(M)$ be the space of compactly supported $p$-forms on $M$. For $\alpha, \beta \in A^p_c(M)$, let

$$(\alpha, \beta)_g := \star_g (\alpha \wedge \star_g \beta)$$

be a standard pointwise inner product on forms, where $\star_g$ is the Hodge star operator with respect to $g$. Then for every $\alpha, \beta \in A^p_c(M)$, we define the $L^2_f$ inner product by

$$\langle \alpha, \beta \rangle_{L^2_f} := \int_M (\alpha, \beta)_g \, dvol_f.$$

The completion of $A^p_c(M)$ with respect to the norm $\| \cdot \|_{L^2_f}$ is denoted by $L^2_f A^p(M)$. Then the exterior derivative $d : A^p_c(M) \to A^{p+1}_c(M)$ is densely defined operator on $L^2_f A^p(M)$, and it has the formal adjoint $\delta_f$ of $d$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2_f}$ on $A^{p+1}_c(M)$. Now we define the weighted Laplacian acting on
\[ A^p(M) \text{ by } d\delta + \delta d \text{ which is formally self-adjoint and densely defined on } L^2 d^p(M). \]

Note that \( d\delta + \delta d \) has the unique closed extension which is denoted by \( \Delta_f \) and it is self-adjoint. A \( p \)-form \( \alpha \in A^p(M) \) is called \( L^2 f \)-harmonic if \( \alpha \in L^2 d^p(M) \) and satisfies \( \Delta_f \alpha = 0 \). The space of all such \( p \)-forms is denoted by

\[ \mathcal{H}^p_f(M) := \{ \alpha \in L^2 d^p(M) \mid \Delta_f \alpha = 0 \}. \]

It is known that \( \mathcal{H}^p_f(M) \subseteq A^p(M) \), namely, \( L^2 f \)-harmonic forms are smooth. See [1] for details.

### 3. Weighted harmonic forms and topology

In this section, we assume that \( M \) is oriented connected complete Riemannian manifold, but not necessarily be a translator.

The \( p \)-th de Rham cohomology group with compact support on \( M \) is defined by

\[ H^p_c(M) = \frac{\{ \alpha \in A^p_c(M) \mid d\alpha = 0 \}}{dA^{p-1}_c(M)}. \]

We also consider the space of \( L^2 f \) closed forms:

\[ L^2_c Z^p(M) := \{ \alpha \in L^2 c A^p(M) \mid d\alpha = 0 \}, \]

where it is understood that \( d\alpha = 0 \) holds weakly. Then we define the reduced \( L^2 f \) cohomology by

\[ L^2_c H^p(M) := \frac{L^2_c Z^p(M)}{dA^{p-1}_c(M)}, \]

where \( dA^{p-1}_c(M) \) is the closure of \( dA^{p-1}_c(M) \) with respect to the norm \( || \cdot ||_{L^2_f} \). The reduced \( L^2_f \) cohomology on a complete Riemannian manifold is isomorphic to the space of \( L^2 f \)-harmonic forms (see [1]):

\[ \mathcal{H}^p_f(M) \cong L^2_c H^p(M). \]

#### 3.1. Weighted \( L^2 \) harmonic forms and weighted Sobolev inequality

First, we consider the following approximation theorem.

**Lemma 3.1.** Let \( \alpha \in L^2_c Z^1(M) \cap A^1(M) \) and suppose that \( \alpha \) is zero in \( L^2_f H^1(M) \), namely, there is a sequence of smooth functions \( (u_i)_i \) such that

\[ du_i \to \alpha \quad (i \to \infty) \text{ in } L^2_f, \]

then there exists \( u \in C^\infty(M) \) such that \( \alpha = du \).

**Proof.** By the Poincaré duality, it is enough to show that

\[ \int_M \alpha \wedge \eta = 0 \]

for every closed form \( \eta \in A^{n-1}_c(M) \). For such \( \eta \), define a 1-form \( \theta \in A^1(M) \) by \( \theta := (-1)^{n-1} e^f * g \eta \). Then

\[ \int_M \alpha \wedge \eta = \int_M \alpha \wedge *_g (e^{-f} \theta) = \int_M (\alpha, \theta)_g d\text{vol}_f \]

\[ = \lim_{i \to \infty} \int_M (du_i, \theta)_g d\text{vol}_f = \lim_{i \to \infty} \int_M du_i \wedge \eta = \lim_{i \to \infty} \int_M d(u_i \eta) = 0. \]

\[ \square \]
If the weighted Sobolev inequality holds on \( M \), we obtain the following result as an application of the approximation theorem (Lemma 3.1).

**Proposition 3.2.** Let \( M \) be an \( n \)-dimensional complete Riemannian manifold. Suppose that \( M \) supports the weighted \( L^1 \) Sobolev inequality, that is, there exists a constant \( C > 0 \) such that for every compactly supported nonnegative function \( h \in C_c^\infty(M) \),

\[
\|h\|_{L^1_f} \leq C \|dh\|_{L^1_f}.
\]

Then the natural map

\[
H^1_c(M) \to L^2_f H^1(M) \cong H^1_f(M)
\]

is injective.

**Proof.** It is known by Impera-Rimoldi \([5]\) that every end has infinite \( f \)-volume as a consequence of the weighted \( L^1 \) Sobolev inequality. Also, applying the \( L^1 \) Sobolev inequality to the function \( h = v^{2n} \), \( v \in C_c^\infty(M) \), we obtain the weighted \( L^2 \) Sobolev inequality

\[
\|v\|_{L^2_f} \leq \mu(n) \|dv\|_{L^2_f},
\]

where \( \nu = n + 1 > 2 \) and \( \mu(n) > 0 \) is a constant depending only on \( n \).

Now, take a closed 1-form \( \alpha \in A^1_c(M) \) which is zero in \( L^2_f H^1(M) \). Then there exists a sequence \((u_i)_i \in C_c^\infty(M)\) such that \( du_i \to \alpha \) in \( L^2_f \). Hence by Lemma 3.1 there exists a function \( u \in C^\infty(M) \) such that \( \alpha = du \), and by the weighted \( L^2_f \) Sobolev inequality, \( u_i \to u \) in \( L^2_f \). Since \( du = \alpha = 0 \) in \( M \setminus \text{supp} \alpha \), \( u \) is constant on each connected component of \( M \setminus \text{supp} \alpha \). However, by the weighted \( L^2 \) Sobolev inequality, \( u \) must be zero on each unbounded connected component of \( M \) since each end of \( M \) has infinite \( f \)-volume. Hence \( u \) has compact support and \( \alpha \) is zero in \( H^1_c(M) \). \( \square \)

### 3.2. Weighted \( L^2 \) harmonic forms and cycles

Let \( Z \) be a \( p \)-dimensional oriented, connected, compact manifold without boundary and \( \phi : Z \to M^n \) be an embedding. The pair \((Z, \phi)\) is called a \( p \)-cycle on \( M \). We say that an \((n-1)\)-cycle (or codimension one cycle) \( Z \) does not disconnect \( M \) if \( M \setminus \phi(Z) \) is connected. We consider a relation between \( H^1_c(M) \) and codimension one cycles.

**Proposition 3.3.** Assume that there exists a codimension one cycle \( Z \) which does not disconnect \( M \). Then there exists a closed 1-form \( \alpha \in A^1_c(M) \) and a 1-cycle \( \gamma \) on \( M \) such that

\[
\int_\gamma \alpha = 1.
\]

Therefore \( H^1_c(M) \neq \{0\} \).

**Proof.** This proof is due to the lecture note by Carron \([2]\). Consider an embedding \( F : Z \times [-1,1] \to M \) with \( F_p(0) = F(p,0) = \phi(p) \) for any \( p \in Z \), and set \( N_Z = F(Z \times [-1,1]) \). Then \( M \setminus N_Z \) is connected since \( M \setminus Z \) is connected by the assumption. Now define a function \( \rho : Z \times [-1,1] \to \mathbb{R} \) by

\[
\rho(p,t) = \rho(t) = \begin{cases} 1 & (1/2 \leq t \leq 1), \\ 0 & (-1 \leq t \leq -1/2). \end{cases}
\]
Clearly, $d\rho$ has a compact support on $Z \times (-1, 1)$. Hence we can extend $d(F^{-1})^* \rho$ to a closed 1-form $\alpha$ defined on whole $M$ with compact support in $N_Z$.

For a fixed point $p_0 \in Z$, we can take a continuous curve $c : [0, 1] \to M \setminus N_Z$ joining $F(p_0, 1)$ and $F(p_0, -1)$ since $M \setminus N_Z$ is connected. Define a closed curve (a 1-cycle) $\gamma$ on $M$ by

$$
\gamma(t) = \begin{cases} F_{p_0}(t) := F(p_0, t) & (-1 \leq t \leq 1), \\ c(t-1) & (1 \leq t \leq 2). \end{cases}
$$

It is not difficult to see that

$$
\int_\gamma \alpha = 1,
$$

hence $\alpha$ is not exact. □

Combining Proposition 3.2 and Proposition 3.3, we immediately conclude the following.

**Corollary 3.4.** Assume that $M$ supports the weighted $L^1$ Sobolev inequality in Proposition 3.2 and admits a codimension one cycle which does not disconnect $M$, then $H^1_f(M) \neq \{0\}$.

In [5], Impera and Rimoldi showed that every complete translator $x : M^n \to \mathbb{R}^{n+1}, n \geq 2$ supports the weighted $L^1$ Sobolev inequality. Hence, in particular, we conclude the following.

**Corollary 3.5.** If a complete translator $x : M^n \to \mathbb{R}^{n+1}$ admits a codimension one cycle which does not disconnect $M$, then $H^1_f(M) \neq \{0\}$.

3.3. **Another method.** We may adopt another proof of Corollary 3.5 without the Sobolev inequality. The following proposition is essentially due to Gaffney [4] where he used the heat equation method. Most of his method can be applicable to our weighted Laplacian case straightforwardly since the method is based on general theory for self-adjoint operators. The only thing we need to do is to modify the proof such as Lemma 3.1 when we use the Poincaré duality.

**Proposition 3.6.** For any closed $p$-form $\alpha \in \mathcal{A}_p^c(M)$ and any $p$-cycle $Z$ on $M$,

$$
\int_Z \alpha = \int_Z H_f \alpha,
$$

where $H_f \alpha$ denotes the $f$-harmonic part of $\alpha$.

Combining Proposition 3.2 and Proposition 3.6 we recover the result by Dodziuk [3] in our weighted case. Note that $M$ does not need to be a translator in the following result.

**Corollary 3.7.** Assume that $M$ admits a codimension one cycle $Z$ which does not disconnect $M$, then $H^1_f(M) \neq \{0\}$.

4. **$f$-stability and weighted $L^2$ harmonic 1-forms**

In this section, we prove the main theorem:

**Theorem 4.1.** Any complete $f$-stable translator $x : M^n \to \mathbb{R}^{n+1}$ admits no codimension one cycle which does not disconnect $M$. 
where $f$ is the shape operator of the hypersurface. Note that $\Delta f = 0$, we have
\[ 2|\Delta f\xi| - 2|\nabla\xi|^2 = \Delta f|\xi|^2 = -2Ric_f(\xi, \xi) - 2|\nabla\xi|^2, \]
where $Ric_f := Ric + Hess_f$ is Bakry-Émery Ricci tensor on $M$. Set
\[ K(\xi) := |\nabla\xi|^2 - |\nabla\xi|^2, \]
\[ W(\xi) := |A|^2|\xi|^2 - (S(\xi), S(\xi)), \]
where $S$ is the shape operator of the hypersurface. Note that $K(\xi) \geq 0$ by Kato’s inequality. Also $W(\xi) \geq 0$ in general. Using the Gauss equation, we compute
\[ (2) \quad |\xi|\Delta f|\xi| = -Ric_f(\xi, \xi) - K(\xi) = (S(\xi), S(\xi)) - K(\xi) = |A|^2|\xi|^2 - W(\xi) - K(\xi). \]
Now we take a cutoff function $\eta \in C^\infty(M)$ which has the property
- (a) $0 \leq \eta \leq 1$,
- (b) $\eta \equiv 1$ on $B_R$, $\eta \equiv 0$ outside $B_R$,
- (c) $|\nabla\eta|^2 \leq \frac{C}{R^2}$, $C$ is a constant independent of $R$,
where $B_R$ is a geodesic ball on $M$ with radius $R > 0$. Consider a variation vector field as $V = \eta|\xi|\nu$, where $\nu$ is a unit normal of the hypersurface. From the stability inequality \[ \Box \], the equality \[ \Box \] and the Stokes theorem \[ \langle \Delta f u, v \rangle_{L_2^f} = \langle du, dv \rangle_{L_2^f} \] for $u, v \in C^\infty_0(M)$, we compute
\[ 0 \leq \int_M |\nabla^2 V|^2 - |A|^2|V|^2 \, dvol_f \]
\[ = \int_M |\nabla(\eta|\xi|)|^2 - |A|^2\eta^2|\xi|^2 \, dvol_f \]
\[ = \int_M \eta|\xi|\Delta f(\eta|\xi|) - |A|^2\eta^2|\xi|^2 \, dvol_f \]
\[ = \int_M \eta|\xi|(\eta\Delta f|\xi| + |\xi|\Delta f \eta - 2\langle \nabla\eta, \nabla|\xi| \rangle) - |A|^2\eta^2|\xi|^2 \, dvol_f \]
\[ = \int_M \eta^2(|\xi|\Delta f|\xi| - |A|^2|\xi|^2) - 2\eta|\xi|(\nabla|\xi|, \nabla\eta) + (|\xi|^2\eta)\Delta f \eta \, dvol_f \]
\[ = \int_M -\eta^2(W(\xi) + K(\xi)) + |\xi|^2|\nabla\eta|^2 \, dvol_f \]
\[ \leq \int_{B_R} -(W(\xi) + K(\xi)) \, dvol_f + \frac{C}{R^2} \int_{B_R} |\xi|^2 \, dvol_f. \]
Letting $R \to \infty$ and using $\alpha \in L^2_\infty A^1(M)$, $W \equiv 0$ and $K \equiv 0$ on whole $M$. Since $\xi$ is not identically zero, we choose a point $p \in M$ where $|\xi(p)| \neq 0$. We may take a
sufficiently small neighborhood $U$ of $p$ on which $|\xi| > 0$ holds. Let \( \{e_1, \ldots, e_n\} \) be an orthonormal frame on $U$ such that $e_1 = \xi/|\xi|$. Then $W \equiv 0$ implies
\[
|A|^2 = \langle S(e_1), S(e_1) \rangle = \langle A(e_1, e_1), A(e_1, e_1) \rangle = a_{11}^2 + \cdots + a_{1n}^2,
\]
where $a_{ij} = A(e_i, e_j)$. Since $A$ is symmetric, $a_{ij} = 0$ except for $a_{11}$. It follows that the scalar curvature of $M$ must be zero. However it is known that a zero scalar curvature translator is only a hyperplane or a grim reaper cylinder by Martín, Savas-Halilaj and Smoczyk (Corollary 2.1 in [7]). This contradicts our assumption since both of them never have codimension one cycle which does not disconnect $M$ by Jordan-Brouwer separation theorem.

\[\square\]

**Remark 4.2.** Our proof of the main theorem may be used to prove Proposition 1.3 by a slightly different way. In fact, if a complete Riemannian manifold $M$ has at least two ends, then $H^2_1(M) \neq \{0\}$ (see the lecture note by Carron [2] for the proof). In particular, for a translator $x : M^n \to \mathbb{R}^{n+1}$, $H^2_1(M) \neq \{0\}$ by Proposition 3.2. However it leads to a contradiction for an $f$-stable translator by the same way as Theorem 1.1.

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