ON THE CONVERGENCE OF THE DRAINAGE NETWORK WITH BRANCHING

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ABSTRACT. The Drainage Network is a system of coalescing random walks exhibiting long range dependence before coalescence that was introduced by Gangopadhyay, Roy and Sarkar [12]. Coletti, Fontes and Dias [5] proved its convergence to the Brownian Web under diffusive scaling. In this work we introduce a perturbation of the system allowing branching of the random walks with low probabilities varying with the scaling parameter. According to the specification of the branching probabilities, we show that this drainage network with branching either converges to the Brownian Web or consists of a tight family such that any weak limit point contains a Brownian Net. In the latter case, we conjecture that the limit is indeed the Brownian Net.

Keywords: System of coalescing random walks, Drainage Network, Drainage Network with branching, Brownian Web, Brownian Net.

Contents

1. Introduction .............................................. 2
2. Brownian Web and Brownian Net ......................... 3
   2.1. The space of compact sets of paths .................. 3
   2.2. The Brownian Web and its dual .................... 4
   2.3. Convergence criteria for the Brownian Web ......... 5
   2.4. The construction of the Brownian Net ............... 7
   2.5. Convergence criteria for the Brownian Net ...... 9
3. Drainage Network with branching ........................ 11
   3.1. Model description and main results ................ 11
   3.2. Dual process for Drainage Network with branching 15
4. Estimates for coalescence times .......................... 17
5. Convergence of the finite-dimensional distributions ... 22
   5.1. Part I: Only one pair \( (L_t^{(n)}, R_t^{(n)}) \) 23

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1. Introduction

Here we study convergence in distribution for a diffusively scaled system of non-crossing one-dimensional branching-coalescing random walks starting at each point in the space and time lattice \( \mathbb{Z} \times \mathbb{Z} \) that presents long range dependence before coalescence. To not get repetitive, when we make mention to the convergence of a system of random walks, we always consider one-dimensional random walks and diffusive space-time scaling. For systems of coalescing random walks without branching and independence before coalescence, the limiting system is the so called Brownian Web introduced by Fontes et al \[8, 9\], although formal descriptions of systems of coalescing Brownian motions had previously been considered, initially by Arratia \[1, 2\] and then by Tóth and Werner \[22\]. Since \[8, 9\] convergence in distribution of systems of coalescing random walks to the Brownian Web and its variants have been an object of extensive study as we can see for instance in \[4, 7, 10, 13, 15, 17\].

Motivated by an application that appears in Scheidegger \[18\] about drainage pattern into an intramontane trench which is related to river networks, Roy, Saha, and Sarkar in \[12\] introduced the Drainage Network, see also \[16\]. It arises as a natural model for systems of coalescing random walks with dependence before coalescence. The model can be described as follows: Each site in \( \mathbb{Z}^2 \) is independently open or closed with a given probability \( p \in (0, 1) \); from each site \( (x, y) \in \mathbb{Z}^2 \) departs a directed edge towards the nearest open site in \( \mathbb{Z} \times \{y + 1\} \); If we have two nearest open sites, necessarily of the form \( (x + z, y + 1) \) and \( (x - z, y + 1) \) for some \( z \in \mathbb{Z} \), then we choose one of them with equal probability. Considering the first coordinate as space and the second one as time, we obtain a system of coalescing random paths called the Drainage Network. Its convergence to Brownian Web was proved first by Coletti, Fontes and Dias \[5\] and after that by Coletti and Valle \[6\] for a more general version of the model that allows paths to cross each other.

The convergence of systems of branching-coalescing random walks have been later studied by sun and Swart \[21\]. They considered systems of coalescing random walks evolving independently before coalescence with a superposed dynamics that allows branching of paths...
with a small probability inversely proportional to the scaling parameter. They also characterized the diffusive limit as a system of branching-coalescing Brownian motions which they called the Brownian Net.

Considering again the Drainage Network, it is natural to consider an extension of the Drainage Network where branching can occur: when there are two nearest open sites as destination for an edge departing from a given site, we allow both edges to exist in the system with some given probability $\epsilon_n$, where $n$ is a scaling parameter. We call $\epsilon_n$ the branching probability. Trajectories of the Drainage Network with branching also exhibit long range dependence. In this work, we study the convergence of the Drainage Network with branching under diffusive scaling to the Brownian Web or Net under specific conditions for the branching probability. We show that based on the specification of $\epsilon_n$, we can have convergence to the Brownian Web or we can have a tight family such that any weak limit point contains a Brownian Net. This latter case corresponds to the one where $\epsilon_n$ is inversely proportional to $n$ and we conjecture that the limit is indeed the Brownian Net, but we were not able to prove it in this paper.

Our result contributes to the understanding of the universality class related to the Brownian Net, that is, we believe that like the Brownian Web, the convergence for Brownian Net is robust and all we need is the asymptotic independence of the trajectories in the system and proper moment conditions.

This work is organized as follows: In Section 2 we describe the Brownian Web and Brownian Net. In Section 3 we describe the Drainage Network with Branching and its dual, also stating our main results. In Section 4 some basic properties of the Drainage Network with branching are obtained. Finally, in Sections 5, 6 and 7 the main theorems are proved.

2. Brownian Web and Brownian Net

2.1. The space of compact sets of paths.

The Brownian Web and the Brownian Net are random elements of a proper metric space introduced in [8, 9]. It is a space of sets of space-time continuous paths on a compactification of the space-time plane $\mathbb{R}^2$ endowed with the Hausdorff topology based on the uniform metric on the compactification. We follow the definition in [21] which is equivalent but slightly different from that in [10], for details see [21, Appendix].

Let $\mathbb{R}^2$ be the compactification of the space-time plane $\mathbb{R}^2$ under the metric $\rho$ where:

$$\rho((x_1, t_1), (x_2, t_2)) = |\tanh(t_1) - \tanh(t_2)| \lor \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right|,$$

(2.1)

and $\tanh(x)$ is the hyperbolic tangent of $x$: $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. More precisely, consider $([-\infty, \infty] \times (-\infty, \infty)) \cup \{(\ast, \pm\infty)\}$, endowed with a topology such that $(x_n, t_n) \to (\ast, \pm\infty)$ whenever $t_n \to \pm\infty$. Then $\mathbb{R}^2$ can be thought as the continuous image of this space under
the map
\[(x, t) \in [-\infty, \infty] \times (-\infty, \infty) \mapsto \left(\frac{\tanh(x)}{1 + |t|}, \tanh(t)\right), \quad (2.2)\]
\[(*, -\infty) \mapsto (0, -1) \text{ and } (*, \infty) \mapsto (0, 1).\]

Note that this mapping gives a region contained in the square \([-1, 1] \times [-1, 1]\).

A path \(\pi\) in \(\mathbb{R}^2\) with starting time denoted by \(\sigma_\pi \in [-\infty, \infty]\), is a map \(\pi: [\sigma_\pi, \infty) \rightarrow [-\infty, \infty] \cup \{\ast\}\) such that \(\pi(\infty) = \ast, \pi(\sigma_\pi) = \ast\) if \(\sigma_\pi = -\infty\) and \(t \mapsto (\pi(t), t)\) is a continuous map from \([\sigma_\pi, \infty]\) to \((\mathbb{R}^2, \rho)\). Then define \(\Pi\) to be the space of all paths \(\pi\) in \(\mathbb{R}^2\) endowed with the metric \(d\), where \(d(\pi_1, \pi_2)\) is given by

\[|\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})| \vee \sup_{t \geq (\sigma_{\pi_1}, \hat{\sigma}_{\pi_2})} \left|\frac{\tanh(\pi_1(t \vee \sigma_{\pi_1})) - \tanh(\pi_2(t \vee \sigma_{\pi_2}))}{1 + |t|}\right|. \quad (2.3)\]

We have that \((\Pi, d)\) is a complete separable metric space.

Finally, we are able to define the space \((\mathcal{H}, d_\mathcal{H})\), where the Brownian Web and Net will be defined. Consider \(K \subset \Pi\) and let \(\mathcal{H}\) denote the space of compact subsets of \((\Pi, d)\), under the Hausdorff metric \(d_\mathcal{H}\), that is:

\[d_\mathcal{H}(K_1, K_2) = \sup_{\pi_1 \in K_1} \inf_{\pi_2 \in K_2} d(\pi_1, \pi_2) \vee \sup_{\pi_2 \in K_2} \inf_{\pi_1 \in K_1} d(\pi_1, \pi_2), \quad K_1, K_2 \in \mathcal{H}. \quad (2.4)\]

We have that \((\mathcal{H}, d_\mathcal{H})\) is also a complete separable metric space.

Letting \(\mathcal{B}_\mathcal{H}\) denote the Borel \(\sigma\)-algebra associated with the metric \(d_\mathcal{H}\), we will construct the Brownian Web \(\mathcal{W}\) and Brownian Net \(\mathcal{N}\) as random elements defined in \((\mathcal{H}, \mathcal{B}_\mathcal{H})\).

### 2.2. The Brownian Web and its dual.

The Brownian Web \(\mathcal{W}\) is a random element of \((\mathcal{H}, \mathcal{B}_\mathcal{H})\) characterized by the following:

**Proposition 2.1.** ([9, Theorem 2.1]) There exists an \((\mathcal{H}, \mathcal{B}_\mathcal{H})\)-valued random variable \(\mathcal{W}\), called the standard Brownian Web, whose distribution is uniquely determined by the following properties:

- (a) From any deterministic point \(z \in \mathbb{R}^2\), there is almost surely a unique path \(\pi_z \in \mathcal{W}\) starting from \(z\);
- (b) For any finite deterministic set of points \(z_1, \ldots, z_k \in \mathbb{R}^2\), the collection \((\pi_{z_1}, \ldots, \pi_{z_k})\) is distributed as coalescing standard Brownian motions;
- (c) For any deterministic countable dense subset \(D \subset \mathbb{R}^2\), almost surely, \(\mathcal{W}\) is the closure of \(\{\pi_z : z \in D\}\) in \((\Pi, d)\).

The characterization in Proposition 2.1 can be extended to allow the Brownian paths to have a fixed diffusion coefficient \(\lambda^2 \neq 1\) and a drift \(b \neq 0\). The only difference is in the property (b), where the coalescing Brownian motions may have a diffusion coefficient distinct from one and a non-zero drift. We denote by \(\mathcal{W}_{\lambda, b}\) the Brownian Web with diffusion coefficient \(\lambda^2 > 0\) and drift \(b \in \mathbb{R}\) and \(\mathcal{W}_{\lambda, b} = \mathcal{W}_{\lambda}\) if \(b = 0\), see [21, Theorem 1.5] for a formal
construction. As a matter of fact, one can show that \( W_{\lambda,b} \) can be obtained as the image of \( \mathcal{W} \) by a proper map on \((\mathcal{H}, d_{\mathcal{H}})\). This map is naturally obtained from

\[
(\pi(t))_{t \geq \sigma_\pi} \in \Pi \mapsto (\lambda \pi(t) + b(t - \sigma_\pi))_{t \geq \sigma_\pi} \in \Pi.
\]

The Brownian Web \( \mathcal{W} \) has a dual process \( \hat{\mathcal{W}} \) which is called the dual Brownian Web. \( \hat{\mathcal{W}} \) is a collection of coalescing paths running backward in time which is uniquely determined by the restriction that these paths cannot cross any path from \( \mathcal{W} \).

Let us describe the space where \( \hat{\mathcal{W}} \) takes its values. Given a set \( A \subset \mathbb{R}^2 \), denote \( -A = \{-z : z \in A\} \). Identifying each path \( \pi \in \Pi \) with its graph as a subset of \( \mathbb{R}^2 \), \( \hat{\pi} = -\pi \) defines a path running backward in time, with starting time \( \hat{\sigma}_\pi = -\sigma_\pi \). Let \( \hat{\Pi} = -\Pi \) denote the set of all these backward paths endowed with a metric \( \hat{d} \) inherited from \((\Pi, d)\) under the sign change mapping. Let \( \hat{\mathcal{H}} \) be the space of compact subsets of \((\hat{\Pi}, \hat{d})\) endowed with the Hausdorff metric \( d_{\mathcal{H}} \) and Borel \( \sigma \)-algebra \( \mathcal{B}_{\mathcal{H}} \). For any \( K \in \mathcal{H} \), \(-K\) denote the set \( \{ -\pi : \pi \in K \} \in \hat{\mathcal{H}} \).

The next proposition characterizes the joint law of the Brownian Web \( \mathcal{W} \) and its dual \( \hat{\mathcal{W}} \) as random elements taking values in \( \mathcal{H} \times \hat{\mathcal{H}} \) equipped with the product \( \sigma \)-algebra.

**Proposition 2.2.** ([20, Theorem 2.4]) There exists an \( \mathcal{H} \times \hat{\mathcal{H}} \)-valued random element \((\mathcal{W}, \hat{\mathcal{W}})\), called the double Brownian Web (with \( \hat{\mathcal{W}} \) called the dual Brownian Web), whose distribution is uniquely determined by the following properties:

(a) \( \mathcal{W} \) and \( \hat{\mathcal{W}} \) are both distributed as the standard Brownian Web;

(b) Almost surely, no path \( \pi_z \in \mathcal{W} \) crosses any path \( \hat{\pi}_z \in \hat{\mathcal{W}} \), i.e., there are no \( z = (x,t) \) and \( \hat{z} = (\hat{x},\hat{t}) \) with \( t < \hat{t} \) such that \((\pi_z(s_1) - \hat{\pi}_z(s_1))(\pi_z(s_2) - \hat{\pi}_z(s_2)) < 0 \) for some \( t < s_1 < s_2 < \hat{t} \).

Furthermore, for each \( z \in \mathbb{R}^2 \), \( \hat{\mathcal{W}}(z) \) a.s. consists of a single path \( \hat{\pi}_z \) which is the unique in \( \hat{\Pi} \) that does not cross any path in \( \mathcal{W} \). Thus \( \hat{\mathcal{W}} \) is a.s. determined by \( \mathcal{W} \) and vice versa.

Analogously, we can define the double Brownian Web with diffusion coefficient \( \lambda^2 \) and drift \( b \), which we denote by \((\mathcal{W}_{\lambda,b}, \hat{\mathcal{W}}_{\lambda,b})\).

### 2.3. Convergence criteria for the Brownian Web.

The convergence criterion for the Brownian Web were originally described in [9, Theorem 2.2], except for a tightness condition which we call below Condition (T). This condition is unnecessary for systems with non-crossing paths (Tightness for these systems is proved in [9, Proposition B2]). The condition (T) appears in [13, Theorem 1.4] for the first time.

Consider a sequence of \((\mathcal{H}, \mathcal{B}_{\mathcal{H}})\)-valued random variables \((\mathcal{Z}_n)_{n \in \mathbb{N}}\). There are four conditions that, if satisfied, guarantees that \((\mathcal{Z}_n)_{n \in \mathbb{N}}\) converges to a Brownian Web \( \mathcal{W}_\lambda \):

**Condition (T):** [Tightness criterion] The law of a sequence of \((\mathcal{H}, \mathcal{B}_{\mathcal{H}})\)-valued random variables \((\mathcal{Z}_n)_{n \in \mathbb{N}}\) is tight if for all \( L > 0, T > 0 \) and \( \eta > 0 \),

\[
\lim_{\delta \downarrow 0} \delta^{-1} \lim_{n \to \infty} \sup_{(x,t) \in [-L,L] \times [-T,T]} P(\mathcal{Z}_n \in A_{\delta,\eta}(x,t)) = 0,
\]
where \(A_{\delta,\eta}(x,t)\) is the event (in \(\mathcal{B}_H\)) that \(K \in \mathcal{H}\) contains some path which intersects the rectangle \([x-\frac{\eta}{2}, x+\frac{\eta}{2}] \times [t, t+\frac{\delta}{2}]\), and at a later time, intersects the left or right boundary of a larger rectangle \([x-\eta, x+\eta] \times [t, t+\delta]\).

**Condition (I):** There exists \(\pi_{n,z} \in \mathcal{Z}_n\) for each \(z \in \mathbb{R}^2\), such that for any deterministic \(z_1, \ldots, z_k \in \mathbb{R}^2\), \((\pi_{n,z_i})_{1 \leq i \leq k}\) converge in distribution to coalescing Brownian motions with diffusion coefficient \(\lambda^2\) starting at \((z_i)_{1 \leq i \leq k}\).

**Condition (B1):** For all \(t > 0\)

\[
\limsup_{\delta \downarrow 0} \limsup_{n \to \infty} \sup_{a,t_0 \in \mathbb{R}} P[\eta_{Z_n}(t_0, t; a, a+\delta) \geq 2] = 0,
\]

where given a \((\mathcal{H}, \mathcal{B}_H)\)-valued random element \(X, t_0 \in \mathbb{R}, t > 0, a < b, \eta_X(t_0, t; a, b) = |\{\pi(t_0 + t) : \pi \in X, \pi(t_0) \in [a,b]\}|\), which counts the number of distinct paths in \(X\) at time \(t_0 + t\), among all paths that start from interval \([a,b]\) at time \(t_0\).

**Condition (B2):** For all \(t > 0\)

\[
\limsup_{\delta \downarrow 0} \frac{1}{\delta} \limsup_{n \to \infty} \sup_{a,t_0 \in \mathbb{R}} P[\eta_{Z_n}(t_0, t; a, a+\delta) \geq 3] = 0.
\]

During this work, we will need to use slightly different conditions in place of (B1) and (B2), which we call respectively (B1*) and (B2*).

**Condition (B1*):** If \(Z\) is any subsequential weak limit of \(Z_n\), we have that for all \(t > 0\)

\[
\limsup_{\delta \downarrow 0} \sup_{a,t_0 \in \mathbb{R}} P[\eta_{Z}(t_0, t; a, a+\delta) \geq 2] = 0.
\]

**Condition (B2*):** If \(Z\) is any subsequential weak limit of \(Z_n\), we have that for all \(t > 0\)

\[
\limsup_{\delta \downarrow 0} \frac{1}{\delta} \sup_{a,t_0 \in \mathbb{R}} P[\eta_{Z}(t_0, t; a, a+\delta) \geq 3] = 0.
\]

Conditions (B1*) and (B2*) are more convenient when the discrete system can have more than one path starting from each vertex. However, (B1*) does not imply (B1) and (B2*) does not imply (B2), only the opposite holds. Following the proof of [9, Lemma 5.1], we see that (B1*) and (B2*) can indeed replace (B1) and (B2) to guarantee that any limit point is a system of coalescing Brownian motions with the minimum number of paths, thus it should be the Brownian Web.

To conclude, it will also be useful in our proofs that Condition (B2) can be replaced by a condition called (E′) described below, [13, Theorem 1.4 and Lemma 6.1]. It was introduced in [13] to prove Condition (B2) for systems that allow crossings of their paths.

**Condition (E′):** Let \(Z_{t_0}^n\) be the subset of paths in \(Z_n\) which starts before or at time \(t_0\). If \(Z_t^{t_0}\) is any subsequential limit of \(Z_{t_0}^n\) for any \(t_0 \in \mathbb{R}\), then \(\forall t, a, b \in \mathbb{R}\) with \(t > 0\) and \(a < b\),
where we have:

\[ E[\hat{\eta}_Z(t_0, t, a, b)] \leq E[\hat{\eta}_\mathcal{W}(t_0, t, a, b)] = \frac{b - a}{\sqrt{\pi t}}, \]

where

\[ \hat{\eta}_Z(t_0, t; a, b) = |\{\pi(t_0 + t) \cap (a, b) : \pi \in Z, \pi(t_0) \in \mathbb{R}\}|, \]

which counts the number of points in interval \((a, b)\) touched at time \(t_0 + t\) by paths in \(Z\) starting before or at time \(t_0\).

### 2.4. The construction of the Brownian Net.

Here we define the Brownian Net which generalizes the Brownian Web by allowing paths to branch. It was introduced by Rongfeng Sun and J. M. Swart in [21]. The first step is to describe what is called the left-right Brownian Web. Consider a collection of space-time random paths characterized as a random element \((l_z, r_z)_{z \in \mathbb{R}^2}\) of \(\mathcal{H} \times \mathcal{H}\) where \(l_z\) and \(r_z\) start both at \(z\) for each \(z \in \mathbb{R}^2\). We call \(l_z\) an \(l\)-path (left path) and \(r_z\) an \(r\)-path (right path). Also consider the following finite dimensional distributions: For any two finite collection of points \((z_i)_{1 \leq i \leq k}\) and \((z'_i)_{1 \leq i \leq k'}\) in \(\mathbb{R}^2\)

- The paths \((l_{z_1}, \ldots, l_{z_k}, r_{z'_1}, \ldots, r_{z'_{k'}})\) evolve independently until they meet each other.
- The \(l\)-paths \((l_{z_1}, \ldots, l_{z_k})\) coalesce when they meet and the same is true for the \(r\)-paths \((r_{z'_1}, \ldots, r_{z'_{k'}})\).
- Each pair of left-right paths \((l_{z_i}, r_{z'_j})\) solves the following system of SDEs:

\[
\begin{align*}
    dL_t &= I_{\{L_t \neq R_t\}} dB^l_t + I_{\{L_t = R_t\}} dB^*_t - dt, \\
    dR_t &= I_{\{L_t \neq R_t\}} dB^r_t + I_{\{L_t = R_t\}} dB^*_t + dt,
\end{align*}
\]

where the \(l\)-path \(L\) and the \(r\)-path \(R\) are driven by independent Brownian motions \(B^l\) and \(B^r\) when they are apart and driven by the same Brownian motion \(B^*\) (independent of \(B^l\) and \(B^r\)) when they coincide. Furthermore, \(L\) and \(R\) have the restriction that \(L_t \leq R_t\) for all \(t \geq \inf\{u \geq \sigma_L \lor \sigma_R : L_u \leq R_u\}\) where \(\sigma_L\) and \(\sigma_R\) being the starting times of \(L\) and \(R\).

The pair of SDEs (2.5) has a unique solution and the above properties uniquely determine the joint law of \((l_{z_1}, \ldots, l_{z_k}, r_{z'_1}, \ldots, r_{z'_{k'}})\), which are called \textit{left-right coalescing Brownian motions}.

Extending the starting points to a countable dense set in \(\mathbb{R}^2\) and then taking the closure of the resulting set of \(l\)-paths and \(r\)-paths a.s. determines a random element of \(\mathcal{H}^2\) denoted by \((\mathcal{W}^l, \mathcal{W}^r)\), which is called the \textit{left-right Brownian Web}.

The left-right Brownian Web is characterized together with its dual by the following result, presented in [20, Theorem 3.2] and based in the theorem that gave the definition of this object for the first time, presented in [21, Theorem 1.5].

**Proposition 2.3. (Theorem 1.5 of [21])** There exists an \(\mathcal{H}^2\)-valued random variable \((\mathcal{W}^l, \mathcal{W}^r)\), called the (standard) left-right Brownian Web, whose distribution is uniquely determined by the following properties:
(i) For each deterministic \( z \in \mathbb{R}^2 \), \( \mathcal{W}^l \) and \( \mathcal{W}^r \) almost surely contain a single path each, that starts from the vertex \( z \);

(ii) For any finite deterministic set of points \( z_1, \ldots, z_k, z'_1, \ldots, z'_{k'} \in \mathbb{R}^2 \), the collection of paths \((l_{z_1}, \ldots, l_{z_k}, r'_{z'_1}, \ldots, r'_{z'_{k'}})\) is distributed as a family of left-right coalescing Brownian motions.

(iii) For any deterministic countable dense sets \( D^l, D^r \subset \mathbb{R}^2 \),

\[
\mathcal{W}^l = \{l_z : z \in D^l\} \quad \text{and} \quad \mathcal{W}^r = \{r_z : z \in D^r\} \quad \text{a.s.}
\]

Recall that there exists a dual left-right Brownian Web \((\widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r) \in \mathcal{H}^2\), such that \((\mathcal{W}^l, \mathcal{W}^r)\) (resp. \((\mathcal{W}^r, \mathcal{W}^l)\)) is distributed as \((\mathcal{W}, \mathcal{W})\) tilted with drift \(-1\) (resp. \(+1\)). From Section 1.8 in [21], we have that \(-(\mathcal{W}^l, \mathcal{W}^r, \widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r))\) and \((\widehat{\mathcal{W}}^l, \widehat{\mathcal{W}}^r, \mathcal{W}^l, \mathcal{W}^r))\) are identically distributed.

We can extend Proposition 2.3 to the non standard case. For fixed constants \( \lambda > 0 \) and \( b > 0 \), we denote by \((\mathcal{W}^l_{\lambda,b}, \mathcal{W}^r_{\lambda,b})\) the left-right Brownian Web with diffusion coefficient \( \lambda^2 \) and drift coefficient \( b \). Here we consider that the independent Brownian motions \( B^l, B^r \) and \( B^s \) have diffusion coefficient \( \lambda^2 \) and that in place of (2.5) we have

\[
\begin{align*}
\text{d}L_t &= I_{(L_t \neq R_t)} dB^l_t + I_{(L_t = R_t)} dB^s_t - b \text{d}t, \\
\text{d}R_t &= I_{(L_t \neq R_t)} dB^r_t + I_{(L_t = R_t)} dB^s_t + b \text{d}t.
\end{align*}
\]

This same extension can be done with the double left-right Brownian Web (i.e. the joint process of the left-right Brownian Web and its dual) \((\mathcal{W}^l_{\lambda,b}, \mathcal{W}^r_{\lambda,b}, \widehat{\mathcal{W}}^l_{\lambda,b}, \widehat{\mathcal{W}}^r_{\lambda,b})\).

Now we are ready to construct the Brownian Net \( \mathcal{N}_{\lambda,b} \) by allowing paths hopping between paths in the left-right Brownian Web \((\mathcal{W}^l_{\lambda,b}, \mathcal{W}^r_{\lambda,b})\). If two paths \( \pi_1 \) and \( \pi_2 \) satisfy \( \pi_1(t) = \pi_2(t) \), we say that a path \( \pi \) is obtained by hopping from \( \pi_1 \) to \( \pi_2 \) at time \( t \) when \( \pi = \pi_1 \) on \([\sigma_{\pi_1}, t]\) and \( \pi = \pi_2 \) on \([t, \infty)\). We also say that a time \( t > \sigma_{\pi_1} \cup \sigma_{\pi_2} \) is a crossing time between \( \pi_1 \) and \( \pi_2 \) only if these two paths cross each other at this time, i.e., there exist times \( t^- < t^+ \) satisfying \((\pi_1(t^-) - \pi_2(t^-))(\pi_1(t^+) - \pi_2(t^+)) < 0\) and

\[
\begin{align*}
t &= \inf_{s \in (t^-, t^+)} \{(\pi_1(t^-) - \pi_2(t^-))(\pi_1(s) - \pi_2(s)) < 0\}.
\end{align*}
\]

Given a set of paths \( K \in \mathcal{H} \), denote by \( \mathcal{H}_{\text{cross}}(K) \) the set of paths obtained by hopping a finite number of times among paths in \( K \) at crossing times. The Brownian Net is a \((\mathcal{H}, \mathcal{B}_\mathcal{H})\)-valued random variable that can be constructed by setting \( \mathcal{N}_{\lambda,b} = \mathcal{H}_{\text{cross}}(\mathcal{W}^l_{\lambda,b} \cup \mathcal{W}^r_{\lambda,b}) \).

There are several ways to characterize the Brownian Net: hopping characterization, wedge characterization and mesh characterization (see [21]), but we will focus only in the first two characterizations which are directly related to the convergence study in this work. We begin with the hopping characterization. The following result is from [21, Theorem 1.3].

**Proposition 2.4. (Theorem 1.3 of [21])** For every \( \lambda > 0 \) and \( b > 0 \), there exists an \((\mathcal{H}, \mathcal{B}_\mathcal{H})\)-valued random variable \( \mathcal{N}_{\lambda,b} \), called Brownian Net with diffusion coefficient \( \lambda^2 \) and drift coefficient \( b \), whose distribution is uniquely determined by the following properties:
(a) For each $z \in \mathbb{R}^2$, $\mathcal{N}_{\lambda,b}$ a.s. contains a unique $l$-path $l_z$ and $r$-path $r_z$ that start at $z$;
(b) For any finite deterministic set of points $z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{R}^2$, the random vector of paths $(l_{z_1}, \ldots, l_{z_k}, r_{z'_1}, \ldots, r_{z'_k})$ is distributed as a family of left-right coalescing Brownian motions with diffusion coefficient $\lambda^2$ and drift coefficient $b$;
(c) For any deterministic countable dense sets $D^l, D^r \subset \mathbb{R}^2$,
\[
\mathcal{N}_{\lambda,b} = \mathcal{H}_{\text{cross}}(\{l_z : z \in D^l\} \cup \{r_z : z \in D^r\}) \quad \text{a.s.}
\]

The wedge construction is based on the existence of certain forbidden regions in the space-time plane where the Brownian Net paths cannot enter. These regions are called wedges and they are determined by the dual left-right Brownian Web $(\hat{W}^l_{\lambda,b}, \hat{W}^r_{\lambda,b})$ as follows:

**Definition 2.5.** Let $(\hat{W}^l_{\lambda,b}, \hat{W}^r_{\lambda,b}, \hat{W}^l_{\lambda,b}, \hat{W}^r_{\lambda,b})$ be a double left-right Brownian Web. For any path $\hat{r} \in \hat{W}^r_{\lambda,b}$ and $\hat{l} \in \hat{W}^l_{\lambda,b}$ that are ordered with $\hat{r}(s) < \hat{l}(s)$ at the time $s = \hat{\sigma}_z \wedge \hat{\sigma}_l$, let $T = \sup\{\hat{r}(t) = \hat{l}(t)\}$ be the first hitting time of $\hat{r}$ and $\hat{l}$ (possibly equals $-\infty$, in case they never meet). We call the open set:
\[
\Psi = \Psi(\hat{r}, \hat{l}) = \{(x, u) \in \mathbb{R}^2 : T < u < s, \hat{r}(u) < x < \hat{l}(u)\}
\]
a wedge of $(\hat{W}^l_{\lambda,b}, \hat{W}^r_{\lambda,b})$ with left boundary $\hat{r}$, right boundary $\hat{l}$ and bottom point $z = (\hat{r}(T), T) = (\hat{l}(T), T)$. A path $\pi \in \Pi$ is said to enter a wedge from outside if there exist times $t, s$ with $t > s \geq \sigma_\pi$ such that $(\pi(s), s) \notin \overline{\Psi}$ and $(\pi(t), t) \in \Psi$.

Now we are ready to present the wedge characterization of the Brownian Net. The following result is from [21, Theorem 1.10].

**Proposition 2.6.** (Theorem 1.10 of [21]) Let $(\hat{W}^l_{\lambda,b}, \hat{W}^r_{\lambda,b}, \hat{W}^l_{\lambda,b}, \hat{W}^r_{\lambda,b})$ be a double left-right Brownian Web. Then almost surely,
\[
\mathcal{N}_{\lambda,b} = \{\pi \in \Pi : \pi \text{ does not enter any wedge of } (\hat{W}^l_{\lambda,b}, \hat{W}^r_{\lambda,b}) \text{ from outside}\}
\]
is the Brownian Net associated with $(\hat{W}^l_{\lambda,b}, \hat{W}^r_{\lambda,b})$, i.e., $\mathcal{N}_{\lambda,b} = \mathcal{H}_{\text{cross}}(\hat{W}^l_{\lambda,b} \cup \hat{W}^r_{\lambda,b})$.

### 2.5. Convergence criteria for the Brownian Net.

Convergence to Brownian Net so far has only been established for random sets of paths with the non-crossing property. Now we will describe a set of sufficient conditions to prove convergence to the Brownian Net. This list of conditions was presented in [20], based on the proof that branching-coalescing simple random walk paths with asymptotically vanishing branching probability converge to Brownian Net, presented in [21].

Consider a sequence of $(\mathcal{H}, \mathcal{B}_\mathcal{H})$-valued random variables $(\mathcal{Y}_n)_{n \in \mathbb{N}}$. Based on the existence of subsets of non-crossing paths $\hat{W}^l_n, \hat{W}^r_n \subset \mathcal{Y}_n$, there are five conditions that, if satisfied, guarantees that $(\mathcal{Y}_n)_{n \in \mathbb{N}}$ converges to the Brownian Net $\mathcal{N}_{\lambda,b}$. 
The first condition is related to the non-crossing restriction:

- **Condition (C):** No path $\pi \in \mathcal{Y}_n$ crosses any $l \in W^l_n$ from right to left (i.e. there are no $s < t$ such that $\pi(s) > l(s)$ and $\pi(t) < l(t)$) and no path $\pi \in \mathcal{Y}_n$ crosses any $r \in W^r_n$ from left to right.

The second condition ensures that $(W^l_n, W^r_n)_{n \in \mathbb{N}}$ is a tight family and that any subsequential weak limit contains a copy of the left-right Brownian Web. It is called Condition $(\mathcal{I}_\mathcal{W})$:

- **Condition $(\mathcal{I}_\mathcal{W})$:** There exist $l_{n,z} \in W^l_n$ and $r_{n,z} \in W^r_n$ for each $z \in \mathbb{R}^2$, such that for any deterministic $z_1, \ldots, z_k, z'_1, \ldots, z'_k \in \mathbb{R}^2$, $(l_{n,z_1}, \ldots, l_{n,z_k}, r_{n,z'_1}, \ldots, r_{n,z'_k})$ converges in distribution to a random vector of paths $(l_{z_1}, \ldots, l_{z_k}, r_{z'_1}, \ldots, r_{z'_k})$ distributed as a family of left-right coalescing Brownian motions with diffusion coefficient $\lambda^2$ and drift coefficient $b$.

Note that, combining the tightness of $(W^l_n, W^r_n)_{n \in \mathbb{N}}$ with Condition (C) implies that $(\mathcal{Y}_n)_{n \in \mathbb{N}}$ is also a tight family. Indeed Condition (C) ensures that, almost surely, the modulus of continuity of paths in $\mathcal{Y}_n$ can be bounded by the modulus of continuity of paths in $W^l_n \cup W^r_n$, whose starting points become dense in $\mathbb{R}^2$ as $n \to \infty$ by Condition $(\mathcal{I}_\mathcal{W})$. So tightness of $(\mathcal{Y}_n)_{n \in \mathbb{N}}$ is a consequence of conditions (C) and $(\mathcal{I}_\mathcal{W})$.

The third condition ensures that any subsequential weak limit of $(\mathcal{Y}_n)_{n \in \mathbb{N}}$ contains not only a copy of the left-right Brownian Web but also a copy of the Brownian Net constructed by hopping among paths in $W^l \cup W^r$ at crossing times. This is called **Condition (H):**

- **Condition (H):** Almost surely, $\mathcal{Y}_n$ contains all paths obtained by hopping among paths in $W^l_n \cup W^r_n$ at crossing times.

Based on the wedge characterization of the Brownian Net, the last two conditions impose a limitation on the number of paths of any subsequential weak limit of $(\mathcal{Y}_n)_{n \in \mathbb{N}}$. These will be called conditions $(U'_\mathcal{W})$ and $(U''_\mathcal{W})$.

- **Condition $(U'_\mathcal{W})$:** There exist $\widehat{W}^l_n, \widehat{W}^r_n \in \mathcal{H}$, whose starting points become dense in $\mathbb{R}^2$ as $n \to \infty$, such that a.s. paths in $W^l_n$ and $\widehat{W}^l_n$ (resp. paths in $W^r_n$ and $\widehat{W}^r_n$) do not cross.

- **Condition $(U''_\mathcal{W})$:** For any weak limit point $(\mathcal{Y}, W^l, W^r, \widehat{W}^l, \widehat{W}^r)$ of $(\mathcal{Y}_n, W^l_n, W^r_n, \widehat{W}^l_n, \widehat{W}^r_n)$ and for any deterministic countable dense set $D \subset \mathbb{R}^2$, a.s. paths in $\mathcal{Y}$ do not enter any wedge of $\widehat{W}^l(D), \widehat{W}^r(D)$ from outside.

In [20] $(U'_\mathcal{W})$ and $(U''_\mathcal{W})$ are merged into a single condition denoted by $(U_\mathcal{W})$. Here we will be able to verify $(U'_\mathcal{W})$ but not $(U''_\mathcal{W})$ for the DNB. This is related to the conjecture mentioned in Section 1 and it will be discussed later. We point out that to verify $(U''_\mathcal{W})$, it is enough to show that paths in $\mathcal{Y}_n$ do not enter wedges of $\widehat{W}^l_n, \widehat{W}^r_n$ from outside, and when a pair of paths in $\widehat{W}^l_n, \widehat{W}^r_n$ converges to a pair of dual left-right coalescing Brownian motions, the associated first meeting time of the pair in $\widehat{W}^l_n, \widehat{W}^r_n$ also converges, which implies that the associated wedge converges.

Now we can state the convergence result:
Proposition 2.7. ([20, Theorem 6.11]) Let \((Y_n)_{n \in \mathbb{N}}\) be a sequence of \((\mathcal{H}, \mathcal{B}_\mathcal{H})\)-valued random variables which satisfy conditions (C), (I\(_N\)), (H), (U\(_N\)\') and (U\(_N\)\'') described above. Then \((Y_n)\) converges in distribution to the Brownian Net \(N_{\lambda, b}\).

To conclude this section, we state a weaker result that we will need.

Proposition 2.8. Let \((Y_n)_{n \in \mathbb{N}}\) be a sequence of \((\mathcal{H}, \mathcal{B}_\mathcal{H})\)-valued random variables which satisfy conditions (C), (I\(_N\)), (H), and (U\(_N\)\') described above. Then \((Y_n)\) is tight and any subsequential weak limit contains a copy of the Brownian Net \(N_{\lambda, b}\).

The proof of Proposition 2.8 follows directly from the proof of Proposition 2.7 in [20]. Indeed condition (U\(_N\)\'\') is the condition that guarantees that any weak limit point of \((Y_n)_{n \in \mathbb{N}}\) does not have more paths than \(N_{\lambda, b}\) and so it must be identically distributed to \(N_{\lambda, b}\).

3. Drainage Network with Branching

3.1. Model description and main results.

We will begin with an informal description of the model: Consider the lattice \(\mathbb{Z}^2\) where the first coordinate represents space and the second represents time. Suppose that each vertex can be independently either open with probability \(p\) or closed with probability \(1 - p\), for some fixed \(p \in (0, 1)\). Now we consider a random graph with vertex set \(\mathbb{Z}^2\) such that from each vertex \((x, t)\) departs at most two directed edges according to the following rule: when there is a unique \(y\) such that \((y, t + 1)\) is the closest open vertex to \((x, t)\) at time \(t + 1\), then \((x, t) \mapsto (y, t + 1)\) is the unique edge departing from \((x, t)\). In case of non-uniqueness of the nearest open vertex, we have two open vertices \((y, t)\) and \((y', t)\), where \(y - x = x - y'\), then \((x, t) \mapsto (y, t + 1)\) and \((x, t) \mapsto (y', t + 1)\) are the edges departing from \((x, t)\). In this way, from the collection of directed paths in the graph, we obtain a system of coalescing-branching paths. We call these collection of directed paths, or equivalently the random graph, the full Drainage Network with Branching (DNB).

The full DNB which described above is one particular case of what we call a DNB, which can have fewer paths. Specifically we fix a probability \(\epsilon \in [0, 1]\) and we change the graph by removing edges in the following way: independently for each edge \((x, t)\) which has two departing edges, we keep both edges on the graph with probability \(\epsilon\), otherwise we choose uniformly only one of them to be kept and we remove the other one. For \(\epsilon = 1\), we have the full DNB. The DNB extends the usual Drainage Network, which is the case \(\epsilon = 0\), by allowing the paths to branch (see Figure 1 for an example). As we mentioned in the introduction, the Drainage Network was introduced in [12] to represent a drainage system where we have sources of a liquid that flows to a specific direction through empty spaces and it is motivated by studies about drainage pattern into an intramontane trench. There are even verifications of empirical predictions about river network models, see for instance [16]. However we do not know about a similar approach related to the Brownian Net which is connected with other important theoretical systems as the Howitt-Warren flows, see [19].
From these considerations, we believe that the DNB is a natural model related to the recent convergence studies associated to the universality class of the Brownian Web and Net.

In Figure 1 we have the graph illustration for an example of DNB considering parameters $p = 0.5$ and $\epsilon = 0.5$. Note for example that vertex $(0, 0)$ has two edges departing from it, so a branch occurs at this vertex for this realization of the system. Vertex $(4, 1)$ had originally two edges in the full DNB, but one of them was removed from the system.

Now we present a formal description of the DNB with parameters $p$ and $\epsilon$. From now on, we suppose that $p \in (0, 1)$ is fixed for the entire paper.

Let $(\omega(z))_{z \in \mathbb{Z}^2}$ be a family of independent Bernoulli random variables with parameter $p$ and $(\theta(z))_{z \in \mathbb{Z}^2}$ be a family of independent and identically distributed random variables on $\{-1, 0, 1\}$ that depends on a parameter $\epsilon \in (0, 1)$ and have the following probability function:

$$P(\theta(z) = 0) = \epsilon, \quad P(\theta(z) = -1) = P(\theta(z) = 1) = \frac{1 - \epsilon}{2}.$$  

We consider that these two families are independent of each other. Also as a reference for future use, we introduced the notation $\mathcal{F}_t = \sigma\{(\omega(z), \theta(z)), z = (z_1, z_2), z_1 \in \mathbb{Z}, z_2 \leq t\}$, $t \in \mathbb{Z}$, which represents the filtration generated by the environment $(\omega, \theta)$.

Every vertex $z \in \mathbb{Z}^2$ will be classified as open or closed if $\omega(z) = 1$ or $\omega(z) = 0$ respectively. Denote by $V$ the set of open vertices: $V = \{z \in \mathbb{Z}^2 : \omega(z) = 1\}$. Moreover, we say that a vertex $z = (z_1, z_2)$ is above $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ if $z_2 > \tilde{z}_2$, immediately above if $z_2 = \tilde{z}_2 + 1$, at the right of $\tilde{z}$ if $z_1 > \tilde{z}_1$ and at the left if $z_1 < \tilde{z}_1$.

For all $z \in V$ we consider $h(z, i), i \in \{-1, 1\}$ as the nearest open vertex which is immediately above $z$ when it is uniquely defined, otherwise $h(z, -1)$ is the nearest open vertex immediately above and at the left of $z$; and $h(z, 1)$ is the nearest open vertex immediately above and at the right of $z$.
above and at the right of $z$. Using the function $h$ we define $\Gamma^l(z)$ and $\Gamma^r(z)$ as follows:

$$\Gamma^l(z) = \begin{cases} h(z, -1), & \text{if } \theta(z) \in \{-1, 0\}, \\ h(z, 1), & \text{if } \theta(z) = 1; \end{cases} \quad \text{and} \quad \Gamma^r(z) = \begin{cases} h(z, -1), & \text{if } \theta(z) = -1, \\ h(z, 1), & \text{if } \theta(z) \in \{0, 1\}. \end{cases}$$

Note that $\Gamma^l(z) \neq \Gamma^r(z)$ only when we have a branching on vertex $z$, i.e., $\theta(z) = 0$.

Now for $z \in V$ and $\tilde{a} = (a_j)_{j \geq 1}$, $a_j \in \{l, r\}$ for each $j \geq 1$, we can associate a continuous random path $(X_t^{z, \tilde{a}})_{t \geq 0}$, by making $X_0^{z, \tilde{a}} = z$ and $X_t^{z, \tilde{a}} = \Gamma^{a_n} \circ \Gamma^{a_{n-1}} \circ \ldots \circ \Gamma^{a_0}(z)$, and connecting consecutive vertices \{X_{j-1}^{z, \tilde{a}}, X_j^{z, \tilde{a}}\}, $j \in \mathbb{N}$, through linear interpolation.

Let $\mathcal{G} = (V, E)$ be the random directed graph with set of vertices $V$ and edges $E = \{(z, \Gamma^l(z)), (z, \Gamma^r(z)) : z \in V\}$. Also denote by $A = \{\tilde{a} = (a_j)_{j \geq 1} : a_j \in \{l, r\}\}$ and put $\mathcal{X}_t = \{(X_t^{z, \tilde{a}}, s)_{s \geq z} : z \in V, \tilde{a} \in A\}$. The random set of paths $\mathcal{X}_t$ is called the Drainage Network with branching parameter $\epsilon$ and $\mathcal{G} = (V, E)$ is its associated directed graph. Note that two paths in $\mathcal{X}_t$ will coalesce when they meet each other in the sense that they coincide from that time on.

Fix a sequence \{\epsilon_n\}_{n \geq 1} in $(0, 1)$ and let $\mathcal{X}^\epsilon_n = \{(\frac{z_1}{n}, \frac{z_2}{n}) \in \mathbb{R}^2 : (z_1, z_2) \in \mathcal{X}_n\}$, for $n \geq 1$, be the diffusively rescaled Drainage Network with branching parameter $\epsilon_n$. When the explicit value of $\epsilon_n$ is not relevant we will usually omit the index of $\mathcal{X}^\epsilon_n$.

**Remark 3.1.** Note that paths in the DNB have identically distributed increments which have finite moments of any order. Indeed the absolute value of the increments are stochastically dominated by a geometric distribution with parameter $p$, denoted here Geo$(p)$. The verification of the last assertion is simple. Once at site $(x, t)$, the size of the next increment is the minimum between the distance at time $t+1$ between $x$ and the nearest open site at its right-hand side and the nearest open site at its left-hand side. These distances are iid Geo$(p)$ and we can take any of them as an upper bound for the increment size.

**Remark 3.2.** Note that the probability of branching in an open vertex of the DNB depends on the parameter $p$, since the branching can occur only if we have a tie in the nearest open vertices at the next time level. It is straightforward to see that the probability of having this tie for an open vertex is $\frac{p(1-p)}{2-p}$, and then, the probability of having a branching occurring in a vertex is $\frac{p(1-p)}{2-p} \epsilon$.

Before we state our main results we need to define some special subsets of paths in $\mathcal{X}^\epsilon$:

An $l$-path in $\mathcal{X}^\epsilon$ is a path obtained by always choosing the leftmost edge when a branching occurs along the path in the DNB. Analogously an $r$-path in $\mathcal{X}^\epsilon$ is a path obtained by always choosing the rightmost edges. Let $W^l_n \subset \mathcal{X}^\epsilon$ be its subset of $l$-paths and $W^r_n \subset \mathcal{X}^\epsilon$ be the subset of $r$-paths. By this definition we have that no path $\pi \in \mathcal{X}^\epsilon$ crosses any path $l \in W^l_n$ from right to left, i.e., $\pi(s) > l(s)$ and $\pi(t) < l(t)$ for some $s < t$, and no path $\pi \in \mathcal{X}^\epsilon$ crosses any path $r \in W^r_n$ from left to right. For example, in Figure 1 the $l$-path that starts on vertex $(2, 1)$ is the one that passes by vertices $(2, 2), (1, 3), (0, 4), (1, 5)$ and the $r$-path that starts from this same vertex is the one that passes by vertices $(2, 2), (3, 3), (3, 4), (2, 5)$.

The main objective of our study is to describe the asymptotic behavior of the diffusively rescaled DNB when $\epsilon_n = b n^{-\alpha}$ for different values of $\alpha > 0$, where $b$ is a positive constant.
In more generality, we may consider \((\epsilon_n)_{n \geq 1}\) such that \(\lim_{n \to \infty} \epsilon_n n^\alpha = b > 0\), but to simplify our exposition, we will do the proofs assuming \(\epsilon_n = b n^{-\alpha}\), since we consider that this generalization is straightforward.

**Remark 3.3.** If we have \(\epsilon_n = b n^{-\alpha}\) for \(b > 0\) and \(\alpha \in [0, 1)\), then the process \(X^n_{\epsilon_n}\) does not converge in distribution under diffusive scaling. In this case, \(\epsilon_n\) diverges to infinity when \(n \to \infty\), which means that the lattice \(\mathbb{Z}^2\) is compressed faster than the reduction of the branching probability. So, if \(X^n_{\epsilon_n}\) had a weak limit, it would be concentrated on paths with drift \(\infty\) or \(-\infty\) which is not possible.

**Remark 3.4.** The Brownian Web and Brownian Net that appear as limit processes in all theorems of this section will have diffusion coefficient equal to \(\lambda^2 = \lambda_p^2\), which is the variance of an increment of a DNB path. This variance finite since the increments are stochastically bounded by \(\text{Geom}(p)\). If one wishes to have these results for the standard Brownian Web and Net with diffusion coefficient equal to 1, it can be achieved dividing the space coordinate by \(\lambda\) in the definition of \(X^n_{\epsilon_n}\).

**Theorem 3.1.** If \(\epsilon_n = b n^{-\alpha}\) for some \(b > 0\) and \(\alpha > 1\), then \(X^n_{\epsilon_n}\) converges in distribution in \(\mathcal{H}\) to the Brownian Web \(\mathcal{W}_\lambda\) when \(n \to \infty\).

**Remark 3.5.** The case of \(\alpha = \infty\) (which means \(\epsilon_n = 0\) for all \(n \in \mathbb{N}\)) is equivalent to the usual Drainage Network where no branching occurs. For this process, the convergence to the Brownian Web has been established in [5].

The following two theorems are devoted to the study of the limit behavior of the DNB when the branching parameter is of order \(\frac{1}{n}\). We use the standard notation "\(\Rightarrow\)" for convergence in distribution.

**Theorem 3.2.** Let \(W^l_n\) (resp. \(W^r_n\)) be the set of \(l\)-paths (resp. \(r\)-paths) of a diffusively rescaled DNB with parameter \(\epsilon_n = b n^{-1}\) for some \(b > 0\). There exist \(\hat{W}^l_n\) and \(\hat{W}^r_n\) dual processes of \(W^l_n\) and \(W^r_n\) respectively such that
\[
(W^l_n, W^r_n, \hat{W}^l_n, \hat{W}^r_n) \Rightarrow (W^l_{\lambda,bp}, W^r_{\lambda,bp}, \hat{W}^l_{\lambda,bp}, \hat{W}^r_{\lambda,bp}) \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad n \to \infty,
\]
where \(bp = \frac{b(1-p)}{(2-p)^2}\).

**Theorem 3.3.** If we have \(X^n_{\epsilon_n}\) the diffusively rescaled Drainage Network with branching parameter \(\epsilon_n = b n^{-1}\) for \(b > 0\), then \(X^n_{\epsilon_n}\) is tight and any subsequential limit contains the Brownian Net \(\mathcal{N}_{\lambda,bp}\) with \(bp = \frac{b(1-p)}{(2-p)^2}\).

Following the convergence criteria in Subsection 2.5 we will show that conditions (C), (I\(_N\)), (H) and (U\(_N\)) holds for the DNB when \(\epsilon_n = b n^{-1}\) to prove Theorem 3.3. We conjecture that \(X^n_{\epsilon_n}\) converges to \(\mathcal{N}_{\lambda,bp}\), but we would need to show that condition (U\(_N^\prime\)) holds to prove
it and we were not able to prove it. In Appendix C, we talk about this difficulty and which details remain pending to show the convergence to Brownian Net.

Theorem 3.1 is proved in Section 6. Theorems 3.2 and 3.3 are proved in Section 7, although the main condition to prove Theorem 3.3 is developed in Section 5.

In the next subsection, we will define the dual process for the DNB, which is mentioned in Theorem 3.2 and have an important role in our convergence study.

3.2. Dual process for Drainage Network with branching.

For the process $X^n$ we can describe a dual process that we denote by $\hat{X}^n$, with paths that follow the reverse direction in time and satisfy the following restrictions:

- The configuration of $\hat{X}^n$ is unique and totally determined by the realization of the process $X^n$;
- $\hat{X}^n$ also has subsets of dual $l$-paths and dual $r$-paths that we denote by $\hat{W}^l_n$ and $\hat{W}^r_n$ respectively. Dual $l$-paths in $\hat{W}^l_n$ cannot cross $l$-paths in $W^l_n$ and dual $r$-paths in $\hat{W}^r_n$ cannot cross $r$-paths in $W^r_n$.

First we will construct $\hat{X}_\varepsilon$, the dual process of $X_\varepsilon$ and after that, we define $\hat{X}^n$, the dual of the diffusively rescaled Drainage Network $X^n$. To construct $\hat{X}_\varepsilon$ we will adapt some ideas presented in [16] for the case without branching.

Recall from Section 3.1 that $G = (V,E)$ is the random directed graph associated to $X_\varepsilon$. The vertices of dual processes will be given by the mid-points between two consecutive vertices in $V$ on each line of time. We denote this random set of vertices by $\hat{V}$. For a more formal description, let us define for $z = (z_1,z_2) \in \mathbb{Z}^2$:

$$K^r(z) = \inf\{k \geq 1 : (z_1+k,z_2) \in V\},$$

$$K^l(z) = \inf\{k \geq 1 : (z_1-k,z_2) \in V\},$$

$$r(z) = (z_1+K^r(z),z_2) \text{ and } l(z) = (z_1-K^l(z),z_2).$$

Notice that $r(z)$ and $l(z)$ denote the nearest open vertex respectively at the right-hand side of $z$ and at the left-hand side of $z$. For all $z \in V$, let $\hat{r}(z) = \left(z_1 + \frac{K^r(z)}{2}, z_2\right)$ denote the dual neighbor at the right of $z$ and $\hat{l}(z) = \left(z_1 - \frac{K^l(z)}{2}, z_2\right)$ denote the dual neighbor at the left of $z$. Now we can define the set of vertices in dual process as:

$$\hat{V} = \{\hat{r}(z) : z \in V\}.$$

**Remark 3.6.** Note that the right dual neighbor of a vertex $z \in V$ is equal to the left dual neighbor of the vertex $r(z)$ that is also in $V$, so we can define $\hat{V}$ considering $\hat{l}(z)$ for all $z \in V$ instead of $\hat{r}(z)$ for all $z \in V$.

**Remark 3.7.** Since $K^r(z)$ and $K^l(z)$ can be odd numbers, the first coordinate of dual vertices can be non-integer, so $\hat{V}$ is a subset of the lattice $\frac{1}{2}\mathbb{Z} \times \mathbb{Z}$. 
Now, let us construct the edge set $\hat{E}$ of the dual graph. From each vertex $\hat{z} \in \hat{V}$ departs either one or two edges connecting it to a vertex immediately below (since this process evolves backward in time). If there exists a vertex $\hat{z}' \in \hat{V}$ at time level $\hat{z}_2 - 1$ such that the line that connect the points $\hat{z}$ and $\hat{z}'$ does not cross any edge in $E$ and $\hat{z}'$ is the nearest vertex in $\hat{V}$ with this property, then $(\hat{z}, \hat{z}')$ is the unique edge in $\hat{E}$ departing from $\hat{z}$. But due to branching in the DNB, some vertices in $\hat{V}$ are not able to reach any other vertex immediately below without crossing any path of $\mathcal{X}$, or equivalently an edge of $E$. In these cases, the path of the dual process also branches. This leads to a pair of edges departing from $\hat{z}$, one connecting it to the closest vertex to the right immediately below and another connecting it to the closest vertex to the left immediately below. When such branching occurs, a path from the dual crosses a path from the DNB, but we do not have crossings between l-paths or between r-paths. Moreover, with our definition, branchings of the dual are in one to one correspondence with branchings of the DNB.

For a formal description of $\hat{E}$, define for all $\hat{z} \in \hat{V}$:

$$a^l(\hat{z}) = \sup \{ k \in \mathbb{Z} : (k, \hat{z}_2 - 1) \in V, \Gamma^l \{(k, \hat{z}_2 - 1)\} < \hat{z}_1 \},$$

$$a^r(\hat{z}) = \inf \{ k \in \mathbb{Z} : (k, \hat{z}_2 - 1) \in V, \Gamma^r \{(k, \hat{z}_2 - 1)\} > \hat{z}_1 \}.$$

If $a^l(\hat{z}) \neq a^r(\hat{z})$, then we set

$$\hat{\Gamma}^l(\hat{z}) = \hat{\Gamma}^r(\hat{z}) = \left( \frac{a^l(\hat{z}) + a^r(\hat{z})}{2}, \hat{z}_2 - 1 \right) \in \hat{V}.$$ 

In this case we have an unique edge departing from $\hat{z}$ which is $(\hat{z}, \hat{\Gamma}^r(\hat{z})) \in \hat{E}$. This edge does not cross any edge of $\mathcal{G}$. Note that, by definition, $(a^l(\hat{z}), \hat{z}_2 - 1), (a^r(\hat{z}), \hat{z}_2 - 1)$ are the nearest vertices in $V$ respectively at the left-hand side and right-hand side of the dual vertex $\hat{\Gamma}^r(\hat{z})$.

If $a^l(\hat{z}) = a^r(\hat{z}) = a$, then for every vertex $\hat{z}' = (\hat{x}, \hat{z}_2 - 1)$ in $\hat{V}$, $\hat{x} \in \frac{1}{2}\mathbb{Z}$, the edge $(\hat{z}, \hat{z}')$ crosses either $((a, \hat{z}_2 - 1), \Gamma^l(a, \hat{z}_2 - 1)) \in E$ or $((a, \hat{z}_2 - 1), \Gamma^r(a, \hat{z}_2 - 1)) \in E$. In this case we set $\hat{\Gamma}^l(\hat{z}) = (\hat{r}(a, \hat{z}_2 - 1), \hat{z}_2 - 1) \in \hat{V}$ and $\hat{\Gamma}^r(\hat{z}) = (\hat{l}(a, \hat{z}_2 - 1), \hat{z}_2 - 1) \in \hat{V}$. Note that $\hat{\Gamma}^l(\hat{z})$ and $\hat{\Gamma}^r(\hat{z})$ are the nearest vertices in $\hat{V}$ to the right-hand side and left-hand side, respectively, of the vertex $(a, \hat{z}_2 - 1)$. It means that a branching occurs in the dual process at vertex $\hat{z}$. Also note that since the dual paths flow in the reverse direction of time, the sense of left and right is reversed in relation to the DNB paths. For example, if we look to $\hat{z} = (2, 3)$ in Figure 2, we have that $a^l(\hat{z}) = a^r(\hat{z}) = 2$, $\hat{\Gamma}^l(\hat{z}) = (\frac{3}{2}, 2)$ and $\hat{\Gamma}^r(\hat{z}) = (1, 2)$. Finally, the edge set of the dual graph $\mathcal{G} = (\hat{V}, \hat{E})$ is given by $\hat{E} = \hat{E}^l \cup \hat{E}^r$ where:

$$\hat{E}^l = \{ \langle \hat{z}, \hat{\Gamma}^l(\hat{z}) \rangle : \hat{z} \in \hat{V} \} \quad \text{and} \quad \hat{E}^r = \{ \langle \hat{z}, \hat{\Gamma}^r(\hat{z}) \rangle : \hat{z} \in \hat{V} \}.$$

Now we can define a backward in time continuous random path $(\hat{X}_{t,j}^{\hat{z},\hat{d}})_{t \geq 0}$ starting from vertex $\hat{z} \in \hat{V}$ by making $\hat{X}_{t}^{\hat{z},\hat{d}} = \hat{\Gamma}^{d_n} \circ \hat{\Gamma}^{d_{n-1}} \circ \ldots \hat{\Gamma}^{d_1}(\hat{z})$ with $\hat{d} = d_1, \ldots, d_n \in \{l, r\}$ for $n \in \mathbb{N}$ and doing linear interpolation between all pair of consecutive vertex $\{\hat{X}_{j+1}^{\hat{z},\hat{d}}, \hat{X}_{j}^{\hat{z},\hat{d}}\}$ for $j = 0, 1 \ldots n - 1$. 


In Figure 2 we have the graph illustration of the dual process related to the DNB example presented in Figure 1. Note that between two open vertices of the DNB we always have one vertex of the dual exactly in the middle and the direction of the edges (arrows), indicating that the dual flow in the reverse time direction. Also note that a dual path branches at one vertex if, and only if, in front of it we have an open vertex of DNB where a branching also occurs. So branchings for both systems are in one to one correspondence.

\[ \hat{X}_n = \hat{\mathcal{X}}^n = \{ (\frac{z_1}{n}, \frac{z_2}{n^2}) : (z_1, z_2) \in \hat{\mathcal{X}}_n \}, \]

for \( n \in \mathbb{N} \) be the dual of the diffusively rescaled Drainage Network with branching parameter \( \epsilon_n \). We will denote by \( \hat{W}_n^l \subset \hat{\mathcal{X}}^n \) the subset of the dual l-paths, which contains all the paths obtained by always choosing the left option when the dual branches and by \( \hat{W}_n^r \subset \hat{\mathcal{X}}^n \) the subset of the dual r-paths, which contains all paths obtained by always choose the right option when the dual branches. Note that we do not have crossings between any path of \( \hat{W}_n^l \) with any path of \( \hat{W}_n^r \).

### 4. Estimates for coalescence times

This section is devoted to obtaining an upper bound for the probability that either two l-paths or two r-paths do not coalesce until time \( t \). This bound is fundamental to prove all theorems presented in the previous section.

Consider two l-paths \( X^u_t, X^v_t \in W^l \) starting from points \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \); and define the process \( (Z^{u,v}_t)_{t \geq u_2 \lor v_2} \) as being the distance between the position of these two paths at time \( t \), i.e.,

\[ Z^{u,v}_t = |X^u_t - X^v_t|, \quad t \geq u_2 \lor v_2. \]
Note that due to translation invariance of the DNB, the distribution of $Z_{u,v}^{t}$ depends on $u$ and $v$ only through the initial distance between the paths at time $(u_2 \vee v_2): k = |X^u_{(u_2 \vee v_2)} - X^v_{(u_2 \vee v_2)}|$. Then let us consider that both paths start at time zero, i.e., $u_2 = v_2 = 0$, and with initial distance $k$ between them. We write $Z_{u,v}^k$ to simplify the notation.

Denote by $\tau_k$ the time until coalescence of the two $l$-paths starting at distance $k$ from each other, i.e.

$$\tau_k = \min\{ t \geq 0 : Z_{u,v}^{k,t} = 0 \}.$$

**Remark 4.1.** Note that we can define an analogous process $Z_{u,v}^{k,t}$ and a coalescence time $\tau_k$ for a pair $r$-paths at initial distance $k$ (we use the same notation). They have the same distribution as the respective variables defined for $l$-paths at initial distance $k$.

Finally, we can present the main result of this section.

**Lemma 4.1.** There exists a constant $C_0 > 0$ such that for every $t > 0$ and $k \in \mathbb{N}$ we have:

$$P(\tau_k > t) \leq \frac{C_0 k}{\sqrt{t}},$$

where this coalescence time can refer to a pair of either $l$-paths or $r$-paths.

For the proof of Lemma 4.1 we will make use of the following result from [7]:

**Proposition 4.2.** ([7, Theorem 5.2]) Let $\{G_l : l \geq 0\}$ be a filtration and $\{Y_l : l \geq 0\}$ be a $G_l$-adapted discrete-time stochastic process taking values in $\mathbb{R}_+$. Let $\nu^Y = \inf\{l \geq 1 : Y_l = 0\}$ be the first time that the process $Y_l$ reaches zero. Suppose that:

(i) For any $l \geq 0$, a.s. $E[(Y_{l+1} - Y_l)|G_l] \leq 0$.

(ii) There exist constants $C_1, C_2 > 0$ such that for any $l \geq 0$, a.s. on the event $\{Y_l > 0\}$, we have

$$E[(Y_{l+1} - Y_l)^2 |G_l] \geq C_1 \quad \text{and} \quad E[|Y_{l+1} - Y_l|^3 |G_l] \leq C_2.$$

Then $\nu^Y < \infty$ almost surely. Further, there exists a constant $C_3 > 0$ such that for any $y > 0$ and any integer $n$,

$$P(\nu^Y > n|Y_0 = y) \leq \frac{C_3 y}{\sqrt{n}}.$$

We will prove Lemma 4.1 using Proposition 4.2.

**Proof of Lemma 4.1.** For this proof, we can consider that $\tau_k$ refers to coalescence between $l$-paths, since the proof considering $r$-paths would be identical. Also recall that we are supposing that $u_2 = v_2 = 0$. We also consider $u_1 > v_1$, so that $u$ is at the left-hand side of $v$.

We have to verify that our distance process $Z_{u,v}^{k,t}$ and the coalescence time $\tau_k$ satisfy the conditions of Proposition 4.2. Note that $Z_{u,v}^{k,t}$ only take values in $\mathbb{R}_+$, because we can not have crossings between $l$-paths. If the two conditions of Proposition 4.2 hold, we have proved Lemma 4.1.
We have that $Z^k_t$ is a Markov chain with state space $\mathbb{Z}^+ \cup \{0\}$ which has state 0 as the only absorbent state, since two paths coalesce when they meet for the first time. The processes $(X^u_t)_{t \geq 0}$ and $(X^v_t)_{t \geq 0}$ have iid symmetric increments. Besides that, the increments of $X^u_{t+1} - X^v_u$ and $X^v_{t+1} - X^u_t$ are also identically distributed (but not independent). Recalling that $\mathcal{F}_t$ is the filtration generated by the environment $(\omega, \theta)$, as defined in Section 3.1, we have:

$$E[Z^k_{t+1} | \mathcal{F}_t] = E[|X^u_{t+1} - X^v_t| | \mathcal{F}_t] = E[X^u_{t+1} - X^v_t | \mathcal{F}_t] = X^u_t - X^v_t = Z^k_t,$$

which implies that the process $Z^k_t$ is a non negative martingale. Then, for any $l \geq 0$, $E[(Z^k_{t+1} - Z^k_t) | \mathcal{F}_t] = 0$, so the condition (i) of Proposition 4.2 is satisfied.

About the first inequality on condition (ii) of Proposition 4.2, note that the increments of $Z^k_t$ are not spatially homogeneous, but given $\mathcal{F}_t$ and that $Z^k_t > 0$, it is always possible to assure that $(Z^k_{t+1} - Z^k_t)^2 \geq 1$ with a convenient choice of open and closed vertices at time $l + 1$. To achieve that, it is enough to have an open vertex at position $(X_t^u, l + 1)$ and a closed vertex at position $(X_t^v, l + 1)$. So, by Markov property of $Z^k_t$, we have that $E[(Z^k_{t+1} - Z^k_t)^2 | \mathcal{F}_t] \geq p(1 - p) = C_1$ for any $l \geq 0$.

The second inequality on condition (ii) of Proposition 4.2 is a consequence of the independence of the increments of $(X^u_t)_{t \geq 0}$ and $(X^v_t)_{t \geq 0}$ and additionally the finite moments property of these increments, see Remark 3.1:

$$E[|Z^k_{t+1} - Z^k_t|^3 | \mathcal{F}_t] \leq 4(E[|X^u_{t+1} - X^u_t|^3 | \mathcal{F}_t] + E[|X^v_{t+1} - X^v_t|^3 | \mathcal{F}_t]) = 8E[|X^u_{t+1} - X^u_t|^3 | \mathcal{F}_t] = C_2,$$

where $C_2$ depends only on $p$. □

From Lemma 4.1 we can state the following corollary, that gives an estimate for crossing times between an l-path and an r-path which starts on the left-hand side of the l-path.

**Corollary 4.3.** Let $L^u_t \in W^l$ starting from $u = (u_1, u_2)$ and $R^v_t \in W^r$ starting from $v = (v_1, v_2)$ with $u_1 - v_1 = k > 0$ and $\tau_k^c = \min\{t \geq 0 : L^u_t \leq R^v_t\}$ the time until we have a crossing between $L^u_t$ and $R^v_t$. There exists a constant $C_0 > 0$ such that for every $t > 0$ and $k \in \mathbb{N}$ we have:

$$P(\tau_k^c > t) \leq P(\tau_k > t) \leq \frac{C_0 k}{\sqrt{t}}.$$

**Proof.** Let $R^v_t \in W^r$ be the r-path that starts from $u = (u_1, u_2)$. Since $L^u_t$ can never be at right-hand side of $R^v_t$, it will be squeezed between $R^v_t$ and $R^u_t$, which implies that it has to cross $R^v_t$ before or at the same time of the coalescence between $R^v_t$ and $R^u_t$, which gives the first inequality of the corollary. The last inequality follows from Lemma 4.1. □

**Remark 4.2.** Suppose that the processes $X^u$ and $X^v$ that appear in the definition of $Z^k$ are independent. More precisely, assume that the DNB paths $X^u$ and $X^v$ are evolving according to different environments. We have that both Lemma 4.1 and Corollary 4.3 also holds in this case. To argue that, we use a result from [13, Lemma 2.2] that obtain the same upper bound presented in Lemma 4.1 for the first meeting time between two independent random walks that can cross each other. In [13, Lemma 2.2], the increments of the random walks
have mean zero and finite second moment, but it is straightforward to note from the proof that it is enough to have that the difference between the increments of the two random walks has mean zero instead of having mean zero for the increments of both paths.

Now we will state an equivalent result for coalescence times considering paths from the dual, which we prove in Subsection A.2. First let us give some notation for the dual case.

Recalling the notation of Section 3.2, fix \( \hat{z} \in \hat{V} \) and set recursively
\[
\hat{\Gamma}_0(\hat{z}) = \hat{z}, \quad \hat{\Gamma}_1(\hat{z}) = \hat{\Gamma}(\hat{z}) \quad \text{and} \quad \hat{\Gamma}_{k+1}(\hat{z}) = \hat{\Gamma}(\Gamma_k(\hat{z})), \text{ for } k \geq 1.
\]
Let \( \hat{\gamma}_k(\hat{z}) \) denote the first coordinate of \( \hat{\Gamma}_k(\hat{z}) \).

Also define \( \hat{\Gamma}_k^r(\hat{z}) \) and \( \hat{\gamma}_k^r(\hat{z}) \) analogously using the \( \hat{r} \) instead of \( \hat{\Gamma} \). So, \( \hat{\gamma}_t^r(\hat{u}) \) and \( \hat{\gamma}_t^r(\hat{v}) \) denote the position of the dual paths respectively at dual times \( \hat{u}_2 - t \) and \( \hat{v}_2 - t \) and starting from the dual vertices \( \hat{u} \) and \( \hat{v} \). Define the process \((\hat{Z}^u_\hat{v}, \hat{t})_{t \geq 0}\) as being the distance between the position of these two dual paths after \( t \) steps, starting to compute it at time \((\hat{u}_2 \wedge \hat{v}_2)\), i.e.
\[
\hat{Z}^u_\hat{v}_t = |\hat{\gamma}_t^u(\hat{u}_2 - (\hat{u}_2 \wedge \hat{v}_2)) - \hat{\gamma}_t^\hat{v}(\hat{v}_2 - (\hat{u}_2 \wedge \hat{v}_2))|, \quad t \geq 0.
\]
Let us denote by \( \tau_{\hat{u}, \hat{v}} \) the time until coalescence between the two dual \( l \)-paths starting from \( \hat{u} \) and \( \hat{v} \), i.e.
\[
\tau_{\hat{u}, \hat{v}} = \min\{t \geq 0 : \hat{Z}^u_\hat{v}_t = 0\}.
\]

Note that, differently of what happened with the DNB paths, in this case, the process \( \hat{Z}^u_\hat{v}_t \) still depends on the specific position of the dual vertices \( \hat{u} \) and \( \hat{v} \) even if we know the initial distance between them. It occurs because the distribution of the increments of dual paths depends if they are in an integer position or not (see Figure 2). Even if we know that the initial distance is a non-integer, the distribution of \( \hat{Z}^u_\hat{v}_t \) also depends on which one of either \( \hat{u} \) or \( \hat{v} \) is a non-integer.

There is an argument that allows us to obtain the bound for the coalescence time considering only the case where \( \hat{u} \) and \( \hat{v} \) are integers, thus avoiding the problem of dealing with the distinct distributions for \( \hat{Z}^u_\hat{v}_t \). To argue that, we will show an environment configuration that assures that \( \hat{Y}^t_{\hat{u}_1}(\hat{u}) \) and \( \hat{Y}^t_{\hat{v}_1}(\hat{v}) \) are integers, given that at least one between \( \hat{Y}^t_{\hat{u}_1}(\hat{u}) \) and \( \hat{Y}^t_{\hat{v}_1}(\hat{v}) \) is a non integer.

To simplify the notation, due to the time homogeneity of the dual process, we can consider that both dual paths start at time zero, i.e., \( \hat{u}_2 = \hat{v}_2 = 0 \). First suppose that \( \hat{Y}^t_{\hat{u}_1}(\hat{u}) = y_u \) is an integer, \( \hat{Y}^t_{\hat{v}_1}(\hat{v}) = y_v \) is a non integer and \( y_u < y_v \) (besides that, by the construction rule of the dual paths, we need to have that \( |y_v - y_u| \geq 1 \)). Consider the environmental event \((y_u, -t - 1)\) is closed, \((y_u - 1, -t - 1)\) is open, \((y_v - 0.5, -t - 1)\) is open, \((y_v + 0.5, -t - 1)\) is closed and \((y_v + 1.5, -t - 1)\) is open. In this event, \((\hat{y}_u - t, y_u - t - 1)\) and \((y_v - t, (y_v + 0.5, -t - 1))\) are in \( \hat{E} \), and \( \hat{Y}^t_{\hat{u}_1}(\hat{u}) = y_u, \hat{Y}^t_{\hat{v}_1}(\hat{v}) = y_v + 0.5 \). Both dual paths are at integer positions at time \( t + 1 \). The event has probability either \( p^3(1 - p)^2 \) if \( y_u + 1 < y_v - 0.5 \) or \( p^3(1 - p)^2 \) if \( y_u + 1 = y_v - 0.5 \).

Now suppose that both \( \hat{Y}^t_{\hat{u}_1}(\hat{u}) = y_u \) and \( \hat{Y}^t_{\hat{v}_1}(\hat{v}) = y_v \) are non integers and \( y_u < y_v \). Consider the event \((y_u - 1.5, -t - 1)\) is open, \((y_u - 0.5, -t - 1)\) is closed, \((y_u + 0.5, -t - 1)\)
is open, \((y_v - 0.5, -t - 1)\) is open, \((y_v + 0.5, -t - 1)\) is closed and \((y_v + 1.5, -t - 1)\) is open. In this event, \(((u, -t), (y_u - 0.5, -t - 1))\) and \(((v, -t), (y_v + 0.5, -t - 1))\) are in \(E_t\), and \(\hat{Y}_{t+1}^u(u) = y_u - 0.5, \hat{Y}_{t+1}^v(v) = y_v + 0.5\). Both dual paths are at integer positions at time \(t + 1\). The event has probability either \(p^4(1 - p)^2\) if \(y_u + 0.5 < y_v - 0.5\) or \(p^3(1 - p)^2\) if \(y_u + 0.5 = y_v - 0.5\).

Considering the previous events, we have that the probability of having both dual l-paths at integer positions at time \(t + 1\), given that at least one of them is at a non-integer position at time \(t\), is bounded from below by \(p^4(1 - p)^2\). Denote by

\[
\hat{\nu}_{u, \hat{v}} = \inf\{t \geq 0 : \hat{Y}_t^u(\hat{u}) \in \mathbb{Z} \text{ and } \hat{Y}_t^v(\hat{v}) \in \mathbb{Z}\}.
\]

Hence, by the Markov property of the dual l-paths (in the reverse direction of time), we have that \(\hat{\nu}_{u, \hat{v}}\) is stochastically dominated by a random variable \(\hat{\nu}\) that has geometric distribution with parameter \(p^4(1 - p)^2\). Then:

\[
P(\hat{\nu}_{u, \hat{v}} > t) = P((\hat{\nu}_{u, \hat{v}} - \hat{\nu}_{u, \hat{v}}) + \hat{\nu}_{u, \hat{v}} > t) \leq P(\hat{\nu}_{u, \hat{v}} - \hat{\nu}_{u, \hat{v}} > t/2) + P(\hat{\nu}_{u, \hat{v}} > t/2).
\]

By the stochastic domination mentioned before and supposing that a coalescence time bound as the one that appears in Lemma 4.1 holds for dual l-paths starting at integer positions (this coalescence time bound is proved in Appendix A.2), we have that \(P(\hat{\nu}_{u, \hat{v}} > t/2) \leq C e^{-ct}\). Now, recalling that \(k\) denotes the initial distance between the two dual l-paths, we have

\[
P \left( \frac{\hat{\nu}_{u, \hat{v}} - \hat{\nu}_{u, \hat{v}}}{2} \right) \leq \frac{C}{\sqrt{t}} \mathbb{E}[\hat{Z}_{\hat{\nu}_{u, \hat{v}}}] = \frac{C}{\sqrt{t}} \left( k + \sum_{m=1}^{\infty} \mathbb{E} \left[ \hat{Z}_{\hat{\nu}_{u, \hat{v}}} - k \right] | \hat{\nu}_{u, \hat{v}} = m \right) \mathbb{P}(\hat{\nu}_{u, \hat{v}} = m) \leq \frac{C}{\sqrt{t}} \left( k + \sum_{m=1}^{\infty} c m \mathbb{P}(\hat{\nu}_{u, \hat{v}} = m) \right) = \frac{C}{\sqrt{t}} \left( k + c \mathbb{E}[\hat{\nu}_{u, \hat{v}}] \right),
\]

where \(c\) a upper bound on the conditional mean of an increment of the distance between two dual l-paths, we mean here that one can find \(c\) that does not depend on the initial distance between both paths. It is straightforward to compute \(c\) and we leave the details to the reader. Thus we only need to consider that both dual paths start from integer positions.

Denote by \(\hat{Z}_t^k\) the distance after \(t\) steps between the position of two dual paths that start in integer positions at time zero with initial distance \(k\), and by \(\hat{\tau}_k\) the coalescence time of these two dual paths, i.e:

\[
\hat{Z}_t^k = |\hat{Y}_t^u((0, 0)) - \hat{Y}_t^v((k, 0))|, \ t \geq 0, \ \text{and} \ \hat{\tau}_k = \min\{t \geq 0 : \hat{Z}_t^k = 0\}.
\]

**Remark 4.3.** Recall that the dual paths flows in the reverse direction of time. We put the notation in such way to deal only with positive and crescent values for the index of process \(\hat{Z}_{\hat{\nu}}\), but the coalescence between two dual paths starting from the dual vertices \(\hat{u}\) and \(\hat{v}\) will actually happen at time \((\hat{u}_2 \wedge \hat{v}_2) - \hat{\tau}_{\hat{u}, \hat{v}}\).
Remark 4.1 is also applied here. We are considering only dual l-paths, but everything would be analogous for r-paths.

Now we are finally ready to present an equivalent of Lemma 4.1 for dual paths.

**Lemma 4.4.** There exists a constant $\hat{C}_0 > 0$ such that for every $t > 0$ and $k \in \mathbb{N}$ we have:

$$P(\hat{\tau}_k > t) \leq \frac{\hat{C}_0 k}{\sqrt{t}},$$

where this coalescence time can refer to two dual l-paths or to two dual r-paths.

To prove Lemma 4.4 we cannot just use Proposition 4.2 directly as we did for the DNB paths, because, as pointed before, the distance between two dual paths has different behavior according to the different possibilities for these paths as to be in integer positions or not. To deal with this problem, our strategy will be based in the following idea: We define specific regeneration times where both dual paths are in integer positions, then we show that these times appear with enough frequency allowing us to simplify our problem considering the distance between the two dual paths only in these convenient regeneration times. The proof is presented in Section A.2.

## 5. Convergence of the finite-dimensional distributions

This section is devoted to prove that $(I_N)$ holds for the DNB when the branching parameter is $\epsilon_n = b n^{-1}$, for $b > 0$. So we begin by stating the main result of this section.

**Remark 5.1.** Whenever we have a path $\pi$ that starts at time $\sigma_\pi$, we can consider that $\pi_t = \pi_{\sigma_\pi}$ for $t < \sigma_\pi$. It does not affect any convergence result and simplifies the notation when we have paths starting from different times.

**Proposition 5.1.** Let $\epsilon_n = b n^{-1}$. Consider the sequences of points $z^{(n)}_1, \ldots, z^{(n)}_k$ and $\tilde{z}^{(n)}_1, \ldots, \tilde{z}^{(n)}_k$ in $\mathbb{Z}^2$ where $z^{(n)}_i = (x^{(n)}_i, s^{(n)}_i)$ with $\left(\frac{1}{n} x^{(n)}_i, \frac{1}{n} s^{(n)}_i\right) \to (x_i, s_i) = z_i \in \mathbb{R}^2$ for $i = 1, \ldots, k$; and $\tilde{z}^{(n)}_j = (\tilde{x}^{(n)}_j, \tilde{s}^{(n)}_j)$ with $\left(\frac{1}{n} \tilde{x}^{(n)}_j, \frac{1}{n} \tilde{s}^{(n)}_j\right) \to (\tilde{x}_j, \tilde{s}_j) = \tilde{z}_j \in \mathbb{R}^2$ for $j = 1, \ldots, \tilde{k}$.

Denote by $l^{(n)}_i$ the $l$-path of the DNB with branching parameter $\epsilon_n$, starting from $z^{(n)}_i$ and by $r^{(n)}_j$ the $r$-path of the DNB with branching parameter $\epsilon_n$, starting from $\tilde{z}^{(n)}_j$. Then the following convergence holds:

$$\left(\frac{l^{(n)}_1(t) n^2}{n}, \ldots, \frac{l^{(n)}_k(t) n^2}{n}, \frac{r^{(n)}_1(t) n^2}{n}, \ldots, \frac{r^{(n)}_\tilde{k}(t) n^2}{n}\right)_{t \in \mathbb{R}} \Rightarrow (l_1(t), \ldots, l_k(t), r_1(t), \ldots, r_\tilde{k}(t))_{t \in \mathbb{R}},$$

as random elements of the space of continuous functions $C(\mathbb{R}, \mathbb{R}^{k+\tilde{k}})$ endowed with the uniform norm topology, where $(l_1, \ldots, l_k, r_1, \ldots, r_\tilde{k})$ is a collection of left-right coalescing Brownian motions (as defined in Section 2.4), starting from $(z_1, \ldots, z_k, \tilde{z}_1, \ldots, \tilde{z}_\tilde{k})$. 
Before we prove Proposition 5.1, which is one of the main results in this paper, we will focus on proving this convergence only considering a single left-right pair of paths. Some of our arguments in this first stage are similar to the ones that we find in [21] to prove an equivalent condition for simple random walks with branching which are independent before coalescence. But the dependence of paths in the DNB implies that we need a new approach to prove convergence. After that, we generalize our ideas to more than one pair of paths and for this stage, we make use of some ideas presented in [6] to prove an equivalent condition for the generalized Drainage Network without branching.

**Proposition 5.2.** Let \( \epsilon_n = bn^{-1} \). Consider sequences \((x^{(n)})_{n \geq 1}\) and \((y^{(n)})_{n \geq 1}\) in \( \mathbb{Z} \) such that \( \lim_{n \to \infty} x^{(n)}/n = x \) and \( \lim_{n \to \infty} y^{(n)}/n = y \), \( x, y \in \mathbb{R} \). Denote by \( (L_t^{(n)})_{t \geq 0} \) the \( l \)-path of \( X_{\epsilon_n}^n \) starting from \((x^{(n)}, 0)\) and by \( (R_t^{(n)})_{t \geq 0} \) the \( r \)-path of \( X_{\epsilon_n}^n \) starting from \((y^{(n)}, 0)\). Then the following convergence holds:

\[
\left( \frac{1}{n} L_{tn^2}, \frac{1}{n} R_{tn^2} \right)_{t \geq 0} \overset{n \to \infty}{\longrightarrow} (L_t, R_t)_{t \geq 0},
\]

where \((L_t, R_t)_{t \geq 0}\) is the unique solution of (2.5) with initial state \((L_0, R_0) = (x, y)\) under the constraint that \(L_t \leq R_t\) for all \(t \geq T = \inf\{s \geq 0 : L_s = R_s\}\).

The entire proof of the previous two propositions will be divided in five subsections, where in each subsection we consider a more general case. Sections 5.1 and 5.2 together will give the proof of Proposition 5.2 and Section 5.3 will give the proof of Proposition 5.1.

5.1. **Part I: Only one pair** \((L_t^{(n)}, R_t^{(n)})_{n \in \mathbb{N}}\) with \( R_0^{(n)} \geq L_0^{(n)} \).

First we will state and prove a result that assures that the family of joint processes \((L_t^{(n)}, R_t^{(n)})_{n \in \mathbb{N}}\) is tight.

**Lemma 5.3.** Consider \((L_t^{(n)})_{t \geq 0}\) and \((R_t^{(n)})_{t \geq 0}\) as defined in Proposition 5.2. We have that \((L_t^{(n)})_{t \geq 0}\) and \((R_t^{(n)})_{t \geq 0}\) individually converges weakly under diffusive scaling to a Brownian motion with diffusion coefficient \(\lambda_p^2\) and drift \(-b_p, b_p\), respectively, where \(\lambda_p^2\) is defined in Remark 3.4 and \(b_p = \frac{\beta(1-p)}{2(2-p)^2}\).

**Proof.** As we show below, these convergences are a consequence of Donsker’s Theorem. Indeed we can write the sequence of increments of \((L_t^{(n)})_{t \geq 0}\) as a sum of two sequences, one with mean zero that converges to a Brownian motion under diffusive scaling by a direct application of Donsker’s Theorem and other that converges in probability to the drift function \(t \mapsto -b_p t\) under diffusive scaling by the Law of Large Number. An analogous proof holds for \((R_t^{(n)})_{t \geq 0}\) for the drift function \(t \mapsto b_p t\), and hence the proof will be done only for \((L_t^{(n)})_{t \geq 0}\).

More specifically, \(L_t^{(n)}\) is equal in distribution to \(L_t^{a(n)} + L_t^{b(n)}\), where: \(L_t^{a(n)}\) jumps using the same rule of a path from the Drainage Network without branching linearly interpolated. \(L_t^{b(n)}\) is a linearly interpolated jump process that only jumps in times where \(L_t^{a(n)}\) jumps
to the right, and at these times, $L_t^{b(n)}$ will have probability $\frac{bp(1-p)}{2n(2-p)}$ to do a jump to the left with a length equal to twice of the jump length of $L_t^{a(n)}$ and probability $1 - \frac{bp(1-p)}{2n(2-p)}$ to stay in the same position. Indeed $\frac{bp(1-p)}{2n(2-p)}$ is the probability that a branching occurs in the DNB from a given vertex $(x,t)$ and, through a uniform choice, the right-hand side is chosen as the destination from that vertex. It can be computed as:

(branching parameter) $\times$ (side choice probability) $\times$ (branching probability in full-DNB)

$$= \epsilon_n \times \frac{1}{2} \times P((x,t+1) \text{ is closed}) \times \text{(probability of equality between two iid Geom(p))}$$

$$= \frac{b}{2n} \times (1-p) \times \frac{p}{2-p} = \frac{bp(1-p)}{2n(2-p)}$$

So $L_t^{b(n)}$ is a drift correction for paths in Drainage Network without branching which generates paths identically distributed to $l$-paths in the DNB.

Both process $L_t^{a(n)}$ and $L_t^{b(n)}$ have independent increments with finite variance. We have that $L_t^{a(n)}$ converges to a Brownian motion under diffusive scaling by Donsker’s Theorem. Moreover $L_t^{b(n)} + \frac{b\epsilon}{n}$ is a martingale for any fixed $n \in \mathbb{N}$, where $b_p = \frac{b(1-p)}{(2-p)^2}$. Indeed we have that $-\frac{b_p}{n}$ is the mean value of the increments of $L_t^{b(n)}$. The mean value is computed as follows:

$$- \sum_{k=1}^{\infty} 2k \text{ (probability that } ((x,t), (x+\pm k,t+1)) \in E \text{ and edge } ((x,t), (x+k,t+1)) \text{ is chosen)}$$

$$= - \sum_{k=1}^{\infty} 2k \epsilon_n \times \frac{1}{2} \times P((x,t+1) \text{ is closed}) \times \text{(probability that two iid Geom(p) are equal to k)}$$

$$= - \frac{b(1-p)}{n} \sum_{k=1}^{\infty} k(1-p)^{2(k-1)} p^2 = - \frac{b(1-p)}{n(2-p)^2} = - \frac{b_p}{n}.$$ 

Now denote by $(Z_j^{b(n)})_{j \geq 1}$ the increments of $L_t^{b(n)}$ at integer times. Note that $Z_j^{b(n)} | Z_j^{a(n)} \neq 0$ is stochastically dominated by $2W$ with $W \sim Geom(p)$, which implies that

$$Var \left( Z_j^{b(n)} \right) \leq E[(Z_j^{b(n)})^2] = E[(Z_j^{b(n)})^2 | Z_j^{b(n)} \neq 0] P(Z_j^{b(n)} \neq 0) \leq \frac{2b(1-p)}{np} \to 0$$

as $n \to \infty$.

Apply Doob maximal inequality and we have for any fixed $t > 0, \delta > 0$ and $n$ sufficiently large depending on $\delta$ that

$$P \left( \sup_{0 \leq s \leq t} \left[ \frac{1}{n} L_{\left[sn^2\right]}^{b(n)} + b_p s \right] \geq \delta \right) \leq P \left( \max_{0 \leq k \leq \lfloor mn^2 \rfloor} \left[ \frac{1}{n} \sum_{j=1}^{k} Z_j^{b(n)} + \frac{b_p}{n} \right] \geq \frac{\delta}{2} \right)$$
\[ \leq \frac{C}{\delta^2 n^2} \text{Var} \left( \sum_{j=1}^{\lfloor tn^2 \rfloor} Z_j^{\text{b}(n)} \right) \leq \frac{C t}{\delta^2} \left( 1 + \frac{1}{n^2} \right) \text{Var} \left( Z_j^{\text{b}(n)} \right) \to 0, \text{ as } n \to \infty. \]

Hence,
\[ \left( L_t^{\text{b}(n)} \right)_{t \geq 0} \text{ converges under diffusive scaling to the deterministic path } t \mapsto -b_p t \quad (5.1) \]
and the convergence holds in probability on every bounded interval \([0, T], T \geq 0. \, \square \]

Now we will state and prove a result that gives some control on the probability of occurrence of long jumps in the paths of the DNB, which will be required further ahead.

Lemma 5.4. Let \( \pi_z(t) \), for \( t > \sigma_\pi \) be a random path in DNB that starts from an open vertex \( z \) at time \( \sigma_\pi \). We have that, for any fixed \( s > 0 \), \( g \geq 1 \) and \( n \geq 1 \),
\[ P \left( \sup_{\sigma_\pi < t \leq \sigma_\pi + sn^2} |\pi_z(t) - \pi_z(t - 1)| \geq g \right) \leq 2sn^2 e^{-c(g-1)}, \]
for some \( c = c(p) > 0 \). In particular, for any \( \gamma > 0 \), the probability that a path in the DNB makes a jump of size greater than \( g(n) = n^\gamma \) in a time interval of order \( O(n^2) \) goes to zero as \( n \to \infty. \)

Proof. Let \( Y_t = |\pi_z(t) - \pi_z(t - 1)| \), for \( t \in \mathbb{N} \) and \( t \geq \sigma_\pi \). First, since the distribution of the path increments is invariant in time and space, we can assume that \( \pi_z(t) \) starts at time 0 (i.e. \( \sigma_\pi = 0 \)). Now, since \( Y_t \preceq G \sim \text{Geom}(p) \) and using Markov property, for any fixed \( s > 0 \)
\[ P \left( \sup_{0 < t \leq sn^2} Y_t \geq g \right) \leq (\lfloor s \rfloor + 1)n^2 P(G \geq \lceil g \rceil) \leq 2sn^2 (1 - p)^{g-1}. \]
To conclude the proof just take \( c = -\log(1 - p). \, \square \]

Corollary 5.5. For any \( K > -2/\log(1 - p) \), the probability that a path in the DNB makes a jump of size greater than \( K \log(n) \) in a time interval of order \( O(n^2) \) goes to zero as \( n \to \infty. \)

For the remain of this section, \( \kappa > -2/\log(1 - p) \) is a fixed positive real number. For any fixed \( s > 0 \), we have by Corollary 5.5 that the probability that \( L_t^{\text{b}(n)} \) or \( R_t^{\text{b}(n)} \) makes any jump of size greater than \( \kappa \log(n)/2 \) in the time interval \([0, sn^2]\) converges to zero as \( n \) goes to infinity. So, we can construct our arguments conditioning to the event that those long jumps do not happen, and we will assume that from now on.

Our initial strategy would be to represent \((L_t^{\text{b}(n)}, R_t^{\text{b}(n)})_{t \geq 0}\) as the solution of a difference equation, which in the limit yields an SDE with an unique solution that also solves the
Lemma 2.2 of Proposition 5.6 defines the following equations:

\begin{align}
(i) \quad &dL_t = d\tilde{B}_T^l + dB^s_{S_t} - b_p dt, \\
(ii) \quad &dR_t = d\tilde{B}_T^r + dB^s_{S_t} + b_p dt, \\
(iii) \quad &T_t + S_t = t \quad \forall t > 0, \\
(iv) \quad &\int_0^t I_{(L_s < R_s)} dS_s = 0, \quad \forall t > 0, \\
\end{align}

(5.2)

where $\tilde{B}_T^l$, $\tilde{B}_T^r$ and $B^s_{S_t}$ are independent Brownian motions. Under the restriction that $L_t \leq R_t$ for all $t \geq 0$, (5.2) is equivalent to (2.5) due to Lemma 2.2 of [21] that we present just below.

**Proposition 5.6.** (Lemma 2.2 of [21]) Denote $R_{\leq}^2 = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$.

(a) There is a one-to-one correspondence in law between weak $R_{\leq}^2$-valued solutions of (2.5) and weak $R_{\leq}^2$-valued solutions of (5.2).

(b) For each initial state $(L_0, R_0) \in R_{\leq}^2$, equation (5.2) has a pathwise unique solution.

(c) Solutions to (5.2) satisfy $S_t = \int_0^t I_{(L_s = R_s)} ds$,

\[ S_t = \sup_{0 \leq s \leq T_t} \left( \frac{1}{2} (L_0 + \tilde{B}_s^l - R_0 - \tilde{B}_s^r) - s \right) \quad a.s., \]

and $\lim_{t \to \infty} T_t = \infty$.

However, comparing to [21], in our case, due to the dependence between $L_t^{(n)}$ and $R_t^{(n)}$, it is harder to represent $(L_t^{(n)}, R_t^{(n)})_{t \geq 0}$ as the solution of a convenient difference equation. So, to have some control on this dependence, we will create an auxiliary process $\tilde{R}_t^{(n)}$ that will evolve in the same way as $R_t^{(n)}$ when it is far from $L_t^{(n)}$ and will evolve according to a different rule when it is near to $L_t^{(n)}$. The process $\tilde{R}_t^{(n)}$ will be created in such a way that we can represent $(L_t^{(n)}, \tilde{R}_t^{(n)})_{t \geq 0}$ as the solution of a difference equation that converges to the solution of (5.2) and that the difference between $R_t^{(n)}$ and $\tilde{R}_t^{(n)}$ becomes negligible under diffusive scaling.

The process $\tilde{R}_t^{(n)}$ will be constructed through a coupling that uses a convenient alternation between independent environments. We will condition on the event that $L_t^{(n)}$, $R_t^{(n)}$ and $\tilde{R}_t^{(n)}$ do not make jumps of size $\kappa \log(n)/2$ in some time interval $[0, sn^2]$. Thus essentially we will use the fact that, if two paths of the DNB are far from each other (by a distance of $n^{3/4} \gg \kappa \log(n)$ at least), then with high probability these two paths will only need to observe two disjoint sets of vertices in the next time level to make their jump decision, independently from each other.

Recall the definition of the environment in Section 3.1. We say that $(\omega(z))_{z \in \mathbb{Z}^2}$ and $(\theta(z))_{z \in \mathbb{Z}^2}$ define the $(\omega, \theta)$-environment. $(L_t^{(n)}, R_t^{(n)})_{t \geq 0}$ evolves according to the $(\omega, \theta)$-environment. Let $\tilde{\omega} = (\tilde{\omega}(z))_{z \in \mathbb{Z}^2}$ and $\tilde{\theta} = (\tilde{\theta}(z))_{z \in \mathbb{Z}^2}$ be independent families of random
variables identically distributed as $\omega$ and $\theta$ respectively, and also independent of them. Then $(\tilde{\omega}, \tilde{\theta})$ is a new environment. The process $\tilde{R}^{(n)}_t$ will evolve almost as a path of a DNB, and its jump choices will be obtained from an alternation between the $(\omega, \theta)$-environment and the $(\tilde{\omega}, \tilde{\theta})$-environment. Hence we can also use Corollary 5.5 to have that $\tilde{R}^{(n)}_t$ does not make a jump of size greater than $\kappa \log(n)/2$ in time interval $[0, sn^2]$ with a probability that goes to one as $n$ goes to infinity. Additionally the distance between $\tilde{R}^{(n)}_t$ and $L^{(n)}_t$ does not make a jump of size greater than $\kappa \log(n)$ in time interval $[0, sn^2]$ also with a probability that goes to one as $n$ goes to infinity.

Let us denote by $\Delta^{(n)}_t$ the distance between $\tilde{R}^{(n)}_t$ and $L^{(n)}_t$. We construct $(\tilde{R}^{(n)}_t)_{t \geq 0}$ according to the steps $(\tilde{R}.1)$, $(\tilde{R}.2)$ and $(\tilde{R}.3)$ described below (recall that $\kappa$ is fixed above):

$(\tilde{R}.1)$ $\tilde{R}^{(n)}_t$ jumps according to $(\omega, \theta)$ ($\tilde{R}^{(n)}_t$ and $R^{(n)}_t$ move together) until $\Delta^{(n)}_t$ becomes smaller than $n^{\frac{3}{2}}$. After that $\tilde{R}^{(n)}_t$ starts to jump according to $(\tilde{\omega}, \tilde{\theta})$ as a path of DNB. If $\Delta^{(n)}_t$ is already smaller than $n^{\frac{3}{2}}$, we have that $\tilde{R}^{(n)}_t$ immediately starts to evolve according to $(\omega, \theta)$. After that, either $\Delta^{(n)}_t$ becomes greater than $n^{\frac{3}{2}}$, then go to step $(\tilde{R}.2)$, or $\Delta^{(n)}_t$ becomes smaller than $\kappa n^{\frac{3}{2}} \log(n)^2$, then go to step $(\tilde{R}.3)$.

$(\tilde{R}.2)$ After step $(\tilde{R}.1)$, if $\Delta^{(n)}_t$ is greater than $n^{\frac{3}{2}}$, either $\Delta^{(n)}_t$ becomes smaller than $n^{\frac{3}{2}}$ again, then return to step $(\tilde{R}.1)$, or $\tilde{R}^{(n)}_t$ returns to jump according to $(\omega, \theta)$ when it meets $R^{(n)}_t$ in a position at distance greater than $n^{\frac{7}{8}}$ from $L^{(n)}_t$. In the latter case $\tilde{R}^{(n)}_t$ and $R^{(n)}_t$ return to move together until $\Delta^{(n)}_t$ becomes smaller than $n^{\frac{3}{2}}$ again, then also return to step $(\tilde{R}.1)$.

$(\tilde{R}.3)$ Here we impose a non-crossing rule between $\tilde{R}^{(n)}_t$ and $L^{(n)}_t$. After step $(\tilde{R}.1)$ or the event described at the end of this step $(\tilde{R}.3)$ itself, $\Delta^{(n)}_t$ is smaller than $\kappa n^{\frac{3}{2}} \log(n)^2$. If we have an attempt of crossing between $\tilde{R}^{(n)}_t$ and $L^{(n)}_t$, we will force $\tilde{R}^{(n)}_t$ to stop in the same position of $L^{(n)}_t$. Whenever $\Delta^{(n)}_t$ becomes lesser than $\kappa n^{\frac{3}{2}} \log(n)^2$ (including an attempt of crossing), $\tilde{R}^{(n)}_t$ will begin to make the same movements as those of $L^{(n)}_t$, keeping $\Delta^{(n)}_t$ constant. They remain evolving like this until $L^{(n)}_t$ decides to branch, which increases $\Delta^{(n)}_t$ (note that it is the same behavior that $\tilde{R}^{(n)}_t$ and $L^{(n)}_t$ have when they meet each other, but with $\tilde{R}^{(n)}_t$ we enforce this behavior before they meet each other). After that, $\tilde{R}^{(n)}_t$ returns to jump according to $(\tilde{\omega}, \tilde{\theta})$ (independently of $L^{(n)}_t$), even if $\Delta^{(n)}_t$ continues lesser than $\kappa n^{\frac{3}{2}} \log(n)^2$. We write $\tilde{R}^{(n)}_t \approx L^{(n)}_t$ when $\tilde{R}^{(n)}_t$ is making the same jumps as $L^{(n)}_t$. From the moment $\tilde{R}^{(n)}_t$ and $L^{(n)}_t$ start to jump independently again, either $\Delta^{(n)}_t$ decreases of any amount, then restart step $(\tilde{R}.3)$ (we will have $\tilde{R}^{(n)}_t \approx L^{(n)}_t$ again), or $\Delta^{(n)}_t$ becomes greater than $\kappa n^{\frac{3}{2}} \log(n)^2$, then return to step $(\tilde{R}.1)$.

Note that the interaction rule of step $(\tilde{R}.3)$ creates a sort of barriers whenever the paths get closer which tries to prevent an attempt of crossing (that is possible due to independence).
To illustrate the relation between $\tilde{R}(n)_t$ and $L(n)_t$, we have Figure 3, where we considered two different scenarios that can happen after we have $\tilde{R}(n)_t \approx L(n)_t$.

In the scenario (a) of Figure 3, we start at step $(\tilde{R}.1)$. At time $\tau_1$ the distance $\Delta(n)_t$ becomes smaller than $\kappa n^{1/4} \log(n)^2$ and we pass to step $(\tilde{R}.3)$. At time $\tau_2$ path $L(n)_t$ branches, thereafter $\Delta(n)_t$ increases to $\kappa n^{1/4} \log(n)^2$ and we go back to step $(\tilde{R}.1)$. At time $\tau_3$ we have $\tilde{R}(n)_t \approx L(n)_t$ again, when $\Delta(n)_t$ becomes lesser than $\kappa n^{1/4} \log(n)^2$ a second time.

In the scenario (b) of Figure 3, again we start at step $(\tilde{R}.1)$. At time $\tau_1$ the distance $\Delta(n)_t$ becomes smaller than $\kappa n^{1/4} \log(n)^2$ and we pass to step $(\tilde{R}.3)$. At time $\tau_2$ path $L(n)_t$ branches, thereafter $\Delta(n)_t$ continues lesser than $\kappa n^{1/4} \log(n)^2$ and $\tilde{R}(n)_t$ and $L(n)_t$ are moving independently. At time $\tau_3$ the distance $\Delta(n)_t$ decreases and we restart step $(\tilde{R}.3)$, so $\tilde{R}(n)_t \approx L(n)_t$ again. What have occurred at times $\tau_2$ and $\tau_3$ occurs again respectively at times $\tau_4$ and $\tau_5$.

**Figure 3.** Example of the interaction between $\tilde{R}(n)_t$ and $L(n)_t$ considering scenarios (a) and (b). The left picture of each scenario shows the evolution of $\tilde{R}(n)_t$ and $L(n)_t$, while the right picture shows the evolution of the difference $\Delta(n)_t = \tilde{R}(n)_t - L(n)_t$ which is constant whenever $\tilde{R}(n)_t \approx L(n)_t$. The dotted line indicates the distance $\kappa n^{1/4} \log(n)^2$ at right-hand side of $L(n)_t$. if $\tilde{R}(n)_t$ enters the gray area or makes a jump to the left inside it, we begin to have $\tilde{R}(n)_t \approx L(n)_t$ until $L(n)_t$ chooses to branch. The random times $\tau_i$, for i odd, indicate the times where we begin to have $\tilde{R}(n)_t \approx L(n)_t$ and the random times $\tau_i$, for i even, indicate the times where $\tilde{R}(n)_t$ returns to move independently of $L(n)_t$.

We will denote by $J(n)$ the set of integer times in which $\tilde{R}(n)_t \approx L(n)_t$. 
To prepare ourselves to write the system of difference equations for the pair \((L_t, \tilde{R}_t)\), we need to define some auxiliary processes. Let \((V_t^l)_{t \in \mathbb{N}}, (V_t^r)_{t \in \mathbb{N}}\) and \((V_t)_{t \in \mathbb{N}}\) be independent discrete-time symmetric Markov chains defined in \(\mathbb{Z}\), starting at the origin at time zero and with the following transition probabilities:

\[
P_v(x, x) = p, \quad \text{and} \quad P_v(x, y) = p(1 - p)2^{(y-x)}, \quad \forall y \neq x,
\]

where \(p\) is the probability parameter of the DNB. The transition probabilities of these processes coincide with those of a path from the Drainage Network without branching.

For \(\alpha = l, s\), let \((D_t^{(n), \alpha, -})_{t \in \mathbb{N}}\) be a process that can move only in times where \(V_t^\alpha\) jumps to a position at its right, and at these times, \(D_t^{(n), \alpha, -}\) will have probability \(\frac{bp(1 - p)}{2n(2 - p)}\) to do a jump to the left with a length equal to twice of the length of the jump of \(V_t^\alpha\) and probability \(1 - \frac{bp(1 - p)}{2n(2 - p)}\) to remain in the same position (see the proof of Lemma 5.3). Likewise, for \(\alpha = r, s\), let \((D_t^{(n), \alpha, +})_{t \in \mathbb{N}}\) be a process that can move only in the times where \(V_t^\alpha\) jumps to a position at its left, and at these times, \(D_t^{(n), \alpha, +}\) will have probability \(\frac{bp(1 - p)}{2n(2 - p)}\) to do a jump to the right with a length equal to twice of the length of the jump of \(V_t^\alpha\) and probability \(1 - \frac{bp(1 - p)}{2n(2 - p)}\) to remain in the same position. The choices are independent for \(\alpha = r, l, s\), therefore \(D_t^{(n), l, -}, D_t^{(n), r, +}\) and \((D_t^{(n), s, -}, D_t^{(n), s, +})\) are independent processes.

Recalling that we are conditioning on the event that \(L_t^{(n)}\) and \(R_t^{(n)}\) do not make jumps of size greater than \(\kappa \log(n)/2\), the pair \((L_t^{(n)}, \tilde{R}_t^{(n)})_{t \geq 0}\) can be constructed as the solution of:

\[
\begin{align*}
(i) \quad L_t^{(n)} &= V_{T_t^{(n)}} + D_{T_t^{(n)}}^{(n), l, -} + V_{S_t^{(n)}}^s + D_{S_t^{(n)}}^{(n), s, -}, \\
(ii) \quad \tilde{R}_t^{(n)} &= V_{T_t^{(n)}}^r + D_{T_t^{(n)}}^{(n), r, +} + V_{S_t^{(n)}}^s + D_{S_t^{(n)}}^{(n), s, +} + \sum_{j=1}^t U_j^{(n)}, \\
(iii) \quad T_t^{(n)} &= \sum_{s=0}^{t-1} I_{\{s \notin J_t^{(n)}\}}, \\
(iv) \quad S_t^{(n)} &= \sum_{s=0}^{t-1} I_{\{s \in J_t^{(n)}\}},
\end{align*}
\]

where \(U_j^{(n)}, j \in \mathbb{N}\), are random variables that for each \(j\) assumes the value of

\[
\left(\left[ L_j^{(n)} - (\tilde{R}_{j-1}^{(n)} + (V_{T_j^{(n)}}^r + D_{T_j^{(n)}}^{(n), r, +}) - (V_{T_{j-1}^{(n)}}^r + D_{T_{j-1}^{(n)}}^{(n), r, +})) \right] I_{\{\tilde{R}_{j-1}^{(n)} > L_{j-1}^{(n)} \cap (j-1) \notin J_t^{(n)}\}} \right)_+,
\]

where \((x)_+\) denotes the positive part of \(x\). The random variables \(U_j^{(n)}\) and the last term in the equation (ii) are here to deal with the corrections in \(\tilde{R}_t^{(n)}\) that we have to make whenever it attempts to cross \(L_t^{(n)}\). At these crossing attempts, the correction puts \(\tilde{R}_j^{(n)}\) on the same position of \(L_j^{(n)}\).
We define $L_{i}^{(n)}$, $\tilde{R}_{i}^{(n)}$, $V_{i}^{\alpha}$, $D_{i}^{(n),l,-}$, $D_{i}^{(n),r,+}$, $D_{i}^{(n),s,\pm}$, $T_{i}^{(n)}$, $S_{i}^{(n)}$ and $U_{j}^{(n)}$ at non integer times by linear interpolation. The rescaled processes then satisfy the following equations:

\[
(i) \quad \frac{1}{n} L_{t}^{(n)} = \frac{1}{n} V_{t}^{l} + \frac{1}{n} D_{t}^{(n),l,-} + \frac{1}{n} V_{t}^{s} + \frac{1}{n} D_{t}^{(n),s,-},
\]

\[
(ii) \quad \frac{1}{n} \tilde{R}_{t}^{(n)} = \frac{1}{n} V_{t}^{r} + \frac{1}{n} D_{t}^{(n),r,+} + \frac{1}{n} V_{t}^{s} + \frac{1}{n} D_{t}^{(n),s,+} + \frac{1}{n} \sum_{j=1}^{tn^2} U_{j}^{(n)},
\]

\[
(iii) \quad \frac{1}{n^2} T_{t}^{(n)} + \frac{1}{n^2} S_{t}^{(n)} = t,
\]

\[
(iv) \quad \int_{0}^{t} \left\{ \frac{1}{n} \tilde{R}_{sn^2}^{(n)} - \frac{1}{n} L_{sn^2}^{(n)} \geq \kappa \log(n)^{(n)} \frac{3}{4} \right\} d \left( \frac{1}{n^2} S_{sn^2}^{(n)} \right) = 0. \tag{5.5}
\]

Applying Donsker’s Theorem and the same arguments used in the proof of Lemma 5.3, we have that

\[
\left( \frac{1}{n} V_{tn^2}^{l}, \frac{1}{n} V_{tn^2}^{r}, \frac{1}{n} V_{tn^2}^{s}, \frac{1}{n} D_{tn^2}^{(n),l,-}, \frac{1}{n} D_{tn^2}^{(n),r,+}, \frac{1}{n} D_{tn^2}^{(n),s,-}, \frac{1}{n} D_{tn^2}^{(n),s,+} \right)_{t \geq 0} \quad \tag{5.6}
\]

converge in distribution when $n \to \infty$ to $(B_{i}^{l}, B_{i}^{r}, B_{i}^{s}, b_{p}, b_{p}, b_{p}, b_{p}, b_{p})_{t \geq 0}$, where $B_{i}^{l}$, $B_{i}^{r}$ and $B_{i}^{s}$ are independent Brownian motions with diffusion coefficient $\lambda_{p}^{2}$, $b_{p} = \frac{b(1-p)}{(2-p)^{p}}$, as we verified in Lemma 5.3. This convergence is the same obtained in [21] to prove Proposition 5.2 in the case of independent paths.

We still have to analyze the limit behavior of the last term in (ii) of (5.5), but our next lemma give us that this term must converge to zero in probability and consequently does not affect the limit behavior of $\frac{1}{n} \tilde{R}_{tn^2}^{(n)}$.

**Lemma 5.7.** Considering $U_{j}^{(n)}$ as defined in (5.4), we have that:

\[
\frac{1}{n} \sum_{j=1}^{tn^2} U_{j}^{(n)} \xrightarrow{n \to \infty} 0 \text{ in probability.}
\]

The proof of Lemma 5.7 will be postponed. It will be presented immediately before Part II of this section.

Now note that, since $t \mapsto \frac{1}{n^2} T_{tn^2}^{(n)}$ and $t \mapsto \frac{1}{n^2} S_{tn^2}^{(n)}$ increases with slope at most 1, the laws of $\left\{ \frac{1}{n^2} T_{tn^2}^{(n)} \right\}_{t \geq 0}$ and $\left\{ \frac{1}{n^2} S_{tn^2}^{(n)} \right\}_{t \geq 0}$ are tight. By the convergence of the 7-tuple in (5.6) and Lemma 5.7, we have that $\left\{ \left( \frac{1}{n} L_{tn^2}^{(n)}, \frac{1}{n} \tilde{R}_{tn^2}^{(n)} \right)_{t \geq 0} \right\}_{n \in \mathbb{N}}$ are also tight and
consequently, for \( n \in \mathbb{N} \), the laws of the 11-tuple which consists of the 7-tuple in (5.6) joint with
\[
\left( \frac{1}{n} L_{tn^2}^{(n)}, \frac{1}{n} \tilde{R}_{tn^2}^{(n)}, \frac{1}{n^2} T_{tn^2}^{(n)}, \frac{1}{n^2} S_{tn^2}^{(n)} \right)_{t \geq 0}
\]
are tight. Let us prove that all limit point process of subsequences has the same distribution and to achieve that, let us consider a subsequence that converge in distribution to \( \tilde{\rho} \)

For this subsequence, by Skorohod’s Representation Theorem, we can couple this 11-tuple for \( n \in \mathbb{N} \) and the limiting process in (5.7), such that the convergence is almost surely.

Assuming this coupling, we claim that \((L_t, R_t, T_t, S_t)_{t \geq 0}\) solves equation (5.2) and consequently is uniquely determined in law by Proposition 5.6. In fact (i), (ii) and (iii) of (5.2) follows immediately from making \( n \) goes to infinity in (i), (ii) and (iii) of (5.5) respectively. Now to verify that (iv) of (5.2) also holds, let us choose a continuous non-decreasing function \( \rho_\delta : [0, \infty) \mapsto \mathbb{R} \), for each \( \delta > 0 \), such that \( \rho_\delta(u) = 0 \) for \( u \leq \delta \) and \( \rho_\delta(u) = 1 \) for \( u \geq 2\delta \). Then, since \( \rho_\delta \) is bounded, by (5.5) and our convergence assumption, we have that
\[
0 = \lim_{n \to \infty} \int_0^t \left\{ \frac{1}{n} \tilde{R}_{sn^2}^{(n)} - \frac{1}{n} L_{sn^2}^{(n)} \right\} d \left( \frac{1}{n^2} S_{sn^2}^{(n)} \right) 
\]
\[
\geq \lim_{n \to \infty} \int_0^t \rho_\delta \left( \frac{1}{n} \tilde{R}_{sn^2}^{(n)} - \frac{1}{n} L_{sn^2}^{(n)} \right) d \left( \frac{1}{n^2} S_{sn^2}^{(n)} \right) = \int_0^t \rho_\delta (\tilde{R}_s - L_s) dS_s, \forall \delta > 0.
\]

Letting \( \delta \downarrow 0 \), by Dominated Convergence Theorem, we have (iv) of (5.2). It gives us that the limit in distribution of any subsequence of \( \left( \frac{1}{n} L_{tn^2}^{(n)}, \frac{1}{n} \tilde{R}_{tn^2}^{(n)} \right) \) is a left-right Brownian motion, which implies that \((L_t^{(n)}, \tilde{R}_t^{(n)})_{t \geq 0}\) converges to a left-right Brownian motion under diffusive scaling.

Now it remains to verify that \((L_t^{(n)}, R_t^{(n)})_{t \geq 0}\) and \((L_t^{(n)}, \tilde{R}_t^{(n)})_{t \geq 0}\) have this same limit and to deal with it we have the following lemma, which completes the proof of Part I.

**Lemma 5.8.** Considering \( L_t^{(n)}, R_t^{(n)} \) and \( \tilde{R}_t^{(n)} \) as defined above, we have that the pairs \((L_t^{(n)}, R_t^{(n)})_{t \geq 0}\) and \((L_t^{(n)}, \tilde{R}_t^{(n)})_{t \geq 0}\) converge weakly to the same limit under diffusive scaling.

**Proof.** First note that the processes \( R_t^{(n)} \) and \( \tilde{R}_t^{(n)} \) are equal except in some specific time windows (depending on the distances between \( R_t^{(n)} \) and \( L_t^{(n)} \), and between \( \tilde{R}_t^{(n)} \) and \( L_t^{(n)} \)).

Also note that for any \( \beta < 1 \), paths that are at distance smaller than \( n^\beta \) from each other are shrunken to the same path under diffusive scaling, thus we do not need to worry with time windows where \( \tilde{R}_t^{(n)} \) are at distance smaller than \( n^\beta \) from \( R_t^{(n)} \). Therefore our strategy here will be to show that we can fix an exponent \( \beta < 1 \) such that the time windows where \( \tilde{R}_t^{(n)} \) can be at a distance greater than \( n^\beta \) from \( R_t^{(n)} \) almost surely shrink to zero under diffusive scaling. This is enough to assure that \((L_t^{(n)}, R_t^{(n)})\) and \((L_t^{(n)}, \tilde{R}_t^{(n)})\) have the same limit.
Recall steps (R.1), (R.2) and (R.3), in the definition of \((\tilde{R}_t^{(n)})\). Fix \(\frac{7}{8} < \beta < 1\) and \(s > 0\). Denote by \([\tilde{\xi}_k^{(n)}, \tilde{\xi}_k^{(n)}], 1 \leq k \leq N_{\beta}^{(n)}\),
\[
\tilde{\xi}_1^{(n)} < \tilde{\Omega}_1^{(n)} < \cdots < \tilde{\xi}_k^{(n)} < \tilde{\Omega}_k^{(n)} < \cdots < \tilde{\xi}_{N_{\beta}^{(n)}}^{(n)} < \tilde{\Omega}_{N_{\beta}^{(n)}}^{(n)};
\]
where \(N_{\beta}^{(n)} := \sup\{k \geq 1 : \tilde{\xi}_k^{(n)} < sn^2\}\), such that \(\tilde{\xi}_k^{(n)} \leq [sn^2] \leq \tilde{\Omega}_k^{(n)} \) or \(\tilde{\Omega}_k^{(n)} < [sn^2]\), the sequence of time windows defined according to the following rules:
- \(\tilde{\xi}_k^{(n)} < sn^2\) is a time where \(R_t^{(n)}\) and \(\tilde{R}_t^{(n)}\) stop to evolve together (that is, \(\tilde{R}_t^{(n)}\) begins to evolve according to \((\tilde{\omega}, \theta)\));
- \(\tilde{\Omega}_k^{(n)}\) is a time where \(\tilde{R}_t^{(n)}\) and \(\tilde{R}_t^{(n)}\) meet each other at a distance greater than \(n^\frac{\beta}{2}\) from \(L_t^{(n)}\), returning to evolve together;
- In \([\tilde{\xi}_k^{(n)}, \tilde{\xi}_k^{(n)}]\), either \(R_t^{(n)}\) or \(\tilde{R}_t^{(n)}\) reaches a distance greater than \(n^\beta\) from \(L_t^{(n)}\).

Now denote by \(\tilde{T}_{\beta, n}\) the amount of time inside \([\tilde{\xi}_k^{(n)}, \tilde{\xi}_k^{(n)}]\) where \(\tilde{R}_t^{(n)}\) is at a distance greater than \(n^\beta\) from \(R_t^{(n)}\). It is enough to show that, for some \(\gamma < 2\),
\[
\lim_{n \to \infty} P\left(\sum_{k=1}^{N_{\beta}^{(n)}} \tilde{T}_{\beta, n} > n^\gamma\right) = 0.
\]

First we need some control on \(N_{\beta}^{(n)}\), when \(n\) goes to infinity. We will assume without loss of generality that \(\tilde{R}_t^{(n)}\) reaches the distance \(n^\beta\) from \(L_t^{(n)}\) before \(R_t^{(n)}\), but the argument would be the same in the other scenario. Given that the distance between \(\tilde{R}_t^{(n)}\) and \(L_t^{(n)}\) is greater than \(n^\beta\), and recalling that we are assuming that these paths do not make any jump of size greater than \(\kappa \log(n)/2\), we have that \(\Delta_t^{(n)} - \tilde{R}_t^{(n)}\) is a random walk with drift \(\frac{2b_p}{n}\) (twice the drift of each path). Note that for any fixed \(n\), \(\Delta_t^{(n)}\) is a Markov process with independent increments and positive drift. Besides that, \(\Delta_t^{(n)}\) converges under diffusive scaling to a Brownian motion with drift \(2b_p > 0\). Now let us define the events
- \(\tilde{E}_n = \{\Delta_t^{(n)}\) reaches the position \(n\) before it reaches the position \(n^{\frac{3}{2}}\) given that \(\Delta_0^{(n)} = n^\beta\}\),
- \(\tilde{E}_2, n = \{\Delta_t^{(n)}\) never reaches the position \(n^{\frac{3}{2}}\) given that \(\Delta_0^{(n)} = n\}\),
- \(\tilde{E}_n = \{\Delta_t^{(n)}\) never reaches the position \(n^{\frac{3}{2}}\) after it reaches a position grater than \(n^\beta\}\).

For \(n\) fixed, let us \(\tau_{E}^{(n)} = \inf_{t>0}\{\Delta_t^{(n)} \notin (n^{\frac{3}{2}}, n)\}\). Now we denote by \(\xi_{-}^{(n)}\) and \(\xi_{+}^{(n)}\) the overshoot distribution of \(\Delta_t^{(n)}\) when it crosses the position \(n^{\frac{3}{2}}\) from right to left and when it cross the position \(n\) from left to right, respectively (i.e. the distribution of the distance between \(\Delta_t^{(n)}\) and this specific position that it needs to cross). Note that \(\Delta_t^{(n)}\) is a \(F_t\)-submartingale and \(\tau_{E}^{(n)}\) is an almost surely finite stopping time. Thus \((\Delta_t^{(n)}\)\) is bonded
in $L^2$, then uniformly integrable. The Optional Stopping Theorem gives us that

$$E[\Delta_0^{(n)}] \leq E \left[ \Delta_0^{(n)} \right]_{\bar{F}_k},$$

which implies

$$n^\beta \leq (n^{\frac{3}{4}} - \tilde{C}_-^{(n)})(1 - P(\bar{E}_n)) + (n + \tilde{C}_+^{(n)})P(\bar{E}_n) \leq n^\frac{3}{4}(1 - P(\bar{E}_n)) + (n + \tilde{C}_+^{(n)})P(\bar{E}_n),$$

where $\tilde{C}_-^{(n)} = E[\xi^{(n)}_\gamma] > 0$ for all $n \in N$ and $\tilde{C}_+^{(n)} = E[\xi^{(n)}_\beta] \leq \frac{5}{p}$ for $n$ large enough by Claim B.1 in Appendix B. Thus

$$P(\bar{E}_n) \geq \frac{n^\beta - n^{\frac{3}{4}}}{n - n^{\frac{3}{4}} + \tilde{C}_+^{(n)}} \geq \frac{n^\beta - n^{\frac{3}{4}}}{n - n^{\frac{3}{4}} + \frac{5}{p}} = \frac{1}{n^{1-\beta}} \left( \frac{1 - n^{\frac{3}{4} - \beta}}{1 - n^{\frac{3}{4} - \frac{5}{p}}} \right) \geq \frac{1}{2n^{1-\beta}}.$$

for large values of $n$. Concerning the event $\bar{E}_{2,n}$, by Donsker’s and Portmanteau Theorems, we have that

$$\liminf_{n\to\infty} P(\bar{E}_{2,n}) \geq \liminf_{n\to\infty} P \left( \inf_{t \geq 0} \inf \Delta_t^{(n)} > n^{\frac{3}{4}} \Delta_0^{(n)} = n \right) \geq P \left( \inf_{t>0} B_{t}^{(0,0),2b_p,2\lambda_p^2} > -\frac{1}{2} \right) = 1 - \exp \left\{ -\frac{\sqrt{2b_p}}{\lambda_p} \right\} = \frac{1}{C_1} > 0, \quad (5.8)$$

where $B_{s}^{z,\mu,\sigma^2}$ is a Brownian motion with drift $\mu$ and diffusion $\sigma^2$, starting at point $z$. The equality (5.8) occurs because $-(\inf_{t \geq 0} B_{t}^{(0,0),\mu,\sigma^2})$ when $\mu > 0$ has exponential distribution with parameter $\frac{2\mu}{\sigma^2}$ (see e.g. section 6.8 of [14]). The result in (5.8) gives that $P(\bar{E}_{2,n}) \geq \tilde{C}_1^{-1}$ for large values of $n$.

So, by the Markov property, we have that $P(\tilde{F}_n) \geq \frac{1}{2n^{1-\beta}}$ for large values of $n$. Again by the Markov property, $\tilde{N}_\beta^{(n)}$ is stochastically bounded from above by a geometric random variable $G_F^{(n)}$ with parameter $\frac{1}{2C_1n^{1-\beta}}$ for large values of $n$. Now we can write

$$P \left( \sum_{k=1}^{\tilde{N}_\beta^{(n)}} T_k^{\beta,(n)} > n^\gamma \right) = \sum_{m=1}^{\infty} P \left( \sum_{k=1}^{\tilde{N}_\beta^{(n)}} T_k^{\beta,(n)} > n^\gamma \middle| \tilde{N}_\beta^{(n)} = m \right) P(\tilde{N}_\beta^{(n)} = m)$$

which is bounded above by

$$\sum_{m=1}^{\infty} P \left( \sum_{k=1}^{G_F^{(n)}} T_k^{\beta,(n)} > n^\gamma \middle| G_F^{(n)} = m \right) P(G_F^{(n)} = m) \leq \sum_{m=1}^{\infty} m P \left( \frac{T_1^{\beta,(n)} > \frac{n^\gamma}{m}}{m} \right) P(G_F^{(n)} = m)$$
\[ \sum_{m=1}^{\left\lfloor n^{1-\beta \log(n)^2} \right\rfloor} m P(G_F^{(n)} = m) P\left( \hat{T}_1^{\beta,n} > \frac{n^\gamma 1}{n^{1-\beta} \log(n)^{2}} \right) + \sum_{m=[n^{1-\beta \log(n)^2}]}^{\infty} m P(G_F^{(n)} = m) \]

\[ \leq E[G_F^{(n)}] P\left( \hat{T}_1^{\beta,n} > \frac{n^\gamma 1}{n^{1-\beta} \log(n)^{2}} \right) + \sum_{m=[n^{1-\beta \log(n)^2}]}^{\infty} m P(G_F^{(n)} \geq m) \]

\[ \leq 2C_1 n^{1-\beta} P\left( \hat{T}_1^{\beta,n} > \frac{n^\gamma 1}{n^{1-\beta} \log(n)^{2}} \right) + 2C_1 n^{1-\beta} \left( 1 - \frac{1}{2C_1 n^{1-\beta}} \right)^{n^{1-\beta \log(n)^2}} \]  

(5.9)

Note that the rightmost term in (5.9) converges to zero as \( n \) goes to infinity and then, to finish the proof, we will show that

\[ P\left( \hat{T}_1^{\beta,n} > \frac{n^{\gamma + \beta - 1}}{\log(n)^2} \right) < \frac{1}{n^{1-\beta} \log(n)}, \]

for large values of \( n \) and some \( \gamma < 2 \). To prove it, we will choose proper disjoint time windows \([\xi_{1,j}^{(n)}, \mathcal{G}_{1,j}^{(n)}] \subset [\xi_{1}^{(n)}, \mathcal{G}_{1}^{(n)}], j = 1, ..., \nu^{(n)}\), such that:

\[ \mathcal{G}_{1}^{(n)} - \xi_{1}^{(n)} \geq \sum_{j=1}^{\nu^{(n)}} (\mathcal{G}_{1,j}^{(n)} - \xi_{1,j}^{(n)}) = \sum_{j=1}^{\nu^{(n)}} \left( \hat{T}_j^{(n),A} + \hat{T}_j^{(n),B} \right) \geq \sum_{j=1}^{\nu^{(n)}} \hat{T}_j^{(n),B} \geq \hat{T}_1^{\beta,n}. \]  

(5.10)

Define:

- \( \nu^{(n)} \) is a random variable that counts the number of times in time window \([\xi_{1,j}^{(n)}, \mathcal{G}_{1,j}^{(n)}]\) that either \( R_t^{(n)} \) or \( \bar{R}_t^{(n)} \) reaches a distance greater than \( 2n^{\frac{\gamma}{2}} \) from \( L_t^{(n)} \) and meet each other, after they have been together at some distance smaller than \( n^{\frac{\gamma}{2}} \) from \( L_t^{(n)} \), until this meeting occurs at a distance greater than \( n^{\frac{\gamma}{2}} \) from \( L_t^{(n)} \). It will be clear just below why we have chosen \( 2n^{\frac{\gamma}{2}} \) instead of \( n^{\frac{\gamma}{2}} \), but it is the reason why \([\xi_{1,j}^{(n)}, \mathcal{G}_{1,j}^{(n)}], j = 1, ..., \nu^{(n)}\), is not a partition of \([\xi_{1}^{(n)}, \mathcal{G}_{1}^{(n)}]\). Recall that we will only be interested in time windows where either \( R_t^{(n)} \) or \( \bar{R}_t^{(n)} \) reaches a distance greater than \( n^{\beta} \) from \( L_t^{(n)} \).

- \( \hat{T}_j^{(n),A}, j = 1, ..., \nu^{(n)} \), describe the length of each time window that starts when \( R_t^{(n)} \) and \( \bar{R}_t^{(n)} \) reaches together a distance smaller than \( n^{\frac{\gamma}{2}} \) from \( L_t^{(n)} \) and ends when either \( R_t^{(n)} \) or \( \bar{R}_t^{(n)} \) reaches a distance greater than \( n^{\frac{\gamma}{2}} \) from \( L_t^{(n)} \). Recall that in this time window \( \bar{R}_t^{(n)} \) is evolving according to \((\tilde{\omega}, \tilde{\theta})\).

- \( \hat{T}_j^{(n),B}, j = 1, ..., \nu^{(n)} \), describe the length of each time window that starts immediately after the end of the respective time window related to \( \hat{T}_j^{(n),A} \), i.e. when either \( \bar{R}_t^{(n)} \) or \( R_t^{(n)} \) reaches a distance greater than \( n^{\frac{\gamma}{2}} \) from \( L_t^{(n)} \) and ends when they meet each other for the first time at a distance greater than \( n^{\frac{\gamma}{2}} \) from \( L_t^{(n)} \), thus coalescing.
We will obtain an upper bound on the distribution of $\nu^{(n)}$. For $j = 1, ..., \nu^{(n)}$, we have

$$\max \left( |\tilde{R}_{1,j}^{(n)} - L_{1,j}^{(n)}|, |\tilde{R}_{1,j}^{(n)} - L_{1,j}^{(n)}| \right) \leq 2n\bar{\tau},$$

$$\max \left( |\tilde{R}_{1,j}^{(n)} - L_{1,j}^{(n)}|, |\tilde{R}_{1,j}^{(n)} - L_{1,j}^{(n)}| \right) > 2n\bar{\tau},$$

$$\mathcal{T}_{1,j}^{(n)} = \inf \left\{ t > 0 : \max \left( |\tilde{R}_{t}^{(n)} - L_{t}^{(n)}|, |\tilde{R}_{t}^{(n)} - L_{t}^{(n)}| \right) > 2n\bar{\tau} \right\}.$$ 

Moreover we clearly must have either $|\tilde{R}_{t}^{(n)} - L_{t}^{(n)}| \leq n\bar{\tau}$ or $|\tilde{R}_{t}^{(n)} - L_{t}^{(n)}| > n\bar{\tau}$ and, in the former case, $j = \nu^{(n)}$ and $\mathcal{G}_{1,j}^{(n)} = \mathcal{G}_{1}^{(n)}$. Based on this description and the independence of the increments of paths in DNB, we claim that $\nu^{(n)}$ is stochastically bounded from above by a geometric random variable with parameter 0.5. To prove it, let $L, R$ and $R'$ be respectively an l-path starting in 0, an r-path starting in 0 and an r-path starting at 2 at time 0, and let $\tau$ be the coalescing time between $R$ and $R'$. It is equivalent to say that $\tau$ is the coalescing time between $R - L$ and $R' - L$, which are positively drifted random walks starting respectively at positions 0 and $2n\bar{\tau}$. By stationarity of the environment and symmetry properties, we have that the event

$$\{ R - L$ and $R' - L$ meet each other for the first time at distance greater than $n\bar{\tau} \}$$

has probability greater or equal to 0.5. This last assertion proves the previous claim.

Now note that according to the definition of $[\mathcal{T}_{1,j}^{(n)}, \mathcal{G}_{1,j}^{(n)}]$, we can have either $\tilde{R}_{t}^{(n)}$ or $\tilde{R}_{t}^{(n)}$ at a distance greater than $n\beta$ from $L_{t}^{(n)}$ only during $\tilde{T}_{j}^{(n),B}$, so we do not need to worry about $\tilde{T}_{j}^{(n),A}$. Also note that if the meeting between $\tilde{R}_{t}^{(n)}$ or $\tilde{R}_{t}^{(n)}$ that is registered by $\nu^{(n)}$ occurs before neither $\tilde{R}_{t}^{(n)}$ nor $\tilde{R}_{t}^{(n)}$ reaches a distance greater than $n\beta$ from $L_{t}^{(n)}$, then we have a time window where these paths will be shrunk to the same path under diffusive scaling. So, we only need to consider the terms of the sum in (5.10) where either $\tilde{R}_{t}^{(n)}$ or $\tilde{R}_{t}^{(n)}$ has reached a distance greater than $n\beta$ from $L_{t}^{(n)}$ before they coalesce. Denote by $\mathcal{I}^{(n)}$ the set of indexes of these terms in (5.10). Note that our definition of $[\mathcal{T}_{1}^{(n)}, \mathcal{G}_{1}^{(n)}]$ implies that $\mathcal{I}^{(n)}$ is not empty. Then, denoting by $\tilde{G}$ the Geom(0.5), we can write

$$P \left( \tilde{T}_{1}^{(n),B} > \frac{n^{\gamma + \beta - 1}}{\log(n)^{2}} \right) \leq P \left( \sum_{j=1}^{\nu^{(n)}} \tilde{T}_{j}^{(n),B} > \frac{n^{\gamma + \beta - 1}}{\log(n)^{2}} \right)$$

$$= \sum_{k=1}^{\infty} P \left( \sum_{j=1}^{\nu^{(n)}} \tilde{T}_{j}^{(n),B} > \frac{n^{\gamma + \beta - 1}}{\log(n)^{2}} | \nu^{(n)} = k \right) P(\nu^{(n)} = k)$$
which is bounded above by
\[
\sum_{k=1}^{\infty} P\left( \sum_{j=1}^{k} \frac{\tilde{G}}{G = k} \right) P(\tilde{G} = k) \leq \sum_{k=1}^{\infty} kP\left( \tilde{T}_{1}^{(n)} > \frac{n^{\gamma+\beta-1}}{\log(n)^{2}} \right) P(\tilde{G} = k).
\]

Thus
\[
P\left( \tilde{T}_{1}^{\beta,(n)} > \frac{n^{\gamma+\beta-1}}{\log(n)^{2}} \right) \leq E[\tilde{G}] P\left( \tilde{T}_{1}^{(n)} > \frac{n^{\gamma+\beta-1}}{\log(n)^{3}} \right) + \sum_{k=\lceil \log(n) \rceil}^{\infty} kP(\tilde{G} = k)
\]
\[
\leq 2P\left( \tilde{T}_{1}^{(n)} > \frac{n^{\gamma+\beta-1}}{\log(n)^{3}} \right) + 2^{-\log(n)+2} \left( \frac{\log(n) + 1}{2} \right). \quad (5.11)
\]

Since
\[
\lim_{n \to \infty} n^{1-\beta} 2^{-\log(n)+2} \left( \frac{\log(n) + 1}{2} \right) = 0,
\]
we do not need to worry with the rightmost term in (5.11) and to conclude we show that
\[
P\left( \tilde{T}_{j}^{(n),B} > \frac{n^{\gamma+\beta-1}}{\log(n)^{3}} \right) < \frac{1}{2n^{1-\beta} \log(n)},
\]
for large values of \( n \).

The time window of type \( \tilde{T}_{j}^{(n),B} \) is a coalescence time and we will use estimates like the one in Lemma 4.1 to control the tail of its distribution. We start this time window as soon as \( \tilde{R}_{i}^{(n)} \) or \( \tilde{R}_{t}^{(n)} \) reach a distance greater than \( n^{\gamma} \) from \( L_{i}^{(n)} \), which means that at this moment, the distance between \( \tilde{R}_{i}^{(n)} \) and \( R_{i}^{(n)} \) is at most \( n^{\gamma} \) plus some almost surely finite value \( \delta^{(n)} \) that depends on the overshoot distribution of \( \tilde{R}_{i}^{(n)} \) or \( R_{i}^{(n)} \) when one of them surpasses the distance \( n^{\gamma} \) from \( L_{i}^{(n)} \).

By a rough estimate the distribution of \( \delta^{(n)} \) is bounded by the sum of two random variables (not necessarily independent) one representing the overshoot distribution for \( \kappa_{+}^{(n)} \) of \( R_{i}^{(n)} - L_{i}^{(n)} \) and one the overshoot distribution for \( \tilde{R}_{i}^{(n)} - L_{i}^{(n)} \). So, by Claim B.1 in Appendix B and Lemma 2.6 of [13], \( \sup_{n \geq 1} E[\delta^{(n)}] \leq C' \). Then, since \( \tilde{T}_{j}^{(n),B} \) is a coalescence time between two r-paths that are evolving in different environments, we have by Remark 4.2 and Chebyshev inequality that, for large values of \( n \),
\[
P\left( \tilde{T}_{j}^{(n),B} > \frac{n^{\gamma+\beta-1}}{\log(n)^{3}} \right)
\]
\[
= P\left( \tilde{T}_{j}^{(n),B} > \frac{n^{\gamma+\beta-1}}{\log(n)^{3}} \Big| \delta^{(n)} < 2n^{\gamma} \right) P\left( \delta^{(n)} < 2n^{\gamma} \right) + P\left( \tilde{T}_{j}^{(n),B} > \frac{n^{\gamma+\beta-1}}{\log(n)^{3}} \Big| \delta^{(n)} \geq 2n^{\gamma} \right) P\left( \delta^{(n)} \geq 2n^{\gamma} \right)
\]
\[
\leq \frac{C_{0} \left( 2n^{\gamma} + 2n^{\gamma} \right) \log(n)^{3}}{n^{\gamma+\beta-1}} + E[\delta^{(n)}] \leq \frac{4C_{0}n^{\gamma} \log(n)^{3}}{2n^{\gamma}} + C'.
\]

(5.12)
To conclude, we just need to choose $\gamma$ and $\beta$ in such a way that the first term on the right hand side of (5.12) become smaller than $0.5n^{\beta-1}\log(n)^{-1}$ for large values of $n$ (recalling that $\gamma < 2$ and $\frac{7}{8} < \beta < 1$). It can be achieved for example if we take $\gamma = \frac{47}{24}$ and $\beta = \frac{23}{24}$, completing the proof.

From the proof of Lemma 5.8, we can state the following claim, which we use later.

Claim 5.1. For any $s > 0$, the amount of time where $|R_t^{(n)} - \tilde{R}_t^{(n)}| > n^{\frac{23}{24}}$ in the time interval $[0, sn^2]$ is almost surely $o(n^{\frac{23}{24}})$.

Proof of Lemma 5.7. We have a sum of non-negative terms, thus it is enough to show that for any fixed $t > 0$ and any $\epsilon > 0$:

$$\limsup_{n \to \infty} P\left(\frac{1}{n} \sum_{j=1}^{nt^2} U_j^{(n)} > \epsilon\right) = 0.$$ 

Here we put $\Gamma_{n,t}$ as the event that neither $L_t^{(n)}$ nor $\tilde{R}_t^{(n)}$ make a jump of size greater than $\kappa \log(n)/2$ in time interval $[0, tn^2]$. By Corollary 5.5, it is enough to prove that

$$\limsup_{n \to \infty} P\left(\frac{1}{n} \sum_{j=1}^{nt^2} U_j^{(n)} > \epsilon ; \Gamma_{n,t}\right) = 0,$$

Note that the distribution of $U_j^{(n)}$ on the event $\{U_j^{(n)} > 0\}$ depends on how far the paths $L_j^{(n)}$ and $\tilde{R}_j^{(n)}$ are from each other before the attempt of crossing. However, when $L_j^{(n)}$ and $\tilde{R}_j^{(n)}$ are evolving independently, we have that $\Delta_t^{(n)} = \tilde{R}_j^{(n)} - L_j^{(n)}$ is a random walk with increments that have finite absolute third moment, which follows from Remark 3.1. Then, by Lemma 2.6 of [13], there exists $C' = C'(p)$ such that

$$E[U_j^{(n)}|U_j^{(n)} > 0, L_{j-1}^{(n)}, \tilde{R}_{j-1}^{(n)}] \leq C'.$$

Conditioning on the event that suppresses long jumps, the conditional expectation of $U_j^{(n)}$ only decreases. Thus the previous claims and the non-crossing rule of steps $(\tilde{R}.1)$, $(\tilde{R}.2)$ and $(\tilde{R}.3)$ guarantee that (enlarging $C'$ if necessary)

$$E[U_j^{(n)}|U_j^{(n)} > 0, \Gamma_{n,t}] \leq C'.$$

Applying Markov’s inequality:

$$P\left(\frac{1}{n} \sum_{j=1}^{nt^2} U_j^{(n)} > \epsilon ; \Gamma_{n,t}\right) = P(\Gamma_{n,t}) P\left(\frac{1}{n} \sum_{j=1}^{nt^2} U_j^{(n)} > \epsilon | \Gamma_{n,t}\right) \leq \frac{P(\Gamma_{n,t})}{n\epsilon} E\left(\sum_{j=1}^{nt^2} U_j^{(n)} | \Gamma_{n,t}\right).$$

(5.13)
Now, denote by \( N_u^{t,(n)} \) the random variable that counts the number of attempts of crossings between \( L_t^{(n)} \) and \( R_t^{(n)} \) before time \( tn^2 \) (so, the only cases where \( U^{(n)}_j \) have a positive value). Given \( N_u^{t,(n)} \) and \( i_1 < ... < i_{N_u^{t,(n)}} \leq tn^2 \), define the event

\[
J(i_1, ..., i_{N_u^{t,(n)}}) = \{ U_{i_1} > 0, ..., U_{i_{N_u^{t,(n)}}} > 0 \}.
\]

Then write the expectation in (5.13) as

\[
E \left( \sum_{i_1 < ... < i_{N_u^{t,(n)}}} P(J(i_1, ..., i_{N_u^{t,(n)}}) | N_u^{t,(n)}, \Gamma_{n,t}) E \left( \sum_{j=1}^{tn^2} U_j^{(n)} | J(i_1, ..., i_{N_u^{t,(n)}}, \Gamma_{n,t}) \right) \right),
\]

where

\[
E \left( \sum_{j=1}^{tn^2} U_j^{(n)} | J(i_1, ..., i_{N_u^{t,(n)}}, \Gamma_{n,t}) \right) \leq \sum_{k=1}^{N_u^{t,(n)}} E \left( U_{i_k}^{(n)} | U_{i_k}^{(n)} > 0, N_u^{t,(n)}, \Gamma_{n,t} \right)
\]

\[
\leq C' E(N_u^{t,(n)} | \Gamma_{n,t}).
\]

Returning to (5.13), we obtain that

\[
P \left( \frac{1}{n} \sum_{j=1}^{tn^2} U_j^{(n)} > \epsilon ; \Gamma_{n,t} \right) \leq \frac{C'}{n\epsilon} E(N_u^{t,(n)} | \Gamma_{n,t}).
\]

Therefore we need to show that \( E(N_u^{t,(n)} | \Gamma_{n,t}) \) is \( o(n) \). We are going to prove a stronger result, that \( E(N_u^{t,(n)} | \Gamma_{n,t}) \) converges to zero as \( n \) goes to infinity.

Given \( \Gamma_{n,t} \), the process \( \Delta^{(n)}_t \) does not make any jump of size greater than \( \kappa \log(n) \) in the interval \([0, tn^2] \). It means that as soon as \( \Delta^{(n)} \) becomes smaller than \( \kappa n^{\frac{1}{2}} \log(n)^2 \), it needs to cross towards 0, one by one, each interval of size \( \kappa \log(n) \), until it could reach 0. So, \( \Delta^{(n)} \) has to visit at least \( n^{\frac{1}{2}} \log(n) \) positions at its left before it can reach zero.

When \( \Delta^{(n)} \leq \kappa n^{\frac{1}{2}} \log(n)^2 \), \( \Delta^{(n)} \) jumps to the left, it remains in the same position for some time and, after that, it necessarily jumps to the right. So, let us define an auxiliary process \( \Delta^{(n)}_u \) based on \( \Delta^{(n)} \) that starts as soon as \( \Delta^{(n)} \) becomes smaller than \( \kappa n^{\frac{1}{2}} \log(n)^2 \), at the same position of \( \Delta^{(n)} \), and evolves in the following way:

(i) We only consider increments of time for \( \Delta^{(n)}_u \) at moments when \( \Delta^{(n)} \) can jump, i.e., if \( \Delta^{(n)} \leq \kappa n^{\frac{1}{2}} \log(n)^2 \) and we do not have \( \tilde{R}^{(n)} \approx L^{(n)} \) as in step (R.3).

(ii) Whenever \( \Delta^{(n)} \) jumps to zero, \( \Delta^{(n)}_u \) also jumps to zero and stops. It does not move anymore.

(iii) Whenever \( \Delta^{(n)} \) jumps from a position less than or equal to \( \kappa n^{\frac{1}{2}} \log(n)^2 \) to a position greater than \( \kappa n^{\frac{1}{2}} \log(n)^2 \), then if \( \Delta^{(n)}_u \) has not stopped yet, it jumps to \( \kappa n^{\frac{1}{2}} \log(n)^2 \).
(iv) Except for (ii), whenever $\Delta^{(n)}$ jumps from a position greater than $\kappa n^{\frac{1}{4}} \log(n)^2$ to a position less than or equal to $\kappa n^{\frac{1}{4}} \log(n)^2$, then $\Delta^{(n)}_u$ jumps to the same destination of $\Delta^{(n)}$.

(v) Except for (ii), if $\Delta^{(n)} \leq \kappa n^{\frac{1}{4}} \log(n)^2$, whenever $\Delta^{(n)}$ jumps to the left, the next movement that changes the value of $\Delta^{(n)}$ is necessarily a jump to the right after a period with $\tilde{R}^{(n)} \approx L^{(n)}$ when $L^{(n)}$ branches. In this case $\Delta^{(n)}_u$ makes a single jump equal to the sum of these two jumps of $\Delta^{(n)}$. Recall that $\Delta^{(n)} \geq 0$ by the non-crossing rule and equally $\Delta^{(n)}_u \geq 0$.

(vi) Whenever $\Delta^{(n)}$ makes jumps to the right that are not related to a branching of $L^{(n)}_t$ or stay in the same position without having $\tilde{R}^{(n)} \approx L^{(n)}$, and $\Delta^{(n)}_u$ has not stopped yet, except for the rule in (iii), $\Delta^{(n)}_u$ simply jumps equally to $\Delta^{(n)}$.

By construction, we can verify the following properties:

- If $\Delta^{(n)}_u$ starts at a position greater than zero, then the probability that $\Delta^{(n)}$ reaches zero in time interval $[0, tn^2]$ is equal to the probability that $\Delta^{(n)}_u$ reaches zero in this time interval.
- The process $\Delta^{(n)}_u$ is a Markov process while evolving inside the interval $[0, \kappa n^{\frac{1}{4}} \log(n)^2]$, except for the times where it stops at position $\kappa n^{\frac{1}{4}} \log(n)^2$ when $\Delta^{(n)}$ becomes greater than this same amount. But, given $\Gamma_{n,t}$, after one time unit, $\Delta^{(n)}_u$ returns to evolve as a Markov Process from a random position in $[\kappa n^{\frac{1}{4}} \log(n)^2 - \kappa \log(n), \kappa n^{\frac{1}{4}} \log(n)^2]$.
- Considering (v) and (vi), a typical jump of $\Delta^{(n)}_u$ is distributed as a typical jump of $\Delta^{(n)}$ before period with $\tilde{R}^{(n)} \approx L^{(n)}$ plus an independent positive jump of size greater or equal to one that is counted only when $\Delta^{(n)}$ jumps to the left and is related to exit from condition $\tilde{R}^{(n)} \approx L^{(n)}$. From this we have that $\Delta^{(n)}_u$ has a positive drift that is bounded from below by $\frac{p(1-p)}{2}$ for any $n \in \mathbb{N}$.

By the above properties, when $\Delta^{(n)}_u$ is at a position greater than or equal to $j\kappa \log(n)$ for $j = 1, \ldots, n^{\frac{1}{4}} \log(n)$, it has probability $p_u^{(n)}$ of never visiting $(j-1)\kappa \log(n)$ and due to the translation invariance of $\Delta^{(n)}_u$, this probability does not depend on $j$. Besides that, since $\Delta^{(n)}_u$ has a positive drift that does not converge to zero as $n$ goes to infinity, there exists a probability $p_u > 0$ such that $p_u^{(n)} \geq p_u$ for any value of $n$. Thus

$$P \left( \inf_{0 \leq s \leq tn^2} \Delta^{(n)}_u = 0 \mid \Gamma_{n,t} \right) \leq tn^2 \left( 1 - p_u \right)^{n^{\frac{1}{4}} \log(n)}.$$

Hence, we can write

$$E(N^{t,(n)}_u \mid \Gamma_{n,t}) \leq tn^2 P \left( \inf_{0 \leq s \leq tn^2} \Delta^{(n)}_u = 0 \mid \Gamma_{n,t} \right) \leq t^2 n^{4} \left( 1 - p_u \right)^{n^{\frac{1}{4}} \log(n)},$$

which converges to zero as $n \to \infty$ for any fixed $t > 0$, concluding the proof. □
5.2. Part II: Only one pair \((L^{(n)}, R^{(n)})_{n \in \mathbb{N}}\) with \(R^{(n)}_0 < L^{(n)}_0\). Before we prove the convergence in this case, we will state Lemma 5.9 and Lemma 5.10 that will be useful here and in the following scenarios. These lemmas are proved in Appendix B.

**Lemma 5.9.** Let \(l^{(n)}_x\) be an \(l\)-path that starts from \((x, s)\) and \(\tilde{l}^{(n)}_x\) be another \(l\)-path, evolving in the same environment, that starts from \((x + n^{\gamma_1}, s)\), for some \(\gamma_1 > 0\). Denote by \(r^{(n)}_x\) the coalescence time between \(l^{(n)}_x\) and \(\tilde{l}^{(n)}_x\). We have that the following limit holds:

\[
\lim_{n \to \infty} P \left( \sup_{n \leq t \leq s + r^{(n)}_x} \left| l^{(n)}_x(t) - \tilde{l}^{(n)}_x(t) \right| > n^{\gamma} \right) = 0, \quad \text{for all } x \in \mathbb{Z} \text{ and } \gamma > \gamma_1.
\]

This statement also holds if \(l^{(n)}_x\) and \(\tilde{l}^{(n)}_x\) are evolving according to independent environments (in this case, we can think of \(r^{(n)}_x\) as the first meeting time). Besides that, this statement also holds if we replace the \(l\)-paths by \(r\)-paths.

**Lemma 5.10.** Let \(r^{(n)}_x\) be an \(r\)-path that starts from \((x, s)\) and \(l^{(n)}_x\) be an \(l\)-path that starts from \((x + n^{\gamma_1}, s)\), for some \(\gamma_1 > 0\), both evolving according to the same environment. Now let \(\tilde{r}^{(n)}_x\) be an \(r\)-path that also starts from the same point \((x, s)\), but evolves in an independent environment until it crosses \(l^{(n)}_x\) for the first time. After that, \(\tilde{r}^{(n)}_x\) starts to evolve in the same environment of \(r^{(n)}_x\) and \(l^{(n)}_x\). Also define \(T^{(n)}_x\) as the amount of time after \(s\) that we have to wait until \(r^{(n)}_x\) and \(\tilde{r}^{(n)}_x\) meet each other for the first time at the right hand side of \(l^{(n)}_x\). Therefore the distribution of \(T^{(n)}_x = T^{(n)}_x\) does not depend on \(x\) or \(s\), moreover

\[
\lim_{n \to \infty} P(T^{(n)} > 2n^{2\gamma}) = 0, \quad \text{for any } \gamma > \gamma_1.
\]

By Remark 4.2, we have that Lemma 5.10 also holds if \(r^{(n)}_x\) and \(\tilde{r}^{(n)}_x\) continue evolving considering independent environments after they cross \(l^{(n)}_x\).

Now note that since \(R^{(n)}\) has positive drift and \(L^{(n)}\) has negative drift, they will cross each other at some stopping time \(S^{(n)}\) that is almost surely finite for any \(n \geq 1\). Besides that, as verified in Part I, \(\left\{ \frac{1}{n} l^{(n)}_x \right\}_{t \geq 0} \) are tight and then, we can take a subsequence that converges weakly to some limiting process \((L_t, R_t)_{t \geq 0}\). By Skorohod’s Representation Theorem, we can couple these processes for \(n \in \mathbb{N}\) and the limit process such that the convergence holds almost surely.

Let \(\overline{S}^{(n)}\) be the random time when \(R^{(n)}\) and \(L^{(n)}\) cross each other. Assuming the coupling, we have that \(\frac{1}{n^2} \overline{S}^{(n)} \to \overline{S}\), \(\frac{1}{n} R^{(n)}_0 \to R_{\overline{S}}\) and \(\frac{1}{n} L^{(n)}_0 \to L_{\overline{S}}\) almost surely as \(n\) goes to infinity, where \(\overline{S}\) is the crossing time between the limit paths \(L\) and \(R\). This is a consequence of the uniform convergence of the subsequence of \(\left\{ \frac{1}{n} l^{(n)}_x \right\}_{t \geq 0} \) that we got from Skorohod’s Representation Theorem. Then, by the interaction rule between
$R^{(n)}$ and $L^{(n)}$, we have that once they meet each other, $R^{(n)}$ can not be at the left-hand side of $L^{(n)}$ anymore.

So, we can divide our proof in two parts: one considering $R_t^{(n)}$ and $L_t^{(n)}$ for $t \in [0, S^{(n)})$ and the other when $t \geq S^{(n)}$. About this second scenario, we already have verified in Subsection 5.1 that the convergence holds when $R^{(n)}$ and $L^{(n)}$ start from fixed vertices such that $R_0^{(n)} \geq L_0^{(n)}$. The difference here is that these paths start at a random time $\tilde{S}^{(n)}$, but they are $\mathcal{F}_t$-stopping times and by the Strong Markov Property of $(L_t^{(n)}, R_t^{(n)})_{t \geq 0}$ the weak convergence holds in the time interval $[\tilde{S}, \infty]$.

For $t < S^{(n)}$, first note that $S^{(n)}$ is finite almost surely, as a consequence of Corollary 4.3. Indeed, the convergence that we mentioned above implies that $S^{(n)}$ is of order $n^2$, when $R_t^{(n)}$ and $L_t^{(n)}$ start at distance of order $n$. Besides that, given any initial distance between $R^{(n)}$ and $L^{(n)}$, Corollary 4.3 gives us an estimate on the tail of the distribution of $S^{(n)}$.

Together with Lemma 5.4, we have that $R_t^{(n)}$ and $L_t^{(n)}$ will not make jumps of size greater than $n^{\frac{1}{2}}$ in the time interval $[0, S^{(n)})$ with a probability that converges to one as $n \to \infty$.

Now denote by $\mathcal{T}^{(n)}$ the first time that $R^{(n)}$ and $L^{(n)}$ reach a distance smaller than $n^{\frac{1}{2}}$ from each other and let us couple $R^{(n)}$ with another path $\overline{R}^{(n)}$. The path $\overline{R}^{(n)}$ is equal to $R^{(n)}$ until $\mathcal{T}^{(n)}$. Note that, at this moment and conditioning on the event that $R^{(n)}$ and $L^{(n)}$ do not make jumps of size greater than $\kappa \log(n)$, $R^{(n)}$ is still on the left-hand side of $L^{(n)}$. Then $\overline{R}^{(n)}$ begins to evolve according to an independent environment that uses the families of variables $\tilde{\omega}(z)$ and $\tilde{\theta}(z)$ for $z \in \mathbb{Z}^2$, until $\overline{R}^{(n)}$ meets $R^{(n)}$ at right-hand side of $L^{(n)}$ for the first time. At this time, $\overline{R}^{(n)}$ returns to be equal to $R^{(n)}$. Denote by $S^{(n)}_c$ the first time where $\overline{R}^{(n)}$ is at right-hand side of $L^{(n)}$.

Since $\left\{ \left( \frac{1}{n} R_{t}^{(n)}, \frac{1}{n} R_{t}^{(n)} \right)_{t \geq 0} \right\}_{n \in \mathbb{N}}$ are tight, we can take a subsequence that converges weakly to some limiting process $(\overline{R}_t, R_t)_{t \geq 0}$. By Skorohod’s Representation Theorem, we can couple these processes for $n \in \mathbb{N}$ and the limit process such that the convergence holds almost surely. Also denote by $\tilde{S}_c$ the time where $\overline{R}$ crosses $L$.

Again conditioning on the event that $R^{(n)}$ and $L^{(n)}$ do not make jumps of size greater than $\kappa \log(n)$, we have that $\overline{R}^{(n)}$ and $L^{(n)}$ are independent paths until $\overline{R}^{(n)}$ meets $R^{(n)}$ at right of $L^{(n)}$. Then, by Lemma 5.3, $(\overline{R}_t^{(n)}, L_t^{(n)})_{t < \tilde{S}_c^{(n)}}$ converges weakly to $(B_t^l, B_t^l)_{t < \tilde{S}_c}$ under diffusive scaling, where $B_t^l$ and $B_t^l$ are independent Brownian motions with diffusion coefficient $\lambda_p$ and drift $b_p$ and $-b_p$ respectively.

To conclude, we have to show that $(R^{(n)}, L^{(n)})$ and $(\overline{R}^{(n)}, L^{(n)})$ converge weakly under diffusive scale to the same limit. Recalling the definition of $\mathcal{T}^{(n)}$ from Lemma 5.10, write

$$P\left( \sup_{\mathcal{T}^{(n)} \leq t \leq \mathcal{T}^{(n)} + \mathcal{T}^{(n)}} \left| R_t^{(n)} - \overline{R}_t^{(n)} \right| > n^{\frac{3}{2}} \right)$$
as
\[
P\left( \sup_{T^{(n)} \leq t \leq T^{(n)} + T^{(n)}} \left| R_t^{(n)} - \overline{R}_t^{(n)} \right| > n^{\frac{3}{8}} \right) \leq P\left( T^{(n)} > 2n^{\frac{3}{8}} \right) + P\left( \sup_{T^{(n)} \leq t \leq T^{(n)} + 2n^{\frac{3}{8}}} \left| R_t^{(n)} - \overline{R}_t^{(n)} \right| > n^{\frac{3}{8}} \right)
\]
\[
\leq P\left( T^{(n)} > 2n^{\frac{3}{8}} \right) + P\left( \sup_{T^{(n)} \leq t \leq T^{(n)} + 2n^{\frac{3}{8}}} \left| R_t^{(n)} - R_{T^{(n)}}^{(n)} \right| + \left| \overline{R}_t^{(n)} - \overline{R}_{T^{(n)}}^{(n)} \right| > n^{\frac{5}{8}} \right). \tag{5.14}
\]

The first term in (5.14) converges to zero as \( n \) goes to infinity by Lemma 5.10 (choosing \( \gamma_1 = \frac{3}{4} \) and \( \gamma = \frac{1}{4} \)). About the second term in (5.14), by Lemma 5.3 we have that both \( R^{(n)} \) and \( \overline{R}^{(n)} \) converge under diffusive scaling to Brownian motions with drift \( b_p \). Since \( 5/6 > 4/5 \), by the Strong Markov property, this second term also converges to zero as \( n \) goes to infinity, which let us conclude that \( (R^{(n)}, L^{(n)}) \) and \( (\overline{R}^{(n)}, L^{(n)}) \) converge weakly under diffusive scaling to the same limit.

Finally, the proofs presented in Sections 5.1 and 5.2 together gives us the proof of Proposition 5.2.

\[5.3 \text{ Part III: } k \text{ l-paths and } \tilde{k} \text{ r-paths.} \quad \]

Now we will assume that we have \( k \) sequences of l-paths \( (l_i^{(n)})_{i=1}^{k} \) of the DNB starting from \( z_i^{(n)} = (x_i^{(n)}, 0) \) with \( n^{-1}z_i^{(n)} \) converging to \( z_i = (x_i, 0) \) and \( \tilde{k} \) sequences of r-paths \( r_{\tilde{j}}^{(n)} \) starting from \( \tilde{z}_{\tilde{j}}^{(n)} = (\tilde{x}_{\tilde{j}}^{(n)}, 0) \), with \( n^{-1}\tilde{z}_{\tilde{j}}^{(n)} \) converging to \( \tilde{z}_{\tilde{j}} = (\tilde{x}_{\tilde{j}}, 0) \), for \( j = 1, \ldots, \tilde{k} \). Based on the non-crossing property, we assume the worst scenery, without loss of generality, which is all r-paths starting at left of all l-paths, i.e., \( \tilde{x}_{\tilde{k}}^{(n)} < \ldots < \tilde{x}_{\tilde{j}}^{(n)} < x_{1}^{(n)} < \ldots < x_{k}^{(n)} \).

The main strategy is to couple our collection of l-paths and r-paths with a collection of paths that are conditionally independent given the non-occurrence of long jumps. This coupling should have the property that the time windows where the collections differ are negligible under diffusive scaling. So we start by doing a slight change in paths \( (l_i^{(n)})_{i=1}^{k-1} \).

We denote these slightly changed paths by \( (\tilde{l}_i^{(n)})_{i=1}^{k-1} \) which constructed in the following way: \( \tilde{l}_i^{(n)} \) is equal to \( l_i^{(n)} \) until it reaches a distance smaller than \( n^{\frac{3}{4}} \) from another l-path at its right-hand side. When it happens, \( \tilde{l}_i^{(n)} \) begins to evolve according to a different environment \( (\omega, \theta_i) \) (which is a independent copy of \( (\omega(z), \theta(z)) \)) until it attempts to cross with this another l-path, at this moment the paths will coalesce at the position of the l-path with the greater index. Note that in a system with independent paths until coalescence, we may have crossings and coalescence only occurs when the paths meet each other at the same position. This is not the case here, however both the systems will have the same asymptotic behavior due to the coalescence times estimates.
Now we have to replace the r-paths with different versions, which will be denoted by \((\tau^{(n)}_j)_{j=1}^k\). Let \((\overline{\omega}_j(z), \overline{\theta}_j(z))_{z \in \mathbb{Z}^2}, j = 1, \ldots, \tilde{k}\) and \((\omega_i(z), \theta_i(z))_{z \in \mathbb{Z}^2}, i = 1, \ldots, k\), be iid environments which are also independent of \((\omega, \theta)\). The environments \((\omega_i, \theta_i)\) will be used to construct \(\overline{I}^{(n)}_j\). The environments \((\overline{\omega}_j, \overline{\theta}_j)\) will be used to construct \(\tau^{(n)}_j\). Now define, 
\[
\mathcal{F}_t = \sigma((\omega(z), \theta(z)), (\overline{\omega}_j(z), \overline{\theta}_j(z)), (\omega_i(z), \theta_i(z)), z_2 \leq t),
\]
for \(t \geq 0\), and consider the \(\mathcal{F}_t\)-stopping times 
\[
\mathcal{T}^{(n)}_{i,j} = \inf \{t \geq 0 : (\overline{I}^{(n)}_i(t) - \tau^{(n)}_j(t)) \leq n^{\frac{3}{4}}\}, \ i = 1, \ldots, k, \text{ and } j = 1, \ldots, \tilde{k},
\]
i.e. \(\mathcal{T}^{(n)}_{i,j}\) is the first time that \(\tau^{(n)}_j\) reaches a distance smaller than \(n^{\frac{3}{4}}\) to the left of \(\overline{I}^{(n)}_i\) and \(\mathcal{S}^{(n)}_{i,j}\) is the first time that either \(\tau^{(n)}_j\) meets or crosses \(\overline{I}^{(n)}_i\). Note that for any fixed \(n \in \mathbb{N}\) and \(i = 1, 2, \ldots, k\), by the assumption on the starting points, \(\mathcal{T}^{(n)}_{i,1} \leq \mathcal{T}^{(n)}_{i,2} \leq \ldots \leq \mathcal{T}^{(n)}_{i,k}\).

Each one of the paths \(\tau^{(n)}_1, \ldots, \tau^{(n)}_{\tilde{k}}\) will be constructed according to the following steps (and is illustrated in Figure 4)

(i) Each \(\tau^{(n)}_j\) are equal to \(r^{(n)}_j\) until they reach a stopping time \(\overline{I}^{(n)}_i\) or until two paths among \((\tau^{(n)}_j)_{j=1}^\tilde{k}\) reach a distance smaller than \(n^{\frac{3}{4}}\) from each other.

(ii) Whenever any \(\tau^{(n)}_j\) reaches a distance smaller than \(n^{\frac{3}{4}}\) from another r-path \(\tau^{(n)}_l\) that is at its left and is evolving according to the same environment, \(\tau^{(n)}_j\) begin to evolve according a different environment using families of variables \((\overline{\omega}_j(z))_{z \in \mathbb{Z}^2}\) and \((\overline{\theta}_j(z))_{z \in \mathbb{Z}^2}\) (independent copies respectively of \(\omega\) and \(\theta\)). After that, when either \(\tau^{(n)}_j\) meets or crosses \(\tau^{(n)}_l\) for the first time, they coalesce and continue to evolve according to the environment that \(\tau^{(n)}_l\) is considering.

(iii) Whenever \(\tau^{(n)}_j\) reaches a stopping time \(\overline{I}^{(n)}_{i,j}\) for \(i = 1, \ldots, k\), it will start to evolve according to the environment \((\overline{\omega}_j(z), \overline{\theta}_j(z))\) until it reaches a \(\mathcal{F}_t\)-stopping time \(\mathcal{S}^{(n)}_{i,j}\) defined below. The only exception to this rule is the situation described in (v).

\[
\mathcal{S}^{(n)}_{i,j} = \inf \{t \geq 0 : \tau^{(n)}_i(t) \geq \overline{I}^{(n)}_i(t)\}, \ i = 1, \ldots, k, \text{ and } j = 1, \ldots, \tilde{k},
\]
i.e. \(\mathcal{S}^{(n)}_{i,j}\) is the first time that either \(\tau^{(n)}_j\) meets or crosses \(\overline{I}^{(n)}_i\). Note that for any fixed \(n \in \mathbb{N}\) and \(i = 1, 2, \ldots, k\), by the assumption on the starting points, \(\mathcal{S}^{(n)}_{i,1} \leq \mathcal{S}^{(n)}_{i,2} \leq \ldots \leq \mathcal{S}^{(n)}_{i,k}\). Also, for any fixed \(n \in \mathbb{N}\) and \(j = 1, 2, \ldots, \tilde{k}\), \(\mathcal{T}^{(n)}_{i,j} \leq \mathcal{T}^{(n)}_{2,j} \leq \ldots \leq \mathcal{T}^{(n)}_{k,j}\) and \(\mathcal{S}^{(n)}_{i,j} \leq \mathcal{S}^{(n)}_{2,j} \leq \ldots \leq \mathcal{S}^{(n)}_{k,j}\). Some of these stopping times (from the same vector) can be equal, which happens, for example, if some l-paths coalesce before they get close to the same \(\tau^{(n)}_j\). Moreover, \(\mathcal{T}^{(n)}_{i,j} \leq \mathcal{S}^{(n)}_{i,j}\) for any pair \((i, j)\) with \(i = 1, \ldots, k\) and \(j = 1, \ldots, \tilde{k}\).
(iv) Whenever $\tau_j^{(n)}$ reaches a stopping time $\tilde{\tau}_i^{(n)}$, $\tau_j^{(n)}$ will begin to evolve in the same environment of $\tilde{\tau}_i^{(n)}$ until one of the three following possibilities happens:

1. Another r-path $\tilde{\tau}_i^{(n)}$ crosses $\tilde{\tau}_i^{(n)}$ and stay between $\tau_j^{(n)}$ and $\tilde{\tau}_i^{(n)}$, within a distance smaller than $n\frac{3}{2}$ from $\tilde{\tau}_i^{(n)}$. In this scenario, $\tau_j^{(n)}$ begin to evolve in the same environment of $\tilde{\tau}_i^{(n)}$ and $\tau_j^{(n)}$ automatically begin to evolve according to the environment $(\tilde{\omega}_j(z), \tilde{\theta}_j(z))$ because we have the situation described in the step (ii).

2. $\tau_j^{(n)}$ meets $\tau_l^{(n)}$ in the same environment. In this case we return to have $\tau_j^{(n)}$ equal to $\tau_l^{(n)}$.

3. $\tau_j^{(n)}$ reaches a distance smaller than $n\frac{3}{2}$ to the left of another l-path $\tilde{\tau}_m^{(n)}$, $m \geq i + 1$. In this case we return to step (iii). The only exception to this rule is the situation described in (v).

(v) Whenever $\tau_j^{(n)}$ returns to a distance smaller than $n\frac{3}{2}$ from the nearest l-path $\tilde{\tau}_i^{(n)}$ at its left and there are not any other r-path $\tilde{\tau}_i^{(n)}$ between $\tau_j^{(n)}$ and $\tilde{\tau}_i^{(n)}$, $\tau_j^{(n)}$ returns to evolve in the same environment of $\tilde{\tau}_i^{(n)}$, like in step (iv), if they were not already in the same environment.

To illustrate an example of this construction, we have Figure 4 considering only one alternative r-path $\tilde{\tau}(n)$. Note that when $\tilde{\tau}(n)$ hits the stopping time $\tilde{T}_1^{(n)} = \tilde{T}_1^{(n)}$, it begins to evolve independently of $\tau(n)$ (which is represented by the dashed line) and this event initiate the step (iii). At time $\tilde{S}_1^{(n)} = \tilde{S}_1^{(n)}$, $\tilde{\tau}(n)$ cross $\tilde{T}_1^{(n)}$ and starts to evolve in the same environment of $\tilde{T}_1^{(n)}$, which initiate the step (iv), and then we will wait for either the Scenario 1 or Scenario 3 to happen. At time $\tilde{T}_3^{(n)} = \tilde{T}_3^{(n)}$, $\tilde{\tau}(n)$ meets $\tau(n)$ in a distance greater than $n\frac{3}{2}$ from any l-path at their right-hand side, so $\tilde{\tau}(n)$ returns to be equal to $\tau(n)$ and it means that the event described in Scenario 3 happened before Scenario 1. At time $\tilde{T}_3^{(n)} = \tilde{T}_3^{(n)} = \tilde{T}_2^{(n)} = \tilde{T}_3^{(n)}$, $\tau(n)$ reaches a distance smaller than $n\frac{3}{2}$ from an l-path at its right and we return to the step (iii).

From $(\tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)}, \tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)})$, we will construct the process $(\tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)}, \tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)})$. They will evolve equally, except that the paths $(\tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)}, \tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)})$ can not make jumps of size greater than $n\frac{3}{2}$. So both process will be equal until the first time where one path among $(\tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)}, \tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)})$ makes a jump of size greater than $n\frac{3}{2}$. At this moment, the associated path in $(\tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)}, \tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)})$ will jump only $n\frac{3}{2}$ and both process will no longer be equal, despite that both paths continue to evolve according to the steps described for $(\tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)}, \tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)})$.

From $(\tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)}, \tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)})$ we define, $\tilde{T}_{i,j}^{(n)}$ and $\tilde{S}_{i,j}^{(n)}$ in the same way that we defined, $\tilde{T}_{i,j}^{(n)}$ and $\tilde{S}_{i,j}^{(n)}$ from $(\tau_1^{(n)}, \ldots, \tau_k^{(n)}, \tilde{\tau}_1^{(n)}, \ldots, \tilde{\tau}_k^{(n)})$, which are also $\tilde{F}_t$-stopping times.
Figure 4. Example of the evolution of \( r^{(n)} \). The gray areas indicates the regions near to any \( l_j^{(n)} \), \( j = 1, 2, 3 \) such that if \( r^{(n)} \) enters there, its behavior will change. The dashed lines indicate the evolution of \( r^{(n)} \) when it is not equal to \( r^{(n)} \). The random times in the left-hand side of the figure indicates when something relevant happens in our construction.

Now fix \( s > 0 \) and let us define the following events:

\[
A_{n,s} = \left\{ \max_{1 \leq i \leq k} \sup_{0 \leq t \leq sn^2} \left| l_i^{(n)}(t) - \bar{\tau}_i^{(n)}(t) \right| \leq n^{\frac{7}{18}} \text{ and } \max_{1 \leq j \leq \tilde{k}} \sup_{0 \leq t \leq sn^2} \left| r_j^{(n)}(t) - \bar{\tau}_j^{(n)}(t) \right| \leq n^{\frac{14}{18}} \right\}
\]

\[
B_{n,s} = \bigcap_{i=1}^{k} \bigcap_{j=1}^{\tilde{k}} \left\{ \bar{l}_i^{(n)}(t) \text{ and } \bar{\tau}_j^{(n)}(t) \text{ do not make any jump of size greater than } \frac{n^{\frac{3}{2}}}{2} \text{ in } [0, sn^2] \right\}
\]

Our approach will be the following, similar to the one used in the proof of Theorem 8 of [5]: Prove that conditioned to \( A_{n,s} \) and \( B_{n,s} \), the processes \((\bar{l}_1^{(n)}, \ldots, \bar{l}_k^{(n)}, \bar{r}_1^{(n)}, \ldots, \bar{r}_{\tilde{k}}^{(n)})\) and \((\bar{\tau}_1^{(n)}, \ldots, \bar{\tau}_k^{(n)}, \bar{\tau}_1^{(n)}, \ldots, \bar{\tau}_{\tilde{k}}^{(n)})\) have the same weak limit under diffusive scaling and show that the probabilities of the events \( A_{n,s}^{c} \) and \( B_{n,s}^{c} \) goes to zero as \( n \) goes to infinity, which will allow us to replace the original paths by \((\bar{l}_1^{(n)}, \ldots, \bar{l}_k^{(n)}, \bar{r}_1^{(n)}, \ldots, \bar{r}_{\tilde{k}}^{(n)})\) in our proof.

Let us begin with \( B_{n,s}^{c} \). Since the probability that \( \bar{l}_i^{(n)} \) makes a jump of size greater than \( n^{\frac{3}{2}} \) does not depend on \( i \) and is equal to the probability of \( \bar{\tau}_j^{(n)} \) makes a jump of size
greater than $n^{\frac{3}{2}}$, for any $j$, we can apply Lemma 5.4 and have that

$$P(\mathcal{B}_{n,s}^c) \leq (k + \tilde{k})P\left(\tau_1^{(n)} \text{ makes a jump of size greater than } n^{\frac{3}{2}}/2 \text{ in } sn^2 \text{ attempts}\right)$$

$$\leq (k + \tilde{k})sn^2P(\tau_1^{(n)} \text{ makes a jump of size greater than } n^{\frac{3}{2}}/2 \text{ in one attempt})$$

$$\leq 2(k + \tilde{k})sn^2 \exp\left\{-\frac{c}{2} \left(n^{\frac{3}{2}} - 2\right)\right\}.$$  \hspace{1cm} (5.15)

which converges to zero as $n$ goes to infinity.

To deal with $\mathcal{A}_{n,s}^c$, we will write $\mathcal{A}_{n,s}^c = \mathcal{A}_{n,s}^c \cup \tilde{A}_{n,s}^c$ where:

$$\mathcal{A}_{n,s}^c = \left\{ \max_{1 \leq i \leq k} \sup_{0 \leq t \leq sn^2} |l_i^{(n)}(t) - \tilde{l}_i^{(n)}(t)| > n^{\frac{7}{8}} \right\},$$

$$\tilde{A}_{n,s}^c = \left\{ \max_{1 \leq j \leq k} \sup_{0 \leq t \leq sn^2} |r_j^{(n)}(t) - \tilde{r}_j^{(n)}(t)| > n^{\frac{14}{15}} \right\}.$$

About $\mathcal{A}_{n,s}^c$, we can write $P(\mathcal{A}_{n,s}^c)$ as

$$P\left(\bigcup_{i=1}^{k} \left\{ \sup_{0 \leq t \leq sn^2} |l_i^{(n)}(t) - \tilde{l}_i^{(n)}(t)| > n^{\frac{7}{8}} \right\} \right) \leq kP\left( \sup_{0 \leq t \leq sn^2} |l_1^{(n)}(t) - \tilde{l}_1^{(n)}(t)| > n^{\frac{7}{8}} \right).$$  \hspace{1cm} (5.16)

Now applying Lemma 5.9 with $\gamma_1 = \frac{3}{4}$ and $\gamma = \frac{7}{8}$, we have that the right hand side of the expression (5.16) converges to zero as $n$ goes to infinity, since $\tilde{l}_1^{(n)}$ and $\tilde{l}_1^{(n)}$ will be different only in a time window that starts when $\tilde{l}_1^{(n)}$ reaches a distance smaller than $n^{\frac{3}{2}}$ from another $l$-path $\tilde{l}_2^{(n)}$ and ends when $\tilde{l}_1^{(n)}$ and $\tilde{l}_2^{(n)}$ coalesce.

About $\tilde{A}_{n,s}^c$, note that $r_j^{(n)}(t)$ and $\tilde{r}_j^{(n)}(t)$ can be different for two reasons: Either because $\tilde{r}_j^{(n)}(t)$ reaches a distance smaller than $n^{\frac{3}{2}}$ from another $\tilde{l}_i^{(n)}(t)$ or because $\tilde{r}_j^{(n)}(t)$ reaches a distance smaller than $n^{\frac{3}{2}}$ from an $l$-path $\tilde{l}_i^{(n)}$ at its right. We can control the probability of have $\tilde{A}_{n,s}^c$ happening due to the first reason using the same calculation done in (5.16). Now note that each $r_j^{(n)}(t)$ and $\tilde{r}_j^{(n)}(t)$ are different due to the second reason only during a maximum of $k$ time windows that start at stopping times $T_{i,j}^{(n)}$ and have some length $T_{i,j}^{(n)}$ with the same distribution of $T^{(n)}$ defined in Lemma 5.10 and then, applying this lemma, we can bound the probability of $\tilde{A}_{n,s}^c$ occurring due to the second reason by

$$P\left(\bigcup_{j=1}^{k} \bigcup_{i=1}^{k} \left\{ \sup_{T_{i,j}^{(n)} \leq t \leq T_{i,j}^{(n)} + T_{i,j}^{(n)}} \left| r_j^{(n)}(t) - \tilde{r}_j^{(n)}(t) \right| > n^{\frac{14}{15}} \right\} \right).$$
which is bounded above by
\begin{align}
\tilde{k}kP \left( \sup_{T_{1,1}^{(n)} \leq t \leq T_{1,1}^{(n)} + \tau_{1,1}^{(n)}} \left| r_1^{(n)}(t) - \tilde{r}_1^{(n)}(t) \right| > n^{14/13} \right) \\
\leq \tilde{k}kP \left( \sup_{T_{1,1}^{(n)} \leq t \leq T_{1,1}^{(n)} + \tau_{1,1}^{(n)}} \left| r_1^{(n)}(t) - \tilde{r}_1^{(n)}(t) \right| > n^{4/13}, \tau_{1,1}^{(n)} > 2n^{13/13} \right) \\
+ \tilde{k}kP \left( \sup_{T_{1,1}^{(n)} \leq t \leq T_{1,1}^{(n)} + \tau_{1,1}^{(n)}} \left| r_1^{(n)}(t) - \tilde{r}_1^{(n)}(t) \right| > n^{14/13}, \tau_{1,1}^{(n)} \leq 2n^{13/13} \right) \\
\leq \tilde{k}kP \left( \frac{\tau_{1,1}^{(n)}}{n^{14/13}} > 2n^{13/13} \right) + \tilde{k}kP \left( \sup_{T_{1,1}^{(n)} \leq t \leq T_{1,1}^{(n)} + 2n^{13/13}} \left| r_1^{(n)}(t) - \tilde{r}_1^{(n)}(t) \right| > n^{14/13} \right). 
\tag{5.17}
\end{align}

The first term in (5.17) converges to zero, as \( n \) goes to infinity, by Lemma 5.10 (choosing \( \gamma_1 = \frac{3}{4} \) and \( \gamma = \frac{13}{14} \)). About the second term in (5.17), we have by Lemma 5.3 that both \( r_1^{(n)} \) and \( \tilde{r}_1^{(n)} \) converge under diffusive scale to Brownian motions with diffusion coefficient \( \lambda_p \) and drift \( b_p \). So, by strong Markov property, this second term also converges to zero as \( n \) goes to infinity and hence \( P(A_{s,s}^{(n)}) \) goes to zero as \( n \) goes to infinity.

Now let us fix an uniformly continuous bounded function \( H : C([0, s], \mathbb{R}^{k+k}) \rightarrow \mathbb{R} \). Denoting by \((L_1, \ldots, L_k, R_1, \ldots, R_k)\) the collection of left-right coalescence Brownian motions starting from \((z_1, \ldots, z_k, \tilde{z}_1, \ldots, \tilde{z}_k)\), then the desired convergence is equivalent to show that:

\[
\left| E \left[ H \left( \frac{l_1^{(n)}(sn^2)}{n}, \ldots, \frac{l_k^{(n)}(sn^2)}{n}, \frac{r_1^{(n)}(sn^2)}{n}, \ldots, \frac{r_k^{(n)}(sn^2)}{n} \right) \right] - E[H(L_1, \ldots, L_k, R_1 \ldots R_k) \right] \right|
\]

converges to zero as \( n \rightarrow \infty \). To prove it, we will make use of the auxiliary processes \((\bar{r}_1^{(n)}, \ldots, \bar{r}_k^{(n)}, \bar{l}_1^{(n)}, \ldots, \bar{l}_k^{(n)})\) and \((\tilde{l}_1^{(n)}, \ldots, \tilde{l}_k^{(n)}, \tilde{r}_1^{(n)}, \ldots, \tilde{r}_k^{(n)})\) to bound the expression above by

\[
\left| E \left[ H \left( \frac{l_1^{(n)}(sn^2)}{n}, \ldots, \frac{l_k^{(n)}(sn^2)}{n}, \frac{r_1^{(n)}(sn^2)}{n}, \ldots, \frac{r_k^{(n)}(sn^2)}{n} \right) \right] - E \left[ H \left( \frac{\bar{r}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\bar{r}_k^{(n)}(sn^2)}{n}, \frac{\bar{l}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\bar{l}_k^{(n)}(sn^2)}{n} \right) \right] \right| \\
+ E \left[ H \left( \frac{\tilde{l}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\tilde{l}_k^{(n)}(sn^2)}{n}, \frac{\tilde{r}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\tilde{r}_k^{(n)}(sn^2)}{n} \right) \right] - E \left[ H \left( \frac{l_1^{(n)}(sn^2)}{n}, \ldots, \frac{l_k^{(n)}(sn^2)}{n}, \frac{r_1^{(n)}(sn^2)}{n}, \ldots, \frac{r_k^{(n)}(sn^2)}{n} \right) \right] \\
+ E \left[ H \left( \frac{\tilde{l}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\tilde{l}_k^{(n)}(sn^2)}{n}, \frac{\tilde{r}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\tilde{r}_k^{(n)}(sn^2)}{n} \right) \right] - E[H(L_1, \ldots, L_k, R_1 \ldots R_k)]
\tag{5.18}
\]

The first term in the right hand side of (5.18) is bounded above by

\[
\leq E \left[ \left| H \left( \frac{l_1^{(n)}(sn^2)}{n}, \ldots, \frac{l_k^{(n)}(sn^2)}{n}, \frac{r_1^{(n)}(sn^2)}{n}, \ldots, \frac{r_k^{(n)}(sn^2)}{n} \right) - H \left( \frac{\bar{l}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\bar{l}_k^{(n)}(sn^2)}{n}, \frac{\bar{r}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\bar{r}_k^{(n)}(sn^2)}{n} \right) \right| I_{A_{s,s}^{(n)}} \right] + \\
+ E \left[ \left| H \left( \frac{l_1^{(n)}(sn^2)}{n}, \ldots, \frac{l_k^{(n)}(sn^2)}{n}, \frac{r_1^{(n)}(sn^2)}{n}, \ldots, \frac{r_k^{(n)}(sn^2)}{n} \right) - H \left( \frac{\tilde{l}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\tilde{l}_k^{(n)}(sn^2)}{n}, \frac{\tilde{r}_1^{(n)}(sn^2)}{n}, \ldots, \frac{\tilde{r}_k^{(n)}(sn^2)}{n} \right) \right| I_{A_{s,s}^{(n)}} \right].
\tag{5.19}
\]
Now note that, on the event $A_{n,s}$, we have that $\sup_{0 \leq t \leq sn^2} \left| \frac{f_i^{(n)}(t)}{n} - \frac{x_i^{(n)}(t)}{n} \right| \to 0$ as $n$ goes to infinity, for $i = 1, \ldots, k$, and $\sup_{0 \leq t \leq sn^2} \left| \frac{\delta_j^{(n)}(sn^2)}{n} - \frac{x_j^{(n)}(sn^2)}{n} \right| \to 0$ as $n$ goes to infinity, for $j = 1, \ldots, \tilde{k}$. Then, since $H$ is a continuous function, we have that the first term in the right hand side of (5.19) converges to zero as $n$ goes to infinity. Besides that, since $P(A_{n,s}^c) \to 0$ as $n$ goes to infinity and $H$ is a bounded function, we have that the second term in the right hand side of (5.19) also converges to zero as $n$ goes to infinity.

About the second term in the right hand side of (5.18), note that conditioned to the occurrence of $B_{n,s}$, we have that the processes $(\hat{l}^{(n)}_1, \ldots, \hat{l}^{(n)}_k, \hat{r}^{(n)}_1, \ldots, \hat{r}^{(n)}_k)$ and $(\hat{l}^{(n)}_1, \ldots, \hat{l}^{(n)}_k, \tilde{r}^{(n)}_1, \ldots, \tilde{r}^{(n)}_k)$ are equal in $[0, sn^2]$. So, we can write:

$$\left| E \left[ H \left( \frac{\hat{l}^{(n)}_1(sn^2)}{n}, \ldots, \frac{\hat{l}^{(n)}_k(sn^2)}{n}, \frac{\hat{r}^{(n)}_1(sn^2)}{n}, \ldots, \frac{\hat{r}^{(n)}_k(sn^2)}{n} \right) - E \left[ H \left( \frac{\hat{l}^{(n)}_1(sn^2)}{n}, \ldots, \frac{\hat{l}^{(n)}_k(sn^2)}{n}, \frac{\tilde{r}^{(n)}_1(sn^2)}{n}, \ldots, \frac{\tilde{r}^{(n)}_k(sn^2)}{n} \right) \right] \right| =$$

$$= E \left[ \left| H \left( \frac{\hat{l}^{(n)}_1(sn^2)}{n}, \ldots, \frac{\hat{l}^{(n)}_k(sn^2)}{n}, \frac{\hat{r}^{(n)}_1(sn^2)}{n}, \ldots, \frac{\hat{r}^{(n)}_k(sn^2)}{n} \right) - H \left( \frac{\hat{l}^{(n)}_1(sn^2)}{n}, \ldots, \frac{\hat{l}^{(n)}_k(sn^2)}{n}, \frac{\tilde{r}^{(n)}_1(sn^2)}{n}, \ldots, \frac{\tilde{r}^{(n)}_k(sn^2)}{n} \right) \right| 1_{B_{n,s}} \right] \leq 2(k + \tilde{k})sn^2 \exp \left\{ -\frac{c}{2} \left( \frac{3}{4} \right) \right\} \|H\|_{\infty},$$

which converges to zero as $n$ goes to infinity since $H$ is a bounded function.

Now it only remains to deal with the last term in the right hand side of (5.18). To show that this term converges to zero, we will use the same argument of [21] to prove an equivalent result for simple coalescing random walks with branching (Proposition 5.2 of that article). First, note that, for any fixed $t \in [0, sn^2]$, we have that all l-paths $\hat{l}_i, i = 1, \ldots, k$ are independent until they meet or try to cross each other, i.e. when they coalesce. Now we will argue that the jump probabilities of $\hat{r}_j^{(n)}(t)$, for any $j = 1, \ldots, \tilde{k}$ given the position of the nearest l-path at its left-hand side, which we denote by $\hat{l}_m$, is independent of any other l-path $\hat{l}_i, i = 1, \ldots, k$ that have not coalesced yet with $\hat{l}_m$:

- For all $i < m$, with $\hat{l}_i^{(n)}(t) \neq \hat{l}_m^{(n)}(t)$, we have that given $\hat{l}_m^{(n)}(t)$, $\hat{r}_j^{(n)}(t)$ jumps independently of $\hat{l}_i^{(n)}(t)$ because if $\hat{l}_i^{(n)}(t)$ is at a distance greater than $n^{3/2}$ from $\hat{r}_j^{(n)}(t)$, we have independence due to the fact that the paths $\hat{r}_j^{(n)}$ and $\hat{l}_i^{(n)}$ can not make jumps of size greater than $n^{3/2}/2$. If $\hat{l}_i^{(n)}(t)$ is at a distance smaller than $n^{3/2}$ from $\hat{r}_j^{(n)}(t)$, it is also at a distance smaller than $n^{3/2}$ from $\hat{l}_m^{(n)}(t)$ and consequently, $\hat{l}_i^{(n)}(t)$ is evolving according to a different environment. Note that if $\hat{l}_i^{(n)}(t)$ would make a jump trying to cross $\hat{r}_j^{(n)}(t)$, then $\hat{l}_i^{(n)}(t)$ would also try to cross $\hat{l}_m^{(n)}(t)$ and hence these two l-paths would coalesce.
• For all \( i > m \), with \( \tilde{l}^{(n)}(t) \neq \tilde{l}^{(m)}(t) \), we have that \( \tilde{r}^{(n)}(t) \) is independent of \( \tilde{l}^{(m)}(t) \) because if \( \tilde{l}^{(m)}(t) \) is at a distance smaller than \( n^2/2 \) from \( \tilde{l}^{(n)}(t) \), then \( \tilde{r}^{(n)}(t) \) is evolving according to a different environment. If \( \tilde{l}^{(n)}(t) \) is at a distance greater than \( n^2/2 \) from \( \tilde{r}^{(n)}(t) \), we have independence due to the fact that the paths \( \tilde{r}^{(n)} \) and \( \tilde{l}^{(n)} \) can not make jumps of size greater than \( n^2/2 \).

Besides that, we also have independence between pairs of r-paths \( \tilde{r}_j, j = 1, \ldots, \tilde{k} \). When these r-paths are at a distance greater than \( n^2/4 \) of each other, we have independence because \( \tilde{r}_j \) do not make jumps of size greater than \( n^2/2 \) and when they are at a distance smaller than \( n^2/4 \) of each other, we also have independence because they will be evolving in different environments.

So, due to the Markov property of the paths and the independence mentioned above, we have that \((\tilde{l}^{(n)}_1, \ldots, \tilde{l}^{(n)}_k, \tilde{r}^{(n)}_1, \ldots, \tilde{r}^{(n)}_k)\) can be constructed inductively by concatenating independent evolutions of sets of paths, where each set consists of either a single l-path, a single r-path or a pair of left-right paths, like was done in [21].

Before we describe the construction, for a fixed \( n \), let us define a non decreasing sequence of stopping times \( \tilde{\tau}^{(n)}_j \) where \( \tilde{\tau}^{(n)}_1 \) means the first time where either a coalescence between two l-paths occurs or a crossing between one r-path and an l-path occurs. \( \tilde{\tau}^{(n)}_2 \) means the second time where either a coalescence between two l-paths occurs or a crossing between one r-path and one l-path occurs, and so on. Since we have at most \( k + \tilde{k} - 2 \) coalescence events and at most \( k\tilde{k} \) crossings, the sequence \( \tilde{\tau}^{(n)}_j \) will have at most \( k\tilde{k} + k + \tilde{k} - 2 \) stopping times.

We first construct the system up to \( \tilde{\tau}^{(n)}_1 \). The paths are initially ordered as \( \tilde{r}^{(n)}_1, \ldots, \tilde{r}^{(n)}_k \), \( \tilde{l}^{(n)}_1, \ldots, \tilde{l}^{(n)}_k \), so, we can partition them as \( \{\tilde{r}^{(n)}_1\} \ldots \{\tilde{r}^{(n)}_k\} \{\tilde{l}^{(n)}_1\} \ldots \{\tilde{l}^{(n)}_k\} \) and each partition evolve independently until they hit \( \tilde{\tau}^{(n)}_1 \). When it happens, we have a change in this partition. For example, if \( \tilde{\tau}^{(n)}_1 \) is related to a crossing event between \( \tilde{r}^{(n)}_k \) and \( \tilde{l}^{(n)}_1 \), we will begin to have the partition \( \{\tilde{r}^{(n)}_1\} \ldots \{\tilde{r}^{(n)}_{k-1}\} \{\tilde{l}^{(n)}_1\} \{\tilde{l}^{(n)}_2\} \ldots \{\tilde{l}^{(n)}_k\} \). If \( \tilde{\tau}^{(n)}_1 \) is related to a coalescence event, let us say between \( \tilde{l}^{(n)}_2 \) and \( \tilde{l}^{(n)}_3 \) for example, we will have the partition \( \{\tilde{r}^{(n)}_1\} \ldots \{\tilde{r}^{(n)}_{k-1}\} \{\tilde{l}^{(n)}_1\} \{\tilde{l}^{(n)}_3\} \{\tilde{l}^{(n)}_4\} \ldots \{\tilde{l}^{(n)}_k\} \). After that, we continue to let the partition elements evolve independently until they hit \( \tilde{\tau}^{(n)}_2 \), when we have another change in the partition. This procedure continues in the same way, changing the partition whenever the system hits a stopping time \( \tilde{\tau}^{(n)}_j \). In addition, note that some of the stopping times \( \tilde{\tau}^{(n)}_j \) can be equal. Since the sequence \( \tilde{\tau}^{(n)}_j \) has at most a finite number of stopping times, we change the partition only a finite number of times and it eventually leads to a single pair \( \{\tilde{l}^{(n)}_k, \tilde{r}^{(n)}_k\} \).

So, by Markov property of our system and the observation that the stopping times \( \tilde{\tau}^{(n)}_j \) used in the construction is almost surely continuous functionals on \( \Pi^{k+\tilde{k}} \) with respect
to the law of independent evolution of paths in different partition elements, we can describe the weak limit under diffusive scale according to the limit of each partition element. By Proposition 5.2 we have that the partition elements composed by a pair of l-path and r-path converge to a pair of left-right coalescing Brownian motion. By Donsker Theorem we have that the partition elements composed by a single l-path or r-path converge to a Brownian motion with drift $-b_p$ and $+b_p$ respectively.

**Remark 5.2.** If we have paths starting from different times, we just need to add more stopping times to the sequence $\tau_j^{(n)}$ to include the times where a new path starts in the system. The argument can still be used, but requires an adjustment, because whenever the process hits a time when a new path starts to move, the partition will change too, like when we have crossings or coalescence events. We begin the process at time $t_0^{(n)} = \min(z_{2,1}^{(n)}, \ldots, z_{2,k}^{(n)}, z_{1,1}^{(n)}, \ldots, z_{1,k}^{(n)})$.

Finally, the proofs presented in Section 5.3 give us the proof of Proposition 5.1, since the general case will follow by induction.  

### 6. Proof of Theorem 3.1

In this section we prove Theorem 3.1. To achieve that, we will use the convergence criterion to the Brownian Web described in Section 2.3. Thus we need to prove conditions: (I), (T), (B1) and (B2). Obviously in this section we only have $\alpha > 1$.

**Proof of Theorem 3.1.** We begin with condition (I). To prove it, let us consider the sequences of points $\pi_1^{(n)}, \ldots, \pi_k^{(n)} \in \mathbb{Z}^2$, where $\pi_i^{(n)} = (x_i^{(n)}, s_i^{(n)})$ with $(\frac{1}{n}x_i^{(n)}, \frac{1}{n}s_i^{(n)}) \to (x_i, s_i) = z_i \in \mathbb{R}^2$ for $i = 1, \ldots, k$ and a sequence of paths $\pi_{z_i^{(n)}}^{(n)}, \ldots, \pi_{z_k^{(n)}}^{(n)} \in \mathcal{X}_n$ starting respectively from the points $(\frac{1}{n}x_1^{(n)}, \frac{1}{n}s_1^{(n)}), \ldots, (\frac{1}{n}x_k^{(n)}, \frac{1}{n}s_k^{(n)})$. Our strategy will be to prove that

$$
\max \sup_{1 \leq i \leq k} d(\pi_{z_i^{(n)}}^{(n)}(t), \pi_{z_i^{(n)}}^{(n)}(T)) \to 0 \text{ in probability, as } n \to \infty, \forall T > 0,
$$

(6.1)

where the metric $d$ is defined in Section 2.1 and, for $i = 1, \ldots, k$, $\pi_{z_i^{(n)}}^{(n)}(t)$ is the diffusively rescaled path of the usual Drainage Network model (without branching) starting from $(\frac{1}{n}x_i^{(n)}, \frac{1}{n}s_i^{(n)})$, constructed on the same environment of the DNB and thus naturally coupled with it. It will be enough to prove that (6.1) holds to obtain condition (I) because we already know from [5] that the usual Drainage Network model converges to the Brownian Web.

Since the distance between one path from DNB starting from $z_i^{(n)}$ and the path from the usual Drainage Network starting from this same point is bounded from above by the distance between the l-path and the r-path that start from $z_i^{(n)}$, it is enough to show that

$$
\max \sup_{1 \leq i \leq k} d(l_{z_i^{(n)}}^{(n)}(t), r_{z_i^{(n)}}^{(n)}(t)) \to 0 \text{ in probability, as } n \to \infty
$$

(6.2)

where $l_{z_i^{(n)}}^{(n)} \in W_n^l$ and $r_{z_i^{(n)}}^{(n)} \in W_n^r$. 


Let us denote \( S_{Tn^2}^{(n)} = r_{z_i}^{(n)}(T) - l_{z_i}^{(n)}(T) \). We can write \( S_{Tn^2}^{(n)} = \sum_{j=1}^{Tn^2} Z_{j}^{(n)}z_{i}^{(n)}, \) where \( Z_{j}^{(n)}z_{i}^{(n)} \) is the increment of the distance between the two paths \( r_{z_i}^{(n)} \) and \( l_{z_i}^{(n)} \) at time \( j \). It is simple to check that (6.2) follows if

\[
P \left( \max_{1 \leq i \leq k} \max_{1 \leq j \leq Tn^2} S_{j}^{z_i^{(n)}} \geq n^\gamma \right) \rightarrow 0.
\]

for some \( \gamma < 1 \).

Since the average displacement of \( r_{z_i}^{(n)} \) and \( l_{z_i}^{(n)} \) in each step is respectively \( \frac{b_{\alpha}}{n^\alpha} \) and \( -\frac{b_{\alpha}}{n^\alpha} \), we have \( E(Z_{j}^{(n)}z_{i}^{(n)}) = 2b_{\alpha}, \) recalling that \( b_{\alpha} = \frac{b(1-p)}{(2-p)^2} \) and it was calculated in the proof of Lemma 5.3. Consequently, we also have \( E(S_{Tn^2}^{z_i^{(n)}}) = 2b_{\alpha}Tn^{2-\alpha}. \) Furthermore \( S_{Tn^2}^{z_i^{(n)}} \) is a non-negative submartingale, since the paths \( r_{z_i}^{(n)} \) and \( l_{z_i}^{(n)} \) have independent increments, they can not cross each other and the distance between them will increase by \( \frac{2b_{\alpha}}{n^\alpha} \) in average, after each step. So, we can apply Doob’s inequality and write

\[
P \left( \max_{1 \leq i \leq k} \max_{1 \leq j \leq Tn^2} S_{j}^{z_i^{(n)}} \geq n^\gamma \right) \leq kP \left( \max_{1 \leq i \leq Tn^2} S_{j}^{z_i^{(n)}} \geq \frac{n^\gamma}{k} \right) \leq \frac{k^2E(S_{Tn^2}^{z_i^{(n)}})}{n^\gamma} = \frac{2k^2b_{\alpha}Tn^{2-\alpha}}{n^\gamma},
\]

for any \( \gamma > 0. \) Since \( \alpha > 1, \) we just need to fix \( \gamma \in (2 - \alpha, 1) \) to conclude the proof of (6.1).

**Remark 6.1.** Note that (6.3) gives us a result stronger than condition (I). We have that the set of all paths that start from any fixed point \( z \) almost surely converge to a single path when \( n \) goes to infinity because all these paths will be squeezed between \( r_{z}^{(n)} \) and \( l_{z}^{(n)} \).

About the condition (T), we do not need to prove the general criterion described in Section 2.3, because we can use the fact that we do not have crossings between pairs of l-paths and between pairs of r-paths (same argument of [21] to prove tightness in their case). If we begin considering only the r-paths or the l-paths, we have a system of non-crossing paths and by [9, Proposition B2], the tightness of \( (W_n^l \cup W_n^r) \) follows from condition (I).

Now we can use the tightness of \( (W_n^l \cup W_n^r) \) to argue that \( X_n^{\alpha} \) is also tight. Recall the map presented in (2.2) and the metric space \( \Pi \) introduced in Section 2.1. By Arzela-Ascoli, we note that a set \( K \subset \Pi \) is precompact if and only if the set of functions defined by the images of the graphs of \( \pi \in K \) under the map described in (2.2) is equicontinuous. It means that the modulus of continuity of \( K, \)

\[
m_K(\delta) = \sup \left\{ \left| \frac{\tanh(\pi(t))}{1 + |t|} - \frac{\tanh(\pi(s))}{1 + |s|} \right| : \pi \in K, \ s, t \geq \sigma, \ |\tanh(t) - \tanh(s)| \leq \delta \right\} ,
\]

converges to zero as \( \delta \rightarrow 0. \)
To control $m_{\mathcal{X}_n}(\delta)$, note that for any $\pi^{(n)} \in \mathcal{X}^n$, the jump of $\pi^{(n)}$ at any time $s \geq \sigma_{\pi^{(n)}}$, $s \in \frac{1}{\pi^2} \mathbb{Z}$, is equal to the jump of the l-path in $W^l_n$ or the r-path in $W^r_n$ that starts at time $s$ from the vertex that $\pi^{(n)}$ visits at time $s$. Then, $m_{\mathcal{X}_n} (\delta) \leq m_{W^l_n \cup W^r_n} (\delta)$, concluding that (T) holds. A proof that the modulus of continuity of the system is bounded by the modulus of continuity of the collection of l-paths and r-paths is given in [21, Lemma 4.6].

Now we have to deal with the condition (B1). As mentioned in Section 2.3, we will prove a slightly different condition (B1*) that it is also enough to achieve the convergence for the Brownian Web and it will be more convenient in our case. So, denoting by $\mathcal{X}$ a weak limit of $\mathcal{X}_n$, we have to show that

$$\sup_{a,t \in \mathbb{R}} P[\eta_\mathcal{X}(t_0, t; a, a + \delta) \geq 2] \xrightarrow{\delta \to 0} 0, \quad \forall t > 0,$$

recalling that $\eta_\mathcal{X}(t, h; a, b) = |\{\pi(t + h) : \pi \in \mathcal{X}, \pi(t) \in [a, b]\}|$, $t \in \mathbb{R}, h > 0, a < b$. By translation invariance, it is enough to show that

$$P(\eta_\mathcal{X}(0, t; 0, \delta) \geq 2) \xrightarrow{\delta \to 0} 0.$$

Our strategy here uses the same arguments of [5] to prove condition (B1) for the usual Drainage Network model together with the statement (6.3). Note that by (6.3), all paths starting from (0, 0) converge to a single path under diffusive scaling when $n$ goes to infinity, and the same happens with the paths starting from $\delta$ at time zero. So, we have $\eta_\mathcal{X}(0, t; 0, \delta) \geq 2$ if and only if these two paths have not met up to time $t$ and by condition (I), we have that these two limit paths will be distributed as coalescing Brownian motions. So, using the reflection principle, we can write

$$P(\eta_\mathcal{X}(0, t; 0, \delta) \geq 2) = 1 - 2P\left(\inf_{s \in [0, t]} |B^{(0,0),2\delta_\mathcal{X}}_s - B^{(0,0),\lambda_\mathcal{X}}_s| > 0\right) = 1 - 2P\left(\inf_{s \in [0, t]} B^{(0,0),2\lambda_\mathcal{X}}_s \leq -\delta\right) =$$

$$1 - 2P\left(B^{(0,0),2\lambda_\mathcal{X}}_t \leq -\delta\right) = 2\Phi\left(\frac{\delta}{\sqrt{2t\lambda_\mathcal{X}}}\right) - 1 \xrightarrow{\delta \to 0} 0, \quad (6.5)$$

where $(B^{\sigma^2}_s)_{s \geq 0}$ is a Brownian motion with diffusion $\sigma^2$, starting at point $z$ and $\Phi(\cdot)$ is the standard normal distribution function.

To conclude, we will deal with the condition (B2*). So, we will prove the following statement:

$$\frac{1}{\delta} \sup_{a,t \in \mathbb{R}} P[\eta_\mathcal{X}(t_0, t; a, a + \delta) \geq 3] \xrightarrow{\delta \to 0} 0, \quad \forall t > 0,$$

which by translation invariance is equivalent to

$$\frac{1}{\delta} P[\eta_\mathcal{X}(0, t; 0, \delta) \geq 3] \xrightarrow{\delta \to 0} 0, \quad \forall t > 0. \quad (6.6)$$

Now, let us define $\eta_\mathcal{X}^*(0, t; 0, \delta) = \eta_\mathcal{X}(0, t; 0, \delta) - \eta_\mathcal{R}(0, t; 0, \delta)$, where $\mathcal{R}$ is a weak limit of $W^r_n$ such that $\mathcal{R} \subset \mathcal{X}$. We have that (6.6) occurs if the following three statements are true:

1. $P[\{\eta_\mathcal{R}(0, t; 0, \delta) = 1\} \cap \{\eta_\mathcal{X}^*(0, t; 0, \delta) \geq 2\}] = 0, \forall \delta > 0$;
(ii) \( \frac{1}{\delta} P[\{ \eta_R(0,t;0,\delta) = 2 \} \cap \{ \eta^*_\mathcal{X}(0,t;0,\delta) \geq 1 \}] \xrightarrow{\delta \to 0} 0; \)

(iii) \( \frac{1}{\delta} P[\eta_R(0,t;0,\delta) \geq 3] \xrightarrow{\delta \to 0} 0. \)

Now we will prove each one of these three statements in the following order: (i),(iii),(ii). This order are more convenient according to our proof strategy.

Proof of (i): Note that, on the event \( \{ \eta_R(0,t;0,\delta) = 1 \} \), we know that the pair of r-paths in \( \mathcal{X} \) that start from \( (0,0) \) and \( (\delta,0) \) coalesce before time \( t \). It implies that the only possibility to have \( \{ \eta^*_\mathcal{X}(0,t;0,\delta) \geq 2 \} \) would be if two or more paths in \( \mathcal{X} \) that start from \( (0,0) \) are at positive distance from each other at time \( t \). But by (6.4) we know that \( \mathcal{X} \) will have only one path starting from \( (0,0) \) almost surely, which concludes the proof of (i).

□

Proof of (iii) : Since in this case we are dealing only with r-paths, we have that (iii) is consequence of the following statement:

\[
\frac{1}{\delta} \limsup_{n \to \infty} P[\eta_{W_n^r}(0,tn^2;0,\delta n) \geq 3] \xrightarrow{\delta \to 0} 0, \tag{6.7}
\]

which is a consequence of Portmanteau Theorem, since \( \{ K \in \mathcal{H} : \eta_K(0,t;0,\delta) \geq 3 \} \), are open sets. The proof of (6.7) follows the same steps of the proof of condition (B2) in [5], for the Drainage Network without branching.

Proof of (ii): We need to prove that

\[
\delta^{-1} P[\{ \eta_R(0,t;0,\delta) = 2 \} \cap \{ \eta^*_\mathcal{X}(0,t;0,\delta) \geq 1 \}] \xrightarrow{\delta \to 0} 0.
\]

Note that the occurrence of event \( \{ \eta_R(0,t;0,\delta) = 2 \} \) implies that exists a real number \( j \in (0,\delta) \) such that every r-path in \( \mathcal{X} \) that visits a vertex in \( [0,j] \) at time 0 will coalesce before time \( t \) and will be at the same position at time \( t \) (let us say position \( A \)); and every r-path in \( \mathcal{X} \) that visit a vertex in \( (j,\delta] \) at time 0 will coalesce before time \( t \) and will be at the same position at time \( t \), which is different from \( A \) (let us say position \( B \)).

Now by (6.2) we have that the l-path in \( \mathcal{X} \) that visits the vertex \( (0,0) \) will a.s. be at position \( A \) at time \( t \) and that the l-path in \( \mathcal{X} \) that visits the vertex \( (\delta,0) \) will a.s. be at position \( B \) at time \( t \). Then, to have the events \( \{ \eta_R(0,t;0,\delta) = 2 \} \) and \( \{ \eta^*_\mathcal{X}(0,t;0,\delta) \geq 1 \} \) happening together, we need to have at least one l-path that visits a vertex in \( (0,\delta) \) at time 0 and that will be in a position different than \( A \) and \( B \), let us say position \( C \) at time \( t \). Note that \( C \in [A,B] \) as a consequence of (6.4), see Figure 5. So, if we denote by \( \mathcal{L} \) a weak limit of \( W_n^l \) such that \( \mathcal{L} \subset \mathcal{X} \), we can write:

\[
\frac{1}{\delta} P[\{ \eta_R(0,t;0,\delta) = 2 \} \cap \{ \eta^*_\mathcal{X}(0,t;0,\delta) \geq 1 \}] \leq \frac{1}{\delta} P[\{ \eta_C(0,t;0,\delta) \geq 3 \} \tag{6.8}
\]

To conclude, the right term in (6.8) converges to zero as \( \delta \to 0 \) as already proved in statement (iii) (in fact the proof of statement (iii) was done for the weak limit of r-paths and not l-paths, but the proof for l-paths would be analogous). □
Figure 5. Illustration of what needs to happen with the l-paths and r-paths in $\mathcal{X}$ to have the events \( \{ \eta_R(0, t; 0, \delta) = 2 \} \) and \( \{ \eta_X^*(0, t; 0, \delta) \geq 1 \} \) happening together. The full lines represent r-paths and dashed lines represent l-paths.

7. Proof of Theorems 3.2 and 3.3

This section is devoted to proving Theorems 3.2 and 3.3.

In the proof of Theorem 3.2, we follow the same steps of the proof of [21, Theorem 5.3]. We will need the next result, which is equivalent to [16, Theorem 2.9] for the Drainage Network without branching:

**Proposition 7.1.** In the conditions of Theorem 3.2, we have that \((W^L_n, \hat{W}^L_n) \Rightarrow (W^L_{\lambda,b}, \hat{W}^L_{\lambda,b})\) and \((W^R_n, \hat{W}^R_n) \Rightarrow (W^R_{\lambda,b}, \hat{W}^R_{\lambda,b})\).

We will prove Proposition 7.1 after proving Theorem 3.2.

**Proof of Theorem 3.2.** By Proposition 7.1 we have that \(\{(W^L_n, W^R_n, \hat{W}^L_n, \hat{W}^R_n)\}_{n \in \mathbb{N}}\) is tight. Let \((X^L, X^R, Z^L, Z^R)\) be a subsequential weak limit of this sequence. Then, again by Proposition 7.1, we have that \((X^L, Z^L)\) is distributed as \((W^L_{\lambda,b}, \hat{W}^L_{\lambda,b})\) and \((X^R, Z^R)\) is distributed as \((W^R_{\lambda,b}, \hat{W}^R_{\lambda,b})\). Recall that \((W^L_n, W^R_n)\) and \((W^L_{\lambda,b}, W^R_{\lambda,b})\) determine their dual processes, which are respectively \((\hat{W}^L_n, \hat{W}^R_n)\) and \((\hat{W}^L_{\lambda,b}, \hat{W}^R_{\lambda,b})\). Then, if we have that

\[
(W^L_n, W^R_n) \Rightarrow (W^L_{\lambda,b}, W^R_{\lambda,b}),
\]

it follows that \((X^L, X^R, Z^L, Z^R)\) has the same distribution as \((W^L_{\lambda,b}, W^R_{\lambda,b}, \hat{W}^L_{\lambda,b}, \hat{W}^R_{\lambda,b})\), which proves Theorem 3.2. \( \square \)

To prove (7.1), we have to verify that \((X^L, X^R)\) satisfies conditions (i), (ii) and (iii) in Proposition 2.3. Conditions (i) and (iii) follow from Proposition 7.1. In addition to that, by condition (I_{\mathcal{X}}) (Proposition 5.1) we have that \((X^L, X^R)\) also satisfies condition (ii), consequently \((X^L, X^R)\) has the same distribution as the left-right Brownian Web \((W^L_{\lambda,b}, W^R_{\lambda,b})\).

**Proof of Proposition 7.1.** We will only prove Proposition 7.1 for the system with l-paths, but the proof for the r-paths is analogous. Let us begin by introducing some maps. Let \(\phi : \Pi \to \Pi\) such that, for any path \(\pi_t \in \Pi\), we have:

\[
\phi(\pi_t) = \pi_t + b_p t, \text{ for any } t \geq \sigma_\pi.
\]
Now let $\varphi : \mathcal{H} \to \mathcal{H}$ be such that, for any $K \in \mathcal{H}$,
\[ \varphi(K) = \{ \phi(\pi), \pi \in K \}. \]
Analogously, let $\varphi_2 : \mathcal{H} \times \hat{\mathcal{H}} \to \mathcal{H} \times \hat{\mathcal{H}}$ be such that, for any $(K, \hat{K}) \in \mathcal{H} \times \hat{\mathcal{H}}$,
\[ \varphi_2(K, \hat{K}) = \{ (\phi(\pi), \phi(\hat{\pi})), \pi \in K, \hat{\pi} \in \hat{K} \}. \]
These three maps are homeomorphisms on their respective domains.

Note that the map $\varphi_2$, when applied to the set of $l$-paths and its dual, removes the drift of all these paths; and when applied to $(W_{\lambda,b}, \hat{W}_{\lambda,b})$ gives us the Brownian Web together with its dual. The main idea of the proof, following an approach used in [11] to prove the main theorem of that article, will be to verify that
\[ \varphi_2(W_{n,l}, \hat{W}_{n,l}) \implies (W_\lambda, \hat{W}_\lambda), \text{ as } n \to \infty. \] (7.2)
Since the map $\varphi_2$ is a homeomorphism on $\mathcal{H} \times \hat{\mathcal{H}}$, we have that (7.2) implies the statement of the Proposition 7.1. To prove that (7.2) holds, we will begin by proving that
\[ \varphi(W_{n,l}) \implies W_\lambda, \text{ as } n \to \infty. \] (7.3)
To verify that (7.3) holds, we will follow the converge criterion to Brownian Web presented in Section 2.3. So, according to that criterion, since we do not have crossings among the paths of $\varphi(W_{n,l})$, it is only necessary to prove conditions (I) and (E'), also described in Section 2.3. To simplify the notation, we denote $W_{n,l} = \varphi(W_{n,l})$ and $(W_{n,l}, \hat{W}_{n,l}) = \varphi_2(W_{n,l}, \hat{W}_{n,l})$.

Here we decided to use condition (E') instead (B2*) as we did in the proof of Theorem 3.1. Condition (E') should be simpler to verify following arguments in [13] for systems of random walks, like $W_{n,l}$, which have one path starting from each vertex. In the case of Theorem 3.1 we were dealing with all the paths in the DNB and thus we had more than one path starting from each vertex.

We already have condition (I) as a consequence of Proposition 5.1.

As in Section 2.3, denote by $W_{n,l,0}$ the subset of paths in $W_{n,l}$ which starts before or at time $t_0$ and let $Z_{t_0}$ be a subsequential limit of $W_{n,l,0}$ for any fixed $t_0 \in \mathbb{R}$. Recalling the condition (E'), we have to prove that
\[ E[\hat{n}_{Z_{t_0}}(t_0, t, a, b)] \leq E[\hat{n}_{W}(t_0, t, a, b)] = \frac{b - a}{\sqrt{\pi t}}, \]
where $\hat{n}_{Z_{t_0}}(t_0, t, a, b)$, for $a < b$, is the number of distinct points in $(a, b) \times \{ t_0 + t \}$ that are occupied by a path of $Z_{t_0}$ that started before or at time $t_0$.

The verification of condition (E') follows the same arguments presented in [13, Section 6]. The aim is to show that [13, Lemmas 6.2 and 6.3] also hold in our case, since it was proved in that paper that these two lemmas imply condition (E') for general models of coalescing random walks, in particular to the DNB, see also [5] and [6].

Fix $\delta > 0$ and denote by $Z_{t_0}(t_0 + \delta)$ the intersection of paths in $Z_{t_0}$ with the time line $t_0 + \delta$. To prove that [13, Lemma 6.2] holds in our case we need to show that $Z_{t_0}(t_0 + \delta)$ is almost surely locally finite for any $\delta > 0$. To prove that [13, Lemma 6.3] holds in our case
we need to show that the set of paths in $\mathcal{Z}_{t_0}$ truncated before time $t_0 + \delta$, for any $\delta > 0$ is distributed as coalescing Brownian motions starting from the random set $\mathcal{Z}_{t_0}(t_0 + \delta) \subset \mathbb{R}^2$.

We can argue that $\mathcal{Z}_{t_0}(t_0 + \delta)$ is almost sure locally finite for any $\delta > 0$ as a consequence of Lemma 4.1. Define the event

$$O_t = \{\text{There is an l-path that starts from a point } z \in \mathbb{Z} \text{ at time } 0 \text{ that visits the origin at time } t\}.$$  

One can show that

$$P(O_t) \leq \frac{C}{\sqrt{t}},$$  (7.4)

and it gives the locally finiteness property of $\mathcal{Z}_{t_0}(t_0 + \delta)$ ([13, Lemma 6.2]). The statement in (7.4) is equivalent to [6, Lemma 3.3] for the generalized Drainage Network and [13, Lemma 2.7] for independent random walks. The proof that (7.4) holds given the upper bound of Lemma 4.1 is analogous to the proof of [13, Lemma 2.7].

Now, since we have locally finiteness of $\mathcal{Z}_{t_0}(t_0 + \delta)$, condition (I) implies that $\mathcal{Z}_{t_0}^{(t_0+\delta)r}$ is distributed as coalescing Brownian motions starting from the random set $\mathcal{Z}_{t_0}(t_0 + \delta)$, where $\mathcal{Z}_{t_0}^{(t_0+\delta)r}$ is the set of paths in $\mathcal{Z}_{t_0}$ which start before or at time $t_0$ and truncated before time $t_0 + \delta$. Then, we get the statement of [13, Lemma 6.3] and condition (E') from the fact that a family of coalescing Brownian motions starting from a random locally finite subset of the real line is stochastically dominated by a system of coalescing Brownian motions starting from every point in $\mathbb{R}$ at time $t_0 + \delta$ and taking limit when $\delta \to 0$. Hence, (7.3) holds.

The next step will be to show that (7.2) holds using that $\varphi(W_n^l) \Rightarrow W$ as $n \to \infty$. For that step, we will use an approach similar to the one used in [16, Theorem 2.9] to prove an equivalent result for a Drainage Network model without branching. The main idea here is to use the following result:

**Proposition 7.2.** ([16, Proposition 2.7]) Let $(\mathcal{W}, \mathcal{Z})$ be a $(\mathcal{H} \times \hat{\mathcal{H}}, \mathcal{B}_H \times \mathcal{B}_{\hat{H}})$-valued random variable with $\mathcal{W}$ distributed as the Brownian Web. If the following conditions are satisfied:

(i) For any deterministic $(x,t) \in \mathbb{R}^2$, there exists a path $\hat{\pi}^{(x,t)} \in \mathcal{Z}$ starting at $(x,t)$ and going backward in time almost surely;

(ii) Paths in $\mathcal{Z}$ do not cross paths in $\mathcal{W}$ almost surely. That is, there does not exist any $\pi \in \mathcal{W}$, $\hat{\pi} \in \mathcal{Z}$ and $t_1, t_2 \in (\sigma_{\pi}, \sigma_{\hat{\pi}})$ such that $(\hat{\pi}(t_1) - \pi(t_1))(\hat{\pi}(t_2) - \pi(t_2)) < 0$ almost surely;

(iii) Paths in $\mathcal{Z}$ and paths in $\mathcal{W}$ do not coincide over any time interval almost surely. That is, for any $\pi \in \mathcal{W}$ and $\hat{\pi} \in \mathcal{Z}$, does not exist a pair of points $t_1 < t_2$ with $\sigma_{\pi} \leq t_1 < t_2 \leq \sigma_{\hat{\pi}}$ such that $\hat{\pi}_t = \pi_t$ for all $t \in [t_1, t_2]$ almost surely;

Then, $\hat{\mathcal{W}}$ almost surely.

Since $\hat{W}_n^l$ consists only of paths that can not cross each other, Lemma A.1 in Appendix A implies tightness of the family $\{\hat{W}_n^l\}_{n \in \mathbb{N}}$ (according to [9, Proposition B.2]). So, the joint families $\{(W_n^l, \hat{W}_n^l)\}_{n \in \mathbb{N}}$ and $\{(W_n^l, W_n^l)\}_{n \in \mathbb{N}}$ are also tight since both have tight coordinates. Then, to prove Proposition 7.1 it is enough to show that for any subsequential limit $(\mathcal{W}, \mathcal{Z})$ of
{(W_n, \hat{W}_n)}_{n \in \mathbb{N}}$, the random variable $Z$ satisfies the conditions given in Proposition 7.2. By Skorohod’s representation theorem, we may assume that the convergence to $(W, Z)$ happens almost surely.

By Lemma A.1 for any deterministic $(x, t) \in \mathbb{R}^2$ there exists a path $\hat{\pi} \in Z$ starting at $(x, t)$ going backward in time almost surely, which gives us (i) of Proposition 7.2.

Since we are assuming the almost surely convergence of $\{(W_n, \hat{W}_n)\}_{n \in \mathbb{N}}$ to $(W, Z)$, if a dual path in $Z$ crosses a path in $W$, there exists a dual path in $\hat{W}^l_n$ which crosses a path in $W^l_n$, for some $n \geq 1$, which, by construction, is a contradiction. Hence, the paths in $Z$ do not cross paths in $W$ almost surely, which gives us (ii) in Proposition 7.2.

Now, to verify condition (iii), for $\delta > 0$ and each integer $m \geq 1$, define $A_{\delta,m} = \{\text{there exist paths } \pi \in W, \hat{\pi} \in Z \text{ with } \sigma_\pi, \sigma_{\hat{\pi}} \in (-m, m), \text{ and there exists } t_0 \text{ such that } \sigma_\pi < t_0 < t_0 + \delta < \sigma_{\hat{\pi}} \text{ with } -m < \pi(t) = \hat{\pi}(t) < m \text{ for all } t \in [t_0, t_0 + \delta]\}$. It is enough to show that for any fixed $\delta > 0$ and for $m \geq 1$, we have $P(A_{\delta,m}) = 0$. Note that the $P(A_{\delta,m}) = 0$ if and only if the equivalent event considering $\pi \in W^l_0$ and $\hat{\pi} \in Z^l$ also have probability zero, where $(W^l_0, Z^l)$ is a subsequential weak limit of $\{(W_n^l, \hat{W}_n^l)\}_{n \in \mathbb{N}}$. Let us call this alternative version of event $A_{\delta,m}$ by $\overline{A}_{\delta,m}$.

In [16, Theorem 2.9 and Lemma 2.11] is is proved that $P(A_{\delta,m}) = 0$ considering $(W, Z)$ as a subsequential weak limit of paths from the Drainage Network without branching and its dual under diffusive scaling. The proof here that $P(\overline{A}_{\delta,m}) = 0$ is analogous since the only difference is that the paths from $W^l_0$, have a drift to the left, which implies that paths from $Z^l$ will go back in time with a drift to the right, but the behavior of the distance between them is the same. It concludes the proof of Proposition 7.1 and part I.

Proof of Theorem 3.3 By Proposition 2.8 we need to show that conditions (C), (I’), (H) and (U’)$^r$, hold for $X^l_0$. Condition (I’$^l$) was already proved in Section 5. Conditions (C), (H) and (U’$^r$) follow directly from the construction of the DNB and its dual.

Appendix A. Auxiliary results, proofs and additional discussions

Here we prove two useful results for the dual DNB. The first result states that single l-paths and single r-paths from the dual converge to Brownian motions with drift. The second one gives an upper bound on the probability that two dual l-paths or two dual r-paths do not coalesce until time $t$ (analogous to the one obtained for non dual paths in Lemma 4.1).

A.1. Convergence of one l-path or r-path from the dual.

**Lemma A.1.** For any deterministic point $(x, t) \in \mathbb{R}^2$, there exists a sequence $(x_n, t_n)$ converging to $(x, t)$, a sequence of l-paths $\hat{X}^l_n \subset \hat{W}^l_n$ starting from $(x_n, t_n)$ and a sequence of r-paths $\hat{X}^r_n \subset \hat{W}^r_n$ starting from $(x_n, t_n)$ which converge to Brownian motions starting at $x$ at time $t$, evolving backward in time, with drift $+b_p$ and $-b_p$ respectively, when $n \to \infty$ and the branching parameter is $\epsilon_n = bn^{-1}$.
Recall the construction and notation from Section 3.2. By translation invariance, it is enough to consider \((x, t) = (0, 0)\) and \((x, t) = \left(\frac{1}{2}, 0\right)\) depending if either \(x \in \mathbb{Z}\) or \(x \notin \mathbb{Z}\), but since the proof of case \((x, t) = \left(\frac{1}{2}, 0\right)\) would be analogous, we will consider only the case \((x, t) = (0, 0)\). Also recall that \(\hat{Y}^l_k(z)\) and \(\hat{Y}^r_k(z)\) denote the positions respectively of the dual \(l\)-path and the dual \(r\)-path that start from the dual vertex \(\hat{z}\) after \(t\) steps, i.e., at time \(\hat{z}_2 - t\).

**Proposition A.2.** For any \(\hat{z} \in \hat{V}\), \((\hat{Y}^l_k(\hat{z}))_{k \geq 0}\) and \((\hat{Y}^r_k(\hat{z}))_{k \geq 0}\) are time homogeneous Markov processes with respect to the filtration \(\mathcal{F}_k = \sigma((\omega(v), \theta(v)) : v \in \mathbb{Z}^2, \hat{z}_2 \geq v_2 \geq \hat{z}_2 - k)\).

**Proof.** We will present the proof for \((\hat{Y}^l_k(\hat{z}))_{k \geq 0}\) and the proof for \((\hat{Y}^r_k(\hat{z}))_{k \geq 0}\) is analogous.

We need the specification of the transition probabilities of \((\hat{Y}^l_k(\hat{z}))_{k \geq 0}\). Recall that \(\hat{Y}^l_k(z)\) assumes values in \(\frac{1}{2}\mathbb{Z}\). The description of the transition probabilities depends if either the current value of the process is an integer or not, so we study each case separately.

(i) If the current value of \(\hat{Y}^l_k(\hat{z})\) is non-integer: In this case we cannot have a vertex from \(V\) in front of \((\hat{Y}^l_k(\hat{z}), \hat{z}_2 - k)\) since \((\hat{Y}^l_k(\hat{z}), \hat{z}_2 - (k + 1)) \notin \mathbb{Z}^2\). This implies that the dual path can not be blocked by paths in the DNB that have just branched, so we also can not have a branching occurring on the dual process at this vertex. Thus this dual vertex will be connected to the nearest one that it can reach without cross a path from DNB.

It is straightforward to verify from definitions that if \(z \in \hat{V}\) and \(\hat{z} \in \hat{V}\) satisfy that either \(z_2 \neq \hat{z}_2\) or \(z_2 = \hat{z}_2\) and \(z_1 > \hat{z}_1\), then \(K^r(z)\) and \(K^l(z)\) are i.i.d. Geom\((p)\). To simplify notation, let \((G_1, G_2)\) be a random vector distributed as i.i.d. Geom\((p)\).

For \(u \notin \mathbb{Z}\) and \(v \in \frac{1}{2}\mathbb{Z}\) we have:

\[
P\left(\hat{Y}^l_{k+1}(\hat{z}) - \hat{Y}^l_k(\hat{z}) = v \mid \hat{Y}^l_k(\hat{z}) = u\right) = P\left(K^r\left(u - \frac{1}{2}, z_2 - (k + 1)\right) - K^l\left(u + \frac{1}{2}, z_2 - (k + 1)\right) = 2v\right) = P\left(G_1 - G_2 = 2v\right).
\]

(ii) If the current value of \(\hat{Y}^l_k(\hat{z})\) is an integer: In this case we have three distinct scenarios which are (a) either we do not have a vertex from \(V\) in front of \((\hat{Y}^l_k(\hat{z}), \hat{z}_2 - k)\) or (b) that vertex exists and the DNB branches (branching is possible since the dual vertex is the midpoint between two consecutive open vertices) or (c) that vertex exists and the DNB does not branch. Situations, (a), (b) and (c) are formally described just below. To help the reader, in Figure 6 we exemplify the occurrence of the situations (a), (b) and (c).

(a) The vertex \((\hat{Y}^l_k(\hat{z}), \hat{z}_2 - (k + 1))\) is not in \(V\) (not open): This event occurs with probability \((1 - p)\) and, for \(u \in \mathbb{Z}\) and \(v \in \frac{1}{2}\mathbb{Z}\), we have that

\[
P\left(\hat{Y}^l_{k+1}(\hat{z}) - \hat{Y}^l_k(\hat{z}) = v, (\hat{Y}^l_k(\hat{z}), \hat{z}_2 - (k + 1)) \notin V \mid \hat{Y}^l_k(\hat{z}) = u\right)
\]
\[\begin{align*}
= & \quad P(\omega(u, z_2 - (k+1)) = 0) \\
= & \quad P\left(\hat{Y}_{k+1}^t(z) - \hat{Y}_k^t(z) = v \mid \hat{Y}_k^t(z) = u, \omega(u, z_2 - (k+1)) = 0\right) \\
= & \quad (1 - p)P(K^r(u, z_2 - (k+1)) - K^l(u, z_2 - (k+1)) = 2v) \\
= & \quad (1 - p)P(G_1 - G_2 = 2v). \\
\end{align*}\]

(b) The vertex \((\hat{Y}_k^t(z), \hat{z}_2 - (k+1))\) is in \(V\) and a branching of the DNB occurs at this vertex. This event occurs with probability \(p\epsilon_n\) and it implies that the dual process also branches at vertex \((\hat{Y}_k^t(z), \hat{z}_2 - k)\). Also recall that we are dealing with dual \(l\)-paths that must jump to the left when a branching occurs. Thus for \(u \in \mathbb{Z}\) and \(v \in \frac{1}{2}\mathbb{Z}\), put \(\tilde{z}_k = (u, z_2 - (k+1))\) and write

\[P\left(\hat{Y}_{k+1}^t(z) - \hat{Y}_k^t(z) = v, \omega(\tilde{z}_k) = 1, \theta(\tilde{z}_k) = 0 \mid \hat{Y}_k^t(z) = u\right) = P(\omega(\tilde{z}_k) = 1)P(\theta(\tilde{z}_k) = 0) \quad P\left(\hat{Y}_{k+1}^t(z) - \hat{Y}_k^t(z) = v \mid \hat{Y}_k^t(z) = u, \omega(\tilde{z}_k) = 1, \theta(\tilde{z}_k) = 0\right) \quad p\epsilon_nP(K^r(\tilde{z}_k) = 2v) = p\epsilon_nP(G_1 = 2v).\]

Note that if \(v \leq 0\), the second term is equal to zero since the dual \(l\)-path always choose the left option in case when a branching occurs. This is the only term that would change in the proof for \((\hat{Y}_k^r(z))_{k \geq 0}\) and it would be replaced by \(p\epsilon_nP(G_1 = -2v)\).

(c) The vertex in position \((\hat{Y}_k^t(z), \hat{z}_2 - (k+1))\) is in \(V\) but we do not have a branching of the DNB at this vertex: This event occurs with probability \(p(1 - \epsilon_n)\) and here the dual process have conditional probability 1/2 of being forced to go to left (transition to a higher value) and conditional probability 1/2 of being forced to go to the right (transition to a lower value) to avoid a crossing with paths of the DNB. Thus for \(u \in \mathbb{Z}\) and \(v \in \frac{1}{2}\mathbb{Z}\), put \(\tilde{z}_k = (u, z_2 - (k+1))\) and write

\[P\left(\hat{Y}_{k+1}^t(z) - \hat{Y}_k^t(z) = v, \omega(\tilde{z}_k) = 1, |\theta(\tilde{z}_k)| = 1 \mid \hat{Y}_k^t(z) = u\right) = P(\omega(\tilde{z}_k) = 1)P(|\theta(\tilde{z}_k)| = 1) \quad P\left(\hat{Y}_{k+1}^t(z) - \hat{Y}_k^t(z) = v \mid \hat{Y}_k^t(z) = u, \omega(\tilde{z}_k) = 1, \theta(\tilde{z}_k) = 0\right)\]

which is equal to

\[p(1 - \epsilon_n)\left(\frac{P(K^r(\tilde{z}_k) = 2v)}{2} + \frac{P(K^l(\tilde{z}_k) = 2v)}{2}\right) = \frac{p}{2}(1 - \epsilon_n)P(G_1 = |2v|).\]

From situations (a), (b) and (c) above, for \(u \in \mathbb{Z}\) and \(v \in \frac{1}{2}\mathbb{Z}\) we have:

\[P\left(\hat{Y}_1^t(z) - \hat{Y}_0^t(z) = v \mid \hat{Y}_0^t(z) = u\right) = \]

\[= (1 - p)P(G_1 - G_2 = 2v) + p\epsilon_nP(G_1 = 2v) + \frac{p}{2}(1 - \epsilon_n)P(G_1 = |2v|).\]

\(\Box\)
Figure 6. Example of vertices (marked with a square) from Figure 2 where the situations (a), (b) and (c) described in the proof of proposition A.2 occurs.

Note that the increments of \( (\hat{Y}_l^k(z))^k \geq 0 \) do not have mean zero:

\[
E\left( \hat{Y}_{k+1}^l(z) - \hat{Y}_k^l(z) \bigg| \hat{Y}_k^l(z) \right) = \begin{cases} 
0, & \text{if } \hat{Y}_k^l(z) / \notin Z, \\
\frac{E(G)}{2}, & \text{if } \hat{Y}_k^l(z) \in Z, \end{cases} \quad \forall k \geq 0.
\tag{A.1}
\]

For each fixed \( \hat{z} \in \hat{V} \) and \( n \geq 1 \), we consider the following process:

\[
M_{n,0}(\hat{z}) = \hat{z}, \quad M_{n,k}(\hat{z}) = \hat{Y}_k^l(\hat{z}) - \sum_{j=1}^{k-1} \frac{\epsilon_n}{2} \mathbb{I}_{\{Z\}}(\hat{Y}_j^l(\hat{z})), \quad k \geq 1.
\]

For any \( \hat{z} \in \hat{V} \cap Z^2 \), we have that the process \( (M_{n,k}(\hat{z}))_{k \geq 0} \) is an \( L^2 \)-martingale with respect to the reversed time filtration \( \hat{F}_k = \sigma(\{\omega(z), \theta(z) : z \in Z \text{ with } z_2 \geq \hat{z}_2 - k\}) \). By the invariance under integer translations of the system, we can suppose that \( (0, 0) \in \hat{V} \). Write \( M_{n,j} = M_{n,j}(0, 0) \) and \( \hat{Y}_j = \hat{Y}_j(0, 0) \) for the sake of simplicity and define:

\[
s_k^2 = E(M_{n,k}^2) = \sum_{j=1}^{k} E[(M_{n,j} - M_{n,j-1})^2], \quad \sigma_k^2 = \sum_{j=1}^{k} E[(M_{n,j} - M_{n,j-1})^2 | \hat{F}_{j-1}],
\]

\[
\mathcal{M}_{n,k}(t) = s_k^{-1} \left( M_{n,j} + \frac{(M_{n,j+1} - M_{n,j})(ts_k^2 - s_j^2)}{(s_{j+1}^2 - s_j^2)} \right),
\]

for \( k \in \mathbb{N}, t \in [0, \infty) \) and \( s_j^2 \leq ts_k^2 < s_{j+1}^2 \). So, by definition, \( (\mathcal{M}_{n,k}(t))_{t \geq 0} \) is the linearly interpolated normalization of the martingale \( (M_{n,k}(t))_{t \geq 0} \) and note that since \( M_{n,j}(\hat{z}) \) is a time homogeneous process, we have the same values of \( s_k^2 \) and same distribution of \( \sigma_k^2 \) for any vertex \( \hat{z} \in \hat{V} \cap Z^2 \).

**Proposition A.3.** The process \( \mathcal{M}_{n,k}(t) \) converges in distribution to a standard Brownian motion when \( k \to \infty \).

For the proof of Proposition A.3 we will make use of the following result:

**Proposition A.4.** (\cite[Theorem 3]{3}) Let \( \{C, \mathcal{B}, P_w\} \) be the probability space where \( C = C[0, 1] \) with the sup norm topology, \( \mathcal{B} \) being the Borel \( \sigma \)-field generated by open sets in \( C \) and \( P_w \) the Wiener measure. Let \( \{P_k\} \) be the sequence of probability measures on \( \{C, \mathcal{B}\} \) determined by
Before we proceed, note that since \( \{M_{n,k}(t), 0 \leq t \leq 1\} \). Then \( P_k \to P_w \) weakly as \( k \to \infty \) if the following two conditions hold:

\[
\begin{align*}
(i) \quad & \frac{\sigma^2_k}{s_k^2} \overset{P}{\to} 1 \text{ as } k \to \infty; \\
(ii) \quad & \text{Lindeberg condition:}
\end{align*}
\]

\[
\frac{1}{s_k^2} \sum_{j=1}^{k} E \left[ (M_{n,j} - M_{n,j-1})^2 I_{\{M_{n,j} - M_{n,j-1} \geq \epsilon_n, s_k\}} \right] \to 0 \text{ as } k \to \infty \text{ for all } \epsilon_n > 0.
\]

\textbf{Proof of Proposition A.3.} For this proof we will verify that \( \{M_{n,k}(t), 0 \leq t \leq 1\} \) satisfies the two sufficient conditions of Proposition A.4. For condition (ii), using the bounds on the probability function of \( (\hat{Y}_j(t))_{k \geq 0} \) obtained in Proposition A.2, we can check that the increments \( M_{n,j} - M_{n,j-1} \) have uniformly bounded variance. So \( s_k^2 \) is of order \( k \) and (ii) follows. For condition (i), first, we need to do some auxiliary calculus using results obtained in the proof of Proposition A.2:

\[
E \left[ (M_{n,j} - M_{n,j-1})^2 \right] \overset{F_{j-1}, \hat{Y}_j \in \{Z + \frac{1}{2}\}}{=} \frac{1}{4} [E(G_1^2) + E(G_2^2) - 2E(G_1)E(G_2)] = \frac{1-p}{2p^2}.
\]

(A.2)

Now the same computation for the case where \( \hat{Y}_{j-1} \) is an integer:

\[
E \left[ (M_{n,j} - M_{n,j-1})^2 \right] \overset{F_{j-1}, \hat{Y}_j \in \{Z\}}{=} E \left[ (\hat{Y}_j - \hat{Y}_{j-1} - \frac{\epsilon_n}{2})^2 \right] \overset{\hat{Y}_{j-1} \in \{Z\}}{=}
\]

\[
\frac{\epsilon_n^2}{4} - \epsilon_n E \left[ \left( \hat{Y}_j - \hat{Y}_{j-1} \right) \left( \hat{Y}_j - \hat{Y}_{j-1} \right) \left. \right| \hat{Y}_{j-1} \in \{Z\} \right] + E \left[ \left( \hat{Y}_j - \hat{Y}_{j-1} \right) \left( \hat{Y}_j - \hat{Y}_{j-1} \right) \left. \right| \hat{Y}_{j-1} \in \{Z\} \right]
\]

\[
= \frac{\epsilon_n^2}{4} - \epsilon_n \frac{\epsilon_n}{4} + \frac{1-p}{4} E \left[ (G_1 - G_2)^2 \right] + \frac{p}{4} E \left[ G^2 \right]
\]

\[
= - \frac{\epsilon_n^2}{4} + \frac{1-p}{4} \left( \frac{2(1-p)}{p^2} \right) + \frac{p}{4} \left( \frac{2-p}{p^2} \right) = \frac{2(1-p)^2 + p(2-p)}{4p^2} - \frac{\epsilon_n^2}{4}.
\]

(A.3)

Before we proceed, note that since \( (\hat{Y}_j)_{j \geq 0} \) is a time homogeneous Markov process, we can create a discrete homogeneous Markov chain \( (Z_j)_{j \geq 0} \) with space state \( \{0, 1\} \) that describes if either \( \hat{Y}_j \) is integer \( (Z_j = 1) \) or not \( (Z_j = 0) \). Since \( (Z_j)_{j \geq 0} \) is an irreducible and aperiodic,
it converges to a unique invariant distribution that we will denote by \( \pi = (\pi(0), \pi(1)) \). Now we are ready to verify (i):

\[
\frac{\sigma^2}{k} = \frac{1}{k} \sum_{j=1}^{k} E[(M_{n,j} - M_{n,j-1})^2 | \hat{F}_{j-1}] =
\]

\[
= \frac{1}{k} \sum_{j=1}^{k} \left[ \frac{1 - p}{2p^2} \mathbb{1}_{\{z + 1/2\}} (\hat{Y}_{j-1}^l) + \left( \frac{2(1-p)^2 + p(2-p)}{4p^2} - \frac{\epsilon_n^2}{4} \right) \mathbb{1}_{\{z\}} (\hat{Y}_{j-1}^l) \right] \overset{a.s.}{\rightarrow} (1 - \frac{p}{2p^2}) \pi(0) + \left( \frac{2(1-p)^2 + p(2-p)}{4p^2} - \frac{\epsilon_n^2}{4} \right) \pi(1),
\]

where the last equality is justified by \( (A.2), (A.3) \) and this convergence is justified by the Ergodic Theorem. We also have:

\[
\frac{s^2}{k} = \frac{1}{k} \sum_{j=1}^{k} E[(M_{n,j} - M_{n,j-1})^2] = \frac{1}{k} \sum_{j=1}^{k} E \left[ \left( M_{n,j} - M_{n,j-1} \right)^2 \mid \hat{F}_{j-1} \right] =
\]

\[
= E \left[ \frac{\sigma^2}{k} \right] \overset{a.s.}{\rightarrow} \left( \frac{1 - p}{2p^2} \right) \pi(0) + \left( \frac{2(1-p)^2 + p(2-p)}{4p^2} - \frac{\epsilon_n^2}{4} \right) \pi(1),
\]

where this convergence is justified by Dominated Convergence Theorem, since \( \frac{\sigma^2}{k} \) is dominated by the maximum between \( \frac{1 - p}{2p^2} \) and \( \left( \frac{2(1-p)^2 + p(2-p)}{4p^2} - \frac{\epsilon_n^2}{4} \right) \). Since \( \sigma^2/k \) and \( s^2/k \) converge a.s. to the same limit, (i) is satisfied and it concludes the proof. \( \square \)

**Remark A.1.** The Proposition A.3 can be adapted for the dual r-paths. We just need to define the process \( \bar{M}_{n,k}(t) \) using the martingale \( \bar{M}_{n,k}(\hat{z}) = \hat{Y}_{k}^r(\hat{z}) + \sum_{j=1}^{k-1} \frac{\epsilon_n^2}{2} \mathbb{1}_{\{z\}} (\hat{Y}_{j}^r(\hat{z})) \) instead of \( M_{n,k}(\hat{z}) \) and its proof would be analogous.

**Remark A.2.** The proof of Proposition A.3 considering that the initial vertex \( \hat{z} \) has a non integer first coordinate would be analogous.

Now we are ready to present the proof of Lemma A.1.

**Proof of Lemma A.1.** This proof will be done only for \( \hat{X}_n^l \subset \hat{W}_n^l \) since the proof for the case \( \hat{X}_n^r \subset \hat{W}_n^r \) is analogous.

By the invariance under integer translations of the system, if the starting point of the dual l-path \( \hat{X}_n^l \) has an integer first coordinate, we can assume that this path is starting from \( (0,0) \). In the case where the starting point has a non integer first coordinate, we can restart the path \( \hat{X}_n^l \) when it reaches an integer position for the first time and the number of steps that we need to wait until it happens is stochastically dominated by a geometric distribution.
with a parameter that does not depend on $n$. So, in both cases, we can assume that $\hat{X}_n^l$ starts from $(0,0)$.

Now recall that we are assuming that $\epsilon_n = b n^{-1}$ and so we have:

$$\frac{1}{n} \sum_{j=1}^{n^2-1} \frac{\epsilon_n}{2} I\{Z\} \left( \hat{Y}_j^l \right) = \frac{b}{n^2} \sum_{j=1}^{n^2-1} I\{Z\} \left( \hat{Y}_j^l \right) \xrightarrow{a.s.} b \pi(1),$$

where $\pi(1)$ is the same as defined in the proof of Proposition A.3 and this convergence is justified by the Ergodic Theorem.

Now since we have from Proposition A.3 that $M_{n,k}(t)$ converges to a standard Brownian motion, it follows from Slutsky’s Theorem, and the same arguments used in proof of Lemma 5.3, that $\hat{X}_n^l$ converges to a Brownian motion with drift, going backward in time.

To conclude, we will argue that this drift is equal to $b_p$. Since l-paths from dual never crosses l-paths from DNB, we have that $\hat{X}_n^l$ is always compressed between two l-paths from DNB, one that visits a position at the right of $x_n$ at time $t_n$ and other that visits a position at the left of $x_n$ at time $t_n$. By Lemma 5.3 we have that these two l-paths converge to Brownian motions with drift $-b_p$, under diffusive scale, which implies that limiting process of $\hat{X}_n^l$, given that it exists, has drift $b_p$. □

A.2. Estimates for coalescence times on dual.

This subsection is devoted to prove Lemma 4.4. Before we present the proof, recall the notation of Section 4. Our strategy will consist in two steps: First, we will analyze specific random times where both dual paths are in integer positions and show that these times almost surely appear with enough frequency to allow us to do the proof considering the distance between the two dual paths only in these convenient random times. For this first part, we will prove Proposition A.5. In the second part we will show that, restricted to these random times, Proposition 4.2 holds.

**Proposition A.5.** Let $\hat{\pi}_1$ and $\hat{\pi}_2$ be two dual l-paths starting from different integer points $x_1$ and $x_2$ at the same time, with $x_2 > x_1$. There exists a sequence of random times $\{\hat{S}_n : n \geq 0\}$ such that:

(a) $\hat{\pi}_1(\hat{S}_n) \in \mathbb{Z}$ and $\hat{\pi}_2(\hat{S}_n) \in \mathbb{Z}$ for all $n \geq 0$;

(b) The sequence $\{\hat{S}_{n+1} - \hat{S}_n : n \geq 0\}$ is independent and identically distributed;

(c) There exists a constant $C_s$ such that $\mathbb{E}[|\hat{S}_1 - \hat{S}_0|] \leq C_s$;

(d) For any $l \geq 0$, a.s. $\mathbb{E}[|((\hat{\pi}_2(\hat{S}_{l+1}) - \hat{\pi}_1(\hat{S}_{l+1})) - (\hat{\pi}_2(\hat{S}_l) - \hat{\pi}_1(\hat{S}_l))| |\hat{\pi}_2(\hat{S}_l) - \hat{\pi}_1(\hat{S}_l)] \leq 0.$

**Proof.** First recall that if at least one of $\hat{\pi}_1$ and $\hat{\pi}_2$ is not in an integer position, the amount of time required until we have both $\hat{\pi}_1$ and $\hat{\pi}_2$ in an integer position is stochastically dominated by a geometric random variable $G_s$, with parameter $p^4(1-p)^2$, as we argued in Section 4.

So, define $\hat{S}_0$ as the first time such that both $\hat{\pi}_1$ and $\hat{\pi}_2$ are in an integer position. After that, if both of them continues in integer positions, we have $\hat{S}_1 = \hat{S}_0 - 1$. Otherwise, if at
least one of them jump to a non integer position, the remaining amount of time until both 
$\hat{\pi}_1$ and $\hat{\pi}_2$ reach integer positions is a random variable $T$, which is stochastically dominated by $G_s$, and in this case, we have $\hat{S}_1 = \hat{S}_0 - 1 - T$. Now, we can define $\hat{S}_n$ for any $n \geq 1$ following this procedure.

In this way, we have the statement $(a)$ of the proposition by construction. Statement $(b)$ follows from Markov property of the system. Statement $(c)$ holds since we can take $C_s = E[G_s] + 1 = p^{-4}(1 - p)^{-2} + 1$.

Now it remains statement $(d)$. First note that since $x_2 > x_1$, $\hat{\pi}_2(\hat{S}_l) - \hat{\pi}_1(\hat{S}_l) = 0$, it means that these two paths have already coalesced and then the distance between them will remain constant zero forever. So, we do not need to worry about this situation. If we have $\hat{\pi}_2(\hat{S}_l) - \hat{\pi}_1(\hat{S}_l) > 0$, we can divide our analysis in two cases depending if either these paths coalesce or not until time $\hat{S}_{l+1}$. In the case where they coalesce, we have that $\hat{\pi}_2(\hat{S}_{l+1}) - \hat{\pi}_1(\hat{S}_{l+1}) = 0$ and consequently the distance between them will reduce. For the case where they do not coalesce, note that we do not have information about the environment between the times $\hat{S}_l$ and $\hat{S}_{l+1}$; and that both are dual l-paths, so in average both paths have the same behavior. Because of that, we expect to see the same value for the distance at time $\hat{S}_{l+1}$ that we have in $\hat{S}_l$. It implies that the expectation value in part $(d)$ has to be less or equal than zero.

□

Remark A.3. All statements in the Proposition A.5 are also valid for dual r-paths in the same way and the proof would be analogue.

Now we are ready to present the proof of Lemma 4.4.

Proof of Lemma 4.4. For this proof, we can consider that $\hat{\tau}_k$ refers to coalescence between l-paths since the proof considering r-paths would be exactly the same.

First recall the following definition, from Section 4:

$$\hat{Z}_k^t = |\hat{Y}_t^l((0, 0)) - \hat{Y}_t^l((k, 0))|, \ t \geq 0.$$  

The main idea of this proof will be to verify that our distance process $\hat{Z}_k^t$ and the coalescence time $\hat{\tau}_k^t$ satisfy the conditions of Proposition 4.2 when we consider the process $\hat{Z}_k^t$ only at times $\hat{S}_n$ as defined in Proposition A.5. To simplify the notation, from now on we will keep the notation $\hat{Z}_k^t$, but what we will actually analyze here is the sequence $\hat{Z}_k^{S_n}$.

Proposition A.5 will allow us to extend our conclusions to the whole process $\hat{Z}_k^t$, that is, if we prove the statement of Lemma 4.4 considering the dual paths only in the times $S_n$, we will have it valid for the entire time line too.

Now denote $\hat{\mathcal{F}}_t = \sigma\{(\omega(z), \theta(z)), z = (z_1, z_2), z_1 \in \mathbb{Z}, z_2 \geq t\}, t \in \mathbb{Z}$. Note that $\hat{Z}_k^t$ only take values in $\mathbb{R}_+$ because we can not have a crossing between l-paths and since $\hat{\tau}_k$ is the first time that this process reaches zero, if all two conditions of Proposition 4.2 holds, considering the filtration $\hat{\mathcal{F}}_t$, we will immediately have Lemma 4.4 proved. So, from now on, we will prove these conditions.

Condition (i) of Proposition 4.2 follows directly from $(d)$ in Proposition A.5.
About the first inequality of condition (ii) of Proposition 4.2, note that the increments of \( \hat{Z}^k_t \) are not spatially homogeneous, but given \( \hat{F}_t \) and that \( \hat{Z}^k_t > 0 \), it is always possible to assure that \((\hat{Z}^k_{t+1} - \hat{Z}^k_t)^2 \geq 1\) with a convenient choice of a finite number of open and closed vertices at time \( S_{t+1} \).

About the second part of condition (ii), we can write:

\[
E[|\hat{Z}^k_{t+1} - \hat{Z}^k_t|^3 |\hat{F}_t] = E_{S_j - S_{j+1}} \left[ E \left[ |\hat{Z}^k_{j+1} - \hat{Z}^k_j|^3 |\hat{F}_j, S_j - S_{j+1} \right] \right] \leq 2E(S_j - S_{j+1})E \left[ |\hat{Y}^l_{j+1}(\tilde{u}) - \hat{Y}^l_j(\tilde{u})|^3 |\hat{F}_j, S_j - S_{j+1} \right] \leq 2C_s E[|G|^3],
\]

(A.4)

where \( G \) is a \text{geom}(p) and \( C_s \) is the same constant that appears in Proposition A.5. The last inequality of (A.4) holds due to Proposition A.5 and the fact that the increments of a dual path is stochastically dominated by a \text{geom}(p). Hence, since the third absolute moment of the geometric distribution is finite, we have that condition (ii) is also satisfied.

**APPENDIX B. PROOFS OF SOME AUXILIARY RESULTS FOR SECTION 5.**

This subsection is devoted to prove some statements and lemmas of Section 5. So, the notations and definitions used here are the same from Section 5. We begin by proving the upper bound of \( E[\xi_+^{(n)}] \) that was used in the proof of Lemma 5.8.

**Claim B.1.** Let \( \xi_+^{(n)} \) be the overshoot distribution that is defined during the proof of Lemma 5.8. There exists \( N \in \mathbb{N} \) such that \( E[\xi_+^{(n)}] \leq \frac{5}{p}, \forall n \geq N \).

**Proof.** Recall that \( \xi_+^{(n)} \) is the overshoot distribution of \( \Delta_1^{(n)} = \hat{R}_t^{(n)} - L_t^{(n)} \) when it crosses the position \( n \) from left to right at a time \( t \). To simplify the notation, let us say here that it will happen at time \( t = 1 \). This simplification can be done because of the strong Markov property. We also suppress \( n \) writing simply \( \hat{R}_t^{(n)} = R_t \) and \( L_t^{(n)} = L_t \).

By Lemma 5.4 and assuming that \( n \) is large enough, we can consider that \( n - n^\frac{3}{4} \leq R_0 - L_0 \leq n \). Denote by \( B_+ \) the event \( \Delta_1^{(n)} > n \). First note that if \( L_1 > R_0 - \frac{n}{2} \), then vertex \((L_1, 1)\) is open and either \( L_1, R_1 \in [R_0 - \frac{n}{2}, R_0 + \frac{n}{2}] \) or \( L_1 = R_1 \).

Thus \( R_1 - L_1 \leq n \) and \( B_+ \) does not happen. Analogously, if \( R_1 < L_0 + \frac{n}{2} \), then \( B_+ \) does not happen either. Put \( A_+ = \{ L_1 < R_0 - \frac{n}{2}, R_1 > L_0 + \frac{n}{2} \} \), then \( B_+ \subset A_+ \).

We also define \( X_+ \) the distance between \( \hat{R}_0 - n \) and the first open vertex at the left hand side of \( R_0 - n \) at time \( t = 1 \) and \( Y_+ \) as the distance between

\[
a = \max \left( R_0, L_1 + n, \left( n - (R_0 - 2L_0) \right) + X_+ \right)_{L_1 < R_0 - n} - 1
\]

and the first open vertex at the right hand side of \( a \) at time \( t = 1 \). Given \( B_+ \), and thus also \( A_+ \), our strategy from here will be to also conditioning on the position of \( L_1 \). Specifically it is straightforward to see that

\[
\xi_+^{(n)} \leq Y_+ \text{ on } L_1 = j \text{ for } R_0 - n \leq j \leq R_0 - \frac{n}{2},
\]
and
\[ \xi_+^{(n)} \leq 2X_- + Y_+ \text{ on } L_1 < R_0 - n. \]
The second inequality above requires \( R_0 - L_0 \geq \frac{n}{2} \) which is less restrictive than our hypothesis. Therefore
\[
E[\xi_+^{(n)}] \leq \sum_{j=R_0-n}^{R_0-\frac{n}{2}} E[Y_+|A_+, B_+, L_1 = j]P(L_1 = j|A_+, B_+) \\
+ E[2X_- + Y_+|A_+, B_+, L_1 < R_0 - n]P(L_1 < R_0 - n|A_+, B_+). \tag{B.1}
\]

In Figure 7 we have an illustration of the occurrence of event \( B_+ \) according to either \( L_1 \) be smaller than \( R_0 - n \) (scenario (a)) or not (scenario (b)).

![Figure 7](image-url)

**Figure 7.** Illustration of event \( B_+ \). In scenario (a) we have the situation where \( L_1 < R_0 - n \) and here \( \xi_+^{(n)} = x + y \). In scenario (b) we have the situation where \( R_0 - n \leq j \leq R_0 - \frac{n}{2} \) and here \( \xi_+^{(n)} = y - x \).

On the event \( A_+ \cap B_+ \cap \{L_1 = j\} \), \( R_0 - n \leq j \leq R_0 - \frac{n}{2} \), the configuration on vertices \((x, 1) \in \mathbb{Z}^2 \) such that \( R_0 \leq j + n \leq x \leq 2R_0 - j \) is distributed as the conditional of i.i.d. Bernoulli with parameter \( p \) given that at least one of these Bernoullis is one. This gives that
\[
E[Y_+|A_+, B_+, L_1 = j] \leq E[\text{Geom}(p)|A_+, B_+, L_1 = j] = \frac{1}{p}. \tag{B.2}
\]

Now write \( P(X_- > x|A_+, B_+, L_1 < R_0 - n) \) as
\[
P(B_+, A_+|X_- > x, L_1 < R_0 - n)P(X_- > x|L_1 < R_0 - n)P(L_1 < R_0 - n) \\
\geq \frac{P(B_+, A_+|L_1 < R_0 - n)P(L_1 < R_0 - n)}{2P(Geom(p) > x)},
\]
where the first inequality follows since conditioned to the occurrence of \( \{L_1 < R_0 - n\} \), \( R_t \) jumps to the right or stays in the same position at time 1 is enough to assure that \( B_+ \cap A_+ \) happens, which implies that \( P(B_+, A_+|L_1 < R_0 - n) \geq \frac{1}{2} \). Therefore
\[
E[X_-|A_+, B_+, L_1 < R_0 - n] \leq 2 \frac{1}{p}. \tag{B.3}
\]

Going back to (B.1), from (B.3) and (B.2) we obtain that \( E[\xi_+^{(n)}] \leq 5/p \). □
Now we present the proof of Lemmas 5.9 and 5.10.

**Proof of Lemma 5.9.** We will only consider l-paths, but the proof for r-paths is analogous. By the time homogeneity of paths in the DNB, we can assume that $s = 0$.

Now fix $\gamma_2$ such that $\gamma_1 < \gamma_2 < \gamma$ and then, we can write:

$$
P \left( \sup_{0 \leq t \leq \tau_x^{(n)}} \left| \tilde{r}_x^{(n)}(t) - l_x^{(n)}(t) \right| > n^\gamma \right) = P \left( \sup_{0 \leq t \leq \tau_x^{(n)}} \left| l_x^{(n)}(t) - l_x^{(n)}(t) \right| > n^\gamma; \tau_x^{(n)} > n^{2\gamma_2} \right)
$$

$$
+ P \left( \sup_{0 \leq t \leq \tau_x^{(n)}} \left| \tilde{r}_x^{(n)}(t) - l_x^{(n)}(t) \right| > n^\gamma; \tau_x^{(n)} \leq n^{2\gamma_2} \right)
$$

which is bounded from above by

$$
P \left( \tau_x^{(n)} > n^{2\gamma_2} \right) + P \left( \sup_{0 \leq t \leq n^{2\gamma_2}} \left| \tilde{l}_x^{(n)}(t) - l_x^{(n)}(t) \right| > n^\gamma \right). \tag{B.4}
$$

About the first term in (B.4), we have by Lemma 4.1 (and Remark 4.2 if $l_x^{(n)}$ and $\tilde{l}_x^{(n)}$ are evolving according to independent environments) that:

$$
P \left( \tau_x^{(n)} > n^{2\gamma_2} \right) \leq \frac{C_0 n^{\gamma_1}}{n^{2\gamma_2}},
$$

which converges to zero as $n \to \infty$. About the second term in (B.4), we have by Lemma 5.3 that both $l_x^{(n)}$ and $\tilde{l}_x^{(n)}$ converge under diffusive scale to Brownian motions with drift $-b_p$ and diffusion coefficient $\lambda_p^2$. So, this second term also converges to zero as $n \to \infty$. \hfill \Box

**Proof of Lemma 5.10.** First note that $T_x^{(n)}$ does not depend on $x$ or $s$ due to the translation invariance in time and space of the DNB paths.

To prove the limit, let us decompose $T^{(n)} = \tilde{T}^{(n)} + \tilde{T}^{(n)}$, where $\tilde{T}^{(n)}$ is the amount of time that we have to wait until both $r_x^{(n)}$ and $\tilde{r}_x^{(n)}$ cross $l_x^{(n)}$, and $\tilde{T}^{(n)}$ is the amount of time that until $r_x^{(n)}$ and $\tilde{r}_x^{(n)}$ meet each other for the first time after they cross $l_x^{(n)}$.

Now fix $\gamma_2$ and $\gamma_3$ such that $\gamma_1 < \gamma_2 < \gamma_3 < \gamma$. About $\tilde{T}^{(n)}$, recall that $\tau_c^k$ is the time until we have a crossing between an l-path at the right-hand side of an r-path initially at distance $k$ from each other. By Corollary 4.3 and Remark 4.2, we have that

$$
P \left( \tilde{T}^{(n)} > n^{2\gamma_2} \right) \leq P \left( \tilde{T}^{(n)} > n^{2\gamma_2} \right) \leq 2P \left( \tau_{n\gamma_1}^c > n^{2\gamma_2} \right) \leq \frac{2C_0 n^{\gamma_1}}{n^{2\gamma_2}} = \tilde{C}_1 n^{\gamma_1 - \gamma_2},
$$

which converges to zero as $n$ goes to infinity.

About $\tilde{T}^{(n)}$, denote by $D_t^{(n)}$ the distance between $r_x^{(n)}$ and $\tilde{r}_x^{(n)}$ at time $t$. We have by Lemma 5.3 that both $r_x^{(n)}$ and $\tilde{r}_x^{(n)}$ converge under diffusive scaling to Brownian motions with drift $-b_p$ and diffusion coefficient $\lambda_p^2$. Then

$$
P \left( D_t^{(n)} > n^{\gamma_3}, T^{(n)} \leq n^{2\gamma_2} \right) \leq P \left( \sup_{0 \leq t \leq n^{2\gamma_2}} D_t^{(n)} > n^{\gamma_3} \right) \to 0 \text{ as } n \to \infty.$$
So, by Lemma 4.1, we have:

\[
P\left(\tilde{T}^{(n)} > n^{2\gamma}\right) = P\left(\tilde{T}^{(n)} > n^{2\gamma} \mid D^{(n)}_{\tilde{T}^{(n)}} < n^{\gamma_3}\right) P\left(D^{(n)}_{\tilde{T}^{(n)}} < n^{\gamma_3}\right) \\
+ P\left(\tilde{T}^{(n)} > n^{2\gamma} \mid D^{(n)}_{\tilde{T}^{(n)}} \geq n^{\gamma_3}\right) P\left(D^{(n)}_{\tilde{T}^{(n)}} \geq n^{\gamma_3}\right) \\
\leq P\left(\tilde{T}^{(n)} > n^{2\gamma} \mid D^{(n)}_{\tilde{T}^{(n)}} < n^{\gamma_3}\right) + P\left(D^{(n)}_{\tilde{T}^{(n)}} \geq n^{\gamma_3}\right) \\
\leq C_0 n^{\gamma_3 - \gamma} + P\left(D^{(n)}_{\tilde{T}^{(n)}} \geq n^{\gamma_3}, \tilde{T}^{(n)} \leq n^{2\gamma_2}\right) + P\left(\tilde{T}^{(n)} > n^{2\gamma_2}\right) \\
\leq C_0 n^{\gamma_3 - \gamma} + P\left(D^{(n)}_{\tilde{T}^{(n)}} \geq n^{\gamma_3}, \tilde{T}^{(n)} \leq n^{2\gamma_2}\right) + \tilde{C}_1 n^{\gamma_1 - \gamma_2} \to 0 \text{ as } n \to \infty,
\]
which completes the proof. 

\[\square\]

**APPENDIX C. ON THE CONDITION \((U''_N)\)**

To prove that the DNB with branching parameter \(\epsilon_n = b n^{-1}\) converges in distribution under diffusive scaling to the Brownian Net still remains to verify that any subsequential limit from Theorem 3.3 does not have more paths than the Brownian Net. The strategy we know so far follows from [21] and consists of verifying that conditions \((U'_N)\) and \((U''_N)\).

We have proved \((U'_N)\), but to prove condition \((U''_N)\), we need to verify that for any limit point \((X, W^l, W^r, \tilde{W}^l, \tilde{W}^r)\) of \((X_n, W^l_n, W^r_n, \tilde{W}^l_n, \tilde{W}^r_n)\) and for any deterministic countable dense set \(D \subset \mathbb{R}^2\), a.s. paths in \(X\) do not enter any wedge of \((\tilde{W}^l(D), \tilde{W}^r(D))\) from outside. This last condition was not proved, but we conjecture that \((U''_N)\) holds for the DNB.

Let us make some considerations about \((U''_N)\). The arguments used in [21] to prove it for independent random walks with branching are the following:

Let \(D^l, D^r \subset \mathbb{R}^2\) be deterministic countable dense sets. For each \(z \in D^l\) (resp. \(z \in D^r\), fix a sequence \(z_n \in \mathbb{Z}^2_{\text{even}}\) (resp. \(\tilde{z}_n \in \mathbb{Z}^2_{\text{even}}\)) such that \(z_n \to z\) (resp. \(\tilde{z}_n \to \tilde{z}\)) under diffusive scaling and let \((\tilde{l}^{(n)}_{z_n})_{n \geq 1}\) (resp. \((\tilde{r}^{(n)}_{\tilde{z}_n})_{n \geq 1}\) be a sequence of dual \(l\)-paths (resp. dual \(r\)-paths) with branching parameter \(o(n^{-1})\) starting from \(z_n\) (resp. \(\tilde{z}_n\)). Also let \(\tau(\tilde{\pi}_1, \tilde{\pi}_2)\) denote the "possibly infinite" first meeting time of dual paths \(\tilde{\pi}_1, \tilde{\pi}_2\). Using independence, it is straightforward to show that for any \(z \in D^l\) and \(\tilde{z} \in D^r\), \((\tilde{l}^{(n)}_{z_n}, \tilde{r}^{(n)}_{\tilde{z}_n}, \tau(\tilde{l}^{(n)}_{z_n}, \tilde{r}^{(n)}_{\tilde{z}_n}))\) jointly converges under diffusive scaling to \((\tilde{l}_z, \tilde{r}_z, \tau(\tilde{l}_z, \tilde{r}_z))\) where \((\tilde{l}_z, \tilde{r}_z)\) is a pair of left-right Brownian motions.

Let \(\mathcal{N}^*\) be a weak limit of the system of independent random walks defined in [21]. By Skorohod’s representation theorem, we can construct a coupling such that the previous convergences occur almost surely and we will assume such coupling from now on. If \(\pi \in \mathcal{N}^*\) enters a wedge \(\psi(\tilde{r}, \tilde{l})\) from outside, then by [21, Lemma 3.4(b)], \(\pi\) must enter some skeletal wedge \(\psi(\tilde{r}_z, \tilde{l}_z)\) from outside, with \(z \in D^l\) and \(\tilde{z} \in D^r\). By the a.s. convergence of the system to \(\mathcal{N}^*\), there exist \(\pi^{(n)}\) such that \(\pi^{(n)} \to \pi\) under diffusive scaling. By the a.s. convergence
of $\hat{r}^{(n)}_{\infty}$, $\hat{r}^{(n)}_{\infty}$ to $\hat{L}_z$, $\hat{r}_z$ and the convergence of their first meeting time, for $n$ large enough, $\pi^{(n)}$ must enter a discrete wedge from outside, which is impossible.

So, the two key ingredients of their proof are: (1) the convergence of $(\hat{L}^{(n)}_{\infty}, \hat{r}^{(n)}_{\infty}, \tau(\hat{L}^{(n)}_{\infty}, \hat{r}^{(n)}_{\infty}))$; (2) [21, Lemma 3.4(b)] which ensures that a path can only enter a wedge from the outside if it also enters a discrete version of this wedge from outside. It is worth noting that (1) is straightforward to verify for systems with independence before meeting times or coalescence. This follows from the fact that two independent Brownian paths with opposite drift will cross each other immediately after they meet, and this implies that pairs of random walks converging to the pair of Brownian motions will also cross each other (thus they meet) when they get close to each other by a distance negligible under diffusive scaling.

However, in our case, we do not have independence between paths and we were not able to prove the convergence of the first meeting time between one dual $l$-path and one dual $r$-path. Without that, it seems possible that two dual limiting paths $\hat{r}_z$ and $\hat{L}_z$ can create a wedge, although existing sequences of paths $\hat{r}^{(n)}_{\infty}$ and $\hat{L}^{(n)}_{\infty}$ that converge to them and never meet each other. It would mean that we do not create a discrete version of the wedge and we cannot guarantee that do not exist paths in $\mathcal{X}$ which enter the wedge from outside.

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