Endpoint Strichartz estimates with angular integrability and some applications

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Abstract. The endpoint Strichartz estimate \( \| e^{it\Delta} f \|_{L_t^2 L_x^\infty} \lesssim \| f \|_{L^2} \) is known to be false in two space dimensions. Taking averages spherically on the polar coordinates \( x = \rho \omega, \rho > 0, \omega \in S^1 \), Tao showed a substitute of the form \( \| e^{it\Delta} f \|_{L_t^2 L_x^\infty} \lesssim \| f \|_{L^2} \). Here we address a weighted version of such spherically averaged estimates. As an application, the existence of solutions for the inhomogeneous nonlinear Schrödinger equation is shown for \( L^2 \) data.

1. Introduction

The physical interpretation of the Schrödinger equation \( i \partial_t u + \Delta u = 0 \) is that \( |u(x, t)|^2 \) is the probability density for finding a quantum particle at place \( x \in \mathbb{R}^n \) and time \( t \in \mathbb{R} \). This leads us to think that \( L^2(\mathbb{R}^n) \) will play a distinguished role. Indeed, the Schrödinger propagator \( e^{it\Delta} \), which gives a formula for the solution, is an isometry on \( L^2 \). That is, \( \| e^{it\Delta} f \|_{L_x^2} = \| f \|_{L^2} \) for any fixed \( t \). But interestingly, when averages on time are also made, a much richer \( L^p \) integrability can be observed. This space-time integrability known as Strichartz estimates has been extensively studied over the last several decades and is now completely understood as follows (see [11,19,23,26]):

\[
\| e^{it\Delta} f \|_{L_t^q L_x^r} \lesssim \| f \|_{L^2}
\]

if and only if \( (q, r) \) is Schrödinger-admissible, i.e., \( q \geq 2, 2/q + n/r = n/2 \) and \( (q, r, n) \neq (2, \infty, 2) \).

The endpoint case \( q = 2 \) is known to be false in two space dimensions [23], in which case Tao [27] showed a substitute of the form

\[
\| e^{it\Delta} f \|_{L_t^q L_x^\infty L_{\omega}^2} \lesssim \| f \|_{L^2}
\]

by taking averages spherically on the polar coordinates \( x = \rho \omega, \rho > 0, \omega \in S^1 \). Tao’s result was extended to higher dimensions in [15]. More general spherically averaged estimates involving \( L_t^q L^r \) \( L_{\rho}^2 L_{\omega}^2 \) were also studied (see [12–14]).

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1.1. Endpoint estimates

In this paper, we are concerned with a weighted version of the spherically averaged estimates, which involves weighted mixed norms with the angular variable treated in a different $L^p$ space than the radial variable:

$$\| f(x) \|_{L^pL^k_\omega(|x|^{-r})} = \left( \int_0^\infty \| \rho^{-\gamma} f(\rho \omega) \|_{L^k_\omega(S^{n-1})} \rho^{n-1} d\rho \right)^{1/r}$$

where $1 \leq r, k \leq \infty$ and $\gamma \geq 0$. Particularly when $r = k$, this norm coincides with the weighted $L^r$ norm, $\| f(x) \|_{L^r(|x|^{-r})}$. Our first result is the following endpoint Strichartz estimates with angular integrability.

**Theorem 1.1.** Let $n \geq 3$ and $0 \leq \gamma \leq 1$. Assume that

$$\frac{1}{r} = \frac{n-2}{2n} + \frac{\gamma}{n} \quad \text{and} \quad \frac{1}{r} - \frac{\gamma}{2(n-1)} \leq \frac{1}{k} \leq \frac{1}{r}. \quad (1.2)$$

Then, we have

$$\| |x|^{-\gamma} e^{it\Delta} f \|_{L^q^rL^r_\rhoL^k_\omega} \lesssim \| f \|_{L^2}. \quad (1.3)$$

**Remark 1.2.** One can also trivially obtain further estimates (1.3) for $(1/r, 1/k)$ contained in the closed quadrangle with vertices $A, D, C, B$ in Fig. 1, using the inclusion of $L^k$ spaces on the compact set $S^{n-1}$. As will be seen later (Sect. 4), this trivial region is also needed for obtaining some applications to nonlinear equations described below.

We shall give more details about the region of $(1/r, 1/k)$ for which the theorem holds; the region is given by the closed triangle with vertices $A, E, D$. Especially when $\gamma \to 0$, $(1/r, 1/k)$ goes to the point $A$ and (1.3) boils down to the endpoint case $q = 2$ of the classical estimates (1.1). The segment $[E, D]$ corresponds to the case $\gamma = 1$. The lower and upper bounds of $1/k$ in (1.2) determine the segments $[A, E]$ and $[A, D]$, respectively.

In view of interpolation between (1.3) and the trivial estimate $\| e^{it\Delta} f \|_{L^\infty_tL^2_x} \lesssim \| f \|_{L^2}$ (point $F$), we have

$$\| |x|^{-\gamma} e^{it\Delta} f \|_{L^q^rL^r_\rhoL^k_\omega} \lesssim \| f \|_{L^2} \quad (1.4)$$

for $(q, r, k)$ contained in the closed tetrahedron with vertices $A, E, D, F$ with

$$\gamma = \frac{2}{q} - n\left(\frac{1}{2} - \frac{1}{r}\right). \quad (1.5)$$

This simply recovers the previous result of Ozawa and Rogers [24] in which the non-endpoint case where $q > 2$ in (1.4) was obtained using Pitt’s inequalities and weighted versions of the Hausdorff–Young inequality. In this regard, the theorem fills the gap $q = 2$ in this result. We also refer the reader to [4,9,18] for some related works on more general estimates with angular regularity. For the optimality of (1.4) particularly when $r = k$, we refer the reader to [22].
1.2. Applications

Now we would like to give some applications of the endpoint estimates (1.3) to the inhomogeneous nonlinear Schrödinger equation (INLS)

\[
\begin{cases}
    i \partial_t u + \Delta u = \lambda |x|^{-\alpha} |u|^{\beta} u, & (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
    u(x, 0) = u_0(x) \in L^2,
\end{cases}
\]  

(1.6)

where $0 < \alpha < 2$, $\beta > 0$ and $\lambda = \pm 1$. Here, the case $\lambda = 1$ is defocusing, while the case $\lambda = -1$ is focusing. This model arises in nonlinear optics and plasma physics for the propagation of laser beams in an inhomogeneous medium [1,28]. The equation enjoys the scale invariance $u(x, t) \mapsto u_\delta(x, t) = \delta^{\frac{2-\alpha}{\beta n}} u(\delta x, \delta^2 t)$ for $\delta > 0$, and

\[\| u_{\delta,0} \|_{L^2} = \delta^{\frac{2-\alpha}{\beta n} - \frac{2}{n}} \| u_0 \|_{L^2}\]

with rescaled initial data $u_{\delta,0}$. If $\beta = (4 - 2\alpha)/n$, the scaling preserves the $L^2$ norm of $u_0$ and (1.6) goes by the name of the mass-critical (or $L^2$-critical) INLS.

This critical case remained unsolved and was recently solved by the authors [20]. (For initial data in other Sobolev spaces $H^s$, see, e.g., [3,6,7,10,16,21].) But here, we provide more information on the solution with respect to angular integrability.

To apply the endpoint estimates (1.3) to the nonlinear problems, one needs inhomogeneous estimates which can be obtained by the standard Christ–Kiselev lemma. However, the lemma misses the following double $L^2$ type estimates:
Theorem 1.3. Let $n \geq 3$ and $0 < \gamma < 1$. Then, we have

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s)ds \right\|_{L^2_t L^r\rho L^k_\omega(|\cdot|^{-r\gamma})} \lesssim \|F\|_{L^2_t L^\rho\omega L^k_\omega(|\cdot|^{-r\gamma})}$$

(1.7)

if

$$\frac{1}{r} = \frac{n-2}{2n} + \frac{\gamma}{n} \quad \text{and} \quad \frac{1}{r} - \frac{\gamma}{2(n-1)} < \frac{1}{k} < \frac{1}{r} \leq \frac{1}{k} \leq 1.$$  

(1.8)

Here, the region of $(1/r, 1/k)$ and $(1/r, 1/\tilde{k})$ is given by the open triangle with vertices $A, E, D$ and the closed quadrangle with vertices $A, D, C, B$ without the boundaries $[A, B], [D, C]$, respectively. The weights $|x|^{-\gamma}$ in the weighted estimates allow us to handle the singularity $|x|^{-\alpha}$ in the nonlinearity effectively. As a result, we obtain the following well-posedness result.

Theorem 1.4. Let $n \geq 3$, $0 < \alpha < 2$ and $\beta = (4-2\alpha)/n$. If $\|u_0\|_{L^2}$ is assumed to be small, then there exists a unique solution to (1.6)

$$u \in C_t([0, \infty); L^2_x) \cap L^2_t([0, \infty); L^r\rho L^k_\omega(|\cdot|^{-\alpha r/2}))$$

for $(r, k)$ satisfying

$$\frac{1}{r} = \frac{n-2}{2n} + \frac{\alpha}{2n} \quad \text{and} \quad \frac{1}{r} - \frac{\alpha}{4(n-1)} < \frac{1}{k} < \frac{1}{r}.$$  

(1.9)

Furthermore, the solution scatters in $L^2$, i.e., there exists $\varphi \in L^2$ such that

$$\lim_{t \to \infty} \|u(t) - e^{it\Delta} \varphi\|_{L^2_x} = 0.$$  

We note in passing that similar results can be obtained by treating the non-endpoint estimates (1.4). However, we do not pursue this issue here.

The paper is organized as follows. In Sect. 2, we prove the endpoint estimates in Theorem 1.1 by making use of interpolation together with the Sobolev embedding on the unit sphere $S^{n-1}$. Section 3 is devoted to proving the inhomogeneous estimates in Theorem 1.3. Here, we basically adopt the bilinear interpolation argument developed by Keel and Tao [19], but we need to modify the argument to make it applicable to the weighted setting. In Sect. 4, we finally apply the estimates to obtain Theorem 1.4.

Throughout this paper, the letter $C$ stands for a positive constant which may be different at each occurrence. We also denote $A \lesssim B$ to mean $A \leq CB$ with unspecified constants $C > 0$.

2. Endpoint estimates

In this section, we prove Theorem 1.1 by interpolating between the estimates on the point $A$ and the segment $[E, D]$. As mentioned in Remark 1.2, the estimate on
the point \( A \) becomes equivalent to the endpoint case \( q = 2 \) of the classical estimates (1.1). Hence, we only need to obtain (1.3) on the segment \([E, D]\); this is the case \( r = 2, \gamma = 1 \) corresponding to

\[
\|x|^{-1} e^{it \Delta} f\|_{L_t^2 L_x^2 L_\omega^k} \lesssim \|f\|_{L^2}
\]

for \( \frac{n-2}{2(n-1)} \leq \frac{1}{k} \leq \frac{1}{2} \). From [17] (see (1.2) there), we first recall

\[
\|\Lambda^{\frac{1}{2}} |x|^{-1} e^{it \Delta} f\|_{L_t^2 L_x^2 L_\omega^2} \lesssim \|f\|_{L^2}
\]

(2.1)

where \( \Lambda := \sqrt{1 - \Delta_\omega} \) for the Laplace–Beltrami operator \( \Delta_\omega \) on the unit sphere \( S^{n-1} \), \( n \geq 3 \). Then by (2.1), it is enough to show

\[
\|x|^{-1} e^{it \Delta} f\|_{L_t^2 L_x^2 L_\omega^k} \lesssim \|\Lambda^{(\frac{1}{2} - \frac{1}{k})} |x|^{-1} e^{it \Delta} f\|_{L_t^2 L_x^2 L_\omega^2}
\]

(2.2)

for \( \frac{n-2}{2(n-1)} \leq \frac{1}{k} \leq \frac{1}{2} \). For this, we first apply the Sobolev embedding\(^1\) on \( S^{n-1} \),

\[
\|f\|_{L^p_\omega} \lesssim \|\Lambda^{(n-1)(\frac{1}{2} - \frac{1}{p})} f\|_{L^2_\omega}, \quad 2 \leq p < \infty,
\]

(2.3)

to see

\[
\|x|^{-1} e^{it \Delta} f\|_{L_t^2 L_x^2 L_\omega^k} \lesssim \|\Lambda^{(n-1)(\frac{1}{2} - \frac{1}{k})} |x|^{-1} e^{it \Delta} f\|_{L_t^2 L_x^2 L_\omega^2}
\]

(2.4)

for \( 2 \leq k < \infty \). We then use the inclusion \( L^p_\omega \subseteq L^2_\omega \) for \( p \geq 2 \) and (2.3) again to get

\[
\|\Lambda^{(n-1)(\frac{1}{2} - \frac{1}{k})} g\|_{L^2_\omega} \lesssim \|\Lambda^{(n-1)(\frac{1}{2} - \frac{1}{k})} f\|_{L^2_\omega} \lesssim \|\Lambda^{\frac{1}{2}} g\|_{L^2_\omega}
\]

(2.5)

with \( \frac{1}{p} = \frac{2n-3}{2(n-1)} - \frac{1}{k} \) and \( 2 \leq p < \infty \). Now the desired estimate (2.2) follows from combining (2.4) and (2.5) with \( g = |x|^{-1} e^{it \Delta} f \). Note that the conditions on \( k \) and \( p \) here determine \( \frac{n-2}{2(n-1)} \leq \frac{1}{k} \leq \frac{1}{2} \) we wanted.

3. Inhomogeneous estimates

Next, we prove the inhomogeneous estimates (1.7) in Theorem 1.3 by adopting the bilinear interpolation argument developed by Keel and Tao [19]. We need to modify the argument to make it applicable to the present setting with weights.

Instead of (1.7), we shall show a stronger estimate which is given by replacing \( \int_0^t \) in (1.7) by \( \int_{-\infty}^t \):

\[
\left\| \int_{-\infty}^t e^{i(t-s) \Delta} F(\cdot, s) \, ds \right\|_{L_t^2 L_x^r L_\omega^k(|\cdot|^{-\gamma})} \lesssim \|F\|_{L_t^2 L_x^r L_\omega^k(|\cdot|^{-\gamma})}
\]

(3.1)

\(^1\)See, for example, Lemma 7.1 in [18]
for $r, k, \tilde{k}, \gamma$ given as in the theorem. To deduce (1.7) from (3.1), first decompose the $L_t^2$ norm in the left-hand side of (1.7) into two parts, $t \geq 0$ and $t < 0$. Then, the latter can be reduced to the former by a change of variables $t \mapsto -t$, and so we only need to consider the first part $t \geq 0$. But, since $[0, t) = (-\infty, t) \cap [0, \infty)$, by applying (3.1) with $F$ replaced by $\chi_{[0, \infty)}(s)F$, the first part follows directly.

Then by duality, we may show the following bilinear form estimate:

$$|T(F, G)| \lesssim \|F\|_{L_t^2 L_r^\rho L_k^\omega} \|G\|_{L_t^2 L_r^\rho L_k^\omega}$$

where

$$T(F, G) := \int_{-\infty}^{\infty} \int_{s < t} \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle_x ds \, dt.$$

Here, $(\cdot, \cdot)$ denotes the usual inner product on $L^2$. Indeed, by duality, (3.2) implies (3.1) with $\tilde{k}'$ replaced by $k'$. Since $k' \leq \tilde{k}'$, (3.1) then follows. By duality and symmetry, we also note that (3.2) implies

$$\left\| \int_{-\infty}^{\infty} e^{i(t-s)\Delta} F(\cdot, s) ds \right\|_{L_t^2 L_r^\rho L_k^\omega} \lesssim \|F\|_{L_t^2 L_r^\rho L_k^\omega}$$

with the same $\gamma$, $r$, $k$, from which we can alternatively prove (1.3) on the open triangle with vertices $A$, $E$, $D$ in Fig. 1 using the $TT^*$ argument.

Let us now show (3.2). We first decompose the integral region $\Omega = \{(s, t) \in \mathbb{R}^2 : s < t\}$ dyadically away from the singularity $t = s$. Indeed, we break $\Omega$ into a series of time-localized regions using a Whitney-type decomposition (see [25] or [8]); let $Q_j$ be the family of dyadic squares in $\Omega$ whose side length is dyadic number $2^j$ for $j \in \mathbb{Z}$. Each square $Q = I \times J \in Q_j$ has the property that

$$2^j \sim |I| \sim |J| \sim \text{dist}(I, J)$$

and $\Omega = \bigcup_{j \in \mathbb{Z}} \bigcup_{Q \in Q_j} Q$ where the squares $Q$ are essentially disjoint. Now we may write

$$T(F, G) = \sum_{j \in \mathbb{Z}} T_j(F, G),$$

where

$$T_j(F, G) := \sum_{Q \in Q_j} \int_{I \times J} \int_{s \in I} \langle e^{-is\Delta} F(s), e^{-it\Delta} G(t) \rangle_x ds \, dt.$$

We then obtain the desired estimate (3.2) by making use of the bilinear interpolation between its time-localized estimates in the following proposition which will be proved later.
Figure 2. The range of \((1/a, 1/\tilde{a})\) for which (3.5) holds

**Proposition 3.1.** Let \(n \geq 3\) and \(0 < \gamma < 1\). Assume that \(2 \leq a, \tilde{a} < \infty\), \(1/a - \gamma/(2(n-1)) \leq 1/b \leq 1/a\) and \(1/\tilde{a} - \gamma/(2(n-1)) \leq 1/\tilde{b} \leq 1/\tilde{a}\).

Then, we have

\[
|T_j(F, G)| \lesssim 2^{-j\beta(a, \tilde{a})} \|F\|_{L^2_t L^a_{\rho} L^b_{\omega}(|\cdot|^{\rho \gamma})} \|G\|_{L^2_t L^\tilde{a}_{\rho} L^\tilde{b}_{\omega}(|\cdot|^{\tilde{\rho} \gamma})} \tag{3.5}
\]

for all \(j \in \mathbb{Z}\) and all \((1/a, 1/\tilde{a})\) in a neighborhood of \((1/r, 1/r)\) (see Fig. 2) with

\[
1/r = n - 2/(2n) + \gamma/n \quad \text{and} \quad \beta(a, \tilde{a}) = -1 + n/2 - n/(2a) - n/(2\tilde{a}) + \gamma.
\]

From making use of the bilinear interpolation between the estimates (3.5), we shall now deduce

\[
\sum_{j \in \mathbb{Z}} |T_j(F, G)| \lesssim \|F\|_{L^2_t L^a_{\rho} L^b_{\omega}(|\cdot|^{\rho \gamma})} \|G\|_{L^2_t L^\tilde{a}_{\rho} L^\tilde{b}_{\omega}(|\cdot|^{\tilde{\rho} \gamma})} \tag{3.6}
\]

which clearly implies (3.2).

Indeed, from the proposition we have the following three estimates:

\[
|T_j(F, G)| \lesssim 2^{-j\beta(r_0, r_0)} \|F\|_{L^2_t L^r_{\rho} L^{r_0}_{\omega}(|\cdot|^{\rho \gamma})} \|G\|_{L^2_t L^\tilde{r}_{\rho} L^{\tilde{r}_0}_{\omega}(|\cdot|^{\tilde{\rho} \gamma})},
\]

\[
|T_j(F, G)| \lesssim 2^{-j\beta(r_1, r_0)} \|F\|_{L^2_t L^r_{\rho} L^{r_0}_{\omega}(|\cdot|^{\rho \gamma})} \|G\|_{L^2_t L^\tilde{r}_{\rho} L^{\tilde{r}_0}_{\omega}(|\cdot|^{\tilde{\rho} \gamma})},
\]

\[
|T_j(F, G)| \lesssim 2^{-j\beta(r_1, r_0)} \|F\|_{L^2_t L^r_{\rho} L^{r_0}_{\omega}(|\cdot|^{\rho \gamma})} \|G\|_{L^2_t L^\tilde{r}_{\rho} L^{\tilde{r}_0}_{\omega}(|\cdot|^{\tilde{\rho} \gamma})},
\]
where, for a sufficiently small $\varepsilon > 0$ and $i = 0, 1$,
\[
\frac{1}{r_0} = \frac{n - 2}{2n} + \frac{\gamma}{n - \varepsilon}, \quad \frac{1}{r_1} = \frac{n - 2}{2n} + \frac{\gamma}{n + 2\varepsilon}
\]
and
\[
\frac{1}{r_i} - \frac{\gamma}{2(n - 1)} \leq \frac{1}{r_i} \leq \frac{1}{r_i}
\]
(3.7)

Next, we define the vector-valued bilinear operator $B$ by
\[
B(F, G) = \{ T_j(F, G) \}_{j \in \mathbb{Z}}.
\]

Then, the above three estimates are rewritten as
\[
\|B(F, G)\|_{\ell^0_\infty} \lesssim \|F\|_{L^2 r_0' L^0_\infty(| \cdot | r_0' \gamma)} \|G\|_{L^2 r_0' L^0_\infty(| \cdot | r_0' \gamma)},
\]
\[
\|B(F, G)\|_{\ell^{\beta_1}_\infty} \lesssim \|F\|_{L^2 r_0' L^0_\infty(| \cdot | r_0' \gamma)} \|G\|_{L^2 r_0' L^0_\infty(| \cdot | r_0' \gamma)},
\]
\[
\|B(F, G)\|_{\ell^{\beta_0}_\infty} \lesssim \|F\|_{L^2 r_0' L^0_\infty(| \cdot | r_0' \gamma)} \|G\|_{L^2 r_0' L^0_\infty(| \cdot | r_0' \gamma)},
\]
respectively, with $\beta_0 = \beta(r_0, r_0)$ and $\beta_1 = \beta(r_0, r_1) = \beta(r_1, r_0)$.

Here, $\ell_q^s$ denotes a weighted sequence space defined for $s \in \mathbb{R}$ and $1 \leq q \leq \infty$ with the norm
\[
\|\{x_j\}_{j \geq 0}\|_{\ell^s_q} = \left\{ \left( \sum_{j \geq 0} |x_j|^q \right)^{1/q} \right\} \quad \text{if } q \neq \infty,
\]
\[
\sup_{j \geq 0} |x_j| \quad \text{if } q = \infty.
\]

Applying the following lemma with $p = q = 2$ and $\theta_0 = \theta_1 = 1/3$, we now get
\[
B : (A_0, A_1)_{\frac{1}{2}, 2} \times (B_0, B_1)_{\frac{1}{2}, 2} \to (\ell^0_\infty, \ell^1_\infty)_{\frac{1}{2}, 1}
\]
(3.8)

with $A_0 = B_0 = L^2 r_0' L^0_\infty(| \cdot | r_0' \gamma)$ and $A_1 = B_1 = L^2 r_1' L^0_\infty(| \cdot | r_1' \gamma)$.

**Lemma 3.2.** ([2], Section 3.13, Exercise 5(b)) For $i = 0, 1$, let $A_i, B_i, C_i$ be Banach spaces and let $T$ be a bilinear operator such that $T : A_0 \times B_0 \to C_0$, $T : A_0 \times B_1 \to C_1$, and $T : A_1 \times B_0 \to C_1$. Then, one has
\[
T : (A_0, A_1)_{\theta, p} \times (B_0, B_1)_{\theta, q} \to (C_0, C_1)_{\theta, 1}
\]
if $0 < \theta_i < \theta = \theta_0 + \theta_1 < 1$ and $1/p + 1/q \geq 1$ for $1 \leq p, q \leq \infty$. Here, $(\cdot, \cdot)_{\theta, p}$ denotes the real interpolation functor.

Finally, we shall apply the real interpolation space identities in the following lemma (see [5] and [2]).

**Lemma 3.3.** Let $0 < \theta < 1$. If $1 \leq p_0, p_1 < \infty$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$, then
\[
(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta, q} = \left\{ \begin{array}{ll}
L^p((A_0, A_1)_{\theta, p}) & \text{if } q = p, \\
L^{p, q}(A) & \text{if } A_0 = A_1 = A,
\end{array} \right.
\]
for Banach spaces $A_0, A_1$. If $s_0, s_1 \in \mathbb{R}, s_0 \neq s_1$ and $s = (1 - \theta)s_0 + \theta s_1$, then
\[
(\ell^{s_0}_\infty, \ell^{s_1}_\infty)_{\theta, 1} = \ell^s_1.
\]
Indeed, applying the lemma, we first see \((\ell_{\epsilon_0}^{\beta_0}, \ell_{\epsilon\infty}^{\beta_1})_{\frac{3}{2}, 1} = \ell_1^0\) and
\[
(L_t^2 L_{\rho}^{r_0} L_{\omega}^{k_0} (\cdot \cdot | \cdot |^2), L_t^2 L_{\rho}^{r_1} L_{\omega}^{k_1} (\cdot \cdot | \cdot |^2))_{\frac{3}{2}, 2} = L_t^2 ((L_{\rho}^{r_0} L_{\omega} (\cdot \cdot | \cdot |^2), L_{\rho}^{r_1} L_{\omega} (\cdot \cdot | \cdot |^2))_{\frac{1}{2}, 2}).
\]
Since \(r' < 2\), the first identity in (3.9) has not been applied inside the \(L_t^2\) space any more. Instead, we will make use of the second one, and hence we must take \(k' = k_0' = k_1'\), but this is possible because (3.7) holds for a sufficiently small \(\epsilon > 0\) if
\[
\frac{1}{r} - \frac{r'}{2(n - 1)} < \frac{1}{k} < \frac{1}{r}
\]
which is exactly consistent with that in the assumption (1.8). Since we may write
\[
\|f\|_{L_{\rho}^{r'} L_{\omega}^{k'} (| \cdot |^r')} = \|F\|_{L_{\rho}^{r'} L_{\omega}^{k'}},
\]
with \(\tilde{\rho} = \rho^n/n\) and \(F(\tilde{\rho}, \omega) = (n\tilde{\rho})^{\gamma/n} f((n\tilde{\rho})^{1/n} \omega)\), we are indeed reduced to showing
\[
L_t^2 ((L_{\rho}^{r_0} L_{\omega}^{k_0} L_{\rho}^{r_1} L_{\omega}^{k_1})_{\frac{3}{2}, 2}) = L_t^2 L_{\rho}^{r_0} L_{\omega}^{k_0} \supset L_t^2 L_{\rho}^{r'} L_{\omega}^{k'}
\]
which follows directly from applying the second identity in (3.9) and then using the embedding property of Lorentz spaces, \(L_{\rho}^{r'} \subset L_{\rho}^{r_0} L_{\omega}^{k_0}\) for \(r' < 2\). Combining (3.8) with the resulting real interpolation spaces, we now get
\[
B : L_t^2 L_{\rho}^{r'} L_{\omega}^{k'} (| \cdot |^r') \times L_t^2 L_{\rho}^{r'} L_{\omega}^{k'} (| \cdot |^r') \rightarrow \ell_1^0
\]
which is equivalent to the desired estimate (3.6). This completes the proof.

3.1. Time-localized estimates

This subsection is devoted to proving the time-localized estimate (3.5) in Proposition 3.1. Let us first set
\[
T_{j,Q}(F, G) := \int_{t\in J} \int_{s\in I} \langle e^{-i s \Delta} F(s), e^{-i t \Delta} G(t)\rangle_x ds dt
\]
for each square \(Q = I \times J \in Q_j\). Then, we only need to show
\[
|T_{j,Q}(F, G)| \lesssim 2^{-j\beta(a, \tilde{a})} \|F\|_{L_t^2(I; L_{\rho}^{r'} L_{\omega}^{k'} (| \cdot |^r'))} \|G\|_{L_t^2(J; L_{\rho}^{r'} L_{\omega}^{k'} (| \cdot |^r'))}
\]
(3.11)
to get (3.5). Using the fact that for each \(I\) there are at most a fixed finite number of intervals \(J\) which satisfy (3.3) and they are all contained in a neighborhood of \(I\) of size \(O(2^J)\), we indeed get
\[
\sum_{Q \in Q_j} |T_{j,Q}(F, G)| \lesssim 2^{-j\beta(a, \tilde{a})} \sum_{Q \in Q_j} \|F\|_{L_t^2(I; L_{\rho}^{r'} L_{\omega}^{k'} (| \cdot |^r'))} \|G\|_{L_t^2(J; L_{\rho}^{r'} L_{\omega}^{k'} (| \cdot |^r'))}
\]
\[ \leq 2^{-j\beta(a, \tilde{a})} \left( \sum_{Q \in \mathcal{Q}_j} \| F \|^2_{L^2_t(J; L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma))} \right)^{\frac{1}{2}} \]

\[ \cdot \left( \sum_{Q \in \mathcal{Q}_j} \| G \|^2_{L^2_t(J; L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma))} \right)^{\frac{1}{2}} \]

\[ \lesssim 2^{-j\beta(a, \tilde{a})} \| F \|^2_{L^2_t(\mathbb{R}; L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma))} \| G \|^2_{L^2_t(\mathbb{R}; L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma))} \]

as desired.

From now on, we shall show (3.11) for the following exponents (see Fig. 2):

(a) \((a)\) \(a = \tilde{a} = \frac{8}{3\gamma} := \lambda, \ b = \tilde{b}\) (point \(A\)),

(b) \((b)\) \(2 \leq a < r = \frac{2n}{n-2+2\gamma}, \ \tilde{a} = 2\) (segment \((B, D)\)),

(c) \((c)\) \(a = 2, \ 2 \leq \tilde{a} < r = \frac{2n}{n-2+2\gamma}\) (segment \((C, D)\)),

in which \(b\) and \(\tilde{b}\) are also given to hold (3.4). The proposition will then follow by interpolation and the fact that \(2 < r < \infty\).

To show the first case \((a)\), we recall the following time decay estimates (see Proposition 4.2 in [24]):

\[ \| e^{it\Delta} u_0 \|_{L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma)} \lesssim |t|^{-n(\frac{1}{\gamma} - \frac{1}{a})} \| u_0 \|_{L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma)} \] (3.12)

where \(2 \leq a \leq b < \infty\) and \((2n-1)(\frac{1}{a} - \frac{1}{b}) \leq \gamma < \frac{n}{a}\). Since this estimate does not hold for \(a = \infty\), we cannot take the origin instead of the point \(A\) in Fig. 2.

Hence, we need to carefully choose the point \(A\) near the origin by observing, from the condition \(\gamma < \frac{n}{a}\), the fact that the more nearer we take the point \(A\) to the origin, the higher the admissible dimension is. The point \(A = (\frac{3\gamma}{8}, \frac{3\gamma}{8})\) would suffice to cover all dimensions \(n \geq 3\). Now we use Hölder’s inequality and (3.12) to obtain

\[ |T_{j, Q}(F, G)| \leq \int_{I_j} \int_I \| \rho^{-\gamma} e^{i(t-s)\Delta} F(s) \|_{L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma)} \| \rho^\gamma G(t) \|_{L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma)} \| \frac{J}{\rho^\gamma} L^b_{\gamma}(\cdot|\cdot^\gamma) \| \frac{\omega(t)}{R} ds dt \]

\[ \lesssim \int_{I_j} \int_I |t-s|^{-n(\frac{1}{\gamma} - \frac{1}{a})} \| F(s) \|_{L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma)} \| G(t) \|_{L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma)} \| \frac{J}{\rho^\gamma} L^b_{\gamma}(\cdot|\cdot^\gamma) \| \frac{\omega(t)}{R} ds dt \]

\[ \lesssim 2^{-jn(\frac{1}{\gamma} - \frac{1}{a}) - \gamma} \int_{I_j} \int_I \| F(s) \|_{L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma)} \| G(t) \|_{L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma)} \| \frac{J}{\rho^\gamma} L^b_{\gamma}(\cdot|\cdot^\gamma) \| \frac{\omega(t)}{R} ds dt \]

where \(\lambda \leq b < \infty\) and \((2n-1)(1/\lambda - 1/b) \leq \gamma\) are required. This requirement is the same as the condition (3.4) with \(a = \lambda\). By applying Hölder’s inequality again in each \(t\) and \(s\), we get

\[ |T_{j, Q}(F, G)| \lesssim 2^{-j\beta(\lambda, \lambda)} \| F \|^2_{L^2_t(J; L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma))} \| G \|^2_{L^2_t(J; L^p_{\rho} L^b_{\gamma}(\cdot|\cdot^\gamma))}, \]

as desired.
Now it remains to show the second case (b). (The case (c) is shown clearly in the same way.) By bringing the $s$-integration inside the inner product in (3.10) and then applying Hölder’s inequality in $\omega$, $\rho$ and $t$ in turn, we first see that

$$
|T_{j}(F, G)| \leq \int J \left( \left\| \int_{I} e^{i(t-s)\Delta} F(s) ds \bigg\|_{x} \bigg\|_{x} \right\|_{x} \right) \ dt
$$

$$
\leq \left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} \chi_{I}(s) F(s) ds \right\|_{L_{x}^{\tilde{q}} L_{\rho}^{\frac{3}{2}} L_{T}^{\frac{2}{2r}}(|\cdot|^{-2\gamma})} \ \|G\|_{L_{t}^{\tilde{q}}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))}
$$

(3.13)

where $\tilde{q} > 2$ is given so that (1.5) holds for $(q, r) = (\tilde{q}, 2)$. Then, by using the $TT^{*}$ version of (1.4), we have

$$
\left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} \chi_{I}(s) F(s) ds \right\|_{L_{x}^{\tilde{q}} L_{\rho}^{\frac{3}{2}} L_{T}^{\frac{2}{2r}}(|\cdot|^{-2\gamma})} \lesssim \|F\|_{L_{t}^{\tilde{q}}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))} \|G\|_{L_{t}^{\tilde{q}}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))},
$$

(3.14)

for $q > 2$ given so that (1.5) holds for $(q, r) = (q, a)$. (Note here that there can exist such $q, \tilde{q} > 2$ for $0 < \gamma < 1$.) By combining (3.13) and (3.14), we now get

$$
|T_{j}(F, G)| \lesssim \|F\|_{L_{t}^{\tilde{q}}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))} \|G\|_{L_{t}^{\tilde{q}}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))}.
$$

Using Hölder’s inequality in $t$ since $q' < 2$, and then using the identity (1.5) for $(q, r) = (q, a)$, we estimate

$$
\|F\|_{L_{t}^{\tilde{q}}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))} \lesssim 2^{j\left(\frac{1}{2} - \frac{1}{q'}\right)} \|F\|_{L_{t}^{2}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))}
$$

$$
= 2^{j\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{2}\right)} \|F\|_{L_{t}^{2}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))}
$$

and similarly

$$
\|G\|_{L_{t}^{\tilde{q}}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))} \lesssim 2^{j\left(\frac{1}{2} - \frac{2}{2}\right)} \|G\|_{L_{t}^{2}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))}.
$$

Therefore, we get

$$
|T_{j}(F, G)| \lesssim 2^{-j\beta(a, 2)} \|F\|_{L_{t}^{2}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))} \|G\|_{L_{t}^{2}(J; L_{\rho}^{2} L_{\omega}^{2}(|\cdot|^{2\gamma}))}
$$

as desired.

4. The well-posedness in $L^{2}$

In this section, we prove Theorem 1.4 making use of the weighted Strichartz estimates in Theorems 1.1 and 1.3.
4.1. Nonlinear estimates

The following nonlinear estimates play a key role in the proof.

**Lemma 4.1.** Let \( n \geq 3, 0 < \alpha < 2 \) and \( \beta = (4 - 2\alpha)/n \). Assume that the exponents 
\((1/r, 1/k)\) satisfy all the conditions given as in Theorem 1.4. Then, there exist certain exponents 
\((1/r, 1/\widetilde{k})\) in the open quadrangle with vertices \( A, D, C, B \) in Fig. 1, for which

\[
\| |x|^{-\alpha} |u|^\beta v \|_{L^2_\rho([0,\infty); L^k_\omega(|\cdot|^{-\gamma})]} \leq \| u \|_{L^\infty_\rho([0,\infty); L^2_\omega(|\cdot|^{-\gamma})]} \| v \|_{L^2_\rho([0,\infty); L^\tilde{k}_\omega(|\cdot|^{-\gamma})]} \]

holds with \( \gamma = \alpha/2 \).

**Proof of Lemma 4.1.** Let us first write the conditions on 
\((r, k; \gamma)\) as 
\[
0 < \gamma < 1, \quad \gamma = 1 - n \left( \frac{1}{2} - \frac{1}{r} \right), \quad 0 < \frac{1}{r} - \frac{1}{k} < \frac{\gamma}{2(n-1)},
\]
and set, for \( 1 < \tilde{k} < \infty \), 
\[
\frac{1}{r'} = \frac{\beta}{2} + 1, \quad \frac{1}{\tilde{k}'} = \frac{\beta}{2} + \frac{1}{k}, \quad \text{and} \quad \gamma - \alpha = -\gamma
\]
with which we use Hölder’s inequality to obtain

\[
\| |x|^{-\alpha} |u|^\beta v \|_{L^2_\rho([0,\infty); L^\tilde{k}_\omega(|\cdot|^{-\gamma})]} = \| |x|^\gamma |u|^\beta v \|_{L^2_\rho([0,\infty); L^\tilde{k}^\prime_\omega(|\cdot|^{-\gamma})]} \\
= \| |x|^{-\gamma} |u|^{\beta} v \|_{L^2_\rho([0,\infty); L^{\tilde{k}^\prime + 2\rho + 2}_\omega(|\cdot|^{-\gamma}))} \\
\leq \| u \|_{L^\infty_\rho([0,\infty); L^2_\omega(|\cdot|^{-\gamma})]} \| v \|_{L^2_\rho([0,\infty); L^\tilde{k}^\prime_\omega(|\cdot|^{-\gamma})]}
\]

as desired in the lemma.

Now we only need to check the condition (1.9). Using the first and second conditions of (4.3), it is not difficult to check that (4.2) may be replaced by the last one in (4.1).

We insert \( \gamma = \alpha/2 \) from the last one in (4.3) into (4.1). Then, it follows that

\[
0 < \alpha < 2, \quad \frac{1}{r} = \frac{n - 2}{2n} + \frac{\alpha}{2n}, \quad \frac{1}{r} - \frac{\alpha}{4(n-1)} < \frac{1}{k} < \frac{1}{r}
\]
where the last two conditions are exactly the same as in (1.9). Note here that the second condition of (4.4) combined with the first one of (4.3) implies \( \beta = (4 - 2\alpha)/n \). \( \square \)
4.2. Contraction mapping

By Duhamel’s principle, the solution of the Cauchy problem (1.6) can be written as

$$
\Phi(u) := e^{it\Delta}u_0 - i\lambda \int_0^t e^{i(t-s)\Delta}F(u)ds
$$

(4.5)

where \( F(u) = |\cdot|^{-\alpha}|u(\cdot, s)|^\beta u(\cdot, s) \). For suitable values of \( M, N > 0 \), we suffice to show that \( \Phi \) defines a contraction on

$$
X(T, M, N) = \left\{ u \in C_1((0, \infty); L^2_x(\mathbb{R}^N)) \cap L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2})) : 
\sup_{t \in [0, \infty)} \| u \|_{L^2_t} \leq N, \quad \| u \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))} \leq M \right\}
$$

equipped with the metric

$$
d(u, v) = \sup_{t \in [0, \infty)} \| u - v \|_{L^2_t} + \| u - v \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))}
$$

where \((r, k)\) is given as in Theorem 1.4. To do so, we need some inhomogeneous estimates to control the Duhamel term in (4.5). In our case, the estimates in Theorem 1.3 are enough.

Now we show that \( \Phi \) is well defined on \( X \). In other words, for \( u \in X \)

$$
\sup_{t \in [0, \infty)} \| \Phi(u) \|_{L^2_t} \leq N \quad \text{and} \quad \| \Phi(u) \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))} \leq M.
$$

Using Plancherel’s theorem, the adjoint version of (1.3) for \( \gamma = \alpha/2 \) combined with Remark 1.2, and Lemma 4.1 in turn, we see

$$
\sup_{t \in [0, \infty)} \| \Phi(u) \|_{L^2_t} \leq C\| u_0 \|_{L^2} + C \left\| \int_{-\infty}^{\infty} e^{-is\Delta} \chi[0, r](s) F(u)ds \right\|_{L^2_t}
\leq C\| u_0 \|_{L^2} + C\| F(u) \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))}
\leq C\| u_0 \|_{L^2} + CN^\beta M.
$$

On the other hand, applying (1.7) with \( \gamma = \alpha/2 \) to (4.5), and then using Lemma 4.1, we obtain

$$
\| \Phi(u) \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))}
\leq \| e^{it\Delta}u_0 \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))} + C\| F(u) \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))}
\leq \| e^{it\Delta}u_0 \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))} + CN^\beta M.
$$

Here, we observe that

$$
\| e^{it\Delta}u_0 \|_{L^2_t((0, \infty); L^r_p L^k_\omega(|\cdot|^{-\alpha r/2}))} \leq \varepsilon
$$
for some sufficiently small \( \varepsilon > 0 \) chosen later, if \( \|u_0\|_{L^2} \) is small (see (1.3) with \( \gamma = \alpha/2 \)). We therefore get \( \Phi(u) \in X \) for \( u \in X \) if

\[
C\|u_0\|_{L^2} + CN^\beta M \leq N \quad \text{and} \quad \varepsilon + CN^\beta M \leq M. \quad (4.6)
\]

Next, we show that \( \Phi \) is a contraction. Namely, for \( u, v \in X \)

\[
d(\Phi(u), \Phi(v)) \leq \frac{1}{2}d(u, v).
\]

As before, we see

\[
d(\Phi(u), \Phi(v)) = \sup_{t \in [0, \infty)} \|\Phi(u) - \Phi(v)\|_{L^2_t} + \|\Phi(u) - \Phi(v)\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))}
\]

\[
\leq C\|F(u) - F(v)\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))}.
\]

We will show

\[
\|F(u) - F(v)\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))} \leq CN^\beta \|u - v\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))}
\]

which is reduced to showing

\[
\|x^{-\alpha}\|_{L_t^2([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))} \leq N^\beta \|u - v\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))}
\]

and

\[
\| \|x^{-\alpha}\|_{L_t^2([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))} \leq N^\beta \|u - v\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))},
\]

by the following simple inequality:

\[
(|u|^\beta u - |v|^\beta v) \leq C(|u|^\beta + |v|^\beta)|u - v|.
\]

We apply Lemma 4.1 with \( v \) replaced by \( |u - v| \) to obtain

\[
\|x^{-\alpha}\|_{L_t^2([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))}
\]

\[
\leq \|u\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))} \leq \|v\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))}
\]

\[
\leq N^\beta \|u - v\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))}.
\]

Similarly, we get (4.7). Hence, we obtain

\[
d(\Phi(u), \Phi(v)) \leq CN^\beta \|u - v\|_{L^2_t([0, \infty); L^r_p L^{k}_{\omega}(|\cdot|^{-ar/2}))}
\]

\[
\leq CN^\beta d(u, v).
\]

Now by taking \( N = 2C\|u_0\|_{L^2} \) and \( M = 2\varepsilon \), and then choosing \( \varepsilon > 0 \) small enough such that (4.6) holds and \( CN^\beta \leq 1/2 \), it follows that \( \Phi \) is a contraction on \( X \).
Finally, we show the scattering property. Using (4.5) and following the argument above, one can easily see that
\[
\|e^{-it_2 \Delta} u(t_2) - e^{-it_1 \Delta} u(t_1)\|_{L^2_x} = \left\| \int_{t_1}^{t_2} e^{-is \Delta} F(u) \, ds \right\|_{L^2_x} \\
\lesssim \|F(u)\|_{L^2_t([t_1, t_2]; L^p L^k_{\omega}(\cdot; \cdot^{\frac{\alpha r}{2}}))} \\
\lesssim \|u\|^\beta_{L^\infty_t([t_1, t_2]; L^2_x)} \|u\|_{L^2_t([t_1, t_2]; L^p L^k_{\omega}(\cdot; \cdot^{\frac{\alpha r}{2}}))} \to 0
\]
as \(t_1, t_2 \to \infty\). This yields that
\[
\varphi := \lim_{t \to \infty} e^{-it \Delta} u(t)
\]
exists in \(L^2\). In addition, one has
\[
u(t) - e^{it \Delta} \varphi = i \lambda \int_{t}^{\infty} e^{i(t-s) \Delta} F(u) \, ds,
\]
and therefore
\[
\|u(t) - e^{it \Delta} \varphi\|_{L^2_x} = \left\| \int_{t}^{\infty} e^{i(t-s) \Delta} F(u) \, ds \right\|_{L^2_x} \\
\lesssim \|F(u)\|_{L^2_t([t, \infty); L^p L^k_{\omega}(\cdot\cdot^{\frac{\alpha r}{2}}))} \\
\lesssim \|u\|^\beta_{L^\infty_t([t, \infty); L^2_x)} \|u\|_{L^2_t([t, \infty); L^p L^k_{\omega}(\cdot\cdot^{\frac{\alpha r}{2}}))} \to 0
\]
as \(t \to \infty\). This completes the proof.

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