COLIMITS IN THE CORRESPONDENCE BICATEGORY

SULIMAN ALBANDIK AND RALF MEYER

Abstract. We interpret several constructions with C*-algebras as colimits in the bicategory of correspondences. This includes crossed products for actions of groups and crossed modules, Cuntz-Pimsner algebras of proper product systems, direct sums and inductive limits, and certain amalgamated free products.

1. Introduction

A basic idea of noncommutative geometry is to replace ordinary quotient spaces by noncommutative generalisations. For instance, let a group $G$ act on a space $X$. The orbit space $X/G$ is often badly behaved as a topological space. In noncommutative geometry, it is replaced by the crossed product C*-algebra \( C_0(X) \rtimes G \). We may view the action of $G$ on $X$ as a diagram of topological spaces. The quotient space is the colimit of this diagram. We will exhibit the crossed product for a group action as a colimit as well, in an appropriate bicategory of C*-algebras.

As this motivating example shows, our bicategorical colimit construction leads to noncommutative C*-algebras even when we start with a diagram of locally compact spaces.

The most concrete description of bicategories involves objects, arrows, and 2-arrows, the composition of arrows and the horizontal and vertical composition of 2-arrows. We shall emphasise a more conceptual definition: in a bicategory, sets of arrows between objects are replaced by categories of arrows, and the composition becomes a bifunctor. Associativity and unitality may hold exactly (strict 2-categories or just 2-categories) or only up to natural equivalences of categories that satisfy suitable coherence conditions (weak 2-categories or bicategories, see [2, 10]). We shall mostly work in the bicategory \( \text{Corr} \) of C*-algebra correspondences. This is introduced by Landsman in [9] and studied in some depth in [6].

For simplicity, we also consider the bicategory \( \mathcal{E}^*(2) \), which is introduced in [6] [2.1.1]. Its objects are C*-algebras, its arrows $A \to B$ are nondegenerate *-homomorphisms $A \to \mathcal{M}(B)$, where $\mathcal{M}(B)$ denotes the multiplier algebra, and its 2-arrows $f_1 \Rightarrow f_2$ for nondegenerate *-homomorphisms $f_1, f_2: A \subseteq \mathcal{M}(B)$ are unitary multipliers $u \in \mathcal{U}(B)$ with $uf_1(a)u^* = f_2(a)$ for all $a \in A$. Since unitaries are invertible, the arrows $A \to B$ and 2-arrows between them in $\mathcal{E}^*(2)$ form a groupoid, not just a category.

By the way, we may also restrict to nondegenerate *-homomorphisms $A \to B$; this is like restricting to proper correspondences. Since a non-unital C*-algebra contains no unitary elements, our bicategory depends on using unitary multipliers. We need nondegeneracy for our arrows so that they act on unitary multipliers.

What are diagrams in categories and their colimits? Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A diagram in $\mathcal{D}$ of shape $\mathcal{C}$ is a functor $\mathcal{C} \to \mathcal{D}$. Such diagrams are again the objects of a category $\mathcal{D}^\mathcal{C}$, with natural transformations between functors as arrows. Any object $x$ of $\mathcal{D}$ gives rise to a “constant” diagram $\text{const}_x: \mathcal{C} \to \mathcal{D}$ of shape $\mathcal{C}$.

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The colimit \( \text{colim} \ F \) of a diagram \( F : C \to D \) is an object of \( D \) with the following universal property: there is a natural bijection between arrows \( \text{colim} \ F \to x \) in \( D \) and natural transformations \( F \Rightarrow \text{const}_x \) for all objects \( x \) of \( D \). In brief,

\[
\text{colim} \ F, x \cong \text{colim} F = \text{const}_x.
\]

Now let \( C \) and \( D \) be bicategories. As before, a diagram in \( D \) of shape \( C \) is a functor \( C \to D \), as defined in, say, \( [10] \). The functors \( C \to D \) are the objects of a bicategory \( \mathcal{D} \); its arrows and 2-arrows are the transformations between functors and the modifications between transformations, see \( [2, 10] \). These definitions are repeated in our main reference \( [6] \) in Definition 4.1 and in §4.2 and §4.3.

Thus \( \mathcal{D}(F_1, F_2) \) for two diagrams \( F_1 \) and \( F_2 \) is now a category, not just a set, with transformations \( F_1 \Rightarrow F_2 \) as objects and modifications between them as arrows. Similarly, for two objects \( x_1 \) and \( x_2 \) of \( D \), there is a category \( \mathcal{D}(x_1, x_2) \) of arrows \( x_1 \to x_2 \) and 2-arrows between them. Once again, there is a constant diagram \( \text{const}_x \) of shape \( C \) for any object \( x \) of \( D \). The bicategorical colimit is defined by the same condition \( [14] \), now interpreting \( \cong \) as a natural equivalence of categories. An object \( \text{colim} F \) of \( D \) with this property is unique up to equivalence if it exists.

What do these definitions mean if \( C = G \) is a group and \( D \) is the bicategory \( G^*(2) \) described above? First, diagrams in \( G^*(2) \) are the twisted group actions in the sense of Bushy and Smith: this is observed in \( [6] \). Transformations between such diagrams are also described there. In particular, a transformation \( F \Rightarrow \text{const}_p \) is a covariant representation of the twisted \( G \)-action corresponding to \( F \) in the multiplier algebra of \( D \). A modification is a unitary intertwiner between two covariant representations. Hence the colimit and the crossed product for the twisted action are characterised by the same universal property, forcing them to be isomorphic. As a result, if we replace the category of spaces and maps by the bicategory \( G^*(2) \), we are led to enlarge the class of group actions to twisted actions, and the crossed product construction appears as the natural analogue of a “quotient” in our bicategory.

Here we interpret many interesting constructions with \( C^* \)-algebras as colimits. Thus our new point of view unifies several known constructions with \( C^* \)-algebras. Most proofs are as trivial as above: we merely make the universal property that defines the bicategorical colimit explicit in a particular case and recognise the result as the definition of a familiar \( C^* \)-algebraic construction.

We mostly work with the correspondence bicategory \( \text{Corr} \), which is defined in \( [6] \, §2.2 \). Let \( A \) and \( B \) be \( C^* \)-algebras. A correspondence from \( A \) to \( B \) is a Hilbert \( B \)-module \( \mathcal{E} \) with a nondegenerate *-homomorphism from \( A \) to the \( C^* \)-algebra of adjointable operators on \( \mathcal{E} \). An isomorphism between two such correspondences is a unitary operator intertwining the left \( A \)-actions. We let \( \text{Corr}(A, B) \) be the groupoid of correspondences from \( A \) to \( B \) and their isomorphisms. The composition is given by the bifunctors

\[
\text{Corr}(B, C) \times \text{Corr}(A, B) \to \text{Corr}(A, C), \quad (\mathcal{E}, F) \mapsto F \otimes_B \mathcal{E}.
\]

This is associative and monoidal up to canonical isomorphisms, which are part of the bicategory structure (see \( [6] \)). A correspondence \( \mathcal{E} \) from \( A \) to \( B \) is proper if the left \( A \)-module structure is through a map \( A \to \mathbb{K}(\mathcal{E}) \) to the \( C^* \)-algebra of compact operators. Thus proper correspondences with isomorphisms between them form a subbicategory \( \text{Corr}_{\text{prop}} \) of \( \text{Corr} \). Our main results will only hold for diagrams of proper correspondences, that is, functors to \( \text{Corr}_{\text{prop}} \).

Groups are categories with only one object. At the other extreme are discrete categories. These are categories where all arrows are identities, that is, sets viewed as categories. Colimits in this case are also called coproducts. Whereas coproducts need not exist in \( G^*(2) \), they are given by the \( C_0 \)-direct sum in the correspondence bicategory \( \text{Corr} \); this statement is a standard additivity result about Hilbert module
representations of $C_0$-direct sums. The nonexistence of coproducts in $\mathcal{C}^*(2)$ is one reason to prefer the correspondence bicategory $\mathbf{Corr}$. Moreover, since $\mathcal{C}^*(2)$ is a subbicategory of $\mathbf{Corr}$, we get more diagrams in $\mathbf{Corr}$ than in $\mathcal{C}^*(2)$.

A functor $G \to \mathbf{Corr}$ for a group $G$ is equivalent to a saturated Fell bundle over $G$ (see [1]). The colimit for such a functor is the full $C^*$-algebra of sections of the corresponding Fell bundle. Crossed modules are a 2-categorical generalisation of groups. Their actions on $C^*$-algebras by automorphisms or correspondences have been introduced in [4,6]. Once again, the universal property of the colimit is the same as that for the appropriate analogue of the crossed product in this context.

What happens for non-reversible dynamical systems? Let $P$ be a monoid, that is, a category with a single object. A functor $P \to \mathbf{Corr}$ is the same as an essential product system over the opposite monoid $P^{op}$. The change of direction comes from [1,2], where we tensor in reverse order to conform to the usual conventions of composing maps. Colimits for product systems are remarkable because the universal property we get is not always but often equivalent to a standard one. More precisely, if the product system is proper, that is, all left actions in the product system are through compact operators, then the colimit of the corresponding diagram exists and is isomorphic to the Cuntz–Pimsner algebra of the product system. We get the “absolute” Cuntz–Pimsner algebra, not the popular modification by Kasparov, and we get there directly and never see the Cuntz–Toeplitz algebra along the way. This result on Cuntz–Pimsner algebras is the main idea of [11]. We had originally planned [1] as an applications section inside this article. We were, however, convinced by C-algebra colleagues to write down those results separately, to make them accessible without category theory background.

Readers familiar with free products of C$^*$-algebras may have been surprised that the bicategory $\mathcal{C}^*(2)$ is not closed under coproducts: already in the usual category of C$^*$-algebras with $*$-homomorphisms, there is a coproduct, namely, the free product. This does not cooperate with unitary multipliers, however, and fails to satisfy the universal property for a coproduct in $\mathcal{C}^*(2)$ or $\mathbf{Corr}$. This situation clears up when we consider pushouts. Given two nondegenerate $*$-homomorphisms $B_1 \leftarrow A \to B_2$, their coproduct in $\mathbf{Corr}$ or $\mathcal{C}^*(2)$ is the amalgamated free product $B_1 \ast_A B_2$. Free products without amalgamation occur in the highly degenerate case $A = 0$.

Even more fundamental than pushouts are coequalisers. These are colimits of diagrams of the shape $\mathcal{E}_1, \mathcal{E}_2$: $A \rightrightarrows B$. For instance, if $A = B = \mathbb{C}$ and $\mathcal{E}_i = \mathbb{C}^{n_i}$ for $i = 1,2$, then the coequaliser is the universal C$^*$-algebra generated by elements $u_{ij}$ for $1 \leq j \leq n_1$, $1 \leq k \leq n_2$, subject to the relations

$$\sum_j u_{ij} u_{kj}^* = \delta_{i,k}, \quad \sum_i u_{ij}^* u_{ik} = \delta_{j,k}$$

for all $1 \leq i, k \leq n_1$ or all $1 \leq j, k \leq n_2$, respectively. If $n_1 = n_2$, then this is the C$^*$-algebra $U_n^{nc}$ introduced by Brown and studied further by McClanahan [4,11]. This example shows that coequalisers, even of very small diagrams, need not be particularly well-behaved C$^*$-algebras. An explanation for this may be that all colimits may be reduced to coproducts and a coequaliser, so coequalisers are already the most general types of colimits.

Another situation we treat are inductive limits: the inductive limit of a chain of $*$-homomorphisms is also a colimit in $\mathbf{Corr}$, even if some of these $*$-homomorphisms are degenerate.

We also prove one general result here: any diagram of proper correspondences, indexed by any bicategory, has a colimit. We describe this colimit by generators and relations, with the known construction of Cuntz–Pimsner algebras of product systems as a model case. This model case also shows that something may go wrong for diagrams involving non-proper correspondences.
2. **Colimits in Bicategories**

Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. An object of $\mathcal{D}^\mathcal{C}$ is a functor $\mathcal{C} \to \mathcal{D}$; it consists of several objects, arrows and 2-arrows in $\mathcal{D}$. In the constant diagram, $\text{const}_x : \mathcal{C} \to \mathcal{D}$, all these objects are the same object $x$ of $\mathcal{D}$, all the arrows are the identity on $x$, and all 2-arrows are the identity 2-arrow on $\text{id}_x$.

For instance, functors $G \to \mathcal{C}^\ast(2)$ for a group $G$ are identified with Busby–Smith twisted actions of $G$ on $\mathcal{C}^\ast$-algebras in [6, §3.1.1]. The constant diagram $\text{const}_A$ for a $\mathcal{C}^\ast$-algebra $A$ is the trivial $G$-action on $A$, with trivial twists. Functors $G \to \text{Corr}$ are identified with saturated Fell bundles in [6, §3.1.1]. A constant diagram $\text{const}_A$ in $\text{Corr}$ corresponds to the constant Fell bundle with all fibres equal to $A$ and the constant multiplication and involution.

**Definition 2.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A cone over $F$ is an object $x$ of $\mathcal{D}$ with a transformation $\vartheta_x : F \to \text{const}_x$; a colimit of $F$ is a universal cone over $F$, that is, an object $x$ of $\mathcal{D}$ with a transformation $\vartheta_x : F \to \text{const}_x$, such that composition with $\vartheta_x$ induces equivalences of categories

$$\mathcal{D}(x, y) \xrightarrow{\cong} \mathcal{D}^\mathcal{C}(F, \text{const}_y)$$

for all objects $y$ of $\mathcal{D}$.

If we are given natural equivalences $\mathcal{D}(x, y) \cong \mathcal{D}^\mathcal{C}(F, \text{const}_y)$, then the identity map in $\mathcal{D}(x, x)$ gives a transformation $\vartheta_x : F \to \text{const}_x$, which is determined uniquely up to isomorphism; naturality forces the equivalences $\mathcal{D}(x, y) \cong \mathcal{D}^\mathcal{C}(F, \text{const}_y)$ to be composition with $\vartheta_x$. Hence a colimit may also be defined as an object $x$ of $\mathcal{D}$ with natural equivalences of categories $\mathcal{D}(x, y) \cong \mathcal{D}^\mathcal{C}(F, \text{const}_y)$.

**Proposition 2.2.** The colimit is functorial: a transformation $\Phi : F_1 \to F_2$ induces an arrow $\text{colim}\, \Phi : \text{colim}\, F_1 \to \text{colim}\, F_2$, and a modification $\Phi_1 \to \Phi_2$ induces a 2-arrow $\text{colim}\, \Phi_1 \to \text{colim}\, \Phi_2$, and these constructions are compatible with the composition bifunctor for transformations.

**Proof.** Let $(x_1, \vartheta_1)$ and $(x_2, \vartheta_2)$ be colimits of $F_1$ and $F_2$, respectively. Transformations may be composed, so $\vartheta_2 \circ \Phi$ is an object of $\mathcal{D}^\mathcal{C}(F, \text{const}_{x_2})$. By the definition of the colimit, there is an arrow $\text{colim}\, \Phi : x_1 \to x_2$ with $\vartheta_2 \circ \Phi = (\text{colim}\, \Phi) \circ \vartheta_1$, and this arrow is unique up to equivalence. Similarly, a modification $\Phi_1 \to \Phi_2$ induces a modification $\vartheta_2 \circ \Phi_1 \to \vartheta_2 \circ \Phi_2$, which gives a 2-arrow $\text{colim}\, \Phi_1 \to \text{colim}\, \Phi_2$. Thus we get a functor $\mathcal{D}^\mathcal{C}(F_1, F_2) \to \mathcal{D}(\text{colim}\, F_1, \text{colim}\, F_2)$. It is routine to check that this functor, up to equivalence, does not depend on choices and that the construction is compatible with the composition bifunctors in $\mathcal{D}^\mathcal{C}$ and $\mathcal{D}$.

**Corollary 2.3.** Any two colimits of the same diagram are canonically equivalent. □

The equivalences in $\mathcal{C}^\ast(2)$ are the $\ast$-isomorphisms, those in $\text{Corr}$ are the imprimitivity bimodules. Hence colimits in $\mathcal{C}^\ast(2)$ are unique up to isomorphism if they exist, whereas colimits in $\text{Corr}$ are only unique up to Morita–Rieffel equivalence.

3. **Coproducts and Products**

Coproducts are colimits of diagrams indexed by a category with only identity morphisms. Such a diagram is simply a map from some index set $I$ to the objects of the category. The following proposition shows that the usual $C_0$-direct sum of $\mathcal{C}^\ast$-algebras is both a coproduct and a product of the set of objects $(A_i)_{i \in I}$ in $\text{Corr}$. (We do not consider limits in this article because it seems rare that they exist in $\text{Corr}$. We only mention the result on products because its proof and statement is so similar to the description of coproducts.)
Proposition 3.1. Let $A_i$ for $i \in I$ and $B$ be $C^*$-algebras. Then
\[
\text{Corr} \left( \bigoplus_{i \in I} A_i, B \right) \cong \prod_{i \in I} \text{Corr}(A_i, B),
\]
\[
\text{Corr} \left( B, \bigoplus_{i \in I} A_i \right) \cong \prod_{i \in I} \text{Corr}(B, A_i).
\]

Proof. Given correspondences $\mathcal{E}_i : A_i \to B$, we may form the Hilbert $B$-module $\bigoplus_{i \in I} \mathcal{E}_i$ and equip it with a nondegenerate left action of $\bigoplus_{i \in I} A_i$ to get a correspondence from $\bigoplus_{i \in I} A_i$ to $B$. Isomorphisms of correspondences $\mathcal{E}_i \to \mathcal{E}_i'$ may be put together to an isomorphism of correspondences $\bigoplus_{i \in I} \mathcal{E}_i \to \bigoplus_{i \in I} \mathcal{E}_i'$. Thus we get a functor
\[
\prod_{i \in I} \text{Corr}(A_i, B) \to \text{Corr} \left( \bigoplus_{i \in I} A_i, B \right).
\]

To show that \((3.2)\) is an equivalence, consider a correspondence $\mathcal{E}$ from $\bigoplus_{i \in I} A_i$ to $B$. Since the left action is nondegenerate, it extends to an action of the multiplier algebra of $\bigoplus_{i \in I} A_i$. (The product is taken in the category of $C^*$-algebras, so it contains only bounded families.) In particular, $\mathcal{M}(\bigoplus_{i \in I} A_i)$ contains an orthogonal projection $p_i$ onto the $i$th summand for each $i \in I$. We have strict convergence $\sum_{i \in I} p_i = 1$. The projections $p_i$ act by orthogonal projections on $\mathcal{E}$. Let $\mathcal{E}_i := p_i \mathcal{E}$ be their images; these are Hilbert submodules on which $A_i$ acts nondegenerately, respectively. Thus $\mathcal{E}_i$ is a correspondence from $A_i$ to $B$. Since $\sum_{i \in I} p_i = 1$, we have $\bigoplus_{i \in I} \mathcal{E}_i = \mathcal{E}$. Thus $\mathcal{E}$ belongs to the essential range of the functor \((3.2)\). Furthermore, since any intertwining operator between two correspondences commutes with the left action of the multiplier algebra and hence with the projections $p_i$, it comes from a family of intertwining operators on the summands $\mathcal{E}_i$; this shows that the functor \((3.2)\) is fully faithful. Hence \((3.2)\) is an equivalence of groupoids. This yields the first isomorphism, showing that $\bigoplus_{i \in I} A_i$ is a coproduct of $(A_i)_{i \in I}$ in $\text{Corr}$.

Now consider a family of correspondences $\mathcal{E}_i$ from $B$ to $A_i$. Let $\bigoplus_{i \in I} \mathcal{E}_i$ be the set of all families $(\xi_i)_{i \in I}$ with $\xi_i \in \mathcal{E}_i$ and $(i \mapsto \|\xi_i\|) \in C_0(I)$. This is a Hilbert module over $\bigoplus_{i \in I} A_i$ by the pointwise operations. The left actions of $B$ on the Hilbert modules $\mathcal{E}_i$ give a nondegenerate left action of $B$ on $\bigoplus_{i \in I} \mathcal{E}_i$. Thus we get a correspondence from $B$ to $\bigoplus_{i \in I} A_i$. This construction is clearly natural with respect to isomorphisms of correspondences and hence gives a functor
\[
\prod_{i \in I} \text{Corr}(B, A_i) \to \text{Corr} \left( B, \bigoplus_{i \in I} A_i \right).
\]

Take a correspondence $\mathcal{E}$ from $B$ to $\bigoplus_{i \in I} A_i$. For each $i \in I$, $\mathcal{E}_i := \mathcal{E} \cdot A_i \subseteq \mathcal{E}$ is a correspondence from $B$ to the ideal $A_i$ in $\bigoplus_{j \in I} A_j$. Since these ideals are orthogonal, we have $\mathcal{E} \cong \bigoplus_{i \in I} \mathcal{E}_i$. Thus $\mathcal{E}$ belongs to the essential range of \((3.3)\). Since the decomposition $\mathcal{E} \cong \bigoplus_{i \in I} \mathcal{E}_i$ is natural, the functor \((3.3)\) is fully faithful. \qed

Proposition 3.1 works because we may take direct sums of correspondences to make things orthogonal. In the category of $C^*$-algebras with $^*$-homomorphisms as morphisms, coproducts are free products, which are highly noncommutative. Since the coproduct in $\text{Corr}$ is unique up to isomorphism in $\text{Corr}$, that is, Morita–Rieffel equivalence, the free product is not a coproduct in $\text{Corr}$ any more. The reason is that it is not compatible with isomorphisms of correspondences: for a coproduct, we allow different unitaries $\mathcal{E}_i \cong \mathcal{E}_i'$ for all $i \in I$. Orthogonality of the $\mathcal{E}_i$ allows us
to put two unrelated unitaries together. In the 2-category \( \mathcal{E}^*(2) \), coproducts do not exist in general for this reason: there are no orthogonal direct sums in \( \mathcal{E}^*(2) \), and free products do not behave well with respect to 2-arrows.

**Example 3.4.** We prove formally that the coproduct of two copies of \( \mathcal{C} \) in \( \mathcal{E}^*(2) \) does not exist. Let \( B \) be a \( \mathcal{C}^* \)-algebra. There is a unique arrow \( \mathcal{C} \rightarrow B \), namely, the unit map of \( \mathcal{M}(B) \). Thus there is a unique transformation from our coproduct diagram to \( \text{const}_B \), given by the unit map on both copies of \( \mathcal{C} \). A modification on this unique transformation is given by two unitaries \( u_1, u_2 \in \mathcal{M}(B) \), one for each copy of \( \mathcal{C} \), subject to no conditions. If we also take \( B = \mathcal{C} \), then our groupoid of transformations is the two-torus group \( \mathbb{T}^2 \).

Now assume that the \( \mathcal{C}^* \)-algebra \( A \) were a coproduct of \( \mathcal{C} \) and \( \mathcal{C} \) in \( \mathcal{E}^*(2) \). Then the groupoid or arrows \( A \rightarrow \mathcal{C} \) would be equivalent to \( \mathbb{T}^2 \). Its objects are nonzero characters \( A \rightarrow \mathcal{C} \) and its arrows are unitaries in \( \mathcal{C} \) acting on characters by conjugation, that is, trivially. So we get a disjoint union of some copies of the group \( \mathbb{T}^2 \), one for each character of \( A \). But this is never equivalent to \( \mathbb{T}^2 \) because the groups \( \mathbb{T} \) and \( \mathbb{T}^2 \) are not isomorphic. To see the latter, observe that \( \mathbb{T} \) has exactly one element of order 2, namely, \( -1 \), while \( \mathbb{T}^2 \) has exactly three of them, namely, \( (-1, +1), (-1, -1), (+1, -1) \).

The category \( \mathcal{C}orr \) has more diagrams than \( \mathcal{E}^*(2) \). Proposition 3.1 and Example 3.4 show that some very simple diagrams have a colimit in \( \mathcal{C}orr \), but not in \( \mathcal{E}^*(2) \). In the following, we therefore mostly study colimits in \( \mathcal{C}orr \).

Next we clarify the role of free products in our theory. We show that amalgamated free products are pushouts in \( \mathcal{C}orr \), but only under a nondegeneracy assumption; this rules out, in particular, free products without any amalgamation. Indeed, in the most degenerate case where we amalgamate over 0, Proposition 3.1 shows that the coproduct is the \( \mathbb{C}_0 \)-direct sum and not the free product.

### 3.1. Pushouts

A pushout in \( \mathcal{C}orr_{\text{prop}} \) is a colimit of a diagram of the form

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B_1 \\
E_1 & \downarrow & \\
E_2 & \xrightarrow{\varphi_2} & B_2,
\end{array}
\]

where \( A, B_1 \) and \( B_2 \) are \( \mathcal{C}^* \)-algebras and \( E_1 \) and \( E_2 \) are proper correspondences, without further data or conditions.

One extreme case is \( A = 0 \), where the pushout degenerates to a coproduct; this gives the direct sum \( B_1 \oplus B_2 \) by Proposition 3.1. Here we consider the opposite extreme case, where \( E_1 \) and \( E_2 \) are associated to nondegenerate \( * \)-homomorphisms \( A \rightarrow B_1, A \rightarrow B_2 \); that is, \( E_i = B_i \) with \( A \) acting by \( a \cdot b := \varphi_i(a) \cdot b \) for \( i = 1, 2 \).

**Proposition 3.5.** Let \( A, B_1 \) and \( B_2 \) be \( \mathcal{C}^* \)-algebras and let \( \varphi_1 : A \rightarrow B_1 \) and \( \varphi_2 : A \rightarrow B_2 \) be nondegenerate \( * \)-homomorphisms. The amalgamated free product \( B_1 \ast_A B_2 \) is also a pushout in \( \mathcal{C}orr \).

**Proof.** When we turn the \( * \)-homomorphism \( \varphi_i \) for \( i = 1, 2 \) into a correspondence \( E_i \), we take the right ideal \( \varphi_i(A) \cdot B_1 \), viewed as a Hilbert \( B_i \)-module, and equipped with the left action of \( A \) through \( \varphi_i \). Our nondegeneracy assumption means that \( E_i = B_i \) as a right Hilbert \( B_i \)-module. Furthermore, we remark that \( \varphi_i(A) \subseteq \mathbb{K}(E_i) = B_i \) by assumption, so the \( E_i \) are proper correspondences. We will see later that properness is crucial to get colimits.

Let \( D \) be a \( \mathcal{C}^* \)-algebra. A transformation in \( \mathcal{C}orr \) from our pushout diagram to the constant diagram on \( D \) is given by correspondences \( F_1 : B_1 \rightarrow D, F_2 : B_2 \rightarrow D \).
and an isomorphism

$$U : \mathcal{F}_1 \cong B_1 \otimes_{B_i} \mathcal{F}_1 \to B_2 \otimes_{B_i} \mathcal{F}_2 \cong \mathcal{F}_2$$

of correspondences from $A$ to $D$. That is, $U$ is a unitary operator $\mathcal{F}_1 \to \mathcal{F}_2$ that intertwines the left actions of $A$ given by composing the actions of $B_i$ with the $\ast$-homomorphisms $\varphi_i$. Here we have used the nondegeneracy of $\varphi_i$ to identify $\mathcal{E}_i = B_i$ as Hilbert $B_i$-modules.

A modification from $(\mathcal{F}_i, U)$ to $(\mathcal{F}_i', U')$ is given by isomorphisms of correspondences $V_i : \mathcal{F}_i \to \mathcal{F}_i'$ for $i = 1, 2$ that intertwine $U$ and $U'$.

Every such transformation is isomorphic to one where $\mathcal{F}_i = \mathcal{F}_2$ as right Hilbert $D$-modules and $U$ is the identity operator: the identity on $\mathcal{F}_1$ and $U : \mathcal{F}_1 \to \mathcal{F}_2$ is an invertible modification. Hence restricting to transformations with $\mathcal{F}_i = \mathcal{F}_2$ and $U = \text{id}$ gives an equivalent groupoid. So it does not change the colimit. The intertwining condition for modifications now simply says that the unitaries $\mathcal{F}_i \to \mathcal{F}_i'$ for $i = 1, 2$ are the same unitary, so we only have a single unitary that intertwines the actions of both $B_1$ and $B_2$, and hence the actions of $A$.

If $\mathcal{F}_1 = \mathcal{F}_2$ and $U = \text{id}$, then both $B_1$ and $B_2$ act on the same Hilbert module, and the two actions composed with $\varphi_i$ coincide on $A$; thus we get an action of the amalgamated free product $B_1 \ast_A B_2$ on $\mathcal{F}_1$. Since both $B_1$ and $B_2$ act nondegenerately, so does $B_1 \ast_A B_2$. Hence we get a correspondence $B_1 \ast_A B_2 \to D$.

Conversely, a correspondence $B_1 \ast_A B_2 \to D$ gives a Hilbert module $\mathcal{F}$ with a nondegenerate left action of $B_1 \ast_A B_2$. Since $A \cdot B_i = B_i$, the embedding $A \to B_1 \ast_A B_2$ is nondegenerate, so the action of $A$ on $\mathcal{F}$ is nondegenerate, and then so are the actions of $B_i$. Thus we get a transformation from the pushout diagram to the constant diagram on $D$ with $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$ and $U$ the identity. Thus we have found an equivalence between the groupoid of natural transformations and modifications and the groupoid of correspondences from $B_1 \ast_A B_2$ to $D$. This proves that $B_1 \ast_A B_2$ is a colimit.

**Corollary 3.6.** Let $\mathcal{E}_i$ be proper, full correspondences from $A$ to $B_i$ for $i = 1, 2$. The pushout in $\text{Corr}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ is the amalgamated free product $K(\mathcal{E}_1) \ast_A K(\mathcal{E}_2)$.

**Proof.** Since $\mathcal{E}_i$ is full, it provides a Morita–Rieffel equivalence between $K(\mathcal{E}_i)$ and $B_i$. Hence the diagrams in $\text{Corr}$ given by $\mathcal{E}_1$ and $\mathcal{E}_2$ and by the $\ast$-homomorphisms $A \to K(\mathcal{E}_i)$ for $i = 1, 2$ from the left $A$-module structures on $\mathcal{E}_i$ are isomorphic. The latter diagram has $K(\mathcal{E}_1) \ast_A K(\mathcal{E}_2)$ as a colimit by Proposition 3.5. Since the construction of colimits is functorial by Proposition 2.2, this is also a colimit of the original diagram. □

### 3.2. An example of a coequaliser.

A **coequaliser** is a colimit of a diagram consisting of two parallel arrows $\alpha_1, \alpha_2 : A_1 \rightrightarrows A_2$. Colimits of arbitrary diagrams may be reduced to coproducts and coequalisers. Hence coequalisers already show the largest complexity among all colimits. That they may be rather complicated is shown by the following example.

**Example 3.7.** Consider the coequaliser of the following diagram:

$$\begin{array}{ccc}
\mathbb{C}^m & \xrightarrow{\alpha_1} & \mathbb{C}^n \\
\alpha_2 & \xrightarrow{} & \mathbb{C}^n \\
\end{array}$$

A transformation from the above diagram to the constant diagram on a $C^*$-algebra $D$ is given by a Hilbert $D$-module $\mathcal{F}$ and a unitary operator

$$U : \mathcal{F}^m \cong \mathbb{C}^m \otimes_{\mathbb{C}} \mathcal{F} \xrightarrow{\cong} \mathbb{C}^m \otimes_{\mathbb{C}} \mathcal{F} \cong \mathcal{F}^m.$$
We may write $U$ as a matrix $U = (u_{i,j})$ with $u_{i,j} \in \mathcal{B}(\mathcal{F})$ for $1 \leq i \leq n, 1 \leq j \leq m$. The operator $U$ is unitary if and only if

$$
\sum_{k=1}^{m} u_{i_1,k} u_{i_2,k}^* = \delta_{i_1,i_2}, \quad \sum_{k=1}^{n} u_{j_1,k}^* u_{j_2,k} = \delta_{j_1,j_2}
$$

for all $1 \leq i_1,i_2 \leq n, 1 \leq j_1,j_2 \leq m$. Hence the coequaliser of (3.8) is the universal $C^*$-algebra generated by the elements $u_{i,j}$ for $i = 1, \ldots, n, j = 1, \ldots, m$ that satisfy (3.9). For $m = n$, this is the $C^*$-algebra $U_n^{nc}$ introduced by Lawrence Brown [3] and studied further by Kevin McClanahan, who showed that $U_n^{nc}$ has no projections ([11, Corollary 2.7]) and is KK-equivalent to $C^*(Z) \cong C(T)$ ([12, Proposition 5.5]).

4. Colimits for group and crossed module actions

We now consider colimits where $\mathcal{C}$ is a group $G$ or a crossed module. We consider both target bicategories $\mathcal{E}^r(2)$ and $\mathsf{Corr}$. In all these cases, the identification of the colimit with an appropriate “crossed product” is a mere reformulation of results in [4][6]. Hence we will be rather brief. These results are trivial, but they are important motivation for us to look at colimits in bicategories.

To make the results below look more surprising, we briefly consider the colimit for a group action in the usual category of $C^*$-algebras and $^*$-homomorphisms, without any 2-arrows. A group action by automorphisms is, indeed, the same as a functor from $G$ to the category of $C^*$-algebras, given by a $C^*$-algebra $A$ and $\alpha_g \in \text{Aut}(A)$ satisfying $\alpha_g \alpha_h = \alpha_{gh}$. A cone over this diagram is a $C^*$-algebra $B$ with a $^*$-homomorphism $f : A \to B$ such that $f \circ \alpha_g = f$ for all $g \in G$. Thus $f$ vanishes on the ideal $I_A$ generated by $\alpha_g(a) - a$ for all $g \in G, a \in A$. Indeed, the quotient map $A \to A/I_A$ is the universal cone. Hence the colimit is $A/I_A$. This is very often zero, and certainly not an object worth studying.

When working in a bicategory, we replace the condition $f \circ \alpha_g = f$ by extra data, say, by a unitary $u_g$ with $u_g f(a) u_g^* = f(\alpha_g(a))$ for all $a \in A$. Thus the bicategorical colimit is larger than $A$, very much unlike $A/I_A$ above.

The objects of $\mathcal{E}^r(2)^G$ are described concretely in [6] §3.1.1] as Busby–Smith twisted actions of $G$; those of $\mathsf{Corr}^G$ are equivalent to saturated Fell bundles over $G$. The transformations in $\mathcal{E}^r(2)^G$ and $\mathsf{Corr}^G$ are described concretely in [6] §3.2; modifications in $\mathcal{E}^r(2)^G$ and $\mathsf{Corr}^G$ are described concretely in [6] §3.3]. Results in [6] immediately give the following proposition:

**Proposition 4.1.** Let $G$ be a group and let $\alpha : G \to \text{Aut}(A)$ and $\omega : G \times G \to U(A)$ be a Busby–Smith twisted action of $G$ on a $C^*$-algebra $A$. The crossed product $A \rtimes_{\alpha,\omega} G$ is a colimit of the functor $F : G \to \mathcal{E}^r(2)$ associated to $(A,\alpha,\omega)$.

**Proof.** Let $D$ be a $C^*$-algebra. The functor $\text{const}_D : G \to \mathcal{E}^r(2)$ corresponds to the trivial action of $G$ on $D$. A transformation from $F$ to $\text{const}_D$ is equivalent to a covariant representation of $(A,G,\alpha,\omega)$ in $\mathcal{M}(D)$, that is, a nondegenerate representation $\varrho : A \to \mathcal{M}(D)$ and a map $\pi : G \to \mathcal{U}(D)$ satisfying $\pi_g \varrho(a) \pi_g^* = \varrho(\alpha_g(a))$ for all $g \in G, a \in A$ and $\pi_{g_1} \pi_{g_2} = \varrho(\omega(g_1,g_2)) \pi_{g_1 g_2}$ for all $g_1, g_2 \in G$ (see [6] Example 3.8)). Modifications between such transformations are the same as unitary equivalences between covariant representations by [6] Example 3.13).

The crossed product is defined to be universal for covariant representations. That is, there is a bijection between transformations from $F$ to $\text{const}_D$ and morphisms from $A \rtimes_{\alpha,\omega} G$ to $D$; the modifications between the transformations corresponding to covariant representations $(\varrho,\pi)$ and $(\varrho',\pi')$ are unitary multipliers $u$ of $D$ with $u \varrho(a) u^* = \varrho'(a)$ for all $a \in A$ and $u \pi(g) u^* = \pi'(g)$ for all $g \in G$. These are exactly...
the unitaries that intertwine the induced representations of $A \rtimes_{\alpha,\omega} G$. Thus the groupoids $\mathcal{C}^*(2)^G(F, \text{const}_D)$ and $\mathcal{C}^*(2)(A \rtimes_{\alpha,\omega} G, D)$ are naturally isomorphic. \qed

For group actions by correspondences, that is, saturated Fell bundles, the sectional $\mathcal{C}^*$-algebra plays the role of the crossed product:

**Proposition 4.2.** Let $G$ be a group and let $(A_g)_{g \in G}$ be a saturated Fell bundle over $G$, viewed as a functor $F: G \rightarrow \mathcal{Corr}$. The sectional $\mathcal{C}^*$-algebra of $(A_g)_{g \in G}$ is a colimit of $F$.

**Proof.** Let $D$ be a $\mathcal{C}^*$-algebra. Then $\text{const}_D$ corresponds to the constant Fell bundle with fibres $D$, which describes the trivial action of $G$ on $D$. Transformations to $\text{const}_D$ in $\mathcal{Corr}^G$ are in bijection with pairs $(E, \pi)$, where $E$ is a Hilbert $D$-module and $\pi: \bigsqcup_{g \in G} A_g \rightarrow \mathbb{B}(E)$ is a nondegenerate Fell bundle representation (see the discussion before [6, Definition 3.12]). Modifications between such transformations are equivalent to unitary intertwiners between Fell bundle representations.

The sectional $\mathcal{C}^*$-algebra $C := \mathcal{C}^*(A_g)_{g \in G}$ is defined as the $\mathcal{C}^*$-completion of the convolution algebra of sections of the Fell bundle. By definition, representations of a Fell bundle integrate to $*$-representations of this sectional $\mathcal{C}^*$-algebra, and all representations of $C$ come from Fell bundle representations. Furthermore, a Fell bundle representation is nondegenerate if and only if the resulting representation of $C$ is nondegenerate. A nondegenerate representation of $C$ on a Hilbert $D$-module is the same as a correspondence from $C$ to $D$. Furthermore, an operator intertwines the Fell bundle representations if and only if it intertwines the resulting representations of $C$, that is, is an isomorphism of correspondences. Hence the groupoids $\mathcal{Corr}(F, \text{const}_D)$ and $\mathcal{Corr}(C, D)$ are naturally isomorphic. \qed

Summing up, we merely have to inspect the description of transformations and modifications between functors $G \rightarrow \mathcal{E}^*(2)$ or $G \rightarrow \mathcal{Corr}$ in [6] to see that the colimit in either case is the crossed product or Fell bundle section algebra, respectively.

Now let $\mathcal{CM}$ be a crossed module; that is, $\mathcal{CM}$ consists of two groups $G$ and $H$ with homomorphisms $\partial: H \rightarrow G$ and $\epsilon: G \rightarrow \text{Aut}(H)$, such that $\partial(\epsilon_g(h)) = g\partial(h)g^{-1}$ and $\epsilon_{gh}(k) = hkh^{-1}$ for all $g \in G$, $h, k \in H$.

Strict actions of crossed modules on $\mathcal{C}^*$-algebras and crossed products for such actions are defined in [5]. These are more special than functors $\mathcal{CM} \rightarrow \mathcal{E}^*(2)$, which are discussed in [6] §4.1.1. Functors $\mathcal{CM} \rightarrow \mathcal{Corr}$ are described in [4] Theorem 2.11, generalising the notion of a saturated Fell bundle from groups to crossed modules. The crossed product for a functor $F: \mathcal{CM} \rightarrow \mathcal{Corr}$ is defined in [4] Definition 2.8 by a universal property and identified more concretely in [4] Proposition 2.17.

**Proposition 4.3.** The crossed product for a crossed module action by correspondences is a colimit in $\mathcal{Corr}$.

**Proof.** Let $F: \mathcal{CM} \rightarrow \mathcal{Corr}$ be a functor. As in the group case, the proof is by making explicit what transformations $F \rightarrow \text{const}_D$ and modifications between them are and observing that the resulting universal property for the colimit is the same one as the defining universal property of the crossed product. Since this is routine checking, we omit further details. \qed

5. A SINGLE ENDOmorphISM

Before we study colimits of arbitrary shape, we look at an important special case: let $C$ be the monoid $(\mathbb{N}, +)$, viewed as a category with a single object.

A functor $C \rightarrow \mathcal{Corr}$ is given by a $\mathcal{C}^*$-algebra $A$, correspondences $\mathcal{E}_n: A \rightarrow A$ for $n \in \mathbb{N}$ and isomorphisms of correspondences $\mu_{n,m}: \mathcal{E}_n \otimes_A \mathcal{E}_m \cong \mathcal{E}_{n+m}$ for all $n, m \in \mathbb{N}$, such that $\mathcal{E}_0$ is the identity correspondence, $\mu_{0,m}$ and $\mu_{n,0}$ are the canonical
transforms, and the multiplication maps $\mu_{n,m}$ are associative in a suitable sense. This is a special case of Proposition 5.1 below.

A transformation between such diagrams $(A, E_n, \mu_{n,m})$, $(B, F_n, v_{n,m})$, is given by a correspondence $G : A \to B$ and isomorphisms

\begin{equation}
E_n \otimes A G \xrightarrow{w_n} G \otimes B F_n
\end{equation}

for all $n \in \mathbb{N}$, subject to compatibility conditions with the $\mu_{n,m}$ and $v_{n,m}$ for $n, m \in \mathbb{N}$ and the condition that $w_0$ should be the canonical isomorphism (see Proposition 6.3). A modification between two such transformations, $(G, w_n)$ and $(G', w'_n)$, is given by an isomorphism of correspondences $G \to G'$ intertwining the $w_n$ and $w'_n$ in the obvious sense (see also Proposition 6.5).

This data can be simplified because the monoid $(\mathbb{N}, +)$ is freely generated by 1 in $\mathbb{N}$. For a functor $\mathbb{N} \to \text{Corr}$, it is enough to give $A$ and a single correspondence $E = E_1$, with no further data or conditions. We may extend this to a functor in the above sense by letting $E_n := E^\otimes n$ for $n \in \mathbb{N}$ (understood to be the identity correspondence if $n = 0$), and letting $\mu_{n,m}$ be the canonical map (this is the identity map up to the associators, which we have dropped from our notation). The conditions on the $\mu_{n,m}$ ensure that any functor is isomorphic to one of this form.

Next, a transformation is specified by a correspondence $G$ and an isomorphism

$$w = w_1 : E \otimes A G \cong G \otimes B F,$$

with no condition on $w$: iteration of $w_1$ provides the isomorphisms $w_n$, for $n \in \mathbb{N}$ as in (5.1), and the compatibility conditions for the $w_n$ say that any transformation is generated from $w_1$ in this way. Finally, for a modification, it is enough to require the intertwining condition for $w_1$, then the condition follows for $w_n$ for all $n \in \mathbb{N}$. In brief, the bicategory of functors $\mathbb{N} \to \text{Corr}$ is equivalent to the following simpler bicategory:

1. objects are given by a $C^*$-algebra $A$ and a correspondence $E \to E$;
2. arrows $(A, E) \to (B, F)$ are given by a correspondence $G : A \to B$ and an isomorphism of correspondences $w : G \otimes B F \cong E \otimes A G$;
3. 2-arrows $(G, w) \to (G', w')$ are isomorphisms $x : G \to G'$ with $(\text{id}_E \otimes A x) \circ w = w \circ (x \otimes B \text{id}_F)$.

We may use the simplified data to describe colimits as well, which only require equivalences of categories.

We now analyse transformations from $(A, E)$ to a constant diagram $\text{const}_D$. First, $\text{const}_D = (D, D)$, where the second $D$ means the identity correspondence on $D$. Hence the isomorphism $w$ in a transformation may also be viewed as an isomorphism $G \cong E \otimes A G$; here we use the canonical isomorphism $G \otimes D D \cong G$.

Roughly speaking, we want to turn an isomorphism $w : G \cong E \otimes A G$ into a representation of a $C^*$-algebra on $G$. The necessary work is carried out in [1]. First, the isomorphism $w : G \cong E \otimes A G$ is turned into a “representation” $E \to \mathbb{B}(G)$ by sending $\xi \in E$ to the operator

$$G \ni \eta \mapsto w^*(\xi \otimes \eta) \in G.$$

This is a representation of the Hilbert module $E$ in the standard sense, satisfying an extra nondegeneracy condition corresponding to the surjectivity of $w^*$. This extra nondegeneracy condition is equivalent to the Cuntz–Pimsner covariance condition provided $E$ is a proper correspondence by [1] Proposition 2.5. This leads to the following theorem:

**Theorem 5.2.** Let $E : A \to A$ be a proper correspondence. The Cuntz–Pimsner algebra of $E$ is a colimit of the corresponding diagrams $(\mathbb{N}, +) \to \text{Corr}_{\text{prop}}$ and $(\mathbb{N}, +) \to \text{Corr}$. 
Proof. The Cuntz–Pimsner algebra \( \mathcal{O}_E \) is characterised by the universal property that \(*\)-homomorphisms \( \mathcal{O}_E \to D \) for a \( C^* \)-algebra \( D \) are in bijection with pairs \((\varphi, \vartheta)\), where \( \varphi : A \to D \) is a \(*\)-homomorphism and \( \vartheta : \mathcal{E} \to D \) is a Cuntz–Pimsner covariant representation of \( \mathcal{E} \) (see [14, Theorem 3.12]). In particular, \( A \subseteq \mathcal{O}_E \), and inspection shows that this embedding is nondegenerate, that is, \( A \cdot \mathcal{O}_E \) is dense in \( \mathcal{O}_E \). It follows that the \(*\)-homomorphism \( \mathcal{O}_E \to \mathcal{B}(F) \) associated to \( \varphi : A \to \mathcal{B}(F) \) and \( \vartheta : \mathcal{E} \to \mathcal{B}(F) \) is nondegenerate if and only if \( \varphi \) is nondegenerate. Thus a correspondence from \( \mathcal{O}_E \) to \( D \) is the same as a correspondence \((\mathcal{F}, \varphi)\) from \( A \) to \( D \) with a map \( \vartheta : \mathcal{E} \to \mathcal{B}(F) \) which, together with \( \varphi \), is a Cuntz–Pimsner covariant representation.

Since \( \mathcal{E} \) is proper, the Cuntz–Pimsner covariance condition for \( \vartheta \) holds if and only if \( \vartheta \) is “nondegenerate” in the sense that the closed linear span of \( \vartheta(\mathcal{E}) \cdot (\mathcal{F}) \) is \( \mathcal{F} \) (see [1, Proposition 2.5]). Such nondegenerate correspondences are in bijection with isomorphisms of correspondences \( \mathcal{E} \otimes \mathcal{F} \cong \mathcal{F} \) by [1, Proposition 2.3]. So a correspondence from \( \mathcal{O}_E \) to \( D \) is the same as a correspondence \( \mathcal{F} \) from \( A \) to \( D \) with an isomorphism of correspondences \( \mathcal{E} \otimes_A \mathcal{F} \cong \mathcal{F} \). These are exactly the simplified transformations of functors \((\mathbb{N}, +) \to \mathcal{Corr}\), by the discussion above the theorem.

Isomorphisms of correspondences \( \mathcal{O}_E \to D \) are the same as unitaries \( \mathcal{F} \to \mathcal{F}' \) that intertwine the left actions of \( A \) and \( \mathcal{E} \). Intertwining the left actions of \( A \) means that they are isomorphisms of correspondences from \( A \) to \( D \), and intertwining the representations of \( \mathcal{E} \) means that they are modifications between the corresponding transformations of functors \((\mathbb{N}, +) \to \mathcal{Corr}\). Hence the groupoid of correspondences \( \mathcal{O}_E \to D \) and their isomorphisms is naturally isomorphic to the groupoid of simplified transformations \((A, \mathcal{E}) \to \text{const}_D \) and their modifications. This says that \( \mathcal{O}_E \) has the universal property of a colimit in \( \mathcal{Corr} \).

A correspondence \( \mathcal{F} : \mathcal{O}_E \to D \) is proper if and only if the corresponding representation \( \varphi \) of \( A \) has image in the compact operators, \( \varphi(A) \subseteq \mathcal{K}(\mathcal{F}) \); this is because \( A \cdot \mathcal{O}_E = \mathcal{O}_E \). Hence \( \mathcal{O}_E \) has the universal property of a colimit in \( \mathcal{Corr}_{\text{prop}} \) as well. \( \square \)

Note that the colimit is the Cuntz–Pimsner algebra right away, the Cuntz–Toeplitz algebra plays no role; this is because of the built-in nondegeneracy properties of \( \mathcal{Corr} \).

Following Muhly and Solel [13] and Katsura [8], many authors have modified the definition of the Cuntz–Pimsner algebra by requiring the Cuntz–Pimsner covariance condition only on a suitable ideal in \( \varphi^{-1}(\mathcal{K}(\mathcal{E})) \). Such modifications are particularly popular if the left action of \( A \) on \( \mathcal{E} \) is not faithful because in that case, the unmodified Cuntz–Pimsner algebra may well be zero. The colimit construction, however, singles out the unmodified Cuntz–Pimsner algebra.

Unlike the Cuntz–Pimsner condition, “nondegeneracy” is not a relation that we may impose on a bunch of generators. This is why there need not be a universal \( C^* \)-algebra for nondegenerate representations, but there is always one for Cuntz–Pimsner covariant representations. The two properties are only equivalent if \( \mathcal{E} \) is proper. This is the reason why we only understand colimits for diagrams of proper correspondences.

It seems likely that the colimit of the diagram \((\mathbb{N}, +) \to \mathcal{Corr}\) given by the endomorphism \( \mathcal{P}(\mathbb{N}) \) of \( \mathcal{C} \) does not exist. In the following, we therefore restrict attention to colimits of diagrams of proper correspondences.

6. Category-shaped diagrams and product systems

We have examined enough examples that it makes sense to spell out what functors, transformations, and modifications \( \mathcal{C} \to \mathcal{Corr} \) mean for an arbitrary category \( \mathcal{C} \).
We are particularly interested in transformations to a constant functor, which lead to the description of the colimit of a diagram.

6.1. Functors, transformations and modifications. The objects of $\text{Corr}^C$ are functors $C \to \text{Corr}$; arrows are transformations between such functors, and 2-arrows are modifications. We describe these things more concretely and then explain briefly how to compose transformations. These definitions are summarised succinctly in [10]. They are worked out for $\text{Corr}(2)^C$ in [6 §4], even for an arbitrary bicategory $C$. The definitions simplify if $C$ is a category because part of the data does not occur any more. The following propositions already contain these simplifications. We omit the (rather trivial) proofs. Readers that do not care much about bicategory theory could take the following propositions as definitions.

**Proposition 6.1.** A functor $C \to \text{Corr}$ consists of

- $C^*$-algebras $A_x$ for all objects $x$ of $C$;
- correspondences $\mathcal{E}_g: A_x \to A_y$ for all arrows $g: x \to y$ in $C$;
- isomorphisms of correspondences $\mu_{g,h}: \mathcal{E}_g \otimes A_x \to \mathcal{E}_h$ for all pairs of composable arrows $g: y \to z$, $h: x \to y$ in $C$;

such that

1. $\mathcal{E}_{1_x}$ is the identity correspondence on $A_x$ for all objects $x$ of $C$;
2. $\mu_{1_y,g}: \mathcal{E}_g \otimes A_y \to \mathcal{E}_g$ and $\mu_{g,1_x}: \mathcal{E}_g \otimes A_x \to \mathcal{E}_g$ are the canonical isomorphisms for all arrows $g: x \to y$ in $C$;
3. for all composable arrows $g_{01}: x_0 \to x_1$, $g_{12}: x_1 \to x_2$, $g_{23}: x_2 \to x_3$, the following diagram commutes:

   \[
   \begin{array}{ccc}
   (\mathcal{E}_{g_{01}} \otimes A_{x_1}) & \xrightarrow{\mu_{g_{12},g_{01}} \otimes A_{x_2}} & \mathcal{E}_{g_{12}} \otimes A_{x_2}
   \\ \downarrow{\text{id}_{\mathcal{E}_{g_{01}}} \otimes A_{x_1}} & & \downarrow{\text{id}_{\mathcal{E}_{g_{12}}} \otimes A_{x_2}}
   \\ \mathcal{E}_{g_{01}} \otimes A_{x_1} & \xrightarrow{\mu_{g_{23},g_{12}} \otimes A_{x_1}} & \mathcal{E}_{g_{12}} \otimes A_{x_2}
   \end{array}
   \]

   Here $g_{02} := g_{12} \circ g_{01}$, $g_{13} := g_{23} \circ g_{12}$, and $g_{03} := g_{23} \circ g_{12} \circ g_{01}$.

   The diagram (6.2) commutes automatically if one of the arrows $g_{01}, g_{12}$ or $g_{23}$ is an identity arrow.

**Proposition 6.3.** Let $(A^0_x, \mathcal{E}_g^0, \mu^0_{g,h})$ and $(A^1_x, \mathcal{E}_g^1, \mu^1_{g,h})$ be two functors from $C$ to $\text{Corr}$. A transformation between them consists of

- correspondences $\gamma_x$ from $A^0_x$ to $A^1_x$ for all objects $x$ of $C$;
- isomorphisms of correspondences $V_g: \gamma_x \otimes A^1_x \to \mathcal{E}_g \otimes A^1_x$, $\gamma_x$ for all arrows $g: x \to y$ in $C$;

such that

1. $V_{1_x}: \gamma_x \otimes A^1_x \to A^0_x \otimes A^0_x$ is the canonical isomorphism through $\gamma_x$ for each object $x$ in $C$;
(2) for each pair of composable arrows $g: y \to z$, $h: x \to y$ in $C$, the following diagram commutes:

$$
\begin{array}{cccc}
V_h \otimes A^1_y & \text{id}_{E^1_g} & \gamma_x \otimes A^1_y & E^0_g \otimes A^0_g \\
\gamma_x \otimes A^1_y & \text{id}_{E^1_y} & \gamma_y \otimes A^1_x & E^0_y \otimes A^0_y \\
\text{id}_{\gamma_x} & u^1_{g,h} & \gamma_x \otimes A^1_y & E^0_y \otimes A^0_y \\
\gamma_x \otimes A^1_y & \text{id}_{E^1_y} & \gamma_y \otimes A^1_x & E^0_y \otimes A^0_y \\
V_{gh} & \text{id}_{\gamma_y} & V_{gh} & \\
\end{array}
$$

The diagram (6.4) commutes automatically if $g$ or $h$ is an identity arrow.

**Proposition 6.5.** Let $(A^0_x, E^0_x, \mu^0_{x,y})$ and $(A^1_x, E^1_x, \mu^1_{x,y})$ be functors from $C$ to $\mathbf{Corr}$ and let $(\gamma_x^1, V^1_x)$ and $(\gamma_x^2, V^2_x)$ be transformations between them. A modification from $(\gamma_x^1, V^1_x)$ to $(\gamma_x^2, V^2_x)$ consists of isomorphisms of correspondences $W_x: \gamma_x^1 \to \gamma_x^2$ for all objects $x$ in $C$ such that the diagrams

$$
\begin{array}{c}
\gamma_x^1 \otimes A^1_y E^1_g W_x \otimes A^1_x \text{id}_{E^1_y} \gamma_x^2 \otimes A^1_y E^1_g \\
V^1_g \otimes A^0_y \gamma_x^1 \gamma_x^2 \otimes A^1_y \gamma_x^1 \otimes A^0_y V^1_g \\
E^0_y \otimes A^0_y \gamma_x^1 \text{id}_{E^0_y} \gamma_x^2 \otimes A^1_y \gamma_x^1 \otimes A^0_y V^1_g \\
\end{array}
$$

 commute for all arrows $g: x \to y$ in $C$. This diagram commutes automatically if $g$ is an identity arrow.

The composition of transformations is defined as follows. Describe functors $C \to \mathbf{Corr}$ by $(A^0_x, E^0_x, \mu^0_{x,y})$, $(A^1_x, E^1_x, \mu^1_{x,y})$ and $(A^2_x, E^2_x, \mu^2_{x,y})$, and transformations between them by $(\gamma_x^1, V^1_x)$ and $(\gamma_x^2, V^2_x)$ as above. The product is given by the correspondences $\gamma_x^{12} := \gamma_x^1 \otimes A^1_y \gamma_x^2$ from $A^0_x$ to $A^2_x$ for objects $x$ of $C$ and by the isomorphisms of correspondences

$$
V^{12}_x: \gamma_x^{12} \otimes A^2_x E^2_g = \gamma_x^1 \otimes A^1_y \gamma_x^2 \otimes A^2_x E^2_g \text{id}_{\gamma_x^1 \otimes A^1_y \gamma_x^2} \otimes A^2_x V^{12}_x \gamma_x^1 \otimes A^2_x \gamma_x^2 \\
$$

for arrows $g: x \to y$ in $C$. These $(\gamma_x^{12}, V^{12}_x)$ indeed form a transformation. General bicategory theory predicts that this composition turns $\mathbf{Corr}^C$ into a bicategory again, and this is routine to check by hand.

To understand the above definitions, consider the special case where $C$ has only one object, that is, $C$ is a monoid. Then we may drop all indices $x$ above: a functor provides a single $C^*$-algebra $A$, a transformation a single correspondence $\gamma$, and a modification a single isomorphism $W$. Furthermore, all arrows in $C$ are composable, and there is only one identity morphism. Simplifying the data in Proposition 6.5 accordingly, the result is very close to a product system in the notation of Fowler [4].

There are only two differences. First, we require all left actions on Hilbert modules to be nondegenerate (or “essential”), whereas Fowler is careful to avoid this assumption. Secondly, we multiply in the opposite order, $E_h \otimes A E_g \to E_{gh}$, which corresponds to the composition of *-homomorphisms. As a result, functors $M \to \mathbf{Corr}$ for a monoid $M$ are the same as essential product systems over the opposite monoid $M^\text{op}$.

When we pass from monoids to categories, the only change is that we get more than one $C^*$-algebra: one for each object of the category.
Nondegeneracy of the left actions on correspondences is necessary for unit arrows in $\mathfrak{Corr}$ to work as expected: otherwise we would not get a bicategory. The order reversal comes in because when we pass from $\ast$-homomorphisms to correspondences, the composition of $\ast$-homomorphisms becomes the reverse-order tensor product. With our convention, monoid actions by $\ast$-endomorphisms become actions by correspondences of the same monoid. The same order-reversal also appears when translating between actions of a group by correspondences and saturated Fell bundles over the group. It is the reason why $g^{-1}$ appears in the correspondence between functors $G \to \mathfrak{Corr}$ and saturated Fell bundles over $G$ in the proof of [6, Theorem 3.3].

6.2. Colimits. Let $C$ be a category, let $(A_x, \mathcal{E}_g, \mu_{g,h})$ describe a functor $F : C \to \mathfrak{Corr}$ as in Proposition 6.1, and let $D$ be a $C^\ast$-algebra. We first describe the constant functor $\text{const}_D : C \to \mathfrak{Corr}$. Then we specialise the description of transformations and modifications to the case of a constant target. We use this to describe the colimit of a proper product system by generators and relations.

Definition 6.7. Let $D$ be a $C^\ast$-algebra. The constant functor $\text{const}_D : C \to \mathfrak{Corr}$ maps all objects $x$ of $C$ to $D$, all arrows $g$ in $C$ to the identity correspondence on $D$, and all pairs $g, h$ to the canonical isomorphism $D \otimes_D D = D$.

A transformation from the functor given by $(A_x, \mathcal{E}_g, \mu_{g,h})$ to $\text{const}_D$ is given by correspondences $\gamma_x$ from $A_x$ to $D$ for all objects $x$ of $C$ and isomorphisms of correspondences

\[ \gamma_x : x \to \mathcal{E}_g \otimes_{A_y} \gamma_y \quad \text{for all arrows } g : x \to y \text{ in } C, \]

such that $V_{1_x}$ for an object $x$ is the canonical isomorphism and the diagrams

\[ \begin{array}{ccc}
V_{gh} & \xrightarrow{\gamma_x} & \mathcal{E}_{gh} \otimes_{A_x} \gamma_z \\
\gamma_x & \xrightarrow{\mu_{g,h} \otimes_{A_y} 1} & \mathcal{E}_h \otimes_{A_y} \mathcal{E}_g \otimes_{A_y} \gamma_z \\
\text{id}_{\gamma_x} & \downarrow & \gamma_x \\
\mathcal{E}_{gh} \otimes_{A_x} \gamma_z & \xrightarrow{\mu_{g,h} \otimes_{A_y} 1} & \mathcal{E}_h \otimes_{A_y} \mathcal{E}_g \otimes_{A_y} \gamma_z
\end{array} \]

for composable arrows $g : y \to z$, $h : x \to y$ in $C$ commute. Here we simplified the data in Proposition 6.3 using the canonical isomorphisms $\gamma_x \otimes_D D \cong \gamma_x$ for all $x$; we may, of course, drop the identity arrow on $\gamma_x$ and redraw this diagram as a commuting square:

\[ \begin{array}{ccc}
\gamma_x & \xrightarrow{V_{gh}} & \mathcal{E}_g \otimes_{A_y} \gamma_y \\
\gamma_x & \xrightarrow{\mu_{g,h} \otimes_{A_y} 1} & \mathcal{E}_h \otimes_{A_y} \mathcal{E}_g \otimes_{A_y} \gamma_z \\
\gamma_x & \xrightarrow{\text{id}_{\gamma_x}} & \gamma_x
\end{array} \]

This diagram commutes automatically if $g$ or $h$ is an identity arrow.

If $(\gamma_x^1, V_g^1)$ and $(\gamma_x^2, V_g^2)$ are two such transformations, then a modification between them is given by isomorphisms of correspondences

\[ W_x : \gamma_x^1 \to \gamma_x^2 \quad \text{for all objects } x \text{ of } C, \]
such that the diagrams

\[
\begin{array}{ccc}
\gamma_x^1 & \xrightarrow{W_x} & \gamma_x^2 \\
V_g^1 & \xleftarrow{\mathcal{E}_g \otimes A_y} & V_g^2
\end{array}
\]

(6.9)

\[
\begin{array}{c}
\text{id}_{\mathcal{E}_y} \otimes A_y
\end{array}
\]

\[
\begin{array}{c}
\gamma_y^1
\end{array}
\]

\[
\begin{array}{c}
\gamma_y^2
\end{array}
\]

\[
\begin{array}{c}
\mathcal{E}_y \otimes A_x
\end{array}
\]

\[
\begin{array}{c}
\gamma_x^1
\end{array}
\]

\[
\begin{array}{c}
V_g^1
\end{array}
\]

\[
\begin{array}{c}
W_y
\end{array}
\]

\[
\begin{array}{c}
V_g^2
\end{array}
\]

\[
\begin{array}{c}
\mathcal{E}_g \otimes A_x
\end{array}
\]

\[
\begin{array}{c}
\gamma_x^2
\end{array}
\]

commute for all arrows \( g: x \to y \) in \( C \). This diagram commutes automatically if \( g \) is an identity arrow.

The colimit for a functor \( F: C \to \text{Corr} \) is, by definition, a \( C^* \)-algebra \( B \) such that, for each \( C^* \)-algebra \( D \), the groupoid of correspondences \( B \to D \) and isomorphisms of correspondences between them is naturally equivalent to the groupoid of transformations \( F \to \text{const}_D \) and modifications between them.

**Proposition 6.10.** There is a bijection between transformations \( F \to \text{const}_D \) and the following set of data:

- Hilbert \( D \)-modules \( \gamma_x \) for objects \( x \) of \( C \);
- nondegenerate *-homomorphisms \( \varphi_x: A_x \to \mathbb{B}(\gamma_x) \) for objects \( x \) of \( C \);
- linear maps \( S_g: \mathcal{E}_g \to \mathbb{B}(\gamma_y, \gamma_x) \) for arrows \( g: x \to y \) in \( C \);

such that

1. For each arrow \( g: x \to y \), \( S_g \) is \( A_x \)-\( A_y \)-linear, compatible with inner products, and nondegenerate:
   a. \( S_g(a_1 \xi a_2) = \varphi_x(a_1)\mathcal{E}_g(\xi)\varphi_y(a_2) \) for \( a_1 \in A_x, a_2 \in A_y \);
   b. \( S_g(\xi_1^* S_g(\xi_2^*) = \varphi_y(\xi_1, \xi_2 A_y) \) for all \( \xi_1, \xi_2 \in \mathcal{E}_g \);
   c. The closed linear span of \( S_g(\mathcal{E}_g) \cdot \gamma_y \) is \( \gamma_x \);
2. \( S_{1_x} = \varphi_x: A_x \to \mathbb{B}(\gamma_x) \) for all objects \( x \);
3. For each pair of composable arrows \( g: y \to z, h: x \to y \) in \( C \), \( \xi, \eta \in \mathcal{E}_h \), we have \( S_h(\eta)S_g(\xi) = S_{gh}(\mu_{gh}(\eta \otimes \xi)) \).

Let \( (\gamma_x, \varphi_x, S^1_x) \) and \( (\gamma^2_x, \varphi^2_x, S^2_x) \) be two such collections. Then modifications between the corresponding transformations are in natural bijection with families of unitaries \( W_x: \gamma^1_x \to \gamma^2_x \) such that \( W_x \varphi_x(a) = \varphi_x(a) W_x \) for all objects \( x \) and all \( a \in A_x \) and \( \gamma^1_x \mathcal{E}_g(\xi) = S_g(\xi) \mathcal{E}_g \) for all arrows \( g: x \to y \) in \( C \) and all \( \xi \in \mathcal{E}_g \).

**Proof.** Let \( (\gamma_x, V_g) \) as in Proposition 6.3 describe a transformation from \( F \) to \( \text{const}_D \). The left \( A_x \)-module structure on \( \gamma_x \) is through a nondegenerate *-homomorphism \( \varphi_x: A_x \to \mathbb{B}(\gamma_x) \), and when we record this as extra data, we may forget the left module structure on \( \gamma_x \) and view it simply as a Hilbert \( D \)-module. We also replace the unitary \( V^*_g: \mathcal{E}_g \otimes A_y \gamma_y \to \gamma_x \) by the linear map \( S_g: \mathcal{E}_g \to \mathbb{B}(\gamma_y, \gamma_x) \) defined by \( S_g(\xi)(\eta) := V^*_g(\xi \otimes \eta) \). The map \( S_g \) satisfies (a)–(c) in (1) and, conversely, maps \( S_g \) with these three properties are in bijection with isomorphisms of correspondences \( V^*_g \); this is proved in [1] Proposition 2.3].

To give a transformation, the unitaries \( V_g \) for arrows \( g \) in \( C \) must also satisfy the two conditions in Proposition 6.3. The first one describes \( V_{1_x} \), and it gives our condition (2) when we translate it into \( S_{1_x} \). The second condition in Proposition 6.3 is the commuting diagram \( [6.3] \) that relates \( V_g \) and \( V_h \) to \( V_{gh} \). This is equivalent to

\[
V^*_g(\eta \otimes V^*_g(\xi \otimes \zeta)) = V^*_g(\mu_{gh}(\eta \otimes \xi \otimes \zeta))
\]

for all \( \xi \in \mathcal{E}_g, \eta \in \mathcal{E}_h, \xi \in \gamma_x \). This is, in turn, equivalent to

\[
S_h(\eta)S_g(\xi)(\zeta) = S_{gh}(\mu_{gh}(\eta \otimes \xi))(\zeta)
\]

so that we get condition (3) above. All these steps may be reversed. So a family \( (\gamma_x, \varphi_x, S_g) \) with the properties (1)–(3) always comes from a unique transformation.
Theorem 6.12. Let $\text{completion of quotient of }^\ast\text{take the universal }C$\hbox{}$a$\hbox{}$\in C$\hbox{}plies $\text{Definition 6.11.}$

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this right ideal as a Hilbert module over $p$

The nondegeneracy condition (1),(c) in Proposition 6.10 is the only one with an unusual form, which we cannot impose as a relation on generators of a universal $C^\ast$-algebra. If each $E_g$ is proper, then this condition is equivalent to a Cuntz–Pimsner covariance condition for each $E_g$; this is, at first sight, slightly more general than Theorem 6.12 because we are dealing with a correspondence between two different $C^\ast$-algebras. All proofs carry over to this case, however, and we can now write down a candidate for the colimit using generators and relations:

Definition 6.11. Let $O(A_x,E_g,\mu_{y,h})$ be the universal $C^\ast$-algebra generated by the $C^\ast$-algebra $\bigoplus_x A_x$ and symbols $S_g(\xi)$ for arrows $g: x \to y$ in $C$ and $\xi \in E_g$, subject to the following relations:

(1) the relations in the $C^\ast$-algebra $\bigoplus_x A_x$ hold, $E_g \ni \xi \mapsto S_g(\xi)$ is linear for each arrow $g$, and $S_{1_x}(a) = a$ for all $a \in A_x$ and all $x$;

(2) if $g: x \to y$, $\xi \in E_g$, $a \in A_x$, then

$$S_g(\xi)a = \begin{cases} S_g(\xi a) & \text{if } \xi = y, \\ S_g(\xi a) & \text{if } \xi \neq y, \end{cases} \quad aS_g(\xi) = \begin{cases} S_g(a\xi) & \xi = x, \\ S_g(a\xi) & \xi \neq x. \end{cases}$$

(3) if $g: x \to y$, $\xi_1,\xi_2 \in E_g$, then $S(\xi_1)^\ast S(\xi_2) = (\xi_1,\xi_2)_{A_y} \subseteq A_y$;

(4) for $g: x \to y$ and $a \in A_x$ with $\varphi_{E_g}(a) \in \mathbb{K}(E_g)$ and for $\xi,\eta \in E_g$, the norm of $a - \sum_{j=1}^n S(\xi_j)^\ast S(\eta_j)^\ast$ is at most the norm of $\varphi_{E_g}(a) - \sum_{j=1}^n |\xi_j|/|\eta_j|$ in $\mathbb{K}(E_g)$; here $\varphi_{E_g}: A_x \to \mathbb{B}(E_g)$ denotes the left action;

(5) $S_h(\eta)S_g(\xi) = S_{gh}(\mu_{y,h}(\eta \otimes \xi))$ for all $\xi \in E_g$, $\eta \in E_h$.

It is clear that there is a universal $C^\ast$-algebra satisfying these relations. First, take the universal $^\ast$-algebra $U_1$ on the set of generators. Secondly, let $U_2$ be the quotient of $U_1$ by the ideal generated by the conditions (1)–(3) and (5). Thirdly, take the supremum of all $C^\ast$-seminorms on $U_2$ that satisfy (4). This is the maximal $C^\ast$-seminorm on $U_1$ that satisfies (4). The maximum exists because there is a unique $C^\ast$-seminorm on the $^\ast$-subalgebra $\bigoplus_x A_x \subseteq U_2$ and $\|S_g(\xi)\| = \|\xi\|$ for any $C^\ast$-seminorm on $U_2$ by condition (3). Finally, $O(A_x,E_g,\mu_{y,h})$ is the (Hausdorff) completion of $U_2$ in this $C^\ast$-seminorm.

Theorem 6.12. Let $C$ be a category and let $(A_x,E_g,\mu_{y,h})$ give a functor $F: C \to \text{Corr}$. Assume that $E_g$ is a proper correspondence for each arrow $g: x \to y$. Then the $C^\ast$-algebra $O(A_x,E_g,\mu_{y,h})$ is a colimit of $F$ both in $\text{Corr}$ and in $\text{Corr}_\text{prop}$.

Proof. We abbreviate $O := O(A_x,E_g,\mu_{y,h})$. Condition (1) in Definition 6.11 gives a $^\ast$-homomorphism $f: \bigoplus_x A_x \to O$ and linear maps $S_g: E_g \to O$. Condition (2) implies $A_x S_g(\xi) A_y = S_g(\xi)$ and hence $A_x S_g(\xi)^\ast A_y = S_g(\xi)^\ast$. Since all elements in $O$ may be approximated by noncommutative polynomials in elements of $S_g(\xi)$, $S_g(\xi)^\ast$ for arrows $g$ and $A_x$ for objects $x$, this implies that the $^\ast$-homomorphism $f: \bigoplus_x A_x \to O$ is nondegenerate.

Let $p_x \in \mathcal{M}(\bigoplus A_y)$ be the projection onto $A_x$ and let $\gamma_x^O := f(p_x)O$; we view this right ideal as a Hilbert module over $O$. Let $A_x$ act on $\gamma_x^O$ on the left via multiplication through $f$. This is nondegenerate, so $\gamma_x^O$ becomes a correspondence from $A_x$ to $O$. We may identify $f(p_x) \cdot \gamma_x^O$ with $\mathbb{K}(|\gamma_y^O,\gamma_x^O|) \subseteq \mathbb{B}(|\gamma_y^O,\gamma_x^O|)$.

Condition (2) in Definition 6.11 implies $S_g(\xi) \subseteq f(p_x) \cdot O \cdot f(p_y)$ for $g: x \to y$. Conditions (2) and (3) say that $S_g: E_g \to \mathbb{K}(|\gamma_y^O,\gamma_x^O|)$ is a representation of the correspondence $E_g$. They provide an isometric embedding of correspondences $V_g^O: E_g \otimes A_y \gamma_y^O \to \gamma_x^O$ by the proof of [1] Proposition 2.3.
Our next goal is to show that this isometry is unitary or, equivalently, $S_g(E_g) \circ \gamma^O$ spans a dense subspace of $\gamma^O$. This argument is essentially the same as for one direction in [1, Proposition 2.5]. It is the place where we need the correspondences $\epsilon_g$ to be proper, that is, $\varphi_{E_g}(A_x) \subseteq B(E_g)$. Let $(u_i)_{i \in I}$ be an approximate unit in $A_x$.

For each $i \in I$ and $\epsilon > 0$ there is a finite-rank operator $T = \sum_{i=1}^k |\xi_i\rangle \langle \eta_i|$ on $E_g$ with $\|\varphi_{E_g}(u_i) - T\| < \epsilon$. Condition (4) ensures that

$$\left\| \sum_{i=1}^k S_g(\xi_i)S_g(\eta_i)^* - u_i \right\| < \epsilon$$

as well. Thus we may approximate $u_x$ by elements belonging to $S_g(E_g)S_g(E_g)^* x \subseteq S_g(E_g)^*\gamma^O_y$ for any $x \in \mathcal{O}$. Since the left action of $A_x$ on $\gamma^O_x$ is non-degenerate, this shows that $S_g(E_g)\gamma^O_x$ spans a dense subspace of $\gamma^O_x$, as desired.

We have verified the critical condition (1),(c) in Proposition 6.10 for the correspondences $\gamma^O_x$ for $x \in C^O$ and the maps $S_g : E_g \to B(\gamma^O_x, \gamma^O_x)$. The remaining conditions are built into our relations very directly. So this data comes from a transformation $(\gamma^O_x, V_g)$ from our diagram $F \to \text{const}_O$.

Now let $F$ be a correspondence from $O$ to a $C^*$-algebra $D$. Then the correspondences $F_x := \gamma^O_x \otimes_O F$ from $A_x$ to $D$ and the isomorphisms of correspondences $V_g \otimes_O \text{id}_F : E_g \otimes A_x F_y \to F_x$ clearly form a transformation $F \to \text{const}_D$. We claim that this construction gives an equivalence of groupoids between the groupoid of correspondences $O \to D$ and the groupoid of transformations $F \to \text{const}_D$.

Let $(\gamma_x, S_g)$ be the data of a transformation to $\text{const}_D$ for some $C^*$-algebra $D$. Let $\gamma := \bigoplus_x \gamma_x$ with the canonical representation of $\bigoplus_x A_x$, as in Proposition 5.1. Also map $S_g(\xi) \in B(\gamma_x, \gamma_y)$ to an operator on $\gamma$ that vanishes on $\gamma_z$ for $z \neq y$. We claim that this defines a $\gamma^O_x$-homomorphism $\varphi : O \to B(\gamma)$, which is nondegenerate because already its restriction to $\bigoplus_x A_x$ is nondegenerate. We want to use the universal property of $O$, of course. All conditions except the fourth one are evident. To check that one, we copy the other half of the proof of [1, Proposition 2.5].

Let $g : x \to y$ be an arrow, let $a \in A_x, \xi, \eta \in E_g$, and let $C > 0$ be strictly bigger than the norm of $\varphi_{E_g}(a) - \sum |\xi_i\rangle \langle \eta_i|$. It is convenient to use that the map $|\xi_i\rangle \langle \eta_i| \mapsto S_g(\xi)S_g(\eta)^*$ induces a $\gamma^O_x$-homomorphism $\varphi_{g} : \mathcal{K}(E_g) \to \mathcal{B}(\gamma_x)$. This is nondegenerate because Proposition 6.10 gives $\mathcal{K}(E_g)\gamma_x = S_g(E_g)S_g(E_g)^*\gamma_x \supseteq S_g(E_g)\gamma_y \supseteq \gamma_x$.

Since $aS_g(\xi) = S_g(\varphi_{E_g}(a)\xi)$ for all $a \in A_x$, we get $a^* \subseteq S_g(\varphi_{E_g}(a))$ for all $a \in A_x, \xi \in \mathcal{K}(E_g)D = D$. Thus the direct action of $A_x$ is equal to $\varphi_{g}(\varphi_{E_g}(a))$. This easily implies the norm estimate (4) in Definition 6.11. Hence we get the desired nondegenerate $\gamma^O_x$-homomorphism $O \to B(\gamma)$, so $\gamma$ becomes a correspondence from $O$ to $D$. By construction, the transformation $(\gamma^O_x \otimes \gamma, V_g \otimes \gamma)$ associated to this correspondence $\gamma$ is the transformation given by the original data $(\gamma_x, S_g)$.

Let $(\gamma^1_x, S^1_g)$ and $(\gamma^2_x, S^2_g)$ be transformations $F \to \text{const}_D$. Form the associated correspondences $\gamma^1$ and $\gamma^2$ from $O$ to $D$. A family of isomorphisms of correspondences $W_x : \gamma^1_x \to \gamma^2_x$ gives a unitary operator $\bigoplus W_x : \gamma^1 \to \gamma^2$ that intertwines the left actions of $\bigoplus A_x \subseteq O$. Conversely, any such operator $\gamma^1 \to \gamma^2$ commutes with the projections $I_{A_x}$ and therefore decomposes as $\bigoplus W_x$ for isomorphisms of correspondences $W_x : \gamma^1_x \to \gamma^2_x$. The operators $W_x$ form a modification if and only if they also intertwine the actions of $S^1_g(\xi)$ and $S^2_g(\xi)$ for all $\xi \in E_g$ and all arrows $g$ in $C$. Since these elements together with $\bigoplus A_x$ generate $O$, this is equivalent to intertwining the representations of $O$. Thus modifications between functors $F \to \text{const}_D$ are in bijection with isomorphisms of the associated correspondences $O \to D$. Hence we have an equivalence of groupoids $\text{Corr}^O(F, \text{const}_D) \cong \text{Corr}(O, D)$. If the correspondences $\gamma_x$ are proper, then $\bigoplus A_x \to \mathcal{K}(\gamma)$ and hence $O \to \mathcal{K}(\gamma)$ because $\bigoplus A_x \to O$ is nondegenerate. Thus we get a proper correspondence from $O$
to $D$. The converse also holds because the correspondences $\gamma_x^O$ are proper. Hence the equivalence above restricts to $\text{Corr}^C_{\text{prop}}(F, \text{const}_D) \cong \text{Corr}^D_{\text{prop}}(O, D)$, that is, $O$ is also a colimit in the subcategory $\text{Corr}^D_{\text{prop}}$.

Let us return to the notationally easier case where $C$ has only one object, that is, $C$ is a monoid $P$. By Proposition 6.11 a functor $P \to \text{Corr}$ is the same as an essential product system over the opposite monoid $P^{\text{op}}$.

**Theorem 6.13.** Let $P$ be a monoid and let $P^{\text{op}}$ be its opposite monoid. View a proper, essential product system over $P^{\text{op}}$ as a functor $P \to \text{Corr}^P_{\text{prop}}$. The Cuntz–Pimsner algebra of the product system is the colimit of this functor $P \to \text{Corr}^P_{\text{prop}}$ both in $\text{Corr}^P_{\text{prop}}$ and in $\text{Corr}$.

**Proof.** The quickest proof is by inspecting the description of the colimit given by Theorem 6.12 and Definition 6.11 and observing that this $C^*$-algebra is also universal for Cuntz–Pimsner covariant representations of the product system. □

### 6.3. Colimits over bicategories

If $C$ is a category, then diagrams $C \to \text{Corr}^P_{\text{prop}}$ have a colimit by Theorem 6.12 and Definition 6.11 and observing that this $C^*$-algebra is also universal for Cuntz–Pimsner covariant representations of the product system. □

If $C$ is a strict 2-category, its arrows and objects form a category $C_1$, and a functor $F: C \to \text{Corr}$ contains a functor $C_1 \to \text{Corr}$; the latter is given by $C^*$-algebras $A_x$ for objects $x$ of $C$, correspondences $E_g$ from $A_x$ to $A_y$ for arrows $g: x \to y$ in $C$, isomorphisms of correspondences $\mu_{g,h}: E_h \otimes A_x \to E_g$ for composable arrows $g: y \to z$ and $h: x \to y$, subject to the conditions in Proposition 6.1. In addition, a functor $F: C \to \text{Corr}$ also provides isomorphisms of correspondences $\nu_a: E_g \to E_h$ for 2-arrows $a: g \Rightarrow h$, which are compatible with horizontal and vertical composition.

We refer to [3] §4.1 for the details, which play no role in the following.

Describe two functors $F_i: C \to \text{Corr}$ for $i = 0, 1$ by the data $(A_x^i, E_g^i, \mu_{g,h}^i, \nu_a^i)$ as above. A transformation $\Phi: F_0 \to F_1$ between them restricts to a transformation between their restrictions to $C_1$ and thus provides correspondences $\gamma_x: A_x^0 \to A_x^1$ and isomorphisms of correspondences $V_g: \gamma_x \otimes A_x^1 \to \gamma_x \otimes A_x^0$, $\nu_a^0 \otimes A_y^0 \to \nu_a^1 \otimes A_y^1$ for arrows $g: x \to y$ in $C$, subject to the conditions in Proposition 6.3. To be a transformation on the level of $C$, we need no extra data, but extra conditions: the diagrams

$$
\begin{array}{c}
\gamma_x \otimes A_x^1 \to \gamma_x \otimes A_x^0 \\
\downarrow \mu_{g,h}^1 \otimes A_x^1 \downarrow \mu_{g,h}^0 \otimes A_x^0 \quad \Phi_g = \nu_a^1 \otimes A_y^1 \Phi_a \\
E_g^0 \otimes A_y^0 \gamma_y \to E_g^1 \otimes A_y^1 \gamma_y
\end{array}
$$

(6.14)

commute for all 2-arrows $a: g \Rightarrow h$ in $C$, for parallel arrows $g, h: x \equiv y$. This diagram commutes automatically if $a$ is an identity 2-arrow.

A modification between two transformations $\Phi_1, \Phi_2: F_0 \to F_1$ is defined exactly as in Proposition 6.5 there is no extra data and no extra condition to be a modification on the level of $C$.

**Definition 6.15.** Let $(A_x, E_g, \mu_{g,h}, \nu_a)$ describe a functor from the 2-category $C$ to $\text{Corr}$. The Cuntz–Pimsner algebra $\mathcal{O}(A_x, E_g, \mu_{g,h}, \nu_a)$ is defined as the quotient of $\mathcal{O}(A_x, E_g, \mu_{g,h})$ (see Definition 6.11) by the relations $S_h(\nu_a(\xi)) = S_g(\xi)$ for all 2-arrows $a: g \Rightarrow h$ and all $\xi \in E_g$. 
Theorem 6.16. Let $\mathcal{C}$ be a (strict) 2-category and let $(A_x, \mathcal{E}_g, \mu_{g,h}, v_a)$ give a functor $F: \mathcal{C} \to \text{Corr}_{\text{prop}}$. The $C^*$-algebra $\mathcal{O}(A_x, \mathcal{E}_g, \mu_{g,h}, v_a)$ is a colimit of $F$ both in $\text{Corr}$ and in $\text{Corr}_{\text{prop}}$.

Proof. Let $F_1: \mathcal{C}_1 \to \text{Corr}_{\text{prop}}$ denote the restriction of a diagram to the arrows and objects in $\mathcal{C}$. A transformation $F \to \text{const}_D$ is also a transformation $F_1 \to \text{const}_D$, and the modifications are the same in both cases. Hence the universal $C^*$-algebra for transformations $F \to \text{const}_D$ is a quotient of the one for transformations $F_1 \to \text{const}_D$. The extra relations that we need to divide out are exactly the relations $S_h(v_a(\xi)) = S_g(\xi)$ for all 2-arrows $a: g \Rightarrow h$ and all $\xi \in \mathcal{E}_g$: this is exactly what is needed to make the diagrams commute. \hfill \Box

If $\mathcal{C}$ is only a bicategory, then functors $\mathcal{C} \to \text{Corr}$ look the same as above, except that now the “category” $\mathcal{C}_1$ is no longer associative: it is only associative up to 2-arrows, which we have forgotten by taking $\mathcal{C}_1$. The descriptions of functors, transformations, and modifications do not use the associators, however, and the proof of Theorem 6.12 also extends to non-associative “categories.” This is why everything works literally the same way for bicategories.

7. Inductive limits

Let $\mathcal{C}$ be the partially ordered set $(\mathbb{N}, \leq)$ viewed as a category, that is, with a unique arrow $m \to n$ if $m \leq n$ and no arrow otherwise. Diagrams indexed by $\mathcal{C}$ are called inductive systems, and their colimits are also called inductive limits.

Such a diagram in $\text{Corr}$ is given by $C^*$-algebras $A_n$, correspondences $\mathcal{E}_n^n: A_m \to A_n$ for $m \leq n$, and isomorphisms of correspondences $\mu_{m,n,k}: \mathcal{E}_m^m \otimes A_n \cong \mathcal{E}_n^n$ for all $m \leq n \leq k$, subject to the following conditions. First, $\mathcal{E}_m^m \cong A_m$ and $\mu_{m,n,k}$ has to be the canonical isomorphism if $m = n$ or $n = k$. Secondly, the maps $\mu_{m,n,k}$ are “associative” (view them as multiplication maps).

We may, however, simplify this data considerably, up to isomorphism of diagrams: It is enough to specify $C^*$-algebras $A_n$ and correspondences $\mathcal{E}_n^{n+1}$ for $n \in \mathbb{N}$, with no constraints on the $\mathcal{E}_n^{n+1}$. We may extend this to a diagram as above by taking

$$\mathcal{E}_m^m \cong \mathcal{E}_m^{m+1} \otimes_{A_{m+1}} \mathcal{E}_{m+2}^{m+2} \otimes_{A_{m+2}} \cdots \otimes_{A_{n-1}} \mathcal{E}_n^{n-1}$$

for $m \leq n$ (the empty tensor product is interpreted as $A_n$ for $m = n$) and letting $\mu_{m,n,k}$ be the canonical isomorphisms. Conversely, any diagram is isomorphic to one of this form.

Let $(A_n, \mathcal{E}_n^n, \mu_{m,n,k})$ and $(B_n, \mathcal{F}_n^n, v_{m,n,k})$ be such diagrams. We simplify transformations between them in a similar way. By definition, such a transformation is given by correspondences $G_n: A_n \to B_n$ and isomorphisms of correspondences

$$w_{m,n}: \mathcal{E}_m^m \otimes_{A_m} G_n \cong G_n \otimes_{B_m} \mathcal{F}_m^m$$

for all $m \leq n$, subject to compatibility conditions with $\mu_{m,n,k}$ and $v_{m,n,k}$ for all $m \leq n \leq k$, and the condition that $w_{m,n}$ be the canonical isomorphism. It suffices, however, to specify only the isomorphisms $w_{n,n+1}$ for $n \in \mathbb{N}$, without any further condition on them.

Finally, a modification between two such transformations, $(G_n, w_{n,n+1})$ and $(G'_n, w'_{n,n+1})$, is given by isomorphisms of correspondences $x_n: G_n \to G'_n$ such that $w_{m,n} \circ (\text{id}_{\mathcal{E}_m^m} \otimes A_n)$ and $w'_{m,n}$ for all $m \leq n$; but these conditions hold for all $m \leq n$ once they hold for all $m \leq n$ and $n = m + 1$.

The simplifications above say that the bicategory of functors $\mathcal{C} \to \text{Corr}$ is equivalent to the bicategory of simplified functors with simplified transformations and modifications. In particular, for colimits it does not make a difference whether we work with full or simplified diagrams.
Our general existence theorem shows that any inductive system of proper correspondences has a colimit in $\text{Corr}$. We claim that for an inductive system of $\ast$-homomorphisms in the usual sense, this colimit is the same as the usual inductive limit in the category of $C^\ast$-algebras. Thus we consider a diagram
\begin{equation}
A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \cdots \xrightarrow{\varphi_{n-1}} A_n \xrightarrow{\varphi_n} \cdots,
\end{equation}
where the $A_n$ are $C^\ast$-algebras and the $\varphi_n$ are $\ast$-homomorphisms. Let $A_\infty$ be the inductive limit $C^\ast$-algebra of this diagram in the usual sense, and let $\varphi_\infty^n: A_n \to A_\infty$ be the canonical $\ast$-homomorphisms.

**Proposition 7.2.** The $C^\ast$-algebra $A_\infty$ with the maps $\varphi_\infty^n$ is also a colimit of \[(7.1)\] in $\text{Corr}$ and $\text{Corr}$.

**Proof.** Let $D$ be a $C^\ast$-algebra and let $E_\infty: A_\infty \to D$ be a correspondence. For $n \in \mathbb{N}$, we define a correspondence $E_n := A_n \otimes_{\varphi_\infty^n} E_\infty: A_n \to D$. These correspondences together with the canonical isomorphisms
\begin{equation}
A_n \otimes_{\varphi_\infty^n} E_{n+1} \cong A_n \otimes_{\varphi_\infty^n} A_{n+1} \otimes_{\varphi_\infty^{n+1}} E_\infty \cong A_n \otimes_{\varphi_\infty^{n+1}} \varphi_\infty^n E_\infty \cong E_n
\end{equation}
give a transformation from \[(7.1)\] to const$_D$. Clearly, an isomorphism of correspondences $E_\infty \to E'_\infty$ induces a modification between these associated transformations, so we get a functor from the groupoid of correspondences $A_\infty \to D$ to the groupoid of transformations in $\text{Corr}$ from the diagram \[(7.1)\] to the constant diagram on $D$. We claim that this functor is an equivalence of groupoids.

Let the correspondences $E_n: A_n \to D$ and the isomorphisms of correspondences $\mu_n: A_n \otimes_{\varphi_\infty^n} E_{n+1} \to E_n$ form a transformation from \[(7.1)\] to the constant diagram on $D$. We are going to construct a correspondence $E_\infty: A_\infty \to D$.

If $a \in \ker \varphi_\infty^n \subseteq A_n$, then $a \otimes_{\varphi_\infty^n} \xi = 0$ for all $\xi \in E_{n+1}$ and hence $ab \otimes_{\varphi_\infty^n} \xi = a \otimes_{\varphi_\infty^n} b \xi = 0$ for all $b \in A_n$, $\xi \in E_{n+1}$. Since $A_n \otimes_{\varphi_\infty^n} E_{n+1} \cong E_n$, ker $\varphi_\infty^n$ acts trivially on $E_n$. Similarly, the kernel of $\varphi_\infty^{n+m}: A_n \to A_{n+m}$ acts trivially on $E_n$ because $E_n \cong A_n \otimes_{\varphi_\infty^n} E_{n+1} \otimes_{\varphi_\infty^{n+1}} \cdots \otimes_{\varphi_\infty^{n+m-1}} E_{n+m}$. The union of these kernels is dense in the kernel of $\varphi_\infty^n$, which therefore also acts trivially on $E_n$. Thus we may turn $E_n$ into a correspondence $E'_n$ from $A'_n := A_n/\ker \varphi_\infty^n$ to $D$. The maps $\varphi_\infty^n$ become embeddings $A'_n \to A'_n \cdot E'_n$ and the isomorphisms $\mu_n: A_n \otimes_{\varphi_\infty^n} E_{n+1} \to E_n$ induce isomorphisms $E'_n \cong A'_n \otimes_{\varphi_\infty^n} E'_{n+1}$. We use these isomorphisms and the embeddings $A'_n \to A'_n \cdot E'_n$ to view $E'_n$ as a subspace of $E'_{n+1}$ for each $n$.

Let $E_\infty := \underset{n \to \infty}{\lim} E'_n$. Then $E_\infty$ is a Hilbert $D$-module and the $C^\ast$-algebras $A_n'$ act on $E_\infty$ because $A'_n \cdot E_\infty = E'_n \subseteq E_\infty$. The left action of $A_\infty$ is nondegenerate because $A_\infty \cdot E_\infty$ contains $A_n' \cdot E_\infty = E'_n$ for each $n \in \mathbb{N}$, and these subspaces are dense in $E_\infty$. Thus $E_\infty$ is a correspondence from $A_\infty$ to $D$.

This construction is inverse to the one above because $E_n \cong A_n \otimes_{\varphi_\infty^n} E_\infty$. Hence $A_\infty$ has the universal property of the colimit. \[\qed\]

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E-mail address: albandik@uni-goettingen.de

E-mail address: rmeyer2@uni-goettingen.de

Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3–5, 37073 Göttingen, Germany