Nonequilibrium Transport through Double Quantum Dots: Exact Results near Quantum Critical Point

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We study a double quantum dot in the regime where each dot carries a spin-1/2. This system is described by the 2-impurity Kondo model, having a non-Fermi liquid fixed point for a critical value of the inter-impurity coupling. The Hamiltonian describing the vicinity of the critical point, including the relevant potential scattering perturbations, can be cast in quadratic form. This allows us to predict a universal scaling function for the finite temperature nonlinear conductance along the crossover from the critical point to the surrounding stable fixed points.

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Introduction. It is now well established that quantum dots (QDs) behave as Kondo impurities at low temperatures [1, 2]. So far, most experiments focused on the study of Fermi-liquid states, with regular thermodynamic as well as nonequilibrium transport properties [3] at low temperatures. However, recently a controlled QD setup was constructed experimentally to access the two-channel Kondo fixed point [4]. Remarkably, the conductance of this fixed point matches a non-Fermi-liquid (NFL) scaling function over a broad range of energy.

Another simple impurity model showing NFL behavior, similar to the two-channel Kondo model, is the two-impurity Kondo model (2IKM), consisting of two impurity spins that are coupled to conduction electrons and, at the same time, interact with each other through an exchange interaction \( K \). Jones et. al. [5] observed that in the 2IKM a NFL quantum critical point (QCP) at \( K = K_c \) separates a local singlet from a Kondoscreened phase. The exact critical behavior was found using conformal field theory methods in [6, 7]. Georges and Meir [8] and Zaránd et. al. [9] proposed different QD realizations of this critical point.

In this paper we study a series double quantum dot realizing the 2IKM when each dot behaves as an effective spin [see Fig. (1)]. Ref. [8] studied this system in a wide range of parameters at zero temperature using a slave boson mean field theory. Here we concentrate on the vicinity of the critical point and predict exact full crossover functions for the nonlinear conductance from the QCP to the surrounding manifold of fixed points at finite temperature and source drain voltage. This crossover is calculated taking into account all three relevant perturbations at the QCP. The first perturbation is associated with \( K-K_c \). Whereas typically potential scattering is a marginal perturbation in quantum impurity models, in the 2IKM it leads to two additional relevant operators at the critical point [6, 7, 9].

Our results are obtained using a convenient basis in which the QCP is described by a quadratic Hamiltonian. This Hamiltonian follows from Gan’s theory for the 2IKM [10]. While this theory was derived for a spin anisotropic version of the 2IKM, it was argued that it describes correctly also the spin SU(2) symmetric critical point [10]. We substantiate this statement further by showing explicitly that the operators at the critical point have the same form as in the conformal field theory [6].

In our crossover calculation of the conductance we include not only finite temperature, but also address the non-equilibrium problem at finite source drain voltage. Originally this problem was addressed using perturbation theory [11], valid at energy scales large compared to the Kondo temperature \( T_K \). To address the crossover to low energies different nonperturbative techniques were adopted. Ref. [12] studied transport through a 1-channel Kondo impurity using abelian bosonization; however exact results were obtained only for a specific point in the parameters space (Toulouse limit). Another important development in this direction was the application of the Bethe-ansatz and finding of many-body scattering states [13, 14].

\[ L \]
\[ \uparrow \]
\[ S_L \]
\[ \downarrow \]
\[ S_R \]
\[ R \]

FIG. 1: Double quantum dot system.

Model. Double QDs have already been proven to display rich Kondo physics in competition with inter-dot exchange interaction [8, 9, 10]. A multilevel single QD near a singlet triplet transition is also described by the 2IKM [19].

Our model consists of two leads, denoted L and R (or \( i = 1 \) or 2 respectively), attached to two QDs which behave as effective spins \( S_L, S_R \) due to large Coulomb energy \( U \). After the standard “unfolding transformation” [20], one obtains a description of the model in terms of 4 left moving Dirac fermions \( \psi_{i\alpha}(x), i = L, R, \alpha = \uparrow, \downarrow, \ x \in \{-\infty, \infty\} \). We use a convention in which \( h = v_F = 1 \), where \( v_F \) is the Fermi velocity. At the moment let us assume particle-hole symmetry. The interaction Hamil-
tonian generated to leading order in the tunneling amplitudes $t_L, t_R, t_{LR}$ takes the form

$$H_K = J_L \tilde{s}_{LL} \cdot \tilde{S}_L + J_R \tilde{s}_{RR} \cdot \tilde{S}_R + K \tilde{S}_L \cdot \tilde{S}_R, \quad (1)$$

with $\tilde{s}_{ij} = \psi_i^\alpha \sigma_3^d \bar{\psi}_j^\beta$. It is found that the only nontrivial part of the BC reads $\chi_2(0^-) = -\chi_2(0^+)$. This follows by identifying $\chi_2$ with the Majorana fermion of the Ising model in [21].

At $t_{LR} = 0$ the Kondo effect takes place separately between the left lead and left dot, and between the right lead and right dot. The system is insulating in this case. This Kondo-screened phase persists at positive nonzero $K$ up to $K = K_c$, above which a local singlet forms between the two impurities [2, 3]. The critical value of $K$ occurs at $K_c \sim T_K$. Here we assume roughly equal Kondo temperatures for left and right sides, $T_{KL} \approx T_{KR} \equiv T_K$, but most of our results do not depend on that assumption, as discussed below. The fixed point at $K = K_c$ has NFL properties and is unstable with respect to an operator with anomalous dimension $\Delta = 1/2$ associated with the perturbation $K - K_c$. However for the general case when particle-hole and parity ($L \leftrightarrow R$) symmetries are broken the QCP is unstable against two additional relevant operators identified with potential scattering [3, 4]. It was found that while intra lead potential scattering terms $V_{LL} (\psi_L \psi_L + V_{RR} \psi_R \psi_R)$ are marginal at the QCP, inter-lead direct tunneling terms

$$H_{PS} = V_{LR} (\psi_L \psi_R + h.c.) \quad (2)$$
correspond to two relevant $\Delta = 1/2$ operators. In our model $|V_{LR}| \sim t_L t_L t_R t_R / U^2$. When parity is broken $V_{LR}$ can be complex.

**Fixed point Hamiltonian.** We bosonize the original theory and introduce four left moving bosonic fields $\phi_{ja}$ satisfying $\psi_{ja} \sim e^{i\phi_{ja}}$. Subsequently 4 linear bosonic combinations are defined, corresponding to charge, spin, flavor, and difference of spin between the flavors: $\phi_c = \frac{1}{2} \sum_j \phi_{ja}, \phi_s = \frac{1}{2} \sum_j (\sigma^0)_{ja} \phi_{ja}, \phi_f = \frac{1}{2} \sum_j (\sigma^2)_{ja} \phi_{ja}$, and $\phi_{d,j} = \frac{1}{2} \sum_{j} (\sigma^0)_{ja} \phi_{ja}$. Here $\vec{d}$ and $\vec{f}$ are Pauli matrices acting on the spin and $L/R$ space respectively. Since the exponentials of the new bosons have dimension $\Delta = 1/2$, we can define new fermions $\psi_A = e^{i\phi_A}$, $A = c, s, f, X$. Taking the real and imaginary parts of those fermions we obtain 8 Majorana fermions

$$\chi_{2} = \frac{\psi^{1}_{i} + \psi_{i}^{3}}{\sqrt{2}}, \quad \chi_{2}^{\dagger} = \frac{\psi^{1}_{i} - \psi_{i}^{3}}{\sqrt{2}}. \quad (5)$$

This is denoted as the $SO(8)$ Majorana representation of the free fermions, due to the $SO(8)$ symmetry group at the trivial fixed point $J = 0$.

At the trivial fixed point $J = 0$ the free Majorana fermions have trivial boundary condition (BC) $\chi_{2}^{(+)}(0^-) = \chi_{2}^{(0)}$, $i = 1, 2$. $A = c, s, f, X$, implying the continuity of those fields at the boundary $x = 0$. The fermions are still free at the critical point and obey the BC established in Ref. [2], using a Bose-Ising representation for the free fermions. It was found that the nontrivial BC occurs in the Ising sector of the theory. By relating this representation with the $SO(8)$ Majorana representation, it is found that the only nontrivial part of the BC reads $\chi_2(0^-) = -\chi_2(0^+)$. This follows by identifying $\chi_2$ with the Majorana fermion of the Ising model in [21].

It is our main point in this section to argue that for energy scales well below $T_K$ the corrections to the free Hamiltonian of the QCP take a simple quadratic form in the $SO(8)$ representation:

$$\delta H = i \sum_{j=1}^{3} \lambda_j \chi_j(0)a, \quad (3)$$

$$\lambda_1 = c_1 \frac{K - K_c}{\sqrt{T_K}}, \quad (\lambda_2, \lambda_3) = c_2 \sqrt{T_K} \psi_{LR}(ReV_{LR}, ImV_{LR}),$$

where $\chi_1(x) = \chi_2^x(x) \text{sgn}(x), \chi_2(\chi_3) = (\chi_{1}^{T}, \chi_{2}^{T}), c_1$ and $c_2$ are constant factors of order 1, and $a$ is a local Majorana fermion $a^2 = 1/2$. At $\lambda_2 = \lambda_3 = 0$, Eq. (3) corresponds to the correction to the fixed point Hamiltonian of the particle-hole and parity symmetric model, Eq. (4). Using conformal field theory methods it was shown that the relevant operator in this case is equivalent to a magnetic field acting on the boundary spin of a quantum Ising chain. This is consistent with the $j = 1$ term of Eq. (3), since one can identify $\chi_1(x)$ with the Ising fermion [21] and write the boundary spin of the Ising model as $\sigma_{B} = i \chi_{1}(x = 0)a$. In this analogy $\chi_1$ corresponds to the magnetic field acting on the boundary spin.

In addition we will clarify that the $j = 1$ term of Eq. (3) is consistent with Gan’s theory for the anisotropic Toulouse limit of the 2IKM [10]. This theory uses the $SO(8)$ Majorana representation, and the two impurity spins turn into a local fermion $d$, where $\{d, d\dagger\} = 1$. Defining two Majorana fermions $a = \frac{d - d\dagger}{\sqrt{2}}$ and $b = \frac{d + d\dagger}{\sqrt{2}}$, the boundary part of Gan’s Hamiltonian reads

$$\delta H_G = 2i \sqrt{T_K}(0) \chi_{2}^{X}(0)b - i(K - K_c)ab. \quad (4)$$

In general $\delta H_G \neq \delta H|_{\lambda_2 = \lambda_3 = 0}$, however we shall show that the two coincide for energy scales $\ll T_K$. To see this suppose $K = K_c$ and consider a mode expansion

$$\chi_{2}^{X}(x) = \sum_{k} \left( \chi_{k}^{X}(x) \psi_{k} + h.c. \right), \quad b = \sum_{k} (u_{k} \psi_{k} + h.c.),$$

where $\{\psi_{k}, \psi_{k}\dagger\} = \delta(k - k\prime), \{\psi_{k}, \psi_{k}\dagger\} = 0$, and where initially we choose $\Lambda \gg T_K$ as an ultraviolet cutoff. Solving Schrödinger’s equation for the wave functions $\phi_{k}(x)$ and $u_{k}$ one finds $\phi_{k}(x) = e^{ikx} \left[ \theta(x) \chi_{2}^{X}(x) + \theta(-x) \chi_{2}^{X}(x) \right], \phi_{k}(0) = \frac{1}{2} (\phi_{k}^{(+)} + \phi_{k}^{(-)}), u_{k} = \frac{1}{2} \sqrt{T_K} \phi_{k}(0), \phi_{k}^{(-)} / \phi_{k}^{(+)} = e^{2\delta}, \tan \delta = \frac{2T_{K}}{\sqrt{2}}$. One should normalize $\phi_{k}^{(+)} = \frac{1}{\sqrt{2}}$, where $\ell$ is the size of the system. While at $T_K = 0$ we have the BC $\chi_{2}^{X}(0^{+}) = \chi_{2}^{X}(0^{-})$, we see from the wave function that the effect of the first term in $\delta H_G$ is to modify this BC to $\chi_{2}^{X}(0^{+}) = -\chi_{2}^{X}(0^{-})$ for energies $\ll T_K$. The key observation is that the following operator identity holds.
Physically this means that at energy scales below $T_K$ the local operator $b$ is absorbed into the field $\chi_2^X$ and changes its BC. Using the operator identity Eq. (4), we see that the term $\propto K - K_c$ in $\delta H_{\text{G}}$ is equivalent to the $j = 1$ term in $\delta H$. This establishes the connection between Gan’s theory and the boundary Ising model arising from the conformal field theory solution, showing that Gan’s anisotropic theory describes correctly also the vicinity of the isotropic fixed point.

When particle-hole and parity symmetries are broken Gan’s theory predicts two additional relevant $\Delta = 1/2$ operators which take the form of the $j = 2, 3$ terms of Eq. (3). The presence of these two terms is consistent with the conformal field theory: Consider the relevant potential scattering terms $\psi^\dagger L \psi_R + \psi^\dagger_R \psi_L$ and $i(\psi^\dagger_L \psi_R - \psi^\dagger_R \psi_L)$ in Eq. (2), which have the SO(8) representation $i\chi_2^X \chi_2^X$ and $i\chi_2^X \chi_2^Y$ respectively. We are interested in the form these operators take near the NFL critical point. This can be obtained, using conformal field theory methods, by applying double fusion in the Ising sector. This operation acts on $\chi_2^X$ only, and transforms it into the identity operator. To maintain the bosonic nature of the Hamiltonian a local dimension 0 Majorana fermion must replace $\chi_2^X$. To account for the correct ground state degeneracy of the QCP one requires this local Majorana fermion to coincide with the same operator $a$ which couples to $\chi_2^X$ at $K \neq K_c$. Therefore the two relevant potential scattering terms must have the form of the $j = 2, 3$ terms of $H_{\text{QCP}}$.

Conductance. With the fixed point Hamiltonian Eq. (4) we proceed to calculate the conductance of the double dot along the crossover from the critical point to the surrounding fixed points, as function of temperature $T$, source drain voltage $V$, and the three coupling constants $\lambda_1$, $\lambda_2$, and $\lambda_3$. As we shall see, the crossover is controlled by an energy scale

$$T^* = \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \quad (6)$$

Our result, Eq. (5), is valid for any $T/T^*$ and $eV/T^*$, as long as $eV, T, T^* \ll \min\{T_{LR}, T_{RR}\}$.

Schiller and Hershfield [12] studied the related problem of transport through a single QD with effective spin $\tilde{S}$. They considered the anisotropic Kondo interaction of the form $H_{\text{Kondo}} = \sum_{i,j=L,R} \sum_{\eta=x,y,z} t_{ij}^\eta \psi_{ij}^\dagger \chi_2^\eta S^\eta \psi_{ij}$ with $J_z = J_z = 0$. Using the SO(8) representation, they obtained a quadratic theory [Eq. (3.15)] at zero magnetic field $B = 0$ for the case $J_z = 0$, $J_z^{\text{LR}} = J_z^{\text{RR}} = 2\pi$, with a boundary term

$$\delta H_{\text{BD}} = \frac{i}{\sqrt{2\pi}} [ J_z^2 \chi_2^X b + J_z^{\text{LR}} \chi_2^L a + J_z^{\text{RR}} \chi_2^R a ], \quad (7)$$

where $J_z^\pm = (J_z^{\text{LR}} \pm J_z^{\text{RR}})/2$ and $\alpha^{-1}$ is an ultraviolet momentum cutoff. Comparing $\delta H_{\text{BD}}$ to $\delta H_{\text{G}}$ in Eq. (4), the coupling $J_z^2 \chi_2^X b$ is the only term involving $b$. For either the double or single QD the current operator is given by $I = -i[H, Y]$ where $Y = \int_{-\infty}^\infty dx (\psi_{ij}^\dagger L \psi_{ij} - \psi_{ij}^\dagger R \psi_{ij}) = \int_{-\infty}^\infty dx \psi_F^\dagger \psi_F$. We see that the current operator involves the flavor fermions, and hence the first term $\propto J_z$ in Eq. (7) does not contribute to the conductance, since the flavor fermions are decoupled from $b$. The operator $a$ in Eq. (7) couples linearly to bulk Majorana fields at $x = 0$. Comparing to Eq. (8), it is easy to see that under the replacements $\frac{\lambda_1}{\sqrt{2\pi}} \rightarrow \lambda_1$ and $\frac{\lambda_3}{\sqrt{2\pi}} \rightarrow \sqrt{\lambda_2^2 + \lambda_3^2}$, the two systems have the same current as function of temperature and voltage. We shall not repeat the calculation of [12] and present the result for the nonlinear conductance, $G = dI/dV$, in terms of the parameters of our model

$$G = G_0 F\left[ \frac{T}{T^*}, \frac{eV}{T^*} \right], \quad G_0 = \frac{2e^2}{\hbar} T_{LR}^*$$

$$F[t, v] = \frac{1}{2t} \text{Re} \psi^\dagger \left( \frac{1}{2} + \frac{1}{2t} + \frac{iv}{2nt} \right), \quad (8)$$

where $T_{LR}^* = \lambda_2^2 + \lambda_3^2$ and $\psi(z)$ is the digamma function. We see that the energy scale $T^*$ defined in Eq. (3) determines the crossover scale. At $K = K_c$, $T^* \rightarrow T_{LR}^*$. The fact that $T^*$ is quadratic in $K - K_c$ and in $V_{LR}$ is a signature of a NFL; these coupling constants have fractional renormalization group scaling dimensions 1/2. In Fig. (2) we plot the conductance as function of $eV/T^*$ for various values of $T/T^*$. At $T = 0$, $G_{0} = (1 + (eV/\pi T)^2)^{-1}$ and for $V = 0, T \ll T^*$, $G/G_0 \rightarrow 1 - T^2/3T^*^2$.

FIG. 2: Scaling function for the nonlinear conductance.

Note that the current vanishes when $V_{LR} = 0$. This occurs since the corresponding operators in Eq. (2) are the only terms in our model which do not conserve $Y$. At the fixed point we expect additional terms not conserving $Y$, such as the marginal operators $\chi_2^L \chi_2^X$. However the amplitude for such terms transferring charge between the leads is small, and their effect can be neglected.

Experimentally it is convenient to tune $K$ through $K_c$ by varying $t_{LR}$, and observe the evolution of a conductance peak as function of energy scales $T$ and $eV$. 

\[ \text{FIG. 2: Scaling function for the nonlinear conductance.} \]
At $K = K_c$ there is a residual energy scale $\tilde{T}_B$ due to potential scattering. Using $|V_{LR}| \sim \frac{|ieV|}{U} \frac{\nu J}{T}$ and $K \sim \frac{t^2}{U} \sim T_K$ we see that for a typical QD the energy scale $\tilde{T}_B$ is in the range of validity of our theory, $\tilde{T}_B \sim (\nu J)^2 T_K^2 / U \ll T_K \frac{\nu J}{T}$. At low energies $T$, $eV \ll T^*$, we have $F \rightarrow 1$ and $G \rightarrow G_0$. Thus $G$ exhibits a peak in the low $T$ linear conductance with height $2e^2/h$ at $K = K_c$, and width in $K - K_c$ which scales with $V_{LR}$. Such a peak structure was discussed in [8]; here we give its precise form. Intra-lead potential scattering $V_{LL}$ and $V_{RR}$ are marginal perturbations; they reduce the conductance at the peak to $G_{peak} = 2e^2/h [1 - O(\nu J^2)]$. Now consider the regime $T, eV \gg T^*$. In this case as $t_{LR}$ is varied the system passes through the critical region at $K = K_c$ where the system does not flow away from the NFL fixed point due to any of the relevant perturbations. The conductance now becomes:

$$G \rightarrow \frac{e^2}{h} \frac{T^*}{T} \text{Re} \psi \left( \frac{1}{2} + \frac{i eV}{2 \pi T} \right) = \frac{e^2}{h} \frac{T^*}{T} \frac{\pi^2}{2 \cosh^2 \frac{eV}{2T}}. \quad (9)$$

The width of the peak in $G$ as a function of source-drain voltage, $V$, now scales with $T$.

Although we assumed $T_{KL} \approx T_{KR}$ above, in fact, most of our results don’t depend on that assumption since we have already taken into account all relevant and marginal operators allowed in the effective Hamiltonian when parity is broken. Only the precise values of $T^*$, $T_{LR}^*$ and $K_c$ change in the asymmetric case, and the necessary conditions to approach the critical region becomes $T^* \ll T$, $eV \ll \{T_{RR}, T_{KL}\}$. In particular, note that the $T = 0$ linear conductance has the value $2e^2/h$, for $K = K_c$, independent of asymmetry in that limit. This is in striking contrast to the $T = 0$ linear conductance through a single Kondo impurity which is suppressed by a factor of $2|t_{LR}|/(|t_L|^2 + |t_R|^2)$. In the strongly asymmetric case with $T_{KR} \ll T_{KL}$, we find, up to logarithmic corrections, that

$$T^* \approx (K - K_c)^2 T_{KR} / T_{KL}^2 + |\nu V_{LR}|^2 T_{KR}, \quad (10)$$

$$T_{LR}^* \approx |\nu V_{LR}|^2 T_{KR}, \quad K_c \approx T_{KL} \text{[3]}. \quad (10)$$

As pointed out by Schiller and Herschfield [12], their single quantum dot model has the peculiarity of being related to the two-channel Kondo NFL fixed point. This may be largely a consequence of fine tuning to the Toulouse limit $J_{z}^R = 0$, $J_{z}^{LL} = J_{z}^{RR} = 2\pi$. The two-channel and the two-impurity NFL critical points are closely related. The double dot system discussed here provides a method of getting this NFL behavior in an experimentally feasible way rather than by the unrealistic fine tuning of parameters.

Our system differs from the proposal of Zar´ and et al. [3], in the way in which the leads couple to the interacting system. In [3] only one linear combination of the source and drain leads acts as a Kondo-screening channel, whereas the second combination acts like a scanning tunneling microscope tip. A third lead is needed as a second Kondo screening channel. On the other hand the standard double dot system considered here has only two leads and can be therefore easier to construct experimentally. In [3] exotic temperature dependence for the conductance $\sqrt{T}$ arises from the irrelevant operator. However we find that the (same) irrelevant operator gives regular quadratic corrections in $eV$ and in $T$ for our system [24]. On the other hand the NFL signatures are apparent from the relevant operators with dimension $1/2$ which can be addressed analytically in our configuration.

Conclusion. In this paper we found new exact results for the 2IKM in a double QD. We derived full crossover formulas for the conductance from the NFL critical point to the stable fixed points in a 3-dimensional parameter space including $K - K_c$ and the two additional relevant potential scattering operators. We clarified that this crossover is described by a quadratic Hamiltonian.

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