Relaxed multibang regularization for the combinatorial integral approximation*

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Abstract

Multibang regularization and combinatorial integral approximation decompositions are two actively researched techniques for integer optimal control. We consider a class of polyhedral functions that arise particularly as convex lower envelopes of multibang regularizers and show that they have beneficial properties with respect to regularization of relaxations of integer optimal control problems. We extend the algorithmic framework of the combinatorial integral approximation such that a subsequence of the computed discrete-valued controls converges to the infimum of the regularized integer control problem.

1 Introduction

We consider the following class of integer optimal control problems:

\[
\begin{align*}
\inf_v F(v) + R(v) \\
\text{s.t. } v \in L^\infty(\Omega, \mathbb{R}^m) \text{ and } v(x) \in \{\nu_1, \ldots, \nu_M\} =: V \text{ for almost all (a.a.) } x \in \Omega.
\end{align*}
\]

(P)

Here, \( \Omega \) is a bounded domain. The optimized function \( v \) is called the control input of the problem and may attain only the finite number of values in the set \( V \subset \mathbb{R}^m \). The function \( R : L^\infty(\Omega, \mathbb{R}^m) \to \mathbb{R} \) is a regularizer for the control input and is of the form \( R(v) = \int_\Omega g(v(x)) \, dx \) for a proper convex lower semicontinuous function \( g : \text{conv} V \to \mathbb{R} \). The function \( F : L^\infty(\Omega, \mathbb{R}^m) \to \mathbb{R} \) is convex and maps weakly-*-convergent sequences in \( L^\infty(\Omega, \mathbb{R}^m) \) to convergent sequences in \( \mathbb{R} \) (weakly-*-sequentially continuous function). The set \( V \) is called the set of bangs and has the cardinality \( |V| = M \).

Apart from the discreteness constraint \( v(x) \in \{\nu_1, \ldots, \nu_M\} \), this setting is typical for PDE-constrained optimization, and a usual choice for \( F \) is the composition of a convex function with the solution operator of some initial or boundary value problem. We relax the constraint \( v(x) \in \{\nu_1, \ldots, \nu_M\} \) to \( v(x) \in \text{conv}\{\nu_1, \ldots, \nu_M\} \), where \( \text{conv} \) denotes the convex hull operator, and obtain the continuous relaxation of \( (P) \):

\[
\begin{align*}
\min_v F(v) + R(v) \\
\text{s.t. } v \in L^\infty(\Omega, \mathbb{R}^m) \text{ and } v(x) \in \text{conv}\{\nu_1, \ldots, \nu_M\} \text{ for a.a. } x \in \Omega.
\end{align*}
\]

(R)

The combinatorial integral approximation [32, 17] divides the decomposition of the solution process of \( (P) \) into two steps.

1. Solve the continuous relaxation \( (R) \) of \( (P) \).

2. Solve an approximation problem (also called rounding problem) to approximate the solution (control) computed in the first step in the weak-*-topology.

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We use $\min$ in the definition of $\langle R \rangle$ because we generally seek for or assume settings such that $\langle R \rangle$ admits a minimizer. Let $\inf \langle P \rangle$ denote the infimal value of $\langle P \rangle$ and $\min \langle R \rangle$ denote the minimal value of $\langle R \rangle$. If the identity

$$\min \langle R \rangle = \inf \langle P \rangle$$

holds, $\inf \langle P \rangle$ can be approximated arbitrarily close with the combinatorial integral approximation decomposition, for example, by following the algorithmic framework in [24].

Recent research on the combinatorial integral approximation has focused on the types of dynamical systems for which the required properties of $F$ are satisfied [24, 23, 19], as well as algorithmic improvements for the second step [13, 24, 38] [2, 3, 17]. All of these articles analyze the setting $R = 0$. A regularization of $\langle P \rangle$ is either not considered or included only in the second step. The reason is that many common choices for regularizers, such as $R(v) = \int_\Omega \|v(x)\|^2 dx$, exhibit strict convexity and thus are generally incompatible with the identity (1.1). In fact, the following proposition holds.

**Proposition 1.1.** Let $g : \text{conv} V \to [0, \infty)$ be strictly convex and continuous. Let $R(v) := \int_\Omega g(v(x)) \, dx$. Let $w \in L^\infty(\Omega, \mathbb{R}^n)$ and $\text{conv} V$-valued. Let $A \subset \Omega$ be a set of strictly positive measure with $w(x) \notin V$ for a.a. $x \in A$. Let $(v^n)_n \subset L^\infty(\Omega, \mathbb{R}^m)$ satisfy $v^n(x) \in V$ for a.a. $x \in \Omega$ and all $n \in \mathbb{N}$, and $v^n \overset{\Delta}{\to} v$ for some $v \in L^\infty(\Omega, \mathbb{R}^m)$. Then,

$$R(v) < \lim \inf R(v^n).$$

**Proof.** The proof of Proposition 1.1 is deferred to Appendix A.1.

This implies that if $R$ is induced by a strictly convex function $g$, we have

$$\inf \langle P \rangle > \min \langle R \rangle$$

unless a solution of $\langle R \rangle$ is already discrete-valued. This is closely related to the fact that for solutions of $\langle R \rangle$ that are not $V$-valued a.e., we cannot expect $v^n \to v$ if the $v^n$ are $V$-valued even if (1.1) holds and $v^n \overset{\Delta}{\to} v$; see also [9, Cor. 11].

We build on the ideas and analysis presented in [9] and consider functions $R$ of the form $R(v) = \int_\Omega g(v(x)) \, dx$, where $g$ is not strictly convex but has a polyhedral epigraph instead. Such functions are relaxations of multibang regularizers and arise as their convex lower envelopes; see [7, 8, 9]. If the set $\{ \nu_i, g(v_i) \}_{i \in \{1, \ldots, M\}}$ is the set of vertices (extremal points) of the epigraph of $g$, the identity (1.1) still holds.

To use these regularizers in the combinatorial integral approximation, we extend the algorithmic framework from [24]. We alleviate the nonsmoothness of the regularizers by means of Moreau envelopes, for which we obtain $\Gamma$-convergence. The extended algorithm produces a sequence of discrete-valued controls that admits at least one weak$^*$-cluster point. All weakly$^*$-convergent subsequences are minimizing sequences of $\langle P \rangle$.

We structure the remainder of the article as follows. In Section 2 we introduce and analyze the relaxed multibang regularization. In Section 3 we provide the extended algorithmic framework and prove convergence. In Section 4 we present two examples to validate our analysis computationally.

**Notation.** For $n \in \mathbb{N}$, we define $[n] := \{1, \ldots, n\}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. For a subset $A \subset \Omega$, we denote its relative complement with respect to $\Omega$, by $A^c := \Omega \setminus A$. The Lebesgue measure is denoted by the symbol $\lambda$. For two subsets $A, B \subset \mathcal{V}$ of a vector space $\mathcal{V}$, we define the Minkowski sum $A + B := \{a + b : a \in A, b \in B\}$. We abbreviate the feasible set of the optimization problem $\langle R \rangle$ by $\mathcal{F}_{\langle R \rangle}$, specifically $\mathcal{F}_{\langle R \rangle} := \{v \in L^\infty(\Omega, \mathcal{V}) \mid v(x) \in \text{conv} \{\nu_1, \ldots, \nu_M\} \text{ for a.a. } x \in \Omega\}$. For a set $A$, we denote its binary-valued indicator function by the symbol $\chi_A$.

### 2 Relaxed Multibang Regularization

In this section, we introduce and analyze relaxed multibang regularizers and the relationship between $\langle R \rangle$ and $\langle P \rangle$. First, we consider scalar-valued controls in Section 2.1. We show that convex piecewise affine functions avoid the problem shown in Proposition 1.1. Instead, the identity (1.1) holds. Second, we introduce relaxed multibang regularizers for vector-valued controls in Section 2.2. Their integrands are convex polyhedral functions.
with bounded domain that are characterized as minimum values of pointwise-defined linear programs (LPs). This characterization is important for the algorithmic framework in Section 3.

In Section 2.3 we analyze the measurability of the pointwise-defined functions that minimize the LPs, and we provide a constructive proof of the identity (2.1). In Section 2.4 we show that smoothing of relaxed multibang regularizers with Moreau envelopes implies $\Gamma$-convergence.

### 2.1 Scalar-Valued Controls

Let $[\nu_1, \nu_M] \subset \mathbb{R}$ and $\nu_1 \leq \ldots \leq \nu_M$, in particular conv $V = [\nu_1, \nu_M]$. Let $\gamma > 0$. The convex lower envelope (Fenchel biconjugate) of the multibang regularizer for a one-dimensional real domain is given explicitly in [8] and is

$$R(v) = \int_{\Omega} g(v(x)) \, dx,$$

where

$$g(u) := \begin{cases} \frac{2}{\gamma}((\nu_i + \nu_{i+1})u - \nu_i \nu_{i+1}) & \text{for } u \in [\nu_i, \nu_{i+1}] \text{ for } i = 1, \ldots, M - 1, \\ \infty & \text{for } u \notin [\nu_1, \nu_M]. \end{cases}$$

Let $v^n(x) \in \{\nu_1, \ldots, \nu_M\}$ a.e. for all $n \in \mathbb{N}$, and let $v^n \xrightarrow{\ast} v$ in $L^\infty(\Omega)$. Then,

$$R(v^n) = \int_{\Omega} g(v^n(x)) \, dx = \frac{1}{\gamma} \sum_{i=1}^{M} \int_{\Omega} A_i^n g(\nu_i) \, dx \rightarrow \sum_{i=1}^{M} \int_{\Omega} \alpha_i(x) g(\nu_i) \, dx,$$

where $A_i^n = \{x \in \Omega \mid v^n(x) = \nu_i\}$. The function $\alpha : \Omega \rightarrow [0, 1]^M$ is well defined by virtue of the Lyapunov convexity theorem [21, 34] and satisfies $\sum_{i=1}^{M} \alpha_i(x) = 1$ a.e. The uniqueness of the weak-$^\ast$-limit gives $\sum_{i=1}^{M} \alpha_i(x) \nu_i = v(x)$ a.e.

For a.a. $x \in \Omega$, there is $i \in [M - 1]$ such that $v(x) \in [\nu_i, \nu_{i+1}]$. Assume that $\alpha$ encodes the corresponding convex coefficients, specifically,

$$v(x) = \alpha_i(x) \nu_i + \alpha_{i+1}(x) \nu_{i+1} \text{ and } \alpha_i(x) + \alpha_{i+1}(x) = 1 \quad (2.2)$$

and $\alpha_j(x) = 0$ for $j \notin \{i, i + 1\}$ for a.a. $x \in \Omega$ such that $v(x) \in [\nu_i, \nu_{i+1}]$.

Then, we may insert the definition of $g$ and obtain for a.a. $x \in \Omega$ that

$$\frac{2}{\gamma} (\alpha_i(x) g(\nu_i) + \alpha_{i+1}(x) g(\nu_{i+1})) = \alpha_i(x) (\nu_i + \nu_{i+1}) \nu_i - \alpha_i(x) \nu_i \nu_{i+1}$$

$$+ \alpha_{i+1}(x) (\nu_i + \nu_{i+1}) \nu_{i+1} - \alpha_{i+1}(x) \nu_i \nu_{i+1}$$

$$= (\nu_i + \nu_{i+1}) v(x) - \nu_i \nu_{i+1} = \frac{2}{\gamma} g(v(x)).$$

Inserting this identity into (2.1), we obtain

$$R(v^n) \rightarrow R(v).$$

Choosing the convex coefficients $\alpha_i(x)$ such that $\alpha_i(x) > 0$ holds only for the neighboring bangs $\nu_i$ and $\nu_{i+1}$ of $v(x)$ is the key for the convergence to the desired limit. This is illustrated in Figure 1. Convex combinations of neighboring bangs $u_i$ enable the evaluation of the regularizer to commute with the evaluation of the convex combination of the $u_i$. This is not the case if convex combinations of non-neighboring bangs are used, as indicated by the dotted green line in Figure 1 that lies strictly above $g$ in $(\nu_1, \nu_3)$.

We summarize these considerations in the following proposition.

**Proposition 2.1.** Let $v^n \xrightarrow{\ast} v$ in $L^\infty(\Omega)$ with $v(x) \in [\nu_i, \nu_M]$ for a.a. $x \in \Omega$. For a.a. $x \in \Omega$ assume that $v(x) \in [\nu_i, \nu_{i+1}]$ for some $i \in [M - 1]$ implies $v^n(x) \in \{\nu_i, \nu_{i+1}\}$ for all $n \in \mathbb{N}$. Then, $R(v^n) \rightarrow R(v)$.

**Proof.** The claim follows directly from the considerations above. □
Therefore, the function $R$ allows the identity (1.1) to be preserved. However, this does not mean that $R$ is a weak-*-norm continuous function.

The articles [7, 8] provide evidence that the regularizer $R$ promotes $\{v_1, \ldots, v_M\}$-valued solutions of (R). It can be interpreted as a generalization of the $L^1$-regularization to promote discrete-valued controls with more than three bangs ($M > 3$) [8] Sect. 3. The function $g$ above is the convex lower envelope of the function

$$g_0(u) := \frac{\gamma}{2} |u|^2 + \beta \prod_{i=1}^M |u - v_i|^0 + \delta_{[v_1,v_M]},$$

(2.3)

where $\delta_{[v_1,v_M]}$ is the $\{0, \infty\}$-valued indicator function of the set $[v_1,v_M]$, and $|0|^0 = 0$ and $|u|^0 = 1$ for $u \neq 0$; see [8] Sect. 3. However, our arguments do not use this particular structure, and the following generalization is straightforward.

**Corollary 2.2.** Let $g : [v_1, v_M] \to \mathbb{R}$ be a positive, piecewise affine and convex function with the kinks connecting the affine pieces at $\{v_1, \ldots, v_M\}$. Let $R(v) := \int_\Omega g(v(x)) \, dx$. Let $v^n \overset{\star}{\rightharpoonup} v$ in $L^\infty(\Omega)$ with $v(x) \in [v_1, v_M]$ for a.a. $x \in \Omega$. For a.a. $x \in \Omega$ assume that $v(x) \in [v_i, v_{i+1}]$ for some $i \in \{1, \ldots, M-1\}$ implies $v^n(x) \in \{v_i, v_{i+1}\}$ for all $n \in \mathbb{N}$. Then, $R(v^n) \to R(v)$.

**Proof.** Since $g : [v_1, v_M] \to \mathbb{R}$ is a piecewise affine and convex function, its Clarke subdifferential $\partial^c g$ is

$$\partial^c g(u) = \begin{cases} 
\{L_i\} & \text{if } u \in (v_i, v_{i+1}) \text{ for some } i \in [M-1], \\
[-\infty, L_i] & \text{if } u = v_i, \\
[L_i, L_{i+1}] & \text{if } u = v_i \text{ for some } i \in [M-1], \\
[L_M, \infty] & \text{if } u = v_M
\end{cases}$$

for some $-\infty < L_1 \leq \ldots \leq L_M < \infty$. Thus, $g$ has a constant slope of $L_i$ on $(v_i, v_{i+1})$ for $i \in [M-1]$. Combining this with the prerequisite that $v(x) \in [v_i, v_{i+1}]$ for some $i \in [M-1]$ implies $v^n(x) \in \{v_i, v_{i+1}\}$ for all $n \in \mathbb{N}$, we again obtain that the evaluation of $g$ and the evaluation of the convex combination commute. The rest of the proof follows with the arguments above.

**Remark 2.3.** The assumption that for a.a. $x \in \Omega$ the inclusion $v(x) \in [v_i, v_{i+1}]$ implies $v^n(x) \in \{v_i, v_{i+1}\}$ for all $n \in \mathbb{N}$ constrains the algorithms that compute the discrete-valued controls $v^n$ from a given $v$ in the second step of the combinatorial integral approximation.

Therefore, instead of computing the discrete-valued controls $v^n$ directly, binary-valued approximations $\omega^n$ of the convex coefficient function $\alpha$ are computed. Hence, the convergence $R(v^n) \to R(v)$ hinges on satisfying (2.2) in this algorithmic framework. This enters our analysis in the general definition of $g$ in Definition 2.4 and the proof of Lemma 2.11.

![Figure 1: Convex lower envelope of a multibang regularizer and convex combinations $u = \alpha v_2 + (1 - \alpha) v_3$.](image-url)
2.2 Vector-Valued Controls

In [9], the multibang regularizer for vector-valued admissible controls is introduced as the choice \( g := \alpha c + \delta v \), where \( c \) is a positive strictly convex lower semicontinuous function and \( \delta v \) denotes the \( \{0, \infty\} \)-valued indicator function of the set \( V \). Relaxations of multibang regularizers are then defined as the convex lower envelopes of such functions.

We take a different approach and define the considered class of regularizers for multidimensional controls geometrically. While this also yields polyhedral functions, it directly fits the algorithmic framework of the combinatorial integral approximation.

We recall that a function \( f : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\} \) is called polyhedral if its epigraph

\[
\text{epi } f = \{(u, r) \in \mathbb{R}^m \times \mathbb{R} | f(u) \leq r\}
\]

is a convex polyhedron. Next, we introduce the general class of regularizers, on which we focus in the remainder of the article.

**Definition 2.4.** Let \( V \subset \mathbb{R}^m \), and let \( \{g_1, \ldots, g_M\} \subset [0, \infty) \) satisfy the identity \( \text{ext } (\text{conv } \{(\nu_i, g_i) | i \in [M]\}) + \{(0, r) | r \geq 0\}) = \{(\nu_i, g_i) | i \in [M]\} \). Let \( g : \mathbb{R}^m \to [0, \infty] \) be defined through

\[
g(u) := \begin{cases} 
\min \left\{ \sum_{i=1}^{M} \alpha_i g_i | \sum_{i=1}^{M} \alpha_i \nu_i = u, \sum_{i=1}^{M} \alpha_i = 1, \alpha \geq 0 \right\} & \text{if } u \notin \text{conv } V, \\
\infty & \text{if } u \in \text{conv } V.
\end{cases}
\]

Then, we call the function \( R(v) := \int_{\Omega} g(v(x)) \, dx \) a relaxed multibang regularizer.

The positive scalars \( g_i \) may be interpreted as a means to encode a preference of the different discrete control values \( \nu_i \). We establish well-definedness and basic properties of relaxed multibang regularizers in the proposition below.

**Proposition 2.5.** 1. Let \( g \) be as in Definition 2.4. Then, \( g \) is well defined and polyhedral.

2. Let \( g \) be as in Definition 2.4. Then, \( g \) is Lipschitz continuous.

3. Let \( g : \mathbb{R}^m \to [0, \infty] \) be polyhedral with nonempty bounded domain. Then, there exist \( \{\nu_1, \ldots, \nu_M\} \subset \mathbb{R}^m \) and \( \{g_1, \ldots, g_M\} \subset [0, \infty) \) such that \( g \) can be stated in the form of Definition 2.4.

4. The function \( R \) in Corollary 2.2 is a relaxed multibang regularizer.

**Proof.**

1. \( u \in \text{conv } V \) can be represented as a convex combination of the \( \nu_i \). Thus the feasible set of the LP defining \( g(u) \) is nonempty. Moreover, the feasible set is a polytope and the LP admits a minimizer. Consequently, \( g(u) \) is well defined.

To show that \( g \) is polyhedral, we prove that \( \text{epi } g = \text{conv } \{(\nu_i, g_i) | i \in [M]\} + \{(0, r) | r \geq 0\} \). The inclusion \( \subset \) holds because \( (u, g(u)) \in \text{conv } \{(\nu_i, g_i) | i \in [M]\} \) for \( g(u) < \infty \). To assert the inclusion \( \supset \), we consider \( (u, s) \in \text{conv } \{(\nu_i, g_i) | i \in [M]\} \) and \( r \geq 0 \). Then \( (u, s) \) can be represented as a convex combination of the \( (\nu_i, g_i) \), and the definition of \( g(u) \) gives \( g(u) \leq s \leq s + r \). Thus \( (u, s) + (0, r) \in \text{epi } g \).

2. This follows from the Lipschitz continuity of LPs with respect to changes in the right-hand side in monotonic norms; see [22].

3. We use the inner description of the convex polyhedron \( \text{epi } g \) and write \( \text{epi } g = P + C \), where \( P \) is a polytope and \( C \) a finitely generated convex cone. Because the domain of \( g \) is bounded and \( g \geq 0 \), it follows that we may choose \( C = \{(0, r) | r \geq 0\} \) and that \( \text{epi } g \) is pointed. Thus, \( \text{epi } g \) has uniquely determined extremal points such that \( P = \text{conv } \text{ext } \text{epi } g \). Consequently, the claim follows by setting \( M := |\text{ext } \text{epi } g| \) and defining the \( (\nu_i, g_i) \) as the extremal points of \( \text{epi } g \).

4. Because the function \( g \) in Corollary 2.2 is piecewise affine and convex, it is polyhedral with domain \([\nu_1, \nu_M]\), which proves the claim.

**Remark 2.6.** The assertions of Proposition 2.5 still hold if the relaxed identity

\[
\text{ext } \text{conv } \{(\nu_i, g_i) | i \in [M]\} = \{(\nu_i, g_i) | i \in [M]\}
\]
Lemma 2.10. We show an auxiliary lemma.

Remark 2.7. For \( u \in \text{conv } V \), we consider minimizers \( \alpha(u) \) of \( g(u) \), that is, \( \alpha(u) = \arg \min \{ \sum_{i=1}^{M} \alpha_i g_i(u) \mid u = \sum_{i=1}^{M} \alpha_i v_i, \sum_{i=1}^{M} \alpha_i = 1 \} \).

The minimizers are unique in the setting of Corollary 2.2 but this is not always true. For example, consider \( V = \{ (1 0)^T, (0 1)^T, (-1 0)^T, (0 -1)^T \} \) and \( g_i = 1 \) for \( i \in \{1, \ldots, 4\} \). Then, several convex combinations exist for \( u = (0 0)^T \), and \( \sum_{i=1}^{M} \alpha_i g_i = 1 \) holds for all of them.

2.3 Selection Functions for Convex Coefficients

The algorithms for the second step of the combinatorial integral approximation operate on functions of convex coefficients \( \alpha : \Omega \to [0,1]^M \) with \( v(x) = \sum_{i=1}^{M} \alpha_i(x) v_i \) a.e. instead of the function \( v : \Omega \to \mathbb{R}^m \) directly. Consequently, for a given control \( v : \Omega \to \text{conv } V \), we need to recover convex coefficients \( \alpha(x) \in \mathbb{R}^M \) for a.a. \( x \in \Omega \) that minimize the LPs defining \( g(v(x)) \).

We desire a measurable function \( \alpha : \Omega \to [0,1]^M \). Because the LPs defining \( g(v(x)) \) for \( x \in \Omega \) do not necessarily have unique minimizers and a selection function \( \alpha(x) \) is not readily available in closed form, the existence and computation of a measurable selection function \( \alpha \) are not immediate. We consider the set-valued optimal policy function \( G : \text{conv } V \Rightarrow \mathbb{R}^M \), which is defined as

\[
G(u) := \arg \min \left\{ \sum_{i=1}^{M} \alpha_i g_i \mid u = \sum_{i=1}^{M} \alpha_i v_i, \sum_{i=1}^{M} \alpha_i = 1, \alpha \geq 0 \right\} \tag{2.4}
\]

for all \( u \in \text{conv } V \). Thus, we require a measurable selector function \( \tilde{g} : \text{conv } V \Rightarrow \mathbb{R}^M \) for \( G \). We recall that the multifunction \( G \) is weakly measurable if the set \( \{ u \in \text{conv } V \mid G(u) \cap U \neq \emptyset \} \) is Borel measurable for all open sets \( U \subset \mathbb{R}^M \). The following abstract result follows from the literature.

Lemma 2.8. The multifunction \( G \) is weakly measurable and admits a measurable selector \( \phi : \text{conv } V \Rightarrow \mathbb{R}^M \).

Proof. We combine the Kuratowski–Ryll–Nardzewski measurable selection theorem [20] with the fact that the set \( G(u) \) depends continuously on the input vector \( u \) [8].

The measurability of the multifunction \( G \) allows us to prove measurability for a large class of possible selector functions.

Lemma 2.9. Let \( \eta : \mathbb{R}^M \to \mathbb{R} \) be strictly convex and lower semi-continuous. Then, the function

\[
\tilde{g}(u) := \arg \min \{ \eta(\alpha) \mid \alpha \in G(u) \}
\]

is single-valued and measurable. In particular, \( \tilde{g} \) defined as \( \tilde{g}(u) := \arg \min \{ \|\alpha\|^2_2 \mid \alpha \in G(u) \} \) for \( u \in \text{conv } V \) is measurable.

Proof. We observe that \( G \) is weakly measurable (Lemma 2.8) and compact-valued. Then, we apply [15 Thm.2 & 3] with the choices \( u := -\eta \) and \( f := \tilde{g} \). The optimization problem \( \min \{ \eta(\alpha) \mid \alpha \in G(u) \} \) has a unique solution for \( u \in \text{conv } V \). The existence of a minimizer follows from compactness of \( G(u) \) and lower semi-continuity of \( \eta \). The uniqueness follows from the strict convexity of \( \eta \). Therefore, the selector \( \tilde{g} \) is single-valued and can be interpreted as a function \( \tilde{g} : \text{conv } V \Rightarrow \mathbb{R}^M \).

2.4 Proof of the Identity \( \text{(1.1)} \)

We prove the identity \( \text{(1.1)} \) for relaxed multibang regularizers by showing that for a \( \text{conv } V \)-valued control \( v \), there exist \( V \)-valued controls \( v^n \) such that \( R(v) = \lim \inf R(v^n) \) holds. Before this result is proven in Lemma 2.11 we show an auxiliary lemma.

Lemma 2.10. Let \( R(v) := \int_0^1 g(g_i(v(x))) \, dx \) be a relaxed multibang regularizer. Then, the function \( g \) defined in Definition 2.4 and \( G \) defined in (2.4) satisfy \( g(u_i) = g_i \) and \( G(u_i) = \{ e_i \} \) for \( i \in [M] \), where \( e_i \) is the \( i \)th canonical unit vector in \( \mathbb{R}^M \).
Proof. Let \( i \in [M] \). The vector \( e_i \) is feasible for the LP defining \( g(\nu_i) \) with objective value \( g(\nu_i) \leq g_i \). Let \( \alpha \in G(\nu_i) \) and \( \alpha \neq e_i \). We proceed by contradiction and distinguish the cases \( g(\nu_i) = g_i \) and \( g(\nu_i) < g_i \).

Let \( \sum_{i=1}^{M} \alpha_i \nu_i = \nu_i \) and \( \sum_{i=1}^{M} \alpha_i g_i = g_i \). Then, \( (\nu_i, g_i) \notin \text{ext} \left( \text{conv} \{ (\nu_i, g_i) \mid i \in [M] \} + \{ (0, r) \mid r \geq 0 \} \right) \), a contradiction to Definition 2.4.

Let \( \sum_{j=1}^{\nu_i} \alpha_j \nu_j = \nu_i \) and \( \sum_{j=1}^{M} \alpha_j g_j < g_i \). Then, we can write \( g_i = \eta \left( \sum_{j=1}^{\nu_i} \alpha_j \nu_j \right) + (1 - \eta)2g_i \) for some \( \eta \in (0, 1) \). Clearly \( \eta \sum_{j=1}^{\nu_i} \alpha_j \nu_j + (1 - \eta) \nu_i = \nu_i \). Again, we obtain \( (\nu_i, g_i) \notin \text{ext} \left( \text{conv} \{ (\nu_i, g_i) \mid i \in [M] \} + \{ (0, r) \mid r \geq 0 \} \right) \), which contradicts Definition 2.4. Thus, \( e_i \) is the only feasible point for the LP that defines \( g(\nu_i) \), and we deduce \( g(\nu_i) = g_i \) and \( G(\nu_i) = \{ e_i \} \).

Lemma 2.11. Let \( R \) be a relaxed multibang regularizer. Let \( v \in L^\infty(\Omega, \mathbb{R}^m) \) with \( R(v) < \infty \). Then, there exists a sequence of functions \( (v^n)_n \subset L^\infty(\Omega, \mathbb{R}^m) \) that are \( V \)-valued such that \( v^n \xrightarrow{n} v \) in \( L^\infty(\Omega, \mathbb{R}^m) \) and

\[
R(v) = \lim_{n \to \infty} R(v^n).
\]

Proof. Because \( R(v) < \infty \), we can assume that \( v(x) \in \text{conv} V \) a.e. Lemma 2.8 gives the existence of a measurable selector function \( \phi \) for the LP solution sets that defines the values of the integrand \( g \) of \( R \). Because \( v \) is measurable and \( \text{conv} V \)-valued a.e., the function \( \alpha : \Omega \to \mathbb{R}^M \) defined as \( \alpha(x) := \phi(v(x)) \) a.e. is measurable and in \( L^\infty(\Omega, \mathbb{R}^M) \).

Definition 3.1 gives that \( \alpha(x) \geq 0 \), \( \sum_{i=1}^{M} \alpha_i(x) = 1 \), and \( v(x) = \sum_{i=1}^{M} \alpha_i(x) \nu_i \). We can apply the multidimensional sum-up rounding algorithm on a sequence of suitably refined grids (see [24]) to obtain a sequence of binary-valued functions \( \omega^n : \Omega \to \{0, 1\}^M \) such that \( \sum_{i=1}^{M} \omega^n_i(x) = 1 \) holds a.e. and for all \( n \). Then, \( \omega^n \xrightarrow{n} \alpha \) follows from the analysis in [24]. We define \( v^n(x) := \sum_{i=1}^{M} \omega^n_i(x) \nu_i \) a.e. and for all \( n \), and we deduce

\[
R(v) = \int_{\Omega} g(v(x)) \, dx = \int_{\Omega} \sum_{i=1}^{M} g_i \alpha_i(x) \, dx
\]

\[
= \sum_{i=1}^{M} g_i \int_{\Omega} \alpha_i(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{M} g_i \int_{\Omega} \omega^n_i(x) \, dx = \lim_{n \to \infty} \int_{\Omega} \sum_{i=1}^{M} g_i \omega^n_i(x) \, dx.
\]

Thus, it remains to show that

\[
\int_{\Omega} \sum_{i=1}^{M} g_i \omega^n_i(x) \, dx = \int_{\Omega} g \left( \sum_{i=1}^{M} \nu_i \omega^n_i(x) \right) \, dx = R(v^n). \tag{2.5}
\]

For a.a. \( x \in \Omega \), there exists \( i \in [M] \) such that \( \omega^n_i(x) = 1 \) and \( \omega^n_j(x) = 0 \) for \( j \neq i \). This implies

\[
g(v^n(x)) = g(\omega^n_i(x) \nu_i) = g(\nu_i) \quad \text{Lebesgue} \implies g_i = \sum_{i=1}^{M} g_i \omega^n_i(x). \tag{2.6}
\]

Integrating (2.6) over \( \Omega \) yields (2.5), which closes the argument.

Having obtained Lemma 2.11, we are prepared to prove the identity (1.1).

Theorem 2.12. Let \( \Omega \) be a bounded domain. Let \( F : L^\infty(\Omega, \mathbb{R}^m) \to \mathbb{R} \) be weakly-* sequentially continuous. Let \( V = \{ \nu_1, \ldots, \nu_M \} \subset \mathbb{R}^m \) and \( \{ g_1, \ldots, g_M \} \subset [0, \infty) \) satisfy the assumptions of Definition 2.4 and let \( R \) be defined as in Definition 2.4. Then, \( R \) admits a minimizer, and the identity (1.1) holds.
Remark 2.13. Consider, for example, the version \[34, \text{Thm. 3}\] of the convexity theorem—we have that the feasible set \( F \) is weakly-compact. For \( g \) defined in Definition 2.4, \( \text{epi} \, g \) is a convex polyhedron and therefore closed. Thus, \( g \) is a convex proper continuous function with bounded domain. Consequently, \( R : L^\infty(\Omega, \mathbb{R}^m) \to [0, \infty] \) is weakly*-sequentially lower semicontinuous (see, e.g., \[11, \text{Thm. 5.14}\]) and bounded from below. The existence of a minimizer for \( (R) \) follows with the direct method of calculus of variations.

Let \( v \) be a minimizer of \( (R) \). Lemma 2.11 gives that there exists a sequence of functions \((v^n)_n\) that are feasible for \( (R) \) such that \( R(v) = \lim R(v^n) \). The sequence \((v^n)_n\) provided by Lemma 2.11 satisfies \( v^n \rightharpoonup v \). The claim follows from the weak*-sequential continuity of \( F \) and \( R(v) = \lim R(v^n) \).

Remark 2.14. The key step to prove the identity \((1.1)\) is the identity
\[
\int_{\Omega} g(v(x)) \, dx = \int_{\Omega} \sum_{i=1}^{M} g_i(x) \, dx
\]
in Lemma 2.11. Here, the assumptions on the pairs \((\nu_i, g_i)\) and pointwise the definition of \( g \) as the minimum of an LP allow again that evaluating convex combinations of the \( \nu_i \) and evaluating \( g \) commute. Computing suboptimal convex combinations would result in a gap between \( R(v) \) and \( \lim \inf R(v^n) \); see again Figure 1.

2.5 Smoothing of \( R \)

We recall that the Moreau envelope of a proper lower semicontinuous function \( f : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\} \) is defined as
\[
(e_\gamma f)(x) := \inf \left\{ f(y) + \frac{1}{2\gamma}||x - y||^2 \left| y \in \mathbb{R}^m \right. \right\}
\]
for \( \gamma > 0 \). Let \( R \) given as \( R(v) = \int_{\Omega} g(v(x)) \, dx \) be a relaxed multibang regularizer. Then for \( \gamma > 0 \), we define the function
\[
R_\gamma(v) := \int_{\Omega} (e_\gamma g)(v(x)) \, dx.
\]
Moreover, we define the following smoothed control problems, \( (R_\gamma) \) for \( \gamma > 0 \),
\[
\min_{v} \mathcal{F}(v) + R_\gamma(v)
\]
s.t. \( v \in L^\infty(\Omega, \mathbb{R}^m) \) and \( v(x) \in \text{conv}\{\nu_1, \ldots, \nu_M\} \) for a.a. \( x \in \Omega \),
\[
(R_\gamma)
\]
and set \( (R_0) := (R) \). We briefly summarize the resulting convexity and differentiability properties and the convergence of minimizers of the \( (R_\gamma) \) to minimizers of \( (R) \) for \( \gamma \downarrow 0 \) below.

The basic properties of the Moreau envelope yield the following properties.

Proposition 2.14. 1. Let \( u \in \text{conv} \, V \). Then, \( (e_\gamma g)(u) \uparrow g(u) \) for \( \gamma \downarrow 0 \).

2. Let \( v \in \mathcal{F}(R) \). Then, \( R_\gamma(v) \uparrow R(v) \) for \( \gamma \downarrow 0 \).

3. Let \( \gamma > 0 \). Then, the function \( e_\gamma g : \text{conv} \, V \to \mathbb{R} \) is convex and continuously differentiable with the Lipschitz-continuous derivative.

4. Let \( \gamma > 0 \), and let \( p \geq 2 \). Then, \( R_\gamma : L^p(\Omega, \mathbb{R}^m) \to \mathbb{R} \) is differentiable with derivative \( R_\gamma'(v) \, d = \int_{\Omega} (e_\gamma g)'(v(x)) \, d(x) \, dx \) for \( d \in L^p(\Omega, \mathbb{R}^m) \).

Proof. The first claim follows from basic properties of the Moreau envelope \[27, \text{Sect. 1.G}\]. The second claim follows from the Beppo–Levi monotone convergence theorem and the first claim. The third claim follows from \[27, \text{Thm. 2.26}\] with the fact that the proximal mapping is firmly nonexpansiveness. The fourth claim follows from the chain rule and differentiability properties of Nemitskij operators \[12, \text{Thm. 7}\].
Proposition 2.15. Let $(\gamma^n)_n \subset [0, \infty)$ satisfy $\gamma^n \downarrow 0$. Then, the sequence of functionals $F + R_{\gamma^n}$ on $F_0 \subset L^\infty(\Omega, \mathbb{R}^m)$ is $\Gamma$-convergent with limit $F + R$ with respect to weak-* convergence in $L^\infty$.

Proof. Because $F$ is sequentially weak-*-sequentially continuous, it suffices to prove that $R$ is a $\Gamma$-limit for the sequence $(R_{\gamma^n})_n$. Let $L > 0$ denote the Lipschitz constant of $g$, which exists by virtue of Proposition 2.5. Then, we obtain

$$R(w) - R_{\gamma^n}(w) = \int_\Omega g(w(x)) - (e_{\gamma^n} g)(w(x)) \, dx \leq \frac{1}{2} \lambda(\Omega)L^2 \gamma^n$$ (2.7)

for all $w \in F_0$, where we defer the proof to Section A.2. Let $v^n, v \in F_0$ for all $n \in \mathbb{N}$ be such that $v^n \rightharpoonup^* v$ in $L^\infty(\Omega, \mathbb{R}^m)$. Because $R$ is weak-* sequentially lower semicontinuous, it follows that

$$R(v) \leq \liminf_n R(v^n) \leq \liminf_n R_{\gamma^n}(v^n) + \frac{1}{2} \lambda(\Omega)L^2 \gamma^n = \liminf_n R_{\gamma^n}(v^n),$$

where the second inequality follows from (2.7) with the choice $w = v^n$.

By virtue of Proposition 2.14, it follows that

$$R_{\gamma^1}(v) \leq R_{\gamma^n}(v) \leq R_{\gamma^{n+1}}(v) \leq R(v)$$

for all $n \in \mathbb{N}$. Combining these estimates, we obtain $\Gamma$-convergence. \hfill \qed

Corollary 2.16. Let $(\gamma^n)_n \subset [0, \infty)$ satisfy $\gamma^n \downarrow 0$, and let $(\varepsilon^n)_n \subset [0, \infty)$ satisfy $\varepsilon^n \to 0$. Let $v, v^n \in F_0$ for $n \in \mathbb{N}$ satisfy $v^n \rightharpoonup^* v$ and

$$F(v^n) + R_{\gamma^n}(v^n) < \varepsilon^n + \inf_{(R_{\gamma^n})} F + R_{\gamma^n}.$$

Then, $v$ is a minimizer of \( R \).

Proof. This follows with a standard proof from $\Gamma$-convergence; see, for example, [10]. \hfill \qed

Remark 2.17. This approach differs from the Moreau–Yosida regularization performed in [9]. Therein, the authors work with a Moreau envelope of the convex conjugate of the relaxed multibang regularizer $R$, specifically $(e_R R^*)^\ast(v) = R(v) + \frac{\alpha}{2} \|v\|^2$. They improve the convergence of the sequence of minimizing controls for \( R \) to norm-convergence. This is enabled by the strict convexity that is added by the term $\frac{\alpha}{2} \|v\|^2$. The resulting regularized multibang regularizer is nondifferentiable, and nonsmooth techniques are necessary for the optimization; see also [9] Rem. 2.3.

3 Algorithmic Framework

We use the properties of relaxed multibang regularizers and the optimization problems \( R \) and \( R\ast \) to formulate an algorithmic framework to compute minimizing sequences of controls for \( F \).

In Section 3.1 we provide the necessary definitions to formulate the second step of the combinatorial integral approximation decomposition, namely, so-called rounding algorithms and the grids they operate on. In Section 3.2 we integrate these concepts with the findings from Section 2 into an algorithmic framework, for which we prove well-definedness and asymptotics. Practical considerations on the solution of the involved optimization problems are given in Section 3.3.

3.1 Rounding Algorithms

We introduce the concepts of rounding grid and of order-conserving dissection [24]. A rounding grid is a partition of the domain $\Omega$. An order-conserving domain dissection is a sequence of refined rounding grids that satisfies certain regularity properties.

Definition 3.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. We call a finite partition $\mathcal{T} = \{T_1, \ldots, T_N\} \subset 2^{\Omega}$ of $\Omega$ into $N \in \mathbb{N}$ grid cells a rounding grid. We denote its maximum grid cell volume by $\Delta_T = \max \{\lambda(T_i) \mid i \in [N]\}$.

We call a sequence of rounding grids $(\mathcal{T}^n)_n \subset 2^{\mathcal{B}(\Omega)}$ with $\mathcal{T}^n = \{T^1_n, \ldots, T^N_n\}$ and corresponding maximum grid cell volumes for all $n \in \mathbb{N}$ an order-conserving domain dissection of $\Omega$ if
1. $\Delta T^n \to 0$,

2. for all $n$ and all $i \in [N^n-1]$, there exist $1 \leq j < k \leq N^n$ such that $\bigcup_{i=j}^{k} T_i^n = T_i^{n-1}$, and

3. the cells $T_j^n$ shrink regularly; that is, there exists $C > 0$ such that for each $T_j^n$ there exists a Ball $B_{j}^{n}$ such that $T_j^n \subset B_{j}^{n}$ and $\lambda(T_j^n) \geq C \lambda(B_{j}^{n})$.

Definition 3.1 requires that the maximum volume of the grid cells of a rounding grid vanish over the refinements. Definition 3.1 requires that a grid cell of a rounding grid be decomposed into finitely many grid cells in the next rounding grid and that the order of the grid cells of a partition be conserved by the grid cells in which it is decomposed in all later iterations. Definition 3.1 requires that the grid cells shrink regularly. In particular, their shape cannot degenerate, and their eccentricity remains bounded over the iterations.

**Example 3.2.** Definition 3.1 is, for example, satisfied by uniform refinements of a uniform mesh, where the order of the grid cells is induced by the course of approximants of space-filling curves through the grid cells.

We introduce the terms **binary** and **relaxed control** to denote output and input functions of the rounding algorithms.

**Definition 3.3.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. We call measurable functions $\omega : \Omega \to \{0, 1\}^M$ such that $\sum_{i=1}^{M} \omega_i(x) = 1$ holds a.e. **binary controls**. We call measurable functions $\alpha : \Omega \to [0, 1]^M$ such that $\sum_{i=1}^{M} \alpha_i(x) = 1$ holds a.e. **relaxed controls**.

Rounding algorithms give rise to approximations of relaxed controls by binary controls in pseudometrics that are induced by rounding grids.

**Definition 3.4.** Let $T$ be a rounding grid. We define its **induced pseudometric** as

$$d_T(\alpha, \beta) := \max \left\{ \left\| \int_{T_i}^{N} \alpha(x) - \beta(x) \, dx \right\|_\infty \mid k \in [N] \right\}$$

for relaxed controls $\alpha$ and $\beta$.

It is straightforward that $d_T(\alpha, \beta)$ is a pseudometric on $L^\infty(\Omega, \mathbb{R}^M)$. One can show that the corresponding sequence of induced pseudometrics for an order-conserving domain dissection equips the space $L^\infty(\Omega, \mathbb{R}^M)$ with a Hausdorff topology and that convergence with respect to this topology implies weak-*-convergence and convergence in $H^{-1}(\Omega, \mathbb{R}^M)$.

With respect to the nullspace of the pseudometric $d_T$, we note that $d_T$ cannot distinguish functions that have the same average value per grid cell. Therefore, approximations of control functions by their average per grid cell will be important in the algorithmic framework. For a measurable control function $v : \Omega \to \text{conv} V$ and a rounding grid $T$, we define its average per grid cell $\bar{v} : \Omega \to \text{conv} V$ as

$$\bar{v} := \sum_{i=1}^{N} \chi_{T_i} \frac{1}{\lambda(T_i)} \int_{T_i} v(x) \, dx.$$  \hspace{1cm} (3.1)

Rounding algorithms map relaxed controls to binary controls and satisfy certain properties. Because our analysis depends on this functional nature and the particular choice is of lesser importance, we introduce them as functions below.

**Definition 3.5.** A rounding algorithm is a function $\mathcal{R}A$ that maps a rounding grid $T$ and a relaxed control $\alpha$ to a binary control $\omega$; that is, $\omega = \mathcal{R}A(T, \alpha)$. In particular, there exists a constant $\theta > 0$ such that $d_T(\alpha, \omega) \leq \theta \Delta_T$ holds for all relaxed controls $\alpha$, all rounding grids $T$, and $\omega = \mathcal{R}A(T, \alpha)$.

**Remark 3.6.** Definition 3.5 is satisfied for several algorithms used in the combinatorial integral approximation framework such as sum-up rounding (SUR), next-forced rounding (NFR), or the optimization-based approaches presented in [16, 3].
3.2 Main Algorithm

We now introduce our algorithmic framework as Algorithm 1.

Algorithm 1 Approximation of (P)

Input: Order-conserving sequence of rounding grids $(T^n)_n$.
Input: Rounding algorithm $\mathcal{R}A$.
Input: Null sequence $e^n \downarrow 0$.
Input: Null sequence $\gamma^n \downarrow 0$.

1. $e^0 \leftarrow 0$
2. for $n = 0, \ldots, \#$ do
3. Compute $v^n$ such that $F(v^n) + R_{\gamma^n}(v^n) < e^n + \inf_{R_{\gamma^n}} F + R_{\gamma^n}$.
4. for $k = 1, \ldots$ do
5. Compute avg. per grid cell $v^n$ (see (3.1)) from $v^n$ on rounding grid $T^{c^n}$.
6. if $\|v^n - v^n\|_{L^2(\Omega)} < e^n$ then
7. break
8. end if
9. $e^n \leftarrow e^n + 1$
10. end for
11. Compute $\alpha^n(x) := \arg\min\{\|a\|_2^n \mid a \in G(\bar{v}^n(x))\}$ with $G$ defined in (2.4).
12. Compute binary control $\omega^n = \mathcal{R}A(T^{c^n}, \alpha^n)$.
13. Compute $\{\nu_1, \ldots, \nu_M\}$-valued control $\bar{v} := \sum_{i=1}^M \omega^n_i \nu_i$.
14. $e^{n+1} \leftarrow e^n$.
15. end for

Before proving the asymptotics of Algorithm 1 we argue that its steps are well defined in the proposition below. The critical steps that require consideration are the finite termination of the for loop beginning in Line 4 and the computation of $\alpha^n$ in Line 11.

Proposition 3.7. Let $R$ be a relaxed multibang regularizer. Let the inputs of Algorithm 1 be given. For all iterations $n \in \mathbb{N}$ of Algorithm 1 it holds that

1. the for loop starting in Line 4 terminates after finitely many iterations and
2. $\alpha^n$ is a uniquely defined relaxed control.

Proof. Definition 3.1 implies that order-conserving domain dissections satisfy the prerequisites of the Lebesgue differentiation theorem [33, Chap. 3, Cor. 1.6 & 1.7]. Thus $\bar{v}^n \rightarrow v^n$ pointwise a.e. for $k \rightarrow \infty$. Because $v^n(x) \in \text{conv}\{\nu_1, \ldots, \nu_M\}$ a.e., which translates to $\bar{v}^n$ computed in the $k$th iteration, Lebesgue’s dominated convergence theorem gives $\bar{v}^n \rightarrow v^n$ in $L^2(\Omega)$ for $k \rightarrow \infty$. Since $e^n > 0$ for all $n \in \mathbb{N}$, the termination criterion $\|v^n - v^n\|_{L^2(\Omega)}$ is satisfied after a finite number of iterations of the inner loop. Lemma 2.9 gives that $\min\{\|a\|_2^n \mid a \in G(u)\}$ has a unique minimizer that satisfies $\sum_{i=1}^M a_i = 1$ and $a \geq 0$ for all $u \in \text{conv} V$. Moreover, the function $\bar{v}^n$ is piecewise constant on the cells of a finite decomposition of $\Omega$. Third, we observe that $\alpha^n$ is measurable from the fact that $\bar{v}^n$ is measurable and from Lemma 2.9

Now, we state the main result of the article, the convergence of the iterates produced by Algorithm 1. Then, we prove an auxiliary lemma before proving the theorem.

Theorem 3.8. Let $R$ be a relaxed multibang regularizer. Let the inputs of Algorithm 1 be given. Then, Algorithm 1 produces an infinite sequence of iterates $\bar{v}^n$, $\bar{v}^n, \bar{v}^n \in L^\infty(\Omega, \mathbb{R}^m)$ such that $v^n$ admits a weak-$^*$-cluster point and all weak-$^*$-cluster points of $v^n$ satisfy the following:

1. there is a subsequence $\bar{v}^{n_k} \rightharpoonup v^*$ that minimizes (R),
2. $\bar{v}^{n_k} \rightharpoonup v^*$,
3. $\bar{v}^{n_k} \rightharpoonup v^*$,
4. $F(\hat{v}^{nk}) \to F(v^*)$,
5. $R(\hat{v}^{nk}) \to R(v^*)$, and in particular
6. $(\hat{v}^{nk})_k$ is a minimizing sequence for $(P)$.

Before providing the proof of Theorem 3.8, we prove an auxiliary lemma to show that $R(\hat{v}^n)$ and $R(\hat{v}^n)$ are close for $n \to \infty$.

**Lemma 3.9.** Let $R$ be a relaxed multibang regularizer. Let the inputs of Algorithm 1 be given. There exists $C > 0$ such that for $\hat{v}^n$ and $\check{v}^n$ computed by Algorithm 1 lines 4 to 7 from $v^n \in \mathcal{F}$, it holds that $|R(\hat{v}^n) - R(\check{v}^n)| \leq C \Delta T_{n,n}$. The constant $C > 0$ does not depend on the rounding grids $T^n$.

**Proof.** Because we consider a fixed iteration $n$, we omit the index $n$ for the functions as well as the index $c^n$ for the rounding grid in this proof. From the construction of $\alpha$ in Line 11 we obtain

$$R(\hat{v}) = \int_{\Omega} g(\hat{v}(x)) \, dx = \int_{\Omega} \sum_{i=1}^{M} \omega_i(x)g_i \, dx.$$

Because $\omega = R.A(T, \alpha)$, we have that $\omega$ is a binary control, which implies

$$R(\hat{v}) = \int_{\Omega} g(\hat{v}(x)) \, dx = \int_{\Omega} \sum_{i=1}^{M} \omega_i(x)g_i \, dx = \int_{\Omega} \sum_{i=1}^{M} \omega_i(x)g_i \, dx,$$

where the last equality follows from Lemma 2.9.

We conclude

$$|R(\hat{v}) - R(\check{v})| \leq \sum_{i=1}^{M} |g_i| \left| \int_{\Omega} \omega_i(x) - \omega_i(x) \, dx \right| \leq \sum_{i=1}^{M} |g_i| d_T(\alpha, \omega),$$

where the first inequality follows from the triangle inequality and the second inequality by definition of $d_T$. Because of Definition 3.5, the claim follows with the choice $C := \sum_{i=1}^{M} |g_i| \theta$. \qed

We are ready to prove our main result.

**Proof of Theorem 3.8.** For the statements of Theorem 3.8 to be well defined, we need to ensure that the inner loop (indexed by $k$) of Algorithm 1 terminates finitely for all $n \in \mathbb{N}$. This follows from Proposition 3.7. The existence of weak-$^*$-cluster points follows from the boundedness of the sequence. We prove the claims one by one.

1. The minimization property of weakly-$^*$-convergent subsequences follows directly from Corollary 2.16.
2. follows from the finite termination of the inner loop, in particular $\|\check{v}^{nk} - v^{nk}\|_{L^2(\Omega)} \to 0$, and $v^{nk} \rightharpoonup v^*$.
3. We consider the construction of $\hat{v}^{nk}$ and obtain

$$\hat{v}^{nk} = \sum_{i=1}^{M} \omega_i^{nk} \nu_i = \sum_{i=1}^{M} (\omega_i^{nk} - \alpha_i^{nk}) \nu_i + \sum_{i=1}^{M} \alpha_i^{nk} \nu_i = \sum_{i=1}^{M} (\omega_i^{nk} - \alpha_i^{nk}) \nu_i + \check{v}^{nk}.$$  

Then, we test with $f \in L^1(\Omega, \mathbb{R}^m)$ and apply [24] Lem. 4.4 to obtain that the difference term vanishes weakly-$^*$ for $k \to \infty$. We deduce

$$\lim_{k \to \infty} \langle v^{nk}, f \rangle_{L^1, L^\infty} = \lim_{k \to \infty} \langle \check{v}^{nk}, f \rangle_{L^1, L^\infty} = \langle v^*, f \rangle_{L^1, L^\infty}.$$  

4. follows from 3 and the weak-$^*$-sequential continuity of $F$.
5. We estimate

$$|R(\hat{v}^{nk}) - R(\check{v}^n)| \leq |R(\hat{v}^{nk}) - R(\check{v}^{nk})| + |R(\check{v}^{nk}) - R(v^{nk})| + |R(v^{nk}) - R(v^*)|$$

by means of the triangle inequality. Because the rounding algorithm is executed on grids of an order-conserving domain dissection and $c^{nk} \to \infty$, the first term tends to zero by virtue of Lemma 3.9. The finite termination of
the inner loop and \( \varepsilon \to 0 \) imply that the second term tends to zero. It remains to show \( |R(v^n_k) - R(v^*)| \to 0 \).

To this end, we estimate

\[
|R(v^n_k) - R(v^*)| \leq \left| R_{\gamma_k}(v^n_k) - R(v^n_k) \right| + \left| R_{\gamma_k}(v^n_k) - R(v^*) \right|
\]

by means of the triangle inequality. The first term tends to zero by virtue of (2.7). The second term tends to zero by combining \( \text{[4]} \) with \( \text{[9]} \).

### 3.3 Practical Considerations

The fact that we require \( v^n \) to be \( \varepsilon^n \)-optimal for \( \text{[R]} \) allows us to replace the infinite-dimensional optimization problem \( \text{[R]} \) by successively refined discretized finite-dimensional problems using, for example, the argument in \( \text{[14]} \).

In Algorithm \( \text{[1]} \) in line 11, the bilevel optimization problem

\[
\min \|a\|^2_2 \text{ s.t. } a \in \text{arg min} \left\{ \sum_{i=1}^{M} a_i g_i \left| \sum_{i=1}^{M} a_i \nu_i = u, \sum_{i=1}^{M} a_i = 1, a \geq 0 \right. \right\}
\]

has to be solved for different inputs \( u \in \text{conv} V \). Proposition \( \text{[2.5]} \) gives that the lower-level problem is an LP that has a nonempty bounded feasible set. Consequently, strong duality for LPs holds, and we may rewrite (3.2) equivalently as

\[
\min_{a, \hat{a}, z} a^T a \text{ s.t. } \begin{cases} a^T c - y^T u - z = 0, \\ Aa = u, \\ \mathbb{1}_M^T a = 1, \\ y^T A + z \mathbb{1}_M^T \leq c^T \\ a_i \geq 0 \quad \text{for } i \in \{1, \ldots, M\}, \\ y_i \in \mathbb{R} \quad \text{for } i \in \{1, \ldots, m\}, \\ z \in \mathbb{R}. \end{cases}
\]

In this problem formulation, \( A \in \mathbb{R}^{m \times M} \) is the matrix that consists of the vectors \( \nu_i \) for \( i \in [M] \) as columns, \( c \in \mathbb{R}^M \) is the vector that consists of the scalars \( g_i \) for \( i \in [M] \) as components, and \( \mathbb{1}_M \) is the vector in \( \mathbb{R}^M \) that has the entry 1 in all components. Moreover, this is a convex quadratic program with \( M + m + 1 \) variables, which can be solved with standard quadratic programming techniques.

The function \( \bar{v}^n \) is constant per grid cell on the finitely many grid cells that constitute a rounding grid. Therefore, the quadratic program (3.3) has only to be evaluated finitely many times per iteration. In fact, the proof of Theorem 3.8 can be carried out without the intermediate step of computing \( \bar{v}^n \). However, defining the function \( \alpha^n \) as the solution of (2.4) or (3.3) with \( u = v^n(x) \) for a.a. \( x \in \Omega \) cannot be implemented directly.

Regarding Remark 2.17, we note that one can replace the smoothed regularization and the smooth optimization in Algorithm 1 \( \text{[1]} \) in line 3 by the nonsmooth problems and the semismooth Newton method presented in \( \text{[9]} \).

It is not necessary for our analysis to use Moreau envelopes to approximate relaxed multibang regularizers smoothly. We provide an alternative for the scalar-valued case \( m = 1 \) in Section B in closed form, which is also amenable to our analysis.

### 4 Computational Examples

We provide two examples to demonstrate the efficacy of the algorithmic framework and validate our arguments computationally. First, we apply Algorithm 1 \( \text{[1]} \) to a signal reconstruction problem with a one-dimensional control input that has been studied in \( \text{[6]} \) \( \text{[19]} \). Second, we apply Algorithm 1 \( \text{[1]} \) to the Lotka–Volterra problem from the benchmark library \( \text{[MINTOC]} \) \( \text{[29]} \). This problem has been used frequently to evaluate algorithms for the combinatorial integral decomposition \( \text{[28]} \) \( \text{[31]} \) \( \text{[2]} \) \( \text{[3]} \). We modify the problem such that it has a two-dimensional control input.
For both examples, we have used rounding algorithms that have the additional property that for all grid cells $T \in \mathcal{T}^{c^n}$ it holds that $\int_T \alpha^n(x) \, dx = 0$ implies $\int_T \omega^n(x) \, dx = 0$ to the rounding algorithm. This can prevent $\omega$ from spontaneously switching on a control value that is far away from the values of the continuous relaxation at this spot.

4.1 Signal Reconstruction Problem

The signal reconstruction problem from [19 Sect. 5] with relaxed multibang regularizer is

$$\inf_{y,v} \int_{t_0}^{t_f} (y(t) - f(t))^2 \, dt + \eta \int_{t_0}^{t_f} g(v(t)) \, dt =: J(v)$$

(SPP)

\[\begin{align*}
\text{s.t.} \quad & y = k \ast v, \\
& v(t) \in \{v_1, \ldots, v_M\} \text{ for a.a. } t \in (t_0, t_f),
\end{align*}\]

where

$$v_1 = -1, \quad v_2 = -0.25, \quad v_3 = 0, \quad v_4 = 0.35, \quad v_5 = 1$$

and $g$ is defined as in Definition 2.4 with

$$g_1 = 1, \quad g_2 = 0.125, \quad g_3 = 0, \quad g_4 = 0.175, \quad g_5 = 1.$$

We choose $\eta = 0.01$. The continuous relaxation of (SPP) is convex. To discretize the convolution, we use a Gaussian-Legendre quadrature. We use piecewise constant ansatz functions for the control of the continuous relaxation. We smooth the function $g$ as proposed in Section 5. Scipy’s [35] implementation of L-BFGS-B [5, 36] is used to solve the smoothed continuous relaxation. We use both the optimization-based approach SCARP [2, 4] and SUR in the version of [18] as rounding algorithms in Algorithm 1. We choose $\theta = 2$, for which the optimization-based approach SCARP satisfies the prerequisites of Definition 3.5 with the same constant as SUR. This follows from the bounds proven in [25].

We show how $\inf$ (SPP) gets approximated from below by the minimizers of the smoothed continuous relaxation and with the iterates produced in Algorithm 1. In [13], we use the same fine discretization and high accuracy for Algorithm 1 in [8] for all iterations. We have run Algorithm 1 for nine iterations. We provide the values of $\Delta^n, N^n, \varepsilon^n, \gamma^n$ as well as the relative difference in the objective between the overestimator $J(\hat{v}^n)$ and the underestimator $J_{n,\varepsilon}(v^n)$, the infimum for the current discretization and smoothing. The relative objective error tends to zero over the iterations, which indicates that the computed iterates $v^n$ of the discretized continuous relaxations tend to a minimizer and the discrete-valued iterates $\hat{v}^n$ converge weakly-* to the same minimizer.

The smoothed relaxed and discrete controls for iterations 1, 5, and 9 are depicted in Figure 2. The first row with the bottom row, one can observe the reduced switching behavior that is due to the use of SCARP instead of SUR. The objective values for the smoothed relaxed controls and discrete controls are displayed over the iterations in Figure 3. As predicted by our analysis, the gap between the lower bound given by $\min \{R_{\gamma}\}$ and the upper bound given by the objective value of the discrete control tends to zero.

4.2 Lotka–Volterra Problem

The Lotka–Volterra problem [29] with relaxed multibang regularizer for a two-dimensional discrete control input is

$$\inf_{y,v} \int_{t_0}^{t_f} \left\| y(t) - (1, 1)^T \right\|^2_2 + \eta \int_{t_0}^{t_f} g(v(t)) \, dt =: J(v)$$

\[\begin{align*}
\text{s.t.} \quad & \dot{y}_1(t) = y_1(t) - y_1(t)y_2(t) - y_1(t)v_1(t) \text{ for a.a. } t \in (t_0, t_f), \\
& \dot{y}_2(t) = -y_2(t) + y_1(t)y_2(t) - y_2(t)v_2(t) \text{ for a.a. } t \in (t_0, t_f), \\
& y(t_0) = (0.5, 0.7)^T, \\
& v(t) \in \{v_1, v_2, v_3, v_4, v_5\} \text{ for a.a. } t \in (t_0, t_f),
\end{align*}\]

(LVP)
Table 1: Output of nine iterations of Algorithm 1 applied to the signal reconstruction problem with objective $J$ ($J_{\gamma}$ for the smoothed problem).

| It. | $N^n$ | $\Delta^n$ | $\epsilon^n$ | $\gamma^n$ | $\frac{J(v^n)-J_{\gamma n}(v^n)}{J_{\gamma n}(v^n)}$ |
|-----|-------|------------|--------------|------------|------------------------------------------------|
| 1   | 16    | 1.2500e-01 | 1.000e+00    | 4.000e-01 | 2.3736e+00                                      |
| 2   | 32    | 6.2500e-02 | 5.000e-01    | 2.000e-01 | 7.5314e-01                                      |
| 3   | 64    | 3.1250e-02 | 2.500e-01    | 1.000e-01 | 4.7885e-01                                      |
| 4   | 128   | 1.5625e-02 | 1.250e-01    | 5.000e-02 | 1.9752e-01                                      |
| 5   | 256   | 7.8125e-03 | 6.250e-02    | 2.500e-02 | 8.4494e-02                                      |
| 6   | 512   | 3.9062e-03 | 3.125e-02    | 1.250e-02 | 3.7259e-02                                      |
| 7   | 1024  | 1.9531e-03 | 1.563e-02    | 6.250e-03 | 1.4573e-02                                      |
| 8   | 2048  | 9.7656e-04 | 7.813e-03    | 3.125e-03 | 6.4613e-03                                      |
| 9   | 4096  | 4.8828e-04 | 3.906e-03    | 1.563e-03 | 3.1779e-03                                      |

Figure 2: Computed controls iteration in iterations $n = 1, 5, 9$ (left to right) by solving the smoothed continuous relaxation in line 3 (top), executing SUR in line 13 (center), and executing SCARP in line 13 (bottom).
Deriving a closed-form expression for the Moreau envelopes of $g$ is difficult. Therefore, we have computed the values of $g$ and $g_\gamma$ on a fine grid that discretizes $\text{conv} V = [0,0.4] \times [-0.1,0.1]$ and interpolated them. Figure 4 shows the function $g$ and its Moreau envelopes $g_\gamma$ for the choices $\gamma = 5 \cdot 10^{-3}$ and $\gamma = 10^{-3}$.

The Lotka–Volterra problem is nonconvex, and we cannot expect more than convergence of the solution of the continuous relaxation to a local minimizer. Therefore, we disregard the global optimality condition implied by Algorithm 1 ln. 3 for this problem. Again, we use the optimization-based approach SCARP [2, 3] as rounding algorithm in Algorithm 1. With respect to Definition 3.5 we have chosen the parameter value $\theta = 2$. We mimic driving $\varepsilon^n$ to zero by refining the discretization in every iteration. We optimize over piecewise constant ansatz functions for the controls and use the control discretization grid as rounding grid. Therefore, we always have $\| v^n - \bar{v}^n \|_{L^2} = 0$. 

We choose $\eta = 0.005$. To discretize the initial value problem and the objective and to solve the continuous relaxation, we use the software package CasADi [1] with IPOPT [37] as the nonlinear programming solver.
We have run the algorithm for six iterations. We provide the values of $\Delta^n$, $N^n$, $\gamma^n$ as well as the relative difference in the objective between the overestimator $J(\hat{v}^n)$ and the underestimator $J_{\gamma^n}(v^n)$ for the current discretization. The relative difference tends to zero over the iterations, which indicates that the computed iterates $v^n$ of the discretized continuous relaxations tend to a local minimizer and corresponding weak-$^*$-convergence of the discrete-valued iterates $\hat{v}^n$. The $L^2$-difference between relaxed and discrete control $\|v^n - \hat{v}^n\|_{L^2}$ decreases with less smoothing over the iterations, indicating that the relaxed multibang regularizer promotes discrete-valued controls as desired. The relaxed and discrete controls in iterations $2$, $4$ and $6$ are displayed in Figure 5.

Table 2: Output of six iterations of Algorithm 1 applied to (LVP).

| Iteration | $N^n$    | $\Delta^n$ | $\gamma^n$ | $\frac{J(v^n) - J_{\gamma^n}(v^n)}{J_{\gamma^n}(v^n)}$ | $\|v^n - \hat{v}^n\|_{L^2}$ |
|-----------|----------|------------|------------|------------------------------------------------|-----------------------------|
| 1         | 16       | 7.5000e-01 | 3.1250e-01 | 2.9629e+00                                        | 4.0609e-01                  |
| 2         | 32       | 3.7500e-01 | 6.2500e-02 | 9.5257e-01                                        | 4.7660e-01                  |
| 3         | 64       | 1.8750e-01 | 1.2500e-02 | 5.0805e-01                                        | 3.9329e-01                  |
| 4         | 128      | 9.3750e-02 | 2.5000e-03 | 3.9579e-02                                        | 2.4475e-01                  |
| 5         | 256      | 4.6875e-02 | 5.0000e-04 | 2.4117e-02                                        | 2.1208e-01                  |
| 6         | 512      | 2.3438e-02 | 1.0000e-04 | 6.9964e-03                                        | 1.8743e-01                  |

Figure 5: Components (1 green, 2 blue) of the controls $v^n$ (dotted) and $\hat{v}^n$ (solid) produced by Algorithm 1 in iterations $n = 2, 4, 6$.

5 Conclusion

Relaxed multibang regularizers are suitable for regularizing the integer optimal control problems that can be treated with the combinatorial integral approximation decomposition. They can be integrated in the algorithmic framework, and their nonsmoothness can be alleviated by using Moreau envelopes. Two test problems demonstrate the efficacy of the extended algorithmic framework.
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A Auxiliary Proofs

A.1 Proof of Proposition 1.1

Proof. Since the Carathéodory conditions are satisfied for $g$, the Nemytskii operator induced by $g$ is a bounded and continuous map from $L^2(\Omega, \mathbb{R}^m)$ to $L^2(\Omega)$ and thus $R \in C(L^2(\Omega, \mathbb{R}^m), [0, \infty))$; see [26, Sect. 10.3.4].

We split $R(w)$ as follows:

$$R(w) = \int_{A} g(w(x)) \, dx + \int_{A^c} g(w(x)) \, dx$$

Let $x \in \Omega$. Let $i \in [M]$ and $n \in \mathbb{N}$. Then, we can define

$$\beta^n_i(x) := \begin{cases} 1 & \text{if } w^n(x) = w_i, \\ 0 & \text{else.} \end{cases}$$

Since the $w^n$ are measurable, so are the $\beta^n$. The sequence $(\beta^n)_n$ is bounded in the space $L^\infty(\Omega, \mathbb{R}^M)$ and thus admits a weakly-*-convergent subsequence with limit $\alpha \in L^\infty(\Omega, \mathbb{R}^M)$. Moreover, since $w^n = \sum_{i=1}^{M} \beta^n_i w_i$, we obtain

$$w^n = \sum_{i=1}^{M} \beta^n_i w_i \to \sum_{i=1}^{M} \alpha_i w_i = w,$$

where the last identity follows from the uniqueness of the limit. From the fact that the $\beta^n$ are $\{0,1\}^M$-valued, we deduce that $0 \leq \alpha_i(x)$ and $\sum_{i=1}^{M} \alpha_i^n(x) = 1$ for a.a. $x \in \Omega$. In words, $\alpha$ constitutes a vector of convex coefficients a.e. Moreover, the fact that the $\beta^n$ are $\{0,1\}^M$-valued also implies $g(w^n(x)) = \sum_{i=1}^{M} \beta^n_i(x) g(w_i)$ for a.a. $x \in \Omega$.

Thus, we may conclude

$$R(w^n) = \sum_{i=1}^{M} \int_{\Omega} \beta^n_i(x) g(w_i) \, dx \to \sum_{i=1}^{M} \int_{\Omega} \alpha_i(x) g(w_i) \, dx.$$ (A.1)

Let $D \subset \Omega$ be measurable. Then,

$$\sum_{i=1}^{M} \int_{D} \alpha_i(x) g(w_i) \, dx \geq \int_{D} g(w(x)) \, dx$$ (A.2)

because $\alpha$ constitutes a vector of convex coefficients a.e., and $g$ is convex.

Now, we note that $\alpha(x) \notin \{0,1\}^M$ for a.a. $x \in A$ because $w(x) \in \text{conv } V \setminus V$ for a.a. $x \in A$. We show that there exists $\varepsilon > 0$ such that

$$\sum_{i=1}^{M} \alpha_i(x) g(w_i) > g \left( \sum_{i=1}^{M} \alpha_i(x) w_i \right) + \varepsilon = g(w(x)) + \varepsilon$$

for all $x \in B$ for a set $B \subset A$ such that $\lambda(B) > 0$. To see this, we assume the converse and obtain that

$$\sum_{i=1}^{M} \alpha_i(x) g(w_i) = g \left( \sum_{i=1}^{M} \alpha_i(x) w_i \right) = g(w(x))$$
holds for a.a. \( x \in A \). But since \( g \) is strictly convex, this means that \( \alpha(x) \in \{0, 1\}^M \) for a.a. \( x \in A \), which is a contradiction.

We conclude that
\[
\int_{B} \sum_{i=1}^{M} \alpha_i(x) g(w_i) > \int_{B} g(w(x)) \, dx + \varepsilon \lambda(B).
\]
Combining this estimate with (A.2) for the choice \( D = B^c \) and inserting both estimates into (A.1), we get
\[
\lim \inf \, R(w^n) = \sum_{i=1}^{M} \int_{B^c} \alpha_i(x) g(w_i) \, dx + \sum_{i=1}^{M} \int_{B} \alpha_i(x) g(w_i) \, dx
\]
\[
> \int_{B^c} g(w(x)) \, dx + \int_{B} g(w(x)) \, dx + \varepsilon \lambda(B)
\]
\[
> \int_{\Omega} g(w(x)) \, dx.
\]

This concludes the proof.

We note that the Lyapunov convexity theorem (see [21, 34]) asserts that such a sequence \((w^n)\) exists for all functions \( w \) as in Proposition 1.1.

### A.2 Proof of (2.7)

**Proof.** Let \( w \in F(\Omega) \) and \( \gamma > 0 \) be given. Then, we obtain
\[
R(w) - R_\gamma(w) = \int_{\Omega} d(w(x)) \, dx
\]
with
\[
d(u) := g(u) - \inf \left\{ g(y) + \frac{1}{2\gamma} \|u - y\|_2^2 \mid y \in \text{conv}\{\nu_1, \ldots, \nu_M\} \right\}.
\]
For \( u \in \text{conv}\{\nu_1, \ldots, \nu_M\} \), we consider \( d(u) \) and rewrite it as
\[
d(u) = \sup \left\{ g(u) - g(y) - \frac{1}{2\gamma} \|u - y\|_2^2 \mid y \in \text{conv}\{\nu_1, \ldots, \nu_M\} \right\}.
\]
Proposition 2.5 gives that \( g \) is Lipschitz continuous with constant \( L \), and we obtain
\[
d(u) \leq \sup \left\{ L \|u - y\|_2 - \frac{1}{2\gamma} \|u - y\|_2^2 \mid y \in \text{conv}\{\nu_1, \ldots, \nu_M\} \right\} \leq \frac{L^2 \gamma}{2},
\]
where the second inequality follows from the maximization of a parabola with negative curvature. Inserting this estimate into (A.3) yields (2.7).

### B Alternative Smoothing in the One-Dimensional Case

Let \( R \) be a relaxed multibang regularizer for discrete- and scalar-valued controls; that is, \( R(v) := \int_{\Omega} g(v(x)) \, dx \) for control functions \( v \in F(\Omega) \). Here, we consider the set of feasible controls for \( R \):
\[
F_{\Omega} = \{ v \in L^2(\Omega) \mid v(x) \in [\nu_1, \nu_M] \text{ for a.a. } x \in \Omega \}
\]
with scalars \( \nu_1 < \ldots < \nu_M \). We assume that \( g : [\nu_1, \nu_M] \to \mathbb{R} \) is a positive, continuous, montonously increasing, piecewise affine convex function with \( g(\nu_1) = 0 \) and that \( g(\nu_M) = B \in \mathbb{R} \).

It is straightforward to generalize the following ideas if the monotonicity assumption is dropped or \( g(\nu_1) \) is allowed to be nonzero; but this restriction simplifies the remainder significantly, and we believe that it also helps get a good intuition.
The Clarke subdifferential $\partial^c g$ of $g$ is

$$\partial^c g(u) = \begin{cases} 
\{L_1\} & \text{if } u = \nu_1, \\
\{L_i\} & \text{if } u \in (\nu_i, \nu_{i+1}) \text{ for some } i \in [M-1], \\
\{L_{M-1}\} & \text{if } u = \nu_M, \\
[L_{i-1},L_i] & \text{if } u = \nu_i \text{ for some } i \in [M-1]. 
\end{cases}$$

for positive slopes $0 < L_1 < \ldots < L_{M-1}$. We observe that $\partial^c g$ is almost everywhere single-valued. Thus, we may interpret $\partial^c g$ as an $L^\infty$-function, which we denote by $g'$ because for this $L^\infty$-function we still have $g(u) = \int_{\nu_1}^u g'(w) \, dw$ by the fundamental theorem of Lebesgue integral calculus. In other words, the function $g$ is absolutely continuous. For $v \in \mathcal{F}[\nu_1,\nu_2]$ we have $g(v(x)) \leq B$ for a.a. $x \in \Omega$, and thus $R(v) \leq B\lambda(\Omega)$. For $i \in [M]$, we define $g_i := g(\nu_i)$.

Now, we define differentiable convex underestimators of $g$. For $0 < \gamma < \min\{\nu_{i+1} - \nu_i \mid i \in [M-1]\}$, we define the $C^1 \cap W^{2,\infty}$-function $g_\gamma : [\nu_1,\nu_M] \to \mathbb{R}$

$$g_\gamma(u) := \int_{\nu_1}^u g'_\gamma(w) \, dw \text{ for } u \in [\nu_1,\nu_M],$$

where $g'_\gamma$ is defined as

$$g'_\gamma(w) := \begin{cases} 
L_1 & \text{if } w \in [\nu_1,\nu_2), \\
L_i - L_{i-1} & (s - \nu_i) \text{ if } w \in [\nu_i,\nu_i + \gamma) \text{ for some } i \in [M-1] \setminus \{1\}, \\
L_i & \text{if } w \in [\nu_i + \gamma,\nu_{i+1}) \text{ for some } i \in [M-1], \\
L_{M-1} & \text{if } w = \nu_M 
\end{cases}$$

for all $w \in [\nu_1,\nu_M]$. We show an example for $g$ with underestimators and their derivatives in Figure 6.

Following our notation, we define the smoothed regularizer as

$$R_\gamma(v) := \int_\Omega g_\gamma(v(x)) \, dx \quad \text{(B.1)}$$

for all $v \in \mathcal{F}[\nu_1,\nu_2]$. We summarize the properties of the relationship between $g_\gamma$ and $g$ in the proposition below.

**Proposition B.1.** Let $0 < \gamma < \min\{\nu_{i+1} - \nu_i \mid i \in [M-1]\}$. It holds that

$$\begin{align*}
\|g - g_\gamma\|_{C([0,T])} &= 0.5(L_{M-1} - L_1)\gamma, \\
\|g' - g'_\gamma\|_{L^1([0,T])} &= 0.5(L_{M-1} - L_1)\gamma, \\
\|g' - g'_\gamma\|_{L^p([0,T])} &= \left(\sum_{i=1}^{M-2} (L_{i+1} - L_i)^p \frac{p}{p+1} \gamma\right)^{\frac{1}{p}} \text{ for } p \in (1,\infty), \\
\|g' - g'_\gamma\|_{L^\infty([0,T])} &= \max\{L_{i+1} - L_i \mid i \in [M-2]\}.
\end{align*}$$

Figure 6: $g$ and $\partial^c g$ as well as their approximations $g_\gamma$ and $g'_\gamma$ for the choices $\gamma \in \{2^{-1}, 2^{-2}, 2^{-3}\}$. 
Proof. By construction of $g_\gamma$, we have $g(u) \geq g_\gamma(u)$ for all $u \in [\nu_1, \nu_M]$. We consider the difference $d_\gamma(u) := g(u) - g_\gamma(u)$ for $u$ in different intervals. For $i \in [M]$, let $d_i^\gamma := g(\nu_i) - g_\gamma(\nu_i)$.

Because $g(u) = g_\gamma(u)$ for all $u \in [\nu_1, \nu_2]$ it holds that $d_1^\gamma = d_2^\gamma = 0$. Let $i \in \{2, \ldots, M-1\}$. For $u \in [\nu_i, \nu_{i+1}]$ it holds that

$$d_i^\gamma(u) = d_i^\nu + \int_{\nu_i}^{\min\{u, \nu_i+\gamma\}} L_i - L_{i-1} - \left(\frac{L_i - L_{i-1}}{\gamma}\right) (w - \nu_i) \, dw$$

$$= d_i^\nu + (L_i - L_{i-1}) \min\{u - \nu_i, \gamma\} - \frac{L_i - L_{i-1}}{2\gamma} \min\{u - \nu_i, \gamma\}^2.$$

Thus, $d_i^\gamma$ is a continuous monotonously nondecreasing function, and we obtain $\sup_u d_i^\gamma(u) = d_i^\gamma(\nu_M)$. Using the iterative description of $d_i^\gamma$ derived above and the assumption $\gamma < \min\{\nu_{i+1} - \nu_i | i \in [M-1]\}$, we can compute

$$d_i^\gamma(\nu_M) = \sum_{i=1}^{M-2} (L_{i+1} - L_i)\gamma - \frac{L_{i+1} - L_i}{2\gamma} \gamma^2 = \frac{L_{M-1} - L_1}{2\gamma}.$$

This implies $\|g - g_\gamma\|_C = 0.5(L_{M-1} - L_1)\gamma$. Since $g' \geq g'_\gamma$ almost everywhere, it also holds that $\|g' - g'_\gamma\|_{L^1} = 0.5(L_{M-1} - L_1)\gamma$. Let $p \in (1, \infty)$. Then a reasoning similar to the above yields

$$\|g' - g'_\gamma\|_{L^p([\nu_1, \nu_M])} = \left( \sum_{i=1}^{M-2} (L_{i+1} - L_i)^p \frac{p}{p+1} \right)^{\frac{1}{p}}.$$

By construction $\|g' - g'_\gamma\|_{L^\infty([\nu_1, \nu_2])} = 0$ and $\|g' - g'_\gamma\|_{L^\infty([\nu_{i-1}, \nu_i])} = L_i - L_{i-1}$ for all $i \in \{2, M-1\}$, which yields the last claim.

We obtain the following corollary, which establishes the properties of the Moreau envelope from Proposition \ref{prop:smoothness} for $R_\gamma$ and $g_\gamma$ as well.

**Corollary B.2.**

1. For all $u \in \text{conv} V$ it holds that $g_\gamma(u) \uparrow g(u)$ for $\gamma \downarrow 0$.

2. For all $v \in \mathcal{F}([R])$, it holds that $R_\gamma(v) \uparrow R(v)$ for $\gamma \downarrow 0$.

3. For all $\gamma > 0$, the function $g_\gamma : \text{conv} V \to \mathbb{R}$ is convex and continuously differentiable with Lipschitz-continuous derivative.

4. Let $p \geq 2$. Then, $R_\gamma : L^p(\Omega, \mathbb{R}^n) \to \mathbb{R}$ is differentiable with derivative $R'_\gamma(v) = \int_\Omega g'_\gamma(v(x))^T d(x) \, dx$.

**Proof.** The claims follow along the lines of the proof of Proposition \ref{prop:smoothness}.

**Proposition B.3.** Let $(\gamma^n)_n \subset [0, \infty)$ satisfy $\gamma^n \downarrow 0$. Then, the sequence of functionals $F + R_{\gamma^n}$ on $\mathcal{F}([R]) \subset L^\infty(\Omega, \mathbb{R}^n)$ is $\Gamma$-convergent with limit $F + R$ with respect to weak-*-convergence in $L^\infty$.

**Proof.** The claim follows along the lines of the proof of Proposition \ref{prop:gamma-convergence}. The estimates on $g - g_\gamma$ can be taken directly from Proposition \ref{prop:gamma-convergence}.

**Theorem B.4.** Let $R$ be a relaxed multibang regularizer. Let the inputs of Algorithm [1] be given. Let $\nu_1 < \ldots < \nu_M$. Then, the assertions of Theorem \ref{thm:smoothness} hold true if the smoothing of $R$ is performed as defined in (B.1) instead of using the Moreau envelope.

**Proof.** This follows along the lines of the proof of Theorem \ref{thm:smoothness} with the considerations above to replace the estimates on the Moreau envelope.
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