The geometry of Riemannian curvature radii

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Abstract

We study the geometric structures associated with curvature radii of curves with values on a Riemannian manifold \((M, g)\). We show the existence of sub-Riemannian manifolds naturally associated with the curvature radii and we investigate their properties. In the particular case of surfaces these sub-Riemannian structures are of Engel type. The main character of our construction is a pair of global vector fields \(f_1, f_2\), which encodes intrinsic information on the geometry of \((M, g)\).

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1 Introduction

1.1 From contact elements to curvature radii

The space of contact elements, first introduced by S. Lie (see Gei08 pages 6-11, or Mon02 pages 78-80) as a geometric tool for studying differential equations, marks the birth of contact geometry. Let us recall that given a 2-dimensional
Riemannian manifold \((M, g)\), the space of (oriented) contact elements of \(M\) is the unit tangent bundle \(UM\), i.e. the set of couples \((x, v)\) where \(x \in M\), \(v \in T_x M\) and \(|v| = 1\). Any regular \(M\)-valued curve can be lifted to a \(UM\)-valued one through the maps

\[
\ell_{±} : \gamma \mapsto \left(\gamma, \pm \frac{\dot{\gamma}}{|\dot{\gamma}|}\right).
\]  

(1.1)

There exists a contact distribution \(\Delta\) over \(UM\) naturally associated to the lifts \(\ell_{±}\). At a point \((x, v) \in UM\) such distribution is defined as

\[
\Delta_{(x,v)} = \langle \{v\} \rangle \oplus T_v(U_x M),
\]

where we have denoted \(U_x M = \{v \in T_x M : |v| = 1\}\). This distribution characterizes the images of \((1.1)\), in the sense that a curve \((\gamma, v) : [0, 1] \to UM\) is in the image of one of such lifts if and only if it is tangent to \(\Delta\) and \(\gamma\) is a regular curve. In this paper we study a second order generalization of the space of contact elements, in particular, given a Riemannian manifold \((M, g)\), we study

\[
\ell_{±} : \gamma \mapsto \left(\gamma, \pm \frac{\dot{\gamma}}{|\dot{\gamma}|}, R_g(\gamma)\right),
\]  

(1.2)

where \(R_g(\gamma)\) is the radius of curvature of \(\gamma\) (the definition is recalled below \((1.4)\)). Such second order construction has affinities with Cartan’s prolongation of contact structures (see 6.3 of [Mon02]). Recall that, given a Riemannian manifold \((M, g)\) (throughout the whole paper we assume \(\dim(M) > 1\)), the geodesic curvature of a regular curve \(\gamma : [0, T] \to M\) is defined as

\[
\kappa_g(\gamma) := \frac{|\pi_{\dot{\gamma}}(D_t \dot{\gamma})|}{|\dot{\gamma}|^2},
\]  

(1.3)

where \(D_t\) denotes the covariant derivative along \(\gamma\), and \(\pi_{\dot{\gamma}} : T_{\gamma(t)} M \to T_{\dot{\gamma}(t)} M\) denotes the orthogonal projection to \(\{\dot{\gamma}(t)\}^\perp\). If the geodesic curvature is never vanishing, we can define the radius of curvature of \(\gamma\), computed with respect to \(g\), as

\[
R_g(\gamma) := \frac{1}{\kappa_g(\gamma)} \frac{\pi_{\dot{\gamma}}(D_t \dot{\gamma})}{|\pi_{\dot{\gamma}}(D_t \dot{\gamma})|}.
\]  

(1.4)

With the sole purpose of simplifying the exposition and the expression of certain equations, we study the following modified versions of \((1.2)\)

\[
\ell_{±} : \gamma \mapsto \left(\gamma, \pm \frac{1}{|\gamma|} R_g(\gamma), \frac{|R_g(\gamma)|}{|\gamma|} \dot{\gamma}, R_g(\gamma)\right).
\]  

(1.5)

We define the manifold of curvature radii of \((M, g)\), denoted with \(\mathcal{R}(M, g)\), as the space of triples \((x, V, R)\) such that \(x \in M\), \(R \in T_x \setminus \{0\}\) and \(V \in \{R\}^\perp\), \(|V| = |R|\). The map \((1.5)\) lifts regular \(M\)-valued curves with never vanishing geodesic curvature to \(\mathcal{R}(M, g)\)-valued ones. The first central result of this work is the following theorem.
Theorem 1.1. Let \((M, g)\) be a smooth \(n\)-dimensional Riemannian manifold, and let \(\mathcal{R}(M, g)\) be the corresponding manifold of curvature radii. There exists a smooth, rank-\(n\) distribution, \(\mathcal{D}(M, g)\), over \(\mathcal{R}(M, g)\) with the property that a smooth curve \((\gamma, V, R) : [0, 1] \to \mathcal{R}(M, g)\) is in the image of one of the lifts (1.3) if and only if it is tangent to \(\mathcal{D}(M, g)\) and \(\gamma\) is a regular curve. A local basis for \(\mathcal{D}(M, g)\) is given by \(n\) local vector fields \(\{f_1, \ldots, f_n\}\), which can be characterized in terms of the ODEs satisfied by their integral curves:

\[
\begin{align*}
\dot{x} &= V, \\
D_t R &= -V, \\
\dot{R} &= R, \\
\dot{V} &= V,
\end{align*}
\]  

(1.6)

\[j = 3, \ldots, n,\] where \(\{b_3(x,V,R), \ldots, b_n(x,V,R)\}\) is a norm-\(|R|\) local orthogonal basis of \(\{R,V\}^\perp\). The distribution \(\mathcal{D}(M, g)\) is bracket generating of step 3, it holds:

\[
\begin{align*}
\mathcal{D}(M, g) &= \langle \{f_1, \ldots, f_n\} \rangle, \\
\mathcal{D}^2(M, g) &= \mathcal{D}(M, g) \oplus \langle \{f_{jk}\}_{k=2}^n \rangle, \\
\mathcal{D}^3(M, g) &= \mathcal{D}^2(M, g) \oplus \langle \{f_{jkl}\}_{k=2}^n \rangle = T\mathcal{R}(M, g).
\end{align*}
\]  

(1.7)

The fields \(f_1, f_2\) described in equation (1.6) are closely related to the geometry of \((M, g)\). Indeed, as shown in Section 4, many geometric features of the original Riemannian manifold can be recovered considering their Lie brackets. The first bracket \([f_1, f_2]\) gives us back the geodesic flow of \((M, g)\) (see Proposition 1.1); in a way these fields factorize the geodesic flow. Moreover we can read the Riemann curvature tensor in the structure constant of the frame (1.6) (see Proposition 4.3).

1.2 Metric structures

The knowledge of the curvature radii of curves of a Riemannian manifold \((M, g)\), is enough to characterize the metric \(g\) up to a homothetic transformation, indeed, as shown in Section 4, two Riemannian manifolds having the same curvature radii are homothetic (see Definition 4.2 for the precise statement).

Theorem 1.2. Two Riemannian manifolds \((M, g)\) and \((N, \eta)\) have the same curvature radii if and only if they are homothetic.

It is then natural to endow the distribution \(\mathcal{D}(M, g)\) with a metric which is invariant under the action of the homothety group of \((M, g)\). In Section 3 we show the existence of a family of metrics \(\eta_{a,b}\) on \(\mathcal{D}(M, g)\), parametrized by two real numbers \(a\) and \(b\), having this invariance property. The triple \((\mathcal{R}(M, g), \mathcal{D}(M, g), \eta_{a,b})\) is a sub-Riemannian manifold \([ABB21, Mon02]\), which we denote with \(\mathcal{R}_{a,b}(M, g)\). For any \((\gamma, R, V)\) in the image of the lift (1.5) the metric \(\eta_{a,b}\) satisfies the following equation

\[
\left[ \frac{d}{dt} (\gamma, V, R) \right]_{\eta_{a,b}}^2 = a^2 |\gamma|^2 + b^2 |D_t R|^2.
\]

(1.8)
The central result regarding these metrics is stated in the following theorem.

**Theorem 1.3.** Let \((M, g)\) be a Riemannian manifold, let \(a, b \in \mathbb{R}, b > 0\), and let \(\mathcal{R}_{a,b}(M, g)\) be the corresponding sub-Riemannian manifold of curvature radii. The following map is a group isomorphism

\[
\text{Homothety}(M, g) \rightarrow \text{Isometry}(\mathcal{R}_{a,b}(M, g))
\]

\[
\varphi \mapsto (\varphi \ast \varphi)_{|\mathcal{R}(M, g)}.
\]

(1.9)

In particular, if \((N, \eta)\) is another Riemannian manifold, \(\mathcal{R}_{a,b}(M, \eta)\) is isometric to \(\mathcal{R}_{a,b}(M, g)\) if and only \((N, \eta)\) and \((M, g)\) have the same curvature radii.

In Section 5 we show that the sub-Riemannian manifolds \(\mathcal{R}_{a,b}(\mathbb{R}^2, g_e)\), where \(g_e\) is the standard Euclidean metric, are all isometric to left-invariant sub-Riemannian structures on the group orientation preserving homothetic transformations of \((\mathbb{R}^2, g_e)\), and we give a characterization of their geodesics. These geometries are related to the left-invariant sub-Riemannian structure on the group of rigid motions of \(\mathbb{R}^2\) ([Ard+21], [MS10], [Sac10], [Sac11]).

1.3 Structure of the paper

In Section 2 we describe the sub-Riemannian manifold of curvature radii in the 2-dimensional case. In Section 3 we generalize the constructions of Section 2 to an arbitrary Riemannian manifold \((M, g)\) and we prove Theorems 1.1, 1.2, 1.3.

In Section 4 we show that taking Lie brackets of the fields \(f_1, f_2\), mentioned in the abstract, we can reconstruct the geodesic flow of the original Riemannian manifold and its Riemann curvature tensor. Finally in Section 5 we study the manifold of curvature radii of \(\mathbb{R}^2\) endowed with an homothetic invariant metric.

2 Curvature Radii on surfaces

Let \((M, g)\) be a 2-dimensional Riemannian manifold, which for simplicity we assume to be oriented. Let \(\gamma : [0, 1] \rightarrow M\) be an arc-length parametrized curve, then its curvature can be computed as

\[
k_g(\gamma) = |D_t \gamma|,
\]

(2.1)

and, provided that \(k_g(\gamma)\) is never vanishing, its radius of curvature is

\[
R_g(\gamma) = \frac{1}{k_g(\gamma)^2} D_t \gamma.
\]

(2.2)

For every \(t \in [0, 1]\), the curvature radius of \(\gamma\) at the point \(\gamma(t)\) is a non-zero tangent vector, for this reason we define the manifold of curvature radii of \((M, g)\) as \(\mathcal{R}(M) = TM \setminus s_o\), where \(s_o\) is the zero-section of \(TM\). Every regular curve with non-vanishing curvature can be lifted to a \(\mathcal{R}(M)\)-valued curve through the radius of curvature map

\[
\gamma \mapsto (\gamma, R_g(\gamma)).
\]

(2.3)
We would like to find a distribution $\mathcal{D}(M,g)$ over $\mathcal{R}(M)$ characterizing the image of such lift, in the sense that a curve $(\gamma, R) : [0,1] \to \mathcal{R}(M)$ is in the image of (2.3) if and only if it is tangent to $\mathcal{D}(M,g)$ and $\gamma$ is regular. To build such a distribution at a point $(x_0, R_0) \in \mathcal{R}(M)$ we collect all the velocities of all radii of curvature going through this point, and we take the vector space generated by them

$$\mathcal{D}(M,g)(x_0, R_0) := \left\{ \left( \frac{d}{dt}(\gamma, R_g(\gamma))(0) : \gamma(0) = x_0, R_g(\gamma)(0) = R_0 \right) \right\}. \quad (2.4)$$

Since $M$ is oriented we have a complex structure

$$\iota_s : TM \setminus s_o \to TM \setminus s_o$$

$$(x, R) \mapsto (x, R^{\iota s})$$

where $R^{\iota s}$ is the unique vector orthogonal to $R$, positively oriented with it, satisfying $|R|_g = |R^{\iota s}|_g$.

**Proposition 2.1.** The collection of vector spaces defined in (2.4) is a smooth Engel distribution over $\mathcal{R}(M)$ (for basic facts on Engel distributions see for instance [Bry+91], chapter 2). Moreover a curve $(\gamma, R) : [0,1] \to \mathcal{R}(M)$ is in the image of the lift (2.3) if and only if it is tangent to $\mathcal{D}(M,g)$ and $\gamma$ is a regular curve. A basis for $\mathcal{D}(M,g)$ is given by two vector fields $f_1^g, f_2^g$, which are characterized in terms of the ODEs satisfied by their integral curves as

$$f_1^g : \begin{cases} \dot{x} = R^{\iota s}, \\ D_t R = -R^{\iota s}, \end{cases} \quad f_2^g : \begin{cases} \dot{x} = 0, \\ \dot{R} = R, \end{cases} \quad (2.6)$$

where $D_t R$ denotes the covariant derivative of $R(t)$ along the curve $x(t)$.

**Proof.** It is a special case of Theorem 1.41 which is proved in Section 3. \[\square\]

What is the geometric significance of the fields $f_1^g, f_2^g$? It is quite explicit that the integral curves of $f_2^g$ are dilatation of $R$ in a fixed fiber, i.e. curves of the kind $t \mapsto (x_0, e^t R_0)$, whereas if $(\gamma, R) : [0,1] \to \mathcal{R}(M)$ is an integral curve of $f_1^g$, as a consequence of Proposition 2.1 $R$ is the radius of curvature of $\gamma$, and hence $\kappa_g(\gamma) = 1/|R|_g$. On the other hand according to (2.6) $D_t R = -R^{\iota s}$, therefore

$$\frac{d}{dt} |R|^2_g = -2(R, R^{\iota s})_g = 0,$$

hence $\kappa_g(\gamma)$ is constant. Thus the integral curves of $f_1^g$ are exactly lifts of curves with constant geodesic curvature. Homothetic transformations preserve the curvature radius map. It is then natural to endow $\mathcal{D}(M,g)$ with a metric which is invariant under the action of the homothety group of $(M,g)$. For every $a, b \in \mathbb{R}, b \neq 0$, we define a metric $\eta_{a,b}^g$ on $\mathcal{D}(M,g)$ by declaring the fields

$$f_1^{g,a,b} := \frac{1}{\sqrt{a^2 + b^2}} f_1^g,$$

$$f_2^{g,a,b} := \frac{1}{b} f_2^g,$$  \quad (2.7)
an orthonormal frame. A simple computation, which is a particular case of (3.47), shows that if \((\gamma, R)\) is an admissible curve, then the metric \(\eta_{a,b}^s\) satisfies the following equation

\[
\left| \frac{d}{dt}(\gamma, R)\right|_{\eta_{a,b}^s}^2 = a^2 \frac{|\gamma|^2}{|R|^2} + b^2 \frac{|D_t R|^2}{|R|^2}.
\]

The homogeneity of the right hand side shows the homothetic invariant nature of the metric \(\eta_{a,b}^s\). The triple \((\mathcal{R}(M), \mathcal{D}(M, g), \eta_{a,b}^s)\) defines a sub-Riemannian manifold, which we denote with \(\mathcal{R}_{a,b}(M, g)\).

3 The sub-Riemannian manifold of curvature radii

We would like to extend the construction of \(\mathcal{R}_{a,b}(M, g)\), presented for surfaces in Section 2, to an arbitrary Riemannian manifold. One of the advantages of working with surfaces is that, given a regular curve \(\gamma\) with never vanishing geodesic curvature, the direction of \(\dot{\gamma}\) is uniquely determined by the radius of curvature. This is no more true when \(\dim(M) \geq 3\), and we need to keep track of the velocity’s direction in some way. For this reason we cannot define the manifold of curvature radii as \(TM \setminus s_o\) anymore. Instead we define it as a subset of \(TM \oplus TM\).

Definition 3.1. Let \((M, g)\) be a Riemannian manifold. The manifold of curvature radii of \((M, g)\), denoted by \(\mathcal{R}(M, g)\), is defined by

\[
\mathcal{R}(M, g) := \{(x, V, R) \in TM \oplus TM : \langle R, V \rangle_g = 0, |R|_g = |V|_g > 0\}. \quad (3.1)
\]

From now on, when the metric \(g\) we are referring to is clear from the context, we drop the \(g\) subscript, for instance we denote \(|\cdot| = |\cdot|_g, \perp_g\) and so on. One can check that, if \(M\) is an \(n\)-dimensional manifold, then \(\mathcal{R}(M, g)\) is a smooth embedded sub-manifold of \(TM \oplus TM\) of dimension \(3n - 2\). Moreover \(\mathcal{R}(M, g)\) has the structure of a \(S^{n-2}\) bundle over \(TM \setminus s_o\), where the fiber at \((x, R) \in TM \setminus s_o\) is the sphere of radius \(|R|\) contained in the plane \(R^2\),

\[
\mathcal{R}(M, g)_{(x, R)} = \{V \in T_xM : V \perp R, |V| = |R|\} \simeq S^{n-2}. \quad (3.2)
\]

As expected, if \(n = 2\) these spheres have dimension 0, and \(\mathcal{R}(M, g)\) reduces to a 0-dimensional fibration over \(TM \setminus s_o\). If \(M\) is a surface, oriented for simplicity, we have

\[
\mathcal{R}(M, g) \simeq TM \setminus s_o \cup TM \setminus s_o.
\]

In general \(\mathcal{R}(M, g)\) has also the structure of a fiber bundle over \(M\), whose fibers are

\[
\mathcal{R}(M, g)_x = \{(V, R) \in T_xM \oplus T_xM : \langle R, V \rangle = 0, |R| = |V| > 0\}.
\]

There exist two lifts which allows us to map regular curves \(\gamma : [0, 1] \to M\) with never vanishing geodesic curvature, to \(\mathcal{R}(M, g)\)-valued curves

\[
c_\gamma : \gamma \mapsto \left(\gamma, \pm |R_g(\gamma)| \frac{\dot{\gamma}}{|\dot{\gamma}|}, R_g(\gamma)\right), \quad (3.3)
\]
where $R_q$ is the map defined in (1.3). We would like to find a distribution characterizing the image of the lift (3.3), i.e. a distribution $\mathcal{D}(M,g)$ such that $(\gamma, V, R) : [0,1] \rightarrow \mathcal{R}(M,g)$ is the lift of some curve $\gamma$ if and only if it is tangent to $\mathcal{D}(M,g)$, and $\gamma$ is regular. To construct this distribution at $q_0 := (x_0, V_0, R_0) \in \mathcal{R}(M,g)$ we collect the velocities of all such curves going through this point, and we take the vector space generated by them:

$$\mathcal{D}(M,g)_{q_0} := \left\{ \left\{ \frac{d}{dt} (\gamma, R, V) : (\gamma, V, R)(0) = q_0, (\gamma, V, R) \text{ as in (3.3)} \right\} \right\}. \quad (3.4)$$

Before moving to the proof of Theorem 1.1, let us fix the notation

$$f_{i_1i_2\ldots i_k} = [f_{i_1}, [f_{i_2}, \ldots f_{i_k}]] \ldots,$$

to indicate the iterated brackets of the fields $f_1, \ldots, f_n$. Moreover we will often make use of the abbreviation

$$X \partial_x = \sum_{i=1}^n X^i \partial_{x_i}. \quad (3.5)$$

where $(x^1, \ldots, x^n)$ are local coordinates and $X \in \mathbb{R}^n$ is the vector $X = (X^1, \ldots, X^n)$.

**Proof.** (Theorem 1.1) The vector fields in (1.6) are, by definition, local sections of $T(TM \oplus TM)$. We need to show that they are actually tangent to $\mathcal{R}(M,g) \subset TM \oplus TM$. Let $(x_0, V_0, R_0) \in \mathcal{R}(M,g)$ and let $(x, V, R)(t)$ be an integral curve of $f_i$ with initial point $(x_0, V_0, R_0)$. We have to show that $(V(t), R(t)) = 0$ and that $|V(t)| = |R(t)|$ for each $t$ such that the flow of $f_i$ is defined. Let us start with $f_1$, any of its integral curves satisfies

$$\begin{align*}
\dot{x} &= V, \\
D_1 R &= -V, \\
D_1 V &= R.
\end{align*}$$

Therefore we have

$$\frac{d}{dt} (R, V) = |R|^2 - |V|^2, \quad \frac{d}{dt} |R|^2 - |V|^2 = -4(R, V),$$

but $|R(0)|^2 - |V(0)|^2 = 0 = (R(0), V(0))$, because $(x, V, R)(0) \in \mathcal{R}(M,g)$, then from uniqueness of ODEs solutions we obtain $|V(t)| - |R(t)| = (V(t), R(t)) \equiv 0$. Let $(x, R, V)(t)$ be an integral curve of $f_2$ with initial point in $\mathcal{R}(M,g)$, then from (1.6) we compute

$$\frac{d}{dt} \left( |R|^2 - |V|^2 \right) = 2 \left( |R|^2 - |V|^2 \right), \quad \frac{d}{dt} (R, V) = 2(R, V).$$

Since $(\gamma, V, R)(0) \in \mathcal{R}(M,g)$, we deduce $|V(t)| - |R(t)| = (V(t), R(t)) \equiv 0$. Let $(\gamma, R, V)(t)$ be an integral curve of $f_j$, $j = 3, \ldots, n$, then from (1.6) we have

$$\frac{d}{dt} \left( |R|^2 - |V|^2 \right) = 0, \quad \frac{d}{dt} (R, V) = 0.$$
hence \( f_j \) is tangent to \( \mathcal{R}(M, g) \).

Now we claim that a curve \((\gamma, V, R) : [0, 1] \rightarrow \mathcal{R}(M, g)\) is in the image of one of the lifts \((3.3)\) if and only if it is tangent to \(\{(f_1, \ldots, f_n)\}\), and \(\gamma\) is regular. Notice that, by definition of \(\mathcal{D}(M, g)\), this would imply \(\mathcal{D}(M, g) = \{(f_1, \ldots, f_n)\}\). To prove our claim we have to show that \((\gamma, V, R)\) is in the image of one of the lifts \((3.3)\) if and only if there exist \(n\) smooth functions, \(u_1, \ldots, u_n : [0, 1] \rightarrow \mathbb{R}\), with \(u_1 \neq 0\), such that

\[
\frac{d}{dt}(\gamma, R, V) = u_1 f_1 + \cdots + u_n f_n,
\]

or in other words, according to \((1.6)\)

\[
\begin{aligned}
\dot{\gamma} &= u_1 V, \\
D_t R &= -u_1 V + u_2 R + u_3 b_3 + \cdots + u_n b_n, \\
D_t V &= u_1 R + u_2 V.
\end{aligned}
\tag{3.6}
\]

Assume that \(\sigma : (\gamma, V, R) : [0, 1] \rightarrow \mathcal{R}(M, g)\) is in the image of \((3.3)\), then, by definition \(R = R_\sigma(\gamma)\) and \(\dot{\gamma} = u_1 V\), for some never vanishing smooth function \(u_1 : [0, 1] \rightarrow \mathbb{R}\). Since the condition \(\dot{\sigma} \in \{(f_1, \ldots, f_n)\}\) is independent of the parametrization, without loss of generality we may assume \(|\dot{\gamma}| = 1\), i.e. \(u_1 = 1/|R|\).

Since \(\gamma\) is arc-length parametrized, as a consequence of formulae \((1.4)\) and \((1.3)\), we have

\[
D_t \dot{\gamma} = \frac{1}{|R|^2} R.
\]

On the other hand, since \(\dot{\gamma} = 1/|R| V\), we have

\[
D_t \dot{\gamma} = D_t V = \frac{d}{dt} \left( \frac{1}{|R|} \right) V + \frac{1}{|R|} D_t V,
\]

therefore

\[
D_t V = \frac{1}{|R|} R - \frac{|R|}{|R|} \frac{d}{dt} \left( \frac{1}{|R|} \right) V = u_1 R + u_2 V,
\]

where we have denoted \(u_2 := -|R| \frac{d}{dt} \left( \frac{1}{|R|} \right) = \frac{d}{dt} \log |R|\). It remains to compute the covariant derivative of \(R\) along \(\gamma\). Notice that \(\{V(t), R(t), b_3(t), \ldots, b_n(t)\}\) is a basis for \(T_{\gamma(t)} M\) for every \(t \in [0, 1]\), therefore there exist \(\lambda_1, \ldots, \lambda_n : [0, 1] \rightarrow \mathbb{R}\) smooth functions such that

\[
D_t R = \lambda_1 V + \lambda_2 R + \lambda_3 b_3 + \cdots + \lambda_n b_n.
\]

To prove that \((3.6)\) holds we just have to show that \(\lambda_1 = -u_1, \lambda_2 = u_2\). We begin with \(\lambda_1:\)

\[
\lambda_1 = (D_t R, \frac{V}{|V|^2}) = -\frac{1}{|R|^2} (R, D_t V) = -\frac{1}{|R|^2} (R, u_1 R) = -u_1.
\]

Concerning the second coefficient \(\lambda_2\) we have

\[
\lambda_2 = (D_t R, \frac{R}{|R|^2}) = \frac{1}{2|R|^2} \frac{d}{dt} |R|^2 = \frac{1}{2} \frac{d}{dt} \log |R|^2 = \frac{d}{dt} \log |R| = u_2.
\]
Conversely let $\sigma = (\gamma, V, R)$ be a curve satisfying (3.6), with $u_1 \neq 0$. We want to show that $R = R_g(\gamma)$, i.e. that $R$ is the radius of curvature of $\gamma$. Exploiting (3.6) we compute
\begin{align*}
\pi_{\gamma} D_t \dot{\gamma} = \pi_{\gamma} (u_1 D_t V + \dot{u}_1 V) = u_1^2 R,
\end{align*}
then, by definition (1.4), the radius of curvature of $\gamma$ satisfies
\begin{align*}
R_g(\gamma) &= \frac{\pi_{\gamma} D_t \dot{\gamma}}{|\pi_{\gamma} D_t \dot{\gamma}|} = \frac{R}{\kappa_g(\gamma) |\dot{\gamma}|}.
\end{align*}
On the other hand combining (1.3) with (3.8), and considering that $|R| = |V|$, we deduce
\begin{align*}
\kappa_g(\gamma) &= \frac{|\pi_{\gamma} D_t \dot{\gamma}|}{|\dot{\gamma}|^2} = \frac{u_1^2 |R|}{u_1^2 |V|^2} = \frac{1}{|R|}.
\end{align*}
Substituting (3.10) into (3.9) we obtain $R_g(\gamma) = R$, as requested.

We now show that $D(M, g)$ is an equiregular bracket generating distribution of step 3. To do this we need to be more precise about the definition of the local fields $f_3, \ldots, f_n$ in equation (1.6), in particular given $(x, V, R) \in R(M, g)$ we have to make a choice of a local basis for $\{R, V\}^\perp$. Let $\mathcal{U} \subset M$ be an open subset and let $E_1, \ldots, E_n$ be a local orthonormal frame for $g$ on $\mathcal{U}$. For every $i \neq j$ we define the following two-form $\omega_{ij} \in \Omega^2(\mathcal{U})$:
\begin{align*}
\omega_{ij}(R, V) := \det \begin{pmatrix} (R, E_i)_g & (V, E_i)_g \\ (R, E_j)_g & (V, E_j)_g \end{pmatrix}
\end{align*}
and the following open subset of $TM \oplus TM$
\begin{align*}
\mathcal{U}_{ij} := \{(x, V, R) : \omega_{ij}(R, V) \neq 0\}.
\end{align*}
By construction, the sets $\{V_{ij} := \mathcal{U}_{ij} \cap R(M, g)\}_{i \neq j}$, constitute an open cover for $R(M, g) \cap T\mathcal{U} \oplus T\mathcal{U}$. Moreover the vectors
\begin{align*}
\{R, V, E_3(x), \ldots, \dot{E}_i(x), \ldots, \dot{E}_j(x), \ldots, E_n(x)\},
\end{align*}
are linearly independent for every $(x, V, R) \in V_{ij}$. We define
\begin{align*}
\{e_3(x, V, R), \ldots, e_n(x, V, R)\}
\end{align*}
to be the norm-$|R|$ basis of $\{R, V\}^\perp$ obtained by applying the Gram-Schmidt algorithm to the vectors
\begin{align*}
\{R, V, E_3(x), \ldots, \dot{E}_i(x), \ldots, \dot{E}_j(x), \ldots, E_n(x)\},
\end{align*}
and, on $V_{ij}$, we set
\begin{align*}
f_k := e_k \partial_R = \sum_{\mu=1}^n e_k^\mu \partial_{R^\mu}, \ k = 3, \ldots, n.
\end{align*}
Defined in this way, the local fields $e_k$ satisfy two useful properties:

$$e_k(x, \lambda R, \lambda V) = \lambda e_k(x, V, R),$$  \hspace{1cm} (3.12)

for every $\lambda > 0$ and

$$e_k(x, \cos \theta R - \sin \theta V, \sin \theta R + \cos \theta V) = e_k(x, V, R),$$  \hspace{1cm} (3.13)

for every $\theta \in [0, 2\pi]$. Let $(x^\mu)$ be local coordinates on $M$ and let $(x^\mu, R^\mu, V^\mu)$ be the local coordinates induced by $(x^\mu)$ on $TM \oplus TM$. Let $\Gamma_{\alpha\beta}^\gamma$ be the corresponding Christoffel symbols of the metric $g$. Given $X, Y \in \mathbb{R}^n$, we denote with $\Gamma(X, Y)$ the row vector

$$\Gamma(X, Y) = \sum_{\alpha, \beta = 1}^{n} (\Gamma_{\alpha\beta}^1 X^\alpha Y^\beta, \ldots, \Gamma_{\alpha\beta}^n X^\alpha Y^\beta),$$  \hspace{1cm} (3.14)

then, making use of the notation [3.5], the vector fields $f_1, f_2$ read

$$f_1 = V \partial_x - (V + \Gamma(V, R)) \partial_R + (R - \Gamma(V, V)) \partial_V,$$

$$f_2 = R \partial_R + V \partial_V,$$

and their commutator reads

$$[f_2, f_1] = [R \partial_R + V \partial_V, V \partial_x - (V + \Gamma(V, R)) \partial_R + (R - \Gamma(V, V)) \partial_V]$$

$$= - \Gamma(V, R) \partial_R + R \partial_V + V \partial_x - (V + \Gamma(V, R)) \partial_R - 2 \Gamma(V, V) \partial_V$$

$$+ (V + \Gamma(V, R)) \partial_R - (R - \Gamma(V, V)) \partial_V$$

$$= V \partial_x - \Gamma(V, V) \partial_R - \Gamma(V, V) \partial_V.$$

We define the vector field $X_{12}$ as

$$X_{12} := f_1 - f_2 = - V \partial_R + R \partial_V.$$  \hspace{1cm} (3.17)

We define

$$f_{121} := [f_{12}, f_1] = [X_{12}, f_1],$$  \hspace{1cm} (3.18)

and we notice that this vector field satisfies

$$\pi_*^M f_{121} = \pi_*^M [X_{12}, f_1] = [R \partial_V, V \partial_x] = R \partial_x.$$  \hspace{1cm} (3.19)

Finally, to span the whole tangent space, we need the following vector fields from the third layer

$$f_{1k1} = [[f_1, f_k], f_1], \ k = 3, \ldots, n.$$

We start by calculating the field $f_{1k} := [f_1, f_k]$. Let $e_k(t) = e_k(x(t), R(t), V(t))$, where $(x(t), R(t), V(t))$ is an integral curve of $f_1$. Then, according to [14,10], we have

$$\langle D_t e_k, R \rangle = - \langle e_k, D_t R \rangle = \langle e_k, V \rangle = 0,$$

$$\langle D_t e_k, V \rangle = - \langle e_k, D_t V \rangle = - \langle e_k, R \rangle = 0,$$

$$F_k^i = \frac{1}{|R|^2} \langle D_t e_k, e_i \rangle.$$
Therefore
\[ f_1(e_k) = \sum_{i=3}^{n} F^i_k e_i - \Gamma(V, e_k), \]  
(3.20)

and hence we have
\[ f_1 = [f_1, f_k] = [V \partial_x - (V + \Gamma(V, R)) \partial_R + (R - \Gamma(V, V)) \partial_V, e_k \partial_V] \]
\[ = f_1(e_k) \partial_R + \Gamma(V, e_k) \partial_R - e_k \partial_V \]
\[ = \left( \sum_{i=3}^{n} F^i_k e_i - \Gamma(V, e_k) \right) \partial_R + \Gamma(V, e_k) \partial_R - e_k \partial_V \]  
(3.21)

Observe that the field \( f_{1k1} \) satisfies
\[ \pi^* M f_{1k1} = \pi^* M [f_{1k}, f_1] = \pi^* M \left[ \sum_{i=3}^{n} F^i_k e_i \partial_R - e_k \partial_V, V \partial_x \right] = -e_k \partial_x. \]  
(3.22)

Now we claim that the following \( 3n - 2 \) vector fields
\[ f_1, \ldots, f_n, f_{12}, f_{13}, \ldots, f_{1n}, f_{121}, f_{131}, \ldots, f_{1n1}, \]
are linearly independent. Notice that, since \( X_{12} = f_1 + f_{12} \), we may equivalently claim that
\[ f_1, \ldots, f_n, X_{12}, f_{13}, \ldots, f_{1n}, f_{121}, f_{131}, \ldots, f_{1n1}, \]  
(3.23)

are linearly independent. Consider a vanishing linear combination, with coefficients in \( C^\infty(\mathcal{R}(M, g)) \), of (3.23):
\[ \sum_{k=1}^{n} a_k f_k + a_{12} X_{12} + \sum_{k=3}^{n} a_{1k} f_{1k} + a_{121} f_{121} + \sum_{k=3}^{n} a_{1k1} f_{1k1} = 0. \]  
(3.24)

If we apply \( \pi^* M \) to both sides of (3.24), since according to (1.6), (3.17), (3.21), the fields \( f_2, \ldots, f_n, X_{12}, f_{13}, \ldots, f_{1n} \) have \( x \)-component equal to zero, we are left with
\[ \pi^* M \left( a_1 f_1 + a_{121} f_{121} + \sum_{k=3}^{n} a_{1k1} f_{1k1} \right) = 0, \]
which thanks to (3.15), (3.19) and (3.22) translates to
\[ a_1 V + a_{121} R - \sum_{k=3}^{n} a_{1k1} e_k = 0, \]  
(3.25)

therefore \( a_1 = a_{121} = a_{131} = \cdots = a_{1n1} = 0 \) and equation (3.24) becomes
\[ \sum_{k=2}^{n} a_k f_k + a_{12} X_{12} + \sum_{k=3}^{n} a_{1k} f_{1k} = 0. \]  
(3.26)
If we compute the V-component of both sides of (3.20), according to (3.15), (3.17), (3.21) we get

\[ a_2 V + a_{12} R - \sum_{k=3}^{n} a_{1k} e_k = 0, \]

thus \( a_2 = a_{12} = a_{123} = \cdots = a_{12n} = 0 \), and equation (3.20) reads

\[ \sum_{k=3}^{n} a_k f_k = 0, \] (3.27)

but \( f_3, \ldots, f_n \) are linearly independent by definition, therefore also \( a_3 = \cdots = a_n = 0 \). We have just proved that \( \mathcal{D}^3(M, g) = T\mathcal{R}(M, g) \). It remains to show that \( \text{rank} \mathcal{D}^2(M, g) < \text{rank} T\mathcal{R}(M, g) = 3n - 2 \). Notice that

\[ \mathcal{D}^2(M, g) = \mathcal{D}(M, g) + \langle \{[f_1, f_j]\}_{j=2}^{n} \rangle + \langle \{[f_i, f_j]\}_{i,j=2}^{n} \rangle. \]

We claim that the distribution \( \langle \{f_2, \ldots, f_n\} \rangle \) is integrable. If that is the case then \( \{[f_i, f_j]\}_{i,j=2}^{n} \subset \mathcal{D}(M, g) \) and as a consequence

\[ \mathcal{D}^2(M, g) = \mathcal{D}(M, g) + \{[f_1, f_j]\}_{j=2}^{n}, \] (3.28)

therefore, since \( f_1, \ldots, f_n, f_{12}, \ldots, f_{1n} \) are linearly independent, we deduce

\[ \text{rank} \mathcal{D}^2(M, g) = \text{rank} \mathcal{D}(M, g) + \text{rank} \langle \{f_1, f_j\}_{j=2}^{n} \rangle = 2n - 1 < 3n - 2. \] (3.29)

To prove our claim observe that, for \( j = 2, \ldots, n, \ [f_2, f_j] = 0 \), indeed \( e^t f_2(x, V, R) = (x, e^t R, e^t V) \), hence by construction \( e_j( e^t f_2(x, V, R) ) = e^t e_j(x, V, R) \), or in other words \( f_2(e_j) = e_j \), but then

\[ [f_2, f_j] = f_2(e_j) \partial_R - e_j \partial_R = 0. \]

Now consider \( f_j, f_i \) with \( i, j \geq 3 \), to lighten the notation in the following we have denoted \( e_{ij}(t) = e_j( e^t (x_0, V_0, R_0) ) \), \( V(t) = V( e^t (x_0, V_0, R_0) ) \) and \( R(t) = R( e^t (x_0, V_0, R_0) ) \). Let us compute

\[ \langle \dot{e}_{ij}, R \rangle = -\langle e_{ij}, \dot{R} \rangle = -\langle e_{ij}, e_{ii} \rangle = 0, \]
\[ \langle \dot{e}_{ij}, V \rangle = -\langle e_{ij}, \dot{V} \rangle = 0, \]

and let us denote \( a_{ijk} = \langle \dot{e}_{ij}, e_k \rangle / |R|^2 \), then we have

\[ f_j(e_i) = \dot{e}_{ij} = \sum_{k=3}^{n} a_{ijk} e_k, \]

meaning that

\[ [f_i, f_j] = (f_i(e_j) - f_j(e_i)) \partial_R = \sum_{k=3}^{n} (a_{ijk} - a_{ijk}) e_k \partial_R = \sum_{k=3}^{n} (a_{ijk} - a_{ijk}) f_k, \]
and this concludes the proof of the claim. Equation (3.28), together with the fact that the fields \( \{ f_j \}_{j=1}^n, \{ f_{1k} \}_{k=2}^n, \{ f_{1k1} \}_{k=3}^n \) are a basis of \( TR(M, g) \), gives us the following local characterization of \( D(M, g) \)'s flag:

\[
\begin{align*}
D(M, g) &= (f_1, \ldots, f_n), \\
D^2(M, g) &= D(M, g) \oplus \{ f_{1k} \}_{k=2}^n, \\
D^3(M, g) &= D^2(M, g) \oplus \{ f_{1k1} \}_{k=3}^n = TR(M, g).
\end{align*}
\]

In particular we have

\[
\text{Growth } D(M, g) = (n, 2n - 1, 3n - 2).
\]

\[
(3.31)
\]

We obtain the following corollary.

**Corollary 3.1.** The following inclusions hold:

\[
\begin{align*}
f_{jk} &\in D(M, g), \ j, k = 2, \ldots, n, \\
f_{j1k} &\in D^2(M, g), \ j, k = 2, \ldots, n.
\end{align*}
\]

\[
(3.32)
\]

**Proof.** We have already proved the first inclusion of (3.32) since we have shown that \( \{ f_2, \ldots, f_n \} \) is integrable. To prove the second inclusion observe that the kernel of \( \pi^M \) has dimension \( 3n - 2 - n = 2n - 2 \). On the other hand equations (1.6), (3.17), (3.21), imply that the following \( 2n - 2 \) linearly independent local smooth sections of \( D^2(M, g) \)

\[
f_2, f_3, \ldots, f_n, X_{12}, f_{13}, \ldots, f_{1n},
\]

(3.33)

are contained in \( \ker \pi^M \), and hence they constitute a basis for it:

\[
\ker \pi^M = \{ f_2, f_3, \ldots, f_n, X_{12}, f_{13}, \ldots, f_{1n} \}.
\]

The distribution \( \ker \pi^M \) is integrable: if \( X, Y \) are sections of \( \ker \pi^M \), then they are \( \pi^M \)-related to the zero-section of \( TM \), and therefore also their bracket is so. Thanks to the integrability of \( \ker \pi^M \), recalling that \( X_{12} = f_1 + f_{12} \), we deduce that \( f_{j1k} \in \mathcal{D}^2(M, g), \ j, k = 2, \ldots, n. \)

\[
(3.33)
\]

If \( (M, g) \) is a Riemannian surface, which for simplicity we assume to be orientable, then the fields \( f_1, f_2 \) defined in (1.6) can be related with the fields \( f_1^g \), \( f_2^g \) defined in (2.6). Indeed, in this case \( \mathcal{R}(M, g) \) has two connected components:

\[
\mathcal{R}(M, g) = \{ (x, R, R^t) : (x, R) \in TM \setminus s_o \} \cup \{ (x, R, -R^t) \mid (x, R) \in TM \setminus s_o \}.
\]

The projection \( p : \mathcal{R}(M, g) \to TM \setminus s_o \) restricted to the first of such components is a diffeomorphism satisfying

\[
p_*f_1 = f_1^g, \ p_*f_2 = f_2^g.
\]

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This fact, combined with Theorem 1.1, constitutes a proof of Propositions 2.1. Focusing back on the general case, we would like to define a metric on $\mathcal{D}(M,g)$ that preserves the symmetries of the curvature radii lift

$$R_g : \{ \gamma \in C^2([0,1], M) : \gamma \text{ regular}, \kappa_g(\gamma) \neq 0 \} \to C^0([0,1], T\!M)$$

$$\gamma \mapsto R_g(\gamma),$$

hence before defining such a metric, we need to have a better understanding of how much information is encoded in the mapping (3.34).

**Definition 3.2.** Let $(M,g)$ and $(N,\eta)$ be Riemannian manifolds, we say that they have the same curvature radii if and only if there exists a diffeomorphism $\varphi : (M,g) \to (N,\eta)$ such that

$$R_\eta \circ \varphi = \varphi_\ast R_g.$$  \hspace{1cm} (3.35)

**Remark 3.1.** Equality (3.35) is meant as an equality of maps, therefore the diffeomorphism $\varphi$ of Definition 3.2 maps the domain of $R_g$ to the one of $R_\eta$. This implies that $\gamma : [0,1] \to M$ is the reparametrization of a geodesic of $(M,g)$ if and only if $\varphi \circ \gamma$ is the reparametrization of a geodesic of $(N,\eta)$. In particular

$$\kappa_g(\gamma) \equiv 0 \iff \kappa_\eta(\varphi \circ \gamma) \equiv 0.$$  \hspace{1cm} (3.36)

Definition 3.2 gives a precise meaning to the statement of Theorem 1.2, which we now prove.

**Proof.** (Theorem 1.2) It is sufficient to prove the theorem for two Riemannian metrics $\eta, g$ on the same manifold $M$. Assume first that $(M,g)$ and $(M,\eta)$ are homothetic manifolds, then $\eta = \lambda g$ for some $\lambda > 0$. The two metrics have the same Levi-Civita connection $\nabla$, moreover for every $X,Y \in T\!M$, $X \perp Y$ if and only if $X \perp g Y$, therefore we simply denote $\perp = \perp g = \perp_\eta$. Let $\gamma : [0,T] \to M$ be a regular curve, then the curvature computed with respect to $\eta$ can be easily related to the curvature computed with respect to $g$:

$$\kappa_\eta(\gamma) = \frac{|\pi_\eta(D_\gamma^2)\gamma|_{\lambda g}}{|\gamma(t)|_{\lambda g}^2} = \frac{1}{\sqrt{\lambda}} \frac{|\pi_g(D_\gamma^2)\gamma|_g}{|\gamma(t)|_g^2} = \frac{1}{\sqrt{\lambda}} \kappa_g(\gamma).$$

Therefore $k_\eta(\gamma)$ is never vanishing if and only if also $k_g(\gamma)$ is so, and in that case using equation (1.3) we deduce $R_\eta(\gamma) = R_g(\gamma)$.

Conversely, assume that $(M,g)$ and $(M,\eta)$ have the same curvature radii, i.e. (3.33) holds and

$$R_g(\gamma) = R_\eta(\gamma)$$  \hspace{1cm} (3.37)

for any regular curve $\gamma$ with never vanishing geodesic curvature. Observe that for any $X,Y \in T_x M$, $X \perp g Y$ if and only if there exists $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\dot{\gamma}(0) = X$ and $R_g(\gamma)(0) = Y$, but then, since $R_g = R_\eta$, we obtain $\perp g = \perp_\eta$. According to Remark 3.1 we have

$$D^g_\gamma \dot{\gamma} \propto \dot{\gamma} \iff D^g_\gamma \dot{\gamma} \propto \dot{\gamma}.$$
Hence for any curve $\gamma : [0,1] \to M$ the following proportionality relationship holds
\[ D_t^\| \dot{\gamma} - D_t^\| \dot{\gamma} \propto \dot{\gamma}, \] (3.38)
and in particular
\[ \pi_{\gamma_1} D_t^\| \dot{\gamma} = \pi_{\gamma_1} D_t^\| \dot{\gamma}. \] (3.39)

In light of (3.39), condition (3.37) reduces to
\[ \kappa_g(\gamma)|\pi_{\gamma_1} D_t^\| \dot{\gamma}|_g = \kappa_\eta(\gamma)|\pi_{\gamma_1} D_t^\| \dot{\gamma}|_\eta, \]
which, by definition of geodesic curvature, in turn is equivalent to
\[ \frac{|\pi_{\gamma_1} D_t^\| \dot{\gamma}|_g}{|\gamma|_g} = \frac{|\pi_{\gamma_1} D_t^\| \dot{\gamma}|_\eta}{|\gamma|_\eta}. \] (3.40)

For any $X, Y \in T_xM$, $X$ is perpendicular to $Y$ if and only if there exists $\gamma : [0,1] \to M$ satisfying
\[ \dot{\gamma}(0) = X, \quad \pi_{\gamma_1} D_t^\| \dot{\gamma}(0) = Y. \]
Hence equation (3.40) implies that
\[ \frac{|Y|_g}{|X|_g} = \frac{|Y|_\eta}{|X|_\eta}, \quad \forall X \perp Y, \] (3.41)
which in turn implies that the two metrics are conformally related: there exists a smooth function $f : M \to \mathbb{R}$ such that
\[ \eta = e^{2f}g. \] (3.42)

Since the metrics are conformal, their Levi-Civita connection difference tensor can be easily computed
\[ T(X, Y) := \nabla^\eta_X Y - \nabla^g_X Y = X(f)Y + Y(f)X - \langle X, Y \rangle_g \text{grad}_g f. \]
On the other hand thanks to equation (3.38) we know that
\[ T(X, X) \propto X, \]
and hence for any $Y \perp X$ we have
\[ 0 = \langle T(X, X), Y \rangle_g = -|X|^2_g (\text{grad}_g f, Y)_g = -|X|^2_g df(Y). \]
Since $X, Y$ are arbitrary orthogonal vectors, this implies that $f$ is constant. \(\square\)

The curvature radius lift $R_g$ of a Riemannian manifold $(M, g)$ is a complete homothety invariant of the metric $g$. It is then natural to define on $\mathcal{D}(M, g)$ a metric which is invariant under homothetic transformations. For every $a, b \in \mathbb{R}$, $b \neq 0$, we define a metric $\eta_{a,b}$ on $\mathcal{D}(M, g)$ by declaring the fields
\[ f^a_{1} := \frac{1}{\sqrt{a^2 + b^2}} f_1, \]
\[ f^a_{j} := \frac{1}{b} f_j, \quad j = 2, \ldots, n, \] (3.43)
a local orthonormal frame. For any \((\gamma, R, V)\) in the image of the lift \(\gamma, R, V\) in the image of the lift (3.3) the metric \(\eta_{a,b}\) satisfies the following equation

\[
\frac{d}{dt} \left( \frac{d(\gamma, V, R)}{dt} \right)_{\eta_{a,b}} = a^2 \frac{\dot{\gamma}^2}{|R|^2} + b^2 \frac{|D_t R|^2}{|R|^2}.
\] (3.44)

To prove (3.44) recall that according to Theorem 1.1 there exists \(u_1, \ldots, u_n : [0,1] \to \mathbb{R}\) such that

\[
\frac{d}{dt} (\gamma, V, R) = u_1 f_1 + u_2 f_2 + \ldots + u_n f_n,
\] (3.45)

or equivalently according to (3.46),

\[
\dot{\gamma} = \frac{u_1}{\sqrt{a^2 + b^2}} V, \\
D_t R = -\frac{u_1}{\sqrt{a^2 + b^2}} V + \frac{u_2}{b} R + \frac{u_3}{b} e_3 + \ldots + \frac{u_n}{b} e_n, \\
D_t V = \frac{u_1}{\sqrt{a^2 + b^2}} R + \frac{u_2}{b} V.
\] (3.46)

Hence

\[
a^2 \frac{\dot{\gamma}^2}{|R|^2} + b^2 \frac{|D_t R|^2}{|R|^2} = a^2 \frac{u_1^2}{|R|^2(a^2 + b^2)} |R|^2 + b^2 \frac{u_2^2}{|R|^2(a^2 + b^2)} |R|^2 + \ldots + \frac{u_n^2}{b^2} |R|^2.
\] (3.47)

The sub-Riemannian manifold \((R(M, g), D(M, g), \eta_{a,b})\) is called the sub-Riemannian manifold of curvature radii of \((M, g)\) and it is denoted with \(R(a,b)(M, g)\). We now prove the main result regarding these metrics, Theorem 1.3.

Proof. (Theorem 1.3) We begin by making sure that the map (1.9) is well-defined. Let \(\varphi : (M, g) \to (M, g)\) be a homothety of Riemannian manifolds. By construction \(\varphi_* \Phi : TM \oplus TM \to TM \oplus TM\) maps \(R(M, g)\) to \(R(M, g)\), this is a simple consequence of the fact that \(\varphi_*\) preserves orthogonality and the ratios between norms of vectors. To prove that \(\Phi := \varphi_* \Phi\) is a sub-Riemannian isometry we show that \(\Phi_* f_i = f_i, i = 1, 2,\) and that the fields \(\{\Phi_* f_3, \ldots, \Phi_* f_n\}\) are related to \(\{f_3, \ldots, f_n\}\) by an orthogonal transformation. Let \((\gamma, V, R) : [0, T] \to R(M, g)\) be an integral curve of \(f_1\) then we have

\[
\begin{align*}
\dot{\gamma} &= V, \\
D_t R &= -V, \\
D_t V &= R,
\end{align*}
\]
while the image under $\Phi$ of such curve, $\Phi(\gamma, R, V) = (\varphi \circ \gamma, \varphi \ast R, \varphi \ast V) =: (\bar{\gamma}, \bar{R}, \bar{V})$, satisfies
\[
\dot{\bar{\gamma}} = \frac{d}{dt} \varphi(\gamma) = \varphi \ast \dot{\gamma} = \varphi \ast V = \bar{V},
\]
\[
D_t \bar{R} = \nabla_{\bar{V}} \bar{R} = \varphi \ast \nabla_V R = -\varphi \ast V = -\bar{V},
\]
\[
D_t \bar{V} = \nabla_{\bar{V}} \bar{V} = \varphi \ast \nabla_V V = \varphi \ast R = \bar{R}.
\]
Summarizing we have
\[
\begin{cases}
\dot{\bar{\gamma}} = \bar{V}, \\
D_t \bar{R} = -\bar{V}, \\
D_t \bar{V} = \bar{R},
\end{cases}
\]
hence $\Phi \ast f_1 = f_1$. The case of $f_2$ is analogous. Since $\varphi$ is a homothety it sends local orthogonal basis of $TM$ to local orthogonal basis, and if $X, Y \in T_x M$ have the same norm, then so do $\varphi \ast X, \varphi \ast Y$. Given $(x, V, R) \in \mathcal{R}(M, g)$, let $\{e_3(x, V, R), \ldots, e_n(x, V, R)\}$ be the local orthogonal basis of $\{R, V\}^\perp$ constructed in the proof of Theorem 1.1 then both $\{\varphi \ast e_j(x, V, R)\}_{j=3}^n$ and $\{\varphi \ast e_j(x, V, R)\}_{j=3}^n$ are orthogonal basis of $\{\varphi \ast R, \varphi \ast V\}^\perp$, therefore they are related by an orthogonal transformation: there exists $O \in O(n-2)$ such that
\[
\varphi \ast e_j(x, V, R) = O^i_j e_i(\varphi(x), \varphi \ast V, \varphi \ast R).
\]
We claim that $\ker \pi_*^\mathcal{M}$ having maximal rank $2n - 2$. On the other hand, as shown in the proof of Theorem 1.1, the equations (1.10), (3.17), (3.21), imply that the following $2n - 2$ linearly independent smooth local sections of $\mathcal{D}^2(M, g)$

$$f_2, f_3, \ldots, f_n, X_{12}, f_{13}, \ldots, f_{1n},$$

(3.50)

are contained in $\ker \pi_*^\mathcal{M}$, and hence they constitute a basis for it:

$$\ker \pi_*^\mathcal{M} = \{ (f_2, f_3, \ldots, f_n, X_{12}, f_{13}, \ldots, f_{1n}) \}.$$

We claim that $\ker \pi_*^\mathcal{M} \subset \mathcal{D}^2(M, g)$ is the unique integrable sub-bundle of $\mathcal{D}^2$ having maximal rank $2n - 2$. We have already noticed that $\ker \pi_*^\mathcal{M}$ is integrable in the proof of Theorem 1.1. Let $\mathcal{D}' \subset \mathcal{D}^2(M, g)$ be an integrable distribution and let $X, Y$ be smooth sections of $\mathcal{D}'$. Then there exists some smooth functions $\alpha_1, \ldots, \alpha_{2n-1}, \beta_1, \ldots, \beta_{2n-1}$ such that

$$X = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_{n+1} X_{12} + \alpha_{n+2} f_{13} + \cdots + \alpha_{2n-1} f_{1n},$$

$$Y = \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_{n+1} X_{12} + \beta_{n+2} f_{13} + \cdots + \beta_{2n-1} f_{1n}.$$

On the other hand, since $\mathcal{D}' \subset \mathcal{D}^2(M, g)$ is integrable, it holds

$$[X, Y] \equiv 0 \mod \mathcal{D}^2(M, g),$$

(3.51)

which translates to

$$(\alpha_1 \beta_{n+1} - \beta_1 \alpha_{n+1})[f_1, X_{12}] + \sum_{k=3}^{n} (\alpha_1 \beta_{n+k-1} - \beta_1 \alpha_{n+k-1})[f_1, f_{1k}] \equiv 0 \mod \mathcal{D}^2(M, g).$$

Since $[f_1, X_{12}]$, $[f_1, f_{1k}]$, $k = 3, \ldots, n$, are linearly independent mod $\mathcal{D}^2(M, g)$, we have

$$\alpha_1 \beta_{n+k-1} - \beta_1 \alpha_{n+k-1} = 0, \ k = 2, \ldots, n.$$  

(3.52)

If $\alpha_1 \equiv \beta_1 \equiv 0$ then $X, Y \in \ker \pi_*^\mathcal{M}$. Otherwise if, for instance, $\alpha_1 \neq 0$, then

$$\beta_{n+k-1} = \frac{\alpha_1}{\alpha_k} \alpha_{n+k-1}$$

for $k = 2, \ldots, n$ and setting

$$Z = \alpha_{n+1} X_{12} + \alpha_{n+2} f_{13} + \cdots + \alpha_{2n-1} f_{1n},$$

we deduce that

$$X = \sum_{i=1}^{n} \alpha_i f_i + Z, \ Y = \sum_{i=1}^{n} \beta_i f_i + \frac{\beta_1}{\alpha_1} Z.$$  

(3.53)

As a consequence

$$\mathcal{D}' \subset \{ (f_1, f_2, \ldots, f_n, Z) \},$$

therefore rank $\mathcal{D}' \leq n + 1$. Observe that it is not possible that rank $\mathcal{D}' = n + 1$, because otherwise $\mathcal{D}'$ would contain $\mathcal{D}(M, g)$, contradicting the hypothesis of integrability. We deduce that

$$\text{rank } \mathcal{D}' \leq n < 2n - 2, \ \forall \ n > 2,$$

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proving our claim. Since \( \Phi \) is an isometry, it preserves every layer of \( \mathcal{D}(M, g) \)'s flag, in the sense that

\[
\Phi_* \mathcal{D}_i(M, g) = \mathcal{D}_i(M, g), \quad i = 1, 2, 3,
\]

therefore since \( \ker \pi_*^M \subset \mathcal{D}^2(M, g) \) is the unique integrable sub-bundle of \( \mathcal{D}^2(M, g) \) having maximal rank \( 2n - 2 \), we deduce that

\[
\Phi_* \ker \pi_*^M = \ker \pi_*^M, \quad (3.54)
\]

or equivalently

\[
\ker (\pi^M \circ \Phi)_* = \ker \pi_*^M. \quad (3.55)
\]

Both \( \ker \pi_*^M \) and \( \mathcal{D}(M, g) \) are invariant under \( \Phi_* \), therefore also their intersection is so, hence

\[
\Phi_*(\langle f_2, \ldots, f_n \rangle) = \Phi_*(\ker \pi_*^M \cap \mathcal{D}(M, g)) = \ker \pi_*^M \cap \mathcal{D}(M, g) = \langle f_2, \ldots, f_n \rangle,
\]

but then, since \( \Phi_* \) is a sub-Riemannian isometry preserving \( \langle f_2, \ldots, f_n \rangle \), it must preserve also its orthonormal complement, namely \( \Phi_* f_1 = \pm f_1 \). To show that \( \Phi_* f_2 = \pm f_2 \), consider the following linear map

\[
L : \ker \pi_*^M \cap \mathcal{D}(M, g) \rightarrow TM
\]

\[
X \mapsto \pi_*^M [f_1, X],
\]

and notice that, according to (3.16) and (3.21), \( \ker L = \langle f_3, \ldots, f_n \rangle \). On the other hand, according to (3.55)

\[
\pi_*^M [f_1, X] = 0 \iff \pi_*^M \circ \Phi_* [f_1, X] = 0,
\]

which, taking into account the fact that \( \Phi_* f_1 = \pm f_1 \), translates to

\[
\pi_*^M [f_1, X] = 0 \iff \pi_*^N [f_1, \Phi_* X] = 0,
\]

or more simply to \( L = \Phi_* \ker L \), thus \( \Phi_*(\langle f_3, \ldots, f_n \rangle) = \langle f_3, \ldots, f_n \rangle \). So far we have deduced that \( \Phi_*(\langle f_1, f_3, \ldots, f_n \rangle) = \langle f_1, f_3, \ldots, f_n \rangle \), taking the orthonormal complement of this last equation with respect to the sub-Riemannian metric, we deduce that \( \Phi_* f_2 = \pm f_2 \). A direct consequence of equation (3.55) is that

\[
\Phi^x(x, V, R) = \Phi^x(x, R', V'), \quad \forall (x, V, R), (x, R', V') \in \mathcal{R}(M, g), \quad (3.56)
\]

therefore the following map is well defined diffeomorphism

\[
\varphi : M \rightarrow M
\]

\[
x \mapsto \Phi^x(x, V, R) = \pi_*^M \circ \Phi(x, V, R).
\]

Let \( (\gamma, V, R) \) be an integral curve of \( f_1 \), then, since \( \Phi_* f_1 = \pm f_1 \), it holds

\[
\begin{cases}
\Phi^x = \pm \Phi^V, \\
D_t \Phi^R = \pm \Phi^V, \\
D_t \Phi^V = \pm \Phi^R.
\end{cases}
\quad (3.57)
\]

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but we know that $\Phi^x = \varphi$, hence the first equation of (3.57) reads
\[ \varphi_x V = \varphi_\gamma = \dot{\Phi}^x = \pm \Phi V, \]
consequently $\Phi V = \pm \varphi_x V$. Now consider the vector field $f_{21} = [f_1, f_1]$, recall that, according to (3.16), its integral curves satisfy
\[
\begin{cases}
\dot{x} = V, \\
D_t R = 0, \\
D_t V = 0,
\end{cases}
\tag{3.58}
\]
on the other hand since $\Phi \circ f_1 = \pm f_1$, $\Phi \circ f_2 = \pm f_2$, it also holds $\Phi \circ f_{12} = \pm f_{12}$ thus
\[ \nabla_V V = 0 \iff \nabla_\varphi \varphi, V = 0 \]
or equivalently
\[ \nabla = \varphi^* \nabla . \tag{3.59} \]
Exploiting (3.59) together with the third equation in (3.57) we deduce
\[ \pm \Phi^R = D_t \Phi V = \nabla_\varphi \varphi, V = \varphi_\gamma \nabla_V V = \varphi_\gamma D_t V = \varphi_\gamma R, \]
from which we conclude
\[ \Phi(x, V, R) = (\varphi(x), \pm \varphi_\gamma, \pm \varphi_\gamma V) = \pm \varphi_\gamma \oplus \varphi_\gamma (x, V, R) . \tag{3.60} \]
Substituting (3.60) in the second equation of (3.57) we find that we have to discard the minus sign and we are left with
\[ \Phi(x, V, R) = (\varphi(x), \varphi, V, \varphi, R) . \tag{3.61} \]
Let $(\gamma, V, R) : [0, 1] \to R(M, g)$ be a curve tangent to $D(M, g)$ with $\gamma$ regular, then according to Theorem 1.1 $R = R_g(\gamma)$. On the other hand since $\Phi$ preserves $D(M, g)$, also $(\varphi \circ \gamma, \varphi_\gamma R, \varphi_\gamma V)$ is tangent to $D(M, g)$ and hence, according to Theorem 1.1 we have
\[ R_g(\varphi_\gamma) = \varphi_\gamma R_g(\gamma) . \]
On the other hand, since $\varphi : (M, \varphi^* g) \to (M, g)$ is an isometry, it holds
\[ R_g(\varphi_\gamma) = \varphi_\gamma R_{\varphi^* g}(\gamma) , \]
therefore $R_g = R_{\varphi^* g}$ and, according to Theorem 1.2 $\varphi : (M, g) \to (M, g)$ is a homothety.
Assume now that $n = \dim M = 2$. Consider the following linear map
\[ P : D(M, g) \to \mathcal{D}^3(M, g)/\mathcal{D}^2(M, g) \]
\[ X \mapsto [X, [f_1, f_2]] \mod \mathcal{D}^2(M, g) . \tag{3.62} \]
Observe that the kernel of (3.62), is invariant under pushforwards of sub-Riemannian isometries, in the sense that
\[ \Phi_\gamma \ker P = \ker P , \]
\[ \ \]
therefore
\[ \Phi_* \langle \{ f_2 \} \rangle = \langle \{ f_2 \} \rangle, \]
and since \( \Phi \) is an isometry we deduce that
\[ \Phi_* f_1 = \pm f_1, \quad \Phi_* f_2 = \pm f_2. \quad (3.63) \]
We claim that \( \Phi_* f_2 = f_2 \). Assume by contradiction that \( \Phi_* f_2 = -f_2 \) and consider the following couple of vector fields
\[ X = f_1 + f_{12}, \quad Y = f_2. \]
Since \([ f_2, f_{12} ] = f_{12}\), we have
\[ [X, Y] = [f_1, f_2] - f_{12} - f_{12} = 0. \]
On the other hand since \( \Phi_* f_1 = \pm f_1, \quad \Phi_* f_2 = -f_2 \) we find
\[ 0 = \Phi_* [X, Y] = [\pm f_1 \mp f_{12}, -f_2] = \mp 2 f_{12} \neq 0, \]
which is a contradiction. Combining \( \Phi_* f_2 = f_2 \) and \( \Phi_* f_1 = \pm f_1 \) we deduce that
\[ \Phi_* X_{12} = \pm X_{12}, \]
therefore, since
\[ \ker \pi_*^M = \langle \{ f_2, X_{12} \} \rangle, \]
we obtain
\[ \ker \pi_*^M = \ker (\pi_*^M \circ \Phi)_*. \]
The remaining part of the proof is identical to the one of the case \( n > 2 \). In particular we can conclude by repeating verbatim what is written between equation (3.56) and the discussion of the 2-dimensional case.

**4 The fields \( f_1, f_2 \)**

Given a Riemannian manifold \( (M, g) \), the vector fields \( f_1, f_2 \) defined in (1.6) are global sections of \( T(TM @ TM) \), which restrict to vector fields on \( \mathcal{R}(M, g) \). As a consequence of Theorem (1.1) we know that they are metric invariants of \( (M, g) \). Actually they are even invariant under homothetic transformation. In the current section we show that some classical metric invariants can be recovered from the iterated Lie brackets of \( f_1, f_2 \). Indeed, as the next proposition shows, the field \([ f_1, f_2 ]\) already gives us a complete description of the geodesics of \( (M, g) \); in this sense the fields \( f_1, f_2 \) give us a factorization of the geodesic flow.

**Proposition 4.1.** Let \( (M, g) \) be a Riemannian manifold, let \( x \in M \) and let \( \exp_{x}^{(M, g)} \) be the corresponding exponential map at \( x \). Let \( f_{21} = [f_2, f_1] \) and

\[ \text{...} \]
\(f_{121} = [f_1, f_2]\), then, for every \((x, V, R) \in TM \oplus TM\), the following formulae hold

\[
\begin{align*}
\pi^M \circ e^{tf_{21}}(x, V, R) &= \exp^{(M,g)}_x(tV), \\
\pi^M \circ e^{tf_{121}}(x, V, R) &= \exp^{(M,g)}_x(tR),
\end{align*}
\]

(4.1) for every \(t \in \mathbb{R}\) such that the flow of the fields is defined.

**Proof.** According to equation (3.16), the vector field \(f_{21}\) reads

\[
f_{21} = [f_2, f_1] = V \partial_x - \Gamma(V, R) \partial_R - \Gamma(V, V) \partial_V,
\]

where the symbols \(\Gamma\) are the ones defined in equation (3.14). Consequently, any integral curve of \(f_{21}\) satisfies

\[
\begin{align*}
\dot{x} &= V, \\
D_t R &= 0, \\
D_t V &= 0.
\end{align*}
\]

(4.2) Therefore the curve \(x(t) = \pi^M \circ e^{t[f_2, f_1]}(x_0, V_0, R_0)\) is the unique geodesic with initial point \(x_0\) and initial velocity \(V_0\).

Recall the vector field \(X_{12}\) defined in equation (3.17) and observe that \(f_{121} = [f_1, f_21] = [f_1, f_1 - X_{12}] = -[f_1, X_{12}]\), therefore

\[
egin{align*}
f_{121} &= [V \partial_x - (V + \Gamma(R, V)) \partial_R + (R - \Gamma(V, V)) \partial_V, V \partial_R - R \partial_V] \\
&= [V \partial_x - \Gamma(R, V) \partial_R - \Gamma(V, V) \partial_V, V \partial_R - R \partial_V] \\
&= \Gamma(R, V) \partial_V - \Gamma(V, V) \partial_R + \Gamma(V, V) \partial_R + R \partial_x \\
&- \Gamma(R, R) \partial_R - 2\Gamma(R, V) \partial_V \\
&= R \partial_x - \Gamma(R, R) \partial_R - \Gamma(R, V) \partial_V.
\end{align*}
\]

Hence any integral curve of \(f_{121}\) satisfies

\[
\begin{align*}
\dot{x} &= R, \\
D_t R &= 0, \\
D_t V &= 0,
\end{align*}
\]

(4.3) concluding the proof.

If we consider another layer of the Lie algebra generated by \(f_1, f_2\), the components of Riemann curvature tensor appear.

**Proposition 4.2.** Let \((M, g)\) be a Riemannian manifold, the integral curves of the vector field \(f_{1121} := [f_1, f_{121}]\) satisfy

\[
\begin{align*}
\dot{x} &= -V, \\
D_t R &= -\nabla^V(V, R)R, \\
D_t V &= -\nabla^V(V, R)V,
\end{align*}
\]

(4.4) where \(\nabla^V : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)\) is the Riemann curvature tensor of \((M, g)\).

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Proof. Given a vector field $X$ we denote with $X^x$ its $x$-component, with $X^R$ its $R$-component and with $X^V$ its $V$-component, meaning that $X = X^x \partial_x + X^R \partial_R + X^V \partial_V$. We compute (4.4) component by component:

$$f_{1121} = f_1(f_{121}^x) - f_{121}(f_1^x) = f_1(R) - f_{121}(V)$$

$$= - V - \Gamma(R, V) + \Gamma(R, V) = - V. \quad (4.5)$$

Concerning the $R$-component we compute

$$f_1(f_{121}^R) = (V \partial_x - (V + \Gamma(R, V)) \partial_R + (R - \Gamma(V, V)) \partial_V)(- \Gamma(R, R))$$

$$= - V \partial_x \Gamma(R, R) + 2 \Gamma(V, R) + 2 \Gamma(\Gamma(R, V), R),$$

and

$$f_{121}(f_1^R) = (R \partial_x - \Gamma(R, R) \partial_R - \Gamma(R, V) \partial_V)(- V - \Gamma(R, V))$$

$$= - R \partial_x \Gamma(R, V) + \Gamma(\Gamma(R, V), V) + \Gamma(R, V) + \Gamma(R, \Gamma(R, V)),$$

therefore

$$f_{1121}^R = f_1(f_{121}^R) - f_{121}(f_1^R)$$

$$= \Gamma(R, V) - V \partial_x \Gamma(R, R) + R \partial_x \Gamma(R, V)$$

$$- \Gamma(V, \Gamma(R, R)) - \Gamma(\Gamma(R, V), R) - \Gamma(R, \Gamma(R, V)) \quad (4.6)$$

The quantity $f_1(f_{121}^V)$ can be computed as

$$f_1(f_{121}^V) = (V \partial_x - (V + \Gamma(R, V)) \partial_R + (R - \Gamma(V, V)) \partial_V)(- \Gamma(R, V))$$

$$= - V \partial_x \Gamma(R, V) + \Gamma(\Gamma(R, V), V) + \Gamma(\Gamma(R, V), V) - \Gamma(R, R) + \Gamma(R, \Gamma(V, V),$$

whereas $f_{121}(f_1^V)$ satisfies

$$f_{121}(f_1^V) = (R \partial_x - \Gamma(R, R) \partial_R - \Gamma(R, V) \partial_V)(R - \Gamma(V, V))$$

$$= - R \partial_x \Gamma(V, V) - \Gamma(R, R) + 2 \Gamma(\Gamma(R, V), V),$$

hence

$$f_{1121}^V = f_1(f_{121}^V) - f_{121}(f_1^V)$$

$$= \Gamma(V, V) - V \partial_x \Gamma(R, V) + R \partial_x \Gamma(V, V)$$

$$- \Gamma(V, \Gamma(R, V)) + \Gamma(R, \Gamma(V, V)) \quad (4.7)$$

$$= \Gamma(V, V) - \mathcal{R}^V(R, V) V.$$  

Equations (4.5), (4.6) and (4.7) together imply that any integral curve of $f_{1121}$ satisfies the ODEs system (4.4). $\square$

As stated in Theorem 1.1 the fields

$$\{f_1, \ldots, f_n, f_{12}, \ldots, f_{1n}, f_{121}, \ldots, f_{1n1}\} \quad (4.8)$$

constitute a local basis for $T \mathcal{R}(M, g)$, therefore there exists some real valued smooth functions $\{c_1, \ldots, c_{3n-2}\}$ on $\mathcal{R}(M, g)$ such that

$$f_{1121} = \sum_{i=1}^n c_i f_i + \sum_{i=2}^n c_{n+i-1} f_{1i} + \sum_{i=2}^n c_{2n+i-2} f_{1i1}. \quad (4.9)$$
Proposition 4.3. Let \((M, g)\) be a Riemannian manifold, then the function \(c_1 : \mathcal{R}(M, g) \to \mathbb{R}\) defined in equation (11.9) is a homothetic invariant of \((M, g)\), which can be expressed in terms of sectional curvatures as
\[
c_1(x, V, R) = |R|^2 \sec(R, V). \tag{4.10}
\]
In particular if \((M, g)\) is a Riemannian surface, then
\[
c_1(x, V, R) = |R|^2 K(x), \tag{4.11}
\]
where \(K\) is the Gaussian curvature of \((M, g)\).

Proof. From equations (4.4) and (4.2) it follows that
\[
f_{1121} + f_{21} = -\mathcal{R}^V (V, R) R \partial_R - \mathcal{R}^V (V, R) V \partial_V, \tag{4.12}
\]
therefore
\[
f_{1121} + f_{21} = -\frac{1}{|R|^2} (\mathcal{R}^V (V, R) R) R \partial_R - \frac{1}{|R|^2} (\mathcal{R}^V (V, R) V) R \partial_V
\]
\[= |R|^2 \sec(R, V)(-V \partial_R + R \partial_V) - X,
\]
where \(\{e_3, \ldots, e_n\}\) is the basis of \(\{R, V\}^\perp\) described in the proof of Theorem 1.1 and we have denoted
\[
X = \frac{1}{|R|^2} \sum_{i=3}^n (\langle \mathcal{R}^V (V, R) R, e_j \rangle e_j \partial_R + \langle \mathcal{R}^V (V, R) V, e_j \rangle e_j \partial_V).
\]
Observe that
\[
-V \partial_R + R \partial_V = f_1 - f_{21},
\]
hence
\[
f_{1121} + f_{21} = |R|^2 \sec(R, V)(f_1 - f_{21}) - X. \tag{4.13}
\]
The unique expression of the vector field \(X\) in terms of the basis \(\{V_1, V_2\}\) is a linear combination which does not involve \(f_1\), hence we deduce
\[
c_1(x, V, R) = |R|^2 \sec(R, V). \tag{4.14}
\]
\[\square\]

The fields \(f_1, f_2\) allow us to characterize the homotheties of Riemannian surfaces with one synthetic equation. Let \(M\) be a smooth manifold and let \(X \in \mathfrak{X}(M)\) be a vector field, which we assume to be complete for simplicity. The family of maps \(\{e^*_t X\}_{t \in \mathbb{R}}\) constitutes a one-parameter group of diffeomorphisms of \(TM\); we denote its infinitesimal generator with \(\tilde{X}\).
Proposition 4.4. Let \((M, g)\) be a Riemannian surface and let \(f_1\) be the local vector field over \(TM \setminus s_o\) defined in (2.6). A vector field \(X \in \mathfrak{X}(M)\) is the infinitesimal generator of a one-parameter group of homotheties if and only if

\[
[X, f_1] = 0.
\]

\((4.15)\)

Proof. A vector field \(X \in \mathfrak{X}(M)\) satisfies (4.15) if and only if

\[
e^{tX} f_1 = f_1
\]

for every \(t \in \mathbb{R}\). On the other hand, any vector field \(X \in \mathfrak{X}(M)\) satisfies

\[
e^{tX} f_2 = f_2,
\]

indeed

\[
e^{sX} \circ e^{tf_2}(x, R) = e^{sX}(x, e^t R) = (e^{sX} x, e^t e^{sX} R) = (e^{sX} x, e^t e^{sX} R) = e^{sf_2} \circ e^{sX}(x, R).
\]

Therefore \(X\) satisfies (4.15) if and only if \(e^{tX}\) is an isometry of \(\mathcal{R}_{a,b}(M, g)\) for each \(t \in \mathbb{R}\), hence, by Theorem 1.3, \(X\) satisfies (4.15) if and only if \(e^{tX}\) is a one-parameter group of homotheties of \((M, g)\).

The results obtained so far can be used to produce a flatness theorem for Riemannian surfaces having a 4-dimensional Lie algebra of homothetic vector fields.

Theorem 4.1. Let \((M, g)\) be a 2-dimensional Riemannian manifold and let \(L \subset \mathfrak{X}(M)\) be the corresponding Lie algebra of homothetic vector fields. Then \(\dim(L) \leq 4\), and \((M, g)\) is flat if equality is achieved.

Proof. Let \(\mathcal{L} \subset \mathfrak{X}(\mathcal{R}(M))\) be the Lie algebra of isometric vector fields for the manifold of curvature radii \(\mathcal{R}(M, g)\). By Theorem 1.3 this Lie algebra is isomorphic to the one of homothetic vector fields of \((M, g)\). Let \(X \in \mathcal{L}\) be a vector field vanishing at some point \((x, R) \in \mathcal{R}(M)\). We claim that \(X\) is identically zero. Indeed, since \(X\) is isometric, by Proposition 4.4 it satisfies \([X, f_1] = [X, f_2] = 0\). Since \(\mathcal{D} = \text{span}\{f_1, f_2\}\) is bracket generating, for any \((x', R') \in \mathcal{R}\) there exist some real numbers \(s_1, \ldots, s_k\) such that

\[
(x', R') = e^{s_1 f_1} \circ \cdots \circ e^{s_k f_k} (x, R), \quad i_1, \ldots, i_k \in \{1, 2\},
\]

\((4.18)\)

consequently

\[
X(x', R') = e^{s_1 f_1} \circ \cdots \circ e^{s_k f_k} X(x, R) = 0.
\]

Let \(X_1, \ldots, X_5 \in \mathcal{L}\) and \((x, R) \in \mathcal{R}(M)\). The manifold of curvature radii is 4-dimensional, hence there exists a linear combination of \(X_1, \ldots, X_5\) vanishing at \((x, R)\), and thus vanishing everywhere. We deduce that \(\dim \mathcal{L} \leq 4\). Assume now that \(\dim \mathcal{L} = 4\) and let \(X_1, \ldots, X_4\) be a basis for \(\mathcal{L}\). By the above argument \(X_1, \ldots, X_4\) are linearly independent at every point of \(\mathcal{R}(M)\). Thus they can
be used to produce local coordinates in the neighbourhood of every point, by
means of the map
\[(s_1, \ldots, s_4) \mapsto e^{s_1 X_1} \circ \cdots \circ e^{s_4 X_4}(x, R).\]

This implies that the structure coefficients of the frame \(f_1, f_2, f_{12}, f_{121}\) are con-
stants (Theorem 4.3 implies that the latter is a basis for \(\mathcal{TR}(M)\)). In particular
there exist real constants \(c_1, c_2, c_{12}, c_{121}\) such that
\[f_{121} = [f_1, f_{121}] = c_1 f_1 + c_2 f_2 + c_{12} f_{12} + c_{121} f_{121}.\]

Now we exploit Proposition 4.3, telling us that \(c_1(x, R) = |R|^2 K(x)\), where \(K(x)\)
is the Gaussian curvature of \((M, g)\). Such a function is constant on a fixed fiber
of the manifold of curvature radii \(\mathcal{R}(M) = TM \setminus s_0\) if and only if it is identically
zero. It follows that \(K = 0\). \(\square\)

5 Similarity transformations of the plane

In this section we show that the sub-Riemannian manifolds \(\mathcal{R}_{a,b}(\mathbb{R}^2, g_e)\), where
\(g_e\) is the standard Euclidean metric, are isomorphic to left invariant sub-Riemannian
structures on the group of orientation preserving similarities of \(\mathbb{R}^2\), which we
denote with \(G\). Moreover we give a characterization of the sub-Riemannian geodesics of \(\mathcal{R}_{0,1}(\mathbb{R}^2, g_e)\) in terms of the euclidean curvature of their projections
to a plane. Such characterization in terms of curvature is similar to the one of Euler elastic curves ([13]), which are projection of normal extremal trajectories of the nilpotent Engel group ([AS11]). All the results of this section follow from straightforward computations, therefore the proofs are omitted. Let \((y_1, y_2, R_1, R_2)\) be global coordinates for \(T\mathbb{R}^2 \setminus \{0\} = \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{0\}\). It is conve-
nient to define a new set of coordinates \((\theta, r, x_1, x_2)\) as
\[(y_1, y_2, R_1, R_2) \mapsto (x_1 + r \cos \theta, x_2 + r \sin \theta, -r \cos \theta, -r \sin \theta). \quad (5.1)\]

In coordinates \((y, R)\) a point in \(\mathcal{R}_{a,b}(\mathbb{R}^2, g_e)\) is interpreted as the curvature
radius of some curve going through \(y\). In the new coordinates \((\theta, r, x_1, x_2)\) we inter-
pret a point in \(\mathcal{R}_{a,b}(\mathbb{R}^2, g_e)\) as the osculating circle of such curve, having
radius \((r \cos \theta, r \sin \theta)\) and center \((x_1, x_2)\). Observe that the data of a homothetic transformation, which in the case of \((\mathbb{R}^2, g_e)\) consists in a composition of rotations, dilations and translation, can be encoded into an osculating circle, i.e. a point of \(\mathcal{R}_{a,b}(\mathbb{R}^2, g_e)\): given a circle \((\theta, r, x_1, x_2)\) we can dilate by \(r\), rotate
by \(\theta\) and translate by \((x_1, x_2)\). We have a diffeomorphism
\[F : \mathcal{R}(\mathbb{R}^2) \to G \]
\[(\theta, r, x_1, x_2) \mapsto Q := \begin{pmatrix} r \cos \theta & -r \sin \theta & x_1 \\ r \sin \theta & r \cos \theta & x_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.2)\]

which allows us to push forward the sub-Riemannian structure of \(\mathcal{R}_{a,b}(\mathbb{R}^2, g_e)\),
to \(G\). The resulting sub-Riemannian structure is left invariant.
Proposition 5.1. The frame \((f_1, f_2)\) can be written in coordinates \((\theta, r, x_1, x_2)\) as
\[
f_1 = -\frac{\partial}{\partial \theta}, \quad f_2 = r \frac{\partial}{\partial r} - r \cos \theta \frac{\partial}{\partial x_1} - r \sin \theta \frac{\partial}{\partial x_2}.
\] (5.3)
Both of these vector fields are pushforwarded to left invariant vector fields by the map (5.2):
\[
(F_* f_1)(Q) = -Q \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (F_* f_2)(Q) = Q \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (5.4)

Remark 5.1. The geometry of \(G\) is reminiscent of the left invariant sub-Riemannian structure on the group of rigid motions of the plane, related to ‘bicycling mathematics’, which has been studied in [Ard+21], [MS10], [Sac10], [Sac11]. The group of rigid motions of \(\mathbb{R}^2\) can be described as
\[
SE_2 = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & x_1 \\ \sin \theta & \cos \theta & x_2 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in [0, 2\pi], (x_1, x_2) \in \mathbb{R}^2 \right\}.
\]
In coordinates \((\theta, x_1, x_2)\) we can define a left-invariant sub-Riemann structure on \(SE_2\) by declaring the fields
\[
X_1 = \cos \theta \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial x_2}, \quad X_2 = \frac{\partial}{\partial \theta},
\] (5.5)
an orthonormal generating family. There exists a submersion
\[
P : G \to SE_2
\]
\[
(\theta, r, x_1, x_2) \mapsto (\theta, x_1, x_2),
\] (5.6)
satisfying
\[
P_* \frac{1}{r} f_2 = -X_1, \quad P_* f_1 = -X_2.
\] (5.7)
Let \(h_1, h_2 : T^* \mathcal{R}(\mathbb{R}^2) \to \mathbb{R}\) be the Hamiltonian functions associated with \(f_1, f_2\) and let \(p_\theta, p_r, p_{x_1}, p_{x_2}\) be the canonical momenta associated with the coordinates \(\theta, r, x_1, x_2\), then
\[
h_1 = -p_\theta, \quad h_2 = rp_r - r \cos \theta p_{x_1} - r \sin \theta p_{x_2}.
\]
The sub-Riemannian Hamiltonian of \(\mathcal{R}_{0,1}(\mathbb{R}^2, g_e)\) reads
\[
2H = h_1^2 + h_2^2 + p_\theta^2 + (rp_r - r \cos \theta p_{x_1} - r \sin \theta p_{x_2})^2.
\] (5.8)
If we make the change of coordinates \(\rho = \log r\) the Hamiltonian becomes
\[
2H = p_\rho^2 + (p_\rho - e^\rho \cos \theta p_{x_1} - e^\rho \sin \theta p_{x_2})^2.
\] (5.9)
The next proposition characterizes normal extremal trajectories in terms of the euclidean curvature of their projection to the \((\rho, \theta)\)-plane.

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Proposition 5.2. The following quantities

\[ \epsilon = \sqrt{p_{x_1}^2 + p_{x_2}^2}, \quad \alpha = \text{arg} (p_{x_1} + ip_{x_2}), \]

are first integrals of the sub-Riemannian Hamiltonian \(5.9\). A curve \((\rho, \theta) : [0, 1] \to \mathbb{R}^2\) is the projection of a normal extremal trajectory \((\rho, \theta, x_1, x_2) : [0, 1] \to \mathbb{R}^4\), if and only if its Euclidean curvature \(\kappa\) satisfies

\[ \kappa(\rho, \theta) = \epsilon e^\rho \sin(\theta - \alpha). \]

There are no strictly abnormal extremals.

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