A slightly different view of complexity

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Abstract. This paper presents a different approach to understanding the concept of complexity and its relation to various phenomena in the field of physics [1-6]. A unique definition of complexity does not yet exist. Complexity Cmp was calculated in three systems simulated in computational experiments. In the first simulation a body in free fall was observed while falling towards body of much greater mass. Second observed system was two bodies with different temperatures, and third system was crystals in crystallization chamber. Complexity increases when the temperatures of two bodies are equalized, when one body falls to another and the crystal grows from the solution. In crystal growth theory, there are also still some concerns about the growth of some types of crystals.

1. Introduction

Unique view of the establishment of thermodynamic equilibrium, the establishment of a mechanical equilibrium and crystal growth. Complexity Cmp was calculated in three systems simulated in computational experiments. In the first simulation a body in free fall was observed while falling towards body of much greater mass. Second observed system was two bodies with different temperatures, and third system was crystals in crystallization chamber. The complexity of Cmp increases in the equilibrium of the temperature of the two bodies, when one body falls to another and the crystal grows from the solution.

2. Cmp

We divide time series \( A_1, A_2, ..., A_{110} \) into eleven subseries. First ten subseries are

\[
F_{1,j} = A_j, \quad F_{2,j} = A_{j+10}, \quad ..., \quad F_{10,j} = A_{j+90}
\]

(1)

where \( j = 1, 2, ..., 10 \). The eleventh subseries we represent as linear combination of previous ones.

\[
A_{101} = \sum_{i=1}^{10} c_i F_{i,1}, \quad A_{102} = \sum_{i=1}^{10} c_i F_{i,2}, \quad ..., \quad A_{110} = \sum_{i=1}^{10} c_i F_{i,10}
\]

(2)

We now make permutations inside first ten subseries and get new subseries

\[
F'_{1,1} = A_{10}, \quad F'_{1,j} = A_{j-1}
\]

\[
F'_{2,1} = A_{20}, \quad F'_{2,j} = A_{j+9}
\]

\[
F'_{3,1} = A_{30}, \quad F'_{3,j} = A_{j+19}
\]

\[
..., \quad F'_{10,1} = A_{100}, \quad F'_{10,j} = A_{j+89}
\]

(3)
where \( j = 2, 3, \ldots, 10 \).

The equations

\[
A_{101} = \sum_{i=1}^{10} c_i F'_{i1}, \quad A_{102} = \sum_{i=1}^{10} c_i F'_{i2}, \quad \ldots, \quad A_{110} = \sum_{i=1}^{10} c_i F'_{i10}
\]  \( (4) \)

yield new constants of the linear combination. Using 2-norm of vectors, we define complexity:

\[
Cmp = -\ln \frac{\|c_{1}c_{2}c_{10} - (c_{1}c_{2}c_{10})\|}{\|c_{1}c_{2}c_{10}\|}
\]  \( (5) \)

By solving the differential equation of nonlinear oscillations

\[
\frac{dv}{dt} = -\sum_{j=0}^{N} \frac{x^{j+1}}{2j+1}, \quad v = \frac{dx}{dt}
\]  \( (6) \)

we obtain \( x(0.1k), (k = 1, 2, 3, \ldots) \). For certain initial conditions and certain values of \( N \) we calculate mean value of \( Cmp \) and the standard deviation (Table 1) by taking many series into consideration:

\[
1 + p \leq k \leq 110 + p, \quad p = 0, 1, \ldots, 99
\]  \( (7) \)

**Table 1.** \( Cmp \) of various series calculated by solving differential equation of motion (6). Initial values are (a) \( x(0) = -1.5 \) and \( v(0) = 0.4 \); (b) \( x(0) = 1.2 \) and \( v(0) = 0.5 \); (c) \( x(0) = 0.92 \) and \( v(0) = -0.01 \), (d) \( x(0) = -1.22 \) and \( v(0) = 0.27 \); (e) \( x(0) = 1.3 \) and \( v(0) = 0.85 \).

| \( N \) | \( (a) \) | \( (b) \) | \( (c) \) | \( (d) \) | \( (e) \) |
| --- | --- | --- | --- | --- | --- |
| 0 | -17±1 | -16±1 | -16±1 | -16±1 | -16±1 |
| 1 | -15±1 | -13±1 | -14±1 | -15±1 | -13±1 |
| 2 | -9±1 | -11±1 | -13±1 | -12±2 | -11±2 |
| 3 | -6±2 | -1±1 | -12±1 | -9±1 | -11±1 |
| 4 | -4±2 | -7±2 | -12±1 | -7±2 | -5±2 |

\( Cmp \) is not a perfect measure of complexity, but a significant increase in the complexity of the differential equation of motion will be accompanied by a rise in \( Cmp \).

For forced oscillations described by the differential equation of motion

\[
\frac{dv}{dt} = -x - \sum_{j=0}^{N} \frac{\sin(j\omega t - 0.8j)}{j+1}, \quad v = \frac{dx}{dt}
\]  \( (8) \)

\( Cmp \) is calculated for series \( x(0.1k) \) for \( 1 + p \leq k \leq 110 + 9, \quad p = 0, 1, \ldots, 99 \) (Table 2.).
Table 2. $Cmp$ of various series calculated by solving differential equation of motion (8). Initial values are (a) $\omega=4.9$, $x(0)=0.75$ and $v(0)=-1.4$; (b) $\omega=5.5$, $x(0)=-1.6$ and $v(0)=0.3$; (c) $\omega=5.1$, $x(0)=1.1$ and $v(0)=3.8$; (d) $\omega=7.14$, $x(0)=7.4$ and $v(0)=-2.1$; (e) $\omega=3.0 \times x(0)=0$ and $v(0)=4.8$.

| N  | (a)  | (b)  | (c)  | (d)  | (e)  |
|----|------|------|------|------|------|
| 0  | -16±1| -16±1| -16±2| -16±1| -16±1|
| 1  | -13±1| -13±1| -13±1| -13±1| -15±1|
| 2  | -13±1| -13±1| -12±1| -13±1| -12±1|
| 3  | -12±1| -12±1| -12±1| -14±1| -10±1|
| 4  | -5±1 | -6±1 | -5±1 | -6±1 | -9±1 |
| 5  | -4±1 | -6±1 | -5±1 | -6±2 | -7±1 |
| 6  | -4±1 | -5±2 | -5±1 | -6±1 | -6±1 |

Solution of Schrödinger’s equation for quantum harmonic oscillator contains Hermite polynomial. When calculating Hermite polynomial of the seventh order

$$H_7(x) = -1680x + 3360x^3 - 1344x^5 + 128x^7$$

(9)

four additions and sixteen multiplications are performed. Number of additions and multiplications (complexity of the calculations) increases with order of the polynomial. By calculating complexity $Cmp$ we find

$$Cmp\left(H_j(x)\right) < Cmp\left(H_{j+1}(x)\right)$$

$$7 \leq j \leq 88, \ x = kh \ (k = 1,2,...,110), \ 1.5 \leq h \leq 7.5$$

(10)

When calculating Laguerre polynomial of the forth order

$$L_4(x) = 1 - 4x + 3x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4$$

(11)

four additions and ten multiplications are performed. Number of additions and multiplications increases with order of the polynomial. By calculating complexity $Cmp$ we find

$$Cmp\left(L_j(x)\right) < Cmp\left(L_{j+15}(x)\right)$$

$$2 \leq j \leq 40, \ x = kh \ (k = 1,2,...,110), \ 1.5 \leq h \leq 2.5$$

(12)

When calculating Legendre polynomial of the eight order

$$P_8(x) = \frac{35}{128} - \frac{315}{32}x^2 + \frac{3465}{44}x^4 - \frac{3003}{32}x^6 + \frac{6435}{128}x^8$$

(13)

four additions and twenty multiplications are performed. Number of additions and multiplications increases with order of the polynomial. By calculating complexity $Cmp$ we find

$$Cmp\left(P_j(x)\right) < Cmp\left(P_{j+1}(x)\right)$$

$$7 \leq j \leq 90, \ x = kh \ (k = 1,2,...,110), \ 1.5 \leq h \leq 5.5$$

(14)
3. Economy and complexity Cmp

While observing quarterly GDP in UK from 1955 until 2014 it is calculated that $Cmp = -2.366$ for years 1955-1984 and $Cmp = -0.827$ for years 1985-2014.

4. Mechanical equilibrium

$Cmp$ is calculated for series of velocity values of nonlinear oscillator.

$$v_j = \sqrt{5.1 - U(b - jh)}$$

$$U(b - jh) = \sum_{k=1}^{5} \frac{(b-2j)^{2k}}{26.31^k}$$

$$j = 1, 2, 3, ..., 110.$$  

Values of $h$ are such that:

0.0097 ≤ $h$ ≤ 0.0098, 1.1 ≤ $b$ ≤ 5 (left part of Figure 1.),

0.000097 ≤ $h$ ≤ 0.000098, 0.011 ≤ $b$ ≤ 0.05 (right part of Figure 1.).

![Figure 1](image1.png)

**Figure 1.** $Cmp$ calculated for series of velocities for nonlinear oscillator (15). Tendency towards equilibrium point $(b - 110h = 0)$ is followed with tendency towards higher complexity. $Cmp$ is not a perfect measure of complexity. Instabilities are particularly noticeable when complexity is very high and when complexity is very small (in this example complexity is very small)

Let’s observe free fall of body towards body of much greater mass. Path that lighter body passes in time $(k + N)h$ is

$$x_{k+N} = \frac{g}{2} (k + N)^2 h^2; \quad k = 1, 2, ..., 110$$

where $h$ is time step.

We are calculating $Cmp$ of position series for 0.00487731 ≤ $h$ ≤ 0.00487831. Complexity of this time series will be denoted with $Cmp_N$. For these values of $h$, we are obtaining

$$-7.23 \leq Cmp_0 - Cmp_{1320} \leq -0.25$$

By increasing $N$ we are obtaining series of values of coordinates which correspond to smaller distances between bodies (Table 3.). When calculating $Cmp$, system of independent linear equations needs to be solved. This means that series needs to be non-trivial enough. For many values of $h$, $Cmp$ cannot be calculated. Time step $h$ cannot be too long, because $g$ needs to remain constant. Time step $h$ also cannot be very short – we need to consider motion on significantly different heights to see noticeable change in $Cmp$. 


Table 3. In free fall lighter body is approaching equilibrium point and complexity is mostly increasing. Cmp is calculated for various values of $N$ and various values of time step $h$ from indicated interval. Similar results are obtained for many values of $h$ outside of this interval.

| $N$   | Probability that $Cmp_0 < Cmp_N$ |
|-------|----------------------------------|
| 330   | 0.96                             |
| 440   | 0.98                             |
| 770   | 0.92                             |
| 1320  | 1                                |

5. Thermodynamic equilibrium
We bring two equal bodies of different temperatures into contact. We are looking at a chain of 110 atoms, normal to the touch surface of the bodies; 55 atoms are in one and 55 are in another body. The system goes towards thermodynamic equilibrium. When the bodies are in non-equilibrium, atomic coordinates are given by

$$x_j = b_j + A_1 q_j, \quad 1 \leq j \leq 55$$

$$x_j = b_j + A_2 q_j, \quad 56 \leq j \leq 110$$

(18)

$A_1 q_j$ and $A_2 q_j$ are elongations, $q_j$ are randomly chosen numbers between -1 and 1, and $A_1$ and $A_2$ are amplitudes. When the equilibrium is established, the coordinates of the atom are

$$x_j = b_j + A_1 q_j, \quad 1 \leq j \leq 110, A = \frac{\sqrt{A_1^2 + A_2^2}}{\sqrt{2}}$$

(19)

We calculate with various realizations of random numbers. $Cmp$ is calculated along with standard deviation (Table 4.).

Table 4. When establishing thermodynamic equilibrium $Cmp$ increases.

| $b$     | $A_1$ | $A_2$ | Non-equilibrium $Cmp$ | Equilibrium $Cmp$ |
|---------|-------|-------|-----------------------|-------------------|
| 4.3567  | 1.0   | 0.01  | -7±1                  | -3±1              |
| 4.4852  | 1.15  | 0.014 | -6±1                  | -3±1              |
| 3.1773  | 0.85  | 0.011 | -6±1                  | -3±1              |

6. Crystal growth
Under the same external conditions, different crystals of the same material grow at different rates. Also, under the same external macroscopic conditions, specific crystal faces grow at different rates. That phenomenon is called growth rate dispersion (GRD). Growth rate dispersion occurs for a number of inorganic and organic crystals generated by primary or secondary nucleation. Growth rate dispersion is still not understood well [7-10].

Growth rate dispersion affects crystal size distribution. If we change the growth temperature or the solution supersaturation we are altering crystal growth rate (or size) distribution. Investigation of sodium chlorate crystals was performed in Laboratory of crystal growth, University of Belgrade.
Several experiments were performed, short time experiments with duration of about four hours, and long time experiments with duration of about twenty hours. Experimental setup is described in [11]. Analar grade of this system (99% purity) was used. Crystals grew from aqueous solutions prepared by equilibrating an excess of crystal with distilled water for 3 days at saturation temperature. Concentrations were calculated from empirical formulas (20) and (21).

\[ C_1 = 0.2267T(\text{C}) + 44.38(g\text{NaClO}_3/100\ g\ \text{solution}) \] \[ C_2 = 0.5088 \cdot 10^{-4}T^2(\text{C}^2) + 0.6816 \cdot 10^{-2}T(\text{C}) + 0.08033(kg\text{NaClO}_3/kg\text{H}_2\text{O}) \] (21)

Crystals were nucleated, grown and observed in a crystallization cell described in Figure 2. Crystals were nucleated in the cell by introducing air bubbles through a needle into the cell. Dimensions of the observed crystals were measured by a digital optical microscope (Nikon SMZ800) supplied with camera (Luminera, Infinity 1) using transmitted light. During each growth run, 15–33 crystal nuclei.

![Figure 2. Crystallization cell: a) real experiment [11], b) schematic draw used in computational experiment](image)

In this paper we present possible explanation of such crystal behavior. We calculate complexity and probability of nucleation of crystal, or complexity and the tendency of the system towards equilibrium. Let’s imagine the direction that goes through the system (crystallization cell). A specific point on that direction has a coordinate \( x \) (nucleation point). The diagram represents the assumed probability that at point \( x \) crystallization will occur after a certain time. Crystallization begins at places of maximum probabilities. Then there are places with smaller probability. The less likely it is, we wait longer for crystallization to occur.
Figure 3. a) Probability distribution of low complexity; b) Probability distribution of high complexity
Figure 4. Distributions with five maxima, each of them having one of two forms. Tenderness to the greater complexity is the tendency of distribution that contains both broad and narrow maxima, but narrow maxima are more dominant.
The results of computer experiments shown above are in accordance also with the experimentally obtained crystals. During experimental work with sodium chlorate crystals, which were dissolved and subsequently refaced, it was observed that not all faces grow at the same time (Figure 5). It occurred that some of the faces did not grow, whereas the neighbor faces of the same crystals grew. The dissolution and refaceting of the crystals did not affect the probability of the nongrowing face appearance and the reduction of the range of face growth rates. Nongrowing faces are very stable; they start to grow at a relatively high supersaturation, face by face or several faces simultaneously. The order of the start of the growth is random, subsequently grown neighbor or opposite faces.

![Figure 5. Time evolution of one Sodium Chlorate crystal after dissolution [14]](image)

The procedure explained in the first part, which refers to the probability of nucleation, can be used as a possibility to explain why, after refaceting, some faces of crystals continue to grow after a certain time, while others grow all the time. As mentioned earlier what is more likely we wait shorter, while what is less likely we wait longer. This can be used to explain the behavior of facet growth after refaceting. The probability that all four facets appear at the same time is more probable and it represents a lower complexity (Figure 3a), while in the real experiment the facets appear after a certain time, which represents a higher complexity (Figure 3b). It should also be noted here that in the real experiment the crystals in which all four facets grow after refaceting dominate which is also in accordance with the results of the computer simulations presented above.

7. Conclusion

*Cmp* measures the non-triviality of the data connection that describes the system. This non-triviality is called complexity, and the generally accepted definition of complexity does not exist. We notice that there is a tendency of *Cmp* increase while reaching mechanical equilibrium, for example with the harmonic oscillator and when one body falls to another, and tendency of *Cmp* increase while going towards thermodynamic equilibrium. It seems that the experiment with the crystals growth and unusual crystal behavior after refaceting could be interpreted as a tendency towards equilibrium and greater *Cmp*.

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