On the existence of Kundt’s metrics with compact sections of null hypersurfaces

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Abstract

It is shown that Kundt’s metric for vacuum cannot be constructed when two-dimensional space-like sections of null hypersurfaces are compact, connected manifolds with no boundary unless they are tori or spheres, i.e. higher genus $g \geq 2$ is excluded by vacuum Einstein equations. The so-called basic equation (resulting from Einstein equations) is examined. This is a non-linear PDE for unknown covector field and unknown Riemannian structure on the two-dimensional manifold. It implies several important results derived in this paper. It arises not only for Kundt’s class but also for degenerate Killing horizons and vacuum degenerate isolated horizons.

1 Introduction

Let us consider a null hypersurface in a Lorentzian spacetime $M$ which is a three-dimensional submanifold $S \subset M$ such that the restriction $g_{ab}$ of the spacetime metric $g_{\mu\nu}$ to $S$ is degenerate.

We shall often use adapted coordinates, where coordinate $x^3 = u$ is constant on $S$. Coordinates on $S$ will be labeled by $a, b = 0, 1, 2$ and sometimes coordinate $x^0$ will be denoted by $v$, finally, coordinates on $B_v := \{x \in S \mid x^0 = v = \text{const}\}$ will be labeled by $A, B = 1, 2$. Spacetime coordinates will be labeled by Greek characters $\alpha, \beta, \mu, \nu$.

The non-degeneracy of the spacetime metric implies that the metric $g_{ab}$ induced on $S$ from the spacetime metric $g_{\mu\nu}$ has signature $(0, +, +)$. This means that there is a non-vanishing null-like vector field $X^a$ on $S$, such that its four-dimensional embedding $X^\mu$ to $M$ (in adapted coordinates $X^3 = 0$) is orthogonal to $S$. Hence, the covector $X_\nu = X^\mu g_{\mu\nu} = X^a g_{av}$ vanishes on vectors tangent to $S$ and, therefore, the following identity holds:

$$X^a g_{ab} \equiv 0.$$ (1)

It is easy to prove (cf. [3]) that integral curves of $X^a$, after a suitable reparameterization, are geodesic curves of the spacetime metric $g_{\mu\nu}$. Moreover, any null hypersurface $S$ may always be embedded in a one-parameter congruence of null hypersurfaces.

We assume that topologically we have $S = I \times S^2$ where $I \subset \mathbb{R}^1$ is a real interval. Since our considerations are purely local, we fix the orientation of the $\mathbb{R}^1$ component and assume that null-like vectors $X$ describing degeneracy of the metric $g_{ab}$ of $S$ will be always compatible with this orientation. Moreover, we shall always use coordinates such that the coordinate $x^0$ increases in the direction of $X$, i.e., inequality $X(x^0) = X^0 > 0$ holds. In these
coordinates degeneracy fields are of the form $X = f(\partial_0 - n^A \partial_A)$, where $f > 0$, $n_A = g_{0A}$ and we rise indices with the help of the two-dimensional matrix $\tilde{g}^{AB}$, inverse to $g_{AB}$.

If by $\lambda$ we denote the two-dimensional volume form on each surface $x^0 = \text{const}$:

$$\lambda := \sqrt{\det g_{AB}},$$

then for any degeneracy field $X$ of $g_{ab}$ the following object

$$v_X := \frac{\lambda}{X(x^0)},$$

is a well defined scalar density on $S$ according to [4]. This means that $v_X = v_X dx^0 \wedge dx^1 \wedge dx^2$ is a coordinate-independent differential three-form on $S$. However, $v_X$ depends upon the choice of the field $X$.

It follows immediately from the above definition that the following object:

$$\Lambda = v_X X$$

is a well defined (i.e., coordinate-independent) vector density on $S$. Obviously, it does not depend upon any choice of the field $X$:

$$\Lambda = \lambda(\partial_0 - n^A \partial_A).$$

Hence, it is an intrinsic property of the internal geometry $g_{ab}$ of $S$. The same is true for the divergence $\partial_a \Lambda^a$, which is, therefore, an invariant, $X$-independent, scalar density on $S$. Mathematically (in terms of differential forms), the quantity $\Lambda$ represents the two-form:

$$L := \Lambda^a (\partial_a \lrcorner dx^0 \wedge dx^1 \wedge dx^2),$$

whereas the divergence represents its exterior derivative (a three-from): $dL := (\partial_a \Lambda^a) dx^0 \wedge dx^1 \wedge dx^2$. In particular, a null surface with vanishing $dL$ is called a non-expanding horizon (see [1]).

The examples of spacetimes obeying Einstein equations suggest that non-expanding horizons are rather isolated objects. In this paper we consider the problem of existence of one-parameter congruence of local non-expanding horizons. The family of null hypersurfaces which are simultaneously non-expanding horizons leads to the algebraically special spacetimes so called non-diverging solutions or Kundt’s class of metrics (see chapter 31 in [5]).

2 Topological rigidity

Following chapter 31 of [5] let us consider the line element in the following form:

$$ds^2 = g_{AB} dx^A dx^B - 2du (dv + m_A dx^A + H du), \quad A, B = 1, 2$$

or equivalently (see (31.6) in [5]) in a complex notation:

$$ds^2 = 2P^{-2} d\zeta d\bar{\zeta} - 2du (dv + W d\zeta + \bar{W} d\bar{\zeta} + H du)$$

We assume that two-dimensional section (parameterized by coordinates $x^1, x^2$ or $\zeta, \bar{\zeta}$) of null hypersurfaces $u = \text{const}$ is a compact, connected manifold $B$ with no boundary. The extrinsic curvature $l_{ab} = -\frac{1}{2} \hat{\epsilon}_{X} g_{ab}$ of the null hypersurface $u = \text{const}$ vanishes because $\partial_\nu g_{AB} = 0$ (cf. [3]).

1We assume that coordinates $u$ and $v$ are only in small neighborhood and $M$ is constructed locally around given sphere.
**Theorem 1.** For any Riemannian metric \( g_{AB} \) on a two-dimensional, compact, connected manifold with no boundary and genus \( g \geq 2 \) the vacuum Einstein equations imply no solutions for the line element (4) which describes Kundt’s class of metrics.

**Proof.** Einstein-Maxwell equations for Kundt’s metrics split into system of non-linear two-dimensional partial differential equations (eqs (31.21) in [5])

\[
\begin{align*}
(P^2 W_v)_\xi - \frac{1}{2} (W_v)^2 &= 0 , \\
\Phi_{1, \xi} &= W_v \Phi_1 , \\
\Delta \ln P + \frac{1}{2} P^2 \left( W_{v \xi} + W_{v \bar{\xi}} - 2 W_{v \bar{V}_v} \right) &= 2 \kappa_0 |\Phi_1|^2,
\end{align*}
\]

The remaining Einstein-Maxwell equations (see 31.22-25 in [5]) reduce to polynomial dependence on \( v \) and linear problems on \( B \) if we assume that \( u \)-dependence is given.

In the case of vacuum \(^2\) the equations (6) imply the following basic equation on \( B \):

\[
\omega_{A||B} + \omega_{B||A} + 2 \omega_{A\omega B} = R_{AB} ,
\]

where \( \omega_A \) corresponds to \( \partial_v W , || \) denotes covariant derivative with respect to the metric \( g_{AB} \) and \( R_{AB} \) is its Ricci tensor. The equation (7) is a starting point of our considerations and it is a special case of (3.7) in [1], if we assume that \( \tilde{S}_{AB} \) vanishes.

The traceless part of (7) reads

\[
\omega_{A||B} + \omega_{B||A} - g_{AB} \omega_C ||_C = -2 \omega A \omega_B + g_{AB} \omega^C \omega_C
\]

and for the trace we get

\[
\omega^A ||_A = K - \omega^A \omega_A ,
\]

where \( K := \frac{1}{2} \tilde{g}^{-1}_{AB} R_{AB} \) is the Gaussian curvature of \( B \) and \( \tilde{g}^{AB} \) is the two-dimensional inverse metric.

Let us notice that eq. (9) and Gauss-Bonnet theorem

\[
2 - 2g = \frac{1}{2\pi} \int_B \lambda K
\]

exclude immediately the case \( g \geq 2 \) because

\[
0 \leq \int_B \omega_A \omega^A = \int_B \lambda K < 0
\]

which is impossible.

Moreover, for \( g = 1 \) equation (9) and (10) imply that on a torus the vector field \( \omega^A \) is vanishing and we obtain the following result:

**Theorem 2.** For any Riemannian metric \( g_{AB} \) on a two-dimensional torus equation (7) possesses only trivial solutions \( \omega^A \equiv 0 \equiv K \) and the metric \( g_{AB} \) is flat.

The Theorems [1] and [2] do not cover the most interesting case \( B = S^2 \). We would like to add some more observations which are valid in general case before we restrict ourselves to the case when manifold \( B \) is a sphere.

Contracting equation (8) with \( \omega^A \omega_B \), we obtain the following identity:

\[
\omega^B (\omega^A \omega_A) ||_B = \omega^A \omega_B ||_B - (\omega^A \omega_A)^2 .
\]

\(^2\)It is enough to assume that \( \Phi_1 = 0 \).
Using (9) and (11), one can check the following equality
\[
\omega_B^A \alpha = (2\alpha + 1) \omega_A \omega_A^{\alpha + 1} + (\alpha + 1) \omega_A^{\alpha + 1} K,
\] (12)
which finally implies one-parameter family of integral identities
\[
\frac{2\alpha + 1}{\alpha + 1} \int_B \lambda F^{\alpha + 1} = \int_B \lambda K F^\alpha,
\] (13)
where \(F := \omega_A \omega_A\) and \(\lambda := \sqrt{\det g_{AB}}\).

Suppose \(\omega_A\) has only finite set of critical points which are isolated. Then using eq. (12) for \(\alpha = -\frac{1}{2}\) we obtain
\[
\left[ \frac{\omega_B^A}{\sqrt{\omega_A \omega_A}} \right]_{||B} = \frac{1}{2} \frac{K}{\sqrt{\omega_A \omega_A}}.
\]
(14)

Surrounding critical points of \(\omega_A\) by small circles and passing to the limit (i.e. shrinking circles to critical points) one can check that
\[
\int_B \frac{\lambda K}{\sqrt{\omega_A \omega_A}} = 0,
\]
which is a special case of (13) for \(\alpha = -\frac{1}{2}\).

Now, let us restrict ourselves to the case \(B = S^2\). From Gauss-Bonnet theorem we have
\[
\int_{S^2} \lambda K = 4\pi > 0.
\]
Hence the condition (14) implies that \(K\) must be negative on some open subset of \(S^2\). The above considerations can be summarized by the following

**Theorem 3.** There are no solutions of equation (7) with the following properties:

- \(\omega_A = 0\) only at finite set of points,
- \(B\) is a sphere with non-negative Gaussian curvature.

When two-dimensional surface \(B\) is a sphere then the corresponding null hypersurface is the non-expanding horizon. A one-parameter family of non-expanding horizons is still possible in the case of vacuum Einstein equations but the Gaussian curvature has to be negative on some domains or the vector field \(\omega_A\) vanishes on infinite set of points.

Let us observe that
\[
\int_{S^2} \lambda \omega_A \omega_A = \int_{S^2} \lambda K = 4\pi
\]
implies that \(\omega_A \neq 0\) on an open subset. On the other hand, we show in the next Section an example of solutions for Einstein-Maxwell equations with \(\omega_A \equiv 0\), which means that some arguments used above are no longer true for non-vacuum solutions.

### 3 Axially symmetric solutions for Einstein-Maxwell equations

The two-dimensional part of Einstein-Maxwell equations (6) can be written as follows
\[
\omega_{A||B} + \omega_{B||A} - g_{AB} \omega_C ||C = -2\omega_A \omega_B + g_{AB} \omega_C \omega_C,
\]
\[
f_{A||B} + f_{B||A} - g_{AB} f_C ||C = -2f_A \omega_B - 2f_B \omega_A + 2g_{AB} f_C \omega_C,
\]
\[
K - \omega_A ||A - \omega_A \omega_A = \kappa_0 f_A f_A,
\]
(16)
where \( f_A := F_{vA} \) is a covector on \( B \) corresponding to \( \Phi_1 \). The all objects in (10), namely \( \omega_A, f_A \) and \( g_{AB} \), do not depend on \( v \).

Some arguments from the proof of Theorem 1 can be generalized to the case with Maxwell field. In particular, the equation (12) takes now the following form:

\[
\left[ \omega^B (\omega^A \omega_A)^\alpha \right]_{||B} = -(2\alpha + 1) (\omega^A \omega_A)^{\alpha + 1} + (\alpha + 1) (\omega^A \omega_A)^\alpha (K - \kappa_0 f_A f^A),
\]

and we get the same integral identity (13) for functions \( F = \omega^A \omega_A \) and \( K - \kappa_0 f_A f^A \) instead of \( K \). Moreover, for genus \( g \neq 0 \) we can repeat the arguments and we get

\[
4\pi(1 - g) = \int_B \lambda K = \int_B \lambda (\omega^A \omega_A + \kappa_0 f_A f^A),
\]

which implies \( g = 1, \omega_A \equiv 0, f_A \equiv 0 \) and finally \( K = 0 \). This result can be described as follows

**Theorem 4.** For any Riemannian metric \( g_{AB} \) on a two-dimensional, compact, connected manifold with no boundary and genus \( g \geq 1 \) equations (10) have no solutions for \( g \geq 2 \) and for \( g = 1 \) they possess only trivial solutions i.e. \( \omega_A \equiv 0, f_A \equiv 0 \) and flat metric \( g_{AB} \).

The Theorem 4 together with the assumption of non-triviality of \( f_A \) restricts ourselves to \( B = S^2 \). Moreover, let us assume that \( \omega_A \equiv 0 \). Hence, equations (16) reduce to the following system of equations on \( S^2 \):

**18**

\[
f_A||B + f_B||A - g_{AB} f_C||C = 0,
\]

**19**

\[
K = \kappa_0 f_A f^A
\]

The equation (18) simply means that \( f^A \) is a conformal vector field on \( S^2 \). Moreover, the metric \( g_{AB} \) is always conformally related to a round unit sphere metric \( h_{AB} \) i.e.

\[
\frac{g_{AB}}{\Omega^2} = \frac{h_{AB}}{\Omega^2}, \quad \Omega > 0
\]

and eq. (19) reduces to

\[
-\Delta_h \ln \Omega = \kappa_0 \Omega^2 h_{AB} f^A f^B - 1,
\]

where \( \Delta_h \) is the Laplace-Beltrami operator on \( S^2 \) with respect to the metric \( h_{AB} \).

The construction of all axially symmetric solutions of equations (18) and (20) can be obtained as follows: In the coordinate system \((\theta, \phi)\) such that \( h_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta d\phi^2 \) the axially symmetric solutions of (18) belong to the following two-dimensional family of conformal vector fields:

\[
f^\theta = -a \sin \theta, \quad f^\phi = b
\]

and \( h_{AB} f^A f^B = (a^2 + b^2) \sin^2 \theta \). Hence, if we assume \( \partial_\phi \Omega = 0 \), the equation (20) simplifies to the following form

\[
\frac{d}{dx} \left[ (1 - x^2) \frac{d \ln \Omega}{dx} \right] = 1 - d(1 - x^2)\Omega^4,
\]

where \( d := \kappa_0 (a^2 + b^2) \) is a positive real constant and new coordinate \( x := \cos \theta \). A general solution of (21)

\[
\Omega^4 = \frac{4 \beta^2 c(1 - x^2)^{3-2}}{d [2c(1 + x)^3 + (1 - x)^3]^2}
\]
becomes regular and positive for $\beta = 2$ and
\[ \Omega^2 = \frac{4}{2c(1 + x)^2 + (1 - x)^2} \sqrt{\frac{c}{d}} \]
is an admissible conformal factor for any positive constant $c$.

The above result can be extended to the full space-time Einstein-Maxwell solution in the Kundt’s form similar to (31.57) in [5] or rather to (31.55) with $G^0 = \Phi_2 = \partial_u(\ln P) = \partial_u\Phi_1 = W = 0$. In this simple case the equations (31.56) reduce to $\Delta H^0 = 0$ hence $H^0 = \text{const.}$.

4 Other facts resulting from basic equation

We start again with equation (7):
\[ \omega^A || B + \omega_B ||^A + 2 \omega^A \omega_B = R^A_B = \frac{1}{2} R \delta^A_B, \tag{22} \]
where $\omega^A$ is now a vector, $\omega_B = g_{AB} \omega^B$, $||$ denotes covariant derivative with respect to the metric $g_{AB}$ and $R_{AB}$ is its Ricci tensor. The above equation appears not only in the context of Kundt’s class, it also arises in the study of vacuum degenerate isolated horizons [1], [6], [7]. Moreover, any degenerate Killing horizon also implies this equation [2]. Hence, it is important to explore properties of this equation. We already know that for axial symmetry and spherical topology there is a unique solution – extremal Kerr (see [7]). Moreover, when one-form $\omega_B dx^B$ is closed (e.g. static degenerate horizon [2]) there are no solutions of (7). However, in general, the space of solutions is not known.

The traceless part of (22) reads
\[ \omega_A || B + \omega_B || A - g_{AB} \omega^C || C = -2 \omega_A \omega_B + g_{AB} \omega^C \omega_C \]
and for the trace we get
\[ \omega^A ||^A = K_g - \omega^A \omega_A, \tag{24} \]
where $K_g := \frac{1}{2} \tilde{g}^{AB} R_{AB}$ is the Gaussian curvature of $g$ and $\tilde{g}^{AB}$ is the two-dimensional inverse metric.

Let us notice that eq. (24) enables one to perform conformal transformation which leads to non-negative curvature. More precisely, let us choose $\alpha$ such that
\[ \Delta_g \alpha = \omega^A || A \]
then from (24) we get
\[ K_g - \Delta_g \alpha = \omega^A \omega_B g_{AB}, \tag{25} \]

Now, we define
\[ h_{AB} := \exp(2\alpha) g_{AB} \]
and hence
\[ \exp(2\alpha) K_h = K_g - \Delta_g \alpha \]
\footnote{We remind that $\Omega$ corresponds to $P$ and the vector field $f^A$ to $\Phi_1$.}
and finally
\[ K_h = \exp(-4\alpha)\omega^A\omega^B h_{AB} \]
is non-negative. Moreover, traceless part (23) is invariant with respect to conformal rescaling (26) of the metric \( g \):
\[ \Gamma^A_{BC}(h) = \Gamma^A_{BC}(g) + \delta^A_B\alpha_C + \delta^A_C\alpha_B - g_{BC}\alpha_\omega D^A \]
\[ \omega^A||B + \omega_B||A - \delta^A_B\omega^C||C = \nabla_B(h)\omega^A + \nabla^A(h)(h_{BC}\omega^C) - \delta^A_B\nabla_C(h)\omega^C \]
hence we get
\[ \nabla_B(h)\omega^A + \nabla^A(h)(h_{BC}\omega^C) - \delta^A_B\nabla_C(h)\omega^C = -2\omega^A\omega_B + \delta^A_B\omega^C\omega_C \]
\[ = \exp(-2\alpha)(-2\omega^A h_{BC}\omega^C + \delta^A_B h_{CD}\omega^C\omega^D) \] (27)
Contracting equation (24) with \( \omega^A\omega^B \), we obtain the following identity:
\[ \omega^B\nabla_B \left( \|\omega\|^2 \right) = \|\omega\|^2\nabla_B\omega^B - \exp(-2\alpha) \left( \|\omega\|^2 \right)^2, \] (28)
where \( \|\omega\| := \sqrt{h_{AB}\omega^A\omega^B} \).
The last equality implies that when \( \|\omega\| > 0 \) there are no solutions of equation (22). More precisely, we have:
\[ \nabla_B \left( \omega^B\|\omega\|^{-2} \right) = \exp(-2\alpha) = \frac{\sqrt{K_h}}{\|\omega\|} > 0, \] (29)
and integrating the above equality we get a contradiction. This is not surprising because any vector field on a sphere vanishes at least at one point.

### 4.1 Integrability conditions

Equation (22) written in the following equivalent form:
\[ \omega^A||B = f\varepsilon^A B + \frac{1}{4} R g_{AB} - \omega_A\omega_B, \] (30)
where \( f := \frac{1}{2}\omega^A||B \varepsilon^{AB} \) is an unknown function on a sphere, implies as follows:
\[ \omega^A||BC\varepsilon^{BC} = -f_A - 3f\omega_A + \frac{1}{4} \varepsilon^{AB} \left( R^{||B} + R\omega^B \right). \]
Moreover, definition of curvature gives
\[ \omega^A||BC\varepsilon^{BC} = R^D_{ABC}\omega^D\varepsilon^{BC} \]
where
\[ R^D_{ABC} = \frac{1}{2} R \left( \delta^D_B g_{AC} - \delta^D_C g_{AB} \right) \]
hence
\[ R^D_{ABC}\varepsilon^{BC} = R\varepsilon^D_A. \]
Using the above formulae and the identity
\[ f^||_{AB} \varepsilon^{AB} = 0 \]
we can derive the following integrability condition:
\[ \frac{1}{4} R^{||A} + 2(R\omega^A)||A = 6f^2 + \frac{3}{8} R(R - 12\omega_A\omega^A). \] (31)
Equation (31) implies that there exists non-empty open subset where \( 12\omega_A\omega^A > R > 0 \).
4.2 Transformation to linear problem

Let us denote
\[ \Phi_A := \frac{\omega_A}{\omega_B \omega_B}. \]

For any domain where \( \omega_B \omega_B > 0 \) equation (30) implies
\[ \Phi_A|C \varepsilon^{AC} = \left( \frac{\omega_A}{\omega_B \omega_B} \right)_{|C} \varepsilon^{AC} = 0 \] (32)

which simply means that the one-form \( \Phi_A dx^A \) is closed and locally there exists coordinate \( \Phi \) such that
\[ d\Phi = \Phi_A dx^A. \]

Moreover, from (30) we get
\[ \Phi_A|A = 1 \] (33)

hence the potential \( \Phi \) is a solution of the Poisson’s equation:
\[ \Delta \Phi = 1. \] (34)

Remark If we choose one isolated point where \( \omega \) vanishes then for a given metric \( g \) we have unique solution of the above Laplace-Beltrami equation (Green function in the enlarged sense). For more isolated points we can take linear combination of such solutions. More precisely, let \( G_{x_0} \) be a unique solution (for a given metric \( g \)) of the equation (34) on \( S^2 - \{x_0\} \). If \( c_0 + c_1 + \ldots + c_n = 1 \) (where \( c_i \in \mathbb{R} \)) then \( \Phi = c_0 G_{x_0} + c_1 G_{x_1} + \ldots + c_n G_{x_n} \) is a solution of (34) on \( S^2 - \{x_0, x_1, \ldots, x_n\} \) and \( \omega \) vanishes at the points \( x_0, x_1, \ldots, x_n \).

4.3 Solution of the problem with axial symmetry

Let us consider axially symmetric two-metric on a sphere in the following form:
\[ g = 2m^2 \left[ A^{-1}(\theta)d\theta^2 + A(\theta) \sin^2 \theta d\phi^2 \right] \] (35)

where \( A : [0, \pi] \to \mathbb{R} \) is a positive smooth function with boundary values \( A(0) = A(\pi) = 1 \) and positive constant \( m^2 \) controls the total volume of a sphere. Eq. (35) implies that \( \lambda = \sqrt{\text{det} g_{AB}} = 2m^2 \sin \theta \). From (33) we get
\[ \partial_A (\lambda \Phi^A) = \lambda \]

hence
\[ \lambda \Phi^\theta = -2m^2 (\cos \theta + C) \]

where \( C \) is a constant of integration. Moreover, from (32) we obtain \( \partial_\theta \Phi_\phi = 0 \) and
\[ \Phi_\phi = 2m^2 \alpha \]

with arbitrary constant \( \alpha \). The equation (34), in terms of \( \Phi^A \), takes the following form:
\[ \partial_A \left( \frac{\lambda \Phi^A}{\Phi_B \Phi_B} \right) + \frac{\lambda}{\Phi_B \Phi_B} - \lambda K = 0. \] (36)
The square of vector $\Phi A$:

$$\Phi^2 A = 2m^2 \frac{(\cos \theta + C)^2 + \alpha^2}{A \sin^2 \theta},$$

Gaussian curvature:

$$\lambda K = \frac{1}{2} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{1}{\partial \theta} (A \sin^2 \theta) \right]$$

and the equation (36) imply that the function $A$ obeys the following linear ODE:

$$\frac{d}{dx} \left[ (x + C) y \right] + \frac{y}{(x + C)^2 + \alpha^2} + \frac{1}{2} \frac{d^2 y}{dx^2} = 0 \quad (37)$$

where $x := \cos \theta$ and $y := A \sin^2 \theta$. For $\alpha = 0$ we get

$$\frac{d^2 y}{dx^2} \left[ (x + C) y \right] = 0$$

with a general solution $y = \frac{ax + b}{x^2 + \alpha^2}$. However, in the case $\alpha = 0$ the function $A = \frac{y}{x^2 + \alpha^2}$ can not be regular at both points $+1$ and $-1$ simultaneously. Nonexistence of regular solutions for $\alpha = 0$ confirms the main result of [2] because $\Phi_\phi = 0$ gives $\omega_\phi = 0$ which obviously implies $d \omega = 0$.

For $\alpha \neq 0$ we take a new variable $t := \frac{x + C}{\alpha}$ and the equation (37) takes the form

$$\frac{d}{dt} \left[ \frac{d}{dt} (ty) - \frac{2y}{1 + t^2} \right] = 0$$

with the following general solution

$$y = \frac{at + b(t^2 - 1)}{t^2 + 1},$$

(38)

with arbitrary constants $a, b$. The solution (38) gives the following form of the function $A$:

$$A = \frac{y}{1 - x^2} = \frac{a \alpha(x + C) + b[(x + C)^2 - \alpha^2]}{(1 - x^2)[(x + C)^2 + \alpha^2]}.$$

Regularity of $A$ at $x = \pm 1$ implies that $C^2 = 1 - \alpha^2$ (hence $0 < |\alpha| \leq 1$) and $\Phi \alpha + 2C = 0$ which gives

$$A = \frac{-b}{[(x + C)^2 + \alpha^2]}.$$

Moreover, $A(\pm 1) = 1$ implies $b = -2$, $\alpha = 1$, $C = 0$ hence

$$A = \frac{2}{1 + x^2} = \frac{2}{1 + \cos^2 \theta},$$

and finally

$$g = 2m^2 \left[ \frac{1 + \cos^2 \theta}{2} d\theta^2 + \frac{2 \sin^2 \theta}{1 + \cos^2 \theta} d\phi^2 \right]$$

(39)

and

$$\omega^\theta = -\frac{\sin \theta \cos \theta}{m^2(1 + \cos^2 \theta)^2}, \quad \omega^\phi = \frac{1}{2m^2(1 + \cos^2 \theta)},$$

(40)
which corresponds to extremal Kerr with mass $m$ and angular momentum $m^2$.

It is worth to notice that the solution (10) in terms of $\Phi_A$ has a simple and natural form. More precisely, equations (32) and (33) extended through the “poles” are the following:

$$\Phi_A|_{C\varepsilon^AC} = 4\pi m^2 (\delta_{\theta=\pi} - \delta_{\theta=0}),$$  

$$\Phi^A|_{A} = 1 - 4\pi m^2 (\delta_{\theta=\pi} + \delta_{\theta=0}),$$

where by $\delta_p$ we denote a Dirac delta at point $p$ and $8\pi m^2 (= \int \lambda)$ is a total volume of the sphere (39).

Let $G_p$ be a Green function satisfying

$$\begin{cases}
\Delta G_p = 1 - 8\pi m^2 \delta_p, \\
\int \lambda G_p = 0.
\end{cases}$$

The potentials $\Phi$, $\tilde{\Phi}$ for the covector field $\Phi_A$ defined (up to a constant) as follows

$$\Phi_A = \partial_A \Phi + \varepsilon^A_B \partial_B \tilde{\Phi}$$

take a simple form

$$\Phi = \frac{1}{2} (G_{\theta=0} + G_{\theta=\pi})$$

$$\tilde{\Phi} = \frac{1}{2} (G_{\theta=0} - G_{\theta=\pi})$$

because equations (41), (42) and (44) imply

$$\Delta \Phi = 1 - 4\pi m^2 (\delta_{\theta=\pi} + \delta_{\theta=0}),$$

$$\Delta \tilde{\Phi} = 4\pi m^2 (\delta_{\theta=\pi} - \delta_{\theta=0}).$$

Moreover, the Green functions for extremal Kerr (39) are given in the explicit form:

$$G_{\theta=0} = 4m^2 \left[ \frac{1}{2} \sin^2 \frac{\theta}{2} + \frac{1}{8} \sin^2 \theta - \log(\sin \frac{\theta}{2}) + \frac{1}{3} \right],$$

$$G_{\theta=\pi} = 4m^2 \left[ \frac{1}{2} \cos^2 \frac{\theta}{2} + \frac{1}{8} \sin^2 \theta - \log(\cos \frac{\theta}{2}) + \frac{1}{3} \right].$$

5 Conclusions

We have discussed some geometric consequences of the basic equation (7) appearing in the context of Kundt’s class metrics and degenerate (extremal) horizons. We have obtained several important results like topological rigidity of the horizon (Section 2 and 3), integrability conditions and transformation to linear problem which simplifies the proof of the uniqueness of extremal Kerr for axially symmetric horizon (Section 4). However, the problem of the existence of non-symmetric solutions to the basic equation remains opened.

A Extremal Kerr

For extremal Kerr we have

$$h = \left( 1 - \frac{1}{2} \sin^2 \theta \right)^2 d\theta^2 + \sin^2 \theta d\varphi^2, \quad \lambda_h := \sqrt{\det h_{AB}} = \frac{1 + \cos^2 \theta}{2} \sin \theta$$
\[ \exp(-2\alpha) = \frac{4m^2}{1 + \cos^2 \theta}, \quad K_h = \frac{4 \sin^2 \theta}{(1 + \cos^2 \theta)^3} \]

\[ \omega^\theta = -\frac{\sin \theta \cos \theta}{m^2(1 + \cos^2 \theta)^2}, \quad \omega^\varphi = \frac{1}{2m^2(1 + \cos^2 \theta)}, \quad \|\omega\| = \frac{\sin \theta}{2m^2 \sqrt{1 + \cos^2 \theta}} \]

Equation (29) for extremal Kerr takes a simple form:

\[ \lambda_h \omega^\theta \|\omega\|^{-2} = -2m^2 \cos \theta, \quad \lambda_h \exp(-2\alpha) = \lambda_h \frac{\sqrt{K_h}}{\|\omega\|} = 2m^2 \sin \theta. \]

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