JULIA SETS WITH AHLFORS-REGULAR CONFORMAL DIMENSION ONE

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Abstract. For a post-critically finite hyperbolic rational map \( f \), we show that the Julia set \( J_f \) has Ahlfors-regular conformal dimension one if and only if \( f \) is a crochet map, i.e., there is an \( f \)-invariant graph \( G \) containing the post-critical set such that \( f|_G \) has topological entropy zero.

We use finite subdivision rules to obtain graph virtual endomorphisms, which are 1-dimensional simplifications of post-critically finite rational maps, and approximate the asymptotic conformal energies of the graph virtual endomorphisms to estimate the Ahlfors-regular conformal dimensions. In particular, we develop an idea of reducing finite subdivision rules and prove the monotonicity of asymptotic conformal energies under the decomposition of rational maps.

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1. INTRODUCTION

Julia sets are fractals defined by dynamical properties of iterations of rational maps. The dynamics of rational maps tends to be more complicated as their Julia sets are more intricate. Hausdorff dimension is a classical invariant of fractals and has been widely used as a measurement of the complexity of Julia sets. However Hausdorff dimension is sometimes too sensitive to understand topological properties of the dynamics of rational maps. Ahlfors-regular conformal dimension is a variant of Hausdorff dimension that is less sensitive to geometric deformations. For example, rational maps in the same hyperbolic component have the same Ahlfors-regular conformal dimension while Hausdorff dimensions vary. Julia sets of post-critically finite rational maps have Ahlfors-regular conformal dimension between 1 and 2. A Julia set has the maximal Ahlfors-regular conformal dimension, \( \text{ARC}. \dim(J_f) = 2 \), if and only if \( J_f \) is the whole Riemann sphere. However the other extreme case, \( \text{ARC}. \dim(J_f) = 1 \), includes various Julia sets, such as post-critically finite polynomials or Newton maps, by contrast with Hausdorff dimension. In this paper, we characterize post-critically finite hyperbolic rational maps whose Julia sets have Ahlfors-regular conformal dimension one.
Theorem A. For a hyperbolic post-critically finite rational map $f$, the Ahlfors-regular conformal dimension of the Julia set $J_f$ is one if and only if $f$ is a crocheted map.

Let us see more details and backgrounds on the terminologies written in Theorem A.

**Hyperbolic post-critically finite rational maps.** For a rational map $f : \hat{\mathbb{C}} \to \mathbb{C}$, a point $z \in \hat{\mathbb{C}}$ is critical if $f'(z) = 0$, or, equivalently, $f$ is not a local homeomorphism near $z$. Denote by $\text{Crit}(f)$ the set of all critical points of $f$. To control the orbits of critical points, we may assume that the post-critical set $P_f$, defined by

$$P_f = \{ f^n(c) \mid c \in \text{Crit}(f) \text{ and } n > 0 \},$$

is a finite set. We say that $f$ is post-critically finite if $P_f$ is finite. One reason for the popularity of post-critically finite rational maps is Thurston’s characterization, which gives a one-to-one correspondence between post-critically finite rational maps and certain homotopy classes of post-critically finite topological branched coverings, see Theorem 2.4. Thus topological ideas can be well applied to investigate the dynamical properties of post-critically finite rational maps.

A rational map $f : \hat{\mathbb{C}} \to \mathbb{C}$ is hyperbolic if every critical point is attracted to an attracting periodic cycle. The set of hyperbolic rational maps of degree $d$, denoted by $\mathcal{H}_d$, is open in the set of all rational maps of degree $d$, denoted by $\text{Rat}_d$. It is a famous conjecture in complex dynamics that $\mathcal{H}_d$ is dense in $\text{Rat}_d$. Each connected component of $\mathcal{H}_d$ is called a hyperbolic component. By [MnSS83], if two rational maps $f$ and $g$ are in the same hyperbolic component, then their Julia sets $J_f$ and $J_g$ are quasi-symmetrically equivalent with respect to the spherical metric. Moreover, if the Julia sets of a hyperbolic component are connected, then the hyperbolic component contains a unique rational map that is post-critically finite, see [McM88, Corollary 3.6] and [Mil12, Corollary 5.2]. To sum up, a conjecturally generic (an element in an open and dense subset in this context) connected Julia set is quasi-symmetrically equivalent to the Julia set of a hyperbolic post-critically finite rational map.

**Ahlfors-regular conformal dimension.** Two metric spaces $(X, d_X)$ and $(Y, d_Y)$ are quasi-symmetric (or quasi-symmetrically equivalent) if there is a distortion function $\eta : [0, \infty) \to \mathbb{R}$ that is a homeomorphism such that, for every distinct $x, y, z \in X$, we have

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta \left( \frac{d_X(x, y)}{d_X(x, z)} \right).$$

We write $X \sim_{q.s.} Y$ if $X$ and $Y$ are quasi-symmetric. Quasi-symmetric classes of Julia sets have been studied in diverse perspectives, e.g., quasi-symmetries of Julia sets [BLM16, LM18, BF15, QYZ19] and quasi-symmetric uniformizations of Julia sets [QY21, Bon11, QYZ19].

For a compact metric space $X$, we denote its Hausdorff dimension by $\text{H.dim}(X)$. A compact metric space $X$ is Ahlfors-regular if there is a uniform constant $C > 1$ such that, for every $x \in X$ and $0 < r < \text{diam}(X)$, we have

$$\frac{1}{C} \cdot r^d < \mathcal{H}^d(B(x, r)) < C \cdot r^d,$$

where $B(x, r)$ is the ball of radius $r$ with center $x$, $d = \text{H.dim}(X)$, and $\mathcal{H}^d$ is the $d$-dimensional Hausdorff measure.

The (resp. Ahlfors-regular) conformal dimension of a compact metric space $X$, denoted by $\text{C.dim}(X)$ (resp. $\text{ARC.dim}(X)$), is the infimum of the Hausdorff dimension of $Y$ satisfying $X \sim_{q.s.} Y$ (resp. with $Y$ Ahlfors-regular). Thus $\text{C.dim}(X)$ and $\text{ARC.dim}(X)$ are invariants of the quasi-symmetric class of the metric space $X$. We say that the (Ahlfors-regular) conformal dimension is
attained if it is equal to the minimum of Hausdorff dimension in the quasi-symmetric class. See [HP09, HP12, HP14] for some previous works on the Ahlfors-regular conformal dimension of Julia set.

The notion of conformal dimension was introduced by P. Pansu [Pan89] in the study of negatively curved rank one symmetric spaces of non-compact type and their boundaries. Then the idea of conformal dimension was applied in the study of Gromov hyperbolic spaces. For a Gromov hyperbolic space $X$, its boundary at infinity $\partial_X X$ has a metric, called a visual metric, that is determined up to quasi-symmetry. In this respect, taking the infimum of Hausdorff dimension over a quasi-symmetric class is a natural way to obtain an invariant of $X$ or $\partial_X X$. See [MT10] for a comprehensive account of conformal dimension.

Applications of conformal dimensions in realization problems. For a motivation of studying conformal dimensions, let us review some applications of conformal dimensions to the realization problems in geometric group theory and complex dynamics.

J. W. Cannon conjectured that for a Gromov hyperbolic group $G$, if $\partial_X G$ is homeomorphic to the 2-sphere then $G$ is virtually isomorphic to the fundamental group of a compact hyperbolic 3-manifold [Can91, Conjecture 11.34]. M. Bonk and B. Kleiner proved a partial result that if $\text{ARC} \cdot \dim(\partial_X G)$ is equal to 2 and attained, then $G$ is virtually isomorphic to the fundamental group of a compact hyperbolic 3-manifold [BK05].

The realization problems in complex dynamic ask what topological branched self-coverings of the 2-sphere are topologically conjugate (sometimes up to homotopy) to the dynamics of rational maps. Thurston’s characterization is one of the most famous realization theorems in complex dynamics. In [HP14], P. Haïssinsky and K. Pilgrim showed that for a coarse expanding conformal map $f$ with the repellor $X$ equal to $S^2$, $\text{ARC} \cdot \dim(X) = 2$ and attained if and only if $f$ is topologically conjugate to a semi-hyperbolic rational map, which is an analogue of the Bonk-Kleiner’s theorem. However, there are many coarse expanding conformal maps that are not topologically conjugate to rational maps, which give counter examples to an analogue of the Cannon’s conjecture for coarse expanding conformal maps.

More recently, P. Haïssinsky showed that for a Gromov hyperbolic group $G$, if $\text{ARC} \cdot \dim(\partial_X G) < 2$ then $G$ is virtually isomorphic to a convex cocompact Kleinian group [Hai15]. Using a similar technique, P. Haïssinsky announced a result that a coarse expanding conformal map is topologically conjugate to a rational map if the Ahlfors-regular conformal dimension of its repellor is less than 2. A summary of its proof is in [Hai18].

By [BD18], if a post-critically finite topological branched covering $f$ of the sphere does not have a Levy cycle, then the Julia set $\mathcal{J}_f$ is still well-defined. Then, Conjecture 1.3 together with Theorem 1.2 implies that if $f$ is of hyperbolic-type and $\text{ARC} \cdot \dim(\mathcal{J}_f) < 2$ then $f$ is realized as a rational map.

Attainment of $\text{ARC} \cdot \dim \mathcal{J}_f = 1$. As stated in the previous paragraph, rigidity theorems follow from the assumptions that Ahlfors-regular conformal dimensions are attained. The rigidity for the attainment of $\text{ARC} \cdot \dim \mathcal{J}_f = 1$ was obtained by the author and A. Wu [PW22].

**Theorem 1.1** ([PW22, Theorem C]). Let $f$ be a semi-hyperbolic rational map of degree $d$ with a connected Julia set $\mathcal{J}_f$. If $\text{ARC} \cdot \dim(\mathcal{J}_f) = 1$ and attained, then $\mathcal{J}_f$ is quasi-symmetric to $S^1$ or $[0,1]$. Moreover, if $f$ is post-critically finite, then $f$ is $z^d$, $1/z^d$, the degree-$d$ Chebyshev polynomial or the degree-$d$ negated Chebyshev polynomial up to conjugation by Möbius transformations.

Computation of conformal dimensions. Unlike the case of negatively curved symmetric spaces [Pan89], explicit values of conformal dimensions of Julia sets or boundaries of Gromov hyperbolic
groups are mostly unknown. For a post-critically finite rational map \( f \), the Julia set \( J_f \) has Ahlfors-regular conformal dimension between 1 and 2. Arc dim(\( J_f \)) is equal to two if and only if \( J_f = \hat{\mathbb{C}} \). The other extreme case, when Arc dim(\( J_f \)) = 1, is the subject of this article. Before this work, it was shown that Arc dim(\( J_f \)) = 1 if \( f \) is a semi-hyperbolic polynomial [Car12, Kin17] or the mating of the Rabbit and the Basilica polynomials [PT21], see Figure 1.

**Decomposition of branched coverings.** To obtain lower bounds of Ahlfors-regular conformal dimensions of Julia sets, we use an idea of decomposition of branched coverings, which was rigorously formulated in [Pil03]. Let \( f : (S^2, A) \propto \) be a post-critically finite branched covering and \( \Gamma \) be a completely \( f \)-invariant multicurve, i.e., every essential connected component of \( f^{-1}(\Gamma) \) is contained in \( \Gamma \) up to homotopy (backward invariant) and every component of \( \Gamma \) is homotopic to a component of \( f^{-1}(\Gamma) \) (forward invariant), see Section 2. By pinching the sphere along \( \Gamma \), we decompose the sphere into several small spheres. There is an induced dynamics among the small spheres. The first return maps of periodic small spheres define post-critically finite branched coverings \( f_i : (S^2(i), A(i)) \propto \), which are called the small branched coverings of \( (f, \Gamma) \). See Section 6.6 for details.

**Crochet maps.** A post-critically finite rational map \( f \) is a crochet map if there exists an \( f \)-invariant graph \( G \) so that \( P_f \subset G \) and the topological entropy of \( f|_G : G \to G \) is zero. For example, every post-critically finite polynomial \( f \) is a crochet map because its spider [Poi09, BFH92] is an \( f \)-invariant graph with topological entropy zero. Also, critically fixed rational maps [CGN+15, Hlu19], (second iterates of) critically fixed anti-rational maps [Gey20], post-critically finite Newton maps [DMRS19, LMS15], and matings of two post-critically finite polynomials one of which has core entropy zero, Proposition 7.3, are crochet maps.

Crochet maps were recently introduced by Dudko-Hlushchanka-Schleicher as elementary building blocks of rational maps, together with Sierpiński carpet maps, in their decomposition theory [DHS22]. We call a rational map a Sierpiński carpet map, if its Julia set is homeomorphic to the Sierpiński carpet. They proved that if \( f \) is a post-critically finite rational map with non-empty Fatou set, then there is a canonical completely \( f \)-invariant multicurve \( \Gamma \) such that each small rational map of the \( \Gamma \)-decomposition is either a Sierpiński carpet map or a crochet map. They also proved that if \( f \) is not a crochet map, then either (1) \( \Gamma \) is a Cantor multicurve or (2) the \( \Gamma \)-decomposition has at least one small Sierpiński carpet map. We say that \( \Gamma \) is a Cantor multicurve if the number of essential components of \( f^{-n}(\Gamma) \) isotopic to elements of \( \Gamma \) grows exponentially fast. See Section 7 for details. We sketch the idea of the crochet decomposition in the appendix for the sake of helping the reader understand crochet maps as a byproduct of a decomposition theory, which is canonical in the sense of Dudko-Hlushchanka-Schleicher.

There is intuitive evidence that every non-crochet map has conformal dimension strictly greater than one. Suppose that \( f \) is not a crochet map so that we are in case either (1) or (2) of the previous paragraph. There is a natural map into \( J_f \) either from a Cantor set times the circle, \( \mathcal{C} \times S^1 \), in the case (1), or from a Sierpiński carpet Julia set in the case (2). \( \mathcal{C} \times S^1 \) is a well-known example having conformal dimension strictly greater than one [MT10]. A self-similar Sierpiński carpet also has conformal dimension strictly greater than one by [Mac10]. We use the theory of asymptotic conformal energies to prove this rigorously.

**Asymptotic \( p \)-conformal energies.** One major technique applied in the proof of Theorem A is the theory of \( p \)-conformal energies introduced by D. Thurston [Thu19a, Thu19b, Thu20]. For every post-critically finite hyperbolic-type topological branched self-covering of the sphere \( f : (S^2, A) \propto \)
Figure 1. The mating of the Rabbit and the Basilica polynomials, \( f(z) \approx \frac{z^2 + (0.5 - 0.866)}{z^2 + (0.5 + 0.866)} \), with an \( f \)-invariant graph with topological entropy zero drawn. The graph is obtained from the Hubbard tree of the Basilica polynomial and the spider of the Rabbit polynomial, see Proposition 7.3.

and \( p \in [1, \infty] \), we define the asymptotic \( p \)-conformal energy \( E^p(f) \in \mathbb{R}_+ \), see Section 6. Below is a summary of known properties of the asymptotic \( p \)-conformal energies.

**Theorem 1.2.** For every post-critically finite hyperbolic-type topological branched self-covering of the sphere \( f : (S^2, A) \preceq \), we have the following properties.

1. \( E^2(f) < 1 \) if and only if \( f \) is combinatorially equivalent to a rational map [Thu20].
2. \( E^p(f) < 1 \) if and only if \( f \) does not have a Levy cycle [BD18, Nek14].
3. As a function of \( p \), the \( E^p(f) \) is monotonically decreasing, i.e., \( E^p(f) \leq E^q(f) \) if \( p > q \) [Thu20].
4. For \( p_* = \text{ARC}. \dim(\mathcal{J}_f) \), we have \( E^{p_*}(f) = 1 \) [PT21].

**Remark.** As for (2), the equivalence between \( E^\infty(f) < 1 \) and having contracting biset is almost immediate by [Nek14], and the equivalence between having contracting biset and the non-existence of Levy cycles is by [BD18]. See [Par21, Proposition 7.21] for more details.

The following conjecture, together with (4) in Theorem 1.2, implies that Ahlfors-regular conformal dimensions of Julia sets can be determined by asymptotic conformal energies.

**Conjecture 1.3** ([PT21, Conjecture 1.1]). For a post-critically finite hyperbolic-type branched covering \( f : (S^2, A) \preceq \), the asymptotic \( p \)-conformal energy \( E^p(f) \) is either constant or strictly decreasing for \( p \in [1, \infty] \).

Our second main result is the monotonicity of conformal energies under decompositions by multicurves.

**Theorem B** (Monotonicity of conformal energies). Suppose \( f : (S^2, A) \preceq \) is a post-critically finite hyperbolic-type branched covering without a Levy cycle. Let \( \Gamma \) be a completely \( f \)-invariant multicurve such that

\[
\{ f_i : (S^2(i), A(i)) \preceq \mid i = 1, 2, \ldots, n \}
\]
is the set of small branched coverings of \((f, \Gamma)\) with the first return times \(\{\tau_i\}_{i=1,2,\ldots,n}\). Then, for every \(p \in [0, \infty]\), we have
\[
E^p(f) \geq \max \left\{ \left\{ (E^p(f_i))^{1/\tau_i} \mid 1 \leq i \leq n \right\}, \ (\lambda_p(\Gamma))^{1/p} \right\}.
\]
Moreover, we have \(\text{ARC}\dim(\mathcal{J}_f) \geq Q(\Gamma)\).

**Remark.** The inequality \(\text{ARC}\dim(\mathcal{J}_f) \geq Q(\Gamma)\) was previously known [HP08, Theorem 1.5].

The number \(\lambda_p(\Gamma)\) can be viewed as the asymptotic \(p\)-conformal energy \(E^p(f|_{\mathcal{U}} : \mathcal{U})\) of the restricted dynamics. Then \(Q(\Gamma)\) is determined by \(\lambda_Q(\Gamma) = 1\). See Section 2 and Section 6.6 for details.

The next conjecture follows from Conjecture 1.3 and Theorem B.

**Conjecture C.** In the setting of Theorem B, we also have
\[
\text{ARC}\dim(\mathcal{J}_f) \geq \max \{ \text{ARC}\dim(\mathcal{J}_{f_i}) \mid 1 \leq i \leq n \}.
\]

**Sullivan’s dictionary.** There is an analogy, called Sullivan’s dictionary, between complex dynamics and Kleinian groups. In particular, Julia sets are compared with limit sets. Theorems in this article are comparable with the work on hyperbolic groups by Carrasco-Mackay [CM22].

**Theorem 1.4** ([CM22, Theorem 1.1]). Let \(G\) be a hyperbolic group that is not virtually free. Suppose that there is a graph of groups decomposition of \(G\), with vertex groups \(\{G_i\}\) and elementary edge groups. Then we have
\[
\text{ARC}\dim(\partial_x G) = \max \{\{1\} \cup \{ \text{ARC}\dim(\partial_x G_i) \mid |G_i| = \infty \}\}.
\]

Theorem B and Conjecture C can be compared with Theorem 1.4. However, the inequality in Conjecture C may be strict; some matings of post-critically finite polynomials (\(\text{ARC}\dim = 1\)) yield Sierpiński carpet Julia sets (\(\text{ARC}\dim > 1\)).

Like Theorem A, there is also a characterization of hyperbolic groups whose boundaries have conformal dimension one.

**Theorem 1.5** ([CM22, Corollary 1.2]). Let \(G\) be a hyperbolic group that is not virtually free and has no 2-torsion. Then, \(\text{ARC}\dim(\partial_x G) = 1\) if and only if the \(G\) has a hierarchical decomposition over elementary edge groups so that every vertex group is either elementary or virtually Fuchsian.

**Finite subdivision rules.** A finite subdivision rule is a version of Markov partition for branched coverings of the sphere. It consists of a CW-complex structure on the sphere, called a subdivision complex, and a description on how the CW-complex changes by the pull-back of a branched self-covering of the sphere, which is called a subdivision map. It was originally developed to understand the boundaries of Gromov hyperbolic groups in an attempt to prove the Cannon’s conjecture [CFP01]. Later, it turned out that finite subdivision rules are a useful combinatorial tool to describe the dynamics of rational maps [CFKP03, CFP07].

Finite subdivision rule is also very useful to detect the existence of Levy cycles [FPP18, FPP20]. Non-expanding spines are subgraphs of the dual 1-skeletons of subdivision complexes where the dynamics are not expanding. By [BD18], the non-existence of Levy cycles is equivalent to having an expanding dynamical system. Based on this, the author showed that a subdivision map does not have a Levy cycle if and only if the non-expanding spines are homotopically trivial [Par20]. The homotopical triviality is a key property used in the proof of \(\text{ARC}\dim(\mathcal{J}_f) = 1\) for any crochet map \(f\).
In this article, we use finite subdivision rule to construct graph virtual endomorphisms, which is necessary to calculate asymptotic conformal energies, see Section 6. A vertex in a subdivision complex is a *Julia vertex* if its forward orbit does not contain any periodic critical point. Julia vertices are difficult to manage in our construction, so we would like to remove them as many as possible. For this reason, we simplify finite subdivision rules by collapsing some parts of the sphere which have many Julia vertices, Theorem 4.1 and Corollary 5.3.

**Non-hyperbolic post-critically finite crochet maps.** The proof in this paper relies on two results: asymptotic $p$-conformal energies and the decomposition theory in [DHS22]. The decomposition theory works for non-hyperbolic rational maps as well. However, it is still in progress by D. Thurston, to generalize the theory of asymptotic $p$-conformal energies to the case of non-hyperbolic rational maps. We expect that the proof in this paper would still work for non-hyperbolic crochet maps with the generalized notion of asymptotic $p$-conformal energies.

Below is a brief summary of key ideas in the proof of Theorem A and relevant sections.

**Proof of “crochet map ⇒ ARC.dim=1”**. We will show that if $f$ is a crochet map, then $E^1(f) = 1$ and $E^p(f) < 1$ for every $p \in (1, \infty]$. Then by Theorem 1.2 we have $\text{ARC.dim}(\mathcal{J}_f) = 1$.

By Dudko-Hlushchanka-Schleicher [DHS22], a crochet map is combinatorially equivalent to a subdivision map of a finite subdivision rule $\mathcal{R}$ with polynomial growth of edge subdivisions. The equality $E^1(f) = 1$ easily follows from the condition that edges subdivide polynomially fast.

To show $E^p(f) < 1$ for $p > 1$ needs a more careful estimate. We first reduce the finite subdivision rule $\mathcal{R}$ to a finite subdivision rule having isolated Julia vertices. In the reduction process, we use a semi-conjugacy from a topological branched covering without Thurston obstruction to the rational map, which was originally introduced by Rees and Shishikura [Shi00] and further developed later by Cui-Peng-Tan [CPT12]. Then we use a fact that the non-expanding spine is homotopically trivial to show $E^p(f) < 1$.

**Application of asymptotic conformal energies**

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**Proof of “crochet map ⇐ ARC.dim=1”**. If $f$ is not a crochet map, then by [DHS22] there exists an $f$-invariant multicurve $\Gamma$ such that either (i) $\Gamma$ is Cantor multicurve or (ii) at least one small rational map, say $f_i$, of $f$ has a Sierpiński carpet Julia set, see Theorem 7.5. We have $Q(\Gamma) > 1$ for the case (i) and $E^1(f_i) > 1$ for the case (ii) by [PT21]. Then $\text{ARC.dim}(\mathcal{J}_f) > 1$ follows from the monotonicity theorem, Theorem B, and properties (3) and (4) in Theorem 1.2.

**Monotonicity theorem (Theorem B)**

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**Acknowledgment.** This work is based on author’s Ph.D. thesis at Indiana University. The author particularly thanks Dylan Thurston and Kevin Pilgrim for their constant support and encouragement throughout the graduate school years. The author also thanks Laurent Bartholdi, Mario Bonk, Dzmitry Dudko, Mikhail Hlushchanka, Jeremy Kahn, and Dierk Schleicher for useful conversations.

The author was supported by the Simons Foundation Institute Grant Award ID 507536 while the author was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI.
2. Obstructions of branched coverings

In this section, we review two geometrization theorems of branched coverings of the sphere, Thurston’s characterization of conformal geometry [DH93] and Bartholdi-Dudko’s characterization of expanding geometry [BD18], and then also review properties of invariant multicurves, which play important roles in both characterizations.

A continuous map \( f : S^2 \to S^2 \) is a topological branched covering if it is locally \( z \mapsto z^d \) for some integer \( d > 0 \) under suitable local coordinates. A point \( x \in S^2 \) is a critical point of \( f \) if \( f \) is not locally injective at \( x \). The critical set \( \text{Crit}(f) \) of \( f \) is the set of all critical points, and its forward orbit \( P_f := \bigcup_{k=1}^\infty f^k(\text{Crit}(f)) \) is the post-critical set. If \( P_f \) is finite, we call \( f \) a post-critically finite branched covering.

Definition 2.1 (Combinatorial equivalence). Two branched coverings \( f : (S^2, A) \circlearrowleft \) and \( g : (S^2, B) \circlearrowleft \) are combinatorially equivalent (by \( \phi_0 \) and \( \phi_1 \)) if there exist homeomorphisms \( \phi_0, \phi_1 : (S^2, A) \to (S^2, B) \) such that (1) \( \phi_0(A) = \phi_1(A) = B \), (2) \( \phi_1 \) is homotopic relative to \( A \) to \( \phi_0 \), and (3) the following diagram commutes.

\[
\begin{array}{ccc}
(S^2, A) & \xrightarrow{\phi_0} & (S^2, B) \\
\downarrow f & & \downarrow g \\
(S^2, A) & \xrightarrow{\phi_1} & (S^2, B)
\end{array}
\]

Fatou and Julia marked points, hyperbolic-type branched coverings. Let \( f : (S^2, A) \circlearrowleft \) be a branched covering. A marked point \( a \in A \) is a Fatou point if \( f^k(a) \) is a periodic critical point for some \( k \geq 0 \). Otherwise \( a \) is a Julia point. The branched covering \( f : (S^2, A) \circlearrowleft \) is of hyperbolic-type if every marked point is a Fatou point. We remark that the hyperbolicity depends not only on the covering map but also on the postcritical set of marked points.

Let \( A \subset S^2 \) be a finite set. For \( a \in A \), a simple closed curve \( \gamma \) is a peripheral loop of \( a \) if \( \gamma \) bounds a disk whose intersection with \( A \) is \( \{a\} \). A multicurve \( \Gamma \) of \( (S^2, A) \) is a collection of disjoint essential simple closed curves which are pairwise non-isotopic relative to \( A \).

Invariance of graphs and multicurves. Let \( f : (S^2, A) \circlearrowleft \) be a branched covering.

- A graph \( G \subset S^2 \) is forward \( f \)-invariant up to isotopy relative to \( A \) if there exist a subgraph \( H \) of \( f^{-1}(G) \) and a homeomorphism \( \phi : S^2 \to S^2 \) such that \( \phi(H) = G \) and \( \phi \) is isotopic to the identity map relative to \( A \).
- A graph \( G \) is forward \( f \)-invariant (or shortly \( f \)-invariant) if \( f(G) \subset G \).
- A multicurve \( \Gamma \) on \( (S^2, A) \) is forward \( f \)-invariant up to isotopy relative to \( A \) if it is so as a graph, i.e., for every \( \gamma \in \Gamma \), there exists \( \gamma' \) such that \( \gamma \) is isotopic to a connected component of \( f^{-1}(\gamma') \).
- A multicurve \( \Gamma \) is backward \( f \)-invariant up to isotopy relative to \( A \), or \( f \)-stable, if every connected component of \( f^{-1}(\gamma) \) for \( \gamma \in \Gamma \) is either isotopic to an element of \( \Gamma \) or inessential relative to \( A \).
- A multicurve \( \Gamma \) is completely \( f \)-invariant up to isotopy relative to \( A \) if \( \Gamma \) is both forward and backward invariant under \( f \) up to isotopy relative to \( A \).
When the set of marked points $A$ is understood, we omit “relative to $A$”. Multicurves are always considered up to isotopy relative to $A$, so we also omit “up to isotopy” for multicurves. For graphs which are not multicurves, we usually consider $f$-invariant graphs.

**Definition 2.2** (Levy cycle). Let $f : (S^2, A) \to$ be a branched covering. A multicurve $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of $(S^2, A)$ is a _Levy cycle_ (or _Levy obstruction_) if, for every $i \pmod n$, there is a connected component $\gamma_i'$ of $f^{-1}(\gamma_i)$ that is isotopic relative to $A$ to $\gamma_i$ such that $\deg(f|_{\gamma_i'}) = 1$.

Let $f : (S^2, A) \to$ be a branched covering and $\Gamma$ be a multicurve of $(S^2, A)$. For $p \in [1, \infty]$, the _linear $p$-transformation of $\Gamma$_ is a linear map $f_{p, \Gamma} : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ defined by

$$f_{p, \Gamma}[\gamma_i] = \sum_{\gamma_j \in f^{-1}(\gamma_i)} \left( \sum_{\gamma \sim \gamma_i \in f^{-1}(\gamma_i)} (\deg(f|_\gamma : \gamma \to \gamma_i))^{1-p} \right)[\gamma_j],$$

where $[\gamma_i]$ means the isotopy class of $\gamma_i$ and the inner sum is taken over connected components $\gamma$ of $f^{-1}(\gamma_i)$ that are isotopic to $\gamma_j$. We define the inner sum to be zero if there is no such $\gamma$. If $p = \infty$, then we take the value of the limit $\lim_{p \to \infty}$. Denote by $\lambda_p(\Gamma)$ the leading eigenvalue of $f_{p, \Gamma}$, which is a non-negative real number by the Perron-Frobenius Theorem.

$$\left( \begin{array}{cccc} A_{p,1} & \ast & \cdots & \ast \\ 0 & A_{p,2} & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{p,k-1} \\ 0 & 0 & \cdots & 0 & A_{p,k} \end{array} \right)$$

Under a suitable choice of ordered basis, $f_{p, \Gamma}$ can be represented as an upper block triangular matrix whose entries are non-negative integers as above. Let $A_{p,1}, A_{p,2}, \ldots, A_{p,k}$ be the square matrices on the diagonal such that $\lambda_{p,i}$ be the leading eigenvalue of $A_{p,i}$. Then $\lambda_p(\Gamma) = \max_i \lambda_{p,i}$. Each $A_{p,i}$ is either zero or irreducible. For each $A_{p,i}$, there is a corresponding sub-multicurve $\Gamma_i$ of $\Gamma$, possibly $\Gamma_i = \Gamma$ when $k = 1$, such that $f_{p, \Gamma_i} = A_{p,i}$. Note that $A_{p,i}$ is irreducible for some $p \in [1, \infty]$ if and only if $A_{p,i}$ is irreducible for every $p \in [1, \infty)$. We say that $\Gamma_i$ is _irreducible_ if $A_{p,i}$ is irreducible for $p \in [1, \infty)$.

It follows from definition that every $A_{1,i}$ and $A_{\infty,i}$ consist of non-negative integers. Hence, for $p \in \{1, \infty\}$, if $A_{p,i}$ is non-zero, then $\lambda_{p,i} \geq 1$, see [Par20, Lemma 8.1]. If $A_{\infty,i}$ is non-zero, then $\Gamma_i$ contains a Levy cycle. There are two cases [HP08, Lemma A.2]:

1. If $A_{p,i} = 0$ for every $1 \leq i \leq k$, then $f_{p, \Gamma}$ is nilpotent and $\lambda_p(\Gamma) = 0$ for every $p \in [1, \infty]$;
2. If at least one $A_{p,i}$ is irreducible, then $\lambda_1(\Gamma) \geq 1$ and
   1. $\lambda_{\infty}(\Gamma) = 0$ and $\lambda_p(\Gamma)$ is strictly decreasing in $p$, or
   2. $\lambda_{\infty}(\Gamma) \geq 1$ and $\Gamma$ contains a Levy cycle.

Assume that $\Gamma$ does not contain a Levy cycle but contains an irreducible sub-multicurve. It follows from (2) above that there is a unique $Q(\Gamma) \geq 1$ with $\lambda_{Q(\Gamma)}(\Gamma) = 1$. We say that $Q(\Gamma)$ is the _critical exponent_ of $\lambda_p(\Gamma)$ as a function of $p$. The critical exponent $Q(\Gamma)$ can also be defined as the conformal dimension of a Cantor circle associated to the pair $(f, \Gamma)$. See [PT21].

**Definition 2.3** (Thurston obstructions and $p$-superadditive multicurve). Let $f : (S^2, A) \to$ be a post-critically finite branched covering. For $p \in [1, \infty]$, a multicurve $\Gamma$ of $(S^2, A)$ is _$p$-superadditive_ if $\lambda_p(\Gamma) \geq 1$. In particular, a $2$-superadditive multicurve is conventionally called a _Thurston obstruction_.

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By (2-ii) above, a multicurve $\Gamma$ is $\infty$-superadditive if and only if $\Gamma$ contains a Levy cycle.

**Remark.** The condition $\lambda_2(\Gamma) \leq 1$ corresponds to the subadditivity of moduli of annuli. Thus we call the reversed inequality the superadditivity.

The $2$-superadditive multicurves obstruct conformal structures.

**Theorem 2.4 (Characterization of rational maps [DH93]).** Let $f : (S^2, A) \rhd$ be a post-critically finite branched covering that is not doubly covered by a torus endomorphism. Then $f : (S^2, A) \rhd$ is combinatorially equivalent to a post-critically finite rational map if and only if $f : (S^2, A) \rhd$ does not have a Thurston obstruction. Moreover, such a rational map is unique up to conjugation by Möbius transformation.

The $\infty$-superadditive multicurves obstruct expanding structures. We define expanding property of post-critically finite topological branched coverings of the sphere as follows.

**Definition 2.5 (Böttcher expanding map, local rigidity).** A post-critically finite branched covering $f : (S^2, A) \rhd$ is Böttcher (metrically) expanding if there is a length metric $\mu$ on $S^2 \setminus A^\infty$, where $A^\infty$ is the set of periodic Fatou points, satisfying the following properties.

- For every rectifiable curve $\gamma : [0, 1] \to S^2 \setminus A^\infty$, the length of any lift of $\gamma$ through $f$ is strictly less the length of $\gamma$, and
- (Local rigidity near critical cycles) $f$ is locally rigid near critical cycles, i.e., for every $a \in A^\infty$, the first return map of $f$ near $a$ is locally conjugate to $z \mapsto z^{\deg_a(f^n)}$, where $n$ is the period of $a$.

A post-critically finite rational map is a Böttcher expanding map because (1) Böttcher coordinates yield the local rigidity near critical cycles and (2) the conformal metric is expanding.

For a Böttcher expanding map $f$, its Julia set $J_f$ is defined as the repellor and enjoys many properties which are satisfied by Julia sets of rational maps [BD18].

**Theorem 2.6 (Characterization of expanding maps [BD18]).** Let $f : (S^2, A) \rhd$ be a post-critically finite branched covering which is not doubly covered by a torus endomorphism. Then $f : (S^2, A) \rhd$ is combinatorially equivalent to a Böttcher expanding map $F : (S^2, B) \rhd$ if and only if $f$ does not have a Levy cycle. Moreover, if exists, the Böttcher expanding map is unique up to topological conjugation in the combinatorial equivalence class.

**Levy and Thurston obstructions for different marked points.** In general, the more additional marked points are, the more likely marked branched coverings are obstructed. When being in the preimage of original marked points, however, additional marked points do not give rise to new obstructions.

**Lemma 2.7.** Let $f : (S, A) \rhd$ be a marked branched covering. For every $n \geq 1$, if $\Gamma$ is an irreducible multicurve of $f : (S^2, f^{-n}(A)) \rhd$, then $\Gamma$ is also an irreducible multicurve of $f : (S^2, A) \rhd$, i.e., every simple closed curve $\gamma \in \Gamma$, which is essential relative to $f^{-n}(A)$, is also essential relative to $A$.

**Proof.** Let $\Gamma$ be an irreducible multi-curve of $f : (S^2, f^{-n}(A)) \rhd$. By the irreducibility, for every $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such that a connected component of $f^{-n}(\gamma)$ is isotopic to $\gamma'$ relative to $f^{-n}(A)$. In particular, the connected component is essential relative to $f^{-n}(A)$. It follows from the homotopy lifting property that $\gamma$ is essential relative to $A$. \qed
Proposition 2.8. Let \( f : S^2 \to S^2 \) be a post-critically finite branched covering that is not doubly covered by a torus endomorphism. Suppose \( A \) and \( B \) are sets of marked points, i.e., \( f(A) \cup P_f \subset A \) and \( f(B) \cup P_f \subset B \). Then the following are satisfied.

1. For \( n \geq 1 \), \( f^n : (S^2, A) \leq \) has a Thurston obstruction (resp. a Levy cycle) of if and only if \( f : (S^2, A) \leq \) has a Thurston obstruction (resp. a Levy cycle).

2. Suppose \( B \subset A \). If \( f : (S^2, B) \leq \) has a Thurston obstruction (resp. a Levy cycle), then \( f : (S^2, A) \leq \) has a Thurston obstruction (resp. a Levy cycle).

3. Suppose \( B \subset f^{-k}(A) \) for some \( k \geq 0 \). If \( f : (S^2, B) \leq \) has a Thurston obstruction (resp. a Levy cycle), then \( f : (S^2, A) \leq \) has a Thurston obstruction (resp. a Levy cycle).

4. Suppose \( B \subset f^{-k}(A) \) for some \( k \geq 0 \) and \( B \supset A \). Then \( f : (S^2, A) \leq \) has a Thurston obstruction (resp. a Levy cycle) if and only if \( f : (S^2, B) \leq \) has a Thurston obstruction (resp. a Levy cycle).

5. For every set of marked points \( A \) such that \( f : (S^2, A) \leq \) is of hyperbolic-type, the branched covering \( f : (S^2, A) \leq \) has a Thurston obstruction (resp. a Levy cycle) if and only if \( f : (S^2, P_f) \leq \) has a Thurston obstruction (resp. a Levy cycle).

Proof. We prove the statements for Thurston obstructions. The same argument works for Levy cycles.

1. It is clear from definition that every Thurston obstruction \( \Gamma \) of \( f : (S^2, A) \leq \) is also a Thurston obstruction of \( f^n : (S^2, A) \leq \). Conversely, if \( f : (S^2, A) \leq \) does not have a Thurston obstruction, then \( f : (S^2, A) \leq \) is combinatorially equivalent to a rational map \( F : (\hat{C}, A') \leq \) for a set of marked point \( A' \supset P_f \) with \( |A| = |A'| \) by Theorem 2.4. Then \( f^n : (S^2, A) \leq \) is combinatorially equivalent to a rational map \( F^n : (\hat{C}, A') \leq \). It follows from Theorem 2.4 again that \( f^n : (S^2, A) \leq \) does not have a Thurston obstruction.

2. (2) is immediate; (3) The case \( B = f^{-k}(A) \) follows from Lemma 2.7. The general case follows from (2) and the case \( B = f^{-k}(A) \); (4) follows from (2) and (3); (5) follows from (4) and the fact that if \( f : (S^2, A) \leq \) is of hyperbolic-type then \( A \subset f^{-k}(P_f) \) for some \( k \geq 0 \).

Thus, extending the set of marked points \( B \) within the preimage \( f^{-k}(A) \) does not impact on the existence of a Thurston obstruction or a Levy cycle.

3. Finite subdivision rules

A finite subdivision rule is a recursive formula to generate a sequence of subdivisions of a given CW-complex. In this section, we review basic definitions of finite subdivisions rules and introduce new notions, such as bands, which will be used in subsequent sections. Then we define directed graphs encoding the dynamics of subdivisions. We also discuss a reduction of finite subdivision rule, Theorem 3.12, which is crucial in the proof of main theorem.

3.1. Definitions of finite subdivision rules. A finite subdivision rule \( \mathcal{R} \) consists of

- a finite CW-complex structure \( S_{\mathcal{R}} \) of the sphere \( S^2 \),
- a subdivision \( \mathcal{R}(S_{\mathcal{R}}) \) of \( S_{\mathcal{R}} \), and
- a continuous map \( f : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}} \) that is a cellular homeomorphism, i.e., it sends an open cell onto an open cell homeomorphically. We call \( f \) the subdivision map of \( \mathcal{R} \).

See Figure 7 as an example.

We usually consider \( \mathcal{R}(S_{\mathcal{R}}) \) and \( S_{\mathcal{R}} \) as different CW-complex structures on the same sphere \( S^2 \).

The finite subdivision rule \( \mathcal{R} \) is orientation preserving (resp. reversing) if \( f \) is orientation preserving...
(resp. reversing). In this paper, we always assume $\mathcal{R}$ is orientation preserving. The subdivision map $f$ is a post-critically finite branched covering of the sphere with $P_f \subset \text{Vert}(S_\mathcal{R})$.

We refer to closed 0-cells, 1-cells, and 2-cells as vertices, edges, and tiles respectively.

**Subdivision complexes.** Consider a subdivision map $f : \mathcal{R}(S_\mathcal{R}) \to S_\mathcal{R}$. By subdividing the range $S_\mathcal{R}$ to $\mathcal{R}(S_\mathcal{R})$ and pulling back via $f$, we have a further subdivision $\mathcal{R}^2(S_\mathcal{R})$ of the domain $\mathcal{R}(S_\mathcal{R})$. By iterating this process, we have the level-$n$ subdivision complex $\mathcal{R}^n(S_\mathcal{R})$ for every $n \geq 0$. A level-$n$ vertex, a level-$n$ edge, or a level-$n$ tile is a vertex, an edge, or a tile of $\mathcal{R}^n(S_\mathcal{R})$.

**$\mathcal{R}$-complexes.** A CW-complex $X$ is a $\mathcal{R}$-complex if there is a continuous map $h : X \to S_\mathcal{R}$ that is cellularly homeomorphic. By pulling back the level-$n$ subdivision complex $\mathcal{R}(S_\mathcal{R})$ via $h$, we have the level-$n$ subdivision complex $\mathcal{R}^n(X)$ of $X$.

**Edge and tile types.** For $n \geq 2$, a polygon (or $n$-gon) $X$ is a CW-complex whose underlying space is homeomorphic to the closed 2-disk such that $X$ has only one 2-cell and $n$ edges and vertices on the boundary. Let $t$ be a closed level-0 tile. There is a characteristic map $\phi_t : t \to t$ where $t$ is a polygon. We refer to $t$ as the tile type of $t$. Edge types are similarly defined as the domains of characteristic maps for level-0 edges. Note that every edge or tile type is a $\mathcal{R}$-complex. Hence the level-$n$ subdivision $\mathcal{R}^n(e)$ and $\mathcal{R}^n(t)$ of an edge type $e$ and a tile type $t$ are well-defined. We can think of level-$n$ subdivisions $\mathcal{R}^n(e)$ and $\mathcal{R}^n(f)$ as the images of $\mathcal{R}^n(e)$ and $\mathcal{R}^n(t)$ under characteristic maps.

There is a 1-1 correspondence between tile (resp. edge) types and level-0 tiles (resp. edges). We often use them interchangeably, but when they need to be distinguished we write bold letters for edge or tile types and normal letters for edges or tiles in the subdivision complex.

For a level-0 tile $t$, a level-$n$ tile $t^n$ is of type $t$ (or type $t$) if $f^n(t^n) = t$. A level-$n$ edge $e^n$ is of type $e$ (or type $e$) if $f^n(e^n) = e$ where $e$ is a level-0 edge. Here we use level-0 edges or tiles even if we say about their types. This could be confusing, but it would much simplify notations in later sections. If $\deg(f) = d$, then for every $n \geq 0$ there are $d^n$ many level-$n$ tiles or edges of the same type. If $t^n$ is of type $t$, then there is a characteristic map $\phi_{t^n} : t \to t^n$ so that

$$f^n \circ \phi_{t^n} = \phi_t : t \to t \subset S_\mathcal{R}.$$  

For $n > m \geq 0$, a level-$n$ tile $t^n$ is a subtile of a level-$m$ tile $t^m$ if $t^n \subset t^m$, and a level-$n$ edge $e^n$ is a subedge of a level-$m$ edge $e^m$ if $e^n \subset e^m$.

As written above, our convention is to use superscripts to indicate the levels of edges and tiles. Since level-0 objects are frequently used, we often omit the superscript $0$ for level-0 objects.

**Bands and bones.** Bands and bones are used to define non-expanding spines which detect the existence of Levy cycles.

**Bands:** A level-$n$ band is a triple $b^n = (t^n; e^n_1, e^n_2)$ where (i) $t^n$ is a level-$n$ tile and (ii) $e^n_1$ and $e^n_2$ are boundary edges of $t^n$. We allow $e^n_1 = e^n_2 (= e^n)$ only when two boundary edges are identified to $e^n$ by the characteristic map $\phi_{t^n}$. The edges $e^n_1$ and $e^n_2$ are called the sides of the band $b^n$. There are many level-$n$ bands whose sides are not subedges of level-0 edges, which are not of our interest. So we always assume that (iii) $e^n_1$ and $e^n_2$ are subedges of level-$0$ edges unless otherwise stated.

**Bones:** The bone of a band $b^n = (t^n; e^n_1, e^n_2)$, denoted by bone($b^n$), is a homotopy class (or, ambiguously, a representative of the class) of curves properly embedded in $t^n$ whose endpoints are in the interiors of $e^n_1$ and $e^n_2$. When $e^n_1 = e^n_2$, we require that the curve is closed.

**Subbands:** For $n > m \geq 0$, a level-$n$ band $(t^n; e^n_1, e^n_2)$ is a subband of a level-$m$ band $(t^m; e^m_1, e^m_2)$ if $t^n$ is a subtile of $t^m$ and $e^n_i$ is a subedge of $e^m_i$ for $i \in \{1, 2\}$. 
Lemma 3.4 (Uniqueness of recurrent paths). Let \( G \) be a finite directed graph and \( v \in \text{Vert}(G) \). Suppose \( P(v, n) \) grows polynomially fast about \( n \) and \( v \) is a recurrent vertex. Then, for every \( n \geq 0 \), there exists a unique recurrent path of length \( n \) that starts from \( v \).
Proof. By Proposition 3.3, there is only one cycle passing through \( v \). Hence there is only one path that starts from \( v \) and supported in the cycle. \( \square \)

**Preorder on the vertex set** \( \text{Vert}(G) \).

**Definition 3.5** (Preorder). For a set \( X \), a **preorder** is a binary relation \( \leq \) satisfying the following.

1. (Reflexivity) For every \( x \in X \), \( x \leq x \).
2. (Transitivity) For \( x, y, z \in X \), if \( x \leq y \) and \( y \leq z \), then \( x \leq z \).

We say that \( x \) and \( y \) are **equivalent** with respect to the preorder and write \( x \simeq y \) if \( x \leq y \) and \( x \geq y \).

There is a natural preorder on \( \text{Vert}(G) \) defined by the following.

For \( v, w \in \text{Vert}(G) \), \( v \leq w \) if and only if there is a path from \( v \) to \( w \).

It is immediate that \( v \simeq w \) if and only if they belong to the same cycle.

**Definition 3.6** (Ideals, radical ideals). Let \( G \) be a directed graph.

- A subset \( X \subseteq \text{Vert}(G) \) is an **ideal** if the following condition hold: If \( v \in X \) and \( v \leq w \), then \( w \in X \), i.e., if there is a path from \( v \) to \( w \), then \( w \in X \).
- For an ideal \( X \), a vertex \( v \in \text{Vert}(G) \) is a **tail** of \( X \) if there exists \( N \geq 1 \) such that every path of length \( \geq N \) starting from \( v \) has its terminal vertex in \( X \). Denote by \( \text{Tail}(X) \) the set of tails of \( X \). By definition, we have \( X \subseteq \text{Tail}(X) \), and it is easy to show that \( \text{Tail}(X) \) is also an ideal. If \( X = \text{Tail}(X) \), then we say that \( X \) is a **radical ideal**.

3.2.2. **Directed graphs from subdivisions.** There are several directed graphs encoding the dynamics of finite subdivision rules.

A **directed graph** \( \mathcal{E}_R \) of edge subdivisions is defined in such a way that

- \( \text{Vert}(\mathcal{E}_R) \) is the set of all level-0 edges (we use bracket \([e]\) to distinguish vertices of \( \mathcal{E}_R \) from actual edges of \( S_R \)), and
- every edge of \( \mathcal{E}_R \) from \([e]\) to \([e']\) corresponds to a level-1 subedge \( e^1 \) of \( e \) with \( f(e^1) = e' \).

Similarly, a **directed graph** \( \mathcal{T}_R \) of tile subdivisions is defined in such a way that

- \( \text{Vert}(\mathcal{T}_R) \) is the set of all level-0 tiles (we use bracket \([t]\) to distinguish vertices of \( \mathcal{T}_R \) from actual tiles of \( S_R \)), and
- every edge of \( \mathcal{T}_R \) from \([t]\) to \([t']\) corresponds to a level-1 subtile \( t^1 \) of \( t \) with \( f(t^1) = t' \).

A **directed graph** \( \mathcal{B}_R \) of bands is defined in such a way that

- \( \text{Vert}(\mathcal{B}_R) \) is the set of all level-0 bands (we use bracket \([b]\) to distinguish vertices of \( \mathcal{B}_R \) from actual bands of \( S_R \)), and
- every edge of \( \mathcal{B}_R \) from \([b]\) to \([b']\) corresponds to a level-1 subband \( b^1 \) of \( b \) with \( f(b^1) = b' \).

Let \( e \) be a level-0 edge. Every level-\( n \) subedge \( e^n \) of \( e \) is bijectively corresponded to a path of length \( n \) in \( \mathcal{E}_R \) starting from \([e]\). The terminal vertex of the path is the type of \( e^n \). There are similar 1-1 correspondences for tiles and bands as well.

**Definition 3.7** (Recurrent edges and bands). For \( n > 0 \), a level-\( n \) subedge (resp. subband) of a level-0 edge \( e \) (resp. band \( b \)) is **recurrent** if the corresponding path in \( \mathcal{E}_R \) (resp. in \( \mathcal{B}_R \)) is recurrent. A level-0 edge \( e \) (resp. band \( b \)) is **recurrent** if there is a cycle in \( \mathcal{E}_R \) (resp. \( \mathcal{B}_R \)) that passes through \([e]\) (resp. \([b]\)).
3.3. Growth rate of edge subdivisions.

**Definition 3.8** (Growth rate of edge subdivisions). For a level-0 edge $e$ of a finite subdivision rule $R$, the *exponential growth rate* of the subdivision of the edge $e$ is a number $\rho(e) \geq 1$ defined by

$$\rho(e) \coloneqq \lim_{n \to \infty} \left( |R^n(e)| \right)^{\frac{1}{n}}.$$

We say that the edge $e$ has *sub-exponential growth rate of subdivisions* if $\rho(e) < 1$. A finite subdivision rule $R$ has *sub-exponential growth rate of edge subdivisions* if every edge type has sub-exponential growth rate of subdivisions. By Proposition 3.9, we can substitute the word “sub-exponential” for “polynomial”.

Recall we defined the directed graph of edge subdivisions $E$ in the previous section.

**Proposition 3.9.** A finite subdivision rule $R$ has sub-exponential growth of edge subdivisions if and only if the cycles in $E$ are disjoint. In this case, the growth rate of $\# \{\text{level-}n\, \text{subedges of} \, e\}$ is polynomial for each level-0 edge $e$.

**Proof.** It is straightforward from Proposition 3.3 and the correspondence between paths of length $n$ in $E$ and level-$n$ subedges of level-0 edges. □

Let $f^{(1)} : (S_R)^{(1)} \to (S_R)^{(1)}$ be the restriction of $f : R(S_R) \to S_R$ to the 1-skeleton $(S_R)^{(1)}$ of the level-0 complex. Then $f^{(1)}$ is a Markov map, in a sense that each edge locally homeomorphically maps to a union of edges. The adjacency matrix of the directed graph of edge subdivision $E$ coincides with the incidence matrix of the Markov map $f^{(1)}$.

**Proposition 3.10.** A finite subdivision rule $R$ has polynomial growth rate of edge subdivisions if and only if $h_{\text{top}}(f^{(1)}) = 0$, where $h_{\text{top}}$ means topological entropy.

**Proof.** See [Par20, Proposition 9.3]. □

3.4. Quotients of finite subdivision rules. In this subsection, we simplify a finite subdivision rule $R$ by collapsing some edge and tile types. By collapsing an edge type $e$ (resp. a tile type $t$), we mean collapsing all the edges of type $e$ (resp. tiles of type $t$).

Suppose we collapse an edge type $e$. Let $f(e)$ be a union of edges of type $e_1, e_2, \ldots, e_k$. Since we also want to have an induced map on the quotient space, the edges of type $e_1, e_2, \ldots, e_k$ have to be collapsed accordingly. Hence we have to collapse all edge types in an ideal of $E_R$ at the same time. Now we suppose that we collapse the edge types $e_1, e_2, \ldots, e_k$ which forms an ideal of $E_R$, and there is an edge $e'$ of type $e'$ such that $f^n(e')$ is a union of edges whose types are supposed to be collapsed for some $n > 0$, i.e., $[e']$ is in the tail of the ideal. Then we should collapse the edge type $e'$ to make the induced map locally injective. Therefore, a radical ideal is an appropriate collection of edge types to collapse at the same time. The same argument also works for tiles.

**Definition 3.11** (Collapsible subcomplexes). Let $R$ be a finite subdivision rule. A subcomplex $X$ of the level-0 complex $S_R$ is *collapsible* if

- the edges and tiles in $X$ form radical ideals in $E_R$ and $T_R$ respectively, and
- every connected component of $X$ is simply connected.

We also say that the collection of edge types and tile types is *collapsible* if they are edge and tile types of a collapsible subcomplex of $S_R$. We write $X_E$ and $X_T$ for the collections of edges and tiles in $X$, which are radical ideals of the directed graphs $E_R$ and $T_R$ of edge and tile subdivisions respectively.
Quotient finite subdivision rules. Let $X$ be a collapsible subcomplex of $S_{\mathcal{R}}$. Every possibly non-simple closed curve in $X$ is homotopically trivial in $X$. By the homotopy lifting property, every connected component of $f^{-1}(X)$ is also simply connected. We define $S_{\mathcal{R}'}$ (resp. $\mathcal{R}'(S_{\mathcal{R}})$) to be the quotient CW-complex of $S_{\mathcal{R}}$ (resp. $\mathcal{R}(S_{\mathcal{R}})$) obtained by collapsing each connected component of $X$ (resp. $f^{-1}(X)$) to a point. Let $g_0 : S_{\mathcal{R}} \rightarrow S_{\mathcal{R}'}$ and $g_1 : \mathcal{R}(S_{\mathcal{R}}) \rightarrow \mathcal{R}'(S_{\mathcal{R}})$ be the quotient maps. Then there exists a map $f' : \mathcal{R}'(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}'}$ such that $f' \circ g_1 = g_0 \circ f$. It is easy to show that the CW-complexes $S_{\mathcal{R}'}$ and $\mathcal{R}'(S_{\mathcal{R}})$ together with the map $f' : \mathcal{R}'(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}'}$ form a finite subdivision rule $\mathcal{R}'$. We call the new finite subdivision rule $\mathcal{R}'$ a quotient finite subdivision rule of $\mathcal{R}$ by the collapsible subcomplex $X$.

**Theorem 3.12** (Quotient finite subdivision rules). Let $\mathcal{R}$ be a finite subdivision rule with a subdivision map $f : \mathcal{R}(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}}$. Let $\mathcal{R}'$ be the quotient finite subdivision rule of $\mathcal{R}$ by a collapsible subcomplex $X$ such that $f' : \mathcal{R}'(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}'}$ is the subdivision map of $\mathcal{R}'$. Let $g_0 : S_{\mathcal{R}} \rightarrow S_{\mathcal{R}'}$ and $g_1 : \mathcal{R}(S_{\mathcal{R}}) \rightarrow \mathcal{R}'(S_{\mathcal{R}})$ be the quotient maps such that $g_0 \circ f = f' \circ g_1$.

- The directed graph $\mathcal{E}_{\mathcal{R}'}$ (resp. $\mathcal{T}_{\mathcal{R}'}$) is obtained by removing from $\mathcal{E}_{\mathcal{R}}$ (resp. $\mathcal{T}_{\mathcal{R}}$) the vertices in $X_E$ (resp. $X_T$) and the edges incident to the vertices in the ideal $X_E$ (resp. $X_T$). Here $X_E$ and $X_T$ are considered as subsets of Vert($\mathcal{E}_{\mathcal{R}}$) and Vert($\mathcal{T}_{\mathcal{R}}$).
- Let $A \subset \text{Vert}(S_{\mathcal{R}})$ be a set of marked points of the subdivision map $f : \mathcal{R}(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}}$ and $A' := g_0(A) \subset \text{Vert}(S_{\mathcal{R}'}$) be the corresponding set of marked points of the subdivision map $f' : \mathcal{R}'(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}'}$. If $g_0|_A : A \rightarrow A'$ is bijective, then there exists homeomorphisms $\phi_0, \phi_1 : (S^2, A) \rightarrow (S^2, A')$ which are homotopic to $g_0$ relative to $A$ such that $f : (S^2, A) \xrightarrow{\phi_0} (S^2, A') \xleftarrow{\phi_1} f'$ which are combinatorially equivalent through $(\phi_0, \phi_1)$.

**Proof.** All but the combinatorial equivalence are straightforward from the construction. Since $g_0|_A$ is a bijection, the $g_0$ and $g_1$ are homotopic relative to $A$. Define $\phi_0 : (S^2, A) \rightarrow (S^2, A')$ to be a homeomorphism homotopic relative to $A$ to $g_0$. Then we can find $\phi_1$ by lifting $\phi_0$ through the coverings maps $f$ and $f'$.

3.5. Shifts and powers of finite subdivision rules. Let $\mathcal{R}$ be a finite subdivision rule with a subdivision map $f : \mathcal{R}(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}}$.

**Definition 3.13** (Shifts). For $k \geq 1$, the $k^{th}$-shift of $\mathcal{R}$, denoted by $\mathcal{R}[k]$, is a finite subdivision rule whose level-$0$ and -1 complexes are equal to $\mathcal{R}^k(S_{\mathcal{R}})$ and $\mathcal{R}^{k+1}(S_{\mathcal{R}})$ respectively, and the subdivision map $f[k]$ is equal to $f : \mathcal{R}^{k+1}(S_{\mathcal{R}}) \rightarrow \mathcal{R}^k(S_{\mathcal{R}})$.

**Definition 3.14** (Powers). For $k \geq 1$, the $k^{th}$-power of $\mathcal{R}$, denoted by $\mathcal{R}^k$, is a finite subdivision rule whose level-$0$ and -1 complexes are equal to $S_{\mathcal{R}}$ and $\mathcal{R}^k(S_{\mathcal{R}})$ respectively, and the subdivision map is $f^k : \mathcal{R}^k(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}}$.

We remark that shifts and powers do not essentially change subdivision maps. For example, whether edges subdivide exponentially fast is invariant under shifts and powers. Thus we may replace our subdivision rules by their shifts or powers especially when we discuss Julia sets or the existence of obstructions, which are invariant under iterates.

4. Non-expanding spines

Let $\mathcal{R}$ be a finite subdivision rule. The level-$n$ non-expanding spine $N^n$ is roughly speaking the union of bones of level-$n$ recurrent subbands. Since level-$n$ recurrent bones may intersect each other very complicaedly, we trim their union to a simpler graph with an appropriate train-track structure.

Three highlights of this section are
• Theorem 4.1, which states that the homotopical triviality (relative to the marked points) of the non-expanding spines $N^n$ is equivalent to the non-existence of Levy cycles;

• Proposition 4.10, which relates the level-$n$ non-expanding spine $N^n$ to a subgraph $N^m$ of the dual 1-skeleton of the level-$n$ subdivision complex $R^n(S_R)$;

• Proposition 4.13, which states that every simple closed curve in non-expanding spines is originated from a periodic Julia vertex.

4.1. Definition of non-expanding spines. We first define the non-expanding spine of each tile and glue them to obtain the non-expanding spine of a finite subdivision rule.

Train-track. Let $G$ be a graph $G$. For any $v \in \text{Vert}(G)$, denote by $D_v(G)$ the set of germs of homeomorphic curves starting at $v$. Any element of $D_v(G)$ is called a direction at $v$. For any small neighborhood $U$ of $v$, $|D_v(G)|$ is the number of components of $U \setminus \{v\}$. A train-track structure on $G$ is an assignment $\tau$ of an equivalence class of $D_v(G)$ for each $v \in \text{Vert}(G)$. Each equivalence class of $D_v(G)$ is called a gate at $v$. A train-track structure is discrete if every gate has only one direction. A graph with a train-track structure $(G, \tau)$ is called a train-track.

Let $(S, A)$ be a surface with a finite set of marked points. Let $T = (G, \tau)$ be a train-track with $G \subset S \setminus A$. We say that a closed curve $\gamma : S^1 \to S \setminus A$ is carried by $T$ if $\gamma$ is homotopic relative to $A$ to a locally homeomorphic curve $\gamma' : S^1 \to G$ satisfying the following: For every $v \in \text{Vert}(G)$ and $t \in \gamma^{-1}(\{v\})$, the two directions at $v$ defined by $\gamma'$ near $t$ belong to different gates at $v$. One can easily generalize this definition to train-tracks $T = (G, \partial G, \tau)$ properly embedded in marked surfaces with boundary $(S, \partial S, A)$ and non-closed curves $\gamma : (I, \partial I) \to (S, \partial S)$.

Non-expanding spines of tiles. Let $P$ be a polygon. We consider curves and trees properly embedded in the polygon. By a homotopy, we mean a homotopy on $P$ that preserve edges and vertices respectively.

Let $t$ be a level-0 tile. We will assume that $t$ is a disk, which is not true if boundary edges of $t$ are identified by the characteristic map $\phi : t \to t$. If $t$ is not a disk, then we do every thing in the same manner in the tile type $t$, which is a disk, and consider the $\phi_t$-image in the subdivision complex.

The boundary $\partial t$ is also an $R$-complex, so we have level-$n$ subdivisions $R^n(\partial t)$. At level-$n$, we consider a polygon, denoted by $(t, R^n(\partial t))$, whose underlying disk is $t$ and the vertex set equals to $\text{Vert}(R^n(\partial t))$. The bone of each level-$n$ recurrent band is a homotopy class of curves properly embedded in $(t, R^n(\partial t))$. For each boundary edge $e^n$, we choose a point $m_{e^n}$, called the midpoint of $e^n$. Then we choose a representative of the bone of each level-$n$ recurrent band $b^n$, denoted by bone($b^n$), in such a way that the endpoints of bone($b^n$) are midpoints of the side edges.

Consider the union of the representatives of bones of level-$n$ recurrent bands of $t$ as a graph. The level-$n$ bones satisfy the crossing condition: If bones of two recurrent bands $b^n = (t^n; e^n_1, e^n_2)$ and $b'^n = (t'^n; e'^n_1, e'^n_2)$ cross, then the four bands $(t^n; e^n_1, e'^n_2)$, $(t'^n; e^n_1, e'^n_2)$, $(t^n; e^n_2, e'^n_1)$, and $(t'^n; e^n_2, e'^n_1)$ are also recurrent [Par20, Lemma 6.11]. Hence the graph can be decomposed into mutually non-crossing complete subgraphs such that each complete subgraph has either two vertices or at least four vertices [Par20, Lemma 6.9]. We transform each complete subgraph with at least four vertices into a star-like tree whose leaves are the vertices of the subgraph. So far, we have obtained a union of star-like trees $G_1, G_2, \ldots, G_k$ each of which has either exactly two or at least four leaves so that any pair $G_i$ and $G_j$ are disjoint or intersect in the boundary $\partial t$.

Lastly, we define a train-track structure on the union of mutually non-crossing complete graphs. First, on the center $v$ of each star-like tree with at least four leaves, we assign the discrete train-track structure. Second, if $v \in \partial t$ is an intersection of some graphs, say $G_1, G_2, \ldots, G_k$, then we
zip the $k$ edges up a little so that the new vertex has two gates. More precisely, let $e_1, e_2, \ldots, e_k$ be the edges of $G_i$’s which meet at $v$. After the zip-up, we have $k$ edges $e_1', e_2', \ldots, e_k'$ meeting at a vertex $v'$ and one new edge $e'$, whose endpoints are $v$ and $v'$. Denote by $N^n(t)$ the result, which is a train-track.

**Remark.** For a technical reason, when $N^n(t)$ has a connected component which is a triangle, which consists of three star-like trees with two leaves, then we further transform it into a tripod. We will discuss it again in Section 4.2.

![Figure 2. Transformation from a union of complete graphs to a train-track [Par20, Figure 8]. The left picture is the union of level-$n$ recurrent bones. The middle picture indicates the union of star-like trees. The right picture shows the train-track obtained after zipping-ups.](image)

**Non-expanding spines of finite subdivision rules.** The level-$n$ non-expanding spine $N^n$ is defined as the union of $N^n(t)$ over all level-0 tiles $t$’s.

Let $A \subset \text{Vert}(S_R)$ be a set of marked points, i.e., $f(A) \cup P_f \subset A$. The level-$n$ non-expanding spine $N^n$ is essential relative to $A$ if it carries a (possibly non-simple) closed curve which is not homotopic relative to $A$ to either a point or some iterate of a peripheral loop of a Julia point in $A$.

**Caution.** A peripheral loop of a Fatou point in $A$ is essential in our definition, which is different from a conventional definition of the essentiality of curves in punctured surfaces.

**Theorem 4.1** ([Par20, Theorem 6.21, Theorem 8.6]). Let $R$ be a finite subdivision rule with a subdivision map $f : \mathcal{R}(S_R) \to S_R$ and $A \subset \text{Vert}(S_R)$ be a set of marked points. Suppose that $f : (S^2, A) \curvearrowright$ is not doubly covered by a torus endomorphism. Then $f : (S^2, A) \curvearrowright$ has a Levy cycle if and only if the level-$n$ non-expanding spine $N^n$ is essential relative to $A$ for every $n \geq 0$. Moreover, if $R$ has polynomial growth of edge subdivisions, the following are equivalent.

1. $f : (S^2, A) \curvearrowright$ has a Levy cycle.
2. The level-0 non-expanding spine $N^0$ is essential relative to $A$.
3. $f : (S^2, A) \curvearrowright$ has a Thurston obstruction.

A distinctive property for the case when edges subdivide polynomially fast is that recurrent subedges and subbands are unique at each level, in the sense of the following lemma.

**Lemma 4.2.** Let $R$ be a finite subdivision rule.

- For any $n \geq 0$ and a level-0 recurrent edge $e$ with polynomial growth of subdivisions, there is a unique level-$n$ subedge of $e$ that is recurrent.
- For any $n \geq 0$ and a level-0 recurrent band $(t; e_1, e_2)$, if $e_1$ and $e_2$ have polynomial growth of subdivisions, then there exists a unique level-$n$ subband of $(t; e_1, e_2)$ that is recurrent.
Proof. It is straightforward from Lemma 3.4. □

The condition (2) in Theorem 4.1 is immediate from the following proposition.

**Proposition 4.3 ([Par20, Proposition 8.5]).** Let $R$ be a finite subdivision rule with polynomial growth rate of edge subdivisions. Let $A \subset \text{Vert}(S_R)$ be a set of marked points. For every $n \geq 0$, the level-$n$ non-expanding spine $N^n$ is isotopic relative to $A$ to $N^0$ as train-tracks, i.e., there is a homeomorphism $\phi : N^n \to N^0$ preserving the train-track structures so that $\phi$ extends to a homeomorphism of the sphere which is homotopic relative to $A$ to the identity map.

**Proof.** It follows from Lemma 4.2 that there is a 1-1 correspondence between level-$n$ recurrent bands and level-0 recurrent bands. Hence $N^0$ and $N^n$ are made of the same number of curves which are glued in the same manner. The isotopy is almost immediate because if $b^n$ is a subband of $b^m$ for $n > m \geq 0$, $\text{bone}(b^n)$ and $\text{bone}(b^m)$ are isotopic in the level-$m$ subdivision complex. See [Par20, Proposition 8.5] for more details. □

The equivalence between the existence of a Levy cycle and of a Thurston obstruction follows from the graph intersection obstruction theorem [Par20, Theorem 7.6].

**Corner bands and non-expanding peripheral cycles.** If a set of marked point $A \subset \text{Vert}(S_R)$ contains a periodic Julia vertex $v$, then the non-expanding spine $N^n$ always contains a loop peripheral to $v$.

**Definition 4.4 (Corner band).** A level-$n$ band $b^n = (t^n; e_1^n, e_2^n)$ is a corner band if the two sides $e_1^n$ and $e_2^n$ have a common endpoint $v$. The common endpoint $v$ is called the corner vertex of the band $b^n$. For $m > n$, a level-$m$ subband $b^m = (t^m; e_1^m, e_2^m)$ of a corner band $b^n$ is a corner subband if $b^m$ is also a corner band. In this case, $b^m$ and $b^n$ share the corner vertex $v$.

![Diagram](image.png)

(A) A corner band and a corner subband  (B) A non-expanding peripheral cycle

**Figure 3**

**Proposition 4.5.** The following properties about corner bands are satisfied.

1. Let $(t^n; e_1^n, e_2^n)$ be a level-$n$ corner band with the corner vertex $v$. For every $m > n$, there exists at most one level-$m$ corner subband $(t^m; e_1^m, e_2^m)$ of $(t^n; e_1^n, e_2^n)$.
2. Let $(t^n; e_1^n, e_2^n)$ be a level-$n$ corner band with the corner vertex $v$. If $(t^n; e_1^n, e_2^n)$ is recurrent, then $v$ is a periodic point of $f$. That is, only periodic vertices can be corner vertices.
(3) Let \( v \in \text{Vert}(S_R) \) be a Fatou vertex. Then, there exists at least one level-0 corner band \((t; e_1, e_2)\) with the corner vertex \( v \) such that \((t; e_1, e_2)\) does not have a level-\( n \) corner subband for any sufficiently large \( n > 0 \).

(4) Let \( v \in \text{Vert}(S_R) \) be a periodic Julia vertex. Then, for every \( n > 0 \), every level-0 corner band \((t; e_1, e_2)\) with the corner vertex \( v \) has a unique level-\( n \) corner subband. Moreover, these level-\( n \) corner bands are all recurrent.

Proof. (1) and (2) are immediate from definitions; (3) Since \( v \) is a Fatou vertex, its forward image contains a periodic critical point. Then the number of level-\( n \) corner bands whose corner vertex is \( v \) grows exponentially fast about \( n \); (4) Since \( f \) yields a map from \( S_R^{(1)} \) to itself, for every \( w \in \text{Vert}(S_R) \), we have \( \deg(w) \leq \deg(f(w)) \cdot \deg_w(f) \), where \( \deg_w(f) \) means the local degree of \( f \) at \( w \) and \( \deg(w) \) and \( \deg(f^n(w)) \) mean the degrees as vertices of \( S_R^{(1)} \). Hence \( \deg(f^n(v)) = \deg(v) \) for every \( n \geq 0 \), i.e., the degree of \( v \) in a graph \( R^n(S_R)^{(1)} \) is independent of \( n \). Then every level-0 corner band of \( v \) has a corner subband at every level. The uniqueness follows from (1), and the recurrence follows from \( v \) being periodic.

**Definition 4.6 (Non-expanding peripheral cycles).** For every \( n \geq 0 \) and every periodic Julia vertex \( v \in \text{Vert}(S_R) \), there is a cycle \( C^n \) in the level-\( n \) non-expanding spine \( N^n \) which consists of the bones of the level-\( n \) corner bands of \( v \) discussed in Proposition 4.5-(4). We call the cycle \( C^n \) the level-\( n \) non-expanding cycle peripheral to \( v \).

By Proposition 4.5-(2), any non-periodic Julia vertex does not have a peripheral cycle.

4.2. Non-expanding spines when edges subdivide polynomially fast. In this subsection, we show the following: When edges of a finite subdivision rule \( R \) subdivide polynomially fast, (1) the underlying graph of the level-\( n \) non-expanding spine \( N^n \) is a subgraph of the dual 1-skeleton of \( R^n(S_R) \) (Proposition 4.10) for any sufficiently large \( n > 0 \) and (2) \( N^n \) has the discrete train-track structure (Proposition 4.8).

We first see the following special property for finite subdivision rules with polynomial growth rate of edge subdivisions.

**Proposition 4.7 (Transitivity of recurrent subbands).** Suppose \( R \) is a finite subdivision rule with polynomial growth rate of edge subdivisions. Then the recurrence of subbands are transitive. More precisely, for \( i \in \{0, 1, 2\} \), let \( e_i^0 \) be a level-0 edge and \( e_i^n \) be the unique level-\( n \) recurrent subedge of \( e_i^0 \). Suppose that \( b_1^n = (t^n; e_0^n, e_1^n) \) and \( b_2^n = (t^n; e_0^n, e_2^n) \) are level-\( n \) recurrent bands. Then the level-\( n \) subband \( b_3^n = (t^n; e_1^n, e_2^n) \) is also recurrent.

Proof. By the assumption, for every \( m > n \), there exists level-\( m \) recurrent bands \( b_1^m = (t^m; e_0^m, e_1^m) \) and \( b_2^m = (t^m; e_0^m, e_2^m) \) which are subbands of \( b_1^n \) and \( b_2^n \). It suffices to show that for some \( m > n \) the level-\( m \) subband \( b_3^m = (t^m; e_1^m, e_2^m) \) of \( b_3^n = (t^n; e_1^n, e_2^n) \) has the same type as \( b_3^n = (t^n; e_1^n, e_2^n) \).

In the directed graph of edge subdivisions \( E_R \), the vertex \( [e_i^0] \) is contained in a unique cycle \( C_i \) for \( i \in \{0, 1, 2\} \). Let \( p \) be the least common multiple of the periods of \( C_0, C_1, \) and \( C_2 \). Then, for \( m = n + p \), \( e_i^m \) and \( e_i^n \) are of the same type for \( i \in \{0, 1, 2\} \). Then the bands \( b_1^m \) and \( b_2^m \) have the same type, which implies that \( b_3^m \) is recurrent.

**Transform triangles into tripods and discrete train-track structure.** Recall that for a level-0 tile \( t \) the non-expanding spine \( N^n(t) \) is decomposed into star-like trees with at least four leaves and or with exactly two leaves. Theses star-like trees may intersect in their leaves. If \( R \) has polynomial growth rate of edges subdivisions, Proposition 4.7 implies that each connected component of \( N^n(t) \) is either a star-like tree or a triangle, which consists of three star-like trees.
with two leaves. We transform each triangle into a tripod. This does not change the set of curves carried by $N^n(t)$ so that we can still use this transformed non-expanding spine for our purpose, such as Theorem 4.1.

We remark that this transformation does not work in general. For example, suppose that $N^n(t)$ contains two triangles which share a side $s$. If we transform two triangles into tripods, then we obtain a star-like tree with four leaves. Then the curve connecting two vertices of the triangles that crosses $s$ is carried by the new star-like tree but not carried by $N^n(t)$.

![Figure 4. Transformation of a triangle into a tripod](image)

The next proposition is immediate from the discussion above.

**Proposition 4.8** (Discrete train-track). Let $R$ be a finite subdivision rule with polynomial growth rate of edge subdivisions. Then every connected component of the level-$n$ non-expanding spine $N^n$ (as a graph) is a star-like tree, which has the discrete train-track structure. Hence $N^n$ also has the discrete train-track structure, and the set of curves carried by $N^n$ is the same as the curves supported in $N^n$ up to homotopy.

**Dual recurrent skeletons.** There are some subtleties when we identify non-expanding spines with subgraphs of dual 1-skeletons of subdivision complexes.

**Definition 4.9** (Dual recurrent skeletons). The level-$n$ dual recurrent skeleton $N^m$ is a subset of the dual 1-skeleton of $R^n(S_R)$ consisting of the dual edges of level-$n$ recurrent edges. An edge component of $N^m$ is a connected component of $N^m$ consisting of a single dual edge.

We equip $N^m$ with a metric in such a way that every edge of $N^m$ has length 1 and intersects its dual edge at the midpoint. The 1/2-truncation of $N^m$ is the closure of the removal of the closed 1/2-neighborhood of the set of leaves of $N^m$. Note that every edge component of $N^m$ vanishes in the 1/2-truncation of $N^m$.

**Proposition 4.10.** Let $R$ be a finite subdivision rule with polynomial growth rate of edge subdivisions. There is a non-negative integer $K \leq 2 \cdot |\text{Edge}(S_R)|^2$ such that for every $n \geq K$, the level-$n$ non-expanding spine $N^n$ is the 1/2-truncation of the level-$n$ dual recurrent skeleton $N^m$. In particular, the inclusion $N^n \hookrightarrow N^m \setminus \{\text{the edge components}\}$ is a deformation retract.

We will prove Proposition 4.10 after Lemma 4.11. Recall that if a level-0 edge $e$ is recurrent and subdivides polynomially fast, then the corresponding vertex $[e] \in \text{Vert}(E_R)$ belongs to a unique cycle. We call the length of the cycle the period of recurrency of the edge $e$.

**Lemma 4.11** (Adjacency of dual edges). For a finite subdivision rule $R$, the following properties hold.

1. Every level-$n$ band bijectively corresponds to the adjacency of duals of level-$n$ subedges of level-0 edges.
2. Under the bijection of (1), level-$n$ recurrent bands (possibly not surjectively) correspond to the adjacencies of recurrent level-$n$ edges.
(3) If $\mathcal{R}$ has polynomial growth rate of edge subdivisions, then the correspondence in (2) is surjective for any sufficiently large $n > 0$. To be more precise, suppose $e_1$ and $e_2$ are level-0 recurrent edges with polynomial growth of subdivisions. Let $p$ be the least common multiple of the periods of recurrence of $e_1$ and $e_2$. If $n \geq 2p$ and the duals of $e_1^n$ and $e_2^n$ are adjacent, then the band $b^n = (t^n; e_1^n, e_2^n)$ corresponding to the adjacency of $e_1^n$ and $e_2^n$ is recurrent.

Proof. (1) is immediate and (2) follows from a fact that sides of a level-$n$ recurrent band are level-$n$ recurrent subedges.

Let us show (3). Having polynomial growth rate of subdivisions, each $e_i$ for $i = 1, 2$ has a unique recurrent subedge at each level. If $b^0$ is recurrent, $b^0$ has a unique recurrent subband at each level whose sides are the unique recurrent subedges of $e_1$ and $e_2$. Hence it suffices to show that if duals of $e_1^n$ and $e_2^n$ are adjacent for $n(\geq 2p)$, then $b^0$ is recurrent.

For any $k \geq 0$ with $k \leq 2p \leq n$, we define $b^k = (t^k; e_1^k, e_2^k)$ by the level-$k$ band that contains $b^0$ as a subband. In particular, $e_i^{kp}$ is of type $e_i$ for $i, k \in \{1, 2\}$.

Case 1: $b^0$ is the only level-0 band having sides $\{e_1, e_2\}$.

In this case, $b^0$ is of type $b^0$ and $b^0$ is recurrent.

Case 2: There exists a level-0 band $b^0 = (t'; e_1, e_2)$ such that $t \neq t'$.

Since $b^0$ and $b'^0$ are the only level-0 bands which have $e_1$ and $e_2$ as their sides, $b^0$ and $b'^0$ are of type either $b_0$ or $b'_0$.

If $b^0$ is of type $b^0$, then $b^0$ is recurrent.

Suppose both $b^p$ and $b'^p$ are of type $b^0$. Then $b^0$ is a recurrent band with period recurrency $p$.

Let $b^p = (t^p; e_1^p, e_2^p)$ be the level-$p$ recurrent subband of $b'$. Note that $b^p$ and $b'^p$ share the sides $e_1^p$ and $e_2^p$, which are the unique level-$p$ recurrent subedges of $e_1$ and $e_2$. Since $b^p$ and $b'^p$ are of type $b^0$, i.e., $f^p(b^p) = f^p(b'^p) = b^0$, $f^p$ cannot be orientation preserving on $t^p \cup t'^p$. See Figure 5.

If $b^0$ is of type $b^0$ and $b'^0$ is of type $b^0$, then, by a similar reasoning, $f^p$ cannot be orientation preserving. 

\end{proof}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Figure 5}
\end{figure}

Proof of Proposition 4.10. Define $K > 0$ by

$$K := \max \{\text{lcm}(\text{per}(e_1), \text{per}(e_2)) \mid e_1 \text{ and } e_2 \text{ are recurrent level-0 edges}\},$$

where $\text{per}(e_i)$ means the period of recurrence of $e_i$, i.e., the length of the cycle containing $[e_i]$ in the directed graph of edges subdivisions $\mathcal{E}_\mathcal{R}$. Since $\text{per}(e_i) \leq |\text{Edge}(\mathcal{R})|$ for every level-0 recurrent edge $e_i$, we have $K \leq |\text{Edge}(\mathcal{R})|^2$.

It follows from Lemma 4.11 that for any $n \geq K$ every level-$n$ recurrent band bijectively corresponds to the adjacency of dual edges of recurrent level-$n$ subedges. Let $e^n$ be a level-$n$ recurrent subedge and $e^{*n}$ be its dual. There are three cases.

\begin{itemize}
\item $e^n$ is not a side of any recurrent band: $e^{*n}$ is an edge component of $N^n$.
\end{itemize}
The next lemma works in a general non-dynamical setting. Let $S$ be a surface and $\mathcal{T}$ be a finite CW-complex structure on $S$. Let $\gamma$ be a simple closed curve such that $\gamma_1, \ldots, \gamma_k$ be the segments of $\gamma$ divided by $\mathcal{T}$. A segment $\gamma_i$ is a corner segment if there exists a connected component $D$ of $S \setminus (\gamma_i \cup \mathcal{T}^{(1)})$ such that $\partial D \cap \operatorname{Vert}(\mathcal{T})$ is a singleton $\{v\}$. We call $v$ the corner vertex of $\gamma_i$.

Lemma 4.15. Let $S$ be a surface and $\mathcal{T}$ be a finite CW-complex structure on $S$. Suppose there is a subset $J$ of $\operatorname{Vert}(\mathcal{T})$ which is isolated in a sense that for any $v \in J$, every vertex adjacent to $v$ is not in $J$. Let $\alpha$ be a simple closed curve in $G$ which bounds a disk $U$ such that $U \cap \operatorname{Vert}(\mathcal{T}) \subset J$. Then there exists $v \in U$ satisfying one of the following.

(i) $\alpha$ is a peripheral loop of $v$ consisting of all the corner segments of $v$,
(ii) $\deg(v) > 1$ and $\alpha$ contains all but one corner segments of $v$, or
(iii) $\deg(v) = 1$.

Here $\deg(v)$ is the number of edges incident to $v$ in $\mathcal{T}^{(1)}$. Moreover, if $\alpha$ is a simple closed curve in the dual 1-skeleton of $\mathcal{T}$, then $\alpha$ is the peripheral loop.
Proof. We first define a tree in $U$ that is a deformation retract relative to $J$. Let $t$ be a closed 2-cell and $V$ be a connected component $U \cap t$. Let $c_1, \ldots, c_k$ be the connected components of $\nabla \cap \partial t$. If $k = 1$, then we do nothing. Suppose $k \geq 2$. Since every vertex $w \in \text{Vert}(\mathcal{T})$ adjacent to a vertex $v \in J$ is not in $J$, each $c_i$ contains at most one vertex in $\text{Vert}(\mathcal{T})$, which is actually in $J$. For each $c_i$, we choose one point $v_i \in c_i$ so that $v_i \in \text{Vert}(\mathcal{T})$ if $c_i \cap \text{Vert}(\mathcal{T}) \neq \emptyset$ and $v_i$ is an interior point of an 1-cell of $\mathcal{T}$ otherwise. There is a star-like tree $T_V \subset V$ of degree $k$ which is a deformation retract of $V$ relative to $\{v_1, \ldots, v_k\}$ so that the set of leaves of $T_V$ is $\{v_1, \ldots, v_k\}$.

![Figure 6](image)

**Figure 6.** The left is a star-like tree $H$, and the right is an example of the case when $U$ contains a degree one vertex. The bold parts are the star-like trees, and the dotted curves indicate the simple closed curve $\alpha$. Solid line segments are edges of $\mathcal{T}$.

See Figure 6. There are two starlike trees, one of which is a tripod (deg = 3) and the other is a curve (deg = 2). When a segment $c_i$ is shared by connected components of $U \cap t$ and $U \cap t'$, we choose the same point $v_i$ in the interior of $c_i$ so that $v_i$ is a joint of adjacent star-like trees. By joining all the star-like trees, we have a connected graph $H$ which is a deformation retract of $U$ relative to $J$ such that $H$ is a tree and $H \supset U \cap \text{Vert}(\mathcal{T})$.

If $H = \emptyset$, then $\alpha$ is a peripheral loop. Suppose $H \neq \emptyset$. Since $H$ is a tree, it has a leaf $v \in U \cap \text{Vert}(\mathcal{T})$. Suppose not (iii), i.e., $\text{deg}(v) \geq 2$. It follows from $U \cap \text{Vert}(\mathcal{T}) \subset J$ that $\alpha$ intersects every edge incident to $v$. Then there exist $\text{deg}(v) - 1$ corner segments of $\alpha$ around $v$. The only corner segment that is not in $\alpha$ is the corner segment contained in $t$ where $t$ is the tile containing a small neighborhood of $v$ in $H$. Hence $\alpha$ satisfies (ii).

Suppose that $\alpha$ is in the dual 1-skeleton of $\mathcal{T}$. It is easy to show that if $\alpha$ has $\text{deg}(v) - 1$ corner segments of $v$, then $\alpha$ must have the other corner segment. Hence $\alpha$ is a peripheral loop of $v$. □

**Proof of Proposition 4.13.** For $A := P_I \cup \text{Vert}_F(S_R)$, the branched covering $f : (S^2, A) \subset$ with the set of marked points $A$ does not have a Levy cycle by Proposition 2.8. It follows from Theorem 4.1 and Proposition 4.10 that, for a sufficiently large $n$, every connected component of the level-$n$ dual recurrent skeleton $N^n$ is either homotopically trivial or peripheral to a Julia vertex relative to $A$.

Let $\alpha$ be a simple closed curve in $N^n$. Then $\alpha$ bounds a disk $U_n$ such that $U_n \cap \text{Vert}(S_R) \subset \text{Vert}_I(S_R)$ and $U_n$ contains at most one vertex in $A$. It follows from Lemma 4.15 that $\alpha_n$ is the non-expanding peripheral cycle of a periodic Julia vertex. □

Recall that an edge $e$ is a Julia edge if it does not contain any Julia vertices in its subdivisions $\mathcal{R}^n(e)$. If $\mathcal{R}$ does not have any Julia edges, then we can make $\mathcal{R}$ have isolated Julia vertices by adding more vertices.

**Proposition 4.16.** Let $\mathcal{R}$ be a finite subdivision rule. Suppose that $\mathcal{R}$ does not have a Julia edge. Then, by adding some Fatou vertices to the vertex set of the level-0 complex $S_R$ if necessary, we can
make \( R \) have isolated Julia vertices. Moreover, this vertices adding process preserves the property that edges subdivide polynomially fast.

**Proof.** Suppose that there is a level-0 edge \( e \) of \( R \) whose endpoints are Julia vertices. Since \( R \) is assumed not to have any Julia edge, \( R^n(e) \) has a Fatou vertex \( w \) for some \( n > 0 \). Because \( f^n(w) \in \text{Vert}(S_R) \), the forward orbit of \( w \) is a finite set. Adding the forward orbit of \( w \) as level-0 vertices of \( R \), we can remove the adjacency of Julia vertices, which are endpoints of \( e \). Iterating this process, we obtain a finite subdivision rule with isolated Julia vertices.

Having polynomial growth rate of edge subdivisions is equivalent to having zero topological entropy, see Proposition 3.10. Since adding more vertices does not change the topological dynamics, we still have polynomial growth rate of edge subdivisions after adding vertices. \( \square \)

5. **Semi-conjugacy from subdivision maps to expanding maps**

The purpose of this section is to show that if a finite subdivision rule \( R \) has polynomial growth rate of edge subdivisions, then we can additionally assume that \( R \) has isolated Julia vertices. By Proposition 4.16, it suffices to show that we may assume that there are no Julia edges, see Corollary 5.3 for a precise statement. Our strategy is to collapse all the Julia edges of \( R \) to get a new finite subdivision rule \( R' \) which does not have any Julia edges. In this process, we show the following two properties: (1) The quotient space is still the sphere and (2) the subdivision maps of \( R \) and \( R' \) are combinatorially equivalent. We use a theory of semi-conjugacy (from topological branched self-coverings to rational maps) to show both claims.

**Remark.** It seems to be also possible to directly show that the finite subdivision rules of crochet maps constructed in [DHS22] have isolated Julia vertices, which is what we need in this article. However, the idea of simplification of finite subdivision rules by collapsing Julia edges subdividing polynomially fast may be applied in a broader context.

5.1. **Semi-conjugacy to Böttcher expanding maps.** Rees [Ree92] and Shishikura [Shi00] introduced the idea of semi-conjugacy to show the topological conjugacies between topological matings of polynomials, which are the quotients of the formal matings by the external ray equivalence relations, and the corresponding rational maps. Later, Cui-Peng-Tan further developed the idea and showed that if a topological branched covering \( f \) is combinatorially equivalent to a rational map \( F \), then there is a semi-conjugacy from \( f \) to \( F \) [CPT12].

Theorem 5.1 follows from [CPT12, Theorem 1.1, Corollary 1.2]. We replace rational maps by Böttcher expanding maps, but the arguments in [CPT12] still work for Böttcher expanding maps. In [CPT12], the conformal structure of the Riemann sphere was used only for (1) the existence of Böttcher coordinates near periodic Fatou points and (2) the expanding property of the conformal metric, both of which are still satisfied by Böttcher expanding maps. More precisely,

- in [CPT12], they use post-critically finite branched coverings on the Riemann sphere \( \hat{\mathbb{C}} \) that are holomorphic near critical cycles. Given a post-critically finite branched covering (on the topological sphere) which is locally rigid near critical cycles, we may define a holomorphic structure on the sphere so that the branched covering is holomorphic near critical cycles;
- the orbifold conformal metric in [CPT12, Section 2] can be replaced by the Riemannian orbifold metric in [BD18]. Then we still have the expansion property of homotopic lengths of paths.

**Theorem 5.1** (Semi-conjugacies to expanding maps). Let \( f : (S^2, A) \to \) be a post-critically finite branched covering which is locally rigid near critical cycles. Supposed \( f \) is combinatorially...
equivalent to a Böttcher expanding map $F : (S^2, B) \preceq \rightarrow \phi_0, \phi_1 : (S^2, A) \rightarrow (S^2, B)$, i.e., $F \circ \phi_0 = \phi_1 \circ f$. Let $F_f$ and $J_f$ be the Fatou set and the Julia set of $f$ respectively. Then there exists a semi-conjugacy $h : (S^2, A) \rightarrow (S^2, B)$ from $f$ to $F$ such that the following properties are satisfied.

- $h^{-1}(W)$ is a singleton for $w \in F_f$ and a full continuum for $w \in J_f$.
- For $x, y \in S^2$ with $F(x) = y$, the $h^{-1}(x)$ is a connected component of $f^{-1}(h^{-1}(y))$. Moreover, the degree of the map $f : h^{-1}(x) \rightarrow h^{-1}(y)$ is equal to $\deg_x(f)$; more precisely, for every $w \in h^{-1}(y)$ we have
  \[
  \sum_{z \in h^{-1}(x) \cap f^{-1}(w)} \deg z f = \deg_x(F).
  \]
- If $E \subset S^2$ is a continuum, then $h^{-1}(E)$ is a continuum.
- $f(h^{-1}(E)) = h^{-1}(F(E))$ for every $E \subset S^2$.
- $f^{-1}(\hat{E}) = f^{-1}(E)$ for every $E \subset S^2$, where $\hat{E} := h^{-1}(h(E))$.
- $h$ is homotopic relative to $A$ to $\phi_0$

5.2. Semi-conjugacy from subdivision maps. Recall that an edge (resp. a tile) is a Julia edge (resp. a Julia tile) if its subdivisions do not have any Julia vertices, see Definition 3.1.

Let $f : (S^2, A) \preceq \rightarrow$ be a subdivision map and consider the quotient map $h$ in Theorem 5.1. The quotient map $h$ maps every Julia edge or tile into the Julia set of a rational map $F$. Since the dynamics on the Julia set is expanding, every Julia edge or tile is either subdividing exponentially fast or collapsed to a point by the quotient map. Let us state this idea more rigorously as follows.

**Theorem 5.2.** Let $f : (S^2, A) \preceq \rightarrow$ be a subdivision map of a finite subdivision rule $R$ such that it does not have a Levy cycle. Let $g : (S^2, B) \preceq \rightarrow$ be a Böttcher expanding map combinatorially equivalent to $f$, which exists by Theorem 2.6. Assume that $f$ is locally rigid near critical cycles. For the semi-conjugacy $h : (S^2, A) \rightarrow (S^2, B)$ from $f$ to $g$ defined by Theorem 5.1, the following are satisfied.

- For every level-0 Julia edge $e$ with polynomial growth rate of subdivisions, $h(e)$ is a single point. Hence every level-$n$ edge of type $e$ in $R^n(S_R)$ also maps to a single point by $h$.
- If there is a Jordan curve $\gamma$ which consists of Julia edges of any level that are subdividing polynomially fast, then one of its two Jordan domains collapses to a point by $h$. The closure $D_{\gamma}$ of the collapsing Jordan domain consists of Julia tiles such that either $D_{\gamma} \cap A = \emptyset$ or $D_{\gamma} \cap A$ is a single point, which is a Julia vertex.

**Proof.** First we suppose that a level-0 edge $e$ is periodic, i.e., $f^n$ maps $e$ to $e$ homeomorphically for some $n > 0$. Since $g^n(h(e)) = h(f^n(e)) = h(e)$, by the expanding property of $g$, $h(e)$ is a single point. Since every periodic edge collapses to a point by $h$, we obtain $g^n(h(e)) = h(f^n(e)) = h(e)$ for edges $e$ with linear growth rate of subdivisions by a similar reasoning. Then by induction we can show that every level-0 edge with polynomial growth of subdivisions collapses to a point by $h$. The second part follows from Theorem 5.1.

**Collapsible subcomplex generated by Julia edges subdividing polynomially fast.** Let $f$ and $h$ be as in Theorem 5.2. Let $X_f$ be the collection of level-0 Julia edges with polynomial growth rate of subdivisions. Suppose some edges in $X_f$ form a Jordan curve $\gamma$. By Theorem 5.2, $h(\gamma)$ is a single point $w \in J_g$. It follows from Theorem 5.1 that $h^{-1}(w)$ is a full continuum. Then, for one of the two closed Jordan domains of $\gamma$, say $D_\gamma$, we have $h(D_\gamma) = w$. Hence $D_\gamma$ consists of Julia tiles, which are collapsed by $h$. Let $X_f$ be the level-0 Julia tiles that are collapsed by $h$ to
a point, and define $X^j := X^j_E \cup X^j_R$. Then $X^j$ is a collapsible subcomplex, see Definition 3.11. We call $X^j$ the collapsible subcomplex generated by Julia edges subdividing polynomially fast. The next corollary immediately follows from Theorem 5.2.

**Corollary 5.3.** Let $\mathcal{R}$ be a finite subdivision rule with a subdivision map $f : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$. Let $A \subset \text{Vert}(S_{\mathcal{R}})$ be a set of marked points of $f$. Suppose that $f : (S^2, A) \overset{\sim}{\to}$ does not have a Levy cycle. Let $\mathcal{R}'$ be the quotient finite subdivision rule of $\mathcal{R}$ by the collapsible subcomplex generated by Julia edges with polynomial growth of subdivisions. Let $f' : \mathcal{R}'(S_{\mathcal{R}'}) \to S_{\mathcal{R}'}$ be the subdivision map of $\mathcal{R}'$ and $A'$ be the image of $A$ under the quotient map. Then we have

- $\mathcal{R}'$ has Julia edges with polynomial growth rate of subdivisions, and
- $f' : (S^2, A') \overset{\sim}{\to}$ is combinatorially equivalent to $f : (S^2, A) \overset{\sim}{\to}$.

In addition, if $\mathcal{R}$ has polynomial growth rate of edge subdivisions, we have

- $\mathcal{R}'$ has polynomial growth rate of edge subdivisions, and
- (after adding more points as vertices of $\mathcal{R}'$ as in Proposition 4.16) $\mathcal{R}'$ has isolated Julia vertices.

**Proof.** Since every Julia edge with polynomial growth of subdivisions are collapsed, $\mathcal{R}'$ does not have such an edge. The combinatorial equivalence between $f$ and $f'$ follows from Theorem 5.2.

Assume that $\mathcal{R}$ has polynomial growth rate of edges subdivisions. Collapsing a collapsible subcomplex does not increase the growth rate of subdivisions, thus $\mathcal{R}'$ also has polynomial growth of edge subdivisions. Having isolated Julia vertices follows from Proposition 4.16. \qed

6. **Asymptotic $p$-conformal energy and Ahlfors-regular conformal dimension**

We have two objectives in this section: (1) We review the theory of conformal energies and its application in estimating conformal dimensions of Julia sets (Sections 6-6.2) and (2) establish the monotonicity of conformal energies under the decomposition of branched coverings (Theorem 6.22).

6.1. **Asymptotic $p$-conformal energies.** Let us review the theory of conformal energies introduced by D. Thurston in [Thu19a],[Thu20].

6.1.1. **Conformal graphs and energies.**

**Conformal graphs.** Let $G$ be a finite graph. We always assume that a linear structure is given to each edge, i.e., we fix a homeomorphism from each edge $e$ to $[0, 1]$ so that piecewise-linear (PL) maps between finite graphs are well-defined.

For $p \in (1, \infty]$, a $p$-length $\alpha$ on $G$ is a function $\alpha : \text{Edge}(G) \to \mathbb{R}_{>0}$. If $p = \infty$ we allow $\alpha(e) = 0$ for $e \in \text{Edge}(G)$. For $p \in (1, \infty]$, a $p$-conformal graph, denoted by $(G, \alpha)$, is a graph $G$ equipped with a $p$-length $\alpha$. In particular, if $\alpha$ is an $\infty$-length, then $(G, \alpha)$ is also called a length graph.

For the case $p = 1$, we assign a weight function to $G$ which is a weight function $w : \text{Edge}(G) \to \mathbb{R}_{\geq 0}$, which is also called an 1-length. We call $(G, w)$ a weighted graph or 1-conformal graph.

**Singular and regular values.** Let $G$ and $H$ be finite connected graphs. Suppose $\phi : H \to G$ is a PL map. A point $x \in H$ is a singular point if it is a vertex of $H$, a preimage of a vertex of $G$, a point where the linearity of $\phi$ breaks, or $\phi$ is constant on a small neighborhood of $x$. Otherwise $x \in H$ is a regular point. The image of a singular point is a singular value, and a point $y \in G$ is a regular value if it is not a singular value. There are only finitely many singular values.
Let \((G, \alpha)\) be a \(p\)-conformal graph and \((H, \beta)\) be a \(q\)-conformal graph with \(p \geq q\). For any PL map \(\phi : (H, \beta) \to (G, \alpha)\), we define conformal energies \(E_p^p(\phi)\) and \(E_p^q[\phi]\). In this paper we only consider two cases: \(p = q\) or \(1 = q < p\). See [Thu19a] for general cases.

**Conformal energies when \(p = q\).** First assume that \(p \in (1, \infty]\). Let \(\phi : (H, \beta) \to (G, \alpha)\) be a PL map between \(p\)-conformal graphs. We define

\[
\text{Fill}_p^p(\phi)(y) = \sum_{x \in \phi^{-1}(y)} |f'(x)|^{p-1},
\]

where \(y \in G\) is a regular value. Here \(f'\) is evaluated by considering \(\alpha\) and \(\beta\) as actual lengths of edges. Although \(\text{Fill}_p^p(\phi)\) is not defined on singular values, we simply say that \(\text{Fill}_p^p(\phi) : H \to \mathbb{R}_{\geq 0}\) is a function defined on \(H\).

When \(q = 1\), let \(\phi : (H, w_H) \to (G, w_G)\) be a PL map between two weighted graphs. The function \(\text{Fill}_1^1(\phi) : H \to \mathbb{R}_{\geq 0}\) is defined by

\[
\text{Fill}_1^1(\phi)(y) = \sum_{x \in \phi^{-1}(y)} w_H(x) \frac{w_G(y)}{w_G(y)}.
\]

The \(p\)-conformal energy of \(\phi\), denoted by \(E_p^p(\phi)\), is defined by

\[
E_p^p(\phi) := \text{ess. sup}_{y \in G} \left(\text{Fill}_p^p(\phi)(y)\right)^{1/p}.
\]

We often consider the infimum in the homotopy class

\[
E_p^p[\phi] := \inf_{\psi \sim \phi} E_p^p(\psi),
\]

which is also referred to as the \(p\)-conformal energy of \(\phi\).

**Conformal energies when \(1 = q < p\).** Let \((H, w)\) be a weighted graph and \((G, \alpha)\) be a \(p\)-conformal graph for \(p > 1\). Let \(\phi : H \to G\) be a PL map. We first define the multiplicity function \(n_\phi : G \to \mathbb{R}_{\geq 0}\) by

\[
n_\phi(y) = \sum_{x \in \phi^{-1}(y)} w(x).
\]

Then the conformal energy \(E_1^1\) is defined by

\[
E_1^1(\phi) := \|n_\phi\|_{p'} = \left(\int_G n_\phi w_H \frac{(p-1)}{p} \right)^{(p-1)/p},
\]

where \(p' := p/(p-1)\). We often use the infimum in the homotopy class

\[
E_1^1[\phi] := \inf_{\psi \sim \phi} E_1^1(\psi).
\]

It is immediate from definitions that the conformal energies are sub-multiplicative about compositions.

**Proposition 6.1 (Sub-multiplicativity).** For \(1 \leq p \leq q \leq r \leq \infty\), let \(F, G,\) and \(H\) be \(p\)-, \(q\)-, and \(r\)-conformal graphs respectively. Suppose that there are PL maps \(f : F \to G\) and \(g : G \to H\). Then, we have

\[
E_r^r[g \circ f] \leq E_r^q[g] \cdot E_q^p[f].
\]
Energy minimizers and taut maps. Let $\phi : H \to G$ be a PL map where $H$ is a $q$-conformal graph and $G$ is a $p$-conformal graph. The $\phi$ is an energy minimizer if $E_p^q[\phi] = E_p^q(\phi)$. An energy minimizer always exists [Thu19a, Theorem 6].

Suppose that $(H, w)$ be a weighted graph. For an interior point $y$ of an edge of $G$, we define
\[ n_{[\phi]}(y) = \inf_{\psi \sim \phi} n_{\psi}(y), \]
where the infimum is taken over $\psi$ for which $y$ is a regular value. From the definition, $n_{[\phi]}$ is constant on the interior of each edge. A map $\phi : H \to G$ is taut if $n_{[\phi]}(y) = n_{\phi}(y)$ for every $y$ in the interior of an edge of $G$. It is immediate from definitions that a taut map is an $E_p^1$-energy minimizer. A taut map always exists in the homotopy class [Thu19a, Theorem 3].

Proposition 6.2 ([Thu19a, Theorem 6]). For every $p \in [1, \infty]$ and any continuous map $f : G \to H$ between two $p$-conformal graphs, we have
\[ E_p^p[f] = \sup_{[c] : W \to G} \frac{E_p^1[f \circ c]}{E_p^1[c]}, \]
where $W$ is a weighted graph and $c : W \to G$ is a PL map.

Proof. See [Thu19a, Theorem 6].

(1) and (4) in the next proposition implies that the conformal energy is not changed by extension of range.

Proposition 6.3. Fix $p \in [1, \infty]$. Let $G$ be a $p$-conformal graph. Let $H \subset G$ be a $p$-conformal homotopically non-trivial subgraph such that $\iota : H \hookrightarrow G$ be the embedding. Then we have the following properties.

1. [Thu19a, Lemma 3.43] For every weighted graph $W$ and a continuous map $f : W \to H$, the $f$ is taut if and only if $\iota \circ f$ is taut.
2. $E_p^1[\iota] = 1$.
3. $E_p^1[f] = E_p^1[\iota \circ f]$.
4. For every $p$-conformal graph $F$ and a continuous map $g : F \to H$, we have $E_p^p[g] = E_p^p[\iota \circ g]$.

Proof. (1) We refer to [Thu19a, Lemma 3.43]; (2) It is immediate that $1 \leq E_p^p[\iota] \leq E_p^p(\iota) = 1$; (3) Since a taut map always exists in the homotopy class, we may assume that there is a taut map $f : W \to H$. It follows from (1) that $\iota \circ f$ is also taut. Then we have
\[ E_p^1(\iota \circ f) = E_p^1[\iota \circ f] \leq E_p^p[\iota] \cdot E_p^1[f] = E_p^p[\iota] \cdot E_p^1(f). \]
We have $E_p^1(\iota \circ f) = E_p^1(f) \neq 0$ from definition of $E_p^1$ and $H$ being homotopically non-trivial. Thus $1 \leq E_p^p[\iota]$. Then the above inequality is the equality by (2) and we have $E_p^1[f] = E_p^1[\iota \circ f]$; (4) By Theorem 6.2, we have
\[ E_p^p[\iota \circ g] = \sup_{[c] : W \to F} \frac{E_p^1[\iota \circ g \circ c]}{E_p^1[c]} = \sup_{[c] : W \to F} \frac{E_p^1[g \circ c]}{E_p^1[c]} = E_p^p[g]. \]
We use (3) for the middle equality.

Reduced maps. For a point $x$ in a graph $G$, a direction $d$ at $x$ is a germ of homeomorphisms $\gamma : [0, 1] \to G$ with $\gamma(0) = x$. Directions at $x$ are bijectively corresponded to connected components of $U \setminus \{x\}$ where $U$ is a small neighborhood of $x$. Let $Z \subset G$ be a connected subset. A direction from $Z$ is a direction $d$ at a boundary point $x \in \partial Z$ such that $d$ is not toward $Z$. 
Suppose $\phi : H \to G$ is a continuous map between finite graphs. Let $x \in G$ be a point and $Z \subset H$ be a connected component of $\phi^{-1}(x)$. Let $D$ be the directions from $Z$. Then $\phi(D)$ is a set of directions at $x$. The connected component $Z$ is a dead end if $\phi(D)$ consists of only one direction at $x$. The map $\phi$ is reduced if it does not have any dead end.

If $\phi : H \to G$ has a dead end $Z$, we can pull the $\phi$-image of $Z$ in the direction $\phi(D)$, via a homotopy, to remove the dead end. This way, we can homotope any map $\phi : H \to G$ to a reduced map with conformal energies non-increased [Thu19a, Propositions 3.4, 3.5]. Hence we may assume that every energy minimizer is reduced.

Conformal energies for marked graphs. A marked graph $(G,A)$ is a pair of a finite graph $G$ and its finite subset $A$. Elements in $A$ are called marked points. A continuous map $\phi : (H,B) \to (G,A)$ between two marked graphs is always assumed to satisfy $\phi(B) \subset A$. By a homotopy between two maps $\phi, \psi : (H,B) \to (G,A)$ we mean a homotopy relative to $B$. The definitions and properties of conformal energies still work for marked graphs and relative homotopies. See [Thu19a] for details. In this paper, we use marked graphs only in Lemma 8.3 and its application in Section 8.3.

6.1.2. Graph virtual endomorphisms and asymptotic conformal energies.

**Definition 6.4** (Graph virtual endomorphism). A graph virtual endomorphism is a quadruple $\mathcal{G} = (G_0,G_1,\pi,\phi)$ where $\pi : G_1 \to G_0$ is a covering map of finite degree on each connected component and $\phi : G_1 \to G_0$ is a continuous function. We often write $\pi,\phi : G_1 \to G_0$ to indicate the virtual endomorphism. If $G_0$ and $G_1$ are connected, then we say that $\mathcal{G}$ is connected.

In this paper, all but the graph virtual endomorphisms of stable multicurves, which will be introduced in Section 6.6, are assumed to be connected.

**Iterates of virtual endomorphisms.** Let $\pi,\phi : G_1 \to G_0$ be a graph virtual endomorphism. Pulling back the covering map $\pi : G_1 \to G_0$ through $\phi : G_1 \to G_0$, we have a covering map $\pi^n_1 : G_2 \to G_1$. Lifting the continuous map $\phi : G_1 \to G_0$ through the coverings $\pi$ and $\pi^2$, we have $\phi^n_1 : G_2 \to G_1$. Let $\pi^n_0 = \pi$ and $\phi^n_0 = \phi$. By iterating the process, for every $n \geq 0$, we have $G_n$ and $\pi^n_0, \phi^n_0 : G_n \to G_{n-1}$. For every $n > m \geq 0$, we define

$$\pi^n_m := \pi^{m+1}_n \circ \cdots \circ \pi^{-1}_{n-2} \circ \pi^{n-1}_{n-1} : G_n \to G_m$$

and

$$\phi^n_m := \phi^{m+1}_n \circ \cdots \circ \phi^{-1}_{n-2} \circ \phi^{n-1}_{n-1} : G_n \to G_m.$$

Assume that $G_0$ is equipped with a $p$-length $\alpha$ for $p \in [1,\infty]$. For every $n \geq 1$, we equip $G_n$ with a $p$-length that is the lift of $\alpha$ through $\pi^n_0$. Then we can evaluate $E_p[\phi^n_0]$ for each $n \geq 1$.

Note that we use superscripts for the levels of domains and subscripts for the levels of ranges.

**Definition 6.5** (Asymptotic conformal energies). Let $\mathcal{G} = (G_0,G_1,\pi,\phi)$ be a graph virtual endomorphism. For $p \in [1,\infty]$, the asymptotic $p$-conformal energy $E^p(\mathcal{G})$ of $\mathcal{G}$ is defined by

$$E^p(\mathcal{G}) := \lim_{n \to \infty} \sqrt[p]{E^p_p[\phi^n_0]}.$$

There are two issues in the above definition: the existence of the limit and the independence of the choice of $p$-lengths on $G_0$.

First, assume that a $p$-length of $G_0$ is given. For every $n \geq 0$ and $k > 0$, the $\phi^{n+k} : G_{n+k} \to G_n$ is the lift of $\phi^k : G_k \to G_0$ through the coverings $\pi^{n+k}_0$ and $\pi^k_0$. By the homotopy lifting property, we have

$$E^p_p[\phi^{n+k}] \leq E^p_p[\phi^k].$$
Hence, for every \( n, k \geq 1 \), we have

\[
E_p^p[\phi_n^{n+k}] = E_p^p[\phi_n^{n+k} \circ \phi_0^n] \leq E_p^p[\phi_0^n] \cdot E_p^p[\phi_0^n].
\]

Then the limit in Definition 6.5 exists by Fekete’s lemma.

Second, a different choice of \( p \)-length changes \( E_p^p[\phi_0^n] \) within constant multiplications that are independent of \( n \). Thus the limit is unchanged.

**Homotopy invariance of conformal energies.**

**Definition 6.6** (Homotopy morphisms). Let \( \mathcal{G} = (G_0, G_1, \pi_G, \phi_G) \) and \( \mathcal{H} = (H_0, H_1, \pi_H, \phi_H) \) be two graph virtual endomorphisms. A homotopy morphism from \( \mathcal{G} \) to \( \mathcal{H} \) is a pair of maps \( \theta_0 : G_0 \to H_0 \) and \( \theta_1 : G_1 \to H_1 \) satisfying \( \theta_0 \circ \pi_G = \pi_H \circ \theta_1 \) and \( \theta_0 \circ \phi_G = \phi_H \circ \theta_1 \), where \( \sim \) means a homotopy. We write \( \theta : \mathcal{G} \to \mathcal{H} \) to indicate the homotopy morphism.

Let \( \theta : \mathcal{G} \to \mathcal{H} \) be a homotopy morphism as in Definition 6.6. By the homotopy lifting property, we have a sequence of maps \( \{\theta_n : G_n \to H_n\}_{n \geq 0} \) such that for every \( m > n \geq 0 \) we have

\[
\theta_n \circ (\phi_G)_n^m \sim (\phi_H)_n^m \circ \theta_m.
\]

**Definition 6.7** (Homotopy equivalence). Two graph virtual endomorphisms \( \mathcal{G} = (G_0, G_1, \pi_G, \phi_G) \) and \( \mathcal{H} = (H_0, H_1, \pi_H, \phi_H) \) are homotopy equivalent if there exist morphisms \( \theta : \mathcal{G} \to \mathcal{H} \) and \( \eta : \mathcal{H} \to \mathcal{G} \) such that \( \eta_0 \circ \theta_0 \sim id_{H_0} \) and \( \eta_0 \circ \theta_0 \sim id_{G_0} \), and \( \phi_H \circ \theta_1 \sim \theta_0 \circ \phi_G \) and \( \phi_G \circ \eta_1 \sim \eta_0 \circ \phi_H \).

The conformal energies are invariant under homotopy equivalence. More precisely, the terms in the limit in Definition 6.5 differ by constant multiplications and additions, which vanish as \( n \) tends to the infinity.

**Proposition 6.8** (Homotopy invariance of conformal energies). Let \( \mathcal{G} = (G_0, G_1, \pi_G, \phi_G) \) and \( \mathcal{H} = (H_0, H_1, \pi_H, \phi_H) \) be homotopically equivalent graph virtual endomorphisms. Then, for every \( p \in [1, \infty) \), we have \( E^p(\mathcal{G}) = E^p(\mathcal{H}) \).

### 6.2. Julia sets of virtual endomorphisms.

Let \( \mathcal{G} = (G_0, G_1, \pi, \phi) \) be a connected graph virtual endomorphism, i.e., \( G_0 \) and \( G_1 \) are connected. We say that \( \mathcal{G} \) is recurrent if \( \phi_* : \pi_1(G_1) \to \pi_1(G_0) \) is surjective and \( \mathcal{G} \) is forward-expanding (or backward-contracting) if \( E^p(\mathcal{G}) < 1 \). We define the Julia set \( J_\mathcal{G} \) of \( \mathcal{G} \) as the inverse limit of the system

\[
\{\phi_m^n : G_n \to G_m \mid n > m \geq 0\}.
\]

If \( \mathcal{G} \) is recurrent and forward-expanding, then the Julia set \( J_\mathcal{G} \) is locally connected and has a visual metric \( d_{\text{vis.}} \), which is Ahlfors-regular and defined up to quasi-symmetry. More precisely, the visual metric is determined up to snowflake equivalence, and any snowflake map is a quasi-symmetry.

**Definition 6.9** (Ahlfors-regular gauge of Julia set). The quasi-symmetric class of Ahlfors-regular metrics on the Julia set \( J_\mathcal{G} \) containing visual metrics is called the Ahlfors-regular gauge of \( J_\mathcal{G} \).

The covering map \( \pi \) induces a self-covering map of \( J_\mathcal{G} \), which we also denote by \( \pi \). If \( \mathcal{G} = (G_0, G_1, \pi_G, \phi_G) \) and \( \mathcal{H} = (H_0, H_1, \pi_H, \phi_H) \) are homotopically equivalent graph endomorphisms, then their self-coverings maps on the Julia sets \( \pi_\mathcal{G} : J_\mathcal{G} \cong \) and \( \pi_\mathcal{H} : J_\mathcal{H} \cong \) are topologically conjugate. Furthermore, the topological conjugacy preserves the Ahlfors-regular gauges of \( J_\mathcal{G} \) and \( J_\mathcal{H} \).

**Definition 6.10** (Ahlfors-regular conformal dimension of Julia set). For a connected recurrent forward-expanding graph virtual endomorphism \( \mathcal{G} \), we define the Ahlfors-regular conformal dimension of the Julia set \( J_\mathcal{G} \), denoted by \( \text{ARC. dim}(J_\mathcal{G}) \), to be the infimum of the Hausdorff dimensions with respect to the metrics in the Ahlfors-regular gauge of \( J_\mathcal{G} \).
**Theorem 6.11** (Pilgrim-D. Thurston [PT21]). Let $\mathcal{G} = (G_0, G_1, \pi, \phi)$ be a connected recurrent forward-expanding graph virtual endomorphism. For $p_* = \text{ARC}. \dim (\mathcal{J}_G)$, we have $E^{p_*}(\mathcal{J}_G) = 1$.

### 6.3. Shift and power.

**Definition 6.12** (Shift). Let $\mathcal{G} = (G_0, G_1, \pi, \phi)$ be a graph virtual endomorphism. For an integer $k \geq 0$, the $k^{th}$ shift $\mathcal{G}_k$ of $\mathcal{G}$ is defined by $\mathcal{G}_k = (G_k, G_{k+1}, \pi_k^{k+1}, \phi_k^{k+1})$.

**Proposition 6.13.** Let $\mathcal{G}$ be a graph virtual endomorphism and $\mathcal{G}_k$ be its $k^{th}$-shift for some integer $k \geq 0$. For every $p \in [1, \infty]$, we have

$$E^p(\mathcal{G}) = E^p(\mathcal{G}_k).$$

**Proof.** For every $n > 0$,

$$E^p_0[\phi_0^{n+k}] \leq E^p_0[\phi_k^{n+k}] \cdot E^p_0[\phi_0^k] \leq E^p_0[\phi_0^n] \cdot E^p_0[\phi_0^k].$$

The last inequality follows from (1). Because

$$E^p(\mathcal{G}_k) = \lim_{n \to \infty} \sqrt[n]{E^p_0[\phi_k^{n+k}]},$$

we can obtain the conclusion from the above inequalities. \(\Box\)

**Definition 6.14** (Power). Let $\mathcal{G} = (G_0, G_1, \pi, \phi)$ be a graph virtual endomorphism. For an integer $k \geq 0$, the $k^{th}$-power $\mathcal{G}^k$ of $\mathcal{G}$ is defined by $\mathcal{G}^k = (G_0, G_k, \pi_0^k, \phi_0^k)$.

The following proposition is immediate from the definitions.

**Proposition 6.15.** Let $\mathcal{G}$ be a graph virtual endomorphism and $\mathcal{G}^k$ be the $k^{th}$-power of $\mathcal{G}$ for some integer $k \geq 1$. For every $p \in [1, \infty]$, we have

$$E^p(\mathcal{G}) = (E^p(\mathcal{G}^k))^{1/k}. $$

### 6.4. Monotonicity of conformal energies: Graph virtual endomorphisms.

**Theorem 6.16** (Monotonicity of conformal energies of graph virtual endomorphisms). Let $\mathcal{G} = (G_0, G_1, \pi_G, \phi_G)$ and $\mathcal{H} = (H_0, H_1, \pi_H, \phi_H)$ be graph virtual endomorphisms. Suppose that there is a homotopy morphism $\theta : \mathcal{H} \to \mathcal{G}$ such that $\theta_0 : H_0 \to G_0$ is an embedding (so that $\theta_n : H_n \to G_n$ is also an embedding for every $n \geq 0$). Then, for every $p \in [1, \infty]$,

$$E^p(\mathcal{H}) \leq E^p(\mathcal{G}).$$

**Proof.** Since $\theta_n : H_n \to G_n$ is an embedding, $E^p_0[\theta_n] = 1$ for every $n \in \mathbb{N}$ by Proposition 6.3. It follows that

$$E^p_0[(\phi_G)_0^n \circ \theta_n] \leq E^p_0[(\phi_G)_0^n] \cdot E^p_0[\theta_n] = E^p_0[(\phi_G)_0^n].$$

On the other hand, it also follows from Proposition 6.3 that

$$E^p_0[(\phi_H)_0^n] = E^p_0[\theta_0 \circ (\phi_H)_0^n] = E^p_0[(\phi_G)_0^n \circ \theta_n].$$

Therefore, for every $n \geq 0$, we have $E^p_0[(\phi_G)_0^n] \leq E^p_0[(\phi_G)_0^n]. \Box$
6.5. **Virtual endomorphisms of post-critically finite branched coverings.** Suppose that \( f: (S^2, A) \Rightarrow \) is a post-critically finite hyperbolic-type branched covering. Let a finite graph \( G_0 \) be a spine of \( (S^2, A) \), i.e., \( G_0 \subset S^2 \backslash A \) is a deformation retract of \( S^2 \backslash A \). Define \( G_1 := f^{-1}(G_0) \). Then \( G_1 \) is a spine of \( (S^2, f^{-1}(A)) \). Since \( f(A) \subset A \), we have \( A \subset f^{-1}(A) \). The inclusion \( S^2 \backslash f^{-1}(S) \hookrightarrow S^2 \backslash A \) defines a homotopy class of continuous maps \( \{ \phi : G_1 \to G_0 \} \), where \( \phi \) is a representative. Then the quadruple \( \mathcal{G}_f := (G_0, G_1, f, \phi) \) is a virtual endomorphism of \( f \).

There are different choices of the spine \( G_0 \) and the representative \( \phi \), but all the choices give rise to the same graph virtual endomorphism up to homotopy equivalence.

For \( \mathcal{G}_f \) given as above, the \( n^{th} \)-iterate \( G_n \) is easily obtained as the \( n^{th} \)-preimage \( f^{-n}(G_0) \). The homotopy class \( [\phi^0]_n \) is induced from the inclusion \( S^2 \backslash f^{-n}(A) \hookrightarrow S^2 \backslash A \).

**Remark.** We may also define a graph virtual endomorphism of a post-critically finite branched covering that is not of hyperbolic-type in the same way. However, its asymptotic conformal energy is always greater than or equal to one; for any covering that is not of hyperbolic-type in the same way. Hence, its asymptotic conformal energy is invariant under the choice of the set of marked points \( A \). Hence we simply denote by \( \overline{E}^p(f) \) the asymptotic \( p \)-conformal energy of \( f: (S^2, A) \Rightarrow \).

**Proposition 6.17** (Independence of the choice of marked points). Suppose \( f: (S^2, A) \Rightarrow \) is a hyperbolic-type post-critically finite branched covering and \( \mathcal{G} \) is its graph virtual endomorphism. Let \( \mathcal{H} \) be graph virtual endomorphisms of \( f: (S^2, P_f) \Rightarrow \). Then \( \overline{E}^p(\mathcal{G}) = \overline{E}^p(\mathcal{H}) \) for every \( p \in [1, \infty] \).

**Proof.** Since \( P_f \subset A \), we can take \( \mathcal{G} \) and \( \mathcal{H} \) in such a way that, for every \( n \geq 0 \), \( H_n \subset G_n \). It follows from Theorem 6.16 that \( \overline{E}^p(\mathcal{G}) \geq \overline{E}^p(\mathcal{H}) \).

Since \( A \) is a set of Fatou points, there exists an integer \( k \geq 0 \) such that \( P_f \subset A \subset f^{-k}(P_f) \). Then \( \mathcal{G} \) and \( \mathcal{H} \) can be chosen so that there is an embedding of \( \mathcal{G} \) into the \( k \)-th shift \( \mathcal{H}[k] \) of \( \mathcal{H} \). Then we have

\[
\overline{E}^p(\mathcal{G}) \leq \overline{E}^p(\mathcal{H}[k]) = \overline{E}^p(\mathcal{H}).
\]

\[\square\]

The equivalence between the non-existence of a Levy cycle and \( \overline{E}^{\mathbb{C}} < 1 \) is mentioned in [Thu20].

**Proposition 6.18.** Let \( f: (S^2, A) \Rightarrow \) be a hyperbolic-type post-critically finite branched covering. Then \( f: (S^2, A) \Rightarrow \) does not have a Levy cycle if and only if \( \overline{E}^{\mathbb{C}}(f) < 1 \).

**Proof.** See [Par21, Proposition 7.21] for details. \[\square\]

If \( f \) is a rational map, then the self-covering \( f: J_f \Rightarrow \) is topologically conjugate to the self-covering \( f: J_{g_f} \Rightarrow \). Moreover, the Ahlfors-regular gauge of \( J_{g_f} \) contains the pull-back of the spherical metric on \( J_f \) through the topological conjugacy [HP09].

6.6. **Monotonicity of conformal energies: Branched coverings.**
Decomposition along multicurves. Let $\Gamma$ be a multicurve of $(S^2, A)$ which is completely $f$-invariant up to isotopy, i.e., the collections of the homotopy classes of essential components appear in $f^{-1}(\Gamma)$ and in $\Gamma$ are the same. We split the sphere into small spheres by pinching along $\Gamma$. More precisely, we remove $\Gamma$ from $S^2$, compactify each connected component $X$ with boundary circles, and collapse each boundary circle to a point. We refer to the compactified spheres as the small spheres of the decomposing multicurve $\Gamma$.

Then the dynamics $f: (S^2, A) \curvearrowright$ descents to the dynamics on the union of small spheres as follows: Let $X$ be a connected component of $S^2 \setminus \Gamma$ and $\hat{X}$ be the sphere that is the compactification of $X$. There exists a unique connected component $Y'$ of $S^2 \setminus f^{-1}(\Gamma)$ such that $X$ and $Y'$ are homotopic relative to $A$. Let $Y$ be a connected component of $S^2 \setminus \Gamma$ such that $Y = f(Y')$ and $\hat{Y}$ be the sphere that is the compactification of $Y$. Using the map $f \circ Y' : Y' \to Y$ and the homotopy between $X$ and $Y$, we have a map $\hat{X} \to \hat{Y}$. In this way, we have an induced map among the small spheres.

Let $F$ be the induced map among the small spheres of $\Gamma$. A small sphere $S$ is periodic if $F^k(S) = S$ for some $k > 0$. Let $\{S^2(i) \mid i = 1, 2, \ldots, n\}$ be the set of periodic small spheres. The first return time $\tau_i$ of a periodic small sphere $S^2(i)$ is the smallest positive integer $k$ satisfying $F^k(S^2(i)) = S^2(i)$. For each $i$, we define the first return map $f_i$ of $S^2(i)$ by

$$f_i := F^{\tau_i} = S^2(i) \curvearrowright.$$ 

In this article, we discard small spheres with $\deg(f_i) = 1$ and assume that $\deg(f_i) > 1$.

To each periodic small sphere $S^2(i)$ we assign a set of marked points $A(i)$ as follows: Let $U(i)$ be the connected component of $S^2 \setminus \Gamma$ such that $U(i)$ is the compactification of $U(i)$ with boundary circles and there is a quotient map $q_i : U(i) \to S^2(i) = U(i)/\sim$ which collapses every boundary circles to a point. The set $A(i) \subset S^2(i)$ is defined as the union of $q_i(A \cap U(i))$ and $q_i(\partial U(i))$. Thus every point in $A(i)$ comes from $A$ or $U$. It is easy to show that $f_i$ is post-critically finite and $P_{f_i} \subset A(i)$. Hence $f_i : (S^2(i), A(i)) \curvearrowright$ is also a post-critically finite branched covering, and we call it a small branched covering of the $\Gamma$-decomposition of $f$.

If there is a Thurston obstruction $\Gamma$ of a small branched covering $f_i : (S^2(i), A(i)) \curvearrowright$, then $\Gamma$ lifts to a Thurston obstruction of $f : (S^2, A) \curvearrowright$. It follows that if $f : (\hat{\mathcal{C}}, A) \curvearrowright$ is a post-critically finite rational map, then its small branched coverings are combinatorially equivalent to rational maps. In this case, after taking a homotopy and giving an appropriate complex structure on $S^2(i)$, we may assume that the small branched coverings are post-critically finite rational maps $f_i : (\hat{\mathcal{C}}(i), A(i)) \curvearrowright$, which we refer to as small rational maps of $\Gamma$-decomposition of $f$.

We refer the reader to [Pil03] for details of the decomposition of branched coverings.

**Lemma 6.19.** Let $f : (S^2, A) \curvearrowright$ is a post-critically finite branched covering of hyperbolic-type. Suppose $\Gamma$ is a multicurve which is completely $f$-invariant up to isotopy such that $\Gamma$ does not contain a Levy cycle. Then every small branched covering of the $\Gamma$-decomposition of $f$ is also of hyperbolic-type.

**Proof.** Let $f_i : (S^2(i), A(i)) \curvearrowright$ be a small branched covering of $(f, \Gamma)$. Let $p \in A(i)$. If $p$ comes from $U(i) \cap A$, then $p$ is a Fatou point because every point in $A$ is a Fatou point of $f : (S^2, A) \curvearrowright$. Assume $p$ is the quotient of an element of $\Gamma$. If $p$ is periodic under $f_i$, then the periodic cycle containing $p$ is a critical cycle because $\Gamma$ does not contains a Levy cycle. Hence every element $A(i)$ is a Fatou point of $f_i : (S^2(i), A(i)) \curvearrowright$. 

**Remark.** If $\Gamma$ has a Levy cycle, then some small branched covering $f_i : (S^2(i), A(i)) \curvearrowright$ may have a Julia marked point $p \in A(i)$. But $f_i : (S^2(i), P_{f_i}) \curvearrowright$ is still of hyperbolic-type.
Let $\Gamma$ be a post-critically finite hyperbolic-type branched covering. Suppose $\Gamma$ is a multicurve of $(S^2, A)$ that is completely $f$-invariant up to isotopy such that $\Gamma$ does not contain a Levy cycle. Let $f_i : (S^2, A(i)) \varsubsetneq$ be a small branched covering of the $\Gamma$-decomposition of $F$ with the first return time $\tau_i$. Then for any $p \in [1, \infty]$ we have

$$(E^p(f^{\tau_i}))^{1/\tau_i} \leq E^p(f).$$

**Proof.** Let us take a spine $H_0$ of $(S^2(i), A(i))$. For the corresponding component $U(i)$ of $S^2 \setminus \Gamma$, $H_0$ lifts to a spine of $U(i)$, which we also denote by $H_0$. We extend $H_0$ to a spine $G_0$ of $(S^2, A)$ so that there is an embedding $\iota_0 : H_0 \to G_0$. Then we have graph virtual endomorphisms $\mathcal{G} = (G_0, G_1, f, \phi)$ of $f : (S^2, A) \varsubsetneq$ and $\mathcal{H} = (H_0, H_1, f_i, \phi_i)$ of $f_i : (S^2(i), A(i)) \varsubsetneq$ such that $\iota : \mathcal{H} \to \mathcal{G}_{\Gamma}$ is a morphism with $\iota_0 : H_0 \to G_0$ being an embedding. Then the inequality follows from Proposition 6.15 and Theorem 6.16. □

**Virtual endomorphism of stable multicurves.** Let $f : (S^2, A) \varsubsetneq$ be a hyperbolic-type post-critically finite possibly peripheral multicurve and $\Gamma$ be an $f$-stable possibly peripheral multicurve of $(S^2, A)$. i.e., we allow a component of $\Gamma$ to be peripheral to $A$. Denote $\Gamma_0 = \Gamma$ and let $\Gamma_1$ be the collection of connected components of $f^{-1}(\Gamma)$ that are essential relative to $A$. The restriction $f_{\Gamma} := f|_{\Gamma_1} : \Gamma_1 \to \Gamma_0$ is a covering map on each connected component. Since $\Gamma$ is $f$-stable, every connected component $\gamma'$ of $\Gamma_1$ is isotopic relative to $A$ to a connected component $\gamma$ of $\Gamma_0$. Define $\phi_{\Gamma} : \Gamma_1 \to \Gamma_0$ in such a way that the $\gamma'$ is mapped to $\gamma$ through the isotopy. Then the quadruple $\mathcal{G}_{\Gamma} = (\Gamma_0, \Gamma_1, f_{\Gamma}, \phi_{\Gamma})$ is the graph virtual endomorphism of $\Gamma$. This is the only graph virtual endomorphism in this article whose graph is disconnected.

**Proposition 6.21.** Let $f : (S^2, A) \varsubsetneq$ be a post-critically finite hyperbolic-type branched covering. Let $\Gamma$ be an $f$-stable possibly peripheral multicurve and $\mathcal{G}_{\Gamma} = (\Gamma_0, \Gamma_1, f_{\Gamma}, \phi_{\Gamma})$ be its graph virtual endomorphism. Then, for every $p \in [1, \infty]$, we have

$$E^p(\mathcal{G}_{\Gamma}) = (\lambda_p(\Gamma))^{1/p},$$

where $\lambda_p(\Gamma)$ is the maximal eigenvalue of the $p$-transformation of $\Gamma$.

**Proof.** Since $E^p$ is independent of the length of edges, we may assume that all the circles in $\Gamma$ have the same length. Assume that $\phi_{\Gamma}$ has constant derivative on every circle.

Let $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ and consider it as an ordered basis of $\mathbb{R}^\Gamma$. Let $A_{p, \Gamma}$ be the matrix representative of the $p$-transformation $f_{\Gamma}$ with respect to this ordered basis. For every $k > 0$ and $x \in \gamma_i$, the Fill$^p((\phi_{\Gamma})^k_0 : \Gamma_k \to \Gamma_0)(x)$ is equal to the $i^{th}$-row sum of $(A_{p, \Gamma})^k$. It follows that $E^p((\phi_{\Gamma})^k_0)$ is the $p^{th}$-root of the maximal row sum of $(A_{p, \Gamma})^k$. Then the conclusion follows from the fact that the maximal row sum of $(A_{p, \Gamma})^k$ grows exponentially fast about $k$ with base $\lambda_p(\Gamma)$. □

We remark that the definitions of the decomposition along $\Gamma$ and small branched coverings still work even if $\Gamma$ has a peripheral component. Recall that if $\Gamma$ does not have a Levy cycle, then $\lambda_p(\Gamma)$ is strictly decreasing about $p$ so that the critical exponent $Q(\Gamma)$ is determined by $\lambda_{Q(\Gamma)}(\Gamma) = 1$.

**Theorem 6.22** (Monotonicity of conformal energies). Let $f : (S^2, A) \varsubsetneq$ be a post-critically finite hyperbolic-type branched covering without a Levy cycle. Let $\Gamma$ be a multicurve which possibly has a simple closed curve peripheral to $A$. Suppose that $\Gamma$ is completely $f$-invariant up to isotopy such that

$$\{f_i : (S^2(i), A(i)) \varsubsetneq \mid i = 1, 2, \ldots, n\}$$
is the set of small branched coverings of the $\Gamma$-decomposition with the first return times $\{\tau_i\}_{i=1,2,\ldots,n}$. Then, for every $p \in [0, \infty]$, we have
\[\mathcal{E}^p(f) \geq \max\left\{ \left(\mathcal{E}^p(f_i)\right)^{1/\tau_i} \mid 1 \leq i \leq n, \ (\lambda_p(\Gamma))^{1/p} \right\} .\]
Moreover, we have $\text{ARC} \cdot \dim(J_f) \geq Q(\Gamma)$.

Proof. The first inequality follows from Proposition 6.20 and Proposition 6.21. Since $\Gamma$ does not have a Levy cycle, $\lambda_p(\Gamma)$ is strictly decreasing. Then the last inequality follows from Theorem 6.11. □

7. Crochet maps

In this section, we define crochet maps and discuss some examples, such as matings (Proposition 7.3). Then we state Dudko-Hlushchanka-Schleicher’s decomposition theorem (Theorem 7.5), and show one direction of Theorem A (Corollary 7.8).

Definition 7.1 (Fatou quotient). Let $f$ be a post-critically finite rational map with non-empty Fatou set. We define an equivalence relation $\sim_F$ as the closure of a relation defined by
\[x \sim y \in \hat{\mathbb{C}} \text{ if } x \text{ and } y \text{ are in the same Fatou component}.
The quotient space $\hat{\mathbb{C}}/\sim_F$ is called the Fatou quotient of the Riemann sphere. Alternatively, we can obtain the same quotient space by collapsing every Fatou component to a point and then taking the Hausdorff quotient.

Definition 7.2 (Crochet maps). Let $f$ be a post-critically finite rational map with non-empty Fatou set. Then $f$ is a crochet map if there exists a forward $f$-invariant graph $G$ containing $P_f$ such that $h_{\text{top}}(f|_G) = 0$, where $h_{\text{top}}$ means topological entropy.

Below is a list of families of crochet maps and their invariant graphs.

- Post-critically finite polynomials and spiders [HS94].
- Critically fixed rational maps and Tischler graphs [CGN+15, Hlu19].
- Critically fixed anti-rational maps and Tischler graphs [Gey20, LLM22].
- Post-critically finite Newton maps and extended Newton graphs [DMRS19, LMS15]. We need a slight modification in this case. In these articles, invariant graphs, called extended Newton graphs, are defined as the extensions of Newton graphs by using Hubbard trees for renormalizable parts. We can also extend the Newton graphs by using spider graphs, and this extension yields invariant graphs with topological entropy zero.
- Matings of post-critically finite polynomials at least one of whose core entropy is zero, Proposition 7.3.

Proposition 7.3 (Crochet matings). Let $f$ and $g$ be post-critically finite polynomials of the same degree $d$. Suppose that the Böttcher coordinate for the external Fatou components are fixed so that the mating of $f$ and $g$ is uniquely defined. Assume that $f$ and $g$ are mateable and $F$ is the corresponding degree $d$ rational map. If $f$ (or $g$) has core entropy zero, then $F$ is a crochet map.

Proof. Let $S_g$ be the spider of $g$ and $\Theta(g)$ be the set of external angles whose external rays are in $S_g$. Let $M$ be the set of landing points of external rays of $f$ whose angles are in $-\Theta(g)$. Let $T_f$ be the $f$-invariant tree defined as the regulated hull of $M \cup P_f \cup \text{Crit}(f)$. By [Ree92, Shi00], $F$ is topologically conjugate to the quotient of the formal mating of $f$ and $g$ by the ray-equivalence classes. Hence, $S_g \cup T_f$ yields a connected $F$-invariant graph $G$ containing $P_f$. It is not hard to show that both $S_g$ and $T_f$ has entropy zero. Hence $G$ also has topological entropy zero. □
Remark. Even if both polynomials \( f \) and \( g \) have positive core entropy, their mating may be a crochet map, e.g., the mating of the Airplane polynomial \((\approx z^2 - 1.75488)\) and the Kokopelli polynomial \((\approx z^2 - 0.15652 + 1.03225i)\).

**Definition 7.4 (Cantor multicurve).** Let \( f : (S^2, A) \leftrightarrow \) be a post-critically finite branched covering. A multicurve \( \Gamma \) of \((S^2, A)\) is a Cantor multicurve if there is \( \gamma \in \Gamma \) such that the number of connected components of \( f^{-n}(\Gamma) \) that are isotopic to \( \gamma \) relative to \( A \) grows exponentially fast about \( n \).

It follows from the definitions that a multicurve \( \Gamma \) is a Cantor multicurve if and only if the Perron-Frobenius eigenvalue of the linear 1-transformation \( f_{1,\Gamma} : \mathbb{R}^{\Gamma} \to \mathbb{R}^{\Gamma} \) is greater than one, i.e., \( \lambda_1(\Gamma) > 1 \). Hence, if \( \Gamma \) does not contain a Leev cycle, it is also equivalent to \( Q(\Gamma) > 1 \).

**Theorem 7.5 (Dudko-Hlushchanka-Schleicher [DHS22]).** Let \( f \) be a post-critically finite rational map with non-empty Fatou set. Then there exists a completely \( f \)-invariant multicurve \( \Gamma \) so that any small rational map of the \( \Gamma \)-decomposition is either a Sierpiński carpet map or a crochet map. Moreover, if \( f \) is not a crochet map then either

- \( \Gamma \) is a Cantor multicurve, or possibly non-exclusively
- there exists a small rational map of the \( \Gamma \)-decomposition that is a Sierpiński carpet map.

Also the following are equivalent.

1. For every pair of Fatou components \( U \) and \( V \), their centers are connected by a curve \( \gamma \) that intersects the Julia set \( \mathcal{J}_f \) in a countable set.
2. The Fatou quotient \( \hat{\mathcal{C}}/\sim_F \) is a singleton.
3. \( f \) is a crochet map.

We will briefly sketch the proof in Appendix.

**Lemma 7.6.** If a post-critically finite rational map \( f \) is a crochet map then \( f \) is a subdivision map of a finite subdivision rule with polynomial growth of edge subdivisions.

**Proof.** If \( f \) is a crochet map, then there is a forward \( f \)-invariant graph \( G \) with \( P_f \subset G \). Then we define a finite subdivision rule \( R \) in such a way that the 1-skeleton of the level-0 subdivision complex \( S_R^{(1)} \) is equal to \( G \). The \( h_{\text{top}}(f|_G) = 0 \) is equivalent to having polynomial growth of edge subdivisions. \( \square \)

**Proposition 7.7 ([PT21, Theorem C]).** Let \( f \) be a post-critically finite hyperbolic rational map such that the Julia set \( \mathcal{J}_f \) is a Sierpiński carpet. Then \( \overline{E}^{\text{H}}(f) > 1 \).

**Proof.** We sketch the idea of the proof in [PT21]. Since \( \mathcal{J}_f \) is a Sierpiński carpet, the Fatou quotient \( \hat{\mathcal{C}}/\sim_F \) is a sphere \( S^2 \). Let \( F : (S^2, P_F) \leftrightarrow \) be the induced branched covering. By [GHMZ18], \( F \) is an expanding Thurston map. It follows from [BM17, Theorem 15.1], that there is a Jordan curve \( C \supset P_f \) such that \( F^k(C) = C \) for some \( k > 0 \). By replacing \( F \) by \( F^k \), we may assume that \( F(C) = C \).

We define a finite subdivision rule \( R \) in such a way that \( S_R^{(1)} = C \) and \( F \) is the subdivision map. A technical part is to prove, by using the expanding property, that, for any \( p \in P_F \), the number of simple closed curves in \( R^n(S_R)^{(1)} \) peripheral to \( p \) increases exponentially fast about \( n \). Then, by taking a spine \( G_0 \subset S^2 \backslash P_f \) containing a peripheral loop of \( p \), we can show that \( \overline{E}^{\text{H}}(F) > 1 \). \( \square \)

**Corollary 7.8.** Let \( f \) be a post-critically finite hyperbolic rational map. If \( f \) is not a crochet map, then \( \text{ARC. dim}(\mathcal{J}_f) > 1 \).
Proof. By Theorem 7.5, there is a multicurve $\Gamma$ which is completely invariant up to isotopy such that either $\Gamma$ is a Cantor multicurve or there exists a small rational map of the $\Gamma$ decomposition, say $g$, such that the Julia set $J_g$ is a Sierpiński carpet.

If $\Gamma$ is a Cantor multicurve, then $Q(\Gamma) > 1$. Thus $\text{ARC. dim}(J_f) > 1$ by Theorem 6.22.

Suppose that such a small rational map $g$ exists. By Lemma 6.19, the $g$ is a hyperbolic post-critically finite rational map. By Theorem 6.22 and Proposition 7.7, we have

$$\bar{E}^1(f) \geq \bar{E}^1(g) > 1.$$ 

Then $\text{ARC. dim}(J_f) > 1$ follows from Theorem 6.11 and the fact that $\bar{E}^p(f)$ is monotonically decreasing in $p$, Theorem 1.2. $\square$

8. Asymptotic conformal energies of crochet maps

We complete the proof of Theorem A by showing that the Julia sets of crochet maps have Ahlfors-regular conformal dimension one. Let us discuss a simple example to see the ideas of complicated approximations we perform in this section.

![Figure 7](image_url)

**Figure 7.** See Example 8.1 for a description.

**Example 8.1.** See Figure 7. The level-0 complex $S_\mathcal{R}$ consists of two tiles. The subdivision map $f : \mathcal{R}(S_\mathcal{R}) \to S_\mathcal{R}$ is a hyperbolic-type degree 4 branched self-covering. Non-expanding spines $N^0$ and $N^1$ are drawn in bold line. The complement of the non-expanding spines in the dual-1 skeletons $G_0$ and $G_1$ are drawn in dotted line.

The natural representative $\phi^1 : 0 : G_0 \to G_1$ is the identity the bold tripod and collapsing three dotted hexagons on the right to the center of the dotted tripod on the left.

Assume that all the edges have $p$-length one. The conformal energy $E^p_\mathcal{R}(\phi^1_0)$ is equal to one because of the identity part. However, we can pull some lengths from the collapsed hexagons to reduce the conformal energy. This local deformation will be described in Section 8.2.

In general, there are two more difficulties to manage which Example 8.1 does not have.
• First, more than one edges $e_1, e_2, \ldots, e_k$ may be homeomorphically mapped onto an edge $e$ by $\phi_0^n$. We will assign different lengths to different edges in such a way that all but one edges, $e_i$ with $i > 1$, are so longer than $e$ that only one edge $e_1$ contributes the most to the conformal energy over $e$. This length will be defined in Section 8.3 and called a $K$-expanding $p$-length.

• Second, the subdivision complex $S_R$ may have a Julia vertex. Then the level-0 dual 1-skeleton $G_0$ is a spine of $S^2 \setminus \text{Vert}(S_R)$ while we need a spine of $S^2 \setminus \text{Vert}_F(S_R)$. To get a spine $H_0$ of $S^2 \setminus \text{Vert}_F(S_R)$, at each cycle of Julia vertex we remove the longest edge. This process is manageable when $R$ has isolated Julia vertices, which we may assume thanks to Corollary 5.3.

8.1. Dual graph virtual endomorphisms of finite subdivision rules. We consider subdivision complexes $R^n(S_R)$’s of different levels $n$’s as different CW-complex structures on the same underlying sphere $S^2$. Define the dual 1-skeleton of $S_R$ as a graph embedded in $S^2$ transversely to the 1-skeleton of $S_R$. For $n \geq 0$, define $G_n$ to be the preimage $f^{-1}(G_0)$. Then, $G_n$ is the dual 1-skeleton of $R^n(S_R)$ and transverse to the 1-skeleton of $R^n(S_R)$. It follows from the below diagram that we have a homotopy class $[\phi_0^n] : G_n \to G_0$. The vertical arrows are embeddings, which are also homotopy equivalences.

The restriction of the subdivision map $f|_{G_1} : G_1 \to G_0$ is a covering. For $\phi := \phi_0^1$, we call the graph virtual endomorphism $G = (G_0, G_1, f, \phi)$ the dual graph virtual endomorphism of the finite subdivision rule $R$.

A post-critically finite branched covering $f : (S^2, \text{Vert}(S_R)) \to (S^2, \text{Vert}(S_R))$ is of hyperbolic-type if and only if every point in $\text{Vert}(S_R)$ is a Fatou vertex. In the case, the dual graph virtual endomorphism $G$ is a graph virtual endomorphism of $f : (S^2, \text{Vert}(S_R)) \to (S^2, \text{Vert}(S_R))$. In general, however, $\text{Vert}(S_R)$ may contain Julia vertices so that $f : (S^2, \text{Vert}(S_R)) \to (S^2, \text{Vert}(S_R))$ is not of hyperbolic-type. We use Corollary 5.3 to remove as many Julia vertices as possible with the combinatorial equivalence class of $f : (S^2, P_f) \to (S^2, P_f)$ unchanged. Then we remove a few more edges from $G_0$ near Julia vertices to obtain a spine of $(S^2, \text{Vert}_F(S_R))$, where $\text{Vert}_F(S_R) \subset \text{Vert}(S_R)$ is the subset of Fatou vertices.

Natural representative $\phi_0^n : G_n \to G_0$. Let $v$ be any vertex of $G_n$. There exists a level-$n$ tile $t^n$ that is dual to $v$. Let $t$ be the level-0 tile containing $t^n$ as a subtile and $w \in \text{Vert}(G_0)$ be the dual vertex of $t$. Then we define $\phi_0^n(v) = w$. Since $R^n(S_R)$ is a subdivision of $S_R$, a pair of adjacent vertices $v$ and $v'$ of $G_n$ are mapped to a pair of adjacent vertices or the same vertex. More precisely, if the level-$n$ tiles that are dual to $v$ and $v'$ are subtiles of the same level-0 tile $t$, then $v$ and $v'$ are mapped to the same vertex, which is dual to $t$. Otherwise, $v$ and $v'$ are mapped to adjacent vertices of $G_0$. Hence, $\phi_0^n$ extends to $G_n$ in such a way that every edge is collapsed to a vertex or mapped to an edge homeomorphically. We call this map $\phi_0^n$ the natural representative of the homotopy class $[\phi_0^n : G_n \to G_0]$ defined in the dual graph virtual endomorphism. The same construction applies to obtain a natural representative $\phi_m^n : G_n \to G_m$ for any $n > m \geq 0$.

An upper bound of $E_1$. Let $\phi_0^n : G_n \to G_0$ be the natural representative. Let $e \in \text{Edge}(S_R)$ be a level-0 edge and $e^* \in \text{Edge}(G_0)$ be its dual. The edges of $G_n$ which are homeomorphically...
mapped onto $e^*$ by $\phi^n_0$ are duals to the level-$n$ subedges of $e$. Hence we have

$$E^1(\phi^n_0) = \max_{e \in \text{Edge}(S_R)} |R^n(e)|.$$  

The exponential growth rate of the subdivision of the edge $e$ is a number $\rho(e) \geq 1$ defined by

$$\rho(e) := \lim_{n \to \infty} \left( |R^n(e)| \right)^{\frac{1}{n}}.$$  

An edge $e$ has polynomial growth rate of subdivisions if and only if $\rho(e) = 1$. If there exists a path from $[e]$ to $[e']$ in $E$, or equivalently $[e] \leq [e']$ with respect to the preorder on Vert($E$), then $\rho(e) \geq \rho(e')$. The following proposition is immediate from from Equations (2) and (3).

**Proposition 8.2.** Let $R$ be a finite subdivision rule and $G$ be the dual graph virtual endomorphism. Then we have

$$1 \leq E^1(G) \leq \max_{e \in \text{Edge}(S_R)} \rho(e).$$  

In particular, if $R$ has polynomial growth of edge subdivisions, then $E^1(G) = 1$.

### 8.2. Local deformations on trees.

**$(\epsilon, \epsilon')$-Deformation.** Let $\phi : (G, \alpha) \to (H, \beta)$ be a PL-map between $p$-conformal graphs. Let $v \in \text{Vert}(H)$ and choose a connected component $Z$ of $\phi^{-1}(v)$. Under suitable subdivisions of $G$ and $H$, we may assume that $Z$ is a subgraph of $G$ and every edge of $G$ incident to $Z$ is homeomorphically mapped with the constant derivative onto an edge of $H$ incident to $v$. Assuming $\phi$ to be reduced, we have at least two edges incident to $v$ which are images of edges incident to $Z$.

Let $\{e_0(=e), e_1, e_2, \ldots, e_k\} \subset \text{Edge}(H)$ be the set of edges incident to $v$. For every $i \in \{0, 1, \ldots, k\}$, let $\{e_{i,1}, e_{i,2}, \ldots, e_{i,p_i}\} \subset \text{Edge}(G)$ be the set of edges incident to $Z$ so that $\phi(e_{i,j}) = e_i$ for every $j \in \{1, 2, \ldots, p_i\}$. We have $p_i = 0$ if there is no such $e_{i,j}$'s. Let $v_{i,j} := Z \cap e_{i,j}$ be the leaf of $Z$ that is an endpoint of $e_{i,j}$.

The $(\epsilon, \epsilon')$-deformation $\phi(\epsilon, \epsilon')$ is, roughly speaking, pulling the map at $v$, $\phi|_{Z} : Z \to \{v\}$, toward the edge $e$. See Figure 8. For a precise construction, let us take small positive real numbers $\epsilon, \epsilon' > 0$. Let $v_i$ be the point in $e$ with $\beta(v, v_i) = \epsilon$ and $v'_{i,j,\epsilon'}$ be the point in $e'_{i,j}$ with $\alpha(v'_{i,j}, v'_{i,j,\epsilon'}) = \epsilon'$ for every $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, p_i\}$. Here $p$-lengths $\alpha$ and $\beta$ are used as metrics on graphs. The deformation $\phi(\epsilon, \epsilon')$ is defined in such a way that

1. $\phi(\epsilon, \epsilon') = \phi$ on $G \setminus Y$, where $Y$ the union of $Z$ and the edges incident to $Z$,
2. $\phi(\epsilon, \epsilon')(v'_{i,j,\epsilon'}) = v$ for each $i \in \{1, 2, \ldots, k\}$,
3. $\phi(\epsilon, \epsilon')(Z) = v_e$, and
4. $\phi(\epsilon, \epsilon')$ is constant on intervals in $Y$ divided by $\{v'_{i,j,\epsilon'} | 1 \leq i \leq k, 1 \leq j \leq p_i\}$.

We remark that the $(\epsilon, \epsilon')$-deformation depends on the choices of $v$, a connected component $Z$ of $\phi^{-1}(v)$, and an edge $e \in \text{Edge}(H)$ incident to $v$. When a specification is required, we call it the $(\epsilon, \epsilon')$-deformation at $Z$ toward $e$.

**Shortening boundary edges of trees.** An edge of a graph $G$ is an internal edge if its any endpoint is not a boundary vertex of $G$. Otherwise, the edge is a boundary edge.

**Lemma 8.3.** Let $T$ be a tree and $\partial T$ be the set of leaves. Let $\alpha$ be a p-length on $T$ for $p \in (1, \infty]$. Let $X$ be the set of boundary edges of $T$. For any small $\epsilon > 0$, $\alpha_{X, \epsilon}$ is another $p$-length on $T$ defined by

$$\alpha_{X, \epsilon}(e) = \begin{cases} \alpha(e) & \text{if } e \notin X \\ \alpha(e) - \epsilon & \text{if } e \in X \end{cases}.$$
Then,

\[ E_p^p([\text{id} : (T, \partial T, \alpha) \to (T, \partial T, \alpha_e)]) < 1. \]

Moreover, the same statement holds even if one boundary edge is excluded from \( X \).

Recall that \( E_p^p([\text{id} : (T, \partial T, \alpha) \to (T, \partial T, \alpha_e)]) \) is the infimum of \( E_p^p \) energies over the homotopy class relative to \( \partial T \).

**Proof.** It suffices to show the statement when \( X \) does not contain one boundary edge \( e \). Let \( v \) is the leaf of \( T \) that is incident to \( e \). For \( w \neq v \in \text{Vert}(T) \), the combinatorial distance between \( v \) and \( w \) is the minimal number of edges in paths joining \( v \) and \( w \). For any integer \( i > 0 \), let \( \{v_{i,1}, \ldots v_{i,n_i}\} \) be the set of internal vertices whose combinatorial distance from \( v \) is equal to \( i \). We define \( v_{0,1} := v \). Since \( T \) is a tree, for every \( v_{i,j} \), there exists a unique \( j' \in \{1, 2, \ldots, n_{i-1}\} \) such that \( v_{i-1,j'} \) and \( v_{i,j} \) are connected by an edge, say \( e_{i,j} \). Let \( k \) be the maximal integer satisfying \( n_k \neq 0 \). We perform \((\epsilon_i, \epsilon'_i)\)-deformation to the identity map \( \text{id}_T \) at every vertex \( v_{i,j} \) toward \( e_{i,j} \) and denote the deformed map by \( \text{id}_{(\epsilon,\epsilon')} \). For an appropriate choice of small numbers \( \epsilon_1 < \epsilon'_1 < \epsilon_2 < \epsilon'_2 < \cdots < \epsilon_{n_k} < \epsilon'_{n_k} \), we have \( E_p^p(\text{id}_{(\epsilon,\epsilon')}) < 1. \)

**8.3. Proof of ARC.dim(the Julia set of a crochet map)=1.** Suppose \( f \) is a hyperbolic crochet map. By Lemma 7.6 and Corollary 5.3, we may assume that \( f \) is the subdivision map of a finite subdivision rule \( \mathcal{R} \) which has polynomial growth rate of edge subdivisions and has isolated
Julia vertices. Let $G_0$ and $G_1$ be the dual 1-skeletons of $S_R$ and $R(S_R)$ such that $\phi : G_1 \to G_0$ be the natural representative in the homotopy class.

**Total order and $K$-expanding $p$-length.** Recall that the directed graph of edge subdivisions $E_R$ defines a preorder on the set of level-0 edges (Edge$(S_R)$, $\leq$). Since distinct cycles in $E_R$ are vertex-disjoint, replacing $R$ by $R^k$ for some $k > 0$ if necessary, we may assume that every cycle of $E_R$ has period one, i.e., every cycle is a loop. It follows that $e \approx e'$ if and only if $e = e'$, thus $\leq$ is a partial order. By the order extension principle [Szp30], the partial order $\leq$ extends to a total order $\leq^*$ on $E_R$. By abusing notation, we denote also by $\geq$ the extended total order on Edge$(S_R)$. By the dual correspondence between Edge$(S_R)$ and Edge$(G_0)$, we also have a total order $\geq$ on Edge$(G_0)$. For $n \geq 0$, we refer to an edge of $G_n$ as a recurrent edge if its dual is a level-$n$ recurrent subedge of a level-0 edge.

Under the assumptions in the previous paragraph, we have the following property: For a recurrent edge $e \in$ Edge$(G_0)$, there exists a unique recurrent edge $e' \in$ Edge$(G_1)$ that is mapped onto $e$ by $\phi$ such that the types of the dual edges of $e$ and $e'$ are the same. We call $e'$ the (unique) recurrent $\phi$-preimage of $e$. Recall that the level-0 dual recurrent skeleton $N_0^0$ is the union of level-0 recurrent edges in $G_0$, and the level-1 dual recurrent skeleton $N_1^1$ is the union of the recurrent $\phi$-preimages of the recurrent edges in $G_0$.

For $K > 1$ and $p > 1$, a $p$-length $\alpha_K$ on $G_0$ is $K$-expanding if $\alpha_K(e_1) > K \cdot \alpha_K(e_2)$ for every $e_1, e_2 \in$ Edge$(S_R)$ with $e_1 > e_2$. We assume that $G_0$ is equipped with a $K$-expanding $p$-length $\alpha_K$, and $G_1$ is equipped with the lifted $p$-length, which we also denote by $\alpha_K$. We assume that the derivative $\phi'$ is constant on each edge of $G_1$. Let $e \in$ Edge$(G_0)$ and $e' \in$ Edge$(G_1)$ with $\phi(e') = e$. Then we have

- $\alpha_K(e') > K \cdot \alpha_K(e)$ if $e' \not\in$ Edge$(N_1^0)$, and
- $\alpha_K(e') = \alpha_K(e)$ if $e' \in$ Edge$(N_1^0)$, i.e., the $e'$ is the unique recurrent $\phi$-preimage of $e$.

We remark, for a better understanding of the rest part, that when $K$ is sufficiently large the $p$-conformal energy $E_p^p$ is almost determined by the $E_p^p$ on $N_1^1$.

**Construction of graph virtual endomorphisms from the dual 1-skeletons.** For each Julia vertex $v \in$ Vert$(S_R)$, let $E_v := \{e_{v,1}, e_{v,2}, \ldots, e_{v,k_v}\} \subset G_0$ be the set of edges dual to the edges of $S_R$ incident to $v$ such that $e_i \leq e_j$ for every $i \geq j$. Since $R$ has isolated Julia vertices, we obtain a subgraph $H_0 \subset G_0$, which is a deformation retract of $S^2 \setminus$ Vert$_F(S_R)$ by removing from $G_0$ the edge $e_{v,1}$, which is the longest edge in $E_v$, for every Julia vertex $v \in$ Vert$(S_R)$.

We define a continuous map $\rho : G_0 \to H_0$ in such a way that $\rho|_{H_0} = \text{id}_{H_0}$ and $\rho$ maps $e_{v,1}$ to $\bigcup_{i=2}^{k_v} e_{v,i}$ for every Julia vertex $v \in$ Vert$(S_R)$. Let $H_1 := f^{-1}(H_0)$. Then $f, \rho \circ \phi|_{H_1} : H_1 \to H_0$ is a graph virtual endomorphism of a branched covering $f : (S^2, \text{Vert}_F(S_R)) \to$. See Figure 9. Since the $\rho$ sends the longest edge $e_{v,1}$ to the union of the other edges $\bigcup_{i=2}^{k_v} e_{v,i}$, we have

\begin{equation}
E_p^p(\rho) < \left( \frac{L}{K} \right)^{p-1} + 1 \right)^{1/p},
\end{equation}

where $L$ is the maximal degree of Julia vertices in $S_R$.

By Proposition 4.10, replacing $R$ by the $k$th-shift $R_k$ for some $k > 0$ if necessary, we may assume that for $i \in \{0, 1\}$ the level-$i$ non-expanding spine $N_i$ is the $1/2$-truncation of the level-$i$ dual recurrent skeleton $N_i^0$. It follows from Proposition 4.13 that $F_i := N_i \cap H_i$ is tree-like for $i \in \{0, 1\}$, i.e., every connected component is a tree.
To show $\text{ARC. dim}(\mathcal{F}_f) = 1$, by Theorem 6.11, it suffices to show that $E^p_0[\rho \circ \phi|_{H_1}] < 1$ for every $p > 1$. Since we have

$$E^p_0[\rho \circ \phi|_{H_1}] \leq E^p_0[\rho] \circ E^p_0[\phi|_{H_1}],$$

by the above inequality (4), it is enough to show

$$E^p_0[\phi|_{H_1} : H_1 \to G_0] < \left( \left( \frac{L}{K} \right)^{p-1} + 1 \right)^{-1/p}$$

with respect to a $K$-expanding $p$-length $\alpha_K$ for a sufficiently large $K$. Before starting a complicated analysis, let us summarize conditions we have from the above discussion.

- For $i \in \{0, 1\}$, $H_i \subset G_0$ such that $f|_{H_i} : H_1 \to H_0$ is a covering map,
- There is a total order $\geq$ on $\text{Edge}(G_0)$,
- The derivative of $\phi : G_1 \to G_0$ is constant on each edge with respect a $p$-length $\alpha_K$,
- For $e \in \text{Edge}(G_0)$ and $e' \in \text{Edge}(G_1)$ with $\phi(e') = e$,
  - $\alpha_K(e') > K \cdot \alpha_K(e)$ if $e' \notin \text{Edge}(N_{e}^1)$, and
  - $\alpha_K(e') = \alpha_K(e)$ if $e' \in \text{Edge}(N_{e}^1)$ (and such an $e'$ is unique and called the recurrent $\phi$-preimage of $e$),
- The map $\phi|_{N^1} : N^1 \to N_{G_0} \subset G_0$ is a homeomorphism sending each edge onto an edge such that its restriction $\phi|_{F_1} : F_1 \to F_0$ is also a homeomorphism.

**Proof of Crochet $\Rightarrow$ ARC.dim=1.** Let $N := \max_{e \in \text{Edge}(G_0)} \# \{ e' \in \text{Edge}(G_1) \mid \phi(e') = e \}$ and $M$ be the maximal degree of vertices of $G_0$. Since $G_1$ is a covering space of $G_0$, $M$ is also the maximal degree of vertices of $G_1$.

Recall that $\text{Fill}^p(\phi)(y) := \sum_{x \in \phi^{-1}(y)} \phi'(y)^{p-1}$ and $E^p_0(\phi) := (\|\text{Fill}^p\|_x)^{1/p}$. Since $\phi$ maps each edge to an edge or a vertex, $\text{Fill}^p(\phi) : G_0 \to \mathbb{R}$ is constant on each edge. Moreover,

- $1 \leq \text{Fill}^p(\phi)(x) \leq 1 + \frac{N}{K^{p-1}}$ if $x \in G_0$ is on a recurrent edge, and
- $0 < \text{Fill}^p(\phi)(x) \leq \frac{N}{K^{p-1}}$ otherwise.

The same inequalities hold for $\phi|_{F_1}$. We will let $K$ be very large to the extent that $\frac{N}{K^{p-1}}$ can be negligible.

One may observe that $E^p_0(\phi|_{F_1}) = 1$ and $E^p_0(\phi|_{H_1 \setminus F_1}) \leq N/K^{p-1}$. Since $F_1$ is tree-like, we will slightly deform the map $\phi$ near $F_1$ using Lemma 8.3 to make the energy $E^p_0$ on the entire graph $H_1$ strictly less than one.

Let us describe the deformation more precisely. For any $v \in \partial F_0$, let $e_v$ be the boundary edge of $F_0$ incident to $v$ and $Z_v$ be the connected component of $\phi^{-1}(v)$ that adjacent to $F_1$. For small $\epsilon > 0$, we perform the $(\epsilon, \epsilon_1)$-deformation to $\phi|_{H_1}$ at every $Z_v$ of toward $e_v$ where $e_v$ is the edge of $F_0$ incident to $\phi(v)$, and denote the deformed map by $\phi|_{H_1, (\epsilon, \epsilon_1)}$. Define a subgraph $X$ of...
$H_1$ by

$$X := F_1 \cup \bigcup_{v \in \partial F_0} Z_v \cup \bigcup_{v \in \partial F_0} \{\text{edges of } H_1 \text{ adjacent to } Z_v\}.$$ 

Now $\phi|_{H_1, (\epsilon, \epsilon_1)}$ maps $F_1$ onto $F_0 \setminus N_\epsilon(\partial F_0, \alpha_K)$ so that all the internal edges are mapped isometrically but all the boundary edges are contracted. Here $N_\epsilon(\partial F_0, \alpha_K)$ means the $\epsilon$-neighborhood of $\partial F_0$ in $H_0$ with respect to $\alpha_K$. By applying Lemma 8.3 to every connected component of $F_1$, which is a tree, we obtain $\psi : H_1 \to G_0$ which is homotopic to $\phi|_{H_1} : H_1 \to G_0$ such that

- $\psi = \phi|_{H_1}$ on $H_1 \setminus X$,
- $\psi = \phi|_{H_1, (\epsilon, \epsilon_1)}$ on $X \setminus F_1$, and
- $\text{Fill}^p(\psi|_{F_1}) < 1 - \epsilon_2$ for some $\epsilon_2$ which depends on $\epsilon$ and $\epsilon_1$.

Note that the $E_p^\epsilon(\psi)$ depends on $\alpha_K$. Lastly, we show $E_p^\epsilon(\psi) < 1$ for a sufficiently large $K$. We approximate $\text{Fill}^p(\psi)(x)$ from above where $x \in e$ and $e \in \text{Edge}(H_0)$. When $e$ is recurrent, let $e', e'_1, \ldots, e'_m$ indicate the edges of $H_1$ which are homeomorphically mapped to $e$ by $\phi$ such that $e'$ is the recurrent $\phi$-preimage of $e$. If $e$ is not recurrent, then $e'_1, \ldots, e'_m$ indicate such edges and we do not have $e'$. Then, for every $i \in \{1, 2, \ldots, m\}$, we have $\alpha_K(e)/\alpha_K(e'_i) < 1/K$.

There are four cases depending on $e$ and $x$. Recall that, for two points $v, w$ on a tree, we denote by $[v, w]$ the geodesic path joining $v$ to $w$. In cases 1 and 2 (when $e$ is recurrent), we adopt the notations used in the description of $(\epsilon, e')$-deformations with $e'$ being replaced by $\epsilon_1$. See Figure 8 and Figure 10. For example, in this case

- $p_0 = 1$ and $e'_{0,1} = e'$,
- $e_1, e_2, \ldots, e_k \in \text{Edge}(H_0)$ are edges incident to $v$,
- edges $e'_1, e'_2, \ldots, e'_{i, p_k}$ are the edges mapped to $e_i$ so that $v'_{i,j}$ is the endpoint of $e'_{i,j}$ that maps to $v$, and
- $v_{i,j, \epsilon_1}$ is the point on $e'_{i,j}$ with $\alpha_K(v'_{i,j}, v_{i,j, \epsilon_1}) = \epsilon_1$.

To avoid confusion, we remark that $e'_1, e'_2, \ldots, e'_m$ do not appear in the $(\epsilon, e')$-deformation or Figure 8.

![Figure 10](image)

**Figure 10.** Near a point $v$ in $\partial F_0$. $F_0$ and $F_1$ are drawn in bold line. The deformation from $\phi$ to $\psi$ changes $\phi$ only near $F_1 \cup \bigcup_{v \in \partial F_0} Z_v$. An edge $e$ on $F_0$ covers the cases 1 and 2.

Recall that $N := \max_{e \in \text{Edge}(G_0)} \#\{e' \in \text{Edge}(G_1) \mid \phi(e') = e\}$ and $M$ is the maximal degree of vertices of $G_0$.

**Case 1:** $e$ is a recurrent edge and $x \in e \cap N_\epsilon(\partial F_0, \alpha_K)$, i.e., $x$ is in $[v, v_\epsilon]$ in Figure 8.
We have \( m < N \) and \( p_i \leq N \) so that \( \sum_{i=1}^{k} p_i \leq MN \). Hence

\[
\text{Fill}^p(\psi)(x) = \sum_{i=1}^{k} \sum_{j=1}^{p_i} \left( \frac{\alpha_K([v_{i,j}, \lambda_{i,j}])}{\alpha_K([v_{i,j}, \lambda_{i,j}])} \right)^{p-1} + \sum_{i=1}^{m} \left( \frac{\alpha_K(e)}{\alpha_K(e_i)} \right)^{p-1} < MN \left( \epsilon_{\epsilon_1} \right)^{p-1} + \frac{N}{K^{p-1}}
\]

**Case 2**: \( e \) is a recurrent edge and \( x \in e \setminus N_\epsilon(\partial F_0, \alpha_K) \).

By the construction of \( \psi \), we have

\[
\text{Fill}^p(\psi)(x) = \text{Fill}^p(\psi|_F)(x) + \text{Fill}^p(\psi|_{H_1 \setminus F})(x) < 1 - \epsilon_2 + \sum_{i=1}^{m} \left( \frac{\alpha_K(e_i)}{\alpha_K(e)} \right)^{p-1} < 1 - \epsilon_2 + \frac{N}{K^{p-1}}
\]

**Case 3**: \( e \) is not a recurrent edge and any of its endpoints is not in \( \partial F_0 \).

We have

\[
\text{Fill}^p(\psi)(x) = \sum_{i=1}^{m} \left( \frac{\alpha_K(e_i)}{\alpha_K(e)} \right)^{p-1} < \frac{N}{K^{p-1}}
\]

**Case 4**: \( e \) is not a recurrent edge and at least one of its endpoints is in \( \partial F_0 \).

After reordering the indices, we may assume that (1) both endpoints of \( e_i \) are in \( \partial F_1 \) for \( 1 \leq i \leq l_1 \), (2) only one endpoint of \( e_i \) is in \( \partial F_1 \) for \( l_1 + 1 \leq i \leq l_1 + l_2 \), and (3) no endpoints of \( e_i \) are in \( \partial F_1 \) for \( l_1 + l_2 < l \leq m \). We have

\[
\text{Fill}^p(\psi)(x) = \sum_{i=1}^{l_1} \left( \frac{\alpha_K(e_i)}{\alpha_K(e_{i-2})} \right)^{p-1} + \sum_{i=l_1+1}^{l_1+l_2} \left( \frac{\alpha_K(e_i)}{\alpha_K(e_{i-\epsilon_1})} \right)^{p-1} + \sum_{i=l_1+l_2+1}^{m} \left( \frac{\alpha_K(e_i)}{\alpha_K(e_{i})} \right)^{p-1} < \frac{N}{(K/2)^{p-1}}
\]

where the inequality holds when \( \epsilon_1 \) is small enough so that \( 4 \cdot \epsilon_1 < K \cdot \alpha_K(e) \) for every edge \( e \). Since \( \epsilon, \epsilon_1, \) and \( \frac{\epsilon_2}{\epsilon_1} \) can be chosen arbitrary small and \( K \) can be chosen arbitrary large, the upper bounds in cases 1,3,4 are arbitrarily close to zero. Suppose \( \epsilon, \epsilon_1, \) and \( K \) are chosen so that \( \text{Fill}^p(\psi) < 1 \) in the cases 1,3,4. Note that \( \epsilon_2 \) is determined by \( \epsilon \) and \( \epsilon_1 \). Then, in case 2 we can further increase \( K \) so that

\[
\text{Fill}^p(\psi) < 1 - \epsilon_2 + \frac{N}{K^{p-1}} < 1 - \left( \frac{L}{K} \right)^{p-1} < \frac{1}{1 + (L/K)^{p-1}}.
\]

Since \( \psi \) is homotopic to \( \phi|_{H_1} \), we have

\[
E_p^p[\rho \circ \phi|_{H_1}] \leq E_p^p[\rho] \cdot E_p^p[\phi|_{H_1}] < 1
\]

\( \square \)

**Appendix A. Crochet decomposition**

In the appendix, we briefly summarize a part of Dudko-Hlushchanka-Schleicher’s work on the crochet decomposition [DHS22], focusing on the application in the present article. The author credits D. Dudko, M. Hlushchanka, and D. Schleicher for all the ideas and examples in the appendix, which are written based on the conversations with them.
Theorem A.1 (=Theorem 7.5, Dudko-Hlushchanka-Schleicher). Let $f$ be a post-critically finite rational map with non-empty Fatou set. Then there exists a completely $f$-invariant multicurve $\Gamma$ so that any small rational map of the $\Gamma$-decomposition is either a Sierpiński carpet map or a crochet map. Moreover, if $f$ is not a crochet map then

- $\Gamma$ is a Cantor multicurve, or possibly non-exclusively
- there exists a small rational map of the $\Gamma$-decomposition that is a Sierpiński carpet map.

Also the following are equivalent:

1. For every pair of Fatou components $U$ and $V$, their centers are connected by a curve $\gamma$ that intersects the Julia set $J_f$ in a countable set.
2. The Fatou quotient $\hat{\mathbb{C}}/\sim_F$ is a singleton.
3. $f$ is a crochet map.

Levy arcs and adjacency of Fatou components. Let $f$ be a post-critically finite rational map. By an arc of $(\hat{\mathbb{C}}, P_f)$ we mean an arc $\alpha : I \to \hat{\mathbb{C}}$ so that $\alpha((0,1)) \subset \hat{\mathbb{C}}\setminus P_f$ and $\alpha((0,1)) \subset P_f$. A Levy arc is an arc $\alpha$ satisfying the following: There exists a finite sequence of arcs $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha_{i+1}$ is homotopic relative to $P_f$ to a lifting of $\alpha_i$ through $f$ for every $i \mod n$. The following proposition is well-known, see [Pil94, Corollary 5.18] or see [BD18, Proposition 1.3] together with [GHMZ18].

Proposition A.2. For a post-critically finite rational map $f$ with non-empty Fatou set, the Julia set $J_f$ is a Sierpiński carpet if and only if $f$ does not have a Levy arc.

Proof. Suppose that two Fatou components $U$ and $V$ are adjacent, i.e., $U \cap V \neq \emptyset$. Then there exists a periodic or preperiodic arc $\gamma$ consisting of internal rays of $U$ and $V$ [Ros17]. The forward image of this arc contains a periodic arc $\gamma'$ which is either (1) an arc connecting the centers of periodic Fatou components or (2) an internal arc of a periodic Fatou component. In either case, the endpoints of $\gamma'$ are in the post-critical set.

Conversely, suppose $\alpha$ is a Levy arc. By the expanding property of conformal metric, we may assume that $\alpha$ is preserved by $f^n$ without any homotopy, i.e., $f^n(\alpha) = \alpha$. It is easy to show that $\alpha$ has a Julia fixed point $x$ and $\alpha$ a Levy arc of type (1) or (2) in the previous paragraph depending on whether $x$ is an interior point or not. □

Directed graphs of multicurve. Let $f : (S^2, A) \to (S^2, A)$ be a branched covering and $\Gamma$ be a multicurve of $(S^2, A)$. The directed graph of $\Gamma$, denoted by diGraph($\Gamma$), is defined as the directed graph whose adjacency matrix is the 1-linear transformation $f_{1,\Gamma}$. In other words, vertices of diGraph($\Gamma$) have bijective correspondence with elements of $\Gamma$, and each directed edge from $\gamma$ to $\gamma'$ corresponds to a connected component of $f^{-1}(\gamma)$ that is homotopic relative to $A$ to $\gamma'$.

Definition A.3 (Strongly connected components). Let $G$ be a directed graph.

- A subset $X \subset \text{Vert}(G)$ is strongly connected if for every pair $v \neq w \in X$ there exists a path from $v$ to $w$ and also there exists a path from $w$ to $v$, i.e., $v \simeq w$ with respect to the preorder on Vert($G$) defined in Definition 3.5. We say that a singleton $\{v\}$ is strongly connected if there is an edge from $v$ to itself. A strongly connected component is a maximal strongly connected subset of Vert($G$).
- A strongly connected component is called a unicycle if it consists of a single cycle and called a bicycle otherwise.
- A strongly connected component $X$ is primitive if there are no paths to $X$ from other strongly connected components.
Let $\Gamma$ be a multicurve of $f: (S^2, A) \not\subset$. We extend the above definitions for any sub-multicurve $\Gamma' \subseteq \Gamma$ by considering $\Gamma'$ as a subset of $\text{Vert}(\text{diGraph}(\Gamma))$.

By definition $\Gamma$ is a Cantor multicurve if and only if it has at least one bicycle.

**Extension of entropy zero invariant graphs.** Suppose $\mathcal{F}_f$ is not a Sierpiński carpet so that there are Levy arcs by Proposition A.2. By the expanding property of conformal metric, we may assume that the Levy arcs can be chosen in the homotopy classes so that each of them consists of one or two internal rays of Fatou components. Let $G^0$ be the union of the Levy arcs and $P_f$. Remark that we allow a single vertex as a connected component of $G^0$. Let $G_1^0, G_2^0, \ldots, G_k^0$ be the connected components of $G^0$. For any $n \geq 0$ we inductively define $(G_{i}^n)_{i=1}^{k}$ as the connected component of $f^{-1}(G^n)$ containing $G_i^0$ and define $G^{n+1} := \bigcup_{i=1}^{k} G_{i}^{n+1}$. Then $G_{i}^{n+1} \supseteq G_{i}^{n}$ and possibly $G_{i}^{n} = G_{j}^{n}$ for any $n \geq 0$ and $i \neq j$. We define a cluster $K_i$ as the closure of the union of Fatou components intersecting $\bigcup_{n \geq 0} G_{i}^{n}$.

(E1) We firstly extend $G^0$ by replacing it with $G^m$ for a sufficiently large $m$ so that we can assume $G_{i}^{n} \not\subseteq G_{j}^{n}$ for any $i \neq j$ and $n \geq 0$.

Like the case of adjacent periodic Fatou components, if $K_i \cap K_j \neq \emptyset$, then one can show that $K_i \cap K_j$ has a periodic or preperiodic point. For every unordered pair $(i, j)$ with $K_i \cap K_j \neq \emptyset$, we pick a periodic point $p_{ij} \in K_i \cap K_j$. Let $P$ be the union of points in the periodic cycles of $p_{ij}$ for any $(i, j)$ with $K_i \cap K_j \neq \emptyset$. For each $i$, we extend $G_i^0$ to $G_i^0$ by connecting $G_i^0$ to $(P \cap K_i)$ with regulated paths within $K_i$. Though $G_i^0 \cap G_j^0 = \emptyset$, we have $G_i^0 \cap G_j^0 \neq \emptyset$. Let $H^0$ be the union of $G_i^0$'s and $H_1^0, H_2^0, \ldots, H_k^0$ be the connected components of $H^0$. By appropriately choosing paths joining $G_i$ to $(P \cap K_i)$, we may assume that $f(H^0) = H^0$. It is easy to show the topological entropy of $f|_{H^0} : H^0 \not\subset$ is zero. We call this process obtaining $H^0$ from $G^0$ an extension.

(E2) We secondly extend $G^0$ to $H^0$ so that we remove intersections between clusters $K_i$’s by combining them to the same cluster.

We note that there are more technical difficulties in choosing regulated paths joining $G_i$ to $(P \cap K_i)$. One might think that it would be similar to taking the regulated hull of an invariant cluster. We call the multicurve of maximal clusters and iterations.

We can repeat the extension process described above. Since (E1) and (E2) in the previous paragraph strictly decrease the number of connected components, the extension stops in finite steps.

Denote by $G$ the maximally extended graph. Let $G_1, G_2, \ldots, G_n$ be the connected components of $G$ and $K_1, K_2, \ldots, K_n$ be the corresponding clusters. We call $K := \bigcup_{1 \leq i \leq n} K_i$ the maximal cluster of $f$. Since $G$ is maximally extended, $K$’s are pairwise disjoint and $G_i$ is a deformation retract of $K_i$ relative to $P_f$. Define a multicurve $\Gamma$ as the multicurve consisting of the boundary components of a small neighborhood of $G$ that are essential relative to $P_f$. Since clusters are pairwise disjoint, one can show that $\Gamma$ is completely $f$-invariant so that we can consider the $\Gamma$-decomposition. We call $\Gamma$ the multicurve of maximal clusters of $f$.

Recall that each small sphere $\hat{C}_U$ in the $\Gamma$-decomposition comes from a connected component $U$ of $C \setminus \Gamma$. If $U$ contains a connected component of $G$, then the corresponding small rational map $f_U : \hat{C}_U \not\subset$ is a crochet map; we can extend $G \cap U$ to obtain an entropy zero invariant graph of $f_U$ containing $P_{f_U}$. If $U \cap G = \emptyset$, however, then $f_U$ may be a crochet map or a Sierpiński carpet map or even neither. Hence we further decompose $f_U$ along its multicurve of maximal clusters.

Let $\Gamma_0$ be the multicurve of maximal clusters of $f$. Inductively we define $\Gamma_{n+1}$ as the union of multicurves of maximal clusters of small rational maps in the $\Gamma_n$-decomposition that do not
intersect the maximal clusters of the small rational maps. We can consider $\Gamma_n$'s as multicurves in the original sphere $(\hat{C}, P_f)$. Let $C_n := \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_n$. Since $|P_f| < \infty$, this iteration stops in finite steps. $C_n$ is forward invariant but possibly not backward invariant. So we define a completely invariant multicurve $C$ as the collection of homotopy classes in $f^{-n}(C_m)$ for sufficiently large $n, m > 0$. By the construction, for every small rational map $g$ of the $C$-decomposition, the multicurve of maximal cluster for $g$ is empty.

Using Proposition A.2, one can show the following: For any post-critically finite rational map $h$, the multicurve of maximal cluster of $h$ is empty if and only if $h$ is either a crochet map (when the cluster is too large) or a Sierpiński carpet map (when the cluster is trivial). Therefore every small rational map of the $C$-decomposition is either a Sierpiński carpet map or a crochet map, which proves the first part of Theorem A.1. To have the properties for non-crochet maps, however, we need to reduce $C$.

**Small crochet maps adjacent via primitive unicycles.** Let $\Gamma$ be a completely invariant multicurve of $f : (S^2, A) \subset$. Fix an element $\gamma \in \Gamma$. Let $U_1$ and $U_2$ be connected components of $S^2 \setminus \Gamma$ that are incident to $\gamma$, i.e., $\gamma$ is a boundary component of both $U_1$ and $U_2$. In the case, we say that $U_1$ and $U_2$ are adjacent. Let $S_1$ and $S_2$ be the small spheres associated to $U_1$ and $U_2$ respectively. Similarly, we say that $S_1$ and $S_2$ are adjacent to each other and are incident to $\gamma$.

Suppose additionally that $\gamma$ belongs to a primitive unicycle $C$ of diGraph($\Gamma$). Then both $S_1$ and $S_2$ are periodic. If both small maps $f_1 : S_1 \subset$ and $f_2 : S_2 \subset$ are crochet maps, then we combine them to a larger crochet map. To formulate this idea, we take a sub-multicurve $\Gamma' \subset \Gamma$ generated by all the strongly connected components but $C$ and consider the $\Gamma'$-decomposition. Then there is a small map $g : S' \subset$ so that $S'$ is defined from the connected component $U'$ of $\hat{C} \setminus \Gamma'$ with $U' \supset U_1 \cup U_2$. One can show that $g$ is a crochet map by extending entropy zero invariant graphs of $f_1$ and $f_2$. In the rigorous proof in [DHS22], we need at least one of $f_1$ and $f_2$ to be $F_\gamma$-vacant crochet, although we skip the relevant details in this article.

From the multicurve $C$ we had obtained, we iteratively remove primitive unicycles that are incident to small crochet maps on both sides and denote by $C_{cro}$ the result multicurve. We call $C_{cro}$ the crochet multicurve of $(f, P_f)$.

Now we prove the second part of Theorem A.1. Suppose that $f$ is not a crochet map. Consider the multicurve $C_{cro}$ and suppose that no strongly connected components of $C_{cro}$ are bicycles. Then every (primitive) strongly connected component is a unicycle. By the last reduction procedure, there exists a small Sierpiński carpet map incident to any primitive unicycle.

**Equivalent characterizations of crochet maps.** Let us discuss the last part of Theorem A.1. These equivalences are not necessary in our discussion, but the condition (2) is a very useful criterion to determine whether a rational map is a crochet map when its Julia set is drawn.

(1) $\Rightarrow$ (2) is immediate. (3) $\Rightarrow$ (1) is also easily obtained because $h_{top}(f|_G) = 0$ implies the intersection $G \cap J_f$ is a countable set. The proof of (2) $\Rightarrow$ (3) is more involved and we do not prove it in this article. Instead, we state some evidence on this implication. Suppose $f$ is not a crochet map. Then $f$ has either a small Sierpiński carpet map or a Cantor multicurve. A small Sierpiński carpet map yields a sphere in the quotient $\hat{C}/ \sim_F$, which is intuitive because collapsing the closure of every connected component of the complement of a Sierpiński carpet yields a sphere. On the other hand, a Cantor multicurve yields an arc in $\hat{C}/ \sim_F$. A Cantor multicurve gives rise to an uncountable family of possibly intersecting closed curves in the Julia set. These curves are projected to an arc in $\hat{C}/ \sim_F$, see Figure 11. It would be also helpful to recall the following fact:
Let $X$ be the standard ternary Cantor set. The quotient space of $X$ that is obtained by identifying the two endpoints of each open interval we removed is homeomorphic to a closed interval.

Figure 11. $f(z) \approx 0.128262 \cdot z^3 + 1/z^3$. This is an example from [DHS22]. 0 and $\infty$ are critical points of degree 3 and $f(0) = \infty$. The other six critical points are of degree 2 and prefixed points with preperiod 2. Let $\gamma$ be a simple closed curve that is the boundary of a small neighborhood of the Fatou component of 0. Then $f^{-1}(\gamma)$ consists of two simple closed curves both homotopic to $\gamma$ relative to $P_f$. Thus $\{\gamma\}$ is a Cantor multicurve. The Julia set $J_f$ is the union of uncountably many simple closed curves homotopic to $\gamma$. The Fatou quotient $\hat{\mathbb{C}}/\sim_f$ can be identified with a line segment in on the real line.

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