EXISTENCE OF LIMITING DISTRIBUTION FOR AFFINE PROCESSES

PENG JIN*, JONAS KREMER, AND BARBARA RÜDIGER

Abstract. In this paper, sufficient conditions are given for the existence of limiting distribution of a conservative affine process on the canonical state space $\mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$, where $m, n \in \mathbb{Z}_{\geq 0}$ with $m + n > 0$. Our main theorem extends and unifies some known results for OU-type processes on $\mathbb{R}^n$ and one-dimensional CBI processes (with state space $\mathbb{R}_{\geq 0}$). To prove our result, we combine analytical and probabilistic techniques; in particular, the stability theory for ODEs plays an important role.

1. Introduction

Let $D := \mathbb{R}_{\geq 0}^m \times \mathbb{R}^n$, where $m, n \in \mathbb{Z}_{\geq 0}$ with $m + n > 0$. Roughly speaking, an affine process with state space $D$ is a time-homogeneous Markov process $(X_t)_{t \geq 0}$ taking values in $D$, whose log-characteristic function depends in an affine way on the initial value of the process, that is, there exist functions $\phi, \psi = (\psi_1, \ldots, \psi_{m+n})$ such that

$$E\left[e^{\langle u, X_t \rangle} \mid X_0 = x\right] = e^{\phi(t,u) + \langle \psi(t,u), x \rangle},$$

for all $u \in i\mathbb{R}^{m+n}$, $t \geq 0$ and $x \in D$. The general theory of affine processes was initiated by Duffie, Pan and Singleton [9] and further developed by Duffie, Filipović, and Schachermayer [8]. In the seminal work of Duffie et al. [8], several fundamental properties of affine processes on the canonical state space $D$ were established. In particular, the generator of $D$-valued affine processes is completely characterized through a set of admissible parameters, and the associated generalized Riccati equations for $\phi$ and $\psi$ are introduced and studied. The results of [8] were further complemented by many subsequent developments, see, e.g., [3, 7, 14, 16, 18].

Affine processes have found a wide range of applications in finance, mainly due to their computational tractability and modeling flexibility. Many popular models in finance, such as the models of Cox et al. [5], Heston [13] and Vasicek [25], are of affine type. Moreover, from the theoretical point of view, the concept of affine processes enables a unified treatment of two very important classes of continuous-time

*Peng Jin is partially supported by the STU Scientific Research Foundation for Talents (No. NTF18023).

Date: December 14, 2018.

2010 Mathematics Subject Classification. Primary 60J25; Secondary 60G10.

Key words and phrases. affine process, limiting distribution, stationary distribution, generalized Riccati equation.
Markov processes: OU-type processes on \( \mathbb{R}^n \) and CBI (continuous-state branching processes with immigration) processes on \( \mathbb{R}^m_{\geq 0} \).

In this paper, we are concerned with the following question: when does an affine process converge in law to a limit distribution? This problem has already been dealt with in the following situations:

- Sato and Yamazato [23] provided conditions under which an OU-type process on \( \mathbb{R}^n \) converges in law to a limit distribution, and they identified this type of limit distributions with the class of operator self-decomposable distributions of Urbanik [24];
- without a proof, Pinsky [22] announced the existence of a limit distribution for one-dimensional CBI processes, under a mean-reverting condition and the existence of the log-moment of the Lévy measure from the immigration mechanism. A recent proof appeared in [20, Theorem 3.20 and Corollary 3.21] (see also [15, Theorem 3.16]). A stronger form of this result can be found in [17, Theorem 2.6];
- Glasserman and Kim [12] proved that affine diffusion processes on \( \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \) introduced by Dai and Singleton [6] have limiting stationary distributions and characterized these limits;
- Barczy, Döring, Li, and Pap [2] showed stationarity of an affine two-factor model on \( \mathbb{R}^m_{\geq 0} \times \mathbb{R} \), with one component being the \( \alpha \)-root process.

Our motivation for this paper is twofold. On the one hand, we would like to formulate a general result for affine processes with state space \( D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \), which unifies the above mentioned results; on the other hand, our result should also provide new results for the unsolved cases where \( D = \mathbb{R}^m_{\geq 0} \) (\( m \geq 2 \)) and \( D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \) (\( m \geq 1, n \geq 1 \)). As our main result (see Theorem 2.6 below), we give sufficient conditions such that an affine process \( X \) with state space \( D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \) converges in law to a limit distribution as time goes to infinity, and we also identify this limit through its characteristic function. Using a similar argument as in [15], we will show that the limit distribution is the unique stationary distribution for \( X \).

The rest of this paper is organized as follows. In Section 2 we recall some definitions regarding affine processes and present our main theorem, whose proof we defer to Section 4. In Section 3 we deal with the large time behavior of the function \( \psi \) and show that \( \psi(t,u) \) converges exponentially fast to 0 as \( t \) goes to infinity. Finally, we prove our main theorem in Section 4.

2. Preliminaries and main result

2.1. Notation. Let \( \mathbb{N}, \mathbb{Z}_{\geq 0}, \mathbb{R} \) denote the sets of positive integers, non-negative integers and real numbers, respectively. Let \( \mathbb{R}^d \) be the \( d \)-dimensional (\( d \geq 1 \)) Euclidean space and define

\[
\mathbb{R}^d_{\geq 0} := \{ x \in \mathbb{R}^d : x_i \geq 0, \ i = 1, \ldots, d \}
\]

and

\[
\mathbb{R}^d_{> 0} := \{ x \in \mathbb{R}^d : x_i > 0, \ i = 1, \ldots, d \}.
\]
For \( x, y \in \mathbb{R} \), we write \( x \wedge y := \min\{x, y\} \). By \( \langle \cdot, \cdot \rangle \) and \( \|x\| \) we denote the inner product on \( \mathbb{R}^d \) and the induced Euclidean norm of a vector \( x \in \mathbb{R}^d \), respectively. For a \( d \times d \)-matrix \( A = (a_{ij}) \), we write \( A^\top \) for the transpose of \( A \) and define \( \|A\| := (\text{trace}(A^\top A))^{1/2} \). Let \( \mathbb{C}^d \) be the space that consists of \( d \)-tuples of complex numbers. We define the following subsets of \( \mathbb{C}^d \):

\[
\mathbb{C}^d_{\leq 0} := \{ u \in \mathbb{C}^d : \text{Re} \, u_i \leq 0, \ i = 1, \ldots, d \}
\]

and

\[
i\mathbb{R}^d := \{ u \in \mathbb{C}^d : \text{Re} \, u_i = 0, \ i = 1, \ldots, d \}.\]

The following sets of matrices are of particular importance in this work:

- \( \mathbb{M}^-_d \) which stands for the set of real \( d \times d \) matrices all of whose eigenvalues have strictly negative real parts. Note that \( A \in \mathbb{M}^-_d \) if and only if \( \|\exp\{tA\}\| \to 0 \) as \( t \to \infty \);

- \( \mathbb{S}^+_d \) (resp. \( \mathbb{S}^{++}_d \)) which stands for the set of all symmetric and positive semidefinite (resp. positive definite) real \( d \times d \) matrices.

If \( A = (a_{ij}) \) is a \( d \times d \)-matrix, \( b = (b_1, \ldots, b_d) \in \mathbb{R}^d \) and \( I, J \subset \{1, \ldots, d\} \), we write \( A_{IJ} := (a_{ij})_{i \in I, j \in J} \) and define \( b_I := (b_i)_{i \in I} \).

Let \( U \) be an open set or the closure of an open set in \( \mathbb{R}^d \). We introduce the following function spaces: \( C^k(U) \), \( C^\infty_k(U) \), and \( C^\infty(U) \) which denote the sets of \( \mathbb{C} \)-valued functions on \( U \) that are \( k \)-times continuously differentiable, that are \( k \)-times continuously differentiable with compact support, and that are smooth, respectively. The Borel \( \sigma \)-Algebra on \( U \) will be denoted by \( \mathcal{B}(U) \).

Throughout the rest of this paper, let \( D := \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \), where \( m, n \in \mathbb{Z}_{\geq 0} \) with \( m + n > 0 \). Note that \( m \) or \( n \) may be \( 0 \). The set \( D \) will act as the state space of affine processes we are about to consider. The total dimension of \( D \) is denoted by \( d = m + n \). We write \( \mathcal{B}_b(D) \) for the Banach space of bounded real-valued Borel measurable functions \( f \) on \( D \) with norm \( ||f||_\infty := \sup_{x \in D} |f(x)| \).

For \( D \), we write

\[
I := \{1, \ldots, m\} \quad \text{and} \quad J := \{m + 1, \ldots, m + n\}
\]

for the index sets of the \( \mathbb{R}^m_{\geq 0} \)-valued components and the \( \mathbb{R}^n \)-valued components, respectively. Define

\[
\mathcal{U} := C^m_{\geq 0} \times i\mathbb{R}^n \triangledown \{ u \in \mathbb{C}^d : \text{Re} \, u_I \leq 0, \ \text{Re} \, u_J = 0 \}.
\]

Note that \( \mathcal{U} \) is the set of all \( u \in \mathbb{C}^d \), for which \( x \mapsto \exp\{\langle u, x \rangle\} \) is a bounded function on \( D \).

Further notation is introduced in the text.

### 2.2. Affine processes on the canonical state space.

Affine processes on the canonical state space \( D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \) have been systematically studied in the well-known work [5]. We remark that affine processes considered in [5] are in full generality and are allowed to have explosions or killings. In contrast to [5], in this paper we restrict ourselves to conservative affine processes. In terms of terminology and notation, we mainly follow, instead of [5], the paper by Keller-Ressel and Mayerhofer [10], where only the conservative case was considered.
Let us start with a time-homogeneous and conservative Markov process with state space $D$ and semigroup $(P_t)$ acting on $\mathcal{B}_d(D)$, that is,

$$P_t f(x) = \int_D f(\xi)p_t(x,d\xi), \quad f \in \mathcal{B}_d(D).$$

Here $p_t(x,\cdot)$ denotes the transition kernel of the Markov process. We assume that $p_0(x,\{x\})=1$ and $p_t(x,D)=1$ for all $t \geq 0$, $x \in D$.

Let $(X, (\mathbb{P}_x)_{x \in D})$ be the canonical realization of $(P_t)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$, where $\Omega$ is the set of all càdlàg paths in $D$ and $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. Here $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by $X$ and $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$. The probability measure $\mathbb{P}_x$ on $\Omega$ represents the law of the Markov process $(X_t)_{t \geq 0}$ started at $x$, i.e., it holds that $X_0 = x$, $\mathbb{P}_x$-almost surely. The following definition is taken from [16, Definition 2.2].

**Definition 2.1.** The Markov process $X$ is called affine with state space $D$, if its transition kernel $p_t(x,A) = \mathbb{P}_x(X_t \in A)$ satisfies the following:

(i) it is stochastically continuous, that is, $\lim_{s \to t} p_s(x,\cdot) = p_t(x,\cdot)$ weakly for all $t \geq 0$, $x \in D$, and

(ii) there exist functions $\phi : \mathbb{R}_{\geq 0} \times U \to \mathbb{C}$ and $\psi : \mathbb{R}_{\geq 0} \times U \to \mathbb{C}^d$ such that

\[
\int_D e^{\langle w, \xi \rangle} p_t(x,d\xi) = \mathbb{E}_x \left[ e^{\langle X_t, u \rangle} \right] = \exp \left\{ \phi(t,u) + \langle x, \psi(t,u) \rangle \right\}
\]

for all $t \geq 0$, $x \in D$ and $u \in U$, where $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$.

The stochastic continuity in (i) and the affine property in (ii) together imply the following regularity of the functions $\phi$ and $\psi$ (see [18, Theorem 5.1]), i.e., the right-hand derivatives

\[
F(u) := \frac{\partial}{\partial t} \phi(t,u) \bigg|_{t=0^+} \quad \text{and} \quad R(u) := \frac{\partial}{\partial t} \psi(t,u) \bigg|_{t=0^+}
\]

exist for all $u \in U$, and are continuous at $u = 0$. Moreover, according to [8, Proposition 7.4], the functions $\phi$ and $\psi$ satisfy the semi-flow property:

\[
\phi(t+s,u) = \phi(t,u) + \phi(s,\psi(t,u)) \quad \text{and} \quad \psi(t+s,u) = \psi(s,\psi(t,u)),
\]

for all $t, s \geq 0$ with $(t+s,u) \in \mathbb{R}_{\geq 0} \times U$.

**Definition 2.2.** We call $(a, \alpha, b, \beta, m, \mu)$ a set of admissible parameters for the state space $D$ if

(i) $a \in \mathbb{S}_d^+$ and $a_{kl} = 0$ for all $k \in I$ or $l \in I$;

(ii) $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_i = (\alpha_{i,kl})_{1 \leq k, l \leq d} \in \mathbb{S}_d^+$ and $\alpha_{i,kl} = 0$ if $k \in I \setminus \{i\}$ or $l \in I \setminus \{i\}$;

(iii) $m$ is a Borel measure on $D \setminus \{0\}$ satisfying

\[
\int_{D \setminus \{0\}} \left( 1 \land \|\xi\|^2 + \sum_{i \in I} (1 \land \xi_i) \right) m(d\xi) < \infty;
\]
(iv) \( \mu = (\mu_1, \ldots, \mu_m) \) where every \( \mu_i \) is a Borel measure on \( D \setminus \{0\} \) satisfying

\[
(2.4) \quad \int_{D \setminus \{0\}} \left( \|\xi\| \wedge \|\xi\|^2 + \sum_{k \in I \setminus \{i\}} \xi_k \right) \mu_i(\mathrm{d}\xi) < \infty.
\]

(v) \( b \in D; \)

(vi) \( \beta = (\beta_{ki}) \in \mathbb{R}^{d \times d} \) with \( \beta_{ki} = \int_{D \setminus \{0\}} \xi_k \mu_i(\mathrm{d}\xi) \geq 0 \) for all \( i \in I \) and \( k \in I \setminus \{i\} \), and \( \beta_{ki} = 0 \) for all \( k \in I \) and \( i \in J \);

We remark that our definition of admissible parameters is a special case of [8, Definition 2.6], since we require here that the parameters corresponding to killing are constant 0; moreover, the condition in (iv) is also stronger as usual, i.e., we assume that the first moment of \( \mu_i \)'s exists, which, by [8, Lemma 9.2], implies that the affine process under consideration is conservative. However, we should remind the reader that (2.4) is not a necessary condition for conservativeness. In fact, an example of a conservative affine process on \( \mathbb{R}_\geq 0 \), which violates (2.4), is provided in [21] Section 3.

We write "(2.4)" is given in [21, Section 3].

Theorem 2.3. Let \((a, \alpha, b, \beta, m, \mu)\) be a set of admissible parameters in the sense of Definition 2.2. Then there exists a (unique) conservative affine process \(X\) with state space \(D\) such that its infinitesimal generator \(A\) operating on a function \(f \in C^2_c(D)\) is given by

\[
Af(x) = \sum_{k,l=1}^d \left( a_{kl} + \sum_{i=1}^m \alpha_{i,kl}x_i \right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle b + \beta x, \nabla f(x) \rangle
\]

\[
+ \int_{D \setminus \{0\}} \left( f(x + \xi) - f(x) - \langle \nabla f(x), \xi \rangle \mathbb{1}_{\{\|\xi\| \leq 1\}}(\xi) \right) m(\mathrm{d}\xi)
\]

\[
+ \sum_{i=1}^m x_i \int_{D \setminus \{0\}} \left( f(x + \xi) - f(x) - \langle \nabla f(x), \xi \rangle \right) \mu_i(\mathrm{d}\xi)
\]

where \(x \in D, \nabla := (\partial_{x_k})_{k \in J}\). Moreover, (2.1) holds for some functions \(\phi(t, u)\) and \(\psi(t, u)\) that are uniquely determined by the generalized Riccati differential equations: for each \(u = (v, w) \in \mathbb{C}^m_{\leq 0} \times i\mathbb{R}^n\),

\[
\partial_t \phi(t, u) = F(\psi(t, u)), \quad \phi(0, u) = 0,
\]

\[
\partial_t \psi^I(t, u) = R^I \left( \psi^I(t, u), e^{\beta J^I} w \right), \quad \psi^I(0, u) = v
\]

\[
\psi^J(t, u) = e^{\beta J^J} w,
\]
where
\[ F(u) = \langle u, au \rangle + \langle b, u \rangle + \int_{D \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle u, \xi \rangle \right) m(d\xi) \]
and \( R^I = (R_1, \ldots, R_m) \) with
\[ R_i(u) = \langle u, \alpha_i u \rangle + \sum_{k=1}^d \beta_{ki} u_k + \int_{D \setminus \{0\}} \left( e^{\langle u, \xi \rangle} - 1 - \langle u, \xi \rangle \right) \mu_i(d\xi), \quad i \in I. \]

**Remark 2.4.** If an affine process \( X \) with state space \( D \) and a set of admissible parameters \((a, \alpha, b, \beta, m, \mu)\) satisfy a relation as in Theorem 2.3, then we say that \( X \) is an affine process with admissible parameters \((a, \alpha, b, \beta, m, \mu)\).

The following lemma is a consequence of the condition (iv) in Definition 2.2.

**Lemma 2.5.** Let \( X \) be an affine process with state space \( D \) and admissible parameters \((a, \alpha, b, \beta, m, \mu)\). Let \( R \) and \( \psi \) be as in Theorem 2.3. For each \( i \in I \) it holds that \( R_i \in C^1(U) \) and \( \psi_i \in C^1(\mathbb{R}_{\geq 0} \times U) \).

To see that Lemma 2.5 is true, we only need to apply Lemmas 5.3 and 6.5 of [8].

### 2.3. Main result

Our main result of this paper is the following.

**Theorem 2.6.** Let \( X \) be an affine process with state space \( \mathbb{R}_m^m \times \mathbb{R}^n \) and admissible parameters \((a, \alpha, b, \beta, m, \mu)\) in the sense of Definition 2.2. If
\[ \beta \in \mathbb{M}_d^- \quad \text{and} \quad \int_{\{\|\xi\| > 1\}} \log \|\xi\| m(d\xi) < \infty, \]
then the law of \( X_t \) converges weakly to a limiting distribution \( \pi \), which is independent of \( X_0 \) and whose characteristic function is given by
\[ \int_D e^{\langle u, x \rangle} \pi(dx) = \exp \left\{ \int_0^\infty F(\psi(s, u)) ds \right\}, \quad u \in U. \]

Moreover, the limiting distribution \( \pi \) is the unique stationary distribution for \( X \).

**Remark 2.7.** In virtue of the definition of admissible parameters, we can write \( \beta \in \mathbb{R}^{d \times d} \) in the following way:
\[ \beta = \begin{pmatrix} \beta_{II} & 0 \\ \beta_{JI} & \beta_{JJ} \end{pmatrix}, \]
where \( \beta_{II} \in \mathbb{R}^{m \times m}, \beta_{JI} \in \mathbb{R}^{n \times m} \) and \( \beta_{JJ} \in \mathbb{R}^{n \times n} \). It is easy to see that \( \beta \in \mathbb{M}_d^- \) is equivalent to the fact that \( \beta_{II} \in \mathbb{M}_m^- \) and \( \beta_{JJ} \in \mathbb{M}_n^- \).

We now make a few comments on Theorem 2.4. To our knowledge, Theorem 2.6 seems to be the first result towards the existence of limiting distributions for affine processes on \( D \) in such a generality. It includes many previous results as special cases. In particular, it covers [12, Theorem 2.4] for affine diffusions, and partially extends [23, Theorem 4.1] for OU-type processes and [22, ...]
Corollary 2] for 1-dimensional CBI processes. However, we are not able to show \( \int_{\{\|\xi\|>1\}} \log \|\xi\| m(\text{d}\xi) < \infty \), provided that \( \beta \in \mathbb{M}^{-d}_a \) and the stationarity of \( X \) is known.

Our strategy of proving Theorem 2.6 is as follows. Clearly, to prove the weak convergence of the distribution of \( X_t \) to \( \pi \), it is essential to establish the pointwise convergence of the corresponding characteristic functions, i.e.,

\[
\mathbb{E}_x \left[ e^{\langle X_t, u \rangle} \right] = \exp \left\{ \int_0^\infty F(\psi(s,u)) \text{d}s \right\} \quad \text{as } t \to \infty.
\]

We will proceed in two steps. In the first step, we prove that for each \( u \in \mathcal{U} \), \( \psi(t,u) \) converges to zero exponentially fast. For \( u \) in a small neighborhood of the origin, this convergence follows by a fine analysis of the generalized Riccati equations (2.5), (2.7) and an application of the linearized stability theorem for ODEs. Then, by some probabilistic arguments, we show that \( \psi(t,u) \) reaches every neighborhood of the origin for large enough \( t \). The essential observation here is the tightness of the laws of \( X_t, t \geq 0 \). This is a simple consequence of the uniform boundedness for the first moment of \( X_t, t \geq 0 \), which we show in Proposition 3.8. We thus obtain the desired convergence speed of \( \psi(t,u) \to 0 \) by the semi-flow property (2.3). In the second step, we show that

\[
(2.9) \quad \phi(t,u) = \int_0^t F(\psi(s,u)) \text{d}s \to \int_0^\infty F(\psi(s,u)) \text{d}s \quad \text{as } t \to \infty.
\]

Since \( \psi(s,u) \to 0 \) exponentially fast as \( s \to \infty \), we will see that the convergence in (2.9) is naturally connected with the condition \( \int_{\{\|\xi\|>1\}} \log \|\xi\| m(\text{d}\xi) < \infty \).

Finally, the stationarity of \( \pi \) can be derived using the semi-flow property.

3. Large time behavior of the function \( \psi(t,u) \)

In this section we consider an affine process \( X \) with admissible parameters \((a, \alpha, b, \beta, m, \mu)\) and assume that

\[
(3.1) \quad a = 0, \ b = 0, \ m = 0.
\]

In particular, we have \( F \equiv 0 \) as well as \( \phi \equiv 0 \). We will show that if \( \beta \in \mathbb{M}^{-d}_a \), then \( \psi(t,u) \to 0 \) exponentially fast as \( t \to \infty \).

Remark 3.1. The assumption that \( a = 0, \ b = 0 \) and \( m = 0 \) is not essential. Indeed, Proposition 3.10, as the main result of this section, remains true if we drop Assumption (3.1). This follows from the following observation: when we study the properties of the function \( \psi(t,u) \), the parameters \( a, b \) and \( m \) do not play a role.

3.1. Uniform boundedness for the first moment of \( X_t, t \geq 0 \). The aim we pursue in this subsection is to establish the uniform boundedness for the first moment of \( X_t, t \geq 0 \). We start with some approximations of \( X \), which were introduced in (4). For \( K \in (1, \infty) \), let

\[
\mu_{K,i}(\text{d}\xi) := 1_{\{\|\xi\| \leq K\}}(\xi) \mu_i(\text{d}\xi),
\]
Proposition 6.1 and noting that $\mu$ similarly, the same inequality holds for $\psi$ and $u$.

In view of the formula (6.16) in the proof of [8, Proposition 6.1], we have for some positive constant $(3.4)$

$$\sup_{t \in [0,T]} \| \psi_K(t, u) \|^2 \leq \sup_{t \in [0,T]} \left( \|v\|^2 + c_1 \int_0^t \left( 1 + \|e^{\beta_j s} w \|^2 \right) ds \right) \times \exp \left\{ c_1 \int_0^t \left( 1 + \|e^{\beta_j s} w \|^2 \right) ds \right\} \leq \left( \|v\|^2 + c_1 \int_0^T \left( 1 + \|e^{\beta_j s} w \|^2 \right) ds \right) \times \exp \left\{ c_1 \int_0^T \left( 1 + \|e^{\beta_j s} w \|^2 \right) ds \right\},$$

for some positive constant $c_1$. Moreover, by checking carefully the proof of [8, Proposition 6.1] and noting that $\mu_{K,i} \leq \mu$, we can actually choose $c_1$ in such a way that it depends only on the parameters $\alpha, \beta, \mu$. So $c_1$ is independent of $K$.

Similarly, the same inequality holds for $\psi$:

$$\sup_{t \in [0,T]} \| \psi(t, u) \|^2 \leq \left( \|v\|^2 + c_1 \int_0^T \left( 1 + \|e^{\beta_j s} w \|^2 \right) ds \right)$$
\[ \times \exp \left\{ c_1 \int_0^T \left( 1 + \|e^{\beta \mathbf{J}_s w}\|_2^2 \right) ds \right\}. \]

According to Lemma 2.5, the mapping \( u \mapsto R^I(u) : U \to \mathbb{C}^m \) is locally Lipschitz continuous. Therefore, for each \( L > 0 \), there exists a constant \( c_2 = c_2(L) > 0 \) such that
\[ (3.5) \quad \|R_i(u_1) - R_i(u_2)\| \leq c_2 \|u_1 - u_2\|, \quad \text{for all } i \in I \text{ and } \|u_1\|, \|u_2\| \leq L. \]

In addition, it is easy to see that for \( u \in U \),
\[ (3.6) \quad \|R_i(u) - R_{K,i}(u)\| = \int_{\{\|\xi\| > K\}} \left| e^{(u, \xi)} - 1 - (u, \xi) \right| \mu_i(d\xi) \leq \int_{\{\|\xi\| > K\}} 2\mu_i(d\xi) + \|u\| \int_{\{\|\xi\| > K\}} \|\xi\| \mu_i(d\xi) \leq \varepsilon_K (1 + \|u\|) \]
where \( \varepsilon_K := \sum_{i=1}^m \int_{\{\|\xi\| > K\}} (2 + \|\xi\|) \mu_i(d\xi) \). Note that \( \varepsilon_K \to 0 \) as \( K \to \infty \) by dominated convergence.

Let
\[ g_K(t) := \|\psi^I(t, u) - \psi^I_K(t, u)\|, \quad t \in [0, T]. \]

By \[ 3.2 \] and \[ 3.3 \], we have
\[ (3.7) \quad g_K(t) \leq \left\| \int_0^t R^I \left( \psi^I(s, u), e^{\beta \mathbf{J}_s w} \right) ds - \int_0^t R^I_K \left( \psi^I_K(s, u), e^{\beta \mathbf{J}_s w} \right) ds \right\| \]
\[ + \sum_{i=1}^m \int_0^t \left\| R_i \left( \psi^I(s, u), e^{\beta \mathbf{J}_s w} \right) - R_i \left( \psi^I_K(s, u), e^{\beta \mathbf{J}_s w} \right) \right\| ds. \]

In virtue of \[ 3.4 \], there exists a constant \( c_3 = c_3(T) > 0 \) such that
\[ \sup_{K \in [1, \infty)} \sup_{s \in [0, T]} \|\psi^I_K(s, u)\| \leq c_3 < \infty, \]
which implies
\[ (3.8) \quad \sup_{K \in [1, \infty)} \sup_{s \in [0, T]} \left\| \left( \psi^I_K(s, u), e^{\beta \mathbf{J}_s w} \right) \right\| \leq c_4 < \infty. \]

So, for \( 0 < s \leq T \), we get
\[ (3.9) \quad \left\| R_i \left( \psi^I(s, u), e^{\beta \mathbf{J}_s w} \right) - R_i \left( \psi^I_K(s, u), e^{\beta \mathbf{J}_s w} \right) \right\| \leq c_5 \|\psi^I(s, u) - \psi^I_K(s, u)\| \]
from \[ 3.5 \], and obtain
\[ (3.10) \quad \left\| R_i \left( \psi^I_K(s, u), e^{\beta \mathbf{J}_s w} \right) - R_{K,i} \left( \psi^I_K(s, u), e^{\beta \mathbf{J}_s w} \right) \right\| \leq \varepsilon_K (1 + c_6) \]
from \[ 3.6 \] and \[ 3.8 \]. Here, \( c_5, c_6 > 0 \) are constants not depending on \( K \).
Combining (3.7), (3.9) and (3.10) yields, for \( t \in [0, T] \),
\[
g_K(t) \leq c_5 m \hat{g} t_0 \psi_I(s,u) - \psi_I K(s,u) ds + m \varepsilon_K (1 + c_6) t.
\]
Gronwall’s inequality implies
\[
g_K(t) \leq m \varepsilon_K (1 + c_6) t + m^2 \varepsilon_K (1 + c_6) c_5 \int_0^t s e^{c_5 m (t-s)} ds
\]
\[
\leq m \varepsilon_K (1 + c_6) (T + c_5 m T^2 e^{c_5 m T}), \quad t \in [0, T].
\]
Since \( \varepsilon_K \to 0 \) as \( K \to \infty \), we see that \( g_K(t) \to 0 \) and thus
\[
\psi_I K(t,u) \to \psi_I (t,u), \quad \text{for all } t \in [0, T].
\]
\( \square \)

For \( K \in (1, \infty) \), the generator \( A_K \) of \( (X_{K,t})_{t \geq 0} \) is given by
\[
A_K f(x) = \sum_{k,l=1}^d \left( \sum_{i=1}^m \alpha_{i,kl} x_i \right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle \beta x, \nabla f(x) \rangle
\]
\[
+ \sum_{i=1}^m x_i \int_{D \setminus \{0\}} (f(x + \xi) - f(x) - \langle \nabla f(x), \xi \rangle) \mu_{K,i} (d\xi),
\]
defined for every \( f \in C^2_c(D) \).

To avoid the complication of discussing the domain of definition for the generator \( A_K \), we introduce the operator \( A^\flat_K \), which was also used in [8].

**Definition 3.3.** If \( f \in C^2_c(D) \) is such that for all \( x \in D \),
\[
\sum_{i=1}^m \int_{D \setminus \{0\}} |f(x + \xi) - f(x) - \langle \nabla f(x), \xi \rangle| \mu_{K,i} (d\xi) < \infty,
\]
then we say that \( A^\flat_K f \) is well-defined and let
\[
A^\flat_K f(x) := \sum_{k,l=1}^d \left( \sum_{i=1}^m \alpha_{i,kl} x_i \right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle \beta x, \nabla f(x) \rangle
\]
\[
+ \sum_{i=1}^m x_i \int_{D \setminus \{0\}} (f(x + \xi) - f(x) - \langle \nabla f(x), \xi \rangle) \mu_{K,i} (d\xi),
\]
for \( x \in D \).

It is easy to see that if \( f \in C^2_c(D) \) has bounded first and second order derivatives, then \( A^\flat_K f \) is well-defined.

Recall that the matrix \( \beta \) can be written as in (2.8). We define the following matrices
\[
M_1 := \int_0^\infty e^{t \beta_{1i}} e^{t \beta_{1i}} dt \quad \text{and} \quad M_2 := \int_0^\infty e^{t \beta_{ij}} e^{t \beta_{jj}} dt.
\]
Since $\beta_{ij} \in M_n$ and $\beta_{ij} \in M_n^-$, the matrices $M_1$ and $M_2$ are well-defined. Moreover, we have that $M_1 \in \mathbb{S}^+_n$ and $M_2 \in \mathbb{S}^+_n$. In the following we will often write $x = (y, z) \in \mathbb{R}_+^m \times \mathbb{R}^n$ for $x \in D$. For $y_1, y_2 \in \mathbb{R}_+^m$ and $z_1, z_2 \in \mathbb{R}^n$, we define

$$
\langle y_1, y_2 \rangle_I := \int_0^\infty \langle e^{t\beta_{11}}y_1, e^{t\beta_{11}}y_2 \rangle \, dt \quad \text{and} \quad \langle z_1, z_2 \rangle_J := \int_0^\infty \langle e^{t\beta_{22}}z_1, e^{t\beta_{22}}z_2 \rangle \, dt.
$$

It is easily verified that $\langle \cdot, \cdot \rangle_I$ and $\langle \cdot, \cdot \rangle_J$ define inner products on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Moreover, we have that

$$
\langle y_1, y_2 \rangle_I = y_2^T M_1 y_1 = \langle y_1, M_1 y_2 \rangle \quad \text{and} \quad \langle z_1, z_2 \rangle_J = z_2^T M_2 z_1 = \langle z_1, M_2 z_2 \rangle.
$$

The norms on $\mathbb{R}^m$ and $\mathbb{R}^n$ induced by the scalar products $\langle \cdot, \cdot \rangle_I$ and $\langle \cdot, \cdot \rangle_J$ are denoted by

$$
\|y\|_I := \sqrt{\langle y, y \rangle_I} \quad \text{and} \quad \|z\|_J := \sqrt{\langle z, z \rangle_J},
$$

respectively.

In the following lemma we construct a Lyapunov function $V$ for $(X_{K,t})_{t \geq 0}$. Note that the definition of $V$ does not depend on $K$.

**Lemma 3.4.** Assume $m \geq 1$ and $n \geq 1$. Suppose that $\beta \in M_n^+$. Let $V \in C^2(D, \mathbb{R})$ be such that $V > 0$ on $D$ and

$$
V(x) = (\langle y, y \rangle_I + \varepsilon \langle z, z \rangle_J)^{1/2}, \quad \text{whenever} \ x = (y, z) \in \mathbb{R}_+^m \times \mathbb{R}^n \text{ with } \|x\| > 2.
$$

Here $\varepsilon > 0$ is some small enough constant. Then $A^{\varepsilon}_K V$ is well-defined and $V$ is a Lyapunov function for $(X_{K,t})_{t \geq 0}$, that is, there exist positive constants $c$ and $C$ such that

$$
A^{\varepsilon}_K V(x) \leq -c V(x) + C, \quad \text{for all } x \in D.
$$

Moreover, the constants $c$ and $C$ can be chosen to be independent of $K$.

**Proof.** For $x_1 = (y_1, z_1) \in \mathbb{R}_+^m \times \mathbb{R}^n$ and $x_2 = (y_2, z_2) \in \mathbb{R}_+^m \times \mathbb{R}^n$, we define

$$
\langle x_1, x_2 \rangle_\beta := \langle y_1, z_1 \rangle_I + \varepsilon \langle y_2, z_2 \rangle_J,
$$

where $\varepsilon > 0$ is a small constant to be determined later. Set $\tilde{V}(x) := (\langle x, x \rangle_\beta)^{1/2}$, $x \in D$. Then $\tilde{V}$ is smooth on $\{x \in D : \|x\| > 1\}$. By the extension lemma for smooth functions (see [13] Lemma 2.26), we can easily find a function $V \in C^\infty(D, \mathbb{R})$ such that $V > 0$ on $D$ and $V(x) = \tilde{V}(x) = (\langle x, x \rangle_\beta)^{1/2}$ for $\|x\| > 2$. So for all $x = (y, z) \in \mathbb{R}_+^m \times \mathbb{R}^n$ with $\|x\| > 2$, we have

$$
\nabla V(y, z) = V(y, z)^{-1} \left( \begin{array}{c} M_1 y \\ \varepsilon M_2 z \end{array} \right)
$$

and

$$
\nabla^2 V(y, z) = \left( \begin{array}{ccc} M_1 V(y, z)^{-1} M_1 y & -\varepsilon (M_1 y)(M_2 z)^\top \\ -\varepsilon (M_1 y)(M_2 z)^\top V(y, z)^2 & V(y, z)^{-1} \end{array} \right).
$$
We write $A^K_t V = D V + J^K_t V$, where
\begin{equation}
DV(x) := \sum_{k,l=1}^{d} \langle \alpha_{k,l}, x \rangle \frac{\partial^2 V(x)}{\partial x_k \partial x_l} + \langle \beta x, \nabla V(x) \rangle,
\end{equation}
\begin{equation}
J^K_t V(x) := \sum_{i=1}^{m} x_i \int_{D_1 \setminus \{0\}} (V(x + \xi) - V(x) - \langle \nabla V(x), \xi \rangle) \mu_{K,i}(d\xi).
\end{equation}

We now estimate $DV(x)$ and $J^K_t V(x)$ separately. Let us first consider $DV(x)$. We may further split $DV(x)$ into the drift part and the diffusion part.

**Drift.** Recall that $\beta_{IJ} = 0$. Consider $x = (y, z)$ with $\|x\| > 2$. It follows from (3.11) that
\begin{equation*}
\langle \beta x, \nabla V(x) \rangle = \left( \begin{array}{c} \beta_{I1} y \\ \beta_{J1} y + \beta_{JJ} z \end{array} \right) = \left( \begin{array}{c} V(y, z)^{-1} M_1 y \\ V(y, z)^{-1} \varepsilon M_2 z \end{array} \right).
\end{equation*}

The first and the third inner product on the right-hand side may be estimated similarly. Namely, we have
\begin{equation*}
V(y, z)^{-1} \langle \beta_{I1} y, M_1 y \rangle = \frac{1}{2} V(y, z)^{-1} y^T (M_1 \beta_{I1} + \beta_{I1}^T M_1) y.
\end{equation*}

The definition of $M_1$ implies
\begin{equation*}
M_1 \beta_{I1} + \beta_{I1}^T M_1 = \int_0^\infty \left( e^{t \beta_{I1}} e^{t \beta_{I1}} \beta_{I1} + \beta_{I1}^T e^{t \beta_{I1}} e^{t \beta_{I1}} \right) dt = \int_0^\infty \left( \frac{d}{dt} e^{t \beta_{I1}} e^{t \beta_{I1}} \right) dt = e^{t \beta_{I1}} e^{t \beta_{I1}} \bigg|_{t=0} = -I_m,
\end{equation*}
where $I_m$ denotes the $m \times m$ identity matrix. Hence
\begin{equation*}
V(y, z)^{-1} \langle \beta_{I1} y, M_1 y \rangle = -\frac{1}{2} V(y, z)^{-1} y^T y.
\end{equation*}

Since all norms on $\mathbb{R}^m$ are equivalent, we have
\begin{equation*}
y^T y \leq -c_1 y^T M_1 y = -c_1 (y, y)_I \leq -c_1 \|y\|^2_I,
\end{equation*}
for some positive constant $c_1$ that is independent of $K$. So
\begin{equation}
V(y, z)^{-1} \langle \beta_{I1} y, M_1 y \rangle \leq -c_1 \|y\|^2 V(y, z)^{-1}.
\end{equation}

In the very same way we obtain
\begin{equation}
V(y, z)^{-1} \langle \beta_{JJ} z, \varepsilon M_2 z \rangle \leq -c_2 \|z\|^2 V(y, z)^{-1},
\end{equation}
for some constant $c_2 > 0$. To estimate the remaining term, we can use Cauchy-Schwarz inequality to obtain
\begin{equation*}
|V(y, z)^{-1} \langle \beta_{JJ} y, \varepsilon M_2 z \rangle| \leq \varepsilon V(y, z)^{-1} \|\beta_{JJ} y\| \|M_2 z\| \leq c_3 \varepsilon V(y, z)^{-1} \|y\| \|z\|,
\end{equation*}
for some constant $c_3 > 0$. Using the fact that all norms on $\mathbb{R}^d$ are equivalent, we get
\[
|\langle \beta x, \nabla V(x) \rangle| \leq c_4 \left( \frac{\sqrt{\varepsilon} \langle y, z \rangle}{\sqrt{\varepsilon} (\langle y, z \rangle + \varepsilon)} \right) \leq c_4 \sqrt{\varepsilon} \|y\|.
\]
(3.17)

Combining (3.15), (3.16) and (3.17), we obtain
\[
\langle \beta x, \nabla V(x) \rangle \leq -c_5 \|y\|^2 V(y, z)^{-1} - \varepsilon c_2 \|z\|^2 V(y, z)^{-1} + c_4 \sqrt{\varepsilon} \|y\| \leq -c_5 V(y, z) + c_4 \sqrt{\varepsilon} V(y, z),
\]
where $c_5 := c_1 \land c_2 > 0$. Since $c_4$ and $c_5$ depend only on $\beta$ but not on $\varepsilon$, by choosing $\varepsilon = \varepsilon_0 > 0$ sufficiently small, we get
\[
(\beta x, \nabla V(x)) \leq -c_6 V(x), \quad x \in D \quad \text{with } \|x\| > 2.
\]
(3.18)

From now on we take $\varepsilon = \varepsilon_0$ as fixed. In particular, the upcoming constants $c_7 - c_{11}$ may depend on $\varepsilon$.

**Diffusion.** By (3.12), we have
\[
\frac{\partial^2 V(x)}{\partial x_k \partial x_l} \leq \frac{c_7}{V(x)}, \quad \text{for all } \|x\| > 2, \ k, l \in \{1, \ldots, d\},
\]
(3.19)

where $c_7 > 0$ is a constant. This implies
\[
\sup_{x \in D} \left| x_i \frac{\partial^2 V(x)}{\partial x_k \partial x_l} \right| < \infty, \quad \text{for all } i \in I \text{ and } k, l \in \{1, \ldots, d\}.
\]

We conclude that
\[
\left| \sum_{k,l=1}^d \left( \sum_{i \in I} \alpha_{i,k} x_i \right) \frac{\partial^2 V(x)}{\partial x_k \partial x_l} \right| \leq c_8, \quad \text{for all } x \in D,
\]
(3.20)

where $c_8 > 0$ is a constant.

Turning to the jump part $J_K$, we define for $i \in I$ and $k \in \mathbb{N},$
\[
J_{k,i,*} V (x) := x_i \int_{\{0 < \|\xi\| < k\}} (V(x + \xi) - V(x) - \langle \nabla V(x), \xi \rangle) \mu_{K,i} (d\xi),
\]
and
\[
J_{k,i} V (x) := x_i \int_{\{|\xi| \geq k\}} (V(x + \xi) - V(x) - \langle \nabla V(x), \xi \rangle) \mu_{K,i} (d\xi).
\]

So $J_K V(x) = \sum_{i \in I} (J_{k,i,*} V(x) + J_{k,i} V(x)).$

**Big jumps.** By the mean value theorem, we get
\[
|J_{k,i} V (x)| \leq \|x_i\| \int_{\{|\xi| \geq k\}} (\|\nabla V\|_\infty \|\xi\| + \|\nabla V(x)\| \|\xi\|) \mu_i (d\xi) \leq 2 \|x\| \|\nabla V\|_\infty \int_{\{|\xi| \geq k\}} \|\xi\| \mu_i (d\xi)
\]
(3.21)
\[ \leq c_9 (1 + V(x)) \int_{\{||\xi|| \geq k\}} ||\xi|| \mu_i (d\xi) < \infty, \]

where we used that \( ||\nabla V||_\infty = \sup_{x \in D} ||\nabla V(x)|| < \infty \), as a consequence of (3.11). Hence, by dominated convergence, we can find large enough \( k = k_0 > 0 \) such that

\[ |J_{k_0,i} V(x)| \leq \frac{1}{2} c_6 (1 + V(x)), \quad x \in D. \]

**Small jumps.** To estimate the small jump part, we apply (3.19) and the mean value theorem, yielding for \( ||x|| > 3k_0 \),

\[
|J_{k_0,i} V(x)| \leq \left| x_i \int_{\{0 < ||\xi|| < k_0\}} \left( \int_0^1 (\nabla V(x + r\xi) - \nabla V(x,\xi)) \right) dr \eta_{k,i} (d\xi) \right| \\
\leq ||x|| \sup_{\tilde{x} \in B_{k_0}(x)} \|\nabla^2 V(\tilde{x})\| \int_{\{0 < ||\xi|| < k_0\}} \|\xi\|^2 \mu_i (d\xi) \\
\leq c_7 ||x|| \sup_{\tilde{x} \in B_{k_0}(x)} \frac{1}{V(\tilde{x})} \int_{\{0 < ||\xi|| < k_0\}} \|\xi\|^2 \mu_i (d\xi) \\
\leq c_{10} \frac{||x||}{||x|| - k_0} \leq 2c_{10} < \infty,
\]

with some positive constant \( c_{10} \) not depending on \( K \). Here \( B_{k_0}(x) \) denotes the ball with center \( x \) and radius \( k_0 \). Note that \( J_{k_0,i} V(x) \) is continuous in \( x \in D \). Hence, we conclude that

\[ |J_K V(x)| \leq \frac{1}{2} c_6 V(x) + c_{11}, \quad x \in D. \]

Combining the latter inequality with (3.18) and (3.20), we obtain the desired result, namely,

\[ A_K V(x) = D V(x) + J_K V(x) \leq \frac{1}{2} c_6 V(x) + c_{12}, \quad x \in D. \]

**Remark 3.5.** For the function \( V \) defined in the last lemma, we can easily find positive constants \( c_1, c_2, c_3, c_4 \) such that for all \( x \in D \),

\[ V(x) \leq c_1 ||x|| + c_2 \quad \text{and} \quad ||x|| \leq c_3 V(x) + c_4. \]

**Proposition 3.6.** Assume \( m \geq 1 \) and \( n \geq 1 \). Suppose that \( \beta \in M_d^+ \). Let \( c, C \) and \( V \) be the same as in Lemma 3.4. Then

\[ \mathbb{E}_x |V(X_{K,t})| \leq e^{-cd} V(x) + c^{-1} C \quad \text{for all} \ K \geq 1, \ x \in D \text{ and } t \in \mathbb{R}_{\geq 0}. \]

**Proof.** Let \( x \in D, \ K \geq 1 \) and \( T > 0 \) be fixed. The proof is divided into three steps.

**Step 1:** We show that

\[ \sup_{t \in [0,T]} \mathbb{E}_x \left[ ||X_{K,t}||^2 \right] < \infty. \]
Since $\mu_{K,i}$ has compact support, it follows that $\int_{\{\|\xi\|>1\}} \|\xi\|^k \mu_{K,i}(d\xi) < \infty$ for all $k \in \mathbb{N}$. By [8] Lemmas 5.3 and 6.5, we know that $\psi_K \in C^2(\mathbb{R}_+ \times \mathcal{U})$. Moreover, by [8] Theorem 2.16, we have

$$E_x [\|X_{K,t}\|^2] = -\sum_{i=1}^d \left( \langle x, \partial^2_{\lambda_i} \psi_K(t, i\lambda) |_{\lambda=0} \rangle + \langle x, \partial_{\lambda_1} \psi_K(t, i\lambda) |_{\lambda=0} \rangle \right),$$

where the right-hand side is a continuous function in $t \in [0, T]$. So (3.25) follows.

**Step 2:** We show that

$$\sup_{t \in [0, T]} E_x [V(X_{K,t})] < \infty.$$ 

In fact, (3.26) follows from (3.23) and (3.25).

**Step 3:** We show that (3.24) is true. It follows from [8] Theorem 2.12 and [8, Chap.4, Lemma 3.2] that

$$f(X_{K,t}) - f(X_{K,0}) - \int_0^t \mathcal{A}_K f(X_{K,s}) ds, \quad t \in \mathbb{R}_{\geq 0},$$

is a $\mathbb{P}_x$-martingale for every $f \in C^2_c(D)$. Note that $V$ belongs to $C^2(D)$ but does not have compact support. Let $\varphi \in C^\infty(\mathbb{R}_{\geq 0})$ be such that $1_{[0,1]} \leq \varphi \leq 1_{[0,2]}$, and define $(\varphi_j)_{j \geq 1} \subset C^\infty_c(D)$ by $\varphi_j(y) := \varphi(\|y\|^2/j^2)$. Then

$$\varphi_j(y) = 1 \text{ for } \|y\| \leq j \text{ and } \varphi_j(y) = 0 \text{ for } \|y\| > \sqrt{2}j,$$

and $\varphi_j \to 1$ as $j \to \infty$. For $j \in \mathbb{N}$, we then define

$$V_j(y) := V(y)\varphi_j(y), \quad y \in D.$$ 

So $V_j \in C^2_c(D)$. In view of (3.27) and [10] Chap.4, Lemma 3.2, it follows that

$$e^{ct}V_j(X_{K,t}) - V_j(X_{K,0}) - \int_0^t e^{cs} \mathcal{A}_K V_j(X_{K,s}) ds - \int_0^t e^{cs} V_j(X_{K,s}) ds, \quad t \in \mathbb{R}_{\geq 0},$$

is a $\mathbb{P}_x$-martingale, and hence

$$e^{ct}E_x [V_j(X_{K,t})] - V_j(x) = E_x \left[ \int_0^t e^{cs} \left( \mathcal{A}_K V_j(X_{K,s}) + c V_j(X_{K,s}) \right) ds \right].$$

Now, a simple calculation shows

$$\|\nabla \varphi_j(y)\| \leq \frac{2\|y\|}{j} \|\varphi'\|_{\infty} \leq \frac{2c_1\|y\|}{j^2},$$

for some constant $c_1 > 0$. Therefore, by (3.28), we get

$$\|\nabla V_j(y)\| = 1_{\{\|y\| \leq \sqrt{2}j\}} \|\nabla \varphi_j(y)\| V(y) + V(y) \nabla \varphi_j(y)\| \leq 1_{\{\|y\| \leq \sqrt{2}j\}} \left( \|\nabla V\|_{\infty} + c_2 (1 + \|y\|) \frac{2c_1\|y\|}{j^2} \right) \leq c_3 \frac{(1 + j)j}{j^2}.$$
where \( c_2 \) and \( c_3 \) are positive constants. A similar calculation yields that there exists a constant \( c_4 > 0 \) such that
\[
\|\nabla^2 \varphi_j(y)\| \leq c_4 \frac{\|y\|^2 + j^2}{j^4}.
\]

So
\[
\|\nabla^2 V_j(y)\| \leq \mathbb{I}_{\{\|y\| \leq \sqrt{j}\}} \left( \|\nabla^2 V\|_{\infty} + 2\|\nabla V\|_{\infty} \|\nabla \varphi_j(y)\| + \|V(y)\| \|\nabla^2 \varphi_j(y)\| \right)
\leq \mathbb{I}_{\{\|y\| \leq \sqrt{j}\}} \left( c_5 + \frac{c_6 \|y\|}{j^2} + c_7 (1 + \|y\|) \frac{\|y\|^2 + j^2}{j^4} \right)
\leq c_8 \frac{1 + j + j^2}{j^2},
\]

(3.29)

where \( c_5, c_6, c_7, c_8 > 0 \) are constants. Define \( D V_j \) and \( J_K V_j \) similarly as in (3.13) and (3.14), respectively. It holds obviously that
\[
|D V_j(y)| \leq c_9 \|y\| \left( \|\nabla V_j\|_{\infty} + \|\nabla^2 V_j\|_{\infty} \right), \quad y \in D.
\]

Similarly as in (3.21) and (3.22), we have that for all \( y \in D \),
\[
|J_K V_j(y)| \leq c_{10} \|y\| \sum_{i=1}^{m} \left( \|\nabla V_j\|_{\infty} \int_{\{\|\xi\| \geq 1\}} \|\xi\| \mu_i (d\xi) 
+ \|\nabla^2 V_j\|_{\infty} \int_{\{0 < \|\xi\| < 1\}} \|\xi\|^2 \mu_i (d\xi) \right).
\]

Using (3.25), (3.29) and the above estimates for \( D V_j \) and \( J_K V_j \), we obtain
\[
|A_K V_j(y)| \leq c_{11} (1 + \|y\|), \quad y \in D,
\]

(3.30)

where \( c_{11} > 0 \) is a constant not depending on \( j \). The dominated convergence theorem implies \( \lim_{j \to \infty} A_K V_j(y) = A_K^t V(y) \) for all \( y \in D \). By (3.26), (3.30) and again dominated convergence, it follows that
\[
e^{-ct} \mathbb{E}_x \left[ V(\hat{X}_{K,t}) - V(x) \right] = \mathbb{E}_x \left[ \int_0^t e^{cs} \left( A_K^s V(\hat{X}_{K,s}) + cV(\hat{X}_{K,s}) \right) ds \right].
\]

Applying Lemma 3.4 yields
\[
e^{-ct} \mathbb{E}_x \left[ V(\hat{X}_{K,t}) - V(x) \right] \leq \mathbb{E}_x \left[ \int_0^t e^{cs} \right] \leq c^{-1} C e^{ct},
\]

which implies
\[
\mathbb{E}_x \left[ V(\hat{X}_{K,t}) \right] \leq e^{-ct} V(x) + c^{-1}, \quad \text{for} \ t \in [0, T].
\]

Since \( x \in D, K \geq 1 \) and \( T > 0 \) are arbitrary, the assertion follows. \( \square \)

Arguing similarly as in Lemma 3.4 and Proposition 3.6, we obtain also an analog result for the case where \( m \geq 1 \) and \( n = 0 \).

**Proposition 3.7.** Assume \( m \geq 1 \) and \( n = 0 \). Suppose that \( \beta \in M_d^- \). Let \( V \in C^2(D, \mathbb{R}) \) be such that \( V > 0 \) on \( D \) and
\[
V(x) = \langle x, x \rangle^{1/2}, \quad \text{whenever} \ \|x\| > 2.
\]
Then $\mathcal{A}_{K}^{u} V$ is well-defined and there exist positive constants $c$ and $C$, independent of $K$, such that

$$\mathcal{A}_{K}^{u} V(x) \leq -cV(x) + C, \quad \forall x \in D.$$ 

Moreover, for all $K \geq 1$, $t \geq 0$ and $x \in D$, it holds that

$$\mathbb{E}_{x} [V (X_{K,t})] \leq e^{-ct} V(x) + c^{-1} C.$$ 

We are now ready to prove the uniform boundedness for the first moment of $X_{t}$, $t \geq 0$.

**Proposition 3.8.** Let $X$ be an affine process satisfying (3.1). Suppose that $\beta \in \mathbb{M}_{d}^{\geq}$. Then

$$\sup_{t \geq 0} \mathbb{E}_{x} [\|X_{t}\|] < \infty \quad \text{for all} \quad x \in D.$$ 

**Proof.** If $m = 0$ and $n \geq 1$, then $(X_{t})_{t \geq 0}$ degenerates to a deterministic motion governed by the vector field $x \rightarrow \beta x$. In this case we have

$$X_{t} = e^{\beta t} X_{0},$$ 

so (3.31) follows from the assumption that $\beta \in \mathbb{M}_{d}^{\geq}$.

For the case where $m \geq 1$, by Propositions 3.6 and 3.7, we have

$$\mathbb{E}_{x} [V (X_{K,t})] \leq e^{-ct} V(x) + c^{-1} C, \quad \text{for all} \quad K \geq 1, x \in D \text{ and } t \in \mathbb{R}_{\geq 0},$$

where $c, C > 0$ are constants not depending on $K$.

Let $x \in D$ be fixed and assume without loss of generality that $X_{0} = x$ a.s. In view of Lemma 3.2 and Skorokhod’s representation theorem (see, e.g., [10, Chap.3, Theorem 1.8]), there exist some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which $(X_{K,t})_{K \geq 1}$ and $\tilde{X}_{t}$ are defined such that $X_{K,t}$ and $\tilde{X}_{t}$ have the same distributions as $X_{K,t}$ and $X_{t}$, respectively, and $\tilde{X}_{K,t} \rightarrow \tilde{X}_{t}$ $\mathbb{P}$-almost surely as $K \rightarrow \infty$. Hence $V(\tilde{X}_{K,t}) \rightarrow V(\tilde{X}_{t})$ $\mathbb{P}$-almost surely as $K \rightarrow \infty$. By (3.32) and Fatou’s lemma, we have

$$\mathbb{E}_{x} [V (X_{t})] = \mathbb{E} \left[ V \left( \tilde{X}_{t} \right) \right] = \liminf_{K \rightarrow \infty} \mathbb{E} \left[ V \left( \tilde{X}_{K,t} \right) \right]$$

$$\leq \liminf_{K \rightarrow \infty} \mathbb{E}_{x} [V (X_{K,t})] \leq e^{-ct} V(x) + c^{-1} C$$

for all $t \geq 0$. By (3.23), the assertion follows. □

**3.2. Exponential convergence of $\psi(t, u)$ to zero.** In this subsection we study the convergence speed of $\psi(t, u) \rightarrow 0$ as $t \rightarrow \infty$.

**Lemma 3.9.** Suppose that $\beta \in \mathbb{M}_{d}^{\geq}$. There exist $\delta > 0$ and constants $C_{1}, C_{2} > 0$ such that for all $u \in \mathcal{U}$ with $\|u\| < \delta$,

$$\|\psi(t, u)\| \leq C_{1} \exp \left\{-C_{2} t\right\}, \quad t \geq 0.$$ 

**Proof.** For $u \in \mathcal{U}$, we can write $u = (v, w) \in \mathbb{C}^{m}_{\geq 0} \times i \mathbb{R}^{n}$ and further $v = x + iy$ and $w = iz$, where $x \in \mathbb{R}^{m}_{\geq 0}$, $y \in \mathbb{R}^{m}$ and $z \in \mathbb{R}^{n}$. Therefore,

$$\psi(t, u) = \psi(t, v, w) = \left( \psi^{f}(t, x + iy, iz) \right)^{i e \beta t}.$$
Recall that \( \psi(t, x, y, z) := \begin{pmatrix} \text{Re } \psi^f(t, x + iy, iz) \\ \text{Im } \psi^f(t, x + iy, iz) \end{pmatrix} e^{\beta_j t} \), \( t \geq 0 \).

For \( x \in \mathbb{R}^m, y \in \mathbb{R}^m, \) and \( z \in \mathbb{R}^n, \) we define
\[
\tilde{\psi}(t, x, y, z) := \begin{pmatrix} \text{Re } \psi^f(t, x + iy, iz) \\ \text{Im } \psi^f(t, x + iy, iz) \end{pmatrix} e^{\beta_j t} z, \quad t \geq 0.
\]

Recall that \( \psi^f(t, u) \) satisfies the Riccati equation
\[
\partial_t \psi^f(t, v, w) = R^f \left( \psi^f(t, v, w), e^{\beta_j t} w \right), \quad \psi^f(0, v, w) = v.
\]

So
\[
\partial_t \tilde{\psi}(t, x, y, z) = \begin{pmatrix} \partial_t \text{Re } \psi^f(t, x + iy, iz) \\ \partial_t \text{Im } \psi^f(t, x + iy, iz) \end{pmatrix} e^{\beta_j t} z
\]
\[
= \begin{pmatrix} \text{Re } R^f \left( \psi^f(t, x + iy, iz), e^{\beta_j t} z \right) \\ \text{Im } R^f \left( \psi^f(t, x + iy, iz), e^{\beta_j t} z \right) \end{pmatrix}
\]
\[
= \begin{pmatrix} \text{Re } R^f \left( \psi^f(t, x + iy, iz) + i \text{Im } \psi^f(t, x + iy, iz), e^{\beta_j t} z \right) \\ \text{Im } R^f \left( \psi^f(t, x + iy, iz) + i \text{Im } \psi^f(t, x + iy, iz), e^{\beta_j t} z \right) \end{pmatrix}
\]
\[
= \begin{pmatrix} \text{Re } R^f (\theta + i \eta, i \zeta) \\ \text{Im } R^f (\theta + i \eta, i \zeta) \end{pmatrix}
\]
\[
=: \tilde{R}(\theta, \eta, \zeta),
\]

where the map \( \mathbb{R}^m_+ \times \mathbb{R}^m \times \mathbb{R}^n \to (\theta, \eta, \zeta) \to \tilde{R}(\theta, \eta, \zeta) \) is \( C^1 \) by \[8\] Lemma 5.3.

Hence \( \tilde{\psi}(t, x, y, z) \) solves the equation
\[
\partial_t \tilde{\psi}(t, x, y, z) = \tilde{R}(\theta, \eta, \zeta), \quad t \geq 0, \quad \psi(0, x, y, z) = (x, y, z).
\]

Similarly to \[8\] p.1011, (6.7)), we have, for \( u = (x + iy, iz), \)
\[
\text{Re } R_t (x + iy, iz) = \alpha_{i,i} x_i^2 - \langle \alpha_i \text{Im } u, \text{Im } u \rangle + \sum_{k=1}^m \beta_{ki} x_k
\]
\[
+ \int_{D \setminus \{0\}} \left( e^{\langle \xi, x \rangle} \cos(\text{Im } u, \xi) - 1 - \langle \xi, x \rangle \right) \mu_i (d\xi)
\]
and
\[
\text{Im } R_t (x + iy, iz) = 2 \alpha_{i,i} x_i y_i + \langle \beta_{ii}, y \rangle + \langle \beta_{ji}, z \rangle
\]
\[
+ \int_{D \setminus \{0\}} \left( e^{\langle \xi, x \rangle} \sin(\text{Im } u, \xi) - \langle \text{Im } u, \xi \rangle \right) \mu_i (d\xi).
\]

Since \( \tilde{R} : \mathbb{R}^m_+ \times \mathbb{R}^m \to \mathbb{R}^{2m} \) is \( C^1 \), so
\[
\left\| \tilde{R}(\theta, \eta, \zeta) - D \tilde{R}(0)(\theta, \eta, \zeta) \right\|.
\]
According to (3.35) and (3.36), we see that $D$ holds. Here, (3.37)

$$
\sup_{0 \leq r \leq 1} \left\| D\tilde{R}(r(\vartheta,\eta,\zeta)) - D\tilde{R}(0) \right\| \cdot \left\| (\vartheta,\eta,\zeta)^\top \right\|
$$

(3.37)

From (3.37) it follows that $\ast$ where

By assumption, we know that $\beta$ as shown in the proof of [26, VII. Stability Theorem, p.311], we can find constants $\delta, c$ such that

$$
\beta_{II} = \left( \begin{array}{ccc}
\beta_{II}^T & 0 & 0 \\
0 & \beta_{IIJ} & * \\
0 & 0 & \beta_{IJ}
\end{array} \right)
$$

where $*$ is a $(m \times n)$-matrix. By the Riccati equation (3.35) for $\tilde{\psi}$, we can write

$$
\partial_t \tilde{\psi}(t, x, y, z) = D\tilde{R}(0)\tilde{\psi}(t, x, y, z) + \left(\tilde{R} \left(\tilde{\psi}(t, x, y, z)\right) - D\tilde{R}(0)\tilde{\psi}(t, x, y, z)\right).
$$

From (3.37) it follows that

$$
\lim_{\|\vartheta,\eta,\zeta\| \to 0} \frac{\left\| \tilde{R}(\vartheta,\eta,\zeta) - D\tilde{R}(0) (\vartheta,\eta,\zeta)^\top \right\|}{\| (\vartheta,\eta,\zeta) \|} = 0.
$$

By assumption, we know that $\beta_{II} \in M_m^-$ and $\beta_{IJ} \in M_n^-$, which ensures $D\tilde{R}(0) \in M_{2m+n}^-$. Now, an application of the linearized stability theorem (see, e.g., [26, VII. Stability Theorem, p.311]) yields that $\tilde{\psi}$ is asymptotically stable at $0$. Moreover, as shown in the proof of [26, VII. Stability Theorem, p.311], we can find constants $\delta, c_1, c_2 > 0$ such that

$$
\|\tilde{\psi}(t, x, y, z)\| \leq c_1 e^{-c_2 t}, \quad \forall \ t \geq 0, (x, y, z) \in B_\delta(0) \cap \mathbb{R}^m \times \mathbb{R}^m+n,
$$

where $B_\delta(0)$ denotes the ball with center $0$ and radius $\delta$. By the definition of $\tilde{\psi}$, the latter inequality implies that (3.33) is true. The lemma is proved.

Next, we extend the estimate in Lemma 3.9 to all $u \in \mathcal{U}$.

**Proposition 3.10.** Let $X$ be an affine process satisfying (3.1). Suppose that $\beta \in M_d^-$. Then for every $u \in \mathcal{U}$, there exist positive constants $c_1, c_2$, which depend on $u$, such that

$$
\|\psi(t, u)\| \leq c_1 \exp \{-c_2 t\}, \quad t \geq 0.
$$

**Proof.** Our proof is inspired by the proof of [12, Theorem 2.4]. By Proposition 3.8 we have $\sup_{t \in \mathbb{R}_+} \mathbb{E}_x ||X_t|| < \infty$ for all $x \in D$. Then for $M > 0$,

$$
\mathbb{P}_x (||X_t|| > M) \leq \frac{\mathbb{E}_x [||X_t||]}{M} \leq \sup_{t \geq 0} \frac{\mathbb{E}_x [||X_t||]}{M},
$$

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which implies 
\[
\sup_{t \geq 0} \mathbb{P}_x (\|X_t\| > M) \rightarrow 0 \quad \text{as} \quad M \rightarrow \infty.
\]

We see that under \( \mathbb{P}_x \), the sequence \( \{X_t, t \geq 0\} \) is tight. Consider an arbitrary subsequence \( \{X_{t^\nu}\} \). Then it contains a further subsequence \( \{X_{t^\nu}\} \) converging in law to some limiting random vector, say \( X^a \). Since \( X_{t^\nu} \) converges weakly to \( X^a \) as \( t^\nu \rightarrow \infty \), Lévy’s continuity theorem implies that the characteristic function of \( X_{t^\nu} \) converges pointwise to that of \( X^a \), namely,
\[
\lim_{t^\nu \rightarrow \infty} \mathbb{E}_x [\exp \{\langle u, X_{t^\nu}\rangle\}] = \mathbb{E} [\exp \{\langle u, X^a\rangle\}], \quad \text{for all} \quad u \in \mathcal{U}.
\]

We know by Proposition 3.9 that the original sequence \( \{X_i\} \) satisfies
\[
\lim_{t \rightarrow \infty} \mathbb{E}_x [\exp \{\langle u, X_t\rangle\}] = \lim_{t \rightarrow \infty} \exp \{\langle x, \psi(t, u)\rangle\} = 1
\]
for all \( u \in \mathcal{U} \) with \( \|u\| < \delta \). As a consequence, we get
\begin{equation}
\mathbb{E} [\exp \{\langle u, X^a\rangle\}] = 1, \quad \text{for all} \quad u \in \mathcal{U} \quad \text{with} \quad \|u\| < \delta.
\end{equation}

We claim that \( X^a = 0 \) almost surely. To prove this, we consider an arbitrary \( z \in \mathbb{R}^d \) with \( z \neq 0 \). Then there exists an \( u_0 \in \mathbb{R}^d \) with \( \|u_0\| < \delta \) such that \( 0 < \langle u_0, z \rangle < \pi/6 \), and hence \( 0 < \cos((u_0, z)) < 1 \). Continuity of cosine implies that there exists an \( \epsilon > 0 \) such that \( 0 \notin B_\epsilon(z) := \{y \in \mathbb{R}^d : \|y - z\| < \epsilon\} \) and \( 0 < \cos((u_0, y)) < 1 \) for all \( y \in B_\epsilon(z) \). Suppose that \( \mathbb{P} (X^a \in B_\epsilon(z)) > 0 \). It follows that
\[
\mathbb{E} [\cos (\langle u_0, X^a\rangle) 1_{\{X^a \in B_\epsilon(z)\}}] < \mathbb{P} (X^a \in B_\epsilon(z)),
\]
which in turn implies
\[
\text{Re} \mathbb{E} [\exp \{i\langle u_0, X^a\rangle\}] = \mathbb{E} [\cos (\langle u_0, X^a\rangle)] \\
\leq \mathbb{E} [\cos (\langle u_0, X^a\rangle) 1_{\{X^a \in B_\epsilon(z)\}}] \\
+ \mathbb{E} [\cos (\langle u_0, X^a\rangle) 1_{\{X^a \notin B_\epsilon(z)\}}] \\
< \mathbb{P} (X^a \in B_\epsilon(z)) + \mathbb{P} (X^a \notin B_\epsilon(z)) \\
= 1,
\]
a contradiction to (3.38). We conclude that \( \mathbb{P}(X^a \in B_\epsilon(z)) = 0 \). Since \( z \neq 0 \) is arbitrary, \( X^a \) must be 0 almost surely. Now we have shown that every subsequence of \( \{X_i\} \) contains a further subsequence converging weakly to \( \delta_0 \), so the original sequence \( \{X_i\} \) must converge to \( \delta_0 \) weakly. In view of this, we now denote \( X^a \) by \( X_\infty \) which is 0 almost surely. We have thus shown that for all \( x \in D \) and \( u \in \mathcal{U} \),
\begin{equation}
\exp \{\langle x, \psi(t, u)\rangle\} = \mathbb{E}_x [\exp \{\langle u, X_t\rangle\}] \rightarrow 1 \quad \text{as} \quad t \rightarrow \infty.
\end{equation}

From the above convergence of \( \exp \{\langle x, \psi(t, u)\rangle\} \) to 1, we infer that for each \( i = 1, \ldots, d \),
\begin{equation}
\text{Re} \psi_i(t, u) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\end{equation}

Moreover, we must have \( \sup_{i \in [0, \infty)} |\psi_i(t, u)| \leq C \) for some constant \( C = C(u) < \infty \), otherwise, by continuity, \( \text{Im} \psi_i(t, u) \) hits the set \( \{2k\pi + \pi/2 : k \in \mathbb{Z}\} \) infinitely many times as \( t \rightarrow \infty \), so \( \sin (\text{Im} \psi_i(t, u)) = 1 \) infinitely often, contradicting the fact that \( \exp \{\langle x, \psi(t, u)\rangle\} \rightarrow 1 \) for all \( x \in D \).
Let $z, z' \in \mathbb{C}$ be two different accumulation points of $\{\psi_1(t, u), t \geq 0\}$ as $t \to \infty$, that is, we can find sequences $t_n, t_n' \to \infty$ such that $\psi_1(t_n, u) \to z$ and $\psi_1(t_n', u) \to z'$. Using once again the convergence in (3.39), we obtain that $z = i2\pi k_1$ and $z' = i2\pi k_2$ for some $k_1, k_2 \in \mathbb{Z}$. By (3.40) and a similar argument as in the last paragraph, $\psi_1(t, u)$ is not allowed to fluctuate between $z$ and $z'$, showing that $z = z'$. So $z = i2\pi k_1$ is the only accumulation point of $\{\psi_1(t, u), t \geq 0\}$, and $\psi_1(t, u) \to z = i2\pi k_1$ as $t \to \infty$. Moreover, we must have $k_1 = 0$, otherwise for some $x \in D$ we get $\exp \{x_1 2\pi ik_1\} \neq 1$, which is impossible due to (3.39). We conclude that

$$\psi_1(t, u) \to 0 \quad \text{as } t \to \infty \text{ for all } u \in \mathcal{U}.$$  

In the same way it follows that $\psi_i(t, u) \to 0$ as $t \to 0$ for all $i = 2, \ldots, d$ and $u \in \mathcal{U}$.

Finally, we prove that the convergence of $\psi(t, u)$ to zero as $t \to \infty$ is exponentially fast. Since $\psi(t, u)$ converges to 0 as $t \to \infty$, there exists a $t_0 > 0$ such that $\|\psi(t_0, u)\| < \delta$. Combining Lemma 3.9 with the semi-flow property of $\psi$, we conclude that

$$\|\psi(t + t_0, u)\| = \|\psi(t, \psi(t_0, u))\| < c_1 e^{-\delta t}, \quad t \geq 0,$$

for some positive constants $c_1$ and $c_2$. Hence,

$$\|\psi(t, u)\| < c_3 e^{-\delta t}, \quad t \geq t_0.$$

Since $\sup_{t \in [0, t_0]} \|\psi(t, u)\| < c_4$, where $c_4 > 0$ is a constant, it follows that

$$\|\psi(t, u)\| < c_5 e^{-\delta t}, \quad t \geq 0,$$

with another constant $c_5 > 0$. This completes our proof. \qed

4. Proof of the main result

In this section we will prove Theorem 2.6

Let $X$ be an affine process with state space $D$ and admissible parameters $(a, \alpha, b, \beta, m, \mu)$. Recall that $F(u)$ is given by (2.4). We start with the following lemma.

**Lemma 4.1.** Suppose $\beta \in M_d$ and $\int_{\{|\xi| > 1\}} \log |\xi| m(d\xi) < \infty$. Then

$$\int_0^\infty |F(\psi(s, u))| \, ds < \infty \quad \text{for all } u \in \mathcal{U}.$$  

**Proof.** Let $u \in \mathcal{U}$ be fixed. By Remark 3.1 and Proposition 3.10, we can find constants $c_1, c_2 > 0$ depending on $u$ such that

$$|\psi(s, u)| < c_1 e^{-c_2 s}, \quad s \geq 0.$$  

It is clear that finiteness of $\int_0^\infty |F(\psi(s, u))| \, ds$ depends only on the jump part of $F$. We denote

$$\mathcal{I}(u) = \int_0^\infty \int_{\{0 < |\xi| \leq 1\}} \left| e^{\langle \xi, \psi(s, u) \rangle} - 1 - \langle \psi^J(s, u), \xi \rangle \right| m(d\xi) \, ds$$

$$+ \int_0^\infty \int_{\{|\xi| > 1\}} \left| e^{\langle \xi, \psi(s, u) \rangle} - 1 \right| m(d\xi) \, ds$$

where $e^{\langle \psi^J(s, u), \xi \rangle}$ is the exponential of the jump part of $\psi(s, u)$. Then

$$\mathcal{I}(u) \leq \int_0^\infty \int_{\{0 < |\xi| < 1\}} \left| e^{\langle \xi, \psi(s, u) \rangle} - 1 - \langle \psi^J(s, u), \xi \rangle \right| m(d\xi) \, ds$$

$$+ \int_0^\infty \int_{\{|\xi| > 1\}} \left| e^{\langle \xi, \psi(s, u) \rangle} - 1 \right| m(d\xi) \, ds.$$  

as $t \to \infty$. Therefore, we conclude that

$$\|\psi(t, u)\| \to 0 \quad \text{as } t \to \infty \text{ for all } u \in \mathcal{U}.$$  

In the same way it follows that $\psi_i(t, u) \to 0$ as $t \to \infty$ for all $i = 2, \ldots, d$ and $u \in \mathcal{U}$.
Note that $t$ variables.

Let us define $I := \mathbb{I}(u) + \mathbb{I}^+(u)$.

With the latter fact in mind, we start with the big jumps. We can apply Fubini’s theorem to get

$$\mathbb{I}(u) = \int_{\{\|\xi\|>1\}} \int_0^\infty \left|e^{\langle \xi, \psi(s, u) \rangle} - 1 \right| ds m(d\xi).$$

Let us define $I_1(\xi) := \int_0^{\infty} |\exp\{\psi(s, u), \xi\} - 1| ds$. For $\|\xi\| > 1$, by a change of variables $t := \exp\{-c_2 s\} \|\xi\|$, we get $ds = -c_2^{-1} t^{-1} dt$, and hence

$$I_1(\xi) = \frac{1}{c_2} \int_0^{\|\xi\|} \frac{1}{t} \left|e^{\langle \xi, \psi(s^{-1}(t, u)) \rangle} - 1 \right| dt$$

$$= \frac{1}{c_2} \int_0^{\|\xi\|} \frac{1}{t} \left|e^{\langle \xi, \psi(s^{-1}(t, u)) \rangle} - 1 \right| dt$$

$$\leq \frac{1}{c_2} \int_0^{1} \frac{1}{t} \left|e^{\langle \xi, \psi(s^{-1}(t, u)) \rangle} - 1 \right| dt + \frac{1}{c_2} \int_1^{\|\xi\|} \frac{2}{t} dt$$

$$=: I_2(\xi) + I_3(\xi).$$

Note that

$$\left|e^{\langle \xi, \psi(s^{-1}(t, u)) \rangle} - 1 \right| = \left|\int_0^1 e^{r\langle \xi, \psi(s^{-1}(t, u)) \rangle} \langle \xi, \psi(s^{-1}(t, u)) \rangle dr \right|$$

$$\leq \left|\langle \xi, \psi(s^{-1}(t, u)) \rangle \right|.$$

Using (4.1), we obtain

$$I_2(\xi) \leq \frac{1}{c_2} \int_0^1 \frac{1}{t} \left|\langle s^{-1}(t, u), \xi \rangle \right| dt$$

$$\leq \frac{1}{c_2} \int_0^1 \frac{1}{t} \left|\psi(s^{-1}(t, u)) \right| \|\xi\| dt$$

$$\leq \frac{1}{c_2} \int_0^1 \frac{1}{t} e^{-c_2 s^{-1}(t) \|\xi\|} dt.$$

Since $s^{-1}(t) = \log(t\|\xi\|^{-1})(-c_2)^{-1}$, it follows that

$$I_2(\xi) \leq \frac{1}{c_2} \int_0^1 c_1 dt = \frac{c_1}{c_2}.$$

On the other hand, it is easy to see that

$$I_3(\xi) \leq \frac{2}{c_2} \log \|\xi\|,$$

Having established the latter inequalities, we conclude that

$$\mathbb{I}(u) \leq \int_{\{\|\xi\|>1\}} (I_2(\xi) + I_3(\xi)) m(d\xi)$$

$$\leq \int_{\{\|\xi\|>1\}} \left(\frac{c_1}{c_2} + \frac{2}{c_2} \log \|\xi\| \right) m(d\xi)$$

$$= \frac{c_1}{c_2} m(\{\|\xi\| > 1\}) + \frac{2}{c_2} \int_{\{\|\xi\|>1\}} \log \|\xi\| m(d\xi).$$
Because the Lévy measure $m(d\xi)$ integrates $\mathbb{1}_{\{\|\xi\|>1\}} \log \|\xi\|$ by assumption, we see that

\begin{equation}
\mathcal{I}^*(u) < \infty.
\end{equation}

We now turn to $\mathcal{I}_b(\xi)$. We can write

\begin{align*}
\exp\langle \xi, \psi(s, u) \rangle - 1 - \langle \psi^I(s, u), \xi \rangle &\leq \exp\langle \xi, \psi(s, u) \rangle - 1 - \langle \psi^I(s, u), \xi \rangle \\
&\leq (c_1 + c_2^2) e^{-c_2 s} (\|\xi_t\| + (\|\xi_t\| + \|\xi_J\|) \|\xi_J\|)
\end{align*}

Noting (4.1) and $\text{Re}(\langle \xi, \psi(s, u) \rangle) \leq 0$, we deduce that for $\|\xi\| \leq 1$ and $s \geq 0$,

\begin{equation}
\begin{align*}
\exp\langle \xi, \psi(s, u) \rangle - 1 - \langle \psi^I(s, u), \xi \rangle &\leq \|\psi^I(s, u)\| \|\xi_t\| + \|\psi(s, u)\| \|\xi_t\| \|\psi^I(s, u)\| \|\xi_J\| \\
&\leq (c_1 + c_2^2) e^{-c_2 s} (\|\xi_t\| + (\|\xi_t\| + \|\xi_J\|) \|\xi_J\|)
\end{align*}
\end{equation}

So

\[
\mathcal{I}_b(u) \leq (c_1 + c_2^2) \int_0^\infty e^{-c_2 s} ds \int_{\{0 < \|\xi\| \leq 1\}} \left(2 \|\xi_t\| + \|\xi_J\|^2\right) m(d\xi) < \infty,
\]

where the finiteness of the integral on the right-hand side follows by Definition 2.2 (iii). Since (4.2) holds, it follows that

\[
\int_0^\infty |F(\psi(s, u))| ds \leq \mathcal{I}_b(u) = \mathcal{I}_b(u) + \mathcal{I}^*(u) < \infty.
\]

The lemma is proved. \hfill \square

We are now ready to prove our main result.

**Proof of Theorem 2.6** Recall that the characteristic function of $X_t$ is given by

\[
\mathbb{E}_x \left[ e^{i(u, X_t)} \right] = \exp \{ \phi(t, u) + \langle x, \psi(t, u) \rangle \}, \quad (t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}.
\]

Using Remark 3.1 and Theorem 3.10 and Lemma 4.1, we have that $\psi(t, u) \to 0$ and

\[
\phi(t, u) = \int_0^t F(\psi(s, u)) ds \to \int_0^\infty F(\psi(s, u)) ds, \quad as \ t \to \infty.
\]

We now verify that $\int_0^\infty F(\psi(s, u)) ds$ is continuous at $u = 0$. It is easy to see that $\int_0^T F(\psi(s, u)) ds$ is continuous at $u = 0$. It suffices to show that the convergence \[ \lim_{T \to \infty} \int_0^T F(\psi(s, u)) ds = \int_0^\infty F(\psi(s, u)) ds \]

is uniform for $u$ in a
small neighborhood of 0. By (3.33), there exist $\delta > 0$ and constants $c_1, c_2 > 0$ such that for all $B_4(0) \cap \mathcal{U}$,
\[ \|\psi(t, u)\| \leq c_1 \exp \{-c_2 t\}, \quad t \geq 0. \]

Define
\[ I_T(u) = \int_T^{\infty} \int_{\{0 < \|\xi\| \leq 1\}} \left| e^{(\xi, \psi(s, u))} - 1 - (\psi'(s, u), \xi) \right| m(\xi) \, ds \]
\[ + \int_T^{\infty} \int_{\{1 < \|\xi\| \leq K\}} \left| e^{(\xi, \psi(s, u))} - 1 \right| m(\xi) \, ds \]
\[ + \int_T^{\infty} \int_{\{\|\xi\| > K\}} \left| e^{(\xi, \psi(s, u))} - 1 \right| m(\xi) \, ds \]
\[ =: I_{s, T}(u) + I_T(u) + I_{s, T}^{**}(u), \]
where $K > 0$. Let $\varepsilon > 0$ be arbitrary. By Fubini's theorem,
\[ I_{s, T}^{**}(u) = \int_{\{\|\xi\| > K\}} \int_T^{\infty} \left| e^{(\xi, \psi(s, u))} - 1 \right| ds m(\xi). \]

Set $I_1(\xi) := \int_T^{\infty} \left| \exp\{\langle \psi(s, u), \xi \rangle \} - 1 \right| ds$. As in the proof of Lemma 4.1, we introduce a change of variables $t := \exp \{-c_2(s - T)\} \|\xi\|$ and obtain for $\|\xi\| > 1$,
\[ I_1(\xi) = \frac{1}{c_2} \int_0^{\|\xi\|} \frac{1}{t} \left| e^{(\xi, \psi(s^{-1}(t), u))} - 1 \right| \, dt \]
\[ \leq \frac{1}{c_2} \int_0^{\|\xi\|} \frac{1}{t} \left[ e^{(\xi, \psi(s^{-1}(t), u))} - 1 \right] \, dt + \frac{1}{c_2} \int_1^{\|\xi\|} \frac{2}{t} \, dt \]
\[ \leq \frac{1}{c_2} \int_0^1 \frac{c_1}{t} e^{-c_2 s^{-1}(t)} \|\xi\| \, dt + \frac{2}{c_2} \log \|\xi\| \]
\[ \leq \frac{1}{c_2} \int_0^1 c_1 e^{-c_2 T} \, dt + \frac{2}{c_2} \log \|\xi\|. \]

So
\[ I_{s, T}^{**}(u) \leq \int_{\{\|\xi\| > K\}} \left( \frac{c_1}{c_2} e^{-c_2 T} + \frac{2}{c_2} \log \|\xi\| \right) m(\xi) \, d\xi \]
\[ \leq \frac{c_1}{c_2} m(\{\|\xi\| > K\}) + \frac{2}{c_2} \int_{\{\|\xi\| > K\}} \log \|\xi\| \, m(\xi) \, d\xi. \]

We now choose $K > 0$ large enough such that $I_{s, T}^{**}(u) < \varepsilon/3$.

For $I_T(u)$, by (4.4), we have
\[ I_1(\xi) = \frac{1}{c_2} \int_0^{\|\xi\|} \frac{1}{t} \left| e^{(\xi, \psi(s^{-1}(t), u))} - 1 \right| \, dt \]
\[ \leq \frac{1}{c_2} \int_0^{\|\xi\|} \frac{c_1}{t} e^{-c_2 s^{-1}(t)} \|\xi\| \, dt \]
\[ \leq \frac{1}{c_2} \int_0^{\|\xi\|} c_1 e^{-c_2 T} \, dt \]
\[ \leq \frac{c_1}{c_2} e^{-c_2 T} \|\xi\|, \]

which implies

\[ I^*_T(u) \leq \int_{\{1 < \|\xi\| \leq K\}} \left( \frac{c_1}{c_2} e^{-c_2 T} \|\xi\| \right) m(d\xi) \]
\[ \leq \frac{c_1}{c_2} e^{-c_2 T} \int_{\{1 < \|\xi\| \leq K\}} \|\xi\| m(d\xi) \to 0, \quad \text{as } T \to \infty. \]

So we find \( T_1 > 0 \) such that for \( T > T_1, I^*_T(u) < \varepsilon/3 \). It follows from (4.3) that

\[ I^*_T(u) \leq \left( \frac{c_1}{c_2} + c_1^2 \right) \int_{0 < \|\xi\| \leq 1} \left( 2 \|\xi_r\| + \|\xi_j\|^2 \right) m(d\xi) \to 0, \quad \text{as } T \to \infty. \]

Hence there exists \( T_2 > T_1 \) such that for \( T > T_2, I^*_T(u) < \varepsilon/3 \). Finally, we get for \( T > T_2, \)

\[ \int_T^\infty |F(\psi(s,u))| ds \leq I_{s,T}(u) + I^*_T(u) + I^{**}_T(u) < \varepsilon. \]

Moreover, the particular choice of above \( K, T_1, T_2 \) do not depend on \( u \in B_\delta(0) \cap \mathcal{U} \). We thus obtain the desired uniform convergence and further the continuity of \( \int_0^\infty F(\psi(s,u)) ds \) at \( u = 0 \).

By Lévy’s continuity theorem, the limiting distribution of \( X_t \) exists and we denote it by \( \pi \). The limiting distribution \( \pi \) has characteristic function

\[ \int_D e^{(u,x)} \pi(dx) = \exp \left\{ \int_0^\infty F(\psi(s,u)) ds \right\}. \]

We now verify that \( \pi \) is the unique stationary distribution. We start with the stationarity. Suppose that \( X_0 \) is distributed according to \( \pi \). Then, for any \( u \in \mathcal{U}, \)

\[ \mathbb{E}_\pi \left[ \exp \{\langle u, X_t \rangle \} \right] = \int_D \exp \{\phi(t,u) + \langle x, \psi(t,u) \rangle \} \pi(dx) \]
\[ = e^{\phi(t,u)} \int_D \exp \{\langle x, \psi(t,u) \rangle \} \pi(dx) \]
\[ = e^{\phi(t,u)} \int_D e^{\langle x, \eta \rangle} \pi(dx), \]

where we substituted \( \eta := \psi(t,u) \) in the last equality. Note that the integral on the right-hand side of the last equality is the characteristic function of the limit distribution \( \pi \). Therefore, using the semi-flow property of \( \psi \) in (2.3), we have

\[ \mathbb{E}_\pi \left[ \exp \{\langle u, X_t \rangle \} \right] = e^{\phi(t,u)} \exp \left\{ \int_0^\infty F(\psi(s,\eta)) ds \right\} \]
\[ = e^{\phi(t,u)} \exp \left\{ \int_0^\infty F(\psi(s,\psi(t,u))) ds \right\} \]
\[ = e^{\phi(t,u)} \exp \left\{ \int_0^\infty F(\psi(t+s,u)) ds \right\} \]
\[ = e^{\phi(t,u)} \exp \left\{ \int_0^\infty F(\psi(s,u)) ds \right\}. \]
So, by the generalized Riccati equation (2.5) for $\phi$,
$$
\mathbb{E}_\pi \left[ \exp \{ \langle u, X_t \rangle \} \right] = \exp \left\{ \int_0^\infty F(\psi(s, u)) \, ds \right\} = \int_D e^{\langle x, u \rangle} \pi(dx).
$$

Hence $\pi$ is a stationary distribution for $X$.

Finally, we prove the uniqueness of stationary distributions for $X$. We proceed as in [15, p.80]. Suppose that there exists another stationary distribution $\pi'$. Let $X_0$ be distributed according to $\pi'$. Recall that for all $u \in \mathcal{U}$, $\psi(t, u) \to 0$ as $t \to \infty$ in virtue of Theorem 3.10 and, by Lemma 4.1, $\phi(t, u) \to \int_0^\infty F(\psi(t, u)) \, ds$ as $t \to \infty$. Hence, by dominated convergence,
$$
\int_D e^{\langle x, u \rangle} \pi'(dx) = \lim_{t \to \infty} \mathbb{E}_{\pi'} \left[ \exp \{ \langle u, X_t \rangle \} \right]
= \lim_{t \to \infty} \int_D \exp \{ \phi(t, u) + \langle x, \psi(t, u) \rangle \} \pi'(dx)
= \int_D \exp \left\{ \int_0^\infty F(\psi(s, u)) \, ds \right\} \pi'(dx)
= \exp \left\{ \int_0^\infty F(\psi(s, u)) \, ds \right\} = \int_D e^{\langle x, u \rangle} \pi(dx).
$$

So $\pi = \pi'$.

\[ \square \]

Acknowledgement. We would like to thank Martin Friesen for several helpful discussions.

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(Peng Jin) Department of Mathematics, Shantou University, Shantou, Guangdong 515063, China
E-mail address: pjin@stu.edu.cn

(Jonas Kremer) Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42119 Wuppertal, Germany
E-mail address: jkremer@uni-wuppertal.de

(Barbara Rüdiger) Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42119 Wuppertal, Germany
E-mail address: ruediger@uni-wuppertal.de