Quantum Circuits and $Spin(3n)$ Groups

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Abstract

All quantum gates with one and two qubits may be described by elements of $Spin$ groups due to isomorphisms $Spin(3) \simeq SU(2)$ and $Spin(6) \simeq SU(4)$. However, the group of $n$-qubit gates $SU(2^n)$ for $n > 2$ has bigger dimension than $Spin(3n)$. A quantum circuit with one- and two-qubit gates may be used for construction of arbitrary unitary transformation $SU(2^n)$. Analogously, the ‘$Spin(3n)$ circuits’ are introduced in this work as products of elements associated with one- and two-qubit gates with respect to the above-mentioned isomorphisms.

The matrix tensor product implementation of the $Spin(3n)$ group together with relevant models by usual quantum circuits with $2n$ qubits are investigated in such a framework. A certain resemblance with well-known sets of non-universal quantum gates (e.g., matchgates, noninteracting-fermion quantum circuits) related with $Spin(2n)$ may be found in presented approach. Finally, a possibility of the classical simulation of such circuits in polynomial time is discussed.

Keywords: quantum computation, matchgates, spin groups, polynomial time
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1 Introduction

An example of representation of quantum gates with $Spin$ groups and Clifford algebras was considered in earlier work [1]. Similar approach was also discussed due to relation of matchgates and noninteracting-fermion quantum circuits [2, 3, 4, 5, 6, 7]. It was also found in a broader context, that the $Spin$ groups can be also related with so-called holographic algorithms [8, 9], but this issue is beyond the scope of the presented work.

Such a representation corresponds to the non-universal set of quantum one- and two-qubit gates generating the subgroup isomorphic to $Spin(2n)$ for a quantum circuit with $n$ qubits. It was shown directly in [1] and also follows from other works [3, 5, 6] due to definition of $Spin(2n)$ [10, 11, 12].

Relation with the physical fermions is not obvious from such a construction with $Spin(2n)$, e.g., for one qubit $Spin(2)$ is simply one-parameter group. On the other hand, there is an
isomorphism between Spin(3) and the group of one-qubit gates SU(2) and it has the direct relation with a physical implementation of a single qubit by a spin-half particle.

The isomorphism [11, 12] between Spin(6) and the group of two-qubit gates SU(4) is less trivial and does not have clear physical implications. A similar relation between group of n-qubit gates SU(2^n) and Spin(3n) may not exist for n > 2, because dimensions of such groups are 4^n − 1 and 3n(3n − 1)/2 respectively.

Let us consider 3n elements \( e_j, j = 1, \ldots, 3n \) (used further for construction of the Clifford algebra \( \mathcal{C}(3n) \) Eq. (2) and the Spin(3n) group) together with the subdivision:

\[
e_\nu^{(l)} = e_{3(l-1)+\nu},
\]

(1)

where \( l = 1, \ldots, n \) and \( \nu = 1, 2, 3 \).

An idea to identify \( \nu \) with the index of a line in some circuit might be more clear further in Sec. 2.4 with rewriting \( e_\nu^{(l)} \) as Eq. (32) and in Sec. 4 due to possibility to associate each \( \nu \) with pair of lines in quantum circuits with 2n qubits.

A set with \( k \) indexes \( l_1 < l_2 < \cdots < l_k \) corresponds to \( 3k \) elements \( e_\nu^{(l_1)}, \ldots, e_\nu^{(l_k)} \) those may be used for construction of subalgebra \( \mathcal{C}(3k) \subset \mathcal{C}(3n) \) and subgroup Spin(3k) \( \subset \) Spin(3n). An element of the subgroup is considered further as an analogue of a gate with \( k \) lines. For \( k = 1, 2 \) there are isomorphisms with the groups of quantum one- and two-qubit gates, because Spin(3) \( \simeq \) SU(2) and Spin(6) \( \simeq \) SU(4).

The composition of such analogues of one- and two-line gates may be considered as some ‘Spin(3n) circuits.’ Formally, any quantum circuit composed only from one- and two-qubit gates would define a ‘Spin(3n) circuits’ by remapping of all such gates using above-mentioned isomorphisms into elements of the group Spin(3n).

Despite of the formal isomorphisms for gates with fixed one or two lines, results of composition with different lines may not be isomorphic, because quantum one- and two-qubit gates may produce whole group SU(2^n), unlike analogous gates in ‘Spin(3n) circuits.’

The plan of the paper. In the Sec. 2 construction of ‘Spin(3n) circuits’ is discussed using rather abstract and general mathematical structures. Results and methods from the Sec. 2 are revisited and become more descriptive in the next two sections. The Sec. 3 uses more understanding models with matrices and Sec. 4 illustrates some important results using quantum circuits with 2n qubits.

Finally, in Sec. 5 the effective simulation of ‘Spin(3n) circuits’ by classical computers is discussed using the model with 2n qubits from Sec. 4 and methods of simulating quantum circuits with matchgates developed earlier [5, 6].

2 Structure of Spin(3n) groups

2.1 Clifford algebras and Spin groups

Let us recall some preliminaries [10, 11, 12, 13, 14]. Real Clifford algebra is defined by \( n \) generators \( e_j \) with properties

\[
e_j^2 = -1, \quad e_j e_k = -e_k e_j \quad (j \neq k),
\]

(2)

often written in a single equation \( e_j e_k + e_k e_j = -2\delta_{jk} 1 \). Different products of the generators \( e_j \) is a basis of the (universal) real Clifford algebra \( \mathcal{C}(n) \) with dimension \( 2^n \).
A similar equation $e_j e_k + e_k e_j = 2 \delta_{jk} 1$ defines $\mathcal{C}(n)$. In the most general case a real Clifford algebra is defined by a quadratic form $g$ with the matrix $g_{jk}$

$$e_j e_k + e_k e_j = -2 g_{jk} 1.$$  

(3)

For a diagonal matrix $g$ with the signature $(m,n-m)$ Eq. (3) defines the Clifford algebra $\mathcal{C}(m,n-m)$, e.g., $\mathcal{C}(n) = \mathcal{C}(0,n)$.

The Clifford conjugation may be defined on the generators by the equation

$$\bar{e}_j e_j = 1$$  

(4a)

and extended on the whole algebra by the property

$$\bar{ab} = \bar{b} \bar{a}.$$  

(4b)

The complex Clifford algebra $\mathcal{C}(n, \mathbb{C})$ also is defined by Eq. (2), but in such a case any signatures are equivalent due to the possibility of substitutions $e_k \mapsto ie_k$.

Let us consider the $n$-dimensional subspace $V$ of the Clifford algebra $\mathcal{C}(n)$ with elements

$$v \in V, \quad v = \sum_{k=1}^{n} v_k e_k.$$  

(5)

By the definition [10, 11, 12] the Spin$(n)$ group is generated by all possible products with even number of $v \in V$ normalized by the condition

$$\sum_{k=1}^{n} v_k^2 = 1.$$  

(6)

Real Clifford algebras $\mathcal{C}(n, \mathbb{C})$ also may be used for an analogous definition of the Spin$(n)$ group, yet $\mathcal{C}(n, \mathbb{C})$ and $\mathcal{C}(n)$ are different algebras [10, 11]. The property is important for some constructions below.

A basic property of the Spin groups: Let $S \in$ Spin$(n)$, $v \in V$ Eq. (5) and

$$v' = SvS^{-1},$$  

(7)

then $v' \in V$ and

$$v' = \sum_{k=1}^{n} v'_k e_k, \quad v'_k = \sum_{j=1}^{n} R^{kj}_S v_j, \quad R_S \in SO(n).$$  

(8)

Any rotation $R \in SO(n)$ may be represented in such a way. Due to Eq. (7) both $S$ and $-S$ correspond to the same $R_S \in SO(n)$ and so, it is $2 \rightarrow 1$ covering homomorphism.

Thus, dimensions of Spin$(n)$ and SO$(n)$ are the same, $n(n-1)/2$. On the other hand, any product with even number of $e_k$ belongs to Spin$(n)$. The linear span of such products has dimension $2^{n-1}$ and corresponds to an even subalgebra of $\mathcal{C}(n)$ denoted further as $\mathcal{C}^0(n)$.

Due to such a complicated structure of the Spin groups it may be more convenient sometimes to use Lie algebras spin$(n)$. The elements of spin$(n)$ are linear combinations of products $e_j e_k$ equipped with the Lie bracket operation

$$[a, b] = ab - ba.$$  

(9)

The spin$(n)$ is isomorphic with the Lie algebra so$(n)$ of the orthogonal group SO$(n)$ [10, 12].
2.2 Spin(3) group

Clifford algebras with three generators may be considered as building blocks in many constructions used in this work. The Spin(3) group may be constructed both from $\mathcal{O}(3)$ and $\mathcal{C}^+(3)$, but the algebras are not equivalent. In both cases an element of the Spin group is represented as

$$r_0 \mathbf{1} + r_1 \epsilon_{23} + r_2 \epsilon_{31} + r_3 \epsilon_{12}, \quad r_k \in \mathbb{R}, \quad \sum_{k=0}^{3} r_k^2 = 1,$$

(10)

where $\epsilon_{jk} \equiv \epsilon_j \epsilon_k$. Both for $\mathcal{O}(3)$ and $\mathcal{C}^+(3)$

$$(\epsilon_{jk})^2 = \epsilon_j \epsilon_k \epsilon_j \epsilon_k = -\epsilon_j \epsilon_k \epsilon_k \epsilon_j = -\epsilon_j^2 \epsilon_k^2 = -1.$$

(11)

Let us recall that the quaternions $\mathbb{H}$ [10, 11, 12, 13, 14] are defined by relations

$$q = q_0 \mathbf{1} + q_1 i + q_2 j + q_3 k, \quad q_k \in \mathbb{R},$$

$$ij = -ji = k, \quad ti = -it = j, \quad jk = -kj = i, \quad i^2 = j^2 = k^2 = -1.$$

(12)

Note: The quaternions formally correspond to an universal Clifford algebra $\mathcal{C}^0(3)$ with only two generators $i$, $j$ and $k = ij$. Omitting the claim about universality they are treated sometimes as a Clifford algebra with three quaternionic units $i$, $j$, $k$ satisfying Eq. (2).

The multiplicative norm of a quaternion $|q|$ is defined as

$$|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

(13)

The group of quaternions with the unit norm is isomorphic with $SU(2)$ [10, 11, 12, 13, 14].

The substitutions

$$i = \epsilon_{23}, \quad j = \epsilon_{31}, \quad k = \epsilon_{12}$$

(14)

in Eq. (10) are in agreement with properties Eq. (12) of quaternions. They relate to the quaternionic representation of group $SU(2) \simeq Spin(3)$ and to the isomorphisms $\mathcal{O}_0(3) \simeq \mathcal{C}^0(3) \simeq \mathbb{H} \simeq \mathcal{O}(2)$.

A representation of the algebra $\mathcal{O}(3)$ may be described by the double quaternions $2\mathbb{H}$ with generators expressed as the matrices

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad e_3 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

(15)

The definition below is useful for description of $\mathcal{C}^+(3)$.

**Definition 1.** The decomplexification [13] (realification) $\mathcal{A}_R$ of the $n$-dimensional complex algebra $\mathcal{A}$ with the basis $a_k, \quad k = 1, \ldots, n$ is the $2n$-dimensional real algebra with the basis $b_k = a_k, \quad b_{k+n} = ia_k$.

The eight-dimensional real Clifford algebra $\mathcal{C}^+(3)$ is isomorphic with the decomplexification of the Pauli algebra $M(2, \mathbb{C})$ of complex $2 \times 2$ matrices, $\mathcal{C}^+(3) \simeq M(2, \mathbb{C})_R$. It may be considered in such a way due to the commutative element

$$\iota = \epsilon_{123} \equiv \epsilon_1 \epsilon_2 \epsilon_3, \quad \iota^2 = -1, \quad \iota e_k = \epsilon_k \iota$$

(16)
corresponding to an imaginary unit. A similar method does not work with \( \mathcal{C}\ell(3) \) where \((e_1 e_2 e_3)^2 = 1\).

It may be written

\[
\forall z \in \mathcal{C}\ell_+(3), \quad z = x + i y, \quad x, y \in \mathcal{C}\ell^0(3) \simeq \mathbb{H}.
\]  

(17)

Eq. (17) illustrates isomorphisms

\[
M(2, \mathbb{C})_R \simeq \mathcal{C}\ell_+(3) \simeq \mathbb{C}\ell \otimes \mathbb{H}.
\]  

(18)

In such a complex representation the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]  

(19)

may be used as generators of the algebra \( \mathcal{C}\ell_+(3) \)

\[
e_1 = \sigma_1, \quad e_2 = \sigma_2, \quad e_3 = \sigma_3.
\]  

(20)

Finally, elements of \( \mathcal{C}\ell^0_+(3) \simeq \mathbb{H} \) used for construction of \( \text{Spin}(3) \) Eq. (10) may be written as

\[
s_1 \equiv e_{23} = i\sigma_1, \quad s_2 \equiv e_{31} = i\sigma_2, \quad s_3 \equiv e_{12} = i\sigma_3.
\]  

(21)

The elements Eq. (21) are basis of the Lie algebra \( \text{spin}(3) \) and correspond to the representation of \( \text{spin}(3) \simeq \text{su}(2) \) via anti-Hermitian matrices. It may be checked that the Hermitian conjugate of the matrix in such a representation is in agreement with the Clifford conjugation Eq. (4), \( \bar{e}_{jk} = -e_{jk} \).

\textbf{Note:} For construction of the group \( SU(n) \) in physical applications Hermitian matrices \( H^\dagger = H \) are often used together with the equation \( U = e^{iH} \) for \( U \in SU(n) \) group and the commutator \( i[A, B] \) instead of Eq. (9).

Sometimes it may produce some difficulties, \( e.g. \), in construction of \( SU(2) \) with quaternions defined by relations Eq. (12) without an element representing a commutative imaginary unit.

So, the more common description with the relation between Lie algebras and Lie groups expressed as \( G = e^A \) may be appropriate. In such a definition the Lie algebra \( su(n) \) is represented by anti-Hermitian matrices \( A^\dagger = -A \) with Lie brackets Eq. (9) and the relation \( U = e^A \) for \( U \in SU(n) \).

\subsection*{2.3 Different tensor products}

The consideration of \( \mathcal{C}\ell_+(3) \) as the Pauli algebra of 2 \( \times \) 2 complex matrices may be used for the description of quantum gates [15].

The group of quantum \( n \)-qubit gates \( SU(2^n) \) may be expressed using \( 2^n \times 2^n \) complex matrices \( M(2^n, \mathbb{C}) \) represented in turn as the complex tensor product

\[
M(2^n, \mathbb{C}) = M(2, \mathbb{C}) \otimes \cdots \otimes M(2, \mathbb{C}).
\]  

(22)
Due to Eq. (18) an analogue of Eq. (22) for the real tensor product may be written
\[ M(2^n, \mathbb{C}) \cong \mathbb{C}_\mathbb{R} \otimes \mathbb{H} \otimes \cdots \otimes \mathbb{H} = \mathbb{C}_\mathbb{R} \otimes \mathbb{H}^\otimes n. \] (23)

Some subtleties may exist here, because a tensor product of 8D real algebras \( \mathcal{O}_+(3) \) has the real dimension \( 8^n = 2^{3n} \), but \( M(2^n, \mathbb{C}) \) has dimension \( 2^{2n} \) as the complex algebra and \( 2^{2n+1} \) as the real one.

Let us recollect a method to get rid of the extra dimensions [15]. Each term in the real tensor product of \( \mathcal{O}_+(3) \) has its own imaginary unit
\[ \iota_k = 1 \otimes \cdots \otimes 1 \otimes \iota \otimes 1 \otimes \cdots \otimes 1, \] (24)
where \( \iota \) is defined by Eq. (16). It is possible to introduce a projector
\[ P_\iota = \frac{1}{2n-1} \prod_{k=2}^{n} (1 - \iota_1 \iota_k). \] (25)

It is called the correlator in [15] with the properties
\[ \iota_k P_\iota = \iota_1 P_\iota = \iota_k \iota_j P_\iota = -P_\iota, \] (26)
so, all the complex units may be ‘aggregated’ by the projector.

In fact, the method is very general and may be used for tensor products of real spaces with complex structures of any dimension.

However, it may not be applied to the description of the structure of \( \text{Spin}(3n) \) groups and Clifford algebras \( \mathcal{O}_+(3n) \) discussed in this section. Indeed, the method works because all \( \iota_k \) are commutative \( \iota_k \iota_j = \iota_j \iota_k \) due to the definition Eq. (24), but for the description of \( \text{Spin}(3n) \) circuits instead of \( \iota_k \) should be used
\[ \iota^{(k)} = \iota^{(k)}_{123} = \iota^{(k)}_1 \iota^{(k)}_2 \iota^{(k)}_3, \] (27)
where \( \iota^{(k)}_\nu \) are defined by Eq. (1). Unlike \( \iota_k \), elements \( \iota^{(k)} \) anticommute: \( \iota^{(k)} \iota^{(j)} = -\iota^{(j)} \iota^{(k)} \), \( k \neq j \) and a product such as Eq. (25) is not a projector with the desired properties Eq. (26).

Despite this problem, the Clifford algebra \( \mathcal{O}_+(3n) \) as a real linear space may be constructed as a tensor product of \( n \) copies of \( \mathcal{O}_+(3) \). It follows from the general property of Clifford algebras [10, 12]. The difference with the usual tensor product is the definition of multiplication discussed further.

**Definition 2.** An algebra \( \mathcal{A} \) is called \( \mathbb{Z}_2 \)-graded, if it may be decomposed into the direct sum of even and odd linear subspaces \( \mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1 \) with the property
\[ a \in \mathcal{A}^j, b \in \mathcal{A}^k \implies ab \in \mathcal{A}^{(j+k) \mod 2}. \] (28)
Here only an even subspace \( \mathcal{A}^0 \) is a subalgebra.

**Definition 3.** For two \( \mathbb{Z}_2 \)-graded algebras \( \mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1 \) and \( \mathcal{B} = \mathcal{B}^0 \oplus \mathcal{B}^1 \) the \( \mathbb{Z}_2 \)-graded tensor product is defined
\[ (a \otimes b)(a' \otimes b') = (-1)^{jk}(aa') \otimes (bb') \] (29)
if \( b \in \mathcal{B}^j \) and \( a' \in \mathcal{A}^k \). The Eq. (29) may be extended on arbitrary elements of the algebras due to distributivity. It is also called the skew tensor product and denoted as \( \hat{\otimes} \) [12].
Clifford algebras are \( \mathbb{Z}_2 \)-graded due to the decomposition into subspaces generated by products with odd and even number of generators and so the skew tensor product may be defined as well.

### 2.4 Structure of \( C\ell_+(3n) \)

The skew tensor product is important for the description of Clifford algebras, because \( C\ell(n + m) \simeq C\ell(n) \otimes C\ell(m) \), \( C\ell_+(n + m) \simeq C\ell_+(n) \otimes C\ell_+(m) \) and for complex case \( C\ell(n + m, \mathbb{C}) \simeq C\ell(n, \mathbb{C}) \otimes C\ell(m, \mathbb{C}) \) \cite{10, 12}. The repetition of the skew tensor product may be used for construction of \( C\ell_+(3n) \) from \( C\ell_+(3) \)

\[
C\ell_+(3n) \simeq C\ell_+(3) \otimes \cdots \otimes C\ell_+(3). \tag{30}
\]

A similar equation could be also written for \( C\ell(3n) \) and \( C\ell(3) \). The analogue of Eq. (22) is also relevant

\[
C\ell(2n, \mathbb{C}) \simeq C\ell(2, \mathbb{C}) \otimes \cdots \otimes C\ell(2, \mathbb{C}). \tag{31}
\]

For both \( C\ell(3n) \) and \( C\ell_+(3n) \) generators \( c_{\nu}^{(l)} \) Eq. (1) may be represented as

\[
c_{\nu}^{(l)} = 1 \otimes \cdots \otimes 1 \otimes c_{\nu} \otimes 1 \otimes \cdots \otimes 1 \tag{32}
\]

Instead of Eq. (27) for \( \iota^{(k)} \) may be used

\[
\iota^{(k)} = 1 \otimes \cdots \otimes 1 \otimes \iota \otimes 1 \otimes \cdots \otimes 1 \tag{33}
\]

It was already mentioned that the \( Spin(3n) \) group used to be represented via \( C\ell_0^n(3n) \), but an equivalent construction with \( C\ell_3^n(3n) \) may be more desirable here due to relation of \( C\ell_+(3) \) with the Pauli algebra Eq. (20).

The representation of \( Spin(3n) \) with the skew tensor product justifies the idea of ‘\( Spin(3n) \) circuits’, because it has more direct analogy with usual quantum circuits than a rather formal \( 3 \times n \) subdivision Eq. (1) in the introduction, Sec. 1.

Any quantum circuit with \( n \) qubits corresponds to an element of \( SU(2^n) \) or some \( 2^n \times 2^n \) complex matrix from \( M(2^n, \mathbb{C}) \) represented as the complex tensor product of \( n \) Pauli algebras \( M(2, \mathbb{C}) \) Eq. (22).

Due to the isomorphism \cite{10, 11} \( M(2^n, \mathbb{C}) \simeq C\ell(2n, \mathbb{C}) \) and Eq. (31) the complex \( \mathbb{Z}_2 \)-graded tensor product may be used as well, but the situation is more difficult for real algebras.

It was mentioned \( M(2, \mathbb{C})_{\mathbb{R}} \simeq C\ell_+(3) \) Eq. (18), but the usual tensor product of \( n \) copies of \( C\ell_+(3) \) has the real dimension \( 2^{3n} \), and it may be ‘aggregated’ into \( M(2^n, \mathbb{C})_{\mathbb{R}} \) Eq. (23) with the application of \( \text{the correlator} \ P \), Eq. (25).

The \( C\ell_+(3n) \) may be represented as the skew tensor product of \( n \) algebras \( C\ell_+(3) \) Eq. (30) with the real dimension \( 2^{3n} \), but the structure is more complicated, because the ‘imaginary units’ \( \iota^{(k)} \) Eqs. (27, 33) with different \( k \) anticommute.

Now Eq. (17) and Eq. (18) should be applied to the skew tensor product Eq. (30) and any \( \iota \in C\ell_+(3n) \) may be expressed as a composition

\[
\iota = \sum_K c_K \iota_K, \tag{34}
\]
where \( c_K \) are products of \( \iota^{(k)} \), generating \( 2^n \)-dimensional (sub)algebra isomorphic with \( \mathcal{C}(n) \) and \( h_K \) are elements of \( 2^{2n} \)-dimensional (sub)algebra

\[
\frac{\mathcal{O}^0_+ (3) \otimes \cdots \otimes \mathcal{O}^0_+ (3)}{2^n} \simeq \mathbb{H}^\otimes n.
\] (35)

In Eq. (35) a special symbol for the skew tensor product is redundant, because for even subalgebras it coincides with the usual tensor product due to Eq. (29).

All \( c_K \) in Eq. (34) are products of \( \iota^{(k)} \) and commute with \( h_K \) from Eq. (35). So, for any \( l, l' \in \mathcal{C}_+ (3n) \)

\[
\mathcal{U}' = \sum_K c_K h_K \sum_J c'_J h'_J = \sum_{K, J} (c_K c'_J)(h_K h'_J).
\] (36)

Due to Eq. (36) the decomposition Eq. (34) satisfies the formal definition of the tensor product of algebras \([12, 14]\) and so

\[
\mathcal{O}^0_+ (3n) \simeq \mathcal{O}(n) \otimes \mathbb{H}^\otimes n.
\] (37)

The direct consequence of the same constructions is

\[
\mathcal{O}^0_+ (3n) \simeq \mathcal{O}^0 (n) \otimes \mathbb{H}^\otimes n,
\] (38)

because the term \( \mathbb{H}^\otimes n \) Eq. (35) belongs to the even subalgebra and so the number of multipliers \( \iota^{(k)} = \iota_{123}^{(k)} \) should be also even in \( \mathcal{O}^0_+ (3n) \).

The quantum circuits with \( n \) qubits are described by the group \( SU (2^n) \subset M (2^n, \mathbb{C}) \) and the resembling expression Eq. (23) for \( M (2^n, \mathbb{C})_R \) is useful for the comparison with the structure of \( \mathcal{O}_+ (3n) \).

### 2.5 Spin(6) group

Due to Eq. (38)

\[
\mathcal{O}^0_+ (6) \simeq \mathcal{O}^0 (2) \otimes \mathbb{H}^\otimes 2 \simeq \mathbb{C}_R \otimes \mathbb{H} \otimes \mathbb{H} \simeq M (4, \mathbb{C})_R,
\] (39)

where \( \mathcal{O}^0 (2) \simeq \mathbb{C}_R \) with respect to the ‘imaginary unit’

\[
\mathbf{i} = \iota^{(12)} \equiv \iota^{(1)} \iota^{(2)}, \quad \mathbf{i}^2 = -1.
\] (40)

In fact, \( \mathbf{i} \) is the product of all six generators of the Clifford algebra and it commutes with elements of the even subalgebra.

Let us illustrate the isomorphism \( Spin(6) \simeq SU (4) \) already mentioned in the introduction, Sec. 1. It is convenient to use the Lie algebra \( spin(6) \) for the description of the structure of the groups. The basis of the Lie algebra \( spin(n) \) — are products of pairs of generators \([10]\).

The basis of \( spin(6) \) includes fifteen such pairs. The six products \( \iota^{(l)}_{jk}, 1 < j < k < 3, \ l = 1, 2 \) are corresponding to the couple of different \( spin(3) \) subalgebras. The structure of \( Spin(3) \) was already discussed in Sec. 2.2 and with the notation used there in Eq. (21) the six elements may be rewritten as \( s^{(l)}_j, \ j = 1, 2, 3, \ l = 1, 2 \).

Other nine products \( \iota^{(1)}_{jk} \iota^{(2)}_k, \ j, k = 1, 2, 3 \) may be rewritten as \( \mathbf{i} s^{(1)}_j s^{(2)}_k, \ j, k = 1, 2, 3 \), where an ‘imaginary unit’ \( \iota = \iota^{(12)} \) was already introduced above Eq. (40).

Let us show \( spin(6) \simeq su(4) \). The basis of the Lie algebra \( su(4) \) — are anti-Hermitian \( 4 \times 4 \) matrices (see \emph{Note} in Sec. 2.2). Let us use for such a purpose the representation with
the tensor products of Pauli matrices. The isomorphism may be directly shown with the map from the basis of $spin(6)$ into anti-Hermitian matrices

$$s_j^{(1)} \mapsto i\sigma_j \otimes 1, \quad s_j^{(2)} \mapsto 1 \otimes i\sigma_j \quad (j = 1, 2, 3),$$

$$i\sigma_j^{(1)} i\sigma_k^{(2)} \mapsto i\sigma_j \otimes \sigma_k \quad (j, k = 1, 2, 3).$$

Eq. (41a) corresponds to Eq. (21) for two $spin(3)$ subalgebras and Eq. (41b) represents an ‘entanglement’.

### 2.6 $Spin(3n)$ circuits

Let us consider decomposition of the $Spin(3n)$ group. The basis of the Lie algebra $spin(3n)$ may be represented as

$$e_j^{(l)} e_k^{(m)} = \epsilon_{3l-1-j}^{(m)} e_{3m+1+k}.$$ 

An element of the Lie group $Spin(3n)$ may be expressed as an exponent of the linear combination of Eq. (42) or composed from a product of

$$U_z^{(lm)} = \exp(\epsilon_j^{(l)} \epsilon_k^{(m)}).$$

Such approach is well known in the Lie-algebraic description of (non)universal sets of quantum gates [1, 16, 17, 18].

For fixed $l \neq m$ Eq. (43) describes elements from a subgroup of two-line gates $G^{(lm)} \simeq Spin(6)$. The elements Eq. (43) with $l = m$ are from a subgroup of one-line gates $G^{(l)} \simeq Spin(3)$. The one- and two-line gates are enough for construction of the group $Spin(3n)$, because the basis of the Lie algebra $spin(3n)$ includes only the terms such as Eq. (42).

A more general analogue of $n$-line gates for $n > 2$ for $Cl_+ (3n)$ would include some subgroup of invertible elements of the algebra. In fact, the group of usual quantum $n$-qubit gates $SU(2^n)$ is isomorphic with a subgroup of such a group, because $M(2^n, \mathbb{C})$ may be considered as a subalgebra of $Cl_+(3n)$ due to Eq. (23) together with Eq. (37) and an inclusion $\mathbb{C} \mathbb{R} = Cl(1) \subset Cl(n)$.

On the other hand, any $n$-line gate composed from one- and two-line gates due to the structure of $Cl_+(3n)$ is from a subgroup isomorphic with $Spin(3n)$. Let us compare dimensions of the groups:

$$\dim SU(2^n) = 4^n - 1, \quad \dim Spin(3n) = \frac{3n(3n-1)}{2}.$$ 

The dimensions are not equal for $n > 2$ and due to Eq. (44) for $Spin(3n)$ the growth is quadratic with respect to $n$, versus the exponential one for quantum circuits. The comparison of dimensions for $n = 1, \ldots, 5$ is represented in the table below.

| $n$ | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| $\dim SU(2^n)$ | 3 | 15 | 63 | 255 | 1023 |
| $\dim Spin(3n)$ | 3 | 15 | 36 | 66 | 105 |

The case $n = 3$ may be considered for the illustration of a difference for the compositions of two-line gates. It is convenient to use Lie algebras, because the structure of products of elements of a Lie group is clear from the bracket operation Eq. (9) [1, 16, 17].

Let us compare $spin(9)$ and the Lie algebra $su(8)$ of the Lie group $SU(2^3)$ of quantum three-qubit gates. The Lie algebra $su(8)$ may be again represented using tensor products with Pauli matrices to comparison with analogues of Eq. (41).
Let us use notation
\[ \sigma_j:1 = \bigotimes_{i=1}^{l} \mathbf{1} \otimes \sigma_j \otimes \bigotimes_{n-l} \mathbf{1} \] (46)

Eq. (41a) describes one-line gates and only consideration of Eq. (41b) is not trivial. So, it is necessary to consider elements of \( su(8) \) such as
\[ i\sigma_{j;1}\sigma_{k;2}, \quad i\sigma_{j;2}\sigma_{k;3} \quad (j, k, j', k' = 1, 2, 3). \] (47)

An analogue of Eq. (47) for inclusion of two copies of \( \text{spin}(6) \) into \( \text{spin}(9) \) may be written as
\[ \tilde{\ell}\tilde{\sigma}_j^{(1)} \tilde{\sigma}_k^{(2)}, \quad \tilde{\ell}\tilde{\sigma}_j^{(2)} \tilde{\sigma}_k^{(3)} \quad (j, k, j', k' = 1, 2, 3), \] (48)
where \( \tilde{\ell} \) is introduced below in Eq. (49). Despite of the isomorphism \( su(4) \simeq \text{spin}(6) \) expressed by Eq. (41), the essential difference between Eq. (48) and Eq. (47) is the pair of anticommuting ‘imaginary units’
\[ \tilde{\ell} = \ell^{(12)}, \quad \tilde{\ell}' = \ell^{(23)} \equiv \ell^{(2)}\ell^{(3)}, \quad \tilde{\ell}\tilde{\ell}' = -\tilde{\ell}'\tilde{\ell}. \] (49)

Structures of Lie brackets Eq. (9) becomes different due to the anticommuting elements.

For \( su(8) \) the brackets of elements Eq. (47) are zero iff \( k = j' \), but for \( k \neq j' \) the commutator generates an element of third order \( i\sigma_{j;1}\sigma_{1;2}\sigma_{k;3} \), where \( k \neq l \neq j' \). Conversely, for \( \text{spin}(9) \) the brackets of elements Eq. (48) are zero iff \( k \neq j' \), but for \( k = j' \) the commutator is an element of the second order \( \ell^{(13)}\ell^{(1)}\ell^{(3)} \).

Eq. (47) and relevant Eq. (48) correspond to the consideration of two-line gates from \( G^{(12)} \) and \( G^{(23)} \), but taking into account \( G^{(13)} \) or arbitrary triple of indexes \( G^{(l,m,n)} \) may be performed in the similar way.

## 3 Matrix tensor product representation

### 3.1 Clifford algebras \( \mathcal{C}(2n, \mathbb{C}) \) and \( \mathcal{C}(4n, \mathbb{C}) \)

The real Clifford algebra was revisited above in the Sec. 2. They have rather irregular structures and may be isomorphic with algebras of real, complex, quaternionic matrices and also with doubles of such algebras [11, 12], e.g., see Eq. (15) for \( \mathcal{C}(3) \simeq \mathbb{H}^2 \). It may be convenient to include them as subalgebras into complex Clifford algebras with even dimensions \( \mathcal{C}(2n, \mathbb{C}) \) isomorphic with algebra \( M(2^n, \mathbb{C}) \) of \( 2^n \times 2^n \) complex matrices [10, 11].

Generators of Clifford algebras \( \mathcal{C}(2n, \mathbb{C}) \simeq M(2^n, \mathbb{C}) \) in the Jordan-Wigner representation may be expressed as the tensor products of the Pauli matrices [10, 20, 21]

\[ e_{2k-1} = i\sigma_1 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \quad (50a) \]
\[ e_{2k} = i\sigma_1 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}, \quad (50b) \]

where \( k = 1, \ldots, n \).

For construction of \( \mathcal{C}(4n, \mathbb{C}) \) may be used an analogue of Eq. (50) with Dirac 4×4 matrices

\[ \gamma^0 = -i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma = -i \begin{bmatrix} 0 & \sigma \\ \sigma & 0 \end{bmatrix}, \quad \gamma_5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (51) \]
where $\sigma, \gamma$ denote $\sigma_j, \gamma^j$ with $j = 1, 2, 3$ [19].

Let us also rewrite Eq. (51) with tensor products of Pauli matrices

$$\gamma^0 = -i\sigma_1 \otimes 1, \quad \gamma = \sigma_2 \otimes \sigma, \quad \gamma_5 = \sigma_3 \otimes 1.$$  \hfill (52)

Let us consider

$$e_j^{[k]} = \gamma_5 \otimes \cdots \otimes \gamma_5 \otimes \gamma^j \otimes 1 \otimes \cdots \otimes 1,$$ \hfill (53)

where $k = 1, \ldots, n, j = 0, \ldots, 3$.

The elements Eq. (53) are anticommutative and they define a Clifford algebra with $(e_0^{[k]})^2 = -1$ and $(e_j^{[k]})^2 = 1, j \neq 0$. The generators $e_0^{[k]}$ and $ie_j^{[k]} (j = 1, 2, 3)$ may be used for construction of $\Cl(4n, \mathbb{C}) \simeq M(4^n, \mathbb{C})$.

Eq. (53) may be also rewritten using decompositions Eq. (52) with Pauli matrices and notation Eq. (46)

$$e_0^{[k]} = -i(\prod_{l=1}^{k-1} \sigma_{3;2l-1}) \sigma_{1;2k-1}$$ \hfill (54a)

$$e_j^{[k]} = (\prod_{l=1}^{k-1} \sigma_{3;2l-1}) \sigma_{2;2k-1}\sigma_{j;2k} \quad (j = 1, 2, 3).$$ \hfill (54b)

The generators Eq. (53) corresponds to the universal Clifford algebra $\Cl(4n, \mathbb{C})$ with dimension $2^{4n}$, because they generate the complete basis of the algebra $M(4^n, \mathbb{C})$ of $4^n \times 4^n$ complex matrices. Indeed, the Dirac matrices $\gamma^j, j = 0, \ldots, 3$ together with products may be used as the basis of $M(4, \mathbb{C})$ and so the basis of $M(4^n, \mathbb{C})$ may be constructed from $2^{4n}$ different products of elements

$$\gamma^{j;k} = \underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes \gamma^j \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-k},$$ \hfill (55)

with $k = 1, \ldots, n$ and $j = 0, \ldots, 3$. Any element $\gamma^{j;k}$ may be expressed as a product of $e_j^{[k]}$

$$\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes \gamma_5 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-k} = -i e_0^{[k]} e_1^{[k]} e_2^{[k]} e_3^{[k]}, \quad \gamma^{j;k} = e_j^{[k]} \prod_{l=1}^{k-1} (\sigma_{3;2l-1} - i e_0^{[l]} e_1^{[l]} e_2^{[l]} e_3^{[l]}).$$ \hfill (56)

and so, the basis of $M(2^{2n}, \mathbb{C}) \simeq \Cl(4n, \mathbb{C})$ is also generated by $e_j^{[k]}$ Eq. (53).

### 3.2 Clifford algebra $\Cl_+(3n)$

**Definition 4.** Let us introduce $\Cl_+(3n)$ as subalgebra of $\Cl(4n, \mathbb{C}) \simeq M(4^n, \mathbb{C})$ with $3n$ generators $e_j^{[k]} (j = 1, 2, 3$ and $k = 1, \ldots, n$) represented by Eq. (53) or Eq. (54b).

An analogue of Eq. (27) may be written using Eq. (54b)

$$e^{[k]} = e_1^{[k]} e_2^{[k]} e_3^{[k]} = i(\prod_{l=1}^{k-1} \sigma_{3;2l-1}) \sigma_{2;2k-1}.$$ \hfill (57)
Let us note, that all Pauli matrices in Eq. (57) have odd positions in the decomposition Eq. (46). On the other hand, Eq. (54b) consists of $\mathcal{V}^{[k]}$ multiplied on a Pauli matrix in the even position

$$e_j^{[k]} = -i\mathcal{V}^{[k]}\sigma_{j;2k} \quad (j = 1, 2, 3).$$

Let us consider complex subalgebras $M_o$ and $M_e$ of $M(4^n, \mathbb{C})$ generated by products of elements Eq. (46) with Pauli matrices in odd and even positions respectively. Both subalgebras are isomorphic with $M(2^n, \mathbb{C})$ represented as the tensor products with $n$ complex $2 \times 2$ matrices.

Elements $\mathcal{V}^{[k]}$ are anticommutative and generate a subalgebra of $M_o$ isomorphic with $\mathfrak{C}(n)$. With respects to isomorphisms $M_o \simeq M(2^n, \mathbb{C}) \simeq \mathfrak{C}(2n, \mathbb{C})$, elements $\mathcal{V}^{[k]}$ Eq. (57) correspond to $n$ generators Eq. (50b).

Such a decomposition of elements from $\mathfrak{C}(3n)$ on $\mathfrak{C}(n) \subset M_o$ and $M_e \simeq M(2^n, \mathbb{C})$ is a complex analogue of Eq. (37) from Sec. 2.4.

### 3.3 Spin(3) and Spin(6) groups

The analogues of equations for $\text{Spin}(3)$ groups from Sec. 2.2 with generators $e_j \equiv e_j^{[1]}$ from definition 4 are rather straightforward. The products of two generators $e_{jk} \equiv e_j e_k$ defined in Eq. (21) may be represented using $4 \times 4$ matrices Eq. (51) and tensor products Eq. (52)

$$g_l = i e_l^{[1]} = i 1 \otimes \sigma_i = i\sigma_{i;2} \quad (l = 1, 2, 3).$$

Let us now represent the $\text{Spin}(6)$ group using the Clifford algebra $\mathfrak{C}(6)$. In agreement with definition 4

$$\mathfrak{C}(6) \subset \mathfrak{C}(8, \mathbb{C}) \simeq M(16, \mathbb{C})$$

and $e_j^{[k]}, k = 1, 2, j = 1, 2, 3$ may be written as tensor products of two $4 \times 4$ matrices Eq. (53)

$$e_j^{[1]} = \gamma^j \otimes 1, \quad e_j^{[2]} = \gamma_5 \otimes \gamma^j$$

and rewritten with Pauli matrices using Eq. (54b)

$$e_j^{[1]} = \sigma_{2;1} \sigma_{j;2}, \quad e_j^{[2]} = \sigma_{3;1} \sigma_{2;3} \sigma_{j;4}.$$

Similarly with Sec. 2.5 here is again convenient to consider the Lie algebra spin(6) with the basis defined by six products $e_{jk}^{[l]} \equiv e_{j}^{[l]} e_{k}^{[l]}$, $1 < j < k < 3$, $l = 1, 2$ together with nine products $e_j^{[1]} e_k^{[2]}, j, k = 1, 2, 3$.

The basis may be written down using definition 4 and Eq. (46). The first six products describe two spin(3) subalgebras similarly with Eq. (59)

$$g_j^{[1]} = i\sigma_{j;2}, \quad g_j^{[2]} = i\sigma_{j;4},$$

where $j = 1, 2, 3$. Other nine products are

$$e_j^{[1]} e_k^{[2]} = i\sigma_{1;1} \sigma_{j;2} \sigma_{2;3} \sigma_{k;4} = -i\sigma_{1;1} \sigma_{2;3} g_j^{[1]} g_k^{[2]},$$

where $j, k = 1, 2, 3$. The structure of Eq. (60) resembles Eq. (41) used in Sec. 2.5 to illustrate the isomorphism $\text{Spin}(6) \simeq SU(4)$. It is also revised below in the Sec. 4.2.
3.4 Spin(3n) group

The group $\text{Spin}(3n)$ may be constructed with the method already discussed in Sec. 2.6. The Lie algebra $\text{spin}(3n)$ is used for construction of the group. The basis of $\text{spin}(3n)$ is $\epsilon_j^{[l]} \epsilon_k^{[m]}$, $j, k = 1, 2, 3$.

The matrices $\epsilon_j^{[l]}$, $j = 1, 2, 3$ in representations Eq. (53) or Eq. (54b) are Hermitian, but products of such elements are anti-Hermitian matrices,

$$ (\epsilon_j^{[l]} \epsilon_k^{[m]})^\dagger = \epsilon_k^{[m] \dagger} \epsilon_j^{[l] \dagger} = \epsilon_k^{[m]} \epsilon_j^{[l]} = -\epsilon_j^{[l]} \epsilon_k^{[m]}.$$

Any element of $\text{Spin}(3n)$ is the exponent of the linear combination of $\epsilon_j^{[l]} \epsilon_k^{[m]}$. Such an exponent of an anti-Hermitian matrix is unitary.

It was already mentioned in Sec. 2.6, any composition of one- and two -line gates $U_{\epsilon}^{[lm]} = \exp(\varepsilon \epsilon_j^{[l]} \epsilon_k^{[m]})$. is an element of the group $\text{Spin}(3n)$. It may be more common for physical applications to introduce Hamiltonians

$$ H_{jk}^{[lm]} = i \epsilon_j^{[l]} \epsilon_k^{[m]} $$

and rewrite Eq. (61) as

$$ U_{\epsilon}^{[lm]} = \exp(-i \tau H_{jk}^{[lm]}).$$

4 Quantum circuits representation

4.1 Quantum circuits model of Spin(3n)

The quantum circuits framework for $\mathcal{C}^+(3n)$ and $\text{Spin}(3n)$ may be derived from the matrix tensor product representation discussed in the Sec. 3. Such a circuit model may be used both for the definition of gates and states.

The construction of a quantum circuit with $2n$ qubits for modeling of the group $\text{Spin}(3n)$ using $\mathcal{C}(4^n, \mathbb{C}) \cong M(4^n, \mathbb{C})$ is rather straightforward. It was already shown in Sec. 3.4 that all elements of the group $\text{Spin}(3n)$ are unitary matrices and so it is a subgroup of the group $SU(2^{2n})$ of quantum gates with $2n$ qubits.

The decomposition on $M_o$ and $M_e$ mentioned in Sec. 3.2 corresponds to partitions with $n$ qubits in odd and even position respectively. It may be convenient sometimes to reorder qubits into even and odd subsystems using rearrangement

$$(1, 2, \ldots, 2n) \rightarrow (2, 4, \ldots, 2n), (1, 3, \ldots, 2n - 1).$$

4.2 Isomorphism of $\text{Spin}(6)$ and $SU(4)$

The structure of the $\text{Spin}(6)$ group may be modeled using a quantum circuit with four qubits. The element of $\text{Spin}(6)$ may be represented as the exponent of the linear combination of matrices Eq. (60). Let us write Eq. (60) for Hamiltonians of quantum gates Eq. (62)

$$ H_j^{[1]} = 1 \otimes \sigma_j \otimes 1 \otimes 1 = \sigma_j; 2, \quad H_j^{[2]} = 1 \otimes 1 \otimes 1 \otimes \sigma_j = \sigma_j; 4,$$

$$ H_{jk}^{[12]} = \sigma_1 \otimes \sigma_j \otimes \sigma_2 \otimes \sigma_k = \sigma_1; 1 \sigma_j; 2 \sigma_2; 3 \sigma_k; 4$$

(65a)

(65b)
and rewrite that for reordering of qubits into even and odd subsystems Eq. (64)

\[
H_{j1}^{[1]} = \sigma_j \otimes 1 \otimes 1 \otimes 1, \quad H_{j2}^{[2]} = 1 \otimes \sigma_j \otimes 1 \otimes 1, \quad (66a)
\]

\[
H_{jk}^{[12]} = \sigma_j \otimes \sigma_k \otimes \sigma_1 \otimes \sigma_2. \quad (66b)
\]

An arbitrary Hamiltonian \( H \) representing \( \text{Spin}(6) \) is the linear combination of 15 terms Eq. (66) with real coefficients. Let us write \( H = H_1 + H_2 \), with \( H_1 \) and \( H_2 \) are corresponding to Eq. (66a) and Eq. (66b) respectively, then

\[
H_1 = H'_1 \otimes 1 \otimes 1, \quad H_2 = H'_2 \otimes \sigma_1 \otimes \sigma_2, \quad (67)
\]

where \( H'_1 \) and \( H'_2 \) are two-qubits Hamiltonians (on the even subsystem).

Let us now compare Eq. (66) with a basis of Hamiltonians for a system with two qubits

\[
H'_{j1} = \sigma_j \otimes 1, \quad H'_{j2} = 1 \otimes \sigma_j, \quad H'_{jk} = \sigma_j \otimes \sigma_k. \quad (68)
\]

The Hamiltonians Eq. (68) coincide with the first two terms in Eq. (66). The decomposition \( H = H_1 + H_2 \) together with Eq. (67) may be used to define the Hamiltonian \( H' = H'_1 + H'_2 \) on a system with two qubits. Due to one-to-one correspondence between Eq. (68) and Eq. (66), any \( H' \) may be constructed in such a way from some \( H \) and vice versa.

Let us consider action of Hamiltonians Eq. (66) on a system of four qubits decomposed into even and odd subsystems \( |\Psi_e\rangle |\Upsilon_o\rangle \), there \( |\Psi_e\rangle \) is an arbitrary state of two qubits and \( |\Upsilon_o\rangle \) is an eigenstate with unit eigenvalue of the operator \( \sigma_1 \otimes \sigma_2 \). Composing eigenvectors of \( \sigma_1, \sigma_2 \) with equal eigenvalues \( \pm 1 \) it may be obtained

\[
|\Upsilon_o^{++}\rangle = \frac{1}{2} ((0) + (1))((0) + (i|1)) , \quad |\Upsilon_o^{--}\rangle = \frac{1}{2} ((0) - (1))((0) - (i|1)).
\]

A linear combination of the states also may be used

\[
|\Upsilon_o\rangle = \alpha |\Upsilon_o^{++}\rangle + \beta |\Upsilon_o^{--}\rangle, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (69)
\]

Let us consider the Hamiltonian \( H = H_1 + H_2 \) introduced above with Eq. (67), then

\[
H_1 (|\Psi_e\rangle |\Upsilon_o\rangle) + H_2 (|\Psi_e\rangle |\Upsilon_o\rangle) = (H' |\Psi_e\rangle) |\Upsilon_o\rangle. \quad (70)
\]

where \( H' = H'_1 + H'_2 \) is the two-qubit Hamiltonian also defined earlier using Eq. (67). It may be derived directly from Eq. (70), that a quantum gate corresponding to \( \text{Spin}(6) \) for such a state acts as usual quantum gate on the first two qubits:

\[
e^{-iH'_\tau} (|\Psi_e\rangle |\Upsilon_o\rangle) = (e^{-iH'_\tau} |\Psi_e\rangle) |\Upsilon_o\rangle. \quad (71)
\]

It ensures one-to-one correspondence \( SU(4) \cong \text{Spin}(6) \) between the arbitrary gate on two qubits and the \( \text{Spin}(6) \) gate.

### 4.3 Decomposition of Spin(3n)

It was already discussed in Sec. 2.6 that despite of isomorphism of \( \text{Spin}(6) \) with group \( SU(4) \) of two-qubit gates, \( \text{Spin}(3n) \) circuits composed from such \( \text{Spin}(6) \) gates may have only quadratic dimension with respect to \( n \).
An analysis with the quantum circuits model is very similar. In simplest case \( \text{Spin}(9) \) group may be represented by six qubits reordered into even and odd subsystems with three qubits in each using Eq. (64). The Hamiltonians corresponding to overlapped two-line gates Eq. (48) in Sec. 2.6 may be constructed using Eq. (66b)

\[
\begin{align*}
H_{jk}^{[12]} &= (\sigma_j \otimes \sigma_k \otimes 1) \otimes (\sigma_1 \otimes \sigma_2 \otimes 1), \\
H_{j'k'}^{[23]} &= (1 \otimes \sigma_{j'} \otimes \sigma_{k'}) \otimes (1 \otimes \sigma_1 \otimes \sigma_2).
\end{align*}
\]

(72)

Due to Eq. (72) the commutator of \( H_{jk}^{[12]} \), \( H_{j'k'}^{[23]} \) is nonzero only for \( k = j' \) and produces Hamiltonian of ‘second order’

\[
H_{jk'}^{[13]} = i\epsilon_j^{[1]} \epsilon_k^{[3]} = (\sigma_j \otimes 1 \otimes \sigma_{k'}) \otimes (\sigma_1 \otimes \sigma_3 \otimes \sigma_2).
\]

(73)

In more general case the situation is similar and the group \( \text{Spin}(3n) \) is generated by Hamiltonians Eq. (62) with two Pauli matrices in the ‘primary’ (even) subsystem. The number of Pauli matrices in the ‘auxiliary’ (odd) subsystem for \( H_{lm}^{[lm]} \), \( l < m \) is \( m - l + 1 \). Such a Hamiltonian has structure resembling Eq. (73) with \( m - l - 1 \) matrices \( \sigma_3 \) inserted between \( \sigma_1 \) and \( \sigma_2 \).

5 Classical simulation

5.1 General methods

An idea of the classical simulation of the \( \text{Spin}(3n) \) circuit used here is analogous with the approach used in [5, 6] for \( \text{Cl}(2n) \) and \( \text{Spin}(2n) \).

Few distinctions between ‘\( \text{Spin}(3n) \) circuits’ and models related with \( \text{Cl}(2n) \) may be analyzed using 3n generators \( \epsilon_j^{[k]} \) of \( \mathcal{O}(3n) \) defined by Eq. (53) or Eq. (54b) and 2n generators \( \epsilon_k \) of \( \mathcal{O}(2n) \) from Eq. (50).

The product \( i\epsilon_{2k-1} \epsilon_{2k} = \sigma_{3;k} \) (denoted in [5, 6] as \( Z_k \)) is an action of \( \sigma_3 \) on the qubit with index \( k \). It may be compared with \( \sigma_{3;2k} = i\epsilon_1^{[k]} \epsilon_2^{[k]} \), where an even index \( 2k \) corresponds to the initial order of qubits without any reordering. Other Pauli matrices for qubits with even indexes may be expressed as well: \( \sigma_{1;2k} = i\epsilon_2^{[k]} \epsilon_1^{[k]} \), \( \sigma_{2;2k} = i\epsilon_3^{[k]} \epsilon_1^{[k]} \). An expression for qubits with odd indexes is different and may be written using \( \gamma_5 \) from Eq. (52) \( \sigma_{3;2k-1} = -i\epsilon_1^{[k]} \epsilon_2^{[k]} \epsilon_3^{[k]} \).

Due to such a property the variation of setup used in [5, 6] is:

1. the ‘\( \text{Spin}(3n) \) circuit’ with \( 2n \) qubits
2. the input state is any product state
3. the output is a measurement of a single qubit:
   (a) arbitrary for even indexes
   (b) in the computational basis for odd indexes

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Let us first consider qubits with even indexes. Without the lost of generality only measurements in computational basis may be discussed, because any one-qubit gate may be implemented on the even qubit and it may be used for a measurement in another basis.

For such a simplified case methods from [5, 6] may be applied with minimal modifications. Let us consider $U = U_1 U_2 \cdots U_N$ representing element of $Spin(3n)$ group as a circuit with $N$ gates

$$|\Psi_U\rangle = U |\Psi\rangle = U_1 U_2 \cdots U_N |\Psi\rangle. \quad (74)$$

If $p_0^{(2k)}$ and $p_1^{(2k)}$ are probabilities of outcomes of measurements in the computational basis for a qubit with an index $2k$

$$p_0^{(2k)} - p_1^{(2k)} = \langle \Psi_U | \sigma_{3;2k} | \Psi_U \rangle = \langle \Psi | U^\dagger e_1^{[k]} e_2^{[k]} U |\Psi\rangle. \quad (75)$$

Let us note $U^\dagger e_1^{[k]} e_2^{[k]} U = U^\dagger e_1^{[k]} e_2^{[k]} U^\dagger e_2^{[k]} U$ and use a standard property of Spin groups Eq. (8), rewritten as

$$U^\dagger e_j^{[k]} U = \sum_{j', k'} R^{[kk']}_{j j'} e_{j'}^{[k']}, \quad (76)$$

where $R^{[kk']}_{j j'}$ denotes $3n \times 3n$ orthogonal matrix with elements $R_{3(k-1)+j, 3(k'-1)+j'}$.

If the operator $U$ in Eq. (74) corresponds to a decomposition of $Spin(3n)$ circuits on a sequence of $N$ gates, the matrix also may be presented as a product $R = R_1 R_2 \cdots R_N$ with each term corresponding to a gate in the sequence and it may be computed in time $\text{poly}(n, N)$.

Eq. (75) may be rewritten

$$p_0^{(2k)} - p_1^{(2k)} = \langle \Psi | \left( \sum_{j', k'} R^{[kk']}_{j j'} e_{j'}^{[k']} \right) \left( \sum_{j', k'} R^{[kk']}_{j j'} e_{j'}^{[k']} \right) |\Psi\rangle

= \sum_{j', k', j'', k''} \left( R^{[kk']}_{j j'} R^{[kk'']}_{j'' j''} \langle \Psi | e_{j'}^{[k']} e_{j''}^{[k'']} |\Psi\rangle \right). \quad (77)$$

For the sum of $(3n)^2$ elements $\langle \Psi | i e_j^{[k']} e_{j''}^{[k'']} |\Psi\rangle$ with $|\Psi\rangle = |\psi_1\rangle \cdots |\psi_{2n}\rangle$ and product operators

$$e_{j'}^{[k']} e_{j''}^{[k'']} = s_{j' j''} \otimes \cdots \otimes s_{2n j' j''} \quad (78)$$

each element is a product of $2n$ factors $\langle \psi_m | s_{m j' j''} |\psi_m\rangle$, $m = 1, \ldots, 2n$.

Thus, the result of a measurement of a single qubit with an even index may be computed in time $\text{poly}(n, N)$, where $n$ and $N$ are numbers of qubits and gates respectively. Here the resemblance with the approach [5, 6] to matchgates is quite clear.

For qubits with odd indexes the difference is rather not essential. Instead of Eq. (75), for a qubit with an index $2k - 1$ should be used

$$p_0^{(2k-1)} - p_1^{(2k-1)} = -\langle \Psi | U^\dagger i e_0^{[k]} e_1^{[k]} e_2^{[k]} U |\Psi\rangle. \quad (79)$$

Only $e_0^{[k]}$ is not affected by the $Spin(3n)$ group and modifications of other three elements $e_j^{[k]}$ ($j = 1, 2, 3$) are described by Eq. (76). Instead of Eq. (77), similar sum with $(3n)^3$ terms should be written

$$p_0^{(2k-1)} - p_1^{(2k-1)} = \sum_{k_1, k_2, k_3, j_1, j_2, j_3} \left( R^{[kk_1]}_{j_1 j_1} R^{[kk_2]}_{j_2 j_2} R^{[kk_3]}_{j_3 j_3} \langle \Psi | i e_0^{[k]} e_1^{[k]} e_2^{[k]} e_{j_1}^{[k_1]} e_{j_2}^{[k_2]} e_{j_3}^{[k_3]} |\Psi\rangle \right). \quad (80)$$
For the product states $|\Psi\rangle$ and operators $c_{i_1}^{[1]} c_{j_1}^{[1]} c_{i_2}^{[2]} c_{j_2}^{[k_2]}$, each $(\Psi| \ldots |\Psi)$ again may be expressed by multiplication of $2n$ terms $(v_m| \ldots |v_m)$. 

The consideration shows, the result of a measurement in the computational basis of a single qubit with odd index may be again computed in time $\text{poly}(n,N)$.

5.2 Two-qubit gates

5.2.1 Hamiltonians

In the proof of effective classical simulation discussed in Sec. 5.2 was successfully used known approach [5, 6], but the model itself has new properties. A promising achievement is possibility to implement arbitrary two-qubit gate. Let us discuss that before consideration of general case with many qubits.

Such an attainment in comparison with matchgate model is associated with improved control over ‘primary’ qubits with even indexes accompanied by separation of ‘auxiliary’ qubits with odd indexes. The example with two qubits implemented by circuit with two additional auxiliary qubits was discussed in Sec. 4.2. The model allows us to apply arbitrary unitary gate on pair of qubit with fixing auxiliary qubits in undisturbed state $|\Psi_o\rangle$.

The Hamiltonians of two-qubit gates used in Sec. 4.2 are revisited here before discussion in Sec. 5.2.2 about possibility to work directly with unitary gates. Let’s use an opportunity of generators. The $e_{j}$ are operators on four qubits that should be rewritten after reordering Eq. (64) to apply notation in agreement with expressions for two-qubit Hamiltonians such as Eq. (68).

Let us rewrite Eq. (68) using Eq. (65)

\begin{align}
\begin{array}{ll}
s_j^{[1]} & \longleftrightarrow -iH_{j,1} = -i\sigma_j \otimes 1, \\
c_j^{[1]} c_k^{[2]} & \longleftrightarrow -iH_{j,k} = -i\sigma_j \otimes \sigma_k,
\end{array}
\end{align}

(81a)

where $s_j^{[l]}$, $l = 1, 2$ are products of two generators $c_p^{[l]} c_k^{[l]}$, $j \neq p \neq k$ introduced in Sec. 2.2, Eq. (21) and reused in Sec. 3.3, Eq. (60a). Thus, Eq. (81) describe $3 + 3 + 9 = 15$ different products with two generators of Clifford algebra $\mathcal{C}_4(6)$.

Eq. (81) may be also rewritten using notation $\sigma_j^{(1)} = \sigma_j \otimes 1$, $\sigma_j^{(2)} = 1 \otimes \sigma_j$.

\begin{align}
\begin{array}{ll}
c_j^{[1]} c_k^{[1]} & \longleftrightarrow -\sigma_j^{(1)} \sigma_k^{(1)}, \\
c_j^{[2]} c_k^{[2]} & \longleftrightarrow -\sigma_j^{(2)} \sigma_k^{(2)},
\end{array}
\end{align}

(82a)

The imaginary unit multiplier presenting in Eq. (82b), but missing in Eq. (82a) is important. Formally, Eq. (82) are in complete agreement with Eq. (81) due to law of multiplication of Pauli matrices and it illustrates the fact that such correspondence is defined only for pairs of generators. The $c_j^{[l]}$ are operators on four qubits that should be rewritten after reordering Eq. (64) as

\begin{align}
\begin{array}{ll}
c_j^{[1]} = (\sigma_j \otimes 1) \otimes (\sigma_2 \otimes 1), \\
c_j^{[2]} = (1 \otimes \sigma_j) \otimes (\sigma_3 \otimes \sigma_2)
\end{array}
\end{align}

and, so, the generators should not be confused with elements such as $\sigma_j^{(l)}$.

It is clear from Eq. (81) that expression $\exp(-i\tau)$ used for construction of unitary gates contains only real coefficients after rewriting with products of two generators. However, $4 \times 4$ unit matrix $1 \otimes 1$ is not presented in Eq. (81) or Eq. (82) and all fifteen matrices described
by the equations are traceless. The traceless Hamiltonians corresponds to unitary matrices with unit determinant, i.e., to the group $SU(4)$.

Let’s describe the method of construction of element of $Spin(6)$ group for a quantum gate with known representation as $U = \exp(-iH\tau)$ with some real $\tau$ and Hermitian $H$.

The matrix $H$ may be considered traceless without lost of generality by transition

$$H \rightarrow H - \left(\frac{\text{Tr}(H)}{4}\right) \mathbf{1} \otimes \mathbf{1}.$$ 

Let’s enumerate Hamiltonians in Eq. (81) with single index: $H_J$, $J = 1, \ldots, 15$. It may be simply checked that $\text{Tr}(H_JH_K) = 4\delta_{JK}$. Due to that property coefficients of decomposition of a traceless Hamiltonian $H$ using basis Eq. (81) may be expressed as

$$H = \sum_{J=1}^{15} h_J H_J, \quad h_J = \frac{\text{Tr}(H_JH)}{4},$$ (83)

where any index $J$ is associated with known product of two generators. It was mentioned earlier that such products correspond to Lie algebra of $spin(6)$ and so such algorithm produces element $S_H \in spin(6)$ for any traceless Hamiltonian $H$. Now, element $S_H \in Spin(6)$ group may be expressed as $S_H = \exp(S_H\tau)$.

### 5.2.2 Gates

It may look more convenient instead of Hamiltonians used in Sec. 4.2 and Sec. 5.2.1 to work directly with unitary gates. Sometimes exponential representation also can be useful, e.g., any operator $\mathbf{J}$ with property $\mathbf{J}^2 = -\mathbf{1}$ complies with a simple equation

$$\exp(\tau\mathbf{J}) = \cos(\tau)\mathbf{1} + \sin(\tau)\mathbf{J},$$

and together with standard property of exponent for commuting operators $\mathbf{J}$, $\mathbf{K}$

$$\exp(\mathbf{J} + \mathbf{K}) = \exp(\mathbf{J})\exp(\mathbf{K})$$

it can be applied to some interesting examples.

However, a method of direct construction of element $S \in Spin(6)$ from a quantum gate $U \in SU(4)$ is required in more general case. The reason to avoid matrices from $U(4)$ with non-unit determinants was illustrated above in Sec. 5.2.1 and to exploit isomorphism $Spin(6) \simeq SU(4)$ a gate should be tuned using a phase multiplier: $U' = \det(U)^{-1/4}U$.

The unitary gate must be mapped into element of $Spin(6)$ group and such element may contain products with any even number of generators. Thus, it is necessary to consider 32 such products instead of only 15 discussed below and already used in Eq. (83). Such consideration includes unit, product of all six generators denoted earlier by $\mathbf{i}$ in Eq. (40), fifteen products with two generators already used earlier and fifteen products with four generators. Two last numbers are the same, because any product of four generators can be expressed as a pair multiplied on $\mathbf{i}$.

In such a way the Spin(6) group is represented as some subspace of algebra $\mathbb{C}^{0}_{+}(6)$ already discussed in Sec. 2.5. A possible confusion may appear because the 32-dimensional real algebra $\mathbb{C}^{0}_{+}(6)$ used for construction of $Spin(6)$ should be mapped into algebra of all $4 \times 4$ complex matrices.
It was already discussed earlier in Sec. 2.5 with basic idea to use \( \mathbb{i}^2 = -1 \) commuting with all elements of \( C^n_1(6) \) as an imaginary unit. Such general approach is in agreement with Eq. (81) of Eq. (82), because they lead to

\[
\mathbb{i} = (e_1^{[1]} e_2^{[1]}) (e_3^{[1]} e_1^{[2]}) (e_2^{[2]} e_3^{[2]}) \leftrightarrow (-i \sigma_3 \otimes \mathbf{1})(-i \sigma_3 \otimes \sigma_1)(-\mathbf{1} \otimes i \sigma_1) = i \mathbf{1} \otimes \mathbf{1}.
\]

Relation between products with four and two generators obtained by multiplication on \( \mathbb{i} \) was already mentioned above. Thus, the fourfold products correspond to Eq. (81) without imaginary units.

Let’s finally describe the algorithm for the map \( U \in SU(4) \rightarrow Spin(6) \). The unit matrix together with fifteen matrices Eq. (81) may be used as a basis \( C^n_1 \) of Eq. (82), because they lead to

\[
U = \sum_{J=0}^{15} u_J U_J, \quad u_J = \text{Tr}(H_J U)/4
\]

Real and imaginary parts of \( u_0 \) corresponds to unit and \( \mathbb{i} \) respectively. For \( J \geq 1 \) real part of \( u_J \) is responsible for the same pair of generators as \( h_j \) in Eq. (83) and imaginary part of \( u_J \) conforms to product of four generators obtained from this pair by multiplication on \( \mathbb{i} \).

After construction of \( S \in Spin(6) \) using methods discussed above, matrix \( R \in SO(6) \) is defined similarly with Eq. (76)

\[
S^{-1} e_j^{[k]} S = \sum_{j'=1}^{2} \sum_{k'=1}^{2} R_{jj'}^{[kk']} e_j^{[k']}. \quad (85)
\]

The expression Eq. (86) below for six-dimensional vector affected by such rotations makes more clear structure of double indexes used in Eq. (85)

\[
v = v_1^{[1]} e_1^{[1]} + v_2^{[1]} e_2^{[1]} + v_3^{[1]} e_3^{[1]} + v_1^{[2]} e_1^{[2]} + v_2^{[2]} e_2^{[2]} + v_3^{[2]} e_3^{[2]}. \quad (86)
\]

For clarification, in Appendix A is presented a program for computer algebra system calculating elements of \( Spin(6) \) group and rotation for given \( 4 \times 4 \) matrix of two-qubit gate.

Techniques developed here for six generators \( e_j^{[1]} \) and \( e_j^{[2]} \) are simply generalized for any pair \( e_j^{[k']} \) and \( e_j^{[k'']} \) and two-qubit gates on ‘primary’ (even) indexes \( 2k' \) and \( 2k'' \). Any two-qubit gate on given indexes is mapped into rotation of 6D subspace similar with Eq. (86)

\[
v = v_1^{[k']} e_1^{[k']} + v_2^{[k']} e_2^{[k']} + v_3^{[k']} e_3^{[k']} + v_1^{[k'']} e_1^{[k'']} + v_2^{[k'']} e_2^{[k'']} + v_3^{[k'']} e_3^{[k'']} \quad (87)
\]

### 5.3 Simulation of \( Spin(3n) \) circuits with suitable states

The improved control over ‘primary’ qubits with even indexes can be considered as essential contribution of presented model and it is acceptable for a while to avoid detailed consideration of qubits with odd indexes Eq. (79) and Eq. (80).

The method to save states of the ‘auxiliary’ qubits used above to exclude them from consideration does not work for \( n > 2 \). Let’s consider Eq. (72) for illustration of the case \( n = 3 \). Operators acting on three auxiliary qubits for \( H^{[12]} \) and \( H^{[23]} \) are anticommute in
agreement with Eq. (49) and common eigenstate $|\Upsilon\rangle$ (with nonzero eigenvalue) cannot exist for them.

Thus, without requirement about specific states for qubits with odd indexes, initial state may be chosen similarly with matchgate circuits $|\Psi_0\rangle = |00\ldots 0\rangle$ [6]. Probabilities Eq. (77) for given initial state are directly calculated using values

$$p^{[k'k'']}_j = \langle \Psi_0 | i \epsilon_j^{[k]} \epsilon_j^{[k'']} | \Psi_0 \rangle.$$  

(88)

For $k' < k''$ Eq. (55) together with Eq. (52) provide expressions such as

$$i \epsilon_j^{[k']} \epsilon_j^{[k'']} = \left(\frac{1}{2(k' - 1)} \otimes \cdots \otimes 1 \otimes \sigma_1 \otimes \sigma_j \otimes \sigma_3 \otimes 1 \otimes \cdots \otimes \sigma_2 \otimes \sigma_{j''} \otimes 1 \otimes \cdots \otimes 1\right).$$  

(89)

Eq. (88) for the product states includes factors $\langle 0 | \sigma_j | 0 \rangle$ and $\langle 0 | \sigma_j | 0 \rangle$ equal to zero and so for $k' \neq k''$, $\mu_j^{[k'k'']} = 0$. The factors are due to qubits with odd indexes and, moreover, the expression vanishes for initial states of ‘auxiliary’ qubits either $|0\rangle$ or $|1\rangle$.

For $k' = k''$

$$i \epsilon_j^{[k']} \epsilon_j^{[k']} = \left(\frac{1}{2(k' - 1)} \otimes \cdots \otimes 1 \otimes i \sigma_j \otimes i \sigma_{j''} \otimes 1 \otimes \cdots \otimes 1\right).$$  

(90)

Thus, $\mu_j^{[k'k'']} = \langle 0 | i \sigma_j | 0 \rangle = i \delta_{j'j''} - \epsilon_{j'j''3}$, where $\epsilon_{abc}$ is totally antisymmetric Levi-Civita symbol (permutation tensor). Formal expression for both cases $k' = k''$ and $k' \neq k''$ may be written as

$$\mu_j^{[k'k'']} = (i \delta_{j'j''} - \epsilon_{j'j''3}) \delta_{k'k''}.$$  

(91)

However, the term with imaginary unit in Eq. (91) is redundant and it does not produce any contribution in final expression Eq. (77) because

$$\sum_{j',k',j'',k''} R_{1j'}^{[kk']} R_{2j''}^{[kk'']} \epsilon_{j'j''3} \delta_{k'k''} = \sum_{j',k'} R_{1j'}^{[kk']} R_{2j'}^{[kk']} = 0.$$  

(92)

Indeed, in ‘plain’ notation $R_{jj'}^{[kk']} = R_{3(k-1)+j3(k'-1)+j'}$ is $3n \times 3n$ orthogonal matrix and Eq. (90) corresponds to scalar product of two different columns

$$\sum_{m=1}^{3n} R_{3(k-1)+1,m} R_{3(k'-1)+2,m} = 0.$$  

for the matrix of rotation $R$, but they are orthogonal and the scalar product is zero.

The term with $j' = j''$, $k' = k''$ corresponds to product of two equal generators $\epsilon_j^{[k]} \epsilon_j^{[k]} = 1$ and could be from very beginning excluded from consideration for any initial state $|\Psi\rangle$.

Without this vanished term Eq. (77) may be rewritten for $|\Psi_0\rangle$

$$p_0^{(2k)} - p_1^{(2k)} = - \sum_{k',k'',j',j''} R_{1j'}^{[kk']} R_{2j''}^{[kk'']} \epsilon_{j'j''3} \delta_{k'k''} = \sum_{k'=1}^{n} (R_{12}^{[kk']} R_{21}^{[kk'']} - R_{22}^{[kk']} R_{11}^{[kk']}).$$  

(93)

Other initial states may be considered as well, because any one-qubit transformation can be implemented for ‘primary’ qubits by $Spin(3n)$ circuit and used for altering $|\Psi_0\rangle$ in a
desired way. The derivation above shows also that the Eq. (91) is valid only for initial states of ‘auxiliary’ qubits either \(|0\rangle\) or \(|1\rangle\).

Techniques developed in Sec. 5.2 work for six generators \(e_j^{[{k'}]}\) and \(e_j^{[{k''}]}\) with any pair of indexes \(k'\) and \(k''\) and provide method to map any two-qubit gate into 6D rotation on subspace Eq. (87). It is true for arbitrary states of ‘auxiliary’ qubits, yet \(|\Upsilon_o\rangle\) were used earlier for convenience to avoid entanglement between ‘primary’ and ‘auxiliary’ qubits.

In presented method of simulation the final matrix \(R\) is obtained as composition of rotations on the 6D subspaces and with the known matrix \(R\) probabilities for measurement of ‘primary’ qubits are calculated using Eq. (91). Exertion of Eq. (80) for ‘auxiliary’ qubits with chosen initial states is not discussed in presented work.

6 Conclusions

This work was devoted to relations between \(\text{Spin}(3\,n)\) groups and quantum circuits. Any transformation of one or two qubits may be described by \(\text{Spin}(3)\) or \(\text{Spin}(6)\) groups respectively. Such a property together with well-known correspondence between group \(\text{Spin}(2\,n)\) and some non-universal quantum circuits with \(n\) qubits causes a natural question about similar relations for \(\text{Spin}(3\,n)\) groups.

The connection between quantum circuits with \(n\) qubits and \(\text{Spin}(3\,n)\) may be illustrated using the Clifford algebra \(\mathcal{Cl}_+(3)\) isomorphic with the Pauli algebra. On the one hand, the complex tensor product of \(n\) such algebras Eq. (22) is a standard tool for the description of the quantum circuits. On the other hand, the real skew tensor product of the same algebras Eq. (30) is very natural for the construction of Clifford algebra \(\mathcal{Cl}_+(3\,n)\) and \(\text{Spin}(3\,n)\) group.

However, in the usual tensor product all complex structures for different factors may be merged together, but in the skew tensor product different complex units are anti-commuting and form a new term identified with \(\mathcal{Cl}(n)\) in Eq. (37). The distinction between quantum circuits with \(n\) qubits and \(\text{Spin}(3\,n)\) group becomes natural with such a difference of the tensor products.

The description of \(\text{Spin}(3\,n)\) by quantum circuits with \(2n\) qubits in Sec. 4 provides an alternative approach. Here additional \(n\) qubits are used for implementation of the anti-commuting structure.

The quantum circuit model also may be used for a ‘physical’ explanation of the complexity reduction for ‘\(\text{Spin}(3\,n)\) circuits.’ New anti-commutative terms change commutation relations for Hamiltonians implementing partially overlapped gates and (in agreement with Lie-algebraic approach to quantum circuits) prevent construction of gates of higher order from simpler gates, see Sec. 4.3.

Methods of classical simulations of these quantum circuits with \(2n\) qubits are discussed in the last section. Due to such possibility \(\text{Spin}(3\,n)\) group is associated with the new kind of quantum circuits which may be effectively simulated on a classical computer.

Such a circuit uses only \(n\) (‘primary’) qubits with even indexes as direct carriers of quantum information. Any gate with a single ‘primary’ qubit may be realized, but an arbitrary entangled transformation for two ‘primary’ qubits also involves a pair of (‘auxiliary’) qubits with odd indexes and formally corresponds to a four-qubit gate, see Eq. (65).

For any pair of ‘primary’ qubits the accompanying action may be dropped by the specific choice of a state for the ‘auxiliary’ pair Eq. (71). However, for the composition of gates with overlapped ‘primary’ qubits such side-effects may not be omitted, just because they ensure the significant reduction of the complexity.
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A Computer algebra program

The OpenAxiom CAS program* for conversion of a two-qubit gate into element of \( Spin(6) \) group is presented below.

Function \texttt{MatClp6} calculates element of \( S \in Spin(6) \) group for \( 4 \times 4 \) unitary matrix of a two-qubit gate, after that action of \( SO(6) \) group is calculated using map \( v \mapsto S^{-1}vS \).

-- File SP6QC.input

\begin{verbatim}
K := Fraction Polynomial Integer
CK := Complex Fraction Integer
MK := Matrix Complex Fraction Integer
MK2 := SquareMatrix(2,CK)
MK4 := SquareMatrix(4,CK)

-- Pauli matrixes
sg0 : MK2 := matrix[[1,0],[0,1]]
sg1 : MK2 := matrix[[0,1],[1,0]]
sg2 : MK2 := matrix[[0,-%i],[%i,0]]
sg3 : MK2 := matrix[[1,0],[0,-1]]

-- Tensor products of Pauli matrixes
se00 : MK4 := tensorProduct(sg0,sg0)
se10 : MK4 := tensorProduct(sg1,sg0)
se20 : MK4 := tensorProduct(sg2,sg0)
se30 : MK4 := tensorProduct(sg3,sg0)
se01 : MK4 := tensorProduct(sg0,sg1)
se02 : MK4 := tensorProduct(sg0,sg2)
se03 : MK4 := tensorProduct(sg0,sg3)

-- Writing them into array
se := vector([se10,se20,se30,se01,se02,se03])

-- Definition of Clifford algebra Cl_+(6)
Clp6 := CliffordAlgebra(6, K, quadraticForm diagonalMatrix[1,1,1,1,1,1])

-- Definition of generators
ep1 : Clp6 := e(1)
ep2 : Clp6 := e(2)
ep3 : Clp6 := e(3)
ep4 : Clp6 := e(4)
ep5 : Clp6 := e(5)
ep6 : Clp6 := e(6)

-- Writing them into array
ep := vector([ep1, ep2, ep3, ep4, ep5, ep6])
\end{verbatim}

*Latest version of Axiom CAS may not work with the program due to a technical issue with a maximal number of arguments in LISP functions and OpenAxiom 1.4.1 was used instead.
-- Definition of product of all generators: e1 e2 e3 e4 e5 e6
epi := ep1*ep2*ep3*ep4*ep5*ep6

-- Conversion of complex number: a + b i -> a + b e1 e2 e3 e4 e5 e6
CClp6 : (CK) -> Clp6
CClp6 z == real(z)+imag(z)*epi

-- Conversion of 4x4 complex matrices into real Clifford algebra Cl_+(6)
-- SU(4) matrix maps into element of Spin(6) group
MatClp6 : (MK4) -> Clp6
MatClp6 Mat ==
  Cl : Clp6 := CClp6(trace(Mat*se00)/4)
  for i in 2..6 repeat
    for j in 1..i-1 repeat
      if quo(i-1,3) = quo(j-1,3) then
        Cl := Cl + CClp6(trace(Mat*se(i)*se(j))/4)*ep(i)*ep(j)
      else
        Cl := Cl - CClp6(%i*trace(Mat*se(i)*se(j))/4)*ep(i)*ep(j)
  Cl

-- Unitary matrix
Uc := matrix[[1,0,0,0],[0,1,0,0],[0,0,0,-1],[0,0,1,0]]

-- Element of Spin(6) group
SPc := MatClp6(Uc)

-- Element of SO(6)
vec := a*ep1+b*ep2+c*ep3+d*ep4+e*ep5+f*ep6

-- Rotation of vec
recip(SPc) * vec * SPc

-- End of file SP6QC.input --