KMS STATES ON THE SQUARE OF WHITE NOISE ALGEBRA

Luigi Accardi, Grigori Amosov, Uwe Franz

Abstract. It was shown in [AFS00] that there are only three types of irreducible unitary representations \( \theta \) of \( \mathfrak{sl}_2 \). Using the Schurmann triple one can associate with each \( \theta \) a number of representations of the square of white noise (SWN) algebra \( \mathcal{A} \). However, in analogy with the Boson, Fermion and q-deformed case, we expect that some interesting non irreducible representations of \( \mathfrak{sl}_2 \) may result in GNS representations of KMS states associated with some evolutions on \( \mathcal{A} \). In the present paper determine the structure of the \( \ast \)-endomorphisms of the SWN algebra, induced by linear maps in the 1–particle Hilbert algebra, we introduce the SWN analogue of the quasifree evolutions and find the explicit form of the KMS states associated with some of them.

1. The square of white noise and its representations.

Definition 1 The square of white noise (SWN) algebra \( \mathcal{A} \) over the Hilbert algebra \( \mathcal{K} = L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) (see [ALV99, AFS00]), is the unital \( \ast \)-Lie algebra with generators 1 (central element), \( b_{\phi}, b_{\phi}^+, n_{\phi}, \phi \in \mathcal{K} \), which are linearly independent (in the sense that \( f_0 + b_{f_1} + b_{f_2} + n_{f_3} = 0 \) with \( f_0 \in \mathbb{C} \) and \( f_j \in \mathcal{K} \) (\( j = 1, 2, 3 \)) if and only if \( f_1 = f_2 = f_3 = 0 \)) and relations

\[
\begin{align*}
[b_{\phi}, b_{\psi}^+] &= \gamma \langle \phi, \psi \rangle + 1 + n_{\phi \psi}, \quad (\text{SWN}_1) \\
[n_{\phi}, b_{\psi}] &= -2b_{\phi \psi}, \quad (\text{SWN}_2) \\
[n_{\phi}, b_{\psi}^+] &= 2b_{\phi \psi}, \quad (\text{SWN}_3) \\
(b_{\phi})^* &= b_{\phi}^+, \quad (n_{\phi})^* = n_{\overline{\phi}}, \quad (\text{SWN}_4) \\
[n_{\phi}, n_{\psi}] &= [b_{\phi}^+, b_{\psi}^+] = [b_{\phi}, b_{\psi}] = 0 \quad (\text{SWN}_5)
\end{align*}
\]

where \( \gamma \) is a fixed strictly positive real parameter (coming from the renormalization), \( \langle \cdot, \cdot \rangle \) is an inner product in \( \mathcal{K} \) and \( \phi, \psi \in \mathcal{K} \). Furthermore \( b^+ \) and \( n \) are linear and \( b \) is anti-linear in the functions from \( \mathcal{K} \) and we assume that

\[ X_{\phi} = 0 \quad \text{if and only if} \quad \psi = 0 \quad (\text{SWN}_6) \]
Remark 1 The involution on $K$ will be indifferently denoted $f \mapsto f^*$ or $f \mapsto f^T$. Following [AFS00] all the irreducible unitary representations of $A$ can be obtained as follows. Consider the Lie algebra $\mathfrak{sl}_2$ with generators $\{B^+, B^-, M\}$, relations

$$[B^-, B^+] = M, \ [M, B^\pm] = \pm B^\pm,$$

and involution $(B^-)^* = B^+$, $M^* = M$. Let $\rho$ be an irreducible representation of $\mathfrak{sl}_2$ and therefore of its universally enveloping unital algebra $\mathcal{U}(\mathfrak{sl}_2)$ in a Hilbert space $\mathcal{H}_0$ and $\eta$ be a 1–cocycle for $\rho$. Define $L(uv) = \langle \eta(u^*), \eta(v) \rangle$, $u, v \in \mathcal{U}_0(\mathfrak{sl}_2)$, where $\mathcal{U}_0(\mathfrak{sl}_2)$ is the algebra $\mathcal{U}(\mathfrak{sl}_2)$ without unit. Then $(\rho, \eta, L)$ is called a Schürmann triple and a representation $\pi$ of $A$ in the Hilbert space $\Gamma(\mathcal{H}_0 \otimes L^2(\mathbb{R}))$ can be obtained from the relations

\[
\begin{align*}
\pi(b_{X[s,t]}) &= \Lambda_{st}(\rho(B^-)) + A_{st}^*(\eta(B^-)) + A_{st}(\eta(B^+)) + L(B^-)(t-s)\text{Id}, \\
\pi(b^+_{X[s,t]}) &= \Lambda_{st}(\rho(B^+)) + A_{st}^*(\eta(B^+)) + A_{st}(\eta(B^-)) + L(B^+)(t-s)\text{Id}, \\
\pi(n_{X[s,t]}) &= \Lambda_{st}(\rho(M)) + A_{st}^*(\eta(M)) + A_{st}(\eta(M)) + (L(M) - \gamma)(t-s)\text{Id},
\end{align*}
\]

(Rep)

where $\Lambda_{st} = \Lambda_t - \Lambda_s$, $A_{st}^* = A_t^* - A_s$, $A_{st} = A_t - A_s$ are the conservation, creation and annihilation processes on the symmetric Fock space $\Gamma(\mathcal{H}_0 \otimes L^2(\mathbb{R}))$ satisfying the relations (see [Par92, Mey95])

\[
\begin{align*}
[A_t(\phi), A_s^*(\psi)] &= (t \wedge s) \langle \phi, \psi \rangle, \\
[A_t(\phi), A_s(\psi)] &= [A_s^*(\phi), A_t^*(\psi)] = 0, \\
[\Lambda_t(X), \Lambda_s(Y)] &= \Lambda_{t\wedge s}([X,Y]), \\
[\Lambda_t(X), A_s(\phi)] &= -A_{t\wedge s}(X^*\phi), [\Lambda_t(X), A_s^*(\phi)] = A_{t\wedge s}^*(X\phi),
\end{align*}
\]

(CCR)

$\chi_I$ ($I \subset \mathbb{R}$) denotes the multiplication operator by the characteristic function of $I$ ($= 1$ on $I$ and $= 0$ outside $I$), $\gamma$ is the same as in $(\text{SWN}_1)$. Conversely, all the irreducible representations of $A$ arise in this way. We shall say that the representation $\rho$ is associated with the representation $\pi$ anche the cocycle $\eta$. Hence to classify the representations of $A$ one needs to investigate the representations of $\mathcal{U}(\mathfrak{sl}_2)$ and their cocycles.
The contents of the present paper is the following. In Section 2 we introduce the quasifree evolutions on $\mathcal{A}$. Then we obtain a classification of these evolutions and prove their spatiality. In Section 3 we consider the KMS states associated with a special subclass of the quasifree evolutions on $\mathcal{A}$.

2. Endomorphisms of the SWN algebra.

Definition 2 A Hilbert algebra is a *-algebra $\mathcal{K}$, not necessarily with unit, endowed with a scalar product $\langle f, g \rangle \in \mathbb{C}$ satisfying $\langle f, gh \rangle = \langle g^* f, h \rangle$ ($f, g, h \in \mathcal{K}$). A Hilbert algebra endomorphism (resp. automorphism) of $\mathcal{K}$ is a *-endomorphism (resp. *-automorphism) $T$ of the *-algebra structure

$$T(\phi \psi) = T(\phi)T(\psi), \quad (T(\phi))^* = T(\overline{\phi})$$

which is also an isometry (resp. unitary operator), of the pre-Hilbert space structure in the sense that, $\forall \phi, \psi \in \mathcal{K}$

$$\langle T(\phi), T(\psi) \rangle = \langle \phi, \psi \rangle \quad (1)$$

Theorem 1. Let $T^1, T^2, T^3$ be linear operators on $\mathcal{K}$. Define a map $\tau'$ acting on the generators $b, b^+, n$ by the formula

$$1 \rightarrow 1; \quad b_\phi \rightarrow b_{T^1 \phi}, \quad b^+_\phi \rightarrow b^+_{T^2 \phi}, \quad n_\phi \rightarrow n_{T^3 \phi}. \quad (2)$$

The map $\tau'$ can be extended to a *-endomorphism of $\mathcal{A}$ if and only if there exist a Hilbert algebra endomorphism $T$ of $\mathcal{K}$ and a real-valued function $\alpha$ on $\mathbb{R}$, such that, for any $\phi \in \mathcal{K}$,

$$T^3(\phi) = T(\phi) \quad (3)$$

$$T^1(\phi) = T^2(\phi) = e^{i\alpha T}(\phi) \quad (4)$$

The endomorphism $\tau'$ is a an automorphism if and only if $T$ is an automorphism.

Proof. Because of the linear independence of the generators and of $(SWN_6)$, it follows from the relation $(SWN_1)$ that

$$(T^1)^*T^2 = Id, \quad T^3(\overline{\phi} \psi) = \overline{T^1(\phi)T^2(\psi)}, \quad (5)$$
where the first identity in condition (5) has to be interpreted in the sense of (1). From (SWN2) and (SWN6) it follows that

\[ T^1(\overline{\phi}\psi) = \overline{T^3(\phi)}T^1(\psi), \quad (6) \]

from (SWN3) and (SWN6), that

\[ T^2(\phi\psi) = T^3(\phi)T^2(\psi) \quad (7) \]

and from (SWN4) and (SWN6), that

\[ T_0 := T^1 = T^2, \quad \overline{T^3(\phi)} = T^3(\overline{\phi}), \quad (8) \]

for any \( \phi, \psi \in \mathcal{K} \).

Therefore (5),(6),(7),(8) are respectively equivalent to:

\[ T_0^*T_0 = id \quad (9) \]

\[ T^3(\overline{\phi}) = T^3(\phi) \quad (10) \]

\[ T^3(\overline{\phi}\psi) = \overline{T_0(\phi)}T_0(\psi) \quad (11) \]

\[ T_0(\phi\psi) = T^3(\phi)T_0(\psi) \quad (12) \]

From (11) with \( \phi = \psi \), we deduce that for any \( \psi \in \mathcal{K} \)

\[ T^3(|\psi|^2) = |T_0(\psi)|^2 \quad (13) \]

Again from (11), with \( \psi \) replaced by \( \psi\chi \) we obtain

\[ T^3(\overline{\phi}\psi\chi) = \overline{T_0(\phi)}T_0(\psi\chi) \]

and, from (12) this is equal to

\[ \overline{T_0(\phi)}T^3(\psi)T_0(\chi) \]

Choosing \( \phi = \chi \) and using (13) we deduce

\[ T^3(|\phi|^2\psi) = |T_0(\phi)|^2T^3(\psi) = T^3(|\phi|^2)T^3(\psi) \]

Since any positive element in \( \mathcal{K} \) can be written in the form \( |\phi|^2 \) for some \( \phi \in \mathcal{K} \), this implies that, for any positive element \( \phi \) in \( \mathcal{K} \)

\[ T^3(\phi\psi) = T^3(\phi)T^3(\psi) \quad (14) \]
Since any $\phi \in K$ is linear combination of positive elements we conclude that (14) holds for any $\phi, \psi \in K$.

Combining (13) and (14) we conclude that, for any $\psi \in K$

$$|T^3(\psi)|^2 = |T_0(\psi)|^2$$

Thus for each $\psi \in K$ there exists a real valued measurable function $\alpha^\psi$ such that

$$T_0(\psi) = e^{i\alpha^\psi} T^3(\psi)$$

(15)

Finally (15) and (9) imply that, for any $\phi \in K$

$$\langle T^3(\phi), T^3(\phi) \rangle = \langle T_0(\phi), T_0(\phi) \rangle = \langle \phi, \phi \rangle$$

and, by polarization, this implies that $T^3$ is isometric. Thus $T^3$ is a Hilbert algebra endomorphism. Let us denote

$$T := T^3$$

If for some $t > 0$, supp $(\phi) \subseteq [-t, t]$ then (15) implies that

$$e^{\alpha^\phi} T(\phi) = T_0(\phi) = T_0(\phi \chi_{[-t,t]}) = T(\phi) T_0(\phi \chi_{[-t,t]}) =$$

$$e^{i\alpha_{\chi_{[-t,t]}}} T(\phi) T(\chi_{[-t,t]}) = e^{i\alpha_{\chi_{[-t,t]}}} T(\phi)$$

Thus, for any $\phi \in K$ with supp $(\phi) \subseteq [-t, t]$, one has, on supp $(T(\phi))$:

$$\alpha^\phi = \alpha_{\chi_{[-t,t]}} ; \text{ a.e.}$$

(16)

In particular, if $s \leq t$

$$\alpha_{\chi_{[-s,s]}} = \alpha_{\chi_{[-t,t]}} ; \text{ a.e. on supp } T(\chi_{[-s,s]})$$

(17)

Since $T$ is an endomorphism of $K$, $T(\chi_{[-t,t]})$ is a self–adjoint projection in $K$, hence it has the form

$$T(\chi_{[-t,t]}) = \chi_{I_t}$$

for some measurable subset $I_t \subseteq \mathbb{R}$. By the isometry property one can suppose that $t \mapsto I_t$ is increasing:

$$s \leq t \Rightarrow I_s \subseteq I_t ; \text{ a.e.}$$

Denote

$$\chi I := \sup \chi_{I_t} = \chi_{\omega_{\omega_{>0}}}$$
since the union can be taken on any sequence increasing to $+\infty$, one can assume that $I$ is measurable up to a set of measure zero. Moreover for any sequence $t_n \uparrow +\infty$ the function

$$\alpha := \lim \alpha_{\chi_{[-t_n,t_n]}}$$

is well defined on $I$ and measurable up to a sub–set of measure zero of $I$. Because of (16) for any function $\varphi$ with bounded support one has

$$T_0(\varphi) = e^{i\alpha} T(\varphi) = e^{i\alpha} T(\varphi)$$

Since $T_0$ is isometric and the functions with bounded support are dense in $\mathcal{K}$, it follows that

$$T_0 = e^{i\alpha} T$$

(18)

Conversely, if $T : \mathcal{K} \to \mathcal{K}$ is a Hilbert algebra endormorphism, $\alpha : \mathbb{R} \to \mathbb{R}$ a measurable function and $T_0, T_1, T_2$ are defined by (18), (8) respectively, then the map $\tau'$, defined by (2) preserves the commutation relations of the SWN, hence is a $*$–Lie algebra endomorphism.

Finally it is clear that $\tau'$ will be an automorphism (i.e. surjective endomorphism) if and only if $T$ is on to, i.e. unitary.

**Example.** A non surjective endomorphism of $\mathcal{A}$

Consider the following isometry on $L^2(\mathbb{R})$:

$$V f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ f(x-1) & \text{if } x \in [1, +\infty) \\ 0 & \text{if } x \in (-1,0] \\ f(x+1) & \text{if } x \in (-\infty, -1] \end{cases}$$

(19)

One has:

$$\langle V f, V g \rangle = \int [V f(x)]^* [V g(x)] dx$$

$$= \int_1^\infty \bar{f}(x-1) g(x-1) dx + \int_{-\infty}^{-1} \bar{f}(x+1) g(x+1) dx =$$

$$= \int_0^\infty \bar{f}(y) g(y) dy + \int_{-\infty}^0 \bar{f}(y) g(y) dy = \int_{-\infty}^{+\infty} \bar{f}(y) g(y) dy = \langle f, g \rangle_{L^2}$$

Thus $V$ is isometric:

$$V^* V = 1$$
However

\[ \text{Range } V = \{ f \in L^2(\mathbb{R}) : f(x) = 0 \text{ if } x \in [-1, 1] \} \]

which is a proper subspace of \( L^2(\mathbb{R}) \).

**Remark 2** Notice that \( V \) maps \( L^2 \cap L^\infty(\mathbb{R}) =: \mathcal{K} \) into itself and clearly induces a Hilbert algebra endomorphism of \( \mathcal{K} \). Let us denote it \( T^3 \). Notice however that, if \( \chi \) is any characteristic function of a bounded subset in \( \mathbb{R} \), then

\[ T^3(\chi) \leq \chi_{(-1,1)^c} \]

Therefore

\[ \bar{\chi} = \sup \{ T^{3}(\chi) : \chi \text{ projectors on bounded subsets of } \mathbb{R} = \chi_{(-1,1)^c} < 1 \]
operator $\mathcal{U}$ in $\mathcal{H}$ implementing $\tau'$. Every automorphism of $\mathcal{K}$ can be continued to an automorphism of the von Neumann algebra $L^\infty(\mathbb{R})$ that is unitary implementable. Therefore there is a unitary operator $U$ acting in $L^2(\mathbb{R})$ such that $T(x) = UxU^*$, $x \in \mathcal{K}$. The formula $UE(f \otimes \phi) = e(f \otimes U\phi)$, $f \in \mathcal{H}_0$, $\phi \in \Gamma(L^2(\mathbb{R}))$, defines a unitary operator $U = \Gamma(1 \otimes U)$ on the exponential vectors $e(f \otimes \phi)$, $f \in \mathcal{H}_0$, $\phi \in \Gamma(L^2(\mathbb{R}))$, and therefore on the whole of $\mathcal{H}$. The property $\pi(\tau'(x)) = U\pi(x)U^*$, $x \in \mathcal{A}$, holds. One can define $\tau(x) = \tau_T(x) = UxU^*$, $x \in \mathcal{B}(\mathcal{H})$. \thickbox

Proposition 2. In the above notations let $\alpha$ be a locally constant function vanishing outside a bounded interval and let $(\tau_\varepsilon)$ be the 1-parameter automorphism group of $\mathcal{A}$ associated to the pairs $(1, \varepsilon\alpha)$, $\varepsilon \in \mathbb{R}$. Then for any $x \in \mathcal{A}$

$$\tau_\varepsilon(\pi(x)) = e^{i\varepsilon H}\pi(x)e^{-i\varepsilon H}$$

(21)

Proof. For each $x \in \mathcal{A}$ denote

$$x(\varepsilon) = \tau_\varepsilon(x)$$

Choosing $x = b^+_{\psi}$ ($\psi \in \mathcal{K}$) one has

$$\partial_\varepsilon \pi(b^+_{\psi}(\varepsilon)) = \partial_\varepsilon \pi(b^+_{e^{i\varepsilon\alpha}\psi}) = \pi(b^+_{e^{i\varepsilon\alpha}\psi}) = i\alpha \pi(b^+_{\psi}(\varepsilon))$$

where in the last identity we have used the fact that, because of the independent increment property, it is sufficient to consider the case $\alpha = \text{constant}$ on the support of $\psi$. 

8
In our assumptions the operator $H = H(\alpha)$ is self-adjoint and
\[ C^+_\psi(\varepsilon) := e^{i\varepsilon H} (b^+_\psi) e^{-i\varepsilon H} \]
satisfies the equation
\[ \partial_{\varepsilon} C^+_\psi(\varepsilon) = i e^{i\varepsilon H} [H, \pi(b^+_\psi)] e^{-i\varepsilon H} = i e^{i\varepsilon H} \pi([n_{\alpha/2}, b^+_\psi]) e^{-i\varepsilon H} \]

It follows that $\tau_\varepsilon(b^+_\psi))$ and $C^+_\psi(\varepsilon)$ satisfy the same ordinary differential equation with the same initial condition. Since $\alpha$ is constant, in both cases the unique solution is
\[ \tau_\varepsilon(\pi(b^+_\psi)) = C^+_\psi(\varepsilon) = \pi(b^+_\psi_{\varepsilon\alpha\psi}) \]

Similarly one verifies that
\[ \tau_\varepsilon(\pi(b^+_\psi)) = \pi(b^+_\psi_{\varepsilon\alpha\psi}) ; \quad \tau_\varepsilon(\pi(n^+_\psi)) = \pi(n^+_\psi) \]
and from this (21) follows.

3. KMS states associated with quasifree evolutions on the SWN algebra

Let $\tau'_t = (\tau'_t)_{t \in \mathbb{R}}$ be a group of type (B) quasifree automorphisms on $\mathcal{A}$ defined by
\[ \tau'_t(b^0) = \lambda^{-it} b^0, \quad \tau'_t(b^+_0) = \lambda^{it} b^+_0, \]
\[ \tau'_t(n^0) = n^0, \quad t \in \mathbb{R}, \quad 0 < \lambda < 1. \]
Pick up the representations $\rho^\pm$ of $\mathcal{U}(\mathfrak{sl}_2)$ in $l^2$ introduced in [AFS00] as
\[ \rho^+(B^+) e_n = \rho^-(B^-) e_n = \sqrt{(n+1)(n+2)} e_{n+1}, \]
\[ \rho^+(B^-) e_n = \rho^-(B^+) e_n = \sqrt{n(n+1)} e_{n-1} \]
\[ \rho^\pm(M) e_n = \pm(2n+2) e_n, \]
where $\{e_0, e_1, \ldots, e_n, \ldots\}$ is an orthonormal basis of $l^2$. Then define a pair of states $\phi^\pm = \phi^+_\lambda$ on $\mathcal{U}(\mathfrak{sl}_2)$ by the formula
\[ \phi^+_\lambda(\cdot) = (1 - \lambda) \sum_{n=0}^{+\infty} \lambda^n (e_n, \rho^\pm(\cdot) e_n). \]
In the representation \( \pi = \rho^+ \otimes \rho^- \) of \( \mathcal{U}(\mathfrak{sl}_2) \otimes \mathcal{U}(\mathfrak{sl}_2) \) in the Hilbert space \( \mathcal{H}_\phi = l^2 \otimes l^2 \) we get

\[
\phi_\lambda^+(x) = (\psi_\lambda, \pi(x \otimes 1)\psi_\lambda),
\phi_\lambda^-(x) = (\psi_\lambda, \pi(1 \otimes x)\psi_\lambda), \quad \phi_\lambda(x \otimes y) = (\psi_\lambda, \pi(x \otimes y)\psi_\lambda)
\]

\( x, y \in \mathcal{U}(\mathfrak{sl}_2) \),

where \( \psi_\lambda = \sqrt{1 - \lambda} \sum_{n=0}^{+\infty} \lambda^ne_n \otimes e_n \in \mathcal{H}_\phi \). So one can consider \( \pi \) as the GNS representation associated with the state \( \phi_\lambda \). Put \( B_1^+ = \pi(B^+ \otimes 1) \), \( B_1^- = \pi(B^- \otimes 1) \), \( M_1 = \pi(M \otimes 1) \) and \( B_2^+ = \pi(1 \otimes B^+) \), \( B_2^- = \pi(1 \otimes B^-) \), \( M_2 = \pi(1 \otimes M) \). Every triple \((B_1^+, B_1^-, M_1)\) satisfies the relations of \( \mathfrak{sl}_2 \) by the definition.

**Proposition 3.** The pairwise commuting operators \((B_1^+, B_1^-, M_1)\) and \((B_2^+, B_2^-, M_2)\) satisfy the following relations

\[
B_1^- \psi_\lambda = \sqrt{\lambda} B_2^- \psi_\lambda, \quad B_1^+ \psi_\lambda = \frac{1}{\sqrt{\lambda}} B_2^+ \psi_\lambda, \quad M_1 \psi_\lambda = -M_2 \psi_\lambda.
\]

**Proof.** The identities

\[
B_1^+ e_n \otimes e_n = \pi(B^+ \otimes 1)e_n \otimes e_n = \rho^+(B^+)e_n \otimes e_n = \sqrt{(n+1)(n+2)}e_{n+1} \otimes e_n = e_{n+1} \otimes \rho^-(B^-)e_{n+1} = \pi(1 \otimes B^+)e_{n+1} \otimes e_{n+1} = B_2^+ e_{n+1} \otimes e_{n+1}
\]

imply

\[
B_1^+ \psi_\lambda = \sqrt{1 - \lambda} \sum_{n=0}^{+\infty} \lambda^ne_n \otimes e_n = \sqrt{1 - \lambda} \sum_{n=1}^{+\infty} \lambda^{n-1} B_2^+ e_n \otimes e_n = \frac{1}{\sqrt{\lambda}} B_2^+ \psi_\lambda,
\]

where we used the formula \( B_2^+ e_0 \otimes e_0 = e_0 \otimes \rho^-(B^+)e_0 = 0 \). The remaining equalities can be proved in the same way. \( \Box \)
Consider the Schurmann triple \((\pi, \eta, \tilde{\phi}_\lambda)\) consisting of the representation \(\pi\) of \(U(sl_2) \otimes U(sl_2)\), the trivial cocycle \(\eta(x) = \pi(x)\psi_\lambda\) and the conditionally positive functional \(\tilde{\phi}_\lambda(x) = (\psi, (\pi(x) - \varepsilon(x))\psi)\), \(x \in U(sl_2) \otimes U(sl_2)\), where \(\varepsilon\) is a counit. The restrictions of \((\pi, \eta, \tilde{\phi}_\lambda)\) give us two Schurmann triples \((\pi_{\pm}, \eta_{\pm}, \tilde{\phi}_{\lambda, \pm})\) consisting of representations \(\pi_{\pm}\) of \(U(sl_2)\) by \((B^+_1, B^-_1, M_1)\) and \((B^+_2, B^-_2, M_2)\) correspondingly such that \(\pi_{+}(x) = \pi(x \otimes \mathbf{1})\), \(\pi_{-}(x) = \pi(\mathbf{1} \otimes x)\), \(\eta_{\lambda, \pm}(x) = \pi_{\pm}(x)\psi_\lambda\), \(x \in U(sl_2)\). Then we can define the Levy processes \(j_{st}\) over \(U(sl_2) \otimes U(sl_2)\) associated with \((\pi, \eta, \tilde{\phi}_\lambda)\) and two Levy processes \(j_{st}^{\pm}\) over \(U(sl_2)\) associated with the Schurmann triples \((\pi_{\pm}, \eta_{\lambda, \pm}, \tilde{\phi}_{\lambda, \pm})\) correspondingly such that

\[
j_{st}(x) = \Lambda_{st}(\pi(x)) + A_{st}(\eta(x)) + A_{st}(\eta(x^*)) + (t - s)\tilde{\phi}_\lambda(x)Id,
\]

\[
j_{st}^{\pm}(B^\pm) = \Lambda_{st}(B^\pm) + A_{st}^*(B^\pm_1\psi_\lambda) + A_{st}(B^\pm_1\psi_\lambda) + (t - s)(\psi_\lambda, B^\pm_1\psi_\lambda)Id,
\]

\[
j_{st}^{\pm}(M) = \Lambda_{st}(M_1) + A_{st}^*(M_1\psi_\lambda) + A_{st}(M_1\psi_\lambda) + (t - s)(\psi_\lambda, M_1\psi_\lambda)Id,
\]

\[
j_{st}(B^\pm) = \Lambda_{st}(B^\pm) + A_{st}^*(B^\pm_1\psi_\lambda) + A_{st}(B^\pm_2\psi_\lambda) + (t - s)(\psi_\lambda, B^\pm_2\psi_\lambda)Id,
\]

\[
j_{st}(M) = \Lambda_{st}(M_2) + A_{st}^*(M_2\psi_\lambda) + A_{st}(M_2\psi_\lambda) + (t - s)(\psi_\lambda, M_2\psi_\lambda)Id.
\]

One can associate with the Levy processes given above two representations \(\theta_{\pm} = \theta_{\pm}^{(A)}\) of the SWN algebra \(A\) and a representation \(\theta\) of the tensor product of two SWN algebras \(A \otimes A\) in the same Fock space \(\mathcal{H} = \Gamma(\mathcal{H}_\phi \otimes L^2)\) such that

\[
\theta_{\pm}(b_{[s,t]}) = j_{st}^{\pm}(B^-), \quad \theta_{\pm}(b_{[s,t]}^+) = j_{st}^{\pm}(B^+),
\]

\[
\theta_{\pm}(n_{[s,t]}) = j_{st}^{\pm}(M) - \gamma(t - s)Id,
\]

\[
\theta(x \otimes y) = \theta_{+}(x)\theta_{-}(y), \quad x, y \in A.
\]

**Proposition 4.** The maps \(\theta_{\pm}\) define two representations of the SWN algebra \(A\) in the Hilbert space \(\mathcal{H} = \Gamma(\mathcal{H}_\phi \otimes L^2)\) such that \(\theta_{\pm}(x)\) and \(\theta_{-}(y)\) are commuting for all \(x, y \in A\).

**Proof.**

Notice that \(\theta_{+}(x)\) and \(\theta_{-}(y)\) commute if \(x, y \in A\) because \(B^-_1, B^+_1, M_1\) and \(B^-_2, B^+_2, M_2\) commute. Hence we need to prove
only that $\theta_+$ and $\theta_-$ are representations. Using the CCR relations (see the Introduction) we get

$$[\theta_+ (b_{[s,t]}), \theta_+ (b^+_{[s,t]})] = \Lambda_{st} (M_1) + A^\ast_{st} (M_1 \psi_\lambda) + A_{st} (M_1 \psi_\lambda) + (\psi_\lambda, M_1 \psi_\lambda) =$$

$$\theta_+ (n_{[s,t]}) + \gamma (t-s) I d,$$

$$[\theta_+ (n_{[s,t]}), \theta_+ (b^+_{[s,t]})] = \pm \Lambda_{st} (B_1^\pm) \pm A^\ast_{st} (B_1^\pm \psi_\lambda) \mp A_{st} (B_1^\pm \psi_\lambda) \pm (\psi_\lambda, B_1^\pm \psi_\lambda) =$$

$$\theta_+ (b^+_{[s,t]}).$$

The remaining formulae can be proved analogously. □

Define a linear map $\omega_\lambda : A \to C$, by the formula

$$\omega_\lambda (x) = (\Omega, \theta_+ (x) \Omega).$$

**Theorem 2.** $\omega_\lambda$ is a KMS state associated with the evolution $\tau'$, i.e.

$$\omega_\lambda (xy) = \omega_\lambda (x \tau'_t (y))$$

for all elements $x, y \in A$ which are analytic with respect to $\tau'$.

Proof.

Notice that $\tau'_t (b^\pm_{x_{[s,t]}}) = \lambda \pm 1 b^\pm_{x_{[s,t]}}$. Hence it is sufficient to prove only that

$$\omega_\lambda (b^\pm_{x_{[s,t]}} y) = \lambda \pm 1 \omega (y b^\pm_{x_{[s,t]}}), \ y \in A.$$

Given $x = \theta_+ (y) \in \theta_+ (A)$ one can obtain

$$\omega_\lambda (b^+_{x_{[s,t]}} y) = (\Lambda_{st} (B_1^-) \Omega, x \Omega) +$$

$$(A^\ast_{st} (B_1^- \psi_\lambda) \Omega, x \Omega) + (A_{st} (B_1^+ \psi_\lambda) \Omega, x \Omega) + (t-s) (\psi_\lambda, B_1^+ \psi_\lambda) \omega_\lambda (x) \equiv S.$$

Notice that

$$(\psi_\lambda, B^+_2 \psi_\lambda) = (\psi_\lambda, 1 \otimes \rho^- (B^+) \psi_\lambda) =$$

$$(1 - \lambda) \left( \sum_{n=0}^{+\infty} \lambda^n e_n \otimes e_n, \sum_{n=0}^{+\infty} \lambda^n e_n \otimes \sqrt{n (n+1)} (n+1) e_{n-1} \right) = 0.$$ 

Hence

$$0 = (\psi_\lambda, B^+_2 \psi_\lambda) = (\psi_\lambda, B^+_2 \psi_\lambda) = (B^+_2 \psi_\lambda, \psi_\lambda) = (\psi_\lambda, B^-_2 \psi_\lambda).$$
Using the last relation, Proposition 4 and the equality \( \Lambda_{st}(x)\Omega = A_{st}(\xi)\Omega = 0, \ x \in U(\mathfrak{sl}_2), \ \xi \in \mathcal{H} \), we get the following expression,

\[
S = \sqrt{\lambda}(A^*_{st}(B_2^- \psi \lambda)\Omega, x\Omega) + (t - s)(\psi \lambda, B_2^+ \psi \lambda)\omega_\lambda(x) = \\
\sqrt{\lambda}(\Lambda_{st}(B_2^-)\Omega, x\Omega) + \\
(A^*_{st}(B_2^- \psi)\Omega, x\Omega) + (A_{st}(B_2^+ \psi)\Omega, x\Omega) + (t - s)(\psi, B_2^- \psi)\omega_\lambda(x) = \\
\sqrt{\lambda}(\theta_-(b_{X_{[s,t]}})\Omega, x\Omega) = \sqrt{\lambda}(\Omega, x\theta_-(b_{X_{[s,t]}}^+)\Omega) = \lambda\omega_\lambda(yb_{X_{[s,t]}}^+) 
\]

The equality

\[
\omega_\lambda(b_{X_{[s,t]}}x) = \frac{1}{\lambda}\omega_\lambda(xb_{X_{[s,t]}}), \ x \in \mathcal{A},
\]

can be checked in the same way. \( \square \)

In every representation \( \theta \) of the algebra \( \mathcal{A} \) there are sufficiently many hermitian operators \( x \in \theta(\mathcal{A}) \). One can apply the functions \( f \in L^\infty \) to these operators and consider the von Neumann algebra \( \mathcal{M} = \theta(\mathcal{A})'' \wedge \mathcal{B(\mathcal{H})} \) generated by the all \( f(x) \in \mathcal{B(\mathcal{H})} \). The von Neumann algebra \( \mathcal{M} \) generated by the irreducible representation \( \theta \) is of type I. Let \( \mathcal{M} = \mathcal{M}_+ \) be the von Neumann algebras generated by the representations constructed from the KMS state associated with the evolution \( \tau'_\lambda \) on \( \mathcal{A} \). These representations are not irreducible. In particular, the algebras \( M_+ \) and \( M_- \) commute. One might expect \( \mathcal{M} \) to be of type III_{\lambda}, \( 0 < \lambda < 1 \) (see \([C73]\)).

**Acknowledgments**

Grigori Amosov is grateful to Professor Luigi Accardi for kind hospitality during his visit at the Centro Vito Volterra of Universita di Roma Tor Vergata.

**References**

[ALV99] L. Accardi, Y.G. Lu, I.V. Volovich, *White noise approach to classical and quantum stochastic calculi*, Centro Vito Volterra, Universita di Roma “Tor Vergata”, Preprint 375, 1999.

[AFS00] L. Accardi, U. Franz, M. Skeide, *Renormalized squares of white noise and other non-gaussian noises as Levy processes on real Lie algebras*, Centro Vito Volterra, Universita di Roma “Tor
Vergata", Preprint 423, 2000, Commun. Math. Phys. 228 (2002) 123-150.

[Par92] K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Birkhauser, 1992.

[Mey95] P.-A. Meyer, *Quantum Probability for Probabilists, Lecture Notes in Math.*, Vol. 1538, Springer-Verlag, Berlin, 2nd edition, 1995.

[C73] A. Connes, *Une classification des facteurs de type III*, Ann. Sci. Ecole Norm. Sup. 6 (1973) 133-252.