MODIFIED $6j$-SYMBOLS AND 3-MANIFOLD INVARIANTS

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Abstract. We show that the renormalized quantum invariants of links and graphs in the 3-sphere, derived from tensor categories in [6], lead to modified $6j$-symbols and to new state sum 3-manifold invariants. We give examples of categories such that the associated standard Turaev-Viro 3-manifold invariants vanish but the secondary invariants may be non-zero. The categories in these examples are pivotal categories which are neither ribbon nor semi-simple and have an infinite number of simple objects.

Dedicated to Jose Maria Montesinos on the occasion of his 65th birthday

INTRODUCTION

The numerical $6j$-symbols associated with the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ were first introduced in theoretical physics by Eugene Wigner in 1940 and Giulio (Yoel) Racah in 1942. They were extensively studied in the quantum theory of angular momentum, see for instance [4]. In mathematics, $6j$-symbols naturally arise in the study of semisimple monoidal tensor categories. In this context, the $6j$-symbols are not numbers but rather tensors on 4 variables running over certain multiplicity spaces. The $6j$-symbols have interesting topological applications: one can use them to write down state sums on knot diagrams and on triangulations of 3-manifolds yielding topological invariants of knots and of 3-manifolds, see [9], [11], [2].

The aim of this paper is to introduce and to study “modified” $6j$-symbols. This line of research extends our previous paper [6] where we introduced so-called modified quantum dimensions of objects of a monoidal tensor category. These new dimensions are particularly interesting when the usual quantum dimensions are zero since the modified dimensions may be non-zero. The standard $6j$-symbols can be viewed as a far-reaching generalization of quantum dimensions of objects. Therefore it is natural to attempt to “modify” their definition following the ideas of [6]. We discuss here a natural setting for such a modification based on a notion of an ambidextrous pair. In this setting, our constructions produce an interesting (and new) system of tensors. These tensors share some of the properties of the standard $6j$-symbols including the fundamental Biedenharn-Elliott identity. We

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suggest a general scheme allowing to derive state-sum topological invariants of links in 3-manifolds from the modified 6j-symbols.

As an example, we study the modified 6j-symbols associated with a version of the category of representations of $sl_2(\mathbb{C})$. More precisely, we introduce a Hopf algebra $\tilde{U}_q(sl(2))$ and study the monoidal category of weight $\tilde{U}_q(sl(2))$-modules at a complex root of unity $q$ of odd order $r$. This category has an infinite number of simple objects parametrized by elements of $\mathbb{C}/2r\mathbb{Z} \approx \mathbb{C}^\ast$. The standard quantum dimensions of these objects and the standard 6j-symbols are generically zero.

We show that the modified quantum dimensions are generically non-zero. As an application, we construct state-sum topological invariants of triples $(\text{a closed oriented 3-manifold } M, \text{a link in } M, \text{an element of } H^1(M;\mathbb{C}^\ast))$.

Our techniques are similar to those used by Kashaev [7] and later Baseilhac and Benedetti [3] who derived 3-manifold invariants from the category of modules over the Borel subalgebra of $U_q(sl_2(\mathbb{C}))$ at a root of unity $q$. It is interesting to understand exact connections between these constructions; we are currently working on this question.

1. Pivotal tensor categories

We recall the definition of a pivotal tensor category, see for instance, [1]. A tensor category $\mathcal{C}$ is a category equipped with a covariant bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ called the tensor product, an associativity constraint, a unit object $\mathbb{I}$, and left and right unit constraints such that the Triangle and Pentagon Axioms hold. When the associativity constraint and the left and right unit constraints are all identities we say that $\mathcal{C}$ is a strict tensor category. By MacLane’s coherence theorem, any tensor category is equivalent to a strict tensor category. To simplify the exposition, we formulate further definitions only for strict tensor categories; the reader will easily extend them to arbitrary tensor categories.

A strict tensor category $\mathcal{C}$ has a left duality if for each object $V$ of $\mathcal{C}$ there are an object $V^\ast$ of $\mathcal{C}$ and morphisms

$$b_V : \mathbb{I} \rightarrow V \otimes V^\ast \quad \text{and} \quad d_V : V^\ast \otimes V \rightarrow \mathbb{I} \quad (1)$$

such that

$$(\text{Id}_V \otimes d_V)(b_V \otimes \text{Id}_V) = \text{Id}_V \quad \text{and} \quad (d_V \otimes \text{Id}_{V^\ast})(\text{Id}_{V^\ast} \otimes b_V) = \text{Id}_{V^\ast}.$$ 

A left duality determines for every morphism $f : V \rightarrow W$ in $\mathcal{C}$ the dual (or transposed) morphism $f^\ast : W^\ast \rightarrow V^\ast$ by

$$f^\ast = (d_W \otimes \text{Id}_{V^\ast})(\text{Id}_{W^\ast} \otimes f \otimes \text{Id}_{V^\ast})(\text{Id}_{W^\ast} \otimes b_V),$$

and determines for any objects $V, W$ of $\mathcal{C}$, an isomorphism $\gamma_{V, W} : W^\ast \otimes V^\ast \rightarrow (V \otimes W)^\ast$ by

$$\gamma_{V, W} = (d_W \otimes \text{Id}_{(V \otimes W)^\ast})(\text{Id}_{W^\ast} \otimes d_V \otimes \text{Id}_W \otimes \text{Id}_{(V \otimes W)^\ast})(\text{Id}_{W^\ast} \otimes \text{Id}_{V^\ast} \otimes b_{V \otimes W}).$$
Similarly, \( \mathcal{C} \) has a **right duality** if for each object \( V \) of \( \mathcal{C} \) there are an object \( V^\bullet \) of \( \mathcal{C} \) and morphisms

\[
b'_V : I \to V^\bullet \otimes V \quad \text{and} \quad d'_V : V \otimes V^\bullet \to I
\]

such that

\[
(Id_{V^\bullet} \otimes d'_V)(b'_V \otimes Id_{V^\bullet}) = Id_{V^\bullet} \quad \text{and} \quad (d'_V \otimes Id_V)(Id_V \otimes b'_V) = Id_V.
\]

The right duality determines for every morphism \( f : V \to W \) in \( \mathcal{C} \) the dual morphism \( f^\bullet : W^\bullet \to V^\bullet \) by

\[
f^\bullet = (Id_{V^\bullet} \otimes d'_W)(Id_{V^\bullet} \otimes f \otimes Id_{W^\bullet})(b'_V \otimes Id_{W^\bullet}),
\]

and determines for any objects \( V, W \), an isomorphism \( \gamma'_{V,W} : W^\bullet \otimes V^\bullet \to (V \otimes W)^\bullet \) by

\[
\gamma'_{V,W} = (Id_{(V \otimes W)^\bullet} \otimes d'_V)(Id_{(V \otimes W)^\bullet} \otimes Id_V \otimes d'_W \otimes Id_{V^\bullet})(b'_V \otimes W^\bullet \otimes Id_{W^\bullet} \otimes Id_{V^\bullet}).
\]

A **pivotal category** is a tensor category with left duality \( \{b_V, d_V\}_V \) and right duality \( \{b'_V, d'_V\}_V \) which are compatible in the sense that \( V^\ast = V^\bullet, f^\ast = f^\bullet \), and \( \gamma_{V,W} = \gamma'_{V,W} \) for all \( V, W, f \) as above.

A tensor category \( \mathcal{C} \) is said to be an **Ab-category** if for any objects \( V, W \) of \( \mathcal{C} \), the set of morphism \( \text{Hom}(V, W) \) is an additive abelian group and both composition and tensor multiplication of morphisms are bilinear. Composition of morphisms induces a commutative ring structure on the abelian group \( K = \text{End}(I) \). The resulting ring is called the **ground ring** of \( \mathcal{C} \). For any objects \( V, W \) of \( \mathcal{C} \) the abelian group \( \text{Hom}(V, W) \) becomes a left \( K \)-module via \( kf = k \otimes f \) for \( k \in K \) and \( f \in \text{Hom}(V, W) \). An object \( V \) of \( \mathcal{C} \) is **simple** if \( \text{End}(V) = K \text{Id}_V \).

### 2. Invariants of Graphs in \( S^2 \)

From now on and up to the end of the paper the symbol \( \mathcal{C} \) denotes a pivotal tensor Ab-category with ground ring \( K \), left duality \( [1] \), and right duality \( [2] \).

A morphism \( f : V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_m \) in \( \mathcal{C} \) can be represented by a box and arrows as in Figure 1. Here the plane of the picture is oriented counterclockwise, and this orientation determines the numeration of the arrows attached to the bottom and top sides of the box. More generally, we allow such
boxes with some arrows directed up and some arrows directed down. For example, if all the bottom arrows in the above picture and redirected upward, then the box represents a morphism $V_1^* \otimes \cdots \otimes V_n^* \to W_1 \otimes \cdots \otimes W_m$. The boxes as above are called coupons. Each coupon has distinguished bottom and top sides and all incoming and outgoing arrows can be attached only to these sides.

By a graph we always mean a finite graph with oriented edges (we allow loops and multiple edges with the same vertices). By a $C$-colored ribbon graph in an oriented surface $\Sigma$, we mean a graph embedded in $\Sigma$ whose edges are colored by objects of $C$ and whose vertices lying in $\text{Int} \, \Sigma = \Sigma - \partial \Sigma$ are thickened to coupons colored by morphisms of $C$. The edges of a ribbon graph do not meet each other and may meet the coupons only at the bottom and top sides. The intersection of a $C$-colored ribbon graph in $\Sigma$ with $\partial \Sigma$ is required to be empty or to consist only of vertices of valency 1.

We define a category of $C$-colored ribbon graphs $\text{Gr}_C$. The objects of $\text{Gr}_C$ are finite sequences of pairs $(V, \varepsilon)$, where $V$ is an object of $C$ and $\varepsilon = \pm$. The morphisms of $\text{Gr}_C$ are isotopy classes of $C$-colored ribbon graphs $\Gamma$ embedded in $\mathbb{R} \times [0,1]$. The (1-valent) vertices of such $\Gamma$ lying on $\mathbb{R} \times 0$ are called the inputs. The colors and orientations of the edges of $\Gamma$ incident to the inputs (enumerated from the left to the right) determine an object of $\text{Gr}_C$ called the source of $\Gamma$. Similarly, the (1-valent) vertices of $\Gamma$ lying on $\mathbb{R} \times 1$ are called the outputs; the colors and orientations of the edges of $\Gamma$ incident to the outputs determine an object of $\text{Gr}_C$ called the target of $\Gamma$. We view $\Gamma$ as a morphism in $\text{Gr}_C$ from the source object to the target object. More generally, formal linear combinations over $K$ of $C$-colored ribbon graphs in $\mathbb{R} \times [0,1]$ with the same input and source are also viewed as morphisms in $\text{Gr}_C$. Composition, tensor multiplication, and left and right duality in $\text{Gr}_C$ are defined in the standard way, cf. [10]. This makes $\text{Gr}_C$ into a pivotal Ab-category.

The well-known Reshetikhin-Turaev construction defines a $K$-linear functor $G : \text{Gr}_C \to C$ preserving both left and right dualities. This functor is compatible with tensor multiplication and transforms an object $(V, \varepsilon)$ of $\text{Gr}_C$ to $V$ if $\varepsilon = +$ and to $V^*$ if $\varepsilon = -$. The definition of $G$ goes by splitting the ribbon graphs into “elementary” pieces, cf. [10]. The left duality in $C$ is used to assign morphisms in $C$ to the small right-oriented cup-like and cap-like arcs. The right duality is used to assign morphisms in $C$ to the small left-oriented cup-like and cap-like arcs. Invariance of $G$ under plane isotopy is deduced from the conditions $f^* = f^\bullet$ and $\gamma_{V,W} = \gamma'_{V,W}$ in the definition of a pivotal tensor category.

Under certain assumptions, one can thicken the usual (non-ribbon) graphs in $S^2$ into ribbon graphs. The difficulty is that thickening of a vertex to a coupon is not unique. We recall a version of thickening for trivalent graphs following [10].

By basic data in $C$ we mean a family $\{V_i\}_{i \in I}$ of simple objects of $C$ numerated by elements of a set $I$ with involution $I \to I$, $i \mapsto i^*$ and a family of isomorphisms
\{w_i : V_i \to V_i^*\}_{i \in I}$ such that $\text{Hom}_C(V_i, V_j) = 0$ for distinct $i, j \in I$ and

$$d_{V_i}(w_{i^*} \otimes \text{Id}_{V_i}) = d_{V_i^*}((\text{Id}_{V_i} \otimes w_i)) : V_i \otimes V_i \to I$$

for all $i \in I$. In particular, $V_i$ is not isomorphic to $V_j$ for $i \neq j$.

For any $i, j, k \in I$, consider the multiplicity module

$$H_{ij} = \text{Hom}(I, V_i \otimes V_j \otimes V_k).$$

The $K$-modules $H_{ij}, H_{ji}, H_{ki}$ are canonically isomorphic. Indeed, let $\sigma(i, j, k)$ be the isomorphism

$$H_{ij} \to H_{ji}, \quad x \mapsto d_{V_i} \circ (\text{Id}_{V_i^*} \otimes x \otimes \text{Id}_{V_i}) \circ \delta_{V_i^*}.$$

Using the functor $G : \text{Gr}_C \to C$, one easily proves that

$$\sigma(k, i, j) \sigma(j, k, i) \sigma(i, j, k) = \text{Id}_{H_{ij}}.$$

Identifying the modules $H_{ij}, H_{ji}, H_{ki}$ along these isomorphisms we obtain a symmetrized multiplicity module $H(i, j, k)$ depending only on the cyclically ordered triple $(i, j, k)$. This construction may be somewhat disturbing, especially if some (or all) of the indices $i, j, k$ are equal. We give therefore a more formal version of the same construction. Consider a set $X$ consisting of three (distinct) elements $\{a, b, c\}$ with cyclic order $a < b < c < a$. For any function $f : X \to I$, the construction above yields canonical $K$-isomorphisms

$$H_{f(a), f(b), f(c)} \cong H_{f(b), f(c), f(a)} \cong H_{f(c), f(a), f(b)}$$

of the modules determined by the linear orders on $X$ compatible with the cyclic order. Identifying these modules along these isomorphisms we obtain a module $H(f)$ independent of the choice of a linear order on $X$ compatible with the cyclic order. If $i = f(a), j = f(b), k = f(c)$, then we write $H(i, j, k)$ for $H(f)$.

By a labeling of a graph we mean a function assigning to every edge of the graph an element of $I$. By a trivalent graph we mean a (finite oriented) graph whose vertices all have valency 3. Let $\Gamma$ be a labeled trivalent graph in $S^2$. Using the standard orientation of $S^2$ (induced by the right-handed orientation of the unit ball in $\mathbb{R}^3$), we cyclically order the set $X_v$ of 3 half-edges adjacent to any given vertex $v$ of $\Gamma$. The labels of the edges determine a function $f_v : X_v \to I$ as follows: if a half-edge $e$ adjacent to $v$ is oriented towards $v$, then $f_v(e) = i$ is the label of the edge of $\Gamma$ containing $e$; if a half-edge $e$ adjacent to $v$ is oriented away from $v$, then $f_v(e) = i^*$. Set $H_v(\Gamma) = H(f_v)$ and $H(\Gamma) = \otimes_v H_v(\Gamma)$ where $v$ runs over all vertices of $\Gamma$.

Consider now a labeled trivalent graph $\Gamma \subset S^2$ as above, endowed with a family of vectors $h = \{h_v \in H_v(\Gamma)\}_v$, where $v$ runs over all vertices of $\Gamma$. We thicken $\Gamma$ into a $C$-colored ribbon graph on $S^2$ as follows. First, we insert inside each edge $e$ of $\Gamma$ a coupon with one edge outgoing from the bottom along $e$ and with one edge outgoing from the top along $e$ in the direction opposite to the one on $e$. If $e$
is labeled with $i \in I$, then these two new (smaller) edges are labeled with $V_i$ and $V_{i^*}$, respectively, and the coupon is labeled with $w_i : V_i \to V_{i^*}$ as in Figure 2.

![Figure 2:](image)

Next, we thicken each vertex $v$ of $\Gamma$ to a coupon so that the three half-edges adjacent to $v$ yield three arrows adjacent to the top side of the coupon and oriented towards it. If $i, j, k \in I$ are the labels of these arrows (enumerated from the left to the right), then we color this coupon with the image of $h_v$ under the natural isomorphism $H_v(\Gamma) \to H_{ijk}$. Denote the resulting $C$-colored ribbon graph by $\Omega_{\Gamma, h}$. Then $G(\Gamma, h) = G(\Omega_{\Gamma, h})$ is an isotopy invariant of the pair $(\Gamma, h)$ independent of the way in which the vertices of $\Gamma$ are thickened to coupons.

In the next section we describe a related but somewhat different approach to invariants of colored ribbon graphs and labeled trivalent graphs in $S^2$.

### 3. Cutting of graphs and ambidextrous pairs

Let $T \subset S^2$ be a $C$-colored ribbon graph and let $e$ be an edge of $T$ colored with a simple object $V$ of $C$. Cutting $T$ at a point of $e$, we obtain a $C$-colored ribbon graph $T_V$ in $\mathbb{R} \times [0, 1]$ with one input and one output such that the edges incident to the input and output are oriented downward and colored with $V$. In other words, $T_V \in \text{End}_{Gr_C}((V, +))$. Note that the closure of $T_V$ (obtained by connecting the endpoints of $T_V$ by an arc in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ disjoint from the rest of $T_V$) is isotopic to $T$ in $S^2$. We call $T_V$ a cutting presentation of $T$ and let $\langle T_V \rangle \in K$ denote the isotopy invariant of $T_V$ defined from the equality $G(T_V) = \langle T_V \rangle \text{Id}_V$.

In the following definition and in the sequel, a ribbon graph is trivalent if all its coupons are adjacent to 3 half-edges.

**Definition 1.** Let $\{V_i, w_i\}_{i \in I}$ be basic data in $C$. Let $I_0$ be a subset of $I$ invariant under the involution $i \mapsto i^*$ and $d : I_0 \to K$ be a mapping such that $d(i) = d(i^*)$ for all $i \in I_0$. Let $\mathcal{T}_{I_0}$ be the class of $C$-colored connected trivalent ribbon graphs in $S^2$ such that the colors of all edges belong to the set $\{V_i\}_{i \in I}$ and the color of at least one edge belongs to the set $\{V_i\}_{i \in I_0}$. The pair $(I_0, d)$ is trivalent-ambidextrous if for any $T \in \mathcal{T}_{I_0}$ and for any two cutting presentations $T_{V_i}, T_{V_j}$ of $T$ with $i, j \in I_0$,

$$d(i)\langle T_{V_i} \rangle = d(j)\langle T_{V_j} \rangle.$$  

(4)
To simplify notation, we will say that the pair \((I_0, d)\) in Definition \([1]\) is \(t\)-ambi. For a \(t\)-ambi pair \((I_0, d)\) we define a function \(G' : T_{I_0} \to K\) by

\[
G'(T) = d(i) \langle T_{V_i} \rangle,
\]

where \(T_{V_i}\) is any cutting presentation of \(T\) with \(i \in I_0\). The definition of a \(t\)-ambi pair implies that \(G'\) is well defined.

The invariant \(G'\) can be extended to a bigger class of \(C\)-colored ribbon graphs in \(S^2\). We say that a coupon of a ribbon graph is straight if both its bottom and top sides are incident to exactly one arrow. We can remove a straight coupon and unite the incident arrows into a (longer) edge, see Figure 3. We call this operation straightening. A \(\text{quasi-trivalent ribbon graph}\) is a ribbon graph in \(S^2\) such that straightening it at all straight coupons we obtain a trivalent ribbon graph.

![Figure 3: Straightening a quasi-trivalent ribbon graph](image)

**Lemma 2.** Let \((I_0, d)\) be a \(t\)-ambi pair in \(C\) and \(\overline{T}_{I_0}\) be the class of connected quasi-trivalent ribbon graphs in \(S^2\) such that the colors of all edges belong to the set \(\{V_i\}_{i \in I}\), the color of at least one edge belongs to the set \(\{V_j\}_{j \in I_0}\), and all straight coupons are colored with isomorphisms in \(C\). Then Formula \(5\) determines a well-defined function \(G' : \overline{T}_{I_0} \to K\).

**Proof.** Given an endomorphism \(f\) of a simple object \(V\) of \(C\), we write \(\langle f \rangle\) for the unique \(a \in K\) such that \(f = a \text{Id}_V\). Observe that if \(f : V \to W\) and \(g : W \to V\) are homomorphisms of simple objects and \(f\) is invertible, then

\[
\langle fg \rangle = \langle gf \rangle.
\]

To see this, set \(a = \langle fg \rangle \in K\) and \(b = \langle gf \rangle \in K\). Then

\[
a \text{Id}_W = fg(f^{-1}) = f(gf)f^{-1} = bf^{-1} = b \text{Id}_W.
\]

Thus, \(a = b\).

Consider now a \(C\)-colored quasi-trivalent ribbon graph \(T\) in \(S^2\) and a straight coupon of \(T\) such that the arrow adjacent to the bottom side is outgoing and colored with \(V_i\), \(i \in I_0\) while the arrow adjacent to the top side is incoming and colored with \(V_j\), \(j \in I_0\). By assumption, the coupon is colored with an isomorphism \(V_i \to V_j\). Formula \(6\) implies that \(\langle T_{V_i} \rangle = \langle T_{V_j} \rangle\), where \(T_{V_i}\) is obtained by cutting \(T\) at a point of the bottom arrow and \(T_{V_j}\) is obtained by cutting \(T\) at a point of the top arrow. By the properties of a \(t\)-ambi pair, \(d(i) = \cdots\)
Hence Formula (5) yields the same element of $K$ for these two cuttings. The cases where some arrows adjacent to the straight coupon are oriented up (rather than down) are considered similarly, using the identity $d(i) = d(i^*)$. Therefore cutting $T$ at two different edges that unite under straightening, we obtain on the right-hand side of (5) the same element of $K$. When we cut $T$ at two edges which do not unite under straightening, a similar claim follows from the definitions. ☐

We can combine the invariant $G'$ with the thickening of trivalent graphs to obtain invariants of trivalent graphs in $S^2$. Suppose that $\Gamma \subset S^2$ is a labeled connected trivalent graph such that the label of at least one edge of $\Gamma$ belongs to $I_0$. We define

$$G'(\Gamma) \in H(\Gamma)^* = \text{Hom}_K(H(\Gamma), K)$$

as follows. Pick any family of vectors $h = \{h_v \in H_v(\Gamma)\}_v$, where $v$ runs over all vertices of $\Gamma$. The $C$-colored ribbon graph $\Omega_{\Gamma,h}$ constructed at the end of Section 2 belongs to the class $T_{I_0}$ defined in Lemma 2. Set

$$G'(\Gamma)(\otimes_v h_v) = G'(\Omega_{\Gamma,h}) \in K.$$ 

By the properties of $G'$, the vector $G'(\Gamma) \in H(\Gamma)^*$ is an isotopy invariant of $\Gamma$. Both $H(\Gamma)$ and $G'(\Gamma)$ are preserved under the reversion transformation inverting the orientation of an edge of $\Gamma$ and replacing the label of this edge, $i$, with $i^*$. This can be easily deduced from Formula (3).

4. Modified 6j-symbols

Let $C$ be a pivotal tensor Ab-category with ground ring $K$, basic data $\{V_i, w_i : V_i \to V_i^*\}_{i \in I}$, and t-ambi pair $(I_0, d)$. To simplify the exposition, we assume that there is a well-defined direct summation of objects in $C$. We define a system of tensors called modified 6j-symbols. Let $i, j, k, l, m, n$ be six elements of $I$ such that at least one of them is in $I_0$. Consider the labeled trivalent graph $\Gamma = \Gamma(i, j, k, l, m, n) \subset \mathbb{R}^2 \subset S^2$ given in Figure 4. By definition,

$$H(\Gamma) = H(i, j, k^*) \otimes_K H(k, l, m^*) \otimes_K H(n, l^*, j^*) \otimes_K H(m, n^*, i^*).$$ 

We define the modified 6j-symbol of the tuple $(i, j, k, l, m, n)$ to be

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = G'(\Gamma) \in H(\Gamma)^* = \text{Hom}_K(H(\Gamma), K).$$

(7)

It follows from the definitions that the modified 6j-symbols have the symmetries of an oriented tetrahedron. In particular,

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & k^* & i^* \\ m & n & l \end{vmatrix} = \begin{vmatrix} k & l & m \\ n^* & i & j^* \end{vmatrix}.$$
These equalities hold because the labeled trivalent graphs in $S^2$ defining these 6j-symbols are related by isotopies and reversion transformations described above.

To describe $H(\Gamma)^*$, we compute the duals of the symmetrized multiplicity modules. For any indices $i, j, k \in I$ such that at least one of them lies in $I_0$, we define a pairing

$$(x, y)_{ijk} : H(i, j, k) \otimes_K H(k^*, j^*, i^*) \to K$$

by

$$(x, y)_{ijk} = \mathbb{G}'(\Theta)(x \otimes y),$$

where $x \in H(i, j, k) = H^{ijk}$ and $y \in H(k^*, j^*, i^*) = H^{k^* j^* i^*}$ and $\Theta = \Theta_{i,j,k}$ is the theta graph with vertices $u, v$ and three edges oriented from $v$ to $u$ and labeled with $i, j, k$, see Figure 5. Clearly,

$$H_u(\Theta) = H^{ijk} = H(i, j, k) \quad \text{and} \quad H_v(\Theta) = H^{k^* j^* i^*} = H(k^*, j^*, i^*)$$

so that we can use $x, y$ as the colors of $u, v$, respectively. It follows from the definitions that the pairing $(., .)_{ijk}$ is invariant under cyclic permutations of $i, j, k$ and $(x, y)_{ijk} = (y, x)_{k^* j^* i^*}$ for all $x \in H(i, j, k)$ and $y \in H(k^*, j^*, i^*)$.

We now give sufficient conditions for the pairing $(., .)_{ijk}$ to be non-degenerate. We say that a pair $(i, j) \in I^2$ is good if $V_i \otimes V_j$ splits as a finite direct sum of some $V_k$’s (possibly with multiplicities) such that $k \in I_0$ and $d(k)$ is an invertible element of $K$. By duality, a pair $(i, j) \in I^2$ is good if and only if the pair $(j^*, i^*)$ is good. We say that a triple $(i, j, k) \in I^3$ is good if at least one of the pairs

Figure 4: $\Gamma(i, j, k, l, m, n)$

Figure 5: $\Theta_{i,j,k}$
(i, j), (j, k), (k, i) is good. Clearly, the goodness of a triple (i, j, k) is preserved under cyclic permutations and implies the goodness of the triple (k*, j*, i*). Note also that if a triple (i, j, k) is good, then at least one of the indices i, j, k belongs to I₀ so that we can consider the pairing (8).

**Lemma 3.** If the triple (i, j, k) ∈ I³ is good, then the pairing [8] is non-degenerate.

**Proof.** For any i, j, k ∈ I, consider the K-modules

\[ H_{ij}^k = \text{Hom}(V_k, V_i ⊗ V_j) \quad \text{and} \quad H_{ij}^k = \text{Hom}(V_i ⊗ V_j, V_k). \]

Consider the homomorphism

\[
H_{ij}^k \rightarrow H_{ij}^{k*}, \ y \mapsto (y \otimes w_k^{-1}) b_{V_k}. \tag{9}
\]

It is easy to see that this is an isomorphism. Composing with the canonical isomorphism \( H_{ij}^{k*} \rightarrow H(i, j, k*) \) we obtain an isomorphism \( a_{ij}^k : H_{ij}^k \rightarrow H(i, j, k*) \).

Similarly, we have an isomorphism

\[
a_{kij}^l : H_{ij}^k \rightarrow H_{kij}^{l*} = H(k, j*, i*), \ x \mapsto (x \otimes w_j^{-1} \otimes w_i^{-1})(\text{Id}_{V_i} \otimes b_{V_j} \otimes \text{Id}_{V_i^*}) b_{V_i}. \]

Let \( (,)^{ij}_k : H_{ij}^k \otimes_K H_{ij}^k \rightarrow K \) be the bilinear pairing whose value on any pair \( (y \in H_{ij}^k, x \in H_{ij}^k) \) is computed from the equality

\[
(y, x)^{ij}_k \text{Id}_{V_k} = d(k)xy : V_k \rightarrow V_k.
\]

It follows from the definitions that under the isomorphism

\[
a_{ij}^k \otimes a_{kij}^l : H_{ij}^k \otimes_K H_{ij}^k \rightarrow H(i, j, k*) \otimes_K H(k, j*, i*)
\]

the pairing \((,)^{ij}_k\) is transformed into \((,)^{ijk*}_k\).

We can now prove the claim of the lemma. By cyclic symmetry, it is enough to consider the case where the pair (i, j) is good. Then \( H_{ij}^k \) and \( H_{ij}^k \) are free K-modules of the same finite rank and the pairing \((,)^{ij}_k\) is non-degenerate. Therefore the pairing \((,)^{ijk*}_k\) is non-degenerate. \(\Box\)

We say that a 6-tuple \( (i, j, k, l, m, n) \in I^6 \) is good if all the triples

\[(i, j, k*), (k, l, m*), (n, l*, j*), (m, n*, i*) \tag{10}
\]

are good. This property of a 6-tuple is invariant under symmetries of an oriented tetrahedron acting on the labels of the edges. The previous lemma implies that if \( (i, j, k, l, m, n) \) is good, then the ambient module of the modified 6j-symbol [11] is

\[
H(k, j*, i*) \otimes_K H(m, l*, k*) \otimes_K H(j, l, n*) \otimes_K H(i, n, m*). \tag{11}
\]
5. Comparison with the standard $6j$-symbols

We compare the modified $6j$-symbols defined above with standard $6j$-symbols derived from decompositions of tensor products of simple objects into direct sums. We keep notation of Section 4 and begin with two simple lemmas.

**Lemma 4.** For any $i, j, k, l, m \in I$, the formula $(f, g) \mapsto (f \otimes \text{Id}_V)g$ defines a $K$-homomorphism

$$H^{ij}_k \otimes_K H^{kl}_m \to \text{Hom}(V_m, V_i \otimes V_j \otimes V_l).$$

(12)

If $(i, j)$ is good, then the direct sum of these homomorphisms is an isomorphism

$$\bigoplus_{k \in I} H^{ij}_k \otimes_K H^{kl}_m \to \text{Hom}(V_m, V_i \otimes V_j \otimes V_l).$$

(13)

**Proof.** The first claim is obvious; we prove the second claim. If $(i, j)$ is good, then for every $k \in I$, the $K$-modules $H^{ij}_k$ and $H^{k}_{ij}$ are free of the same finite rank. Fix a basis $\{\alpha_{k,r}\}_{r \in R_k}$ of $H^{ij}_k$ where $R_k$ is a finite indexing set. Then there is a basis $\{\alpha^{k,r}\}_{r \in R_k}$ of $H^{ij}_k$ such that $\alpha^{k,r} \alpha_{k,s} = \delta_{r,s} \text{Id}_{V_k}$ for all $r, s \in R_k$. Clearly,

$$\text{Id}_{V_i \otimes V_j} = \sum_{k \in I, r \in R_k} \alpha_{k,r} \alpha^{k,r}.$$ 

We define a $K$-homomorphism $\text{Hom}(V_m, V_i \otimes V_j \otimes V_l) \to \bigoplus_{k \in I} H^{ij}_k \otimes H^{kl}_m$ by

$$f \mapsto \sum_{k \in I, r \in R_k} \alpha_{k,r} \otimes_K (\alpha^{k,r} \otimes \text{Id}_V)f.$$ 

This is the inverse of the homomorphism (13). $\square$

Similarly, we have the following lemma.

**Lemma 5.** For any $i, j, l, m, n \in I$, the formula $(f, g) \mapsto (\text{Id}_V \otimes f)g$ defines a $K$-homomorphism

$$H^{ji}_n \otimes_K H^{lm}_m \to \text{Hom}(V_m, V_i \otimes V_j \otimes V_l).$$

(14)

If the pair $(j, l)$ is good, then the direct sum of these homomorphisms is an isomorphism

$$\bigoplus_{n \in I} H^{ji}_n \otimes_K H^{lm}_m \to \text{Hom}(V_m, V_i \otimes V_j \otimes V_l).$$

(15)

Suppose that both pairs $(i, j)$ and $(j, l)$ are good. Composing the isomorphism (13) with the inverse of the isomorphism (15) we obtain an isomorphism

$$\bigoplus_{k \in I} H^{ij}_k \otimes_K H^{kl}_m \to \bigoplus_{n \in I} H^{ji}_n \otimes_K H^{lm}_m.$$ 

(16)
Restricting to the summand on the left-hand side corresponding to \( k \) and projecting to the summand on the right-hand side corresponding to \( n \) we obtain a homomorphism

\[
\begin{array}{c}
\left\{ \begin{array}{ccc}
  i & j & k \\
  l & m & n
\end{array} \right\} : H_k^{ij} \otimes_K H_{m}^{kl} \to H_n^{jl} \otimes_K H_{m}^{in}.
\end{array}
\] (17)

This is the (standard) \(6j\)-symbol determined by \( i, j, k, l, m, n \). We emphasize that it is defined only when the pairs \((i, j)\) and \((j, l)\) are good. Note that this \(6j\)-symbol is equal to zero unless both \( k \) and \( n \) belong to \( I_0 \).

The following equality is an equivalent graphical form of the same definition. It indicates that the composition of the homomorphisms (17) and (14) summed up over all \( n \in I \) is equal to the homomorphism (12):

\[
\frac{\begin{array}{c}
\begin{array}{c}
  i \\
  j \\
  k
\end{array}
\end{array}}{\begin{array}{c}
\begin{array}{c}
  l \\
  m \\
  n
\end{array}
\end{array}} = \sum_{n \in I} \frac{\begin{array}{c}
\begin{array}{c}
  i \\
  j \\
  n
\end{array}
\end{array}}{\begin{array}{c}
\begin{array}{c}
  l \\
  m \\
  n
\end{array}
\end{array}} \circ \left\{ \begin{array}{ccc}
  i & j & k \\
  l & m & n
\end{array} \right\}.
\]

Our next aim is to relate the \(6j\)-symbol (17) to the modified \(6j\)-symbol (7).

Recall the isomorphism \(a_{ij}^k: H_k^{ij} \to H(i, j, k^*)\) introduced in the proof of Lemma 3. Using these isomorphisms, we can rewrite the \(6j\)-symbol (17) as a homomorphism

\[
\begin{array}{c}
\left\{ \begin{array}{ccc}
  i & j & k \\
  l & m & n
\end{array} \right\} : H(i, j, k^*) \otimes_K H(k, l, m^*) \to H(j, l, n^*) \otimes_K H(i, n, m^*)
\end{array}
\] (18)

Since \((j, l)\) is good, \(H(j, l, n^*)^* = H(n, l^*, j^*)\). Assuming that \((i, n)\) is good, we write \(H(i, n, m^*)^* = H(m, n^*, i^*)\) and consider the homomorphism

\[
\left\{ \begin{array}{ccc}
  i & j & k \\
  l & m & n
\end{array} \right\}^\sigma : H(i, j, k^*) \otimes_K H(k, l, m^*) \otimes_K H(n, l^*, j^*) \otimes_K H(m, n^*, i^*) \to K
\]

adjoint to (18). This homomorphism has the same source module as the modified \(6j\)-symbol (7).

**Lemma 6.** For any \(i, j, k, l, m, n \in I\) such that the pairs \((i, j), (j, l), (i, n)\) are good and \(m, n \in I_0\),

\[
\left\{ \begin{array}{ccc}
  i & j & k \\
  l & m & n
\end{array} \right\}^\sigma = d(n) \left| \begin{array}{ccc}
  i & j & k \\
  l & m & n
\end{array} \right|.
\] (19)

**Proof.** Pick any \(x_1 \in H_k^{ij}, x_2 \in H_m^{kl}\). By Lemma 5,

\[
(x_1 \otimes \text{Id}_{V_i})x_2 = \sum_{n' \in I} \sum_{r \in R_{n'}} (\text{Id}_{V_i} \otimes y_{1,n',r}^{n',r})y_{2,n',r}^{n',r},
\]

where \(y_{1,n',r}^{n',r} \in H_{n'}^{ij}, y_{2,n',r}^{n',r} \in H_{m'}^{in},\) and \(R_{n'}\) is a finite set of indices. By definition,

\[
\left\{ \begin{array}{ccc}
  i & j & k \\
  l & m & n
\end{array} \right\}(x_1 \otimes x_2) = \sum_{r \in R_n} y_{1,n,r}^{n,r} \otimes y_{2,n,r}^{n,r}.
\]
Under our assumptions on $i, j, k, l, m, n$, the pairings $(, )_n^{jl}$ and $(, )_m^{jn}$ introduced in the proof of Lemma 3 are non-degenerate, and we use them to identify $H_n^{jl} = (H_n^{jl})^*$ and $H_m^{jn} = (H_m^{jn})^*$. Consider the homomorphism

$$\left\{ \begin{array}{c} i \\ j \\ k \\ l \\ m \\ n \end{array} \right\} \rightarrow H_n^{ij} \otimes_K H_m^{kl} \otimes_K H_n^{jl} \otimes_K H_m^{jn} \rightarrow K$$

(adjoint to $\left\{ \begin{array}{c} i \\ j \\ k \\ l \\ m \\ n \end{array} \right\}$). The computations above show that

$$\left\{ \begin{array}{c} i \\ j \\ k \\ l \\ m \\ n \end{array} \right\} \left( x_1 \otimes x_2 \otimes x_3 \otimes x_4 \right) = \sum_{r \in R_n} (y_1^{n,r}, x_3)_n^{jl} (y_2^{n,r}, x_4)_m^{jn}$$

for any $x_3 \in H_n^{jl}$ and $x_4 \in H_m^{jn}$.

Consider the $C$-colored ribbon graphs $\Gamma_1, \Gamma_2', \Gamma_3^{n', r}$ in Figure 6 (it is understood that an edge with label $s \in I$ is colored with $V_s$). It is clear that

$$G' (\Gamma_1) = \sum_{n' \in I, r \in R_{n'}} G' (\Gamma_2') = \sum_{n' \in I, r \in R_{n'}} \delta_{n,n'} d(n)^{-1} (y_1^{n,r}, x_3)_n^{jl} G' (\Gamma_3^{n', r})$$

where the second equality follows from the fact that $xy = d(n)^{-1} (y, x) \text{Id}_{V_n}$ for $x \in H_n^{jl}$ and $y \in H_n^{jl}$. Similarly, $G' (\Gamma_3^{n', r}) = (y_2^{n', r}, x_4)_m^{jn}$ (here we use the inclusion $m \in I_0$). Therefore

$$d(n) G' (\Gamma_1) = \left\{ \begin{array}{c} i \\ j \\ k \\ l \\ m \\ n \end{array} \right\} \left( x_1 \otimes x_2 \otimes x_3 \otimes x_4 \right).$$

Rewriting this equality in terms of the symmetrized multiplicity modules, we obtain the claim of the lemma. \[ \square \]
We say that a 6-tuple \((i, j, k, l, m, n) \in I^6\) is strongly good if \(m, n \in I_0\) and the pairs \((i, j), (j, l), (i, n), (k, l)\) are good. A strongly good 6-tuple \((i, j, k, l, m, n)\) is good in the sense of Section 4, so that both associated \(6j\)-symbols \(\{i, j, k\} \in 6\) and \(\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}\) lie in the \(K\)-module \((11)\). Lemma 6 yields equality \((19)\) understood as an equality in this \(K\)-module.

Unfortunately, the notion of a strongly good 6-tuple is not invariant under symmetries of an oriented tetrahedron. To make it symmetric, one has to add more conditions on the labels. We say that a tuple \((i, j, k, l, m, n) \in I^6\) is admissible if all the indices \(i, j, k, l, m, n\) belong to \(I_0\) and the pairs \((i, j), (j, l), (i, n), (k, l), (j, k^*), (k^*, i), (l, m^*), (m^*, k), (n, l^*), (j^*, n), (m, n^*), (i^*, m)\) are good. Admissible 6-tuples are strongly good, and the notion of an admissible 6-tuple is preserved under the symmetries of an oriented tetrahedron.

6. Properties of the modified \(6j\)-symbols

Given a good triple \((i, j, k) \in I^3\) and a tensor product of several \(K\)-modules such that among the factors there is a matched pair \(H(i, j, k), H(k^*, j^*, i^*)\), we may contract any element of this tensor product using the pairing \((8)\). This operation is called the contraction along \(H(i, j, k)\) and denoted by \(*_{ijk}\). For example, an element \(x \otimes y \otimes z \in H(i, j, k) \otimes_k H(k^*, j^*, i^*) \otimes_k H\), where \(H\) is a \(K\)-module, contracts into \((x, y)z \in H\).

**Theorem 7** (The Biedenharn-Elliott identity). Let \(j_0, j_1, \ldots, j_8\) be elements of \(I\) such that the tuples \((j_1, j_2, j_5, j_8, j_0, j_7)\) and \((j_5, j_3, j_6, j_4, j_0, j_8)\) are strongly good, \(j_7, j_8 \in I_0\), and the pair \((j_2, j_3)\) is good. Set
\[
J = \{j \in I \mid H_{j_0}^{j_2j_3} \neq 0\} \subset I_0.
\]

If the pairs \((j_1, j)\) and \((j, j_4)\) are good for all \(j \in J\), then all 6-tuples defining the \(6j\)-symbols in the following formula are strongly good and
\[
\sum_{j \in J} d(j) *_{j_2j_3j_4} *_{j_1j_4j_5} *_{j_3j_4j_5} \left( \begin{array}{ccc} j_1 & j_2 & j_5 \\ j_3 & j_6 & j \\ j_4 & j_0 & j_7 \end{array} \right) \otimes \begin{array}{ccc} j_1 & j_2 & j_5 \\ j_4 & j_0 & j_7 \end{array} \right) = *_{j_0j_3j_8}^j \left( \begin{array}{ccc} j_5 & j_3 & j_6 \\ j_4 & j_0 & j_8 \end{array} \right) \right).
\]

Here both sides lie in the tensor product of six \(K\)-modules
\[
H(j_6, j_3, j_5^*), H(j_5, j_2, j_1^*), H(j_0, j_4, j_6^*), H(j_1, j_7, j_0^*), H(j_2, j_8, j_7^*), H(j_3, j_4, j_8^*).
\]

**Proof.** The claim concerning the strong goodness follows directly from the definitions. We verify \((23)\). Recall that if \((i, j)\) is a good pair, and \(V_k\) is a summand of \(V_i \otimes V_j\), then \(d(k)\) is invertible. Therefore if \(d(j_7)\) is not invertible, then \(V_{j_7}\) cannot be a summand of \(V_{j_2} \otimes V_{j_8}\), and so both sides of \((23)\) are equal to 0. Similarly
if $d(j_8)$ is not invertible, then $V_{j_8}$ can not be a summand of $V_{j_8} \otimes V_{j_8}$, and so both sides of (23) are equal to 0. We assume from now on that $d(j_7)$ and $d(j_8)$ are invertible in $K$.

The Pentagon Axiom for the tensor multiplication in $C$ implies that

$$
\sum_{j \in J} \left( I_{j_0}^{j_1j_7} \otimes \left\{ \begin{array}{c} j_2 \ j_3 \ j_7 \\ j_4 \ j_7 \ j_8 \end{array} \right\} \right) \left( \left\{ \begin{array}{c} j_1 \ j \ j_6 \\ j_4 \ j_0 \ j_7 \end{array} \right\} \otimes I_{j_2j_3}^{j_7j_6} \right) \left( I_{j_6j_4}^{j_8} \otimes \left\{ \begin{array}{c} j_1 \ j_2 \ j_5 \\ j_4 \ j_6 \ j \end{array} \right\} \right)
= \left( \left\{ \begin{array}{c} j_1 \ j_2 \ j_5 \\ j_8 \ j_0 \ j_7 \end{array} \right\} \otimes I_{j_8}^{j_2j_5} \right) P_{23} \left( \left\{ \begin{array}{c} j_5 \ j_3 \ j_6 \\ j_4 \ j_0 \ j_8 \end{array} \right\} \otimes I_{j_1j_2}^{j_6j_5} \right)
$$

(24)

where $I_i^k$ is the identity automorphism of $H_i^k$, $I_{j}^{j_2j_5}$ is the identity automorphism of $H_{j_2j_5}^j$, and $P_{23}$ is the permutation of the second and third factors in the tensor product (cf. Theorem VI.1.5.1 of [10]). Both sides of (24) are homomorphisms

$$
H_{j_0}^{j_6j_4} \otimes H_{j_6}^{j_2j_5} \otimes H_{j_5}^{j_1j_2} \to H_{j_0}^{j_1j_7} \otimes H_{j_7}^{j_2j_8} \otimes H_{j_8}^{j_3j_4}.
$$

We can rewrite all $6j$-symbols in (24) in the form (18). Note that the homomorphism (18) carries any $x \in H(i,j,k) \otimes K H(k,l,m)$ to

$$
*_{ijk} *_{klm} \left( \left\{ \begin{array}{c} i \ j \ k \\ l \ m \ n \end{array} \right\} \otimes x \right) = d(n) *_{ijk} *_{klm} \left( \left\{ \begin{array}{c} i \ j \ k \\ l \ m \ n \end{array} \right\} \otimes x \right),
$$

where we suppose that the tuple $(i,j,k,l,m,n)$ is strongly good and consider both $6j$-symbols as vectors in the module (11).

Denote by $C_j$ the tensor product of the three $6j$-symbols in the $j$-th term of the sum on the left hand side of (23). Denote by $D$ the tensor products of the two $6j$-symbols on the right hand side of (23). We can rewrite (24) as

$$
\sum_{j \in J} d(j_8) d(j_7) d(j) *_{j_2j_3j_7} *_{j_4j_2j_5} *_{j_6j_4j_6} *_{j_1j_2j_5} *_{j_3j_3j_6} (C_j \otimes x)
= d(j_8) d(j_7) *_{j_1j_2j_5} *_{j_3j_6j_6} *_{j_3j_3j_6} (D \otimes x)
$$

for all $x \in H(j_6,j_4,j_0^*) \otimes H(j_5,j_3,j_0^*) \otimes H(j_1,j_2,j_0^*)$. Since $d(j_7)$ and $d(j_8)$ are invertible elements of $K$, the previous equality implies that

$$
\sum_{j \in J} d(j) *_{j_2j_3j_7} *_{j_4j_2j_5} *_{j_6j_4j_6} (C_j) = *_{j_3j_6j_6} (D).
$$

This proves the theorem. □

**Theorem 8** (The orthonormality relation). Let $i,j,k,l,m,p$ be elements of $I$ such that $k,m \in I_0$ and the pairs $(i,j)$, $(j,l)$, $(p,l)$, $(k,l)$ are good. Set

$$
N = \{ n \in I \mid H_n^{jl} \neq 0 \text{ and } H_m^{lm} \neq 0 \} \subset I_0.
$$
If the pair \((i, n)\) is good for all \(n \in N\), then both 6-tuples defining the 6j-symbols in the following formula are good (in fact, the first one is strongly good) and

\[
d(k) \sum_{n \in N} d(n) *_{inm^*} *_{jln^*} \left( \begin{array}{ccc} i & j & p \\ l & m & n \end{array} \right) \otimes \left( \begin{array}{ccc} k & j^* & i \\ n & m & l \end{array} \right) = \delta_{k,p} \text{Id}(i, j, k^*) \otimes \text{Id}(k, l, m^*),
\]

where \(\delta_{k,p}\) is the Kronecker symbol and \(\text{Id}(a, b, c)\) is the canonical element of \(H(a, b, c) \otimes_K H(c^*, b^*, a^*)\) determined by the duality pairing.

**Proof.** Consider any \(i, j, l, m \in I\) such that the pairs \((i, j)\) and \((j, l)\) are good and consider the associated isomorphism (16). Restricting the inverse isomorphism to the summand in the source corresponding to \(n \in I\) and projecting into the summand in the target corresponding to \(k \in I\) we obtain a homomorphism

\[
\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}_{\text{inv}} : H^j_k \otimes_K H^i_n \to H^i_j \otimes_K H^k_l.
\] (25)

These homomorphisms corresponding to fixed \(i, j, l, m\) and various \(k, n \in I\) form a block-matrix of the isomorphism inverse to (16). Therefore,

\[
\sum_{n \in I} \left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}_{\text{inv}} \circ \left\{ \begin{array}{ccc} i & j & p \\ l & m & n \end{array} \right\} = \delta^k_p(i, j, l, m)
\] (26)

where \(\delta^k_p(i, j, l, m)\) is zero if \(k \neq p\) and is the identity automorphism of \(H^i_j \otimes_K H^k_l\) if \(k = p\).

Switching to the symmetrized multiplicity modules as in Section 5, we can rewrite (25) as a homomorphism

\[
H(j, l, n^*) \otimes_K H(i, n, m^*) \to H(i, j, k^*) \otimes_K H(k, l, m^*).\] (27)

Since \((i, j)\) is good, \(H(i, j, k^*) = H(k, j^*, i^*)\). Assuming that \((k, l)\) is good, we can write \(H(k, l, m^*) = H(m, l^*, k^*)\). Consider the homomorphism

\[
\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}_{\text{inv}}^\sigma : H(j, l, n^*) \otimes_K H(i, n, m^*) \otimes_K H(k, j^*, i^*) \otimes_K H(m, l^*, k^*) \to K
\]

adjoint to (27). This homomorphism has the same source module as the 6j-symbol

\[
\left| \begin{array}{ccc} k & j^* & i \\ n & m & l \end{array} \right|,
\]

where we suppose that \(k \in I_0\).

The argument similar to that in Lemma 6 (using the graphs \(\Gamma_4\) and \(\Gamma_5\) in Figure 7) shows that if the pairs \((i, j), (j, l), (k, l)\) are good and \(k, m \in I_0\), then

\[
\left\{ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right\}_{\text{inv}}^\sigma = d(k) \left| \begin{array}{ccc} k & j^* & i \\ n & m & l \end{array} \right|.
\] (28)
Assuming additionally that the pair \((i, n)\) is good, we can view both 6\(j\)-symbols in (28) as vectors in

\[
H(n, l^\ast, j^\ast) \otimes_K H(m, n^\ast, i^\ast) \otimes_K H(i, j, k^\ast) \otimes_K H(k, l, m^\ast).
\]

Then the homomorphism (27) carries any \(y \in H(j, l, n^\ast) \otimes_K H(i, n, m^\ast)\) to

\[
*_{jln^\ast} *_{imm^\ast} \left( \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right)_{\text{inv}} \otimes y = d(k) *_{jln^\ast} *_{imm^\ast} \left( \begin{array}{ccc} k & j^\ast & i \\ n & m & l \end{array} \right) \otimes y.
\]

Now we can rewrite (26) as

\[
\sum_{n \in N} d(n) d(k) *_{jln^\ast} *_{imm^\ast} *_{ijk^\ast} *_{klm^\ast} \left( \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right) \otimes x = \delta_{k,p} x
\]

for all \(x \in H(i, j, k^\ast) \otimes_K H(k, l, m^\ast)\). This implies the claim of the theorem. \(\square\)

7. AMBIDENTROUS OBJECTS AND STANDARD 6\(j\)-SYMBOLS

To apply the results of the previous sections, we must construct a pivotal Ab-category \(\mathcal{C}\) with basic data \(\{V_i, w_i\}_{i \in I}\) and a t-ambi pair \((I_0, d)\). In this and the next sections, we give examples of such objects using the technique of ambidextrous objects introduced in [6]. We first briefly recall this technique.

7.1. Ambidextrous objects. Let \(\mathcal{C}\) be a (strict) ribbon Ab-category, i.e., a (strict) pivotal tensor Ab-category with braiding and twist. We denote the braiding morphisms in \(\mathcal{C}\) by \(c_{V,W} : V \otimes W \to W \otimes V\) and the duality morphisms \(b_V, d_V, b'_V, d'_V\) as in Section 1. We assume that the ground ring \(K\) of \(\mathcal{C}\) is a field.

For an object \(J\) of \(\mathcal{C}\) and an endomorphism \(f\) of \(J \otimes J\), set

\[
\text{tr}_L(f) = (d_J \otimes \text{Id}_J) \circ (\text{Id}_{J^\ast} \otimes f) \circ (b'_J \otimes \text{Id}_J) \in \text{End}(J),
\]

\[
\text{tr}_R(f) = (\text{Id}_J \otimes d'_J) \circ (f \otimes \text{Id}_{J^\ast}) \circ (\text{Id}_J \otimes b_J) \in \text{End}(J).
\]

![Figure 7:](image_url)
An object \( J \) of \( \mathcal{C} \) is \textit{ambidextrous} if \( \text{tr}_L(f) = \text{tr}_R(f) \) for all \( f \in \text{End}(J \otimes J) \).

Let \( \text{Rib}_C \) be the category of \( \mathcal{C} \)-colored ribbon graphs and let \( F : \text{Rib}_C \to \mathcal{C} \) be the usual ribbon functor (see [10]). Let \( T_V \) be a \( \mathcal{C} \)-colored \((1,1)\)-ribbon graph whose open string is oriented downward and colored with a simple object \( V \) of \( \mathcal{C} \). Then \( F(T_V) \in \text{End}_C(V) = K \text{Id}_V \). Let \( < T_V > \in K \) be such that \( F(T_V) = < T_V > \text{Id}_V \). For any objects \( V, V' \) of \( \mathcal{C} \) such that \( V' \) is simple, set

\[
S'(V, V') = \left\langle \begin{array}{c} V' \\ V \end{array} \right\rangle \in K.
\]

Fix basic data \( \{V_i, w_i\}_{i \in I} \) in \( \mathcal{C} \) and a simple ambidextrous object \( J \) of \( \mathcal{C} \). Set

\[
I_0 = I_0(J) = \{ i \in I : S'(J, V_i) \neq 0 \}.
\]

For \( i \in I_0 \), set

\[
d_J(i) = \frac{S'(V_i, J)}{S'(J, V_i)} \in K.
\]

We view \( d_J(i) \) as the modified quantum dimension of \( V_i \) determined by \( J \).

**Theorem 9 (§5).** Let \( L \) be a \( \mathcal{C} \)-colored ribbon \((0,0)\)-graph having an edge \( e \) colored with \( V_i \) where \( i \in I_0 \). Cutting \( L \) at \( e \), we obtain a colored ribbon \((1,1)\)-graph \( T^e \) whose closure is \( L \). Then the product \( d_J(i) < T^e > \in K \) is independent of the choice of \( e \) and yields an isotopy invariant of \( L \).

**Corollary 10.** The pair \((I_0 = I_0(J), d_J)\) is a tambi pair in \( \mathcal{C} \).

**7.2. Example: The standard quantum 6j-symbols.** Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \). Let \( q \) be a primitive complex root of unity of order \( 2r \), where \( r \) is a positive integer. Let \( U_q(\mathfrak{g}) \) be the Drinfeld-Jimbo \( \mathbb{C} \)-algebra associated to \( \mathfrak{g} \). This algebra is presented by the generators \( E_k, F_k, K_k, K_k^{-1} \), where \( k = 1, \ldots, m \) and the usual relations. Let \( \widehat{U}_q(\mathfrak{g}) \) be the Hopf algebra obtained as the quotient of \( U_q(\mathfrak{g}) \) by the two-sided ideal generated by \( E_k^r, F_k^r, K_k^r - 1 \) with \( k = 1, \ldots, m \). It is known that \( \widehat{U}_q(\mathfrak{g}) \) is a finite dimensional ribbon Hopf algebra. The category of \( \widehat{U}_q(\mathfrak{g}) \)-modules of finite complex dimension is a ribbon Ab-category. It gives rise to a ribbon Ab-category \( \mathcal{C} \) obtained by annihilating all negligible morphisms (see Section XI.4 of [10]).

Let \( I \) be the set of weights belonging to the Weyl alcove determined by \( \mathfrak{g} \) and \( r \). For \( i \in I \), denote by \( V_i \) the simple weight module with highest weight \( i \). By [12], there is an involution \( I \to I, i \to i^* \) and morphisms \( \{w_i : V_i \to V_i^*\}_{i \in I} \) satisfying Equation (2). Then \( \mathcal{C} \) is a modular category with basic data \( \{V_i\}_{i \in I} \).

Let \( J = \mathbb{C} \) be the unit object of \( \mathcal{C} \). By Lemma 1 of [6], the object \( J \) is ambidextrous and by Corollary 10 the pair \((I_0 = I_0(J), d_J)\) is a tambi pair. The general theory of [6] implies that \( I_0 = I \) and \( d_J \) is the usual quantum dimension.
We can apply the techniques of Sections 4 and 5 to the basic data \( \{ V_i, w_i \}_{i \in I} \), and the t-ambi pair \(( I = I_0(J), d_J \)\). It is easy to check that the corresponding modified 6j-symbols are the usual quantum 6j-symbols associated to \( \mathfrak{g} \). Note that here all pairs \(( i, j ) \in I^2 \) are good.

8. Example: Quantum 6j-symbols from \( U_q^H(\mathfrak{sl}(2)) \) at roots of unity

In this section we consider a category of modules over the quantization \( \bar{U}_q^H(\mathfrak{sl}(2)) \) of \( \mathfrak{sl}(2) \) introduced in \[6\]. The usual quantum dimensions and the standard 6j-symbols associated to this category are generically zero. We equip this category with basic data and a t-ambi pair leading to non-trivial 6j-symbols.

Set \( q = e^{k \pi i / r} \in \mathbb{C} \), where \( r \) is a positive integer and \( k \) is an odd integer coprime with \( r \). We use the notation \( q^x \) for \( e^{x k \pi i / r} \), where \( x \in \mathbb{C} \) or \( x \) is an endomorphism of a finite dimensional vector space.

Let \( U_q^H(\mathfrak{sl}(2)) \) be the standard quantization of \( \mathfrak{sl}(2) \), i.e. the \( \mathbb{C} \)-algebra with generators \( E, F, K, K^{-1} \) and the following defining relations:

\[
K K^{-1} = K^{-1} K = 1, \quad K E K^{-1} = q^2 E, \quad K F K^{-1} = q^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.
\]

(31)

This algebra is a Hopf algebra with coproduct \( \Delta \), counit \( \varepsilon \), and antipode \( S \) defined by the formulas

\[
\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},
\]

\[
\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1,
\]

\[
S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}.
\]

Let \( U_q^H(\mathfrak{sl}(2)) \) be the \( \mathbb{C} \)-algebra given by the generators \( E, F, K, K^{-1}, H \), relations (31), and the following additional relations:

\[
HK = KH, \quad HK^{-1} = K^{-1} H, \quad [H, E] = 2E, \quad [H, F] = -2F.
\]

The algebra \( U_q^H(\mathfrak{sl}(2)) \) is a Hopf algebra with coproduct \( \Delta \), counit \( \varepsilon \), and antipode \( S \) defined as above on \( E, F, K, K^{-1} \) and defined on \( H \) by the formulas

\[
\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H.
\]

Following [6], we define \( \bar{U}_q^H(\mathfrak{sl}(2)) \) to be the quotient of \( U_q^H(\mathfrak{sl}(2)) \) by the relations \( E^r = F^r = 0 \). It is easy to check that the operations above turn \( \bar{U}_q^H(\mathfrak{sl}(2)) \) into a Hopf algebra.

Let \( V \) be a \( \bar{U}_q^H(\mathfrak{sl}(2)) \)-module. An eigenvalue \( \lambda \in \mathbb{C} \) of the operator \( H : V \rightarrow V \) is called a weight of \( V \) and the associated eigenspace \( E_\lambda(V) \) is called a weight space. We call \( V \) a weight module if \( V \) is finite-dimensional, splits as a direct sum of weight spaces, and \( q^H = K \) as operators on \( V \).
Given two weight modules \( V \) and \( W \), the operator \( H \) acts as \( H \otimes 1 + 1 \otimes H \) on \( V \otimes W \). So, \( E_\lambda(V) \otimes E_\mu(W) \subset E_{\lambda+\mu}(V \otimes W) \) for all \( \lambda, \mu \in \mathbb{C} \). Moreover, \( q^{\Delta(H)} = \Delta(K) \) as operators on \( V \otimes W \). Thus, \( V \otimes W \) is a weight module.

Let \( \mathcal{C}^H \) be the tensor Ab-category of weight \( U_q^H(\mathfrak{sl}(2)) \)-modules. By Section 6.2 of [6], \( \mathcal{C}^H \) is a ribbon Ab-category with ground ring \( \mathbb{C} \). In particular, for any object \( V \) in \( \mathcal{C}^H \), the dual object and the duality morphisms are defined as follows: \( V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C}) \) and

\[
\begin{align*}
b_v : \mathbb{C} &\rightarrow V \otimes V^* \text{ is given by } 1 \mapsto \sum v_j \otimes v_j^*, \\
d_v : V^* \otimes V &\rightarrow \mathbb{C} \text{ is given by } f \otimes w \mapsto f(w), \\
d'_v : V \otimes V^* &\rightarrow \mathbb{C} \text{ is given by } v \otimes f \mapsto f(K^{1-r}v), \\
b'_v : \mathbb{C} &\rightarrow V^* \otimes V \text{ is given by } 1 \mapsto \sum v_j^* \otimes K^{r-1}v_j, \\
\end{align*}
\]

where \( \{v_j\} \) is a basis of \( V \) and \( \{v_j^*\} \) is the dual basis of \( V^* \).

For an isomorphism classification of simple weight \( U_q(\mathfrak{sl}(2)) \)-modules (i.e., modules on which \( K \) acts diagonally), see for example [8], Chapter VI. This classification implies that simple weight \( U_q^H(\mathfrak{sl}(2)) \)-modules are classified up to isomorphism by highest weights. For \( i \in \mathbb{C} \), we denote by \( V_i \) the simple weight \( U_q^H(\mathfrak{sl}(2)) \)-module of highest weight \( i + r - 1 \). This notation differs from the standard labeling of highest weight modules. Note that \( V_{-r+1} = \mathbb{C} \) is the trivial module and \( V_0 \) is the so called Kashaev module.

We now define basic data in \( \mathcal{C}^H \). Set \( I = \mathbb{C} \) and define an involution \( i \mapsto i^* \) on \( I \) by \( i^* = i \) if \( i \in \mathbb{Z} \) and \( i^* = -i \) if \( i \in \mathbb{C} \setminus \mathbb{Z} \).

**Lemma 11.** There are isomorphisms \( \{w_i : V_i \rightarrow (V_i^*)^*\}_{i \in I} \) satisfying (3).

**Proof.** If \( i \in \mathbb{C} \setminus \mathbb{Z} \), then \( V_i^* \cong V_{-i} \), see [6]. We take an arbitrary isomorphism \( V_i \rightarrow (V_i)^* = V_{-i} \) for \( w_i \) and choose \( w_i \) so that it satisfies (3). If \( i \in \mathbb{Z} \), then \( i = i^* \) and there is a unique integer \( d \) such that \( 1 \leq d \leq r \) and \( d \equiv i \) (mod \( r \)). Let \( v_0 \) be a highest weight vector of \( V_i \). Set \( v_j = F^j v_0 \) for \( j = 1, \ldots, d-1 \). Then \( \{v_0, \ldots, v_{d-1}\} \) is a basis of \( V_i \) and in particular, \( \dim V_i = d \). Let \( \{v_j^*\} \) be the basis of \( V_i^* \) dual to \( \{v_j\} \) so that \( v_j^*(v_k) = \delta_{j,k} \). Define an isomorphism \( w_i : V_i \rightarrow (V_i)^* \) by \( v_0 \mapsto v_{d-1}^* \).

To verify (3), it suffices to check that the morphism \( (w_i^{-1})^*w_i : V_i \rightarrow V_i^{**} = V_i \) is multiplication by \( K^{1-r} \). To do this it is enough to calculate the image of \( v_0 \). A direct calculation shows that

\[
S(F)^{d-1}v_0 = (-1)^{d-1}q^{-(d-1)}v_{d-1} = q^{r(d-1)-(d-1)}v_{d-1}.
\]

It follows that \( F^{d-1}v_{d-1}^* = q^{r(1-d)}v_0^* \). Thus for \( k = 0, \ldots, d-1 \),

\[
(w_i^{-1})^*(v_{d-1}^*)(v_k^*) = v_{d-1}^*(w_i^{-1}(v_k^*)) = \delta_{k,0}q^{-(r-1)(d-1)}.
\]
In other words, \( (w_i^{-1})^* (v_{d-1}^*) = q^{-(r-1)(d-1)} v_0^* \). Under the standard identification \( V_i^{**} = V_i \), we obtain

\[
(w_i^{-1})^* w_i (v_0) = (w_i^{-1})^* (v_{d-1}^*) = q^{-(r-1)(d-1)} v_0.
\]

The proof is completed by noticing that \( K^{1-r} v_0 = q^{-(r-1)(d-1)} v_0 \). \( \square \)

Set \( I_0 = (\mathbb{C} \setminus \mathbb{Z}) \cup r \mathbb{Z} \subset I = \mathbb{C} \). We say that a simple weight \( \tilde{U}_q^H(\mathfrak{sl}(2)) \)-module \( V_i \) is typical if \( i \in I_0 \). By [6], every typical module \( J = V_i \) is ambidextrous and the set \( I_0(J) \) defined by (29) is equal to \( I_0 \). Formula (30) defines a function \( d_J : I_0 \to \mathbb{C} \). As shown in [6], up to multiplication by a non-zero complex number, \( d_J \) is equal to the function \( d : I_0 \to \mathbb{C} \) defined by

\[
d(k) = \frac{1}{\prod_{j=0}^{r-2} \{ k - j - 1 \}} \text{ for } k \in I_0
\]

(here \( \{ a \} = q^a - q^{-a} \) for all \( a \in \mathbb{C} \)). Note in particular that

\[
d(k) = d(k + 2r)
\]

for all \( k \in I_0 \). Theorem [9] implies the following lemma.

**Lemma 12.** \( (I_0, d) \) is a tambi pair in \( C^H \).

We can apply the techniques of Sections [4] and [5] to the category \( C^H \) with basic data \( \{ V_i, w_i \}_{i \in I} \) and the tambi pair \( (I_0, d) \). This yields a 6j-symbol \( \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \) for all indices \( i, j, k, l, m, n \in I = \mathbb{C} \) with at least one of them in \( I_0 \). In the case \( i, j, k, l, m, n \in \mathbb{C} \setminus \mathbb{Z} \) this 6j-symbol is identified with a complex number: for any numbers \( i, j, k \in \mathbb{C} \setminus \mathbb{Z} \) we have that \( \dim H(i, j, k) = 0 \) or \( \dim H(i, j, k) = 1 \). In the last case, there is a natural choice of an isomorphism \( H(i, j, k) \cong \mathbb{C} \). Hence the 6j-symbols can be viewed as taking value in \( \mathbb{C} \) and can be effectively computed via certain recurrence relations forming a “skein calculus" for \( C^H \)-colored ribbon trivalent graphs. These relations and the resulting explicit formulas for the 6j-symbols will be discussed in [5]. We formulate here one computation.

**Example 13.** For \( r = 3 \) and \( q = e^{i\pi/3} \) we can compute \( \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \) for all \( i, j, k, l, m, n \in \mathbb{C} \setminus \mathbb{Z} \) as follows.

1. If \( k = i + j + 2, \ l = n - j - 2, \) and \( m = n + i + 2, \) then

\[
\begin{array}{ccc} i & j & k \\ l & m & n \end{array} = \{ n + 1 \} \{ n + 2 \}.
\]

2. If \( k = i + j, \ l = n - j, \) and \( m = n + i + 2, \) then

\[
\begin{array}{ccc} i & j & k \\ l & m & n \end{array} = \{ j + 2 \} \{ n + 1 \}.
\]
If \( k = i + j, \ l = n - j, \) and \( m = n + i, \) then
\[
\begin{vmatrix}
  i & j & k \\
  l & m & n \\
\end{vmatrix}
= - \left( q^{i+j-n} + q^{j-n-i} + q^{n-i-j} + q^{i+j-n} + q^{j-n-i} + q^{n-i-j} \right).
\]

For tuples \( i, j, k, l, m, n \) \( \in \mathbb{C} \setminus \mathbb{Z} \) obtained from the tuples in (1), (2), (3) by tetrahedral permutations, the value of the 6j-symbol is computed by (1), (2), (3). In all other cases
\[
\begin{vmatrix}
  i & j & k \\
  l & m & n \\
\end{vmatrix} = 0.
\]

Note finally that in the present setting the admissibility condition of Section 5 is generically satisfied.

9. Example: Quantum 6j-symbols from \( U_q(\mathfrak{sl}(2)) \) at roots of unity

In this section we construct a category of \( U_q(\mathfrak{sl}(2)) \)-modules which is pivotal but \( \textit{a priori} \) not ribbon. We use the results of Section 8 to provide this category with basic data and a t-ambi pair.

Let \( r, q, \) and \( U_q(\mathfrak{sl}(2)) \) be as in Section 8. Assume additionally that \( r \) is odd and set \( r' = \frac{r-1}{2}. \) Let \( \bar{U}_q(\mathfrak{sl}(2)) \) be the Hopf algebra obtained as the quotient of \( U_q(\mathfrak{sl}(2)) \) by the relations \( E^r = F^r = 0. \)

A \( \bar{U}_q(\mathfrak{sl}(2)) \)-module \( V \) is a \textit{weight module} if \( V \) is finite dimensional and \( K \) act diagonally on \( V. \) We say that \( v \in V \) is a \textit{weight vector} with \textit{weight} \( \bar{\lambda} \in \mathbb{C}/2r\mathbb{Z} \) if \( Kv = q^{\bar{\lambda}}v. \) Let \( \mathcal{C} \) be the tensor Ab-category of weight \( \bar{U}_q(\mathfrak{sl}(2)) \)-modules (the ground ring is \( \mathbb{C} \)).

Lemma 14. The category \( \mathcal{C} \) is a pivotal Ab-category with duality morphisms (32).

Proof. It is a straightforward calculation to show that the left and right duality are compatible and define a pivotal structure on \( \mathcal{C}. \) \( \square \)

If \( V \) is a weight \( \bar{U}_q(\mathfrak{sl}(2)) \)-module containing a weight vector \( v \) of weight \( \bar{\lambda} \) such that \( Ev = 0 \) and \( V = U_q(\mathfrak{sl}(2))v, \) then \( V \) is a \textit{highest weight module} with \textit{highest weight} \( \bar{\lambda}. \) The classification of simple weight \( U_q(\mathfrak{sl}(2)) \)-modules (see Chapter VI of [3]) implies that highest weight \( \bar{U}_q(\mathfrak{sl}(2)) \)-modules are classified up to isomorphism by their highest weights. For any \( \bar{i} \in \mathbb{C}/2r\mathbb{Z}, \) we denote by \( V_{\bar{i}} \) the highest weight module with highest weight \( \bar{i} + r - 1, \) where \( r - 1 \in \mathbb{C}/2r\mathbb{Z} \) is the projection of \( r - 1 \) to \( \mathbb{C}/2r\mathbb{Z}. \) The following lemma yields basic data in \( \mathcal{C}. \)

Lemma 15. Set \( I^C = (\mathbb{C} \setminus \mathbb{Z})/2r\mathbb{Z} \) and define an involution \( \bar{i} \mapsto \bar{i}^* \) on \( I^C \) by \( \bar{i}^* = -\bar{i}. \) Then there are isomorphisms \( \{w_i : V_{\bar{i}} \to (V_{\bar{i}})^*\}_{i \in I^C} \) satisfying Equation (3).

Proof. The lemma follows as in the proof of Lemma 11. \( \square \)
Theorem 16. Let \( d : I^C \rightarrow \mathbb{C} \) be the function
\[
d(\tilde{k}) = \frac{1}{\prod_{j=0}^{\tilde{k}-2} \{ \tilde{k} - 1 - j \}}
\] for any \( \tilde{k} \in I^C \). Then the pair \((I_0 = I^C, d)\) is tambi in \( \mathcal{C} \).

A proof of Theorem 16 will be given at the end of the section using the following preliminaries. Recall the category \( \mathcal{C}^H = \mathcal{U}_q^H(\mathfrak{sl}(2))\)-mod from Section 8. Let \( \varphi : \mathcal{C}^H \rightarrow \mathcal{C} \) be the functor which forgets the action of \( H \). Consider the functors \( G_C \) and \( G_{C^H} \) associated to \( \mathcal{C} \) and \( \mathcal{C}^H \), respectively (see Section 2). The functor \( \varphi : \mathcal{C}^H \rightarrow \mathcal{C} \) induces (in the obvious way) a functor \( \varphi_{Gr} : Gr_{C^H} \rightarrow Gr_C \) such that the following square diagram commutes:
\[
\begin{array}{ccc}
Gr_{C^H} & \xrightarrow{\varphi_{Gr}} & Gr_C \\
G_{C^H} \downarrow & & \downarrow G_C \\
\mathcal{C}^H & \xrightarrow{\varphi} & \mathcal{C}
\end{array}
\]

We say that the highest weight \( \tilde{U}_q(\mathfrak{sl}(2)) \)-module \( V_i \) is typical if \( \tilde{i} \in I^C \cup \{ \tilde{0}, \tilde{r} \} \). In other words, \( V_i \) is typical if its highest weight is in \( I^C \cup \{ \tilde{1}, 2\tilde{r} \} \subset \mathbb{C}/2r\mathbb{Z} \). A typical module \( V_i \) has dimension \( r \) and its weight vectors have the weights \( \tilde{i} + 2\tilde{k} \) where \( k = -r', -r'+1, \ldots, r' \). It can be shown that if a typical module \( V \) is a submodule of a finite dimensional \( \tilde{U}_q(\mathfrak{sl}(2)) \)-module \( W \), then \( V \) is a direct summand of \( W \). Therefore, if \( V_i \) and \( V_j \) are typical modules such that \( \tilde{i} + \tilde{j} \notin \mathbb{Z}/2r\mathbb{Z} \), then
\[
V_i \otimes V_j \cong \bigoplus_{k=-r'}^{r'} V_{i+j+2k}.
\]

As in Section 8 for any \( \tilde{i} \in \mathbb{C} \), we denote by \( V_i \) a weight module in \( \mathcal{C}^H \) of highest weight \( i + r - 1 \). Clearly, \( \varphi(V_i) \cong V_i \) for all \( \tilde{i} \in \mathbb{C} \), where \( \tilde{i} = i \mod 2r\mathbb{Z} \) and \( \cong \) denotes isomorphism in \( \mathcal{C} \). To specify an isomorphism \( \varphi(V_i) \cong V_i \), we fix for each \( \tilde{i} \in \mathbb{C} \) a highest weight vector in \( V_i \in \mathcal{C}^H \) and we fix for each \( \tilde{a} \in \mathbb{C}/2r\mathbb{Z} \) a highest weight vector in \( V_a \in \mathcal{C} \). Then for every \( \tilde{a} \in \mathbb{C} \), there is a canonical isomorphism \( \varphi(V_i) \rightarrow V_i \) in \( \mathcal{C} \) carrying the highest weight vector of \( V_i \) into the highest weight vector of \( V_i \). To simplify notation, we write in the sequel \( \varphi(V_i) = V_i \) for all \( \tilde{i} \in \mathbb{C} \).

We say that a triple \((i, j, k) \in \mathbb{C}^3\) has height \( i + j + k \in \mathbb{C} \). For \( i, j, k \in \mathbb{C} \), set
\[
H^{ijk} = Hom_{\mathcal{C}^H}(\mathbb{I}, V_i \otimes V_j \otimes V_k) \quad \text{and} \quad H^{ijk}_k = Hom_{\mathcal{C}^H}(V_k, V_i \otimes V_j).
\]

Lemma 17. Let \( i, j, k \in \mathbb{C} \setminus \mathbb{Z} \). If the height of \((i, j, k)\) does not belong to the set \( \{ -2r', -2r'+2, \ldots, 2r' \} \), then \( H^{ijk} = 0 \). If the height of \((i, j, k)\) belongs to the set \( \{ -2r', -2r'+2, \ldots, 2r' \} \), then the composition of the homomorphisms
\[
H^{ijk} = Hom_{\mathcal{C}^H}(\mathbb{I}, V_i \otimes V_j \otimes V_k) \xrightarrow{\varphi} Hom_{\mathcal{C}}(\mathbb{I}, \varphi(V_i) \otimes \varphi(V_j) \otimes \varphi(V_k))
\]
is an isomorphism.

Proof. We begin with a simple observation. Since \( i \in \mathbb{C} \setminus \mathbb{Z} \), the character formula for \( V_i \) is \( \sum_{l=-r'} u^{i+2l} \) where the coefficient of \( u^a \) is the dimension of the \( a \)-weight space. Therefore, the character formula for \( V_i \otimes V_j \) is

\[
\left( \sum_{l=-r'} u^{i+2l} \right) \left( \sum_{m=-r'} u^{j+2m} \right) = \sum_{l,m=-r'} u^{i+j+2l+2m}.
\] (38)

As we know, Formula (9) defines an isomorphism \( H^{ijk} \simeq H^{-ijk} \). We claim that \( \dim(H^{-ijk}) = 1 \) if the height of \( (i,j,k) \) belongs to \( \{-2r', -2r' + 2, \ldots, 2r'\} \) and \( H^{-ijk} = 0 \), otherwise. To see this, we consider two cases.

Case 1: \( i + j \in \mathbb{Z} \). Since \( k \notin \mathbb{Z} \), the height of \( (i,j,k) \) is not an integer and does not belong to the set \( \{-2r', -2r' + 2, \ldots, 2r'\} \). Equation (38) implies that all the weights of \( V_i \otimes V_j \) are integers. Since \( k \notin \mathbb{Z} \), we have \( H^{-ijk} = 0 \).

Case 2: \( i + j \notin \mathbb{Z} \). It can be shown that if \( V \in \mathcal{C}^H \) is a typical module which is a sub-module of a module \( W \in \mathcal{C}^H \), then \( V \) is a direct summand of \( W \). Combining with Equation (38) we obtain that

\[
V_i \otimes V_j \simeq \bigoplus_{l=-r'} V_{i+j+2l}.
\] (39)

Therefore \( H^{-ijk} \neq 0 \) if and only if \( i + j + k \in \{-2r', -2r' + 2, \ldots, 2r'\} \). In addition, Formula (39) implies that if \( H^{-ijk} \neq 0 \) then \( \dim(H^{-ijk}) = 1 \). This proves the claim above and the first statement of the lemma.

To prove the second part of the lemma, assume that the height of \( (i,j,k) \) is in \( \{-2r', -2r' + 2, \ldots, 2r'\} \). It is enough to show that the homomorphism

\[
\operatorname{Hom}_{\mathcal{C}^{H'}}(\mathbb{I}, V_i \otimes V_j \otimes V_k) \xrightarrow{\varphi} \operatorname{Hom}_{\mathcal{C}}(\mathbb{I}, \varphi(V_i) \otimes \varphi(V_j) \otimes \varphi(V_k))
\]

is an isomorphism. This homomorphism is injective by the very definition of the forgetful functor. So it suffices to show that the domain and the range have the same dimension. From the claim above, \( \dim(H^{-ijk}) = 1 \) and thus the domain is one-dimensional. The assumption on the height of \( (i,j,k) \) combined with Equation (37) implies that \( \dim \operatorname{Hom}_{\mathcal{C}}(\varphi(V_{-k}), \varphi(V_i) \otimes \varphi(V_j)) = 1 \). Thus, the range is also one-dimensional. This completes the proof of the lemma.

We say that a triple \( i, j, k \in \mathbb{C}/2r\mathbb{Z} \) has integral height if \( i + j + k \in \mathbb{Z}/2r\mathbb{Z} \) and \( i + j + k \) is even. In this case the height of the triple \( (i,j,k) \) is the unique (even) \( n \in \{-2r', -2r' + 2, \ldots, 2r'\} \) such that \( i + j + k = n \ (\text{mod} \ 2r) \).

By an \( I^C \)-colored ribbon graph we mean a \( C \)-colored ribbon graphs such that all edges are colored with modules \( V_i \) with \( i \in I^C = (\mathbb{C} \setminus \mathbb{Z})/2r\mathbb{Z} \). To each \( I^C \)-colored
trivalent coupon in $S^2$ we assign a triple of elements of $I^c$: each arrow attached to the coupon contributes a term $\varepsilon i \in I^c$ where $V_i$ is the color of the arrow and $\varepsilon = +1$ if the arrow is oriented towards the coupon, and $\varepsilon = -1$. Such a coupon has integral height if the associated triple has integral height. We say that a $I^c$-colored trivalent ribbon graph in $S^2$ has integral heights if all its coupons have integral height. The total height of an $I^c$-colored trivalent ribbon graph with integral heights is defined to be the sum of the heights of all its coupons. This total height is an even integer.

**Lemma 18.** Let $T$ be a $I^c$-colored trivalent ribbon graph in $S^2$. If $T$ has a coupon which does not have integral height, then $G_C(T_V) = 0$ for any cutting presentation $T_V$ of $T$.

**Proof.** Let $(i, j, k)$ be the triple associated with a coupon of $T$ which does not have integral height. An argument similar to the one in the proof of Lemma 17 shows that $\text{Hom}_C(\mathbb{Z}, V_i \otimes V_j \otimes V_k) = 0$. This forces $G_C(T_V) = 0$ for any cutting presentation $T_V$ of $T$. \hfill \Box

**Lemma 19.** Let $T$ be a connected $I^c$-colored trivalent ribbon graph in $S^2$ with no inputs and outputs and with integral heights. There exists a $I^c_H$-colored trivalent ribbon graph $\hat{T}$ such that $\varphi_{Gr}(\hat{T}) = T$ if and only if the total height of $T$ is zero. If the total height of $T$ is non-zero, then $G_C(T_V) = 0$ for any cutting presentation $T_V$ of $T$.

**Proof.** Given a CW-complex space $X$ and an abelian group $G$, we denote by $C_n(X; G)$ the abelian group of cellular $n$-chains of $X$ with coefficients in $G$.

Consider a connected $I^c$-colored trivalent ribbon graph $S$ with integral heights and possibly with inputs and outputs. Let $|S|$ be the underlying 1-dimensional CW-complex of $S$ and $|\partial S| \subset |S|$ be the set of univalent vertices (corresponding to the inputs and outputs of $S$). The coloring of $S$ determines a 1-chain $\tilde{c} \in C_1(|S|; \mathbb{C}/2r\mathbb{Z})$. Consider also the 0-chain

$$c_0 = \sum_{x \in |\partial S|} C_x x \in C_0(|\partial S|; \mathbb{C}/2r\mathbb{Z}),$$

where $C_x \in \mathbb{C}/2r\mathbb{Z}$ is the label of the only edge of $S$ adjacent to $x$ if this edge is oriented towards $x$ and minus this label otherwise. The heights of the coupons of $S$ form a 0-chain $w \in C_0(|S| \setminus |\partial S|; 2\mathbb{Z})$. Clearly,

$$\partial \tilde{c} = c_0 + w \pmod{2r\mathbb{Z}}. \quad (40)$$

Suppose that the 0-chain $c_0$ lifts to a certain 0-chain $b \in C_0(|\partial S|; \mathbb{C})$ such that $[b + w] = 0 \in H_0(|S|; \mathbb{C}) = \mathbb{C}$. We claim that then the 1-chain $\tilde{c}$ lifts to a 1-chain $c \in C_1(|S|, \mathbb{C})$ such that $\partial c = b + w$. Indeed, pick any $c' \in C_1(|S|; \mathbb{C})$ such that $\tilde{c} = c' \pmod{2r\mathbb{Z}}$. Set $x = \partial c' - b - w \in C_0(|S|; \mathbb{C})$. By (10), $x \in C_0(|S|; 2r\mathbb{Z})$. \hfill \Box
Clearly, \( [x] = -[b + w] = 0 \in H_0(|S|; 2r\mathbb{Z}) \). Then \( x = \partial\delta \) for some \( \delta \in C_1(|S|; 2r\mathbb{Z}) \) and \( c = c' - \delta \) satisfies the required conditions.

Let us now prove the first statement of the lemma. By assumption, \( |\partial T| = 0 \). As above, the coloring of \( T \) determines a 1-chain \( \tilde{c} \in C_1(|T|; \mathbb{C}/2r\mathbb{Z}) \). Let \( w \in C_0(|T|; 2\mathbb{Z}) \) be the 0-chain formed by the heights of the coupons of \( T \). The total height of \( T \) is \( [w] \in H_0(|T|; 2\mathbb{Z}) = 2\mathbb{Z} \). If \( [w] = 0 \), then the preceding argument (with \( b = 0 \)) shows that \( \tilde{c} \) lifts to a 1-chain \( c \in C_1(|T|; \mathbb{C}) \) such that \( \partial c = w \). The chain \( c \) determines a coloring of all edges of \( T \) by the modules \( \{ V_i \mid i \in \mathbb{C} \} \). Clearly, the height of any coupon is equal to the corresponding value of \( w \). In particular, all these heights belong to the set \( \{-2r', -2r' + 2, \ldots, 2r'\} \). By Lemma 17, this coloring of edges can be uniquely extended to a \( C^H \)-coloring \( \tilde{T} \) of our graph such that \( \varphi_{Gr}(\tilde{T}) = T \). Conversely, if \( \varphi_{Gr}(\tilde{T}) = T \) then with the same notation \( w = \partial c \) and thus the total height of \( T \) is zero.

Suppose now that \( [w] \neq 0 \). Consider the cutting presentation \( T_V \) where \( V = V_{\tilde{\alpha}} \) with \( \tilde{\alpha} \in \mathbb{C} \setminus \mathbb{Z} \mod 2r\mathbb{Z} \) is the color of an edge of \( T \). Here \( |\partial T_V| \) is the two-point set formed by the extremities of \( T_V \) and \( c_0 \) is the 0-chain \( \tilde{\alpha} \times (\text{the input vertex minus the output vertex}) \).

As
\[
[w] = [\partial \tilde{c}] - [c_0] = 0 \in H_0(|T_V|; \mathbb{C}/2r\mathbb{Z}),
\]
we have \( [w] \in 2r\mathbb{Z} \). Pick \( \alpha \in \mathbb{C} \setminus \mathbb{Z} \) such that \( \tilde{\alpha} = \alpha \mod 2r\mathbb{Z} \). Consider the 0-chain
\[
b = \alpha \times (\text{the input vertex minus the output vertex}) - [w] \times (\text{the output vertex}).
\]
Clearly, \( [w+b] = 0 \in H_0(|T_V|; \mathbb{C}) \). The argument at the beginning of the proof and Lemma 17 imply that there is \( \tilde{T} \in \text{Hom}_{Gr_{c,H}}(V_{\alpha}, V_{\alpha+[w]}) \) such that \( \varphi_{Gr}(\tilde{T}) = T_V \).

But then \( C_{c,H}(\tilde{T}) = 0 \) because it is a morphism between non-isomorphic simple modules. Therefore \( C_c(T_V) = C_c(\varphi_{Gr}(\tilde{T})) = \varphi(C_{c,H}(\tilde{T})) = 0 \).

**9.1. Proof of Theorem 16.** Let \( T_{V_i} \) and \( T_{V_j} \) be cutting presentations of a connected \( I^c \)-colored trivalent ribbon graph \( T \) in \( S^2 \) with no inputs and outputs. We claim that
\[
d(\tilde{i}) < G_c(T_{V_i}) > = d(\tilde{j}) < G_c(T_{V_j}) >.
\]
(41)

By the previous two lemmas it is enough to consider the case where all coupons of \( T \) have integral height and the total height of \( T \) is zero. By Lemma 19 there is a \( C^H \)-colored trivalent ribbon graph \( \tilde{T} \) such that \( \varphi_{Gr}(\tilde{T}) = T \). Cutting \( \tilde{T} \) at the same edges, we obtain \( C^H \)-colored graphs \( \tilde{T}_{V_i} \) and \( \tilde{T}_{V_j} \) carried by \( \varphi_{Gr} \) to \( T_{V_i} \) and \( T_{V_j} \), respectively. From Diagram (36) we have \( < G_{c,H}(\tilde{T}_{V_i}) > = < G_c(T_{V_i}) > \) and \( < G_{c,H}(\tilde{T}_{V_j}) > = < G_c(T_{V_j}) > \). Lemma 12 implies that \( d(i) < G_{c,H}(\tilde{T}_{V_i}) > = d(j) < G_{c,H}(\tilde{T}_{V_j}) > \). Since \( d(i) = d(\tilde{i}) \) and \( d(j) = d(\tilde{j}) \), these formulas imply Formula (41).
Remark 20. The category $\mathcal{C}$ with basic data $\{V_i, w_i\}_{i \in I}$ and t-ambi pair $(I^C = (\mathbb{C} \setminus \mathbb{Z})/2r\mathbb{Z}, d)$ determines modified $6j$-symbols $\begin{vmatrix} i & \tilde{j} & \tilde{k} \\ \tilde{l} & \tilde{m} & \tilde{n} \end{vmatrix}$ for $i, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n} \in I^C$.

Using Lemma 17, one observes that as in Section 8 the $6j$-symbols of this section can be viewed as taking values in $\mathbb{C}$. The values of these $6j$-symbols are essentially the same as the values of the $6j$-symbols derived from the category $\mathcal{C}^H$. More precisely, let $\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n} \in I^C$. If the total height of $\Gamma(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n})$ is non-zero, then $\begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ \tilde{l} & \tilde{m} & \tilde{n} \end{vmatrix} = 0$. If the total height of $\Gamma(\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n})$ is zero, then for some lifts $i, j, k, l, m, n \in \mathbb{C} \setminus \mathbb{Z}$ of $\tilde{i}, \tilde{j}, \tilde{k}, \tilde{l}, \tilde{m}, \tilde{n}$,

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} \tilde{i} & \tilde{j} & \tilde{k} \\ \tilde{l} & \tilde{m} & \tilde{n} \end{vmatrix}_{\mathcal{C}^H}.$$

Example 15 computes these $6j$-symbols for $r = 3$ and $q = e^{i\pi/3}$.

10. Three-manifold invariants

In this section we derive a topological invariant of links in closed orientable 3-manifolds from a suitable pivotal tensor Ab-category. We also show that Sections 7.2 and 9 yield examples of such categories.

10.1. Topological preliminaries. Let $M$ be a closed orientable 3-manifold and $L$ a link in $M$. Following [3], we use the term quasi-regular triangulation of $M$ for a decomposition of $M$ as a union of embedded tetrahedra such that the intersection of any two tetrahedra is a union (possibly, empty) of several of their vertices, edges, and (2-dimensional) faces. Quasi-regular triangulations differ from usual triangulations in that they may have tetrahedra meeting along several vertices, edges, and faces. Nevertheless, the edges of a quasi-regular triangulation have distinct ends. A Hamiltonian link in a quasi-regular triangulation $\mathcal{T}$ is a set $\mathcal{L}$ of unoriented edges of $\mathcal{T}$ such that every vertex of $\mathcal{T}$ belongs to exactly two edges of $\mathcal{L}$. Then the union of the edges of $\mathcal{T}$ belonging to $\mathcal{L}$ is a link $L$ in $M$. We call the pair $(\mathcal{T}, \mathcal{L})$ an H-triangulation of $(M, L)$.

Proposition 21 ([3], Proposition 4.20). Any pair (a closed connected orientable 3-manifold $M$, a non-empty link $L \subset M$) admits an H-triangulation.

10.2. Algebraic preliminaries. Let $\mathcal{C}$ be a pivotal tensor Ab-category with ground ring $K$, basic data $\{V_i, w_i : V_i \rightarrow V_i^*\}_{i \in I}$, and t-ambi pair $(I_0, d)$. As in Section 8.1, we assume that the ground ring $K$ of $\mathcal{C}$ is a field. To define the associated 3-manifold invariant we need the following requirements on $\mathcal{C}$. Fix an abelian group $G$. Suppose that $\mathcal{C}$ is $G$-graded in the sense that for all $g \in G$, we have a class $\mathcal{C}_g$ of object of $\mathcal{C}$ such that

(1) $I \in \mathcal{C}_0$, 

We shall assume that $G$ contains a set $X$ with the following properties:

(1) $X$ is symmetric: $-X = X$,
(2) $G$ can not be covered by a finite number of translated copies of $X$, in other words, for any $g_1, \ldots, g_n \in G$, we have $\bigcup_{i=1}^n (g_i + X) \neq G$,
(3) if $g \in G \setminus X$, then the set $I_g$ is a finite subset of $I_0$, $d(I_g) \subset K^*$, and every object of $C_g$ is isomorphic to a direct sum of a finite family of objects $\{V_i | i \in I_g\}$.

Note that the last condition and the definition of a basic data imply that for $g \in G \setminus X$, every simple object of $C_g$ is isomorphic to $V_i$ for a unique $i \in I_g$.

Finally, we assume to have a map $b : I_0 \rightarrow \mathbb{K}$ such that

(1) $b(i) = b(i^*)$, for all $i \in I_0$,
(2) for any $g, g_1, g_2 \in G \setminus X$ with $g + g_1 + g_2 = 0$ and for all $j \in I_g$,

$$b(j) = \sum_{j_1 \in I_{g_1}, j_2 \in I_{g_2}} b(j_1) b(j_2) \dim(H(j, j_1, j_2)).$$

Denoting by $b_g$ the formal sum $b_g = \sum_{j \in I_g} b(j) V_j$, one obtains $b_{g_1} \otimes b_{g_2} = b_{g_1 + g_2}$ whenever $g_1, g_2, g_1 + g_2 \in G \setminus X$. The map $g \mapsto b_g$ can be seen as a “representation” of $G \setminus X$ in the Grothendieck ring of $C$.

10.3. A state sum invariant. We start from the algebraic data described in the previous subsection and produce a topological invariant of a triple $(M, L, h)$, where $M$ is a closed connected oriented 3-manifold, $L \subset M$ is a non-empty link, and $h \in H^1(M, G)$.

Let $(T, L)$ be an $H$-triangulation of $(M, L)$. By a $G$-coloring of $T$, we mean a $G$-valued 1-cocycle $\Phi$ on $T$, that is a map from the set of oriented edges of $T$ to $G$ such that

(1) the sum of the values of $\Phi$ on the oriented edges forming the boundary of any face of $T$ is zero and
(2) $\Phi(-e) = -\Phi(e)$ for any oriented edge $e$ of $T$, where $-e$ is $e$ with opposite orientation.

Each $G$-coloring $\Phi$ of $T$ represents a cohomology class $[\Phi] \in H^1(M, G)$.

A state of a $G$-coloring $\Phi$ is a map $\varphi$ assigning to every oriented edge $e$ of $T$ an element $\varphi(e)$ of $I^{\Phi(e)}$ such that $\varphi(-e) = \varphi(e)^*$ for all $e$. The set of all states of $\Phi$ is denoted $\text{St}(\Phi)$. The identities $d(\varphi(e)) = d(\varphi(-e))$ and $b(\varphi(e)) = b(\varphi(-e))$ allow us to use the notation $d(\varphi(e))$ and $b(\varphi(e))$ for non-oriented edges.
We call a $G$-coloring of $(\mathcal{T}, \mathcal{L})$ admissible if it takes values in $G \setminus X$. Given an admissible $G$-coloring $\Phi$ of $(\mathcal{T}, \mathcal{L})$, we define a certain partition function (state sum) as follows. For each tetrahedron $T$ of $\mathcal{T}$, we choose its vertices $v_1, v_2, v_3, v_4$ so that the (ordered) triple of oriented edges $(\overrightarrow{v_1v_2}, \overrightarrow{v_1v_3}, \overrightarrow{v_1v_4})$ is positively oriented with respect to the orientation of $M$. Here by $\overrightarrow{v_1v_2}$ we mean the edge oriented from $v_1$ to $v_2$, etc. For each $\varphi \in \text{St}(\Phi)$, set

$$|T|_\varphi = \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \text{ where } \begin{cases} i = \varphi(v_2v_1), & j = \varphi(v_3v_2), & k = \varphi(v_4v_3) \\ l = \varphi(v_4v_1), & m = \varphi(v_4v_3), & n = \varphi(v_4v_2). \end{cases}$$

This 6$j$-symbol belongs to the tensor product of 4 multiplicity modules associated to the faces of $T$ and does not depend on the choice of the numeration of the vertices of $T$ compatible with the orientation of $M$. This follows from the tetrahedral symmetry of the (modified) 6$j$-symbol discussed in Section 4. Note that any face of $T$ belongs to exactly two tetrahedra of $\mathcal{T}$, and the associated multiplicity modules are dual to each other. The tensor product of the 6$j$-symbols $|T|_\varphi$ associated to all tetrahedra $T$ of $\mathcal{T}$ can be contracted using this duality. We denote by $\text{cntr}$ the tensor product of all these contractions. Let $\mathcal{T}_1$ be the set of unoriented edges $T$ and let $\mathcal{T}_3$ the set of tetrahedra of $\mathcal{T}$. Set

$$TV(\mathcal{T}, \mathcal{L}, \Phi) = \sum_{\varphi \in \text{St}(\Phi)} \prod_{e \in \mathcal{T}_3} d(\varphi(e)) \prod_{e \in \mathcal{L}} b(\varphi(e)) \text{ cntr } \bigotimes_{T \in \mathcal{T}_3} |T|_\varphi \in K.$$  

**Theorem 22.** $TV(\mathcal{T}, \mathcal{L}, \Phi)$ depends only on the isotopy class of $L$ in $M$ and the cohomology class $[\Phi] \in H^1(M, G)$. It does not depend on the choice of the $H$-triangulation of $(M, L)$ and on the choice of $\Phi$ in its cohomology class.  

A proof of this theorem will be given in the next section.  

**Lemma 23.** Any $h \in H^1(M, G)$ can be represented by an admissible $G$-coloring on an arbitrary quasi-regular triangulation $\mathcal{T}$ of $M$.  

**Proof.** Take any $G$-coloring $\Phi$ of $\mathcal{T}$ representing $h$. We say that a vertex $v$ of $\mathcal{T}$ is bad for $\Phi$ if there is an oriented edge $e$ in $\mathcal{T}$ outgoing from $v$ such that $\Phi(e) \in X$. It is clear that $\Phi$ is admissible if and only if $\Phi$ has no bad vertices. We show how to modify $\Phi$ in its cohomology class to reduce the number of bad vertices. Let $v$ be a bad vertex for $\Phi$ and let $E_v$ be the set of all oriented edges of $\mathcal{T}$ outgoing from $v$. Pick any

$$g \in G \setminus \left( \bigcup_{e \in E_v} (\Phi(e) + X) \right).$$

Let $c$ be the $G$-valued 0-cochain on $\mathcal{T}$ assigning $g$ to $v$ and 0 to all other vertices. The 1-cocycle $\Phi + \delta c$ takes values in $G \setminus X$ on all edges of $\mathcal{T}$ incident to $v$ and takes the same values as $\Phi$ on all edges of $\mathcal{T}$ not incident to $v$. Here we use the fact that the edges of $\mathcal{T}$ are not loops which follows from the quasi-regularity of $\mathcal{T}$.  

The transformation $\Phi \mapsto \Phi + \delta c$ decreases the number of bad vertices. Repeating this argument, we find a 1-cocycle without bad vertices. □

We represent any $h \in H^1(M, G)$ by an admissible $G$-coloring $\Phi$ of $T$ and set

$$TV(M, L, h) = TV(T, \mathcal{L}, \Phi) \in K.$$  

By Theorem 22, $TV(M, L, h)$ is a topological invariant of the triple $(M, L, h)$.

10.4. Example. Set $G = \mathbb{C}/2\mathbb{Z}$ and let $\mathcal{C}$ be the pivotal tensor Ab-category with basic data and t-ambi pair defined in Section 9. This category is $G$-graded: for $\pi \in G$, the class $C_\pi$ consists of the modules on which the central element $K^r \in U_q(\mathfrak{sl}(2))$ act as the scalar $q^r$. Set

$$X = \mathbb{Z}/2\mathbb{Z} \subset \mathbb{C}/2\mathbb{Z} = G.$$  

It is easy to see that $G, X$, and the constant function $b = r^{-2}$ satisfy the requirements of Section 10.2 (cf. (37)). The constructions above derive from this data a state-sum topological invariant of links in 3-manifolds.

The original Turaev-Viro-type state sum construction of 3-manifold invariants can not be applied to $\mathcal{C}$ because $\mathcal{C}$ contains infinitely many isomorphism classes of simple objects and is not semi-simple. Moreover, the standard quantum dimensions of simple objects in $\mathcal{C}$ are generically zero and hence the standard $6j$-symbols associated with $\mathcal{C}$ are generically equal to zero.

11. Proof of Theorem 22

Throughout this section, we keep notation of Theorem 22. We begin by explaining that any two $H$-triangulations of $(M, L)$ can be related by elementary moves adding or removing vertices, edges, etc. We call an elementary move positive if it adds edges and negative if it removes edges.

The first type of elementary moves are the so-called $H$-bubble moves. The positive $H$-bubble move starts with a choice of a face $F = v_1v_2v_3$ of $T$ such that at least one of its edges, say $v_2v_3$, is in $\mathcal{L}$. Consider two tetrahedra of $T$ meeting along $F$. We unglue these tetrahedra along $F$ and insert a 3-ball between the resulting two copies of $F$. We triangulate this 3-ball by adding a vertex $v$ at its center and three edges $vv_1$, $vv_2$, $vv_3$. The edge $v_2v_3$ is removed from $\mathcal{L}$ and replaced by the edges $\{vv_2, vv_3\}$. This move can be visualized as in the transformation of Figure 8a (where the bold (green) edges belong to $\mathcal{L}$). The inverse move is the negative $H$-bubble move.

The second type of elementary moves is the $H$-Pachner $2 \leftrightarrow 3$ moves shown in Figure 8b. It is understood that the newly added edge on the right is not an element of $\mathcal{L}$. The negative $H$-Pachner move is allowed only when the edge common to the three tetrahedra on the right is not in $\mathcal{L}$. 


Proposition 24 ([3], Proposition 4.23). Let $L$ be a non-empty link in a closed connected orientable 3-manifold $M$. Any two $H$-triangulations of $(M, L)$ can be related by a finite sequence of $H$-bubble moves and $H$-Pachner moves in the class of $H$-triangulations of $(M, L)$.

We will need one more type of moves on $H$-triangulations of $(M, L)$ called the $H$-lune moves. It is represented in Figure 8c where for the negative $H$-lune move we require that the disappearing edge is not in $L$. The $H$-lune move may be expanded as a composition of $H$-bubble moves and $H$-Pachner moves (see [3], Section 2.1), but it will be convenient for us to use the $H$-lune moves directly.

The following lemma is an algebraic analog of the $H$-bubble move.

Lemma 25. Let $g_1, g_2, g_3, g_4, g_5, g_6 \in G \setminus X$ with $g_3 = g_1 + g_2$, $g_6 = g_2 + g_4$ and $g_5 = g_1 + g_6$. If $i \in I^{g_1}$, $j \in I^{g_2}$, $k \in I^{g_3}$, then

$$d(k) \sum_{l \in I^{g_4}, m \in I^{g_5}, n \in I^{g_6}} d(n) b(l) b(m) *_{klm^*} *_{inn^*} *_{jmn^*} \left( \left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right| \otimes \left| \begin{array}{ccc} k & j & i \\ n & m & l \end{array} \right| \right)$$

$$= b(k) \operatorname{Id}(i, j, k^*)$$ \hspace{1cm} (42)

Proof. We can apply the orthonormality relation (Theorem 8) to the tuple consisting of $i, j, k$, arbitrary $l \in I^{g_4}$, $m \in I^{g_5}$, and $p = k$. Note that $k, m \in I_0$ and the pair $(i, j)$ is good because $V_i \otimes V_j \in C_{g_1 + g_2} = C_{g_3}$ and $g_3 \notin X$. Similarly, the pairs $(j, l)$ and $(k, l)$ are good. Analyzing the grading in $C$, we observe that the set $N$ appearing in the orthonormality relation is a subset of $I^{g_6}$. Moreover, if $n \in I^{g_6} \setminus N$ then

$$\left| \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right| = 0$$

as $H(j, l, n^*) = 0$ or $H(i, n, m^*) = 0$ for such $n$. If we multiply the orthogonality relation by $b(l) b(m)$, apply $*_{klm^*}$, and sum over all pairs $(l, m) \in I^{g_4} \times I^{g_5}$ we

\[\text{(a) } H\text{-bubble move}\]

\[\text{(b) } H\text{-Pachner move}\]

\[\text{(c) } H\text{-lune move}\]

Figure 8: Elementary moves
obtain that the left hand side of (42) is equal to
\[
\sum_{(l, m) \in I^4 \times I^5} b(l) b(m) *_{klm^*} (\text{Id}(i, j, k^*) \otimes \text{Id}(k, l, m^*))
\] (43)

Finally, using the equality \( *_{klm^*} (\text{Id}(k, l, m^*)) = \dim(H(k, l, m^*)) \) and the relations satisfied by \( b \), we obtain that the expression (43) is equal to \( b(k) \text{Id}(i, j, k^*) \).

**Lemma 26.** Let \( \Phi \) be an admissible \( G \)-coloring of \( T \). Suppose that \((T', L')\) is an \( H \)-triangulation obtained from \((T, L)\) by a negative \( H \)-Pachner, \( H \)-lune or \( H \)-bubble move. Then \( \Phi \) restricts to an admissible \( G \)-coloring \( \Phi' \) of \( T' \) and
\[
TV(T, L, \Phi) = TV(T', L', \Phi').
\] (44)

**Proof.** The values of \( \Phi' \) form a subset of the set of values of \( \Phi \), and therefore the admissibility of \( \Phi \) implies the the admissibility of \( \Phi' \).

The rest of the proof is similar to the one in [11, Section VII.2.3]. In particular, we can translate the \( H \)-Pachner, \( H \)-lune, and \( H \)-bubble move into algebraic identities: the Biedenharn-Elliott identity, the orthonormality relation and Equation (42), respectively. The first two identities require certain pairs of indices to be good. To see that all relevant pairs in this proof are good we make the following observation. If \( e_1, e_2, e_3 \) are consecutive oriented edges of a 2-face of \( T \), then \( \Phi(e_1) + \Phi(e_2) = \Phi(e_3) \in G \setminus X \), and the semi-simplicity of \( C_{\Phi(e_3)} \) implies that the pair \((\varphi(e_1), \varphi(e_2))\) is good for any \( \varphi \in \text{St}(\Phi) \). Translating into the language of 6\( j \)-symbols, we obtain that all the 6\( j \)-symbols involved in our state sums are admissible.

We will now give a detailed proof of (44) for a negative \( H \)-bubble move. Let \( \varphi' \in \text{St}(\Phi') \) and \( S \subset \text{St}(\Phi) \) be the set of all states \( \varphi \) of \( \Phi \) extending \( \varphi' \). It is enough to show that the term \( TV_{\varphi'} \) of \( TV(T', L', \Phi') \) associated to \( \varphi' \) is equal to the sum \( TV_S \) of the terms of \( TV(T, L, \Phi) \) associated to the states in the set \( S \). Here it is equivalent to work with the positive \( H \)-bubble move, which we do for convenience.

Recall the description of the positive bubble given at the beginning of this section. Let \( v, v_1, v_2, v_3 \) (resp. \( F = v_1v_2v_3 \)) be the vertices (resp. face) given in this description (see Figure 9). Set \( i = \varphi'(v_3v_1), j = \varphi'(v_1v_2) \) and \( k = \varphi'(v_3v_2) \).

![Figure 9: \( T_1 \cup T_2 \) colored by \( \varphi \in S \)]
Let $T_1$ and $T_2$ be the two new tetrahedra of $T$ and $f_1, f_2, f_3$ be their faces $vv_2v_3, vv_3v_1, vv_1v_2$. A state $\varphi \in S$ is determined by the values $l, m, n$ of $\varphi$ on the edges $v_2v, v_3v, v_1v$, respectively (see Figure 9).

As explained above, in the bubble move two tetrahedra meeting along $F$ are unglued and a 3-ball is inserted between the resulting two copies $F_1$ and $F_2$ of $F$. Note that $F_1$ and $F_2$ are faces of $T$. Also this 3-ball is triangulated with the two tetrahedra $T_1$ and $T_2$.

For a fixed state on a triangulation, denote by $*_f$ the contraction along a face $f$. One can write

$$TV_{\varphi} = *_F(b(\varphi'(v_3v_2))X)$$

where $X$ is all the factors of the state sum for $\varphi'$ except for $*_F$ and $b(\varphi'(v_3v_2))$. Since all $\varphi \in S$ restrict to $\varphi'$, we have that $TV_S$ is equal to

$$*_F*_{F_2}\left(X \otimes \sum_{\varphi \in S} d(\varphi(v_3v_2))d(\varphi(v_1v))b(\varphi(v_2v))b(\varphi(v_3v))*_f_1*_f_2*_f_3(|T_1|_{\varphi} \otimes |T_2|_{\varphi})\right)$$

where the additional factors come from the triangulation of the 3-ball and the fact that the link goes through $v_3v, vv_2$ instead of $v_3v_2$. Here, the contraction $*_F*_{F_2}$ is equal to $*_ij_k*_{ij_k*}$. On the other hand, one has $*_F(X) = *_{ij_k*}(X) = *_{ij_k*} *_{ij_k*} (X \otimes \text{Id}(i, j, k^*))$. Hence the equality $TV_{\varphi'} = TV_S$ follows from

$$\sum_{\varphi \in S} d(\varphi(v_3v_2))d(\varphi(v_1v))b(\varphi(v_2v))b(\varphi(v_3v))*_f_1*_f_2*_f_3(|T_1|_{\varphi} \otimes |T_2|_{\varphi})$$

$$= \text{Id}(i, j, k^*)b(\varphi'(v_3v_2))$$

which is exactly the identity established in Lemma 25. \hfill \Box

Let $T_0$ be the set of vertices of $T$. Let $\delta$ be the coboundary operator from the $G$-valued 0-cochains on $T$ to the $G$-valued 1-cochains on $T$.

**Lemma 27.** Let $v_0 \in T_0$ and $c : T_0 \rightarrow G$ be a map such that $c(v) = 0$ for all $v \neq v_0$ and $c(v_0) \notin X$. If $\Phi$ and $\Phi + \delta c$ are admissible $G$-colorings of $T$, then $TV(T, \mathcal{L}, \Phi) = TV(T, \mathcal{L}, \Phi + \delta c)$

**Proof.** In the proof, we shall use the language of skeletons of 3-manifolds dual to the language of triangulations (see, for instance [10, 3]). A skeleton of $M$ is a 2-dimensional polyhedron $P$ in $M$ such that $M \setminus P$ is a disjoint union of open 3-balls and locally $P$ looks like a plane, or a union of 3 half-planes with common boundary line in $\mathbb{R}^3$, or a cone over the 1-skeleton of a tetrahedron. A typical skeleton of $M$ is constructed from a triangulation $T$ of $M$ by taking the union $P_T$ of the 2-cells dual to its edges. This construction establishes a bijective correspondence $T \leftrightarrow P_T$ between the quasi-regular triangulations $T$ of $M$ and the skeletons $P$ of $M$ such that every 2-face of $P$ is a disk adjacent to two distinct components of $M \setminus P$. To specify a Hamiltonian link $L$ in a triangulation $T$, we provide
some faces of $P_T$ with dots such that each component of $M - P_T$ is adjacent to precisely two (distinct) dotted faces. These dots correspond to the intersections of $L$ with the 2-faces. The notion of a $G$-coloring on an $H$-triangulation $T$ of $(M, L)$ can be rephrased in terms of $P_T$ as a 2-cycle on $P_T$ with coefficients in $G$, that is a function assigning an element of $G$ to every oriented 2-face of $P_T$ such that opposite orientations of a face give rise to opposite elements of $G$ and the sum of the values of the function on three faces of $P_T$ sharing a common edge and coherently oriented is always equal to zero. The notion of a state on $T$ can be also rephrased in terms of $P_T$. The state sum $TV(T, L, \Phi)$ can be rewritten in terms of $P = P_T$ in the obvious way, and will be denoted $TV(P, \Phi)$ in the rest of the proof. The moves on the $H$-triangulations may also be translated to this dual language and give the well-known Matveev-Piergallini moves on skeletons adjusted to the setting of Hamiltonian links, see [3]. We shall use the Matveev-Piergallini moves on dotted skeletons dual to the $H$-Pachner move and to the $H$-lune moves. Instead of the dual $H$-bubble move we use the so-called $b$-move $P \to P'$. The $b$-move adds to a dotted skeleton $P \subset M$ a dotted 2-disk $D \subset M$ such that the circle $\partial D$ lies on a dotted face $f$ of $P$, bounds a small 2-disk $D' \subset f$ containing the dot of $f$, and the 2-sphere $D \cup D'$ bounds an embedded 3-ball in $M$ meeting $P$ solely along $D'$. Note that a dual $H$-bubble move on dotted skeletons is a composition of a $b$-move with a dual $H$-lune move.

We apply the $b$-move to the polyhedron $P = P_T$ as follows. Consider the open 3-ball of $M - P_T$ surrounding the vertex $v_0$. Assume first that the closure $B$ of this open ball is an embedded closed 3-ball in $M$. The 2-sphere $\partial B$ meets $L$ at two dots arising as the intersection of $P$ with the two edges of $L$ adjacent to $v_0$. We call these dots the south pole and the north pole of $B$. We apply the $b$-move $P \to P' = P \cup D$ at the south pole of $B$. Here $D \subset B$ is a 2-disk such that $D \cap P = \partial D = \partial D'$, where $D'$ is a small disk in $P$ centered at the south pole and contained in a face $f$ of $P$. The given $G$-coloring $\Phi$ of $P$ induces a $G$-coloring $\Phi'$ of $P'$ which coincides with $\Phi$ on the faces of $P$ distinct from $f$ and assigns to the faces $f - D', D, D'$ of $P'$ the elements $\varphi(f), g = c(v_0) \notin X, \varphi(f) - g$ of $G$, respectively. Here the orientation $D'$ is induced by the one of $M$ restricted to $B$ and $f - D', D$ are oriented so that $\partial D' = \partial D = -\partial(f - D')$ in the category of oriented manifolds. Next, we push the equatorial circle $\partial D$ towards the north pole of $\partial B$. This transformation changes $P'$ by isotopy in $M$ and a sequence of Matveev-Piergallini moves dual to the $H$-Pachner move and to the $H$-lune moves. This is accompanied by the transformation of the $G$-coloring $\Phi'$ of $P'$ which keeps the colors of the faces not lying on $\partial B$ or lying on $\partial B$ to the north of the (moving) equatorial circle $\partial D$ and deduces $g$ from the $\Phi$-colors for the faces lying on $\partial B$ to the south of $\partial D$. The color of the face $\text{Int} D$ remains $g$ throughout the transformations. These $G$-colorings are admissible because all colors of the faces of the southern hemisphere are given by $\Phi + \delta c$ whereas the
colors of the faces of the northern hemisphere are given by \( \Phi \). Hence, by Lemma 26 \( TV(P', \Phi') \) is preserved through this isotopy of \( \partial D \) on \( \partial B \). Finally, at the end of the isotopy, \( \partial D \) becomes a small circle surrounding the northern pole of \( \partial B \). Applying the inverse \( b \)-move, we now remove \( D \) and obtain the skeleton \( P \) with the \( G \)-coloring \( \Phi + \delta(c) \). Hence \( TV(P, \Phi) = TV(P, \Phi + \delta(c)) \) which is equivalent to the claim of the lemma. In the case where the 3-ball \( B \) is not embedded in \( M \), essentially the same argument applies. The key observation is that since \( T \) is a quasi-triangulation, the edges of \( T \) adjacent to \( v_0 \) are not loops, and therefore the ball \( B \) does not meet itself along faces of \( P \) (though it may meet itself along vertices and/or edges of \( P \)). \( \square \)

**Lemma 28.** If \( \Phi \) and \( \Phi' \) are two admissible \( G \)-colorings of \( T \) representing the same class in \( H^1(M; G) \), then \( TV(T, \mathcal{L}, \Phi) = TV(T, \mathcal{L}, \Phi') \).

**Proof.** As \( \Phi \) and \( \Phi' \) represent the same cohomology class, \( \Phi' = \Phi + \delta c_1 + \cdots + \delta c_n \) where \( c_i : T_0 \to G \) is a 0-cochain taking non-zero value at a single vertex \( v_i \) for all \( i = 1, \ldots, n \). We prove the desired equality by induction on \( n \). If \( n = 0 \) then \( \Phi' = \Phi \) and the equality is clear. Otherwise, let \( E_1 \) be the set of (oriented) edges of \( T \) beginning at \( v_1 \). Pick any

\[
g \in G \setminus \left[ X \cup \bigcup_{e \in E_1} (\Phi(e) + X) \cup \bigcup_{e \in E_i} (c_1(v_1) + \Phi'(e) + X) \right].
\]

Let \( c : T_0 \to G \) be the map given by \( c(v_1) = g \) and \( c(v) = 0 \) for all \( v \neq v_1 \). Then \( \Phi + \delta c \) and \( \Phi + \delta c + \delta c_2 + \cdots + \delta c_n = \Phi' + \delta(c - c_1) \) are admissible colorings. Lemma 27 and the induction assumption imply that

\[
TV(T, \mathcal{L}, \Phi) = TV(T, \mathcal{L}, \Phi + \delta c) = TV(T, \mathcal{L}, \Phi + \delta c + \delta c_2 + \cdots + \delta c_n) = TV(T, \mathcal{L}, \Phi').
\]

\( \square \)

**Theorem 29.** Let \((T, \mathcal{L})\) and \((T', \mathcal{L}')\) be two \( H \)-triangulations of \((M, L)\) such that \((T', \mathcal{L}')\) is obtained from \((T, \mathcal{L})\) by a single \( H \)-Pachner move, \( H \)-bubble move, or \( H \)-lune move. Then for any admissible \( G \)-colorings \( \Phi \) and \( \Phi' \) on \( T \) and \( T' \) respectively, representing the same class in \( H^1(M; G) \),

\[
TV(T, \mathcal{L}, \Phi) = TV(T', \mathcal{L}', \Phi').
\]

**Proof.** For concreteness, assume that \( T' \) is obtained from \( T \) by a negative move. The admissible \( G \)-coloring \( \Phi \) of \( T \) restricts to an admissible \( G \)-coloring \( \Phi'' \) of \( T' \) which represent the same class in \( H^1(M; G) \). Now Lemma 26 implies that \( TV(T, \mathcal{L}, \Phi) = TV(T', \mathcal{L}', \Phi'') \) and Lemma 28 implies that \( TV(T', \mathcal{L}', \Phi'') = TV(T', \mathcal{L}', \Phi') \). \( \square \)
Proof of Theorem\textup{22}. From Proposition\textup{24} we know that any two $H$-triangulation of $(M, L)$ are related by a finite sequence of elementary moves. Then the result follows from the Theorem\textup{29} by induction on the number of moves. □

12. Totally symmetric 6$j$-symbols

Let $\mathcal{C}$ be a pivotal tensor Ab-category with ground ring $K$, basic data $\{V_i, w_i : V_i \to V_i^*\}_{i \in I}$, and t-ambi pair $(I_0, d)$. Recall that the associated modified 6$j$-symbols have the symmetries of an oriented tetrahedron. The modified 6$j$-symbols are totally symmetric if they are invariant under the full group of symmetries of a tetrahedron. More precisely, suppose that for every good triple $i, j, k \in I$, we have an isomorphism $\eta(i, j, k) : H(i, j, k) \to H(k, j, i)$ satisfying the following conditions:

$$\eta(i, j, k) = \eta(j, k, i) = \eta(k, i, j),$$

$$\eta(k, j, i) \circ \eta(i, j, k) = \text{id} : H(i, j, k) \to H(i, j, k)$$

and

$$(, kji)(\eta(i, j, k) \otimes \eta(k^*, j^*, i^*)) = (, ijk) : H(i, j, k) \otimes_K H(k^*, j^*, i^*) \to K$$

where $(, )_{ijk}$ is the pairing (8). We say that the modified 6$j$-symbols are totally symmetric if for any good tuple $(i, j, k, l, m, n) \in I^6$

$$\eta(i, j, k, l, m, n) \circ [\eta(k^*, j, i) \otimes \eta(m^*, l, k) \otimes \eta(j^*, l^*, n) \otimes \eta(i^*, n^*, m)] = [j, i, k, l^*, m^*, n^*].$$

For ribbon $\mathcal{C}$, the associated modified 6$j$-symbols are totally symmetric (see\textup{[10]}, Chapter VI). The isomorphisms $\eta(i, j, k)$ in this case are determined by so-called half-twists. A half-twist in $\mathcal{C}$ is a family $\{\theta'_i \in K\}_{i \in I}$ such that for all $i \in I$, we have $\theta'_i^* = \theta'_i$ and the twist $V_i \to V_i$ is equal to $(\theta'_i)^2 \text{id}_{V_i}$.

Lemma 30. The modified 6$j$-symbols defined in Section\textup{9} are totally symmetric.

Proof. The category $\mathcal{C}^H$ is ribbon with half-twist $(\theta'_i)_{i \in I} = \left(\left(\frac{q}{q^2}\right)^2 - \left(\frac{r}{q^2}\right)^2\right)_{i \in I}$, see\textup{[6]}. This gives rise to a family of isomorphisms

$$\eta(i, j, k) : H_{\mathcal{C}^H}(i, j, k) \to H_{\mathcal{C}^H}(k, j, i)$$

making the modified 6$j$-symbols associated with $\mathcal{C}^H$ totally symmetric. Using the isomorphisms provided by Lemma\textup{17}, one can check that this family induces a well-defined family of isomorphisms

$$\eta(i, j, k) : H_{\mathcal{C}}(i, j, k) \to H_{\mathcal{C}}(k, j, i).$$

The latter family makes the modified 6$j$-symbols associated with $\mathcal{C}$ totally symmetric. □

Remark 31. For a category with totally symmetric 6$j$-symbols, the construction of Section\textup{10} may be applied to links in non-oriented closed 3-manifolds.
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