Two-dimensional relativistic hydrogenic atoms: A complete set of constants of motion

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(Dated: September 11, 2008)

The complete set of operators commuting with the Dirac Hamiltonian and exact analytic solution of the Dirac equation for the two-dimensional Coulomb potential is presented. Beyond the eigenvalue \( \mu \) of the operator \( J_z \), two quantum numbers \( \eta \) and \( \kappa \) are introduced as eigenvalues of hermitian operators \( P = \beta \sigma_z \) and \( K = \beta (\sigma_z l_z + 1/2) \), respectively. The classification of states according to the full set of constants of motion without referring to the non-relativistic limit is proposed. The linear Paschen-Back effect is analyzed using exact field-free wave-functions as a zero-order approximation.

PACS numbers: 03.65.Pm, 03.65.Ge, 31.15.-p, 68.65.Fg
Keywords: Dirac equation, analytic solution

I. INTRODUCTION

Low-dimensional quantum systems have been the focus of extensive theoretical investigations in the last decades. Technological advances in semiconductor physics and recent developments in nanostructure technology provide techniques of creating low-dimensional structures like superlattices, quantum dots, quantum wires or quantum wells [1]. The most representative analogues of hydrogen-like systems in the world of semiconductors are hydrogenic donors being the bound states of conduction electron and a donor impurity [2] and Wannier-Mott excitons formed by an electron and a hole [3]. After the renormalization of Coulomb potential by introducing of dielectric constant and replacing the electron mass by effective one, the atomic objects in two-dimensional structures can be treated as 2-D hydrogenic like atoms. At this stage, the 2-D hydrogen problem determines a leading approximation for study of hydrogen type bound states in extreme anisotropic crystals, in which the \( z \)-component of a diagonal anisotropic mass tensor is much larger then the two remaining ones [4]. The non- and weak-relativistic approaches are usually considered as sufficiently good approximations to the realistic description of the 2-D objects in solid matter. However, in a searching of the quantum-mechanical properties of the 2-D systems interacting with low-dimensional gauge fields [4, 7], the complete relativistic theory is inevitable.

The quantum mechanical two-dimensional central problem, with the Coulomb potential \(-Z/\rho\) has been solved by many authors. In the nonrelativistic theory the analytic solution can be derived in strict analogy to the three-dimensional Coulomb problem, after the separation of the Schrödinger equation in polar coordinates [8, 9] or in parabolic coordinates [10]. The solution of the relativistic 2-D hydrogen-like problem has been obtained in the framework of the two-component approach [11]. Although the formalism based on the two-dimensional representation of gamma matrices gives correct formula for energy levels, it does not provide a good background for analysis other observables. Alternatively, standard Dirac-Pauli representation of the Dirac matrices can be used. The four-component analytic solution of the Dirac equation with the Hamiltonian

\[
H = c \mathbf{\alpha} \cdot \mathbf{p} + \beta c^2 - \frac{Z}{\rho}
\]

in two spatial dimensions has been obtained by Guo et al. [12]. The authors have investigated two decoupled eigenstates of \( H \)

\[
\Psi^{(1)} = \begin{bmatrix} f_1(\rho)e^{(\mu-1/2)\phi} \\ 0 \\ 0 \\ ig_1(\rho)e^{(\mu+1/2)\phi} \end{bmatrix}, \tag{2}
\]

\[
\Psi^{(2)} = \begin{bmatrix} 0 \\ f_2(\rho)e^{(\mu+1/2)\phi} \\ ig_2(\rho)e^{(\mu-1/2)\phi} \\ 0 \end{bmatrix}, \tag{3}
\]

where \( \mu \) is the eigenvalue of the \( z \)-component of the total angular momentum \( j_z \) defined as:

\[
j_z = l_z + \frac{1}{2} \sigma_z, \tag{4}
\]

with

\[
\sigma_z = \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix}, \tag{5}
\]

and

\[
l_z = -i \partial / \partial \phi. \tag{6}
\]

In non-relativistic limit the states [2] and [3] describe an electron with spin \textit{up} and \textit{down}, respectively. Without any additional information about other conserved quantities the radial amplitudes of these two decoupled states...
have to be determined by two different sets of radial equations \[12\]. It seems a little undesirable, in particular if we appeal to the three-dimensional theory where only one pair of radial equations appears for all quantum states.

The purpose of this paper is the analysis of integrals of motion of the two-dimensional relativistic hydrogen atom and classification of the states based on good quantum numbers. These goals are achieved by introducing, into Dirac Hamiltonian, hermitian operators associated with conserved quantities. Another important aspect of the presented approach is the possibility of determining all radial functions from one system of radial equations.

II. EXACT SOLUTION FOR FIELD-FREE ATOM

We have found that two linearly-independent states \[2\] and \[3\] are the eigenstates of an operator

\[ P = \beta \sigma_z', \]

which commute with the Hamiltonian \[11\]. It follows from \[2\] that \( P \) is an involution \( (P^2 = 1) \) and has two eigenvalues \( \eta = \pm 1 \). In the non-relativistic limit the different signs of \( \eta \) correspond to states with opposite spin directions. Moreover, it can be directly verified, by computing relevant commutators, that beyond operators \( j_z \) and \( P \) there exists an operator \( K \)

\[ K = \beta (\sigma_z' l_z + \frac{1}{2}), \]

which commutes with \( H \) and both operators \( P, j_z \). The eigenvalue of \( K \) can be referred to the Dirac quantum number \( \kappa \). To determine a physical meaning of quantum numbers \( \kappa, \mu \), let us derive a relation between \( \kappa \) and \( \mu \). If we consider the square of \( K \),

\[ K^2 = (\sigma_z' l_z)^2 + \sigma_z' l_z + \frac{1}{4} = (l_z + \frac{1}{2} \sigma_z')^2, \]

we obtain

\[ K^2 = j_z^2, \]

which means that eigenvalues of \( K \) satisfy the relation

\[ \kappa = \pm \mid \mu \mid. \]

We note that operators \( K, P \) and \( j_z \) are not independent. They fulfil the relation

\[ K = P j_z, \]

which gives a similar relation for eigenvalues

\[ \kappa = \mu \eta. \]

It appears from \[13\] that the upper and lower signs in \[11\] distinguish between two different physical situations, when, in non-relativistic limit, spin is parallel or antiparallel to the total angular momentum.

Now we are in a position to solve the 2-D Coulomb problem in strict analogy to 3-D case and perform the classification of states free of non-relativistic quantum numbers. To this end we introduce to the Dirac equation quantum numbers associated with the complete set of commuting operators \( (H, K, j_z) \).

In atomic units and polar coordinates \((\rho, \phi)\) Hamiltonian \[11\] can be written in the form

\[ H = c(\alpha_\rho \rho_\rho + \frac{i}{\rho} \alpha_\rho \sigma_z' l_z) + \beta c^2 - \frac{Z}{\rho}, \]

where

\[ \alpha_\rho = \begin{bmatrix} 0 & \sigma \cdot \hat{\rho} \\ \sigma \cdot \hat{\rho} & 0 \end{bmatrix}, \sigma \cdot \hat{\rho} = \begin{bmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{bmatrix}, \]

and

\[ p_\rho = -i \frac{\partial}{\partial \rho}. \]

Taking into account the algebraic properties of matrices \( \alpha, \beta \) and definition \[8\] of \( K \) we obtain

\[ H = c(\alpha_\rho \pi_\rho + \frac{i}{\rho} \alpha_\rho \beta K) + \beta c^2 - \frac{Z}{\rho}, \]

where the radial momentum \( \pi_\rho \) is defined as

\[ \pi_\rho = -i(\frac{\partial}{\partial \rho} + \frac{2}{\rho}). \]

In the representation in which operators \( H, j_z \) and \( K \) are diagonal, energy levels are determined by radial part of the wave function only. The pertinent radial Dirac equation takes the form

\[ [c(\alpha_\rho \pi_\rho + \frac{i}{\rho} \alpha_\rho \beta) + \beta c^2 - \frac{Z}{\rho}] R = WR, \]

where \( W \) denotes energy and \( Z \) is the charge of the Coulomb field. Since \( \alpha_\rho \) and \( \beta \) fulfil relations

\[ \alpha_\rho^2 = \beta^2 = 1, \quad \alpha_\rho \beta + \beta \alpha_\rho = 0 \]

they can be represented by two dimensional hermitian matrices

\[ \alpha_\rho = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

According to Eq. \[21\] the radial function \( R \) has two component, which for convenience we take in the form

\[ R(\rho) = \frac{1}{\rho^{1/2}} \begin{bmatrix} F(\rho) \\ G(\rho)Z/c \end{bmatrix}. \]

Introducing two new variables

\[ r = Z \rho, \quad E = \frac{W - c^2}{Z^2} \]
and substituting (21) into (19) leads to the wave equation for the electron moving in the two-dimensional Coulomb field in the form of the pair of \( \kappa \)-dependent radial equations:

\[
\frac{dG}{dr} + \frac{\kappa}{r} G + \left( \frac{1}{r} + E \right) F = 0, \tag{24}
\]

\[
\frac{dF}{dr} - \frac{\kappa}{r} F - \left[ \frac{1}{r} + E \right] + 2 \right) G = 0, \tag{25}
\]

where \( \lambda = \left( \frac{Z}{e} \right)^2 \).

Since asymptotic solutions of Eqs. (24), (25) decay exponentially we try to find the solution of radial equations (24) and (25) in the form

\[
F = r^\gamma e^{-\alpha r} \sum_{i=0}^{\infty} a_i r^i, \quad G = r^\gamma e^{-\alpha r} \sum_{i=0}^{\infty} b_i r^i, \tag{26}
\]

where

\[
\alpha = \sqrt{-E(2 + \lambda E)}. \tag{27}
\]

Substituting expansions (26) into equations (24), (25) we obtain the linear relations between the expansion coefficients

\[
(i + \gamma - \kappa) a_i - \lambda b_i = a a_{i-1} + (2 + \lambda E) b_{i-1}, \tag{28}
\]

\[
a_i + (i + \gamma + \kappa) b_i = -E a_{i-1} + \alpha b_{i-1}. \tag{29}
\]

For \( i = 0 \), we have

\[
(\gamma - \kappa) a_0 - \lambda b_0 = 0, \quad a_0 + (\gamma + \kappa) b_0 = 0. \tag{30}
\]

Since both \( a_0 \) and \( b_0 \) are different from zero, the secular determinant must vanish, which leads to

\[
\gamma = \pm \sqrt{\kappa^2 - \lambda}, \tag{31}
\]

and

\[
b_0 = -\frac{a_0}{\gamma + \kappa}. \tag{32}
\]

For \( i > 0 \), the expanding coefficients can be calculated iteratively from the relations

\[
b_i = \frac{(i + \gamma - \kappa + \alpha/E) a_i}{i(i + 2\gamma)}, \tag{33}
\]

\[
a_i = w_i - (i + \gamma + \kappa) b_i, \tag{34}
\]

where

\[
w_i = -E a_{i-1} + \alpha b_{i-1}. \tag{35}
\]

The condition of square integrability of the wave-function allows only the upper sign in Eq. (31) and requires the termination of power series (26) at a some power \( n' \)

\[
a_{n'+1} = 0, \quad b_{n'+1} = 0, \quad a_{n'} \neq 0, \quad b_{n'} \neq 0, \tag{36}
\]

which leads to the condition

\[
Ea_{n'} = \alpha b_{n'}. \tag{37}
\]

Dividing (34) by (33) for \( i = n' \) and taking into account (27) and (37) leads to the equation for \( E \)

\[
\sqrt{-E(2 + \lambda E) + 1} = 1 + \lambda E. \tag{38}
\]

Solving this equation and substituting

\[
\gamma = \sqrt{\kappa^2 - \lambda}, \tag{39}
\]

we obtain

\[
E = \frac{1}{\lambda} \left[ 1 + \frac{\lambda}{(n' + \sqrt{\kappa^2 - \lambda})^{1/2} - 1} \right]. \tag{40}
\]

In order to compare the expression (40) with non-relativistic one, we define the principal quantum number as follows

\[
n = n' + | \kappa | + 1/2. \tag{41}
\]

Since \( n' \geq 0 \) we must have \( | \kappa | \leq (n - 1/2) \). However for \( n' = 0 \), the number \( \kappa \) must have positive value only. The absence of the \( \kappa < 0 \) for \( n' = 0 \) follows from equations (32) and (37), which both imply

\[
(\gamma + \kappa) > 0. \tag{42}
\]

According to (39) \( \gamma \) is a real number smaller than \( | \kappa | \) and inequality (42) can be satisfied only if \( \kappa \) is positive. Therefore \( \kappa \) must fulfills relation

\[
| \kappa - 1/2 | \leq n - 1. \tag{43}
\]

Finally, energy levels are given through

\[
E_{n\kappa} = \frac{1}{\lambda} \left[ 1 + \frac{\lambda}{(n - | \kappa | - 1/2 + \sqrt{\kappa^2 - \lambda})^{1/2} - 1} \right]. \tag{44}
\]

In the non-relativistic limit (\( \lambda \to 0 \)) we obtain

\[
E_n = \frac{2}{(2n - 1)^2}. \tag{45}
\]

The complete spin-space-dependence of wave-functions is given by

\[
\Psi_{n\kappa\mu}(r, \phi) = \left[ f_{n\kappa}(r) \Omega_{\kappa\mu}(\phi) \right], \tag{46}
\]

where

\[
n = 1, 2, 3, \ldots, \tag{47}
\]

\[
\kappa = 1/2, -1/2, 3/2, -3/2, \ldots, (n - 1/2), \tag{48}
\]

and

\[
\mu = \pm \kappa. \tag{49}
\]
The cylindrical spinor $\Omega_{\kappa\mu}(\phi)$ is defined as
\[
\Omega_{\kappa\mu}(\phi) = \left[ \frac{\kappa + \mu e^{i(\mu - 1/2)\phi}}{2\mu} - \frac{\kappa - \mu e^{i(\mu + 1/2)\phi}}{2\mu} \right],
\]
and radial amplitudes have the form
\[
f_{\kappa\kappa}(r) = r^{\gamma - 1/2}e^{-\alpha r} \sum_{i=0}^{-n-1/2} a_i r^i,
\]
\[
g_{\kappa\kappa}(r) = r^{\gamma - 1/2}e^{-\alpha r} \sum_{i=0}^{-n-1/2} b_i r^i,
\]
with coefficients determined by relations (32)-(34).

Alternatively, in a similar way as in 3-D case, the solution of radial equations (24) and (25) may be expressed in terms of confluent hypergeometric functions
\[
F(r) = r^{\gamma}e^{-\alpha r}[(\kappa + \frac{1}{\alpha})F_1(r) - n'F_2(r)],
\]
\[
G(r) = \frac{E}{\alpha}r^{\gamma}e^{-\alpha r}[(\kappa + \frac{1}{\alpha})F_1(r) + n'F_2(r)],
\]
where
\[
F_1(r) = F_1(-n', 2\gamma + 1; 2\alpha r),
\]
\[
F_2(r) = F_1(1 - n', 2\gamma + 1; 2\alpha r).
\]

In order to introduce the classification scheme based on spectroscopic notation it is useful to define an orbital quantum number as
\[
l = |\kappa - \frac{1}{2}|.
\]

Note that this quantity according to (13) satisfies the inequality $l \leq (n - 1)$.

In Table II we display, as an example, the lowest few states with principal quantum numbers $n = 1, 2, 3$. We can see that for a given $n$, the states with the same $|\kappa|$ are degenerate and the lowest energy corresponds to minimal value of $|\kappa|$. According to (19), for each value of $\kappa$ there are two possible values of $\mu$. Therefore, the degree of degeneracy of the $n\kappa$-th energy level is 2 for $\kappa = (n - 1/2)$ and 4 for $|\kappa| < (n - 1/2)$, respectively. It is worth to point out that, in a contrast to the three-dimensional case, due to the equality $|\kappa| = |\mu|$, the states with $|\mu| < |\kappa|$ do not occur.

III. LINEAR PASCHEN-BACK EFFECT

Let us consider now the Dirac Hamiltonian describing transversal motion of an electron around a fixed center of Coulomb field with charge $Z$ and in a static uniform magnetic field. In atomic units the relativistic Hamiltonian can be written in the form
\[
H = c\alpha \cdot (p + A) + 2\alpha^2 - \frac{Z}{\rho}.
\]

Taking into account the standard four-dimensional Dirac-Pauli representation of the Dirac matrices and the vector potential $A = B \times \rho/2$, for $B = Bz$ perpendicular to the plane of transversal motion of the electron, we can write
\[
\alpha \cdot (p + A) = \alpha_\rho p_\rho + i\alpha_\rho \sigma_\rho^Z \left( \frac{l_z}{\rho} + \frac{B\rho}{2} \right).
\]

Introducing into Hamiltonian $H$ both operators $P$ and $K$, defined in a previous section, we obtain
\[
H = c(\alpha_\rho p_\rho - i\alpha_\rho \beta K + \frac{i}{2}B\rho\alpha_\rho \beta P) + 2\alpha^2 - \frac{Z}{\rho}.
\]

and appropriate radial Dirac equation in the form
\[
[c(\alpha_\rho p_\rho + i\kappa \rho \alpha_\rho \beta + \frac{iB}{2} \rho \alpha_\rho \beta) + 2\alpha^2 - \frac{Z}{\rho}R] = WR.
\]

Substituting (24) into (61) and changing variables according to (22) and (23) and introducing a new variable $B := B/Z^2$ leads to the wave equation, for the electron moving in the superposition of the two-dimensional Coulomb field and constant homogeneous magnetic field, in the form of the pair of $\kappa$- and $\eta$-dependent radial equations
\[
\frac{dG}{dr} + \left( \frac{\kappa}{r} + \frac{1}{2} \eta B r \right)G + \left( \frac{1}{r} + E \right)F = 0,
\]
\[
\frac{dF}{dr} - \left( \frac{\kappa}{r} + \frac{1}{2} \eta B r \right)F - \left[ \lambda \left( \frac{1}{r} + E \right) + 2 \right]G = 0.
\]

The complete spin-space description of eigenfunctions is the same as for the field-free atom. The only difference is in dependence of radial functions on both quantum numbers $\kappa$ and $\mu$. In consequence, magnetic energy shift may depend on the symmetry of the states. We investigate this problem in the first-order perturbation approach. In order to apply the perturbation formalism we rewrite Eqs. (62) and (63) in $2 \times 2$ matrix form
\[
(h^{(0)} + B^{(1)} - ES)\Phi = 0,
\]
where
\[
h^{(0)} = \begin{bmatrix} -1/2 & -d/dr - \kappa/r \\ d/dr - \kappa/r & -(2 + \lambda/r) \end{bmatrix},
\]
\[
h^{(1)} = \begin{bmatrix} 0 & -\eta r/2 \\ -\eta r/2 & 0 \end{bmatrix},
\]
\[
S = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix},
\]
\[ \Phi(r) = \begin{bmatrix} F(r) \\ G(r) \end{bmatrix}. \]  

Perturbation expansions for energy and wave-function
\[ E = \sum_{i=0}^{\infty} E_i B^i, \quad \Phi = \sum_{i=0}^{\infty} \Phi_i B^i \]  
lead to the following perturbation equations
\[ h^{(0)} \Phi^{(n)} + h^{(1)} \Phi^{(n-1)} - \sum_{i=0}^{n} E^{(i)} S \Phi^{(n-i)} = 0. \]  

The zero-order equation \((n = 0)\)
\[ (h^{(0)} - E^{(0)} S) \Phi^{(0)} = 0 \]  
is equivalent to the system of radial equations \((23)\) and \((24)\) for field-free atom. Under the condition of orthogonality
\[ \langle \Phi^{(0)}, S \Phi^{(i)} \rangle = 0, \]  
for \(i > 0\), the \(n\)-th order energy correction can be written in the form
\[ E^{(n)} = \frac{\langle \Phi^{(0)}, h^{(1)} \Phi^{(n-1)} \rangle}{\langle \Phi^{(0)}, S \Phi^{(0)} \rangle}. \]  

The calculation of the first-order energy correction
\[ E^{(1)} = -\eta \frac{\langle F, rG \rangle}{\langle F, F \rangle + \lambda \langle G, G \rangle} \]  

can be performed in closed form using radial functions \((63)\) and \((64)\). For \(n' = 0\) \((\kappa = | \mu |)\), we obtain
\[ E^{(1)} = \frac{\mu}{4\kappa}(2\gamma + 1). \]  

Taking into account that hypergeometric functions \((63)\) and \((64)\) depend only on \(| \kappa |\), the dependence of functions \(F\) and \(G\) on \(\kappa\) is due to the factor \((\kappa + 1/2)\) in front of \(F_1\) in \((63)\) and \((64)\). It means that for \(n' > 0\) the first-order energy corrections have the general form
\[ E^{(1)} = -\eta \frac{a \kappa + b}{c \kappa + d} = \mu A_1 + \frac{\kappa}{\mu} A_2, \]  
where
\[ A_1 = \frac{a d - b c}{\kappa^2 c^2 - d^2}, \quad A_2 = \frac{b d - \kappa^2 a c}{\kappa^2 c^2 - d^2}. \]  

Radial integrals \(a, b, c, d\) depend only on \(\kappa^2\) and \(n'\) and are given in Appendix A. It appears from Eqs. \((75)\) and \((76)\) that the \(\kappa\)- and \(\mu\)-degeneracy of field-free levels is completely removed by external magnetic field. Table \(A\) lists the first-order energy shifts of levels corresponding to states with \(n = 1, 2, 3\), presented in Table \(I\). We can see that linear corrections \(E^{(1)}\) essentially depend on the sign of both quantum numbers \(\kappa\) and \(\mu\). Calculating the non-relativistic limit of \((75)\) and \((76)\) we obtain the values of energy corrections \(E_N^{(1)} = \lim_{\lambda \to 0} E^{(1)}\), which consist with the non-relativistic result
\[ E_N^{(1)} = \frac{1}{2}(m + 2m_s), \]
where \(m\) and \(m_s\) are the eigenvalues of \(I_z\) and \(s_z = \frac{1}{2} \sigma_z\), respectively. The non-relativistic formula \((78)\) may be derived from the Schrödinger-Pauli equation in a similar way as in the 3-D case (see for example \([14]\)).

**IV. CONCLUDING REMARKS**

The exact solution of 2-D hydrogen problem which have been presented in this paper is consistent with fundamental principles of quantum mechanics. Due to introducing good quantum numbers into wave equation, the problem of energy spectrum is solved exactly in close analogy to the 3-D case. Moreover, the wave-functions are classified according to the complete set of constants of motion and determined by one \(\kappa\)-dependent system of radial equations. Although, the energy spectrum of field-free central problem depends only on \(| \kappa | = | \mu |\), the full \(\kappa\)- and \(\mu\)-dependence appears when the external magnetic field is applied.

**APPENDIX A**

Radial integrals occurring in the expression of the first-order magnetic energy correction \((76)\) are defined as follows
\[ a = \frac{2E}{\alpha^2} K_1, \quad b = \frac{E}{\alpha} (\kappa^2 + \frac{1}{\alpha^2}) K_1 - n'^2 K_2, \]  
\[ c = \frac{2}{\alpha}(1 + \frac{\lambda E^2}{\alpha^2}) I_1 + 2n' (\frac{\lambda E^2}{\alpha^2} - 1) I_{12}, \]  
\[ d = (1 + \frac{\lambda E^2}{\alpha^2}) [(\kappa^2 + \frac{1}{\alpha^2}) I_1 + n'^2 I_2] + 2n' (\frac{\lambda E^2}{\alpha^2} - 1) I_{12}, \]

where
\[ K_i = \int_0^{\infty} r^{2\gamma+1} e^{-2\sigma r} F_i^2 dr, \]  
\[ I_i = \int_0^{\infty} r^{2\gamma} e^{-2\sigma r} F_i^2 dr, \]  
\[ I_{12} = \int_0^{\infty} r^{2\gamma} e^{-2\sigma r} F_1 F_2 dr. \]
TABLE I: Relativistic quantum numbers, spectroscopic notation and energies for bound states with \( n = 1, 2, 3 \) for the two-dimensional hydrogen atom. The values of energy are computed with \( c = 137.0359976 \) \[13\].

| \( n \) | \( n' = n - |\kappa| - 1/2 \) | \( \kappa \) | \( l = |\kappa - 1/2| \) | Notation | Energy   |
|-------|-------------------|--------|------------------|---------|----------|
| 1     | 0                 | 1/2    | 0                | 1s\(_{1/2}\) | -2.000006514052 |
| 2     | 1                 | 1/2    | 0                | 2s\(_{1/2}\) | -0.222234057055  |
| 2     | 1                 | -1/2   | 1                | 2p\(_{1/2}\) | -0.222234057055  |
| 2     | 0                 | 3/2    | 1                | 2p\(_{3/2}\) | -0.22223537086   |
| 3     | 2                 | 1/2    | 0                | 3s\(_{1/2}\) | -0.080002897124  |
| 3     | 2                 | -1/2   | 1                | 3p\(_{1/2}\) | -0.080002897124  |
| 3     | 1                 | 3/2    | 1                | 3p\(_{3/2}\) | -0.080000624824  |
| 3     | 1                 | -3/2   | 2                | 3d\(_{3/2}\) | -0.080000624824  |
| 3     | 0                 | 5/2    | 2                | 3d\(_{5/2}\) | -0.080000170405  |

TABLE II: First-order magnetic corrections \( E^{(1)} \) to the energies for states with principal quantum numbers \( n = 1, 2, 3 \) of the two-dimensional relativistic hydrogen atom. The upper and lower signs in the front of \( E^{(1)} \) are referred to the two values of \( \mu = \pm |\kappa| \), respectively. The numbers in brackets are the powers of 10 by which the entries are multiplied. In the column 5 nonrelativistic values of linear magnetic corrections are given.

| \( n' \) | \( \kappa \) | State  | \( E^{(1)} \) | \( E_{\mu}^{(1)} \) |
|---------|--------|--------|--------------|--------------|
| 0       | 1/2    | 1s\(_{1/2}\) | ±0.49997337  | ±0.5         |
|         | 3/2    | 2p\(_{3/2}\) | ±0.99999112  | ±1           |
|         | 5/2    | 3d\(_{5/2}\) | ±1.49999467  | ±1.5         |
| 1       | 1/2    | 2s\(_{1/2}\) | ±0.49999704  | ±0.5         |
|         | -1/2   | 2p\(_{1/2}\) | ±2.9586[-6]  | 0            |
|         | 3/2    | 3p\(_{3/2}\) | ±0.99999680  | ±1           |
|         | -3/2   | 3d\(_{3/2}\) | ±0.49999680  | ±0.5         |
| 2       | 1/2    | 3s\(_{1/2}\) | ±0.49999899  | ±0.5         |
|         | -1/2   | 3p\(_{1/2}\) | ±1.0651[-6]  | 0            |

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