A Dimension-Insensitive Algorithm for Stochastic Zeroth-Order Optimization

Hongcheng Liu · Yu Yang

Received: date / Accepted: date

Abstract This paper concerns a convex, stochastic zeroth-order optimization (S-ZOO) problem. The objective is to minimize the expectation of a cost function whose gradient is not directly accessible. For this problem, traditional optimization algorithms mostly yield query complexities that grow polynomially with dimensionality (the number of decision variables). Consequently, these methods may not perform well in solving massive-dimensional problems arising in many modern applications. Although more recent methods can be provably dimension-insensitive, almost all of them require arguably more stringent conditions such as everywhere sparse or compressible gradient. In this paper, we propose a sparsity-inducing stochastic gradient-free (SI-SGF) algorithm, which provably yields a dimension-free (up to a logarithmic term) query complexity in both convex and strongly convex cases. Such insensitivity to the dimensionality growth is proven, for the first time, to be achievable when neither gradient sparsity nor gradient compressibility is satisfied. Our numerical results demonstrate a consistency between our theoretical prediction and the empirical performance.

Keywords stochastic optimization · zeroth-order method · high dimensionality · sparsity

Mathematics Subject Classification (2010) 90C15 · 90C25 · 90C26

1 Introduction

For many modern optimization problems, the (stochastic) gradient can be hardly available. This happens, for instance, when the objective function ad-
mits no known explicit form, or the (stochastic) gradient is too expensive
to compute. Applications of this type render many efficient and thus popular
algorithms, such as the stochastic first-order methods, no longer directly appli-
cable. As a remedy, zeroth-order optimization (ZOO), also known as black-box
or derivative-free optimization [5], has attracted much research interest.

In this paper, we propose a novel zeroth-order method to solve a stochastic
ZOO (S-ZOO) problem with the following formulation:

$$\min_{x \in \mathbb{R}^d} \{ F(x) := \mathbb{E}[f(x, \xi)] \}, \quad (1)$$

where $\xi$ is a random vector of problem parameters whose probability distri-
bution $\mathbb{P}$ is supported on a measurable set $\Theta \subseteq \mathbb{R}^q$, and $f : \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}$ is
deterministic and measurable. Denote by $x^* \in \mathbb{R}^d$ an optimal solution to (1).

Here, the dimensionality of the problem $d$ is assumed, without loss of gener-
ality, to satisfy $d \geq 3$ throughout this paper. In addition, it is assumed that
$f(\cdot, \xi)$ is everywhere continuously differentiable for almost every $\xi \in \Theta$, $F$
is convex, and the expectation $\mathbb{E}[f(x, \xi)] = \int_{\Theta} f(x, \xi) \, d\mathbb{P}(\xi)$ is well defined and
finite-valued for every $x \in \mathbb{R}^d$. Given $\xi$, let $\nabla f(\cdot, \xi)$ be the gradient of $f(\cdot, \xi)$. $\|\cdot\|_1$ and $\|\cdot\|_2$ are the 1-norm and 2-norm, respectively. Furthermore, we impose
the following assumptions hereafter for some known constant $R \geq 1$.

Assumption 1 It is possible to generate independent and identically dis-
tributed (iid) realizations, $\xi_1, \xi_2, \ldots$, of the random vector $\xi$.

Assumption 2 There is a stochastic zeroth-order oracle that returns the value
of $f(x, \xi)$ for a given input point $(x, \xi) \in \mathbb{R}^d \times \Theta$.

Assumption 3 For every $x \in \mathbb{R}^d$, it holds that
$$\nabla F(x) = \mathbb{E}[\nabla f(x, \xi)] \quad \text{and} \quad \mathbb{E} \left[ \| \nabla f(x, \xi) - \nabla F(x) \|^2 \right] \leq \sigma^2 \text{ for some } \sigma > 0.$$  

Assumption 4 There exists a constant $L > 0$, such that
$$\| \nabla f(x_1, \xi) - \nabla f(x_2, \xi) \| \leq L \| x_1 - x_2 \|,$$
for all $x_1, x_2 \in \mathbb{R}^d$ and almost every $\xi \in \Theta$.

Assumption 5 Problem (1) admits a bounded optimal solution such that $\{ x : \|x\|_1 \leq R \} \cap \arg \min_{x \in \mathbb{R}^d} F(x) \neq \emptyset$.

Assumptions 1 through 5 above are common in the ZOO literature (See
[10, 3, 24]). Assumption 3 allows for the availability of a simulator to generate
sample scenarios of the random vector $\xi$. Assumption 2 concerns the algorithmic
oracle. By this assumption, we may only have access to noisy objective values $f(x, \xi)$, i.e., inexact zeroth-order information of $F$, for a given tuple of
function input $(x, \xi)$. No higher-order information, such as gradient or hessian,
is available. Assumption 3 stipulates that $\nabla f(\cdot, \xi)$ is an unbiased estimator
of $\nabla F$ with a bounded variance. Assumption 4 requires $f(\cdot, \xi)$ to be differentiable and its gradient to be Lipschitz continuous. A well-known inequality as
an immediate result of this assumption is that, for almost every \(\xi \in \Theta\), and for all \(x, y \in \mathbb{R}^d\):

\[
f(x, \xi) \leq f(y, \xi) + \langle \nabla f(y, \xi), x - y \rangle + \frac{L}{2} \|x - y\|^2.
\]

(2)

Assumption 5 imposes the boundedness of an optimal solution. While this assumption also holds for many problems in practice, we are particularly interested in scenarios where the problem dimensionality \(d\) is very large compared to \(R\); that is, \(R \ll d\). In this case, Assumption 5 is also referred to as the weak sparsity condition by [20,16] in statistics and inverse problems, which is an extension to the conventional sparsity. Indeed, when the optimal solution \(x^*\) is a sparse element of a hypercube (that is, \(x^* \in [-r, r]^d : \|x^*\|_0 = s \ll d\) for some non-negative integer \(s\) and some scalar \(r > 0\)), we may see that weak sparsity easily holds when the traditional sparsity holds, as \(R = s \cdot r \ll d\).

However, our results to be presented subsequently may not be advantageous when \(R\) is large or even comparable with \(d\). Admittedly, one may argue that, regardless of how large \(R\) is, we can always introduce a change of variables \(z := x/C_R\), for some quantity \(C_R\) dependent only on \(R\), such that \(z\) becomes the actual vector of decision variables and \(\|z\|_1\) is small (thus, weak sparsity still holds for \(z\)). However, readers are reminded that such rescaling may undesirably affect both the variance \(\sigma^2\) in Assumption 3 and the Lipschitz constant \(L\) in Assumption 4 — both \(\sigma^2\) and \(L\) will grow polynomially in \(C_R\) after the rescaling.

Some of our results will be additionally contingent upon the assumptions of strong convexity and (the traditional) sparsity as below:

**Assumption 6** Function \(F(\cdot)\) is strongly convex with modulus \(\mu > 0\).

**Assumption 7** Problem (1) admits a finite, \(s\)-sparse optimal solution. More specifically, there exists \(x^* \in \{x : \|x\|_1 \leq R\} \cap \arg\min_{x \in \mathbb{R}^d} F(x)\) such that \(\|x^*\|_0 \leq s\), for some \(s : 1 \leq s \ll d\).

Strongly convex functions under Assumption 6 have been frequently studied in function minimization. Assumption 7 is the conventional sparsity condition, which is a more stringent requirement than Assumption 5. This condition holds for many modern statistical and machine learning problems as discussed, e.g., by [20,6,9,4]. Sparsity and its benefit in decision-making and optimization problems have been discussed by much, and growingly more, literature, e.g., in [13,15,7]. Exploiting sparsity in stochastic optimization has also been studied by [17]. Problem (1), even under both Assumptions 6 and 7 additionally, has a wide spectrum of applications, such as simulation-based optimization [24], parameter tweaking of deep learning models [27], and optimal therapeutic designs [18].

Effective algorithmic paradigms for solving (1) are available in the rich ZOO literature, including pattern search [30,11,21], random search [28], and

\footnote{Here \(\|\cdot\|_0\) denotes the number of nonzero entries of \(\cdot\).}
bayesian optimization [19], among many others (see [14] for an excellent review). Among the existing ZOO methods, the gradient estimation-based ZOO framework discussed by seminal works such as [24, 29, 13, 10] is closely related to this current work.

Despite numerous results on ZOO, a persistent challenge, as pointed out by [5, 3], is that the performance of almost all existing ZOO algorithms deteriorates rapidly as the problem dimensionality $d$ increases. In particular, for convex S-ZOO with a potentially nonsmooth cost function (a more general setting than ours in terms of Assumption 4 above), a randomized gradient-free algorithm achieves a complexity of $O(d^2/\epsilon^2)$-many queries of the zeroth-order oracles according to [24]. If the smoothness condition as in Assumption 4 holds, [10] provides a rate of $O(D_0d/\epsilon^2)$, where $D_0$ is the squared Euclidean distance between the initial solution and the optimal solution. Some analysis on the performance lower bound [12] indicates that the rate by [10] is already optimal without additional regularity assumptions on the objective function $F$. These complexity results suggest the potential inefficiency of existing ZOO algorithms for high-dimensional applications, where the number of decision variables can be in millions, billions, or even more. On the other hand, such high-dimensional problems are emerging rapidly in, e.g., data science, deep learning, and imaging, due to the ever-increasing demand for higher resolution and improved comprehensiveness in an optimized system.

Although several promising high-dimensional ZOO paradigms have been proposed recently, e.g., by [32, 3, 5, 2], the corresponding ZOO theories are based on some arguably restrictive assumptions. Indeed, while query complexities that are (notably) logarithmic in $d$ have been achieved by [32, 3, 2], their results are based on the assumption that $\nabla F$, the gradient of $F$, is everywhere $s$-sparse for some $s \ll d$. This means that there are always no more than $s$-many nonzero components in the gradient vector $\nabla F(x)$, for any choice of $x$. Some results by [3] further require that the optimal solution $x^*$ is sparse. The assumption of sparse gradient, according to [5], is comparatively stringent. In relaxing this assumption, [5] has developed the zeroth-order regularized optimization (ZORO) method, which is effective when the gradient is dense and satisfies a compressibility condition proposed therein. Additionally, [5] imposes a more specific problem structure than [1]—the random noise in evaluating the zeroth-order information is additive. Namely, it is assumed that $f(x) = F(x) + u$ for some random variable $u \in \mathbb{R}$ with a bounded support $\mathcal{U}$.

In contrast to the aforementioned methods, this paper presents a novel, sparsity-inducing stochastic gradient-free (SI-SGF) algorithm, which can effectively reduce the query complexity in terms of the dependence on the dimension $d$, even when most of the aforementioned assumptions, i.e., sparse gradient, compressible gradient, or additive randomness, are absent. Imposed instead in this work is the more common and more easily verifiable assump-

---

2 The query complexity of the zeroth-order oracle refers to the number of calls to the zeroth-order oracle required to achieve a desired accuracy $\epsilon > 0$.

3 A more general problem with a known, nonsmooth regularization term has also been considered by [3].
Table 1 Comparison of query complexity results. The “Assumption” column presents conditions other than Assumptions 1 through 5 which are standard to the convex ZOO literature. $D_0 := \|x_1^* - x^*\|^2$ measures the squared distance between the initial solution and an optimal solution. Although $D_0 \sim O(d)$ in general, it can be $O(s)$ when $x^*$ has only $s$-many nonzero components and the initial solution is chosen to be sparse (e.g., the initial solution can be the all-zero vector). “$s$-sparse gradient” refers to the assumption that the gradient has no more than $s$-many nonzero components everywhere, and “$x^*$ is $s$-sparse” means that the optimal solution has no more than $s$-many nonzero components.

| Algorithms | Complexity | Assumption |
|------------|------------|------------|
| [24]       | $O\left(\frac{d^2}{\epsilon^2}\right)$ | No additional assumption |
|            | $\text{Cost function can be nonsmooth}$ | |
| [10]       | $O\left(\frac{DD_0}{\epsilon^2}\right) = O\left(\frac{d^2}{\epsilon^2}\right)$ | No additional assumption |
| [10]       | $O\left(\frac{DD_0}{\epsilon^2}\right) = O\left(\frac{d^2}{\epsilon^2}\right)$ | $x^*$ is $s$-sparse |
| [32]       | $O\left(\frac{s^2}{\epsilon^2} \ln d\right)$ | $s$-sparse gradient | Bounded 1-norm of gradient |
|            | Additive randomness | Function sparsity | $\|x^*\|_1 \leq R$ |
| [5]        | $O\left( s \cdot \ln d \cdot \ln \left(\frac{1}{\epsilon}\right)\right)$ | Compressible gradient | Bounded 1-norm of Hessian |
|            | Additive randomness | Coercivity | $\|x^*\|_1 \leq R$ |
| [3,2]      | $O\left(\frac{D_0s^2}{\epsilon^2} + \frac{D_0s^2}{\epsilon^2} \ln d\right)$ | $s$-sparse gradient | $x^*$ is $s$-sparse |
|            | $= O\left(\frac{s^3}{\epsilon^2} + \frac{s^2}{\epsilon^2} \ln d\right)$ | | |
| Proposed   | $O\left(\frac{(D_0+R)s^3}{\epsilon^2} \ln d\right)$ | $\|x^*\|_1 \leq R$ | $x^*$ is $s$-sparse |
|            | $= O\left(\frac{(s+R)^2}{\epsilon^2} \ln d\right)$ | | Strong convexity |

More specifically, our main result in Theorem 1 only requires a weak sparsity assumption as in Assumption 5 which holds even if $x^*$ is dense. When $R$ therein is dimension-independent, we prove that the SI-SGF can yield a dimension-free (up to a logarithmic term) query complexity. A significant acceleration is further achieved when $x^*$ is sparse (as in Assumption 7) and the objective function of (1) is strongly convex (as in Assumption 6). Table 1 summarizes the complexity results and assumptions for the proposed SI-SGF and several important alternatives. Although the complexity rates by [3,2] can be more appealing than ours in terms of the desired accuracy $\epsilon$, the proposed SI-SGF is perhaps the first algorithm that can be shown to achieve dimension-insensitive query complexities, when gradient is neither sparse nor compressible.
Note that our results do not contradict with the lower performance bounds by [8] (in Propositions 1 and 2 therein) for a convex ZOO. While these lower bounds are tight when the domain is an $\ell_2$-ball, the problem of interest under Assumption 5 concerns a special case of their results; that is, when the domain is an $\ell_1$-ball. In our case, the lower performance bounds by [8] actually becomes "0". Furthermore, our research is focused on making use of some special and important problem structures in accelerating S-ZOO. Exploiting special problem structures to outperform the worst-case theoretical lower bounds is fairly common in the optimization literature (e.g., in [23]). Although we hypothesize that our complexity results are optimal under our setting, we leave the investigation of this hypothesis for future research.

1.1 Outline

The rest of the paper is organized as follows. In section 2, we provide some preliminaries on gradient approximation via randomized smoothing. Section 3 presents the proposed algorithm. Section 4 presents our main complexity results on the SI-SGF in both convex and strongly convex cases. A preliminary numerical study is included in Section 5. Finally, Section 6 concludes the paper. Some proofs and auxiliary results are provided in Appendix A.

1.2 Notations

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the collection of all real numbers and non-negative real numbers, respectively. For any vector $\mathbf{x} := (x_1, \cdots, x_d)^\top \in \mathbb{R}^d$, we sometimes use $(x_i)$ to denote $(x_1, \cdots, x_d)^\top$ for convenience, and $\mathbf{x}^\top$ to denote its transpose. The cardinality of a set $S$ is denoted by $|S|$ and $\mathbf{x}_S = (x_i : i \in S)$ is the sub-vector of $\mathbf{x}$ that only consists of components in the index set $S$. $\mathbf{1}$ and $\mathbf{0}$ are all-one and all-zero vectors of proper dimensions, respectively. $\nabla F(\mathbf{x})$ is the gradient of $F$ at $\mathbf{x}$ and $\nabla_S F(\mathbf{x})$ is the subvector of $\nabla F(\mathbf{x})$ that only consists of entries from the index set $S$. The set of integers $\{1, 2, \cdots, K\}$ is denoted by $[K]$. $\lceil \cdot \rceil$ represents the smallest integer no smaller than "\cdot". $N_d(\mathbf{x}, \Sigma)$ is the $d$-variate normal distribution with mean $\mathbf{x} \in \mathbb{R}^d$ and covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$. Lastly, $N(0, 1)$ is the standard normal distribution.

2 Gradient approximation via randomized smoothing

In this section, we provide some preliminaries on how to approximate the gradient of the objective function using only zeroth-order oracles through a randomized smoothing scheme. Many results below are based on the existing analyses by [24][5][10].

To approximate the gradient of $f(\cdot, \xi)$ with respect to $\mathbf{x}$, denoted by $\nabla f(\mathbf{x}, \xi)$, we propose to follow a similar approach as discussed by [24][5], using
the finite-difference-like formula below.

$$G^\delta(x, \xi) := \frac{f(x + \delta u, \xi) - f(x, \xi)}{\delta},$$  \hspace{1cm} (3)$$

where $u = (u_i : i = 1, ..., d)$ has iid entries with $u_i \in \{-1, 1\}$, for all $i$, following a discrete uniform distribution. Hereafter, we denote by $E_u$ the expectation over $u$ and, in contrast, by $E$ the expectation over $\xi$. By the definition of $u$, we have

$$f^\delta(x, \xi) := E_u[f(x + \delta u, \xi)] = \frac{1}{2^d} \sum_{u \in \{-1, 1\}^d} f(x + \delta u, \xi).$$  \hspace{1cm} (4)$$

By the probability mass function of $u$, we have that

$$\nabla f^\delta(x, \xi) = \frac{1}{2^d} \sum_{u \in \{-1, 1\}^d} \nabla f(x + \delta u, \xi) = E_u[\nabla f(x + \delta u, \xi)].$$  \hspace{1cm} (5)$$

Since $E[\nabla f(\cdot, \xi)] = \nabla E[f(\cdot, \xi)]$ in our settings, we have

$$E[\nabla f^\delta(\cdot, \xi)] = \nabla E[f^\delta(\cdot, \xi)].$$  \hspace{1cm} (6)$$

The lemma below provides a characterization on how the randomized smoothing scheme can be effective in approximating both the zeroth- and first-order information of $f(\cdot, \xi)$.

Lemma 1 Under Assumption 4, the below statements hold for any $\delta > 0$:

(a). Let $f^\delta$ be defined as in (4). Then, for any $x \in \mathbb{R}^d$ and almost every $\xi \in \Theta$,

$$|f^\delta(x, \xi) - f(x, \xi)| \leq \frac{L}{2}d\delta^2.$$  \hspace{1cm} (7)$$

(b). For any $v, x \in \mathbb{R}^d$, and almost every $\xi \in \Theta$,

$$\left| E_u \left[ \frac{f(x + \delta u, \xi) - f(x, \xi)}{\delta} \cdot u^T v \right] - \langle \nabla f(x, \xi), v \rangle \right| \leq \frac{L\delta d^{3/2}}{2} \|v\|. $$

Proof The proof of Part (a) is similar to that in [24], except that $u$ therein follows a different distribution. In view of $E_u[u] = 0$, we obtain that, for almost every $\xi \in \Theta$,

$$|f^\delta(x, \xi) - f(x, \xi)| = \left| \frac{1}{2^d} \sum_{u \in \{-1, 1\}^d} \{f(x + \delta u, \xi) - f(x, \xi) - \delta \langle \nabla f(x, \xi), u \rangle \} \right|$$

$$\leq \frac{1}{2^d} \sum_{u \in \{-1, 1\}^d} \left| \{f(x + \delta u, \xi) - f(x, \xi) - \delta \langle \nabla f(x, \xi), u \rangle \} \right|$$

$$\leq \frac{1}{2^d} \sum_{u \in \{-1, 1\}^d} \frac{L}{2} \delta^2 \|u\|^2,$$  \hspace{1cm} (7)$$
where the inequality in (7) follows from the Lipschitz continuity of $\nabla f(\cdot, \xi)$. The results in Part (a) immediately follows from the above in view of $\|u\|^2 = d$, as per the underlying distribution of $u$.

For Part (b), similarly, since $|f(x + \delta u, \xi) - f(x, \xi) - \langle \nabla f(x, \xi), \delta u \rangle | \leq \frac{L}{2} \|\delta u\|^2$, we have

\[
\left| \mathbb{E}_u \left[ \frac{f(x + \delta u, \xi) - f(x, \xi)}{\delta} \cdot u^\top v \right] - \mathbb{E}_u \left[ \langle \nabla f(x, \xi), u \rangle \cdot u^\top v \right] \right| \leq L \delta \|u\|^2 \|u\| \|v\|.
\]

In view of $\|u\|^2 = d$, and $\mathbb{E}_u \left[ \langle \nabla f(x, \xi), u \rangle \cdot u^\top v \right] = \langle \nabla f(x, \xi), v \rangle$, we then immediately have the desired result. \(\square\)

With (4) and (5), it is easy to verify the following properties.

(a) For almost every $\xi \in \Theta$, because $\nabla f(\cdot, \xi)$ is $L$-Lipschitz continuous, so is $\nabla f^\delta(\cdot, \xi)$.

(b) Because $\mathbb{E}[f(x, \xi)]$ is convex and continuously differentiable in $x$, so is $F^\delta(x) := \mathbb{E}[f^\delta(x, \xi)]$.

(c) By the convexity of $F(\cdot)$, we have

\[
F^\delta(x) = \mathbb{E} \{ \mathbb{E}_u[f(x + \delta u, \xi)] \} = \mathbb{E}_u \{ \mathbb{E}[f(x + \delta u, \xi)] \}
= \mathbb{E}_u[F(x + \delta u)] \geq F(x) + \mathbb{E}_u[\langle \delta u, \nabla F(x) \rangle] = F(x).
\]

(d) Consider the case where $F(x) = \mathbb{E}[f(x, \xi)]$ is strongly convex in $x$ with modulus $\mu$. $F^\delta(x)$ must also be strongly convex. Further invoking Assumption 4 we have, for all $x_1, x_2 \in \mathbb{R}^d$,

\[
F^\delta(x_1) - F^\delta(x_2) \geq \langle \nabla F^\delta(x_2), x_1 - x_2 \rangle + \frac{\mu}{2} \|x_1 - x_2\|^2.
\]

3 The Proposed Sparsity-Inducing Stochastic Gradient-Free (SI-SGF) Algorithm

Our proposed method is shown in Algorithm 1. At each iteration, it calls the subroutine in Algorithm 2. In particular, at the $k$-th iteration of Algorithm 1, $M > 0$ is the mini-batch size, $\gamma_k > 0$ is the step size, and $U_k > 0$ is a parameter input to Algorithm 2. Given the parameter $U \leftarrow U_k$, Algorithm 2 takes the input $x \leftarrow \mathbf{x}^k - \gamma_k g_k(\mathbf{x}^k)$ and outputs $v$, which is assigned to $x^{k+1}$ in Algorithm 1 i.e., $x^{k+1} \leftarrow v$. 


Algorithm 1 Sparsity-inducing stochastic gradient-free (SI-SGF) algorithm.

**Initialization:** Set hyper-parameters \( \{ \gamma_k \} \), \( M \), \( \{ U_k \} \), and \( K \). Let \( x^1 \) be a feasible solution such that \( \|x^1\|_1 \leq R \) and \( \|x^1\|_0 \leq \frac{2R}{\gamma_1} \) (e.g., \( x^1 := 0 \)).

for \( k = 1, \ldots, K \), do

Step 1. Generate a sample mini-batch of size \( M \), \( (\xi^{k,1}, \ldots, \xi^{k,M}) \), and compute
\[
g_{\delta}^k(x^k) := \frac{1}{M} \sum_{m=1}^{M} \left[ f(x^k + \delta u^{k,m}, \xi^{k,m}) - f(x^k, \xi^{k,m}) \right],
\]
where \( \{ u^{k,m} \} \) are iid random realizations of the \( d \)-variate random vector each entry of which follows a discrete uniform distribution on \( \{-1, 1\} \).

Step 2. Invoke the subroutine in Algorithm 2, with input \( x^k - \gamma k g_{\delta}^k(x^k) \), parameter \( U_k \), and output \( x^{k+1} \).

Output: \( x^Y \) for a random \( Y \), which has a discrete distribution on \( [K] \) with a probability mass function \( P[Y = k] = \frac{\gamma_{k-1}}{\sum_{k=1}^{K} \gamma_{k-1}} \).

Note that the output of the algorithm above is a randomly drawn element from the algorithm’s solution sequence. This output scheme follows [10]. We describe alternative output schemes, which tend to exhibit stronger empirical performance, in Section 4.3.

The design of Algorithm 1 mimics a standard stochastic first-order method (S-FOM), such as in [10], except for two differences. First, we follow [24, 5, 10] in approximating the stochastic gradient of the S-FOM by a randomized estimator as discussed in Section 2 above. Second, we invoke a subroutine to perform sparse projection at each iteration. The pseudo-code of the this subroutine is presented in Algorithm 2 below.

Algorithm 2 Per-iteration subroutine of SI-SGF.

**Input:** \( x = (x_i) \) and parameter \( U \).

Step 1. Let \( x_+ = (\max\{x_i, 0\}) \) and \( x_- = (\max\{-x_i, 0\}) \). Sort the components of the vector \( \tilde{x} = [x_+; x_-] \) in a descending order, and let \( (\tilde{x}(i)) \) denote the sorted vector.

Below, \( z(i) \) and \( v(i) \) follow the same indexing of components as \( \tilde{x}(i) \).

Step 2. Calculate \( z = (z_i) \in \mathbb{R}^{2d} \), for \( i = 1, \cdots, 2d \), by
\[
z_i = \begin{cases} \tilde{x}_i, & \text{if } \tilde{x}_i \geq U; \\ 0, & \text{otherwise}. \end{cases}
\]

Step 3. If \( 1^T z \leq R \), set \( \tilde{v} = z \).

Else compute \( \bar{v} = (\bar{v}_i) \), for \( i = 1, \cdots, 2d \), by
\[
\bar{v}(i) = \begin{cases} \tilde{x}(i) + \tau, & \text{if } i \leq \rho; \\ 0, & \text{otherwise}, \end{cases}
\]
where \( \tau = \frac{R - \sum_{i=1}^{\rho} \tilde{x}(i)}{\rho} \) and \( \rho = \max \left\{ j : \tilde{x}(j) + \frac{R - \sum_{i=1}^{j} \tilde{x}(i)}{j} \geq U \right\} \).

Output: \( v = (\bar{v}_i : i = 1, \ldots, d) - (\bar{v}_i : i = d + 1, \ldots, 2d) \).
As mentioned, Algorithm 2 equivalently solves a sparse projection problem, whose exact formulation is made explicit in the proposition below. Recall that, for Algorithm 2, $U$ is a user-specified parameter and $x$ is the input.

Algorithm 2 involves $O(d \ln d)$-many arithmetic operations and thus is a reasonably efficient. In comparison, the randomized smoothing scheme in [3] yields at least $O(d)$-many arithmetic operations.

In the pseudo-codes above, we specify that Algorithm 2 should run for $k = 1, \ldots, K$. Yet, in implementation, the algorithm may terminate at the $(K-1)$-th iteration, because the output of the algorithm relies only on results from the first $(K-1)$-many iterations. The $K$-th iteration is used only for our subsequent theoretical analysis, which happens to involve $x^{K+1}$.

**Proposition 1** Let $a, \lambda, \gamma$ be arbitrarily chosen positive scalars such that $\gamma \geq 2a$ and $a\lambda = U$. Let $v$ denote the output of Algorithm 2. Then, we have:

a. For all $i = 1, \ldots, d$, either $|v_i| \geq U$ or $v_i = 0$.

b. Moreover, $v$ is the optimal solution to the following optimization problem.

$$\min_{v' \in \mathbb{R}^d} \left\{ \frac{1}{2\gamma} \|v' - x\|^2 + \sum_{i=1}^{d} \left[ \frac{a\lambda - |v_i|}{a} \right] \cdot |v'_i| : \|v'\|_1 \leq R \right\},$$

(10)

**Proof** See Appendix A.1.

\[\square\]

4 Main complexity results for the SI-SGF

In this section, we present our main complexity results for the SI-SGF in solving (1) when $F$ is convex or strongly convex in Sections 4.1 and 4.2 respectively. In both cases, we prove that SI-SGF is dimension-insensitive. Section 4.3 presents an alternative output scheme, which is potentially more practical than the default random output in Algorithm 1. For all the proofs, we mostly focus on the orders of complexity rates and the constants involved may not have been optimized.

4.1 Complexity of the SI-SGF in solving convex S-ZOO problems

The complexity analysis relies on the following technical lemma, whose proof is postponed till Section A.2 of the Appendix.

**Lemma 2** Suppose that Assumption 3 holds. Let $(\xi^m : m = 1, \ldots, M)$ be a sample mini-batch of the random parameters $\xi$ in Problem (1) and $u^m$ be a vector of iid symmetric Bernoulli random variables; that is, they are uniformly...
distributed random variables on \([-1, 1]\). For any \(x \in \mathbb{R}^d\), it holds that

\[
\mathbb{E}_{\mathcal{V}} \left[ \max_{S \subseteq \{1, \ldots, d\}} \left\| \sum_{m=1}^{M} \frac{f(x + \delta u^m, \zeta^m) - f(x, \xi^m)}{M\delta} u^m_S - \sum_{m=1}^{M} \nabla_S f(x, \xi^m) \right\|^2 \right] \leq \frac{L^2 \delta^2 d^2}{a\lambda} + \frac{772R \cdot \ln d}{a\lambda} \cdot \sigma^2 + \frac{\|\nabla F(x)\|^2}{M},
\]

where \(\mathcal{V} := ((\zeta^m, u^m) : m = 1, \ldots, M)\).

**Proof** See Section A.2. \(\Box\)

Let the parameters of Algorithms 1 and 2 be set as follows.

\[
U_k = U = a\lambda, \quad \lambda = \frac{200L}{K\varpi}, \quad a = \frac{1}{100L}, \quad \gamma_k = \gamma = \frac{1}{50L}, \quad \text{for } k = 0, \ldots, K,
\]

\[
\delta \leq \frac{\theta}{Kd}, \quad M = \left\lceil \frac{50K^2\varpi \max\{1, \sigma^2\}}{L^2} \cdot \ln d \right\rceil,
\]

where \(\theta > 0\) and \(\varpi > 0\) are some user-specified hyper-parameters. Now we are ready to present the main result for convex S-ZOO problems.

**Theorem 1** Suppose that Assumptions 1 through 5 hold. Given that the hyper-parameters are set as in (11) and that \(K \geq 30L^2R\), there exists a constant \(C_1 > 0\) such that the output solution of Algorithm 1 satisfies

\[
\mathbb{E} \left[ F(x^Y) - F(x^*) \right] \leq \frac{C_1L}{K} \cdot \frac{\|x^Y - x^*\|^2}{K} + \frac{C_1LR}{K} \cdot (1 + \varpi^{-1} + \varpi\theta^2)
\]

\[
\quad + \frac{C_1L}{K^2} \cdot \left( \frac{\theta^2}{d} + \frac{\varpi^{-1}}{\ln d} \right),
\]

where \(\mathbb{E}\) is the expectation taken over all the random variables in Algorithm 2.

**Proof** Firstly, for some \(a\) and \(\lambda\) such that \(a \cdot \lambda = U\) and \(a \leq \frac{\varpi}{2}\), Proposition 1 (therein with \(x := x^k - \gamma_k g_k(x^k), \gamma := \gamma_k\) and \(v := x^{k+1} - (x^{k+1})\)) implies that Algorithm 2 computes an optimal solution \(x^{k+1}\) to the following optimization problem:

\[
\min_{v \in \mathbb{R}^d} \left\{ \frac{1}{2\gamma_k} \|v - x^k + \gamma_k g_k(x^k)\|^2 + \sum_{i=1}^{d} \frac{[a\lambda - |v_i|]_+}{a} |v_i'| : \|v\|_1 \leq R \right\}.
\]

The first-order necessary optimality conditions of the problem above yield that, for all \(x : \|x\|_1 \leq R\):

\[
\left( \frac{1}{\gamma_k} (x^{k+1} - x^k + \gamma_k g_k(x^k)) + g^{k+1}, x - x^{k+1} \right) \geq 0,
\]

(13)
where \( \varrho^{k+1} = \left( \frac{a\lambda - |x_i^{k+1}|}{a} : \Gamma_{x_i^{k+1}} : i = 1,\ldots,d \right) \) and \( \Gamma_{x_i^{k+1}} \) is a subgradient of the absolute value function \(| \cdot |\) at \( x_i^{k+1} \). If we plug in \( x := x^* \in \arg \min_{x \in \mathbb{R}^d} F(x) \cap \{ x : \|x\|_1 \leq R \} \) and invoke the convexity of \(| \cdot |\), we have

\[
\langle \varrho^{k+1}, x^* - x^{k+1} \rangle \leq \sum_{i=1}^{d} \frac{a\lambda - |x_i^{k+1}|}{a} (|x_i^*| - |x_i^{k+1}|).
\]

Let \( \lambda_i := \frac{[a\lambda - |x_i^{k+1}|]}{a} \). As per Lemma 1, we have \( \lambda_i = \frac{[a\lambda - |x_i^{k+1}|]}{a} = 0 \) for all \( i \) such that \( x_i^{k+1} \neq 0 \) and \( \lambda_i = \frac{[a\lambda - |x_i^{k+1}|]}{a} = \lambda \) for all \( i \) such that \( x_i^{k+1} = 0 \). Therefore, we may continue from (13) above (with \( x := x^* \) therein) to obtain

\[
\|x^{k+1}\|^2 - (x^{k+1})^T x^k + (x^*)^T (x^* - x^{k+1}) \\
\leq \langle \gamma_k g_k^S(x^k), x^* - x^{k+1} \rangle + \gamma_k \sum_{i=1}^{d} \lambda_i (|x_i^*| - |x_i^{k+1}|). \quad (14)
\]

Notice that

\[
\frac{1}{2} \|x^{k+1} - x^*\|^2 - \frac{1}{2} \|x^k - x^*\|^2 \\
= \|x^{k+1}\|^2 + \langle x^k - x^{k+1}, x^* \rangle - \frac{1}{2} \|x^k - x^{k+1}\|^2 - (x^{k+1})^T x^k.
\]

Let \( S_k := \{ i : x_i^k - x_i^{k+1} \neq 0 \} \), \( g(x^k) := \frac{1}{M} \sum_{m=1}^{M} \nabla f(x^k, \xi^{k,m}) \), and, thus, \( g_S(x^k) = \frac{1}{M} \sum_{m=1}^{M} \nabla S f(x^k, \xi^{k,m}) \). By Lemma 1, we know that \( \|x^k\|_0 \leq \frac{R}{\sqrt{d}} \) and \( \|x^{k+1}\|_0 \leq \frac{R}{\sqrt{d}} \). This comes immediately from the observation that, due to Lemma 1, \( |x_i^{k+1}| \geq U \) if \( x_i^k \neq 0 \) (and \( |x_i^{k+1}| \geq U \) if \( x_i^{k+1} \neq 0 \), as well as \( \|x^k\|_1 \leq R \) (and \( \|x^{k+1}\|_1 \leq R \), respectively). Consequently, \( |S_k| \leq \frac{2R}{\sqrt{d}} = \frac{2R}{\sqrt{d}} \).
In view of (14), we obtain from the above that

\[
\frac{1}{2} ||x^{k+1} - x^*||^2 - \frac{1}{2} ||x^k - x^*||^2
\]

\[
\leq \langle \gamma_k g^k_\delta(x^k), x^* - x^{k+1} \rangle - \frac{1}{2} ||x^{k+1} - x^k||^2 + \gamma_k \sum_{i=1}^d \lambda_i (|x_i^*| - |x_i^{k+1}|)
\]

\[
\leq \langle \gamma_k g^k_\delta(x^k), x^* - x^k \rangle + \langle \gamma_k g^k_\delta(x^k), x^k - x^{k+1} \rangle - \frac{1}{2} ||x^{k+1} - x^k||^2
\]

\[
+ \gamma_k \sum_{i=1}^d \lambda_i (|x_i^*| - |x_i^{k+1}|)
\]

\[
= \langle \gamma_k g^k_\delta(x^k), x^* - x^k \rangle + \langle \gamma_k g^k_{S_k}(x^k), x^k - x^{k+1} \rangle - \frac{1}{2} ||x^{k+1} - x^k||^2
\]

\[
+ \gamma_k \sum_{i=1}^d \lambda_i (|x_i^*| - |x_i^{k+1}|)
\]

(15)

\[
\leq \langle \gamma_k g^k_\delta(x^k), x^* - x^k \rangle + \frac{\gamma^2}{2\eta} ||g^k_{S_k}(x^k)||^2 + \frac{\eta}{2} ||x^{k+1} - x^k_{S_k}||^2
\]

\[
- \frac{1}{2} ||x^{k+1} - x^k||^2 + \gamma_k \sum_{i=1}^d \lambda_i (|x_i^*| - |x_i^{k+1}|)
\]

(16)

\[
\leq \langle \gamma_k g^k_\delta(x^k), x^* - x^k \rangle + \frac{\eta}{2} ||x^{k+1} - x^k||^2 - \frac{1}{2} ||x^{k+1} - x^k||^2
\]

\[
+ \frac{\gamma^2}{2\eta} ||g^k_{S_k}(x^k) - g_{S_k}(x^k) + g_{S_k}(x^k) - \nabla S_k F(x^k) + \nabla S_k F(x^k)||^2
\]

\[
+ \gamma_k \sum_{i=1}^d \lambda_i (|x_i^*| - |x_i^{k+1}|)
\]

(17)

where the (15) is by the definition of \(g_{S_k}(x^k)\) and (16) is due to \(||a||^2 + ||b||^2 \geq ||a+b||^2\) for arbitrary vectors \(a, b \in \mathbb{R}^d\). As we set \(\gamma_k = \gamma\), and \(\eta = 1\), we can continue from the above to obtain

\[
\frac{1}{2} ||x^{k+1} - x^*||^2 - \frac{1}{2} ||x^k - x^*||^2
\]

\[
\leq \langle \gamma g^k_\delta(x^k), x^* - x^k \rangle + \gamma \sum_{i=1}^d \lambda_i |x_i^*| + \frac{3\gamma^2}{2} ||\nabla F(x^k)||^2
\]

\[
+ \frac{3\gamma^2}{2} ||g^k_{S_k}(x^k) - g_{S_k}(x^k)||^2 + \frac{3\gamma^2}{2} ||g(x^k) - \nabla F(x^k)||^2.
\]
By Part (a) of Lemma 1, we may immediately obtain $|F^S(x) - F(x)| \leq \frac{L}{2} d \delta^2$. Further invoking Eq. [8], and the convexity of $F$, we know that

$$
E_{((k,m) : m=1, \ldots, M)} \left[ \|g_{x_k}^k(x^k) - g_{x_k}^S(x^k) \|^2 \right] 
= E_{((k,m) : m=1, \ldots, M)} \left[ M^{-1} \sum_{m=1}^M \frac{F(x^k + \mu \cdot u_{k,m}^m)}{\delta} - F(x^k) \cdot u_{k,m}^m \cdot x^k - x^k \right] 
\leq E_{((k,m) : m=1, \ldots, M)} \left[ M^{-1} \sum_{m=1}^M \langle \nabla F(x^k + \mu \cdot u_{k,m}^m), u_{k,m}^m \rangle \cdot (u_{k,m}^m, x^k - x^k) \right] 
= \langle \nabla F^S(x^k), x^k - x^k \rangle \leq F^S(x^k) - F^S(x^k) \leq F(x^k) - F(x^k) + \frac{L}{2} \delta^2 d. \quad (19)
$$

Meanwhile, we can obtain an upper bound on $\|g_{x_k}^S(x^k) - g_{x_k}^S(x^k)\|^2$ in (18) by invoking Lemma 2. More specifically, we have

$$
|S_k| \leq \frac{2d^2}{R} \quad E_{((k,m) : m=1, \ldots, M)} \left[ \max_{S \subseteq \{1, \ldots, d\} : |S| \leq \frac{2d^2}{R}} \|g_{x_k}^S(x^k) - g_{x_k}^S(x^k)\|^2 \right] 
\leq \frac{L^2 d^2 R}{a \lambda} + \frac{772 R \cdot \ln d}{a \lambda} \cdot \sigma^2 + \|\nabla F(x^k)\|^2. \quad (20)
$$

Combining the above with (18), (19), and Assumption 3 and taking expectation with respect to $\mathcal{W} = ((k,m) : k = 1, \ldots, K, m = 1, \ldots, M)$, we obtain

$$
\frac{1}{2} E_{\mathcal{W}} \left[ \|x^{k+1} - x^*\|^2 \right] - \frac{1}{2} E_{\mathcal{W}} \left[ \|x^k - x^*\|^2 \right] 
\leq E_{\mathcal{W}}[\gamma F(x^*)] - E_{\mathcal{W}}[\gamma F(x^k)] + \frac{\gamma L \delta^2 d}{2} + \gamma \lambda \|x^*\|^2 + \frac{3\gamma^2}{2} E_{\mathcal{W}} \left[ \|\nabla F(x^k)\|^2 \right] 
+ \frac{3\gamma^2}{2} \left( \frac{L^2 d^2 R}{a \lambda} + \frac{772 R \cdot \ln d}{a \lambda} \cdot \sigma^2 + \|\nabla F(x^k)\|^2 \right) + \frac{3\gamma^2 \sigma^2}{2M}. \quad (21)
$$

By the well-known inequality for convex and smooth function (with $L$-Lipschitz gradient), we have $F(x) - F(x^*) - \langle \nabla F(x^*), x - x^* \rangle \geq \frac{1}{2\delta^2} \|\nabla F(x) - \nabla F(x^*)\|^2$ for any $x \in \mathbb{R}^d$. As $\nabla F(x^*) = 0$, we then have $F(x^k) - F(x^*) \geq \frac{1}{2\delta^2} \|\nabla F(x^k) - \nabla F(x^*)\|^2 = \frac{1}{2\delta^2} \|\nabla F(x^k)\|^2$. This, combined with (21), leads to

$$
\frac{1}{2} E_{\mathcal{W}} \left[ \|x^{k+1} - x^*\|^2 \right] - \frac{1}{2} E_{\mathcal{W}} \left[ \|x^k - x^*\|^2 \right] 
\leq \left( \gamma - 3L\delta^2 - \frac{2316 L R \gamma^2 \ln d}{a \lambda M} \right) (F(x^*) - E_{\mathcal{W}}[F(x^k)]) + \frac{\gamma L \delta^2 d}{2} + \gamma \lambda \|x^*\|^2 
+ \frac{3\gamma^2}{2a \lambda} L^2 \delta^2 d^2 R + \frac{158 L \gamma^2 \sigma^2}{a \lambda M} \ln d + \frac{3\gamma^2 \sigma^2}{2M}.
$$
Applying the above inequality recursively for all \( k = 1, \ldots, K \), and summing them up, we obtain

\[
\sum_{k=1}^{K} \left[ \frac{1}{2} \mathbb{E}_W [\|x^{k+1} - x^*\|^2] - \frac{1}{2} \mathbb{E}_W [\|x^k - x^*\|^2] \right]
\leq K \left( \gamma - 3\gamma^2 L - \frac{2316LR\gamma^2 \ln d}{a\lambda M} \right) (F(x^*) - \mathbb{E}_W[F(x^*)]) + K\gamma L\delta^2 d R + K\gamma \lambda \|x^*\|_1 + \frac{3K\gamma^2 R^2}{2a\lambda} L^2 \delta^2 d^2 R + \frac{1158R\gamma^2 \sigma^2 K}{a\lambda M} \ln d + \frac{3\gamma^2 \sigma^2 K}{2M}.
\]

By some simplification and the definition of \( E \), which is the expectation over all the random variables in Algorithm 1, we have

\[
\frac{1}{2} \mathbb{E}_W [\|x^{K+1} - x^*\|^2] - \frac{1}{2} \mathbb{E}_W [\|x^1 - x^*\|^2]
\leq K \left( \gamma - 3\gamma^2 L - \frac{2316LR\gamma^2 \ln d}{a\lambda M} \right) (F(x^*) - \mathbb{E}[F(x^*)]) + K\gamma \lambda \|x^*\|_1 + \frac{3K\gamma^2 R^2}{2a\lambda} L^2 \delta^2 d^2 R + \frac{1158R\gamma^2 \sigma^2 K}{a\lambda M} \ln d + \frac{3\gamma^2 \sigma^2 K}{2M}.
\]

By properly choosing parameters, to be elaborated later, we can ensure that \( \alpha := \gamma - 3\gamma^2 L - \frac{2316R\gamma^2 \ln d}{a\lambda M} \geq 0 \). After some rearrangement, we obtain

\[
\mathbb{E}_W [F(x^Y) - F(x^*)] \leq \mathbb{E}_W [\|x^1 - x^*\|^2] + \frac{\lambda \gamma \|x^*\|_1}{2K\alpha} + \frac{3\gamma^2 R^2}{2a\alpha \lambda} L^2 \delta^2 d^2 R + \frac{\gamma L\delta^2 d}{2a\alpha} + \frac{1158R\gamma^2 \sigma^2}{a\alpha \lambda M} \ln d + \frac{3\gamma^2 \sigma^2 K}{2a\alpha M}.
\]

Let \( \lambda = \frac{200L}{K\alpha}, \gamma = \frac{1}{100L}, a = \frac{\gamma}{2} = \frac{1}{100L}, M = \left\lceil \frac{50K^2 \pi \max(1, \sigma^2)}{L^2} \ln d \right\rceil \), and \( K \geq L^2 R \). Thus, \( 1 - 3\gamma L - \frac{2316LR\gamma^2 \ln d}{a\lambda M} \geq 1 - \frac{3}{50} \frac{2316}{5000} = 0.4768 \) and \( \alpha = \gamma - 3\gamma^2 L - \frac{96R\gamma^2}{a\lambda M} \geq \frac{1}{100L} \). We obtain the desired result by plugging the above into (23), while recalling that \( \delta \leq \frac{\sigma}{K^2} \) and \( x^1 \) is deterministic. \( \square \)

**Remark 1** We would like to make a few remarks on the above result.

- The algorithm parameters can be chosen with more flexibility to achieve the promised query complexity than what is given in (11). In fact, if we set any of \( \lambda, a, \gamma, \delta, \) or \( M \) to be some constant multiple of the current value, the same complexity rate can be achieved.
- To obtain an \( \epsilon \)-suboptimal solution, (12) indicates an iteration complexity of \( O\left( \frac{(D_o + R)L}{\epsilon} \right) \), where \( D_o := \|x^1 - x^*\|^2 \). Since each per-iteration subroutine invokes \( O\left( \frac{(D_o + R)\gamma^2 \ln d}{\epsilon} \right) \)-many queries of the stochastic zeroth-order
oracle, the total number of calls to the oracle is

\[ O \left( \frac{(D_0 + R)^3 L \sigma^2 \cdot \ln d}{\epsilon^3} \right), \tag{24} \]

which is dimension-free up to a logarithmic term, when \( R, \epsilon, L, \) and \( \sigma \) are fixed.

By (24), we know that the proposed SI-SGF algorithm tends to be more effective when \( R \) is small. In particular, when \( x^1 = 0 \) and there exists an \( s \)-sparse solution (formalized in Assumption 7) for some \( s \) such that \( 1 \leq s \ll d \), then (24) immediately reduces to

\[ O \left( \frac{s^3 L \sigma^2 \ln d}{\epsilon^3} \right). \tag{25} \]

As a benchmark, the iteration complexity of the randomized stochastic gradient free (RSGF) algorithm for zeroth-order optimization in [10] is

\[ O \left( \frac{d D_0 \sigma^2}{\epsilon^2} \right) = \begin{cases} 
O \left( \frac{d^2 \sigma^2}{\epsilon^2} \right) & \text{In general;} \\
O \left( \frac{s d \sigma^2}{\epsilon^2} \right) & \text{If solution } x^* \text{ is } s\text{-sparse.} \end{cases} \tag{26} \]

Thus, if \( d \gg \frac{s^2}{\epsilon^2} \), the rate in (25) is significantly more appealing than (26).

As in Table 1, compared to the state-of-the-art algorithm for high-dimensional S-ZOO in [32,5,3,2], the proposed algorithm does not rely on any assumption of sparse gradient, compressible gradient, or additive randomness. Instead, we only require that the optimal solution is (approximately) sparse; that is \( \|x^*\|_1 \leq R \) for some small \( R \), and \( R \) indeed can be small when \( x^* \) is sparse. To our knowledge, Theorem 1 is the first result for dimension-insensitive S-ZOO under the relatively weak assumption of (weakly) sparse optimal solution.

Under the same assumptions and parameter settings as in Theorem 1, by Markov’s inequality, we further obtain that \( \forall \epsilon > 0 \), there exists a constant \( C_1 > 0 \) such that

\[ \text{Prob} \left[ F(x^Y) - F(x^*) \leq \epsilon \right] \geq 1 - \frac{C_1 L \|x^1 - x^*\|^2}{K \epsilon} - \frac{C_1 L R}{K \epsilon} \cdot (1 + \varpi^{-1} + \varpi^2) - \frac{C_1 L}{K \epsilon} \cdot \left( \frac{\theta^2}{d} + \frac{\varpi^{-1}}{\ln d} \right). \tag{27} \]

The implementation of the algorithm does not rely on the knowledge of the true sparsity-level \( s \) of an optimal solution. Instead, an over-estimate of its \( \ell_1 \)-norm will suffice to set the hyper-parameters of Algorithm 1.

Finally, the effectiveness of the proposed algorithm depends on machine precision \( \hat{\epsilon} \), the relative approximation error due to rounding in floating point arithmetic. In particular, it is implicitly required that, to implement the SI-SGF, the quantity \( \delta \) cannot be smaller than \( \hat{\epsilon} \). In other words, it is required that \( \frac{\theta}{K \delta} \geq \delta > \hat{\epsilon} \). For the double precision on a 32-bit computer,
A Dimension-Insensitive Algorithm for Stochastic Zeroth-Order Optimization 17

\[ \hat{\epsilon} = 2^{-52} \approx 10^{-16}, \] which requires that \( Kd < \frac{\hat{\epsilon}}{\theta} \approx 10^{16} \cdot \theta. \] Thus, in spite of the worst-case dimension-insensitive complexity of the proposed SI-SGF, there is an upper limit on the admissible problem dimensionality. This limit is less stringent when the machine precision improves.

4.2 Complexity of the SI-SGF in solving strongly convex S-ZOO problems

We now consider solving a strongly convex S-ZOO problem. The assumptions on the strong convexity of \( F(\cdot) \) and the sparsity of an optimal solution are formalized in Assumptions 6 and 7, respectively. Before presenting our main result for strongly convex S-ZOO problems, we recall the assumption that \( R \geq 1. \)

**Theorem 2** Suppose that Assumptions 1 through 7 hold, and the hyper-parameters in Algorithm 1 are set as follows:

\[
\gamma_k = \frac{2}{\mu \cdot (k + \frac{100 L}{\mu})}, \quad a_k = \frac{\gamma_k - 1}{2}, \quad U_k = a_k \cdot \lambda, \quad \text{for } k = 1, \ldots, K;
\]

\[
\delta = \frac{\theta}{K^{1.5} d}, \quad \lambda = \frac{200 L}{K \omega}, \quad M = \lceil 50 K^3 \omega \max\{1, \sigma^2\} \cdot \mu \cdot L^{-3} \cdot \ln d \rceil,
\]

where \( K \geq \frac{1}{\sqrt{\mu}} L^{3/2} R^{1/2} \) with \( \omega > 0 \) and \( \theta > 0 \) being some user-specified hyper-parameters. Then, the output of Algorithm 1 satisfies that

\[
E \left[ F(x^Y) - F(x^*) \right] \leq C_2 \cdot \frac{L^2}{K^2 \mu} \left[ s + R \cdot (1 + \theta^2 \omega \cdot \mu) + \|x_1 - x^*\|^2 + \frac{\mu \theta^2}{K L d} + \frac{1}{\omega^2 R} \right],
\]

for some constant \( C_2 > 0. \)

**Proof** By the same argument as in deriving (17), we obtain

\[
\frac{1}{2} \|x^{k+1} - x^*\|^2 - \frac{1}{2} \|x^k - x^*\|^2
\]

\[
\leq (\gamma_k g^k(x^k), x^* - x^k) + \frac{\eta - 1}{2} \|x^{k+1} - x^k\|^2
\]

\[
+ \frac{\gamma_k}{2\eta} \|g^k_S(x^k) - g_S(x^k) + g_S(x^k) - \nabla S_k F(x^k) + \nabla S_k F(x^k)\|^2.
\]

\[+ \gamma_k \sum_{i=1}^d \lambda_i |x^*_i| - \gamma_k \sum_{i=1}^d \lambda_i |x^k_{i+1}|.\]

As \( 0 \leq \lambda_i \leq \lambda \) for all \( i \), we have \( \gamma_k \sum_{i=1}^d \lambda_i (|x^*_i| - |x^k_{i+1}|) \leq \gamma_k \sum_{i: x_i \neq 0} \lambda_i |x^k_{i+1} - x^k_i| \leq \gamma_k \sqrt{\bar{d}} \|x^{k+1} - x^*\|. \) Let \( S_k := \{ i : x^k_i - x^k_{i+1} \neq 0 \}. \)
Then,
\[
\frac{1}{2} \| x^{k+1} - x^* \|^2 - \frac{1}{2} \| x^k - x^* \|^2 \\
\leq (\gamma_k g^k(x^k), x^* - x^k) + \frac{\eta - 1}{2} \| x^{k+1} - x^k \|^2 + \gamma_k \lambda \sqrt{s} \| x^{k+1} - x^k \|^2 \\
+ \frac{3\gamma_k^2}{2\eta} \| S_k(x^k) - g_S(x^k) \|^2 + \frac{3\gamma_k^2}{2\eta} \| g(x^k) - \nabla F(x^k) \|^2 + \frac{3\gamma_k^2}{2\eta} \| \nabla F(x^k) \|^2.
\]

We claim that \(|S_k| \leq \frac{2R}{a_{k+1}\lambda} \). To see this, we observe that \(\| x^k \|_0 \leq \frac{R}{a_{k}\lambda} \) and \(\| x^{k+1} \|_0 \leq \frac{R}{a_{k+1}\lambda} \). Actually, by Proposition 1 \(|x^k| \geq U_k = a_k\lambda \geq a_{k+1}\lambda = U_{k+1} \) if \(x^k \neq 0 \) (and \(|x^{k+1}| \geq U_{k+1} \) if \(x^{k+1} \neq 0 \), as well as \(\| x^k \|_1 \leq R \) (and \(\| x^{k+1} \|_1 \leq R \)). Consequently, \(|S_k| \leq \frac{R}{U_k + \frac{R}{U_{k+1}} \leq \frac{2R}{a_{k+1}\lambda} \).

Consider Lemma 2 as in deriving (20) (where we let \(a \) therein be \(a_{k+1} \)). We then have \(E_{\nu,t^k} \left[ \| g_{S_k}(x^k) - g_{S_k}(x^k) \|^2 \right] \leq \frac{L^2\sigma^2 d^2 R + 772\sigma^2 R \ln d + \sigma^2 + \| \nabla F(x^k) \|^2}{a_{k+1}\lambda} \), where \(U^k := ((u_{k,m}, \xi_{k,m}^k) : m = 1, ..., M) \). Meanwhile, observe that \(E_{\nu,t^k} \left[ \| g_{S_k}(x^k) - \nabla S_k F(x^k) \|^2 \right] \leq \frac{\sigma^2}{M} \). We may then continue to obtain
\[
\begin{align*}
\frac{1}{2} E_{\nu,t^k} \left[ \| x^{k+1} - x^* \|^2 \right] - \frac{1}{2} E_{\nu,t^k} \left[ \| x^k - x^* \|^2 \right] \\
\leq E_{\nu,t^k} \left[ (\gamma_k g^k(x^k), x^* - x^k) \right] + \frac{\eta - 1}{2} E_{\nu,t^k} \left[ \| x^{k+1} - x^k \|^2 \right] \\
+ \frac{3\gamma_k^2}{2\eta} \left( \frac{L^2\sigma^2 d^2 R}{a_{k+1}\lambda} + \frac{772\sigma^2 R \ln d}{a_{k+1}\lambda} + \frac{\sigma^2 + \| \nabla F(x^k) \|^2}{M} \right) + \frac{3\gamma_k^2}{2\eta} \cdot \frac{\sigma^2}{M} \\
+ \frac{3\gamma_k^2}{2\eta} E_{\nu,t^k} \left[ \| \nabla F(x^k) \|^2 \right] + \frac{\gamma_k \lambda \sqrt{s}}{2\eta} E_{\nu,t^k} \| x^{k+1} - x^* \| \\
\leq E_{\nu,t^k} \left[ (\gamma_k g^k(x^k), x^* - x^k) \right] + \frac{\eta - 1}{2} E_{\nu,t^k} \left[ \| x^{k+1} - x^k \|^2 \right] \\
+ \frac{3\gamma_k^2}{2\eta} \left( \frac{L^2\sigma^2 d^2 R}{a_{k+1}\lambda} + \frac{772\sigma^2 R \ln d}{a_{k+1}\lambda} + \frac{\sigma^2}{M} \right) \\
+ \frac{3\gamma_k^2}{2\eta} \left( 1 + \frac{772\sigma^2 R \ln d}{a_{k+1}\lambda M} \right) E_{\nu,t^k} \left[ \| \nabla F(x^k) \|^2 \right] + \frac{\gamma_k \lambda \sqrt{s}}{2\eta} E_{\nu,t^k} \| x^{k+1} - x^* \|.
\end{align*}
\]

Similar to (19) and in view of Part (b) of Lemma 1 and the strong convexity of \(F \) (with modulus \(\mu \)), we know that
\[
E_{\nu,t^k}((\xi_{k,m}, u_{k,m}) : m = 1, ..., M) \left[ (g^k_{k}(x^k), x^* - x^k) \right] \leq \left( \nabla F^S(x^k), x^* - x^k \right)
\]

Strong convexity and \(\frac{\mu}{2} \| x^* - x^k \|^2 \)

and \(\frac{\mu}{2} \| x^* - x^k \|^2 \)

Since \(F \) is convex and \(\nabla F \) is Lipschitz continuous, we have \(F(x^k) - F(x^*) - \langle \nabla F(x^*), x^k - x^* \rangle \geq \frac{1}{2\eta} \| \nabla F(x^*) - \nabla F(x^k) \|^2 = \frac{1}{2\eta} \| \nabla F(x^k) \|^2 \). Therefore,
we obtain from (30) that

$$\frac{1}{2} \mathbb{E}_{t^k} \left[ \|x^{k+1} - x^*\|^2 \right] - \frac{1}{2} \mathbb{E}_{t^k} \left[ \|x^k - x^*\|^2 \right]$$

$$\leq - \frac{\gamma_k \mu}{2} \mathbb{E}_{t^k} [\|x^* - x^k\|^2] + \frac{\gamma_k \lambda^2 d^2}{2} + \frac{\eta - 1}{2} \mathbb{E}_{t^k} [\|x^{k+1} - x^k\|^2]$$

$$+ \frac{3\gamma_k^2}{2\eta} \left( \frac{L^2 \delta^2 d^2 R}{a_{k+1} \lambda} + \frac{772 R \cdot \ln d}{a_{k+1} \lambda} \cdot \frac{\sigma^2}{M} \right) + \gamma_k \lambda \sqrt{s} \cdot \mathbb{E}_{t^k} [\|x^{k+1} - x^*\|]$$

$$+ \left[ \frac{3\gamma_k^2}{\eta} \left( 1 + \frac{772 R \cdot \ln d}{a_{k+1} \lambda M} \right) \right] L - \gamma_k \cdot \mathbb{E}_{t^k} [F(x^k) - F(x^*)].$$

By setting $\eta = 1/2$, we reduce the above inequality to

$$\frac{1}{2} \mathbb{E}_{t^k} \left[ \|x^{k+1} - x^*\|^2 \right] - \frac{1}{2} \|x^k - x^*\|^2$$

$$\leq - \frac{\gamma_k \mu}{2} \|x^* - x^k\|^2 + \frac{\gamma_k \lambda^2 d^2}{2} - \frac{1}{4} \mathbb{E}_{t^k} [\|x^{k+1} - x^k\|^2]$$

$$+ \frac{3\gamma_k^2}{2} \left( \frac{L^2 \delta^2 d^2 R}{a_{k+1} \lambda} + \frac{772 R \cdot \ln d}{a_{k+1} \lambda} \cdot \frac{\sigma^2}{M} \right) + \gamma_k \lambda \sqrt{s} \cdot \mathbb{E}_{t^k} [\|x^* - x^{k+1}\|]$$

$$+ \left( 6\gamma_k^2 L + \frac{4632 \gamma_k^2 LR}{a_{k+1} \lambda M} - \gamma_k \right) \cdot [F(x^k) - F(x^*)].$$

By strong convexity, we have $\|x^* - x^k\| \leq \sqrt{\frac{2}{\mu} [F(x^k) - F(x^*)]}$. In view of

$$\mathbb{E}_{t^k} [\|x^* - x^{k+1}\|] - \mathbb{E}_{t^k} [\|x^k - x^{k+1}\|] \leq \mathbb{E}_{t^k} [\|x^k - x^*\|],$$

thus

$$\frac{1}{2} \mathbb{E}_{t^k} \left[ \|x^{k+1} - x^*\|^2 \right] - \frac{1}{2} \|x^k - x^*\|^2 \leq - \frac{\gamma_k \mu}{2} \cdot \|x^* - x^k\|^2 + \frac{\gamma_k \lambda^2 d^2}{2}$$

$$+ \gamma_k \lambda \sqrt{s} \cdot \left( \sqrt{\frac{2}{\mu} [F(x^k) - F(x^*)]} \right) + \frac{3\gamma_k^2}{2} \left( \frac{L^2 \delta^2 d^2 R}{a_{k+1} \lambda} + \frac{772 R \cdot \ln d}{a_{k+1} \lambda} \cdot \frac{\sigma^2}{M} \right)$$

$$+ \left( 6\gamma_k^2 L + \frac{4632 \gamma_k^2 LR}{a_{k+1} \lambda M} - \gamma_k \right) \cdot [F(x^k) - F(x^*)] + \gamma_k \lambda^2 s \cdot \sqrt{\frac{2}{\mu} [F(x^k) - F(x^*)]}.$$
Multiplying both sides by $\frac{k + \lceil \frac{100L}{\mu \varpi} \rceil}{\gamma_k}$ and taking expectation with respect to $\mathcal{W} := \{(\xi^{k,m}, \mathbf{u}^{k,m}) : k = 1, \ldots, K, m = 1, \ldots, M\}$, we have

$$k + \left( k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil \right) \frac{1}{2\gamma_k} \mathbb{E}_\mathcal{W}[\|x^{k+1} - x^*\|^2] - \left( k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil \right) \frac{(1 - \gamma_k \mu)}{2\gamma_k} \mathbb{E}_\mathcal{W}[\|x^k - x^*\|^2]$$

$$\leq \left( k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil \right) \frac{L \delta^2 d}{2} + \gamma_k \lambda^2 s + \gamma_k \left( 3L^2 \delta^2 d^2 \frac{R}{a_{k+1} \lambda} + \frac{2316 R \cdot \ln d}{a_{k+1} \lambda} \cdot \sigma^2 \right)$$

$$+ \left( k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil \right) \left( 6\gamma_k L + \frac{4632\gamma_k LR}{a_{k+1} \lambda M} - 1 \right) \mathbb{E}_\mathcal{W}[F(x^k) - F(x^*)]$$

$$+ \left( k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil \right) \lambda \sqrt{s} \cdot \mathbb{E}_\mathcal{W} \left( \sqrt{\frac{2}{\mu}} [F(x^k) - F(x^*)] \right).$$

(31)

Note that \(k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil (1 - \gamma_k \mu) = \mu \cdot \left(k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil \right)(k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil - 1) = k - 1 + \frac{100L}{\mu \varpi}\) and \((k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil) \gamma_k \leq \frac{2}{\mu}\) by the selection of \(\gamma_k\) as in (28). Therefore, we may invoke (31) recursively and sum them up to obtain

$$\mu \left( K + \frac{100L}{\mu \varpi} \right) \left( K + \left\lceil \frac{100L}{\mu \varpi} \right\rceil + 1 \right) \cdot \mathbb{E}_\mathcal{W}[\|x^{K+1} - x^*\|^2]$$

$$- \frac{\mu}{4} \left\lceil \frac{100L}{\mu \varpi} \right\rceil \cdot \left( \left\lceil \frac{100L}{\mu \varpi} \right\rceil + 1 \right) \mathbb{E}_\mathcal{W}[\|x^1 - x^*\|^2]$$

$$- \sum_{k=1}^{K} \left( k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil \right) \left( \frac{L \delta^2 d}{2} + \lambda \sqrt{s} \cdot \mathbb{E}_\mathcal{W} \left( \sqrt{\frac{2}{\mu}} [F(x^k) - F(x^*)] \right) \right)$$

$$\leq \sum_{k=1}^{K} \left( k + \left\lceil \frac{100L}{\mu \varpi} \right\rceil \right) \left( \frac{3L^2 \delta^2 d^2 R \gamma_k}{a_{k+1} \lambda} + \frac{2316 \gamma_k R \cdot \ln d}{a_{k+1} \lambda} \cdot \sigma^2 \right) + \frac{2K \lambda^2 s}{\mu}.$$

Recall that \(\lambda = 200 \varpi^{-1} \lambda^{-1} L\), and \(M = [50 \varpi K^3 \max\{1, \sigma^2\} \cdot L^{-3} \mu \cdot \ln d]\), and \(K \geq L^{3/2} R^{1/2} / \sqrt{d}\), then we have \(\frac{4632 L R}{\lambda M} \leq 0.47 L R / K^2 \mu \leq 0.47\). Furthermore, \(6\gamma_k L \leq 0.12\) by the selection of \(\gamma_k\). Thus, \(6\gamma_k L + \frac{4632\gamma_k LR}{a_{k+1} \lambda M} - 1 \leq -0.41\).

Likewise, \(\frac{2316 \gamma_k R \cdot \ln d}{a_{k+1} \lambda M} \leq 0.24 L^2 R / K^2 \mu\) and \(\frac{3L^2 \delta^2 d^2 \gamma_k}{a_{k+1} \lambda} = 0.03 \varpi L \delta^2 RK d^2\). Conse-
Initially, we define the transition probability for \( Y \) as
\[
\left( K + \left[ \frac{100L}{\mu \varpi} \right] \right) \mu \left( K + \left[ \frac{100L}{\mu \varpi} \right] + 1 \right) \mathbb{E}_W [\| x^{K+1} - x^* \|^2] \\
- \frac{\mu}{4} \left[ \frac{100L}{\mu \varpi} \right] \left( \left[ \frac{100L}{\mu \varpi} \right] + 1 \right) \mathbb{E}_W [\| x^1 - x^* \|^2] \\
+ \sum_{k=1}^{K} 0.41 \left( k + \left[ \frac{100L}{\mu \varpi} \right] \right) \mathbb{E}_W [F(x^k) - F(x^*)] \\
\leq \sum_{k=1}^{K} \left( k + \left[ \frac{100L}{\mu \varpi} \right] \right) \left\{ \frac{L \delta^2 d}{2} + \lambda \sqrt{\gamma} \cdot \mathbb{E}_W \left[ \sqrt{\frac{2}{|\mu|} (F(x^k) - F(x^*))} \right] \right\} \\
+ \sum_{k=1}^{K} \left( k + \left[ \frac{100L}{\mu \varpi} \right] \right) \left( 0.03 \varpi L \delta^2 RKd^2 + \frac{0.24L^2R}{K^2 \mu} \right) + \frac{4 \times 10^4 \cdot L^2 s}{K \mu \varpi^2}.
\]  

(32)

By the definition of \( Y \) and \( E \), we know that
\[
\mathbb{E} [F(x^Y) - F(x^*)] = \sum_{k=1}^{K} \frac{\gamma_{k-1}^{-1}}{\sum_{k=1}^{K} \gamma_{k-1}} \mathbb{E}_W [F(x^k) - F(x^*)].
\]

Similarly,
\[
\sqrt{\mathbb{E} [F(x^Y) - F(x^*)]} \geq \mathbb{E} \sqrt{F(x^Y) - F(x^*)} \\
= \sum_{k=1}^{K} \frac{\gamma_{k-1}^{-1}}{\sum_{k=1}^{K} \gamma_{k-1}} \mathbb{E}_W \left[ \sqrt{F(x^k) - F(x^*)} \right].
\]

Recall that \( \gamma_k \) as given in (28), and thus \( \sum_{k=1}^{K} \gamma_{k-1}^{-1} = \frac{K}{2} \left( \left[ \frac{100L}{\mu \varpi} \right] + \frac{K+1}{2} \right) \) and \( \frac{\gamma_{k-1}^{-1}}{\sum_{k=1}^{K} \gamma_{k-1}} = \frac{k+\left[ \frac{100L}{\mu \varpi} \right]}{K \left( \left[ \frac{100L}{\mu \varpi} \right] + \frac{K+1}{2} \right)} \). We may then simplify (32) into
\[
\frac{\mu}{4} \left( K + \left[ \frac{100L}{\mu \varpi} \right] \right) \left( K + \left[ \frac{100L}{\mu \varpi} \right] + 1 \right) \mathbb{E}_W [\| x^{K+1} - x^* \|^2] \\
- \frac{\mu}{4} \left[ \frac{100L}{\mu \varpi} \right] \left( \left[ \frac{100L}{\mu \varpi} \right] + 1 \right) \mathbb{E}_W [\| x^1 - x^* \|^2] \\
+ 0.205K \left( K + 2 \left[ \frac{100L}{\mu \varpi} \right] + 1 \right) \mathbb{E} [F(x^Y) - F(x^*)] \\
\leq \frac{K}{2} \left( K + 2 \left[ \frac{100L}{\mu \varpi} \right] \right) \left\{ \frac{L \delta^2 d}{2} + \lambda \sqrt{\gamma} \cdot \mathbb{E} [F(x^Y) - F(x^*)] \right\} \\
+ \frac{K}{2} \left( K + 2 \left[ \frac{100L}{\mu \varpi} \right] \right) \left( 0.03 \varpi L \delta^2 RKd^2 + \frac{0.24L^2R}{K^2 \mu} \right) + \frac{4 \times 10^4 \cdot L^2 s}{K \mu \varpi^2}.
\]
By rearranging the items and plugging in $\lambda = \frac{200L}{K}$ from (28), we have

\[
0.205K \left( K + 2 \frac{100L}{\mu \varpi} + 1 \right) E \left[ F(x^Y) - F(x^* \phi) \right]
\leq \frac{\mu}{4} \left[ \frac{100L}{\mu \varpi} \right] \left( \left[ \frac{100L}{\mu \varpi} \right] + 1 \right) E_{W \phi} \left[ ||x^1 - x^*||^2 \right]
+ \frac{K}{2} \left( K + 2 \frac{100L}{\mu \varpi} \right) \frac{L \delta^2 d}{2}
+ 100K \left( K + 2 \frac{100L}{\mu \varpi} \right) \frac{1}{K} \sqrt{\frac{\sigma}{\mu}} \sqrt{2E \left[ F(x^Y) - F(x^*) \right]}
+ \frac{1}{2} K \left( K + 2 \frac{100L}{\mu \varpi} \right) \left( \left( \frac{100L}{\mu \varpi} \right) + 1 \right) E_{W \phi} \left[ ||x^1 - x^*||^2 \right]
+ 0.03 \varpi L \delta^2 RK d^2 + \frac{0.24 L^2 R}{K^2 \mu} + \frac{8 \times 10^4 \cdot L^2 s}{K \mu \varpi^2}.
\]

Dividing both sides by $\frac{K}{2} \left( K + 2 \frac{100L}{\mu \varpi} + 1 \right)$, we then have

\[
0.41E \left[ F(x^Y) - F(x^*) \right]
\leq \frac{200L}{K} \sqrt{\frac{\sigma}{\mu}} \cdot \sqrt{2E \left[ F(x^Y) - F(x^*) \right]} + \frac{L \delta^2 d}{2}
+ \frac{K}{2} \left( K + 2 \frac{100L}{\mu \varpi} + 1 \right) \frac{1}{\mu} \left[ \frac{100L}{\mu \varpi} \right] \left( \left[ \frac{100L}{\mu \varpi} \right] + 1 \right) E_{W \phi} \left[ ||x^1 - x^*||^2 \right]
+ 0.03 \varpi L \delta^2 RK d^2 + \frac{0.24 L^2 R}{K^2 \mu} + \frac{8 \times 10^4 \cdot L^2 s}{K \mu \varpi^2} \left( K + 2 \right) \frac{100L}{\mu \varpi} + 1 \right).
\]

We can view the above equation as a quadratic inequality with the unknown variable $\sqrt{E\left[ F(x^Y) - F(x^*) \right]}$. Solving this inequality and after some simplification, we obtain, for some constant $c_3 > 0$, $E\left[ F(x^Y) - F(x^*) \right] \leq c_3 \left[ \frac{L^2 (s + R + D_0)}{K \mu} + \frac{L^2 s}{K \mu} \right] E_{W \phi} \left[ ||x^1 - x^*||^2 \right] + L \delta^2 d + L \delta^2 d^2 RK \varpi + \frac{L^2 s}{K \mu + \varpi}$. Recall that $R \geq 1$, $\delta \leq \frac{\theta}{K \mu + \varpi}$ and $x^1$ is deterministic, then it directly leads to the desired result.

\[ \square \]

**Remark 2** Below, we would like to make a few remarks on the above theorem.

For these remarks, we assume w.l.o.g. that $\mu \leq 1$ and $\sigma \geq 1$.

- Let $D_0 := ||x^1 - x^*||^2$, which can be $O(s)$ under Assumption 7, if $||x^1||_0 \leq s$.

  (For instance, $x^1$ can be an all-zero vector.) Compared to (24), For any $\varepsilon \in (0, L^-1]$, Theorem 2 implies a significantly sharper query complexity $O \left( \frac{L^2 (s + R + D_0)^2}{\mu \varpi} \ln d \right)$.

  To see this, note that the iteration complexity is $O \left( \sqrt{\frac{L^2}{\mu} (s + R + D_0)^2} \right)$, and the per-iteration query complexity is $M = O \left( \frac{(s + R + D_0)^{3/2} \varpi^2}{L^2 s \mu} \ln d \right)$.

  Once again, the query complexity is independent of dimensions $d$, up to a logarithmic term, when $\sigma$, $s$, $L$, and $\mu$ are fixed.
By a similar argument as used in deriving (27), there exists some universal constant \( C_3 > 0 \), such that

\[
\text{Prob} \left[ F(x^Y) - F(x^*) \leq \varepsilon \right] \geq 1 - C_2 \cdot \frac{L^2}{K^2 \mu} \left[ s + R \cdot (1 + \theta^2 \varpi \cdot \mu) + \|x^1 - x^*\|^2 + \frac{\mu \theta^2}{KLd} + \frac{1}{\varpi^2 K} \right],
\]

for any \( \varepsilon > 0 \), under the same set of assumptions as in Theorem 2.

Similar to the convex S-ZOO case, to implement Algorithm 1 for solving the strongly convex S-ZOO, we do not need to know the sparsity-level \( s \) of the optimal solution. Actually, the algorithm can automatically exploit the sparsity, provided a coarse over-estimate of \( R \), i.e., the \( \ell_1 \)-norm of the optimal solution, is available.

Also similar to the convex S-ZOO case, the effectiveness of the proposed algorithm depends on machine precision, \( \hat{\epsilon} \). For the double precision on a 32-bit computer with \( \hat{\epsilon} = 2^{-52} \approx 10^{-16} \), it is stipulated that \( K d^{1.5} < \frac{\theta}{\hat{\epsilon}} \approx 10^{16} \); namely, there can be an upper limit on the admissible problem dimensionality for the proposed SI-SGF.

The algorithm parameters can be more flexible than (28) to achieve the promised query complexity. In fact, if we choose any of \( \lambda, a, \gamma, \delta \), or \( M \) to be some constant multiple of their current values, the same complexity rate holds.

### 4.3 Alternative schemes for algorithm output

Algorithm 1 relies on a simple and randomized criterion to determine the output \( x^Y \) from the sequence \( \{x^k\} \), with the index \( Y \) randomly chosen as per a pre-defined discrete distribution. We may also use two alternative output schemes (AOS). The first AOS generates \( x^{k^*} \) as follows.

\[
k^* \in \arg \min \left\{ M^{-1} \sum_{m=1}^{M} f(x^k, \xi^{k,m}) : k = 1, ..., K \right\}.
\]

When the above set is not a singleton, \( k^* \) is selected arbitrarily from the set.

Intuitively, this AOS outputs the solution with the smallest in-sample cost calculated on a mini-batch among all the solutions generated from iterations 1 to \( K \). Our numerical experiments in Section 5 show that the AOS tends to yield better solution quality in practice than the default randomized output scheme in Algorithm 1. The corollary below provides a theoretical guarantee of the AOS’s effectiveness.

**Corollary 1** Let \( \theta \) and \( \varpi \) in (11) and (28) be some universal constants. For any \( \varepsilon > 0 \), there exists some constant \( C_4 > 0 \) such that the following hold.
(a) Under the same setting as in Theorem 1, it holds with probability at least 
\[ 1 - \frac{C\varepsilon}{\epsilon}, \left( \frac{L_k^2}{K} + \frac{L\|x^* - x^k\|^2}{K} + \frac{1 + L/K + LR}{K} \right) \] 
that \( F(x^k) - F(x^*) \leq \varepsilon \).

(b) Under the same setting as in Theorem 2, \( F(x^k) - F(x^*) \leq 3\varepsilon \) with 
probability at least \( 1 - \frac{C\varepsilon}{\epsilon} \left( L + s + R + \|x^* - x^*\|^2 \right) \).

Proof Because \( M^{-1} \sum_{m=1}^{M} f(x^k, \xi^k, m) \leq M^{-1} \sum_{m=1}^{M} f(x^Y, \xi^Y, m) \), we have

\[
F(x^k) - F(x^Y) 
\leq F(x^k) - F(x^Y) + M^{-1} \sum_{m=1}^{M} f(x^Y, \xi^Y, m) - M^{-1} \sum_{m=1}^{M} f(x^k, \xi^k, m) 
\leq \sqrt{\left( F(x^k) - M^{-1} \sum_{m=1}^{M} f(x^Y, \xi^Y, m) \right)^2 + \left( F(x^Y) - M^{-1} \sum_{m=1}^{M} f(x^k, \xi^k, m) \right)^2} 
\leq 2 \max_{k=1,\ldots,K} \sqrt{\left( F(x^k) - M^{-1} \sum_{m=1}^{M} f(x^k, \xi^k, m) \right)^2},
\]

for \( k = 1, \ldots, K \). By Markov’s inequality, for any \( \varepsilon > 0 \), it holds with probability at least \( 1 - \frac{C\varepsilon}{\epsilon} \) that

\[
\sqrt{\left( F(x^k) - M^{-1} \sum_{m=1}^{M} f(x^k, \xi^k, m) \right)^2} \leq \varepsilon.
\]

Furthermore, by union bound and De Morgan’s law, we then have

\[
\max_{k=1,\ldots,K} \sqrt{\left( F(x^k) - M^{-1} \sum_{m=1}^{M} f(x^k, \xi^k, m) \right)^2} \leq \varepsilon 
\] 

with probability at least \( 1 - \frac{K\varepsilon^2}{M\epsilon} \). This combined with (35) implies that \( F(x^k) - F(x^Y) \leq 2\varepsilon \) 
with probability at least \( 1 - \frac{K\varepsilon^2}{M\epsilon} \). In view of (27) and (33) as well as the choices of \( M \) 
in (11) and (28), we then have the desired results in (a) and (b), respectively.

From this corollary, we know that the AOS is provably effective. It is also worth noticing that adopting this output scheme would incur almost no additional computational cost in Algorithm 1. Our numerical results presented subsequently indicate the effectiveness compared with the output scheme originally in Algorithm 1.

The second AOS is commonly utilized in the literature [22]. At the end of iteration \( K \), Algorithm 1 yields \( \bar{x} \), which is the weighted average of the whole solution sequence calculated as

\[
\bar{x} := \sum_{k=1}^{K} \frac{\gamma_{k-1}}{\sum_{k=1}^{K} \gamma_{k-1}} x^k.
\]

To understand the effectiveness of this AOS, we observe that \( E_Y[x^Y] = \bar{x} \), where \( Y \) and \( x^Y \) are defined as in Theorems 1 and 2 and \( E_Y \) denotes the
expectation taken over \( Y \). Since \( F \) is convex, the above leads to \( \mathbb{E}_Y [F(Y)] \geq F(\mathbb{E}_Y [x^Y]) = F(x) \). This, combined with Theorems 1 and 2 immediately shows the effectiveness of the AOS in (36).

5 Numerical Results

We conducted preliminary experiments on a stochastic quadratic programming problem modified from [24]. More specifically, we focused on solving the following optimization problem:

\[
 f_d(x; (\omega_i, \nu_i)) = \frac{1}{2} x_i^2 + \sum_{i=1}^{d-1} \frac{1}{2} (x_{i+1} - x_i - C_{i+1} + C_i)^2 \\
 + \frac{1}{2} x_d^2 + \sum_{i=1}^{d} \omega_i \cdot \nu_i \cdot x_i, \quad (37)
\]

where \( x_i \) is the \( i \)-th entry of \( x \), \( C_i = 1.5 \) for all \( i \in \{2, 6, 9\} \), and \( C_i = 0 \) for all other \( i \) (that is, \( i \notin \{2, 6, 9\} \)), for each \( i \), \( \omega_i \) is a standard normal random variable, and \( \nu := \{\nu_i\} \) is a vector of random variables such that exactly three components of it take value 1. That means, each \( \nu \) is randomly drawn from \( \mathcal{V} := \{\nu_i \in \{0, 1\} : \sum_{i=1}^{d} \nu_i = 3\} \) with equal probability of \( \frac{1}{|\mathcal{V}|} \). By construction, the optimal solution of the problem is verifiably \( x^* = [0; 1.5; 0; 0; 1.5; 0; 0; 1.5; 0; \ldots; 0] \), which is indeed a sparse solution, the corresponding optimal objective value is 0, and \( \sigma^2 = 3 \). The suboptimality gap, in this case, is the objective value of the output solution.

In all our experiments, we set the budget of the maximum number of zeroth-order oracle calls to be 320,000. We tested different mini-batch sizes \( M \) for SI-SGF under both convex and strongly convex settings in different problem cases. Correspondingly, the maximum iteration count was

\[
 K = \left\lfloor \frac{320,000}{M} \right\rfloor. \quad (38)
\]

We experimented with the following algorithms and configurations:

- SI-SGF for the convex settings with \( \varpi = 5 \), \((\lambda, \gamma, a)\) as per (11) in Theorem 1 and \( M \) determined empirically in the sequel:
  - si-sgfR: SI-SGF with randomized output scheme as in Algorithm 1
  - si-sgf*: SI-SGF with output scheme as in (34)
  - si-sgfA: SI-SGF with output scheme as in (36)
- SI-SGF for the strongly convex settings with \( \varpi = 5 \), \((\lambda, \gamma_k, a_k)\) selected as per (28) in Theorem 2 and \( M \) determined empirically below:
  - si-sgfR*: SI-SGF with randomized output scheme as in Algorithm 1
  - si-sgf*: SI-SGF with output scheme as in (34)
  - si-sgfA*: SI-SGF with output scheme as in (36)
- Benchmark algorithm:
sgf: The SGF from [10] with the best combination of the output scheme and the mini-batch size. More precisely, for each problem instance, all combinations of the three aforementioned output schemes and the candidate mini-batch sizes (to be detailed below) were compared. Among them, the one with the best quality in terms of the expected cost function $F$ was selected as the sgf’s output solution. The step sizes for SGF with $M = 1$ were chosen as per Corollary 3.3 of [10] with $\bar{D} = 1.5$ therein. Denote this value by $\gamma_{SGF}$. Then, for SGF with other mini-batch sizes $M$, the step sizes were selected to be $\gamma_{SGF} \cdot M$.

All the algorithms above were initialized with an all-zero vector, whose objective value was 6.75.

Note that the alternative dimension-insensitive S-ZOO algorithms by [5] and [2] cannot be applied to our settings directly, because the problem considered here does not satisfy the additive structure or the assumption of everywhere sparse gradient.

The first experiment was to determine the mini-batch size $M$ for SI-SGF and understand how the performance of the algorithms above changes as the mini-batch size $M$ varies. To this end, we fixed $d$ and $\delta$ to be $2^{15}$ and $10^{-7}$, respectively. This dimensionality was intentionally chosen to be larger than a tenth of the total budget of calls to the zeroth order oracle. For each choice of $M$, we performed ten random replications. Mean values and standard deviations of the suboptimality gaps are shown in both Figure 1.(a) and Table 2 (in the supplemental material). As shown therein, ’si-sgf’, ’si-sgf-s’, and ’si-sgfs’ were relatively insensitive to different choices of $M$, especially when $M \geq 100$. Some deterioration in the performance of these three variants has been observed for scenarios with larger values of $M$. Recall that $K$, the maximum iteration number, was decreasing in $M$ as per (38) to maintain the same maximum number of queries. Thus, the above observed deterioration was believed to be the result of smaller values of $K$. Other variants of the SI-SGF were comparatively more sensitive to the changes in $M$. Yet, in almost all test cases, all variants of SI-SGF outperformed the benchmark ’sgf’. We further evaluated the average suboptimality gaps across all three output schemes for the SI-SGF, and picked the mini-batch sizes that led to the best performance. As in Figure 1.(b) and Table 2 the best mini-batch sizes in this test were 160 and 280, respectively, for SI-SGF under convex and strongly convex settings.

The second set of experiments was to test the algorithms when the dimensionality $d$ belonged to $\{2^k : 6 \leq k \leq 21\}$. We set $\delta = 10^{-7}$, and the mini-batch sizes were 160 and 280 (as selected above) for the SI-SGF in convex and strongly convex settings, respectively. For each case, the benchmark ’sgf’ reported in this test was the best suboptimality gap achieved by the SGF’s output among all combinations of the three different output schemes and the mini-batch sizes of 1, 160, and 280. For each case, five random replications were performed. Mean values and standard deviations of the resulting suboptimality gaps out of these replications are reported in both Table 3 (in the supplemental material of this paper) and in Figure 2. It can be seen from
Fig. 1  Comparisons in the average suboptimality gaps (over ten random replications) of all the algorithms for different mini-batch sizes $M$. Subplot (a) shows the mean suboptimality gaps of each of the algorithms. Subplot (b) shows the comparisons among ‘si-sgf’, ‘si-sgfA’, and ‘sgf’. Here, ‘si-sgf’ refers to the average of the mean suboptimality gaps generated by ‘si-sgfR’, ‘si-sgf∗’, and ‘si-sgfA’; ‘si-sgfA’ refer to the average of the mean suboptimality gaps generated by ‘si-sgfAR’, ‘si-sgfA∗’, and ‘si-sgfA2’.

subplots (a)-(c) of this figure, as the dimensionality $d$ increased exponentially above $10^{14}$, the performance of the benchmark ‘sgf’ deteriorated rapidly and then plateaued as the suboptimality gap got closer to 6.75, which is the objective value of the initial solution. In contrast, the proposed SI-SGF under both convex and strongly convex settings for all three output schemes was significantly insensitive to the increase of dimensions. These observations agreed with our theoretical results that the SI-SGF, under both convex and strongly convex settings, are provably dimension-insensitive.

Figure 2.(d). compares different variants of SI-SGF. Each curve therein shows the “ratio of gaps”, that is, the ratio of the suboptimality gaps incurred by an SI-SGF variant of interest to that of ‘si-sgfA∗’ when $d$ increased. If any point is above the line of $y = 1$, then that SI-SGF variant performed worse than ‘si-sgfA∗’ in the corresponding case of $d$. As can be seen from Subplot (d), ‘si-sgfA∗’ yielded the best overall performance among all the variants. Both ‘si-sgf’ and ‘si-sgfA’ were competitive against ‘si-sgfA∗’, yet ‘si-sgfA∗’ was noticeably better when $d \geq 2^{19}$. The rest of the variants were non-trivially less competitive in almost all the cases of $d$.

The last experiment was focused on the sensitivity of the algorithms to the hyper-parameter $\delta$, which is used in the randomized smoothing scheme for gradient estimation. We set $d = 2^{15}$ and tested the scenarios with the value of $\delta$ ranging from $10^{-7}$ to $10^{-3}$. The mini-batch sizes were 160 and 280, respectively, for SI-SGF under convex and strongly convex settings according to the first experiment above. Figure 3 and Table 4 (in the supplemental material of this paper) summarize the results. Our benchmark ‘sgf’, again, denotes the best suboptimality gap achieved by the SGF’s output among all combinations of the three different output schemes and the mini-batch sizes of 1, 160, and 280. Each entry in this table reports the mean and standard deviation of suboptimality gaps generated by different algorithms with different $\delta$ over 10 random replications. As one can see, the results were comparable
Fig. 2 (a). The mean suboptimality gaps of ‘si-sgf$^R$’ and ‘si-sgf$s^R$’, i.e., SI-SGF in both convex and strongly settings with randomized output scheme, in comparison with ‘sgf’. (b). The mean suboptimality gaps of ‘si-sgf$^\ast$’ and ‘si-sgf$s^\ast$’, i.e., SI-SGF in both convex and strongly settings with output scheme as in (34), in comparison with ‘sgf’. (c). The mean suboptimality gaps of ‘si-sgf$^A$’ and ‘si-sgf$s^A$’, i.e., SI-SGF in both convex and strongly settings with output scheme as in (36), in comparison with ‘sgf’. (d). The ratios between the mean suboptimality gaps of different variants of SI-SGF and that of ‘si-sgf$s^\ast$’. For subplots (a)-(c), 6.75 was the objective value of the initial solution to all algorithms.

when $\delta \leq 10^{-6}$ for all the algorithms. However, significant performance deterioration was observed, when $\delta$ became larger. The canonical SGF appeared to be the most insensitive towards $\delta$; the deterioration did not happen until $\delta$ was $10^{-3}$. In contrast, for ‘si-sgf$s^R$’, the observed deterioration started as $\delta$ became no less than $10^{-5}$. For all other variants of SI-SGF, the deterioration started when $\delta$ turned no less than $10^{-4}$. Nonetheless, in all cases, the variants of SI-SGF significantly outperformed the SGF in terms of the suboptimality.

6 Concluding remarks

This paper presents a sparsity-inducing stochastic gradient-free (SI-SGF) algorithm for solving high-dimensional S-ZOO problems. By exploiting (weak) sparsity, the proposed algorithm is significantly less sensitive to the increase of dimensionality. In contrast to all existing dimension-insensitive S-ZOO paradigms, our theories do not require the (sometimes critical) assumptions
such as everywhere sparse or compressible gradient. Our numerical results indicate that the proposed SI-SGF is a promising approach and can potentially outperform the baseline stochastic gradient-free algorithms.

A Technical proofs

A.1 Proof of Proposition

Proof By construction in Step 3 of Algorithm 2, it must hold that either \( \tilde{v}_i \geq U \) or \( \tilde{v}_i = 0 \) for all \( i = 1, \ldots, 2d \). This immediately implies that either \( |v_i| \geq U \) or \( v_i = 0 \) for \( i = 1, \ldots, d \), which shows Part (a) of the proposition.

The proof for Part (b) is divided into two steps below.

Step 1. In this step, we would like to first show that \( \tilde{v} \) as in Algorithm 2, is a KKT point of

\[
\min_{z \in \mathbb{R}^{2d}} \left\{ \frac{1}{2\gamma} \|z - \tilde{x}\|^2 + \sum_{i=1}^{2d} P_{\lambda}(z_i) : \mathbf{1}^\top z \leq R \right\},
\]

where \( P_{\lambda}(\theta) := \int_{\theta}^{0} \left[ a\lambda - t \right]_+ dt \) for arbitrary values of \( a, \lambda, \) and \( \gamma \) such that \( U = a\lambda \) and \( a \leq \frac{\gamma}{2} \). More explicitly, we will show that there exist some \( \beta \) and \( (\mu_i) \) such that \( \tilde{v} = (\tilde{v}_i) \) satisfies the following nonlinear system:

\[
\frac{1}{\gamma} (\tilde{v}_i - \tilde{x}_i) + \frac{[a\lambda - \tilde{v}_i]}{\alpha} + \beta - \mu_i = 0, \quad i = 1, \ldots, 2d; \\
\tilde{v}_i \geq 0, \quad \mu_i \geq 0, \quad i = 1, \ldots, 2d; \\
\mu_i \cdot \tilde{v}_i = 0, \quad i = 1, \ldots, 2d; \\
\beta \geq 0, \quad \beta \left( \sum_{i=1}^{2d} \tilde{v}_i - R \right) = 0, \quad \sum_{i=1}^{2d} \tilde{v}_i \leq R.
\]

Let \( z \) be the result computed in Step 2 of Algorithm 2. We consider two cases below.

Case (i): Consider the case where \( \mathbf{1}^\top z \leq R \):

According to Algorithm 2, \( \tilde{v}_i = z_i \) for \( i = 1, \ldots, 2d \). Then we can set \( \beta = 0 \), and let \( \mu_i = \begin{cases} \frac{\lambda - \tilde{x}_i}{\alpha}, & \text{if } \tilde{x}_i < U; \\ 0, & \text{otherwise;} \end{cases} \) for all \( i : 1 \leq i \leq 2d \). By Steps 2 and 3 of Algorithm 2, we evidently have \( \mu_i \cdot \tilde{v}_i = 0 \) for all \( i \) and \( \mathbf{1}^\top \tilde{v} \leq R \). Thus, the above construction immediately leads to the third and the last lines of (40). Below, we will prove the first two lines of (40).
For all $i$ such that $\overline{x}_i \geq U$, by the construction both in the above and in Steps 2 and 3 of Algorithm 2, we have $\overline{v}_i = z_i = \overline{x}_i \geq U$, $\beta = 0$ and $\mu_i = 0$. Thus, the second line of (40) holds for all $i$ such that $\overline{x}_i \geq U$. Similarly, $\frac{1}{\gamma} (\overline{v}_i - \overline{x}_i) + \frac{[\alpha \lambda - \overline{v}_1]}{\alpha} + \beta - \mu_i = \frac{1}{\gamma} (\overline{v}_i - \overline{x}_i) + \beta - \mu_i = 0$, which shows that the first line of (40) holds for $i$ such that $\overline{x}_i \geq U$.

For all $i$ such that $\overline{x}_i < U$, we have $\overline{v}_i = z_i = 0$, $\beta = 0$, and $\mu_i = \lambda - \frac{\overline{x}_i}{\gamma}$. As a result, $\frac{1}{\gamma} (\overline{v}_i - \overline{x}_i) + \frac{[\alpha \lambda - \overline{v}_1]}{\alpha} + \beta - \mu_i = -\frac{1}{\gamma} \overline{x}_i + \lambda + \beta - \mu_i = 0$, which shows that the first line of (40) holds for $i$ such that $\overline{x}_i < U$. In view of $\gamma \geq 2a$ and $U = a \lambda$, we have $\lambda - \overline{x}_i \geq U - \overline{x}_i > 0$. Thus, $\mu_i = \lambda - \frac{\overline{x}_i}{\gamma} \geq 0$. This, combined with $\overline{v}_i = 0$ as proven above, leads to the satisfaction of the second line in (40) for all $i$: $\overline{x}_i < U$.

Case (ii): Consider the case where $1^\top \overline{z} > R$.
By Step 1 of Algorithm 2, $(\overline{x}(i))$ is the vector after sorting the components of $\overline{z}$ in a non-increasing order; that is, $\overline{x}(1) \geq \overline{x}(2) \geq \cdots \geq \overline{x}(2d)$. Also recall that $(\tilde{v}(i))$ is the vector following the same index order as in $(\overline{x}(i))$. We let

$$\beta = -\frac{\tau}{\gamma}, \text{ and } \mu(i) = \begin{cases} 0, & i = 1, \cdots, \rho, \\ \lambda - \frac{\overline{x}(i) + \tau}{\gamma}, & \text{otherwise,} \end{cases}$$

in the KKT conditions (40).

We first check the feasibility of $\overline{v}$. We claim that $\overline{v}(i) \geq 0$ for all $i = 1, \cdots, 2d$. To see this, by the construction in Step 3 of Algorithm 2, $\overline{v}(i) = \overline{x}(i) + \tau \geq \overline{x}(\rho) + \tau = \tilde{v}(\rho) \geq U = a \lambda > 0$ for $i = 1, \cdots, \rho$. Meanwhile, $\overline{v}(i) = 0$ for all $i > \rho$. By the same observation about $\overline{v}(i)$ above, we have $\sum_{i=1}^{2d} \overline{v}(i) = \sum_{i=1}^{\rho} \tilde{v}(i)$. Combining this with the relationship that $\overline{v}(i) = \overline{x}(i) + \tau$, we then have

$$\sum_{i=1}^{2d} \overline{v}(i) = \sum_{i=1}^{\rho} (\overline{x}(i) + \tau) = \rho \tau + \sum_{i=1}^{\rho} \overline{x}(i) = R - \rho \tau + \sum_{i=1}^{\rho} \overline{x}(i) = R,$$

where the second last equality is due to how $\tau$ is constructed in Step 3 of Algorithm 2.

Therefore, $\overline{v}$ is a feasible solution to (40), namely, the last relationship in the fourth line of (40) holds. By the same reasoning, we immediately have the second relationship in the fourth line of (40) to be satisfied.

By construction, $\overline{v}(i) = 0$ for all $i > \rho$ and $\mu(i) = 0$ for all $i \leq \rho$. We then have the third line of (40) to hold.

For $i = 1, \cdots, \rho$, we have $\overline{v}(i) = \overline{x}(i) + \tau \geq U = a \lambda$, $\beta = -\frac{\tau}{\gamma}$, and $\mu(i) = 0$, thus $\frac{1}{\gamma} (\overline{v}(i) - \overline{x}(i)) + \frac{[\alpha \lambda - \overline{v}_1]}{\alpha} + \beta - \mu(i) = \frac{1}{\gamma} (\overline{v}(i) - \overline{x}(i)) + \beta = 0$. For $i = \rho + 1, \cdots, 2d$, we have $\overline{v}(i) = 0$, $\beta = -\frac{\tau}{\gamma}$, $\mu(i) = \lambda - \frac{\overline{x}(i) + \tau}{\gamma}$, and consequently, $\frac{1}{\gamma} (\overline{v}(i) - \overline{x}(i)) + \frac{[\alpha \lambda - \overline{v}_1]}{\alpha} + \beta - \mu(i) = -\frac{1}{\gamma} \overline{x}(i) + \lambda + \beta - \mu(i) = 0$, which implies that the first line of (40) holds.

By the above choices of parameters, it is also easy to verify that $\mu(i) \cdot \overline{v}(i) = 0$ for all $i$, which immediately leads to the third line of (40).

To finally verify that $\overline{v}$ is a KKT point, it suffices to show that $\beta \geq 0$ and $\mu(i) \geq 0$ for $i = \rho + 1, \cdots, 2d$. Between them, we first show $\beta \geq 0$. To that end, it suffices to prove $\tau < 0$ by contradiction. For this purpose, we suppose $\tau \geq 0$. Let $k := \max\{i : \overline{x}(i) \geq U\}$. Then, by construction, $\overline{x}(k) + \tau \geq \overline{x}(k) \geq U = a \lambda$ and $1^\top \overline{z} = \sum_{i=1}^{k} \overline{x}(i)$. By definition of $\rho$, we have $\rho \geq k$. Since $\{\overline{x}(i)\}$ is a descent sequence, we have $\overline{v}(i) = \overline{x}(i) + \tau \geq \overline{x}(\rho) + \tau \geq U = a \lambda > 0$ for all $i \leq \rho$. Recall that we are considering the case where $1^\top \overline{z} > R$. This contradicts with (43) as $R = \sum_{i=1}^{2d} \overline{v}(i) = \sum_{i=1}^{\rho} \overline{v}(i) \geq k \tau + \sum_{i=1}^{\rho} \overline{z}(i) \geq \sum_{i=1}^{k} \overline{z}(i) = 1^\top \overline{z}$. We have thus proven $\tau < 0$, which evidently leads to $\beta = -\frac{\tau}{\gamma} \geq 0$ in view of (41).

To show $\mu(i) \geq 0$, we recall the construction of $\mu_i$ by (41). If $\overline{x}(\rho + 1) + \tau \leq 0$, then by the positiveness of $\lambda$ and $\gamma$ and the fact that $\{\overline{x}(i)\}$ is a descent sequence, it is easy to see $\mu(i) \geq 0$ for all $i$. Therefore, we only need to consider the case when $\overline{x}(\rho + 1) + \tau > 0$ below.

Let $\tau' = \frac{1}{\rho^2} \left( R - \sum_{i=1}^{\rho + 1} \overline{z}(i) \right)$, then $\tau' = \frac{1}{\rho^2} \left( R - \sum_{i=1}^{\rho + 1} \overline{z}(i) \right) - \frac{1}{\rho} \left( R - \sum_{i=1}^{\rho} \overline{z}(i) \right) = -\frac{R - \rho \overline{z}(\rho + 1) + \sum_{i=1}^{\rho} \overline{z}(i)}{\rho},$ where the last equality is due to $\tau = R - \sum_{i=1}^{\rho} \overline{z}(i)$ as
in Algorithm 1. As a result, \( \bar{x}_{(\rho+1)} + \tau - (\tau' - \tau) = \bar{x}_{(\rho+1)} + \tau' + \frac{\bar{x}_{(\rho+1)} + \tau}{\rho} \Rightarrow \bar{x}_{(\rho+1)} + \tau = \frac{\tau'}{\tau} (\bar{x}_{(\rho+1)} + \tau') \). Thus, the definition of \( \rho \) we have \( \bar{x}_{(\rho+1)} + \tau' \leq \bar{x}_{(\rho+1)} + \tau < U = a\lambda \). In view of \( \gamma \geq 2a \) and \( \frac{e^{\frac{1}{2a}}}{2} \leq 1 \), we further have \( \frac{\bar{x}_{(\rho+1)} + \tau}{\rho} \leq \frac{\bar{x}_{(\rho+1)} + \tau'}{\rho} = e^{\frac{1}{2a}} \frac{\bar{x}_{(\rho+1)} + \tau'}{2a} < \lambda \). Since \( \{\bar{x}_{(\rho)}\} \) is a descent sequence, we have \( \bar{x}_{(\rho)} + \tau \leq \bar{x}_{(\rho+1)} + \tau < \gamma \lambda \) for \( i = \rho + 1, \ldots, d \), which, combined with (41), proves the non-negativeness of \( \mu_{(i)} \).

In sum, we have proven that \( \bar{\nu} \) is a KKT point of (39).

**Step 2.** In this step of the proof, we will show that the output of Algorithm 2 \( \nu \), is the optimal solution to (10). We first observe that \( \bar{\nu} \) is the optimal solution to following convex problem, because its KKT conditions at solution \( \bar{\nu} \) coincide with those of (39).

\[
\min_{\nu \in \mathbb{R}^d} \left\{ \frac{1}{2\gamma} \| \nu - \bar{x} \|^2 + \sum_{i=1}^{d} \frac{|a\lambda - \bar{v}_i|}{a} \cdot \nu_i : 1^T \nu \leq R \right\}:
\] (43)

We claim that if \( \bar{x}_{i} = 0 \) then \( \bar{v}_i = 0 \) for any \( i \). To see this, suppose the \( l \)-th entry, \( \bar{x}_l \), of \( \bar{x} \), equals 0. let \( \bar{\nu} \) be a feasible solution to (43) and \( \bar{\nu}'_l \) be the \( l \)-th entry of \( \bar{\nu} \). Suppose that \( \bar{\nu}'_l \neq 0 \) and it must be that \( \bar{\nu}'_l > 0 \) by its definition. Then \( \bar{\nu}'' = \bar{\nu} - \bar{\nu}'_l \) is a strictly better solution than \( \bar{\nu} \) in terms of the objective value.

We also claim that \( \bar{v}_i \) and \( \bar{v}_{i+d} \) cannot be nonzero simultaneously since at least one of \( \bar{x}_i \) and \( \bar{x}_{i+d} \) is zero. To see this, recall that \( x_i \) is the \( i \)-th entry of \( x \) for \( i = 1, \ldots, d \). Then, by definition, \( \bar{x}_i = \max\{0, x_i\} \) and \( \bar{x}_{i+d} = \max\{0, -x_i\} \) for all \( i = 1, \ldots, d \). Therefore, it must hold that \( \bar{x}_i \cdot \bar{x}_{i+d} = 0 \) for all \( i = 1, \ldots, d \). We have shown that \( \bar{x}_i = 0 \Rightarrow \bar{v}_i = 0 \) for any \( i \), thus, at least one of \( \bar{v}_i \) and \( \bar{v}_{i+d} \) must be zero. By construction, \( \bar{v}_i = \bar{v}_i - \bar{v}_{i+d} \) for all \( i \). We thus have

\[
\begin{align*}
\nu_i = \bar{v}_i &\geq 0, \quad \text{and} \quad \bar{v}_{i+d} = 0, & \quad \text{if} \quad x_i \geq 0, \\
\nu_i = -\bar{v}_i &\leq 0, \quad \text{and} \quad \bar{v}_{i+d} = 0, & \quad \text{if} \quad x_i < 0,
\end{align*}
\]

which directly gives rise to

\[
\begin{align*}
|\nu_i| &= \bar{v}_i = \bar{v}_i + \bar{v}_{i+d}, & \quad \text{if} \quad x_i \geq 0, \\
|\nu_i| &= \bar{v}_{i+d} = \bar{v}_i + \bar{v}_{i+d}, & \quad \text{if} \quad x_i < 0,
\end{align*}
\]

Thus, \( \bar{\nu} \) is the optimal solution to

\[
\min_{\nu \in \mathbb{R}^d} \sum_{i=1}^{d} \frac{1}{2\gamma} (w_i - \max\{0, x_i\})^2 + \sum_{i=1}^{d} \frac{1}{2\gamma} (w_{i+d} - \max\{0, -x_i\})^2 \\
+ \sum_{i=1}^{d} \frac{|a\lambda - |\nu_i| |}{a} \cdot (w_i + w_{i+d})
\]

s.t. \( 1^T \nu \leq R; \quad \nu \geq 0. \)

Equivalently, \( \nu \) is the optimal solution to \( \min_{\nu' \in \mathbb{R}^d} \left\{ \sum_{x_i \geq 0} \frac{1}{2\gamma} (\nu'_i - x_i)^2 + \sum_{x_i < 0} \frac{1}{2\gamma} (\nu'_i - x_i)^2 + \sum_{i=1}^{d} \frac{|a\lambda - |\nu'_i| |}{a} \cdot |\nu'_i| : \|\nu'\|_1 \leq R \right\} \), which immediately leads to the desired result. \( \square \)
A.2 Proof of Lemma 2

Proof Observe that bounding the left-hand-side of the desired inequality can be reduced to bounding $\Delta_1$ and $\Delta_2$ below:

$$
\frac{1}{2} \mathbb{E}_{\omega M} \left[ \max_{\beta \subseteq \{1, \ldots, d\}; \|\beta\| \leq \frac{2M}{\delta}} \left\| \frac{1}{M} \sum_{m=1}^{M} \left( \frac{f(x + \delta u^m, \xi^m) - f(x, \xi^m)}{\delta} - \nabla f(x, \xi^m) \right) w^m_{\beta} - \frac{1}{M} \sum_{m=1}^{M} \nabla_{\beta} f(x, \xi^m) \right\|^2 \right]
$$

$$
\leq \mathbb{E}_{\omega M} \left[ \max_{\beta \subseteq \{1, \ldots, d\}; \|\beta\| \leq \frac{2M}{\delta}} \left\| \frac{1}{M} \sum_{m=1}^{M} \left( \frac{f(x + \delta u^m, \xi^m) - f(x, \xi^m)}{\delta} - \nabla f(x, \xi^m) \right) w^m_{\beta} - \frac{1}{M} \sum_{m=1}^{M} u^m_{\beta} \nabla f(x, \xi^m) \right\|^2 \right]
$$

$$
+ \mathbb{E}_{\omega M} \left[ \max_{\beta \subseteq \{1, \ldots, d\}; \|\beta\| \leq \frac{2M}{\delta}} \left\| \frac{1}{M} \sum_{m=1}^{M} \nabla_{\beta} f(x, \xi^m) - \frac{1}{M} \sum_{m=1}^{M} u^m_{\beta} \nabla f(x, \xi^m) \right\|^2 \right]
$$

(i) To bound $\Delta_1$, by Jensen’s inequality,

$$
\Delta_1 \leq \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_{\omega M} \left[ \max_{\beta \subseteq \{1, \ldots, d\}; \|\beta\| \leq \frac{2M}{\delta}} \left\| \left( \frac{f(x + \delta u^m, \xi^m) - f(x, \xi^m)}{\delta} - \nabla f(x, \xi^m) \right) w^m_{\beta} \right\|^2 \right]
$$

$$
\leq \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}_{\omega M} \left[ \max_{\beta \subseteq \{1, \ldots, d\}; \|\beta\| \leq \frac{2M}{\delta}} \left\| \left( \frac{f(x + \delta u^m, \xi^m) - f(x, \xi^m) - \nabla f(x, \xi^m)}{\delta} \right) w^m_{\beta} \right\|^2 \right]
$$

$$
\leq \frac{L^2 \sigma^2}{4} \frac{d}{\delta^2} \frac{1}{M} \sum_{m=1}^{M} |\beta|^4 \|\nabla f(x, \xi^m)\|^4
$$

where the last line results from three observations: (i) $\|w^m_{\beta}\|^2 = |\beta|$; (ii) the Lipschitz continuity of $\nabla f(x, \xi^m)$ implies that (as per (2)) $|f(x + \delta u^m, \xi^m) - f(x, \xi^m) - \nabla f(x, \xi^m) u^m_{\beta}| \leq \frac{L^2 \sigma^2 u^m_{\beta}^2}{\delta^2}$ for almost every $\xi^m$ and every $m = 1, \ldots, M$; and (iii) $\|u^m\|^2 = d$.

(ii) Let $\Xi := \{\xi^m : 1 \leq m \leq M\}$. To bound $\Delta_2$, we may invoke Lemma 3 in Section A.3 below, together with the independence between $\{u^m\}_{1 \leq m \leq M}$ and $\nabla f(x, \xi^m)$, to obtain

$$
\mathbb{E}_{\Xi} \left[ \mathbb{E}_{\omega M} \left[ \max_{1 \leq m \leq M} \left\{ \frac{1}{M} \sum_{m=1}^{M} \left( \nabla f(x, \xi^m) - u^m_{\beta} \nabla f(x, \xi^m) \right) \right\}^2 \right] \right]
$$

$$
\leq \mathbb{E}_{\Xi} \left[ 193 \sum_{m=1}^{M} \|\nabla f(x, \xi^m)\|^2 \cdot \ln d \right] \leq 193 \sigma^2 + \frac{\|\nabla F(x)\|^2}{\delta^2} \cdot \ln d,
$$

where the last inequality is immediately from Assumption 3. Therefore, $\Delta_2 \leq \frac{386 R}{\lambda} \cdot \ln d \cdot \frac{\sigma^2 + \|\nabla F(x)\|^2}{M^2}$. Combining (i) and (ii) above immediately leads to the desired result. \hfill \square

A.3 An auxiliary lemma

**Lemma 3** Let $\{u^m : m = 1, \ldots, M\}$ be an independent sequence of $d$-dimensional random vectors whose entries are $\alpha$ symmetric Bernoulli random variables. Consider a given sequence $\{\nu^m : m = 1, \ldots, M\} \subseteq \mathbb{R}^d$. Let $u^m$ and $\nu^m$ be the $i$-th entries of $u^m$ and $\nu^m$, respectively.
respective. Then $Z_i := \left[ M^{-1} \sum_{m=1}^M \left( u_i^m - u_i^m (u_i^m)^\top v_i^m \right) \right]^2$ for any $i \in \{1, \ldots, d\}$ is a subexponential random variable. Furthermore, $E \left[ \max_{1 \leq i \leq d} Z_i \right] \leq \frac{193 \sum_{m=1}^M \|v_i^m\|^2}{M^2} \cdot \ln d$.

Proof We will first examine the random variable $v_i^m - u_i^m (u_i^m)^\top v_i^m$ for a given vector $v_i^m = (v_i^m) \in \mathbb{R}^d$ and a given index $i \in \{1, \ldots, d\}$. Observe that $u_i^m \in \{-1, 1\} \implies (u_i^m)^2 = 1$, we thus have that $v_i^m - u_i^m (u_i^m)^\top v_i^m = v_i^m - (u_i^m)^2 \cdot v_i^m - \sum_{i \neq i} u_i^m \cdot u_i^m \cdot v_i^m = \sum_{i \neq i} u_i^m \cdot v_i^m$. Thus, $Z_i = \left[ M^{-1} \sum_{m=1}^M \left( v_i^m \sum_{i \neq i} u_i^m v_i^m \right) \right]^2$. Below, we prove that $\left[ M^{-1} \sum_{m=1}^M \left( u_i^m \sum_{i \neq i} u_i^m v_i^m \right) \right]^2$ is a subexponential random variable.

Because $\{u_i^m\}$ are i.i.d symmetric Bernoulli random variables, by Hoeffding’s inequality (See Theorem 2.2.2 of [3]), and the fact that $\text{Prob}(u_i = 1) = \text{Prob}(u_i = -1) = 0.5$, we have

$$\text{Prob}\left( \sum_{i \neq i} u_i^m v_i^m \geq t \right) \leq \exp \left( -\frac{t^2}{2 \sum_{i \neq i} (v_i^m)^2} \right)$$

and likewise, $\text{Prob}(\sum_{i \neq i} u_i^m v_i^m \leq -t) \leq \exp \left( -\frac{t^2}{2 \sum_{i \neq i} (v_i^m)^2} \right)$.

Therefore, $u_i^m \sum_{i \neq i} u_i^m v_i^m$ is a subgaussian random variable.

By a well-known property of a subgaussian random variable (as in Lemma 1.5 by [25]), it holds that

$$E \left[ \exp \left\{ \tau \left( \sum_{i \neq i} u_i^m v_i^m \right) \right\} \right] \leq \exp \left\{ 4 \|v_i^m\|^2 \tau^2 \right\}, \quad \text{for any } \tau \in \mathbb{R}. \quad (44)$$

In view of the fact that $\{u_i^m \sum_{i \neq i} u_i^m v_i^m : m = 1, \ldots, M\}$ is a sequence of independent random variables, we obtain from [24] that $E \left[ \exp \left\{ \tau \cdot M^{-1} \sum_{m=1}^M \left( u_i^m \sum_{i \neq i} u_i^m v_i^m \right) \right\} \right] \leq \exp \left\{ 4 \sum_{m=1}^M \|v_i^m\|^2 \tau^2 \right\}$. By a well-known relationship between subgaussian and subexponential random variables (as in Lemma 1.12 by [25]), we then have that $Z_i$ is subexponential in the sense that, for all $|\tau| \leq \frac{128 \sum_{m=1}^M \|v_i^m\|^2}{M^2}$

$$E[\exp(\tau Z_i - \tau E[Z_i])] \leq \exp \left\{ 128 \tau^2 \left( 8 \sum_{m=1}^M \|v_i^m\|^2 \right)^2 \right\} \quad (45)$$

which immediately leads to the desired result in the first part of this lemma.

Below we show the second part of the lemma. For any $\tau : 0 < \tau \leq \frac{128 \sum_{m=1}^M \|v_i^m\|^2}{M^2}$, we have $E[\max_{1 \leq i \leq d}(Z_i - E[Z_i])] \leq \frac{1}{8} \ln E[\exp(\tau \cdot \max_{1 \leq i \leq d}(Z_i - E[Z_i]))] \leq \frac{1}{8} \ln E[\exp(\tau \cdot (Z_i - E[Z_i]))]$.

By [24], $E[\max_{1 \leq i \leq d}(Z_i - E[Z_i])] \leq \frac{1}{8} \ln(d \cdot \exp(128 \tau^2 (8 \sum_{m=1}^M \|v_i^m\|^2)^2)) = \frac{1}{8} \ln d + 128 \tau^2 \cdot (8 M^{-2} \sum_{m=1}^M \|v_i^m\|^2)^2$. We may as well let $\tau = \frac{\ln d}{128 \sum_{m=1}^M \|v_i^m\|^2}$. Therefore,

$$E \left[ \max_{1 \leq i \leq d} Z_i \right] - \max_{1 \leq i \leq d} \left[ E[Z_i] \right] \leq E \left[ \max_{1 \leq i \leq d} (Z_i - E[Z_i]) \right] \leq \frac{128 \sum_{m=1}^M \|v_i^m\|^2}{M^2} \cdot \ln d + \frac{64 \sum_{m=1}^M \|v_i^m\|^2}{M^2}. \quad (46)$$
Because \( u_{i}^{m_1} \) and \( u_{i}^{m_2} \) are centered at zero, we know (also by the independence of the random variables) that
\[
E[u_{i}^{m_1} u_{i}^{m_2} v_{i_1}^{m_1} v_{i_2}^{m_2} u_{i_1}^{m_1} u_{i_2}^{m_2}] = 0, \quad \forall (i_1, i_2, m_1, m_2) : i_1 \neq i_2 \text{ or } m_1 \neq m_2. \tag{47}
\]
Evidently, for any \( \varepsilon : 1 \leq \varepsilon \leq d \), it holds that
\[
E[Z_i] = E \left[ M^{-2} \sum_{m_1, m_2} \sum_{i_1, i_2} u_{i_1}^{m_1} u_{i_2}^{m_2} v_{i_1}^{m_1} v_{i_2}^{m_2} u_{i_1}^{m_1} u_{i_2}^{m_2} \right].
\]

Combining (46), (48), and the assumption that \( d \geq 3 \), we have
\[
E[Z_i] \leq \frac{128 \Sigma_{m=1}^{M} ||u^{m}||^2}{M^2} \ln d + \frac{18 \Sigma_{m=1}^{M} ||u^{m}||^2}{M^2} \leq \frac{193 \Sigma_{m=1}^{M} ||u^{m}||^2}{M^2} \ln d, \text{ which completes the proof.} \tag*{\Box}
\]

References

1. A. Agarwal, O. Dekel, and L. Xiao. Optimal algorithms for online convex optimization with multi-point bandit feedback. In COLT, pages 28–40. Citeseer, 2010.
2. K. Balasubramanian and S. Ghadimi. Zeroth-order (non)-convex stochastic optimization via conditional gradient and gradient updates. In Proceedings of the 32nd International Conference on Neural Information Processing Systems, pages 3459–3468, 2018.
3. K. Balasubramanian and S. Ghadimi. Zeroth-order nonconvex stochastic optimization: Handling constraints, high-dimensionality and saddle-points. arXiv preprint arXiv:1809.06474, 2018.
4. P. J. Bickel, Y. Ritov, and A. B. Tsybakov. Simultaneous analysis of lasso and dantzig selector. The Annals of Statistics, 37(4):1705–1732, 2009.
5. H. Cai, D. McKenzie, W. Yin, and Z. Zhang. Zeroth-order regularized optimization (zoro): Approximately sparse gradients and adaptive sampling. arXiv preprint arXiv:2003.13901, 2020.
6. E. Candes and T. Tao. The dantzig selector: Statistical estimation when p is much larger than n. The annals of Statistics, 35(6):2313–2351, 2007.
7. P. S. Cho and M. H. Phillips. Reduction of computational dimensionality in inverse radiotherapy planning using sparse matrix operations. Physics in Medicine & Biology, 46(5):N117, 2001.
8. J. C. Duchi, M. I. Jordan, M. J. Wainwright, and A. Wibisono. Optimal rates for zero-order convex optimization: The power of two function evaluations. IEEE Transactions on Information Theory, 61(5):2788–2806, 2015.
9. J. Fan and R. Li. Variable selection via nonconcave penalized likelihood and its oracle properties. J. Amer. Statist. Assoc., 96(456):1348–1360, 2003.
10. S. Ghadimi and G. Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization, 23(4):2341–2368, 2013.
11. R. Hooke and T. A. Jeeves. “direct search” solution of numerical and statistical problems. Journal of the ACM (JACM), 8(2):212–229, 1961.
12. K. G. Jamieson, R. D. Nowak, and B. Recht. Query complexity of derivative-free optimization. In Proceedings of the 25th International Conference on Neural Information Processing Systems-Volume 2, pages 2672–2680, 2012.
13. W. C. Jordan and S. C. Graves. Principles on the benefits of manufacturing process flexibility. Management science, 41(4):577–594, 1995.
14. J. Larson, M. Menickelly, and S. M. Wild. Derivative-free optimization methods. arXiv preprint arXiv:1904.11585, 2019.
15. A. N. Letchford and A. Ouikil. Exploiting sparsity in pricing routines for the capacitated arc routing problem. Computers & Operations Research, 36(7):2320–2327, 2009.
16. Y. Li and G. Raskutti. Minimax optimal convex methods for poisson inverse problems under $\ell_p$-ball sparsity. IEEE Transactions on Information Theory, 64(8):5498–5512, 2018.
17. H. Liu, X. Wang, T. Yao, R. Li, and Y. Ye. Sample average approximation with sparsity-inducing penalty for high-dimensional stochastic programming. Mathematical programming, 178(1):69–108, 2019.
18. A. L. Marsden, J. A. Feinstein, and C. A. Taylor. A computational framework for derivative-free optimization of cardiovascular geometries. Computer methods in applied mechanics and engineering, 197(21-24):1890–1905, 2008.
19. J. Mockus. Bayesian approach to global optimization: theory and applications, volume 37. Springer Science & Business Media, 2012.
20. S. N. Negahban, P. Ravikumar, M. J. Wainwright, and B. Yu. A unified framework for high-dimensional analysis of $m$-estimators with decomposable regularizers. Statistical science, 27(4):538–557, 2012.
21. J. A. Nelder and R. Mead. A simplex method for function minimization. The computer journal, 7(4):308–313, 1965.
22. A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. SIAM Journal on optimization, 19(4):1574–1609, 2009.
23. Y. Nesterov. Smooth minimization of non-smooth functions. Mathematical programming, 103(1):127–152, 2005.
24. Y. Nesterov and V. Spokoiny. Random gradient-free minimization of convex functions. Foundations of Computational Mathematics, 17(2):527–566, 2017.
25. P. Rigollet. 18. s997: High dimensional statistics. chapter 1: Sub-gaussian random variables. (Lecture Notes). Cambridge, MA, USA: MIT Open-Course-Ware, 2015.
26. R. Y. Rubinstein and D. P. Kroese. Simulation and the Monte Carlo method, volume 10. John Wiley & Sons, 2016.
27. J. Snoek, H. Larochelle, and R. P. Adams. Practical bayesian optimization of machine learning algorithms. Advances in neural information processing systems, 25:2951–2959, 2012.
28. F. J. Solis and R. J.-B. Wets. Minimization by random search techniques. Mathematics of operations research, 6(1):19–30, 1981.
29. J. C. Spall. An overview of the simultaneous perturbation method for efficient optimization. Johns Hopkins apl technical digest, 19(4):482–492, 1998.
30. V. Torezan. On the convergence of pattern search algorithms. SIAM Journal on optimization, 7(1):1–25, 1997.
31. R. Vershynin. High-dimensional probability: An introduction with applications in data science, volume 47. Cambridge university press, 2018.
32. Y. Wang, S. Du, S. Balakrishnan, and A. Singh. Stochastic zeroth-order optimization in high dimensions. In International Conference on Artificial Intelligence and Statistics, pages 1356–1365. PMLR, 2018.
Table 2  Comparison of suboptimality gap when the mini-batch size $M$ increases. Here, the budget of the total number of zeroth-order oracle calls is 160,000, the dimension $d = 2^{15}$ = 32,768, the iteration count $K = \left\lfloor \frac{320,000}{M} \right\rfloor$, and $\delta = 1 \times 10^{-7}$. For each value of $M$, numbers in the first row are the average suboptimality gaps out of ten random replications while those in second (behind the “±”-signs) are the standard deviations. “$\cdot$” means “$\times 10^\cdot$”.

| $M$  | si-sgf$^R$ | si-sgf$^*$ | si-sgf$^A$ | si-sgfs$^R$ | si-sgfs$^*$ | si-sgfs$^A$ | sgf |
|------|------------|------------|------------|------------|------------|------------|-----|
| 40   | 1.3e-1     | 1.1e-1     | 1.7e-1     | 5.5e-1     | 3.0e-1     | 3.8e-1     | ±1.5 |
|      | ±3.0e-2    | ±4.7e-3    | ±1.7e-3    | ±5.3e-1    | ±3.0e-2    | ±5.6e-3    | ±7.6e-3 |
| 100  | 4.5e-1     | 3.3e-2     | 1.2e-1     | 9.8e-2     | 8.8e-2     | 9.6e-2     | ±1.4 |
|      | ±1.2       | ±3.1e-3    | ±2.0e-3    | ±7.6e-3    | ±6.4e-3    | ±1.6e-3    | ±4.8e-3 |
| 160  | 1.0e-1     | 3.1e-2     | 1.5e-1     | 9.0e-2     | 5.1e-2     | 5.6e-2     | ±1.4 |
|      | ±1.5e-1    | ±3.3e-3    | ±2.1e-3    | ±1.1e-1    | ±4.9e-3    | ±1.1e-3    | ±6.6e-3 |
| 220  | 4.0e-1     | 3.3e-2     | 2.0e-1     | 6.7e-2     | 4.2e-2     | 4.2e-2     | ±1.4 |
|      | ±3.9e-1    | ±1.9e-3    | ±3.0e-3    | ±6.7e-2    | ±2.6e-3    | ±1.4e-3    | ±7.9e-3 |
| 280  | 7.6e-1     | 3.8e-2     | 2.4e-1     | 4.4e-2     | 3.5e-2     | 3.6e-2     | ±1.5 |
|      | ±1.6       | ±1.3e-3    | ±4.0e-3    | ±1.1e-2    | ±1.5e-3    | ±4.7e-4    | ±6.4e-3 |
| 340  | 1.5        | 4.4e-2     | 2.9e-1     | 4.8e-2     | 3.7e-2     | 3.6e-2     | ±1.5 |
|      | ±2.3       | ±1.5e-3    | ±2.5e-3    | ±1.1e-2    | ±1.5e-3    | ±8.5e-4    | ±8.7e-3 |
| 400  | 6.2e-1     | 5.3e-2     | 3.5e-1     | 5.8e-2     | 4.5e-2     | 4.0e-2     | ±1.5 |
|      | ±1.4       | ±1.8e-3    | ±2.3e-3    | ±2.3e-2    | ±1.4e-3    | ±1.1e-3    | ±6.9e-3 |
| 460  | 7.1e-1     | 6.6e-2     | 4.1e-1     | 3.9e-1     | 5.4e-2     | 4.5e-2     | ±1.5 |
|      | ±9.7e-3    | ±1.3e-3    | ±4.1e-3    | ±1.0       | ±2.1e-3    | ±1.3e-3    | ±7.4e-3 |
| 520  | 6.6e-1     | 6.8e-2     | 4.9e-1     | 1.3e-1     | 2.5e-2     | 5.0e-2     | ±1.5 |
|      | ±1.2       | ±2.0e-3    | ±3.6e-3    | ±2.0e-1    | ±1.3e-3    | ±1.3e-3    | ±9.1e-3 |
| 580  | 2.9e-1     | 7.5e-2     | 5.6e-1     | 3.9e-1     | 2.1e-2     | 5.7e-2     | ±1.5 |
|      | ±6.6e-1    | ±2.6e-3    | ±3.4e-3    | ±1.1       | ±2.6e-3    | ±2.6e-3    | ±5.7e-3 |
| 640  | 7.1e-1     | 8.9e-2     | 6.4e-1     | 5.7e-2     | 2.5e-2     | 6.6e-2     | ±1.5 |
|      | ±1.0       | ±1.7e-3    | ±5.3e-3    | ±5.8e-2    | ±1.9e-3    | ±2.7e-3    | ±7.1e-3 |
| 700  | 1.0        | 1.1e-1     | 7.3e-1     | 4.7e-1     | 2.8e-2     | 7.4e-2     | ±1.5 |
|      | ±8.6e-1    | ±3.4e-3    | ±6.1e-3    | ±1.3       | ±8.3e-4    | ±1.1e-3    | ±4.4e-3 |
| 760  | 1.2        | 1.2e-1     | 8.1e-1     | 6.0e-1     | 3.0e-2     | 8.2e-2     | ±1.5 |
|      | ±1.2       | ±1.2e-3    | ±4.3e-3    | ±1.1       | ±1.3e-3    | ±1.4e-3    | ±8.1e-3 |
| 820  | 9.3e-1     | 1.4e-1     | 8.9e-1     | 1.6e-1     | 3.2e-2     | 8.8e-2     | ±1.5 |
|      | ±1.6       | ±1.5e-3    | ±5.6e-3    | ±1.9e-1    | ±1.3e-3    | ±1.5e-3    | ±6.6e-3 |
| 880  | 1.4        | 1.6e-1     | 9.9e-1     | 2.2e-1     | 3.4e-2     | 9.7e-2     | ±1.5 |
|      | ±1.8       | ±4.8e-3    | ±1.1e-2    | ±4.8e-1    | ±1.5e-3    | ±3.4e-3    | ±7.2e-3 |
Table 3 Comparison of suboptimality gap when \( d \) increases exponentially. \( \delta = 10^{-7} \). The budget of the total number of zeroth-order oracle calls is 160,000. For all variants of \( si-sgf \) and \( si-sgf's \), the mini-batch sizes \( M \) are chosen as 160 and 280, respectively. Correspondingly, iteration count \( K = \lceil \frac{320,000}{d M} \rceil \). For each value of \( d \), numbers in the first row are the average suboptimality gaps out of five random replications while those in second (behind the ‘+’ signs) are the standard deviations. ‘\( \times 10^k \)’ means ‘\( \times 10^k \)’. The numbers in bold refer to the smallest average suboptimality gaps for the same \( d \).

| \( d \) | \( \text{si-sgf}^R \) | \( \text{si-sgf}^* \) | \( \text{si-sgf}^A \) | \( \text{si-sgf's}^R \) | \( \text{si-sgf's}^* \) | \( \text{si-sgf's}^A \) | \( \text{sgf} \) |
|---|---|---|---|---|---|---|---|
| 2^6 | 4.2e-2 | 3.2e-2 | 8.1e-2 | 1.5e-2 | 1.5e-2 | 2.5e-2 | 2.2e-3 |
| ±1.4e-2 | ±7.0e-3 | ±2.2e-3 | ±3.1e-3 | ±3.6e-3 | ±1.3e-3 | ±7.4e-4 |
| 2^7 | 3.5e-2 | 3.0e-2 | 8.1e-2 | 3.2e-2 | 1.9e-2 | 2.6e-2 | 3.9e-3 |
| ±7.4e-3 | ±4.7e-3 | ±8.0e-4 | ±1.5e-2 | ±3.2e-3 | ±1.7e-3 | ±6.1e-4 |
| 2^8 | 3.9e-2 | 3.7e-2 | 8.2e-2 | 4.0e-2 | 2.7e-2 | 2.7e-2 | 1.0e-2 |
| ±1.1e-2 | ±4.9e-3 | ±1.4e-3 | ±1.3e-2 | ±3.1e-3 | ±1.9e-3 | ±8.1e-4 |
| 2^9 | 7.6e-2 | 3.7e-2 | 8.4e-2 | 4.9e-2 | 3.7e-2 | 2.6e-2 | 2.2e-2 |
| ±3.8e-2 | ±2.1e-3 | ±1.8e-3 | ±1.0e-2 | ±2.2e-3 | ±1.2e-3 | ±6.5e-4 |
| 2^{10} | 5.0e-1 | 4.1e-2 | 8.5e-2 | 5.6e-2 | 4.5e-2 | 2.5e-2 | 4.4e-2 |
| ±9.9e-1 | ±2.3e-3 | ±1.2e-3 | ±6.3e-3 | ±1.7e-3 | ±1.5e-3 | ±4.9e-4 |
| 2^{11} | 6.9e-2 | 3.7e-2 | 8.8e-2 | 2.1e-1 | 4.4e-2 | 2.4e-2 | 7.2e-2 |
| ±4.1e-2 | ±2.8e-3 | ±2.1e-3 | ±3.4e-1 | ±2.2e-3 | ±6.7e-4 | ±1.0e-3 |
| 2^{12} | 2.1e-1 | 3.4e-2 | 9.8e-2 | 1.1e-1 | 4.1e-2 | 2.5e-2 | 8.7e-2 |
| ±3.0e-1 | ±2.9e-3 | ±1.5e-3 | ±1.4e-1 | ±2.1e-3 | ±7.6e-4 | ±1.9e-3 |
| 2^{13} | 1.8e-1 | 3.3e-2 | 1.1e-1 | 4.2e-2 | 3.6e-2 | 2.5e-2 | 1.4e-1 |
| ±2.8e-1 | ±4.7e-3 | ±1.1e-3 | ±7.4e-3 | ±1.4e-3 | ±1.3e-3 | ±1.2e-3 |
| 2^{14} | 2.6e-1 | 2.7e-2 | 1.5e-1 | 5.5e-1 | 3.4e-2 | 3.0e-2 | 4.5e-1 |
| ±3.6e-1 | ±2.2e-3 | ±1.0e-3 | ±1.1 | ±1.1e-3 | ±6.1e-4 | ±5.0e-3 |
| 2^{15} | 3.3e-2 | 3.0e-2 | 1.6e-1 | 3.9e-2 | 3.4e-2 | 3.5e-2 | 1.5 |
| ±2.4e-2 | ±2.8e-3 | ±3.5e-3 | ±1.0e-2 | ±1.3e-3 | ±8.2e-4 | ±4.8e-3 |
| 2^{16} | 8.0e-2 | 3.5e-2 | 1.9e-1 | 1.1 | 3.6e-2 | 4.3e-2 | 3.0 |
| ±7.0e-2 | ±4.1e-3 | ±1.8e-3 | ±2.3 | ±1.9e-3 | ±8.6e-4 | ±3.4e-3 |
| 2^{17} | 7.0e-1 | 3.8e-2 | 2.2e-1 | 5.9e-2 | 4.1e-2 | 5.1e-2 | 4.4 |
| ±1.1 | ±3.7e-3 | ±1.0e-3 | ±2.5e-2 | ±1.6e-3 | ±1.4e-3 | ±3.4e-3 |
| 2^{18} | 3.7e-1 | 4.3e-2 | 2.6e-1 | 7.3e-2 | 4.3e-2 | 6.1e-2 | 5.5 |
| ±4.6e-1 | ±4.2e-3 | ±2.5e-3 | ±3.6e-2 | ±3.4e-3 | ±9.9e-4 | ±2.8e-3 |
| 2^{19} | 5.9e-1 | 5.6e-2 | 3.0e-1 | 1.3 | 4.7e-2 | 7.2e-2 | 6.1 |
| ±9.7e-1 | ±2.6e-3 | ±3.6e-3 | ±2.8 | ±4.3e-3 | ±1.4e-3 | ±9.2e-4 |
| 2^{20} | 1.6e-1 | 7.0e-2 | 3.5e-1 | 3.5e-1 | 5.2e-2 | 8.6e-2 | 6.4 |
| ±7.4e-2 | ±1.2e-3 | ±3.3e-3 | ±4.1e-1 | ±1.7e-3 | ±1.7e-3 | ±8.3e-4 |
| 2^{21} | 4.5e-1 | 9.0e-2 | 4.0e-1 | 2.5e-1 | 5.8e-2 | 9.8e-2 | 6.6 |
| ±7.5e-1 | ±3.3e-3 | ±5.2e-3 | ±4.0e-1 | ±2.6e-3 | ±2.1e-3 | ±3.3e-4 |
Table 4 Comparison of suboptimality gap when $\delta$ increases exponentially. Here, the budget of the total number of zeroth-order oracle calls is 160,000, and the dimension $d = 2^{15} = 32,768$. For all variants of $si-sgf$ and $si-sgfs$, the mini-batch sizes $M$ are chosen as 160 and 280, respectively. Correspondingly, iteration count $K = \left\lfloor \frac{320,000}{M} \right\rfloor$. For each value of $\delta$, numbers in the first row are the average suboptimality gaps out of ten random replications while those in second (behind the “±”-signs) are the standard deviations. “e•” means “×10•”.\

| $\delta$ | $si-sgf^R$ | $si-sgf^*$ | $si-sgf^A$ | $si-sgfs^R$ | $si-sgfs^*$ | $si-sgfs^A$ | $sgf$ |
|----------|----------|----------|----------|----------|----------|----------|------|
| $10^{-8}$ | 3.1e-1  | 2.9e-2  | 1.6e-1  | 3.8e-2  | 3.6e-2  | 3.6e-2  | 1.6  |
|          | ±5.8e-1 | ±1.9e-3 | ±2.3e-3 | ±4.1e-3 | ±1.5e-3 | ±7.2e-4 | ±1.9e-1 |}

| $10^{-7}$ | 3.9e-1  | 2.8e-2  | 1.6e-1  | 4.5e-2  | 3.6e-2  | 3.6e-2  | 1.6  |
|          | ±7.4e-1 | ±2.5e-3 | ±2.0e-3 | ±1.3e-2 | ±2.4e-3 | ±9.7e-4 | ±8.2e-2 |}

| $10^{-6}$ | 9.9e-2  | 3.3e-2  | 1.6e-1  | 3.9e-2  | 3.5e-2  | 3.5e-2  | 1.6  |
|          | ±9.5e-2 | ±4.7e-3 | ±1.1e-3 | ±4.6e-3 | ±3.2e-3 | ±9.1e-4 | ±1.6e-1 |}

| $10^{-5}$ | 1.1e-1  | 3.0e-2  | 1.6e-1  | 8.6e-2  | 3.6e-2  | 3.7e-2  | 1.5  |
|          | ±1.1e-1 | ±4.0e-3 | ±2.4e-3 | ±7.3e-2 | ±5.8e-4 | ±1.1e-3 | ±4.9e-2 |}

| $10^{-4}$ | 6.2e-1  | 1.1e-1  | 3.2e-1  | 1.4e-1  | 1.1e-1  | 1.5e-1  | 1.6  |
|          | ±6.3e-1 | ±8.1e-3 | ±6.5e-3 | ±1.7e-2 | ±4.6e-3 | ±2.3e-3 | ±6.4e-2 |}

| $10^{-3}$ | 6.8     | 6.6     | 6.7     | 6.9     | 6.4     | 6.7     | 6.7  |
|          | ±2.8e-2 | ±5.2e-2 | ±1.6e-3 | ±1.5e-1 | ±1.5e-1 | ±6.1e-3 | ±2.6e-4 |