Is there a “Charge - Magnet Paradox”

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In this paper it is shown that in the approach to special relativity in which an independent physical reality is attributed to the four-dimensional (4D) geometric quantities, the invariant special relativity (ISR), there is no recently presented paradox that in a static electric field a magnetic dipole moment (MDM) is subject to a torque in some frames and not in others. Hence, in the ISR, there is no need either for the change of the Lorentz force, but as a 4D geometric quantity, or for the introduction of some “hidden” 3D quantities. Furthermore, in the ISR, contrary to all previous approaches, an electrically neutral current-loop in its rest frame possesses not only a MDM $m$, but also an electric dipole moment (EDM) $p$ and a stationary permanent magnet possesses not only an intrinsic magnetization $M$ but also an intrinsic electric polarization $P$. Hence, in a static electric field, both, a current-loop and a permanent magnet experience the Lorentz force $K_L$ and the torque $N$ (bivector), i.e., the “space-space” part $N_s$ and the “time-space” part $N_t$ of $N$, in all relatively moving inertial frames. The quantities $m, p, M, P, K_L, N, N_s, N_t$ are the 4D geometric quantities.

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1. Introduction

In a recent article [1] it is argued that in the presence of the magnetization $M$ and the electric polarization $P$ the usual expression for the Lorentz force with the 3-vectors fails to accord with the principle of relativity, because it leads to an apparent paradox involving a magnetic dipole moment (MDM) $m$ in the presence of an electric field $E$; in a static electric field a MDM $m$ is subject to a torque $T$ in some frames and not in others. In this notation all 3-vectors are designated in boldface type. In [1] it is argued that the Lorentz force should be replaced by the Einstein-Laub law, which predicts no torque $T$ in all frames. In the following, we shall partly rely on the results and the explanations from [2]. In Section 9.1 in [2], Mansuripur’s paper [1], the highlight of it [3] and some of the critics of it [4] are considered. Mansuripur’s response to the critics is given in [5]. In [1, 3-5], all quantities $E, B, P, T$, etc. are the three-dimensional (3D) vectors and it is considered that their transformations (they will be called the usual transformations (UT)) are the relativistically correct Lorentz transformations (LT) (boosts). Here, in the whole paper, under the name LT we shall only consider boosts.
Here, in Section 2, the UT of the 3-vectors $E$ and $B$, Eq. (1), and their derivation, Eq. (3), are objected. According to these UT $E'$ for one inertial observer is “seen” as slightly changed $E$ and an induced $B$ for a relatively moving inertial observer. The similar UT hold for other 3-vectors, $P$ and $M$, the EDM $p$ and MDM $m$, see Eq. (2), for the torque 3-vector $T$ and another 3D vector $R$, Eq. (4), etc. In all papers [1, 3-5] the induced EDM $p$, Eq. (2), for the moving MDM $m$, leads to the interaction with the electric field $E$ and to the offending torque $T$, i.e., to the violation of the principle of relativity and to the above mentioned paradox. In Section 2, it is also shown that the derivation of the UT as in (3) is not possible if instead of Einstein’s synchronization a nonstandard “radio,” “r” synchronization, is used, see Eqs. (5) and (7). Also, the “r” synchronization is briefly described.

In this paper (and in Section 9.2 in [2]) it is shown that in the recently developed geometric approach to special relativity (SR), i.e., in the invariant SR (ISR), in which an independent physical reality is attributed to the 4D geometric quantities and not, as usual, to the 3D quantities, the principle of relativity is naturally satisfied and there is no paradox. Hence, there is no need either for the change of the expression for the Lorentz force, but as a 4D geometric quantity, or for the introduction of some “hidden” 3D quantities. For a brief review of the ISR see, e.g., [2] and references therein. In some papers, e.g., [6-9], the ISR is called the “True transformations relativity.”

In the ISR, which is perfectly suited to the symmetry of the 4D spacetime, we deal either with the abstract, coordinate-free, 4D geometric quantities, e.g., vectors (4-vectors in the usual notation) $E(x), B(x), \ldots \ (x$ is the position vector), or with the 4D coordinate-based geometric quantities (CBGQs) comprising both components and a basis, e.g., $E = E^\nu \gamma_\nu$. The CBGQs are invariant under the passive LT, see, e.g., Eq. (11). Such a 4D geometric quantity represents the same physical quantity for relatively moving inertial observers. It is not the case in all usual approaches that deal with the 3D quantities and their UT or, as in the usual covariant approaches, e.g., [10], with the components of tensors, which are implicitly taken in the standard basis, see below.

In Section 3, the primary quantity for the electromagnetic field, the bivector $F$ is introduced and its decomposition into the vectors $E$ and $B$ is presented in Eqs. (14) and (15). The vectors $E$ and $B$ are then derived from $F$ and $v$, the velocity vector of the observers who measure $E$ and $B$ fields, Eqs. (16) and (17). Similar equations are presented for the generalized magnetization-polarization bivector $\mathcal{M}$ and its “time-space” and “space-space” parts, the polarization vector $P$ and the magnetization vector $M$, respectively, Eqs. (20) and (21), for the dipole moment bivector $D$ and its “time-space” and “space-space” parts, the EDM vector $p$ and MDM vector $m$, respectively, Eqs. (22) and (23), for the 4D angular momentum bivector $J$ and its “time-space” and “space-space” parts, the angular momentum vectors $J_t$ and $J_s$, respectively, Eqs. (24) - (26), for the torque bivector $N$ and its “time-space” and “space-space” parts, the torque vectors $N_t$ and $N_s$, respectively, Eqs. (27) - (29). The same equations as for $J$ and $J_s$, $J_t$, hold also for the spin bivector $S$ as the intrinsic angular momentum and its usual “space-space” intrinsic angular momentum vector $S$. 

2
and its “time-space” intrinsic angular momentum vector \( Z \), but, the velocity vector \( v \) of observers that appears in Eqs. (24) - (26) is replaced by \( u \), the velocity vector of the particle, see Section 8 in [2] and references therein. Furthermore, in Section 3, the LT of the electric field vector are given by Eq. (10) and their derivation is given by Eq. (15). In the 4D spacetime, in contrast to the UT of the 3D quantities, the mathematically correct LT transform, e.g., the electric field vector again to the electric field vector; there is no mixing with the magnetic field vector. The same LT hold for all other vectors, \( x, P, M, p, m, J_t, J_s, N_t, N_s, S, Z, \) etc.

In this paper, the treatment of the interaction between a static electric field and a permanent magnet will be as in Section 9.2 in [2], i.e., very similar to the treatments of Jackson’s paradox [11] and the Trouton-Noble paradox [12, 13]. For simplicity, mainly the standard basis \( \{\gamma_\mu; 0, 1, 2, 3\} \) of orthonormal vectors, with \( \gamma_0 \) in the forward light cone, will be used. The \( \gamma_k \) \((k = 1, 2, 3)\) are spacelike vectors. As already stated, we use the term vector for the usual 4-vector, but as a 4D geometric quantity. The \( \{\gamma_\mu\} \) basis corresponds to the specific system of coordinates with Einstein’s synchronization [14] of distant clocks and Cartesian space coordinates \( x^i \). Using the 4D geometric quantities it is shown, e.g., in [6] and [2], that an electrically neutral current-loop (superconducting) in its rest frame possesses not only a MDM \( m \), but also an EDM \( p \). In the second paper in [6], the incorrect quadrupole field of the stationary current loop from the published version is replaced by the dipole field. Similarly, it is shown in Section 8 in [2] that a stationary permanent magnet possesses not only an intrinsic magnetization \( M \) but also an intrinsic electric polarization \( P \). That result was derived using the generalized Uhlenbeck-Goudsmit hypothesis from [15] according to which the connection between the dipole moment bivector \( D \) and the spin bivector \( S \) is given as \( D = g_\mathbf{S} S \), Eq. (31). From (31) and using the decompositions of \( D \) (22) and \( S \), the same as (20), we write the connections between \( m \) and \( p \) and the corresponding intrinsic angular momentums, vectors, \( S \) and \( Z \) as Eq. (32). Hence, in a static electric field, both, a current-loop and a permanent magnet experience the Lorentz force (vector) \( K_L \) and the torque (bivector) \( N \) in all relatively moving inertial frames and there is no paradox.

In Section 4, the main results are obtained. First, the most general expression for the Lorentz force density \( k_L \) as an abstract vector is presented in Eq. (35). In the 4D spacetime, it is relativistically correct expression and there is no need to change it. Furthermore, if \( k_L \) is written as a CBGQ in the standard basis it becomes Eq. (36). In \( S’ \), the rest frame of the magnet, it is given by the relation (37). That \( k_L \) from (36) and also from (37) is, as any other CBGQ, invariant under the passive LT, which means that it is not as in [1, 3-5] different for relatively moving inertial observers. The integrated torque \( N \) as a CBGQ is given by Eq. (38) in \( S’ \) and by Eq. (39) in \( S \), the lab frame. The expression (39) is obtained by the use of the LT from Eq. (38). It is shown in (10) that \( N \) in \( S’ \), Eq. (38), is equal to \( N \) in \( S \), Eq. (39). This result directly proves that with the use of the 4D torque \( N \) there is no paradox. Of course, the same result is obtained dealing with the “space-space” torque \( N_s \) and the “time-space” torque \( N_t \), Eqs. (41), (42) and (43).
In Sections 5 - 5.6.2.2 we have discussed some other differences between the conventional formulation of SR that deals with the 3-vectors and their UT and this formulation, the ISR, that deals with 4D geometric quantities. In Section 5.1, it is argued that usual Maxwell’s equations (with 3-vectors), Eqs. (1-4) in [1], have to be replaced by Eqs. (48) and (49), which are invariant under the LT. In Section 5.2, it is briefly explained that, contrary to the assertion from [1, 5], the Lorentz force law is compatible with momentum conservation laws if all quantities are the 4D geometric quantities. In Section 5.3, it is explained that Eq. (51) is the relativistic version of Newton’s law and not, as argued in [1], Eq. (50). In Section 5.4, the charge densities for a moving or stationary infinite wire with a steady current are investigated. It is explained that the Lorentz contraction is not well-defined in the 4D spacetime and that only the rest length is properly defined quantity. Therefore, in the 4D spacetime, the charge density of moving charges is meaningless and only the current density vector $j$ is a well-defined physical quantity. Moreover, the usual conclusion that magnetism is a relativistic phenomenon is not correct in the 4D spacetime, because it is derived using the Lorentz contraction and the definition of charge in terms of 3D quantities [54]. The charge defined by Eq. (54) is not invariant under the LT and in the 4D spacetime that definition has to be replaced by the relativistically correct definition (55). The current density vector $j$, as a CBGQ in the rest frame of the wire, is determined by Eq. (60). It is mentioned that the result (60) leads to the existence of an external electric field outside a stationary wire with a steady current. In Section 5.5, another paradox from [1] is examined in detail. In [1], it is argued that a current-carrying wire in the presence of a constant, uniform electric field experiences a Lorentz force in some frames and not in others. This paradox is again obtained using the 3-vectors and the UT (1), i.e., the UT of the 3-force. In [1], it is stated that the SR is not violated because there is “an increase in the mass of the wire” under the action of the $E$ field, what is, according to [1], in agreement with “the relativistic version of Newton’s law,” i.e., with Eq. (50). In [1], such an explanation is considered to be the resolution of the paradox. However, it is shown in that section 5.5 that again there is no paradox if the 4D geometric quantities are used, i.e., if the Lorentz force density $k_L$ is given by Eq. (62). In Sections 5.6 - 5.6.2.2, the electromagnetic field of a point charge in uniform motion is examined. That example very nicely illustrates the essential difference between the conventional SR and the ISR. First, we present the expressions for $F$ as an abstract bivector (65) and as a CBGQ (66). That $F$ in (65) and (66) depends only on the velocity vector $u_Q$ of the charge $Q$ and not on $v$. It is argued in that section that the 3-vectors $E'$ and $E$, $H$ that are defined by Eq. (11) and by Eqs. (12a), (12b) in [1], respectively, have to be replaced by $F$ defined by (65) and its representation (66), i.e., by the CBGQs defined by (68) and (70). They are all the 4D geometric quantities that properly transform under the LT. Then, $E$ and $B$ as abstract vectors are given by Eq. (72) and as CBGQs by Eq. (73). $E$ and $B$ from (72) and from (73) depend on two velocity vectors $u_Q$ and $v$. This enable us to compare the usual expressions for the 3-vectors $E'$ and $E$, $H$ from [1] with $E$, $B$ fields in the case when the observers who measure fields are at rest, $v = c\gamma_0'$, in
and $A$ grade of the highest-grade terms respectively of the above series; $A \langle \ldots \rangle$ be seen from Eq. (76), both relatively moving inertial observers. In the second case (Section 5.6.2.2), as can be seen from Eq. (74), the components $E^\mu$ and $B^\mu$ in $S'$ are $B = B^{\mu \gamma'_\mu} = 0$, whereas the components $E^\mu$ of $E$ in $S'$ ($E = E^{\mu \gamma'_\mu}$) agree with the usual result, e.g., with the components of the 3-vector $E$ given by Eq. (11) in [1]. However, as seen from Eq. (75), the components $E^\mu$ and $B^\mu$ in $S$, which are obtained by the LT (10) from those in (74), significantly differ from the components of the 3-vectors $E$ and $H$ given by Eqs. (12a), (12b) in [1], which are obtained by the UT (1). Particularly, since the magnetic field vector transforms by the LT again to the magnetic field vector, $B$ remains zero for all relatively moving inertial observers. In the second case (Section 5.6.2.2), as can be seen from Eq. (76), both 4D vector fields $E$ and $B$ are different from zero in $S$, the lab frame, in which the charge $Q$ is moving. Now, the components $E^\mu$ and $B^\mu$ in $S$ agree with the components of the 3-vectors $E$ and $H$ given by Eqs. (12a), (12b) in [1]. However, as seen from Eq. (77), the components $E^\mu$ and $B^{\mu \gamma'_\mu}$ in $S'$, the rest frame of the charge $Q$, which are obtained by the LT (10) from those in (76), are completely different than the components of the 3-vectors $E'$ and $H'$ ($H' = 0$) given by Eq. (11) in [1]. Particularly, in (77), $B^{\mu \gamma'_\mu}$ different from zero and, of course, the same holds for all relatively moving inertial observers. Thus, in sections 5.6.2.1 and 5.6.2.2, it is proved that the usual expressions with the 3-vectors, Eqs. (11), (12a) and (12b) in [1], are not equivalent to the expressions with the 4D geometric quantities, the abstract vectors $E$ and $B$ given by Eq. (72), or CBGQs which are given by Eq. (73). This means that the usual expressions for the electric and magnetic fields as the 3-vectors are not relativistically correct in the 4D spacetime. Observe that all four expressions for $E$ and $B$, (72), (73), (76) and (77), give the same $F$ from (66), if they are introduced into $F$ from (15). Remember that $F$ from (65) and (66) does not contain the velocity of the observer $v$, but only the velocity vector $u_Q$ of the charge $Q$. This proves that the electromagnetic field, here the bivector $F$, is the primary quantity for the whole electromagnetism and not the electric and magnetic fields.

In Section 6 the conclusions are briefly exposed.

In this paper, the presentation will be in the geometric algebra formalism, see a brief summary in Section 2 in [2] and references therein. Here, for the reader’s convenience, we shall write all equations with the CBGQs in the standard basis and only some of them with the abstract multivectors. Hence, the geometric algebra product is written by simply juxtaposing multivectors $AB$. The geometric product of a grade-$r$ multivector $A_r$ with a grade-$s$ multivector $B_s$ decomposes into $A_r B_s = \langle AB \rangle_r^{s+} + \langle AB \rangle_r^{s-2} \cdots + \langle AB \rangle_r^{-s}$. The inner and outer (or exterior) products are the lowest-grade and the highest-grade terms respectively of the above series; $A_r \cdot B_s \equiv \langle AB \rangle_r^{|r-s|}$ and $A_r \wedge B_s \equiv \langle AB \rangle_r^{s+}$. For vectors $a$ and $b$ we have: $ab = a \cdot b + a \wedge b$, where $a \cdot b \equiv (1/2)(ab + ba)$, $a \wedge b \equiv (1/2)(ab - ba)$. Usually the above mentioned stan-
standard basis is introduced. The generators of the spacetime algebra (the Clifford algebra generated by Minkowski spacetime) are taken to be four basis vectors \{\gamma_\mu\}, \mu = 0..3, satisfying \gamma_\mu \cdot \gamma_\nu = \eta_\mu\nu = \text{diag}(+-++)$. The basis vectors \gamma_\mu generate by multiplication a complete basis for the spacetime algebra: $1$, $\gamma_\mu$, $\gamma_\mu \wedge \gamma_\nu$, $\gamma_\mu \gamma_5$, $\gamma_5$ ($2^4 = 16$ independent elements). $\gamma_5$ is the right-handed unit pseudoscalar, $\gamma_5 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$. Any multivector can be expressed as a linear combination of these 16 basis elements of the spacetime algebra.

2. 3-vectors E, B, P, .. and their UT

In all conventional formulations of the relativistic electrodynamics, e.g., [14], [10], the UT of the 3-vectors E and B, and also P and M, p and m, are considered to be the LT. These UT are given by, e.g., [14], the last equations in §6, II. Electrodynamical Part, or [10], Eq. (11.149) for E and B or Eq. (11.148) for components implicitly taken in the standard basis and for a boost along the $x^1$ axis they are

$$
E_1 = E'_1, \quad E_2 = \gamma(E'_2 + \beta B'_3), \quad E_3 = \gamma(E'_3 - \beta B'_2),
B_1 = B'_1, \quad B_2 = \gamma(B'_2 - \beta E'_3), \quad B_3 = \gamma(B'_3 + \beta E'_2). \quad (1)
$$

The essential feature is that, e.g., the transformed B is expressed by the mixture of B' and $\gamma E'$, and similarly for E. The same holds for the UT of P and M and also for EDM p and MDM m. Thus, if a permanent magnetization M' is viewed from a moving frame it produces an electric polarization $P = \gamma U \times M'/c^2$. It is also argued that a neutral stationary current loop with a magnetic moment m' in its rest frame $S'$, acquires an electric dipole moment

$$
p = \beta \times m'/c \quad (2)
$$

if it is moving with uniform 3-velocity $U (\beta = U/c)$ relative to the laboratory frame $S$. It is always assumed that in the rest frame of the neutral current loop the electric moment p' is zero, p' = 0. For more detail see Sections 3.1 and 3.2 in [2] and references therein. It is visible, e.g., from Griffiths and Hnizdo (GH) [4], that the offending torque T from their Eq. (5) is obtained from the term $p \times E$, where $p$ is the induced EDM, their Eq. (4), which means that the violation of the principle of relativity is a direct consequence of the UT for the 3-vectors p and m, i.e., P and M. In all papers in [1, 4] and [5] the same induced EDM p appears as a result of the UT for p and m.

In [10], the six independent components of the electromagnetic field tensor $F^{\alpha\beta}$ (only components in the standard basis) are identified to be six components of the 3-vectors E and B in both relatively moving inertial frames of reference,

$$
E'_i = F'^{i0}, \quad E_i = F^{i0}; \quad B'_i = (1/2c)\varepsilon_{ikl}F^{dlk}, \quad B_i = (1/2c)\varepsilon_{ikl}F^{dlk}. \quad (3)
$$

This means that the UT of the components of E and B are derived assuming that they transform under the LT as the components of $F^{\alpha\beta}$ transform, Eq. (11.148) in [10], i.e., Eq. (1) here. (Note that in [3] the components of the
Here, they are written for the motion along the $x$ in the moving frame are expressed by the mixture of the components of $R$ of (4) that corresponds to $T$ and of another 3D vector $R$ are identified with the “space-space” and “time-space” components respectively of the torque four-tensor $N_{\alpha\beta}$ in both relatively moving inertial frames of reference, see Cross [4] and Section 9.1 in [2]. This yields the UT for the components $T_i$ of $T$ and $R_i$ of $R$, which, as can be seen from (4), are the same as the UT for $B_i$ and $E_i$, respectively.

\[
T_1 = T'_1, \quad T_2 = \gamma(T'_2 - \beta R'_3), \quad T_3 = \gamma(T'_3 + \beta R'_2), \\
R_1 = R'_1, \quad R_2 = \gamma R'_2 + \beta T'_2, \quad R_3 = \gamma R'_3 - \beta T'_2. \tag{4}
\]

Here, they are written for the motion along the $x^1$ axis. Note that the component in [4] that corresponds to $T^{x2}$ in Cross [4] is $T_2 = -\gamma \beta R_3$. The components $T_i$ in the moving frame are expressed by the mixture of the components of $T'$ and of $R'$ from the rest frame. This causes that the components of $T$ will not vanish in the $S$ frame even if they vanish in the $S'$ frame, i.e., that there is a “charge-magnet paradox” in all usual approaches to SR that deal with the 3-vectors or with components implicitly taken in the standard basis. The same holds for the components of the angular momentum $L$, $L = r \times p$, with $T = dL/dt$, and of another 3-vector $K$. It is assumed that they transform as the “space-space” and “time-space” components respectively of the usual covariant angular momentum four-tensor $J^{\alpha\beta}$, see [16] and Section 3 in [11]. These UT of the components of $L$ are the same as the UT (4) but with $L_i, K_i$ replacing $T_i, R_i$, respectively. Observe that in [16], and also in [1, 3-5], only the “space-space” components of $J^{\alpha\beta} (L_i)$ and $N^{\alpha\beta} (T_i)$ are considered to be the physical angular momentum and torque respectively, because they are associated with actual rotation in the 3D space of the object. On the other hand, the “time-space” components of $J^{\alpha\beta} (K_i)$ and $N^{\alpha\beta} (R_i)$ are not considered to be of the same physical nature as $L_i$ and $T_i$. In all usual treatments it is considered that $K_i$ and $R_i$ are not the physical angular momentum and torque respectively, because they are not associated with any overt rotation in the 3D space of the object, see, particularly, GH [4] and Jackson’s paper [16].

However, if one does not use Einstein’s synchronization but, e.g., a drastically different “radio,” “r” synchronization, then, as will be shown below, it is not possible to make the identification of the components of, e.g., the 3-vectors $E$ and $B$ with the components $F^{\alpha\beta}$, i.e., $L_i, K_i$ with the components $J^{\alpha\beta}$, or $T_i, R_i$ with the components $N^{\alpha\beta}$. For the “r” synchronization, see also Section 3.1 in [2] and references therein. The “r” synchronization is commonly used in everyday life and not Einstein’s synchronization. If the observers who are at different distances from the studio clock set their clocks by the announcement
from the studio then they have synchronized their clocks with the studio clock according to the “r” synchronization. Thus, there is an absolute simultaneity in the “r” synchronization. If, e.g., the components $F^{\alpha\beta}$ of $F$ are transformed by the transformation matrix $R^\mu_\nu$ to the $\{r_\mu\}$ basis, then it is obtained that

$$F^{r \alpha}_{\mu 10} = F_{10} - F_{12} - F_{13}. \tag{5}$$

The same equation holds for $J_{r \alpha}^{10}, N_{r \alpha}^{10}, \ldots$. In the transformation matrix $R^\mu_\nu$, which connects the components from the $\{\gamma_\mu\}$ basis with the components from the $\{r_\mu\}$ basis, the only components that are different from zero are

$$R^\mu_\mu = -R^0_1 = 1. \tag{6}$$

It is visible from (5) that the “time-space” component in the $\{r_\mu\}$ basis is expressed by the mixture of the “time-space” component and the “space-space” components from the $\{\gamma_\mu\}$ basis. Hence, in the $\{r_\mu\}$ basis (with the “r” synchronization) it holds that $F_{r \alpha}^{10} = E_1 + cB_3 - cB_2$. The same identification as in $\{\gamma_\mu\}$ basis, yields that, e.g., the component $E_{1r}$ in the $\{r_\mu\}$ basis is expressed as the combination of $E_i$ and $B_i$ components from the $\{\gamma_\mu\}$ basis,

$$E_{1r} = F_{r \alpha}^{10} = E_1 + cB_3 - cB_2. \tag{7}$$

This means that the usual identifications, e.g., Eq. (11.137) in [10], or those ones in Cross [4], are meaningful only if the $\{\gamma_\mu\}$ basis is chosen, i.e., if the Minkowski metric is used. But, every synchronization is only a convention and physics must not depend on conventions.

We note that in the $\{r_\mu\}$ basis the usual time and space components of the position vector $x$ cannot be separated. The connections between the components of $x$ in both bases are given by the relations

$$x^i_r = x^i - x^1 - x^2 - x^3, \quad x^i_r = x^i. \tag{8}$$

It is visible from (8) that in the $\{r_\mu\}$ basis the space-time split (3+1) of the 4D spacetime is not possible. Hence, specifically, in the 4D spacetime, the usual translation in the 3D space has not some definite physical meaning. However, if the 4D geometric quantities are used, i.e., in the ISR, the physics does not depend on conventions, since the abstract geometric quantity, the vector $x$, can be decomposed in both bases and it holds that

$$x = x^\mu r_\mu = x^\mu r_\mu. \tag{9}$$

Thus, as already stated in Section 1, in the 4D spacetime the physical quantities are not correctly represented only by components, but a basis must be included.

Furthermore, in the 4D spacetime, it is meaningless to consider that, e.g., the 3D $T$ and $L$ are physical torque and angular momentum, respectively, whereas $R$ and $K$ are not of the same physical nature. In the same way as in (7), the “time-space” component of $J_{r \alpha}^{10} (K_{1r})$ (or the same for $N_{r \alpha}^{10} (R_{1r})$) in the $\{r_\mu\}$
basis will be expressed in terms of the “time-space” component ($K_1$) and the “space-space” components ($L_i$) from the \{$\gamma_\mu$\} basis. Hence, in the 4D spacetime, as mentioned above for the usual translation in the 3D space, the usual rotation, i.e., an overt rotation in the 3D space has not a definite physical meaning. In the 4D spacetime, the correctly defined 4D angular momentum is the bivector $J$ given by Eq. (24) and the correctly defined 4D torque is the bivector $N$ given by Eq. (27), which are connected by the relation (30), $N = dJ/d\tau$. In the 4D spacetime, they completely describe all phenomena connected with a rotation.

3. Vectors $E$, $B$, $P$, .. and their LT

Recently, in [17-22], it is proved both in the tensor formalism and in the geometric algebra formalism, that the UT of the 3-vectors $E$ and $B$ ARE NOT the LT and also the correct LT of the 4D geometric quantities that represent the electric and magnetic fields are derived. In the 4D spacetime, the correct LT always transform the 4D algebraic object representing, e.g., the electric field only to the electric field; there is no mixing with the magnetic field. For a review see [2]. In [17-22], see also Section 5 in [2], the LT of the components $E^\mu$ (in the \{$\gamma_\mu$\} basis) of the vector $E = E^\mu \gamma_\mu$ are given as

$$E^0 = \gamma(E^0 - \beta E^1), \quad E^1 = \gamma(E^1 - \beta E^0), \quad E^{2,3} = E^{2,3},$$  \hspace{1cm} (10)

for a boost along the $x^1$ axis. As mentioned above any CBGQ is unchanged under the LT, i.e., it holds that

$$E = E^{\mu} \gamma_\mu = E^{\mu} \gamma'_\mu = E^{\mu} r_\mu = E^{\mu} r'_\mu,$$  \hspace{1cm} (11)

where the primed quantities in both bases \{$\gamma_\mu$\} and \{$r_\mu$\} are the Lorentz transforms of the unprimed ones; for the LT in the \{$r_\mu$\} basis see [7]. The same LT hold for any other vector, e.g., $x$, $B$, $P$, $M$, EDM $p$ and MDM $m$, the torque vectors $N_s$ and $N_t$, see below, etc. The equation (11) shows that in the 4D spacetime the vector $E$ is the same 4D quantity for all relatively moving inertial frames of reference and for all systems of coordinates chosen in them. This is an essential difference relative to all usual approaches with the 3-vectors and their UT; the 3-vectors $B'$ and $B$ that are connected by the UT (11) are completely different quantities in the 4D spacetime. Note that the same holds for the usual covariant approach, e.g., from [10], that deals with components implicitly taken in the standard basis, i.e., $F^{\alpha\beta} \neq F^{\alpha\beta}$; they are different quantities in the 4D spacetime. The components do not contain the whole information about some physical quantity; a basis must be included.

A short derivation of the LT (10) is presented in [22] and it will be briefly repeated here. It is proved in [12] that in the 4D spacetime the primary physical quantity for the whole electromagnetism is the electromagnetic field (bivector) $F$. There, an axiomatic geometric formulation of electromagnetism is developed in which there is only one axiom, the field equation for $F$

$$\partial \cdot F + \partial \wedge F = j/\varepsilon_0 c,$$  \hspace{1cm} (12)
magnetic fields have to be represented by the 4D geometric quantities, i.e., as in [24], a mathematical argument is presented according to which the electric and magnetic vectors can be reinvented and generalized in terms of 4D geometric quantities in [17-22]. In a given inertial frame will determine which three components are independent. Namely, these definitions are mathematically correct definitions, which are first given (only in the component form) by Minkowski in Section 11.6 in [23] and reinvented and generalized in terms of 4D geometric quantities in [17-22]. In [24], a mathematical argument is presented according to which the electric and magnetic fields have to be represented by the 4D geometric quantities, i.e., as in

\[ \partial_\alpha F^{\alpha\beta\gamma\delta} - \partial_\alpha *F^{\alpha\beta\gamma\delta} = (1/\varepsilon_0c)\gamma_5\gamma_\beta, \]  

(13)

where \( \varepsilon^{\alpha\beta\gamma\delta} \) is the totally skew-symmetric Levi-Civita pseudotensor and \( \gamma_5 \), as already stated, is the right-handed unit pseudoscalar \( \gamma_5 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \) and \( *F^{\alpha\beta} = (1/2)\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \) is the usual dual tensor. The usual covariant form of Eq. (13), i.e., only the basis components in the \( \{\gamma_\mu\} \) basis, are two equations, the equation with sources \( \partial_\alpha F^{\alpha\beta\gamma\delta} = j_{\beta}/\varepsilon_0c \), and that one without sources \( \partial_\alpha *F^{\alpha\beta} = 0 \). It can be seen from [12] that the bivector field \( F \) yields the complete description of the electromagnetic field. For the given sources the expression for \( F \) can be found from Eqs. (7) and (8) in [12] and there is no need to introduce either the field vectors or the potentials. However, the field vectors can be introduced using a mathematical theorem that any second rank antisymmetric tensor can be decomposed into two space-like vectors and a unit time-like vector (the velocity vector/c). Hence, \( F \) can be decomposed as

\[ F = (1/c)E \wedge v + (IB) \cdot v, \]  

(14)

where the unit pseudoscalar \( I \) is defined algebraically without introducing any reference frame. If \( I \) is represented in the \( \{\gamma_\mu\} \) basis it becomes \( I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_5 \). If that equation for \( F \) is written with the CBGQs in the \( \{\gamma_\mu\} \) basis it becomes

\[ F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \quad F^{\mu\nu} = (1/c)(E^\mu v^\nu - E^\nu v^\mu) + \varepsilon^{\mu\nu\alpha\beta}v_\alpha B_\beta, \]  

(15)

where \( \gamma_\mu \wedge \gamma_\nu \) is the bivector basis. Observe that bivector \( F \) is the same 4D quantity for relatively moving inertial observers and for all bases chosen by them, i.e., the relation like (11) holds for \( F \) as well, \( F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = ... \). The vectors \( E \) and \( B \) are defined in terms of \( F \) and \( v \), the velocity vector of a family of observers who measure \( E \) and \( B \) fields in the following way

\[ E = (1/c)F \cdot v, \quad B = -(1/c^2)I(F \wedge v). \]  

(16)

We write them as the CBGQs in the \( \{\gamma_\mu\} \) basis

\[ E = E^\mu \gamma_\mu = (1/c)F^{\mu\nu}v_\nu \gamma_\mu, \quad B = B^\mu \gamma_\mu = (1/2c^2)\varepsilon^{\mu\nu\alpha\beta}F_{\nu\alpha}v_\beta \gamma_\mu. \]  

(17)

Since \( F \) is antisymmetric it holds that \( E^\mu v_\mu = B^\mu v_\mu = 0 \), only three components of \( E \) and \( B \) in any basis are independent. However, it does not mean that three spatial components of \( E \), or \( B \), are necessarily independent components. Namely \( E \) and \( B \) depend not only on \( F \) but on \( v \) as well. The form of \( v \) in a given inertial frame will determine which three components are independent. These definitions are mathematically correct definitions, which are first given (only in the component form) by Minkowski in Section 11.6 in [23] and reinvented and generalized in terms of 4D geometric quantities in [17-22]. In [24], a mathematical argument is presented according to which the electric and magnetic fields have to be represented by the 4D geometric quantities, i.e., as in
Minkowski’s Section 11.6 and in [17-22]. Namely, it is explained in [24] that the number of variables on which a vector field depends, i.e., the dimension of its domain is essential for the number of components of that vector field. Hence, the usual time-dependent \( \mathbf{E}(r,t), \mathbf{B}(r,t) \) cannot be the 3-vectors, since they are defined on the spacetime. Therefore, we use the term “vector” for the correctly defined geometric quantity, which is defined on the spacetime, e.g., \( E(x), B(x), P(x), M(x), \) etc. An incorrect expression, the 3-vector or the 3D vector, will still remain for the usual \( \mathbf{E}(r,t), \mathbf{B}(r,t) \) from the conventional formulations of the electromagnetism, e.g., [14], [10], [25-30], ... . However, it has to be noted that in the 4D spacetime we always have to deal with correctly defined vectors \( E(x), B(x), P(x), M(x), \) etc. even in the usual static case, i.e., if the usual 3D fields \( \mathbf{E}(r), \mathbf{B}(r) \) do not explicitly depend on the time \( t \). The reason is that in the 4D spacetime there is no static case, i.e., there is no electrostatic and magnetostatic.

The LT mix the time and space components, which means that the LT cannot transform the spatial coordinates from one frame only to spatial coordinates in a relatively moving inertial frame of reference. What is static case for one inertial observer is not more static case for relatively moving inertial observer, but a time dependent case. Furthermore, as explained above in Section 2, if an observer uses the “r” synchronization and not the standard Einstein’s synchronization, then as seen from (8) the space and time are not separated and the usual 3D vector \( \mathbf{r} \) is meaningless. Hence, if the principle of relativity has to be satisfied and the physics must be the same for all inertial observers and for \( \{ \gamma_\mu \}, \{ r_\mu \}, \{ \gamma'_\mu \} \), etc. bases which they use, then the properly defined quantity is the position vector \( x, \) \( x = x^\mu \gamma_\mu = x^\mu \gamma'_\mu = x^\mu r_\mu = x^\mu r'_\mu, \) and not \( \mathbf{r} \) and \( t \). Consequently, in the 4D spacetime, e.g., the electric field is properly defined as the vector \( E(x) \) for which the relation (11) holds.

The frame of “fiducial” observers for which \( v = c \gamma_0 \), with the \( \{ \gamma_\mu \} \) basis in it, will be called the \( \gamma_0 \)-frame. This is not any kind of a preferred frame, because any inertial frame can be chosen to be the \( \gamma_0 \)-frame. In the \( \gamma_0 \)-frame \( E^0 = B^0 = 0 \) and \( E^i = F^{0i}, B^i = (1/c)\epsilon^{ijk}F_{kj} \); the same components as in, e.g., Eq. (11.137) in [10]. However, in any other inertial frame, the “fiducial” observers are moving, and \( v = v^0 \gamma_0 = c\gamma_0 = v^\mu \gamma'_\mu \). For the “fiducial” observers, \( v^\mu = c\gamma_\mu \) and \( E^\mu = F^{\mu
u} \gamma_{0\nu} \). It is proved by Minkowski in Section 11.6 in [23], and reinvented in [17-22], see also Section 5 in [2], that in the mathematically correct procedure for the derivation of the LT of \( E \) both \( F \) and the velocity vector \( v \) have to be transformed by the LT, e.g., as shown in [22], for the LT from the \( \gamma_0 \)-frame

\[
E = E^\mu \gamma_\mu = [(1/c)F^{0i}v_0]_i = E'^\mu \gamma'_\mu = [(1/c)F'^{\mu\nu}v'_\nu]_\gamma'_\mu. \tag{18}
\]

Hence, the components \( E^\mu \) transform by the LT again to the components \( E'^\mu \) of the same electric field vector, i.e., the above quoted LT (10) of the components \( E^\mu \) are obtained. The main point is that the transformed components \( E'^\mu \) are not determined only by \( F'^{\mu\nu} \), as in all usual approaches, e.g., Eqs. (11.147) and (11.148) in [10], but also by \( v^\mu \). In the third paper in [24] Oziewicz, from the mathematical point of view, nicely explains the results obtained in my papers [17-22]. (The references in the quoted part refer to the mentioned Oziewicz’s
paper.) He states: “Minkowski [1], and then Ivetić [7-10], observed correctly that if a Lorentz transformation is an isomorphism of a vector space, then the entire algebra of tensor fields must be Lorentz-covariant. .... An active Lorentz transformation must act on all tensor fields, including an observer’s time-like vector field.” This means that if the Lorentz transformation is applied to $E$ from (16), $E = (1/c)F \cdot v$, then, from the mathematical point of view, it is necessary that the Lorentz transformation acts not only on $F$ but on $v$, i.e., “an observer’s time-like vector field”, as well.

It can be easily checked, see Section 6 in [2], that the UT, Eq. (11.148) in [10], i.e., Eq. (1) here, will be obtained if only the components $F^{\mu\nu}$ are transformed but not the components $v^\mu$. Hence, from the above mentioned Oziewicz’s mathematical argument the UT cannot be - the LT.

Similarly, the bivector $\mathcal{M}$ can be decomposed as

$$\mathcal{M} = P \wedge u/c + (MI) \cdot u/c^2, \quad (19)$$

or, as a CBGQ, it is written as

$$\mathcal{M} = (1/2)\mathcal{M}^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad \mathcal{M}^{\mu\nu} = (1/c)(P^\nu u^\mu - P^\mu u^\nu) + (1/c^2)\varepsilon^{\mu\nu\alpha\beta} M_\alpha u_\beta. \quad (20)$$

The vectors $P(x)$ and $M(x)$ are determined by $\mathcal{M}(x)$ and the unit time-like vector $u/c$, where $u$ is identified with bulk velocity vector of the medium in spacetime

$$P = (1/c)\mathcal{M}^{\mu\nu} u_\nu \gamma_\mu, \quad M = (1/2)\varepsilon^{\mu\nu\alpha\beta} \mathcal{M}_{\alpha\nu} u_\beta \gamma_\mu, \quad (21)$$

with $P^\mu u_\mu = M^\mu u_\mu = 0$. It is visible from (21) that $P$ and $M$ depend not only on $\mathcal{M}$ but on $u$ as well, see also Section 4 in [2]. In the same way, the bivector $D$ as the primary physical quantity for the dipole moments can be written as

$$D = (1/2)D^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad D^{\mu\nu} = (1/c)(p^\nu u^\mu - p^\mu u^\nu) + (1/c^2)\varepsilon^{\mu\nu\alpha\beta} m_\alpha u_\beta, \quad (22)$$

see also [22] and Section 4 in [15]. Then, one finds that $p$ and $m$ are determined by the bivector $D$ and the velocity vector of the particle $u$ as

$$p = (1/c)D^{\mu\nu} u_\nu \gamma_\mu, \quad m = (1/2)\varepsilon^{\mu\nu\alpha\beta} D_{\alpha\nu} u_\beta \gamma_\mu, \quad (23)$$

with $p^\mu u_\mu = m^\mu u_\mu = 0$. In the particle’s rest frame (the $S'$ frame) and the $\gamma_\mu$ basis, $u = c\gamma_0$, which yields that $p^0 = m^0 = 0$, $p^i = D^{i0}$, $m^i = (c/2)\varepsilon^{0ijk} D_{jk}$. Therefore $p$ and $m$ can be called the “time-space” part and the “space-space” part, respectively, of $D$. The quotation marks are written because the relation, e.g., $p^i = D^{i0}$, holds only in the $\{\gamma_\mu\}$ basis but not in other bases, e.g., in the $\{\gamma_\mu\}$ basis.

In the 4D spacetime, the primary physical quantity for the 4D angular momentums is the bivector $J$,

$$J = (1/2)J^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu. \quad (24)$$

It can be decomposed into the “space-space” and the “time-space” angular momentum vectors $J_s$ and $J_t$ respectively and the velocity vector $v$ of a family of observers who measures $J_s$ and $J_t$. The components $J^{\mu\nu}$ are given as.
\[ J^{\mu\nu} = (1/c)[(J_t^\mu v^\nu - J_t^\nu v^\mu) + \varepsilon^{\mu\nu\alpha\beta} J_s,\alpha v^\beta]. \]  

Then, the vectors \( J_s \) and \( J_t \) are derived from \( J \) and the velocity vector \( v \) as

\[ J_t = (1/c)J^{\mu\nu} v_\nu \gamma_\mu, \quad J_s = (1/2c)\varepsilon^{\mu\nu\alpha\beta} J_{\alpha\nu} v_\beta \gamma_\mu, \]

with \( J_t^\mu v_\mu = J_t^\nu v_\nu = 0 \). \( J_s \) and \( J_t \) depend not only on \( J \) but also on \( v \). In the \( \gamma_0 \)-frame \( J_t^\mu = 0 \) and only the spatial components remain \( J_s^\mu = (1/2)\varepsilon^{\mu jk} J_j^k \). \( J_t^i = J^{i0} \). \( J_s^i \) and \( J_t^i \) correspond to the components of the 3-vectors \( L \) and \( K \) that are introduced, e.g., in [16]. However, as already stated above, in [16], as in all usual treatments including [1, 4, 10, 25-30], it is considered that only \( L \) is a physical quantity whose components transform according to the UT, e.g., Eq. (11) in [16], i.e., the same as \( J \) but with \( L_i, K_i \) replacing \( T_i, R_i \), respectively.

Furthermore, see [11-13] and Sections 9.1 and 9.2 in [2], the torque bivector \( N \) as a CBGQ is given as

\[ N = (1/2)N^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad N^{\mu\nu} = x^\mu K_L^\nu - x^\nu K_L^\mu, \]

where \( K_L \) is the Lorentz force vector. The decomposition of \( N \) into the “space-space” and the “time-space” vectors \( N_s \) and \( N_t \) respectively is given as

\[ N^{\mu\nu} = (1/c)[(N_s^\mu v^\nu - N_s^\nu v^\mu) + \varepsilon^{\mu\nu\alpha\beta} N_s,\alpha v^\beta]. \]

The “time-space” torque \( N_t \) and the “space-space” torque \( N_s \)

\[ N_t = (1/c)N^{\mu\nu} v_\nu \gamma_\mu, \quad N_s = (1/2c)\varepsilon^{\mu\nu\alpha\beta} N_{\alpha\nu} v_\beta \gamma_\mu, \]

are determined by \( N \) and the velocity vector \( v \) of a family of observers who measures \( N_s \) and \( N_t \). It holds that \( N_s^\mu v_\mu = N_t^\mu v_\mu = 0 \). In the \( \gamma_0 \)-frame, \( v^\mu = (c, 0, 0, 0) \), \( N_s^0 = N_t^0 = 0 \) and only the spatial components \( N_s^i \) and \( N_t^i \) remain, \( N_s^i = (1/2)\varepsilon^{ijk} N_j^k \), \( N_t^i = N^{i0} \). Both vectors \( N_s \) and \( N_t \) are in the same measure physical 4D torques, which, only if taken together, are equivalent to the 4D torque \( N \). \( N \) is connected with the angular momentum bivector \( J \) as

\[ N = dJ/d\tau, \]

where \( \tau \) is the proper time.

In the following we shall also need an important relation, the generalized Uhlenbeck-Goudsmit hypothesis, which is explained in detail in [15]. In the same way as \( J \) is the primary physical quantity for the 4D angular momentums the spin bivector \( S \) (four-tensor \( S^{ab} \) in [15]) is the primary quantity with definite physical reality for the intrinsic angular momentums. It can be decomposed into the usual “space-space” intrinsic angular momentum vector \( S \), the “time-space” intrinsic angular momentum vector \( Z \) and the unit time-like vector \( u/c \), where \( u \) is the velocity vector of the particle. The relations are the same as \( (25) \) and \( (26) \), but \( J, J_s, J_t \) and \( v \) are replaced by \( S, S, Z \) and \( u \), respectively. Then, in [15], the usual connection between the 3-vectors \( m \) and \( S, m = \gamma_S S \), is formulated.
as the generalized Uhlenbeck-Goudsmit hypothesis; the dipole moment bivector \( D \) is proportional to the spin bivector \( S \)

\[
D = g_S S. \tag{31}
\]

Furthermore, in [15], using the decompositions of \( D \) (22) and \( S \), the same as (25), we have formulated the connections between the dipole moments, vectors, \( m \) and \( p \) and the corresponding intrinsic angular momentums, vectors, \( S \) and \( Z \), respectively, as

\[
m = c g_S S, \quad p = g_S Z. \tag{32}
\]

Hence, in [15], a fundamentally new result is obtained only from a relativistically correct treatment of physical quantities \( D \) and \( S \), i.e., that any fundamental particle has not only the usual spin vector \( S \) and the corresponding intrinsic MDM \( m \), but also another spin vector \( Z \) and the corresponding intrinsic EDM \( p \), whose magnitude is \((1/c)\) of that for \( m \). In the particle’s rest frame and the \( \{\gamma \}' \mu \) basis,

\[
u = c \gamma_0', \quad p^0' = m^0 = 0, \quad p^i = g_S Z'^i, \quad m^i = c g_S S'^i. \]

Hence, comparing this last relation with \( m = \gamma_S S \), it is visible that \( g_S = \gamma_S/c \). As shown in Section 8 in [2] these results yield that in the same way as the MDMs determine the magnetization \( M \) of a stationary permanent magnet the EDMs determine its polarization \( P \), which induces an electric field outside a permanent magnet, moving or stationary.

This discussion explicitly shows that from the ISR viewpoint the derivation of the transformations of the 3-vectors \( E \) and \( B \) from [10] is not mathematically correct, i.e., Eqs. (11.148), (11.149) in [10] are not the LT but the UT. The same holds for the UT of the 3-vectors \( P \) and \( M \), \( p \) and \( m \), \( R \) and \( T \), \( K \) and \( L \).

Hence, from the ISR viewpoint, all “resolutions” of Mansuripur’s paradox from [1] and [3-5] are not relativistically correct, because they are based on the use of the 3D quantities and their UT, see also Section 9.1 in [2]. In addition, it is worth mentioning that all treatments from [1, 3-5] are meaningless if only the Einstein synchronization is replaced by the “r” synchronization. This conclusion simply follows already from the above mentioned equations for \( F_{10}^{(5)} \) and the expression for \( E_{1r}^{(7)} \).

4. With the 4D torques there is no “Charge - Magnet Paradox”

We consider the system from [1], but, without loss of generality, the electric charge will be substituted by a uniform electric field. The common rest frame of the source of the electric field (a point charge \( q \) in [1], i.e., \( Q \) in this paper) and of the permanent magnet will be denoted as \( S' \), whereas the lab frame, in which the \( S' \) frame moves with uniform velocity \( V = V \gamma_1 \) along the common \( x^1, x'^1 \) axes, will be denoted as \( S \). Hence, in \( S' \), only the component \( F_{10}^{(10)} \) of \( F^\mu_\nu \) is different from zero. From the point of view of the ISR it would be more appropriate to exclusively deal with the primary quantity \( F \), i.e., \( F^\mu_\nu \), as in the treatment of the Trouton-Noble paradox in [12]. However, for the reader’s convenience and for an easier comparison with the usual treatments from [1] and [3-5] we shall explicitly work with quantities that are derived by a correct
mathematical procedure from $F$ and $v$ according to (17), i.e., with the vectors $E$ and $B$. Observe, as shown in Section 5.6 below, that $E$ and $B$ are different for different choices of the velocity vector of the observer $v$, but $F$ is the same for all these choices of $v$. $F$ is independent of $v$ and that fact shows that $F$ is the primary quantity for the electromagnetism and not the electric and magnetic fields. This is completely different than in all conventional formulations of the electromagnetism, e.g., [1], [3-5], [10], [16], [25-30], in which the 3-vectors $E$ and $B$, i.e., their components implicitly taken in the standard basis, are considered to be the primary quantities, whereas the components $F_{\mu\nu}$ are determined in terms of $E_{x,y,z}$ and $B_{x,y,z}$, i.e., as already stated, they are identified to be six components of the 3-vectors $E$ and $B$ in all relatively moving inertial frames of reference, according to (3). Obviously, the same consideration holds for the bivector $M$ of reference, according to (3). This is completely different than in all conventional formulations of the electromagnetism and not the electric and magnetic fields. From (14) and of $M$, $k_L$, as an abstract vector, becomes

$$k_L = (1/c) F \cdot j = (1/c) F^{\mu\nu} j_{\nu\gamma} \mu,$$  

where the total current density vector $j$ is $j = j^{(C)} + j^{(M)}$; $j^{(C)}$ is the conduction current density of the free charges and $j^{(M)}$ is the magnetization-polarization current density of the bound charges, $j^{(M)} = -c \partial \mathcal{M} = -c \partial \cdot \mathcal{M}$ ($\partial \cdot \mathcal{M} = 0$, since $j^{(M)}$ is a vector). If written as a CBGQ $j^{(M)}$ is

$$j^{(M)} = -c \partial \mathcal{M} ^{\mu\nu} \gamma_{\mu\nu}.$$  

In the considered case it is taken that $j^{(C)} = 0$. Using the decompositions of $F$ and $M$, $k_L$, as an abstract vector, becomes

$$k_L = (1/c^2)(E \wedge v) \cdot [-(\partial \cdot P)u + (u \cdot \partial) P + (1/c) [u \wedge (\partial \wedge M)] I],$$  

where it is taken in the decomposition of $F$ that in the considered case $B = 0$. In contrast to all previous treatments with the UT, according to the LT $B$ is always $= 0$ and therefore there is no reason for the appearance of the paradox. It is visible that the expression for $k_L$ contains two velocity vectors, $v$ - the velocity vector of the observers who measure $E$ and $B$ fields (from (14), i.e., (15)), and $u$ - the velocity vector of the permanent magnet, i.e., of the electric current loop (from (20)). This $k_L$ is relativistically correct expression and it does not need any change. If $k_L$ is written as a CBGQ in the standard basis it becomes

$$k_L = \frac{1}{(c^2)} \left[ (\partial_{\mu} P^{\mu}) [ (E^{\mu} u_{\nu}) v_{\rho} \omega (v^{\sigma} u_{\sigma}) E_{\rho} ] - (u^{\mu} \partial_{\nu} P^{\mu}) [ E_{\nu} v^{\rho} - v_{\nu} E_{\rho} ] \right] + \frac{1}{c^2} \epsilon^{\mu \nu \alpha \beta} u_{\mu} (\partial_{\alpha} M_{\beta}) [ E_{\nu} v^{\rho} - v_{\nu} E_{\rho} ] \gamma_{\rho} = k_L \gamma_{\rho}.$$  

As any other CBGQ $k_L$ from (36) is invariant under the LT; it is the same 4D quantity for all relatively moving inertial observers. Here, we write $k_L$
in the $S'$ frame, i.e., for the case that $u = v = c\gamma'_0$ and accordingly that $E^0 = P^0 = M^0 = 0$. Then, $k_L$ as a CBGQ in the $\{\gamma'_\mu\}$ basis is

$$
k_L = (-E'^k\partial'_0 P'_k + (1/c)e^{0jkl}E'_j\partial'_k M'_l)\gamma'_0 - E'^0(\partial'_k P'^{0k})\gamma'_k.
$$

(37)

In the usual approaches with the 3-vectors and their UT, e.g., in [1] and in GH [4], the Lorentz 3-force density is zero in the $S'$ frame; there is no $\gamma'_0$ term and there is no $P$. The components $k'^\mu_L$ correspond to the time and spatial components of $f^\alpha$ from Cross [4], i.e., to $f^0 = (1/c)\mathbf{E}(\partial\mathbf{P}/\partial t + \nabla \times \mathbf{M})$ and $f^i = -E^i(\nabla \mathbf{P})$. But, the components $k'^\mu_L$ are multiplied by the unit basis vectors $\gamma'_\mu$ in order to form the CBGQ $k'^\mu_L\gamma'_\mu$, i.e., a representation in the standard basis of a vector $k_L$, whereas the 3-vector, e.g., $\mathbf{E}$ is constructed from the components $E_{x,y,z}$ and the unit 3-vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. In the 4D spacetime there is no room for the 3-vectors; they cannot correctly transform under the LT. It is not correct to write the components of some 4D CBGQ in terms of the 3-vectors like in Cross [4], Vanzella [4], Barnett [4], etc. as in almost all textbooks that treat SR, e.g., [10, 25 - 30].

For the sake of brevity we shall explicitly write the results for $N, N_s$ and $N_l$, for others see Section 9.2 in [2]. The torque density $n$ is $n = (1/2)n^\mu\nu\gamma_\mu \wedge \gamma_\nu, n^\mu\nu = x^\mu k'_L - x^\nu k'_L$, where $k'^\mu_L$ is given by Eq. (36), $n$, as a CBGQ in the standard basis is given by Eq. (69) in [2]. In the $S'$ frame it is given by Eq. (70) in [2] and it is $\neq 0$, whereas in the approaches with the 3-vectors $n$ is zero in the $S'$ frame.

In the common rest frame $S'$, in which $\nu^\mu = (c, 0, 0, 0)$, the integrated torque $N$ as a CBGQ is given as

$$
N = -(1/c)E'^1 m^2(\gamma'_0 \wedge \gamma'_3) - E'^3 p^3 (\gamma'_1 \wedge \gamma'_3),
$$

(38)

where $m$ is the MDM vector and $p$ is the EDM vector. In the considered case, $E = E'^1\gamma'_1, m = m^2\gamma'_2$ and $p = p^3\gamma'_3$. The quantities in (38) are all properly defined in the 4D spacetime and they properly transform under the LT. Hence, in $S$, the lab frame, the torque $N$ can be obtained by the LT from $S'$ and it is

$$
N = (-E^3 m^2/c + \beta E^1 p^3)(\gamma_0 \wedge \gamma_3) + (\beta E^1 m^2/c - E^3 p^3)(\gamma_1 \wedge \gamma_3).
$$

(39)

The LT of the components of the electric field vector $E$, Eq. (10), are used to derive that $E'^1 = (1/\gamma)E^1$. It can be seen from Eqs. (38) and (39) that the 4D torque $N$ is the same 4D geometric quantity for all relatively moving inertial observers, i.e., it holds that

$$
N = (1/2)N'^{\mu\nu} \gamma'_\mu \wedge \gamma'_\nu = (1/2)N^\mu\nu \gamma_\mu \wedge \gamma_\nu
$$

(40)

and there is no paradox. In the same way as in (11) for $E$ the bivector $N$ will be the same 4D quantity for all bases, e.g., the $\{r_\mu\}$ basis, and not only for the standard basis.
Let us determine $N_s$ and $N_t$, which are both equally well physical, as 4D CBGQs in the common rest frame $S'$. They are

$$N_s = N_s^\mu \gamma_\mu = (1/c) E^1 p^3 v_0, \quad N_t = N_t^\mu \gamma_\mu = -(1/c^2) E^1 m^2 v_0 \gamma_3, \quad (41)$$

where $N_t$ in (41) corresponds to the expression $R = m \times E = -m E \vec{y}$ in Cross [4] that describes the interaction of the magnetic moment with the electric field in the rest frame $S'$. Remember that in Cross [4] the rest frame is with unprimed quantities and the motion is along the $x^3$ axis. Moreover, Cross deals with components $N^{\mu\nu}$ and $J^{\mu\nu}$ implicitly taken in the standard basis and not with the CBGQs. Contrary to his assertion, the transformations of his $R$ and $T$, i.e., of the corresponding components $N^{01}$ and $N^{12}$ respectively are not the LT but the UT given by Eq. (4). Thus his $T$ and $R$ are not relativistically correct and they are completely different than $N_s$ and $N_t$. It is visible from (38) that $N_t$ in (41) comes from the “time-space” component $N^{03}$ and in this geometric approach it exists even if $p^{\mu\nu}$ would be zero. $N_s$ in (41) does not appear in any previous paper since it emerges from the existence of the EDM $p$ for a stationary permanent magnet. It describes the interaction of the EDM $p$ of the stationary permanent magnet with the electric field $E$ in the rest frame $S'$. It comes from the “space-space” component $N^{12}$ in (38). In the usual formulation with the 3-vectors it would correspond to the usual 3D torque $T = p \times E$, but, in contrast to all previous formulations, this torque is in the rest frame $S'$. Then, we determine $N_s$ and $N_t$ in $S$, e.g., using the LT of all quantities which determine $N_s$ and $N_t$ in (41).

$$N_s = N_s^\mu \gamma_\mu = (1/\gamma) E^1 p^3 \gamma_2, \quad N_t = -(1/c \gamma) E^1 m^2 \gamma_3. \quad (42)$$

Observe that $N_s$ ($N_t$) transforms under the LT as every vector transforms, i.e., as in (10), which means that components $N_s^\mu$ of $N_s$ transform to the components $N_s^\mu$ of the same torque $N_s$ in the $S$ frame; there is no mixing with the components of $N_t$. These LT of the components of $N_s$ ($N_t$) are obtained in the same way as the LT of $B$ ($E$) are obtained, i.e., that both $N$ and $v$ from the definitions of $N_s$ ($N_t$) (20) are transformed by the LT. This is in a sharp contrast to the UT of the 3D torque $T$, Eq. (1), in which the transformed components $T_i$ are expressed by the mixture of components $T^i_k$ of the 3-vector $T'$ and of components $R_i$ of another 3-vector $R'$ from the rest frame. These UT of $T_i$ (and $R_i$) can be obtained in such a way that only $N$ from the definitions of $N_s$ ($N_t$) (20) is transformed by the LT, but not the velocity of the observer $v$.

It is worth noting that the CBGQs $N_s^\mu \gamma_\mu$ and $N_s^\mu \gamma_\mu$ are the same quantity $N_s$ in $S'$ and $S$ frames, and the same for $N_t$,

$$N_s = N_s^\mu \gamma_\mu = N_s^{\mu\nu} \gamma_\mu, \quad N_t = N_t^\mu \gamma_\mu = N_t^{\mu\nu} \gamma_\mu. \quad (43)$$

The relation (43) holds in the same measure for all bases and not only for the standard basis, as in (11). This again shows, as in [11-13], that in the approach with the 4D torques $N_s$ and $N_t$ the principle of relativity is naturally satisfied and there is no paradox. Observe that $N_s$ is always determined by the interaction of
the EDM $p$ and $E$, whereas $N_t$ is determined by the interaction of $m$ and $E$. In this geometric approach there is no need either for the “hidden” 3D mechanical angular momentum or for the “hidden” 3D torque.

Let us examine what would be if it is taken that, as in the usual treatments [1, 3-5], a permanent magnet possesses only a MDM $m$ and not an EDM $p$. Note, that in our approach there is $p \neq 0$ and the assumption that $p = 0$ is only taken for some comparison with the usual approaches. However, it is worth mentioning that even in this case $p = 0$ we deal with correctly defined vectors in the 4D spacetime and with their LT and not with the 3-vectors and their UT. Then, for $p = 0$, as can be seen from (38), the “space-space” component $N'_{13}$ is zero, but the “time-space” component $N'_{03}$ is different from zero. As already stated, in the conventional treatments only the “space-space” components $N''_{ij}$ are considered to be physical, i.e., that they are three components of the 3D torque $T'$, which is connected with the usual 3D rotation. But, in this geometric approach, as explained at the end of Section 2, the usual rotation in the 3D space has not a definite physical meaning. In the 4D spacetime, only the whole $N$ given by (35) does have a definite physical reality. In $S$, the torque $N$ is given by Eq. (39) and for $p = 0$ both the “space-space” component $N_{13}$ and the “time-space” component $N_{03}$ are different from zero. They are both determined by the interaction of the magnetic moment $m$ with the electric field $E$. As can be easily seen, Eq. (40) holds in this case too and there is no paradox.

It is visible from (41) and (42) that for $p = 0$ the “space-space” part of $N$, the torque vector $N_s$ is always zero, $N_s = 0$, but $N_t$ is different from zero, $N_t \neq 0$, and it is always the same 4D geometric quantity, which means that again there is no paradox. Remember that only if $N_s$ and $N_t$ are taken together then they are equivalent to the primary physical quantity for the 4D torques, to the bivector $N$. It is, as explained above, different from zero and it is the same 4D quantity for all relatively moving inertial frames of reference. Also, as in the case with $p \neq 0$, there is no need for the change of the expression for the Lorentz force, but as a 4D geometric quantity, or for the introduction of some “hidden” 3D quantities.

5. Another differences in the treatments with 4D geometric quantities and with the 3D quantities

The preceding consideration clearly shows that, as in the case with Jackson’s paradox [11] and the Trouton-Noble paradox [12, 13], Mansuripur’s paradox with the torque appears because of the use of the 3D quantities and their UT. But, it is visible from the relations for $N$ (35), (39), (40) and those for $N_s$ and $N_t$ (41), (42), (43) that there is no paradox if an independent physical reality is attributed to the 4D geometric quantities and if their LT are used. According to that in the ISR there is no need to introduce some “hidden” quantities. These “hidden” quantities are introduced in different ways in almost all papers in [4]. In the 4D spacetime, they are without well-defined physical meaning. Simply, they are an artifact of the use of the 3D quantities and their UT.
In the ISR, it is proved that there is a true agreement, independent of the chosen inertial reference frame and of the chosen system of coordinates in it, with different experiments, e.g., the motional electromotive force in [18], the Faraday disk in [19], the Trouton-Noble experiment in [12, 13] and also in [8, 9], the well-known experiments that test SR: the “muon” experiment, the Michelson-Morley-type experiments, the Kennedy-Thorndike-type experiments and the Ives-Stilwell-type experiments. This true agreement with experiments directly proves the physical reality of the 4D geometric quantities. It is also shown in the mentioned papers that the agreement between the experiments and Einstein’s formulation of SR [14] is not a true agreement since it depends on the chosen synchronization. Remember, as already stated several times, the conventional SR deals with the synchronously defined spatial length, i.e., the Lorentz contraction, see also Appendix in [2], then with the conventional dilation of time and also with the UT of the components of the 3-vectors $E$ and $B$. Particularly, this is explicitly shown in [9] in which both Einstein’s synchronization and the “$r$” synchronization are used throughout the paper. As can be seen from [6-9], contrary to the generally accepted opinion in the conventional SR, the relativity of simultaneity, the Lorentz contraction and the time dilation are not well-defined in the 4D spacetime. They are not the intrinsic relativistic effects since they depend on the chosen synchronization. However, as already stated, every synchronization is only a convention and physics must not depend on conventions.

In the following we shall examine several other ambiguities in the conventional treatments of SR and present how they are removed in this approach with the 4D geometric quantities.

5.1 Electromagnetic field equations for moving media

In all usual approaches, e.g., [10], [25-30], including [1, 3-5], it is supposed that Maxwell’s equations with the 3-vectors, both in the vacuum and in a moving medium, are covariant under the LT. However, for the vacuum, it is proved in [19] that it is not true, because the transformations of the 3-vectors $E$ and $B$ are not the LT but the relativistically incorrect UT and the Lorentz invariant field equations with $E$ and $B$ are presented, Eqs. (39) and (40) in that paper (Eqs. (27-29) in [2]). Moreover, in [12], an axiomatic geometric formulation of electromagnetism in vacuum is developed which has only one axiom, the field equation for $F$, Eq. (12) here. If it is written with the CBGQs in the standard basis it becomes Eq. (13). Its generalization to a magnetized and polarized moving medium with $M(x)$ is presented in [31]. It is

$$\partial (\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c; \quad \partial \cdot (\varepsilon_0 F + \mathcal{M}) = j^{(C)}/c, \quad \partial \wedge F = 0. \quad (44)$$

If written with the CBGQs in the standard basis that equation becomes

$$\partial_\alpha (\varepsilon_0 F^\alpha + \mathcal{M}^\alpha)_{\gamma\beta} - \partial_\alpha (\varepsilon_0^* F^{\alpha} + \mathcal{M}^* \gamma\beta\gamma\beta = c^{-1} j^{(C)\beta}\gamma\beta, \quad (45)$$
which can be separated into two equations, the equation with sources

\[ \partial_\alpha (\varepsilon_0 F^{\alpha\beta} + M^{\alpha\beta}) \gamma_\beta = c^{-1} j^{(C)\gamma_\beta} \]  

(46)

and the equation without sources, which is the same as in the vacuum

\[ \partial_\alpha * F^{\alpha\beta} \gamma_5 \gamma_\beta = 0. \]  

(47)

In [31], the equation (44) with \( F(x) \) and \( M(x) \) is also written in terms of vectors \( E, B \) and \( P, M \). If written with \( E, B, P \) and \( M \) as CBGQs in the standard basis the equation with sources (46) becomes

\[ \partial_\alpha \{ \varepsilon_0 [\delta^{\alpha\beta}_{\mu\nu} E^\mu v^\nu + \varepsilon^{\alpha\beta\mu\nu} v_\mu B_\nu] + [\delta^{\alpha\beta}_{\mu\nu} P^\mu u^\nu + (1/c) \varepsilon^{\alpha\beta\mu\nu} M_\mu u_\nu] \} \gamma_\beta = j^{(C)\beta \gamma_\beta}, \]

(48)

where \( \delta^{\alpha\beta}_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu \) and the equation without sources (47) becomes

\[ \partial_\alpha (c \varepsilon^{\alpha\beta\mu\nu} B^\mu v^\nu + \varepsilon^{\alpha\beta\mu\nu} E_\mu v_\nu) \gamma_5 \gamma_\beta = 0. \]  

(49)

In the ISR Eqs. (48) and (49) are fundamental equations for moving media and they replace all usual Maxwell’s equations (with 3-vectors) for moving media, thus Eqs. (1-4) in [1] as well. Observe, as pointed out in [31], that, in contrast to all conventional formulations of the field equations for moving media, Eq. (48) contains two different velocity vectors, \( v \) - the velocity of the observers and \( u \) - the velocity of the moving medium, which come from the decompositions of \( F \) and \( M \), Eqs. (15) and (20), respectively. Therefore, in the general case, i.e., for \( u \neq v \), it is not possible to introduce the electric and magnetic excitations \( D \) and \( H \), where

\[ D = \varepsilon_0 E + P \]  

and

\[ H = (1/\mu_0) B - M. \]

The mentioned introduction of \( D \) and \( H \) is possible if only one velocity, the velocity of the medium \( u \), is taken into account, or the case \( u = v \) is considered, or both decompositions (15) and (20) are made with the same velocity vector, either \( u \) or \( v \), but that last case has not a proper physical interpretation. This means that, in the general case \( u \neq v \), Eqs. (1-4) from [1] with 3-vectors \( D \) and \( H \) are not possible to derive in a mathematically correct way from Eqs. (48) and (49).

All this is discussed in detail in Sections 6 and 7 in [31]. There, Eqs. (48) and (49) with 4D geometric quantities \( E, B, P \) and \( M \) are compared with the usual Maxwell equations with the 3-vectors \( E, B, D, H \), which are the same as Eqs. (1-4) in [1]. It is shown, as for the vacuum in [19], that in the 4D spacetime Eqs. (48) and (49) are not equivalent to the usual Maxwell equations with the 3-vectors and their UT. The equations (48) and (49) hold for all relatively moving inertial observers and for all bases used by them, which is not the case for, e.g., Eqs. (1-4) in [1].

Recently, in the same formulations with the 4D geometric quantities, the constitutive relations and the magnetoelectric effect for moving media are investigated in detail in [32].

The axiomatic geometric formulation of electromagnetism is also presented in the modern textbook on classical electrodynamics [33], which uses the calculus of exterior forms. The formulation from [33] deals with the electromagnetic
The excitation tensor $\mathcal{H}$, which is decomposed into the electric $D$ and magnetic excitations $H$ and with field equations for them as the primary equations. As discussed above it is not correct in the general case, $u \neq v$. Furthermore, Hehl and Obukhov, [33], introduce six axioms (the charge conservation, the postulated expression for the Lorentz force law, ..) and from them the field equations for $\mathcal{H}$ and $F$ are derived. It can be seen that all axioms from [33] can be derived from only one axiom, the field equation for $F$. For vacuum, this is explicitly shown in [12]. As pointed out in [31], the generalization to a moving medium can be obtained simply replacing $F$ by $F + \mathcal{M}/\varepsilon_0$. Some other ambiguities and shortcomings in the formulation from [33] are discussed in detail in [32].

5.2 The Lorentz force and momentum conservation laws

It is also argued in [1, 5] that the usual Lorentz force is incompatible with momentum conservation laws and has to be replaced by the Einstein-Laub law; all with the 3-vectors. However, for the electromagnetic momentum that correctly transforms under the LT see Sections 4 and 5 - 5.3 in [34] (only components) and for the more general expressions with the 4D geometric quantities see Section 2.6 in [12]. There, in Eqs. (37) - (43) in [12], a basis-free expression for the most important quantity for the momentum and energy of the electromagnetic field, the stress-energy vector $T(n)$, then the expressions for the energy density $U$, the Poynting vector $S$ and the momentum density $\mathcal{M}$ and the Lorentz force $K_L$ are directly derived from the field equation for $F$ and they are written exclusively in terms of $F$. The notation is as in [12]. Furthermore, the local conservation laws are also directly derived from that field equation for $F$ and presented in Section 2.7 in [12], see Eqs. (48) - (51). As mentioned in the preceding section, the generalization of these relations to a moving medium is obtained replacing $F$ by $F + \mathcal{M}/\varepsilon_0$ and it is briefly discussed in Section 2 in [31]. The Lorentz force law is completely compatible with momentum conservation laws, but all quantities have to be the 4D geometric quantities.

5.3 “The relativistic version of Newton’s law”

In [1], and also in the well-known textbooks, e.g., [10], [25-30], it is also stated that the equation $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ is “the relativistic version of Newton’s law.” However, as shown in [11] and also in [20], the equation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad \mathbf{p} = m\gamma_u \mathbf{u}$$

is not the relativistic equation of motion since, contrary to the common assertions, it is not covariant under the LT. Any 3D quantity cannot correctly transform under the LT; it is not the same quantity for relatively moving observers in the 4D spacetime. Instead of the equation with the 3D quantities one
has to use the equation of motion with 4D geometric quantities, Eq. (10) in [11],
\[ K = dp/d\tau, \quad p = mu, \]  
(51)
where \( p \) is the proper momentum vector and \( \tau \) is the proper time, a Lorentz scalar. In the 4D spacetime \( p, \tau \) and \( K \) from (51) are the correctly defined quantities and not the 3D \( p, t \) and the 3-force \( F \). The Lorentz force \( K_L \) can be defined in terms of \( F \), or using the decomposition of \( F \) in terms of \( E \) and \( B \), as

\[ K_L = \left( \frac{q}{c} \right) \mathbf{F} \cdot \mathbf{u}, \quad K_L = \left( \frac{q}{c} \right) \left[ \frac{1}{c} \mathbf{E} \wedge \mathbf{v} + \mathbf{F} \cdot \mathbf{v} \right] \cdot \mathbf{u}. \]  
(52)

If written as a CBGQ in the standard basis \( K_L \) becomes

\[ K_L = \left( \frac{q}{c} \right) F^{\mu \nu} u_\nu \gamma_\mu, \quad K_L = \left( \frac{q}{c^2} \right) \left[ (\nu^\mu u_\nu) E^\mu + \varepsilon^{\lambda \mu \nu \rho} v_\lambda u_\nu c B_\rho - (E^\nu u_\nu) v^\mu \right] \gamma_\mu. \]  
(53)

Particularly, the Lorentz force ascribed by an observer comoving with a charge, \( u = v \), i.e., if the charge and the observer world lines coincide, then \( K_L \) is purely electric, \( K_L = qE \). In Section 6.1 in [12], under the title “The Lorentz force and the motion of charged particle in the electromagnetic field \( F \)” the definition of \( K_L \) in terms of \( F \) is exclusively used (\( K_L = \left( \frac{q}{c} \right) F \cdot u \)) without introducing the electric and magnetic fields. The quantities \( K \) (\( K_L \)), \( p \), \( u \) transform in the same way, like any other vector, i.e., according to the LT (for the components in the standard basis they are the same as the above mentioned LT of \( E^\mu \)) and not according to the awkward UT of the 3-force \( F \), e.g., Eqs. (12.66) and (12.67) in [25], and the 3-momentum \( p \), i.e., the 3-velocity \( u \).

5.4 The charge densities in an infinite wire with a steady current.

Is magnetism a relativistic phenomenon?

In [1], Mansuripur mentions and “resolves” yet another apparent conflict with relativity, i.e., another paradox, that refers to the force on a current-carrying wire. In order to show that in this case too there is no paradox we shall need to know the charge densities for a moving or stationary infinite wire with a steady current. They are considered, e.g., in the well-known textbooks [25-30]. In [25], in Section 12.3.1 under the title “Magnetism as a Relativistic Phenomenon” it is assumed, as in all other conventional treatments, that Clausius’ hypothesis holds, i.e., that such stationary wire with a steady current is globally and locally charge neutral; \( \rho' = \rho'_0 + \rho'_\perp = \rho_0 - \rho_0 = 0 \). \( \rho'_0 \) is the charge density of the stationary ions, which is the same as in that wire but without current, \( \rho'_0 \), whereas \( \rho'_\perp \) is the charge density of the moving electrons that is taken to be the same as in that charge neutral wire without current, \( -\rho_0 \). In \( S \), in which the wire is moving, in Section 12.3.1 in [25], it is argued: “Conclusion: As a result of unequal Lorentz contraction (my emphasis) of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.” The same conclusion is obtained in, e.g., [26-30]. The net charge density of the moving wire with steady current \( \rho \neq 0 \)
sets up an external electric field. The essential point is that in the conventional formulation of SR the charge density of the moving charges is considered to be a well-defined quantity. Simply, it is increased by the Lorentz factor \( \gamma \) because of the Lorentz contraction of the moving length or volume and because it is assumed that the charge defined by Eq. (54) is the Lorentz invariant charge.

In the above example, e.g., \( \rho_+ = \gamma \rho_0 \). Purcell, Section 5.9 in [26] and Griffiths, Section 12.3.1 in [25], calculate the external electric field for the moving current-carrying wire and they also determine the electrical force on the test charge \( q \) in the rest frame of that test charge; it is denoted as \( S \) in [25]. In that frame \( S \) the current-carrying wire is moving; the wire is charged and \( q \) is at rest. Then, they argue that if there is a force on \( q \) in one frame, the \( S \) frame, there must be the force in the rest frame of the wire; it is denoted as \( S \) in [25] and it is our \( S' \) frame. They calculate that non-electrical force in the rest frame of the wire, where the wire is supposed to be neutral, using the UT of the force 3-vector, Eqs. (12.65)-(12.67) in [25]. It is stated in [25]: “Taken together, then, electrostatics and relativity imply the existence of another force. This “other force” is, of course, magnetic.” It is written in the form of Eq. (12.85) in [25], which is the same as the usual expression for the Lorentz force exerted by the magnetic field of a long, straight wire on a moving charge \( q \). According to that result, the authors of [26] and [25], and also many other authors of textbooks and papers who used the consideration from [26], concluded that magnetism is a relativistic phenomenon. But, in the 4D spacetime, such a conclusion is not relativistically correct. As explained below, in the 4D spacetime, the Lorentz contraction has not well-defined physical meaning. Furthermore, in [26] and [25] as in all other usual treatments, the conventional definition of charge in terms of 3D quantities is used in which the values of the charge density \( \rho(\mathbf{r}, t) \) are taken simultaneously at some \( t \) for all \( \mathbf{r} \) in the 3D volume \( \mathcal{V}(t) \) over which \( \rho \) is integrated,

\[
Q = \int_{\mathcal{V}(t)} \rho(\mathbf{r}, t)d\mathcal{V}.
\]  

(54)

The same equation is supposed to be valid in some relatively moving inertial frame of reference with primed quantities replacing the unprimed ones, see Eqs. (50, 51) in Section 7.1 in [2] and references therein. But, observe that \( t' \) in \( S' \) is not connected in any way with \( t \) in \( S \). Contrary to the generally accepted opinion, the charge \( Q \) defined in such a way is not invariant under the LT. Instead of that usual definition with 3D quantities we deal with the 4D quantities. The total electric charge \( Q \) in a three-dimensional hypersurface \( \mathcal{H} \) (with two-dimensional boundary \( \delta \mathcal{H} \)) is defined as a Lorentz scalar by the equation

\[
Q_{\mathcal{H}} = (1/c) \int_{\mathcal{H}} \mathbf{j} \cdot d\mathcal{H},
\]  

(55)

where \( \mathbf{j} \) is the current density vector and the vector \( n \) is the unit normal to \( \mathcal{H} \).

Many years ago, it was shown by Rohrlich [35] and Gamba [36] that the Lorentz contraction has nothing to do with the LT. It is, according to Rohrlich [35], an “apparent” transformation (AT) that does not refer to the same physical
quantity in the 4D spacetime, whereas the transformations that refer to the same 4D physical quantity were called the “true transformations”. The LT are the “true transformations”. Two spatial lengths that are synchronously determined for the observers, the rest length and the Lorentz contracted length, are two different quantities in the 4D spacetime and accordingly they cannot be connected by the LT, see Section 4.1 and Fig. 3 in [7] and compare it with the spacetime length, Section 3.1 and Fig. 1 in [7], which is the same 4D quantity for all relatively moving inertial observers. The LT do not connect two spatial lengths taken alone, i.e., in the 4D spacetime, two relatively moving observers cannot compare spatial lengths taken alone. Rohrlich’s and Gamba’s ideas are properly mathematically formulated using 4D geometric quantities in [6-9]; for the proof of the relativistic incorrectness of the Lorentz contraction see also Appendix in [2]. In [7-9] it is proved that the time dilation is also an AT, which has nothing in common with the LT and that both the Lorentz contraction and the time dilation are not the intrinsic relativistic effects. Note that the UT and the UT of P and M, EDM p and MDM m are also the AT and, as shown in [11], the same holds for the transformations of other 3D quantities, like the AT of the 3D force, e.g., Eqs. (12.65)-(12.67) in [25], the AT of the 3D torque, the AT of the 3D angular momentum, e.g., in [16], etc.

In the 4D spacetime the properly defined 4D geometric quantities are the position vectors \( x_A, x_B \) of the events A and B, respectively, the distance vector \( l_{AB} = x_B - x_A \) between two events and the spacetime length \( l, l = \sqrt{g_{\mu\nu} l_{\mu AB,\nu}} \), which is, e.g., for a moving rod, \( l = L_0 \), where \( L_0 \) is the rest length. The spacetime length \( l \) is a Lorentz scalar, see, e.g., [7].

As already mentioned, the LT cannot transform the spatial (temporal) distance between two events again to the spatial (temporal) distance. Hence, in the 4D spacetime, the Lorentz contracted length is meaningless and only the rest length is a well-defined quantity. In [6], an apparent relativistic paradox is investigated both with the 4D geometric quantities and with the conventional SR. That paradox is connected with the Lorentz contraction and it often appears in different textbooks and papers under the different names, e.g., in [37], and the same in [6], it is called “Car and garage paradox,” whereas in [25] it is called “the barn and ladder paradox”, etc. It is shown in [6] that, in contrast to [37], [25], etc., i.e., to the conventional formulation of SR, i.e., Einstein’s formulation of SR, there is no paradox in the formulation of SR with 4D geometric quantities, i.e., in the ISR. In the Lorentz contraction the relatively moving observers make the same measurements (synchronously determine the spatial length), but they do not look at the same 4D quantity, i.e., at the same set of events. On the other hand, as already stated, the LT refer to the same 4D quantity, i.e., to the same set of events. It can be easily seen from Section 4.1 and Fig. 3 in [7], or from Appendix in [2], that in the Lorentz contraction the relatively moving observers do not look at the same set of events. This means that the Lorentz contraction has nothing to do with the LT, i.e., with the SR, which is the theory of the 4D spacetime with the pseudo-Euclidean geometry. Only the transformations that leave the pseudo-Euclidean geometry of the 4D spacetime unchanged are the relativistically correct transformations. The time dilation
and the Lorentz contraction are not such type of transformations, whereas the LT belong to that category of transformations.

According to this discussion it is clear that the assertion from [26] and [25] that magnetism is a relativistic phenomenon is meaningless in the 4D spacetime. That conclusion is obtained using the Lorentz contraction and the definition of charge in terms of 3D quantities [24], which are not well-defined in the 4D spacetime. Moreover, the axiomatic geometric formulation from [12] explicitly shows that the electromagnetic field \( F \) is the primary quantity, which means that the whole electromagnetism is a relativistic phenomenon.

The fact that only the rest length is properly defined entails that in the 4D spacetime it is not possible to give a definite physical meaning to the charge density of moving charges. In the 4D spacetime the usual charge density \( \rho \) and the usual current density \( j \) as a 3-vector are not well-defined physical quantities, but it is only the current density vector \( j \), or as a CBGQ, e.g., in the standard basis, \( j = j^\mu \gamma_\mu \). Hence, as shown in Sections 3 and 3.1 in [6], or in Section 7.1 in [2], in order to determine the current density vector \( j^\mu \) in some inertial frame of reference in which the charges are moving we first have to determine that vector in the rest frame of the charges, where the spatial components \( j^i \) are zero and only the temporal component \( j^0 \neq 0 \) and then to transform by the LT so determined \( j^\mu \gamma_\mu \) to the considered inertial frame of reference. This means that in order to determine the current density vector \( j \) in some inertial frame of reference for an infinite wire with a steady current we first have to determine the current density vectors \( j^\mu \gamma_\mu \) and \( j^\mu \gamma_\mu \) for positive and negative charges, respectively, in their rest frames. Then, they have to be transformed by the LT to the given inertial frame of reference. Thereby, in the rest frame of the wire, the \( S' \) frame, the positive charges are at rest and \( j^\mu \gamma_\mu \) as a CBGQ is

\[
j_+ = j^\mu \gamma_\mu = (c\rho_0')\gamma_0' + 0\gamma_1'. \tag{56}
\]

The negative charges are moving in \( S' \) and, according to the above discussion, we first have to write the current density vector of the electrons in the frame, let us denote it as \( S_e \), in which the spatial components of the vector \( j_- (j_- = j^\mu_\gamma_\mu) \); \( j^\mu_- \) are the components of \( j_- \) in the standard basis and in the \( S_e \) frame, whereas \( \gamma_\mu_\gamma_\mu \) are the unit vectors in \( S_e \) (zero, \( j^\mu_- = 0 \)). In the usual notation, in \( S_e \), the drift velocity 3-vector of the electrons is zero. Hence, the temporal component \( j^0_- \) in \( S_e \) is a well-defined quantity. Remember that the electrons, in average, are not moving in \( S_e \), which means that the situation for the electrons is the same as in that wire but without any current, i.e., \( j^0_\epsilon = c\rho_\epsilon = -c\rho_0' \); \( \rho_\epsilon \) is the same as the proper charge density of the electrons \( -\rho_0' \). Observe that such result is completely different than the Clausius hypothesis. The current density vector of the electrons \( j_- \) as a CBGQ in \( S_e \) is

\[
j_- = j^\mu_\epsilon_\gamma_\mu = (c\rho_0')\gamma_\epsilon_0 + 0\gamma_\epsilon_1. \tag{57}
\]

Then, by means of \( j^\mu_\epsilon_\gamma_\mu \) and the LT one finds the current density vector of the electrons in \( S' \), the rest frame of the wire, i.e., the lab frame, as

\[
j_- = j^\mu_\epsilon_\gamma_\mu = (-c\gamma_\epsilon\rho_0')\gamma_0' + (-c\gamma_\epsilon\beta_\epsilon\rho_0')\gamma_1'. \tag{58}
\]
where $\gamma_e = (1 - \beta_e^2)^{-1/2}$ and $\beta_e = v_d/c$, $v_d$ is the usual drift velocity of the electrons in that stationary wire with current. Similarly, $j_+$ as a CBGQ in $S_e$ can be determined using the LT of quantities from (56). This yields that $j_+$ in $S_e$ is

$$j_+ = j_{e,+}^\mu \gamma_{e,\mu} = (c \gamma_e \rho_0') \gamma_e,0 + (-c \gamma_e \beta_e \rho_0') \gamma_e,1$$

(59)

It can be seen from (56) and (59) that $j_+ = j_{e,+}^\mu \gamma_{e,\mu} = j_{e,+}^\mu + \gamma_{e,\mu}$ and the same for $j_-$ using (57) and (58). The total current density vector in $S'$ is $j = j'^\mu \gamma_{\mu}$, where the components in the standard basis are $j'^\mu = j_{e,+}^\mu + j_{e,-}^\mu$. As we know, $j_{e,+}^\mu = (c \rho_0', 0)$ and $j_{e,-}^\mu$ are given by (58), which yields that $j$ as a CBGQ in $S'$, the rest frame of the wire, is

$$j = j'^\mu \gamma_{\mu} = c(1 - \gamma_e) \rho_0' \gamma_0' + (-c \gamma_e \beta_e \rho_0') \gamma_1$$

(60)

the temporal component $j'^0$ is not zero. Observe that it will again hold that $j = j'^\mu \gamma_{\mu} = j_{e,+}^\mu \gamma_{e,\mu}$, where the quantities in $S'$ and in $S_e$ are connected by the LT. Then, the result (60) causes that, in contrast to the usual approaches, there is an external electric field not only outside moving wire with a steady current but also outside that stationary wire. All this, together with the expression for the external electric field from a stationary wire with a steady current, is already discussed in a slightly different way in Sections 3 - 3.3 in [6] and 7.1 in [2].

An infinite wire is not a physical system and therefore the above results are generalized to a current loop in Sections 4 in [6] and discussed also in Section 7.1 in [2]. It is shown there that the external electric field exists not only for a moving current loop, as in the usual approaches, but for the stationary current loop as well. Such a current loop, moving or stationary, always behaves at points far from that current loop like an electric dipole, but as a 4D geometric quantity. In Section 7.2 in [2] different experiments for the detection of that dipole electric field from a stationary current loop are discussed.

5.5 Is there a conflict with relativity for the force on a current-carrying wire

Having determined the charge densities in a current-carrying wire we turn to the investigation of the force on such a current-carrying wire. Mansuripur [1] considers “a thin, straight, charge-neutral wire carrying a constant, uniform current density $J_{\text{free}}$ along $x'$ in the presence of a constant, uniform $E$ field (also along $x'$)” Remember that in [1] the $S'$ frame moves along the $z$ axis, as seen by a stationary observer in $S$. It is argued in [1] that in its rest frame (the $S'$ frame) the wire does not experience a Lorentz force, but if seen by the stationary observer in the $S$ frame it does experience a Lorentz force along the $z$ axis. Let us try to explain how this result is obtained. In $S'$ the Lorentz force density

$$f' = \rho' E' + J' \times B'$$

(61)

is zero because, for a stationary wire with a steady current it is assumed that $\rho' = 0$ (Clausius’ hypothesis, see, as mentioned above, Sections 3 and 3.1 in [6]
or 7.1 and 7.2 in [2] and references therein) and $\mathbf{B}' = 0$. In $S$, the components $j^\mu$ are $\rho = 0$, $j_x = j'_x$, $j_y = j_z = 0$. According to the UT (11) the components of the 3-vector $\mathbf{E}$ are $E_x = \gamma E'_x$, $E_y = E_z = 0$ and there is an induced component of the magnetic field 3-vector, i.e., $B_y = -\gamma \beta E'_x$, $B_x = B_z = 0$, which yields that there is $f_z = j_x B_y \neq 0$. It is clear that again the real cause of the existence of the paradox in that example is the use of the 3-vectors and their UT. How does Mansuripur “resolve” this paradox? He argues “… special relativity is not violated here because the energy delivered by the $E$ field at the rate of $\mathbf{E} \cdot \mathbf{J}_{\text{free}}$ to the current causes an increase in the mass of the wire. (my emphasis)” Seen by the observer in the $xyz$ frame, the wire has a nonzero (albeit constant) velocity along $z$, and, therefore, its relativistic momentum $\mathbf{p}$ increases with time, not because of a change of velocity but because of a change of mass. The observed electromagnetic force in the moving frame thus agrees with the relativistic version of Newton’s law.” From the point of view of the ISR such a “resolution” contains a wealth of relativistically incorrect quantities and explanations. Again, the “resolution” exclusively deals with the 3-vectors, $\mathbf{E}$, $\mathbf{J}$, $\mathbf{p}$ and their UT. Furthermore, in the 4D spacetime only the rest mass is well defined quantity and thus there is not “a change of mass.” Also, as explained above, $\mathbf{F} = \frac{\text{d}\mathbf{p}}{\text{d}t}$ is not the relativistic equation of motion.

Barnett [4], in his Comment on [1], uses the similar argument as above but for the “resolution” of Mansuripur’s paradox. He declares: “In Mansuripur’s magnetic-dipole thought experiment there is no change in the velocity of the dipole because there is no net force acting on it, but there is a change in the moment of inertia and this balances exactly the torque derived from (1).” (my emphasis) The equation (1) in Barnett’s paper is the usual expression for the Lorentz force density, $\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$. The torque obtained in Barnett’s paper is the 3D torque and it arises, as in all usual approaches including all papers [1, 3-5], from the use of the UT of the 3D quantities according to which, Barnett [4]: “The moving magnetic dipole, moreover, acquires some electric dipole character by virtue of its motion.” In order to cancel this offending torque he argues that the time component of the force as a four-vector “the $\mathbf{J} \cdot \mathbf{E}$ term produces a change to the moment of inertia of the dipole, …” All objections that we have presented above hold in the same measure for Barnett’s “resolution” of Mansuripur’s paradox. In the 4D spacetime there are no 3D quantities, the components (even in the standard basis) of some 4-vector, i.e., vector on the 4D spacetime, cannot be written in terms of the 3-vectors, the UT are not the LT, there is no change to the moment of inertia, etc. Barnett, as in almost all usual covariant approaches, e.g., [10, 25 - 30], considers that components implicitly taken in the standard basis are, e.g., four-vector, or, more generally tensors. Components are numbers depending on the chosen basis and mathematically they are not tensors. It has to be pointed out once again that in the 4D spacetime there are no 3D force $\mathbf{F}$, 3D magnetization $\mathbf{M}$, 3D torque $\mathbf{T}$, 3D acceleration $\mathbf{a}$, etc. The LT cannot transform the 3D quantities. They are transformations of the 4D geometric quantities that are properly defined on the 4D spacetime.

Now, let us examine the force on a current-carrying wire for the case con-
sidered in [1] if it is treated with 4D geometric quantities. The Lorentz force density \( k_L \) as a correctly defined abstract vector is \( k_L = (1/c)F \cdot j \), or as a CBGQ it is given by (33). Inserting the decomposition of \( F \) (15) into (33) \( k_L \) becomes

\[
k_L = (1/c^2)[(v'^\mu j_\nu)E^{\mu} + \varepsilon^{\lambda\mu\rho\nu}v_{\lambda j_\rho}cB_\rho - (E'^\nu j_\nu)v^\mu]\gamma_\mu. \tag{62}
\]

That expression is correct in all bases and all quantities in (62) properly transform under the LT. \( k_L \) from (62) replaces the usual expression with the 3D quantities, like, e.g., (61). In \( S' \), the rest frame of the wire, which is taken to be the \( \gamma_0 \)-frame, \( v'^\mu = (c, 0, 0, 0) \) and \( E'^\mu = (0, E'^{1}, 0, 0) \) and the components of current density vector are \( j'^\mu = (j'^0, j'^1, 0, 0) \), where \( j'^0 \) and \( j'^1 \) are given by (60). \( j'^0 = c(1 - \gamma_\epsilon)\rho'_0 \), \( j'^1 = -c\gamma_\epsilon\beta\gamma c\rho'_0 \). The magnetic field vector is zero, \( B = B'^\mu j'_\mu = 0 \). There remains zero in all relatively moving inertial frames of reference. Inserting these components into (62) it is obtained that

\[
k_L = k_L^0 \gamma'_\mu = (1/c^2)[-cE'^{1}j'_1\gamma'_0 + cE'^{1}j'_0\gamma'_1 + 0\gamma'_2 + 0\gamma'_3]. \tag{63}
\]

In \( S' \), the Lorentz force density is not zero; there are both the temporal component and a spatial component. This result is essentially different than in [1]. All quantities in (63) are properly defined quantities on the 4D spacetime, which correctly transform under the LT. \( k_L \) given by (62), as a CBGQ, is an invariant quantity, i.e., it will be the same as in (63) for all relatively moving inertial frames of reference, thus for the \( S' \) frame from [1] as well, \( k_L^0 \gamma'_\mu = k_L^0 \gamma_\mu \).

Let us explicitly determine \( k_L \) in the \( S' \) frame from [1]. This can be made in different ways, e.g., by the LT of all quantities in (63) from \( S' \) to \( S \), or by the LT of the Lorentz force density vector, i.e., of \( k_L^0 \gamma'_\mu \) to \( \gamma'_\mu \) and \( \gamma_\mu \) to \( \gamma_\mu \), or to simply introduce into (62) the quantities from the \( S' \) frame. We shall use the third possibility. In \( S \), the “fiducial” observers are moving and \( v'^\mu = (\gamma c, 0, 0, \beta \gamma c) \). In the same way we find that \( E'^\mu = (0, E^1 = E'^1, 0, 0) \) and \( j'^\mu = (j'^0 = \gamma j'^0, j'^1 = j'^1, 0, j'^3 = \beta \gamma j'^0) \), and of course, \( B'^\mu = (0, 0, 0, 0) \).

Note that the electric field vector transforms by the LT, as in (10), again to the electric field vector and the same for the magnetic field vector. This yields that

\[
k_L = k_L^0 \gamma_\mu = (1/c^2)[-c\gamma E^1j_1\gamma_0 + E^1c\gamma(j_0 + \beta j_3)\gamma_1 + 0\gamma_2 - c\beta \gamma E^1j_1\gamma_3], \tag{64}
\]

where \( \gamma = (1 - \beta^2)^{-1/2} \) and \( \beta = V/c \). It can be easily shown that \( k_L^0 \gamma_\mu \) from (64) is \( k_L^0 \gamma'_\mu \) from (63); there is no paradox and there is no need for “a change of mass.”

Let us suppose for a moment that \( j'^0 = 0 \), as in the usual approaches. Then, from (64) it follows that \( k_L = k_L^0 \gamma_\mu = (-1/c)(E^1 j_1)\gamma_0' \). It corresponds to \( J \cdot E \) term from Barnett’s paper [4], but, remember, that we deal with correctly defined quantities in the 4D spacetime and with their LT and not with the 3D quantities and their UT, i.e., the AT. Hence, in \( S \), the LT of \( k_L \gamma_\mu \) give that \( k_L = k_L^0 \gamma_\mu = (-1/c)[\gamma E^1j_1\gamma_0 + \beta \gamma E^1j_1\gamma_3] \) and again it holds \( k_L = k_L^0 \gamma_\mu = k_L^0 \gamma'_\mu \); it is the same quantity for relatively moving inertial observers and there is no paradox. It is visible that \( k_L \) again contains the temporal component as
in $S'$, but also a spatial component, which is in the $\gamma_3$ direction. Observe that the magnetic field vector is again zero both in $S'$ and $S$ as in (65) and (66), $B = B^\mu \gamma_\mu' = B^\mu \gamma_\mu = 0$, but, nevertheless, there is $\gamma_3$ component.

### 5.6 The electromagnetic field of a point charge in uniform motion

#### 5.6.1 The bivector $F$ for an uniformly moving charge

Furthermore, as already stated, in this geometric approach to SR, i.e., in the ISR, the electromagnetic field $F$ yields the complete description of the electromagnetic field and, in principle, there is no need to introduce either the field vectors $E$ and $B$ or the 4D potential $A$. For the given sources Eq. (12), or Eq. (13), can be solved to give the electromagnetic field $F$. The expression for $F$ for an arbitrary motion of a point charge is given in [12] by Eq. (11) and, particularly, by Eq. (12) for a charge $Q$ moving with constant velocity vector $\mathbf{u}$. It is

$$F(x) = G(x \wedge (u_Q/c)), \quad G = kQ/\sqrt{|x \wedge (u_Q/c)|^3},$$  \hspace{1cm} (65)

where $k = 1/4\pi\varepsilon_0$. $G$ is a number, a Lorentz scalar, whereas the geometric character of $F$ is contained in $x \wedge (u_Q/c)$. $F$ from (65) can be written as a CBGQ in the standard basis, $F = (1/2)F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$.

$$F = G(1/c)x^\mu u_Q^\nu(\gamma_\mu \wedge \gamma_\nu), \quad G = kQ[(x^\mu u_Q^\nu)^2 - c^2 x^\mu x_\mu]^{3/2}. \hspace{1cm} (66)$$

The basis components $F^{\mu\nu}$ are determined as $F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F$, $F^{\mu\nu} = G(1/c)(x^\mu u_Q^\nu - x^\nu u_Q^\mu)$. \hspace{1cm} (67)

In Section 4, for simplicity, we have dealt with a uniform electric field. However, as already discussed at the beginning of Section 4, in the ISR it would be more appropriate to exclusively deal with the primary quantity $F$, i.e., $F^{\mu\nu}$.

Let us write the expression for $F$ (66) in the $S'$ frame in which the charge $Q$ is at rest, i.e., $u_Q/c = \gamma_0$ with $\gamma_0^\mu = (1, 0, 0, 0)$. Then, $F = (1/2)F^{\mu\nu}\gamma_\mu' \wedge \gamma_\nu'$ and

$$F = F^{i0}(\gamma_i' \wedge \gamma_0) = Gx^i(\gamma'_i \wedge \gamma_0), \quad G = kQ/(x^i x_i')^{3/2}. \hspace{1cm} (68)$$

In $S'$ and in the standard basis, the basis components $F^{\mu\nu}$ of the bivector $F$ are obtained from (67) and they are:

$$F^{i0} = -F^{0i} = kQx^i/(x^i x_i')^{3/2}, \quad F^{ij} = 0. \hspace{1cm} (69)$$

In the same way we find the expression for $F$ (66) in the $S$ frame in which the charge $Q$ is moving, i.e., $u_Q = u_Q^\mu \gamma_\mu$ with $u_Q^\mu/c = (\gamma_Q, \gamma_Q\beta_Q, 0, 0)$. Then

$$F = G\gamma_Q[(x^1 - \beta_Q x^0)(\gamma_1 \wedge \gamma_0) + x^2 (\gamma_2 \wedge \gamma_0) + x^3 (\gamma_3 \wedge \gamma_0) - \beta_Q x^2 (\gamma_1 \wedge \gamma_2) - \beta_Q x^3 (\gamma_1 \wedge \gamma_3)] \hspace{1cm} (70)$$

$$G = kQ/|\gamma_Q|^2 (x^1 - \beta_Q x^0)^2 + (x^2)^2 + (x^3)^2)^{3/2}. \hspace{1cm} (70)$$
In $S$ and in the standard basis, the basis components $F^{\mu\nu}$ of the bivector $F$ are again obtained from (67) and they are

$$
F^{10} = G\gamma_Q(x^1 - \beta Q x^0), \quad F^{20} = G\gamma_Q x^2, \quad F^{30} = G\gamma_Q x^3, \\
F^{21} = G\gamma_Q\beta Q x^2, \quad F^{31} = G\gamma_Q \beta Q x^3, \quad F^{32} = 0.
$$

The expression for $F$ as a CBGQ in the $S$ frame can be found in another way as well, i.e., to make the LT of the quantities from Eq. (68). Observe that the CBGQs from (68) and (70), which are the representations of the bivector $F$ in $S'$ and $S$ respectively, are equal, $F$ from (68) = $F$ from (70); they are the same quantity $F$ from (65) for observers in $S'$ and $S$.

In Section 2 it is explained that in the usual approaches to the relativistic electrodynamics the components of the 3-vectors $E$ and $B$ in $S'$ and $S$ are identified with the components $F^{\alpha\beta}$ implicitly taken in the standard basis and that the UT of the components of $E$ and $B$ are derived assuming that they transform under the LT as the components $F^{\alpha\beta}$ transform. Then, the 3-vectors $E'$, $B'$ in $S'$ and $E$, $B$ in $S$ are constructed in the same way, i.e., multiplying the components $E_{x,y,z}$, $B_{x,y,z}$ by the unit 3-vectors $i$, $j$, $k$ and the components $E_{x,y,z}$, $B_{x,y,z}$ by the unit 3-vectors $i'$, $j'$, $k'$ respectively. Such a procedure yields the UT of the 3-vectors $E$ and $B$, Eq. (11). The mathematical incorrectness of that procedure can be nicely illustrated comparing Eqs. (68) and (69) with Eq. (11) from [1] and Eqs. (70) and (71) with Eqs. (12a) and (12b) in [1].

The equations (68) and (71) reveal that the physical quantities are not only the components (69) and (71), respectively, but these components multiplied by the bivector bases in $S'$ and $S$. Only if components and bases are taken together like in (68) and (70), these CBGQs represent the same quantity, $F$ from (65). In the 4D spacetime, all quantities in (68) and (70) are correctly defined and they properly transform under the LT; the CBGQ from (70) is the LT of the CBGQ from (68).

The situation is completely different in Eqs. (11), (12a) and (12b) in [1]. The components of the 3-vector $E'$ in Eq. (11) in [1] are the same as the components in (69), but they are multiplied by the unit 3-vectors $i'$, $j'$, $k'$: $E'$ is a geometric quantity in the 3D space. Mathematically, this is an incorrect step; the components of the 4D geometric quantity are multiplied by the unit vectors from the 3D space to form a 3D vector. Similarly, the components of the 3-vectors $E$ and $B$ in Eqs. (12a) and (12b) in [1] are the same as the components in (71), but the same remarks about the mathematical incorrectness of the construction of $E$ and $B$ hold in this case as well. It is stated in [1]: “When the above E field (E’ in Eq. (11), my remark) is Lorentz transformed to the xyz frame, the resulting fields will be (E and H in Eqs. (12a) and (12b), my remark).” The LT always act on the 4D spacetime and consequently they cannot transform the unit 3-vectors $i'$, $j'$, $k'$ into the unit 3-vectors $i$, $j$, $k$. Moreover, there is not any kind of transformations which transform the 3-vectors from one 4D frame to the 3-vectors from relatively moving 4D frame. Furthermore, as explained at the end of Section 2, if instead of the standard basis the observers
use the \{r_{\mu}\} basis with the “radio” synchronization then the space-time split of the 4D spacetime is not possible and the identification of the components of \(E\) and \(B\) with the components \(F^{\alpha\beta}\) is meaningless. This consideration, once again, explicitly shows that in the 4D spacetime there is no room for the 3-vectors and accordingly that the transformations that transform the \(E'\) field given by Eq. (11) into \(E\) and \(H\) fields given by Eqs. (12a) and (12b) in [1] have nothing in common with the relativistically correct LT as the transformations that are defined on the 4D spacetime.

In the 4D spacetime it is appropriate to deal with the abstract \(F\) defined by (65), or with its representations, the CBGQs, defined by (68) and (70), but not with \(E'\) and \(E, H\) that are defined by Eq. (11) and by Eqs. (12a), (12b) in [1], respectively.

5.6.2 The vectors \(E\) and \(B\) for a charge \(Q\) moving with constant velocity \(u_Q\)

Instead of to deal exclusively with \(F\) we can construct in a mathematically correct way vectors \(E\) and \(B\) for a charge \(Q\) moving with constant velocity \(u_Q\). The vectors \(E\) and \(B\) are explicitly observer dependent, i.e., dependent on \(v\).

For the same \(F\) the vectors \(E\) and \(B\) will have different expressions depending on the velocity of observers who measure them. Using (16) and \(F\) from (65) we find the expressions for \(E\) and \(B\) in the form

\[
\begin{align*}
E &= (G/c^2)[(u_Q \cdot v)x - (x \cdot v)u_Q], \\
B &= (-G/c^3)I(x \wedge u_Q \wedge v).
\end{align*}
\]

(72)

It is worth mentioning that \(E\) and \(B\) from (72) depend on two velocity vectors \(u_Q\) and \(v\), whereas the 3-vectors \(E\) and \(B\) depend only on the 3-velocity of the charge \(Q\). If the world lines of the observer and the charge \(Q\) coincide, \(u_Q = v\), then (72) yields that \(B = 0\) and only an electric field (Coulomb field) remains.

It can be seen that if \(E\) and \(B\) from (72) are introduced into \(F\) from (14) then they will yield \(F\) defined by (66), which contains only \(u_Q\), the velocity of the charge \(Q\) and not the velocity of the observer \(v\). This result directly proves that the electromagnetic field \(F\) is the primary quantity from which the observer dependent \(E\) and \(B\) are derived.

If the CBGQs are used then the expressions for \(E\) and \(B\), Eq. (17), and that one for \(F\) (68) yield \(E\) and \(B\), Eq. (72), written as CBGQs in the standard basis

\[
\begin{align*}
E &= E^\mu\gamma_\mu = (G/c^2)[(u_Q^\nu v_\nu)x^\mu - (x^\nu v_\nu)u_Q^\mu]\gamma_\mu, \\
B &= B^\mu\gamma_\mu = (G/c^3)\varepsilon^{\mu\nu\alpha\beta}x_\nu u_\alpha v_\beta\gamma_\mu.
\end{align*}
\]

(73)

If \(E\) and \(B\) from (73) are introduced into \(F\) from (15) then they will yield \(F\) as the CBGQ that is defined by (68). Again, although \(E\) and \(B\) as the CBGQs from (73) depend not only on \(u_Q\) but on \(v\) as well the electromagnetic field \(F\) from (68) does not contain the velocity of the observer \(v\).
5.6.2.1 The “fiducial” observers are in $S'$, $v = c\gamma_0'$, which is the rest frame of the charge $Q$

Let us take that the observers who measure $E$, $B$ fields are at rest, $v = c\gamma_0'$, in the rest frame of the charge $Q$, $u_Q = v = c\gamma_0'$. This means that the $S'$ frame is the $\gamma_0$-frame; the “fiducial” observers with the $\{\gamma_\mu\}$ basis. It follows from (73) that

$$E = E^\mu \gamma_\mu' = G x^\mu \gamma_1', \quad E^0 = 0, \quad G = kQ/(x^i x^j)^{3/2}; \quad B = B^\mu \gamma_\mu' = 0. \quad (74)$$

This result agrees with the usual result, e.g., Eq. (11) in [1]. Now comes the essential difference relative to all usual approaches. In order to find the representations of $E$ and $B$ in $S$, i.e., the CBGQs $E^\mu \gamma_\mu$ and $B^\mu \gamma_\mu$, we can either perform the LT of $E^\mu \gamma_\mu'$ and $B^\mu \gamma_\mu'$ that are given by (74), or simply to take in (73) that both the charge $Q$ and the “fiducial” observers are moving relative to the observers in $S$; $\nu^\mu = u^\mu_Q = (\gamma_Q c, \beta_Q \gamma_Q c, 0, 0)$. This yields the CBGQs $E^\mu \gamma_\mu$ and $B^\mu \gamma_\mu$, in $S$ with the condition that the “fiducial” observers are in $S'$, $v = c\gamma_0'$, which is the rest frame of the charge $Q$, $u_Q = c\gamma_0'$.

$$E = E^\mu \gamma_\mu = G[\beta_Q \gamma_Q x^1 (1 - \beta_Q x^0) \gamma_1 + \gamma_2^2 (1 - \beta_Q x^0) \gamma_1 + x^2 \gamma_2 + x^3 \gamma_3], \quad B = B^\mu \gamma_\mu = 0, \quad (75)$$

where $G$ is that one from (70). The result (75) significantly differs from the result obtained by the UT, Eqs. (12a), (12b) in [1]. Under the LT the electric field vector transforms again to the electric field vector and the same for the magnetic field vector. It is worth mentioning that, in contrast to the conventional results, it holds that $E^\mu \gamma_\mu'$ from (74) is $= E^\mu \gamma_\mu$ from (73); they are the same quantity $E$ for all relatively moving inertial observers. The same holds for $B$, $B^\mu \gamma_\mu'$ from (74) is $= B^\mu \gamma_\mu$ from (73) and they are $= 0$ for all observers. Furthermore, observe that in $S'$ there are only the spatial components $E^i$, whereas in $S$ there is also the temporal component $E^0$ as the consequence of the LT.

5.6.2.2 The “fiducial” observers are in $S$, $v = c\gamma_0$, in which the charge $Q$ is moving

Now, let us take that the “fiducial” observers are in $S$, $v = c\gamma_0$, in which the charge $Q$ is moving, $u^\mu_Q = (\gamma_Q c, \beta_Q \gamma_Q c, 0, 0)$. In contrast to the previous case, both $E$ and $B$ are different from zero. The expressions for the CBGQs $E^\mu \gamma_\mu$ and $B^\mu \gamma_\mu$ in $S$ can be simply obtained from (73) taking in it that $v = c\gamma_0$ and $\nu^\mu_Q = \gamma_Q c \gamma_0 + \beta_Q \gamma_Q c \gamma_1$. This yields that $E^0 = B^0 = 0$ (from $v = c\gamma_0$) and the spatial parts are

$$E = E^i \gamma_i = G \gamma_Q [(x^1 - \beta_Q x^0) \gamma_1 + x^2 \gamma_2 + x^3 \gamma_3],$$

$$B = B^i \gamma_i = (G/c)[0 \gamma_1 - \beta_Q \gamma_Q x^3 \gamma_2 + \beta_Q \gamma_Q x^2 \gamma_3], \quad (76)$$

where $G$ is again as in (70). The 4D vector fields $E$ and $B$ from (76) can be compared with the usual expressions for the 3D fields $\mathbf{E}$ and $\mathbf{B}$ of an uniformly
moving charge, e.g., from Eqs. (12a), (12b) in [1]. It is visible that they are similar, but $E$ and $B$ in (65) are the 4D fields and all quantities in (65) are correctly defined in the 4D spacetime, which transform by the LT, whereas the fields in Eqs. (12a), (12b) in [1] are the 3D fields that transform according to the UT.

In order to find the representations of $E$ and $B$ in $S'$, i.e., the CBGQs $E^\mu_\gamma$, and $B^\mu_\gamma$, we can either perform the LT of $E^\mu_\gamma_\mu$ and $B^\mu_\gamma_\mu$ that are given by (76), or simply to take in (75) that relative to $S'$ the “fiducial” observers are moving with $v = v^\mu_\gamma_\mu$, $v^\mu = (c\gamma_\mu Q, -\beta Q\gamma_Q c, 0, 0)$, and the charge $Q$ is at rest relative to the observers in $S'$, $u^{\mu}_Q = (c, 0, 0, 0)$. This yields the CBGQs $E^\mu_\gamma_\mu$ and $B^\mu_\gamma_\mu$ in $S'$ with the condition that the “fiducial” observers are in $S$, $v = c\gamma_0$,

$$E = E^\mu_\gamma_\mu = G_\gamma Q[-\beta Q x^1_0^\gamma_0^\gamma_0 + x^2_0^\gamma_1^\gamma_1 + x^3_0^\gamma_2^\gamma_2 + x^3_3^\gamma_3^\gamma_3],$$

$$B = B^\mu_\gamma_\mu = (G/c)(0^\gamma_0_0 + 0^\gamma_1_1 - \beta Q\gamma_Q x^3_2^\gamma_2^\gamma_2 + \beta Q\gamma_Q x^3_2^\gamma_2^\gamma_2),$$

(77)

where $G$ is again as in (74). Again, as in the case that $v = c\gamma_0$, it holds that $E^\mu_\gamma_\mu$ from (76) is $= E^\mu_\gamma_\mu$ from (77); they are the same quantity $E$ for all relatively moving inertial observers. The same holds for $B^\mu_\gamma_\mu$ from (76) which is $= B^\mu_\gamma_\mu$ from (77) and they are both different from zero. Note that in this case there are only the spatial components $E^i$ in $S$, whereas in $S'$ there is also the temporal component $E^0$ as the consequence of the LT. It is visible from (77) that if the $\gamma_0$-frame is the lab frame ($v = c\gamma_0$) in which the charge $Q$ is moving then $E^\mu_\gamma_\mu$ and $B^\mu_\gamma_\mu$ in the rest frame of the charge $Q$, the $S'$ frame, are completely different than those from (74); in (77) $B^\mu_\gamma_\mu$ is different from zero and the representation of $E$ contains also the term $E^0_\gamma_0$.

It has to be emphasized that all four expressions for $E$ and $B$, (74), (75), (76) and (77), are the special cases of $E$ and $B$ given by (73), i.e., they are different representations (CBGQs) of $E$ and $B$ from (72). They all give the same $F$ from (66), which is the representation (CBGQ) of $F$ given by the basis free, abstract, bivector (65).

6. Conclusions

The whole consideration shows that in the ISR, i.e., in the approach with 4D geometric quantities, the principle of relativity is naturally satisfied and there is no trouble with any quantity and no paradox, i.e., that the ISR is perfectly suited to the symmetry of the 4D spacetime, which is not the case with the conventional SR, e.g., [10, 25 - 30], [1, 3-5], that deals with the 3D quantities and their UT or, as in the usual covariant approaches, with components implicitly taken in the standard basis.

In the 4D spacetime, as seen from the treatment of Mansuripur’s paradox that is presented here and from the similar treatments of Jackson’s paradox [11] and the Trouton-Noble paradox [12, 13], the physical angular momentum is not the 3-vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and the physical torque is not $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ with $\mathbf{T} = d\mathbf{L}/dt$, but the physical quantities are the 4D geometric quantities, $J$, i.e., $J_s$ and $J_t$.
taken together, which are given by Eqs. (24) - (26), then $N_s$, i.e., $N_s$ and $N_t$ taken together, which are given by Eqs. (27) - (29). The relation $T = dL/dt$ describes the usual 3D rotation, but in the 4D spacetime it is without well-defined physical sense and it is replaced with mathematically and relativistically correct relation (30), $N = dJ/d\tau$. According to the UT (4) the components of the 3D torque $T$, $T_i$, i.e., the “space-space” components of $N_{\mu\nu}$ in one frame are expressed by the mixture of the components of $T'$, $T'_i$, and of another 3D quantity $R'$, $R'_i$, i.e., the “time-space” components of $N'_{\mu\nu}$ from a relatively moving frame. Furthermore, if instead of Einstein’s synchronization one uses a nonstandard “radio” synchronization then, even in one frame, e.g., the “time-space” components of $N_{\mu\nu}$ in the $\{r_\mu\}$ basis (with the “radio” synchronization) are expressed by the mixture of the “time-space” components and the “space-space” components from the $\{\gamma_\mu\}$ basis, see similar equations for $F_{\mu\nu}$, Eqs. (5) and (7). This proves that the 3D quantities $T$ and $L$ and the usual 3D rotation are not physically well-defined in the 4D spacetime. Therefore, all treatments and all “resolutions” of Mansuripur’s paradox from [1, 3-5] are not relativistically correct in the 4D spacetime.

Regarding the measurements of the 4D quantities, it has to be pointed out that in the usual approaches with the 3D quantities, e.g., in the usual 3D rotation, the experimentalists measure only three components of the 3D torque $T$ and three components of the 3D angular momentum $L$ in both frames $S'$ and $S$. In the 4D spacetime, the experimentalists have to measure all six independent components of $N_{\mu\nu}$ (or, equivalently, three independent components of $N_{\mu}^s$ and three independent components of $N_{\mu}^t$), and also of $J_{\mu\nu}$ ($J_{\mu}^s$ and $J_{\mu}^t$), in both frames $S'$ and $S$. The observers in relatively moving inertial frames of reference, here in $S'$ and $S$, are able to compare only such complete set of data which corresponds to the same 4D geometric quantity. It is shown in Section 2.5 in [12] how $F$ can be experimentally determined using the definitions of the Lorentz force (with $F$) from Eqs. (52) and (53).

It is worth mentioning that different experiments for the detection of the electric field from a stationary current loop are discussed in Section 7.2 in [2]. Recently, the most promising experiments with cold ions are proposed in [38]. It is suggested in [2] that they could be also used for the detection of the electric field from a stationary permanent magnet.

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