A REMARK ON THE SCATTERING THEORY FOR THE 2D RADIAL FOCUSING INLS

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Abstract. We consider the scattering results of the radial solutions below the ground state to the focusing inhomogeneous nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u + |x|^{-b}|u|^p u = 0 \]

in two dimension, where \(0 < b < 1\) and \(2 - b < p < \infty\). We use a modified version of Arora-Dodson-Murphy’s approach [1] to give a new proof that extends the scattering results of [10] and avoids concentration compactness.

Key Words: Schrödinger equation; Scattering theory.

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1. Introduction

We consider the Cauchy problem of the inhomogeneous nonlinear Schrödinger equation (INLS)

\[
\begin{align*}
\partial_t u + \Delta u + |x|^{-b}|u|^p u &= 0, \\
0(t, x) &= u_0(x) \in H^1(\mathbb{R}^2),
\end{align*}
\]

with \(0 < b < 1\) and \(2 - b < p < \infty\). The equation (1.1) arises naturally in nonlinear optics for the propagation of laser beams, see [3] [4].

The equation (1.1) is \(H^s(\mathbb{R}^2)\)-critical in the sense that the \(H^s(\mathbb{R}^2)\) norm of initial data is invariant under the standard scaling

\[ u_\lambda(t, x) = \lambda^{\frac{2-b}{2}} u(\lambda^2 t, \lambda x), \]

where \(s_p = \frac{d}{2} - \frac{2-b}{p}\). The solutions to equation (1.1) conserve the mass, defined by

\[ M(u) := \int_{\mathbb{R}^2} |u|^2 dx = M(u_0), \]

and the energy, defined as the sum of the kinetic and potential energies:

\[ E(u) := \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 - \frac{1}{p+2}|x|^{-b}|u|^{p+2} dx = E(u_0). \]

From \(2 - b < p < \infty\), we have \(0 < s_p < 1\), which implies that the equation (1.1) is mass supercritical and energy subcritical.

Now, we recall the well-posedness theory of the equation (1.1). In [14], Genoud and Stuart proved that solution to the Cauchy problem of (1.1) is locally well-posed in \(H^1(\mathbb{R}^d)\) for \(0 < b < \min\{2, d\}\). More recently, Guzmán [11] established the local well-posedness of (1.1) based on Strichartz estimates. More precisely, for \(d \geq 4\) with \(0 < b < 2\) or \(d = 1, 2, 3\) with \(0 < b < \frac{d}{3}\), the Cauchy problem (1.1) is locally well-posed in \(H^1(\mathbb{R}^d)\). Dinh [13] extended Guzmán’s results to larger regions.
This equation admits a global nonscattering solution of the form \( u(t) = e^{it}Q \), where \( Q \) is the ground state solution to the elliptic equation
\[
-\Delta Q + Q - |x|^{-b}|Q|^p Q = 0.
\]
The existence of the ground state is proved in Genoud [11,12], Genoud and Stuart [14], while the uniqueness is handled in Yangida [19]. Genoud [13]. Moreover, the global well-posedness and scattering theory of solutions to (1.1) under the ground states \( Q \) have been studied in [2,9,10], which extended the results of [7,16] for nonlinear Schrödinger equation with \(|u|^pu\) type nonlinearities.

In this paper, we consider the mass-supercritical case in two spatial dimension. We will give another proof of the following scattering results.

**Theorem 1.1.** Let \( 0 < b < 1 \) and \( 2 - b < p < \infty \). Suppose \( u_0 \) is radial and such that
\[
M(u_0)^{1-s_p}E(u_0)^{s_p} < M(Q)^{1-s_p}E(Q)^{s_p},
\]
and
\[
\|u_0\|_{L^2}^{1-s_p}\|\nabla u_0\|_{L^2}^{s_p} < \|Q\|_{L^2}^{1-s_p}\|\nabla Q\|_{L^2}^{s_p}.
\]
Then the solution \( u \) to equation (1.1) with initial data \( u_0 \) is globally well-posed and scatters, that is, there exist \( u_{\pm} \in H^1(\mathbb{R}^2) \), such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta}u_{\pm}\|_{H^1(\mathbb{R}^2)} = 0.
\]

By the concentration compactness/rigidity method, Farah-Guzmán [10] proved Theorem 1.1 for the case \( 0 < b < \frac{2}{3} \). However, the well-posedness theory of (1.1) for \( 0 < b < 1 \) has been proved by [5]. Then, inspired by the new approach of [1,6], we present another proof that avoids the concentration compactness. Our proof is based on a Virial/Morawetz estimate, the radial Sobolev embedding and a scattering criterion that we will establish.

The rest of this paper is organized as follows: In section 2, we set up some notations and recall some basic properties for the equation (1.1). We will prove a new scattering criterion for (1.1) in section 3. In section 4, via the Morawetz identity, we will establish the Virial/Morawetz estimates and then show that the solution satisfies the scattering criterion of Lemma 3.1, thereby completing the proof of Theorem 1.1.

We conclude the introduction by giving some notations which will be used throughout this paper. We always use \( X \lesssim Y \) to denote \( X \leq CY \) for some constant \( C > 0 \). Similarly, \( X \lesssim_n Y \) indicates there exists a constant \( C := C(u) \) depending on \( u \) such that \( X \leq C(u)Y \). We also use the big-oh notation \( O \), e.g. \( A = O(B) \) indicates \( C_1B \leq A \leq C_2B \) for some constants \( C_1,C_2 > 0 \). The derivative operator \( \nabla \) refers to the spatial variable only. We use \( L^r(\mathbb{R}^2) \) to denote the Banach space of functions \( f : \mathbb{R}^2 \to \mathbb{C} \) whose norm
\[
\|f\|_r := \|f\|_{L^r} = \left( \int_{\mathbb{R}^2} |f(x)|^r dx \right)^{\frac{1}{r}}
\]
is finite, with the usual modifications when \( r = \infty \). For any non-negative integer \( k \), we denote by \( H^{k,r}(\mathbb{R}^2) \) the Sobolev space defined as the closure of smooth compactly supported functions in the norm \( \|f\|_{H^{k,r}} = \sum_{|\alpha| \leq k} \|\partial_x^{\alpha}f\|_{L^r} \), and we denote it by \( H^k \) when \( r = 2 \). For a time slab \( I \), we use \( L^q_t(I;L^r_x(\mathbb{R}^2)) \) to denote the space-time norm
\[
\|f\|_{L^q_tL^r_x(I \times \mathbb{R}^2)} = \left( \int_I \|f(t,x)\|^q_{L^r_x} dt \right)^{\frac{1}{q}}
\]
with the usual modifications when $q$ or $r$ is infinite, sometimes we use $\|f\|_{L^q(I;L^r)}$ or $\|f\|_{L^q(I;L^r)}$ for short.

2. Preliminaries

We start this section by introducing some notations used throughout the paper. We say the pair $(q,r)$ is $L^2$-admissible or simply admissible pair if it satisfies the condition $\frac{2}{q} = \frac{d}{2} - \frac{d}{r}$ and

$$
\begin{align*}
2 \leq r &\leq \frac{2d}{d-2}, \quad d \geq 3; \\
2 \leq r &< \infty, \quad d = 2; \\
2 \leq r &\leq \infty, \quad d = 1.
\end{align*}
$$

(2.2)

For $s > 0$, we also say the pair $(q,r)$ is $\dot{H}^s$-admissible if $\frac{2}{q} = \frac{d}{2} - \frac{d}{r} - s$ and

$$
\begin{align*}
\frac{2d}{d-2s} &\leq r \leq \left(\frac{2d}{d-2s}\right)^{-}, \quad d \geq 3; \\
\frac{2}{1-s} &\leq r \leq \left(\frac{2}{1-s}\right)^{+}, \quad d = 2; \\
\frac{2}{d-2s} &\leq r \leq \infty, \quad d = 1.
\end{align*}
$$

(2.3)

Here, $a^-$ is a fixed number and slightly smaller than $a$ (i.e., $a^- = a - \epsilon$, where $\epsilon$ is small enough) and, we define $a^+$ in a similar way. Moreover, we denote $(a^+)'$ is the number such that

$$
\frac{1}{a} = \frac{1}{(a^+)} + \frac{1}{a^+},
$$

that is, $(a^+)' = \frac{a^+}{a - a^+}$. Finally, we say that $(q,r)$ is $\dot{H}^{-s}$-admissible if $\frac{2}{q} = \frac{d}{2} - \frac{d}{r} + s$ and

$$
\begin{align*}
\left(\frac{2d}{d-2s}\right)^+ &\leq r \leq \left(\frac{2d}{d-2s}\right)^-, \quad d \geq 3; \\
\left(\frac{2}{1-s}\right)^+ &\leq r \leq \left(\frac{2}{1-s}\right)^+, \quad d = 2; \\
\left(\frac{2}{d-2s}\right)^+ &\leq r \leq \infty, \quad d = 1.
\end{align*}
$$

(2.4)

Given $s \in \mathbb{R}$, let $\Lambda_s = \{(q,r) : (q,r) \text{ is } \dot{H}^{-s} \text{ -admissible}\}$. We define the following Strichartz norm

$$
\|u\|_{\dot{S}(\dot{H}^s,I)} = \sup_{(q,r) \in \Lambda_s} \|u\|_{L^q_t L^r_x(I;\mathbb{R}^2)}
$$

and dual Strichartz norm

$$
\|u\|_{S'(\dot{H}^{-s},I)} = \inf_{(q,r) \in \Lambda_s} \|u\|_{L^q_t L^r_x(I;\mathbb{R}^2)^\prime}.
$$

If $s = 0$, we shall write $S'(\dot{H}^0,I) = S'(L^2,I)$, $\dot{S}(\dot{H}^0,I) = S(L^2,I)$. If $I = \mathbb{R}$, we will often omit $I$. Now, we recall the radial Sobolev embedding and the Strichartz estimates.

**Lemma 2.1** (Radial Sobolev embedding[1][18].) For radial $f \in H^1(\mathbb{R}^2)$, then

$$
|||x|^2f||_{L^\infty(\mathbb{R}^2)} \lesssim ||f||_{H^1(\mathbb{R}^2)}.
$$

**Lemma 2.2** (Strichartz estimates[17]). The following estimates hold.

(i)(linear estimate).

$$
\|e^{it\Delta}f\|_{\dot{S}(\dot{H}^s)} \leq C\|f\|_{\dot{H}^s}.
$$

(ii)(inhomogeneous estimates).

$$
\left\| \int_0^t e^{i(t-s)\Delta}g(\cdot,s)ds \right\|_{\dot{S}(\dot{H}^s)} \leq C\|g\|_{S'(\dot{H}^{-s},I)}.
$$
Next, we recall some interpolation estimates for the nonlinear term.

**Lemma 2.3.** Let \(2 - b < p < \infty\) and \(0 < b < 2\). If \(s_p < 1\), then for \(j \in \{1, 2\}\), there exist \(\theta_j \in (0, p)\) sufficiently small so that the following hold true:

\[
\| |x|^{-\theta_j} |u|^p v \|_{S'(H^{-s_p}, I)} \leq C \| \|u\|_{L^p_{x,t}S(I \times \mathbb{R}^2)} \| u \|_{S(H^{s_p}, I)} \| v \|_{S(H^{s_p}, I)},
\]

\[
\| |x|^{-\theta_j} |u|^p v \|_{S'(L^2, I)} \leq C \| \|u\|_{L^p_{x,t}S(I \times \mathbb{R}^2)} \| u \|_{S(H^{s_p}, I)} \| v \|_{S(L^2, I)},
\]

where \(C > 0\) is some constant independent of \(u\).

**Proof.** For a sufficiently small number \(\theta > 0\), we define the following numbers (depending on \(p\) and \(b\))

\[
\hat{q} = \frac{2p(p + 2 - \theta)}{p(p + b) - \theta(p - 2 + b)}, \quad \hat{r} = \frac{2p(p + 2 - \theta)}{(p - \theta)(2 - b)},
\]

and

\[
\hat{a} = \frac{p(p + 2 - \theta)}{(p + b - \theta) - (2 - b)(1 - \theta)}, \quad \hat{d} = \frac{p(p + 2 - \theta)}{2 - b}.
\]

It is easy to see that \((\hat{q}, \hat{r})\) is \(L^2\)-admissible, \((\hat{a}, \hat{d})\) is \(H^{s_p}\)-admissible and \((\hat{a}, \hat{d})\) is \(H^{-s_p}\)-admissible. Moreover, we observe that

\[
1 \leq \frac{\hat{a}}{\hat{a}} + \frac{1}{\hat{d}} = \frac{2}{\hat{d}}.
\]

By Hölder and Sobolev (see [13] for details), we have

\[
\| |x|^{-\theta} |u|^p v \|_{L^\infty} \lesssim \| u \|_{H^1} \| u \|_{L^\infty} \| v \|_{L^\infty},
\]

so that (2.5) and (2.6) follow. \(\square\)

**Lemma 2.4** ([5], Lemma 6.4). Let \(b \in (0, 1)\) and \(2 - b < p < \infty\). Then there exist \((p_1, q_1)\), \((p_1, q_2) \in \Lambda_0\) satisfying \(2p + 2 > p_1, p_2\), and

\[
\| \nabla (|x|^{-b} |u|^p u) \|_{S(L^2, I)} \lesssim (\| u \|_{L^p_{x,t}S(I \times \mathbb{R}^2)} + \| u \|_{L^p_{x,t}S(I \times \mathbb{R}^2)}) \| \nabla u \|_{S(L^2, I)},
\]

where \(m_1 = \frac{\alpha p_1}{p_1 - 2}\) and \(m_2 = \frac{\alpha p_2}{p_2 - 2}\).

Denote \(C_{p,d,b} = \frac{\frac{d}{d - \theta} + b}{p + 2 - \frac{d}{d - \theta} + b}\), then we have the following Gagliardo-Nirenberg inequality.

**Proposition 2.5** ([5]). Let \(\frac{d - 2b}{2} < p < \frac{d - 2b}{d - 2}\) and \(0 < b < \min\{2, d\}\), then we have

\[
\int_{\mathbb{R}^d} |x|^{-b} |u|^{p + 2} dx \leq C_0 \| \nabla u \|_{L^2}^{\frac{d}{d - \theta} + b} \| u \|_{L^2}^{p + 2 - \left(\frac{d}{d - \theta} + b\right)}.
\]

Here the sharp constant \(C_0 > 0\) is explicitly given by

\[
C_0 = C_{p,d,b}^{\frac{d - 2b - 2b}{2}} \frac{p + 2}{(\frac{d}{d - \theta} + b)} \| Q \|_{L^2}^p,
\]

where \(Q\) is the unique non-negative, radially-symmetric, decreasing solution of the equation

\[
\Delta Q - Q + |x|^{-b} |Q|^p Q = 0.
\]

Moreover the solution \(Q\) satisfies the following relations

\[
\| \nabla Q \|_{L^2} = C_{p,d,b}^{\frac{1}{\theta}} \| Q \|_{L^2},
\]

and

\[
\int_{\mathbb{R}^d} |x|^{-b} |Q|^{p + 2} dx = C_{p,d,b} \| Q \|_{L^2}^2.
\]
Then, the inequality \( R \) may have such that
\[
\int_{\alpha}^{\beta} f(t) dt < \infty
\]
and such that
\[
(1.1)
\]
In this section, we prove a new scattering criterion for the solution \( s \) of the equation (1.1).

**Lemma 3.1.** Let \( f, \phi \in L^p(\mathbb{R}^2) \) for some \( 0 < \alpha < 2 \). Indeed, for fixed \( \epsilon \leq \epsilon(E) > 0 \) and \( R = R(E) > 0 \) such that if
\[
(3.10)
\]
and
\[
(3.11)
\]
then \( u \) is a radial solution to (1.1). Indeed, for fixed \( \epsilon < \epsilon(E) > 0 \) and \( R = R(E) > 0 \) such that
\[
(3.12)
\]
In fact, the inequality \( (1.1) \) follows from \( (3.11) \). Indeed, for fixed \( R \gg 1 \), we may have
\[
R^{-b} \int_0^T \int_{|x| \leq R} |u|^2 \geq T \alpha,
\]
By the mean value theorem of calculus on interval \([0, T]\), there exists time sequence \( \{t_n\} \to \infty \)
such that
\[
\int_{|x| \leq R} |u(t_n)|^{p+2} dx \to 0, \quad n \to \infty.
\]
Then, the inequality \( (3.10) \) follows from this estimate and the fact that
\[
\|u\|_{L^p_\alpha(|x| \leq R)} \leq R^{1 - \frac{2}{p+2}} \|u\|_{L^{p+2}_2(|x| \leq R)}.
\]
**Proof.** First, we claim that it suffices to show
\[
(3.12)
\]
In fact, by a standard continuity argument and Lemma 2.3, the conclusion follows if one has
\[
\|u\|_{S(H^{\gamma_1})} < \infty,
\]
where \((m_1, q_1)\) and \((m_2, q_2)\) are given in Lemma 2.3. However, these two estimates are the consequences of the interpolation between (3.12) and the assumption (3.10).
Thus, by interpolation, we get $q,r$ for any $(q, r)$, then $(3.13)$.

We rewrite $e^{i(t-T_0)\Delta}u(T_0)$ as

$$e^{i(t-T_0)\Delta}u(T_0) = e^{it\Delta}u_0 - iF_1(t) - iF_2(t),$$

where

$$F_j(t) := \int_{I_j} e^{i(t-s)\Delta}(|x|^{-b}|u|^p u)(s)\,ds, \quad j = 1, 2,$$

with $I_1 = [0, T_0 - \epsilon^{-\theta}]$ and $I_2 = [T_0 - \epsilon^{-\theta}, T_0]$. Here, $\theta > 0$ will be chosen later. Let $T_0$ be large enough, then we have

$$\|e^{it\Delta}u_0\|_{S(H^s_p, (T_0, \infty))} \ll 1.$$

Hence, it remains to show

$$\|F_j(t)\|_{S(H^s_p, (T_0, \infty))} \ll 1, \quad \text{for } j = 1, 2.$$

**Estimation of $F_1(t)$:** We may use the dispersive estimate, Hölder’s inequality, and the assumption (3.11) to obtain

$$\left\| \int_{0}^{T_0-\epsilon^{-\theta}} e^{i(t-s)\Delta} |x|^{-b} |u|^p u \, ds \right\|_{L^\infty_t} \lesssim \int_{0}^{T_0-\epsilon^{-\theta}} |t-s|^{-1} \| |x|^{-b} |u|^p u \|_{L^1_t} \, ds$$

$$\lesssim \int_{0}^{T_0-\epsilon^{-\theta}} |t-s|^{-1} \left( \| |x|^\frac{b}{p+1} u \|_{L^{p+2}_t(B_1)} \right) \left( \| |x|^\frac{b}{2} u \|_{L^{\infty}_t(B_1)} \right) \, ds$$

$$+ \int_{0}^{T_0-\epsilon^{-\theta}} |t-s|^{-1} \left( \| |x|^\frac{b}{p+2} u \|_{L^\infty_t(B_1)} \right) \left( \| |x|^\frac{b}{2} u \|_{L^{p+2}_t(B_1)} \right) \, ds$$

$$\lesssim T_0^{-\frac{\theta}{2}} \epsilon^{\frac{\theta}{2}},$$

which yields

$$\left\| \int_{0}^{T_0-\epsilon^{-\theta}} e^{i(t-s)\Delta} |x|^{-b} |u|^p u \, ds \right\|_{L^\infty_t L^p_{x,t}} \lesssim T_0^{-\frac{\theta}{2}} \epsilon^{\frac{\theta}{2}}.$$

On the other hand, we may rewrite $F_1$ as

$$(3.13) \quad F_1(t) = e^{i(t-T_0+\epsilon^{-\theta})\Delta}u(T_0 - \epsilon^{-\theta}) - e^{it\Delta}u_0.$$

For any $(q, r) \in \Lambda_s$, we define the pair $(q_0, r_0)$ by

$$q_0 = (1 - s_p)q, \quad r_0 = (1 - s_p)r,$$

then $(q_0, r_0) \in \Lambda_0$. By Strichartz and the equality (3.13), we have

$$\|F_1(t)\|_{L^q_y L^r_t} \lesssim 1.$$

Thus, by interpolation, we get

$$\|F_1(t)\|_{S(H^s_p, (T_0, \infty))} \lesssim (T_0^3 \epsilon^\frac{1}{2})^{\frac{1}{1+s_p}}.$$
**Estimation of $F_2(t)$:** By the inequality (2.5), Sobolev and radial Sobolev embedding, we get

$$
\|F_2(t)\|_{S(\dot{H}^{s_p}, [T_0, \infty))} \lesssim \|x^{-\eta} |u|^{p} \|u\|^{s_p}_r \relphantom{\|x^{-\eta} |u|^{p} \|u\|^{s_p}_r} \\
\lesssim \|u\|^{p+1- \eta}_{L^{\infty}_{t} L^{s_p}_{x}(I_{2})} \\
\lesssim \sup_{(q, r) \in \Lambda_{s_p}} \|u\|^{p+1- \eta}_{L^{q}_{t} L^{r}_{x}(I_{2})} e^{-(p+1- \eta)\theta q}. 
$$

(3.14)

By the assumption (3.10), we may choose $T > T_0$ so that

$$
\int \chi \left(\frac{x}{R}\right) |u(T, x)|^2 < \epsilon^2.
$$

(3.15)

Here $\chi_R(x) := \chi \left(\frac{x}{R}\right)$ for $R > 0$, where $\chi(x)$ is a radial smooth function such that

$$
\chi(x) = \begin{cases} 
1, & |x| \leq \frac{1}{2}, \\
0, & |x| > 1.
\end{cases}
$$

Using the identity

$$
\partial_t |u|^2 = -2 \nabla \cdot \text{Im}(\overline{u} \nabla u)
$$

together with (3.15), integration by parts, and Cauchy-Schwarz, we can deduce

$$
\left| \partial_t \int_{I_2} \chi_R |u|^2 ds \right| \lesssim \frac{1}{R}.
$$

Thus, for $R \gg \epsilon^{-(2+\theta)}$, we find

$$
\|\chi_R u\|_{L^{\infty}_{t} L^{r}_{x}(I_{2} \times \mathbb{R}^{2})} \lesssim \epsilon.
$$

Using the radial Sobolev inequality and choosing $R$ large enough, we have

$$
\|u\|_{L^{r}_{t} L^{s_p}_{x}} \lesssim \|\chi_R u\|_{L^{\infty}_{t} L^{2}_{x} L^{r}_{x}(I_{2})} + \|1 - \chi_R\|_{L^{\infty}_{t}} \|u\|_{L^{2}_{t} L^{s_p}_{x}} \lesssim \epsilon^{\frac{1}{p}}.
$$

where the time-space norms are over the region $I_{2} \times \mathbb{R}^{2}$.

By the definition of $\Lambda_{s_p}$, there exists $\delta, c > 0$ such that $\frac{2}{1-s_p} + \delta \leq r \leq c$. Thus, by (3.14), we have

$$
\|F_2(t)\|_{S(\dot{H}^{s_p}, [T_0, \infty))} \lesssim \sup_{r} \|u\|^{p+1- \eta}_{L^{r}_{t} L^{s_p}_{x}(I_{2})} e^{-(p+1- \eta)\theta q} \\
\lesssim \epsilon^{(p+1- \eta)\left(\frac{1}{r} - \frac{1}{2}\right)}.
$$

Choosing $\theta = \frac{1}{c}$, $\gamma$ such that $\epsilon = T^{-\epsilon(1+\gamma)}$ and $\gamma + \alpha < 1$, we get

$$
\|e^{i(t-T_0)A} u(T_0)\|_{S(\dot{H}^{s_p}, [T_0, \infty))} \ll 1.
$$

The proof is completed. 

□
Throughout this section, we suppose \( u(t) \) is a solution to equation (1.1) satisfying the hypotheses of Theorem 1.1. In particular, we have that \( u \) is global and uniformly bounded in \( H^1 \). Furthermore, we will see that there exists \( \delta > 0 \) so that

\[
\sup_{t \in \mathbb{R}} \| u(t) \|_{L^2}^{1-s_p} \| \nabla u(t) \|_{L^2}^{s_p} < (1 - 2\delta) \| Q \|_{L^2}^{1-s_p} \| \nabla Q \|_{L^2}^{s_p}.
\]

We will prove the following Morawetz estimates.

**Proposition 4.1 (Morawetz estimates).** Let \( d = 2, 0 < b < 2, 2 - b < p < \infty \) and \( u \) be a solution to the focusing equation (1.1) on the space-time slab \([0, T] \times \mathbb{R}^d\). Then there exists constant \( 0 < \beta < 1 \) such that

\[
\int_0^T \int_{\mathbb{R}^d} \left| x \right|^{-b} |u(t, x)|^{p+2} dx dt < T^\beta.
\]

Using Proposition 4.1, Remark 3.1, rescaling, and the scattering criterion in Section 3, we can quickly prove Theorem 1.1.

Next, we prove Proposition 4.1 by a Morawetz identity. First, we recall the necessary coercivity property, that is, the inequality (4.16) holds on large balls.

**Lemma 4.2 (Coercivity on balls).** There exists \( R = R(\delta, M(u), Q) > 0 \) sufficiently large so that

\[
\sup_{t \in \mathbb{R}} \| \chi_R u \|_{L^2}^{1-s_p} \| \chi_R u \|_{\dot{H}^1}^{s_p} < (1 - \delta) \| Q \|_{L^2}^{1-s_p} \| Q \|_{\dot{H}^1}^{s_p}.
\]

In particular, there exists \( \delta' \) so that

\[
\int |\nabla (\chi_R u)|^2 dx = \frac{b + p}{p + 2} \int |x|^{-b} |\chi_R u|^{p+2} dx \geq \delta' \int |x|^{-b} |u|^{p+2}.
\]

**Proof.** The proof follows from the conservations of mass and energy and the Gagliardo-Nirenberg inequality \((\text{2.8})\). We refer to \([1]\) for an analogous proof. \(\square\)

By a direct computation, we have the following Virial/Morawetz identity.

**Lemma 4.3 (Virial/Morawetz identity).** Let \( u \) be the solution of (1.1) and \( a(x) \) be a smooth function. We denote the Morawetz action \( M_a(t) \) by

\[
M_a(t) = 2 \int_{\mathbb{R}^2} \nabla a(x) \text{Im}(\bar{u} \nabla u)(x) dx.
\]

Then we have

\[
\frac{d}{dt} M_a(t) = -\int \Delta^2 a(x) |u|^2 dx + 4 \int \partial_{jk} a(x) \text{Re}(\partial_k u \partial_j \bar{u}) dx
\]

\[
-\frac{2p}{p + 2} \int \Delta a(x) |x|^{-b} |u|^{p+2} dx + \frac{4}{p + 2} \int \nabla a(x) \cdot \nabla (|x|^{-b}) |u|^{p+2} dx,
\]

where the repeated indices are summed.

Using this identity, we may prove proposition 4.1 as following:
Proof. Let $R \gg 1$ to be chosen later. We take $a(x)$ to be a radial, smooth function satisfying

\begin{equation}
\begin{aligned}
a(x) = \begin{cases}
|x|^2, & \text{for } |x| \leq R; \\
3R|x|, & \text{for } |x| > 2R,
\end{cases}
\end{aligned}
\end{equation}

and when $R < |x| \leq 2R$, for any multiindex $\gamma$ there hold

\[ \partial_r a \geq 0, \quad \partial_{rr} a \geq 0 \quad \text{and} \quad |\partial^\gamma a| \lesssim R |x|^{-|\gamma|+1}. \]

Here $\partial_r$ denotes the radial derivative. Under these conditions, the matrix $(a_{jk})$ is non-negative. And, it is easy to verify that

\[
\begin{cases}
  a_{jk} = 2\delta_{jk}, & \Delta a = 4, & \Delta \Delta a = 0, & \text{for } |x| \leq R, \\
  a_{jk} = \frac{4R}{|x|} \delta_{jk} - \frac{x_j x_k}{|x|^2}, & \Delta a = \frac{3R}{|x|}, & \Delta \Delta a = \frac{3R}{|x|^2}, & \text{for } |x| > 2R.
\end{cases}
\]

Thus, we can divide $\frac{dM_a(t)}{dt}$ as follows:

\begin{equation}
\begin{aligned}
\frac{dM_a(t)}{dt} &= 8 \int_{|x| \leq R} |\nabla u|^2 - \frac{p + b}{p + 2} |x|^{-b} |u|^{p+2} dx \\
&\quad + \int_{|x| > 2R} \frac{-6pR}{|x|^{p+1}} |u|^{p+2} dx + \int \frac{-12Rb}{(p+2)|x|^{p+1}} |u|^{p+2} dx \\
&\quad + \int_{|x| > 2R} \frac{3R}{|x|^3} |u|^2 dx + \int_{|x| > 2R} \frac{12R}{|x|} |\nabla u|^2 dx \\
&\quad + \int_{R < |x| \leq 2R} 4Re\bar{u}_j a_{ij} u_j + O\left(\frac{R}{|x|^{b+1}} |u|^{p+2} + \frac{R}{|x|^3} |u|^2 \right) dx,
\end{aligned}
\end{equation}

where $\nabla$ denotes the angular derivative and subscripts denote partial derivatives. This implies

\begin{equation}
\int_{|x| < R} |\nabla u|^2 dx = \frac{p + b}{p + 2} \int_{|x| \leq R} |x|^{-b} |u|^{p+2} dx \lesssim \frac{dM_a(t)}{dt} + \frac{1}{R^\alpha},
\end{equation}

where $\alpha := \min\{2, b + \frac{2}{p}\}$. From the identity

\begin{equation}
\int \chi_R^2 |\nabla u|^2 = \int |\nabla (\chi_R u)|^2 + \chi_R \Delta (\chi_R) |u|^2 dx,
\end{equation}

we have

\[ \|\chi_R u\|_{H_1^2} \lesssim \|u\|_{H_1^2} + \frac{1}{R^2}. \]

Thus, by the radial Sobolev inequality, the Sobolev embedding and Lemma 4.2, we get

\begin{equation}
\int_{|x| \leq \frac{R}{2}} |x|^{-b} |u|^{p+2} dx \lesssim \frac{dM_a(t)}{dt} + \frac{1}{R^\alpha},
\end{equation}

From this inequality and the radial Sobolev inequality, we have

\[ \int_{\mathbb{R}^2} |x|^{-b} |u|^{p+2} dx \lesssim \frac{dM_a(t)}{dt} + \frac{1}{R^\alpha}. \]

Note that from the uniform $H^1$-bounds for $u$, and the choice of the weight function $a$, we have

\[ \sup_{t \in [0, T]} |M_a(t)| \lesssim R. \]

Then, we can use the fundamental theorem of calculus on an interval $[0, T]$ to obtain

\[ \int_0^T \int_{\mathbb{R}^2} |x|^{-b} |u|^{p+2} dx dt \lesssim R + \frac{T}{R^\alpha}. \]

The conclusion follows if we take $R = T^{1/\alpha}$. \qed
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