ON THE DISTRIBUTION OF ADDITION CHAINS

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Abstract. In this paper, we study the theory of addition chains producing any given number \( n \geq 3 \). With the goal of estimating the partial sums of addition chains, we introduce the notion of the determiners and the regulators of an addition chain and prove the following identities

\[
\sum_{j=2}^{\delta(n)+1} s_j = 2(n-1) + (\delta(n) - 1) + a_{\delta(n)} - r_{\delta(n)+1} + \int_{2}^{\delta(n)-1} \sum_{2 \leq j \leq t} r_j dt.
\]

where

\[
2, s_3 = a_3 + r_3, \ldots, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)}, s_{\delta(n)+1} = a_{\delta(n)+1} + r_{\delta(n)+1} = n
\]

are the associated generators of the chain \( 1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1} = n \) of length \( \delta(n) \). Also we obtain the identity

\[
\sum_{j=2}^{\delta(n)+1} a_j = (n-1) + (\delta(n) - 1) + a_{\delta(n)} - r_{\delta(n)+1} + \int_{2}^{\delta(n)-1} \sum_{2 \leq j \leq t} r_j dt.
\]

1. Introduction

The notion of an addition chain producing \( n \geq 3 \), introduced by Arnold Scholz, is a sequence of numbers of the form

\[1, 2, \ldots, s_{k-1}, s_k = n\]

where each term in the sequence is generated by adding two earlier terms and the terms are allowed to be homogeneous. The number of terms in the sequence determines the length of the chain. The length of the smallest such chain producing \( n \) is the shortest length of the addition chain. It is a well-known problem to determine the length of the shortest addition chain producing numbers \( 2^n - 1 \) of special forms. More formally the conjecture states

Conjecture 1.1. Let \( \iota(n) \) for \( n \geq 3 \) be the shortest addition chain producing \( n \), then the inequality is valid

\[\iota(2^n - 1) \leq n - 1 + \iota(n).\]

The conjecture was studied fairly soon afterwards by Alfred Brauer, who obtained some weaker bounds \[1\]. There had also been an amazing computational work in verifying the conjecture \[2\]. In this paper, we study the partial sums of the addition chain producing \( n \geq 3 \). We obtain the following results in this paper.

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Theorem 1.1. Let $1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1}$ be an addition chain producing $n \geq 3$ of length $\delta(n)$ with associated generators

$$2, s_3 = a_3 + r_3, \ldots, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)}, s_{\delta(n)+1} = a_{\delta(n)+1} + r_{\delta(n)+1} = n.$$  

Then we have

$$\sum_{j=2}^{\delta(n)+1} s_j = 2(n-1) + (\delta(n) - 1) + a_{\delta(n)} - r_{\delta(n)+1} + \int_2^{\delta(n)-1} \sum_{2 \leq j \leq t} r_j dt.$$  

2. The regulators and determiners of an addition chain

In this section we recall the notion of an addition chain and introduce the notion of the generators of the chain and their accompanying determiners and regulators.

Definition 2.1. Let $n \geq 3$, then by an addition chain of length $k - 1$ producing $n$, we mean the sequence

$$1, 2, \ldots, s_{k-1}, s_k$$

where each term $s_j$ ($j \geq 3$) in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

where $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the $i^{th}$ generator of the chain for $2 \leq i \leq k$. We call $a_i$ the determiner and $r_i$ the regulator of the $i^{th}$ generator of the chain. We call the sequence $(r_i)$ the regulators of the addition chain and $(a_i)$ the determiners of the chain for $2 \leq i \leq k$. We call the subsequence $(s_j)$ for $2 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a truncated addition chain producing $n$.

Theorem 2.2. Let $1, 2, \ldots, s_{k-1}, s_k$ be an addition chain producing $n \geq 3$ with associated generators

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$  

Then the following relation for the regulators

$$\sum_{j=2}^{k} r_j = n - 1$$

hold.

Proof. First we notice that $r_k = n - a_k$. It follows that

$$r_k + r_{k-1} = n - a_k + r_{k-1}$$

$$= n - (a_{k-1} + r_{k-1}) + r_{k-1}$$

$$= n - a_{k-1}.$$  

Again we obtain from the following iteration

$$r_k + r_{k-1} + r_{k-2} = n - a_{k-1} + r_{k-2}$$

$$= n - (a_{k-2} + r_{k-2}) + r_{k-2}$$

$$= n - a_{k-2}.$$  

By iterating downwards in this manner the relation follows. □
Corollary 2.1. Let \( 1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1} \) be an addition chain producing \( n \geq 3 \) with associated generators
\[
2 = 1 + 1, \ldots, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)}, s_{\delta(n)+1} = a_{\delta(n)+1} + r_{\delta(n)+1} = n.
\]
If we denote the length of the chain by \( \delta(n) \), then we have the inequality
\[
\frac{n - 1}{\inf \{r_j\}_{j=2}^{\delta(n)+1}} \leq \delta(n) \leq \frac{n - 1}{\sup \{r_j\}_{j=2}^{\delta(n)+1}}.
\]

Proof. The result follows by applying the relation in Theorem 2.2 and noting that the number of regulators \( r_j \) in the chain with multiplicity counts as the length of the chain. \( \square \)

Theorem 2.3. Let \( 1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1} \) be an addition chain producing \( n \geq 3 \) of length \( \delta(n) \) with associated generators
\[
2, s_3 = a_3 + r_3, \ldots, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)}, s_{\delta(n)+1} = a_{\delta(n)+1} + r_{\delta(n)+1} = n.
\]
Then we have
\[
\delta(n) \sum_{j=2}^{\delta(n)} a_j = (\delta(n) - 1) + \sum_{j=2}^{\delta(n)-1} (\delta(n) - j)r_j.
\]

Proof. This identity is easily established by induction on an iteration on the sum. We first notice that
\[
a_2 + a_3 = a_2 + a_2 + r_2 = 2a_2 + r_2.
\]
Again it follows that
\[
a_2 + a_3 + a_4 = 2a_2 + r_2 + a_4 = 3a_2 + 2r_2 + r_3.
\]
By inducting on this iteration we can now write
\[
\delta(n) \sum_{j=2}^{\delta(n)} a_j = (\delta(n) - 1)a_2 + (\delta(n) - 2)r_2 + (\delta(n) - 3)r_3 + \cdots + 2r_{\delta(n)-2} + r_{\delta(n)-1}
\]
thereby establishing the identity. \( \square \)

Remark 2.4. The above result ensures that we can write the partial sums of the determiners \( a_j \) of the generators of any addition chain as a linear combination of the regulators \( r_j \). We leverage this identity to obtain an identity relating the partial sums of the determiners to the length of the chain and a certain integral of regulators of the chain. We state the result in the following manner as follows.

Theorem 2.5. Let \( 1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1} \) be an addition chain producing \( n \geq 3 \) of length \( \delta(n) \) with associated generators
\[
2, s_3 = a_3 + r_3, \ldots, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)}, s_{\delta(n)+1} = a_{\delta(n)+1} + r_{\delta(n)+1} = n.
\]
Then we have the identity
\[
\sum_{j=2}^{\delta(n)+1} a_j = (n - 1) + (\delta(n) - 1) + a_{\delta(n)} - r_{\delta(n)+1} + \int_2^{\delta(n)-1} \sum_{2 \leq j \leq t} r_j dt.
\]

**Proof.** Let \(1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1}\) be an addition chain producing \(n \geq 3\) with associated generators
\[
2, s_3 = a_3 + r_3, \ldots, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)}, s_{\delta(n)+1} = a_{\delta(n)+1} + r_{\delta(n)+1} = n.
\]
We consider the identity
\[
\sum_{j=2}^{\delta(n)} a_j = (\delta(n) - 1) + \sum_{j=2}^{\delta(n)-1} (\delta(n) - j) r_j
\]
in Theorem 2.3. By applying partial summation on the sum on the right-hand side of the expression we have
\[
\sum_{j=2}^{\delta(n)-1} (\delta(n) - j) r_j = \sum_{j=2}^{\delta(n)-1} r_j + \int_2^{\delta(n)-1} \sum_{2 \leq j \leq t} r_j dt
\]

\[
= n - 1 - r_{\delta(n)} - r_{\delta(n)+1} + \int_2^{\delta(n)-1} \sum_{2 \leq j \leq t} r_j dt
\]
by appealing to Theorem 2.2 and the result follows by arranging the terms in the expression and appealing to Theorem 2.2. \(\square\)

**Remark 2.6.** Next we relate the length of an additive chain function to the corresponding argument \(n \geq 3\) in the following inequality. We obtain the following weaker but much less crude bound as follows.

**Proposition 2.1.** Let \(1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1} = n\) be an addition chain producing \(n \geq 3\) with associated generators
\[
2 = 1 + 1, \ldots, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)}, s_{\delta(n)+1} = a_{\delta(n)+1} + r_{\delta(n)+1} = n.
\]
Then the inequality
\[
\sum_{j=2}^{\delta(n)+1} s_j \geq 2n - 2
\]
hold.

**Proof.** We first write
\[
\sum_{j=2}^{\delta(n)+1} s_j = \sum_{j=2}^{\delta(n)+1} a_j + \sum_{j=2}^{\delta(n)+1} r_j
\]

\[
\geq 2 \sum_{j=2}^{\delta(n)+1} r_j
\]

\[
\geq 2(n - 1)
\]
by Theorem 2.2. \(\square\)
Remark 2.7. We improve on the estimate of the partial sums of an additive chain producing \( n \geq 3 \) in the following result. It has become so clear that we need to a larger extent the argument and the length of an additive chain to determine very roughly the value of their partial sums.

**Theorem 2.8.** Let \( 1, 2, \ldots, s_{\delta(n)}, s_{\delta(n)+1} = n \) be an addition chain producing \( n \geq 3 \) of length \( \delta(n) \) with associated generators

\[
2, s_3 = a_3 + r_3, \ldots, s_{\delta(n)} = a_{\delta(n)} + r_{\delta(n)}, s_{\delta(n)+1} = a_{\delta(n)+1} + r_{\delta(n)+1} = n.
\]

Then we have

\[
\sum_{j=2}^{\delta(n)+1} s_j = 2(n - 1) + (\delta(n) - 1) + a_{\delta(n)} + r_{\delta(n)+1} + \int_2^{\delta(n)-1} \sum_{2 \leq j \leq t} r_j dt.
\]

**Proof.** Under the main assumption we have, by appealing to Theorem 2.2

\[
\sum_{j=2}^{\delta(n)+1} r_j = n - 1.
\]

By adding the partial sums of the determiners of the same scale to the earlier sum we have by appealing to Theorem 2.3

\[
\sum_{j=2}^{\delta(n)+1} s_j = \sum_{j=2}^{\delta(n)} a_j + a_{\delta(n)+1} + \sum_{j=2}^{\delta(n)+1} r_j
\]

\[
= (n - 1) + (\delta(n) - 1) + a_{\delta(n)+1} + \sum_{j=2}^{\delta(n)-1} (\delta(n) - j)r_j
\]

\[
= 2(n - 1) + (\delta(n) - 1) + a_{\delta(n)+1} - r_{\delta(n)} - r_{\delta(n)+1} + \int_2^{\delta(n)-1} \sum_{2 \leq j \leq t} r_j dt
\]

by an application of partial summation. \( \square \)

**References**

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