DETERMINING ANOMALIES IN A SEMILINEAR ELLIPTIC EQUATION BY A MINIMAL NUMBER OF MEASUREMENTS

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ABSTRACT. We are concerned with the inverse boundary problem of determining anomalies associated with a semilinear elliptic equation of the form $-\Delta u + a(x, u) = 0$, where $a(x, u)$ is a general nonlinear term that belongs to a Hölder class. It is assumed that the inhomogeneity of $f(x, u)$ is contained in a bounded domain $D$ in the sense that outside $D$, $a(x, u) = \lambda u$ with $\lambda \in C$. We establish novel unique identifiability results in several general scenarios of practical interest. These include determining the support of the inclusion (i.e. $D$) independent of its content (i.e. $a(x, u)$ in $D$) by a single boundary measurement; and determining both $D$ and $a(x, u)|_{\partial D}$ by $M$ boundary measurements, where $M \in N$ signifies the number of unknown coefficients in $a(x, u)$. The mathematical argument is based on microlocally characterising the singularities in the solution $u$ induced by the geometric singularities of $D$, and does not rely on any linearisation technique.

Keywords: semilinear elliptic PDE; inverse boundary problem; nonlinear inclusion; minimal measurement; singularities.

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1. Introduction

1.1. Mathematical setup and summary of major findings. Initially focusing on the mathematics, but not the physics, we introduce the forward boundary value problem associated with a semilinear elliptic equation:

$$\Delta u + a(x, u) = 0 \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = \psi,$$

where

1. $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$, and $\psi \in H^{1/2}(\partial \Omega)$;
2. there is a bounded Lipschitz domain $D \Subset \Omega$ such that $\Omega \setminus D$ is connected, and a constant $\lambda \in C$ such that

$$a(x, u) = (f(x, u) - \lambda u)\chi_D + \lambda u, \quad x \in \Omega. \quad (1.2)$$

That is, $a(x, u) = f(x, u)$ in $D$, whereas $a(x, u) = \lambda u$ in $\Omega \setminus D$. Furthermore, we suppose that $a(x, u)$ is $C^1$-continuous with respect to $u$ for a fixed $x \in \Omega$ and $\partial_\nu a(x, u) \in L^\infty(\Omega)$.

3. $f(x, z) : (x, z) \in D \times \mathbb{C} \mapsto \mathbb{C}$ fulfils the following admissibility conditions:

(a) For $u(\cdot) \in H^1(\Omega)$, $f(x, u(\cdot)) \in L^2(\Omega)$;
(b) $f(x, z)$ is $C^\gamma$-continuous, $\gamma \in (0, 1)$, with respect to $(x, z) \in D \times \mathbb{C}$;
(c) $f(x, z)$ fulfills that for a proper $\psi \in H^{1/2}(\partial \Omega)$, there exists a solution $u \in H^1(\Omega)$ to (1.1).

In such a case, we say that $f$ belongs to the admissible class $\mathcal{A}$ and write $f \in \mathcal{A}$ or $(D; f) \in \mathcal{A}$ to signify the support of the inhomogeneity of $f$ is $D$.

In what follows, we assume that $\lambda$ is known, which characterises the homogeneous space $\Omega \setminus D$, whereas $(D; f)$ is unknown, which is referred to as an anomalous inhomogeneous inclusion. In this paper, we aim to study the following inverse boundary problem:

$$\Lambda_{D, f}(\psi) := (\psi|_{\partial \Omega}, \partial_n u|_{\partial \Omega}), \quad \psi \in H^{1/2}(\partial \Omega) \text{ fixed} \longrightarrow D \quad \text{independent of } f, \quad (1.3)$$
where \( u \in H^1(\Omega) \) is a solution to (1.1), and \( \nu \in S^{n-1} := \{ x \in \mathbb{R}^n; |x| = 1 \} \) is the exterior unit normal vector to \( \partial \Omega \). In the physical context, \( D \) signifies the support of the anomalous inhomogeneity whereas \( f \) characterises its physical content. Hence, the inverse problem (1.3) is concerned with recovering the location and shape of the anomalous inhomogeneity independent of its content. It is also referred to as the inverse inclusion problem in the theory of inverse problems. Furthermore, we also study the following inverse boundary problem:

\[
\Lambda_{D,f}(\psi_j) := (\psi_j|_{\partial \Omega}, \partial_n u_j|_{\partial \Omega}), \; \psi_j \in H^{1/2}(\partial \Omega), \; j = 1, \ldots, N \in \mathbb{N} \rightarrow \text{both } D \text{ and } f, \quad (1.4)
\]

where \( u_j \in H^1(\Omega) \) is a solution to (1.1) associated with the boundary data \( u_j|_{\partial \Omega} = \psi_m \). Here, \( N \in \mathbb{N} \) signifies the number of unknown coefficients of \( f(x, u) \), say e.g. \( f(x, u) = \sum_{j=1}^{N} \lambda_j u^j \) with \( \lambda_j \in \mathbb{C} \). That is, for the inverse problem (1.4), we aim at recovering both the support and its physical content of the inhomogeneous inclusion by \( N \) boundary measurements. It can be verified that both inverse problems (1.3) and (1.4) are formally determined; that is, the cardinalities of the unknown inclusion and the known boundary data are equal. By cardinality, we mean the number of independent variables in a quantity. Hence, we refer to them as inverse problems with a minimal number of measurements, or simply minimal boundary measurements.

It is emphasised that we only assume the existence of a solution to (1.1) and do not assume the uniqueness of the solution. That is, there might exist multiple solutions to (1.1). Associated with a single \( \psi \in H^{1/2}(\partial \Omega) \), \( \Lambda_{\psi} \) is referred to as a single pair of Cauchy data, or a single boundary measurement. Throughout, we always assume that \( \psi \) is properly chosen such that (1.1) has a solution \( u \in H^1(\Omega) \). By the admissibility of \( f \), one can easily infer from the standard interior regularity estimate for elliptic PDEs that \( u \in H^2(\Omega') \) for any \( \Omega' \subseteq \Omega \) (cf. [25]).

For the inverse inclusion problem (1.3), we mainly consider its unique identifiability issue. That is, we aim at establishing the sufficient conditions under which \( D \) can be uniquely determined by \( \Lambda_{D,f}(\psi) \) in the sense that if two admissible inclusions \( (D_m; f_m), \) \( m = 1, 2 \), produce the same boundary measurement, i.e. \( \Lambda_{D_1,f_1}(\psi) = \Lambda_{D_2,f_2}(\psi) \) associated with a fixed \( \psi \in H^{1/2}(\partial \Omega) \), then one has \( D_1 = D_2 \). The main results that we establish in this paper for the inverse problem (1.3) can be roughly summarised as follows:

1. Under a generic condition, a local unique identifiability result is established showing that the difference of the supports of two nonlinear anomalies cannot possess corner or conic singularities;
2. If certain a-priori information is available on \( D \), say e.g. it is a convex polygon or polyhedron or of a corona-shape, it can be uniquely determined.
3. In several practical scenarios, say e.g. nonlinear anomalies are embedded in linear anomalies in a layered manner or certain multi-layered/nest nonlinear anomalies, we show that under generic conditions, one can determine the support of each layer by a single measurement within convex polygon/polyhedral geometries.

Similarly, for the inverse problem (1.4), we establish unique identifiability results in three scenarios:

1. If \( f(x, u) = \sum_{j=1}^{N} \lambda_j u^j \) with \( \lambda_j \in \mathbb{C} \) and \( D \) is of polygonal/polyhedral or corona-shape, then under generic conditions, we can establish the unique identifiability result in determining both \( D \) and \( f \) by using \( N \) measurements.
2. If the anomalous inclusion is of a layered/nest structure with \( f \) in each layer of the form given in (1) above (distinct among different layers), we can establish the unique identifiability result in determining both \( D \) and \( f \) by using minimal boundary measurements.
1.2. Physical motivation and background discussion. In the physical context, the PDE system (1.1) can be used to describe several physical problems of practical importance, especially in the wave scattering theory. For example, if one takes
\[
\lambda = k^2 \quad \text{with} \quad k \in \mathbb{R}_+; \quad f(x, u) = k^2 q_1(x)u \quad \text{with} \quad q_1 \in L^\infty(D),
\]
(1.5) (1.1) is the classical Helmholtz system, which describes the transverse time-harmonic electromagnetic scattering when \( n = 2 \) \([19]\), and the time-harmonic acoustic scattering when \( n = 3 \) \([3]\). In the physical setup, \( k \in \mathbb{R}_+ \) is the wavenumber and \( q_1 \) characterises the medium content of an inhomogeneity \( D \). In nonlinear optics or acoustics \([6]\), \( f(x, u) \) can be of a more general form than that in (1.5), say e.g. \( f(x, u) = k^2 q_1(x)u + q_2(x)u^2 \) to characterise the nonlinear effect. In a similar manner, (1.1) can also be used to describe the Schrödinger equation that governs the quantum scattering (cf. \([13]\)). On the other hand, we note that the well-posedness of the elliptic system (1.1) has been extensively studied in the literature: in the linear case, the well-posedness is well understood \([21, 25]\); and in the nonlinear case, the well-posedness can be achieved in many generic setups (cf. \([15]\) and the references cited therein) and in particular, if smallness is imposed on the solution, which in many situations of practical interest is equivalent to imposing smallness on the boundary input \( \psi \), the well-posedness of (1.1) can also be guaranteed; see e.g. \([18]\) where the nonlinear term \( f(x, u) \) is assumed to belong to a certain analytic class. Since our focus is the inverse problems (1.3) and (1.4), and also in order to appeal for a general study, we always assume the well-posedness of the forward problem (1.1). Nevertheless, for self-containedness as well as our use, we establish the well-posedness for small solutions of the forward problem (1.1) when \( f(x, u) \) is only assumed to belong to the Hölder class.

The inverse inclusion problem (1.3) is a longstanding problem in the theory of inverse problems, but mainly restricted to linear mediums. We refer to \([23, 24]\) for recent progress in electrostatics, \([2–4, 7, 8, 22]\) in inverse acoustic scattering, \([5, 10]\) in inverse electromagnetic scattering and \([1, 11]\) in inverse elastic scattering. To our best knowledge, there is no result available for the inverse inclusion problem (1.3) associated with general nonlinear anomalies. On the other hand, we note that recently there are many studies on the inverse boundary problem of recovering \( f \) by knowledge of \( \Lambda(\psi) \) associated with all \( \psi \in H^{1/2}(\partial\Omega) \); that is, infinitely/uncountably many boundary measurements are needed. We refer to \([14, 16–18, 27]\) and the references cited therein for related results. It is pointed out that in all of those inverse problem studies, \( f(x, u) \) is usually required to possess higher regularities than the Hölder one required in the current article. By aiming at recovering the support of the anomaly, but not its physical content, we can work with merely Hölder continuous nonlinearities. Moreover, it is emphasised that we only make use of a single boundary measurement. If \( f(x, u) \) is of a particular (still general) form, we can determine both the support and its physical content of the anomalous inclusion by a minimal number of boundary measurements. Nevertheless, it is also pointed out that we require that \( D \) is of polygonal/polyhedral or corona-shape since the corner or conic singularities are essentially needed in our mathematical argument. The mathematical arguments are based on microlally characterising the singularities in a quantitative manner of the solution \( u \) to (1.1) induced by the geometric singularities in \( f(x, u) \). Finally, we would like to emphasise that the results obtained in this paper include the relevant ones for linear mediums as special cases, and moreover our study indicates that the nonlinear effect can induce new phenomena that are of both theoretical and practical interest.

In summary, we list the major contributions of this work in what follows.

1. We establish local and global uniqueness results in determining certain general nonlinear anomalies in several separate cases by minimal boundary measurements. These results are highly interesting, in particular in the following two aspects. First,
to our best knowledge, this is first result in the literature concerning the shape
determination of general nonlinear anomalies by a single measurement. The existing
studies are mainly devoted to the determination of linear anomalies. Second, there
are many existing studies on inverse problems for nonlinear differential equations,
but most of them make use of infinitely many measurements.

(2) In achieving the results in (1), we need to impose “strong” a-priori information
on the target anomaly in that either its support or its physical content belongs to
certain admissible classes. Nevertheless, on the one hand, these admissible classes
are general enough to include some physically important cases, and on the other
hand, they are good examples to verify that in the theory of inverse problems, the
a-priori information can bring beneficial advantages to the inversion process.

(3) It is also worth noting that our results include many existing studies for linear anom-
alias as special cases. Moreover, they extend and generalise the relevant studies in
that our results show that the nonlinearities can leverage certain technical restric-
tions in the linear counterpart and can help identify the anomalies; see Remark 2.6
for more relevant discussion.

The rest of the paper is organised as follows. In Section 2, we present the unique iden-
tifiability results for general anomalies including local uniqueness results with corner/conic
singularities and a global unique result within polygonal/polyhedral or corona geometry. In
Section 3, we present unique identifiability results for inverse problem (1.4) with a single-
layer structure. Section 4 is devoted to deriving unique identifiability results in determining
layered anomalies.

2. DETERMINING SUPPORTS OF ANOMALOUS INCLUSIONS BY A SINGLE MEASUREMENT

In this section, we consider the inverse boundary problem (1.3) in determining the sup-
port of an anomalous inclusion independent of its physical content by a single boundary
measurement.

2.1. Local uniqueness results. First, we introduce the geometric setup of our study. For
a given point $x_0 \in \mathbb{R}^n$, $n = 2, 3$, we let $v_0 = y_0 - x_0$ where $y_0 \in \mathbb{R}^n$ is fixed. Set

$$
S_{x_0, \theta_0} := \{ y \in \mathbb{R}^n \mid 0 \leq \angle(y - x_0, v_0) \leq \theta_0 \} \quad (\theta_0 \in (0, \pi/2)),
$$

(2.1)

which is a strictly convex conic cone with the apex $x_0$ and an opening angle $2\theta_0 \in (0, \pi)$ in
$\mathbb{R}^n$. Here $v_0$ is referred to be the axis of $C_{x_0, \theta_0}$. Define the truncated conic cone as

$$
S_{x_0, \theta_0}^{bh} := S_{x_0, \theta_0} \cap B_h(x_0),
$$

(2.2)

where $B_h(x_0)$ is an open ball centered at $x_0$ with the radius $h \in \mathbb{R}_+$. When $n = 2$, $S_{x_0}^{bh}$ is a
sectorial corner with the apex $x_0$ and an opening angle $2\theta_0 \in (0, \pi)$.

We also introduce a polyhedral corner in $\mathbb{R}^3$ as follows. Assume that $K_{x_0; e_1, \ldots, e_{\ell}}$ is a
polyhedral cone with the apex $x_0$ and edges $e_j$ ($j = 1, \ldots, \ell$, $\ell \geq 3$), where $e_j$, $j = 1, 2, \ldots, \ell$
are mutually linearly independent vectors in $\mathbb{R}^3$. Throughout of this paper we always
suppose that $K_{x_0; e_1, \ldots, e_{\ell}}$ is strictly convex, which implies that it can be fitted into a conic
cone $S_{x_0, \theta_0}$ with an opening angle $\theta_0 \in (0, \pi/2)$, where $S_{x_0, \theta_0}$ is defined in (2.1). Given a
constant $h \in \mathbb{R}_+$, we define the truncated polyhedral corner $K_{x_0}^{bh}$ as

$$
K_{x_0}^{bh} = K_{x_0; e_1, \ldots, e_{\ell}} \cap B_h(x_0).
$$

(2.3)

Throughout the rest of the paper, we denote

$$
C_h := S_{x_0, \theta_0}^{bh} \quad \text{or} \quad K_{x_0}^{bh}
$$

(2.4)
as a corner in $\mathbb{R}^n$ ($n = 2, 3$) with the apex $x_0$, where $S_{x_0, \theta_0}^h$ and $K_{x_0}^h$ are defined in (2.2) and (2.3) respectively. The schematic illustration of a conic and polyhedral corner is displayed in Figure 1.

![Illustrations of conic and polyhedral corner](Image)

**Figure 1.** Illustrations of conic and polyhedral corner

**Lemma 2.1.** Suppose that $\tau \in \mathbb{R}_+$ and $C_h$ is defined in (2.4). For $x \in \mathbb{R}^n$ ($n = 2, 3$), let

$$u_0(x) = e^{\tau(d + i\alpha) \cdot x},$$

where $d \cdot d^\perp = 0$ with $d, d^\perp \in S^{n-1}$, then $\Delta u_0 = 0$ in $\mathbb{R}^n$. There exist unit vectors $d, d^\perp \in S^{n-1}$ and a positive number $\zeta$ depending on $C_h$ satisfying

$$-1 < d \cdot \hat{x} \leq -\zeta < 0 \quad \text{for all} \quad x \in C_h, \quad \text{and} \quad d \cdot d^\perp = 0,$$

where $\hat{x} = \frac{x}{|x|}$. Furthermore, for sufficient large $\tau$, it holds that

$$\left| \int_{C_h} u_0(x) dx \right| \leq C_{C_h} \tau^{-n} + O \left( \tau^{-1} e^{-\frac{1}{2} \zeta \tau} \right),$$

$$\left| \int_{C_h} |x|^\alpha u_0(x) dx \right| \leq \tau^{-(\alpha+n)} + \frac{1}{\tau} e^{-\frac{1}{2} \zeta \tau}, \quad \forall \alpha \in \mathbb{R}_+,$$

$$\|u_0\|_{H^1(\partial C_h \cap \partial B_h(x_0))} \leq (2\tau^2 + 1)^{\frac{1}{2}} e^{-\frac{1}{2} \zeta \tau},$$

$$\|\partial_n u_0\|_{L^2(\partial C_h \cap \partial B_h(x_0))} \leq \sqrt{2} e^{-\frac{1}{2} \zeta \tau},$$

where $C_{C_h}$ is a positive constant not depending on $\tau$. Here "$\lesssim$" means that we neglect the generic constant $C$ associated with the principle term with respect to $\tau$ in the upper bounds of (2.8), (2.9) and (2.10) respectively, where $C$ does not depend on $\tau$.

**Proof.** Since $d \perp d^\perp$, one knows that $\Delta u_0 = 0$ in $\mathbb{R}^n$. Without loss of generality, in the following we assume that the apex $x_0$ of $C_h$ is the origin. In view of the convexity of $C_h$ defined (2.4), there exists a vector $d \in S^{n-1}$ satisfying (2.6).

In what follows, we only prove the cases that $C_h$ is a sectorial corner in $\mathbb{R}^2$ and $C_h$ is a conic corner in $\mathbb{R}^3$ respectively. The case that $C_h$ is a polyhedral corner in $\mathbb{R}^3$ can be proved similarly and we only remark it at the end of the proof.

For a fixed $\alpha \in \mathbb{R}_+$, if $\Re \mu \geq 2\alpha/e$, where $\mu \in \mathbb{C}$, it yields that $r^\alpha \leq e^{R \mu \tau / 2}$. Hence we have

$$\int_{\varepsilon}^{\infty} r^\alpha e^{-\mu r} dr \leq \int_{\varepsilon}^{\infty} e^{-R \mu \tau / 2} dr = \frac{2}{R \mu} e^{-R \mu \tau / 2}.$$  \hspace{1cm} (2.11)

where $\varepsilon \in \mathbb{R}_+$ is fixed. Using Laplace transform, one can derive that

$$\int_{0}^{\varepsilon} r^\alpha e^{-\mu r} dr = \frac{\Gamma(\alpha + 1)}{\mu^{\alpha+1}} + \int_{\varepsilon}^{\infty} r^\alpha e^{-\mu r} dr,$$  \hspace{1cm} (2.12)

where $\Gamma$ is the Gamma function.
Case 1: \( C_h \) is a sectorial corner. Write \( x = (x_1, x_2) \in \mathbb{R}^2 \) in the polar coordinates as \( x = (r \cos \theta, r \sin \theta) \), where \( r \geq 0 \) and \( \theta \in [0, 2\pi) \). Let \( \Gamma^+_h \) be two edges of \( C_h \). Set

\[
\Gamma^+_h = \{ x \in \mathbb{R}^2 \mid x = r(\cos \theta_M, \sin \theta_M) \}, \quad \Gamma^-_h = \{ x \in \mathbb{R}^2 \mid x = r(\cos \theta_m, \sin \theta_m) \},
\]

where \( r \in [0, h] \) with \( h \in \mathbb{R}_+ \), \( \theta_M, \theta_M \in [0, 2\pi) \) and \( \theta_M - \theta_m = 2\theta_0 \). Here \( 2\theta_0 \) is the opening angle of \( C_h \), where \( \theta_0 \in (0, \pi/2) \). Using the polar-coordinate transformation and (2.12), it can be obtained that

\[
\int_{\mathcal{C}_h} e^{\rho x} dx = \int_{\mathcal{C}_h} e^{-\tau (d + id^\perp) \cdot x} dx = \frac{\Gamma(2)}{\tau^2} \int_{\theta_m}^{\theta_M} \frac{1}{\left( d \cdot \hat{x} + id^\perp \cdot \hat{x} \right)^2} d\theta - \int_{\theta_m}^{\theta_M} I_r d\theta,
\]

where \( I_r = \int_0^\infty e^{-\tau (d + id^\perp) \cdot x} r dr \). Hence, it can be directly calculated that

\[
\left| \int_{\theta_m}^{\theta_M} \frac{1}{\left( d \cdot \hat{x} + id^\perp \cdot \hat{x} \right)^2} d\theta \right| = \frac{\theta_M - \theta_m}{\left| d \cdot \hat{x}(\theta) + id^\perp \cdot \hat{x}(\theta) \right|^2} \geq \frac{\theta_M - \theta_m}{2}
\]

by using the integral mean value theorem. For sufficiently large \( \tau \), according to (2.6) and (2.11), we have the following integral inequality

\[
\left| \int_{\mathcal{C}_h} e^{\rho x} dx \right| \geq \frac{\Gamma(2)(\theta_M - \theta_m)}{2\tau^2} - \left| \int_{\theta_m}^{\theta_M} I_{R}\! d\theta \right| \geq C_{\mathcal{C}_h} \frac{\theta_M - \theta_m}{\tau^2} - \frac{2}{\zeta} e^{-\frac{\theta_M - \theta_m}{\tau}}.
\]

Therefore, we prove (2.7) for \( n = 2 \), where \( C_{\mathcal{C}_h} = \theta_M - \theta_m = 2\theta_0 \) in (2.7).

By adopting a similar argument for (2.7) when \( C_h \) is a sectorial corner, for (2.8) we have

\[
\int_{\mathcal{C}_h} |x|^a u_0 dx = \frac{\Gamma(\alpha + 2)}{\tau^{\alpha + 2}} \int_{\theta_m}^{\theta_M} \left( \frac{1}{\left( d \cdot \hat{x} + id^\perp \cdot \hat{x} \right)^2} + I_R \right) d\theta,
\]

where we utilize (2.12) and \( I_R = \int_0^\infty r^{\alpha + 2} e^{\tau (d \cdot \hat{x} + id^\perp \cdot \hat{x})} dr \). Using (2.11), we can prove (2.8).

By using the polar-coordinate transformation and (2.6), we have the following inequality:

\[
\| u_0 \|_{L^2(\partial \mathcal{C}_h \cap \partial B_h(0))} = \left( \int_{\theta_m}^{\theta_M} e^{2h\tau (d \cdot \hat{x})} d\theta \right)^{\frac{1}{2}} \leq (\theta_M - \theta_m)^{\frac{1}{2}} e^{-\frac{\theta_M - \theta_m}{\tau}}.
\]

In view of (2.5) and (2.16), one can directly verify that

\[
\| u_0 \|_{H^1(\partial \mathcal{C}_h \cap \partial B_h(0))} = \left( \| u_0 \|_{L^2(\partial \mathcal{C}_h \cap \partial B_h(0))}^2 + \| \tau (d + id^\perp) u_0 \|_{L^2(\partial \mathcal{C}_h \cap \partial B_h(0))}^2 \right)^{\frac{1}{2}} \\
\leq (2\tau^2 + 1)^{\frac{1}{2}} \| u_0 \|_{L^2(\partial \mathcal{C}_h \cap \partial B_h(0))} \leq (2\tau^2 + 1)^{\frac{1}{2}} e^{-\frac{\theta_M - \theta_m}{\tau}}, \quad n = 2, 3.
\]

Furthermore, by virtue of (2.6) and Cauchy-Schwarz inequality, it yields that

\[
\| \partial_x u_0 \|_{L^2(\partial \mathcal{C}_h \cap \partial B_h(0))} \leq \| \nabla u_0 \|_{L^2(\partial \mathcal{C}_h \cap \partial B_h(0))} \leq \sqrt{2}\tau e^{-\frac{\theta_M - \theta_m}{\tau}}.
\]

Therefore we obtain the estimates (2.9) and (2.10) as \( \tau \to \infty \), respectively.

Case 2: \( C_h \) is a conic corner in \( \mathbb{R}^3 \). Recall that \( C_h \) has the opening angle \( 2\theta_0 \), which is defined in (2.1). By virtue of (2.12), it yields that

\[
\int_{\mathcal{C}_h} e^{\tau (d + id^\perp) \cdot x} dx = I_1 + \int_0^{2\pi} d\varphi \int_0^{h_0} I_{R_2} \sin \theta d\theta,
\]

where

\[
I_1 = \int_0^{2\pi} \int_0^{h_0} \left( \frac{\Gamma(3)}{\tau^3 (d \cdot \hat{x} + id^\perp \cdot \hat{x})^3} \right) \sin \theta d\varphi d\theta, \quad I_{R_2} = \int_h^\infty r^2 e^{\tau (d \cdot \hat{x} + id^\perp \cdot \hat{x})} dr.
\]
By the integral mean value theorem, one can arrive at that

\[
I_1 = \frac{\Gamma(3)}{\tau^3} \int_0^{2\pi} \frac{1}{\tau^3} (d \cdot \hat{x}(\varphi, \theta \xi) + id^+ \cdot \hat{x}(\varphi, \theta \xi))^3 \, d\varphi \int_0^{\theta_0} \sin \theta_0 d\theta
= \frac{2\pi \Gamma(3)(1 - \cos \theta_0)}{\tau^3} \frac{1}{(d \cdot \hat{x}(\varphi, \theta \xi) + id^+ \cdot \hat{x}(\varphi, \theta \xi))^3}.
\]

For sufficient large \( \tau \), from (2.11), one has

\[
|I_{r_2}| = \left| \int_0^\infty r^2 e^{r\tau(d \cdot \hat{x} + id^+ \cdot \hat{x})} \, dr \right| \leq \frac{2}{\zeta \tau} e^{-\frac{1}{2} \zeta \chi h},
\]

which implies that

\[
\int_0^{2\pi} I_{r_2} \sin \theta d\theta \leq \int_0^{\theta_0} |I_{r_2}| \, d\varphi \leq \frac{4\pi \theta_0}{\zeta \tau} e^{-\frac{1}{2} \zeta \chi h}. \tag{2.18}
\]

From (2.17) and (2.18), using Cauchy-Schwarz inequality and (2.6), one can prove (2.7) for the case that \( C_h \) is a conic corner, where \( C_h = \sqrt{2\pi}(1 - \cos \theta_0) \) in (2.7).

For (2.8), from (2.12), it yields that

\[
\int_{C_h} |x|^a u_0 d\mathbf{x} = \frac{\Gamma(n + 3)}{\tau^{n+3}} \int_0^{2\pi} \frac{1}{\tau^3} \left( d \cdot \hat{x} + id^+ \cdot \hat{x} \right)^3 + I_R \sin \theta d\theta,
\]

where \( I_R = \int_0^\infty r^{a+3} e^{r\tau(d \cdot \hat{x} + id^+ \cdot \hat{x})} \, dr \). In view of (2.11) we obtain (2.8).

For (2.9), according to polar coordinate transformation and (2.6), one has

\[
\|u_0\|_{L^2(\partial C_h \cap \partial B_h(0))} = \left( \int_0^{\theta_0} \int_0^{2\pi} e^{2h\tau(d \cdot \hat{x})} \, d\varphi \, d\theta \right)^{\frac{1}{2}} \leq \left( \frac{2\pi \theta_0}{\zeta \tau} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \zeta \chi h},
\]

which can be used to derive (2.9) immediately. Similarly, using (2.6) and Cauchy-Schwarz inequality, one can show that (2.10) is valid for the case that \( C_h \) is a conic corner.

Finally, the case that \( C_h \) is a polyhedral corner in \( \mathbb{R}^3 \) can be proved in a similar manner; see also [2, Lemma 3.4].

The proof is complete. \( \square \)

A main auxiliary theorem is given as follows.

**Theorem 2.2.** Let \( (D; f) \in \mathcal{A} \) and \( C_h \) be a corner. Consider the following system of differential equations for \( u \in H^{2}_{loc}(C_h) \) and \( v \in H^{1}_{loc}(C_h) \):

\[
\begin{cases}
\Delta u + f(x, u) = 0 & \text{in } C_h, \\
\Delta v + \lambda v = 0 & \text{in } C_h, \\
v = u, \quad \partial_n u = \partial_n v & \text{on } \partial C_h \cap \partial B_h,
\end{cases}
\]

where \( \nu \) is the exterior unit normal vector to \( \partial D \). Then one has

\[
\lambda u(x_0) - f(x_0, u(x_0)) = 0, \tag{2.20}
\]

where \( x_0 \) is the apex of \( C_h \).

**Proof.** Since \( \Delta \) is invariant under rigid motions, without loss of generality, we assume that the apex \( x_0 \) of \( C_h \) coincides with the origin. By virtue of Green’s formula and (2.19), we have the following integral identity:

\[
\int_{C_h} (\lambda v - f(x, u)) u_0 \, d\mathbf{x} = \int_{\partial C_h \cap \partial B_h(0)} u_0 \partial_n (u - v) - (u - v) \partial_n u_0 \, d\mathbf{\sigma}, \tag{2.21}
\]

where \( u_0 \) is defined in (2.5). According to Sobolev's embedding theorem, we have \( u, v \in C^{3}(C_h) \) (\( \beta \in (0, 1] \) for \( n = 2 \) and \( \beta \in (0, 1/2] \) for \( n = 3 \)) since \( u, v \in H^{2}(C_h) \). By further
using the Hölder continuity of $f(x, \cdot)$ and the transmission conditions, we can derive the following expansions:

$$F(x) := \lambda v - f(x, u) = \lambda u(0) - f(0, u(0)) + \delta_v(x) + \delta f_{f(u)}(x),$$

$$|\delta f_{f(u)}(x)| \leq \|f(x, u)\|_{C^0(C_h)}|x|^\alpha, \quad |\delta_v(x)| \leq \|v\|_{C^0(C_h)}|x|^\alpha,$$  \hspace{1cm} (2.22)

where $\alpha = \min\{\beta, \gamma\} \in (0, 1)$ depending on the Hölder indices $\gamma$ and $\beta$.

Combining (2.22) with (2.21), one can show that

$$(\lambda u(0) - f(0, u(0))) \int_{C_h} u_0(x)dx = - \int_{C_h} (\delta_v(x) + \delta f_{f(u)})u_0dx$$

$$+ \int_{\partial C_h \cap \partial B_h(0)} u_0\partial_v(u - v) - (u - v)\partial_vu_0d\sigma.$$  \hspace{1cm} (2.23)

By virtue of (2.22) and (2.8), one has

$$\left| \int_{C_h} (\delta_v(x) + \delta f_{f(u)})u_0dx \right| \leq (\|v\|_{C^0(C_h)} + \|f\|_{C^0(C_h)}) \int_{C_h} |x|^\alpha |u_0|dx$$

$$\lesssim \tau^{-(\alpha + n)} + \frac{1}{\tau^{1+\frac{1}{2}\zeta \hbar}}, \quad n = 2, 3.$$  \hspace{1cm} (2.24)

According to the trace theorem and the fact that $u, v \in H^1(C_h)$, from (2.9) and (2.10), we can deduce that

$$\left| \int_{\partial C_h \cap \partial B_h(0)} u_0\partial_v(u - v)d\sigma \right| \leq \|u_0\|_{H^2(\partial C_h \cap \partial B_h(0))} \|\partial_v(u - v)\|_{H^{-\frac{1}{2}}(\partial C_h \cap \partial B_h(0))}$$

$$\leq C\|u_0\|_{H^1(\partial C_h \cap \partial B_h(0))}\|u - v\|_{H^1(C_h)}$$

$$\lesssim (2\tau^2 + 1)^{\frac{1}{2}} e^{-\frac{1}{2}\zeta \hbar \tau},$$  \hspace{1cm} (2.25)

$$\left| \int_{\partial C_h \cap \partial B_h(0)} (u - v)\partial_vu_0d\sigma \right| \leq \|\partial_vu_0\|_{L^2(\partial C_h \cap \partial B_h(0))}\|u - v\|_{L^2(\partial C_h \cap \partial B_h(0))}$$

$$\leq C\|u - v\|_{H^1(C_h)}\|\partial_vu_0\|_{L^2(\partial C_h \cap \partial B_h(0))},$$

$$\leq \sqrt{2C}\tau e^{-\frac{1}{2}\zeta \hbar \tau},$$  \hspace{1cm} (2.26)

as $\tau \to \infty$, where $C$ is a generic constant originating from the trace theorem.

Substituting (2.7), (2.8), (2.25) and (2.26) into (2.23), one has

$$\left( C_h \tau^{-n} + O \left( \tau^{-1} e^{-\frac{1}{2}\zeta \hbar \tau} \right) \right) |\lambda u(0) - f(0, u(0))| \lesssim \tau^{-(\alpha + n)} + (1 + \tau) e^{-\frac{1}{2}\zeta \hbar \tau} + \frac{1}{\tau^{1+\frac{1}{2}\zeta \hbar}},$$  \hspace{1cm} (2.27)

as $\tau \to \infty$. Multiplying $\tau^n$ on both sides of (2.27) and letting $\tau \to \infty$, then we can derive (2.20). We complete the proof of Theorem 2.2. \hfill \Box

We can show a local unique recovery result for the inverse problem (1.3). Before that, we introduce an admissibility condition for $\psi$.

**Assumption A.** We say that $\psi \in H^{1/2}(\partial \Omega)$ is admissible and write $\psi \in \mathcal{B}$ if the solution to (1.1) fulfills:

$$\lambda u(x_c) - f(x, u(x_c)) \neq 0, \quad \lambda u(x) - f(x, u(x)) \neq 0, \quad \forall x \in \Omega \setminus \overline{D},$$  \hspace{1cm} (2.28)

where $x_c \in \partial D$ satisfies $D \cap B_h(x_c) = C_h$ defined in (2.4) for a sufficient small $h \in \mathbb{R}_+$. It is emphasised that in Section 5, we shall show that Assumption A can hold in a certain generic scenario of practical interest.
Theorem 2.3. Let \((D_j; f_j) \in \mathcal{A}, j = 1, 2\), and suppose that
\[
\Lambda_{D_1, f_1}(\psi) = \Lambda_{D_2, f_2}(\psi) \quad \text{for a fixed } \psi \in \mathcal{B}.
\] (2.29)
Then \(D_1 \Delta D_2\) cannot possess a corner on \(\partial G\), where \(G\) is the connected component of \(\Omega \setminus (D_1 \cup D_2)\) that connects to \(\partial \Omega\).

Proof. By contradiction and also noting that \(\Delta\) is invariant under rigid motion, without loss of generality, we assume that there exists a corner \(C_h\) defined (2.4) satisfying \(D_2 \cap B_h(0) = C_h \Subset \Omega \setminus D_1\), where \(0 \in \partial D_2\). Let \(u_j\) be the wave field to the scattering problem (1.1) associated with \(D_j, j = 1, 2\). By virtue of (2.29), using the fact that \(u_j\) is real analytic in \(\Omega \setminus (D_1 \cup D_2)\), from unique continuation principle, it yields that
\[
\begin{cases}
\Delta u_2 + f_2(x, u_2) = 0 & \text{in } C_h, \\
\Delta u_1 + \lambda u_1 = 0 & \text{in } C_h, \\
u_2 = u_1, & \partial_n u_2 = \partial_n u_1 \quad \text{on } \partial C_h \setminus \partial B_h.
\end{cases}
\] (2.30)
Since \(f_2 \in \mathcal{A}\), according to Theorem 2.2, it arrives that \(\lambda u_2(0) - f_2(0, u(0)) = 0\), which contradicts to (2.28).

The proof is complete. \(\square\)

2.2. Global unique identifiability results. If we impose certain a-prior knowledge on the inclusion, we can establish the global uniqueness in determining the shape of the inclusion by a single measurement in the following two theorems by utilizing Theorem 2.3 and contradiction arguments.

Theorem 2.4. Let \((D; f) \in \mathcal{A}\), where \(D\) is a convex polygon in \(\mathbb{R}^2\) or a convex polyhedron in \(\mathbb{R}^3\). Then \(D\) is uniquely determined by a single boundary measurement \(\Lambda_{D, f}(\psi)\) with a fixed \(\psi \in \mathcal{B}\).

In the following we introduce an admissible class \(\mathcal{T}\) of corona shapes, which shall be used in Theorem 2.5. The schematic illustration of corona-shape scatterers is displayed in Figure 3.

Definition 2.1. Let \(\bar{D}\) be a convex bounded Lipschitz domain with a connected complement \(\mathbb{R}^3 \setminus \bar{D}\). If there exist finitely many strictly convex conic cones \(S_{x_j, \theta_j} (j = 1, 2, \ldots, \ell, \ell \in \mathbb{N})\) defined in (2.1) such that

![Figure 2. Schematic illustration of corona-shape scatterers.](image-url)
(a) the apex $x_j \in \mathbb{R}^3 \setminus \overline{D}$ and let $S_{x_j,\theta_j}^* = S_{x_j,\theta_j} \setminus \overline{D}$ respectively, where the apex $x_j$ belongs to the strictly convex bounded conic corner of $S_{x_j,\theta_j}^*$;

(b) $\partial S_{x_j,\theta_j}^* \cap \partial S_{x_j,\theta_j} \subset \partial \overline{D}$ and $\cap_{j=1}^\ell \partial S_{x_j,\theta_j}^* \setminus \partial S_{x_j,\theta_j} = \emptyset$;

then $D := \cup_{j=1}^\ell S_{x_j,\theta_j} \cup \overline{D}$ is said to belong to a class $T$ of corona shape.

**Theorem 2.5.** Suppose that $D_m,m = 1,2$ belong to the admissible class $T$ of corona shape, where

$$D_m = \cup_{j(m)=1}^{\ell(m)} C_{x_j(m),\theta_j(m)} \cup \overline{D}_m, \quad m=1,2.$$  \hspace{1cm} (2.31)

Consider the scattering problem (1.1) associated with $(D_m,f_m) \in \mathcal{A}, m = 1,2$. If the following conditions:

$$\Lambda_{D_1,f_1}(\psi) = \Lambda_{D_2,f_2}(\psi) \quad \text{for a fixed } \psi \in \mathcal{B}$$  \hspace{1cm} (2.32a)

$$\overline{D}_1 = \overline{D}_2,$$ \hspace{1cm} (2.32b)

$$\theta_{j(1)} = \theta_{j(2)} \quad \text{for } i(1) \in \{1,\ldots,\ell(1)\} \text{ and } j(2) \in \{1,\ldots,\ell(2)\} \text{ when } x_{i(1)} = x_{j(2)}, \quad (2.32c)$$

then $\ell(1) = \ell(2)$, $x_{i(1)} = x_{j(2)}$ and $\theta_{j(1)} = \theta_{j(2)}$, where $j(m) = 1,\ldots,\ell(m), m = 1,2$. Namely, one has $D_1 = D_2$.

**Proof.** We prove this theorem by contradiction. Suppose that $D_1 \neq D_2$, due to (2.32b) and (2.32c), without loss of generality one concludes that there exists a conic corner $S_{x_c,\theta_c}^h \subset D_2 \setminus \overline{D}_1$. Under (2.32a), by virtue of Theorem 2.3, we get the contradiction. \hfill $\square$

**Remark 2.6.** In Theorems 2.4 and 2.5, a single boundary measurement $\Lambda_{D,f}(\psi)$ can uniquely determine the inclusion $D$ under certain a-prior knowledge on $D$, where $\psi \in \mathcal{B}$. Namely, if $\psi \in \mathcal{B}$, the the admissible condition $\lambda u(x_c) - f(x,u(x_c)) \neq 0$ is fulfilled, where $x_c$ is an apex of $D$ and $u$ is the solution to (1.1) associated with $\psi$. The aforementioned admissible condition covers the corresponding admissible assumption for previous uniquely shape determination of a convex polygonal or polyhedral or corona-shape acoustic medium scatter $D$ by a single far-field measurement in inverse acoustic scattering problems (cf. [8, Theorem 4.1]) and [9, Theorems 5.2, 5.3 and Corollary 5.5], where the medium parameter $f(x,u)$ characterizing $D$ is linear with respect to the total wave field $u$, namely $f(x,u) = q u$ with $q \in L^\infty(D)$. Indeed, the admissible assumption in [8, 9] is $(q(x_c) - \lambda)u(x_c) \neq 0$, where $q$ is Hölder continuous near the neighborhood of $x_c$. On the other hand, the nonlinearities can leverage certain technical restrictions in the linear counterpart and can help identify the anomalies. For example, when $f(x,u) = \lambda u + q(x)u^2$, where $f$ has the same linear term as the background medium configuration, the admissible condition (2.28) turns out to be $q(x_c)u^2(x_c) \neq 0$. Therefore, for this specific form of $f(x,u)$ characterizing the anomalous inclusion $D$, although the linear term in $f(x,u)$ cannot contribute to the shape determination of $D$, the nonlinear term in $f(x,u)$ can help one to identify $D$ by a single boundary measurement $\Lambda_{D,f}(\psi)$ under the admissible condition $q(x_c)u^2(x_c) \neq 0$, where $D$ is a convex polygon or polyhedron or corona-shape inclusion with certain a-prior knowledge described in Theorem 2.5.

3. Determining both supports and contents of anomalous inclusions

In this section, we consider the inverse boundary problem (1.4) in determining both the support and its physical content of an anomalous inclusion by a minimal number of boundary measurements. Throughout the present section, we consider $a(x,u)$ in (1.1) of
the following form:

\[ a(x, u) = \left( \sum_{j=1}^{N} \lambda_j u^j - \lambda u \right) \chi_D + \lambda u \chi_{\Omega}, \quad x \in \Omega, \]  

(3.1)

where \( \lambda_j \in \mathbb{C} \). That is, the inhomogeneity inside \( D \) is given by

\[ f(x, u) = \sum_{j=1}^{N} \lambda_j u^j, \quad \lambda_j \in \mathbb{C}. \]  

(3.2)

Next, we shall show that an anomalous inclusion of the form \((D; f)\) with \( D \) being a convex polygon/polyhedron or an admissible corona shape and \( f \) of the form (3.2) can be uniquely determined uniquely determined by \( N \) properly chosen boundary measurements. To that end, we introduce the following admissibility condition on the boundary inputs.

**Assumption B.** Let \((D; f)\) be described above. We say that \( \psi_j \in H^{1/2}(\partial \Omega), j = 1, 2, \ldots, N, \) are admissible and write \( \psi_j \in \mathcal{H} \) if the corresponding solutions to (1.1), written as \( u_{\psi_j} \) in what follows, fulfil the following condition:

\[ \lambda u_{\psi_j}(x_c) - f(x_c, u_{\psi_j}(x_c)) \neq 0, \quad 1 \leq j \leq N; \quad \prod_{1 \leq i \leq j \leq N} \left( u_{\psi_j}(x_c) - u_{\psi_i}(x_c) \right) \neq 0, \]  

(3.3)

where \( x_c \in \partial D \) satisfies \( D \cap B_h(x_c) = C_h \) defined in (2.4) for a sufficient small \( h \in \mathbb{R}_+ \).

Similar to Assumption A, we shall show in Section 5 that Assumption B can hold in a certain generic scenario of practical interest.

**Theorem 3.1.** Let \((D; f) \in \mathcal{A}, \) where \( D \) is a convex polygon in \( \mathbb{R}^2 \) or a convex polyhedron in \( \mathbb{R}^3 \). Assume that \( f \) is of the form (3.2). Then both \( D \) and \( f \) are uniquely determined by \( N \) boundary measurements \( \Lambda_{D, f}(\psi_j) \) with \( \psi_j \in \mathcal{H}, j = 1, 2, \ldots, N. \)

Assume that \( D_m, m = 1, 2, \) are two an admissible corona shape as described in Definition 2.1, where \( D_m \) is defined by (2.31). Suppose that

\[ f_m(x, u) = \sum_{j=1}^{N_m} \lambda_{j,m} u^j, \quad \lambda_{j,m} \in \mathbb{C}, \quad N_m \in \mathbb{N}. \]  

(3.4)

If the assumption (2.32b), (2.32c) and

\[ \Lambda_{D_1, f_1}(\psi_j) = \Lambda_{D_2, f_2}(\psi_j) \quad \text{for a fixed } \psi \in \mathcal{H}, \quad j = 1, \ldots, \max\{N_1, N_2\}. \]  

(3.5)

are fulfilled, then \( D_1 = D_2, \) \( N := N_1 = N_2 \) and \( \lambda_{j,(1)} = \lambda_{j,(2)}, j = 1, \ldots, N. \)

In order to prove Theorem 3.1, we first derive an auxiliary lemma.

**Lemma 3.2.** Let \( f_m \in \mathcal{A}, m = 1, 2, \) and \( C_h \) be a corner. Consider the following system of differential equations for \( u \in H^2_{\text{loc}}(C_h) \) and \( v \in H^2_{\text{loc}}(C_h) \):

\[
\begin{cases}
\Delta u + f_1(x, u) = 0 & \text{in } C_h, \\
\Delta v + f_2(x, v) = 0 & \text{in } C_h, \\
u = v, \quad \partial_n u = \partial_n v & \text{on } \partial C_h \setminus \partial B_h,
\end{cases}
\]  

(3.6)

where \( v \) is the exterior unit normal vector to \( \partial D \). Then one has

\[ f_1(x_0, u(x_0)) - f_2(x_0, v(x_0)) = 0, \]  

(3.7)

where \( x_0 \) is the apex of \( C_h \).
Proof. The proof of this lemma is similar to that for Theorem 2.2. We sketch the argument in what follows. Let $u_0$ be given (2.5) by letting $\lambda = 0$. By virtue of (3.6) and Green’s formula, one has

$$
\int_{C_h} (f_1(x, u) - f_2(x, v)) u_0 \, dx = \int_{\partial C_h \setminus \partial B_h(x_0)} u_0 \partial_v (u - v) - (u - v) \partial_v u_0 \, d\sigma. \tag{3.8}
$$

By Sobolev’s embedding theorem, we have $u, v \in C^\beta(C_h)$ ($\beta \in (0, 1]$ for $n = 2$ and $\beta \in (0, 1/2]$ for $n = 3$) since $u, v \in H^2(C_h)$. By further using the Hölder continuity of $f_m(x, \cdot)$, $m = 1, 2$, it yields that following expansions:

$$
F(x) := f_1(x, u) - f_2(x, v) = f_1(x_0, u(x_0)) - f_2(x_0, v(x_0)) + \delta f_1(x, u) - f_2(x, v)(x),
$$

$$
|\delta f_1(x, u) - f_2(x, v)(x)| \leq \|F(x)\|_{C^\alpha(C_h)} |x|^\alpha, \tag{3.9}
$$

where $\alpha \in (0, 1)$ depending on the Hölder indices $\gamma$ and $\beta$.

In view of (3.8) and (3.9), by virtue of (2.7), we can follow the similar argument in the proof of Theorem 2.2 to prove this lemma.

Proof of Theorem 3.1. Let $(D_m; f_m)$, $m = 1, 2$, be two anomalous inclusions as described in the statement of the theorem. Assume that

$$
f_m(x, u) = \sum_{j=1}^{N_m} \lambda_j^{(m)} u^j, \quad \lambda_j^{(m)} \in \mathbb{C}, \quad m = 1, 2. \tag{3.10}
$$

By introducing zero coefficients if necessary, we can assume that $N_1 = N_2$ and set $N := N_1 = N_2$. We also assume that

$$
\Lambda_{D_1, f_1}(\psi_j) = \Lambda_{D_2, f_2}(\psi_j), \quad \psi_j \in \mathcal{H}, \quad j = 1, 2, \ldots, N. \tag{3.11}
$$

First, by following a similar argument to the proofs of Theorems 2.4 and 2.5, and using the first admissibility condition in (3.3), one can show that

$$
D_1 = D_2. \tag{3.12}
$$

Set $D = D_1 = D_2$ and let $C_h$ be a corner on $\partial D$ with the apex being $x_0$. By (3.11), we have

$$
\begin{cases}
\Delta u_{\psi_j}^{(1)} + f_1(x, u_{\psi_j}^{(1)}) = 0 & \text{in } C_h, \\
\Delta u_{\psi_j}^{(2)} + f_2(x, u_{\psi_j}^{(2)}) = 0 & \text{in } C_h, \\
u_{\psi_j}^{(1)} = u_{\psi_j}^{(2)}, \quad \partial_v u_{\psi_j}^{(1)} = \partial_v u_{\psi_j}^{(2)} & \text{on } \partial C_h \setminus \partial B_h,
\end{cases} \tag{3.13}
$$

for $j = 1, 2, \ldots, N$, where $u_{\psi_j}^{(m)}$ signifies the solution to (1.1) associated with $f_m$ and $\psi_j$, $m = 1, 2$ and $1 \leq j \leq N$. By Lemma 3.2, we readily have

$$
\sum_{j=1}^{N} \lambda_j^{(1)} [u_{\psi_j}^{(1)}(x_0)]^j - \sum_{j=1}^{N} \lambda_j^{(2)} [u_{\psi_j}^{(2)}(x_0)]^j = 0, \quad i = 1, 2, \ldots, N. \tag{3.14}
$$

On the other hand, by (3.13), we note that

$$
u_{\psi_j}^{(1)}(x_0) = u_{\psi_j}^{(2)}(x_0) := u_{\psi_j}(x_0), \quad i = 1, 2, \ldots, N. \tag{3.15}
$$

By combining (3.14) and (3.15), we readily have

$$
\sum_{j=1}^{N} (\lambda_j^{(1)} - \lambda_j^{(2)}) [u_{\psi_j}(x_0)]^j = 0, \tag{3.16}
$$

which together with the second admissibility condition in (3.3) readily yields that

$$
\lambda_j^{(1)} = \lambda_j^{(2)}, \quad j = 1, 2, \ldots, N.
$$
The unique determination for the support and its physical content of an admissible inclusion \((D; f)\) of corona shape as described in Definition 2.1 by \(N\) measurements can be proved in a similar way, where \(N\) is an a-prior parameter of \(f\) with the form (3.2).

The proof is complete. \(\square\)

4. Determining emdeded nonlinear anomalies

In this section we consider the determination of the shape and physical parameters of the embedded nonlinear anomalies by minimal measurements, which have a polygonal or polyhedral nest structure. We first introduce several definitions.

**Definition 4.1.** \(D\) is said to have a polygonal-nest or polyhedral-nest partition if there exist \(\Sigma_\ell, \ell = 1, 2, \ldots, N, N \in \mathbb{N}\), such that each \(\Sigma_\ell\) is an open convex simply-connected polygon or polyhedron and

\[
\Sigma_N \subset \Sigma_{N-1} \subset \cdots \subset \Sigma_2 \subset \Sigma_1 = D.
\] (4.1)

![Figure 3. Schematic illustration of the polygonal-nest structure.](image)

In the follow two definitions, we introduce an anomalous inclusion possessing a polygonal-nest or polyhedral-nest structure of the class \(A\) or \(B\), respectively.

**Definition 4.2.** Let \((D; f) \in A\) be an anomalous inclusion. It is said to possess a polygonal-nest or polyhedral-nest structure of the class \(A\) if the following conditions are fulfilled:

1. \(D\) has a polygonal-nest or polyhedral-nest partition as described in Definition 4.1;
2. each \(\Sigma_\ell\) is an anomalous inclusion such that

\[
f(x, u)|_{U_\ell} = \sum_{j=1}^{M_\ell} \lambda_j^{(\ell)} u^j, \quad \lambda_j \in \mathbb{C}, \quad U_\ell := \Sigma_\ell \setminus \Sigma_{\ell+1}, \quad M_\ell \in \mathbb{N}, \quad 1 \leq \ell \leq N,
\] (4.2)

where for any \(\ell \in \{1, \ldots, N - 1\}\), it holds that

\[
\sum_{j=1}^{M_\ell} \lambda_j^{(\ell)} t^j \neq \sum_{j=1}^{M_{\ell+1}} \lambda_j^{(\ell+1)} t^j, \quad t \in \mathbb{C}.
\]

**Definition 4.3.** Let \((D; f) \in A\) be an anomalous inclusion. It is said to possess a polygonal-nest or polyhedral-nest structure of the class \(B\) if the following conditions are fulfilled:
(1) $D$ has a polygonal-nest or polyhedral-nest partition as described in Definition 4.1;  
(2) each $\Sigma_\ell$ is an anomalous inclusion such that
\[
 f(x, u) \bigg|_{U_\ell} = \lambda_\ell u, \quad \lambda_\ell \in \mathbb{C}, \quad U_\ell := \Sigma_\ell \setminus \Sigma_{\ell+1}, \quad 1 \leq \ell \leq N - 1,
\]
where
\[
 \lambda_\ell \neq \lambda_{\ell+1}, \quad \ell = 1, \ldots, N - 2.
\]  

We shall give the unique shape and physical parameter determination for two admissible classes introduced in Theorems 4.1 and 4.2 by minimal boundary measurements under the following admissible assumption. 

**Assumption C.** Let $(D; f)$ be described in Definition 4.2, where $f$ has the form (4.2). We say that $\psi_j^{(\ell)} \in H^{1/2}(\partial \Omega)$, $j = 1, 2, \ldots, M_\ell$, $\ell = 1, \ldots, N$, are admissible and write $\psi_j^{(\ell)} \in \mathcal{C}$ if the corresponding solutions to (1.1), written as $u_{\psi_j^{(\ell)}}$ in what follows, fulfill the following condition:
\[
 \lambda u_{\psi_j^{(\ell)}}(y_c) - f(y_c, u_{\psi_j^{(\ell)}}(y_c)) \bigg|_{U_1} \neq 0, \quad 1 \leq j \leq M_1, \quad \forall y_c \in \mathcal{V}(\partial \Sigma_1),
\]
\[
 \sum_{m=1}^{M_\ell} \lambda_{\ell}^{(\ell)} u_{m}^{(\ell)}(x_c) - f(x_c, u_{\psi_j^{(\ell)}}(x_c)) \bigg|_{U_\ell} \neq 0, \quad 1 \leq j \leq M_\ell, \quad \forall x_c \in \mathcal{V}(\partial \Sigma_\ell), \quad \ell = 2, \ldots, N - 1,
\]
\[
 \prod_{1 \leq \ell \leq M_\ell} \left( u_{\psi_j^{(\ell)}}(x_c) - u_{\psi_j^{(\ell)}}(x_c) \right) \neq 0, \quad \forall x_c \in \mathcal{V}(\partial \Sigma_\ell), \quad \ell = 1, \ldots, N,
\]  

where $\mathcal{V}(\partial \Sigma_\ell)$ is the vertex set of $\Sigma_\ell$, $\ell = 1, \ldots, N$.

Similar to Assumptions A and B before, we shall show in Section 5 that Assumption C can hold in a certain generic scenario of practical interest.

We are now in a position to present the main theorem of this section. 

**Theorem 4.1.** Let $(D; f) \in \mathcal{A}$, where $D$ has a polygonal-nest structure in $\mathbb{R}^2$ or polyhedral-nest structure in $\mathbb{R}^3$ of the class $A$. Assume that $f$ is of the form (4.2). Then both $D$ and $f$ are uniquely determined by $\sum_{\ell=1}^{N} M_\ell$ boundary measurements $\Lambda_{D,f}(\psi_j^{(\ell)})$ with $\psi_j^{(\ell)} \in \mathcal{C}$, $j = 1, 2, \ldots, M_\ell$ and $\ell = 1, \ldots, N$.

**Proof.** Assume that $(D_m; f_m)$ $(m = 1, 2)$ are two anomalous inclusions as described in the statement of the theorem. Namely,
\[
 \Sigma_{N_1, (m)} \subseteq \Sigma_{N_1-1, (m)} \subseteq \cdots \subseteq \Sigma_{2, (m)} \subseteq \Sigma_{1, (m)} = D_m,
\]
\[
 f_m(x, u) \bigg|_{U_{\ell, (m)}} = \sum_{j=1}^{M_{\ell, (m)}} \lambda_j^{(\ell, (m))} u^j, \quad \lambda_j^{(\ell, (m))} \in \mathbb{C}, \quad M_{\ell, (m)} \in \mathbb{N}, \quad 1 \leq \ell \leq N_m,
\]  

where each $\Sigma_{\ell, (m)}$ is an open convex simply-connected polygon or polyhedron, and $U_{\ell, (m)} := \Sigma_{\ell, (m)} \setminus \Sigma_{\ell+1, (m)}$. Without loss of generality we assume that $N_1 \leq N_2$. By introducing zero coefficients if necessary, in view of (4.6), one can readily know that
\[
 f_m(x, u) \bigg|_{U_{\ell, (m)}} = \sum_{j=1}^{M_\ell} \lambda_j^{(\ell, (m))} u^j, \quad \lambda_j^{(\ell, (m))} \in \mathbb{C}, \quad 1 \leq \ell \leq N_1,
\]
where $M_\ell = \max\{M_{\ell, (1)}, M_{\ell, (2)}\}$. 

Suppose that
\[ \Lambda_{D_1,i_1}(\psi_j^{(\ell)}) = \Lambda_{D_2,i_2}(\psi_j^{(\ell)}) \] for \( \psi_j^{(\ell)} \in \mathcal{C}, \) \( j = 1, \ldots, M_\ell, \) \( \ell = 1, \ldots, N_1. \) (4.8)

In the following we prove this theorem by mathematical induction. Under the assumption (4.8), according to Theorem 3.1, it holds \( \partial D_1 = \partial D_2, \) which implies that \( \partial \Sigma_1,(-1) = \partial \Sigma_1,(-2). \) Furthermore, from Theorem 3.1, one can claim that
\[ \lambda_{j,(1)}^{(1)} = \lambda_{j,(2)}^{(1)}, \quad j = 1, 2, \ldots, M_1. \]

Let \( u^{(m)}_{\psi_j^{(\ell)}} \) be the solution of (1.1) associated with \( (D_m; f_m) \) and \( \psi_j^{(\ell)}. \) Hence by unique continuation, one has
\[ u_j^{(1)}|_{U_1} = u_j^{(2)}|_{U_1} \quad \text{in} \ U_1 = \Sigma_1 \setminus \Sigma_2. \]

Suppose that there exists an index \( n_* \in \mathbb{N}\setminus\{1\} \) such that
\[ \partial \Sigma_\ell := \partial \Sigma_\ell,(1) = \partial \Sigma_\ell,(2), \quad \lambda_{j,(1)}^{(\ell)} = \lambda_{j,(2)}^{(\ell)}, \quad j = 1, \ldots, M_\ell, \quad \ell = 2, \ldots, n_* - 1. \] (4.9)

Therefore we can recursively prove that
\[
\begin{align*}
& u_j^{(1)}|_{U_\ell} = u_j^{(2)}|_{U_\ell} \quad \text{in} \ U_\ell = \Sigma_\ell \setminus \Sigma_{\ell+1}, \quad j = 1, 2, \ldots, M_{\ell+1}, \quad \ell = 1, 2, \ldots, n_* - 2, \\
& u_j^{(1)}|_{U_{n_*-1,(m)}} = u_j^{(2)}|_{U_{n_*-1,(m)}} \quad \text{in} \ U_{n_*-1,(m)} = \Sigma_{n_*-1} \setminus \Sigma_{n_*,(m)}, \quad j = 1, \ldots, M_{n_*}, \quad m = 1, 2, 
\end{align*}
\]
by using (4.8).

Assume that \( \partial \Sigma_{n_*,(1)} \neq \partial \Sigma_{n_*,(2)}. \) By the convexity of \( \Sigma_{n_*,(m)} (m = 1, 2), \) without loss of generality, we can suppose that there exists a polyhedral corner \( \mathcal{C}_h \) with the apex \( x_0, \) satisfying \( \mathcal{C}_h \subset \Sigma_{n_*,(1)} \setminus \Sigma_{n_*,(2)}. \) According to (4.9) and (4.10), it yields that
\[
\begin{align*}
& \Delta u_j^{(1)}(n_*) + \sum_{j=1}^{M_{n_*}} \lambda_{j,(1)}^{(n_*)} u_j^{(1)}(n_*) = 0 \quad \text{in} \ \mathcal{C}_h, \\
& \Delta u_j^{(2)}(n_*) + \sum_{j=1}^{M_{n_*}} \lambda_{j,(1)}^{(n_*)} u_j^{(2)}(n_*) = 0 \quad \text{in} \ \mathcal{C}_h, \\
& u_j^{(1)}(n_*) = u_j^{(2)}(n_*) = 0 \quad \text{on} \ \partial \mathcal{C}_h \setminus \partial B_h, 
\end{align*}
\]
where \( u_j^{(m)}(n_*) \in H^2(\mathcal{C}_h) \) by noting interior elliptic regularity. Using Lemma 3.2, one has
\[
\sum_{j=1}^{M_{n_*}} \lambda_{j,(1)}^{(n_*)} u_j^{(1)}(n_*) = 0
\]
which contradicts to the second admissible condition in (4.5). Therefore, it is ready to know that \( \partial \Sigma_{n_*} := \partial \Sigma_{n_*,(1)} = \partial \Sigma_{n_*,(2)} \).

Let \( \mathcal{C}_h \) be a corner on \( \partial \Sigma_{n_*} = \partial \Sigma_{n_*,(1)} = \partial \Sigma_{n_*,(2)} \) with the apex \( x_0. \) According to (4.10), it yields that
\[
\begin{align*}
& \Delta u_j^{(1)}(n_*) + \sum_{j=1}^{M_{n_*}} \lambda_{j,(1)}^{(n_*)} u_j^{(1)}(n_*) = 0 \quad \text{in} \ \mathcal{C}_h, \\
& \Delta u_j^{(2)}(n_*) + \sum_{j=1}^{M_{n_*}} \lambda_{j,(2)}^{(n_*)} u_j^{(2)}(n_*) = 0 \quad \text{in} \ \mathcal{C}_h, \\
& u_j^{(1)}(n_*) = u_j^{(2)}(n_*) = 0 \quad \text{on} \ \partial \mathcal{C}_h \setminus \partial B_h, 
\end{align*}
\]
for \( i = 1, 2, \ldots, M_{n*} \), where \( u_{\psi_i(n*)}^{(m)} \) signifies the solution to (1.1) associated with \( f_m \) and \( \psi_{i}^{(n*)} \), and \( 1 \leq i \leq M_{n*} \). Using Lemma 3.2, one readily has

\[
\sum_{j=1}^{M_{n*}} \left( \lambda_{j,1}^{(n*)} - \lambda_{j,2}^{(n*)} \right) \left[ u_{\psi_i^{(n*)}}^{(1)}(x_0) \right]^{j} = 0, \quad i = 1, 2, \ldots, M_{n*}.  \tag{4.12}
\]

by noting the transmission condition in (4.11). By virtue of the third admissible condition in (4.5) together with (4.12), it arrives that

\[
\lambda_{j,1}^{(n*)} = \lambda_{j,2}^{(n*)}, \quad j = 1, 2, \ldots, M_{n*}.
\]

Moreover, by (4.8) one conclude that

\[
u_{\psi_i^{(n*)+1}}^{(1)}(U_{n*+1,m}) = u_{\psi_i^{(n*)+1}}^{(2)}(U_{n*+1,m}) \quad \text{in} \quad U_{n*,(m)} = \Sigma_{n*} \setminus \Sigma_{n*+1,(m)}, \quad j = 1, \ldots, M_{n*+1, m = 1, 2}.
\]

We can prove \( N_1 = N_2 \) by using the contradiction. Indeed, we assume that \( N_1 < N_2 \). Therefore, there exits a corner point \( x_c \in \partial \Sigma_{N_1+1,(2)} \) lying inside of \( \Sigma_{N_1,(2)} \). From Lemma 3.2, we can prove that

\[
\sum_{j=1}^{M_{N_1}} \lambda_{j}^{(N_1)} \left( u_{\psi_i^{(N_1)}}^{(2)}(x_c) \right) - f(x_c, u_{\psi_i^{(N_1)}}^{(2)}(x_c)) \bigg|_{U_{N_1+1,(2)}} = 0, \quad 1 \leq i \leq M_{N_1},
\]

which contradicts to the second admissible condition of (4.5).

The proof is complete. \( \square \)

In the next theorem we prove that an anomalous inclusion possessing a polygonal-nest structure in \( \mathbb{R}^2 \) or polyhedral-nest structure in \( \mathbb{R}^3 \) of the class \( \mathcal{B} \) can be uniquely determined by a single boundary measurement fulfilling Assumption D introduced below.

**Assumption D.** Let \((D; f)\) be described in Definition 4.3, where \( f \) has the form (4.3). We say that \( \psi \in H^{1/2}(\partial \Omega) \) is admissible and write \( \psi \in \mathcal{D} \) if the corresponding solutions to (1.1), written as \( u_{\psi} \), in what follows, fulfill the following condition:

\[
\lambda u_{\psi}(y_c) - f(y_c, u_{\psi}(y_c)) \bigg|_{U_{\ell}} \neq 0, \quad \forall y_c \in \mathcal{V}(\partial \Sigma_1),
\]

\[
\lambda_{\ell-1} u_{\psi}(x_c) - f(x_c, u_{\psi}(x_c)) \bigg|_{U_{\ell}} \neq 0, \quad \forall x_c \in \mathcal{V}(\Sigma_{\ell}), \quad \ell = 2, \ldots, N - 1,
\]

\[
\lambda u_{\psi}(x_c) - f_N(x_c, u_{\psi}(x_c)) \neq 0, \quad \forall x_c \in \mathcal{V}(\partial \Sigma_N),
\]

\[
u_{\psi}(x_c) \neq 0, \quad \forall x_c \in \mathcal{V}(\partial \Sigma_{\ell}), \quad \ell = 1, \ldots, N,
\]

where \( \mathcal{V}(\partial \Sigma_{\ell}) \) is the vertex set of \( \Sigma_{\ell}, \ell = 1, \ldots, N \).

In Section 5, we shall show that in a certain generic scenario of practical interest that Assumption D can hold.

**Theorem 4.2.** Let \((D; f) \in \mathcal{A} \), where \( D \) has a polygonal-nest structure in \( \mathbb{R}^2 \) or polyhedral-nest structure in \( \mathbb{R}^3 \) of the class \( \mathcal{B} \). Assume that \( f \) is of the form (4.3). Then both \( D \) and the physical parameters of \( f \lambda_\ell \) \( (\ell = 1, \ldots, N - 1) \) are uniquely determined by a single boundary measurement \( \Lambda_{D,f}(\psi) \) with \( \psi \in \mathcal{D} \).

**Proof.** We sketch the proof of this theorem by modifying necessary parts of the proof of Theorem 4.1. By contradiction, suppose that there exist two anomalous inclusions \((D_m; f_m)\) described by the statement of this theorem such that

\[
\Lambda_{D_1,f_1}(\psi) = \Lambda_{D_2,f_2}(\psi) \quad \text{for} \quad \psi \in \mathcal{C}, \tag{4.14}
\]

where \( \mathcal{C} \) is the class of admissible anomalies.
where
\[ \Sigma_{N_1(m)} \subseteq \Sigma_{N_1-1(m)} \subseteq \cdots \subseteq \Sigma_{2(m)} \subseteq \Sigma_{1(m)} = D_m, \]
\[ f_m(x, u)|_{U_\ell,m} = \lambda_{\ell,m}u, \quad \lambda_{\ell,m} \in \mathbb{C}, \quad 1 \leq \ell \leq N_m. \tag{4.15} \]
Here each \( \Sigma_{\ell,m} \) is an open convex simply-connected polygon or polyhedron, and \( U_{\ell,m} := \Sigma_{\ell,m} \setminus \Sigma_{\ell+1,m} \).

Using the first admissible condition in (4.13), under (4.14), from Theorem 3.1, we can obtain that \( \partial \Sigma_1 = \partial \Sigma_2 \). Once the unique shape determination of \( \partial \Sigma_1 \) is derived, by using (3.2) and noting the fourth admissible condition in (4.13), we can prove that \( \lambda_1(1) = \lambda_1(2) \).

Therefore, one has \( u_1^{(1)}|_{U_1} = u_2^{(2)}|_{U_1} \) in \( U_1 = \Sigma_1 \setminus \Sigma_2 \) by unique continuation principle, where \( u_\psi^{(m)} \) is the solution to (1.1) associated with \((D_m; f_m)\) and \( \psi, m = 1, 2 \).

Following a similar argument in the proof of Theorem 4.1, by virtue of Lemma 3.2 and (4.13) we can prove that
\[ N_1 = N_2 := N, \quad \Sigma_{\ell,(1)} = \Sigma_{\ell,(2)}, \quad \lambda_{\ell,(1)} = \lambda_{\ell,(2)}, \quad \ell = 1, \ldots, N. \]

The proof is complete. \( \square \)

5. Discussion on admissibility conditions

In this section we shall show that the technical Assumptions A, B, C and D introduced in the previous sections can be fulfilled under generic scenarios.

Recall that \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^n \) \((n = 2, 3)\), and \( D \) is a bounded Lipschitz domain such that \( D \Subset \Omega \) and \( \Omega \setminus \overline{D} \) is connected. For illustrative purpose, we consider some specific nonlinear Helmholtz equations that arise in the time-harmonic wave scattering theory; see also our discussion in Introduction. It is emphasised that one can derive similar results in other setups by following similar arguments as discussed in what follows.

Let \( u \in H^1(\Omega) \) be the solution to
\[ \begin{cases} \Delta u + a(x, u) = 0 & \text{in } \Omega, \\ u = \psi(\varepsilon, k, d) & \text{on } \partial \Omega, \end{cases} \tag{5.1} \]
where \( a(x, u) = (f(x, u) - k^2 u)\chi_D + k^2 u \) with \( k \in \mathbb{R}_+ \cup \{0\} \) and \( \psi(x; \varepsilon, k, d) := \varepsilon e^{ikx \cdot d} \) with \( d \in \mathbb{R}^{n-1} \) and \( \varepsilon \in \mathbb{R}_+ \) with \( \varepsilon \ll 1 \).

In the following proposition, when \( f(x, u) \) takes the form (3.2), we shall prove that the solution \( u \) to (5.1) can be decomposed as \( u = \psi(\varepsilon, k, d) + v \), where \( v \) can be viewed as a small perturbation.

**Proposition 5.1.** Consider the semilinear elliptic equation (5.1), where
\[ f(x, u) = \sum_{j=1}^N \lambda_j u^j, \quad \lambda_j \in \mathbb{C}. \]
Suppose that \( u \in H^1(\Omega) \) is the solution to (5.1), which satisfies \( u = \psi(x; \varepsilon, k, d) + v \). Then \( v \in H^1(\Omega) \) fulfills
\[ \begin{cases} \Delta v + k^2 \varepsilon \chi_{\Omega \setminus D} = (k^2 \psi - \sum_{j=1}^N \lambda_j u^j)\chi_D & \text{in } \Omega, \\ v = 0 & \text{on } \partial \Omega. \end{cases} \tag{5.2} \]
Furthermore, if
\[ k = \mathcal{O}(\varepsilon^{\zeta_1}) \quad \text{and} \quad |\lambda_1| = \mathcal{O}(\varepsilon^{\zeta_2}), \quad \zeta_j \in \mathbb{R}_+, \quad j = 1, 2, \tag{5.3} \]
then it holds that
\[ \|v\|_{H^1(\Omega)} = o(\varepsilon), \]  
where \( \lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0. \)

**Proof.** Since \( \Delta \psi(x; \varepsilon, k, d) + k^2 \psi(x; \varepsilon, k, d) = 0 \) in \( \Omega \), it is ready to see that \( v \) fulfills (5.2). According to Proposition 5.6, one has
\[ \|u\|_{H^1(\Omega)} \leq C \|\psi\|_{H^{\frac{1}{2}}(\Omega)} = O(\varepsilon). \]  
By elliptic regularity of (5.2), using (5.5) and (5.3), it yields that
\[ \|v\|_{H^1(\Omega)} \leq k^2 \|\psi\|_{H^1(\Omega)} + \sum_{j=1}^{N} |\lambda_j| \|u\|^j_{H^1(\Omega)} = o(\varepsilon) \]
which completes the proof of this proposition. \( \square \)

In the following we show that the admissible conditions introduced in previous sections can be fulfilled for the case that the nonlinear anomaly \( f(x, u) \) is characterized by (5.1). Under this situation, Assumption A can be implied by Assumption B directly. Hence we first consider Assumption B in the proposition below.

**Proposition 5.2.** Suppose that \( f(x, u) \) in (5.1) takes the form (3.2) and the set \( \{\varepsilon_1, \ldots, \varepsilon_N\} \) is pairwise different, where \( \varepsilon_j \in \mathbb{R}_+ \) with \( \varepsilon_j \ll 1 \) and \( N \) is an index related to \( f(x, u) \). Let \( u_j \) be the solution to (5.1) associated with \( \psi(x; \varepsilon_j, k, d) = \varepsilon_j e^{ikx - d} \), then Assumption B is fulfilled under the condition (5.3) and \( k^2 \neq \lambda_1. \)

**Proof.** Under the assumption (5.3), for \( x_\varepsilon \in \partial D \), from Proposition 5.1, it can be derive that
\[ k^2 u_j(x_\varepsilon) - f(x_\varepsilon, u_j(x_\varepsilon)) = (k^2 - \lambda_1) \psi(x_\varepsilon; \varepsilon_j, k, d) + k^2 v_j + R, \]  
where \( u_j = \psi(x_\varepsilon; \varepsilon_j, k, d) + v_j \) and \( R = -\sum_{\ell=2}^{N} \lambda_j (\psi(x_\varepsilon; \varepsilon_j, k, d) + v)^\ell. \) Here \( v_j \) satisfies (5.2) and (5.4). Therefore, from (5.6), one can readily know that
\[ k^2 u_j(x_\varepsilon) - f(x_\varepsilon, u_j(x_\varepsilon)) = (k^2 - \lambda_1) \psi(x_\varepsilon; \varepsilon_j, k, d) + o(\varepsilon) \neq 0 \]
by noting \( k^2 \neq \lambda_1 \) and \( \psi(x_\varepsilon; \varepsilon_j, k, d) \neq 0. \)

Since \( \varepsilon_i \neq \varepsilon_j \) for any \( i \neq j \) satisfying \( i, j \in \{1, \ldots, N\} \), from Proposition 5.1, we can obtain that
\[ \prod_{1 \leq i \leq j \leq N} (u_j(x_\varepsilon) - u_i(x_\varepsilon)) = \prod_{1 \leq i \leq j \leq N} \left[(\varepsilon_j - \varepsilon_i) e^{ikx - d} + v_j(x_\varepsilon) - v_i(x_\varepsilon)\right] \neq 0 \]
by virtue of \( \|v_i\|_{H^1(\Omega)} = o(\varepsilon_i) \) and \( \|v_j\|_{H^1(\Omega)} = o(\varepsilon_j). \)

Therefore, the two admissible conditions in Assumption B are fulfilled. \( \square \)

We can adopt a similar argument for proving Proposition 5.1 to validate the following proposition.

**Proposition 5.3.** Consider the semilinear elliptic equation (5.1), where the anomalous inclusion \( D \) possesses a polygonal-nest or polyhedral-nest structure of the class \( \mathcal{A} \) described by Definition 4.2 satisfying
\[ \Sigma_N \Subset \Sigma_{N-1} \Subset \cdots \Subset \Sigma_2 \Subset \Sigma_1 = D. \]  
and \( f(x, u) \) is characterized by (4.2). Here \( \Sigma_i \) is a polygon in \( \mathbb{R}^2 \) or a polyhedron in \( \mathbb{R}^3, \) \( \ell \in \{1, \ldots, N\}. \) Suppose that \( u \in H^1(\Omega) \) is the solution to (5.1), which satisfies \( u = \)
\( \psi(x; \varepsilon, k, d) + v. \) Then \( v \in H^1(\Omega) \) fulfills
\[
\begin{cases}
\Delta v + k^2 v \chi_{D} = k^2 \psi \chi_{D} - N \sum_{\ell = 1}^{M_\varepsilon} M_{j_\ell} \lambda_\ell^{(\varepsilon)} u^j \chi_{U_\ell} - N \sum_{j = 1}^{M_N} \lambda_j^{(N)} u^j \chi_{\Sigma_N}, & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(5.8)

where \( U_\ell := \Sigma_{\ell} \setminus \Sigma_{\ell + 1}, M_\varepsilon \in \mathbb{N}, \) and \( 1 \leq \ell \leq N - 1. \) Furthermore, if
\[
k = \mathcal{O}(\varepsilon^{\zeta_\varepsilon}) \quad \text{and} \quad |\lambda_\ell^{(\varepsilon)}| = \mathcal{O}(\varepsilon^{\zeta_\varepsilon}), \quad \ell \in \{1, \ldots, N\},
\]
(5.9)

where \( \zeta_j \in \mathbb{R}_+ \) for \( j \in \{0, 1, \ldots, N\}, \) then it yields that
\[
\|v\|_{H^1(\Omega)} = o(\varepsilon).
\]
(5.10)

Similarly, consider the semilinear elliptic equation (5.1), where the anomalous inclusion \( D \) possesses a polygonal-nest or polyhedral-nest structure of the class \( B \) described by Definition 4.3 satisfying
\[
\Sigma_N \subseteq \Sigma_{N-1} \subseteq \ldots \subseteq \Sigma_2 \subseteq \Sigma_1 = D.
\]

and \( f(x, u) \) is characterized by (4.3). Suppose that \( u \in H^1(\Omega) \) is the solution to (5.1), which satisfies \( u = \psi(x; \varepsilon, k, d) + v. \) Then \( v \in H^1(\Omega) \) fulfills
\[
\begin{cases}
\Delta v + k^2 v \chi_{\Omega, D} = k^2 \psi \chi_{\Omega, D} - \sum_{\ell = 1}^{N-1} \lambda_\ell^{(N)} u^j \chi_{U_\ell} - f_N(x, u) \chi_{\Sigma_N}, & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(5.11)

where \( U_\ell := \Sigma_{\ell} \setminus \Sigma_{\ell + 1}, 1 \leq \ell \leq N - 1, \) and
\[
f_N(x, u) = \sum_{j = 1}^{N} \lambda_j^{(N)} u^j, \quad \lambda_j^{(N)} \in \mathbb{C}.
\]

Furthermore, if
\[
k = \mathcal{O}(\varepsilon^{\zeta}), \quad |\lambda_\ell| = \mathcal{O}(\varepsilon^{\zeta}), \quad \ell \in \{1, \ldots, N - 1\}, \quad \text{and} \quad |\lambda_1^{(N)}| = \mathcal{O}(\varepsilon^{\zeta_N}),
\]
(5.12)

where \( \zeta_j \in \mathbb{R}_+ \) for \( j \in \{0, 1, \ldots, N\}, \) then it yields that
\[
\|v\|_{H^1(\Omega)} = o(\varepsilon).
\]

Assumption C can be satisfied under generic conditions introduced in the following proposition.

**Proposition 5.4.** Assume that an anomalous inclusion \( D \subset \Omega \) possesses a polygonal-nest or polyhedral-nest structure of the class \( A \) described by Definition 4.2, namely (5.7) holds. Suppose that \( f(x, u) \) in (5.1) is characterized by (4.2) and the set \( \mathcal{E} := \cup_{\ell = 1}^{N} \{ \varepsilon_1^{(\ell)}, \ldots, \varepsilon_{M_\ell}^{(\ell)} \} \) is pairwise different, where \( \varepsilon_\ell^{(\varepsilon)} \in \mathbb{R}_+ \) with \( \varepsilon_\ell^{(\varepsilon)} \ll 1. \) Let \( u^{(\varepsilon)} \) be the solution to (5.1) associated with \( \psi^{(\varepsilon)} := \psi(x; \varepsilon^{(\varepsilon)}, k, d) = \varepsilon_\ell^{(\varepsilon)} e^{ix \cdot d}, \) then Assumption C is fulfilled under the condition (5.9) and
\[
k^2 \neq \lambda^{(1)}_1 \) \text{ and } \lambda^{(\ell-1)}_1 \neq \lambda^{(\ell)}_1 \text{ with } \ell \in \{2, \ldots, N\}.
\]
(5.13)

**Proof.** Let \( v^{(\varepsilon)} := u^{(\varepsilon)} - \psi^{(\varepsilon)} \). According to Proposition 5.3, \( v^{(\varepsilon)} \) fulfills
\[
\begin{cases}
\Delta v^{(\varepsilon)} + k^2 v^{(\varepsilon)} \chi_{\Omega, D} = k^2 \psi^{(\varepsilon)} \chi_{D} - \sum_{\ell = 1}^{M_\varepsilon} \sum_{m = 1}^{M_\ell} \lambda_\ell^{(\varepsilon)} u^j \chi_{U_\ell} - \sum_{m = 1}^{M_N} \lambda_m^{(N)} u^j \chi_{\Sigma_N}, & \text{in } \Omega, \\
v^{(\varepsilon)} = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( U_\ell := \Sigma_\ell \setminus \Sigma_{\ell+1} \) and \( 1 \leq \ell \leq N - 1 \). Under (5.9) one has

\[
\left\| v_{\psi_j}^{(\ell)} \right\|_{H^1(\Omega)} = o(\varepsilon_j^{(\ell)}),
\]

(5.14)

Recall that \( V(\partial \Sigma_\ell) \) is the vertex set of \( \Sigma_\ell \), \( \ell = 1, \ldots, N \). For the three conditions in Assumption C, it can be directly obtained that

\[
k^2 u_{\psi_j}^{(\ell)}(y_c) - f(y_c, u_{\psi_j}^{(\ell)}(y_c))|_{U_1} = \left( k^2 - \lambda_1^{(1)} \right) \psi_j^{(1)} + \left( k^2 - \lambda_1^{(1)} \right) v_{\psi_j}^{(\ell)}(y_c)
- \sum_{m=2}^{M_\ell} \lambda_m^{(\ell-1)} u_{\psi_j}^{(\ell)}(y_c) = \left( \lambda_1^{(\ell-1)} - \lambda_1^{(1)} \right) \psi_j^{(\ell)} + o(\varepsilon_j^{(\ell)}) \neq 0, \quad 1 \leq j \leq M_1, \quad \forall y_c \in V(\partial \Sigma_1),
\]

\[
\sum_{m=1}^{M_\ell} \lambda_m^{(\ell-1)} u_{\psi_j}^{(\ell)}(x_c) = f(x_c, u_{\psi_j}^{(\ell)}(x_c))|_{U_\ell} = \left( \lambda_1^{(\ell-1)} - \lambda_1^{(1)} \right) \psi_j^{(\ell)} + o(\varepsilon_j^{(\ell)})
\times v_{\psi_j}^{(\ell)}(x_c) + \sum_{m=2}^{M_\ell+1} \sum_{m=2}^{M_\ell} \lambda_m^{(\ell-1)} u_{\psi_j}^{(\ell)}(x_c) - \sum_{m=2}^{M_\ell+1} \lambda_m^{(\ell)} u_{\psi_j}^{(\ell)}(x_c) = \left( \lambda_1^{(\ell-1)} - \lambda_1^{(1)} \right) \psi_j^{(\ell)}(x_c) + o(\varepsilon_j^{(\ell)})
\neq 0, \quad 1 \leq j \leq M_\ell, \quad \forall x_c \in V(\partial \Sigma_\ell), \quad \ell = 2, \ldots, N - 1,
\]

\[
\prod_{1 \leq i \leq j \leq M_\ell} \left( u_{\psi_j}^{(\ell)}(x_c) - u_{\psi_i}^{(\ell)}(x_c) \right) = \prod_{1 \leq i \leq j \leq N} \left[ (\varepsilon_j^{(\ell)} - \varepsilon_i^{(\ell)}) e^{ikx \cdot d} + v_{\psi_j}^{(\ell)}(x_c) - v_{\psi_i}^{(\ell)}(x_c) \right] 
\neq 0, \quad \forall x_c \in V(\partial \Sigma_\ell), \quad \ell = 1, \ldots, N,
\]

by using (5.13) and (5.14). □

Using Proposition 5.3 and following a similar argument for Proposition 5.4, we can show that Assumption D can be satisfied under certain generic scenarios in the following proposition. The detailed proof of this proposition is omitted.

**Proposition 5.5.** Assume that an anomalous inclusion \( D \Subset \Omega \) possesses a polygonal-nest or polyhedral-nest structure of the class \( B \) described by Definition 4.3, namely (5.7) holds. Suppose that \( f(x, u) \) in (5.1) is characterized by (4.3), where

\[
f(x, u)|_{U_\ell} = \lambda_\ell u, \quad \lambda_\ell \in \mathbb{C}, \quad U_\ell := \Sigma_\ell \setminus \Sigma_{\ell+1}, \quad \lambda_\ell \neq \lambda_{\ell+1}, \quad 1 \leq \ell \leq N - 1,
\]

\[
f(x, u)|_{\Sigma_N} = f_N(x, u) = \sum_{j=1}^{M_N} \lambda_j^{(N)} u^j, \quad \lambda_j^{(N)} \in \mathbb{C}, \quad M_N \in \mathbb{N}.
\]

Let \( u_\psi \) be the solution to (5.1) associated with \( \psi(x; \varepsilon, k, d) = \varepsilon e^{ikx \cdot d} \), then Assumption D is fulfilled under the conditions (5.12)

\[
k^2 \neq \lambda_1^{(1)} \text{ and } \lambda_{N-1} \neq \lambda_1^{(N)}.
\]

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Moreover, we assume that the nonlinear term \( a(x,u) \) is \( C^1 \)-continuous with respect to \( u \) for a fixed \( x \in \Omega \) and \( \partial_ua(x,u) \in L^\infty(\Omega) \). Moreover, we assume that the nonlinear term \( a \) satisfies the following two conditions:

\[
a(x,0) = 0,
\]

\[
\text{the map } v \mapsto \Delta v + \partial_ua(\cdot,u)v \text{ is injective on } H_0^1(\Omega). \tag{5.17}
\]

Indeed, from (5.16), one can directly know that \( u \equiv 0 \) is a solution of (1.1) when the Dirichlet data is zero. The condition (5.17) guarantees that the linearized equation of (1.1) at \( u \equiv 0 \) is well-posed. The next result considers mappings between Banach spaces which are Fréchet differentiable. We refer the reader to [12, Section 1.1] and [26, Section 10] for basics about Fréchet differentiability.

**Proposition 5.6.** (Well-posedness of the semilinear elliptic boundary value problem (1.1) with small boundary data) Let \( \Omega \subset \mathbb{R}^n \), \( n = 2,3 \) be a bounded Lipschitz domain and let \( Q \) be the semilinear elliptic operator given by (5.15) satisfying (5.16) and (5.17). There exist constants \( \delta, C > 0 \) such that for any \( \psi \) in the set

\[
U_\delta := \{ h \in H^{1/2}(\partial\Omega); \|h\|_{H^{1/2}(\partial\Omega)} < \delta \},
\]

there is a solution \( u = u_\psi \) of

\[
\begin{cases}
\Delta u + a(x,u) = 0 & \text{in } \Omega, \\
u = \psi & \text{on } \partial\Omega,
\end{cases}
\]

which satisfies

\[
\|u\|_{H^1(\Omega)} \leq C\|\psi\|_{H^{1/2}(\partial\Omega)}.
\]

The solution \( u_\psi \) is unique within the class \( \{ w \in H^1(\Omega); \|w\|_{H^1(\Omega)} \leq C\delta \} \).

**Proof.** We adopt the implicit function theorem in Banach spaces [26, Theorem 10.6] to prove the existence. Introduce the following map

\[
F : H^{1/2}(\partial\Omega) \times H^1(\Omega) \to H^{-1}(\Omega) \times H^{1/2}(\partial\Omega), \quad F(\psi,u) = (Q(u), u|_{\partial\Omega} - \psi).
\]

We first show that the image of \( F \) belongs to \( Z \). Recall that \( a(x,u) \) defined in (1.2) is \( C^1 \)-continuous with respect to \( u \) for a fixed \( x \in \Omega \). One know that

\[
u \to a(x,u)
\]

maps \( H^1(\Omega) \) to \( L^2(\Omega) \). Since \( u \in H^1(\Omega), Q(u) \in H^{-1}(\Omega) \) is defined in (5.15) and \( (u|_{\partial\Omega} - \psi) \in H^{1/2}(\partial\Omega) \). Therefore \( F \) is well defined.

Next, we prove that \( F \) is continuously differentiable. Recall that \( a(x,u) \) is \( C^1 \)-continuous with respect to \( u \) for a fixed \( x \in \Omega \), which implies that

\[
a(x,u + v) = a(x,u) + \partial_ua(x,u)v + \|v\|_{H^1(\Omega)}\delta(v), \quad \lim_{\|v\|_{H^1(\Omega)} \to 0} \delta(v) = 0.
\]

Therefore, it yields that

\[
F(\psi + \varphi, u + v) = F(\psi,u) + (\Delta v + \partial_ua(x,u)v|_{\partial\Omega} - \varphi) + (\|v\|_{H^1(\Omega)}\delta(v), 0). \tag{5.19}
\]

Noting that \( a(x,u) \) is \( C^1 \)-continuous with respect to \( u \) for a fixed \( x \in \Omega \), we can conclude that \( F \) is continuously differentiable from \( H^{1/2}(\partial\Omega) \times H^1(\Omega) \) to \( H^{-1}(\Omega) \times H^{1/2}(\partial\Omega) \).
From \((5.19)\), the linearization of \(F\) at \((0,0)\) is
\[
D_u F|_{(0,0)}(v) = (\Delta v + \partial_u a(x,0)v, v|_{\partial \Omega}).
\]

In the following we show that \(D_u F|_{(0,0)}\) is a homeomorphism from \(H^1(\Omega)\) to \(H^{-1}(\Omega) \times H^{1/2}(\partial \Omega)\) under the condition \((5.17)\). To this end, consider the following Dirichlet boundary value problem
\[
\begin{aligned}
\Delta v + \partial_u a(x,0)v &= f & \text{in } \Omega, \\
v &= \varphi & \text{on } \partial \Omega,
\end{aligned}
\]
\((5.20)\)
where \(f \in H^{-1}(\Omega)\) and \(\varphi \in H^{1/2}(\partial \Omega)\). Suppose that there exists a solution to \((5.20)\), then the solution is unique by using \((5.17)\). Therefore, utilizing Fredholm alternative (cf. [28, Proposition 1.9]), one can show that there exist a solution to \((5.20)\) in \(H^1(\Omega)\) for any source in \(H^{-1}(\Omega)\) and the Dirichlet data in \(H^{1/2}(\partial \Omega)\).

According to the implicit function theorem in Banach spaces [26, Theorem 10.6], we know that there is a \(\varepsilon > 0\) and an open ball \(B_\varepsilon = B(0,\varepsilon) \subset H^{1/2}(\partial \Omega)\) and a continuously differential mapping \(T : B_\varepsilon \rightarrow H^1(\Omega)\) such that
\[
F(\psi, T(\psi)) = (0,0)
\]
under the condition \(\|\psi\|_{H^{1/2}(\partial \Omega)} \leq \delta\). Since \(T\) is Lipschitz continuous and \(T(0) = 0\), \(u = T(\psi)\) satisfies
\[
\|u\|_{H^1(\Omega)} \leq C\|\psi\|_{H^{1/2}(\partial \Omega)}.
\]
Furthermore, one can claim that \(u = T(\psi)\) is the unique solution to \(F(\psi, S(\psi)) = (0,0)\) under the assumption \(\|\psi\|_{H^{1/2}(\partial \Omega)} \leq \delta\) by necessarily refining the parameter \(\delta\), where \(\|u\|_{H^1(\Omega)} \leq C\delta\).

The proof is complete. \(\square\)

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