On the residual resistivity near a two dimensional metamagnetic quantum critical point

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The behavior of the residual (impurity-dominated) resistivity is computed for a material near a two dimensional quantum critical point characterized by a divergent $q = 0$ susceptibility. A singular renormalization of the amplitude for back-scattering of an electron off of a single impurity is found. When the correlation length of the quantum critical point exceeds the mean free path, the singular renormalization is found to convert the familiar ‘Altshuler-Aronov’ logarithmic correction to the conductivity into a squared-logarithmic form. Impurities can induce unconventional Friedel oscillations, which may be observable in scanning tunnelling microscope experiments. Possible connections to the metamagnetic quantum critical end point recently proposed for the material $\text{Sr}_3\text{Ru}_2\text{O}_7$ are discussed.

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\section{I. INTRODUCTION}

The interplay of quantum mechanics, disorder and reduced dimensionality is a central question in condensed matter physics. The issue becomes particularly interesting for materials near a quantum critical point (i.e. one produced at temperature $T = 0$ by variation of a control parameter such as pressure or chemical composition). Near such a critical point, quantum fluctuations may become particularly strong, and may interact with randomness in an important way. In this paper, we consider the heretofore little studied question of the effect of quantum critical fluctuations on the residual resistivity of a metal. We consider two dimensional systems in which the criticality involves long wavelength fluctuations and find that the critical fluctuations lead to a singular, and possibly divergent, renormalization of the amplitude for an electron to backscatter off of an isolated impurity atom. For a system with a nonvanishing density of impurities, we find that this physics leads to a strengthening of the ‘Altshuler-Aronov’ correction to the conductivity from $\ln(1/T)$ to $\ln^2(1/T)$.

Renormalization of some electron-impurity vertices is expected in materials near density wave transitions; for example, near a charge density wave transition, an impurity will produce density fluctuations whose component at the ordering wavevector will diverge as the transition is approached (for an experimental demonstration, see [2]). However, we show here that long-wavelength spin fluctuations can drastically increase backscattering by spinless impurities. Our work is related to a previous analysis, by Altshuler, Ioffe, Larkin, and one of us, of the staggered susceptibility of a model of electrons interacting with gauge fluctuations [3].

Our work is motivated by recent experiments on the bilayer ruthenate, $\text{Sr}_3\text{Ru}_2\text{O}_7$, which apparently can at ambient pressure be tuned through a quantum critical point by variation of applied magnetic field [4-7]. $\text{Sr}_3\text{Ru}_2\text{O}_7$ has a layered crystal structure and highly anisotropic conductivity properties characterized by a very low in-plane (‘$ab$’) resistivity ($\rho_{ab} \sim 5 \mu\Omega \cdot \text{cm}$ as $T \to 0$) and a much higher out of plane (‘$c$-axis’) resistivity ($\rho_c \sim 10^3 \mu\Omega \cdot \text{cm}$ as $T \to 0$) [8], so its electronic properties are to a good approximation two dimensional. At ambient pressure and no applied magnetic field the material is paramagnetic, but with a magnetic susceptibility $\chi$ which is strongly $T$-dependent (for $T \gtrsim 10^3 \text{K}$) and at low $T$ strongly enhanced above the band theory value [4]. As applied magnetic field $H$ is increased at low $T$ ($T < 2 \text{K}$) resistivity measurements indicate that the material passes near to a critical point [4, 5]. In the vicinity of this critical point the differential magnetic susceptibility $\chi_{\text{diff}} = dM/dH$ becomes large and the residual resistivity $\rho(T) \to \lim T \to 0$ exhibits a pronounced cusp [6].

The natural interpretation is that the transition observed in $\text{Sr}_3\text{Ru}_2\text{O}_7$ is metamagnetic in nature. Metamagnetic materials exhibit a phase diagram in the field ($H$)-temperature ($T$) plane characterized by a line of first order transitions across which the magnetization jumps. The first order line ends in a critical point at temperature $T_M$ and field $H_M$. In $\text{Sr}_3\text{Ru}_2\text{O}_7$ parameters are apparently such that at ambient pressure $T_M$ is very near to $T = 0$ (indeed for one field orientation it can be made to vanish exactly) so that one has a quantum critical end point [4, 5, 6, 8, 9]. A renormalization group theory of quantum critical end-points and their application to $\text{Sr}_3\text{Ru}_2\text{O}_7$ was given by [9]; in this paper we extend their theory to include electron-impurity coupling and use it to calculate the residual resistivity. In addition we make a few remarks about the applicability of our calculations to more general classes of critical points.

The rest of this paper is organized as follows. Section II presents the critical theory, section III gives the analysis of the renormalization of the Born approximation amplitude for an electron to scatter off of an isolated impurity, section IV considers the single impurity problem beyond the Born approximation and applies the results to $\text{Sr}_3\text{Ru}_2\text{O}_7$, section V treats the extension to
a non-vanishing density of impurities and section VI is a conclusion and discussion of implications for other experiments.

II. METAMAGNETIC QUANTUM CRITICALITY

Ref [3] employed the standard approach of Hertz [9] and Millis [10] which involves integrating out electronic degrees of freedom to obtain an effective field theory with overdamped bosonic excitations, which is then analysed. This approach is presently the subject of debate [1]. On the experimental side, the theory is apparently inconsistent with data obtained on a variety of materials with antiferromagnetic critical points, and on the theoretical side the validity of integrating out gapless electronic degrees has been questioned [12, 13, 14], both two dimensional antiferromagnets and for ferromagnetic transitions involving an order parameter with continuous symmetry and time reversal invariance in the disordered phase. However, the assumptions made in [12, 13] do not apply to metamagnetic transitions, which involve an Ising order parameter and an explicitly broken time reversal symmetry, nor do the Fermi surface nesting complications of antiferromagnetic transitions [14] apply here, because the order parameter is centered at \( q = 0 \).

The order parameter for a metamagnetic critical point occurring at a field \( H_M \) is the difference of the local magnetization density \( M(x, \tau) \) (which we write here in space and imaginary time) from the average value \( M^* \) produced by the field \( H_M \) at \( T = 0 \). \( (M^* \approx 5 \mu_B \text{ for } Sr_3Ru_2O_7) \).

It is convenient to measure the field and magnetization fluctuations per \( Ru \) via

\[
\phi(x, \tau) = \frac{M(x, \tau) - M^*}{M_{sat}},
\]

\[
h = \frac{(H - H_M)M_{sat}}{E_0},
\]

where \( M_{sat} \) is the high field saturation magnetization. In a clean (non-disordered) system the action describing the fluctuations of \( \phi \) was argued [8] to be (\( \beta = 1/T \))

\[
S = \frac{E_0}{2} \int \frac{d^2x}{a^2} \int_0^\beta d\tau \left[ 2h\phi + \zeta_0^2 (\nabla \phi)^2 + \frac{1}{2} \phi^4 \right] + S_{dyn},
\]

with \( \phi(q, \nu) = E_0 a^{-2} \int d^2x \int_0^\beta d\tau e^{i(q \cdot x - i\nu \tau)} \phi(x, \tau) \) and, in a clean system,

\[
S_{dyn} = \frac{a^2}{2E_0} \int \frac{d^2q}{(2\pi)^2} T \sum_{\nu \nu} \left| \frac{\nu}{vq} \right| \left| \phi(q, \nu) \right|^2,
\]

where \( \nu = 2\pi n T \). For \( Sr_3Ru_2O_7, \ E_0 \approx 7000 K \) [8] \; a value for \( \xi_0 \) has not been established. A natural guess would be something of the order of the in-plane lattice constant \( a \approx 4 \AA \). However in many ferromagnetic materials band theory suggests that the momentum dependence of the magnetic polarizability is very weak, suggesting \( \xi_0 < a \); while if the considerations of [12] are relevant a considerably larger value might be appropriate.

A value for \( v \) has also not yet been established: the natural expectation is that it is of order the planar band Fermi velocity \( v_F \approx 1eV - A \). The theory is above its upper critical dimension so the nonlinear term is a ‘dangerously’ irrelevant operator and may be treated by standard means [8, 10].

We have written a two dimensional theory. At some scale a crossover to three dimensional behavior will occur, but present neutron experiments have so far been unable to observe any correlations along the c-axis [15]. Because the transition takes place in a magnetic field, the fluctuating field has Ising symmetry and time reversal invariance is explicitly broken. Neither ‘precession’ terms in the dynamics nor the anomalous \( |q| \) momentum dependence proposed in [12] will occur.

The action given in Eq 3 describes the critical fluctuations. In the absence of critical fluctuations we take the action for electrons to be

\[
S_{el} = \frac{a^2}{E_0} \int \frac{d^2x}{a^2} \int_0^\beta d\tau \psi^\dagger \left( \nu \omega_n + \xi_p \right) \psi \left( p, i\omega_n \right),
\]

where \( \psi \) represents the electrons with the “spin” index \( \alpha, \xi_p \) is the electron dispersion and \( \omega_n = (2n+1)\pi T \). The coupling between the electrons and the critical fluctuations can be written as

\[
S_{\phi - \psi} = g \int \frac{d^2x}{a^2} \int E_0 d\tau \psi \sigma_z \psi \phi,
\]

where \( \sigma_z \) is the z-component of the Pauli matrix.

To obtain the propagator \( D \) describing the metamagnetic fluctuations we expand the action about the mean field value \( \phi_0 \) given by \( \partial S/\partial \phi = 0 \) and read off the quadratic term, obtaining

\[
D(q, i\nu_n) = \frac{1}{\nu_n + \xi_0^2 q^2 + \hbar^2/3}
\]

The coupling constant \( g \) may be determined because the damping term in \( S_{dyn} \) arises from processes in which a critical fluctuation decays into one particle-hole pair [4, 10] and is

\[
g^2 = \frac{4\pi^2}{a^2 E_0 v S_F}\]

with \( S_F \) the length of the Fermi surface in two dimensions \( (S_F = 2\pi p_F \text{ for a circular Fermi surface}) \).

Using these definitions we find that the one-loop self energy at \( T = 0 \), shown in Fig 1a, is

\[
\Sigma_1(p, i\omega) = \frac{2iv_F}{S_F \xi_0^2} \int_0^\infty dx \ln \left( \frac{\omega_0}{x^3 + \hbar^2/3} \right).
\]
Notice that as \( \omega \to 0 \) at fixed \( h \neq 0 \), \( \Sigma(p, i\omega) = \frac{2\pi q_0}{4\pi F} \frac{1}{\sqrt{p^2 + q_0^2}} \) while as \( h \to 0 \) at fixed \( \omega \) we have \( \Sigma(p, i\omega) = \frac{1}{4\pi F \xi_0 (\omega/\xi_0)} \).

In spatial dimension \( d > 2 \) the fact that the theory is above the upper critical dimension guarantees that a one-loop (Migdal-like) approximation yields an asymptotically exact approximation to the self energy, but in \( d = 2 \) this is not the case [16] [17] [18]. While higher corrections such as those shown in Fig 1b do not change the powers, they do introduce a dependence on momentum \( \delta p = p - p_F \) and induce a dependence on \( \frac{\omega}{\omega^*(h)} \) with \( \omega^*(h) = vh/\xi_0 \). The scaling function for the self-energy has only been computed in large \( N \) and small \( N \) expansions where \( N \) is the number of order parameter components. Unfortunately, in the metamagnetic problem of interest here \( N = 1 \).

III. ELECTRON-IMPURITY VERTEX: BORN APPROXIMATION

![Diagram](image)

FIG. 2: Diagrams corresponding to the renormalization of the impurity scattering amplitude. (a) the bare vertex (b) the lowest order correction (c) the sum of the ladder diagrams (d) an example of higher order corrections.

We now consider the effect of critical fluctuations upon the amplitude, \( A_{p_1 p_2} \) for an electron initially in a state of momentum \( p_1 \) to scatter off of an isolated impurity atom into a state of momentum \( p_2 \). The basic electron-impurity vertex is shown in Fig 2a; we assume that this is structureless; corresponding to scattering amplitude independent of \( p_{1.2} \). The first correction due to critical fluctuations is shown in Fig 2b, and is found to be divergent when \( p_{1.2} \) are on the Fermi surface and are such that the Fermi surface tangents at these two points are parallel. For a circular Fermi surface this condition corresponds to \( p_1 + p_2 = 0 \) and \( |p_1| = p_F \). In this case, choosing the \( x \) coordinate to be parallel to \( p_1 \) and \( p_{1.2} \) to be on the Fermi surface we have \( \varepsilon_{p_1 + q} = v_F (q_x + q_y^2/2q_0) \) and \( \varepsilon_{p_2 + q} = v_F (-q_x + q_y^2/2q_0) \) with \( q_0 \) a quantity of the order of \( p_F \) parametrizing the curvature of the Fermi surface. If \( p_2 = -p_1 + p \) with \( |p| \ll |p_1| \), we find that the the correction shown in Fig 2b is

\[
A_1(p, h) = A_0 I \left( \frac{2\pi q_0}{S_F} \right) \ln \left( \frac{1}{\max[|h|^{2/3}, (p_\psi)^2]} \right)
\]

with \( I(b) = \frac{2h}{\pi} \int_0^\infty dy \left( \frac{y}{1+y^2} \right) \left( \frac{b}{1+y^2} \right)^{1/2} \) and \( A_0 \) is the bare scattering amplitude: \( I(1) \approx 0.23 \); \( I(10) \approx 0.5 \) and \( I(b) \) is an increasing function of \( b \). Notice that if \( \pi - \theta \) is the angle between \( p_1 \) and \( p_2 \), then \( p_\psi \approx 0 \).

A very similar result was obtained in the context of the \( U(1) \) gauge theory of the ‘RVB’ regime of the two-dimensional \( t-J \) model [14], where the \( 2p_F \) spin susceptibility was considered. The \( U(1) \) gauge theory possesses an interaction (mediated by an internal gauge field) with a very similar mathematical structure to our interaction \( D(q, iv_n) \) (Eq. [10]), except that in the \( U(1) \) problem gauge invariance dictates that in the RVB regime the mass (\( h \) in our notations) vanishes. Further, in the vertex computation two additional minus signs occur (but compensate each other), one from the fact that gauge interaction involves currents which are oppositely directed at momentum transfer \( 2p_F \) and one from the transverse nature of the gauge interaction.

Higher order diagrams such as those shown in Fig. 2c may be evaluated similarly; we find that the leading behavior of the \( n^{th} \) order term is

\[
A_n = A_0 \frac{1}{n!} I \left( \frac{2\pi q_0}{S_F} \right) \ln \left( \frac{1}{\max[|h|^{2/3}, (p_\psi)^2]} \right)^n
\]

so that finally we obtain

\[
A(\theta, h) \sim A_0 \left( \max[|h|^{2/3}, (p_\psi)^2]^{-\psi}
\]

with \( \psi = I \left( \frac{S_F}{2\pi q_0} \right) \) at one loop order.

We should further consider the effect of higher order diagrams, such as those shown in Fig 2d. These do not change the basic power counting, and may therefore be expected not to alter the basic power law we have found; they will however change the numerical value of the exponent. According to the result of small \( N \) expansion [14], the effect of ‘crossed diagrams’ is to increase the exponent. We conclude that the Born-approximation amplitude for an electron to back-scatter off of an impurity suffers a singular renormalization near criticality.
IV. BEYOND BORN APPROXIMATION; RESISTIVITY

The previous section presented the renormalization of the leading-order term in the electron-impurity vertex. In this section we consider effects arising when the Born approximation is not justified, either because the initial scattering amplitude is not small or because the renormalizations increase an initially weak interaction beyond the regime of validity of the Born approximation. Corrections to the Born approximation result for the electron self energy correspond to multiple scattering of the electron off of the same impurity, and are represented diagrammatically in Fig. 3. Within the ‘non-crossing’ approximation used in previous sections we find that the leading renormalization near criticality comes from the correcting each impurity vertex individually. The series for the self energy may be summed by defining the $T$-matrix which for incident electron energies very close to the fermi surface becomes

$$T(\theta - \theta', h) = A(\theta - \theta', h) - i \int \frac{d\theta_1}{2\pi} N(\theta_1) A(\theta_1, h) T(\theta_1 - \theta', h)$$

(13)

with $N_0 = \int \frac{d^2 k}{(2\pi)^2} \delta(\epsilon_p)$ the single-spin fermi surface density of states. Assuming for simplicity a circular fermi surface with density of state $N_0$ we may solve the equation by resolving $T$ and $A$ into their angular components $T_m = \int \frac{d\theta}{2\pi} T(\theta, h) e^{im\theta}$ and $A_m = \int \frac{d\theta}{2\pi} A(\theta, h) e^{im\theta}$ so that

$$T_m = \frac{A_m}{1 + i A_m N_0}. \quad \text{(14)}$$

We note that the sign of $T_m$ and therefore the sign of the angular momentum-$m$ channel phase shift alternates with $m$, so that the Friedel sum rule is straightforwardly satisfied.

$$\times \quad + \quad \times \quad + \quad \times \quad + \quad \cdots$$

FIG. 3: Diagrams representing multiple scattering of electrons off of an impurity. The shaded triangle represents the renormalization of the Born scattering amplitude given by Fig.2c.

The final result depends on the parameter $g_0 = A_0 N_0$ and on the exponent $\psi$. If $\psi < 1/2$, then none of the $A_m$ or $T_m$ are divergent as $h \to 0$. The distance to criticality, parametrized by $h$, controls the number $m_{\text{max}}$ of angular momentum channels for which $A_m$ and $T_m$ are non-negligible ($m_{\text{max}} \sim h^{-1/3}$). On the other hand, if $\psi > 1/2$, then in each angular momentum channel the amplitude $A$ diverges as $h \to 0$ so that the resulting phase shift saturates at $\pi/2$ in each channel, and again the number of channels which are relevant diverges rather strongly as $h \to 0$.

The contribution of this scattering amplitude to the probability that an electron is scattered through an angle $\theta$, $W(\theta)$, is given by $|T(\theta)|^2$. The impurity scattering rate revealed by the resistivity, $\Gamma_{ir} = \int \frac{d\theta}{2\pi} (1 + \cos(\theta)) W(\theta)$ (note that we have defined angular coordinates so that $\theta = 0$ corresponds to backscattering). Because we are concerned with a scattering amplitude strongly peaked about the back-scattering direction, the $\cos(\theta)$ factor is unimportant. Then $\Gamma_{ir}$ is given by the imaginary part of the diagonal $T$-matrix $\Gamma_{ir} \sim \text{Im} T(0; h)$ in the usual approximation. Therefore, we obtain

$$\rho_{\text{res}} \sim \text{Im}|T_0| + 2 \sum_{m=1}^{\infty} \text{Im} T_m$$

(15)

As noted above, within this approximation the resistivity diverges as criticality is approached, but the approximation itself breaks down when the mean free path implied by Eq.15 becomes smaller than the correlation length. The question of a divergent resistivity is further examined in section VII.

V. APPLICATION TO DATA

In this section we attempt to relate calculations of the $h$ dependence of $\rho_{\text{res}}$ to data obtained on Sr$_3$Ru$_2$O$_7$. We first note that it is sensible to discuss the scattering from an isolated impurity only if the elastic mean free path $l$ is greater than the correlation length $\xi = \xi_0 h^{-1/3}$. The observed low-T resistivity of clean samples of Sr$_3$Ru$_2$O$_7$ is of the order of $5 \mu \Omega \cdot \text{cm}$ corresponding to a $p_F l \approx 250$, i.e. to an $l$ of the order of $300 - 400 A$. The appropriate value of $\xi_0$ is not known at present, but if it is of the order of the lattice constant or smaller then almost all the available data are in the regime in which the calculation applies.

Sr$_3$Ru$_2$O$_7$ presents an interesting issue in modelling. If the estimates presented in [8] are correct, then even in zero field the material is close to a ferromagnetic critical point, so that renormalizations of impurity vertices could be substantial, even at vanishing applied field (provided, of course, that a sufficiently wide regime exists in
Representative results are shown in Fig.4. The values at applied field $H = 0$ depend on the behavior of the theory at higher energy scales, which is not known for the reasons given above. However, the change in behavior near the metamagnetic field (about 7T for the parameters used here) should be reliable. We observe that there is some interplay between exponent and initial scattering strength but that sharpness and relative height of the resistivity peak depend most strongly on the exponent. Comparison to data [3] shows that our calculation is consistent with a moderate bare scattering amplitude and an exponent $\psi \approx 3/4$. We also note that the rather small resistivities observed in experimental systems suggest either that the initial scattering amplitude is weak or that the renormalizations associated with the nearby ferromagnetic quantum critical point are not large, perhaps for the reasons given in [12].

VI. FRIEDEL OSCILLATIONS

The physics which gives rise to this correction to the conductivity also gives rise to a singular behavior [3] in the susceptibility at wavevector $q = 2p_F$ and it is proportional to $(2p_F - q)^{1-3\psi}$; this leads to a change in amplitude and distance dependence of the Friedel oscillations induced by an impurity in two dimensions. We find that if $\psi < 1/3$ the Friedel oscillations decay as $\cos(2p_F R)/R^{2-3\psi}$ while it is $\cos(2p_F R)/R^{4+3\psi}$ when $\psi > 1/3$. Notice that it decays slower (faster) than the standard $1/R^2$ form if $1/6 < \psi < 1/2$ ($\psi < 1/6$ or $\psi > 1/2$). At non-zero field, the spin up and spin down Fermi surfaces are characterized by different Fermi wave vectors so we predict two superposed oscillations emerging from each impurity site, each decaying with a characteristic power. These oscillations should be observable in STM experiments. We note, however, that the effect discussed here is a long wavelength, low energy effect. It does not imply that the density modulations are greatly enhanced near to an impurity; only that they decay much less (or more) rapidly with distance than in a non-critical material.

It is also interesting to consider the situation at $H = 0$, i.e. near to a 2D ferromagnetic transition, and indeed we note that for the Sr$_3$Ru$_2$O$_7$ parameters given in [3], $H = 0$ corresponds to $h \sim 10^{-3}$ so one might expect the enhancements to be noticeable even at zero field. The Friedel oscillations from a non-magnetic impurity would indeed be long ranged, however it is interesting to note that the RKKY interactions are suppressed (in a system with Heisenberg symmetry) because the spin commutation relations lead to a minus sign in the renormalization of $\sigma_z$ vertex by $\sigma_x$ or $\sigma_y$ fluctuations.

We also note that field-dependent STM studies might present an interesting test of the result [12] that for 2D Heisenberg materials the leading momentum dependences is $|q|$. In this case, the renormalizations we have discussed would not exist. As $H$ is increased, the effects

FIG. 4: Residual resistivity calculated for different exponents and initial scattering amplitudes. The curves for the two larger exponents are scaled (by the factors indicated) to coincide with the $H = 0$ values calculated for the smallest exponent. Dashed lines, $\psi = 1$; heavier solid lines, $\psi = 0.75$; light solid lines, $\psi = 0.5$. Three panels show the results of different initial scattering amplitude (a) $g_0 = 0.5$ (well below unitarity limit for initial point-impurity scattering amplitude) (b) $g_0 = 1.0$ (roughly half way to unitarity limit) (c) $g_0 = 5.0$ (close to unitarity limit).
that produce $|q|$ are believed to be cut off, and the unconventional Friedel oscillations should reappear.

Finally, we observe that the presence of the Friedel oscillations implies that the state of a critical system is in some sense a random charge density wave, characterized by infinite ranges of charge oscillations emerging from the various impurity sites. The resistivity and other properties of such a state are an interesting issue for future research.

VII. NONVANISHING IMPURITY DENSITY

The preceding treatment is valid for correlation lengths less than the mean free path. We now consider what happens when parameters are tuned so that the correlation length $\xi$ exceeds the mean free path $l$. We first note that the problem has two energy scales: the impurity scattering rate $\tau^{-1} = v_F/l$ and the characteristic quantum critical frequency $\omega^* = \nu\xi^2/\xi^3$ so that near criticality, when $\xi > \xi_0$ we have $\omega^* \tau < 1$.

For length scales longer than the mean free path the dynamic term in the action is modified to be

$$S_{\text{dyn, dirty}} = \frac{a^2}{2E_0} \int \frac{d^2 q}{(2\pi)^2} T \sum_{\nu_n} \left| \frac{\nu_n}{D q^2} \phi(q, \nu_n) \right|^2$$

with $D'$ of the order of the diffusion constant $D = v_F^2 \tau/2$. We note $D'/D = v/v_F$ from the definition of $g^2$ in Eq.8. Here $1/\tau = 2\pi u^2 N_0$ is the impurity scattering rate, $u^2$ is the renormalized squared impurity scattering strength (renormalized due to the effects discussed in section IV), and $N_0$ is the density of states at the Fermi energy. Thus the dynamical critical exponent is $z = 4$ in this case and the nonlinearity in the critical theory remains dangerously irrelevant.

The leading order correction to the conductivity is given by the sum of two diagrams in Fig.5, where the wavy line represents the metamagnetic fluctuations with the propagator $D(q, i\nu_n) = 1/(i\nu_n + q^2 \xi_0^2 + h^2/3)$. Notice that there exist three more diagrams at the same order, but they cancel each other. Each interaction vertex that represents the coupling between the electrons and the bosonic mode is renormalized by the Diffuson correction, $(|\nu_n| + D q^2)^{-1} \tau^{-1}$. After evaluating these diagrams we find that the leading correction to the dc conductivity at nonzero temperature is given by (we assume $T < 1/\tau$)

$$\delta\sigma = -\frac{e^2}{\hbar} \frac{\lambda}{4\pi^2} \ln \left( \frac{\lambda D}{T \xi_0^2} \right)$$

for $T > T_h$ (17)

$$\delta\sigma = -\frac{e^2}{\hbar} \frac{\lambda}{3\pi^2} \ln \left( \frac{1}{h} \right) \ln \left( \frac{\lambda D}{T \xi_0^2} \right)$$

for $T < T_h$, (18)

where $\lambda = v_F/v$ and $T_h = h^{3/2} \lambda D/\xi_0^2$. Notice that these results are valid as far as we are in the perturbative regime, i.e., $\delta\sigma \ll \sigma_0 = \frac{e^2}{\pi^2} kF l$.

The physical content of Eqs (7),(8) is that as field is varied at a fixed, low, temperature the resistivity will initially increase as the critical point is approached, but sufficiently close to the critical point (i.e. when $h < h_T$ with $h_T = (\xi_l/\xi_0)^{3/4}$) resistivity will level off at a temperature dependent but field-independent value. Fig. 6 presents a qualitative sketch of the predicted behavior, including both the 'single-impurity' regime of the previous sections and the 'Altshuler-Aronov' regime considered in this section.

![FIG. 6: Schematic behavior of resistivity as a function of $h$. Here $h_l = (\xi_l/D)^3$ is the crossover field at which the mean free path becomes less than the correlation length and the single-impurity results of section IV break down, while $h_T = (\xi_l/\xi_0)^{3/4}$ is the field scale beyond which the field dependence is cut off by thermal effects. If $h_T < h < h_l$, Eq.8 applies and the resistivity increases logarithmically as $h$ is decreased. When $h < h_T$, Eq.7 applies and the resistivity becomes a temperature dependent constant.

This result should be compared with the well-known behavior of the quantum correction to the conductivity, $\delta\sigma = -\frac{e^2}{\pi^2} \ln(1/T\tau)$, in weakly disordered interacting two dimensional electron systems [1]. The notable differences are i) $\delta\sigma$ in the present case has more singular temperature dependence when $T > T_h$ and at the critical point ($h = 0$). ii) If $T < T_h$, the coefficient of the logarithmic term explicitly depends on $h$ and it increases as the critical point is approached until it is cut off by temperature.

Now let us estimate the size of the perturbative quantum correction in $Sr_3Ru_2O_7$. We assume that $D' \approx D$.
and $\delta \sigma / \sigma$ is of the order of the in-plane lattice constant $\sim 4\AA$. Using $v_F \sim 1eV - A$ and $l \sim 300 - 400\AA$, we get $1/\tau \sim 30K$ and $D \sim 45 - 60cm^2/sec$. Let us consider $h \sim 10^{-6}$, then $T_h \sim 10^{-4}K$. In this case, $T > T_h$ for experimentally relevant temperature range and Eq. [17] should be used. Using the leading order result, $\sigma_0 = \frac{e^2 \tau}{F l}$, in two dimensions the relative size of the correction can be estimated as $\delta \sigma / \sigma_0 \sim 0.07$ at $T = 2K$ and $\delta \sigma / \sigma_0 \sim 0.1$ at $1K$. Thus the relative correction is only $7-10\%$. Given that the residual resistivity is about $5\mu \Omega - cm$, it will be hard to see the effect of these corrections. If the material were more dirty, say $k_F l \sim 50$, we would get $1/\tau \sim 150 - 200K$ and $T_h \sim 10^{-4}K$. Similar estimation would predict that the relative correction is $30-50\%$ at $T = 0.1 - 2K$. It would be interesting to test these predictions in more dirty samples.

To conclude this section we note that the scattering time $\tau$ is the impurity scattering rate, as renormalized by critical fluctuations at scales less than the (renormalized) $l$. We also observe that our result is perturbative in both the disorder strength and the interaction. Presumably when the resistivity becomes of the order of the Mott value a crossover to insulating behavior occurs. The insulating state should presumably be interpreted in terms of the random charge density wave state discussed at the end of Section VI, but the issue deserves more careful investigation.

VIII. CONCLUSION

In summary, the effect of critical fluctuations on the residual resistivity is studied near a two dimensional metamagnetic quantum critical point. When the correlation length is smaller than the mean free path, the critical fluctuations induce a singular renormalization of the amplitude of the back-scattering off an impurity of an electron. This leads to a pronounced cusp in the residual resistivity near the metamagnetic critical point in accordance with the experimental results. When the correlation length becomes larger than the mean free path, the critical fluctuations convert the ‘Aronov-Altshuler’ logarithmic correction to the conductivity to the more singular squared-logarithmic behavior near the critical point. Our results imply a divergent resistivity at criticality. We argued that the state which gives rise to this divergence is some sort of random charge density wave, but a detailed investigation of its properties has not been performed.

Our results may have broader implications. The apparently successful comparison of our calculation to data suggests that the singular “$2p_F$” renormalization discovered in the gauge theory context in [3] is more than a theoretical curiosity, and therefore motivates examination of other systems where $2p_F$ effects might be important; for example in the transresistance of bilayer $\nu = 1/2$ quantum Hall systems. Work in this direction is in progress.

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