THE DISCRETE ANALOGUE OF THE OPERATOR $\frac{d^{2m}}{dx^{2m}}$ AND ITS PROPERTIES
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Abstract

In this paper the discrete analogue $D_{m}[\beta]$ of the differential operator $\frac{d^{2m}}{dx^{2m}}$ is constructed and its some new properties are proved.

Key words and phrases: Discrete function, discrete analogue of the differential operator, Euler polynomial.

1 Main results.

First S.L.Sobolev [1] studied construction and investigated properties of the operator $D_{hH}^{(m)}[\beta]$, which is inverse of the convolution operator with function $G_{hH}^{(m)}[\beta] = h^{n}G_{m}(hH\beta)$. The function $D_{hH}^{(m)}[\beta]$ of discrete variable, satisfying the equality

$$h^{n}D_{hH}^{(m)}[\beta] * G_{hH}^{(m)}[\beta] = \delta[\beta]$$

is called by the discrete analogue of the polyharmonic operator $\Delta^{m}$. S.L.Sobolev suggested an algorithm for finding function $D_{hH}^{(m)}[\beta]$ and proved several properties of this function. In one dimensional case, i.e. the discrete analogue of the operator $\frac{d^{2m}}{dx^{2m}}$ was constructed by Z.Zh.Zhamalov [2, 3]. But there the form of this function was written with $m + 1$ unknown coefficients. In works [4, 5] these coefficients were found, hereunder the discrete analogue of the operator $\frac{d^{2m}}{dx^{2m}}$ was constructed completely.

In this paper we give the results of works [4-7], concerning to construction of the discrete analogue $D_{m}[\beta]$ of the operator $\frac{d^{2m}}{dx^{2m}}$, and discovery of its properties, which early were not known.

Following statements are valid.

Theorem 1. The discrete analogue of the differential operator $\frac{d^{2m}}{dx^{2m}}$ have following form

$$D_{m}[\beta] = \frac{(2m - 1)!!}{2^{m}h^{2m}} \begin{cases} \sum_{k=1}^{m-1} \frac{(1 - \lambda_{k})^{2m+1}\lambda_{k}^{\beta}}{\lambda_{k}E_{2m-1}(\lambda_{k})} & \text{for } |\beta| \geq 2, \\ 1 + \sum_{k=1}^{m-1} \frac{(1 - \lambda_{k})^{2m+1}}{E_{2m-1}(\lambda_{k})} & \text{for } |\beta| = 1, \\ -2^{2m-1} + \sum_{k=1}^{m-1} \frac{(1 - \lambda_{k})^{2m+1}}{\lambda_{k}E_{2m-1}(\lambda_{k})} & \text{for } \beta = 0, \end{cases}$$

(1)

where $E_{\alpha}(\lambda)$ is the Euler polynomial of degree $\alpha$, $\lambda_{k}$ are the roots of the Euler polynomial $E_{2m-2}(\lambda)$, in module less than unity, i.e. $|\lambda_{k}| < 1$, $h$ is the step of the lattice.
Property 1. The discrete analogue $D_m[\beta]$ of the differential operator of order $2m$ have representation

$$D_m[\beta] = \frac{(2m - 1)!}{h^{2m}} \Delta_2^{[m]}[\beta] * \sum_{k=1}^{m-1} \frac{\lambda_k^{\beta+m-2}}{E_{2m-2}(\lambda_k)},$$

where $\Delta_2^{[m]}[\beta] = \sum_{k=-m}^{m} (-1)^{k+m} \binom{2m}{m+k} \delta[\beta - k]$ is symmetric difference of order $2m$.

Property 2. The operator $D_m[\beta]$ and monomials $[\beta]^k = (h\beta)^k$ are connected as

$$\sum_{\beta} D_m[\beta][\beta]^k = \begin{cases} 0 & \text{for } 0 \leq k \leq 2m - 1, \\ (2m)! & \text{for } k = 2m, \end{cases}$$

(2)

$$\sum_{\beta} D_m[\beta][\beta]^k = \begin{cases} 0 & \text{for } 2m + 1 \leq k \leq 4m - 1, \\ \frac{h^{2m}(4m)!B_{2m}}{(2m)!} & \text{for } k = 4m. \end{cases}$$

(3)

Property 3. The operator $D_m[\beta]$ and the function $\exp(2\pi ihp\beta)$ connected as

$$\sum_{\beta} D_m[\beta] \exp(2\pi ihp\beta) =$$

$$\frac{(-1)^m 2^{2m}(2m - 1)!h^{-2m} \sin^{2m}(\pi hp)}{2 \sum_{k=0}^{m-2} a_k^{(2m-2)} \cos 2\pi hp(m - 1 - k) + a_{m-1}^{(2m-2)}},$$

where

$$a_k^{(2m-2)} = \sum_{j=0}^{k} (-1)^{j} \binom{2m}{j} (k + 1 - j)^{2m-1}$$

are the coefficients of the Euler polynomial $E_{2m-2}(\lambda)$.

2 Lemmas.

As known, Euler polynomials $E_k(\lambda)$ have following form

$$\lambda E_k(\lambda) = (1 - \lambda)^{k+2} D^k \frac{\lambda}{(1 - \lambda)^2},$$

(4)

where

$$D = \frac{d}{d\lambda}, \quad D^k = \frac{d}{d\lambda} D^{k-1}.$$

In [8] was shown, that all roots $\lambda_j^{(k)}$ of the Euler polynomial $E_k(\lambda)$ are real, negative and different:

$$\lambda_1^{(k)} < \lambda_2^{(k)} < \ldots < \lambda_k^{(k)} < 0.$$
Furthermore, the roots, equal standing from the ends of the chain (5) mutually inverse:

$$\lambda_j^{(k)} \cdot \lambda_{k+1-j}^{(k)} = 1.$$  \hspace{1cm} (6)

If we denote $E_k(\lambda) = \sum_{s=0}^{k} a_s^{(k)} \lambda^s$, then the coefficients $a_s^{(k)}$ of Euler polynomials, as this was shown by Euler himself, are expressed by formula

$$a_s^{(k)} = \sum_{j=0}^{s} (-1)^j \binom{k+2}{j} (s+1-j)^{k+1}.$$  \hspace{1cm} (7)

From the definition $E_k(\lambda)$ follow following statement.

**Lemma 1.** For polynomial $E_k(\lambda)$ following recurrence relation is valid

$$E_k(\lambda) = (k\lambda + 1)E_{k-1}(\lambda) + \lambda(1-\lambda)E'_{k-1}(\lambda),$$ \hspace{1cm} (8)

where $E_0(\lambda) = 1, k = 1, 2, ...$.

**Lemma 2.** The polynomial $E_k(\lambda)$ satisfies the identity

$$E_k(\lambda) = \lambda^k E_k \left( \frac{1}{\lambda} \right),$$ \hspace{1cm} (9)

or otherwise $a_s^{(k)} = a_{k-s}^{(k)}, s = 0, 1, 2, ..., k$.

**Proof of lemma 1.** From (4) we can see, that

$$E_{k-1}(\lambda) = \lambda^{-1}(1-\lambda)^{k-1}D^{k-1} \frac{\lambda}{(1-\lambda)^2}.$$  \hspace{1cm} (10)

Differentiating by $\lambda$ the polynomial $E_{k-1}(\lambda)$, we get

$$E'_{k-1}(\lambda) = -(1-\lambda)^k \lambda^{-2}(k\lambda + 1)D^{k-1} \frac{\lambda}{(1-\lambda)^2} + \frac{E_k(\lambda)}{\lambda(1-\lambda)}.$$ \hspace{1cm} (11)

Hence and from (9) we obtain, that

$$(k\lambda + 1)E_{k-1}(\lambda) + \lambda(1-\lambda)E'_{k-1}(\lambda) = (k\lambda + 1)\lambda^{-1}(1-\lambda)^{k+1}D^{k-1} \frac{\lambda}{(1-\lambda)^2} -$$

$$-(1-\lambda)^{k+1}\lambda^{-1}(k\lambda + 1)D^{k-1} \frac{\lambda}{(1-\lambda)^2} + E_k(\lambda) = E_k(\lambda).$$

So, lemma 1 is proved.

**Proof of lemma 2.** The lemma we will prove by induction method. When $k = 1$ from (4) we find

$$E_1(\lambda) = \lambda + 1.$$  

We suppose, that when $k \geq 1$ the equality $a_n^{(k-1)} = a_{k-1-n}^{(k-1)}, n = 0, 1, ..., k-1$ is fulfilled. We assume, that $a_n^{(k-1)} = 0$ for $n < 0$ and $n > k - 1$. 

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From (7) we have
\[ a_s^{(k)} = (s + 1)a_s^{(k-1)} + (k - s + 1)a_{s-1}^{(k-1)}, \]
then, using assumption of induction, we get
\[ a_k^{(k)} = (k - s + 1)a_{k-s}^{(k-1)} + (s + 1)a_{k-s-1}^{(k-1)} = (k - s + 1)a_{s-1}^{(k-1)} + (s + 1)a_s^{(k-1)} = a_s^{(k)}, \]
and lemma 2 is proved.

3 Proof of theorem 1

For this we will use function
\[ G_m(x) = \frac{x^{2m-1} \text{sign} x}{2 \cdot (2m - 1)!}. \]
To this function we correspond following function of discrete argument:
\[ G_m[\beta] = \frac{(h\beta)^{2m-1} \text{sign}(h\beta)}{2 \cdot (2m - 1)!}. \]
Here we must find such function \( D_m[\beta] \), which satisfies the equality
\[ hD_m[\beta] * G_m[\beta] = \delta[\beta]. \] (10)

According to the theory of periodic generalized functions and Fourier transformation in them instead of function \( D_m[\beta] \) it is convenient to search harrow shaped function [1]
\[ \overrightarrow{D}_m(x) = \sum_\beta D_m[\beta] \delta(x - h\beta). \]

The equality (10) in the class of harrow shaped functions goes to equation
\[ h \overrightarrow{D}_m(x) * \overrightarrow{G}_m(x) = \delta(x), \] (11)
where
\[ \overrightarrow{G}_m(x) = \sum_\beta G_m[\beta] \delta(x - h\beta). \]

It is known [1], that the class of harrow shaped functions and the class of functions of discrete variables are isomorphic. So instead of function of discrete argument \( D_m[\beta] \) it is sufficiently to investigate the function \( \overrightarrow{D}_m(x) \), defining from equation (11).

Later on we need following well known formulas of Fourier transformation:
\[ F[f(p)] = \int f(x) \exp(2\pi ipx) dx, \]
\[ F^{-1}[f(p)] = \int f(x) \exp(-2\pi ipx) dx, \]
\[
F[f(x) \ast \varphi(x)] = F[f(x)] \cdot F[\varphi(x)],
\]
\[
F[\delta(x)] = 1.
\]

Applying to both parts of (11) Fourier transformation, we get
\[
F[D_m(x)] \cdot F[h \overrightarrow{G}_m(x)] = 1.
\] (12)

Fourier transform of \(h \overrightarrow{G}_m(p)\) is well known periodic function, given in \(R\) with period \(h^{-1}\)
\[
F[h \overrightarrow{G}_m(x)] = \frac{(-1)^m}{(2\pi)^{2m}} \sum_{\beta} \frac{1}{|p - h^{-1}\beta|^{2m}}, \quad p \neq h^{-1}\beta.
\] (13)

This formula is obtained from the equalities
\[
F[G_m(p)] = \frac{(-1)^m}{(2\pi)^{2m}} \frac{1}{|p|^{2m}} \quad (\text{[1, p. 729]})
\]
and
\[
\overrightarrow{G}_m(x) = G_m(x) \sum_{\beta} \delta(x - h\beta).
\]

Hence, taking into account (12), we get
\[
F[D_m(p)] = \left[\frac{(-1)^m}{(2\pi)^{2m}} \sum_{\beta} \frac{1}{|p - h^{-1}\beta|^{2m}}\right]^{-1}.
\] (14)

The main properties of this function in multidimensional case, appearing in construction of discrete analogue of the polyharmonic operator, were investigated in [1].

We give some of them, which we will use later on.

1. Zeros of the function \(F[D_m(p)]\) are the points \(p = h^{-1}\beta\).
2. The function \(F[D_m(p)]\) is periodic with period \(h^{-1}\), real and analytic for all real \(p\).

The function \(F[D_m(p)]\) can be represented in the form of Fourier series
\[
F[D_m(p)] = \sum_{\beta} \hat{D}_m[\beta] \exp(2\pi ih\beta p),
\] (15)
where
\[
\hat{D}_m[\beta] = \int_0^{h^{-1}} F[D_m(p)] \exp(-2\pi ih\beta p) dp.
\] (16)

Applying inverse Fourier transformation to the equality (15), we get harrow shaped function
\[
\overrightarrow{D}_m(x) = \sum_{\beta} \hat{D}_m[\beta] \delta(x - h\beta).
\] (17)
Thus, $\hat{D}_m[\beta]$ is searching function $D_m[\beta]$ of discrete argument or discrete analogue of the operator $\frac{d^{2m}}{dx^{2m}}$. For finding the function $\hat{D}_m[\beta]$ calculation of the integral (16) inadvisable. We will find it by following way.

By virtue of known formula

\[
\sum_{\beta} \frac{1}{(p - \beta)^2} = \frac{\pi^2}{\sin^2 \pi p}
\]

and from the formula (13) we get

\[
F[h \stackrel{\rightarrow}{G}_1 (p)] = -\frac{1}{(2\pi)^2} \sum_{\beta} \frac{1}{(p - h^{-1}\beta)^2} = \frac{-h^2}{4 \sin^2 \pi ph}.
\]

Hence by differentiating we have

\[
\frac{d}{dp} F[h \stackrel{\rightarrow}{G}_1 (p)] = \frac{2}{(2\pi)^2} \sum_{\beta} \frac{1}{(p - h^{-1}\beta)^3}.
\]

Thus continuing further, we obtain

\[
\frac{d^{2m-2}}{dp^{2m-2}} F[h \stackrel{\rightarrow}{G}_1 (p)] = -\frac{(2m - 1)!}{(2\pi)^{2m}} \sum_{\beta} \frac{1}{(p - h^{-1}\beta)^{2m}} =
\]

\[
= (-1)^{m-1}(2m - 1)!(2\pi)^{2m-2} F[h \stackrel{\rightarrow}{G}_m (p)].
\]

So,

\[
F[h \stackrel{\rightarrow}{G}_m (p)] = \frac{(-1)^m h^2}{2^{2m}\pi^{2m-2}(2m - 1)!} \frac{d^{2m-2}}{dp^{2m-2}} \left( \frac{1}{\sin^2 \pi hp} \right).
\]

Consider, the expression

\[
\frac{d^{2m-2}}{dp^{2m-2}} \left( \frac{1}{\sin^2 \pi hp} \right).
\]

Using

\[
\sin \pi hp = \frac{\exp(\pi ihp) - \exp(-\pi ihp)}{2i},
\]

we have

\[
\frac{d^{2m-2}}{dp^{2m-2}} \left( \frac{\exp(\pi ihp) - \exp(-\pi ihp)}{2i} \right) = -4 \frac{d^{2m-2}}{dp^{2m-2}} \left( \frac{\exp(2\pi ihp)}{(\exp(2\pi ihp) - 1)^2} \right).
\]

We will do change of variables $\lambda = \exp(2\pi ihp)$, then in view of that

\[
\frac{d}{dp} = \frac{d\lambda}{dp} \frac{d}{d\lambda} \text{ and } \frac{d}{dp} = 2\pi i h \lambda \frac{d}{d\lambda},
\]

we get

\[
\frac{d^{2m-2}}{dp^{2m-2}} = (2\pi i h)^{2m-2} D^{2m-2},
\]
where

\[ D = \frac{d}{d\lambda}, \quad D^{2m-2} = \frac{d}{d\lambda} D^{2m-3}. \]

Thus,

\[ F[h \overset{\leftrightarrow}{G}_m (p)] = \frac{h^{2m}}{(2m - 1)!} D^{2m-2} \frac{\lambda}{(1 - \lambda)^2}. \]

Hence in virtue of (4) we have

\[ F[h \overset{\leftrightarrow}{G}_m (p)] = \frac{h^{2m} \lambda E_{2m-2} (\lambda)}{(2m - 1)! (1 - \lambda)^{2m}}. \tag{18} \]

From (18), according to (12), we obtain

\[ F[\overset{\leftrightarrow}{D}_m (p)] = \frac{(2m - 1)! (1 - \lambda)^{2m}}{h^{2m} \lambda E_{2m-2} (\lambda)}. \tag{19} \]

Now in order to obtain Fourier-series expansion, we will do following.

We divide the polynomial \((1 - \lambda)^{2m}\) to the polynomial \(\lambda E_{2m-2} (\lambda)\):

\[ \frac{(1 - \lambda)^{2m}}{\lambda \sum_{s=0}^{2m-2} a_s (2m-2) \lambda^s} = \lambda - 2m - a_{2m-3}^{(2m-2)} + \frac{P_{2m-2} (\lambda)}{\lambda E_{2m-2} (\lambda)}, \tag{20} \]

where \(P_{2m-2} (\lambda)\) is a polynomial of degree \(2m - 2\). It is not difficult to see, that the rational fraction \(\frac{P_{2m-2} (\lambda)}{\lambda E_{2m-2} (\lambda)}\) is proper fraction, i.e. degree of the polynomial \(P_{2m-2} (\lambda)\) is less than degree of the polynomial \(\lambda E_{2m-2} (\lambda)\). Since the roots of the polynomial \(E_{2m-2} (\lambda)\) are real and different, then the rational fraction \(\frac{P_{2m-2} (\lambda)}{\lambda E_{2m-2} (\lambda)}\) is expanded to the sum of elementary fractions. Searching expansion has following form

\[ \frac{P_{2m-2} (\lambda)}{\lambda E_{2m-2} (\lambda)} = \frac{A_0}{\lambda} + \sum_{k=1}^{m-1} \frac{A_{1,k}}{\lambda - \lambda_{1,k}} + \sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda - \lambda_{2,k}}, \tag{21} \]

where \(A_0, A_{1,k}, A_{2,k}\) are unknown coefficients, \(\lambda_{1,k}\) are the roots of the polynomial \(E_{2m-2} (\lambda)\), in modulus less than unity, and \(\lambda_{2,k}\) are the roots of the polynomial \(E_{2m-2} (\lambda)\), in modulus greater than unity. By (21) the equality (20) takes the form

\[ \frac{(1 - \lambda)^{2m}}{\lambda \sum_{s=0}^{2m-2} a_s (2m-2) \lambda^s} = \lambda - 2m - a_{2m-3}^{(2m-2)} + \frac{A_0}{\lambda} + \]

\[ + \sum_{k=1}^{m-1} \frac{A_{1,k}}{\lambda - \lambda_{1,k}} + \sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda - \lambda_{2,k}}. \tag{22} \]

Reducing to the common denominator and omitting it, we get

\[ (1 - \lambda)^{2m} = \lambda^2 E_{2m-2} (\lambda) - \lambda (2m + a_{2m-3}^{(2m-2)}) E_{2m-2} (\lambda) + \]

\[ \vdots \]
+ A_0 E_{2m-2}(\lambda) + \sum_{k=1}^{m-1} \frac{A_{1,k} \lambda E_{2m-2}(\lambda)}{\lambda - \lambda_{1,k}} + \sum_{k=1}^{m-1} \frac{A_{2,k} \lambda E_{2m-2}(\lambda)}{\lambda - \lambda_{2,k}}. \quad (23)

Assuming in the equality (23) consequently \( \lambda = 0 \), \( \lambda = \lambda_{1,k} \) and \( \lambda = \lambda_{2,k} \), we find

\[ 1 = E_{2m-2}(0) A_0; \quad (1 - \lambda_{1,k})^{2m} = \lambda_{1,k} E'_{2m-2}(\lambda_{1,k}) A_{1,k}; \]

\[ (1 - \lambda_{2,k})^{2m} = \lambda_{2,k} E'_{2m-2}(\lambda_{2,k}) A_{2,k}. \]

Hence

\[ A_0 = 1; \quad A_{1,k} = \frac{(1 - \lambda_{1,k})^{2m}}{\lambda_{1,k} E'_{2m-2}(\lambda_{1,k})}; \quad A_{2,k} = \frac{(1 - \lambda_{2,k})^{2m}}{\lambda_{2,k} E'_{2m-2}(\lambda_{2,k})}. \]

Using (6), we have

\[ A_{1,k} = \frac{(1 - \frac{1}{\lambda_{1,k}})^{2m}}{\lambda_{1,k}^{-1} E'_{2m-2}(\frac{1}{\lambda_{1,k}})} = \frac{(\lambda_{2,k} - 1)^{2m}}{\lambda_{2,k}^{2m-1} E'_{2m-2}(\frac{1}{\lambda_{2,k}})}. \]

In virtue of (7) we obtain

\[ \frac{1}{\lambda_{2,k}} \left( 1 - \frac{1}{\lambda_{2,k}} \right) E'_{2m-2}(\frac{1}{\lambda_{2,k}}) = E_{2m-1}(\frac{1}{\lambda_{2,k}}), \]

\[ \lambda_{2,k}(1 - \lambda_{2,k}) E'_{2m-2}(\lambda_{2,k}) = E_{2m-1}(\lambda_{2,k}), \]

hence

\[ E'_{2m-2}(\frac{1}{\lambda_{2,k}}) = \frac{E_{2m-1}(\frac{1}{\lambda_{2,k}}) \lambda_{2,k}^2}{\lambda_{2,k} - 1}, \]

\[ E'_{2m-2}(\lambda_{2,k}) = \frac{E_{2m-1}(\lambda_{2,k})}{\lambda_{2,k}(1 - \lambda_{2,k})}. \]

From here application of the lemma 2 gives

\[ A_{1,k} = \frac{-A_{2,k}}{\lambda_{2,k}^2}, \quad A_{1,k} = \frac{(1 - \lambda_{1,k})^{2m+1}}{E_{2m-1}(\lambda_{1,k})}. \quad (24) \]

Since \( |\lambda_{1,k}| < 1 \) and \( |\lambda_{2,k}| > 1 \), then

\[ \sum_{k=1}^{m-1} \frac{A_{1,k}}{\lambda - \lambda_{1,k}} \quad \text{and} \quad \sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda - \lambda_{2,k}} \]

can be represented as Laurent series on the circle \( |\lambda^2| = 1 \):

\[ \sum_{k=1}^{m-1} \frac{A_{1,k}}{\lambda - \lambda_{1,k}} = \frac{1}{\lambda} \sum_{k=1}^{m-1} \frac{A_{1,k}}{\lambda - \lambda_{1,k}} = \frac{1}{\lambda} \sum_{k=1}^{m-1} A_{1,k} \sum_{\beta=0}^{\infty} \left( \frac{\lambda_{1,k}}{\lambda} \right)^\beta, \quad (25) \]

\[ \sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda - \lambda_{2,k}} = -\sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda_{2,k}(1 - \lambda/\lambda_{2,k})} = -\sum_{k=1}^{m-1} A_{2,k} \sum_{\beta=0}^{\infty} \left( \frac{\lambda}{\lambda_{2,k}} \right)^\beta. \quad (26) \]
Putting (25), (26) to (22) and taking into account $\lambda = \exp(2\pi ihp)$ from (19), (20), we obtain

$$F[\overset{\leftarrow}{D}_m(p)] = \frac{(2m-1)!}{h^{2m}} \left[ \exp(2\pi ihp) - 2m - a^{(2m-2)}_{2m-3} + \exp(-2\pi ihp) + \sum_{k=1}^{m-1} \left( A_{1,k} \sum_{\beta=0}^{\infty} \lambda_{1,k}^{\beta} \exp(-2\pi ihp(\beta + 1)) - A_{2,k} \lambda_{2,k}^{-1} \sum_{\beta=0}^{\infty} \left( \frac{1}{\lambda_{2,k}} \right)^{\beta} \exp(2\pi ihp\beta) \right) \right].$$

Thus, searching Fourier series for $F[\overset{\leftarrow}{D}_m(p)]$ have following form

$$F[\overset{\leftarrow}{D}_m(p)] = \sum_{\beta} D_m[\beta] \exp(2\pi ih\beta p),$$

where

$$D_m[\beta] = \frac{(2m-1)!}{h^{2m}} \left\{ \begin{array}{ll}
\sum_{k=1}^{m-1} A_{1,k} \lambda_{1,k}^{-\beta-1} & \text{for } \beta \leq -2,
1 + \sum_{k=1}^{m-1} A_{1,k} & \text{for } \beta = -1,
-2^{2m-1} - \sum_{k=1}^{m-1} A_{2,k} \lambda_{2,k}^{-1} & \text{for } \beta = 0,
1 - \sum_{k=1}^{m-1} A_{2,k} \lambda_{2,k}^{-2} & \text{for } \beta = 1,
- \sum_{k=1}^{m-1} A_{2,k} \lambda_{2,k}^{-\beta-1} & \text{for } \beta \geq 2.
\end{array} \right\}$$

With the help (24) the function $D_m[\beta]$ we rewrite in the form

$$D_m[\beta] = \frac{(2m-1)!}{h^{2m}} \left\{ \begin{array}{ll}
\sum_{k=1}^{m-1} \frac{(1 - \lambda_{1,k})^{2m+1} \lambda_{1,k}^{\beta}}{\lambda_{k} E_{2m-1}(\lambda_{1,k})} & \text{for } |\beta| \geq 2,
1 + \sum_{k=1}^{m-1} \frac{(1 - \lambda_{1,k})^{2m+1}}{E_{2m-1}(\lambda_{1,k})} & \text{for } |\beta| = 1,
-2^{2m-1} + \sum_{k=1}^{m-1} \frac{(1 - \lambda_{1,k})^{2m+1}}{\lambda_{1,k} E_{2m-1}(\lambda_{1,k})} & \text{for } \beta = 0.
\end{array} \right\}$$

We note, that

$$D_m[\beta] = D_m[-\beta].$$

Theorem 1 is proved completely.
4 Proofs of properties.

Proof of property 1.
Following is takes placed

\[ F[\Delta_2^{[1]}(p)] = -4 \sin^2 \pi ph. \]

Indeed,

\[ \Delta_2^{[1]}(x) = \delta(x + 1) - 2 \delta(x) + 2 \delta(x - 1) = \delta(x + h) - 2\delta(x) + \delta(x - h). \]

By definition of Fourier transformation we have

\[ F[\Delta_2^{[1]}(p)] = \int \exp(2\pi ipx) \Delta_2^{[1]}(x) dx = -4 \sin^2 \pi ph. \]

Hence consequently we obtain

\[ F[\Delta_2^{[m]}(p)] = F[\Delta_2^{[1]}(p)] \ast \Delta_2^{[1]}(p) \ast \Delta_2^{[1]}(p) = (-4)^m \sin^{2m} \pi ph. \quad (27) \]

Immediately we have

\[ \frac{(1 - \lambda)^2}{-4\lambda} = \sin^{2m} \pi ph. \quad (28) \]

By virtue of (27) and (28) the formula (19) takes form

\[ F[\overline{D_m}(p)] = \frac{(2m - 1)!}{h^{2m}} \frac{\lambda^{m-1}}{E_{2m-2}(\lambda)} F[\Delta_2^{[m]}(p)]. \quad (29) \]

Now expanding rational fraction \( \frac{\lambda^{m-1}}{E_{2m-2}(\lambda)} \) to the sum of elementary fractions, we have

\[ \frac{\lambda^{m-1}}{E_{2m-2}(\lambda)} = \sum_{k=1}^{m-1} \left[ \frac{B_{1,k}}{\lambda - \lambda_{1,k}} + \frac{B_{2,k}}{\lambda - \lambda_{2,k}} \right], \quad (30) \]

where

\[ B_{1,k} = \frac{\lambda^{m-1}}{E'_{2m-2}(\lambda_{1,k})}, \quad B_{2,k} = \frac{\lambda^{m-1}}{E'_{2m-2}(\lambda_{2,k})}. \]

Since \( |\lambda_{1,k}| < 1 \) and \( |\lambda_{2,k}| > 1 \), then expanding \( \frac{B_{1,k}}{\lambda - \lambda_{1,k}} \) and \( \frac{B_{2,k}}{\lambda - \lambda_{2,k}} \) to the Laurent series on the circle \( |\lambda| = 1 \), we find

\[ \frac{B_{1,k}}{\lambda - \lambda_{1,k}} = \frac{1}{\lambda} \cdot \frac{\lambda B_{1,k}}{1 - \lambda_{1,k}} = \frac{B_{1,k}}{\lambda} \sum_{\beta=0}^{\infty} \left( \frac{\lambda_{1,k}}{\lambda} \right)^{\beta}, \quad (31) \]

\[ \frac{B_{2,k}}{\lambda - \lambda_{2,k}} = -\frac{B_{2,k}}{\lambda_{2,k}(1 - \lambda/\lambda_{2,k})} = -\frac{B_{2,k}}{\lambda_{2,k}} \sum_{\beta=0}^{\infty} \left( \frac{\lambda}{\lambda_{2,k}} \right)^{\beta}. \quad (32) \]
On the strength of (6), (7), (8), (31), (32) the equality (30) takes the form

\[
\frac{\lambda^{m-1}}{E_{2m-2}(\lambda)} = \sum_{k=1}^{m-1} \frac{\lambda_k^{m-2}}{E'_{2m-2}(\lambda_k)} \sum_{\beta} \lambda_k^{\beta} \lambda^\beta, \quad \lambda_k = \lambda_{1,k}.
\]

So, from (29) we get

\[
F[D_m(x)] = \frac{(2m-1)!}{h^{2m}} \sum_{k=1}^{m-1} \frac{\lambda_k^{m-2}}{E'_{2m-2}(\lambda_k)} \sum_{\beta} \lambda_k^{\beta} \lambda^\beta F[D_{\Delta}^m(x)].
\]  

(33)

Applying to (33) inverse Fourier transformation, after some simplifications we obtain

\[
D_m(x) = \frac{(2m-1)!}{h^{2m}} \sum_{\beta} D_{\Delta}^m(\beta) * \sum_{k=1}^{m-1} \frac{\lambda_k^{\beta+m-2}}{E'_{2m-2}(\lambda_k)} \delta(x - h\beta) = \sum_{\beta} D_m(\beta) \delta(x - h\beta).
\]  

(34)

According to definition of harrow shaped functions from (34) we get the statement of property 1.

**Proof of property 2.**

The equality (2) proved in [1].

Here we will prove (3).

From (15), (17) we obtain

\[
F[D_m(x)] = \sum_{\beta} D_{m}(\beta) \exp(2\pi i h\beta p).
\]  

(35)

Using expansion of \(\exp(2\pi i h\beta p)\) and by formula (2) in cases \(0 \leq k \leq 2m - 1\) and \(k = 2m\), we will have

\[
F[D_m(x)] = \sum_{\beta} D_{m}(\beta) \sum_{k=0}^{\infty} \frac{(2\pi i h\beta p)^k}{k!} = \sum_{\beta} D_{m}(\beta) \sum_{k=2m+1}^{\infty} \frac{(2\pi i h\beta p)^k}{k!} + (2\pi i p)^{2m}.
\]  

(36)

On the other hand, from (14) we get

\[
F[D_m(x)] = (2\pi i)^{2m} \left[ \sum_{\beta} \frac{1}{(p - h^{-1}\beta)^{2m}} \right]^{-1}.
\]  

(37)

Thus, on the strength of (36) and (37) we have

\[
\sum_{\beta} D_{m}(\beta) \sum_{k=2m+1}^{\infty} \frac{(2\pi i h\beta p)^k}{k!} + (2\pi i p)^{2m} = (2\pi i)^{2m} \left[ \sum_{\beta} \frac{1}{(p - h^{-1}\beta)^{2m}} \right]^{-1}.
\]
or

\[ \sum_{\beta} D_m[\beta] \sum_{k=2m+1}^{\infty} \frac{(2\pi ihp\beta)^k}{k!} = -(2\pi i p)^{2m} \left( 1 - \sum_{\gamma} \frac{(ph-\gamma)^{2m}}{(ph-\gamma)^{2m}} \right)^{-1}. \]  

(38)

Consider

\[ \psi(h, p, m) = 1 - \left( \sum_{\gamma} \frac{(ph)^{2m}}{(ph-\gamma)^{2m}} \right)^{-1} = \]

\[ = 1 - \left[ 1 + (ph)^{2m} \sum_{\gamma=1}^{\infty} \left( \frac{1}{(ph-\gamma)^{2m}} + \frac{1}{(ph+\gamma)^{2m}} \right) \right]^{-1} = \]

\[ = 1 - \left[ 1 + (ph)^{2m} \sum_{\gamma=1}^{\infty} \gamma^{-2m} \left( (1 - \frac{ph}{\gamma})^{-2m} + (1 + \frac{ph}{\gamma})^{-2m} \right) \right]^{-1}. \]

Hence choosing \( p \) such, that

\[ Q = (ph)^{2m} \sum_{\gamma=1}^{\infty} \gamma^{-2m} \left( (1 - \frac{ph}{\gamma})^{-2m} + (1 + \frac{ph}{\gamma})^{-2m} \right) < 1, \]

expanding the fraction \( \frac{1}{1+Q} \) to the series of geometric progression, we have

\[ \psi(h, p, m) = (ph)^{2m} \sum_{\gamma=1}^{\infty} \gamma^{-2m} \left( (1 - \frac{ph}{\gamma})^{-2m} + (1 + \frac{ph}{\gamma})^{-2m} \right) + \]

\[ + O((hp)^{4m}) = 2(ph)^{2m} \sum_{\gamma=1}^{\infty} \gamma^{-2m} + O(h^{2m+1}). \]  

(39)

The left part of (38) we rewrite in the form

\[ \sum_{\beta} h^{2m} D_m[\beta] \sum_{k=2m+1}^{\infty} h^{k-2m} \frac{(2\pi ip\beta)^k}{k!} = \sum_{n=1}^{\infty} h^n \mu(p, n, m), \]  

(40)

where \( \mu(p, n, m) = \sum_{\beta} h^{2m} D_m[\beta] \frac{(2\pi i p\beta)^{2m+n}}{(2m+n)!} \) does not depend on \( h \), since \( h^{2m} D_m[\beta] \) does not depend \( h \), that clear from (1).

Comparing right hand sides of (39) and (40), from (38) we have \( \mu(p, n, m) = 0 \) for \( n = 1, 2, ..., 2m-1 \), i.e. for \( k = 2m+1, ..., 4m-1 \)

\[ \mu(p, 2m, m) = \sum_{\beta} h^{2m} D_m[\beta] \frac{(2\pi i p\beta)^{4m}}{(4m)!} = (-1)^{m+1} 2(2\pi)^2 p^{4m} \sum_{\gamma=1}^{\infty} \gamma^{-2m}. \]

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Hence after some calculations and by virtue of the equality
\[
\sum_{\gamma=1}^{\infty} \gamma^{-2m} = \frac{(-1)^{m-1}(2\pi)^{2m}B_{2m}}{2 \cdot (2m)!}
\]
we get
\[
\sum_{\beta} D_m[\beta][\beta]^{4m} = \frac{h^{2m}(4m)!B_{2m}}{(2m)!},
\]
which proves the property 2.

**Proof of property 3.** From (19) and (35) we have
\[
\sum_{\beta} D_m[\beta] \exp(2\pi ihp\beta) = \frac{(2m - 1)!(1 - \lambda)^{2m}}{h^{2m}\lambda E_{2m-2}(\lambda)}.
\]
Using (8), (40) and \(\lambda = \exp(2\pi ihp\beta)\), after some simplifications we get property 3.

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