Interacting Scalar and Electromagnetic Fields in $f(R, T)$ Theory of Gravity

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Within the scope of $f(R, T)$ gravity we have studied the interacting scalar and electromagnetic fields in a Bianchi type I universe. It was found that if the study is confined to the case $f(R, T) = R + \lambda f(T)$, the system is completely given by the equations similar to Einstein gravity. Moreover, the present study imposes some severe restrictions on the field equations as well.

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I. INTRODUCTION

The existence of dark matter in the Universe as well as the recent observational data [1–7] supporting the accelerated mode of expansion of the Universe pose a fundamental theoretical challenge to the Einstein theory of gravity. One of the possible ways to explain the observations is the modification of the Einstein gravity in such a way that it would give the gravitational alternative to DE. The modified theories of gravity justify the unification of DM and DE, transition from deceleration to acceleration epoch of the universe, description of hierarchy problem, dominance of effective DE, which help to solve the coincidence problem and many more. Currently there are a number of candidates for DE such as cosmological constant, quintessence, Chaplygin gas, k-essence, spinor field, tachyon etc. Another approach is to modify the general relativity itself. This is known as modified gravity. One of such theoretical model is known as f(R) gravity, in which the standard Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar R. This model has been extensively used in recent time and it was found that the late-time acceleration of the Universe can be explained within this theory [8]. Recently a generalization of f(R, T) theory of gravity was proposed by Harko et al [9], where T is the trace of stress-energy tensor. After this paper was published in 2011, many authors have investigated different problems within the scope of f(R, T) theory. The purpose of this note is to study the system of interacting scalar and electromagnetic field within the framework of this new theory and see if this new model can improve the results obtained earlier for a standard Einstein-Hilbert model [10].

II. BASIC EQUATIONS

Following [9] let us consider the action of the form

\[ S = \frac{1}{2\kappa} \int f(R, T) \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x, \]  

(2.1)

with R being the Ricci scalar curvature and \( T = g^{\mu\nu} T_{\mu\nu} \), where the energy-momentum tensor \( T_{\mu\nu} \) is given by

\[ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}} = L_m g_{\mu\nu} - 2 \frac{\partial L_m}{\partial g^{\mu\nu}}. \]  

(2.2)

It should be noted that the basic equations given here are almost the same as in [9], though at places there are some small differences. That is why I will derive the equations in details and underline the differences in due course.

Variation of action (2.1) with respect to \( g^{\mu\nu} \) gives

\[ \delta S = \frac{1}{2\kappa} \int \left( f_R \delta R + f_T \frac{\delta T}{\delta g^{\mu\nu}} \delta g^{\mu\nu} + \frac{1}{\sqrt{-g}} f \sqrt{-g} + \frac{2\kappa}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}} \right) \sqrt{-g} d^4x. \]  

(2.3)

Further taking into account that \( R = g^{\mu\nu} R_{\mu\nu} \) and \( \nabla_{\rho} g^{\mu\nu} = 0 \) one finds that

\[ \delta R = (R_{\mu\nu} + g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) \delta g^{\mu\nu}, \]  

(2.4)

where \( \Box = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \).

On the other hand

\[ \frac{\delta T}{\delta g^{\mu\nu}} = \frac{\delta (g^{\alpha\beta} T_{\alpha\beta})}{\delta g^{\mu\nu}} = \frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}} T_{\alpha\beta} + g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu}, \]  

(2.5)

where we define

\[ \Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}, \]  

(2.6)
Here let us make a remark on $\frac{\delta g^{\alpha \beta}}{\delta g^{\mu \nu}}$. Here we define

$$\frac{\delta g^{\alpha \beta}}{\delta g^{\mu \nu}} = \delta^\alpha_\mu \delta^\beta_\nu. \quad (2.7)$$

Like this case we come to the expression

$$\frac{\delta g^{\alpha \beta}}{\delta g^{\mu \nu}} T_{\alpha \beta} = T_{\mu \nu}, \quad (2.8)$$

Alternatively we can define

$$\frac{\delta g^{\alpha \beta}}{\delta g^{\mu \nu}} = \frac{1}{2} \left[ \delta^\alpha_\mu \delta^\beta_\nu + \delta^\alpha_\nu \delta^\beta_\mu \right], \quad (2.9)$$

which will leave the results unaltered. In [9], the authors name $\frac{\delta g^{\alpha \beta}}{\delta g^{\mu \nu}} = \delta^\alpha_\mu \delta^\beta_\nu$ the Generalized Kronecker Symbol, which in fact is totally antisymmetric (see Mathematical encyclopedia)

$$\frac{\delta g^{\alpha \beta}}{\delta g^{\mu \nu}} = \delta^\alpha_\mu \delta^\beta_\nu = \left[ \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu \right], \quad (2.10)$$

and in no way can give the results they have. In this case for example we have $\frac{\delta g^{\alpha \beta}}{\delta g^{\mu \nu}} T_{\alpha \beta} = 0$.

Hence after the integration we get the following equation

$$f_R(R, T) R_{\mu \nu} - \frac{1}{2} f(R, T) g_{\mu \nu} + \left( g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu \right) f_R(R, T)$$

$$= \kappa T_{\mu \nu} - f_T(R, T) \left( T_{\mu \nu} + \Theta_{\mu \nu} \right) \quad (2.11)$$

Introducing $G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R$ this equation can be rewritten as

$$f_R(R, T) G_{\mu \nu} - \frac{1}{2} \left( f(R, T) - R f_R(R, T) \right) g_{\mu \nu} + \left( g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu \right) f_R(R, T)$$

$$= \kappa T_{\mu \nu} - f_T(R, T) \left( T_{\mu \nu} + \Theta_{\mu \nu} \right) \quad (2.12)$$

Taking the trace of (2.11) one finds

$$\Box f_R(R, T) = \frac{2}{3} f(R, T) - \frac{1}{3} f_R(R, T) R + \kappa - \frac{1}{3} f_T(R, T) T - \frac{1}{3} f_T(R, T) \Theta. \quad (2.13)$$

Then inserting (2.13) into (2.11) one finds

$$f_R(R, T) G_{\mu \nu} + \frac{g_{\mu \nu}}{6} \left[ f(R, T) + f_R(R, T) R \right] - \nabla_\mu \nabla_\nu f_R(R, T)$$

$$= (\kappa - f_T(R, T)) \left[ T_{\mu \nu} - \frac{1}{3} g_{\mu \nu} T \right] - f_T(R, T) \left[ \Theta_{\mu \nu} - \frac{1}{3} g_{\mu \nu} \Theta \right]. \quad (2.14)$$

It should be noted that the Eq. (2.14) can be used only in case $f_R(R, T) \neq \text{const.}$, otherwise, i.e., if $f_R(R, T) = \text{const.}$, the corresponding equation will be

$$f_R(R, T) R_{\mu \nu} - \frac{1}{2} f(R, T) g_{\mu \nu} = \kappa T_{\mu \nu} - f_T(R, T) \left( T_{\mu \nu} + \Theta_{\mu \nu} \right) \quad (2.15)$$
And finally, taking into account that
\[ \nabla_\mu \nabla_\nu f(R, T) = \frac{\partial^2 f_R(R, T)}{\partial x^\mu \partial x^\nu} + \Gamma^\alpha_{\mu\nu} \frac{\partial f_R(R, T)}{\partial x^\alpha} \]
we rewrite (2.14) in the form
\[
G_{\mu\nu} = \frac{1}{f_R(R, T)} \left[ \frac{\partial^2 f_R(R, T)}{\partial x^\mu \partial x^\nu} + \Gamma^\alpha_{\mu\nu} \frac{\partial f_R(R, T)}{\partial x^\alpha} \right] - \frac{g_{\mu\nu}}{6f_R(R, T)} [f(R, T) + f_R(R, T)R]
+ \frac{\kappa - f_T(R, T)}{f_R(R, T)} \left[ T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T \right] - \frac{f_T(R, T)}{f_R(R, T)} \left[ \Theta_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \Theta \right]
\] (2.16)
Further taking into account that
\[
\frac{\partial f_R(R, T)}{\partial x^\alpha} = \frac{\partial f_R(R, T)}{\partial R} \frac{dR}{dx^\alpha} + \frac{\partial f_R(R, T)}{\partial T} \frac{dT}{dx^\alpha}
\]
and
\[
\frac{\partial^2 f_R(R, T)}{\partial x^\mu \partial x^\nu} = \frac{\partial^2 f_R(R, T)}{\partial R^2} \frac{dR}{dx^\mu} \frac{dR}{dx^\nu} + \frac{\partial^2 f_R(R, T)}{\partial T \partial R} \frac{dT}{dx^\mu} \frac{dR}{dx^\nu} + \frac{\partial f_R(R, T)}{\partial R} \frac{d^2 R}{dx^\mu dx^\nu}
+ \frac{\partial^2 f_R(R, T)}{\partial T^2} \frac{dT}{dx^\mu} \frac{dT}{dx^\nu} + \frac{\partial f_R(R, T)}{\partial R} \frac{d^2 T}{dx^\mu dx^\nu}
+ \Gamma^\alpha_{\mu\nu} \left[ \frac{\partial f_R(R, T)}{\partial R} \frac{dR}{dx^\alpha} + \frac{\partial f_R(R, T)}{\partial T} \frac{dT}{dx^\alpha} \right] - \frac{g_{\mu\nu}}{6f_R(R, T)} [f(R, T) + f_R(R, T)R]
+ \frac{\kappa - f_T(R, T)}{f_R(R, T)} \left[ T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T \right] - \frac{f_T(R, T)}{f_R(R, T)} \left[ \Theta_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \Theta \right]
\] (2.17)
Divergence of (2.12) leads to
\[
(\nabla^\mu f_R(R, T)) G_{\mu\nu} - \frac{1}{2} (\nabla_\nu f(R, T) - (\nabla_\nu R) f(R, T) - R \nabla_\nu f_R(R, T)) + (\nabla_\nu \Box - \Box \nabla_\nu) f_R(R, T)
= (\kappa - f_T(R, T)) \nabla^\mu T_{\mu\nu} - (\nabla^\mu f_T(R, T)) (T_{\mu\nu} + \Theta_{\mu\nu})
= f_T(R, T) \nabla^\mu \Theta_{\mu\nu},
\] (2.18)
where we used the fact that \( g_{\mu\nu} \nabla^\mu f(R, T) = \nabla_\nu f(R, T) = f_R \nabla_\nu R + f_T \nabla_\nu T = f_R g_{\mu\nu} \nabla^\mu R + f_T \nabla_\nu T, \quad \nabla^\mu G_{\mu\nu} = 0, \quad \nabla^\mu g_{\mu\nu} = 0 \) and
\[
(\nabla_\nu \Box - \Box \nabla_\nu) f_R(R, T) = - (\nabla^\mu f_R(R, T)) R_{\mu\nu}.
\] (2.19)
The relation (2.19) follows from the fact that
\[
(\nabla_\nu \Box - \Box \nabla_\nu) S = g^{\alpha\beta} (\nabla_\alpha \nabla_\beta S - \nabla_\alpha \nabla_\beta \nabla_\nu S)
= g^{\alpha\beta} (\nabla_\alpha S_{\beta} + \nabla_\beta S_{\alpha} - \nabla_{\alpha\nu} S_{\beta})
= g^{\alpha\beta} R^\eta_{\beta\alpha\nu} S_{\eta} = - R^\eta_{\nu} S_{\eta}.
\] (2.20)
where we used the fact that for any scalar $S$

$$\nabla_\beta \nabla_\nu S = \nabla_\nu \nabla_\beta S, \quad (2.21)$$

and for any vector $S_{\alpha}^{\beta}$

$$\nabla_\nu \nabla_\alpha S_{\beta}^{\gamma} - \nabla_\alpha \nabla_\nu S_{\beta}^{\gamma} = R^{\eta}_{\beta \alpha \nu} S;_{\eta} = -R^{\eta}_{\beta \nu \alpha} S;_{\eta}. \quad (2.22)$$

We also denote $\nabla_\beta S = S_{\beta}$. After a little manipulation from (2.18) we find

$$(\kappa - f_T(R, T)) \nabla_\mu T_{\mu \nu} = (\nabla_\mu f_T(R, T)) (T_{\mu \nu} + \Theta_{\mu \nu}) + f_T(R, T) \nabla_\mu \Theta_{\mu \nu} - \frac{1}{2} f_T \nabla_\nu T. \quad (2.23)$$

which in view of $\nabla_\mu f_T(R, T) = f_T(R, T) \nabla_\mu \ln f_T(R, T)$ gives

$$\nabla_\mu T_{\mu \nu} = \frac{f_T(R, T)}{\kappa - f_T(R, T)} \left[ (T_{\mu \nu} + \Theta_{\mu \nu}) \nabla_\mu \ln f_T(R, T) + \Theta_{\mu \nu} - \frac{1}{2} \nabla_\nu T \right]. \quad (2.24)$$

Note that unlike many authors here we have an additional term $-\frac{1}{2} \nabla_\nu T$ which comes from $\nabla_\nu f(R, T)$.

Let us now calculate the tensor $\Theta_{\mu \nu}$. Varying (2.2) with respect to metric function we find

$$\frac{\delta T_{\alpha \beta}}{\delta g^\mu \nu} = \frac{\delta g_{\alpha \beta}}{\delta g^\mu \nu} L_m + g_{\alpha \beta} \frac{\partial L_m}{\partial g^\mu \nu} - 2 \frac{\partial^2 L_m}{\partial g_{\alpha \beta} \partial g^\mu \nu}$$

$$= \frac{\delta g_{\alpha \beta}}{\delta g^\mu \nu} L_m + \frac{1}{2} g_{\alpha \beta} [g_{\mu \nu} L_m - T_{\mu \nu}] - 2 \frac{\partial^2 L_m}{\partial g_{\alpha \beta} \partial g^\mu \nu}. \quad (2.25)$$

From the condition $g_{\alpha \sigma} g^{\sigma \beta} = \delta^\beta_{\alpha}$, we have

$$\frac{\delta g_{\alpha \beta}}{\delta g^\mu \nu} = -g_{\alpha \eta} g_{\beta \tau} \frac{\delta g^{\eta \tau}}{\delta g^\mu \nu}. \quad (2.26)$$

Further using the definition (2.7), we find

$$\Theta_{\mu \nu} = g^{\alpha \beta} \frac{\delta T_{\alpha \beta}}{\delta g^\mu \nu} = -2 T_{\mu \nu} + g_{\mu \nu} L_m - 2 g^{\alpha \beta} \frac{\partial^2 L_m}{\partial g^\mu \nu \partial g_{\alpha \beta}}. \quad (2.27)$$

Further on account of $g_{\mu \nu} L_m = T_{\mu \nu} + 2 \frac{\partial L_m}{\partial g^\mu \nu}$ we write

$$\Theta_{\mu \nu} = -T_{\mu \nu} + 2 \left[ \frac{\partial L_m}{\partial g^\mu \nu} - g^{\alpha \beta} \frac{\partial^2 L_m}{\partial g^\mu \nu \partial g_{\alpha \beta}} \right]. \quad (2.28)$$
III. MATTER FIELD LAGRANGIAN

In what follows we consider the electromagnetic field with induced massive term given by the Lagrangian [13]

$$\mathcal{L} = \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \varphi_{,\eta} \varphi_{,\eta} \mathcal{G} \right), \quad \mathcal{G} = (1 + \Phi(I)), \quad I = A_{\mu} A^{\mu}. \quad (3.1)$$

It should be noted that since the early days of elementary particle physics, attempts were undertaken to construct a divergence-free theory. A nonlinear modification of Maxwell theory was proposed by Mie[11, 12], with the nonlinear electric current of the form

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Then for $\Theta_{\mu \nu}$ one gets

$$\Theta_{\mu \nu} = -T_{\mu \nu} + (\mathcal{G} - I \mathcal{G}_1) \varphi_{,\mu} \varphi_{,\nu} - I \mathcal{G}_{11} \varphi_{,\eta} \varphi_{,\eta} A_{\mu} A_{\nu}, \quad (3.4)$$

with

$$T_{\mu \nu} = \left[ F_{\mu \alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \right] + \left[ \frac{1}{2} g_{\mu \nu} - \mathcal{G}_1 A_{\mu} A_{\nu} \right] \varphi_{,\alpha} \varphi_{,\alpha} - \mathcal{G}_1 \varphi_{,\mu} \varphi_{,\nu}. \quad (3.5)$$

From (3.4) and (3.5) one finds

$$\Theta = -T + (\mathcal{G} - I \mathcal{G}_1 - I^2 \mathcal{G}_{11}) \varphi_{,\alpha} \varphi_{,\alpha} = -I^2 \mathcal{G}_{11} \varphi_{,\alpha} \varphi_{,\alpha} \quad (3.6)$$

and

$$T = (\mathcal{G} - I \mathcal{G}_1) \varphi_{,\alpha} \varphi_{,\alpha} \quad (3.7)$$

In literature it is customary to consider a few cases such as

(i) $f(R, T) = R + \lambda f(T),$

(ii) $f(R, T) = f_1(R) + f_2(T),$

(iii) $f(R, T) = f_1(R) + f_2(T) f_3(T).$

In this paper we consider only the simplest case setting $f(R, T) = R + \lambda f(T).$

In this case from (2.11) we find

$$G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \propto T_{\mu \nu} - \lambda (\Theta_{\mu \nu} + T_{\mu \nu}) + \frac{1}{2} g_{\mu \nu} \lambda T. \quad (3.8)$$

From this equation one can easily obtain

$$R_{\mu}^{\nu} = \propto \left( T_{\mu}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} T \right) - \lambda \left( \Theta_{\mu}^{\nu} + T_{\mu}^{\nu} \right) + \frac{1}{2} \lambda \delta_{\mu}^{\nu} \Theta. \quad (3.9)$$

We consider the BI metric in the form

$$d^2 s = e^{2\alpha} dt^2 - e^{2\beta_1} dx^2 - e^{2\beta_2} dy^2 - e^{2\beta_3} dz^2. \quad (3.10)$$
The metric functions $\alpha, \beta_1, \beta_2, \beta_3$ depend on $t$ only and obey the coordinate condition

$$\alpha = \beta_1 + \beta_2 + \beta_3. \quad (3.11)$$

Note that the Ricci tensor for the Bianchi type-I model given in the form (3.10) has only nonzero diagonal components. So here we write only the diagonal components of the system of equations (3.9), while the off-diagonal components we use later in order to find the relations between metric functions and the components of the vector potential. On account of (3.11) we find

$$e^{-2\alpha} \left( \dot{\alpha} - \dot{\alpha}^2 + \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2 \right) = \kappa \left( T_0^0 - \frac{1}{2} T \right) - \lambda \left( \Theta_0^0 + \Theta_T^0 \right) + \frac{1}{2} \lambda \Theta, \quad (3.12a)$$

$$e^{-2\alpha} \dot{\beta}_1 = \kappa \left( T_1^1 - \frac{1}{2} T \right) - \lambda \left( \Theta_1^1 + \Theta_T^1 \right) + \frac{1}{2} \lambda \Theta, \quad (3.12b)$$

$$e^{-2\alpha} \dot{\beta}_2 = \kappa \left( T_2^2 - \frac{1}{2} T \right) - \lambda \left( \Theta_2^2 + \Theta_T^2 \right) + \frac{1}{2} \lambda \Theta, \quad (3.12c)$$

$$e^{-2\alpha} \dot{\beta}_3 = \kappa \left( T_3^3 - \frac{1}{2} T \right) - \lambda \left( \Theta_3^3 + \Theta_T^3 \right) + \frac{1}{2} \lambda \Theta, \quad (3.12d)$$

where over dot means differentiation with respect to $t$.

Variation of (3.1) with respect to electromagnetic field gives

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left( \sqrt{-g} F_{\mu \nu} \right) - (\varphi, \varphi) \partial_\nu A^\mu = 0, \quad \partial_\mu = \frac{d\mathcal{G}}{dt}. \quad (3.13)$$

The scalar field equation corresponding to the Lagrangian (3.1) has the form

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu \nu} \varphi \mathcal{G} \right) = 0. \quad (3.14)$$

We consider the case when the electromagnetic and scalar fields are the functions of $t$ only. Taking this in mind we choose the vector potential in the following way:

$$A_\mu = (0, A_1(t), A_2(t), A_3(t)). \quad (3.15)$$

In this case the electromagnetic field tensor $F_{\mu \nu}$ has three non-vanishing components, namely

$$F_{01} = \dot{A}_1, \quad F_{02} = \dot{A}_2, \quad F_{03} = \dot{A}_3. \quad (3.16)$$

On account of (3.15) and (3.16) we now have

$$I = -\dot{A}_1^2 e^{-2\beta_1} - \dot{A}_2^2 e^{-2\beta_2} - \dot{A}_3^2 e^{-2\beta_3}, \quad (3.17)$$

$$F_{\mu \nu} F^{\mu \nu} = -2 e^{-2\alpha} \left( \dot{A}_1^2 e^{-2\beta_1} + \dot{A}_2^2 e^{-2\beta_2} + \dot{A}_3^2 e^{-2\beta_3} \right). \quad (3.18)$$

Let us now solve the scalar field equation. Taking into account that $\varphi = \varphi(t)$, from the scalar field equation one finds

$$\varphi = \varphi_0, \quad \Rightarrow \varphi_\mu \varphi^\mu = \varphi_0^2 e^{-2\alpha}, \quad \varphi_0 = \text{const.} \quad (3.19)$$

On account of (3.16) and (3.19) from (3.13) for electromagnetic field we find

$$\frac{d}{dt} \left( \dot{A}_1^2 e^{-2\beta_1} \right) - \varphi_0^2 \partial_t \dot{A}_1^2 e^{-2\beta_1} = 0, \quad (3.20a)$$

$$\frac{d}{dt} \left( \dot{A}_2^2 e^{-2\beta_2} \right) - \varphi_0^2 \partial_t \dot{A}_2^2 e^{-2\beta_2} = 0, \quad (3.20b)$$

$$\frac{d}{dt} \left( \dot{A}_3^2 e^{-2\beta_3} \right) - \varphi_0^2 \partial_t \dot{A}_3^2 e^{-2\beta_3} = 0, \quad (3.20c)$$
where we set $P(I) = 1/\mathcal{G}(I)$.

The nonzero components of the energy momentum tensor of material fields. In view of (3.19) from (3.5) we find

\[
T_0^0 = \left[ \frac{\phi_0^2 P}{2} + \frac{1}{2} \left( \dot{A}_1^2 e^{-2\beta_1} + \dot{A}_2^2 e^{-2\beta_2} + \dot{A}_3^2 e^{-2\beta_3} \right) \right] e^{-2\alpha},
\]

(3.21a)

\[
T_1^1 = \left[ -\frac{\phi_0^2 P}{2} + \frac{1}{2} \left( \dot{A}_1^2 e^{-2\beta_1} - \dot{A}_2^2 e^{-2\beta_2} - \dot{A}_3^2 e^{-2\beta_3} \right) + \phi_0^2 P \dot{A}_1^2 e^{-2\beta_1} \right] e^{-2\alpha},
\]

(3.21b)

\[
T_2^2 = \left[ -\frac{\phi_0^2 P}{2} + \frac{1}{2} \left( \dot{A}_2^2 e^{-2\beta_2} - \dot{A}_3^2 e^{-2\beta_3} - \dot{A}_1^2 e^{-2\beta_1} \right) + \phi_0^2 P \dot{A}_2^2 e^{-2\beta_2} \right] e^{-2\alpha},
\]

(3.21c)

\[
T_3^3 = \left[ -\frac{\phi_0^2 P}{2} + \frac{1}{2} \left( \dot{A}_3^2 e^{-2\beta_3} - \dot{A}_1^2 e^{-2\beta_1} - \dot{A}_2^2 e^{-2\beta_2} \right) + \phi_0^2 P \dot{A}_3^2 e^{-2\beta_3} \right] e^{-2\alpha},
\]

(3.21d)

\[
T_2^1 = (\dot{A}_1 \dot{A}_2 + \phi_0^2 P \dot{A}_1 \dot{A}_2) e^{-2\alpha - 2\beta_1},
\]

(3.21e)

\[
T_3^2 = (\dot{A}_2 \dot{A}_3 + \phi_0^2 P \dot{A}_2 \dot{A}_3) e^{-2\alpha - 2\beta_2},
\]

(3.21f)

\[
T_1^3 = (\dot{A}_3 \dot{A}_1 + \phi_0^2 P \dot{A}_3 \dot{A}_1) e^{-2\alpha - 2\beta_3}.
\]

(3.21g)

From (3.21) one also finds

\[
T = \left[ -\phi_0^2 P + \phi_0^2 P \left( \dot{A}_1^2 e^{-2\beta_1} + \dot{A}_2^2 e^{-2\beta_2} + \dot{A}_3^2 e^{-2\beta_3} \right) \right] e^{-2\alpha} = -\phi_0^2 [P + IP] e^{-2\alpha}.
\]

(3.22)

From (3.4) we now find

\[
\Theta_0^0 = -T_0^0 + (\mathcal{G} - \mathcal{G} \mathcal{L}) \phi_0^2 e^{-2\alpha} = -T_0^0 + \phi_0^2 (P + IP) e^{-2\alpha},
\]

(3.23a)

\[
\Theta_1^1 = -T_1^1 + \frac{\phi_0^2}{\mathcal{G}^2} I \mathcal{G}_\mathcal{L} \dot{A}_1^2 e^{-2(\alpha + \beta_1)},
\]

(3.23b)

\[
\Theta_2^2 = -T_2^2 + \frac{\phi_0^2}{\mathcal{G}^2} I \mathcal{G}_\mathcal{L} \dot{A}_2^2 e^{-2(\alpha + \beta_2)},
\]

(3.23c)

\[
\Theta_3^3 = -T_3^3 + \frac{\phi_0^2}{\mathcal{G}^2} I \mathcal{G}_\mathcal{L} \dot{A}_3^2 e^{-2(\alpha + \beta_3)},
\]

(3.23d)

\[
\Theta_2^1 = -T_2^1 + \frac{\phi_0^2}{\mathcal{G}^2} I \mathcal{G}_\mathcal{L} \dot{A}_1 \dot{A}_2 e^{-2(\alpha + \beta_1)},
\]

(3.23e)

\[
\Theta_3^2 = -T_3^2 + \frac{\phi_0^2}{\mathcal{G}^2} I \mathcal{G}_\mathcal{L} \dot{A}_2 \dot{A}_3 e^{-2(\alpha + \beta_2)},
\]

(3.23f)

\[
\Theta_1^3 = -T_1^3 + \frac{\phi_0^2}{\mathcal{G}^2} I \mathcal{G}_\mathcal{L} \dot{A}_3 \dot{A}_1 e^{-2(\alpha + \beta_3)}.
\]

(3.23g)

From (3.21) one also finds

\[
\Theta = -\frac{\phi_0^2}{\mathcal{G}^2} I^2 \mathcal{G}_\mathcal{L} e^{-2\alpha} = -\phi_0^2 I^2 \left[ \frac{2}{P} P^2 - P_{II} \right] e^{-2\alpha}.
\]

(3.24)

Taking into account that the BI metric given by (3.10) has only nonzero diagonal components
from (3.8) we have the following constraints:

\[ \alpha T_2^1 - \lambda (\Theta_2^1 + T_2^1) = \left[ \kappa \left( \dot{A}_1 \dot{A}_2 + \phi_0^2 P_1 A_1 A_2 \right) - \frac{\lambda}{\kappa} \frac{\phi_0^2}{\gamma^2} I G_{11} A_1 A_2 \right] e^{-2(\alpha + \beta_1)} = 0, \quad (3.25a) \]

\[ \alpha T_3^2 - \lambda (\Theta_3^2 + T_3^2) = \left[ \kappa \left( \dot{A}_2 \dot{A}_3 + \phi_0^2 P_1 A_2 A_3 \right) - \frac{\lambda}{\kappa} \frac{\phi_0^2}{\gamma^2} I G_{11} A_2 A_3 \right] e^{-2(\alpha + \beta_2)} = 0, \quad (3.25b) \]

\[ \alpha T_1^3 - \lambda (\Theta_1^3 + T_1^3) = \left[ \kappa \left( \dot{A}_3 \dot{A}_1 + \phi_0^2 P_1 A_3 A_1 \right) - \frac{\lambda}{\kappa} \frac{\phi_0^2}{\gamma^2} I G_{11} A_3 A_1 \right] e^{-2(\alpha + \beta_3)} = 0. \quad (3.25c) \]

From (3.25) one finds

\[ \frac{\dot{A}_1 A_2}{A_1 A_2} = \frac{\dot{A}_2 A_3}{A_2 A_3} = \frac{\dot{A}_3 A_1}{A_3 A_1} = \frac{\dot{A}_1}{A_1} = \frac{\dot{A}_2}{A_2} = \frac{\dot{A}_3}{A_3} = \frac{\dot{A}}{A}, \quad (3.27) \]

that leads to the following relations between the three components of vector potential:

\[ A_1 = A, \quad A_2 = C_{21} A, \quad A_3 = C_{31} A, \quad (3.28) \]

with \( C_{21} \) and \( C_{31} \) being constants of integration. In view of (3.27) we find

\[ A_1^2 e^{-2\beta_1} + A_2^2 e^{-2\beta_2} + A_3^2 e^{-2\beta_3} = \left( A_1^2 e^{-2\beta_1} + A_2^2 e^{-2\beta_2} + A_3^2 e^{-2\beta_3} \right) \frac{A_1^2}{A^2} = \left( \frac{\phi_0^2 P_1 - \lambda}{\kappa} I \left[ \frac{2}{P} P_1^2 - P_{11} \right] \right) I, \quad (3.29) \]

where we have used the relation (3.26). On account of (3.29) further we find

\[ T_0^0 = \left[ \frac{1}{2} \phi_0^2 (P + IP_1) - \frac{\lambda}{2\kappa} \phi_0^2 I^2 \left[ \frac{2}{P} P_1^2 - P_{11} \right] \right] e^{-2\alpha}, \quad (3.30a) \]

\[ T_1^1 = -T_0^0 + \frac{\lambda}{\kappa} \phi_0^2 I \left[ \frac{2}{P} P_1^2 - P_{11} \right] A_1^2 e^{-2(\alpha + \beta_1)}, \quad (3.30b) \]

\[ T_2^2 = -T_0^0 + \frac{\lambda}{\kappa} \phi_0^2 I \left[ \frac{2}{P} P_1^2 - P_{11} \right] A_2^2 e^{-2(\alpha + \beta_2)}, \quad (3.30c) \]

\[ T_3^3 = -T_0^0 + \frac{\lambda}{\kappa} \phi_0^2 I \left[ \frac{2}{P} P_1^2 - P_{11} \right] A_3^2 e^{-2(\alpha + \beta_3)}. \quad (3.30d) \]

Inserting (3.30), (3.23) and (3.24) into (3.12) for the metric functions one finds:

\[ \bar{\alpha} - \dot{\alpha}^2 + \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2 = (\kappa - \lambda) \phi_0^2 P_1 + \lambda \phi_0^2 I^2 \left[ \frac{2}{P} P_1^2 - P_{11} \right], \quad (3.31a) \]

\[ \dot{\beta}_1 = 0, \quad (3.31b) \]

\[ \dot{\beta}_2 = 0, \quad (3.31c) \]

\[ \dot{\beta}_3 = 0. \quad (3.31d) \]

From (3.31b), (3.31c) and (3.31d) we find

\[ \beta_1 = b_1 t + \beta_{10}, \quad \beta_2 = b_2 t + \beta_{20}, \quad \beta_3 = b_3 t + \beta_{30}. \quad (3.32) \]
Here $b_i$ and $\beta_{i0}$ are integration constants. It should be noted that in order to maintain the same scaling along all the axes, the constants $\beta_{i0}$ should be the same, hence, without losing generality one can set $\beta_{i0} = 0$. Let us note that the Eqs. (3.31a) - (3.31c) coincide with those obtained in [10], while the (3.31d) has the additional term, namely $\lambda \varphi_0^2 P_2^2 P_2^2 G_{II}$. In case of $\lambda = 0$ we come to the standard Einstein case, which is obvious. Moreover, for $G_{II} = 0$, i.e. $G = C_1 I + C_2$ be a linear function of invariant $I$ we get the analogous result.

Now let us go back to the electromagnetic field equations (3.20), which can be arranged as

\[
\frac{\dot{A}_1}{A_1} + \left(\frac{\dot{A}_1}{A_1}\right)^2 - 2\left(\frac{\dot{A}_1}{A_1}\right)\beta_1 - \varphi_0^2 P_I = 0, \tag{3.33a}
\]

\[
\frac{\dot{A}_2}{A_2} + \left(\frac{\dot{A}_2}{A_2}\right)^2 - 2\left(\frac{\dot{A}_2}{A_2}\right)\beta_2 - \varphi_0^2 P_I = 0, \tag{3.33b}
\]

\[
\frac{\dot{A}_3}{A_3} + \left(\frac{\dot{A}_3}{A_3}\right)^2 - 2\left(\frac{\dot{A}_3}{A_3}\right)\beta_3 - \varphi_0^2 P_I = 0. \tag{3.33c}
\]

In view of (3.26) from (3.33) we conclude that

\[
\dot{\beta}_1 = \dot{\beta}_2 = \dot{\beta}_3, \tag{3.34}
\]

which is equivalent to $b_1 = b_2 = b_3 = b$ in (3.32), i.e.

\[
\beta_1 = \beta_2 = \beta_3 = bt. \tag{3.35}
\]

As one sees from (3.35) we have isotropy at any given time.

Now taking into account that both $A_2$ and $A_3$ can be expressed in term of $A_1$ one could solve only one of the three equations of (3.20). In view of (3.28) and (3.35) let us first rewrite (3.17) as follows

\[
I = -QA e^{-2bt}, \quad Q = [1 + C_{21}^2 + C_{31}^2]. \tag{3.36}
\]

In view of (3.26) and (3.35) the equation for $A$ now reads

\[
A\ddot{A} + \dot{A}^2 - 2bA\dot{A} - \frac{\lambda}{2} \varphi_0^2 I \left[ \frac{2}{P_I} P_I^2 - P_{II} \right] A^2 = 0. \tag{3.37}
\]

In case of $\lambda = 0$ or $G_{II} = \left[ \frac{2}{P_I} P_I^2 - P_{II} \right] = 0$, the equation (3.37) can be easily solved [10]. In this case we have

\[
A\ddot{A} + \dot{A}^2 - 2bA\dot{A} = 0. \tag{3.38}
\]

with the solution

\[
A = \sqrt{D^2 e^{2bt} - C/b}, \tag{3.39}
\]

with $D$ and $C$ being the constants of integration. Inserting $A$ from (3.39) into (3.26) we find

\[
\frac{dP}{dI} = -\frac{1}{\varphi_0^2} \left(\frac{\dot{A}}{A}\right)^2 = -\frac{1}{\varphi_0^2} \frac{b^2 Q^2 D^4}{I^2}. \tag{3.40}
\]

The second equality was obtained using (3.36). From (3.40) one finds that

\[
P = \frac{b^2 Q^2 D^4}{\varphi_0^2 I} + C_1, \tag{3.41}
\]
with $C_1$ being some integration constant. Note that the above result is valid for both $\lambda = 0$ or $\mathcal{G}_{II} = \left[ \frac{2}{P} P_I^2 - P_{II} \right] = 0$. In case of $\lambda = 0$ the equation (3.31a) takes the form

$$\ddot{\alpha} - \alpha^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = \kappa \phi_0^2 [P + IP],$$

(3.42)

Now taking into account that $\alpha = \beta_1 + \beta_2 + \beta_3 = 3\beta = 3bt$, from (3.42) we find

$$6b^2 = -\kappa \phi_0^2 C_1.$$

(3.43)

That is in case of Einstein theory we get the foregoing solution. Now let us see, what happens if we consider $f(R, T)$ theory. In this case the solution (3.39) occurs due to

$$\frac{2}{P} P_I^2 - P_{II} = 0,$$

(3.44)

which leads to

$$P = \frac{1}{C_2 + C_3},$$

(3.45)

where $C_2$ and $C_3$ are integration constants. Comparing (3.41) and (3.45) we conclude that $C_1 = C_3 = 0$ and $C_2 = \phi_0^2 / b^2 Q^2 D^4$. That is in this case we have $P = \frac{b^2 Q^2 D^4}{\phi_0^2}$. Now inserting $\alpha, \beta_i$'s and $P$ into (3.42) we find

$$6b^2 = 0.$$

(3.46)

It means in this case the spacetime becomes flat. Hence we see that the consideration of $f(R, T) = R + \lambda f(T)$ leads to the flat spacetime.

Let us consider the case when $P = I^n$. In this case on account of (3.36) equation (3.37) can be written as

$$A \ddot{A} + \dot{A}^2 - 2bA\ddot{A} + \frac{\lambda}{\kappa} \phi_0^2 n(n+1)P n^2 e^{2bt} = 0.$$

(3.47)

On account of (3.36) the foregoing equation can be rewritten as

$$\ddot{X} + p \dot{X} + qX^ne^{2(1-n)bt} = 0,$$

(3.48)

where we denote $X = A^2$, $p = -2b$ and $q = (-1)^n \frac{2\lambda}{\kappa} \phi_0^2 n(n+1)Q^{n-1}$. In case of $n = 1$ from (3.48) we find

$$\ddot{X} + p \dot{X} + qX = 0,$$

(3.49)

which is the equation for free oscillation. Depending on $p$ and $q$ (3.49) allows following three solutions:

$$X = \begin{cases} C_1 e^{-\frac{D - \lambda}{2} t} + C_2 e^{-\frac{D + \lambda}{2} t} & D^2 > 0 \\ e^{-\frac{\sqrt{D}}{2}} \left( C_1 \cos \frac{D t}{2} + C_2 \sin \frac{D t}{2} \right) & D^2 < 0 \\ e^{-\frac{\sqrt{D}}{2}} \left( C_1 t + C_2 \right) & D^2 = 0, \end{cases}$$

(3.50)

where we denote $D^2 = p^2 - 4q$. On account of $q = -4\lambda \phi_0^2 / \kappa$ in this case we have $D^2 = 4 \left( b^2 + 4\lambda \phi_0^2 / \kappa \right)$.

On the other hand inserting $P = I^n$ into (3.31a) we find

$$6b^2 = \phi_0^2 (1 + n) [\lambda (1 + n) - \kappa] I^n.$$

(3.51)

Since $b$ is a constant, this (3.51) can be valid only if $n = 0$ which gives $6b^2 = \phi_0^2 [\lambda - \kappa]$. It means in the case considered we have $P = 1 \Rightarrow \mathcal{G} = 1$, i.e. we come to the case of minimal coupling.
IV. CONCLUSION

Within the scope of $f(R, T)$ theory of gravity an interacting system of scalar and electromagnetic fields has been thoroughly investigated. As one can expect, the case with $f(R, T) = R + \lambda f(T)$ leaves the system qualitatively same though at places bring some additional restrictions on the solutions.

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