DIFFERENTIAL GEOMETRY OF WEIGHTINGS

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Abstract. We describe the notion of a weighting along a submanifold $N \subseteq M$, and explore its differential-geometric implications. This includes a detailed discussion of weighted normal bundles, weighted deformation spaces, and weighted blow-ups. We give a description of weightings in terms of subbundles of higher tangent bundles, which leads us to notions of multiplicative weightings for Lie algebroids and Lie groupoids.

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1. Introduction

The technique of assigning weights to local coordinate functions is used in many areas of mathematics, such as singularity theory, microlocal analysis, sub-Riemannian geometry, or algebraic geometry, under various terminologies. A weighting along a submanifold $N \subseteq M$ may be described in terms of a filtration

$$C^\infty_M = C^\infty_{M,(0)} \supseteq C^\infty_{M,(1)} \supseteq \cdots$$

of the sheaf of functions whose $i$-th term is thought of as the sheaf of functions vanishing to weighted order $i$ along $N$. See Section 2 for a precise definition. The idea for such `quasi-homogeneous structures’ in terms of filtrations goes back to Melrose [39].

As an example, letting $N = \{0\}$ be the origin in $M = \mathbb{R}^3$, one can consider a weighting where the coordinate functions $x, y, z$ are assigned weights $1, 2, 3$ respectively. The ideal of functions of filtration degree four is then generated by the monomials $x^4, x^2y, xz, yz, y^2, z^2$. 
A weighting along a submanifold $N \subseteq M$ determines a decreasing filtration of the normal bundle

$$\nu(M, N) = F_{-r} \supseteq \cdots \supseteq F_{-1} \supseteq F_0 = 0;$$

here $r$ is the order of the weighting (an upper bound for the weights of local coordinate functions), and $F_{-i}$ is spanned by normal directions annihilating all functions of filtration degree $i + 1$. It turns out that weightings of order $r = 2$ are completely determined by the subbundle $F = F_{-1}$. However, for $r > 2$ knowledge of the $F_{-i}$ does not suffice: intuitively, these subbundles only give first-order information whereas the weighting requires higher order information. This can be made precise using jet bundles. Letting $T_r M = J^r_0(\mathbb{R}, M)$ be the $r$-th tangent bundle, we will prove that an order $r$ weighting is equivalent to a certain subbundle

$$Q \subseteq T_r M$$

along $N \subseteq M$. Theorem 8.4 gives an intrinsic characterization of subbundles corresponding to weightings. The new description of weightings has some advantages; for example, one is led to notions of multiplicative weightings on Lie groupoids and infinitesimally multiplicative weightings on Lie algebroids.

One of our main observations is that an order $r$ weighting along a submanifold determines a fiber bundle over $N$, which we call the weighted normal bundle:

$$\nu_W(M, N) \to N.$$ 

It admits an algebraic definition as the character spectrum of the associated graded algebra of the algebra of smooth functions, or an alternative description as a subquotient $Q/\sim$ of the $r$-th tangent bundle $T_r M$. (For a trivial weighting, the latter is the usual definition $\nu(M, N) = TM|_N/TN$ of the normal bundle.) The weighted normal bundle is not naturally a vector bundle, but it comes with an action of the multiplicative monoid $(\mathbb{R}, \cdot)$, making into a graded bundle in the sense of Grabowski-Rotkiewicz [23]. It is non-canonically isomorphic to the associated graded bundle of $\nu(M, N)$, for the filtration described above. In concrete examples, the weighted normal bundle often has additional structure as a bundle of nilpotent Lie groups, or as a bundle of homogeneous spaces of nilpotent Lie groups.

The properties of the ordinary normal bundle extend to the weighted setting. For example, a function $f \in C^\infty(M)$ of filtration degree $i$ canonically determines a function $f^{[i]} \in C^\infty(\nu_W(M, N))$ on the weighted normal bundle, homogeneous of degree $i$. Similarly, one has weighted homogeneous approximations of more general tensor fields, such as differential forms and vector fields. There is also a weighted deformation space

$$\delta_W(M, N) = \nu_W(M, N) \sqcup (M \times \mathbb{R}^\times) \xrightarrow{\pi} \mathbb{R},$$

allowing to interpolate between a tensor field of given filtration degree with its weighted homogeneous approximation.

As one important application of the deformation spaces, one obtains a simple coordinate-free construction of weighted blow-ups of $M$ along $N$, extending many of the properties of ordinary blow-ups along submanifolds. Recall that weighted blow-ups in algebraic geometry have recently been used to great effect to give a short proof of Hironaka’s desingularization theorem [1, 35].
More generally, one can consider *multi-weightings* for a finite collection of submanifolds $N_1, \ldots, N_d$. We shall require that these have *clean intersection*, in the sense that near every $m \in M$ the intersection of submanifolds passing through $m$ is modeled by an intersection of coordinate subspaces. There are corresponding multi-weighted normal bundles $\nu_W(M, N_1, \ldots, N_d)$ and deformation spaces

$$\delta_W(M, N_1, \ldots, N_d) \xrightarrow{\pi} \mathbb{R}^d.$$  

For any multi-weighting, one obtains a total weighting along the intersection $N_1 \cap \cdots \cap N_d$. If $d \geq 2$, the total weighting is non-trivial even if the original multi-weighting is just given by the order of vanishing on the $N_\alpha$’s. As a special case, every nested sequences of submanifolds $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r$ determines a weighting.

Weightings, and the more general multi-weightings, appear in many contexts: We have already mentioned that a weighting of order $r = 2$ amounts to the choice of a subbundle of the normal bundle. As a simple example, given submanifolds $N_1, N_2 \subseteq M$ with clean intersection $N$, the total weighting for the corresponding trivial bi-weighting is given by the subbundle $(TN_1|_N + TN_2|_N)/TN \subseteq \nu(M, N)$. In symplectic geometry, given an isotropic submanifold $N$ of a symplectic manifold $(M, \omega)$, the symplectic normal bundle $TN^\omega/TN \subseteq \nu(M, N)$ defines a 2nd order weighting. In [37], this is used to prove a version of the Weinstein isotropic embedding theorem, where the ‘local model’ does not involve choices. Collections of cleanly intersecting submanifolds, and the corresponding multi-weightings, arise in the theory of manifolds with corners (e.g., [39]), as well as in Lie theory (e.g., [5]). Another source of examples are manifolds with *Lie filtrations*, given by a filtration of the tangent bundle such that the induced filtration of sections is compatible with Lie brackets. These so-called *filtered manifolds* have been studied deeply in recent years, owing to their significance in the theory of hypo-elliptic operators. See [14, 26, 41, 42, 47, 48, 49], for example. The *osculating tangent bundle* and *osculating tangent groupoid* (using the terminology from [49]) may be regarded as the weighted normal bundle and deformation space of the diagonal $M \subseteq M \times M$, for a suitable weighting defined by the Lie filtration. Similarly, the normal bundles for filtered submanifolds, as in [26], are realized as weighted normal bundles.

The full details of these applications to filtered manifolds, and a further generalization to singular Lie filtrations, will be given in a separate article. Forthcoming work of Daniel Hudson uses the weighted deformation spaces to develop a theory of weighted conormal distributions, complementing the work of Van Erp-Yuncken [48] and Debord-Skandalis [17]. For future work, we envisage applications to normal forms for vector fields and Poisson structures in weighted settings (see e.g. [19, Section 2.2] or [33]).

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2. Weightings along submanifolds

2.1. Definitions. We start out with a definition of weightings on \( n \)-dimensional manifolds in terms of a local model.

**Definition 2.1.** A weight sequence is a non-decreasing sequence of non-negative integers
\[
0 \leq w_1 \leq w_2 \leq \cdots \leq w_n.
\]
The weight sequence is equivalently determined by the numbers \( 0 \leq k_0 \leq k_1 \leq \cdots \leq n \), where
\[
a \leq k_i \iff w_a \leq i.
\]
That is, \( k_i = \# \{ a \mid w_a \leq i \} \). An upper bound \( r \in \mathbb{N} \) of the weight sequence we be called called its order. (We don’t insist that this is the least upper bound, hence, a weight sequence of order \( r \) may also be regarded as a weight sequence of order \( r' \geq r \).) By definition, \( k_r = n \).

Given an open subset \( U \subseteq \mathbb{R}^n \) and any \( i \geq 0 \), let \( C^\infty(U)_i \) be the ideal generated by monomials
\[
x^s = x_1^{s_1} \cdots x_n^{s_n}, \quad s = (s_1, \ldots, s_n)
\]
with \( s \cdot w \equiv \sum_{a=1}^{n} s_a w_a \geq i \). These ideals determine a filtration of the algebra of smooth functions
\[
C^\infty(U) = C^\infty(U)_0 \supseteq C^\infty(U)_1 \supseteq \cdots,
\]
compatible with the algebra structure. Some immediate observations:

(a) In the definition of \( C^\infty(U)_i \), we need only consider monomials \( x^s \) such that \( s \) is minimal relative to the condition \( s \cdot w \geq i \); in particular, we may demand that \( s_a = 0 \) for \( w_a = 0 \).

(b) If \( \mathbb{R}^{k_0} \cap U = \emptyset \), the filtration is trivial: \( C^\infty(U)_i = C^\infty(U) \) for all \( i \).

**Definition 2.2.** An order \( r \) weighting on a \( C^\infty \) manifold \( M \) is a filtration of its structure sheaf by ideals,
\[
C^\infty_M = C^\infty_{M,(0)} \supseteq C^\infty_{M,(1)} \supseteq \cdots
\]
with the property that every point of \( M \) has an open neighborhood \( U \), with coordinates \( x_a \), such that the filtration of \( C^\infty_M(U) = C^\infty(U) \) is given by (3). A morphism of weighted manifolds is a smooth map \( \phi: M \to M' \) such that the pullback map \( \phi^* : C^\infty_{M'} \to C^\infty_M \) preserves filtrations.

We refer to the special coordinates in this definition as weighted coordinates. Note that the coordinate function \( x_a \in C^\infty(U) \) has filtration degree \( w_a \). Observe also that an order \( r \) weighting may be regarded as an order \( r' \) weighting, for any \( r' \geq r \).

**Remark 2.3.** After completing a first version of this paper, we learned the concept of weightings had been introduced in lecture notes of Richard Melrose from 1996, under the name of quasi-homogeneous structure. See [39, Proposition 1.15.1]. The details of Melrose’s approach were further developed in the recent Ph.D. thesis of Malte Behr [5].

**Lemma 2.4.** The ideal \( C^\infty_{M,(1)} \) is the ideal sheaf \( \mathcal{I} = \mathcal{I}_N \) of a closed submanifold \( N \subseteq M \). A morphism of weighted manifolds \( \phi: M \to M' \) takes \( N \) into the corresponding closed submanifold \( N' \subseteq M' \).
Proof. Let \( N \subseteq M \) be the closed subset of all points where the filtration (8) is non-trivial. Given \( m \in N \), choose weighted coordinates \( x_a \) on an open neighborhood \( U \) of \( m \). From the definition of the filtration in coordinates, we see that \( C^\infty(U)_m \) is the ideal of functions vanishing on \( \mathbb{R}^{k_0} \cap U \). In particular, the weighted coordinates give submanifold charts for \( N \). The last claim is immediate from \( \phi^* C^\infty_{M,(1)} \subseteq C^\infty_{M,(1)} \).

If a closed submanifold \( N \subseteq M \), with ideal sheaf \( I \), is given in advance, we refer to a weighting with \( C^\infty_{M,(1)} = I \) as a weighting along \( N \), and to \((M,N)\) as a weighted manifold pair.

Remark 2.5. The requirement that the submanifold \( N \) be closed is sometimes inconvenient. One can give a more flexible definition, for arbitrary immersions \( i: N \to M \), as a filtration of the inverse image sheaf \( i^{-1}C^\infty_{M} \). Alternatively, given any \( m \in N \), one can replace \( N \) with a suitable neighborhood of \( m \), and \( M \) with a neighborhood of \( i(m) \), to reduce to the case that \( N \) is closed.

From the local coordinate description of the filtration, we see that functions of filtration degree \( i > r \) are sums of products of functions of filtration degrees \( i_\nu \leq r \), with \( \sum i_\nu = i \). Hence, an order \( r \) weighting of \( M \) is fully determined by the ideals \( C^\infty_{M,(i)} \) with \( i \leq r \):

\[
i > r \Rightarrow C^\infty_{M,(i)} \subseteq I^2.
\]

Recall that \( I/I^2 \) is the sheaf of sections of the conormal bundle \( \text{ann}(TN) \subseteq T^*M|_N \).

Proposition 2.6. An order \( r \) weighting of \((M,N)\) determines a filtration of the normal bundle \( \nu(M,N) = TM|_N/TN \),

\[
\nu(M,N) = F_{-r} \supseteq \cdots \supseteq F_{-1} \supseteq F_0 = N,
\]

by subbundles \( F_{-i} \) of dimension \( k_i \), with the property that for all \( i \geq 1 \), the quotient

\[
C^\infty_{M,(i)}/(C^\infty_{M,(i)} \cap I^2)
\]

is the sheaf of sections of \( \text{ann}(F_{-i+1}) \subseteq \nu(M,N)^* \).

Proof. The quotient sheaf (7) vanishes over \( M - N \). At points \( m \in N \), we may choose local weighted coordinates \( x_a \) on an open neighborhood \( U \) of \( m \). The space of sections of (7) over \( N \cap U \) is then spanned by \( dx_a|_{N\cap U} \) with \( w_a \geq i \), i.e., \( a > k_{i-1} \).

The converse does not hold: A filtration of \( C^\infty_M \), with \( C^\infty_{M,(1)} = I \) and \( C^\infty_{M,(r+1)} \subseteq I^2 \), and such that the quotients (7) define subbundles of the conormal bundle, need not correspond to a weighting.

Example 2.7. Let \( A_i = C^\infty_{M,(i)} \) be given by

\[
A_1 = I, \ A_2 = A_3 = I^2, \ A_4 = A_5 = I^3, \ldots \n\]

This is a multiplicative filtration with the property that the quotients (7) are zero for \( i > 1 \), and equal to \( I/I^2 \) for \( i = 1 \). Clearly, this filtration does not correspond to a weighting.

The following proposition gives an additional condition ruling out such examples. Similar conditions are stated in [5, 39].
Proposition 2.8. Let \( N \subseteq M \) be a closed submanifold, with vanishing ideal \( \mathcal{I} \), and let

\[
C_M^\infty = C_M^\infty(0) \supseteq C_M^\infty(1) \supseteq \cdots
\]

be a multiplicative filtration with \( C_M^\infty(1) = \mathcal{I} \) and \( C_M^\infty(r+1) \subseteq \mathcal{I}^2 \). Then (8) defines a weighting of order \( r \) along \( N \) if and only if the following two conditions hold:

(a) For \( i \geq 1 \), the quotient \( C_M^\infty(i)/(C_M^\infty(i) \cap \mathcal{I}^2) \) is the sheaf of sections of a vector bundle.

(b) For \( i \geq 2 \),

\[
C_M^\infty(i) \cap \mathcal{I}^2 = \sum_{1 \leq j < i} C_M^\infty(i,j) \cdot C_M^\infty(i-j).
\]

Proof. Clearly, the filtration defined by a weighting has these properties. Conversely, if these conditions are satisfied, Condition (a) gives a filtration of the conormal bundle by subbundles

\[
\nu(M,N)^* = E_1 \supseteq E_2 \supseteq \cdots \supseteq E_r \supseteq E_{r+1} = 0,
\]

where \( E_i \) has \( C_M^\infty(i)/(C_M^\infty(i) \cap \mathcal{I}^2) \) as its space of sections. Put \( k_i = \dim M - \text{rank}(E_{i+1}) \), so that \( \dim N = k_0 \leq k_1 \leq \cdots \leq k_r = n \), and define a weight sequence by letting \( w_a = i \) for \( k_{i-1} < a \leq k_i \).

Given a point \( m \in N \), choose functions \( x_a \in C^\infty(U)^{(r)} \) for \( k_{r-1} < a \leq n \), defined on an open neighborhood \( U \) of \( m \), such that their images under the quotient map define a frame for \( E_r \) over \( U \cap N \). Taking \( U \) smaller if needed, extend to a collection of functions \( x_a \in C^\infty(U)^{(r-1)} \) for \( k_{r-2} < a \leq n \) such that their images under the quotient map define a frame for \( E_{r-1} \) over \( U \cap N \). Proceeding in this way, we obtain local coordinates \( x_1, \ldots, x_n \) near \( m \), with the property \( x_a \in C^\infty(U)^{(i)} \) for \( k_{i-1} < a \leq k_i \). Define a weighting on \( U \) along \( N \cap U \) where the coordinate function \( x_a \) has weight \( w_a \).

Let \( C_U^\infty = B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots \) be the filtration defined by this weighting, and let \( A_i = C_U^\infty(i) \). We shall argue by induction that \( A_i = B_i \) for all \( i \). By assumption, \( A_1 = B_1 = \mathcal{I}|_U \).

Suppose \( i > 1 \) and \( B_j = A_j \) for \( j < i \). By condition (b) and the inductive hypothesis,

\[
B_i \cap \mathcal{I}^2 = \sum_{1 \leq j < i} B_j \cdot B_{i-j} = \sum_{1 \leq j < i} A_j \cdot A_{i-j} = A_i \cap \mathcal{I}^2.
\]

Note that \( B_i \) is generated by this intersection, together with the coordinate functions \( x_a \) with \( k_{i-1} < a \leq k_i \). Hence \( B_i \subseteq A_i \). To prove equality, it suffices to show that the induced map \( B_i/(B_i \cap \mathcal{I}^2) \rightarrow A_i/(A_i \cap \mathcal{I}^2) \) is an isomorphism. But this is clear since both are identified with the sheaf of sections of \( E_i|_U \). \( \square \)

2.2. Examples, basic constructions.

Examples 2.9. Let \( N \subseteq M \) be a closed submanifold.

(a) A weighting of order \( r = 1 \) along \( N \) is the trivial weighting along \( N \), given by

\[
C_M^\infty_M(i) = \mathcal{I}^i,
\]

the ideal of functions vanishing to order \( i \) on \( N \) (in the usual sense).

(b) A weighting of order \( r = 2 \) along \( N \) is fully determined by the subbundle \( F = F_{-1} \subseteq \nu(M,N) \): Let \( \mathcal{J} \subseteq \mathcal{I} \) be the functions \( f \) such that \( f|_N = 0 \) and \( df|_{\tilde{F}} = 0 \), where \( \tilde{F} \subseteq TM|_N \) is the pre-image of \( F \). Then [37]

\[
C_M^\infty_M(2i+1) = \mathcal{I}\mathcal{J}^i, \quad C_M^\infty_M(2i) = \mathcal{J}^i.
\]
(c) Consider a nested sequence of submanifolds of \( M \)
\[ M = N_{r+1} \supseteq N_r \supseteq \cdots \supseteq N_1 = N, \]
and let \( I_i \subseteq \mathcal{I} \) be the vanishing ideal of \( N_i \). Then
\[ C^\infty_{M,(i)} = \sum_{\ell \geq 1} \sum_{i_1 + \cdots + i_{\ell} = i} I_{i_1} \cdots I_{i_{\ell}} \]
for \( i \geq 1 \) defines an order \( r \) weighting along \( N \). Near any point \( m \in N \) we may choose coordinates \( x_1, \ldots, x_n \), that are submanifold coordinates for each of the \( N_i \)'s; these coordinates will serve as weighted coordinates. The corresponding filtration (6) of the normal bundle is given as \( F_{-i} = TN_{i+1}|_N/TN \).

(d) Let \( N_1, N_2 \subseteq M \) be closed submanifolds with clean intersection. That is, \( N = N_1 \cap N_2 \) is a submanifold, with \( TN = TN_1 \cap TN_2 \). Then \( C^\infty_M \) acquires a bi-filtration by ideals \( I_{(i_1,i_2)} \), whose local sections are the functions vanishing to order \( i_1 \) on \( N_1 \) and to order \( i_2 \) on \( N_2 \). Let
\[ C^\infty_{M,(i)} = \sum_{i_1+i_2=i} I_{(i_1,i_2)} \]
Using local coordinates adapted to the clean intersection (see [46]), one verifies that this is a weighting of order \( r = 2 \) along \( N = N_1 \cap N_2 \). The corresponding subbundle \( F \subseteq \nu(M,N) \) is \((TN_1|_N + TN_2|_N)/TN\); this subbundle, and hence the weighting, is trivial if and only if the intersection is transverse. This example generalizes to finite collections of closed submanifolds \( N_1, \ldots, N_r \subseteq M \) with clean intersection (see Section 9.1 below).

(e) (Linear weightings.) Let \( V \to M \) be a vector bundle, and \( W \to N \) a subbundle along \( N \subseteq M \). We adopt the Grabowski-Rotkiewicz approach [23] to vector bundles in terms of their scalar multiplication; thus, \( W \) is characterized as a submanifold invariant under scalar multiplication. A weighting of \( V \) along \( W \) will be called linear if the corresponding filtration of \( C^\infty_{V} \) is \((\mathbb{R},\cdot)-\)invariant. It induces a unique weighting of \( M \) along \( N \), in such a way that the bundle projection and inclusion of units are morphisms of weighted manifolds.

(f) (Lie filtrations.) A Lie filtration of a manifold \( M \) is a filtration of the tangent bundle by subbundles,
\[ TM = H_{(-r)} \supseteq \cdots H_{(-1)} \supseteq H_{(0)} = 0_M \]
such that the resulting filtration of vector fields preserves brackets. Such a Lie filtration canonically determines a weighting of \( M \times M \) along the diagonal \( M \). Given a ‘filtered submanifold’ \( N \subseteq M \) [26, Section 7], one obtains a weighting of \( M \) along \( N \). These examples will be discussed in detail in [34], with further generalizations to singular Lie filtrations.

Some basic constructions with weightings:

(a) **Products.** Given weightings of \((M,N)\) and \((M',N')\), we obtain a weighting of the product \((M \times M', N \times N')\), with \( C^\infty_{M \times M',(i)} \) the ideal generated by the sum of all \( C^\infty_{M,(i_1)} \otimes C^\infty_{M',(i_2)} \) with \( i_1 + i_2 = i \).
(b) **Pullbacks.** Suppose we are given a weighting along \( N \subseteq M \), and let \( \varphi : M' \to M \) be a smooth map transverse to \( N \). Then there is a canonically induced weighting of \( M' \) along \( N' = \varphi^{-1}(N) \), with \( C^\infty_{M',(i)} \) the ideal generated by \( \varphi^*C^\infty_{M,(i)} \). (To see that this is a weighting, let \( m' \in N' \), and choose weighted coordinates \( x_a \) near \( m = \varphi(m') \). The pullbacks \( \varphi^*x_a \) of coordinate functions with \( a > k_0 \) may be completed to a coordinate system near \( m' \), and these will then serve as weighted coordinates near \( m' \).) As a special case, given a submanifold \( \Sigma \subseteq M \) transverse to \( N \), one obtains a weighting of \( \Sigma \) along \( \Sigma \cap N \).

### 2.3. Filtrations on forms and vector fields.

The filtration of \( C^\infty_M \) extends uniquely to a filtration of the sheaf \( \Omega^\bullet_M \) of differential forms,

\[
\Omega^\bullet_M = \Omega^\bullet_{M,(0)} \supseteq \Omega^\bullet_{M,(1)} \supseteq \cdots
\]

compatible with the algebra structure, in such a way that the de Rham differential preserves the filtration degree. In weighted coordinates on \( U \subseteq M \), the space \( \Omega^q(U)_{(i)} \) consists of forms

\[
\sum_{a_1 < \cdots < a_q} f_{a_1 \cdots a_q} dx_{a_1} \wedge \cdots dx_{a_q}
\]

such that \( f_{a_1 \cdots a_q} \) has filtration degree \( i - w_{a_1} - \cdots - w_{a_q} \).

Similarly, the sheaf of vector fields \( \mathcal{X}_M \) inherits a filtration by \( C^\infty_M \)-submodules, where a vector field \( X \) has filtration degree \( i \) if the Lie derivative \( L_X \) on functions raises the filtration degree by \( i \). This filtration starts in degree \( -r \):

\[
\mathcal{X}_M = \mathcal{X}_{M,(-r)} \supseteq \mathcal{X}_{M,(-r+1)} \supseteq \cdots
\]

In weighted coordinates on \( U \subseteq M \), the space \( \mathcal{X}(U)_{(i)} \) consists of vector fields

\[
\sum_a f_a \frac{\partial}{\partial x_a}
\]

such that \( f_a \) has filtration degree \( i + w_a \). These filtrations on vector fields and forms are compatible with all the operations from Cartan’s calculus (brackets, contractions, Lie derivatives, and so on).

The sheaf \( \mathcal{X}_{M,(0)} \) is closed under brackets. Its local sections are the vector fields preserving the filtration of \( C^\infty_M \), and so are the infinitesimal automorphisms of the weighting. Using weighted coordinates on \( U \subseteq M \), we see that vector fields in \( \mathcal{X}(U)_{(0)} \) are tangent to \( N \cap U \), and that the restriction map \( \mathcal{X}(U)_{(0)} \to \mathcal{X}(N \cap U) \) is surjective. Hence, we obtain a surjective sheaf map \( \mathcal{X}_{M,(0)} \to \mathcal{X}_N \).

### 2.4. Other settings.

The definition of weightings was formulated in such a way that it extends to various other kinds of structure sheaves. In particular, for a complex manifold \( M \) one defines a holomorphic weighting in terms of a decreasing filtration of the sheaf \( \mathcal{O}_M \) of holomorphic functions, given by the local model from Section 2.1 in holomorphic coordinates. The degree 1 part of such a filtration is the vanishing ideal of a complex submanifold \( N \). The filtration of the structure sheaf determines filtrations on the sheaves of holomorphic differential forms and holomorphic vector fields.
For much of the present paper, we will restrict the discussion to the $C^\infty$ category. Having partitions of unity at our disposal, we may avoid the use of sheaves, and simply work with the filtration of global functions,

$$C^\infty(M) = C^\infty(M)_{(0)} \supseteq C^\infty(M)_{(1)} \supseteq \cdots.$$ 

In Section 9.2, we will make a few more comments on the extension of our results to the holomorphic context.

3. Graded bundles (after Grabowski-Rotkiewicz)

In Section 4, we will define the weighted normal bundle for a given weighting along a submanifold. We shall see that the weighted normal bundle is not naturally a vector bundle; rather, it is a graded bundle in the sense of Grabowski-Rotkiewicz [24]. To prepare, let us review the definition and basic properties of graded bundles, following [8, 9, 22, 24].

3.1. Definition and examples. A graded bundle may be defined as a special case of a non-negatively graded supermanifold [11, 36] in which no fermionic variables are present. In this paper, we will work with a simple definition of graded bundles due to Grabowski-Rotkiewicz [24], solely in terms of scalar multiplications.

**Definition 3.1.** [24] A graded bundle is a manifold $E$ with a smooth action

$$\kappa: \mathbb{R} \times E \to E; \ (t, x) \mapsto \kappa_t(x)$$

of the monoid $(\mathbb{R}, \cdot)$ of real numbers, $\kappa_{t_1 t_2} = \kappa_{t_1} \circ \kappa_{t_2}$, $\kappa_1 = \text{id}_E$. The Euler vector field of a graded bundle $E$ is the vector field $\mathcal{E}$ with flow $s \mapsto \kappa_{\exp(-s)}$. A morphism of graded bundles $\varphi: E' \to E$ is a smooth map intertwining the $(\mathbb{R}, \cdot)$-actions.

For any graded bundle, the map $\kappa_0: E \to E$ is a smooth projection, hence its range is a closed submanifold $N \subseteq E$. In fact, the projection $\kappa_0$ makes $E$ into a locally trivial fiber bundle over $N$. A graded subbundle is an $(\mathbb{R}, \cdot)$-invariant submanifold $E' \subseteq E$; it is a graded bundle in its own right, with the inclusion map a morphism of graded bundles.

**Examples 3.2.**

(a) A vector bundle is a graded bundle with the special property that the map $E \to TE$, $x \mapsto \frac{d}{dt}|_{t=0}\kappa_t(x)$ is injective. Conversely, if this condition is satisfied, then $\kappa_t$ is the scalar multiplication for a unique vector bundle structure on $E$. [24].

(b) Every negatively graded vector bundle

$$W = W^{-1} \oplus \cdots \oplus W^{-r} \to N$$

is a graded bundle, by letting $\kappa_t$ be multiplication by $t^i$ on $W^{-i}$. The corresponding Euler vector field is $\mathcal{E} = \sum_{i=1}^{r} i\mathcal{E}_i$, where $\mathcal{E}_i$ is the usual Euler vector field of the vector bundle $W^{-i}$.

(c) Let $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}^{-i}$ be a negatively graded Lie algebra. Then $\mathfrak{g}$ is nilpotent, and the exponential map gives a global diffeomorphism with the corresponding simply connected nilpotent Lie group $G$. The $(\mathbb{R}, \cdot)$-action on $\mathfrak{g}$ (as in (b)) is by Lie algebra morphisms, hence it exponentiates to an action by Lie group morphisms, making $G \to \text{pt}$ into a graded Lie group. These have appeared in the literature under the name of homogeneous Lie group (see the monograph [21]); if $\mathfrak{g}^{-1}$ generates $\mathfrak{g}$ as a Lie algebra, they are also called Carnot groups. Given a graded Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, with corresponding subgroup $H \subseteq G$, the homogeneous space $G/H \to \text{pt}$ becomes a graded bundle.
(d) More generally, one can consider graded Lie groupoids: that is, Lie groupoids with an action of \((\mathbb{R}, \cdot)\) by Lie groupoid morphisms. (For a vector bundle, regarded as a Lie groupoid, this recovers the notion of graded vector bundle.)

(e) The bundle of \(r\)-velocities or \(r\)-th tangent bundle of a manifold \(M\) was introduced by Ehresmann [20] as the bundle of \(r\)-jets of curves,

\[ T_rM = J_r^0(\mathbb{R}, M) = \{ j_0^r(\gamma) \mid \gamma: \mathbb{R} \to M \}. \]

It is a graded bundle \(T_rM \to M\), with the \((\mathbb{R}, \cdot)\)-action coming from the action on the domain of such paths: \(\kappa_t(j_0^r(\gamma)) = j_0^r(\gamma_t)\) with \(\gamma_t(s) = \gamma(ts)\).

(f) According to Grabowski-Rotkiewicz [23], double vector bundles may be defined as manifolds \(D\) with two vector bundle structures \(D \to A\) (‘horizontal’) and \(D \to B\) (‘vertical’), in such a way that the corresponding scalar multiplications \(\kappa_t^h, \kappa_t^v\) commute. In this case, \(A, B\) are themselves vector bundles, over a common base \(N\). Double vector bundles become graded bundles of order 2 for the scalar multiplication \(\kappa_t = \kappa_t^h \circ \kappa_t^v\). More generally, one can consider multi-vector bundles.

For every graded bundle \(E \to N\), there is a canonical weighting of order \(r\) along \(N\) given by

\[ C^\infty(E)_{(i)} = \{ f \in C^\infty(E) \mid \kappa_t^i f = O(t^i) \}. \]

The graded bundle coordinates on \(E\) serve as weighted coordinates.

3.2. Linear approximation. Given a graded bundle \(E \to N\), the normal bundle \(\nu(E,N)\) becomes a graded bundle, with scalar vector multiplication \(\nu(\kappa_t)\) obtained by applying the normal bundle functor to \(\kappa_t: E \to E\). This linear approximation \(E_{\text{lin}} = \nu(E,N)\) is naturally a graded vector bundle,

\[ E_{\text{lin}} = \bigoplus_{i \geq 1} E_{\text{lin}}^{-i}, \]

with \(E_{\text{lin}}^{-i}\) the sub-vector bundle on which \(\nu(\kappa_t)\) acts as scalar multiplication by \(t^i\). (Given a linear \((\mathbb{R}, \cdot)\)-action on a real vector space \(E\), the complexified \((\mathbb{C}, \cdot)\)-action on \(E \otimes_{\mathbb{R}} \mathbb{C}\) restricts to a \(U(1)\)-action; the fact that it extends to a \((\mathbb{C}, \cdot)\)-action means that negative weights cannot occur.)

 Morphisms of graded bundles \(E \to E'\) induce morphisms of graded vector bundles \(E_{\text{lin}} \to E'_{\text{lin}}\). We say that the graded bundle \(E\) has order \(r\) if \(E_{\text{lin}}^{-i} = 0\) for \(i > r\).

Examples 3.3. The linear approximations for some of our examples are as follows:

(a) For a graded vector bundle \(W\), we have a canonical identification \(W_{\text{lin}} = W\).

(b) For a graded Lie group \(G\), we recover the Lie algebra \(G_{\text{lin}} = \mathfrak{g}\). The linear approximation of group multiplication \(G \times G \to G\) is the addition \(\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\).

(c) For the \(r\)-th tangent bundle, we have that \((T_rM)_{\text{lin}} = TM \oplus \cdots \oplus TM\). See Section 7.1.3 below.

(d) For a double vector bundle \(D\) with side bundles \(A, B\) over a base \(N\), let \(\text{core}(D) \subseteq D\) be the subset on which \(\kappa_t^h, \kappa_t^v\) coincide. This is a vector bundle over \(N\), and one finds that

\[ D_{\text{lin}} = (A \oplus B) \oplus \text{core}(D). \]

with \(A \oplus B\) in degree \(-1\) and \(\text{core}(D)\) in degree \(-2\).

Grabowski-Rotkiewicz established the following result, which may be seen as an analogue of Batchelor’s theorem [4] for supermanifolds:
Theorem 3.4. [24] For every graded bundle $E \to N$ there exists a (non-canonical) isomorphism of graded bundles

\begin{equation}
E_{\text{lin}} \cong E
\end{equation}

whose linear approximation is the identity.

We refer to any such isomorphism as a linearization of $E$. When proving facts about graded bundles, it is often helpful to choose such a linearization to reduce to the case of a graded vector bundle.

Remark 3.5. There is an obvious version of graded bundles in the holomorphic or analytic categories. (See [29] for some details.) In these settings, linearizations exist near any given $m \in N$, but need not exist globally.

3.3. Graded bundle coordinates. Let $\pi: E \to N$ be a graded bundle of order $r$, with $\dim E = n$. Put $k_0 = \dim N$, $k_i = \dim E^{k_i}_{\text{lin}}$ for $i > 0$, and let $w_1 \leq \cdots \leq w_n = r$ be the corresponding weight sequence, given by $w_a = i$ for $k_i \leq a < k_{i+1}$. Coordinates $x_a$, $a = 1, \ldots, \dim E$ defined on $\pi^{-1}(O)$ for open subsets $O \subseteq N$, are called graded bundle coordinates if $x_a$ for $k_i \leq a < k_{i+1}$ are homogeneous of degree $i$. To construct such coordinates, choose a linearization, and pick vector bundle coordinates for $E_{\text{lin}}$ compatible with the grading. In graded bundle coordinates, the Euler vector field of $E \to N$ is given by

$$
\mathcal{E} = \sum_a w_a x_a \frac{\partial}{\partial x_a}.
$$

3.4. Polynomial functions, forms and vector fields. Given a graded bundle $E \to N$, with Euler vector field $\mathcal{E}$, let $C^\infty(E)^{|k|}$ be the space of smooth functions $f$ that are homogeneous of degree $k$, i.e. $\mathcal{E} f = k f$. These spaces are non-zero only if $k$ is a nonnegative integer, and their direct sum is a graded algebra $C^\infty_{\text{pol}}(E)$, called the polynomial functions on $E$. As shown in [22], one recovers $E$ from this algebra as the character spectrum

\begin{equation}
E = \text{Hom}_{\text{alg}}(C^\infty_{\text{pol}}(E), \mathbb{R});
\end{equation}

here the $(\mathbb{R}, \cdot)$-action on algebra morphisms is given by duality with the $(\mathbb{R}, \cdot)$-action on polynomials. Similarly, we can consider vector fields or differential forms of homogeneity $k$, making

$$
\mathfrak{x}_{\text{pol}}(E) = \bigoplus_{k \geq -r} \mathfrak{x}(E)^{|k|}, \quad \Omega_{\text{pol}}(E) = \bigoplus_{k \geq 0} \Omega(E)^{|k|}
$$

into a graded Lie algebra and a bigraded algebra, respectively. The gradings are compatible with the $C^\infty_{\text{pol}}(E)$-module structure, as well as the operations from Cartan’s calculus (Lie derivatives, Lie brackets, contractions, differentials). Note that the grading of $\mathfrak{x}_{\text{pol}}(E)$ starts in degree $-r$; the part of strictly negative degree is a nilpotent Lie subalgebra

\begin{equation}
\mathfrak{x}_{\text{pol}}(E)^- = \bigoplus_{1 \leq i \leq r} \mathfrak{x}(E)^{|-i|}.
\end{equation}

The vector fields in this subalgebra annihilate $C^\infty(E)^{[0]} \cong C^\infty(M)$, hence they are all vertical. We have $\mathfrak{x}(E)^{|-r|} \cong \Gamma(E_{\text{lin}}^{-r})$ via restriction of vector fields to $M \subseteq E$, and identifying $TE|_M = \cdots$
$TM \oplus E_{\text{lin}}$. Analyzing fiberwise (and using graded bundle coordinates), one finds that the vector fields in (13) are all complete. Hence, this subalgebra exponentiates to a nilpotent group of fiber-preserving diffeomorphisms of $E$. Observe also that

$$\mathfrak{X}(E)^[0] = \text{aut}_{\text{gr}}(E)$$

is the Lie algebra of infinitesimal graded bundle automorphisms of $E$, given by $\kappa_t$-invariant vector fields.

4. Weighted normal bundle

The normal bundle of a closed submanifold can be defined in several equivalent ways: (i) as the character spectrum of the associated graded algebra of $C^\infty(M)$, using the filtration by the powers of the vanishing ideal $I$ of $N$, (ii) as a subquotient of the tangent bundle $\nu(M, N) = TM|_N/TN$, (iii) in terms of its space of sections, $\Gamma(\nu(M, N)) = \mathfrak{X}(M)/\mathfrak{X}(M, N)$ where $\mathfrak{X}(M, N)$ is the subspace of vector fields tangent to $N$. Each of these descriptions admits a generalization to the weighted setting. In this section we follow approach (i), and also give a coordinate description. Later, we shall explain versions of (ii) and (iii).

4.1. Definition of the weighted normal bundle. Let $(M, N)$ be a manifold pair with a weighting of order $r$. Consider the corresponding filtration of the algebra of smooth functions, and let $\text{gr}(C^\infty(M))$ be the associated graded algebra, with graded components

$$(14) \quad C^\infty(M)_{(i)}/C^\infty(M)_{(i+1)}.$$ 

For $f \in C^\infty(M)_{(i)}$, its image in (14) will be denoted $f[i]$, and is called the $(i)$-th homogeneous approximation of $f$. If $i = 0$, this is the restriction $f|_N \in C^\infty(N)$.

**Definition 4.1** (Weighted normal bundle). The *weighted normal bundle* for an order $r$ weighting along $N \subseteq M$ is the character spectrum,

$$\nu_W(M, N) = \text{Hom}_{\text{alg}}(\text{gr}(C^\infty(M)), \mathbb{R}).$$

By definition, $f[i]$ for $f \in C^\infty(M)_{(i)}$ is a function on the weighted normal bundle. The smooth structure is uniquely specified by the requirement that all such functions are smooth. More precisely:

**Theorem 4.2.**  
(a) The weighted normal bundle $\nu_W(M, N)$ has a unique structure as a graded bundle over $N$, of dimension equal to that of $M$, in such a way that

$$C^\infty_{\text{pol}}(\nu_W(M, N)) = \text{gr}(C^\infty(M)).$$

as graded algebras.

(b) Given weighted coordinates $x_1, \ldots, x_n$ on $U \subseteq M$, the homogeneous approximations $(x_a)^{[w_a]}$ for $a = 1, \ldots, n$ serve as graded bundle coordinates on $\nu_W(M, N)|_{U \cap N}$.

**Proof.** This may be proved by using the general criterion in Haj-Higson [26], or by direct construction of coordinates, as follows. Let $p: \nu_W(M, N) \rightarrow N$ be the projection induced on the algebra side by the inclusion $C^\infty(N) \rightarrow \text{gr}(C^\infty(M))$ as the degree 0 summand. For any open subset $U \subseteq M$, we have that $p^{-1}(U) = \nu_W(U, U \cap N)$. Given weighted coordinates $x_1, \ldots, x_n$ on $U \subseteq M$, let $y_a = (x_a)^{[w_a]}$ be their homogeneous approximations. For $a \leq k_0$ we have $y_a = x_a|_{U \cap N}$, hence $y_1, \ldots, y_{k_0}$ serve as coordinates on $U_1 = U \cap N$. Let $\overline{U_1} \subseteq \mathbb{R}^{k_0}$ be the
image of this coordinate map. The graded algebra \( \text{gr}(\mathcal{C}^\infty(U)) \) is generated, as an algebra over its degree 0 summand \( \mathcal{C}^\infty(U \cap N) \), by its elements \( y_a \) for \( a > k_0 \). Hence,

\[
(y_1, \ldots, y_n): p^{-1}(U) \to \mathbb{R}^n
\]
is a bijection onto an open subset of the form \( \overline{U}_1 \times \mathbb{R}^{n-k_0} \). We will take these as graded bundle charts. In particular, this specifies a \( \mathcal{C}^\infty \)-structure on \( p^{-1}(U) \). Let us show that if \( f \in \mathcal{C}^\infty(U)_{(i)} \), then \( f[i]: p^{-1}(U) \to \mathbb{R} \) is smooth. It suffices to check for functions of the form

\[
f(x_1, \ldots, x_n) = \chi(x_1, \ldots, x_n)x_1^{s_1} \cdots x_n^{s_n} \quad \text{with} \quad \chi \in \mathcal{C}^\infty(U)
\]
and \( s \cdot w = \sum s_aw_a = i \). For such a function,

\[
f[i](y_1, \ldots, y_n) = \chi(y_1, \ldots, y_{k_0}, 0, \ldots, 0) y_1^{s_1} \cdots y_n^{s_n},
\]
which is evidently smooth, and is homogeneous of degree \( i \). This also show that for a different set \( x'_1, \ldots, x'_n: U' \to \mathbb{R} \) of weighted coordinates, the ‘transition function’ is smooth: taking \( x'_a \) as \( f \in \mathcal{C}^\infty(U \cap U')_{(w_0)} \), we see that \( y'_a \) is smooth as a function of \( y_1, \ldots, y_n \). This completes the construction of a \( \mathcal{C}^\infty \)-structure for which the functions \( f[i] \) with \( f \in \mathcal{C}^\infty(M)_{(i)} \) are all smooth. The uniqueness assertion is clear.

Note that for a weighted coordinate chart, the algebra \( \text{gr}(\mathcal{C}^\infty(U)) \) coincides with the corresponding algebra for the trivial weighting on \( U \cap N \) – only the grading is different. This shows that locally, the weighted normal bundle is isomorphic to the usual normal bundle. However, this identification depends on the choice of coordinates, and the transition functions for the two bundles are different.

**Example 4.3.** Suppose \( x, y, z \) are weighted coordinates of weights \( 0, 1, 3 \) on \( \mathbb{R}^3 \), and consider the coordinate change (defined near the origin)

\[
x' = \sin(x) \exp(yz), \quad y' = y \exp(xy), \quad z' = 3z + \sin^3(xy).
\]
The corresponding coordinate change on the weighted normal bundle reads as

\[
(x')^{[0]} = \sin(x^{[0]}), \quad (y')^{[1]} = y^{[1]}, \quad (z')^{[3]} = 3z^{[3]} + (x^{[0]})^3(y^{[1]})^3.
\]
The coordinate change for the usual normal bundle is given by a similar formula, omitting the ‘non-linear’ term \((x^{[0]})^3(y^{[1]})^3\).

The construction of the weighted normal bundle is functorial: Any morphism of weighted manifold pairs \( \varphi: (M, N) \to (M', N') \) gives a morphism of filtered algebras \( \mathcal{C}^\infty(M') \to \mathcal{C}^\infty(M) \), hence of the associated graded algebras. It therefore determines a map of weighted normal bundles, \( \nu(\varphi): \nu\mathcal{W}(M, N) \to \nu\mathcal{W}(M', N') \). To check that this map is smooth, it suffices to show that the pullback of smooth functions on \( \nu\mathcal{W}(M', N') \) is again smooth. Let \( f \in \mathcal{C}^\infty(M')_{(i)} \), with corresponding homogeneous function \( f[i] \) on \( \nu\mathcal{W}(M', N') \). Then \( (\nu(\varphi))^*f[i] = (\varphi^*f)[i] \) is smooth, as required.

### 4.2. Properties of the weighted normal bundle

Recall that the weighting along \( N \subseteq M \) determines a filtration (6) of the (usual) normal bundle \( \nu(M, N) \) by subbundles \( F_{-i} \). The associated graded bundle has its components in negative degrees, \( \text{gr}(\nu(M, N)) = \bigoplus_i F_{-i}/F_{-i+1} \).

**Proposition 4.4.** There is a canonical identification of graded vector bundles over \( N \),

\[
\nu\mathcal{W}(M, N)_{\text{lin}} = \text{gr}(\nu(M, N)).
\]

Here \( \nu\mathcal{W}(M, N)_{\text{lin}} \) is the linear approximation of the graded bundle \( \nu\mathcal{W}(M, N) \).
Proof. The space of polynomial functions of degree $k$ on $\nu_M^i(M, N)_{\text{lin}}$ is given by
\[(C^\infty(M)_{(i)} \cap \mathcal{T}^k)/(C^\infty(M)_{(i)} \cap \mathcal{T}^{k+1} + C^\infty(M)_{(i+1)} \cap \mathcal{T}^k).\]
The space of polynomial functions of degree $k$ on $\operatorname{gr}(\nu(M, N))^i$ has exactly the same description. $\square$

**Proposition 4.5.** For a graded bundle $E \to N$, with its canonical weighting along $N \subseteq M$, there is a canonical identification
\[\nu_M(E, N) = E.\]

**Proof.** This follows from the equality (12) together with $\operatorname{gr}(C^\infty(E)) = C^\infty_{\text{pol}}(E)$ (by definition of the weighting). $\square$

### 4.3. Homogeneous approximations.

The filtration of functions on $M$ induces filtrations on all spaces of tensor fields on $M$, and we obtain corresponding homogeneous approximations. In particular, there is an algebra isomorphism
\[(15) \quad \operatorname{gr}(\mathcal{X}(M)) \to \mathcal{X}_{\text{pol}}(\nu_M(M, N)),\]
defined by the map taking a $p$-form $\alpha$ of filtration degree $i$ to the form $\alpha^{[i]} \in \mathcal{O}^p(\nu_M(M, N))^{[i]}$, homogeneous of degree $i$. Similarly, if $X$ is a vector field of filtration degree $i$ (i.e., its action on functions by Lie derivative raises the filtration degree by $i$), we obtain a homogeneous approximation $X^{[i]}$ of homogeneity $i$, resulting in a Lie algebra isomorphism
\[(16) \quad \operatorname{gr}(\mathcal{X}(M)) \to \mathcal{X}_{\text{pol}}(\nu_M(M, N)).\]

All of these isomorphisms are compatible with the usual operations from Cartan's calculus; for example, $(d\alpha)^{[j]} = d(\alpha^{[j]})$ and $(\iota_X \alpha)^{[i+j]} = \iota_X^{[i]} \alpha^{[j]}$ for $X \in \mathcal{X}(M)_{(i)}$ and $\alpha \in \mathcal{O}(M)_{(j)}$.

### 5. Weighted deformation space

The deformation space for a manifold pair $(M, N)$ is a manifold $\delta(M, N)$ with a submersion onto the real line, such that the zero fiber is the normal bundle $\nu(M, N)$, while all other fibers are copies of $M$. The manifold structure is such that the directions normal to $N$ are ‘magnified’ as one approaches the zero fiber. Excellent introductions to deformation spaces may be found in [26, 28]. An important special case is Connes’ tangent groupoid [15], which may be seen as the deformation space $\delta(M \times M, M)$ of the pair groupoid $M \times M$ with respect to the diagonal; here $\nu(M \times M, M) \cong TM$. In this section, we generalize to the weighted setting.

#### 5.1. Constructions.

Let $(M, N)$ be a weighted manifold pair, with an order $r$ weighting. As in [27], and extending the algebraic definition of the weighted normal bundle $\nu_M(M, N)$, there is an algebraic definition of the **weighted deformation space** as the character spectrum
\[(17) \quad \delta_W(M, N) = \operatorname{Hom}_{\text{alg}}(\operatorname{Rees}(C^\infty(M)), \mathbb{R})\]
of the **Rees algebra** of the filtered algebra $C^\infty(M)$,
\[(18) \quad \operatorname{Rees}(C^\infty(M)) = \left\{ \sum_{i \in \mathbb{Z}} z^{-i} f_i : f_i \in C^\infty(M)_{(i)} \right\} \subseteq C^\infty(M)[z, z^{-1}].\]

For $f \in C^\infty(M)_{(i)}$, we denote by
\[\tilde{f}^{[i]} : \delta_W(M, N) \to \mathbb{R},\]
The function defined by the Laurent polynomial $z^{-i}f$. If $f$ has filtration degree $i$, and $g$ has filtration degree $j$, then
\[ \tilde{fg}^{[i+j]} = \tilde{f}^{[i]}g^{[j]} \]

Let
\[ \pi: \delta_W(M, N) \to \mathbb{R} \]
be the map given by evaluation on the Laurent polynomial ‘$z$’; thus $\pi = \tilde{1}[-1]$. The map $\pi$ is surjective; its fibers are
\begin{equation}
\pi^{-1}(t) = \begin{cases} 
\nu_W(M, N) & t = 0, \\
M & t \neq 0.
\end{cases}
\end{equation}

This follows since an algebra morphism $\text{Rees}(C^\infty(M)) \to \mathbb{R}$ taking $z$ to $t$ is the same as an algebra morphism from the quotient by the ideal $(z - t)\text{Rees}(C^\infty(M))$; this quotient is equal to $\text{gr}(C^\infty(M))$ if $t = 0$ and to $C^\infty(M)$ if $t \neq 0$. For $f \in C^\infty(M)_t$, we obtain
\[ \tilde{f}^{[i]}: \delta_W(M, N) \to \mathbb{R}, \quad \tilde{f}^{[i]}|_{\pi^{-1}(t)} = \begin{cases} 
f^{[i]} & t = 0, \\
t^{-i}f & t \neq 0.
\end{cases}
\]

**Theorem 5.1.** The space $\delta_W(M, N)$ has a unique $C^\infty$ structure, as a manifold of dimension $\dim(M) + 1$, in such a way that the functions given by evaluation on elements of $\text{Rees}(C^\infty(M))$ are all smooth. In terms of this manifold structure, the map
\[ \pi: \delta_W(M, N) \to \mathbb{R} \]
is a surjective submersion, and the identifications (19) are diffeomorphisms.

**Proof.** The construction is a straightforward extension from the unweighted case (see [26, 28]). For weighted coordinates $x_1, \ldots, x_n$ on open subsets $U \subseteq M$, we shall take the functions
\[ y_a = \tilde{x}_a^{[w_a]}, \quad a = 1, \ldots, n, \]

and the variable ‘$t$’ (given by the projection $\pi$) as coordinates on $\delta_W(U, U \cap N)$. Denote by $\overline{U} \subseteq \mathbb{R}^n$ the image of $U$ under the coordinate map $(x_1, \ldots, x_n): U \to \mathbb{R}^n$. Since
\[ y_a = \tilde{x}_a^{[w_a]} = \begin{cases} 
x_a^{[w_a]} & t = 0, \\
t^{-w_a}x_a & t \neq 0.
\end{cases}
\]
we see that the map $(y_1, \ldots, y_n, t): \delta_W(U, U \cap N) \to \mathbb{R}^{n+1}$ is a bijection onto the open subset
\[ \{ (y_1, \ldots, y_n, t) \in \mathbb{R}^{n+1} | (t^{-w_1}y_1, \ldots, t^{-w_n}y_n) \in \overline{U} \}. \]

Clearly, a covering of $M$ by weighted coordinate charts $U$ determines a covering of $\delta_W(M, N)$ by charts $\delta_W(U, U \cap N)$.

We need to show that weighted coordinate changes on $M$ gives rise to smooth coordinate changes on the deformation space $\delta_W(M, N)$. By considering the components of a weighted coordinate change, it is enough to show that if $U$ is the domain of a set of weighted coordinates $x_a$, and $f \in C^\infty(U)$ is a function of filtration degree $i$, then the restriction of $\tilde{f}^{[i]}$ to $\delta_W(U, U \cap U)$ is smooth. We may assume that $f(x_1, \ldots, x_n) = \chi(x_1, \ldots, x_n)x^s$ with $w \cdot s \geq i$ and $\chi \in C^\infty(U)$. Expressed in deformation space coordinates, the function $\tilde{f}^{[i]}$ is given for $t \neq 0$ by
\[ \tilde{f}^{[i]}(y_1, \ldots, y_n, t) = t^{-i}f(x_1, \ldots, x_n) = t^{-i}f(t^{w_1}y_1, \ldots, t^{w_n}y_n) = t^{w \cdot s - i}\chi(t^{w_1}y_1, \ldots, t^{w_n}y_n)y^s. \]
We see that this function extends smoothly to $t = 0$. The limiting function is zero if $w \cdot s > i$ (so that $f$ actually has filtration degree $i + 1$), and for $w \cdot s = i$ is given by $\chi(y_1, \ldots, y_{k_0}, 0, \ldots, 0) y^s$. In both cases, this agrees with

$$\bar{f}^{[i]}(y_1, \ldots, y_n, 0) = f^{[i]}(y_1, \ldots, y_n),$$

as required.

The remaining claims are immediate from the coordinate description.

Remark 5.2. The construction may be adapted to the analytic or holomorphic settings, by working with the sheaf of algebras $\text{Rees}(C^\infty_M)$. For $N = \text{pt}$, a weighted deformation to the normal cone was used in Kaveh’s work on toric degenerations [30].

The construction of weighted deformation spaces is functorial: a morphism of weighted pairs $\varphi: (M, N) \to (M', N')$ defines a smooth map of deformation spaces

$$\delta_W(\varphi): \delta_W(M, N) \to \delta_W(M', N'),$$

given on the open piece by $M \times \mathbb{R}^x \to M' \times \mathbb{R}^x$, $(m, t) \mapsto (\varphi(m), t)$, and on the zero fiber by $\nu_W(\varphi): \nu_W(M, N) \to \nu_W(M', N')$. (In terms of the algebraic definition, it is induced by the morphism of Rees algebras $\text{Rees}(C^\infty(M')) \to \text{Rees}(C^\infty(M))$ given by the pullback of functions.) The smoothness of $\delta_W(\varphi)$ follows since for any $g \in C^\infty(M')_{(i)}$, the pullback of $\bar{g}^{[i]}$ is the smooth function $\tilde{g} \circ \tilde{\varphi}^{[i]}$, while the pullback of the function ‘$t$’ on $M'$ (given by the projection $\pi$ to $\mathbb{R}$) is the function ‘$t$’ on $M$. As a special case, we can apply this to the map of manifold pairs $(N, N) \to (M, N)$, where $(N, N)$ has the trivial weighting. Since $\delta(N, N) = N \times \mathbb{R}$, this gives a natural embedding

$$N \times \mathbb{R} \to \delta_W(M, N).$$

5.2. Basic properties. Here are some other important aspects of the weighted deformation space.

(a) If $E \to N$ is a graded bundle, with its canonical weighting, then $\delta_W(E, N) \cong E \times \mathbb{R}$.

(b) The map

$$\delta_W(M, N) \to M,$$

given on $M \times \mathbb{R}^x$ by projection to the first factor and on $\nu_W(M, N)$ by bundle projection to $N$, followed by inclusion, is smooth. This follows from the local coordinate description, or since its composition with any $f \in C^\infty(M)$ is the smooth function $\bar{f}^{[0]}$.

(c) The space $\delta_W(M, N)$ comes with an action $u \mapsto \tilde{\kappa}_u$ of the non-zero scalars, in such a way that each $\bar{f}^{[i]}$ is homogeneous of degree $i$. The action of $u \in \mathbb{R}^x$ is defined by the algebra morphism $\sum_{i \in \mathbb{Z}} z^{-i} f_i \mapsto \sum_{i \in \mathbb{Z}} u^i z^{-i} f_i$ of the Rees algebra. The action map

$$\tilde{\kappa}: \mathbb{R}^x \times \delta_W(M, N) \to \delta_W(M, N), \quad (u, p) \mapsto \tilde{\kappa}_u(p)$$

is smooth, since its composition with $\bar{f}^{[i]}$, for a function $f$ of filtration degree $i$, is the smooth function $(u, p) \mapsto \bar{f}^{[i]}(u \cdot p) = u^i \bar{f}^{[i]}(p)$, and its composition with $\pi$ is the smooth function $(u, p) \mapsto u^{-1} \cdot \pi(p)$.

(d) The vector field

$$\Theta \in \mathfrak{X}(\delta_W(M, N))$$

with flow $\tau \mapsto \tilde{\kappa}_e^{\tau}$ is given by $-E$ on $\nu_W(M, N)$ and by $t \frac{\partial}{\partial t}$ on $M \times \mathbb{R}^x$. 
5.3. **Homogeneous interpolations.** For \( f \in C^\infty(M)_{(i)} \), the function \( \bar{f}^{[i]} \in C^\infty(\delta_W(M, N)) \) interpolates between the original function \( f \) at \( t = 1 \) and its homogeneous approximation \( f^{[i]} \) at \( t = 0 \). As already mentioned, it is homogeneous of degree \( i \) for the \( \mathbb{R}^\times \)-action. We get similar interpolations for all tensor fields of a given filtration degree. For example, given a vector field \( X \) of filtration degree \( i \geq -r \), then the vector field \( t^{-1}X \) on the open subset \( M \times \mathbb{R}^\times \) extends to a vector field \( \bar{X}^{[i]} \) on \( \delta_W(M, N) \). If \( \alpha \in \Omega(M) \) has filtration degree \( i \) then \( t^{-1}\alpha \) extends to a form \( \bar{\alpha}^{[i]} \). These extensions are homogeneous of degree \( i \), and along the zero fiber \( \nu_W(M, N) \), one obtains the homogeneous approximations \( \bar{X}^{[i]} \), respectively \( \alpha^{[i]} \). The Lie derivative \( \mathcal{L}_\Theta \) acts as \(-i\) on these extensions, for example,

\[
\mathcal{L}_\Theta \bar{\alpha}^{[i]} = -i\bar{\alpha}^{[i]}
\]

if \( \alpha \) has filtration degree \( i \). Once again, the extensions are compatible with the Cartan calculus. For example, in order to prove \( \mathcal{L}_{\bar{X}^{[0]}} \bar{\alpha}^{[j]} = (\mathcal{L}_X \bar{\alpha})^{[i+j]} \), it suffices to remark that the formula holds over the open set \( M \times \mathbb{R}^\times \).

**Example 5.3.** Let \( \pi \in \mathfrak{X}^2(M) \) be a Poisson bivector field, that is, the Schouten bracket with itself is zero: \([\pi, \pi] = 0\). If \( \pi \) has filtration degree \( i \) for a given weighting of \((M, N)\), then we obtain a bivector field on the deformation space, homogeneous of degree \( i \),

\[
\bar{\pi}^{[i]} \in \mathfrak{X}^2(\delta_W(M, N)),
\]

which is given by \( t^{-1}\pi \) on all non-zero fibers and by the homogeneous approximation \( \pi^{[i]} \) on the zero fiber \( \nu_W(M, N) \). Both \( \bar{\pi}^{[i]} \) and \( \pi^{[i]} \) are again Poisson structures, since Schouten bracket of the \( i \)-homogeneous approximations is the \( 2i \)-homogeneous approximation of the Schouten bracket \([\pi, \pi]\), hence is zero.

5.4. **Euler-like vector fields.**

**Definition 5.4.** A vector field \( X \in \mathfrak{X}(M) \) is called (weighted) Euler-like for the weighted manifold pair \((M, N)\) if it has filtration degree \( 0 \), with homogeneous approximation \( X^{[0]} \) the Euler vector field on the weighted normal bundle.

Recall that in graded bundle coordinates, the Euler vector field on a graded bundle with weights \( w_1, \ldots, w_n \) is given by \( E = \sum_{a=1}^n w_a \partial_{y^a} \).

**Definition 5.5.** A smooth map \( \varphi: \nu_W(M, N) \supseteq O \to M \), defined on a star-shaped open neighborhood \( O \) of the zero section \( N \), is called a weighted tubular neighborhood embedding if it is a morphism of weighted manifolds, with homogeneous approximation the identity map of \( \nu_W(M, N) \).

Here we are using the natural identification \( \nu_W(E, N) = E \) (see Proposition 4.5) for \( E = \nu_W(M, N) \). In analogy with the key observation in [10] (see also [7, 16, 26]), we have:

**Theorem 5.6.** Every Euler-like vector field \( X \) for the weighted manifold pair \((M, N)\) defines a unique maximal tubular neighborhood embedding \( \varphi: \nu_W(M, N) \supseteq O \to M \) such that \( \varphi^*(X|_{\varphi(O)}) = E|_O \). If \( X \) is complete, one can take \( O = \nu_W(M, N) \).

Here the uniqueness part means that any other tubular neighborhood embedding with these properties is obtained by restriction to an open subset of \( O \). The proof is parallel to the unweighted case [26] (cf. [7]): One observes that the vector field \( \frac{1}{2}(X^{[0]} + \Theta) \) is well-defined; the diffeomorphism \( \varphi \) is obtained by applying its flow to the zero fiber \( \nu_W(M, N) \).
Remark 5.7. Note that the weighting for a manifold pair \((M, N)\) may be recovered from the corresponding Euler-like vector fields \(X\). Indeed, letting \(s \mapsto \Phi^X_s\) be the flow of \(X\) (defined on \(O\), for all \(s \geq 0\)), the maps \(\lambda_t = \Phi^X_{-\log(t)}\) extend smoothly to \(t = 0\), and \(f\) has filtration degree \(i\) if and only if \(\lambda^*_t f = O(t^i)\).

Theorem 5.6 shows that weighted tubular neighborhood embeddings and weighted Euler-like vector fields are (essentially) the same thing. This is very useful for normal form problems. For example, given a tensor field \(\alpha\) (for example, a differential form) of filtration degree \(i\), one may ask for a weighted tubular neighborhood embedding \(\varphi\) taking \(\alpha\) to its homogeneous approximation \(\alpha^{[i]}\). Using the theorem, this amounts to constructing a weighted Euler-like vector field \(X\) satisfying \(\mathcal{L}_X \alpha = i \alpha\). Indeed, the pullback of this equation under the resulting \(\varphi\) reads as \(\mathcal{L}_{\mathcal{E} \varphi^* \alpha} = i \varphi^* \alpha\), which means that \(\varphi^* \alpha\) is homogeneous of degree \(i\), and hence coincides with \(\alpha^{[i]}\). See [37] for some applications of this technique, in the case \(r = 2\).

6. Weighted blow-ups

Given a submanifold \(N \subseteq M\), its real blow-up \(\text{Bl}(M, N)\) is a manifold with boundary, obtained by replacing \(N\) with the sphere bundle of its normal bundle. Real blow-ups along submanifolds are widely used in the analysis of singular spaces, pioneered in the work of Melrose [40]. One can also consider projective blow-ups (using the projectivization of the normal bundle) avoiding the introduction of a boundary, but often introducing orientability issues. In this section, we will generalize to the weighted setting. Of course, weighted blow-ups are widely used in algebraic geometry, and have also appeared in \(C^\infty\)-differential geometry (notably in the work of Melrose and collaborators, e.g., [32, 39]). However, we are not aware of a published account of a general coordinate-free framework.

6.1. Blow-up

Given any graded bundle \(E \to N\), we can form its sphere bundle

\[ S(E) = (E - N)/\mathbb{R}_{>0}, \]

where \(\mathbb{R}_{>0}\) is the multiplicative group of positive real numbers. This is indeed a fiber bundle with fibers diffeomorphic to spheres: After choice of a linearization \(E \cong E_{\text{lin}}\), and fixing a Euclidean fiber metric on each graded summand, we see that each \(\mathbb{R}_{>0}\)-orbit intersects the unit sphere, consisting of vectors \(v \in E \cong E_{\text{lin}}\) with \(||v|| = 1\), in a unique point. We may interpret \(S(E)\) as the set of ‘closed rays’ \(\mathbb{R}_{>0} \cdot v, v \notin N\).

Remark 6.1. By contrast, the projectivization \(P(E) = (E - N)/\mathbb{R}^\times\) may have orbifold singularities: in fact, it is smooth if and only if all weights from the weight sequence are odd, or all are even. In terms of a linearization \(E \cong E_{\text{lin}}\), this follows because nonzero vectors in \(E_{\text{lin}}^{-i}\) have trivial stabilizer if \(i\) is odd, and stabilizer \(\mathbb{Z}_2\) is \(i\) is even.

Given a weighted manifold pair \((M, N)\), we wish to define a real weighted blow-up

\[ \text{Bl}_W(M, N) = S(\nu_W(M, N)) \sqcup (M - N) \]

as a manifold with boundary. It is convenient to consider its ‘doubled’ version \(\hat{\text{Bl}}_W(M, N)\) as a manifold without boundary. The following description of weighted blow-ups in terms of weighted deformation spaces extends a description in [18, Definition 2.9].
Proposition 6.2. The action of \( \mathbb{R}_{>0} \) on the subset \( \delta_W(M, N) - (N \times \mathbb{R}) \) is free and proper. Hence, the quotient by this action,
\[
\tilde{\text{Bl}}_W(M, N) = (M - N) \sqcup S(\nu_W(M, N)) \sqcup (M - N),
\]
inherits a manifold structure, with a \( \mathbb{Z}_2 \)-action interchanging the two copies of \( M - N \) and preserving \( S(\nu_W(M, N)) \).

Proof. It is obvious that the action on this subset is free. To see that it is proper (and hence, a principal action), it suffices to construct local slices for the action. Consider local weighted coordinates \( x_1, \ldots, x_n \) on \( U \subseteq M \), and the resulting coordinates \( y_1, \ldots, y_n, t \) on \( \delta_W(U, N \cap U) \). The complement of \( (N \cap U) \times \mathbb{R} \) in this coordinate chart is characterized by the condition that at least one of the \( y_a \)'s with \( a > k_0 \) is non-zero. Define a covering of this complement by invariant open subsets \( V_{k_0+1}^+, V_{k_0+1}^- \ldots, V_n^+, V_n^- \), where \( V_a^\pm \) are defined by the condition \( \pm y_a > 0 \). On \( V_a^\pm \), the \( \mathbb{R}_{>0} \)-action has a slice given by the subset where \( y_a = \pm 1 \).

The manifold with boundary \( \tilde{\text{Bl}}_W(M, N) \) is realized as the image of \( \pi^{-1}(\mathbb{R}_{\geq 0}) - (N \times \mathbb{R}_{\geq 0}) \) under the quotient map.

Remark 6.3. The action of \( \mathbb{R}_{>0} \) on \( \delta_W(M, N) - (N \times \{0\}) \) is also free. However, it is not proper, and the quotient space fails to be a manifold.

6.2. Blow-down. Observe that the canonical map \( \delta_W(M, N) \to M \) is \( \mathbb{R}^k \)-invariant. It hence descends to a smooth blowdown map
\[
\tilde{\text{Bl}}_W(M, N) \to M.
\]
On each of the two copies of \( M - N \), the blow-down map is the obvious inclusion, and on the exceptional divisor it is the bundle projection \( S(\nu_W(M, N)) \to N \) followed by inclusion \( N \to M \).

Proposition 6.4. The blow-down map is a morphism of weighted manifold pairs,
\[
(\tilde{\text{Bl}}_W(M, N), S(\nu_W(M, N))) \to (M, N)
\]
using the trivial weighting for the pair \( (\tilde{\text{Bl}}_W(M, N), S(\nu_W(M, N))) \). Every Euler-like vector field \( X \in \mathfrak{X}(M) \), with respect to the given weighting of \( (M, N) \), lifts to an Euler-like vector field on \( \tilde{\text{Bl}}_W(M, N) \), with respect to the trivial weighting .

Proof. It is enough to prove the second claim, since weightings are uniquely determined by Euler-like vector fields. Using the tubular neighborhood embedding defined by \( X \), we may assume that \( M = E \) is a graded bundle, with \( X = \mathcal{E} \) the Euler vector field of \( E \). Hence, in graded bundle coordinates \( x_1, \ldots, x_n \),
\[
X = \sum_a w_a x_a \frac{\partial}{\partial x_a}.
\]
The vector field \( \tilde{X}^{[0]} \), written in coordinates \( y_1, \ldots, y_n, t \), is given by a similar expression \( \tilde{X}^{[0]} = \sum_a w_a y_a \frac{\partial}{\partial y_a} \). We want to express this vector field in a chart \( V_c^\pm \), with coordinates
\[
z_a = y_a y_c^{-w_a/w_c}, \quad a \neq c, \quad z_c = t y_c^{1/w_c}, \quad t.
\]
(Note that in the new coordinates, the action of \( \mathbb{R}_{>0} \) on the \( z_a \)'s becomes trivial.) An elementary calculation gives \( \tilde{X}^{[0]} = z_c \frac{\partial}{\partial z_c} \), confirming the claim. \( \square \)
6.3. Further remarks. Some further comments on weighted blow-ups:

(a) Tensor fields on $M$ lift to the weighted blow-up if and only if they are of filtration degree 0. For example, if $X \in \mathfrak{X}(M)$ has filtration degree 0, then the vector field $\tilde{X}^{[0]}$ on $\delta(M, N) - (N \times \mathbb{R})$ is $\mathbb{R}^\times$-invariant, and so descends. Similarly, if $\alpha \in \Omega(M)_{(0)}$ then $\tilde{\alpha}^{[0]}$ on $\delta(M, N) - (N \times \mathbb{R})$ is $\mathbb{R}^\times$-basic, and so it descends.

(b) We defined the smooth structure on the weighted blow-up in terms of the deformation space. One can also proceed in the other direction, viewing $\delta_W(M, N)$ as a submanifold of a weighted blow-up. In fact, there is a canonical decomposition

$$\tilde{\text{Bl}}_W(M \times \mathbb{R}, N \times \{0\}) = \delta_W(M, N) \sqcup \tilde{\text{Bl}}_W(M, N) \sqcup \delta_W(M, N),$$

with $\tilde{\text{Bl}}_W(M, N)$ embedded as a hypersurface and the two copies of the deformation space embedded as open subsets.

(c) Suppose $\varphi: (M, N) \to (M', N')$ is a morphism of weighted manifold pairs. In order to lift to the blow-up manifolds, it is necessary to remove a certain closed subset: we obtain a smooth lift $\tilde{\text{Bl}}^0_W(M, N) \to \text{Bl}_W(M', N')$ where

$$\tilde{\text{Bl}}^0_W(M, N) = (\delta_W(M, N) - \delta_W(\varphi)^{-1}(N' \times \mathbb{R})) / \mathbb{R}_{>0} \subseteq \tilde{\text{Bl}}_W(M, N).$$

(d) Given a linear weighting of a vector bundle $V \to M$ along a subbundle $W \to N$, our constructions give vector bundles

$$\nu_W(V, W) \to \nu_W(M, N), \ \delta_W(V, W) \to \delta_W(M, N), \ \tilde{\text{Bl}}^0_W(V, W) \to \tilde{\text{Bl}}_W(M, N),$$

where $\tilde{\text{Bl}}^0_W(V, W) \subseteq \tilde{\text{Bl}}_W(V, W)$ is the quotient of $\delta_W(V, W)|_{\delta_W(M, N) - N \times \mathbb{R}}$ by $\mathbb{R}^\times$.

7. Weightings as subbundles of the $r$-th tangent bundle

For $r > 2$, the filtration of the normal bundle $\nu(M, N)$ does not suffice to describe a weighting along $N \subseteq M$. Intuitively, the subbundles $F_{-i}$ in (6) only give first-order information, whereas the weighting requires higher order information. This motivates working with jet spaces. We will see that order $r$ weightings are in 1-1 correspondence with certain subbundles $Q$ of the $r$-th tangent bundle $T_rM$, and the weighted normal bundle is a quotient of that subbundle. For a trivial weighting, we have $Q = TM|_N$, and we recover the usual description of the normal bundle as a quotient $TM|_N/TN$.

7.1. Higher tangent bundles. We will need additional background material on higher tangent bundles $T_rM$.

7.1.1. Algebraic definition. In Example 3.2(e) we recalled the definition of $T_rM$ as the space $J^r_0(\mathbb{R}, M)$ of $r$-jets of curves. Similar to the description of tangent vectors as derivations $v: C^\infty(M) \to \mathbb{R}$, there is a more algebraic approach to $T_rM$, which will be more convenient for our purposes. See [31, Chapter VIII] for detailed discussions and proofs. Consider the truncated polynomial algebra

$$A_r = \mathbb{R}[\epsilon]/(\epsilon^{r+1}),$$

equipped with the grading where $\epsilon^i$ has degree $-i$. The $r$-th tangent bundle is the space of algebra morphisms,

$$T_rM = \text{Hom}_{\text{alg}}(C^\infty(M), A_r).$$
Elements of \( T_r M \) may be written \( u = \sum_{i=0}^{r} u_i \epsilon^i \), with linear maps \( u_i : C^\infty(M) \to \mathbb{R} \). Here \( u_0 \) is an algebra morphism (hence is given by evaluation at some point \( m \in M \)), \( u_1 \) is a derivation with respect to \( u_0 \) (hence is given by a tangent vector based at \( m \)), and so on. The identification of the jet space \( J^r_0(\mathbb{R}, M) \) with the space \( \text{Hom}_{\text{alg}}(C^\infty(M), \mathbb{A}_r) \) is the map taking \( j^r_0(\gamma) \) to the algebra morphism

\[
C^\infty(M) \to \mathbb{A}_r, \quad f \mapsto \sum_{i=0}^{r} \frac{d^i}{dt^i} \bigg|_{t=0} f(\gamma(t)) \epsilon^i.
\]

Put differently, this is the algebra morphism \( f \mapsto j^r_0(f \circ \gamma) \), where we identify \( J^r_0(\mathbb{R}, M) \cong \mathbb{A}_r \).

**Remark 7.1.** The same terminology ‘\( r \)-fold tangent bundle’ if often used for the bundle

\[
T^r M = \text{Hom}_{\text{alg}}(C^\infty(M), \mathbb{A}_1 \otimes \cdots \otimes \mathbb{A}_1).
\]

We shall call the latter the \( r \)-fold tangent bundle since it is isomorphic to the iteration \( TT \cdots TM \). The symmetric group \( S_r \) acts on \( \mathbb{A}_1 \otimes \cdots \otimes \mathbb{A}_1 \) by permutation of the factors. Its fixed point set is \( \mathbb{A}_r \), embedded by the map taking the generator \( \epsilon \) of \( \mathbb{A}_r \) to the sum \( \sum_i \epsilon_i \) of the generators for various copies of \( \mathbb{A}_1 \). This gives an embedding \( T_r M \to T^r M \) as the ‘symmetric part’ of the \( r \)-fold tangent bundle.

In the algebraic picture, the \((\mathbb{R}, \cdot, \cdot)\)-action on \( T_r M \) is given by \( \kappa_t : \sum u_j \epsilon^j \mapsto \sum_j u_j t^j \epsilon^j \) (cf. Example 3.2(b)). It extends further to a monoid action of

\[
\Lambda_r = \text{Hom}_{\text{alg}}(\mathbb{A}_r, \mathbb{A}_r),
\]

by composition, \( \Psi u = \Psi \circ u \). (See [29] for a detailed discussion of the \( \Lambda_2 \)-action on \( T_2 M \).) As a vector space, \( \Lambda_r \cong \epsilon \mathbb{A}_r = \mathbb{A}_r^{-} \), since any element \( \Psi \in \Lambda_r \) is uniquely determined by the image of \( \epsilon \), and since the degree 0 part of \( \Psi(\epsilon) \) must vanish due to \( \Psi(\epsilon)^{r+1} = \Psi(\epsilon^{r+1}) = 0 \). In the jet picture, \( \Lambda_r = J^r_0(\mathbb{R}, \mathbb{R})^{\text{op}} \) acting on \( J^r_0(\mathbb{R}, M) \) by composition.

The quotient maps

\[
p_r^* : T_r M \to T_s M
\]

for \( r \geq s \) are induced by the algebra morphisms \( \mathbb{A}_r \to \mathbb{A}_s \), and in particular \( p_0^* \) is the base projection \( T_0 M \to M \). The \( r \)-th tangent lift of a smooth map \( F : M \to M' \) is the morphism of graded bundles \( T_r F : T_r M \to T_r M' \), taking \( u \in T_r M \) to \( u \circ F^* \), where \( F^* \) is the pullback of functions. In particular, diffeomorphisms of \( M \) act on \( T_r M \) by graded bundle automorphisms.

### 7.1.2. Lift of functions.

Every function \( f \in C^\infty(M) \) determines a sequence of functions \( f^{(i)} \in C^\infty(T_r M)^{[i]} \), homogeneous of degree \( i \), taking \( u = \sum_{j=0}^{r} u_j \epsilon^j \in T_r M \) to

\[
f^{(i)}(u) = u_i(f).
\]

In the jet picture, the map takes \( u = j^r_0(\gamma) \) to \( \frac{1}{i!} \frac{d^i}{dt^i} \bigg|_{t=0} f(\gamma(t)) \). In particular, \( f^{(0)} \) is the pullback \((p_0^*)f \), while \( f^{(1)} \) is the pullback under \( p_1^* \) of the exterior differential \( df \), regarded as a function on \( TM \). Under multiplication of functions, one has the product rule

\[
(fg)^{(i)} = \sum_{j=0}^{i} f^{(j)} g^{(i-j)}.
\]

If \( x_a \) are local coordinates on \( U \subseteq M \), then the functions

\[
x_a^{(i)}, \quad a = 1, \ldots, n, \quad i = 0, \ldots, r
\]
serve as graded bundle coordinates on $T_r U \subseteq T_r M$. The function $f^{(i)}$ is a polynomial in the variables $x^{(j)}_a$ with $0 < j \leq i$. In low degrees,

$$f^{(1)} = \sum_a \frac{\partial f}{\partial x_a} x^{(1)}_a,$$

$$f^{(2)} = \sum_a \frac{\partial f}{\partial x_a} x^{(2)}_a + \frac{1}{2} \sum_{a_1, a_2} \frac{\partial^2 f}{\partial x_{a_1} \partial x_{a_2}} x^{(1)}_{a_1} x^{(1)}_{a_2},$$

$$f^{(3)} = \sum_a \frac{\partial f}{\partial x_a} x^{(3)}_a + \sum_{a_1, a_2} \frac{\partial^2 f}{\partial x_{a_1} \partial x_{a_2}} x^{(2)}_{a_1} x^{(1)}_{a_2} + \frac{1}{6} \sum_{a_1, a_2, a_3} \frac{\partial^3 f}{\partial x_{a_1} \partial x_{a_2} \partial x_{a_3}} x^{(1)}_{a_1} x^{(1)}_{a_2} x^{(1)}_{a_3};$$

these expressions are found as the Taylor coefficients of $t \mapsto f(x(t))$. As one may see from these formulas, the $C^{\infty}(M)$-module of homogeneous polynomials spanned by functions of the form $f^{(i)}$ is a proper submodule of $C^{\infty}(T_r M)^{[i]}$ if $n \geq 2$. For example, $x^{(1)}_1 x^{(2)}_2$ is not in the span, since the formula for $f^{(3)}$ only gives symmetric combinations such as $x^{(1)}_1 x^{(2)}_2 + x^{(1)}_2 x^{(2)}_1$.

7.1.3. Lifts of vector fields. As observed by A. Morimoto [43], every vector field $X \in \mathfrak{X}(M)$ gives rise to vector fields $X^{(-i)} \in \mathfrak{X}(T_r M)^{[-i]}$, for $i = 0, \ldots, r$, on the $r$-th tangent bundle. These are uniquely determined by their actions on lifts of functions $f \in C^{\infty}(M)$, as follows:

$$X^{(-i)} f^{(\ell)} = (X f)^{(\ell-i)}$$

(25)

The vector field $X^{(0)}$ is the $r$-th tangent lift, while $X^{(-i)}$ for $0 < i \leq r$ are the vertical lifts. In local coordinates,

$$\left( \sum_{a=1}^n f_a \frac{\partial}{\partial x_a} \right)^{(-i)} = \sum_{k=1}^r \sum_{a=1}^n \frac{f_a^{(k-i)}}{\partial x^{(k)}}.$$ \hspace{1cm} (26)

The product rule $(f X)^{(-i)} = \sum_{k=1}^r f^{(k-i)} X^{(-k)}$ shows that the maps $X \mapsto X^{(-i)}|_M$ are $C^{\infty}(M)$-linear, since $f^{(j)}|_M = 0$ for $j > 0$. This realizes the isomorphism of vector bundles

$$T(T_r M)|_M = TM \oplus (T_r M)|_{lin} = TM^{\oplus (r+1)}$$ \hspace{1cm} (27)

The lifts satisfy bracket relations

$$[X^{(-i)}, Y^{(-j)}] = [X, Y]^{(-i-j)},$$ \hspace{1cm} (28)

where the right hand side is zero if $i + j > r$.

7.2. The graded subbundle defined by a weighting. The following result shows that an order $r$ weighting along a submanifold $N \subseteq M$ determines a graded subbundle of the $r$-th tangent bundle, and may be recovered from that subbundle. Let

$$Q = \{ u \in T_r M | \forall f \in C^{\infty}(M)^{(i)}, \ j < i \leq r \Rightarrow f^{(j)}(u) = 0 \}.$$ \hspace{1cm} (29)

Denote by $F_{-i} \subseteq TM|_N$ the pre-images of the subbundles $F_{-i} \subseteq \nu(M, N)$ from Proposition 2.6.

**Theorem 7.2** (Weightings as subbundles of $T_r M$). (a) The subset $Q \subseteq T_r M$ is a graded subbundle $Q \to N$ of dimension $k_0 + \ldots + k_r$. In fact, $Q$ is $\Lambda_r$-invariant.
(b) The linear approximation is
\[ Q_{\text{lin}} = \tilde{F}_{-1} \oplus \cdots \oplus \tilde{F}_{-r}, \]
as a graded subbundle of \((T_r M)_{\text{lin}} = TM^{r}.\) In particular, \(Q_{\text{lin}}^{r} = TM_{|N}.\)

(c) The weighting is recovered from \(Q\) as follows:
\[ C^\infty(M)_{(i)} = \{ f \in C^\infty(M) | \forall j < i: f^{(j)} \text{ vanishes on } Q \} \]
for \(i \leq r.\)

Note that for the case of a trivial weighting \((r = 1),\) the graded subbundle \(Q \subseteq T_1 M\) is the restriction \(TM_{|N} \subseteq TM.\)

The proof of Theorem 7.2 is based on the following local coordinate description of \(Q:\)

**Lemma 7.3.** In weighted coordinates \(x_a \in C^\infty(U),\) the set \(Q \cap T_r U\) is given by the system of equations
\[
x_a^{(0)} = 0 \quad \text{for } w_a > 0,
\]
\[
x_a^{(1)} = 0 \quad \text{for } w_a > 1,
\]
\[
\ldots
\]
\[
x_a^{(r-1)} = 0 \quad \text{for } w_a > r - 1.
\]

Hence, the collection of all \(x_a^{(j)}\) with \(w_a \leq j\) serve as coordinates on \(Q \cap T_r U.\)

**Proof.** Let \(Q' \subseteq T_r U\) be the subset given by this system of equations. We claim \(Q \cap T_r U = Q'.\)

For the inclusion \(\subseteq,\) note that if \(w_a > j,\) then \(x_a\) has filtration degree \(j + 1,\) and so \(x_a^{(j)}\) must vanish on \(Q \cap T_r U.\) For the opposite inclusion \(\supseteq,\) consider the monomials \(x^s = x_1^{s_1} \cdots x_n^{s_n}\) for multi-indices \(s = (s_1, \ldots, s_n).\) By induction on \(\sum_a s_a,\) and using the product rule (24), the function \((x^s)^{(j)}\) vanishes on \(Q'\) for \(\sum_a s_a w_a > j.\) By definition, the space of functions of filtration degree \(j + 1\) is spanned by products \(x^s g\) with \(\sum_a s_a w_a > j;\) hence, using the product rule again, it follows that \(f^{(j)}\) vanishes on \(Q'.\) \(\square\)

**Proof of Theorem 7.2.** (a) The local coordinate description shows that \(Q\) is a graded subbundle, of the stated dimension. Let \(\Psi \in A_r, u \in Q.\) To show \(\Psi u \in Q,\) let \(f \in C^\infty(M)_{(i)}\) with \(i \leq r\) be given. We have, using (23),
\[
\sum_{j=0}^{r} f^{(j)}(\Psi u) \epsilon^j = (\Psi u)(f) = \Psi(u(f)) = \sum_{j=1}^{r} f^{(j)}(u)(\Psi(\epsilon))^j.
\]

On the right hand side of this equality, since \(\Psi(\epsilon) \in \epsilon A_r,\) the coefficients of \(\epsilon^j\) with \(j < i\) are zero. Hence, the same is true for the left hand side, proving that \(j < i \Rightarrow f^{(j)}(\Psi u) = 0.\) We conclude \(\Psi u \in Q.\)

(b) We shall use the description of \(Q_{\text{lin}}\) in terms of \(TQ_{|N} = TN \oplus Q_{\text{lin}},\) where the second summand is embedded as the tangent space to the fibers of \(Q \rightarrow N.\) Similarly \(T(T_r M)_{|M} = TM \oplus (T_r M)_{\text{lin}}.\) In terms of weighted coordinates over \(U \subseteq M,\)
\( T(T_r M)_{\text{lin}}|_U \) is spanned by all \( \frac{\partial}{\partial x_a^j} \) with \( 0 \leq j \leq r \) and \( a = 1, \ldots, n \), while

\[
T(Q \cap T_r U) = \text{span} \left\{ \frac{\partial}{\partial x_a^j} \mid w_a \leq j \right\}.
\]

The identification \( TM \cong T(T_r M)^{-j} \) from (27) is given by \( X \mapsto X^{(-j)} \). Since

\[
(30) \quad \frac{\partial}{\partial x_a^j} = (\frac{\partial}{\partial x_a})^{(-j)},
\]

it follows that the pre-image of \( Q^{(-j)}_{\text{lin}}|_{N \cap U} \) in \( TM|_{N \cap U} \) is spanned by all \( \frac{\partial}{\partial x_a} \) with \( w_a \leq j \). But this is exactly the subbundle \( \tilde{E}_j \). (See the proof of Proposition 2.6.)

(c) Suppose that \( f^{(j)} \) vanishes on \( Q \), for all \( j < i \). We want to show that \( f \) has filtration degree \( i \). Using weighted coordinates, and using a Taylor expansion on the coordinates \( x_a \) with \( w_a > 0 \), this is equivalent to showing that all derivatives

\[
\frac{\partial^p f}{\partial x_{a_1} \cdots \partial x_{a_p}}
\]

with \( w_{a_1} > 0 \) and \( w_{a_1} + \cdots + j_{a_p} < i \) vanish along \( N \). Let \( j_1 = w_{a_1}, \ldots, j_p = w_{a_p} \), and put \( j = j_1 + \cdots + j_p \). Then the derivative above is (up to a positive multiple) the coefficient of

\[
x_{a_1}^{(j_1)} \cdots x_{a_p}^{(j_p)}
\]

in \( f^{(j)} \). Since the \( x_a^{(k)} \) with \( k \leq w_a \) are the coordinates on \( Q \) the coefficient must be zero, as claimed. \( \square \)

Suppose \( (M, N) \) and \( (M', N') \) are manifold pairs, with weightings of order \( r, r' \). Raising one of the orders if needed, we may assume \( r = r' \). As a simple consequence of Theorem 7.2, we see that a map \( \varphi: M \rightarrow M' \) is a morphism of weighted manifolds if and only if the map \( T_r \varphi: T_r M \rightarrow T_r M' \) satisfies \( (T_r \varphi)(Q) \subseteq Q' \).

### 7.3. The weighted normal bundle as a subquotient

Let \( Q \subseteq T_r M \) be the graded subbundle associated to the weighting. We shall give a description of the weighted normal bundle as a quotient of \( Q \) under an equivalence relation.

**Theorem 7.4.** The weighted normal bundle for an order \( r \) weighting along \( N \) is canonically the quotient

\[
\nu_W(M, N) = Q/ \sim
\]

under the equivalence relation,

\[
q_1 \sim q_2 \iff f^{(i)}(q_1) = f^{(i)}(q_2) \quad \text{for all} \quad f \in C^\infty(M)_{(i)}, \ 0 \leq i \leq r.
\]

**Proof.** By definition of \( Q \), if \( f \in C^\infty(M) \) has filtration degree \( i+1 \), then its \( i \)-th lift \( f^{(i)} \) vanishes on \( Q \). Furthermore, if \( f \) has filtration degree \( i \) and \( g \) has filtration degree \( j \), then

\[
(fg)^{(i+j)}|_Q = f^{(i)}|_Q g^{(j)}|_Q
\]

by the product rule (24), since all the other terms vanish on \( Q \). This defines a map of graded algebras

\[
\text{gr}(C^\infty(M)) \rightarrow C^\infty_{\text{pol}}(Q),
\]

and dually (applying $\text{Hom}_{\text{alg}}(\cdot, \mathbb{R})$) a morphism of graded bundles $Q \to \nu_W(M, N)$. We claim that the fibers of this map are exactly the fibers of the equivalence relation (31). This will be evident from a coordinate description. Recall that in local weighted coordinates, $Q$ is given by the equations $x_a^{(i)} = 0$ for $i < w_a$. Hence, $x_a^{(i)}$ for $i \geq w_a$ serve as coordinates on $Q$. The following lemma shows that the quotient map simply forgets the coordinates with $i > w_a$, and the remaining coordinates $x_a^{(w_a)}$ descend to the standard coordinates on $\nu_W(M, N)$.

\textbf{Lemma 7.5.} Let $x_1, \ldots, x_n$ be weighted coordinates on $U \subseteq M$. For $q_1, q_2 \in Q \cap T_iU$, 
\begin{equation}
q_1 \sim q_2 \iff x_a^{(w_a)}(q_1) = x_a^{(w_a)}(q_2) \quad \text{for } a = 1, \ldots, n.
\end{equation}

\textbf{Proof.} The direction $\Rightarrow$ follows from (31) by putting $f = x_a$. Suppose conversely that $q_1, q_2 \in Q$ satisfy the condition on the right hand side of (32). To prove $q_1 \sim q_2$, we want to show that $f(i)(q_1) = f(i)(q_2)$ for all $f \in C^\infty(U)_{(i)}$ with $0 \leq i \leq r$. By the coordinate description of the filtration, it suffices to consider functions of the form $f = \chi x_{a_1} \cdots x_{a_p}$ with $w_{a_1} + \cdots + w_{a_p} \geq i$ and $\chi \in C^\infty(U)$. By the product rule (24),
\begin{equation}
f(i) = \sum \chi^{(j)}(j_{a_1}^{(j_1)} \cdots x_{a_p}^{(j_p)})
\end{equation}
where the sum is over $j_1, \ldots, j_p, j \geq 0$ with $j_1 + \cdots + j_p + j = i$. Such a term vanishes on $Q$ unless $j_\nu \geq w_{a_\nu}$ for all $\nu$. So, only terms with $j_\nu = w_{a_\nu}$ will remain after restriction:
\begin{equation}
f(i)|_Q = (\chi^{(0)} x_{a_1}^{(w_{a_1})} \cdots x_{a_p}^{(w_{a_p})})|_Q.
\end{equation}
By assumption, the terms $x_{a_\nu}^{(w_{a_\nu})}$ take on the same values at $q_1, q_2$. On the other hand, $\chi^{(0)}$ is the pullback of $\chi$, and (32) for $i = 0$ shows that $q_1, q_2$ have the same base point in $Q_0 = N$. Hence $\chi^{(0)}$ also takes on the same value at $q_1, q_2$. By (31), we conclude $q_1 \sim q_2$. \hfill $\square$

\textbf{Remark 7.6.} For a trivial weighting $r = 1$, we have $Q = TM|_N \subseteq TM$. Here, the $i = 0$ part of (31) says that the tangent vectors $q_1, q_2$ have the same base point in $N$, and the $i = 1$ part says that they agree on the vanishing ideal of $N$, i.e., their difference is tangent to $N$. This correctly gives the equivalence relation for $\nu(M, N)$ as a quotient $TM|_N/TN$.

\subsection*{7.4. Definition of weighted normal bundle in terms of filtration of vector fields.}

Given a weighting on $M$ along $N$, consider the resulting filtration on the Lie algebra of vector fields. The associated graded algebra is realized as polynomial vector fields on the weighted normal bundle, by the isomorphism (16). We see that the sheaf gr($\mathcal{X}(M)$)$^-$, given as the sum of components of strictly negative degree, is the sheaf of sections of a negatively graded Lie algebra bundle $\mathfrak{g} \to N$. Weighted coordinates $x_a$ over $U \subseteq M$ define a local frame for $\mathfrak{g}$, given by all
\begin{equation}
x^a \frac{\partial}{\partial x_a}|_{N \cap U}
\end{equation}
with $w \cdot s - w_a < 0$ and $a > k_0$, and where $s$ ranges over multi-indices with $s_b = 0$ for $w_b = 0$. Similarly, the subsheaf gr($\mathcal{I} \mathcal{X}_M$)$^-$ (where $\mathcal{I} \subseteq C^\infty_M$ is the vanishing ideal of $N$) is the sheaf of sections of a sub-Lie algebra bundle $\mathfrak{l} \to N$. A local frame for $\mathfrak{l}$ is obtained from that for $\mathfrak{g}$ by the additional condition $s \neq 0$. Let $L \subseteq K$ be the bundles, over $N$, of nilpotent Lie groups exponentiating these bundles of Lie algebras.

The isomorphism (16) identifies sections of $\mathfrak{g}$ with polynomial vector fields on $\nu_W(M, N)$ of strictly negative degree; the sections of $\mathfrak{l}$ are polynomial vector fields which furthermore vanish
along $N$. Note that $\mathfrak{X}_{\text{pol}}(\nu_{\mathcal{W}}(M, N))^{-}$ acts fiberwise transitively on $\nu_{\mathcal{W}}(M, N)$, with stabilizer along $N$ the polynomial vector fields vanishing along $N$. It follows that $K$ acts fiberwise transitively on $\nu_{\mathcal{W}}(M, N)$, with stabilizer along $N$ the subbundle $L$. To summarize:

**Proposition 7.7.** Given a weighted manifold pair $(M, N)$, the $C_{N}^{\infty}$-modules

$$\text{gr}(\mathcal{I}\mathfrak{X}_{M}^{-}) \subseteq \text{gr}(\mathfrak{X}_{M}^{-})^{-}$$

are the sheaves of sections of negatively graded Lie algebra bundles $I \subseteq \mathfrak{k}$ over $N$. Letting $L \subseteq K$ be the corresponding bundles of nilpotent Lie groups, we have that

$$\nu_{\mathcal{W}}(M, N) = K/L.$$  

In the case of a trivial weighting ($r = 1$), the filtration on vector fields starts with $\mathfrak{X}_{M,(-1)} = \mathfrak{X}_{M}$, while $\mathfrak{X}_{M,(0)}$ are vector fields tangent to $N$. Hence, in this case we directly have $\mathfrak{k} = T M|_{N}/TN$ (with zero bracket) and $I = 0$.

8. A necessary and sufficient condition

8.1. Some examples. In the previous section, we saw that a weighting of order $r$ determines a graded subbundle $Q \subseteq T_{r}M$, and is uniquely determined by $Q$. However, we did not specify which graded subbundles correspond to weightings. In particular, $\Lambda_{r}$-invariance does not suffice:

**Example 8.1.** The graded subbundle $Q \subseteq T_{2}\mathbb{R}^{2}$ along $N = \mathbb{R}$ defined by $x_{2} = 0$, $x_{1}^{(1)} = x_{2}^{(2)} = 0$ is $\Lambda_{2}$-invariant, and satisfies $Q_{\text{lin}}^{-2} = T M|_{N}$, but it does not correspond to a weighting since $Q_{\text{lin}}^{-1}$ is the zero bundle (and so does not contain $TN$).

We might add the condition that the summands of $Q_{\text{lin}} \subseteq (T_{r}M)_{\text{lin}} = T M^{\leq r}$ define a filtration of $T M|_{N}$, with $Q_{\text{lin}}^{-r} = T M|_{N}$, but this still is not enough:

**Example 8.2.** The graded subbundle $Q \subseteq T_{4}\mathbb{R}^{3}$ along $N = \{0\}$, given by the equations

$$x_{1} = x_{2} = x_{3} = 0, \quad x_{3}^{(1)} = 0, \quad x_{3}^{(2)} = 0, \quad x_{3}^{(3)} = x_{1}^{(1)} x_{2}^{(2)} - x_{1}^{(2)} x_{2}^{(1)},$$

is $\Lambda_{4}$-invariant, and defines a filtration of $T M|_{N}$, but it does not correspond to a weighting. Indeed, if $Q$ were the graded subbundle defined by a weighting, then the prescription from Theorem 7.2 would show that $x_{1}, x_{2}$ have filtration degree 1 (but not 2) while $x_{3}$ has filtration degree 3 (but not 4). However, the graded subbundle defined by this weighting is given by the equations $x_{1} = x_{2} = x_{3} = 0$, $x_{3}^{(1)} = 0$, $x_{3}^{(2)} = 0$, and so is strictly larger than $Q$.

8.2. Infinitesimal transitivity. The key properties of graded subbundles corresponding to weightings come from the following result.

**Proposition 8.3.** Let $Q \subseteq T_{r}M$ be the graded subbundle defined by a weighting of order $r$.

(a) The lift $X^{-}(-i)$ of a vector field $X \in \mathfrak{X}(M)$ is tangent to $Q$ if and only if $X$ has filtration degree $-i$.

(b) Vector fields of the form $X^{-}(-i)$ with $0 \leq i \leq r$ and $X \in \mathfrak{X}(M)_{(-i)}$ span the tangent bundle of $Q$ everywhere.
Proof. We use local weighted coordinates $x_a$ on $U \subseteq M$. A vector field $X = \sum f_a \frac{\partial}{\partial x_a} \in \mathfrak{X}(U)$ has filtration degree $-i$ if and only if each $f_a$ has filtration degree $w_a - i$. On the other hand, $X^{(-i)}$ is given in coordinates $x_a^{(k)}$ by (26). Since
\[ T(Q \cap T_r U) = \text{span} \left\{ \frac{\partial}{\partial x_a} \right\}_{w_a \leq j}, \]
we see that $X^{(-i)}$ is tangent to $Q$ if and only if the coefficients $f_a^{(j-i)}$ with $j < w_a$ vanish on $Q$. This is equivalent to saying that each $f_a$ has filtration degree $w_a - i$, proving (a). Part (b) is also clear from local coordinates, using (30).

Recall that there is a natural vector bundle action of $TM \to M$ on $T_r M \to M$. In terms of the description $T_r M = \text{Hom}_{\text{alg}}(C^\infty(M), A_r)$, the action of a tangent vector $v \in TM|_m$ (regarded as a linear map $C^\infty(M) \to \mathbb{R}$ with the usual derivation property) on $u \in T_r M|_m$ is given by
\[ u \mapsto u + v \epsilon^r. \]
Note that the action of $X \in \mathfrak{X}(M) = \Gamma(TM)$ is the time-1 flow of the $r$-th vertical lift $X^{(-r)}$. If $Q \subseteq T_r M$ corresponds to a weighting, we saw that all such lifts are tangent to $Q$ (since $\mathfrak{X}(M) = \mathfrak{X}(M)_{(-r)}$). That is, $Q$ is invariant under the $TM$-action on $T_r M$.

We shall also need the following natural action of the algebra $A_r$ on the tangent spaces of $T_r M$. This action was observed by Koszul; for detailed discussions see the articles of A. Morimoto [44] and Okassa [45]. One possible description uses the identification
\[ T(T_r M) = \text{Hom}_{\text{alg}}(C^\infty(M), A_r \otimes A_1). \]
The action of $x \in A_r$ on this space is by composition with the algebra endomorphism of $A_r \otimes A_1$, given on generators by $\epsilon \otimes 1 \mapsto \epsilon \otimes 1$, $1 \otimes \epsilon \mapsto x \otimes \epsilon$. For a more explicit description, let us compute the action of $\epsilon$ on $X^{(-i)}|_u \in T(T_r M)|_u$, for $X \in \mathfrak{X}(M)$ and $u \in T_r M$. By definition (see (25)), $X^{(-i)}|_u$ is the algebra morphism
\[ f \mapsto \sum_j f^{(j)}(u)(\epsilon^j \otimes 1) + \sum_j (X f)^{(j-i)}(u)(\epsilon^j \otimes \epsilon). \]
Hence $\epsilon \cdot X^{(-i)}|_u$ is the algebra morphism
\[ f \mapsto \sum_j f^{(j)}(u)(\epsilon^j \otimes 1) + \sum_j (X f)^{(j-i)}(u)(\epsilon^{j+1} \otimes \epsilon). \]
Shifting indices in the second sum, this is recognized as $X^{(-i-1)}|_u$. This shows that
(33) \[ \epsilon \cdot X^{(-i)} = X^{(-i-1)}. \]
Consider again a graded subbundle $Q \subseteq T_r M$ defined by a weighting. Since $\mathfrak{X}(M)_{(-i)} \subseteq \mathfrak{X}(M)_{(-i-1)}$, the proposition shows that if $X^{(-i)}$ is tangent to $Q$ then so is (33). Since the tangent bundle is spanned by such lifts, we conclude that $TQ \subseteq T(T_r M)|_Q$ is invariant under the action of $A_r$. 

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8.3. **Necessary and sufficient conditions.** We are now in position to formulate a necessary and sufficient condition characterizing weightings:

**Theorem 8.4.** A graded subbundle $Q \subseteq T_r M$ corresponds to a weighting if and only if the following three conditions are satisfied.

(i) $Q$ is $TM$-invariant,

(ii) $TQ$ is $k_r$-invariant,

(iii) $TQ$ is spanned by lifts $X^{(-i)}$ of vector fields $X$ such that $X^{(-i)}$ is tangent to $Q$.

The necessity of these conditions was proved above. For the other direction, suppose that a graded subbundle $Q \subseteq T_r M$ satisfying the conditions of the theorem is given. We want to construct local weighted coordinates $x_a$ such that $Q$ is defined by the equations from Lemma 7.3. The proof (modeled after the construction of 'privileged coordinates' in Choi-Ponge [13, 12], which in turn was motivated by Bellaïche [6]) is rather long, and will be given in the final section of this paper.

8.4. **Application: Singular Lie filtrations.** Using Theorem 8.4, we obtain new examples of weightings arising from *singular Lie filtrations*. We will give a detailed discussion elsewhere [34], but here is a quick preview. A singular Lie filtration on a manifold $M$ is a filtration $\mathcal{X}_M = H_{-r} \supseteq \cdots \supseteq H_0$ of the sheaf of vector fields by locally finitely generated $C^\infty_M$-submodules, compatible with brackets in the sense that $[H_{-i}, H_{-j}] \subseteq H_{-i-j}$. Examples of singular Lie filtrations abound: they include the much-studied *regular* Lie filtrations, with $H_{-i}$ the sheaves of sections of vector bundles (see references given in the introduction), but also many examples from sub-Riemannian geometry [6] or examples related to singular hypo-elliptic operators [2].

Let $E \subseteq \mathcal{X}_{T_r M}$ be the $C^\infty_{T_r M}$-submodule spanned by lifts $X^{(-i)}$ of vector fields $X \in H_{-i}(U)$. The bracket condition implies that $E$ is involutive, hence defines a singular foliation of $T_r M$ (in the sense of Androulidakis-Skandalis [3]). Suppose $Q \subseteq T_r M$ is a closed leaf. Then $Q$ satisfies the conditions of Theorem 8.4: Condition (iii) holds by construction, (ii) holds because the $H_{-i}$ define a filtration, and (i) holds since $H_{-r} = \mathcal{X}_M$. Hence, $Q$ defines a weighting of $M$ along $N = Q \cap M$ (which is itself a leaf of the singular foliation $H_0$).

8.5. **Application: Multiplicative weightings.** Viewing weightings in terms of subbundles of the $r$-th tangent bundle suggests notions of *multiplicative weightings* on Lie groupoids and Lie algebroids. Given a Lie groupoid $G \rightrightarrows M$ and a subgroupoid $H \rightrightarrows M$, we say that a weighting of $G$ along $H$ is *multiplicative* if the corresponding subbundle $Q \subseteq T_r G$ is a Lie subgroupoid of $T_r G \rightrightarrows T_r M$. The units of this subgroupoid are then a graded subbundle $Q_N \subseteq T_r M$ defining a weighting of $M$ along $N$. We obtain groupoids

$$\nu_W(G, H) \rightrightarrows \nu_W(M, N), \quad \delta_W(G, H) \rightrightarrows \delta_W(M, N), \quad \widehat{Bl}_W(G, H) \rightrightarrows \widehat{Bl}_W(M, N)$$

where the weighted blow-up groupoid is defined as the quotient of the restriction of $\delta_W(G, H)$ to $\delta_W(M, N) - (N \times \mathbb{R})$ by the action of $\mathbb{R}^\times$. (Recall that the restriction of a groupoid to a subset of its units is the set of arrows for which both source and target are in that subset.) See [18, 25] for blow-ups of Lie groupoids in the unweighted setting.
Similarly, given a Lie algebroid $A \Rightarrow M$ with a Lie subalgebroid $B \Rightarrow N$, we can define \textit{infinitesimally multiplicative} weightings of $A$ along $B$ by the condition that the corresponding graded subbundle $Q \subseteq T_B A$ is a Lie subalgebroid $Q \Rightarrow Q_N$ of $T_B A \Rightarrow T_B M$. Given such a weighting, we obtain new Lie algebroids

$$\nu_W(A, B) \Rightarrow \nu_W(M, N), \quad \delta_W(A, B) \Rightarrow \delta_W(M, N), \quad \mathcal{B}_W(A, B) \Rightarrow \mathcal{B}_W(M, N).$$

As an example, any order $r$ weighting of $M$ along $N$, defined by a graded subbundle $Q_N \subseteq T_N M$, the pair groupoid Pair$(M) = M \times M \Rightarrow M$ has a multiplicative weighting along Pair$(N) \Rightarrow N$, defined by the graded subbundle

$$\text{Pair}(Q_N) \subseteq \text{Pair}(T_r M) = T_r(\text{Pair}(M)).$$

Likewise, the tangent bundle $TM \Rightarrow M$ acquires an infinitesimally multiplicative weighting along $TN \Rightarrow N$, with the corresponding graded subbundle $TQ \subseteq T(T_r M) \cong T_r(TM)$. One has an isomorphism $\nu_W(TM, TN) \cong T \nu_W(M, N)$, by an argument extending that in [10, Appendix].

9. \textbf{Further Generalizations}

9.1. \textbf{Multi-weightings.} The theory of weightings on manifolds extends to \textit{multi-weightings}, with weight sequences $w_1, \ldots, w_n \in \mathbb{Z}^{d_0}$ for some $d > 1$. We declare that a monomial $x^s = x_1^{s_1} \cdots x_n^{s_n}$ with $s = (s_1, \ldots, s_n)$ has multi-weight $s \cdot w = \sum_a s_a w_a$. For $U \subseteq \mathbb{R}^n$ open, we take $C^\infty(U)_{(i)}$ to be the ideal spanned by all monomials $x^s$ such that $s \cdot w \geq i$ (using the partial ordering given by the componentwise inequality). Generalizing the case $d = 1$, this may be used as a local model to define multi-weightings on manifolds in terms of a filtration of the function sheaf by ideals $C^\infty_{M,(i)}$.

\textbf{Example 9.1.} Let $N_1, \ldots, N_d \subseteq M$ be closed submanifolds, intersecting cleanly in the sense that every point of $M$ is contained in a coordinate chart in which the submanifolds look like coordinate subspaces, that is, subspaces of $\mathbb{R}^n$ given by the vanishing of some of the coordinate functions. Then there is a \textit{trivial multi-weighting}, where $C^\infty_{M,(i_1, \ldots, i_d)}$ consists of all smooth functions vanishing to order $i_\alpha$ on $N_\alpha$, for $\alpha = 1, \ldots, d$.

Let $e_\alpha \in \mathbb{Z}^d$ denote the $\alpha$-th standard basis vector. For a $d$-fold multi-weighting, we obtain (single) weightings of $M$ given $C^\infty_{M,(i_e_\alpha)}$. The closed submanifolds

$$N_1, \ldots, N_d \subseteq M$$

defined by these weightings intersect cleanly, as defined above. (Using multi-weighted coordinates, these submanifolds look like coordinate subspaces.) Conversely, the collection of these weightings determines the multi-weighting via $C^\infty_{M,(i)} = \bigcap_\alpha C^\infty_{M,(i_e_\alpha)}$. We may thus regard a $d$-fold multi-weighting as a collection of $d$ weightings with some compatibility condition.

One defines $d$-fold \textit{multi-graded bundles} [23, 24] as manifolds $E$ with smooth actions of the monoid $(\mathbb{R}^d, \cdot) = (\mathbb{R}, \cdot) \times \cdots \times (\mathbb{R}, \cdot)$. Put differently, $E$ comes equipped with $d$ commuting scalar multiplications $\kappa^1, \ldots, \kappa^d$. The linear approximation $E_{\text{lin}} = \nu(E, N)$ of a multi-graded bundle is a multi-graded \textit{vector bundle}. Multi-graded bundles have a canonical multi-weighting, described by the obvious generalization of (10). An example of a multigraded bundle is

$$T_{r_1, \ldots, r_d} M = J_{0}^{r_1, \ldots, r_d}(\mathbb{R}^d, M) = \text{Hom}(C^\infty(M), \mathcal{A}_{r_1} \otimes \cdots \otimes \mathcal{A}_{r_d}).$$
For instance, $T_{1,1}M \cong TTM$ is the double tangent bundle.

The construction of weighted normal bundle admits a straightforward generalization to multi-weightings. Given a multi-weighting of $M$ defined by a multi-filtration of $C^\infty(M)$, let $\text{gr}(C^\infty(M))$ be the associated multi-graded algebra, with summands

$$
C^\infty(M)_{(i)}/\sum_{j>1} C^\infty(M)_{(j)}.
$$

The character spectrum

$$
\nu_\mathcal{V}(M, N_1, \ldots, N_d) = \text{Hom}_{\text{alg}}(\text{gr}(C^\infty(M)), \mathbb{R})
$$

is then a multi-graded bundle over the intersection $N = N_1 \cap \cdots \cap N_d$. (Multi-graded coordinates on this bundle are constructed from multi-weighted coordinates on $M$, by a straightforward generalization from the $d=1$ case.) Likewise, we obtain $d$-fold deformation spaces

$$
\pi : \delta_\mathcal{V}(M, N_1, \ldots, N_d) \to \mathbb{R}^d.
$$

It admits an algebraic definition as the character spectrum of the $\mathbb{Z}^d$-graded Rees algebra of $C^\infty(M)$, consisting of of $d$-variable Laurent polynomials

$$
\sum_{i_1, \ldots, i_d \in \mathbb{Z}} z_1^{-i_1} \cdots z_d^{-i_d} f_{i_1, \ldots, i_d}, \quad f_{i_1, \ldots, i_d} \in C^\infty(M)_{(i_1, \ldots, i_d)};
$$

the natural inclusion $\mathbb{R}[z_1, \ldots, z_d] \to \text{Rees}(C^\infty(M))$ defines the projection $\pi$. The fibers of the projection $\pi$ are themselves multi-weighted normal bundles (for smaller $d$). For instance, in the case of a bi-weighting,

$$
\pi^{-1}(t_1, t_2) =
\begin{cases}
\nu_\mathcal{V}(M, N_1, N_2) & t_1 = t_2 = 0 \\
\nu_\mathcal{V}(M, N_1) & t_1 = 0, \ t_2 \neq 0 \\
\nu_\mathcal{V}(M, N_2) & t_1 \neq 0, \ t_2 = 0 \\
\nu(M, \emptyset) = M & t_1, t_2 \neq 0
\end{cases}
$$

\begin{example}
For a trivial multi-weighting along cleanly intersecting submanifolds $N_1, \ldots, N_d$, we obtain a $d$-fold vector bundle (called the $d$-fold normal bundle)

$$
\nu(M, N_1, \ldots, N_d) \to N,
$$

and a corresponding $d$-fold deformation space. See \cite{38, 46} for $d = 2$. In \cite{46}, it is shown that the double deformation space may also be obtained by iteration; the argument generalizes to $d > 2$.

Given a $d$-fold multi-weighting of $M$, let $r_1, \ldots, r_d$ be the orders of the resulting weightings along the submanifolds $N_1, \ldots, N_d$, and $Q_1 \subseteq T_{r_1}M, \ldots, Q_d \subseteq T_{r_d}M$ the corresponding graded subbundles. By iterated tangent prolongation, these give rise to multi-graded subbundles $Q'_a \subseteq T_{r_1, \ldots, r_d}M$ with clean intersection. The weighted normal bundle $\nu_\mathcal{V}(M, N_1, \ldots, N_d)$ is realized as a quotient of a multi-graded subbundle $Q = Q'_1 \cap \cdots \cap Q'_d \subseteq T_{r_1, \ldots, r_d}M$, similar to the $d = 1$ case.

\begin{remark}
To any multi-weighting we may associate its total weighting along the intersection $N = N_1 \cap \cdots \cap N_d$, by taking $C^\infty(M)_{(j)}$ to be the sum of all $C^\infty(M)_{(j)}$ such that $|i| \geq j$. Note that a trivial multi-weighting may result in a non-trivial total weighting. Examples 2.9(c) and
\end{remark}
2.9(d) are special cases of this construction (note that a nested sequence of submanifolds has clean intersection). The weighted normal bundle for the total weighting is obtained by passing to the diagonal \((\mathbb{R}, \cdot)\)-action as in Example 3.2 (f) above.

9.2. The holomorphic setting. In this paper, we developed the theory of weightings in the category of \(C^\infty\)-manifolds. Working with sheaves, the definition generalizes to complex manifolds, where it leads to a concept of holomorphic weighted normal bundle \(\nu_W(M,N) \to N\) with an associated holomorphic weighted deformation space \(\delta_W(M,N)\). While the main ideas of this extension are reasonably straightforward, there are some aspects deserving special attention. Note first that \(29\) describes the theory of graded bundles in the complex setting, using holomorphic actions of the monoid \((\mathbb{C}, \cdot)\) of complex numbers. One important difference to the \(C^\infty\)-context is Theorem 3.4 – that is, a holomorphic graded bundle is not, in general, globally isomorphic to its linear approximation.

An example of a holomorphic graded bundle is the holomorphic \(r\)-th tangent bundle \(T_rM\). This may be defined using holomorphic jets (see \([29]\)), or algebraically using the holomorphic counterpart to definition 4.1. A weighting is then characterized as a certain holomorphic graded subbundle \(Q \subseteq T_rM\), and the weighted normal bundle is a quotient of the latter. Holomorphic weighted Euler-like vector fields and the associated weighted tubular neighborhood embeddings only exist locally, in general – in fact, the deformation space serves as a substitute for tubular neighborhood embeddings in some situations. Finally, in the complex setting there is no good analogue of the ‘spherical’ weighted blow-up discussed in Section 6; on the other hand, the projective weighted blow-ups always acquire orbifold singularities.

10. Proof of Theorem 8.4

This section is devoted to a proof of Theorem 8.4, characterizing the subbundles \(Q \subseteq T_rM\) corresponding to weightings. The approach is inspired by the construction of privileged coordinates in \([6, 13, 12]\), and some of the techniques in \([26]\).

10.1. Filtration of vector fields. Let \(N \subseteq M\) be a closed submanifold. We denote by \(\mathcal{I} \subseteq C^\infty_M\) the ideal sheaf of functions vanishing on \(N\). Let \(Q \subseteq T_rM\) be a graded subbundle along \(N \subseteq M\). Define a filtration \(C^\infty_M = C^\infty_{M,0} \supseteq C^\infty_{M,1} \supseteq \cdots\), by the prescription from Theorem 7.2: For \(1 \leq i \leq r\),

\[
C^\infty(U)_{(i)} = \{ f \in C^\infty(U) \mid \forall j < i: f^{(j)} \text{ vanishes on } Q \cap T_rU \};
\]

for \(i > r\), we define \(C^\infty(U)_{(i)}\) as sums of products of functions such that the filtration degrees add to \(i\). As usual, the filtration of the algebra of functions determines a filtration of vector fields \(\mathfrak{X}_M = \mathfrak{X}_{M,-r} \supseteq \mathfrak{X}_{M,-r+1} \supseteq \cdots\), where \(X \in \mathfrak{X}(U)\) has filtration degree \(j\) if it takes \(C^\infty(U)_{(i)}\) to \(C^\infty(U)_{(i+j)}\). Note that \(C^\infty(U)_{(1)}\) are the functions vanishing on \(U \cap N\).

Let us now impose the first two conditions from Theorem 8.4: thus \(Q\) is invariant under the bundle action of \(TM\) and its tangent bundle is \(K_r\)-invariant. Define a sheaves of vector fields by

\[
\mathcal{K}_{-i}(U) = \{ X \in \mathfrak{X}(U) \mid X^{(-i)} \text{ is tangent to } Q \cap T_rU \}.
\]

Proposition 10.1. The sheaves \(\mathcal{K}_{-i}\) are \(C^\infty_M\)-submodules, with \([\mathcal{K}_{-i}, \mathcal{K}_{-j}] \subseteq \mathcal{K}_{-i-j}\). They define a filtration of the sheaf of vector fields

\[
\mathfrak{X}_M = \mathcal{K}_{-r} \supseteq \mathcal{K}_{-r+1} \supseteq \cdots \supseteq \mathcal{K}_0.
\]
The submodule $\mathcal{K}_{-i}$ is contained in $\mathfrak{x}_{M,(-i)}$, for $i = 0, \ldots, r$.

**Proof.** From $(fX)^{(-i)} = \sum_{j=0}^{r-i} f^{(j)}X^{(-i-j)}$ we see that the $\mathcal{K}_{-i}$ are $C^\infty_M$-submodules, and $[X^{(-i)}, Y^{(-j)}] = [X, Y]^{(-i-j)}$ implies compatibility with Lie brackets. Furthermore, $\mathfrak{x}_M = \mathcal{K}_{-r}$ by $TM$-invariance of $Q$, and $\mathcal{K}_{-i} \subseteq \mathcal{K}_{-i-1}$ by $A_r$-invariance of $TQ$. It remains to show that $\mathcal{K}_{-i} \subseteq \mathfrak{x}_{M,(-i)}$. Given $X \in \mathcal{K}_{-i}(U)$ and $f \in C^\infty(U)_{(j)}$, we have to show $Xf \in C^\infty(U)_{(j-i)}$. We may assume $j \geq i$. Let $i \leq \ell < j$. By (25),

$$(Xf)^{(-\ell)} = X^{(-\ell)}f^{(\ell)}.$$  

But $f^{(\ell)}$ vanishes on $Q$ for $\ell < j$, and $X^{(-i)}$ is tangent to $Q \cap T_iU$. It hence follows that $(Xf)^{(-\ell)}$ vanishes on $Q \cap T_iU$. This means that $Xf$ has filtration degree $j - i$. \hfill $\square$

**Remark 10.2.** Proposition 8.3 shows that for graded subbundles $Q \subseteq T_iM$ defined by a weighting, the submodules $\mathcal{K}_{-i}$ coincide with $\mathfrak{x}_{M,(-i)}$ for $i = 0, \ldots, r$. On the other hand, Example 8.2 shows that in general, the inclusion $\mathcal{K}_{-i} \subseteq \mathfrak{x}_{M,(-i)}$ is strict.

**Lemma 10.3.** For $0 \leq j \leq i \leq r$, multiplication by $C^\infty_{M,(j)}$ takes $\mathcal{K}_{-i}$ to $\mathcal{K}_{-i+j}$.

**Proof.** Let $g \in C^\infty(U)_{(j)}$ and $X \in \mathcal{K}_{-i}(U)$. We have to show that $(gX)^{(-i+j)} = \sum_\ell g^{(\ell)}X^{(-i-\ell+j)}$ is tangent to $Q \cap T_iU$. For $\ell < j$ the function $g^{(\ell)}$ vanishes on $Q \cap T_iU$, while for $\ell \geq j$ we have that $-i - \ell + j \leq -i$, which implies that $X^{(-i-\ell+j)}$ is tangent to $Q \cap T_iU$. \hfill $\square$

Let $DO_M$ be the sheaf of scalar differential operators on $M$, and $DO^p_M$ the subsheaf of differential operators of order at most $p$. (That is, $DO^p(U)$ are scalar differential operators of order $\leq p$ acting on $C^\infty(U)$.) The filtration of vector fields, given by the submodules $\mathcal{K}_{-i}$, extends to a filtration on each $DO^p(M)$.

**Definition 10.4.** We say that $D \in DO(U)$ has $Q$-weight $\ell \leq 0$ if it can be written as a linear combination of operators

$$X_p \cdots X_1,$$

with $X_p \in \mathcal{K}_{-j_p}(U)$ and $-j_1 + \ldots + j_p = \ell$. Let $DO^p(U)_\ell$ be the space of differential operators of order $\leq p$ and $Q$-weight $\ell$.

For fixed $p$, the filtration starts by $Q$-weight starts in degree $-rp$:

$$DO^p_M = DO^p_{M,-rp} \supseteq \cdots \supseteq DO^p_{M,0} \supseteq 0.$$  

As a consequence of Lemma 10.1, and a straightforward induction, the filtration is compatible with products: If $D_1, D_2$ have $Q$-weights $\ell_1, \ell_2$, respectively, then $D_1 \circ D_2$ has $Q$-weight $\ell_1 + \ell_2$. Lemma 10.1 also shows that differential operators of $Q$-weight $\ell$ act on functions as operators of filtration degree $\ell$.

**10.2. Consequences of transitivity.** In terms of the submodules $\mathcal{K}_{-i}$, property (iii) from Theorem 8.4 says that the tangent bundle of $Q \cap T_iU$ is spanned by all $X^{(-i)}$ with $i = 0, \ldots, r$ and $X \in \mathcal{K}_{-i}(U)$. In other words, the action of the Lie subalgebra

$$\left\{ \sum_{i=0}^r X_i \otimes e^i : X_i \in \mathcal{K}_{-i}(U) \right\} \subseteq \mathfrak{x}(U) \otimes A_r$$

on $\mathfrak{x}_{M,(-i)}$ is tangent to $Q \cap T_iU$. It hence follows that $Xf$ has filtration degree $j - i$. \hfill $\square$
given by $\sum_{i=0}^{r} X_i \otimes e^i \mapsto \sum_{i=0}^{r} X_i^{(-i)}$ is infinitesimally transitive. Let us assume this transitivity property from now on. Define $\widetilde{F}_{-i} = Q_{\text{fin}}$ for $i = 1, \ldots, r$ and $\widetilde{F}_0 = TN$. Thus

$$TQ|_N = F_0 \oplus \cdots \oplus F_{-r},$$

as a subbundle of $T(T_r M) = TM \oplus (r+1)$. Transitivity of the action implies that $\widetilde{F}_{-i}$ is spanned by the restriction of $K_{-i}$ to $N$. We hence obtain a filtration by subbundles,

$$TM|_N = \widetilde{F}_{-r} \supseteq \widetilde{F}_{-r+1} \supseteq \cdots \supseteq \widetilde{F}_{-1} \supseteq \widetilde{F}_0 = TN.$$  

(37)

Letting $k_i = \text{rank}(\widetilde{F}_{-i})$ we obtain a unique weight sequence $0 \leq w_1 \leq \cdots \leq w_n \leq r$ such that $w_a < i \iff a < k_i$. We may characterize the filtration of functions in terms of the filtration of vector fields by $K_{-i} \subseteq X_M$.

**Lemma 10.5.** Let $1 \leq i \leq r$. A function $f \in C^\infty(U)$ has filtration degree $i$ if and only if

$$Df|_N = 0$$

for all $D \in \text{DO}(U)$ of $Q$-weight $\ell$, with $\ell + i > 0$.

**Proof.** Suppose that $f \in C^\infty(U)$ has filtration degree $i$. To show $Df|_N = 0$ for all differential operators of $Q$-weight $\ell > -i$, it suffices to check for $D = X_p \cdots X_1$ with $X_\nu \in K_{-j_\nu}$ where $-(j_1 + \cdots + j_\nu) > -i$. By Lemma 10.1, vector fields in $K_{-j}$ lower the filtration degree on functions by $j$. Hence, if $f$ has filtration degree $i$, then $Df$ has filtration degree $i - (j_1 + \cdots + j_\nu) > 0$, and so $Df|_N = 0$.

For the converse, suppose that $f \in C^\infty(U)$ satisfies the conditions of the lemma, for some given $i$. To show that $f$ has filtration degree $i$, we need to show that $f^{(j)}$ vanishes on $Q$ for all $j < i$. By the transitivity assumption, the action of the nilpotent Lie subalgebra

$$\{ \sum_{\ell=1}^{r} X_\ell \otimes e^\ell : X_i \in K_{-i}(U) \}$$

by vertical vector fields $\sum_{\ell=1}^{r} X_\ell^{(-\ell)}$ is infinitesimally transitive on the fibers of $Q \cap T^\nu U \to N \cap U$. The elements $\exp(\sum_{\ell=1}^{r} X_\ell \otimes e^\ell)$, $X_i \in K_{-i}(U)$ of the corresponding nilpotent group act by the time-one flow of $\sum_{\ell=1}^{r} X_\ell^{(-\ell)}$. By the Baker-Campbell-Hausdorff formula, we may write

$$\exp(\sum_{\ell=1}^{r} X_\ell \otimes e^\ell) = \exp(\sum_{\ell=j+1}^{r} X_\ell^{j+1} \otimes e^\ell) \exp(\sum_{\ell=1}^{j} X_\ell \otimes e^\ell)$$

with new elements $X_\ell' \in K_{-\ell}(U)$. The action of $\exp(\sum_{\ell=j+1}^{r} X_\ell' \otimes e^\ell)$ on $T_j U$ preserves the fibers of $p_j^* : T_j U \to T_j U$, and $f^{(j)}$ is constant along those fibers. Hence, to show that $f^{(j)}$ vanishes on $Q \cap T_j U$, it suffices to show that $(\Phi^Z_\ell)^* f^{(j)}|_{N \cap U} = 0$ for all vector fields of the form

$$Z = \sum_{\ell=1}^{j} X_\ell^{(-\ell)}, \quad X_\ell \in K_{-\ell}(U),$$

with $\Phi^Z_\ell$ denoting the flow of $Z \in \mathfrak{X}(T_j U)$. Note that the flow $\Phi^Z_\ell$ preserves the spaces of polynomials of any given degree $\leq d$, since the Lie derivative $\mathcal{L}_Z$ does. Hence $(\Phi^Z_\ell)^*$ is given
on fiberwise polynomial functions, such as \( f^{(j)} \), by a truncated exponential series
\[
(\Phi_t^Z)^* f^{(j)} = \exp(tL_Z) f^{(j)} = \sum_{p=0}^j \frac{t^p}{p!} (L_Z)^p f^{(j)}.
\]
We have
\[
L_Z f^{(j)} = \sum_{j_1=1}^j (L_{X_{j_1}} f)^{(j-j_1)},
\]
\[
(L_Z)^2 f^{(j)} = \sum_{j_1=1}^j \sum_{j_2=1}^{j-j_1} (L_{X_{j_2}} L_{X_{j_1}} f)^{(j-j_1-j_2)},
\]
and so on. Upon restriction to \( U \subseteq T_rU \), only the terms with \( j_1 + j_2 + \ldots + j_p = j \) remain:
\[
(L_Z)^p f^{(j)}|_U = \sum_{j_1+\ldots+j_p=j} L_{X_{j_p}} \cdots L_{X_{j_1}} f.
\]
By assumption, this expression vanishes along \( N \cap U \), and so \( f^{(j)} \) vanishes on \( Q \cap T_rU \).

10.3. Local frames. By an adapted local frame over an open subset \( U \subseteq M \), we mean a frame \( V_1, \ldots, V_n \in \mathfrak{X}(U) \) of \( TM|_U \), such that for all \( i \leq r \),
\[
V_1, \ldots, V_{k_i} \in \mathcal{K}_{-i}(U),
\]
and such that the restrictions \( V_a|_N \) for \( a \leq k_0 \) commute (recall that vector fields in \( \mathcal{K}_0(U) \) are tangent to \( N \cap U \)). Given \( m \in M \), one may construct an adapted local frame over a neighborhood \( U \) of \( m \): Start with a frame of \( \tilde{F}_0 = TN \), given by a collection of commuting vector fields on \( N \). Extend to a local frame for \( TM|_N \to N \) near \( m \), adapted to the filtration by subbundles \( \tilde{F}_{-i} \). Then use \( \tilde{F}_{-i} = \mathcal{K}_{-i}|_N \) to extend these sections to a local frame for \( TM \to M \) near \( m \).

The lifts \( V^{(-i)}_a \in \mathfrak{X}(T_rU) \) for \( i = 0, \ldots, r \) and \( a = 1, \ldots, n \) are a frame for \( T(T_rU) \). Those satisfying the extra condition \( a \leq k_i \) (i.e., \( v_a \leq i \)) restrict to a frame for \( T(Q \cap T_rU) \): they are tangent to \( Q \) by definition of \( \mathcal{K}_{-i} \), and they span the tangent bundle by dimension count. Every \( D \in DO(U) \) can be uniquely written as a finite sum
\[
D = \sum_{|s| \leq p} f_s V^s,
\]
using multi-index notation \( s = (s_1, \ldots, s_n) \), \( |s| = \sum_a s_a \) with \( V^s = V_1^{s_1} \cdots V_n^{s_n} \).

Lemma 10.6. If \( D \in DO(U) \) has \( Q \)-weight \( \ell \), then the functions \( f_s \) defined by the standard form (39) have filtration degree \( \ell + w \cdot s \).

Proof. Let \( D \) be a differential operator of \( Q \)-weight \( \ell \leq 0 \). If \( D \) has order 0 (so that it is given by a function, acting by multiplication) the statement is obvious: Every function has filtration degree 0, hence also filtration degree \( \ell \leq 0 \). Consider next the case that \( D \) is a vector
field $X \in \mathcal{K}_{-i}(U)$ with $\ell = -i$. Writing $X = \sum_{a=1}^{n} f_a V_a$ we want to show that the coefficient functions $f_a \in C^\infty(U)$ have filtration degree $w_a - i$. By definition of $\mathcal{K}_{-i}$, the vector field

$$X^{(-i)} = \sum_{a=1}^{n} \sum_{j=0}^{r-i} f_a^{(j)} V_a^{(-i-j)}$$

is tangent to $Q \cap T_r U$. But $V_a^{(-i-j)}$ belongs to the frame for $T(Q \cap T_r U)$ if and only if $w_a \leq i + j$. Hence, $X^{(-i)}$ is tangent to $Q \cap T_r U$ if and only if $f_a^{(j)}|_Q = 0$ for $w_a > i + j$, which is the case if and only if $f_a$ has filtration degree $w_a - i$. This proves the claim for $D$ of order $\leq 1$.

To prove the general case, it hence suffices to show that if the statement holds for differential operators $D_1, D_2$ of $Q$-weights $\ell_1, \ell_2$, then it also holds for the product $D = D_1 \circ D_2$, as a differential operator of $Q$-weight $\ell = \ell_1 + \ell_2$. This involves re-arranging the terms in $D_1 \circ D_2$ to bring them to the standard form $\sum_{|\alpha| \leq p} f_\alpha V^\alpha$. It is enough to check this on generators for the algebra $DO(U)$. If $a > b$ (so that $V_a \circ V_b$ is in the ‘wrong’ order), we have

$$V_a \circ V_b = V_b \circ V_a + [V_a, V_b].$$

Here $X = [V_a, V_b] \in \mathcal{K}_{-w_a-w_b}$, and (as shown above) the coefficients $f_\alpha^{(c)}$ in the bracket relation $[V_a, V_b] = \sum c f_\alpha^{(c)} V_c$ have filtration degree $w_c - w_a - w_b$. On the other hand, if $f \in C^\infty(U)$ and $V_a \in \mathcal{K}_{-w_a}(U)$, consider $V_a \circ f$ as a product of differential operators of $Q$-weights $\ell_1 = -w_a$ and $\ell_2 = 0$. Its standard form reads as

$$V_a \circ f = f V_a + V_a(f).$$

As noted above, for functions (such as $V_a(f)$) the claim is obvious. On the other hand, since $\mathcal{K}_{-w_a}(U)$ is a $C^\infty(U)$-module, the first term is a differential operator of $Q$-weight $-w_a$, and it has the required form since the coefficient function $f$ has filtration degree $\ell + w \cdot s = -w_a + w_a = 0$.

This allows us to reformulate Lemma 10.5 as follows:

**Lemma 10.7.** A function $f \in C^\infty(U)$ has filtration degree $i$ if and only if

$$(V^s f)|_N = 0$$

for all multi-indices $s$ with $w \cdot s < i$. (It suffices to verify this condition for multi-indices with $s_a = 0$ for $a \leq k_0$.)

**Proof.** The condition is necessary, since the differential operator $V^s$ has $Q$-weight $-w \cdot s$, and hence has filtration degree $-w \cdot s$ as an operator on functions. Conversely, suppose that the condition is satisfied. Lemma 10.5 shows that $f$ has filtration degree $i$ if and only if $D f|_N = 0$ for all differential operators $D$ of $Q$-weight $\ell > -i$. Write any such $D$ in its standard form (39) where $f_s$ has filtration degree $\ell + w \cdot s$. For coefficients $s$ with $\ell + w \cdot s > 0$, already $f_s$ vanishes on $N$. For coefficients with $\ell + w \cdot s \leq 0$, the assumption guarantees that $(V^s f)|_N = 0$. Hence $(D f)|_N = 0$ as required. 

10.4. **Adapted coordinates.** After these preparations, we get to the last stage of the proof, using the frame $\{V_a\}$ on a neighborhood $U$ of a given point $m \in N$ to construct adapted coordinates, so that $Q \cap T_r U$ is given by the equations from Lemma 7.3. We observe that this is equivalent to $x_a$ having the expected filtration degrees (using the filtration (34) on functions):
Lemma 10.8. Suppose \( U \subseteq M \) is an open neighborhood of \( m \), with coordinate functions \( x_a \in C^\infty(U) \), such that \( x_a \) has filtration degree \( w_a \). Then the manifold \( Q \cap T_y U \) is given by the equations
\[
x_a^{(i)} = 0 \quad \text{for} \quad i = 0, \ldots, r, \quad a > k_i.
\]
In particular, \( Q \cap T_y U \) corresponds to a weighting of \( (U, N \cap U) \).

Proof. By definition, the coordinate function \( x_a \) has filtration degree \( w_a \) if and only if the function \( x_a^{(i)} \) vanishes on \( Q \cap T_y U \) for all \( a > k_i \), i.e. \( w_a > i \). Since \( \dim Q = k_0 + \ldots + k_r \), it follows that \( Q \cap T_y U \) is given exactly by the vanishing of these functions.

Lemma 10.9. After replacing \( U \) with a smaller neighborhood of \( m \), if necessary, there are local coordinates \( x_1, \ldots, x_n \) on \( U \) such that each \( x_a \) has filtration degree \( w_a \), and
\[
(V_a x_b)|_{N \cap U} = \delta_{ab}.
\]

Proof. Since the restrictions \( V_a|_N \), \( 1 \leq a \leq k_0 \) are commuting vector fields on \( N \) (near \( m \)), we can choose functions \( y_a \in C^\infty(U) \) for \( 1 \leq a \leq k_0 \) such that
\[
(V_a y_b)|_{N \cap U} = \delta_{ab} \quad \text{for} \quad a, b \leq k_0.
\]
Complete to a collection of functions \( y_a \in C^\infty(U) \), \( 1 \leq a \leq n \), with \( y_a|_{N \cap U} = 0 \) for \( a > k_0 \), such that (40) holds for all \( 1 \leq a, b \leq n \). Our goal is to replace the \( y_a \) with new coordinates \( x_a \), satisfying analogous properties, such that the \( x_a \) have filtration degrees \( w_a \). For \( a \leq k_0 \) we will put \( x_a = y_a \). For \( a > k_0 \), we will look for a coordinate change of the form
\[
x_a = y_a + \sum \chi_{au} y^u, \quad a > k_0
\]
(\( u \) is an index with \( |u| = \sum u_a \geq 2 \)). \( y_u \) has filtration degree \( w_u \), \( a \geq |u| \), \( a \geq k_0 \) if and only if \((V_a y_b)|_N = \delta_{ab}\)). By Lemma 10.7, the coordinate function \( x_a \) has filtration degree \( w_a \) if and only if \((V_s x_a)|_N = 0\) for all \( s \). Again, we need only consider multi-indices \( s \) where \( s_a = 0 \) for \( a \leq k_0 \). For any such \( s \), \((V_s x_a)|_N = (V_s y_a)|_N + \sum (V_s (\chi_{au} y^u))|_N\).

If \( |u| > |s| \) then \((V_s (\chi_{au} y^u))\) is a polynomial of degree \( > 0 \) in the normal coordinates, and so vanishes on \( N \). For \( |u| = |s| \), we obtain a non-zero term only when \( u = s \), and by applying all derivatives to \( y^u \). Letting \( c_s = (V_s y^u)|_N \) (a positive constant), we therefore obtain the condition
\[
0 = (V_s y_a)|_N + c_s \chi_{as}|_N + \sum_{u: |u| < |s|} (V_s (\chi_{au} y^u))|_N.
\]
That is,
\[
\chi_{as} = -\frac{1}{c_s} (V_s y_a|_N + \sum_{u: |u| < |s|} (V_s (\chi_{au} y^u))|_N).
\]
This gives a recursive formula for the desired coordinate change.
This completes the proof of Theorem 8.4, giving a characterization of weightings in terms of a graded subbundle $Q \subseteq T r M$.

REFERENCES

1. D. Abramovich, M. Temkin, and J. Wodarczyk, *Functorial embedded resolution via weighted blowings up*, 2020.
2. I. Androulidakis, O. Mohsen, R. Yuncken, and E.van Erp, in preparation.
3. I. Androulidakis and G. Skandalis, *The holonomy groupoid of a singular foliation*, J. Reine Angew. Math. 626 (2009), 1–37.
4. M. Batchelor, *The structure of supermanifolds*, Trans. Amer. Math. Soc. 253 (1979), 329–338.
5. M. Behr, *Quasihomogeneous blow-ups and pseudodifferential calculus on SL(n)*, Ph.D. thesis, Universität Oldenburg, 2021.
6. A. Bellaïche, *The tangent space in sub-Riemannian geometry*, Sub-Riemannian geometry, Progr. Math., vol. 144, Birkhäuser, Basel, 1996, pp. 1–78.
7. F. Bischoff, H. Burzyn, H. Lima, and E. Meinrenken, *Deformation spaces and normal forms around transverseals*, Compositio Math. 156 (2020), 697–732.
8. A.J. Bruce, K. Grabowska, and J. Grabowski, *Linear duals of graded bundles and higher analogues of (Lie) algebroids*, J. Geom. Phys. 101 (2016), 71–99.
9. , *Introduction to graded bundles*, Note Mat. 37 (2017), no. suppl. 1, 59–74.
10. H. Bursztyn, H. Lima, and E. Meinrenken, *Splitting theorems for Poisson and related structures*, J. Reine Angew. Math. 754 (2019), 281–312.
11. A. Cattaneo and F. Schätz, *Introduction to supergeometry*, Rev. Math. Phys. 23 (2011), no. 6, 669–690.
12. W. Choi and R. Ponge, *Privileged coordinates and nilpotent approximation for Carnot manifolds, II. Carnot coordinates*, J. Dyn. Control Syst. 25 (2019), no. 4, 631–670.
13. , *Privileged coordinates and nilpotent approximation of Carnot manifolds, I. General results*, J. Dyn. Control Syst. 25 (2019), no. 1, 109–157.
14. , *Tangent maps and tangent groupoid for Carnot manifolds*, Differential Geom. Appl. 62 (2019), 136–183.
15. A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.
16. N. V. Dang, *The extension of distributions on manifolds, a microlocal approach*, Ann. Henri Poincaré 17 (2016), no. 4, 819–859.
17. C. Debord and G. Skandalis, *Lie groupoids, pseudodifferential calculus and index theory*, Advances in Noncommutative Geometry. On the Occasion of Alain Connes’ 70th Birthday., Springer International Publishing, 2019, arXiv: 1907.05258.pdf, pp. 245–289.
18. , *Blowup constructions for Lie groupoids and a Boutet de Monvel type calculus*, Münster Journal of Mathematics 14 (2021), 1–40, arXiv:1705.09588.
19. J.-P. Dufour and N.T. Zung, *Poisson structures and their normal forms*, Progress in Mathematics, vol. 242, Birkhäuser Verlag, Basel, 2005.
20. C. Ehresmann, *Prolongements des catégories différentiables*, Topologie et Géométrie Différentielle (Séminaire Ehresmann, Vol. VI, 1964), Inst. Henri Poincaré, Paris, 1964, p. 8.
21. V. Fischer and M. Ruzhansky, *Quantization on nilpotent Lie groups*, Progress in Mathematics, vol. 314, Birkhäuser/Springer, 2016.
22. J. Grabowski, M. Jóźwikowski, and M. Rotkiewicz, *Duality for graded manifolds*, Rep. Math. Phys. 80 (2017), no. 1, 115–142.
23. J. Grabowski and M. Rotkiewicz, *Higher vector bundles and multi-graded symplectic manifolds*, J. Geom. Phys. 59 (2009), no. 9, 1285–1305.
24. , *Graded bundles and homogeneity structures*, J. Geom. Phys. 62 (2012), no. 1, 21–36.
25. M. Gualtieri and S. Li, *Symplectic groupoids of log symplectic manifolds*, Int. Math. Res. Not. IMRN (2014), no. 11, 3022–3074.
26. A.R. Haj Saeedi Sadegh and N. Higson, *Euler-like vector fields, deformation spaces and manifolds with filtered structure*, Documenta Mathematica 23 (2018), 293–325.
27. P. J. Higgins and K. Mackenzie, *Algebraic constructions in the category of Lie algebroids*, J. Algebra **129** (1990), 194–230.

28. N. Higson, *The tangent groupoid and the index theorem*, Quanta of maths, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, RI, 2010, pp. 241–256.

29. M. Jóźwikowski and M. Rotkiewicz, *A note on actions of some monoids*, Differential Geom. Appl. **47** (2016), 212–245.

30. K. Kaveh, *Toric degenerations and symplectic geometry of smooth projective varieties*, J. Lond. Math. Soc. (2) **99** (2019), no. 2, 377–402.

31. I. Kolář, P. Michor, and J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993.

32. C. Kottke and R. Melrose, *Generalized blow-up of corners and fiber products*, Trans. Amer. Math. Soc. **367** (2015), no. 1, 651–705.

33. C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke, *Poisson structures*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 347, Springer, Heidelberg, 2013.

34. Y. Loizides and E. Meinrenken, *Singular Lie filtrations and weightings*, In preparation.

35. M. McQuillan and G. Marzo, *Very fast, very functorial, and very easy resolution of singularities*, 2019.

36. R. Mehta, *Supergroupoids, double structures, and equivariant cohomology*, 2006, Ph.D. thesis, University of California, Berkeley, p. 133.

37. E. Meinrenken, *Euler-like vector fields, normal forms, and isotropic embeddings*, Indagationes Mathematicae **32** (2021), 224–245.

38. E. Meinrenken and J. Pike, *The Weil algebra for double Lie algebroids*, Int. Math. Res. Not. IMRN (2021), no. 11, 8550–8622.

39. R. B. Melrose, *Differential analysis on manifolds with corners*, Manuscript, available at http://www-math.mit.edu/ rbm/book.html.

40. ____, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, vol. 4, A K Peters, Ltd., Wellesley, 1993.

41. O. Mohsen, *Index theorem for inhomogeneous hypoelliptic differential operators*, Preprint, 2020, arXiv 2001.00488.pdf.

42. ____, *On the deformation groupoid of the inhomogeneous pseudo-differential calculus*, Bull. Lond. Math. Soc. **53** (2021), no. 2, 575–592.

43. A. Morimoto, *Liftings of tensor fields and connections to tangent bundles of higher order*, Nagoya Math. J. **40** (1970), 99–120.

44. ____, *Prolongation of connections to bundles of infinitely near points*, Journal of Differential Geometry **11** (1976), no. 4, 479 – 498.

45. E. Okassa, *Prolongement des champs de vecteurs à des variétés de points proches*, Ann. Fac. Sci. Toulouse Math. (5) **8** (1986/87), no. 3, 349–366.

46. J. Pike, *Weil algebras and double Lie algebroids*, 2020, Ph.D. thesis, University of Toronto.

47. E. van Erp, *The index of hypoelliptic operators on foliated manifolds*, J. Noncommut. Geom. **5** (2011), no. 1, 107–124.

48. E. van Erp and R. Yuncken, *On the tangent groupoid of a filtered manifold*, Bull. Lond. Math. Soc. **49** (2017), no. 6, 1000–1012.

49. ____, *A groupoid approach to pseudodifferential calculi*, J. Reine Angew. Math. **756** (2019), 151–182.