Energy-momentum tensor on the lattice: recent developments

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- Noether current for the translational invariance
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- We have enough motivation to investigate EMT on the lattice.
Energy-momentum tensor (EMT) $T_{\mu\nu}(x)$

- Noether current for the translational invariance $\Leftarrow$ broken on the lattice!
- Generates the Poincaré symmetry and the dilatation
- Its components: energy, momentum, stress densities, and pressure
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- (Quasi-)Conformal field theory, IR fixed point, large anomalous dimension, (quasi-)dilaton
- Of course, source of the gravity
- We have enough motivation to investigate EMT on the lattice.
Talks on the shear viscosity

- Braguta, “Temperature dependence of shear viscosity in $SU(3)$-gluodynamics,” 7/29, 17:50– this afternoon, Building 32 Room 1015.
- Pasztor, “Viscosity of the pure $SU(3)$ gauge theory revisited,” 7/29, 18:10– this afternoon, Building 32 Room 1015.
Related investigations in the early days

- K. Fujikawa, “Chiral and conformal anomalies in lattice gauge theory,” Z. Phys. C 25, 179 (1984).
- A. S. Kronfeld and D. M. Photiadis, “Phenomenology on the lattice: composite operators in lattice gauge theory,” Phys. Rev. D 31, 2939 (1985).
- G. Martinelli and C. T. Sachrajda, “A lattice calculation of the pion’s form-factor and structure function,” Nucl. Phys. B 306, 865 (1988).
WT relation on the lattice (Caracciolo et. al. (1990))
WT relation on the lattice (Caracciolo et. al. (1990))

- Would-be translation ($\delta_\xi A_\mu(x) = \xi_\nu(x) F_{\nu\mu}(x)$)

$$
\delta_\xi U(x, \mu) = \xi_\nu(x) \hat{F}_{\nu\mu}(x) U(x, \mu)
$$
WT relation on the lattice (Caracciolo et. al. (1990))

- Would-be translation ($\delta_\xi A_\mu(x) = \xi_\nu(x) F_{\nu\mu}(x)$)

$$\delta_\xi U(x, \mu) = \xi_\nu(x) \hat{F}_{\nu\mu}(x) U(x, \mu)$$

- Since the lattice action is not translational invariant,

$$\left\langle \left[ \hat{\partial}_\mu \hat{T}_{\mu\nu}^{\text{naive}}(x) + a\hat{X}_\nu(x) \right] \hat{\mathcal{O}}(y) \right\rangle = - \left\langle \frac{\delta}{\delta \xi_\nu(x)} \delta_\xi \hat{\mathcal{O}}(y) \right\rangle,$$

where \( \hat{T}_{\mu\nu}^{\text{naive}} = \hat{T}_{\mu\nu}^{[1]} + \hat{T}_{\mu\nu}^{[3]} \) and

\[
\hat{T}_{\mu\nu}^{[1]} = \frac{1}{g_0^2} (1 - \delta_{\mu\nu}) \sum_\rho \hat{F}_{\mu\rho}^a \hat{F}_{\nu\rho}^a, \quad \text{(only off-diagonal comps.)}
\]

\[
\hat{T}_{\mu\nu}^{[3]} = \frac{1}{g_0^2} \delta_{\mu\nu} \left( \sum_\rho \hat{F}_{\mu\rho}^a \hat{F}_{\nu\rho}^a - \frac{1}{4} \sum_{\rho\sigma} \hat{F}_{\rho\sigma}^a \hat{F}_{\rho\sigma}^a \right), \quad \text{(only diagonal comps.)}
\]
Due to the radiative corrections (under hypercubic symmetry),

\[
a \hat{X}_\nu(x) = \left( \frac{Z_T}{Z_\delta} - 1 \right) \hat{\partial}_\mu \hat{T}^{[1]}_{\mu \nu}(x) + \left( \frac{Z_T Z_t}{Z_\delta} - 1 \right) \hat{\partial}_\mu \hat{T}^{[3]}_{\mu \nu}(x) \\
+ \frac{Z_T Z_s}{Z_\delta} \hat{\partial}_\mu \hat{T}^{[2]}_{\mu \nu}(x) + \frac{1}{Z_\delta} a \hat{R}_\nu(x),
\]

where

\[
\hat{T}^{[2]}_{\mu \nu} = \frac{1}{4g_0^2} \delta_{\mu \nu} \sum_{\rho \sigma} \hat{F}^a_{\rho \sigma} \hat{F}^a_{\rho \sigma}, \quad \text{(only diagonal comps.)}
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Inserting this into

\[
\left\langle \left[ \hat{\partial}_\mu \hat{T}_{\mu\nu}^{\text{naive}}(x) + a\hat{X}_\nu(x) \right] \hat{O}(y) \right\rangle = - \left\langle \frac{\delta}{\delta \xi_\nu(x)} \delta_\xi \hat{O}(y) \right\rangle,
\]
Properly normalized EMT

The identity becomes

\[ Z_T \left\langle \hat{\partial}_\mu \left[ \hat{T}^{[1]}_{\mu\nu}(x) + Z_T \hat{T}^{[3]}_{\mu\nu}(x) + Z_s \hat{T}^{[2]}_{\mu\nu}(x) \right] \hat{O}(y) \right\rangle \]

\[ = - \left\langle Z_\delta \frac{\delta}{\delta \xi}(x) \delta_\xi \hat{O}(y) + a\hat{R}_\nu(x)\hat{O}(y) \right\rangle \]

contact term
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\]

- Restoration of the translational invariance says that \( Z_\delta \) can be chosen so that

\[
a \rightarrow 0 \quad - \delta(x - y) \left\langle \partial_\nu \mathcal{O}(y) \right\rangle + \cdots .
\]
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\[ Z_T \langle \hat{\partial}_\mu \left[ \hat{T}^{[1]}_{\mu\nu}(x) + z_T \hat{T}^{[3]}_{\mu\nu}(x) + z_s \hat{T}^{[2]}_{\mu\nu}(x) \right] \hat{\mathcal{O}}(y) \rangle \]

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The properly normalized EMT is given by

\[ \hat{T}_{\mu\nu}(x) = Z_T \left[ \hat{T}^{[1]}_{\mu\nu}(x) + z_T \hat{T}^{[3]}_{\mu\nu}(x) + z_s \hat{T}^{[2]}_{\mu\nu}(x) \right] - \text{VEV}. \]
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\[ Z_T \left< \hat{\partial}_\mu \left[ \hat{T}^{[1]}_{\mu\nu}(x) + Z_T \hat{T}^{[3]}_{\mu\nu}(x) + Z_s \hat{T}^{[2]}_{\mu\nu}(x) \right] \hat{O}(y) \right> \]

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Determine \( Z_T, z_t, \) and \( z_s. \)
Set $y \neq x$ and conservation law determines $z_t$ and $z_s$. The overall factor $Z_T$ may be fixed by the rest energy $-\hat{T}_{00}$ of a hadronic state (Caracciolo-Curci-Menotti-Pelissetto, Caracciolo-Menotti-Pelissetto)
Various strategies

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- Matching to **bulk thermodynamic quantities**; $Z_Tz_T$ and $Z_Tz_s$ only (Meyer, Hübner-Karsch-Pica, . . .)
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Use gradient flow for the probe operator $\hat{O}(y)$ (Del Debbio-Patella-Rago, Capponi-Del Debbio-Patella-Rago, Capponi-Rago-Del Debbio-Ehret-Pellegrini)
Various strategies

- Set $y \neq x$ and \textbf{conservation law} determines $z_t$ and $z_s$. The overall factor $Z_T$ may be fixed by the rest energy $-\hat{T}_{00}$ of a hadronic state (Caracciolo-Curci-Menotti-Pelissetto, Caracciolo-Menotti-Pelissetto)

- Matching to \textbf{bulk thermodynamic quantities}; $Z_T z_T$ and $Z_T z_s$ only (Meyer, Hübner-Karsch-Pica, . . .)

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- Use \textbf{gradient flow for the probe operator} $\hat{O}(y)$ (Del Debbio-Patella-Rago, Capponi-Del Debbio-Patella-Rago, Capponi-Rago-Del Debbio-Ehret-Pellegrini)

- Use \textbf{gradient flow for EMT itself}; perturbative matching to EMT with dimensional regularization (H.S., Makino-H.S., FlowQCD Collaboration, WHOT QCD Collaboration)
Free energy with a shifted boundary condition

\[
f(L_0, \xi) = -\frac{1}{V} \ln \text{Tr} \left[ e^{-L_0 (H - i\xi \cdot P)} \right], \quad \phi(L_0, x) = \phi(0, x - L_0 \xi).
\]
Free energy with a shifted boundary condition

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Derivative wrt \(\xi_i\) gives rise to the \(i\)-th momentum:

\[ \frac{\partial}{\partial \xi_i} f(L_0, \xi) = -\frac{1}{V} iL_0 \langle P_i \rangle_{\xi} = -L_0 \langle T_{0i} \rangle_{\xi}. \]
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Shifted boundary condition

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\[ L_0 \xi \]

\[ L_0 \sqrt{1 + \xi^2} \]

\[ L/\sqrt{1 + \xi^2} \]

- From the underlying SO(4) covariance, for \( L \to \infty \),

\[ \langle T_{0i} \rangle_\xi = \frac{\xi_i}{1 - \xi_i^2} \langle T_{00} - T_{ii} \rangle_\xi. \]
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\[ \langle T_{0i} \rangle_\xi = \frac{\xi_i}{1 - \xi^2} \langle T_{00} - T_{ii} \rangle_\xi \leftrightarrow \text{determines } z_T \]
Very accurate determination of $Z_T(g_0^2)$ and $z_T(g_0^2)$ (plaquette action, clover $\hat{F}_{\mu\nu}$)

**Fit formulas**

$$Z_T(g_0^2) = \frac{1 - 0.4457 g_0^2}{1 - 0.7165 g_0^2} - 0.2543 g_0^4 + 0.4357 g_0^6 - 0.5221 g_0^8,$$

$$z_T(g_0^2) = \frac{1 - 0.5090 g_0^2}{1 - 0.4789 g_0^2}.$$
Resulting very accurate entropy density \( \sim \langle T_{0i} \rangle_{\xi} \) to \( T \sim 7.5 T_c \)
Updated very accurate entropy density $\sim \langle T_{0i}\rangle \xi$ to $T \sim 250 T_c$

Pepe, “Thermodynamics of strongly interacting plasma with high accuracy,” 7/29, 16:30– this afternoon, Building 32 Room 1015
The WT relation

\[
\langle \hat{\partial}_\mu \left[ Z_6 \hat{T}^{[1]}_{\mu\nu}(x) + Z_3 \hat{T}^{[3]}_{\mu\nu}(x) + 4Z_1 \hat{T}^{[2]}_{\mu\nu}(x) \right] \hat{O}(y) \rangle
\]

\[
= - \left\langle Z_\delta \frac{\delta}{\delta \xi_\nu(x)} \delta_x \hat{O}(y) + a\hat{R}_\nu(x) \hat{O}(y) \right\rangle_{\text{contact term}}
\]

\[
= - \delta_{x,y} \langle \hat{\partial}_\nu \hat{O}(y) \rangle + \cdots
\]
The WT relation

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\left\langle \hat{\partial}_\mu \left[ Z_6 \hat{T}^{[1]}_{\mu\nu}(x) + Z_3 \hat{T}^{[3]}_{\mu\nu}(x) + 4Z_1 \hat{T}^{[2]}_{\mu\nu}(x) \right] \hat{\mathcal{O}}(y) \right \rangle
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\]

\[
= - \delta_{x,y} \left\langle \hat{\partial}_\nu \hat{\mathcal{O}}(y) \right \rangle + \cdots .
\]

Use the gradient flow (Narayanan-Neuberger (2006), Lüscher 2009)

\[
\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x), \quad B_\mu(t = 0, x) = A_\mu(x)
\]

for the probe operator \( \hat{\mathcal{O}}(y) \). For example,

\[
\hat{\mathcal{O}}^{[\alpha]}_\nu(t, y) = \hat{\partial}_\rho \hat{T}^{[\alpha]}_{\rho\nu}(y)_{\text{flowed gauge field}}
\]
Then, because of the UV finiteness of the gradient flow (Lüscher-Weisz 2011), no contact term as $a \to 0$,

\[
\left\langle \hat{\partial}_\mu \left[ Z_6 \hat{T}^{[1]}_{\mu\nu}(x) + Z_3 \hat{T}^{[3]}_{\mu\nu}(x) + 4Z_1 \hat{T}^{[2]}_{\mu\nu}(x) \right] \hat{O}^{[\alpha]}(t, y) \right\rangle
\]

\[
= - \left\langle Z_\delta \frac{\delta}{\delta \xi_{\nu}(x)} \delta \xi \hat{O}^{[\alpha]}(t, y) \right\rangle - \left\langle a\hat{R}_{\nu}(x)\hat{O}^{[\alpha]}(t, y) \right\rangle \xrightarrow{\to 0} \text{known}
\]

\[
= - \left\langle \hat{\partial}_\nu \hat{O}^{[\alpha]}(y) \right\rangle, \text{ when integrated over } x.
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\left\langle \delta_{\mu} \left[ Z_6 \hat{T}^{[1]}_{\mu\nu}(x) + Z_3 \hat{T}^{[3]}_{\mu\nu}(x) + 4Z_1 \hat{T}^{[2]}_{\mu\nu}(x) \right] \hat{O}^{[\alpha]}(t, y) \right\rangle \\
= - \left\langle Z_\delta \frac{\delta}{\delta \xi_{\mu}(x)} \delta_{\xi} \hat{O}^{[\alpha]}(t, y) \right\rangle - \left\langle a\hat{R}_{\nu}(x) \hat{O}^{[\alpha]}(t, y) \right\rangle_{\to 0} \\
= - \left\langle \hat{\partial}_{\nu} \hat{O}^{[\alpha]}(y) \right\rangle, \text{ when integrated over } x.
\]

This can be used to determine the ratios

\[
\frac{Z_6}{Z_\delta}, \quad \frac{Z_3}{Z_\delta}, \quad \frac{Z_1}{Z_\delta},
\]

and the multiplicative factor for the translation,

\[
Z_\delta.
\]
\( Z_3/Z_\delta \) as a function of \( c = \sqrt{8t/aL} \):

\[
\begin{align*}
L=12, \beta=5.8506 \\
L=16, \beta=6.0056 \\
L=24, \beta=6.2670 \\
L=32, \beta=6.4822
\end{align*}
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\[ Z_6/Z_\delta \] as a function of \( c = \sqrt{8t}/aL \):

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L=12, \beta &= 5.8506 \\
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\end{align*}
\]

Also \( Z_\delta \); Capponi, 7/27, 9:20–Building B2a Room 2077.
Application of this strategy in 3D $\lambda\phi^4$-theory.

Ehret, “Renormalisation of the scalar energy-momentum tensor with the Wilson flow,” 7/27, 9:00– Building B2a Room 2077
Universal formula via gradient flow (H.S., Makino-H.S., 2013–)

- Gauge flow, fermion flow (flow time $t$) (Lüscher 2009–)

\[
\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x), \quad B_\mu(t = 0, x) = A_\mu(x), \\
\partial_t \chi(t, x) = \Delta \chi(t, x), \quad \chi(t = 0, x) = \psi(x),
\]

Composite operators of flowed fields are automatically renormalized ones (Lüscher-Weisz 2011).

Strategy.

\[\]
Universal formula via gradient flow (H.S., Makino-H.S., 2013–)

- **Gauge flow, fermion flow** (flow time $t$) (Lüscher 2009–)
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- Strategy
  
  regularization independent

  a composite operator of flowed fields

  EMT in dimensional regularization

  Lattice
Universal formula for EMT

Such an operator can be constructed by the small flow-time $t \to 0$ expansion (Lüscher-Weisz 2011)
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For the system containing fermions,

$$T_{\mu\nu}(x) = \lim_{t \to 0} \left\{ c_1(t) G^a_{\mu\rho}(t, x) G^a_{\nu\rho}(t, x) + \left[ c_2(t) - \frac{1}{4} c_1(t) \right] \delta_{\mu\nu} G^a_{\rho\sigma}(t, x) G^a_{\rho\sigma}(t, x) + c_3(t) \hat{\chi}(t, x) \left( \gamma_\mu \overleftarrow{D}_\nu + \gamma_\nu \overleftarrow{D}_\mu \right) \hat{\chi}(t, x) + [c_4(t) - 2c_3(t)] \delta_{\mu\nu} \hat{\chi}(t, x) \overleftarrow{D} \hat{\chi}(t, x) + c_5(t) m \hat{\chi}(t, x) \hat{\chi}(t, x) - \text{VEV} \right\},$$

and
the coefficients can be determined a priori (asymptotic freedom for $t \to 0$):

\begin{align*}
c_1(t) &= \frac{1}{\bar{g}^2} - b_0 \ln \pi - \frac{1}{(4\pi)^2} \left[ \frac{7}{3} C_2(G) - \frac{3}{2} T(R) N_f \right], \\
c_2(t) &= \frac{1}{8} \frac{1}{(4\pi)^2} \left[ \frac{11}{3} C_2(G) + \frac{11}{3} T(R) N_f \right], \\
c_3(t) &= \frac{1}{4} \left\{ 1 + \frac{\bar{g}^2}{(4\pi)^2} C_2(R) \left[ \frac{3}{2} + \ln(432) \right] \right\}, \\
c_4(t) &= \frac{1}{8} d_0 \bar{g}^2, \\
c_5(t) &= -\frac{\bar{m}}{m} \left\{ 1 + \frac{\bar{g}^2}{(4\pi)^2} C_2(R) \left[ 3 \ln \pi + \frac{7}{2} + \ln(432) \right] \right\}.
\end{align*}

where

\[ \bar{g} = \bar{g}(1/\sqrt{8t}), \quad \bar{m} = \bar{m}(1/\sqrt{8t}), \]

are running parameters in the MS scheme.
Universal formula for EMT

- Universal formula,

\[ T_{\mu\nu} = \lim_{t \to 0} \text{(known quantity in any lattice transcription)}, \]

however, holds only when the regulator is removed.
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Continuum limit \( a \to 0 \) first, then small flow time limit \( t \to 0 \) next.
Universal formula for EMT

- Universal formula,

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however, holds only when the regulator is removed.

- Continuum limit \( a \to 0 \) first, then small flow time limit \( t \to 0 \) next.

- The physical flow time is restricted flow below:

\[ a \ll \text{smearing radius of the flow} \simeq \sqrt{8t} \]
Universal formula, 

\[ T_{\mu \nu} = \lim_{t \to 0} (\text{known quantity in any lattice transcription}), \]

however, holds only when the regulator is removed.

- Continuum limit \( a \to 0 \) first, then small flow time limit \( t \to 0 \) next.
- The physical flow time is restricted flow below:

\[ a \ll \text{smearing radius of the flow} \approx \sqrt{8t} \]

- Extrapolation to \( t \to 0 \) is a possible source of the systematic error.
Pure $SU(3)$. $a = 0.041 - 0.11$ fm, $N_s = 32$, $N_T = 6 - 10$, $\sim 300$ configs.
Pure $SU(3)$. $a = 0.013–0.061 \text{ fm}$, $N_s = 64–128$, $N_T = 12–24$, $\sim 1000–2000$ configs.
Pure $SU(3)$. $a = 0.013–0.061$ fm, $N_s = 64–128$, $N_T = 12–24$, ~ 1000–2000 configs.
Pure $SU(3)$. $a = 0.013–0.061$ fm, $N_s = 64–128$, $N_T = 12–24$, $\sim 1000–2000$ configs.

Works very well!
Pure $SU(3)$. $a = 0.017$ fm, $N_s = 96$, $T = 1.66 T_c$, $\sim 50000$ configs.

\[
\frac{1}{T^5} \int d^3x \, \langle \delta T_{00}(x, \tau) \delta T_{00}(0) \rangle
\]
Pure $SU(3)$. $a = 0.017$ fm, $N_s = 96$, $T = 1.66 T_c$, $\sim 50000$ configs.

$$\frac{1}{T^5} \int d^3 x \left\langle \delta T_{00}(x, \tau)\delta T_{00}(0) \right\rangle$$

Very clean signal. Indication of the energy conservation!
Pure $SU(3)$. $a = 0.019 \text{ fm}$, $N_s = 64$, $T = 2.2 T_c$, $\sim 50000$ configs.

$$\frac{1}{T^5} \int d^3 x \  \langle \delta T_{\mu \nu}(x, \tau) \delta T_{\mu \nu}(0) \rangle$$

No useful signal...
\( N_f = 2 + 1 \) QCD. \( a = 0.070 \text{ fm}, \frac{m_\pi}{m_\rho} \simeq 0.63, \frac{m_\eta_{ss}}{m_\phi} \simeq 0.74, \)
\( N_S = 32, N_T = 4–56, \sim 100–1000 \text{ configs}. \)

Figure: \[ (e - 3p) / T^4, T = 232 \text{ MeV} \]  
Figure: \[ (e + p) / T^4, T = 232 \text{ MeV} \]
\( N_f = 2 + 1 \) QCD. \( a = 0.070 \text{ fm}, m_\pi/m_\rho \approx 0.63, m_{\eta ss}/m_\phi \approx 0.74, N_s = 32, N_\tau = 4–56, \sim 100–1000 \text{ configs.} \)

**Figure:** \( (e - 3p)/T^4, T = 279 \text{ MeV} \) **Figure:** \( (e + p)/T^4, T = 279 \text{ MeV} \)
\( N_f = 2 + 1 \) QCD. \( a = 0.070 \text{ fm}, \frac{m_\pi}{m_\rho} \simeq 0.63, \frac{m_{\eta_{ss}}}{m_\phi} \simeq 0.74, N_S = 32, N_T = 4–56, \sim 100–1000 \text{ configs.} \)

**Figure:** Black: T. Umeda et al. [WHOT-QCD Collaboration] (2012)
Related presentations

- Kanaya, “Equation of state in (2 + 1)-flavor QCD with gradient flow,” 7/27, 11:30– Building 32 Room 1015
- Ejiri, “Determination of latent heat at the finite temperature phase transition of $SU(3)$ gauge theory,” 7/29, 16:50– this afternoon, Building 32 Room 1015
I hope I convinced you that...
I hope I convinced you that... there have been rapid developments recently for this *old, but important* problem in lattice field theory, with encouraging results.
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So far, tests/applications of new ideas are limited mostly to bulk thermodynamics (and a few on viscosities)...
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there have been rapid developments recently for this old, but important problem in lattice field theory, with encouraging results.

So far, tests/applications of new ideas are limited mostly to bulk thermodynamics (and a few on viscosities).

Remembering possible vast applications, spin/momentum structure, (quasi-)conformal field theory, large anomalous dimension, gravity, . . . (mainly related to correlation functions), we expect much to be explored.
It is interesting that we have a closed universal expression of EMT.
It is interesting that we have a **closed universal** expression of EMT.

Bulk thermodynamics shows encouraging results.
It is interesting that we have a closed universal expression of EMT.
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As a by-product, correlation functions are typically quite clean!
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Bulk thermodynamics shows encouraging results.

As a by-product, correlation functions are typically quite clean!

Still, we need to understand/reduce the systematic error associated with the $t \to 0$ extrapolation.