First and Second Order Semi-strong Interaction in Reaction-Diffusion Systems

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Jens Rademacher
Quasi-stationary sharp interfaces

Prototype: Allen-Cahn model for phase separation

\[ V_t = \varepsilon^2 V_{xx} + V(1 - V^2), \]

\[ x \in \mathbb{R}, \quad 0 < \varepsilon \ll 1. \]

Interface/front: on small scale \( y = x/\varepsilon \) as \( \varepsilon \to 0. \)
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Weak interface interaction: through exponentially small tails – motion is exponentially slow in \( \varepsilon^{-1} \). Carr & Pego, Fusca & Hale.

More general: Ei; Sandstede; Promislow; Zelik & Mielke.
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More general: Ei; Sandstede; Promislow; Zelik & Mielke.

Global dynamics: motion gradient-like and coarsening.
Quasi-stationary ‘semi-sharp’ interfaces

Weak coupling to linear equation (FitzHugh-Nagumo type system):

\[
\begin{align*}
\partial_t U &= \partial_{xx} U - U + V \\
\partial_t V &= \varepsilon^2 \partial_{xx} V + V(1 - V^2) + \varepsilon U.
\end{align*}
\]
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Interface: problem on both slow/large \( x \)-scale and fast/small \( y \)-scale.

Multiple steady patterns: replacing \( \varepsilon U \) by \( \varepsilon g(U) \) arbitrary

singularities can be imbedded in existence problem [manuscript].
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Stability: Evans function in singular limit (‘NLEP’) [Doelman, Gardner, Kaper]; for this system: van Heijster’s results. (For other singular perturbation regime: SLEP method [Nishiura, Ikeda & Fuji, 80-90’s])
Semi-strong interaction

Interface motion: Now of order $\varepsilon^2$.  

$\varepsilon = 0.01, T = 2000$  

$\varepsilon = 0.005, T = 8000$
Semi-strong interaction

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Semi-strong interaction laws: Leading order form

$$\frac{d}{dt} r_j = -\varepsilon^2 \langle u_{0,j}, \partial_y v_0 \rangle / \| \partial_y v_0 \|^2_2, \quad u_{0,j} = a_j(r_1, \ldots, r_N)$$
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Rigorously [Doelman, van Heijster, Kaper, Promislow]
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Rigorously [Doelman, van Heijster, Kaper, Promislow]

Strong interaction: numerics as in scalar case, monotone & coarsening
The large and small scale problem

\[
\partial_t U = \partial_{xx} U - U + V
\]

\[
\partial_t V = \varepsilon^2 \partial_{xx} V + V (1 - V^2) + \varepsilon U.
\]
The large and small scale problem

\[ \partial_t U = \partial_{xx} U - U + V \]
\[ \partial_t V = \varepsilon^2 \partial_{xx} V + V(1 - V^2) + \varepsilon U. \]

Large scale: Assume stationary to leading order in \( \varepsilon \)

\[ 0 = \partial_{xx} U_0 - U_0 + V_0 \]
\[ 0 = V_0(1 - V_0^2). \]
The large and small scale problem

\[ \partial_t U = \partial_{xx} U - U + V \]
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\[ 0 = \partial_{xx} U_0 - U_0 + V_0 \]
\[ 0 = V_0(1 - V_0^2). \]

Small scale: \( y = x/\varepsilon \)

\[ \varepsilon^2 \partial_t u = \partial_{yy} u - \varepsilon^2 (u + v) \]
\[ \partial_t v = \partial_{yy} v + v(1 - v^2) + \varepsilon u. \]
The large and small scale problem

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\begin{align*}
\partial_t U &= \partial_{xx} U - U + V \\
\partial_t V &= \varepsilon^2 \partial_{xx} V + V(1 - V^2) + \varepsilon U.
\end{align*}
\]

**Large scale:** Assume stationary to leading order in \( \varepsilon \)

\[
0 = \partial_{xx} u_0 - u_0 + v_0 \\
0 = v_0(1 - v_0^2).
\]

**Small scale:** \( y = x/\varepsilon \), assume stationary to leading order

\[
0 = \partial_{yy} u_0 \\
0 = \partial_{yy} v_0 + v_0(1 - v_0^2).
\]
A 3-component FHN-type system

\[ \tau \partial_t U = \partial_{xx} U - U + V \]
\[ \theta \partial_t W = \partial_{xx} W - W + V \]
\[ \partial_t V = \varepsilon^2 \partial_{xx} V + V(1 - V^2) + \varepsilon(\gamma + \alpha U + \beta W). \]

Front patterns studied in semi-strong regime by van Heijster (with Doelman, Kaper, Promislow; also in 2D with Sandstede).

Already single front behaves different from Allen-Cahn: ‘butterfly catastrophe’ and Hopf bifurcation.

[Chirilius-Bruckner, Doelman, van Heijster, R.; manuscript]
Is this typical for localised solutions?

Consider \((u, v) \in \mathbb{R}^{N+M}\) and systems of the form

\[
\begin{align*}
\partial_t u &= D_u \partial_{xx} u + F(u, v; \varepsilon) \\
\partial_t v &= \varepsilon^2 D_v \partial_{xx} v + G(u, v; \varepsilon)
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\]
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**Fronts:** localisation to jump in \(v\) as \(\varepsilon \to 0\).
Is this typical for localised solutions?

Consider \((u, v) \in \mathbb{R}^{N+M}\) and systems of the form

\[
\begin{align*}
\frac{\partial_t u}{\partial t} &= D_u \partial_{xx} u + F(u, v; \varepsilon) \\
\frac{\partial_t v}{\partial t} &= \varepsilon^2 D_v \partial_{xx} v + G(u, v; \varepsilon)
\end{align*}
\]

**Fronts:** localisation to jump in \(v\) as \(\varepsilon \to 0\).

**Pulse:** localisation to Dirac mass in \(v\) as \(\varepsilon \to 0\).
‘Semi-sharp’ pulses / spikes

A major motivation for semi-strong regime: Pulse motion and pulse-splitting in Gray-Scott model.
Numerics and asymptotic matching by Reynolds, Pearson & Ponce-Dawson in early 90’s. Continued by Osipov, Doelman, Kaper, Ward, Wei, ...

Weak interaction: ‘edge splitting’
Semi-strong interaction: ‘$2^n$-splitting’
Example: simplified Schnakenberg model

\[ \partial_t U = \partial_{xx} U + \alpha - V \]

\[ \partial_t V = \varepsilon^2 \partial_{xx} V - V + UV^2. \]

Leading order existence, stability, interaction?
Two regimes within semi-strong regime

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha - v \\
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\end{align*}
\]

For Dirac-mass on \( x \)-scale set: \( u = \hat{u}, \ v = \varepsilon^{-1} \hat{v} \rightarrow \)

\[
\begin{align*}
\partial_t \hat{u} &= \partial_{xx} \hat{u} + \hat{\alpha} - \varepsilon^{-1} \hat{v} \\
\partial_t \hat{v} &= \varepsilon^2 \partial_{xx} \hat{v} - \hat{v} + \varepsilon^{-1} \hat{u} \hat{v}^2.
\end{align*}
\]

We will see that here motion is order \( \varepsilon \).
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\end{align*}
\]

We will see that here motion is order \(\varepsilon\).

Embedded motion of order \(\varepsilon^2\) analogous to front for \(\alpha = \sqrt{\varepsilon \hat{\alpha}}:\)
\(u = \sqrt{\varepsilon}\hat{u}, \ v = \sqrt{\varepsilon^{-1}}\hat{v} \rightarrow \)

\[
\begin{align*}
\partial_t \hat{u} &= \partial_{xx} \hat{u} + \hat{\alpha} - \varepsilon^{-1}\hat{v} \\
\partial_t \hat{v} &= \varepsilon^2 \partial_{xx} \hat{v} - \hat{v} + \hat{u}\hat{v}^2.
\end{align*}
\]
Generally: Two regimes within semi-strong regime

\[ \partial_t u = \partial_{xx} u + \alpha - u - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]
Generally: Two regimes within semi-strong regime

\[ \partial_t u = \partial_{xx} u + \alpha - u - uv^2 \]
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Case \( \alpha = \hat{\alpha} = O(1) \): \( u = \hat{u}, v = \hat{v}/\varepsilon \rightarrow \text{‘1st order standard form’} \)

\[ \partial_t \hat{u} = \partial_{xx} \hat{u} + \hat{\alpha} - \varepsilon^{-1} (\hat{u} + \varepsilon^{-1} \hat{u} \hat{v}^2) \]
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Generally: Two regimes within semi-strong regime

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Case \( \alpha = \sqrt{\varepsilon} \hat{\alpha} \): \( u = \sqrt{\varepsilon} \tilde{u}, \, v = \tilde{v}/\sqrt{\varepsilon} \rightarrow '2nd order standard form' \)

\[ \partial_t \tilde{u} = \partial_{xx} \tilde{u} + \tilde{\alpha} - \varepsilon^{-1}(\tilde{u} + \tilde{u} \tilde{v}^2) \]
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\end{align*}
\]

Can distinguish interaction type in general systems via ‘standard forms’

[R. SIADS ’13]
1st order semi-strong interaction

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]

\[ \varepsilon = 0.01, \quad T = 200 \]
\[ \varepsilon = 0.005, \quad T = 400 \]
\[ \alpha = 2.95 \]
Asymptotics for 1st order interaction

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha - uv^2 \\
\partial_t v &= \varepsilon^2 \partial_{xx} v - v + uv^2.
\end{align*}
\]

Expand \( u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2) \), \( v = v_0 + \mathcal{O}(\varepsilon) \).
Asymptotics for 1st order interaction

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
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Expand \( u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2) \), \( v = v_0 + \mathcal{O}(\varepsilon) \)

Large scale: \( V_0 = 0 \), \( 0 = \partial_{xx} \hat{U}_0 + \hat{\alpha} \rightarrow \text{parabola} \)
Asymptotics for 1st order interaction

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Small scale: \( \hat{u}_0 = 0 , \ \partial_{yy} \hat{u}_1 = \hat{u}_1 \hat{v}_0^2 \)

('core problem') \( \partial_{yy} \hat{v}_0 = \hat{v}_0 - \hat{u}_1 \hat{v}_0^2. \)
Asymptotics for 1st order interaction

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
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('core problem') \( \partial_{yy} \hat{v}_0 = \hat{v}_0 - \hat{u}_1 \hat{v}_0^2. \)

Matching:

\[ \hat{U}_0(x_j) = 0 (!) \]
\[ \partial_y \hat{u}_1(\pm\infty) = \partial_x \hat{U}_0(x_j \pm 0) \]
\[ \hat{v}_0(\pm\infty) = 0. \]
Asymptotics for 1st order interaction

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]

Expand \( u = u_0 + \varepsilon u_1 + O(\varepsilon^2) \), \( v = v_0 + O(\varepsilon) \)

Large scale: \( V_0 = 0 \), \( 0 = \partial_{xx} \hat{U}_0 + \hat{\alpha} \rightarrow \) parabola

Small scale: \( \hat{u}_0 = 0 \), \( \partial_{yy} \hat{u}_1 = \hat{u}_1 \hat{v}_0^2 \)

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⇒ one parameter missing \rightarrow allow for \( dx_j/dt = \varepsilon c + O(\varepsilon^2). \)
Asymptotics for 1st order interaction

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\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha - uv^2 \\
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\]

Expand \( u = u_0 + \varepsilon u_1 + O(\varepsilon^2), \ v = v_0 + O(\varepsilon) \)

Large scale: \( V_0 = 0, \ 0 = \partial_{xx} \hat{U}_0 + \hat{\alpha} \rightarrow \text{parabola} \)

Small scale: \( \hat{u}_0 = 0, \ \partial_{yy} \hat{u}_1 = \hat{u}_1 \hat{v}_0^2 \)

('core problem') \( \partial_{yy} \hat{v}_0 = \hat{v}_0 - \hat{u}_1 \hat{v}_0^2 + c \partial_y \hat{v}_0 \).

Matching:

\[
\begin{align*}
\hat{U}_0(x_j) &= 0 \ (!) \\
\partial_y \hat{u}_1(\pm \infty) &= \partial_x \hat{U}_0(x_j \pm 0) \\
\hat{v}(\pm \infty) &= 0.
\end{align*}
\]

\( \Rightarrow \) one parameter missing \( \rightarrow \) allow for \( \frac{dx_j}{dt} = \varepsilon c + O(\varepsilon^2). \)
Large and small scale problems

\[ \hat{U}_0(x_j) = 0 \]

\[ \partial_y \hat{u}_1(\pm \infty) = \partial_x \hat{U}_0(x_j \pm 0) \]
Large and small scale problems

\[ \hat{U}_0(x_j) = 0 \]
\[ \partial_y \hat{u}_1(\pm \infty) = \partial_x \hat{U}_0(x_j \pm 0) \]

Existence problem local:

nearest neighbor coupling

⇒ use single small scale problem, parameters \( c, p_{\pm} := \partial_x \hat{U}_0(x_j \pm 0) \).

\[ \partial_{yy} \hat{u}_1 = \hat{u}_1 \hat{v}_0^2 \]
\[ \partial_{yy} \hat{v}_0 = c \partial_y \hat{v}_0 + \hat{v}_0 - \hat{u}_1 \hat{v}_0^2. \]

Motion law not projection onto translation mode!
Small-scale pulse manifold

Numerically compute by continuation in $p_s = p_+ - p_-$, $p_a = p_+ + p_-$:

- $c > 0$
- $c < 0$
- $c = 0$

- $p_s$
- $p_a = 0$
Small-scale pulse manifold

Numerically compute by continuation in \( p_s = p_+ - p_- \), \( p_a = p_+ + p_- \):

Redced 1-pulse motion towards symmetric configuration for \( p_s < p_s^* \). Pulse-splitting near \( p_a = 0 \) for \( p_s > p_s^* \).
Small-scale pulse manifold

Numerically compute by continuation in $p_s = p_+ - p_-$, $p_a = p_+ + p_-$:

Reduced 1-pulse motion towards symmetric configuration for $p_s < p_s^*$. Pulse-splitting near $p_a = 0$ for $p_s > p_s^*$. Monotonicity of $c$ in $p_a \Rightarrow$ Abstract theorem applies: e.g. largest pulse distance is Lyapunov functional (until splitting). [R. SIADS ‘13]
Existence & stability map for pulse patterns

Solve boundary value problem formulation for eigenfunctions again by numerical continuation:

\[ c < 0 \]

Hopf stable region

\[ c = 0 \]
Crossing the boundary: Pulse-replication

Numerics by J. Ehrt (WIAS/HU)
1st and 2nd order semi-strong interaction

with M. Wolfrum & J. Ehrt
(WIAS/HU)

\[ \partial_t u = \partial_{xx} u + \alpha - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v - v + uv^2. \]

1st order semi-strong:
velocity \( c = \mathcal{O}(\varepsilon) \), \( \alpha = 0.9 \).

2nd order semi-strong:
velocity \( c = \mathcal{O}(\varepsilon^2) \), \( \alpha = 1.3\sqrt{\varepsilon} \).

Small ‘production’: slow motion and coarsening,
Large ‘production’: fast motion and splitting
Stability boundary in 2nd order case

PDE numerics when crossing boundary: annihilation (‘overcrowding’)

Numerics delicate: delayed Hopf-bifurcation...
Crossing unstable region

\[ \alpha = 1.2 \]

\[ x \]

\[ v(x) \]
Crossing unstable region

Only for relatively large $\varepsilon$: bifurcation (appears to be) subcritical.
A class of examples

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha - \mu u + \gamma v - uv^2 \\
\partial_t v &= \varepsilon^2 \partial_{xx} v + \beta - v + u v^2,
\end{align*}
\]
A class of examples

\[
\begin{align*}
\partial_t u &= \partial_{xx} u + \alpha - \mu u + \gamma v - uv^2 \\
\partial_t v &= \varepsilon^2 \partial_{xx} v + \beta - v + uv^2,
\end{align*}
\]

Schnakenberg model: \( \mu = \gamma = 0 \),

Gray-Scott model: \( \alpha = \mu, \gamma = \beta = 0 \),

Brusselator model: \( \alpha = \mu = 0 \).
A class of examples

\[ \partial_t u = \partial_{xx} u + \alpha - \mu u + \gamma v - uv^2 \]
\[ \partial_t v = \varepsilon^2 \partial_{xx} v + \beta - v + uv^2, \]

Schnakenberg model: \( \mu = \gamma = 0 \),
Gray-Scott model: \( \alpha = \mu, \gamma = \beta = 0 \),
Brusselator model: \( \alpha = \mu = 0 \).

Scalings:

1st order semi-strong: \( v = \varepsilon^{-1} \hat{v} \)
2nd order semi-strong: \( \alpha = \varepsilon^{1/2} \hat{\alpha}, u = \varepsilon^{1/2} \hat{u}, v = \varepsilon^{-1/2} \hat{v} \)
Literature

Second order case \((\alpha = \varepsilon \tilde{\alpha})\):
- Existence and stability from Doelman-Kaper ‘normal form’ approach.
- For fronts in FHN-type system: van Heijster.
- Interaction laws rigorously for FHN-type and Gierer-Meinhardt model variants [Doelman, Kaper, Promislow; van Heijster; Bellsky].

First order case \((\alpha = \mathcal{O}(1))\):
- Numerically: ‘core problem’ existence up to critical value (fold) – proof?
- Rich solution set [Doelman, Kaper, Peletier ’06].
- Interaction laws: asymptotics for Schnakenberg model [Ward et al].
- Proofs?
Model independent view

Semi-strong interaction can occur for $0 < \varepsilon \ll 1$ in systems of the form

\[
\begin{align*}
\partial_t u &= D_u \partial_{xx} u + F(u, v; \varepsilon) \\
\partial_t v &= \varepsilon^2 D_v \partial_{xx} v + G(u, v; \varepsilon) v + \varepsilon E(u, v; \varepsilon)
\end{align*}
\]

Pulse: localisation to Dirac mass in $v$ as $\varepsilon \to 0$.

Fronts: localisation to jump in $v$ as $\varepsilon \to 0$. 

![Graphs illustrating pulse and fronts behavior](image-url)
Model independent view

Semi-strong interaction can occur for $0 < \varepsilon \ll 1$ in systems of the form

$$
\begin{align*}
\partial_t u &= D_u \partial_{xx} u + F(u, v; \varepsilon) \\
\partial_t v &= \varepsilon^2 D_v \partial_{xx} v + G(u, v; \varepsilon)v + \varepsilon E(u, v; \varepsilon)
\end{align*}
$$

**Pulse:** localisation to Dirac mass in $v$ as $\varepsilon \to 0$.

**Fronts:** localisation to jump in $v$ as $\varepsilon \to 0$.

Expand and apply natural constraints to obtain boundedness as $\varepsilon \to 0$:

$$
\begin{align*}
\partial_t u &= D_u \partial_{xx} u + H(u, v; \varepsilon) + \varepsilon^{-1}(F^s(u, v) + \varepsilon^{-1}F^f(u, v)u)v, \\
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$$
Model independent view

Semi-strong interaction can occur for $0 < \varepsilon \ll 1$ in systems of the form

\[
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\]

Second order semi-strong interaction: $F^f \equiv G^f \equiv 0$. 
Summary

- Semi-strong interaction comes in different types.
- Unified framework for fronts, pulses and 1st, 2nd order interaction.
- Can read off the laws of motion (formally).
- In 1st order case: conditions for gradient-like pulse interaction.

[R. SIADS '13]
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Sidenote: In semi-strong regime also rich single interface bifurcations & pencil and paper analysis possible also for nonlocalized solutions...
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Thank you!