One-in-Two-Matching Problem is NP-complete

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Abstract. 2-dimensional Matching Problem, which requires to find a matching of left- to right-vertices in a balanced 2n-vertex bipartite graph, is a well-known polynomial problem, while various variants, like the 3-dimensional analogoue (3DM, with triangles on a tripartite graph), or the Hamiltonian Circuit Problem (HC, a restriction to “unicyclic” matchings) are among the main examples of NP-hard problems, since the first Karp reduction series of 1972. The same holds for the weighted variants of these problems, the Linear Assignment Problem being polynomial, and the Numerical 3-Dimensional Matching and Travelling Salesman Problem being NP-complete.

In this paper we show that a small modification of the 2-dimensional Matching and Assignment Problems in which for each \( i \leq n/2 \) it is required that either \( \pi(2i - 1) = 2i - 1 \) or \( \pi(2i) = 2i \), is a NP-complete problem. The proof is by linear reduction from SAT (or NAE-SAT), with the size \( n \) of the Matching Problem being four times the number of edges in the factor graph representation of the boolean problem. As a corollary, in combination with the simple linear reduction of One-in-Two Matching to 3-Dimensional Matching, we show that SAT can be linearly reduced to 3DM, while the original Karp reduction was only cubic.

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1. Introduction

Consider the problems encodable in the framework of the theory of Turing machines (say, the problems which could be translated into a computer program). Say that an instance of a problem has size \( N \) if it can be encoded into a sequence of \( N \) bits, within a given dictionary depending only on the problem. In a few words, Complexity Theory deals with the classification of these problems according to the asymptotic behaviour in \( N \) of the time of solution in the most difficult instance of size at most \( N \). This kind of analysis is called worst-case analysis, and is sided by an average-case analysis, where the time of the solution is the average in a given measure over the ensemble of instances of size \( N \).

We can restrict our attention, from this large class of problems, to problems of a special kind, such that, for any size \( N \), a known finite set \( S_N \) of “feasible solutions” (or “configurations”) exist, and, given an instance, the problem is to find the solution minimizing a certain integer-valued cost function (or objective function); or, if the cost function is valued in \{True, False\}, to find a solution evaluated to True or a certificate that all solutions are evaluated to False. Problems in the first and second class are called respectively optimization and decision problems. If, for each feasible solution, the time required for the evaluation of its cost is finite, any of these problems allows at least for one finite-time algorithm, the one which just tries sequentially all the feasible solutions.

The class \( P \) is the class of problems such that the worst-case complexity is bounded by a given polynomial in \( N \). Up to exotic cases, you can think to the class of problems such that, for each problem, a given algorithm has been explicitly provided, and has been proven to have a worst-case complexity bounded by the given polynomial. This class has a pleasant property: suppose that two problems \( A \) and \( B \) are given such that:
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- problem B is known to be of polynomial complexity, with a certain polynomial \( P_B(N) \);
- an encoding map exists, which encodes a size-\( N \) instance of problem \( A \) into an instance of problem \( B \) of size at most \( Q(N) \), in a time \( o(P_B(Q(N))) \);
- a decoding map exists, which decodes a whatever optimal solution of an instance of problem \( B \), obtained from the encoding above, into an optimal solution of the original problem \( A \), in a time \( o(P_B(Q(N))) \);

Then the algorithm consisting of the encoding, followed by problem-\( B \) algorithm, and finally the decoding, results into a polynomial-time algorithm for problem \( A \), with polynomial \( P_A(N) = P_B(Q(N)) \). This idea leads to the concept of polynomial reduction of problems.

In the theory of Turing machines, the traditional paradigm is sided by the definition of non-deterministic Turing machines [1][2]. The reader used to programming can think to these machines as ordinary computer programs, with the formal rule that the time of a forked process is the maximum of the two forked times (as if the forked processes were running on different processors), instead of the sum of the two times (as if the forked processes were running sequentially on a unique processor).

The class \( \text{NP} \) of problems is the class such that the complexity, calculated within the rule above, is polynomial (indeed, \( \text{NP} \) stands for non-deterministic-polynomial-time). The idea of polynomial reduction holds also for this class of problems, as it can be applied separately to each branch of the process.

Then, the crucial theorem by Cook [3] states that any \( \text{NP} \) problem can be polynomially reduced to the Boolean Satisfiability Problem, i.e. to the decision problem given by \( n \) boolean variables \( u_i \) (i.e. literals) on the alphabet \{True, False\}, and a statement in disjunctive normal form with \( m \) clauses (i.e. a string of the kind \((u_1 \lor u_2 \lor \overline{u_7}) \land (u_2 \lor u_3 \lor u_9 \lor u_{13}) \land \ldots \) with \( m \) parentheses), a solution being a boolean assignment to the \( n \) variables which satisfies the statement. Despite many efforts, it has not been determined whether the class \( \text{NP} \) coincides with the class \( \text{P} \) (\( \text{P} = \text{NP} \)), or is definitely wider (\( \text{P} \subset\subset \text{NP} \)), although the present intuition is maybe in the direction of stating that \( \text{P} \subset\subset \text{NP} \). Thus, understanding the roots of complexity commons to problems in the \( \text{NP} \) class, in order to compare them to the characteristics of the \( \text{P} \) class, has become an important direction of research of last decades.

In particular, since every \( \text{NP} \) problem is polynomially equivalent to Boolean Satisfiability, it is also polynomially equivalent to any other problem which encodes Boolean Satisfiability through a (chain of) polynomial reductions. This induces the concept of \( \text{NP-completeness} \) for problems with this characteristic: via Cook theorem and the corresponding chain of reductions, a hypothetic polynomial-time algorithm for any \( \text{NP-complete} \) problem would automatically provide a polynomial-time algorithm for all problems in the \( \text{NP} \) class, thus proving that \( \text{P} = \text{NP} \). Since the first papers by Karp [4][5], the family of \( \text{NP-complete} \) problems has grown, up to lead to the famous large collection in Garey and Johnson book [1], and, as can be easily argued, has still grown in the 25 following years [6].

1.1. Motivations and a digression

In this paper we describe a new \( \text{NP-complete} \) problem, the One-in-Two Matching, and give a polynomial reduction from Boolean Satisfiability. A detailed description, section by section, of the contents of the paper, is postponed to the conclusions, in section 7 while here we mostly concentrate on the motivations.

What is the reason, in present days, for such a work? Of course, the fact is interesting by itself, in the idea of making the list of \( \text{NP-complete} \) problems still wider (a larger list of \( \text{NP-complete} \) problems gives a larger number of possible starting points for a reduction proof, and thus makes easier the task of determining whether a new problem is \( \text{NP-complete} \)). As we will see, as a side result we show that One-in-Two Matching Problem induces a chain of reductions from Boolean Satisfiability to the important 3-Dimensional Matching Problem (3DM in Garey-Johnson [1]) which avoids the complicated and size-demanding original reduction by Karp [4].

Another motivation, which was indeed the original one, is that One-in-Two Matching and Assignment give a hint on the structural reason why Hamiltonian Circuit (HC) and
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Travelling Salesman (TSP) are NP-complete, although their analogue Matching and Assignment are polynomial, and intrinsically simple for what concerns the energetic landscape of configurations [1] [3] [4]. Indeed, configurations of HC and TSP are permutations \( \pi \) composed of a single cycle. It is trivial to impose that \( \pi \) has no fixed points (just taking infinite weights on the diagonal), while the remaining restriction to have no cycles of length \( 2 \leq \ell < n \) must be at the root of the complexity discrepancy among the two problems.

In the definition of One-in-Two problems, put at infinity the non-diagonal block elements, \( w_{2i-1,2i} = w_{2i,2i-1} = +\infty \), make the diagonal weight very favourite, \( w_{ii} \rightarrow \Delta \) with \( \Delta \rightarrow +\infty \), and then transpose all the row pairs \( (2i - 1, 2i) \). Then, the One-in-Two constraints is equivalent to forbid all the length-2 cycles among \( 2i - 1 \) and \( 2i \) (which are a small subset of all the cycles forbidden in TSP). On the other side, the matching \( \pi(2i) = 2i - 1, \pi(2i - 1) = 2i \) constitutes the obvious ground state of pure Assignment.

So, although random TSP instances have good heuristics and efficient approximants, based on the connection with Assignment [10] [11], this new ensemble of random TSP instances would be hard in the average case, as, instead of just condensing a few \( (\mathcal{O}(\ln n)) \) relevant long cycles, it must first choice how to disentangle \( \mathcal{O}(n) \) robust short cycles. Roughly speaking, as a cycle of length \( \ell \) can be broken in \( \ell \) points, and the set of cycle lengths is a partition of \( n \), the average complexity of this procedure scales with \( \exp(n/\ell \ln 7) \), and has a finite maximum for \( \ell = \mathcal{O}(1) \). Similar arguments are depicted at the end of section 1.3 [12].

Indeed, the existence of a NP-completeness proof for One-in-Two problems suggests that this narrow subset of the set of extra constraints of HC and TSP already contains the core of extra complexity of these problems. Also remark that, although the chain of reductions from SAT to Hamiltonian Circuit (3-SAT \( \rightarrow \) Vertex-Covering \( \rightarrow \) HC) is beautiful and elegant [1], our direct reduction to One-in-Two Matching is much cheaper.

A posteriori, maybe the main motivation for this work lies in how similar the problem is to situations arising in real-world experiments, to be compared to graph-, number- or set-theoretical problems, which require a level of mathematical abstraction. This makes One-in-Two Matching particularly suitable for a Quantum Adiabatic Algorithm (QAA) implementation [13] [14] [15] [16] [17].

Quantum Adiabatic techniques are an alternative to more traditional Quantum Computing ideas. The former would solve a given NP decision problem, measuring observables on a quantum system which directly encodes the desired objective function (more precisely, a final-time Hamiltonian would reproduce the objective function, while an initial-time Hamiltonian is such that the ground state can be prepared, and the intermediate evolution is sufficiently slow that, by the quantum adiabatic theorem, with finite probability the final state encodes a solution of the problem).

The latter, instead, would implement a quantum algorithm in a sequence of logical gates, in a kind of “quantum logic circuit”, employing quantum bits (qubits, i.e. two-state quantum systems) interacting only locally, in the single gate. On one side, the existence of Quantum Error Correction techniques is what makes the mainstream Quantum Computing idea promising, as it would protect us from the unavoidable decoherence of single qubits (anyhow, cfr. [18] for Error Correction on QAA). The drawback is the requirement of a large number of identical quantum gates, and quantum analog of electric wires, to be arranged in a complex circuit, all within a support that does not add too much decoherence: while modern “classic” electronics allows for extremely complicated circuit patterns, we lack for a quantum device with a similar flexibility.

Although the possibility that a QAA paradigm, as well as any other classical or quantum gedankenexperiment, could solve NP complete problems in polynomial time, is quite remote [17], it has shown promising features on less ambitious tasks (cfr. mainly [10]). So, the Quantum Adiabatic idea remains a promising field, provided that the mapping to the physical device is done starting from an “experimentally simple” NP decision problem. We can outline some points which could describe this vague idea:

**modularity:** the easiest realizations are the ones in which it is required to design a very small number of elementary objects, arranged in a regular pattern. This *excludes* problems encoded on general graphs, unless we allow to deal with the (generally much larger) array which encodes the graph structure, i.e. for example the adjacency matrix of the graph, or the incidence matrix
between vertices and edges. (So, the adjacency structure of a graph, required in a graph-theoretical NP-complete problem, would be not cheaper than our array structure).

**low dimensionality:** two-dimensional arrays of electronical devices are experimentally much more accessible than three-dimensional ones, and infinitely more accessible than higher-dimensional ones. Conversely, one-dimensional arrays, with elements having a short-range interaction, are probably insufficient, as the corresponding ground state is at all times a tree state \[20\]. Our array would be two-dimensional. Cfr. \[16\] for a related discussion.

**reusability:** the “hard and expensive” experimental work for producing the device should allow to build a machine which could encode any instance of the chosen NP-complete problem, up to a given size. Then, there should exist a relatively “cheap and fast” machine, allowing to set up the specific instance, such that a given device can be used many times for different instances, and also for the many variants of the instance coming from the small changes of parameters (for example, variations in time) in the real-life problem one is interested into. In our case, the set of all constraints should be implemented on the device, and the set of weights should be encoded in tunable local “external fields”, of some physical nature, on the entries of the grid.

**small encoding:** the theoretical idea of polynomial reduction could be uneffective in real-life applications for large polynomials (large coefficients and/or large order). One should search for a NP-complete problem whose reduction from the original SAT universal problem is as small as possible, and it is strongly advised to be linear in the encoding size of the SAT instance, i.e. approximatively the number of logic operators (AND and OR) in the logic statement. We will see how we have a linear encoding from generic SAT, and, in section \[4\] how the encoding could be still improved in real-life applications.

**no fine-tuning:** the Hamiltonian should not reproduce the cost function through an exact cancellation of dominant terms, or a fine-tuning of contributions coming from different physical terms (say, like reproducing a 2-SAT clause by mean of a two-body interaction among two spins, and the contribution of an external magnetic field with a specific fine-tuned value which makes \((T,T), (T,F)\) and \((F,T)\) of degenerate cost), because this is unlikely to be faithfully reproduced by the experimental device. The unavoidable degeneracies of the cost function in a complex decision problem must derive uniquely from symmetry properties of the device components. This point will be shortly discussed in section \[6\].

**locality:** it is severely hard to reproduce non-local constraints, like the one of Hamiltonian Path or of Number Partitioning Problems, in a physical system, just because most of the interactions under experimental control have a short range. Our new One-in-Two constraint is highly local, as it involves occupation numbers of two \((i,i)−(i+1,i+1)\) neighbours on the grid; the traditional Matching constraint can be reproduced by many mechanisms, of which the most simple is just creating at the beginning one “row”-particle for each row (and resp. for columns), and implement a dynamic with conserved number of particles (cfr. section \[6\]).

So, as it will be clear from the paper, contrarily to many other NP-complete problems, our One-in-Two Matching seems to meet a large number of these requirements, and it is maybe not hazardous to auspicate that it could be a viable candidate for a large-size experimental implementation of the Quantum Adiabatic Algorithm, if any will ever be pursued.‡

### 1.2. Definition of 2-Dimensional Matching and Linear Assignment Problems

Given an unoriented bipartite graph \(G(V_1, V_2; E)\), with \(V_1\) and \(V_2\) being the sets of the two kinds of vertices, both of size \(n\), and \(E \subseteq V_1 \times V_2\) the set of edges, the Matching Problem \[22\] asks for a spanning subgraph \(G(V_1, V_2; M)\), \(M \subseteq E\) with cardinality \(n\), such that each vertex has degree exactly 1, or for a certificate that such a subgraph does not exist. A set \(M\) satisfying this requirement is

‡ It is worth remarking that QAA procedure on small instances has already been tried in \[19\].
called a perfect matching over $G$. Here and in the following, we describe a problem in the scheme

| Problem name: | feasible solution; condition to satisfy. |
|---------------|------------------------------------------|

then the scheme corresponding to Matching is

### Matching Problem:

$G(V_1, V_2; E)$

with $|V_1| = |V_2| = n$.

$M \subseteq E$; $|M| = n$, $\forall v \in V_1 \cup V_2 \ deg_M(v) = 1$.

This problem is a specific case of the more general Linear Assignment Problem, in which integer weights $w(e)$ are associated to the edges, a threshold value $k$ is given, and the search is restricted to perfect matchings $M$ such that the sum of weights on the edges of $M$ is smaller than $k$.

### Linear Assignment Problem:

$G(V_1, V_2; E)$

with $|V_1| = |V_2| = n$.

$w : E \rightarrow \mathbb{Z}$;

$k \in \mathbb{Z}$.

$M \subseteq E$; $|M| = n$, $\forall v \in V_1 \cup V_2 \ deg_M(v) = 1$,

$\sum_{e \in M} w(e) \leq k$.

More precisely, the weights could be also infinite, i.e. $w : E \rightarrow \mathbb{Z} \cup \{+\infty\}$, with the natural formal rules $n + \infty = +\infty + n = +\infty + \infty = +\infty$ and $+\infty > k$. Then, one can assume without loss of generality that $G$ is the complete balanced bipartite graph.

In traditional notations, Matching Problem is resumed in the case $w(e) = 1$ for edges in $E(G)$ and $w(e) = 0$ otherwise, $\leq$ being replaced by $\geq$, and $k = n$.

A convenient representation of these problems is via the $n \times n$ matrix $W = \{w_{ij}\}$ of the weights. A feasible matching $M$ is then described by a matrix $X = \{x_{ij}\}$, with $x_{ij} = 0, 1$ and exactly one element equal to 1 per row and per column, such that $x_{ij} = 1$ if edge $e = (i, j)$ is in $M$. The cost function for $M$ is restated into

$$C_W(X) = \sum_{i,j} w_{ij} x_{ij} = \text{tr} W X^T.$$  

(1)

Another convenient representation of feasible matchings is via permutations in the symmetric group over $n$ elements, $\pi \in \mathcal{S}_n$, where $\pi(i) = j$ if edge $(i, j)$ is in $M$. In this notation the cost function reads

$$C_W(\pi) = \sum_i w_{i\pi(i)}.$$  

(2)

An example of problem instance and solution could be (on the left, items $w_{i\pi(i)}$ are written in bold)

$$W = \begin{pmatrix} 3 & 7 & 2 & 4 & 1 & 1 \\ 1 & 6 & 1 & 7 & 8 & 2 \\ 6 & 3 & 2 & 5 & 6 & 3 \\ 4 & 2 & 8 & 6 & 2 & 5 \\ 5 & 5 & 1 & 6 & 3 & 4 \\ 4 & 9 & 8 & 1 & 4 & 3 \end{pmatrix} \text{ with } k = 15; \quad \{\pi(i)\}_{i=1,...,6} = \{3, 1, 2, 5, 6, 4\}; \quad C(\pi) = 13.$$

Given a whatever Assignment instance $W$, many algorithms allow to find in polynomial time the optimal assignment $\pi^*$, and its cost $C^* = C(\pi^*)$, for example the Hungarian Algorithm [21][22].

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$\S$ Equivalently, one can restrict to consider the complete balanced bipartite graph with $2n$ vertices, $K_{n,n}$, and set $w(e) = +\infty$ for edges in $E(K_{n,n}) \smallsetminus E(G)$.
1.3. Definition of One-in-Two Matching and One-in-Two Assignment Problems

Now we can define the One-in-Two Matching and Assignment Problems as the variant of Matching (resp. Assignment) Problem in which, assumed that the dimension 2n of the matrix is even, the set of allowed partitions \( \pi \) is restricted to include only the ones such that, for each \( i = 1, \ldots, n \), either \( \pi(2i-1) = 2i-1 \) or \( \pi(2i) = 2i \).

Thus the description of One-in-Two Matching and Assignment could be resumed in the tables

One-in-Two Matching Problem:

\[
G(V_1, V_2; E) \\
\text{with } |V_1| = |V_2| = n; \\
\text{partition of } \{V_1; V_2\} \text{ into quadruplets } q = (v, v'; u, u').
\]

\[
M \subseteq E; \\
|M| = n, \ \forall v \in V_1 \cup V_2 \ \deg_M(v) = 1, \\
\forall q \ \left((v, u) \in M \right) \lor \left((v', u') \in M\right).
\]

One-in-Two Assignment Problem:

\[
G(V_1, V_2; E) \\
\text{with } |V_1| = |V_2| = n; \\
\text{partition of } \{V_1; V_2\} \text{ into quadruplets } q = (v, v'; u, u'); \\
w : E \to \mathbb{Z}; \\
k \in \mathbb{Z}.
\]

\[
M \subseteq E; \\
|M| = n, \ \forall v \in V_1 \cup V_2 \ \deg_M(v) = 1, \\
\forall q \ \left((v, u) \in M \right) \lor \left((v', u') \in M\right), \\
\sum_{e \in M} w(e) \leq k.
\]

Remark that, when proven that One-in-Two Assignment is NP-complete, we will also have a proof that the variant with \( ((v, u) \in M) \lor ((v', u') \in M) \) instead of \( ((v, u) \in M) \lor ((v', u') \in M) \) is NP-complete. Indeed, given an instance of the OR problem, shifting the diagonal weights to \( w_{ii} \to w_{ii} + \Delta \), and \( k \to k + n\Delta \), we have that \( C_W(M) - k = n'\Delta \), with \( n' \) the number of blocks with two matched elements, and in the limit \( \Delta \to -\infty \) we recover the analogous XOR problem. On the contrary, in the limit \( \Delta \to -\infty \) the problem becomes trivial.

The costs of the diagonal elements do not play any role, and can be fixed to zero. Indeed, if for some \( i \leq n \) we have \( w_{2i,2i} = w_{2i-1,2i-1} = +\infty \), no finite-cost assignment exists, while if only one of the two is infinite (say, \( w_{2i-1,2i-1} \)), we are forced to fix the permutation on the other one \( (w_{2i,2i}) \), and thus the elements with \( i \) or \( j \) equal to \( 2i \) never play a role: we would have had an identical cost function if \( w_{2i-1,2i-1} = 0 \), and \( w_{2i,j} = w_{j,2i} = +\infty \) for each \( j \neq 2i \):

\[
W = \begin{pmatrix}
+\infty & w_{12} & w_{13} & \cdots \\
w_{21} & +\infty & w_{23} & \cdots \\
w_{31} & w_{32} & +\infty & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
0 & +\infty & w_{13} & \cdots \\
+\infty & w_{22} & +\infty & \cdots \\
w_{31} & +\infty & w_{33} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Thus, without loss of generality, we can assume that \( w_{ii} \) is finite for each \( i \). Then, from the invariance of Linear Assignment, one easily convince himself that an equivalent instance (i.e. an instance with identical cost function, up to an overall constant) can be produced, with \( w_{ii} = 0 \) for each \( i \leq 2n \), and that the values \( w_{2i,2i-1} \) and \( w_{2i-1,2i} \) never appear in allowed matchings. For this reason, we will assume in the following that \( w_{ii} = 0 \), and denote the four elements in the \( n \) diagonal blocks of size 2 with special symbols * and \( \cdot \), instead that with a weight value. For example, an instance with \( 2n = 6 \) could be

\[
W = \begin{pmatrix}
* & \cdot & 2 & 4 & 1 & 1 \\
\cdot & * & 1 & 7 & 8 & 2 \\
3 & 3 & * & 6 & 3 \\
4 & 2 & \cdot & * & 2 & 5 \\
5 & 5 & 1 & 6 & * & \cdot \\
4 & 9 & 8 & 1 & \cdot & *
\end{pmatrix}
\]

with \( k = 10 \);

The choice of representation with \(*)s\) is done for mnemonic reasons: at sight, one knows that a valid matching should use exactly one \(*) per block. For example, a valid matching with weight 9 could be
the following (on the right side, elements \(i\) such that \(\pi(i) = i\) are underlined in order to highlight the one-in-two constraint satisfaction)

\[
\begin{array}{cccc}
* & 2 & 4 & 1 \\
* & 1 & 7 & 8 \\
3 & 3 & * & 6 \\
4 & 2 & * & 2 \\
5 & 5 & 1 & 6 \\
4 & 9 & 8 & 1 \\
\end{array}
\]

\(\{\pi(i)\}_{i=1,\ldots,6} = \{3, 2; 6, 4; 5, 1\}; \quad C(\pi) = 9.\)

Clearly, a one-in-two matching can be described by a choice of the elements kept fixed by the permutation (i.e. the \(*\) chosen in each block), times an allowed choice of assignment in the \(n\)-dimensional minor matrix resulting from the removal of the fixed rows and columns. Thus, allowed matchings are in bijection with pairs \((\vec{\sigma}, \pi)\), where \(\vec{\sigma} \in \{0, 1\}^n\) and \(\pi \in S_n\), with all \(\pi(i) \neq i\). For example, the previous configuration could be described as \((\vec{\sigma}, \{\pi(i)\}_{i=1,\ldots,3}) = ((0, 0, 1), \{2, 3, 1\})\).

This fact suggests a naïve interpretation for the potential hardness of this variant of Assignment: for any choice of fixed elements (the \(*\)'s), the problem of finding the optimal assignment is polynomial (just put \(+\infty\) on the remaining \(*\)'s, and use Hungarian Algorithm on the resulting Linear Assignment instance), nonetheless one should perform a search among these \(2^n\) possible choices, which, in absence of a sufficiently strong correlation or a skill mathematical structure, could make the search exponential in size. We will come back on this point in section 5.

2. Proof of linear reduction from Boolean Satisfiability problems

Here we prove that also One-in-Two Matching, less general w.r.t. the analogue One-in-Two Assignment, allows for linear reduction from arbitrary instances of SAT, 3-SAT or NAE-3-SAT Problems.

A SAT (or NAE-SAT) instance with \(n\) literals and \(m\) clauses can be encoded into a bipartite graph \(G(V \ell, V_c; E)\), with \(V \ell\) being the set of literals \(\{u_i\}_{i=1,\ldots,n}\) and \(V_c\) the set of clauses \(\{C_a\}_{a=1,\ldots,m}\), and a map \(s : E \rightarrow \{\pm 1\}\) which states whether the literal enters negated or not, i.e. if \(u_i \in C_a\) then \(s(i, a) = +1\), while if \(u_i \in C_a\) then \(s(i, a) = -1\).

A satisfiability instance can be presented in the form of a factor graph, i.e. the bipartite graph above, where vertices in \(V \ell\) are denoted by small circles, and vertices in \(V_c\) by small squares, and an edge is drawn in solid line if \(s(e) = +1\), and dashed if \(s(e) = -1\). The instance is satisfied by a given boolean assignment if a local constraint is satisfied on each clause vertex. For the SAT problem, given a clause with \(k\) incident variables, the number of solid-edge true neighbours plus dashed-edge false neighbours must be at least 1. For NAE-SAT it must be at least 1, and at most \(k - 1\).

In order to perform our reduction, it is easier to first perform a decoration on the graph: for each variable, introduce an auxiliary variable per incident edge (in a sense, the literal “as seen from the clause”), then substitute the original variable node by a “consistency check” clause (drawn as a small triangle), which ensures that the boolean values on the copies of the variable coincide. A small example of SAT factor graph, and the corresponding decorated graph, could be the following:

In our reduction, we have two \(2 \times 2\) blocks \((*, *)\) for each edge \((i, a) \in E(G)\), and thus the entries of the matrix \(W\) are labeled by an index \((i, a)_{1,2}^{\pm}\), where \((1, 2)\) stands for the first and second block, and \(\pm\) stands for the first and second index inside the block. Suppose to order arbitrarily the edges
for helping visualization, we will assume that the entries of $W$ are ordered as

\[
((i_1, a_1), (i_1, a_1), \ldots, (i_k, a_k), (i_1, a_1), (i_1, a_1), \ldots, (i_k, a_k), (i_k, a_k)),
\]

that is, first all the 1-blocks, then, in the same order, all the 2-blocks.

The choice of $*$s in the matching corresponds to the sequence of boolean assignments for the literals, “as they are seen from the clause”, i.e. for the literals in the decorated instance. So we need a “truth-setting” structure, which ensures that all these values coincide (in other words, implements the “consistency check” clause), and a “satisfaction-testing” structure, which checks that each boolean clause in the original formula is satisfied. The truth-setting structure is encoded in the set of 1s in the entry pairs with $i = j$, while the satisfaction-testing structure is encoded in the set of 1s in the entry pairs with $a = b$. More precisely, entries $w = 1$ can appear in the off-block matrix elements at index pairs $((i, a)_{\alpha}, (j, b)_{\beta})$ only in one of the two cases:

- $i = j$, $\alpha = 2$ and $\beta = 1$ (truth-setting structure)
- $a = b$, $\alpha = 1$ and $\beta = 2$ (satisfaction-testing structure)

We start describing the truth-setting structures. For each variable $i$, call $A(i)$ the set of adjacent clauses, and choose an arbitrary cyclic ordering on this set. For $i \in V_t$ and $a, b \in A(i)$ we state

\[
w_{(i, a)_{\alpha}, (i, b)_{\beta}} = \begin{cases} 1 & a = b, \quad \sigma = \tau = –, \quad \alpha = 2, \quad \beta = 1; \\
0 & a = b – 1, \quad \sigma = \tau = +, \quad \alpha = 2, \quad \beta = 1; \\
0 & \text{otherwise.}
\end{cases}
\]

that is, the minor of $W$ restricted to indices with fixed $i$ looks like

\[
W_{\text{fixed } i} = \begin{pmatrix} I_\tau & 0 \\ W' & I_\sigma \end{pmatrix};
\]

\[
I_\tau = \begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & * \\ \end{pmatrix}, \quad W' = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & 0 \\ \end{pmatrix}.
\]

Remark that in all rows with sub-index 2 and columns with sub-index 1 (i.e. all rows ands columns in $W'$) we have exactly one allowed entry beyond the $*$ element on the diagonal. In order to see how the truth-setting procedure works, consider what happens if we choose the top-left $*$ in the first block (i.e. we choose $\sigma(i, a) = 1$ in the string of $\sigma(i, a)$ for the “variables seen from the clauses”). At the beginning the matrix is (a circle means “element chosen in the matching”, a bar means “element not chosen in the matching”)

\[
W = \begin{pmatrix} \circ & \ast & \ast & \ast & \cdots \\ \ast & \circ & \ast & \ast & \cdots \\ 1 & 1 & \circ & \ast & \cdots \\ \ast & \ast & \ast & \ast & \cdots \\ \end{pmatrix}
\]
One-in-Two-Matching Problem is NP-complete

while, after six logical implication we have

\[ W = \begin{pmatrix}
    0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

Remarking that the choice of 1s in the matrix has a cyclic ordering w.r.t. clause indices, with a simple induction one deduces that the boolean assignments are \( \sigma_{i,a_k} = 1 \) for all clauses \( a_k \) incident with variable \( i \). A similar statement can be done in the case \( \sigma_{i,a_1} = 0 \) (just exchange circles with bars in the diagrams above). So, we have determined that only consistent boolean assignment of literals are allowed.

Now we can build the structures for the satisfaction testing. We are interested in the restriction of \( W \) to a given fixed index \( a \), which is of the form

\[ W|_{\text{fixed } a} = \begin{pmatrix}
    I_l & W''_l \\
    0 & I_s
\end{pmatrix}. \tag{8} \]

We describe the matrix \( W'' \) for a clause of length \( k \) with all unnegated literals, \( C_a = u_{i_1} \lor \ldots \lor u_{i_k} \). All the other cases can be trivially inferred \( \| \). Choose a whatever literal index (say, \( i_1 \)) among the neighbours of the clause \( a \), and set

\[ w_{(i,a_1),(j,a_1)} = \begin{cases} 
    1 & \text{if } i = j \text{ and } \sigma = \tau = -1; \\
    1 & \text{if } i = j \neq i_1 \text{ and } \sigma = \tau = +1; \\
    1 & \text{if } i = i_1, j \neq i_1, \sigma = +1 \text{ and } \tau = -1; \\
    1 & \text{if } i \neq i_1, j = i_1, \sigma = -1 \text{ and } \tau = +1; \\
    0 & \text{otherwise}
\end{cases} \tag{9} \]

that is, in an extensive representation of the matrix,

\[ W'' = \begin{pmatrix}
    0 & 0 & 0 & 1 & 0 & 1 & \ldots \\
    0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
    0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
    0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
    0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
    0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \tag{10} \]

which indeed makes the game, as one can easily check. Indeed, if all literals are negated, we are left with matrix minor

\[ W''_{(F,F,\ldots,F)} = \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix}, \]

which clearly does not allow for any valid matching, while, if the first literal is true we have

\[ W''_{(T,\ldots)} = \begin{pmatrix}
    1 \\
    0 \\
    0
\end{pmatrix}, \]

\( \| \) Define the transposition matrix \( (T^{(k)})_{ij} = 1 \) if \( i = j \notin \{2k, 2k-1\} \), if \( i = 2k \text{ and } j = 2k-1 \text{ or if } i = 2k-1 \text{ and } j = 2k \), and zero otherwise. I.e., matrix \( T^{(k)} \), acting on the left, transpose rows \( 2k-1 \text{ and } 2k \), while acting on the right transpose the corresponding columns. Then, if matrix \( W'' \) encodes a clause involving \( (u_1,\ldots,u_k,\ldots,u_\ell) \), the matrix \( T^{(k)} W'' W^{(k)} \) encodes the same clause on literals \( (u_1,\ldots,\bar{u}_k,\ldots,u_\ell) \).
which allows at least for the diagonal matching, \( \pi(j) = j \) for all \( j = 1, \ldots, k \), and if the first literal is false, but one of the others (say, the \( h \)-th) is true, we have

\[
W''_{(F, \ldots, T, \ldots)} = \begin{pmatrix}
0 & \cdot & 1 & \cdot \\
\cdot & I_{h-2} & 0 & 0 \\
1 & 0 & 1 & 0 \\
\cdot & 0 & 0 & I_{k-h}
\end{pmatrix},
\]

which allows at least for the matching \( \pi(1) = h, \pi(h) = 1 \) and \( \pi(j) = j \) otherwise. Similarly, for a NAE-k-SAT clause \( C_a = (u_{i_1} \lor \ldots \lor u_{i_k}) \land (\overline{u_{i_1}} \lor \ldots \lor \overline{u_{i_k}}) \) we can choose

\[
w'_{(i,a), (j,a)} = \begin{cases}
1 & \text{if } i = j \neq i_k \text{ and } \sigma = \tau = -1; \\
& \text{if } i = j \neq i_1 \text{ and } \sigma = \tau = +1; \\
& \text{if } i = i_1, j \neq i_1, \sigma = +1 \text{ and } \tau = -1; \\
& \text{if } i \neq i_1, j = i_1, \sigma = -1 \text{ and } \tau = +1; \\
& \text{if } i = i_k, j \neq i_k, \sigma = -1 \text{ and } \tau = +1; \\
& \text{if } i \neq i_k, j = i_k, \sigma = +1 \text{ and } \tau = -1; \\
0 & \text{otherwise}
\end{cases}
\]

\[
W'' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & \ldots & 0 & 0
\end{pmatrix}
\]

This completes the reduction proof for SAT and NAE-SAT (and then, in particular, for 3-SAT and NAE-3-SAT). Indeed, the proposed encodings of a SAT and NAE-SAT clause are a special case of the most general \( k \)-literal clause, in which the number of true literals must be in the range \( \{h_{\min}, \ldots, h_{\max}\} \), (also having as a special case the 1-in-3-SAT problem, which is NP-complete \( \mathbb{II} \)), for which a general encoding is shown in section \( \mathbb{III} \).

It is common in reduction proofs that multiple appearance of a literal in a clause requires some care (truth-setting and satisfaction-testing structures could damagefully interfere), and one should make some standard comment on the fact that this case can be excluded with small effort from any SAT instance. This does not happen in our case. One can understand this from the fact that the 1s in the two structures appear in different rows and columns, and logical implications which allow to test the performance of the structures involve only these rows/columns.

Also remark that the reduction is linear not only in matrix size w.r.t. the original factor-graph size (the dimension \( n \) of the Matching matrix is four times the number of edges in the factor graph), but also in the number \( N \) of non-zero entries in the matrix, which is indeed very sparse. Each literal of coordination \( k \) requires a truth-setting structure with \( 2k \) entries, while each clause of length \( k \) requires a satisfaction-testing structure with \( 4k - 3 \) entries (\( 6k - 8 \) for a NAE clause), thus we have that, for a factor graph \( G(V_e, V_c; E) \)

\[
N = 6|E(G)| - 3|V_e(G)| \quad \text{SAT problem;}
\]
\[
N = 8|E(G)| - 8|V_e(G)| \quad \text{NAE-SAT problem.}
\]

Finally, we also remark that the instances of One-in-Two Matching obtained as reduction from Satisfiability Problems (or, as we will see in section \( \mathbb{III} \) from problems with clauses having more generic truth tables) are of a particular kind: the 1s are contained only in the top-right and bottom-left \( 2n \times 2n \) quadrants (the matrices \( W' \) and \( W'' \)). We call Bipartite One-in-Two Matching this specialized problem.
3. Proof of linear reduction from One-in-Two Matching to 3 Dimensional Matching

We recall the definition of the 3-Dimensional Matching Problem [1]:

**3-Dimensional Matching Problem:**

\[
G(V_1, V_2, V_3; T) \quad \text{with} \quad T \subseteq V_1 \times V_2 \times V_3 \\
\text{and} \quad |V_1| = |V_2| = |V_3| = n. \quad M \subseteq T; \\
|M| = n, \quad \forall v \in V_1 \cup V_2 \cup V_3 \quad \deg_T(v) = 1.
\]

The object \(G\) could be called a “triptartite hypergraph”: indeed, instead of edges, it contains hyperedges with three endpoints, one for each set of vertices. Equivalently to what has been done in section 1.2, another representation turns out to be useful, in which the allowed elements for a matching are encoded in an array of zeroes and ones. Now the array is three-dimensional. Thus, define \(W = \{w_{ijk}\}_{i,j,k=1,...,n}\) such that \(w_{ijk} = 1\) if \((i, j, k) \in T\), with \(i \in V_1\), \(j \in V_2\) and \(k \in V_3\), and \(w_{ijk} = 0\) otherwise. A feasible matching \(M\) is then described by an array \(X = \{x_{ijk}\}\), with \(x_{ijk} = 0, 1\), such that \(x_{ijk} = 1\) if the triangle \(t = (i, j, k)\) is in \(M\), and then exactly one element of the array is equal to 1 per \(i, j\) or \(k\) fixed, i.e.

\[
\forall i = 1, \ldots, n \quad \sum_{j,k} x_{ijk} = 1; \\
\forall j = 1, \ldots, n \quad \sum_{i,k} x_{ijk} = 1; \\
\forall k = 1, \ldots, n \quad \sum_{i,j} x_{ijk} = 1.
\]

The cost function for \(M\) is then restated into

\[
C_W(X) = \sum_{i,j,k} w_{ijk} x_{ijk},
\]

and \(X\) is a valid matching if \(C_W(X) = n\). The numerical version is defined accordingly, where the weights \(w_{ijk}\) are integers, and we have a threshold value for the cost [14].

Now we describe the reduction from One-in-Two Matching to 3DM. An identical reduction goes from One-in-Two Assignment to Numerical 3DM. Call \(W = \{w_{ij}\}\) our One-in-Two Matching instance of dimension \(2n\) to be encoded, with

\[
w_{ij} = \begin{cases} 
* & i = j \leq 2n; \\
0, & i = 2h, j = 2h - 1, h \leq n \\
& \text{or } i = 2h - 1, j = 2h, h \leq n; \\
0 & \text{otherwise.}
\end{cases}
\]

and \(W^{(3)}\) our suggested output 3DM instance, also of dimension \(2n\). Our formal reduction is, calling \(A(k) = \{2(k - n) - 1, 2(k - n)\}\) for \(k = n + 1, \ldots, 2n,\)

\[
w_{ijk}^{(3)} = \begin{cases} 
1 & i = j = 2k - 1, \\
0 & \text{otherwise.}
\end{cases}
\]

or, more pictorially, call \(\vec{e}_i\) and \(\vec{w}_i\) the vectors

\[
\vec{e}_i = (0, \ldots, 0, 1, \ldots, 0, 0, \ldots, 0); \\
\vec{w}_{2i-1} = (w_{2i-1,1}, \ldots, w_{2i-1,2i-2}, 0, 0, w_{2i-1,2i-1} + 1, \ldots, w_{2i-1,2n}); \\
\vec{w}_{2i} = (w_{2i,1}, \ldots, w_{2i,2i-2}, 0, 0, w_{2i,2i+1} + 1, \ldots, w_{2i,2n}).
\]
then \( W^{(3)} \), written as a matrix on indices \((i, k)\), of vectors on index \(j\), looks like

\[
W^{(3)} = \begin{pmatrix}
\vec{e}_1 & 0 & \ldots & \vec{w}_1 & 0 & \ldots \\
\vec{e}_2 & 0 & \ldots & \vec{w}_2 & 0 & \ldots \\
0 & \vec{e}_3 & \ldots & 0 & \vec{w}_3 & \ldots \\
0 & \vec{e}_4 & \ldots & 0 & \vec{w}_4 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
\end{pmatrix}.
\]  

Indeed, remark that in the planes \((i, j)\) for \(k = 1, \ldots, n\) there are only two allowed entries, whose \((i, j)\) coordinates correspond to the ones of the \(\ast\)s in the \(k\)-th block of the original instance. Mimicking the One-in-Two constraint, for each block \(k\) we are forced to choose a value \(\sigma_k \in \{0, 1\}\), with (say) 0 and 1 selecting respectively the entry with even and odd indices. Then, in all \((i, j)\) layers with \(k = n + 1, \ldots, n\) there are only two non-empty \(i\)-rows, (which are empty in all the other layers with \(k = n + 1, \ldots, 2n\)).

Because of the choice of the vector \(\vec{\sigma}\), exactly one of them is now forbidden. So, the three-dimensional constraint of choosing one element for each index \(i, j\) and \(k\) is at this point reduced to a two-dimensional matching constraint, as there is a bijection between unmatched layers \(k\) and non-empty unmatched rows \(i\). I.e. the 3-dimensional \(2n \times 2n \times 2n\) array of equation (18) is now restricted to the \(n \times n \times n\) array

\[
W^{(3)}(\vec{\sigma}) = \begin{pmatrix}
w_{2-\sigma(1)}^{(\vec{\sigma})} & 0 & 0 & \ldots \\
0 & w_{4-\sigma(2)}^{(\vec{\sigma})} & 0 & \ldots \\
0 & 0 & w_{6-\sigma(3)}^{(\vec{\sigma})} & \ldots \\
\vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}; \quad (w_j^{\sigma(\vec{\sigma})})_j = (\vec{w}_i)_{2j-\sigma(j)}.
\]  

It is easily understood that the set of forbidden indices \(j\) after the choice of vector \(\vec{\sigma}\) and the disposition of the entries \(w_{ij}\) are in agreement with the picture of section 1.5, where a vector \(\vec{\sigma}\) determines a \(n\)-dimensional minor of the original \(2n\)-dimensional instance, with forbidden entries on the diagonal.

As a corollary of the construction we have that the reduction is linear not only for what concerns the size of the array (an \(n \times n\) One-in-Two instance goes into an \(n \times n \times n\) 3DM array), but also on the number of non-zero entries in the instance \(\bullet\), which is proportional to the length of the bit-encoding of the instance.

Putting together this result with the one of section \(\mathbb{2}\) we have in turn a linear reduction from SAT problems to 3DM,

\[
\mathcal{N}_{3DM} = 10|E(G)| - 3|V_0(G)| \quad \text{SAT problem;}
\]

\[
\mathcal{N}_{3DM} = 12|E(G)| - 8|V_0(G)| \quad \text{NAE-SAT problem;}
\]

which is much more economic of the cubic one first presented in Karp seminal ’72 paper \(\mathbb{4}\) (see also \(\mathbb{1}\)), and maybe technically simpler.

4. Encoding of more complex boolean patterns

We have seen how generic boolean disjunctive-form expressions containing SAT and NAE-SAT clauses can be encoded into a One-in-Two Matching instance of dimension four times the number of edges in the factor-graph representation of the boolean formula.

We can be more ambitious, and try to encode more complex boolean structures, in order to further reduce the output size of One-in-Two instances for real-life problems. For example, One-in-Three-SAT Problem is another NP-complete problem, where the truth tables over three literals only contain certain triplets of combinations (say, \((TFF)\), \((FTF)\) and \((FFT)\)) \(\mathbb{1}\). A One-in-Three clause can clearly be encoded with five 3-SAT clauses (or three 3-SAT and one NAE-3-SAT), but

\(\bullet\) More precisely, the cardinality of \(T\) for 3DM equals the one of \(E\) for One-in-Two Matching, plus the \(2n\) “deterministic” entries of vectors \(\vec{e}_i\), i.e. inside a factor 2 if we understand that 3DM instances having planes with only one valid entry can be trivially reduced in size.
this would require 15 (or 12) edges in the factor graph. If one could produce an encoding for One-in-Three clauses directly with a 6-dimensional matrix, one would save a factor 5 (or 4) in the final dimension of the matrix. This is the spirit in which, in section 2 we designed explicitly compact matrices for NAE-SAT clauses, instead of converting them into pairs of opposite SAT clauses.

Of course this project can not be fulfilled for arbitrary clauses of generic length: it is easily estimated that the number of inequivalent truth tables for length-\(k\) clauses diverges in \(k\) much more severely than the number of inequivalent \(2k \times 2k\) matrices of zeroes and ones. More precisely, the number of possible truth tables is \(2^{2^k}\). Then, we must consider gauge orbits (or classes, \(C\)) w.r.t. the symmetries of the clause, which are relabeling of literal indices (a group \(S_k\)), times independent negations of the literals (a group \((\mathbb{Z}_2)^k\)). If a truth table \(T\) has a residual symmetry group \(G(T)\), as the orbits are a quotient, their size \(g(C)\) equals the ratio between the number of elements of \(S_k \times (\mathbb{Z}_2)^k\), and of \(G(T)\)

\[
\#\{C\} = \sum_c 1 = \sum_c g(C) = \sum_c g(C) \frac{|G(C)|}{2^k k!} \geq \sum_c g(C) \frac{1}{2^k k!} = \frac{1}{k!} \#\{T\} = \frac{1}{k!} 2^{2^k - k}, \tag{20}
\]

this estimate being asymptotically very reliable \(^+\), while the number of different matrices, also in the generous bound which neglects gauge orbits and trivially equivalent patterns, is at most \(2^{4k^2}\). So, our idea of encoding in small structures more complex boolean patterns should be intended as

(i) determine the encoding for special families of generic-length clauses, as we already did for SAT, NAE-SAT and truth-setting structures.

(ii) determine the encoding for all the clause classes up to a given reasonable length. This is done in Appendix A, up to size 4. For size 5 our estimate tells us that the number of classes is higher than \(10^9\), and one should be very motivated to pursue this full classification. For size 6 there are more than \(4 \cdot 10^{14}\) classes, so that this classification is structurally unfeasible.

(iii) determine the encoding for specially-important clauses (as, for example, the 5-clause used for summing integers).

(iv) suggest heuristics for determining a compact encoding for new clause patterns, not discussed at point (iii), which could emerge in a given real-life application.

In this section we develop the idea of point (ii), for four classes of clauses of generic length \(k\):

- **2-false**: clauses in which only two literal assignments for \((u_1, \ldots, u_k)\) evaluate to False;
- **range-T**: clauses in which \((u_1, \ldots, u_k)\) evaluates to True if the number of True literals is in a range \([h_{\text{min}}, \ldots, h_{\text{max}}]\), and variants in which each literal could enter negated;
- **binary threshold**: clauses in which \((u_1, \ldots, u_k)\), interpreted as the binary number \(n(u) = 2^{k-1} u_1 + 2^{k-2} u_2 + \ldots\), evaluates to True if \(n(u) \leq f\), with \(f\) a threshold value;
- **binary distinct \((q\text{-colouring)}\)**: clauses in which \((u_1, \ldots, u_k, u_{h+1}, \ldots, u_{2h})\), interpreted as the set of two binary numbers \(n_1(u) = 2^{h-1} u_1 + 2^{h-2} u_2 + \ldots + u_h, n_2(u) = 2^{h-1} u_{h+1} + 2^{h-2} u_{h+2} + \ldots + u_{2h}\), evaluates to True if \(n_1(u) \neq n_2(u)\). Allows to encode \(q\)-colouring for \(q = 2^h\), and if \(q\) is not a power of 2, for \(2^{h-1} < q < 2^h\), jointly with a “binary threshold” clause per colour-variable.

\(^+\) Indeed, from a refined analysis,

\[
\#\{C\} - \frac{2^k}{2k!} = \frac{1}{2^k k!} \sum_{f \in S_k \times (\mathbb{Z}_2)^k} \left( \sum_{c : f \in G(C)} g(C) \right), \tag{21}
\]

and, calling \(R_f\) the term in parenthesis, we have \(R_f < 2^{\frac{2}{k}2^k}\) for simple \((ij)\) transpositions in \(S_k\), \(R_f < 2^{\frac{2}{k}2^k}\) for permutations \((ij)(k)\) in \(S_k\), \(R_f < 2^\frac{2}{k}2^k\) for \(\mathbb{Z}_2\) symmetry, and for permutations \((ijk)\) in \(S_k\), and so on. So, the first correction is of order \(2^{-\frac{2}{k}2^k} \sim \exp(-0.173 \cdot 2^k)\).
4.1. 2-False clauses

We expect that clauses in which almost all choices of literals are evaluated to True, or to False, are simpler to encode. Given a clause, call $T$ the set of True assignments of literals, and $|T|$ its size. The cases $|T| = 0, 1, 2^k$ are trivial, and the case $|T| = 2^k - 1$ is SAT. The case $|T| = 2$ corresponds either to the truth-setting structure (if the two True vectors are opposite on the hypercube $\{\text{True}, \text{False}\}^k$), or otherwise to a clause which is trivial in the sense of Rule 4 of Appendix A, and for which an encoding is easily deduced, combining a truth-setting structure on $k'(< k)$ blocks, with a bunch of $k - k'$ diagonal $(0 \ 0)$ blocks.

So, the first non-trivial class of clauses we will try to encode is the set of clauses for which $|T| = 2^k - 2$, of which NAE-SAT is the special case in which the two False vectors are opposite. We call them “2-false” clauses. Without loss of generality, we can assume that the two False vectors are $(FF\cdots F)$ and $(F\cdots FT\cdots T)$, where the number of True’s is $h = 1, \ldots, k$, with $h = k$ corresponding to NAE-SAT. Actually, $h = 1$ is trivial in the sense of Rule 3 of Appendix A. For $1 \leq h \leq k - 1$, an encoding is given by the matrix below

$$W = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix} \ \{h\}
$$

Indeed, for each possible choice of the literals, we have either an easy certificate of absence of any matching, or a matching which is valid at sight, according to the table (notation: $X \cdots X$ means a sequence of all $X$, while $\ldots$ means arbitrary literals in the interval)

\[
\begin{array}{c|c|c|c|c}
(F \cdots F \ | \ F \cdots F) & 1st row = \bar{0} \\
(F \cdots F \ | \ T \cdots T) & 1st column = \bar{0} \\
(T \ | \ \ldots \ ) & \pi(i) = i; \\
(F_{i-th} \cdots T \ | \ \ldots \ ) & \pi(1) = i, \ \pi(i) = 1, \ \pi(j) = j \ \text{o.w.} \\
(F \cdots F \ | \ F_{i-th}T \cdots F) & \pi(1) = i, \ \pi(i) = i + 1, \ \pi(i + 1) = 1, \ \pi(j) = j \ \text{o.w.} \\
(F \cdots F \ | \ F_{i-th} T F \cdots T) & \pi(1) = i + 1, \ \pi(i + 1) = i, \ \pi(i) = 1, \ \pi(j) = j \ \text{o.w.}
\end{array}
\]

Also remark that the number of 1s required in the encoding, $4k + 2h - 5 (< 6k - 7)$, is linear in $k$.

4.2. Range-T clauses

Another family of clauses we can fully classify is what we call “range-T” clauses, i.e. $k$-literal clause, in which the number of true literals must be in the range of values $\{h_{\min}, \ldots, h_{\max}\}$, for generic $0 \leq h_{\min} \leq h_{\max} \leq k$. Special cases are SAT ($h_{\min} = 1, h_{\max} = k$), NAE-SAT ($h_{\min} = 1, h_{\max} = k - 1$), and One-in-Three-SAT ($k = 3, h_{\min} = h_{\max} = 1$). Call $I$ a matrix of all $1 \ 0$ blocks on the diagonal, and 0 outside, $I_T$ (resp. $I_F$) a matrix of all $1 \ 0$ (resp. $0 \ 1$) blocks on the diagonal, and 0 outside; then call $J$ a (possibly rectangular) matrix of all 1s, and $J_{FT}$ (resp. $J_{TF}$) a (possibly rectangular) matrix of all $1 \ 0$ (resp. $0 \ 1$) blocks.
Then, for given $h_{1,2,3}$, $h_1 + h_2 + h_3 = k$, consider the matrix

$$W = \begin{pmatrix}
I_T & J_{FT} & J_{FT} \\
J_{FT} & I & J_{FT} \\
J_{FT} & J_{FT} & I_F
\end{pmatrix}
\begin{array}{c}
\{ h_1 \\
\{ h_2 \\
\{ h_3
\end{array}$$

For a given assignment $\vec{r}$ of literals, call $h_1', \ldots, h_3'$ the number of true assignments in the three blocks. Also call $h''_3 = h_1 - h_1'$. Because of permutation invariance inside the first block, we can equivalently assume they are the first ones of each block, and find a $k \times k$ reduced matrix

$$W_{\vec{r}} = \begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & 0 & J & 0 & 0 \\
0 & J & I & 0 & 0 \\
0 & 0 & 0 & I & J \\
0 & J & 0 & I & 0 \\
0 & 0 & 0 & 0 & I
\end{pmatrix}
\begin{array}{c}
\{ h_1' \\
\{ h_2' \\
\{ h_3' \\
\{ h_1'' \\
\{ h_2'' \\
\{ h_3''
\end{array}$$

Up to a permutation both on rows and columns

$$W_{\vec{r}} = \begin{pmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & J & 0 \\
0 & 0 & 0 & J & 0 \\
0 & J & 0 & 0 & 0 \\
0 & 0 & J & 0 & I \\
0 & 0 & 0 & 0 & I
\end{pmatrix}
\begin{array}{c}
\{ h_1' \\
\{ h_2' \\
\{ h_3' \\
\{ h_1'' \\
\{ h_2'' \\
\{ h_3''
\end{array}$$

Then, if $h''_1 > h''_2 + h''_3$ or $h''_3 > h''_2 + h''_1$ we have a certificate that no matching exists, because the rows corresponding resp. to entries $h''_1$ or $h''_3$ have too many empty columns. Conversely, if none of the two facts above happens, calling $h'' = h''_1 + h''_2 + h''_3$ and $h = \min(h''_1, h''_3)$, the permutation $\pi(h'' - \ell) = h'' + \ell + 1$, $\pi(h'' + \ell + 1) = h'' - \ell$ for $\ell = 0, \ldots, h - 1$ and $\pi(j) = j$ otherwise is a valid matching.

Thus the conditions for an assignment of literals to be evaluated to True are

$$h''_1 \leq h''_2 + h''_3; \quad h''_3 \leq h''_1 + h''_2; \quad (22)$$

which can be restated as

$$h'' \geq h_1; \quad h'' \leq k - h_3; \quad (23)$$

and, as $h''$ is the number of True literals, we recognize the “range-T” constraint, with $h_{\min} = h_1$ and $h_{\max} = k - h_3$. Remark that our choice of matrix encoding for SAT and NAE-SAT coincides with the general recipe described here.

4.3. Binary threshold clauses

A third class of clauses we encode is the “binary threshold” clause. For a set of $k$ literals $(u_1, \ldots, u_k)$, denote $u$ the binary number $[u_1 \cdots u_k] = 2^{k-1}u_1 + 2^{k-2}u_2 + \ldots$. The clause evaluates to True if $u \leq q$, with $q = [q_1 \cdots q_k]$ a threshold value with binary entries $q_i$. The clause is trivially reduced if $q < 2^{k-1}$, but, as our proof of the encoding is inductive, we consider the general $q < 2^k$ case.

A simple computer procedure which would determine if $u \leq q$ is the following *

```
for(i=1,...,k-1) 
  if(q[i]==1 AND u[i]==0) Return[Yes];
  if(q[i]==0 AND u[i]==1) Return[No]; 
if(q[k] > u[k]) Return[Yes];
else Return[No];
```

* “Return[x]” means “quit the whole procedure, giving x as result”.

---

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else Return[No];
```

* “Return[x]” means “quit the whole procedure, giving x as result”.
We will show that our matrix encoding reproduces this code. Consider the shortcuts

\[ 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad 1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad A(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad A(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] (24a)

\[ B(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad B(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad C(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \] (24b)

and denote \( A_i = A(q_i) \), \( B_i = B(q_i) \), \( C = C(q_k) \). A valid encoding is given by the matrix

\[
\begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 & B_1 \\
1 & A_2 & 0 & 0 & B_2 \\
0 & 1 & A_3 & \ddots & 0 & B_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & A_{k-1} & B_{k-1} \\
0 & \cdots & 0 & 0 & 1 & C
\end{pmatrix}
\] (25)

where the matrix minor associated to a binary number is the one in which the odd (resp. even) rows and columns are deleted if the binary digit is 1 (resp. 0).

As anticipated, we prove this inductively. First notice that \( C \) encodes the clause with \( k = 1 \): if \( q = 1 \) every literal assignment is valid, while if \( q = 0 \) only \( u = 0 \) is accepted. This encodes the third “if” of the program.

Then, in the induction assume that the minor of (25) in which the first two rows and columns are removed encodes the clause on the \( k - 1 \) digits \([q_2 \cdots q_k]\). Then, if \( q_1 = 1 \) we have

\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 1 & 1 \\
0 & 1 & A_2 & 0 & 0 & B_2 \\
1 & 1 & \ddots & \ddots & 0 & B_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & A_{k-1} & B_{k-1} \\
0 & \cdots & 0 & 0 & 1 & C
\end{pmatrix}
\] (26)

If \( u_1 = 1 \), the first row of the reduced matrix is \( (1,0,\ldots,0) \), so that one is forced to have \( \pi(1) = 1 \), and the existence of a valid matching is reduced to the constraint \([u_2 \cdots u_k] \leq [q_2 \cdots q_k]\). If \( u_1 = 0 \), the first row of the reduced matrix is \( (0,0,\ldots,1) \), and, as, regardless to \([u_2,\ldots,u_k]\), we have a whole subdiagonal filled with 1s, the permutation \( \pi(1) = k \) and \( \pi(i) = i - 1 \) for \( i = 2,\ldots,k \) is a valid matching. This encodes the first “if” of the program. Instead, if \( q_1 = 0 \) we have

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & 1 & A_2 & 0 & 0 & B_2 \\
1 & 1 & \ddots & \ddots & 0 & B_3 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & A_{k-1} & B_{k-1} \\
0 & \cdots & 0 & 0 & 1 & C
\end{pmatrix}
\] (27)

If \( u_1 = 1 \), the first row of the reduced matrix is \( (0,0,\ldots,0) \), so that no matching is possible. If \( u_1 = 0 \) the first row of the reduced matrix is \( (1,0,\ldots,0) \), and again one is forced to have \( \pi(1) = 1 \), and the existence of a valid matching is reduced to the constraint \([u_2 \cdots u_k] \leq [q_2 \cdots q_k]\). This encodes the second “if” of the program, and completes the proof. The number of 1s in the matrix is \( 5k - 4 + 2 \sum_i q_i (\leq 7k - 4) \), again only linear in \( k \).

4.4. Binary distinct clauses

Finally, we will encode into a matrix with \( k \) blocks the clause which compares two binary numbers with \( h = k/2 \) digits (“binary-distinct” clause). More precisely, given the literals \((u_1,\ldots,u_h,u_{h+1},\ldots,u_k)\), associate the two binary numbers

\[
n_1(u) = [u_1 \cdots u_h] = 2^{h-1}u_1 + 2^{h-2}u_2 + \cdots + u_h, \tag{28a}
\]

\[
n_2(u) = [u_{h+1} \cdots u_{2h}] = 2^{h-1}u_{h+1} + 2^{h-2}u_{h+2} + \cdots + u_{2h}, \tag{28b}
\]
and say that the clause is satisfied whenever \( n_1(u) \neq n_2(u) \). This clause is an “OR” of binary “XOR”s, i.e. \( u \) evaluates to
\[
C(u) = \bigvee_{i=1,\ldots,h} (u_i \oplus u_{h+i}),
\]
and is encoded by the matrix (it is intended that matrix entries are 0 where not specified)
\[
\begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
\cdots & \cdot & \cdot & \cdot \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
\cdots & \cdot & \cdot & \cdot \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Indeed, the only part of the matrix sensible to literal assignment is the diagonal of the top-right quadrant, which has 1s in correspondence of the unequal-digit indices, and thus is totally empty iff the two binary integers are equal. In this latter case we have an easy certificate, for example the first \( h \) rows have complexively only \( h-1 \) columns not totally empty. Conversely, if the \( i \)-th element of the diagonal is 1 (that is, if \( u_i \neq u_{h+i} \)), a valid permutation is
\[
\begin{align*}
\pi(i) &= h + i \\
\pi(2h) &= 1 \\
\pi(j) &= j + 1 & 1 \leq j < i, \quad h + i \leq j < 2h; \\
\pi(j) &= j & i < j < h + i.
\end{align*}
\]
This proves the encoding. Again remark the linear number of 1s: \( 7k - 8 \).

5. The roots of NP-hardness: a negative remark

An astonishing fact of Complexity Theory is the frequent similarity in the pictorial description of pairs of problems, which are indeed one proven to be polynomial, the other to be NP-complete. Examples are Chinese Postman vs. Travelling Salesman, or Arc Covering vs. Vertex Covering, or Min-Cut-Max-Flow vs. Max-Cut-Max-Flow. In other cases, a natural threshold value seems to exists, for example, 2-SAT, 2-Coloring, Minimum Spanning Tree and 2-Dimensional Matching are polynomial, while \( K \)-SAT, \( K \)-Coloring, Minimum Spanning (\( K \))-hypertree and \( K \)-Dimensional Matching are NP-hard for any \( K \geq 3 \). A generic lesson is that we lack for a reasonable intuition of which problems are hard, and which not.

On the other side, a hypothetic problem of cost minimization, in which an instance consists of a random extraction of \( 2^n \) i.i.d. “costs” variables for the \( 2^n \) feasible solutions in \( \{0,1\}^n \) (called REM, Random Energy Model [23, 24, 25]), would certainly be exponentially hard: no strategy is possible, behind the trivial one of just reading all the costs, and finding the smallest one, and this would take \( 2^n \) operations. Similarly, a decision problem in which each of the \( 2^n \) configurations is a solution independently from the others, with probability \( p = 2^{-\alpha n} \), \( 0 < \alpha < 1 \), would be exponentially hard: no strategy is possible, behind the trivial one of just searching sequentially for a solution, the average number of operations required being \( 2^{(1-\alpha)n} \).

Of course, these problems are out of the NP class, as the mere encoding of the instance is exponential in size (if you want, this is the content of the celebrated Shannon entropy theorem). This “physical” intuition corresponds to a well-known statement of Complexity Theory, namely that \( P \neq \text{NP} \) “in an oracle setting” [26].

The widespread idea behind (average-case) NP-hardness is that, although an ensemble of random instances of a given NP-complete decision problem only visits an infinitesimal fraction of the \( 2^{(2^n)} \) possible SAT-UNSAT truth tables (a subset of the ones which can be optimally encoded
with a polynomial entropy, as the polynomial encoding of the instance is a bound for this value), it is possible that they are both “sufficiently well distributed” and “sufficiently typical”, such that the resulting ensemble of tables is “well approximated” by the one of the REM (or “oracle”) ensemble.

Some interesting efforts have been recently done in the direction of proving that certain NP-complete problems have statistical properties strongly similar (although not identical) to the ones of REM [27, 28, 29], and this fact has been interpreted as a hint toward NP-completeness. Some other observations have been raised on the fact that the same kind of properties arises also in well-known polynomial problems, such as the determination of the rank of a matrix [30].

An observation of this last nature can be done also in this case. That is, we want to state that not only One-in-Two Matching is formulated in an apparently similar way to other Matching problems which are indeed polynomial, but also the statistical properties of the spectra of costs for random instances of these problems share many common features.

We know that pure Matching is polynomial. We have just proven that One-in-Two-Matching is NP-complete, and we gave at the end of section 1 [13] an argument for this fact: changing also a single “spin” determination \( \sigma \), translates into the substitution of a whole row and a whole column in the induced matching subproblem. What “almost always” \( \$ \) happens for a random instance of, say, i.i.d. real positive entries \( w_{ij} \) with measure \( p(w) = e^{-w} \), is that at fixed \( \sigma \) the optimal permutation \( \pi^*(\sigma) \) is such that \( C_W(\sigma, \pi^*(\sigma)) \simeq \zeta(2) \) [31, 32, 33] and all the summands are of order \( 1/N \), the largest one being of order \( \ln(N)/N \). Then, taking a \( \sigma' \) which differs from \( \sigma \) on a single entry, resamples both a whole row and a whole column. In correspondence with the old permutation we will now have a huge energy, \( \langle C_W(\sigma', \pi^*(\sigma)) \rangle \simeq \zeta(2) + 2 \), well inside the high-density part of the spectrum. Then, correlations among optimal permutations \( \pi^*(\sigma) \) and \( \pi^*(\sigma') \), and among energies \( C_W(\sigma, \pi^*(\sigma)) - \zeta(2) \) and \( C_W(\sigma', \pi^*(\sigma')) - \zeta(2) \), can be studied, also by mean of the exact results by Nair, Prabhakar and Sharma [34, 35] (this being the subject of a paper in preparation [36]).

Then, it can be seen that similar results arise from the analysis of a further variant of the problem, that we can call “One-in-Four” Matching. We restrict the ensemble of valid matchings to permutations \( \pi \) such that, for each \( i = 1, \ldots, n \), either \( \pi(2i-1) \) or \( \pi(2i) \) are in \( \{2i-1, 2i\} \), i.e. in the graphical language of section 2 [2] we have four \( * \)s in the four entries of the \( 2 \times 2 \) diagonal blocks. For this problem, we can describe the configurations by mean of triplets \( (\sigma, \tau, \pi) \), with \( \sigma, \tau \in \{0, 1\}^n \) and \( \pi \in S_n \) with no fixed points, and we can repeat the argument above, such that the best over \( \pi \) of \( (\sigma, \tau) \) pairs differing also for a single entry are strongly decorrelated, as the sub-instances differ by a whole row or a whole column. Nonetheless, One-in-Four Matching is trivially reduced to Matching, and the same holds for Assignment. Indeed, consider now classes of configurations \( (\sigma, \tau, \pi) \) sharing the same \( \pi \). The set of \( 2^{2n} \) costs for various \( (\sigma, \tau) \) is

\[
\left\{ \sum_i w_{2i-\sigma(i), 2\pi(i) - \tau(\pi(i))} \right\}_{\sigma, \tau \in \{0, 1\}^n}
\]

and is trivially minimized, as each variable \( \sigma_i \) or \( \tau_j \) enters only once in the cost function

\[
\min_{\sigma, \tau \in \{0, 1\}^n} \left( \sum_i w_{2i-\sigma(i), 2\pi(i) - \tau(\pi(i))} \right) = \sum_i \min_{\sigma, \tau \in \{0, 1\}} w_{2i-\sigma, 2\pi(i) - \tau}
\]

(31)

that is, the minimization problem is reduced to a traditional assignment problem, where \( 2 \times 2 \) non-diagonal blocks are replaced by the minimum weight among the four entries, and the diagonal blocks by infinite weights.

The same construction can not be repeated for One-in-Two Assignment, as the spin variables enter in the cost function in different points.

Another derivation of this property is given by the reduction of One-in-Four Assignment to a Min-Cut-Max-Flow Problem (cfr. [37] for all definitions), which is a polynomial problem. In a similar fashion to the mechanism above, the reduction can be done because of the factorization of

\( \$ \) In the sense of probability, that is, for \( N \) the size of the problem, always up to a probability \( p_N \), such that \( p_N \to 0 \) for \( N \to \infty \).
the choices over $\sigma$ and $\tau$, the corresponding network being

![Network Diagram]

all the edges have capacity 1, all the edges except the ones in the middle layer have cost zero, and, for the middle layer, an edge connects the $i$-th vertex on the left to the $j$-th vertex on the right only if $\lceil i/2 \rceil \neq \lceil j/2 \rceil$, and in this case it has cost $w_{ij}$. Checking that this instance of Min-Cost-Max-Flow reproduces the One-in-Four Assignment, where the maximum flow is $n$, is a simple exercise, while the procedure of equation (31) corresponds to the “resolution” of the Min-Cost-Max-Flow problem on the subgraphs.

6. A Quantum Adiabatic Algorithm for One-in-Two Matching

Here we describe a quantum Hamiltonian reproducing a Quantum Adiabatic Algorithm for One-in-Two Matching or Assignment, which is a very natural one for condensed-matter systems. We assume our instance is composed of a matrix of size $2n \times 2n$, with weights $w_{ij}$ out of the diagonal blocks, all much larger than zero (this is always the case, up to a trivial gauge transformation on the instance).

Consider two layers, each consisting of $2n \times 2n$ discrete positions on a square lattice. We call them the row- and the column-layers. We should imagine them geometrically “facing”, like two plates of a condenser.

In each row of the row-layer there is one particle, which could jump from one column to a neighbouring one. If particle in row $i$ is in layer position $(ij)$, we say that it is in the state $|j\rangle_{r(i)}$.

Similarly, in the column-layer we have one particle per column, which can hop on neighbouring rows, and the state of column $j$ is $|i\rangle_{c(j)}$ if the particle occupies position $i$. We call $b_{j,i}$, $b_{j,i}^\dagger$ the quantum oscillators for this layer.

A kinetic “diffusion” term would have the form $a_{i,j}^\dagger a_{i,j} + a_{i,j-1}^\dagger a_{i,j} + b_{j,i}^\dagger b_{j,i} + b_{j,i+1}^\dagger b_{j,i} + b_{j,i-1}^\dagger b_{j,i}$. So the purely kinetic part of the Hamiltonian, $\hat{H}_0$, is

$$\hat{H}_0 = \sum_{i,j} \left( a_{i,j+1}^\dagger a_{i,j} + a_{i,j-1}^\dagger a_{i,j} + b_{j,i+1}^\dagger b_{j,i} + b_{j,i-1}^\dagger b_{j,i} \right).$$

On the boundaries of the chains, you can assume either open or periodic boundary conditions (the first one is more “physical”, the second one makes the analysis simpler, anyhow the difference is negligible). Then, under this Hamiltonian, the particles are non-interacting. At sufficiently low temperatures, scaling with $n^{-2}$, the single-particle system (say, of row $i$), up to an exponentially small fraction of the time, will occupy the ground state with zero momentum, $(2n)^{-1/2} \sum_j |j\rangle_{r(i)}$, as it has a polynomially-small but finite gap, given by the square of the smallest non-zero momentum on the lattice.

The ground state of the system under the Hamiltonian $\hat{H}_0$ is

$$|\psi\rangle_{GS} = (2n)^{-2n} \left( |1\rangle + \cdots + |2n\rangle \right)_{r(1)} \otimes \cdots \otimes \left( |1\rangle + \cdots + |2n\rangle \right)_{r(2n)} \otimes \left( |1\rangle + \cdots + |2n\rangle \right)_{c(1)} \otimes \cdots \otimes \left( |1\rangle + \cdots + |2n\rangle \right)_{c(2n)}.$$

(32)
and it is a reasonable initial state, for a quantum adiabatic paradigm in which the coordinate-space representation reproduces the set of feasible solutions, although it is probably a poor initial state for a traditional quantum computing approach, as it is a “tree state” with $O(n^2 \ln n)$ symbols (cfr. [20], ch. 13).

Up to now, we have one particle per row on the row-layer, and similarly on the column-layer, but we have not implemented neither the matching constraint, that row-particle $i$ is in column $j$ iff column-particle $j$ is in row $i$, nor the one-in-two constraint.

At this aim we introduce an interaction Hamiltonian $\tilde{H}_1$, which couples the particles on the two layers. Furthermore, we will have special terms on the diagonal $2 \times 2$ blocks. Introduce the shortcut

$$\left( a^\dagger a b^\dagger b \right)_{i,j} := a^\dagger_{i,j} a_{i,j} b^\dagger_{i,j} b_{i,j}; \quad \left( a^\dagger a b^\dagger b \right)_{i,i} := \left( a^\dagger a b^\dagger b \right)_{i,i}.$$  \hspace{1cm} (34)

For pairs $(ij)$ not in a diagonal block, we would have a term of the form $-w_{ij}(a^\dagger a b^\dagger b)_{i,j}$, while for a block on entries $\{2i - 1, 2i\}$ we would have a combination of the form $W ((a^\dagger a b^\dagger b)_{2i} - c)((a^\dagger a b^\dagger b)_{2i-1} - c)$, with $c$ a value between 0 and 2 (for example, 1, but notice how the fine-tuning of $c$ exactly equal to 1 is not required), and $W$ an energy value larger than the typical $w_{ij}$. Remark that the procedure works also if $W$ and $c$ are slightly varying from block to block, as expected from an experimental realization.

Summarizing, the Hamiltonian $\tilde{H}_1$ (notation: $\sum_{i,j}^t$ means “pairs $(ij)$ not in a diagonal block”, i.e. such that $[\frac{1}{i} \neq \frac{1}{j}]$)

$$\tilde{H}_1 = -\sum_{i,j}^t w_{ij}(a^\dagger a b^\dagger b)_{i,j} - W \sum_{i=1}^n ((a^\dagger a b^\dagger b)_{2i} - c)((a^\dagger a b^\dagger b)_{2i-1} - c)$$  \hspace{1cm} (35)

is diagonal in coordinate space, as it only involves “number” operators, and has the optimal one-in-two matchings as ground states. Indeed, violating a matching constraint would have a finite positive cost $\sim w_{ij}$, while violating a one-in-two constraint would have a cost $\sim W$. Finally, for a valid matching, the energy of the state reproduces exactly the cost function.

For $w_{ij}$ not scaling with $n$ (which is the proper choice in order to have an extensive energy), in the case of matching we have a gap of order 1, while in the case of random assignment we have a gap only algebraically small in $n$, and not exponentially (cfr. [31, 32]).

So, the adiabatic interpolation, for time $t \in [0,1]$, given by $\hat{H}_t = (1-t)\hat{H}_0 + t\hat{H}_1$, could be a good prescription for a quantum adiabatic algorithm for the one-in-two matching, as, at least at the two endpoints, there are no exponentially-small energy gaps. A further analysis of what happens at intermediate times would be too long and difficult, and out of the aims of this paper.

7. Conclusion and perspectives

In this paper we introduced a problem, the One-in-Two Matching, which lies in between the (polynomial) Matching Problem and the (NP-complete) Hamiltonian Circuit Problem (HC). Despite, at a first glance, it could seem much more similar to Matching than to HC, we prove that indeed it is NP-complete (section 3). We thus expect that it captures the core reasons of complexity of HC, although largely simplifying the model.

The reduction is relatively simple and specially compact. As a side result, combination with a reduction proof from One-in-Two Matching to 3-Dimensional Matching (3DM) given in section 4 provides a simple reduction proof for 3DM, which could be preferred for pedagogical purposes to the original one by Karp in 1972.

Everything above applies to the “weighted” variants of the problems: (One-in-Two) Matching $\rightarrow$ (One-in-Two) Assignment, Hamiltonian Circuit $\rightarrow$ Travelling Salesman, 3DM $\rightarrow$ numerical 3DM.

In the introduction, and in section 6 we also give very preliminary arguments towards the fact that One-in-Two Matching is a viable candidate for an experimental realization of a Quantum Adiabatic Algorithm protocol, devoted to the solution of NP-complete problems. In two words, the main advantages are the structural simplicity of the problem formulation, and the unusual compactness of the reduction.
At the aim of still improving this second point for real-life applications (in which short clauses could appear, that, when reduced to SAT, would largely increase the instance size), in section 4 we give compact encodings of a larger family of clauses, including SAT, NAE-SAT, One-in-Three-SAT, Colouring and much more. Furthermore, in Appendix A we find an encoding for all clauses involving up to four literals.

Further directions of investigation will include a better understanding of the statistical properties of Random One-in-Two Assignment, at the light of the results of Nair, Prabhakar and Sharma [34, 35], and also with Cavity Method techniques [38], in relation to the important question of determining the geometrical characteristics of the hard instances of TSP [12, 36].

Appendix A. Truth-table dictionary

In this Appendix we classify the encoding of all truth tables up to 4 literals. Given \( k \) literals \( u_1, \ldots, u_k \), in principle there are \( 2^{2^k} \) possible truth tables, corresponding to the set of subsets of \{True, False\}^k (the satisfying assignments of the literals). More precisely, a clause \( C_T \) is identified by the set \( T \) of literal configurations evaluated to True

\[
T = \{ \bar{\tau} \} \subseteq \{\text{True, False}\}^k; \quad C_T(\bar{\tau}) = \text{True} \iff \bar{\tau} \in T. \quad (A.1)
\]

Still in section 4 we discuss the “gauge” invariance which allows to consider truth tables inside the same class as equivalent, this fact allows to reduce the number of tables to be analyzed, for a fixed value of the length \( k \). This number is further reduced by the fact that certain tables are “trivial” inside a boolean decision problem, in a sense which is depicted by the following rules:

**Rule 1**: The clause is always UNSAT, i.e. \( T = \emptyset \). The instance is determined to be unsatisfiable.

**Rule 2**: The clause is always SAT, i.e. \( T = \{\text{True, False}\}^k \). One can reduce to a smaller instance, where this clause is removed.

**Rule 3**: The clause can be encoded in logical form with no use of a literal (say, \( u_i \)). On the truth table, this means that, defining the symmetry operation

\[
R_i(\tau_1, \ldots, \tau_k) = (\tau_1, \ldots, \overline{\tau_i}, \ldots, \tau_k),
\]

\( T = R_i(T) \). The clause is equivalent to a smaller one, and thus the whole instance can be reduced in size.

**Rule 4**: The clause forces the assignment of a given literal (say, \( u_i = \text{True} \), or \( u_i = \text{False} \)). One can simply regard \( u_i \) as a synonymus of the fixed boolean variable, and the instance is reduced in size (the new instance has one literal less, and all the clauses in which \( u_i \) appears are shorter).

**Rule 5**: The clause forces the relative assignment of a given pair of literals (say, \( u_i \lor u_j = \text{True} \), or \( u_i \lor \overline{u_j} = \text{False} \)). One can simply regard \( u_j \) as a synonymus of \( u_i \) (or of \( \overline{u_i} \)), and the instance is reduced in size (the new instance has one literal less, and all the clauses in which both \( u_i \) and \( u_j \) appear are shorter).

Actually, Rule 5 is not so rigorous: sometimes synonima could be useful in certain constructions, and for example we made large use of them in our reduction proof from SAT to One-in-Two Matching (and we also described the “consistency check” clause, which indeed violates Rule 5 on all the pairs).

So, in our enumeration we will include all clauses violating Rule 5, although in this case we will point out the fact.

We make a complete enumeration “by hand” up to length 3, where the truth tables are easy to handle. We skip both clauses which are trivial w.r.t. rules 1-3, and clauses of the kind “range-T”, for which a general recipe has already be given in section 4.

For length 4, the manual enumeration would have been a formidable task, so we preferred to use a randomized search method: we just tried to fill randomly matrices \( W \), with a given density, and classified the resulting clauses, up to when each class had at least one representative.††

†† Although this was not assured to happen, we were lucky, and found a representative per each class in a few hours of computer time.
Another difficulty is that, as the same truth tables are difficult to handle, our results would be hardly fruible if the user would not have a recipe to identify his clause of interest in our list. For this reason, we decided an easy algorithm for the determination of an “algebraic signature” of the gauge orbit of a given truth table. This signature is an integer number, and thus, as in a “dictionary”, decides the sorting of the entries of our enumeration. For fruibility to the user, also the clauses violating rules 3 and 4 are listed, with a side annotation. On the other side, the cases $|T| \leq 2$ and $|T| \geq 14 = 2^4 - 2$, which fall in the general framework of section 3 have been omitted.

More detailed explanations are given below.

Appendix A.1. Clauses of length 2

The cases $|T| = 0, 1, 4$ are trivial. The case $|T| = 3$ is SAT. The case $|T| = 2$, which is both NAE-SAT and XOR, i.e. truth-setting structure, is already in the framework of section 2. So, there are no new cases.

Appendix A.2. Clauses of length 3

The cases $|T| = 0, 1, 8$ are trivial. The cases $|T| = 2, 6, 7$ have been studied in sections 2 and 3.

Now consider the case $|T| = 3$, and call $T = \{\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3\}$. The set of “distances”, where the distance is defined as $d(\bar{\tau}, \bar{\tau}') = \#\{i : \tau_i \neq \tau_i'\}$, can form the triangles $(1,1,2)$, $(2,2,2)$ and $(1,2,3)$. The clause $(1,1,2)$ violates Rule 4, because leaves a sub-hypercube (just a square in this case) outside. The clause $(2,2,2)$ is the One-in-Three clause, which is a special case of range-T clause. The clause $(1,2,3)$ violates Rule 5. An example is $(u_1 \lor u_2) \land (u_1 \lor u_2 \lor u_3)$, where violation of Rule 5 is made explicit. Anyhow, we give a matrix encoding in Table A1.

The case $|T| = 5$ has a similar case study: if we consider the complementary of $T$, it has cardinality three, and with the same reasoning above we have the three cases with sets of distances $(1,1,2), (2,2,2)$ and $(1,2,3)$. The difference is that now all these cases are new.

In the case $|T| = 4$, either all the four vectors are at distance 2 (and in this case we deal with the important XOR clause) or there are at least two neighbouring vectors (say, $\bar{\tau}_{1,2}$). Then, six vertices remain on the cube, of which four are neighbours of $\bar{\tau}_{1,2}$, so either $\bar{\tau}_{1,4}$ fill the pair of non-neighbouring sites, and we have the trivial clause $(u_1 \lor u_2)$, which violates many rules, or there is a vertex with two neighbours, say, $\bar{\tau}_1$ is at distance one from $\bar{\tau}_2$ and $\bar{\tau}_3$. If it is at distance one also from $\bar{\tau}_4$, we have a “At-least-2-in-3” clause, i.e. a special case of range-T clause with $k = h_{\text{max}} = 3$ and $h_{\text{min}} = 2$, for which we have a general recipe. Otherwise, if $\bar{\tau}_4$ is at the last vertex of the square with vertices $\bar{\tau}_{1,2,3}$, we have a trivial clause of length 1. There are only two other choices left, either the pairs of neighbouring vertices form an open path of length 3 (the “first-F-then-T” clause, $(\bar{\tau}_1 \land \bar{\tau}_2) \lor (u_2 \land u_3)$, with truth table $\{(TTT), (FTT), (FFT), (FFF)\}$), or the last vertex $\bar{\tau}_4$ is antipodal to $\bar{\tau}_1$.

This completes the case study for $k = 3$: we have three new cases at $|T| = 4$ and three at $|T| = 5$, and a case at $|T| = 3$ which violates Rule 5. The suggested encoding matrices are listed in Table A1.

Appendix A.3. Clauses of length 4

Each entry of our dictionary is encoded in the form

$|T|$  Algebraic signature  Truth table  Encoding matrix $W$  $\#\{w_{ij} = 1\}$  (Comments)

The Algebraic signature allows to determine the gauge class of a given truth table. This is the reason why the truth tables are sorted according to that. In a sense, it is a univoque name for a clause: if you want to encode a given truth table with 4 literals and $|T|$ True assignments into a One-in-Two pattern, and this truth table does not follow into one of the general classes studied in sections 2 and 3, you should calculate the algebraic signature (a simple and fast algorithmic procedure), and then search the corresponding entry in this dictionary; then, the other fields of the entry will allow you to construct the proper matrix.
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The choice is, of course, arbitrary. We choose decreasing real numbers in \([0, 1]\), with alternating
signs, in order to have determinants of the same order of magnitude. Then, we build the square matrix $M$ of size $|T|$, with $M_{i,j} = r(d(i, j))$. Just to reduce computation times, if $|T| > 2^k - 1$ we use instead of $T$ its complementary set. The chosen “algebraic signature” of truth table $T$ is the integer part of $10^{18} \det M(T)$.

The truth table is simply encoded in a compact way, the $2^{16}$ possible vectors $\vec{f}$ being ordered lexicographically, and the table $T$ being denoted by the integer $n(T) \in \{ 0, \ldots , 2^{16} - 1 \}$ corresponding to the binary string of vectors evaluated to True

$$n(T) = \sum_{\vec{f} \in T} 2^n(\vec{f}) , \quad n'(\vec{f}) = \sum_{i=1}^{k} 2^{k-i} \delta_{\vec{f}, \text{True}} . \quad (A.3)$$

The representative of the class is chosen in arbitrary way. Indeed, it comes out that it is the one with lowest index $n(T)$, among the $T$s in the same class.

Similarly, the encoding matrix is written in a more compact way as a vector of integers, where each row of 0s and 1s, of length $2k = 8$, is replaced by the corresponding integer in $\{ 0, \ldots , 2^8 - 1 \}$. Just for statistic reasons, the number of 1s in the matrix is indicated.

Finally, if the clause is trivial in the sense of the rules 3-5 above (failures of rules 1 and 2 are excluded trivially), we report the fact, and the rule it fails (if it is trivial for more than one reason, we choose a reason of smaller rule index). Here the dictionary follows:

| $A$ | $B$ | $C$ | $D$ | $E$ |
|-----|-----|-----|-----|-----|
| $F$ | $G$ | $H$ | $I$ | $J$ |

| $K$ | $L$ | $M$ | $N$ | $O$ |
|-----|-----|-----|-----|-----|
| $P$ | $Q$ | $R$ | $S$ | $T$ |

$$A \lor B \lor C \lor D \lor E \lor F \lor G \lor H \lor I \lor J \lor K \lor L \lor M \lor N \lor O \lor P \lor Q \lor R \lor S \lor T$$

| $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ |
|-------|-------|-------|-------|-------|
| $u_6$ | $u_7$ | $u_8$ | $u_9$ | $u_{10}$ |

Finally, if the clause is trivial in the sense of the rules 3-5 above (failures of rules 1 and 2 are excluded trivially), we report the fact, and the rule it fails (if it is trivial for more than one reason, we choose a reason of smaller rule index). Here the dictionary follows:
One-in-Two-Matching Problem is NP-complete
One-in-Two-Matching Problem is NP-complete
Acknowledgments

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