Deviation from Tri-bimaximal Mixings in Two Types of Inverted Hierarchical Neutrino Mass Models

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Abstract

An attempt is made to explore the possibility for deviations of solar mixing angle ($\theta_{12}$) from tri-bimaximal mixings, without sacrificing the predictions of maximal atmospheric mixing angle ($\theta = \pi/4$) and zero reactor angle ($\theta_{13} = 0$). We find that the above conjecture can be automatically realised in the inverted hierarchical neutrino mass model having 2-3 symmetry, in the basis where charged lepton mass matrix is diagonal. For the observed ranges of $\Delta m^2_{21}$ and $\Delta m^2_{23}$, we calculate the predictions on $\tan^2 \theta_{12} = 0, 0.5, 0.45, 0.35$ for different input values of the parameters in the neutrino mass matrix. We also observe a possible crossing over from one type of inverted hierarchical model having same CP parity (Type-IHA) to other type having opposite CP parity (Type-IHB). Such neutrino mass matrices can be obtained from the canonical seesaw formula using diagonal form of Dirac neutrino mass matrix and non-diagonal texture of right-handed Majorana mass matrix, and may have important implications in model building using discrete as well as non-abelian symmetry groups.

Keywords: inverted hierarchical, tri-bimaximal mixings, solar mixing angle, neutrino mass matrix.

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1 Introduction

Current observational data\[1\] on neutrino oscillations indicates a clear departure from the tri-bimaximal mixings or Harrison-Perkins-Scott (HPS) mixing pattern\[2\]. The most recent SNO experimental determination\[3\] of solar angle gives $\tan^2 \theta_{12} = 0.45^{+0.09}_{-0.08}$ compared with $\tan^2 \theta_{12} = 0.50$ in HPS scheme. There is no strong claim for a substantial departure from the maximal atmospheric mixing ($\tan^2 \theta_{23} = 1$), and also from the zero reactor angle ($\sin \theta_{13} = 0$). Only upper bound for $\sin \theta_{13}$ is known at the moment and future measurements may possibly give a very small value so small that we can approximate it by zero\[4\]. This does not yet contradict with the non-observation of Dirac CP phase angle. There are discussions\[5\] on the experimental requirements for mass hierarchy measurements for $\theta_{13} = 0$.

Conditions of maximal atmospheric mixing ($\theta_{23} = \pi/4$) and exact zero of reactor angle ($\theta_{13} = 0$), are in fact necessary and sufficient conditions\[4-9\] for the leptonic mixing matrix obtained from the diagonalization of the left-handed Majorana neutrino mass matrix having a 2-3 symmetry. Here the 2-3 symmetry simply implies an invariance under the simultaneous permutation of the second and third rows as well as the second and third columns\[4\]. In the basis where charged lepton mass matrix is diagonal, a 2-3 symmetry can be generally realised in three degenerate models, and two inverted hierarchical models of neutrino mass patterns\[10\]. However, normal hierarchical model generally does not possess a 2-3 symmetry and hence it does not predict maximal atmospheric mixing as well as exact zero reactor angle\[11\]. The degenerate models on the other hand give either very small ($\tan^2 \theta_{12} \approx 0.27$) or maximal solar mixings\[10\]. Further constraints such as zero determinant\[12,13,4\] or zero trace\[14\] of the neutrino mass matrix, which lead to other interesting properties, are not considered in the present analysis. This freedom allows us to consider larger value of non-zero mass eigenvalue $m_3$ within the framework of inverted hierarchical model.

A general form of inverted hierarchical mass matrix $m_{LL}$ having 2-3 symmetry, can be written as\[4\],

$$m_{LL} = \begin{pmatrix} m_{11} & m_{12} & m_{12} \\ m_{12} & m_{22} & m_{23} \\ m_{12} & m_{23} & m_{22} \end{pmatrix}$$ (1)
which is diagonalised by the relation $m_{LL} = UDU^\dagger$ where $U$ is given by

$$U = \begin{pmatrix} c_{12} & -s_{12} & 0 \\ \frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (2)$$

Here we have $c_{12} = \cos\theta_{12}$ and $s_{12} = \sin\theta_{12}$; and $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. In the basis where charged lepton mass matrix is diagonal, $U$ in eq.(2) is identified as the MNS mixing matrix $U_{MNS}$ where the solar mixing angle $\theta_{12}$ is arbitrary, atmospheric mixing angle $\theta_{23}$ is maximal and reactor angle $\theta_{13}$ is exactly zero. For bimaximal mixings, we choose $c_{12} = 1/\sqrt{2}$ and $s_{12} = 1/\sqrt{2}$, leading to $\tan^2\theta_{12} = 1.0$, whereas for tri-bimaximal mixings we have $c_{12} = \sqrt{\frac{2}{3}}$ and $s_{12} = 1/\sqrt{3}$, leading to $\tan^2\theta_{12} = 0.5$.

We are now interested in the present analysis to investigate the condition for fixing the arbitrary solar mixing angle to its tri-bimaximal value and also for a possible gradual departure from this tri-bimaximal value, without sacrificing maximal atmospheric mixing and exact zero value of reactor angle. We calculate here three neutrino mass eigenvalues consistent with the present observational data.

We confine our analysis in inverted hierarchical neutrino mass model having a 2-3 symmetry in eq.(1). We organise the paper as follows. In section 2 we present an analytic solution for the inverted hierarchical model and find out condition for lowering the solar mixing angle. This is followed by a crossing over from Type-IHA to Type-IHB inverted hierarchical models. Numerical results are presented in section 3. We conclude in section 4 with a summary and discussions.

## 2 Analysis of Inverted hierarchical model

We have in general two types of inverted hierarchical models based on the relative sign of the first two mass eigenvalues $m_1$ and $m_2$ : Type-IHA for same CP parity $(m_1, m_2, m_3)$ and Type-IHB for opposite CP parity $(m_1, -m_2, m_3)$. Type-IHB is considered to be more stable under radiative corrections in MSSM [17-19], whereas Type-IHA is found to be more stable under the presence of left-handed Higgs triplet term in type-II seesaw mechanism[20]. For our present analysis we will not address the issue of stability of neutrino mass model. Instead, we explore the properties of these two types of inverted hierarchical mass matrices and their interconnection.
2.1 Inverted hierarchy of Type-IHA

We start with a specific form[10] of Type-IHA mass matrix having 2-3 symmetry (1) as,

$$M_{IHA} = \begin{pmatrix} 1 - 2\epsilon & -\epsilon & -\epsilon \\ -\epsilon & \frac{1}{2} & \frac{1}{2} - \eta \\ -\epsilon & \frac{1}{2} - \eta & \frac{1}{2} \end{pmatrix} m_0, \tag{3}$$

where the symmetry breaking parameters are $\epsilon, \eta << 1$ and $m_0 = 0.05\text{eV}$ as input value[10]. Such left-handed Majorana mass matrix can be realised in the canonical seesaw formula using a generalised diagonal form of Dirac mass matrix $m_{LR}$ and non-diagonal form of right-handed Majorana mass matrix $M_{RR}$ given by[10]

$$m_{LR} = \begin{pmatrix} \lambda^m & 0 & 0 \\ 0 & \lambda^n & 0 \\ 0 & 0 & 1 \end{pmatrix} m_f, \tag{4}$$

and

$$M_{RR} = \begin{pmatrix} (1 + 2\epsilon)\lambda^{2m} & \epsilon\lambda^{m+n} & \epsilon\lambda^n \\ \epsilon\lambda^{m+n} & \frac{1}{2}\lambda^{2m} & (1 - \frac{1}{2}\eta)\lambda^n \\ \epsilon\lambda^{m} & (1 - \frac{1}{2}\eta)\lambda^n & \frac{1}{2}\eta \end{pmatrix} v_0. \tag{5}$$

where $m_0 = m^2_f/v_0$. The pair of $(m, n)$ can take different sets of values which can represent mass matrices belonging to up-quark, down-quark, charged-lepton or any diagonal form of Dirac neutrino.

$M_{IHA}$ in eq.(3) can be reduced to the zeroth order texture[16] when $\epsilon = \eta = 0$,

$$M_{IHA}^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} m_0, \tag{6}$$

This has a degeneracy in the first two mass eigenvalues, $(1,1,0)m_0$, and such degeneracy makes the solar mixing angle $\theta_{12}$ arbitrary, and may have infinite values lying between 0 and $\frac{\pi}{4}$. Once the degeneracy is removed as in eq.(3), the solar angle is then fixed at a particular value. Such freedom in fixing the solar angle does not destroy 2-3 symmetry of the mass matrix, and it depends absolutely on the choice of input values of $\eta$ and $\epsilon$, without disturbing the predictions on atmospheric angle $\theta_{23} = \frac{\pi}{4}$ and reactor angle $\theta_{13} = 0$. Diagonalizing (3) we obtain the three mass eigenvalues,

$$m_1 = (2 - 2\epsilon - \eta - y)\frac{m_0}{2}, \quad m_2 = (2 - 2\epsilon - \eta + y)\frac{m_0}{2}, \quad m_3 = \eta m_0 \tag{7}$$
where

$$y^2 = 12\epsilon^2 - 4\epsilon\eta + \eta^2. \quad (8)$$

The solar mixing is now fixed by the relation[7],

$$\tan 2\theta_{12} = \frac{2\sqrt{2}}{2 - \frac{\eta}{\epsilon}} \quad (9)$$

As long as \((2 - \frac{\eta}{\epsilon}) \leq 1\), we have tri-bimaximal (or HPS) mixing and its possible deviation to lower values. For tri-bimaximal mixing, we find two solutions of eq.(7), for \(\eta/\epsilon\) at 1 and 3 respectively. Similarly, for deviation from tri-biimaximal solar mixing to lower values, we have the corresponding values of this ratios as \(\eta/\epsilon < 1\) and \(\eta/\epsilon > 3\). Once the solar angle is fixed, then the range of \(\eta\) or \(\epsilon\) can be solved through a search programme. Table-1 represents a summary of our results.

### 2.2 Crossing over from Type-IHA to Type-IHB

One particular observation for the solution of eq.(3) in the range \(\epsilon > 0.5\), is the crossing over from inverted hierarchical model with same CP parity (Type-IHA) to inverted hierarchical model with opposite CP parity (Type-IHB). To demonstrate this interconnection, we start with the most general form of mass matrix[8] belonging to Type-IHB,

$$M_{IHB} = \begin{pmatrix} \delta_1 & 1 & 1 \\ 1 & \delta_2 & \delta_3 \\ 1 & \delta_3 & \delta_2 \end{pmatrix} m'_0. \quad (10)$$

The zeroth order mass matrix of eq.(10) has the form[16]

$$M^0_{IHB} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} m'_0, \quad (11)$$

which has non-degenerate eigenvalues \((1, -1, 0)m'_0\) compared to that of eq.(6). In addition, the solar mixing is now fixed at the maximal value \((\theta_{12} = \pi/4)\) unlike eq.(4) where it can have any arbitrary value. The diagonalization of (10) leads to the following mass eigenvalues,

$$m_1 = (\delta_1 + \delta_2 + \delta_3 + x)\frac{m'_0}{2}, \quad m_2 = (\delta_1 + \delta_2 + \delta_3 - x)\frac{m'_0}{2}, \quad m_3 = (\delta_2 - \delta_3)m'_0 \quad (12)$$
where
\[ x^2 = 8 + (\delta_1^2 + \delta_2^2 + \delta_3^2) - 2\delta_1\delta_2 - 2\delta_1\delta_3 + 2\delta_2\delta_3; \quad (13) \]
and the solar mixing,
\[ \tan 2\theta_{12} = \frac{2\sqrt{2}}{(\delta_1 - \delta_2 - \delta_3)}. \quad (14) \]
Type-IHB in (10) generally predicts nearly maximal solar mixing and it has been considered almost impossible to tone down the solar angle without corrections from charged lepton mass matrix[21]. Here we wish to show that this notion is not completely true and can be avoided through the following transformation,
\[ \delta_1 = 2(1 - \frac{1}{2\epsilon}), \quad \delta_2 = -\frac{1}{2\epsilon}, \quad \delta_3 = (\frac{\eta}{\epsilon} - \frac{1}{2\epsilon}), \quad m'_0 = m_0(-\epsilon) \quad (15) \]
where the three parameters \( \delta_1, \delta_2, \delta_3 \) are now replaced by only two parameters \( \eta, \epsilon \). From eq.(15) it is easy to see that inverted hierarchical model (Type-IHB) in eq.(10) is now equivalent to Type-IHA in eq.(3), and their expressions for eigenvalues and solar angle in eqs.(7)-(9) and eqs.(12) - (14) respectively, are also equivalent. As emphasised earlier, such cross-over will take place only when the value of \( \epsilon > 0.5 \) and this is shown in Table-1.

3 Numerical calculations and results

We follow two important steps for carrying out numerical estimations. In the first step, we choose a specific value of solar mixing \( \tan^2 \theta_{12} \) via eq.(9) or (14), and then solve for possible values of the ratio \( \eta/\epsilon \). In the second step we take up a particular value of this ratio \( \eta/\epsilon \), and find out the ranges of either \( \eta \) or \( \epsilon \) for the given ranges of \( \Delta m^2_{21} \) and \( \Delta m^2_{23} \) which are consistent with observational data[1].

For a demonstration, we present here the numerical estimations for the value of solar mixing \( \tan^2 \theta_{12} = 0.45 \) which in turn corresponds to two values of \( r = \eta/\epsilon \) at \( r = 0.840498455 \) and \( r = 3.1594955 \) derived from eq.(9). The first value leads to two ranges of \( \eta \) as (A): \( 0.00398619 \leq \eta \leq 0.00527107 \) and (B): \( 0.586525 \leq \eta \leq 0.58781 \), whereas second one has only one range of \( \eta \) as (C): \( -0.0193327 \leq \eta \leq -0.0147069 \). As discussed before, case (B) belongs to inverted hierarchy (Type-IHB) and cases (A,C) belong to
Type-IHA. However the expressions for eigenvalues and solar mixing angle are the same. We use standard procedure to estimate neutrino masses and mixings[10,11].

Case (A): Type-IHA where mass eigenvalues are of the form $(m_1, m_2, m_3)$.
Using the best value $\eta = 0.00462863$ and $\epsilon = 0.0055071$, we have

$$m_{LL} = M_{IHA} = \begin{pmatrix} 0.0494493 & -0.000275351 & -0.000275351 \\ -0.000275351 & 0.025 & 0.0247686 \\ -0.000275351 & 0.0247686 & 0.026 \end{pmatrix} \quad (16)$$

Diagonalising the above mass matrix we have three mass eigenvalues,

$$m_i = (0.0491881, 0.0500298, 0.00231432) \text{eV}, \quad i = 1, 2, 3.$$ leading to $\Delta m_{21}^2 = 8.35 \times 10^{-5}\text{eV}^2$ and $\Delta m_{23}^2 = 2.5 \times 10^{-3}\text{eV}^2$. The MNS mixing matrix is extracted as,

$$U_{MNS} = \begin{pmatrix} 0.830455 & 0.557086 & -2.5 \times 10^{-18} \\ 0.393919 & -0.58722 & -0.707107 \\ -0.393919 & -0.58722 & 0.70717 \end{pmatrix} \quad (17)$$

which gives $\tan^2 \theta_{12} = 0.45$, $\tan^2 \theta_{23} = 1$ and $\sin \theta_{13} = 0$.

Case (B): Type-IHB where the mass eigenvalues are of the form $(-m_1, m_2, m_3)$.
For the best input value $\eta = 0.5871675$ and $\epsilon = 0.6985945$ we have

$$m_{LL} = M_{IHB} = \begin{pmatrix} -0.0198595 & -0.0349297 & -0.0349297 \\ -0.0349297 & 0.025 & -0.00435837 \\ -0.0349297 & -0.00435837 & 0.026 \end{pmatrix}, \quad (18)$$

Diagonalising the above mass matrix we have three mass eigenvalues,

$$m_i = (-0.0529967, 0.0537789, 0.0293584) \text{eV}, \quad i = 1, 2, 3.$$ leading to $\Delta m_{21}^2 = 8.35 \times 10^{-5}\text{eV}^2$ and $\Delta m_{23}^2 = 2.03 \times 10^{-3}\text{eV}^2$. The model is quite different from degenerate model where the overall magnitude of neutrino masses, is of the order of 0.4eV. The MNS mixing matrix is extracted as,

$$U_{MNS} = \begin{pmatrix} -0.830455 & 0.557086 & -4.4 \times 10^{-17} \\ -0.393919 & -0.58722 & -0.707107 \\ -0.393919 & -0.58722 & 0.70717 \end{pmatrix}, \quad (19)$$
which gives \( \tan^2 \theta_{12} = 0.45, \tan^2 \theta_{23} = 1 \) and \( \sin \theta_{13} = 0 \).

**Case (C):** Type-IHA where the mass eigenvalues are of the form \((m_1, m_2, -m_3)\).

For the best input value \( \eta = -0.0170198 \) and \( \epsilon = -0.005386865 \) we have,

\[
\begin{pmatrix}
0.0505387 & 0.000269343 & 0.000269343 \\
0.000269343 & 0.025 & 0.025851 \\
0.000269343 & 0.025851 & 0.025
\end{pmatrix}
\]

Diagonalising the above mass matrix we have three mass eigenvalues

\[
m_i = (0.0502832, 0.0511065, -0.00085099) \text{eV}, \quad i = 1, 2, 3.
\]

leading to \( \Delta m^2_{21} = 8.3 \times 10^{-5} \text{eV}^2 \) and \( \Delta m^2_{23} = 2.6 \times 10^{-3} \text{eV}^2 \). The MNS mixing matrix is extracted as,

\[
U_{MNS} = \begin{pmatrix}
0.830455 & -0.557086 & -1.5 \times 10^{-18} \\
-0.393919 & -0.58722 & -0.707107 \\
-0.393919 & -0.58722 & 0.70717
\end{pmatrix}
\]

which gives \( \tan^2 \theta_{12} = 0.45, \tan^2 \theta_{23} = 1 \) and \( \sin \theta_{13} = 0 \).

We present our calculations in Table-1 for all possible ranges of \( \eta \) and \( \epsilon \) leading to different values of solar mixing \( \tan^2 \theta_{12} \) at 0.5 for tri-bimaximal mixing, and then 0.45 and 0.35 as possible deviations from tri-bimaximal mixing. The present analysis shows enormous provision for lowering the solar mixing angle without sacrificing predictions on \( \tan^2 \theta_{23} = 1 \) and \( \sin \theta_{13} = 0 \).

It is interesting to note that only two parameters \( \eta \) and \( \epsilon \) play the key roles in the whole analysis. For each value of \( \tan^2 \theta_{12} \) we have three solutions corresponding to two values of \( \eta/\epsilon \), and every solution has a particular range of \( \eta \). These values satisfy observed ranges of \( \Delta m^2_{21} \) and \( \Delta m^2_{23} \).

As a representative example, we present in Fig.1 the graphical solution of the ratio \( \eta/\epsilon \) corresponding to \( \tan^2 \theta_{12} = 0.45 \). In Figs. 2-4 we summarise all the results of the calculation corresponding to \( \tan^2 \theta_{12} = 0.45 \) case. In particular, Fig.2 presents a graphical summary of the predictions on \( \Delta m^2_{21} \) and \( \Delta m^2_{23} \), and a corresponding correlation graph between them for the valid range \( 0.003986 \leq \eta \leq 0.005271 \). Similarly, Figs. 3 and 4 for \( 0.586525 \leq \eta \leq 0.58781 \) and \( -0.019333 \leq \eta \leq -0.014707 \) ranges, respectively.
4 Summary and discussions

We summarise the main points in this work. The inverted hierarchical neutrino mass matrix having 2-3 symmetry, has great potential to give tribimaximal mixings and also possible deviations of solar mixing angle, without sacrificing predictions on maximal atmospheric mixing angle and zero reactor angle. We do not take corrections from charged lepton mass matrix and this helps to maintain 2-3 symmetry of the neutrino mass matrix. Such neutrino mass matrices are generated from seesaw formula using diagonal form of Dirac mass matrix and non-diagonal form of right handed Majorana mass matrix. Two types of inverted hierarchical models are found to be crossed over at a particular range of input parameters. The present analysis though phenomenological, may have important implications in model buildings[22] on tri-bimaximal mixings and its possible deviations, based on various discrete symmetries as well as non-Abelian gauge groups, and also on the experimental requirements for mass hierarchy measurements at zero reactor angle[5].

Table-1: Prediction of the solar mixing angle and its deviation from tri-bimaximal mixings, along with the predictions on solar and atmospheric mass-squared differences.

| $\tan^2 \theta_{12}$ | $\eta/\epsilon$ | range of $\eta$ | $\Delta m_{21}^2 (10^{-3}eV^2)$ | $\Delta m_{23}^2 (10^{-3}eV^2)$ |
|---------------------|-----------------|-----------------|--------------------------|--------------------------|
| 0.5                 | 1.0             | 0.004835 − 0.006395 | 7.20 − 9.50 | 2.50 − 2.50 |
| 0.5                 | 1.0             | 0.66072 − 0.66183 | 9.50 − 7.20 | 1.41 − 1.41 |
| 0.5                 | 3.0             | −0.01871 − 0.01423 | 9.41 − 7.20 | 2.63 − 2.60 |
| 0.45                | 0.84049         | 0.003986 − 0.005271 | 7.17 − 9.50 | 2.5 − 2.5 |
| 0.45                | 0.84049         | 0.586525 − 0.58781 | 9.5 − 7.2 | 2.03 − 2.03 |
| 0.45                | 3.15949         | −0.019333 − 0.014707 | 9.41 − 7.20 | 2.63 − 2.60 |
| 0.35                | 0.44620         | 0.002002 − 0.0026463 | 7.20 − 9.50 | 2.51 − 2.51 |
| 0.35                | 0.44620         | 0.36217 − 0.362811 | 9.5 − 7.2 | 4.01 − 4.01 |
| 0.35                | 3.55380         | −0.020592 − 0.015666 | 9.41 − 7.20 | 2.63 − 2.60 |

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Figure 1: Two solutions of $\tan^2 \theta_{12} = 0.45$ corresponding to $(\eta/\epsilon)$ equals to 0.84049 and 3.15949 respectively.
Figure 2: Predictions on $\Delta m_{21}^2$ in the unit ($10^{-5}\text{eV}^2$) and $\Delta m_{23}^2$ in the unit ($10^{-3}\text{eV}^2$) for the value $\tan^2\theta_{12} = 0.45$ in the range: $0.003986 \leq \eta \leq 0.005271$ and the corresponding correlation graph.

Figure 3: Predictions on $\Delta m_{21}^2$ in the unit ($10^{-5}\text{eV}^2$) and $\Delta m_{23}^2$ in the unit ($10^{-3}\text{eV}^2$) for the value $\tan^2\theta_{12} = 0.45$ in the valid range: $0.586525 \leq \eta \leq 0.58781$ and the corresponding correlation graph.
Figure 4: Predictions on $\Delta m_{21}^2$ in the unit $(10^{-5}eV^2)$ and $\Delta m_{23}^2$ in the unit $(10^{-3}eV^2)$ for the value $\tan^2 \theta_{12} = 0.45$ in the valid range: $-0.019333 \leq \eta \leq -0.014707$ and the corresponding correlation graph.