Limit linear series for curves of compact type with three irreducible components

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Abstract

Our aim in this work is to study exact Osserman limit linear series on curves of compact type \( X \) with three irreducible components. This case is quite different from the case of two irreducible components studied by Osserman. For instance, for curves of compact type with two irreducible components, every refined Eisenbud-Harris limit linear series has a unique exact extension. But, for the case of three irreducible components, this property is no longer true. We find a condition characterizing when a given refined Eisenbud-Harris limit linear series has a unique exact extension. To do this, it is necessary to understand how to construct exact extensions. We find a constructive method, which describes how to construct all exact extensions of refined limit linear series. By our method, we get that every refined limit linear series has at least one exact extension.

1 Introduction

In Algebraic Geometry, the theory of linear series on smooth curves is closely related to that of Abel maps. The fibers of Abel maps consist precisely of complete linear series.

For curves of compact type, (Eisenbud and Harris 1986) developed the theory of limit linear series as an analogue of linear series. This theory is very powerful for degeneration arguments on curves. The idea is to analyze how linear series degenerate when a family of smooth curves degenerates to a compact type curve. Eisenbud and Harris approached this situation by considering only the possible limit line bundles with nonnegative multidegree and degree \( d \) on one irreducible component of the curve. (Osserman 2006a) developed a new and more functorial construction for the theory of limit linear series. The basic idea is to consider all possible limit line bundles with nonnegative multidegree. Thus, Osserman limit linear series carry more information about limit line bundles. This new theory has a generalization to higher rank vector bundles (Osserman 2014).

Abel maps for curves of compact type have been studied by (Coelho and Pacini 2010). Recently, for curves of compact type \( X \) with two irreducible components, (Esteves and Osserman 2013) related limit linear series to fibers of Abel maps via the definition of limit linear series by (Osserman 2006a). They studied the notion of exact limit linear series. These contain in particular all limits of linear series on the generic fiber in a regular smoothing family.

Also, for curves of compact type \( X \) with two irreducible components, (Osserman 2006b) studied the space of limit linear series corresponding to a given Eisenbud-Harris limit linear series. He obtained an upper bound for the dimension of that space. Using this result, he also obtained a simple proof of the Brill-Noether theorem using only the limit linear series theory.

Our aim in this work is to study exact limit linear series on curves of compact type \( X \) with three irreducible components. This case is quite different from the case of two
irreducible components. For instance, for curves of compact type with two irreducible components, (Osserman 2006a) showed that every refined Eisenbud-Harris limit linear series has a unique exact extension. But, for the case of three irreducible components, this property is no longer true.

We will study the case of exact limit linear series which are obtained as the unique exact extension of a refined Eisenbud-Harris limit linear series. For curves $X$ consisting of a chain of smooth curves, (Osserman 2014) studied the notion of chain adaptable refined limit linear series. He showed that every chain adaptable refined limit linear series has a unique extension. In particular, for our case of curves of compact type $X$ with three irreducible components, chain adaptable refined limit linear series are an example of refined Eisenbud-Harris limit linear series with a unique extension. It turns out that, by our Theorem 5.3 every refined Eisenbud-Harris limit linear series having a unique extension is chain adaptable. We should mention that, for our case of curves of compact type with three irreducible components, it is easy to see that every chain adaptable refined limit linear series has a unique extension (we give a simple proof of that in Theorem 5.3), but proving that the unique extension property implies the chain adaptable condition is not easy at all. We really need to understand how to construct extensions (in Section 4 we describe a method for the construction of all exact extensions).

We now explain the contents of the paper in more detail, especially the statement of our main theorem, which is Theorem 5.3. We begin with the notation of a limit linear series. In (Estes and Osserman 2013), a limit linear series of degree $d$ and dimension $r$ on a curve of compact type $X$ with two irreducible components $Y$ and $Z$, meeting transversally at a point $P$, is a collection $g := (L, V_0, \ldots, V_d)$, where $L$ is an invertible sheaf on $X$ of degree $d$ on $Y$ and degree $0$ on $Z$, and $V_i$ is a vector subspace of $H^0(X, L^i)$ of dimension $r + 1$, for each $i = 0, \ldots, d$, where $L^i$ is the invertible sheaf on $X$ with restrictions $L|_Y(-iP)$ and $L|_Z(iP)$, and these vector subspaces are linked by certain natural maps between the sheaves $L^i$. Thus, a limit linear series is defined by a collection of pairs $(L^i, V_i)$, for each $i = 0, \ldots, d$, and we can use the notation $\{(L^i, V_i)\}_i$, where $0 \leq i \leq d$. For each $i = 0, \ldots, d$, the invertible sheaf $L^i$ has multidegree $d_i := (d - i, i)$. So, for each $i = 0, \ldots, d$, setting $d := (d - i, i)$, $L_d = L^i$ and $V_d = V_i$, a limit linear series can also be denoted by a collection $\{(L_d, V_d)\}_d$, where $d \geq 0$ of total degree $d$. We will use this notation for the case of three irreducible components.

In this work, $X$ will denote the union of three smooth curves $X_1, X_2$ and $X_3$, such that $X_1$ and $X_2$ meet transversally at a point $A$, and $X_2$ and $X_3$ meet transversally at a point $B$, with $A \neq B$. A limit linear series of degree $d$ and dimension $r$ on $X$ is a collection $g := \{(L_d, V_d)\}_d$, where $d \geq 0$ of total degree $d$, each $L_d$ is an invertible sheaf of multidegree $d$, and $V_d$ is a vector subspace of $H^0(X, L_d)$ of dimension $r + 1$, for each $d$, where $L_d$ is the invertible sheaf on $X$ with restrictions $L|_{X_1}(-(d-i)A)$, $L|_{X_2}((d-i)A-IB)$ and $L|_{X_3}(IB)$, with $d = (i, d-i-l, l)$ and $L := L_{(d, 0, 0)}$, and the subspaces $V_d$ are linked by certain natural maps between the sheaves $L_d$; see Subsection 2.11 for the precise definition of limit linear series.

We restrict our attention to the case of exact limit linear series which are obtained as the unique exact extension of a refined Eisenbud-Harris limit linear series. Given a refined limit linear series, for any $i \geq 0$ and $l \geq 0$ such that $i + l \leq d$ there is a natural space, here denoted $K_d$, satisfying that, for every extension $g := \{(L_d, V_d)\}_d$, $K_d$ contains $V_{(i, d-i-l, l)}$. 


It turns out that these spaces are very important to understand when the extension is unique (spaces with the same property as the $K_d$ appeared in (Osserman 2006a) and (Osserman 2014), for curves of compact type with two irreducible components and for curves of compact type with more than two components, respectively).

We recall the analogous spaces for the case of curves of compact type $X$ with two components. Given an invertible sheaf $L$ on $X$ of degree $d$ on $Y$ and degree $0$ on $Z$, for each nonnegative multidegree $d := (i, d - i)$, let $L_d$ be the invertible sheaf on $X$ with restrictions $L|_Y(-(d-i)P)$ and $L|_Z((d-i)P)$, and let $V_Y$ and $V_Z$ be vector subspaces of $H^0(L_{(d,0)})$ and $H^0(L_{(0,d)})$, respectively, of dimension $r + 1$, such that $\{(L_{(d,0)}|_Y, V_Y)\}$ is a refined Eisenbud-Harris limit linear series. In this situation, (Osserman 2006a) showed that the unique exact extension is given as follows:

(*) For each nonnegative multidegree $d := (i, d - i)$, $V_d$ is the space of sections $s$ of $H^0(X, L_d)$ whose image in $H^0(X, L_{(d,0)})$ belongs to $V_Y$ and vanishes at $P$ with order at least $d - i$, and whose image in $H^0(X, L_{(0,d)})$ belongs to $V_Z$ and vanishes at $P$ with order at least $i$.

Now, in our case of three irreducible components, consider the analogous situation:

given an invertible sheaf $L$ on $X$ of multidegree $(d, 0, 0)$, for each nonnegative multidegree $d := (i, d - i - l, l)$, let $L_d$ be the invertible sheaf on $X$ with restrictions $L|_{X_1}(-(d-i-l)A)$, $L|_{X_2}((d-i-l)A-lB)$, and $L|_{X_3}(lB)$, and let $V_{X_1}$, $V_{X_2}$, and $V_{X_3}$ be vector subspaces of $H^0(L_{(d,0,0)})$, $H^0(L_{(0,d,0)})$, and $H^0(L_{(0,0,d)})$, respectively, of dimension $r + 1$, such that $\{(L_{(d,0,0)}|_{X_1}, V_{X_1}|_{X_1})\}$ is a refined Eisenbud-Harris limit linear series. For each $i, l$, define $K_d$ as the natural generalization of (*).

More specifically:

For each nonnegative multidegree $d := (i, d - i - l, l)$, $K_d$ is the space of sections $s$ of $H^0(X, L_d)$ whose image in $H^0(X, L_{(d,0,0)})$ belongs to $V_{X_1}$ and vanishes at $A$ with order at least $d - i$, whose image in $H^0(X, L_{(0,d,0)})$ belongs to $V_{X_2}$ and vanishes at $A$ with order at least $i$ and vanishes at $B$ with order at least $l$, and whose image in $H^0(X, L_{(0,0,d)})$ belongs to $V_{X_3}$ and vanishes at $B$ with order at least $d - l$; see Subsection 3.1 for the precise definition of the spaces $K_d$.

The difference with the case of two irreducible components is the fact that in our case the spaces defined above does not necessarily have dimension $r + 1$.

In Section 4, we describe a method for the construction of all exact extensions of refined limit linear series. As a consequence, we get that every refined linear series has at least one exact extension. We use the method of Section 4 to understand when a refined limit linear series has a unique exact extension. Our Theorem 5.3 says that, for a refined limit linear series:

There is a unique exact extension if and only if

$$\dim K_d = r + 1 \text{ if } i + l \leq d, \quad b_{j-1} < i \leq b_j, \quad b_{k-1} < l \leq b_k \text{ and } j + k \leq r + 1,$$

where $b_0, \ldots, b_r$ are the orders of vanishing at $A$ and $b_0', \ldots, b_r'$ are the orders of vanishing at $B$, all orders correspond to the linear series on $X_2$. Moreover, this condition is also equivalent to the existence of a unique extension.

It follows from the exact sequence defining $K_d$ (see Subsection 3.1), that the condition in Theorem 5.3 is equivalent to the following condition:
\[ \dim V_{X_2}(-iA-lB) = r + 1 - j - k \]
if \( i + l \leq d, b_{j-1} < i \leq b_j, b'_{k-1} < l \leq b'_k \) and \( j + k \leq r + 1 \). This condition means exactly that our refined limit linear series is chain adaptable.

For compact type curves with two irreducible components, given an exact limit linear series, (Esteves and Osserman 2013) associated a closed subscheme of the fiber of the corresponding Abel map. In a subsequent work, for compact type curves with three irreducible components, we will give a description of this closed subscheme when the underlying exact limit linear series is the unique extension of a refined limit linear series.

Our techniques can be generalized to the case of compact type curves with an arbitrary number of irreducible components (work in progress).

## 2 Preliminaries

2.1. (Limit linear series) Throughout this article, \( X \) will denote the union of three smooth curves \( X_1, X_2 \) and \( X_3 \), such that \( X_1 \) and \( X_2 \) meet transversally at a point \( A \), and \( X_2 \) and \( X_3 \) meet transversally at a point \( B \), with \( A \neq B \). If \( Y \) is a reduced union of some components of \( X \), we get the following exact sequence

\[ 0 \to \mathcal{L}|_{Y'}(-Y \cap Y') \to \mathcal{L} \to \mathcal{L}|_Y \to 0, \]

for any invertible sheaf \( \mathcal{L} \) on \( X \). Let \( \mathcal{L} \) be an invertible sheaf on \( X \) of degree \( d \) on \( X_1 \) and degree 0 on \( X_2 \) and \( X_3 \). For any \( i \geq 0 \) and \( l \geq 0 \) such that \( i + l \leq d \), let \( \mathcal{L}_{(i,d-i-l,l)} \) be the invertible sheaf on \( X \) with restrictions \( \mathcal{L}|_{X_1}(-(d-i)A), \mathcal{L}|_{X_2}((d-i)A-lB) \) and \( \mathcal{L}|_{X_3}(lB) \). Note that \( \mathcal{L}_{(i,d-i-l,l)} \) has multidegree \((i,d-i-l,l)\). For any \( i \geq 0 \) and \( l \geq 0 \) such that \( i + l \leq d \), let \( \bar{d} := (i, d-i-l, l) \) and set

\[ \bar{d} := \begin{cases} 
(i - 1, d - i - l + 1, l) & \text{if } q = 1, \\
(i + 1, d - i - l - 2, l + 1) & \text{if } q = 2, \\
(i, d - i - l + 1, l - 1) & \text{if } q = 3.
\end{cases} \]

Whenever \( \bar{d} \geq 0 \), there are natural maps

\[ \varphi_{\bar{d}d} : \mathcal{L}_{\bar{d}} \to \mathcal{L}_{\bar{d}}|_{X_{\bar{q}}} = \mathcal{L}_{\bar{d}}|_{X_{\bar{q}}}(-X_{\bar{q}} \cap X_{\bar{q}'} \cap X_{\bar{q}}) \hookrightarrow \mathcal{L}_{\bar{d}}, \]

\[ \varphi_{\bar{d}d} : \mathcal{L}_{\bar{d}} \to \mathcal{L}_{\bar{d}}|_{X_{\bar{q}}} = \mathcal{L}_{\bar{d}}|_{X_{\bar{q}}}(-X_{\bar{q}} \cap X_{\bar{q}}') \hookrightarrow \mathcal{L}_{\bar{d}}, \]

where the first map in each composition is the restriction map and the last maps are the natural inclusions. Note that the compositions \( \varphi_{\bar{d}d}\varphi_{\bar{d}d} \) and \( \varphi_{\bar{d}d}\varphi_{\bar{d}d} \) are zero.

If \( Y \) is a subcurve of \( X \), for any \( \bar{d} \) and for any subspace \( V \subseteq H^0(X, \mathcal{L}) \), we denote by \( V^Y \) the subspace of \( V \) of sections that vanish on \( Y \). If \( Y \) is an irreducible component of \( X \), we denote by \( \mathcal{L}_Y \) the invertible sheaf \( \mathcal{L}_{\bar{d}} \) where the component of \( \bar{d} \) corresponding to \( Y \) is equal to \( d \) and the other components of \( \bar{d} \) are 0. Also, to ease notation, let \( \mathcal{L}_{\bar{d}} := \mathcal{L}_{(i,d-i-l,l)}. \)

Fix integers \( d \) and \( r \). A limit linear series on \( X \) of degree \( d \) and dimension \( r \) is a collection consisting of an invertible sheaf \( \mathcal{L} \) on \( X \) of degree \( d \) on \( X_1 \) and degree 0 on \( X_2 \) and \( X_3 \),
and vector subspaces $V_d \subseteq H^0(X, \mathcal{L}_d)$ of dimension $r + 1$, for each $d := (i, d - i - l, l) \geq 0$, such that $\varphi_{d, \bar{d}}(V_d) \subseteq V_{\bar{d}}$ and $\varphi_{d, \bar{d}}(V_{\bar{d}}) \subseteq V_d$ for each $d \geq \bar{d}$, whenever $d \geq 0$.

Given a limit linear series, if $Y$ is an irreducible component of $X$, we denote by $V_Y$ the corresponding subspace of $H^0(X, \mathcal{L}_Y)$. Also, we denote by $V_{\bar{d}}$ the corresponding subspace of $H^0(X, \mathcal{L}_{\bar{d}})$.

The conditions $\varphi_{d, \bar{d}}(V_d) \subseteq V_{\bar{d}}$ and $\varphi_{d, \bar{d}}(V_{\bar{d}}) \subseteq V_d$ are called the linking condition, and we say that $V_d$ and $V_{\bar{d}}$ are linked by the maps $\varphi_{d, \bar{d}}$ and $\varphi_{d, \bar{d}}$.

Note that $\varphi_{d, \bar{d}} : V_d \to V_{\bar{d}}$ has kernel $V_d^{X_q, 0}$ and image contained in $V_{\bar{d}}^{X_q, 0}$. Analogously, the map $\varphi_{d, \bar{d}} : V_{\bar{d}} \to V_d$ has kernel $V_{\bar{d}}^{X_q, 0}$ and image contained in $V_d^{X_q, 0}$.

A limit linear series $\{(\mathcal{L}_d, V_d)\}_{d}$ is called exact if

$$\text{Im} (\varphi_{d, \bar{d}} : V_d \to V_{\bar{d}}) = V_d^{X_q, 0} \text{ and } \text{Im} (\varphi_{d, \bar{d}} : V_{\bar{d}} \to V_d) = V_{\bar{d}}^{X_q, 0}$$

for each $d := (i, d - i - l, l) \geq 0$, whenever $d \geq 0$.

**Remark 2.2.** Note that, from the construction of the invertible sheaves $\mathcal{L}_d$ and the maps $\varphi_{d, \bar{d}}$, we have that, for $\bar{d} \neq q$, $s \in H^0(\mathcal{L}_d)$ vanishes on $X_{\bar{d}}$ if and only if $\varphi_{d, \bar{d}}(s) \in H^0(\mathcal{L}_{\bar{d}})$ vanishes on $X_q$. In particular, given a limit linear series $\{(\mathcal{L}_d, V_d)\}_{d}$, we get natural inclusions, for $\bar{d} \neq q$:

$$V_d/V_d^{X_q, 0} \hookrightarrow V_{\bar{d}}/V_{\bar{d}}^{X_q, 0}.$$ 

Also, if $s \in H^0(\mathcal{L}_d)$ and $\varphi_{d, \bar{d}}(s) \in H^0(\mathcal{L}_{\bar{d}})$ vanishes on $X_q$, then $\varphi_{d, \bar{d}}(s)$ vanishes on $X_q \cup X_{\bar{d}} = X$, as $\varphi_{d, \bar{d}}(H^0(\mathcal{L}_d))$ is contained in the kernel of the map $H^0(\mathcal{L}_d) \to H^0(\mathcal{L}_{\bar{d}}|_{X_{\bar{d}}})$, and hence $\varphi_{d, \bar{d}}(s) = 0$, which implies that $s$ vanishes on $X_q$. Thus, given a limit linear series $\{(\mathcal{L}_d, V_d)\}_{d}$ we get the natural inclusion

$$V_d/V_d^{X_q, 0} \hookrightarrow V_{\bar{d}}/V_{\bar{d}}^{X_q, 0}.$$ 

### 3 The kernel $K_{il}$

**3.1. (The kernel $K_{il}$)** Let $\mathcal{L}$ be an invertible sheaf on $X$ of degree $d$ on $X_1$ and degree 0 on $X_2$ and $X_3$. For any $i \geq 0$ and $l \geq 0$ such that $i + l \leq d$, recall that $\mathcal{L}_i$ denotes the invertible sheaf on $X$ with restrictions $\mathcal{L}|_{X_1} = -(d - i)A$, $\mathcal{L}|_{X_2} = -(d - i)A - lB$ and $\mathcal{L}|_{X_3} = lB$.

Note that

$$\mathcal{L}_i|_{X_1} = \mathcal{L}_{X_1}|_{X_1}(-(d - i)A), \mathcal{L}_i|_{X_2} = \mathcal{L}_{X_2}|_{X_2}(-(d - i)A - lB), \mathcal{L}_i|_{X_3} = \mathcal{L}_{X_3}|_{X_3}(-(d - l)B).$$

Then, we get the natural exact sequence:

$$0 \to H^0(\mathcal{L}_i) \to H^0(\mathcal{L}_{X_1}|_{X_1}(-(d - i)A)) \oplus H^0(\mathcal{L}_{X_2}|_{X_2}(-(d - i)A - lB))$$

$$\oplus H^0(\mathcal{L}_{X_3}|_{X_3}(-(d - l)B)) \to k \oplus k,$$

where the first summand in $k \oplus k$ corresponds to the point $A$ and the second summand corresponds to the point $B$. The last map in the exact sequence will be denoted $ev_{il}$. 

Let $V_{X_1}, V_{X_2}, V_{X_3}$ be $r + 1$-dimensional subspaces of $H^0(\mathcal{L}_{X_1}), H^0(\mathcal{L}_{X_2})$ and $H^0(\mathcal{L}_{X_3})$, respectively, such that they satisfy the linking condition. Assume that the associated Eisenbud-Harris limit linear series on $X$ is refined. Call $\mathfrak{h}$ this limit linear series. For any $i \geq 0$ and $l \geq 0$ such that $i + l \leq d$, we define $K_{il}$ by the exact sequence:

$$0 \to K_{il} \to V_{X_1}(-(d-i)A) \oplus V_{X_2}(-iA-lB) \oplus V_{X_3}(-(d-l)B) \to k \oplus k$$

Thus

$$K_{il} = (\alpha_{1d})^{-1}(V_{X_1}(-(d-i)A)) \cap (\alpha_{2d})^{-1}(V_{X_2}(-iA-lB)) \cap (\alpha_{3d})^{-1}(V_{X_3}(-(d-l)B))$$

where $d := (i, d - i - l, l)$, and the natural map

$$\alpha_{qd}: H^0(\mathcal{L}_d) \to H^0(\mathcal{L}_{X_q})$$

for each $q = 1, 2, 3$. Denote by $b_0, \ldots, b_r$ the orders of vanishing of $V_{X_3}$ at $A$, and denote by $b_0', \ldots, b_r'$ the orders of vanishing of $V_{X_2}$ at $B$. Also, let $a_0, \ldots, a_r$ denote the orders of vanishing of $V_{X_1}$ at $A$, and $c_0, \ldots, c_r$ the orders of vanishing of $V_{X_3}$ at $B$. Throughout this article, the data of this subsection will remain fixed.

**Remark 3.2.** Note that, if $\{(\mathcal{L}_{d}, V_{d})\}_d$ is a limit linear series which is an extension of $\mathfrak{h}$, then $V_{il} \subseteq K_{il}$ for any $i \geq 0$ and $l \geq 0$ such that $i + l \leq d$.

Indeed, let $d := (i, d - i - l, l)$. By the linking condition, we have that $\alpha_{1d}(V_{d}) \subseteq V_{X_1}$. Since $\text{Im}(\alpha_{1d}) \subseteq H^0(\mathcal{L}_{X_1}|_{X_1}(-(d-i)A))$, $\alpha_{1d}(V_{d}) \subseteq H^0(\mathcal{L}_{X_1}|_{X_1}(-(d-i)A))$, and hence $\alpha_{1d}(V_{d}) \subseteq V_{X_1}(-(d-i)A)$. Thus $V_{d} \subseteq (\alpha_{1d})^{-1}(V_{X_1}(-(d-i)A))$. Analogously, we have $V_{d} \subseteq (\alpha_{2d})^{-1}(V_{X_2}(-iA-lB))$ and $V_{d} \subseteq (\alpha_{3d})^{-1}(V_{X_3}(-(d-l)B))$. It follows that $V_{d} \subseteq K_{il}$.

**Remark 3.3.** By abuse of notation, denote the restriction of $ev^{il}$ to the vector subspace

$$V_{X_1}(-(d-i)A) \oplus V_{X_2}(-iA-lB) \oplus V_{X_3}(-(d-l)B)$$

by $ev^{il}$ as well. Notice that

1. $(1, 0) \in \text{Im}(ev^{il})$ if $V_{X_1}(-(d-i)A) \neq V_{X_1}(-(d-i+1)A)$ or $V_{X_2}(-iA-lB) \neq V_{X_2}(-(i+1)A-lB)$,

and

2. $\text{Im}(ev^{il}) \subseteq \{0\} \oplus k$ if $V_{X_1}(-(d-i)A) = V_{X_1}(-(d-i+1)A)$ and $V_{X_2}(-iA-lB) = V_{X_2}(-(i+1)A-lB)$.

Analogously, we have that

(i) $(0, 1) \in \text{Im}(ev^{il})$ if $V_{X_3}(-(d-l)B) \neq V_{X_3}(-(d-l+1)B)$ or $V_{X_2}(-iA-lB) \neq V_{X_2}(-iA-(l+1)B)$,

and

(ii) $\text{Im}(ev^{il}) \subseteq k \oplus \{0\}$ if $V_{X_3}(-(d-l)B) = V_{X_3}(-(d-l+1)B)$ and $V_{X_2}(-iA-lB) = V_{X_2}(-iA-(l+1)B)$.
Remark 3.4. Let $C$ be a smooth curve, $L$ an invertible sheaf on $C$ of degree $d$, and $V \subseteq H^0(L)$ a linear series. Let $r + 1 := \dim V$, and let $Q_1, Q_2 \in C$ distinct points. Let $e_1, \ldots, e_r$ be the orders of vanishing of $V$ at $Q_1$, and $e'_1, \ldots, e'_r$ the orders of vanishing of $V$ at $Q_2$. Then $e_j + e'_k \leq d$ if $j + k \leq r$. Furthermore, $\dim V(-e_jQ_1 - e'_kQ_2) \geq r + 1 - (j + k)$ for any $j, k$.

Indeed,

$$\dim V(-e_jQ_1 - e'_kQ_2) = \dim V(-e_jQ_1) + \dim V(e'_kQ_2) - \dim (V(-e_jQ_1) + V(e'_kQ_2)) \geq \dim V(-e_jQ_1) + \dim V(e'_kQ_2) - (r + 1) = (r + 1 - j) + (r + 1 - k) - (r + 1) = r + 1 - (j + k).$$

Thus, if $j + k \leq r$, then $\dim V(-e_jQ_1 - e'_kQ_2) \geq 1$, which implies $h^0(L(-e_jQ_1 - e'_kQ_2)) \geq 1$, and hence $\deg(L(-e_jQ_1 - e'_kQ_2)) \geq 0$, i.e. $e_j + e'_k \leq d$.

Proposition 3.5. The following statements hold:

1. $\dim K_{i,l} \geq r + 1$. Furthermore, $\dim K_{i,l} = r + 1$ if $i \leq b_0$ or $l \leq b'_0$.
2. The subspaces $K_{i,l} \subseteq H^0(Q_{i,l})$ satisfy the linking condition.

Proof. We will first prove that $\dim K_{i,l} \geq r + 1$. There are five cases to consider.

Case 1: If $i = b_j$ and $l = b'_{k}$ for some $j, k$.

Consider the exact sequence

$$0 \to K_{i,l} \to V_{X_1}(-(d-i)A) \oplus V_{X_2}(-iA-lB) \oplus V_{X_3}(-(d-l)B) \to k \oplus k$$

Since

$$V_{X_1}(-(d-i)A) \neq V_{X_1}(-(d-i + 1)A) \text{ and } V_{X_3}(-(d-l)B) \neq V_{X_3}(-(d-l+1)B),$$

as $d-i = d-b_j = a_{r-j}$ is an order of vanishing of $V_{X_1}$ at $A$ and $d-l = d-b'_k = c_{r-k}$ is an order of vanishing of $V_{X_3}$ at $B$, we have that $\text{Im}(ev^{i,l}) = k \oplus k$, by Remark 3.3 and hence

$$\dim K_{i,l} = \dim V_{X_1}(-(d-i)A) + \dim V_{X_2}(-iA-lB) + \dim V_{X_3}(-(d-l)B) - 2 = (r + 1 - (r - j)) + \dim V_{X_2}(-iA-lB) + (r + 1 - (r - k)) - 2 = j + k + \dim V_{X_2}(-iA-lB) \geq j + k + (r + 1 - (j + k)) = r + 1,$$

where in the last inequality we used Remark 3.4.

Case 2: If $b_{j-1} < i < b_j$ and $l = b'_{k}$ for some $j, k$.

Since

$$V_{X_1}(-(d-i)A) = V_{X_1}(-(d-i + 1)A), V_{X_2}(-iA-lB) = V_{X_2}(-(i+1)A-lB) \text{ and } V_{X_3}(-(d-l)B) \neq V_{X_3}(-(d-l+1)B),$$

as $a_{r-j} < d-i < a_{r+1-j}$ is not an order of vanishing of $V_{X_1}$ at $A$, $b_{j-1} < i < b_j$ is not an order of vanishing of $V_{X_2}$ at $A$ and $d-l = c_{r-k}$ is an order of vanishing of $V_{X_3}$ at $B$, we have that $\text{Im}(ev^{i,l}) = \{0\} \oplus k$, by Remark 3.3 and hence

$$\dim K_{i,l} = \dim V_{X_1}(-(d-i)A) + \dim V_{X_2}(-iA-lB) + \dim V_{X_3}(-(d-l)B) - 1 = (r + 1 - (r + 1 - j)) + \dim V_{X_2}(-iA-lB) + (r + 1 - (r - k)) - 1 = j + k + \dim V_{X_2}(-iA-lB) = j + k + \dim V_{X_2}(-b_jA-lB) \geq j + k + (r + 1 - (j + k)) = r + 1,$$
where in the last equality we used that $V_{X_1}(-iA) = V_{X_2}(-b_jA)$, as $b_{j-1} < i < b_j$, and in the last inequality we used Remark 3.4.

Case 3: If $i = b_j$ and $b_{k-1}' < l < b_k'$ for some $j, k$.

This case is analogous to Case 2.

Case 4: If $b_{j-1} < i < b_j$ and $b_{k-1}' < l < b_k'$ for some $j, k$.

Since

$$V_{X_1}(-(d - i)A) = V_{X_1}(-(d - i + 1)A), \quad V_{X_3}(-(iA - lB) = V_{X_3}(-(i + 1)A - lB),$$

$$V_{X_2}(-(iA - lB) = V_{X_2}(-(iA - (l + 1)B) \text{ and } V_{X_3}(-(d - l)B) = V_{X_3}(-(d - l + 1)B),$$

as $d - i$ is not an order of vanishing of $V_{X_1}$ at $A$, $i$ is not an order of vanishing of $V_{X_2}$ at $A$, $l$ is not an order of vanishing of $V_{X_2}$ at $B$ and $d - l$ is not an order of vanishing of $V_{X_3}$ at $B$, we have that $\text{Im}(ev^d) = \{0\} \oplus \{0\}$, by Remark 3.3 and hence

$$\dim K_{il} = \dim V_{X_1}(-(d - i)A) + \dim V_{X_2}(-(iA - lB) + \dim V_{X_3}(-(d - l)B)$$

$$= j + k + \dim V_{X_3}(-(iA - lB))$$

$$= j + k + \dim V_{X_2}(-b_jB) \geq j + k + (r + 1 - (j + k)) = r + 1,$$

where in the last inequality we used Remark 3.4 and in the last equality we used that $V_{X_2}(-(iA) = V_{X_2}(-b_jA)$ and $V_{X_2}(-(lB) = V_{X_2}(-b_k'B)$. 

Case 5: If $i < b_0$ or $i > b_r$ or $l < b_0'$ or $l > b_r'$.

We will only prove the stated inequality in the case $i < b_0$, as the other cases are analogous. Suppose $i < b_0$. Then $d - i > a_r$, and hence $V_{X_1}(-(d - i)A) = 0$. Also, we have $V_{X_2}(-(iA - lB) = V_{X_2}(-(i + 1)A - lB)$, as $i$ is not an order of vanishing of $V_{X_2}$ at $A$.

Suppose first that $l = b_k'$ for some $k$. Then $V_{X_3}(-(d - l)B) \neq V_{X_3}(-(d - l + 1)B)$, and it follows from the exact sequence defining $K_{il}$ that

$$\dim K_{il} = \dim V_{X_1}(-(d - i)A) + \dim V_{X_2}(-(iA - lB) + \dim V_{X_3}(-(d - l)B) - 1$$

$$= 0 + \dim V_{X_2}(-(iA - lB) + (r + 1 - (r - k)) - 1$$

$$= k + \dim V_{X_2}(-(iA - lB))$$

$$= k + \dim V_{X_2}(-(lB)) = k + (r + 1 - k) = r + 1.$$

We used above that $V_{X_2}(-(iA) = V_{X_2}$, as $i < b_0$.

Suppose $b_{k-1}' < l < b_k'$ for some $k$. Then $V_{X_3}(-(d - l)B) = V_{X_3}(-(d - l + 1)B)$ and $V_{X_2}(-(iA - lB) = V_{X_2}(-(iA - (l + 1)B)$, and hence

$$\dim K_{il} = \dim V_{X_1}(-(d - i)A) + \dim V_{X_2}(-(iA - lB) + \dim V_{X_3}(-(d - l)B)$$

$$= k + \dim V_{X_2}(-(iA - lB))$$

$$= k + \dim V_{X_2}(-(lB)) = k + (r + 1 - k) = r + 1.$$

Now, suppose $l < b_0'$. Then $V_{X_3}(-(iA - lB) = V_{X_3}(-(iA - (l + 1)B)$. On the other hand, $d - l > c_r$, and hence $V_{X_3}(-(d - l)B) = 0$. It follows that

$$\dim K_{il} = \dim V_{X_1}(-(d - i)A) + \dim V_{X_2}(-(iA - lB) + \dim V_{X_3}(-(d - l)B)$$

$$= \dim V_{X_2}(-(iA - lB))$$

$$= \dim V_{X_2} = r + 1.$$

We used above that $V_{X_2}(-(iA) = V_{X_2}$ and $V_{X_2}(-(lB) = V_{X_2}$.

Finally, suppose $l > b_r'$. Then
\[ V_{X_2}(-iA - lB) = V_{X_3}(-iA - (l + 1)B) \] and \[ V_{X_3}(-(d - l)B) = V_{X_3}(-(d - l + 1)B). \]

On the other hand, since \( l > b'_{r} \), we have \( d - l < c_{0} \), and hence \( V_{X_3}(-(d - l)B) = V_{X_3} \).

Then

\[
\dim K_{il} = \dim V_{X_1}(-(d - i)A) + \dim V_{X_2}(-iA - lB) + \dim V_{X_3}(-(d - l)B) \\
= 0 + 0 + (r + 1) = r + 1,
\]

We used above that \( V_{X_2}(-lB) = 0 \), as \( l > b'_{r} \). This finishes the proof of the stated inequality.

Now, we will prove that \( \dim K_{il} = r + 1 \) if \( i \leq b_{0} \) or \( l \leq b_{0} \). We will only prove the stated equality in the case \( i \leq b_{0} \), as the other case is analogous. Notice that, in Case 5, we saw \( \dim K_{il} = r + 1 \) if \( i < b_{0} \). Thus, it remains to show the stated equality in the case \( i = b_{0} \). Assume \( i = b_{0} \).

Suppose first that \( l = b'_{l} \) for some \( k \). Notice that, in Case 1, for \( j = 0 \), the equality holds in \( \dim V_{X_2}(-(iA - lB) \geq r + 1 - (j + k) \), as \( V_{X_2}(-(iA - A)B) = V_{X_2}(-iB) \). Thus \( \dim K_{il} = r + 1 \).

An analogous reasoning works for the case \( b'_{k-1} < l < b'_{k} \). Now, suppose \( l < b'_{0} \). In Case 5 we saw \( \dim K_{il} = r + 1 \) if \( i < b_{0} \). Analogously, we can show that \( \dim K_{il} = r + 1 \) if \( l < b'_{0} \).

Finally, suppose that \( l > b'_{r} \). Then

\[ V_{X_3}(-(d - l)B) = V_{X_3}(-(d - l + 1)B), \ V_{X_2}(-(iA - lB)) = 0 \] and \( V_{X_3}(-(d - l)B) = V_{X_3} \).

On the other hand, since \( i = b_{0} \), we have \( d - i = a_{r} \). It follows that

\[
\dim K_{il} = \dim V_{X_1}(-(d - i)A) + \dim V_{X_2}(-iA - lB) + \dim V_{X_3}(-(d - l)B) - 1 \\
= (r + 1 - r) + 0 + (r + 1) - 1 = r + 1.
\]

This finishes the proof of the stated equality.

Now, we will prove the statement 2 of the proposition. Keep the notation of multidegrees \( \underline{d} \) and \( \underline{d} \) used in Section 2. We will only prove the linking condition for \( q = 1 \), as the proofs for \( q = 2, 3 \) are analogous. We will first prove that \( \varphi_{\underline{d},\underline{d}}(K_{il}) \subseteq K_{i-1,l} \). (Recall that, for \( q = 1, \underline{d} = (i - 1, d - i - l + 1, l) \).) Let \( s \in K_{il} \). We have

\[
(\alpha_{\underline{d},\underline{d}} \circ \varphi_{\underline{d},\underline{d}})(s) = (\alpha_{\underline{d},\underline{d}} \circ \varphi_{\underline{d},\underline{d}} \circ \varphi_{\underline{d},\underline{d}})(s) = (\alpha_{\underline{d},\underline{d}} \circ 0)(s) = 0 \in V_{X_1}(-(d - i + 1)A). \tag{1}
\]

On the other hand, \( s \in (\alpha_{\underline{d},\underline{d}})^{-1}(V_{X_2}(-(iA - lB))) \subseteq (\alpha_{\underline{d},\underline{d}})^{-1}(V_{X_2}(-(i - 1)A - lB)) \), as \( s \in K_{il} \). Then

\[
(\alpha_{\underline{d},\underline{d}} \circ \varphi_{\underline{d},\underline{d}})(s) = \alpha_{\underline{d},\underline{d}}(s) \in V_{X_1}(-(i - 1)A - lB). \tag{2}
\]

Also, since \( s \in K_{il} \), \( s \in (\alpha_{\underline{d},\underline{d}})^{-1}(V_{X_3}(-(d - l)B)) \), and hence

\[
(\alpha_{\underline{d},\underline{d}} \circ \varphi_{\underline{d},\underline{d}})(s) = \alpha_{\underline{d},\underline{d}}(s) \in V_{X_3}(-(d - l)B). \tag{3}
\]

It follows from (1), (2) and (3) that \( \varphi_{\underline{d},\underline{d}}(s) \in K_{i-1,l} \). This proves that \( \varphi_{\underline{d},\underline{d}}(K_{il}) \subseteq K_{i-1,l} \).

Now, we will prove that \( \varphi_{\underline{d},\underline{d}}(K_{i-1,l}) \subseteq K_{il} \). Let \( s \in K_{i-1,l} \). Then
\[ s \in (\alpha_{\overline{d}l})^{-1}(V_{X_1}(-(d-i+1)A)) \subseteq (\alpha_{\overline{d}l})^{-1}(V_{X_1}(-(d-i)A)), \]
and hence
\[ (\alpha_{\overline{d}l} \circ \varphi_{\overline{d}d})(s) = \alpha_{\overline{d}l}(s) \in V_{X_1}(-(d-i)A). \quad (4) \]
On the other hand,
\[ (\alpha_{\overline{d}l} \circ \varphi_{\overline{d}d})(s) = (\alpha_{\overline{g}l} \circ \varphi_{\overline{g}d} \circ \varphi_{\overline{d}d})(s) = (\alpha_{\overline{g}l} \circ \varphi_{\overline{d}d})(s) = \varphi_{\overline{d}d}(0)(s) = 0 \in V_{X_2}(-iA-lB), \quad (5) \]
and analogously
\[ (\alpha_{\overline{d}l} \circ \varphi_{\overline{d}d})(s) = (\alpha_{\overline{d}l} \circ \varphi_{\overline{d}d} \circ \varphi_{\overline{d}d})(s) = (\alpha_{\overline{d}l} \circ \varphi_{\overline{d}d})(s) = \varphi_{\overline{d}d}(0)(s) = 0 \in V_{X_3}(-(d-l)B). \quad (6) \]
It follows from (3), (5) and (6) that \( \varphi_{\overline{d}d}(s) \in K_{il} \). This proves that \( \varphi_{\overline{d}d}(K_{i-1,l}) \subseteq K_{il} \), which finishes the proof of the proposition.

**Proposition 3.6.** The following statements hold:

1. For any \( i \geq 1 \) and \( l \geq 1 \) such that \( i+l \leq d \),
\[ \varphi_{\overline{dd}'}(K_{il}) = K_{X_{i-1,l-1}^l}, \]
where \( \overline{d} := (i, d-i-l, l) \) and \( \overline{d}^\prime := (i-1, d-i-l+2, l-1) \).

2. For any \( i \geq 1 \) and \( l \geq 0 \) such that \( i+l \leq d \),
\[ \varphi_{\overline{dd}''}(K_{il}) = K_{X_{i-1,l}^l}, \]
where \( \overline{d} := (i, d-i-l, l) \) and \( \overline{d}'' := (i-1, d-i-l+1, l) \).

3. For any \( i \geq 0 \) and \( l \geq 1 \) such that \( i+l \leq d \),
\[ \varphi_{\overline{dd}'}(K_{il}) = K_{X_{i-1,l-1}^l}, \]
where \( \overline{d} := (i, d-i-l, l) \) and \( \overline{d} := (i, d-i-l+1, l-1) \).

**Proof.** We will first see how the statements 2 and 3 imply the statement 1. Let \( i \geq 1 \) and \( l \geq 1 \) such that \( i+l \leq d \). Let \( s' \in K_{X_{i-1,l-1}^l} \). Then \( s' \in K_{X_{i-1,l-1}^l} \). But, by the statement 3 of the proposition, \( \varphi_{\overline{dd}''}(K_{i-1,l}) = K_{X_{i-1,l-1}^l} \), so \( s'' = \varphi_{\overline{dd}''}(s') \) for some \( s'' \in K_{i-1,l} \). As \( \varphi_{\overline{dd}''}(s'') = s' \in K_{X_{i-1,l-1}^l} \subseteq K_{X_{i-1,l-1}^l} \), it follows from Remark 2.2 that \( s'' \in K_{i-1,l} \). Then by the statement 2 of the proposition, \( s'' = \varphi_{\overline{dd}''}(s) \) for some \( s \in K_{il} \). Thus
\[ s' = \varphi_{\overline{dd}''}(s'') = \varphi_{\overline{dd}''} \circ \varphi_{\overline{dd}'}(s) = \varphi_{\overline{dd}'}(s) \in \varphi_{\overline{dd}'}(K_{il}). \]
This proves that \( K_{X_{i-1,l-1}^l} \subseteq \varphi_{\overline{dd}'}(K_{il}) \). But, it follows from Proposition 3.5, item 2, that \( \varphi_{\overline{dd}'}(K_{il}) \subseteq K_{X_{i-1,l-1}^l} \), so \( \varphi_{\overline{dd}'}(K_{il}) = K_{X_{i-1,l-1}^l} \).

It remains to show the statements 2 and 3. We will only prove the statement 2, as the statement 3 is analogous.

By abuse of notation, we denote the restriction of \( ev_{il} \) to the vector subspace
\[ V_{X_1}(-(d-i)A) \oplus V_{X_2}(-iA-lB) \oplus V_{X_3}(-(d-l)B) \]
by $ev^d$ as well. It follows from the exact sequence defining $K_d$ that
\[ \dim K_d = \dim V_{X_1}(-(d-i)A) + \dim V_{X_2}(-iA - lB) + \dim V_{X_3}(-(d-l)B) - \dim \text{Im}(ev^d). \]
On the other hand, the exact sequence defining $K_d$ induces the following exact sequence
\[ 0 \to K_d^{X_i,0} \to V_{X_1}(-(d-i)A) \oplus \{0\} \oplus \{0\} \to k \oplus k \]
Then $K_d^{X_i,0} \cong V_{X_1}(-(d-i+1)A)$, so $\dim \varphi_d^\prime(K_d) = \dim K_d - \dim V_{X_1}(-(d-i+1)A)$. Thus
\[ \dim \varphi_d^\prime(K_d) = \dim V_{X_1}(-(d-i)A) + \dim V_{X_2}(-iA - lB) + \dim V_{X_3}(-(d-l)B) \]
\[ - \dim \text{Im}(ev^d) - \dim V_{X_1}(-(d-i+1)A). \quad (7) \]
On the other hand, the exact sequence
\[ 0 \to K_{i-1,l}^{X_i,0} \to \{0\} \oplus V_{X_2}(-(i-1)A - lB) \oplus V_{X_3}(-(d-l)B) \to k \oplus k \]
implies
\[ \dim K_{i-1,l}^{X_i,0} = \dim V_{X_2}(-(i-1)A - lB) + \dim V_{X_3}(-(d-l)B) - \dim \text{Im}(ev_i^{l-1}) \]
where $ev_i^{l-1}$ is the restriction of $ev^d_{i-1,l}$ to $\{0\} \oplus V_{X_2}(-(i-1)A - lB) \oplus V_{X_3}(-(d-l)B)$. But, it follows from Proposition 3.7 item 2, that $\varphi_d^{\prime\prime}(K_d) \subseteq K_{i-1,l}^{X_i,0}$, so from (7) and (8), we have that $\varphi_d^{\prime\prime}(K_d) = K_{i-1,l}^{X_i,0}$ if and only if
\[ \dim \text{Im}(ev_i^{l-1}) - (\dim V_{X_2}(-(i-1)A - lB) - \dim V_{X_2}(-iA - lB)) \]
\[ = \dim \text{Im}(ev^d) - (\dim V_{X_1}(-(d-i)A) - \dim V_{X_1}(-(d-i+1)A)) \quad (9) \]
By checking cases $i = b_j$, $i \neq b_j$, $l = b_k'$ and $l \neq b_k'$, we see that both sides of (9) are equal to $\dim V_{X_3}(-(d-l)B) - \dim V_{X_3}(-(d-l+1)B)$. Thus (9) is true, and hence $\varphi_d^{\prime\prime}(K_d) = K_{i-1,l}^{X_i,0}$, proving the statement 2 of the proposition. This finishes the proof of the proposition.

**Proposition 3.7.** The following statements hold:

1. For any $i \geq 1$ and $l \geq 0$ such that $i+l \leq d$, the following statements are equivalent:
   (i) $\varphi_d^{\prime\prime}(K_{i+1,l}) \neq K_{i,l}^{X_i,0}$, where $\underline{d} := (i, d - i - l, l)$ and $\underline{d}'' := (i-1, d - i - l + 1, l)$.
   (ii) $i-1$ is an order of vanishing of $V_{X_2}$ at $A$ and $i-1$ is not an order of vanishing of $V_{X_3}(-(lB))$ at $A$.

2. For any $i \geq 0$ and $l \geq 1$ such that $i+l \leq d$, the following statements are equivalent:
   (i) $\varphi_d^{\prime\prime}(K_{i,l-1}) \neq K_{i,l}^{X_i,0}$, where $\underline{d} := (i, d - i - l, l)$ and $\underline{d} := (i, d - i - l + 1, l - 1)$.
   (ii) $l-1$ is an order of vanishing of $V_{X_2}$ at $B$ and $l-1$ is not an order of vanishing of $V_{X_2}(-iA)$ at $B$.

**Proof.** We will only prove the statement 1. (The statement 2 is analogous.) Suppose first that (i) holds. The exact sequence
\[ 0 \to K_d^{X_i,0} \to V_{X_1}(-(d-i)A) \oplus \{0\} \oplus \{0\} \to k \oplus k \]
implies that $\alpha_{1d}|_{K_{il}^{X_{il},0}} : K_{il}^{X_{il},0} \to V_{X_{i}}(-d - i + 1)A)$ is an isomorphism. On the other hand, it follows from Proposition 3.3 item 2, that $\varphi_{d''}^{i}(K_{i-1,l}) \subseteq K_{il}^{X_{il},0}$. Then, we have that $\varphi_{d''}^{i}(K_{i-1,l}) = K_{il}^{X_{il},0}$ if and only if $\alpha_{1d}(\varphi_{d''}^{i}(K_{i-1,l})) = \alpha_{1d}(K_{il}^{X_{il},0})$, i.e., if and only if $\alpha_{1d''}(K_{i-1,l}) = V_{X_{i}}(-(d - i + 1)A)$. By hypothesis, $\varphi_{d''}^{i}(K_{i-1,l}) \neq K_{il}^{X_{il},0}$, so $\alpha_{1d''}(K_{i-1,l})$ is a proper subspace of $V_{X_{i}}(-(d - i + 1)A)$. The exact sequence defining $K_{i-1,l}$ induces the following exact sequence

$$0 \to K_{i-1,l} \to \alpha_{1d''}(K_{i-1,l}) \oplus V_{X_{2}}(-(i - 1)A - lB) \oplus V_{X_{3}}(-(d - l)B) \to k \oplus k$$

By abuse of notation, we denote the restriction of $ev^{i-1,l}$ to the vector subspace

$$V_{X_{1}}(-(d - i + 1)A) \oplus V_{X_{2}}(-(i - 1)A - lB) \oplus V_{X_{3}}(-(d - l)B)$$

by $ev^{i-1,l}$ as well, and let $\overline{ev}^{i-1,l}$ be the restriction of $ev^{i-1,l}$ to the vector subspace $\alpha_{1d''}(K_{i-1,l}) \oplus V_{X_{2}}(-(i - 1)A - lB) \oplus V_{X_{3}}(-(d - l)B)$. We have

$$\dim K_{i-1,l} = \dim V_{X_{1}}(-(d - i + 1)A) + \dim V_{X_{2}}(-(i - 1)A - lB) + \dim V_{X_{3}}(-(d - l)B) - \dim \text{Im}(ev^{i-1,l})$$

and also

$$\dim K_{i-1,l} = \dim \alpha_{1d''}(K_{i-1,l}) + \dim V_{X_{2}}(-(i - 1)A - lB) + \dim V_{X_{3}}(-(d - l)B) - \dim \text{Im}(\overline{ev}^{i-1,l}).$$

Therefore

$$\dim V_{X_{1}}(-(d - i + 1)A) - \dim \alpha_{1d''}(K_{i-1,l}) = \dim \text{Im}(ev^{i-1,l}) - \dim \text{Im}(\overline{ev}^{i-1,l}), \quad (10)$$

and since $\dim \text{Im}(ev^{i-1,l}) - \dim \text{Im}(\overline{ev}^{i-1,l}) \leq \dim \text{Im}(ev^{i-1,l}) \leq 2$, it follows that

$$\dim V_{X_{1}}(-(d - i + 1)A) - 2 \leq \dim \alpha_{1d''}(K_{i-1,l}) \leq \dim V_{X_{1}}(-(d - i + 1)A) - 1,$$

as $\alpha_{1d''}(K_{i-1,l})$ is a proper subspace of $V_{X_{1}}(-(d - i + 1)A)$. Thus, there are two cases to consider.

**Case 1:** If $\dim \alpha_{1d''}(K_{i-1,l}) = \dim V_{X_{1}}(-(d - i + 1)A) - 1$.

It follows from (10) that $\dim \text{Im}(\overline{ev}^{i-1,l}) = \dim \text{Im}(ev^{i-1,l}) - 1$. We will first prove that $i - 1 = b_{j}$ for some $j$. Suppose by contradiction that $i - 1$ is not an order of vanishing of $V_{X_{2}}$ at $A$. Then $\dim \text{Im}(ev^{i-1,l}) \leq 1$, and hence $\dim \text{Im}(\overline{ev}^{i-1,l}) \leq 0$. So $\dim \text{Im}(\overline{ev}^{i-1,l}) = 0$ and $\dim \text{Im}(ev^{i-1,l}) = 1$. Now, since $i - 1$ is not an order of vanishing of $V_{X_{2}}$ at $A$, $\dim \text{Im}(ev^{i-1,l}) = 1$ implies that $l = b'_{k}$ for some $k$, which implies that $\text{Im}(\overline{ev}^{i-1,l}) \supseteq \{0\} \oplus k$, and hence $\dim \text{Im}(\overline{ev}^{i-1,l}) \geq 1$, a contradiction. Thus $i - 1 = b_{j}$ for some $j$.

Now, we will prove that $i - 1$ is not an order of vanishing of $V_{X_{2}}(-lB)$ at $A$. Suppose first that $l = b'_{k}$ for some $k$. Since $i - 1 = b_{j}$ and $l = b'_{k}$, we have $\dim \text{Im}(ev^{i-1,l}) = 2$. Then $\dim \text{Im}(\overline{ev}^{i-1,l}) = 1$. Since $l = b'_{k}$, $\text{Im}(\overline{ev}^{i-1,l}) \supseteq \{0\} \oplus k$, and it follows from dimension considerations that $\text{Im}(\overline{ev}^{i-1,l}) = \{0\} \oplus k$. This implies that all sections of $\alpha_{1d''}(K_{i-1,l}) \subseteq V_{X_{1}}(-(d - i + 1)A)$ and all sections of $V_{X_{2}}(-(i - 1)A - lB)$ vanish at $A$, i.e.,
\( \alpha_{1g''}(K_{i-1,l}) \subseteq V_{X_1}(-(d-i+2)A) \) and \( V_{X_2}(-(i-1)A-lB) = V_{X_2}(-iA-lB) \).

Thus \( i-1 \) is not an order of vanishing of \( V_{X_2}(-lB) \) at \( A \). In addition, since \( \alpha_{1g''}(K_{i-1,l}) \) is contained in \( V_{X_1}(-(d-i+2)A) \), it follows from dimension considerations that \( \alpha_{1g''}(K_{i-1,l}) \) is equal to \( V_{X_1}(-(d-i+2)A) \).

Now, assume \( l \) is not an order of vanishing of \( V_{X_2} \) at \( B \). Then \( \dim \text{Im}(ev_i^{i-1,l}) = 1 \), as \( i-1 = b_j \), and hence \( \text{Im}(ev_i^{i-1,l}) = \{0\} \oplus \{0\} \). This implies that all sections of \( \alpha_{1g''}(K_{i-1,l}) \subseteq V_{X_1}(-(d-i+1)A) \) and all sections of \( V_{X_2}(-(i-1)A-lB) \) vanish at \( A \). It follows that \( \alpha_{1g''}(K_{i-1,l}) = V_{X_1}(-(d-i+2)A) \) and \( i-1 \) is not an order of vanishing of \( V_{X_2}(-lB) \) at \( A \).

Case 2: If \( \dim \alpha_{1g''}(K_{i-1,l}) = \dim V_{X_1}(-(d-i+1)A) - 2 \).

It follows from (10) that \( \dim \text{Im}(ev_i^{i-1,l}) = \dim \text{Im}(ev_i^{i-1,l}) - 2 \). As \( \dim \text{im}(ev_i^{i-1,l}) \leq 2 \), we have \( \text{im}(ev_i^{i-1,l}) = \{0\} \oplus \{0\} \) and \( \dim \text{im}(ev_i^{i-1,l}) = 2 \). Since \( \dim \text{im}(ev_i^{i-1,l}) = 2 \), it follows that \( i-1 = b_j \) and \( l = b_k^j \) for some \( j, k \). Now, as \( l = b_k^j \), we get \( \dim \text{im}(ev_i^{i-1,l}) \geq 1 \), a contradiction. Thus, the only case can happen is Case 1, and hence (ii) holds.

Suppose now that (ii) holds. Define \( K' \subseteq H^0(\mathcal{L}_{g''}) \) by the exact sequence

\[
0 \to K' \to V_{X_1}(-(d-i+2)A) \oplus V_{X_2}(-(i-1)A-lB) \oplus V_{X_3}(-(d-l)B) \to k \oplus k.
\]

By abuse of notation, we denote the restriction of \( ev_i^{i-1,l} \) to the vector subspace

\[
V_{X_1}(-(d-i+1)A) \oplus V_{X_2}(-(i-1)A-lB) \oplus V_{X_3}(-(d-l)B)
\]

by \( ev_i^{i-1,l} \) as well, and let \( \tilde{ev_i}^{i-1,l} \) be the restriction of \( ev_i^{i-1,l} \) to the vector subspace \( V_{X_1}(-(d-i+2)A) \oplus V_{X_2}(-(i-1)A-lB) \oplus V_{X_3}(-(d-l)B) \). We have that \( K' \subseteq K_{i-1,l} \), as \( V_{X_1}(-(d-i+2)A) \subseteq V_{X_1}(-(d-i+1)A) \). On the other hand, it follows from the definition of \( K' \) that \( \alpha_{1g''}(K') \subseteq V_{X_1}(-(d-i+2)A) \), and hence \( \alpha_{1g''}(K') \neq V_{X_1}(-(d-i+1)A) \).

Now, recall that \( \varphi_{g''}(K_{i-1,l}) = K_{i-1,l}^{X_1,0} \) if and only if \( \alpha_{1g''}(K_{i-1,l}) = V_{X_1}(-(d-i+1)A) \). Therefore, to prove (i), we need only show that \( \alpha_{1g''}(K_{i-1,l}) \neq V_{X_1}(-(d-i+1)A) \). For this, it suffices to show that \( K_{i-1,l} = K' \).

Since \( K' \subseteq K_{i-1,l} \), we need only prove that \( \dim K' = \dim K_{i-1,l} \). We have

\[
\dim K_{i-1,l} = \dim V_{X_1}(-(d-i+1)A) + \dim V_{X_2}(-(i-1)A-lB) + \dim V_{X_3}(-(d-l)B) - \dim \text{im}(ev_i^{i-1,l})
\]

and also

\[
\dim K' = \dim V_{X_1}(-(d-i+2)A) + \dim V_{X_2}(-(i-1)A-lB) + \dim V_{X_3}(-(d-l)B) - \dim \text{im}(\tilde{ev_i}^{i-1,l}).
\]

Therefore \( \dim K' = \dim K_{i-1,l} \) if and only if

\[
\dim V_{X_1}(-(d-i+1)A) - \dim V_{X_1}(-(d-i+2)A) = \dim \text{im}(ev_i^{i-1,l}) - \dim \text{im}(\tilde{ev_i}^{i-1,l}),
\]

i.e., if and only if \( \dim \text{im}(\tilde{ev_i}^{i-1,l}) = \dim \text{im}(ev_i^{i-1,l}) - 1 \), as \( i-1 = b_j \) for some \( j \). There are two cases to consider.

Case 1: If \( l = b_k^j \) for some \( k \).
Since \( i - 1 = b_j \) and \( l = b'_k \), \( \dim(\ev^{i-1,l}) = 2 \). On the other hand, since \( i - 1 \) is not an order of vanishing of \( V_{X_2}(-lB) \) at \( A \), we have \( V_{X_2}(-(i-1)A-lB) = V_{X_2}(-iA-lB) \), i.e., all sections of \( V_{X_2}(-(i-1)A-lB) \) vanish at \( A \). Then \( \Im(\tilde{\ev}^{i-1,l}) \subseteq \{0\} \oplus k \). But, since \( l = b'_k, \Im(\tilde{\ev}^{i-1,l}) \supseteq \{0\} \oplus k \), and hence \( \Im(\tilde{\ev}^{i-1,l}) = \{0\} \oplus k \). Therefore \( \dim \Im(\tilde{\ev}^{i-1,l}) = 1 \), and thus \( \dim \Im(\tilde{\ev}^{i-1,l}) = \dim \Im(\ev^{i-1,l}) - 1 \).

**Case II:** If \( l \) is not an order of vanishing of \( V_{X_2} \) at \( B \).

Since \( i - 1 = b_j \), \( \dim(\ev^{i-1,l}) = 1 \). As in Case I, \( \Im(\tilde{\ev}^{i-1,l}) \subseteq \{0\} \oplus k \). But, since \( l \) is not an order of vanishing of \( V_{X_2} \) at \( B \), \( \Im(\tilde{\ev}^{i-1,l}) \subseteq k \oplus \{0\} \), and hence we have \( \Im(\tilde{\ev}^{i-1,l}) = \{0\} \oplus \{0\} \). So \( \dim \Im(\tilde{\ev}^{i-1,l}) = 0 = \dim \Im(\ev^{i-1,l}) - 1 \). This finishes the proof of the proposition. \( \square \)

**Remark 3.8.** Let \( V_1, V_2 \) and \( V_3 \) vector subspaces of a \( N \)-dimensional vector space \( V \). We will say that \( V_1 \) **distributes over** \( V_2 \) and \( V_3 \) if \( V_1 \cap (V_2 + V_3) = V_1 \cap V_2 + V_1 \cap V_3 \). Note that this notion is symmetric on \( V_1, V_2 \) and \( V_3 \).

Indeed, since \( V_1 \cap V_2 + V_1 \cap V_3 \subseteq V_1 \cap (V_2 + V_3) \), we have that \( V_1 \cap (V_2 + V_3) = V_1 \cap V_2 + V_1 \cap V_3 \) is equivalent to \( \dim V_1 \cap (V_2 + V_3) = \dim (V_1 \cap V_2 + V_1 \cap V_3) \). We have

\[
\dim V_1 \cap (V_2 + V_3) = \dim (V_1 \cap V_2 + V_1 \cap V_3) \text{ if and only if }
\]

\[
\dim V_1 + \dim (V_2 + V_3) - \dim (V_1 + V_2 + V_3) = \dim V_1 \cap V_2 + \dim V_1 \cap V_3 - \dim V_1 \cap V_2 \cap V_3.
\]

This is equivalent to

\[
\dim V_1 + \dim V_2 + \dim V_3 - \dim V_2 \cap V_3 - \dim (V_1 + V_2 + V_3) = \dim V_1 \cap V_2 + \dim V_1 \cap V_3 - \dim V_1 \cap V_2 \cap V_3, \text{ i.e.,}
\]

\[
\dim (V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim V_1 \cap V_2 - \dim V_1 \cap V_3 - \dim V_2 \cap V_3 + \dim V_1 \cap V_2 \cap V_3. \tag{11}
\]

Now, just notice that (11) is symmetric on \( V_1, V_2 \) and \( V_3 \). Thus, the following statements are equivalent:

1. \( V_1 \) distributes over \( V_2 \) and \( V_3 \).
2. \( V_2 \) distributes over \( V_1 \) and \( V_3 \).
3. \( V_3 \) distributes over \( V_1 \) and \( V_2 \).

**Proposition 3.9.** For any \( i \geq 0 \) and \( l \geq 0 \) such that \( i + l \leq d \):

\[
K_{il}^{X_1} \cap (K_{il}^{X_2} + K_{il}^{X_3}) = K_{il}^{X_1} + K_{il}^{X_2} + K_{il}^{X_3} \text{ for distinct } q_1, q_2, q_3.
\]

**Proof.** By Remark 3.8, it is enough to prove the case \( q_m = m \) for \( m = 1, 2, 3 \). Via the injective map in the exact sequence defining \( K_{il} \), we can see \( K_{il} \) as a subspace of \( V_{X_1}(-(d-i)A) \oplus V_{X_2}(-iA-lB) \oplus V_{X_3}(-(d-l)B) \). It follows from the exact sequence defining \( K_{il} \) that

\[
K_{il}^{X_2} = V_{X_1}(-(d-i+1)A) \oplus \{0\} \oplus V_{X_3}(-(d-l+1)B),
\]

\[
K_{il}^{X_3} = \{0\} \oplus \{0\} \oplus V_{X_3}(-(d-l+1)B),
\]

\[
K_{il}^{X_1} = \{0\} \oplus V_{X_2}(-(i+1)A-(l+1)B) \oplus \{0\}
\]

and

\[
K_{il}^{X_i} = V_{X_1}(-(d-i+1)A) \oplus \{0\} \oplus \{0\}.
\]
Also, by checking cases \( i = b_j, i \neq b_j, l = b'_k \) and \( l \neq b'_k \), we get
\[
\dim K^{X_1,0}_d = \dim V_{X_2}(-(i + 1)A - lB) + \dim V_{X_3}(-(d - l + 1)B),
\]
\[
\dim K^{X_2,0}_d = \dim V_{X_1}(-(d - i + 1)A) + \dim V_{X_2}(-iA - (l + 1)B)
\]
and \( \dim K_d = \dim V_{X_1}(-(d - i + 1)A) + \dim V_{X_2}(-iA - lB) + \dim V_{X_3}(-(d - l + 1)B) \).

Then
\[
\dim(K^{X_2,0}_d + K^{X_3,0}_d) = \dim V_{X_2}(-(i + 1)A - (l + 1)B) + \dim V_{X_3}(-(d - l + 1)B).
\]

In particular, \( \dim(K^{X_2,0}_d + K^{X_3,0}_d) \geq \dim K_d - 1 \), and equality holds if and only if \( l \) is an order of vanishing of \( V_{X_2}(-(iA) \) at \( B \). On the other hand, we have
\[
\dim(K^{X_2,0}_d + K^{X_3,0}_d) = \dim V_{X_2}(-(i + 1)A - (l + 1)B) + \dim V_{X_3}(-(d - l + 1)B),
\]
so, by dimension considerations, we have that \( K^{X_2,0}_d + K^{X_3,0}_d = K^{X_1,0}_d \) if and only if
\[
V_{X_2}(-(i + 1)A - lB) = V_{X_2}(-(i + 1)A - (l + 1)B).
\]
In this case, the statement of the proposition holds.

Suppose now that \( V_{X_2}(-(i + 1)A - lB) \neq V_{X_2}(-(i + 1)A - (l + 1)B) \). Then, we have
\[
V_{X_2}(-iA - lB) \neq V_{X_2}(-iA - (l + 1)B),
\]
and hence \( \dim(K^{X_2,0}_d + K^{X_3,0}_d) = \dim K_d - 1 \). Notice that
\[
\dim(K^{X_2,0}_d + K^{X_3,0}_d) = \dim V_{X_2}(-(i + 1)A - (l + 1)B) + \dim V_{X_3}(-(d - l + 1)B)
\]
\[
= \dim V_{X_2}(-(i + 1)A - lB) - 1 + \dim V_{X_3}(-(d - l + 1)B)
\]
\[
= \dim K^{X_1,0}_d - 1.
\]
Now, since \( K^{X_2,0}_d + K^{X_3,0}_d \subseteq K^{X_1,0}_d \cap (K^{X_2,0}_d + K^{X_3,0}_d) \) and \( \dim(K^{X_2,0}_d + K^{X_3,0}_d) = \dim K_d - 1 \), it suffices to show that \( K^{X_1,0}_d \) is not contained in the space \( K^{X_2,0}_d + K^{X_3,0}_d \). Suppose by contradiction that \( K^{X_1,0}_d \subseteq K^{X_2,0}_d + K^{X_3,0}_d \). Notice that
\[
K^{X_3,0}_d \subseteq V_{X_1}(-(d - i)A) \oplus V_{X_2}(-iA - (l + 1)B) \oplus \{0\}.
\]
It follows that
\[
K^{X_2,0}_d + K^{X_3,0}_d \subseteq V_{X_1}(-(d - i)A) \oplus V_{X_2}(-iA - (l + 1)B) \oplus V_{X_3}(-(d - l + 1)B),
\]
and hence \( K^{X_1,0}_d \subseteq V_{X_1}(-(d - i)A) \oplus V_{X_2}(-iA - (l + 1)B) \oplus V_{X_3}(-(d - l + 1)B) \). So, since \( K^{X_1,0}_d \subseteq \{0\} \oplus V_{X_2}(-(i + 1)A - lB) \oplus V_{X_3}(-(d - l + 1)B) \), we get
\[
K^{X_1,0}_d \subseteq \{0\} \oplus V_{X_2}(-(i + 1)A - (l + 1)B) \oplus V_{X_3}(-(d - l + 1)B).
\]
Then
\[
\dim K^{X_1,0}_d \leq \dim V_{X_2}(-(i + 1)A - (l + 1)B) + \dim V_{X_3}(-(d - l + 1)B)
\]
\[
= \dim V_{X_2}(-(i + 1)A - lB) - 1 + \dim V_{X_3}(-(d - l + 1)B)
\]
\[
= \dim K^{X_1,0}_d - 1,
\]
a contradiction. So the statement of the proposition is shown. \( \square \)
Proposition 3.10. For any $i \geq 0$ and $l \geq 0$ such that $i + l \leq d$, the following statements hold:
1. $\dim(K_{d}^{X_{i},0} + K_{d}^{X_{i+1},0}) \geq \dim K_{d} - 1$ for any $q_{1} \neq q_{2}$.
2. $\dim K_{d}^{X_{i},0} + \dim K_{d}^{X_{i+1},0} \geq 2(\dim K_{d} - 1)$.

Proof. By the proof of Proposition 3.9, the statement 1 holds for $q_{1} = 2$ and $q_{2} = 3$. The proofs of the other cases are analogous. As for the statement 2, putting together the following equalities

$$K_{d}^{X_{i},0} = V_{X_{i}}(-(d - i + 1)A) \oplus \{0\} \oplus V_{X_{3}}(-(d - l + 1)B),$$

$$\dim K_{d}^{X_{i},0} = \dim V_{X_{3}}(-(i + 1)A) + \dim V_{X_{3}}(-(d - l + 1)B),$$

$$\dim K_{d}^{X_{i+1},0} = \dim V_{X_{3}}(-(d - i + 1)A) + \dim V_{X_{3}}(-(i - A - (l + 1)B)),$$

and $\dim K_{d} = \dim V_{X_{3}}(-(d - i + 1)A) + \dim V_{X_{3}}(-(i - A - (l + 1)B)) + \dim V_{X_{3}}(-(d - l + 1)B)$,

and the inequalities

$$\dim V_{X_{3}}(-(i + 1)A - lB) \geq \dim V_{X_{3}}(-iA - lB) - 1,$$

and $\dim V_{X_{3}}(-iA - (l + 1)B) \geq \dim V_{X_{3}}(-iA - lB) - 1$,

we get $\dim K_{d}^{X_{i},0} + \dim K_{d}^{X_{i+1},0} + \dim K_{d}^{X_{i+2},0} \geq 2(\dim K_{d} - 1)$, and equality holds if and only if the equality holds in the two inequalities above. \[\square\]

Proposition 3.11. For any $i \geq 0$ and $l \geq 0$ such that $i + l \leq d$:

$$\dim K_{d}^{X_{i},0} + \dim K_{d}^{X_{i+1},0} + \dim K_{d}^{X_{i+2},0} = 2(\dim K_{d} - 1) \text{ if and only if } i \text{ is an order of vanishing of } V_{X_{2}}(-lB) \text{ at } A \text{ and } l \text{ is an order of vanishing of } V_{X_{2}}(-iA) \text{ at } B. \text{ In this case, we have that } K_{d}^{X_{i},0}, K_{d}^{X_{i+1},0} \text{ and } K_{d}^{X_{i+2},0} \text{ are proper subspaces of } K_{d}.$$

Proof. We have seen in the proof of Proposition 3.10 that

$$\dim K_{d}^{X_{i},0} + \dim K_{d}^{X_{i+1},0} + \dim K_{d}^{X_{i+2},0} = 2(\dim K_{d} - 1)$$

if and only if

$$\dim V_{X_{3}}(-(i + 1)A - lB) = \dim V_{X_{3}}(-iA - lB) - 1,$$

and $\dim V_{X_{3}}(-iA - (l + 1)B) = \dim V_{X_{3}}(-iA - lB) - 1,$

i.e., if and only if $i$ is an order of vanishing of $V_{X_{2}}(-lB)$ at $A$ and $l$ is an order of vanishing of $V_{X_{2}}(-iA)$ at $B$.

Now, suppose that $i$ is an order of vanishing of $V_{X_{2}}(-lB)$ at $A$ and $l$ is an order of vanishing of $V_{X_{3}}(-iA)$ at $B$. By the proof of Proposition 3.9, $K_{d}^{X_{i},0}$ has dimension $\dim K_{d} - 1$ if and only if $l$ is an order of vanishing of $V_{X_{3}}(-lB)$ at $A$. So, by the hypothesis on $l$, we have $\dim(K_{d}^{X_{i},0} + K_{d}^{X_{i+1},0}) = \dim K_{d} - 1$, and hence $K_{d}^{X_{i},0}$ and $K_{d}^{X_{i+1},0}$ are proper subspaces of $K_{d}$. Analogously, since $i$ is an order of vanishing of $V_{X_{2}}(-lB)$ at $A$, we have $\dim(K_{d}^{X_{i+1},0} + K_{d}^{X_{i+2},0}) = \dim K_{d} - 1$, and hence $K_{d}^{X_{i+1},0}$ is a proper subspace of $K_{d}$. \[\square\]
4 Constructing exact extensions

We will describe a method for the construction of exact extensions. Furthermore, this method allows us to construct any exact extension. The main result of this section is the following proposition, which is the fundamental statement for our method.

Proposition 4.1. For any \( i \geq 0 \) and \( l \geq 0 \) such that \( i + l \leq d \), let \( d := (i, d - i - l, l) \), and let \( d'' := (i - 1, d - i - l + 1, l) \) if \( i > 0 \). Then, the following statements hold:

1. Let \( i, l \) be positive integers such that \( i + l = d \).
   Let \( d := (i - 1, d - i - l + 2, l - 1) \). Let \( V_{d'} \) and \( V_{d''} \) be \( r + 1 \)-dimensional subspaces of \( \mathbb{K}_{i-1,l-1} \) and \( \mathbb{K}_{i-1,l} \), respectively, such that
   \[
   \varphi_{d',d''}(V_{d'}) = V_{d'}^{X,0} \quad \text{and} \quad \varphi_{d',d''}(V_{d''}) = V_{d''}^{X,0}.
   \]
   Set \( \beta := \dim V_{d'}^{X,0} - \dim (V_{d'}^{X,0} \oplus V_{d''}^{X,0}) \). Then, for any linearly independent elements \( u_1, \ldots, u_\beta \in V_{d'}^{X,0} \) such that \( V_{d'}^{X,0} = (V_{d'}^{X,0} \oplus V_{d''}^{X,0}) \oplus \langle u_1, \ldots, u_\beta \rangle \), and for any elements \( v_1, \ldots, v_\beta \in V_{d''} \) such that \( \varphi_{d',d''}(v_1) = u_1, \ldots, \varphi_{d',d''}(v_\beta) = u_\beta \), the subspace
   \[
   V_d := \varphi_{d',d''}(V_{d'}) + \langle v_1, \ldots, v_\beta \rangle \subseteq K_d
   \]
is \( r + 1 \)-dimensional, and
   \[
   \varphi_{d',d''}(V_{d'}) = V_{d'}^{X,0}, \quad \varphi_{d',d''}(V_{d''}) = V_{d''}^{X,0} \quad \text{and} \quad \varphi_{d',d''}(V_{d'}) = V_{d'}^{X,0}.
   \]

2. Let \( i, l \) be positive integers such that \( i + l \leq d - 1 \).
   Let \( d := (i - 1, d - i - l + 2, l - 1) \) and \( d'' := (i, d - i - l - 1, l + 1) \). Let \( V_{d'}, V_{d''} \) and \( V_{d'''} \) be \( r + 1 \)-dimensional subspaces of \( \mathbb{K}_{i-1,l-1}, \mathbb{K}_{i-1,l} \) and \( \mathbb{K}_{i,l+1} \), respectively, such that
   \[
   \varphi_{d',d'''}(V_{d'}) = V_{d'}^{X,0}, \quad \varphi_{d',d'''}(V_{d''}) = V_{d''}^{X,0}, \quad \varphi_{d',d'''}(V_{d''}) = V_{d''}^{X,0} \quad \text{and} \quad \varphi_{d',d'''}(V_{d''}) = V_{d''}^{X,0}.
   \]
   Set \( \beta := \dim V_{d'}^{X,0} - \dim (V_{d'}^{X,0} \oplus V_{d''}^{X,0}) \). Then, for any linearly independent elements \( u_1, \ldots, u_\beta \in V_{d'}^{X,0} \) such that \( V_{d'}^{X,0} = (V_{d'}^{X,0} \oplus V_{d''}^{X,0}) \oplus \langle u_1, \ldots, u_\beta \rangle \), and for any elements \( v_1, \ldots, v_\beta \in V_{d''} \) such that \( \varphi_{d',d'''}(v_1) = u_1, \ldots, \varphi_{d',d'''}(v_\beta) = u_\beta \), the subspace
   \[
   V_d := (\varphi_{d',d''}(V_{d'}) + \varphi_{d',d'''}(V_{d''}))) + \langle v_1, \ldots, v_\beta \rangle \subseteq K_d
   \]
is \( r + 1 \)-dimensional, and
   \[
   \varphi_{d',d''}(V_{d'}) = V_{d'}^{X,0}, \quad \varphi_{d',d''}(V_{d''}) = V_{d''}^{X,0} \quad \text{and} \quad \varphi_{d',d''}(V_{d'}) = V_{d'}^{X,0}.
   \]

3. Let \( 0 < i < d \) and \( l = 0 \).
   Let \( d := (i, d - i - l - 1, l + 1) \). Let \( V_{d'} \) and \( V_{d''} \) be \( r + 1 \)-dimensional subspaces of \( \mathbb{K}_{i-1,l} \) and \( \mathbb{K}_{i,l+1} \), respectively, such that
\[ \varphi_{d''}^{X_0}(V_{d''}) = V_{d''}^{X_2,0} \] and \[ \varphi_{d''}^{X_0}(V_{d''}) = V_{d''}^{X_3,0}. \]

Set \( \beta := \dim V_{d''}^{X_1,0} - \dim (V_{d''}^{X_2,0} \oplus V_{d''}^{X_3,0}) \). Then, for any linearly independent elements \( u_1, \ldots, u_\beta \in V_{d''}^{X_1,0} \) such that \( V_{d''}^{X_1,0} = (V_{d''}^{X_2,0} \oplus V_{d''}^{X_3,0}) \oplus (u_1, \ldots, u_\beta) \), and for any elements \( v_1, \ldots, v_\beta \in K_{d''} \) such that \( \varphi_{d''}(v_1) = u_1, \ldots, \varphi_{d''}(v_\beta) = u_\beta \), the subspace

\[ V_d := \varphi_{d''}^{X_0}(V_{d''}) + \langle v_1, \ldots, v_\beta \rangle \subseteq K_{d''} \]

is \( r + 1 \)-dimensional, and

\[ \varphi_{d''}^{X_0}(V_d) = V_d^{X_2,0}, \quad \varphi_{d''}^{X_0}(V_d) = V_d^{X_3,0} \quad \text{and} \quad \varphi_{d''}^{X_0}(V_d) = V_d^{X_1,0}. \]

**Proof.** We will first prove the statement 2. Consider the following diagram

\[ \begin{array}{c}
K_{i-1,l-1} \\
\varphi_{d''}^{X_1} \downarrow \quad \varphi_{d'}^{X_1} \\
K_i \quad \varphi_{d' d''}^{X_1} \downarrow \varphi_{d''}^{X_1} \\
K_{i,l+1} \\
\varphi_{d''}^{X_1} \downarrow \varphi_{d''}^{X_1}
\end{array} \]

Notice that, by Proposition 3.6, elements \( v_1, \ldots, v_\beta \in K_{d''} \) exist satisfying that \( \varphi_{d''}(v_1) = u_1, \ldots, \varphi_{d''}(v_\beta) = u_\beta \). Since all sections of \( \varphi_{d''}^{X_0}(V_{d''}) \subseteq H^0(L_d) \) vanish on \( X_2 \),

\[ \varphi_{d''}^{X_0}(V_d) \cap \varphi_{d''}^{X_0}(V_{d''}) \subseteq \varphi_{d''}^{X_0}(V_{d''})^{X_2,0} = \varphi_{d''}^{X_0}(V_{d''}^{X_2,0}) = \varphi_{d''}^{X_0}((\varphi_{d''}^{X_0}(V_{d''}))) = \varphi_{d''}^{X_0}(V_{d''}), \]

where in the first equality we used Remark 2.2, and in the second equality we used that

\[ \varphi_{d''}^{X_0}(V_{d''}) = V_{d''}^{X_2,0}. \]

On the other hand, \( \varphi_{d''}^{X_0}(V_{d''}) = \varphi_{d''}^{X_0}(\varphi_{d''}^{X_0}(V_{d''})) \subseteq \varphi_{d''}^{X_0}(V_{d''}) \), as \( \varphi_{d''}^{X_0}(V_{d''}) \subseteq V_{d''} \). Analogously, \( \varphi_{d''}^{X_0}(V_{d''}) = \varphi_{d''}^{X_0}(\varphi_{d''}^{X_0}(V_{d''})) \subseteq \varphi_{d''}^{X_0}(V_{d''}) \), as \( \varphi_{d''}^{X_0}(V_{d''}) \subseteq V_{d''} \). It follows that \( \varphi_{d''}^{X_0}(V_{d''}) \subseteq \varphi_{d''}^{X_0}(V_{d''}) \cap \varphi_{d''}^{X_0}(V_{d''}) \), and hence

\[ \varphi_{d''}^{X_0}(V_{d''}) = \varphi_{d''}^{X_0}(V_{d''}) \cap \varphi_{d''}^{X_0}(V_{d''}). \]

Then

\[ \dim(\varphi_{d''}^{X_0}(V_{d''}) + \varphi_{d''}^{X_0}(V_{d''})) = \dim \varphi_{d''}^{X_0}(V_{d''}) + \dim \varphi_{d''}^{X_0}(V_{d''}) - \dim \varphi_{d''}^{X_0}(V_{d''}) = (r + 1 - \dim V_{d''}^{X_2,0}) + (r + 1 - \dim V_{d''}^{X_3,0}) - (r + 1 - \dim V_{d''}^{X_1,0}) = r + 1 - (\dim V_{d''}^{X_2,0} - (\dim V_{d''}^{X_1,0} - \dim V_{d''}^{X_3,0})). \] \hfill (12)

On the other hand, since \( \varphi_{d''}^{X_0}(V_{d''}) = V_{d''}^{X_2,0} \), we have

\[ \varphi_{d''}^{X_0}(V_{d''}) = (\varphi_{d''}^{X_0}(V_{d''}))^{X_1,0} = V_{d''}^{X_2,0} \]

where in the first equality we used Remark 2.2. Then

\[ \dim V_{d''}^{X_1,0} - \dim V_{d''}^{X_3,0} = \dim V_{d''}^{X_2,0}. \] \hfill (13)

Also, since \( \varphi_{d''}^{X_0}(V_{d''}) = V_{d''}^{X_1,0} \), we have

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\[ \varphi_{d',d''}(V_{d'}^{X_1}) = (\varphi_{d',d''}(V_{d'}))^{X_1} = (V_{d'}^{X_1})^{X_1} = V_{d'}^{X_1}, \]

and hence
\[ \dim V_{d'}^{X_1} - \dim V_{d'}^{X_1} = \dim V_{d'}^{X_1}. \]

It follows from (12), (13) and (14) that
\[ \dim(\varphi_{d',d''}(V_{d'}) + \varphi_{d',d''}(V_{d''})) = r + 1 \] (15)

Then, to prove that \( \dim V_{d} = r + 1 \), it suffices to show that
\[ (\varphi_{d',d''}(V_{d'}) + \varphi_{d',d''}(V_{d''})) \cap \langle v_1, \ldots, v_\beta \rangle = 0. \]

Notice that
\[ \varphi_{d,d''}(\varphi_{d',d''}(V_{d'}) + \varphi_{d',d''}(V_{d''})) = \varphi_{d,d''}(\varphi_{d',d''}(V_{d'})) + \varphi_{d,d''}(\varphi_{d',d''}(V_{d''})) \]
\[ = \varphi_{d,d''}(\varphi_{d',d''}(V_{d'})) + \varphi_{d',d''}(\varphi_{d',d''}(V_{d''})) \]
\[ = V_{d'}^{X_1} + V_{d''}^{X_1} \] (16)

and
\[ \varphi_{d,d''}((\langle v_1, \ldots, v_\beta \rangle) = \langle u_1, \ldots, u_\beta \rangle. \] (17)

Then
\[ \varphi_{d,d''}((\varphi_{d',d''}(V_{d'}) + \varphi_{d',d''}(V_{d''})) \cap \langle v_1, \ldots, v_\beta \rangle) \subseteq (V_{d'}^{X_1} + V_{d''}^{X_1}) \cap \langle u_1, \ldots, u_\beta \rangle = 0, \]

where in the last equality we used that \( V_{d'}^{X_1} = (V_{d'}^{X_1} \oplus V_{d''}^{X_1}) \oplus \langle u_1, \ldots, u_\beta \rangle \). Therefore
\[ \varphi_{d,d''}((\varphi_{d',d''}(V_{d'}) + \varphi_{d',d''}(V_{d''})) \cap \langle v_1, \ldots, v_\beta \rangle) = 0. \]

On the other hand, as \( u_1, \ldots, u_\beta \) are linearly independent and \( \varphi_{d,d''}(v_1) = u_1, \ldots, \varphi_{d,d''}(v_\beta) = u_\beta \), it follows that
\[ \varphi_{d,d''} |_{\langle v_1, \ldots, v_\beta \rangle} : \langle v_1, \ldots, v_\beta \rangle \to \langle u_1, \ldots, u_\beta \rangle \]
is an isomorphism. So \( (\varphi_{d',d''}(V_{d'}) + \varphi_{d',d''}(V_{d''})) \rangle \langle v_1, \ldots, v_\beta \rangle = 0 \), and hence
\[ V_{d} = (\varphi_{d',d''}(V_{d'}) + \varphi_{d',d''}(V_{d''})) \oplus \langle v_1, \ldots, v_\beta \rangle \]
is \( r + 1 \)-dimensional. Since \( V_{d'} \) and \( V_{d''} \) are subspaces of \( K_{i-1,i-1} \) and \( K_{i+1,i+1} \), respectively, we have \( \varphi_{d',d''}(V_{d'}) \subseteq K_{i1} \) and \( \varphi_{d',d''}(V_{d''}) \subseteq K_{i1} \). Thus, as \( \langle v_1, \ldots, v_\beta \rangle \subseteq K_{i1} \) as well, it follows that \( V_{d} \subseteq K_{i1} \).

Now, it follows from (16) and (17) that
\[ \varphi_{d,d''}(V_{d}) = (V_{d'}^{X_1} + V_{d''}^{X_1}) \oplus \langle u_1, \ldots, u_\beta \rangle = V_{d'}^{X_1}. \] (18)

On the other hand, since \( \varphi_{d',d''}(V_{d'}) \subseteq \varphi_{d',d''}(V_{d'}) \subseteq V_{d'}, \) it follows that
\[ \dim V_{d''}^{X_{1,0}} + \dim \varphi'_{d''}(V_{d''}) = \dim V_d = r + 1, \]

so

\[ \dim \varphi'_{d''}(V_{d''}) = r + 1 - \dim V_{d''}^{X_{1,0}} = \dim V_d - \dim V_{d''}^{X_{1,0}} = \dim V_d^{X_{1,0}}, \]

where the last equality follows from (18). Thus, as \( \varphi'_{d''}(V_{d''}) \subseteq V_d^{X_{1,0}}, \) it follows that

\[ \varphi'_{d''}(V_{d''}) = V_d^{X_{1,0}}. \]

Now, we will show that \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}} \) and \( \varphi_{d''}(V_{d''}) = V_d^{X_{2,0}}. \) We have

\[ \varphi'_{d''}(V_{d''}) = \varphi'_{d''} \varphi_{d''}(V_{d''}) = \varphi'_{d''} (V_d^{X_{1,0}}) = (\varphi'_{d''} (V_{d''}))^{X_{1,0}} = (V_d^{X_{2,0}})^{X_{1,0}} = V_d^{X_{2,0}}, \]

where the second equality follows from (18), and in the fourth equality we used that \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}}. \) Since \( \varphi'_{d''}(V_{d''}) \subseteq V_d, \) it follows that

\[ \dim V_d^{X_{2,0}} + \dim \varphi'_{d''}(V_{d''}) = \dim V_d = r + 1, \]

and hence

\[ \dim \varphi'_{d''}(V_{d''}) = r + 1 - \dim V_d^{X_{2,0}} = \dim V_d - \dim V_d^{X_{2,0}} = \dim V_d^{X_{2,0}}, \]

where in the last equality we used that \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}}. \) Since \( \varphi'_{d''}(V_{d''}) \subseteq V_d^{X_{2,0}}, \) we get \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}}. \) The proof of the equalities \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}} \) and \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}} \) is analogous to that of \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}} \) and \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}}. \) This proves statement 2.

Now, we will prove the statement 1. Notice that (14) holds, as \( \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}}. \) Then

\[ \dim \varphi'_{d''}(V_{d''}) = r + 1 - \dim V_d^{X_{2,0}} \]

\[ = r + 1 - (\dim V_{d''}^{X_{1,0}} - \dim V_{d''}^{X_{2,0}}). \] (19)

Since \( i + l = d, \) we have \( d'' = (i, 0, l), \) and hence \( V_{d''}^{X_{2,0}} = 0. \) Thus

\[ V_{d''}^{X_{1,0}} = \langle V_{d''}^{X_{2,0}} \oplus V_{d''}^{X_{3,0}} \oplus \langle u_1, \ldots, u_\beta \rangle \rangle = V_{d''}^{X_{2,0}} \oplus \langle u_1, \ldots, u_\beta \rangle \). (20)

It follows from (19) and (20) that \( \dim \varphi'_{d''}(V_{d''}) = r + 1 - \beta. \) To prove that \( \dim V_d = r + 1, \) it suffices to show that

\[ \varphi'_{d''}(V_{d''}) \cap \langle v_1, \ldots, v_\beta \rangle = 0. \]

We have

\[ \varphi'_{d''}(\varphi'_{d''}(V_{d''})) = \varphi'_{d''}(V_{d''}) = V_d^{X_{2,0}}. \] (21)

Then

\[ \varphi'_{d''}(\varphi'_{d''}(V_{d''}) \cap \langle v_1, \ldots, v_\beta \rangle) \subseteq V_d^{X_{2,0}} \cap \langle u_1, \ldots, u_\beta \rangle = 0, \]

where in the last equality we used (20). Thus \( \varphi'_{d''}(\varphi'_{d''}(V_{d''}) \cap \langle v_1, \ldots, v_\beta \rangle) = 0. \) Reasoning as in the proof of the statement 2, we get \( \varphi'_{d''}(V_{d''}) \cap \langle v_1, \ldots, v_\beta \rangle = 0, \) and hence
\[ V_d = \varphi_{d,d'}(V_d) \oplus \langle v_1, \ldots, v_\beta \rangle \]

is \( r + 1 \)-dimensional. Reasoning as in the proof of the statement 2, we get \( V_d \subseteq K_{d'} \).

Now, it follows from (20) and (21) that
\[
\varphi_{d,d''}(V_d) = V_d^{X_{i},0} + \langle u_1, \ldots, u_\beta \rangle = V_d^{X_{i},0}.
\]

The proofs of the equalities \( \varphi_{d,d'}(V_d) = V_d^{X_{i},0} \), \( \varphi_{d,d'}(V_d) = V_d^{X_{i},0} \) and \( \varphi_{d,d''}(V_d) = V_d^{X_{i},0} \)
are the same as in the proof of the statement 2. So the statement 1 is shown.

Now, we will prove the statement 3. Notice that (13) holds, as \( \varphi_{d'',d'''}(V_d') = V_d^{X_{i},0} \). Therefore
\[
\dim \varphi_{d'',d'''}(V_d') = r + 1 - \dim V_d^{X_{i},0} = r + 1 - (\dim V_d^{X_{i},0} - \dim V_d^{X_{i},0}).
\]  

(22)

Since \( l = 0 \), we have \( d'' = (i - 1, d - i + 1, 0) \), and hence \( V_d^{X_{i},0} = 0 \). Then
\[ V_d^{X_{i},0} = (V_d^{X_{i},0} \oplus V_d^{X_{i},0}) + \langle u_1, \ldots, u_\beta \rangle = V_d^{X_{i},0} \oplus \langle u_1, \ldots, u_\beta \rangle. \]

(23)

It follows from (22) and (23) that \( \dim \varphi_{d'',d'''}(V_d') = r + 1 - \beta \). To prove that \( \dim V_d = r + 1 \), it suffices to show that
\[ \varphi_{d'',d'''}(V_d') \cap \langle v_1, \ldots, v_\beta \rangle = 0. \]

We have
\[
\varphi_{d'',d'''}(\varphi_{d'',d'''}(V_d')) = \varphi_{d'',d'''}(V_d') = V_d^{X_{i},0}.
\]  

(24)

Then
\[ \varphi_{d'',d'''}(\varphi_{d'',d'''}(V_d')) \subseteq V_d^{X_{i},0} \cap \langle u_1, \ldots, u_\beta \rangle = 0, \]

where in the last equality we used (23). So \( \varphi_{d'',d'''}(\varphi_{d'',d'''}(V_d') \cap \langle v_1, \ldots, v_\beta \rangle) = 0 \), and reasoning as in the proof of the statement 2, we get \( \varphi_{d'',d'''}(V_d') \cap \langle v_1, \ldots, v_\beta \rangle = 0 \), and hence
\[ V_d = \varphi_{d'',d'''}(V_d') \oplus \langle v_1, \ldots, v_\beta \rangle \]
is \( r + 1 \)-dimensional. Reasoning as in the proof of the statement 2, we get \( V_d \subseteq K_{d'} \) as well.

Now, it follows from (23) and (24) that
\[
\varphi_{d',d'''}(V_d) = V_d^{X_{i},0} + \langle u_1, \ldots, u_\beta \rangle = V_d^{X_{i},0}.
\]

Reasoning as in the proof of the statement 2, we get the remaining equalities. So the statement 3 is shown, proving the proposition. \( \square \)

Now, we describe the method for the construction of exact extensions \( \{ (L_d, V_d) \}_{d} \). According to Remark 3.2 we necessarily have to do the construction in such a way that \( V_d \) to be contained in \( K_{d'} \) for any \( i \geq 0 \) and \( l \geq 0 \) such that \( i + l \leq d \). The idea is first to construct the subspaces \( V_d \) for \( i = 0 \), then for \( i = 1 \), and so on, until \( i = d - 1 \). For each \( i \geq 1 \), we first construct the subspaces \( V_d \) for \( l = d - i \), then for \( l = d - i - 1 \), and so on, until \( l = 0 \).

Suppose inductively that, for \( i \geq 1 \), the \( r + 1 \)-dimensional subspaces
\[ V_{i-1,l} \subseteq K_{i-1,l} \]

have been constructed for \( l = d - (i - 1), \ldots, l = 0 \) in such a way that
\[
\varphi_{d',d''}(V_{d'}) = V_{d',0}^{X_{i,0}} \text{ and } \varphi_{d',d''}(V_{d''}) = V_{d''}^{X_{i,0}} \text{ for } l = 1, \ldots, d - (i - 1),
\]
where, as usual, \( d' := (i - 1, d - (i - 1) - l, l - 1) \) and \( d'' := (i - 1, d - (i - 1) - l, l) \).

We say that the subspaces \( V_{i-1,l} \) satisfy the vertical exactness property.

Then we will construct \( r + 1 \)-dimensional subspaces \( V_{d,i} \subseteq K_{d} \) for \( d = d - i, \ldots, l = 0 \) in such a way that

(i) The subspaces \( V_{d,i} \) satisfy the vertical exactness property.

(ii) \[
\varphi_{d',d''}(V_{d'}) = V_{d'}^{X_{i,0}}, \varphi_{d',d''}(V_{d''}) = V_{d''}^{X_{i,0}} \text{ for } l = 1, \ldots, d - i, \text{ and }
\]
\[
\varphi_{d',d''}(V_{d'}) = V_{d'}^{X_{i,0}}, \varphi_{d',d''}(V_{d''}) = V_{d''}^{X_{i,0}} \text{ for } l = 0, \ldots, d - i,
\]
where \( d := (i, d - i - l, l), d' := (i - 1, d - (i - 1) - l + 1, l - 1) \) and \( d'' := (i - 1, d - (i - 1) - l, l) \).

We inductively do the construction as follows:

Step 1. For \( l = d - i \), the subspace \( V_{d,i} \) is the subspace \( V_{d} \) defined in Proposition 4.1 item 1.

Step 2. For \( l = d - i - 1, \ldots, l = 1 \), the subspace \( V_{d,i} \) is the subspace \( V_{d} \) defined in Proposition 4.1 item 2.

Step 3. For \( l = 0 \), the subspace \( V_{d,i} \) is the subspace \( V_{d} \) defined in Proposition 4.1 item 3.

By Proposition 4.1, the subspaces \( V_{d,i} \) satisfy the properties (i) and (ii). Thus, it remains to construct the \( r + 1 \)-dimensional subspaces \( V_{d,0} \subseteq K_{d} \) to satisfy the vertical exactness property, and verify that the exact limit linear series \( \{ (L_{d}, V_{d}) \}_{d} \) that we construct is in fact an extension.

By Proposition 3.5 item 1, \( \dim K_{d,i} = r + 1 \) if \( i \leq b_{0} \) or \( l \leq b_{0} \). Since \( V_{X_{1}}, V_{X_{2}} \) and \( V_{X_{3}} \) are linked, we have \( V_{X_{1}} \subseteq K_{d,0}, V_{X_{2}} \subseteq K_{d,0} \) and \( V_{X_{3}} \subseteq K_{d,0} \). It follows from dimension considerations, that \( V_{X_{1}} = K_{d,0}, V_{X_{2}} = K_{d,0} \) and \( V_{X_{3}} = K_{d,0} \). On the other hand, \( \dim K_{d,0} = r + 1 \) if \( 0 \leq l \leq d \). Thus, we define \( V_{d,0} := K_{d,0} \) for any nonnegative integer \( l \leq d \). (Note that, for \( l = 0 \) and \( l = d \), the definition coincides with our fixed subspaces \( V_{X_{2}} \) and \( V_{X_{3}} \).)

Now, we will prove that
\[
\varphi_{d'',d''}(V_{d''}) = V_{d''}^{X_{3,0}}, \varphi_{d'',d''}(V_{d''}) = V_{d''}^{X_{3,0}} \text{ for } l = 0, \ldots, d - 1,
\]
where \( d := (0, d - l, l) \) and \( d'' := (0, d - l - 1, l + 1) \). (Observe that this notation corresponds to the notation in Proposition 4.1 for \( i = 0 \).)

Let \( l \) be a nonnegative integer such that \( l \leq d \). By Proposition 3.6 item 3, we have \( \varphi_{d'',d''}(K_{0,l+1}) = K_{d'',0}^{X_{3,0}} \). Then \( \dim K_{0,l+1}^{X_{3,0}} = \dim K_{0,l+1}^{X_{3,0}} = \dim K_{0,l+1}^{X_{3,0}} \). On the other hand, since \( \dim K_{0,l+1} = r + 1 = \dim K_{0,l} \), we have \( \dim K_{0,l+1}^{X_{3,0}} = \dim K_{0,l+1}^{X_{3,0}} = \dim K_{0,l} \). Now, \( \dim K_{0,l+1}^{X_{3,0}} = \dim \varphi_{d'',d''}(K_{0,l}) = \dim K_{0,l+1}^{X_{3,0}} \). Hence
\[
\dim \varphi_{d'',d''}(K_{0,l}) = \dim K_{0,l+1}^{X_{3,0}},
\]
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and since $\varphi_{d,d''}(K_{0}) \subseteq K_{1,0}^{X_{i},0}$, we have $\varphi_{d,d''}(K_{0}) = K_{1,0}^{X_{i},0}$, proving that the subspaces $V_{0} = K_{0}$ satisfy the vertical exactness property.

Thus, we construct subspaces $V_{1} \subseteq K_{0}$ for $i = 0, \ldots, d - 1$ and $l = 0, \ldots, d - i$. Now, since $\dim K_{0} = r + 1$ for $i = 0, \ldots, d - 1$, we have that, by dimension considerations, $V_{0} = K_{0}$ for $i = 0, \ldots, d - 1$. On the other hand, the subspaces $\{K_{i}\}_{i=0,\ldots,d}$ satisfy the following exactness property analogous to that of the subspaces $K_{0}$

$$\varphi_{d''}(K_{i-1,0}) = K_{0}^{X_{i},0} \text{ and } \varphi_{d'}(K_{0}) = K_{1,0}^{X_{i},0} \text{ for } i = 1, \ldots, d,$$

where $d := (i, d - i, 0)$ and $d'' := (i - 1, d - i + 1, 0)$.

Thus, since $V_{X_{i}} = K_{i}^{0}$, we get an exact limit linear series $\{(\mathcal{L}_{d}, V_{d})\}_{d}$ which is an extension of $L$.

Now, let $\{(\mathcal{L}_{d}, V_{d})\}_{d}$ be any exact extension. We will prove that the subspaces $V_{d}$ are constructed by our method. Since $\dim K_{0} = r + 1$ if $0 \leq l \leq d$, we have $V_{0} = K_{0}$ for each integer $l$ such that $0 \leq l \leq d$. Let $0 < i < d$, $0 \leq l \leq d - i$, and $d := (i, d - i - l, l)$. We will show that $V_{d}$ is constructed by our method if $l > 0$ and $i + l \leq d - 1$. (The proofs of the cases $i + l = d$, $l = 0$ are analogous.) Keep the notation of multidegrees used in Proposition 4.1. By Remark 3.2 we have that $V_{d'}$, $V_{d''}$ and $V_{d'''}$ are $r + 1$-dimensional subspaces of $K_{d}$, $K_{i-1,l-1}$, $K_{l+1,i}$ and $K_{i+l+1}$, respectively. Set $\beta := \dim V_{d'}^{i} - \dim (V_{d''}^{i} \oplus V_{d'''}^{i})$. As $V_{d} \supseteq \varphi_{d}^{i}(V_{d})$ and $V_{d} \supseteq \varphi_{d''}^{i}(V_{d})$, it follows that $V_{d} \supseteq \varphi_{d}^{i}(V_{d}) + \varphi_{d''}^{i}(V_{d})$. By the proof of the statement 2 of Proposition 4.1 we have \(\dim (\varphi_{d}^{i}(V_{d}) + \varphi_{d''}^{i}(V_{d})) = r + 1 - \beta\), so

$$V_{d} = (\varphi_{d}^{i}(V_{d}) + \varphi_{d''}^{i}(V_{d})) \oplus \langle v_{1}, \ldots, v_{\beta} \rangle,$$

for some $v_{1}, \ldots, v_{\beta} \in K_{d}$ which are linearly independent.

Now, let $u_{i} := \varphi_{d}^{i}(v_{1}), \ldots, u_{\beta} := \varphi_{d''}^{i}(v_{\beta})$. By the proof of the statement 2 of Proposition 4.1 we have

$$\varphi_{d}^{i}(\varphi_{d}^{i}(V_{d}) + \varphi_{d''}^{i}(V_{d})) = V_{d''}^{i} \oplus V_{d'}^{i},$$

and hence $V_{d'}^{i} = \varphi_{d}^{i}(V_{d}) = (V_{d''}^{i} + V_{d'}^{i}) + \langle u_{1}, \ldots, u_{\beta} \rangle$. Since

$$\beta = \dim V_{d''}^{i} - \dim (V_{d''}^{i} \oplus V_{d'}^{i}),$$

we necessarily have that $u_{1}, \ldots, u_{\beta}$ are linearly independent and

$$V_{d''}^{i} = (V_{d''}^{i} \oplus V_{d'}^{i}) \oplus \langle u_{1}, \ldots, u_{\beta} \rangle.$$

This proves that our method constructs any exact extension.

## 5 Unique exact extension

In this section, we will show the conditions under which the exact extension is unique, and in this case, we will describe the scheme $P(g)$ for such a unique extension. Keeping the notation of multidegrees used in Proposition 4.1 we have the following lemma.
Lemma 5.1. If \( \varphi_{\underline{d},\underline{d}'}(K_{i-1,l}) = K_{il}^{X_l,0} \), then \( \dim K_{il} = \dim K_{i-1,l} \).

Proof. It follows from the hypothesis that
\[
\dim K_{i-1,l}^{X_l,0} + \dim K_{il}^{X_l,0} = \dim K_{i-1,l}.
\]
On the other hand, by Proposition 3.6 item 2, we have \( \varphi_{\underline{d},\underline{d}'}(K_{il}) = K_{il}^{X_l,0} \), and hence
\[
\dim K_{il}^{X_l,0} + \dim K_{i-1,l}^{X_l,0} = \dim K_{il}.
\]
Thus \( \dim K_{il} = \dim K_{i-1,l} \).

\[\square\]

Lemma 5.2. Let \( i \geq 0 \) and \( l \geq 0 \) such that \( i + l \leq d \). Then, the following statements hold:
1. If \( i > 0 \), then \( \dim K_{il} \geq \dim K_{i-1,l} \).
2. If \( l > 0 \), then \( \dim K_{il} \geq \dim K_{i,l-1} \).

Proof. We will only prove the statement 1, as the proof of the statement 2 is analogous. Let \( \underline{d} := (i, d - i - l, l) \) and \( \underline{d}'' := (i - 1, d - i - l + 1, l) \). By Proposition 3.6 item 2, we have \( \varphi_{\underline{d},\underline{d}'}(K_{il}) = K_{i-1,l}^{X_l,0} \). It follows that
\[
\dim K_{il}^{X_l,0} + \dim K_{i-1,l}^{X_l,0} = \dim K_{il}.
\]
On the other hand, since \( \varphi_{\underline{d},\underline{d}'}(K_{i-1,l}) \subseteq K_{il}^{X_l,0} \), we have
\[
\dim K_{i-1,l} = \dim K_{i-1,l}^{X_l,0} + \dim \varphi_{\underline{d},\underline{d}'}(K_{i-1,l}) \leq \dim K_{i-1,l}^{X_l,0} + \dim K_{il}^{X_l,0}.
\]
It follows that \( \dim K_{il} \geq \dim K_{i-1,l} \), proving the statement 1 of the lemma.

\[\square\]

Theorem 5.3. The following statements are equivalent:
1. \( \mathfrak{h} \) has a unique exact extension.
2. \( \dim K_{il} = r + 1 \) if \( i + l \leq d \), \( b_{j-1} < i \leq b_j \), \( b'_{k-1} < l \leq b'_k \), and \( j + k \leq r + 1 \).
3. \( \mathfrak{h} \) has a unique extension.

Proof. First, for a fixed integer \( j \) such that \( 1 \leq j \leq r \), let us consider the following statement:
4. \( \dim K_{il} = r + 1 \) if \( i + l \leq d \), \( b_{j-1} < i \leq b_j \), \( b'_{k-1} < l \leq b'_k \), and \( j + k \leq r + 1 \).

We will prove that, for that fixed integer \( j \), the statement 4 implies the following statement:
5. Let \( e_{j,0}, \ldots, e_{j,r-j} \) be the orders of vanishing of \( V_{X_l}(-b_jA) \) at \( B \). Then \( e_{j,r-j} = b'_{r-j} \).

Indeed, assume statement 4 holds. By the proof of the first four cases of Proposition 3.7, we have that, for \( b_{j-1} < i \leq b_j \) and \( b'_{k-1} < l \leq b'_k \) such that \( i + l \leq d \), \( \dim K_{il} = r + 1 \) if and only if \( \dim V_{X_l}(-iA - lB) = r + 1 - j - k \). On the other hand, by Remark 3.4, \( b_j + b'_{r-j} \leq d \). Thus \( \dim K_{b_j,b'_{r-j}} = r + 1 \), and it follows that
\[
\dim V_{X_l}(-b_jA - b'_{r-j}B) = r + 1 - j - (r - j) = 1.
\]
If $b_j + b'_{r-j} + 1 \leq d$, then, by hypothesis, $\dim K_{b_i} b'_{r-j}, b'_{r-j} + 1 = r + 1$, as $b'_{r-j} < b'_{r-j} + 1 \leq b'_{r+1-j}$.

So $\dim V_{X_2}(-b_j A - (b'_{r-j} + 1)B) = r + 1 - j - (r + 1 - j) = 0$, and it follows that $b'_{r-j} = e_{r-j}$.

Now, if $b_j + b'_{r-j} + 1 > d$, then $V_{X_2}(-b_j A - (b'_{r-j} + 1)B) = 0$ as well, and hence $b'_{r-j} = e_{r-j}$.

So statement 5 holds.

Now, we will prove that the statement 1 implies the statement 2. Assume statement 1 holds. Let $\{ (\mathcal{L}_d, V_d) \} \subseteq \mathbb{R}$ be the unique exact extension. We claim that, for $0 < i < d$ and $0 \leq l \leq d - i$,

$$K_d^{X_i,0} = \varphi_d^\beta, (V_d')$$ if $\beta > 0$, where

$$d := (i, d - i - l, l), d' := (i - 1, d - i - l + 1, l)$$ and $\beta := \dim V_{d'}^{X_i,0} - \dim (V_d^{X_i,0} \oplus V_{d'}^{X_i,0})$.

Indeed, we will prove the claim for $l > 0$ and $i + l \leq d - 1$, as the remaining cases are analogous. Assume $\beta > 0$. In Section 4 we saw that

$$V_d = \left( \varphi_d^\beta, (V_d') \oplus (v_1, \ldots, v_\beta) \right),$$

where $d := (i, d - i - l, l), d' := (i - 1, d - i - l + 1, l)$ and $v_1, \ldots, v_\beta \in K_d$ satisfy that $V_{d'}^{X_i,0} = (V_d^{X_i,0} \oplus V_{d'}^{X_i,0}) \oplus \{ u_1, \ldots, u_\beta \}$, where $u_1 := \varphi_d^{\beta^d', (v_1)}, \ldots, u_\beta := \varphi_d^{\beta^d', (v_\beta)}$.

Suppose that $K_d^{X_i,0}$ is not contained in $V_d$. Let $\tilde{v} \in K_d^{X_i,0} \setminus V_d$ and set

$$\tilde{V}_d := (\varphi_d^{\beta^d', (V_d')} \oplus \varphi_d^{\beta^d', (V_d''}) \oplus \{ v_1 + \tilde{v}, \ldots, v_\beta \}.$$ We have that $\tilde{V}_d \subseteq K_d$ and $\varphi_d^{\beta^d', (v_1 + \tilde{v})} = u_1, \ldots, \varphi_d^{\beta^d', (v_\beta)} = u_\beta$. Now, as $v_1 \in V_d$ and $\tilde{v} \notin V_d$, it follows that $v_1 + \tilde{v} \notin V_d$, and hence $\tilde{V}_d \neq V_d$. However, by the method of Section 4 this allows us to construct an exact extension which is different from the unique exact extension, a contradiction. Thus $K_d^{X_i,0} \subseteq V_d$, and hence $K_d^{X_i,0} = V_d^{X_i,0} = \varphi_d^{\beta, (V_d')}$. So our claim is established.

Now, we will prove the statement 2 by induction on $j$. Let $l_0$ be the largest order of vanishing of $V_{X_2}(-b_1 A) \setminus \setminus B$. Notice that $b_j + l_0 \leq d$, as $V_{X_2}(-b_1 A - l_0 B) \neq 0$. Also, we have $l_0 \leq l$, as $l_0$ is necessarily an order of vanishing of $V_{X_2}$ at $\setminus B$. By definition of $l_0$, and since $V_{X_2}(-b_0 (b_0 + 1) A) = V_{X_2}(-b_1 A) = 0$, we get

$$\dim V_{X_2}(-(b_0 + 1) A - l_0 B) = 1 \text{ and } \dim V_{X_2}(-(b_0 + 1) A - (l_0 + 1) B) = 0.$$ Thus, since $K_d^{X_i,0} = V_{X_2}(-(i + 1) A - (l + 1) B)$ for any $i \geq 0$ and $l \geq 0$ such that $i + l \leq d$, we get

$$\dim K_{b_0, l_0 - 1}^{X_i,0} - \dim K_{b_0, l_0}^{X_i,0} = 1 - 0 = 1 \text{ if } l_0 > 0. \quad (25)$$

Now, set $i := b_0 + 1$ and $l := l_0$, keep the notation of multidegrees used in Proposition 11 and set $\beta := \dim V_{d'}^{X_i,0} - \dim (V_d^{X_i,0} \oplus V_{d'}^{X_i,0})$. We will prove that $\beta > 0$. Suppose first that $l > 0$. It follows from the proofs of the statements 1 and 2 of Proposition 11 that $\beta = \dim V_{d'}^{X_i,0} - \dim V_d^{X_i,0}$. On the other hand, $\dim K_{b_0, l_0} = r + 1$ and $\dim K_{b_0, l_0 - 1} = r + 1$, so $V_d' = K_{b_0, l_0}$ and $V_d' = K_{b_0, l_0 - 1}$. Therefore, by (25),

$$\beta = \dim V_{d'}^{X_i,0} - \dim V_d^{X_i,0} = \dim K_{b_0, l_0 - 1}^{X_i,0} - \dim K_{b_0, l_0}^{X_i,0} = 1,$$
so $\beta > 0$. Suppose now that $l = 0$. It follows from the proof of the statement 3 of Proposition 4.1 that $\beta = \dim V^{X,0}_{\bar{d},0} - \dim V^{X,0}_{\bar{d},0}$. On the other hand, as we saw, $V^{g''}_{\bar{d}} = K_{b_0,l_0}$. Since $\dim K_{b_0,l_0} = 0$ and

$$
\dim K_{b_0,l_0}^{X,0} = \dim V_{X_3}(-(b_0 + 1)A - l_0B) + \dim V_{X_3}^{\prime\prime}(-(d - l_0 + 1)B) = 1 + \dim V_{X_3}^{\prime\prime}(-(d - 0 + 1)B) = 1,$$

we get

$$
\beta = \dim V_{d',0}^{X,0} - \dim V_{d''_0}^{X,0} = \dim K_{b_0,l_0}^{X,0} - \dim K_{b_0,l_0}^{X,0} = 1 - 0 = 1.
$$

Thus, in any case, $\beta > 0$. It follows from the claim that $K_{d''_0}^{X,0} = \varphi_{d''_0}^{\bar{d}}(V^{g''}_{\bar{d}})$, and hence $K_{d''_0}^{X,0} = \varphi_{d''_0}^{\bar{d}}(K_{i-1,i})$. Then, by lemma 5.1 we get

$$
\dim K_{d''_0} = \dim K_{i-1,i} = \dim K_{b_0,l_0} = r + 1.
$$

On the other hand, notice that, for $l > 0$, $V_{X_3}(-(b_0 + 1)A - l_0B)$ if and only if $V_{X_3}^{\prime\prime}(-(d - l_0 + 1)B)$, i.e., and only if $V_{X_3}^{\prime\prime}(-(d - l_0 + 1)B) = 0$, that is, $l > l'$. Thus, $b_0$ is an order of vanishing of $V_{X_3}^{\prime\prime}$ if $0 < l < l'$. Then, by Proposition 3.7 item 1, if $l_0 < l < l'$ and $i + l < d, K_{d''_0}^{X,0} = \varphi_{d''_0}^{\bar{d}}(K_{i-1,i})$, where $\bar{d} := (i, d - i - l, \tilde{l})$ and $d''_0 := (i - 1, d - i - \tilde{l} + 1, \tilde{l})$. It follows from lemma 5.1 that

$$
\dim K_{d''_0} = \dim K_{i-1,i} = \dim K_{b_0,l_0} = r + 1.$$

Therefore, by (26), (27), lemma 5.2 item 2 and Proposition 3.5 item 1,

$$
\dim K_{d''_0} = r + 1 \text{ if } 0 \leq l \leq l' \text{ and } i + l \leq d.
$$

On the other hand, by Proposition 3.7 item 1, if $b_0 + 1 < \tilde{l} \leq b_1, \tilde{l} \geq 0$ and $i + \tilde{l} \leq d, K_{d''_0}^{X,0} = \varphi_{d''_0}^{\bar{d}}(K_{i-1,i})$, where $\bar{d} := (i, d - \tilde{l} - i, \tilde{l})$ and $d''_0 := (i, d - \tilde{l} - i, \tilde{l})$). Then, it follows from lemma 5.1 that

$$
\dim K_{d''_0} = \dim K_{i-1,i} = \ldots = \dim K_{b_0,l_0} = r + 1,$$

if $b_0 + 1 < \tilde{l} \leq b_1, 0 \leq \tilde{l} \leq l'$ and $i + \tilde{l} \leq d$. Thus, (28) and (29) prove the case $j = 1$. Now, suppose by induction that, for a certain $2 \leq j < r,$

$$
\dim K_{d''_0} = r + 1 \text{ if } i + \tilde{l} \leq d, b_{j-1} < \tilde{l} \leq b_j, b_{j-1} < \tilde{l} \leq b_k \text{ and } j + k \leq r + 1.
$$

Then, since the statement 4 implies the statement 5, $l_{j-1} := b_{j-1}$ is the largest order of vanishing of $V_{X_3}(-(b_j A)$ at $B$. Let $l_j$ be the largest order of vanishing of $V_{X_3}(-(b_{j+1} A)$ at $B$. Notice that $b_{j+1} + l_j \leq d$, as $V_{X_3}(-(b_{j+1} A - l_j B) \neq 0$. Also, we have $l_j \leq l_{j-1}$, i.e., $l_j \leq l'_{j-1}$. By definition of $l_j$, and since $V_{X_3}(-(b_j + 1) A) = V_{X_3}(-(b_{j+1} A)$, we get

$$
\dim V_{X_3}(-(b_j + 1) A - l_j B) = 1 \text{ and } \dim V_{X_3}(-(b_j + 1) A - (l_j + 1) B) = 0,$$
and hence
\[ \dim K_{b_j,l_j}^{X_2} - \dim K_{b_j,l_j}^{X_2} = 1 - 0 = 1 \quad \text{if } l_j > 0. \]  
(30)

Now, we proceed as in the first case. Set \( i := b_j + 1 \) and \( l := l_j \), keep the notation of multidegrees used in Proposition 1, and set \( \beta := \dim V_{d''}^{X_1} - \dim (V_{d''}^{X_1} \oplus V_{d''}^{X_2}) \). We will show that \( \beta > 0 \). Suppose first that \( l > 0 \). As we saw, \( \beta = \dim V_{d''}^{X_1} - \dim V_{d''}^{X_2} \). On the other hand, by induction, \( \dim K_{b_j,l_j} = r + 1 \) and \( \dim K_{b_j,l_j-1} = r + 1 \), as \( b_j + l_j \leq d \) and \( l_j \leq b_{r-j} \), so \( V_{d''} = K_{b_j,l_j} \) and \( V_{d''} = K_{b_j,l_j-1} \). Therefore, by (30),
\[ \beta = \dim V_{d''}^{X_1} - \dim V_{d''}^{X_2} = \dim K_{b_j,l_j}^{X_1} - \dim K_{b_j,l_j}^{X_2} = 1, \]
so \( \beta > 0 \). Suppose now that \( l = 0 \). We have \( \beta = \dim V_{d''}^{X_1} - \dim V_{d''}^{X_2} \) and \( V_{d''} = K_{b_j,l_j} \). Since \( \dim K_{b_j,l_j}^{X_2} = 0 \) and
\[ \dim K_{b_j,l_j}^{X_1} = \dim V_{X_3}(-(b_j+1)A - l_jB) + \dim V_{X_3}(-(d-l_j+1)B) \]
\[ = 1 + \dim V_{X_3}(-(d-0+1)B) = 1, \]
we get
\[ \beta = \dim V_{d''}^{X_1} - \dim V_{d''}^{X_2} = \dim K_{b_j,l_j}^{X_1} - \dim K_{b_j,l_j}^{X_2} = 1 - 0 = 1. \]
Thus, in any case, \( \beta > 0 \). Reasoning as in the case \( j = 1 \), we get
\[ \dim K_{i,l} = \dim K_{b_j,l_j} = r + 1. \]  
(31)

On the other hand, notice that, for \( i > l_j \), \( V_{X_3}(-(b_jA - lB)) = V_{X_3}(-(b_j+1)A - lB) \) if and only if \( V_{X_3}(-(b_jA - lB)) = 0 \), i.e., if and only if \( l > l_{j-1} = b_{r-j} \). Thus, \( b_j \) is an order of vanishing of \( V_{X_3}(-lB) \) if \( l < \tilde{l} \leq b_{r-j} \). Then, by Proposition 4, item 1, if \( l < \tilde{l} \leq b_{r-j} \) and \( i + \tilde{l} \leq d \), \( K_{i,l}^{X_1} = \phi_{d''}(K_{i-1,l}) \), where \( \tilde{d}_i := (i, d-i-\tilde{l}, \tilde{l}) \) and \( \tilde{d}'' := (i-1, d-i-\tilde{l}+1, \tilde{l}) \). By induction and lemma 5.1
\[ \dim K_{i,l} = \dim K_{i-1,l} = \dim K_{b_j,l} = r + 1 \]  
if \( l_j < \tilde{l} \leq b_{r-j} \) and \( i + \tilde{l} \leq d \).  
(32)

By (31) and (32),
\[ \dim K_{i,l} = r + 1 \]  
if \( 0 \leq \tilde{l} \leq b_{r-j} \) and \( i + \tilde{l} \leq d \).

Reasoning as in the case \( j = 1 \), we get
\[ \dim K_{i,l} = r + 1 \]  
if \( b_j < \tilde{l} \leq b_{j+1}, \) \( 0 \leq \tilde{l} \leq b_{r-j} \) and \( i + \tilde{l} \leq d \).

This finishes the proof of the statement 2.

Now, we will prove that the statement 2 implies the statement 3. Assume statement 2 holds. Let \( \{ (L_{\tilde{d}}, V_{\tilde{d}}) \}_d \) be an extension. Then, since \( b_j + b_{r-j} \leq d \), we have
\[ V_{d''} = K_{i,l} \]  
if \( 0 < j \leq r, \) \( b_{j-1} < i \leq b_j \) and \( 0 \leq l \leq b_{r-j}. \)
Also,

\[ V_d = K_d \text{ if } i \leq b_0 \text{ or } l \leq b'_0. \]

On the other hand, since the statement 4 implies the statement 5, for each \( 0 < j \leq r \), \( b'_{r-j} \) is the largest order of vanishing of \( V_{X_2}(-(b_j A)) = V_{X_2}(-(b_{j-1} + 1)A) \) at \( B \). Then \( V_{X_2}(-(b_{j-1} + 1)A - (b'_{r-j} + 1)B) = 0 \), and hence \( V_{X_2}(-(i + 1)A - (l + 1)B) = 0 \) if \( i \geq b_{j-1} \) and \( l \geq b'_{r-j} \). It follows that \( K_i^{X_2,0} = 0 \) if \( i \geq b_{j-1}, l \geq b'_{r-j} \) and \( i + l \leq d \). Thus, for \( i > b_{j-1}, l > b'_{r-j} \) and \( i + l \leq d \), \( V_{i,j}^{X_2,0} = 0 \), implying that

\[ \varphi(i-1,d-i+l+1,l,d-l-i)(V_{i-1,l-1}) = V_d \text{ if } i > b_{j-1}, l > b'_{r-j} \text{ and } i + l \leq d. \]

(In this case, \( V_d = V_d^{X_2,0} \).) It follows that the extension is unique, proving the statement 3.

Finally, by the method of Section 4, there exists at least one exact extension, so the statement 3 implies the statement 1. This finishes the proof of the theorem. \( \square \)

**Remark 5.4.** Suppose that \( h \) has a unique exact extension and let \( \{(L_d, V_d)\}_d \) be its unique exact extension. Note that, in the proof of Theorem 5.3, we saw that \( V_d = V_d^{X_2,0} \) for any \( d = (i, d - i - l, l) \) with \( i > b_{j-1}, l > b'_{r-j} \) and \( i + l \leq d \).

**Remark 5.5.** Suppose that \( h \) has a unique exact extension and let \( \{(L_d, V_d)\}_d \) be its unique exact extension. It follows from the proof of Theorem 5.3 that, for \( 0 \leq j \leq r \), \( b_0, \ldots, b'_{r-j} \) are the orders of vanishing of \( V_{X_2}(-(b_j A)) \) at \( B \). Analogously, for \( 0 \leq k \leq r \), \( b_0, \ldots, b_{r-k} \) are the orders of vanishing of \( V_{X_2}(-(b'_k B)) \) at \( A \).

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