Local conditional entropy in measure for covers with respect to a fixed partition

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Abstract

In this paper we introduce two measure theoretical notions of conditional entropy for finite measurable covers conditioned to a finite measurable partition and prove that they are equal. Using this we state a local variational principle with respect to the notion of conditional entropy defined by Misiurewicz (1976 Stud. Math. 55 176–200) for the case of open covers. This in particular extends the work done in Romagnoli (2003 Ergod. Theor. Dynam. Syst. 23 1601–10), Glasner and Weiss (2006 Handbook of Dynamical Systems vol 1B (Amsterdam: Elsevier)) and Huang et al (2006 Ergod. Theor. Dynam. Syst. 26 219–45).

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1. Introduction

Topological dynamics and ergodic theory exhibit a remarkable parallelism. It is usual to find counterparts in both theories as for instance transitivity, weak mixing and strong mixing and also the ones we study in this paper, this is topological entropy and measure theoretical entropy. Although, even if the notions and results are similar the methods to prove them are quite different. In many cases the natural connection between these notions is a variational principle that relates the suprema over all invariant measures to the topological notion. For the case of $\mathbb{Z}$ actions measure-theoretical entropy for an invariant measure was introduced in 1958 in [K] and topological entropy in 1965 in [AKM]. In 1969 and 1970 Goodman [G] and Goodwyn [GW] in two separate papers proved the first variational principle. Misiurewicz
introduced the notion of topological conditional entropy in [M] and the variational principle was established by Ledrappier in [L].

This paper is inserted in the so called local entropy theory for topological dynamical systems that started in the early 90’s with the work of François Blanchard (see [B] and [BL]). Nowadays this theory is interesting by itself and has also proven to be fundamental to many other related areas. For instance topological Pinsker factors, zero entropy factors, disjointness theorems, characterizations of positive entropy, entropy pairs and tuples and so on.

In [BGH] while extending the notion of topological entropy pairs to a measure theoretical setting for a topological dynamical system \((X, T)\) the authors proved a variational inequality for open covers. More precisely, for every open cover there exists a \(T\)-invariant measure \(\mu\) such that the topological entropy of the cover is bounded above by the \(\mu\)-entropy of every partition finer than the cover.

In [R] the author gave a new approach to the inequality given in [BGH] extending the measure theoretical notions from partitions to covers and thus proposing two measure theoretical notions of entropy for open covers denoted \(h^+_\mu\) and \(h^-_\mu\). In doing so a first local variational principle for \(h^+_\mu\) was stated and proven. Namely, for every open cover there exists a \(T\)-invariant measure \(\mu\) such that the topological entropy is equal to the \(h^+_\mu\)-entropy of the cover. The notion \(h^-_\mu\) verified most of the relevant properties of the usual notion of entropy for measurable partitions. It is also proven that the infimum that appears in the definition of \(h^-_\mu\) was attained. This fact implies very good properties, for instance that \(h^-_\mu\) is preserved by measure-theoretic extensions and that as a function of \(\mu\), \(h^-_\mu\) is upper semicontinuous (see [HY]). The main unsolved result in [R] was if these two notions were equal.

In [HMRY] the authors proved that these notions were positive simultaneously and, using the Jewett–Krieger theorem, they proved that the equality between \(h^+_\mu\) and \(h^-_\mu\) is equivalent to the existence of a local variational principle for \(h^+_\mu\) (similar to the one proven for \(h^-_\mu\)). In [Gw] such a variational principle was established for \(h^+_\mu\) and thus the equality holds. An alternative definition for measure entropy for covers was given in [S] giving a new proof of the variational principle and at the same time proving the equality of \(h^+_\mu\) and \(h^-_\mu\) once again.

In [HY] the authors applied the same ideas to extend the notion of topological pressure to a measure theoretical setting proving again a local variational principle of the same topological pressure and in [HYZ] a similar one in a conditional version with respect to a factor.

For the existence of a variational principle the requirement of the cover to be open is unavoidable. This was well known even in the first known proofs of the global variational principle. The global bound of the variational principle fails even for nice non open covers as is the case of closed covers where for example the existence of one non recurrent point implies that the suprema of the entropy over all closed covers is infinite. This was reviewed and explained by Goodwyn in [Gw2] (see example 3) although it was already mentioned by him in [Gw]. However the equality between \(h^+_\mu\) and \(h^-_\mu\) can be stated in general for measurable covers extending the topological notion to this kind of covers since it is turns out to be mainly combinatorial.

Since then, the research has mainly focused in extending the notions of entropy and pressure to more general actions. In [HYZ2] they consider the case of countable discrete amenable groups by using the tool of Følner sequences making the techniques and the proofs very similar to the case of \(\mathbb{Z}\) actions. The case of continuous bundle random dynamical systems of an infinite countable discrete amenable group action was proven in [DZ] and for sofic group actions in [Z].

It is important to notice that the main idea that gives the natural bridge between the measurable and the topological notions in every one of the versions of variational principles discussed
so far is the one given in [R]. This is, to extended the measurable formulas that apply for partitions to covers by considering the infima over all partitions finer than the cover. Then prove for measurable covers that the natural ‘+’ and ‘−’ definitions are equal for all measurable covers. Finally use this fact to prove a variational principle for open covers for the suitable topological equivalent.

Unfortunately the real local variational problem is still open, this is, with respect to a fixed open cover but conditioned to another fixed open cover. The missing ingredient is a clear way to extend the notion with respect to the conditioning partition. For a local conditional definition extending the definition with respect to the conditioning variable by taking infima as for the first partition is not the way to go. Considering the alternative definition of conditional entropy as an average of the entropy of the first partition with respect to the induced measures over the atoms of the conditioning partition and using infima in a clever way solves the issue giving once again two different definitions that turn out to be the same. In this paper we address this for the first time proving a local conditional version with respect to a finite measurable partition.

More precisely, in this paper we propose two notions of conditional measure theoretical entropy for measurable finite covers conditioned to a fixed measurable partition that extends the notions $h^-_{\mu}$ and $h^+_{\mu}$ given in [R] and [HYZ] proving that they still coincide for every finite measurable cover. The work in [R] is just conditioning with respect to the trivial partition $\beta = \{X\}$ and in the setting of [HYZ] this work can be seen as extending the conditioning to the case of a non dynamical invariant $\sigma$-algebra that does not give rise to a factor as required in their work. In the case of finite open covers we prove that they satisfy a local variational principle with respect to the notion of conditional entropy defined by Misiurewicz in [M] once again extending the variational principles proven in [R] and [HYZ].

2. Basic definitions and results

Let us introduce the basic notation and definitions used in this article. For more details on measure theoretical and topological entropy including the most recent results of the beginning of the year 2000 we refer to the deeply insightful and inspirational textbook by Tomasz Downarowicz [D].

As it is standard in the theory we will define a series of notions as infima that turn out to be also limits using the classic subadditive lemma that we state without proof:

**Lemma 1 ([D]).** For any subadditive sequence $\{a_N\}_{N \in \mathbb{N}} \subseteq \mathbb{R}^+$, this is $a_{N+M} \leq a_N + a_M$ for any $N, M \in \mathbb{N}$ we have:

$$\lim_{N \to \infty} \frac{1}{N} a_N = \inf_{N \in \mathbb{N}} \frac{1}{N} a_N.$$  

$(X, T)$ is called a *topological dynamical system* (TDS) when $X$ is a compact metric space and $T : X \to X$ a homeomorphism. A TDS $(X, T)$ is 0-*dimensional* when the space $X$ has a countable open-closed (clopen) topological basis. The set $\mathcal{M}_T(X)$ of $T$-invariant Borel probability measures is a non empty convex weakly compact set. We denote $\mathcal{M}_T^e(X)$ the set of ergodic measures. A *measure theoretic dynamical system* (MDS), $(X, \mathcal{B}, \mu, T)$, is a probability space $(X, \mathcal{B}, \mu)$ and a bi-measurable bijection $T : X \to X$ that preserves the measure $\mu$. So in every TDS $(X, T)$ there is a natural family of MDS indexed by the set $\mathcal{M}_T(X)$ when one considers $\mathcal{B}$ as the $\sigma$-algebra of the Borel sets of $X$. 


In this article a \textit{cover} of \( X \) is a finite cover of Borel subsets of \( X \). Such a cover is said to be \textit{open} if all its elements are open sets. We consider a \textit{partition} as a finite measurable cover by pairwise disjoint sets. Denote \( \mathcal{P}_X \) the set of finite measurable partitions of \( X \), \( \mathcal{C}_X \) the set of finite covers of \( X \) and \( \mathcal{C}_X^o \) the set of finite open covers of \( X \). To simplify notation we will always assume by default that \( \alpha, \beta, \text{etc...} \) refer to elements in \( \mathcal{P}_X \) and \( \mathcal{U}, \mathcal{V}, \text{etc...} \) refer to elements in \( \mathcal{C}_X \).

Given two covers \( \mathcal{U}, \mathcal{V} \in \mathcal{C}_X \), \( \mathcal{U} \) is said to be \textit{finer} than \( \mathcal{V} \) \( (\mathcal{U} \succeq \mathcal{V}) \) if for every \( U \in \mathcal{U} \), there is \( V \in \mathcal{V} \) such that \( U \subseteq V \). This gives a partial order relation for both \( \mathcal{C}_X \) and \( \mathcal{P}_X \). \( \mathcal{U} \cup \mathcal{V} = \{ U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V} \} \). It is clear that \( \mathcal{U} \cup \mathcal{V} \succeq \mathcal{U} \cup \mathcal{V} \succeq \mathcal{V} \). However, \( \mathcal{V} \succeq \mathcal{U} \) does not imply that \( \mathcal{U} \cup \mathcal{V} = \mathcal{V} \). Given integers \( M \leq N \) and \( \mathcal{U}, \mathcal{V} \in \mathcal{C}_X \), one sets, \( \mathcal{U}_M^N = \bigvee_{m=M}^N T^{-m} \mathcal{U} \).

In general one can define the \textit{static \( \mu \)-conditional entropy of} \( \alpha \in \mathcal{P}_X \) \textit{conditioned by} a sub \( \sigma \)-algebra \( \mathcal{A} \subseteq \mathcal{B} \) as:

\[
H_\mu(\alpha|\mathcal{A}) := \sum_{\alpha \in \mathcal{A}} \int_X \phi(X) \left( \mathbb{E}_\mu \left( \mathbb{I}_\mathcal{A} \right) (\omega) \right) \, d\mu(\omega).
\]

(2)

Where \( \phi : [0, 1] \to \mathbb{R}^+ \) is defined as \( \phi(x) := -x \log x \) for \( x > 0 \) and \( \phi(0) = 0 \).

Given \( \alpha, \beta \in \mathcal{P}_X \) the \textit{static \( \mu \)-entropy of} \( \alpha \) \textit{conditioned by} \( \beta \) is given by \( H_\mu(\alpha|\beta) = H_\mu(\alpha|\overline{\beta}) \) where \( \overline{\beta} \) is the \( \sigma \)-algebra generated by \( \beta \). In this case there are simpler formulas to compute it, for instance for any \( \beta \in \mathcal{P}_X \):

\[
H_\mu(\alpha|\beta) := H_\mu(\alpha \vee \beta) - H_\mu(\beta) = \sum_{B \in \beta} \mu(B) H_\mu(\alpha|B)
\]

(3)

where \( \mu_B \) denotes the conditional measure induced by \( \mu \) on \( B \) (zero if \( \mu(B) = 0 \)). The function \( H_\mu(\alpha|A) \) is concave on \( \mathcal{M}_T(X) \) (since \( -x \log x \) is concave).

\textbf{Remark.} The formula \( H_\mu(\alpha|\beta) \) can be applied to any disjoint family of measurable sets even if it does not cover \( X \).

Given \( \alpha, \beta \in \mathcal{P}_X \) or any sub-\( \sigma \)-algebra \( \mathcal{A} \subseteq \mathcal{B} \) by lemma 1 the \textit{\( \mu \)-entropy of} \( \alpha \) \textit{conditioned to} \( \beta \in \mathcal{P}_X \) (or some \( T \)-invariant \( \sigma \)-algebra \( \mathcal{A} \)) \textit{with respect to} \( T \) is well defined as:

\[
\begin{align*}
_h_\mu(\alpha|\beta, T) & := \lim_{N \to \infty} \frac{1}{N} H_\mu(\alpha|\beta^N) = \inf_{N \in \mathbb{N}} \frac{1}{N} H_\mu(\alpha|\beta^N), \\
_h_\mu(\alpha|\mathcal{A}, T) & := \lim_{N \to \infty} \frac{1}{N} H_\mu(\alpha|\mathcal{A}) = \inf_{N \in \mathbb{N}} \frac{1}{N} H_\mu(\alpha|\mathcal{A}).
\end{align*}
\]

(4)

When \( \beta = \{X\} \) this yields the standard \textit{\( \mu \)-entropy of} \( \alpha \) \textit{with respect to} \( T \).

For any \( \mathcal{U} = \{ U_1, \ldots, U_M \}, \mathcal{V} = \{ V_1, \ldots, V_M \} \in \mathcal{C}_X \) define \( \mu(\mathcal{U} \Delta \mathcal{V}) = \sum_{m=1}^M \mu(U_m \Delta V_m) \). This definition gives a notion of distance between covers and partitions that is compatible with the conditional measure entropy. The proof is the following lemma taken from Peter Walter’s classic textbook [W].

\textbf{Lemma 2 ([W])}. Fix \( M \in \mathbb{N} \) and \( \epsilon > 0 \) then there exists \( \delta > 0 \) such that \( \forall \alpha, \alpha', \beta \in \mathcal{P}_X \) with \( |\alpha| = |\alpha'| = M \) if \( \mu(\alpha \Delta \alpha') < \delta \) one has that \( |h_\mu(\alpha|\beta, T) - h_\mu(\alpha'|\beta, T)| \leq \epsilon \).

We use the same definitions of chapter 6.3 in [D] but for \( \mathcal{U} \in \mathcal{C}_X \) and \( \beta \in \mathcal{P}_X \):
\[ N(\mathcal{U}|\beta) := \max_{B \in \beta} N(\mathcal{U} \cap B), \quad N(\mathcal{U} \cap B) := \min\{|V|/|V| \subseteq \mathcal{U}, B \subseteq \bigcup_{V \in V} V\}. \] \hspace{1cm} (5)

From the same ideas used in fact 6.3.2 in [D] and other simple calculations:

**Proposition 3.** Let \((X, T)\) be a TDS and \(\mathcal{U}, \mathcal{V} \in \mathcal{C}_X\) and \(\beta, \gamma \in \mathcal{P}_X\) then:

1. \(N(\mathcal{U}|\beta) = 0\) iff \(\beta \supseteq \mathcal{U}\).
2. \(N(T^{-1}\mathcal{U}|T^{-1}\beta) = N(\mathcal{U}|\beta)\).
3. \(N(\mathcal{U}|\beta) \geq N(\mathcal{V}|\beta)\) if \(\mathcal{U} \supseteq \mathcal{V}\).
4. \(N(\mathcal{U}|\beta) \leq N(\mathcal{U}|\gamma)\) for \(\beta \supseteq \gamma\).
5. \(N(\mathcal{U} \vee \mathcal{V}|\beta) \leq N(\mathcal{U}|\beta) \cdot N(\mathcal{V}|\beta)\).

**Proof.** Part (2) is trivial and for all the others use fact 6.3.2. with \(F = X\) and consider \(\mathcal{V}\) and \(\mathcal{W}\) as partitions when required. \(\square\)

Proposition 3 parts (2), (4) and (6) show that \(\{\log N(\mathcal{U}_0^{N-1}|\beta_0^{N-1})\}_{X \in \mathcal{N}}\) is a subadditive sequence, and using lemma 1 we obtain the combinatorial (topological if \(\mathcal{U}\) is open) entropy of the cover \(\mathcal{U} \in \mathcal{C}_X\) conditioned by partition \(\beta \in \mathcal{P}_X\) with respect to \(T\) as:

\[ h(\mathcal{U}|\beta, T) := \lim_{N \to \infty} \frac{1}{N} \log N(\mathcal{U}_0^{N-1}|\beta_0^{N-1}) = \inf_{N \in \mathbb{N}} \frac{1}{N} \log N(\mathcal{U}_0^{N-1}|\beta_0^{N-1}). \] \hspace{1cm} (6)

For the case of open covers with \(\beta = \{X\}\) we recover the classical topological entropy.

### 3. New definitions and main theorems

As stated in the introduction the main idea is to extend real valued functions from \(\mathcal{P}_X\) to \(\mathcal{C}_X\) by taking for each \(\mathcal{U} \in \mathcal{C}_X\) the infima over the set \(\{\alpha \supseteq \mathcal{U}\}\). In our case the functions are order preserving and using lemma 2 in [HMRY] we can replace this set for a simpler one. This is for \(\mathcal{U} = \{U_1, \ldots, U_M\}, \mathcal{U}^* = \{\alpha = \{A_1, \ldots, A_M\} \in \mathcal{P}_X/\forall m \in \{1, \ldots, M\}, A_m \subseteq U_m\}\). An even simpler subset of \(\mathcal{U}^*\) defined in [R] will be also useful, this is \(\text{Ext}(\mathcal{U})\) the set of partitions of the form \(\{U_1, U_2 \setminus U_1, \ldots, U_d \setminus (U_1 \cup \cdots \cup U_{d-1})\}\) where \(\{U_1, \ldots, U_d\}\) is an ordering of \(\mathcal{U}\).

In [R] we defined for \(\mathcal{U} \in \mathcal{C}_X^\alpha\) and \(\mu \in \mathcal{M}\_T(X)\), \(H_\mu(\mathcal{U}) := \inf_{\alpha \supseteq \mathcal{U}} H_\mu(\alpha)\). We extend this idea for every \(\mathcal{U} \in \mathcal{C}_X\) and \(\beta \in \mathcal{P}_X\) first for every \(\mathcal{B} \in \beta\) as \(H_{\mu_\mathcal{B}}(\mathcal{U})\) (even if \(\mathcal{B}\) is not \(T\)-invariant) and using the decomposition shown in (3) we define:

\[ H_\mu(\mathcal{U}|\beta) := \sum_{B \in \beta} \mu(B) H_{\mu_\mathcal{B}}(\mathcal{U}), \quad H^*_\mu(\mathcal{U}|\beta) := \inf_{\alpha \supseteq \mathcal{U}} H_\mu(\alpha|\beta). \] \hspace{1cm} (7)

It is straightforward that \(H_\mu(\mathcal{U}|\beta) \leq H^*_\mu(\mathcal{U}|\beta)\) and when \(\beta = \{X\}\) and \(\mathcal{U} \in \mathcal{C}_X^\alpha\) we recover in both cases the definition given in [R] since \(H^*_\mu(\mathcal{U}|\{X\}) = H_\mu(\mathcal{U}|\{X\}) = H_\mu(\mathcal{U})\). As it turns out we can prove that they are actually always equal.

**Lemma 4.** For any \(\mathcal{U} \in \mathcal{C}_X, \beta \in \mathcal{P}_X\), \(H_\mu(\mathcal{U}|\beta) = H^*_\mu(\mathcal{U}|\beta)\).

**Proof.** Fix \(\mathcal{U} \in \mathcal{C}_X\) and \(\beta \in \mathcal{P}_X\), we only need to prove that \(H^*_\mu(\mathcal{U}|\beta) \leq H_\mu(\mathcal{U}|\beta)\). From fact (8) for any \(B \in \beta\) there exists \(\alpha_B \in \mathcal{P}_X\) such that \(H_{\mu_\mathcal{B}}(\mathcal{U}) = H_{\mu_\mathcal{B}}(\alpha_B)\).
Define $\alpha = \bigvee_{B \in \beta} [\alpha B \cap B] \cup \{B\}$. By definition for any any $\tilde{\alpha} \succeq \mathcal{U}$ and $B \in \beta$, $H_{\mu_B}(\tilde{\alpha}) \geq H_{\mu_B}(\alpha B) = H_{\mu_B}([\alpha B \cap B] \cup \{B\}) = H_{\mu_B}(\alpha)$ so multiplying by $\mu(B)$ and adding up over $B \in \beta$ we conclude that $H_\mu(\mathcal{U} | \beta) = H_\mu(\alpha | \beta) = H_\mu(\mathcal{U} | \beta)$. \hfill \Box

**Remark.** From now we will use as definition of $H_\mu(\mathcal{U} | \beta)$ the most suitable of these two formulas as needed in the proofs.

Now we prove that the basic properties of conditional entropy extend to $H_\mu(\mathcal{U} | \beta)$. From [R] as mentioned in fact 8.3.5 in [D]:

$$H_\mu(\mathcal{U}) = \min_{\alpha \in \text{Ext}(\mathcal{U})} H_\mu(\alpha).$$  

Notice that this is also true for $H_{\mu_B}(\mathcal{U})$ and any $B \in \beta$ since the proof does not require $\mu$ to be $T$-invariant.

**Proposition 5.** Let $(X, \mathcal{B}, \mu, T)$ be a MDS and $\mathcal{U}, \mathcal{V}, W \in \mathcal{C}_X$ and $\alpha, \beta, \gamma \in \mathcal{P}_X$ then

1. $0 \leq H_\mu(\mathcal{U} | \beta) \leq \log N(\mathcal{U} | \beta)$ and $H_\mu(\mathcal{U} | \beta) = 0$ if $\beta \succeq \mathcal{U}$.
2. $H_\mu(T^{-1}\mathcal{U} | T^{-1}\beta) = H_\mu(\mathcal{U} | \beta)$.
3. $H_\mu(\mathcal{U} | \beta) \leq H_\mu(\mathcal{U} | \gamma)$ for $\beta \succeq \gamma$.
4. $H_\mu(\mathcal{U} \cup \mathcal{V} | \beta) = H_\mu(\mathcal{U} | \beta) + H_\mu(\mathcal{V} | \beta)$.

**Proof.** Fix $\mathcal{U}, \mathcal{V}, W \in \mathcal{C}_X$ and $\alpha, \beta, \gamma \in \mathcal{P}_X$.

1. For any $B \in \beta$, $H_{\mu_B}(\alpha) \leq \log |\alpha \cap B|$ and $\inf_{\alpha \in \mathcal{U}} |\alpha \cap B| = N(\mathcal{U} \cap B)$. So:

$$H_\mu(\mathcal{U} | \beta) = \sum_{B \in \beta} \mu(B) H_{\mu_B}(\mathcal{U}) \leq \sum_{B \in \beta} \mu(B) \log N(\mathcal{U} \cap B) \leq \log N(\mathcal{U} | \beta).$$

If $\beta \succeq \mathcal{U}$, $H_{\mu_B}(\mathcal{U}) \leq H_{\mu_B}(\beta) = 0$ for any $B \in \beta$.

2. For any $B \in \beta$ from equation (8):

$$H_{\mu_{T^{-1}\beta}}(T^{-1}\mathcal{U}) = \min_{\alpha \in \text{Ext}(T^{-1}\mathcal{U})} H_{\mu_{T^{-1}\beta}}(\alpha) = \min_{\alpha \in T^{-1}\text{Ext}(\mathcal{U})} H_{\mu_{T^{-1}\beta}}(\alpha),$$

$$= \min_{\alpha \in \text{Ext}(\mathcal{U})} H_{\mu_{T^{-1}\beta}}(T^{-1}\alpha) = \min_{\alpha \in \text{Ext}(\mathcal{U})} H_{\mu_B}(\alpha) = H_{\mu_B}(\mathcal{U}).$$

Adding up over all $B \in \beta$ and since $\mu(T^{-1}B) = \mu(B)$ we conclude the result.

3. Let $\beta = \{B_1, \ldots, B_N\}$ and $\gamma = \{C_1, \ldots, C_M\}$ if $\beta \succeq \gamma$ then there exists a partition $\{K_1\}_{n=1}^{N}$ of $\{1, \ldots, N\}$ such that $B_n = \bigcup_{k \in K_n} C_k$ for any $1 \leq n \leq N$.

It is easy to prove that when $A$ and $B$ are disjoint then:

$$\mu(A \cup B) H_{\mu_{A \cup B}}(\alpha) \geq \mu(A) H_{\mu_A}(\alpha) + \mu(B) H_{\mu_B}(\alpha) \geq \mu(A) H_{\mu_A}(\alpha).$$

Thus if $A \subseteq B$ then $\mu(A) H_{\mu_A}(\mathcal{U}) \leq \mu(B) H_{\mu_B}(\mathcal{U})$.

So for any $1 \leq n \leq N$:

$$\mu(B_n) H_{\mu_{B_n}}(\mathcal{U}) \geq \sum_{k \in K_n} \mu(C_k) H_{\mu_{C_k}}(\mathcal{U}).$$

Adding up over $n$ concludes the result.

4. For any $\alpha \succeq \mathcal{U}$ and $\gamma \succeq \mathcal{V}$ from basic properties of $H_\mu$ for partitions:
\[ H_\mu(\mathcal{U} \vee \mathcal{V}|\beta) \leq H_\mu(\alpha \vee \gamma|\beta) \leq H_\mu(\alpha|\beta) + H_\mu(\gamma|\beta). \]

Taking infima over all \( \alpha \geq \mathcal{U} \) and \( \gamma \geq \mathcal{V} \) concludes the result.

This allows us to define two measure theoretical notions of dynamical entropy for a measurable cover conditioned to a fixed partition.

Proposition 5 parts (2)–(4) imply that the sequence \( \{H_\mu(\mathcal{U}_0^{N-1}|\beta_0^{N-1})\}_{N \in \mathbb{N}} \) is subadditive, and so by lemma 1 we define two notions of \( \mu \)-entropy of the cover \( \mathcal{U} \in \mathcal{C}_X \) conditioned by the partition \( \beta \in \mathcal{P}_X \) with respect to \( T \) as:

\[
\begin{align*}
      h_\mu^- (\mathcal{U}|\beta, T) := \lim_{N \to \infty} \frac{1}{N} H_\mu(\mathcal{U}_0^{N-1}|\beta_0^{N-1}) &= \inf_{N \in \mathbb{N}} \frac{1}{N} H_\mu(\mathcal{U}_0^{N-1}|\beta_0^{N-1}), \\
      h_\mu^+ (\mathcal{U}|\beta, T) := \inf_{\alpha \geq \mathcal{U}} h_\mu(\alpha|\beta, T). 
\end{align*}
\]

Notice that when \( \beta = \{X\} \) we recover the notions in [R].

The two main results of this paper can now be easily stated as:

1. For every open cover and every measurable partition in a TDS there always exists an invariant measure such that the conditional topological entropy of the cover with respect to the partition coincides with the measure conditional entropy of the cover with respect to the partition.

2. For every measurable cover and every measurable partition in a TDS both notions of conditional measure entropy of the cover with respect to the partition coincide.

The techniques and proofs rely strongly but not entirely on the work done in [R, Gw, HMRY] and [HYZ]. These results can be decomposed in the proof of three main theorems:

**Theorem 6.** For every TDS \( (X, T) \), \( \mathcal{U} \in \mathcal{C}_X \) and \( \beta \in \mathcal{P}_X \) there exists \( \mu \in \mathcal{M}_T(X) \) such that \( h_\mu^+ (\mathcal{U}|\beta, T) \geq \hat{h}(\mathcal{U}|\beta, T) \) and \( h_\mu^- (\mathcal{U}|\beta, T) = \hat{h}(\mathcal{U}|\beta, T) \).

**Theorem 7.** For every TDS \( (X, T) \), \( \mathcal{U} \in \mathcal{C}_X \), \( \beta \in \mathcal{P}_X \) we have that:

\[
\inf_{\alpha \geq \mathcal{U}} \sup_{\mu \in \mathcal{M}_T(X)} h_\mu(\alpha|\beta, T) \leq \hat{h}(\mathcal{U}|\beta, T).
\]

**Theorem 8.** For every TDS \( (X, T) \), \( \mathcal{U} \in \mathcal{C}_X \), \( \beta \in \mathcal{P}_X \) and \( \forall \mu \in \mathcal{M}_T(X) \), \( h_\mu^+ (\mathcal{U}|\beta, T) = h_\mu^- (\mathcal{U}|\beta, T) \).

These three theorems combined yield the following theorem that summarizes the minimax property behind them:

**Theorem 9.** For every TDS \( (X, T) \), \( \mathcal{U} \in \mathcal{C}_X \) and \( \beta \in \mathcal{P}_X \) there exists \( \mu \in \mathcal{M}_T(X) \) such that:

\[
\hat{h}(\mathcal{U}|\beta, T) = \sup_{\mu \in \mathcal{M}_T(X)} \inf_{\alpha \geq \mathcal{U}} h_\mu(\alpha|\beta, T) = \inf_{\alpha \geq \mathcal{U}} \sup_{\mu \in \mathcal{M}_T(X)} h_\mu(\alpha|\beta, T).
\]

**Proof of theorem 9.** Define:

\[
\begin{align*}
\breve{h}(\mathcal{U}|\beta, T) := &\ \sup_{\mu \in \mathcal{M}_T(X)} \inf_{\alpha \geq \mathcal{U}} h_\mu(\alpha|\beta, T) = \sup_{\mu \in \mathcal{M}_T(X)} h_\mu^+ (\mathcal{U}|\beta, T), \\
\check{h}(\mathcal{U}|\beta, T) := &\ \inf_{\alpha \geq \mathcal{U}} \sup_{\mu \in \mathcal{M}_T(X)} h_\mu(\alpha|\beta, T).
\end{align*}
\]
Clearly $\hat{h}(U|\beta, T) \leq h(U|\beta, T)$. By theorem 6 $h(U|\beta, T) \leq \hat{h}(U|\beta, T)$ and by theorem 7 $\hat{h}(U|\beta, T) \leq h(U|\beta, T)$. So $h(U|\beta, T) = \hat{h}(U|\beta, T) = h(U|\beta, T)$. □

In what follows we prove theorems 6–8. The first step is to prove theorem 6 that requires some combinatorial arguments and basic properties of these notions with respect to factors and other classic results. The next step is to prove theorems 7 and 8. To do so we rely heavily in the Jewett–Krieger theorem and thus we need to prove an Ergodic Decomposition for these new notions.

4. Basic combinatorial properties and proof of theorem 6

In this section we prove theorem 6. The first step is an extremely useful lemma, first stated in [R], that will be used many times in the sequel. It proves several properties of $h^+_\mu$ and $h^-_\mu$ and a formula to relate them.

**Lemma 10.** Let $(X, B, \mu, T)$ a MDS and $U \in C_X$ and $\beta \in \mathcal{P}_X$. Then:

1. $h^-_\mu(U|\beta, T) \leq h^+_\mu(U|\beta, T)$ and $h^-_\mu(U|\beta, T) \leq h(U|\beta, T)$.
2. $\forall M \in \mathbb{N}$, $h^-_\mu(U|\beta, T) = \frac{1}{M} h^+_\mu(U^{M-1}_0|\beta^{M-1}_0, T^M)$.
3. $h^-_\mu(U|\beta, T) = \lim_{M \to \infty} \frac{1}{M} h^+_\mu(U^{M-1}_0|\beta^{M-1}_0, T^M)$.

**Proof.** Fix $U \in C_X$ and $\beta \in \mathcal{P}_X$.

1. By proposition 5 part (1), $H_\mu(U|\beta) \leq \log N(U|\beta)$ so:

   $$h^-_\mu(U|\beta, T) = \lim_{N \to \infty} \frac{1}{N} H_\mu(U^{N-1}_0|\beta^{N-1}_0),$$

   $$\leq \lim_{N \to \infty} \frac{1}{N} \log N(U^{N-1}_0|\beta^{N-1}_0) = h(U|\beta, T).$$

   For any $N \in \mathbb{N}$, $H_\mu(U^{N-1}_0|\beta^{N-1}_0) \leq H^*_\mu(U^{N-1}_0|\beta^{N-1}_0) \leq \inf_{\alpha \geq U} H_\mu(\alpha^{N-1}_0|\beta^{N-1}_0).$ Dividing by $N$ and taking limit proves that $h^-_\mu(U|\beta, T) \leq h^+_\mu(U|\beta, T)$.

2. For every $M \in \mathbb{N}$ just by definition:

   $$h^-_\mu(U|\beta, T) = \lim_{N \to \infty} \frac{1}{NM} H_\mu(U^{NM-1}_0|\beta^{NM-1}_0),$$

   $$= \frac{1}{M} \left( \sum_{n=0}^{N-1} \frac{1}{N} H_\mu(U^{nM}_0|\beta^{nM}_0) \right),$$

   $$= \frac{1}{M} h^+_\mu(U^{M-1}_0|\beta^{M-1}_0, T^M).$$

3. From parts (1) and (2) for every $M \in \mathbb{N}$;

   $$h^-_\mu(U|\beta, T) = \frac{1}{M} h^+_\mu(U^{M-1}_0|\beta^{M-1}_0, T^M) \leq \frac{1}{M} h^+_\mu(U^{M-1}_0|\beta^{M-1}_0, T^M).$$

   Also for every $N \in \mathbb{N}$ using proposition 5 parts (3) and (4):
\[
\frac{1}{M} h_\mu^T((U_0^{M-1}|\beta_0^{M-1},T^M) = \lim_{N \to \infty} \frac{1}{NM} \inf_{\alpha \geq \mu_{U_0}} H_\mu \left( \sum_{n=0}^{N-1} T^{-nM} \alpha | \beta_0^{N-1} \right), \\
\leq \lim_{N \to \infty} \frac{1}{NM} \inf_{\alpha \geq \mu_{U_0}} \sum_{n=0}^{N-1} H_\mu \left( T^{-nM} \alpha | T^{-nM} \beta_0^{M-1} \right), \\
\leq \frac{1}{M} \inf_{\alpha \geq \mu_{U_0}} H_\mu \left( \alpha | \beta_0^{M-1} \right) = \frac{1}{M} H_\mu \left( U_0^{M-1}|\beta_0^{M-1} \right).
\]

Taking limit as \( M \) goes to infinity in this equation and equation (10) concludes the proof.


\[\square\]

**Proposition 11.** Let \((X,T)\) and \((Y,S)\) be TDS, \(\mu \in \mathcal{M}_T(X)\) and \(\nu \in \mathcal{M}_S(Y).\) Let \(\varphi:(X,T,\mu) \to (Y,S,\nu)\) be a measure-theoretical factor map, \(U \in \mathcal{C}_Y\) and \(\beta \in \mathcal{P}_Y.\) Then \(h_\varphi(U|\beta), S = h_\varphi((\varphi^{-1}U|\varphi^{-1}\beta))\) and \(h(U|\beta, S) = h((\varphi^{-1}U|\varphi^{-1}\beta), T),\)

**Proof.** Fix \(U \in \mathcal{C}_Y\) and \(\beta \in \mathcal{P}_Y.\) For every \(N \in \mathbb{N}, N((\varphi^{-1}U_0^{N-1}|\varphi^{-1}\beta_0^{N-1}) = N(U_0^{N-1}|\beta_0^{N-1})\) so taking log diving by \(N\) and taking limit as \(N\) tends to infinity proves that \(h(U|\beta, S) = h((\varphi^{-1}U|\varphi^{-1}\beta), T),\)

Since \(\varphi^{-1} \text{Ext}(U) = \text{Ext}(\varphi^{-1}U),\) for any measurable set \(B \mu_{\varphi^{-1}B} \circ \varphi^{-1} = \nu_B\) from equation (8):

\[H_\varphi(U) = \min_{\alpha \in \text{Ext}(U)} H_\mu \varphi^{-1}\alpha = \min_{\alpha \in \text{Ext}(U)} H_\mu \varphi^{-1}\alpha,
\]

\[= \min_{\alpha \in \varphi^{-1} \text{Ext}(U)} H_\mu \varphi^{-1}\alpha = \min_{\alpha \in \text{Ext}(\varphi^{-1}U)} H_\mu \varphi^{-1}\alpha = H_\mu \varphi^{-1}U.
\]

Taking the sum over all \(B \in \beta\) we conclude that:

\[H_\mu((\varphi^{-1}U|\varphi^{-1}\beta) = H_\nu(U|\beta).
\]

Using equation (11) since \(S \circ \varphi = \varphi \circ T\) for any \(N \in \mathbb{N}\) we have that:

\[H_\mu \left( \sum_{n=0}^{N-1} S^{-n}U | \sum_{n=0}^{N-1} S^{-n} \beta \right) = H_\mu \left( \sum_{n=0}^{N-1} S^{-n}U | \sum_{n=0}^{N-1} S^{-n} \beta \right).
\]

Dividing and taking the limit when \(N\) goes to infinity concludes the proof.

\[\square\]

**Lemma 12.** Let \((X,T)\) a TDS, \(U \in \mathcal{C}_X\) and \(\beta \in \mathcal{P}_X.\) For every family \(\{\alpha_i\}_{i=1}^K\) of finite partitions finer than \(U,\) for every \(N \in \mathbb{N} \) choose \(B \in \beta_0^{N-1}\) such that the maximum of \(N(U_0^{N-1}|\beta_0^{N-1})\) is attained.

There exists a finite subset \(P \subseteq B\) such that every element of \(\langle \alpha_i \rangle_0^{N-1}\) contains at most one point of \(P\) for every \(1 \leq l \leq K\) and \(|P| \geq \frac{N(U_0^{N-1}|\beta_0^{N-1})}{K}.\)

**Proof.** Choose \(U \in \mathcal{C}_X, \beta \in \mathcal{P}_X, N \in \mathbb{N}\) and \(B \in \beta_0^{N-1}\) as stated.

For \(x \in X\) and \(l \leq 1 \leq K,\) let \(A_l(x)\) be the element in \(\langle \alpha_i \rangle_0^{N-1}\) that contains \(x.\) Take \(x_1 \in B.\)

If \(B \subseteq \bigcup_{l=1}^K A_l(x)\) then \(P = \{x_1\}.\) Every element in \(\langle \alpha_i \rangle_0^{N-1}\) contains at most one point of
For every $1 \leq l \leq K$ and $|P|K \geq N(U_{0}^{N_l} \mid \beta_{0}^{N_l})$.

If $B \subseteq \bigcup_{i=1}^{K} A_{i}(x_{1})$ let $B_{l} = B \setminus \bigcup_{i=1}^{K} A_{i}(x_{1})$ and take $x_{2} \in B_{l}$. If $B_{1} \subseteq \bigcup_{i=1}^{K} A_{i}(x_{2})$ then $P = \{x_{1} \cdot x_{2}\}$. Every element in $(\alpha_{l})_{i}^{N_{l}}$ contains at most one point of $P$ for every $1 \leq l \leq K$ and $|P|K \geq N(U_{0}^{N_l} \mid \beta_{0}^{N_l})$.

Otherwise, $B_{2} = B_{1} \setminus \bigcup_{n=1}^{n} A_{l}(x_{2})$. Since this is a finite procedure we obtain a set $\{x_{1}, \ldots, x_{m}\}$ such that $B \subseteq \bigcup_{n=1}^{n} \bigcup_{j=1}^{n} A_{l}(x_{j})$.

Let $P = \{x_{1}, \ldots, x_{m}\}$. By construction $(\alpha_{l})_{i}^{N_{l}}$ contains at most one point of $P$ for every $1 \leq l \leq K$ and $|P|K = |\bigcup_{n=1}^{n} \bigcup_{j=1}^{n} A_{l}(x_{j})| \geq N(U_{0}^{N_l} \mid \beta_{0}^{N_l})$.

Following the ideas stated in [BGH] and using the results that we just proved in this section we can now prove the first theorem.

**Proof of theorem 6.** First we prove in the 0-dimensional case that for every $U = \{U_{1}, \ldots, U_{d}\} \in \mathcal{C}_{X}$ and $\beta \in \mathcal{P}_{\mu}$ there exists $\mu \in \mathcal{M}_{T}(X)$ such that $h^{T_{n}}_{\mu}(U \mid \beta, T) \geq h(U \mid \beta, T)$.

The set of clopen sets of a zero dimensional set $X$ is a countable set, thus the family of partitions in $\mathcal{U}$ consisting of clopen sets is countable and lets enumerate it as $\{\alpha_{l} / l \in \mathbb{N}\}$.

Now fix $n \in \mathbb{N}$ and use lemma 12 with $N = n^{2}$, $K = n$ and the family $\{T^{-n} \alpha_{l}\}_{l=1}^{n}$ applied to the cover $T^{-n} \mathcal{U}$ and the partition $T^{-n} \beta$. We obtain $C_{n} \in \beta_{n}^{2+n-1}$ and $P_{n} \subseteq C_{n}$ such that $|P_{n}| \geq N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}})$ and every element of $(\alpha_{l})_{i}^{N_{l}}$ contains at most one point of $P_{n}$ for every $1 \leq l \leq n$. To simplify the size of the formulas denote $N = n^{2} + n$.

Denote as $\nu_{n}$ the counting measure over $P_{n}$ and choose $1 \leq i, l \leq n$. By definition and basic properties:

$$H_{T_{n}}((\alpha_{l})_{i}^{N_{l}} \mid \beta_{0}^{N_{l}}) \geq H_{T_{n}}((\alpha_{l})_{i}^{N_{l}} \mid \beta_{0}^{N_{l}}) - H_{T_{n}}(\beta_{0}^{N_{l}}),$$

$$= H_{T_{n}}((\alpha_{l})_{i}^{N_{l}}) - H_{T_{n}}(\beta_{0}^{N_{l}}),$$

$$\geq H_{T_{n}}((\alpha_{l})_{i}^{N_{l}}) - \log |\beta_{0}^{N_{l}}|.$$  \hspace{1cm} (13)

From proposition 3 parts (2), (4) and (6):

$$N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}}) \leq N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}}) \cdot N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}}),$$

$$\leq N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}}) \cdot N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}}),$$

$$\leq d^{n} \cdot N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}}).$$  \hspace{1cm} (14)

Since $T^{-i}((\alpha_{l})_{i}^{N_{l}} \geq (\alpha_{l})_{i}^{N_{l}}$ then every element contains at most one atom of the discrete measure $\nu_{n}$ and so using equations (13) and (14):

$$H_{T_{n}}((\alpha_{l})_{i}^{N_{l}} \mid \beta_{0}^{N_{l}}) \geq H_{\nu_{n}}(T^{-i}(\alpha_{l})_{i}^{N_{l}}) - \log |\beta_{0}^{N_{l}}|,$$

$$\geq \log \left[\frac{N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}})}{n}\right] - \log |\beta_{0}^{N_{l}}|,$$

$$\geq \log \left[\frac{N(U_{0}^{N_{l}} \mid \beta_{0}^{N_{l}})}{nd^{n}}\right] - \log |\beta_{0}^{N_{l}}|.$$  \hspace{1cm} (15)
Fix $m \in \mathbb{N}$ with $m \leq n$ and decompose $N = km + b$ with $0 \leq b \leq m - 1$. Then:

\[ H_{T^{\nu_n}}((\alpha_0)^{N-1}|\beta_0^{N-1}) = H_{T^{\nu_n}} \left( \bigvee_{j=0}^{k-1} T^{-mj}(\alpha_0)^{m-1} \cap (\alpha_j)^{N-1}|\beta_0^{N-1} \right), \]

\[ \leq \sum_{j=0}^{k-1} H_{T^{\nu_n}}(T^{-mj}(\alpha_0)^{m-1}|\beta_0^{N-1}) + H_{T^{\nu_n}}((\alpha_j)^{N-1}), \]

\[ \leq \sum_{j=0}^{k-1} H_{T^{\nu_n}}(T^{-mj}(\alpha_0)^{m-1}|T^{-mj}\beta_0^{m-1}) + \log |(\alpha_j)^{N-1}|, \]

\[ \leq \sum_{j=0}^{k-1} H_{T^{\nu_n}}((\alpha_0)^{m-1}|\beta_0^{m-1}) + m \log d. \]

Adding up for every $0 \leq i \leq m - 1$ from equation (15) we obtain:

\[ m \log \left[ \frac{\mathcal{N}(\mathcal{U}_0^{N-1}|\beta_0^{N-1})}{nd^m} \right] - m \log |\beta_0^{m-1}| \leq \sum_{i=0}^{m-1} H_{T^{\nu_n}}((\alpha_0)^{N-1}|\beta_0^{N-1}), \]

\[ \leq \sum_{i=0}^{m-1} \sum_{j=0}^{k-1} H_{T^{\nu_n}}((\alpha_i)^{m-1}|\beta_0^{m-1}) + m^2 \log d, \]

\[ \leq \sum_{i=0}^{N-1} H_{T^{\nu_n}}((\alpha_0)^{m-1}|\beta_0^{m-1}) + m^2 \log d. \quad (16) \]

Denote as $\mu_n$ the Cesaro mean measure of $(T^{\nu_n})_{j=0}^{N-1}$. By concavity and using equation (16) we get for any $0 \leq l \leq n$:

\[ H_{\mu_n}((\alpha_0)^{m-1}|\beta_0^{m-1}) \geq \frac{1}{N} \sum_{j=0}^{N-1} H_{T^{\nu_n}}((\alpha_j)^{m-1}|\beta_0^{m-1}), \]

\[ \geq \frac{m}{N} \left( \log \left[ \frac{\mathcal{N}(\mathcal{U}_0^{N-1}|\beta_0^{N-1})}{nd^m} \right] - \log |\beta_0^{m-1}| \right). \quad (17) \]

We can assume by taking a subsequence that there exists $\mu \in \mathcal{M}(X)$ such that in the weak-* topology $\lim_{n \to \infty} \mu_n = \mu$. Also by construction $\mu \in \mathcal{M}_T(X)$. Clearly:

\[ h(\mathcal{U}|\beta, T) = \lim_{n \to \infty} \frac{1}{N} \left( \log \left[ \frac{\mathcal{N}(\mathcal{U}_0^{N-1}|\beta_0^{N-1})}{nd^m} \right] - m \log d - \log |\beta_0^{m-1}| \right). \quad (18) \]

Since all elements of $(\alpha_0)^{m-1}$ are clopen by upper semicontinuity and using equations (17) and (18) divided by $m$ we obtain for any $l, m \in \mathbb{N}$:

\[ \frac{1}{m} H_{\mu}((\alpha_0)^{m-1}|\beta_0^{m-1}) \geq \limsup_{n \to \infty} \frac{1}{m} H_{\mu_n}((\alpha_0)^{m-1}|\beta_0^{m-1}) = h(\mathcal{U}|\beta, T). \quad (19) \]
Taking limit as \( m \) tends to infinity proves that \( h^\mu_{p}(\alpha|\beta, T) \geq h(U|\beta, T) \). Since \( \{\alpha\}_{I \in \mathbb{N}} \) is in \( L^1_{\nu} \)-dense in the set of Borel partitions finer than \( U \) this proves that \( h_{p}^{\mu}(U|\beta, T) \geq h(U|\beta, T) \). By lemma 10 part (3) this proves that \( h_{p}^{\mu}(U|\beta, T) \geq h(U|\beta, T) \) but since for any \( \mu \in \mathcal{M}_{Y}(X) \), \( h_{p}^{-}(U|\beta, T) \leq h(U|\beta, T) \) we conclude that for the 0-dimensional case \( h_{p}^{-}(U|\beta, T) = h(U|\beta, T) \).

As in [BGH] we use the existence of a zero dimensional extension \((Y,S)\) of \((X,T)\), this is a continuous surjective map \( \varphi : Y \to X \) such that \( \varphi \circ S = T \circ \varphi \) where \((Y,S)\) is a zero-dimensional TDS.

Choose \( U \in C_{X} \), \( \beta \in \mathcal{P}_{X} \) and apply the previous case for \((Y,S)\), the cover \( \varphi^{-1}U \) and the partition \( \varphi^{-1}\beta \) to obtain \( \nu \in \mathcal{M}_{S}(Y) \) such that \( h_{p}^{-}(\varphi^{-1}U|\varphi^{-1}\beta,S) = h(\varphi^{-1}U|\varphi^{-1}\beta,S) \). Define \( \mu = \nu \circ \varphi \) from proposition 11 we conclude that:

\[
\begin{align*}
  h_{p}^{-}(U|\beta,S) &= h_{p}^{-}(\varphi^{-1}U|\varphi^{-1}\beta,S) = h(\varphi^{-1}U|\varphi^{-1}\beta,S) = h(U|\beta,T).
\end{align*}
\]

(20)

In order to prove the inequality for \( h_{p}^{\mu} U|\beta,T \) just notice that it is always true that \( h_{p}^{\mu}(U|\beta,T) \geq h_{p}^{-}(U|\beta,T) \). \( \square \)

5. Ergodic decomposition and proof of theorems 7 and 8

There are several approaches to define the ergodic decomposition of the measure theoretical entropy and we choose the simple and clear approach taken in [D]. The idea is to use the notion of disintegration of the measure \( \mu \) with respect to a Borel sub-\( \sigma \)-algebra \( D \subseteq B_{X} \). There is a one to one correspondence between a sub-\( \sigma \)-algebra \( D \) and a measure theoretical factor \((Y,D',\nu)\) where the elements \( y \in Y \) are the atoms of \( D \) and the map \( \varphi : X \to Y \) is defined by inclusion with \( \varphi^{-1}D' = D \) and \( \nu = \mu \circ \varphi^{-1} \).

For any \( B \in B \), the function \( \mathbb{E}_{\mu}(\mathbb{1}_{B}|D) : X \to \mathbb{R} \) for \( \nu \)-almost every \( y \in Y \), is \( \mu \)-almost surely constant in \( \varphi^{-1}y \in Y \) so we can consider that \( \mathbb{E}_{\mu}(\mathbb{1}_{B}|D) \) is defined over \( Y \). Using this we define the disintegration of \( \mu \) as the \( \nu \)-almost everywhere defined assignment \( y \to \mu_{y}(\cdot) := \mathbb{E}_{\mu}(\mathbb{1}_{\cdot}|D)(y) \in \mathcal{M}(X) \) where \( \mu_{y} \) is a probability measure on \( B \) supported by the atom \( \{y\} \). By simple properties of the conditional expectation:

\[
\begin{align*}
  \forall B \in B, \mu(B) = \mathbb{E}_{\nu}(\mathbb{E}_{\mu}(\mathbb{1}_{B}|D)) = \int_{Y} \mu_{y}(B) \, d\nu(y) = \mathbb{E}_{\nu}(\mathbb{E}_{\mu}(\mathbb{1}_{B}|D)).
\end{align*}
\]

(21)

In our case we consider \( D = \{B \in B / T^{-1}B = B\} \). For the case of any \( T \)-invariant \( \sigma \)-algebra we can also define a factor \((Y,D,\nu,S)\) such that for any \( y \in Y \), \( \mu_{Sy} = \mu_{y} \circ T^{-1} \). In our choice of \( D \) we also have that for any \( y \in Y \), \( \mu_{y} \in \mathcal{M}_{S}(X) \).

From theorem 2.6.4 in [D] for any \( \alpha \in \mathcal{P}_{X} \):

\[
\begin{align*}
  h_{\mu}(\alpha,T) = \int_{Y} h_{\mu_{y}}(\alpha,T) \, d\nu(y).
\end{align*}
\]

(22)

For any \( \beta \in \mathcal{P}_{X} \) since \( h_{\mu}(\alpha|\beta, T) = h_{\mu}(\alpha \vee \beta, T) - h_{\mu}(\beta, T) \) we also have that:

\[
\begin{align*}
  h_{\mu}(\alpha|\beta, T) = \int_{Y} h_{\mu_{y}}(\alpha|\beta, T) \, d\nu(y).
\end{align*}
\]

(23)

Using the same ideas given in [HMRY] we prove that:
Lemma 13. For any $U \in \mathcal{C}_X$, $\beta \in \mathcal{P}_X$

\[
h^+_\mu(U|\beta, T) = \int_Y h^+_\mu(U|\beta, T) \, d\nu(y), \quad h^-_\mu(U|\beta, T) = \int_Y h^-_\mu(U|\beta, T) \, d\nu(y).
\]  

(24)

Proof. Let $U = \{U_1, \ldots, U_M\} \in \mathcal{C}_X$ and $\beta \in \mathcal{P}_X$. Since $X$ is a compact metric space there exists a sequence $\{\alpha_k\}_{k \in \mathbb{N}}$ in $U^*$ that is $L^1_\nu$-dense in $U^*$ for every $\eta \in \mathcal{M}(X)$.

So in particular we have that for any $\eta \in \mathcal{M}_T(X)$:

\[
h^+_\eta(U|\beta, T) = \inf_{k \in \mathbb{N}} h^+_\eta(\alpha_k|\beta, T).
\]  

(25)

Denote for any $k \in \mathbb{N}$, $\alpha_k = \{A^k_1, \ldots, A^k_M\}$ where $A^k_\mu \subseteq U_m$ for any $1 \leq m \leq M$. By equations (23) and (25) and Fatou’s lemma we have that:

\[
h^+_\mu(U|\beta, T) = \inf_{k \in \mathbb{N}} h^+_\mu(\alpha_k|\beta, T) = \inf_{k \in \mathbb{N}} \int_Y h^+_\mu(\alpha_k|\beta, T) \, d\nu(y),
\]

\[
\geq \int_Y \inf_{k \in \mathbb{N}} h^+_\mu(\alpha_k|\beta, T) \, d\nu(y) = \int_Y h^+_\mu(U|\beta, T) \, d\nu(y).
\]  

(26)

For every $\epsilon > 0$ and $n \in \mathbb{N}$ define $B^n_\epsilon = \{y \in Y/h^+_\mu(\alpha_k|\beta, T) < h^+_\mu(U|\beta, T) + \epsilon\}$. By equation (25) we know that $\mu(Y \Delta \bigcup_{n \in \mathbb{N}} B^n_\epsilon) = 0$ and so there exists $\{Y_n\}_{n \in \mathbb{N}}$ a $\nu$-partition of $X$, $\mu(Y_n) > 0$ and a subsequence of partitions $\{\alpha_{k_n}\}_{k \in \mathbb{N}}$ such that for any $n \in \mathbb{N}$ and $\nu$-almost every $y \in Y_n$, $h^+_\mu(\alpha_{k_n}|\beta, T) < h^+_\mu(U|\beta, T) + \epsilon$.

Now define for any $n \in \mathbb{N}$ a measure $\mu_n \in \mathcal{M}_T(X)$ as:

\[
\forall B \in \mathcal{B}, \mu_n(B) = \frac{1}{\mu(Y_n)} \int_Y \mu(B \cap Y_n) \, d\nu(y).
\]  

(27)

By definition:

\[
h^+_\mu(\alpha_{k_n}|\beta, T) = \frac{1}{\mu(Y_n)} \int_Y h^+_\mu(\alpha_{k_n}|\beta, T) \, d\nu(y),
\]

\[
\leq \frac{1}{\mu(Y_n)} \int_Y h^+_\mu(U|\beta, T) \, d\nu(y) + \epsilon.
\]  

(28)

Notice that for any $n, m \in \mathbb{N}$, $\mu_n(Y_m) = \mathbbm{1}_{(m=m)}$. For any $1 \leq m \leq M$ define $A_m = \bigcup_{n \in \mathbb{N}} (Y_n \cap A^k_m)$ and $\alpha = \{A_1, \ldots, A_M\} \in U^*$. By construction and equation (28):

\[
h^+_\mu(U|\beta, T) \leq h^+_\mu(\alpha|\beta, T) = \sum_{n \in \mathbb{N}} \mu(Y_n) h^+_\mu(\alpha, T),
\]

\[
= \sum_{n \in \mathbb{N}} \mu(Y_n) h^+_\mu(\alpha_{k_n}, T) \leq \int_Y h^+_\mu(U|\beta, T) \, d\nu(y) + \epsilon.
\]  

(29)

Since this is true for any $\epsilon > 0$ this proves the reverse inequality for $h^+_\mu(U|\beta, T)$. Finally using lemma 10 part (3) proves the result for $h^-_\mu(U|\beta, T)$. □
The final step in this procedure is to prove equality between \( h^+_\mu \) and \( h^-_\mu \). To do so we state the universal version of Rohlin’s lemma given in [Gw].

**Lemma 14 ([Gw]).** Let \((X, T)\) an invertible TDS. Then \( \forall N \in \mathbb{N} \) and \( \epsilon > 0 \), there exists \( D \in \mathcal{B}(X) \) such that \( D, TD, \ldots, T^{N-1}D \) are pairwise disjoint and \( \mu(\bigcup_{n=0}^{N-1} T^nD) > 1 - \epsilon \) for every non atomic measure \( \mu \in \mathcal{M}_f^\epsilon(X) \).

**Proof of theorem 7.** By lemma 13 we just need to prove it for ergodic measures moreover just for non atomic ergodic measures since atomic ergodic measures have zero measure theoretical entropy.

For any \( k \in \mathbb{N} \) let \( \mathcal{U} = \{U_1, \ldots, U_k\} \) and for any \( N \in \mathbb{N} \) let \( \alpha' \succeq U_0^{N-1} \) such that \( \mathcal{N}(\alpha'|\beta_0^{N-1}) = \mathcal{N}(U_0^{N-1}|\beta_0^{N-1}) \). Every element of \( \alpha' \) is of the form

\[
P = P_{i_0i_1\ldots i_{k-1}} \subseteq \bigcap_{j=0}^{N-1} T^{-j}U_{i_j}.
\]

For any \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) big enough such that:

\[
\mathcal{N}(U_0^{N-1}|\beta_0^{N-1}) \leq e^{N(h(\mathcal{U}|\beta)+\epsilon)},
\]

\[
-\frac{1}{N} \log \frac{1}{N} - \left(1 - \frac{1}{N}\right) \log \left(1 - \frac{1}{N}\right) < \epsilon.
\]

Let \( 0 < \delta < 1 \) small enough so that \( 2\sqrt{\delta} \log 2k < \epsilon \).

By lemma 14 \( \exists D \subseteq X \) with \( D, TD, \ldots, T^{N-1}D \) pairwise disjoint such that \( \mu(\bigcup_{n=0}^{N-1} T^nD) > 1 - \delta \) for every \( \mu \in \mathcal{M}_f^\epsilon(X) \) non atomic. Define the partition of \( D \), \( \alpha'_D = \{A' \cap D : A' \in \alpha'\} \).

We use the partition \( \alpha'_D \) to define a partition of cardinality at most \( 2k \) where we define \( k \) atoms \( \{A_1, \ldots, A_k\} \) over the set \( \bigcup_{n=0}^{N-1} T^nD \) assigning to the set \( A_i \) all sets

\[
T^iP \quad \text{con} \quad P = P_{ih\ldots i_{k-1}} \quad \text{where} \quad \imath_j = \imath_i \quad \text{and} \quad j = 0, \ldots, N-1.
\]

On the rest of the space \( \alpha \) we use any partition that refines \( \mathcal{U} \) that can be done with at most \( k \) atoms. By construction it is not difficult to see that \( \alpha_0^{N-1} \cap D \succeq \alpha'_D \). So, for any \( B \in \beta_0^{N-1} \):

\[
|\alpha_0^{N-1} \cap D \cap B| \leq |\alpha'_D \cap B| \leq |\alpha' \cap B|,
\]

\[
\leq \mathcal{N}(\alpha'|\beta_0^{N-1}) = \mathcal{N}(U_0^{N-1}|\beta_0^{N-1}).
\]

Let us fix \( \mu \in \mathcal{M}_f^\epsilon(X) \). We will show that

\[
h^-_\mu(\alpha|\beta, T) \leq h(\mathcal{U}|\beta, T) + 3\epsilon.
\]

Let \( E = \bigcup_{i=0}^{N-1} T^nD \) such that \( \mu(E) > 1 - \delta \). For \( n > N \), define:

\[
G_n = \{x \in X/\frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_E(T^i x) > 1 - \sqrt{\delta}\}.
\]
By definition $\int_X \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_E(T^ix) \, d\mu(x) > 1 - \delta$ and since $\frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_E(T^ix) \leq 1$ then:

$$
\mu(G_n) + (1 - \sqrt{\delta})(1 - \mu(G_n)) \geq \int_X \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_E(T^ix) \, d\mu(x) > 1 - \delta \Rightarrow \mu(G_n) > 1 - \sqrt{\delta}.
$$

For $x \in G_n$, define $S_n(x) = \{i \in \{0, 1, \ldots, n-1\} \mid T^ix \in D\}$. By definition for any $i = 0, 1, \ldots, n-1$, $T^ix \notin E$ if and only if $\forall j \in \{0, 1, \ldots, N-1\}, i \notin S_n(x) + j$ thus:

$$
\left| \{0, 1, \ldots, n-1\} \setminus \bigcup_{j=0}^{N-1} (S_n(x) + j) \right| \leq n\sqrt{\delta}. \quad (33)
$$

Let $F_n = \{S_n(x)/x \in G_n\}$. By disjointness of the $T$-iterations of $D$ for any $F = \{s_1, \ldots, s_l\} \in F_n, F \cap (F + i) = \emptyset, i = 1, \ldots, N-1$, so $|F| \leq \frac{n}{N} + 1$. Define $a_n = \left[ \frac{n}{N} \right] + 1$ then:

$$
|F_n| \leq \sum_{j=1}^{a_n} \binom{n}{j} \leq a_n \binom{n}{a_n} \leq n \binom{n}{a_n}.
$$

By Stirling’s formula and equation (30):

$$
\lim_{n \to \infty} \frac{1}{n} \log \left( \binom{n}{a_n} \right) = - \frac{1}{N} \log \left( \frac{1}{N} \right) - \left( 1 - \frac{1}{N} \right) \log \left( 1 - \frac{1}{N} \right) < \epsilon.
$$

This implies that:

$$
\lim_{n \to \infty} \sup_{|F|} \frac{1}{n} \log |F_n| \leq \lim_{n \to \infty} \frac{1}{n} \log \left( \binom{n}{a_n} \right) \leq \epsilon. \quad (34)
$$

For $F \in F_n$ define $B_F = \{x \in G_n/S_n(x) = F\}$ and $\gamma = \{B_F/F \in F_n\}$ a measurable partition of $G_n$.

Let $H_F = \{0, 1, \ldots, n-1\} \setminus \bigcup_{j=0}^{N-1} (F + j)$ by (33), $|H_F| \leq n\sqrt{\delta}$.

For any $B = \bigcap_{i=0}^{N-1} T^{-i}B_i \in \beta_0^{n-1}$ define $\forall j = 1, \ldots, l-1$:

$$
B^j = \bigcap_{i=0}^{n-1} T^{-i}B_{j+i} \quad \text{and} \quad B^{\gamma} = \bigcap_{i=0}^{n-1} T^{-i}B_{n+j+i}.
$$

Then $B = \bigcap_{j=1}^{l-1} T^{-s_j}B(s_j) \cap T^{-s_j}B(s_j) \cap \bigcap_{i \in H_F} T^{-i}B_r$.

Since for all $j = 1, \ldots, l-1$, $B(s_j) \in \beta_j \cap \beta_0^{n-1}$. Since $B_F \subseteq \bigcap_{j=1}^{l-1} T^{-s_j}D$ and equation (31):

\[\text{nonlinearity} \quad (31)\]
\[ |\alpha_0^{-1} \cap B_F \cap B| \leq \left| \bigcap_{j=1}^{f} T^{-j}(\alpha_0^{-1} \cap D \cap B^0) \cup \bigcup_{r \in H_r} T^{-r}(\alpha \cap B_r), \right| \]
\[ \leq \prod_{j=1}^{f} |\alpha_0^{-1} \cap D \cap B^0| \cdot \prod_{r \in H_r} |\alpha \cap B_r|, \]
\[ \leq N(\mathcal{U}_0^{N-1}|\beta_0^{-1})^{-1}|\alpha|^{-H_r} + N \leq N(\mathcal{U}_0^{N-1}|\beta_0^{-1})^{\frac{3}{2}} (2k)^{n \sqrt{\delta} + N}. \]

Then, adding over \( F \in \mathcal{F}_n \), we have for every \( B \in \beta_0^{-1} \)
\[ \sum_{F \in \mathcal{F}_n} |\alpha_0^{-1} \cap B_F \cap B| \leq |\mathcal{F}_n| N(\mathcal{U}_0^{N-1}|\beta_0^{-1})^{\frac{3}{2}} (2k)^{n \sqrt{\delta} + N}. \tag{35} \]

Then by equation (35):
\[ H_{\mu_B}(\alpha_0^{-1} \vee \gamma) \leq \sum_{F \in \mathcal{F}_n, C \in \alpha_0^{-1} \cap H_r} \phi(\mu_B(C)) + \phi(\mu_B(X \setminus G_n)), \]
\[ \leq \log \left( \sum_{F \in \mathcal{F}_n} |\alpha_0^{-1} \cap B_F \cap B| + 1 \right), \]
\[ \leq \log \left( |\mathcal{F}_n| N(\mathcal{U}_0^{N-1}|\beta_0^{-1})^{\frac{3}{2}} (2k)^{n \sqrt{\delta} + N} + 1 \right). \tag{36} \]

A simple calculation shows that:
\[ \mu(B)H_{\mu_B}(\alpha_0^{-1} \cap \{X \setminus G_n\}) = \mu((X \setminus G_n) \cap B)H_{\mu_B((X \setminus G_n) \cap \beta_0^{-1})) + \mu(B)H_{\mu_B}((X \setminus G_n)),\]
\[ \leq \mu((X \setminus G_n) \cap B) \log(2k)^n + \mu(B) \log 2. \tag{37} \]

By equations (36) and (37):
\[ H_\mu(\alpha_0^{-1} | \beta_0^{-1}) \leq H_\mu(\alpha_0^{-1} \vee (\gamma \cup \{X \setminus G_n\}) | \beta_0^{-1}), \]
\[ = \sum_{B \in \beta_0^{-1}} \mu(B) [H_{\mu_B}(\alpha_0^{-1} \vee \gamma) + H_{\mu_B}(\alpha_0^{-1} \cap \{X \setminus G_n\})], \]
\[ \leq \log \left( |\mathcal{F}_n| N(\mathcal{U}_0^{N-1}|\beta_0^{-1})^{\frac{3}{2}} (2k)^{n \sqrt{\delta} + N} + 1 \right) + \sqrt{\delta} \log(2k)^n + \log 2. \tag{38} \]

By the form in which we choose \( \delta \) and equations (34) and (38):
\[ h_\mu(\alpha|\beta, T) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{-1} | \beta_0^{-1}), \]
\[ \leq \limsup_{n \to \infty} \frac{1}{n} \left( \log |\mathcal{F}_n| + \frac{n}{N} \log N(\mathcal{U}_0^{N-1}|\beta_0^{-1}) + (2n \sqrt{\delta} + N) \log 2k + \log 2 \right), \]
\[ \leq \frac{1}{N} \log N(\mathcal{U}_0^{N-1}|\beta_0^{-1}) + 2\epsilon. \]

Finally by equation (30), \( h_\mu(\alpha|\beta, T) \leq h(\mathcal{U}|\beta, T) + 3\epsilon. \)

Since this is true \( \forall \mu \in \mathcal{M}_T(X) \) and \( \epsilon \) is arbitrary we conclude that:
The final step in this procedure is to prove equality between \( h^+_{\mu} \) and \( h^-_{\mu} \). For the proof of this last theorem we need an analogue of lemma 9 in [HMRY]:

**Lemma 15.** For any \( \beta \in P_X \), for every \( M \in \mathbb{N} \) and \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for every pair of measurable covers \( \mathcal{U} = \{ U_1, \ldots, U_M \} \) and \( \mathcal{V} = \{ V_1, \ldots, V_M \} \) such that \( \mu(\mathcal{U} \Delta \mathcal{V}) < \delta \) one has that

\[
| h^+_{\mu}(U|\beta, T) - h^+_{\mu}(V|\beta, T) | \leq \epsilon.
\]

**Proof.** Fix \( M \in \mathbb{N} \) and \( \epsilon > 0 \) and choose \( \delta > 0 \) given by lemma 2. Now fix two measurable covers \( \mathcal{U} = \{ U_1, \ldots, U_M \} \) and \( \mathcal{V} = \{ V_1, \ldots, V_M \} \) such that \( \mu(\mathcal{U} \Delta \mathcal{V}) < \delta \).

First we prove that for every \( \alpha = \{ A_1, \ldots, A_M \} \in \mathcal{U}^* \) there exists \( \alpha' \in \mathcal{V}^* \) such that:

\[
h_{\mu}(\alpha|\beta, T) \geq h_{\mu}(\alpha'|\beta, T) - \epsilon. \tag{39}
\]

Define \( \alpha' = \{ A'_1, \ldots, A'_M \} \) as:

\[
A'_1 = V_1 \setminus \left( \bigcup_{m=1} A_m \cap V_m \right),
\]

\[
A'_m = V_m \setminus \left( \bigcup_{k>m} A_k \cap V_k \cup \bigcup_{1<k<m} A_k' \right) \quad \text{for } 1 < m \leq M.
\]

For \( 1 \leq m \leq M \) by construction \( A_m \cap V_m \subseteq A'_m, A_m \setminus A'_m \subseteq U_m \setminus V_m \) and \( A'_m \setminus A_m \subseteq \bigcup_{k \neq m} U_k \setminus V_k \).

This implies that \( A_m \Delta A'_m \subseteq \bigcup_{m=1} U_m \Delta V_m \) and so \( \mu(\alpha \Delta \beta) < M \cdot \mu(\mathcal{U} \Delta \mathcal{V}) < \delta \) that proves

\[
h_{\mu}(\alpha|\beta, T) \geq h_{\mu}(\alpha'|\beta, T) - \epsilon.
\]

Since for every \( \alpha' \in \mathcal{V} \), \( h_{\mu}(\alpha'|\beta, T) \geq h^+_{\mu}(\mathcal{V}|\beta, T) \) from equation (39) we conclude that

\[
h_{\mu}(\alpha|\beta, T) \geq h^+_{\mu}(\mathcal{V}|\beta, T) - \epsilon.
\]

Taking infima over all \( \alpha \in \mathcal{U}^* \) we conclude that

\[
h^+_{\mu}(\mathcal{U}|\beta, T) \geq h^+_{\mu}(\mathcal{V}|\beta, T) - \epsilon.
\]

Exchanging the roles of \( \mathcal{U} \) and \( \mathcal{V} \) shows that

\[
h^+_{\mu}(\mathcal{V}|\beta, T) \geq h^+_{\mu}(\mathcal{U}|\beta, T) - \epsilon.
\]

And so \( | h^+_{\mu}(\mathcal{U}|\beta, T) - h^+_{\mu}(\mathcal{V}|\beta, T) | \leq \epsilon \). \( \square \)

**Proof of theorem 8.** First consider the case when \((X, T)\) is uniquely ergodic. Fix \( \mathcal{V} \in \mathcal{C}_X \) and define \( R = |\mathcal{V}| \) by lemma 15 there exists \( \delta_1 > 0 \) such that for every \( \mathcal{U}_1 \in \mathcal{C}_X, |\mathcal{U}_1| = R \) and \( \mu(\mathcal{U}_1 \Delta \mathcal{V}) < \delta_1 \) one has:

\[
h^+_{\mu}(\mathcal{V}|\beta, T) \leq h^+_{\mu}(\mathcal{U}_1|\beta, T) + \epsilon. \tag{40}
\]

By lemma 10 part (3) we can choose \( M \in \mathbb{N} \) such that:

\[
\frac{1}{M} h^+_{\mu}(\mathcal{V}_0^{M-1}|\beta_0^{M-1}, T^M) \leq h^+_{\mu}(\mathcal{U}_1|\beta, T) + \frac{\epsilon}{2}. \tag{41}
\]

Once again by lemma 15 there exists \( \delta_2 > 0 \) such that for every \( \mathcal{U}_2 \in \mathcal{C}_X, |\mathcal{U}_2| = R^M \) and \( \mu(\mathcal{U}_2 \Delta \mathcal{V}_0^{M-1}) < \delta_2 \) one has:
\[
\frac{1}{M} h^+_{\mu}(\nu_0^{M-1}|\beta_0^{M-1}, T^M) \leq \frac{1}{M} h^+_{\mu}(U_2|\beta_0^{M-1}, T^M) + \frac{\epsilon}{2}. \tag{42}
\]

Now choose any \( U \in C_\chi \) with \(|U| = R \) by unique ergodicity and theorems 6 and 7 we have that:
\[
h_{\mu}^+(U|\beta, T) = h(U|\beta, T) = h_{\mu}^+(|\beta, T). \tag{43}
\]

Choose \( \delta < \min\{\delta_1, \frac{\epsilon}{2}\} \). By simple calculations:
\[
\mu(U_0^{M-1}\Delta V_0^{M-1}) \leq R\mu(U\Delta V) < \delta_2. \tag{44}
\]

By equations (41) and (42):
\[
\frac{1}{M} h^+_{\mu}(U_0^{M-1}|\beta_0^{M-1}, T^M) \leq \frac{1}{M} h^+_{\mu}(V_0^{M-1}|\beta_0^{M-1}, T^M) + \frac{\epsilon}{2} \leq h_{\mu}^+(V|\beta, T) + \epsilon. \tag{45}
\]

By equations (41)–(43) and lemma 10 part (2):
\[
h_{\mu}^+(V|\beta, T) \leq h_{\mu}^+(U|\beta, T) + \epsilon = h_{\mu}^-(U|\beta, T) + \epsilon = \frac{1}{M} h_{\mu}^-(U_0^{M-1}|\beta_0^{M-1}, T^M) + \epsilon,
\]
\[
\leq \frac{1}{M} h_{\mu}^-(U_0^{M-1}|\beta_0^{M-1}, T^M) + \epsilon \leq h_{\mu}^-(V|\beta, T) + 2\epsilon. \tag{46}
\]

Since this is for any \( \epsilon > 0 \) this proves that \( h_{\mu}^+(V|\beta, T) = h_{\mu}^-(V|\beta, T) \).

Now by lemma 13 we just need to prove the equality for \( \mu \in M_\beta(Y) \).

By the Jewett–Krieger theorem (see [Kr]) there exists a \( \varphi \)-measure theoretical isomorphism with a uniquely ergodic TDS \((Y, S)\). In general \( h_{\mu}^+(\varphi^{-1}U|\varphi^{-1}S) \leq h_{\mu}^+(U|\beta) \) but since \( \varphi \) is an isomorphism it is also true that \( h_{\mu}^+(\varphi^{-1}U|\varphi^{-1}S) \geq h_{\mu}^+(U|\beta) \). By lemma 11 \( h_{\mu}^-(U|\beta) = h_{\mu}^-(\varphi^{-1}U|\varphi^{-1}S) = h_{\mu}^+(\varphi^{-1}U|\varphi^{-1}S) = h_{\mu}^+(U|\beta) \).

\( \square \)

6. Relation with other variational principles

This obviously extends the work done in [R] and [Gw] just by considering the case \( \beta = \{X\} \). Another direct application is the work done in [HYZ]. They consider a fixed factor \( \varphi \) denoted as \((Y, D, S)\) and consider \( B_{\varphi} = \varphi^{-1}B_X \) to define for every \( U \in C_\chi \):
\[
H_{\mu}(U| Y) := H_{\mu}(U| B_{\varphi}), \quad N(U| Y) := \sup_{y \in Y} N(U \cap \varphi^{-1}(y)),
\]
\[
h_{\mu}^-(U| Y, T) := \lim_{N \to \infty} \frac{1}{N} H_{\mu}(U| B_{\varphi}), h_{\mu}^+(U| Y, T) := \inf_{\alpha \geq t} h_{\mu}(\alpha| B_{\varphi}),
\]
\[
h(U| Y, T) := \lim_{N \to \infty} \frac{1}{N} \log N(U| B_{\varphi}).
\]

The notion in this paper allows us to condition with respect to any fixed partition, this is any \( \sigma \)-algebra in the language of [HYZ]. However in their case they require a \( \sigma \)-algebra invariant
with respect to the dynamics (thus giving a factor). Using their work we can only prove the results for partitions that satisfy $T^{-1} \beta = \beta$.

By taking infima over the set of partitions $\{\varphi^{-1} \beta / \beta \in P_Y\}$ by some classic and simple calculations (see remark 4 in [DS] for instance):

$$H_\mu(\mathcal{U}|B_\varphi) = \inf_{\beta \in P_X} H_\mu(\mathcal{U}|\varphi^{-1} \beta), \quad N(\mathcal{U}|Y) = \inf_{\beta \in P_X} N(\mathcal{U}|\varphi^{-1} \beta).$$

So lemma 13 and theorems 6–8 prove for a general non invariant $\sigma$-algebra theorems 3.6, 5.3, 5.4, 6.2 in [HYZ]. Theorem 9 proves a new theorem regarding these notions showing that the infima and the suprema can be reversed.

7. Conclusion and final remarks

The results of this paper show that the variational principle once again is more local than was previously stated, now with respect to the conditioning variable. This takes us one step closer to the ultimate local variational principle, this is, for an open cover conditioned with respect to a fixed open cover. Once again, the technique developed in [R] to define a ‘+’ and ‘−’ definitions of measure theoretical entropy and proving their equality remains crucial and unavoidable and works for all measurable covers.

The requirement of the cover to be open is once more proven to be necessary for the existence of a local variational principle. This fails even for closed covers as the pioneers of these ideas knew from the beginning for the global variational principles. This shows that this is not a merely combinatorial result and the topological structure is crucial to link measure and topology. In the local case, an equality is attained for a specific invariant measure and not just an equality for the suprema over all invariant measures as for the global principles but little is known about that measure. However in many cases for the global variational principles much is known about the measure that attains the equality when it does exist.

The extension of this work for more general actions requires extra work but at least for the case of countable discrete amenable groups by using the tool of Følner sequences it seems clear how to do it.

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