Common eigenfunctions of commuting differential operators of rank 2

V.Oganesyan

Introduction

Let us consider two differential operators

\[ L_n = \sum_{i=0}^{n} u_i(x) \partial_x^i, \quad L_m = \sum_{i=0}^{m} v_i(x) \partial_x^i. \]

If \( L_n \) and \( L_m \) commute, then there is a nonzero polynomial \( R(z, w) \) such that \( R(L_n, L_m) = 0 \) (see [1]). The curve \( \Gamma \) defined by \( R(z, w) = 0 \) is called the spectral curve. If \( L_n \psi = z\psi, \quad L_m \psi = w\psi, \)
then \((z, w) \in \Gamma.\) For almost all \((z, w) \in \Gamma\) the dimension of the space of common eigenfunctions \( \psi \) is the same. The dimension of the space of common eigenfunctions of two commuting differential operators is called the rank. The rank is a common divisor of \( m \) and \( n.\)

If the rank equals 1, then there are explicit formulas for coefficients of commutative operators in terms of Riemann theta-functions (see [2]).

The case when rank is greater than one is much more difficult. The first examples of commuting ordinary scalar differential operators of the nontrivial ranks 2 and 3 and the nontrivial genus \( g=1 \) were constructed by Dixmier [8] for the nonsingular elliptic spectral curve \( w^2 = z^3 - \alpha, \) where \( \alpha \) is arbitrary nonzero constant:

\[ L = (\partial_x^2 + x^3 + \alpha)^2 + 2x, \]
\[ M = (\partial_x^2 + x^3 + \alpha)^3 + 3x\partial_x^2 + 3\partial_x + 3x(x^2 + \alpha), \]

where \( L \) and \( M \) is the commuting pair of the Dixmier operators of rank 2, genus 1. There is an example

\[ L = (\partial_x^2 + x^2 + \alpha)^2 + 2\partial_x, \]
\[ M = (\partial_x^2 + x^2 + \alpha)^3 + 3\partial_x^4 + 3(x^2 + \alpha)\partial_x + 3x, \]

where \( L \) and \( M \) is the commuting pair of the Dixmier operators of rank 3, genus 1.
The general classification of commuting ordinary differential operators of rank greater than 1 was obtained by Krichever [3]. The general form of commuting operators of rank 2 for an arbitrary elliptic spectral curve was found by Krichever and Novikov [4]. The general form of operators of rank 3 for an arbitrary elliptic spectral curve was found by Mokhov [5], [6]. Mironov in [7] constructed examples of operators

\[ L = (\partial_x^2 + A_3 x^3 + A_2 x^2 + A_1 x + A_0)^2 + g(g + 1)A_3 x, \]

\[ M^2 = L^{2g+1} + a_{2g} L^{2g} + \ldots + a_1 L + a_0, \]

where \( a_i \) are some constants and \( A_i, A_3 \neq 0 \), are arbitrary constants. Operators \( L \) and \( M \) are commuting operators of rank 2, genus \( g \).

Examples of commuting ordinary differential operators of arbitrary genus and arbitrary rank with polynomial coefficients were constructed in [11] by Mokhov.

It is proved in [12] that the operators

\[ L = (\partial_x^2 + A x^6 + B x^2)^2 + 16g(g + 1)A x^4, \]

\[ M^2 = L^{2g+1} + a_{2g} L^{2g} + \ldots + a_1 L + a_0, \]

where \( A, B \) are arbitrary constants, \( A \neq 0, a_i \) are some constants, are commuting operators of rank 2.

In this paper we find common eigenfunctions of \( L \) and \( M \). Until now common eigenfunctions of commuting differential operators with polynomial coefficient were not found explicitly.

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**Commuting operators of rank 2**

Consider the operator

\[ L = (\partial_x^2 + V(x))^2 + W(x). \]

We know that ([7]) the operator commutes with an operator \( M \) of order \( 4g + 2 \) with hyperlyptic spectral curve of genus \( g \) and hence is operator of rank 2, if and only if there is a polynomial

\[ Q = z^g + a_1(x) z^{g-1} + a_2(x) z^{g-2} + \ldots + a_{g-1}(x) z + a_g(x) \]
that the following relation is satisfied
\[ Q^{(5)} + 4VQ'' + 6V'Q'' + 2Q'(2z - 2W + V'') - 2QW' \equiv 0, \]
where \( Q' \) means \( \partial_x Q \). The spectral curve has the form
\[ 4w^2 = 4(z - W)Q^2 - 4V(Q')^2 + (Q'')^2 - 2Q'Q'' + 2Q(2V'Q' + 4VQ'' + Q^{(4)}). \]
Common eigenfunctions of \( L \) and \( M \) satisfy the second order differential equation [7]
\[ \psi''(x, P) - \chi_1(x, P)\psi'(x, P) - \chi_0(x, P)\psi(x, P) = 0, \]
where \( \chi_0 \) and \( \chi_1 \) have the form
\[ \chi_1 = \frac{Q'}{Q}, \quad \chi_0 = -\frac{Q''}{2Q} + \frac{w}{Q} - V. \]

**Common eigenfunctions of commuting differential operators of rank 2**

Let me recall some definitions.
Bessel functions \( J_\alpha \) are solutions of the Bessel equation
\[ x^2y'' + xy' + (x^2 - \alpha^2)y = 0. \]
If \( \alpha \) is not integer, then \( J_\alpha, J_{-\alpha} \) satisfy Bessel equation, where
\[ J_\alpha(x) = \frac{x^\alpha}{2^\alpha \Gamma(\alpha + 1)} \left(1 - \frac{x^2}{2^2 1!(\alpha + 1)} + \frac{x^4}{2^4 2!(\alpha + 2)} - \ldots\right) \]
If \( \alpha \) is integer, then \( J_\alpha, J_{-\alpha} \) are not independent solutions. Note that (see [13])
\[ J'_\alpha = \frac{\alpha J_\alpha(x)}{x} - J_{\alpha+1}(x). \]
Functions
\[ Y\alpha(x) = \frac{J_\alpha(x)\cos(\alpha \pi) - J_{-\alpha}(x)}{\sin(\alpha \pi)} \]
are called Bessel functions of the second kind.
Solutions \( H(\alpha, \lambda, \beta, \gamma, \delta, \eta; x) \) of the following equation
\[ y''(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x - 1} + \frac{\alpha + \beta - \gamma - \delta + 1}{x - a}\right)y'(x) + \frac{\alpha \beta x - q}{x(x - 1)(x - a)} y(x) = 0. \]
are called Heun functions. This equation has four regular singular points 0, 1, a, ∞. Confluent Heun equation is obtained from the Heun equation through a confluence process (see [14]), that is, a process where two singularities coalesce. Denote by $CH(\alpha, \beta, \gamma, \delta, \eta; x)$ the solution of the confluent Heun equation

$$y''(x) + \left(\frac{\beta + \gamma - \alpha + 2}{x - 1} + \frac{x\alpha}{x - 1} - \frac{\beta + 1}{x(x - 1)}\right)y'(x) +$$

$$+\left(\frac{\alpha(\beta + \gamma + 2) + 2\delta}{2(x - 1)} - \frac{\alpha(\beta + 1) - \beta(\gamma + 1) - 2\eta - \gamma}{2x(x - 1)}\right)y(x) = 0$$

where

$$y(0) = 1, \quad y'(0) = \frac{\beta(\gamma - \alpha + 1) + \gamma - \alpha + 2}{2(\beta + 1)}.$$  

There is formula for Bessel functions [14]

$$J_\alpha(x) = \frac{x^\alpha(2ix + 1)CH(1, 2\alpha, 1, 0, \frac{1}{2}; -2ix)}{\Gamma(\alpha + 1)2^\alpha e^{ix}}.$$  

We know from [12] that

$$L = \left(\partial_x^2 + Ax^6 + Bx^4\right)^2 + 16g(g + 1)Ax^4$$

commutes with a differential operator $M$ of order $4g + 2$. Let us assume that $B = 0$. If $g = 1$, then spectral curve of commuting pair $L$ and $M$ is equal to

$$w^2 = z(192A + z^2)$$

and differential equation for common eigenfunctions has the form

$$\psi'' - \frac{64Ax^3}{16Ax^4 + z}\psi' - \left(\frac{w - 96Ax^2}{16Ax^4 + z} - Ax^4\right)\psi = 0. \quad (1)$$

Let us suppose that $w = 0$. So, $z = 0, \pm \sqrt{-192A}$. If $z = 0$, then solutions of (1) are

$$x^{\frac{1}{2}}J_{\frac{1}{2}}\left(\frac{x^4\sqrt{A}}{4}\right), \quad x^{\frac{1}{2}}Y_{\frac{1}{2}}\left(\frac{x^4\sqrt{A}}{4}\right).$$

If $z = \pm \sqrt{-192A}$, then solutions of (1) are

$$e^{-\frac{Ax^4}{4\sqrt{x}}CH\left(\frac{z}{32}\sqrt{-\frac{1}{A}}, -\frac{1}{4}, -2, 0, \frac{5}{4}; -\frac{16Ax^4}{z}\right)},$$

$$xe^{-\frac{Ax^4}{4\sqrt{x}}CH\left(\frac{z}{32}\sqrt{-\frac{1}{A}}, 1, -2, 0, \frac{5}{4}; -\frac{16Ax^4}{z}\right)}.$$
If $g = 2$, then spectral curve of commuting operators $L$ and $M$ is equal to

$$w^2 = z(20160A + z^2)(20736A + z^2).$$

If $z = 0$, then equation for common eigenfunctions has the form

$$(4Ax^8 + 35)\psi'' - 32Ax^7\psi' + (147Ax^6 + 4A^2x^{14})\psi = 0. \quad (2)$$

Equation (2) has solutions

$$CH(0, \frac{1}{8}, -2, -\frac{35}{256}, \frac{387}{256}, -\frac{4Ax^8}{35}),$$

$$xCH(0, \frac{1}{8}, -2, -\frac{35}{256}, \frac{387}{256}, -\frac{4Ax^8}{35}).$$

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Department of Geometry and Topology, Faculty of Mechanics and Mathe-
matics, Lomonosov Moscow State University, Moscow, 119991 Russia.

E-mail address: vardin.o@mail.ru