Integrated Adaptive Control and Reference Governors for Constrained Systems With State-Dependent Uncertainties

Pan Zhao, Member, IEEE, Ilya Kolmanovsky, Fellow, IEEE, and Naira Hovakimyan, Fellow, IEEE

Abstract—This article presents an adaptive reference governor (RG) framework for a linear system with matched nonlinear uncertainties that can depend on both time and states, subject to both state and input constraints. The proposed framework leverages an $L_1$ adaptive controller ($L_1$ AC) that compensates for the uncertainties, and provides guaranteed transient performance in terms of uniform bounds on the error between actual states and inputs and those of a nominal (i.e., uncertainty-free) system. The uniform performance bounds provided by the $L_1$ AC are used to tighten the prespecified state and control constraints. A reference governor is then designed for the nominal system using the tightened constraints, which guarantees robust constraint satisfaction. Moreover, the conservatism introduced by constraint tightening can be systematically reduced by tuning some parameters within the $L_1$ AC. Compared with existing solutions, the proposed adaptive RG framework can potentially yield reduced conservatism for constraint enforcement and improved tracking performance due to the inherent uncertainty compensation mechanism. Simulation results for a flight control example illustrate the efficacy of the proposed framework.

Index Terms—Adaptive control, constrained control, safety-critical control, uncertainties.

I. INTRODUCTION

HERE has been a growing interest in developing control methods that can handle state and/or input constraints. Examples of such constraints include actuator magnitude and rate limits, bounds imposed on process variables to ensure safe and efficient system operation, and collision/obstacle avoidance requirements. There are several choices for a control practitioner when dealing with constraints. One choice is to adopt the model predictive control (MPC) framework [1], [2], in which the state and input constraints can be incorporated into the optimization problem for computing the control signals. Another route is to augment a well-designed nominal controller that already achieves high performance for small signals, with constraint handling capability that protects the system against constraint violations in transients for large signals. The second route is attractive to practitioners who are interested in preserving an existing/legacy controller or are concerned with the computational cost, tuning complexity, stability, robustness, certification issues, and/or other requirements satisfactorily addressed by the existing controller. The reference governor (RG) is an example of the second approach. As its name suggests, RG is an add-on scheme for enforcing pointwise-in-time state and control constraints by modifying the reference command in a well-designed closed-loop system. The RG acts like a prefilter that, based on the current value of the desired reference command $r(t)$ and of the states (measured or estimated) $x(t)$, generates a modified reference command $v(t)$ which avoids constraint violations. Since its advent, variants of RGs have been proposed for both linear and nonlinear systems. See the survey paper [3] and references therein. While RG has been extensively studied for systems for which exact dynamic models are available, the design of RG for uncertain systems, i.e., systems with unknown parameters, state-dependent uncertainties, unmodeled dynamics and/or external disturbances, has been less addressed.

A. Related Work

1) Robust RG/MPC: As mentioned in [3], the RG can be straightforwardly modified to handle unmeasured set-bounded disturbances by taking into account all possible realizations of the disturbances when determining the maximal output admissible set [4]. For uncertain systems, various robust or tube MPC schemes have also been proposed [5], [6], [7], [8], [9], [10], [11] and summarized in [12], most of which consider parametric uncertainties and bounded disturbances with only a few exceptions (e.g., [10] and [11]) that consider state-dependent uncertainties. However, robust approaches often lead to conservative performance when the disturbances are large.

2) Adaptive and Disturbance Cancellation-Based RG/MPC: Various adaptive MPC strategies have been proposed for systems with unknown parameters [13], [14], [15] and...
state-dependent uncertainties [16], [17]. In particular, [15] uses an $L_1$ adaptive controller [18] to compensate for matched parametric uncertainties so that the uncertain plant behaves close to a nominal model, and uses robust MPC to handle the error between the combined system consisting of the uncertain plant and the adaptive controller, and the nominal model. To the best of our knowledge, all of the existing adaptive MPC solutions involve propagation of uncertainties along a prediction horizon. Polóni et al. [19] merged a Lyapunov function-based RG with a disturbance-canceling controller based on an input observer to achieve nonconservative treatment of uncertainties. Unfortunately, a bound on the rate of change of the disturbance is needed for the design, which is often difficult to obtain when the disturbance is dependent on states. Additionally, input constraints were not considered in that work. For nonlinear systems with parametric uncertainties, Chakrabarty et al. [20] proposed a parameter-adaptive RG, which leverages Kalman and particle filters to estimate unknown parameters and machine learning tools to learn robust constraint-admission sets. However, the performance depends on the amount and quality of data collected.

3) MPC Under State-Dependent Uncertainties (SDUs): If a system is affected by SDUs, and the states are limited to a compact set, it is always possible to bound the SDU with a worst-case value and to apply the robust approaches (e.g., robust or tube MPC [5], [6], [7]) developed for bounded disturbances. However, by accounting for the state dependence, one can improve performance and reduce conservatism, as demonstrated in robust MPC solutions in [11] and [21]. Adaptive MPC solutions, which account for SDUs have been proposed in [16] and [17]. These solutions essentially rely on computing the uncertainty or state bounds along the prediction horizon using the Lipschitz proprieties of SDUs, and solving a robust MPC problem, using the computed bounds.

4) Robust Adaptive Control Under Constraints: Adaptive control in the presence of input constraints has been studied [22], [23], [24], [25]. Many of these existing solutions consider single input [22], [23], [26] and/or rely on backstepping, therefore requiring the plant to have a special structure, e.g., be in a strict feedback form [25], [26]. There has also been an extensive study of adaptive control of strict- or pure-feedback systems in the presence of state/output constraints [27], [28], [29], [30], which frequently use barrier Lyapunov functions (BLFs) [31]. While input and state/output constraints are usually treated separately in the context of adaptive control, there are also exceptions [32], [33], [34], [35], all of which use BLFs and require the systems to have a special structure such as being in a strict- or pure-feedback form. Finally, we noticed that disturbance observer-based control under input constraints has been studied in [36] and [37], where linear matrix inequality (LMI) conditions were used to guarantee input constraint satisfaction.

B. Contributions
For constrained control under uncertainties, we develop an $L_1$-RG framework for linear systems with matched nonlinear uncertainties that could depend on both time and states, and with both input and state constraints. Our framework leverages an $L_1$ adaptive controller ($L_1$ AC) to compensate for the uncertainties, and to guarantee transient performance in terms of uniform bounds on the error between actual states and inputs and those of a nominal (i.e., uncertainty-free) closed-loop system. These uniform bounds characterize tubes in which actual states and control inputs are guaranteed to stay despite uncertainties. An RG designed for the nominal system with constraints tightened using these uniform bounds guarantees robust constraint satisfaction in the presence of uncertainties. Additionally, we show that these uniform bounds on state and input errors, and thus the conservatism induced by constraint tightening, can be arbitrarily reduced in theory by tuning the filter bandwidth and estimation sample time parameters of the $L_1$ AC. Second, as a separate contribution to $L_1$ adaptive control, we propose a novel scaling technique that allows deriving separate tight uniform bounds on each state and adaptive control input, as opposed to a single bound for all states, or adaptive control inputs in existing $L_1$ AC solutions [18]. The ability to provide such separate tight bounds makes an $L_1$ AC particularly attractive to be integrated with an RG for simultaneous constraint enforcement and improved trajectory tracking. Third, we validate the efficacy of the proposed approach using a flight control example in comparison with existing solutions.

Compared to existing literature, $L_1$-RG has the following novel aspects.
1) Thanks to the inherent uncertainty compensation mechanism, $L_1$-RG simultaneously improves tracking performance and enforces the constraints, while existing robust/disturbance-observer-based RG or robust/adaptive MPC solutions, except a few, such as [15] and [19], focus on constraint satisfaction only.
2) Within $L_1$-RG, the uniform bounds on the state and input errors (used for constraint tightening) and thus the conservatism induced by constraint tightening can be made arbitrarily small in theory, which cannot be achieved by existing methods.
3) $L_1$-RG is able to handle uncertainties that can nonlinearly depend on both time and states. Such a case has not been considered by previous adaptive MPC solutions that are based on uncertainty compensation. For instance, the solution in [15], which also leverages an $L_1$ AC, only treats parametric uncertainties and state constraints.

The rest of this article is organized as follows. Section II formally states the problem. Section III provides an overview of the proposed solution and discusses preliminaries related to RG and $L_1$ AC design. Section IV introduces a scaling technique to derive separate and tight performance bounds for an $L_1$ AC, while Section V presents the synthesis and performance analysis of the proposed $L_1$-RG framework. Section VI includes validation of the proposed $L_1$-RG framework on a flight control problem in simulations. Finally, Section VII concludes this article.

Notations: Let $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{Z}^+_n$ denote the set of real, nonnegative real, and nonnegative integer numbers, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the $n$-dimensional real vector space and the set of real $m$ by $n$ matrices, respectively. $\mathbb{Z}^+_n$ denote the integer sets $\{1, 2, \ldots, n\}$. respectively.
\( I_n \) denotes an identity matrix of size \( n \), and 0 is a zero matrix of a compatible dimension. \( \| \cdot \| \) and \( \| \cdot \|_\infty \) denote the 2-norm and \( \infty \)-norm of a vector or a matrix, respectively. The \( \mathcal{L}_\infty \)- and truncated \( \mathcal{L}_\infty \)-norm of a function \( x : \mathbb{R}_+ \to \mathbb{R}^n \) are defined as \( \| x \|_{\mathcal{L}_\infty} \triangleq \sup_{t \geq 0} \| x(t) \|_\infty \) and \( \| x \|_{\mathcal{L}_\infty,b} \triangleq \sup_{0 \leq t \leq T} \| x(t) \|_\infty \), respectively. The Laplace transform of a function \( x(t) \) is denoted by \( x(s) \triangleq \mathcal{L}[x(t)] \). For a vector \( x, x_i \) denotes the \( i \)th element of \( x \).

Given a positive scalar \( \rho, \Omega(\rho) \triangleq \{ z \in \mathbb{R}^n : \| z \|_\infty \leq \rho \} \) denotes a high-dimensional ball set of radius \( \rho \) and centered at the origin, while its dimension \( n \) can be deduced from the context.

For a high-dimensional set \( \mathcal{X} \), \( \mathcal{int}(\mathcal{X}) \) denotes the interior of \( \mathcal{X} \) and \( \mathcal{X}' \) denotes the projection of \( \mathcal{X} \) onto the \( i \)th coordinate. For given sets \( \mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n, \mathcal{X} \oplus \mathcal{Y} \triangleq \{ x + y : x \in \mathcal{X}, y \in \mathcal{Y} \} \) is the Minkowski set sum and \( \mathcal{X} \ominus \mathcal{Y} \triangleq \{ z : z + y \in \mathcal{X}, \forall y \in \mathcal{Y} \} \) is the Pontryagin set difference.

### II. Problem Statement

Consider an uncertain linear system represented by

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B(u(t) + f(t, x(t))) \\
y(t) &= Cx(t), \quad x(0) = x_0
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), and \( y(t) \in \mathbb{R}^m \) are the state, input, and output vectors, respectively, \( x_0 \in \mathbb{R}^n \) is the initial state vector, \( f(t, x(t)) \in \mathbb{R}^m \) denotes the uncertainty that can depend on both time and states, and \( A, B, C \) are matrices of compatible dimensions. We want to design a control law for \( u(t) \) such that the output vector \( y(t) \) tracks a reference signal \( r(t) \) while satisfying the specified state and control constraints

\[ x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U} \quad \forall t \geq 0 \]

where \( \mathcal{X} \subset \mathbb{R}^n \) and \( \mathcal{U} \subset \mathbb{R}^m \) are prespecified convex and compact sets with 0 in the interior. Note that (2) can also represent constraints on some of the states and/or inputs. Suppose a baseline controller is available and achieves desired performance for the nominal (i.e., uncertainty-free) system given a small desired reference command \( r(t) \) to track. To enforce state and input constraints (2) for the nominal system with larger signals, one can simply leverage the conventional RG, which will generate a modified reference command \( v(t) \) based on \( r(t) \). In such a case, the baseline controller can be selected as

\[ u_b(t) = K_x x(t) + K_v v(t) \]

where \( K_x \) and \( K_v \) are feedback and feedforward gains. For both improved tracking performance and constraint enforcement in the presence of the uncertainty \( f(t, x) \), we leverage an \( \mathcal{L}_1 \) AC. To this end, we adopt a compositional control law

\[ u(t) = u_b(t) + u_a(t) \]

where \( u_a(t) \) is the vector of the adaptive control inputs designed to cancel \( f(t, x) \). With (3), the uncertain system (1) can be rewritten as

\[
\begin{align*}
\dot{x}(t) &= A_m x(t) + B_v v(t) + B(u_a(t) + f(t, x(t))) \\
y(t) &= Cx(t), \quad x(0) = x_0
\end{align*}
\]

where \( A_m \triangleq A + BK_x \) is a Hurwitz matrix and \( B_v \triangleq BK_v \).

The problem to be tackled can be stated as follows: Given an uncertain system (1), a baseline controller (3) and a desired reference signal \( r(t) \), design an RG (for determining \( v(t) \)) and the \( \mathcal{L}_1 \) AC for \( u_a(t) \) such that the output signal \( y(t) \) tracks \( r(t) \) whenever possible, while the state and input constraints (2) are satisfied. We make the following assumption on the uncertainty.

**Assumption 1:** Given a compact set \( \mathcal{Z} \), there exist known positive constants \( L_{f_j} z, l_{f_j} z \) and \( b_{f_j} z \) \( (j \in \mathbb{Z}^n) \) such that for any \( x, z \in \mathcal{Z} \) and \( t, \tau \geq 0 \), the following inequalities hold for each \( j \in \mathbb{Z}^n \):

\[
\begin{align*}
|f_j(t, x) - f_j(\tau, z)| &\leq L_{f_j} z \|x - z\|_\infty + l_{f_j} z |t - \tau| \quad (6a) \\
|f_j(t, x)| &\leq b_{f_j} z \quad (6b)
\end{align*}
\]

where \( f_j(t, x) \) denotes the \( j \)th element of \( f(t, x) \).

**Remark 1:** Assumption 1 indicates that in the compact set \( \mathcal{Z} \), \( f_j(t, x) \) is Lipschitz continuous with respect to \( x \) with a known Lipschitz constant \( L_{f_j} z \), has a bounded rate of variation \( l_{f_j} z \) with respect to \( t \), and is uniformly bounded by a constant \( b_{f_j} z \).

In fact, given the local Lipschitz constant \( L_{f_j} z \) and the bounded rate of variation \( l_{f_j} z, b_{f_j} z \), a uniform bound for \( f_j(t, x) \) in \( \mathcal{Z} \) can always be derived if the bound on \( f_j(t, x^*) \) for an arbitrary \( x^* \) in \( \mathcal{Z} \) and any \( t \geq 0 \) is known. For instance, assuming we know \( |f_j(t, 0)| \leq b_0 \), from (6a), we have that \( |f_j(t, x) - f_j(t, 0)| \leq L_{f_j} z \|x\|_\infty \), which immediately leads to \( |f_j(t, x)| \leq b_0 + L_{f_j} z \max_{x \in \mathcal{X}} \|x\|_\infty \), for any \( x \in \mathcal{Z} \) and \( t \geq 0 \). In practice, some prior knowledge about the uncertainty (e.g., \( f_j \) depends on only a few instead of all states) may be leveraged to obtain a tighter bound than the preceding one, derived using the Lipschitz continuity and triangular inequalities. This motivates the assumption on the uniform bound in (6b).

Under the conditions in Assumption 1, we immediately obtain that for any \( x, z \in \mathcal{Z} \) and \( t, \tau \geq 0 \)

\[
\begin{align*}
\|f(t, x) - f(\tau, z)\|_\infty &\leq L_{f_j} z \|x - z\|_\infty + l_{f_j} z |t - \tau| \quad (7a) \\
\|f(t, x)\|_\infty &\leq b_{f_j} z \quad (7b)
\end{align*}
\]

where

\[
L_{f_j} z = \max_{j \in \mathbb{Z}^n} L_{f_j} z, \quad l_{f_j} z = \max_{j \in \mathbb{Z}^n} l_{f_j} z, \quad b_{f_j} z = \max_{j \in \mathbb{Z}^n} b_{f_j} z.
\]

**Remark 2:** Our choice of making assumptions on \( f_j(t, x) \) instead of on \( f(t, x) \) as in (8) facilitates deriving an individual bound on each state and on each adaptive input (see Section IV for details).

**Remark 3:** In principle, given the uniform bound on \( f(t, x) \) in (7b) obtained from Assumption 1, constraints can be enforced via robust RG or robust MPC approaches that handle bounded disturbances, as discussed in Section I-A. However, when this bound is large, robust approaches can yield overly conservative performance.

### III. Overview and Preliminaries

In this section, we first present an overview of the proposed \( \mathcal{L}_1 \)-RG framework and then introduce some preliminary results that provide a foundation for the \( \mathcal{L}_1 \)-RG framework.
A. Overview of the $\mathcal{L}_1$-RG Framework

Fig. 1 depicts the proposed $\mathcal{L}_1$-RG framework. As shown in Fig. 1, $\mathcal{L}_1$-RG is comprised of two integrated components. The first one is an $\mathcal{L}_1$AC designed to compensate for the uncertainty $f(t,x)$ and to guarantee uniform bounds on the errors between actual states and inputs, and those of a nominal closed-loop system

$$
\dot{x}_n(t) = A_m x_n(t) + B_m v(t), \quad x_n(0) = x_0 \tag{9a}
$$

$$
u_n(t) = K_x x_n(t) + K_v v(t) \tag{9b}
$$

where $x_n$ and $u_n$ are the vectors of nominal states and inputs, respectively. The second component is an RG designed for the nominal system (9a) with tightened constraints computed using the uniform bounds guaranteed by the $\mathcal{L}_1$AC. More formally, we will design the $\mathcal{L}_1$AC to ensure

$$
x(t) - x_n(t) \in \hat{\mathcal{X}}, \quad u(t) - u_n(t) \in \hat{\mathcal{U}} \quad \forall t \geq 0 \tag{10}
$$

where $x(t)$ and $u(t)$ are the vectors of states and of the total control inputs of the closed-loop system (5)

$$
u(t) - u_n(t) = K_x (x(t) - x_n(t)) + u_n(t) \tag{11}
$$

where $u(t)$ is given by (4) and $u_n(t)$ by (9b), and $\mathcal{X}$ and $\mathcal{U}$ are some precomputed hyperrectangular sets dependent on the properties of $f(t,x)$ and of the $\mathcal{L}_1$AC. The details will be given in Theorem 3 in Section III-C. Define

$$
\hat{\mathcal{X}} \triangleq \mathcal{X} \cup \mathcal{X}, \quad \hat{\mathcal{U}} \triangleq \mathcal{U} \cup \mathcal{U}. \tag{12}
$$

Then, for robust constraint enforcement, one just needs to design an RG for the nominal system (9) with tightened constraints given by

$$
x_n(t) \in \hat{\mathcal{X}}_n, \quad u_n(t) \in \hat{\mathcal{U}}_n \quad \forall t \geq 0. \tag{13}
$$

B. Reference Governor Design for a Nominal System

We now introduce the RG for the nominal system (9) to enforce the constraints (13). Hereafter, we use Roman math font to denote the variables (e.g., states, inputs, references) in a discrete-time system. We use the discrete-time RG approach of [3] that uses a discrete-time model

$$
\begin{align*}
x_n(k+1) &= \hat{A}_m x_n(k) + \hat{B}_v v(k), \quad x(0) = x_0 \\
u_n(k) &= K_x x_n(k) + K_v v(k)
\end{align*} \tag{14}
$$

where $x_n(k)$, $v(k)$, and $u_n(k)$ denote the vectors of states, of reference command inputs, and of nominal control inputs, respectively, and $\hat{A}_m$ and $\hat{B}_v$ are computed from $A_m$ and $B_v$ in (9) assuming a sampling time, $T_d$. When doing the discretization, we ensure that the discrete-time system (14) has the same states as the continuous-time system at all sampling instances. This can be achieved by using the zero-order hold discretization, since

$$
v(t) = v(kT_d) \quad \forall t \in [kT_d, (k+1)T_d) \tag{15}
$$

which indicates that $v(t)$ is piecewise constant. The constraints (13) are imposed in discrete-time as

$$
x_n(k) \in \hat{\mathcal{X}}_n, \quad u_n(k) \in \hat{\mathcal{U}}_n \quad \forall k \in \mathbb{Z}_+. \tag{16}
$$

where $\hat{\mathcal{X}}_n$ and $\hat{\mathcal{U}}_n$ are tightened versions of $\mathcal{X}_n$ and $\mathcal{U}_n$, respectively, introduced to avoid intersample constraint violations, and are defined by

$$
\hat{\mathcal{X}}_n \triangleq \mathcal{X} \ominus \{ z \in \mathbb{R}^n : \|z\|_\infty \leq \nu(T_d) \} \tag{17a}
$$

$$
\hat{\mathcal{U}}_n \triangleq \mathcal{U} \ominus \{ z \in \mathbb{R}^m : \|z\|_\infty \leq \|K_x\|_\infty \nu(T_d) \} \tag{17b}
$$

while

$$
\nu(T_d) \triangleq \max_{\tau \in [0,T_d]} \| e^{A_m \tau} - I_n \|_\infty \max_{x \in \mathcal{X}_n, v \in \mathcal{V}} \| x + A_m^{-1} B_v v \|_\infty \tag{18}
$$

with $\mathcal{V}$ denoting the set of all possible reference commands output by the RG.

The following lemma formally guarantees that no intersample constraint violations will happen for the continuous-time system (9) when the constraints for the discrete-time system (21) are satisfied at all sampling instants.

Lemma 1: Consider the continuous-time system (9) and its discrete-time counterpart (14) that has the same states as (9) at all sampling instants. If for the discrete-time system (14)

$$
x_n(k) \in \hat{\mathcal{X}}_n, \quad u_n(k) \in \hat{\mathcal{U}}_n, \quad k \in \mathbb{Z}_+ \tag{19}
$$

with $\hat{\mathcal{X}}_n$ and $\hat{\mathcal{U}}_n$ defined in (17), then (13) holds for the continuous-time system (9).

Proof: See Section A1.

Remark 4: From (17) and (18), we can see that $\hat{\mathcal{X}}_n$ and $\hat{\mathcal{U}}_n$ are close to $\mathcal{X}_n$ and $\mathcal{U}_n$, respectively, when $T_d$ is small. For practical implementation, intersample constraint violations may not be a big concern when $T_d$ is small. In such a case, we can simply set $\hat{\mathcal{X}}_n = \mathcal{X}_n$ and $\hat{\mathcal{U}}_n = \mathcal{U}_n$.

Define

$$
y_n^c(k) \triangleq \begin{bmatrix} x_n(k) \\ K_x x_n(k) + K_v v(k) \end{bmatrix} = \begin{bmatrix} I_n \\ K_x \end{bmatrix} x_n(k) + \begin{bmatrix} 0 \\ K_v \end{bmatrix} v(k) \tag{20}
$$

Then, the constraint (16) can be rewritten as

$$
y_n^c(k) \in \mathcal{Y}_n \triangleq \mathcal{X}_n \times \mathcal{U}_n \quad \forall k \in \mathbb{Z}_+. \tag{21}
$$

where $\times$ denotes the cross product.

Remark 5: In case there are no constraints on certain states and/or inputs, one can remove the rows of $C_v$ and $D_v$, defined in (20) corresponding to these states and/or inputs, and adjust the sets $\hat{\mathcal{X}}_n, \hat{\mathcal{U}}_n$, and $\mathcal{Y}_n$ accordingly.

Similar to most RG schemes, the RG scheme we adopt here computes at each time instant a command $v(k)$ such that, if it is
For a stable proper MIMO system \( O_{\infty} \) as the set of all states \( x_n \) and inputs \( v \), such that the predicted response from the initial state \( x_0 \), and with a constant input \( v \) satisfies the constraint \((21)\), i.e.,

\[
O_{\infty} = \left\{ (x_n, v) : \tilde{y}_n(k|x_n, v) \in \mathcal{Y}_n \quad \forall k \in \mathbb{Z}_+ \right\}
\]  

(22)

where the output prediction \( \tilde{y}_n(k|x_n, v) \) for system \((14)\) is given by

\[
\tilde{y}_n(k|x_n, v) = C_c \hat{A}_m^k x_n + C_c \sum_{j=1}^{k} \hat{A}_m^{j-1} \hat{B}_c v + D_c v
\]

\[
= C_c \hat{A}_m^k x_n + C_c \left( I_n - \hat{A}_m \right)^{-1} \left( I_n - \hat{A}_m^k \right) \hat{B}_c v + D_c v.
\]

(23)

Moreover, the \( \hat{O}_\infty \) is positively invariant, which means that if \((v(k), x_n(k)) \in \hat{O}_\infty\), and \( v(k) \) are applied to the system at time \( k \), then \((v(k), x_n(k+1)) \in \hat{O}_\infty\). Furthermore, if \( \mathcal{Y}_n \) is convex, then \( \hat{O}_\infty \) is also convex.

Remark 6: The process of computing \( k^* \) involves computing sets \( \hat{O}_k \) for \( k = 1, 2, \ldots \), and checking the condition \( \hat{O}_k = \hat{O}_{k+1} \). \( k^* \) is the minimum \( k \) for which this condition holds.

The proposed \( L_1 \)-RG framework can leverage most of the existing RG schemes developed for uncertainty-free systems. As an illustration and demonstration in Section VI, we choose the scalar RG introduced in [40] and [41]. The scalar RG computes at each time instant \( k \) a command \( v(k) \) which is the best approximation of the desired set-point \( r(k) \) along the line segment connecting \( v(k-1) \) and \( r(k) \) that ensures \((v(k), x_n(k)) \in \hat{O}_\infty\).

More specifically, the scalar RG solves at each discrete time \( k \), the following optimization problem:

\[
\kappa(k) = \max_{\kappa \in [0,1]} \kappa
\]

\[
\text{s.t. } v = v(k-1) + \kappa(r(k) - r(k-1))
\]

(27b)

\[
(v, x_n(k)) \in \hat{O}_\infty
\]

(27c)

where \( \kappa(k) \) is a scalar adjustable bandwidth parameter and \( v(k) = v(k-1) + \kappa(k)(r(k) - v(k-1)) \) is the modified reference command to be applied to the system. If there is no danger of constraint violation, \( \kappa(k) = 1 \) and \( v(k) = r(k) \) so that the RG does not interfere with the desired operation of the system. If \( v(k) = r(k) \) would cause a constraint violation, the value of \( \kappa(k) \) is decreased by the RG. In the extreme case, \( \kappa(k) = 0 \), \( v(k) = v(k-1) \), which means that the RG momentarily isolates the system from further variations of the reference command for constraint enforcement. Due to the positive invariance of \( O_{\infty} \), \( v(k) = v(k-1) \) always satisfies the constraints, which ensures recursive feasibility under the condition that at \( t = 0 \) a command \( v(0) \) is known such that \((v(0), x_n(0)) \in \hat{O}_\infty\).

C. \( L_1 \) Adaptive Control Design and Uniform Performance Bounds

We now present an \( L_1 \)-AC that guarantees the bounds in \((10)\), without considering the state and control constraints in \((2)\). We first recall some basic definitions and facts from control theory, and introduce some definitions and lemmas.

Definition 1 [42, Sec. III.F]: For a stable proper MIMO system \( \mathcal{H}(s) \) with states \( x(t) \), inputs \( u(t) \in \mathbb{R}^m \) and outputs \( y(t) \in \mathbb{R}^p \), its \( L_1 \) norm is defined as

\[
\| \mathcal{H}(s) \|_{L_1} = \sup_{x(0) = 0, u(0)} \| y \|_{L_1}.
\]

(28)

The following lemma follows directly from Definition 1.

Lemma 2: For a stable proper MIMO system \( \mathcal{H}(s) \) with states \( x(t) \in \mathbb{R}^n \), inputs \( u(t) \in \mathbb{R}^m \) and outputs \( y(t) \in \mathbb{R}^p \), under zero initial states, i.e., \( x(0) = 0 \), we have \( \| y \|_{L_1, r} \leq \| \mathcal{H}(s) \|_{L_1} \| y \|_{L_1, r} \), for any \( r \geq 0 \). Furthermore, for any matrix \( T \in \mathbb{R}^{q \times p} \), we have \( \| T \mathcal{H}(s) \|_{L_1} \leq \| T \|_{\infty} \| \mathcal{H}(s) \|_{L_1} \).

A unique feature of an \( L_1 \)-AC is a low-pass filter \( C(s) \) (with dc gain \( C(0) = I_m \)) that decouples the estimation loop from the control loop, thereby allowing for arbitrarily fast adaptation without sacrificing the robustness [18]. For simplicity, we can select \( C(s) \) to be a first-order transfer function matrix

\[
C(s) = \text{diag}(C_1(s), \ldots, C_m(s)), \quad C_j(s) = \frac{k_j^2}{s + k_j}, \quad j \in \mathbb{Z}^m
\]

(29)

where \( k_j^2 \) \((j \in \mathbb{Z}^m)\) is the bandwidth of the filter for the \( j \)th input channel. We now introduce a few notations that will be used later

\[
\mathcal{H}_{x_m}(s) \triangleq (sI_n - A_m)^{-1} B, \quad \mathcal{H}_{x}(s) \triangleq (sI_n - A_m)^{-1} B
\]

(30a)

\[
G_{x_m}(s) \triangleq \mathcal{H}_{x_m}(s)(I_m - C(s))
\]

(30b)
where $A_m, B_v$ correspond to system (9) and $B$ to (1). Also, letting $x_m(t)$ be the state of the system $\dot{x}_m(t) = A_m x_m(t)$, $x_m(0) = x_0$, we have $x_m(s) ≜ (s I_n - A_m)^{-1} x_0$. Defining $\rho_m ≜ \| (s I_n - A_m)^{-1} \|_c$, $\max_{x_0 \in \mathbb{R}^n} \| x_0 \|_\infty$, and further considering that $A_m$ is Hurwitz and $X_0$ is compact, we have $\| x_m \|_c \leq \rho_m$ according to Lemma 2.

1) $L_1$ Adaptive Control Architecture: For stability guarantees, the filter $\bar{C}(s)$ in (29) needs to ensure that there exists a positive constant $\gamma$ and a (small) positive constant $\gamma_1$ such that

$$
\| \mathcal{G}_{x_m}(s) \|_{L_1} b_f x_r < \rho_r - \| \mathcal{H}_{x_r}(s) \|_{L_1} v \| v \|_{L_2} - \rho_m \tag{31a}
$$

$$
\| \mathcal{G}_{x_m}(s) \|_{L_1} L_f x_u < 1 \tag{31b}
$$

where

$$
\rho ≜ \rho_r + \gamma \tag{32}
$$

$$
\mathcal{X} ≜ \Omega(\rho_r), \mathcal{X}_\gamma ≜ \Omega(\rho) \tag{33}
$$

Remark 7: We will show in Lemma 3 and Theorem 1 that $\rho_r$ and $\rho$ are actually uniform bounds on the states of the system, respectively.

Remark 8: Note that $\| \mathcal{G}_{x_m}(s) \|_{L_1} \rightarrow 0$, when the bandwidth of the filter $\mathcal{C}(s)$ goes to infinity, i.e., $k_j \rightarrow \infty$ for all $j \in \mathcal{Z}_1^m$. Furthermore, $b_f, \Omega(\rho_r)$ can be bounded using the Lipschitz property (7a) of $f(t, x)$ in $\Omega(\rho_r)$, and $L_f, \Omega(\rho)$ is bounded given any $\rho > 0$. Therefore, (31) can always be satisfied under a sufficiently high bandwidth for $\mathcal{C}(s)$.

A typical $L_1$AC is comprised of three elements, namely, a state predictor, an adaptive law, and a low-pass filtered control law. For system (5), the state predictor is defined by

$$
\dot{x}(t) = A_m x_m(t) + B_f v(t) + B_u u(t) + \hat{\sigma}(t) + A_e \hat{x}(t)
$$

$$
\hat{x}(0) = x_0 \tag{34}
$$

where $\hat{x}(t) = \hat{x}(t) - x(t)$ is the prediction error, $A_e$ is a Hurwitz matrix, $\hat{\sigma}(t)$ is the estimate of $B_f(x(t)$. The estimate $\hat{\sigma}(t)$ is updated by the following piecewise-constant adaptive law (similar to that in [18, Sec. 3.3]):

$$
\begin{cases}
\hat{\sigma}(t) = \hat{\sigma}(iT), & t \in [iT, (i + 1)T) \\
\hat{\sigma}(iT) = -\Phi^{-1}(T)e^{A_e T} \hat{x}(iT)
\end{cases} \tag{35}
$$

where $T$ is the estimation sampling time. Finally, the control law is given by

$$
u_u(s) = -\mathcal{C}(s) \mathcal{L} \left( B^\dagger \hat{\sigma}(t) \right) \tag{36}
$$

$B^\dagger = (B^T B)^{-1} B^T$ is the pseudo-inverse of $B$, and $\Phi(T) ≜ A_e^{-1}(e^{A_e T} - I_n)$. Note that $B^\dagger \hat{\sigma}(t)$ is an estimate of the uncertainty $f(t, x)$. As a result, the control law (36) tries to cancel the estimated uncertainty within the bandwidth of the filter $\mathcal{C}(s)$.

2) Uniform Performance Bounds: We first define some constants

$$
\bar{\alpha}_0(T) ≜ \max_{\bar{r}(t)} \left\| e^{A_e(T - \tau)} B \right\|_c d\tau \tag{37a}
$$

$$
\bar{\alpha}_1(T) ≜ \max_{\bar{r}(t)} \left\| e^{A_e t} \right\|_c \tag{37b}
$$

Further define

$$
\rho_{ur} ≜ \| \mathcal{C}(s) \|_{L_1} b_f, x_r \tag{39}
$$

$$
\gamma_2 ≜ \| \mathcal{C}(s) \|_{L_1} L_f, x_u, \gamma_1 + \| \mathcal{C}(s) B^\dagger (s I_n - A_e) \|_{L_1} \gamma_0(T) \tag{40}
$$

where $\gamma_1$ is introduced in (32). Due to (38) and (31b), we can always select a small enough $T > 0$ such that

$$
\left\| \mathcal{H}_{x_m}(s) \mathcal{C}(s) B^\dagger (s I_n - A_e) \right\|_{L_1} \gamma_0(T) < \gamma_2 \tag{41}
$$

where $\gamma_e$ is defined in (33) and $B^\dagger$ is the pseudo-inverse of $B$.

Following the convention for performance analysis of an $L_1$AC [18], we introduce the following reference system:

$$
\dot{x}_r(t) = A_m x_r(t) + B_f v(t) + B(u(t) + f(t, x_r(t))) \tag{43a}
$$

$$
u_a(s) = -\mathcal{C}(s) \mathcal{L} \left[ f(t, x_r(t)) \right], \quad x_r(0) = x_0. \tag{43b}
$$

Clearly, the control law in the reference system (43) partially cancels the uncertainty $f(t, x_r(t))$ within the bandwidth of the filter $\mathcal{C}(s)$. Moreover, the control law depends on the true uncertainties and is thus not implementable. The reference system is introduced to help characterize the performance of the adaptive closed-loop system, which will be done in four sequential steps: (i) establishing the bounds on the states and inputs of the reference system (Lemma 3); (ii) quantifying the difference between the states and inputs of the adaptive system and those of the reference system (Theorem 1); (iii) quantifying the difference between the states and inputs of the reference system and those of the nominal system (Lemma 5); (iv) based on the results from (ii) and (iii), quantifying the difference between the states and inputs of the adaptive system and those of the nominal system (Theorem 2).

The proofs of these lemmas and theorems mostly follow the typical $L_1$AC analysis procedure [18], and are included in appendices for completeness.

For notation brevity, we define

$$
\eta(t) ≜ f(t, x(t)), \quad \eta_h(t) ≜ f(t, x_r(t)). \tag{44}
$$

To provide an overview, Table 1 summarizes the different (error) systems involved in this section and their related theorems/lemmas, the uniform bounds, the $L_1$AC parameters and
TABLE I
OVERVIEW OF DIFFERENT (ERROR) SYSTEMS INVOLVED IN SECTION III-C, AND THEIR RELATED THEOREM/LEMMA, UNIFORM BOUNDS, L₁ AC PARAMETERS AND CONDITIONS

| (Error) System | Theorem/Lemma | Uniform Bounds on States and Inputs | L₁ AC Parameters | Conditions |
|----------------|---------------|-----------------------------------|------------------|------------|
| 1 Nominal system (9) | Lemma 1 | \(|x_t|_{\mathcal{L}_\infty} \leq \rho_r\) | \(C(s)\) | N/A |
| 2 Reference system (43) | Lemma 3 | \(|x_t|_{\mathcal{L}_\infty} < \rho_r\) | \(\|u_a\|_{\mathcal{L}_\infty} \leq \rho_{ur}\) | \(\rho_r\) |
| 3 Diff. b/t reference and adaptive systems | Theorem 1 | \(|x_t|_{\mathcal{L}_\infty} < \gamma_1\) | \(\|u_a\|_{\mathcal{L}_\infty} \leq \gamma_2\) | \(\rho_r, T, C(s)\) |
| 4 Diff. b/t reference and nominal systems | Lemma 5 | \(|x_t|_{\mathcal{L}_\infty} \leq \|G_{xt}\|_{\mathcal{L}_1} \|b_f, x_r\|\) | \(C(s)\) | \(\rho_r\) |
| 5 Adaptive system (5) and the \(\mathcal{L}_1\) AC | Theorem 1 | \(|x|_{\mathcal{L}_\infty} < \rho\) | \(\|u_a\|_{\mathcal{L}_\infty} \leq \rho_u\) | \(\rho, T, C(s)\) |
| 6 Diff. b/t adaptive and nominal systems | Theorem 2 | \(\|x - x_n|_{\mathcal{L}_\infty} \leq \bar{\rho}\) | \(\|u_a\|_{\mathcal{L}_\infty} \leq \rho_u\) | \(\rho, T, C(s)\) |

\[\bar{\rho} \triangleq \|G_{xt}(s)\|_{\mathcal{L}_1} b_f, x_r + \gamma_1.\] (51)

Remark 10: When the bandwidth of the filter \(C(s)\) goes to infinity, \(\|G_{xt}(s)\|_{\mathcal{L}_1}\) and thus \(\|x - x_n|_{\mathcal{L}_\infty}\) go to 0. This indicates that the difference between the states of the reference system and those of the nominal system can be made arbitrarily small by increasing the filter bandwidth. However, a high-bandwidth filter allows for high-frequency control signals to enter the system under fast adaptation (corresponding to small \(T\)), compromising its robustness. Thus, the filter presents a tradeoff between robustness and performance. More details about the role and design of the filter can be found in [18].

From Theorem 1, Lemma 5, and the application of the triangle inequality, we can obtain uniform bounds on the error between the actual system (5) and the nominal system (9a), formally stated in the following theorem. The proof is straightforward and thus omitted.

Theorem 2: Given the uncertain system (5) subject to Assumption 1 and the reference system (43) subject to the conditions (31a) and (31b) with a constant \(\gamma_1 > 0\) and the \(\mathcal{L}_1\) AC defined via (34)-(36) subject to the conditions (31a) and (31b) with a constant \(\gamma_1 > 0\) and the sample time constraint (42), we have

\[\|x - x_n|_{\mathcal{L}_\infty} \leq \bar{\rho}\] (52)

\[\|u_a\|_{\mathcal{L}_\infty} \leq \rho_u\] (53)

where \(\rho_u\) is defined in (41), and

Remark 11: From Remarks 9 and 10, by decreasing \(T\) and increasing the bandwidth of the filter \(C(s)\), one can make (i) the states of the adaptive system arbitrarily close to those of the nominal system; and (ii) the adaptive inputs \(u_a(t)\) arbitrarily close to \(f(t,x)\), i.e., the true uncertainty, since \(f(t,x)\) is arbitrarily close to \(f(t,x)\) when the error between \(x(t)\) and \(x_r(t)\) is arbitrarily small. From Remarks 9 and 10, by decreasing \(T\) and increasing the bandwidth of the filter \(C(s)\), one can make 1) the states of the adaptive system arbitrarily close to those of the nominal system; and 2) the adaptive inputs \(u_a(t)\) arbitrarily close to \(f(t,x)\), i.e., the true uncertainty, since \(f(t,x)\) is arbitrarily close to \(f(t,x)\) when the error between \(x(t)\) and \(x_r(t)\) is arbitrarily small. In practice, the size of \(T\) is limited by computational hardware. There is a tradeoff between performance and robustness when selecting the bandwidth of \(C(s)\), which, in practice, cannot be

conditions. The proofs for Lemmas 3–5 are given in Appendices A2–A5.

Lemma 3: For the closed-loop reference system in (43) subject to Assumption 1 and the stability condition in (31a), we have

\[\|x_t\|_{\mathcal{L}_\infty} < \rho_r\] (45)

\[\|u_t\|_{\mathcal{L}_\infty} < \rho_{ur}\] (46)

where \(\rho_r\) is introduced in (31a), and \(\rho_{ur}\) is defined in (39).

From (5) and (34), the prediction error dynamics are given by

\[\dot{x}(t) = A_x \dot{x}(t) + \tilde{\sigma}(t) - B f(t, x(t)).\] (47)

The following lemma establishes a bound on the prediction error under the assumption that the actual states and adaptive inputs are bounded.

Lemma 4: Given the uncertain system (5) subject to Assumption 1, the state predictor (34) and the adaptive law (35), if

\[\|x\|_{\mathcal{L}_1} \leq \rho, \|u_a\|_{\mathcal{L}_1} \leq \rho_u\] (48)

with \(\rho\) and \(\rho_u\) defined in (32) and (41), respectively, then

\[\|\tilde{x}\|_{\mathcal{L}_1} \leq \gamma_0(T).\] (49)

Theorem 1: Given the uncertain system (5) subject to Assumption 1 and the reference system (43) subject to the conditions (31a) and (31b) with a constant \(\gamma_1 > 0\) and the sample time constraint (42), we have

\[\|x\|_{\mathcal{L}_\infty} \leq \rho\] (50a)

\[\|u_a\|_{\mathcal{L}_\infty} \leq \rho_{ur}\] (50b)

\[\|x_t - x\|_{\mathcal{L}_\infty} \leq \gamma_1\] (50c)

\[\|u_t - u_a\|_{\mathcal{L}_\infty} \leq \gamma_2\] (50d)

where \(\rho, \gamma_1, \gamma_2, \rho_{ur}\) are defined in (32), (40), and (41), respectively.

Remark 9: For an arbitrarily small \(\gamma_1 > 0\), one can always find a small enough \(T\) such that the constraint (42) is satisfied. According to (40), \(\gamma_2\) depends on \(\gamma_1\) and \(\gamma_0(T)\), and can be made arbitrarily small by reducing \(\gamma_1\) and \(T\). Thus, by reducing \(T\), both \(\gamma_1\) and \(\gamma_2\) can be made arbitrarily small, which indicates that the difference between the inputs and states of the adaptive system and those of the reference system can be made arbitrarily small from Theorem 1.

Lemma 5: Given the reference system (43) and the nominal system (9a), subject to Assumption 1, and the condition (31a), we have

\[\|x_t - x_n|_{\mathcal{L}_\infty} \leq \|G_{xt}\|_{\mathcal{L}_1} b_f, x_r + \gamma_1.\] (51)
arbitrarily high to maintain robustness margins (e.g., against input delays) [18, Sec. 2.2.5].

Remark 12: The transient performance guarantees provided by $L_1$ AC are in the form of $L_\infty$ norms of system states and control inputs. This is different from many other adaptive control schemes that have some form of performance guarantee (mostly in an asymptotic sense). The performance guaranteed by the model reference adaptive control scheme with closed-loop reference models [43] is in the forms of $L_2$ norm of key signals. While for linear systems, such guarantees may also imply transient performance guarantees; for nonlinear systems, such a claim is not true. The $L_\infty$-norm type performance guarantees of $L_1$ AC ensure transient performance bounds.

IV. $L_1$ AC WITH SEPARATE BOUNDS FOR STATES AND INPUTS

In Section III-C, we presented an $L_1$ AC that guarantees uniform bounds on the states and adaptive control inputs of the adaptive system with respect to the nominal system, without consideration of the constraint (2). However, as can be seen from Theorem 2, the uniform bound on $x(t) - x_0(t)$ or $u_0(t)$ is represented by the vector-$\infty$ norm, which always leads to the same bound for all the states, $x_i - x_{n,i}(t) (i \in \mathbb{Z}_1^n)$, or all the adaptive inputs, $u_{a,j} (j \in \mathbb{Z}_1^m)$. The use of vector-$\infty$ norms may lead to conservative bounds for some specific states or adaptive inputs, making it impossible to satisfy the constraints (2) or leading to significantly tightened constraints for the RG design. To reduce such conservatism, this section will present a scaling technique to derive an individual bound for each $x_i(t) - x_{n,i}(t)$ ($i \in \mathbb{Z}_1^n$) and $u_{a,j}(j \in \mathbb{Z}_1^m)$.

From Theorem 2, one can see that the bound on $x(t) - x_0(t)$ or $u_0(t)$ consists of two parts: the first part is $\gamma_1$ or $\gamma_2$ that can be made arbitrarily small by reducing $T$ (see Remark 9), while the second part is a bound on $x_i(t) - x_{n,i}(t)$ ($u_{a,j}(j \in \mathbb{Z}_1^m)$). Next, we will derive an individual bound for each $x_i(t) - x_{n,i}(t)$ ($u_{a,j}(j \in \mathbb{Z}_1^m)$).

Derive separate bounds for states via scaling: For deriving an individual bound for each $x_i(t) - x_{n,i}(t)$, we introduce the following coordinate transformations for the reference system (43) and the nominal system (9a) for each $i \in \mathbb{Z}_1^n$:

$$
\begin{align*}
\tilde{x}_i &= T^i_x x_i, \\
\tilde{x}_{n,i} &= T^i_x x_{n,i}, \\
\tilde{A}^i_n &= T^i_x A_{m}(T^i_x)^{-1}, \\
\tilde{B}^i_v &= T^i_x B_v, \\
\tilde{B}^i_v &= T^i_x B_v,
\end{align*}
$$

(55)

where $T^i_x > 0$ is a diagonal matrix that satisfies

$$
T^i_x[k] = 1, 0 < T^i_x[k] \leq 1 \quad \forall k \neq i
$$

with $T^i_x[k]$ denoting the $k$th diagonal element. Under the transformation (55), the reference system (43) is converted to

$$
\begin{align*}
\dot{\tilde{x}}_i(t) &= \tilde{A}^i_n \tilde{x}_i(t) + \tilde{B}^i_v v(t) + \tilde{B}^i_v u_i(t) + \tilde{f}(t, \tilde{x}_i(t)) \\
\tilde{u}_i(t) &= -C(s)\bar{L} \tilde{f}(t, \tilde{x}_i(t)), \tilde{x}(0) = T^i_x x_0
\end{align*}
$$

(57)

where

$$
\tilde{f}(t, \tilde{x}_i(t)) = f(t, x_i(t)) = f(t, (T^i_x)^{-1} \tilde{x}_i(t)).
$$

(58)

Given a set $\mathcal{Z}$, define

$$
\tilde{Z} \triangleq \{ \tilde{z} \in \mathbb{R}^n : (T^i_x)^{-1} \tilde{z} \in \mathcal{Z} \}.
$$

(59)

Similar to (30), for the transformed reference system (57), we have

$$
\begin{align*}
\mathcal{H}^{i}_{zm}(s) &\triangleq (sI_n - \tilde{A}^i_n)^{-1} \tilde{B}^i_v = T^i_x \mathcal{H}_{zm}(s) \\
\mathcal{H}^{i}_{zv}(s) &\triangleq (sI_n - \tilde{A}^i_n)^{-1} \tilde{B}^i_v = T^i_x \mathcal{H}_{zv}(s) \\
\mathcal{G}^{i}_{zm}(s) &\triangleq \mathcal{H}^{i}_{zm}(s)(I_m - C(s)) = T^i_x \mathcal{G}_{zm}(s)
\end{align*}
$$

(60a)

(60b)

(60c)

where $\mathcal{H}_{zm}, \mathcal{H}_{zv}, \mathcal{G}_{zm}$ are defined in (30). By applying the transformation (55) to the nominal system (9a), we obtain

$$
\begin{align*}
\dot{\tilde{x}}_i(t) &= \tilde{A}^i_n \tilde{x}_i(t) + \tilde{B}^i_v v(t), \tilde{x}_i(0) = T^i_x x_0 \\
\tilde{u}_i(t) &= \tilde{C} \tilde{x}_i(t).
\end{align*}
$$

(61)

Letting $\tilde{x}_i(t)$ be the state of the system $\tilde{x}_i(t) = \tilde{A}^i_n \tilde{x}_i(t)$ with $\tilde{x}_i(0) = \tilde{x}_i(0) = T^i_x x_0$, we have $\tilde{x}_i(s) = (sI_n - \tilde{A}^i_n)^{-1} \tilde{x}_i(0) = T^i_x(sI_n - A_{m})^{-1} \tilde{x}_i(0)$. Defining

$$
\rho^{i}_{m} \triangleq \|T^i_x(sI_n - A_{m})^{-1}\|_{L_1} \max_{x \in \mathcal{X}_n} \|x\|_{\infty}
$$

(62)

and further considering Lemma 2, we have $\|\tilde{x}_i\|_{\mathcal{C}} \leq \rho^{i}_{m}$. Similar to (31a), for the transformed reference system (57), consider the following condition:

$$
\|\mathcal{G}^{i}_{zm}(s)\|_{L_1} b_{j,i} \tilde{x}_i < \rho^{i}_{m} - \|\mathcal{H}^{i}_{zm}(s)\|_{L_1} \|v\|_{\mathcal{C}} - \rho^{i}_{m}
$$

(63)

where $\mathcal{X}_n$ is defined in (33) and $\tilde{\mathcal{X}}_n$ is defined according to (59) and $\rho^{i}_{m}$ is a positive constant to be determined. Then we have the following result.

Lemma 6: Consider the reference system (43) subject to Assumption 1, the nominal system (9a), the transformed reference system (57) and transformed nominal system (61) obtained by applying (55) with any $T^i_x$ satisfying (56). Suppose that (31a) holds with some constants $\rho_{r}$ and $\|v\|_{\mathcal{C}}$. Then, there exists an constant $\rho^{i}_{m} \leq \rho_{r}$ such that (63) holds with the same $\|v\|_{\mathcal{C}}$.

Furthermore

$$
|x_{n,i}(t)| \leq \rho^{i}_{m} \quad \forall t \geq 0
$$

(64)

$$
|x_{n,i}(t)| \leq \|\mathcal{G}^{i}_{zm}(s)\|_{L_1} b_{j,i} \tilde{x}_i \quad \forall t \geq 0
$$

(65)

where we redefine

$$
\mathcal{X}_n^{'} \triangleq \{ z \in \mathbb{R}^n : |z_i| \leq \rho^{i}_{m}, i \in \mathbb{Z}_1^n \}.
$$

(66)

Proof: For any $T^i_x$ satisfying (56) with an arbitrary $i \in \mathbb{Z}_1^n$, we have $\|T^i_x\|_{\infty} = 1$. Therefore, under the transformation (55), considering (60) and (62) and Lemma 2, we have

$$
\begin{align*}
\|\mathcal{H}^{i}_{zm}(s)\|_{L_\infty} &\leq \|T^i_x\|_{\infty} \|\mathcal{H}_{zm}(s)\|_{L_\infty} = \|\mathcal{H}_{zm}(s)\|_{L_\infty} \\
\|\mathcal{H}^{i}_{zv}(s)\|_{L_\infty} &\leq \|T^i_x\|_{\infty} \|\mathcal{H}_{zv}(s)\|_{L_\infty} = \|\mathcal{H}_{zv}(s)\|_{L_\infty} \\
\|\mathcal{G}^{i}_{zm}(s)\|_{L_\infty} &\leq \|T^i_x\|_{\infty} \|\mathcal{G}_{zm}(s)\|_{L_\infty} = \|\mathcal{G}_{zm}(s)\|_{L_\infty}
\end{align*}
$$

(67a)

(67b)

(67c)

$$
\rho^{i}_{m} \leq \|T^i_x\|_{\infty} \rho_{m} = \rho_{m}.
$$

(67d)

It follows from Lemma 3 that: $x_{n,i}(t) \in \mathcal{X}_n^{'}$ for any $t \geq 0$, which, together with (55), implies $\tilde{x}_i(t) \in \mathcal{X}_n^{'}$ for any $t \geq 0$, where $\mathcal{X}_n^{'}$.
is defined via (59). Considering (58) and (59), for any compact set $X_r$, we have

$$b_k x_k = b_f x_r.$$  

(68)

Now suppose that constants $\rho_x$ and $\|v\|_{\mathcal{L}_\infty}$ satisfy (31a). Then, due to (67) and (68), with $\rho_x^2 = \rho_x$ and the same $\|v\|_{\mathcal{L}_\infty}$, (63) is satisfied. Additionally, if (63) holds, by applying Lemma 3 to the transformed reference system (57), we obtain that $\|\hat{x}_i(t)\|_{\mathcal{L}_\infty}$ is bounded for all $t \geq 0$. Since $\hat{x}_i(t) = x_i(t)$ due to the constraint (56) on $T^*_x$, we have Equation (64). Equation (64) is equivalent to $x_r(t) \in X_r$ for any $t \geq 0$, with the redefinition of $X_r$ in (66). Following the proof of Lemma 5, one can obtain $\|\hat{x}-\hat{x}\|_{\mathcal{L}_\infty} \leq \|G_{\bar{x}m}(s)\|_{\mathcal{L}_1} b_f \dot{x}$ where the equality is due to (68). Further considering $\hat{x}_i(t) = x_i(t)$ and $\tilde{x}_n(t) = x_n(t)$ due to the constraint (56) on $T^*_x$, we have (65).

Remark 13: Lemmas 3 and 5 imply $|x_i(t)| \leq \rho_x$ and $|x_i(t) - x_n(t)| \leq \|G_{\bar{x}m}(s)\|_{\mathcal{L}_1} b_f \dot{x}$, respectively, for all $i \in Z^n_1$ and $t \geq 0$. Lemma 6 indicates that $\|\hat{x}_i(t)\|_{\mathcal{L}_\infty}$ is bounded for all $t \geq 0$. So, $x_i(t) - x_n(t)$ is bounded for all $t \geq 0$ and any $i \in Z^n_1$. Therefore, (74) is satisfied. On the other hand, Theorem 1 indicates that $|x_i(t) - x_n(t)| \leq \gamma_1$ for any $t \geq 0$ and any $i \in Z^n_1$. Therefore, (74) is true. On the other hand, Theorem 1 indicates that $|u_{i,j}(t) - u_{n,j}(t)| \leq \gamma_2$ for any $t \geq 0$ and any $j \in Z^n_1$, which, together with (70), leads to (71b).

Remark 14: Theorem 3 provides a way to derive an individual bound on $x_i(t)$, and $x_i(t) - x_n(t)$ for each $i \in Z^n_1$ and $u_j(t) - u_{n,j}(t)$ for each $j \in Z^n_1$ via coordinate transformations. Additionally, similar to the arguments in Remark 11, by decreasing $T$ and increasing the bandwidth of the filter $C(s)$, one can make $\rho_x(i \in Z^n_1)$ arbitrarily small, i.e., making the states of the adaptive system arbitrarily close to those of the nominal system, and make the bounds on $u_{i,j}(t)$ and $u_j(t) - u_{n,j}(t)$ arbitrarily close to the bound on the true uncertainty $f_j(t, x)$ for $x \in X_r$, for each $j \in Z^n_1$.

According to Theorems 2 and 3, the procedure for designing an $L_1$ AC with separate bounds on states and adaptive inputs can be summarized in Algorithm 1.

Remark 15: One can try different $T^*_x$ in step 9 of Algorithm 1 and select the one that yields the tightest bound for the $i$th state.

Remark 16: The conditions (31) and (42) can be quite conservative for some systems, due to the frequent use of inequalities related to the $L_1$ norm (stated in Lemma 2), Lipshitz continuity and matrix/vector norms. As a result, the bandwidth of the filter $C(s)$ computed via (31) could be unnecessarily high, while the sample time $T$ computed via (42) under a given $\gamma_1$ could be unnecessarily small. Based on our experience, assuming that some bounds $\rho_x(i \in Z^n_1)$ and $\rho_x(j \in Z^n_1)$ satisfying (71) are derived under a specific filter $C(s)$ and $T^*$ that satisfy (31) and (42), those bounds will most likely be respected in simulations even if we decrease the bandwidth of $C(s)$ by 1~3 times and/or increase $T^*$ by 1~10 times.

V. $L_1$-RG: Adaptive Reference Governor for Constrained Control Under Uncertainties

Leveraging the uniform bounds on state and input errors guaranteed by the $L_1$ AC, we now integrate the $L_1$ AC and the RG introduced in Section III-B to synthesize the $L_1$-RG framework for simultaneously enforcing the constraints (2) and improving the tracking performance.

A. $L_1$-RG Design

We first make the following assumption.

Assumption 2: $X_r$ and $U_r$ defined by (12), (17), and (71) are nonempty. Furthermore, there exists a known command $v(0)$ such that

$$v(0), x_0) \in \partial_{\infty}$$  

(73)

where $\partial_{\infty}$ is defined in (26).

Remark 17: Considering (26), (73) implies $x_0 \in X_n$ and $u_n(0) = K_x x_0 + K_v v(0) \in U_n$ since $x_n(0) = x_0$ where $X_n$
Algorithm 1: Designing an $L_1$ AC with Separate Bounds.

**Input:** uncertain system (5) subject to Assumption 1, initial parameters $A_r, C(s)$ and $T$ to define an $L_1$ AC, $\gamma_1, X_0$, \(\|v\|_{L_\infty}\), tol

1: **procedure** DECREASEFILTERUNCERTBND(\(C(s), \gamma_1, X_0, \|v\|_{L_\infty}\))
2: while condition (31a) or (31b) does not hold do
3: Increase the bandwidth of $C(s)$ \(\triangleright\) See Remark 8,
4: \[X_r = \Omega(\rho_r) \text{ and } b_{f,x_r}\] will be computed.
5: **end procedure**
6: Set \(b_{f,x_r} = b_{f,x_r}\).
7: **procedure** DERIVESPECSTATEBNDs (\(b_{f,x_r}, \gamma_1, C(s), X_0, \|v\|_{L_\infty}\))
8: \(\text{for } i = 1, \ldots, n \text{ do}\)
9: Evaluate (60) and compute $\hat{\rho}_i$ according to (62)
10: Compute $\hat{\rho}_i^\gamma$ that satisfies (63) \(\triangleright\) Such a $\hat{\rho}_i^\gamma \leq \rho_r$ is guaranteed to exist from Lemma 6
11: Set $\hat{\rho}_i = \hat{\rho}_i^\gamma + \gamma_1, \bar{\rho}_i = \|\hat{G}_{f,\infty}(s)\|_{L_1, b_{f,x_r} + \gamma_1}$
12: **end for**
13: **end procedure**
14: **procedure** DERIVESPECINPUTBNDs (\(X_r, \{\hat{\rho}_i\}_{i \in \mathbb{Z}_+}, \gamma_2, C(s)\))
15: \(\text{for } j = 1, \ldots, m \text{ do}\)
16: Compute $\hat{\rho}_u$ and $\bar{\rho}_u$ according to (72b)
17: **end for**
18: **end procedure**
19: **procedure** DECIDESTATEINPUTBNDs (\(A_r, C(s), T, X_u\))
20: **while** constraint (42) does not hold do
21: Decrease $T \triangleright$ Small $T$ can enforce (42) due to (38),
22: **end while**
23: **end procedure**

**Output:** An $L_1$ AC defined by (34)–(36) with parameters $A_e$ and $C(s)$ and $T, \rho_r$, and $\hat{\rho}_i^\gamma$ for $i \in \mathbb{Z}_+^n, \hat{\rho}_u$ and $\bar{\rho}_u$ for $j \in \mathbb{Z}_1^n$

Algorithm 2: $L_1$-RG Design.

**Input:** An continuous-time uncertain system (5) subject to Assumption 1, constraint sets $X$ and $U$ as in (2), $X_0, V$ (admissible set for $v(t)$), baseline control law in (3), initial parameters $A_r, C(s)$ and $T$ to define an $L_1$ AC, $\gamma_1, T_d$ and $\epsilon$ for RG design, tol

1: **procedure** $L_1$-RG-DESIGNUNDERCONSTRAINTS
2: Compute $\|v\|_{L_\infty}$ given $V$
3: **while** (31a) with $X_0 = \Omega(\rho_r) \cap X$ or (31b) with $X_0 = \Omega(\rho_r + \gamma_1) \cap X$ does not hold for any $\rho_r$ do
4: Increase the bandwidth of $C(s)$ \(\triangleright\) See Remark 8.
5: **end while** \(\triangleright\) $X_r$ and $b_{f,x_r}$ will be computed.
6: Set $b_{f,x_r} = b_{f,x_r}$
7: Run DERIVESPECINPUTBNDs of Algorithm 1 with $b_{f,x_r}$, and obtain $\hat{\rho}_i^\gamma$ and $\bar{\rho}_i$ for $i \in \mathbb{Z}_1^n$
8: Set $X_r = \{z \in \mathbb{R}^n : |z_i| \leq \hat{\rho}_i \} \cap X$ and update $b_{f,x_r}$
9: \[i \text{ if } \hat{b}_{f,x_r} - b_{f,x_r} > \text{tol then}\]
10: \[\text{Set } b_{f,x_r} = b_{f,x_r}, \text{ and go to step 7}\]
11: **end if**
12: \[\text{Set } X_u = \{z \in \mathbb{R}^n : |z_i| \leq \rho_i, \text{ } i \in \mathbb{Z}_+^n \} \cap X\]
13: Run DERIVESPECINPUTBNDs of Algorithm 1 with $C(s)$ from step 5 and $X_r$ from step 8, and obtain $\hat{\rho}_u$ and $\hat{\rho}_u^\gamma$ for $j \in \mathbb{Z}_1^n$
14: Run DECIDESTATEINPUTBNDs of Algorithm 1 with $C(s)$ from step 5 and $X_u$ from step 12, and obtain $T$
15: Compute $\hat{X}$ and $\hat{U}$ with $\{\hat{\rho}_i\}_{i \in \mathbb{Z}_1^n}$ and $\{\hat{\rho}_u^\gamma\}_{j \in \mathbb{Z}_1^n}$ via (71)
16: **end procedure**
17: **procedure** RG-DESIGN
18: Compute $X_u$ and $U$ with $X, U, \hat{X}, \hat{\rho}$, and $\hat{U}$ via (12)
19: Formulate the nominal discrete-time model (14) with the sample time $T_d$
20: Compute $\hat{X}_u$ and $\hat{U}_u$ via (17) \(\triangleright\) With a small $T_d$, one may set $\hat{X}_u = X_u, \hat{U}_u = U_u$ for practical implementation.
21: Compute the set $\hat{O}_\infty$ dependent on $\epsilon$ according to (26)
22: **end procedure**

**Output:** An $L_1$-RG consisting of a RG designed for the nominal system (9a) and an $L_1$ AC to compensate for uncertainties

and $\hat{U}_u$, according to (17), are tightened versions of $X_u$ and $U_u$ that are defined in (12). From Remark 14, with a sufficiently high bandwidth for $C(s)$ and sufficiently small $T$, one can make $X_u$ arbitrarily close to $X$ and make $\hat{U}$ arbitrarily close to the bound set for the true uncertainty in $X$. Additionally, as mentioned in Remark 4, $X_u$ and $U_u$ are close to $X_u$ and $U_u$, respectively, when $T_d$ is small. As a result, with a sufficiently high bandwidth for $C(s)$, and sufficiently small $T$ and $T_d$, Assumption 2 roughly states that the initial state stays in $X$, and the constraint set $\mathcal{U}$ is sufficiently large to ensure enough control authority for tracking an initial reference command $v(0)$ and additionally for compensating the uncertainty in $X$.

Under the preceding assumption, the design procedure for $L_1$-RG is summarized in Algorithm 2. Compared to step 2 of Algorithm 1, we additionally constrain $x_r(t)$ and $\rho_r$ to stay in $X$ for all $t \geq 0$ in step 3 of Algorithm 2. Such constraints can potentially limit the size of uncertainties that need to be compensated and significantly reduce the conservatism of the proposed solution.
We are ready to state the guarantees regarding tracking performance and constraint enforcement provided by $\mathcal{L}_1$-RG.

**Theorem 4:** Consider an uncertain system (5) subject to Assumption 1 and the state and control constraints in (2). Suppose that an $\mathcal{L}_1$AC [defined by (34)–(36)] and a RG are designed by following Algorithm 2. If Assumption 2 holds, then, under the baseline control law (3) and the $\mathcal{L}_1$-RG consisting of the compositional control law (4), the $\mathcal{L}_1$AC and the RG for computing the reference command input $v(t)$ according to (15) and (27), we have

\begin{align}
    x(t) & \in \text{int}(\mathcal{X}), \ u(t) \in \text{int}(\mathcal{U}) \quad \forall t \geq 0 \quad (74) \\
    x(t) - x_n(t) & \in \text{int}(\mathcal{X}) \quad \forall t \geq 0 \quad (75) \\
    y(t) - y_n(t) & \in \{ z \in \mathbb{R}^m : |z_i| \leq \tilde{p}_y \} \quad \forall t \geq 0 \quad (76)
\end{align}

where $x_n(t)$ and $y_n(t)$ are the states and outputs of the nominal system (9) under the reference command input $v(t)$, and

\begin{equation}
    \tilde{p}_y \triangleq \sum_{i=1}^{n} |C[j,i]| \tilde{p}^i \quad \forall j \in \mathbb{Z}_1^n.
\end{equation}

**Proof:** Equation (73) in Assumption 2 implies $(v(0), x_n(0)) \in \mathcal{O}_\infty$ (due to $x_n(0) = x_0$), and $u_n(0) = K_x x_n(0) + K_v v(0) \in \mathcal{U}_n$. Thus, the reference command $v(k)$ produced by (27) ensures $x_n(k) \in \mathcal{X}_n$ and $u_n(k) \in \mathcal{U}_n$ for all $k \in \mathbb{Z}_+$, which, due to Lemma 1, implies

\begin{equation}
    x_n(t) \in \mathcal{X}_n, \quad u_n(t) \in \mathcal{U}_n \quad \forall t \geq 0.
\end{equation}

Compared to lines 2, 15, and 19 of Algorithm 1, we restrain $\mathcal{X}_n$ and $\mathcal{X}_n$ to be subsets of $\mathcal{X}$ in lines 3, 8, and 12 of Algorithm 2. As a result, if (74) and (79) jointly hold, condition (75) holds according to Theorem 3, while (65) holds according to Lemma 6.

We next prove (74) and (79) by contradiction. Assume (74) or (79) do not hold. The initial condition (73) implies $x(0) \in \mathcal{X}_n \subset \text{int}(\mathcal{X})$ and $u(0) \in \mathcal{U}_n \subset \text{int}(\mathcal{U})$. As a result, we have $x(0) \in \mathcal{X}, x_n(0) \in \mathcal{X}_n$ and $u(0) \in \mathcal{U}$. Since $x(t), x_n(t), \text{and} u(t)$ are continuous, there must exist a time instant $\tau$, such that

\begin{align}
    x(t) & \in \text{int}(\mathcal{X}), \quad x_n(t) \in \text{int}(\mathcal{X}_n) \quad \forall t \in [0, \tau) \quad (80a) \\
    x_n(t) & = \text{int}(\mathcal{X}_n) \quad \forall t \in [0, \tau).
\end{align}

Similarly, according to Theorem 3, due to (80) and the definition in (71), we have $x(t) - x_n(t) \in \text{int}(\mathcal{X})$, $u(t) - u_n(t) \in \text{int}(\mathcal{U})$, for any $t \in [0, \tau)$, which, together with (12) and (78), implies

\begin{align}
    x(t) & \in \text{int}(\mathcal{X}), \quad u(t) \in \text{int}(\mathcal{U}) \quad \forall t \in [0, \tau].
\end{align}

We now apply $\mathcal{L}_1$-RG to the longitudinal dynamics of an F-16 aircraft. The model was adapted from [44] with slight modifications to remove the actuator dynamics, in which the state vector $x(t) = \gamma(t), \theta(t), (\alpha(t))^T$ consists of the flight path angle, pitch rate and angle of attack, and the control input vector $u(t) = [\delta_e(t), \delta_f(t)]$ includes the elevator deflection and flap-erion deflection. The output vector is $y(t) = \theta(t), (\gamma(t))^T$, where $\theta(t) = (\gamma(t) + \alpha(t))$ is the pitch angle; the reference input vector is $r(t) = [\theta_e(t), \gamma_e(t)]^T$, where $\theta_e$ and $\gamma_e$ are the commanded pitch angle and flight path angle, respectively. The system is subject to state and control constraints

\begin{align}
    |\alpha(t)| & \leq 4^\circ, \quad |\delta_e(t)| \leq 25^\circ, \quad |\delta_f(t)| \leq 22^\circ
\end{align}

where the state constraint can also be represented as $x(t) \in \mathcal{X} \triangleq [-10^3, 10^3] \times [-10^3, 10^3] \times [-4, 4]$ following the convention in (2). Furthermore, we assume

\begin{equation}
    ||x||_{\infty} \leq 10, \quad x(0) \in \mathcal{X}_0 = \Omega(0.1).
\end{equation}

The open-loop dynamics are given by

\begin{equation}
    \dot{x} =
    \begin{bmatrix}
        0 & 0.0067 & 1.34 \\
        -0.869 & 43.2 & 0 \\
        0.993 & -1.34 & 0
    \end{bmatrix} 
    \begin{bmatrix}
        x \\
        u + f(t, x)
    \end{bmatrix}
    +
    \begin{bmatrix}
        0.169 & 0.252 \\
        -17.3 & -1.58 \\
        0 & -0.169 & -0.252
    \end{bmatrix}
\end{equation}

where $f(t, x) = [-0.8 \sin(0.4 t) - 0.1^2 x - 0.1 - 0.2 \alpha + \gamma]$ is an artificial uncertainty dependent on both time and $\alpha$. The feedback and feedforward gains of the baseline controller (3) are selected to be $K_x = [3.25, 0.891, 7.12; -6.10, -0.898, -10.0]$ and $K_v = [-3.93, 0.679; 2.57, 3.53]$. Via simple calculations, we can see that $f(t, x) \in \mathcal{W} = [-2.4, 2.4] \times [-0.9, 0.9]$ when $x \in \mathcal{X}$ holds.

### A. $\mathcal{L}_1$-RG Design

It can be verified that given any set $\mathcal{Z}$, $L_{f_1, \mathcal{Z}} = 0.2 \max_{\alpha \in \mathcal{Z}} |\alpha|$, $L_{f_2, \mathcal{Z}} = 0.2\delta_c, (\alpha \in \mathcal{Z}) = 0.8 + 0.1 \max_{\alpha \in \mathcal{Z}} (\alpha)^2$, $b_{f, \mathcal{Z}} = 0.1 + 0.2 \max_{\alpha \in \mathcal{Z}} |\alpha|$ satisfy Assumption 1. For the design of the $\mathcal{L}_1$AC in (34)–(36), we select $A_e = -10 I_3$ and parametrize the filter as $C(s) = \frac{k_f}{s + k_f} I_3$, where $k_f > 0$ denotes the bandwidth for both input channels. Table II lists the bounds on $x_i(t) - x_{n,i}(t)$ and $u_j(t) - u_{n,j}(t)$ theoretically computed by

| TABLE II |
| --- |
| **Performance Bounds Obtained Under Different Filter Bandwidth and Sample Time T With and Without (W/O) Scaling** |
| | $k_f = 200$, $T = 10^{-5}$ | $k_f = 10^4$, $T = 10^{-7}$ |
| Var. | W/O scaling | With scaling | W/O scaling | With scaling |
| $\gamma_l, \gamma_r, \delta_c$ | $[41, 1, 1]$ | $[0.015, 0.41, 0.638]$ | $[0.043, 1, 1]$ | $[12, 4.3, 3.5 \times 10^{-2}$ |
| $[\tilde{p}_1, \tilde{p}_2]$ | $[8.15, 9.02]$ | $[4.20, 2.85]$ | $[2.94, 1.69]$ | $[2.51, 1.03]$ |
applying Algorithm 2 under different $\mathcal{C}(s)$ and $T$ with and without using the scaling technique in Section IV. When applying the scaling technique, we set $T^2[k] = 0.01$ for each $i, k \in \mathbb{Z}^3_1$ and $k \neq i$, which satisfies (56). Several observations can be made from Table II. First, by increasing the filter bandwidth $k_f$ and decreasing $T$, we are able to obtain a smaller $\gamma_i$ satisfying (42) and achieve tighter bounds for all states and inputs. In fact, if $k_f = 10^3$ and $T = 10^{-7}$, then $\hat{\rho}_u$ and $\hat{\rho}_u^2$ are fairly close to the bounds on $f_1(t, x)$ and $f_2(t, x)$ for $x \in \mathcal{X}$, respectively, which is consistent with Remark 14. Additionally, with scaling, we could significantly reduce $\hat{\rho}_u$ and $\hat{\rho}_u^2$, the bounds on $\gamma(t) - \gamma_0(t)$ and $\alpha(t) - \alpha_0(t)$, and $\hat{\rho}_x^1$ and $\hat{\rho}_x^2$, the bounds on $\delta_e(t) - \delta_e(u_0, t)$ and $\delta_f(t) - \delta_f(u_0, t)$. Moreover, with $T^3_x$, we can verify that the condition (63) holds with $b_{j, \mathcal{X}}$, as long as $\|v\|_{L_\infty} < 1.868$. As mentioned in Remark 16, the conditions (31) and (42) and the resulting bounds $\hat{\rho}_u$ and $\hat{\rho}_u^2$ could be conservative. As a result, a larger reference command can potentially be allowed in a practical implementation while keeping $x(t)$ to stay in $\mathcal{X}$, as demonstrated in the following simulations. Following Algorithm 2, we used the bounds $\hat{\rho}_u^3$, $\hat{\rho}_u^4$, and $\hat{\rho}_x^3$ obtained for the case when $k_f = 200$ and $T = 10^{-5}$, to tighten the original constraints (83) and then used the tightened constraints to design the RG, for which we chose $T_d = 0.005$. Considering that $T_d$ was small, we did not consider intersample constraint violations and simply set $\tilde{X}_t = X_t$ and $U_t = U_t$ instead of (17). For comparisons, we also designed a robust RG (RRG) that treats the uncertainty $f(t, x)$ as a bounded disturbance $w(t) \in \mathcal{W}$, where $\mathcal{W}$ is introduced below (88). RRG design also uses $O_\infty$ set [defined in (22) for RG design]; however, the prediction of the output, which corresponds to $\hat{y}_e(k|v, x)$ for RG design, becomes a set-valued one, taking into account all possible realizations of the disturbance $w(t)$ (see [3] for details). We additionally designed a standard RG by simply ignoring the uncertainty $f(t, x)$.

### B. Simulation Results

As mentioned in Remark 16, the value of $T$ theoretically computed according to (42) is often unnecessarily small. For the subsequent simulations, we simply adopted an estimation sample time of 1 ms, i.e., $T = 0.001$ s. As one can see in the subsequent simulation results, all the bounds derived in Section VI-A for $k_f = 200$ and $T = 10^{-5}$ still hold. The reference command $r(t)$ was set to be [9,6.5] deg for $t \in [0,7.5]$ s, and [0,0] deg for $t \in [7.5,15]$ s. The results are shown in Figs. 2–4.

In terms of constraint enforcement, Fig. 3 shows that both RRG and $L_1$-RG successfully enforced all the constraints, while violation of the constraints on the state $\alpha(t)$ and the input $\delta_f(t)$ happened under RG. However, from Fig. 2, one can see that the RRG was quite conservative, leading to a large difference between the modified reference and original reference commands and subsequently large tracking errors for both $\theta(t)$ and $\gamma(t)$ throughout the simulation. In comparison, the modified reference command under RG reached the original reference command, leading to better tracking performance. Finally, $L_1$-RG yielded the best tracking performance, driving both $\theta(t)$ and $\gamma(t)$ very close to their commanded values at steady state. While noticeable under RG and RRG, the uncertainty-induced swaying

![Fig. 2. Tracking performance under RG, RRG, and $L_1$-RG.](image1)

![Fig. 3. Trajectories of constrained state and inputs under RG, RRG, and $L_1$-RG.](image2)

![Fig. 4. Actual and estimated uncertainties under $L_1$-RG. The symbol $f_i$ denotes the $i$th element of $f(\phi)$, for $i = 1, 2$.](image3)
in the outputs at steady state was negligible under $L_1$-RG, thanks to the active compensation of the uncertainty by the $L_1$ AC. From Fig. 4, one can see that the estimation of the uncertainty within the $L_1$-RG was quite accurate. We next check whether the derived uniform bounds on the errors in states, $x(t) - x_\alpha(t)$, and on the adaptive inputs, $u_\alpha(t)$, hold in the simulation. It can be seen from Fig. 5 that the bounds on both $u_{a,1}(t)$ and $u_{a,2}(t)$ were respected in the simulation and, moreover, are fairly tight. Fig. 6 reveals that all the actual states under $L_1$-RG were fairly close to their nominal counterparts, and moreover, the bound on $x_i(t) - x_{n,i}(t)$ for each $i \in \mathbb{Z}_1^3$ was respected. Note that $x_n(t)$ in Fig. 6 was produced by applying the same reference command $v(t)$ yielded by $L_1$-RG to the nominal system (9).

VII. CONCLUSION

In this article, we developed $L_1$-RG, an adaptive RG framework, for the control of linear systems with time- and state-dependent uncertainties subject to both state and input constraints. At the core of $L_1$-RG is an $L_1$ adaptive controller that provides guaranteed uniform bounds on the errors between states and inputs of the uncertain system and those of a nominal (i.e., uncertainty-free) system. With such uniform error bounds for constraint tightening, an RG designed for the nominal system with tightened constraints guarantees the satisfaction of the original constraints by the actual states and inputs. Simulation results validate the efficacy and advantages of the proposed approach.

In the future, we will address unmatched uncertainties, and extend the proposed framework to the nonlinear setting leveraging the results in [45] and [46]. Additionally, we would like to extend the proposed solution to adaptive MPC.

APPENDIX A

PROOFS

A. Proof of Lemma 1

Proof: Since the continuous-time system (9) has the same states as the discrete-time system (14) at all sampling instants, if (19) holds for (14), then we have

$$x_n(kT_d) \in \mathcal{X}_n, \quad u_n(kT_d) \in \mathcal{U}_n \quad \forall k \in \mathbb{Z}_+ \quad (86)$$

for (9). Next we analyze the behavior of (9) between adjacent sampling instants. Towards this end, consider any $t = kT_d + \tau$ for some $k \in \mathbb{Z}_+$ and $\tau \in [0, T_d)$. From (9), we have

$$x_n(t) = x_n(kT_d + \tau) = e^{A_m\tau}x_n(kT_d) + \int_{kT_d}^{(k+1)T_d} e^{A_m(t-s)}B_r v(s) ds = e^{A_m\tau}x_n(kT_d) + A^{-1}_m(e^{A_m\tau} - I_n)B_r v(kT_d),$$

where the third equality is due to the fact that $v(kT_d + \tau) = v(kT_d)$ for all $\tau \in [0, T_d)$. As a result, we have

$$x_n(t) - x_n(kT_d) = x_n(kT_d + \tau) - x_n(kT_d) = (e^{A_m\tau} - I_n)(x_n(kT_d) + A^{-1}_mB_r v(kT_d)).$$

Thus, we have

$$\|x_n(t) - x_n(kT_d)\| \leq \nu(T_d) \quad (87a)$$

$$\|u_n(t) - u_n(kT_d)\| \leq \|K_x\| \nu(T_d) \quad (87b)$$

where $\nu(T_d)$ defined in (18), while (87b) is due to the fact that $u_n(t) - u_n(kT_d) = K_x(x_n(t) - x_n(kT_d))$. Considering (17), (86), and (87), we have $x_n(t) \in \mathcal{X}_n$ and $u_n(t) \in \mathcal{U}_n$ for all $t \geq 0$. The proof is complete.

B. Proof of Lemma 3

Proof: Rewriting the dynamics of the reference system in (43) in the Laplace domain yields

$$x_r(s) = G_{xm}(s)\mathcal{L}\{f(t, x_i(t))\} + H_{xv}(s)v(s) + x_{in}(s). \quad (88)$$

Therefore, from Lemma 2, for any $\xi > 0$, we have

$$\|x_r\|_{\mathcal{L}_1[0, \xi]} \leq \|G_{xm}(s)\|_{\mathcal{L}_1} \|\eta\|_{\mathcal{L}_1[0, \xi]} + \|H_{xv}(s)\|_{\mathcal{L}_1} \|v\|_{\mathcal{L}_1} + \|x_{in}\|_{\mathcal{L}_\infty} \quad (89)$$

where $\eta(s)$ is defined in (44). If (45) is not true, since $x_i(t)$ is continuous and $\|x_i(0)\| < \rho_r$, there exists a $\tau > 0$ such that

$$\|x_i(t)\|_{\mathcal{L}_\infty} < \rho_r \quad \forall t \in [0, \tau],$$

and $\|x_i(\tau)\|_{\mathcal{L}_\infty} = \rho_r \quad (90)$

which implies $x_i(t) \in \Omega(\rho_r)$ for any $t \in [0, \tau]$. Further considering (7b) that results from Assumption 1, we have

$$\|\eta\|_{\mathcal{L}_1[0, \xi]} \leq b_f \Omega(\rho_r). \quad (91)$$
Plugging the preceding inequality into (89) leads to
\[ \rho_r \leq \|G_{xm}(s)\|_{\mathcal{L}_1} b_f, \Omega(\rho_r) + \|H_{xv}(s)\|_{\mathcal{L}_1} u \|_{\mathcal{L}_\infty} + \rho_m \] (92)
which contradicts the condition (31a). Therefore, (45) is true. Equation (46) immediately follows from (45) and (43).

C. Proof of Lemma 4

Proof: Due to (48), we have \( x(t) \in \Omega(\rho) \) for any \( t \in [0, \tau] \). Further considering (7b) that results from Assumption 1, we have
\[ \|f(t, x(t))\|_{\infty} = \|\eta(t)\|_{\infty} \leq b_f, \Omega(\rho) \quad \forall t \in [0, \tau]. \] (93)
From (47), for any \( 0 \leq t < T \) and \( i \in Z_0 \), we have
\[ \dot{x}(iT + t) = e^{A_i t} \dot{x}(iT) + \int_{iT}^{iT+\xi} e^{A_i \xi} \dot{x}(iT + \xi)d\xi - \int_{iT}^{iT+\xi} e^{A_i \xi} B\eta(\xi)d\xi = e^{A_i t} \dot{x}(iT) + \int_{0}^{t} e^{A_i (\xi-t)} \dot{x}(iT)d\xi - \int_{0}^{t} e^{A_i (\xi-t)} B\eta(iT + \xi)d\xi. \] (94)
Considering the adaptive law (35), (94) implies
\[ \dot{x}(iT + (i+1)T) = -\int_{0}^{T} e^{A_i (T-\xi)} B\eta(iT + \xi)d\xi. \] (95)
Therefore, for any \( i \in Z_0 \) with \( (i+1)T \leq \tau \), we have
\[ \|\ddot{x}(i+1)T\|_{\infty} \leq \int_{0}^{T} \left\| e^{A_i (T-\xi)} B \right\| \|\eta(iT + \xi)\|_{\infty} d\xi \leq \tilde{o}_0(T)b_f, \Omega(\rho) \] (96)
where \( \tilde{o}_0(T) \) is defined in (37a), and the last inequality is due to (93). Since \( \tilde{x}(0) = 0 \), we therefore have
\[ \|\ddot{x}(iT)\|_{\infty} \leq \tilde{o}_0(T)b_f, \Omega(\rho) \|\eta(T)\|_{\infty} \quad \forall T \leq \tau, i \in Z_0. \] (97)
Now consider any \( t \in [0, T] \) such that \( iT + t \leq \tau \) with \( i \in Z_0 \). From (94) and the adaptive law (35), we have
\[ \|\dot{x}(iT + t)\|_{\infty} \leq \|e^{A_i t}\|_{\infty} \|\ddot{x}(iT)\|_{\infty} + \int_{0}^{t} \left\| e^{A_i (\xi-t)} \Phi^{-1}(T) e^{A_i T}\right\| \|\ddot{x}(iT)\|_{\infty} d\xi \leq \|e^{A_i t}\|_{\infty} \|\ddot{x}(iT)\|_{\infty} + \int_{0}^{t} \left\| e^{A_i (\xi-t)} \Phi^{-1}(T) e^{A_i T}\right\| \|\ddot{x}(iT)\|_{\infty} d\xi \leq \tilde{o}_1(T) + \tilde{o}_2(T) + 1 \] (98)
where \( \tilde{o}_1(T) \) and \( \tilde{o}_2(T) \) are defined in (37a)–(37c), and the last inequality is partially due to the fact that \( \int_{0}^{T} \|e^{A_i (T-\xi)}\|_{\infty} d\xi = \tilde{o}_0(T) \). Equations (97) and (98) imply (49).

D. Proof of Theorem 1

Proof: We first prove (50c) and (50d) by contradiction. Assume (50c) or (50d) do not hold. Since \( \|x_i(0) - x(0)\|_{\infty} = 0 < \gamma_1 \) and \( \|u_i(0) - u(0)\|_{\infty} = 0 < \gamma_2 \), and \( x(t), u(t), x_i(t) \) and \( u_i(t) \) are all continuous, there must exist an instant \( \tau \) such that
\[ \|x_i(\tau) - x(\tau)\|_{\infty} = \gamma_1 \text{ or } \|u_i(\tau) - u(\tau)\|_{\infty} = \gamma_2 \] (99)
while
\[ \|x_i(t) - x(t)\|_{\infty} < \gamma_1, \|u_i(t) - u(t)\|_{\infty} < \gamma_2 \quad \forall t \in [0, \tau). \] (100)
This implies that at least one of the following equalities hold:
\[ \|x_i - x\|_{L_0} = \gamma_1, \|u_i - u\|_{L_0} = \gamma_2. \] (101)
Note that \( \|x_i\|_{L_\infty} \leq \rho_r < \rho \) according to Lemma 3 and \( \|x\|_{L_\infty} \leq \rho_r + \gamma_1 = \rho \) from (101). Further considering (7a) that results from Assumption 1, we have that
\[ \|f(t, x(t)) - f(t, x(t))\|_{\infty} \leq L_f, \Omega(\rho) \|x - x\|_{L_0} \quad \forall t \in [0, \tau]. \] (102)
The control laws in (36) and (43) indicate
\[ u_i(s) - u(s) = -C(s)\Sigma[f(t, x) - B^I \dot{x}(t)] \]
\[ = C(s)\Sigma[f(t, x) - f(t, x)] + C(s)B^I(\dot{x}(t) - \Sigma[f(t, x)]). \] (103)
Equation (47) indicates that
\[ B^I\dot{x}(s) - \Sigma[f(t, x)] = B^I(sI_n - A_e)\dot{x}(s). \] (104)
Considering (5), (36), and (104), we have
\[ x(s) = G_{xm}(s)\Sigma[f(t, x)] + H_{xv}(s)v(s) + x_m(s) \]
\[ - H_{xm}(s)C(s)B^I(sI_n - A_e)\dot{x}(s). \] (105)
which, together with (88), implies
\[ x_i(s) - x(s) = G_{xm}(s)\Sigma[f(t, x) - f(t, x)] + H_{xm}(s)C(s)B^I(sI_n - A_e)\dot{x}(s). \] (106)
Therefore, further considering (102) and Lemma 4, we have
\[ \|x_i - x\|_{L_0} \leq \|G_{xm}\|_{L_1} L_f, \Omega(\rho) \|x - x\|_{L_0} \]
\[ + \|H_{xm}(s)C(s)B^I(sI_n - A_e)\|_{L_1} \gamma_0(T). \]
The preceding equation, together with (31b), leads to
\[ \|x_i - x\|_{L_0} \leq \frac{\|H_{xm}(s)C(s)B^I(sI_n - A_e)\|_{L_1} \gamma_0(T)}{1 - \|G_{xm}\|_{L_1} L_f, \Omega(\rho)} \] (107)
which, together with the sample time constraint (42), implies that
\[ \|x_i - x\|_{L_0} < \gamma_1. \] (108)
On the other hand, it follows from (102) to (104) that:
\[ \|u_i - u\|_{L_0} \leq \|C(s)\|_{L_1} L_f, \Omega(\rho) \|x - x\|_{L_0} \]
\[ + \|C(s)B^I(sI_n - A_e)\|_{L_1} \|\dot{x}\|_{L_0} \]
\[ < \|C(s)\|_{L_1} L_f, \Omega(\rho) \gamma_1 + \|C(s)B^I(sI_n - A_e)\|_{L_1} \gamma_0(T). \] (109)
Further considering the definition in (40), we have
\[ \|u_i - u\|_{L_0} < \gamma_2. \] (109)
Note that (108) and (109) contradict the equalities in (101), which proves (50c) and (50d). The bounds in (50a) and (50b) follow directly from (45), (46), (50c), and (50d) and the definitions of ρ and ρD in (32) and (41). The proof is complete. □

E. Proof of Lemma 5

Proof: From (9a) and (43), we have

\[ x_1(t) - x_1(0) = G_{cm}(s) \mathcal{L} \left[ f(t, x_1) \right] = G_{cm}(s) \mathcal{L} \left[ \eta(t) \right]. \]  

(110)

According to Lemma 3, we have \( x_1(t) \in \Omega(\rho_r) \) for any \( t \geq 0 \). Further considering (7b), which results from Assumption 1, we have \( \| \eta \|_{L_\infty} \leq b_{f, f}(\rho_r) \), which, together with (110), leads to (51). □

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N. Hovakimyan, “Tube-certified trajectory tracking for nonlinear systems
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Pan Zhao (Member, IEEE) received the B.E.
degree in automation and the M.S. degree in
mechatronics engineering from Beihang Univer-
sity, Beijing, China, in 2009 and 2012, respec-
tively, and the Ph.D. degree in mechanical engi-
neering from the University of British Columbia,
Vancouver, Canada, in 2018.

He is currently an Assistant Professor with
the Department of Aerospace Engineering
and Mechanics, University of Alabama (UA),
Tuscaloosa, AL, USA. Before joining UA, he was
a Postdoctoral Researcher with the University of Illinois at Urbana–
Champaign, Urbana, IL, USA. His research interests include robust
adaptive control, autonomy and machine learning with applications to
aerospace, robotics, and sustainable agriculture.

Ilya Kolmanovsky (Fellow, IEEE) received the
M.S. and Ph.D. degrees in aerospace engineer-
ing, and the M.A. degree in mathematics from
the University of Michigan, Ann Arbor, MI, USA,
in 1993, 1995, and 1995, respectively.

He is presently a Professor with the Depart-
ment of Aerospace Engineering, University of
Michigan. His research interests include control
theory for systems with state and control con-
straints, and in control applications to aerospace
and automotive systems.

Dr. Kolmanovsky was the past recipient of the Donald P. Eckman
Award of American Automatic Control Council. He is named as an
inventor on 104 United States patents.

Naira Hovakimyan (Fellow, IEEE) received the
MS degree in applied mathematics from Yere-
van State University, Yerevan, Armenia, and the
Ph.D. degree in physics and mathematics from
the Institute of Applied Mathematics, Russian
Academy of Sciences, Moscow, Russia.

She is currently W. Grafton and Lillian B.
Wilkins Professor of Mechanical Science and
Engineering and the Director of AVIATE Center
of UIUC. She has coauthored two books, 11
patents, and more than 450 refereed publica-
tions.

Dr. Hovakimyan was the recipient of the 2011 AIAA Mechanics and
Control of Flight Award, Humboldt prize for her lifetime achievements in
2014, 2015 SWE Achievement Award, 2017 IEEE CSS Award for Tech-
nical Excellence in Aerospace Controls, 2019 AIAA Pendray Aerospace
Literature Award, and UIUC Engineering Council Award for Excellence
in Advising in 2015 and 2023. She is Fellow of AIAA and ASME, and
the Senior Member of NAI. She is Cofounder and Chief Scientist of
Intelinair. Her work was featured in the New York Times, and on Fox
TV and CNBC.

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