ON A LINEARIZED MULLINS-SEKERKA/STOKES SYSTEM FOR TWO-PHASE FLOWS

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Abstract. We study a linearized Mullins-Sekerka/Stokes system in a bounded domain with various boundary conditions. This system plays an important role to prove the convergence of a Stokes/Cahn-Hilliard system to its sharp interface limit, which is a Stokes/Mullins-Sekerka system, and to prove solvability of the latter system locally in time. We prove solvability of the linearized system in suitable $L^2$-Sobolev spaces with the aid of a maximal regularity result for non-autonomous abstract linear evolution equations.

1. Introduction. We study the following linearized Mullins-Sekerka/Stokes system

$$D_t \Gamma h + b \cdot \nabla h \Gamma - bh + \frac{1}{2} X_0^* ((v^+ + v^-) \cdot n_{\Gamma_t}) + \frac{1}{2} X_0^* (\partial_{n_{\Gamma_t}} \mu) = g \quad \text{on } \Sigma \times (0, T),$$

$$h(., 0) = h_0 \quad \text{in } \Sigma,$$

where for every $t \in [0, T]$, the functions $v^\pm = v^\pm (x, t)$, $p^\pm = p^\pm (x, t)$ and $\mu^\pm = \mu^\pm (x, t)$ for $(x, t) \in \Omega^\pm_T$ with $v^\pm \in H^2(\Omega^\pm (t))$, $p^\pm \in H^1(\Omega^\pm (t))$ and $\mu^\pm \in H^2(\Omega^\pm (t))$ are the unique solutions to

$$\Delta \mu^\pm = a_1 \quad \text{in } \Omega^\pm (t),$$

$$\mu^\pm = X_0^* (\sigma \Delta \Gamma h \pm a_2 h) + a_3 \quad \text{on } \Gamma_1,$$

$$n \cdot \nabla \mu^- = a_4 \quad \text{on } \Gamma_{\mu, 1},$$

$$\mu^- = a_4 \quad \text{on } \Gamma_{\mu, 2},$$

$$-\Delta v^\pm + \nabla p^\pm = a_1 \quad \text{in } \Omega^\pm (t),$$

$$\text{div } v^\pm = 0 \quad \text{in } \Omega^\pm (t),$$

$$|v| = a_2 \quad \text{on } \Gamma_1,$$

$$[2D_s v - p I] n_{\Gamma_t} = X_0^* (a_3 h + a_4 \Delta \Gamma h + a_5 \nabla \Gamma h + a_5) \quad \text{on } \Gamma_1,$$

$$B_j (v^-, p^-) = a_5 \quad \text{on } \Gamma_{S, j}, j = 1, 2, 3.$$

Here $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with smooth boundary, which is the disjoint union of $\Omega^+(t)$, $\Omega^-(t)$ and $\Gamma_t$, where $\Gamma_t = \partial \Omega^+(t)$ is a smoothly evolving $(d - 1)$-dimensional orientable hypersurface. We assume that $\Gamma_t \subseteq \Omega$ for all $t \in [0, T]$.}

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(0, T), i.e., there is no boundary contact and contact angle. Moreover, \( \Gamma_t \) is given for \( t \in [0, T] \) as well as \( a_1, \ldots, a_4, a_1, \ldots, a_6 \) are given for some \( T > 0, \sigma > 0 \) is the surface tension constant and \( D_a \mathbf{v} = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^T) \). Furthermore,

\[
[g](p, t) := \lim_{h \to 0} \left[ g^+(p + n_{\Gamma_t}(p)h, t) - g^-(p - n_{\Gamma_t}(p)h, t) \right] \quad \text{for } p \in \Gamma_t
\]

for suitable functions \( g^\pm \) and \( X_0: \Sigma \times [0, T] \to \Gamma := \bigcup_{t \in [0, T]} \Gamma_t \times \{t\} \) is a suitable diffeomorphism, which is described in Section 2 below. For \( a: \Sigma \times [0, T] \to \mathbb{R}^N, N \in \mathbb{N} \), we define \( X_0^{-1}a: \Gamma \to \mathbb{R}^N \) by

\[
(X_0^{-1}a)(p, t) = a(X_0^{-1}(p, t)) \quad \text{for all } (p, t) \in \Gamma
\]

and for \( b: \Gamma \to \mathbb{R}^N, N \in \mathbb{N} \), we define \( X_0^*b: \Sigma \times [0, T] \to \mathbb{R}^N \) by

\[
(X_0^*b)(s, t) = b(X_0(s, t)) \quad \text{for all } (s, t) \in \Sigma \times [0, T].
\]

This system arises in the construction of approximate solutions in the proof of convergence of a Stokes/Cahn-Hilliard system to its sharp interface limit, which is a Stokes/Mullins-Sekerka system, cf. [2]-[3]. Here \( \mathbf{v}^\pm: \bigcup_{t \in [0, T]} \Omega^\pm(t) \times \{t\} \to \mathbb{R}^d \) and \( p^\pm: \bigcup_{t \in [0, T]} \Omega^\pm(t) \times \{t\} \to \mathbb{R} \) are the velocity and pressure incompressible viscous Newtonian fluids filling the domains \( \Omega^\pm(t) \) at time \( t \), which are separated by the (fluid) interface \( \Gamma_t \). Furthermore, \( h: \Sigma \times [0, T] \to \mathbb{R} \) is a linearized height function that describes the evolution of the interface at a certain order and \( \mu^\pm: \bigcup_{t \in [0, T]} \Omega^\pm(t) \times \{t\} \to \mathbb{R} \) is a linearized chemical potential related to the fluids in \( \Omega^\pm(t) \). If one neglects the terms related to \( \mathbf{v}^\pm, p^\pm \), a similar linearized system arises in the study of the sharp interface limit of the Cahn-Hilliard equation, cf. [6]. Moreover, similar systems arise in the construction of strong solutions for a Navier-Stokes/Mullins-Sekerka system locally in time, cf. [4].

We consider different kinds of boundary conditions for \( \mathbf{v}^- \) and \( \mu^- \) simultaneously. More precisely, we assume that

\[
\partial \Omega = \Gamma_{\mu,1} \cup \Gamma_{\mu,2} = \Gamma_{S,1} \cup \Gamma_{S,2} \cup \Gamma_{S,3},
\]

where \( \Gamma_{\mu,1}, \Gamma_{\mu,2} \) and \( \Gamma_{S,1}, \Gamma_{S,2}, \Gamma_{S,3} \) are disjoint and closed. Moreover, we have

\[
B_1(\mathbf{v}^-, p^-) = \mathbf{v}^- \quad \text{on } \Gamma_{S,1}\n
(B_2(\mathbf{v}^-, p^-))_\tau = \left( (2D_\tau \mathbf{v}^- - p^-) n_{\partial \Omega} \right)_\tau + \alpha_2 \mathbf{v}^- \quad \text{on } \Gamma_{S,2}\n
n_{\partial \Omega} \cdot B_2(\mathbf{v}^-, p^-) = n_{\partial \Omega} \cdot \mathbf{v}^- \quad \text{on } \Gamma_{S,2}\n
B_3(\mathbf{v}^-, p^-) = (2D_\tau \mathbf{v}^- - p^-) n_{\partial \Omega} + \alpha_3 \mathbf{v}^- \quad \text{on } \Gamma_{S,3},
\]

where \( n_{\partial \Omega} \) denotes the exterior normal on \( \partial \Omega \). To avoid a non-trivial kernel in the following we assume that one of the following cases holds true:

\[
|\Gamma_{S,1}| + \alpha_2 |\Gamma_{S,2}| + \alpha_3 |\Gamma_{S,3}| > 0
\]

Then Korn’s inequality yields

\[
\|\mathbf{v}\|_{H^1(\Omega)} \leq C \left( \|D_\tau \mathbf{v}\|_{L^2(\Omega)} + \|\mathbf{v}\|_{L^2(\Gamma_{S,2})} + \alpha_3 \|\mathbf{v}\|_{L^2(\Gamma_{S,3})} \right)
\]

for all \( \mathbf{v} \in H^1(\Omega)^d \) with \( \mathbf{v}|_{\Gamma_{S,1}} = 0, n_{\partial \Omega} \cdot \mathbf{v}|_{\Gamma_{S,2}} = 0 \), cf. [5, Corollary 5.9].

The structure of this contribution is as follows: In Section 2 we summarize some preliminaries on the parametrization of the interface \( \Gamma_t \) and non-autonomous evolution equations. In Section 3 we present and prove our main results on existence and smoothness of solutions to the linearized Mullins-Sekerka system. Finally, in the appendix we prove an auxiliary result on the existence of a pressure.
The results of this paper are extensions of results in the second author’s PhD Thesis.

2. Preliminaries.

2.1. Notation. Throughout this manuscript we denote by $\xi \in C^{\infty}(\mathbb{R})$ a cut-off function such that

$$\xi(s) = 1 \text{ if } |s| \leq \delta, \quad \xi(s) = 0 \text{ if } |s| > 2\delta, \quad \text{and } 0 \geq s\xi'(s) \geq -4 \text{ if } \delta \leq |s| \leq 2\delta. \quad (11)$$

2.2. Coordinates. We parametrize $(\Gamma_t)_{t \in [0,T_0]}$ with the aid of a family of smooth diffeomorphisms $X_0: \Sigma \times [0,T_0] \to \Gamma = \bigcup_{t \in [0,T_0]} \Gamma_t \times \{t\}$. Here either $\Sigma \subseteq \mathbb{R}^d$ is a smooth $(d-1)$-dimensional compact, orientable manifold without boundary, where $d \geq 2$ is allowed, or $d = 2$ and $\Sigma = T^1$. We have included the latter case to cover the setting in [1, 2, 3]. Moreover, $n_{\Gamma_t}(x)$ denotes the exterior normal of $\Gamma_t$ in $x$ with respect to $\Omega^-(t)$ and we denote

$$n(s,t) := n_{\Gamma_t}((X_0(s,t))_1) \quad \text{for all } s \in \Sigma, t \in [0,T_0],$$

where $(X_0(s,t))_1 \in \mathbb{R}^d$ denote the spatial components of $X_0(s,t)$. In the following we will need a tubular neighborhood of $\Gamma_t$: For $\delta > 0$ sufficiently small, the orthogonal projection $P_{\Gamma_t}(x)$ of all

$$x \in \Gamma_t(3\delta) = \{ y \in \Omega : \text{dist}(y,\Gamma_t) < 3\delta \}$$

is well-defined and smooth. Moreover, we choose $\delta$ so small that $\text{dist}(\partial\Omega, \Gamma_t) > 3\delta$ for every $t \in [0,T_0]$. Every $x \in \Gamma_t(3\delta)$ has a unique representation

$$x = P_{\Gamma_t}(x) + r n_{\Gamma_t}(P_{\Gamma_t}(x))$$

where $r = \text{sdist}(\Gamma_t, x)$. Here

$$d_{\Gamma}(x,t) := \text{sdist}(\Gamma_t, x) = \begin{cases} \text{dist}(\Omega^-(t), x) & \text{if } x \notin \Omega^-(t), \\ -\text{dist}(\Omega^+(t), x) & \text{if } x \in \Omega^-(t). \end{cases}$$

For the following we define for $\delta' \in (0,3\delta]$

$$\Gamma(\delta') = \bigcup_{t \in [0,T_0]} \Gamma_t(\delta') \times \{t\}.$$ 

We introduce new coordinates in $\Gamma(3\delta)$ which we denote by

$$X: (-3\delta,3\delta) \times \Sigma \times [0,T_0] \mapsto \Gamma(3\delta) \text{ by } X(r,s,t) := X_0(s,t) + r n(s,t),$$

where

$$r = \text{sdist}(\Gamma_t, x), \quad s = (X_0^{-1}(P_{\Gamma_t}(x),t))_1 := S(x,t),$$

where $(X_0^{-1}(P_{\Gamma_t}(x),t))_1$ denote the components in $\Sigma$ of $X_0^{-1}(P_{\Gamma_t}(x),t)$.

In the case that $h$ is twice continuously differentiable with respect to $s$ and continuously differentiable with respect to $t$, we introduce the notations

$$D_{\Gamma} h(s,t) := \partial_t \left( h(S(x,t),t) \right)|_{x=X_0(s,t)}, \quad \nabla_{\Gamma} h(s,t) := \nabla \left( h(S(x,t),t) \right)|_{x=X_0(s,t)},$$

$$\Delta_{\Gamma} h(s,t) := \Delta \left( h(S(x,t),t) \right)|_{x=X_0(s,t)},$$
where $\nabla$ and $\Delta$ act with respect to $x$. We note that in the case that $d = 2$ and $\Sigma = \mathbb{T}^1$ we have
\[
\begin{align*}
D_t h(s, t) &= (\partial_t + \partial_t S(X_0(s, t)) \cdot \partial_s) h(s, t), \\
\nabla_t h(s, t) &= \nabla S(X_0(s, t)) \partial_s h(s, t), \\
\Delta_t h(s, t) &= \Delta S(X_0(s, t)) \partial_s h(s, t) + |\nabla S(X_0(s, t))|^2 \partial_s^2 h(s, t).
\end{align*}
\]
as in [1, 2, 3].

2.3. Maximal regularity for non-autonomous equations. In order to prove our main result we use the theory of maximal regularity for non-autonomous abstract evolution equations. Therefore, we give a short overview of the basic definitions and results which we will use. These are taken from [7] and all the proofs of the statements can be found in that article.

In this subsection let $X$ and $D$ be two Banach spaces such that $D$ is continuously and densely embedded in $X$.

**Definition 2.1** ($L^p$-maximal regularity). Let $p \in (1, \infty)$.

1. Let $A \in \mathcal{L}(D, X)$. Then $A$ has $L^p$-maximal regularity and we write $A \in \mathcal{MR}_p$ if for some bounded interval $(t_1, t_2) \subset \mathbb{R}$ and all $f \in L^p(t_1, t_2; X)$ there exists a unique $u \in W^{1, p}(t_1, t_2; X) \cap L^p(t_1, t_2; D)$ such that
   \[
   \partial_t u + Au = f \quad \text{a.e. on } (t_1, t_2),
   \]
   \[
   u(t_1) = 0.
   \]

2. Let $T > 0$ and $A : [0, T] \to \mathcal{L}(D, X)$ be a bounded and strongly measurable function. Then $A$ has $L^p$-maximal regularity and we write $A \in \mathcal{MR}_p(0, T)$ if for all $f \in L^p(0, T; X)$ there exists a unique $u \in W^{1, p}(0, T; X) \cap L^p(0, T; D)$ such that
   \[
   \partial_t u + A(t) u = f \quad \text{a.e. on } (0, T),
   \]
   \[
   u(0) = 0.
   \]

It can be shown that if $A \in \mathcal{MR}_p$ for some $p \in (1, \infty)$ then $A \in \mathcal{MR}_p$ for all $p \in (1, \infty)$. Hence, we often simply write $A \in \mathcal{MR}$.

**Definition 2.2** (Relative Continuity).

We say that $A : [0, T] \to \mathcal{L}(D, X)$ is relatively continuous if for each $t \in [0, T]$ and all $\epsilon > 0$ there exist $\delta > 0$, $\eta \geq 0$ such that for all $x \in D$ and for all $s \in [0, T]$ with $|s - t| \leq \delta$ the inequality
\[
\|A(t)x - A(s)x\|_X \leq \epsilon \|x\|_D + \eta \|x\|_X
\]
holds.

**Theorem 2.3.** Let $T > 0$ and $A : [0, T] \rightarrow \mathcal{L}(D, X)$ be a strongly measurable and relatively continuous function. If $A(t) \in \mathcal{MR}$ for all $t \in [0, T]$, then $A \in \mathcal{MR}_p(0, t)$ for every $0 < t \leq T$ and every $p \in (1, \infty)$.

**Proof.** See [7, Theorem 2.7].

A very important tool for proving maximal regularity properties of differential operators are perturbation techniques. Employing these can often help to show maximal regularity for a variety of operators by separating them into a main part (for which maximal regularity can be readily shown) and a perturbation.

In the following we give a perturbation result which is key to many results in the next chapter.
Definition 2.4 (Relatively Close).

Let \( Y \) be a Banach space such that

\[
D \hookrightarrow Y \hookrightarrow X.
\]

We say \( Y \) is close to \( X \) compared with \( D \), if for each \( \epsilon > 0 \) there exists \( \eta \geq 0 \) such that

\[
\|x\|_Y \leq \epsilon \|x\|_D + \eta \|x\|_X \quad \text{for all} \quad x \in D.
\]

Proposition 1. Let \( Y \) be as in Definition 2.4 and let the inclusion \( D \hookrightarrow Y \) be compact. Then \( Y \) is close to \( X \) compared with \( D \).

Proof. See [7, Example 2.9 (d)]. \( \square \)

Theorem 2.5. Let \( T > 0 \) and \( Y \) be a Banach space that is close to \( X \) compared with \( D \). Furthermore, let \( A: [0, T] \rightarrow \mathcal{L}(D, X) \) be relatively continuous and \( B: [0, T] \rightarrow \mathcal{L}(Y, X) \) be strongly measurable and bounded. If \( A(t) \in \mathcal{MR} \) for every \( t \in [0, T] \) then \( A + B \in \mathcal{MR}_p(0, T) \).

Proof. See [7, Theorem 2.11]. \( \square \)

3. Main results. We introduce the space

\[
X_T = L^2(0, T; H^\frac{7}{2}(\Sigma)) \cap H^1(0, T; H^\frac{1}{2}(\Sigma))
\]

for \( T \in (0, \infty) \), where we equip \( X_T \) with the norm

\[
\|h\|_{X_T} = \|h\|_{L^2(0, T; H^\frac{7}{2}(\Sigma))} + \|h\|_{H^1(0, T; H^\frac{1}{2}(\Sigma))} + \|h|_{t=0}\|_{H^1(\Sigma)}.
\]

Theorem 3.1. Let \( T \in (0, T_0] \). Let \( b: \Sigma \times [0, T] \rightarrow \mathbb{R}^d \) and \( b_1, b_2: \Sigma \times [0, T] \rightarrow \mathbb{R} \) be smooth given functions. For every \( g \in L^2(0, T; H^\frac{1}{2}(\Sigma)) \) and \( h_0 \in H^2(\Sigma) \), there is a unique solution \( h \in X_T \) of

\[
D_t h + b \cdot \nabla h + b_1 h + X_0^* \left( \partial_n \mu \right) = g \quad \text{on} \quad \Sigma \times (0, T),
\]

\[
h(., 0) = h_0 \quad \text{on} \quad \Sigma,
\]

where \( \mu|_{\Omega^\pm(t)} \in H^2(\Omega^\pm(t)), \) for \( t \in [0, T] \), is determined by

\[
\Delta \mu^\pm = 0 \quad \text{in} \quad \Omega^\pm(t),
\]

\[
\mu^\pm = X_0^* (\sigma \Delta \Gamma h \pm b_2 h) \quad \text{on} \quad \Gamma_t,
\]

\[
\mathbf{n}|_{\partial \Omega} \cdot \nabla \mu^- = 0 \quad \text{on} \quad \Gamma_{\mu, 1},
\]

\[
\mu^- = 0 \quad \text{on} \quad \Gamma_{\mu, 2}.
\]

Furthermore, the estimates

\[
\sum_{\pm} \|\mu^\pm\|_{L^2(0, T; H^2(\Omega^\pm(t))))} + \|\mu^\pm\|_{L^2(0, T; H^1(\Omega^\pm(t)))} \leq C \|h\|_{X_T},
\]

hold for some constant \( C > 0 \) independent of \( \mu \) and \( h \).

Proof. We may write (13) in abstract form as

\[
\partial_t h + A(t) h = g \quad \text{in} \quad \Sigma \times [0, T],
\]

\[
h(., 0) = h_0 \quad \text{in} \quad \Sigma,
\]

where \( A(t): H^\frac{7}{2}(\Sigma) \rightarrow H^\frac{1}{2}(\Sigma) \) depends on \( t \in [0, T] \). Now we fix \( t_0 \in [0, T] \) and analyze the operator \( A(t_0) \), where we replace \( t \) with the fixed \( t_0 \) in all time dependent coefficients.
In order to understand this operator we define

\[ \mathcal{D}_{t_0} : H^\frac{3}{2}(\Sigma) \to H^\frac{3}{2}(\Gamma_{t_0}) : h \mapsto (X_0^{-1}(\sigma\Delta_t h))(\cdot, t_0), \]
\[ S_{t_0}^N : H^\frac{3}{2}(\Gamma_{t_0}) \to H^2(\Omega^+(t_0)) \times H^2(\Omega^-(t_0)) : f \mapsto (\Delta_N)^{-1} f, \]
\[ B_{t_0} : H^2(\Omega^+(t_0)) \times H^2(\Omega^-(t_0)) \to H^\frac{1}{2}(\Sigma) : (\mu^+, \mu^-) \mapsto (X_0^1(\nabla \mu \cdot \mathbf{n}_{\Gamma_{t_0}}))(\cdot, t_0), \]

where \((\Delta_N)^{-1} f\) is the unique solution \((\mu_N^+, \mu_N^-)\) to

\[ \begin{align*}
\Delta \mu_N^+ &= 0 & \text{in } \Omega^+(t_0), \\
\mu_N^+ &= f & \text{on } \Gamma_{t_0}, \\
\nabla \mu_N^- \cdot \mathbf{n}_{\partial \Omega} &= 0 & \text{on } \partial \Omega.
\end{align*} \tag{16a,b,c} \]

In the literature the concatenation \(B_{t_0} \circ S_{t_0}^N\) is often referred to as the Dirichlet-to-Neumann operator and \(A_0(t_0) := B_{t_0} \circ S_{t_0}^N \circ \mathcal{D}_{t_0}\) is called the Mullins-Sekerka operator. It can be shown that

\[ A_0 : [0, T] \to \mathcal{L}(H^\frac{3}{2}(\Sigma), H^\frac{3}{2}(\Sigma)) \]

has \(L^p\)-maximal regularity, i.e., \(A_0 \in \mathcal{MR}_p(0, T)\). We will not prove this in detail but just give a short sketch describing the essential ideas: first, a reference surface \(\tilde{\Sigma} \subset \subset \Omega\) is fixed such that \(\Gamma_t\) can be expressed as a graph over \(\tilde{\Sigma}\) for \(t\) in some time interval \([\tilde{t}, \tilde{t} + \epsilon] \subset [0, T]\), e.g. one may choose \(\Sigma := \Gamma_0\) and then determine \(\epsilon_0 > 0\) such that \(\Gamma_t\) may be written as graph over \(\Gamma_0\) for all \(t \in [0, \epsilon_0]\), which is possible since \(\tilde{\Sigma}\) is a smoothly evolving hypersurface. Next, a Hanzawa transformation is applied, enabling us to consider (16c) as a system on fixed domains \(\Omega^\pm\) and \(\tilde{\Sigma}\), but with time dependent coefficients (see e.g. [4, Chapter 2.2] or [12, Chapter 4]). Here, \(\Omega^+, \Omega^-\) and \(\tilde{\Sigma}\) denote disjoint sets such that \(\partial \Omega^+ = \tilde{\Sigma}\) and \(\Omega = \Omega^+ \cup \Omega^- \cup \tilde{\Sigma}\) holds and we assume in the following that \(t_0 \in [0, \epsilon_0]\). To be more specific, the Hanzawa transformation results in a system of the form

\[ a(x, t, \nabla x) \tilde{\mu}^\pm = 0 \quad \text{in } \Omega^\pm, \]
\[ \tilde{\mu}^\pm = \tilde{f} \quad \text{on } \tilde{\Sigma}, \]
\[ \nabla \tilde{\mu}^- \cdot \mathbf{n}_{\partial \Omega} = 0 \quad \text{on } \partial \Omega, \]

where \(a\) is the transformed Laplacian, depending smoothly on \(t\) and \(\tilde{f}\) is the transformation of \(f\). Applying the Hanzawa transformation (and the diffeomorphism \(X_0\)) also to the operators \(\mathcal{D}_{t_0}\) and \(B_{t_0}\), we end up with a transformed operator \(\tilde{A}_0(t_0) \in \mathcal{L}(H^\frac{3}{2}(\Sigma), H^\frac{3}{2}(\Sigma))\) and [11, Corollary 6.6.5] implies that \(\tilde{A}_0(t_0)\) has \(L^p\)-maximal regularity. As all involved differential operators and coefficients depend smoothly on \(t\), it is possible to show that \(\tilde{A}_0 : [0, \epsilon_0] \to \mathcal{L}(H^\frac{3}{2}(\Sigma), H^\frac{3}{2}(\Sigma))\) is relatively continuous. Therefore Theorem 2.3 implies \(\tilde{A}_0 \in \mathcal{MR}_p(0, \epsilon_0)\) and, transforming back, also \(A_0 \in \mathcal{MR}_p(0, \epsilon_0)\). Repeating this procedure with a new reference surface \(\Sigma := \Gamma_{t_0}\) and iteratively continuing the argumentation, we end up with \(A_0 \in \mathcal{MR}_p(0, T)\).

We proceed by showing that \(\mathcal{A}(t_0) = A_0(t_0) + \mathcal{B}(t_0)\) holds for some lower order perturbation \(\mathcal{B}\). We introduce

\[ S_{t_0}^{DN} : H^\frac{3}{2}(\Gamma_{t_0}) \to H^2(\Omega^+(t_0)) \times H^2(\Omega^-(t_0)) : f \mapsto (\Delta_{DN})^{-1} f, \]

where \((\mu_{DN}^+, \mu_{DN}^-) := (\Delta_{DN})^{-1} f\) is the unique solution to (16), replacing \(\nabla \mu_N^- \cdot \mathbf{n}_{\partial \Omega} = 0\) on \(\Gamma_{\mu, 2}\) by \(\mu_D = 0\) on \(\Gamma_{\mu, 2}\). Moreover, we write \(S^\Delta_{t_0} := S_{t_0}^{DN} - S_{t_0}^N\) and
observe that the equality

\[ B_{t_0} \circ S_{t_0}^{DN} \circ D_{t_0} \sigma = A_0(t_0) + B_0(t_0) \]  

(17)
is satisfied, where \( B_0(t_0) := B_{t_0} \circ S_{t_0}^{\Delta} \circ D_{t_0} \sigma \). Let \( f \in H^{\frac{3}{2}}(\Gamma_{t_0}) \) be fixed, \((\mu_N^{+}, \mu_N^{-}) := S_{DN} f, (\mu_N^{\pm}, \tilde{\mu}) := S_{t_0} f \) and \( \tilde{\mu}^\pm := \mu_N^\pm - \mu_N^{-} \), implying \((\tilde{\mu}^+, \tilde{\mu}^-) = S_{t_0} f \). Then \( \tilde{\mu}^\pm \in H^2(\Omega^\pm(t_0)) \) solves

\[
\begin{align*}
\Delta \tilde{\mu}^\pm &= 0 \quad \text{in } \Omega^\pm(t_0), \\
\tilde{n}_{\partial \Omega} \cdot \nabla \tilde{\mu}^\pm &= 0 \quad \text{on } \Gamma_{t_0}, \\
\tilde{\mu}^\pm &= \mu_N^{-} \quad \text{on } \Gamma_{\mu,2}
\end{align*}
\]

and elliptic regularity theory implies

\[ \| \tilde{\mu}^- \|_{H^2(\Omega^-(t_0))} \leq C \| \mu_N^- \|_{H^2(\Gamma_{t_0})} \]  

(18)

and \( \tilde{\mu}^+ \equiv 0 \) in \( \Omega^+(t_0) \). For the further argumentation, we show

\[ \| \mu_N^- \|_{H^2(\partial \Omega)} \leq C \| \mu_N^- \|_{H^2(\Gamma_{t_0})}. \]  

(19)

To this end let \( \gamma(x) := \xi(4d_B(x)) \) for all \( x \in \Omega \), where \( \xi \) is a cut-off function satisfying (11). In particular supp\( \gamma \cap \Gamma_t = \emptyset \) for all \( t \in [0, T_0] \) by our assumptions and \( \gamma \equiv 1 \) in \( \partial \Omega(\frac{3}{2}) \). Denoting \( \hat{\mu} := \gamma \mu_N^- \in H^2(\Omega^-(t_0)) \), we compute using \( \Delta \mu_N^- = 0 \) in \( \Omega^-(t_0) \) that \( \hat{\mu} \) is a solution to

\[
\begin{align*}
\Delta \hat{\mu} &= 2\nabla \gamma \cdot \nabla \mu_N^- + (\Delta \gamma) \mu_N^- \quad \text{in } \Omega^-(t_0), \\
\hat{\mu} &= 0 \quad \text{on } \Gamma_{t_0}, \\
\nabla \hat{\mu} \cdot \nu_{\Omega} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

which, again regarding elliptic regularity theory, implies \( \| \hat{\mu} \|_{H^2(\Omega^-(t_0))} \leq C \| \mu_N^- \|_{H^2(\Omega^-(t_0))} \). This is essential in view of (19) as it leads to

\[
\| \mu_N^- \|_{H^2(\partial \Omega)} = \| \hat{\mu} \|_{H^2(\partial \Omega)} \leq C \| \mu_N^- \|_{H^2(\Omega^-(t_0))} \leq C \| \mu_N^- \|_{H^2(\Gamma_{t_0})},
\]

where we used the continuity of the trace operator \( \text{tr}: H^2(\Omega^-(t_0)) \to H^\frac{3}{2}(\partial \Omega^-(t_0)) \) in the first inequality (cf. [10, Theorem 3.37]) and standard estimates for elliptic equations in the second and third inequality.

Let now \( h \in H^\frac{3}{2}(\Sigma) \) and \((\tilde{\mu}^+, \tilde{\mu}^-) := S_{t_0}^\Delta \circ D_{t_0} h \). Our prior considerations enable us to estimate

\[
\| B_{t_0} \circ S_{t_0}^\Delta \circ D_{t_0} h \|_{H^\frac{3}{2}(\Sigma)} \leq C \| \tilde{\mu}^+ \|_{H^2(\Omega^+(t_0))} \leq C \| \mu_N^- \|_{H^\frac{3}{2}(\Gamma_{t_0})} \leq C \| \sigma \Delta \Gamma h \|_{H^\frac{3}{2}(\Sigma)} \leq C \| h \|_{H^\frac{3}{2}(\Sigma)},
\]

where we employed the continuity of the trace in the first line, (18) in the second, (19) in the third and the definition of \( \mu_N^- \) in the fourth. As \( H^\frac{3}{2}(\Sigma) \) is dense in \( H^\frac{3}{2}(\Sigma) \), we may extend \( B_0(t_0) \) to an operator

\[ B_0(t_0): H^\frac{3}{2}(\Sigma) \to H^\frac{3}{2}(\Sigma), \]

(20)

which shows in regard to (17) that we may view \( B_{t_0} \circ S_{t_0}^\Delta \circ D_{t_0} \) as a lower order perturbation of \( A_0(t_0) \).
Next we take care of the term involving \(b_2\) in (14b). For this we consider the operator
\[
B_1(t_0): H^{\frac{7}{2}}(\Sigma) \to H^{\frac{3}{2}}(\Sigma): h \mapsto X_0^\ast(\partial_\nu_{\Gamma_0} \mu_1),
\]
where \(\mu_1^\pm \in H^2(\Omega^\pm(t_0))\) is the solution to
\[
\begin{align*}
\Delta \mu_1^\pm &= 0 & &\text{in } \Omega^\pm(t_0), \\
\mu_1^\pm &= \pm b_2 h & &\text{on } \Gamma_0, \\
\mathbf{n}_\partial \Omega \cdot \nabla \mu_1^- &= 0 & &\text{on } \Gamma_{\mu,1}, \\
\mu_1^- &= 0 & &\text{on } \Gamma_{\mu,2}.
\end{align*}
\]
We estimate
\[
\|B_1(t_0)h\|_{H^{\frac{3}{2}}(\Sigma)} \leq C \left\| \partial_\nu_{\Gamma_0} \mu_1 \right\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq C \left( \|\mu_1^+\|_{H^2(\Omega^+(t_0))} + \|\mu_1^-\|_{H^2(\Omega^-(t_0))} \right) \leq C \|h\|_{H^{\frac{7}{2}}(\Sigma)},
\]
where \(C > 0\) can be chosen independent of \(h\) and \(t_0 \in [0,T]\). Here we again employed the continuity of the trace operator and elliptic theory.

Defining
\[
B(t_0): H^{\frac{7}{2}}(\Sigma) \to H^{\frac{3}{2}}(\Sigma): h \mapsto B(t_0)h := \tilde{b}(.,t_0) : \nabla \Gamma h - b_1(.,t_0)h + (\mathcal{B}_0(t_0) + \mathcal{B}_1(t_0))h,
\]
where \(\tilde{b}\) is chosen such that \(\partial_t h + \tilde{b} \cdot \nabla \Gamma h = D_{t_1} h + b \cdot \nabla \Gamma h\), and using (21) and (20), we find that
\[
\|B(t_0)h\|_{H^{\frac{3}{2}}(\Sigma)} \leq C \|h\|_{H^{\frac{7}{2}}(\Sigma)}.
\]
Thus, we can extend \(B(t_0)\) to a bounded operator \(B(t_0): H^{\frac{3}{2}}(\Sigma) \to H^{\frac{3}{2}}(\Sigma)\). Since \(H^{\frac{3}{2}}(\Sigma)\) is close to \(H^{\frac{7}{2}}(\Sigma)\) compared to \(H^{\frac{3}{2}}(\Sigma)\) as the embedding \(H^{\frac{3}{2}}(\Sigma) \hookrightarrow H^{\frac{7}{2}}(\Sigma)\) is compact, we get due to Theorem 2.5, that \(A = A_0 + B\) has \(L^p\)-maximal regularity for all \(t \in [0,T]\).

By elliptic theory
\[
\|\mu^\pm\|_{H^1(\Omega^\pm(t))} \leq C \|X_0^{s-1}(\sigma \Delta \Gamma h + b_2 h)\|_{H^{\frac{3}{2}}(\Gamma_1)} \leq C \|h\|_{H^{\frac{7}{2}}(\Sigma)}
\]
for almost all \(t \in [0,T]\) and thus
\[
\|\mu^\pm\|_{L^6(0,T;H^1(\Omega^\pm(t)))} \leq C \|h\|_{L^6(0,T;H^{\frac{3}{2}}(\Sigma))} \leq C \|h\|_{X_T}.
\]

\(\square\)

**Theorem 3.2.** Let \(T \in (0,T_0]\) and \(t \in [0,T]\). For every \(f \in L^2(\Omega)^d\), \(s \in H^{\frac{3}{2}}(\Gamma_0)^d\), \(a \in H^{\frac{3}{2}}(\Gamma_1)^d\) and \(g: \partial \Omega \to \mathbb{R}^d\) such that \(g|_{\Gamma_{S,1}} \in H^{\frac{3}{2}}(\Gamma_{S,1})^d\), \(\mathbf{n}_\partial \Omega \cdot g|_{\Gamma_{S,2}} \in H^{\frac{3}{2}}(\Gamma_{S,2})^d\), \((I - \mathbf{n}_\partial \Omega \otimes \mathbf{n}_\partial \Omega)g|_{\Gamma_{S,2}} \in H^{\frac{3}{2}}(\Gamma_{S,2})^d\), \(g|_{\Gamma_{S,3}} \in H^{\frac{3}{2}}(\Gamma_{S,3})^d\) satisfying the compatibility condition
\[
\int_{\Gamma_0} \mathbf{n}_{\Gamma_0} \cdot s \, d\mathcal{H}^{d-1} + \int_{\partial \Omega} \mathbf{n}_\partial \Omega \cdot g \, d\mathcal{H}^{d-1} = 0 \quad \text{if } \Gamma_{S,3} = \emptyset
\]
(22)
Thus, defining $\tilde{w} = w + \nabla q$, the couple $(\tilde{w}, \tilde{p})$ solves
\[-\Delta \tilde{w} + \nabla \tilde{p} = 0 \quad \text{in } \Omega^-(t),
\]
\[\text{div } \tilde{w} = 0 \quad \text{in } \Omega^-(t),
\]
\[\tilde{w} = \mathbf{s} \quad \text{on } \Gamma_t,
\]
\[\tilde{w} = 0 \quad \text{on } \partial \Omega.
\]
and may be estimated by \( s \) in strong norms. Next, let
\[
\tilde{g} := g_j + B_j(\tilde{w}, \tilde{p}) \quad \text{on } \Gamma_{S,j}, j = 1, 2, 3
\]
and \( \tilde{a} := a - (2D_s \tilde{w} - \tilde{p}) n_{\Gamma_j} \in H^{\frac{1}{2}}(\Gamma_j)^d \), where the regularity is due to the properties of the trace operator. Then, for every strong solution \((\tilde{v}^+, \tilde{p}^+)\) of \((23)-(27)\), with \( s \equiv 0 \) and \( g, a \) substituted by \( \tilde{g}, \tilde{a} \), the functions
\[
(\tilde{v}^+, p^+) := (\tilde{v}^+, \tilde{p}^+) \quad \text{and} \quad (\tilde{v}^-, p^-) := (\tilde{v}^-- \tilde{w}, \tilde{p}^- - \tilde{p})
\]
are solutions to the original system \((23)-(27)\). So, we will consider \( s \equiv 0 \) in the following and show existence of strong solutions in that case.

As a starting point, we construct a solution \((v, p) \in V(\Omega) \times L^2(\Omega)\) to the weak formulation
\[
\int_{\Omega} 2D_s v : D_s \psi \, dx - \int_{\Omega} p \, \div \psi \, dx + \int_{\Gamma_{S,2}} \alpha_2 v \cdot \psi \, d\mathcal{H}^{d-1}(s) + \int_{\Gamma_{S,3}} \alpha_3 v \cdot \psi \, d\mathcal{H}^{d-1}(s) = \int_{\Omega} f \cdot \psi \, dx + \int_{\Gamma_1} a \cdot \psi \, d\mathcal{H}^{d-1}(s) - \int_{\partial\Omega} g \cdot \psi \, d\mathcal{H}^{d-1}(s),
\]
for all \( \psi \in H^1(\Omega)^d \) with \( \psi|_{\Gamma_{S,1}} = 0, n \cdot \psi|_{\Gamma_{S,2}} = 0 \), where
\[
V(\Omega) = \{ u \in H^1(\Omega)^d : \div u = 0, u|_{\Gamma_{S,1}} = 0, n \cdot u|_{\Gamma_{S,2}} = 0 \}.
\]

Considering first \( \psi \in V(\Omega) \) and the right hand side as a functional \( F \in (V(\Omega))' \), the Lemma of Lax-Milgram implies the existence of a unique \( v \in V(\Omega) \) solving \((29)\) for all \( \psi \in V(\Omega) \), where the coercivity of the involved bilinear form is a consequence of \((10)\).

Next consider the functional
\[
F(\psi) := -\int_{\Omega} 2D_s v : D_s \psi \, dx - \int_{\Gamma_{S,2}} \alpha_2 v \cdot \psi \, d\mathcal{H}^{d-1}(s) - \int_{\Gamma_{S,3}} \alpha_3 v \cdot \psi \, d\mathcal{H}^{d-1}(s) + \int_{\Omega} f \cdot \psi \, dx + \int_{\Gamma_1} a \cdot \psi \, d\mathcal{H}^{d-1}(s) - \int_{\partial\Omega} g \cdot \psi \, d\mathcal{H}^{d-1}(s),
\]
for all \( \psi \in H^1(\Omega)^d \) with \( \psi|_{\Gamma_{S,1}} = 0, n \cdot \psi|_{\Gamma_{S,2}} = 0 \). Then \( F \) vanishes on \( V(\Omega) \) and by Lemma A.1 in Appendix A there is a unique \( p \in L^2(\Omega) \) with \( \int_{\Omega} p \, dx = 0 \) if \( \Gamma_{S,3} = \emptyset \) such that
\[
F(\psi) = -\int_{\Omega} p \, \div \psi \, dx \quad \text{for all } \psi \in H^1(\Omega)^d \text{ with } \psi|_{\Gamma_{S,1}} = 0, n \cdot \psi|_{\Gamma_{S,2}} = 0.
\]
Hence \((v, p)\) solve \((29)\). Moreover, we obtain the estimate
\[
\|(v, p)\|_{H^1(\Omega) \times L^2(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)} + \|a\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} \right).
\]
We now show higher regularity of \((v, p)\) by localization.

Let \( \eta^\pm \in C^\infty(\Omega) \) be a partition of unity of \( \Omega \), such that the inclusions \( \Omega^+(t) \cup \Gamma_t(\delta) \subset \{ x \in \Omega : \eta^+(x) = 1 \} \) and \( \partial\Omega(\delta) \subset \{ x \in \Omega : \eta^-(x) = 1 \} \) hold. We choose \( \eta^\pm \) such that \( \{ x \in \Omega : \eta^+(x) = 1 \} \) has smooth boundary and define \( U^\pm := \text{supp}(\eta^\pm), \partial U^\pm_0 := \partial U^\pm \setminus \partial\Omega \) and
\[
\hat{U} := \{ x \in \Omega : \eta^+(x) \in (0, 1) \} = \{ x \in \Omega : \eta^-(x) \in (0, 1) \}.
\]
Moreover, we set \( \hat{p}^- := pm^- \) and \( \hat{v}^- := v\eta^- \) in \( \Omega \) and we correct the divergence of \( \hat{v}^- \) with the help of the Bogovskii-operator: Let \( \varphi \in C^\infty_c(\Omega) \) with \( \text{supp}(\varphi) \subset U^+ \setminus \hat{U} \)
and \( \int_{\Omega} \varphi \, dx = 1 \) and set

\[
\hat{g} := \text{div} (\tilde{v}^-) - \varphi \int_{U^+} \text{div} (\tilde{v}^-) \, dx
\]

in \( U^+ \). As \( v \in V(\Omega) \), we have \( \text{div} (\tilde{v}^-) = v \cdot \nabla \eta^- \) and thus \( \hat{g} \in H_0^1 (U^+) \), \( \int_{U^+} \hat{g} \, dx = 0 \). Consequently, [9, Theorem III.3.3] implies that there is \( \tilde{v}^- \in H_0^2 (U^+) \), which we extend onto \( \Omega \) by 0, satisfying

\[
\text{div} \tilde{v}^- = \hat{g} \text{ in } U^+,
\]

\[
\|\tilde{v}^-\|_{H^2(\Omega)} \leq C \|v\|_{H^1(\Omega)}.
\]

Therefore, \( \hat{v}^- := \tilde{v}^- - \tilde{v}^- \) fulfills \( \text{div} \hat{v}^- = 0 \) in \( U^- \) since \( \varphi \equiv 0 \) in that domain. Let now

\[
\psi \in \{ w \in H^1(U^d) : w = 0 \text{ on } \partial U_0^- \setminus S, \, w|_{\Gamma_{S,1}} = 0, \, n \cdot w|_{\Gamma_{S,2}} = 0 \},
\]

then

\[
\int_{U^-} 2D_s \hat{v}^- : D_s \psi - \hat{p}^- \text{ div } \psi \, dx + \int_{\Gamma_{S,2}} \alpha_2 v \cdot \psi \, d\mathcal{H}^{d-1}(s) + \int_{\Gamma_{S,3}} \alpha_3 v \cdot \psi \, d\mathcal{H}^{d-1}(s)
\]

\[
= \int_{U^-} 2D_s \tilde{v}^- : D_s \psi - \hat{p}^- \text{ div } \psi \, dx + \int_{\Gamma_{S,2}} \alpha_2 v \cdot \psi \, d\mathcal{H}^{d-1}(s) + \int_{\Gamma_{S,3}} \alpha_3 v \cdot \psi \, d\mathcal{H}^{d-1}(s)
\]

\[
+ \int_{\Gamma_{S,3}} \alpha_3 v \cdot \psi \, d\mathcal{H}^{d-1}(s) - \int_{U^-} 2D_s \hat{v}^- : D_s \psi \, dx
\]

\[
= \int_{U^-} f \cdot \psi \eta^- \, dx - \int_{\partial \Omega} g \cdot \psi \, d\mathcal{H}^{d-1}(s) + (p \nabla \eta^-) \cdot \psi \, dx
\]

\[
+ \int_{U^-} 2 \text{ div } (D_s \psi) \cdot \psi + (2D_s v \nabla \eta^- - \text{div } (v \otimes \nabla \eta^- + \nabla \eta^- \otimes v)) \cdot \psi \, dx,
\]

where we used the definition of \( \tilde{v}^- \) and \( \hat{p}^- \) in the first equality and integration by parts together with \( \tilde{v}^- \in H_0^2 (U^+) \) and \( \nabla \eta^- = 0 \) on \( U^- \) in the second equality. Additionally, we employed the fact that \( (v, p) \) is the weak solution to \((29)\). Hence, \((\hat{v}^-, \hat{p}^-)\) are a weak solution to the system

\[
-\Delta \hat{v}^- + \nabla \hat{p}^- = \hat{f} \quad \text{in } U^-, \\
\text{div } \hat{v}^- = 0 \quad \text{in } U^-,
\]

\[
\hat{v}^- = 0 \quad \text{on } \partial U_0^-,
\]

\[
B_j (\hat{v}^-, \hat{p}^-) = g \quad \text{on } \Gamma_{S,j}, j = 1, 2, 3,
\]

(32)

where

\[
\hat{f} := f \eta^- + p \nabla \eta^- + 2 \text{ div } (D_s \tilde{v}) + 2D_s v \nabla \eta^- - \text{div } (v \otimes \nabla \eta^- + \nabla \eta^- \otimes v) \in L^2(U^-)
\]

and \( \hat{v}^- \in H_0^2 (\partial U_0^-) \) by the properties of the trace operator. Writing

\[
\bar{g} := \begin{cases} 
\bar{g} \quad \text{on } \Gamma_{S,1}, \\
\alpha_j \tilde{v}^- + \bar{g} \quad \text{on } \Gamma_{S,j}, j = 2, 3,
\end{cases}
\]

using localization techniques and results for strong solutions of the stationary Stokes equation in one phase with inhomogeneous do-nothing boundary condition (cf. Theorem 3.1 in [14]), with Dirichlet boundary condition (cf. [9]) and slip-boundary
Analogously, we define \( \tilde{v}^+ := v \eta^+ \) and \( \tilde{v}^+ \in H^2_0(\tilde{U}) \) as a solution to \( \text{div} \tilde{v}^+ = \text{div} \hat{v}^+ \). Here, we do not need to correct the mean value, since
\[
\int_{\tilde{U}} \text{div} \hat{v}^+ \, dx = \int_{\partial \tilde{U}} v \cdot n_{\partial \tilde{U}} \eta^+ \, d\mathcal{H}^{d-1}(s) = -\int_{\{\eta^+=1\}} \text{div} \, v \, dx = 0.
\]

We set \( \hat{v}^+ := \hat{v}^+ - \tilde{v}^+ \) and \( \hat{p}^+ := p \eta^+ \) and get after similar calculations as before that \( (\hat{v}^+ , \hat{p}^+) \) is a weak solution to the two phase stationary Stokes system
\[
-\Delta \hat{v}^+ + \nabla \hat{p}^+ = \hat{f} \quad \text{in } U^+, \quad \text{(33)}
\]
\[
\text{div} \hat{v}^+ = 0 \quad \text{in } U^+, \quad \text{(34)}
\]
\[
\hat{v}^+ = 0 \quad \text{on } \partial U^+, \quad \text{(35)}
\]
\[
[\hat{v}^+] = 0 \quad \text{on } \Gamma_t, \quad \text{(36)}
\]
\[
[2D_a \hat{v}^+ - \hat{p}^+ I] n_{\Gamma_t} = a \quad \text{on } \Gamma_t, \quad \text{(37)}
\]

where \( \hat{f} \in L^2(U^+) \). Then \( \text{[13, Theorem 1.1]} \) implies \( \hat{v}^+ |_{\Omega^+ (t)} \in H^2(\Omega^+(t)) \) and \( \hat{v}^+ |_{\Omega^+ \setminus \Omega^+(t)} \in H^2(U^+ \setminus \Omega^+(t)) \), and also that the pressure satisfies \( \hat{p}^+ |_{\Omega^+ (t)} \in H^1(\Omega^+(t)) \) and \( \hat{p}^+ |_{\Omega^+ \setminus \Omega^+(t)} \in H^1(U^+ \setminus \Omega^+(t)) \) with estimates in the associated norms. In particular, \( v = \hat{v}^+ \) in \( \Omega^+(t) \) and \( v = \hat{v}^+ + \hat{v}^+ + \hat{v}^- \) in \( \Omega^- (t) \), yielding the desired regularity and (28). To show that \( C > 0 \) may be chosen independently of \( t \in [0, T_0] \), one may make use of extension arguments, see e.g. the proof of \( \text{[1, Lemma 2.10]} \). \( \square \)

**Theorem 3.3.** Let \( T \in (0, T_0] \). Let \( b : \Sigma \times [0, T] \to \mathbb{R}^d \), \( b : \Sigma \times [0, T] \to \mathbb{R} \), \( a_1 : \Omega \times [0, T] \to \mathbb{R} \), \( a_2, a_3, a_5 : \Gamma \to \mathbb{R} \), \( a_4 : \partial \Omega \times [0, T] \to \mathbb{R} \), \( a_1 : \Omega \times [0, T] \to \mathbb{R}^d \), \( a_2, a_3, a_5 : \Gamma \to \mathbb{R} \) and \( a_6 : \partial \Omega \times [0, T] \to \mathbb{R}^d \) be smooth given functions such that
\[
\int_{\Gamma_S} n_{\Gamma_S} \cdot a_2 \, d\mathcal{H}^{d-1} = \int_{\partial \Omega} n_{\partial \Omega} \cdot a_6 \, d\mathcal{H}^{d-1} = 0 \quad \text{if } \Gamma_{S,3} = \emptyset.
\]

For every \( g \in L^2(0, T; H^{\frac{1}{2}}(\Sigma)) \) and \( h_0 \in H^2(\Sigma) \) there exists a unique solution \( h \in X_T \) of
\[
D_{t,t}h + b \cdot \nabla h - bh + \frac{1}{2} X_0^* (v^+ + v^-) \cdot n_{\Gamma_t} + \frac{1}{2} X_0^* (\partial n_{\Gamma_t}, \mu) = g \quad \text{in } \Sigma \times (0, T),
\]
\[
h (., 0) = h_0 \quad \text{in } \Sigma,
\]

where for every \( t \in [0, T] \), the functions \( v^\pm = v^\pm (x, t) \), \( p^\pm = p^\pm (x, t) \) and \( \mu^\pm = \mu^\pm (x, t) \) for \( (x, t) \in \Omega_T^\pm \) with \( v^\pm \in H^2(\Omega^\pm (t)) \), \( p^\pm \in H^1(\Omega^\pm (t)) \) and \( \mu^\pm \in H^2(\Omega^\pm (t)) \)
are the unique solutions to
\[
\Delta \mu^\pm = a_1 \quad \text{in } \Omega^\pm(t),
\]
\[
\mu^\pm = X_0^{-1}(-\sigma \Delta h \pm a_2 h) + a_3 \quad \text{on } \Gamma_t,
\]
\[
n_{\partial \Omega} \cdot \nabla \mu^- = a_4 \quad \text{on } \Gamma_{\mu,1},
\]
\[
\mu^- = a_4 \quad \text{on } \Gamma_{\mu,2},
\]
\[
-\Delta \nu^\pm + \nabla p^\pm = a_1 \quad \text{in } \Omega^\pm(t),
\]
\[
\text{div } \nu^\pm = 0 \quad \text{in } \Omega^\pm(t),
\]
\[
[v] = a_2 \quad \text{on } \Gamma_t,
\]
\[
2D_v \nu - p I | n_{\Gamma_t} = X_0^{-1} (a_3 h + a_4 \Delta h + a_5 \nabla h + a_5) \quad \text{on } \Gamma_t,
\]
\[
B_j(\nu^-, p^-) = a_6 \quad \text{on } \Gamma_{S,j}, j = 1, 2, 3.
\] (46)

Moreover, if \(g, h_0\) and \(b, a_i\) are smooth on their respective domains for \(i \in \{1, \ldots, 5\}, j \in \{1, \ldots, 6\}\), then \(h\) is smooth and \(p^\pm, \nu^\pm\) and \(\mu^\pm\) are smooth on \(\Omega^\pm(t)\).}

Proof: We show this by a perturbation argument. First of all note that we may without loss of generality assume that \(a_1, a_3, a_4, a_1, a_2, a_5, a_6 = 0\) on their respective domains. The above system may be reduced to this case by solving
\[
\Delta \bar{\mu}^\pm = a_1 \quad \text{in } \Omega^\pm(t),
\]
\[
\bar{\mu}^\pm = a_3 \quad \text{on } \Gamma_t,
\]
\[
n \cdot \nabla \bar{\mu}^- = a_4 \quad \text{on } \Gamma_{\mu,1},
\]
\[
\bar{\mu}^- = a_4 \quad \text{on } \Gamma_{\mu,2},
\]
with the help of standard elliptic theory and
\[
-\Delta \bar{\nu}^\pm + \nabla \bar{p}^\pm = a_1 \quad \text{in } \Omega^\pm(t),
\]
\[
\text{div } \bar{\nu}^\pm = 0 \quad \text{in } \Omega^\pm(t),
\]
\[
[\bar{\nu}] = a_2 \quad \text{on } \Gamma_t,
\]
\[
2D_v \bar{\nu} - p I | n_{\Gamma_t} = a_5 \quad \text{on } \Gamma_t,
\]
\[
B_j(\bar{\nu}^-, \bar{p}^-) = a_6 \quad \text{on } \Gamma_{S,j}, j = 1, 2, 3,
\]
with the help of Theorem 3.2 and setting
\[
\hat{g} = g - \frac{1}{2} X_0^*( [\partial_{n_{\Gamma_t}}, \bar{\mu}] + (\bar{\nu}^\pm + \bar{\nu}^-) \cdot n_{\Gamma_t}) .
\]

Now let \(t \in [0, T], h \in H^2(\Sigma)\) and let \(v_h^\pm \in H^2(\Omega^\pm(t))^d, p_h^\pm \in H^1(\Omega^\pm(t))\) be the solution to (42)–(46). Multiplying (42) by \(v_h^\pm\) and integrating in \(\Omega^\pm(t)\) together with integration by parts and the consideration of the boundary values (45) and (46) allows us to deduce
\[
\int_{\Omega^+(t)} 2|D_v v_h^+|^2 dx + \int_{\Omega^-(t)} 2|D_v v_h^-|^2 dx + \sum_{j=2,3} \int_{\Gamma_{S,j}} \alpha_j |v_h^-|^2 d\mathcal{H}^{d-1}(s)
\]
\[
= \int_{\Gamma_t} X_0^{-1} (a_3 h + a_4 \Delta h + a_5 \nabla h) \cdot v_h^- d\mathcal{H}^{d-1}(s).
\] (47)

Hence, by [5, Corollary 5.8] and the continuity of the trace we find
\[
\|v_h^-\|_{H^1(\Omega^-(t))} \leq C\|h\|_{H^2(\Sigma)}
\] (48)
for $C$ independent of $h$ and $t$. [5, Corollary 5.8], also implies
\[
\int_{\Omega_+^{(t)}} 2|D_s v_h^+|^2 \, dx + \int_{\Gamma_1} |v_h^+|^2 \, d\mathcal{H}^{d-1}(s) \geq C\|v_h^+\|^2_{H^1(\Omega^{(t)})},
\]
leading to
\[
\|v_h^+\|_{H^1(\Omega^{(t)})} \leq C\|h\|_{H^2(\Sigma)}
\] (49)
due to $v_h^+ = v_h^-$ on $\Gamma_1$, (48) and (47). Defining
\[
B(t) : H^{\frac{3}{2}}(\Sigma) \rightarrow H^{\frac{1}{2}}(\Sigma) : h \mapsto B(t)h = \frac{1}{2}X_0^*(\{(v_h^+ + v_h^-) \cdot n_{\Gamma_1}\}),
\]
we may use (48) and (49) to confirm
\[
\|B(t)h\|_{H^{\frac{1}{2}}(\Sigma)} \leq C\|h\|_{H^2(\Sigma)}
\] for $C > 0$ independent of $h$ and $t$. As $H^{\frac{3}{2}}(\Sigma)$ is dense in $H^2(\Sigma)$ we can extend $B(t)$ to an operator $B(t) : H^2(\Sigma) \rightarrow H^{\frac{1}{2}}(\Sigma)$ and $H^2(\Sigma)$ is close to $H^{\frac{3}{2}}(\Sigma)$ compared with $H^2(\Sigma)$.

The existence of a unique solution $h \in X_T$ with the properties stated in the theorem is now a consequence of Theorem 2.5. Higher regularity may be shown by localization and e.g. the usage of difference quotients.

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**Appendix A. Existence of a pressure.**

**Lemma A.1.** Let $F \in \{\psi \in H^1(\Omega)^d : \psi|_{\Gamma_{S,1}} = 0, n_{\partial \Omega} \cdot \psi|_{\Gamma_{S,2}} = 0\} \rightarrow \mathbb{R}$ be linear and bounded such that
\[
F(\psi) = 0 \quad \text{for all } \psi \in V(\Omega) = \{\psi \in H^1(\Omega)^d : \text{div } \psi = 0, \psi|_{\Gamma_{S,1}} = 0, n_{\partial \Omega} \cdot \psi|_{\Gamma_{S,2}} = 0\}.
\]
Then there is a unique $p \in L^2(\Omega)$ with $\int_{\Omega} p \, dx = 0$ if $\Gamma_{S,3} = \emptyset$ such that
\[
F(\psi) = -\int_{\Omega} \text{div } \psi \, dx \quad \text{for all } \psi \in H^1(\Omega)^d \text{ with } \psi|_{\Gamma_{S,1}} = 0, n_{\partial \Omega} \cdot \psi|_{\Gamma_{S,2}} = 0.
\]

**Proof.** We will apply the closed range theorem. To this end let
\[
X = \{\psi \in H^1(\Omega)^d : \psi|_{\Gamma_{S,1}} = 0, n_{\partial \Omega} \cdot \psi|_{\Gamma_{S,2}} = 0\},
\]
\[
Y = \left\{ g \in L^2(\Omega) : \int_{\Omega} g(x) \, dx = 0 \text{ if } \Gamma_{S,3} = \emptyset \right\}
\]
and consider
\[
T : X \rightarrow Y : \psi \mapsto -\text{div } \psi.
\]
Then $T$ is onto, which can be seen as follows: Let $g \in Y$.

If $\Gamma_{S,3} \neq \emptyset$, then there is a unique solution $q \in H^1(\Omega)$ of
\[
\Delta q = g \quad \text{in } \Omega,
\]
\[
q|_{\Gamma_{S,3}} = 0 \quad \text{on } \Gamma_{S,3},
\]
\[
n_{\partial \Omega} \cdot \nabla q|_{\Gamma_{S,1} \cup \Gamma_{S,2}} = 0 \quad \text{on } \Gamma_{S,1} \cup \Gamma_{S,2}.
\]
Moreover, using the solvability of the stationary Stokes equation with nonhomogeneous Dirichlet boundary conditions, we find some $w \in H^1(\Omega)$ with $\text{div} \ w = 0$ and

$$w|_{\Gamma} = \nabla q|_{\Gamma_{S,1} \cup \Gamma_{S,2}}, \quad w|_{\Gamma_{S,3}} = 0.$$  

Then $\psi = w - \nabla q \in X$ with $-\text{div} \ \psi = g$.

If $\Gamma_{S,3} = \emptyset$, we have $\int_{\Omega} g(x) \, dx = 0$ and can use the well-known Bogovskii operator to obtain some $\psi \in H^1_0(\Omega)$ with $-\text{div} \ \psi = g$.

Now the closed range theorem implies that $T': Y' \to X'$ is injective and

$$R(T') = N(T)^\circ = \{ F \in X' : F(\psi) = 0 \text{ for all } \psi \in V(\Omega) \}.$$  

This proves the statement of the lemma.

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