DISCRETE CONFORMAL GEOMETRY OF POLYHEDRAL SURFACES AND ITS CONVERGENCE

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ABSTRACT. The paper proves a result on the convergence of discrete conformal maps to the Riemann mappings for Jordan domains. It is a counterpart of Rodin-Sullivan’s theorem on convergence of circle packing mappings to the Riemann mapping in the new setting of discrete conformality. The proof follows the same strategy that Rodin-Sullivan used by establishing a rigidity result for regular hexagonal triangulations of the plane and estimating the quasiconformal constants associated to the discrete conformal maps.

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1. INTRODUCTION

Thurston’s conjecture on the convergence of circle packing mappings to the Riemann mapping is a constructive and geometric approach to the Riemann mapping theorem. The conjecture was solved in an important work by Rodin and Sullivan [34] in 1987. There have been many research works inspired by the work of Thurston and Rodin-Sullivan since then. This paper addresses a counterpart of Thurston’s convergence conjecture in the setting of discrete conformal change of polyhedral surfaces associated to the notion of vertex scaling (Definition 1.1). We prove a weak version of Rodin-Sullivan’s theorem in this new setting. There are still many problems to be resolved in order to prove the full convergence conjecture.

Let us begin with a recall of Thurston’s conjecture and Rodin-Sullivan’s solution. Given a bounded simply connected domain $\Omega$ in the complex plane $\mathbb{C}$, one constructs a sequence of approximating triangulated polygonal disks $(D_n, \mathcal{T}_n)$ whose triangles are equilateral and edge lengths of the triangles tend to zero such that $D_n$ converges to $\Omega$. For each such polygonal disk, by the Koebe-Andreev-Thurston’s existence theorem, there exists a circle packing of the...
unit disk $\mathbb{D}$ such that the combinatorics (or the nerve) of circle packing is isomorphic to the 1-skeleton of the triangulation $T_n$. This produces a piecewise linear homeomorphism $f_n$, called the circle packing mapping, from the polygonal disk $D_n$ to a polygonal disk inside $\mathbb{D}$ associated to the circle packing. Thurston conjectured in 1985 that, under appropriate normalizations, the sequence $\{f_n\}$ converges uniformly on compact subsets of $\Omega$ to the Riemann mapping for $\Omega$. Here the normalization condition is given by choosing a point $p \in \Omega$, a sequence of vertices $v_n$ in $(D_n, T_n)$ such that $\lim_n v_n = p$, and $f_n(v_n) = 0$ such that $f_n'(v_n) > 0$. The Riemann mapping $f$ for $\Omega$ sends $p$ to 0 and $f'(p) > 0$. Rodin-Sullivan’s proof of Thurston’s conjecture is elegant and goes in two steps. In the first step, they show that the circle packing mappings $f_n$ are $K$-quasiconformal for some constant $K$ independent of the indices. In the second step, they show that there is only one hexagonal circle packings of the complex plane up to Moebius transformations. This implies that the limit of the sequence $\{f_n\}$ is conformal.

Circle packing metrics introduced by Thurston [42] can be considered as a discrete conformal geometry of polyhedral surfaces. In recent times, there have been many works on discretization of 2-dimensional conformal geometry ([26], [4], [19], [13], [12], and others). In this paper, we consider the counterpart of Thurston’s conjecture in the setting of discrete conformal change defined by vertex scaling.

To state our main results, let us recall some related material and notations. A compact topological surface $S$ together with a non-empty finite subset of points $V \subset S$ will be called a marked surface. A triangulation $T$ of a marked surface $(S, V)$ is a topological triangulation of $S$ such that the vertex set of $T$ is $V$. We use $E = E(T)$, $V = V(T)$ to denote the sets of all edges and vertices in $T$ respectively. A polyhedral metric $d$ on $(S, V)$, to be called a PL metric on $(S, V)$ for simplicity, is a flat cone metric on $(S, V)$ whose cone points are contained in $V$. We call the triple $(S, V, d)$ a polyhedral surface. The discrete curvature, or simply curvature, of a PL metric $d$ is the function $K : V \rightarrow (-\infty, 2\pi)$ sending an interior vertex $v$ to $2\pi$ minus the cone angle at $v$ and a boundary vertex $v$ to $\pi$ minus the sum of angles at $v$. All PL metrics are obtained by isometric gluing of Euclidean triangles along pairs of edges. If $T$ is a triangulation of a polyhedral surface $(S, V, d)$ for which all edges in $T$ are geodesic, we say $T$ is geometric in $d$ and $d$ is a PL metric on $(S, T)$. In this case, we can represent $d$ by the length function $l_d : E(T) \rightarrow \mathbb{R}_{>0}$ sending each edge to its length. Thus the polyhedral surface $(S, V, d)$ can be represented by $(S, T, l_d)$ where $l_d \in \mathbb{R}_{>0}^E$. We will also call $(S, T, l_d)$ or $l_d$ a PL metric on $T$.

**Definition 1.1.** (Vertex scaling change of PL metrics [26]) Two PL metrics $l$ and $l^*$ on a triangulated surface $(S, T)$ are related by a vertex scaling if there exists a map $w : V(T) \rightarrow \mathbb{R}$ so that if $e$ is an edge in $T$ with end points $v$ and $v'$, then the edge lengths $l(e)$ and $l^*(e)$ are related by

$$l^*(e) = e^{w(v)+w(v')}l(e).$$

We denote $l^*$ by $w*l$ if (1) holds and call $l^*$ obtained from $l$ by a vertex scaling and $w$ a discrete conformal factor.

Condition (1) was proposed in [26] as a discrete conformal equivalence between PL metrics on triangulated surfaces. There are three basic problems related to the vertex scaling. The first is the existence problem. Namely, given a PL metric $l$ on a triangulated closed surface $(S, T)$ and a function $K : V(T) \rightarrow (-\infty, 2\pi)$ satisfying the Gauss-Bonnet condition, is there a PL metric $l^*$ of the form $w*l$ whose curvature is $K$? Unlike Koebe-Andreev-Thurston’s theorem which guarantees the existence of circle packing metrics, the answer to the above existence problem is negative in general. This makes the convergence of discrete conformal
mappings a difficult problem. On the other hand, the uniqueness of the vertex scaled PL metric \( l^* \) with prescribed curvature holds. This was established in an important paper by Bobenko-Pinkall-Springborn [4]. The third is the convergence problem. Namely, assuming the existence of PL metrics with prescribed curvatures, can these discrete conformal polyhedral surfaces approximate a given Riemann surface? The main result of the paper gives a solution to the convergence problem for the simplest case of Jordan domain.

The convergence theorem that we proved is the following. Let \( \Omega \) be a Jordan domain with three points \( p, q, r \) specified in the boundary. By Caratheodory’s extension theorem [32], the Riemann mapping from \( \Omega \) to the unit disk \( \mathbb{D} \) extends to a homeomorphism from the closure \( \overline{\Omega} \) to the closure \( \overline{\mathbb{D}} \). Therefore, there exists a unique homeomorphism \( g \) from \( \overline{\Omega} \) to an equilateral Euclidean triangle \( \Delta ABC \) with vertices \( A, B, C \) such that \( p, q, r \) are sent to \( A, B, C \) and \( g \) is conformal in \( \Omega \). For simplicity, we call \( g \) and \( g^{-1} \) the Riemann mappings for \( (\Omega, (p, q, r)) \).

Given an oriented triangulated polygonal disk \( (D, \mathcal{T}, l) \) and three boundary vertices \( p, q, r \in V \), suppose there exists a PL metric \( l^* = w \ast l \) on \( (D, \mathcal{T}) \) for some \( w : V \to \mathbb{R} \) such that its discrete curvature at all vertices except \( \{p, q, r\} \) are zero and the curvatures at \( p, q, r \) are \( \frac{2\pi}{3} \). Then the associated flat metric on \( (D, \mathcal{T}, l^*) \) is isometric to an equilateral triangle \( \Delta ABC \), i.e., there is a geometric triangulation \( \mathcal{T}' \) of \( \Delta ABC \) such that \( (\Delta ABC, \mathcal{T}', l_{st}) \) is isometric to \( (D, \mathcal{T}, l^*) \). Here and below, if \( \mathcal{T} \) is a geometric triangulation of a domain in the plane, we use \( l_{st} : E(\mathcal{T}) \to \mathbb{R} \) to denote the length of edges \( e \in \mathcal{T} \) in the standard metric on \( \mathbb{C} \). Let \( f : D \to \Delta ABC \) be the piecewise linear orientation preserving homeomorphism sending \( V \) to the vertex set \( V(\mathcal{T}') \) of \( \mathcal{T}' \), and \( p, q, r \) to \( A, B, C \) respectively and being linear on each triangle of \( \mathcal{T} \). We call \( f \) the discrete uniformization map associated to \( (D, \mathcal{T}, l, \{p, q, r\}) \). Note that \( f \) may not exist due to the lacking of existence theorem.

**Theorem 1.2.** Suppose \( \Omega \) is a Jordan domain in the complex plane with three distinct points \( p, q, r \subset \partial \Omega \). Then there exists a sequence \( (\Omega_n, \mathcal{T}_n, l_{st}, (p_n, q_n, r_n)) \) of simply connected triangulated polygonal disks in \( \mathbb{C} \) where \( \mathcal{T}_n \) are triangulations by equilateral triangles and \( p_n, q_n, r_n \) are three boundary vertices such that
1. \( \Omega = \bigcup_{n=1}^{\infty} \Omega_n \) with \( \Omega_n \subset \Omega_{n+1} \), and \( \lim_n p_n = p \), \( \lim_n q_n = q \) and \( \lim_n r_n = r \),
2. discrete uniformization maps associated to \( (\Omega_n, \mathcal{T}_n, l_{st}, (p_n, q_n, r_n)) \) exist and converge uniformly to the Riemann mapping for \( (\Omega, (p, q, r)) \).

In Rodin-Sullivan’s convergence theorem, any sequence of approximating circle packing maps associated to the approximation triangulated polyhedral disks \( \Omega_n \) such that \( \Omega_n \subset \text{int}(\Omega_{n+1}) \) and \( \Omega = \bigcup_n \Omega_n \) converges to the Riemann mapping. Theorem 1.2 is less robust in this aspect since discrete conformal maps may not exist if the triangulations \( \mathcal{T}_n \) are not carefully selected. A stronger version of this conjecture is stated in §7.

The conformality of the limit of the discrete conformal maps in Theorem 1.2 is a consequence of the following result. Recall that a geometric triangulation \( \mathcal{T} \) of polyhedral surface is called Delaunay if the sum of two angles facing each interior edge is at most \( \pi \). Delaunay triangulations always exist for each PL metric on compact surfaces.

**Theorem 1.3.** Suppose \( \mathcal{T} \) is a Delaunay geometric triangulations of the complex plane \( \mathbb{C} \) such that its vertex set is a lattice and \( l_{st} : E(\mathcal{T}) \to \mathbb{R} \) is the edge length function of \( \mathcal{T} \). If \( (\mathbb{C}, \mathcal{T}, w \ast l_{st}) \) is a Delaunay triangulated surface isometric to an open set in the Euclidean plane \( \mathbb{C} \), then \( w \) is a constant function.

We remark that the same result as above for the standard hexagonal lattice has been proved independently by Dai-Ge-Ma [9] in a recent preprint.
Using an important result in [4] that vertex scaling is closely related to hyperbolic 3-dimensional geometry and the work of [13], one sees that Theorem 1.4 implies the following rigidity result on convex hyperbolic polyhedra.

**Theorem 1.4.** Suppose \( L = \mathbb{Z} + \tau \mathbb{Z} \) is a lattice in the plane \( \mathbb{C} \) and \( V \subset \mathbb{C} \) is a discrete set such that there exists an isometry between the boundaries of the convex hulls of \( L \) and \( V \) in the hyperbolic 3-space \( \mathbb{H}^3 \) preserving cell structures. Then \( V \) and \( L \) differ by a complex affine transformation of \( \mathbb{C} \).

This prompts us to propose the following conjecture. A closed set \( X \) in the Riemann sphere is said to be of *circle type* if each connected component of \( X \) is either a point or a round disk. Consider the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \) as the infinity of the (upper-half-space model of) hyperbolic 3-space \( \mathbb{H}^3 \).

**Conjecture 1.5.** For any genus zero connected complete hyperbolic surface \( \Omega \), there exists a circle type closed set \( X \subset \mathbb{C} \cup \{ \infty \} \) such that \( \Omega \) is isometric to the boundary of the convex hull of \( X \) in \( \mathbb{H}^3 \).

**Conjecture 1.6.** Suppose \( X \) and \( Y \) are circle type closed sets in \( \mathbb{C} \) such that boundary of the convex hulls of \( X \) and \( Y \) in \( \mathbb{H}^3 \) are isometric. Then \( X \) and \( Y \) differ by a Möbius transformation.

The paper is organized as follows. §2 recalls the basic material for discrete conformal geometry of polyhedral surfaces. Sections 3 and 4 are devoted to prove Theorem 1.4. The main tools used are a maximum principle, a variational principle for discrete conformal geometry of polyhedral surfaces and spiral hexagonal triangulations derived from linear conformal factors. Section §5 investigates the existence of flat metrics with prescribed boundary curvature on polygonal disks. The main result (Theorem 5.1) is an existence result for vertex scaling equivalence if triangulations of a polyhedral disk are sufficiently fine subdivided. The basic tools used are discrete harmonic functions, their gradient estimates and solutions to ordinary differential equations. We prove the convergence Theorem 1.2 in §6 using the results obtained in §4, §5 and Rado-Palka’s theorem on uniform convergence of Riemann mappings and quasiconformal mappings. §7 discusses a strong version of the convergence of discrete uniformization maps and the motivations for Conjecture 1.5.

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2. POLYHEDRAL METRICS, VERTEX SCALING AND A VARIATIONAL PRINCIPLE

We begin with some notations. Let \( \mathbb{C}, \mathbb{R}, \mathbb{Z} \) be the sets of complex, real, and integers respectively. \( \mathbb{R}_{>0} = \{ t \in \mathbb{R} | t > 0 \} \), \( \mathbb{Z}_{\geq k} = \{ n \in \mathbb{Z} | n \geq k \} \) and \( SS^1 = \{ z \in \mathbb{C} | \|z\| = 1 \} \). We use \( \mathbb{D} \) to denote the open unit disk in \( \mathbb{C} \) and \( \mathbb{H}^n \) to denote the \( n \)-dimensional hyperbolic space.

Given that \( X \) is a compact surface with boundary, its interior is denoted by \( \text{int}(X) \). A graph with vertex set \( V \) and edge set \( E \) is denoted by \( (V, E) \). Two vertices \( i, j \) in a graph \( (V, E) \) are adjacent, denoted by \( i \sim j \), if they are the end points of an edge. If \( i \sim j \), we use \([ij]\) (respectively \(ij\)) to denote an oriented (respectively unoriented) edge from \( i \) to \( j \). An edge *path* joining \( i, j \in V \) is a sequence of vertices \( \{v_0 = i, v_1, ..., v_m = j\} \) such that \( v_k \sim v_{k+1} \). The length of the path is \( m \). The *combinatorial distance* \( d_c(i, j) \) between two vertices in a
connected graph \((V, E)\) is the length of the shortest edge path joining \(i, j\). Suppose \((S, T)\) is a triangulated surface with possibly non-empty boundary \(\partial S\) and possibly non-compact \(S\). Let \(E = E(T), V = V(T)\) be the sets of edges, vertices respectively and \(T^{(1)} = (V, E)\) be the associated graph. A vertex \(v \in V(T) \cap \partial S\) (resp. \(v \in V \cap (S - \partial S)\)) is called a boundary (resp. interior) vertex. Boundary and interior edges are defined in the same way. A PL metric on \((S, T)\) or simply on \(T\) can be represented by a length function \(l : E(T) \to \mathbb{R}_{>0}\) so that if \(e_i, e_j, e_k\) are three edges forming a triangle in \(T\), then the strict triangle inequality holds,

\[(2) \quad l(e_i) + l(e_j) > l(e_k).\]

We will use limits of PL metrics. To this end, we introduce the notion of generalized PL metrics on \((S, T)\). Take three pairwise distinct points \(v_1, v_2, v_3\) in the plane. The convex hull of \(\{v_1, v_2, v_3\}\) is a generalized triangle with vertices \(v_1, v_2, v_3\). We denoted it by \(\Delta v_1v_2v_3\). If \(v_1, v_2, v_3\) are not in a line, then \(\Delta v_1v_2v_3\) is a (usual) triangle. If \(v_1, v_2, v_3\) lie in a line, then \(\Delta v_1v_2v_3\) is a degenerate triangle with the flat vertex at \(v_i\) if \(|v_j - v_i| + |v_k - v_i| = |v_j - v_k|\), \(\{i, j, k\} = \{1, 2, 3\}\). Let \(l_i = |v_i - v_k| \in \mathbb{R}_{>0}\) be the edge length and \(a_i \in [0, \pi]\) be the angle at \(v_i\). Then \(l_i + l_j \geq l_k > 0\) and the angles are given by

\[(3) \quad a_i = \arccos\left(\frac{l_j^2 + l_k^2 - l_i^2}{2l_jl_k}\right).\]

Furthermore, the angle \(a_i = a_i(l_1, l_2, l_3) \in [0, \pi]\) is continuous in \((l_1, l_2, l_3)\). Degenerate triangles are characterized by either having an angle \(\pi\) or the lengths satisfying \(l_i = l_j + l_k\) for some \(i, j, k\).

A generalized PL metric on a triangulated surface \((S, T)\) is represented by an edge length function \(l : E(T) \to \mathbb{R}_{>0}\) so that if \(e_i, e_j, e_k\) are three edges forming a triangle in \(T\), then the triangle inequality holds,

\[(4) \quad l(e_i) + l(e_j) \geq l(e_k).\]

We will abuse the use of terminology and call \(l\) a generalized PL metric on \((S, T)\) or \(T\). The discrete curvature \(K : V(T) \to (-\infty, 2\pi]\) of a generalized PL metric \((S, T, l)\) is defined as follows. If \(v \in V(T)\) is an interior vertex, \(K(v)\) is \(2\pi\) minus the sum of angles of (generalized triangles) at \(v\); if \(v\) is a boundary vertex, \(K(v)\) is \(\pi\) minus the sum of angles at \(v\). Note that the Gauss-Bonnet theorem \(\sum_{v \in V(T)} K(v) = 2\pi \chi(S)\) still holds for a compact surface \(S\) with a generalized PL metric. Clearly the curvature \(K\) and inner angles depend continuously on the length vector \(l \in \mathbb{R}^{E(T)}_{>0}\). A generalized PL metric is called flat if its curvatures are zero at all interior vertices \(v\). A generalized PL metric \((S, T, l)\) (or sometimes written as \((T, l)\)) is called Delaunay if for each interior edge \(e \in E(T)\) the sum of the two angles facing \(e\) is at most \(\pi\). If \((S, T, l)\) is a Delaunay generalized PL metric such that each angle facing a boundary edge is at most \(\pi/2\), then the metric double of \((S, T, l)\) along its boundary is a Delaunay triangulated generalized PL metric surface. Two generalized PL metrics \(l\) and \(\tilde{l}\) on \((S, T)\) are related by a vertex scaling if there is \(w \in \mathbb{R}^V\) so that

\[
\tilde{l}(vv') = e^{w(v) + w(v') - w(v) - w(v')} l(vv')
\]

for all edges \(vv' \in E(T)\). We write \(\tilde{l} = w \ast l\) and call \(w\) a discrete conformal factor.

Two generalized triangles \(\Delta v_1v_2v_3\) and \(\Delta u_1u_2u_3\) are equivalent if there exists an isometry sending \(v_i\) to \(u_i\) for \(i = 1, 2, 3\). The space of all equivalence classes of generalized triangles can be identified with \(\{(l_1, l_2, l_3) \in \mathbb{R}^3_{>0} | l_i + l_j \geq l_k\}\). It contains the space of all equivalence
classes of triangles \( \{ (l_1, l_2, l_3) \in \mathbb{R}^3 \mid l_i + l_j > l_k \} \). Given two generalized triangles \( l = (l_1, l_2, l_3) \) and \( \tilde{l} = (\tilde{l}_1, \tilde{l}_2, \tilde{l}_3) \) there exists \( w = (w_1, w_2, w_3) \in \mathbb{R}^3 \) such that \( \tilde{l}_i = l_i e^{w_j + w_k} \).

The following result was proved in [25, Theorem 2.1] for Euclidean triangles. The extension to generalized triangles is straightforward.

**Proposition 2.1 ([26]).** Let \( \Delta v_1 v_2 v_3 \) be a fixed generalized triangle of edge length vector \( l = (l_1, l_2, l_3) \) and \( w \ast l \) is the edge length vector of a vertex scaled generalized triangle whose inner angle at \( v_i \) is \( a_i(w) \).

(a) For any two constants \( c_i, c_j \), the set \( \{ (w_1, w_2, w_3) \in \mathbb{R}^3 \mid w \ast l \text{ is a generalized triangle and } w_i = c_i, w_j = c_j \} \) is either connected or empty.

(b) If \( (\Delta v_1 v_2 v_3, l) \) is a non-degenerate triangle and \( i, j, k \) distinct, then

\[
\frac{\partial a_i}{\partial w_i} \bigg|_{w=0} = -\frac{\sin(a_i)}{\sin(a_j) \sin(a_k)} < 0, \quad \frac{\partial a_i}{\partial w_j} \bigg|_{w=0} = \frac{\partial a_j}{\partial w_i} \bigg|_{w=0} = \cot(a_k), \quad \text{and} \quad \sum_{j=1}^{3} \frac{\partial a_i}{\partial w_j} = 0.
\]

The matrix \(-[\frac{\partial a_i}{\partial w_j}]_{3 \times 3}\) is symmetric, positive semi-definite with null space spanned by \((1, 1, 1)^T\).

(c) If \( (\Delta v_1 v_2 v_3, l) \) is a degenerate triangle having \( v_3 \) as the flat vertex, then for small \( t > 0 \), \( (\Delta v_1 v_2 v_3, (0, 0, t) \ast l) \) is a non-degenerate triangle. The angle \( a_3(0, 0, t) \) is strictly decreasing in \( t \) for all \( t \) for which \( (0, 0, t) \ast l \) is a generalized triangle. The angles \( a_i(0, 0, t) \), \( i = 1, 2 \), are strictly increasing in \( t \in [0, \epsilon) \) for some \( \epsilon > 0 \).

![Figure 1. Vertex scaling of a triangle](image)

**Proof.** To see part (a), without loss of generality, we may assume \( c_1 \) and \( c_2 \) are the given constants. Then the variable \( w_3 \) is defined by three inequalities \( e^{w_3}(e^{c_1}l_2 + e^{c_2}l_1) \geq e^{c_1+c_2}l_3, e^{c_1+c_2}l_3 \geq e^{w_3}(e^{c_1}l_2 - e^{c_2}l_1) \geq -e^{c_1+c_2}l_3 \). Each of these inequalities defines an interval in \( w_3 \) variable. Therefore the solution space is either the empty set or a connected set.

Part (b) is in [26, Theorem 2.1].

To see (c), due to \( l_3 = l_1 + l_2 \), for small \( t > 0 \), we have \((0, 0, t) \ast l = (e^t l_1, e^t l_2, l_1 + l_2) \in \Delta \): \( \{(x_1, x_2, x_3) \in \mathbb{R}^3_0 \mid x_i + x_j \geq x_k \} \). Now by (5) and the Sine Law, \( \frac{\partial a_3}{\partial w_3}(0, 0, t) = -\frac{\sin(a_3)}{\sin(a_1) \sin(a_2)} < 0 \). Together with part (a), the angle \( a_3(0, 0, t) \) as a function of \( t \) is defined on an interval and is strictly decreasing in \( t \). Since \( \lim_{t \to 0^+} \frac{\partial a_3}{\partial w_3} = \lim_{t \to 0^+} \cot(a_2) = \infty \), due to \( \lim_{t \to 0^+} a_2(t, 0, 0) = 0 \), therefore the result holds for \( a_1 \). By the same argument, the result holds for \( a_2 \).

As a consequence,
Corollary 2.2. Under the same assumption as in Proposition 2.1, if \( w(t) = (w_1(t), w_2(t), w_3(t)) \in \mathbb{R}^3 \) is smooth in \( t \) such that \( w(t) \ast l \) is the edge length vector of a triangle with inner angle \( \alpha_i(t) = a_i(w(t) \ast l) \) at \( v_i \), then

\[
\frac{da_i(t)}{dt} = \sum_{j \sim i} \cot(a_k)\left(\frac{dw_j}{dt} - \frac{dw_i}{dt}\right)
\]

where \( j \sim i \) means \( v_j \) is adjacent to \( v_i \) and \( \{i, j, k\} = \{1, 2, 3\} \).

Write \( w'_j(t) = \frac{dw_j}{dt} \). Indeed by the chain rule and \( \frac{d}{dt} \), we have

\[
\frac{da_i(t)}{dt} = \frac{\partial a_i}{\partial w_i} w'_i + \sum_{j \neq i} \frac{\partial a_i}{\partial w_j} w'_j
\]

\[
= -\sum_{j \neq i} \cot(a_k) w'_i + \sum_{j \sim i} \cot(a_k) w'_j = \sum_{j \sim i} \cot(a_k) (w'_j - w'_i)
\]

Suppose \((S, \mathcal{T}, l)\) is a geometrically triangulated compact polyhedral surface and \( w(t) \in \mathbb{R}^V \) is a smooth path in parameter \( t \) such that \( w(t) \ast l \) is a PL metric on \((S, \mathcal{T})\). Let \( K_i = K_i(t) \) be the discrete curvature at \( i \in V \) and \( \theta_{ij}^k = \theta_{ij}^k(t) \) be the inner angle at the vertex \( i \) in \( \Delta_{ijk} \) in the metric \( w(t) \ast l \). For an edge \([ij]\) in the triangulation \( \mathcal{T} \), define \( \eta_{ij} \) to be \( \cot(\theta_{ij}^k) + \cot(\theta_{ij}^l) \) if \([ij]\) is an interior edge facing two angles \( \theta_{ij}^k \) and \( \theta_{ij}^l \) and \( \eta_{ij} = \cot(\theta_{ij}^k) \) if \([ij]\) is a boundary edge. If \([ij]\) is an interior edge, then \( \eta_{ij} \geq 0 \) if and only if \( \theta_{ij}^k + \theta_{ij}^l \leq \pi \), i.e., the Delaunay condition holds at \([ij]\).

The curvature variation formula is the following.

Proposition 2.3.

\[
\frac{dK_i}{dt} = \sum_{j \sim i} \eta_{ij} (\frac{dw_j}{dt} - \frac{dw_i}{dt})
\]

This follows directly from the Corollary 2.2 since \( K_i = c\pi - \sum_{r,s \in V} \theta_{rs}^i \) where \( c = 1 \) or \( 2 \) and \( \theta_{rs}^i \) are angles at \( i \). Since \( \frac{dK_i}{dt} = -\sum_{r,s \in V} \frac{d\theta_{rs}^i}{dt} \), \( \frac{d}{dt} \), \( \sum_{j \sim i} \frac{\partial a_i}{\partial w_j} w'_j \)

3. A maximum principle, a ratio lemma and spiral hexagonal triangulations

Let \( v_0 \) be an interior point of a star-shaped \( n \)-sided polygon \( P_n \) having vertices \( v_1, \ldots, v_n \) labelled cyclically. The triangulation \( \mathcal{T} \) of \( P_n \) with vertices \( v_0, \ldots, v_n \) and triangles \( \Delta v_0 v_i v_{i+1} \) \((v_{n+1} = v_1)\) is called a star triangulation of \( P_n \). See Figure 2.

Theorem 3.1 (Maximum principle). Let \( \mathcal{T} \) be a star triangulation of \( P_n \) and \( l : E(\mathcal{T}) \to \mathbb{R}_{>0} \) be a generalized Delaunay polyhedral metric on \( \mathcal{T} \). If \( w : \{v_0, v_1, \ldots, v_n\} \to \mathbb{R} \) satisfies

(a) \( w \ast l \) is a generalized Delaunay polyhedral metric on \( \mathcal{T} \),

(b) the curvatures \( K_0(w \ast l) \) of \( w \ast l \) and \( K_0(l) \) of \( l \) at vertex \( v_0 \) satisfy \( K_0(w \ast l) \leq K_0(l) \), and

(c) \( w(v_0) = \max\{w(v_i) | i = 0, 1, \ldots, n\} \),

then \( w(v_i) = w(v_0) \) for all \( i \).
As a convention, if \( x = (x_0, \ldots, x_m) \) and \( y = (y_0, y_1, \ldots, y_m) \) are in \( \mathbb{R}^{m+1} \), then \( x \geq y \) means \( x_i \geq y_i \) for all \( i \). Given \( w : \{v_0, \ldots, v_m\} \to \mathbb{R} \), we use \( w_i = w(v_i) \) and identify \( w \) with \((w_0, w_1, \ldots, w_m) \in \mathbb{R}^{m+1} \). The cone angle of \( w \ast l \) at \( v_0 \) will be denoted by \( \alpha(w) \). Thus Theorem 3.1(b) says \( \alpha(w) \geq \alpha(0) \).

The proof of Theorem 3.1 depends on the following lemma.

**Lemma 3.2.** If \( w : \{v_0, v_1, \ldots, v_n\} \to \mathbb{R} \) satisfies (a), (b), (c) in Theorem 3.1 such that there is \( w_0 < w_0 \), then there exists \( \hat{w} \in \mathbb{R}^{n+1} \) such that

(a) \( \hat{w}_i \geq w_i \) for \( i = 1, 2, \ldots, n \),
(b) \( \hat{w}_i \leq w_0 = w_0 \) for \( i = 1, 2, \ldots, n \),
(c) \( \hat{w} \ast l \) is a generalized Delaunay polyhedral metric on \( T \), and
(d) \( \alpha(\hat{w}) > \alpha(w) \).

Let us first prove Theorem 3.1 using Lemma 3.2.

**Proof.** By replacing \( w \) by \( w - w(v_0)(1, 1, \ldots, 1) \), we may assume that \( w(v_0) = 0 \). Suppose the result does not hold, i.e., there exists \( w \) so that \( w_0 = 0, w_i \leq 0 \) for \( i = 1, 2, \ldots, n \) with one \( w_i = 0 \), and \( w \ast l \) is a generalized Delaunay PL metric on \( T \) so that \( \alpha(w) \geq \alpha(0) \). We will derive a contradiction as follows. By Lemma 3.2, we may assume, after replacing \( w \) by \( \hat{w} \), that

\[
\alpha(w) \geq \alpha(0).
\]

Consider the set

\[
X = \{ x \in \mathbb{R}^{n+1} \mid w \leq x \leq 0, x_0 = 0, \ x \ast l \ \text{is a generalized Delaunay polyhedral metric on } T \}.
\]

Clearly \( w \in X \) and therefore \( X \neq \emptyset \) and \( X \) is bounded. Since inner angles are continuous in edge lengths, we see that \( X \) is a closed set in \( \mathbb{R}^{n+1} \). Therefore \( X \) is compact. Let \( t \in X \) be a maximum point of the continuous function \( f(x) = \alpha(x) \) on \( X \). We claim that \( t = 0 \). To prove this, we assume \( t \neq 0 \) and \( t \leq 0 \). Then by Lemma 3.2, we can find \( \hat{t} \geq t \) such that \( \hat{t}_0 = 0 \) and \( \hat{t} \ast l \) is a generalized Delaunay polyhedral metric on \( T \) with \( \alpha(\hat{t}) > \alpha(t) \). This contradicts the maximality of \( t \). Now for \( t = 0 \), we have

\[
\alpha(0) = \alpha(t) \geq \alpha(w) > \alpha(0)
\]

where the last inequality follows from (9). This is a contradiction. \( \square \)

Now back to the proof of Lemma 3.2.

**Proof.** After replacing \( w \) by \( w - w(v_0)(1, 1, \ldots, 1) \), we way assume \( w_0 = 0 \). Let \( a_i = a_i(w) = a_i(w_0, w_i, w_i+1) \), \( b_i = b_i(w) = b_i(w_0, w_i-1, w_i) \) and \( c_i = c_i(w) = c_i(w_0, w_i, w_i+1) \) be the inner angles \( \angle v_0v_iv_{i+1}, \angle v_0v_{i-1}v_i \) and \( \angle v_0v_i^1v_{i+1} \) in the metric \( w \ast l \) respectively. See Figure 2. Let \( l_i = l(v_0v_i) \) and \( l_{i,i+1} = l(v_i^1v_{i+1}) \) be the edge lengths in the metric \( l \).

Let us begin the proof for the simplest case where all triangles in \( w \ast l \) are non-degenerate (i.e., \( w \ast l \) is a PL metric) and \( w_i < 0 \) for all \( i \geq 1 \). Let \( j \in \{1, 2, 3, \ldots, n\} \) be the index such that \( l(v_0v_j) = \min \{ w \ast l(v_0v_k) \mid k = 1, 2, \ldots, n \} \). It is well known that in a Euclidean triangle \( \triangle ABC \), \( \angle A < \pi/2 \) if \( BC \) is not the unique largest edge. Hence, due to \( w \ast l(v_0v_j) \leq w \ast l(v_0v_{j \pm 1}) \), in the triangles \( \Delta v_0v_jv_{j \pm 1} \), we have

\[
(10) \quad a_j(w) < \pi/2, \quad b_j(w) < \pi/2 \quad \text{and} \quad a_j(w) + b_j(w) < \pi.
\]

Now consider \( \hat{w} = (w_0, w_1, \ldots, w_{j-1}, w_j + t, w_{j+1}, \ldots, w_n) \). For small \( t > 0 \), \( w(t) \ast l \) is still a PL metric since \( w \ast l \) is. We claim \( \hat{w} \ast l \) is still Delaunay for small \( t \). Indeed, by Proposition
2.1] both angles \(a_{j-1} \) and \(b_{j+1} \) decrease in \( t \). On the other hand \(a_{j+1}(w) = a_{j+1}(\hat{w}) \) and \( b_{j-1}(w) = b_{j-1}(\hat{w}) \). Therefore, the Delaunay conditions \( b_{j-1} + a_{j-1} \leq \pi \) and \( b_{j+1} + a_{j+1} \leq \pi \) hold for edges \( v_{0}v_{j} \pm 1 \). The Delaunay condition on the edge \( v_{0}v_{j} \) follows from choice of \( j \) that \( a_{j} + b_{j} < \pi \). Finally, by Proposition 2.1(b), \( \frac{\partial \alpha(w)}{\partial t} = \cot(a_{j}) + \cot(b_{j}) = \frac{\sin(a_{j} + b_{j})}{\sin(a_{j}) \sin(b_{j})} > 0 \). Therefore, for small \( t > 0 \), we have \( \alpha(w) > \alpha(0) \).

In the general case, the above arguments still work.

Let \( J = \{ j \in V | w_{j} < 0 \} \). By assumption, \( J \neq \emptyset \).

**Claim 1.** If \( j \in J \), then \( c_{j}(w) < \pi \) and \( c_{j-1}(w) < \pi \).

We prove \( c_{j-1}(w) < \pi \) by contradiction. Suppose otherwise that \( c_{j-1}(w) = \pi \). Then the triangle \( \Delta v_{0}v_{j}v_{j-1} \) is degenerate in \( w \ast l \) metric, i.e., \( e^{w_{j}+w_{j-1}}l_{j,j-1} = e^{w_{j}}l_{j} + e^{w_{j-1}}l_{j-1} \). Due to \( w_{j} < 0 \) and \( w_{j-1} \leq 0 \), we have

\[
e^{w_{j}+w_{j-1}}l_{j,j-1} = e^{w_{j}}l_{j} + e^{w_{j-1}}l_{j-1} > e^{w_{j}+w_{j-1}}l_{j} + e^{w_{j}+w_{j-1}}l_{j-1} = e^{w_{j}+w_{j-1}}(l_{j} + l_{j-1}).
\]

This shows \( l_{j,j-1} > l_{j} + l_{j-1} \) which contradicts the triangle inequality for \( l \) metric. Therefore \( c_{j-1}(w) < \pi \). By the same argument, we have \( c_{j}(w) < \pi \). This proves Claim 1.

Let \( I = \{ i > 0 | w_{i} = 0 \} \) and

\[
\beta(w) = \sum_{i \in I} (b_{i}(w) + a_{i}(w))
\]

and

\[
\gamma(w) = \sum_{j \in J} (b_{j}(w) + a_{j}(w)).
\]

Note that the cone angle at \( v_{0} \) is

\[
\alpha(w) = \sum_{i=1}^{n} (\pi - a_{i}(w) - b_{i+1}(w)) = \pi n - \beta(w) - \gamma(w).
\]

By the assumption that \( \alpha(w) \geq \alpha(0) \), we have

\[
(11) \quad \beta(w) + \gamma(w) \leq \beta(0) + \gamma(0).
\]

**Claim 2.** If \( I \neq \emptyset \), then \( \beta(w) > \beta(0) \).

Indeed, if \( i \in I \), i.e., \( w_{i} = 0 \), then in the triangle \( \Delta v_{0}v_{i}v_{i+1} \), we have \( w_{0} = 0, w_{i} = 0 \) and \( w_{i+1} \leq 0 \). Since \( \Delta v_{0}v_{i}v_{i-1} \) are generalized triangles in both \( l \) and \( w \ast l \) metrics, by proposition

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**Figure 2.** Star triangulation of an \( n \)-sided polygon

[Diagram of a star triangulation of an \( n \)-sided polygon with labels and angles.]
we see that \( \Delta v_0 v_j v_{j+1} \) is a generalized triangle in \((w_0, ..., w_{i-2}, tw_{i-1}, w_i, ..., w_n) \ast l \) for \( t \in [0, 1] \). By Proposition 2.1, and \( w_{i-1} \leq 0 \), \( b_j(w_0, ..., w_{i-2}, tw_{i-1}, w_i, ..., w_n) \) is increasing in \( t \geq 0 \) and is strictly increasing in \( t \geq 0 \) if \( w_{i-1} < 0 \). Therefore,

\[
b_i(w) = b_i(w_0, w_{i-1}, w_i) \geq b_i(w_0, 0, w_i) = b_i(w_0, w_1, ..., w_{i-2}, 0, w_i, ..., w_n) = b_i(0),
\]

and \( b_i(w) > b_i(0) \) if \( w_{i-1} < 0 \). Apply the same argument to \( \Delta v_0 v_j v_{j+1} \) and \( a_i \), we have \( a_i(w) \geq a_i(0) \) and \( a_i(w) > a_i(0) \) if \( w_{i+1} < 0 \). Therefore \( \beta(w) \geq \beta(0) \). On the other hand, since \( J \neq \emptyset \), there exists an \( i \in I \) so that either \( i - 1 \) or \( i + 1 \) is in \( J \). Say \( i - 1 \in J \), i.e., \( w_{i-1} < 0 \). Then we have \( b_i(w) > b_i(0) \) and \( \beta(w) > \beta(0) \).

By claim 2 and (11), if \( I \neq \emptyset \), we conclude that

(12)

\[
\gamma(w) = \sum_{j \in J} (a_j(w) + b_j(w)) < \gamma(0).
\]

Since \( w \ast l \) and \( l \) are Delaunay, we have \( a_i(w) + b_i(w) \leq \pi \) and \( a_i(0) + b_i(0) \leq \pi \) for all \( i = 1, 2, ..., n \). This implies, by (12), that there exists \( j \in J \) so that

(13)

\[
a_j(w) + b_j(w) < \pi.
\]

If \( I = \emptyset \), let \( j \in J = \{1, 2, 3, ..., n\} \) be the index such that \( w \ast l = \min_{k \in J} \{w \ast l(v_0 v_k) : k = 1, 2, ..., n\} \). Then the same argument used in showing (10) and Claim 1 imply (13) still holds. (Here Claim 1 is used to show that \((w_0, ..., w_{j-1}, w_j + l, w_{j+1}, ..., w_n) \ast l \) is a generalized PL metric for small \( t > 0 \).)

Fix this \( j \in J \) as above. To finish the proof, we will show that there exists a small \( t > 0 \) so that for \( \hat{w} = (w_0, w_1, ..., w_{j-1}, w_j + t, w_{j+1}, ..., w_n) \in \mathbb{R}_{\leq 0}^{n+1} \) the following hold:

(i) \( \hat{w} \ast l \) is a generalized polyhedral metric on \( T \);

(ii) \( \hat{w} \ast l \) satisfies the Delaunay condition;

(iii) \( \alpha(\hat{w}) > \alpha(w) \).

Since \( w_j < 0 \), any \( t \in (0, -w_j) \) will make \( \hat{w} \in \mathbb{R}_{\leq 0}^{n+1} \).

To see part (i), by Claim 1 and (13) which imply \( a_j(w), b_j(w), c_j(w), c_{j-1}(w) < \pi \), the triangle \( \Delta v_0 v_j v_{j+1}, w \ast l \) (or \( \Delta v_0 v_j v_{j-1}, w \ast l \)) is either non-degenerate or is degenerate with \( \pi \)-angle at \( v_j \), i.e., \( b_{j+1}(w) = \pi \) (or \( a_{j-1}(w) = \pi \) respectively). Therefore by Proposition 2.1(c), for small \( t > 0 \), \( \hat{w} \ast l \) is still a generalized PL metric.

To see part (ii), we check the sum of opposite angles at the following edges: \( v_0 v_{j-1}, v_0 v_j \) and \( v_0 v_j \). At the edge \( v_0 v_j \), due to (13) and continuity, we see \( a_j(\hat{w}) + b_j(\hat{w}) < \pi \) for small \( t > 0 \). At the edge \( v_0 v_{j-1} \) (or similarly \( v_0 v_{j+1} \)), by Proposition 2.1(c) that \( a_{j-1}(\hat{w} \ast l) \) and \( b_{j+1}(\hat{w} \ast l) \) are strictly decreasing functions in \( t > 0 \) and \( b_{j-1}(\hat{w}) = b_{j-1}(w) \), we have

\[
a_{j-1}(\hat{w}) + b_{j-1}(\hat{w}) < a_{j-1}(w) + b_{j-1}(w) \leq \pi.
\]

Similarly, we have the Delaunay condition for \( \hat{w} \ast l \) at the edge \( v_0 v_{j+1} \).

Finally, to see (iii), by Proposition 2.1 and (13), we have

\[
\frac{d}{dt} \bigg|_{t=0} \alpha(\hat{w}) = \frac{d}{dt} \bigg|_{t=0} (c_j(\hat{w})) + \frac{d}{dt} \bigg|_{t=0} (c_{j-1}(\hat{w})) = \cot(b_j(\hat{w})) + \cot(a_j(\hat{w})) > 0.
\]

Therefore, for small \( t > 0 \), \( \alpha(\hat{w}) > \alpha(w) \).

\( \square \)

**Lemma 3.3.** Let \((P_N, T)\) be a star triangulation of an \( N \)-gon with boundary vertices \( v_1, ..., v_N \) labelled cyclically and one interior vertex \( v_0 \) and \( l : E(T) \to \mathbb{R}_{> 0} \) be a flat generalized PL metric on \( T \). There is a constant \( \lambda(l) \) depending on \( l \) such that if \((P_N, T, w \ast l)\) with
w : \{v_0, ..., v_N\} \to \mathbb{R} is a generalized PL metric with zero curvature at v_0, then the ratio of edge lengths satisfies

\[ \frac{w \ast l(v_i v_0)}{w \ast l(v_i v_{i+1})} \leq \lambda(l) \]

for all indices.

**Figure 3.** Triangulated hexagon and length ratio

**Proof.** Let \( x_i(w) = w \ast l(v_0 v_i) \) and \( y_i(w) = w \ast l(v_i v_{i+1}) \) be the edge lengths in the metric \( w \ast l \) where \( v_{N+1} = v_1 \). By definition,

\[ \frac{x_{i+2}}{y_{i+1}} = \lambda_i \frac{x_i}{y_i} \]

where \( \lambda_i > 0 \) depends on \( l \). Then

\[ \frac{x_{i+1}}{y_{i+1}} \geq \frac{x_{i+2} - y_{i+1}}{y_{i+1}} = \frac{x_{i+2}}{y_{i+1}} - 1 = \lambda_i \frac{x_i}{y_i} - 1. \]

We prove by contradiction. If the result of lemma 3.3 is not true, than there exists a sequence of conformal factors \( w^{(n)} \) such that

\[ \frac{x_i(w^{(n)})}{y_i(w^{(n)})} \to \infty, \]

for some \( i \). Without loss of generality, assume \( i = 1 \) and then by (16) inductively we have

\[ \frac{x_2(w^{(n)})}{y_2(w^{(n)})} \to \infty, \quad \frac{x_3(w^{(n)})}{y_3(w^{(n)})} \to \infty, \ldots, \quad \frac{x_N(w^{(n)})}{y_N(w^{(n)})} \to \infty. \]

Then the angle \( a_i(w^{(n)}) \) at \( v_0 \) in the triangle \( \Delta v_0 v_i v_{i+1} \) (in \( w^{(n)} \ast l \) metric) converge to 0, for any \( i \). But that contradicts the fact that the curvature \( 2\pi - \sum_{i=1}^N a_i(w^{(n)}) \) at \( v_0 \) is zero. □

The next result concerns linear discrete conformal factor and spiral hexagonal triangulations. It is a counterpart of Doyle spiral circle packing in the discrete conformal setting. Unlike Doyle spiral circle packing, not all choices of linear functions produce generalized PL metrics.

We begin by recalling the developing maps. If \((S, T, l)\) is a flat generalized PL metric on a simply connected surface \( S \) (i.e., \( K_v = 0 \) for all interior vertices \( v \)), then a developing map \( \phi : (S, T, l) \to \mathbb{C} \) for \((T, l)\) is an isometric immersion determined by \( |\phi(v) - \phi(v')| = l(vv') \) for \( v \sim v' \). It is constructed as follows. Fix a generalized triangle \( t \in T \) and isometrically embeds \( t \) to \( \mathbb{C} \). This defines \( \phi|_t \). If \( s \) is a generalized triangle sharing a common edge \( e \) with \( t \), we can extend \( \phi|_t \) to \( \phi|_{t \cup s} \) by isometrically embedding \( s \) to \( \phi(s) \subset \mathbb{C} \) sharing the edge \( \phi(e) \) with \( \phi(t) \) such that \( \phi(s) \) and \( \phi(t) \) are on different sides of \( \phi(e) \). Since the surface is simply...
connected, by the monodromy theorem, we can keep extending \( \phi \) to all triangles in \( \mathcal{T} \) and produce a well defined isometric immersion. As a convention, if \( \tau \) is triangle in \( \mathcal{T} \) and \( l \) is a generalized PL metric on \( \mathcal{T} \), we use \((\tau, l)\) to denote the induced generalized PL metric on \( \tau \).

Given a lattice \( L \) in \( \mathbb{C} \), there exists a Delaunay triangulation \( \mathcal{T}_{st} = \mathcal{T}_{st}(L) \) of \( \mathbb{C} \) with vertex set \( L \) such that the \( \mathcal{T}_{st} \) is invariant under the translation action of \( L \). In particular \( \mathcal{T}_{st} \) descends to a 1-vertex triangulation of the torus \( \mathbb{C}/L \). Therefore, the degree of each vertex \( v \in \mathcal{T}_{st} \) is six, i.e., this triangulation is topologically the same as the standard hexagonal triangulation of \( \mathbb{C} \).

Let \( l_0 : E(\mathcal{T}_{st}) \to \mathbb{R}_{>0} \) be the edge length function of \((\mathbb{C}, \mathcal{T}_{st}(L), d_{st})\) where \( d_{st} \) is the standard flat metric on \( \mathbb{C} \). Let \( \tau \) be a triangle in \( \mathcal{T}_{st} \) with vertices \( 0, u_1, u_2 \). Then \( L = u_1 \mathbb{Z} + u_2 \mathbb{Z} \) and \( \{u_1, u_2\} \) is called a geometric basis of \( L \). Note that two vertices \( v, v' \in \mathcal{T} \) are joint by an edge \( e \in \mathcal{T}_{st} \) if and only if \( v - v' \in \{\pm u_1, \pm u_2, \pm(u_1 - u_2)\} \).

**Proposition 3.4.** Suppose \((\mathbb{C}, \mathcal{T}_{st}, l_0)\) is a hexagonal Delaunay triangulation of the plane with vertex set a lattice \( \mathbb{V} = u_1 \mathbb{Z} + u_2 \mathbb{Z} \) where \( \{u_1, u_2\} \) is a geometric basis. Let \( w : \mathbb{V} \to \mathbb{R} \) be a non-constant linear function \( w(\nu u_1 + m u_2) = \nu \ln(\lambda) + m \ln(\mu), \) \( m, n, \in \mathbb{Z} \), such that \( w * l_0 \) is a generalized Delaunay PL metric on \( \mathcal{T}_{st} \). Then the following hold.

(a) The generalized PL metric \((\mathcal{T}_{st}, w * l_0)\) is flat.

Let \( \phi \) be the developing map for the flat metric \((\mathcal{T}_{st}, w * l_0)\).

(b) If there exists a non-degenerate triangle in the generalized PL metric \( w * l_0 \), then there are two distinct non-degenerate triangles \( \sigma_1 \) and \( \sigma_2 \) in \((\mathcal{T}_{st}, w * l_0)\) such that \( \phi(\text{int}(\sigma_1)) \cap \phi(\text{int}(\sigma_2)) \neq \emptyset \).

(c) Suppose all triangles in \( w * l_0 \) are degenerate. Then there exists an automorphism \( \psi \) of the triangulation \( \mathcal{T}_{st} \) such that \( w(\psi(\nu u_1 + m u_2)) = \nu \ln(\gamma_1(V)) + m \ln(\gamma_2(V)) \) where \( \gamma_i(V) \) are two explicit numbers depending only on \( V \).

We remark that parts (a) and (b) for the lattice \( \mathbb{Z} + e^{2\pi i/3} \mathbb{Z} \) were proved in [43].

**Proof.** Consider two automorphisms \( A \) and \( B \) of the topological triangulation \( \mathcal{T}_{st} \) defined by \( A(v) = v + u_1 \) and \( B(v) = v + u_2 \) for \( v \in \mathbb{V} \). By definition, we have \( AB = BA \) and \( A, B \) generate the group \( < A, B > \cong \mathbb{Z}^2 \) acting on \( \mathcal{T}_{st} \). Any triangle in \( \mathcal{T}_{st} \) is equivalent, under the action of \( < A, B > \), to exactly one of the two triangles \( T_1 \) or \( T_2 \) where the vertices of \( T_1 \) are \( 0, u_1, u_2 \) and the vertices of \( T_2 \) are \( 0, -u_1, -u_2 \). In the generalized PL metric \( w * l_0 \), the maps \( A \) and \( B \) satisfy \( w * l_0(A(e)) = \lambda^2 w * l_0(e) \) and \( w * l_0(B(e)) = \mu^2 w * l_0(e) \) for each edge \( e \in \mathcal{T} \). It follows that for any triangle \( \tau \in \mathcal{T}_{st} \), the generalized triangle \((A(\tau), w * l_0)\) (resp. \((B(\tau), w * l_0)\)) is the scalar multiplication of \((\tau, w * l_0)\) by \( \lambda^2 \) (resp. by \( \mu^2 \)). Hence there are only two similarity types of triangles in \((\mathbb{C}, \mathcal{T}_{st}, w * l_0)\). For each \( v \in \mathcal{V} \), the six angles at \( v \) are congruent to the six inner angles in \( T_1 \) and \( T_2 \) in \( w * l_0 \) metric. Therefore, \((\mathcal{T}, w * l_0)\) is a flat metric. See Figure 4(b).

By the assumption that \( w \) is not a constant, we have \( \lambda, \mu \neq (1, 1) \). Say \( \lambda \neq 1 \). Using the developing map \( \phi \), we have two complex affine maps \( \alpha \) and \( \beta \) of the complex plane \( \mathbb{C} \) such that \( \phi A = \alpha \phi \) and \( \phi B = \beta \phi \). Since \( A \) is a scaling by the factor \( \lambda^2 \neq 1 \) and \( \phi \) is a local isometry, the affine map \( \alpha \) is of the form \( \alpha(z) = \lambda^* z + c \) where \( |\lambda^*| = \lambda^2 \neq 1 \) and \( \alpha \) has a unique fixed point \( p \in \mathbb{C} \). By \( AB = BA \), it follows \( \alpha \beta = \beta \alpha \). Therefore, from \( \beta(p) = \beta \alpha(p) = \alpha \beta(p) \), we conclude \( \beta(p) = p \). After replacing the developing map \( \phi \) by \( \rho \circ \phi \) for an isometry \( \rho \) of \( \mathbb{C} \), we may assume that \( \alpha \) and \( \beta \) both fix \( 0 \), i.e., \( \alpha(z) = \lambda^* z \) and \( \beta(z) = \mu^* z \) are both scalar multiplications. Let \( G = < \alpha, \beta > \) be the abelian group generated by \( \alpha, \beta \) which acts on \( \mathbb{C} \) by scalar multiplications.
To see part (b), let $\Omega$ be the image $\phi(C)$ of the developing map which is invariant under the action of $G$. By the assumption that there are non-degenerate triangles in $(T_{st}, w * l_0)$, the image $\Omega$ has non-empty interior. There are two cases we have to consider. In the first case, there exists a pair of integers $(n, m) \neq (0, 0)$ so that $\alpha^n \beta^m$ is the identity element in the group $G$. In this case, we take $\sigma_1$ to be any non-degenerate triangle and $\sigma_2 = A^n B^m(\sigma_1)$. By definition, we have $\phi(\sigma_1) = \phi(\sigma_2)$. Therefore, the result holds. In the second case that for all $(n, m) \neq (0, 0)$, $\alpha^n \beta^m \neq id$, i.e., the group $G$ is isomorphic to $\mathbb{Z}^2$. Since both $\alpha(z)$ and $\beta(z)$ are scalar multiplications, this implies that the action of the group $G$ on $int(\Omega)$ is not discontinuous. In particular, for any non-empty open set $U \subset \Omega$, there is $\alpha^n \beta^m \in G - \{id\}$ so that $\alpha^n \beta^m(U) \cap U \neq \emptyset$. Take $\sigma_1$ to be a non-degenerate triangle, $U = \phi(int(\sigma_1))$ and $\sigma_2 = A^n B^m(\sigma_1)$. Then we have $\phi(int(\sigma_1)) \cap \phi(int(\sigma_2)) \neq \emptyset$.

To see part (c), since each triangle is degenerate, the inner angles $a, b, c$ and $x, y, z$ of two triangles $T_1$ and $T_2$ are 0 or $\pi$ as shown in Figure 4(b). Composing with an automorphism of $T_{st}$, we may assume that $a = \pi$, and then by the Delaunay condition, $y \neq \pi$.

There are two cases depending on $(x, y, z) = (\pi, 0, 0)$ or $(0, 0, \pi)$. The two cases differ by the automorphism $\rho$ of the lattice $u_1z + u_2z$ and of $T_{st}$ such that $\rho(u_1) = u_2$, $\rho(u_2) = u_2 - u_1$ and $\rho(0) = 0$. Thus it suffices to consider the case: $z = \pi$. Let the lengths of $u_1, u_2$ and $u_2 - u_1$ in $l_0$-metric be $b_1, b_2$ and $b_3$ respectively. The lengths of the corresponding edges in $w * l_0$ metric are $\lambda b_1, \mu b_2$ and $\lambda b_3$. By the same computation, one works out the edge lengths of the triangle with vertices $0, u_2$ and $u_2 - u_1$ in $w * l_0$ metric to be $\frac{\mu^2}{\alpha} b_1, \mu b_2$ and $\frac{\mu}{\lambda} b_3$. See Figure 4(c).

We obtain two equations for edge lengths of degenerate triangles: $\lambda b_1 + \mu b_2 = \lambda \mu b_3$ (due to $a = \pi$) and $\frac{\alpha^2}{\lambda} b_1 = \mu b_2 + \frac{\alpha}{\mu} b_3$ (due to $z = \pi$). See Figure 4(c). These are same as $\lambda b_1 + \mu b_2 = \lambda \mu b_3$ and $\mu b_1 = \lambda b_2 + b_3$. Solving $\mu$ in terms of $\lambda$, we obtain a quadratic equation in $\lambda$:

\begin{equation}
\lambda^2 b_2 b_3 \lambda^2 + (b_3^2 - b_1^2 - b_2^2) \lambda - b_2 b_3 = 0.
\end{equation}

Since $b_i > 0$, this equation has a unique positive solution which we call $\gamma_1(V)$. The solution in $\mu$ is called $\gamma_2(V)$.

4. RIGIDITY OF HEXAGONAL TRIANGULATIONS OF THE PLANE

We begin with,
**Figure 5.** Spiral hexagonal triangulations

**Definition 4.1.** A flat generalized PL metric on a simply connected surface \((X, \mathcal{T}, l)\) with developing map \(\phi\) is said to be embeddable into \(\mathbb{C}\) if for every simply connected finite subcomplex \(P\) of \(\mathcal{T}\), there exists a sequence of flat PL metrics on \(P\) whose developing maps \(\phi_n\) converge uniformly to \(\phi|_P\) and \(\phi_n : P \to \mathbb{C}\) is an embedding.

For instance, all geometric triangulations of open sets in \(\mathbb{C}\) are embeddable. However, the spiral flat triangulations produced in Proposition 3.4 are not embeddable. The main result in this section works for embeddable flat PL metrics only.

The following lemma is a consequence of definition.

**Lemma 4.2.** Suppose \((X, \mathcal{T}, l)\) is a flat generalized PL metric on a simply connected surface with a developing map \(\phi\).

(a) Suppose \(\phi\) is embeddable. If \(t_1, t_2\) are two distinct non-degenerate triangles or two distinct edges in \(\mathcal{T}\), then \(\phi(\text{int}(t_1)) \cap \phi(\text{int}(t_2)) = \emptyset\).

(b) If \(\phi\) is the pointwise convergent limit \(\lim_{n \to \infty} \psi_n\) of the developing maps \(\psi_n\) of embeddable flat generalized PL metrics \((X, \mathcal{T}, l_n)\), then \((X, \mathcal{T}, l)\) is embeddable.

**Proof.** To see (a), if otherwise that \(\phi(\text{int}(t_1)) \cap \phi(\text{int}(t_2)) \neq \emptyset\), then \(\phi\) is not embeddable. Indeed, take \(P\) to be a finite simply connected subcomplex containing \(t_1\) and \(t_2\), then the developing maps \(\phi_n\) defined on \(P\) which converge uniformly to \(\phi|_P\) must satisfy \(\phi_n(\text{int}(t_1)) \cap \phi_n(\text{int}(t_2)) \neq \emptyset\) for \(n\) large. This contradicts that \(\phi_n\) are embedding.

Part (b) follows from the fact that \(\psi_n\) converges to \(\phi\) uniformly on compact subsets and the fact that if \(\lim_{n \to \infty} a_n = a\) and \(\lim_{m \to \infty} b_{n,m} = a_n\), then \(a = \lim_{j \to \infty} b_{j,n,j}\) for some subsequence.

Let \(\mathcal{T}_{st}\) be a hexagonal Delaunay triangulation of the plane \(S = \mathbb{C}\) with vertex set the lattice \(V = \{u_1n + u_2m|n, m \in \mathbb{Z}\}\) and \(l_0 : E(\mathcal{T}_{st}) \to \mathbb{R}_{>0}\) be the edge length function associated to \((S, \mathcal{T}_{st}, d_{st})\). Given a flat generalized PL metric \((S, \mathcal{T}_{st}, l)\), its normalized developing map \(\phi = \phi_1 : S \to \mathbb{C}\) is a developing map such that \(\phi(0) = 0\) and \(\phi(u_1)\) is in the positive x-axis. Suppose \(\{u_1, u_2\}\) is a geometric basis of the lattice \(u_1\mathbb{Z} + u_2\mathbb{Z}\). Two vertices \(v, v'\) are adjacent in \(\mathcal{T}_{st}\), i.e., \(v \sim v'\), if and only if \(v = v' + \delta\) for some \(\delta \in \{\pm u_1, \pm u_2, \pm (u_1 - u_2)\}\). Given two vertices \(v, v' \in V\), the combinatorial distance \(d_c(v, v')\) between \(v, v'\) is the length of the shortest edge path joining them.

The goal of this section is to prove the following stronger version of theorem 1.4.
**Theorem 4.3.** Suppose $(S, \mathcal{T}_{st}, l_0)$ is a hexagonal Delaunay triangulation whose vertex set is a lattice in $\mathbb{C}$ and $(S, \mathcal{T}_{st}, w*l_0)$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{st}$. If $(S, \mathcal{T}_{st}, w*l_0)$ is embeddable into $\mathbb{C}$, then $w$ is a constant function.

We will deduce Theorem 4.3 from Theorem 4.3 in §7. Theorem 4.3 will be proved using several lemmas.

### 4.1. Limits of discrete conformal factors.

The following lemma is a corollary of Theorem 3.1.

**Lemma 4.4.** Suppose $(S, \mathcal{T}_{st}, w*l_0)$ is a flat generalized Delaunay PL metric surface. Then for any $\delta \in V$ and $u : V \to \mathbb{R}$ defined by $u(v) = w(v + \delta) - w(v)$, $u*(w*l_0) = (u + w)*l_0$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{st}$. In particular, if $u(v_0) = \max\{u(v) | v \in V\}$, then $u$ is constant.

The next lemma shows how to produce discrete conformal factors $w$ such that $w(v + \delta) - w(v)$ are constants.

**Lemma 4.5.** Suppose $w*l_0$ is a flat generalized Delaunay PL metric on $\mathcal{T}_{st}$. Then for any $\delta \in \{\pm u_1, \pm u_2, \pm (u_1 - u_2)\}$, there exist $v_n \in V$ such that $w_n \in \mathbb{R}^V$ defined by $w_n(v) = w(v + v_n) - w(v_n)$ satisfies

(a) for all $v \in V$, the following limit exists

$$w_\infty(v) = \lim_{n \to \infty} w_n(v) \in \mathbb{R},$$

(b) $w_n*l_0$ and $w_\infty*l_0$ are flat generalized Delaunay PL metric on $\mathcal{T}_{st}$,

(c) $w_\infty(v + \delta) - w_\infty(v) = a$ for all $v \in V$ where $a = \sup\{w(v + \delta) - w(v) | v \in V\}$,

(d) the normalized developing maps $\phi_{w_n*l_0}$ of $w_n*l_0$ converges uniformly on compact sets in $S$ to the normalized developing map $\phi_\infty$ of $w_\infty*l_0$. In particular, if $(S, \mathcal{T}_{st}, w*l_0)$ is embeddable, then $(S, \mathcal{T}_{st}, w_\infty*l_0)$ is embeddable.

**Proof.** By Lemma 3.3, there is a constant $M = M(V)$ depending only on the lattice $V = u_1\mathbb{Z} + u_2\mathbb{Z}$ such that $a = \sup\{w(v + \delta) - w(v) | v \in V\} \leq M(V)$. Take $v_n \in V$ so that

$$w(v_n + \delta) - w(v_n) \geq a - \frac{1}{n}.$$

By definition,

$$w_n(0) = 0, \quad w_n(\delta) \geq a - \frac{1}{n}, \quad w_n(v + \delta) - w_n(v) \leq a,$$

and

$$\sup\{|w_n(v) - w_n(v')||v \sim v'| < \infty.$$

By Lemma 3.3 if $v \in V$ is of combinatorial distance $m$ to $0$, then, using $w_n(0) = 0$, we have

$$|w_n(v)| \leq mM(V).$$

By (18) and the diagonal argument, we see that there exists a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$ for simplicity, so that $w_n$ converges to $w_\infty \in \mathbb{R}^V$ in the pointwise convergent topology. By lemma 4.4, each $w_n*l_0$ is a flat generalized Delaunay PL metric. By $\lim_{n \to \infty} w_n = w_\infty$ and continuity, we conclude that $w_\infty*l_0$ is again a flat generalized Delaunay PL metric on $\mathcal{T}_{st}$. By (18),

$$w_\infty(\delta) - w_\infty(0) = \max\{w_\infty(v + \delta) - w_\infty(v) | v \in V\}.$$
By Lemma 4.4, we see that conclusion (c) holds. Since the developing map $\phi_{\omega*l_0}$ depends continuously on $\omega \in \mathbb{R}^V$, $\lim_{n \to \infty} \phi_{\omega_n*l_0}(v) = \phi_{\infty}(v)$ for each vertex $v \in V$. On the other hand, a developing map $\phi$ is determined by its restriction to $V$. We see that $\phi_{\omega_n*l_0}$ converges to $\phi_{\infty}$ uniformly on compact subsets of the plane. The last statement follows from Lemma 4.2(b) since each $\phi_{\omega_n*l_0}$ is embeddable by definition.

4.2. Proof of Theorem 4.3 Suppose $\omega*l_0$ is a flat generalized Delaunay PL metric on $T_{st}$ with an embeddable developing map $\phi$. Our goal is to show that $\omega : V \to \mathbb{R}$ is a constant. Suppose otherwise, we will derive a contradiction by showing that the developing map $\phi$ is not embeddable.

Since $\omega$ is not a constant, we can choose $\delta_1 \in \{\pm u_1, \pm u_2, \pm(u_1 - u_2)\}$ such that $a_1 = \sup\{w(v + \delta_1) - w(v) | v \in V\} > 0$. By Lemma 4.5 applied to $\omega*l_0$ and $\delta = \delta_1$, we produce a function $w_{\infty} : V \to \mathbb{R}$ so that $w_{\infty}*(l_0)$ is a flat generalized Delaunay PL metric on $T_{st}$ and $w_{\infty}(v + \delta_1) = w_{\infty}(v) + a_1$ for all $v \in V$. Now applying Lemma 4.5 to $w_{\infty}*l_0$ with $\delta_2 \in \{\pm u_1, \pm u_2, \pm(u_1 - u_2)\} - \{\pm \delta_1\}$, we obtain a second function $\hat{\omega} = (w_{\infty})_{\infty} : V \to \mathbb{R}$ and $b_1 \in \mathbb{R}$ such that $\hat{\omega}*(l_0)$ and $\hat{\omega} \in \mathbb{R}$ for all $v \in V$. This shows that $\hat{\omega} : V \to \mathbb{R}$ is a non-constant affine function, i.e., $\hat{\omega}(n + me\pi/3) = a_2n + b_2m + c_2$ for some $a_2, b_2, c_2 \in \mathbb{R}$.

Let $\phi, \phi_{\infty}$ and $\phi$ be the normalized developing maps for $\omega*l_0, w_{\infty}*l_0$ and $\omega*l_0$ respectively. Since $\phi$ is embeddable, by Lemmas 4.5, $\hat{\omega}$ and $\phi_{\infty}$ are embeddable.

If $\omega*l_0$ contains a non-degenerate triangle, then by Proposition 3.4, there exist two non-degenerate triangles $t_1$ and $t_2$ in $(T_{st}, \omega*l_0)$ so that $\phi(int(t_1)) \cap \phi(int(t_2)) \neq \emptyset$. By Lemma 4.2(a), this contradicts that $\hat{\omega}*(l_0)$ is embeddable.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{Angles $a$ and $z$ are zero in $\hat{\omega}*(l_0)$. Part (b) is the developing image of corresponding set in $\omega*l_0$.}
\end{figure}
Therefore all triangles in the generalized PL metric $\hat{w} * l_0$ are degenerate, i.e., all angles in triangles are either 0 or $\pi$. We will use the same notations used in the proof of Proposition 3.4. By Proposition 3.4(c) and Figure 6 we may assume, after composing with an automorphism of $T_{st}$ and subtracting by a constant, that $\hat{w}(nu_1 + mu_2) = n \ln(\gamma_1(V)) + m \ln(\gamma_2(V))$ where $(\gamma_1(V), \gamma_2(V))$ are given by the solutions of (17) and the angles $a, b, c, x, y, z$ of $T_1$ and $T_2$ are $(a, b, c, x, y, z) = (\pi, 0, 0, 0, 0, \pi)$.

Let $P_1 = u_2 - 2u_1$, $P_2 = u_2 - u_1$, $P_3 = 0$ and $P_4 = u_1$ in $V$. See Figure 6(c). In the case of $a = z = \pi$, we claim that the length $\lambda b_3$ of the edge $P_2P_3$ is strictly less than the sum of the lengths $\lambda b_1$ of the edge $P_3P_4$ and $\mu b_1$ of the edge $P_1P_2$, i.e.,

\[ \lambda b_3 < \lambda b_1 + \mu b_1. \]

Indeed, by the equations $\lambda b_1 + \mu b_2 = \lambda \mu b_3$ and $\mu b_1 = \lambda b_2 + b_3$ derived in the proof of Proposition 3.4, we obtain

\[ \frac{b_3}{b_1} = \frac{\lambda^2 + \mu^2}{(1 + \lambda^2)\mu}. \]

Equation (21) says

\[ \frac{b_3}{b_1} < \frac{\lambda^4 + \mu^2}{\lambda^2 \mu}. \]

Thus it suffices to show that $\frac{\lambda^2 + \mu^2}{(1 + \lambda^2)\mu} < \frac{\lambda^4 + \mu^2}{\lambda^2 \mu}$. This is the same as, $\lambda^2(\lambda^2 + \mu^2) < (1 + \lambda^2)(\lambda^4 + \mu^4)$, i.e., $\lambda^4 + \lambda^2 \mu^2 < \lambda^4 + \lambda^2 \mu^2 + \lambda^6 + \mu^2$. The last inequality clearly holds since both $\lambda$ and $\mu$ are positive.

Now consider the oriented edge path $P_1P_2P_3P_4$ (oriented from $P_2$ to $P_4$) in $T_{st}$ and its image under the developing map $\hat{\phi}$ of $\hat{w} * l_0$ in $C$. By the assumption that $a = z = \pi$, the angles of the polygonal path $\hat{\phi}(P_1P_2P_3P_4)$ at $\hat{\phi}(P_2)$ and $\hat{\phi}(P_3)$ are $2\pi$. See Figure 6(c). Also the sum of the lengths of $\hat{\phi}(P_1P_2)$ and $\hat{\phi}(P_2P_4)$ is larger than the length of $\hat{\phi}(P_2P_3)$ by the claim above. On the other hand, since $\hat{\phi}$ is embeddable, there exists a sequence of flat PL metrics on $T_{st}$ whose developing maps $\phi_n$ are embedding and $\phi_n$ converges uniformly on compact sets to $\hat{\phi}$. This implies, that for $n$ large the two line segments $\phi_n(P_1P_2)$ and $\phi_n(P_3P_4)$ intersect in their interiors. This contradicts the assumption that $\phi_n$ is an embedding.

This ends the proof of Theorem 4.3.

**Remark 4.6.** The above argument also gives a new proof of Rodin-Sullivan’s hexagonal circle packing theorem.

The following will be used to show that the limit of discrete uniformization maps is conformal. Let $B_n(v) = \{i \in V(T_{st}) | d_c(i, v) \leq n\}$ and $B_n(v)$ be the subcomplex of $T_{st}$ whose simplices have vertices in $B_n(v)$.

**Lemma 4.7.** Take the standard hexagonal lattice $V = \mathbb{Z} + e^{2\pi i/3} \mathbb{Z}$ and its associated standard hexagonal triangulation whose edge length function is $l : V \rightarrow \{1\}$. There is a sequence $s_n$ of positive numbers decreasing to zero with the following property. For any integer $n$ and a vertex $v$, there exists $N = N(n, v)$ such that if $m \geq N$ and $(B_m(v), w * l_0)$ is a flat Delaunay triangulated PL surface with embeddable developing map, then the ratio of the lengths of any two edges sharing a vertex in $B_m(v)$ is at most $1 + s_n$. 

The proof of the lemma is exactly the same as that of Rodin-Sullivan [34, pages 353-354] since we have Lemma 3.3 and Theorem 4.3 which play the roles of Rodin-Sullivan’s Ring Lemma and rigidity of hexagonal circle packing in [34, pages 352-353]).

5. Existence of Discrete Uniformization Metrics on Polyhedral Disks with Special Equilateral Triangulations

By a polygonal disk we mean a flat PL surface $(\mathcal{P}, V, d)$ which is isometrically embedded in the complex plane $\mathbb{C}$ and $\mathcal{P}$ is homeomorphic to the closed disk. The goal of this section is to prove the existence of a discrete conformal metric by regular subdividing the given triangulations.

An equilateral triangulation $\mathcal{T}$ of a polyhedral surface is a geometric triangulation whose triangles are equilateral. The edge length function of an equilaterally triangulated connected polyhedral surface will be denoted by the constant function $l_{st} : E(\mathcal{T}) \to \mathbb{R}$. Given an equilateral Euclidean triangle $\Delta \subset \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 1}$, the $n$-th standard subdivision of $\Delta$ is the equilateral triangulation of $\Delta$ by $n^2$ equilateral triangles. See Figure 7. If $\mathcal{T}$ is an equilateral triangulation of a polyhedral surface, its $n$-th standard subdivision, denoted by $\mathcal{T}(n)$, is the equilateral triangulation obtained by replacing each triangle in $\mathcal{T}$ by its $n$-th standard subdivision. We use $V(n)$ to denote $V(\mathcal{T}(n))$.

![Figure 7. The standard subdivisions](image)

The main result of this section is the following theorem.

**Theorem 5.1.** Suppose $(\mathcal{P}, \mathcal{T}, l_{st})$ is a flat polygonal disk with an equilateral triangulation $\mathcal{T}$ such that exactly three boundary vertices $p, q, r$ have curvature $\frac{2\pi}{3}$. Then for sufficiently large $n$, there is discrete conformal factor $w_n : V(n) \to \mathbb{R}$ for the $n$-th standard subdivision $(\mathcal{P}, \mathcal{T}(n), l_{st})$ such that the discrete curvature $K$ of $w_n \ast l_{st}$ satisfies

(a) $K_i = 0$ for all $i \in V(n) - \{p, q, r\}$,

(b) $K_i = \frac{2\pi}{3}$ for all $i \in \{p, q, r\}$, and

(c) there is a constant $\epsilon_0 > 0$ independent of $n$ such that all inner angles of triangles in $(\mathcal{T}(n), w_n \ast l_{st})$ are in the interval $[\epsilon_0, \pi/2 + \epsilon_0]$, the sum of two angles facing each interior edge is at most $\pi - \epsilon_0$ and each angle facing a boundary edge is at most $\pi/2 - \epsilon_0$.

Conditions (a) and (b) imply that the underlying metric space of $(\mathcal{P}, \mathcal{T}(n), w_n \ast l_{st})$ is an equilateral triangle. Condition (c) says that the metric doubles of $(\mathcal{P}, \mathcal{T}(n), l_{st})$ and $(\mathcal{P}, \mathcal{T}(n), w_n \ast l_{st})$ are two Delaunay triangulated polyhedral 2-spheres differing by a vertex scaling.

There are two steps involved in the construction of the discrete conformal factor $w_n$ in Theorem 5.1. In the first step, we produce a discrete conformal factor $w^{(1)} : V(n) \to \mathbb{R}$ such that $w^{(1)}$ vanishes outside the union of combinatorial balls of radius $\lfloor n/3 \rfloor$ (the integral part of $n/3$) centered at non-flat vertices $v \neq p, q, r$ and the discrete curvature $K_i(w^{(1)} \ast l_{st}) = 0$ if
formal factor

\[ w \]

\[ (22) \]
so that curvatures are very small, we use a perturbation argument to show that there is

\[ 5.1. \]
material.

\[ \text{gradient estimates and ordinary differential equations (ODE).} \]
We begin by recalling the related

d \[ \] at vertices defined by

\[ (Green’s identity) \]

The basic tools to be used for proving Theorem 5.1 are discrete harmonic functions, their

\[ \text{gradient estimates and ordinary differential equations (ODE).} \]
We begin by recalling the related material.

\textbf{5.1. Laplace operator on a finite graph.} Given a graph \((V, E)\), the set of all oriented edges in \((V, E)\) is denoted by \(\bar{E}\). If \(i \sim j\) in \(V\), we use \([ij]\) \(\in \bar{E}\) to denote the oriented edge from \(i\) to \(j\). If \(x \in \mathbb{R}^V\) and \(y \in \mathbb{R}^E\), we use \(x_i\) and \(y_{ij}\) to denote \(x(i)\) and \(y([ij])\) respectively. A conductance on \(G\) is a function \(\eta : \bar{E} \rightarrow \mathbb{R}_{\geq 0}\) so that \(\eta_{ij} = \eta_{ji}\).

\textbf{Definition 5.2.} Given a finite graph \((V, E)\) with a conductance \(\eta\), the gradient \(\nabla : \mathbb{R}^V \rightarrow \mathbb{R}^E\)
is the linear map

\[ (\nabla f)_{ij} = \eta_{ij}(f_i - f_j), \]

the Laplace operator associated to \(\eta\) is the linear map \(\triangle : \mathbb{R}^V \rightarrow \mathbb{R}^V\) defined by

\[ (\triangle f)_i = \sum_{j \sim i} \eta_{ij}(f_i - f_j), \]

and the Dirichlet energy of \(f \in \mathbb{R}^V\) on \((V, E, \eta)\) is

\[ \mathcal{E}(f) = \frac{1}{2} \sum_{i \sim j} \eta_{ij}(f_i - f_j)^2. \]

The following is well known (see \([7]\)).

\textbf{Proposition 5.3 (Green’s identity).} Given a finite graph \((V, E)\) with a conductance \(\eta\),

(a) for any subset \(V_0 \subset V\),

\[ \sum_{i \in V_0} f_i(\triangle g)_i - g_i(\triangle f)_i = \sum_{i \in V_0, j \sim i, j \notin V_0} \eta_{ij}(g_i f_j - f_i g_j). \]

(b) \(\sum_{i \in V}(\triangle f)_i = 0.\)

Given a set \(V_0 \subset V\) and \(g : V_0 \rightarrow \mathbb{R}\), the Dirichlet problem asks for a function \(f : V \rightarrow \mathbb{R}\)
so that

\[ (\triangle f)_i = 0, \forall i \in V - V_0, \text{ and } f|_{V_0} = g. \]

The Dirichlet principle states that solutions \(f\) to the Dirichlet problem \((22)\) are the same as minimum points of the Dirichlet energy function restricted to the affine subspace \(\{h \in \mathbb{R}^V | h|_{V_0} = g\}\), i.e.,

\[ \mathcal{E}(f) = \min \{\mathcal{E}(h) | h \in \mathbb{R}^V \text{ and } h|_{V_0} = g\}. \]

In particular, the Dirichlet problem \((22)\) is always solvable.

A subset \(U \subset V\) in a graph \((V, E)\) is called connected if any two vertices \(i, j \in U\) can be
joint by an edge path whose vertices are in \(U\). For instance, a connected graph \((V, E)\) means \(V\) is a connected. The following is well known (see \([7]\)).
Proposition 5.4. Suppose \((V, E)\) is a finite connected graph with a conductance \(\eta_{ij} > 0\) for all edges \([ij]\) and \(V_0 \subset V\). Let \(f\) be a solution to the Dirichlet problem \((22)\). Then,
(a) (Maximum principle) for \(V_0 \neq \emptyset\),
\[
\max_{i \in V} f_i = \max_{i \in V_0} f_i.
\]
(b) (Strong maximum principle) If \(V - V_0\) is connected and \(\max_{i \in V - V_0} f_i = \max_{i \in V_0} f_i\), then \(f|_{V-V_0}\) is a constant function.

5.2. A system of ODE associated to discrete conformal change. Let \((S, \mathcal{T}, l)\) be a compact connected polyhedral surface with discrete curvature \(K^0\). Given a subset \(V_0 \subset V\) and a function \(K^* : V - V_0 \to (-\infty, 2\pi)\), we try to find a function \(w : V \to \mathbb{R}\) such that \(w * l\) is a PL metric whose curvature \(K(w)\) is equal to \(K^*\) on \(V - V_0\) and \(w|_{V_0} = 0\). In the PL metric \(w * l\), let \(\theta^i_{jk} = \theta^i_{jk}(w)\) be the angle at vertex \(i\) in the triangle \(\Delta ijk\) and \(\eta_{ij} = \eta_{ij}(w)\) be \(\cot(\theta^k_{ij}) + \cot(\theta^l_{ij})\) if \([ij]\) is an interior edge and \(\eta_{ij} = \cot(\theta^s_{ij})\) if \([ij]\) is a boundary edge. The associated Laplacian \(\Delta : \mathbb{R}^V \to \mathbb{R}^V\) is \((\Delta f)_i = \sum_{j \sim i} \eta_{ij}(f_i - f_j)\). We will construct \(w\) by choosing a smooth 1-parameter family \(w(t) \in \mathbb{R}^V\) such that \(w(0) = 0\) and \(w(t) * l\) is a PL metric whose curvature \(K_i(t) = K_i(w(t) * l)\) satisfies
\[
\forall i \in V - V_0, \quad K_i(t) = (1 - t)K_i^0 + tK_i^*; \quad \text{and} \quad \forall i \in V_0, \quad w_i(t) = 0.
\]

The required vector \(w\) is defined to be \(w(1)\). Note that by definition \(K(0) = K^0\). Due to the curvature evolution equation \((7)\) that \(\frac{dK_i(t)}{dt} = \sum_{j \sim i} \eta_{ij}(w(t))(w'_i - w'_j)\) where \(w'_i(t) = \frac{d}{dt}w_i(t)\), we obtain the following system of ODE in \(w(t)\) equivalent to \((24)\):
\[
\forall i \in V - V_0, \quad \sum_{j \sim i} \eta_{ij}(w'_i - w'_j) = K^*_i - K_i^0; \quad \forall i \in V_0, \quad w'_i(t) = 0; \quad \text{and} \quad w(0) = 0.
\]

Using \(\Delta f\), we can write Equation \((25)\) as
\[
\forall i \in V - V_0, \quad (\Delta w'_i)_i = K^*_i - K_i^0; \quad \forall i \in V_0, \quad w'_i(t) = 0; \quad \text{and} \quad w(0) = 0.
\]

We will show, under some assumptions on \((\mathcal{T}, l)\), that the solution to \((25)\) exists for all \(t \in [0, 1]\).

Let \(W \subset \mathbb{R}^V\) be the open set
\[
W = \{w \in \mathbb{R}^V | w * l\ is a PL metric on \(\mathcal{T}\) and \(\eta_{ij}(w) > 0\) for all edges \([ij]\)\}.
\]

Lemma 5.5. Suppose \(V_0 \neq \emptyset\) and \(0 \in W\). The initial valued problem \((25)\) defined on \(W\) has a unique solution in a maximum interval \([0, t_0)\) with \(t_0 > 0\) such that if \(t_0 < \infty\), then either \(\lim_{t \to t_0^-} \theta^i_{jk}(w(t)) = 0\) for some angle \(\theta^i_{jk}\) or \(\lim_{t \to t_0^-} \eta_{ij}(w(t)) = 0\) for some edge \([ij]\).

Proof. Indeed Equation \((25)\) can be written as \(Y(w) \cdot w'(t) = \beta\) and \(w(0) = 0\) where \(Y(w)\) is a square matrix valued smooth function of \(w \in W\) and \(w'(t)\) is considered as a column vector. We claim that \(Y(w)\) is an invertible matrix for \(w \in W\). If \(Y(w)\) is invertible, then \((25)\) can be written as \(w'(t) = Y(w)^{-1} \beta\) and by the Picard’s existence theorem, there exists an interval on which the ODE \((25)\) has a solution. Now \(Y(w)\) is invertible if and only if the following system of linear equations has only trivial solution \(x = 0\),
\[
Y(w) \cdot x = 0.
\]

By \((25)\), Equation \((28)\) is the same as \((\Delta x)_i = 0\) for \(i \in V - V_0\) and \(x_i = 0\) for \(i \in V_0\). Furthermore \(w \in W\) implies \(\eta_{ij}(w) > 0\) for all edges \([ij]\). By the maximum principle (Proposition 5.4), we see that \(x = 0\).
If $t_0 < \infty$ and $t \uparrow t_0$, then $w(t)$ leaves every compact set in $W$. For each $\delta > 0$, we claim that $W_{\delta} = \{ w \in W | \theta_{jk} \geq \delta, |w_i| \leq \frac{1}{\delta}, \eta_{ij} \geq \delta \}$ is compact. Clearly $W_\delta$ is bounded by definition. To see that $W_\delta$ is closed in $\mathbb{R}^V$, take a sequence $x_n \in W_\delta$ such that $\lim_{n \to \infty} x_n = y \in \mathbb{R}^V$. Then $y \ast l$ is a generalized PL metric with all angles $\theta_{jk} \geq \delta$. Since each degenerate triangle has an angle which is zero, therefore $y \ast l$ is a PL metric. Also by continuity, we have $\theta_{jk}(y) \geq \delta$, $\eta_{ij}(y) \geq \delta$ and $|y_i| \leq \frac{1}{\delta}$, i.e., $y \in W_\delta$. Since $w(t)$ leaves every $W_\delta$ for each $\delta > 0$, one of the following three occurs: $\liminf_{t \to t_0^+} \theta_{jk}(w(t)) = 0$ for some $\theta_{jk}$, or $\liminf_{t \to t_0^-} \eta_{ij}(w(t)) = 0$ for some edge $[ij]$, or $\limsup_{t \to t_0^-} |w_i(t)| = \infty$ for some $i \in V$. However $\lim \sup_{t \to t_0^-} |w_{i_0}(t)| = \infty$ for one vertex $i_0$ implies that $\liminf_{t \to t_0^-} \theta_{jk}(w(t)) = 0$ for some $\theta_{jk}$. Indeed, if otherwise, $\liminf_{t \to t_0^-} \theta_{jk}(w(t)) \geq \delta > 0$ for all $\theta_{jk}$ for some $\delta$. It is well known that in a Euclidean triangle whose angles are at least $\delta$, the ratio of two edge lengths is at most $\frac{1}{\sin(\delta)}$. Therefore, in each triangle $\Delta v_i v_j v_k$ in $\mathcal{T}$, we have $e^{w_i(t)} \leq e^{w_j(t)} \frac{d(v_i v_k)}{d(v_i v_k) \sin(\delta)}$. Since $w_j(t) = 0$ for $j \in V_0$ and the surface $S$ is connected, we conclude that all $w_k(t), k \in V$, are bounded for all $t$. This contradicts $\lim \sup_{t \to t_0^-} |w_{i_0}(t)| = \infty$. 

\section{5.3. Standard subdivision of an equilateral triangle.}

\textbf{Theorem 5.6.} Let $S = \Delta ABC$ be an equilateral triangle, $\mathcal{T}$ be the $n$-th standard subdivision of $S$ with the associated PL metric $l_{st} : V = V(\mathcal{T}) \to \{1\}$ and $V_0 = \{ v \in V | v$ is in the edge BC of the triangle $\Delta ABC \}$. Given any $\alpha \in \left[ \frac{\pi}{3}, \frac{\pi}{2} \right]$, there exists a smooth family of vectors $w(t) \in \mathbb{R}^V$ for $t \in [0, 1]$ such that $w(0) = 0$ and $w(t) \ast l_{st}$ is a PL metric on $\mathcal{T}$ with curvature $K(t) = K(w(t) \ast l_{st})$ satisfying,

(a) $K_A(t) = -\alpha + (2 + t) \frac{\pi}{3}$ (angle at $A$ is $\alpha + (1 - t) \frac{\pi}{3}$),

(b) $K_i(t) = 0$ for all $i \in V \setminus \{ A \} \cup V_0$,

(c) $w_i(t) = 0$ for all $i \in V_0$,

(d) all inner angles $\theta_{jk}(t)$ in metric $w(t) \ast l_{st}$ are in the interval $\left[ \frac{\pi}{3} - |\alpha - \frac{\pi}{3}|, \frac{\pi}{3} + |\alpha - \frac{\pi}{3}| \right] \subset \left[ \frac{\pi}{6}, \frac{\pi}{2} \right]$,

(e) $\theta_{jk}(t) \leq \frac{50\pi}{120}$ for $i \neq A$,

(f) $|K_i(t) - K_i(0)| \leq \frac{2000}{\sqrt{\ln(n)}}$ for $i \neq A$ and

where

\begin{equation}
\sum_{i \in V_0} |K_i(t) - K_i(0)| \leq \frac{\pi}{6}.
\end{equation}

\textbf{Remark 5.7.} The discrete conformal map from $(\Delta ABC, \mathcal{T}, l_{st})$ to $(\Delta ABC, \mathcal{T}, w(1) \ast l_{st})$ is a discrete counterpart of the analytic function $f(z) = z^{3\alpha/\pi}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{discrete_conformal_maps.png}
\caption{Discrete conformal maps of equilateral triangles and their unions}
\end{figure}
Our proof of Theorem 5.6 relies on the following two lemmas about estimates on discrete harmonic functions on $\mathcal{T}$.

**Lemma 5.8.** Assume $\Delta ABC, n, \mathcal{T}, V_0$ are as given in Theorem 5.6. Let $\tau : \mathcal{T} \to \mathcal{T}$ be the involution induced by the reflection of $\Delta ABC$ about the angle bisector of $\angle BAC$ and $\eta : E \to \mathbb{R}_{\geq 0}$ be a conductance so that $\eta \tau = \eta$ and $\eta_{ij} = \eta_{ji}$. Let $\Delta : \mathbb{R}^V \to \mathbb{R}^V$ be the Laplace operator defined by $(\Delta f)_i = \sum_{j \sim i} \eta_{ij} (f_i - f_j)$. If $f \in \mathbb{R}^V$ satisfies $(\Delta f)_i = 0$ for $i \in V - \{A\} \cup V_0$ and $f|_{V_0} = 0$, then for all edges $[ij]$, the gradient $(\nabla f)_{ij} = \eta_{ij} (f_i - f_j)$ satisfies

$$|\eta_{ij} (f_i - f_j)| \leq \frac{1}{2} |\Delta (f)_A|.$$  

**Lemma 5.9.** Assume $\Delta ABC, n, \mathcal{T}, V_0$ are as given in Theorem 5.6. Let $\eta : E(\mathcal{T}) \to [\frac{1}{\sqrt{M}}, M]$ be a conductance function for some $M > 0$ and $\Delta$ be the Laplace operator on $\mathbb{R}^V$ associated to $\eta$. If $f : V \to \mathbb{R}$ solves the Dirichlet problem $(\Delta f)_i = 0, \forall i \in V - \{A\} \cup V_0, f|_{V_0} = 0$ and $(\Delta f)_A = 1$, then for all $u \in V_0$, $|\Delta f| \leq \frac{20M}{\sqrt{\ln n}}$.

We will prove Lemma 5.8, Lemma 5.9 and Theorem 5.6 in order.

The simplest way to see Lemma 5.8 is to use the theory of electric network. We put a resistance of $\frac{1}{\eta_{ij}}$ Ohms at the edge $[ij]$ (if $\eta_{ij} = 0$, the resistance is $\infty$, or remove edge $[ij]$ from the network). Now place a one-volt battery at vertex $A$ and ground every vertex in $V_0$. Then Kirchhoff’s laws show that the voltage $f_i$ at the vertex $i$ solves the Dirichlet problem $(\Delta f)_i = 0$ for all $i \in V - \{A\} \cup V_0, f_A = 1$ and $f|_{V_0} = 0$. Ohm’s law says $\eta_{ij} (f_i - f_j)$ is the electric current through the edge $[ij]$. Since the resistance is symmetric with respect to the symmetry $\tau$, the currents in the network are the same as the currents in the quotient network $\mathcal{T}/\tau$. In the quotient network $\mathcal{T}/\tau$, there is only one edge $e_A$ from the vertex $A$. Therefore, the current through any edge is at most the current $\frac{1}{2} |\Delta f|_A$ through $e_A$ (in the network $\mathcal{T}/\tau$). This shows $|\eta_{ij} (f_i - f_j)| \leq \frac{1}{2} |\Delta (f)_A|$.

**Proof of lemma 5.8.** Removing all edges $[ij]$ for which $\eta_{ij} = 0$ from the graph $(V, E)$, we obtain a finite collection of disjoint connected subgraphs $\Gamma_1, ..., \Gamma_N$ from $(V, E)$. By construction, the associated Laplace operators on $\Gamma_i$ with conductance $\eta|_{E(\Gamma_i)}$ is the restriction of the Laplace operator $\Delta$ to $V(\Gamma_i)$. By the maximum principle (Proposition 5.4), the function $f|_{V(\Gamma_m)}$ is a constant and (30) holds unless $\Gamma_m$ contains the vertex $A$ and some vertex in $V_0$. Therefore, it suffices to prove the lemma for those edges $[ij]$ in the connected graph $\Gamma_m = (V', E')$ such that $A \in V'$ and $V' \cap V_0 \neq \emptyset$. Let $A_1, A_2 = \tau(A_1)$ be the vertices adjacent to $A$. Since $\tau(A) = A$, $\eta \tau = \eta$ and $V' \cap V_0 \neq \emptyset$, we have $\tau(\Gamma_m) = \Gamma_m$ and $A_1, A_2 \in V'$.

We will work on the graph $\Gamma_m = (V', E')$ from now on. Using the maximum principle for $f - f \tau$, we see that $f = f \tau$. By replacing $f$ by $-f$ if necessary, we may assume that $f_A > 0$. By the maximum principle, we have that $0 \leq f_i < f_A$ for all $i \in V' - \{A\}$.

Take an edge $[ij]$ in the graph $\Gamma_m$. If $\tau(i,j) \neq (i,j)$, then $\tau_i = j$ and $\tau_j = i$. This implies $f_i = f \tau_i = f_j$ and (30) holds. If $\tau(i,j) = (i', j') \neq (i,j)$, say $\tau_i = i', \tau_j = j'$, then $f_i = f_{i'}, f_j = f_{j'}$. We may assume that $f_i \leq f_j$. If $f_i = f_j$, then (30) holds. Hence we may assume $f_i < f_j$. If $j = A$, then $i = A_1$ or $A_2$. Due to $f_{A_1} = f_{A_2}$, then (30) holds. If $j \neq A$, then by the maximum principle applied to $f$ on the subgraph $(V' - \{A\}, E' - \{AA_1, AA_2\})$, we conclude that $f_{A_1} \geq f_j$. Let $U = \{k \in V' - \{A\} | f_k > f_j\}$. By definition, $j, j', A_1, A_2 \in U, i, i', A \notin U$, and $V_0 \cap U = \emptyset$. This shows $(\Delta f)_k = 0$ for all $k \in U$ and hence
\[ \sum_{k \in U} (\Delta f)_k = 0. \] By Green’s formula (5.3),
\[ \sum_{k \in U} (\Delta f)_k = \sum_{k \in U, l \notin U, k \sim l} \eta_{kl} (f_k - f_l) = 0. \]

If \( l \notin U \cup \{A\} \), then by definition \( f_i \geq f_l \). Therefore, if \( k \in U \), \( k \sim l \), and \( l \notin U \cup \{A\} \), then \( f_k > f_i \geq f_l \). This shows,
\[ 0 = \sum_{k \in U, l \notin U, k \sim l} \eta_{kl} (f_k - f_l) \]
\[ = \sum_{k \in U, l \notin U, k \sim l} \eta_{kl} (f_k - f_l) + \sum_{k \sim A} \eta_{kA} (f_k - f_A) \]
\[ \geq (\nabla f)_{jj} + (\nabla f)_{j'j'} - (\Delta f)_A. \]

Therefore, \( |(\Delta f)_A| \geq 2|\nabla f|_{ij} \) since \( (\nabla f)_{ij} = (\nabla f)_{j'j'} \).

**Proof of Lemma 5.9** For the given \( u \in V_0 \), construct a function \( g : V \to \mathbb{R} \) by solving the Dirichlet problem: \( (\Delta g)_i = 0, \forall i \in V - V_0 \), \( g_u = 1 \) and \( g|_{V_0 - \{u\}} = 0 \). By the maximum principle (Proposition 5.4), \( 0 \leq g_i \leq 1 \) for all \( i \). Using Green’s identity that \( \sum_{i \in V} [f_i (\Delta g)_i - g_i (\Delta f)_i] = 0 \), we obtain \( g_A (\Delta f)_A + g_u (\Delta f)_u = 0 \). Since \( (\Delta f)_A = 1 \) and \( g_u = 1 \), we see
\[ (\Delta f)_u = -g_A. \]

**Figure 9. Layers in triangle ABC**

Therefore, it suffices to show that \( |g_A| \leq \frac{20M}{\sqrt{\ln n}} \). For this purpose, take \( k \leq \left[ \frac{n}{2} \right] \) and define
\( U_k = \{ i \in V | d_c(i, u) = k \} \) where \( d_c(i, j) \) is the combinatorial distance in the graph \( T^{(1)} \). Let \( G_k \) be the subgraph of \( T^{(1)} \) whose edges are \([ij]\) where \( i, j \in U_k \). Due to \( k \leq \left[ \frac{n}{2} \right] \), \( U_k \cap V_0 \neq \emptyset \), and \( G_k \) is topologically an arc. By the maximum principle applied to \( g \) on the subgraph whose edges consist of \([ij]\) with \( i, j \in \{ v \in V | d_c(v, u) \geq k \} \), we obtain \( g_A \leq \max_{i \in U_k} g_i \). Let \( v_k \in U_k \) such that \( g_{v_k} = \max_{i \in U_k} g_i \) and edge path \( E_k \) be the shortest edge path in \( G_k \) joining \( v_k \) to a point \( u_k \) in \( V_0 - \{u\} \). By construction \( g_{u_k} = 0 \). Since \( U_k \) contains at most \( 3k + 1 \) vertices, the length of \( E_k \) is at most \( 3k \). The Dirichlet energy \( E(g) \) of \( g \) on \( T^{(1)} \) is given by
\[ E(g) = \frac{1}{2} \sum_{i \sim j} \eta_{ij} (g_i - g_j)^2 \geq \sum_{k=1}^{[n/2]} E_k, \]
where
\begin{equation}
E_k = \frac{1}{2} \sum_{[ij] \in \bar{E}_k} \eta_{ij} (g_i - g_j)^2,
\end{equation}
and \(\bar{E}_k\) be the set of oriented edges in \(E_k\). Suppose \(w_0 = v_k \sim w_1 \sim w_2 \sim \ldots \sim w_{l_k} = u_k\) are the vertices in the edge path \(E_k\) where \(l_k \leq 3k\). Using the Cauchy-Schwartz inequality, we obtain
\begin{equation}
E_k = \sum_{i=1}^{l_k} \eta_{w_i w_{i-1}} (g_{w_i} - g_{w_{i-1}})^2
\geq \frac{1}{M} \sum_{i=1}^{l_k} (g_{w_i} - g_{w_{i-1}})^2
\geq \frac{1}{Ml_k} \left[ \sum_{i=1}^{l_k} (g_{w_i} - g_{w_{i-1}}) \right]^2
\geq \frac{1}{3kM} (g_{v_k} - g_{u_k})^2 = \frac{g_{v_k}^2}{3kM} \geq \frac{g_{v_k}^2}{3kM}.
\end{equation}

By (5.3) and (32), we obtain
\begin{equation}
E(g) \geq \frac{g_A^2}{3M} \sum_{k=1}^{\lceil \frac{n}{2} \rceil} \frac{1}{k} \geq \frac{g_A^2 \ln(n)}{100M}.
\end{equation}

On the other hand, the Dirichlet principle says \(E(g) = \min_{h \in \mathbb{R}^V} \{ \frac{1}{2} \sum_{i \sim j} \eta_{ij} (h_i - h_j)^2 | h_u = 1, h_{|V_0 - \{u\}} = 0 \} \). Take \(h \in \mathbb{R}^V\) to be \(h_u = 1\) and \(h_i = 0\) for all \(i \in V - \{u\}\). We obtain
\[E(g) \leq \frac{1}{2} \sum_{i \sim j} \eta_{ij} (h_i - h_j)^2 \leq 4M.\]
Combining this with (33), we obtain
\[\frac{g_A^2 \ln(n)}{100M} \leq 4M,
\]
i.e.,
\[g_A \leq \frac{20M}{\sqrt{\ln(n)}}.
\]

**Proof of Theorem 5.6** We construct the smooth family \(w(t) \in \mathbb{R}^V\) by solving the system of ordinary differential equations (25) where \((S, T, l) = (\Delta ABC, T, l_{st}), K_{*}|_{V\sim V_0\cup\{A\}} = 0, K_{*} = \pi - \alpha\) and \(w_i(t) = 0\) for \(i \in V_0\). By the assumption that \(\theta_{ijk}(0) = \frac{\pi}{3}\) (i.e., \(T\) is an equilateral triangulation), \(0 \in W\) where the space \(W\) is defined by (27). By Lemma 5.5 there exists a maximum \(s > 0\) such that a solution \(w(t)\) to (25) exists and condition (d) holds for all \(t \in [0, s]\). We claim that \(s \geq 1, w(1)\) exists and \(w(1) * l_{st}\) is a PL metric. In particular, \(w(1) * l_{st}\) satisfies condition (d) and \(w(1) \in W\). Without loss of generality, let us assume that \(s < \infty\). By lemma 5.5 and condition (d), we obtain the following two conclusions:
\begin{equation}
\liminf_{t \to s^{-}} \eta_{ij}(w(t)) = 0 \text{ for some } [ij], \quad \text{or } \limsup_{t \to s^{-}} |\theta_{ijk}(w(t)) - \frac{\pi}{3}| = |\alpha - \frac{\pi}{3}| \text{ for some } \theta_{ijk}.
\end{equation}
The conclusion \( \lim_{t \to s^-} \theta_{jk}(w(t)) = 0 \) is ruled out by condition (d) which implies \( \theta_{jk}(w(t)) \geq \frac{\pi}{6} \).

We prove the claim that \( s \geq 1 \) as follows. Since \( \alpha \in [\frac{\pi}{6}, \frac{\pi}{2}] \), we have \( \frac{\pi}{3} + |\alpha - \frac{\pi}{3}| \leq \frac{\pi}{2} \) and \( \frac{\pi}{3} - |\alpha - \frac{\pi}{3}| \geq \frac{\pi}{6} \). This shows, by (d),

\[
\theta_{jk}(t) \in \left[ \frac{\pi}{6}, \frac{\pi}{2} \right] \quad \text{for all } t \in [0, s).
\]

In particular, \( \cot(\theta_{ik}^k) \geq 0 \) and \( \eta_{ij} \geq \cot(\theta_{ij}^k) \geq 0 \). Hence by definition we have

\[
|((\nabla w')_{ij}| = \eta_{ij}|w_i' - w_j'| \geq \cot(\theta_{ij}^k)|w_i' - w_j'|.
\]

By Lemma 5.8 and the variation formula (7) that \( \frac{dK}{dt} = (\Delta w')_i \), we obtain

\[
2|((\nabla w')_{ij}| \leq |(\Delta w')_A| = |\frac{dK_A}{dt}| = |\alpha - \frac{\pi}{3}|.
\]

This implies, by (6), the following,

\[
|\frac{d\theta_{ij}}{dt}k| \leq \cot(\theta_{ij}^k)|w_i' - w_j'| \leq |((\nabla w')_{jk}| + |(\nabla w')_{ik}| \leq |\alpha - \frac{\pi}{3}|.
\]

Therefore, for all \( t \in [0, s) \),

\[
|\theta_{ij}^k(t) - \frac{\pi}{3}| = |\theta_{ij}^k(t) - \theta_{ij}^k(0)| \leq \int_0^t |\frac{d\theta_{ij}^k}{dt}| \, dt \leq t |\alpha - \frac{\pi}{3}| \leq s |\alpha - \frac{\pi}{3}|.
\]

The above inequality shows that \( s \geq 1 \). Indeed, if otherwise that \( s < 1 \), using (37), we conclude that \( \theta_{ij}^k(t) \in [\frac{\pi}{3} - s |\alpha - \frac{\pi}{3}|, \frac{\pi}{3} + s |\alpha - \frac{\pi}{3}|] \). In particular, \( \lim_{t \to s^-} \eta_{ij}(t) \geq \cot(\frac{\pi}{3} + s |\alpha - \frac{\pi}{3}|) > 0 \) and \( \limsup_{t \to s^-} |\theta_{ij}^k(t) - \pi/3| < |\alpha - \pi/3| \). This contradicts (34).

To see part (e), by (37), if \( t \in [0, \frac{1}{2}] \), we have

\[
|\theta_{ij}^k(t) - \frac{\pi}{3}| \leq \frac{1}{2} |\alpha - \frac{\pi}{3}| \leq \frac{\pi}{12}, \quad \text{i.e.,} \quad \theta_{ij}^k(t) \in \left[ \frac{\pi}{12}, \frac{5\pi}{12} \right].
\]

Now if \( [ij] \) is an interior edge, then for \( t \in [0, \frac{1}{2}] \)

\[
|((\nabla w')_{ij}| = (\cot(\theta_{ij}^k) + \cot(\theta_{ij}^k))|w_i' - w_j'| \\
\geq (1 + \frac{\cot(\theta_{ij}^k)}{\cot(\theta_{ij}^k)}) \cot(\theta_{ij}^k)|w_i' - w_j'| \\
\geq (1 + \cot(\frac{5\pi}{12})) \cot(\theta_{ij}^k)|w_i' - w_j'| \\
\geq \frac{5}{4} \cot(\theta_{ij}^k)|w_i' - w_j'|.
\]

If \( \theta_{jk}^i \) is an angle with \( i \neq A \), then either one of the two edges \( [ij], [ik] \) is an interior edge, or \( i \in \{B, C\} \). In the first case, say \( [ij] \) is an interior edge, using (38) and Lemma 5.8 for \( t \in [0, 1/2] \), we have

\[
|\frac{d\theta_{jk}^i}{dt}| \leq \cot(\theta_{ij}^k)|w_i' - w_j'| + \cot(\theta_{ik}^j)|w_i' - w_k'| \\
\leq \frac{4}{5} |((\nabla w')_{ij}| + |(\nabla w')_{ik}| \\
\leq (\frac{4}{5} + 1) |(\Delta w')_A| = \frac{9}{10} |\alpha - \pi/3| \leq \frac{9}{10} \cdot \frac{\pi}{6} = \frac{3\pi}{20}.
\]
In the second case that $i \in \{B, C\}$, one of the edges $[ij]$ or $[ik]$, say $[ij]$ is in the edge $BC$ of $\Delta ABC$, i.e., $w'_i = w'_j = 0$. Therefore by Lemma 5.8 for $t \in [0, 1/2]$, we have

$$|d\theta_{jk}^i dt| \leq \cot(\theta_{ij}^i)|w'_i - w'_j| + \cot(\theta_{ik}^i)|w'_i - w'_k| \leq |(\nabla w')_{ik}|$$

(40)

$$\leq \frac{|(\nabla w')_{A}|}{2} = \frac{1}{2}|\alpha - \pi/3| \leq \frac{3\pi}{20}.$$

Therefore if $\theta_{jk}^i$ is not the angle at $A$ and $t \in [0, 1]$, by (39) and (5.3), we have

$$|\theta_{jk}^i(t) - \pi/3| = |\theta_{jk}^i(t) - \theta_{jk}^i(0)| \leq \int_0^t |d\theta_{jk}^i dt| dt \leq \int_0^1 |d\theta_{jk}^i dt|dt = \int_0^{1/2} |d\theta_{jk}^i dt|dt + \int_1^{1/2} |d\theta_{jk}^i dt|dt$$

$$\leq \frac{1}{2} \cdot \frac{3\pi}{20} + \frac{1}{2}|\alpha - \pi/3| \leq \frac{3\pi}{40} + \frac{1}{2} \cdot \frac{\pi}{6} = \frac{19\pi}{120}.$$

Therefore, $\theta_{jk}^i(t) \in [\frac{2\pi}{120}, \frac{59\pi}{120}] \subset \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ for all $t \in [0, 1]$. Since conditions (d) and (e) hold for all $t \in [0, 1]$, by definition of $\eta_{ij}$, we see $\liminf_{t \to 1} \eta_{ij}(w(t)) > 0$. Now we prove that $w(1)$ is defined and $w(1) * l_{st}$ is a PL metric. By the estimates above, there exists $\delta > 0$ such that for all $t \in [0, 1]$, $w(t) \in \mathcal{W}_\delta = \{w \in W|\theta_{ij}^t \geq \delta, \eta_{ij} \geq \delta\}$. By Lemma 5.5, the maximum time $t_0$ for which $w(t)$ exists on $[0, t_0)$ must be greater than 0. Therefore, $w(1)$ exists and $w(1) \in W$. Since (d) and (e) are closed conditions, it follows that $w(1) * l_{st}$ satisfies (d) and (e).

Now we prove part (f). By parts (d) and (e), we have $\theta_{jk}^i(t) \in [\frac{\pi}{6}, \frac{59\pi}{120}]$ for $i \neq A$ and $\theta_{jk}^A \in [\frac{\pi}{6}, \frac{\pi}{2}]$. Since the conductance $\eta_{ij}$ is either $\cot(\theta_{ij}^i)$ or a sum $\cot(\theta_{ij}^i) + \cot(\theta_{ij}^j)$, we obtain for all edges $[ij]$ in $\mathcal{T}$, $\eta_{ij}(t) \in [\cot(\frac{59\pi}{120}), 2\cot(\frac{\pi}{6})] \subset [\frac{1}{100}, 100]$. Let $K_i(t)$ be the curvature of the metric $w(t) * l_{st}$ at the vertex $i$. By Lemma 5.9 for $f = \frac{1}{|\alpha - \pi/3|} \frac{dw(t)}{dt}$ and $M = 100$, we conclude that for all $i \in V_0$,

$$\left|\frac{dK_i(t)}{dt}\right| = |(\Delta w')_i| \leq \frac{2000|\alpha - \pi/3|}{\sqrt{\ln(n)}} \leq \frac{2000}{\sqrt{\ln(n)}}.$$

Therefore, $|K_i(t) - K_i(0)| \leq \int_0^t |dK_i(t)/dt| dt \leq \int_0^1 |dK_i(t)/dt| dt \leq \frac{2000}{\sqrt{\ln(n)}}$.

Finally to prove (29), if $\alpha = \pi/3$, then all $w(t) = 0$ and $K(t) = K(0)$ and the result follows. If $\alpha \neq \pi/3$, we first claim that $w_A'(t) \neq 0$ for each $t$. Indeed, if otherwise that $w_A'(t_1) = 0$ for some $t_1$, then by the maximum principle applied to the Dirichlet problem: $(\Delta w')(t_1)_i = 0$ for $i \in V - \{A\} \cup V_0$ and $w'(t_1)_i = 0$ for $i \in V_0 \cup \{A\}$, we conclude $w'_i(t_1) = 0$ for all $i \in V$. In particular, $\alpha - \pi/3 = (\Delta w')_A = 0$ at $t = t_1$ which is a contradiction. Therefore $w'_A(t_1) \neq 0$ and by the maximum principle again $w'_A(t_1)w'_A(t) \geq 0$. Now if $i \in V_0$, then $K_i'(t) = \sum_{j \neq i} \eta_{ji}(w'_i - w'_j) = -\sum_{j \neq i} \eta_{ji}w'_j$. Since $\eta_{ij} \geq 0$, therefore $w'_A(t)K_i'(t) \leq 0$ for $i \in V_0$. It follows that for all $i \in V_0$, $(K_i(t) - K_i(0))w'_A(t) \leq 0$. At the vertex $A$, $|K_A(t) - K_A(0)| = t|\alpha - \pi/3| \leq \frac{\pi}{6}$. Therefore by the Gauss-Bonnet theorem that $K_A(t) + \sum_{i \in V_0} K_i(t) = K_A(t) + \sum_{i \in V} K_i(t) = 2\pi$ and that $K_i(t) - K_i(0)$ have the same signs for $i \in V_0$, we obtain $\sum_{i \in V_0} |K_i(t) - K_i(0)| = |\sum_{i \in V_0} (K_i(t) - K_i(0))| = |K_A(t) - K_A(0)| \leq \frac{\pi}{6}$.\[\square\]
5.4. A gradient estimate of discrete harmonic functions. The proof Theorem \[5.1\] is based on the following estimate. Given a triangulated surface \((S, T), v \in V(T)\) and \(r > 0\), we use \(B_r(v) = \{ j \in V(T) | d_{c}(j, v) \leq r \}\) to denote combinatorial ball of radius \(r\) centered at the vertex \(i\) where \(d_{c}\) is the combinatorial distance on \(T^{(1)}\).

**Proposition 5.10.** Suppose \((P, T', l)\) is polygonal disk with an equilateral triangulation and \(T\) is the \(n\)-th standard subdivision of the triangulation \(T'\) with \(n \geq e^{10^{6}}\). Let \(\eta : E = E(T) \rightarrow [\frac{1}{M}, M]\) be a conductance function and \(\Delta : \mathbb{R}^{V} \rightarrow \mathbb{R}^{V}\) be the associated Laplace operator. Let \(V_{0} \subset V(T)\) be a thin subset such that for all \(v \in V\) and \(m \leq n/2\), \(|B_{m}(v) \cap V_{0}| \leq 10m\).

If \(f : V \rightarrow \mathbb{R}\) satisfies \((\Delta f)_{i} = 0\) for \(i \in V - V_{0}\), \(|(\Delta f)_{i}| \leq \frac{M}{\sqrt{\ln(n)}}\) for \(i \in V_{0}\) and \(\sum_{i \in V_{0}} |(\Delta f)_{i}| \leq M\), then for all edges \([u, v]\) in \(T\),

\[
|f_{u} - f_{v}| \leq \frac{200M^{3}}{\sqrt{\ln(n)}}.
\]

**Proof.** Fix an edge \([u, v]\) in the triangulation \(T\). Construct a function \(g : V = V(T) \rightarrow \mathbb{R}\) by solving the Dirichlet problem \((\Delta g)_{i} = 0\) for \(i \neq u, v\), and \(g_{u} = 1, g_{v} = 0\). By the maximum principle, we have \(0 \leq g_{i} \leq 1\). By the identity \(\sum_{i \in V} (\Delta g)_{i} = 0\) and that \(g\) is not a constant, we obtain \((\Delta g)_{u} = -(\Delta g)_{v} \neq 0\). Using the Green’s identity that \(\sum_{i \in V} (f_{i}(\Delta g)_{i} - g_{i}(\Delta f)_{i}) = 0\) and the assumptions of \(f, g\), we obtain

\[
f_{u}(\Delta g)_{u} + f_{v}(\Delta g)_{v} = \sum_{i \in V_{0}} g_{i}(\Delta f)_{i} = 0.
\]

Since \((\Delta g)_{v} = -(\Delta g)_{u}\), this shows

\[
f_{u} - f_{v} = \frac{1}{(\Delta g)_{u}} \sum_{i \in V_{0}} g_{i}(\Delta f)_{i}.
\]

On the other hand, by the maximum principle \(g_{u} - g_{j} \geq 0\), we have \(|(\Delta g)_{u}| = |\sum_{j \sim u} \eta_{j u} (g_{j} - g_{u})| = \sum_{j \sim u} \eta_{j u} (g_{j} - g_{u}) \geq \frac{1}{M} (g_{u} - g_{v}) = \frac{1}{M} \). Therefore,

\[
(41) \quad |f_{u} - f_{v}| \leq M| \sum_{i \in V_{0}} g_{i}(\Delta f)_{i}|.
\]

To estimate the right-hand side of (41), take \(r = [\sqrt[3]{\ln(n)}]\) and select \(a \notin B_{r}(u)\). Then using \(0 = \sum_{i \in V} (\Delta f)_{i} = \sum_{i \in V_{0}} (\Delta f)_{i}, |g_{i}| \leq 1\), (41) and the Lemma 5.11 below, we obtain

\[
|f_{u} - f_{v}| \leq M| \sum_{i \in V_{0}} g_{i}(\Delta f)_{i}| = M| \sum_{i \in V_{0}} (g_{i} - g_{u})(\Delta f)_{i}| \leq M| \sum_{i \in V_{0}} |(g_{i} - g_{u})|(\Delta f)_{i}| 
\]

\[
\leq M( \sum_{i \in V_{0} \cap B_{r}(u)} |g_{i} - g_{u}|(\Delta f)_{i} + \sum_{i \in V_{0} - B_{r}(u)} |g_{i} - g_{a}|(\Delta f)_{i}) 
\]

\[
\leq M( \frac{2M}{\sqrt{\ln(n)}} |V_{0} \cap B_{r}(u)| + \frac{100M}{\sqrt{\ln(r)}} \sum_{i \in V_{0}} |(\Delta f)_{i}|) 
\]

\[
\leq M[ \frac{20M \sqrt[3]{\ln(n)}}{\sqrt{\ln(n)}} + \frac{100M^{2}}{\ln(3 \sqrt[3]{\ln(n)})}] 
\]

\[
\leq \frac{200M^{3}}{\sqrt{\ln(n)}}.
\]
In the last two steps, we have used $|V_0 \cap B_r(u)| \leq 10r = 10\sqrt{\ln n}$ and $n \geq e^{10^6}$ to ensure

$$\frac{1}{\sqrt{\ln(\sqrt{\ln(n)})}} \geq \frac{\sqrt{\ln(n)}}{\sqrt{\ln(n)}}.$$ 

\[ \square \]

**Lemma 5.11.** Assume $(\mathcal{P}, \mathcal{T}', l, T, E, M, \eta)$ and $\Delta$ are as given in Proposition 5.10 and $g$ is as given in the proof of Proposition 5.10, i.e., $(\Delta g)_i = 0$ for $i \neq u, v$, and $g_u = 1, g_v = 0$. If $100 \leq r \leq \frac{n}{3}$ and $\{a, b\} \cap B_r(u) = \emptyset$, then

$$|g_a - g_b| \leq \frac{100M}{\sqrt{\ln(r)}}.$$ 

The strategy of the proof to Lemma 5.11 is similar to that of Lemma 5.9.

**Proof.** For $k \leq r/3$, let $U_k = \{i \in V | d_c(i, u) = k\}$. Since $T$ is an equilateral triangulation of a flat surface, we have $|U_k| \leq 6k$. Recall that a subset $U$ of $V = V(T)$ is called connected if any two points in $U$ can be joint by an edge path in $T^{(1)}$ whose vertices are in $U$. Each subset $U \subset V$ is a disjoint of connected subsets which are called connected components of $U$. We claim that there exists a connected component $G_k$ of $U_k$ such that $\{a, b\}$ lie in a connected components of $V - G_k$. To see this, note that since $T$ is the $n$-th standard subdivision of $T'$, for all $k \leq r/3 \leq n/9$, the set $B_k(u) = \{i \in V | d_c(i, u) \leq k\}$ is connected and $B_k(u)^c = \{i \in V | d_c(i, u) > k\}$ has at most two connected components which are also connected components of $V - U_k$. If $B_k(u)^c$ is connected, then $U_k$ is connected and we take $G_k = U_k$. If $B_k(u)^c$ has two connected components $R_1$ and $R_2$, then there exists a non-flat boundary vertex $v' \in R_2$ such that $d_c(u, v') \leq 3k \leq r$. This shows that $v' \in B_r(u)$. See Figure 10. The component $R_2$ is contained in $B_r(u)$ due to $d_c(v', u) \leq r$. Since $a, b \notin B_r(u)$, it follows that $a, b$ are in $R_1$. We take $G_k$ to be the connected component of $U_k$ such that $R_1$ is a connected component of $V - G_k$. Therefore, the claim follows.

![Figure 10. Triangulated polygonal disks](image-url)

Let us assume without loss of generality that $g_a \leq g_b$. By the maximum principle applied to $g$ on the connected graph whose vertex set is the connected component of $V - G_k$ containing $\{a, b\}$, there exist two vertices $u_k, u'_k \in G_k$ so that

$$g_{u_k} \geq g_b \text{ and } g_{u'_k} \leq g_a.$$
Let $E_k$ be the shortest edge path with vertices in $G_k$ connecting $u_k$ to $u'_k$ and $\vec{E}_k$ be the set of all oriented edges in $E_k$. The length of $E_k$ is at most $|G_k| \leq 6k$. The Dirichlet energy of $g$ on the graph $\mathcal{T}^{(1)}$ is

$$E(g) = \frac{1}{2} \sum_{i \sim j} \eta_{ij} (g_i - g_j)^2 \geq \frac{1}{2M} \sum_{i \sim j} (g_i - g_j)^2 \geq \frac{1}{2M} \sum_{k=1}^{\lfloor \frac{r}{6} \rfloor} \sum_{[ij] \in E_k} (g_i - g_j)^2.$$

Suppose $w_0 = u_k \sim w_1 \sim w_2 \sim \ldots \sim w_{l_k} = u'_k$ is the edge path $E_k$ where $l_k \leq 6k$. Then by the Cauchy-Schwartz inequality, we have

$$\frac{1}{2} \sum_{[ij] \in E_k} (g_i - g_j)^2 = \sum_{i=1}^{l_k} (g_{w_i} - g_{w_{i-1}})^2 \geq \frac{1}{l_k} \left( \sum_{i=1}^{l_k} (g_{w_i} - g_{w_{i-1}})^2 \right)^{\frac{1}{2}} \geq \frac{1}{l_k} (g_{w_k} - g_{w_0})^2 \geq \frac{(g_a - g_b)^2}{6k}.$$

By Theorem 5.1, we obtain

$$E(g) \geq \frac{1 - \frac{(g_a - g_b)^2}{6}}{1 + \frac{\ln(r)}{100M}}.$$

On the other hand, the Dirichlet principle we have $E(g) \leq \frac{1}{2} \sum_{i \sim j} \eta_{ij} (h_i - h_j)^2$ for any $h \in \mathbb{R}^V$ such that $h_u = 1, h_v = 0$. Take $h$ to be $h_u = 1$ and $h_v = 0, \forall i \in V - \{u\}$. We obtain $E(g) \leq \frac{1}{2} \sum_{i \sim j} \eta_{ij} (h_i - h_j)^2 \leq 6M$. Therefore, $\frac{(g_a - g_b)^2}{100M} \leq 6M$ which implies

$$|g_b - g_a| \leq \frac{100M}{\sqrt{\ln(r)}}. \square$$

5.5. A proof of Theorem 5.1 For simplicity, a boundary vertex $v \in \mathcal{P} - \{p, q, r\}$ with non-zero curvature will be called a corner. Note that corners in $\mathcal{T}$ and its $n$-th standard subdivision $\mathcal{T}^{(n)}$ are the same. In particular, the total number of corners is independent of $n$. Let $V_c$ be the set of all corner vertices. Since $\mathcal{P}$ is embedded in $\mathbb{C}$, given a corner $v \in V_c$, the degree $m$ of $v$ has to be 3, 5 or 6. Consider the combinatorial ball $B_{[n/3]}(v)$ of radius $[n/3]$ centered at a corner $v \in V_c$. By construction $B_{[n/3]}(v) \cap B_{[n/3]}(v') = 0$ for distinct corners $v, v'$. Each $B_{[n/3]}(v)$ is a union of $m - 1[\frac{n}{3}]$-th standard subdivided equilateral triangles $\Delta_1, \ldots, \Delta_{m-1}$ in $\mathcal{T}$. Applying Theorem 5.6, we produce a discrete conformal factor $w(\Delta_i) \in \mathbb{R}^{\mathcal{V}(\Delta_i)}$ for each $\Delta_i$ such that if a vertex $u \in V(\Delta_i) \cap V(\Delta_j)$, then $w_u(\Delta_i) = w_u(\Delta_j)$. In particular there is a well defined discrete conformal factor $w(B_{[n/3]}(v))$ on $B_{[n/3]}(v)$ obtained by gluing these $w(\Delta_i)$. See Figure 8. Define $w^{(1)} : \mathcal{V}(\mathcal{T}^{(n)}) \to \mathbb{R}$ as follows: if $u \in \cup_{v \in V_c} B_{[n/3]}(v)$, then $w_u^{(1)} = w_u(B_{[n/3]}(v))$ for $u \in B_{[n/3]}(v)$ and $w_u^{(1)}(u) = 0$ for $u \notin \cup_{v \in V_c} B_{[n/3]}(v)$. Let $\hat{\ell} = w^{(1)} * \ell$ be the PL metric on $\mathcal{T}^{(n)}$ and $\hat{K}$ be its discrete curvature. Let $K^* : V(n) \to \mathbb{R}$ be defined by $K^*_i = 0$ if $i \notin \{p, q, r\}$, and $K^*_i = \frac{2\pi}{3}$ if $i \in \{p, q, r\}$. By Theorem 5.6, the PL metric $\hat{\ell}$ and $\hat{K}$ satisfy the following:

(a) the curvature $\hat{K}_i = K^*_i$ at all vertices $i$ such that $d_\ell(i, v) \neq [n/3]$ for some corner $v \in V_c$;
(b) $w^{(1)}_i = 0$ for $i \notin \cup_{v \in V_c} B_{[n/3]}(v)$;
(c) all inner angles at a corner $v \in V_c$ are in $[\frac{\pi}{6}, \frac{\pi}{2}]$.
(d) all inner angles at a non-corner vertex are in \([\frac{\pi}{6}, \frac{59\pi}{120}]\);

(e) \(|\hat{K}_i - K_i^*| \leq \frac{4000}{\sqrt{\ln(n)}}\) and \(\sum_{i \in V} |\hat{K}_i - K_i^*| \leq \frac{\pi N}{3}\) where \(N\) is the number of corners in \(\mathcal{P}\).

We will find a discrete conformal factor \(w^{(2)} : V(n) \to \mathbb{R}\) such that \(w^{(2)} \ast \hat{l}\) and its curvature satisfy Theorem 5.1 by solving the following system of ordinary differential equations in \(w(t)\):

\[
dK_i(w(t) \ast \hat{l}) = K_i^* - \hat{K}_i, \forall i \in V(\mathcal{T}(n)) - \{p, q, r\}; \quad w_s(t) = 0, s \in \{p, q, r\}; \quad \text{and} \quad w(0) = 0.
\]

Let \(K(t) = K(w(t) \ast \hat{l})\). Note that \((45)\) and the Gauss-Bonnet formula imply that \(K_p' = K_p^* - \hat{K}_p\). By Lemma 5.5 the solution \(w(t)\) exists on some interval \([0, \epsilon)\). Our goal is to show that for \(n\) large, the solution \(w(t)\) exists on \([0, 1]\). In this case, the conformal factor \(w^{(2)}\) is taken to be \(w(1)\). The required discrete conformal factor \(w_n\) in Theorem 5.1 is taken to be \(w^{(1)} + w^{(2)}\).

Consider the maximum time \(t_0\) such that the solution \(w(t)\) to \((45)\) exists for \(t \in [0, t_0)\) and the PL metrics \(w(t) \ast \hat{l}\) satisfy:

(c') all inner angles at a corner \(v \in V_c\) are in \(\left[\frac{\pi}{6} - \frac{\pi}{1000}, \frac{\pi}{2} + \frac{\pi}{1000}\right]\);

\[(d')\] all inner angles at a non-corner vertex are in \(\left[\frac{\pi}{6} - \frac{\pi}{1000}, \frac{\pi}{2} + \frac{\pi}{1000}\right]\).

Let \(V_0 = \bigcup_{v \in V_c} \{i \in V(n) | d_e(i, v) = [n/3]\}\). By construction, \(|B_\epsilon(i) \cap V_0| \leq 10r\) for all \(r \leq n/3\). Then \(\sum_{i \in V_0} |(\Delta w(t))_i| = \sum_{i \in V_0} |K'_i(t)| \leq \sum_{i \in V_0} |K_i - K_i^*| \leq \frac{\pi N}{3}\) and \(|(\Delta w')_i| = |K'_i(t)| \leq |\hat{K}_i - K_i^*| \leq \frac{4000}{\sqrt{\ln(n)}}\). Choose \(M = \max\{4000, \frac{\pi N}{3}\}\). Then by (c'), (d'), (e) and the formula \(\cot(a) + \cot(b) = \frac{\sin(a+b)}{\sin(a)\sin(b)}\), for all \(t \in [0, t_0)\), we have \(\eta_{ij}(t) = \eta_{ij}(w(t) \ast \hat{l}) \in [\frac{1}{4000}, 4000] \subset [\frac{1}{M}, M]\), \((\Delta w')_i = 0\) for \(i \in V(\mathcal{T}(n)) - V_0\), \(|(\Delta w')_i| \leq \frac{M}{\sqrt{\ln(n)}}\) and \(\sum_{i \in V_0} |(\Delta w')_i| \leq M\). In summary, \(f = w'\) satisfies conditions in Proposition 5.10 for all \(t \in [0, t_0)\). By Proposition 5.10, if \(i \sim j\), then

\[|w'_i(t) - w'_j(t)| \leq \frac{200M^3}{\sqrt{\ln(n)}}\]

On the other hand, by the variation of angle formula (6) and \(M \geq |\cot(\theta_{ij}^k)|\), we have

\[|\frac{d\theta_{ij}^k}{dt}| \leq |\cot(\theta_{ij}^k)(w'_j - w'_k)| + |\cot(\theta_{ik}^l)(w'_l - w'_k)| \leq M(|w'_j - w'_k| + |w'_l - w'_k|) \leq \frac{400M^4}{\sqrt{\ln(n)}}\]

Therefore, for \(t \in [0, t_0)\) and sufficiently large \(n\),

\[
|\theta_{ij}^k(w(t)) - \theta_{ij}^k(0)| \leq \int_0^t |\frac{d\theta_{ij}^k(w(t))}{dt}| dt \leq \frac{400M^4t_0}{\sqrt{\ln(n)}} \leq \frac{\pi t_0}{2000}.
\]

It follows that \(t_0 > 1\) (or \(t_0 = \infty\)) since otherwise, by (46), the choices of angles in (c),(d), (c'), (d') and Lemma 5.5 we can extend the solution \(w(t)\) to \([0, t_0 + \epsilon)\) for some \(\epsilon > 0\) such that (c') and (d') still hold. To be more precise, by Lemma 5.5 on the maximality of \(t_0\), we have either \(\lim \sup_{t \to t_0} |\theta_{ij}^k(t) - \frac{\pi}{3}| = \frac{\pi}{2}\) for an inner angle \(\theta_{ij}^k\) at a corner \(i \in V_c\), or \(\lim \sup_{t \to t_0} \theta_{ij}^k(t) = \frac{\pi}{2} - \frac{\pi}{1000}\), or \(\lim \sup_{t \to t_0} \theta_{ij}^k(t) = \frac{59\pi}{100} + \frac{\pi}{1000}\) for an angle \(\theta_{ij}^k\) at a non-corner vertex \(i\). But, due to (46), none of these conditions holds if \(t_0 \leq 1\). Therefore the solution \(w(1)\) exists. By
construction, the curvature $K(1)$ of $w(1) * \hat{t}$ is $K(0) + \int_0^1 K'(t)dt = \hat{K} + K^* - \hat{K} = K^*$. Furthermore, condition (c) in Theorem 5.1 follows from (c') and (d').

6. A PROOF OF THE CONVERGENCE THEOREM

We will prove the following theorem.

**Theorem 6.1.** Let $\Omega$ be a Jordan domain in the complex plane and \( \{p, q, r\} \subset \partial \Omega \). There exists a sequence of triangulated polygonal disks \( (\Omega_n, T_n, d_{st}, (p_n, q_n, r_n)) \) where $T_n$ is an equilateral triangulation and $p_n$, $q_n$, $r_n$ are three boundary vertices such that
\[(a) \quad \Omega = \bigcup_{n=1}^\infty \Omega_n \text{ with } \Omega_n \subset \Omega_{n+1}, \text{ and } \lim_n p_n = p, \lim_n q_n = q \text{ and } \lim_n r_n = r, \]
\[(b) \quad \text{discrete uniformization maps } f_n \text{ associated to } (\Omega_n, T_n, d_{st}, (p_n, q_n, r_n)) \text{ exist and converge uniformly to the Riemann mapping associated to } (\Omega, (p, q, r)). \]

Before giving the proof, let us recall Rado’s theorem and its generalization to quasiconformal maps. If $\phi: \mathbb{D} \rightarrow \Omega$ is a $K$-quasiconformal map onto a Jordan domain $\Omega$, then $\phi$ extends continuously to a homeomorphism $\overline{\phi}: \mathbb{D} \rightarrow \overline{\Omega}$ between their closures (see [1, corollary on page 301]). If $K = 1$, $\overline{\phi}$ is the Caratheodory extension of the Riemann mapping. A sequence of Jordan curves $J_n$ in $\mathbb{C}$ is said to converge uniformly to a Jordan $J$ curve in $\mathbb{C}$ if there exist homeomorphisms $\phi_n : S^1 \rightarrow J_n$ and $\phi : S^1 \rightarrow J$ such that $\phi_n$ converges uniformly to $\phi$. Rado’s theorem [32] and its extension by Palka [30, corollary 1] states that,

**Theorem 6.2** (Rado, Palka). Suppose $\Omega_n$ is a sequence of Jordan domains such that $\partial \Omega_n$ converges uniformly to $\partial \Omega$. If $f_n : \mathbb{D} \rightarrow \Omega_n$ is a $K$-quasiconformal map for each $n$ such that the sequence $\{f_n\}$ converges to a $K$-quasiconformal map $f : \mathbb{D} \rightarrow \Omega$ uniformly on compact sets of $\mathbb{D}$, then $f_n$ converges to $f$ uniformly on $\mathbb{D}$.

The following compactness result is a consequence of Palka’s theorem ([30, corollary 1]) and Lehto-Virtanen’s work [25, Theorems 5.1, 5.5].

**Theorem 6.3.** Suppose $\Omega_n$ is a sequence of Jordan domains such that $\partial \Omega_n$ converges uniformly to $\partial \Omega$ and $K > 0$ is a constant. Let $p_n, q_n, r_n \in \partial \Omega_n$ and $p, q, r \in \partial \Omega$ be distinct points such that $\lim_n p_n = p, \lim_n q_n = q, \lim_n r_n = r$ and $h_n : \mathbb{D} \rightarrow \Omega_n$ be $K$-quasiconformal maps such that $h_n$ sends $(1, \sqrt{-1}, -1)$ to $(p_n, q_n, r_n)$. Then there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ converging uniformly on $\mathbb{D}$ to a $K$-quasiconformal map $h : \mathbb{D} \rightarrow \Omega$ sending $(1, \sqrt{-1}, -1)$ to $(p, q, r)$.

Now we prove Theorem 6.1.

**Proof.** Given a Jordan domain $\Omega$ with three distinct points $p, q, r$ in $\partial \Omega$, construct a sequence of approximating polygonal disks $\Omega_n$ such that (1) each $\Omega_n$ is triangulated by equilateral triangles of side lengths tending to 0, (2) $\partial \Omega_n$ converges uniformly to the Jordan curve $\partial \Omega$ such that $\Omega_n \subset \Omega_{n+1}$, (3) there are three boundary vertices $p_n, q_n, r_n \subset \partial \Omega_n$ such that $\lim_n p_n = p$, $\lim_n q_n = q$ and $\lim_n r_n = r$, and (4) the curvatures of $\Omega_n$ at $p_n, q_n, r_n$ are $\frac{2\pi}{3}$ and curvatures of $\Omega_n$ at all other boundary vertices are not $\frac{2\pi}{3}$.

By Theorem 5.1, we produce a standard subdivision $T_n$ of $\Omega_n$ and $w_n \in \mathbb{R}^V(T_n)$ such that $(\Omega_n, T_n, w_n * l_{st})$ is isometric to the equilateral triangle $(\Delta ABC, T'_n, l'_n)$ with a Delaunay triangulation $\mathcal{T}'_n$ and $A, B, C$ correspond to $p_n, q_n, r_n$. Let $f_n : (\Delta ABC, T'_n, (A, B, C)) \rightarrow (\Omega_n, T_n, (p_n, q_n, r_n))$ be the associated discrete conformal map and $\hat{f} : (\Delta ABC, (A, B, C)) \rightarrow (\Omega, (p, q, r))$ be the Riemann mapping. We claim that $f_n$ converges uniformly to $\hat{f}$ on $\Delta ABC$. 

...
To establish the claim, first by Theorem 5.1, we know all angles of triangles in the triangulated PL surface $(\Delta ABC, T_n', l_n')$ are at least $\epsilon_0 > 0$. Therefore the discrete conformal maps $f_n$ are K-quasiconformal for a constant $K$ independent of $n$. By Theorem 6.3, it follows that every limit function $g$ of a convergence subsequence $\{f_n\}$ is a K-quasiconformal map from $\text{int}(\Delta ABC)$ to $\Omega$ which extends continuously to $\Delta ABC$ sending $A, B, C$ to $p, q, r$ respectively. We claim that the limit map $g$ is conformal. Indeed, by Lemma 4.7, the discrete conformal map $f_n^{-1}$, when restricted to a fixed compact set $R$ of $\Omega$, maps equilateral triangles in $T_n$ which are inside $R$ to triangles of $T_n'$ that become arbitrarily close to equilateral triangle as $n \to \infty$. Therefore the limit map $g$ of the subsequence $f_{n_i}$ is 1-conformal and therefore conformal in $\text{int}(\Delta ABC)$. The continuous extension of $g$ sends $A, B, C$ to $p, q, r$ respectively by Theorem 6.3. On the other hand, there is only one Riemann mapping $f : \text{int}(\Delta) \to \Omega$ whose continuous extension sends $A, B, C$ to $p, q, r$. Therefore, $g = f$. This shows all limits of convergence subsequences of $\{f_n\}$ are equal $f$. Therefore $\{f_n\}$ converges to $f$ uniformly on compact sets in $\text{int}(\Delta ABC)$. By Theorem 6.2, $f_n$ converges uniformly to $f$. \hfill \Box

7. A CONVERGENCE CONJECTURE ON DISCRETE UNIFORMIZATION MAPS

We discuss a general approximation conjecture and the related topics of discrete conformal equivalence of polyhedral metrics.

7.1. A strong version of convergence of discrete conformal maps. As discussed before, the main drawback of the vertex scaling operation on polyhedral metrics is the lacking of an existence theorem. For instance, given a PL metric on a closed triangulated surface $(S, T, l)$, there is in general no discrete conformal factor $w : V \to \mathbb{R}$ such that the new PL metric $(S, T, w \ast l)$ has constant discrete curvature.

The recent work of [13] established an existence and a uniqueness theorem for polyhedral metrics by allowing the triangulations to be changed.

**Definition 7.1.** (Discrete conformality of PL metrics [13]) Two PL metrics $d, d'$ on $(S, V)$ are discrete conformal if there exist sequences of PL metrics $d_1 = d, ..., d_m = d'$ on $(S, V)$ and triangulations $T_1, ..., T_m$ of $(S, V)$ satisfying

(a) (Delaunay) each $T_i$ is Delaunay in $d_i$,

(b) (Vertex scaling) if $T_i = T_{i+1}$, there exists a function $w : V \to \mathbb{R}$ so that if $e$ is an edge in $T_i$ with end points $v$ and $v'$, then the lengths $l_{d_{i+1}}(e)$ and $l_{d_i}(e)$ of $e$ in $d_i$ and $d_{i+1}$ are related by

$$l_{d_{i+1}}(e) = e^{w(v) + w(v')}l_{d_i}(e),$$

(c) if $T_i \neq T_{i+1}$, then $(S, d_i)$ is isometric to $(S, d_{i+1})$ by an isometry homotopic to the identity in $(S, V)$.

The main theorem proved in [13] is the following.

**Theorem 7.2.** Suppose $(S, V)$ is a closed connected marked surface and $d$ is a PL metric on $(S, V)$. Then for any $K^* : V \to (-\infty, 2\pi)$ with $\sum_{v \in V} K^*(v) = 2\pi \chi(S)$, there exists a PL metric $d^*$, unique up to scaling and isometry homotopic to the identity, on $(S, V)$ such that $d^*$ is discrete conformal to $d$ and the discrete curvature of $d^*$ is $K^*$. Furthermore, the metric $d^*$ can be found using a finite dimensional (convex) variational principle.
There is a close relation between the discrete conformal equivalence defined in Definition 7.1 and convex geometry in hyperbolic 3-space. The first work relating vertex scaling operation and hyperbolic geometry is in the paper by Bobenko-Pinkall-Springborn [4]. They associated each polyhedral metric on \((S, \mathcal{T}, l)\) a hyperbolic metric with cusp end on the punctured surface \(S - V(\mathcal{T})\). However, the Delaunay condition on the triangulation \(\mathcal{T}\) was missing in their definition. The discrete conformal equivalence defined in Definition 7.1 is equivalent to the following hyperbolic geometry construction. Let \((S, V, d)\) be a PL surface. Take a Delaunay triangulation \(\mathcal{T}\) of \((S, V, d)\) and consider the PL metric \(d\) as isometric gluing of Euclidean metrics \(d_\sigma\) and \(d_\tau\) on \((S, V)\) are discrete conformal in the sense of Definition 7.1 if and only if the associated hyperbolic metrics \(d_1^*\) and \(d_2^*\) are isometric by an isometry homotopic to the identity on \(S - V\).

Using this hyperbolic geometry interpretation, one defines the discrete conformal map between two discrete conformally equivalent PL metrics \(d_1^*\) and \(d_2^*\) as follows (see [4] and [13]). The vertical projection of the ideal triangle \(\tau^*\) induces a homeomorphism \(\phi_{d_1} : (S - V, d^*) \to (S - V, d)\). Suppose \(d_1^*\) and \(d_2^*\) are two discrete conformally equivalent PL metrics on \((S, V)\). Then the discrete conformal map from \((S, V, d_1^*)\) to \((S, V, d_2^*)\) is given by \(\phi_{d_2} \circ \psi \circ \phi_{d_1}^{-1}\) where \(\psi : (S, V, d_1^*) \to (S, V, d_2^*)\) is the hyperbolic isometry. Note that in this new setting, discrete conformal maps are piecewise projective instead of piecewise linear.

Theorem 7.2 can be used for approximating Riemann mappings for Jordan domains. Given a simply connected polygonal disk with a PL metric \((D, V, d)\) and three boundary vertices \(p, q, r \in V\), let the metric double of \((D, V, d)\) along the boundary be the polyhedral 2-sphere \((\mathbb{S}^2, V', d^*)\). Using Theorem 7.2 one produces a new polyhedral surface \((\mathbb{S}^2, V', d^*)\) such that: (1) \((\mathbb{S}^2, V', d^*)\) is discrete conformal to \((S, V', \bar{d}^*)\); (2) the discrete curvatures of \(d^*\) at \(p, q, r\) are \(4\pi/3\); (3) the discrete curvatures of \(d^*\) at all other vertices are zero; and (4) the area of \((\mathbb{S}^2, V', d^*)\) is \(\sqrt{3}/2\). Therefore \((S, V', d^*)\) is isometric to the metric double \((\mathcal{D}(\Delta ABC), V'', d'')\) of an equilateral triangle \(\Delta ABC\) of edge length 1. Let \(F\) be the discrete conformal map from \((\mathcal{D}(\Delta ABC), V'', d'')\) to \((\mathbb{S}^2, V', d')\) such that \(F\) sends \(A, B, C\) to \(p, q, r\) respectively. Due to the uniqueness part of Theorem 7.2, we may assume that \(f = F|_{\Delta ABC} : \Delta ABC \to D\) and \(f\) sends \(A, B, C\) to \(p, q, r\) respectively. We call \(f\) the discrete uniformization map associated \((D, V, d, (p, q, r))\).

A strong form of the convergence is the following,

**Conjecture 7.3.** Let \((\Omega, (p, q, r))\) be a Jordan domain in the complex plane with three marked boundary points and \((\Omega_n, T_n, d_{st}, (p_n, q_n, r_n))\) be any sequence of triangulated flat polygonal disks with three marked boundary vertices such that

(a) \(T_n\) is an equilateral triangulation,
(b) \(\partial S_n\) converges uniformly to \(\partial \Omega\),
(c) the edge length of \(T_n\) goes to zero,
Then discrete uniformization maps $f_n$ associated to $(\Omega_n, T_n, d_{st}, (p_n, q_n, r_n))$ converge uniformly to the Riemann mapping associated to $(\Omega, (p, q, r))$.

7.2. Discrete conformal equivalence and convex sets in the hyperbolic 3-space. We now discuss the relationship between discrete conformal equivalence defined in Definition 7.1 and ideal convex sets in the hyperbolic 3-space $\mathbb{H}^3$ and the motivation for Conjectures 1.5 and 1.6.

The classical uniformization theorem for Riemann surfaces follows from the special case that every simply connected Riemann surface is biholomorphic to $\mathbb{C}$, $D$ or $S^2$. The discrete analogous should be the statement that each non-compact simply connected polyhedral surface is discrete conformal to either $(\mathbb{C}, V, d_{st})$ or $(\mathbb{D}, V, d_{st})$ where $V$ is a discrete set and $d_{st}$ is the standard Euclidean metric. Furthermore, the set $V$ is unique up to M"obius transformations. For a non-compact polyhedral surface $(S, V, d)$ with an infinite set $V$, the hyperbolic geometric view point of discrete conformality is a better approach. Namely discrete conformal equivalence between two PL metrics is the same as the Teichmüller equivalence between their associated hyperbolic metrics. For instance, if we take a Delaunay triangulation $T$ of the complex plane $(\mathbb{C}, d_{st})$ with vertex set $V$, then the associated hyperbolic metric $d_{st}$ on $\mathbb{C} - V$ is isometric to the boundary of the convex hull $\partial C_{st}(V)$ in $\mathbb{H}^3$. Therefore, a PL surface $(S, V', d)$ is discrete conformal to $(\mathbb{C}, V, d_{st})$ for some discrete subset $V \subset \mathbb{C}$ if and only if the associated hyperbolic metric $d'$ is isometric to the boundary of the convex hull $\partial C_{st}(V)$. It shows discrete uniformization is the same as realizing hyperbolic metrics as the boundaries of convex hulls (in $\mathbb{H}^3$) of closed sets in $\partial \mathbb{H}^3$. One can formulate the conjectural discrete uniformization theorem as follows. Given a discrete set $V'$ in $\mathbb{C}$ or $D$, let $d$ be the unique conformal complete hyperbolic metric on $\mathbb{C} - V'$ or $D - V'$. Then $d$ is isometric to the boundary of the convex hull of a discrete set $V \subset \mathbb{C}$ or $(\mathbb{C} \cup \{\infty\} - D)$ or $V$ where $V$ is discrete and unique up to M"obius transformations. This is the original motivation for proposing Conjectures 1.5 and 1.6.

These two conjectures bring discrete uniformization close to the classical Weyl problem on realizing surfaces of non-negative Gaussian curvature as the boundaries of convex bodies in the 3-space. In the hyperbolic 3-space $\mathbb{H}^3$, convex surfaces have curvature at least $-1$. The work of Alexandrov [2] and Pogorelov [31] show that for each path metric $d$ on the 2-sphere $SS^2$ of curvature $\geq -1$, there exists a compact convex body, unique up to isometry, in $\mathbb{H}^3$ whose boundary is isometric to $(SS^2, d)$. The interesting remaining cases are non-compact surfaces of genus zero in the hyperbolic 3-space $\mathbb{H}^3$. A theorem of Alexandrov [2] states that any complete surface of genus zero whose curvature is at least $-1$ is isometric to the boundary of a closed convex set in $\mathbb{H}^3$. On the other hand, given a closed set $X \subset \mathbb{C}$, W. Thurston proved that the intrinsic metric on $\partial C_{st}(X)$ is complete hyperbolic (see [10] for a proof). Putting these two theorems together, one sees that each complete hyperbolic metric on a surface of genus zero is isometric to the boundary of the convex hull of a closed set in the Riemann sphere. However, in this generality, the uniqueness of the convex surface is false. Conjectures 1.5 and 1.6 say that one has both the existence and uniqueness if one imposes restrictions to the boundaries of the convex hulls of closed sets.

There are some evidences supporting Conjectures 1.5 and 1.6. The work of Rivin [33] and Schlenker [35] show that Conjectures 1.5 and 1.6 hold if $\Omega$ has finite area (i.e., $X$ is a finite set) or if $\Omega$ is conformal to the 2-sphere with a finite number of disjoint disks removed (i.e., $X$ is a finite disjoint union of round disks). Our recent work [28] shows that Conjectures 1.5 holds for $\Omega$ having countably many topological ends using the work of He-Schramm on K"obe conjecture.
One should compare Conjectures 1.5 and 1.6 with the Köbe circle domain conjecture which states that each genus zero Riemann surface is biholomorphic to the complement of a circle type closed set in the Riemann sphere. The work of He-Schramm [17] shows that Köbe conjecture holds for surfaces with countably many ends and the circle type set is unique up to M"obius transformations. Uniqueness is known to be false for the Köbe conjecture in general. Our recent work [28] shows that the Köbe conjecture is equivalent to Conjecture 1.5. Other related works are [5], [11], [23], [24], [33], [35], [36], and [38].

We end this paper by proposing the following the conjecture. The work of Rodin-Sullivan [34] and Theorem 1.4 show the rigidity phenomena for the two most regular patterns (regular hexagonal circle packing and regular hexagonal triangulation) in the plane. These rigidity results can be used to approximate the Riemann mappings and the uniformization metrics. The third regular pattern in the plane is the hexagonal square tiling in which each square of side length one interests exactly six others. See figure 11.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{Regular hexagonal square tiling}
\end{figure}

Conjecture 7.4. Suppose \( \{S_i \mid i \in I \} \) is a locally finite square tiling of the complex plane \( \mathbb{C} \) such that each square intersects exactly six others. Then all squares \( S_i \) have the same size.

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