A proof of Erdős’s $B + B + t$ conjecture

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Abstract

We show that every set $A$ of natural numbers with positive upper density can be shifted to contain the restricted sumset $\{b_1 + b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\}$ for some infinite set $B \subset A$.

1. Introduction

Looking to connect two major achievements of additive combinatorics from the 1970s, Hindman’s theorem [11] and Szemerédi’s theorem [18], Erdős formulated the following conjecture on multiple occasions.

**Conjecture 1.1** (Erdős [5, Page 305], [6, Pages 57–58], and [7, Page 105]). For any $A \subset \mathbb{N}$ with positive density there exists an infinite set $B \subset A$ and a number $t \in \mathbb{N}$ such that

$$A - t \supset \{b_1 + b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2\}.$$  

This problem was studied by various authors, including Nathanson [17], Kazhdan (see [17, 12]), and Hindman [12, Section 11]. The latter provided several equivalent forms, including a natural reformulation using the Stone-Čech compactification of the integers. A special case of Conjecture 1.1, also conjectured by Erdős, was resolved in [16], asserting that, under the same assumptions, $A$ contains a sumset $B + C = \{b + c : b \in B, c \in C\}$ of two infinite sets $B, C \subset \mathbb{N}$. Further recent progress in this direction has been made in [3, 13, 15], and further history on Conjecture 1.1 and surrounding problems can be found in [12, 16, 17].

Our main theorem resolves Conjecture 1.1. To state our result precisely, recall that a **Følner sequence** $\Phi$ on $\mathbb{N}$ is any sequence $N \mapsto \Phi_N$ of finite subsets of $\mathbb{N}$ with the property that

$$\lim_{N \to \infty} \frac{|\Phi_N \cap (\Phi_N + t)|}{|\Phi_N|} = 1$$

for all $t \in \mathbb{N}$. A set $A \subset \mathbb{N}$ has **positive upper Banach density** if

$$\lim_{N \to \infty} \frac{|A \cap \Phi_N|}{|\Phi_N|} > 0$$

for some Følner sequence $\Phi$. 

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Theorem 1.2. Let \( A \subset \mathbb{N} \) and assume it has positive upper Banach density. Then there exist an infinite set \( B \subset A \) and a shift \( t \in \mathbb{N} \) such that \( A - t \supset \{ b_1 + b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2 \} \).

Note that in the formulation of Theorem 1.2 it is not possible to omit the shift by \( t \) or remove the condition \( b_1 \neq b_2 \) (see the discussion in [16] after Question 6.2). Also, it was observed by Hindman in [12] that writing \( t = 2r + s \) for \( r \in \mathbb{N} \) and \( s \in \{0, 1\} \) and replacing \( B \) by \( B - r \), one obtains the following corollary from Theorem 1.2.

Corollary 1.3. For any \( A \subset \{2n : n \in \mathbb{N}\} \) with positive upper Banach density there exists an infinite set \( B \subset \mathbb{N} \) such that \( A \supset \{ b_1 + b_2 : b_1, b_2 \in B \text{ and } b_1 \neq b_2 \} \).

Our proof of Theorem 1.2 uses ergodic theory and builds on the new dynamical methods developed in [15] to find infinite patterns in sets with positive upper density. To formulate our main dynamical result we recall some basic terminology. By a topological system, we mean a pair \((X, T)\) where \( X \) is a compact metric space and \( T : X \to X \) is a homeomorphism. A system is a triple \((X, \mu, T)\), where \((X, T)\) is a topological system and \( \mu \) is a \( T \)-invariant Borel probability measure on \( X \). The system is ergodic if any \( T \)-invariant Borel subset of \( X \) has either measure 0 or measure 1, and equivalently we say that \( \mu \) is ergodic for \( T \). Given a system \((X, \mu, T)\), a point \( a \in X \) is generic for \( \mu \) along a Følner sequence \( \Phi \), written \( a \in \text{gen}(\mu, \Phi) \), if

\[
\mu = \lim_{N \to \infty} \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \delta_{T^n a}
\]

where \( \delta_x \) is the Dirac measure at \( x \in X \) and the limit is in the weak* topology. This allows us to formulate the main dynamical result used to prove Theorem 1.2.

Theorem 1.4. Let \((X, \mu, T)\) be an ergodic system, let \( a \in \text{gen}(\mu, \Phi) \) for some Følner sequence \( \Phi \), and let \( E \subset X \) be an open set with \( \mu(E) > 0 \). Then there exist \( x_1, x_2 \in X \), \( t \in \mathbb{N} \), and a strictly increasing sequence \( n_1 < n_2 < \ldots \) of integers such that \( x_1 \in E \), \( T^t x_2 \in E \), and \((T \times T)^{n_i}(a, x_1) \to (x_1, x_2)\) as \( i \to \infty \).

The deduction of Theorem 1.2 from Theorem 1.4 is given in Section 2 and the proof of Theorem 1.4 is given in Section 3.

We conclude the introduction with a natural conjecture on a higher order version of our main theorem.

Conjecture 1.5. Let \( A \subset \mathbb{N} \) have positive upper Banach density and let \( k \in \mathbb{N} \). Then there exist an infinite set \( B \subset A \) and a shift \( t \in \mathbb{N} \) such that

\[
A - t \supset \left\{ \sum_{n \in F} n : F \subset B, \ 0 < |F| < k \right\}.
\]

(1.1)

We remark that an example of Straus answering an earlier question of Erdős (see [2, Theorem 2.2] and [12, Theorem 11.6]) shows that \( k \) can not be replaced by infinity in (1.1).
2. Reduction to a dynamical statement

When the set $A$ in Theorem 1.2 is of the form

$$A = \{ n \in \mathbb{N} : \| \theta + n \alpha \|_{\mathbb{R}/\mathbb{Z}} < \varepsilon \}$$

(or, more generally, is a Bohr set), the existence of a set $B \subset \mathbb{N}$ satisfying the conclusion of Theorem 1.2 is connected to the behavior of 3-term arithmetic progressions $\theta, \theta + \beta, \theta + 2\beta$ in $\mathbb{R}/\mathbb{Z}$ (or, more generally, in the underlying group). For arbitrary $A \subset \mathbb{N}$ we bridge the gap between the combinatorial statement Theorem 1.2 and the dynamical statement Theorem 1.4 using a dynamical variant of 3-term progressions defined as follows.

**Definition 2.1.** Given a topological system $(X, T)$, a point $(x_0, x_1, x_2) \in X^3$ is called a 3-term Erdős progression if there exists a strictly increasing sequence $n_1 < n_2 < \cdots$ of integers such that $(T \times T)^{n_i}(x_0, x_1) \to (x_1, x_2)$ as $i \to \infty$.

The role played in this paper by Erdős progressions parallels the role played by Erdős cubes in in [14]. Various other notions of dynamical progressions, for example those in [8, 14, 10], have already been used for related questions, but the one we use does not seem to have been defined previously. We remark that in group rotations all the notions of dynamical progressions agree with the conventional notion of arithmetic progression.

The next result connects the existence of Erdős progressions with sumsets.

**Theorem 2.2.** Fix a topological system $(X, T)$ and open sets $E, F \subset X$. If there exists an Erdős progression $(x_0, x_1, x_2) \in X^3$ with $x_1 \in E$ and $x_2 \in F$, then there exists some infinite set $B \subset \{ n \in \mathbb{N} : T^n x_0 \in E \}$ such that $\{ b_1 + b_2 : b_1, b_2 \in B, b_1 \neq b_2 \}$ is a subset of $\{ n \in \mathbb{N} : T^n x_0 \in F \}$.

**Proof.** Let $c : \mathbb{N} \to \mathbb{N}$ be a strictly increasing sequence such that $(T \times T)^{c(n)}(x_0, x_1) \to (x_1, x_2)$. Since $E$ is a neighborhood of $x_1$, by refining the sequence $c(n)$ we can assume without loss of generality that $\{ c(n) : n \in \mathbb{N} \} \subset \{ n \in \mathbb{N} : T^n x_0 \in E \}$.

We now construct the set $B \subset \{ c(n) : n \in \mathbb{N} \}$ inductively. First choose $b(1)$ in $\{ c(i) : i \in \mathbb{N} \}$ with $T^{b(1)} x_1 \in F$. Note that with this choice of $b(1)$ the set $(T^{-b(1)} F) \times F$ is a neighborhood of $(x_1, x_2)$. Next, choose $b(2)$ in $\{ c(i) : i \in \mathbb{N} \}$ with $b(2) > b(1)$ and $(T \times T)^{b(2)}(x_0, x_1) \in (T^{-b(1)} F) \times F$.

It follows that $T^{b(1)+b(2)} x_0 \in F$ and $x_1 \in T^{-b(2)} F \cap T^{-b(1)} F$.

Supposing that, by induction, we have found $b(1) < \cdots < b(n) \in \{ c(n) : n \in \mathbb{N} \}$ with $x_0 \in \bigcap_{1 \leq i < j \leq n} T^{-b(i)-b(j)} F$ and $x_1 \in \bigcap_{1 \leq i \leq n} T^{-b(i)} F$,
we choose $b(n + 1) \in \{ c(i) : i \in \mathbb{N} \}$ with $b(n + 1) > b(n)$ and
\[
(T \times T)^{b(n+1)}(x_0, x_1) \in \left( \bigcap_{1 \leq i \leq n} T^{-b(i)}F \right) \times F.
\]
This is possible because
\[
\left( \bigcap_{1 \leq i \leq n} T^{-b(i)}F \right) \times F
\]
is a neighborhood of $(x_1, x_2)$ and $(T \times T)^{(n)}(x_0, x_1) \to (x_1, x_2)$ as $n \to \infty$. Together with the inductive hypothesis, this implies
\[
x_0 \in \bigcap_{1 \leq i < j \leq n+1} T^{-b(i)-b(j)}F \quad \text{and} \quad x_1 \in \bigcap_{1 \leq i \leq n+1} T^{-b(i)}F
\]
concluding the induction. Taking $B = \{ b(i) : i \in \mathbb{N} \}$ finishes the proof. \hfill \Box

To deduce Theorem 1.2 from Theorem 1.4 using Theorem 2.2, we use the following version of the Furstenberg correspondence principle.

**Proposition 2.3 ([15, Theorem 2.10]).** Given a set $A \subset \mathbb{N}$ with positive upper Banach density there exists an ergodic system $(X, \mu, T)$, a Følner sequence $\Phi$, a point $a \in \text{gen}(\mu, \Phi)$, and a clopen set $E \subset X$ such that $\mu(E) > 0$ and $A = \{ n \in \mathbb{N} : T^n a \in E \}$.

**Proof of Theorem 1.2 (assuming Theorem 1.4).** Suppose $A \subset \mathbb{N}$ has positive upper Banach density. Our goal is to find an infinite set $B \subset A$ and a shift $t \in \mathbb{N}$ such that $A - t \supset \{ b_1 + b_2 : b_1 \neq b_2 \in B \}$.

Invoking Proposition 2.3 we find an ergodic system $(X, \mu, T)$, a point $a \in \text{gen}(\mu, \Phi)$, a Følner sequence $\Phi$, and a clopen set $E \subset X$ such that $\mu(E) > 0$ and $A = \{ n \in \mathbb{N} : T^n a \in E \}$. Using Theorem 1.4 we find $t \in \mathbb{N}$ and an Erdős progression of the form $(a, x_1, x_2) \in X^3$ such that $x_1 \in E$ and $x_2 \in T^{-t}E$. It now follows from Theorem 2.2, applied with $F = T^{-t}E$, that there exists an infinite set $B \subset \{ n \in \mathbb{N} : T^n a \in E \} = A$ such that
\[
A - t = \{ n \in \mathbb{N} : T^n a \in T^{-t}E \} \supset \{ b_1 + b_2 : b_1, b_2 \in B, b_1 \neq b_2 \},
\]
completing the proof. \hfill \Box

### 3. Proof of the dynamical statement

This section is devoted to the proof of Theorem 1.4. In Section 3.1 we start with some preliminary results describing the relevant factor maps and reduce Theorem 1.4 to the case when these factor maps are continuous, Theorem 3.2. In Sections 3.2 and 3.3 we introduce the measures that capture the statistics of Erdős progressions and study their properties and support. The proof culminates in Section 3.4.
3.1. Continuous factor maps to group rotations

Throughout this section, we make use of two types of factor maps from a system $(X, \mu, T)$ to another system $(Y, \nu, S)$.

- **Measurable factor maps**: a measurable function $\pi: X \to Y$ such that $\pi(\mu) = \nu$ and $\pi \circ T = S \circ \pi$ $\mu$-almost everywhere.

- **Continuous factor maps**: a continuous surjection $\pi: X \to Y$ such that $\pi(\mu) = \nu$ and $\pi \circ T = S \circ \pi$ everywhere.

If there exists a measurable factor map $\pi: X \to Y$, then $(Y, \nu, S)$ is called a factor of $(X, \mu, T)$.

In his proof of Szemerédi's theorem, Furstenberg [8] shows that in order to understand the behavior of 3-term dynamical progressions, it suffices to consider their projections onto the maximal group rotation factor. We use an analogous method to study Erdős progressions.

A **group rotation** is a system of the form $(Z, m, R)$, where $Z$ is a compact abelian group, $m$ is the Haar measure on $Z$, and $R: Z \to Z$ is a rotation of the form $R(z) = z + \alpha$ for a fixed element $\alpha \in Z$. Whenever $(Z, m, R)$ is a group rotation, we assume that the metric on $Z$ is chosen such that $z \mapsto z + w$ is an isometry for all $w \in Z$.

Every ergodic system $(X, \mu, T)$ possesses a maximal group rotation factor called its **Kronecker factor** (see [9, Section 3]). In general, the factor map from an ergodic system $(X, \mu, T)$ onto its Kronecker factor $(Z, m, R)$ is only a measurable factor map. The next lemma, however, shows that in many situations one can assume without loss of generality that the factor map onto the Kronecker factor is continuous, and this is key in our proof of Theorem 1.4.

**Proposition 3.1** ([15, Proposition 3.20]). Let $(X, \mu, T)$ be an ergodic system and let $a \in \text{gen}(\mu, \Phi)$ for some Følner sequence $\Phi$. Then there exists an ergodic system $(\tilde{X}, \tilde{\mu}, \tilde{T})$, a Følner sequence $\Psi$, a point $\tilde{a} \in \tilde{X}$ and a continuous factor map $\tilde{\pi}: \tilde{X} \to X$ such that $\tilde{\pi}(\tilde{a}) = a$ and $\tilde{a} \in \text{gen}(\tilde{\mu}, \Psi)$ and $(\tilde{X}, \tilde{\mu}, \tilde{T})$ has a continuous factor map to its Kronecker factor.

With the help of Proposition 3.1 we can reduce the proof of Theorem 1.4 to the following special case.

**Theorem 3.2.** Let $(X, \mu, T)$ be an ergodic system and assume there is a continuous factor map $\pi$ to its Kronecker. Let $a \in \text{gen}(\mu, \Phi)$ for some Følner sequence $\Phi$, and let $E \subset X$ be an open set with $\mu(E) > 0$. Then there exist $t \in \mathbb{N}$ and an Erdős progression of the form $(a, x_1, x_2) \in X^3$ such that $x_1 \in E$ and $T^t x_2 \in E$.

**Proof** that Theorem 3.2 implies Theorem 1.4. Let $(X, \mu, T)$ be an ergodic system, let $a \in \text{gen}(\mu, \Phi)$ for some Følner sequence $\Phi$ and let $E \subset X$ be open and have positive measure. Let $(\tilde{X}, \tilde{\mu}, \tilde{T})$, $\tilde{a}$ and $\tilde{\pi}$ result from an application of Proposition 3.1 and let $\tilde{E} := \tilde{\pi}^{-1}(E) \subset \tilde{X}$. As $(\tilde{X}, \tilde{\mu}, \tilde{T})$ has a continuous factor map to its Kronecker factor, we can apply Theorem 3.2 to find $t \in \mathbb{N}$ and an Erdős progression $(\tilde{a}, \tilde{x}_1, \tilde{x}_2) \in \tilde{X}^3$ with $\tilde{x}_1 \in \tilde{E}$ and $\tilde{x}_2 \in T^{-t}\tilde{E}$. It is then immediate that $(\tilde{\pi}(\tilde{a}), \tilde{\pi}(\tilde{x}_1), \tilde{\pi}(\tilde{x}_2))$ is an Erdős progression in $X^3$ with $\tilde{\pi}(\tilde{x}_1) \in E$ and $\tilde{\pi}(\tilde{x}_2) \in T^{-t}(E)$. \qed
The proof of Theorem 3.2 is given in Section 3.4.

3.2. The measure on Erdős progressions

The purpose of this section is to introduce and study a natural measure on the space of Erdős progressions. To define these measures, we recall the existence of disintegrations over measurable factor maps.

**Theorem 3.3** (See [4, Theorem 5.14]). Given a measurable factor map \( \pi : X \to Y \) between systems \((X, \mu, T)\) and \((Y, \nu, S)\), there exists a measurable map \( y \mapsto \mu_y \) defined on a full measure subset of \( Y \) and taking values in the space \( \mathcal{M}(X) \) of Borel probability measures on \( X \) with the following properties.

(i) For every bounded, measurable function \( f : X \to \mathbb{R} \), the function

\[
y \mapsto \int_X f \, d\mu_y
\]

is an almost everywhere defined and Borel measurable function on \( Y \) satisfying

\[
\int_D \left( \int_X f \, d\mu_y \right) \, d\nu(y) = \int_{\pi^{-1}(D)} f \, d\mu
\]

for all Borel sets \( D \subseteq Y \).

(ii) For \( \nu \)-almost every \( y \in Y \), we have \( \mu_y(\pi^{-1}(\{y\})) = 1 \).

(iii) Property (i) uniquely determines the map \( y \mapsto \mu_y \) in the sense that if \( y \mapsto \mu'_y \) is another measurable map from \( Y \) to \( \mathcal{M}(X) \) with these properties, then \( \mu_y = \mu'_y \) for \( \nu \)-almost every \( y \in Y \).

(iv) For almost every \( y \in Y \), we have \( T \mu_y = \mu_{Sy} \).

The measurable map \( y \mapsto \mu_y \) from \( Y \) to \( \mathcal{M}(X) \) provided by Theorem 3.3 is known as a disintegration of \( \mu \) over the factor map \( \pi \).

For the rest of this section, we fix an ergodic system \((X, \mu, T)\), let \((Z, m, R)\) denote its Kronecker factor, and further assume that there is a continuous factor map \( \pi : X \to Z \). Moreover, we fix a disintegration \( z \mapsto \eta_z \) of \( \mu \) over \( \pi \) as guaranteed by Theorem 3.3.

**Definition 3.4.** For each \( s \in X \), we define the measure

\[
\sigma_s = \int_Z \eta_z \times \eta_{2z-\pi(s)} \, dm(z) = \int_Z \eta_{\pi(s)+z} \times \eta_{\pi(s)+2z} \, dm(z)
\]

on \( X \times X \).

We think of \( \sigma_s \) as the natural measure to put on the set of pairs \((x_1, x_2) \in X \times X\) such that \((\pi(s), \pi(x_1), \pi(x_2))\) forms a 3-term arithmetic progression in \( Z \). Note also that \( \sigma_s \) is invariant under the transformation \( T \times T^2 \). For the remainder of this section, we use \( \sigma_s \) to denote the measure defined by (3.1).

**Lemma 3.5.** The map \( s \mapsto \sigma_s \) satisfies

\[
\int_X \sigma_s \, d\mu(s) = \mu \times \mu.
\]
Proof. For all \( f, g \in C(X) \), we have

\[
\int_X \int_{X \times X} f \otimes g \, d\sigma_s \, d\mu(s) = \int_X \int_Z \left( \int_X f \, d\eta_z \right) \left( \int_X g \, d\eta_{2z-\pi(s)} \right) \, d\mu(z) \, d\mu(s)
\]

\[
= \int_Z \left( \int_X f \, d\eta_z \right) \left( \int_X g \, d\eta_{2z-\pi(s)} \right) \, d\mu(z)
\]

\[
= \int_Z \left( \int_X f \, d\eta_z \right) \left( \int_X g \, d\mu \right) \, d\mu(z) = \int_{X \times X} f \otimes g \, d(\mu \times \mu),
\]

because \( z \mapsto \eta_z \) is a disintegration of \( \mu \).

\( \square \)

**Lemma 3.6.** Let \( \pi_1 : X \times X \to X \) denote the projection \( (x_1, x_2) \mapsto x_1 \) onto the first coordinate. Then \( \pi_1 \sigma_s = \mu \) for every \( s \in X \).

Proof. For any \( f \in C(X) \), we have

\[
\int_X f \, d(\pi_1 \sigma_s) = \int_{X \times X} (f \otimes 1) \, d\sigma_s
\]

\[
= \int_Z \left( \int_X (f \otimes 1) \, d(\eta_z \times \eta_{2z-\pi(t)}) \right) \, d\mu(z)
\]

\[
= \int_Z \left( \int_X f \, d\eta_z \right) \, d\mu(z) = \int_X f \, d\mu,
\]

as desired.

\( \square \)

**Lemma 3.7.** Let \( \pi_2 : X \times X \to X \) denote the projection \( (x_1, x_2) \mapsto x_2 \) onto the second coordinate. Then \( \frac{1}{2}(\pi_2 \sigma_s + T\pi_2 \sigma_s) = \mu \) for every \( s \in X \).

Proof. Denote by \( 2Z \) the subgroup \( \{z + z : z \in Z\} \) and let \( \xi \) denote its Haar measure. Ergodicity of \( R \) insure that \( Z = (2Z) + R(2Z) \) and that \( m = \frac{1}{2}(\xi + R\xi) \). In particular, for each \( s \in X \) there exists \( w \in Z \) such that either \( \pi(s) = 2w \) or \( \pi(s) = R(2w) \). In the first case

\[
\pi_2 \sigma_s = \int_Z \eta_2(w+z) \, dm(z) = \int_Z \eta_2z \, dm(z) = \int_{2Z} \eta_u \, d\xi(u),
\]

and in the second

\[
\pi_2 \sigma_s = \int_Z \eta_2(w+z)+\alpha \, dm(z) = \int_Z \eta_{2z+\alpha} \, dm(z) = \int_{2Z+\alpha} \eta_u \, d(R\xi)(u).
\]

Since \( T\eta_u = \eta_{Ru} \) and \( R^2\xi = \xi \), it follows that in either case

\[
\frac{1}{2}(\pi_2 \sigma_s + T\pi_2 \sigma_s) = \int_Z \eta_z \, d\frac{1}{2}(\xi + R\xi)(z) = \mu.
\]

\( \square \)

Another important lemma for us asserts that the map \( s \mapsto \sigma_s \) is continuous. This follows, as the proof reveals, from the hypothesis that the factor map \( \pi : X \to Z \) from \( (X, \mu, T) \) onto \( (Z, m, R) \) is continuous.
Lemma 3.8. The map $s \mapsto \sigma_s$ is continuous.

**Proof.** We use a similar argument to that in the proof of [15, Proposition 3.11]. Since continuous functions on $Z$ are uniformly continuous, for any $f, g \in C(Z)$ the map
\[
w \mapsto \int_Z f(w + z) g(w + 2z) \, dm(z)
\]
is continuous. Since $C(Z)$ is dense in $L^2(Z, m)$, the map in (3.2) is continuous even when $f, g \in L^2(Z, m)$. By definition of $\sigma_s$, for any $F, G \in C(X)$ we have
\[
\int_{X \times X} F \otimes G \, d\sigma_s = \int_Z \left( \int_X F \, d\eta_{\pi(s)+z} \right) \left( \int_X G \, d\eta_{\pi(s)+2z} \right) \, dm(z).
\]
Since the map $s \mapsto \pi(s)$ is continuous, it follows from the continuity of (3.2) that the map
\[
s \mapsto \int_{X \times X} F \otimes G \, d\sigma_s
\]
is continuous. Since finite linear combinations of functions of the form $F \otimes G$ are dense in $C(X \times X)$, we conclude that the map $s \mapsto \sigma_s$ is continuous, as desired. \qed

3.3. The support of the measure

Maintaining the same notation as in Section 3.2, we let $(X, \mu, T)$ be an ergodic system, $(Z, m, R)$ its Kronecker factor, and assume that $\pi$ is a continuous factor map from $(X, \mu, T)$ to $(Z, m, R)$. We also continue using $z \mapsto \eta_z$ for the disintegration of $\mu$ with respect to $\pi$.

As in [15, Equation (3.10)], for every $(x_1, x_2) \in X \times X$ we define
\[
\lambda_{(x_1, x_2)} = \int_Z \eta_{z + \pi(x_1)} \times \eta_{z + \pi(x_2)} \, dm(z)
\]
on $X \times X$. As in [15, Proposition 3.11], we have the following properties.

- The map $(x_1, x_2) \mapsto \lambda_{(x_1, x_2)}$ is continuous.
- The map $(x_1, x_2) \mapsto \lambda_{(x_1, x_2)}$ is an ergodic decomposition of $\mu \times \mu$, in the sense that
\[
\int_{X \times X} \lambda_{(x_1, x_2)} \, d(\mu \times \mu)(x_1, x_2) = \mu \times \mu,
\]
and $\lambda_{(x_1, x_2)}$ is ergodic for $T \times T$ for $\mu \times \mu$-almost every $(x_1, x_2) \in X \times X$.
- For every $(x_1, x_2) \in X \times X$, we have that $\lambda_{(x_1, x_2)} = \lambda_{(Tx_1, Tx_2)}$.

**Lemma 3.9.** For any $s \in X$ and for $\sigma_s$-almost every $(x_1, x_2) \in X \times X$, the measures $\lambda_{(s, x_1)}$ and $\lambda_{(x_1, x_2)}$ are equal.

**Proof.** Fix $s \in X$ and consider the set
\[
P_s := \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(s) + z, \pi(x_2) = \pi(s) + 2z \text{ for some } z \in Z\}.
\]
Combining (3.1) and property (ii) of Theorem 3.3 for the disintegration $z \mapsto \eta_z$, it follows.
that \( \sigma_s(P_s) = 1 \) and each \((x_1, x_2) \in P_s \) satisfies
\[
\pi(x_2) - \pi(x_1) = \pi(x_1) - \pi(s).
\]
Thus we have \( \lambda_{(x_1,x_2)} = \lambda_{(s,x_1)} \) by the defining formula (3.3) and translation invariance of \( m \).

In what follows, we denote by \( \text{supp}(\nu) \) the support of a Borel measure \( \nu \).

**Lemma 3.10.** Let \( W \) be a compact metric space, \( \mathcal{M}(W) \) the space of Borel probability measures on \( W \) endowed with the weak* topology, and \( \mathcal{F}(W) \) the space of closed, non-empty subsets of \( W \) endowed with the Hausdorff metric \( H \).

1. The map \( \nu \mapsto \text{supp}(\nu) \) from \( \mathcal{M}(W) \) to \( \mathcal{F}(W) \) is Borel-measurable.
2. If \( x \mapsto \rho_x \) is a measurable map from \( W \) to \( \mathcal{M}(W) \), then \( \{ x \in W : x \in \text{supp}(\rho_x) \} \) is a Borel set.

**Proof.**
1. Combining Theorem 17.14, Lemma 17.5, and Theorem 18.9 in [1], the result follows.
2. The map \( \psi_1(x) = \{ x \} \) from \( W \) to \( \mathcal{F}(W) \) is continuous and hence measurable. By part 1, the map \( \psi_2(x) = \text{supp}(\rho_x) \) from \( W \) to \( \mathcal{F}(W) \) is also measurable. Thus \( \psi(x) = (\psi_1(x), \psi_2(x)) \) from \( W \) to \( \mathcal{F}(W) \times \mathcal{F}(W) \) is measurable. The set \( \Omega = \{ (F_1, F_2) \in \mathcal{F}(W) \times \mathcal{F}(W) : F_1 \cap F_2 \neq \varnothing \} \) is closed, and therefore \( \{ x \in W : x \in \text{supp}(\rho_x) \} = \psi^{-1}(\Omega) \) is Borel.

**Lemma 3.11.** The disintegration \( z \mapsto \eta_z \) satisfies \( \mu(\{ x \in X : x \in \text{supp}(\eta_{\pi(z)}) \}) = 1 \).

**Proof.** Write \( G = \{ x \in X : x \in \text{supp}(\eta_{\pi(z)}) \} \), which is Borel measurable by Lemma 3.10, part 2. Since
\[
\mu(G) = \int_G \eta_z(G) \, dm(z),
\]
it suffices to show \( \eta_z(G) = 1 \) for \( m \)-almost every \( z \in Z \). By Theorem 3.3, part (ii), for almost every \( z \in Z \) we have \( \eta_z(\pi^{-1}(z)) = 1 \). If \( \eta_z(\pi^{-1}(z)) = 1 \), then \( \text{supp}(\eta_z) \subset \pi^{-1}(z) \) because \( \pi^{-1}(z) \) is a closed set of full measure. Thus, for \( m \)-almost every \( z \in Z \), we have \( \text{supp}(\eta_z) \subset \pi^{-1}(z) \) and hence \( \text{supp}(\eta_z) \subset G \). Since \( \text{supp}(\eta_z) \subset G \), we have \( \eta_z(G) \geq \eta_z(\text{supp}(\eta_z)) = 1 \) for \( m \)-almost every \( z \in Z \).

Write
\[
S = \{ (x_1, x_2) \in X \times X : (x_1, x_2) \in \text{supp}(\lambda_{(x_1,x_2)}) \}
\]
and note that Lemma 3.10, part 2, implies that \( S \) is a Borel subset of \( X \times X \). Our goal for the remainder of this section is to show that \( \sigma_s(S) = 1 \) for every \( s \in X \) (see Proposition 3.13).

**Proposition 3.12.** Fix a system \( (X, \mu, T) \) and a continuous factor map \( \pi \) to its Kronecker factor \( (Z, m, T) \). Fix also a disintegration \( z \mapsto \eta_z \) over its Kronecker factor \( (Z, m, R) \).
There is a sequence \(\delta(j) \to 0\) such that for almost every \(x \in X\) the following holds: for every neighbourhood \(U\) of \(x\) we have
\[
\lim_{j \to \infty} \frac{m\left(\{z \in Z : \eta_z(U) > 0\} \cap B(\pi(x), \delta(j))\right)}{m\left(B(\pi(x), \delta(j))\right)} = 1. \tag{3.5}
\]

Proof. Consider the map \(\Phi : Z \to \mathcal{F}(X)\) given by \(\Phi(z) = \text{supp}(\eta_z)\). This map is Borel measurable by Lemma 3.10 as it is the composition of two Borel measurable functions \(z \mapsto \eta_z\) and \(\nu \mapsto \text{supp}(\nu)\). Applying Lusin’s theorem [1, Theorem 12.8] for every \(j \in \mathbb{N}\), there is a closed set \(Z_j \subset Z\) with \(m(Z_j) > 1 - 2^{-j}\) such that \(\Phi|_{Z_j}\) is continuous. By uniform continuity of \(\Phi|_{Z_j}\), there exists a positive number \(\delta(j)\) such that for all \(z_1, z_2 \in Z_j\),
\[
d(x_1, x_2) \leq \delta(j) \quad \Rightarrow \quad H(\Phi(x_1), \Phi(x_2)) < \frac{1}{j}.
\]
Consider the set
\[
K_j = \left\{ z \in Z_j : m\left(B(z, \delta(j)) \cap Z_j\right) > \left(1 - \frac{1}{j}\right) m\left(B(z, \delta(j))\right) \right\}
\]
for each \(j \in \mathbb{N}\). Define
\[
\chi_j(z) = \frac{1}{m(B(0, \delta(j)))} \int_{Z_j} 1_{B(0, \delta(j))}(w - z) \, dm(w)
\]
and note that \(\chi_j(z) \leq 1\) for all \(z \in Z\). Since translations on \(Z\) are isometries, we have
\[
K_j = Z_j \cap \left\{ z \in Z : \chi_j(z) > \left(1 - \frac{1}{j}\right) \right\}. \tag{3.6}
\]
Using Fubini’s theorem, we deduce that
\[
\int_{Z_j} \chi_j(z) \, dm(z) = m(Z_j) > 1 - \frac{1}{2j}, \tag{3.7}
\]
which combined with \(\chi_j(z) \leq 1\) implies that
\[
m\left(\left\{ z \in Z : \chi_j(z) > \left(1 - \frac{1}{j}\right) \right\}\right) \geq 1 - \frac{j}{2j}. \tag{3.8}
\]
Combining (3.6) with (3.7) and (3.8), it follows that \(\sum_{j \in \mathbb{N}} m(Z \setminus K_j) < \infty\).

Let
\[
K = \bigcup_{M \geq 1} \bigcap_{j \geq M} K_j.
\]
Observe that, by the Borel-Cantelli lemma, \(m(K) = 1\). In view of Lemma 3.10, this implies that the set \(L := \{ x \in X : x \in \text{supp}(\eta_{\pi(x)}\} \cap \pi^{-1}(K)\) has \(\mu(L) = 1\). To finish the proof it thus suffices to show that any \(x \in L\) satisfies (3.5).

Fix a point \(x \in L\) and let \(U\) be a neighborhood of \(x\). Let \(z = \pi(x)\). Since \(z \in K\), there exists \(j_0 \in \mathbb{N}\) such that for all \(j \geq j_0\) we have \(z \in K_j\) and \(B(x, 1/j) \subset U\). We claim
that for all \( j \geq j_0 \), we have
\[
B(z, \delta(j)) \cap Z_j \subset H := \{ z \in Z : \eta_s(U) > 0 \}.
\] (3.9)

To verify this claim, let \( z' \in B(z, \delta(j)) \cap Z_j \) be arbitrary. Since \( H(\Phi(z), \Phi(z')) < 1/j \) and \( x \in \Phi(z) \), there exists \( x' \in \Phi(z') \) with \( d(x, x') < 1/j \). From \( d(x, x') < 1/j \) it follows that \( x' \in U \) and using \( x' \in \Phi(z') \) we conclude \( U \cap \Phi(z') \neq \emptyset \). Since \( \Phi(z') = \text{supp}(\eta_{x'}) \), it follows that \( \eta_{x'}(U) > 0 \) and hence that \( z' \in H \), proving that (3.9) holds, as claimed.

Since \( z \in K_j \), it follows from (3.9) and the construction of \( K_j \) that
\[
\frac{m(H \cap B(z, \delta(j)))}{m(B(z, \delta(j)))} \geq 1 - \frac{1}{j}
\]
for all \( j \geq j_0 \). We conclude that
\[
\lim_{j \to \infty} \frac{m(H \cap B(z, \delta(j)))}{m(B(z, \delta(j)))} = 1
\]
and the proof is complete. \( \square \)

**Proposition 3.13.** For every \( s \in X \), the set \( S \) defined in (3.4) satisfies \( \sigma_s(S) = 1 \).

**Proof.** Apply Proposition 3.12 to get a sequence \( \delta(j) \to 0 \) with the properties therein. Let \( L \) denote the set of points satisfying (3.5) which has full \( \mu \)-measure. We conclude from Lemma 3.6 that \( \sigma_s(L \times X) = 1 \) and conclude from Lemma 3.7 that
\[
1 = \mu(L) = \frac{\sigma_s(X \times L) + \sigma_s(X \times T^{-1}L)}{2},
\]
whence \( \sigma_s(X \times L) = 1 \). Thus
\[
\sigma_s(L \times L) = \sigma_s((X \times L) \cap (L \times X)) = 1.
\]

To prove \( \sigma_s(S) = 1 \), it therefore suffices to show \( L \times L \subset S \).

Let \( (x_1, x_2) \in L \times L \). Let \( U_1 \) be a neighborhood of \( x_1 \) and let \( U_2 \) be a neighborhood of \( x_2 \). To show \( (x_1, x_2) \in S \), we have to verify \( \lambda_{(x_1, x_2)}(U_1 \times U_2) > 0 \). For convenience, write \( \beta = \pi(x_2) - \pi(x_1) \). By definition,
\[
\lambda_{(x_1, x_2)} = \int_Z \eta_z \times \eta_{z+\beta} \, dm(z).
\]

Since \( x_1, x_2 \in L \), there exists some \( \delta > 0 \) such that
\[
\frac{m(\{ z \in Z : \eta_{z}(U_1) > 0 \} \cap B(\pi(x_1), \delta))}{m(B(\pi(x_1), \delta))} \geq \frac{3}{4} \quad (3.10)
\]
as well as
\[
\frac{m(\{ z \in Z : \eta_{z}(U_2) > 0 \} \cap B(\pi(x_2), \delta))}{m(B(\pi(x_2), \delta))} \geq \frac{3}{4} \quad (3.11)
\]
Observe that \( \{ z \in Z : \eta_z(U_2) > 0 \} - \beta = \{ z \in Z : \eta_{z+\beta}(U_2) > 0 \} \), and so (3.11) implies

\[
\frac{m(\{ z \in Z : \eta_{z+\beta}(U_2) > 0 \} \cap B(\pi(x_1), \delta))}{m(B(\pi(x_1), \delta))} \geq \frac{3}{4}. \tag{3.12}
\]

Define \( W = \{ z \in Z : \eta_z(U_1) > 0 \text{ and } \eta_{z+\beta}(U_2) > 0 \} \). By (3.10) and (3.12) it follows that \( W \) contains at least one-quarter of the ball \( B(\pi(x_1), \delta) \), which implies \( m(W) > 0 \). Since for all \( z \in W \) one has

\[(\eta_z \times \eta_{z+\beta})(U_1 \times U_2) > 0. \]

and \( m(W) > 0 \), it follows that \( \lambda_{(a_1, z_2)}(U_1 \times U_2) > 0 \) as desired.

\[\square\]

### 3.4. Proof of Theorem 3.2

To prove Theorem 3.2 we need one further lemma.

**Lemma 3.14** ([15, Lemma 3.18]). Let \((X, \mu, T)\) be an ergodic system, let \( a \in \text{gen}(\mu, \Phi) \) for some Følner sequence \( \Phi \). Then there exists a Følner sequence \( \Psi \) such that for \( \mu \)-almost every \( x_1 \in X \) the point \((a, x_1)\) belongs to \( \text{gen}(\lambda_{(a, x_1)}, \Psi) \).

**Proof of Theorem 3.2.** Fix a system \((X, \mu, T)\), \( a \in \text{gen}(\mu, \Phi) \) for some Følner sequence \( \Phi \), and \( E \subset X \) open with \( \mu(E) > 0 \). Assume \((X, \mu, T)\) has a continuous factor map \( \pi \) to its Kronecker factor \((Z, m, R)\). Applying Lemma 3.14 it follows that for \( \mu \)-almost every \( x_1 \in X \), we have \((a, x_1) \in \text{gen}(\lambda_{(a, x_1)}, \Psi) \) for some Følner sequence \( \Psi \). Since, in view of Lemma 3.6, the projection of \( \sigma_a \) onto the first coordinate equals \( \mu \), it follows that for \( \sigma_a \)-almost every \((x_1, x_2) \in X \times X\) we have \((a, x_1) \in \text{gen}(\lambda_{(a, x_1)}, \Psi) \). By Proposition 3.13, \( \sigma_a \)-almost every \((x_1, x_2) \in X \times X\) also has the property that \((x_1, x_2) \in \text{supp}(\lambda_{(x_1, x_2)}) \). Using Lemma 3.9 it follows that \( \sigma_a \)-almost every \((x_1, x_2) \in X \times X\) satisfies \( \lambda_{(x_1, x_2)} = \lambda_{(a, x_1)} \). We conclude that \( \sigma_a \)-almost every \((x_1, x_2) \) satisfies both of the following properties.

- \((x_1, x_2) \in \text{supp}(\lambda_{(a, x_1)}) \)
- \((a, x_1) \in \text{gen}(\lambda_{(a, x_1)}, \Psi) \)

Since orbits of generic points are dense in the support (see, eg., [15, Lemma 2.4]), we deduce that for \( \sigma_a \)-almost every \((x_1, x_2) \in X^2\), the point \((a, x_1, x_2) \in X^3\) is an Erdős progression.

To finish the proof, note that if \((a, x_1, x_2) \in X^3\) is an Erdős progression then \((a, x_1, x_2) \) satisfies the conclusion of Theorem 3.2 if and only if

\[(x_1, x_2) \in E \times T^{-t}E \]

for some \( t \in \mathbb{N} \). Therefore, the proof is complete once we can verify

\[
\sigma_a \left( E \times \left( \bigcup_{t \in \mathbb{N}} T^{-t}E \right) \right) > 0.
\]

Since \( \mu \) is ergodic and \( E \) has positive measure, the union \( \bigcup_{t \in \mathbb{N}} T^{-t}E \) covers all of \( X \) up
to a set of measure 0. Writing
\[ E \times \left( \bigcup_{t \in \mathbb{N}} T^{-t}E \right) = (E \times X) \cap \left( X \times \left( \bigcup_{t \in \mathbb{N}} T^{-t}E \right) \right), \]
we use Lemma 3.7 and then Lemma 3.6 to obtain
\[ \sigma_a \left( E \times \left( \bigcup_{t \in \mathbb{N}} T^{-t}E \right) \right) = \sigma_a (E \times X) = \mu(E) > 0, \]
as desired. \qed

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