3LP: A linear 3D-walking model including torso and swing dynamics

Salman Faraji and Auke J Ijspeert

Abstract
In this paper, we present a new mechanical model for biped locomotion, composed of three linear pendulums (one per leg and one for the whole upper body) to describe stance, swing and torso dynamics. In addition to a double support phase, this model has different actuation possibilities in the swing hip and stance ankle which produce a broad range of walking gaits. Without the need for numerical time-integration, closed form solutions help to find periodic gaits which could simply scale in certain dimensions to modulate the motion online. Thanks to linearity properties, the proposed model can potentially provide a computationally fast platform for model predictive controllers to predict the future and consider meaningful inequality constraints to ensure feasibility of the motion. Such a property comes from describing dynamics with joint torques directly and therefore, reflecting hardware limitations more precisely, even in the very abstract template space. The proposed model produces human-like torque and ground reaction force profiles, and thus, compared to point-mass models, it is more promising for the generation of dynamic walking trajectories. Despite being linear and lacking many features of human walking like center of mass excursion, knee flexion and ground clearance, we show that the proposed model can explain one of the main optimality trends in human walking, i.e. the nonlinear speed-frequency relationship. In this paper, we mainly focus on describing the model and its capabilities, comparing it with human data and calculating optimal human gait variables. Setting up control problems and advanced biomechanical analysis remains for future works.

Keywords
Bipedal walking, template model, human-like, 3D, linear model

1. Introduction
Humanoid robots are challenging to control mainly due to their complex structures and their floating base. During locomotion tasks, these systems introduce another complexity compared to wheeled or flying robots, which is the hybrid nature of stepping where the continuous model changes in each phase. It has always been challenging to balance these robots with only unilateral supporting forces from the environment. Also, creating a sequence of motion, the associated timing and the required control architecture are other important topics in controlling humanoid robots. The main objectives are therefore being human-like, energy efficient, versatile and of course agile like humans. In this paper, we are proposing a new template model that describes the main aspects of walking while being computationally very efficient. Such model can be very useful in modern control architectures from the computational perspective. It can also go beyond conventional template models such as the linear inverted pendulum (LIP) by producing more natural motions and faster walking speeds, resembling human locomotion.

The design of controllers should address many concerns like fast implementation, stability, robustness to unknown model parameters and the degree of dependency on sensory data. Besides, it is desired to handle speed transitions and large disturbance rejection in the same control framework. Candidate methods typically come with proper identification of the basin of attraction regarding system states, actuator limitations and violation of model assumptions. This identification is not always straightforward due to its non-linear nature, though it has been postulated that two steps are enough to stabilize in almost all conditions (Zaytsev et al., 2015). In this regard, model predictive control (MPC) is a powerful framework as it can find optimal policies constrained to certain actuation and state limitations. It can also predict if there is no feasible solution, to let the algorithm take a different decision.

Institute of Bioengineering, EPFL, Switzerland

Corresponding author:
Salman Faraji. EPFL STI IBI-STI BIOROB, Station 9, CH-1015 Lausanne, Switzerland.
Email: salman.faraji@epfl.ch
1.1. Hierarchical controllers

Recently, hierarchical control approaches have become popular, where a simple template model determines the overall dynamics in an abstract way and then, a detailed full-body inverse dynamics controller converts this behavior to individual actuator inputs (Faraji et al., 2014; Feng et al., 2013; Kuindersma et al., 2014). In dynamical systems, prediction of future evolution is mainly sensitive to the model and sensory data precision (Bhounsule et al., 2015). In hierarchical approaches similarly, dynamical matching between the template and the full model is crucial to ensure precise execution of the abstract plan. In this regard, we briefly review relevant template models proposed for walking, identify missing properties and to motivate the new model proposed in this paper.

1.2. Inverted pendulums

One of the earliest template models that roughly explains bipedal mechanics is the inverted pendulum (IP) (McGeer, 1990). In this model, a single mass rolls over a massless stick with a fixed length. IP is widely used to analyze passive walkers (McGeer, 1990) and the energetics of human walking (Kuo et al., 2005). Inspired by IP, many simple robots are built to walk naturally with minimal energy, injected in push-off, swing hip or both (Collins et al., 2005). Later, this model was simplified to the linear inverted pendulum (LIP) (Kajita et al., 2003), favoring analytical solutions instead of numerical integrations. With a proper modulation of zero moment point (ZMP) (Sardain and Bessonnet, 2004), many position controlled humanoid robots like ASIMO (Sakagami et al., 2002) perform walking via inverse kinematics methods. These algorithms are usually able to produce slow to moderate walking speeds. However, robots using the LIP method usually walk with crouched knees to keep the center of mass (CoM) at a constant height. In addition to increasing energy consumption, it is harmful to the robot in the long term and less human-like, although it provides full controllability.

More recent extensions of LIP contain one (Park and Kim, 1998) or two additional masses (Buschmann et al., 2007; Takenaka et al., 2009) in the legs to address swing dynamics while keeping the CoM height constant. For instance, in these extensions, the proposed model is formulated in a 2D space, and parameterized swing trajectories (in the sagittal and vertical planes) are used to generate desired gaits. In the work by Park and Kim (1998) specifically, sinusoidal profiles lead to closed form solutions with a few realistic approximations introduced. In these approaches, despite the advantage of describing ground clearance trajectories with a minimal coupling to the sagittal dynamics, the swing trajectory is yet imposed to the system, which might not match system’s natural dynamics. We look for a more generic model to work at different speeds and frequencies, not relying on parametric trajectories to tune.

1.3. Multi-link pendulums

Apart from these single-mass models, there are other non-linear extensions solved numerically. In the work by Byl and Tedrake (2008) and Asano et al. (2004), the IP model was extended to have two separate masses for each leg as well as a single mass at the hip level. Using similar actuation schemes, this model could produce compass gaits on 2D-constrained robots. In the work by Asano et al. (2004), the same model was modified to have another degree of freedom (DoF) in the swing leg to provide ground clearance. The stance leg, however, always remains straight in this version. In the work by Westervelt et al. (2007), this model was augmented with a torso and later, it was also used by Manchester and Umenberger (2014) to perform natural walking on uneven terrain, using a library of motion primitives. Another model with four masses in the legs, hip and torso was proposed by Gregg and Spong (2009) without any DoF in the stance knee. This model was used to generate walking trajectories with steering properties.

With the aim to remove impacts, a simpler model with two passive springs in the hips was proposed by Gomes and Ruina (2011). These springs are mainly motivated by elastic properties of human muscles. By exploiting torso motions, Gomes could find zero energy gaits. Another interesting complex model was proposed by Gregg et al. (2012) with a pelvis of a certain width in 3D and a mass in the center. The swing leg also had a DoF in the knee. This 3D model takes advantage of a limited transversal wrench in the contact point to facilitate steering. However, finding a periodic gait for such a complicated model is difficult and computationally expensive.

1.4. Spring-loaded pendulums

It is always questionable which template model produces more realistic motion from the viewpoint of geometry, torques or energy. The models above mainly addressed energetic and geometric similarities. However, specific ground reaction force profiles and the elastic behavior observed in human legs are better produced in another category of models based on a spring loaded inverted pendulum (SLIP, Blickhan, 1989). The simplest model in this category is composed of two massless springs (legs) connected to a point mass. Observations indicate a better description of energy exchange in this model over faster walking speeds and running, mimicking compliant properties of human tendons (Blickhan, 1989). Based on this model, Iida et al. (2009) built a hip-actuated robot walking in 2D with various springs, similar to the human muscles. Properties of the passive SLIP itself – without hip actuation – were widely explored later by Rummel et al. (2010). Using the concept of a virtual pivot point (VPP) to stabilize the torso, the model was also extended to have an upper body (Sharbafi and Seyfarth, 2015) which made the motion more human-like.
Fig. 1. Different key models introduced in the literature for walking. In this figure, the model is standing on the left leg, and the right leg is in the swing motion. Solid arrows show the direction of motion and degrees of freedom while gray arrows show actuation torques or push-off forces. For models without swing dynamics, we show the swing leg in gray color to implicitly indicate that the attack angle is a control authority. Note that some of these models are only in 2D while more advanced models are in 3D. Some models simulate the pelvis width, torso dynamics or ground clearance as well. Most of these models produce compass gait. However, some have ankle actuation or arc foot. In the comparison table, we mention important features such as knee flexion (for ground clearance), steering capabilities, 3D formulations, smooth profiles, the inclusion of torso and swing dynamics and linearity. By smoothness, we mean no collision and push-off impulse, but possibly describing the double support phase. Many of these models allow for torso pitch while we keep it fixed in 3LP for simplicity. Note that 3LP can describe steering like our previous work with LIP (Faraji et al., 2014), only if the pelvis width is set to zero. Despite being linear, 3LP offers many features which do not exist in other template models.
In Figure 1, we have briefly shown key models proposed for walking in the literature. Note that in some of them, passive springs are added to the hip actuators for energy storage, similar to humans. In this paper, however, we do not investigate elastic behaviors and energy-saving mechanisms.

1.5. Control difficulty

Except for LIP, all other models presented earlier require numerical integration to obtain time trajectories. Therefore, in a periodic walking paradigm, the Jacobian around a nominal solution linearizes the model and provides the framework for Floquet analysis or discrete controller designs (Rummel et al., 2010). This approach can be used to create an optimal library of primitives (Gregg et al., 2012; Kelly and Ruina, 2015; Manchester and Umenberger, 2014). However, online reaction to disturbances as well as inclusion of feasibility inequality constraints (that are often ignored in calculating a stable basin of attraction) limit the generality of this framework. MPC, on the other hand, is powerful in this regard. However, it requires simple and possibly linear models to facilitate online calculations. The LIP model, therefore, fits best in the MPC framework (Faraji et al., 2014; Herdt et al., 2010). MPC, its simpler version Linear Quadratic Regulator (LQR), and sometimes discretized LQR (DLQR) (Ogata, 1995) controllers are popular in stabilizing walking gaits and recovering large pushes (Byl and Tedrake, 2008; Kelly and Ruina, 2015). With nonlinear models, however, a library of optimal policies is generated offline, or a discrete transition model is considered at specific events like the CoM apex or heel-strike. Such controllers cannot react to perturbations very quickly, since adjusting the plan is not computationally affordable online.

1.6. Why three linear pendulums?

In this paper, we propose a more general version of LIP (Kajita et al., 2003) with three linear pendulums (called 3LP) that capture torso and swing dynamics in 3D. This model allows the prediction of the future at any time in closed form which is favorable by limited computational resources and MPC. Compared to LIP, 3LP-based trajectories are easier to track by inverse dynamics block in a hierarchical controller. In other words, CoM motion becomes more natural for the humanoid robot, since swing and torso dynamics are taken into account. Swing-leg trajectories are also more natural compared to the template models which track an imposed angle of attack with a stiff controller (Byl and Tedrake, 2008; Collins et al., 2005; Kelly and Ruina, 2015). Besides, using 3LP one can define meaningful torque limits in MPC frameworks instead of putting vague timing or step-size limits which do not precisely reflect physical facts about the real hardware. It should be noted, however, that the CoM height in 3LP is constant, similar to LIP.

The 3LP model provides direct access to the hip and ankle torques. These input dimensions let us find various types of gaits with simple closed form expressions. 3LP as a template model is therefore very useful for motion planning. Along with falling dynamics like IP (Kuo et al., 2005) and LIP (Kajita et al., 2001), torso-balancing hip torques are also part of 3LP, as demonstrated by Maufroy et al. (2011). The most outstanding feature of 3LP is in considering swing dynamics in a linear fashion, which allows us to calculate natural cycles. With LIP, however, a reference footstep plan is needed (Faraji et al., 2014). Considering Figure 1 again, few models in the literature consider this integral part of walking. In these models, due to nonlinearity, numerical integration is always needed to search for periodic gaits. In 3LP, however, we do not need to integrate the system or perform numerical optimizations.

This paper merely focuses on introducing the model, formulating equations, finding gaits and comparing them to human gaits. Setting up control problems remain for future works. In the next section, we will explain model details and the assumptions behind them. Next, a method based on geometrical symmetry is introduced to find different periodic gaits. We will demonstrate that for a human-like gait, actuation profiles in 3LP are similar to those in a human. Finally, we show that 3LP, despite being linear, quantitatively explains the main optimality trend in human gaits, i.e. speed-frequency relation.

2. 3LP dynamics

To capture the coupling of swing and torso dynamics, we have added two other pendulums to the standard LIP model, connected with a pelvis of a certain width. In this model, as shown in Figure 2, there are 2-DoF actuators in each hip and ankle. We visualize feet of limited size in Figure 2, only to mention the availability of ankle torques. The upper body (referred to as the torso) and the legs are each represented by a single mass. By construction (assuming ideal controllers), masses stay in horizontal planes of constant height and the torso is always upright without sagittal, frontal or transversal rotations. These assumptions are used in the work by (Gregg et al., 2012) as well to decouple sagittal and lateral dynamics.

Since the torso is connected to an accelerated frame (i.e. pelvis), a balancing torque is always needed to keep the torso upright. This torque is often calculated using the virtual pendulum concept (Sharbafi and Seyfarth, 2015), where the torso can pitch or roll freely. Inspired by rolling contact constraints that produce more human-like gaits (Hamner et al., 2013), we allow for a transversal wrench at the stance foot to help keep the pelvis orientation fixed. In 3LP, we do not consider steering properties as they make the model nonlinear. However, it is practically easy to steer the robot using an inverse dynamics layer, as we showed in our previous work (Faraji et al., 2014), where a simple LIP model was used. There is no need to add
heel-strike and push-off impulses in 3LP since our double support phase smoothly takes care of the contact transition. In SLIP-based models (Sharbafi and Seyfarth, 2015), the compliant springs automatically produce a double support phase and perform a weight transition without impulses. In 3LP, however, switching to double support is triggered when both horizontal components of the swing foot velocity become zero. This assumption is typically used in models with swing dynamics, like in the work by Gomes and Ruina (2011), though unlike 3LP, Gomes and Ruina removes double support for more simplicity (Gomes and Ruina, 2011).

In Figure 2, all external/internal forces and torques, as well as positions, are shown for each interesting point of the model, i.e. contacts, hips and pelvis center. These vector variables in our model are expressed in the Cartesian frame. One can easily write geometric relations as

\[ x_1 = [x_{1,v}, x_{1,v}, 0]^T + h_1 \hat{z} \]
\[ x_2 = x_1 + \frac{w_p d}{2} \hat{y} \]
\[ x_3 = x_1 - \frac{w_p d}{2} \hat{y} \]
\[ X_1 = x_1 + h_4 \hat{z} \]
\[ X_2 = [X_{2,v}, X_{3,v}, 0]^T \]
\[ X_3 = [X_{3,v}, X_{3,v}, 0]^T \]
\[ y_1 = x_1 + h_3 \hat{z} \]
\[ y_2 = x_2 + \frac{h_2}{h_1} (X_2 - x_2) \]
\[ y_3 = x_3 + \frac{h_2}{h_1} (X_3 - x_3) \]  

(1)

where \( w_p \) denotes the pelvis width, phase variable \( d = \pm 1 \) depends on left or right support phases, \( \hat{y} = [0 \ 0 \ 1]^T \) points left, \( \hat{z} = [0 \ 0 \ 1]^T \) points up, \( h_1 \) is the pelvis height, \( h_2 \) is the relative leg mass height, \( h_3 \) is the relative torso mass height and the offset variable \( h_4 \) from the pelvis indicates the point where an external perturbation is applied. These parameters as well as other position variables are shown in Figure 2. We can write total force equations for each mass \( i = 1, 2, 3 \)

\[ m_i (\hat{y} + g) = F_i \]  

(2)

where \( g \) denotes gravity and \( m_2 = m_3 \), assuming a symmetric model. The vectors \( F_i \) and \( f_i \) represent external and internal forces respectively. The total moment equations for each mass \( i = 1, 2, 3 \) are

\[ (X_i - y_i) \times F_i + (x_i - y_i) \times f_i + T_i + \tau_i = 0 \]  

(3)

where vectors \( T_i \) represent external wrenches and \( \tau_i \) represent internal torques, all defined in the Cartesian frame (refer to Figure 2). Finally, we can write total force and moment equations for the mass-less pelvis around the center point

\[ f_1 + f_2 + f_3 = 0 \]
\[ \tau_1 + \tau_2 + \tau_3 + (x_2 - x_1) \times f_2 + (x_3 - x_1) \times f_3 = 0 \]  

(4)

which link all variables together. In these equations, we consider \( X_1, X_2 \) and \( X_3 \) as independent variables and solve for others as dependent variables. Actuation possibilities for 3LP are selected as the stance foot \( T_1 \) and swing hip torques \( \tau_2 \) in the sagittal and lateral planes. Note that \( T_1 \) and \( F_1 \) represent disturbing external forces (at point \( X_1 \) on the torso) which are zero in normal conditions.

Now, differential equations governing the bipedal motion could be obtained for different phases. We consider a full stride, consisting of a double support followed by a single support phase, defined in Table.1. In double support, the weight is transferred from \( F_2 \) to \( F_3 \), and in single support, the leg with variables of subscripts 2 will swing forward. Over the next full stride phase, the supporting leg can be simply changed by altering the sign of variable \( d \).
Equations (2), (3) and (4) are not enough to solve the system completely. We require 12 new equations to find all accelerations in terms of control inputs and vice versa. This number comes from the fact that in total, we have 12 actuation variables in the system (torques in the hips and ankles). In single support, since the system is fully actuated, the additional constraints can simply be added by considering zero contact force in the swing foot and zero acceleration in the stance foot. In double support, however, because of the inherent redundancy, we need to make assumptions to simplify the system.

2.1. Single support

In this phase, the swing foot does not have any external forces and the stance foot is fixed on the ground

\[ F_2 = 0, \quad T_2 = 0, \quad \dot{X}_3 = 0 \]  

(5)

To further simplify the system, we consider two modes of input torques: constant \((U)\) and ramp \((V)\) profiles. Despite being linear, these profiles can still lead to a convincing match with human torque profiles, discussed in the next section. More complex terms might improve the matching precision, but make the equations more complicated. Besides, since there are fundamental geometric differences between the limbs in 3LP and a human (e.g. the number of articulated joints), a higher order terms might lead to an over-fitting which is not desired. Overall, we have eight input parameters \(U\) and \(V\) in the system

\[
\begin{bmatrix}
\tau_{2,y} \\
\tau_{2,x} \\
\tau_{3,y} \\
\tau_{3,x}
\end{bmatrix} = \begin{bmatrix}
U_{h,y} & U_{h,x} \\
U_{a,y} & U_{a,x}
\end{bmatrix} + \frac{t}{t_{ss}} \begin{bmatrix}
V_{h,y} \\
V_{h,x} \\
V_{a,y} \\
V_{a,x}
\end{bmatrix} = U + \frac{t}{t_{ss}} V
\]  

(6)

where \(t_{ss}\) stands for single support duration, \(t\) denotes the time from the beginning of the single support phase and subscripts \(a\) and \(h\) denote the ankle and hip respectively. Equations (5) and (6) together give 12 additional constraints that enable us to solve the system completely. To this end, we define the state vector \(X^{ss}(t)\) and the disturbance vector \(W\) as

\[
\begin{align*}
X^{ss}(t) &= \begin{bmatrix} X_{2,y}(t) & X_{2,x}(t) & X_{1,y}(t) & X_{1,x}(t) & X_{3,y} & X_{3,x} \end{bmatrix}^T \\
W &= \begin{bmatrix} F_{1,y} & F_{1,x} & T_{1,y} & T_{1,x} \end{bmatrix}^T
\end{align*}
\]  

(7)

where \(X_3\) is the fixed contact point location. Using Maple (Monagan et al., 2005), we combine all equations and obtain a linear DAE system symbolically

\[
\frac{d^2}{dt^2} X^{ss}(t) = C_X^{ss} X^{ss}(t) + C_U^{ss} U + C_V^{ss} V + C_W^{ss} W + C_d^{ss} d
\]  

(8)

where constant matrices \(C^{ss}\) merely depend on system parameters (refer to Appendix 2 for further details). In this equation, \(X^{ss}(t)\) is the state vector, \(U\) and \(V\) represent constant and ramp inputs, \(W\) represents perturbations and \(d\) denotes the phase variable. Next, we define the full state vector \(Q^{ss}(t)\) and the constant vector \(R\) as

\[
Q^{ss}(t) = \left[ X^{ss}(t)^T \right]^T \\
R = \left[ U^T \quad V^T \quad W^T \quad d \right]^T
\]  

(9)

Such abstract formulation facilitates the process of finding periodic gaits, discussed in the next section. Assuming constant external forces \(W\), we can solve this system analytically

\[
Q^{ss}(t) = A^{ss}(t) Q^{ss}(0) + B^{ss}(t) R
\]  

(10)

where the time-dependent transition matrices \(A^{ss}(t)\) and \(B^{ss}(t)\) describe state evolution over time. These matrices are in fact very easy to calculate, because of their simple structure

\[
A^{ss}(t) = \sum_{i=1}^{6} A_i^{ss} \gamma_i^{ss}(t) \\
B^{ss}(t) = \sum_{i=1}^{6} B_i^{ss} \gamma_i^{ss}(t)
\]  

(11)

where \(\gamma_i^{ss}(t) = e^{w_i^{ss} t}\) for \(i = 1 \ldots 4\), \(\gamma_5^{ss}(t) = 1\) and \(\gamma_6^{ss}(t) = t\). The four scalars \(w_i^{ss}\) are square roots of the eigenvalues associated to the non-zero part of the \(C_i^{ss}\) matrix (which has four dimensions). These values describe falling and swing dynamics, quantifying instability and divergence properties. A shorter pelvis height, for example, leads to \(w_i^{ss}\) values with larger magnitudes which mean that falling and swing dynamics become faster. \(A_i^{ss}\) and \(B_i^{ss}\) merely depend on constant system parameters, encoding evolution of the initial state and the effect of inputs respectively. Once these individual matrices are calculated offline, \(A^{ss}(t)\) and \(B^{ss}(t)\) can be easily calculated online by few arithmetic operations. Note that in 3LP, lateral and sagittal dynamics are decoupled, but their similarity leads to repeated eigenvalues.

2.2. Double support

In this phase, the two feet are fixed

\[
\dot{X}_3 = 0, \quad \ddot{X}_2 = 0
\]  

(12)

and contact forces are being transferred from \((F_2, M_2)\) to \((F_3, M_3)\). Once \((F_2\) and \(M_2\) become zero, the next single support phase starts, where the leg with subscript 2 performs swing motion. So far, equation (12) give us only four constraints, while eight are still missing.

In double support, we decided a linear transfer of weight from one leg to another. This means the vertical component of the ground reaction force (GRF) in the previous stance leg (which has been stationary during single support) will
go linearly to zero during double support. Such straightforward policy provides simple analytic solutions (compared to quadratic or other forms). Assume that

\[
\begin{bmatrix}
F_{z2} \\ T_{z2}
\end{bmatrix} = \left(1 - \frac{t}{tds} \right) \begin{bmatrix}
\alpha(t) \\ \beta(t)
\end{bmatrix}
\begin{bmatrix}
F_{z3} \\ T_{z3}
\end{bmatrix} = \frac{t}{tds} \begin{bmatrix}
\alpha(t) \\ \beta(t)
\end{bmatrix}
\]

where the minus sign (behind \(\gamma\)) is for symmetry, like before. The function \(\gamma(t)\) plays the same role as \(\gamma(t)\), but in the lateral plane. Now, we can remove \(\gamma(t)\) and \(\zeta(t)\) to make more explicit equations

\[
\tau_{2,y} = \frac{t}{tds} U_{h,y} + \left(1 - \frac{t}{tds}\right) \gamma(t) \quad (13)
\]

\[
\tau_{3,y} = \left(1 - \frac{t}{tds}\right)(U_{h,y} + V_{h,y}) + \frac{t}{tds} \gamma(t)
\]

where \(\tau_{2,y}\) starts from a function \(\gamma(t)\) \(_{t=0}\) in the beginning and converges to \(U_{h,y}\) at the end of the double support phase. Likewise, \(\tau_{3,y}\) starts from \((U_{h,y} + V_{h,y})\) and converges to the same \(\gamma(t)\) \(_{t=tds}\) at the end of the phase. The function \(\gamma(t)\) in fact represents the torque in the stance hip, which is a complex function of other variables in the system. However, we know linear trajectories of swing hip torques from equation (6). Therefore, equation (16) encodes a linear transition policy. Similar rules can be written for the lateral plane

\[
\tau_{2,x} = \frac{t}{tds} U_{h,x} + \left(1 - \frac{t}{tds}\right) \zeta(t) \quad (17)
\]

\[
\tau_{3,x} = \left(1 - \frac{t}{tds}\right)(U_{h,x} + V_{h,x}) + \frac{t}{tds} \zeta(t)
\]

Now, equations (12), (14), (15) and (18) provide the 12 constraints needed to solve the system in the double support phase. Note that except equation (12), the other equations are arbitrary choices to remove the redundancy. One might replace them with other policies for the same purpose. However, our simple assumptions preserve linearity and lead to a convincing match with human torque profiles, discussed in the next sections.

As mentioned earlier, 3LP switches to double support when the swing foot velocity becomes zero. This assumption, however, does not guarantee zero acceleration during the touchdown event. Therefore, a negligible discontinuity of torque and force profiles is unavoidable. In the simplest case, even preserving the continuity of horizontal GRF components lead to terms like \(\alpha(t)\) which are linear, but difficult to solve analytically. With our specific linear transition rules, however, the profiles are almost continuous. Some discontinuities happen only in the transversal torques which are small in magnitude, demonstrated later in Figure 10.

In double support phase, the state variable consists of base positions only

\[
X_{ds}(t) = \begin{bmatrix}
X_{2,x} & X_{2,y} & X_{1,x}(t) & X_{1,y}(t) & X_{3,x} & X_{3,y}
\end{bmatrix}^T
\]

while other vectors are the same as before. Again, we combine all equations and obtain a symbolic linear DAE system for the double support phase

\[
\frac{d^2}{dt^2}X_{ds}(t) = C_{ds}^X X_{ds}(t) + C_{ds}^U U + C_{ds}^V V + C_{ds}^W W + C_{ds}^d d
\]

where constant matrices \(C_{ds}\) merely depend on system parameters (refer to Appendix 2 for further details). Defining the full state vector \(Q(t)\) and the constant vector \(R\) as before, we can obtain a similar system of closed form solutions

\[
Q_{ds}(t) = A_{ds}(t) Q_{ds}(0) + B_{ds}(t) R
\]

where the time-dependent transition matrices \(A_{ds}(t)\) and \(B_{ds}(t)\) describe state evolution over time. Note that since \(X_{2}\) and \(X_{3}\) are constant here, the matrix \(A_{ds}(t)\) is partially diagonal with unit elements, keeping feet positions constant.
The transition matrices have the following simple structure

\[
A^d(t) = \sum_{i=1}^{4} A^d_i \gamma^d_i(t)
\]

\[
B^d(t) = \sum_{i=1}^{4} B^d_i \gamma^d_i(t)
\]

(22)

where \( \gamma^d_i(t) = e^{w^d_i t} \) for \( i = 1 \ldots 2 \), \( \gamma^d_3(t) = 1 \) and \( \gamma^d_4(t) = t \). The two scalars \( w^d_i \) are replicated square roots of the eigenvalues associated with the non-zero part of the \( C^d \) matrix.

2.3. Full stride

Having state transition matrices for both phases, we can now find closed form equations for the full stride phase

\[
Q(t) = A(t) Q(0) + B(t) R
\]

(23)

where \( t \) denotes the time from the beginning of the full stride phase (= beginning of the double support) and \( Q(t) \in \mathbb{R}^{12} \) is defined as

\[
Q(t) = [X(t)^T \dot{X}(t)^T]^T
\]

(24)

and \( X(t) \in \mathbb{R}^6 \) is

\[
X(t) = \begin{bmatrix}
X_{x,3}(t) & X_{y,3}(t) & x_{1,3}(t) & x_{1,y}(t) & X_{3,x} & X_{3,y}
\end{bmatrix}^T
\]

(25)

The transition matrix \( A(t) \) is defined as

\[
A(t) = \begin{cases}
A^d(t) & t \leq t_{ds} \\
A^s(t - t_{ds})A^d(t_{ds}) & 0 < t - t_{ds} \leq t_{ss}
\end{cases}
\]

(26)

and \( B(t) \) is

\[
B(t) = \begin{cases}
B^d(t) & t \leq t_{ds} \\
A^s(t - t_{ds})B^d(t_{ds}) & 0 < t - t_{ds} \leq t_{ss}
\end{cases}
\]

(27)

Note that we have used parameters \( t_{ss} \) and \( t_{ds} \) to calculate transfer matrices. The variable \( t_{ds} \) is crucial for double support calculations, though \( t_{ss} \) only determines the rate of time-increasing input components (\( V \)) in the single support. Therefore, one can easily scale \( t_{ss} \) and \( V \) such that the ratio remains constant in equation (6). The duration of a full stride phase is defined as \( t_{Stride} = t_{ds} + t_{ss} \). Figure 3 provides a demonstration of all transition matrices and timing variables in a full stride phase.

So far in this section, we have found transition matrices for the system and defined a full phase, consisting of a double support followed by a single support. We have also formulated matrices such that they are very fast for online calculation. The matrix \( A(t) \) is used to find the time evolution of our system, especially until the end of a full stride phase where \( t = t_{Stride} \). In the next section, we are going to find different open-loop periodic gaits based on the type of actuation desired.

3. Periodic gaits

Although various types of symmetric and asymmetric gaits can be generated (Rummel et al., 2010), in this paper, we only focus on symmetric gaits observed in normal human walking. In other words, given timing specifications, we find a space of different vectors that produce symmetric gaits. Each of these vectors contains initial conditions and constant inputs. The concept of symmetry could be encoded in a single matrix along with the constraint of zero foot velocity at the end of the stride.

Consider the vectors \( Q(t) \) and \( X(t) \) in equations (24) and (25), respectively. Indeed, zero-velocity constraints imply

\[
[X_{x,3}(t) \ X_{y,3}(t)]^T = \text{const}, \quad 0 \leq t \leq t_{Stride}
\]

\[
[X_{2,x}(0) \ X_{2,y}(0)]^T = 0
\]

\[
[X_{2,x}(t_{Stride}) \ X_{2,y}(t_{Stride})]^T = 0
\]

(28)

The first constraint is automatically satisfied by construction of matrices \( A^s(t) \) and \( A^d \) where we assumed a stationary stance foot (and thus zero acceleration). The other three constraints shall be satisfied in finding periodic gaits.

After a full stride phase, contact positions are exchanged as well, which could be used for visualization and control purposes. Otherwise, the matrix \( A(t) \) is enough for gait generation.

\[
T = \begin{bmatrix}
T_x & 0 \\
0 & I_{6 \times 6}
\end{bmatrix}
\]

(29)
where $T_X \in \mathbb{R}^{6 \times 6}$ is defined as

$$T_X = \begin{bmatrix} \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ 1 & \ldots & 1 \\ \ldots & 1 & \ldots \\ 1 & \ldots & 1 \end{bmatrix} \tag{30}$$

To implement the symmetry concept, we define relative difference vectors between the base, swing foot and stance foot positions. These vectors could be extracted from $Q(t)$ by the following matrix $M \in \mathbb{R}^{12 \times 12}$

$$M = \begin{bmatrix} M_X & 0 \\ 0 & M_X \end{bmatrix} \tag{31}$$

where $0$ and $I$ are zero and identity matrices. Sub-block are also defined as

$$M_X = \begin{bmatrix} I_{6 \times 6} \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ \ldots & 1 & \ldots \\ 1 & \ldots & 1 \end{bmatrix} \tag{32}$$

where $M_X$ extracts difference vectors from the state vector and $M_X$ is used to constrain swing foot velocities to zero. Comparing the difference vectors before and after a full symmetric stride, in the sagittal plane components are equal, and in the lateral plane they are opposite. Besides, we have to encode the velocity constraints of equation (28) as well. So, we define a matrix $S \in \mathbb{R}^{12 \times 12}$

$$O = \text{diag}(1, -1, 1, -1, 0, 0, \ldots, 1, -1, 0, 0, 1, 1) \tag{33}$$

where symmetry conditions are applied to the difference vectors and the base velocity while feet velocities are forced to zero. Now, consider in initial condition vector $Q(0)$ and inputs $R$. The state at the end of a full stride is

$$Q(t_{\text{stride}}) = A(t_{\text{stride}})Q(0) + B(t_{\text{stride}})R \tag{34}$$

Applying contact exchange matrix, the initial state $Q(0)$ for the next phase is

$$Q'(0) = TQ(t_{\text{stride}}) \tag{35}$$

The difference vectors extracted from $Q(0)$ and $Q(t_{\text{stride}})$ should satisfy symmetry conditions while initial and final foot velocities in $Q(0)$ and $Q(t_{\text{stride}})$ should be zero. One can write these conditions as

$$MQ(0) = OMQ(0) \tag{36}$$

Now, for a general vector of initial conditions and actuation parameters packed together

$$Y = [Q^T(0) \ R^T]^T \tag{37}$$

the following equation should hold if $Y$ represents a symmetric periodic gait

$$DY = 0 \tag{38}$$

where

$$D = [-M + OMTA(t_{\text{stride}}) \ OMTB(t_{\text{stride}})] \tag{39}$$

In fact, the matrix $D \in \mathbb{R}^{12 \times 25}$ compares the difference vectors before and after a full stride phase while forcing feet velocities to zero. Note that apart from state vectors, the hip and contact torques $U$ and $V$, the disturbance vector $W$ and the variable $d$ are considered here. However, it is meaningless in practice to consider a periodic gait with constant external disturbance. Initial contact positions $X_3$ would be set to zero to avoid redundant null space dimensions.

Remember that in the previous section, there were two variables to decide: $t_s$ and $t_d$. In this section, we find various types of gaits by selecting different combinations of actuation and timing variables. We do so by considering the matrix $D$ which is, in fact, a function of timing variables. Any periodic solution (a vector containing initial states and actuation inputs) should lie in the null space of the $D$ matrix.

In other words, solution manifolds are found by combining different null vectors of the matrix $D$. A solution contains initial states and actuation parameters where available actuators are swing hip and stance ankle torques (in sagittal and lateral directions), in constant $U$ and time-increasing modes $V$. Note that only columns attributed to non-zero values in the solution vector are selected. In other words, we normally exclude columns related to contact positions $X_3$ and disturbances $W$.

In the rest of this paper, we use human-like body parameters for numerical simulations, where mass distributions and geometries are taken from (De Leva, 1996). Table 2 lists these parameters for two adult-size and kid-size models, used further in this paper.

### Table 2. Parameters for adult-size and kid-size models used in simulations.

| Model       | Adult-size | Kid-size | Unit |
|-------------|------------|----------|------|
| Total mass  | 70         | 30       | kg   |
| Body length | 1.7        | 1.0      | m    |
| $h_1$       | 0.89       | 0.52     | m    |
| $h_2$       | 0.32       | 0.19     | m    |
| $h_3$       | 0.36       | 0.22     | m    |
| $m_1$       | 45.7       | 19.6     | kg   |
| $m_2$       | 12.15      | 5.2      | kg   |
| $w_p/2$     | 0.1        | 0.06     |      |

### 3.1. Pseudo-passive gaits manifold

First, we would like to know whether the system has any pseudo-passive walking pattern or not. By pseudo-passive, we mean a gait in which the swing hip and stance contact torques are zero. The term pseudo indicates that in the
Odic solutions. These solutions are all without swing hip or stance parts to correspond to a null space containing an infinite number of periodic solutions. These solutions are all without swing hip or stance parts. We can simply calculate corresponding timing. This singular value refers to sagittal direction, while absolute singular values of \( D \) for an adult-size model. In these plots, we fix the double support time \( t_{ds} = 0.3 \) s and only change \( t_{gs} \). It is notable that around \( t_{stride} = 0.86 \) s, the system shows zero singular values which correspond to a null space containing an infinite number of periodic solutions. These solutions are all without swing hip or stance contact actuation referred to as pseudo-passive gaits.

stance hip, the actuators might produce or dissipate power and the system is not strictly passive. This term also reflects the fact that in our linear model, the legs are stretched or shortened by prismatic actuators, as part of the model construction. For pseudo-passive gaits, we remove columns associated with \( X_3 \), \( U \), \( V \) and \( W \) in \( D \) and obtain a reduced matrix \( D_{\text{passive}} \in \mathbb{R}^{12 \times 11} \). Figure 4 shows absolute singular values of \( D_{\text{passive}} \) over time.

One can clearly see that there is a time \( t_{\text{relax}} = t_{\text{relax}} = 0.86 \) s where the system shows an additional zero singular value. \( t_{\text{relax}} \) can be found by a simple root-finding algorithm. This singular value refers to sagittal direction, while the other singular value (which is always zero) refers to the lateral direction. We can simply calculate corresponding singular vectors of \( D_{\text{passive}} \) and find a null space manifold. Note that there is only one lateral solution as the value of \( d \) should be ±1. However, the solution in the sagittal plane can be scaled by any arbitrary positive or negative value to obtain different modulated speeds. Therefore, the manifold of pseudo-passive compass gaits, in this case, is only one-dimensional. A demonstration of normal pseudo-passive compass gaits can be found in Figure 6. Also, from Figure 4 we simply conclude that for any stride time other than \( t_{\text{relax}} \), the system cannot demonstrate pseudo-passive gaits.

3.2. Actuated gaits manifold

In this part, we are going to find manifolds of motion which can benefit from swing hip actuation and CoP modulation as well. These inputs of course contain constant and time varying components for both sagittal and lateral dynamics, as discussed in the previous section. With these inputs, we can pump energy into the swing leg and brake at the end of the phase to produce faster swing motions. We can also apply contact torques which modulate the CoP and resemble the fact the CoP in a human goes forward from the heel to the toes during the swing phase. Here we only remove columns associated to \( X_3 \) and \( W \) in \( D \) and obtain a reduced matrix \( D_{\text{active}} \in \mathbb{R}^{12 \times 19} \). The corresponding absolute singular values are shown in Figure 5 over time.

Surprisingly, the system does not have a distinct zero singular value at \( t_{\text{relax}} \) like before. However, it has seven zero singular values that produce a larger null space at any given stride time \( t_{\text{stride}} \). The corresponding actuated gait manifold is not seven-dimensional, however. The variable \( d \) should always be ±1, reducing the total dimensions to six. Besides, one can also choose active actuators and the desired speed to reduce the dimensionality further and find a unique solution.

As demonstrated in pseudo-passive gaits, at \( t_{\text{relax}} \), \( D_{\text{passive}} \) has a certain two-dimensional null space. Now, what if we calculate a seven-dimensional null space using \( t_{\text{relax}} \) and \( D_{\text{active}} \) instead of \( D_{\text{passive}} \)? Could we still find a pseudo-passive gait out of this larger null space? It is in fact possible, even though no distinct zero singular value is observed in Figure 5. The reason is that the rank of actuation space at \( t_{\text{relax}} \) is equal to five in the seven-dimensional null space manifold of \( D_{\text{active}} \). This means if we constrain all of them to zero for pseudo-passive walking, we only lose five ranks. The other two ranks could still be dedicated to the variable \( d \) and the desired speed, like before. So, the null space manifold of actuated gaits already encompasses the one for pseudo-passive gaits and we do not need to calculate them separately. In the next subsection, we are going to show a few examples of walking gaits, using these null spaces.

3.3. Numerical examples

In addition to the pseudo-passive gait which has a certain timing \( t_{\text{relax}} \), we are going to show other same-speed gaits with different timings and actuation patterns. From the singular value analysis, we have seven singular vectors \( n_i, 1 \leq i \leq 7 \) for the matrix \( D_{\text{active}} \). Each of these vectors have similar dimensions with \( Y \) in equation (37), consisting of an initial state, with the contact point \( (X_3) \) at the origin, a resting swing foot \( (X_2 = 0) \), a certain actuation pattern (encoded in \( U \) and \( V \) ) and no disturbance \( (W = 0) \) of course. We pack them together in a matrix \( N = [n_1 \ n_2 \ \ldots \ n_7] \). We also select a smaller yet more human-like choice of \( t_{\text{gs}} = 0.1 \) s (Cappellini et al., 2006), and \( t_{\text{ax}} = 0.6028 \) s calculated by the pseudo-passivity root finding procedure. The choice of speed will be \( v_{\text{des}} = 1 \) m/s.

Given the constant matrix \( N \), we want to find a vector of coefficients \( \delta = [\delta_1 \ \delta_2 \ \ldots \ \delta_7]^T \) that combine the columns in \( N \) and produce unique solutions. We consider...
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Fig. 5. Absolute singular values of the matrix $D_{\text{active}}$ with respect to the stride time $t_{\text{stride}}$, plotted for an adult-size model. In these plots, we have fixed the double support time $t_{\text{ds}} = 0.3$ s. It is notable that the system does not show a unique zero singular value anymore. However, there are always seven default zero singular values that can produce gaits for any choice of $t_{\text{stride}}$. These gaits are indeed actuated with arbitrary swing hip or stance ankle torque profiles.

minimizing squared torques as an estimate of the mechanical power. The quadratic minimization is formulated as

$$
\text{minimize} \quad U^T U + V^T V
$$

s.t.

$$
\begin{bmatrix}
S_{X_{2,x}} \\
S_U \\
S_V \\
S_d
\end{bmatrix}
N \delta =
\begin{bmatrix}
-v_{\text{des}} \times t_{\text{stride}} \\
U \\
V \\
\pm 1
\end{bmatrix}
$$

(40)

where $S_{X_{2,x}}$ selects the row corresponding to the sagittal swing foot position ($X_{2,x}$), $S_U$ and $S_V$ correspond to the constants $U$ and $V$ and $S_d$ corresponds to the constant $d$. The specific way of encoding $v_{\text{des}}$ in the optimization moves the initial swing foot position backward to find gaits with an average velocity equal to $v_{\text{des}}$. Note that the equality constrained quadratic optimization of equation (40) has, in fact, a closed form solution and there is no need to solve it iteratively.

Now consider the following scenarios:

1. **Pseudo-passive walking.** This is calculated as mentioned earlier. In this gait, the hip and ankle torques are all zero.

2. **Long double support.** In this gait, we enforce ankle torques to zero by adding more constraints to the optimization

$$
\begin{bmatrix}
S_{U_a} \\
S_{V_a}
\end{bmatrix}
N \delta = 0
$$

(41)

where $S_{U_a}$ and $S_{V_a}$ select rows corresponding to the stance ankle torques. Keeping the same $t_{\text{stride}}$, we double $t_{\text{ds}}$ and decrease $t_{ss}$ accordingly. Note that now, the walking cannot be pseudo-passive anymore. The optimization, therefore, finds non-zero hip torques to produce the same speed and stride length.

3. **Stage walking.** Here, we constrain the ankle torques to zero like before. Instead of optimizing the hip torques, however, we optimize lateral velocities in the cost function. In this case, the biped walks on a straight line without lateral bounce.

4. **CoP modulation.** Given the length of the feet, the total weight and the timing of single support, we can calculate a constantly increasing ankle torque $\tau_{\text{CoP}}$, acting in the sagittal plane to move the CoP forward. In this scenario, we force the time-increasing sagittal component of the ankle torques to $\tau_{\text{CoP}}$ (and other components to zero) by adding the following constraints to the optimization in equation (40)

$$
\begin{bmatrix}
S_{U_a} \\
S_{V_a}
\end{bmatrix}
N \delta =
\begin{bmatrix}
0 \\
0 \\
\tau_{\text{CoP}}
\end{bmatrix}^T
$$

(42)

The result is a gait with time-increasing ankle torque profiles and minimal hip actuation.

5. **LIP-like.** In this case, keeping the original timing, we change the model of the robot. We transfer most of the...
weight of each leg to the torso, and also move the three masses closer to the pelvis by decreasing $h_2$ and $h_3$. Again, we disable all ankle torques as well.

The 3D geometry of resulting gaits are shown in Figure 7 while a detailed diagram of each stride is shown in Figure 6. The accompanied Multimedia Extension 1 demonstrates different features of 3LP, while Multimedia Extension 2 shows movies of the five previously mentioned scenarios. It can be concluded from Figure 7 that changing different parameters does not have a major effect on the overall geometry of walking. However, since it is important to match dynamics of the full model, we investigate dynamic properties of these walking scenarios as well.

Although CoM trajectories look similar in Figure 7, they have very different characteristics regarding velocity variations, shown in Figure 8. The LIP-like model shows a significant variation in the sagittal velocity. It is not so obvious how swing and torso dynamics affect this motion at first glance. By torso dynamics, we mean the torques required by the stance hip to keep the torso always upright. These torques are not necessarily zero since the pelvis has non-zero accelerations. Therefore, torso-balancing hip torques can affect the CoM motion considerably, especially since the torso is relatively heavy. Moreover, although the swing foot has a smaller weight compared to other parts of the body, it can be seen from Figure 9 that the swing leg has relatively large velocities during single support. Such motion increases the kinetic energy quadratically and therefore results in a significant workflow. In our model, we have described these effects in a simplified and linear fashion, yet capturing important couplings between the three pendulums.

Taking a closer look at Figure 8 reveals that even maximal CoP modulation still does not change velocity profiles considerably. This means the difference between pseudo-passive and LIP-like walking is way larger than that between pseudo-passive and CoP-modulated walking. In other words, CoP authority can at most convert the pseudo-passive gait to the CoP-modulated gait. The available CoP authority is hardly enough to convert the pseudo-passive gait to LIP-like gait, and this gap increases substantially at faster walking speeds. Although here the speed is moderate, we can easily infer that LIP as a template model can only operate in a very limited range of walking speeds. Remember that in fact an inverse dynamics or kinematics approach eventually realizes the template motion with the full model by exploiting all control authorities of the robot (including the CoP). Therefore, this motivates us to not modulate the CoP in a template level and leave the control authority free for the underlying full-body controllers to mimic the template motion as precisely as possible.

In this section, we introduced an easy method to find manifolds of periodic motions without any numerical forward simulation of the system. Once these manifolds were found, we also showed how to find individual solutions, based on the type of actuation and timing desired. We only considered gaits with minimal hip torques here. However, to go further, we would like to investigate the effect of timing and walking speed as well. Such investigation reveals interesting energetic properties of 3LP, discussed in the next section.
Fig. 8. Sagittal vs. lateral CoM velocity trajectories for different scenarios discussed. Note that the LIP-like model produces significant sagittal variations. The pseudo-passive gait shows moderate variations, however, indicating that swing and torso dynamics clearly reduce these variations. Long double support also reduces variations in both directions. By modulating the CoP, although lateral motions remain similar to the pseudo-passive gait, sagittal variations reduce even more, and the motion becomes smoother. Finally, one can see that the stage walking has no lateral motion compared to the other gaits. In general, increasing the double support duration and CoP modulation both have a similar smoothing effect on CoM velocities. However, this does not induce any argument on energy efficiency.

4. Comparison with human data

Compared to LIP, the 3LP model is much more similar to human locomotion, because it describes falling dynamics, swing motion, torso balance, lateral stepping and double support features altogether. In addition to geometric similarities, we investigate ground reaction forces and joint torques to compare the underlying dynamics that result in such geometric similarity. For this purpose, regarding available data from human subjects (Eng and Winter, 1995), we selected similar model parameters and timing, calculated periodic manifolds and found solutions with the same speed and CoP modulation pattern. The resulting trajectories are demonstrated in Figure 10 together with average human profiles. In the following, we discuss various similarities observed in this figure.

4.1. Sagittal dynamics

From the last column of Figure 10, one can observe a good match of hip extensor and ankle plantar flexor torques as well as anterior-posterior (A/P) ground reaction forces. This is despite a relatively fast walking speed and large step sizes (about 80% of the leg length). Note also that the constant and time-increasing components of hip/ankle torques are roughly enough to describe major trends in human curves. Nonlinear profiles in 3LP are, however, related to the stance leg and those degrees of freedom which are not directly controlled by desired input torques. The LIP model does not have hip torques and produces larger A/P GRF, because of the different CoM trajectories shown in Figure 8.

4.2. Vertical GRF

By model construction, the CoM height is constant, and we do not expect two peaks in the vertical GRF profiles, similar to SLIP-based models (Rummel et al., 2010; Sharbafi and Seyfarth, 2015). However, the general trapezoidal shape is preserved, thanks to our double support phase and its linear transition rules. Note that the LIP model can produce a similar profile too. The main consequence of a constant-height profile is walking with crouched knees which looks less human-like compared to many other template models listed in Figure 1.

4.3. Lateral dynamics

In 3LP, we only minimize hip/ankle torques to find a unique solution out of the large manifold of all symmetric periodic gaits. This minimization is not necessarily realistic and human-like, as it leads to wider lateral steps, more bouncing, and often larger transversal ankle torques. A better cost function on energy might produce more human-like gaits, although it is doubted by Workman and Armstrong (1986) that optimal gaits merely depend on energy terms. There might be terms related to balancing performance as well, at least in lower speeds. In faster speeds also (like the
human profiles demonstrated in the work by Eng and Winter, 1995), humans take closer steps laterally compared to our model.

In Figure 7, we see that 3LP keeps general trends like double peaks in the medio-lateral (M/L) GRF, but cannot precisely describe other torque profiles. Humans normally tend to step as close as possible to minimize lateral motions and energy (Kuo, 1999) (similar to stage walking in Figure 7). However, humans swing their foot over an arc shape to avoid self-collision. Such a fine motion is feasible in 3LP but requires a better objective function in equation (40). Note that the sagittal swing motion can influence lateral dynamics as well (Collins et al., 2005; Kuo, 1999), possibly through transversal moments. This might be another reason for the discrepancy observed between different lateral curves since 3LP completely decouples the lateral and sagittal dynamics. In the LIP model, though, there is no pelvis included. However, we consider a gait with the same step-width as 3LP. Although LIP does not have hip torques, M/L GRF forces are still similar to 3LP, as shown in Figure 7.

4.4. Transversal rotation torques

3LP preserves the general trend of transversal torques observed in humans, but not matching precisely, especially in the ankle. One major reason is that arm motions, and pelvic rotations are not considered in the model. Another important reason is wider lateral steps in 3LP compared to humans, which require larger transversal moments. Note that transversal torques are needed to keep the torso upright and straight ahead during the swing phase, compensating the moment produced by the swing leg. In LIP, however, since there is no pelvis and swing leg, we do not expect transversal torques.

4.5. Energetics

Apart from dynamics profiles, it is always interesting to investigate energy flow in the model and compare it with the human. Remember that one of the main motivations behind developing 3LP was to match humanoid dynamics as precisely as possible in the template space. Such matching could be viewed from a power-flow perspective as well, inspired by the fact that humans can walk very efficiently. Compared to LIP, 3LP can additionally describe swing and torso dynamics which are important aspects of locomotion, especially at faster speeds. Since the pelvis has accelerations in sagittal and lateral directions, the whole upper-body requires non-zero hip torques to balance. Our model successfully encodes the influence of these torques on the horizontal motion.

4.5.1. Energy trade-offs. Although the mass of the swing leg is relatively small, its peak velocity is about three times larger than the torso which has a heavier mass. Therefore, the peak kinetic energy of the swing leg is quite comparable with the torso. Such energy comes from both the accelerated pelvis that the swing leg is attached to and swing hip torques that can modulate swing dynamics. In a tradeoff with falling dynamics, this phenomenon can explain optimal speed-frequency relations observed in the metabolic cost of human walking (Bertram, 2005). Although 3LP does not include many important walking determinants like heel-toe motions, knee flexions and CoM excursions (Saunders et al., 1953), surprisingly, it can capture the main optimality trend in human walking.

4.5.2. Comparison method. We consider two parameters of locomotion: stepping frequency and forward speed. Similar to Bertram (2005), we calculate the efficiency of walking (called economy in the original paper) for different speed-frequency combinations. Efficiency is defined as the inverse of the cost of transport, which is the total energy consumed per unit of mass, per unit of distance traveled. There are many ways to calculate such mechanical power in our model. In fact, 3LP does not simulate muscles and the exact geometry of a human and thus, it is unable to precisely reconstruct the human efficiency surface based on the metabolic cost. However, by integrating the positive part of the mechanical power, we can approximate a portion of this energy which still plays a major role in the overall energy landscape according to Anderson and Pandy (2001). We are not able to model other costs like muscle activation, maintenance and shortening heat rates since we do not have muscles in 3LP.

4.5.3. Fixed double support time. We calculate the net positive work divided by the total mass over a grid of different speed-frequency combinations that cover slow to fast walking gaits (Bertram, 2005). Figure 12(a) demonstrates corresponding efficiency contours for the choice of $t_{ds}/t_{stride} = 10\%$. It is surprising that the model shows a peak line within the range of speed-frequencies explored by Bertram (2005). Such peak, in fact, demonstrates the trade-off between swing and falling dynamics. The former increases with higher frequencies, since more energy is pumped into the swing leg and taken out by braking at the end of the swing phase (Doke et al., 2005). The latter, however, increases with step-size which results in larger CoM velocity variations.

We repeated the same process for two other choices of double support time: 20% and 30%. The resulting peak lines are demonstrated in Figure 12(a) as well as the optimal trend of human data (shown in black), taken from the work by Bertram (2005) (the case of constrained speed). Here we use the same average mass (66 kg) and height (1.7 m) of human subjects in the work by Bertram (2005) as well as average mass distributions reported by De Leva (1996). Although 3LP is successful in identifying the trade-off, using the choice of constant double support time ratio, it fails to match the human data.
The optimal choice of double support ratio in 3LP versus human data (Cappellini et al., 2006), over normal walking speed-frequency curves reported by Bertram (2005). In this figure, for each particular choice of speed and frequency, an optimization has found the optimum double support ratio by maximizing walking efficiency. Since the mass and body height of individual subjects were not reported, we used the average body mass 66 kg and height 1.7 m in the work by Bertram (2005). For both human and 3LP data, we also used a linear model to find the trends.

and 3LP, where a line fitted to 3LP values is found as

$$\frac{t_{ds}}{t_{\text{stride}}} = -0.11 v + 0.36$$  \hspace{1cm} (44)
Fig. 12. Efficiency (economy) of walking, i.e. the inverse of cost of transport (CoT) which is calculated as positive work on the CoM over a unit of distance traveled, normalized by body mass. Here, we set the body mass to 66 kg and body height to 1.7 m with normal human-like mass distribution (De Leva, 1996). (a) Contours of 3LP-based walking efficiency, calculated for the choice of \( t_{ds}/t_{stride} = 10\% \). Peaks for this double support ratio as well as other choices of 20% and 30% are plotted versus the optimal relations observed in human. 3LP can demonstrate a peak line, but not matching with human data. (b) 3LP-based efficiency, using the double support ratio of equation (44). Now, 3LP can provide a roughly close prediction. (c) Optimal walking efficiency, where for each choice of speed and frequency, an optimization finds the double support time ratio that maximizes the efficiency. Here, 3LP does not match human trends anymore. (d) The smoothed contours of human walking efficiency, taken from the work by Bertram (2005).

This line is indeed very similar to the optimal trend line in human data, although it predicts shorter double support times in faster speeds. Repeating the same grid search of Figure 12(a) with the particular choice of double support time in equation (44), we obtain Figure 12(b).

4.5.5. Optimal double support time. If we do not limit the optimization to the optimal trend line of human data and perform it everywhere for all combinations of speeds and frequencies, 3LP finds a different trend line, shown in Figure 12(c). This is expected, however, since our simplifying linear assumptions in the double support phase can only approximate the real curves in Figure 10 to some extent. To predict optimal trends in human data without using any information about human walking, we need more complex double support formulations and better optimality criteria. The results shown in Figure 12 are very promising and important, despite the fact that:

- no impact or push-off is considered;
- CoM height is constant;
- no foot clearance is modeled;
- heel-toe motions and knee flexions are not included;
- the torso has no rotation in any direction;
- weight support cost on the stance leg is not considered;
- walking-independent metabolic cost components are not modeled.

Therefore, although the optimal speed-frequency trend is roughly predicted in Figure 12(b), the overall 3LP-based efficiency surface is not matching the human data shown in Figure 12(d). Linear models are conventionally expected to be a linearization of the actual nonlinear system, performing better in near-stance (small step-size) conditions, where the geometry of 3LP is closer to the actual system. 3LP does not model heal-toe-knee motions and intrinsically simplifies them with an extensible prismatic actuator. However, here it appears to demonstrate a similar mechanical energy flow, even with large step sizes. In future works, we would explore other components of walking energy, keeping 3LP as a core model. Inclusion of other costs might reconstruct the actual human efficiency surface more precisely.

5. Conclusion

Compared to most other template models listed in Figure 1, our proposed model considers swing and torso dynamics in
Rather, we focused on biomechanical analysis and hip muscles. Humans increase the frequency as well, which consequently affects swing dynamics and demands more energy from the muscles. The energy flow in 3LP is also more similar to the human, providing more natural motions for humanoids which have similar anthropomorphic features.

In 3LP, the pelvis width parameter has a linear effect on the lateral motion. This parameter is used to find lateral ankle/hip torques and to determine a natural lateral foot placement. This is compared to many other methods like the method proposed by Faraji et al. (2014), where the two feet are forced to be apart to avoid self-collision. In the literature, the timing and footstep locations are imposed without enough knowledge of internal dynamics. In our model, however, since swing dynamics is included and a zero final velocity is assumed for the feet, natural periodic gaits automatically come out of equations. The assumption mentioned relieves the need to calculate impact forces and determines the timing and periodicity conditions as well.

3LP can predict human walking profiles quite well, even in relatively fast speeds where the linearity assumption might not be realistic. Although IP-based models can demonstrate CoM excitations quite well, a nonlinear nature makes them less suitable for highly complex robots that require online planning. There are more advanced versions of IP-based models in the literature, including torso and swing dynamics. However again, nonlinear equations cannot be used in a per-time-step MPC control. The proposed model is based on a reasonable trade-off between geometric and dynamic matching, favoring fast computation properties.

We showed that 3LP could also describe the exchange of energy between swing and falling dynamics, despite a constant CoM height. We would like to mention that the vertical excursion of CoM, even in a very fast walking gait at 2 m/s is still about 5 cm (Gard et al., 2004) which is quite negligible compared to a step-size of about 1 m (pelvis excursion is about 7 cm, however). CoM excursion depends on step-size which does not increase linearly with the walking speed. Humans increase the frequency as well, which consequently affects swing dynamics and demands more energy from the hip muscles.

We did not set up control frameworks in this paper. Rather, we focused on biomechanical analysis and similarities to human locomotion that can be inspiring for generating more precise abstract plans, used to control humanoid robots. In brief, 3LP provides:

- swing dynamics;
- torso balancing torques;
- double support phase;
- hip/ankle actuation possibilities;
- natural lateral motion;
- natural periodic gaits;
- pseudo-passive compass gait;
- computational advantages;
- possibility to consider hip torque limits;
- optimal speed-frequency relation similar to human.

Despite limiting factors such as:

- constant CoM height;
- flat vertical GRF profiles;
- stretched legs;
- no steering capabilities yet;
- no arm motion;
- no torso pitch/roll DoF;

3LP can be extended to have two more degrees of freedom for the torso. It can also include quadratic actuation terms to produce more accurate torque profiles. It should be noted that without a pelvis, steering is still possible as demonstrated in our previous work (Faraji et al., 2014) which is based on LIP. Steering makes 3LP nonlinear, but one can compromise the lateral motion and let inverse dynamics find proper actuation patterns. Without the pelvis in 3LP, one actually needs to impose the lateral bouncing as we did in the LIP-like scenario. An important role of pelvis is, therefore, to produce a natural lateral motion which automatically emerges from the optimization of equation (40). This makes the model more generic without the need to impose the lateral bouncing. Note that the natural step width is not easy to determine, since it depends on the leg length, stepping frequency and available ankle torques.

It is worth mentioning that an imposed step width which is similar to the natural step width (found when including the pelvis) might not produce very different GRF profiles. This is induced from Figure 10 by comparing GRF profiles of the LIP-like scenario and the full 3LP model. As a result, one can easily remove the pelvis and allow for steering with the cost of imposing the step width. This can be realized by finding a natural step width with the pelvis and then imposing it in another version of 3LP which does not have the pelvis anymore. The second model remains natural in terms of GRF, linear and of course suitable for producing steering motions.

In future work, we are going to replace LIP with 3LP in our MPC-based control framework (Faraji et al., 2014). 3LP can be used in both state estimation and planning levels. It can possibly exchange information about the CoM or the feet with the full model. Dynamic equations of 3LP can, therefore, predict future states in a MPC framework or help...
to filter sensory noises by Kalman filtering. The linear equations of 3LP can indeed provide a non-periodic formulation of the system as well, where footsteps might act as inputs to the system instead of the hip torques. This reformulation is similar to planning footsteps in MPC (Faraji et al., 2014) or capturability frameworks (Koolen et al., 2012).

Among all advantages offered by 3LP, we favor its capability to produce more natural motions. In this regard, we can expect our inverse dynamics layer to track the template model more precisely and therefore, being able to produce more human-like motions. Including torso and swing dynamics in template models make them computationally more complicated, but closer to reality. In the literature, complex models are mostly used for offline trajectory generation. In 3LP, however, we partially include these additional dynamic effects while keeping the model still suitable for online control and future prediction. The focus of this paper was to introduce the model and to explore different capabilities. In future, we integrate it with our hierarchical walking controller to achieve more dynamic motions. This paper is accompanied with two multimedia extensions, demonstrating general features of 3LP together with the five gait scenarios discussed. All codes used in this article are available online at http://biorob.epfl.ch/page-99800-en.html.

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6. Appendix 1

Index to multimedia extensions

Archives of IJRR multimedia extensions published prior to 2014 can be found at http://www.ijrr.org, after 2014 all videos are available on the IJRR YouTube channel at http://www.youtube.com/user/ijrrmultimedia

Table of multimedia extensions.

| Extension | Media Type | Description |
|-----------|------------|-------------|
| 1         | Video      | Model properties |
| 2         | Video      | Different walking gaits Model codes |
| 3         | Code       | |

Appendix 2

Model matrices

The parametric matrices $C$ describing 3LP dynamics in single and double support phases are found by Maple. These matrices are sparse and have duplicate elements, since the sagittal and lateral dynamics are similar. Defining auxiliary parameters

\[
\begin{align*}
\beta &= h_3 + h_1 \\
\gamma &= h_2 - h_1 \\
\delta_1 &= m_1 u h_1 + m_2 v^2 \\
\delta_2 &= m_1 u h_1 + 2m_2 v^2 \\
\delta_3 &= -m_1 h_1 + m_2 v - m_2 h_1 \\
\delta_4 &= \left( -1 + \frac{2f}{l_d} \right) h_1 + v
\end{align*}
\]
the matrices in equation (8) take the form

\[
C_{ss}^X = \frac{g}{c_1} \begin{bmatrix}
\left(-\frac{c_2}{h_2} + m_2\nu\right)I & \frac{m_1 h_1 + m_2 h_1 I}{h_2} & 0 \\
m_2 h_1 I & (m_1 h_1 - m_2 \nu)I & 0 \\
0 & 0 & c_3 I
\end{bmatrix}
\]

\[
C_{ss}^U = \frac{h_1}{c_1} \begin{bmatrix}
\frac{c_2}{h_2} I & -\frac{c_2}{h_2} I \\
\frac{h_2}{h_1} I & -I \\
0 & 0
\end{bmatrix}
\]

\[
C_{ss}^V = \frac{t}{t_{ss}} C_{ss}^U
\]

\[
C_{ss}^W = \frac{h_1}{c_1} \begin{bmatrix}
\frac{\nu}{h_2} I & -\frac{\nu}{h_2} I \\
\frac{h_2}{u} I & -I \\
0 & 0
\end{bmatrix}
\]

\[
C_{ss}^d = \frac{g w P}{c_1} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{c_2}{h_2} I & m_2 \nu I & 0
\end{bmatrix}
\]

and the matrices in equation (21) are found as

\[
C_{ps}^d s = \frac{h_1}{c_2} \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\frac{\nu}{h_2} I & 0 \\
\frac{\nu}{h_2} I & \frac{\nu}{h_2} I \\
\frac{h_2}{h_1} I & -I \\
\frac{h_2}{h_1} I & -I \\
0 & 0
\end{bmatrix}
\]

\[
C_{ps}^U = \frac{h_1}{c_2} \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
C_{ps}^W = \frac{h_1}{c_2} \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

where \( I \in \mathbb{R}^{2 \times 2} \) is an identity matrix and \( J = [0 \ 1]^T \). The closed form transition matrices in equations (10) and (21) are then calculated symbolically based on these matrices. Altogether, offline calculations require about 3.7k Floating-Point Operations (FLOPS) in a maple-optimized code which take few microseconds on an average core-i5 computer if implemented in c++. The results are constant matrices in equations (11) and (22) which can be used online with a much faster speed.