ON LOCAL INTEGRABILITY CONDITIONS OF JET GROUPOIDS

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Abstract. A Jet groupoid $R_q$ over a manifold $X$ is a special Lie groupoid consisting of $q$-jets of local diffeomorphisms $X \rightarrow X$. As a subbundle of $J_q(X \times X)$, a jet groupoid can be considered as a nonlinear system of partial differential equations (PDE). This leads to the concept of formal integrability. On the other hand, each jet groupoid is the symmetry groupoid of a geometric object, modelled as a section $\omega$ of a natural bundle $\mathcal{F}$. Using the jet groupoids, we give a local characterisation of formal integrability for transitive jet groupoids in terms of their corresponding geometric objects.

1. Introduction

In two articles [6], Lie introduced pseudogroups and formulated their defining equations as differential invariants of the corresponding pseudogroup. Vessiot continued in [13] with the calculation of necessary conditions for integrability. Later, Pommaret [9] applied Spencer’s approach [10] to pseudogroups and showed that they are all given as symmetry transformations of geometric objects $\omega$ on natural bundles $\mathcal{F}$.

In this paper, we use the language of jet groupoids which provides an efficient language for this theory, including conceptional proofs. A Jet groupoid consists of the algebraic solutions of the pseudogroup equations and is thus also defined by a section of a natural bundle. We express the prolongation and projection of jet groupoids by sections of new natural bundles. All considerations are local. As the main result, we obtain conditions on the sections $\omega$ of $\mathcal{F}$ that classify all formally integrable jet groupoids that can be defined by sections of $\mathcal{F}$.

The last section gives explicit calculations for the well-known example of a Riemannian metric on a two-dimensional manifold using computer algebra.

2. Preliminaries

Before turning to jet groupoids, we shortly introduce the underlying Lie groupoids and their action on fibre bundles. This is done to fix the notation, recent introductory books on Lie groupoids are [7] and [8], which also provide many further references.

Definition 2.1. A groupoid $G$ is a small category with invertible morphisms.
The set of objects, denoted by $G^{(0)}$, is called the base. The set of morphisms is denoted by $G^{(1)}$ and has the projections source and target to the base:

$$s : G^{(1)} \to G^{(0)} : g \mapsto x \quad t : G^{(1)} \to G^{(0)} : g \mapsto y \quad \text{for} \quad g : x \to y \in G^{(1)}.$$ 

Composition of morphisms induces a partial multiplication that is defined whenever source and target match:

$$\mu : G^{(1)} \times_{G^{(0)}} G^{(1)} \to G^{(1)} : (g, h) \mapsto gh$$

with $G^{(1)} \times_{G^{(0)}} G^{(1)} = \{(g, h) \in G^{(1)} \times G^{(1)} | s(g) = t(h)\}$. Via $\iota : G^{(0)} \hookrightarrow G^{(1)} : x \mapsto 1_x$, the base is embedded in $G^{(1)}$ as the identity morphisms. A groupoid is called transitive if $G(x, y) = \{g \in G^{(1)} | s(g) = x, t(g) = y\} \neq \emptyset$ for all $x, y \in G^{(0)}$.

**Definition 2.2.** A groupoid $G$ is called a Lie groupoid if $G^{(0)}$ and $G^{(1)}$ are smooth¹ manifolds where $s$, $t$, $\mu$, $\iota$ and the inversion are smooth, $s$, $t$ being surjective submersions.

In the case of Lie groupoids, $G^{(1)} \times_{G^{(0)}} G^{(1)}$ turns into the fibre product and the isotropy groups $G(x, x)$ are Lie groups [8, Thm 5.4]. Important examples of Lie groupoids are gauge groupoids $PP^{-1} := P \times_H P$ for Lie groups $H$ and principal $H$-bundles $P$.

**Definition 2.3.** Let $G$ be a Lie groupoid and $\pi : \mathcal{F} \to G^{(0)}$ be a fibre bundle. A right groupoid action of $G$ on $\mathcal{F}$ is a smooth map $\mathcal{F} \times_{G^{(0)}} G^{(1)} \to \mathcal{F}$ with $f(ab) = (fa)b$ and $f1_x = f$ whenever $f \in \mathcal{F}_x = \pi^{-1}(x)$ and $a, b \in G^{(1)}$ can be composed.

### 3. Jet Groupoids and Natural Bundles

We will now define jet groupoids as Lie groupoids over a fixed base manifold $G^{(0)} = X$ of dimension $n$. The $q$-th jet bundle $J_q(X \times X)$ over the trivial bundle $X \times X$ provides source $s = pr_1$ and target $t = pr_2$ as projections on the first and second copy of $X$. Restricted to the open subset $\Pi_q \subset J_q(X \times X)$ of invertible jets, the chain rule induces a partial multiplication on $\Pi_q$, which is respected by the natural projections $\pi^{q+r}_q : J_{q+r}(X \times X) \to J_q(X \times X)$.

**Definition 3.1.** $\Pi_q$ is called the full jet groupoid of order $q$ and a jet groupoid $\mathcal{R}_q$ is a subbundle of $\Pi_q$, closed with respect to all groupoid operations.

For the treatment of natural bundles and the projection of jet groupoids, it is helpful to consider the isotropy groups $\Pi_q(x, x)$. They are all isomorphic to $\text{GL}_q = \text{GL}_q(\mathbb{R}^n)$, the Lie group of $q$-jets of diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ leaving the origin fixed. The structure of $\text{GL}_q$ was studied by Terng [12]. By the construction of $\text{GL}_{q+1}$ there is an exact sequence

$$1 \xrightarrow{\cdot} K_{q+1} \xrightarrow{\pi_{q+1}} \text{GL}_{q+1}(\mathbb{R}^n) \xrightarrow{\pi_q} \text{GL}_q(\mathbb{R}^n) \xrightarrow{\cdot} 1$$

deфиниit the normal subgroup $K_{q+1} := \ker(\pi_{q+1}) \leq \text{GL}_{q+1}$. The projection can be identified with $\pi_q^{-1} : \Pi_{q+1}(x, x) \to \Pi_q(x, x)$ for each $x \in X$. $\text{GL}_{q+1}$ is called first principal prolongation of $\text{GL}_q$ in [4] (following [2]).

¹In this paper, smooth means $C^\infty$. 
All jets with fixed target $y_0 \in X$ define a principal $\text{GL}_q$-bundle $P_q := \Pi_q(-, y_0)$. When changing from groupoids to the bundle point of view, left and right $\text{GL}_q \cong \Pi_q(y_0, y_0)$-actions must be swapped to obtain the equations in [6], [13] and (right) principal bundles. We recover $\Pi_q$ as the gauge groupoid $P_q P_q^{-1}$ (via $(g, h) \mapsto gh^{-1}$). The sequence (3.1) implies $P_{q+1}/K_{q+1} \cong P_q$. Writing $K_{q+1}$ for all kernels $\ker(\pi_q^{q+1}) \leq \Pi_{q+1}(x, x)$, we obtain a kind of commutation law $K_{q+1} f_{q+1} = f_{q+1} K_{q+1}$ as sets for all $f_{q+1} \in \Pi_{q+1}$.

**Definition 3.2.** A fibre bundle $\mathcal{F} \to X$ is called *natural bundle* if there exists a $q \in \mathbb{N}$ and a groupoid action of $\Pi_q$ on $\mathcal{F}$. A section $\omega$ of $\mathcal{F}$ is called *geometric object*.

$P_q$ is a natural bundle by right $\Pi_q$-multiplication. In fact, all natural bundles $\mathcal{F}$ with typical fibre $F := \mathcal{F}_{y_0}$ are associated to $P_q$ as $\mathcal{F} \cong P_q \times_{\text{GL}_q} F$. This is done by splitting $u \in \mathcal{F}$ into $u = u_{y_0} f_q$ with $u_{y_0} \in F$ and $f_q \in P_q$, unique up to elements of $\text{GL}_q \cong \Pi_q(y_0, y_0)$.

If not stated otherwise, we assume the natural bundles $\mathcal{F}$ to have fibres $F$ that are homogeneous $\text{GL}_q$-spaces.

**Proposition 3.3.** Each section $\omega$ of a natural bundle $\mathcal{F}$ defines a jet groupoid $\mathcal{R}_q(\omega) = \text{Stab}_q^\omega(\omega)$. Conversely, each transitive jet groupoid $\mathcal{R}_q$ defines a natural bundle $\mathcal{F}$ with section $\omega_0$, such that $\mathcal{R}_q$ is the full symmetry groupoid $\text{Stab}_q^\omega(\omega_0)$ of $\omega_0$.

**Proof.** Define the symmetry groupoid $\text{Stab}_q^\omega(\omega)$ via the $\Pi_q$-action on $\mathcal{F}$

$$\Phi_\omega : \Pi_q \to \mathcal{F} : f_q \mapsto \omega(t(f_q)) f_q$$

as the kernel $\ker_\omega(\Phi_\omega) = \{ f_q \in \Pi_q | \Phi_\omega(f_q) = \omega(s(f_q)) \}$ or by the exact sequence

$$0 \to \text{Stab}_q^\omega(\omega) \to \Pi_q \xrightarrow{\omega \circ s} \mathcal{F}$$

of bundles over $X = s(\Pi_q)$. The $\Pi_q$-action implies that $\text{Stab}_q^\omega(\omega)$ is a groupoid since it is closed under $\mu$, $\iota$ and inversion. As $F$ is homogeneous, $\Phi_\omega$ is surjective and of constant rank. By the implicit function theorem, $\text{Stab}_q^\omega(\omega)$ is a Lie groupoid. Each $f_q \in \Pi_q$ can be modified by $g_q \in \text{GL}_q$ such that $\omega(y) f_q g_q = \omega(x)$, so $\text{Stab}_q^\omega(\omega)$ is transitive.

The transitivity of $\mathcal{R}_q$ implies that all isotropy groups are isomorphic to some $G_q \leq \text{GL}_q$, so choose $y_0 \in X$ and set $\mathcal{F} := \mathcal{R}_q(y_0, y_0) \setminus \Pi_q(-, y_0) \cong P_q \times_{\text{GL}_q} \text{GL}_q / G_q$ with the section

$$\omega_0 : X \to \mathcal{F} : x \mapsto \mathcal{R}_q(y_0, y_0) \mathcal{R}_q(x, y_0) = \mathcal{R}_q(x, y_0).$$

The condition for $r_q \in \text{Stab}_q^\omega(\omega_0)$ is $\omega_0(y) r_q = \omega_0(y)$ or explicitly $\mathcal{R}_q(y_0, y_0) r_q = \mathcal{R}_q(x, y_0)$. This is equivalent to $r_q \in \mathcal{R}_q$. \(\square\)

The groupoids defined by different sections may be the same, e. g. if $\mathcal{F}$ is a vector bundle, $\omega$ and $\lambda \omega$ for a constant $\lambda \neq 0$ describe the same groupoid. Proposition 3.3 can be extended to non-homogeneous fibres $F$ with the additional assumption that $\Phi_\omega$ has constant rank ($\omega(y) f_q = \omega(y)$ implies $\text{im}(\omega) \subseteq \text{im}(\Phi_\omega)$). Vessiot [13] calls the coordinate expressions of $\Phi_\omega(r_q) = \omega(s(r_q))$ Lie form. The sequence (3.2) is due to Pommaret [9].
4. Systems of PDE and Formal Integrability

The next step is to consider a jet groupoid as a system of PDE (see e.g. [3], [5], [10]) in order to study its integrability. The prolongation of a groupoid acting on a fibre bundle was introduced by Ehresmann [2] (see also [4]).

**Definition 4.1.** A subbundle \( \mathcal{R}_q \subseteq J_q(\mathcal{E}) \) of the \( q \)-th order jet bundle of a fibre bundle \( \mathcal{E} \to X \) is called system of PDE and solutions are (local) sections of \( \mathcal{R}_q \). The \( r \)-prolongation is the subset
\[
\mathcal{R}_{q+r} := J_r(\mathcal{R}_q) \cap J_{q+r}(\mathcal{E}), \quad r \in \mathbb{Z}_{\geq 0}
\]
and
\[
\mathcal{R}^{(s)}_{q+r} := \pi_{q+r}^{q+r+s}(\mathcal{R}_{q+r+s}) \subseteq \mathcal{R}_{q+r}, \quad r, s \in \mathbb{Z}_{\geq 0}
\]
is called projection. \( \mathcal{R}_q \) is called formally integrable if \( \mathcal{R}_{q+r} \) is a fibre bundle and the projections \( \pi_{q+r}^{q+r+s} : \mathcal{R}_{q+r+s} \to \mathcal{R}_{q+r} \) are surjective submersions for all \( r, s \in \mathbb{Z}_{\geq 0} \).

An effective criterion to decide the formal integrability of a system of PDE was given by Goldschmidt. It reduces the infinite number of conditions to a single one, once the symbol \( g_q = (S^3T^*X \otimes V(\mathcal{E})) \cap V(\mathcal{R}_q) \) is 2-acyclic. For an introduction to symbols and Spencer cohomology see [3], [5], [10] or [9, ch. 7.2] for details in the case of jet groupoids.

**Theorem 4.2** ([3, Thm 8.1]). Let \( \mathcal{R}_q \subseteq J_q(\mathcal{E}) \) be a system of order \( q \) on \( \mathcal{E} \), such that \( \mathcal{R}_{q+r} \) is a subbundle of \( J_{q+r}(\mathcal{E}) \). If the symbol \( g_q \) is 2-acyclic, \( g_{q+1} \to \mathcal{R}_q \) is a vector bundle and if the map \( \pi_{q+r}^{q+r+1} : \mathcal{R}_{q+r+1} \to \mathcal{R}_q \) is surjective, then \( \mathcal{R}_q \) is formally integrable.

We will now derive the bundles and sections that describe the prolongation \( \mathcal{R}_{q+r}(\omega) \) and the projection \( \mathcal{R}_q^{(1)}(\omega) \) of a jet groupoid \( \mathcal{R}_q(\omega) \). For a short notation, all maps to the base \( X \) (as \( s \), \( t \) or \( \pi \)) will keep their name after prolongating or taking jet bundles. The prolongation \( \mathcal{R}_{q+r}(\omega) \) has an obvious description:

**Proposition 4.3.** Let \( \mathcal{F} \) be a natural bundle of order \( q \) with a section \( \omega \). Then \( J_r(\mathcal{F}) \) is a natural bundle of order \( q + r \) and the prolongation \( \mathcal{R}_{q+r}(\omega) \) of \( \mathcal{R}_q(\omega) \) is the symmetry groupoid \( \text{Stab}_{J_r(\mathcal{F})}(J_r(\omega)) \).

**Proof.** Apply the functor \( J_r(\cdot) \) to the \( \Pi_r \)-action on \( \mathcal{F} \) and use the natural embedding \( \Pi_{q+r} \rightrightarrows J_r(\Pi_q) \) to establish the \( \Pi_{q+r} \)-action on \( J_r(\mathcal{F}) \). Note that the image of this embedding is \( J_r(\Pi_q) \cap \Pi_{q+r} \). As \( \mathcal{R}_q(\omega) \) is defined as \( \ker^{}(\Phi_\omega) \), the exact sequence for \( J_r(\mathcal{R}_q(\omega)) \) is:
\[
0 \to J_r(\mathcal{R}_q(\omega)) \to J_r(\Pi_q) \xrightarrow{\Phi_\omega} J_r(\mathcal{F})
\]
and the intersection \( J_r(\mathcal{R}_q(\omega)) \cap \Pi_{q+r} \) actually is the symmetry groupoid of \( J_r(\omega) \).

The fibres of \( J_r(\mathcal{F}) \) are not necessarily homogeneous, so we cannot assure that \( \mathcal{R}_{q+r}(\omega) \) is still a subbundle of \( \Pi_{q+r} \) or equivalently a Lie groupoid. However if the rank of \( j_r(\Phi_\omega) \) is constant, \( \mathcal{R}_{q+r}(\omega) \) is a Lie groupoid again.
To describe the projections $R_q^{(1)}(\omega)$ we write $J_1(\mathcal{F})$ as a bundle associated to $P_{q+1}$ with fibre $J_1(F) := J_1(\mathcal{F})_{y_0}$. The idea to use fibre $F_1 := J_1(F)/K_{q+1}$ to obtain the associated bundle $\mathcal{F}_1 := P_{q+1} \times_{GL_{q+1}} F_1$ is due to Barakat.

**Proposition 4.4.** $\mathcal{F}_1 \cong P_q \times_{GL_q} F_1$ is a natural bundle of order $q$ and if $I : J_1(\mathcal{F}) \to \mathcal{F}_1$ is the projection, $R_q^{(1)}(\omega)$ is the symmetry groupoid $\text{Stab}_{\mathcal{F}_1}(I(j_1(\omega)))$.

**Proof.** By construction of $K_{q+1}$, the $GL_{q+1}$-action on the fibre $F_1$ factors over $GL_q$ and $P_{q+1}/K_{q+1} \cong P_q$ ensures that $\mathcal{F}_1$ is a natural bundle of order $q$. The preimage $I^{-1}(v)$ of $v \in F_{1y}$ can be written as $u_1 K_{q+1}$ with $u_1 \in I^{-1}(v)$ and $K_{q+1} \subseteq \Pi_{q+1}(y, y')$. The action on $\mathcal{F}_1$ is defined by $\omega v q = u_1 K_{q+1} f_{q+1} = u_1 f_{q+1} K_{q+1}$ and we have the exact sequence for $\text{Stab}_{\mathcal{F}_1}(I(j_1(\omega)))$:

$$0 \rightarrow \text{Stab}_{\mathcal{F}_1}(I(j_1(\omega))) \rightarrow \Pi_q \xrightarrow{I(j_1(\omega))} \mathcal{F}_1.$$

The symmetry condition

$$I(j_1(\omega))(y) f_q = j_1(\omega)(y) f_{q+1} K_{q+1} = j_1(\omega)(x) K_{q+1}$$

is equivalent to the existence of a preimage $r_{q+1} \in R_{q+1}(\omega)$ projecting onto $f_q$. $\square$

We now come to the main result of this article, which is a conceptual proof using groupoids of a theorem implicitly present in [13] and formulated by Pommaret.

**Theorem 4.5.** The projection $\pi_q^{q+1} : R_{q+1}(\omega) \to R_q(\omega)$ is an epimorphism if and only if there is a $\Pi_q$-equivariant section $c : \mathcal{F} \to \mathcal{F}_1$, $c(u f_q) = c(u) f_q$, such that $I(j_1(\omega)) = c(\omega)$. This gives the exact sequence:

$$0 \rightarrow R_q(\omega) \rightarrow \Pi_q \xrightarrow{\Phi_{\omega}} \mathcal{F} \xrightarrow{I \circ j_1} \mathcal{F}_1.$$

**Proof.** Whenever we define an element $a_{q+1}$, $a_{q+1}$ denotes an arbitrary preimage under the appropriate projection $\pi_q^{q+1}$. First assume the existence of $r_{q+1} \in (\pi_q^{q+1})^{-1}(r_q)$ for all $r_q \in R_q(\omega)$. To construct an equivariant section, we define $c(\omega(y)) := j_1(\omega)(y) K_{q+1}$.

For $\omega(y) \neq u \in F_y$ there is a $g_q \in GL_q(\mathbb{R}^n)$ with $u = \omega(y) g_q$, we set $c(u) := j_1(\omega)(y) g_{q+1} K_{q+1}$, which is well-defined due to $g_{q+1} K_{q+1}$ being the whole preimage in $GL_{q+1}(\mathbb{R}^n)$. For each $f_q \in \Pi_q$, we can find $h_q \in GL_q(\mathbb{R}^n)$ with

$$\omega(x) h_q = u f_q = \omega(y) g_q f_q = \omega(y) r_q h_q \quad \text{and} \quad f_q = g_q^{-1} r_q h_q,$$

where the existence of $r_{q+1}$ implies the equivariance:

$$c(u f_q) = j_1(\omega)(x) h_q K_{q+1} = j_1(\omega)(y) g_{q+1} (g_q^{-1} r_{q+1} h_{q+1}) K_{q+1} = c(u) f_q$$

Using the equivariance of $c$ on $c(\omega(y)) r_q$ we obtain

$$j_1(\omega)(y) r_{q+1} K_{q+1} = j_1(\omega)(x) K_{q+1}$$

for an arbitrary preimage $r_{q+1}$. There is a $g_{q+1}$ such that $r_{q+1} = r_{q+1} K_{q+1}$ satisfying

$$j_1(\omega)(y) r_{q+1} = j_1(\omega)(x).$$
which provides a preimage \( r_{q+1} \in \mathcal{R}_{q+1}(\omega) \) for \( r_q \).

Using the \( \mathrm{GL}_q \)-action on the fibre \( F_1 \), all possibilities for equivariant sections \( c \) can be calculated. The resulting integrability conditions \( I(j_1(\omega)) = c(\omega) \) are called Vessiot structure equations. They express the condition that each defining equation for \( \mathcal{R}_{q+1}(\omega) \) where the jets of order \( q+1 \) can be eliminated must be a consequence of the equations for \( \mathcal{R}_q(\omega) \).

If the Vessiot structure equations are fulfilled for a section \( \omega \), \( \mathcal{R}_{q+1}(\omega) \) is transitive and a subbundle of \( \Pi_{q+1} \). Then by [9], the symbol \( q_{q+1} \) is a vector bundle and we can apply Theorem 4.2 which implies formal integrability. Theorem 4.5 can be extended to non-homogeneous fibres \( F \) as long as the section \( \omega \) defines a Lie groupoid.

Starting from an arbitrary transitive jet groupoid \( \mathcal{R}_q \), we have found a natural bundle \( F \) of geometric objects and a special object \( \omega_0 \), such that \( \mathcal{R}_q \times \text{Stab}_{q+1} F(\omega_0) \). It has been shown that the section \( j_r(\omega) \) of \( J_r(F) \) defines the \( r \)-th prolongation \( \mathcal{R}_{q+r}(\omega) \) and that \( I(j_1(\omega)) \) on \( F_1 \) corresponds to the projection \( \mathcal{R}_{q+1}(\omega) \). Based on Theorem 4.2, the projection theorem leads to a check of formal integrability directly on the level of sections \( \omega \) of \( F \). In most cases, the integrability conditions have an immediate geometric interpretation as in the following example.

5. Example

The following calculation is due to Barakat using the MAPLE package jets [1], which contains routines for jet groupoids and natural bundles. It will be used to show explicit examples of the objects in the theoretical part.

```maple
with(jets):
Dimension of the base manifold X and some coordinates:
> n:=2:
> ivar:=[x1,x2]: dvar:=[y1,y2]:
> Ivar:=[phi1,phi2]: Dvar:=[xi1,xi2]:
The jet groupoid expressing the invariance of the flat euclidean metric \( g \) on \( X \):
> (Jac,g):=(matrix(n,n,jetcoor(1,ivar,dvar)), linalg[diag](1$n));
Jac, g :=
[ y1 x1 y1 x2
  y2 x1 y2 x2 ]

> J_:=evalm(linalg[transpose](Jac) &* g &* Jac);
J :=
[ y1 x1 y1 x2 + y2 x1 y2 x2
  y1 x1 y1 x2 + y2 x1 y2 x2
  y1 x2 y1 x2 + y2 x2 y2 x2 ]

> GR_g:= [ J_ [1,1]=1, J_ [1,2]=0, J_ [2,2]=1 ];
GR_g := [ y1 x1 y1 x2 + y2 x1 y2 x2 = 1, y1 x1 y1 x2 + y2 x1 y2 x2 = 0, y1 x2 y1 x2 + y2 x2 y2 x2 = 1 ]

These equations locally define a transitive groupoid \( \mathcal{R}_1(g) \subset \Pi_1 \) with isotropy groups \( O_2(\mathbb{R}) \). They have been constructed by the action of \( \mathrm{GL}_1 \cong \mathrm{GL}(\mathbb{R}^2) \) on the space \( F \) of scalar products on \( \mathbb{R}^2 \). So we start with the natural bundle \( F_g = P_1 \times \mathrm{GL}_1, F \cong S^2 T^* X_{\geq 0} \) of symmetric positive definite 2-forms and the equations for \( \mathcal{R}_1(g) \) are already in Lie form (see section 3 for \( P_q \) and \( \mathrm{GL}_q \)). Define coordinates for \( F_g \) and a section \( \omega \):
> uvar_g:=[u11,u12,u22]: wvar_g:=[omega11,omega12,omega22]:
```
As in [11], the coordinate changes of $\mathcal{F}_g$ are given in the form

$$\hat{x} = \phi(x), \quad u = \Psi(\hat{x} = \phi(x), \hat{u}, \phi(x))$$

(mind the hats in the second equation). For shorter output, jet notation is used for $\phi(x)$ and its derivatives:

$$\begin{align*}
\text{inv}_g & := \text{ezip} (\text{uvar}_g, \text{map} (\text{lhs}, \text{GR}_g)) ; \\
\text{F}_g & := \text{natfin} (\text{inv}_g, \text{ivar}, \text{dvar}, \text{uvar}_g, \text{Ivar}, "") ; \\
\text{eqn2ind} (\text{F}_g, \text{ivar}, \text{Ivar}) ;
\end{align*}$$

The groupoid $\mathcal{R}_1(\omega)$ for a general section in Lie form:

$$\begin{align*}
\text{LieFormG} (\text{F}_g, \text{ivar}, \text{dvar}, \text{Ivar}, \text{wvar}_g) ;
\end{align*}$$

The special section $\omega_0$ for the flat metric $g$:

$$\begin{align*}
\text{omega0} & := \text{map} (\text{rhs}, \text{GR}_g) ;
\end{align*}$$

The application of Theorem 4.5 at this point gives no integrability conditions, although an arbitrary metric should not be integrable. The reason is that the symbol of $\mathcal{R}_1(\omega)$ is not yet 2-acyclic, but $\mathcal{R}_2(\omega)$ has 2-acyclic symbol. We could go on with $J_1(\mathcal{F}_g)$, but in order to keep geometrical interpretation (and short expressions) we also model the Christoffel symbols by plugging the derivatives of the transformed flat metric ($\text{GR}_g$) into:

$$\begin{align*}
\Gamma^{k}_{ij}(x) & = \frac{1}{2} g^{k\mu} (x) \left( \frac{\partial g_{i\mu}}{\partial x^j} (x) + \frac{\partial g_{j\mu}}{\partial x^i} (x) - \frac{\partial g_{ij}}{\partial x^\mu} (x) \right),
\end{align*}$$

which gives the equations for the Christoffel symbols of the flat metric in Lie form:

$$\begin{align*}
\text{dJac} & := \text{linalg} [\text{det}] (\text{Jac}) ; \\
\text{Phi_Gamma} & := [ \\
(\text{y2}[x2]*\text{y1}[x1,x1]-\text{y2}[x1,x1]*\text{y1}[x2]) / \text{dJac}, \\
(\text{y2}[x2]*\text{y1}[x1,x2]-\text{y2}[x1,x2]*\text{y1}[x2]) / \text{dJac}, \\
(\text{y2}[x1,x1]*\text{y1}[x1]-\text{y1}[x1,x1]*\text{y2}[x1]) / \text{dJac}, \\
(\text{y2}[x1,x2]*\text{y1}[x1]-\text{y1}[x1,x2]*\text{y2}[x1]) / \text{dJac}, \\
(\text{y2}[x2]*\text{y1}[x2,x2]-\text{y2}[x2,x2]*\text{y1}[x2]) / \text{dJac}, \\
(\text{y2}[x2,x2]*\text{y1}[x1]-\text{y1}[x2,x2]*\text{y2}[x1]) / \text{dJac} ] ;
\end{align*}$$

The coordinates for the Christoffel symbols (uik stands for $\Gamma^{i}_{jk}$):

$$\begin{align*}
\text{uvar_Gamma} & := [\text{u111}, \text{u112}, \text{u211}, \text{u212}, \text{u122}, \text{u222}] ;
\end{align*}$$

Calculate the natural bundle $\mathcal{F}_\Gamma$ of Christoffel symbols:

$$\begin{align*}
\text{inv_Gamma} & := \text{ezip} (\text{uvar_Gamma}, \text{Phi_Gamma}) ; \\
\text{F_Gamma} & := \text{natfin} (\text{inv_Gamma}, \text{ivar}, \text{dvar}, \text{uvar_Gamma}, \text{Ivar}, "") ;
\end{align*}$$
The result is $\mathcal{F} = \mathcal{F}_g \times_X \mathcal{F}_1 \cong J_1(\mathcal{F}_g)$. Usually, $J_1(\mathcal{F})$ is only an affine bundle over $\mathcal{F}$ and does not split. The fibre $F$ is a homogeneous $GL_2$-space, so each section on $\mathcal{F}$ defines a Lie groupoid.

```
> uvar := [op(uvar_g), op(uvar_Gamma)];
> F := [op(F_g), op(F_Gamma[n+1..-1])];
```

To calculate the projection to the bundle $\mathcal{F}_1$, the vector fields of infinitesimal transformations of $\mathcal{F}$. If $\xi^j(x) \frac{\partial}{\partial x^j}$ is a vector field on $X$, it can be extended to $\mathcal{F}$:

```
> vec := natfin2inf(F, ivar, Ivar, Dvar, "");
```

The above list denotes a vector field. It is read as follows: $[\xi_1, x_1]$ stands for $\xi^j(x) \frac{\partial}{\partial x^j}$ and the complete vector field is obtained by adding up all list entries. Each choice of $\xi^i, \ldots, \xi^j_{x_i, x_j}$ gives an infinitesimal transformation of $\mathcal{F}$. We calculate the coordinates of $\mathcal{F}_1$ that express the projection $I : J_1(\mathcal{F}) \rightarrow \mathcal{F}_1$:

```
> F1 := Ficoor(vec, ivar, Dvar, uvar);
```

```
F1 := [u11 x1, u11 x2, u12 x1, u12 x2, u22 x1, u22 x2,
    u11 x2 - u12 x1, u12 x2 - u22 x1, u22 x2 - u12 x1,
    u22 x2 - u12 x1]
```

All further computations only need the infinitesimal coordinate changes of $\mathcal{F}_1$. Setting zero order jets $\xi^i = 0$ to zero and collecting for higher order jets of $\xi^i$, the list $L_1$ contains a representation of the Lie algebra of $GL_2(\mathbb{R}^2)$ as vertical vector fields on $\mathcal{F}_1$.

```
> inv1 := ezip(vvar, F1);
> vec1 := natinfG(vec, inv1, ivar, uvar, vvar, Dvar);
> L1 := lstvec(sortcon(vec1, [op(uvar), op(vvar)]), ivar, Dvar, "");
```

Before calculating the possible equivariant sections of $\mathcal{F}_1$, we will modify the coordinates of $\mathcal{F}_1$ to obtain a vector bundle atlas. This is achieved by choosing the coordinates for the fibres $\mathcal{F}_1 \rightarrow \mathcal{F}$ to be $K_{q+1}$-invariant (see sequence (3.1) for $K_{q+1}$):

```
> cvar := [c1, c2, c3, c4, c5, c6, c7, c8, c9, c10];
> subv := map(a -> vvar[a] = cvar[a](op(uvar)), [$1..nops(cvar)]):
> cndi := map(i -> subs(subv, invcond(L1[i][n^2+1..-1],
    [lhs(subv[i])-rhs(subv[i]), L1[2][i]], [$1..d1])):
> sol_Gamma := map(ci -> jsolve(cndi[ci], uvar,
    [cvar[ci](op(uvar))], ""), [$1..d1]):
```
The results all depend on arbitrary functions \( F_1(u_{11}, u_{12}, u_{22}) \), which will be set to zero:

\[
\begin{align*}
> \text{sol\_Gamma[1]}; \\
&\quad [c_1(u_{11}, u_{12}, u_{22}, u_{111}, u_{121}, u_{212}, u_{122}) = \\
&\quad \quad 2 u_{111} u_{11} + 2 u_{12} u_{211} + F_1(u_{11}, u_{12}, u_{22}) ]
\end{align*}
\]

\[
> \text{sol\_Gamma := eval(map(a->op(subs(_F1=0,a)),sol\_Gamma))}: \\
\]

The new infinitesimal coordinate changes show the vector bundle structure of \( F_1 \rightarrow F \):

\[
\begin{align*}
> \text{inv1_1 := zip((a,b)->lhs(a) = rhs(a) - rhs(b),inv1,sol\_Gamma)}; \\
> \text{vec1_1:=natinfG(vec,inv1_1,ivar,uvar,vvar,Dvar)}; \\
> \text{L1_1:=lstvec(sortcon(vec1_1,[op(uvar),op(vvar)]),ivar,Dvar,"")}: \\
\]

\[
\begin{align*}
\text{vec1_1 := } & [ . . . \text{vec} . . . ] \\
& [-3 v_1 \xi_{x1} - v_2 \xi_{x2} + 2 v_3 \xi_{x1}, [v_1]], \\
& [-v_1 \xi_{x2} - v_2 \xi_{x1} + 2 v_3 \xi_{x2}, [v_2]], \\
& [-v_1 \xi_{x1} - v_2 \xi_{x2} + 2 v_3 \xi_{x1} - v_4 \xi_{x1}, [v_3]], \\
& [-\xi_{x2} v_2 - v_3 \xi_{x2} - 2 v_4 \xi_{x2} - \xi_{x1} v_4 - v_6 \xi_{x2}, [v_4]], \\
& [-v_3 \xi_{x1} - v_5 \xi_{x2} - v_5 \xi_{x1} - v_6 \xi_{x1}, [v_5]], \\
& [-v_4 \xi_{x2} - v_5 \xi_{x2} - 3 v_6 \xi_{x2}, [v_6]], \\
& [-\xi_{x1} v_7 - \xi_{x1} v_7 - \xi_{x1} v_8 + \xi_{x1} v_9, [v_7]], \\
& [-\xi_{x2} v_7 - 2 \xi_{x2} v_8 + \xi_{x2} v_10, [v_8]], \\
& [\xi_{x1} v_7 - 2 \xi_{x1} v_9 - \xi_{x1} v_10, [v_9]], \\
& [\xi_{x1} v_7 - 2 \xi_{x1} v_9 - \xi_{x2} v_10 - \xi_{x1} v_10, [v_10]]
\end{align*}
\]

We are now able to calculate all equivariant sections \( c \). The infinitesimal conditions for equivariance are obtained by applying all vector fields of the Lie algebra of \( \text{GL}_2 \) to \( v_i - c_i(u) = 0 \) and then substituting \( v_i \rightarrow c_i \). The same method was used for the \( K_2 \)-invariance.

\[
\begin{align*}
> \text{subv_1:=map(a->vvar[a]=cvar[a](op(uvar)),[$1..nops(cvar)])}; \\
> \text{cnd1 := subs(subv_1,invcond(L1_1[1],map(a->lhs(a)-rhs(a),subv_1),L1_1[2])\[1\])}; \\
> \text{cc := jsolve(cnd1,ivar,map(a->a(op(uvar)),cvar),"")}: \\
\]

The Vessiot structure equations show the integrability conditions with the equivariant sections on the right hand side:

\[
> \text{Ves := subs(inv1_1,subs(cc,subv_1))};
\]
\[ \text{Ves} := \begin{align*}
u_{111} - 2u_{111}u_{111} - 2u_{112}u_{211} &= 0, \\
u_{112} - 2u_{111}u_{112} - 2u_{112}u_{212} &= 0, \\
u_{12} - (u_{111} + u_{212})u_{12} - u_{111}u_{112} - u_{22}u_{211} &= 0, \\
u_{122} - (u_{112} + u_{222})u_{12} - u_{22}u_{212} - u_{11}u_{122} &= 0, \\
u_{22} - 2u_{112}u_{12} - 2u_{22}u_{212} &= 0, \\
u_{222} - 2u_{12}u_{122} - 2u_{22}u_{222} &= 0, \\
u_{111}x_1 - u_{112}x_1 - u_{212}u_{12} &= 0, \\
u_{112}x_2 - u_{112}x_2 - (u_{111} + u_{212})u_{12} - u_{112}u_{222} + u_{112}^2 &= -C_1u_{22}, \\
u_{211}x_2 - u_{212}x_1 - u_{122}^2 + u_{112}u_{111} - (u_{112} - u_{222})u_{111} &= -u_{11}C_1, \\
u_{212}x_1 - u_{222}x_1 - u_{12}u_{112} + u_{12}u_{12} &= -C_1u_{12} + \sqrt{-u_{12}^2 + u_{22}u_{11}}, \end{align*} \]

They show that all equivariant sections \( c \) can be parametrised by two constants (\( C_1 \) and \( C_2 \)). The second constant \( C_2 \) is special to the 2-dimensional case and we obtain \( C_2 = 0 \) using the Jacobi conditions in \( [9, \text{Thm. 7.4.8}] \). The first six integrability conditions express the Christoffel symbols in terms of the metric and its first order derivatives (cf. eq. (5.1)):

\[
\begin{align*}
u_{111} &= -\frac{1}{2}\left\{-u_{111}x_1u_{22} + 2u_{12}u_{121} - u_{12}u_{11}x_2\right\}, \\
u_{112} &= -\frac{1}{2}\left\{-u_{112}x_2u_{22} + u_{12}u_{221}\right\}, \\
u_{211} &= \frac{1}{2}\left\{2u_{11}u_{121} - u_{12}u_{111} - u_{11}u_{11}x_2\right\}, \\
u_{212} &= \frac{1}{2}\left\{u_{12}u_{221} - u_{12}u_{112}\right\}, \\
u_{122} &= -\frac{1}{2}\left\{u_{22}u_{12} - 2u_{22}u_{12}x_2 + u_{22}u_{221}\right\}, \\
u_{222} &= \frac{1}{2}\left\{u_{22}u_{22} - 2u_{12}u_{12}x_2 + u_{22}u_{221}\right\}. \end{align*}
\]

Starting with a metric, they are always fulfilled, but an arbitrary section of \( F \) allows to choose metric and Christoffel symbols independently. The last four integrability conditions express components of the Riemann curvature tensor as derivatives of the Christoffel symbols. The equations are equivalent (\( C_2 = 0 \)) to the condition of a metric with constant scalar curvature:

\[
R_{ijkl}^k = \partial_i \Gamma_{lj}^k - \partial_j \Gamma_{li}^k + \Gamma_{lj}^r \Gamma_{ri}^k - \Gamma_{li}^r \Gamma_{rj}^k = C_1(\delta_i^k g_{lj} - \delta_l^k g_{ij}).
\]

The calculations with \texttt{jets} complement the the theory with explicit coordinate changes of the natural bundles \( F \) and \( F_1 \), which are equivalent to the \( \Pi_q \)-action on \( F \). Vessiot’s structure equations can now be used to check the integrability of jet groupoids.

In \([13] \), the structure equations are solved for the constants, which is an alternative choice of coordinates for \( F_1 \). Usually, this leads to larger expressions and hides the geometrical interpretation. If the typical fibre of \( F \) is not homogeneous, the freedom for equivariant sections may extend from constants to smooth invariants.
Vessiot’s structure equations can also be applied to test whether two geometric objects are formally equivalent, which is connected to the integrability of the corresponding groupoids an equivariant sections.

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