ANALYTIC APPROACH TO $S^1$-EQUIVARIANT MORSE INEQUALITIES

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Abstract. It is well known that the cohomology groups of a closed manifold $M$ can be reconstructed using the gradient dynamical of a Morse-Smale function $f: M \to \mathbb{R}$. A direct result of this construction are Morse inequalities that provide lower bounds for the number of critical points of $f$ in term of Betti numbers of $M$. These inequalities can be deduced through a purely analytic method by studying the asymptotic behaviour of the deformed Laplacian operator. This method was introduced by E. Witten and has inspired a numbers of great achievements in Geometry and Topology in few past decades. In this paper, adopting the Witten approach, we provide an analytic proof for; the so called; equivariant Morse inequalities when the underlying manifold is acted on by the Lie group $G = S^1$ and the Morse function $f$ is invariant with respect to this action.

1. Introduction

Classical Morse theory with all its variants are amongst the great achievement of the modern geometry and topology. Its relevance to topology begun by the fundamental observation that the cellular structure of a closed manifold $M$ can be reconstructed through level sets of a Morse function $f$ on $M$. In particular this give a way to reconstruct the cellular chain complex and therefore the cohomology of $M$, as is explained clearly in [8] and [14].

The seminal paper of E. Witten [16] which was inspired by ideas from quantum field theory, shed a new light on Morse theory by providing a new chain complex for computing the cohomology of $M$. This complex; called Morse-Smale-Witten complex; is generated by the critical points of $f$ and graded by their Morse indices, as in the cellular complex. However its differentials are defined through the gradient lines between critical points whose indices differ by one. We refer to [14] for a detailed exposition of this theory. Amongst others, this construction led to the innovation of Floer Homology and solved (partially) the Arnold conjecture, c.f. [13].

An immediate consequence of the reconstruction of the singular cohomology via critical points of a Morse function is the Morse inequalities. Roughly speaking, they provide lower bounds for the number of critical points in term of the Betti numbers of $M$, i.e. the rank of the cohomology spaces of $M$. As far as one is interested in these inequalities rather than the cohomology itself, there is a very elegant and conceptual analytic derivation. This is the Witten idea of deforming the de Rham complex in an appropriate way using the Morse function and then study the asymptotic behaviour of this complex. In this paper we follow Roe’s account of Witten’s approach in [11] chapter 14.

Morse theory can be generalized in other directions. For instance in some situation there is a Lie group $G$ acting on $M$ and preserves $f$. This problem naturally arises in $n$-body problem where the central configurations are critical points of some Morse function which are symmetric with respect to the action of $SO(n)$ (c.f. [10]). Another example is the problem of finding the number of closed geodesics of a Riemannian metric where the Lie group is $S^1$. Actually this last example was amongst the first applications of Morse theory that was worked out by Morse in [9], see also [8] chapter 3. In these cases the critical levels of $f$ are clearly orbits of the action. The Morse-Bott theory ignore the invariance of $f$ under the action and provides lower-bounds for the number of these critical levels (see [5] and [6, page 344]). However; as it is clearly explained in [6] pages 351-355; to get best results one has to assume the $G$-invariance of $f$ and this requires an appropriate cohomology theory that takes the group action into its construction. This is the equivariant cohomology theory which is introduced originally by E. Cartan in 1940s. In this paper we will deal with A. Borel definition of equivariant de Rham complex as is explained; amongst other applications of equivariant cohomology; in [2] and [7]. As in ordinary Morse-Bott theory, the equivariant cohomology may be reconstructed from a cellular...
chain complex generated by critical level of $f$. This is done by A. G. Wasserman in [15 section 4] for a compact Lie group $G$. Nevertheless the Morse-Smale-Witten complex for equivariant cohomology has been worked out recently, for $G = S^1$, by M. J. Berghoff in his PhD thesis [4]. This constructions lead to the equivariant Morse inequalities whose precise statement might be found in [15 page 149] or in [6 page 351].

In this paper we consider also the case $G = S^1$ and prove the equivariant Morse inequalities by adopting the Witten method to deform the Borel complex. This approach will be interesting for those who are interested in analytical methods rather than in topological ones. Moreover this analytic approach is more flexible and can be adopted to different situations. One of the authors has already used this approach to deal with Morse inequalities on manifolds with boundary and, the so called, delocalized Morse inequalities, c.f. [17] and [18].

The structure of the paper is as follows. In section 2 we give the technical definition of the equivariant cohomology theory by introducing the Borel complex. Then we give the precise statement of the equivariant Morse inequalities by stating the main theorem 2.1. In section 3 we establish the Hodge theory for the equivariant complex and introduce the Witten deformation of the Borel complex. Then we prove an infinite number of quite general equivariant analytical Morse inequalities in theorem 3.2. The asymptotic behaviour of these inequalities leads finally to the Morse inequalities. The lemma 3.5 reduces our problem to compute the kernel of some elliptic operators on Euclidian spaces. The computation of these kernels are the subject of section 4 and lead to the proof of the main theorem in last paragraphe of the paper.

2. Equivariant cohomology and main theorem

Let $G = S^1$ be a compact Lie group that acts on a topological space $X$ and let $EG$ be a contractible space that is acted on, freely and continuously, by $G$. Such spaces exist and are unique up to homotopy equivalence and the quotient $BG := EG/G$ is the classifying space of $G$. The diagonal action of $G$ on $X \times EG$ is free and the quotient $X_G := (X \times EG)/G$ is called the homotopy path space. The equivariant cohomology of $X$; that we denote by $H^*_G(X)$; is by definition the singular cohomology (with coefficient in $C$) of the homotopy path space

\begin{equation}
H^*_G(X) = H^*(X_G)
\end{equation}

If $G$ acts trivially then $X_G = X \times BG$ and $H^*_G(X) = H^*(X \times BG) = H^*(X) \otimes H^*(BG)$. In particular the equivariant cohomology of a single point is the cohomology of the group, i.e. $H^*(G; \mathbb{R})$. If the action of $G$ on $X$ is free then $EG \to X_G \to X/G$ is a fibration with contractible fibres which implies $H^*_G(X) = H^*(X/G)$. Actually the equivariant cohomology gives a way to combine the cohomology of the space $X$ and that of $G$ while taking into the account the action of $G$ on $X$. When $X$ is a differential manifold (that we denote by $M$ from now on) and the action is smooth there is a more geometric approach to the construction of the equivariant cohomology. Because our study concerns this formulation, we give a short description of this construction. We consider the simplest case when the Lie group $G$ is just the circle $S^1$.

Let $M$ be a closed smooth manifold of dimension $n$ which is acted on by the group $G = S^1$. This action is supposed to be smooth and not necessarily free. The Lie algebra of $S^1$ is $\mathbb{R}$ with a fixed element 1. This element generates a vector field $v$ over $M$ which is tangent to the orbits and vanishes at fixed points of the action. The vector field $v$ is called the infinitesimal generator of the action and we denote its $t$-time flow by $\phi_t$. Let $\Omega^*_G(M) \subset \Omega^*(M)$ consists of all invariant differential forms $\omega$ satisfying $\phi_t^* (\omega) = \omega$ for $t \in \mathbb{R}$, or equivalently $L_v (\omega) = 0$. Consider the algebra $\mathbb{C}[t] \otimes \Omega^*_G(M)$. This is indeed the algebra of all polynomial function on the Lie algebra $g = \mathbb{R}$ with values in $\Omega^*_G(M)$. Here $t \in g^*$ is dual element corresponding to 1, i.e. $t(1) = 1$. This algebra is graded by the rule $\deg (t^k \otimes \omega) = 2k + \deg (\omega)$. Put $\Omega^*_G(M) := \mathbb{C}[t] \otimes \Omega^*_G(M)$ then the following linear map
It is easy to verify that the formal adjoint of the equivariant exterior derivative

\[ d_{eq} : \Omega^*_eq(M) \to \Omega^*_eq(M) \]

(2.2)

\[ d_{eq}(t^k \otimes \omega) = t^k \otimes d\omega + t^{k+1} \otimes i_\omega \]

is a differential, i.e. \( d_{eq}^2 = 0 \) and increases the degree by one. The equivariant de Rham cohomology groups \( H^*_G(M) \) are the cohomology groups of this graded differential complex. It turns out that these groups are isomorphic to the groups introduced by (2.1) when \( X \) is a smooth manifold.

The cohomology space \( H^*_G(M) \) is a complex finite dimensional vector space. So one can define the equivariant Betti numbers by \( \beta_k^G := \dim H^k_G(M) \). When the action \( S^1 \times M \to M \) has no fixed points, then \( M = M/S^1 \) is a smooth manifold and equivariant cohomology groups of \( M \) are canonically isomorphic to the de-Rham cohomology of \( M \). The equivariant cohomology of a point is just the algebra \( \mathbb{R}[t] \) (with deg \( t = 2 \)) which is isomorphic to the de-Rham cohomology of \( BS^1 = \mathbb{C}P^\infty \) when \( t \) is identified to a symplectic form on \( \mathbb{C}P^\infty \).

Let \( f : M \to \mathbb{R} \) be an invariant smooth function, i.e. \( v.f = 0 \). An orbit \( o \) is critical if one point on it (then all points) is critical point for \( f \). For \( x \in o \) let \( N_x \) stands for the quotient space \( T_xM/T_xo \). For \( x \in M \) and \( X,Y \in T_xM \) the Hessian of \( f \) is a symmetric bi-linear form defined as follows

\[ H_f(X,Y) = X.(Y.f) - (\nabla X Y).f \]

Here \( \nabla \) is the Riemannian connection on \( TM \) associated to a Riemannian metric \( g \). Because \( S^1 \) is compact it is always possible, throgh an averaging procedure, to assume \( g \) be \( S^1 \)-invariant. With this assumption it is true that if \( X \) or \( Y \) belong to \( T_xo \) then \( H_f(X,Y) = 0 \). Therefore the Hessian defines a well defined symmetric bi-linear form on \( N \). Using the Riemannian metric, we can identify \( N_x \) with the orthogonal compliment of \( T_xo \). We denote the restriction of \( H \) to \( N \subset TM \) by \( H_f \). We say a critical orbit \( o \) be transversally non-degenerated (or simply non-degenerated) if \( H_f \) is non-degenerated at any points of \( o \). The Morse index of such orbit is the dimension of the maximal subspace of \( N_x \), on which the Hessian is negative-definite. In the sequel we reserve the notation \( o \) for a non-trivial orbit and we denote a trivial orbit by its geometric image, that is a point \( p \) in \( M \). Let \( c_k \) and \( d_k \) denote respectively the number of critical points and orbits with Morse index \( k \). Our aim is to provide an analytic proof for the following equivariant Morse inequalities via Witten deformation:

**Theorem 2.1.** With \( \hat{c}_k := d_k + c_k + c_{k-2} + c_{k-4} + \ldots \) the following inequalities hold for \( k = 0, 1, 2, \ldots \)

\[ \hat{c}_k - \hat{c}_{k-1} + \cdots + \hat{c}_0 \geq \beta_{eq}^k - \beta_{eq}^{k-1} + \cdots + \beta_{eq}^0, \]

Actually the inequalities for \( k \geq n + 1 \) are the same that the inequality for \( k = n \).

Note that, when the action is free these inequalities reduce to the ordinary Morse inequalities for the function \( \tilde{f} : \tilde{M} \to \mathbb{R} \), while it reduces to the Morse inequalities for the function \( f \) when the action is trivial.

3. **Equivariant Hodge theory and analytic Morse inequalities**

For our purposes in this paper, we need to establish an equivariant version of Hodge theory. The space of \( S^1 \)-invariant differential forms \( \Omega^*_G(M) \) is endowed with an inner product coming from the Riemannian metric \( g \) and its natural lifting to the exterior algebras \( \wedge^*(T_pM) \) for all \( p \in M \). The formal dual of the exterior differential \( d \) is the operator \( d^* \). We define a scalar product on \( \mathbb{C}[t] \otimes \Omega^*_G(M) \) by the bi-linear extension of the following formula

\[ \langle t^i \otimes \omega , t^j \otimes \eta \rangle = \delta_{ij} \langle \omega , \eta \rangle \]

(3.1)

It is easy to verify that the formal adjoint of the equivariant exterior derivative \( d_{eq} \) (see (2.2)) with respect to this scalar product is the linear extension of the following operator of grade \(-1 \), where \( \epsilon_{0i} = 1 - \delta_{0i} \) throughout this paper
The equivariant Laplacian operator is defined by \( \Delta_{eq} := d_{eq}^* d_{eq} + d_{eq} d_{eq}^* \). The following relation is a very direct consequence of definitions and give the action of \( \Delta_{eq} \) on a term like \( t^i \otimes \omega \). Here \( \Delta \) stands for the Laplacian operator on differential forms.

\[
\Delta_{eq}(t^i \otimes \omega) = t^i \otimes (\Delta \omega + v^* \wedge i_v \omega + \epsilon_{0,i} i_v v^* \wedge \omega) + t^{i+1} \otimes (i_v d^* \omega + d^* i_v \omega) + \epsilon_{0,i} t^{i-1} \otimes dv^* \wedge \omega
\]

Clearly \( \Delta_{eq}^k \) may be considered as a differential operator on \( \Omega_{eq}^k(M) \) consisting of all smooth sections of the equivariant exterior algebra \( \wedge_{eq}^k T_p M \) where

\[
\wedge_{eq}^k T_p M := \bigoplus_{i+j=k} t^i \otimes \wedge^j T_p
\]

From this point of view, it is clear that \( \Delta_{eq}^k \) is a second order elliptic differential operator which is formally self adjoint with respect to above scalar product. Note that here \( t \) is merely a label to keep track of the grading and \( \wedge_{eq}^k T_p M \) is a finite dimensional hermitian vector bundle. So, we may apply the ordinary completion procedure to construct the Hilbert spaces \( L^2(M, \wedge_{eq}^k T_p M) \) or Sobolev spaces \( W^\alpha(M, \wedge_{eq}^k T_p M) \). Therefore, exactly as in the classical Hodge theory for the Laplacian (e.g. through the construction of heat operator as in [12, chapter 3]) we have the following isomorphisms for \( k = 0, 1, 2, \ldots \) (see [12, Theorem 1.45])

\[
H^k_G(M) \simeq \ker \Delta_{eq}^k
\]

It is clear from expansion (3.3) that for \( k \geq n-1 \), multiplication by \( t^i \) gives rise to the isomorphism \( \Omega_{eq}^k(M) \simeq \Omega_{eq}^{k+2i}(M) \) and the action of the Laplacian is linear with respect to this multiplication. Therefore \( t^i \otimes \ker \Delta_{eq}^k = \ker \Delta_{eq}^{k+2i} \) which implies, by (3.4), the following results for \( k \geq n-1 \)

\[
H^k_G(M) \simeq H^{k+2i}_G(M)
\]

We use the following lemma to prove that the equivariant Morse inequalities stop beyond order \( n \).

**Lemma 3.1.** For \( k \geq n \) the following equalities hold

\[
\beta_{eq}^k - \beta_{eq}^{k+1} = (-1)^k \chi(M)
\]

where \( \chi(M) \) is the Euler characteristic of \( M \).

**Proof** Using (5.5) it is enough to prove the lemma for \( k = n \). It is clear from (2.2) that \( d_{eq} \) is \( \mathbb{C}[t] \)-linear. However, due to the term \( \epsilon_{0,i} \) in (2.2), the operator \( d_{eq}^* \) is \( \mathbb{C}[t] \)-linear on \( \Omega_{eq}^k \) only for \( k \geq n \). Because \( t \otimes \Omega_{eq}^{n-1}(M) = \Omega_{eq}^{n+1} \) and \( t \otimes \Omega_{eq}^n(M) = \Omega_{eq}^{n+2} \) we can define the following operator

\[
\tilde{D}_{eq} : \Omega_{eq}^n \oplus \Omega_{eq}^{n+1} \to \Omega_{eq}^n \oplus \Omega_{eq}^{n+1}
\]

\[
\tilde{D}_{eq} = \tilde{d}_{eq} + \tilde{d}_eq^*
\]

Here \( \tilde{d}_{eq} := d_{eq} \) and \( \tilde{d}_eq^* := t \otimes d_{eq}^* \) on the first summand while \( \tilde{d}_{eq} := t^{-1} \otimes d_{eq} \) and \( \tilde{d}_eq^* := d_{eq}^* \) on the second summand. The operators \( \tilde{d}_{eq} \) and \( \tilde{d}_eq^* \) are formal adjoint of each other with respect to the inner product (3.1). Therefore the equivariant de Rham operator \( \tilde{D}_{eq} \) is grading reversing and formally self adjoint. It is clear from the definitions that \( \tilde{D}_{eq}^2 = \Delta_{eq} \) therefore \( \tilde{D}_{eq} \) is an elliptic differential operator and its Fredholm index is given by
On the other hand, as differential operators on $\Omega^n$, equality the space of Fredholm operators. Because Fredholm index is homotopy invariant, we get the following equality 

$$\text{ind} \, \bar{\Delta}_eq = \text{ind} \, D = \beta^n + \beta^{n-2} + \cdots - \beta^{n-1} - \beta^{n-3} - \cdots$$

where $\beta^i$ is the $i$-th Betti number of $M$. In above the right hand side equals $(-1)^n \chi(M)$ by definition and this completes the proof of the lemma.

What we need is a deformed version of the above equivariant Laplacian that we introduce here. Given a positive parameter $s$ and using the Morse function $f$, the Witten deformation of $d_{eq}$ is the operator $d_{eq,s}:= e^{-sf}d_{eq}e^{sf} = d_{eq} + sdf \wedge$ and its formal dual is $d^*_{eq,s} = d^*_{eq} + sdf$. The associated deformed equivariant Laplacian $\Delta_{eq,s} := d^*_{eq,s}d_{eq,s} + d_{eq,s}d^*_{eq,s}$ has the following expansion, c.f. [11, Lemma 9.17]

$$\Delta_{eq,s} = \Delta_{eq} + s^2|df|^2 + sH_f$$

Here $H_f$ is the following operator, where $\{e_i\}_{i,j}$ is a local orthonormal base for $TM$ and $L_{e_i}$ and $R_{e_i}$ are respectively the left and right Clifford multiplication by $e_i$ (see [11, page 124])

$$H_f = \sum_{i,j} H_f(e_i,e_j) L_{e_i} R_{e_j}$$

It is clear that $\Omega^*_{eq}(M)$ with the differential $d_{eq,s}$ is a graded differential complex, so we may define the deformed cohomology spaces $H^k_{eq,s}(M)$. Nevertheless these cohomology spaces are not new object and multiplication by $e^{-sf}$ provides the following isomorphisms

$$(3.8) \quad H^k_{eq}(M) \simeq H^k_{eq,s}(M).$$

Using the expansion (3.6) and Duhamel’s formula, one is able to construct the heat operator associated to $\Delta_{eq,s}$ which establishes a Hodge theory and provides the isomorphisms $\ker \Delta^k_{eq,s} \simeq H^k_{eq,s}(M)$. This and (3.8) give the following relation

$$(3.9) \quad \beta^k_{eq} = \dim(\ker \Delta^k_{eq,s})$$

By (3.8), $\Delta^k_{eq,s}$ is a positive and second order elliptic differential operator on $L^2(M, \wedge^k_{eq} TM)$. Therefore given a smooth rapidly decreasing positive function $\phi$ on $\mathbb{R}^{\geq 0}$, the operator $\phi(\Delta^k_{eq,s})$, being a smoothing operator on $L^2(M, \wedge^k_{eq} TM)$ is a trace class operator. Its trace is denoted by

$$\mu^k_{eq,s} = \text{tr} \phi(\Delta^k_{eq,s}) : \quad k = 0, 1, \ldots, n$$

The following equivariant analytic Morse inequalities are our departure point for a proof of theorem 3.1 (see [11] for the non-equivariant version)

**Theorem 3.2** (the analytic equivariant Morse inequalities). With the above notations, the following inequalities hold

$$\mu^k_{eq,s} - \mu^{k-1}_{eq,s} + \cdots \pm \mu^0_{eq,s} \geq \beta^k_{eq} - \beta^{k-1}_{eq} + \cdots \pm \beta^0_{eq}$$

**Proof** If we put $\beta^k_{eq,s} = \dim(\ker \Delta^k_{eq,s})$ then by (3.9) the above inequalities are equivalent to the followings

$$\mu^k_{eq,s} - \mu^{k-1}_{eq,s} + \cdots \pm \mu^0_{eq,s} \geq \beta^k_{eq,s} - \beta^{k-1}_{eq,s} + \cdots \pm \beta^0_{eq,s}$$

The proof of the proposition 14.3 of [11] can be applied literary to the deformed Laplacian and gives these inequalities. For the seek of completeness we give a very brief account of this proof. The
spectrum of the deformed Laplacian $\Delta_{eq,s}^k$ is discrete, so there is a rapidly decreasing function $\tilde{\phi}$ on $\mathbb{R}$ which vanishes on non-zero elements of the spectrum such that $\tilde{\psi}(0) = 1$. Therefore $\beta_{eq,s}^k = \text{tr}(\Delta_{eq,s}^k)$ which implies $\mu_{eq,s}^k - \beta_{eq,s}^k = \text{tr}(\phi - \tilde{\phi})(\Delta_{eq,s}^k)$. The relation $(\phi - \tilde{\phi})(x) = x\psi(x)^2$ defines a rapidly decreasing function $\psi$ on $\mathbb{R}$ and one get the following

$$\mu_{eq,s}^k - \beta_{eq,s}^k = \text{tr}\Delta_{eq,s}^k\psi(\Delta_{eq,s}^k)^2$$

Let $H_j$ denotes the $L^2$-Hilbert space generated by $\Omega_{eq}(M)$. Using the relation $\Delta_{eq,s}^k = d_{eq,s}^*d_{eq,s} + d_{eq,s}d_{eq,s}^*$ one gets easily the following relation

$$\text{tr}\{d_{eq,s}d_{eq,s}^*\psi(\Delta_{eq,s}^j)^2\}_{|_{H_j}} = \text{tr}\{d_{eq,s}^*d_{eq,s}\psi(\Delta_{eq,s}^{j-1})^2\}_{|_{H_{j-1}}}$$

An alternating summation from $j = k$ to $j = 0$ on this relation implies the following relation

$$\mu_{eq,s}^k - \mu_{eq,s}^{k-1} + \cdots + \mu_{eq,s}^0 = \text{tr}\{d_{eq,s}^*d_{eq,s}\psi(\Delta_{eq,s}^k)^2\}_{|_{H_k}}$$

Since $d_{eq,s}^*d_{eq,s}\psi(\Delta_{eq,s}^k)^2$ is a non-negative operator, the right side of the above relation is non-negative in general and this gives the equivariant Morse inequalities.

By the above theorem, to prove the theorem [1] we need just to study the asymptotic behaviour of $\mu_{eq,s}^k$ when $s$ goes toward infinity. Since $\phi$ is rapidly decreasing, the operator $\phi(\Delta_{eq,s}^k)$ is smoothing and has a smooth kernel

$$\phi(\Delta_{eq,s}^k)(\omega)(p) = \int_M K_s^k(p, q)\omega(q) d\mu_{eq}(q)$$

Here $K_s^k(p, q)$ is an element of $\wedge_{eq}T_pM \otimes \wedge_{eq}^*T_{q}^*M$. Therefore we have

$$\mu_{eq,s}^k = \int_M \text{tr}\{K_s^k(p, p)\} d\mu_{eq}(p)$$

In a complement of an open neighbourhood of critical level (i.e. a critical point or orbit) of $f$ we have $|df| \geq c > 0$ and the term $s^2|df|^2$ in (3.60) dominates the other terms when $s$ goes to infinity. Therefore, informally saying, on sections supported in this set the spectrum of $\Delta_{eq,s}^k$ goes to infinity. This makes the following lemma true. The proof of this lemma for the non-equivariant case can be found in [11] Lemma 14.6. This proof uses the finite propagation speed property of the wave operator and Friedrich extension theorem. It can be adopted without problem to our context.

**Lemma 3.3.** When $s$ goes toward infinity, the smoothing kernel $K_s^k(p, q)$ goes uniformly to zero when $p$ or $q$ belong to a complement of an open neighbourhood of the critical levels of $f$. Consequently, by [1], to study the asymptotic behavior of $\mu_{eq,s}^k$ when $s$ goes toward infinity, we need just consider the contribution of points $p$ in an arbitrary small open neighbourhood of critical levels of $f$.

For $\rho > 0$ let $N_\rho(p)$ and $N_\rho(o)$ denote, respectively, the $4\rho$-neighbourhood of the critical point $p$ and the critical orbit $o$. Let also $\phi_p$ and $\phi_o$ denote equivariant non-negative smooth functions on $M$ which are supported, respectively, in $N_{3\rho}(p)$ and in $N_{3\rho}(o)$ such that $\phi_p = 1$ on $N_\rho(p)$ and $\phi_o = 1$ on $N_\rho(o)$. Point-wise multiplication by these functions defines operators on equivariant differential forms. The following corollary comes up as a very direct result of the above lemma:

**Corollary 3.4.** The following relation holds

$$\lim_{s \to \infty} \text{tr} \phi(\Delta_{eq,s}^k) = \lim_{s \to \infty} \text{tr}(\phi_p \phi(\Delta_{eq,s}^k)) + \lim_{s \to \infty} \text{tr}(\phi_o \phi(\Delta_{eq,s}^k))$$

Let $B_{\mathbb{R}^n}(0)$ denote the ball in $\mathbb{R}^n$ with center 0 and radius $a$. It is clear that $N_{4\rho}(p)$ and $B_{\mathbb{R}^n}(0)$ are naturally isometric and in this isometry the point $p$ corresponds to 0. This is also true for $N_{4\rho}(o)$ and $S^1 \times B_{\mathbb{R}^n}^{n-1}(0)$ where $o$ corresponds to $S^1 \times \{0\}$. Let $L_{eq,s}^k$ and $L_{eq,s}^k$ denote, respectively, differential operators on $\Omega_{eq,s}^k(\mathbb{R}^n)$ and $\Omega_{eq,s}^k(S^1 \times \mathbb{R}^{n-1})$ such that with respect to above isometries

$$\Delta_{s|N_{4\rho}(p)}^k = L_{s|B_{\mathbb{R}^n}(0)}^k \quad \text{and} \quad \Delta_{s|N_{4\rho}(o)}^k = L_{s|S^1 \times B_{\mathbb{R}^n}^{n-1}(0)}^k$$
Then; through a standard argument; based on Fourier inversion formula and finite propagation speed of wave operators; the following equalities hold

\[ \phi(\Delta_k^G)(\omega_1) = \phi(L_k^G)(\omega_1) \quad \text{and} \quad \phi(\Delta_k^G)(\omega_2) = \phi(L_k^G)(\omega_2) \]

provided that the Fourier transform \( \hat{\phi} \) of \( \phi \) is supported in \((-\rho, \rho)\) and the support of \( \omega_1 \) and \( \omega_2 \) are included, respectively, in \( B_{3\rho}^n(0) \) and \( S^1 \times B_{3\rho}^{n-1}(0) \). Therefore

\[ \text{tr}(\phi_\rho \phi(\Delta_k^G)) = \text{tr}(\phi_\rho \phi(L_k^G)) \quad \text{and} \quad \text{tr}(\phi_\rho \phi(\Delta_k^G)) = \text{tr}(\phi_\rho \phi(L_k^G)) \]

These equalities and corollary 3.4 together prove the following lemma

**Lemma 3.5.** Provided that the support of \( \hat{\phi} \); the Fourier transform of \( \phi \); is included in a sufficiently small neighbourhood of \( 0 \in \mathbb{R} \) and with above notations, the following relation holds

\[ \lim_{s \to \infty} \mu_k^s = \lim_{s \to \infty} \text{tr} \phi(\Delta_k^G) = \sum_p \sum_{s \to \infty} \text{tr}(\phi_\rho \phi(L_k^G)) + \sum_o \lim_{s \to \infty} \text{tr}(\phi_\rho \phi(L_k^G)) \]

Here \( p \) runs over all critical points of \( f \) while \( o \) runs over all critical orbits of \( f \) and \( L_k^G \) and \( L_k^G \) are the local representation of \( \Delta_k^G \) around \( 4\rho \)-neighbourhood of, respectively, critical points and critical orbits.

4. **Localization on critical levels**

To compute the contribution of critical levels, we need to have a good representation of the deformed equivariant Laplacian operators around them. This is provided by an equivariant version of the Morse lemma that we are going to explain. Let us begin with an equivariant version of the tubular neighbourhood theorem. Suppose that \( G \) be a compact Lie group acting on the closed manifold \( M \) and \( g \) be a Riemannian metric which is invariant under the action. The map \( f_x : G/G_x \to G.x \) given by \( f_x([h]) = h.x \) is a diffeomorphism, where \( G.x \) and \( G_x \) are the orbit and stabilizer of \( x \in M \). For \( h \in G_x \), the derivative \( T_{G.x} : T_{G.x} M \to T_{G.x} M \) is an isometry that maps \( T_{G.x} G.x \) into itself. Therefore, it induces a linear isometry \( \phi(h) : N_x \to N_x \) where \( N_x \subset T_x M \) is the orthogonal complement of \( T_x G.x \). In other words one has an orthonormal representation \( \phi : G_x \to O(N_x) \). The subgroup \( G_x \) has a free action on \( G \times N_x \) given by \( h.(h', v) = (h'h^{-1}, \phi(h)(v)) \). The quotient space is a vector bundle \( \pi : N \to G.x \) whose fibers are isometric to \( N_x \) (here we have used the identification \( G_x = G/G_x \)). The action of \( G \) on \( G \times N_x \); given by \( h.(h', v) = (h'h', v) \); commutes with the action of \( G.x \). Therefore it induces a bundle map on the vector bundle \( N \to G.x \). The equivariant tubular neighbourhood theorem \[\text{[3], page 21}\] asserts that there is an invariant neighbourhood \( W \subset G/G_x \); as the zero section of the bundle \( N \); and an invariant neighbourhood \( U \) of the orbit \( G.x \) and an equivariant diffeomorphism \( \hat{f} : W \to U \) that extends the orbit map \( f \) and makes the following diagram commutative

\[ \begin{array}{ccc}
G/G_x & \xrightarrow{\hat{f}} & G.x \\
\downarrow & & \downarrow \text{id} \\
W \subset N & \xrightarrow{\hat{f}} & U \subset M \\
\end{array} \]

A particular case of this theorem is when \( G_x = G \), i.e. the orbit is a point \( x \). In this case the above theorem provides a coordinates system around \( x \) with respect to which the elements of \( G \) act as orthogonal maps. When \( G = S^1 \) and \( G_x = S^1 \) (i.e. \( G_x = \mathbb{Z}_m \)) then \( N = S^1 \times \mathbb{R}^{n-1} \) and the action of \( S^1 \) factors as \( e^{i\theta} \cdot (e^{iv}, v) = (e^{i(m \theta + v)}, \phi(e^{i\theta})v) \) where \( \phi : S^1 \to SO(n-1) \) is a homomorphism. Although the following lemma should exists in the literatures, we did not find it, so we provide a proof for it.

**Lemma 4.1** (Equivariant Morse Lemma). Let \( x \) be a critical point of the invariant Morse function \( f \) and also a fixed point of the action. There is a coordinate system \((x_1, x_2, \ldots, x_n)\) around \( x \) that maps \( x \) to 0 such that

\[ f(x_1, x_2, \ldots, x_n) = \pm x_1^2 \pm x_2^2 \pm x_{k+1}^2 \pm \cdots \pm x_n^2 \]
and the action of $G$ in this coordinates goes through orthogonal maps on $\mathbb{R}^n$. The case of a critical orbit reduces to this case just by restricting to the normal direction.

**Proof** We use the Moser approach to this problem as it is presented in [1 page 176]. Without loose of generality we may assume $G \subset O(n)$ and 0 is the critical point of $f$ such that $f(0) = 0$. Consider the following differential one-forms on $\mathbb{R}^n$

\[
\omega_0(x) = df(x) ; \quad \omega_1(x) = D^2f(0)(x, .)
\]

Clearly $\omega_1 = dh$ where $h(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0)x_i x_j$. Let $z_t$ be a time-depending smooth vector field around 0 and let $\phi_t$ stands for its flow. The following relations hold where $\omega_t = (1 - t)\omega_0 + t\omega_1$

\[
\frac{d}{dt}\phi_t^*\omega_t = \phi_t^*(L_{\phi_t}\omega_t + \frac{d\omega_t}{dt}) = \phi_t^*d(i_{\phi_t}\omega_t + (h - f)).
\]

Since $df(0) = 0$ and $D^2f(0)$ is non-singular, the one-forms $\omega_t$ are non-degenerate in a neighbourhood of 0. Therefore the algebraic equation $i_{\phi_t}\omega_t + (h - f) = 0$ has unique solution for vector field $z_t$ with $z_t(0) = 0$. Thus for this vector field $\phi_1^*\omega_1 = \phi_0^*\omega_0 = \omega_0$ and $f \circ \phi_1^{-1}(x) = h(x) = \frac{1}{2}\partial_j\partial_j f(0)x_j x_j$. The new point here is that $f$ and $h$ are invariant and since this action is linear $\omega_t$ is also invariant. Therefore, $z_t$ should be invariant and its flow $\Phi_t$ should be equivariant. This gives the equivariant Morse lemma after applying appropriate orthogonal transformation which diagonalize $h$. \hfill \Box

For next uses we need to give explicit expressions for the action and Morse function around both critical points and orbits. Let $p$ be a fixed point of the action of $S^1$ on $M$ and the critical point of Morse function $f$. Then there is a coordinate system $x = (x_1, \ldots, x_{2q-1}, x_{2q}, \ldots, x_n)$ centred at $p$ that satisfies the following conditions.

- In this coordinate system the metric $g$ is given by $g = dx_1^2 + dx_2^2 + \cdots + dx_n^2$.
- The action of $S^1$ is as follows

\[
e^{i\theta}(x_1 + ix_2, \ldots, x_n) = (e^{im_1\theta}(x_1 + ix_2), \ldots, e^{im_n\theta}(x_{2q-1} + ix_{2q}), x_{2q+1}, \ldots, x_n)
\]

where $m_i \in \mathbb{N}$.
- The Morse function takes the following form

\[
f(x_1, x_2, \ldots, x_{2q+1}, \ldots, x_n) = \epsilon_1(x_1^2 + x_2^2) + \cdots + \epsilon_q(x_{2q-1}^2 + x_{2q}^2) + \lambda_{2q+1}x_{2q+1}^2 + \cdots + \lambda_n x_n^2
\]

where $\epsilon_j$’s and $\lambda_j$’s are equal to $\pm 1$ and the total number of occurrence of $-1$ is equal to the Morse index of $p$.

The vector field $v$ and its dual with respect to $g$ take the following form in this coordinate system

\[
v = (-m_1 x_2, m_1 x_1, \ldots, -m_q x_{2q}, m_q x_{2q-1}, 0, \ldots, 0),
\]

\[
v^* = -m_1 x_2 dx_1 + m_1 x_1 dx_2 - \cdots - m_q x_{2q} dx_{2q-1} + m_q x_{2q-1} dx_{2q}
\]

Concerning a non-trivial orbit which is a critical level set of $f$, There is a coordinates system $(\psi, x_1, x_2, \ldots, x_{2q-1}, x_{2q}, x_{2q+1}, \ldots, x_n)$ around it such that $o = (\psi, 0, 0, \ldots, 0)$; where $\psi$ is considered modulo $2\pi$; and the following conditions are satisfied

- The metric has the form $g = s^2du^2 + dx_1^2 + dx_2^2 + \cdots + dx_n^2$.
- The action has the following representation

\[
e^{i\theta}(\psi, x_1 + ix_2, \ldots, x_n) = (m\theta + \psi, e^{im_1\theta}(x_1 + ix_1), \ldots, e^{im_n\theta}(x_{2q-1} + ix_{2q}), x_{2q+1}, \ldots, x_n)
\]

- The function $f$ is given as follows

\[
f(\psi, x_1, \ldots, x_n) = \epsilon_1(x_1^2 + x_2^2) + \cdots + \epsilon_q(x_{2q-1}^2 + x_{2q}^2) + \lambda_{2q+1}x_{2q+1}^2 + \cdots + \lambda_n x_n^2
\]
Here again the restriction of \( f \) to \( \mathbb{R}^n \) factor is a Morse function with critical point 0 and takes the standard form (4.2). With respect to these coordinates the vector field \( v \) and its dual are as follows

\[
(4.7) \quad v = (m, -m_1x_2, m_1x_1, \ldots, -m_qx_{2q}, m_qx_{2q-1}, 0, \ldots, 0),
\]

\[
(4.8) \quad v^* = m dv - m_1x_2 dx_1 + m_1x_1 dx_2 - \cdots - m_qx_{2q} dx_{2q-1} + m_qx_{2q-1} dx_{2q}
\]

In each one of above cases, the Clifford hessian of (3.7) takes the following form (see [11 page])

\[
H = \sum_{i=1}^{n} \lambda_i Z_i \quad Z_i = [dx_i \wedge, dx_i]
\]

where \( \lambda_i = \pm 1 \) is the coefficient of \( x_i \) in (4.2) or (4.6).

**Remark 1.** \( Z_i \) at each point is a diagonalizable linear map on exterior algebra generated by \( dx_i \)'s. An elements of the form \( dx_i \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_q} \) is an eigenvector with eigenvalue 1 if \( i = i_\ell \) for one \( \ell \), otherwise the eigenvalues equals \(-1\).

For next uses we need to review some spectral properties of harmonic oscillator operator

\[
(4.10) \quad -\frac{d^2}{dx^2} + a^2 x^2 ; \quad a > 0
\]

In following remark we summarize those properties of this operator that we will need.

**Remark 2.** The harmonic oscillator operator (4.10) is an unbounded self adjoint operator on \( L^2(\mathbb{R}) \) and provides a spectral resolution for this Hilbert space. The eigenvalues of this operator are \( a(1+2p) \) where \( p = 0, 1, 2, \ldots \). The eigenvector corresponding to the minimal eigenvalue \( a \) is the following function

\[
(4.11) \quad W_a(x) := \sqrt{a^2 \pi} \exp(-ax^2/2)
\]

Given a compactly supported smooth function \( \beta \) on \( \mathbb{R} \) such that \( \beta(0) = 1 \) then

\[
(4.12) \quad \lim_{a \to \infty} \langle \beta(x) W_a(x) \rangle \quad W_a(x) = \beta(0) = 1
\]

Now we turn our attention toward a critical point \( p \). We assume the coordinates introduced just before (4.1) for a \( \epsilon \)-neighbourhood \( N_\epsilon(p) \) of \( p \). Using (3.6), (4.2) and (4.9) the deformed Laplacian \( \Delta^{\text{equiv}}_{\text{eq}}(M) \) coincides with the following operator in a small neighbourhood of \( p \).

\[
(4.13) \quad L_s^k : \Omega^\text{eq}_{\text{eq}}(\mathbb{R}^n) \to \Omega^\text{eq}_{\text{eq}}(\mathbb{R}^n)
\]

\[
L_s^k = \mathbb{H}_s + A_1 + B_1 + A_2 + B_2 + \cdots + A_q + B_q
\]

where

\[
\mathbb{H}_s = \sum_{\ell=2q+1}^{n} \frac{\partial^2}{\partial z_{2\ell-1}^2} + s^2 z_{2\ell}^2 + \lambda_\ell s Z_\ell
\]

and for \( i = 1, \ldots, q \) we have

\[
(4.14) \quad A_i = -\frac{\partial^2}{\partial x_{2i-1}^2} - \frac{\partial^2}{\partial x_{2i}^2} + s^2 x_{2i-1}^2 + s^2 x_{2i}^2 + (\psi_i^* \wedge i_{2i} + \epsilon_{0,i} i_{2i}) \}
\]

\[
(4.15) \quad B_i = s s(Z_{2i-1} + Z_{2i}) + 2t \otimes \psi_{i}^* \wedge \epsilon_{0,i} 2t^{-1} \otimes \psi_{i}^* \wedge
\]

Here \( \epsilon_i = \pm 1 \) is introduced by (4.12) and \( \psi_i = m_i(-x_{2i}, x_{2i-1}) \) and \( \psi_i^* = 2m_i dx_{2i-1} \wedge dx_{2i} \). Because \( \mathbb{H}_s \) is \( \ell \)-linear, we consider \( L_s^k \) as an operator on \( \Omega^\ast(\mathbb{R}^{n-2q}) \otimes \Omega^\text{eq}_{\text{eq}}(\mathbb{R}^{2q}) \). Then \( \mathbb{H}_s \) acts on the first factor and commute with \( A_i \)'s and \( B_i \)'s that act on the second factor. If we consider \( \mathbb{H}_s \) as an operator on \( \Omega^n(\mathbb{R}^{n-2q}) \), then its spectrum; by remarks [1] and [2] consists of the following numbers.
Here $p_\ell = 0, 1, 2, \ldots$ and $q_\ell = \pm 1$. It is clear that the non-zero element of the spectrum are bounded from below by $s$, while $0$ belongs to spectrum only if for all values of $\ell$ we have $\lambda_\ell q_\ell = -1$. This is possible only if the Morse index of the critical point $p = 0$ in $\mathbb{R}^{n-2q}$-direction is equal to $m$. In this case the kernel is generated by the following element

\[
2s \sum_{\ell = 2q+1}^{n} ((1 + 2p_\ell) + \lambda_\ell q_\ell)
\]

(4.16)

$W_s(x_{2q+1}) W_s(x_{2q+2}) \ldots W_s(x_n) \, dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_m}$

Here $W_s$'s are defined by (4.11) and $j$'s are characterized by the property that $\lambda_j = -1$ in (4.2).

Because for $i = 1, 2, \ldots, q$ the operators $A_i + B_i$'s commute with each other, to determine the spectrum of $\sum_i A_i + B_i$ on $\Omega_{eq}^*(\mathbb{R}^2)$ we need just to determine the spectrum of $A_i + B_i$ on $\Omega_{eq}^*(\mathbb{R}^2)$ (here $\mathbb{R}^2 = (x_{2i-1}, x_{2i})$). The operator $B_i$ vanishes on $\mathbb{C}[t] \otimes \Omega_{eq}^2(\mathbb{R}^2)$, while $A_i$ is the sum of two uncoupled harmonic oscillators and a non-negative operator. Therefore; by remark 2 its spectrum is bounded from below by $2s$. Consequently the only invariant subspace that might have a non-zero contribution in $\text{tr} \phi(L_s)$ when $s$ goes to infinity is

\[
\Omega^*(\mathbb{R}^{n-2q}) \otimes \mathbb{C} \Omega_{eq}^{2s}(\mathbb{R}^2) \otimes \Omega_{eq}^{2s}(\mathbb{R}^2) \otimes \cdots \otimes \Omega_{eq}^{2s}(\mathbb{R}^2)
\]

Here the $i$-th $\mathbb{R}^2$ in above is the two dimensional Euclidean space with coordinates $x_{2i-1}$ and $x_{2i}$. To determine the spectrum of $A_i + B_i$ on $\Omega_{eq}^{2s}(\mathbb{R}^2)$ we write this space as the direct sum of following invariant orthogonal subspaces

\[
\Omega_{eq}^{2s}(\mathbb{R}^2) = \Omega_G^0(\mathbb{R}^2) \oplus (\mathbb{C}[t] \otimes \Omega_{eq}^{2s}(\mathbb{R}^2))
\]

On the first summand $\Omega_G^0(\mathbb{R}^2)$ we have $Z_i = -\text{Id}$, so

\[
A_i + B_i = -\frac{\partial^2}{\partial x_{i}^2} - \frac{\partial^2}{\partial y_i^2} + s^2(x_i^2 + y_i^2) - 2\epsilon_i s
\]

(4.18)

The spectrum of this operator on $\Omega_G^0(\mathbb{R}^2)$ consists of numbers $2s(1 + 2p_\ell - \epsilon_i)$ for $p_\ell = 0, 1, 2, \ldots$ (see remark 2) whose non-zero elements go to infinity when $s$ does so. Therefore this is also true for the non-zero elements of its spectrum when restricted to $\Omega_G^0(\mathbb{R}^2)$. On the other hand, again by remarks 1 and 2 just when $\epsilon_i = +1$ the spectrum of $A_i + B_i$ contains 0; by choosing $p_\ell = 0$; and the corresponding eigen-function is the following function which is clearly $S^1$-invariant

\[
W_{i,s} := W_s(x_{2i-1}) W_s(x_{2i})
\]

(4.19)

Here $W$ is defined by (4.11) and we will use this result soon.

By (4.11) and (4.14) the operator $A_i + B_i$ on the second summand in (4.17) is $\mathbb{C}[t]$-linear. Therefore we need just study its spectrum on $\Omega_{eq}^{2s}(\mathbb{R}^2) = \Omega_G^0(\mathbb{R}^2) \otimes H_i$, where $H_i$ is the linear spaces generated by $t$, and $y_i$. Here $y_i = r_i^{-1} dr_i \wedge d\psi_i$ and $(r_i, \psi_i)$ are polar coordinates associated to $(x_i, y_i)$. The operator $A_i + B_i$ on this space has the following expression

\[
A_i + B_i = -\frac{\partial^2}{\partial x_{2i-1}^2} - \frac{\partial^2}{\partial x_{2i}^2} + (s^2 + m_i^2)(x_{2i-1}^2 + x_{2i}^2) + \left[-\frac{2s}{2m_i} \frac{2m_i}{2m_i} \right]
\]

(4.20)

The above matrix acts on $H_i$ and its action commutes with the harmonic oscillator operator defined by other terms. The eigenvalues of the matrix are $\sqrt{s^2 + m_i^2}$ and $-\sqrt{s^2 + m_i^2}$. These correspond, respectively, to the normalized eigenvectors $u_{i,s} := a_i^{-1/2}(m_i t + (\epsilon_i s + \sqrt{s^2 + m_i^2}) \eta_i)$ and $w_{i,s} := b_i^{-1/2}(m_i t + (\epsilon_i s - \sqrt{s^2 + m_i^2}) \eta_i)$ where $a_i = m_i^2 + (\epsilon_i s + \sqrt{s^2 + m_i^2})^2$ and $b_i = m_i^2 + (\epsilon_i s - \sqrt{s^2 + m_i^2})^2$. With respect to the base \{u_{i,s}, w_{i,s}\} the expression (4.20) takes the following form
\( A_i + B_i = -\frac{\partial^2}{\partial x_{2i-1}^2} - \frac{\partial^2}{\partial x_{2i}^2} + (s^2 + m_i^2)(x_{2i-1}^2 + x_{2i}^2) + 2 \left[ \frac{\sqrt{s^2 + m_i^2}}{0} - \frac{0}{\sqrt{s^2 + m_i^2}} \right] \)

The smallest eigenvalue of the harmonic oscillator (whose corresponding eigenfunction is \( S^1 \)-invariant as we have mentioned in above) is 2. Therefore the non-zero eigenvalues of \( A_i + B_i \) go to infinity when \( s \) goes toward infinity. However 0 always belongs to the spectrum of this operator and its corresponding normalized eigenvector is (see (4.11))

\[
\tilde{W}_{i,s}(x_{2i-1}, x_{2i}) = W_s(x_{2i-1})W_s(x_{2i}) \otimes w_{i,s} ; \quad s^2 = s^2 + m_i^2
\]

Therefore the kernel of the restriction of \( A_i + B_i \) to the second summand in (4.17) is generated by \( \tilde{W}_{i,s}(x_{2i-1}, x_{2i}) \).

Now in (4.2) let the number of occurrence of \( \epsilon_i = -1 \) be \( k_1 \) while the number of occurrence of \( \lambda_i = -1 \) be \( k_2 \) (so the Morse index of \( p \) is \( 2k_1 + k_2 \)). Without loss of generality we may assume that \( \epsilon_i \)'s and \( \lambda_i \)'s are sorted increasingly and we consider the operator \( L^k_s \) on \( \Omega^k_{eq}(\mathbb{R}^n) \) where \( k = 2k_1 + k_2 + \ell \).

Following what we have discussed above, the following vectors generate the kernel of \( L^k_s \)

\[
\tilde{W}_{k_1+1,s} \tilde{W}_{k_1+1,s} \tilde{W}_{k_1+1,s} \tilde{W}_{k_1+1,s} \tilde{W}_{k+1,s}(x_{2q+1}) \tilde{W}_s(x_n) \otimes dx_{2q+1} \wedge \cdots \wedge dx_{2q+k_2}
\]

Here \( \tilde{W}_{k_1+1,s} \) is either equal to \( \tilde{W}_{k_1+1,s} \) or equal to \( W_{k_1+1,s} \). When \( s \) goes to infinity the following relations hold

\[
\begin{align*}
\lim \tilde{W}_{i,s} - W_s(x_{2i-1})W_s(x_{2i}) \eta_i &= 0 \quad \text{if } \epsilon_i = -1 \\
\lim \tilde{W}_{i,s} - W_s(x_{2i-1})W_s(x_{2i}) \otimes t &= 0 \quad \text{if } \epsilon_i = +1 \\
W_{i,s}(x_{2i-1}, x_{2i}) &= W_s(x_{2i-1})W_s(x_{2i}) \quad \text{if } \epsilon_i = +1
\end{align*}
\]

Therefore the difference between above vector and the following one goes to zero when \( s \) goes to infinity

\[
\Phi := t^m \otimes W_s(x_1)W_s(x_2) \cdots W_s(x_n) \otimes \eta_1 \wedge \eta_2 \cdots \wedge \eta_{k_1} \wedge dx_{2q+1} \wedge \cdots \wedge dx_{2q+k_2}
\]

Here \( m = k_1 \) is the number of position where \( \epsilon_i = +1 \) and we have chosen the kernel of \( A_i + B_i \) be an element in \( \Omega^k_{eq}(\mathbb{R}^2) \). The exterior-algebraic-grade of this vector is \( 2k_1 + k_2 \) which is equal to the Morse index of the critical point \( p \). Consequently the critical point \( p \) contribute in \( \text{tr} \phi(L^k_s) \) if and only if its Morse index equals \( k, k-2, k-4, \ldots \). This proves the following relation

\[
\lim_{s \to \infty} \sum_p \text{tr} \phi(L^k_s) = c_k + c_{k-2} + c_{k-4} + \ldots
\]

We recall from discussion right before corollary 3.3 the compactly supported smooth functions \( \phi_p \) on \( \mathbb{R}^n \) with \( \phi_p(0) = 1 \). As we have discussed in above, all non-zero eigenvalues of \( L^k_s \) go to infinity with \( s \) and the normalized generator of \( \ker L^k_s \) tends toward orthonormal set of vectors given by (4.22).

This fact and relation (4.12) imply together the following relation

\[
\lim_{s \to \infty} \text{tr} \phi(L^k_s) = \lim_{s \to \infty} \text{tr} \phi_p \phi(L^k_s)
\]

By combining this relation with (4.23) we get the following lemma

**Lemma 4.2.** Let \( \phi_p \) be a compactly supported smooth function on \( \mathbb{R}^n \) with \( \phi_p(0) = 1 \). The following relation holds

\[
\lim_{s \to \infty} \sum_p \text{tr} \phi_p \phi(L^k_s) = c_k + c_{k-2} + c_{k-4} + \ldots
\]

where \( p \) runs over the critical points of the Morse function \( f \) and \( c_i \) denotes the number of critical points with index \( i \).
Now we compute the contribution of the critical orbits in the trace of $\phi(\Delta_{eq}^k)$. In coordinates $(\psi, x_1, x_2, \ldots, x_n)$ introduced in above, the deformed equivariant Laplacian takes the following form when acting on $\Omega_{eq}^k(S^1 \times \mathbb{R}^{n-1})$

$$L_s^k = L_s + T$$

where $L_s$ is the deformed equivariant Laplacian on $\Omega_{eq}^*(\mathbb{R}^n)$ given by (4.13) while $T$ is an operator acting on $\Omega_{eq}^*(S^1)$ given by the following expression

$$T = d\psi \wedge i_{\partial/\partial\psi} + \epsilon_0 i_{\partial/\partial\psi} d\psi \wedge.$$

Here we have used the fact that invariant differential forms on $S^1$ are generated (over $\mathbb{C}$) by $1$ and $d\psi$. We have $\Omega_{eq}^*(S^1 \otimes \mathbb{R}^n) = \Omega_{eq}^*(S^1) \otimes \Omega_{eq}^*(\mathbb{R}^n)$. Here the tensor product is taken over $\mathbb{C}[t]$. Put $\Omega_{eq}^*(S^1) = \mathbb{C}(1) \oplus \Omega_{eq}^{>1}(S^1)$. Then the first summand belongs to $\ker T$, while the restriction of $T$ to the second summand is given by multiplication by $\|\partial/\partial\psi\|^2 = s^2 m^2$.

As we have seen previously, the non-zero elements of the spectrum of $L_s$ go to infinity with $s$ while the kernel of this operator is asymptotically generated by vectors given by (4.22). Therefore $L_s^k$ has a non trivial one-dimensional kernel (generated by $1 \otimes \Phi$) only if the power of $t$ in (4.22) is zero. This is possible only if the Morse index of the critical orbit $o$ is $k$. Therefore if the Morse index of the critical orbit is $k$ then

$$\lim_{s \to \infty} \text{tr} \phi(L_s^k) = 1$$

On the other hand, an argument completely similar to the one that led to (4.24) provides the following relation

$$\lim_{s \to \infty} \text{tr} \phi(L_s^k) = \lim_{s \to \infty} \text{tr} \phi_o \phi(L_s^k)$$

where $\phi_o$ is the function introduced right before corollary 3.4. Summarizing, we have proved the following lemma

**Lemma 4.3.** The following relation holds

$$\lim_{s \to \infty} \sum_{o} \text{tr} \phi_o \phi(L_s^k) = d_k$$

where $o$ runs over the critical orbits of the Morse function $f$ and $d_k$ denotes the number of critical orbits with index $k$ (of course $d_k = 0$ for $k \geq n$).

Now we have every thing to prove the main theorem 2.1.

**Proof of the main theorem 2.1** Actually it follows directly from theorem 3.2 and lemmas 3.5, 4.2 and 4.3. We just need to show that for $k \geq n + 1$ the Morse inequalities do not provide new one and reduce to lower order Morse inequalities. We do this for $k = n$, the general case is similar and follows from (3.5). For this purpose note that $d_k = 0$ for $k \geq n$, therefore by 3.3 and 4.2 we have

$$\lim_{s \to \infty} (\mu_{n+1}^n - \mu_n^k) = (c_{n-1} + c_{n-3} + \ldots) - (c_n + c_{n-2} + \ldots)$$

The right side of this equality is $(-1)^{n-1}$ times the sum of the indices of the vector field $v$ on its singularities which equals $(-1)^{n-1} \chi(M)$, by the Poincare-Hopf theorem. This and lemma 3.1 show that $\tilde{c}_{n+1} - \tilde{c}_n = \beta_{eq}^{n+1} - \beta_{eq}^n$ and completes the proof.
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