Probability Distributions and Hilbert Spaces:
Quantum and Classical Systems

V. I. Man’ko and G. Marmo
Dipartimento di Scienze Fisiche, Università di Napoli “Federico II” and
Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra d’Oltremare, Pad. 20,
80125 Napoli, Italia

Abstract

We use the fact that some linear Hamiltonian systems can be considered as
“finite level” quantum systems, and the description of quantum mechanics in terms
of probabilities, to associate probability distributions with this particular class of
linear Hamiltonian systems.

1 Introduction

Some general structural features of dynamical systems can be exhibited in a study
of the relation between classical and quantum mechanics. The most widely accepted
formal relationship between quantum mechanics and classical mechanics is expressed in
the analogy between commutator brackets of operators on some Hilbert space and Poisson
brackets among functions on some symplectic phase space, along with the analogy between
Heisenberg equations of motion and Hamilton equations of motion. This analogy can be
made more suggestive if operators are replaced by expectation values (quadratic functions
on the Hilbert space) and the commutator is replaced by Poisson brackets among these
quadratic functions \(^1\). It is also possible to show that Schrödinger equations define
Hamiltonian systems on the Hilbert space, where the imaginary part of the scalar product
defines a symplectic structure. These various aspects have been considered in the past \(^2\)
and more recently in connection with Wigner’s problem \(^3\) \(^4\). They can be summarized
in the following way.

On a symplectic vector space, carrier space of a Hamiltonian linear dynamical sys-
tem, one introduces a complex structure compatible with the symplectic structure, i.e.,
their composition, as maps, defines a positive definite inner product on the vector space.
Finite-level quantum systems are shown to be linear Hamiltonian systems which, in addi-
tion, leave invariant a compatible complex structure. Therefore, on a finite-dimensional
Hilbert space quantum systems are a subclass of linear Hamiltonian systems. This char-
acterization carries over to infinite dimensions where the “classical system” has an infinite
numbers of degrees of freedom. By no means, the linear system preserving two compat-
tible structures (quantum system) can be considered as a quantization of the associated
Hamiltonian one (classical system).

Continuing in the spirit of the approach by Moyal \(^5\), Wigner \(^6\), Husimi \(^7\), et al.,
according to which quantum mechanics can be formulated in a natural manner in terms
of functions on the classical phase space, recently \(^8\) a different approach has been pro-
posed dealing with probability distributions rather than quasi-probabilities of previous

\(^1\)on leave from the P. N. Lebedev Physical Institute, Moscow, Russia
approaches. It is to be remarked that this approach differs from the one initiated by Koopman [9], who had shown how the dynamical transformations of classical mechanics considered as measure preserving transformations of the phase space, induce unitary transformations on the Hilbert space of functions which are square integrable with respect to a Liouville measure over the phase space. It is to be stressed that this Hilbert space corresponds not to the space of state vectors in quantum mechanics but to the Hilbert space of operators on state vectors (with the trace of the product of two operators being chosen as the scalar product).

In this paper, we would like to consider classical canonical linear systems identified with the “Hamiltonian part” of quantum systems with a finite number of states. We use previous correspondence with probabilities, to associate probabilities also with classical linear systems. Then we move to infinite dimensions and show how things appear there. This study shows the main difference between classical systems and quantum systems at the level of associated probabilities.

In a future paper, we shall consider how the dynamical evolution and dynamical symmetry groups are represented on these probability distributions.

The state of a classical particle (one degree of freedom) is described by two $c$-number valued functions, position $q$ and momentum $p$ [10]. In classical statistical mechanics, the state of a particle is described by the probability distribution on the particle’s phase space $f(q, p) \geq 0$ such that

$$\int f(q, p) \, dq \, dp = 1.$$  \hfill (1)

In quantum mechanics, pure states of a particle are described by a complex wave function $\Psi(q)$ and mixed normalized states are described by density matrices, $\rho(q, q')$ is the matrix element $\langle q | \hat{\rho} | q' \rangle$ of an Hermitian nonnegative density operator $\hat{\rho}$ with $\text{Tr} \hat{\rho} = 1$. The diagonal elements of the density matrix $\rho(q, q)$ determine the probability distribution in position which for pure state $\rho_\Psi = |\Psi \rangle \langle \Psi |$ is reduced to $\rho_\Psi(q, q) = |\Psi(q)|^2$. The wave function $\Psi(q)$ may be considered as a component $\Psi_q \equiv \Psi(q)$ of the state vector in Hilbert space. The Hilbert space structure is usually considered as appropriate for description of the quantum motion.

On the other hand, Koopman [9] has demonstrated that a Hilbert space structure can be associated to a classical system as well. This approach has been developed in [11]. Also Strocchi [2] has shown that quantum motion (Schrödinger equation [12] for the wave function) may be reformulated as formally classical motion of coupled harmonic oscillators. The linear dynamics of such oscillators and its possible deformations were discussed in [13, 14, 15].

The relation of quantum dynamics to a probability distribution evolution (probability representation of quantum mechanics) was found in [14] in connection with tomography transformation [13, 16] (Radon transform) of the density matrix used for measuring quantum states [14, 17]. In probability representation, a quantum state is described by a family of positive probability distributions of position instead of wave function or density matrix. Due to this the probability representation can be considered as classical-like formulation of quantum mechanics. The connection of irreducible representations of Heisenberg–Weyl group with the tomography transformation was discussed in [18].

The goal of our paper is to clarify some mutual relations between these different aspects of classical and quantum dynamics. We demonstrate that the classical motion of a
system, both with finite and infinite numbers of degrees of freedom, can be associated with
the probability measure (a family of marginal distributions) characteristic for quantum
mechanics.

The paper is organized as follows. In Section 2, we review description of quantum
dynamics in terms of classical formalism. In Section 3, we consider two-level system in
detail. Section 4 is devoted to treating two-level system as a spin-1/2-particle system. In Section 5, the probability representation (tomographic map) of quantum mechanics
of spin-1/2 particle is reviewed. In Section 6, the map of states of two and \( N \) classical
oscillators onto probability distributions is considered. In Section 7, we discuss classical
states of infinite number of oscillators in terms of the marginal distributions and properties
of the probability distributions under action of canonical transform of the oscillators’ phase
space.

## 2 Classical Formalism for Quantum Dynamics

The Schrödinger evolution equation (\( \hbar = 1 \)) for the wave function of a one-dimensional
particle

\[
\dot{\Psi}(q, t) = H \Psi(q, t),
\]

where the Hamiltonian operator is

\[
H = \frac{\hat{p}^2}{2} + V(\hat{q}), \quad \hat{p} = -i \frac{\partial}{\partial q}, \quad \hat{q} = q,
\]
in the coordinate representation for real and imaginary parts of the wave function

\[
\Psi(q, t) \equiv \Psi_q = \frac{1}{\sqrt{2}} (x_q + ip_q); \quad \Psi^*(q, t) \equiv \Psi^*_q = \frac{1}{\sqrt{2}} (x_q - ip_q)
\]
can be rewritten in the form [2]

\[
\frac{i}{\partial t} \left( \begin{array}{c} \Psi_q \\ \Psi^*_q \end{array} \right) = \left( \begin{array}{cc} H_{qq'} & 0 \\ 0 & -H^*_{qq'} \end{array} \right) \left( \begin{array}{c} \Psi_{q'} \\ \Psi^*_{q'} \end{array} \right).
\]

We treat \( H_{qq'} \) as matrix elements of the Hamiltonian in the coordinate representation

\[
\langle q | H | q' \rangle = H_{qq'}
\]

and perform summation (integration) over repeated indices \( q' \). By introducing notation

\[
\xi_q = \left( \begin{array}{c} \Psi_q \\ \Psi^*_q \end{array} \right),
\]

we rewrite the Schrödinger equation in the form

\[
\dot{\xi} = h_{qq'} \xi_q', \quad h_{qq'} = \left( \begin{array}{cc} H_{qq'} & 0 \\ 0 & -H^*_{qq'} \end{array} \right).
\]

The evolution equation can be obtained as a classical equation using the Hamiltonian

\[
\mathcal{H} = \Psi^*_q H_{qq'} \Psi_{q'}
\]
with summation (integration) over repeated indices and with Poisson brackets
\[
\{\Psi_q, \Psi_{q'}\} = \{\Psi_q^*, \Psi_{q'}^*\} = 0; \quad \{\Psi_q, \Psi_{q'}^*\} = i\delta_{qq'}.
\]
(10)
The symbol \(\delta_{qq'}\) is either Kronecker symbol (if \(q\) and \(q'\) are discrete indices) or Dirac delta-function (for continuous indices \(q\) and \(q'\)). Thus, Schrödinger equation is equivalent to the Hamiltonian “classical equation”
\[
\dot{\Psi}_q = \{\mathcal{H}, \Psi_q\}.
\]
(11)
One can see that Schrödinger equations are equivalent to classical equations for a system of coupled oscillators, with coupling of the oscillators being described by matrix elements \(H_{qq'}\). To show this more explicitly, we rewrite the equations for real variables \(x_q, p_q\), for which Poisson brackets are real ones
\[
\{x_q, x_{q'}\} = \{p_q, p_{q'}\} = 0; \quad \{p_q, x_{q'}\} = i\delta_{qq'}.
\]
(12)
Then the Hamiltonian takes the form
\[
\mathcal{H} = \frac{1}{2} Q_q B_{qq'} Q_{q'},
\]
(13)
where
\[
Q_q = \begin{pmatrix} p_q \\ x_q \end{pmatrix} \quad \text{and} \quad B_{qq'} = \frac{1}{2} \begin{pmatrix} H_{qq'} + H_{qq'}^* & i \left( H_{qq'}^* - H_{qq'} \right) \\ i \left( H_{qq'} - H_{qq'}^* \right) & H_{qq'} + H_{qq'}^* \end{pmatrix}.
\]
(14)
Then the evolution equation for the real vector \(Q_q\) reads
\[
\dot{Q}_q = A_{qq'} Q_{q'},
\]
(15)
with summation (integration) over \(q'\) and the matrix \(A_{qq'}\)
\[
A_{qq'} = -\Sigma B_{qq'}; \quad \Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
(16)
We see that the search for the energy levels of the quantum system is equivalent to the search for normal modes and their frequencies for a corresponding classical system of coupled oscillators.

3 Two-Level System

Till now we used infinite system of coupled oscillators and our manipulations were done at a formal level. To clarify the approach for constructing probability distribution associated to a classical state, let us consider a two-level system. In this case, following [3] we have
\[
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} \Psi \\ \Psi^* \end{pmatrix}.
\]
(17)
The Hermitian matrix \(H\) such that
\[
H_{ik} = H_{ki}^* = \begin{pmatrix} a & b_1 + ib_2 \\ b_1 - ib_2 & c \end{pmatrix}; \quad i, k = 1, 2
\]
(18)
is the Hamiltonian and Schrödinger equation looks as
\[ i \dot{\xi} = h \xi; \quad h = \begin{pmatrix} H & 0 \\ 0 & -H^* \end{pmatrix}. \] (19)

Energy levels of a two-level system are determined by matrix elements \( a, b = b_1 + ib_2, c \) of matrix (18) and we arrive at
\[
E_1 = \frac{a + c}{2} + \frac{1}{2} \sqrt{(a + c)^2 + 4(bb^* - ac)}; \\
E_2 = \frac{a + c}{2} - \frac{1}{2} \sqrt{(a + c)^2 + 4(bb^* - ac)}. 
\]

By introducing the real components of the wave function
\[ \Psi_k = \frac{1}{\sqrt{2}} (x_k + ip_k), \] (20)
i.e.,
\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \] (21)
and
\[ \mathbf{Q} = \begin{pmatrix} \mathbf{p} \\ \mathbf{x} \end{pmatrix}, \] (22)
we arrive at the Hamiltonian description of two formally classical coupled oscillators in the form
\[ \mathcal{H} = \frac{1}{2} \sum_{\alpha,\beta=1}^4 Q_{\alpha} B_{\alpha\beta} Q_{\beta}, \] (23)
where 4×4-matrix \( B \) reads
\[ B = \frac{1}{2} \begin{pmatrix} H + H^* & i(H^* - H) \\ -i(H - H^*) & H + H^* \end{pmatrix}. \] (24)

The Schrödinger evolution equation for the two-level system reads
\[ \dot{\mathbf{Q}} = A \mathbf{Q}, \] (25)
where
\[ A = -\Sigma B. \]
Thus, the quantum linear dynamics of two-level system is equivalent to the classical dynamics of two coupled classical oscillators. This means that one can associate to two coupled classical oscillators the two-dimensional Hilbert space of states which formally is equivalent to the artificial spin-1/2 states.

The classical states of two oscillators is described by four real numbers \( q_1, p_1, q_2, p_2 \). The Hilbert structure is introduced by relating these four numbers to “spinors”
\[ \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} q_1 + ip_1 \\ q_2 + ip_2 \end{pmatrix}. \] (26)
On the other hand, the distribution functions which can determine completely the spinors were considered in 20. Thus, we can associate with two linear classical coupled oscillators a positive probability distributions (see below).
4 Two-Level System as Spin-1/2 System

Let us describe now a two-level quantum system and its states in terms of spinorial states. To proceed in this direction, we construct the spinor $\Psi$ as a column consisting of two-state vectors (of the two-level system) $\Psi_1$ and $\Psi_2$ as given by formula (17). Then the density matrix of the two-level system can be constructed for the pure states as

$$\rho_{ik}^{(\Psi)} = \Psi_i \Psi_k^*; \quad i, k = 1, 2. \quad (27)$$

For mixed state, the density operator has the form

$$\hat{\rho} = w_1 \Psi_1 \langle \Psi_1 | + w_2 \Psi_2 \langle \Psi_2 |,$$  \quad (28)

where $w_1$ and $w_2$ are positive probabilities and $w_1 + w_2 = 1$. The generic Hermitian density matrix of the mixed state of the spin-1/2

$$\rho_{ik} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12}^* & \rho_{22} \end{pmatrix} \quad (29)$$

has the property $\rho_{11} + \rho_{22} = 1$.

For pure state, we have

$$\rho_{11}^{(\Psi)} = |\Psi_1|^2; \quad \rho_{22}^{(\Psi)} = |\Psi_2|^2; \quad \rho_{12}^{(\Psi)} = \Psi_1 \Psi_2^* \quad (30)$$

Thus, we can associate to classical two-mode oscillator and to its states (with positions $q_1, q_2$ and momenta $p_1, p_2$) the density matrix of spin-1/2 particle.

5 Tomographic map for spinors

Tomography of spin states was considered in [20, 21]. The marginal distribution which determines the spin state can be constructed using the relation

$$\rho_{mm'} = (D \rho D^\dagger)_{mm'}.$$  \quad (31)

Here $\rho$ is a density matrix in the initial reference frame and $\rho_{mm'}(\varphi, \theta)$ is the density matrix in the rotated reference frame with Euler angles ($\varphi, \theta, \psi$). The matrix $D$ is the matrix of the spinor irreducible representation of the group $SU(2)$ and its matrix elements depend on the Euler angles ($\varphi, \theta, \psi$). In view of the structure of the formula for rotated density matrix, the diagonal elements of the density matrix depend on two angles ($\varphi, \theta$) only determining the direction of the quantization axis. The one-dimensional stability group of $\rho$ in $SU(2)$ determines the quotient $S^2 = SU(2)/U(1)$. The spin projection on this axis has the probability distribution function

$$w (m, \varphi, \theta) = \rho_{mm}(\varphi, \theta). \quad (32)$$

The function $w (m, \varphi, \theta)$ is positive and normalized

$$\sum_{m=-1/2}^{1/2} w (m, \varphi, \theta) = 1.$$
Since the probability distribution function depends on two extra parameters \((\varphi, \theta)\), it is possible to express both diagonal and nondiagonal elements of the initial density matrix which determines the spin quantum state in terms of the distribution function \([20]\):

\[
(-1)^{m_2} \sum_{j_3=0}^{1} \sum_{m_3=-j_3}^{j_3} (2j_3 + 1)^2
\]

\[
\times \frac{1}{2} \sum_{m_1=-1/2}^{1/2} \int \cdots \int \frac{d\omega}{8\pi^2} = \rho_{m_1 m_2}', \quad (33)
\]

where

\[
\int d\omega = \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\psi. \quad (34)
\]

In formula (33), Wigner 3\(j\)-symbols and matrix elements of the irreducible representation of the \(SU(2)\) group \(D_{\text{om}}^{(j)}\) are adopted from \([10]\). Thus, the states of spin-1/2 particle are mapped onto set of marginal probability distributions \(w(m, \varphi, \theta)\).

6 Map of Classical States of Two-Mode Harmonic Oscillators onto Probability Distribution Set

In the previous sections, we showed that the states of a classical two-mode oscillator labelled by four real \(c\)-number functions \(q_1, q_2, p_1, p_2\) are mapped onto state vectors of two-dimensional Hilbert space of spinors

\[
q_1, q_2, p_1, p_2 \longleftrightarrow \Psi.
\]

Spinors, which differ by a common phase factor, i.e., \(\Psi\) and \(e^{i\phi}\Psi\), determine the same quantum state. Thus, we arrive at description of pure quantum state of spin-1/2 in terms of nonnegative Hermitian \(2\times2\) density matrix \(\rho\) with trace equal to unity and condition

\[
\text{Tr} \rho^2 = 1.
\]

This means that states of classical two-mode oscillators are mapped onto a set of density matrices of the pure spin states,

\[
q_1, q_2, p_1, p_2 \rightarrow \rho.
\]

In Section 5, we have shown that the density matrix of spin state determines the probability distribution function

\[
\rho \rightarrow w(m, \varphi, \theta)
\]

of the spin projection \(m\) onto the axis with direction given by angles \(\varphi\) and \(\theta\). Also, given the distribution function \(w(m, \varphi, \theta)\) one determines the density matrix \(\rho\) due to formula (33):

\[
w(m, \varphi, \theta) \rightarrow \rho.
\]
Thus, we have proved that the classical state of a two-mode oscillator, labelled by four real numbers $q_1, q_2, p_1, p_2$ can be associated with probability distribution,

$$q_1, q_2, p_1, p_2 \rightarrow w(m, \varphi, \theta).$$

For the spin $j$, the construction is similar and formulae for spin-1/2 case are modified as follows

$$\Psi^{(j)} = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_{2j+1} \end{pmatrix}.$$ (35)

The density matrix of pure state is given by

$$\rho^{(j)}_{ik} = \Psi_i \Psi_k^*, \quad i, k = 1, 2, \ldots, 2j + 1.$$ (36)

The density operator of a mixed state has the form

$$\hat{\rho}^{(j)} = \sum_{k=1}^{2j+1} \omega_k \langle \Psi_k | \langle \Psi_k |,$$ (37)

where

$$\omega_k \geq 0 \quad \text{and} \quad \sum_{k=1}^{2j+1} \omega_k = 1.$$

The rotated density matrix is given by the relation

$$\rho^{(j)}_{mm'}(\varphi, \theta, \psi) = (D^{(j)} \rho^{(j)} D^{(j)}\dagger)_{mm'}.$$ (38)

Here matrices $D^{(j)}$ are the known matrices of irreducible representation of the group $SU(2)$ with spin $j$. For the case of $j \neq 1/2$, the formula which connects the density matrix with marginal distribution has the natural generalized form of the spin-1/2 case:

$$(-1)^{m_2} \sum_{j_3=0}^{2j} \sum_{m_3=-j_3}^{j_3} (2j_3 + 1)^2$$

$$\times \sum_{m_1=-j}^{j} \int (-1)^{m_1} w(m_1, \varphi, \theta) D^{(j)}_{m_1 m_3}(\varphi, \theta, \psi)$$

$$\times \left( \begin{array}{ccc} \hat{j} & \hat{j} & \hat{j} \\ m_1 & -m_1 & 0 \end{array} \right) \left( \begin{array}{ccc} \hat{j} & \hat{j} & \hat{j} \\ m' & -m' & m' \end{array} \right) \frac{d\omega}{8\pi^2} = \rho^{(j)}_{m'm'},$$ (39)

i.e., in all the cases (where $j = 1/2$ was used) $1/2$ has to be replaced by the value of spin equal to $j$.

Thus, we obtain the generalization of the map of classical states of an $N$-dimensional oscillator onto a set of distributions. We formulate the result: Given $N$ coupled classical oscillators whose state is labeled by $2N$ real number-coordinates on its phase space

$$Q = (q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N).$$

Let us construct complex $N$-vector

$$\Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_N \end{pmatrix}.$$
where
\[ \Psi_k = \frac{1}{\sqrt{2}} (q_k + ip_k); \quad k = 1, \ldots, N. \]
Let us consider now this vector as basis of \( 2j + 1 = N \)-dimensional irreducible representation of the group \( SU(2) \) with spin \( j = (N - 1)/2 \). Then points \( Q \) in the classical phase space are mapped onto the set of probability distribution functions,
\[ Q \rightarrow w^{(j)} (m, \varphi, \theta), \]
where \( m = -j, -j + 1, \ldots, j - 1, j \) and angles \( \varphi, \theta \) on \( S^2 \). The map is invertible up to a common phase factor of the vector \( \Psi \) which means we map one-parameter family of classical states of \( 2N \)-dimensional coupled oscillators
\[ (q_k + ip_k) e^{i\varphi}; \quad k + 1, 2, \ldots, N \]
with common phase, describing a common rotation in phase space of all the constituent oscillators, onto the set of probability distribution functions.

For \( N \) coupled oscillators, one can consider the limit \( N \rightarrow \infty \), which implies limit \( j \rightarrow \infty \).

For spin \( j \), projections \( m = -j, \ldots, j \) mean that the energy spectrum which is mimicked by frequencies of the normal modes is an equidistant one. But this restriction may be easily removed.

7 Map of Infinite-Dimensional Coupled Oscillator onto Distribution Set and Canonical Transformations

We consider now the system of coupled oscillators which is obtained as a limit \( N \rightarrow \infty (j \rightarrow \infty) \) of a finite number of coupled oscillators. To do this, we notice that to achieve the case of nonequidistant frequency spectrum for “coupled classical oscillators” corresponding to nonequidistant energy levels of a quantum system one needs an additional map. This map will transform the wave functions of the system with nonequidistant energy levels into wave functions of the system with equidistant energy levels. The model of spinors is not sensitive to the structure of energy spectrum. The information on the energy-level position is an external one and it can be imposed if the physical interpretation of spinor components and the corresponding marginal distributions are considered.

Another aspect to be mentioned is the behavior of the marginal distribution due to canonical transformation of position and momentum \( q_k, p_k \), i.e., of the real and imaginary parts of the wave functions. First, let us consider linear canonical transformations
\[ Q' = \Lambda Q, \]
where \( \Lambda \in Sp(2N, R) \) is a symplectic real matrix. The Poisson brackets are preserved by this transform. If the matrix \( \Lambda \) has the form
\[ \Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}, \]
the symplectic transform can be written as
\[ \Psi' = u\Psi + v\Psi^*. \]
Here the matrix $u$ and the matrix $v$ are

$$u = \frac{\lambda_1 + i\lambda_2}{2} + \frac{\lambda_4 - i\lambda_3}{2}; \quad u = \frac{\lambda_4 + i\lambda_3}{2} - \frac{\lambda_1 - i\lambda_2}{2}.$$  

Formula (40) shows the action of the linear canonical transform on wave functions in the “classical” formulation of quantum mechanics [2].

The density matrix of the pure state

$$\rho = \Psi \Psi^{\dagger}$$

is transformed into

$$\rho' = u \Psi \Psi^{\dagger} u^{\dagger} + v \Psi^{\ast} \Psi^{\dagger} v^{\dagger} + v \Psi^{\ast} \Psi^{\dagger} u^{\dagger} + u \Psi \Psi^{\dagger} v^{\dagger}.$$ (41)

Formula (41) is an analog of Bogolyubov transform for the density matrix. To show this explicitly, we introduce the matrix

$$\sigma = \Psi \Psi^{\dagger}.$$  

Then the transformed density matrix takes the form

$$\rho' = u \rho u^{\dagger} + v \rho^{\ast} v^{\dagger} + v \sigma^{\ast} u^{\dagger} + u \sigma v^{\dagger}.$$  

For the point canonical transformation defined by

$$\lambda_2 = \lambda_3 = 0; \quad \lambda_1^{-1} = \lambda_4^{\text{tr}},$$

one arrives at

$$u = \frac{\lambda_1 + \lambda_4^{\text{tr}}}{2}; \quad v = \frac{\lambda_1^{-1} - \lambda_1}{2}. \quad (42)$$

Since the marginal distribution is related to the density matrix as diagonal element of the rotated density matrix, we obtain the formula for canonically transformed probability distribution corresponding to the quantum state $\Psi$, namely,

$$w'(m, \varphi, \theta) = \left[ D^{(j)} u \Psi \Psi^{\dagger} u^{\dagger} D^{(j)^{\dagger}} + D^{(j)} v \Psi^{\ast} \Psi^{\dagger} v^{\dagger} D^{(j)^{\dagger}} + D^{(j)} v \Psi^{\ast} \Psi^{\dagger} u^{\dagger} D^{(j)^{\dagger}} + D^{(j)} u \Psi \Psi^{\dagger} v^{\dagger} D^{(j)^{\dagger}} \right]_{mm}. \quad (43)$$

In this way, we have implemented linear canonical transformations on the space of distribution functions.

In infinite-dimensional case, we will use the invertible map

$$w_\Psi(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \exp \left[ \frac{i\mu y^2}{2\nu} - \frac{i\nu X}{\nu} \right] \Psi(y) \, dy \right|^2 \quad (44)$$

of the wave function $\Psi(y)$ onto the marginal probability distribution function $w_\Psi(X, \mu, \nu)$. The function $w_\Psi(X, \mu, \nu)$, with $\mu$ and $\nu$ being real parameters, is positive and normalized,

$$\int w_\Psi(X, \mu, \nu) \, dX = 1$$
for normalized wave functions,

\[ \int |\Psi(y)|^2 dy = 1. \]

The map can be rewritten for density matrix of pure state \( \rho_\Psi(y, y') \)

\[ \rho_\Psi(y, y') = \Psi(y)\Psi^*(y') = xy'p_y' - p_yp_y' + i(x_yp_y' - x_y'p_y) \quad (45) \]

in the form

\[ w_\Psi(X, \mu, \nu) = \frac{1}{2\pi i} \int \rho_\Psi(y, y') \exp \left[ -i \frac{y - y'}{\nu} \left( X - \mu \frac{y + y'}{2} \right) \right] dy dy'. \quad (46) \]

The Wigner–Moyal function \([6, 5]\) is related to the generic density matrix by

\[ W(q, p) = \int \rho \left( q + \frac{u}{2}, q - \frac{u}{2} \right) e^{-ipu} du \quad (47) \]

and

\[ \rho(x, x') = \frac{1}{2\pi} \int W \left( \frac{x + x'}{2}, p \right) e^{ip(x-x')} dp. \quad (48) \]

On the other hand, in the generic case the marginal distribution is related to the Wigner–Moyal function by

\[ w(X, \mu, \nu) = \int W(q, p) e^{-i(kx_\mu - \nu p)} dp \frac{dX}{(2\pi)^2} \quad (49) \]

and

\[ W(q, p) = \frac{1}{2\pi} \int w(X, \mu, \nu) e^{i(x_\mu - \nu p)} dX d\mu d\nu \quad (50) \]

Thus, for pure and mixed states, the probability distribution \( w(X, \mu, \nu) \) is associated to quantum state by means of an invertible integral transform. This property can be used to describe quantum states by positive probability distributions instead of wave functions or density matrices. In the case of classical pure state labeled by a point in the phase space \((x_q, p_q)\), the probability distribution \( w(X, \mu, \nu) \) can be defined, in view of the described chain of maps, both for finite and infinite cases.

### 8 Conclusion

We have shown that classical states labeled by points in the classical phase space of a classical system can be associated with the marginal probability distribution functions. The map constructed uses the map of the points of the classical phase space onto the set of wave functions of a corresponding quantum system. Then one uses the invertible map from wave functions onto probability distributions introduced in the tomography scheme for measuring the quantum states. The established relation clarifies some aspects of the connection between quantum and classical pictures of linear dynamical systems. This gives the possibility to transport the results of classical considerations into quantum pictures and vice versa.
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References

[1] Man’ko, V. I., Marmo, G., and Zaccaria, F., “From equations of motion to canonical commutation relations: classical and quantum systems,” in: Gruber, B., and Ramek, M. (eds.) “Symmetry in Science X. Proceedings of the International Symposium” (Bregenz, Austria, July 1997), (Plenum Press, New York, 1998), p. 223; Man’ko, V. I., Marmo, G., Sudarshan, E. C. G., and Zaccaria, F., “f-Oscillators” in: Atakishiyev, N. M., Seligman, T. H., and Wolf, K.-B. (eds.) “Proceedings of the Fourth Wigner Symposium” (Guadalajara, Mexico, July 1995), (World Scientific, Singapore, 1996), p. 421; Phys. Scr. 55, 528 (1997).

[2] Strocchi, F., Rev. Mod. Phys. 38, 36 (1966).

[3] Man’ko, V. I., Marmo, G., and Zaccaria, F., “q-Nonlinearity, deformations, and Planck distribution,” in: Gruber, B. (ed.) “Symmetry in Science VII. Proceedings of the International Symposium” (Bregenz, Austria, August 1994), (Plenum Press, New York, 1995), p. 341; Man’ko, V. I., Marmo, G., Sudarshan, E. C. G., and Zaccaria, F., J. Mod. Phys. B 11, 1281 (1997).

[4] Lopez-Peña, R., Man’ko, V. I., and Marmo, G., Phys. Rev. A 56, 1126 (1997).

[5] Moyal, J. E., Proc. Cambridge Phylos. Soc. 45, 99 (1949).

[6] Wigner, E., Phys. Rev. 40, 749 (1932).

[7] Husimi, K, Proc. Phys. Math. Soc. Jpn, 23, 246 (1940).

[8] Mancini, S., Man’ko, V. I., and Tombesi, P., Phys. Lett. A 213, 1 (1996).

[9] Koopman, B., Proc. Nat. Acad. Sci. 17, 315 (1931).

[10] Landau, L. D., and Lifshitz, E. M., “Quantum Mechanics” (Pergamon, Oxford, 1979).

[11] Marmo, G., “An assessment of Lagrangian and Hamiltonian formalisms,” PHD Thesis, Università di Napoli (1976).

[12] Schrödinger, E., Ann. Physik, 79, 489 (1926).

[13] Man’ko, V. I., Marmo, G., and Zaccaria, F., Rend. Sem. Math. Univ. Pol. Torino 54, 338 (1996).

[14] Mancini, S., Man’ko, V. I., and Tombesi, P., Found. Phys. 27, 801 (1997); Man’ko, V. I., Rosa, L., and Vitale, P., Phys. Rev. A 57, 329 (1998).

[15] Vogel, K., and Risken, H., Phys. Rev. A 40, 2847 (1989).
[16] Mancini, S., Man’ko, V. I., and Tombesi, P., Quantum Semiclass. Opt. 7, 615 (1995); D’Ariano, G., Mancini, S., Man’ko, V. I., and Tombesi, P., Quantum Semiclass. Opt. 8, 1017 (1996).

[17] Smithey, D. T., Beck, M., Raymer, M. G., and Faridani, A., Phys. Rev. Lett. 70, 1244 (1993).

[18] Wünsche, A., J. Mod. Opt. 44, 2293 (1997).

[19] Man’ko, V. I., and Marmo, G., Phys. Scr. 58, 224 (1998).

[20] Man’ko, V. I., and Man’ko, O. V., JETP 85, 430 (1997).

[21] Dodonov, V. V., and Man’ko, V. I., Phys. Lett. A 239, 335 (1997).

[22] Man’ko, V. I., and Mendes, R. V., “Non-commutative time-frequency tomography of analytic signals,” LANL physics/9712022 Data Analysis, Statistics, and Probability; IEEE Signal Processing (submitted).