ON THE CHARACTERIZATION PROBLEM OF ALEXANDER POLYNOMIALS OF CLOSED 3-MANIFOLDS

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Abstract. We give a characterization for the Alexander polynomials \( \Delta_M \) of closed orientable 3-manifolds \( M \) with first Betti number \( b_1 M = 1 \) and some partial results for the characterization problem in the cases of \( b_1 M > 1 \).

We first prove an analogue of a theorem of Levine: that the product of an Alexander polynomial \( \Delta_M \) with a non-zero trace symmetric polynomial in \( b_1 M \) variables is again an Alexander polynomial of a closed orientable 3-manifold. Using the fact that there exist \( M \) with \( \Delta_M = 1 \) for \( b_1 M = 1, 2, 3 \), we conclude that the symmetric polynomials of non-zero trace in 1, 2 or 3 variables are Alexander polynomials of closed orientable 3-manifolds. When \( b_1 M = 1 \) we prove that non-zero trace symmetric polynomials are the only ones that can arise. Finally, for \( b_1 M \geq 4 \), we prove that \( \Delta_M \neq 1 \) implying that for such manifolds not all non-zero trace symmetric polynomials occur.

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1. Introduction

Let \( M \) be a manifold, \( b_1 M \) its first Betti number. For \( b_1 M \geq 1 \), the Alexander polynomial of \( M \), \( \Delta_M \), is a polynomial invariant in \( b_1 M \) variables first defined by Alexander [2] for \( M = S^3 - K \) a knot complement and more generally for \( M \) having finitely presented \( \pi_1 \) by Fox [3].
In the case of knot complements, a characterization of Alexander polynomials was given by Seifert [14], and later, in the case of link complements, necessary conditions on Alexander polynomials were described by Torres [18] (when $b_1 = 2$, these conditions are insufficient [7], [13]).

In this paper, we consider the characterization problem for Alexander polynomials of closed orientable 3-manifolds, giving a characterization in the case $b_1 M = 1$ and some partial results concerning characterization in the cases of $b_1 M > 1$.

Our first result is the following analog for closed orientable 3-manifolds of a Theorem of Levine (see §3 or [10]), which allows us to produce by multiplication new Alexander polynomials from a given one. By the trace of a polynomial we mean the sum of its coefficients.

**Theorem.** Let $M$ be a closed orientable 3-manifold with $b_1 M = n$ and let $\lambda$ be a symmetric Laurent polynomial in $n$ variables with non-zero trace. Then there exists a 3-manifold $M'$ with $b_1 M' = n$ for which

$$\Delta_{M'} = \lambda \cdot \Delta_M.$$  

This Theorem is proved in §3.

For low Betti numbers $b_1 = 1, 2$ or 3, the following closed orientable manifolds have $\Delta_M = 1$:

- $S^1 \times S^2$, $b_1 = 1$.
- $H_3(\mathbb{R})/H_3(\mathbb{Z})$ = Heisenberg manifold [12], $b_1 = 2$.
- $T^3$ = the 3-torus, $b_1 = 3$.

Combining these examples with the above analog of Levine’s Theorem we have

**Corollary.** Every symmetric Laurent polynomial in 1, 2 or 3 variables having non-zero trace is the Alexander polynomial of a 3-manifold with first Betti number 1, 2 or 3.

It is natural to ask if the statement in the Corollary gives necessary conditions for a characterization for Betti numbers $b_1 = 1, 2$ or 3. In the case $b_1 = 1$, it does. We say that a Laurent polynomial is unit symmetric if it is symmetric after multiplication with a unit of the ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

**Characterization Theorem.** Let $\lambda$ be a Laurent polynomial in 1 variable. Then $\lambda = \Delta_M$ for some closed 3-manifold $M$ with $b_1 M = 1$ if and only if $\lambda$ is unit symmetric and has non-zero trace.

Finally, we consider the case of manifolds with $b_1 \geq 4$. Our main result is the following:

**Theorem.** $\Delta_M \neq 1$ for any closed 3-manifold with $b_1 M \geq 4$.

A consequence of this is the following negative result: not every symmetric polynomial occurs as the Alexander polynomial for a 3-manifold having $b_1 \geq 4$.

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2. Alexander Polynomial

From this point on, all 3-manifolds will be assumed to be orientable.
We begin by reviewing a definition of the Alexander polynomial which employs absolute homology [11], [14]. This definition differs from those that appear in [3] [12] [19] which use either relative homology or Fox calculus. A proof of the equivalence of all three definitions can be found in [1] (the equivalence is in fact implicit in Theorem 2.7 of [11] as well as Theorem 16.5 of [21]).

Let $\psi : G \to H$ be an epimorphism of a finitely presented group $G$ onto a finitely generated free abelian group $H$ of rank $n$. Denote by $\mathbb{Z}[H]$ the group ring associated to $H$ i.e. the ring of formal finite linear combinations

$$n_1 h_1 + \cdots + n_k h_k, \quad n_i \in \mathbb{Z}, \; h_i \in H,$$

and the product in $\mathbb{Z}[H]$ is defined by linear extension of the product of $H$. Note that if we choose a basis $t_1, \ldots, t_n$ of $H$ then we may write elements of $H$ as

$$t^i = t_1^{i_1} \cdots t_n^{i_n}$$

for $i_1, \ldots, i_n \in \mathbb{Z}$, so that the elements of $\mathbb{Z}[H]$ may be viewed as a multivariable Laurent polynomials in the multivariable $t$.

Let $(X, x)$ be a pointed CW-complex whose 0-skeleton consists of the base point $x$, such that $G = \pi_1(X, x)$. Let

$$p_\psi : \tilde{X}_\psi \to X$$

be the normal covering space corresponding to $\psi$: that is, the covering indexed by $\text{Ker}(\psi) \subset G$ with deck group $G/\text{Ker}(\psi) \cong H$.

The **Alexander module** is defined to be

$$A_\psi = H_1(\tilde{X}_\psi) = H_1(\tilde{X}_\psi; \mathbb{Z}),$$

where its structure of $\mathbb{Z}[H]$-module comes from the action of $H$ on $\tilde{X}_\psi$ by deck transformations.

For any finitely presented module $A$ over $\mathbb{Z}[H]$ consider a free resolution

$$\mathbb{Z}[H]^m \xrightarrow{P} \mathbb{Z}[H]^n \to A \to 0.$$  

Such a resolution may be defined using a presentation $A = \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$: we take $\mathbb{Z}[H]^n = \langle x_1, \ldots, x_n | \rangle$, $\mathbb{Z}[H]^m = \langle R_1, \ldots, R_m \rangle$ with $P(R_i) = r_i$. Note that we may assume, without loss of generality, that $m \geq n$. One can represent $P$ by an $m \times n$ matrix also denoted $P$. For each $d = 0, \ldots, n$ we define the **$d$th elementary ideal** $I_d(A) \subset \mathbb{Z}[H]$ to be the ideal generated by the $(n - d) \times (n - d)$ minors of the matrix $P$. These ideals are independent of the resolution of $A$ and form a chain

$$I_0(A) \subset I_1(A) \subset \cdots \subset I_n(A) = \mathbb{Z}[H].$$

The **$d$th order ideal** is the smallest principal ideal containing $I_d(A)$, where any generator $\Delta_d(A)$ of it is called a **$d$th order** of $A$; it is well-defined up to multiplication by units. The $d$th order can also be defined as the greatest common divisor of the $(n - d) \times (n - d)$ minors of the matrix $P$ (which is well-defined since $\mathbb{Z}[H]$ is a unique factorization domain).

When $A = A_\psi$ we denote by

- $P_\psi$ any presentation matrix and call it an **Alexander matrix**.
- $I_0$ the 0th elementary ideal of a presentation matrix $P_\psi$ and call it the **Alexander ideal**.
- $\Delta_0$ a 0th order of $A_\psi$ and call it an **Alexander polynomial**.
Let $M$ be a compact manifold with $\pi_1 M$ finitely presented and with first Betti number $b_1 M = n$. Let $H(M) = H_1 M / \text{Tor}(H_1 M) \cong \mathbb{Z}^n$, where $\text{Tor}(H_1 M)$ is the torsion subgroup of $H_1 M$, and consider the epimorphism $\psi^{\text{fr-ab}} : \pi_1 M \rightarrow H_1 M \rightarrow H(M)$, where the first map is the abelianization and the second map is the projection. The normal covering associated to $K^{\text{fr-ab}} = \text{Ker}(\psi^{\text{fr-ab}})$

$$
p : \hat{M} \rightarrow M
$$

is called the \textbf{universal free abelian cover} of $M$.

**Proposition 1.** Let $H$ be a finitely generated free abelian group and let $\psi : \pi_1 M \rightarrow H$ be an epimorphism with associated covering $p_\psi : \hat{M}_\psi \rightarrow M$. Then there exists a covering $q_\psi : \hat{M} \rightarrow \hat{M}_\psi$ such that $p_\psi \circ q_\psi = p$.

**Proof.** Let $K_\psi = \text{Ker}(\psi)$ which is the image of $\pi_1 \hat{M}_\psi$ by $(p_\psi)_*$. Since $H_1 M$ is the abelianization of $\pi_1 M$, there is an epimorphism $r : H_1 M \rightarrow H \text{ with } \psi = r \circ \psi^{\text{fr-ab}}$. But $H$ is torsion free, therefore all torsion elements of $H_1 M$ are mapped to 0. This means that there exists an epimorphism $r' : H(M) \rightarrow H$ so that $\psi = r' \circ \psi^{\text{fr-ab}}$. We conclude that $K^{\text{fr-ab}} \subset K_\psi$, and the Proposition follows from this. \qed

**Corollary 1.** Any cover of $M$ with deck group $H(M)$ is isomorphic to the universal free abelian cover $p : \hat{M} \rightarrow M$.

When $\psi = \psi^{\text{fr-ab}}$ we denote $A_M$, $I_M$ and $\Delta_M$, and refer to them as the \textbf{Alexander module}, \textbf{ideal} and \textbf{polynomial} of $M$. By Corollary 1 these invariants depend only on $M$.

3. An Analog of a Theorem of Levine for Closed 3-manifolds

First we define three notions of symmetry that will be relevant for us. Let $H \cong \mathbb{Z}^n$, denote $\Lambda = \mathbb{Z}[H]$ the group ring and by $\Lambda^\times$ the group of units. The inversion map in $H$, $h \mapsto h^{-1}$, extends to an automorphism $\iota : \Lambda \rightarrow \Lambda$.

We say that a polynomial $f \in \Lambda$ is

- \textbf{symmetric} if $\iota(f) = f$.
- \textbf{unit symmetric} if there exists a unit $u \in \Lambda^\times$ such that $uf$ is symmetric i.e. $\iota(uf) = uf$.
- \textbf{mod unit symmetric} if $\iota(f) = uf$ for some unit $u \in \Lambda^\times$.

The definitions which we have just described are given in descending order of strength. For example, $t^2 + t + 1$ is unit symmetric but not symmetric. And $t - 1$ is mod unit symmetric but not unit symmetric.

Now we recall Levine’s method for generating link polynomials.

**Theorem 1** (Levine, [10]). Let $\Delta_L$ be the Alexander polynomial of an $n$-component link $L$, and let $\lambda = \sum c_i t^i \in \mathbb{Z}[H]$ satisfy the following conditions:

1. $\lambda$ is symmetric.
2. $\lambda(1, \cdots, 1) = 1$. 


Then there exists an oriented link $L'$ with the same number of components as $L$ such that
\[ \Delta_{L'} = \Delta_L \cdot \lambda. \]

Now we prove an analogue of Levine’s Theorem for closed 3-manifolds. By this we mean the following: let $M$ be a closed 3-manifold with $b_1 M = n$ and let $\lambda$ be a symmetric Laurent polynomial in $n$-variables with non-zero trace. Then we will prove that there exists a closed 3-manifold $M'$ with $b_1 M' = n$ and whose Alexander polynomial is $\lambda \cdot \Delta_M$.

**Proposition 2.** Let $M$ be a 3-manifold with $b_1 M = n$, $S$ a simple closed curve in $M$ which is homotopically trivial and $N$ a tubular neighborhood of $S$. Let $p : \hat{M} \to M$ be the universal free abelian cover and let $\hat{X} = \hat{M} - p^{-1}(N)$. Then we have the following isomorphism of $\Lambda$-modules
\[ H_1(\hat{X}) \cong H_1(\hat{M}) \oplus \Lambda. \]

**Proof.** Consider the pair $(\hat{M}, \hat{X})$ and denote $\hat{N} = p^{-1}(N)$. By excision on the interior of $\hat{X}$, we have that
\[ H_*(\hat{M}, \hat{X}) \cong H_*(\hat{N}, \partial \hat{N}). \]
Since the deck group $H$ acts by homeomorphisms on the pairs $(\hat{M}, \hat{X})$, $(\hat{N}, \partial \hat{N})$, the excision isomorphism is an isomorphism of $\Lambda$-modules. If we let $\hat{N}_0 \supset \partial \hat{N}_0$ be a pair of fixed components of $\hat{N} \supset \partial \hat{N}$ (i.e. a fixed lift of $N \supset \partial N$) then we may write
\[ \hat{N} = \bigcup_{h \in H} h(\hat{N}_0) \supset \partial \hat{N} = \bigcup_{h \in H} h(\partial \hat{N}_0). \]
On the level of homology groups we have then
\[ H_k(\hat{N}, \partial \hat{N}) \cong \bigoplus_{h \in H} H_k(\hat{N}, \partial \hat{N}), \]
a direct sum of copies of $H_k(\hat{N}, \partial \hat{N})$ indexed by $H$. By Lefschetz duality, since $\hat{N} - \partial \hat{N} = N$, we have that
\[ H_k(\hat{N}, \partial \hat{N}) \cong H^{3-k}(N) \cong H^{3-k}(\mathbb{S}^1) = \begin{cases} \mathbb{Z} & \text{if } k = 2, 3, \\ 0 & \text{otherwise.} \end{cases} \]
We note that a generator of $H_2(\hat{N}, \partial \hat{N})$ is given by a disk $D$ whose boundary is a meridian of $N$. Thus as a $\Lambda$-module, we have
\[ H_2(\hat{N}, \partial \hat{N}) \cong \Lambda, \]
with generator a disk in $\hat{N}_0$ whose boundary is a meridian. The generator of $H_2(\hat{M}, \hat{X}) \cong H_2(\hat{N}, \partial \hat{N})$ corresponding to the disk generator $D_0$ of $H_2(\hat{N}, \partial \hat{N})$ is denoted $\nu$. Note that the boundary of $\nu$ is equal to $\partial \hat{D}_0$.

The long exact sequence of the pair $(\hat{M}, \hat{X})$ (an exact sequence of $\Lambda$-modules) can now be written
\[ \cdots \to H_2(\hat{M}, \hat{X}) \cong \Lambda \xrightarrow{\alpha} H_1(\hat{X}) \xrightarrow{\beta} H_1(\hat{M}) \xrightarrow{\gamma} H_1(\hat{M}, \hat{X}) = 0 \to \cdots \]
Then $\gamma$ is the zero map and by exactness $\beta$ is surjective, so that:
\[ H_1(\hat{M}) \cong H_1(\hat{X})/\operatorname{Ker}(\beta). \]
Since, as indicated above, $H_2(M, X) \cong \Lambda$, the map $\alpha$ is of the form
$$\alpha : \Lambda \longrightarrow H_1(X).$$
Thus if we can show that $\alpha$ is injective and that the sequence (3) is split by a $\Lambda$-module map, this will imply that $H_1(X) \cong H_1(M) \oplus \Lambda$.

We begin by showing that $\alpha$ is injective i.e. we show that $\text{Im}(\alpha) \cong \Lambda$. Note first that $\text{Im}(\alpha)$ is generated as a $\Lambda$-module by $\mu = \beta(\nu)$, and $\mu$ is in fact the meridian boundary of $D_0$. Suppose that $\mu$ is a torsion element of $H_1(X)$, i.e $f \mu = 0$ for some $f = \sum c_h \nu \in \Lambda$. We will show that $f = 0$ i.e. that $c_h = 0$ for all $h \in H$. Now $f \mu$ bounds a compact surface in $X$ which when included in $M$ can be filled by meridian disks to form a closed orientable surface $\Sigma$ in $M$. Write $\tilde{S} = \cup \tilde{S}_h$ where $\tilde{S}_h = h \tilde{S}_0$ and $\tilde{S}_0$ is a fixed lift of $S$ to $\tilde{M}$. Since $S$ is homotopically trivial, each lift $\tilde{S}_h$ is also homotopically trivial.

Now we want to show that the short exact sequence is split. Since $\tilde{S}_h$ is homologically trivial for each $h$, it bounds a compact oriented surface $\Sigma_h$ in $\tilde{M}$. Let
$$\phi : H_1(X) \longrightarrow \text{Im}(\alpha) = \Lambda \mu$$
$$l \mapsto \sum_{h \in H} (l \cdot \Sigma_h) h \mu$$
where $l \cdot \Sigma_h$ is the intersection number with $\Sigma_h$. Then since $h \mu$ has intersection number one with $\Sigma_h$ and $h \mu$ has intersection number zero with any $\Sigma_{h'}$ for $h' \neq h$, we have $\phi \circ \gamma(h \mu) = h \mu$. Thus the exact sequence splits, proving the Proposition.

Theorem 2. Let $M$ be a closed 3-manifold with $b_1M = n$, $H \cong \mathbb{Z}^n$ the deck group of its universal free abelian cover and let $\lambda = \Sigma c_I t^I \in \mathbb{Z}[H]$ be a symmetric Laurent polynomial with trace not equal to 0. Then there exists a closed 3-manifold $M'$ with $b_1M' = b_1M$ and having Alexander polynomial
$$\Delta_{M'} = \Delta_M \cdot \lambda.$$  
Proof. Let $\lambda = \Sigma c_I t^I$ be a symmetric Laurent polynomial. We start by constructing a simple closed curve $S \subset M$ associated to the polynomial $\lambda$ with which we will modify $M$.

Let $B \subset M$ be a 3-ball and choose a simple closed curve $S_0 \subset B$ bounding an embedded disk $D$ in $M$. We will modify $S_0$ to obtain a new simple closed curve $S$ that will bound an immersed disk in $M$. For each term pair $\{ t^I, t^{-I} \}$, $I \neq 0$, of the polynomial $\lambda$ having non-zero coefficient $c_I$, pick

1. A point $q_I$ on $S_0$ such that $q_I \notin \{ q_I \}$, if $\{ t^I, t^{-I} \} \neq \{ t^I, t^{-I} \}$.

2. One of the two homology classes associated to $\pm I \in \mathbb{Z}^n \cong H_1(M) / \text{Tor}(H_1(M))$, which we denote $\gamma_{\pm I}$.

Then pick an embedded loop $u_I$ in $M$ based at $q_I$ such that
$$[u_I] = \gamma_{\pm I}.$$
We may assume, after an isotopy, that the loops $u_I$ are disjoint. Consider a segment $\tilde{u}_I$ obtained from $u_I$ by cutting off a small piece of one of its ends, then thicken this segment to a band $\tilde{u}_I \times [0, 1] \approx [0, 1] \times [0, 1]$. See Figure 1. We assume that we have done this in such a way that

- $(\tilde{u}_I \times [0, 1]) \cap D = (\tilde{u}_I \times [0, 1]) \cap S_0 = \{0\} \times [0, 1]$ and that the two end segments $\{0, 1\} \times [0, 1]$ lie in a small neighborhood of $q_I$.
- Each component $[0, 1] \times \{0, 1\}$ is a copy of $u_I$ in the sense that the union of it with a small arc in $S_0$ is isotopic to $u_I$. We may again assume that all such bands are disjoint.

Finally, modify the end of the band as in Figure 1 so that its boundary links according to the coefficient $c_I$. We call the new curve $S$. Note that $S$ bounds an immersed disc in $M$ and is therefore homotopically trivial.

Remove a tubular neighborhood $N$ of $S$ and let $X = M - N$. We construct $M'$ by performing an $m$-surgery on $M$ along $S$, where $m = \text{tr}(\lambda)$, and with respect to a preferred framing $f = (\ell, \mu) \subset \partial N$.

Here, $\ell$ is the preferred longitude – the one characterized by defining a homologically trivial element in $X$ – and $\mu$ is a meridian. We will show that $\Delta_{M'} = \Delta_M \cdot \lambda$.

It will be important for us to have an explicit understanding of the preferred longitude $\ell$. First, consider a tubular neighborhood of the curve as it appears before the linking step; call that curve $S'$. See the first image in Figure 1. Choose a preferred longitude for the original curve $S_0$, cut at the beginning of the band. Continue this longitude along a pair of homotopic segments lying along tubular

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{Modifying $S_0$ to obtain $S$.}
\end{figure}
neighborhoods of the band boundaries. The result will be a curve $\ell'$ which is homologically trivial in the complement of a tubular neighborhood of $S'$.

A tubular neighborhood for $S$ is obtained by cutting the tubular neighborhood of $S'$ just constructed and adding a cylinder that links about $S_0$. If we were to continue $\ell'$ without twisting about this cylindrical piece, we obtain a longitude called the obvious longitude, denoted $l$, which is not homologically trivial. In fact, $l$ bounds an immersed punctured disk, punctured twice for each linking that has been introduced. More specifically, $l$ is homologous in $X$ to

$$\left(\sum_{I \neq 0} c_I \right) \mu$$

where $\mu$ is a meridian of the tubular neighborhood of $S$. The preferred longitude of $S$, denoted $\ell$, is therefore obtained from $l$ by introducing a pair of twists in $l$ (opposite in orientation to the direction of the linkings) for each of the $c_I$ linkings coming from $u_I$. That is,

$$\ell = l - \left(\sum_{I \neq 0} c_I \right) \mu.$$

See Figure 2

![Figure 2. The preferred longitude.](image)

We will now calculate the Alexander module $H_1(\hat{M}')$ as a $\Lambda$-module. Let $\hat{M}$ be the $\mathbb{Z}^n$ cover of $M$, $p$ the covering map so that $\hat{S} = p^{-1}(S)$ and $\hat{N} = p^{-1}(N)$. Let $\hat{X} = \hat{M} - \hat{N}$. By Proposition 2 we have that

$$H_1(\hat{X}) \cong H_1(\hat{M}) \otimes \Lambda$$
as \( \Lambda \)-modules. Then the only new generator in homology which one obtains by removing \( \hat{N} \) corresponds algebraically to the factor \( \Lambda \). In particular, this factor as a \( \Lambda \)-module is cyclic, generated by an element \( \alpha \).

In order to obtain \( \hat{M}' \) from \( \hat{M} \), we will glue in solid tori to \( \hat{X} \), that is, to perform an \( m \)-surgery on each torus component. We will choose a longitude in each torus boundary \( \hat{N} \) coming from the lift of the preferred longitude \( \ell \) of \( N \). For a fixed boundary component of \( \hat{N} \), we denote this longitude by \( \hat{\ell} \) and denote by \( \mu \) a meridian. We let \( \hat{\ell} \) be a lift to \( \hat{N} \) of the obvious longitude \( \ell \) to the same boundary component containing \( \hat{\ell} \). Observe that the homology class of \( \mu \) also generates the factor \( \Lambda \) occurring in \( H_1(\hat{X}) \), so we may assume that \( \mu = \alpha \).

The longitude \( \hat{\ell} \) is not homologically trivial (as \( \ell \) was not) owing to the fact that other lifts of \( N \) link with \( \hat{N} \). Instead, it bounds an immersed punctured disk, in which we have a puncture for each such linking. See Figure 3.

Let \( \lambda_0 = \lambda - c_0 \) so that \( \text{tr}(\lambda_0) = \sum_{t \neq 0} c_t \). Note then that \( \hat{\ell} = \lambda_0 \cdot \mu \) i.e. is the sum of the punctures. Then it follows by our construction of \( \ell \) from \( l \) that

\[
\hat{\ell} = (\lambda_0 - \text{tr}(\lambda_0))\mu = \hat{\ell} - \text{tr}(\lambda_0)\mu.
\]

For the preferred framing \( f = (\ell, \mu) \subset \partial X \) an \( m \)-surgery is given by the formula \( \ell + m\mu \). Thus, in the manifold \( M' \), \( \ell + m\mu \) is trivial. This relation produces the relation in \( \hat{M}' \)

\[
\hat{\ell} + m\mu = (\lambda_0 - \text{tr}(\lambda_0) + m)\mu = 0.
\]
Thus since $m = \text{tr}(\lambda)$ we have $m - \text{tr}(\lambda_0) = c_0$ then this relation becomes

$$\lambda \mu = 0.$$ 

Since this new relation only involves the new generator $\alpha = \mu$, then the presentation of the Alexander module of $M'$ is of the form

$$\langle x_1, \ldots, x_\alpha, \mu| r_1, \ldots, r_b, \lambda \mu \rangle$$

where $\langle x_1, \ldots, x_\alpha| r_1, \ldots, r_b \rangle$ is the presentation for the Alexander module of $M$. It then follows immediately that the presentation matrix for the Alexander module of $M'$ is of the form

$$P' = \begin{pmatrix} P & 0 \\ 0 & \lambda \end{pmatrix}$$

where 0 are zero vectors. It follows that $\Delta_{M'} = \Delta_M \cdot \lambda$.

For low Betti numbers $b_1 = 1, 2$ or $3$, the following closed orientable manifolds have $\Delta_M = 1$ (see [14]):

- $S^1 \times S^2$, $b_1 = 1$.
- $H_3(\mathbb{R})/H_3(\mathbb{Z})$ = Heisenberg manifold, $b_1 = 2$.
- $\mathbb{T}^3$ = the 3-torus, $b_1 = 3$.

Combining these examples with Theorem 2 we have

**Corollary 2.** Every symmetric Laurent polynomial in 1, 2 or 3 variables having non-zero trace is the Alexander polynomial of a 3-manifold with first Betti number 1, 2 or 3.

### 4. Characterization for $b_1M = 1$

In this section we prove the

**Characterization Theorem.** Let $\lambda$ be a Laurent polynomial in 1 variable. Then $\lambda = \Delta_M$ for some closed 3-manifold with $b_1M = 1$ $\iff$ $\lambda$ is unit symmetric and has non-zero trace.

The sufficiency of the condition “$\lambda$ is unit symmetric and has non-zero trace” follows from Corollary 2. Therefore we will dedicate this section to proving necessity, which will be accomplished as follows:

- **§4.1** We recall Blanchfield’s symmetry result, which says that the Alexander polynomial of any closed 3-manifold is mod unit symmetric.
- **§4.2** We show the existence of a “Seifert surface” $\Sigma$ for $M$ and use it to construct the universal infinite cyclic cover $\hat{M}$.
- **§4.3** Using $\Sigma$ we define a “Seifert matrix” for $M$, and use it to show that $\Delta_M$ is unit symmetric.
- **§4.4** We prove in Lemma 3 that the trace of $\Delta_M$ is not 0.
4.1. Blanchfield’s Mod Unit Symmetry Theorem. Let $M$ be a compact and orientable 3-manifold with or without boundary. In [3], a general symmetry result about Alexander ideals was proved. As before, we denote by $\tilde{M}$ the universal free abelian cover and denote by $\Lambda = \mathbb{Z}[H]$ the group ring generated by the deck group $H$ of $\tilde{M} \rightarrow M$. Recall the automorphism $\iota : \Lambda \rightarrow \Lambda$ defined in [2]. We will say that an ideal $a \subset \Lambda$ is symmetric if $\iota(a) = a$.

The following theorem is a special case of Corollary 5.6 of [3].

**Theorem 3.** The principal ideal $(\Delta_M)$ is symmetric.

We have as an immediate corollary:

**Corollary 3.** Any Alexander polynomial $\Delta_M$ of $M$ is mod unit symmetric.

**Proof.** By symmetry $\iota(\Delta_M)) = (\Delta_M)$ so there exists $u \in \Lambda^{\times}$ such that $\iota(\Delta) = u\Delta$ i.e. $\Delta$ is mod unit symmetric. □

4.2. An Analogue of the Seifert Surface Construction. Let $M$ be closed and orientable with $b_1 M = 1$. Recall the map $\psi : \pi_1 M \rightarrow \mathbb{Z}$ defined as the composition

$$\pi_1 M \xrightarrow{\text{abelianization}} H_1(M, \mathbb{Z}) \xrightarrow{\text{projection}} \text{Free}(H_1(M, \mathbb{Z})) \cong \mathbb{Z},$$

where for $A$ an abelian group, $\text{Free}(A) = A/\text{Tor}(A)$. We will usually identify $\text{Free}(A)$ as a subgroup of $A$ by choosing a section of the projection $A \rightarrow \text{Free}(A)$. For an embedded closed oriented surface $\Sigma \subset M$ and a simple closed oriented curve $\gamma \subset M$ we denote by $\gamma \cdot \Sigma \in \mathbb{Z}$ the signed intersection number.

**Theorem 4.** There exists an oriented, embedded and non separating closed surface $\Sigma \subset M$ such that for all $\gamma \in \pi_1 M$,

$$\psi(\gamma) = \gamma \cdot \Sigma.$$

**Proof.** Note that $\psi$ is by definition trivial on $[\pi_1 M, \pi_1 M]$ so that it induces an element of $\text{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$: the projection appearing in [4.2], which is a non trivial homomorphism. The intersection pairing version of Poincaré duality provides in particular a non-degenerate pairing

$$\text{Free}(H_1(M, \mathbb{Z})) \times \text{Free}(H_2(M, \mathbb{Z})) \rightarrow \mathbb{Z}$$

given by the signed intersection number.

See [3]. N.B. $H_2(M, \mathbb{Z}) \cong H_1(M, \mathbb{Z})$ which is free by the universal coefficient theorem. This means that every homomorphism

$$\text{Free}(H_1(M, \mathbb{Z})) \rightarrow \mathbb{Z}$$

is given by the intersection pairing with some element of $H_2(M, \mathbb{Z})$. Thus we may associate to $\psi$ an element $[\Sigma] \in H_2(M, \mathbb{Z})$. We note that there exists a representative $\Sigma \in [\Sigma]$ which is a closed connected embedded surface since $[\Sigma]$ is of co-dimension 1. This surface is orientable because it represents a non-trivial element of $H_2(M, \mathbb{Z})$; any closed non-orientable surface has trivial $H_2$ so could not represent a non-trivial element of $H_2(M, \mathbb{Z})$. Thus we have $\psi(\gamma) = \gamma \cdot \Sigma$ for all $\gamma \in \pi_1 M$. In particular for any curve $\gamma$ with $\psi(\gamma) = 1$ we have $\gamma \cdot \Sigma = 1$. This, along with the fact that $\Sigma$ is orientable, implies that $\Sigma$ is non-separating. For if $M - \Sigma = Y_1 \sqcup Y_2$ is a disjoint union, then since $\Sigma$ is 2-sided, then after moving $\gamma$ by an isotopy, $\gamma - \Sigma = \gamma - \{\text{point}\} \approx (0, 1)$ would connect points of $Y_1$ to points of $Y_2$, which is impossible. □
Since $\Sigma$ is orientable and non-separating, it has a collar which we denote $C(\Sigma)$. Let
\[ X = M - C(\Sigma) = M - \text{int}(C(\Sigma)). \]
Note that $X$ has two boundary components $\Sigma^-$ and $\Sigma^+$.

**Proposition 3.** If $\Sigma$ is of genus $g$ then
\[ H_1(X) \cong H_1(\Sigma) \oplus \text{Tor}(H_1X) \cong \mathbb{Z}^{2g} \oplus \text{Tor}(H_1X). \]
In other words, $\text{Free}(H_1X) \cong \mathbb{Z}^{2g}$.

**Proof.** Observe that the statement of the Proposition is equivalent to showing that
\[ \dim H_1(X; \mathbb{Q}) = 2g. \]
For the remainder of the proof, all homology and cohomology will be with $\mathbb{Q}$ coefficients. Next, we have
\[ H_1(X) \cong H^2(M, \text{int}(C(\Sigma))) \cong H^2(M, \Sigma) \]
where the first isomorphism is by Lefschetz duality (see [16], page 297) and the second isomorphism follows because $\Sigma$ is a deformation retract of $\text{int}(C(\Sigma))$. Since we are working with $\mathbb{Q}$-coefficients, the universal coefficient theorem implies that
\[ H^2(M, \Sigma) \cong H_2(M, \Sigma). \]
(Since $H_1M \cong \mathbb{Q}$ is free, $\text{Ext}(H_1M, \mathbb{Q}) = 0$ which implies $H^2(M, \Sigma) \cong \text{Hom}(H_2(M, \Sigma), \mathbb{Q})$; but $H_2(M, \Sigma)$ is a $\mathbb{Q}$-vector space, so $\text{Hom}(H_2(M, \Sigma), \mathbb{Q}) \cong H_2(M, \Sigma).$) So it will be enough to show that $H_2(M, \Sigma)$ has dimension $2g$.

Let us consider the long exact sequence in homology of the pair $(M, \Sigma)$:
\[ \cdots \rightarrow H_2\Sigma \overset{i_2}{\rightarrow} H_2M \overset{j_2}{\rightarrow} H_2(M, \Sigma) \overset{\partial}{\rightarrow} H_1\Sigma \overset{i_1}{\rightarrow} H_1M \overset{j_1}{\rightarrow} H_1(M, \Sigma) \rightarrow \cdots \]
Notice that
\[ H_2M \cong H^1M \cong H_1M \cong \mathbb{Q}, \]
where the first isomorphism is by Poincaré duality, the second by the universal coefficient theorem and the last one because $b_1M = 1$. Note that $j_1$ is injective, since $\Sigma$ intersects once a generator of $\text{Free}(H_1(M, \mathbb{Z}))$ and therefore when this generator is mapped to $H_1(M, \Sigma)$, it persists.

Thus $\text{Ker}(j_1) = 0 = \text{Im}(i_1)$ by exactness, so that $i_1$ is the zero map. Thus $\text{Ker}(i_1) = H_1\Sigma = \text{Im}(\partial)$ again by exactness. Thus $\partial$ is onto and $H_2(M, \Sigma)$ has dimension $\geq 2g$. On the other hand, $[\Sigma]$ generates an infinite cyclic subgroup of $H_2M$ since it corresponds by duality to $\psi$ which is a free cohomology class. So $i_2$ is injective and $\mathbb{Q} \cong \text{Im}(i_2) = \text{Ker}(j_2)$. Since $\text{Ker}(j_2) \neq 0$, it follows since $H_2M \cong \mathbb{Q}$ that we must have that $\text{Ker}(j_2) = H_2M$. Hence $j_2$ is the 0 map, which implies that $\text{Ker}(\partial) = 0$ i.e. $\partial$ is injective and therefore an isomorphism. \hfill \Box

The argument above can be modified to show that $\partial$ is an isomorphism modulo torsion in $\mathbb{Z}$-coefficients. More precisely,

**Lemma 1.** The homomorphism
\[ \partial : H_2(M, \Sigma; \mathbb{Z}) \rightarrow H_1(\Sigma; \mathbb{Z}) \]
satisfies
1. $\text{Ker}(\partial) \subset \text{Tor}(H_2(M, \Sigma; \mathbb{Z}))$.
2. $\text{Coker}(\partial)$ is a finite group.
Proof. Consider the long exact sequence of the pair appearing in the proof of Proposition 3, but now with \( \mathbb{Z} \)-coefficients. The map \( i_2 \) is still injective and its image is still free (by how we defined \( \Sigma \)), but here we can no longer assert that \( i_2 \) is onto. Nevertheless, this implies that \( \text{Ker}(j_2) = \text{Im}(i_2) \) is free. We claim that this implies that \( \text{Im}(j_2) \) is a finite group. For \( H_2(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \text{Tor}(H_2(M; \mathbb{Z})) \), since we saw in the proof of Proposition 3 that \( H_2(M; \mathbb{Q}) \cong H_2(M; \mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q} \). Therefore, \( \text{Ker}(j_2) \subset \text{Free}(H_2(M; \mathbb{Z})) \cong \mathbb{Z} \) and \( j_2 \) induces a map having domain the finite group \( H_2(M; \mathbb{Z})/\text{Ker}(j_2) \), so \( \text{Im}(j_2) \) is a finite group as claimed. By exactness, \( \text{Ker}(\partial) \) is finite, which proves (1). Now since \( H_2(M, \Sigma; \mathbb{Q}) \cong \mathbb{Q}^g \), \( H_2(M, \Sigma; \mathbb{Z}) \cong \mathbb{Z}^g \oplus \text{Tor}(H_2(M, \Sigma; \mathbb{Z})) \) which by (1) means that \( \text{Ker}(\partial) \subset \text{Tor}(H_2(M, \Sigma; \mathbb{Z})) \). Therefore \( \partial \) restricted to the free part of \( H_2(M, \Sigma; \mathbb{Z}) \cong \mathbb{Z}^g \) maps onto a subgroup of rank \( 2g \) of \( H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^{2g} \). This implies (2).

Example 1. For an example where \( \partial \) is not surjective, consider the case where \( M \) is the mapping torus

\[
\mathbb{T}_A = \mathbb{T}^2 \times [0, 1]/\sim_A, \quad (x, 0) \sim_A (Ax, 1)
\]

associated to the hyperbolic matrix

\[
A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.
\]

Since there is no simple closed curve \( c \subset \mathbb{T}^2 \) such that \( A^n(c) \) is isotopic to \( c \) for some \( n \), we have \( b_1 \mathbb{T}_A = 1 \). The "Seifert surface" \( \Sigma \) in this case is the image of \( \mathbb{T}^2 \times \{0\} \) in \( \mathbb{T}_A \). On the other hand, we know (see [12]) that

\[
\Delta_{\mathbb{T}_A}(t) = \det(A - It) = t^2 - 4t + 1.
\]

By Lemma 3 of §4.4 below, the order of \( \text{Tor}(H_1\mathbb{T}_A) \) is \( |\Delta_{\mathbb{T}_A}(1)| = 2 \). The map \( i_1 : H_1\mathbb{T}^2 \to H_1\mathbb{T}_A \), which is induced by the identification \( \mathbb{T}^2 \approx \Sigma \subset \mathbb{T}_A \) has image \( \text{Tor}(H_1\mathbb{T}_A) \cong \mathbb{Z}/2\mathbb{Z} \): indeed, due to the defining identifications if we denote by \( x = (1,0), y = (0,1) \) the basis of \( \mathbb{Z}^2 = H_1\mathbb{T}^2 \), then \( i_1(x) = x = A(x) = 3x + 2y \) and \( i_1(y) = y = A(y) = x + y \) which implies that \( 2(x + y) = 0 \) and \( x = 0 \) and therefore \( 2y = 0 \). In particular, \( \text{Ker}(i_1) = \text{Im}(\partial) \) is a proper subgroup of \( H_1\mathbb{T}^2 \). Therefore, \( \partial \) is not onto.

We now use the surface \( \Sigma \) to construct the universal infinite cyclic cover of \( M \). Take a countable collection \( \{X_i\}, i \in \mathbb{Z} \), of copies of \( X \), and glue them such that \( \Sigma^+_i \) is identified with \( \Sigma^-_i \) by the "re-gluing" homeomorphisms. Denote the result \( \widehat{M} \).

Theorem 5. \( \widehat{M} \) is a universal infinite cyclic cover of \( M \).

Proof. We show that there exists an infinite cyclic covering \( p : \widehat{M} \to M \). Each point in \( X_i \) is mapped to its counterpart in \( X \subset M \). Now we map \( X \) onto \( \overline{X} = X/\sim \) where \( x \sim y \) if \( x, y \in \partial X \) correspond. The gluing used to define \( \widehat{M} \) is compatible with the relation \( \sim \) so we get a covering map \( \widehat{M} \to \overline{X} \approx M \) which is infinite cyclic. \( \square \)
4.3. An Analogue of the Seifert Matrix. A key fact which we will need in order to prove that $\Delta_M$ is unit symmetric is the existence of a basis of the homology of $X$ dual to a given basis of the homology of $\Sigma$. However, in view of Lemma 1 we will not be able to do this for integral homology, as one does for knot complements in $\mathbb{R}^3$. In order to address this complication, we will work instead with homology with $\mathbb{Q}$-coefficients.

We begin with a notion of linking number valid in $M$. Consider disjoint oriented simple closed curves $l_1, l_2 \subset M$ whose integral homology classes belong to $\text{Tor}(\H_1(M; \mathbb{Z}))$. Then there exists integers $n_1, n_2$ such that $[n_1 l_1] = 0 = [n_2 l_2]$, and therefore there exist compact and oriented surfaces $S_1, S_2 \subset M$ with $\partial S_1 = n_1 l_1$, $\partial S_2 = n_2 l_2$.

**Definition 1.** The linking number of $l_1$ with $l_2$ is

$$\text{lk}(l_1, l_2) = \frac{(n_2 l_2) \cdot S_1}{n_1 n_2} = \frac{l_2 \cdot S_1}{n_1} \in \mathbb{Q}.$$ 

Since we have divided by $n_1 n_2$, the linking number does not depend on the $n_i$ chosen so that $[n_i l_i] = 0$.

**Note 1.** We have $\text{lk}(l_1, l_2) = -\text{lk}(l_2, l_1)$, just as in the case of the classical linking number.

The linking number defined here is for pairs of simple closed curves $l_1$ and $l_2$, and only depends on the isotopy type of $l_1 \cup l_2$. We can in fact extend it bi-linearly to rational multiples $ql_1, q_2 l_2$ by the formula

$$\text{lk}(ql_1, q_2 l_2) := q_1 q_2 \cdot \text{lk}(l_1, l_2).$$

We now choose special generating sets for the homology of the boundary components of $X$.

Fix bases $\{a_i\}, \{a_i^\pm\}$ of $\H_1(\Sigma; \mathbb{Z})$, $\H_1(\Sigma^\pm; \mathbb{Z})$; where $\{a_i^\pm\}$ is a push-off in the $\pm$ normal direction of $\{a_i\}$. When viewed in $M$, they give elements of $\text{Tor}(\H_1 M)$. We can choose $n \in \mathbb{Z}$ so that $\{na_i\}, \{na_i^\pm\}$ are homologous to 0 in $\H_1(M; \mathbb{Z})$ and not just torsion (for example we could take $n = \prod n_i$).

Now let $\{\bar{a}_i\}, \{\bar{a}_i^\pm\}$ be $\{na_i\}, \{na_i^\pm\}$, these elements give bases of $\H_1(\Sigma; \mathbb{Q}), \H_1(\Sigma^\pm; \mathbb{Q})$.

We define a square matrix $V_q = (\bar{v}_{ij})$ by

$$\bar{v}_{ij} = \text{lk}(\bar{a}_i^+, \bar{a}_j^-).$$

Notice that this is well-defined since these curves are torsion as elements of $\H_1(M; \mathbb{Z})$, and therefore their linking numbers are defined.

Also note that

$$\bar{v}_{ij} = \text{lk}(\bar{a}_i, \bar{a}_j^-)$$

so that

$$V_q^T = (\bar{v}_{ij}^T), \quad \text{where} \quad \bar{v}_{ij}^T = \text{lk}(\bar{a}_j, \bar{a}_i^+).$$

We now specify a basis of $\H_1(X; \mathbb{Q})$ dual to the basis $\{\bar{a}_i\}$ of $\H_1(\Sigma; \mathbb{Q})$ with respect to the bilinear pairing $\text{lk}(\cdot, \cdot)$.

**Lemma 2.** There exists a basis $\bar{\beta}_1, \ldots, \bar{\beta}_g$ of $\H_1(X; \mathbb{Q})$ such that viewed in $M$

$$\text{lk}(\bar{a}_i, \bar{\beta}_j) = \delta_{ij}.$$
Proof. The map $\partial : H_2(M, \Sigma; \mathbb{Q}) \to H_1(\Sigma; \mathbb{Q})$ is an isomorphism. If $a_i$ is one of the generators of $H_1(\Sigma; \mathbb{Q})$ specified above, then $\partial^{-1}(a_i)$ is a multiple $n_i S_i$ of a surface $S_i$. We may assume $n_i$ are integers by choosing $n$ large enough, we write $\bar{S}_i = n_i S_i$.

Thus we obtain a generating set $\bar{S}_1, \ldots, \bar{S}_{2g}$ of $H_2(M, \Sigma; \mathbb{Q})$ in which $\bar{S}_i$ is the image in $H_2(M, \Sigma; \mathbb{Q})$ of the class $n_i S_i \in H_2(M, \Sigma; \mathbb{Z})$.

But

$$H^2(M, \Sigma; \mathbb{Q}) \cong H_2(M, \Sigma; \mathbb{Q}) \cong \mathbb{Q}^{2g}$$

where the first isomorphism is a consequence of the universal coefficient theorem (see Corollary 4, page 244 of [16]) and the second isomorphism comes from composition of isomorphisms

$$H_2(M, \Sigma; \mathbb{Q}) \xrightarrow{\partial} H_1(\Sigma; \mathbb{Q}) \cong \mathbb{Q}^{2g}.$$

Therefore, $H^2(M, \Sigma; \mathbb{Q})$ has a basis dual to $\{\bar{S}_i\}$: given by cohomology classes $f_1, \ldots, f_{2g}$ with $f_i(n_j S_j) = \delta_{ij}$. Here we are identifying cohomology classes with functionals of homology.

By Lefschetz duality we have $H_1(X; \mathbb{Q}) \cong H^2(M, \Sigma; \mathbb{Q})$. The duality isomorphism is given by the intersection pairing. Therefore, if $\bar{\beta} \in H_1(X; \mathbb{Q})$ and $\bar{S} \in H^2(M, \Sigma; \mathbb{Q})$ then the $\mathbb{Q}$-Lefschetz duality isomorphism is induced by

$$\bar{S} \mapsto \bar{\beta} \cdot \bar{S}.$$

In particular, if we let $\bar{\beta}_i$ be such that the above function coincides with $f_i$, then we have

$$\bar{\beta}_i \cdot \bar{S}_j = f_i(\bar{S}_j) = \delta_{ij}.$$

But $\partial(\bar{S}_j) = \bar{a}_j$. This implies that $\text{lk}(\bar{\beta}_i, \bar{a}_j) = \delta_{ij}$. \qed

Recall the construction of $\hat{M}$ given at the end of §4.2 of [14]. The Mayer-Vietoris Theorem applied to $\hat{M}$ shows us that as a $\Lambda$-module

$$H_1(\hat{M}; \mathbb{Z}) \cong (H_1(X; \mathbb{Z}) \otimes \Lambda)/\text{relations}$$

where the relations are given by the gluing identifications

$$a_i^- - t a_i^+ = 0, \quad i = 1, \ldots, 2g$$

plus the torsion relations (which do not involve $t$):

$$m_1 \mu_1 = 0, \ldots, m_k \mu_k = 0$$

where the $\mu_i$ are generators of the torsion of $H_1(X; \mathbb{Z})$. Hence the Alexander matrix has the form

$$\begin{pmatrix}
A(t) & 0 & \cdots & 0 \\
0 & m_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & m_k
\end{pmatrix}$$

where $A(t)$ is a $2g \times 2g$ matrix corresponding to the system (4).

Let $\Lambda_\mathbb{Q}$ be the group ring $\mathbb{Q}[t^{\pm 1}]$ with coefficients in $\mathbb{Q}$. Then $H_1(\hat{M}; \mathbb{Q})$ is a $\Lambda_\mathbb{Q}$-module, and its presentation is given by

$$H_1(\hat{M}; \mathbb{Q}) \cong (H_1(X; \mathbb{Z}) \otimes \Lambda\mathbb{Q})/\text{relations}$$

where the relations are

$$n a_i^- - t a_i^+, \quad i = 1, \ldots, 2g,$$
or in other words 
\[ \tilde{a}_i^+ - t\tilde{a}_i^-, \quad i = 1, \ldots, 2g. \]
Thus these are the only relations we have and the \( \mathbb{Q} \) Alexander matrix is 
\[ A_\mathbb{Q}(t) = nA(t). \]

Recall that if \( f(t) = \sum_{i=m}^{n} b_i t^i \) is a Laurent polynomial, where \( b_m, b_n \neq 0 \), then the degree is defined \( \deg(f) = n - m \). This notion of degree is invariant with respect to multiplication by units. If we let \( \Delta_\mathbb{Q}(t) \) be the “Alexander polynomial” associated to \( A_\mathbb{Q}(t) \), then 
\[ \Delta_\mathbb{Q}(t) := \det(A_\mathbb{Q}(t)) = \frac{\nu^{2g}}{m_1 \cdots m_k} \Delta(t). \]
We can see that \( \Delta(t) \) and \( \Delta_\mathbb{Q}(t) \) have the same degree. Our strategy will be to show that \( \Delta_\mathbb{Q}(t) \) has even degree.

**Theorem 6.** \( V_\mathbb{Q} - tV_\mathbb{Q}^T = A_\mathbb{Q}(t) \).

*Proof.* \( H_1(X; \mathbb{Q}) \) has generators the \( \bar{\beta}_i \) and we may write therefore 
\[ \tilde{a}_i^\pm = \sum c_{ij}^\pm \bar{\beta}_j. \]

If we take the linking number both sides of this equation with \( \tilde{a}_j \), we get by Lemma 2 that 
\[ c_{ij}^\pm = -\text{lk}(\tilde{a}_i^\pm, \tilde{a}_j). \]

It follows that the relations (4) may be re-written 
\[ \sum \text{lk}(\tilde{a}_i^-, \tilde{a}_j)\bar{\beta}_j - t \sum \text{lk}(\tilde{a}_i^+, \tilde{a}_j)\bar{\beta}_j = 0. \]

By our definition of \( V_\mathbb{Q} \) and our identification of \( V_\mathbb{Q}^T \) the relations (4) may be re-written 
\[ \sum (\bar{v}_{ij} - t\bar{v}_{ji})\bar{\beta}_j = 0, \quad i = 1, \ldots, 2g. \]

So 
\[ A_\mathbb{Q}(t) = V_\mathbb{Q} - tV_\mathbb{Q}^T, \]
as claimed. \( \Box \)

#### 4.4. Characterization.

We begin with the following corollary of Theorem 6.

**Corollary 4.** \( \Delta_\mathbb{Q}(t) \) is unit symmetric and in particular is of even degree.

*Proof.* By Theorem 6 we have 
\[ \Delta_\mathbb{Q}(t) = \det(V_\mathbb{Q} - tV_\mathbb{Q}^T) = \det(V_\mathbb{Q}^T - tV_\mathbb{Q}) \]
and therefore 
\[ \Delta_\mathbb{Q}(t^{-1}) = \det(V_\mathbb{Q} - t^{-1}V_\mathbb{Q}^T) = \det((-t)^{-1} \cdot (V_\mathbb{Q}^T - tV_\mathbb{Q})) = t^{-2g} \Delta_\mathbb{Q}(t). \]

Then \( f(t) = t^{-g} \Delta_\mathbb{Q}(t) \) is symmetric so \( \Delta_\mathbb{Q}(t) \) is unit symmetric. The last statement follows since odd degree polynomials cannot be unit symmetric (they can be at most mod unit symmetric). \( \Box \)

**Theorem 7.** The Alexander polynomial \( \Delta(t) \) of \( M \) is unit symmetric.

*Proof.* By Corollary 4, \( \Delta(t) \) is of even degree. By Blanchfield we know that any \( \Delta_M(t) \) is mod unit symmetric. Since the degree is even, this implies that it is unit symmetric. \( \Box \)
Lemma 3. For $b_1 M = 1$ the trace of the Alexander polynomial $\Delta_M$ is non-zero and its absolute value is equal to the order of $\text{Tor} H_1(M;\mathbb{Z})$.

Proof. Let $\hat{P}(t)$ be a presentation matrix for the $\Lambda$-module $H_1(\hat{M})$, then $P := \hat{P}(1)$ is a presentation matrix for $G = p_*(H_1\hat{M}) = \text{image of } H_1\hat{M} \text{ in } H_1 M$ by the covering map $p : \hat{M} \rightarrow M$. Since $\hat{M}$ is an infinite cyclic covering of $M$, we have the exact sequence

$$1 \rightarrow p_*(\pi_1(\hat{M})) \subset \pi_1(M) \rightarrow \mathbb{Z} \rightarrow 0.$$ 

By abelianizing $\pi_1 M$, we obtain the sequence

$$0 \rightarrow G \subset H_1(M) \rightarrow \mathbb{Z} \rightarrow 0$$

which is exact: here we are using the fact that

1. The image of the subgroup $p_*(\pi_1(\hat{M})) < \pi_1(M)$ by the abelianization map $\pi_1(M) \rightarrow H_1(M)$ is $G$.

2. The epimorphism $\psi : \pi_1(M) \rightarrow \mathbb{Z}$ satisfies $\ker(\psi) \supset [\pi_1(X), \pi_1(X)]$.

Thus $H_1(M) \approx G \oplus \mathbb{Z}$ as abelian groups so that $\text{Tor}(G) = \text{Tor}(H_1(M))$. But $G$ is a finitely generated abelian group and so is isomorphic to $\mathbb{Z}^r \oplus \mathbb{Z}/n_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k \mathbb{Z}$ for integers $r, n_1, \ldots, n_k$, and $P$ must be equivalent to the diagonal presentation matrix $\text{diag}(n_1, \ldots, n_k)$. Then

$$\Delta_M(1) = \det(P) = n_1 \cdots n_k = |\text{Tor}(H_1(M))|.$$

We can now conclude with the

Proof (Characterization Theorem). Immediate from Theorem 1.2 and Lemma 3.

5. Manifolds with $b_1 > 1$

In this chapter we consider Alexander polynomials of closed 3-manifolds with $b_1 > 1$.

5.1. Manifolds with $b_1 = 2, 3$. As mentioned in the Introduction, the following closed 3-manifolds $M$ have $\Delta_M = 1$:

- $H_3(\mathbb{R})/H_3(\mathbb{Z})$ = Heisenberg manifold [12], $b_1 = 2$.
- $\mathbb{T}^3$ = the 3-torus, $b_1 = 3$.

Applying the generalized Levine’s theorem to the above examples, we have the following corollary:

Corollary 5. Let $M$ be a closed 3-manifold with first Betti number 2 or 3. Then the set of symmetric Laurent polynomials in 2 or 3 variables with $\text{tr} (\lambda) \neq 0$, is contained in the set of Alexander polynomials $\Delta_M$ with first Betti number equal to 2 or 3.
5.2. **Manifolds with** $b_1 \geq 4$. In this section we will prove

**Theorem 8.** $\Delta_M \neq 1$ for any closed 3-manifold with $b_1M \geq 4$.

The proof of this theorem requires several facts which we summarize now. Let $p$ be a prime and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Recall that for a manifold $N$, the mod $p$ Betti numbers are defined

$$b_k(N; \mathbb{F}_p) := \text{rank}(H_k(N; \mathbb{F}_p)).$$

We will use the abbreviated notation

$$d_p(N) := b_1(N; \mathbb{F}_p)$$

for the first mod $p$ Betti number.

**Fact 1.** Let $\tilde{M}_p$ be the finite abelian cover of $M$ associated to the epimorphism

$$\psi_p : \pi_1 M \rightarrow H_1(M; \mathbb{Z}) \rightarrow H_1(M; \mathbb{F}_p).$$

Let $r = d_p(M)$. Using an inequality of Shalen and Wagreich [15], we will deduce in §5.3 that

$$d_p(\tilde{M}_p) \geq \binom{r}{2}.$$

**Fact 2.** Suppose that $\Delta_M = 1$ and let $M' \rightarrow M$ be a finite abelian cover with deck group $\mathbb{F}_{p_1} \oplus \cdots \oplus \mathbb{F}_{p_k}$, $p_1, \ldots, p_k$ primes. Then

a. The torsion subgroup of $H_1(M'; \mathbb{Z})$ is trivial.

b. $b_1M' = b_1M$. This is a consequence of an equality of E. Hironaka [8].

These statements will be proved in §5.4.

Assuming for the moment the above facts, we can now give the

**Proof of Theorem 8.** We consider a cover $\tilde{M}_p$ as in Fact 1 with $p$ prime to the order of the torsion subgroup of $H_1(M'; \mathbb{Z})$ is trivial. Taking $M' = \tilde{M}_p$, it follows from Fact 2. (parts a. and b.) that

$$d_p(\tilde{M}_p) = d_p(M) = r.$$

But the inequality in Fact 1. is satisfied only for $r \leq 3$, since $r < \binom{r}{2}$ for all $r \geq 4$. □

5.3. **$\mathbb{F}_p$-Homology and Finite Cyclic Covers.** In this section we will derive the inequality of Fact 1. above.

Fix a prime $p$ and let $G$ be a group. If $A < G$ is a subgroup then we define

$$G\#A = [G, A]A^p := \langle [g, a]b^p \mid g \in G, a, b \in A \rangle,$$

where $[g, a] = gag^{-1}a^{-1}$ and $\langle X \rangle$ means the group generated by $X$.

We note that if $A < G$ then $G\#A < A$ and $A/(G\#A)$ is an elementary abelian $p$-group i.e a direct sum of copies of $\mathbb{F}_p$. The **mod p lower central series** $\{G_i\}$ of $G$ is defined by

$$G_{i+1} = G\#G_i$$

where $G_0 = G$. By the above comments we have $G_{i+1} < G_i$ and $G_i/G_{i+1}$ is an elementary abelian $p$-group for all $i = 0, 1, 2, \ldots$. See [17].
Let $\Gamma = \pi_1 M$ where $M$ is an orientable closed 3-manifold, and let $\{\Gamma_i\}$ be its mod $p$ lower central series. Let $r = \text{rank}(\Gamma/\Gamma_1)$. The following result appears as Lemma 1.3 in [15].

Theorem 9 (Shalen and Wagreich). $\text{rank}(\Gamma_1/\Gamma_2) \geq \binom{r}{2}$.

Proof. See [15] or [9].

We now give a geometric interpretation of this theorem in terms of first Betti numbers. We start by noting that

Proposition 4. $\Gamma/\Gamma_1 \cong H_1(M;\mathbb{F}_p)$.

Proof. This will follow from showing that $\Gamma_1$ is the kernel of the projection $\psi_p : \Gamma \to H_1(M;\mathbb{F}_p)$. First, note that $\psi_p$ is the composition

$$\Gamma \to H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{F}_p),$$

and the image of $\Gamma_1 = \Gamma \# \Gamma$ by the first map in the composition is $pH_1(M;\mathbb{Z})$ which is the kernel of the second map. Thus $\Gamma_1 \subset \text{Ker}(\psi_p)$. On the other hand, any element $\gamma \in \text{Ker}(\psi_p)$ must belong to a coset of the form $b^p[\Gamma,\Gamma]'$, and thus $\gamma \in \Gamma_1$. □

Let $\tilde{M}_p \to M$ be the finite abelian cover associated to the projection $\psi_p : \Gamma \to H_1(M;\mathbb{F}_p)$. We have $\pi_1\tilde{M}_p \cong \Gamma_1$ and

$$\text{Deck}(\tilde{M}_p/M) \cong \Gamma/\Gamma_1 \cong H_1(M;\mathbb{F}_p) \cong (\mathbb{F}_p)^r.$$

Recall the following notation that was used in the introduction to this chapter

$$d_p(N) := \text{rank } (H_1(N;\mathbb{F}_p)).$$

When $N = M$ we have $d_p(M) = r$. We have the following Corollary to the Theorem of Shalen and Wagreich:

Corollary 6. $d_p(\tilde{M}_p) \geq \binom{r}{2}$.

Proof. We will show that $\text{rank}(\tilde{M}_p) \geq \text{rank}(\Gamma_1/\Gamma_2)$. Applying the analysis of the previous paragraphs to $\tilde{M}_p$ in place of $M$, we know that

$$d_p(\tilde{M}_p) = \text{rank}(H_1(\tilde{M}_p;\mathbb{F}_p)) = \text{rank}(\Gamma_1/(\Gamma_1 \# \Gamma_1)).$$

Now

$$\Gamma_1/(\Gamma_1 \# \Gamma_1) = \Gamma_1/[\Gamma_1,\Gamma_1](\Gamma_1)^p.$$

But $\Gamma_2 = \Gamma \# \Gamma_1 = [\Gamma_1,\Gamma](\Gamma_1)^p$ so that

$$\Gamma_1/\Gamma_2 = \Gamma_1/[\Gamma,\Gamma_1](\Gamma_1)^p.$$

Thus $\Gamma_1/\Gamma_2$ is a quotient of $\Gamma_1/(\Gamma_1 \# \Gamma_1)$ and the Corollary follows. □
5.4. Closed 3-manifolds with $\Delta = 1$. We first recall the following result which relates the order of torsion in the homology of finite abelian covers to values of the Alexander polynomial:

**Theorem 10.** Let $M' \to M$ be a finite abelian cover lying below the universal free abelian cover $\hat{M} \to M$, with deck group $\mathbb{F}_p \oplus \cdots \oplus \mathbb{F}_{p_n}$, where the $p_i$ are primes. Assume that $\Delta_M(t_1, \ldots, t_n)$ has no zero of the form $(\rho_{e_1}^{e_1}, \ldots, \rho_{e_n}^{e_n})$ where $\rho_i$ is a $p_i$th root of unity. Then

$$|\text{Tor}(H_1 M')| = \left| \prod \Delta_M(\rho_{e_1}^{e_1}, \ldots, \rho_{e_n}^{e_n}) \right|$$

where the product is over all $(e_1, \ldots, e_n)$ with $0 \leq e_i < p_i$.

Theorem 10 was first proved in the case of knot complements by Fox (see [5]) and stated in the above generality by Turaev ([20], page 136) but he only provides a proof for cyclic covers. A complete proof can be found in [1]. We have immediately part a. of Fact 2:

**Corollary 7.** Let $M$ be a closed 3-manifold with $\Delta = 1$ and let $M' \to M$ be a finite abelian cover. Then

$$\text{Tor}(H_1 M') = 1.$$ 

In particular, taking $M = M'$, $\text{Tor}(H_1 M) = 1$.

To prove part b. of Fact 2, we will need a formula of E. Hironaka [8], which we describe in our setting. Let $M' \to M$ be a finite cover and assume that $\Delta_M = 1$. Let us denote

- $D = \text{the deck group of } M' \to M$.
- $\Gamma = \pi_1 M$.
- $\alpha : \Gamma \to D$ the projection.

For any group $G$, the character group is denoted

$$\hat{G} = \text{Hom}_{\text{cont}}(G, \mathbb{C}^*)$$

where $\text{Hom}_{\text{cont}}$ means the group of continuous homomorphisms. Write $\hat{1}$ for the trivial character. We recall that $\hat{G}$ is a topological group. In our case, $G = \Gamma$ or $D$, which are discrete groups, so continuous homomorphisms are just homomorphisms. Since $\alpha : \Gamma \to D$ is an epimorphism, there is an induced inclusion

$$\hat{\alpha} : \hat{D} \to \hat{\Gamma}.$$ 

Let $\chi \in \hat{\Gamma}$ and $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. Note that $\chi$ induces a homomorphism of $\Gamma^{ab} \cong \mathbb{Z}^r$ (the last isomorphism is by Corollary [7]). We may then extend $\chi$ linearly to a ring homomorphism $\chi : \Lambda \to \mathbb{C}$.

We now describe the formula of E. Hironaka, following [8]. Before doing so, we remark that the definition of the Alexander polynomial used in [8] is the one formulated using the relative Alexander module

$$A^\text{rel}_M = H_1(\hat{M}, \hat{x}), \quad \hat{x} = p^{-1}(x), x \in M$$

(see [12]). If $P(t_1, \ldots, t_r)$ is a presentation matrix of $A^\text{rel}_M$, the Alexander polynomial is defined in this setting to be a generator of the smallest principal ideal containing the ideal generated by the $(r-1)$ minors of $P(t_1, \ldots, t_r)$. A proof of the equivalence of the relative homology definition with the absolute homology definition can be found in [1] (the equivalence is in fact implicit in Theorem 2.7 of [11] as well as Theorem 16.5 of [21]).
Now given $\chi \in \hat{\Gamma}$ let $P(\chi)$ denote the matrix with complex entries obtained by evaluating each entry of $P(t_1, \ldots, t_r)$ at $\chi$. For each $i$, define

$$V_i = \left\{ \chi \in \hat{\Gamma} \mid \text{rank}(P(\chi)) < r - i \right\}.$$ 

Then Hironaka's formula (see [8], page 16, Proposition 2.5.6.) says that

$$b_1 M' = b_1 M + \sum_{i=1}^{r-1} |\alpha(\hat{D} \setminus \hat{1}) \cap V_i|.$$ 

**Theorem 11.** Suppose that $\Delta_M = 1$ and let $M' \to M$ be any finite abelian cover of $M$ with deck group $D = \mathbb{F}_{p_1} \oplus \cdots \oplus \mathbb{F}_{p_k}$, $p_1, \ldots, p_k$ primes. Then $b_1 M' = b_1 M$.

**Proof.** We claim that $|\alpha(\hat{D} \setminus \hat{1}) \cap V_i| = 0$ for all $i = 1, \ldots, r - 1$. To do this, it is enough to show that

$$|\alpha(\hat{D} \setminus \hat{1}) \cap V_1| = 0,$$

since $V_1 \supset \cdots \supset V_{r-1}$. So we must show that for every character $\chi \in \alpha(\hat{D} \setminus \hat{1})$, $\text{rank } P(\chi) \geq r - 1$.

Let us suppose not, that there exists a $\chi$ with rank $P(\chi) < r - 1$. Then every $r - 1$ minor of $P(\chi)$ is 0. But $P(\chi)$ is obtained by evaluating each polynomial appearing in $P$ at

$$t_1 = \rho_1^{e_1}, \ldots, t_k = \rho_k^{e_k},$$

where $\rho_j = \exp(2\pi i / p_j)$ and the exponents $e_1, ..., e_k$ depend on $\chi$. This implies that

$$\Delta(\rho_1^{e_1}, \ldots, \rho_k^{e_k}) = 0,$$

which contradicts the fact that $\Delta = 1$. Indeed, the greatest common factor of the $(r-1) \times (r-1)$ minors of $P(\chi)$ is $1 = \Delta(\rho_1^{e_1}, \ldots, \rho_k^{e_k})$, so the minors cannot all be 0. This contradicts our hypothesis, and therefore $\text{rank } P(\chi) \geq r - 1$. \qed

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