ROCK, PAPER, SCISSORS, ETC -
THE THEORY OF REGULAR TOURNAMENTS

ETHAN AKIN

Abstract. Following the extension of the classic Rock-Paper-Scissors game of size 3 to Rock-Paper-Scissors-Lizard-Spock of size 5, we consider further extensions to larger odd numbers. Such extensions are modeled by directed graphs called tournaments. The games we will consider are regular tournaments where each strategy beats and is beaten by exactly half of the alternatives. While the theory of tournaments has been well studied, these games, i.e. regular tournaments, have special properties which we consider here. In the process we introduce a number of constructions for such games.

Contents

Introduction
1. Relations and Digraphs
2. Games
3. Group Games
4. Inverting Cycles
5. Interchange Graphs
6. The Double Construction and the Lexicographic Product
7. Bipartite Tournaments and Pointed Games
8. Interchange Graphs, Again
9. Homogeneous Games
10. Games of Size Seven
11. Isomorphism Examples
12. Games of Size Nine
References
Index

Date: June, 2018.

Key words and phrases. tournament, regular tournament, digraph, Eulerian graph, game, group game, homogeneous game, reducible game, Steiner game, interchange graph, tournament double, lexicographic product.

2010 Mathematical Subject Classification 05C20, 05C25, 05C38, 05C45, 05C76.
The classic Rock-Paper-Scissors game of size 3 was extended on the TV show *The Big Bang Theory* to Rock-Paper-Scissors-Lizard-Spock with size 5. The variation was originated by Sam Kass with Karen Bryla. Here we want to consider games of larger size. Thus, we consider a set $I$ of alternative plays. For any pair of distinct plays, $i, j \in I$, one consistently beats the other. We write $i \rightarrow j$ if $i$ beats $j$. Such an arrangement is called a round-robin tournament or just a tournament. The name comes from regarding $I$ as a set of players, instead of plays, and assuming that each pair $i, j$ engages in a single contest. We write $i \rightarrow j$ when $i$ is the winner of the contest between them.

With either interpretation the number of $j \in I$ which are beaten by $i$ is called the score of $i$, denoted $s_i$. The games we want to focus upon are those which are balanced in that every $i \in I$ has the same score. This requires that the number $|I|$ of elements of $I$ be odd and the score $s_i$ equal to $\frac{1}{2}(|I| - 1)$ for all $i$. Thus, if $|I| = 2n + 1$, each $i$ beats $n$ alternatives and is beaten by the $n$ others. Such a tournament is called regular. We will adopt the term game to refer to a regular tournament.

A tournament can be regarded as a directed graph on the set $I$ where between every pair of distinct elements there is exactly one directed edge between them. More generally, for a digraph, between every such a pair there is at most one directed edge.

The theory of digraphs is described in [9] and [17]. A lovely survey of the theory of tournaments is given in [16]. Many computational problems for tournaments are considered in [22]. We will be considering here the special properties and constructions for games, i.e. for regular tournaments. I would like to express my thanks to my colleague W. Patrick Hooper for his helpful insights and enjoyable conversations as these matters developed.

Here is an outline of the work.

**Section 1** We begin with the elementary definitions and results about relations and digraphs. For a finite set $I$ with cardinality $|I|$, a relation $\Pi$ on $I$ is a subset of $I \times I$ with $\Pi^{-1} = \{(j, i) : (i, j) \in I\}$ the reverse relation. A path $[i_1, \ldots, i_n] \in I$ is a sequence with $(i_1, i_2), \ldots, (i_{n-1}, i_n) \in \Pi$. A cycle $\langle i_1, \ldots, i_n \rangle$ is a sequence of distinct elements such that $[i_1, \ldots, i_n, i_1]$ is a -closed- path. For $J \subset I$, the restriction of the relation $\Pi$ to $J$ is the relation $\Pi|J = \Pi \cap (J \times J)$. 

A relation $\Pi$ is a digraph when $\Pi \cap \Pi^{-1} = \emptyset$. For a digraph we write $i \to j$ when $(i, j) \in \Pi$. We let $\Pi(i)$ denote the set of outputs of $i$, i.e. $\{j : (i, j) \in I\}$ so that $\Pi^{-1}(i)$ denote the set of inputs of $i$. A digraph $\Pi$ on $I$ is called Eulerian when for each $i \in I$ the number of inputs equals the number of outputs. When this number is the same for all $i \in I$ then $\Pi$ is called regular. Observe that $\emptyset \subset I \times I$ is an Eulerian digraph.

If $\Pi$ is an Eulerian subgraph of $\Gamma$, then $\Gamma$ is Eulerian if and only if $\Gamma \setminus \Pi$ is Eulerian. Hence, the disjoint union of Eulerian subgraphs is Eulerian. Since a cycle is an Eulerian subgraph, it follows, as was observed by Euler, that a digraph is Eulerian if and only if it can be written as a disjoint union of cycles. Notice that disjoint subgraphs can have vertices in common.

If $\Pi$ and $\Gamma$ are digraphs on $I$ and $J$, then a morphism $\rho : \Pi \to \Gamma$ is a map $\rho : I \to J$ such that for $i, j \in I$ with $\rho(i) \neq \rho(j)$, $i \to j$ in $\Pi$ if and only if $\rho(i) \to \rho(j)$ in $\Gamma$. If $\rho$ is a bijective morphism, then $\rho^{-1} : \Gamma \to \Pi$ is a morphism as well and we call $\rho$ an isomorphism. If, in addition, $\Gamma = \Pi$, then $\rho$ is an automorphism and we let $\text{Aut}(\Pi)$ denote the group of automorphisms of $\Pi$. We write $\bar{\rho}$ for the product $\rho \times \rho : I \times I \to J \times J$. If $\rho$ is a bijection, then it is an isomorphism when $\bar{\rho}(\Pi) = \Gamma$.

Since $i \to j$ implies $j \not\to i$ it follows, Proposition 1.8, that every automorphism of a digraph has odd order. Hence, the order $|\text{Aut}(\Pi)|$ is odd.

**Section 2:** We begin our study of games. For a tournament $\Pi$ on $I$ of size $2n + 1$, the following are equivalent:

- $\Pi$ is Eulerian.
- $\Pi$ is regular.
- For every $i \in I$ the score $s_i = n$.

We will call a regular tournament a *game*. Up to isomorphism, the games of size 3 and of size 5 are unique - Theorem 2.1.

If $J \subset I$ with $|J| = 2k + 1$ and $\Pi$ is a game on $I$ then the restriction $\Pi|J$ is a subgame if the tournament $\Pi|J$ is Eulerian. Equivalently, it is a subgame when $|\Pi(i) \cap J| = k$ for all $i \in J$.

Given a game $\Pi$ on $J$ with $|J| = 2n - 1$ and $K \subset J$ with $|K| = n$ we can choose two new vertices $u, v$ and build the extension of $\Pi$ via $K$ and $u \to v$ to obtain a game $\Gamma$ on $I = J \cup \{u, v\}$ with $u \to v$ and $K = \Gamma(v)$. On the other hand, if $\Gamma$ is a game on $I$ of size $2n + 1$ and $J \subset I$ of size $2n - 1$ is such that the restriction $\Gamma|J$ is a subgame, then with $\{u, v\} = I \setminus J$ and $u \to v$ the game $\Gamma$ is the extension of $\Gamma|J$ via
A game is reducible via \( u \to v \) if and only if there does not exist \( i \in I \) such that \( i \to u \) and \( i \to v \) or an \( i \in I \) such that \( u \to i \) and \( v \to i \). The game is completely reducible when there is an increasing sequence \( I_1 \subset I_2 \subset \cdots \subset I_n = I \) with \( |I_k| = 2k + 1 \) and \( \Gamma|I_k \) a subgame for \( k = 1, \ldots, n \).

It is obvious that if \( \Pi \) is a digraph on \( I \), then it is a subgraph of some tournament on \( I \). A subtler result is Theorem 2.7: If \( \Pi \) is Eulerian, then it is a subgraph of some game on \( I \).

**Section 3:** We introduce the algebraic examples. Let \( G \) be a finite group with odd order \( |G| = 2n + 1 \). Let \( e \) denote the identity element. A game subset \( A \) is a subset of \( G \setminus \{e\} \) of size \( n \) with \( A \) disjoint from \( A^{-1} = \{a^{-1} : a \in A\} \). Equivalently, \( A \subset G \) such that \( G \setminus \{e\} \) is the disjoint union of \( A \) and \( A^{-1} \). Since no element of \( G \) has order 2 the \( n \) pairs \( \{\{a, a^{-1}\} : a \in G \setminus \{e\}\} \) partition the set \( G \setminus \{e\} \). A game subset is obtained by choosing one element from each pair. Thus, there are \( 2^n \) game subsets.

Given a game subset \( A \), we define on \( G \) the game \( \Gamma[A] = \{(i, j) \in G \times G : i^{-1}j \in A\} \). If \( \ell_a \) is the left translation by \( a \in G \), i.e. \( \ell_a(i) = ai \), then it is clear the \( \ell_a \) is an automorphism of \( \Gamma[A] \). Thus, \( G \) acting on itself by left translation is a subgroup of \( Aut(\Gamma[A]) \). Conversely, if \( \Gamma \) is a game on \( G \) such that \( G \) acting by left translation is contained in \( Aut(\Gamma) \), then \( \Gamma = \Gamma[A] \) with \( A = \Gamma(e) \), Theorem 3.4. We call such a game a group game on \( G \). If \( A \subset G \) is a game subset, then \( A^{-1} \) is a game subset with \( \Gamma[A^{-1}] \) equal to the reverse game \( \Gamma[A]^{-1} \). If \( G \) is abelian, then the map \( i \to i^{-1} \) is an isomorphism from a group game onto its reverse game.

For \( \rho \) a permutation of \( G \) with \( \rho(e) = e \) and \( A \) a game subset of \( G \), let \( B = \rho(A) \). Then \( B \) is a game subset and \( \rho : \Gamma[A] \to \Gamma[B] \) is an isomorphism if and only if \( i^{-1}j \in A \) implies \( \rho(i)^{-1}\rho(j) \in B \). In particular, this holds if \( \rho \) is a group automorphism of \( G \) since in that case \( \rho(i)^{-1}\rho(j) = \rho(i^{-1}j) \).

We let \( G^* \) denote the automorphism group of the group \( G \). In Theorem 3.6 we consider the case when \( Aut(\Gamma[A]) = G \). That is, the translations are the only automorphisms of \( \Gamma[A] \). If \( B \) is a game subset with \( \Gamma[B] \) isomorphic to \( \Gamma[A] \), then there is a unique \( \xi \in G^* \) such that \( B = \xi(A) \) and \( \rho = \xi \) is the unique isomorphism \( \rho : \Gamma[A] \to \Gamma[B] \) such that \( \rho(e) = e \). In particular, the set \( \{\xi(A) : \xi \in G^*\} \) is the set of game subsets \( B \) such that \( \Gamma[B] \) is isomorphic to \( \Gamma[A] \). Thus, there are exactly \( |G^*| \) such graph subsets.
In the case when $G$ is cyclic it can be taken to be $\mathbb{Z}_{2n+1}$, the additive group of integers mod $2n + 1$. The automorphisms are multiplications by the units $\mathbb{Z}^*_{2n+1}$ of the ring $\mathbb{Z}_{2n+1}$ and so has order $\phi(2n+1)$ which counts the numbers between 1 and $2n + 1$ which are relatively prime to $2n + 1$. The set $[1, n] = \{1, 2, \ldots, n\}$ is a game subset with $\text{Aut}(\Gamma[[1, n]]) = \mathbb{Z}_{2n+1}$ - Theorem 3.7. For $n > 2$ the number of game subsets, $2^n$ is greater than $\phi(2n + 1)$ and so there exist game subsets $B$ for $\mathbb{Z}_{2n+1}$ with $\Gamma[B]$ not isomorphic to $\Gamma[[1, n]]$.

A Fermat prime $p$ is a prime of the form $2^k + 1$ of which only five are known. Only when $2n + 1$ is a square-free product of Fermat primes is $\phi(2n + 1)$ a power of 2. Otherwise, $\mathbb{Z}^*_{2n+1}$ contains a multiplicative subgroup of odd order.

For an odd order group $G$, if $H \subset G^*$ is a subgroup of odd order, then - Theorem 3.12 - there exists a game subset $A$ of $G$ such that $\rho(A) = A$ for all $\rho \in H$. In that case, $\Gamma[A]$ is a group game such that $\text{Aut}(\Gamma[A])$ contains, in addition to the translations $\ell_a$, the automorphisms $\rho \in H$.

Using this, we construct in Example 3.13 a group game $\Gamma$ on a group $G$ and a group game $\Pi$ on a cyclic group such that for each $i \in G$, the restrictions to $\Gamma(i)$ and $\Gamma^{-1}(i)$ are subgames isomorphic to $\Pi$. Let $p$ be a prime congruent to $-1$ mod 4 and $k$ be an odd number so that $p^k = 2m + 1$ with $m$ odd. Let $G$ be the additive group of the finite field $F$ of order $p^k$ so that $G$ is isomorphic to the product group $(\mathbb{Z}_p)^k$. Let $F'$ be the multiplicative group of nonzero elements of $F$ so that $F'$ is a cyclic group of order $2m$. Let $H = \{a^2 : a \in F'\}$ so that $H$ is a multiplicative cyclic group of order $m$. Regarded as a subset of $G$, $H$ is a game subset and we let $\Gamma = \Gamma[H]$. The required $\Pi$ is a group game on the group $H$.

We close the section by showing that the only group games which are reducible are those isomorphic to $\Gamma[[1, n]]$ on $\mathbb{Z}_{2n+1}$ for some $n$ - Theorem 3.14.

**Section 4.** Suppose that $\Pi$ and $\Gamma$ are tournaments on a set $I$ and that $\rho$ is a permutation of $I$. We define $\Delta(\rho, \Pi, \Gamma) = \{(i, j) \in \Pi : (\rho(j), \rho(i)) \in \Gamma\}$ so that $\Delta(\rho, \Pi, \Gamma)$ is the subgraph of $\Pi$ on which $\rho$ reverses direction. We write $\Delta(\Pi, \Gamma)$ for $\Delta(1_I, \Pi, \Gamma)$.

We say that $\rho$ preserves scores when for all $i \in I$, $|\Pi(i)| = |\Gamma(\rho(i))|$. The permutation $\rho$ preserves scores if and only if $\Delta(\rho, \Pi, \Gamma)$ is Eulerian - Proposition 4.1. If $\Pi$ and $\Gamma$ are games, then any permutation preserves scores. In particular, if $\Pi$ is a game, then $\Gamma$ is a game if and only if $\Delta(\rho, \Pi, \Gamma)$ is Eulerian.
If $\Delta$ is any subgraph of $\Pi$, we define $\Pi/\Delta$ to be $\Pi$ with $\Delta$ reversed, so that $\Pi/\Delta = (\Pi \setminus \Delta) \cup \Delta^{-1}$. If $\Delta = \Delta(\rho, \Pi, \Gamma)$, then $\rho$ is an isomorphism from $\Pi/\Delta$ to $\Gamma$. Thus, if $\Pi$ and $\Gamma$ are games on $I$, then $\Gamma$ can be obtained from $\Pi$ by reversing the Eulerian subgraph $\Delta(\Pi, \Gamma)$. Since an Eulerian graph is a disjoint union of cycles, it follows that $\Gamma$ can be obtained by successively reversing a sequence of cycles. Furthermore, reversing a cycle can be accomplished by reversing a sequence of 3-cycles. Thus, we can obtain the game $\Gamma$ from the game $\Pi$ by reversing a sequence of 3-cycles - Theorem 4.5.

A decomposition for an Eulerian digraph $\Delta$ is a collection of disjoint cycles which covers $\Delta$. It is a maximum decomposition when it is a decomposition of maximum cardinality. We call this maximum cardinality the span of $\Delta$, denoting it by $\sigma(\Delta)$. We call $\beta(\Delta) = |\Delta| - 2\sigma(\Pi)$ the balance invariant of $\Delta$.

Assume that $\Pi$ and $\Gamma$ are games on $I$. If $\Pi'$ is obtained from $\Pi$ by reversing a 3-cycle, then $|\beta(\Delta(\Pi', \Gamma)) - \beta(\Delta(\Pi, \Gamma))| = 1$. Furthermore, there exists a 3-cycle in $\Pi$ such that $|\beta(\Delta(\Pi', \Gamma)) = \beta(\Delta(\Pi, \Gamma))| - 1$. - Theorem 4.7. It follows that $\beta(\Delta(\Pi, \Gamma))$ is the minimum number of 3-cycles which must be reversed in order to obtain $\Gamma$ from $\Pi$.

If $\Pi$ is a game which admits a decomposition by 3-cycles, then such a decomposition is clearly a maximum decomposition. We call such a game a Steiner game. It is a classical result that a set $I$ with $|I| = 2n + 1$ carries some Steiner game if and only if $n$ is congruent to $-1$ mod 3.

Section 5. If $\Pi$ is a game on $I$ with $|I| = 2n + 1$, then $\Delta \mapsto \Pi/\Delta$ is a one-to-one correspondence between the Eulerian subgraphs of $\Pi$-including the empty subgraph- and the set of games on $I$. Thus, the number of Eulerian subgraphs is the same for all games of size $2n + 1$. The number of 3-cycles contained in $\Pi$ is also the same for all games of size $2n + 1$.

Define the interchange graph to be the -undirected- graph with vertices the games on $I$ with $\Pi$, and with $\Gamma$ connected by an edge when each is obtained from the other by reversing a 3-cycle. Thus, the interchange graph is a regular, connected graph.

The distance between two games, $\Pi$ and $\Gamma$, is the length of a path with shortest distance between them. Such a shortest length path is called a geodesic. The distance $d(\Pi, \Gamma)$ is $\beta(\Delta(\Pi, \Gamma))$. If $d(\Pi, \Gamma) = k$ then there are at least $k!$ distinct geodesics between $\Pi$ and $\Gamma$ - Theorem 5.2. There exist examples where there are more than $k!$ geodesics between such games.
In particular, \( d(\Pi, \Pi^{-1}) = \beta(\Pi) \). If \( \Pi \) is a game of size \( 2n - 1 \) and \( \Gamma \) is an extension of \( \Pi \), then \( \beta(\Gamma) \leq \beta(\Pi) + 2n - 1 \) - Lemma 5.6. By induction it follows that if \( \Gamma \) is a completely reducible game of size \( 2n + 1 \), then \( \beta(\Gamma) \leq n^2 \). If \( \Gamma \) is the group game \( \Gamma[[1, n]] \) on \( \mathbb{Z}_{2n+1} \), then \( \beta(\Gamma) = n^2 \) - Theorem 5.7. I conjecture that for any game \( \Gamma \) of size \( 2n + 1 \), \( \beta(\Gamma) \leq n^2 \), and, more generally, that the diameter of the interchange graph is \( n^2 \) for \( I \) with \( |I| = 2n + 1 \).

If \( \Pi \) is a Steiner game, then there is a decomposition by 3-cycles and so the span \( \sigma(\Pi) = n(2n + 1)/3 \). Thus,

\[
d(\Pi, \Pi^{-1}) = \beta(\Pi) = |\Pi| - 2\sigma(\Pi) = n(2n + 1)/3.
\]

In particular, it follows that for \( n > 1 \) then game \( \Gamma[[1, n]] \) is never Steiner.

**Section 6.** If \( \Pi \) is a tournament of size \( n \) on \( I \) we define the double \( 2\Pi \) to be the game of size \( 2n + 1 \) on \( \{0\} \cup I \times \{-1, +1\} \) with \( 2\Pi(0) = I \times \{-1\}, (2\Pi)^{-1}(0) = I \times \{+1\} \). Let \((i, \pm)\) denote \((i, \pm1)\). For \( 2\Pi \), \( i \rightarrow (i, +) \) for all \( i \in I \) and if \( i \rightarrow j \) in \( \Pi \) then

\[
i \rightarrow j, (i, +) \rightarrow (j, +), j \rightarrow (i, +), (j, +) \rightarrow i - .
\]

We let \( \Pi_{\pm} \) denote the restriction of \( 2\Pi \) to \( I \times \{\pm1\} \). Each is clearly isomorphic to \( \Pi \).

Any double is completely reducible and so the only group games which could be isomorphic to a double are the isomorphs of \( \Gamma[[1, n]] \) on \( \mathbb{Z}_{2n+1} \). The game \( \Gamma[[1, n]] \) is indeed isomorphic to the double on its restriction to \( [1, n] = \{1, \ldots, n\} \).

If every \( i \in I \) has both inputs and outputs in \( I \), then any automorphism of \( 2\Pi \) fixes 0 and leaves \( \Pi_- \) and \( \Pi_+ \) invariant. This induces an isomorphism between \( Aut(\Pi) \) and \( Aut(2\Pi) \) - Proposition 6.4. Using this one can construct examples of games which are rigid, i.e. which have trivial automorphism groups and which are not isomorphic to their reverse games.

If \( \Pi \) is itself a game, we can construct other examples by reversing subgames. For example, if \( \Pi \) is a Steiner game then \( 2\Pi/\Pi_+ \) is a Steiner game - Theorem 6.5.

Another construction is the lexicographic product of two digraphs. Let \( \Gamma \) be a digraph on a set \( I \) and \( \Pi \) be a digraph on a set \( J \). Define \( \Gamma \times \Pi \) on the set \( I \times J \) so that for \( p, q \in I \times J \)

\[
p \rightarrow q \iff \begin{cases} p_1 \rightarrow q_1 \text{ in } \Gamma, & \text{or} \\ p_1 = q_1 \text{ and } p_2 \rightarrow q_2 \text{ in } \Pi. \end{cases}
\]
The map $p \to p_1$ is a surjective morphism from $\Gamma \rtimes \Pi$ to $\Gamma$.

If $\Pi$ and $\Gamma$ are games, then $\Gamma \rtimes \Pi$ is a game. The automorphism group $\text{Aut}(\Gamma \rtimes \Pi)$ is the semi-direct product $\text{Aut}(\Gamma) \rtimes [\text{Aut}(\Pi)]^I$ where $\text{Aut}(\Gamma)$ acts on the right by composition on the set of maps from $I$ to $\text{Aut}(\Pi)$ regarded as the product group $[\text{Aut}(\Pi)]^I$ - Theorem 6.9.

With $\Gamma_1$ the game of size 3, we let $\Gamma_k = \Gamma_{k-1} \rtimes \Gamma_1$ so that $\Gamma_k$ is a game on a set of size $3^k$. From the above computation of the automorphism group it follows that $|\text{Aut}(\Gamma_k)| = (3)^{(3^k-1)/2}$. It is known that the order of the automorphism group of a tournament of size $p$ is at most $3^{(p-1)/2}$, which is $3^n$ when $p = 2n + 1$. So $\Gamma_k$ is a game with the automorphism group as large as possible.

Finally, if $\Gamma$ and $\Pi$ are Steiner games, then $\Gamma \rtimes \Pi$ is Steiner.

Section 7: We call a game $\Pi$ on a set $I$ a pointed game when a particular vertex, labeled 0 is singled out. We let $I_0 = \Pi^{-1}(0), I_- = \Pi(0)$ and let $I_\pm$ be the tournament which is the restriction of $\Pi$ to $I_\pm$. If $|I| = 2n+1$ and $\Gamma_+, \Gamma_-$ are arbitrary tournaments on sets of size $n$, then there exists a pointed game $\Pi$ with $\Pi_+$ isomorphic to $\Gamma_+$ and $\Pi_-$ isomorphic to $\Gamma_-$ - Theorem 7.4.

Section 8: Given $I$ with $|I| = 2n+1$, we fix $0 \in I$ and let $I_0 = I \setminus \{0\}$. The map $\Pi \mapsto J = \Pi(0)$ associates to every game a size $n$ subset of $I_0$. The games which map to $J$ are all the pointed games with $I_+ = I_0 \setminus J, I_- = J$. The set of such games forms a convex subset, in the suitable sense, of the interchange graph. From this we obtain the lower bound $\left(\frac{2^n}{n}\right) \cdot 2^{n(n-1)}$ for the number of games on a set $I$ with $|I| = 2n+1$. Dividing by $(2n+1)!$ we obtain a lower bound for the number of isomorphism classes of games of size $2n + 1$.

Section 9: Assume that $H$ is a subgroup of a group $G$ of odd order. A subset $A$ of $G$ is a game subset for the pair $(G, H)$ if $A$ is a game subset for $G$ such that $i \in A \cap G \setminus H$ implies that the double coset $HiH$ is contained in $A$. It then follows that $A_0 = A \cap H$ is a game subset for $H$. The double cosets partition $G$ and $i \notin H$ implies that $HiH$ is disjoint from $H(i^{-1})H$. If we choose one double coset from each such pair and choose a game subset $A_0$ for $H$, then the union is a game subset for the pair.

Let $A$ be a game subset for $(G, H)$. For the homogeneous space of left cosets, $G/H = \{iH : i \in G\}$ define $A/H = \{iH : iH \subset A\}$. The set $\Gamma[A/H] = \{(iH, jH) : i^{-1}jH \in A/H\}$ is a game on $G/H$. For each
If \( H \) is a normal subgroup of \( G \), so that \( \pi: G \to G/H \) is a group homomorphism onto the quotient group, then a subset \( A \) of \( G \) is a game subset for \( (G, H) \) if and only if there exist \( B \) a game subset for \( G/H \) and \( A_0 \) a game subset of \( H \) so that \( A = A_0 \cup \pi^{-1}(B) \). In that case, the games \( \Gamma[A/H] \) and \( \Gamma[B] \) are equal.

On the other hand, assume that \( G \) is a group of odd order acting on a game \( \Pi \) on \( I \). For \( a \in I \) the evaluation map \( \iota_a: G \to I \) is defined by \( \iota_a(g) = g \cdot a \). \( Iso_a = \{ g : g \cdot a = a \} = \iota_a^{-1}\{\{a\}\} \) is a subgroup of \( G \) called the isotropy subgroup of \( a \). Let \( Ga = \iota_a(G) \subset I \) denote the \( G \) orbit of \( a \) and let \( \Pi_a = \Pi \cap (Ga \times Ga) \) be the restriction of \( \Pi \) to \( Ga \). Of course, \( G \) acts transitively on \( I \) exactly when \( Ga = I \) in which case \( \Pi_a = \Pi \).

Let \( H = Iso_a = \iota_a^{-1}\{\{a\}\} \). Choose \( A_0 \) a game subset for \( H \) and let \( A = A_0 \cup \iota_a^{-1}(\Pi(a)) \). The set \( A \subset G \) is a game subset for \( (G, H) \). The restriction \( \Pi_a = \Pi|Ga \) of \( \Pi \) to \( Ga \) is a subgame of \( \Pi \). The map \( \iota_a \) is a morphism from \( \Gamma[A] \) to \( \Pi \) and it factors through the canonical projection \( \pi \) to define the bijection \( \theta_a : G/H \to Ga \) which is an isomorphism from \( \Gamma[A/H] \to \Pi_a \). - Theorem 9.6

Applied with \( G = Aut(\Gamma) \) we see that the restriction of \( \Gamma \) to an \( Aut(\Gamma) \) orbit is a subgame which is isomorphic to a homogeneous game. In particular, \( Aut(\Gamma) \) acts transitively on the vertices if and only if \( \Gamma \) is isomorphic to a homogeneous game.

The lexicographic product of two group games is isomorphic to a group game and the lexicographic product of two homogeneous games is isomorphic to a homogeneous game - Corollary 9.10

Section 10: Every game of size 7 is isomorphic to one of the following three examples - Theorem 10.1.

**Type I**- The group game \( \Gamma_I = \Gamma[[1, 2, 3]] \) on \( \mathbb{Z}_7 \) has \( Aut(\Gamma[[1, 2, 3]]) = \mathbb{Z}_7 \) acting via translation and is reducible via each pair \( i, i + 3 \). The collection \( \{m_a([1, 2, 3]) : a \in \mathbb{Z}_7^*\} \) are the \( 6 = \phi(7) \) game subsets of \( \mathbb{Z}_7 \) whose games are isomorphic to \( \Gamma[[1, 2, 3]] \).

The group game \( \Gamma_I \) is isomorphic to the double \( 2\Pi \) with \( \Pi \) the restriction to \([1, 2, 3] \).
**Type II** - The group \( \Gamma_{II} = \Gamma[[1, 2, 4]] \) is the game on the additive group of the field \( \mathbb{Z}_7 \) with \( H = \{1, 2, 4\} \) the non-trivial, odd order, multiplicative subgroup of \( \mathbb{Z}_7^* \). The two game subsets not of Type I are \([1, 2, 4]\) and \([6, 5, 3] = m_6([1, 2, 4]) = -[1, 2, 4]\).

The game \( \Gamma_{II} \) is not reducible. With \( \Pi \) a 3-cycle, \( \Gamma_{II} \) is isomorphic to the double \( 2\Pi \) with \( \Pi_+ \) reversed.

**Type III** - \( \Gamma_{III} \) is the double of \( 2\Pi \) with \( \Pi \) a 3-cycle. So the automorphism group is isomorphic to that of \( \Pi \) and so is cyclic of order 3.

Every automorphism fixes 0.

The game \( \Gamma_{III} \) is reducible but not via any pair which contains 0.

Since \( \Pi \) is isomorphic to its reversed game, it follows that \( \Gamma_{III} \) is isomorphic to its reversed game as well.

The games of Type II and III are Steiner games.

**Section 11** - We consider various isomorphism examples.

There exist non-isomorphic tournaments with isomorphic doubles.

Every game of size greater than 3 admits non-isomorphic extensions.

There exist reducible games which can be reduced in different ways to get non-isomorphic games.

**Section 12** - With \( 9 = 2 \cdot 4 + 1 \) there are \( 2^4 = 16 \) game subsets. We look at the group games.

Consider the cyclic group \( G = \mathbb{Z}_9 \).

The Type I games come from the \( 6 = \phi(9) \) subsets \( \{m_a(A) : a \in \mathbb{Z}_9^*\} \) with \( A = [1, 4] \) or, equivalently, \( A = \{1, 3, 5, 7\} \).

The Type II games are the 6 subsets \( \{m_a(A) : a \in \mathbb{Z}_9^*\} \) with \( A = \{1, 5, 6, 7\} \). In this case, as for Type I, the automorphism group consists only of the translations by elements of \( \mathbb{Z}_9 \). These group games are not reducible.

The Type III games account for the 4 remaining game subsets. With \( H = \{0, 3, 6\} \) the subgroup of \( G \), there are four game subsets for the pair \((G, H)\). Each game is isomorphic to \( \Gamma_3 \ltimes \Gamma_3 \) with automorphism group \( \mathbb{Z}_3 \ltimes (\mathbb{Z}_3)^{Z_3} \).

If, instead, the group is the product group \( G = \mathbb{Z}_3 \times \mathbb{Z}_3 \) then it is a 2 dimensional vector space over the field \( \mathbb{Z}_3 \). The four one-dimensional subspaces are four subgroups \( H \) of order 3. For each such \( H \) there are four game subsets for the pair \((G, H)\). This accounts for the 16 game subsets. Each game is isomorphic to \( \Gamma_3 \ltimes \Gamma_3 \) with automorphism group \( \mathbb{Z}_3 \ltimes (\mathbb{Z}_3)^{Z_3} \).
1. Relations and Digraphs

We restrict ourselves to finite sets. For a finite set $I$ we will let $|I|$ denote the cardinality of $I$. The symmetric group on $I$, that is, the group of permutations on $I$, is denoted $S(I)$.

Following [1] we call a subset of $I \times I$ a relation on a $I$ with $\Pi^{-1} = \{(i, j) : (j, i) \in \Pi\}$ the reverse relation. A pair $(i, j) \in \Pi$ is an edge in $\Pi$. For $i \in I$, $\Pi(i) = \{j : (i, j) \in \Pi\}$ is the set of outputs of $i$, so that $\Pi^{-1}(i) = \{j : (j, i) \in \Pi\}$ is the set of inputs of $i$. Thus, a function on $I$ is a relation $\Pi$ such that each $\Pi(i)$ is a singleton set, e.g. the identity map $I$ is the diagonal $\{(i, i) : i \in I\}$. We call $i$ a vertex of $\Pi$ when $\Pi(i) \cup \Pi^{-1}(i)$ is nonempty. Thus, $i \in I$ is a vertex of $\Pi$ when it has at least one input or output.

For $J \subset I$, the restriction of $\Pi$ to $J$ is the relation $\Pi|J = \Pi \cap (J \times J)$.

Given relations $\Pi, \Gamma$ on $I$ the composition $\Pi \circ \Gamma = \{(i, j) : \text{there exists } k \text{ such that } (i, k) \in \Gamma, (k, j) \in \Pi\}$. Composition is associative and we inductively define $\Pi^{n+1} = \Pi^n \circ \Pi = \Pi \circ \Pi^n$, for $n \geq 0$, with $\Pi^1 = \Pi$ and $\Pi^0 = I$ and let $\Pi^{-n} = (\Pi^{-1})^n$. We define $\emptyset \Pi = \bigcup_{n=1}^{\infty} = \Pi^n$. Observe that $\emptyset \Pi \Pi^{-1} = (\emptyset \Pi)^{-1}$ and so we may omit the parentheses.

A sequence $[i_0, \ldots, i_n] \in I$ with $n \geq 1$ is a $\Pi$ path from $i_0$ to $i_n$ (or simply a path when $\Pi$ is understood) when $(i_k, i_{k+1}) \in \Pi$ for $k = 0, \ldots, n - 1$. The length of the path is $n$. It is a closed path when $i_n = i_0$. A path is simple when the vertices $i_0, \ldots, i_n$ are distinct and edge-simple when the edges $(i_0, i_1), \ldots, (i_{n-1}, i_n)$ are distinct. Clearly, a simple path is edge-simple, but an edge-simple path may cross itself. An $n$ cycle, denoted $\langle i_1, \ldots, i_n \rangle$, is a closed path $[i_n, i_1, \ldots, i_n]$ such that the vertices $i_1, \ldots, i_n$ are distinct, i.e. $[i_1, \ldots, i_n]$ is a simple path and so the closed path $[i_n, i_1, \ldots, i_n]$ is edge-simple. A path spans $I$ when every $i \in I$ occurs on the path.

Depending on context we will regard a path or a cycle as a sequence of vertices or as a subgraph, i.e. use $[i_0, \ldots, i_n]$ for $\{(i_k, i_{k+1}) : k = 0, \ldots, n - 1\} \subset \Pi$, and similarly, $\langle i_1, \ldots, i_n \rangle = [i_n, i_1, \ldots, i_n] \subset \Pi$.

Notice that $(i, j) \in \Pi^n$ exactly when there is a path from $i$ to $j$ of length $n$.

A relation $\Pi$ on $I$ is reflexive when $1_I \subset \Pi$. It is is symmetric when $\Pi^{-1} = \Pi$. It is transitive when $\Pi \circ \Pi \subset \Pi$ or, equivalently, when $\Pi = \emptyset \Pi$. For any relation $\Pi$, $\emptyset \Pi$ is the smallest transitive relation which contains $\Pi$. Observe that,

$$\emptyset (1_I \cup \Pi) = 1_I \cup \emptyset \Pi.$$

If $\Pi$ is symmetric or transitive, then the reflexive relation $1_I \cup \Pi$ satisfies the corresponding property.
We call \( i \in I \) a recurrent vertex when \( (i, i) \in \mathcal{O}_\Pi \). The relation \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) is an equivalence relation on the set of recurrent vertices. Of course, every \( i \in I \) is recurrent for \( 1_I \cup \Pi \). Since \( \mathcal{O}(1_I \cup \Pi) \cap \mathcal{O}(1_I \cup \Pi)^{-1} = 1_I \cup (\mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1}) \), the recurrent point equivalence classes for \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) are equivalence classes for \( \mathcal{O}(1_I \cup \Pi) \cap \mathcal{O}(1_I \cup \Pi)^{-1} \) and the remaining \( \mathcal{O}(1_I \cup \Pi) \cap \mathcal{O}(1_I \cup \Pi)^{-1} \) equivalence classes are singleton sets.

A non-empty subset \( J \subset I \) is strongly connected in \( I \) when \( J \times J \subset \mathcal{O}_\Pi \) and so \( J \times J \subset \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \). That is, a subset is strongly connected if and only if it is contained in an \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) equivalence class. Thus, the \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) equivalence classes are the maximal strongly connected subsets. We call \( \Pi \) strong when the entire set of vertices of \( \Pi \) is a strongly connected set and so the set of vertices comprises a single \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) equivalence class.

**Lemma 1.1.** Assume that \( \Pi \) is a relation on \( I \) and that \( J \) is a nonempty subset of \( I \). If the restriction \( \Pi|J \) is strong, then \( J \) is strongly connected and so is contained in an \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) equivalence class. Conversely, if \( J \) is an \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) equivalence class, then the restriction \( \Pi|J \) is strong.

**Proof.** The first result is obvious.

Now assume that \( J \) is an \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) equivalence class. If two points \( i, j \in J \) then there is a closed path from \( i \) to \( i \) which passes through \( j \). All of the points on the closed path are \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) equivalent to \( i \) and \( j \) and so are contained in \( J \). Hence, the closed path for \( \Pi \) is a closed path for \( \Pi|J \).

\[ \square \]

**Remark:** Notice that if \( i, j \) are distinct elements of an \( \mathcal{O}_\Pi \cap \mathcal{O}_\Pi^{-1} \) equivalence class, the set \( J = \{i, j\} \) is strongly connected. On the other hand, the restriction \( \Pi|J \) is strong only if \( (i, j) \) and \( (j, i) \) are both elements of \( \Pi \).

A subset \( J \subset I \) is an invariant set for a relation \( \Pi \) on \( I \) when \( \Pi(J) \subset J \) and so \( \mathcal{O}_\Pi(J) \subset J \). A subset \( J \) is invariant for \( \Pi \) if and only if the complement \( I \setminus J \) is invariant for \( \Pi^{-1} \). For any subset \( J \), the set \( J \cup \mathcal{O}_\Pi(J) \) is the smallest invariant set which contains \( J \). It is clear that \( \Pi \) is strongly connected if and only if the set of vertices contains no proper invariant subset. In particular, if \( i, j \) are vertices of \( \Pi \) and there is no path from \( j \) to \( i \) then \( \{j\} \cup \mathcal{O}_\Pi(j) \) is an invariant set which contains \( j \) but not \( i \).

We will call a relation \( \Pi \) on \( I \) a digraph when \( \Pi \cap \Pi^{-1} = \emptyset \). In particular, we have \( \Pi \cap 1_I = \emptyset \). That is, we are interpreting \( \Pi \) as a
directed graph with every pair of distinct elements of $I$ connected by
at most one oriented edge and no element of $I$ is connected by an edge
to itself. We write $i \rightarrow j$ when $(i, j)$ is an edge of the digraph.

We will call a relation $\Pi$ on $I$ an undirected graph when $\Pi \cap 1_I = \emptyset$
and $\Pi = \Pi^{-1}$. That is, we are interpreting $\Pi$ as a graph with every
pair of distinct elements of $I$ connected by at most one unoriented edge
and no element of $I$ is connected by an edge to itself.

A digraph or undirected graph $\Pi$ on $I$ is called bivariante when $I$
is the union of disjoint sets $J, K$ and $\Pi \subset (J \times K) \cup (K \times J)$. That
is, elements of $J$ are connected by an edge only to elements of $K$ and
vice-versa.

A subset of a digraph $\Pi$ is a digraph and we will call it a subgraph of
$\Pi$. Observe that disjoint subgraphs may have vertices in common. We
will call two subgraphs separated when they have no vertices in common.
Of course, separated subgraphs are disjoint. We call $\Pi$ connected
if it cannot be written as the disjoint union of two separated proper
subgraphs.

A digraph $\Pi$ on $I$ is called a round robin tournament, or simply a
tournament, when it is complete on $I$, i.e. when every pair of distinct
elements of $I$ is connected by an edge. Thus, a digraph $\Pi$ is a tournament
when $\Pi \cup \Pi^{-1} \cup 1_I = I \times I$. Clearly, if $\Pi$ is a tournament on $I$ and
$J$ is a nonempty subset of $I$, then the restriction $\Pi|J$ is a tournament
on $J$.

For a tournament $\Pi$ we think of $i \rightarrow j$ to mean $i$ beats $j$. Hence, the
score for $i$ is the cardinality of the output set, $|\Pi(i)|$. The score vector
$(s_1, \ldots, s_p)$ for a tournament on a set $I$ with cardinality $p$, consists of
the scores of the elements, listed in non-increasing order. The score
vector for a tournament was introduced and characterized by Landau
[21] as a tool for his study of animal behavior. It is clear that the sum
of the scores is the number of edges $p(p - 1)/2$.

Clearly, any digraph on $I$ can be extended to occur as a subgraph of
some tournament on $I$.

The following is a sharpening by Moon, [22] Theorem 3, of a result
of Harary and Moser, see [16] Theorem 7.

**Proposition 1.2.** If $\Pi$ is a strong tournament on $I$ with $|I| = p > 1$
and $i \in I$, then for every $\ell$ with $3 \leq \ell \leq p$ there exists a $\ell$-cycle in $\Pi$
passing through $i$.

**Proof.** If $\Pi(i)$ or $\Pi^{-1}(i)$ is empty then the tournament is not strong.
If for no $j_1 \in \Pi(i)$ and $j_2 \in \Pi^{-1}(i)$ is it true that $j_1 \rightarrow j_2$ then the
tournament is not strong. Hence, there is a 3-cycle through $i$. 
Now suppose that $\langle i_1, \ldots, i_r \rangle$ is a cycle through $i$ with $r < p$. We show that we can enlarge the cycle to one of length $r + 1$.

**Case 1** (There exists $j$ not on the cycle but such that both $\Pi(j)$ and $\Pi^{-1}(j)$ meet the cycle): By relabeling we may assume that $i_1 \in \Pi^{-1}(j)$. Let $s$ be the largest integer such that $i_1, \ldots, i_s \in \Pi^{-1}(j)$. By hypothesis, $s < r$ and $i_{s+1} \in \Pi(j)$. Hence, $\langle i_1, \ldots, i_s, j, i_{s+1}, \ldots, i_r \rangle$ is a $r + 1$ cycle.

**Case 2** (For every $j$ not on the cycle either $\Pi(j)$ or $\Pi^{-1}(j)$ does not meet the cycle): Observe that if $\Pi(j)$ does not meet the cycle, then $i_1, \ldots, i_r \in \Pi^{-1}(j)$. Let $A = \{j : i_1, \ldots, i_r \in \Pi^{-1}(j)\}$ and $B = \{j : i_1, \ldots, i_r \in \Pi(j)\}$. By assumption, $r < p$ and $I \setminus \{i_1, \ldots, i_r\} = A \cup B$. Since $\Pi$ is strong, there must exist $u \in A$ and $v \in B$ with $u \rightarrow v$. We may assume, by relabeling, that $i = i_1$. Thus, $\langle i_1, u, v, i_3, \ldots, i_r \rangle$ (omitting $i_2$) is a $r + 1$ cycle which contains $i$.

\[\square\]

A *Hamiltonian cycle* in a digraph is a cycle which passes through every vertex. Proposition 1.2 implies that a tournament admits a Hamiltonian cycle if it is strong. The converse is obviously true.

The opposite extreme of a strong tournament is an *order*. A relation $\Pi$ on $I$ is an order (to be precise, a strict, total order) when $\Pi$ is a transitive tournament. For example, for $[1, p] = \{1, 2, \ldots, p\}$ we let $i \rightarrow j$ when $i < j$ to define the *standard order* on $[1, p]$.

**Proposition 1.3.** For $\Pi$ is a tournament on $I$, with $|I| = p$, the following conditions are equivalent.

(i) $\Pi$ is an order.
(ii) $\Pi$ contains no cycles.
(iii) $\Pi$ contains no 3-cycles.
(iv) No vertex of $\Pi$ is recurrent.
(v) Every equivalence class of $\mathcal{O}(1_I \cup \Pi) \cap \mathcal{O}(1_I \cup \Pi)^{-1}$ is a singleton, i.e. $1_I = \mathcal{O}(1_I \cup \Pi) \cap \mathcal{O}(1_I \cup \Pi)^{-1}$.
(vi) The score vector of $\Pi$ is $(0, 1, \ldots, p - 1)$.
(vii) There is a bijection $k \mapsto i_k$ from $[1, p]$ to $I$ such that $i_k \rightarrow i_\ell$ if and only if $k < \ell$.

**Proof.** Observe that $i \rightarrow j \rightarrow k$ and $i \not\rightarrow k$ implies $\langle i, j, k \rangle$ is a 3-cycle. The equivalences of (i)-(v) are then easy to check. It is obvious that (vii) implies (i) and (vi). For the converse directions we use induction on $p$. 
If $\Pi$ is an order, and $i \in I$ then $\Pi|(I \setminus \{i\})$ is an order and so by induction hypothesis there is a numbering $k \mapsto j_k$ of $I \setminus \{i\}$ according to (vii). Let $k^*$ be the maximum $k$ such that $j_k \rightarrow i$. If there is none such then let $k^* = 0$. By transitivity $j_\ell \rightarrow i$ for $\ell < k^*$. Define $i_k = j_k$ for $k \leq k^*$, $i_{k^*+1} = i$ and $i_k = j_{k-1}$ for $k^* + 1 < k \leq p$.

If $\Pi$ has score vector $(0, 1, \ldots, p-1)$, then let $i \in I$ with score 0. Every $k \rightarrow i$ for $k \neq i$ and so $\Pi'(I \setminus \{i\})$ has score vector $(0, \ldots, p-2)$. By induction hypothesis we have $k \mapsto j_k$ as in (vii) for $\Pi'(I \setminus \{i\})$. Let $i_p = i$ and $i_k = j_k$ for $k < p$.

\[ \square \]

**Remark:** For any tournament $\Pi$ it is clear that $\emptyset \Pi$ induces an order on the set of $\emptyset(1_I \cup \Pi) \cap \emptyset(1_I \cup \Pi)^{-1}$ equivalence classes, with $[i] \rightarrow [j]$ for distinct classes $[i], [j]$ when $(i, j) \in \emptyset \Pi$.

A digraph $\Pi$ on $I$ is Eulerian when for all $i \in I$, $|\Pi(i)| = |\Pi^{-1}(i)|$. That is, each element of $I$ has the same number of inputs and outputs. In [17] such a digraph is called an isograph. In general, for an Eulerian digraph the cardinality $|\Pi(i)| = |\Pi^{-1}(i)|$ may vary with $i$. When it does not, when $k = |\Pi(i)| = |\Pi^{-1}(i)|$ is the same for all $i$, the digraph is called regular or $k$-regular. Obviously, if $i$ is a vertex of a Eulerian graph then neither $\Pi(i)$ nor $\Pi^{-1}(i)$ is empty.

We will call an Eulerian tournament a game, as these are the tournaments which generalize the Rock-Paper-Scissors game. This requires that $I$ have odd cardinality, in which case, a game $\Pi$ is a digraph such that for all $i \in I$, $|\Pi(i)| = |\Pi^{-1}(i)| = n$ with $n = \frac{1}{2}(|I| - 1)$. That is, a game is exactly a regular tournament. The score vector for such a game is given by $s_r = n$ for $r = 1, \ldots, 2n + 1$.

The trivial game has size 1. That is, $I$ is a singleton and the unique digraph is the empty subset of $I \times I$.

A cycle is obviously Eulerian since $|\Pi(i)| = 1 = |\Pi^{-1}(i)|$ for every $i$ on the cycle.

Notice that if a digraph $\Pi$ is a tournament or is Eulerian, then $\Pi^{-1}$ is a digraph satisfying the corresponding property.

**Lemma 1.4.** If $\Pi_1$ is an Eulerian subgraph of a digraph $\Pi$, then $\Pi$ is Eulerian if and only if $\Pi \setminus \Pi_1$ is Eulerian. In particular, the union of disjoint Eulerian graphs on $I$ is Eulerian.

**Proof.** If $\Gamma = \Pi \setminus \Pi_1$ then $\Pi(i)$ is the disjoint union of $\Pi_1(i)$ and $\Gamma(i)$. Hence, $|\Pi(i)| = |\Pi_1(i)| + |\Gamma(i)|$. Similarly, $|\Pi^{-1}(i)| = |\Pi^{-1}_1(i)| + |\Gamma^{-1}(i)|$. By assumption, $|\Pi_1(i)| = |\Pi^{-1}_1(i)|$. So $|\Pi(i)| = |\Pi^{-1}(i)|$ if and only if $|\Gamma(i)| = |\Gamma^{-1}(i)|$. 

The following observation is essentially due to Euler, see, e.g. [17] Theorem 12.5.

**Theorem 1.5.** Any nonempty Eulerian digraph can be written as a disjoint union of cycles.

**Proof.** Let $\Pi$ be a nonempty Eulerian digraph. Let $I$ be the set of vertices, which is nonempty since $\Pi$ is nonempty. Since $\Pi$ is Eulerian, $\Pi(i)$ and $\Pi^{-1}(i)$ are nonempty for every $i$. Beginning with any vertex we can build a simple path $[i_1, \ldots, i_k]$ and continue until $i_p \in \Pi(i_k)$ for some $p < k$ and so, necessarily, $p < k - 1$. Then $\langle i_p, \ldots, i_k \rangle$ is a cycle in $\Pi$.

By Lemma 1.4 $\Pi \setminus \langle i_p, \ldots, i_k \rangle$ is an Eulerian digraph and so, if it is nonempty, it contains a cycle disjoint from $\langle i_p, \ldots, i_k \rangle$.

Continue inductively to exhaust $\Pi$. □

**Remark:** The decomposition of an Eulerian digraph into disjoint cycles is not usually unique.

The following is essentially Theorem 7.4 of [9].

**Theorem 1.6.** Assume that $[i_0, \ldots, i_k]$ with $k > 1$ is a closed edge-simple path for a digraph $\Pi$ (and so $i_k = i_0$). Regarded as a subgraph of $\Pi$, $[i_0, \ldots, i_k]$ is a strong, Eulerian digraph and so is a disjoint union of cycles. Conversely, if $\Pi$ is a connected, Eulerian digraph, then it admits a spanning, closed, edge-simple path. In particular, a connected, Eulerian digraph is strong.

**Proof.** Of course, if the vertices $i_1, \ldots, i_{k-1}$ are distinct, then $[i_0, \ldots, i_k]$ is a single $k$ cycle (and conversely). However, while we are assuming the edges are distinct, the vertices need not be. Nonetheless, the input edge $(i_r-1, i_r)$ for $i_r$ is balanced by the output edge $(i_r, i_{r+1})$. Since the edges are distinct, each vertex has the same number of inputs and outputs.

Since $[i_0, \ldots, i_k]$ is Eulerian, it is a disjoint union of cycles by Theorem 1.5.

Conversely, assume that $\Pi$ is a connected Eulerian digraph and so is a disjoint union of cycles $C_0, \ldots, C_{n-1}$. We prove the existence of the required spanning path by induction on $n$. If $\Pi$ consists of a single cycle, the result is obvious.
Define the reflexive, symmetric relation $R$ on $[0, n] = \{0, \ldots, n - 1\}$ by $(p, q) \in R$ when $C_p$ and $C_q$ have a vertex in common. Thus, $\emptyset R$ is an equivalence relation on $[0, n]$. If $\Pi_1$ is the union of cycles in an $\emptyset R$ equivalence class, then $\Pi_1$ and $\Pi \setminus \Pi_1$ have no vertices in common. Since $\Pi$ is assumed to be connected, it must be the union of a single $\emptyset R$ equivalence class. Let $k$ be the smallest positive integer such that $[0, n] \times [0, n] \subset R^k$ and so for some $p, q \in [0, n]$ $(p, q) \in R^k \setminus R^{k-1}$. For any $q_1 \neq q$ there is an $R$ path from $p$ to $q_1$ with length at most $k$ and $q$ does not lie on such a path. It follows that the Eulerian digraph $\Gamma = \bigcup_{r \neq q} C_r$ is connected and is the union of $n - 1$ disjoint cycles. By induction hypothesis there exists $[i_0, \ldots, i_\ell]$, a closed edge-simple path which spans $\Gamma$. The cycle $C_q = \langle j_1, \ldots, j_r \rangle$ has a vertex in common with $\Gamma$. By relabeling we may assume $i_\ell = i_0 = j_r$. Then $[i_0, \ldots, i_\ell, j_1, \ldots, j_r]$ is a closed, edge-simple path which spans $\Pi$. □

Given a map $\rho : I \to J$ we let $\bar{\rho}$ denote the product map $\rho \times \rho : I \times I \to J \times J$.

**Definition 1.7.** Let $\Pi$ and $\Gamma$ be digraphs on $I$ and $J$, respectively. A morphism $\rho : \Pi \to \Gamma$ is a map $\rho : I \to J$ such that $(\bar{\rho})^{-1}(\Gamma) = \Pi \setminus (\bar{\rho})^{-1}(1_J)$. That is, for $i_1, i_2 \in I$ with $\rho(i_1) \neq \rho(i_2)$ $\rho(i_1) \to \rho(i_2)$ if and only if $i_1 \to i_2$. In particular, if $\rho$ is injective, then it is a morphism if and only if $(\bar{\rho})^{-1}(\Gamma) = \Pi$ and if it is bijective then it is a morphism if and only if $\bar{\rho}(\Pi) = \Gamma$.

Clearly, if $\rho$ is a bijective morphism then $\rho^{-1}$ is a morphism and so $\rho$ is an isomorphism. Two digraphs are isomorphic when each can be obtained from the other by relabeling the vertices.

An automorphism of $\Pi$ is an isomorphism with $\Pi = \Gamma$. We let $Aut(\Pi)$ denote the automorphism group of $\Pi$.

If $J \subset I$ then the inclusion map from $J$ to $I$ is a morphism from the restriction $\Pi|J$ to $\Pi$.

An automorphism $\rho$ of a digraph $\Pi$ is a permutation of the vertices of $\Pi$ and so is a product of disjoint cycles. Observe that if $i \to \rho(i)$ then $\rho(i) \to \rho^2(i)$. So if $\rho$ includes the cycle $(i_1, \ldots, i_k)$ and $i_1 \to i_2$ then $i_2 \to i_3$, ... , $i_k \to i_1$. Thus, $\langle i_1, \ldots, i_k \rangle$ is a $k$-cycle in the digraph. Otherwise, $i_2 \to i_1$ and so $i_1 \to i_k, \ldots, i_3 \to i_2$. In that case $\langle i_k, i_{k-1}, \ldots, i_1 \rangle$ is a $k$-cycle in $\Pi$.

Since $i \to j$ implies that $i \not\leftrightarrow j$, an automorphism can contain no transposition. In fact, it contains no even cycle.
Proposition 1.8. If \( \rho \) is an automorphism of a digraph \( \Pi \), then \( \rho \) is a permutation of odd order. If a pair \( \{i, j\} \) is \( \rho \) invariant then \( \rho \) fixes each element of the pair.

Proof. If \( \rho^{2k} \) is the identity but \( \rho^k \) is not, then for some vertex \( i_1 \), \( \rho^k(i_1) = i_2 \neq i_1 \). Since \( \rho^k(i_2) = i_1 \), the pair \( (i_1, i_2) \) is a transposition for the automorphism \( \rho^k \) which we have seen cannot happen.

If \( \{i, j\} \) is invariant for a permutation then the restriction to \( \{i, j\} \) is either the identity or a transposition and the latter is impossible for an automorphism.

2. Games

Recall that \( \Gamma \) is a game on \( I \) when it is an Eulerian tournament on \( I \). This requires that \( |I| \), the number of vertices of \( I \), be an odd number \( 2n+1 \) so that that each vertex has \( n \) input edges and \( n \) output edges. We call \( |I| = 2n + 1 \) the size of the game. These digraphs are called games, because it is these tournaments which extend the Rock-Paper-Scissors game to larger sets.

A subset \( J \) of cardinality \( |J| = 2k + 1 \) forms a subgame when the restriction \( \Gamma|J \) is Eulerian, i.e. for each \( i \in J \), \( \Gamma(i) \cap J \) has cardinality \( k \).

A 3-cycle is a subgame of size 3. If three vertices do not form a 3-cycle then they form what we will call a straddle. Among the three, one vertex has zero outputs, one has one output and the third has two outputs. That is, the score vector for a straddle is \((0, 1, 2)\) and so a straddle is just an order on three vertices.

Of course, any 3-cycle, i.e. any game of size 3, is isomorphic to the original Rock-Paper-Scissors game. The same uniqueness holds for 5.

Theorem 2.1. Up to isomorphism there is one game \( \Gamma \) of size 5.

Proof. Choose any vertex and label it 0. The pair of outpoints \( \Gamma(0) \) form an edge which we label 1 and 2 with \( 1 \rightarrow 2 \). The input pair \( \Gamma^{-1}(0) \) we label 3 and 4 with \( 3 \rightarrow 4 \). The remaining directions are now determined. We began with \( 3, 4 \rightarrow 0 \rightarrow 1, 2 \)

\[
\begin{align*}
0, 1 & \rightarrow 2 \quad \Rightarrow \quad 2 \rightarrow 3, 4. \\
3 & \rightarrow 0, 4 \quad \Rightarrow \quad 1, 2 \rightarrow 3. \\
2, 3 & \rightarrow 4 \quad \Rightarrow \quad 4 \rightarrow 0, 1.
\end{align*}
\]
• $1 \rightarrow 2, 3 \Rightarrow 0, 4 \rightarrow 1.$

We can diagram the result.

(2.1)

There is a general construction which builds a game of size $2n + 1$
from one of size $2n - 1$.

Let $\Pi$ be a game of size $2n - 1$ on the set of vertices $J$ and $K \subset J$
with $|K| = n$. Let $u, v$ be two additional vertices and let $I = J \cup \{u, v\}$.
Define $\Gamma$ so that $u \rightarrow v$ and $i \rightarrow u$ for all $i \in K$. This requires $u \rightarrow j$
for all $j \in J \setminus K$. Since each vertex in $K$ has now been assigned $n$
outputs, we must have $v \rightarrow i$ for all $i \in K$ and $j \rightarrow v$ for all $j \in J \setminus K$.
Thus, $\Pi$ is a subgame of $\Gamma$. We call $\Gamma$ an extension of $\Pi$ via $u \rightarrow v$ and $K$.

Now assume that $\Gamma$ is a game of size $2n + 1$ on the set of vertices $I$
and $u, v \in I$ such that the restriction $\Pi = \Gamma|J$ with $J = I \setminus \{u, v\}$
is a subgame of $\Gamma$. Assume that $u \rightarrow v$, and let $K = \Gamma^{-1}(u)$. Observe
that if $j \in K$, Then $\Pi^{-1}(j) = \Gamma^{-1}(j) \setminus \{v\}$ has $n - 1$ elements, while
$|\Gamma^{-1}(j)| = n$. Hence, $v \rightarrow j$ for all $j \in K$. Since $n = |K| = |\Gamma(v)|$ it
follows that $\Gamma(v) = K$. Thus, $\Gamma$ is the extension of $\Pi$ via $u \rightarrow v$ and $K = \Gamma^{-1}(u)$. We say that $\Gamma$ is reducible via $\{u, v\}$ when $\Gamma$ restricts to
a subgame on $I \setminus \{u, v\}$.

Notice that the game of size 3 is reducible to the trivial game.

**Proposition 2.2.** Let $\Gamma$ be a game of size $2n + 1$ on the set of vertices $I$.

(a) For all $u, v \in I$, $\Gamma(u) = \Gamma(v)$ or $\Gamma^{-1}(u) = \Gamma^{-1}(v)$ implies $u = v$.

(b) Every edge is contained in at least one 3-cycle.

(c) Every edge is contained in at most $n$ 3-cycles.

(d) For $u, v \in I$ the following seven conditions are equivalent:

(i) $\Gamma$ is reducible via $\{u, v\}$.

(ii) The restriction $\Gamma|(I \setminus \{u, v\})$ is a subgame of $\Gamma$.

(iii) The edge between $u$ and $v$ is contained in $n$ 3-cycles.

(iv) There does not exist $i \in I \setminus \{u, v\}$ such that $i \rightarrow u$ and $i \rightarrow v$. 

(v) There does not exist \( i \in I \setminus \{u, v\} \) such that \( u \to i \) and \( v \to i \).

(vi) \( \Gamma^{-1}(u) \cap \Gamma^{-1}(v) = \emptyset \).

(vii) \( \Gamma(u) \cap \Gamma(v) = \emptyset \).

Furthermore,

(viii) \( \Gamma^{-1}(u) = \Gamma(v) \) if and only if \( u \to v \) and \( \Gamma \) is reducible via \( \{u, v\} \).

(e) If \( \Gamma \) is the extension of \( \Pi \) via \( u \to v \) and \( K \), then the reversed game \( \Gamma^{-1} \) is the extension of \( \Pi^{-1} \) via \( v \to u \) and \( K \).

(f) For all \( u \in I \) there is at most one \( v \) such that \( u \to v \) and \( \Gamma \) is reducible via \( \{u, v\} \), and there is at most one \( v \) such that \( v \to u \) and \( \Gamma \) is reducible via \( \{u, v\} \).

Proof. Assume \( u \to v \) in \( \Gamma \).

(a): \( v \in \Gamma(u) \setminus \Gamma(v) \) and \( u \in \Gamma^{-1}(v) \setminus \Gamma^{-1}(u) \).

(b): \( \Gamma(u) \setminus \{v\} \) contains \( n-1 \) elements. Since \( u \not\in \Gamma(v) \) and the latter contains \( n \) elements, it follows there exists \( w \in \Gamma(v) \cap \Gamma^{-1}(u) \).

So \( u \to v \to w \) is a 3-cycle.

(c): If \( u \to v \) together with \( w \) forms a 3-cycle then \( w \in \Gamma(v) \) and so there are at most \( n \) such \( w \)'s.

(d): (i) \( \iff \) (ii): This is the definition of reducibility.

(i) \( \iff \) (iii): \( \Gamma \) is reducible via \( u \to v \) if and only if \( v \to i \) for all \( i \in K = \Gamma^{-1}(u) \). Furthermore, \( u \to v \) together with \( i \) forms a 3-cycle if and only if \( i \in \Gamma^{-1}(u) \) and \( v \to i \).

(i) \( \iff \) (iv),(v): If \( J = I \setminus \{u, v\} \), then \( i \to u, v \) if and only if \( \Gamma(i) \cap J \) contains only \( n-2 \) elements. Similarly, \( u, v \to i \) if and only if \( \Gamma^{-1}(i) \cap J \) contains only \( n-2 \) elements. In either of these cases, \( \Gamma|J \) is not a subgame.

(iv) \( \iff \) (vi) and (v) \( \iff \) (vii): Obvious.

As we saw above, if \( u \to v \) and then condition (ii) implies \( \Gamma \) is the extension of \( \Gamma|I \setminus \{u, v\} \) via \( u \to v \) and \( K = \Gamma^{-1}(u) \) and so \( K = \Gamma(v) \). Conversely, if \( \Gamma^{-1}(u) = \Gamma(v) \), then \( u \not\in \Gamma^{-1}(u) \) implies \( u \not\in \Gamma(v) \). Since \( v \not\to u \), it follows that \( u \to v \). In addition, condition (vii) clearly holds.

(e): In the extension of \( \Pi^{-1} \) via \( v \to u \) and \( K \) all of the arrows of \( \Gamma \) have been reversed.

(f): If \( \Gamma \) is reducible via \( \{u, v\} \) and if \( u \to v \), then \( u \not\in \Gamma(u) \cup \Gamma(v) \). So if \( \Gamma \) is reducible via \( \{u, v\} \) then from (d)(vi) \( \Gamma(v) = I \setminus (\{u\} \cup \Gamma(u)) \).

By (a) this can be true of at most one \( v \in \Gamma(u) \). For the \( v \to u \) case use (e) and apply the result to the reverse game.

\( \square \)
Corollary 2.3. Any non-trivial game $\Gamma$ is a strong digraph.

Proof. Let $u, v \in I$. If $u \not\rightarrow v$, then $v \rightarrow u$ and the edge is contained in a 3-cycle $\langle u, w, v \rangle$. So there is a path of length 1 or 2 from $u$ to $v$.

□

Corollary 2.4. For $\Pi$ a game on $J$ and $K \subseteq J$ with $|J| = 2n - 1$, $|K| = n$, assume that $\Gamma$ is an extension of $\Pi$ via $u \rightarrow v$ and $K$. Let $i, j \in J$ with $i \rightarrow j$.

(a) $\Gamma$ is reducible via $i \rightarrow u$ if and only if $i \in K$ and $J \setminus K = \Pi^{-1}(i)$, so that $K = \{i\} \cup \Pi(i)$.
(b) $\Gamma$ is reducible via $v \rightarrow i$ if and only if $i \in K$ and $J \setminus K = \Pi(i)$, so that $K = \{i\} \cup \Pi^{-1}(i)$.
(c) $\Gamma$ is not reducible via $u \rightarrow i$ or via $i \rightarrow v$.
(d) $\Gamma$ is reducible via $i \rightarrow j$ if and only if $\Pi$ is reducible via $i \rightarrow j$ and $K \cap \{i, j\}$ is a singleton set.

Proof. It is easy to check that (a), (b) and (d) follow from Proposition 2.2 (d) while (c) follows from Proposition 2.2 (f).

□

For a game $\Pi$ we define the reducibility graph $r\Pi$

\[(2.2) \quad r\Pi = \{ (i, j) \in \Pi : \Pi \text{ is reducible via } i \rightarrow j \}. \]

Of course, if $\Pi$ is not reducible then $r\Pi$ is empty.

Proposition 2.5. Let $\Pi$ be a game on $I$ with $|I| = 2n + 1$.

(a) If $i \in I$ then $(i, j) \in r\Pi$ for at most one $j \in I$ and $(j, i) \in r\Pi$ for at most one $j \in I$.
(b) Let $[i_0, \ldots, i_m]$ be a path in $\Pi$ and let $J = I \setminus \{i_0, \ldots, i_m\}$. The path $[i_0, \ldots, i_m]$ is contained in $r\Pi$ if and only if the following conditions hold:
   (i) If $p, q \in [0, m]$, then $i_p \rightarrow i_q$ in $\Pi$ if and only if $q - p$ is odd and positive or is even and negative.
   (ii) If $j \in J$, and $j \rightarrow i_p$ then $j \rightarrow i_q$ if and only if $q - p$ is even.

When these conditions hold, the path is simple. In addition, $(i_m, i_0) \in r\Pi$ -so that $\langle i_0, \ldots, i_m \rangle$ is a cycle in $r\Pi$- if and only if $J = \emptyset$, or, equivalently, $m = 2n$. In that case, $r\Pi = \langle i_0, \ldots, i_m \rangle$ and the cycle is a Hamiltonian cycle in $\Pi$. 
If $r \Pi$ is not empty and does not consist of a single Hamiltonian cycle on $\Pi$, then it is a disjoint union of separated, simple, non-closed paths.

**Proof.** (a) is a restatement of Proposition 2.2 (f).

(b) Assume that $[i_0, \ldots, i_m]$ is a path in $r \Pi$. For $q > p$ we prove by induction on $q - p$ that $i_p \rightarrow i_q$ if and only if $q - p$ is odd. If $i_p \rightarrow i_{q-1}$ then since $\Pi$ is reducible via $i_{q-1} \rightarrow i_q$ if follows from Proposition 2.2 (d) that $i_q \rightarrow i_p$. Similarly, $i_{q-1} \rightarrow i_p$ implies $i_p \rightarrow i_q$. Condition (i) follows and condition (ii) similarly follows from Proposition 2.2 (d). On the other hand, these conditions imply that $i_p \rightarrow i_{p+1}$ for $p = 0, \ldots, m - 1$ and that that $\Pi$ is reducible via each $i_p \rightarrow i_{p+1}$ by Proposition 2.2 (d) again.

If $m$ is odd, then $i_0 \rightarrow i_m$ and so $(i_m, i_0) \not\in \Pi$. If $m$ is even and $j \in J$, then either $i_0, i_m \rightarrow j$ or $j \rightarrow i_0, i_m$ and so $\Pi$ is not reducible via $i_m \rightarrow i_0$. On the other hand if $m$ is even and $J$ is empty, then condition (i) implies $\Pi$ is reducible via $i_m \rightarrow i_0$. Then (a) implies that every edge of $r \Pi$ lies on the cycle.

(c) If $r \Pi$ is not empty and does not consist of a single Hamiltonian cycle, then by (b) it contains no cycle. By (a) any edge in $r \Pi$ can be extended uniquely to a maximal path in $r \Pi$ and none of the remaining edges of $r \Pi$ has a vertex on the path. Proceed by exhaustion to obtain $r \Pi$ as a separate union of these maximal paths.

Theorem 2.6. Let $\Gamma$ be a game of size $2n + 1$ on the set of vertices $I$. Each vertex $i \in I$ is contained in exactly $n(n + 1)/2$ 3-cycles. The entire game contains $(2n + 1)n(n + 1)/6$ 3-cycles.

**Proof.** There are $n$ vertices in the output set $\Gamma(i)$. Each of these has $n$ output edges and these are all distinct for a total of $n^2$ outputs from these vertices. Between these vertices there are $n(n - 1)/2$ edges each of which is one of the $n^2$ output edges from a vertex of $\Gamma(i)$. The remaining $n(n + 1)/2 = n^2 - [n(n - 1)/2]$ edges terminate at a vertex of $\Gamma^{-1}(i)$ and these are the 3-cycles which contain $i$. Multiplying by the number $2n + 1$ of vertices $i$ we obtain the total number of 3-cycles in $\Gamma$ after we divide by 3 to correct for the triple counting.

The above result (and its proof) comes from Theorem 5.2 of [8]. In general, for a tournament of size $p$ with score vector $s = (s_1, \ldots, s_p)$
the total number $N_s$ of 3-cycles is given by the formula

$$N_s = \frac{p(p-1)(2p-1)}{12} - \frac{1}{2} \sum_{i=1}^{p} s_i^2. \quad (2.3)$$

See [17] Corollary 11.10b or [16] Corollary 6b.

It follows from Theorem 2.6 that in a game with $n > 1$, at least one edge is contained in more than one cycle. For if every edge were contained in exactly one cycle then the number of cycles would be the number of edges divided by 3, i.e. \( n(2n+1)/3 \), but if $n > 1$, then $(n + 1)/2 > 1$.

**Theorem 2.7.** Let $\Pi$ be an Eulerian digraph on a set $I$ with $|I|$ odd. There exists a game $\Gamma$ on $I$ which contains $\Pi$ as a subgraph.

*Proof.* Let $2n + 1 = |I|$ and let $k_i = |\Pi(i)| = |\Pi^{-1}(i)|$ for $i \in I$. We can clearly extend $\Pi$ to some tournament on $I$.

For a tournament $\Gamma$ on $I$ let $s_i = |\Gamma(i)|$, the score of $i$. Let

$$I_+ = \{i \in I : s_i > n\}, \quad I_0 = \{i \in I : s_i = n\}, \quad I_- = \{i \in I : s_i < n\}.$$ 

Call $\sum\{s_i - n : i \in I_+ \cup I_0\}$ the deviation of $\Gamma$. Clearly the deviation is non-negative and since the total sum of the scores is $n(2n+1)$ it follows that $\Gamma$ is a game, i.e. $s_i = n$ for all $i$, if and only if the deviation is zero.

If $\Pi \subset \Gamma$ then for all $i \in I$, then

$$|\Gamma \setminus \Pi)(i)| = s_i - k_i, \quad |\Gamma \setminus \Pi)^{-1}(i)| = 2n - s_i - k_i. \quad (2.5)$$

Thus, $I_+$ is the set of $i \in I$ such that the number of $\Gamma \setminus \Pi$ outputs is greater than the number of $\Gamma \setminus \Pi$ inputs.

Now assume $\Pi \subset \Gamma$ and the deviation of $\Gamma$ is positive. We will show that there exists a tournament $\Gamma'$ containing $\Pi$ and with smaller deviation. Hence, the tournaments which contain $\Pi$ with minimum deviation are the games which contain $\Pi$.

I claim there exists a $\Gamma \setminus \Pi$ path from a vertex in $I_+$ to a vertex in $I_-$. If not, then $\hat{I} = I_+ \cup \partial(\Gamma \setminus \Pi)(I_+)$ is disjoint from $I_-$ and so is contained in $I_+ \cup I_0$. The restriction $\Gamma \setminus \Pi)|\hat{I}$ contains every edge of $\Gamma \setminus \Pi$ from a vertex in $\hat{I} \supset I_+$. Since $I_+$ is assumed to be nonempty it follows that the total number of outputs for $\Gamma \setminus \Pi)|\hat{I}$ is strictly greater than the total number of inputs. But these sums are both equal to the number of edges in $(\Gamma \setminus \Pi)|\hat{I}$. The contradiction establishes the existence of the required path.

Truncate the path to begin at the last occurrence of a vertex in $I_+$ and then terminate at the first occurrence of a vertex in $I_-$. Then
eliminate any intermediate repeated vertices, by removing the piece between the repeats. Thus we obtain a simple $\Gamma \setminus \Pi$ path $[i_1, i_2, \ldots, i_k]$ with $i_1 \in I_+, i_k \in I_-$ and $i_p \in I_0$ for $1 < p < k$.

From $\Gamma$ define $\Gamma'$ by reversing the edges of $[i_1, i_2, \ldots, i_k] \subset \Gamma \setminus \Pi$. This decreases the score of $i_1$ by 1, increases the score of $i_k$ by 1 and leaves every other score unchanged. Thus, the deviation of $\Gamma'$ is one less than that of $\Gamma$. Since the path lies in $\Gamma \setminus \Pi$ it follows that $\Pi \subset \Gamma'$.

\section{Group Games}

We turn now to the algebraic examples. We let $\mathbb{Z}_{2n+1}$ denote the additive group of integers mod $2n + 1$. We label the congruence classes as $0, 1, \ldots, 2n$. So $-k = 2n + 1 - k$ for $k \in [1, 2n] = \{1, \ldots, 2n\}$. We let $\mathbb{Z}^*_{2n+1}$ denote the multiplicative group of units in the ring of integers mod $2n + 1$, i.e. the congruence classes of the integers relatively prime to $2n + 1$.

In general, let $G$ be a group of order $2n + 1$. Notice that if $G$ is abelian and $2n + 1$ is square-free, then $G$ is cyclic and so is isomorphic to $\mathbb{Z}_{2n+1}$. The smallest non-abelian group of odd order is the semi-direct product $\mathbb{Z}_3 \rtimes \mathbb{Z}_7$ with $\mathbb{Z}_3$ regarded as a subgroup of $\mathbb{Z}_7^*$ acting on $\mathbb{Z}_7$ by multiplication. With order less than 20 the groups of odd order are cyclic except for $\mathbb{Z}_3 \times \mathbb{Z}_3$ of order 9.

With $G$ be a group of order $2n + 1$ there is no element of order 2. With $e$ the identity element of $G$, the set $G \setminus \{e\}$ is partitioned by the set of $n$ pairs $\{i, i^{-1}\}$. We will call $A$ a graph subset of $G$ when it is a nonempty subset of $G$ with $A$ disjoint from $A^{-1}$. A game subset $A$ is a graph subset of cardinality $n$ so that $G$ is the disjoint union $A \cup A^{-1} \cup \{e\}$. Thus, a game subset $A$ is obtained by choosing one element from each of the pairs $\{i, i^{-1}\}$. It follows that there $2^n$ game subsets.

If $A$ is a graph subset, then the associated digraph is $\Gamma[A] = \{(i, j) : i^{-1}j \in A\}$. For $k \in G$ define the left translation map $\ell_k$ on $G$ by $\ell_k(i) = ki$. Notice that $\Gamma[A](i) = \ell_i(A) = iA$ for $i \in G$, while $\Gamma[A]^{-1}(i) = \ell_i(A^{-1}) = iA^{-1}$. It follows that $\Gamma[A]$ is Eulerian and if $A$ is a game subset, then $\Gamma[A]$ is a tournament and so is a game. We call a game of this sort a group game.

It is easy to see that $\Gamma[A]$ is connected when $A$ generates the group and that, otherwise, the left cosets of the subgroup generated by $A$ decompose the graph into separate pieces. When $A$ generates $G$ the
digraph $\Gamma[A]$ is the Cayley graph of the group $G$ with respect to the set $A$ of generators.

Notice that if $A$ is a graph subset (or a game subset), then $A^{-1}$ is a graph subset (resp. a game subset) and $\Gamma[A^{-1}]$ is the reversed digraph $\Gamma[A]^{-1}$.

We could define a right hand graph associated with $A$ by $\{(i, j) : ji^{-1} \in A\}$. We do not bother, because it is clear that the map $i \rightarrow i^{-1}$ is an isomorphism from $\Gamma[A]$ onto the right hand game associated with $A^{-1}$. In the abelian case the right hand game for $A$ is the same as $\Gamma[A]$ and so when $G$ is abelian, the map $i \rightarrow i^{-1}$ is an isomorphism from $\Gamma[A]$ to the reversed game $\Gamma[A^{-1}]$.

**Lemma 3.1.** Let $A$ and $B$ be graph subsets of $G$ and let $\rho \in S(G)$, i.e. $\rho$ is a permutation of $G$. The following are equivalent

(i) $\rho$ is an isomorphism from $\Gamma[A]$ to $\Gamma[B]$.
(ii) For all $i, j \in G$  
\[ i^{-1}j \in A \iff \rho(i)^{-1}\rho(j) \in B. \]
(iii) For all $i \in G$, $\rho(iA) = \rho(i)B$.

In particular, if $\rho$ is an isomorphism from $\Gamma[A]$ to $\Gamma[B]$, then $\rho(e) = e$ implies $\rho(A) = B$.

**Proof.** It is obvious that (i) $\iff$ (ii) and (ii) $\iff$ (iii). From (iii), $\rho(e) = e$ implies $\rho(A) = B$.

Let a finite group $T$ act on a finite set $I$. The action is called free when $ti = i$ for some $i \in I$ only when $t$ is the identity element of $T$. The action is called effective when $ti = i$ for all $i \in I$ only when $t$ is the identity element of $T$. Of course, a free action is effective. The action is called transitive if for some $i \in I$, $I = Ti$ where $Ti = \{ti : t \in T\}$ is the $T$ orbit of $i$. In that case, $I = Ti$ for all $i \in T$.

Here are some useful facts about such actions.

**Proposition 3.2.** Let a finite group $T$ act on a finite set $I$.

(a) The group $T$ acts freely on $I$ if and only if for all $i \in I$ the map $t \rightarrow ti$ from $T$ to $I$ is injective. In particular, if $T$ acts freely on $I$ then $|T|$ divides $|I|$.
(b) Any two of the following three conditions implies the third.
\[ \bullet \ \text{The group } T \text{ acts freely on } I. \]
\[ \bullet \ \text{The group } T \text{ acts transitively on } I. \]
\[ \bullet \ \text{The cardinalities } |T| \text{ and } |I| \text{ are equal.} \]
(c) If $T$ is abelian and acts effectively and transitively on $I$, then it acts freely on $I$ and $|T| = |I|$.
(d) Assume that $|I|$ is prime. If $T$ acts transitively on $I$, then $T$ contains a cyclic subgroup $H$ which acts transitively on $I$. If, in addition, the action of $T$ is effective, then $|H| = |I|$.

(e) Assume that $|T| = |I| = k$ and that the group $T$ is cyclic with generator $t$. The group acts freely on $I$ if and only if if regarded as a permutation on $I$, $t$ consists of a single $k$-cycle. If $k$ is prime and the action is non-trivial then it is free.

(f) Assume that $|T| = 2n + 1$ and that $T$ acts freely on $I$. If $A \subset I$ with $|A| = n$, then $t(A) = A$ if and only if $t$ is the identity element of $T$.

(g) If $|T|$ and $|I|$ are odd then the number of $T$ orbits in $I$ is odd.

Proof. (a): If $t_0i = i$ for some $t_0$ not the identity then the map $t \mapsto ti$ is not injective. If $t_0i = t_1i$ with $t_0 \neq t_1$ then $t_1^{-1}t_0i = i$. In any case, $I$ is the disjoint union of the orbits $Ti$. When the action is free, each of these has cardinality $|T|$. So $|I|$ is a multiple of $|T|$.

(b): Assume first that $|T| = |I|$. Since the cardinalities are finite and equal, the map $ev_i : T \to I$ given by $t \mapsto ti$ is injective if and only if it is surjective. On the other hand, if the map is bijective, then $|T| = |I|$.

(c): Assume that $ti = i$. If $j \in J$ then there exists $s \in T$ such that $si = j$ since the action is transitive. Because the group is abelian, $tj = tsi = ssi = si = j$. That is, $tj = j$ for all $j$ and so $t$ is the identity because the action is effective. Then (b) implies that $|T| = |I|$.

(d): Assume that $p = |I|$, which is assumed to be prime. Fix $i \in I$ and let $Iso_i = (ev_i)^{-1} \{i\} = \{t : ti = i\}$. This is a subgroup of $T$ called the *isotropy subgroup* of $i$. Since $ti = si$ if and only if $t^{-1}s \in Iso_i$, it follows that $ev_i$ factors to define an injection from $T/Iso_i$, the set of left cosets $\{tIso_i : t \in T\}$, into $I$. Since the action is transitive, the induced map is a bijection and so the subgroup $Iso_i$ has index $p$. We need the following bit of group theory.

**Lemma 3.3.** If $T$ is a finite group and $J$ is a subgroup with prime index, then there exists a cyclic subgroup $H$ of $T$ such that the restriction to $H$ of the quotient map $T \to T/J$ is surjective.

Proof. Assume that $|T/J| = p$ and that $|J| = p^e a$ with $e \geq 0$ and $a$ relatively prime to $p$. Then $|T| = p^{e+1}a$. By the First Sylow Theorem [15] Theorem 4.2.1, there exists a subgroup $P$ of $T$ with $|P| = p^{e+1}$. It follows that $P$ is not contained in $J$. Let $t \in P \setminus J$. Since $P$ is a $p$-group, $t$ has order a power of $p$. Let $H$ be the cyclic group generated by $t$. Hence, $|H/(H \cap J)|$ is a positive power of $p$ and the restriction of the quotient map factors to an injection from $H/(H \cap J)$ into $T/J$. 
ROCK, PAPER, SCISSORS, ETC

Since $|T/J| = p$, this map is a bijection and so the quotient map takes $H$ onto $T/J$. □

Now apply the Lemma with $J = Iso_i$. If $j \in I$, there exists $t \in T$ such that $ti = j$. There exist $h \in H$ and $s \in Iso_i$ such that $t = hs$. Hence, $hi = hsi = ti = j$. That is, $H$ acts transitively.

If, in addition, the action of $T$ is effective, then the abelian group $H$ acts transitively and effectively and so by (c) the action is free with $|H| = |I|$.

(e): If the action is free and $i \in I$, then $(i ti \ldots ti^{k-1}i)$ is a k-cycle. Conversely, if this is a k-cycle, then the action is transitive and so is free by (b). In any case, the generator $t$ acts as a permutation whose order divides the order of $t$ which is $k$. The order of the generator is the least common multiple of the orders of the cycles contained therein. If $k$ is a prime, then either all cycles have order 1 and the action is trivial or there is a k-cycle and the action is free.

(f): Let $G$ be the subgroup generated by $t$. The order $|G|$ divides $2n + 1$. On the other hand if $A$ is invariant and the action is free then $G$ acts freely on $A$. So by (a), $|G|$ divides $|A| = n$. Since $n$ and $2n + 1$ are relatively prime, $|G| = 1$ and so $t$ is the identity element.

(g): Observe that distinct orbits are disjoint. For $i \in I$, the map $ev_i$ factors to a bijection of the quotient $T/Iso_i$ onto the orbit $Ti$. Hence, $|Ti|$ is odd for every $i$. Thus, the orbits partition the odd cardinality set $I$ into sets of odd cardinality and so there must be an odd number of them. □

**Theorem 3.4.** (a) Let $\Pi$ be a digraph on the set of elements of a group $G$. The group $G$ acting freely on itself by left translation is a subgroup of $Aut(\Pi)$ if and only if there exists a -necessarily unique- graph subset $A$ of $G$ such that $\Pi$ equals $\Gamma[A]$ the associated digraph.

(b) The left translation action of $G$ induces a free action on the collection of translates of game subsets, i.e. for a game subset $A$, $iA = jA$ implies $i = j$.

**Proof.** (a): It is clear from the definition of $\Gamma[A]$ that each left translation is an automorphism. On the other hand, if the translations are automorphisms, then let $A = \Pi(e)$. It is clear that $i \rightarrow j$ if and only if $e = \ell_{i^{-1}}(i) \rightarrow \ell_{i^{-1}}(j) = i^{-1}j$. That is, $i \rightarrow j$ if and only if $i^{-1}j \in A$. Since $\Pi$ is a digraph, $0 \rightarrow i$ implies that $i \rightarrow 0$ and so $0 \rightarrow i^{-1}$. Thus, $A$ is a graph subset and $\Pi = \Gamma[A]$. 
(b): This follows from Proposition 3.2 (f) or from Proposition 2.2 (a) applied to $\Gamma[A]$.

From (a) we obtain the following result of Turner [31].

**Corollary 3.5.** Let $\Pi$ be a digraph on $I$ with $|I| = 2n + 1$ prime. If $\text{Aut}(\Pi)$ acts transitively on the set $I$, then there exists a graph subset $A$ of $\mathbb{Z}_{2n+1}$ such that $\Pi$ is isomorphic to $\Gamma[A]$ on $\mathbb{Z}_{2n+1}$.

**Proof.** By definition $\text{Aut}(\Pi)$ is a subgroup of $S(I)$ and so acts effectively on $I$. By Proposition 3.2 (d) and (c) there exists a cyclic subgroup $H$ of $\text{Aut}(\Pi)$ which acts freely and transitively on $I$ and with $|H| = |I|$. We can identify $H$ with $\mathbb{Z}_{2n+1}$ since it is cyclic and also with $I$ via $ev_i(h) = hi$ for any fixed $i \in I$. Thus, we can regard $\Pi$ as a digraph on $\mathbb{Z}_{2n+1}$ and the translation action is identified with a subgroup of $\text{Aut}(\Pi)$. With these identifications, $\Pi$ becomes a digraph of the form $\Gamma[A]$ by Theorem 3.4 (a).

For any group $G$ we will regard $G$ as a subgroup of $S(G)$ by identifying $i \in G$ with the left translation $\ell_i$. For any graph subset $A$ of a group $G$ we will use this identification to regard $G$ as a subgroup of $\text{Aut}(\Gamma[A])$.

For a finite group $G$ let $G^*$ denote the automorphism group of $G$. That is, $\xi \in G^*$ if and only if $\xi : G \to G$ is a group isomorphism. For the ring of integers mod $2n + 1$ the group of units $\mathbb{Z}_{2n+1}^*$ consists of those $i \neq 0$ which are relatively prime to $2n + 1$ of which there are $\phi(2n + 1)$ (defining the Euler $\phi$-function). For $a \in \mathbb{Z}_{2n+1}$ we let $m_a$ denote multiplication by $a$ so that $m_a(i) = ai$. If $\rho : \mathbb{Z}_{2n+1} \to \mathbb{Z}_{2n+1}$ is an additive group homomorphism and $a = \rho(1)$ then since $i = 1 + 1 \cdots + 1$ ($i$ times) it follows that $\rho(i) = ai = m_a(i)$. In particular, identifying $a \in \mathbb{Z}_{2n+1}^*$ with $m_a$ in the automorphism group of $\mathbb{Z}_{2n+1}$ we regard $\mathbb{Z}_{2n+1}^*$ as the automorphism group of $\mathbb{Z}_{2n+1}$.

**Theorem 3.6.** Let $A$ be a graph subset of $G$ with $\Gamma[A]$ the associated digraph.

(a) If $\xi \in G^*$, then $\xi(A)$ is a graph subset and $\xi$ is an isomorphism from $\Gamma[A]$ to $\Gamma[\xi(A)]$.

(b) The group $\text{Aut}(\Gamma[A])$ is commutative if and only if $G$ is commutative and $\text{Aut}(\Gamma[A]) = G$, i.e. the left translations are the only automorphisms of $\Gamma[A]$.

(c) Assume that $\text{Aut}(\Gamma[A]) = G$. If $B$ is a graph subset with $\Gamma[B]$ isomorphic to $\Gamma[A]$, then there is a unique $\xi \in G^*$ such that $B = \xi(A)$.
and $\rho = \xi$ is the unique isomorphism $\rho : \Gamma[A] \to \Gamma[B]$ such that $\rho(e) = e$. In particular, the set $\{\xi(A) : \xi \in G^*\}$ is the set of graph subsets $B$ such that $\Gamma[B]$ is isomorphic to $\Gamma[A]$. Thus, there are exactly $|G^*|$ such graph subsets.

(d) If $A \subset \mathbb{Z}_{2n+1}$ is a graph subset such that $\text{Aut}(\Gamma[A]) = \mathbb{Z}_{2n+1}$ then the set $\{m_a(A) : a \in \mathbb{Z}_{2n+1}^*\}$ is the set of graph subsets $B$ such that $\Gamma[B]$ is isomorphic to $\Gamma[A]$. Thus, there are exactly $\phi(2n+1)$ such game subsets.

Proof. (a): Since $\xi$ is a bijection with $\xi(e) = e$ and $\xi(i^{-1}) = \xi(i)^{-1}$, it follows that $\xi(A) \cap \xi(A)^{-1} = (A \cap A^{-1}) = \emptyset$ and so $\xi(A)$ is a graph subset.

We have $i^{-1}j \in A$ if and only if $\xi(i^{-1})\xi(j) = \xi(i^{-1}j) \in \xi(A)$. Hence, $\xi$ is an isomorphism from $\Gamma[A]$ to $\Gamma[\xi(A)]$.

(b): As is well-known, if $\rho$ is a permutation of $G$ which commutes with left translations, then $\rho$ is a right translation, because $\rho(i) = \rho(\ell_i(e)) = \ell_i\rho(e) = i\rho(e)$. If $\text{Aut}$ is commutative, then the subgroup $G$ is commutative and every element is a translation, i.e. $\text{Aut} = G$.

(c): By composing with a translation we can assume that $\rho : \Gamma[A] \to \Gamma[B]$ satisfies $\rho(e) = e$. Consider the translation $\ell_{\rho(i)}$ on $\Gamma[B]$. $\rho^{-1} \circ \ell_{\rho(i)} \circ \rho$ is an automorphism of $\Gamma[A]$ and so is a left translation. Since $e \in A$ is mapped to $i$, it is $\ell_i$. That is, $\rho \circ \ell_i = \ell_{\rho(i)} \circ \rho$. Applied to $j$, this says $\rho(ij) = \rho(i)\rho(j)$. That is, $\rho \in G^*$.

If $\hat{\rho} : \Gamma[A] \to \Gamma[B]$ is an isomorphism fixing $e$ then $\rho^{-1} \circ \hat{\rho}$ is an automorphism of $\Gamma[A]$ which fixes $e$. However, the identity is the only such automorphism. Hence, $\hat{\rho} = \rho$.

(d): This is just (c) with $G = \mathbb{Z}_{2n+1}$. Observe that $|G^*| = \phi(2n+1)$.

In $\mathbb{Z}_{2n+1}$ we let $[1, n] = \{1, \ldots, n\}$ and $\text{Odd}_n = \{2k - 1 : k = 1, \ldots, n\}$.

**Theorem 3.7.** The set $[1, n]$ is a game subset of $\mathbb{Z}_{2n+1}$ with $\text{Aut}(\Gamma[[1, n]]) = \mathbb{Z}_{2n+1}$.

Proof. Since $2n + 1 - k = n + (n + 1 - k), -[1, n] = [n + 1, 2n]$ and so $[1, n]$ is a game subset.

If $\rho \in \text{Aut}(\Gamma[[1, n]])$ with $\rho(0) = i$ we may compose with the translation $\ell_i$ to obtain an automorphism which fixes 0. It suffices to show that if $\rho$ is an automorphism which fixes 0 then $\rho$ is the identity.

Let $A = [1, n]$ and $\Gamma = \Gamma[A]$. By Lemma 3.1 $\rho(A + i) = A + \rho(i)$ and since $\rho$ fixes 0, $\rho(A) = A$. 


For $p = 1, \ldots, n$, $A \cap (A + p) = \{i : p < i \leq n\}$ which contains $n - p$ elements.

Thus, $p$ is the unique element $i$ of $A$ such that $A \cap (A + i)$ contains $n - p$ elements. Since $\rho(A \cap (A + i)) = A \cap (A + \rho(i))$, it follows that $\rho$ fixes every element of $[1, n]$. $-(A \cap A + i) = -A \cap (-A + (-i))$. Hence, $-p = 2n + 1 - p$ is the unique element $i$ of $-A$ such that $-A \cap (-A + i)$ contains $n - p$ elements. Thus, $\rho$ fixes every element of $\mathbb{Z}_{2n+1}$.

Notice that $n \in \mathbb{Z}_{2n+1}^*$ and $m_n(2k+1) = n - k \mod 2k + 1$. Hence, $m_n(Odd_n) = [1, n]$. Thus, $Odd_n$ is a game subset of $\mathbb{Z}_{2n+1}$ with $\Gamma[Odd_n]$ isomorphic to $\Gamma[[1, n]]$ via $m_n$.

Observe that for $A = [1, n]$ the tournament which is the restriction $\Gamma[A]|A$ is the standard order on $[1, n]$.

**Proposition 3.8.** If $\Pi$ is a game of size $2n+1$, then the reducibility graph $r\Pi$ is a cycle if and only if $\Pi$ is isomorphic to the group game $\Gamma[[1, n]]$ on $\mathbb{Z}_{2n+1}$.

**Proof.** Using Proposition 2.5 it is easy to check that $\Pi$ is a game with $r\Pi = \langle i_0, \ldots, i_{2n}\rangle$ if and only if $p \mapsto i_p$ is an isomorphism from $\Gamma[Odd_n]$ to $\Pi$.

Recall that there are $2^n$ game subsets of $\mathbb{Z}_{2n+1}$. With $n = 1, 2$, we have $2^n = \phi(2n + 1)$, which must be true since in those cases there is, up to isomorphism, only one game.

**Corollary 3.9.** If $n > 2$ then there exists a game subset $A \subset \mathbb{Z}_{2n+1}$ such that $\Gamma[A]$ is not isomorphic to $\Gamma[[1, n]]$.

**Proof.** By Theorem 3.7 and Theorem 3.6(d) there are $\phi(2n + 1)$ game subsets $A$ such that $\Gamma[A]$ is isomorphic to $\Gamma[[1, n]]$. On the other hand, there are $2^n$ game subsets. For $n > 2$, $\phi(2n + 1) \leq 2n < 2^n$.

**Theorem 3.10.** For a finite group $G$ of order $2n+1$, the following are equivalent.

1. $|G^*|$ is a power of 2.
2. $|G^*|$ divides $2^n$.
3. The order of every element of $G^*$ is a power of 2.
(iv) For every game subset \( A \subset G \), and \( \xi \in G^* \), \( \xi(A) = A \) implies \( \xi = 1_G \).

(v) The group \( G^* \) acts freely on the set of game subsets of \( G \).

Proof. (ii) \( \Rightarrow \) (i): Obvious.

(i) \( \Rightarrow \) (iii): The order of each element divides the order of the group.

(iii) \( \Rightarrow \) (iv): If \( \xi(A) = A \) then \( \xi \) is an automorphism of \( \Gamma[A] \). Since the order of \( \xi \) is a power of 2, the order must be 1 by Proposition 1.8 i.e. \( \xi \) is the identity \( 1_G \).

(iv) \( \Rightarrow \) (v): This is the definition of a free action.

(v) \( \Rightarrow \) (ii): By Proposition 3.2 (a), \( |G^*| \) divides the cardinality of the set of game subsets which is \( 2^n \).

\[ \square \]

A prime of the form \( 2^m + 1 \) is called a Fermat prime. For such a prime, \( m \) itself must be a power of two. To see this, observe that if \( a > 1 \) is an odd divisor of \( m \) then \( x + 1 \) divides \( x^a + 1 \) and the quotient has integer coefficients. Hence, with \( x = 2^{m/a} \) we see that \( 2^{m/a} + 1 \) divides \( 2^m + 1 \). It follows that that \( F_k = 2^k + 1 \) are the only possibilities. Fermat discovered that for \( k = 0, 1, 2, 3, 4 \) these are indeed primes: 3, 5, 17, 257, 65537. See, e. g. [18] Section 2.5. However, these are the only Fermat primes which are known to exist.

Theorem 3.11. For the odd number \( 2n+1 > 1 \) the following conditions are equivalent.

(i) \( \phi(2n+1) \) is a power of 2.

(ii) \( \phi(2n+1) \) divides \( 2^n \).

(iii) The order of every element of \( \mathbb{Z}_{2n+1}^* \) is a power of 2.

(iv) For every game subset \( A \subset \mathbb{Z}_{2n+1} \), and \( a \in \mathbb{Z}_{2n+1}^* \), \( m_a(A) = A \) implies \( a = 1 \).

(v) The group \( \mathbb{Z}_{2n+1}^* \) acts freely on the set of game subsets of \( \mathbb{Z}_{2n+1} \).

(vi) \( 2n + 1 \) is a square-free product of Fermat primes.

Proof. The equivalence of (i)-(v) is the special case of Theorem 3.10 with \( G = \mathbb{Z}_{2n+1} \).

(i) \( \Leftrightarrow \) (vi): Write the prime factorization of \( 2n + 1 \) as \( \Pi_{a=1}^k p_a^{e_a} \) with \( e_a \geq 1 \). Each \( p_a \) is an odd prime and \( \phi(2n + 1) = \Pi_{a=1}^k p_a^{e_a-1}(p_a - 1) \). If \( e_a > 1 \) then \( p_a \mid \phi(2n + 1) \). Hence, \( \phi(2n + 1) \) can be a power of 2 only when \( e_a = 1 \) for all \( a \), i.e. \( \phi(2n + 1) \) is square-free. In that case, \( \phi(2n + 1) = \Pi_{a=1}^k (p_a - 1) \) and so \( \phi(2n + 1) \) is a power of 2 if and only if each \( p_a - 1 \) is a power of 2, i.e. each \( p_a \) is a Fermat prime.

\[ \square \]
If $2n + 1$ is not a square-free product of Fermat primes, then $\mathbb{Z}_{2n+1}^*$ does not act freely on the game subsets and so there exists a game subset $A \subset \mathbb{Z}_{2n+1}$ and $a \in \mathbb{Z}_{2n+1}^*$ with $a \neq 1$ such that $m_a \in \text{Aut}(\Gamma[A])$. However, we can obtain a sharper result.

**Theorem 3.12.** If $G$ is a finite group and $H$ is a non-trivial subgroup of $G^*$ with both $G$ and $H$ of odd order, then there exists a game subset $A \subset G$ such that $H \subset \text{Aut}(\Gamma[A])$.

*Proof.* The group $H$ acts on $G$ and for $i \in G$ we let $Hi$ denote $\{\xi(i) : \xi \in H\}$, the $H$ orbit of $i$ in $G$. Distinct orbits are disjoint. The identity element $e$ is a fixed point, i.e. $\{e\} = He$. If for any $\xi \in G^*$ and $i \neq e$ in $G$, then $\xi(i) = i^{-1}$ implies that $\xi$ has even order. For $i \in G$ has odd order and so $i \neq i^{-1}$. Furthermore, $\xi^2(i) = \xi(i^{-1}) = \xi(i)^{-1} = i$. Thus, $\xi^k(i) = i$ for $k$ even while $\xi^k(i) = i^{-1}$ for $k$ odd. Hence, $\xi$ has even order and so cannot be an element of $H$. Thus, for $i \neq e$ the orbits $Hi$ and $H(i^{-1})$ are disjoint. Furthermore, $j \to j^{-1}$ is a bijection from $Hi$ to $H(i^{-1})$ and so $|Hi| = |H(i^{-1})|$.

Let $i_1, \ldots, i_m \in G \setminus \{e\}$ so that $\{Hi_1 \cup H(i_1^{-1}), \ldots, Hi_m \cup H(i_m^{-1})\}$ is a partition of $G \setminus \{e\}$. Define $A = Hi_1 \cup \cdots \cup Hi_m$ to obtain a game subset which is invariant with respect to the $H$ action. By using $H(i_k^{-1})$ instead of $Hi_k$ for $k$ in an arbitrary subset of $\{1, \ldots, m\}$ we obtain the $2^m$ game subsets which are $H$ invariant.

\[\square\]

In particular, if $p$ is an odd prime which divides $|G^*|$ then there exists $\xi \in G^*$ of order $p$ by the first Sylow Theorem. Theorem 3.12 implies that there exist game subsets $A \subset G$ such that $\xi \in \text{Aut}(\Gamma[A])$. With $H$ the order $p$ cyclic group generated by $\xi$, each orbit $Hi$ has cardinality $p$ or 1. Since $\xi$ is not the identity, not all points are fixed and so at least some orbits have cardinality $p$. If all have cardinality $p$, then $H$ acts freely on $G \setminus \{e\}$ and on $A$. It then follows that $p$ divides $|A| = n$. That is, $p$ is a common factor of $n$ and $|G^*|$ (which is $\phi(2n + 1)$ when $G = \mathbb{Z}_{2n+1}$).

With $G = \mathbb{Z}_{2n+1}$, if $p$ is a prime such that $p^2|2n + 1$, then $p|\phi(2n+1)$, but $p \nmid n$ since $n$ and $2n + 1$ are relatively prime. With $2n + 1 = 21$, $n = 10$ and $\phi(2n + 1) = 12$ with no odd common factor. With $2n + 1 = 35$, $n = 17$ and $\phi(2n + 1) = 24$ which are relatively prime.

We will see below that if $2n + 1$ is composite, then there exists a game subset $A$ such that the inclusion $\mathbb{Z}_{2n+1} \subset \text{Aut}(\Gamma[A])$ is proper and so $\text{Aut}(\Gamma[A])$ is non-abelian. It follows that the only possible numbers $n$ for which it may happen that $\mathbb{Z}_{2n+1} = \text{Aut}(\Gamma[A])$, or, equivalently, that
\(\text{Aut}(\Gamma[A])\) is abelian, for every game subset \(A \subset \mathbb{Z}_{2n+1}\), are those with \(2n + 1\) a Fermat prime. This is trivially true for 3 and 5. We do not know the answer for 17 or the other known Fermat primes.

Using the argument of Theorem 3.12 we can construct an interesting class of examples.

**Example 3.13.** Let \(p\) be a prime number congruent to \(-1\) mod \(4\) so that \(p = 2n + 1\) with \(n\) odd and let \(k\) be an odd number so \(p^k = 2m + 1\) with \(m\) odd. Let \(G\) be the product group \((\mathbb{Z}_p)^k\). In particular, if \(k = 1\) then \(G = \mathbb{Z}_p\) and \(m = n\).

There is a group game \(\Gamma\) on \(G\) and a group game \(\Pi\) on \(\mathbb{Z}_m\) such that for every \(i \in G\) the restrictions of \(\Gamma\) to \(\Gamma(i)\) and to \(\Gamma^{-1}(i)\) are subgames of \(\Gamma\) which are isomorphic to \(\Pi\).

**Proof.** We use some elementary results from the theory of finite fields, see, e.g. [23]. By Theorem 1.2.5 of [23] there exists a finite field \(F\) of order \(p^k\) which is unique up to isomorphism. The additive group \(G\) of the field is a \(\mathbb{Z}_p\) vector space of dimension \(k\) and so we can identify it with \((\mathbb{Z}_p)^k\). Let \(F'\) be the multiplicative group of nonzero elements of \(F\) so that \(F'\) has order \(2m\).

For \(a \in F\) the quadratic equation \(x^2 = a\) has at most two roots in \(F\). Since the characteristic of the field is odd, \(a \neq -a\) if \(a \neq 0\). Hence, the map \(sq\) on \(F\) given by \(sq(a) = a^2\) restricts to a two-to-one map on \(F'\) and so its image on \(F'\) is a subgroup \(H\) of \(F'\) with order \(m\). If \(i \in H\) then the order \(o\) of \(i\) divides \(m\) and so is odd. Hence, the order of \(-i\) is \(2o\) and so \(-i \notin H\). It follows that \(H\) is a game subset of \(G\) and we let \(\Gamma = \Gamma[H]\).

Now \(H\) itself is invariant under the group \(H\). Hence, as in the proof of Theorem 3.12 the multiplications by elements of \(H\) are automorphisms of \(\Gamma\). If we let \(\Pi\) denote the restriction of \(\Gamma\) to \(H = \Gamma(0)\), then \(\Pi\) is a tournament on the multiplicative group \(H\) and the translation maps by elements of \(H\) on \(H\) (by multiplication) are automorphisms of \(\Pi\) because they are restrictions of the automorphisms of \(\Gamma\). It follows from Theorem 3.4 (a) that \(\Pi\) is a group game on \(H\).

Recall that \(\Gamma[-H]\) is the reverse game \(\Gamma[H]^{-1}\). The map \(i \mapsto -i\) is an isomorphism from \(\Gamma[H]\) to \(\Gamma[-H]\) which maps \(-H\) to \(H\). So it maps maps \(\Gamma|(-H) = \Gamma|(\Gamma^{-1}(0))\) isomorphically to \((\Gamma[H])^{-1} = \Pi^{-1}\). But \(\Pi\) is a group game on a commutative group and so it is isomorphic to its reversed game. Thus, \(\Gamma|((\Gamma^{-1}(0)))\) is isomorphic to \(\Pi\).

Translation in \(G\) (by addition) is an automorphism of \(\Gamma\) and so \(\ell_i\) restricts to an isomorphism from \(\Gamma|(\Gamma(0))\) to \(\Gamma|(\Gamma(i))\) and from \(\Gamma|((\Gamma^{-1}(0))\) to \(\Gamma|((\Gamma^{-1}(i))\). Hence, all of these restrictions are isomorphic to \(\Pi\).
Finally, by Theorem 1.2.8 of [23] the multiplicative group $F'$ is cyclic
and so the subgroup $H$ is cyclic. Thus, $H$ is isomorphic to the additive
group $\mathbb{Z}_m$ and we can identify $\Pi$ with a group game on $\mathbb{Z}_m$.

\[\square\]

Remark: If $\rho$ is a nontrivial field automorphism of $F$, then $\rho$
commutes with $sq$ and so leaves $H$ invariant. Hence, $\rho$ is an automorphism
of the group game $\Gamma[H]$. Observe that since $\rho$ is the identity on the sub-
field $\mathbb{Z}_p$, $\rho$ is neither a translation by an element of $G$ nor multiplication
by an element of $F'$.

Finally, we observe that the only group games which are reducible
are those on $\mathbb{Z}_{2^{n+1}}$ which are isomorphic to $\Gamma[Odd_n]$ or, equivalently,
iseomorphic to $\Gamma[[1,n]]$.

**Theorem 3.14.** (a) If $A \subset \mathbb{Z}_{2^{n+1}}$ is a game subset with $\Gamma[A]$ reducible,
then $A = ma(Odd_n)$ for a unique $a \in \mathbb{Z}_{2^{n+1}}$. Conversely, for $a \in \mathbb{Z}_{2^{n+1}}$,
a is the unique element of $\mathbb{Z}_{2^{n+1}}$ such that $\Gamma[ma(Odd_n)]$ is reducible via
$0 \to a$.

(b) If a group $G$ is not cyclic, then for no game subset $A$ of $G$ is
$\Gamma[A]$ reducible.

**Proof.** Assume $A$ is a game subset for a group $G$. If $\Gamma[A]$ is reducible
via $u \to v$, then it is reducible via $e \to u^{-1}v$.

Notice that $e \to a$ if and only if $a \in A$. The following are equivalent.

- $\Gamma[A]$ is reducible via $e \to a$.
- $a \in A$ and $A \cap aA = \emptyset$.
- $A^{-1} = aA$.

Observe that $c \in A \cap aA$ if and only if $e \to c$ and $a \to c$. By Proposition
2.2 (d), the set of such $c$ is empty if and only if $\Gamma[A]$ is reducible via
$\{e, a\}$. If $a \in A$ then $e \not\in aA$ and so $aA = A^{-1}$ if and only if $A$ and $aA$
are disjoint.

From Proposition 2.2 (f), it follows that $\{a \in A : \Gamma[A]$ is reducible
via $e \to a\}$ consists of at most one element.

(a): In this case $G = \mathbb{Z}_{2^{n+1}}$ and so $\Gamma[A]$ is reducible via $e \to a$ if and
only if $-A = a + A$.

The game $\Gamma[Odd_n]$ is reducible via $0 \to 1$ because $Odd_{n+1} = -Odd_n$.
For $a \in \mathbb{Z}_{2^{n+1}}^*$, $m_a$ is an isomorphism from $\Gamma[Odd_n]$ to $\Gamma[m_a(Odd_n)]$ and
so $\Gamma[m_a(Odd_n)]$ is reducible via $0 \to a$.

Now assume that $\Gamma[A]$ is reducible via $0 \to a$. Then it cannot happen
that $i, i + a \in A$ and if $i, i + a$ were in $-A$ then $-(i + a), -i = -(i + a) + a \in A$.
So it cannot happen that $i, i + a \in -A$. 

If \( a \in \mathbb{Z}_{2n+1}^* \) then \( a \) is a generator of the additive group \( \mathbb{Z}_{2n+1} \) and so \( m_a \) is a permutation of \( \mathbb{Z}_{2n+1} \) mapping \( 0, 1, 2, \ldots, 2n \) to \( 0, a, 2a, \ldots, 2na \). Since \( a \in A \), \((2k-1)a \in A, 2ka \in -A \) for \( a = 1, \ldots, n \). That is, \( A = m_a(Odd_n) \).

If \( a \not\in \mathbb{Z}_{2n+1}^* \) then the order of \( a \) is a proper divisor of \( 2n + 1 \) and the cyclic subgroup generated by \( a \) is a proper subgroup of \( \mathbb{Z}_{2n+1} \) and the result follows by the proof in part (b).

(b): Now assume that \( G \) is a general group of odd order, and that the cyclic subgroup \( H \) generated by \( a \) is a proper subgroup of \( G \). This applies to all \( a \in G \) when \( G \) is not cyclic. Let \( 2k + 1 = |H| \) and \( 2n + 1 = |G| \).

In general, if \( \Gamma[A] \) is reducible via \( 0 \to a \) then \( aA \cap A = \emptyset \). Consequently, \( b \in A \) implies \( ab \not\in A \). So for any \( b \in G \), no successive members of the sequence \( b, ab, a^2b, \ldots \) lie in \( A \). This implies that \( |A \cap Hb| \) is at most \( k \). The number of right cosets \( Hb \) is \( (2n + 1)/(2k + 1) \). Since \( k < n \), this would imply that \( n > [(2n + 1)/(2k + 1)] \cdot k > |A| \) which contradicts \( |A| = n \).

\[ \square \]

Notice that \([1, n] + n = [n + 1, 2n] \subset \mathbb{Z}_{2n+1} \) so \( \Gamma[[1, n]] \) is reducible via \( 0 \to n \). Thus, we see again that \([1, n] = m_n(Odd_n) \).

4. Inverting Cycles

While we are most interested in games, the results of this section are equally applicable for more general tournaments and so we will consider the more general case.

Assume that \( \Pi \) and \( \Gamma \) are tournaments on the set \( I \). Let \( \rho \in S(I) \), i.e. a permutation on \( I \). Recall that \( \bar{\rho} = \rho \times \rho \) is the product permutation on \( I \times I \).

We define

\[
\Delta(\rho, \Pi, \Gamma) = \Pi \cap (\bar{\rho})^{-1}(\Gamma^{-1}) = \{(i, j) \in \Pi : (\rho(j), \rho(i)) \in \Gamma\}.
\]

That is, \( \Delta(\rho, \Pi, \Gamma) \) consists of those edges of \( \Pi \) which are reversed by \( \bar{\rho} \). For the special case when \( \rho = 1_I \) we will write

\[
\Delta(\Pi, \Gamma) = \Delta(1_I, \Pi, \Gamma) = \{(i, j) \in \Pi : (j, i) \in \Gamma\} = \Delta(\Gamma, \Pi)^{-1}.
\]
If $\rho_{12} \in S(I)$ is an isomorphism from $\Gamma_1$ to $\Gamma_2$ and $\rho \in S(I)$, then
\begin{equation}
\Delta(\rho, \Pi, \Gamma_1) = \Delta(\rho_{12} \circ \rho, \Pi, \Gamma_2).
\end{equation}

We will say that $\rho \in S(I)$ preserves scores, or is a score-preserving permutation, from $\Pi$ to $\Gamma$ if $|\Gamma(\rho(i))| = |\Pi(i)|$ for all $i \in I$. It then follows that
\[|\Gamma^{-1}(\rho(i))| = |I| - 1 - |\Gamma(\rho(i))| = |I| - 1 - |\Pi(i)| = |\Pi^{-1}(i)|.\]
We will say that $\Pi$ and $\Gamma$ have the same score vector when the identity on $I$ preserves scores from $\Pi$ to $\Gamma$, i.e. $|\Gamma(i)| = |\Pi(i)|$ for all $i \in I$. Between two games on $I$ any permutation preserves scores.

**Proposition 4.1.** Assume that $\Pi$ and $\Gamma$ are tournaments on the set $I$ and $\rho \in S(I)$. The permutation $\rho$ preserves scores from $\Pi$ to $\Gamma$ if and only if $\Delta(\rho, \Pi, \Gamma)$ is Eulerian.

**Proof.** Let $\Delta = \Delta(\rho, \Pi, \Gamma)$. Thus, $\Delta$ is a subgraph of $\Pi$ and $\bar{\rho}(\Delta^{-1})$ is a subgraph of $\Gamma$.

The permutation $\rho$ is an isomorphism from $\Pi \setminus \Delta$ to $\Gamma \setminus \bar{\rho}(\Delta^{-1})$ and is an isomorphism from $\Delta^{-1}$ to $\bar{\rho}(\Delta^{-1}) \subset \Gamma$.

\begin{align}
\Pi(i) &= (\Pi \setminus \Delta)(i) \cup \Delta(i), \\
\Gamma(\rho(i)) &= \rho[(\Pi \setminus \Delta)(i)] \cup \rho[\Delta^{-1}(i)].
\end{align}

The unions are disjoint and the permutation $\rho$ preserves cardinality. So it follows that $|\Pi(i)| = |\Gamma(\rho(i))|$ if and only if $|\Delta(i)| = |\Delta^{-1}(i)|$. 

If $\Delta$ is a subgraph of $\Pi$ then we define $\Pi$ with $\Delta$ reversed by
\begin{equation}
\Pi/\Delta = (\Pi \setminus \Delta) \cup \Delta^{-1}.
\end{equation}

If $\Delta_1$ and $\Delta_2$ are subgraphs of $\Pi$ then clearly
\begin{equation}
\Delta_1 \cap \Delta_2 = \emptyset \implies \Pi/(\Delta_1 \cup \Delta_2) = (\Pi/\Delta_1)/\Delta_2.
\end{equation}
Notice that if, as above, $\Delta_1$ and $\Delta_2$ are disjoint, then $\Delta_2$ is a subgraph of $\Pi/\Delta_1$. Recall that disjoint subgraphs may have vertices in common.

**Proposition 4.2.** Assume that $\Pi$ and $\Gamma$ are tournaments on $I$. If $\Delta$ is a subgraph of $\Pi$, then $\Pi/\Delta = \Gamma$ if and only if $\Delta = \Delta(\Pi, \Gamma)$. In particular,
\begin{equation}
\Delta = \Delta(\Pi, \Pi/\Delta)
\end{equation}
The tournament $\Pi/\Delta$ has the same score vector as that of $\Pi$ if and only if $\Delta$ is Eulerian.
Proof. The first part is obvious. The rest follows from Proposition 4.1.

Corollary 4.3. If $\Pi$ is a game on $I$, $\Delta \subset \Pi$ is Eulerian and $u, v \in I$, then any two of the following conditions implies the third.

(i) The game $\Pi$ is reducible via $\{u, v\}$.
(ii) The game $\Pi/\Delta$ is reducible via $\{u, v\}$
(iii) The graph $\Delta|(I \setminus \{u, v\})$ is Eulerian.

Proof. It is clear that

\[(4.8) \quad (\Pi/\Delta)|(I \setminus \{u, v\}) = (\Pi|(I \setminus \{u, v\}))/((\Delta|(I \setminus \{u, v\}))
\]

Condition (i) is equivalent to the assumption that $\Pi|(I \setminus \{u, v\})$ is a game.

By Equation (4.8), condition (ii) is equivalent to the assumption that $(\Pi|(I \setminus \{u, v\}))/((\Delta|(I \setminus \{u, v\}))$ is a game.

So assuming (i), (ii) $\leftrightarrow$ (iii) by Proposition 4.2. Similarly, assuming (ii), (i) $\leftrightarrow$ (iii) by applying the previous result, replacing $\Pi$ by $\Pi/\Delta$ and $\Delta$ by $\Delta^{-1}$.

It follows from Proposition 4.2 that if $\Pi$ and $\Gamma$ are any two tournaments with the same score vector on $I$, then $\Gamma$ is equal to $\Pi$ with $\Delta$ reversed for $\Delta$ the Eulerian subgraph $\Delta(\Pi, \Gamma)$.

Corollary 4.4. If $\Pi$ is a tournament on $I$, then the number of tournaments on $I$ with the same score vector as $\Pi$ is exactly the number of Eulerian subgraphs of $\Pi$. In particular, the number of Eulerian subgraphs of a tournament depends only on the score vector.

Proof. If $\Gamma$ is a tournament on $I$, then by Proposition 4.2 $\Gamma = \Pi/\Delta$ with $\Delta = \Delta(\Pi, \Gamma)$ and by Proposition 4.1 $\Gamma$ has the same score vector if and only if $\Delta$ is Eulerian. Hence, the set of tournaments $\Gamma$ with the same score vector is in a one-to-one correspondence with the set of Eulerian subgraphs of $\Pi$ via $\Gamma \leftrightarrow \Delta(\Pi, \Gamma)$.

The following is a theorem of Ryser [26], see also [22] Theorem 35 and [7].

Theorem 4.5. Assume that $\Pi$ and $\Gamma$ are tournaments with the same score on $I$. There exists a finite sequence $\Pi_1, \ldots, \Pi_k$ of tournaments on $I$ with $\Pi_1 = \Pi$ and $\Pi_k = \Gamma$ and such that for $p = 1, \ldots, k - 1$, $\Pi_{p+1}$ is
with some 3-cycle reversed. In particular, all of these tournaments have the same score.

If $\Gamma$ is obtained from $\Pi$ by reversing a single cycle of length $\ell$ then a sequence can be chosen with $k = \ell - 2$.

Proof. In one step we can get from $\Pi$ to $\Gamma$ with $\Gamma$ equal to $\Pi$ with the Eulerian subgraph $\Delta(\Pi, \Gamma)$ reversed.

By Theorem 1.5, $\Delta(\Pi, \Gamma)$ is a disjoint union of cycles. From (4.6) and induction it follows that there exists a finite sequence $\Pi_1, \ldots, \Pi_k$ of games on $I$ with $\Pi_1 = \Pi$ and $\Pi_k = \Gamma$ and such that for $p = 1, \ldots, k - 1$ $\Pi_{p+1}$ is $\Pi_p$ with some cycle reversed.

Thus, we are reduced to the case when $\Gamma = \Pi/\Delta$ and $\Delta$ is a cycle $\langle i_1, \ldots, i_\ell \rangle$ with $\ell \geq 3$. We obtain the result by induction on the length $\ell$ of the cycle. If $\ell = 3$, then we get from $\Pi$ to $\Gamma$ immediately by reversing a single 3-cycle. Now assume that $\ell > 3$ and assume that the result holds for shorter cycles.

Case 1 ($i_1 \rightarrow i_3$ in $\Pi$): In this case, $\langle i_1, i_3, \ldots, i_\ell \rangle$ is an $\ell - 1$ cycle in $\Pi$ and so we can get from $\Pi$ to $\tilde{\Gamma} = \Pi/\langle i_1, i_3, \ldots, i_\ell \rangle$ via a sequence of $\ell - 3$ 3-cycles by inductive hypothesis.

Observe that $i_3 \rightarrow i_1$ in $\tilde{\Gamma}$ and so $\langle i_1, i_2, i_3 \rangle$ is a 3-cycle in $\tilde{\Gamma}$. Furthermore, $\Gamma = \Pi/\langle i_1, \ldots, i_\ell \rangle = \tilde{\Gamma}/\langle i_1, i_2, i_3 \rangle$ because the edge between $i_1$ and $i_3$ has been reversed twice. Thus, extending the $\tilde{\Gamma}$ sequence by one we obtain the sequence from $\Pi$ to $\Gamma$.

Case 2 ($i_3 \rightarrow i_1$ in $\Pi$): We proceed as before reversing the order of operations. This time $\langle i_1, i_2, i_3 \rangle$ is a 3-cycle in $\Pi$ and $\langle i_1, i_3, \ldots, i_\ell \rangle$ is an $\ell - 1$ cycle in $\tilde{\Gamma} = \Pi/\langle i_1, i_3, \ldots, i_\ell \rangle$. Now $\Gamma = \tilde{\Gamma}/\langle i_1, i_3, \ldots, i_\ell \rangle$ and so by inductive hypothesis we can get from $\tilde{\Gamma}$ to $\Gamma$ via a sequence of $\ell - 3$ 3-cycles.

\[\square\]

Proposition 4.6. Let $\Pi$ and $\Gamma$ be tournaments on $I$ and $\rho \in S(I)$. If $\Delta$ is a subgraph of $\Pi$, then $\rho$ is an isomorphism from $\Pi/\Delta$ to $\Gamma$ if and only if $\Delta = \Delta(\rho, \Pi, \Gamma)$.

Proof. If $\Delta = \Delta(\rho, \Pi, \Gamma)$, then it is clear that $\rho$ is an isomorphism from $\Pi/\Delta$ to $\Gamma$.

On the other hand, if $\rho$ is an isomorphism from $\Pi/\Delta$ to $\Gamma$, then (1.3) and (4.7) imply that.

\[\Delta = \Delta(\Pi/\Delta) = \Delta(\rho \circ 1_I, \Pi, \Gamma) = \Delta(\rho, \Pi, \Gamma).\]
A decomposition for an Eulerian digraph Π is a pairwise disjoint set of cycles with union Π. A maximum decomposition is a decomposition having maximum cardinality. We call the cardinality of a maximum decomposition the span of Π, denoted σ(Π). The complement in Π of the union of any pairwise disjoint collection of cycles in Π is Eulerian by Lemma 1.4 and so by Theorem 1.5 any such collection can be extended to a decomposition. Thus, the span is the maximum size of such a collection. Furthermore, such a collection has cardinality equal to the span if and only if it is a maximum decomposition.

The balance invariant of Eulerian digraph Π is defined to be

\[ \beta(\Pi) = |\Pi| - 2\sigma(\Pi). \]

That is, \( \beta(\Pi) \) is the number of edges minus twice the size of the maximum decomposition. Observe that if \( \{C_1, C_2, \ldots, C_p\} \) is a maximum decomposition of Π, then

\[ \beta(\Pi) = \sum_{r=1}^{p} \beta(C_r), \]

because if \( C \) is a cycle, \( \beta(C) = |C| - 2 \). Also, it is clear that if \( \{C_1, C_2, \ldots, C_p\} \) is a decomposition of Π then \( \{C_1^{-1}, C_2^{-1}, \ldots, C_p^{-1}\} \) is a decomposition of \( \Pi^{-1} \) and so \( \beta(\Pi) = \beta(\Pi^{-1}) \).

The balance invariant and the following result were shown to me by my colleague Pat Hooper.

**Theorem 4.7.** Let Π and Γ be tournaments with the same score vector on a set I.

(a) If a tournament Γ' is obtained by reversing a 3-cycle in Γ, then

\[ |\beta(\Delta(\Gamma', \Pi)) - \beta(\Delta(\Gamma, \Pi))| = 1. \]

(b) If Π ≠ Γ, then there exists a game Γ' obtained by reversing a 3-cycle in Γ, such that

\[ \beta(\Delta(\Gamma', \Pi)) = \beta(\Delta(\Gamma, \Pi)) - 1. \]

(c) Assume that C is a 3-cycle contained in \( \Delta(\Gamma, \Pi) \) but which is not contained in any maximum decomposition of \( \Delta(\Gamma, \Pi) \). If the tournament Γ' is obtained by reversing the 3-cycle C in Γ, then

\[ \beta(\Delta(\Gamma', \Pi)) = \beta(\Delta(\Gamma, \Pi)) + 1. \]

**Proof.** Let the reverse of the given 3-cycle be \( \langle i_1, i_2, i_3 \rangle \) so that \( \langle i_1, i_2, i_3 \rangle \) is contained in Γ'. Let \( \Delta = \Delta(\Gamma, \Pi) \) and \( \Delta' = \Delta(\Gamma', \Pi) \).
(a): **Case 1**: The 3-cycle is disjoint from $\Delta$ and so its reverse is contained in $\Delta'$.

Clearly, $|\Delta'| = |\Delta| + 3$. Given a maximum decomposition for $\Delta$ we can adjoin the 3-cycle to obtain a decomposition for $\Delta'$. Hence,

$$\sigma(\Delta') \geq \sigma(\Delta) + 1. \quad (4.15)$$

Now take a maximum decomposition for $\Delta'$. If the 3-cycle is one of the cycles of this decomposition then by removing it we obtain a decomposition for $\Gamma$ and so see that $\sigma(\Delta) \geq \sigma(\Delta') - 1$ and so we obtain equality in (4.15). Hence,

$$\beta(\Delta') = |\Delta'| - 2\sigma(\Delta') = |\Delta| + 3 - 2(\sigma(\Delta) + 1) = \beta(\Delta) + 1. \quad (4.16)$$

Thus, (4.12) follows.

Next suppose that $(i_1, i_2)$ is contained in one cycle of the $\Delta'$ decomposition and that $(i_2, i_3)$ and $(i_3, i_1)$ are contained in another. So we can write these cycles $\langle i_2, r_1, \ldots, r_k, i_1 \rangle$ and $\langle i_1, s_1, \ldots, s_\ell, i_2, i_3 \rangle$. In particular, $i_1$ and $i_2$ are vertices of $\Delta$ as well as $\Delta'$. Thus, $[i_2, r_1, \ldots, r_k, i_1]$ and $[i_1, s_1, \ldots, s_\ell, i_2]$ are simple paths in $\Delta$ with no edges in common. It follows from Theorem 1.6 that the closed path $[i_1, s_1, \ldots, s_\ell, i_2, r_1, \ldots, r_k, i_1]$ is an Eulerian subgraph of $\Delta$. Writing it as a disjoint union of cycles and using the remaining cycles of the $\Delta'$ decomposition, we again obtain $\sigma(\Delta) \geq \sigma(\Delta') - 1$ and so (4.12) follows as before.

Now suppose that $(i_1, i_2), (i_2, i_3)$ and $(i_3, i_1)$ are each contained in separate cycles of the $\Delta'$ decomposition. We can write these cycles $\langle i_2, r_1, \ldots, r_k, i_1 \rangle$, $\langle i_1, s_1, \ldots, s_\ell, i_3 \rangle$ and $\langle i_3, t_1, \ldots, t_p, i_2 \rangle$. So $i_1, i_2$ and $i_3$ are vertices of $\Delta$. Again we concatenate simple paths with no edges in common to obtain the closed path

$$[i_2, r_1, \ldots, r_k, i_1, s_1, \ldots, s_\ell, i_3, t_1, \ldots, t_p, i_2].$$

This time we obtain $\sigma(\Delta) \geq \sigma(\Delta') - 2$. Thus, $\sigma(\Delta) + 1 \leq \sigma(\Delta') \leq \sigma(\Delta) + 2$. If $\sigma(\Delta) + 1 = \sigma(\Delta')$ then (4.16) holds and (4.12) follows. If, instead $\sigma(\Delta') = \sigma(\Delta) + 2$ then we obtain

$$\beta(\Delta') = |\Delta'| - 2\sigma(\Delta') = |\Delta| + 3 - 2(\sigma(\Delta) + 2) = \beta(\Delta) - 1, \quad (4.17)$$

which still implies (4.12). This case does occur if the three vertices lie on a cycle in the maximum decomposition of $\Delta$. The length of such a cycle would have to be at least 6.

**Case 2**: The 3-cycle is contained in $\Delta$ and so its reverse is disjoint from $\Delta'$.

This follows from Case 1 by reversing the roles of $\Gamma$ and $\Gamma'$. 
**Case 3:** A single edge of the 3-cycle is contained in $\Delta$ and so the reverse of the other two edges is contained in $\Delta'$.

This time $|\Delta'| = |\Delta| + 1$.

We may suppose that the edge $(i_2, i_1) \in \Delta$. If $\langle i_1, r_1, \ldots, r_k, i_2 \rangle$ is a cycle of a maximum decomposition for $\Delta$ then $\langle i_1, r_1, \ldots, r_k, i_2, i_3 \rangle$ is a cycle in $\Delta'$ and together with the remaining cycles from the $\Delta$ decomposition yields a decomposition of $\Delta'$. Hence, $\sigma(\Delta') \geq \sigma(\Delta)$.

Now take a maximum decomposition for $\Delta'$. If the edges $(i_2, i_3)$ and $(i_3, i_1)$ are contained in a single cycle, $\langle i_1, r_1, \ldots, r_k, i_2, i_3 \rangle$, then reversing the above procedure we obtain a decomposition for $\Delta$ containing $\langle i_1, r_1, \ldots, r_k, i_2 \rangle$. Hence, $\beta(\Delta') = \beta(\Delta) + 1$.

Suppose instead that the edges $(i_2, i_3)$ and $(i_3, i_1)$ are contained separate cycles $\langle i_1, r_1, \ldots, r_k, i_3 \rangle$ and $\langle i_3, s_1, \ldots, s_\ell, i_2 \rangle$ then $[i_1, r_1, \ldots, r_k, i_3, s_1, \ldots, s_\ell, i_2, i_1]$ is a closed path in $\Delta$ with distinct edges and so by Theorem 1.6 again it is a union of cycles. Adjoining the remaining $\Delta'$ cycles we see that $\sigma(\Delta) \geq \sigma(\Delta') - 1$. That is, $\sigma(\Delta) \leq \sigma(\Delta') \leq \sigma(\Delta) + 1$. If $\sigma(\Delta) = \sigma(\Delta')$ then, as before, $\beta(\Delta') = \beta(\Delta) + 1$. If $\sigma(\Delta') = \sigma(\Delta) + 1$, then $\beta(\Delta') = \beta(\Delta) - 1$.

**Case 4:** Two edges of the 3-cycle are contained in $\Delta$ and so the reverse of the other edge is contained in $\Delta'$.

This follows from Case 3 by reversing the roles of $\Gamma$ and $\Gamma'$.

(b) $\Delta$ is nonempty since $\Pi \neq \Gamma$. Among the maximum decompositions for $\Delta$ we choose one such that the shortest cycle has the smallest length possible. We label this shortest cycle $\langle i_1, i_2, \ldots, i_k \rangle$. Thus, $k$ is assumed to be the minimum length of any cycle in $\Delta$ which occurs in a maximum decomposition.

**Claim:** No $(i_p, i_q)$ with $q \neq p \pm 1 \mod k$ lies in $\Delta$.

**Proof.** If it did, then by relabeling we can assume that $p = 1$ and $2 < q < k$. In particular, $k \geq 4$. Then the cycle in the decomposition which contains $(i_1, i_q)$ is of the form $\langle i_q, r_1, \ldots, r_\ell, i_1 \rangle$ with $\ell \geq k - 2 \geq 2$. The union of these two cycles is the same as the union of the cycles $\langle r_1, \ldots, r_\ell, i_1, i_2, \ldots, i_q \rangle$ and $\langle i_1, i_q, i_{q+1}, \ldots, i_k \rangle$. Replacing the initial two cycles by these two we obtain a decomposition of $\Delta$ with cardinality $\sigma(\Delta)$ and so is a maximum decomposition. Furthermore, it contains a cycle of length less than $k$, contradicting the minimality of $k$. \qed
Case 1 \([k = 3]\)
Reverse this 3-cycle to obtain \(\Gamma'\). We see that \(|\Delta'| = |\Delta| - 3\).
The remaining cycles provide a decomposition of \(\Delta'\). Hence, \(\sigma(\Delta') \geq\)
\(\sigma(\Delta) - 1\). It follows that \(\beta(\Delta') \leq \beta(\Delta) - 1\). From \([4,12]\) equality holds.

Case 2: \([k \geq 4 \text{ and } (i_3, i_1) \in \Gamma]\).
From the Claim \((i_3, i_1) \in \Gamma \setminus \Delta\).
Reverse the 3-cycle \(\langle i_1, i_2, i_3 \rangle\). This time \(|\Delta'\| = |\Delta| - 1\).
Now \(\langle i_3, \ldots, i_k, i_1 \rangle\) is a cycle of \(\Delta'\) which, together with the remaining \(\Delta\)
cycles, provides a decomposition for \(\Delta'\). Thus, \(\sigma(\Delta') \geq \sigma(\Delta)\) and so
\(\beta(\Delta') \leq \beta(\Delta) - 1\). Again, from \([4,12]\) equality holds.
In general, by relabeling, Case 2 applies whenever for some \(p = 1, \ldots, k\), \((i_{p+2}, i_p) \in \Gamma\) with \(p + 2\) reduced mod \(k\).
In particular, if \(k = 4\) and \((i_1, i_3) \in \Gamma\) then the cycle is \(\langle i_3, i_4, i_1, i_2 \rangle\) with \((i_1, i_3) \in \Gamma\). Thus, Case 2 applies when \(k = 4\).

Case 3: \([k \geq 5, \text{ and } (i_p, i_{p+2}) \in \Gamma \text{ for } p = 1, \ldots, k \text{ (reducing the indices mod } k)\)]
Observe that, with respect to \(\Gamma\), \(i_5\) is an output for \(i_3\) and \(i_1\) is an input for \(i_3\). So there will exist \(q\) with \(5 \leq q \leq k\) such that \(i_q\) is
an output for \(i_3\) and \(i_{q+1}\) is an input for \(i_3\). By the above remarks
\((i_3, i_q), (i_{q+1}, i_3) \in \Gamma \setminus \Delta\).
Reverse the 3-cycle \(\langle i_3, i_q, i_{q+1} \rangle\). \(|\Delta'| = |\Delta| + 1\). On the other hand,
\(\langle i_3, i_4, \ldots, i_q \rangle\) and \(\langle i_{q+1}, \ldots, i_k, i_1, i_2, i_3 \rangle\) are disjoint cycles of \(\Delta'\) which,
together with the remaining \(\Delta\) cycles, form a decomposition of \(\Delta'\).
Hence, \(\sigma(\Delta') \geq \sigma(\Delta) + 1\). Thus, \(\beta(\Delta') \leq \beta(\Delta) - 1\). Again equality holds.

\(c\) Since \(C\) is contained in \(\Delta(\Gamma, \Pi)\) it follows that \(\Delta(\Gamma', \Pi) = \Delta(\Gamma, \Pi) \setminus C\).
Hence, \(|\Delta(\Gamma', \Pi)| = |\Delta(\Gamma, \Pi)| - 3\). On the other hand, the span
\(\sigma(\Delta(\Gamma', \Pi)) \leq \sigma(\Delta(\Gamma, \Pi)) - 2\). For if we had a decomposition of size at
least \(\sigma(\Delta(\Gamma, \Pi)) - 1\) then adjoining \(C\) we would obtain a decomposition of \(\Gamma\),
containing \(C\) and of size at least \(\sigma(\Delta(\Gamma, \Pi))\) which would have to
be a maximum decomposition, contrary to hypothesis. It thus follows
that \(\beta(\Delta(\Gamma', \Pi)) \geq \beta(\Delta(\Gamma, \Pi)) + 1\) and so equality holds by part \((a)\).

Remark: For a game \(\Pi\) it follows from the Claim in \((b)\) that the
shortest cycle which occurs in a maximum decomposition has length 3.
On the other hand, it can happen that an Eulerian digraph \(\Delta\) contains
a unique 3-cycle and it does not occur in a maximum decomposition. For example, let
\[
\Delta = \langle 0, 1, \ldots, 8 \rangle \cup \langle 6, 3, 0 \rangle = \\
\langle 0, 1, 2, 3 \rangle \cup \langle 3, 4, 5, 6 \rangle \cup \langle 6, 7, 8, 0 \rangle.
\]
(4.18)

From Theorem 4.7 we obtain

**Corollary 4.8.** If Π and Γ are two tournaments on I with the same score vector, then the minimum number of 3-cycle reversal steps needed to get from Γ to Π is \( \beta(\Delta(\Gamma, \Pi)) \).

\[\square\]

We also have

**Corollary 4.9.** If Π and Γ are two tournaments on I with the same score vector, and Π is obtained from Γ by reversing \( k \) 3-cycles, then \( |\Delta(\Gamma, \Pi)| \) and \( \beta(\Delta(\Gamma, \Pi)) \) are both congruent to \( k \) modulo 2.

**Proof.** Obviously, \( |\Delta(\Gamma, \Pi)| \) is congruent to \( \beta(\Delta(\Gamma, \Pi)) \) mod 2 and it is easy to check directly that reversing a 3-cycle changes \( |\Delta(\Gamma, \Pi)| \) by adding or subtracting either 1 or 3. Alternatively, the congruence result follows from (4.13).

\[\square\]

A tournament Γ has a decomposition into cycles if and only if it is Eulerian and so is a game. If the decomposition consists entirely of 3-cycles then the decomposition is clearly maximum.

If I is a set of size \( p \), then a set of Steiner triples for I is a set of three element subsets, called triples, such that each pair of elements is contained in exactly one triple. That is, the \( p(p-1)/2 \) pairs are partitioned into \( p(p-1)/6 \) triples. For each triple \( \{i, j, k\} \) there are two possible orientations defining 3-cycles \( \langle i, j, k \rangle \) or \( \langle k, j, i \rangle \). Choosing the orientations arbitrarily leads to a game and so we obtain \( 2^{p(p-1)/6} \) games on I. Since these are games, the size \( p \) must be odd with \( p(p-1) \) divisible by 6. Thus, \( p \) must be congruent to 1 or 3 modulo 6. Equivalently, if \( p = 2n + 1 \), then \( n \) is congruent to 0 or 1 mod 3 and the \( n(2n+1) \) edges are partitioned into \( n(2n+1)/3 \) triples. The question of which sizes \( p \) admit sets of such triples was raised in 1833 and was solved in 1847 by Rev. T. P. Kirkman \[19\] who showed that these congruence conditions are sufficient as well as necessary. Explicit constructions are given in \[6, 29\] and \[30\]. The latter shows that if \( n \) is congruent to 2 modulo 3, then there exist games of size \( 2n+1 \) with a decomposition consisting of \( \lfloor n(2n+1) - 4 \rfloor / 3 \) 3-cycles and one 4-cycle.
We will call a game a Steiner game when it admits a decomposition consisting of 3-cycles.

5. Interchange Graphs

Recall that an undirected graph on a finite set $X$ is represented by a symmetric relation $R$ on $X$, disjoint from the diagonal. That is, $R = R^{-1}$ and $R \cap 1_X = \emptyset$. There is an edge between $x$ and $y$ when $(x, y), (y, x) \in R$. The distance $d(x, y)$ between vertices $x, y$ of the graph is the smallest $n$ such that there is a path $[x = x_0, \ldots, x_n = y]$. We regard $x = x_0 = x$ as a path of length zero so that $d(x, x) = 0$. A path $[x = x_0, \ldots, x_n = y]$ which achieves this minimum length is called a geodesic connecting $x$ and $y$. It then follows for $0 \leq i \leq j \leq n$ that $[x_i, \ldots, x_j]$ is a geodesic connecting $x_i$ and $x_j$ and the distance $d(x_i, x_j) = j - i$. We define the distance $d(x, y)$ to be infinite if no such path exists. Thus, the graph is connected when every distance is finite. If $d(x, y)$ is finite, then $z \in X$ lies on a geodesic between $x$ and $y$ if and only if $d(x, z) + d(z, y)$.

Given a set $I$ with $|I| = p$ and a score vector $s$, Brualdi and Li [7] define the undirected interchange graph $R$ on the set of tournaments on $I$ with score vector $s$. The tournaments $\Gamma$ and $\Pi$ are connected by an edge in the graph when each can be obtained from the other by reversing a single 3-cycle. That is, when $\Delta(\Gamma, \Pi) = \Delta(\Pi, \Gamma)^{-1}$ is a 3-cycle. By Theorem 4.5 the graph is connected. Since the number of 3-cycles in a tournament depends only on the score vector, see (2.3), the interchange graph is regular with $|R(\Gamma)|$ the number of 3-cycles in $\Gamma$.

By Corollary 4.9 the following conditions are equivalent for tournaments $\Gamma$ and $\Pi$ with the same score vector.

- There exists a path of even length from $\Gamma$ to $\Pi$
- Every path from $\Gamma$ to $\Pi$ has even length.
- The distance $d(\Gamma, \Pi)$ is even.
- $|\Delta(\Gamma, \Pi)|$ is even.
- $\beta(\Delta(\Gamma, \Pi))$ is even.

We then say that $\Gamma$ and $\Pi$ have the same parity. Clearly, an edge in the graph always connects tournaments of opposite parity. Using the two parity classes, we see, as was observed by Brualdi and Li, that the graph is a bipartite graph.
The score vector partitions $I$ into score value subsets. The group of score-preserving permutations is just the product of the permutation groups of each score value subset. In particular, every score-preserving permutation is a product of disjoint score-preserving cycles and of score-preserving transpositions. If $\rho$ is a score-preserving permutation then $\bar{\rho}(\Gamma)$ is another tournament with the same score vector. Recall that $\bar{\rho} = \rho \times \rho$ on $I \times I$. Thus, the score-preserving permutations act on the interchange graph.

**Theorem 5.1.** If $\rho$ is a permutation preserving the score vector of a tournament $\Gamma$ then the tournaments $\Gamma$ and $\bar{\rho}(\Gamma)$ have the same parity if and only if $\rho$ is an even permutation, i.e. a product of an even number of transpositions.

**Proof.** It suffices to show that if $\rho$ is a score-preserving transposition then $\Gamma$ and $\bar{\Gamma}$ have opposite parity.

Suppose that the transposition interchanges vertices $i$ and $j$ with $i \to j$ in $\Gamma$. Let $\Delta = \Delta(\Gamma, \bar{\rho}(\Gamma))$. The edge $(i,j) \in \Delta$ and if $r$ is another vertex which is an input for one of $i$ and $j$ and an output for the other, the the edges between $r$ and both $i$ and $j$ lie in $\Delta$ and these are the only edges in $\Delta$. Thus, $\Delta$ is the union of the 3-cycles $\langle i, j, s \rangle$ and the straddles with $i \to r \to j$. The inputs to $j$ in $\Delta$ are $i$ and the straddle vertices $r$. The outputs of $j$ in $\Delta$ are the cycle vertices $s$. Because $\Delta$ is Eulerian, it follows that if $(i,j)$ is contained in $k+1$ 3-cycles then there are $k$ straddle vertices $r$. Each decomposition for $\Delta$ consists of one 3-cycle and $k$ 4-cycles of the form $\langle j, s, i, r \rangle$. Hence, $\beta(\Delta) = 2k + 1$. Since this is odd, $\Gamma$ and $\bar{\rho}(\Gamma)$ have opposite parity.

It is easy to check that reversing a 3-cycle takes a strong tournament to a strong tournament. In fact, whether the tournaments associated with $s$ are strong or not is detectable by inequalities on the terms of $s$, see, e.g. Theorem 11.13 of [17], or Theorem 9 of [16].

Chen, Chang and Wang [10] studied the interchange graph of the tournaments on $\{0, \ldots, p-1\}$ with score vector $s = (1,1,2,3,\ldots,p-3,p-2,p-2)$. From (2.3) it follows that each game contains exactly $p-2$ 3-cycles. They proved that the interchange graph is a hypercube of dimension $p-2$.

A hypercube of dimension $\ell$ is a graph on the set $\{0,1\}^\ell$ with $\ell$-tuples $x$ and $y$ connected by an edge if they differ in exactly one place. If $x$ and $y$ differ in exactly $k$ places then the distance $d(x,y) = k$ and so the diameter of the hypercube is $\ell$. There are exactly $k!$ geodesics connecting $x$ to $y$ when $d(x,y) = k$, each obtained by choosing an
ordering of the places on which the switches are successively made. In general, this is a lower bound for the number of geodesics.

**Theorem 5.2.** If $\Gamma$ and $\Pi$ are two tournaments with the same score vector and $d(\Gamma, \Pi) = k$ then there are at least $k!$ geodesics connecting $\Gamma$ and $\Pi$.

**Proof.** What we must show is that there are $k$ distinct tournaments $\Gamma'$ such that $d(\Gamma, \Gamma') = 1$ and $d(\Gamma', \Pi) = k - 1$. For then, inductively, there are $(k - 1)!$ geodesics from each $\Gamma'$ to $\Pi$ and each of these extends via the edge $(\Gamma, \Gamma')$ to a geodesic from $\Gamma$ to $\Pi$ for a total of $k!$ geodesics of the latter type.

First we consider the case when $\Delta(\Gamma, \Pi)$ consists of a single cycle $C$ of length $p$ which we will label $\langle 0, \ldots, p - 1 \rangle$ and use the integers mod $p$ as the labels. Then the distance $k$ is $p - 2$. We must find at least $p - 2$ distinct 3-cycles in $\Gamma$ such that the reversal of each of which leads to a tournament $\Gamma'$ with $d(\Gamma', \Pi) = k - 1$. This is trivial if $p = 3$ and so we assume $p \geq 4$.

For each vertex $i$ of $C$ we associate a + if $i - 1 \to i + 1$ in $\Gamma$ and a − if $i + 1 \to i - 1$ in $\Gamma$. If $i$ has a + then $\langle i - 1, i, i + 1 \rangle$ is a 3-cycle in $\Gamma$ and reversing it leads to $\Gamma'$ with $\Delta(\Gamma', \Pi)$ the $p - 1$ cycle $\langle 0, \ldots, i - 1, i + 1, \ldots, p - 1 \rangle$, omitting $i$. We will call this the near cycle associated with the + at $i$.

If $p = 4$ then two vertices have – labels and so there are $p - 2 = 2$ near cycles. So we may assume $p \geq 5$.

For a vertex $i$ a far cycle is $\langle i, j, j + 1 \rangle$ in $\Gamma$ where $i \to j$ and $j + 1 \to i$ with $j \neq i + 1$ and $j + 1 \neq i - 1$. So this far cycle intersects $C$ only at the edge $(j, j + 1)$.

From the proof of Theorem 4.7 (b) reversing a far cycle leads to $\Gamma'$ with $\Delta(\Gamma', \Pi)$ the union of two cycles with only the vertex $i$ in common and with $d(\Gamma', \Pi) = p - 3$.

We show that there are at least $p - 2$ distinct 3-cycles which are either near or far. We will call the near or far cycles the special cycles for $\langle 0, \ldots, p - 1 \rangle$ in the tournament which is the restriction $\Gamma_C = \Gamma \{0, \ldots, p - 1\}$.

For a vertex $i$ we will call the ± labels for the vertices $i - 1, i, i + 1$ the pattern for $i$.

(i) If the pattern for $i$ is +, ±, +, then $i + 2$ is an output for $i$ and $i - 2$ is an input for $i$ and so there is some $j$ between them with $j$ an output for $i$ and $j + 1$ an input, leading to a far cycle for $i$. 


(ii) If the pattern for $i$ is $+, \pm, -$ then $i+2$ and $i-2$ are both inputs for $i$. So there is a far cycle for $i$ unless every $j \neq i-1, i, i+1$ is an input for $i$. In that case the $\Gamma_C$ score of $i$ is 1.

(iii) If the pattern for $i$ is $-, \pm, +$ then $i+2$ and $i-2$ are both outputs for $i$. So there is a far cycle for $i$ unless every $j \neq i-1, i, i+1$ is an output for $i$. In that case the $\Gamma_C$ score of $i$ is $p-1$.

(iv) If the pattern for $i$ is $-, \pm, -$ then it may happen that there is no far cycle for $i$.

If every vertex has a $-$ then there are $p$ near cycles. If every vertex has a $+$ then from type (i) we see that every vertex has at least one far cycle and so there are at least $p$ far cycles.

Now we assume that both $+$ and $-$ labels appear.

For our preliminary estimate we neglect the possibility of scores 1 or $p-1$ and assume that the patterns of types (i), (ii) and (iii) each lead to a far cycle and that type (iv) never does. Thus, every $-$ adjacent to a $+$ leads to both a near and a far cycle and every $+$ adjacent to a $+$ leads to a far cycle.

For a run of $-$'s, each $-$ leads to a near cycle and at each end there is also a far cycle. So the count of cycles is the length of the run plus 1 and plus 1 more if the length of the run is greater than 1.

For a run of $+$'s, each $+$ leads to a far cycle unless the length of the run is 1. Thus, the count of the cycles is the length of the run minus 1 and plus 1 if length of the run is greater than 1.

Notice that because we are on a cycle, the number of $+$ runs is equal to the number of $-$ runs.

Adding these up we obtain as our preliminary estimate the sum of the lengths of the runs, which is $p$, plus the number of runs of either sort which are longer than 1.

Now we must correct for the scores 1 and $p-1$.

Assume first that there is at most one vertex with score 1 and at most one with score $p-1$. For each of these vertices we assumed there was a far cycle where there need not be one. Thus, we correct our preliminary estimate by subtracting 2. Thus, in this case there are at least $p-2$ special cycles.

Now suppose the vertex $i$ has score 1. Every vertex other than $i-1, i, i+1$ is an input to $i$ and so has at least two outputs. If $i$ is associated with $-$ then $i+1$ has two outputs and so the only possibility for another vertex with score 1 is $i-1$. Since the pattern for $i$ is $+, -, -$ the pattern must be $i-1$ is $+, +, -$ if it has score 1. If $i$ is associated with $+$ then only $i+1$ can have score 1 in which case it has pattern
+ , − , − . Observe that in either case, there is a run of +’s and a run of −’s of length greater than one.

Similarly, if the vertex \( i \) has score \( p - 1 \) then only \( i - 1 \) or \( i + 1 \) could have score \( p - 1 \) and which possibility could occur depends on whether \( i \) is associated with − or +. When there are two vertices with score \( p - 1 \) then again there is a run of +’s and a run of −’s of length greater than one.

So if either there is more than one score 1 vertex or more than one score \( p - 1 \) vertex or both, our preliminary estimate is at least \( p + 2 \) for the two long runs. We subtract at most 4 to correct for the 4 far cycles from types (ii) and (iii). Thus again we have at least \( p - 2 \) special cycles.

Finally, notice that if the restriction \( \Gamma_C = \Gamma \mid \{0, \ldots, p - 1\} \) happens to have score vector \((1, 1, \ldots, p - 1, p - 1)\) then there are only \( p - 2 \) cycles in the restriction and these are the \( p - 2 \) special cycles.

Now we return to the general case and suppose that \( C = C_1 \) with \( C_1, C_2, \ldots, C_\ell \) a maximum decomposition of \( \Delta(\Gamma, \Pi) \). Define \( \hat{\Gamma} \) by reversing \( C \) in \( \Gamma \). Thus, \( \Delta(\hat{\Gamma}, \Pi) = \Delta(\Gamma, \Pi) \setminus C \) and \( C_2, \ldots, C_\ell \) is a maximum decomposition for \( \Delta(\hat{\Gamma}, \Pi) \). Hence, \( d(\Gamma, \hat{\Gamma}) = p - 2 \) and \( d(\hat{\Gamma}, \Pi) = d(\Gamma, \Pi) - (p - 2) \). If we reverse a special cycle of \( C \) then we obtain \( \Gamma' \) with \( d(\Gamma, \Gamma') = 1 \) and \( d(\Gamma', \hat{\Gamma}) = p - 3 \). Hence, \( d(\Gamma', \Pi) = d(\Gamma, \Pi) - 1 \).

Since we can rearrange the \( C_r \)’s it follows that by reversing any special cycle in any of the \( C_r \) leads to a \( \Gamma' \) of the required sort. Furthermore, for the cycle \( C_r \) there are at least \( |C_r| - 2 \) special cycles and the sum \( \sum_{r=1}^{\ell} |C_r| - 2 = \beta(\Delta(\Gamma, \Pi)) = d(\Gamma, \Pi) \) which is what we want.

There is, however, a final problem. A 3-cycle may occur as a special cycle in two different \( C_r \)’s leading to double counting.

To cure this, we choose our maximum decomposition with care.

Recall from the Claim in the proof of Theorem 4.7 (b) that if we choose the decomposition so that \( C_1 \) is the cycle of shortest length which occurs in any maximum decomposition, then for vertices \( i, j \) of \( C_1 \) with \( i \rightarrow j \) the edge \( (i, j) \) does not occur in \( \Delta(\Gamma, \Pi) \setminus C_1 \). In particular, no special cycle of \( C_1 \) is contained in \( \Delta(\Gamma, \Pi) \setminus C_1 \). Inductively, we choose the decomposition so that \( C_r \) is the shortest cycle which occurs in any maximum decomposition of \( \Delta(\Gamma, \Pi) \setminus (C_1 \cup \cdots \cup C_{r-1}) \). We thus obtain a maximum decomposition such that no special cycle of \( C_r \) occurs in any \( C_s \) for \( s > r \). Thus, for this decomposition the \( d(\Gamma, \Pi) \) special cycles are all distinct.

\[ \square \]
From the above proof we see that for the cycle $C = \langle 0, \ldots, p - 1 \rangle$ if the tournament $\Gamma|\{0, \ldots, p - 1\}$ has no vertices of score 1 or $p - 1$ then there are at least $p$ special cycles. In particular, this applies if $p = 2k + 1$ and $\Gamma|\{0, \ldots, 2k\}$ is a game.

**Example 5.3.** With $k \geq 3$, let $A$ be a game subset of $\mathbb{Z}_{2k+1}$. Assume that $\Gamma|\{0, \ldots, 2k\}$ is the group game $\Gamma[A]$ and assume that $1 \in A$ so that the cycle $C = \langle 0, \ldots, 2k \rangle$ is contained in $\Gamma[A]$. Since $\mathbb{Z}_{2k+1}$ acts transitively on $\Gamma[A]$ and preserves the cycle $C$, every vertex has the same number of near and far cycles. In particular, every vertex is associated with $+$ if and only if $2 \in A$.

**Example (a) $[A = \text{Odd}_k = \{1, 3, \ldots, 2k - 1\}]$** The vertex 0 is associated with $-$ and so has a near cycle. The far cycles associated with 0 are $\langle 0, j, j + 1 \rangle$ for $j$ an odd number with $3 \leq j \leq 2k - 3$. That is, there are $k - 2$ far cycles for a total of $k - 1$ cycles for each vertex. Thus, the number of special cycles is $(2k + 1)(k - 1)$.

**Example (b) $[A = \{1, 2, 4, \ldots, 2k - 2\}]$** The vertex 0 is associated with $+$ and so has no near cycle. The far cycles associated with 0 are $\langle 0, p, p + 1 \rangle$ for $p$ an even number with $2 \leq p \leq 2k - 2$. That is, there are $k - 1$ far cycles and so the number of special cycles is again $(2k + 1)(k - 1)$.

**Example (c) $[A = [1, k]]$** The vertex 0 is associated with $+$ and so has no near cycle. The unique far cycle associated with 0 is $\langle 0, k, k + 1 \rangle$ and so the number of special cycles is $2k + 1$. Notice that the games in (a) and (c) are isomorphic via the multiplication map $m_k$. However, $m_k$ maps the cycle $C$ to a different Hamiltonian cycle.

**Example (d) $[A = \{1, k+1, \ldots, 2k - 1\}]$** The vertex 0 is associated with $-$ and so has a near cycle. There are no far cycles and so the number of special cycles is again $2k + 1$.

\[\Box\]

**Theorem 5.4.** Let $C = \langle 0, \ldots, p - 1 \rangle$ be a cycle in a tournament $\Gamma$ with $p \geq 4$. If $p = 2k + 1$, then the number of special cycles for $C$ is at most $(2k + 1)(k - 1)$. If $p = 2k$, then the number of special cycles for $C$ is at most $k(2k - 3)$ and the inequality is strict unless $k \equiv 2 \mod 4$.

**Proof.** We use the notation of the proof of Theorem 5.2. In the following table we describe for each pattern for a vertex $i$ the largest possible number of far cycles associated with a vertex. A far cycle is of the form $\langle i, j, j + 1 \rangle$ where $j \neq i - 1, i, i + 1$ and so $i \rightarrow j, j + 1 \rightarrow i$. 

| Pattern | Far Cycles |
|---------|------------|
| $0$     | $\langle 0, j, j + 1 \rangle$ for $3 \leq j \leq 2k - 3$ |
| $2k$    | $\langle 0, p, p + 1 \rangle$ for $2 \leq p \leq 2k - 2$ |
| $1$     | $\langle 0, k, k + 1 \rangle$ |
| $k+1$   | $\langle 1, j, j + 1 \rangle$ for $j \neq 1, 0, k + 1$ |
| $2k-1$  | $\langle k, j, j + 1 \rangle$ for $j \neq k, k - 1, 2k - 1$ |
Vertex Pattern Far Cycle Max if $p = 2k$ if $p = 2k + 1$

| Pattern | Far Cycle | Max |
|---------|-----------|-----|
| $+$ $\pm$ $+$ | $\left\lceil(p - 3)/2 \right\rceil$ | $k - 2$ | $k - 1$ |
| $+$ $\pm$ $-$ | $\left\lceil(p - 4)/2 \right\rceil$ | $k - 2$ | $k - 2$ |
| $-$ $\pm$ $+$ | $\left\lceil(p - 4)/2 \right\rceil$ | $k - 2$ | $k - 2$ |
| $-$ $\pm$ $-$ | $\left\lceil(p - 5)/2 \right\rceil$ | $k - 3$ | $k - 2$ |

It follows that if either every vertex is associated with a $+$ or every vertex is associated with a $-$, then the upper bound of the number of cycles is $2k(k - 2)$ when $p = 2k$ and is $(2k + 1)(k - 1)$ when $p = 2k + 1$. Notice that this case cannot occur when $2k = 4$.

Now assume that both $+$’s and $-$’s occur.

Let $p = 2k + 1$. For a run of $+$’s every vertex may have $k - 1$ far cycles except at the ends and so the number of cycles is $(k - 1)$ times the length of the run minus 1 and minus 1 more if the length of the run is greater than 1. For a run of $-$’s the bound on number of cycles (near and far) is $(k - 1)$ times the length of the run plus 1 and minus 1 if the length of the run is greater than 1. The number of + runs equals the number of $-$ runs and so the total upper bound on the number of cycles is $(2k + 1)(k - 1)$ minus the number of runs of length greater than one. Since $p$ is odd, there is at least one run of length greater than one.

Let $p = 2k$. For a run of $+$’s every vertex may have $k - 2$ far cycles unless the run is a singleton in which case the maximum is $k - 3$. Thus, the bound is $(k - 2)$ times the length of the run minus 1 and plus 1 if the length of the run is greater than 1. For a run of $-$’s the bound on number of cycles (near and far) is $(k - 2)$ times the length of the run plus 1 and plus 1 more if the length of the run is greater than 1. Again the number of + runs equals the number of $-$ runs and so the total upper bound on the number of cycles is $2k(k - 2)$ plus the number of runs of length greater than one. Since such a long run has at least two elements there are at most $k$ such runs. Thus, the upper bound is $2k(k - 2) + k = k(2k - 3)$. Since the number of the two sorts of runs are equal, there are fewer than $k$ such runs if $k$ is odd. Thus, the bound is strict unless $k \equiv 0$ or 2 mod 4. With exactly $k$ long runs the pattern on the cycle consists of pairs of $+$’s alternating with pairs of $-$’s. Now suppose that $k \equiv 0$ mod 4. By relabeling, we may assume that associated with 0, 1, 2, 3 are $-, -, +, +$. It follows that 0 has pattern $+ - +$. Since $k$ is divisible by 4 the vertex at $k$ will have the same pattern. In order to have $k - 2$ far cycles associated with vertex 0, it must happen that the odd vertices (other than $2k - 1$) are outputs of 0 and the even vertices are inputs of 0. In particular, $k \to 0$. 
Similarly, in order to have \( k - 2 \) far cycles associated with vertex \( k \), it must happen that the odd vertices (other than \( k - 1 \)) are outputs of \( k \) and the even vertices are inputs of \( k \). In particular, \( 0 \rightarrow k \). But for a tournament, it cannot happen that both \( 0 \rightarrow k \) and \( k \rightarrow 0 \). Hence, the inequality is strict when \( k \equiv 0 \mod 4 \).

\[\square\]

Example 5.5. The inequality is achieved when \( k \equiv 2 \mod 4 \).

Define the digraph on \( \mathbb{Z} \) by

\[
\begin{align*}
t & \rightarrow t + 1 \\
t & \rightarrow t + 2i, \ t + (2i + 1) \rightarrow t \quad \text{for all } i \geq 1 \text{ and } t \equiv 0, 1 \mod 4 \\
t + 2i & \rightarrow t, \ t \rightarrow t + (2i + 1) \quad \text{for all } i \geq 1 \text{ and } t \equiv 2, 3 \mod 4
\end{align*}
\]

If \( k > 2 \) and \( k \equiv 2 \mod 4 \), then this induces a tournament on \( \mathbb{Z}_{2k} \) for which translation by 4 is an automorphism. Assume that for the cycle \( \langle 0, \ldots, p - 1 \rangle \) with \( p = 2k \) \( \Gamma \{0, \ldots, p - 1\} \) is this tournament. We need only examine the vertices 0, 1, 2, 3. It is easy to see that there are \( 2k(k - 2) + k = k(2k - 3) \) special cycles.

\[\square\]

Now we focus on games on \( I \) with \( |I| = 2n + 1 \). In that case, the score vector is \((n, n, \ldots, n)\) and the entire permutation group on \( I \) acts on the interchange graph. The orbit of a game \( \Gamma \) under the action consists of those games which are isomorphic to \( \Gamma \). Those permutations which map \( \Gamma \) to itself are precisely the automorphisms of \( \Gamma \).

Lemma 5.6. Let \( \Pi \) be a game of size \( 2n - 1 \) on the set of vertices \( J \) and \( K \subset J \) with \( |J| = 2n - 1 \) and \( |K| = n \). If \( \Gamma \) is the extension of \( \Pi \) via \( u \rightarrow v \) and \( K \), so that \( \Gamma \) is a game on \( I = J \cup \{u, v\} \), then

\[
(5.2) \quad \beta(\Gamma) \leq \beta(\Pi) + 2n - 1.
\]

Proof. Let \( K = \{a_0, \ldots, a_{n-1}\}, J \setminus K = \{b_1, \ldots, b_{n-1}\} \). Given a decomposition for \( \Pi \) we build a decomposition for \( \Gamma \).

For \( r = 1, \ldots, n - 1 \) if \( a_r \rightarrow b_r \) in \( \Pi \), then replace the edge \((a_r, b_r)\) by \((a_r, u), (u, b_r)\) to get a cycle in \( \Gamma \) with the length increased by 1. In addition, define the 3-cycle \( D_r = \langle a_r, b_r, v \rangle \). If, instead, \( b_r \rightarrow a_r \) in \( \Pi \), then replace the edge \((b_r, a_r)\) by \((b_r, v), (v, a_r)\) to get a cycle in \( \Gamma \) with the length increased by 1. In addition, define the 3-cycle \( D_r = \langle b_r, a_r, u \rangle \). Finally, define the 3-cycle \( \langle a_0, u, v \rangle \).

Notice that a single cycle \( C \) may contain edges \((a_r, b_r)\) or \((b_r, a_r)\) for more than one \( r \). Thus, instead of two or more cycles each extended in
length by 1 we have a single cycle with the length extended by two or more.

If the original decomposition for $\Pi$ was maximum with cardinality $k$ then $\beta(\Pi) = (n - 1)(2n - 1) - 2k$. The decomposition we have constructed consists of the $k$ extended cycles and $n$ new 3-cycles. Thus, the span of $\Gamma$ is at least $k + n$. Hence, $\beta(\Gamma) \leq n(2n + 1) - 2(k + n) = (n - 1)(2n - 1) - 2k + (2n - 1).

□

Remark: Equality holds in \((5.2)\) if and only if the decomposition for $\Gamma$ is maximum for $\Gamma$.

**Theorem 5.7.** Let $A$ be the game subset $[1, n] \subset \mathbb{Z}_{2n+1}$. For the associated group game $\Gamma[A]$, 
\[
d(\Gamma[A], \Gamma[A]^{-1}) = \beta(\Gamma[A]) = n^2.
\]

**Proof.** On the cycle $C = \langle 0, 1, \ldots, 2n \rangle$ define the group of 1-chains on $C$ to be the free abelian group generated by the edges of $C$ so that an element is a formal sum $\xi = \sum_{r=0}^{2n-1} m_r(r, r + 1)$ with $m_r \in \mathbb{Z}$. The group 0-chains is the free abelian group on the vertices and the boundary map from the 1-chains to the 0-chains is given by $\partial(r, r + 1) = 1(r + 1) - 1(r)$. Clearly, the boundary of a 1-chain is 0 if and only if the chain is a constant multiple of $\sum_{r=0}^{2n-1} 1(r, r + 1)$.

Let $\Gamma$ be any tournament on $\mathbb{Z}_{2n+1}$. Each edge of $\Gamma$ can be written uniquely as $(t, t + s)$ with $t = 0, \ldots, 2n$, $s = 1, \ldots, 2n$ and with addition mod $2n + 1$. Define the associated chain $\xi(t, t + s) = \sum_{r=0}^{s-1} 1(t + r, t + r + 1)$. If $Q$ is a subgraph of $\Gamma$, then the chain $\xi(Q)$ is the sum of $\xi$ applied to the edges of $Q$. Observe first that the coefficients of $\xi(Q)$ are all non-negative. Next, if $Q_1$ and $Q_2$ are disjoint subgraphs, then $\xi(Q_1 \cup Q_2) = \xi(Q_1) + \xi(Q_2)$. Furthermore, if $Q$ is a cycle, then the boundary of $\xi(Q)$ is 0 and so $\xi(Q) = m(Q) \cdot \sum_{r=0}^{2n-1} (r, r + 1)$ with $m(Q) > 0$. The number $m(Q)$ is the number of times the cycle $Q$ wraps around $C$. If $\Pi$ is an Eulerian subgraph, then since it is a disjoint union of cycles, it follows that there exist a positive integer $m = m(\Pi)$ such that $\xi(\Pi) = m(\Pi) \cdot \sum_{r=0}^{2n-1} (r, r + 1)$. Furthermore, for any decomposition of $\Pi$ by disjoint cycles $Q_1, \ldots, Q_k$, $\xi(\Pi) = \xi(Q_1) + \cdots + \xi(Q_k)$ and so $m(\Pi) = m(Q_1) + \cdots + m(Q_k)$. Since each $m(Q_i) \geq 1$, it follows that $k \leq m(\Pi)$ and so the span $\sigma(\Pi)$ is bounded by $m(\Pi)$.

In general, this estimate too crude to be of much use. For example, if $1 \to 0$ in $\Gamma$ then $\xi(1, 0) = \xi(1, 1 + (2n))$ covers the entire cycle except for $(0, 1)$. But for $\Gamma[A]$ it gives us what we need.
To compute \( m(\Gamma[A]) \) we count the edges \((r, r + s)\) such that \( \xi(r, r + s) \) has a coefficient of 1 on \((0, 1)\). These are \((0, 1), (0, 2), \ldots \) with \( s \) translates of the edge \((0, s)\) hitting \((0, 1)\) for \( s = 1, \ldots, n \). Hence, \( m(\Gamma[A]) = 1 + 2 + \cdots + n = n(n + 1)/2 \). Thus, \( \beta(\Gamma[A]) \geq n(2n + 1) - n(n + 1) = n^2 \).

Now we prove that \( \beta(\Gamma[A]) \leq n^2 \) by induction on \( n \). This is trivial for \( n = 1 \).

Now let \( \Pi = \Gamma[[1, n - 1]] \) on \( J = \mathbb{Z}_{2n-1} \) and let \( K = [0, n - 1] \) and let \( \Gamma \) be the extension of \( \Pi \) via \( u \rightarrow v \) and \( K \). Define a map by

\[
\begin{align*}
  i &\mapsto i \quad \text{for } i = 0, \ldots, n - 1, \\
  u &\mapsto n, \\
  i &\mapsto i + 1 \quad \text{for } i = n, \ldots, 2n - 2, \\
  v &\mapsto 2n.
\end{align*}
\]

This is an isomorphism from the extension \( \Gamma \) of \( \Gamma[[1, n - 1]] \) onto \( \Gamma[[1, n]] \).

By induction hypothesis and (5.2)

\[
\beta(\Gamma[[1, n]]) \leq \beta(\Gamma[[1, n - 1]]) + 2n - 1 \leq (n-1)^2 + 2n - 1 = n^2.
\]

Finally, since \( \Gamma = \Delta(\Gamma, \Gamma^{-1}) \) it follows that \( \beta(\Gamma) \) is the distance from a game \( \Gamma \) to its inverse.

\[\square\]

**Remark:** In this case equality holds in (5.5) and so by the Remark after Lemma 5.6 the decompositions constructed from a maximum decomposition of \( \Gamma[[1, n - 1]] \) are maximum decompositions for \( \Gamma[[1, n]] \).

The game \( \Gamma = \Gamma[[1, n - 1]] \) may admit smaller decompositions. For example, if \( 2n + 1 \) is prime, then \{\langle 0, j, 2j, \ldots, 2nj \rangle : j = 1, \ldots, n \} is a decomposition of \( \Gamma \) by \( n \) cycles, each of length \( 2n + 1 \).

I conjecture that for any game \( \Gamma \) on \( I \) with \( |I| = 2n + 1 \), the distance \( d(\Gamma, \Gamma^{-1}) \leq n^2 \). In fact, I suspect that for any pair of games \( \Gamma, \Pi \) on \( I \),

\[
d(\Gamma, \Pi) \leq n^2, \text{ i.e. the diameter of the interchange graph is } n^2. \]

Alon, McDiarmid and Molloy conjecture in [2] that any \( k \)-regular digraph contains a set of \( k(k + 1)/2 \) disjoint cycles. Since a game on \( I \) is \( n \)-regular their conjecture would imply that \( d(\Gamma, \Gamma^{-1}) \leq n^2 \). In addition, if the Eulerian digraph \( \Delta(\Gamma, \Pi) \) happens to be \( k \)-regular then it contains at most \( k(2n + 1) \) edges and, if their conjecture is true, its span is at least \( k(2n + 1)/2 \). So \( d(\Gamma, \Pi) = \beta(\Delta(\Gamma, \Pi)) \) is bounded by \( k(2n + 1) - k(k + 1) = 2nk - k^2 = n^2 - (n - k)^2 \). The diameter conjecture would require \( \beta(\Delta) \leq n^2 \) for every Eulerian graph on at most \( 2n + 1 \) vertices.

The diameter is certainly bounded by \( n(2n - 1) \) because for any Eulerian graph \( \Delta \) on \( I \), \( \beta(\Delta) \leq |\Delta| \cdot \frac{2n-1}{2n+1} \). This follows because \( l_1 + \cdots + \)
\(l_\sigma = |\Delta|\) and each \(l_i \leq 2n+1\), where \(\sigma\) is the span of \(\Delta\) and \(l_1, \ldots, l_\sigma\) are the lengths of the cycles in some maximum decomposition. It follows that \(\sigma \geq |\Delta|/(2n+1)\) and so \(\beta = |\Delta| - 2\sigma\) is at most \(|\Delta| \cdot (1 - \frac{2}{2n+1})\).

If \(n \equiv 0, 1 \mod 3\), then there exist Steiner games \(\Gamma\) on \(I\) for which the maximum decomposition for \(\Gamma\) consists of \(n(2n+1)/3\) 3-cycles. Thus, for a Steiner game any maximum decomposition consists of \(n(2n+1)/3\) cycles and so they all must be 3-cycles. Hence, \(d(\Gamma, \Gamma^{-1}) = \beta(\Gamma) = n(2n+1)/3\). Since \(d(\Gamma[[1, \ldots, n]], \Gamma[[1, \ldots, n]]^{-1}) = n^2\) it is clear that \(\Gamma[[1, \ldots, n]]\) is not a Steiner game. Because the interchange graph is connected, there exists a Steiner game \(\Gamma\) and a 3-cycle \(C\) in \(\Gamma\) such that the game \(\Gamma' = \Gamma/C\) with just \(C\) reversed is not a Steiner game. It follows that \(C\) cannot be an element of any maximum decomposition for \(\Gamma\) because from such a maximum decomposition we would obtain a 3-cycle decomposition for \(\Gamma'\). From Theorem 4.7 (c) applied with \(\Pi = \Gamma^{-1}\) we see that

\[
(5.6) \quad d(\Gamma', \Gamma^{-1}) = 1 + d(\Gamma, \Gamma^{-1}) = 1 + n(2n+1)/3.
\]

That is, in the interchange graph, \(\Gamma'\) is farther away from \(\Gamma^{-1}\) than is its inverse \(\Gamma\).

From the proof of Theorem 5.7 we see that the game \(\Gamma[[1, n]]\) on \(\mathbb{Z}_{2n+1}\) is reducible to \(\Gamma[[1, n]]\) on \(\mathbb{Z}_{2n-1}\). Thus, \(\Gamma[[1, n]]\) is completely reducible, where

**Definition 5.8.** A game \(\Gamma\) on a set \(I\) with \(|I| = 2n+1\) is completely reducible when there is a sequence of subsets \(I_1 \subset I_2 \cdots \subset I_n = I\) with \(|I_k| = 2k + 1\) such that the restriction \(\Gamma|I_k\) is a game on \(I_k\).

**Proposition 5.9.** If \(\Gamma\) is a completely reducible game of size \(2n+1\) then

\[
(5.7) \quad d(\Gamma, \Gamma^{-1}) = b(\Gamma) \leq n^2.
\]

**Proof.** By induction on \(n\) using Lemma 5.6  
\[\square\]
6. The Double Construction and the Lexicographic Product

If $\Pi$ is a tournament on a set $J$ with $|J| = n$, then we define the double of $\Pi$ to be the game $2\Pi$ on $I = \{0\} \cup J \times \{-1,+1\}$. With $2\Pi(0) = J \times \{-1\}$, $(2\Pi)^{-1}(0) = J \times \{+1\}$,

\begin{align}
2\Pi(i+) &= \Pi(i) \times \{+1\} \cup \Pi^{-1}(i) \times \{-1\} \cup \{0\}, \\
2\Pi(i-) &= \Pi(i) \times \{-1\} \cup \Pi^{-1}(i) \times \{+1\} \cup \{i+\},
\end{align}

where we will write $i \pm$ for $(i, \pm 1)$.

That is, if $i \rightarrow j$ in $\Pi$, then in $2\Pi$

\begin{align}
i- &\rightarrow j- \quad i+ \rightarrow j+, \\
&j+ \rightarrow i- \quad j- \rightarrow i+.
\end{align}

In passing we note the following consequence of this construction.

**Proposition 6.1.** If $\Pi$ is a digraph with $n$ vertices, then $\Pi$ is a subgraph of a game of size $2n + 1$.

**Proof.** It is clear that $\Pi$ can be included as a subgraph of some tournament $\Pi_1$ on the set $J$ of the vertices of $\Pi$. So, up to isomorphism, $\Pi$ is a subgraph of the game $2\Pi_1$. 

It is easy to check that $2\Pi$ is reducible via each pair $i- \rightarrow i+$. It reduces to the double of the restriction of $\Pi|\{J \setminus \{i\}\}$. In fact, if $J_1 \subset J$ is nonempty and $\Pi_1$ is the restriction $\Pi|J_1$ then $2\Pi_1$ is a subgame of $2\Pi$.

Thus, a double is completely reducible. It follows from Proposition 5.9 that $d(2\Pi, (2\Pi)^{-1}) \leq n^2$. To see this directly, observe that if $i \rightarrow j$ in $\Pi$ then $\langle i-, j-, i+, j+ \rangle$ is a 4-cycle and $\langle i+, 0, i- \rangle$ is 3-cycle, both in $2\Pi$. Thus, we obtain a decomposition of $2\Pi$ with $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ cycles.

By Proposition 2.2 (f), $i+$ is the only vertex $u$ of $2\Pi$ such that $2\Pi$ is reducible via each pair $i- \rightarrow u$. On the other hand, by Proposition 2.2 (d), $2\Pi$ is reducible via $0 \rightarrow i-$ if and only if $\Pi(i) = \emptyset$, i.e. $i$ has score 0 in $\Pi$. For if $i \rightarrow j$ then $i- \rightarrow j-$ and $0 \rightarrow j-$. Similarly, $2\Pi$ is reducible via $i+ \rightarrow 0$ if and only if $\Pi^{-1}(i) = \emptyset$, i.e. $i$ has score $n-1$ in $\Pi$. If $i \rightarrow j$ is contained in a 3-cycle $\langle i, j, k \rangle$ in $\Pi$, then $2\Pi$ is not reducible via $j+ \rightarrow i-$, because $k- \rightarrow j+, i-$. In general, $2\Pi$ is reducible via $j+ \rightarrow i-$ if and only if for all $k \in J \setminus \{i, j\}$ either $i, j \rightarrow k$ or $k \rightarrow i, j$. 

Example 6.2. Let $\Pi$ be the tournament on $\{1, 2, 3, 4\}$ given by $\langle 2, 3, 4 \rangle \cup (\{1\} \times \{2, 3, 4\})$. Let $\Gamma = 2\Pi$ and $\bar{\Gamma} = \Gamma / \langle 2+, 3+, 4+ \rangle$. The reducibility graphs are given by the paths:

\[
\begin{align*}
r\bar{\Gamma} &= \{[1-, 1+, 0]\}, \quad \text{and} \\
 r\Gamma &= r\bar{\Gamma} \cup \{[p-, p+] : p = 2, 3, 4\}.
\end{align*}
\]

(6.3)

\[
\square
\]

Example 6.3. If $J = [1, n]$ and $i \rightarrow j$ if and only if $i < j$, then $0 \rightarrow 0, i- \rightarrow i$ and $i+ \rightarrow i + n$ is an isomorphism from $2\Pi$ to $\Gamma[[1, n]]$.

Since the $\phi(2n + 1)$ isomorphs of $\Gamma[[1, n]]$, or, equivalently, the isomorphs of $\Gamma[\text{Odd}, n]$, are the only reducible group games, see Theorem 3.14 they are the only group games which can be expressed as doubles.

\[
\square
\]

We let $\Pi_+$ and $\Pi_-$ denote the restrictions of $2\Pi$ to $J \times \{+1\}$ and to $J \times \{-1\}$, respectively. Of course, each of these subgraphs is isomorphic to $\Pi$. In addition, we define $X(2\Pi)$ to be

\[
(6.4) \quad \{ \langle j+, i- \rangle : (i, j) \in \Pi \} \cup \{ \langle j-, i+ \rangle : (i, j) \in \Pi \}.
\]

If $\gamma : \Pi \rightarrow \Pi_1$ is an isomorphism, then we define the isomorphism $2\gamma : 2\Pi \rightarrow 2\Pi_1$ by

\[
(6.5) \quad 2\gamma(0) = 0, \quad \text{and} \quad 2\gamma(j\pm) = \gamma(j)\pm \quad \text{for } j \in J.
\]

The map $\gamma \mapsto 2\gamma$ defines an injective group homomorphism $2 : \text{Aut}(\Pi) \rightarrow \text{Aut}(2\Pi)$.

Proposition 6.4. Let $\Pi$ be a tournament on a set $J$ with $|J| = n$ such that no $i \in J$ has score 0 or $n - 1$. If $\Pi_1$ is a tournament and $\rho : 2\Pi \rightarrow 2\Pi_1$ is an isomorphism then $\rho(0) = 0$ and there exists a unique isomorphism $\gamma : \Pi \rightarrow \Pi_1$ such that $\rho = 2\gamma$. In particular, $2 : \text{Aut}(\Pi) \rightarrow \text{Aut}(2\Pi)$ is an isomorphism.

Proof. Every vertex in $J$ has an input and an output, and so $2\Pi$ is not reducible via any pair which contains 0. On the other hand, $2\Pi$ is reducible via $i- \rightarrow i+$ for all $i \in J$. Since $2\Pi_1$ is reducible via $j- \rightarrow j+$ for all $j \in J_1$ it follows that $\rho(0) = 0$. Hence, $\rho$ restricts to bijection from $(2\Pi)(0) = J \times -1$ to $J_1 \times -1$, yielding an isomorphism from $\Pi_{-}$ to $\Pi_{1-}$. Thus, there is an isomorphism $\gamma : \Pi \rightarrow \Pi_1$ such that $\rho(i-) = \gamma(i)-$ for all $i \in J$.

For each $i \in J$, $2\Pi_1$ is reducible via $\gamma(i)- = \rho(i- \rightarrow \rho(i+)$. On the other hand, $2\Pi_1$ is reducible via $\gamma(i)- \rightarrow \gamma(i)+$. From Proposition 2.2 (f) it follows that $\rho(i+) = \gamma(i)+$. Thus, $\rho = 2\gamma$. 

\[
\square
\]
We will see below in Theorem 11.4 that $2 : \text{Aut}(\Pi) \to \text{Aut}(2\Pi)$ is almost always an isomorphism.

In the special case when $n$ is odd and $\Pi$ is itself a game, we note that $\Pi_+$ and $\Pi_-$ are subgames of $2\Pi$ and so are Eulerian subgraphs. In addition, in this case, $X(2\Pi)$, defined in (6.4), is an Eulerian subgraph. Thus, we can obtain additional examples, by reversing one or more of the three disjoint Eulerian subgraphs $\Pi_+$, $\Pi_-$ and $X(2\Pi)$.

Theorem 6.5. If $\Pi$ is a Steiner game, and $\Gamma = (2\Pi)/\Delta$ with $\Delta$ equal to $\Pi_+$, $\Pi_-$, $X(2\Pi)$, or a union of any two of these, then $\Gamma$ is a Steiner game.

Proof. We will do the case with $\Delta = \Pi_+$ as the others are similar. Notice that if $i \to j$ in $\Pi$ then $j \to i+$, $j+ \to i$ and also $j+ \to i+$ in $\Gamma$. It is this coherence which is the basis of the construction.

Assume that $\langle k, j, i \rangle$ is one of the 3-cycles in a maximum decomposition for $\Pi$. Associated to it we use the following 3-cycles in $\Gamma$

$$
\langle i+, j+, k- \rangle, \langle i-, j+, k+ \rangle, \langle i+, j-, k+ \rangle,
$$

$$
\langle i+, 0, i- \rangle, \langle j+, 0, j- \rangle, \langle k+, 0, k- \rangle,
$$

$$
\langle k-, j-, i- \rangle.
$$

Notice that if the vertex $i$ occurs in two 3-cycles of the decomposition, the vertical edge $(i-, i+)$ occurs in the same 3-cycle $\langle i+, 0, i- \rangle$ associated with both of the decomposition 3-cycles.

Thus, we obtain a decomposition of $\Gamma$ by 3-cycles.

Using the doubling construction we can build some interesting examples.

Call a tournament $\Pi$ rigid when the identity is the only automorphism, i.e. the automorphism group is trivial.

Lemma 6.6. If $\Pi$ is a tournament with score vector $(s_1, \ldots, s_p)$ and for every $k \in \mathbb{N}$, $s_i = k$ for at most two distinct vertices $i$, then $\Pi$ is rigid.

Proof. For any $k$, the set $\{i : s_i = k\}$ is invariant for any automorphism $\rho$ of $\Pi$. So $\rho$ fixes the vertices with a unique score value. If $\{i : s_i = k\}$ consists of exactly two vertices, then $\rho$ fixes each of them by Proposition 1.8. Hence, $\rho$ is the identity.

□
For example, suppose if \( \Pi \) has score vector \((1, 1, 2, \ldots, p-3, p-2, p-2)\), then it is rigid.

Because of Theorem 3.6 we are interested in the case when translations are the only automorphisms of a group game.

**Theorem 6.7.** Let \( A \) be a game subset of a commutative group \( G \) with \( \Gamma[A] \) the associated group game. If the tournament on \( A \) obtained by restricting \( \Gamma[A] \) is rigid, then \( \text{Aut}(\Gamma[A]) = G \), i.e. the left translations are the only automorphisms of \( \Gamma[A] \).

**Proof.** Write \( \Gamma \) for \( \Gamma[A] \). It suffices to show that if \( \rho \) is an automorphism which fixes \( e \), then \( \rho \) is the identity. For such an automorphism \( A = \Gamma(e) \) and \( A^{-1} = \Gamma^{-1}(e) \) are invariant sets and so \( \rho \) restricts to an automorphism of \( \Gamma|A \) and of \( \Gamma|A^{-1} \). Since \( \Gamma|A \) is rigid, \( \rho \) fixes every element of \( A \). Since the group is commutative, \( \Gamma|A^{-1} \) is the reverse game of \( \Gamma|A \) and so is rigid as well. Hence, \( \rho \) fixes every element of \( A^{-1} \) and so is the identity. \( \Box \)

For \( G = \mathbb{Z}_{2n+1} \) the score vector of the restriction of \( \Gamma[A] \) to \( A \) is \((0, 1, 2, \ldots, n-2, n-1)\) for \( A = [1, n] \) and is \((1, 1, 2, 3, \ldots, n-3, n-2, n-2)\) for \( A = [1, n-1] \cup \{n+1\} \) for \( n \geq 4 \). From Lemma 6.6 it follows that \( \text{Aut}(\Gamma[A]) = \mathbb{Z}_{2n+1} \) in each of these cases. Note that the first example provides a reproof of Theorem 3.7.

**Example 6.8.** There exists a game \( \Gamma_1 \) of size 9 which is rigid. There exists a game \( \Gamma_2 \) of size 13 which is rigid and is not isomorphic to its reversed game.

**Proof.** Let \( J_1 = \{1, 2, 3, 4\} \) and define \( \Pi_1 \) to contain the 4-cycle \( \langle 1, 2, 3, 4 \rangle \) and with \( 3 \rightarrow 1, 4 \rightarrow 2 \). The score vector is \((1, 1, 2, 2)\) and so \( \Pi_1 \) is rigid. Since no score value is 0 or 3, it follows from Proposition 6.4 that \( \Gamma_1 = 2\Pi_1 \) is rigid.

We saw above that \( 2(\Pi^{-1}) \) is isomorphic to \( (2\Pi)^{-1} \) for any tournament \( \Pi \). It is easy to check that \( \Pi_1 \) is isomorphic to \( \Pi_1^{-1} \) and so the game \( \Gamma_1 \) is isomorphic to its reversed game.

Now let \( \Pi_0 \) be a game of size 5 on \( \{0, 1, 2, 3, 4\} \). On \( J_2 = \{0, 1, 2, 3, 4, 5\} \) define \( \Pi_2 \) by

\[
\Pi_2(4) = \Pi_0(4) \cup \{5\}, \quad \Pi_2(5) = \{0, 1, 2, 3\}, \\
\Pi_2(i) = \Pi_0(i) \quad \text{for } i = 0, 1, 2, 3.
\]

(6.7)

Thus, the score vector is \((2, 2, 2, 2, 3, 4)\). Since \( \Pi_2^{-1} \) has score vector \((3, 3, 3, 3, 2, 1)\) it follows that \( \Pi_2 \) is not isomorphic to \( \Pi_2^{-1} \).
If \( \rho \) is an automorphism of \( \Pi_2 \) then it must fix the vertices 5 and 4. Hence, \( \Pi_2(5) \) is invariant and so \( \rho \) restricts to an automorphism of \( \Pi_0 \). For the unique game of size 5, the automorphism group is \( \mathbb{Z}_5 \) acting freely by translation. Since \( \rho \) fixes 4, it is the identity.

Since the score values 0 and 6 do not occur for \( \Pi_2 \) it follows from Proposition 6.4 again that \( \Gamma_2 = 2\Pi_2 \) is rigid and is not isomorphic to its reversed game.

□

Another construction is the lexicographic product of two digraphs. The lexicographic product for undirected graphs is described in [27] and [28]. For digraphs it was introduced in [14].

Let \( \Gamma \) be a digraph on a set \( I \) and \( \Pi \) be a digraph on a set \( J \). Define \( \Gamma \Join \Pi \) on the set \( I \times J \) so that for \( p, q \in I \times J \)

\[
  (6.8) \quad p \to q \iff \begin{cases} 
  p_1 \to q_1 \text{ in } \Gamma, & \text{or} \\
  p_1 = q_1 \text{ and } p_2 \to q_2 \text{ in } \Pi.
\end{cases}
\]

The map \( p \to p_1 \) is a surjective morphism from \( \Gamma \Join \Pi \) to \( \Gamma \).

Clearly, if both \( \Pi \) and \( \Gamma \) are Eulerian, or if both are tournaments, then \( \Gamma \Join \Pi \) satisfies the corresponding property. In particular, the lexicographic product of two games is a game. Furthermore, it is clear that

\[
  (6.9) \quad (\Gamma \Join \Pi)^{-1} = (\Gamma^{-1}) \Join (\Pi^{-1}).
\]

For each \( i \in I \), let \( J_i = \{ p : p_1 = i \} \). On each \( J_i \), \( \Gamma \Join \Pi \) restricts to a digraph \( \Pi_i \), so labeled because it is clearly isomorphic to \( \Pi \) via \( p \mapsto p_2 \).

We will call an edge \( (p, q) \) vertical when \( p_1 = q_1 \) and horizontal otherwise, i.e. when \( p_1 \to q_1 \). We will call a subgraph \( \Theta \subset \Gamma \Join \Pi \) vertical (or horizontal) when it contains only vertical (resp. only horizontal) edges. Thus, \( \Theta \) is vertical if and only if it is contained in \( \bigcup_{i \in I} \Pi_i \) and it is horizontal if and only if it is disjoint from \( \bigcup_{i \in I} \Pi_i \).

For \( p \in I \times J \) some of the outputs \( (\Gamma \Join \Pi)(p) \) are in the \( \Pi_{p_1} \) subgame. These the vertical outputs. The remaining - horizontal - outputs are the elements of the \( J_i \)'s with \( i \in \Gamma(p_1) \). That is,

\[
  (6.10) \quad (\Gamma \Join \Pi)(p) = \Pi_{p_1}(p) \cup \bigcup \{ J_i : i \in \Gamma(p_1) \}.
\]

If \( \rho \in Aut(\Gamma) \) and \( \gamma : I \to Aut(\Pi) \) is a map, so that for each \( i \in I \) \( \gamma_i \in Aut(\Pi) \), then we define \( \rho \Join \gamma \in Aut(\Gamma \Join \Pi) \) by

\[
  (6.11) \quad (\rho \Join \gamma)(p) = (\rho(p_1), \gamma_{p_1}(p_2)).
\]
Any permutation $\rho$ of $I$ induces an automorphism of the product group $G^I$ by $(\gamma \circ \rho)_i = \gamma_{\rho(i)}$. This provides a right action of $S(I)$ by group homomorphisms on the product group $G^I$.

Suppose a group $T$ acts on the right by group homomorphisms on a group $K$. We define the semi-direct product $T \ltimes K$ to be $T \times K$ with the multiplication $(t_1, k_1) \cdot (t_2, k_2) = (t_1 t_2, (k_1 \cdot t_2)k_2)$ The group homomorphisms $i : K \to T \ltimes K, j : T \to T \ltimes K$ are defined by $i(k) = (e_T, k), j(t) = (t, e_K)$ where $e_T, e_K$ are the identity elements of $T$ and $K$, respectively, inject $T$ and $K$ as subgroups of $T \ltimes K$. The first coordinate projection $p : T \ltimes K \to T$ is a group homomorphism with $p \circ j$ the identity on $T$ and with $i(K)$ the kernel of $p$. Observe that when the action of $T$ on $K$ is non-trivial, the semi-direct product is non-abelian, because, e.g. $j(t)i(k) = (t, k)$, but $i(k)j(t) = (t, k \cdot t)$.

A short exact sequence is a diagram of group homomorphisms

\[
\begin{array}{c}
K & \overset{i}{\longrightarrow} & G & \overset{p}{\longrightarrow} & T,
\end{array}
\]

where $i$ is an injection onto the kernel of the surjection $p$. We then say that $G$ is an extension of $K$ by $T$.

The short exact sequence splits when there is a homomorphism $j : T \to K$ such that $p \circ j = 1_T$. In that case, $T$ acts on $K$ by $i(k \cdot t) = j(t)^{-1}i(k)j(t)$, and $(t, k) \mapsto j(t)i(k)$ is an isomorphism of $T \ltimes K$ onto $G$.

If $T$ is a subgroup of $S(I)$, then the semi-direct product $T \ltimes G^I$ consists of the set $T \times G^I$ with the group composition $(\rho_1, \gamma_1)(\rho_2, \gamma_2) = (\rho_1 \circ \rho_2, (\gamma_1 \circ \rho_2)\gamma_2)$. This is also called the wreath product of the groups $T$ and $G$.

Thus, we see that $\text{Aut}(\Gamma \ltimes \Pi)$ contains $\text{Aut}(\Gamma) \ltimes \text{Aut}(\Pi)^I$. It is shown in [5] that this is the entire automorphism group. We provide a somewhat simpler proof.

**Theorem 6.9.** If $\Gamma$ and $\Pi$ are tournaments on sets $I$ and $J$, respectively, then the automorphism group of $\Gamma \ltimes \Pi$ is the semi-direct product, i.e.

\[
(6.13) \quad \text{Aut}(\Gamma \ltimes \Pi) = \text{Aut}(\Gamma) \ltimes [\text{Aut}(\Pi)]^I.
\]

**Proof.** We show that if $\theta$ is an automorphism of $\Gamma \ltimes \Pi$ then there exist unique $\rho \in \text{Aut}(\Gamma)$ and $\gamma : I \to \text{Aut}(\Pi)$ such that $\theta = \rho \ltimes \gamma$.

Notice that $p \notin \Pi_{p_1}(p)$ implies for all $p$

\[
(6.14) \quad |\Pi_{p_1}(p)| < |J|.
\]
For a fixed \( p \in I \times J \), let \( i = p_1 \) and \( \tilde{i} = \theta(p)_1 \). From (6.10) and (6.14) we see that
\[
(6.15) \quad |\Gamma(i)| \cdot |J| \leq |(\Gamma \times \Pi)(p)| < (|\Gamma(i)| + 1) \cdot |J|.
\]

Since \( |(\Gamma \times \Pi)(p)| = |(\Gamma \times \Pi)(\theta(p))| \) it follows that \( |\Gamma(i)| = |\Gamma(\tilde{i})| \).

For the product game, assume that \( p \to q \) and consider the intersection of the output sets. If \( q \) is a vertical output, i.e. \( p_1 = q_1 = i \) then
\[
(6.16) \quad (\Gamma \times \Pi)(p) \cap (\Gamma \times \Pi)(q) = [\Pi_i(p) \cap \Pi_i(q)] \cup [\bigcup \{J_k : k \in \Gamma(i)\}].
\]

Thus, if \( q \) is a vertical output, we have as in (6.15)
\[
(6.17) \quad |\Gamma(i)| \cdot |J| \leq |(\Gamma \times \Pi)(p) \cap (\Gamma \times \Pi)(q)| < (|\Gamma(i)| + 1) \cdot |J|.
\]

On the other hand, if \( p \to q \) with \( p_1 \neq q_1 \) and hence \( i = p_1 \to q_1 = j \), then all of the vertical outputs of \( q \) are outputs of \( p \), but \( J_i \) contains no outputs of \( q \). So in that case
\[
(6.18) \quad (\Gamma \times \Pi)(p) \cap (\Gamma \times \Pi)(q) = [\Pi_j(q)] \cup [\bigcup \{J_k : k \in \Gamma(i) \cap \Gamma(j)\}].
\]

Since \( j \in \Gamma(i) \setminus \Gamma(j) \), in this case
\[
(6.19) \quad |(\Gamma \times \Pi)(p) \cap (\Gamma \times \Pi)(q)| < |\Gamma(i)| \cdot |J|.
\]

Since \( \theta \) is an automorphism, it maps \( (\Gamma \times \Pi)(p) \) to \( (\Gamma \times \Pi)(\theta(p)) \) and commutes with intersection. Since \( |\Gamma(i)| = |\Gamma(\tilde{i})| \) it now follows from (6.17) and (6.19) that \( \theta \) maps the vertical outputs of \( p \) to the vertical outputs of \( \theta(p) \).

That is, \( p \to q \) and \( p_1 = q_1 \) implies \( \theta(p) \to \theta(q) \) and \( \theta(p)_1 = \theta(q)_1 \). Since \( \Pi \) is a tournament, it follows that for every \( p \neq q \in J_i \) either \( p \to q \) or \( q \to p \). So we can define \( \rho \) on \( I \) so that \( \theta(p)_i = \rho(p_1) \) and \( \theta \) maps \( J_i \) to \( J_{\rho(i)} \). Hence, we can define \( \gamma_i \) on \( J \) so that \( \theta(p) = (\rho(p_1), \gamma_i(p_2)) \).

That is, \( \theta = \rho \times \gamma \) at least as set maps. Since \( \theta \) is a bijection, \( \rho \) and all the \( \gamma_i \) are bijections. Uniqueness of \( \rho \) and the \( \gamma_i \) is obvious. Finally, it is clear that each \( \gamma_i \) and \( \rho \) preserve the output relation and so are themselves automorphisms.

Using the maps \( i(\gamma) = 1_I \times \gamma \) and \( p(\rho \times \gamma) = \rho \) we obtain the short exact sequence
\[
(6.20) \quad [Aut(\Pi)]^I \longrightarrow Aut(\Gamma \times \Pi) \longrightarrow Aut(\Gamma),
\]
which splits by using \( j(\rho) = \rho \times (1_J)^I \).

It follows that
\[
(6.21) \quad |Aut(\Gamma \times \Pi)| = |Aut(\Gamma)| \cdot |Aut(\Pi)|^{|I|}.
\]
Now let $\Gamma_1$ be the game of size 3 so that $|Aut(\Gamma_1)| = 3$. Inductively for $k = 2, 3, \ldots$ define $\Gamma_k = \Gamma_{k-1} \ltimes \Gamma_1$ so that $\Gamma_k$ is a game on a set of size $3^k$. So from (6.21) it follows that $|Aut(\Gamma_k)|$ is $3^r$ with $r = \sum_{t=1}^{k} 3^{k-t}$. That is,

(6.22) \[ |Aut(\Gamma_k)| = (3^{3^k-1})/2. \]

On the other hand, following [13], Dixon proved in [11] that for a tournament of size $p$ the automorphism group has order at most $(\sqrt{3})^{p-1}$ with the inequality strict unless $p = 3^n$, see also [4].

Thus, the order of the automorphism of a game on a set $I$ of size $2^n + 1$ is at most $3^n$ and the above construction yields a game with the largest possible automorphism group.

Each $\Gamma_k$ is a Steiner game. In fact we have the following, which is essentially Theorem 1.2 of Chapter 8 of [25].

**Theorem 6.10.** If $\Gamma$ and $\Pi$ are Steiner games then the product $\Gamma \ltimes \Pi$ is a Steiner game.

**Proof.** Assume that $\Gamma$ and $\Pi$ are decomposed into 3-cycles. For $p, q, r \in I \times J$ we define the Steiner 3-cycles for the product via four cases.

- $p_1 = q_1 = r_1$ and $\langle p_2, q_2, r_2 \rangle$ is one of the Steiner cycles for $\Pi$,
- $\langle p_1, q_1, r_1 \rangle$ is one of the Steiner cycles for $\Gamma$ and $p_2 = q_2 = r_2$,
- $\langle p_1, q_1, r_1 \rangle$ is one of the Steiner cycles for $\Gamma$ and $\langle p_2, q_2, r_2 \rangle$ is one of the Steiner cycles for $\Pi$,
- $\langle p_1, q_1, r_1 \rangle$ is one of the Steiner cycles for $\Gamma$ and $\langle r_2, q_2, p_2 \rangle$ is one of the Steiner cycles for $\Pi$.

Notice that if $p_1 \rightarrow q_1$ in $\Gamma$ then either $p_2 = q_2$, $p_2 \rightarrow q_2$ or $q_2 \rightarrow p_2$. \[ \square \]

Let $\Theta$ be a tournament on $K$, $\Gamma$ be a digraph on $I$ and $\pi : \Theta \rightarrow \Gamma$ a morphism of digraphs with $\pi : K \rightarrow I$ surjective. For $i \in I$, let define $K_i = \pi^{-1}(i)$ and let $\Pi_i$ be the restriction $\Theta|K_i$, which is a tournament on $K_i$. The morphism lets us regard $\Theta$ as a generalization of the lexicographic product in that for $p, q \in K$:

(6.23) \[ p \rightarrow q \iff \begin{cases} \pi(p) \rightarrow \pi(q) \text{ in } \Gamma, & \text{or} \\ \pi(p) = \pi(q) \text{ and } p \rightarrow q \text{ in } \Pi_{\pi(p)}. \end{cases} \]

Clearly, if there exists a tournament $\Pi$ such that $\Pi_i$ is isomorphic to $\Pi$ for all $i \in I$ then $\Theta$ is isomorphic to $\Gamma \ltimes \Pi$.

If for each $i \in I$, $\gamma_i$ is an automorphism of $\Pi_i$, then $\gamma \in \prod_{i \in I} \gamma_i$ is defined by $g(p) = \gamma_i(p)$ for $p \in K_i$. Thus, as in the lexicographic case
we obtain at least the inclusion of groups

\[(6.24) \prod_{i \in I} \text{Aut}(\Pi_i) \subset \text{Aut}(\Theta).\]

**Theorem 6.11.** Let \(\Theta\) be a tournament on \(K\), \(\Gamma\) be a digraph on \(I\) and \(\pi : \Theta \to \Gamma\) a morphism of digraphs with \(\pi : K \to I\) surjective.

(a) The digraph \(\Gamma\) is a tournament on \(I\).

(b) If \(\Theta\) is a game, then for each \(i \in I\), \(\Pi_i\) is a subgame of \(\Theta\).

(c) For the following three conditions, any two imply the third.

(i) \(\Theta\) is a game.

(ii) \(\Gamma\) is a game.

(iii) \(\Pi_i\) is a game for each \(i \in I\) and for \(i, j \in I\), \(|\Pi_i| = |\Pi_j|\).

**Proof.** (a) If \(i\) and \(j\) are distinct elements of \(I\) then there exist \(p, q \in K\) such that \(\pi(p) = i, \pi(q) = j\). Since \(\Theta\) is a tournament, either \(p \to q\) or \(q \to p\) which implies \(i \to j\) or \(j \to i\).

(b) From (6.23) it follows that for \(p \in K\) with \(\pi(p) = i\)

\[(6.25) \quad \Theta(p) = \Pi_i(p) \cup (\bigcup \{K_j : j \in \Gamma(i)\}).\]

If \(\pi(q) = i\), then \(|\Theta(p)| = |\Theta(q)|\) and (6.25) imply that \(|\Pi_i(p)| = |\Pi_i(q)|\). That is, the scores of all of the elements of the tournament \(\Pi_i\) are the same. Hence, \(\Pi_i\) is a game.

(c) Let \(k_i = |K_i|\) for \(i \in I\). By (b) either (i) or (iii) implies that each \(\Pi_i\) is a game. So (6.25) implies for \(p \in K\) with \(\pi(p) = i\)

\[(6.26) \quad |\Theta(p)| = \frac{k_i - 1}{2} + \sum \{k_j : j \in \Gamma(i)\}).\]

Assume (iii) so that \(k_i = k\) is independent of \(i \in I\). Then (6.26) becomes \(|\Theta(p)| = \frac{k_i - 1}{2} + |\Gamma(i)| \cdot k\). So \(|\Theta(p)|\) is the same for all \(p \in K\) if and only if \(|\Gamma(i)|\) is the same for all \(i \in I\). That is, \(\Theta\) is a game if and only if \(\Gamma\) is a game.

Finally, we assume that \(\Theta\) and \(\Gamma\) are both games and prove (iii). We know from (b) that each \(\Pi_i\) is a game and so it suffices to show that \(k_i\) is independent of \(i \in I\).

Since \(\Theta\) and \(\Gamma\) are games, \(|K| = 2m + 1\) and \(|I| = 2n + 1\) for some natural numbers \(m, n\) and from (6.26) we have

\[(6.27) \quad 2m + 1 = k_i + \sum \{2k_j : j \in \Gamma(i)\} \quad \text{and} \quad 2n + 1 = 1 + 2|\Gamma(i)|\]

for all \(i \in I\).

We require a little linear algebra result.

**Lemma 6.12.** Let \(A = (a_{ij})\) be a square matrix with integer entries. If \(a_{ij}\) is even whenever \(i \neq j\) and \(a_{ii}\) is odd for all \(i\) then \(A\) is nonsingular.
Proof. Using the ring homomorphism \( \mathbb{Z} \rightarrow \mathbb{Z}_2 \) we reduce \( A \mod 2 \) and get the identity matrix over the field \( \mathbb{Z}_2 \). The determinant is 1 in \( \mathbb{Z}_2 \) and so the determinant of \( A \) is congruent to 1 mod 2, i.e. it is odd and so is non-zero.

Define the matrix \( A \) on \( I \times I \) by
\[
(6.28) \quad a_{ij} = \begin{cases} 
2 & \text{if } i \rightarrow j, \\
0 & \text{if } j \rightarrow i, \\
1 & \text{if } i = j.
\end{cases}
\]

Let \( u, q \) be the \( I \times 1 \) matrices with \( u_i = 1 \) and \( q_i = k_i \) for \( i \in K \). From (6.27) we have \( (2^m + 1)u = Aq \) and \( (2^n + 1)u = Au \). Because the matrix \( A \) is nonsingular, we obtain \( q = \frac{2^m+1}{2^n+1}u \). Thus, \( q_i = k_i \) is independent of \( i \).

Example 6.13. There exists a morphism \( \pi : \Theta \rightarrow \Gamma \) with \( \Theta \) but not \( \Gamma \) a game and there exists a 3-cycle \( \Gamma_0 \subset \Gamma \) such that \( \pi^{-1}(\Gamma_0) \) is not a subgame of \( \Theta \).

Proof. Let \( \Gamma_3 \) be the 3-cycle game \( \langle 0, 1, 2 \rangle \) and let \( \Theta = \Gamma_3 \times \Gamma_3 \) which maps to \( \Gamma_3 \) with 3-cycle fibers \( \Pi_i \) for \( i = 0, 1, 2 \). For \( i = 1, 2 \) map \( \Pi_i \) to a single vertex \( a_i \). This maps \( \Theta \) onto a tournament \( \Gamma \) of size 5 with score vector \( (3, 2, 2, 2, 1) \). Select a vertex \( a_0 \in \Pi_0 \) to get a 3-cycle \( \langle a_0, a_1, a_2 \rangle \) in \( \Gamma \) whose preimage in \( \Theta \) is a tournament of size 7 which is not a game.

Now assume that \( \Theta \) is a subgraph of \( \Gamma \times \Pi \). If \( \Theta \) is vertical, then it is a disjoint union of the subgraphs \( \{ \Theta \cap \Pi_i : i \in I \} \). So \( \Theta \) is Eulerian if and only if each \( \Theta \cap \Pi_i \) is Eulerian. If \( \Theta \) is horizontal, then it is a disjoint union of horizontal cycles. Now assume that \( \langle i_1, \ldots, i_n \rangle \) is a cycle in \( \Gamma \). If \( \{ j_1, \ldots, j_n \} \) is an arbitrary sequence of length \( n \) in \( J \), then \( \langle (i_1, j_1), \ldots, (i_n, j_n) \rangle \) is a cycle in \( \Gamma \times \Pi \). Thus, there are \( |J|^n \) distinct cycles which project to \( \langle i_1, \ldots, i_n \rangle \) via \( \pi : \Gamma \times \Pi \rightarrow \Gamma \). Using these
we can construct explicit examples of large numbers of distinct games which are isomorphic.

Assume that \( \langle i_1, \ldots, i_n \rangle \) is a Hamiltonian cycle in the game \( \Gamma \) so that \( n = |I| \), and assume that \( \Pi \) is a group game with \( e \in J \) the identity. For \( i = i_k \) let \( \gamma_i \) be the translation of \( J \) taking \( e \) to \( j_k \) and let \( \rho \) be the identity on \( I \). Then \( \rho \times \gamma_i \) is an automorphism of \( \Gamma \) taking \( \langle (i_1, e), \ldots, (i_n, e) \rangle \) to \( \langle (i_1, j_1), \ldots, (i_n, j_n) \rangle \). So it is an isomorphism from \( \Gamma \times \Pi/\langle (i_1, e), \ldots, (i_n, e) \rangle \) to \( \Gamma \times \Pi/\langle (i_1, j_1), \ldots, (i_n, j_n) \rangle \). In \( \Gamma \times \Pi/\langle (i_1, j_1), \ldots, (i_n, j_n) \rangle \) the edges of the reversed cycle are the only edges reversed by \( \pi \). Hence, the \( |J|/|I| \) games obtained by reversing the distinct cycles are distinct isomorphic games.

In fact, if \( \Gamma \) is a game on \( I \) with \( |I| = 2n + 1 \), then by using the permutations of \( I \) we can construct \( (2n + 1)! \) games. If two of these permutations \( \rho_1, \rho_2 \) map to the same game then they differ by the automorphism \( (\rho_2)^{-1} \rho_1 \) of \( \Gamma \). It follows that \( \Gamma \) is isomorphic to exactly \( (2n + 1)! / |\text{Aut}(\Gamma)| \) distinct games. By the result of Dixon from \( [\Pi] \) quoted above, \( |\text{Aut}(\Gamma)| \leq 3^n \). Consequently, any game of size \( 2n + 1 \) is isomorphic to at least \( (2n + 1)! / 3^n \) distinct games.

If \( G \) is a subgroup of the permutation group \( S(I) \) with order \( |G| \) a power of 2, then by Proposition \ref{prop:automorphisms} the identity is the only element of \( G \) which is an automorphism of any tournament on \( I \). It follows that by applying the elements of \( G \) to any game we obtain \( |G| \) distinct games which are isomorphic.

We can use the above surjective morphism construction to build some illustrative examples of group actions on games.

**Example 6.14.** If \( G \) is a group of odd order and \( n \geq |G| \), then there exists a game \( \Theta \) on a set \( K \) of size \( 2n + 1 \) on which \( G \) acts. The set \( K \) contains two copies of \( G \) on each of which \( G \) acts by translation while the remaining \( 2(n - |G|) + 1 \) points are fixed points of the action.

**Proof.** With \( m = n - |G| \) let \( \Gamma \) be an arbitrary tournament on the set \( I = \{0, 1, \ldots, m\} \). Let \( K_0 = G \) and \( \Pi_0 \) be a group game on \( G \). So \( G \) acts be translation on \( \Pi_0 \). For \( i = 1, \ldots, m \) let \( K_i = \{i\} \) and let \( \Pi_i \) be the trivial -empty- game on the singleton \( K_i \). Let \( K = \bigcup_{i \in I} \{i\} \times K_i \) and let \( \Theta \) be the generalized lexicographic product. From \ref{prop:lexicographic} we see that \( G \) acts on the tournament \( \Theta \). \( K \) contains a copy of \( G \) on which the action is by translation and \( m \) fixed points.

Let \( \Theta \) be the double \( 2\Theta \) on \( K = \{0\} \cup K \times \{-1, +1\} \). By the injection \( 2 : \text{Aut}(\Theta) \to \text{Aut}(2\Theta) \) we obtain the action of \( G \) on \( \Theta \).

\( \square \)
Notice that if $G$ is cyclic, then the translation by a generator is a permutation of $G$ with a cycle of length $|G|$. A fixed point imposes a bound on the length of such a cycle.

**Proposition 6.15.** Let $\Gamma$ be a game on a set $I$ with $|I| = 2n + 1$. If $\rho$ is an automorphism of $\Gamma$ which fixes some vertex, then the length of any cycle in the associated permutation is at most $n$.

**Proof.** Let $i_0$ be a fixed vertex and $(i_1, \ldots, i_m)$ be a cycle in the permutation $\rho$. If $i_0 \to i_1$, then $i_0 = \rho^j(i_0) \to \rho^j(i_1) = i_{j+1}$. Hence, $\{i_1, \ldots, i_m\} \subset \Gamma(i_0)$ and because $\Gamma$ is a game $m \leq n$. Similarly, if $i_1 \to i_0$, then $\{i_1, \ldots, i_m\} \subset \Gamma^{-1}(i_0)$ and so $m \leq n$.

It can happen, however, that a large cyclic group acts effectively on a game.

**Example 6.16.** There exists a game of size 29 on which $\mathbb{Z}_{33}$ acts effectively.

**Proof.** Let $I = \{1, 2\}$ with $\{(1, 2)\} = \Gamma$. Let $K_1 = \mathbb{Z}_3$ and $K_2 = \mathbb{Z}_{11}$ and let $\Pi_1, \Pi_2$ be group games. Again let $\Theta$ be the generalized lexicographic product on $\tilde{K} = (\{1\} \times K_1) \cup (\{2\} \times K_2)$. Thus, $\Theta$ is a tournament on $\tilde{K}$ with $|\tilde{K}| = 14$. Since $\mathbb{Z}_{33}$ is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_{11}$, it acts on $\Theta$. Again let $\Theta = 2\Theta$. If $\rho$ is the generator of $\mathbb{Z}_{33}$, then the associated permutation fixes 0 and contains two 3-cycles and two 11-cycles.

We conclude the section with an obvious remark.

**Proposition 6.17.** Let $G$ act on a game $\Gamma$ if $\Delta \subset \Gamma$ is an Eulerian subgraph which is $G$ invariant, then $G$ acts on the game $\Gamma/\Delta$ with the same action on the vertices.

\[ \square \]

7. Bipartite Tournaments and Pointed Games

Recall that a digraph $\Pi$ on $I$ is bipartite when $I$ is the disjoint union of sets $J, K$ and $\Pi \subset (J \times K) \cup (K \times J)$. A cycle in a bipartite digraph has even length and so a 4-cycle is the cycle of shortest length in a
bipartite digraph. As an Eulerian digraph is a disjoint union of cycles, it follows that a bipartite Eulerian digraph has even cardinality.

We will call $\Pi$ a **bipartite tournament** on the pair $\{J, K\}$ of disjoint sets when $\Pi \cup \Pi^{-1} = (J \times K) \cup (K \times J)$. That is, each $i \in J$ has an edge connecting it to every element of $K$ and vice-versa. Hence, $|\Pi| = |J| \cdot |K|$. We say that two bipartite tournaments $\Pi$ and $\Gamma$ on the pair $\{J, K\}$ have the same **scores** when each vertex $i \in J \cup K$ has the same number of outputs in $\Pi$ and $\Gamma$, and consequently the same number of inputs. The following is the bipartite analogue of Proposition 4.1 and Theorem 4.5.

**Theorem 7.1.** Assume that $\Pi$ and $\Gamma$ are bipartite tournaments on the pair $\{J, K\}$.

(a) The difference graph $\Delta = \Delta(\Pi, \Gamma)$ is Eulerian if and only if $\Pi$ and $\Gamma$ have the same scores.

(b) If $\Pi$ and $\Gamma$ have the same scores, then there exists a finite sequence $\Pi_1, \ldots, \Pi_k$ of bipartite tournaments on $\{J, K\}$ with $\Pi_1 = \Pi$ and $\Pi_k = \Gamma$ and such that for $p = 1, \ldots, k - 1$, $\Pi_{p+1}$ is $\Pi_p$ with some 4-cycle reversed. In particular, all of these bipartite tournaments have the same scores.

If $\Delta$ a single cycle of length $2\ell$ then a sequence can be chosen with $k = \ell - 1$.

**Proof.** (a) is proved exactly as is Proposition 4.1.

(b) As in the proof of Theorem 4.5 we reduce to the case when $\Delta$ is a single cycle whose necessarily even length we write as $2\ell$. We proceed by induction beginning with $\ell = 2$ in which case $\Gamma$ is obtained by reversing $1 = \ell - 1$ 4-cycle.

Assume $\ell > 2$. If $\Delta$ consists of the cycle cycle $\langle i_1, \ldots, i_{2\ell} \rangle$ with $\ell \geq 3$, we use two cases as before.

**Case 1** ($i_1 \to i_4$ in $\Pi$): In this case, $\langle i_1, i_4, \ldots, i_{2\ell} \rangle$ is an $2(\ell - 1)$ cycle in $\Pi$ and so we can get from $\Pi$ to $\tilde{\Gamma} = \Pi/\langle i_1, i_4, \ldots, i_{2\ell} \rangle$ via a sequence of $\ell - 2$ 4-cycles by inductive hypothesis. Then we go from $\tilde{\Gamma}$ to $\Gamma$ by reversing the 4-cycle $\langle i_1, i_2, i_3, i_4 \rangle$ in $\tilde{\Gamma}$. Furthermore, $\Gamma = \Pi/\langle i_1, \ldots, i_{\ell} \rangle = \tilde{\Gamma}/\langle i_1, i_2, i_3, i_4 \rangle$ because the edge between $i_1$ and $i_4$ has been reversed twice.

**Case 2** ($i_4 \to i_1$ in $\Pi$): We proceed as before reversing the order of operations. This time $\langle i_1, i_2, i_3, i_4 \rangle$ is a 4-cycle in $\Pi$ and $\langle i_1, i_4, \ldots, i_{\ell} \rangle$ is an $2(\ell - 1)$ cycle in $\tilde{\Gamma} = \Pi/\langle i_1, i_2, i_3, i_4 \rangle$. Proceed as before.

The following is now obvious.
Corollary 7.2. If $\Pi$ is a bipartite tournament on the pair $\{J,K\}$ then $\Gamma \leftrightarrow \Delta(\Pi, \Gamma)$ is a bijective correspondence between the set of bipartite tournaments $\Gamma$ on the pair $\{J,K\}$ with the same scores as $\Pi$ and the set Eulerian subgraphs of $\Pi$. In particular, the cardinality of the set of Eulerian subgraphs depends only on the set of scores.

We can form the interchange graph on the set of bipartite tournaments on the pair $\{J,K\}$, connecting $\Pi$ and $\Gamma$ by an undirected edge if $\Delta = \Delta(\Pi, \Gamma)$ is a single 4-cycle. In general, it is clear that the distance between $\Pi$ and $\Gamma$ is bounded by $|\Delta| - \sigma(\Delta)$ where, as before, the span $\sigma(\Delta)$ is the size of a maximum decomposition of $\Delta$ by disjoint cycles.

Our brief consideration of bipartite digraphs is motivated by their application to the pointed games which we will now consider.

A pointed game $\Pi$ on $I$ is a game with a chosen vertex which we label 0. Let $I_+ = \Pi^{-1}(0)$ and $I_- = \Pi(0)$. With $I_+, I_-$ fixed we call $\Pi$ a pointed game on the pair $(I_+, I_-)$. We denote by $\Pi_+$ and $\Pi_-$ the tournaments which are the restrictions of $\Pi$ to $I_+$ and $I_-$ respectively. We let $\Xi$ denote $\Pi|_{I_+ \cup I_- \setminus (\Pi_+ \cup \Pi_-)}$. It consists of those edges which connect elements of $I_+$ with those of $I_-$. Thus, $\Xi$ is a bipartite tournament on the pair $\{I_+, I_-\}$. The union $\Pi_+ \cup \Pi_- \cup \Xi$ consists of all edges of $\Pi$ except those which connect to 0.

Recall that a subset $A \subset I$ is invariant for a relation $\Pi$ on $I$ when $\Pi(A) \subset A$, in which case, $B = I \setminus A$ is invariant for $\Pi^{-1}$. We think of the pair $\{B, A\}$ as a splitting for $\Pi$. A nontrivial splitting for a relation $\Pi$ exists if and only $\Pi$ is not strong. Since any game is strong, no game admits a splitting, but the tournaments $\Pi_\pm$ and the bivariate tournament $\Xi$ need not be strong.

Lemma 7.3. (Splitting Lemma) Let $\Pi$ be a pointed game on the pair $(I_+, I_-)$ and $A \subset I \setminus \{0\}$. Let $A_\pm = A \cap I_\pm$. The following are equivalent.

(i) $A$ is invariant for $\Xi^{-1}$.

(ii) $A_\pm$ is invariant for $\Pi_\pm$ and, in addition, either $|A_+| = |A_-|$ or $|A_+| = |A_-| - 1$.

Proof. Let $|I| = 2n + 1$. With $B = I \setminus (A \cup \{0\})$ let $a_\pm = |A_\pm|$ and $b_\pm = |B_\pm|$ with $B_\pm = B \cap I_\pm$. If $2n + 1 = |I|$, then $n = |I_\pm|$ and so $a_\pm + b_\pm = n$. Furthermore, $|A| = a_+ + a_-$ and $|B| = b_+ + b_-$. Let $\Pi|A$ be the restriction of $\Pi$ to $A$.

(i) $\Rightarrow$ (ii): Let $i \in A_+$. By assumption, every $\Pi$ edge between $i$ and an element of $B_-$, i. e. every $\Xi$ edge between $i$ and an element of $B$,
is an output from $i$, accounting for $b_-$ outputs from $i$. There is also one output to 0. Because $\Pi$ is a game there are a total of $n$ outputs from $i$. Thus, in $\Pi\mid A$, the element $i$ has at most $n - b_- - 1 = a_- - 1$ outputs. Thus, the total number of outputs in $\Pi\mid A$ from elements of $A_+$ is bounded by $a_+ \cdot (a_- - 1)$.

Similarly, if $i \in A_-$, then every $\Pi$ edge between $i$ and an element of $B_+$ is an output. Thus, in $\Pi\mid A$ the element $i$ has at most $n - b_+ = a_+$ outputs. Thus, the total number of outputs in $\Pi\mid A$ from elements of $A_-$ is bounded by $a_- \cdot a_+$.

Every $\Pi\mid A$ output comes from an element of $A_+$ or $A_-$. The total number of outputs is the total number of edges which is $\frac{1}{2}(a_+ + a_-)(a_+ + a_- - 1)$. Thus, we obtain

\begin{equation}
\frac{1}{2}(a_+ + a_-)(a_+ + a_- - 1) \leq a_+(a_- - 1) + a_- a_+.
\end{equation}

Furthermore, if any $i \in A_{\pm}$ has an output to $B_{\pm}$, i.e., if either $A_+$ is not $\Pi_+$ invariant or $A_-$ is not $\Pi_-$ invariant, then the inequality is strict. But this inequality can be rewritten as:

\begin{equation}
(a_- - a_+)^2 \leq (a_- - a_+).
\end{equation}

Because $a_+$ and $a_-$ are integers this equation cannot hold strictly and the only way it can hold is if $a_- - a_+$ equals 0 or 1.

(ii) $\Rightarrow$ (i): If $i \in A_+$, then every edge from $B_+$ is an input. In $\Pi\mid A$ $i$ has at most $n - b_+ = a_+$ inputs. Similarly, if $i \in A_-$, then every edge from $B_-$ is an input and it has an input from 0. In $\Pi\mid A$ it has at most $n - b_- - 1 = a_- - 1$ inputs. Again the total number of inputs is the total number of edges. This time we get

\begin{equation}
\frac{1}{2}(a_+ + a_-)(a_+ + a_- - 1) \leq (a_+)^2 + (a_-)^2 - a_-
\end{equation}

and if $A$ is not $\Xi^{-1}$ invariant then this inequality is strict.

This time the inequality can be rewritten as the reverse of (7.2)

\begin{equation}
(a_- - a_+)^2 \geq (a_- - a_+).
\end{equation}

This is always true and can, of course, be strict. We will see below that $A_{\pm}$ can both be $\Pi_{\pm}$ invariant without $\Xi^{-1}$ invariance of $A$.

But if, in addition, $a_- - a_+$ equals 0 or 1, then this inequality is not strict and so $A$ must be $\Xi^{-1}$ invariant.

\[ \square \]

For a pointed game $\Pi$ on $2n+1$ vertices, $\Pi_+$ and $\Pi_-$ are tournaments on $n$ vertices. We now show that these tournaments can be chosen arbitrarily.
Theorem 7.4. Let $\Gamma_+$ and $\Gamma_-$ be tournaments on the set $J$ of size $n$. Let $I_+ = J \times \{+1\}$, $I_- = J \times \{-1\}$ and $I = \{0\} \cup I_- \cup I_+$. There exists a pointed game $\Pi$ on the pair $(I_+, I_-)$ such that with the identification $j \mapsto (j, -1) \Pi_- = \Gamma_-$ and with the identification $j \mapsto (j, +1) \Pi_+ = \Gamma_+$. 

Proof. Let $\Delta = \Delta(\Gamma_-, \Gamma_+)$. We prove the result by induction on $k = |\Delta|$. Notice that we make no assumption about the scores and so $\Delta$ need not be Eulerian.

$(k = 0)$ In this case, $\Gamma_- = \Gamma_+$ and we use $\Pi = 2\Gamma_-$. 

$(k > 0)$ Choose $(i, j) \in \Delta$. Let $\hat{\Delta} = \Delta \setminus \{(i, j)\}$ and $\hat{\Gamma} = \Gamma_- / \hat{\Delta}$. By induction hypothesis there exists a pointed game $\hat{\Pi}$ with $\hat{\Pi} = \Gamma_-$ and $\hat{\Pi}_+ = \hat{\Gamma}$. Let $i+ = (i, +1), j+ = (j, +1)$. Then $(i+, j+)$ is an edge of $\hat{\Pi}_+$ and with the identification above $j \mapsto (j, +1)$ we have $\hat{\Pi}_+/(i+, j+) = \Gamma_+$. 

Let $\hat{\Xi}$ be the bipartite tournament of connections between $I_+$ and $I_-$ in $\hat{\Pi}$. We first show that there exists a $\hat{\Xi}$ path from $j+$ to $i+$. 

Let $A = \{i+\} \cup O\hat{\Xi}^{-1}(i+)$. This is a $\hat{\Xi}^{-1}$ invariant set which contains $i+$. The Splitting Lemma 7.3 implies that $A_+ = A \cap I_+$ is $\hat{\Pi}_+$ invariant. Since $(i+, j+) \in \hat{\Pi}_+$ it follows that $j+ \in A_+ \subseteq A$. This exactly says that there is a $\hat{\Xi}$ path from $j+$ to $i+$. 

By eliminating repeated vertices as usual we get a simple $\hat{\Xi}$ path. Concatenating with the edge $(i+, j+)$ we obtain a cycle $C$ in $\hat{\Pi}$. The cycle is disjoint from $\hat{\Pi}_-$ and does not contain the vertex 0. Furthermore, it intersects $\Pi_+$ only in the edge $(i+, j+)$. It follows that $\Pi = \hat{\Pi}/C$ is a pointed game with $\Pi_- = \Gamma_-$ and $\Pi_+ = \Gamma_+$ as required.

While the input and output tournaments in a game can be arbitrary, this is not true for group games.

Lemma 7.5. Let $A$ be a game subset for the group $\mathbb{Z}_{2n+1}$ with $\Gamma[A]$ the associated game. If $i \in A$ with $i$ relatively prime to $2n + 1$ and the score of $i$ in the tournament $\Gamma[A]|A$ is $n - 1$, i.e. $|A \cap (i + A)| = n - 1$, then $A = m_i([1, n])$.

Proof. First, assume that $i = 1$. Every element of $A$ other than 1 itself is an output of 1. That is, if $a \neq 1$ is an element of $A$, then $a - 1 \in A$. So if $m = \max\{i \in A : i = 1, 2, \ldots, 2n\}$ then, inductively, $m - 1, m - 2, \ldots, 1 \in A$ and $m + 1, \ldots, 2n \not\in A$. Since $|A| = n$, $m = n$. 

In general, if $i$ is relatively prime to $2n + 1$, then multiplication $m_i$ is a group isomorphism on $\mathbb{Z}_{2n+1}$ and with $B = (m_i)^{-1}(A)$ it is an
isomorphism from $\Gamma[B]$ to $\Gamma[A]$ taking $B$ to $A$. So $1 \in B$ with score $n - 1$ for the restriction to $B$. Hence, $B = [1, n]$ and so $A = m_i([1, n])$.

\[\square\]

**Theorem 7.6.** Let $A$ be a game subset for a group $G$ with $|G| = 2n + 1$ prime. If there exists $i \in A$ such that the score of $i$ in the tournament $\Gamma[A]|A$ is $n - 1$, then there is an isomorphism of $G$ with $\mathbb{Z}_{2n+1}$ mapping $i$ to 1 and $A$ to $[1, n]$. In particular, the score vector for the tournament $\Gamma[A]|A$ is $(0, 1, 2, \ldots, n - 1)$.

**Proof.** A group of prime order is cyclic and so $G$ is isomorphic to $\mathbb{Z}_{2n+1}$ by an isomorphism which maps $i$ to some element $j \in [1, 2n]$ and so is relatively prime to $2n + 1$. Apply the previous lemma and compose with $(m_j)^{-1}$. So the score vector of $\Gamma[A]|A$ is the score vector of the restriction to $[1, n]$ of the associated game on $\mathbb{Z}_{2n+1}$.

\[\square\]

**Remarks:** (a) If $2n + 1$ is not prime there can be other games on $\mathbb{Z}_{2n+1}$ with an element of having score $n - 1$ in the tournament $\Gamma[A]|A$. For example, with $2n + 1 = 9$ and $A = \{1, 3, 4, 7\}$ the element 3 has score 3 and the restricted tournament has score vector $(1, 1, 1, 3)$.

(b) It follows that if $2n + 1$ is a prime then a tournament $\Pi$ with score vector $(1, 1, 2, \ldots, n - 3, n - 3, n - 1)$ cannot occur as the restriction $\Gamma[A]|A$ for a game subset $A$ of a group of order $2n + 1$.

From Theorem 7.4 it follows that the number of pointed games $\text{Games}(I_+, I_-)$ on the pair $(I_+, I_-)$ with $|I| = 2n + 1$ is bounded below by the square of the number of tournaments on a set of size $n$. It is bounded above by the number of tournaments on a set of size $2n$, i.e.

\[2^{n(n-1)} \leq |\text{Games}(I_+, I_-)| \leq 2^{n(2n-1)}.\]

It is clear than neither estimate is sharp.

If $\Gamma_+$ and $\Gamma_-$ are tournaments of size $n$, and $\Pi$ is a pointed game with $\Pi_- = \Gamma_-$ and $\Pi_+ = \Gamma_+$ then the scores of the elements of the associated bipartite tournament $\Xi$ are clearly determined by the score vectors of $\Gamma_-$ and $\Gamma_+$. Thus, the number of pointed games $\Pi$ which similarly satisfy $\Pi_- = \Gamma_-$ and $\Pi_+ = \Gamma_+$ is the number of Eulerian subgraphs of $\Xi$ for any one of them.

**Example 7.7.** If $\Gamma = \Gamma_- = \Gamma_+$ and $\Pi = 2\Gamma$, then from the cycles in $\Gamma$ we can obtain cycles in $\Xi$. 
Proof. If $\langle i_1, i_2, \ldots, i_{2\ell} \rangle$ is a cycle in $\Gamma$ of even length, then $\langle i_{2\ell}+, i_{2\ell-1}-, \ldots, i_1- \rangle$ and $\langle i_{2\ell}-, i_{2\ell-1}+, \ldots, i_1+ \rangle$ are disjoint cycles of this same length in $\Xi$.

If $\langle i_1, i_2, \ldots, i_{2\ell+1} \rangle$ is a cycle of odd length in $\Gamma$, then $\langle i_{2\ell+1}+, i_{2\ell}-, \ldots, i_2+, i_1- \rangle$ is a cycle in $\Xi$ of double the length. In addition, $\langle i_{2\ell+1}-, i_{2\ell+1}+, i_{2\ell}-, \ldots, i_2+, i_1- \rangle$ is a cycle in $\Xi$ of double the length. Furthermore, for each $1 \leq k \leq \ell$

$\langle i_{2\ell+1}+, i_{2\ell}-, \ldots, i_{2k}-, i_{2k}+, i_{2k-1}-, \ldots, i_1- \rangle$

and $\langle i_{2\ell+1}-, i_{2\ell}-, \ldots, i_{2k-1}-, i_{2k-1}+, i_{2k-2}-, \ldots, i_1+ \rangle$

are cycles of length $2\ell + 2$ in $\Xi$. Observe that each of these has $\ell + 1$ edges of the form $(i_p+, i_{p-1}-) \pmod{2\ell + 1}$. It follows that no pair of these cycles associated with $\langle i_1, i_2, \ldots, i_{2\ell+1} \rangle$ are disjoint.

If $n = 3$ and $\Gamma = \langle i_1, i_2, i_3 \rangle$, then then for $\Pi = 2\Gamma$ these are the only cycles in $\Xi$. That is, $\Xi$ contains two 6-cycles and three 4-cycles, no two of which are disjoint. Thus, $\Xi$ contains six Eulerian subgraphs (including the empty subgraph). \qed

When a pointed game is the double of $\Pi$ on $I$ then $i- \rightarrow i+$ for all $i \in I$. This convenient pairing need not be possible for all pointed games.

**Example 7.8.** There is a pointed game $\Pi$ on the pair $(I_+, I_-)$ for which there does not exist a bijection $\rho : I_- \rightarrow I_+$ such that $i \rightarrow \rho(i)$ for all $i \in I_-$. Proof. Let $\Gamma$ and $\Theta$ be games on disjoint sets $J$ and $K$ with $|J| = 3$ and $|K| = 5$. With $I = J \cup K$ let $I_\pm = I \times \{\pm 1\}$ with similar notation for $J_\pm, K_\pm \subset I_\pm$.

Let $\Gamma_\pm, \Theta_\pm$ be copies of the games $\Gamma$ and $\Theta$ on $J_\pm$ and $K_\pm$. Define the pointed game $\Pi$ on $\{0\} \cup I_+ \cup I_-$ to be $I_+ \times \{0\} \cup \{0\} \times I_-$ together with

$$2\Gamma \cup \Theta_+ \cup \Theta_- \cup [K_+ \times K_-] \cup [(J_- \cup J_+) \times K_+] \cup [K_- \times (J_- \cup J_+)].$$

(7.6)

We can represent $\Pi$ by the diagram
Since the five elements of $K_-$ have upward outputs only among the three elements of $J_+$, the required bijection cannot exist.

On the other hand, a strengthening of this condition characterizes the games which are doubles.

**Theorem 7.9.** Let $\Pi$ be a pointed game on the $(I_+, I_-)$. If there exists a bijection $\rho : I_- \rightarrow I_+$ such that $i \rightarrow \rho(i)$ and $\Pi$ is reducible via $i \rightarrow \rho(i)$ for all $i \in I_-$, then as a pointed game $\Pi$ is isomorphic to the double $2(\Pi|I_-)$.

*Proof.* By relabeling, we may assume that $I_\pm = J \times \{\pm 1\}$ and $\rho(i-) = i+$. In that case, we will show that $\Pi = 2\Pi_-$ with $\Pi_-$ the game on $J$ which is identified with $\Pi|I_-$ via the identification $i \mapsto i-$. Let $\Delta = \Delta(2\Pi_-, \Pi)$. Since the two games agree on $I_-$, $\Delta$ is disjoint from $\Pi|I_-$. Furthermore, $i- \rightarrow i+$ for all $i \in J$ in both games and so no such edge is in $\Delta$. Furthermore, no edge containing 0 is in $\Delta$. We will show that if $\Delta$ is nonempty, then $\Pi$ is not reducible via $j- \rightarrow j+$ for some $j \in J$.

**Case 1** ($\Delta \cap \Xi \neq \emptyset$): That is, there exists $i \rightarrow j$ in $\Pi_-$ such that either $j+ \rightarrow i-$ or $j- \rightarrow i+$ is in $\Delta$. If $j+ \rightarrow i-$ in $\Delta$ then $i- \rightarrow j+$ in $\Pi$. Since $i- \rightarrow j-$ in $\Pi$ it follows from Proposition 2.2 that $\Pi$ is not reducible via $j- \rightarrow j+$. If $j- \rightarrow i+$ is in $\Delta$, then, since $\Delta$ is Eulerian, there exists an edge to $j-$ in $\Delta$. Since 0 is not a vertex of $\Delta$ and $\Delta$ is disjoint from $\Pi|I_-$, it follows that for some $k \in J$, $k+ \rightarrow j-$ in $\Delta$. As before, $\Pi$ is not reducible via $k- \rightarrow k+$.

**Case 2** ($\Delta \subseteq \Pi|I_+$): There exists $i \rightarrow j$ in $\Pi_-$ such that $i+ \rightarrow j+$ in $\Delta$ and so $j+ \rightarrow i+$ in $\Pi$. Since $\Delta$ is disjoint from $\Xi$, $j+ \rightarrow i-$ in $\Pi$. Hence, $\Pi$ is not reducible via $i- \rightarrow i+$.

Finally, we have some observations about reducibility.
Proposition 7.10. Let \( \Pi \) be a pointed game on the \((I_+, I_-)\). If \( i, j \in I_+ \) or \( i, j \in I_- \), then \( \Pi \) is not reducible via \( \{i, j\} \). If \( i \in I_+ \), then \( \Pi \) is reducible via \( i \rightarrow 0 \) if and only if \( k \rightarrow i \) for all \( k \in I_+ \setminus \{i\} \). If \( i \in I_- \), then \( \Pi \) is reducible via \( 0 \rightarrow i \) if and only if \( k \rightarrow i \) for all \( k \in I_- \).  

Proof. If \( i, j \in I_+ \), then \( i, j \rightarrow 0 \) and so \( \Pi \) is not reducible via \( \{i, j\} \) by Proposition 2.2(b). If \( i, j \in I_- \), then \( 0 \rightarrow i, j \). If \( i, j \in I_+ \) and \( j \rightarrow i \), then since \( j \rightarrow 0 \), \( \Pi \) is not reducible via \( \{0, i\} \). Conversely, if \( i \rightarrow k \) for all \( k \in I_+ \setminus \{i\} \), then \( \Pi(i) = \{0\} \cup (I_+ \setminus \{i\}) \) while \( \Pi(0) = I_- \). So \( \Pi \) is reducible via \( \{i, 0\} \) by Proposition 2.2(b) again. Similarly, for \( i \in I_- \). \( \square \)

Corollary 7.11. If \( \Pi \) is a game, then \( \Gamma = (2\Pi)/\Pi_+ \) is a non-reducible game.

Proof. If \( i \rightarrow j \) in \( \Pi \) then \( i- \rightarrow j-, j- \rightarrow i+, j+ \rightarrow i-, j- \rightarrow i+, i- \rightarrow i+, j- \rightarrow j+ \) in \( \Gamma \). Hence, \( j+ \rightarrow i-, i+, i- \rightarrow j-, i+, j+, i- \rightarrow i+ \), and so \( \Gamma \) is not reducible via any of there pairs. The remaining pairs are excluded by Proposition 7.10. \( \square \)

We close the section with an improvement of Proposition 6.1.

Proposition 7.12. If \( \Pi \) is a digraph with \( n \) vertices, then \( \Pi \) is a subgraph of a game of size \( 2n - 1 \).

Proof. As in the proof of Proposition 6.1 we may assume that \( \Pi \) is a tournament. Let \( \Pi' \) be an order on \( n \) vertices or, more generally, a tournament on \( n \) vertices with a vertex of score \( n - 1 \). By Theorem 7.4 there is a pointed game \( \Gamma \) with \( \Gamma_+ \) isomorphic to \( \Pi' \) and with \( \Gamma_- \) isomorphic to \( \Pi \). If \( u \) is the vertex of \( \Gamma^{-1}(0) \) with score \( n-1 \) in \( \Gamma_+ \), then Proposition 7.10 implies that \( \Gamma \) is reducible via \( u \rightarrow 0 \). The restriction of \( \Gamma \) to the vertices excluding \( u \) and \( 0 \) is a subgame of size \( 2n - 1 \) which contains \( \Gamma_- \). \( \square \)

Notice that if \( \Pi \) itself contains a vertex with score \( n - 1 \) or \( 0 \), then the smallest possible size for a game which contains \( \Pi \) is \( 2(n - 1) + 1 = 2n - 1 \). Thus, if one does not restrict the score vector of the tournament \( \Pi \), Proposition 7.12 is the best possible result.
8. Interchange Graphs, Again

Fix $I = \{0\} \cup I_+ \cup I_-$, disjoint sets with $|I| = n$. Define $I_0 = I \setminus \{0\} = I_+ \cup I_-$. Let $n[I_0]$ denote the set of subsets of $I_0$ of cardinality $n$ so that $|n[I_0]| = \binom{2n}{n}$. If $Games(I)$ is the set of games on $I$ then $\Gamma \mapsto \Gamma(0)$ is a mapping $\pi : Games(I) \to n(I_0)$. If $J_- \in n(I_0)$, and $J_+ = I_0 \setminus J_-$, then a game $\Gamma$ has $\pi(\Gamma) = J_-$ exactly when $\Gamma$ is a pointed game on $(J_+, J_-)$. Furthermore, if $\rho$ is a bijection of $I$ which fixes 0 and maps $I_-$ to $J_-$, then $\rho$ is an isomorphism of any pointed game on $(I_+, I_-)$ onto a pointed game on $(J_+, J_-)$. In particular, for every $J_- \in n[I_0]$ the cardinality of $\pi^{-1}(J_-)$ is that of $Games(I_+, I_-)$. In particular, we have from (7.5)

$$|Games(I)| = \binom{2n}{n} \cdot |Games(I_+, I_-)| \geq \binom{2n}{n} \cdot 2^{n(n-1)}.$$  

Observe that $\binom{2n}{n} \cdot 2^{n(n-1)} = \binom{2n}{n} \cdot \prod_{j=1}^{n-1} 2^{2j}$ we see that this is an improvement on the bound $\prod_{j=1}^{n} \binom{2j}{j}$ given in Theorem 4 of [20]. Note that $\binom{2j}{j} < 2^{2j}$ since the number of subsets of size $j$ is less than the total number of subsets.

Using this inequality we can obtain a lower bound for the number of isomorphism classes of games of a fixed size.

**Proposition 8.1.** Let $IS(n)$ denote the cardinality of the set of isomorphism classes of games on a set of size $2n+1$.

$$IS(n) \geq 2^{n(n-1)} \div [(2n+1) \cdot (n!)^2].$$

If $n \geq 7$, then

$$\ln(IS(n)) \geq n \cdot [\ln n - 2 \ln n].$$

**Proof.** We obtain (8.2) by dividing (8.1) by the order of the permutation group which is $(2n+1)!$. Now we observe that

$$n(n-1) \ln 2 = n^2 \ln 2 - (n+1) \ln 2 + \ln 2,$$

$$\ln(2n+1) \leq \ln 2 + \ln(n+1),$$

$$\ln(n+1) = \ln n + \ln(1 + \frac{1}{n}) \leq \ln n + \frac{1}{n}.$$  

Furthermore,

$$\ln(n!) \leq \int_2^{n+1} \ln t \ dt = (n+1)\ln(n+1) - (n+1) - 2 \ln 2 + 2 \leq n \ln n + 1 + \ln(n+1) - (n+1) - 2 \ln 2 + 2.$$
Putting these together we obtain
\[(8.6)\]
\[
\ln(IS(n)) \geq n \cdot [n \ln 2 - 2 \ln n] + \left[(2 - \ln 2)(n+1) - 3 \ln(n+1) + 4 \ln 2 - 6\right].
\]
The function \( t \mapsto (2 - \ln 2)t - 3 \ln t + 4 \ln 2 - 6 \) is increasing for \( t \geq 3 \) and it is positive for \( t = 8 \) and so for \( t \geq 8 \).

\[\square\]

The set \( n[I_0] \) has a natural undirected graph structure, with \((J_1, J_2)\) an edge if \(|J_1 \cap J_2| = n - 1\). That is, \( J_2 \) is obtained from \( J_1 \) by exchanging the element of \( J_1 \setminus J_1 \cap J_2 \) with the element of \( J_2 \setminus J_1 \cap J_2 \) which is in the complement of \( J_1 \). Each element of \( J_1 \) can be paired up with an element of its complement and each of these \( n^2 \) choices yields a different set \( J_2 \). Thus, \( n[I_0] \) is an \( n^2 \) regular graph. The distance from \( J_1 \) to \( J_2 \) is \( k \) when \(|J_1 \cap J_2| = n - k\). In that case, there are \((k!)^2\) geodesics between \( J_1 \) and \( J_2 \). These are obtained by choosing an ordering on \( J_1 \setminus (J_1 \cap J_2) \) and on \( J_2 \setminus (J_1 \cap J_2) \) for the \( k \) exchanges.

Now suppose \((\Gamma_1, \Gamma_2)\) is an edge in the interchange graph on \( Games(I) \). That is, \( \Gamma_2 \) is obtained from \( \Gamma_1 \) by reversing some 3-cycle \( \langle i, j, k \rangle \). Assume that \( \Gamma_1 \) is a pointed graph on \((I_+, I_-)\). If 0 is not a vertex of the cycle then the cycle intersects either \( \Pi_+ \) or \( \Pi_- \) but not both. It must meet one of them because the bivariate tournament \( \Xi \) contains no 3-cycle. It cannot meet both because a cycle which intersects both \( \Pi_+ \) and \( \Pi_- \) contains at least four vertices. In this case, \( \Gamma_2 \) is also a pointed graph on \((I_+, I_-)\). Thus, \( \pi(\Gamma_1) = \pi(\Gamma_2) \).

If \( 0 \) is a vertex of the cycle then the cycle is \( \langle i, 0, k \rangle \) with \( i \in I_+, k \in I_- \) and \( (k, i) \) an edge in \( \Xi \). In that case, \( \Gamma_2(0) = I_- \cup \{i\} \setminus \{k\} \). That is, \( \pi(\Gamma_2) \) is connected by the edge in \( n[I_0] \) with \( \pi(\Gamma_1) \) via the interchange of \( k \) with \( i \). By Theorem 2.6 the vertex 0 is contained in \( n(n+1)/2 \) 3-cycles. Each of these leads to a different element of \( n[I_0] \). Thus, the edges from \( \Gamma_1 \) project to \( n(n+1)/2 \) of the \( n^2 \) edges from \( I_- \) in \( n[I_0] \).

On the other hand, if \( J = I_- \cup \{i_1\} \setminus \{k_1\} \) for arbitrary \( i_1 \in I_- \), \( k_1 \in I_+ \) then the product of transpositions \((i_1, i)\) and \((k_1, k)\) is a permutation \( \rho \) which induces an isomorphism from \( \Gamma_1 \) to \( \hat{\Gamma}_1 \) which has \( \pi(\Gamma_1) = \pi(\hat{\Gamma}_1) \) and \( \langle i_1, 0, k_1 \rangle \) is a cycle of \( \hat{\Gamma}_1 \).

For an undirected graph \( G \) we will call a set \( T \) of vertices \emph{convex} when for all \( t_1, t_2 \in T \) there exists a geodesic of \( G \) between \( t_1 \) and \( t_2 \) and for every geodesic of \( G \) between \( t_1 \) and \( t_2 \) the vertices are all contained in \( T \).

**Theorem 8.2.** Let \( \Pi \) be a game on a set \( I \) of vertices and let \( Q \subset I \). Define \( Q^* \subset \Pi \) by \( (i, j) \in Q^* \) if and only if \( i \in Q \) or \( j \in Q \) (or both).
Let $Games(Q^*)$ be the set of games $\Gamma$ on $I$ such that $Q^* \subset \Gamma$. That is, every edge which connects to a vertex of $Q$ has the same orientation in $\Gamma$ as in $\Pi$. The set $Games(Q^*)$ is a convex subset of the interchange graph of all games on $I$.

Proof. : Clearly a game $\Gamma$ lies in $Games(Q^*)$ if and only if the Eulerian subgraph $\Delta(\Gamma, \Pi)$ is disjoint from $Q^*$. Recall that by Corollary 4.8 the distance between $\Gamma$ and $\Pi$ is $\beta(\Delta(\Gamma, \Pi))$.

Assume that $\Delta(\Gamma, \Pi)$ is disjoint from $Q^*$ and that we reverse a 3-cycle which meets $Q^*$ to obtain $\Gamma'$ which is not in $Games(Q^*)$. It suffices to show that $\beta(\Delta(\Gamma', \Pi)) = \beta(\Delta(\Gamma, \Pi)) + 1$ for then $\Gamma'$ cannot lie on a geodesic from $\Gamma$ to $\Pi$. We consider the cases from Theorem 4.7.

Let the reverse of the given 3-cycle be $\langle i_1, i_2, i_3 \rangle$ so that $\langle i_3, i_2, i_1 \rangle$ is in $\Gamma$. By assumption at least one of the vertices, say $i_2$, is in $Q$. Hence, $(i_3, i_2), (i_2, i_1) \in Q^*$ and the vertex $i_2$ does not occur in $\Delta(\Gamma', \Pi)$. Hence, $(i_2, i_3), (i_1, i_2) \in \Delta(\Gamma', \Pi)$ and these are the only edges of $\Delta(\Gamma', \Pi)$ which contain the vertex $i_2$. Hence, a cycle which contains $i_2$ from any decomposition for $\Delta(\Gamma', \Pi)$ must contain both these edges.

In the notation of the proof of Theorem 4.7 we first consider Case 1, with the cycle $\langle i_3, i_2, i_1 \rangle$ disjoint from $\Delta$ and with $(i_2, i_3), (i_1, i_2)$ in a single cycle of the maximum decomposition for $\Delta'$. As shown there, $\beta(\Delta') = \beta(\Delta) + 1$.

Alternatively, we could be in Case 3, with $(i_1, i_3) \in \Delta$ and so neither $i_1$ nor $i_3$ is in $Q$. Again since $(i_2, i_3), (i_1, i_2)$ are in a single cycle of the decomposition of $\Delta'$ we again get $\beta(\Delta') = \beta(\Delta) + 1$.

\[\square\]

**Corollary 8.3.** With $I = \{0\} \cup I_+ \cup I_-$, the set $Games(I_+, I_-)$ is a convex subset of the interchange graph of games on $I$. For each $J \in n[I_0]$ the set $\pi^{-1}(J)$ is a convex subset of the interchange graph of games on $I$.

Proof. The above theorem applies with $Q = \{0\}$. \[\square\]

9. Homogeneous Games

For a subgroup $H$ of a group $G$ the double coset of $i \in G$ is the set $HiH$. Clearly, $\{(i, j) \in G \times G : i \in HjH\}$ is an equivalence relation with equivalence classes the double cosets. Of course, $H$ itself is the double coset of the identity element $e$. 
Lemma 9.1. Let $G$ be a finite group with odd order and let $H$ be a subgroup of $G$. Let $i \in G$.

(a) If $i^2 \in H$ then $i \in H$.
(b) If $i \not\in H$ then $i^{-1} \not\in HiH$.

Proof. By Lagrange’s Theorem $H$ and $i$ have odd order.

(a): Since 2 is relatively prime to the order of $i$, $i^2$ is a generator of the cyclic group generated by $i$. Hence, if $i^2 \in H$, then $i$ is in the subgroup generated by $i^2$ which is contained in the subgroup $H$.

(b): Assume $i^{-1} \in HiH$ and so $i^{-1} = h_1h_2$ for some $h_1, h_2 \in H$. Then $(ih_1)(ih_1) = h_2^{-1}h_1 \in H$. So by (a), $ih_1 \in H$ and $i = (ih_1)h_1^{-1} \in H$. □

Definition 9.2. Let $G$ be a finite group with odd order and let $H$ be a subgroup of $G$. A subset $A$ of $G$ is a game subset for $(G, H)$ if it is a game subset of $G$ such that $i \in A \setminus H$ implies $HiH \subset A$.

Theorem 9.3. If $G$ is a finite group with odd order and $H$ is a subgroup of $G$, then there exist game subsets for $(G, H)$.

Proof. By Lemma 9.1 (b) the double cosets $HiH$ and $Hi^{-1}H$ are distinct for all $i \in G \setminus H$. Let $T$ be the set of double cosets other than $H$. We can partition $T$ by pairs $\{HiH, Hi^{-1}H\} : i \in G \setminus H$. Choose $i_1, \ldots, i_k \in G \setminus H$ so that $\{Hi_1H, \ldots, Hi_kH\}$ includes exactly one double coset from each pair. Let $A_0 \subset H$ be a game subset for the odd order group $H$. Let $A = A_0 \cup \bigcup_{p=1}^{k} Hi_pH$. Since $H = A_0 \cup A_0^{-1} \cup \{e\}$ and since $Hi^{-1}H$ consists of the inverses of the elements of $HiH$, it follows that $A$ is a game subset. Clearly, $A \setminus H$ is a union of double cosets.

Remark: It is clear that this construction yields all the game subsets for $(G, H)$. Hence, if $2d$ is the number of double cosets in $G \setminus H$ and $2k + 1$ is the order of $H$, then there are $2^{d+k}$ game subsets for $(G, H)$.

Theorem 9.4. Let $G$ be a finite group with odd order, $H$ be a subgroup of $G$ and $A$ be a game subset for $(G, H)$. Let $G/H$ be the homogeneous space of left cosets, i.e. $G/H = \{iH : i \in G\}$. Define $A/H = \{iH : i \in A\}$.
The set $\Gamma[A/H] = \{(iH, jH) : i^{-1}jH \in A/H\}$ is a game on $G/H$.

For each $k \in G$ the bijection $\ell_k$ on $G/H$ given by $iH \mapsto kiH$ is an automorphism of $\Gamma[A/H]$ and so there is a group homomorphism from $G$ to $\text{Aut}(\Gamma[A/H])$.

The surjection $\pi : G \to G/H$ given by $i \mapsto iH$ is a morphism from $\Gamma[A]$ to $\Gamma[A/H]$.

Proof. Notice first that $iH \subset A$ requires $i \in G \setminus H$ since $A \cap H$ is a proper subset of $H$ (e.g. $e \notin A$). Hence if $i^{-1}jH \subset A$ then with $\hat{i} = ih_1, \hat{j} = jh_2$ then $\hat{i}^{-1}\hat{j}H \subset A$ because $A \setminus H$ is a union double cosets. Thus, $iH \to jH$ if and only if $i^{-1}j \in A \setminus H$. Notice that $iH = \pi(i) = \pi(j) = jH$ if and only if $i^{-1}j \in H$. Thus, we see that $\Gamma[A/H]$ is a tournament and that $\pi$ is a morphism from $\Gamma[A]$ to $\Gamma[A/H]$.

We see that $iH \to jH$ if and only if $j \in i(A \setminus H)$ if and only if $i \in j(A \setminus H)^{-1}$. Hence, the set of inputs and the set of outputs of $iH$ with respect to $\Gamma[A/H]$ both have cardinality $|A \setminus H|/|H|$. Hence, $\Gamma[A/H]$ is Eulerian and so is a game.

Finally, it is clear that $\ell_k$ is an automorphism of $\Gamma[A/H]$.

\[\square\]

Remark: Clearly, $\ell_k$ acts as the identity on $G/H$ if and only if $iki^{-1} \in H$ for all $i \in G$. So the action of $G$ on $G/H$ is effective, i.e. $\ell_k$ acts as the identity only for $k = e$, exactly when $\hat{H} = \bigcap_{i \in G} iHi^{-1}$ is the trivial subgroup, or, equivalently, when $\{e\}$ is the only subgroup of $H$ which is normal in $G$. In general, $G/\hat{H}$ acts effectively on $G/H$ and so injects into $\text{Aut}(\Gamma[A/H])$.

We call the game $\Gamma[A/H]$ a homogeneous game. Of course, a group game is a special case of a homogeneous game with $H$ the trivial subgroup. A game subset for $G$ is a game subset for $(G, \{e\})$.

Theorem 9.5. Let $G$ be a finite group with odd order, $H$ be a normal subgroup of $G$, so that $\pi : G \to G/H$ is a group homomorphism onto the quotient group. A subset $A$ of $G$ is a game subset for $(G, H)$ if and only if there exist $B$ a game subset for $G/H$ and $A_0$ a game subset of $H$ so that $A = A_0 \cup \pi^{-1}(B)$. In that case, the games $\Gamma[A/H]$ and $\Gamma[B]$ are equal.

Proof. When $H$ is normal, a double coset is just a coset. Thus, $A$ is a game subset for $(G, H)$ if and only if $A_0 \cap A$ is a game subset for $H$ and $A \setminus H$ is a union of cosets. Normality of $H$ implies that $(iH)^{-1}jH =
Let $\pi$ be a game on $\Pi$ and let $a \in I$. We say that a group $G$ acts on $\Pi$ when $G$ acts on $I$ and for each $g \in G$, $i \mapsto g \cdot i$ is an automorphism of $\Pi$. Thus, an action of $G$ on $\Pi$ is given by a group homomorphism $G \to \text{Aut}(\Pi)$ and we can identify $g \in G$ with the associated automorphism.

For $a \in I$ the evaluation map $\iota_a : G \to I$ is defined by $\iota_a(g) = g \cdot a$. Let $\text{Iso}_a = \{g : g \cdot a = a\} = \iota_a^{-1}(\{a\})$ be a subgroup of $G$ called the isotropy subgroup of $a$. Let $G a = \iota_a(G) \subset I$ denote the $G$ orbit of $a$ and let $\Pi_a = \Pi \cap (G a \times G a)$ be the restriction of $\Pi$ to $G a$. Of course, $G$ acts transitively on $I$ exactly when $G a = I$ in which case $\Pi_a = \Pi$.

**Theorem 9.6.** Let $\Pi$ be a game on $I$ with $a \in I$. Let $G$ be a finite group of odd order which acts on $\Pi$. For example, $G$ can be any subgroup of $\text{Aut}(\Pi)$. Let $H = \text{Iso}_a = \iota_a^{-1}(\{a\})$. Choose $A_0$ a game subset for $H$ and let $A = A_0 \cup \iota_a^{-1}(\Pi(a))$.

The set $A \subset G$ is a game subset for $(G, H)$. Let $\pi : G \to G/H$ be the canonical projection. The map $\iota_a$ is a morphism from $\Gamma[A]$ to $\Pi$ and it factors through $\pi$ to define $\theta_a : G/H \to I$ which is an injective morphism from $\Gamma[A/H]$ to $\Pi$. The restriction $\Pi \mid G a$ of $\Pi$ to $G a$ is a subgame of $\Pi$ and the bijection $\theta_a : G/H \to G a$ is an isomorphism from $\Gamma[A/H]$ to $\Pi \mid G a$.

**Proof.** Observe first that by Proposition 1.8 every element of $\text{Aut}(\Pi)$ has odd order and so by the first Sylow Theorem, $\text{Aut}(G)$ itself has odd order. By replacing $G$ by its image under the map $G \to \text{Aut}(\Pi)$ we may assume that $G$ is a subgroup of $\text{Aut}(\Pi)$. Technically, we use Theorem 9.4 because, instead of $G$, we are using its quotient by the kernel of this map.

Clearly, for $b \in I$, if $g(a) = b$ then $\iota_a^{-1}(\{b\}) = gH$. Now suppose that $a \to b$ in $\Pi$. For $h \in H$, $h^{-1}(a) = b = g(a)$. Since $h$ acts as an automorphism of $\Pi$, $a = hh^{-1}(a) \to h(b) = h(g(a))$. Hence, $hg \in \iota_a^{-1}(\Pi(a))$. It follows that $\iota_a^{-1}(\Pi(a))$ is a union of $H$ double cosets. Since $g$ is an automorphism, $g^{-1}(a) \to g^{-1}g(a) = a$. Hence, $g^{-1} \notin A$. Finally, if $g \notin H$, i.e. $g(a) \neq a$ then either $a \to g(a)$ and so $g \in A$ or $g(a) \to a$ and, as before, $a \to g^{-1}(a)$ which implies $g^{-1} \in A$. It follows that $A$ is a game subset for $(G, H)$. For $g, k \in G$, $k \to g$ if and only if $k^{-1}g \in A$. So, when $k(a) \neq g(a)$,

\[(9.1) \quad k \to g \iff k^{-1}g \in \iota_a^{-1}(\Pi(a)) \iff a \to k^{-1}g(a) \iff k(a) \to g(a).\]
This says that \( \iota_a \) is a morphism from \( \Gamma[A] \) to \( \Pi \).

Since \( g(a) = b \) implies \( \iota_a^{-1}(\{b\}) = gH \), it follows that \( \iota_a \) factors through \( \pi \) to define the injection \( \theta_a \).

Since \( \pi \) is a surjective morphism from \( \Gamma[A] \) to \( \Gamma[A/H] \) and \( \iota_a \) is a morphism from \( \Gamma[A] \) to \( \Pi \), it easily follows that \( \theta_a \) is a morphism from \( \Gamma[A/H] \) to \( \Pi | Ga \). Since \( \Gamma[A/H] \) is Eulerian and \( \bar{\theta}_a : \Gamma[A/H] \to \Pi | Ga \) is a bijection, it follows that \( \Pi_a \) is Eulerian and so is a subgame of \( \Pi \) with \( \theta_a : G/H \to Ga \) an isomorphism from \( \Gamma[A/H] \) to \( \Pi | Ga \).

We immediately obtain the following.

**Corollary 9.7.** If \( Aut(\Pi) \) acts transitively on the vertices of \( \Pi \), then \( \Pi \) is isomorphic to a homogeneous game.

\( \square \)

In general, for a game \( \Pi \) on \( I \) the restriction of \( \Pi \) to each orbit of the action of \( Aut(\Pi) \) on \( I \) is a subgame isomorphic to a homogeneous game.

**Corollary 9.8.** Assume that \( \xi \) is an automorphism of a game \( \Pi \) on \( I \), so that \( \xi \) is a permutation of \( I \). Assume that \( (a_0, \ldots, a_{2n}) \) is a nontrivial cycle in the permutation \( \xi \), so that \( n > 1 \). The restriction \( \Pi | \{a_0, \ldots, a_{2n}\} \) is a subgame of \( \Pi \) which is isomorphic to a group game on \( \mathbb{Z}_{2n+1} \).

**Proof.** By Proposition 1.8, \( \xi \) has odd order and so every cycle contained in it has odd length. Let \( G \) be the cyclic subgroup of \( Aut(\Pi) \) generated by \( \xi \) so that \( Ga_0 = \{a_0, \ldots, a_{2n}\} \).

It follows from Theorem 9.6 that the restriction is a subgame of \( \Pi \). The map \( i \mapsto a_i \) for \( i = 0, \ldots, 2n \) is a bijection \( \rho : \mathbb{Z}_{2n+1} \to Ga_0 \) which maps the translation \( \ell_1 \) to \( \xi \).

We define the game \( \Gamma \) on \( \mathbb{Z}_{2n+1} \) so that \( \rho \) is an isomorphism from \( \Gamma \) to \( \Pi | Ga \), i.e. \( i \to j \) if and only if \( a_i \to a_j \). Then \( \ell_1 \) is an automorphism of \( \Gamma \) and so the translations of \( \mathbb{Z}_{2n+1} \) are all automorphisms. It follows from Theorem 3.4(a) that \( \Gamma \) is a group game on \( \mathbb{Z}_{2n+1} \).

\( \square \)

**Theorem 9.9.** Let \( G \) be a finite group with odd order, \( H \) be a subgroup of \( G \) and \( A \) be a game subset for \((G, H)\) with \( A_0 = A \cap H \) the game subset of \( H \). The game \( \Gamma[A] \) is isomorphic to the lexicographic product

---

**ROCK, PAPER, SCISSORS, ETC**

---
\[ \Gamma[A/H] \ltimes \Gamma[A_0]. \] In particular, \( \operatorname{Aut}(\Gamma[A]) \) is isomorphic to the semi-direct product \( \operatorname{Aut}(\Gamma[A/H]) \ltimes \operatorname{Aut}(\Gamma[A_0])^{G/H}. \)

**Proof.** Let \( j : G/H \to G \) be a map such that \( \pi \circ j = 1_{G/H} \). So if \( x \in G/H \) then \( x \) is the coset \( j(x)H \). We identify \( G \) with the product \( G/H \times H \) by the bijection \( (x, h) \mapsto j(x)h \). This identifies \( \pi : G \to G/H \) with the first coordinate projection. Notice that we are not assuming that \( H \) is normal and so \( G/H \) need not be a group. Even if it is normal, a group homomorphism splitting \( j \) need not exist. However, we do not need any algebraic conditions on \( j \).

As was observed in the proof of Theorem 9.4, if \( x_1 \neq x_2 \) then \( x_1 \to x_2 \) in \( \Gamma[A/H] \) if and only if \( h_1^{-1}j(x_1)^{-1}j(x_2)h_2 \in A\setminus H \) for some \( h_1, h_2 \in H \) and so for all \( h_1, h_2 \in H \) since \( A \setminus H \) is a union of double cosets. Hence,

\[
(9.2) \quad (x_1, h_1) \to (x_2, h_2) \iff x_1 \to x_2 \quad \text{when} \quad x_1 \neq x_2.
\]

On the other hand, if \( x_1 = x_2 \) then \((j(x_1)h_1)^{-1}(j(x_2)h_2) = h_1^{-1}h_2\). So \((j(x_1)h_1)^{-1}(j(x_2)h_2) \in A\) if and only if \( h_1^{-1}h_2 \in A_0 \). Thus

\[
(9.3) \quad (x_1, h_1) \to (x_2, h_2) \iff h_1 \to h_2 \quad \text{when} \quad x_1 = x_2.
\]

From (6.8) we see that this is exactly the lexicographic product \( \Gamma[A/H] \ltimes \Gamma[A_0] \). The automorphism result then follows from (6.13).

\[ \square \]

**Corollary 9.10.** (a) The lexicographic product of two group games is isomorphic to a group game.

(b) The lexicographic product of two homogeneous games is isomorphic to a homogeneous game.

**Proof.** (a): Let \( A_0 \) be a game subset of a group \( H \) and \( B \) be a game subset of a group \( T \). Let \( G \) be any extension of \( H \) by \( T \). That is, there is a short exact sequence \( H \xrightarrow{i} G \xrightarrow{p} T \). Let \( A = i(A_0) \cup p^{-1}(B) \). For example, we can use the group product \( G = T \times H \).

By Theorems 9.5 and 9.9 \( A \) is a game subset of \( G \) and \( \Gamma[A] \) is isomorphic to \( \Gamma[B] \ltimes \Gamma[A_0] \).

(b): By Corollary 9.7 a game \( \Gamma \) on \( I \) is isomorphic to a homogeneous game if and only if \( \operatorname{Aut}(\Gamma) \) acts transitively on \( I \). If \( p, q \in \Gamma \ltimes \Pi \) and \( \rho \in \operatorname{Aut}(\Gamma), \phi \in \operatorname{Aut}(\Pi) \) satisfy \( \rho(p_1) = q_1 \) and \( \phi(p_2) = q_2 \) then with \( \gamma_i = \phi \) for all \( i \in I \), \( \rho \times \gamma(p) = q \). Thus, \( \operatorname{Aut}(\Gamma \ltimes \Pi) \) acts transitively when \( \operatorname{Aut}(\Gamma) \), and \( \operatorname{Aut}(\Pi) \) do.

\[ \square \]
Remark: If $H, T$ are groups of odd order and $H \xrightarrow{i_1} G_1 \xrightarrow{p_1} T$, $H \xrightarrow{i_2} G_2 \xrightarrow{p_2} T$. are possibly different group extensions, it follows from the above proof that the game $\Gamma[A]$ is isomorphic to $\Gamma[A_2]$ are the same, where $A_\epsilon = i_\epsilon(A_0) \cup p_\epsilon^{-1}(B)$ for $\epsilon = 1, 2$.

**Example 9.11. Commutative group examples.**

With $G = \mathbb{Z}_{2(a+1)(2b+1)}$ we define the injection $\theta : \mathbb{Z}_{2b+1} \rightarrow G$ by $\theta(j) = j(2a + 1)$ for $j = 0, \ldots , 2b$ and let $\pi : \mathbb{Z}_{(2a+1)(2b+1)} \rightarrow \mathbb{Z}_{(2a+1)}$ be the surjection with $\pi(j(2a + 1) + i) = i$ for $i = 0, \ldots , 2a$, $j = 0, \ldots , 2b$.

We identify $\mathbb{Z}_{2b+1}$ with the subgroup $H = \theta(\mathbb{Z}_{2b+1})$ generated by $2a + 1$ in $\mathbb{Z}_{(2a+1)(2b+1)}$ and we identify $\mathbb{Z}_{2a+1}$ with the quotient group $G/H$.

If $B \subset \mathbb{Z}_{2a+1}, A_0 \subset \mathbb{Z}_{2b+1}$ are game subsets then $A = A_0 \cup \pi^{-1}(B)$ satisfies

$$j(2a + 1) + i \in A \iff \begin{cases} i \in B & \text{or} \\ i = 0 \text{ and } j \in A_0. & \end{cases}$$

By Theorems 9.9 and 9.5 $\Gamma[A]$ is the lexicographic product $\Gamma[B] \ltimes \Gamma[A_0]$. Furthermore, the translation map $\ell_1$ on $\mathbb{Z}_{(2a+1)(2b+1)}$ is given by $\rho \cdot \gamma_i$ with $\rho = \ell_1$ on $\mathbb{Z}_{2a+1}$ and $\gamma_i$ equal to $\ell_1$ on $\mathbb{Z}_{2b+1}$ for $i = 2n$ and equal to the identity on $\mathbb{Z}_{2b+1}$ for the remaining $i$.

It follows that if $2n + 1$ is composite $(2a + 1)(2b + 1)$, then there exist game subsets such that the automorphism group of the associated game is non-abelian and so contains $\mathbb{Z}_{2n+1}$ as proper subgroup. Notice that by considering translations alone for $\rho$ and the $\gamma_i$'s we see that the order of $\text{Aut}(\Gamma[B] \ltimes \Gamma[A_0])$ is at least $(2a + 1)(2b + 1)^{2a+1}$. On the other hand, the entire affine group on $\mathbb{Z}_{(2a+1)(2b+1)}$ has order $(2a + 1)(2b + 1) \cdot \phi((2a + 1)(2b + 1)) < (2a + 1)^2(2b + 1)^2$. Note that if $b \geq 1$ and $a \geq 2$ then $2a + 1 < 3^a \leq (2b + 1)^a < (2b + 1)^{2a-1}$. Hence, in these examples there are always automorphisms which are not affine. Observe that if $2a + 1$ and $2b + 1$ are distinct Fermat primes, e.g. 3 and 5, then by Theorem 3.11 the translations are the only affine automorphisms of any $\Gamma[B] \ltimes \Gamma[A_0]$.

If $2a + 1$ and $2b + 1$ are relatively prime then the product group $\mathbb{Z}_{2a+1} \times \mathbb{Z}_{2b+1}$ is isomorphic as a group to $\mathbb{Z}_{(2a+1)(2b+1)}$. If $2a + 1$ and $2b + 1$ are not relatively prime, e.g. if they are equal, then the product group $G = \mathbb{Z}_{2a+1} \times \mathbb{Z}_{2b+1}$ is not isomorphic to $\mathbb{Z}_{(2a+1)(2b+1)}$. Nonetheless, it is an extension of $\mathbb{Z}_{2b+1}$ by $\mathbb{Z}_{2a+1}$ and so has game subsets isomorphic to the lexicographic product $\Gamma[B] \ltimes \Gamma[A_0]$. 
Finally, we note that, by induction, the Steiner game $\Gamma_k$ described at the end of Section 6 is isomorphic to a group game on $\mathbb{Z}_{3^k}$. Define $L_1$ to be the set of natural numbers such that the first nonzero digit in the base three expansion is 1 (rather than 2). Using induction again, one can show that $A = \{i \in L_1 : 0 < i < 3^k\} \subset \mathbb{Z}_{3^k}$ is an example with $\Gamma[A]$ isomorphic to $\Gamma_k$.

10. Games of Size Seven

Now we consider the case $7 = 2 \cdot 3 + 1$.

**TYPE I** - $\Gamma_I = \Gamma[[1, 2, 3]]$ has $Aut(\Gamma[[1, 2, 3]]) = \mathbb{Z}_7$ acting via translation and is reducible via each pair $i, i+3$. The collection $\{m_a([1, 2, 3]) : a \in \mathbb{Z}_7^*\}$ are the $6 = \phi(7)$ Type I game subsets of $\mathbb{Z}_7$ whose games are isomorphic to $\Gamma[[1, 2, 3]]$. See Theorem 3.14 and Corollary 3.9.

The group game $\Gamma_I$ is isomorphic to the double with $\Pi$ the straddle on $[1, 2, 3]$, see Example 6.3.

**Type II** - $\Gamma_{II} = \Gamma[[1, 2, 4]]$ can be described by the following diagram:

$$
\begin{array}{c}
0 \\
\downarrow \quad \downarrow \quad \downarrow \\
1 \quad 2 \quad 4
\end{array}
\begin{array}{c}
\uparrow \quad \uparrow \quad \uparrow \\
3 \quad 6 \quad 5
\end{array}
$$

Clearly, $m_a$ is an automorphism of $\Gamma_{II}$ for $a \in \{1, 2, 4\} \subset \mathbb{Z}_7^*$. Let $\rho$ be an automorphism of $\Gamma_{II}$. By composing with a translation we may assume that $\rho(0) = 0$. Then $\{1, 2, 4\} = \Gamma_{II}(0)$ is $\rho$ invariant. By composing with an element of $G_{[1,2,4]}$ we may assume $\rho(1) = 1$. Then Proposition 1.8 implies that $\rho$ fixes, 2 and 4 as well. From the diagram it then follows that $\rho$ is the identity. Thus, every automorphism is affine, i.e. a composition of a translation and a multiplication by an element of element of $\{1, 2, 4\} \subset \mathbb{Z}_7^*$.

From Theorem 3.14 it follows that $\Gamma_{II}$ is not reducible.

The two Type II game subsets are $[1, 2, 4]$ and $[6, 5, 3] = m_6([1, 2, 4])$. $\Gamma[[6, 5, 3]]$ is the reversed game of $\Gamma[[1, 2, 4]]$ and is isomorphic to it via $m_6 = m_{-1}$.

With $\Pi$ the 3-cycle $\langle 1, 2, 4 \rangle$ game, $\Gamma_{III}$ is isomorphic to $2\Pi/\Pi_+$. That is, the double with the cycle $\langle 3, 6, 5 \rangle$ of the double reversed.
Type III- $\Gamma_{III}$ can be described by the following diagram:

\[
\begin{array}{c}
0 \\
\downarrow \\
\downarrow \\
(1 \rightarrow 2 \rightarrow 4) \\
\uparrow \\
\uparrow \\
\uparrow \\
3 \rightarrow 6 \rightarrow 5 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow 4
\end{array}
\]

With $\Pi$ the 3-cycle $\langle 1, 2, 4 \rangle$ game, $\Gamma_{III}$ is isomorphic to $2\Pi$. Proposition 6.4 implies that $Aut(\Gamma_{III}) = \{m_1, m_2, m_4\}$ with 0 as a fixed point. Thus, $Aut(\Gamma_{III})$ does not act transitively on $\mathbb{Z}_7$.

Since $\Pi$ is isomorphic to its reversed game, it follows that $\Gamma_{III}$ is isomorphic to its reversed game as well.

$\Gamma_{III}$ is reducible but is not reducible via any pair which includes 0.

Theorem 10.1. If $\Gamma$ is a game with 7 vertices then $\Gamma$ is isomorphic to exactly one of $\Gamma_I, \Gamma_{II}$ or $\Gamma_{III}$.

Proof. The three types are distinguished by their automorphism groups and so no two are isomorphic.

We use the labeling procedure as in Theorem 2.1. Choose a vertex and label it 0. The three output vertices in $\Gamma(0)$ form either a 3-cycle or a straddle. Similarly for the three input vertices of $\Gamma^{-1}(0)$.

Case 1 [The inputs and outputs both form straddles]: Label the output vertices 1, 2, 3 with $\langle 1 \rightarrow 2 \rightarrow 4 \rangle$. Each of these receives one input from one of the vertices in $\Gamma^{-1}(0)$. Label by 3 the vertex such that $3 \rightarrow 4$. 3 now has three outputs and so $1, 2 \rightarrow 3$. Label by 5 so that $5 \rightarrow 2$ and so $6 \rightarrow 1$. Now there are two possibilities. Either $3 \rightarrow 5$ which is Type II or $5 \rightarrow 3$ which is Type III.

Case 2 [The inputs form a 3-cycle and the outputs form a straddle, or vice-versa]: By replacing the game by its reverse if necessary we may assume that the outputs form a straddle. Notice that for $\Gamma_{III}$ the
inputs $\Gamma_{III}^{-1}(1)$ form a 3-cycle and the outputs $\Gamma_{III}(1)$ form a straddle. Relabel the vertex 0 of $\Gamma$, calling it 1. Label the vertices of $\Gamma(1)$ as 2, 3, 5 with $5 \to 2, 3$ and $2 \to 3$. Now $5 \to 2, 3$ and $1 \to 5$. Hence, there is one output vertex from 5 among the $\Gamma^{-1}(1)$. Label it 0, so that $5 \to 0$ and choose the remaining two labels so that $\langle 6 \to 0 \to 4 \rangle$ is the input 3-cycle for 1. It suffices to show that the remaining connections are determined by these choices. We began with $0, 4, 6 \to 1 \to 2, 3, 5$.

- $1, 5, 2 \to 3 \Rightarrow 3 \to 6, 0, 4$.
- $3, 5, 6 \to 0 \Rightarrow 0 \to 1, 2, 4$.
- $0, 1, 5 \to 2 \Rightarrow 2 \to 3, 4, 6$.
- $5 \to 0, 2, 3 \Rightarrow 1, 4, 6 \to 5$.
- $4 \to 5, 6, 1 \Rightarrow 0, 2, 3 \to 4$.

This is Type III.

Since $\Gamma_{III}$ is isomorphic to its reversed game, it follows that if the inputs form a straddle and the outputs form a 3-cycle then it is Type III as well.

\[\square\]

From the proof we obtain the following corollary.

**Corollary 10.2.** Let $\Gamma$ be a game on $I$ with $|I| = 7$.

1. If for some $i \in I$ both the input set $\Gamma^{-1}(i)$ and the output set $\Gamma(i)$ form straddles then $\Gamma$ is of Type I, isomorphic to $\Gamma_I$. In that case, for every $j \in I$ the input set and the output set are straddles.
2. The game $\Gamma$ is of Type II, isomorphic to $\Gamma_{II}$, if and only if for every $j \in I$ the input set and the output set are 3-cycles.
3. If for some $i \in I$ either the input set or the output set forms a straddle while the other is a 3-cycle, then $\Gamma$ is of Type III, isomorphic to $\Gamma_{III}$.

\[\square\]

Observe that $\Gamma_{II}$ is obtained from $\Gamma_{III}$ by reversing the upper 3-cycle. It follows from Theorem 6.5 that the games of Type II are Steiner games. In fact, the games of type III are Steiner games as well. To obtain a decomposition by 3-cycles for $\Gamma_{II}$ or $\Gamma_{III}$ we may use the upper 3-cycle and, in addition,

\[\langle 1, 2, 6 \rangle, \langle 2, 4, 5 \rangle, \langle 4, 1, 3 \rangle, \langle 2, 3, 0 \rangle, \langle 4, 6, 0 \rangle, \langle 1, 5, 0 \rangle.\]
Using the results from Section 7 we can compute the number of games on a set $I$ of size seven. Using (8.1) it suffices to compute $|Games(I_+, I_-)|$ with $I$ decomposed as $I_+ \cup \{0\} \cup I_-$. If either $\Pi_+$ or $\Pi_-$ is a straddle then it follows from the Splitting Lemma, and is easy to check directly, that the bivariate tournament $\Xi$ contains no cycles. If both $\Pi_+$ and $\Pi_-$ are 3-cycles, then the number of Eulerian subgraphs of $\Xi$ is the same as the number in the special case when $\Pi$ is the double of a 3-cycle. In Example 7.7 it was shown that $\Xi$ then contains six distinct Eulerian subgraphs (including the empty one). Thus, our lower bound $|Games(I_+, I_-)| \geq 2^3 \cdot 2 = 64$ has to be corrected to account for each of the four cases where $\Pi_+$ and $\Pi_-$ are 3-cycles and so there are six pointed games $\Pi$ instead of one each. That is, $|Games(I_+, I_-)| = 64 + 6 \cdot 4 - 4 = 84$. Finally, from (8.1) it follows that when $|I| = 7$ 

\begin{equation}
|Games(I)| = \binom{6}{3} \cdot 84 = 1680.
\end{equation}

11. Isomorphism Examples

If $\rho : \Pi_1 \rightarrow \Pi_2$ is an isomorphism between digraphs, and $[i_1, \ldots, i_k]$ is a simple path in $\Pi_1$, then $[\rho(i_1), \ldots, \rho(i_k)]$ is a simple path in $\Pi_2$. Since the order is preserved, it follows that if $[j_1, \ldots, j_k]$ is a simple path in $\Pi_2$, then (recall that $\bar{\rho} = \rho \times \rho$) 

\begin{equation}
\bar{\rho}([i_1, \ldots, i_k]) \subset [j_1, \ldots, j_k] \Rightarrow r(i_p) = j_p \text{ for } p = 1, \ldots, k.
\end{equation}

**Proposition 11.1.** (a) If $\rho : \Pi_1 \rightarrow \Pi_2$ is an isomorphism of games, then restricts to an isomorphism between the reducibility digraphs $\rho : r\Pi_1 \rightarrow r\Pi_2$. Furthermore, it maps each maximal simple path in $r\Pi_1$ to a maximal simple path in $r\Pi_2$.

(b) Assume $\rho$ is an automorphism of a game $\Pi$.

(i) If $[i_1, \ldots, i_k]$ is a simple path in $\Pi$ and $\bar{\rho}([i_1, \ldots, i_k]) \subset [i_1, \ldots, i_k]$ then $\rho$ fixes $i_p$ for $p = 1, \ldots, k$. In particular, if $\rho$ maps a maximal simple path of $r\Pi$ to itself, then it fixes every vertex of the path. If $\rho$ fixes some vertex of a maximal simple path of $r\Pi$, then it fixes every vertex of the path.

(ii) If $\Pi$ is reducible via $i \rightarrow j$ and either $i$ or $j$ is fixed by $\rho$ then the other is as well.

**Proof.** (a): This is clear since $\Pi_1$ is reducible via $i \rightarrow j$ if and only if $\Pi_2$ is reducible via $\rho(i) \rightarrow \rho(j)$.
(b)(i): Clearly, \( \rho \) preserves the ordering on the paths. If \( \rho \) fixes a vertex of a maximal simple path of \( r\Pi \), then it maps to itself the maximal simple path of \( r\Pi \) which contains the vertex. Hence, it fixes every vertex on the path.

(ii): If \( \Pi \) is reducible via \( i \rightarrow j \), then the edge \( (i, j) \) lies in a maximal simple path of \( r\Pi \). So if one vertex is fixed, then the other is.

\[ \square \]

If two tournaments are isomorphic, then of course their doubles are isomorphic. If \( \rho : 2\Pi \rightarrow 2\Gamma \) is an isomorphism with \( \rho(0) = 0 \), then \( \rho \) is itself the double of an isomorphism from \( \Pi \) to \( \Gamma \). What happens when \( \rho \) is not an isomorphism of pointed games on \( 0 \)?

Let \( \Pi \) be a tournament on \( I \) with \( |I| = n \). The double \( 2\Pi \) is a game on \( \{0\} \cup I_+ \cup I_- \) with \( I_{\pm} = I \times \{\pm 1\} \). Let \( i, j \in I \) with \( i \rightarrow j \). We recall the following reducibility results which follow from Proposition 2.2.

(i) The game \( 2\Pi \) is reducible via \( i_- \rightarrow i_+ \) and via \( j_- \rightarrow j_+ \). By uniqueness in Proposition 2.2(f) it is not reducible via \( j_- \rightarrow i_+ \). Observe, for example, that \( i_- \rightarrow j_- \) and \( i_- \rightarrow i_+ \).

(ii) Similarly, \( 2\Pi \) is not reducible via \( i_- \rightarrow j_- \) because \( i_-, j_- \rightarrow 0 \), nor is it reducible via \( i_+ \rightarrow j_+ \).

(iii) The game is reducible via \( j_+ \rightarrow i_- \) if and only if for every \( k \in I \setminus \{i, j\} \) either \( i, j \rightarrow k \) or \( k \rightarrow i, j \).

(iv) The game is reducible via \( i_+ \rightarrow 0 \) if and only if \( i \rightarrow k \) for all \( k \in I \setminus \{i\} \) and it reducible via \( 0 \rightarrow i_- \) if and only if \( k \rightarrow i \) for all \( k \in I \setminus \{i\} \).

Now let \( \Gamma \) be a tournament on \( K \) with \( |K| = n \). \( 2\Gamma \) is a game on \( \{0\} \cup K_+ \cup K_- \).

We now describe how \( \rho : 2\Pi \rightarrow 2\Gamma \) can be an isomorphism which is not an isomorphism of pointed games, i.e. \( \rho(0) \neq 0 \). Assume that \( \rho(0) = k_1+ \). Since \( \rho^{-1}(k_1-) \rightarrow 0 \) we have that \( \rho(i_1+) = k_1- \) for some \( i_1 \in I \). Now \( 2\Gamma \) is reducible via \( \rho(i_1-) \rightarrow \rho(i_1+) \) and so \( \rho(i_1-) \in \{0\} \cup K_+ \). We build parallel sequences of distinct elements in the domain and range which are mapped across by \( \rho \).
\begin{equation}
\begin{array}{c}
0 \rightarrow k_1+ \\
\uparrow \uparrow \\
i_1+ \rightarrow k_1- \\
\uparrow \uparrow \\
i_1- \rightarrow k_2+ \\
\uparrow \uparrow \\
i_2+ \rightarrow k_2- \\
\uparrow \uparrow \\
i_2- \rightarrow k_3+
\end{array}
\end{equation}

The way the sequences terminate, as of course they must, is when for some \( m \geq 1 \), \( \rho(i_m-) \) is equal to 0 instead of an element of \( K_+ \). That is,

\begin{equation}
\begin{array}{c}
i_m+ \rightarrow k_m- \\
\uparrow \uparrow \\
i_m- \rightarrow 0
\end{array}
\end{equation}

Notice that \( \rho^{-1}(0) \in I_- \). If we had begun with \( \rho(0) \in I_- \) we would have built the analogous sequence upward.

Since \( 2\Pi \) is reducible via \( i_1+ \rightarrow 0 \), we have \( i_1 \rightarrow j \) for all \( j \in I \setminus \{i_1\} \). We now prove, inductively, that for \( p = 2, \ldots, m, i_p \rightarrow j \) for all \( j \in I \setminus \{i_1, \ldots, i_p\} \). This is because \( 2\Pi \) is reducible via \( i_p+ \rightarrow i_{p-1}- \). By induction hypothesis, \( i_{p-1} \rightarrow j \) for all such \( j \) and so by (iii) above, \( i_p \rightarrow j \).

Similarly, \( j \rightarrow k_m \) for all \( j \in K \setminus \{i_m\} \) and so, inductively, for all \( p = 1, \ldots, m-1, j \rightarrow k_p \) for all \( j \in K \setminus \{i_p, \ldots, i_m\} \).

Let \( J = I \setminus \{i_1, \ldots, i_m\} \) and \( J = K \setminus \{k_1, \ldots, k_m\} \). Notice that \( J_- \subset (2\Pi)^{-1}(i_m+) \) and \( J_+ \subset 2\Pi(i_m+) \) while \( J_- \subset (2\Pi)^{-1}(k_m-) \) and \( J_+ \subset 2\Pi(k_m-) \).

Thus, \( \rho \) restricts to an isomorphism from the tournament \( 2\Pi|(J_+ \cup J_-) \) to the tournament \( 2\Gamma|(J_+ \cup J_-) \) and it takes \( J_+ \) to \( J_+ \) and \( J_- \) to \( J_- \).

If \( \gamma : J \rightarrow J \) is defined by \( \rho(j-) = \gamma(j)- \), then since \( 2\Gamma \) is reducible by \( \rho(j-) \rightarrow \rho(j+) \) and by \( \gamma(j)- \rightarrow \gamma(j)+ \) it follows that \( \rho(j\pm) = \gamma(j\pm) \) for all \( j \in J \). In particular, \( \gamma : \Pi|J \rightarrow \Gamma|J \) is an isomorphism.

Furthermore, \( \Xi = \{i_1, \ldots, i_m\} \times J \subset \Pi \) and \( \Xi = J \times \{k_1, \ldots, k_m\} \subset \Gamma \).
If we define \( \theta: I \rightarrow K \) by \( \theta(i_p) = k_p \) for \( p = 1, \ldots, m \) and \( \theta(j) = \gamma(j) \) for \( j \in J \), then, reversing \( \Xi \) in \( \Pi \), we obtain an isomorphism
\[
\theta: \Pi/\Xi \rightarrow \Gamma.
\]

We can diagram this as follows:
\[
\begin{align*}
\Pi: & \quad i_1 \longrightarrow i_2 \longrightarrow \ldots \longrightarrow i_m \longrightarrow J \\
\Gamma: & \quad k_1 \longrightarrow k_2 \longrightarrow \ldots \longrightarrow k_m \longleftarrow \bar{J}
\end{align*}
\]

Now we use from Proposition 1.3 the equivalent descriptions of an order, i.e. a transitive tournament.

Lemma 11.2. The tournaments \( \Pi \) and \( \Gamma \) are isomorphic if and only if \( \Pi \) is an order.

Proof. If \( m = |I| \), or, equivalently, \( J \) is empty, then \( \Pi \) is an order and \( \theta \) is an isomorphism from \( \Pi \) to \( \Gamma \).

Now assume \( n = |I| > m \).

If \( \Pi \) is an order, then by Proposition 1.3 (f) we can continue the numbering \( i_1, \ldots, i_m \) to \( i_{m+1}, \ldots, i_n \) so that \( i_p \rightarrow i_q \) when \( p < q \). If we reverse \( \Xi = \{i_1, \ldots, i_m\} \times \{i_{m+1}, \ldots, i_n\} \), then the result is again an order with the vertices ordered as \( i_{m+1}, \ldots, i_n, i_1, \ldots, i_m \). By Proposition 1.3 again an order of size \( n \) is unique up to isomorphism. Hence, \( \Gamma \) is isomorphic to \( \Pi \).

Assume instead that \( \Pi \) is not an order. At least one of the \( \mathcal{O}(1_I \cup \Pi) \cap \mathcal{O}(1_I \cup \Pi)^{-1} \) equivalence classes is not a singleton. These are the fat equivalence classes. Recall that \( \mathcal{O}\Pi \) induces an order on the set of \( \mathcal{O}(1_I \cup \Pi) \cap \mathcal{O}(1_I \cup \Pi)^{-1} \) equivalence classes. Obviously the equivalence class of each of the \( i_1, \ldots, i_m \) vertices is a singleton and each lies below all the other classes. Count the classes as in Proposition 1.3 (f) and let \( k^*(\Pi) > m \) be the label of the first fat equivalence class. When we reverse \( \Xi \) all of the \( i_1, \ldots, i_m \) classes are moved above all the other classes in the ordering and the ordering among the remaining classes is unchanged. Hence, \( k^*(\Gamma) = k^*(\Pi) - m \). Since this number is an isomorphism invariant, it follows that \( \Pi \) is not isomorphic to \( \Gamma \).

Example 11.3. There exist non-isomorphic tournaments \( \Pi \) and \( \Gamma \) such that \( 2\Pi \) is isomorphic to \( 2\Gamma \).

Proof. Let \( \Theta \) be a tournament on \( J \) which is not an order. Let \( I = \{i_1, \ldots, i_m\} \cup J \) and on it let \( \Xi \) be the digraph \( \{i_1, \ldots, i_m\} \times J \).
\[
\Pi = \{(i_p, i_q) : 1 \leq p < q \leq m\} \cup \Xi \cup \Theta,
\]
\[
\Gamma = \Pi/\Xi.
\]
Define $\rho : 2\Pi \rightarrow 2\Gamma$ according to the patterns of (11.2) and (11.3) with $k_p = i_p$, and with $\rho(j \pm) = j \pm$ for $j \in J$. By Lemma 11.2 $\Pi$ is not isomorphic to $\Gamma$.

We also obtain the following from Lemma 11.2

**Theorem 11.4.** Let $\Pi$ be a tournament which is not an order. Any automorphism of $2\Pi$ fixes 0 and so the injection $2 : \text{Aut}(\Pi) \rightarrow \text{Aut}(2\Pi)$ is an isomorphism.

□

**Corollary 11.5.** Let $\Pi$ be a tournament which is not an order. If $\Pi$ is a rigid tournament, then $2\Pi$ is a rigid game.

**Proof.** By assumption $\text{Aut}(\Pi)$ is trivial and so $\text{Aut}(2\Pi)$ is trivial by Theorem 11.4.

□

Recall that if $\Pi$ is the standard order on $A = [1, n]$, then the double $2\Pi$ is the group game $\Gamma[A]$ on $\mathbb{Z}_{2n+1}$. Since an order is a rigid tournament by Lemma 6.6 the identity is the only automorphism which fixes 0. On the other hand, the group $\mathbb{Z}_{2n+1}$ acts transitively on $2\Pi = \Gamma[A]$ by translation.

Next we consider the possibility of non-isomorphic extensions of a game.

Observe that the games $\Gamma_{IIII}$ and $\Gamma_I$ of Section 10 are non-isomorphic games of size 7 and both are reducible. Since both reduce to the unique game of size 5, we see that a game can admit non-isomorphic extensions. This phenomenon is quite general.

**Proposition 11.6.** Any game $\Pi$ with size greater than 3 admits non-isomorphic extensions.

**Proof.** Assume that $\Pi$ is a game on $I$ with $|I| = 2n + 1$.

Choosing $K \subset I$ a subset of size $n + 1$ we extend via $u \rightarrow v$ to obtain the game $\Gamma$. Recall that if $i, j \in I$, then Proposition 2.2 implies that $\Pi^{-1}(i) = \Pi(j)$ if and only if $i \rightarrow j$ and $\Pi$ is reducible via $i \rightarrow j$.

**Case 1:** $[\Pi$ is not reducible, i.e. $r\Pi = \emptyset]$ If for $i \in I$ we use $K = I \setminus \Pi(i)$, then by Corollary 2.4(b) $\Gamma$ is reducible via $v \rightarrow i$ as well as $u \rightarrow v$. Since $\Pi$ is not reducible, $\Pi(i) \neq \Pi^{-1}(j)$ for any $j \in I$. So $\Gamma$
is not reducible via \( j \to u \) by Corollary 2.4 (b) again. Since \( \Pi \) is not reducible, \( \Gamma \) is not reducible via any pair \( j_1, j_2 \) in \( I \). Thus, with this choice of \( K \),

\[
(11.7) \quad r\Gamma = [u, v, i].
\]

Similarly, if we use \( K = I \setminus \Pi^{-1}(i) \), we obtain

\[
(11.8) \quad r\Gamma = [i, u, v].
\]

**Case 2:** [\( \Pi \) is reducible, but \( r\Pi \) is not a Hamiltonian cycle] By Proposition 2.5, the reducibility graph is the union of separate maximal simple paths \([i_0, \ldots, i_m]\) with \((i_m, i_0) \notin r\Pi\). Call \( i_0, i_2, \ldots \) the even vertices of the path and \( i_1, i_3, \ldots \) the odd vertices of the path. By Proposition 2.5 again \( \Pi(i_0) \) contains only the odd vertices of the path and it intersects each of the other maximal paths either in the set of its odd vertices or its even vertices.

If we use \( K = I \setminus \Pi(i_0) \) then by Corollary 2.4 (d), we see that \( \Gamma \) is reducible via every edge of \( r\Pi \). In addition, as above it is reducible via \( v \to i_0 \) as well as via \( u \to v \). By maximality for no \( j \in I \) is \((j, i_0) \in r\Pi \) and so, as before \( \Gamma \) is not reducible via \( j \to u \) for any \( j \in I \). Thus, we have

\[
(11.9) \quad r\Gamma = r\Pi \cup [u, v, i_0],
\]

Similarly, if we use \( K = I \setminus \Pi^{-1}(i_m) \), then

\[
(11.10) \quad r\Gamma = r\Pi \cup [i_m, u, v],
\]

**Case 3:** [\( r\Pi \) is a Hamiltonian cycle] In the Hamiltonian case, \( r\Pi = \langle i_0, \ldots, i_{2n} \rangle \) and \( \Pi(i_0) = \Pi^{-1}(i_{2n}) \) is the set of odd vertices. So with \( K = I \setminus \Pi(i_0) = I \setminus \Pi^{-1}(i_{2n}) \) we have

\[
(11.11) \quad r\Gamma = \langle i_0, \ldots, i_{2n}, u, v \rangle.
\]

In all of these cases \(|r\Gamma| = |r\Pi| + 2\).

On the other hand, from Proposition 2.2 (d) again, it follows that for \( i \in I \), \( \Gamma \) is reducible via \( i \to u \) only if \( i \in K \) and \( I \setminus K = \Pi^{-1}(i) \), and \( \Gamma \) is reducible via \( v \to i \) only if \( i \in K \) and \( I \setminus K = \Pi(i) \). If \( I \setminus K \) is not equal to \( \Pi(i) \) or \( \Pi^{-1}(i) \) for any \( i \in I \), then the extension \( \Gamma \) is reducible only by the pair \( u \to v \) as well as possibly by some edges in \( r\Pi \). That is, in that case,

\[
(11.12) \quad r\Gamma \subset r\Pi \cup \{(u, v)\}\]
and so \(|r \Gamma| \leq |r \Pi| + 1\).

Notice that each \(\Pi(i)\) and \(\Pi^{-1}(i)\) is a subset of \(I\) of size \(n\) and there are at most \(2(2n + 1)\) of them. On the other hand, there are a total of \(\binom{2n+1}{n}\) subsets of size \(n\) and for \(n \geq 3\), \(\binom{2n+1}{n} > 4n + 2\). Hence, for \(n \geq 3\) the latter alternative choice is always possible.

In particular, if \(\Pi\) is not reducible, then \(n \geq 3\) and we obtain examples \(\Gamma\) with

\[(11.13) \quad r \Gamma = \{(u, v)\}\]

In the Hamiltonian cycle case, which includes the case \(n = 2\), we can choose \(K = \{i_0, \ldots, i_n\}\). With \(n \geq 2\) the complement \(\{i_{n+1}, \ldots, i_{2n}\}\) is not equal to any \(\Pi(i_p)\) or \(\Pi^{-1}(i_p)\) since it contains both even and odd vertices. Hence, for this extension \(r \Gamma = \{(i_n, i_{n+1}), (i_{2n}, i_0), (u, v)\}\) with \(|r \Gamma| = 3 < 2n + 1 = |r \Pi|\).

\[\square\]

If \(\Pi\) is not reducible, then from the proof of Proposition 11.6 we have the following possibilities for the reducibility graph of \(\Gamma\) the extension of \(\Pi\) on \(I\) via \(K \subset I\) and \(u \rightarrow v\).

\[(11.14) \quad r \Gamma, \quad I \setminus K\]

\((u, v), (v, i) \quad \Pi(i)\]

\((i, u), (u, v) \quad \Pi^{-1}(i)\]

\((u, v) \quad \text{otherwise}\]

In Cases 1-3 of the above proof, we constructed examples which enlarge \(r \Pi\). We pause to consider the opposite extreme.

**Definition 11.7.** A game \(\Pi\) on a set \(I\) with \(|I| = 2n + 1\) is called uniquely reducible when there is a unique subset \(J \subset I\) with \(|J| = 2n - 1\) such that the restriction \(\Pi|J\) is a subgame.

Thus, \(\Pi\) is uniquely reducible when \(r \Pi\) consists of a single edge. For example, a double of a tournament of size at least 2 is never uniquely reducible. In particular, the unique game of size 5, which is the double of a single edge, is not uniquely reducible. Both of the types of reducible games of size 7 are isomorphic to doubles and so are not uniquely reducible.
Proposition 11.8. Let $\Pi$ be a game on $I$ with $|I| = 2n + 1$ and let $K$ be a subset of $I$ with $|K| = n + 1$. The extension $\Gamma$ of $\Pi$ via $K$ and $u \to v$ is uniquely reducible if and only if the following conditions hold.

(i) For every path $[i_0, \ldots, i_m]$ in $r\Pi$ either $\{i_0, \ldots, i_m\}$ is a subset of $K$ or is disjoint from $K$.

(ii) $I \setminus K$ is not equal to $\Pi(j)$ or $\Pi^{-1}(j)$ for any $j \in I$.

Assume $(i_0, i_1) \in r\Pi$ and so $\Pi$ is reducible. If $\Pi$ is uniquely reducible and $K = I \setminus \Pi(i_1) = I \setminus \Pi^{-1}(i_0)$, then $K$ satisfies condition (i) but not condition (ii). For all other cases with $\Pi$ reducible, condition (ii) follows from condition (i).

Proof. By Corollary 2.4 condition (i) is equivalent to non-reducibility via $i \to j$ for $(i, j) \in \Pi$. Condition (ii) is equivalent to non-reducibility via $v \to j$ or $j \to u$ for $j \in K$.

Now assume $(i_0, i_1) \in r\Pi$. For any $j \in I \setminus \{i_0, i_1\}$ either $i_0 \to j \to i_1$ or $i_1 \to j \to i_0$ by Proposition 2.2. Hence, both $\{i_0, i_1\} \cap \Pi(j)$ and $\{i_0, i_1\} \cap \Pi^{-1}(j)$ are singletons as are $\{i_0, i_1\} \cap \Pi(i_0)$ and $\{i_0, i_1\} \cap \Pi^{-1}(i_1)$. Hence, given condition (i) the only possibility with $I \setminus K$ equal to $\Pi(j)$ or $\Pi^{-1}(j)$ is when $K = I \setminus \Pi(i_1) = I \setminus \Pi^{-1}(i_0)$.

Furthermore, if $r\Pi$ contains another edge $(j_0, j_1)$, then $\{j_0, j_1\} \cap \Pi^{-1}(i_0)$ is a singleton (even if $j_1 = i_0$ and $j_0 = i_0$ can’t happen). So if $\Pi$ is not uniquely reducible, then $K = I \setminus \Pi(i_1) = I \setminus \Pi^{-1}(i_0)$ violates condition (i).

If $\{(i_0, i_1)\} = r\Pi$ and $K = I \setminus \Pi(i_1) = I \setminus \Pi^{-1}(i_0)$, then condition (i) is satisfied, but $r\Gamma = [i_0, u, v, i_1]$. \qed

Example 11.9. If $\Pi$ be the double of the three cycle $\langle 1, 2, 3 \rangle$ (so that it is a game of type $\Gamma_{III}$ of Section 10), then $\Pi$ has a uniquely reversible extension.

Proof. $r\Pi = \{(i-, i+) : i = 1, 2, 3\}$. Let

$$K = \{1-, 1+, 2-, 2+\}.$$  

Condition (i) of Proposition 11.8 is obvious and since $\Pi$ is not uniquely reducible, condition (ii) holds as well and implies that $\Gamma$ is uniquely reducible. \qed

Proposition 11.10. If $\Pi$ is a game which is either not reducible or uniquely reducible, then $\Pi$ has a uniquely reducible extension.
Proof. Assume $\Pi$ is a game on $I$. If $\Pi$ is not reducible, then as in the proof of Proposition 11.6 we choose $K$ such that $I \setminus K$ is not equal to $\Pi(i)$ or $\Pi^{-1}(i)$ for any $i \in I$.

If $r\Pi = \{(i_0, i_1)\}$ and $k$ is a vertex of $I \setminus \Pi(i_1) = I \setminus \Pi^{-1}(i_0)$. Then we choose $K$ so that it contains or is disjoint from \{i_0, j_0, k\}. Condition (i) of Proposition 11.8 is obvious and since $I \setminus K$ is not equal to $\Pi(i_1) = \Pi^{-1}(i_0)$, condition (ii) follows and implies that $\Gamma$ is uniquely reducible.

Thus, beginning with the double of a three cycle we can build a totally reducible game $\Pi$ on $I$ with $I_1 \subset I_2 \cdots \subset I_n = I$ with $|I_k| = 2k+1$ such that when $2k+1 > 7$ the subgame $\Pi|I_k$ is uniquely reducible.

Above we saw that games have non-isomorphic extensions. Now we consider the reverse question. Can non-isomorphic games have isomorphic extensions? Equivalently, can a game $\Gamma$ be reduced to two non-isomorphic games. This, of course, requires that the game be reducible via different pairs and so the obvious places to look are at doubles $\Gamma = 2\Pi$ with $\Pi$ a tournament on $I$ with $|I| = n$. For any vertex $i \in I$, $2\Pi$ is an extension of $2(\Pi|I \setminus \{i\})$. So we want a tournament $\Pi$ such that for $i_1, i_2 \in I$, $2(\Pi|I \setminus \{i_1\})$ is not isomorphic to $2(\Pi|I \setminus \{i_2\})$.

Let us first consider when this fails. If $\Pi$ is a homogeneous game, then $\Pi|I \setminus \{i_1\}$ and $\Pi|I \setminus \{i_2\}$ are isomorphic for any pair $i_1$ and $i_2$ since there is an automorphism taking $i_1$ to $i_2$. If $\Pi$ is an order, then $\Pi|I \setminus \{i\}$ is an order for every $i \in I$ and so, despite the rigidity of $\Pi$, all of the $\Pi|I \setminus \{i\}$'s are isomorphic. Isomorphic tournaments have isomorphic doubles.

A more interesting example, is $\Pi$ on $I = \{1, 2, 3, 4, 5, 6\}$ with

\[(11.15) \quad \Pi = \langle 1, 2, 3 \rangle \cup \langle 4, 5, 6 \rangle \cup \{1, 2, 3\} \times \{4, 5, 6\}.\]

The tournaments $\Pi|I \setminus \{1\}$ and $\Pi|I \setminus \{4\}$ are not isomorphic. The former has score vector $(1, 1, 1, 3, 4)$ and the latter has score vector $(0, 1, 3, 3, 3)$. However, as Example 11.3 shows, they have isomorphic doubles. Hence, all of the $2(\Pi|I \setminus \{i\})$'s are isomorphic.

On the other hand, if $\Pi|I \setminus \{i_1\}$ and $\Pi|I \setminus \{i_2\}$ have different score vectors and neither $0$ nor $n-2$ occur among the scores, then, by Proposition 6.4 $2(\Pi|I \setminus \{i_1\})$ is not isomorphic to $2(\Pi|I \setminus \{i_2\})$. It is not hard to construct such examples.

More interesting is the case when $n = 2k+1$ and $\Pi$ is itself a game. In that case, each $\Pi|I \setminus \{i\}$ has score vector $(k-1, \ldots, k-1, k, \ldots, k)$ with
k each of the scores \( k - 1 \) and \( k \). However, if the further restriction of \( \Pi[I \setminus \{i_1\}] \) and \( \Pi[I \setminus \{i_2\}] \) to the vertices with score \( k \) are not isomorphic, then \( \Pi[I \setminus \{i_1\}] \) and \( \Pi[I \setminus \{i_2\}] \) cannot be isomorphic.

**Example 11.11.** Let \( \Pi \) be the game \( \Gamma_{III} \) on \{0, 1, 2, 3, 4, 5, 6\} of Section 10 so that it is the double of the three cycle \( \langle 1, 2, 3 \rangle \). In \( \Pi[I \setminus \{0\}] \) the vertices 1, 2, 3 have score 3 and form a 3-cycle. In \( \Pi[I \setminus \{4\}] \) the vertices 6, 5, 1 have score 3 and form a straddle. The non-isomorphic games \( 2(\Pi[I \setminus \{0\}]) \) and \( 2(\Pi[I \setminus \{4\}]) \) extend via \( (0, -) \to (0, +) \) and via \( (4, -) \to (4, +) \), respectively, to \( 2\Pi \).

\( \square \)

### 12. Games of Size Nine

For the case \( 9 = 2 \cdot 4 + 1 \) we will first describe the isomorphism classes of the group games. There are \( 2^4 = 16 \) game subsets. These are naturally pointed games with tournaments \( \Pi_+, \Pi_- \) each of size 4.

**Proposition 12.1.** Each tournament of size 4 is uniquely determined up to isomorphism by its score vector.

**Proof.** The possible score vectors are:

\[
(12.1) \quad s_1 = (0, 1, 2, 3), \quad s_2 = (1, 1, 1, 3), \quad \bar{s}_2 = (0, 2, 2, 2), \quad s_3 = (1, 1, 2, 2).
\]

Let \( \Theta_p \) be a tournament of size 4 with score vector \( s_p \) for \( p = 1, 2, 3 \). So \( \Theta_2^{-1} \) has score vector \( \bar{s}_2 \).

By Proposition 1.3 a tournament with score vector \( s_1 \) is an order and the order of size 4 is unique up to isomorphism.

If the score vector is \( s_2 \), then the output set of the vertex with score 3 is a 3-cycle. It is obvious that any two such are isomorphic. By using the reverse tournaments we obtain the result for \( \bar{s}_2 \).

Next observe that if \( \Theta \) is a tournament on \( J \) with \( |J| = 2n \) and score vector \( (n - 1, \ldots, n - 1, n, \ldots, n) \), then we there is a game \( \Pi \) of size \( 2n + 1 \) which contains \( \Pi \). If \( u \) is the additional vertex, then \( \Pi(u) \) is the set of vertices of \( J \) with \( \Theta \) score \( n \). Conversely, if we remove a vertex from a game of size \( 2n + 1 \), then we are left with a tournament of size \( 2n \) and with score vector \( (n - 1, \ldots, n - 1, n, \ldots, n) \).

When \( n = 4 \) we apply uniqueness of the game of size 5. If \( \Theta \) and \( \bar{\Theta} \) are tournaments of size 4 with score vectors \( s_3 \), we can adjoin vertices \( u \) and \( \bar{u} \) to obtain games \( \Pi \) and \( \bar{\Pi} \) of size 5. By Theorem 2.4, there
exists a isomorphism between them. Since the game is a group game, \( \mathbb{Z}_5 \) acts transitively on the vertices and so we may assume that the isomorphism takes \( u \) to \( \bar{u} \). It then restricts to an isomorphism from \( \Theta \) to \( \bar{\Theta} \).

\[ \square \]

**Remark:** Notice that an isomorphism between two tournaments \( \Theta_1 \) and \( \Theta_2 \) with score vectors \((n-1, \ldots, n-1, n, \ldots, n)\) extends uniquely to an isomorphism between the games \( \Pi_1 \) and \( \Pi_2 \). Thus, if we begin with non-isomorphic games of size \( 2n+1 \) and we remove an arbitrary vertex from each we obtain non-isomorphic tournaments of size \( 2n \) each with score vector \((n-1, \ldots, n-1, n, \ldots, n)\). Thus, with \( n > 2 \), there are always non-isomorphic tournaments of size \( 2n \) each with score vector \((n-1, \ldots, n-1, n, \ldots, n)\).

In particular, we see that \( \Theta_1 \) and \( \Theta_3 \) are each isomorphic to its reverse tournament.

Let \( \Gamma_3 \) denote the game on \( \mathbb{Z}_3 \) with \( \langle 0, 1, 2 \rangle \).

**Theorem 12.2.** For \( G = \mathbb{Z}_9 \) there are three types of group games.

**TYPE I** (\( A = \{1, 3, 4, 7\} \) or \( [1, 4] \)) - The game \( \Gamma[A] \) is reducible, with \( Aut(\Gamma[A]) = \mathbb{Z}_9 \). The six Type I game subsets \( B \) such that \( \Gamma[B] \) is isomorphic to \( \Gamma[A] \) are the elements of \( \{m_a(A) : a \in \mathbb{Z}_9^*\} \).

**TYPE II** (\( A = \{1, 5, 6, 7\} \)) - The game \( \Gamma[A] \) is not reducible, but has \( Aut(\Gamma[A]) = \mathbb{Z}_9 \). The six Type II game subsets \( B \) such that \( \Gamma[B] \) is isomorphic to \( \Gamma[A] \) are the elements of \( \{m_a(A) : a \in \mathbb{Z}_9^*\} \).

**TYPE III** (\( A_1 = \{1, 3, 4, 7\}, A_2 = \{1, 4, 6, 7\} \)) - There is a non-affine isomorphism between \( \Gamma[A_1] \) and \( \Gamma[A_2] \). The Type III games are isomorphic to the lexicographic product \( \Gamma_3 \ltimes \Gamma_3 \) with automorphism group the semi-direct product \( \mathbb{Z}_3 \ltimes (\mathbb{Z}_3)^3 \). The four Type III game subsets are \( A_1, -A_1, A_2, -A_2 \).

**Proof.** Each group game is a pointed game on the pair \((-A, A)\). We let \( s_A \) and \( s_{-A} \) denote the score vectors of the tournaments \( \Gamma[A]|A \) and \( \Gamma[A]|(-A) \). Since the group is commutative, the tournament \( \Gamma[A]|(-A) \) is the reverse of \( \Gamma[A]|A \).

**TYPE I** - With \( A = [1, 4] \) or \( Odd_4 \) the score vectors are \( s_A = s_{-A} = s_1 = (0, 1, 2, 3) \). These are rigid tournaments and so the only automorphism which fixes 0 is the identity. Hence, as we saw in Theorem 3.7, \( Aut(\Gamma[A]) = \mathbb{Z}_9 \).
TYPE II- With \( A = \{1, 5, 6, 7\} \), \( s_A = s_+ = s_3 = (1, 1, 2, 2) \). By Lemma 6.6 and Theorem 6.7 again have \( \text{Aut}(\Gamma[A]) = \mathbb{Z}_9 \).

By Corollary 3.3 \( \{m_\phi(A) : \phi \in \mathbb{Z}_9^*\} \) are the \( \phi(9) = 6 \) distinct game subsets whose games are isomorphic to \( \Gamma[A] \) for each of these two types.

Type III- With \( A = \{1, 3, 4, 7\} \) or \( \{1, 4, 6, 7\} \) we have \( s_A = s_2 = (1, 1, 1, 3), s_\pi = (\bar{s}_2 = 0, 2, 2, 2) \).

Type III is a special case of Example 9.11. We have

\[
\mathbb{Z}_3 \xrightarrow{\theta} \mathbb{Z}_9 \xrightarrow{\pi} \mathbb{Z}_3 \quad \text{with} \quad \theta(j) = 3j, \quad \pi(3j+i) = i \text{ for } i,j = 0,1,2.
\]

Let \( B = \{1\} \subset \mathbb{Z}_3 \). With \( A_{01} = \{1\}, A_1 = A_{01} \cup \pi^{-1}(B) \) and with \( A_{02} = \{2\}, A_2 = A_{02} \cup \pi^{-1}(B) \). Thus, \( A_1 \) and \( A_2 \) are game subsets for the pair \((\mathbb{Z}_9, \theta(\mathbb{Z}_3))\) and so their inverses \(-A_1, -A_2\) are also game subsets for the pair. Thus, the associated games are all isomorphic to \( \Gamma_3 \times \Gamma_3 \). By Theorem 6.9 the automorphism groups are isomorphic to \( \mathbb{Z}_3 \times [\mathbb{Z}_3]^3 \).

We can describe \( \Gamma[A_1] \) as a 3-cycle of 3-cycles.

\[
\langle (2,5,8) \Rightarrow (0,3,6) \Rightarrow (1,4,7) \rangle.
\]

On the other hand, \( \Gamma[A_2] \) is a 3-cycle of 3-cycles.

\[
\langle (2,8,5) \Rightarrow (0,6,3) \Rightarrow (1,7,4) \rangle.
\]

Clearly, the product of transpositions \( \rho = (8,5)(6,3)(7,4) \) is an isomorphism between them which fixes 0. With \( \phi = m_2 = m_{-1} \) on \( \mathbb{Z}_3 \) and \( \gamma_i = \phi \) for \( i = 0, 1, 2 \), \( \rho = 1_{\mathbb{Z}_3} \times \gamma \).

The group \( \mathbb{Z}_9^* \) is generated by the cyclic groups \( m_4, (m_4)^2 = m_7, (m_4)^3 = m_1 = 1_{\mathbb{Z}_9} \) and \( m_8 = m_{-1}, (m_8)^2 = 1_{\mathbb{Z}_9} \). The cyclic group generated by \( m_4 \) is contained in the automorphism groups of \( \Gamma[A_1] \) and \( \Gamma[A_2] \) while \( m_{-1} \) maps each game to its reverse. In particular, \( \rho \) is not affine.

We can also view Type III by using the construction of 5.12. The set \( H = \{1, 4, 7\} \) is the multiplicative subgroup of \( \mathbb{Z}_9^* \) of order 3. The action of \( H \) fixes 3 and \(-3 = 6\). The four game subsets are obtained by choosing one from each pair of \( H \) orbits: \( \{H, -H\}, \{\{3\}, \{6\}\} \), e.g. \( A_1 = H \cup \{3\}, A_2 = H \cup \{6\} \).

\( \square \)

The Type III cases provide examples of game subsets \( A_1, A_2 \) of \( \mathbb{Z}_9 \) with isomorphic associated games but which are not related by the action of an element of \( \mathbb{Z}_9^* \). This is a special case of the following.

**Example 12.3.** Let \( p \) be an odd prime. There are \( p - 1 \) game subsets \( A_1, \ldots, A_{p-1} \subset \mathbb{Z}_p^2 \).
with isomorphic associated games but with no two related by an element of $\mathbb{Z}_p^*$. 

Proof. Define $\xi : \mathbb{Z}_p \to \mathbb{Z}_p^*$ by $\xi(j) = jp$ and $\pi : \mathbb{Z}_p^* \to \mathbb{Z}_p$ by $\pi(j) = j$, or, equivalently, $\pi(j + kp) = j$. Thus, $\pi$ is a surjective ring homomorphism with kernel $H = \xi(\mathbb{Z}_p)$. Let $K = \pi^{-1}(1) = \{1 + kp\}$, the unique subgroup of $\mathbb{Z}_p^*$ of order $p$, generated by $1 + p$. Each coset $i + H$ is $K$ invariant and if $i \notin H$, then it is a single $K$ orbit. On the other hand, each element of $H$ is fixed by $K$. It follows that a game subset $A$ is $K$ invariant if and only if it is a game subset for the pair $(\mathbb{Z}_p^*, H)$, i.e. if and only if there are game subsets $A_0, B$ of $\mathbb{Z}_p$ such that $A = \xi(A_0) \cup \pi^{-1}(B)$. See Theorem 3.5.

Let $A_{0,1}, B = [1, (p - 1)/2]$ and for $a = 1, \ldots, p - 1$ let $A_{0,a} = m_a(A_{0,1})$. Let $\Pi$ be the game on $\mathbb{Z}_p$ associated with $[1, (p - 1)/2]$. Thus, $\Pi$ is isomorphic to each of the games on $\mathbb{Z}_p$ associated to $B, A_{0,1}, \ldots, A_{0,p-1}$. Define $A_k = A_{0,k} \cup \pi^{-1}(B)$. By Theorem 0.9 the game $\Gamma[A_k]$ on $\mathbb{Z}_p^*$ is isomorphic to the lexicographic product $\Gamma[B] \ltimes \Gamma[A_{0,k}]$ and so is isomorphic to $\Pi \ltimes \Pi$ for all $k$. If we use $\rho$ the identity on $\Gamma[B]$ and $\gamma_i = m_k$ for all $i \in \mathbb{Z}_p$ then $\rho \ltimes \gamma$ is an isomorphism from $\Gamma[A_1] = \Gamma[B] \ltimes \Gamma[A_{0,1}]$ to $\Gamma[A_k] = \Gamma[B] \ltimes \Gamma[A_{0,k}]$.

By Theorem 3.7 the only automorphisms of $\Pi$ are translations by elements of $\mathbb{Z}_p$. This implies that the game subsets $A_{0,1}, \ldots, A_{0,p-1}$ are distinct. Furthermore, $m_a(A_{0,k}) = A_{0,k}$ for $a \in \mathbb{Z}_p^*$ only for $a = 1$.

Now assume that $u \in \mathbb{Z}_p^*$ and that $m_u(A_{k_1}) = A_{k_2}$. The subgroup $H$ is invariant with respect to multiplication by $u$ and so $m_u(\xi(A_{0,k_2})) = \xi(A_{0,k_2})$. Because $\mathbb{Z}_p^* \setminus H$ is invariant as well, $\pi^{-1}(B) = A_{k_1} \cap (\mathbb{Z}_p^* \setminus H) = A_{k_2} \cap (\mathbb{Z}_p^* \setminus H)$ is invariant. This implies that $B$ is invariant with respect multiplication by $\pi(u) \in \mathbb{Z}_p^*$. It follows that $\pi(u) = 1$ and so $u \in K$. Since $K$ fixes every element of $H$, it follows that $\xi(A_{0,k_1}) = \xi(A_{0,k_2})$ and so $k_1 = k_2$.

It follows that distinct subsets $A_{k_1}$ and $A_{k_2}$ are not related by the action of $\mathbb{Z}_p^*$.

Now consider the special case when $p$ is a Fermat prime so that $p - 1$ is a power of 2 and so every element of $\mathbb{Z}_p^* \setminus K$ and every element of $\mathbb{Z}_p^* \setminus \{1\}$ has even order. It follows that $\mathbb{Z}_p^*$ acts freely on the game subsets of $\mathbb{Z}_p$ and $\mathbb{Z}_p^*$ acts freely on the game subsets of $\mathbb{Z}_p^*$ which are not $K$ invariant, see Theorem 3.11.

If $A$ is $K$ invariant then $A = \xi(A_0) \cup \pi^{-1}(B)$ with $A_0$ and $B$ game subsets of $\mathbb{Z}_p$. We obtain $(p - 1)^2$ game subsets $A$ by replacing $A_0, B$ by $m_a(A_0), m_b(B)$ for $a, b \in \mathbb{Z}_p^*$. All of the associated games on $\mathbb{Z}_p^*$ are isomorphic to $\Gamma[B] \ltimes \Gamma[A_0]$. On the other hand, only when $a = b$ is
ξ(A_0) \cup \pi^{-1}(B) related to \xi(m_a(A_0)) \cup \pi^{-1}(m_b(B)) by an element of \mathbb{Z}_{p^2}^*.

Since there are 2^{(p-1)/2} game subsets of \mathbb{Z}_p, it follows that there are 2^{p-1} game subsets of \mathbb{Z}_{p^2} which are \mathcal{K} invariant. These are partitioned into classes of size \((p-1)^2\) all members of which have isomorphic games. Each of these is in turn partitioned into \(p-1\) sets of size \(p-1\) by the action of \mathbb{Z}_{p^2}^*.

\[ \square \]

The following question remains open, as far as I know.

**Question 12.4.** Does there exists a collection of more than \(\phi(2n+1)\) game subsets \(A\) of \(\mathbb{Z}_{2n+1}\) all of whose associated games are isomorphic? In particular, with \(2n+1\) a square-free product of Fermat primes, do there exist game subsets \(A\) and \(B\) of \(\mathbb{Z}_{2n+1}\) which are not related by an element of \(\mathbb{Z}_{2n+1}^*\) but which have isomorphic associated games?

\[ \square \]

Returning to the case with \(2n+1 = 9\) we observe the following.

**Theorem 12.5.** For \(G = \mathbb{Z}_3 \times \mathbb{Z}_3\), \(\Gamma[A]\) is isomorphic to \(\Gamma_3 \ltimes \Gamma_3\) for every game subset \(A\) of \(G\).

**Proof.** The group \(\mathbb{Z}_3 \times \mathbb{Z}_3\) is a two-dimensional vector space over the field \(\mathbb{Z}_3\). The group \(G^*\) of \(2 \times 2\) invertible matrices on \(\mathbb{Z}_3\) acts transitively on the set of nonzero vectors. If \(H\) is a one-dimensional subspace, then there are four game subsets for the pair \((G, H)\). Such a game subset contains a unique affine subspace and it is parallel to \(H\). Thus, the game subsets of \((G, H)\) determine \(H\). The subgroup of matrices which fix \(H\) acts transitively on these four game subsets. If a matrix maps \(H_1\) to \(H_2\) then it maps the game subsets for \((G, H_1)\) to the game subsets for \((G, H_2)\). There are four subspaces \(H\) and so there are 16 game subsets of \((G, H)\) for a suitably chosen one-dimensional subspace \(H\). As there are \(2^4 = 16\) game subsets for \(G\) it follows that all are isomorphic and the associated games are lexicographic products.

\[ \square \]

Finally, we consider a non-trivial automorphism \(\rho\) on a game \(\Gamma\) of size 9. If \(\rho\) has a fixed point, then by Proposition 1.8 and Proposition 6.15 every non-trivial cycle in the permutation \(\rho\) must have length an odd number at most 4 and greater than 1. That is, it has length 3 and so \(\rho\) has order 3. If \(\rho\) consists of a single cycle of length 9, then we can identify the set of vertices with \(\mathbb{Z}_9\) so that \(\rho\) is translation by a
ROCK, PAPER, SCISSORS, ETC

generator. In that case, Theorem 3.4 implies that \( \Gamma \) is isomorphic with one of the group games on \( \mathbb{Z}_9 \) described above. If the permutation has no fixed points and does not consist of a single cycle, then by Proposition 3.2 (g), it contains an odd number of cycles whose lengths sum to 9. So the remaining possibility has \( \rho \) with three 3-cycles and so again \( \rho \) has order 3. For the games of Type III isomorphic to \( \Gamma_3 \times \Gamma_3 \) with automorphism group \( \mathbb{Z}_3 \times [\mathbb{Z}_3]^3 \), it is easy to check that the automorphism group contains permutations of all four types: i.e. of order 9 and of order 3 with exactly one, two or three 3-cycles.

In [8] Chamberland and Herman compute the number of isomorphism classes and associated automorphism groups for games of size 9, 11 and 13. In addition, they provide a beautiful geometric description of the three games of size 7. In addition to the three group games they find seven rigid games and five with automorphism group \( \mathbb{Z}_3 \).

We have seen in Example 6.8 a rigid example of size 9. We begin with the tournament \( \Theta_3 \) of size 4 with score vector \( (1, 1, 2, 2) \), for example, we may use \( \Theta_3 = \Gamma[A]:A \) where \( \Gamma[A] \) is a Type II group game. The tournament \( \Theta_3 \) is rigid and the scores 0 and 3 do not occur. It follows that the double \( 2\Theta_3 \) is a rigid game of size 9.

Let \( \Theta \) be the tournament on \( J = \{1, 2, 3, 4\} \) with score vector \( (1, 1, 1, 3) \) such that \( \langle 2, 3, 4 \rangle \) is the 3-cycle in \( \Theta \). It is clear that \( Aut(\Theta) \) is isomorphic to \( \mathbb{Z}_3 \). Let \( \Gamma = 2\Pi \) and \( \bar{\Gamma} = \Gamma/\langle 2+, 3+, 4+ \rangle \). In Example 6.2 we observed that
\[
\begin{align*}
\rho \Gamma &= [1-, 1+, 0] \cup \{(p-, p+) : p = 2, 3, 4\}, \\
\rho \bar{\Gamma} &= [1-, 1+, 0].
\end{align*}
\]
(12.2)

**Example 12.6.** The games \( \Gamma \) and \( \bar{\Gamma} \) are non-isomorphic games of size 9. Each has automorphism group isomorphic to \( \mathbb{Z}_3 \).

*Proof.* Since an isomorphism associates the reducibility graphs, it follows from (12.2) that \( \Gamma \) and \( \bar{\Gamma} \) are not isomorphic. Since the the reducibility graph is invariant with respect to an automorphism, it is clear that an automorphism of either must map \([1-, 1+, 0]\) so itself and so is fixed on \([1-, 1+, 0]\) by Proposition [1.1]. In particular, it must fix 0. For the double, \( \Gamma = 2\Theta \), we have that the inclusion \( 2 : Aut(\Theta) \to Aut(\Gamma) \) is an isomorphism and so \( \Gamma \) has automorphism group isomorphic to \( \mathbb{Z}_3 \).

Because the cycle \( \langle 2+, 3+, 4+ \rangle \) is invariant with respect to the \( \mathbb{Z}_3 \) action, it follows from Proposition [6.17] that \( \mathbb{Z}_3 \) acts on \( \bar{\Gamma} \). Since an automorphism fixes 0, it restricts to an automorphism of \( \Gamma|J_\perp \) which is isomorphic to \( \Theta \). Hence, the \( \mathbb{Z}_3 \) action includes all of the automorphisms of \( \bar{\Gamma} \).

\( \square \)
We note that by using the construction of Exercise 11.3 we obtain an isomorphism from $\Gamma$ to $\Gamma^{-1}$. The same map induces a isomorphism from $\bar{\Gamma}$ to $\bar{\Gamma}^{-1}$. Finally, we observe that $\Gamma|\Gamma(1+)$ has score $s_2 = (0, 2, 2, 2)$ and $\Gamma|\Gamma^{-1}(1+)$ has score $s_2 = (1, 1, 1, 3)$. Thus, using $1+$ as a base point we obtain a different view of $\Gamma$ as a pointed game.

References

1. E. Akin, *The General Topology of Dynamical Systems*, Graduate Studies in Mathematics, No. 1, Amer. Math. Soc., Providence, RI, 1993.
2. N. Alon, C. McDiarmid and M. Molloy, *Edge disjoint cycles in regular directed graphs*, J. Graph Theory, 22 (1996), 231-237.
3. B. Alspach, *Point-symmetric graphs and digraphs of prime order and transitive permutation groups of prime degree*, J. of Combinatorial Theory 15 (1973), 12-17.
4. B. Alspach and J. L. Bergren, *On the determination of the maximum order of the group of a tournament*, Canad. Math. Bull., 16 (1973), 11-14.
5. B. Alspach, M. Goldberg and J. W. Moon, *The group of the composition of two tournaments*, Math. Magazine, 41 (1968), 77-80.
6. R. C. Bose, *On the construction of balanced incomplete block designs*, Ann. Eugenics, 9 (1939), 353-399.
7. R. A. Bruzli and Q. Li, *The interchange graph of tournaments with the same score vector*, in *Progress in Graph Theory*, Academic Press, New York, NY, 1984, 129-151.
8. M. Chamberland and E. A. Herman, *Rock, paper, scissors meets Borromean rings*, The Math. Intelligencer (to appear).
9. G. Chartrand and P. Zhang, *A First Course in Graph Theory*, Dover Publications, Mineola, NY, 2012.
10. A-H. Chen, J-M. Chang, and Y-L. Wang, *The interchange graphs of tournaments with minimum score vectors are exactly hypercubes*, Graphs and Combinatorics, 25 (2009), 27-34.
11. J. Dixon, *Maximum order of the group of a tournament*, Canad. Math. Bull., 10 (1967), 503-505.
12. P. M. Gibson, *A bound for the number of tournaments with specified scores*, J. Combin. Theory, 36 (1984), 240-243.
13. M. Goldberg and J. W. Moon, *On the maximum order of the group of a tournament*, Canad. Math. Bull., 9 (1966), 563-569.
14. M. Goldberg and J. W. Moon, *On the composition of two tournaments*, Duke Math. J., 37 (1970), 323-332.
15. M. Hall, *The Theory of Groups*, Macmillan, New York, NY, 1959.
16. F. Harary and L. Moser, *The theory of round robin tournaments*, The Amer. Math. Monthly, 73 (1966), 231-246.
17. F. Harary, R. Z. Norman and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs*, John Wiley and Sons, New York, NY, 1965.
18. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford U. Press, London, 1938, 6th Ed. 2008.

19. T. P. Kirkman, *On a problem in combinations*, The Cambridge and Dublin Math. J., 2 (1847), 191-204.

20. W. Honghui and L. Qiao, *On the number of tournaments with prescribed score vector*, Discrete Math., 61 (1986), 213-219.

21. H. G. Landau, *On dominance relations and the structure of animal societies: III, The condition for a score structure*, Bull. of Math. Biophysics, 15 (1953), 143-148.

22. J. W. Moon, *Topics on Tournaments*, Holt, Rinehart and Winston, New York, NY, 1968. Reprinted, Dover Publications, Mineola, NY, 2015.

23. G. L. Mullen and C. Mummert, *Finite Fields and Applications*, Student Mathematical Library No. 41, Amer. Math. Soc., 2007.

24. I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers*, John Wiley and Sons, New York, 1960.

25. H. J. Ryser, *Combinatorial Mathematics*, Carus Math. Monograph No. 14, Math. Assoc. of America, Rahway, NJ, 1963.

26. H. J. Ryser, *Matrices of zeroes and ones in combinatorial mathematics*, in *Recent Advances in Matrix Theory* U. of Wisconsin Press, Madison, 1964, 103-124.

27. G. Sabidussi, *The composition of two graphs*, Duke Math. J., 26 (1959), 693-696.

28. G. Sabidussi, *The lexicographic product of graphs*, Duke Math. J., 28 (1961), 573-578.

29. T. Skolem, *Some remarks on the triple systems of Steiner*, Math. Scand., 6 (1958), 273-280.

30. J. Spencer, *Maximal families of disjoint triples*, J. of Combinatorial Theory 5 (1968), 1-8.

31. J. Turner, *Point-symmetric graphs with a prime number of points*, J. of Combinatorial Theory 3 (1967), 136-145.
Index

2Π, 55
Aut(Π), 17
G/H, 78
G*, 28
Ga, 9, 80
Isoa, 9, 80
T ⊀ K, 60
X(2Π), 56
Δ(Π, Γ), 35
Δ(ρ, Π, Γ), 35
Γ × Π, 7, 59
Γ[A/H], 8, 79
Γ[A], 24
ΩΠ, 11
Π+, 56
Π−, 56
β(Π), 6, 39
β̅, 3
ℓi, 28
σ(Π), 6, 39
rΠ, 21

action
effective, 25, 79
free, 25
transitive, 25
translation, 27
automorphism, 17

balance invariant, 6, 39
bipartite, 66
bipartite graph, 45
bipartite tournament, 67
bivariante, 13

Cayley graph, 25
completely reducible, 4, 54
composition, 11
convex, 76
cycle, 2, 11
far, 46
Hamiltonian, 14
near, 46
special, 46
decomposition, 6
digraph, 2, 3, 12

balance invariant, 39
connected, 13
decomposition, 39
Eulerian, 3, 15
morphism, 3, 17
regular, 3, 15
double, 7, 55
double coset, 77

decorating, 11
horizontal, 59
vertical, 59
effective action, 25, 79
Eulerian, 15
extension, 3, 19
via u → v and K, 19

efficiency, 11
finite field, 33
free action, 25

game, 2, 3, 15, 18
completely reducible, 4, 54
group, 24
homogeneous, 79
pointed, 8, 68
reducible, 4, 19
size, 18
Steiner, 44
trivial, 15
game subset, 4, 24
game subset for (G, H), 78
game subset for the pair (G, H), 8
generalized lexicographic product, 64
game subset for (G, H), 78
game subset for the pair (G, H), 8
generalized lexicographic product, 64
geodesic, 6, 44
graph
bivariante, 13
directed, 12
undirected, 13
graph subset, 24
group game, 4, 24

Hamiltonian cycle, 14
homogeneous game, 79
homogeneous space, 8, 78
horizontal, 59
horizontal outputs, 59
identity map, 11
inputs, 11
interchange graph, 6, 44, 68
invariant set, 12
isomorphism, 17
isotropy subgroup, 9, 26, 80
lexicographic product, 7, 59
generalized, 64
maximum decomposition, 6, 39
morphism, 3, 17
near cycle, 46
order, 14
standard, 14
outputs, 11
parity, 44
path, 2, 11
closed, 11
edge-simple, 11
simple, 11
spanning, 11
pattern, 46
pointed game, 8, 68
preserves scores, 5, 36
reducibility graph, 21
reducible, 4, 19
completely, 4
uniquely, 93
reflexive relation, 11
regular, 15
relation, 2, 11
reflexive, 11
reverse, 11
strong, 12
symmetric, 11
transitive, 11
restriction, 2, 11
reverse, 2
reverse relation, 11
rigid, 7
rigid tournament, 57
score, 13
score vector, 13
score-preserving permutation, 36
semi-direct product, 60
separated subgraphs, 13
short exact sequence, 60
span, 6, 39
special cycle, 46
splitting, 68
standard order, 14
Steiner game, 6, 44
Steiner triples, 43
straddle, 18
strong, 12
strongly connected, 12
subgame, 3, 18
subgraph, 13
symmetric relation, 11
tournament, 2, 13
bipartite, 67
regular, 2
rigid, 57
transitive action, 25
transitive relation, 11
translation action, 27
undirected graph, 13
uniquely reducible, 93
vertex, 11
recurrent, 12
vertical edge, 59
vertical outputs, 59
MATHEMATICS DEPARTMENT, THE CITY COLLEGE, 137 STREET AND CONVENT AVENUE, NEW YORK CITY, NY 10031, USA
E-mail address: ethanakin@earthlink.net