Semi-device-independent quantum key distribution based on a coherence equality

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We introduce the first example of a semi-device-independent quantum key distribution (SDI-QKD) protocol with a classical Alice and Bob. The protocol is based on the Coherence Equality (CE) game recently introduced by del Santo and Dakić, which verifies a coherent quantum superposition of communication trajectories in a de-localized way. We show the protocol to be semi-device-independent since the only trusted operations occur in the users’ labs, and establish security against an adversary with bounded quantum memory. Finally, we recast the setup of the protocol as a counterfactual test of nonlocality, and provide additional insight into the CE game.

INTRODUCTION

It is known that modern cryptosystems based on the hardness of computational assumptions are particularly vulnerable to developments of computational power and new algorithms, especially when considering quantum computers, as demonstrated by the archetypal example of Shor’s algorithm. One possible solution to this problem emerges in the field of quantum cryptography, where one attempts to ensure information security based on the laws of quantum mechanics, so that protocols are resistant to any attack, no matter how much computational power is allowed — a notion coined as unconditional security by Diffie and Hellman in their seminal paper.

An important sub-field of quantum cryptography is that of device-independent (DI) cryptography, which attempts to fulfill the vision of unconditional security in the context of devices that are imperfect or even maliciously designed. Quantum cryptography as originally introduced e.g. BB84 had been predicated on knowing the internal description of the systems being used, such as the source and detectors. This did not account for imperfect devices, and the protocols were shown to be highly vulnerable to noise e.g. imperfect photon sources. Fortunately, Mayers and Yao put forth the concept of self-testing, ensuring that if certain statistical tests are met then the source could be guaranteed to be of adequate use for, say, quantum key distribution (QKD). The notion of self-testing has proven useful for ascertaining specific features of quantum systems just by means of observing classical statistics offered by measurement devices, which themselves might be maliciously designed. Device-independent security is based on such properties certified by the statistics witnessed from self-tests, like the Clauser-Horne-Shimony-Holt (CHSH) inequality which certifies nonlocality. This in turn means that security in a device-independent setting is often represented as a gold standard within quantum cryptography, since it adopts the least possible amount of assumptions needed. For instance, a recent QKD protocol has been proven secure with a linear key rate in the presence of noise, where Alice and Bob’s devices are simply taken as physically isolated black boxes. It should be noted though, for completeness, that device-independent quantum cryptography goes further beyond than just QKD. Some other examples found in the literature are random number generation, position verification, coin flipping, and authorization to private databases.

Even though device-independence is undoubtedly the preferable theoretical scenario, proving unconditional security within it can be an extremely difficult task. As such, one might need to make further assumptions on the security model, on top of the ones established by standard device-independence. Perhaps one might need to partially characterize the nature of the systems, like bounding the dimension of the resource or bounding the expectation value of some appropriate operator e.g. energy, for example, or even to describe the inner workings of just one user but allowing the other to be modelled as a black box, instead of them both. This scenario where additional assumptions are made on the devices, yet one does not need to fully describe them in order to prove security is entitled semi-device-independence (SDI).

Another interesting sub-field branching out of standard quantum cryptography is semi-quantum (SQ) cryptography. Therein one has a vested interest in minimizing the quantum requirements of the systems and/or users involved in the protocol without compromising security. In such semi-quantum protocols, one might want to limit Alice or Bob to a single measurement basis, for instance, or force them to only perform detections and/or reflections of photons. This would render either (or both) Alice and Bob to be classical users whilst still exploiting the necessary quantum features in the protocol which guarantee security. This area has the theoretical motivation of identifying the minimum quantum requirements necessary to obtain a quantum security advantage over standard classical protocols, and the potential practical interest of reducing the technological quantum requirements in experimental implementations. An important observation we should stress though, is that semi-quantum protocols should not be considered universally

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The relationship between these two frameworks has yet to be fully explored and basic questions regarding their interplay remain unknown. One obvious question is if these two frameworks are compatible at all, that is, if there exists any cryptographic protocol which could rightfully claim membership to both, thus existing at the intersection of the SDI and SQ models (see Figure 1). In this work, we precisely answer the previous questions affirmatively by exhibiting a QKD protocol which is both semi-device-independent and semi-quantum. This is true, since we only specify properties of the detection operators used by Alice and Bob, while the source and measurements controlled by the outside servers remain unspecified, and Alice and Bob are entirely classical as they can only attempt detection of the photon used in the protocol. The protocol is based on a Coherence Equality (CE) cooperation game [1], and its security is drawn from the upper bounds restricting the users’ performance in the game, which in turn verifies non-classical quantum behavior [10].

On top of presenting the QKD protocol and proving its security, we discuss the CE game providing further insight into its nature, contrasting it with standard nonlocal tests. In particular, since the quantum-over-classical advantage present in the CE game comes from testing non-null coherence terms, we argue the game can be used for certifying entanglement in Fock space. We also provide an interpretation to the variant of the setup used in the QKD protocol as a counterfactual test of nonlocality, when we relax the single-particle condition. These findings might be of independent interest within areas more related to quantum foundations.

Structure

The paper begins with Section I where some preliminaries are established, namely, a perspective of semi-quantum key distribution (SQKD) and semi-device-independent quantum key distribution (SDI-QKD), which also serves as a motivation for our work; we also provide a brief overview of the notion of security in QKD, the coherence equality (CE) game and an introduction to notation adopted throughout the work. This is followed by the four main sections, II, III, IV and V. Section II introduces the original CE game, and expands on it. Namely, we place the CE game in the context of self-testing and certification of entanglement in Fock space and draw a comparison with the more standard type of nonlocal tests used to certify entanglement. In Section III we introduce the QKD protocol based on the CE game, discuss its status as a semi-device-independent and semi-quantum protocol and characterize the security assumptions of the model. In Section IV we introduce the main technical details needed to prove security, namely the bounds in semidefinite programming [21], and show the security proof of the protocol within the assumptions previously established, namely, that the adversary is memory-bounded. In Section V we use the same setup of the CE game used in the QKD protocol (where the single-particle condition was relaxed) and provide an interpretation of said setup as a counterfactual test of nonlocality. Finally, we present our conclusions in Section VI.

I. PRELIMINARIES

Perspective on SQKD and SDI-QKD

In this section we provide an overview of semi-quantum and semi-device-independent cryptography, in particular the context relevant to our protocol, that is, of quantum key distribution (QKD). For a comprehensive survey of the state-of-the-art in semi-quantum cryptography, we refer the reader to the work of Iqbal and Krawec [21].

Boyer et al. [25] introduced the model of semi-quantum cryptography by demonstrating the robustness of a quantum key distribution protocol where one of the users only has classical capabilities. Any attack that Eve could perform to obtain information about the secret key would necessarily be detectable by Alice and Bob, as they would find errors in their test bits.

 Shortly after, more SQKD protocols were shown to be robust, even for cases where both users have reduced quantum capabilities [26].

More recently, a SQKD protocol was developed where a fully quantum server prepares and measures photons that are sent to users who either choose to attempt detection or reflect the photon back to the server [23]. The protocol was proven secure in the finite-key setting against collective attacks, where each round is attacked separately by Eve attaching an ancilla state, such that by delaying her measurements until a later time she can perform a coherent measurement on the collected quantum side information. This protocol’s setup bears some similarities to ours, and a practical comparison is provided in the conclusions.

In (semi-)device-independent cryptography, the CHSH inequality has been leveraged as a powerful tool to develop protocols in the context of untrusted devices. Mayers and Yao initiated this effort by showing a set of correlations that were sufficient to ensure a source was safe for quantum key distribution [6, 7].

More recently, Pironio et al. [13] proved a quantitative relation between devices that break the CHSH inequality and the randomness of their outputs even in the context of a quantum adversary. Shortly after, Vazirani and

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1 In fact, single-photon detectors, despite being relatively harder to implement with high efficiency, are considered classical resources. In contrast, rotation gates, which can be easier to implement, are still considered quantum operations.
Vidick showed how certifiable random numbers could be produced against a quantum adversary, with no assumptions on the devices apart from a non-signalling principle \cite{27}. Finally, in \cite{9} the same authors also proved the security of a fully device-independent QKD protocol with a linear key rate in the presence of noise. This represents a theoretical conclusion to the question of how much trust must be placed in a description of the devices to perform unconditionally secure quantum key distribution. Further advances have been made in improving key rates and in extending the results from quantum key distribution to conference key agreement \cite{28, 29}.

The difficulty of implementing loophole-free Bell tests has also sustained some interest in QKD protocols that are close to device-independent but more practical. Different relaxations of the demands of device-independence have been developed, such as assuming an upper bound on the dimension of communicated systems in prepare-and-measure protocols \cite{15, 16}, or considering only measurement devices to be untrusted \cite{30, 31}. More recently, another approach to semi-device-independence has been introduced where trusted quantum inputs are considered instead of the usual classical inputs \cite{32}. This flavor of semi-device-independence is of particular importance to our work, as its framework fits naturally with the setup of the QKD protocol here considered, with the added note that Alice and Bob do not provide perfectly trusted inputs to the servers, but instead participate classically in a trusted manner to produce the inputs given to their respective servers. We revisit this perspective more fully in Subsection \ref{subsec:semi-device-independence}.

The frameworks of device-independent and semi-quantum cryptography seem to be, at least superficially, of a conflicting nature. In fact, fully fledged device-independence based on nonlocality, wherein Alice and Bob verify that their correlations break a Bell inequality to check that their outputs are genuinely random and secure against an eavesdropper, are at odds with the setup of semi-quantum cryptography. This is readily noticeable since in the previous example one has to endow Alice and Bob with full quantum privileges in order for them to break locality, such as in the case of CHSH game. Interpreting nonlocality as a particular form of contextuality \cite{33} makes it even more apparent, since contextual phenomena can only arise for scenarios involving various measurement contexts, requiring full quantum capabilities out of the agents. This is not to say that in principle both scenarios are ultimately irreconcilable, and one might ponder if other type of self-tests certifying different kinds of non-classical behavior could exist for setups with reduced quantum requirements.

In this work we precisely explore that idea and using the CE game \cite{11} exhibit a QKD protocol laying at the intersection of the two frameworks of semi-quantum and (semi-)device-independent cryptography. This QKD protocol, although not unconditionally secure (see Section \ref{sec:device-independent}), still represents an important first step in order to better understand the relationship between the two frameworks. It also represents one of the most extreme cases one could think of in reducing quantum requirements, since Alice and Bob can certify a specific type of non-classical behavior as fully classical agents by delegating part of the certification to external servers. The protocol does not have a fully device-independent status as one still needs to model the classical users’ operations, it does have however a semi-device-independent one since the source and measurement servers are untrusted. This means that the QKD protocol is legitimately characterized as being both semi-device-independent and semi-quantum — something which was not known to be possible.

The diagram of Figure \ref{fig:diagram} represents the three distinct protocol’s characteristics which have been previously mentioned, unconditionally secure, (semi-)device-independent and semi-quantum. Our protocol belongs to the overlap between the (semi-)device-independent and semi-quantum regions; the protocol of Vazirani-Vidick \cite{9} is unconditionally secure and (semi-)device-independent. Furthermore, although our protocol is only proved to be secure against a bounded-memory adversary and, thus, it is not unconditionally secure, (see Section \ref{sec:device-independent}) it still represents an important first step in understanding how to merge all three characteristics, which would culminate in a protocol existing in the overlapping three regions.

**Security in quantum key distribution**

In a QKD protocol, the users Alice and Bob and an eavesdropper Eve have access to a quantum state $\rho_{ABE}$. After performing unitary transformations and measurements, they obtain a final state, of which a subsystem $\rho_S$ refers to the secret key $S$.

A key is said to be secure if, from Eve’s perspective, the state is almost indistinguishable from a uniformly
random distribution in the set of keys, as stated in the following definition.

**Definition 1. Universal security in QKD [34]**

Let \( \{|s\rangle\} \) be an orthonormal basis of a Hilbert space \( \mathcal{H}_S \). We say that a key \( S \) is \( \varepsilon \)-secure with respect to \( \mathcal{H}_E \) if

\[
\frac{1}{2} \| \rho_{SE} - \rho_U \otimes \rho_E \|_1 \leq \varepsilon,
\]

where \( \rho_U = \frac{1}{|S|} \sum_{s \in S} |s\rangle \langle s| \) is the fully mixed state on \( \mathcal{H}_S \), \( \| . \|_1 \) is the trace norm.

Since the trace norm does not increase with quantum operations, the security of the key is not reduced by further evolution of the system in which it is used. This is related to the universal composability of a protocol, in which it can be used as a subroutine of a larger protocol and still be secure [35]. If there are no assumptions made on the power of the eavesdropper, then the key is always guaranteed to follow Definition 1 and the protocol is universally composable.

**The Coherence Equality game**

We overview the protocol introduced by del Santo and Dakić – the Coherence Equality (CE) game – but we refer the reader to the original work for further details [1].

Broadly speaking, the CE game is cast within the framework of non-signalling theoretical cooperation games, similarly to nonlocal games [4] — although, as we argue below, the CE game itself should not be interpreted as a nonlocal game (see Sections I and V). In non-signalling cooperation games, spatially distributed players — mediated by an impartial Referee — try to jointly carry out a task with the highest probability of success by exploiting the best strategies available to them, when explicitly prohibited from communicating (non-signalling condition). The setup of the CE game (see Figure 2) is as follows: a source sends a one-particle state to referees \( R_A \) and \( R_B \), which, depending on the self-generated random bit inputs \( x \) and \( y \) respectively, decide to either block the channel or do nothing, passing the state to Alice and Bob, who then perform measurements on the resulting state \( \rho_{xy} \) and output classical bits \( a, b \), respectively. They win if the following equality holds true: \( a \oplus b = x \oplus y \), that is, if the parity of the outputs matches the parity of the inputs. We will use the terms particle and photon interchangeably in the analysis since, according to the work of [1], the best possible strategy requires a boson to be used, of which a photon is the most obvious example for possible implementations.

![Figure 2. Setup of the coherence equality (CE) game. A one-particle state is sent to Alice and Bob, first passing through two referees \( R_A, R_B \) who uniformly at random choose to either block the channel or do nothing, according to bits \( x, y \). Afterwards, Alice and Bob, prohibited from communicating (non-signalling) but possibly using some pre-established entanglement, perform measurements on the received state and output bits \( a, b \) respectively. Notation and parameters](image)

**Protocol.** The protocol consists of \( m \) rounds (with indices in \( [m] \equiv \{1, \ldots, m\} \)), where Alice and Bob uniformly and randomly choose their respective inputs, \( x, y \in \{0,1\} \). The set \( D \) consists of the indices of the rounds where both Alice and Bob chose to attempt detection, i.e. \( x = y = 1 \). To perform the step of classical post-processing, Alice chooses a random subset \( B \) of \( D \) with size \( \gamma |D| \), where \( \gamma > 0 \) is small. With the outcome information of set \( B \), Alice and Bob estimate the single-particle parameter \( d_e \) (Equation 11) and with the totality of the \( m \) rounds they compute the interference term \( I \) for the coherence equality game (Equation 10), given by the correlation function

\[
I := \sum_{x,y} 1_{a \oplus b = x \oplus y} p(ab|xy),
\]

where, to represent if a round satisfies a given predicate function \( F \), we use the indicator function \( 1_F \), which is 1 whenever \( F \) holds and 0 otherwise. Then, for the CE game,

\[
1_{a \oplus b = x \oplus y} := \begin{cases} 1, & a \oplus b = x \oplus y, \\ 0, & \text{otherwise.} \end{cases}
\]

From the observed values of \( I \) and \( d_e \), for a given security parameter \( \mu \), Alice and Bob compute a key of size \( \kappa |D| \)

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2 Note that we make a distinction between the terms non-signalling and nonlocal, which are sometimes used interchangeably. We call any cooperation game a non-signalling game if the players are not allowed to communicate with one another, that is, if the non-signalling condition is enforced, and reserve the term nonlocal for such games where the quantum advantage, if present, is provided explicitly by the nonlocality of the measurement outcomes.
that is secure against a memory bounded eavesdropper with probability \(1 - \mu\), and we show that \(\kappa\) is positive for certain values of \(I\) and \(d_e\).

The CHSH and CE games. In this paper we also discuss the coherence equality game in the context of non-signalling games and nonlocal games. For any given game \(G\), \(\omega^G_1\) and \(\omega^G_2\) represent the highest winning probabilities for quantum and classical agents, respectively. The term \(\Omega_G := \omega^G_1 - \omega^G_2\) is the quantum-over-classical advantage in the game. These terms are optimizations over states and measurements used, and when they are written as dependent on a variable (such as the concurrence \(\mathcal{C}\)), the formula should be taken as optimized over all other variables.

SDP bounds. Using semidefinite programming, we are capable of obtaining upper bounds on the winning probability for certain games by limiting the agents to quantum operations. These bounds are not necessarily tight, but are still useful in bounding an eavesdropper’s information. Using a convergent relaxations method \cite{36}, each tighter bound is associated with an increasing level \(k\). We refer to an upper bound for a certain game obtained in the \(k\)-th level in the relaxation by writing \(\omega^k\) and we similarly write the bound on the guessing probability for outcomes in Alice’s lab as \(p^k(\alpha|x,y)\). In this paper we only present results for values of \(k = 3\) (see Subsection \(\text{II.A}\)), but it is possible that tighter bounds may be obtained using better techniques or equipment, and that the key rate may be improved.

II. THE CE GAME AND ENTANGLEMENT IN FOCK SPACE

In this section, we delve deeper into the nature of the coherence equality (CE) game. In particular, we describe how it appropriately certifies entanglement in Fock space, and how it differs from the more usually studied nonlocal Bell tests.

In the CE game, the photon used as a resource might be prepared in such a way that it is either sent in superposition of trajectories to Alice and Bob’s labs, in which case we say the resource is quantum; or in a fixed trajectory to a single lab, and we say the resource is classical. We can then write the resource’s initial state in second quantization parametrized by \(\theta\),

\[
|\psi\rangle = \cos(\theta) |10\rangle_{AB} + \sin(\theta) |01\rangle_{AB} \in \mathcal{F}^{(1)},
\]

where \(\theta \in [0, \pi/2]\), \(|0\rangle\) is the vacuum state of the electromagnetic (EM) field, and \(|1\rangle = a^\dagger |0\rangle\) an excited mode of that field i.e. a photon, such that both states form a basis in the single-particle Fock space \(\mathcal{F}^{(1)}\). Obviously, if both coefficients are non null, the resource is sent in superposition of trajectories, and if either coefficient is zero it is sent in a fixed trajectory to a single lab depending on the null coefficient.

Consider the initial resource \(\rho_{AB} = |\psi\rangle_\theta \langle \psi|_{AB}\), being sent to referees \(R_A, R_B\) upon which each of them decides to either block the trajectory path leading towards the respective player’s lab (encoded by bit 1), or do nothing (encoded by bit 0). This originates in four overall possibilities which encode bits \(x, y\) (\(x\) for Alice and \(y\) for Bob) on the state \(\rho_{AB}\), such that, the quantum input received by Alice and Bob is \(\rho_{xy} = \mathcal{B}_A^{\mathcal{B}_B}(\mathcal{B}_B^{\mathcal{B}_A}(\rho_{AB}))\), in the case where the path leading towards Alice and Bob is blocked, and \(\mathcal{B}_A^{\mathcal{B}_B}(\rho_{AB}) = \rho_{AB}\) when it isn’t. Thus, the four different quantum inputs Alice and Bob can receive are, \(\rho_{00} = \rho_{AB}\), \(\rho_{01} = \Tr_B(\rho_{AB}) \otimes |0\rangle\langle 0|_B\), \(\rho_{10} = |0\rangle\langle 0|_A \otimes \Tr_A(\rho_{AB})\), \(\rho_{11} = |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B\).

Let \(\Pi_{A/B}^{a/b}\) be the measurements performed by Alice and Bob, respectively. The general expression giving the probability of successfully winning the CE game in terms of the interference terms \(I_{ab} = \sum_{x,y} (-1)^{x+y} p(ab|xy)\) with \(p(ab|xy) = \Tr(\rho_{xy} \Pi_{A/B}^{a/b})\) is,

\[
P_{Q}^{CE} = \frac{1}{2} + \frac{1}{4}(I_{00} + I_{11}).
\]

Also, considering the possibility that each player can perform arbitrary rotations in their lab, \(\Pi_{A/B}^{a/b} = \frac{1}{2}(1 - (-1)^{a/b} \sigma_{A/B})\), which does not impose extra constraints on the set of strategies allowed for a single photon, we get,

\[
P_{Q}^{CE} = \frac{1}{2} + \frac{C(\theta)}{16} \left( |0\rangle\langle A|_1 |1\rangle\langle B|_0 + h.c. \right),
\]

where \(C(\theta) = \sin(2\theta)\) is the concurrence of state \(|\psi\rangle\), defined as \(C(\rho) \equiv \max |\lambda_1 - \lambda_2 - \sqrt{\lambda_3} - \sqrt{\lambda_4}|\), where the \(\lambda_i\)’s are the decreasingly ordered eigenvalues of \(\rho(Y \otimes Y)^* (Y \otimes Y)\), and \(Y\) the Pauli \(Y\)-matrix.

The maximum over the operators is obtained when we set both operators as the Pauli \(X\)-matrix i.e. \(\sigma_A = \sigma_B = X\), yielding,

\[
\omega^CE_\theta = \frac{1}{2} + \frac{C(\theta)}{8}.
\]

The previous expression gives us the maximum probability of Alice and Bob winning the CE game (under optimal operations), as a function of the concurrence of the initial state \(\rho_{AB} = |\psi\rangle_\theta \langle \psi|_{AB}\). It is implicitly assumed that only a single photon is used in a run of the game – the single-particle condition. The classical optimal probability \(\omega^C = 1/2\) is recovered when the interference terms disappear, which we obtain by setting \(\theta\) equal to 0 or \(\pi/2\), corresponding to states \(|10\rangle_{AB}\) and \(|01\rangle_{AB}\), respectively. The highest value will be achieved for a maximum concurrence obtained by setting \(\theta = \pi/4\), which corresponds to a resource initialized in state \((|10\rangle_{AB} + |01\rangle_{AB})/\sqrt{2}\), resulting in the maximum winning probability of 5/8 achieved by a quantum strategy, as found by del Santo and Dakić \cite{1}. 


Device-independent certification of entanglement and self-testing: The device-independent certification of entanglement is intimately associated with the self-testing of nonlocal correlations. This is true since it is known that, at least for pure states, the presence of entanglement is a necessary and sufficient condition for nonlocality, as witnessed in a typical Bell scenario [37]. As such, tests of nonlocality have taken a central role in proving the existence of entanglement in a device-independent manner. Such tests like the CHSH game are traditionally performed with systems of fixed degrees of freedom, characterized by the regular quantum mechanical formalism. This originates an issue — the traditional quantum mechanical formalism is unequipped to handle protocols which make explicit use of the vacuum state as an information carrier. Consequently, such protocols cannot be accurately described within the regular formalism. In these protocols, of which the CE game is clearly an example, the vacuum state should be treated as a legitimate state on equal footing as any other physical state, which requires us to use the formalism of second quantization. Thus, the correct way to understand entanglement in this scenario is not between distinct systems of fixed degrees of freedom, but rather between excitation modes of a unique quantum field. Accordingly, it becomes evident that the resource state as described in Equation 3 is, for non-null values of both coefficients, a manifestly entangled state, and that its entanglement is quantifiable by an appropriate entanglement monotone like the entanglement concurrence $\Omega$. As we expand on below, the CE game then takes a similar role to the CHSH of certifying entanglement by means of testing explicitly non-classical behavior, although the entanglement certification provided by the CE game is not achieved by self-testing nonlocal correlations, and it occurs in Fock space rather than Hilbert space.

As was previously stated, the CE game is cast within the framework of non-signalling cooperation games. In particular the CE game is reminiscent of the quantum generalization of the well known XOR games [39,41]. It is then not surprising that the CE game bears some similarities with nonlocal games, namely, it also exhibits a positive quantum-over-classical advantage which allows a categorical separation between classical and explicitly non-classical quantum resources. Therefore, one can define a relationship respected by all classical strategies, so that its violation certifies the existence of non-classical phenomena exploited as a resource in the game. But what kind of non-classical phenomenon is tested in the CE game? The non-classical feature allowing the violation of the classical bound is the coherent superposition of different directions of communication (i.e., the existence of non-null interference terms), and analogously to the case where the self-testing of nonlocal correlations for pure states is a sufficient condition for certifying entanglement in Hilbert space, so is the case that testing the existence of such interference terms can be used to certify entanglement in Fock space, if some assumptions are met.

Quantifying entanglement in Fock space under the single-particle condition: In [11] it was shown that, assuming the source only shoots a single particle at a time (single-particle condition), the highest winning probability achieved by a classical strategy in the CE game is $\Omega_{\text{CE}} = \frac{1}{2}$ — classical, meaning, having no interference terms and the photon following along a definite trajectory. Furthermore, if Alice and Bob manage to win the game with a higher probability than $1/2$, under such an assumption, they successfully test the existence of non-null interference terms in Equation 5. More precisely, under the single-particle condition, the initial resource can be described by Equation 3 and from Equation 7 we have that the quantum-over-classical advantage is,

$$\Omega_{\text{CE}} = \omega_{\text{q}} - \omega_{\text{c}} = C/8,$$

where $C$ is the concurrence of the resource state; this means the quantum advantage depends explicitly on the entanglement of the initial resource. In fact, the advantage only exists for non-null values of the entanglement concurrence in Fock space, and linearly increases with it. This unequivocally leads us to conclude that testing non-null interference terms by breaking the coherence equality certifies the existence of entanglement in Fock space. Moreover, by analyzing the quantum advantage exhibited when playing the CE game, Alice and Bob cannot only infer qualitatively the presence of entanglement in the resource they shared, but as long as the single-particle condition is met, they can in fact precisely quantify that entanglement which is given by a straightforward linear relationship, Equation 8.

This is analogous to other more familiar self-testing scenarios, for instance in the CHSH game the violation of the Bell bound can also be explicitly quantified as a function of the entanglement concurrence of the shared resource [42],

$$\Omega_{\text{CHSH}} = \omega_{\text{q}} - \omega_{\text{c}} = (\sqrt{1 + C^2} - 1)/4.$$

If the Tsirelson bound [43] of $\cos^2(\pi/8)$ is achieved, then the resource shared between them would have to possess at least one e-bit of information. Similarly, if the quantum upper bound of $5/8$ is achieved in the CE game, under the single photon condition one could also conclude that the resource used had to be a maximally entangled state between the vacuum and single excited modes of the EM field, thus, also possessing one e-bit of information. To reiterate, as the CHSH can be used to certify entanglement between Hilbert spaces with fixed degrees of freedom, so does the CE game can be thought as certifying entanglement in Fock space, inferred from the testing of non-null interference terms.

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3 On occasion, some doubts are expressed against such a claim. A clear exposition of why those doubts are unsubstantiated can be found in SJ Van Enk’s illuminating account [38].
III. SDI-QKD WITH CLASSICAL USERS

Here we will show the SDI-QKD protocol with a classical Alice and Bob, based on the CE game. As previously mentioned, the protocol is semi-device-independent as we only need to specify properties of the detection operators used by Alice and Bob, and the source and measurements controlled by the outside servers remain unspecified. Furthermore, Alice and Bob are entirely classical in the semi-quantum sense, since they can only attempt detection of the photons, which represents a fixed basis measurement.

1. Security assumptions

We assume an adversary with bounded quantum memory to exclude the possibility that they may delay their measurements until after Alice and Bob have shared partial information about their raw key. This allows us to apply the semidefinite programming (SDP) bound directly (see Subsection IVA), as it is valid for any state prepared by the untrusted source, which is equivalent to letting Eve perform any measurement on the state before having any more information. Therefore, either because Eve has access to a small memory or it is not durable, she is expected to have performed her measurements before Alice and Bob share their choice of inputs and partial raw key. We do not assume that the devices behave identically and independently in each round, and allow for internal memory that takes into account all previous rounds. Alice and Bob should also be able to ensure that their devices function as particle detectors, following the requirements given in Equation 17, and to verify from their devices (source and measurement servers) and the source and measurements performed by the servers are spacelike separated in order to respect the non-signalling condition.

2. Setup

Alice and Bob take up the role of the Referees in the original CE game setup, where they are restricted to either attempt detection or do nothing. An untrusted source sends a quantum state to Alice’s and Bob’s labs, and Alice (resp. Bob) randomly chooses to either attempt detection or do nothing (resp. y = 1) or do nothing (x = 0, resp. y = 0). Attempting to detect returns an output α (resp. β) which indicates whether or not the particle was detected. Alice (resp. Bob) then sends their half of the state to the untrusted server S_A (resp. S_B), which output outcomes a (resp. b), as represented in Figure 3.

The precise steps of the protocol are as follows:

1. Take as input parameters µ, η. For each round i ∈ [m], Alice and Bob receive state ρ_i from a source and randomly choose to either detect or do nothing, according to their secret bits x_i, y_i respectively. In case they attempt detection, they receive outcomes α_i, β_i, and every round they receive outcomes a_i, b_i from the untrusted servers.

2. Alice and Bob share their choices of inputs, X and Y respectively, and the outputs A, B from their respective servers S_A and S_B on a public authenticated channel.

3. Alice chooses a random set B as a fraction γ of the rounds in D, where D is the set of rounds i such that x_i = y_i = 1, i.e. when both Alice and Bob chose detection, and share the outcomes α, β of their detection.

4. From their input choices and from the outputs of S_A and S_B, A and B respectively, Alice and Bob calculate the fraction of rounds winning the CE game,

\[ \tilde{P}_{CE} = \frac{1}{4} 1 \sum_{i \in [m]} 1_{a_i \oplus b_i = x_i \oplus y_i} \]

and the fraction of rounds not satisfying the one photon condition, estimated from the public results

\[ 4 \text{ Although we refer to Alice and Bob’s raw keys as } \alpha, \beta, \text{ we point out that these will be complementary keys (when one is 0, the other is typically 1) which means that as a trivial step one of them should flip their bitstring before continuing.} \]
of rounds in $B$,
\[ \hat{d}_\varepsilon \equiv \frac{1}{|B|} \sum_{i \in B} 1_{\alpha_i = \beta_i}. \] (11)

If $\hat{d}_\varepsilon > \eta$, they abort the protocol.

5. Alice and Bob perform information reconciliation, where Bob communicates $\ell = H(1.1\eta)|D| + \log(2/\varepsilon)$ bits of information to Alice.

6. For the security parameter $\mu$, determine $\kappa(\hat{I}, \hat{d}_\varepsilon, \mu)$ such that, with probability at least $1 - \mu$, the protocol is secure and they obtain $\kappa|D| - \ell + O(\log 1/\varepsilon)$ (12) bits of information.

Before proceeding to the security proof, some comments on the nature of the protocol are in order.

3. Relaxing the single-particle condition

In [1] the single-particle condition was implicitly assumed to hold true in their setup. Here we relaxed that assumption by considering the possibility that the source could be untrusted and vulnerable to some error $d_c$, such that, with probability $1 - d_c$, the single-particle condition is met and only one photon shoots out of the source. Accordingly, with probability $d_c$ the condition is violated and multiple (at least two) or no photons shoot out of the source.

In this setting where the single-particle condition is relaxed, one can derive the classical value for the protocol explicitly as a function of the parameter $d_c$. In order to do so, we first notice that if the single-particle condition is verified, the source and servers $S_A, S_B$ cannot win with a higher probability than $1/2$, as was shown in [1]. Moreover, in the case where they break the single-particle condition, the source and servers can win the game deterministically by sending a particle to each lab and learning the inputs $x, y$ directly by simply attempting detection. If the source only breaks the single-particle condition a fraction $d_c$ of the rounds, then the best overall winning probability is

\[ \omega^C_E(d_c) = \frac{1}{2}(1 - d_c) + d_c. \] (13)

We represent the classical value, as given by the previous linear relationship, and a numerical upper bound on the quantum value (from the SDP problem) explicitly as functions of the error $d_c$ in Figure [4] in Subsection [IVA]. The shaded area is the region for which a strategy, should it exist, is necessarily non-classical, meaning that it guarantees the existence of non-null interference terms and consequently non-null entanglement in Fock space.

Notice, though, that unlike the case where the single-particle condition is guaranteed ($d_c = 0$) for which we know the exact quantum value as derived by del Santo and Dakić, for non-null values of $d_c$ we do not have an explicit expression for the quantum value as a function of the concurrence, and cannot in general quantify the amount of entanglement the resource possessed.

4. Semi-device-independence

Fully fledged device-independent certification usually involves modelling the systems as black boxes, wherein the users are only allowed to interact with such boxes (in our case, the servers) by selecting inputs on the boxes and registering the outputs. This form of abstraction is not entirely possible in the case of our protocol, since the inputs given to the servers are actually quantum rather than classical. This characteristic precludes the possibility that the players’ actions could be completely modelled as interactions with black boxes, and so the protocol cannot offer a complete device-independent certification of the tested non-classical correlations. We are, however, in a semi-device-independent scenario, since certain assumptions are explicitly made on the underlying model governing the behavior of the interactions. Namely, the protocol exists within a setup similar to the recently introduced scenario where one does not impose restrictions on the boxes, but rather, one allows for quantum inputs under the assumption that such inputs are prepared in a trusted way [32], in that they are as the players believe them to be. We should note however, that unlike in the trusted quantum input scenario where Alice and Bob are assumed to be quantum parties and the sole agents involved in the preparation of the quantum inputs, in our case Alice and Bob do not provide the trusted inputs themselves to the servers, but instead participate in the preparation of such quantum inputs as classical intermediaries. Obviously, we need to assume that Alice’s and Bob’s operations inside their labs are trusted — in order to model such operations and cast them in a SDP problem to compute the upper bounds on the CE winning probabilities, which guarantee security of the protocol (see Subsection [IVA]) — but do not need to assume trustiness of the whole preparation process. In this setting the protocol can be interpreted as certifying entanglement in a semi-device independent manner, in a similar fashion to the type of self-testing performed with trusted quantum inputs, but here the preparation is separated between an untrusted source and the trusted classical users partaking in the process. Accordingly, the protocol is semi-device-independent, since no other assumptions are made on the nature of the servers themselves or even the photon source, which we assumed to be untrusted by relaxing the single-particle condition.
IV. SECURITY OF THE PROTOCOL

Proof overview

The security proof is analogous to those typically found in the field of device-independent quantum cryptography, in the sense that the security fundamentally stems from the upper bounds restricting the users' performance in a game that verifies quantum behavior [10].

We start by bounding any eavesdropper’s information using the behavior observed by Alice and Bob of the servers and their devices. Using the tools of SDP, we obtain an upper bound on the winning probability for the CE game maximized over all strategies (including those where the single-particle condition is not met), and also an upper bound on the guessing probability for outcomes in Alice’s lab for any given winning probability and probability of observing a single particle. This allows us to establish a bound on the quantum min-entropy of Eve, who may be controlling the source and servers (Theorem 10). Since these results are new, and involve some specific considerations in the way semidefinite programming is applied, we go over these bounds in some detail, explaining some of the steps used.

After bounding the min-entropy in this way, we move on to the problem of noise — here represented by the probability of breaking the single-particle condition, $d_x$ — and its typical QKD solution which is information reconciliation. Alice and Bob share a fraction of their bit strings (Theorem 15), which allows them to obtain an equal key with high probability.

A. Semidefinite programming bounds

A central part of the security proof of our protocol involves establishing a bound on the guessing probability of the key generating events in Alice’s or Bob’s labs, given the behavior exhibited by the untrusted servers. This problem can be stated as an optimization problem in SDP and in this section we give a brief explanation of the techniques used to solve this type of problem [24]. The complete details of the approach used here can be read in the work of Pironio et al. [36].

1. The method of convergent relaxations

The most general case that the technique by Pironio et al. solves is the following: For a set of operators $X$ and polynomials $p$, $q_i$, $r_i$, $s_i$, define the minimum

$$\tilde{p}^* = \min_{\{H, X, \phi\}} \{\phi | p(X) | \phi\}$$

subject to

$$q_i(X) \geq 0 \quad i = 1, \ldots, m_q$$

$$r_i(X) \phi = 0 \quad i = 1, \ldots, m_r$$

$$\langle \phi | s_i(X) | \phi\rangle \geq 0 \quad i = 1, \ldots, m_s,$$

where $q_i(X) \geq 0$ means that $q_i(X)$ is a positive semidefinite operator (see Definition 13 in the Appendix).

This optimization problem can be turned into a sequence of relaxed semidefinite programming problems that result in solutions $\tilde{p}(i), i \in \mathbb{N}$ such that $\tilde{p}(1) \leq \tilde{p}(2) \leq \cdots \leq \tilde{p}^*$ and in the case that the constraints imply that the operators are bounded, then the sequence converges to $\tilde{p}^*$. The dimension of the Hilbert space is not bounded in the optimization problem, as each relaxations increases the dimension of the Hilbert space considered.

2. Implementation

In order to bound the correlations in our precise scenario, we should be able to write all relevant expressions from a finite set of operators $X$. Using polynomials $q_i, r_i, s_i$, all properties of the problem should be able to be written as expressions of the type in equation (14). In our case, the full set of operators is $X = X' \cup (X')^\dagger$, for

$$X' = (D_A^0, D_A^1, D_B^0, D_B^1, M_A^0, M_A^1, M_B^0, M_B^1). \quad (15)$$

There are two types of operators. Detection operators, $D_{A/B}'$, are used in Alice’s/Bob’s labs, where they can attempt to detect the particle. The remaining are measurement operators, $M_{A/B}'$, which in the general case are PVMs. These two groups have different properties, which will appear in the constraints of the SDP problem. Another immediate property of the operators is commutativity between different subspaces.

Particle detectors function as blockers of channels, with the added feature of outputting whether a particle was detected (“1”) or not (“0”). In either case the resulting state is vacuum. We can then represent the operators in second quantization as

$$D_{A/B}^0 = |0\rangle\langle 0|_{A/B}, \quad D_{A/B}^1 = |1\rangle\langle 1|_{A/B}. \quad (16)$$

The relations that these operators follow of relevance to the SDP problem are, for $S \in \{A, B\}$,

$$(D_S^0)^2 = D_S^0, \quad D_S^0 D_S^1 = D_S^1,$$

$$(D_S^1)^2 = 0, \quad D_S^1 D_S^0 = 0,$$

$$D_S^0 (D_S^0)\dagger + (D_S^1)\dagger (D_S^1) = 1_S, \quad (17)$$

as well as the commutation of operators between Alice’s and Bob’s sides.

The value we are optimizing for, within the SDP problem, is the highest likelihood of any outcome in Alice’s lab, in the case $x = 1$, which is the maximum between the quantities

$$P(\alpha = 1|x = 1) = \langle \phi | (D_A^1)\dagger D_A^0 | \phi\rangle, \quad (18)$$

$$P(\alpha = 0|x = 1) = \langle \phi | (D_B^1)\dagger D_B^0 | \phi\rangle. \quad (19)$$

The measurements $M_A^a, M_B^b$, with $a, b \in \{0, 1\}$ used by the servers can be said to follow the properties of
projection operators:
\[
(M^{a/b}_{A/B})^\dagger = M^{a/b}_{A/B}, \quad (M^{a/b}_{A/B})^2 = M^{a/b}_{A/B},
\]
\[
M^0_A + M_A = 1_A, \quad M^1_B + M_B = 1_B.
\] (20)

The use of projections instead of the general POVMs is justified as a POVM can be represented by a projective measurement in a sufficiently large Hilbert space, so no generality is lost \[14\].

The conditional probabilities for the setup can be written as
\[
p(ab|xy) = \sum_{\alpha,\beta} \langle \phi | (D^\alpha_A)^x(D^\beta_B)^y M^a_A M^b_B(D^\beta_B)^y(D^\beta_B)^y | \phi \rangle.
\] (21)

The winning probability for the coherence equality game,
\[
\frac{1}{4} \sum_{abxy} 1_{a=b=x=y} p(ab|xy),
\] (22)
can be written as an expected value over operators in \(X\) by replacing \(p(ab|xy)\) with the expression in Equation (21).

3. Results

In the paper by del Santo and Dakić \[1\], the best strategy presented for the CE game has success probability \(\omega^{CE}_q = 5/8 = 0.625\). Our model contains the one used in the paper, so we can calculate an upper bound on that probability. The best strategy is obtained by the following optimization:

\[
I_{\text{max}}(d_e) = \max \sum_{abxy} 1_{a=b=x=y} P(ab|xy) \quad \text{subject to} \quad \begin{align*}
P(a = \beta | 1, 1) & \leq d_e \\
P(a, b, x, y) & = \left\| M^a_A (D^\alpha_A)^x \otimes M^b_B (D^\beta_B)^y | \phi \rangle \right\|^2
\end{align*}
\] (23)

where \(a, b, \alpha, \beta, x, y \in \{0, 1\}\). The bits \(a\) and \(b\) represent the outcomes of the servers \(S_A\) and \(S_B\), \(x\) and \(y\) represent the choices of Alice and Bob (detect and do nothing, respectively for 1 and 0), and \(\alpha\) and \(\beta\) are the outcomes of the detection of Alice and Bob respectively, and \(\perp\) in case they chose to do nothing.

For this setup, guaranteeing a single photon \((d_e = 0)\), and including the properties between all the operators, we obtain for level \(k = 3\) the maximum value \(\omega^{(3)}_q = 0.625104\), which is very close to the value \(\omega^{CE}_q = 0.625\) obtained in \[1\], and could be improved by doing higher-order relaxations. For higher values of \(d_e\), we observe an increase in the winning probability, as can be seen in Figure 4. This is expected as, for the case \(d_e = 1\), where the server is allowed to send two photons to Alice and Bob, the servers can directly learn the values \(x\) and \(y\) and the game can be won trivially by simply setting \(a = x, b = y\).

Figure 4. Best winning probability for the coherence equality game when the one-photon condition is met with probability \(1 - d_e\), for the case of classical strategies and the first upper bound for quantum strategies. Values of \((P_{\text{win}}, d_e)\) inside the shaded region are only obtained by non-classical strategies.

The safety of the protocol is based on the fact that a set of state and measurements \((\rho, M^a_A, M^b_B)\) picked by the adversary cannot simultaneously be used to win at the CE game and allow for detection events (i.e. the outcomes \(\alpha\) and \(\beta\) to be completely predictable. This property is captured in the following optimization problem:

\[
P^*(I, d_e) = \max \quad P(\alpha | x = 1) \quad \text{subject to} \quad \sum_{abxy} 1_{a=b=x=y} P(ab|xy) \geq I \\
P(\alpha = \beta | 1, 1) \leq d_e \\
P(a, b, \alpha, \beta | x, y) = \left\| M^a_A (D^\alpha_A)^x \otimes M^b_B (D^\beta_B)^y | \phi \rangle \right\|^2,
\] (24)

for which we computed the third relaxation solution \(P^{(3)}(I, d_e)\). We ran the optimization problem on this level for different values of \(I\), starting at the classical value \(I_0 = 2\), for which \(P^{(3)}(I_0, d_e) = 1\). Figure 5 is a graph of the relation between the guessing probability \(P^{(3)}\) and the CE expectation value \(I\), for different values of \(d_e\).

We verify that once we consider values for \(I\) such that classical behavior is not allowed, the result of the detection of a photon by Alice becomes random, and the higher the value of \(I\) the less predictable it is. Also, this relationship is stronger the smaller the value of \(d_e\), the probability of not detecting a single photon. The implication of this bound for the min-entropy is captured by the function \(f(I, d_e) = -\log_2(P(I, d_e))\), plotted in Figure 6.

The solutions to the SDP problems were obtained in Mathematica using the NCAlgebra package \[45\]. Our code is available on Github \[46\].

B. Security proof

In this section, we show that the protocol is secure given the assumptions in Subsection III 1 with a linear
key rate and in the presence of noise. The final result is the following theorem.

**Theorem 2.** Let $m$ be a large enough integer and let $\varepsilon := e^{-c_0 m}$, where $c_0 > 0$ is a small constant. Assume that any eavesdropper has a bounded quantum memory (see Subsection III 1). Then, the protocol described in Subsection III 2 generates a key of length $km$ that is $\varepsilon$-secure: the probability that the users Alice and Bob do not abort and the adversary can obtain information about the key is at most $\varepsilon$. Furthermore, $\kappa$ in principle does not scale with $m$, as it is a function of the probability of winning at the coherence equality game, how much the devices satisfy the one-photon condition, and the security parameter.

**Proof.** Follows directly from Theorems 3 and 11.

The theorem follows from the fact that if the protocol is successful, then Alice and Bob can perform privacy amplification and information reconciliation. We start with the former, and show that if the protocol does not abort, the min-entropy between the users’ and Eve’s systems is such that they can obtain a key of linear size in the number of detection rounds, which will be around a quarter of total rounds.

1. **Min-entropy bound and privacy amplification**

For certain observed values of $d_e$ and $I$, the coherence equality game (extended to more than one particle) played by Alice and Bob can certify that the server is (likely) sending a particle in a superposition to their labs. The fact that a particle sent in superposition cannot be perfectly localized in either lab (as it would be in the classical case) suggests the existence of an upper bound on the probability of guessing in which lab the particle is found, when detection is attempted. We confirm this intuition and obtain a quantitative relation between the relaxed CE game and Eve’s knowledge of which lab finds the particle.

The main result is the following theorem.

**Theorem 3.** **Privacy amplification.** Assume that the protocol does not abort and let $\mu := e^{-c_0 m} > 0$ be a security parameter, for $c_0 > 0$. Then, with probability at least $1 - \mu$, Alice and Bob can perform information reconciliation by sharing $\ell$ bits of information and perform privacy amplification to obtain

$$\kappa |D| - \ell + O(\log 1/\varepsilon)$$

secure bits of information, where $\kappa$ is a constant that only depends on the values observed for $I, d_e$ and the security parameter $\mu$.

**Proof.** Follows from Theorems 4 and 10 and the result on information reconciliation, Theorem 11.

We start by recalling the relation between the min-entropy and the amount of private information that Alice and Bob can extract by classical communication, expressed in the following theorem.

**Theorem 4.** Suppose that there is an information reconciliation protocol requiring at most $\ell$ bits of communication. Then, for any $\varepsilon > 0$, there is a privacy amplification protocol which extracts

$$H_{\min}(\alpha|D)|E'| - \ell + O(\log 1/\varepsilon)$$

bits of key.

To establish how the CE game provides a bound on the min-entropy $H_{\min}(\alpha|D)|E'|$, we start by considering a property of each round of the raw key generation, guaranteed by the SDP bound.

**Lemma 5.** Let $p^{(3)}(I, d_e)$ be the function bounding the guessing probability shown in Figure 5. Then, for any given round $i$ in the protocol, the following three conditions cannot hold at the same time:

![Figure 5. Upper bound on the guessing probability for Alice's (or equivalently, Bob’s) detection outcome for each interference term $I$.](image1)

![Figure 6. Plot of $f(I, d_e) = -\log_2(P(I, d_e))$, where $P$ is an upper bound on the guessing probability for outcome $\alpha$ in Alice’s lab.](image2)
1. The no-signalling condition is satisfied and our assumptions about Alice’s and Bob’s detectors are correct.

2. The set $S_i = (\rho, M_{A_i}^a, M_{B_i}^b)$ of the state and measurements wins at the CE game with probability $P_{win} \geq I/4$, and satisfies the one-photon condition with probability at least $1 - d_c$.

3. In the case that Alice decides to attempt detection $(x = 1)$, there is an outcome $\alpha' \in \{0, 1\}$ such that $p(\alpha = \alpha') > p(3)(I, d_c)$.

Proof. Bound on the detection probability obtained by semidefinite programming (see Section [IV.A]).

The following results are the steps needed to ensure that Alice and Bob have enough statistical information at the end of the protocol to apply the SDP bound, taking into consideration that the devices and the adversary do not necessarily act the same way in every round. In fact, we allow the behavior at round $i$ to be a function of all inputs and outputs up to round $i - 1$, represented by the variable $W_i := (X^{<i}, Y^{<i}, A^{<i}, B^{<i}, \alpha^{<i}, \beta^{<i})$.

Using the Azuma-Hoeffding inequality [18], we see that, for a large number of rounds, the observed behavior of the devices is close to their expected behavior, on average over all rounds.

**Lemma 6.** Let $\hat{I}$ be the observed value for the CE game. Then
\[
\Pr\left(\frac{1}{m} \sum_{i=1}^{m} I(W^i) \leq \hat{I} - \varepsilon \right) \leq \exp\left\{ - \frac{m \varepsilon^2}{32(1 + \omega_q^{CE})^2} \right\}.
\] (27)

Proof. Similar argument to [10] Section A.2]. Consider the random variable
\[
\hat{I}_i = 4 \times 1_{a_i \oplus b_i = x_i \oplus y_i}.
\] (28)

Its expectation conditioned on the past $W^i$ is equal to $E(\hat{I}_i|W^i) = I(W^i)$. The observed value for the game is $\hat{I} = \frac{1}{m} \sum_{i=1}^{m} \hat{I}_i$. Consider now the random variable $Z^k = \sum_{i=1}^{k} (\hat{I}_i - I(W^i))$. It is true that (i) $|Z^k| < \infty$, and that (ii) $E(Z^k|W^i, \ldots, W^j) = E(Z^k|W^j) = Z^j$, for $j \leq k$. Therefore the sequence $\{Z^k : k \geq 1\}$ is a martingale with respect to the sequence $\{Z^k : k \geq 2\}$.

The range of the martingale increments is bounded by $|I_i - I(W^i)| \leq 4(1 + \omega_q^{CE})$. Applying the Azuma-Hoeffding inequality (Theorem [17] in the Appendix) completes the proof.

Now, we wish to show that not only is the observed behavior valid on average over all rounds, but that it remains so when we look only at the rounds where Alice and Bob generate the key, i.e. the rounds in $D$. Since these rounds are chosen uniformly at random, we can apply a Chernoff bound and see that this is indeed true.

**Lemma 7.** Let $D$ be the set of detection rounds used for the raw key and $\hat{I}$ the estimated value for the CE game. Then we have that
\[
\Pr\left(\frac{1}{|D|} \sum_{i \in D} I(W^i) \geq (1 - \delta)(\hat{I} - \varepsilon) \right) \geq 1 - \exp\left( - m \varepsilon^2 \frac{\delta^2}{32(1 + \omega_q^{CE})^2} \right).
\] (29)

Proof. Consider the events, for $\delta > 0$,
\[
E_1 := \frac{1}{m} \sum_{i=1}^{m} I(W^i) > \hat{I} - \varepsilon,
\] (31)
\[
E_2 := \frac{1}{|D|} \sum_{i \in D} I(W^i) \geq (1 - \delta)(\hat{I} - \varepsilon).
\] (32)

We have that $P(E_2) \geq P(E_2 \land E_1) = P(E_2|E_1)P(E_1)$. From Lemma 6 it follows that $P(E_1) \geq 1 - \delta_1$. A bound for $P(E_2|E_1)$ is given by a Chernoff bound,
\[
\Pr\left(\frac{1}{|D|} \sum_{i \in D} I(W^i) \geq (1 - \delta)\left(\frac{1}{m} \sum_{i=1}^{m} I(W^i)\right) \right) \geq 1 - \exp\left\{ - |D| \frac{\delta^2}{2} \left(\frac{1}{m} \sum_{i=1}^{m} I(W^i)\right) \right\}.
\] (33)

Conditioning on $E_1$, we can write
\[
\frac{1}{|D|} \sum_{i \in D} I(W^i|E_1) \geq (1 - \delta)\frac{1}{m} \sum_{i=1}^{m} I(W^i|E_1) \geq (1 - \delta)(\hat{I} - \varepsilon)
\] (35)
and therefore
\[
\Pr\left(\frac{1}{|D|} \sum_{i \in D} I(W^i) \geq (1 - \delta)(\hat{I} - \varepsilon) \mid E_1 \right) \geq 1 - \exp\left\{ - |D| \frac{\delta^2}{2} \left(\frac{1}{m} \sum_{i=1}^{m} I(W^i|E_1)\right) \right\}.
\] (36)

which concludes the proof.

Since our upper bound is a function also of the single-particle probability, we must carry out a similar analysis over $d_c$. There is a small distinction at the end which is that we cannot use the information of all the rounds to estimate the single-particle probability, since that would simply release the entire raw key. Instead, we sacrifice a fraction $\gamma$ chosen randomly from the key, achieve similar conclusions about the rounds $B$ and then use a Chernoff bound in relation to the full detection set $D$. 


Lemma 8. Let \( \hat{d}_\varepsilon \) be the observed value for the one-photon condition, taking the detection values of rounds in \( B \). Then

\[
\Pr \left( \left| \frac{1}{|B|} \sum_{i \in B} d_\varepsilon(W^i) - \hat{d}_\varepsilon \right| \geq \varepsilon \right) \leq 2 \exp \left( - \frac{|B| \varepsilon^2}{8} \right).
\]

Proof. Theorem 17 in the Appendix (Azuma-Hoeffding inequality).

Lemma 9. Let \( B \) be the subset of detection rounds \( D \) used in estimating \( \hat{d}_\varepsilon \), such that \( |B| = \gamma |D| = \gamma m/4 \). Then, for \( \varepsilon, \delta > 0 \),

\[
\Pr \left( \frac{\hat{d}_\varepsilon + \varepsilon}{1 - \delta} > \frac{1}{|D|} \sum_{i \in D} d_\varepsilon(W^i) \right) \geq 1 - \exp \left( - \frac{\delta^2 |B|^2}{2} \right) + \exp \left( - \frac{\gamma m \varepsilon^2}{32} \right).
\]

Proof. Same argument as Lemma 7. Consider the events

\[
F_1 := \frac{1}{|B|} \sum_{i \in B} d_\varepsilon(W^i) - \hat{d}_\varepsilon < \varepsilon, \quad F_2 := \frac{\hat{d}_\varepsilon + \varepsilon}{1 - \delta} > \frac{1}{|D|} \sum_{i \in D} d_\varepsilon(W^i).
\]

The probability \( P(F_1) \geq 1 - \delta_1 \) is given by Lemma 8. Conditioning on \( F_1 \), the probability \( P(F_2|F_1) \) is given by a Chernoff bound:

\[
\Pr \left( \frac{1}{|B|} \sum_{i \in B} d_\varepsilon(W^i) > \frac{1}{|D|} \sum_{i \in D} d_\varepsilon(W^i) \right) \quad (45)
\]

\[
\Pr \left( \frac{\delta^2 |B|^2}{2} \sum_{i \in D} d_\varepsilon(W^i|F_1) \right) \quad (46)
\]

\[
\Pr \left( \frac{\delta^2 |B|^2}{2} (\hat{d}_\varepsilon - \varepsilon) \right) \quad (47)
\]

The last inequality is obtained by noting that

\[
\sum_{i \in D} d_\varepsilon(W^i|F_1) \geq \sum_{i \in B} d_\varepsilon(W^i|F_1) > (\hat{d}_\varepsilon - \varepsilon)|B|.
\]

Theorem 10. Bound on the min-entropy

Let \( \mu > 0 \) be a security parameter and assume that the protocol does not abort, and let \( \bar{d}, \hat{d}_\varepsilon \) be the observed values for the CE game and one-photon probability. Then there exists a function \( f \) and a choice of values \( \bar{I} < \bar{\varepsilon}, \hat{d}_\varepsilon < \bar{d}_\varepsilon \) such that, with probability at least \( 1 - \mu \), we have that

\[
H_{\min}(\alpha_D|ABXY) \geq f(\bar{I}, \hat{d}_\varepsilon)|D|. \quad (48)
\]

Proof. This proof follows along the lines of Section A.2 of [10]. Recalling the relation between min-entropy and guessing probability [17],

\[
H_{\min}(X|Y) = - \log_2 P_{\text{guess}}(X|Y). \quad (49)
\]

We are interested in the min-entropy \( H_{\min}(\alpha_D|ABXY) \) of the detection results \( \alpha \) of rounds in \( D \), given that Eve has access to the strings \( X, Y, A, B \) of inputs and outputs. Consider the strings \( \alpha^d = (\alpha_i)_{i \in [1..d]} \) where \( d \) runs through the indices in the set \( D \). Similarly, \( a^m, b^m, x^m, y^m \) where \( m \) runs through all rounds. We have that

\[
- \log_2 P(\alpha^d|a^m b^m x^m y^m) \quad (50)
\]

\[
= - \log_2 \prod_{i \in D} p(\alpha^i|a^{i-1}b^{i-1}x^{i-1}y^{i-1} \alpha^{i-1}) \quad (51)
\]

\[
= - \log_2 \prod_{i \in D} p(\alpha^i|W^i) \quad (52)
\]

\[
\geq f(I(W^i), d_\varepsilon(W^i)) \quad (55)
\]

This bound is true conditioned on any measurement of an eavesdropper before Alice and Bob share any information about their inputs and outputs, since any outcome of measurement in that case amounts to the preparation of a state to be used by Alice and Bob, and the bound is independent of the state being used. Therefore, in this step we assume that the adversary has a bounded quantum memory and cannot delay her measurements so that they are made after the parameter estimation step. With this caveat in mind, we can write

\[
- \log_2 P(\alpha^d|a^m b^m x^m y^m) \quad (56)
\]

\[
\geq \sum_{i \in D} f(I(W^i), d_\varepsilon(W^i)) \quad (57)
\]

\[
\geq |D| \sum_{i \in D} |I(W^i), \frac{1}{|D|} \sum_{i \in D} d_\varepsilon(W^i)) \quad (58)
\]

The last inequality is deduced using the convexity of \( f \) and Jensen’s inequality for two variables (Theorem 18 in the Appendix).

Let \( \mu > 0 \) be the a security parameter. Using Lemmas 7 and 9 by sharing a fraction \( \gamma > 0 \) of the results of detection rounds, for \( m \) large enough, we can establish values \( \varepsilon, \delta, \varepsilon', \delta' > 0 \) such that, with probability at least \( 1 - \mu \),

\[
\frac{1}{|D|} \sum_{i \in D} d_\varepsilon(W^i) < \hat{d}_\varepsilon + \varepsilon \quad (59)
\]

\[
\frac{1}{|D|} \sum_{i \in D} I(W^i) \geq (1 - \delta')(\bar{I} - \varepsilon') =: \bar{I}. \quad (60)
\]
Since \( f(I, d) \) is increasing with \( I \) and decreasing with \( d \), we have that
\[
H_{\min}(\alpha_d|ABXY) = -\log_2 P(\alpha^d|a^m b^m x^m y^m) \geq |D| f(\hat{I}, \hat{d}) \equiv |D| \kappa,
\]
which concludes the proof.

2. **Max-entropy bound and information reconciliation**

Either from eavesdropping or simply noise in the channels, Alice and Bob’s raw keys will not necessarily match, and so they must reconcile their keys without making them public. By sharing a small fraction at random, they are able to bound the max-entropy between them and with high probability obtain an identical key. Since this analysis is standard in QKD, we simply present the main result and provide a proof in the Appendix.

**Theorem 11. Information reconciliation.** Suppose Alice and Bob do not abort the protocol after Step 4. Then they can perform information reconciliation on their bit strings \( \alpha_D, \beta_D \) sacrificing at most the following amount of bits of information
\[
\ell_{\text{leak}} \leq H(1.1\eta)|D| + \log(2/\varepsilon).
\]

**Proof.** Follows from Theorems 15 and 16 in the Appendix.

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V. **COUNTERFACTUAL NONLOCALITY**

In this section, we show that the setup variant of the CE game considered in the SDI-QKD protocol can be used to counterfactually test nonlocality.

The notion of **source of quantum advantage** can only be appropriately invoked when talking about a particular type of quantum information processing task. This means that one has to know exactly what task is being studied and within which setting, so that we can fully characterize the set of the allowed classical strategies, in order to try and break the bounds permitted by such strategies when exploiting explicitly non-classical quantum behavior. Then, some specific quantum behavior should only be characterized as a source of quantum advantage if it manifests itself within a particular task, allowing the violation of a classical bound for said task.

Take, for instance, the state in Equation 3 which is an entangled pure state in Fock space. As such, it will also necessarily be able to produce nonlocal correlations. That said, **nonlocality** cannot be assumed to be the source powering the quantum advantage of every task where this type of state plays a role, solely by merit of the state being nonlocal. In the CE game, for example, where the state is used, one does not test such nonlocal correlations explicitly even if they are tacitly assumed to be present. The quantum advantage exhibited in the CE game is provided by **non-null interference terms**, which themselves need not be explicitly nonlocal. Thus, **nonlocality** is not the resource powering the advantage in this scenario — at least, not in an explicit manner. In this way, the CE game as originally introduced should not be considered a test of nonlocality, since it does not generally disprove the existence of **local hidden variable** models describing the joint correlations of the measurement outcomes, which themselves might be subject to a factorization under a common mediating hidden variable.

The **source of quantum advantage** is then task dependent, in the sense that, although one might use the same physical state in different types of tests, depending on the tests performed, different aspects of the resource state might come into play. Having said this, we argue that the alternative variant of the CE game used in the QKD protocol (where the single-particle condition can be tested with high enough probability) can be interpreted as a **counterfactual test of nonlocality**.

Counterfactually testing nonlocality: The setup is essentially the same as the one in the QKD protocol (see Figure 3). Alice and Bob self-generate their input bits \( x \) and \( y \), respectively, which are subsequently encoded into the resource state, to be forwarded to the servers which output bits \( a \) and \( b \). Since we are not necessarily interested in minimizing the quantum requirements of Alice and Bob, unlike the QKD protocol, we can assume the servers \( S_A \) and \( S_B \) to be part of Alice’s and Bob’s labs; if one decides to separate the servers from the labs, the setup is exactly the same as in the QKD protocol. The test will go as follows,

1. For each out of \( m \) rounds, Alice and Bob receive state \( \rho \) and randomly choose to either attempt detection or do nothing, according to their uniformly random inputs \( x, y \), subsequently forwarding the resource to their respective measurement servers;

2. Alice and Bob register their personal outputs given by the pairs \( (\alpha, a) \) and \( (\beta, b) \), respectively. The first element of the pair is the bit which corresponds to either observing or not observing the photon, and the second one is the output bit given by the measurement servers.

3. After all rounds are played, Alice and Bob announce their results. From their input choices \( x, y \) and from the outputs \( (\alpha, a) \) and \( (\beta, b) \), Alice and Bob calculate the fraction of rounds winning the CE game,
\[
\hat{P}_{\text{CE}} \equiv \frac{1}{m} \sum_{m} 1_{a \oplus b = x \oplus y},
\]
and the fraction of rounds out of those where \( x = y = 1 \) (rounds \( D \)) that do not satisfy the one photon
condition,
\[ \hat{d}_c \equiv \frac{1}{|D|} \sum_d 1_{\alpha=\beta}. \] (67)

Then from Lemmas 6 and 8 we can conclude, up to negligible error, that if \( \hat{P}_{CE} > \omega_{CE}^c(\hat{d}_c) \) the resources used were entangled in Fock space, as they explicitly exist in the non-classical region of Figure 4. Moreover, in the particular case where the single-particle condition is met up to negligible error we can state the following result.

**Theorem 12.** For \( \hat{d}_c < \eta \), with \( \eta \) arbitrarily close to zero, breaking the classical bound of the CE game (\( \hat{P}_{CE} > 1/2 \)), counterfactually implies that the CHSH Bell-bound could have been broken (\( \max\{P_Q(CHSH)\} > 3/4 \)) with the same resources that were used in the CE game.

**Proof.** Considering that the single-particle condition was confirmed up to negligible error (Lemma 8), then the resource states were adequately represented by Equation 3 and the quantum-over-classical advantage is given explicitly as a function of the entanglement concurrence as \( \Omega_{CE} = C/8 \) (Equation 7). Furthermore, we also know that the quantum-over-classical advantage of the CHSH game is given by \( \Omega_{CHSH} = (\sqrt{1+c^2} - 1)/4 \) (Equation 9). Now one can reason, in a counterfactual manner, that if the states used to play the CE game would have had their correlations transferred to some appropriate ancillary qubit states in Alice’s and Bob’s lab, and CHSH games would have been played with such states instead of the CE game, then the same amount of entangled correlations used to play the CE game would have been present in such ancillary qubits. In fact, by fixing the concurrence of the state we can straightforwardly derive the following expression,

\[ \Omega_{CHSH} = \left( \sqrt{1 + 64 \Omega_{CE}^2} - 1 \right)/4, \] (68)

which is the quantum advantage of the CHSH game given in terms of the quantum advantage of the CE game. It is evident that \( \Omega_{CE} > 0 \) implies \( \Omega_{CHSH} > 0 \), and so it comes directly that when \( \hat{P}_{CE} > 1/2 \), that if the CHSH would have been played with the same resources used in the CE game then \( \max\{P_Q(CHSH)\} > 3/4 \).

Notice that there is no contradiction with our previous stance, since nonlocality is not the explicit source of quantum advantage in the CE game. Nevertheless it does not mean that such information — that the state was nonlocal — could not be available (counterfactually) at the end of the experiment.

**VI. CONCLUSIONS**

We have proposed a QKD protocol with a classical Alice and Bob based on the Coherence Equality (CE) game. The protocol is also semi-device-independent in the sense that we only model the detectors of Alice and Bob, and leave unspecified the nature of the servers and source, which amounts to considering trusted quantum inputs [22]. The protocol makes use of a game which self-tests entanglement in Fock space which, unlike the traditional protocols using the CHSH game, does not explicitly depend on nonlocal correlations, thus enabling one to reduce the quantum requirements of the agents and answering affirmatively a standing open question in the field of semi-quantum cryptography, of whether would it be possible to have semi-device-independent QKD which was also semi-quantum. This provides a first step towards better understanding the interplay of (semi-)device-independent and of semi-quantum cryptography. The protocol security is based on upper bounds restricting the users’ performance in the CE game. Intuitively, the better Alice and Bob play the CE game, the more coherent their resource is, and the more private their measurement outcomes are. Through a series of upper bounds on the maximum guessing probability for the detection outcomes, formalized as SDP problems, we obtain lower bounds on the min-entropy of those results against any state prepared by an eavesdropper (see Subsection III 1) [36]. Thus, we prove security of a linear key rate in the context of noise against an adversary with bounded quantum memory. Furthermore, we also expand on the work of del Santo and Dakić by relaxing the single-particle condition and casting the CE game in the context of self-testing and verification of entanglement in Fock space. Finally, we provide a setup where a variant of the CE game can be reasoned as a counterfactual test of nonlocality.

Our work also provides a practical advantage when compared to similar protocols, namely QKD with classical users implemented in [23]. The major difference being that in our protocol the users do not communicate back to the source but instead send their respective states to spatially separated servers which then perform the measurements. In practical terms, this reduces the expected distance the photon must travel by half, which reduces the impact of noise and in turn can increase the key rate.

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Lemma 6.3.3 in [34].

Proof. 

Regarding the semidefinite programming section of the paper, we provide a definition of a semidefinite operator.

**Definition 13. Positive semidefinite operator**

An operator $V : H_n \rightarrow H_n$ is said to be positive semidefinite, $V \geq 0$, if

$$\langle \varphi | V | \varphi \rangle \geq 0, \quad \forall | \varphi \rangle \in H_n \text{ s.t. } | \varphi \rangle \neq 0.$$  

(69)

**Security proof**

The following results complement the security proof in the main text by justifying the bound on the max-entropy following Alice’s and Bob’s sharing of a random part of their raw keys. The error rate observed in the shared bits in $B$ is used to estimate the error between the remaining portion of the key, using a Chernoff bound.

**Information reconciliation**

**Remark 14.** Let $P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ and $Q_{Y} \in \mathcal{P}(\mathcal{Y})$. Then

$$H_{\text{max}}(P_{XY} | Q_{Y}) = \log \sum_{y \in \mathcal{Y}} Q_{Y}(y) \cdot | \sup(P_{X}^{y})|,$$

(70)

where $P_{X}^{y}$ denotes the function $P_{X}^{y} : x \mapsto P_{XY}(x, y)$. In particular,

$$H_{\text{max}}(P_{XY} | Q_{Y}) = \log \max_{y \in \mathcal{Y}} | \sup(P_{X}^{y})|. $$

(71)

Proof. Remark 3.1.4 in [34].

**Theorem 16.** Suppose Alice and Bob do not abort the protocol after Step 4. Let $D$ be the set of detection rounds. Then, with probability at least $1 - 2e^{-\gamma |D|/250}$,

$$H_{\text{max}}(\beta_{D} | \alpha_{D}) \leq H(1.1\eta |D|).$$

(73)

Proof. Let $X = \gamma |D|\eta_{B}$ be the observed error rate, and $\nu = \gamma |D|\eta_{D}$ the expected value. Since the protocol aborts if $\eta_{B} > \eta$, we have that

$$\Pr(1.1\eta < \eta_{D} \land \neg \text{ aborts}) = \Pr(1.1\eta < \eta_{D} \land \eta_{B} \leq \eta) \leq \Pr(1.1\eta_{B} < \eta_{D} \land 1.1\eta < \eta_{D}).$$

(74)

(75)

(76)

Using a Chernoff bound, we have that

$$\Pr \left( X \leq \frac{1}{1.1} \nu \land 1.1\eta < \eta_{D} \right)$$

(77)

$$\leq \exp \left\{ -\frac{1}{242} \gamma |D| \eta \right\} \leq \exp \left\{ -\frac{1}{250} \gamma |D| \eta \right\}.$$ 

(78)

(79)

Therefore, with probability at least $1 - \exp \left\{ -\frac{1}{250} \gamma |D| \eta \right\}$, the noise rate of $D$ is at most $1.1\eta$. This implies that, for a fixed $\alpha_{D}$, there are at most $2^{\gamma |D|/1.1\eta}$ possible values for $\beta_{D}$. Using Remark 14, we have that $H_{\text{max}}(\beta_{D} | \alpha_{D}) \leq H(1.1\eta |D|).$

**Auxiliary results**

**Theorem 17. Azuma-Hoeffding inequality**

Let $\{X_{i} : i \geq 1\}$ be a martingale and suppose there exists a sequence $\{c_{i} : i \geq 1\}$ such that $|X_{k} - X_{k-1}| \leq c_{k}$, for any $1 \leq k \leq N$. Then, for any $N$,

$$\Pr(X_{N} - X_{1} \geq x) \leq \delta,$$

(80)

$$\Pr(X_{N} - X_{1} \leq -x) \leq \delta,$$

(81)

$$\Pr(|X_{N} - X_{1}| \geq x) \leq 2\delta,$$

(82)

where

$$\delta = \exp \left\{ -\frac{1}{2} x^{2} / \sum_{i=1}^{n} c_{i}^{2} \right\}.$$ 

(83)

**Theorem 18. Jensen’s inequality for two variables**

Let $g(x, y)$ be a convex differentiable function over $x$ and $y$, and $X, Y$ random variables. Then

$$E[g(X, Y)] \geq g(E[X], E[Y]).$$

(84)

Proof. If $g$ is convex and differentiable, then a gradient at any point is also a subgradient. Considering the point $(E[X], E[Y])$, we can write

$$g(X, Y) \geq a + b^{T}(X, Y)$$

(85)
and
\[ g(E[X], E[Y]) = a + b^T(E[X], E[Y])^T, \]  

(86)

for some \( a \in \mathbb{R} \) and \( b \in \mathbb{R}^2 \). Using expectation over equa-

tion and applying linearity of expectation, we obtain
\[ E[g(X, Y)] \geq a + b^T(E[X], E[Y])^T = g(E[X], E[Y]). \]

(87)

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