ON RECTANGULAR DIAGRAMS, LEGENDRIAN KNOTS AND TRANSVERSE KNOTS

HIROSHI MATSUDA AND WILLIAM W. MENASCO

1. INTRODUCTION

A correspondence is studied in [7] between front projections of Legendrian links in \((\mathbb{R}^3, \xi_{\text{std}})\) and rectangular diagrams. In this paper, we introduce braided rectangular diagrams, and study a relationship with Legendrian links in \((\mathbb{R}^3, \xi_{\text{sym}})\). We show Alexander and Markov Theorems for Legendrian links in \((\mathbb{R}^3, \xi_{\text{sym}})\).

We review a relationship between front projections of Legendrian links in \((\mathbb{R}^3, \xi_{\text{std}})\) and rectangular diagrams. The standard contact structure \(\xi_{\text{std}}\) in \(\mathbb{R}^3\) is defined by \(\ker(dz - ydx)\). It is well-known [12] that every Legendrian link in \((\mathbb{R}^3, \xi_{\text{std}})\) has a front projection with transverse double points and cusp singularities. Figure 1 (1) illustrates a front projection of a topologically trivial Legendrian knot. Changing every point with a horizontal tangent in a front projection to a corner, Figure 1 (2), followed by rotating the obtained diagram 45 degree clockwise, we obtain a rectangular diagram, Figure 1 (4).

The intersection of the plane \(\{(x, y, z) \mid y = a\}\) with contact planes, called a characteristic foliation, consists of lines \(z = ax + (\text{constant})\), where \(a \in \mathbb{R}\). The characteristic foliation on the plane \(\{(x, y, z) \mid x = b\}\) consists of lines \(z = (\text{constant})\), where \(b \in \mathbb{R}\). Changing every vertical

![Figure 1](image-url)
arc in a rectangular diagram to a subarc of the line \( \{ (x, -1, z) \mid z = -x + v_i \} \) for some \( v_i \in \mathbb{R} \), every horizontal arc to a subarc of the line \( \{ (x, 1, z) \mid z = x + h_j \} \) for some \( h_j \in \mathbb{R} \), and every corner to a subarc of the line \( \{ (x_i, y, z_j) \mid y \in \mathbb{R} \} \) for some \( x_i \in \mathbb{R} \) and \( z_j \in \mathbb{R} \), we obtain a Legendrian link in \((\mathbb{R}^3, \xi_{\text{std}})\) from a rectangular diagram. We notice that the Legendrian link constructed as above from a rectangular diagram is Legendrian isotopic to the Legendrian link corresponding to the front projection. For example, the diagram in Figure 1 (4) corresponds to a Legendrian knot, illustrated in Figure 1 (3), consisting of the following eight arcs:

\[
\begin{align*}
\{(x, -1, z) \mid x + z = 1, -\varepsilon \leq x \leq 1 - \varepsilon, \varepsilon \leq z \leq 1 + \varepsilon\},
\{(x, 1, z) \mid x - z = -1, -1 - \varepsilon \leq x \leq -\varepsilon, \varepsilon \leq z \leq 1 + \varepsilon\},
\{(x, -1, z) \mid x + z = -1, 1 - \varepsilon \leq x \leq -\varepsilon, -1 + \varepsilon \leq z \leq \varepsilon\},
\{(x, 1, z) \mid x - z = 1, 1 - \varepsilon \leq x \leq -\varepsilon, -1 + \varepsilon \leq z \leq \varepsilon\},
\{(1 - \varepsilon, y, \varepsilon) \mid -1 \leq y \leq 1\},
\{-\varepsilon, y, -1 + \varepsilon\} \mid -1 \leq y \leq 1\},
\end{align*}
\]

where \( \varepsilon \) is a small positive number. The union of these arcs is piecewise Legendrian. We obtain a Legendrian knot by Legendrian-smoothing our edgepath in arbitrarily small neighborhoods around the endpoints of each arc.

In §2, we study a similar correspondence between braided rectangular diagrams and Legendrian links in \((\mathbb{R}^3, \xi_{\text{sym}})\). This leads us to Alexander and Markov Theorems for Legendrian links in \((\mathbb{R}^3, \xi_{\text{sym}})\). This answers Problem 2 in [10], implicitly stated also in [4]. Alexander and Markov Theorems for transverse links in \((\mathbb{R}^3, \xi_{\text{sym}})\) was proved in [1], [10] and [13].

**Theorem 1.1.** [1] (Alexander Theorem for transverse links in \((\mathbb{R}^3, \xi_{\text{sym}})\))

*Any transverse link in \((\mathbb{R}^3, \xi_{\text{sym}})\) is transversely isotopic to a closed braid.*

**Theorem 1.2.** [10], [13] (Markov Theorem for transverse links in \((\mathbb{R}^3, \xi_{\text{sym}})\))

*Two closed braids represent the same transverse link if and only if they are related by positive stabilizations and conjugation in the braid group.*

In §3, we describe a construction of an explicit example of the Etnyre-Honda pair of Legendrian knots [5], which is announced in [9].

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2. ALEXANDER AND MARKOV THEOREMS FOR LEGENDRIAN LINKS IN \((\mathbb{R}^3, \xi_{\text{sym}})\)

Let \(\{H_{\theta} \mid 0 \leq \theta < 2\pi\}\) denote an open book decomposition of \(\mathbb{R}^3\), that is, \(\mathbb{R}^3 \setminus \{\text{z-axis}\}\) is fibered by a collection of half-plane fibers \(H_{\theta}\), where the boundary of \(H_{\theta}\) is the \(\text{z-axis}\). When we use the cylindrical coordinates \((r, \theta, z)\) on \(\mathbb{R}^3\), a fiber \(H_{\theta_0}\) is the set \(\{(r, \theta, z) \mid \theta = \theta_0\}\). An oriented link \(X\) in \(\mathbb{R}^3\) is a closed \(n\)-braid if \(X \subset \mathbb{R}^3 \setminus \{\text{z-axis}\}\) intersects each fiber \(H_{\theta}\) transversely and positively in \(n\) points. Possibly after a small isotopy of \(X\) in \(\mathbb{R}^3 \setminus \{\text{z-axis}\}\), we can consider a regular projection \(\pi: X \to C_1\) given by \((r, \theta, z) \mapsto (1, \theta, z)\), where \(C_1 = \{(r, \theta, z) \mid r = 1\}\). We may assume that the singularities consist of \(\pi(X)\) consists of finitely many transverse double points.

A horizontal arc, \(h \subset C_1\), is an arc with a parameterization \(\{(1, \theta(t), z_0) \mid 0 \leq t \leq 1, \theta(t) \in [\theta_1, \theta_2]\}\), where \(|\theta_1 - \theta_2| < 2\pi\). The horizontal level of \(h\) is a fixed constant \(z_0\), and the angular support of \(h\) is the interval \([\theta_1, \theta_2]\). Horizontal arcs inherit a natural orientation from the forward direction of the \(\theta\) coordinate. A vertical arc, \(v \subset H_{\theta_0}\), is an arc with a parameterization \(\{(r(t), \theta_0, z(t)) \mid 0 \leq t \leq 1, r(0) = r(1) = 1; \text{ and } r(t) > 1, \frac{dz(t)}{dt} \neq 0 \text{ for } t \in (0, 1)\}\), where \(r(t)\) and \(z(t)\) are \(\mathbb{R}\)-valued functions that are continuous on \([0, 1]\) and differential on \((0, 1)\). The angular level of \(v\) is \(\theta_0\), the vertical support of \(v\) is the interval \([z(0), z(1)]\) or \([z(1), z(0)]\).

An oriented link \(X \subset \mathbb{R}^3\) is an arc presentation if \(X = h_1 \cup v_1 \cup \cdots \cup h_n \cup v_n\) satisfies the following conditions:

1. each \(h_i\) (\(i = 1, \cdots, n\)), is an oriented horizontal arc with its inherited orientation agreeing with the orientation of \(X\),
2. each \(v_i\) (\(i = 1, \cdots, n\)), is a vertical arc,
3. the intersection \(h_i \cap v_j\) consists of a point of \(\partial h_i \cap \partial v_j\) for \(i = 1, \cdots, n\) and \(j \equiv \{i - 1, i\}\) (mod \(n\)), and \(h_i \cap v_j = \emptyset\) if \(j \not\equiv \{i - 1, i\}\) (mod \(n\)),
4. the horizontal level of each horizontal arc is distinct, and the angular level of each vertical
arc is distinct,
(5) the orientations of vertical arcs are assigned so as to make the components of $X$ oriented.

A projection of an arc presentation of $X$ onto $C_1$ with over/under informations is called a braided rectangular diagram. A projection $\pi(X)$ without the conditions about orientations in (1) and (5) is called a rectangular diagram. It is proved in [3] and [8] that every link in $\mathbb{R}^3$ has a braided rectangular diagram.

By a contactmorphism $(\mathbb{R}^3, \xi_{\text{std}})$ is equivalent to the symmetric contact structure $(\mathbb{R}^3, \xi_{\text{sym}})$ where $\xi_{\text{sym}} = \ker \alpha_{\text{sym}}$ for $\alpha_{\text{sym}} = dz + xdy - ydx$ (in Euclidean coordinates) or $\alpha_{\text{sym}} = dz + r^2d\theta$ (in cylindrical coordinates). The symmetric contact structure was the structure utilized by D. Bennequin [1] in his classical argument the any transversal knot is transversally isotopic to a braid—the $z$-axis being the designated braid axis.

A rectangular diagram on $C_1$ corresponds to a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$ as follows. First we notice that the characteristic foliation on the cylinder $\{(r, \theta, z) \mid r = r_0\}$ consists of “spiral” curves of slope $\frac{dz}{d\theta} = -r_0^2$, and that the characteristic foliation on the plane $\{(r, \theta, z) \mid \theta = \theta_0\}$ consists of lines $\{(r, \theta_0, z_0) \mid r > 0\}$ for $z_0 \in \mathbb{R}$. Change every horizontal arc in a rectangular diagram to a subarc, a near-horizontal arc, of the characteristic foliation on the cylinder $\{(r, \theta, z) \mid r = r_1\}$ with $r_1$ sufficiently small, and every vertical arc to a subarc, a near-vertical arc, of the characteristic foliation on the cylinder $\{(r, \theta, z) \mid r = r_2\}$ with $r_2$ sufficiently large. Change every corner in a rectangular diagram to a subarc of the line $\{(r, \theta, z) \mid \theta = \theta_0, r > 0\}$ which connects one endpoint of a near-horizontal arc on the cylinder $\{(r, \theta, z) \mid r = r_1\}$ and one endpoint of a near-vertical arc on the cylinder $\{(r, \theta, z) \mid r = r_2\}$. Adjusting the $r$-coordinates of near-horizontal and near-vertical arcs properly and Legendrian smoothing in arbitrarily small neighborhoods of the arc endpoints, we obtain a Legendrian link $L$ in $(\mathbb{R}^3, \xi_{\text{sym}})$ from a braided rectangular diagram. The vertical (resp. angular) support of a near-horizontal (resp. near-vertical) arc of $L$ is the interval in the $z$-coordinate (resp. $\theta$-coordinate) containing the arc. We may Legendrian isotope $L$ so that:

(1) the vertical support of a near-horizontal arc of $L$ is sufficiently small, and is disjoint from each other,
(2) the angular support of a near-vertical arc of $L$ is sufficiently small, and is disjoint from each other.

A Legendrian link $L$ has a horizontal/vertical disjoint property if the above conditions are satisfied.

Next we construct a rectangular diagram from a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$.

**Lemma 2.1.** Let $L$ be a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$. Then $L$ may be Legendrian isotoped to $L'$ so that $L'$ is in in the half-space with $y > 0$. In particular, $L' \cap H_\pi = \emptyset$.

**Proof.** Instead of Legendrian isotoping $L$, we describe a contactomorphism of $(\mathbb{R}^3, \xi_{\text{sym}})$ that takes $L$ to $L'$. Let $f$ be a diffeomorphism of $\mathbb{R}^3$ defined by $(x, y, z) \mapsto (x+K, y+K, z+K(x-y))$, where $K \in \mathbb{R}$. Starting with the Euclidean version of $\alpha_{\text{sym}}$, we then obtain $(\mathbb{R}^3, \xi'_{\text{sym}})$, where $\xi'_{\text{sym}}$ is defined by the kernel of the 1-form $\alpha'_{\text{sym}} = dz + K(x-y) + (x+K)d(y+K) - (y+K)d(x+K) = dz + xdy - ydx$. By the compactness of $L$, for a large enough $K$ the image of $L$ by $f$, $L'$, will be in the half-space $y > 0$. In particular, $L' \cap H_\pi = \emptyset$. □

Let $L$ be a Legendrian link in $(\mathbb{R}^3, \xi_{\text{sym}})$ that is in the half-space having $y > 0$. Then it does not intersect the $z$-axis and the image of $L$ onto $C_1$ by $\pi: (r, \theta, z) \mapsto (1, \theta, z)$, $\pi(L)$ is well-defined. We called $\pi(L)$ the cylinder projection of $L$. We notice that when $L \cap H_\pi = \emptyset$ the front projections of Legendrian links in $(\mathbb{R}^3, \xi_{\text{std}})$ and cylinder projections of Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$ are combinatorial equivalent. We can then adapt the proof of the Reidemeister Theorem for Legendrian links in terms of front projections to the setting of cylinder projections to show the following. See the proof of Theorem B in [12].

**Proposition 2.2.** Let $L_1$ and $L_2$ be Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$ such that each of $L_1$ and $L_2$ are in the half-space having $y > 0$. Let $D_1$ and $D_2$ be the cylinder projections of $L_1$ and $L_2$, respectively. Legendrian links $L_1$ and $L_2$ are Legendrian isotopic in $\mathbb{R}^3 \setminus \{z \text{--axis}\}$ if and only if $D_1$ and $D_2$ are related by regular homotopy and a finite sequence of moves that are obtained from the diagrams in Figure 2 by rotating $\Theta$ degree counterclockwise with $0 < \Theta < 90$.

As we obtain a rectangular diagram from a front projection, we obtain a rectangular diagram on $C_1$ from a cylinder projection by changing every point with $\frac{dz}{d\theta} = -1$ to a corner.
Next we define an operation, called a flip. Let $L = \cup_{i=1}^n (h_i \cup c(h_i) \cup v_i \cup c(v_i))$ be a Legendrian link that corresponds to a rectangular diagram, where $h_i$ (resp. $v_i$) is a near-horizontal (resp. near-vertical) arc, and $c(h_i)$ connects $h_i$ and $v_i$, and $c(v_i)$ connects $v_i$ and $h_{i+1}$. We may assume that a Legendrian link $L$ has a horizontal/vertical disjoint property. Moreover, by a Legendrian isotopy that corresponds to scaling the angle between $v_{i-1}$ and $v_i$ we can assume that the angle between the angular support of $h_i$ is $\pi$. We may Legendrian isotope $L$ so that the $r$-coordinate of $h_i$ goes to 0, and that the $\theta$-coordinates of $c(v_{i-1})$ and $c(h_i)$ remain fixed. Then we have to adjust the lengths of $c(v_{i-1})$ and $c(h_i)$, and the $r$-coordinates of $v_{i-1}$ and $v_i$, and the lengths of $c(h_{i-1})$ and $c(v_i)$. After these Legendrian isotopies of $L$, we may assume that $L$ has a horizontal/vertical disjoint property. We may Legendrian isotope $L$ so that $h_i$ shrinks to a point on the $z$-axis. Therefore the $z$-coordinate of $c(v_{i-1})$ and $c(h_i)$ are the same. We may
further Legendrian isotope \( L \) to \( L' \) so that \( h_i \) passes through the \( z \)-axis to \( h'_i \), that (the angular support of \( h_i \) \( \cap \) (the angular support of \( h'_i \)) = (the \( \theta \)-coordinates of \( c(v_{i-1}) \) and \( c(h_i) \)), where \( h'_i \) is a subarc of \( L' \). By our assumption that the angular support of \( h_i \) is \( \pi \) this isotopy will be Legendrian as \( L \) passes through the \( z \)-axis. This isotopy from \( L \) to \( L' \) is called a flip, and the above argument shows that a flip is a Legendrian isotopy from \( L \) to \( L' \).

**Theorem 2.3.** (Alexander Theorem for Legendrian links in \((\mathbb{R}^3, \xi_{\text{sym}})\))

Every Legendrian link in \((\mathbb{R}^3, \xi_{\text{sym}})\) is Legendrian isotopic to a Legendrian link constructed from a braided rectangular diagram.

**Proof.** Let \( D \) be a rectangular diagram of a Legendrian link \( L \) with a horizontal/vertical disjoint property. Let \( h \) be a near-horizontal arc of \( L \) such that the induced orientation of \( h \) from that of \( L \) disagrees with the forward direction of the \( \theta \)-coordinate. We apply a flip to every such near-horizontal arc \( h \). Then \( L \) corresponds to a braided rectangular diagram. \( \square \)

When a Legendrian link \( L \) intersects the \( z \)-axis during Legendrian isotopy of \( L \) in \((\mathbb{R}^3, \xi_{\text{sym}})\), a neighborhood of the intersection of \( L \) with the \( z \)-axis is horizontal. Therefore we may assume that their rectangular diagrams are related by one flip, so we have the following.

**Theorem 2.4.** (Reidemeister Theorem for Legendrian links in terms of rectangular diagrams)

Let \( L_1 \) and \( L_2 \) be Legendrian links in \((\mathbb{R}^3, \xi_{\text{sym}})\), and let \( D_1 \) and \( D_2 \) be the rectangular diagrams of \( L_1 \) and \( L_2 \), respectively. Legendrian links \( L_1 \) and \( L_2 \) are Legendrian isotopic in \((\mathbb{R}^3, \xi_{\text{sym}})\) if and only if \( D_1 \) and \( D_2 \) are related by a finite sequence of flips and moves in Figure 2 on \( C_1 \).

Next we see how positive and negative transverse push-offs are obtained from a Legendrian link corresponding to a braided rectangular diagram. Let \( L = \bigcup_{i=1}^n (h_i \cup c(h_i) \cup v_i \cup c(v_i)) \) be a Legendrian link in \((\mathbb{R}^3, \xi_{\text{sym}})\), where \( h_i \) (resp. \( v_i \)) denotes a near-horizontal (resp. near-vertical) arc. Let \( \alpha \) be a subarc of \( L \) that is contained in a cylinder \( C_{r_1} = \{(r, \theta, z) \mid r = r_1\} \), so \( \alpha \) is \( h_i \) or \( v_i \). Let \([\theta_1, \theta_2]\) be the angular support of \( \alpha \). Let \( z(\theta) \) be the \( z \)-coordinate at \( \theta \) of the line in the characteristic foliation on \( C_{r_1} \) containing \( \alpha \), defined in the \( \theta \)-interval \([\theta_1 - \varepsilon, \theta_2 + \varepsilon]\), where \( \varepsilon \) is a small positive number. Let \( \delta(\alpha) \) be a neighborhood of \( \alpha \) on \( C_{r_1} \) described as \( \delta(\alpha) = \{(r, \theta, z) \mid \theta_1 - \varepsilon \leq \theta \leq \theta_2 + \varepsilon, z(\theta) - \varepsilon \leq z \leq z(\theta) + \varepsilon\} \). Let \( \Delta(\alpha) \) be a neighborhood
Then we have $tb(L) = \omega(D) - \frac{1}{2}(d(D) + u(D))$ and $r(L) = n(D) + \frac{1}{2}(d(D) - u(D))$, where $u(D)$ (resp. $d(D)$) denotes the number of up (resp. down) corners of $D$, illustrated in Figure 4, and $n(D)$ is the algebraic winding number of $L$ around the $z$-axis, and $\omega(D)$ is the algebraic crossing number of $D$ on $C_1$. 

**Lemma 2.5.** Let $L$ be a Legendrian link in $(\mathbb{R}^3, \xi_{sym})$ represented as a rectangular diagram $D$. Then we have $tb(L) = \omega(D) - \frac{1}{2}(d(D) + u(D))$ and $r(L) = n(D) + \frac{1}{2}(d(D) - u(D))$, where $u(D)$ (resp. $d(D)$) denotes the number of up (resp. down) corners of $D$, illustrated in Figure 4, and $n(D)$ is the algebraic winding number of $L$ around the $z$-axis, and $\omega(D)$ is the algebraic crossing number of $D$ on $C_1$. 

**Figure 4.**
Figure 5.

Proof. Let $v = \frac{\partial}{\partial z}$ be a vector field on $\mathbb{R}^3$. For any Legendrian knot $L$, $v$ is a vector field transverse to $\xi_{\text{sym}}$ along $L$. So we have $tb(L) = \omega(D) - \frac{1}{2}(d(D) + u(D))$.

Let $w = \frac{\partial}{\partial r}$ be a vector field on $\{x \geq \varepsilon\}$, $w = -\frac{\partial}{\partial r}$ a vector field on $\{x \leq -\varepsilon\}$. We define a vector field $w$ on $\{-\varepsilon < x < \varepsilon\}$ by interpolating between these two choices by rotating clockwise in the contact planes. We may Legendrian isotope $L$ so that all the vertical arcs of $D$ are contained in $\{x \geq \varepsilon\}$. Then we have $r(L) = \frac{1}{2}(2n(D) + d(D) - u(D)) = n(D) + \frac{1}{2}(d(D) - u(D))$. □

Similar arguments as above prove the following.

Lemma 2.6. Let $T_+(L)$ be a positive transverse push-off of a Legendrian knot $L$ in $(\mathbb{R}^3, \xi_{\text{sym}})$ corresponding to a braided rectangular diagram $D$. Then we have $sl(T_+(L)) = \omega(D) - n(D)$.

Proposition 4 in [3] and Theorem 2.4 prove the following.

Theorem 2.7. (Markov Theorem for Legendrian links in $(\mathbb{R}^3, \xi_{\text{sym}})$)
Let $D_1$ and $D_2$ be braided rectangular diagrams on $C_1$, and let $L_1$ and $L_2$ be Legendrian links corresponding to $D_1$ and $D_2$, respectively. Two Legendrian links $L_1$ and $L_2$ are Legendrian
isotopic in \((\mathbb{R}^3, \xi_{\text{sym}})\) if and only if \(D_1\) is obtained from \(D_2\) by a finite sequence of moves illustrated in Figure 7.

**Remark 2.8.** Let \(p: \mathbb{R}^3 \rightarrow \mathbb{R}^2\) be a projection onto the \(xy\)-plane, and \(L\) be a Legendrian link in \((\mathbb{R}^3, \xi_{\text{sym}})\). We denote \(p(L)\) with over/under information by \(p'(L)\). Let \(D_1 \rightarrow D_2\) be a move in Figure 6. We note that \(p'(L_1) \rightarrow p'(L_2)\) is a negative Reidemeister move of type I in knot theory, where \(L_i\) \((i = 1, 2)\) is a Legendrian link corresponding to a rectangular diagram \(D_i\). See Figure 6(1). This may be seen as a negative “local-(de)stabilization”. We note also that \(T_+(L_1) \rightarrow T_+(L_2)\) is a transverse isotopy. See Theorem 1.2.

**Remark 2.9.** Let \(D_1 \rightarrow D_2\) be a move in Figure 7 and \(L_i\) \((i = 1, 2)\) be a Legendrian link corresponding to a rectangular diagram \(D_i\). Then \(T_+(L_1) \rightarrow T_+(L_2)\) is a negative (de)stabilization as closed braids, so Theorem 1.2 shows that \(T_+(L_1) \rightarrow T_+(L_2)\) is not a transverse isotopy.

Similar results as Theorems 2.4 and 2.7 are obtained in [11] in terms of rectangular diagrams, also known as grid-link diagrams.
Figure 7.

3. Construction of Etnyre-Honda pair

We start with a standardly embedded torus \( U \) in \( \mathbb{R}^3 \). This torus \( U \) may be described as \( \{(r, \theta, z) \mid (r - 2)^2 + z^2 = 1\} \) in the cylindrical coordinate of \( \mathbb{R}^3 \). Choose two sets of \( pq + p + q \) numbers \( z_1, \ldots, z_{pq+p+q} \) and \( \theta_1, \ldots, \theta_{pq+p+q} \) satisfying the conditions \(-1 < z_1 < \cdots < z_{pq+p+q} < 1 \) and \( 0 < \theta_1 < \cdots < \theta_{pq+p+q} < \pi \). The intersection of \( U \) with the plane \( \{(r, \theta, z) \mid z = z_i\} \) consists of two circles \( L_i \) and \( \ell_i \), where the \( r \)-coordinate of \( L_i \) is larger than that of \( \ell_i \). Let \( M_i \) be the intersection of \( U \) with the plane \( \{(r, \theta, z) \mid \theta = \theta_i\} \), which is a circle. The union of circles \( \bigcup_{i=1}^{pq+p+q}(L_i \cup M_i) \) separate \( U \) into \( (pq+p+q)^2 \) squares. Choose one rectangle \( W_1 \) on \( U \) with \( z_{pq+q} \leq z \leq z_{pq+p+q} \) and \( \theta_1 \leq \theta \leq \theta_{q+1} \) occupying \( pq \) squares. See Figure 8 (2). Choose a rectangle \( W_{k+1} \) on \( U \) occupying \( pq \) squares in \( p \) rows and \( q \) columns so that the upper right corner of \( W_k \) is identified with the lower left corner of \( W_{k+1} \) for \( k = 1, \ldots, pq+p+q \). Isotope \( W_k \) so that \( W_k \cap \{(r, \theta, z) \mid (r - 2)^2 + z^2 = 1\} \) consists of a subarc of each of \( L_i \) and \( L_i+p \). See Figure 8 (1).

Let \( Z_i \) be the subdisc of the plane \( \{(r, \theta, z) \mid z = z_i\} \) with \( \partial Z_i = L_i \). Let \( \mathcal{T} \) denote a torus \( \partial N_\varepsilon((\bigcup_{i=1}^{pq+p+q}Z_i) \cup (\bigcup_{j=1}^{pq+p+q}W_j); S^3) \), that is the boundary of an \( \varepsilon \)-neighborhood of the branched surface \( (\bigcup_{i=1}^{pq+p+q}Z_i) \cup (\bigcup_{j=1}^{pq+p+q}W_j) \) in \( S^3 \). Figure 9 illustrates a tiling obtained from a braid foliation \( \mathcal{T} \cap \{H(\theta)\} \) on \( \mathcal{T} \), where the point with the mark \( +k \) (resp. \( -k \)) represents the intersection of \( \partial N(Z^k) \) and the \( z \)-axis with larger (resp. smaller) \( z \)-coordinate, and the point with the mark \( k+ \) (resp. \( k- \)) represents the hyperbolic singularity on the plane \( \{(r, \theta, z) \mid \theta = \theta_i + \varepsilon\} \) (resp. \( \{(r, \theta, z) \mid \theta = \theta_i - \varepsilon\} \)) corresponding to \( \partial N(W_j) \) (resp. \( \partial N(W_{j+1}) \)). We notice that each of \( G_{++} \) and \( G_{--} \) consists of a circle, where \( G_{++} \) and \( G_{--} \) are the graphs defined in
Let $C_1$ and $C_2$ be the annuli $T \setminus (G_{++} \cup G_{--})$, and let $c_1$ and $c_2$ be the core curve of $C_1$ and $C_2$, respectively. When $p = 2$ and $q = 3$, $c_1$ is a curve of slope $-\frac{2}{11}$ on $T$ with respect to the coordinate system $C'$ in [5], where the boundary of the cabling annulus has slope $\frac{1}{6}$, and the meridian of $T$ has slope $\frac{4}{11}$. Figure [6] illustrates $c_1$ on $T$.

Next we look at $T$ in $(\mathbb{R}^3, \xi_{sym})$. Isotope $T$ so that $L^i$ has sufficiently small $r$ coordinate, and that the hyperbolic singularity on $\partial N(W_i)$ has sufficiently large $r$ coordinate. Each elliptic singularity in the characteristic foliation on $T$ corresponds to an intersection of $T$ with the $z$-axis, which is an elliptic singularity in a braid foliation. Each hyperbolic singularity in the characteristic foliation on $T$ corresponds to a hyperbolic singularity on $T$ in a braid foliation. Thus the characteristic foliation on $T$ is isotopic to the corresponding braid foliation on $T$.

We may use the Giroux Elimination Lemma[6] to isotope $T$ in a small neighborhood of $T$, and we may eliminate $G_{++}$ and $G_{--}$. Then $T$ is a convex torus, and $c_1$ and $c_2$ are Legendrian divides on $T$ with slope $-\frac{2}{11}$ with respect to $C'$. Figure [10] illustrates a train-track on $T$ constructed from $c_1$, $c_2$ and a small arc connecting them. Let $\ell(r, s)$ denote a simple closed curve supported by the train-track with weights $r$, $s$ and $r + s$, as illustrated in the figure.
The topological knot type of $\ell(r, s)$ is a $(2r + s, r + s)$-cable of a $(2, 3)$-torus knot. The Giroux Flexibility Theorem\cite{6} allows one to isotope $T$ in a small neighborhood of $T$ so that $\ell(r, s)$ is a Legendrian ruling curve on $T$ with slope $-\frac{2r+s}{11r+5s}$ with respect to $C'$ on $T$. This Legendrian ruling curve may correspond to a braided rectangular diagram. When $r = s = 1$, $\ell(1, 1)$ is $L_+$ in \cite{5}. A braided rectangular diagram of $\ell(1, 1)$ is obtained from the diagram in Figure 11. It is easy to construct $K_+$ in \cite{5}, as illustrated in Figure 11. A similar proof as in Lemma 6.3 in \cite{5} shows that $\ell(r, s)$ does not admit a Legendrian destabilization.
Figure 10.

Figure 11.

References

[1] D. Bennequin, *Entrelacements et équations de Pfaff*, Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982), 87–161, Astérisque, **107-108**, Soc. Math. France, Paris, 1983.
[2] J. Birman, E. Finkelstein, *Studying surfaces via closed braids*, J. Knot Theory Ramifications 7 (1998), 267–334.

[3] I. A. Dynnikov, *Arc-presentations of links: monotonic simplification*, Fund. Math. 190 (2006), 29–76.

[4] J. Epstein, D. Fuchs, M. Meyer, *Chekanov-Eliashberg invariants and transverse approximations of Legendrian knots*, Pacific J. Math. 201 (2001) 89–106.

[5] J. B. Etnyre, K. Honda, *Cabling and transverse simplicity*, Ann. of Math. 162 (2005) 1305–1333.

[6] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991) 637-677.

[7] H. Matsuda, *Links in an open book decomposition and in the standard contact structure*, Proc. Amer. Math. Sci. 134 (2006) 3697–3702.

[8] W. W. Menasco, *Recognizing when closed braids admit a destabilization, an exchange move or an elementary flype*, math.GT/0507124.

[9] W. W. Menasco, *An addendum on iterated torus knots*, math.GT/0610566.

[10] S. Yu. Orevkov, V. V. Shevchishin, *Markov Theorem for transversal links*, J. Knot Theory Ramifications 12 (2003), 905–913.

[11] P. Ozsváth, Z. Szabó, D. Thurston, *Legendrian knots, transverse knots and combinatorial Floer homology*, math.GT/0611841.

[12] J. Świątkowski, *On the isotopy of Legendrian knots*, Ann. Global Anal. Geom. 10 (1992) 195–207.

[13] N. Wrinkle, *The Markov theorem for transverse knots*, math.GT/0202055.

**Department of Mathematics, Graduate School of Science, Hiroshima University, Hiroshima 739-8526, JAPAN**

*Current address*: Department of Mathematics, Columbia University, New York, NY 10027, USA

*E-mail address*: matsuda@math.sci.hiroshima-u.ac.jp, matsuda@math.columbia.edu

**Department of Mathematics, University at Buffalo, Buffalo, NY 14260**

*E-mail address*: menasco@math.buffalo.edu