Algorithmic homeomorphism of 3-manifolds as a corollary of geometrization

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In this paper we prove two results, one folklore and the other new. The folklore result, which goes back to Thurston, is that the geometrization theorem implies that there is an algorithm for the homeomorphism problem for closed, oriented, triangulated 3-manifolds. We give a self-contained proof, with several variations at each stage, that uses only the statement of the geometrization theorem, basic hyperbolic geometry, and old results from combinatorial topology and computer science. For this result, we do not rely on normal surface theory, methods from geometric group theory, nor methods used to prove geometrization.

The new result is that the homeomorphism problem is elementary recursive, i.e., that the computational complexity is bounded by a bounded tower of exponentials. This result makes essential use of normal surface theory, Mostow rigidity, and improved bounds on the computational complexity of solving algebraic equations.

1. INTRODUCTION

In this paper, we will prove the following two theorems.

Theorem 1.1 (Folklore). Suppose that $M_1$ and $M_2$ are two finite, simplicial complexes that represent closed, oriented 3-manifolds. Then, as a corollary of the geometrization theorem, it is recursive to determine if they are PL-homeomorphic.

Theorem 1.2. The homeomorphism problem for closed, orient­ed 3-manifolds is elementary recursive.

Theorem 1.1 says essentially that the geometrization theorem yields a classification of 3-manifold by the standard of computer science; this is a point that arose in discussions between the author and Bill Thurston. See Section 1.1 for more details.

To support the interpretation that Theorem 1.1 is folklore, we will prove it using only the statement of geometrization, basic facts about hyperbolic geometric triangulations, old results in computability theory, and either the stellar or bistellar moves of Alexander, Newman, and Pachner [3, 35–37]. We avoid Gromov-style geometric group theory [11, 46], which was largely developed after Thurston stated the geometrization conjecture; and normal surface theory [13, 23], which was first developed earlier but is a significant algorithmic theory.

In Theorem 1.2, an algorithm is elementary recursive if its execution time is bounded by a bounded tower of exponentials; for instance, time $O(2^{2^n})$. (See Section 2.2.) In contrast with Theorem 1.1, the proof of Theorem 1.2 does use normal surface theory, as well as Mostow rigidity, and improved bounds on the computational complexity of solving algebraic equations [10]. The connected-sum and JSJ decomposition stages of Theorem 1.2 were partly known. For instance, using similar methods, Mijatović [28, 29] established an elementary recursive bounds on the number of Pachner moves needed to standardize either $S^3$ or a Seifert-fibered space with boundary.

The hyperbolic case of Theorem 1.2 is new. By contrast, Mijatović also established a primitive recursive bound on the number Pachner moves needed to equate two hyperbolic, fiber-free, Haken 3-manifolds. However, primitive recursive is significantly weaker than elementary recursive; the Haken condition is also a significant restriction. Theorem 1.2 also has the advantage of combining a mixed set of methods to handle the full generality of closed, oriented 3-manifolds.

Finally, part of the contrast between Theorem 1.1 and Theorem 1.2 is that it is easy to work too hard to prove former. The standard of a recursive algorithm, which is all that our proof of Theorem 1.1 achieves, does not require any bound on the algorithm’s execution time. (See Proposition 2.3.) Theorem 1.2 is a more explicit result that requires prior bounds on computation time at every stage.

Remark. We leave the non-orientable versions of Theorems 1.1 and 1.2 for future work. This case includes new details such as 3-manifolds with essential, two-sided projective planes and Klein bottles. A more thorough result would also handle compact 3-manifolds with boundary.

1.1. Some history

Ever since Thurston first stated the geometrization conjecture, now the geometrization theorem, he and others interpreted it as a classification of closed 3-manifolds. (As well as compact 3-manifolds, orbifolds, etc.) However, the statement of geometrization is sufficiently complicated that it raises the philosophical question of what is meant by a “classification”. One important interpretation comes from computer science. We can say that closed 3-manifolds are classified if we can:

1. specify every 3-manifold by a finite data structure;
2. list all closed 3-manifolds without repetition;
3. given any 3-manifold $M$, algorithmically identify the standard manifold $M'$ such that $M \cong M'$.

For closed 3-manifolds, condition (1) is addressed by the fact that every 3-manifold has a unique smooth structure and a unique PL structure. As a result, as stated in Theorem 1.1, we can describe a closed 3-manifold as a finite simplicial complex. Unlike in higher dimensions, it is easy to check whether a simplicial complex is a 3-manifold (Section 3). Conditions (2) and (3) are equivalent to an algorithm to determine whether two closed 3-manifolds $M_1$ and $M_2$ are homeomorphic. Condition (3) clearly implies both condition (2) and a

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A homeomorphism algorithm. On the other hand, given a homeo-
omorphism algorithm, and given a 3-manifold \( M \) represented
by a simplicial complex, we can search for the lexicograph-
cally first 3-manifold \( M' \) such that \( M \cong M' \) to satisfy (3). Like-
wise, the list of all lexicographically first representations of
3-manifolds satisfies (2), (Haken calls this remark the “cheap-
phism

searching spaces, it is recursive to calculate the induced isomor-
phism problem

\( A \rightarrow \{ \ast \} \)

is likewise a function

\( F : \Sigma^* \rightarrow \Sigma^* \)

that several published treatments do not distinguish between
orientation-preserving and orientation-reversing homeomor-
phisms of closed, hyperbolic 3-manifolds \( N_1 \) and \( N_2 \). Ac-
tually, calculating orientations does not require much extra
work. For instance, Scott and Short describe an exhaustive
search for isomorphisms \( \pi_1(N_1) \cong \pi_1(N_2) \) between funda-
mental groups. Then, given an isomorphism \( \Gamma_1 \cong \Gamma_2 \) between
any two groups that have finite simplicial complexes as clas-
sifying spaces, it is recursive to calculate the induced isomor-
phism \( H_3(\Gamma_1) \cong H_3(\Gamma_2) \).

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2. COMPUTABILITY

2.1. Recursive and recursively enumerable problems

Let \( \Sigma \) be a finite alphabet and let \( \Sigma^* \) be the set of all finite
words over that alphabet. A **decision problem** is a function

\[ D : \Sigma^* \rightarrow \{ \text{yes, no} \} \]

A **function problem** is likewise a function \( F : \Sigma^* \rightarrow \Sigma^* \), which
is equivalent to many other types of input by some suitable encoding: Finite se-
quences of strings, finite simplicial complexes, etc.

A decision problem or a function problem can be a **promise
problem**, meaning that it is defined only on some subset of
inputs \( P \subseteq \Sigma^* \) which is called a **promise**. Whether two closed
\( n \)-manifolds are PL homeomorphic is an example of a promise
decision problem: The input consists of two simplicial com-
exes that are promised to be manifolds; then the yes/no de-
cision is whether they are homeomorphic. (But see Proposi-
tion 3.1.)

A **decision algorithm** is a mathematical computer program,
which can be modelled by a Turing machine (or some equiva-

tent model of computation), that takes some input \( x \in \Sigma^* \) and
can do one of three things: (1) Terminate with the answer

\( \text{“yes”} \), (2) terminate with the answer “no”, or (3) continue in
an infinite loop. Similarly, a function algorithm can terminate
and report an output \( y \in \Sigma^* \), or it can continue in an infinite
loop. Given a multivalued function \( f \), then a function algo-

\( \{ \ast \} \)

rithm is only required to calculate one of the values of \( f(x) \) on
input \( x \).

A **complexity class or computability class** is some set of de-
cision or function problems, typically defined by the existence
of algorithms of some kind. For example, a decision prob-
lem \( D \) or a function problem \( f \) is **recursive** (or computable or
decidable) if it is computed by an algorithm that always termi-
nates. By definition, the complexity class \( \mathbb{R} \) is the set of
recessive decision problems. By abuse of notation, \( \mathbb{R} \) can also
denote the set of recursive, promise decision problems; or the
set of recursive function problems, with or without a promise.
The following proposition is elementary.

**Proposition 2.1.** If \( D \) is a recursive promise problem, and
if the promise itself is recursive, then \( D \) is a recursive non-
promise problem if we let \( D(x) = \text{no} \) when the promise is not
satisfied.http://nwalsh.com/tex/texhelp/bibtx-12.html

The complexity class \( \mathbb{R} \) is the set of recursively enumer-
able decision problems. These are problems with an algo-

\( \text{“yes”} \) when the answer is yes; but
if the answer is “no”, the algorithm might not terminate. The
complexity class \( \mathbb{R} \) is defined in the same way as \( \mathbb{R} \), but
with yes and no switched. We review the following standard
propositions and theorems.

**Proposition 2.2.** A non-promise decision problem \( D \) is in \( \mathbb{R} \)
if and only if there is an algorithm that lists all solutions to
\( D(x) = \text{yes} \) without repetition.

**Proposition 2.3.** \( \mathbb{R} = \mathbb{R} \cap \text{coR} \)

The proof of Proposition 2.3 is elementary but important:
Given separate \( \mathbb{R} \) algorithms for both the “yes” and “no” an-
wers, we can simply run them in parallel; one of them will
finish. The proposition and its proof reveal the important point that a recursive algorithm might come with no bound whatsoever on its execution time.

**Theorem 2.4 (Turing).** The halting problem is in RE but not in R. In particular, RE ≠ R.

Informally, the halting problem is the question of whether a given algorithm with a given input terminates. Let \( H(x) \) be the halting decision problem. It is easy to show that the halting decision problem \( H(x) \) is RE-complete in the following sense: Given a problem \( D(x) \) in RE, there is a recursive function \( f \) such that \( D(x) = H(f(x)) \). Any other problem in RE with this same property can also be called RE-complete, or halting complete.

**Proposition 2.5.** Let \( G \) be a graph structure on \( \Sigma^* \). If the edge set of \( G \) is recursively enumerable, then so is the set of pairs \( (x, y) \), where \( x \) and \( y \) are vertices in the same connected component of \( G \).

Proposition 2.5 is important for recursively enumerable infinite searches. The interpretation of the proposition, which is conveyed by the proof, is that nested infinite loops can be collapsed into a single infinite loop.

**Proof.** By Proposition 2.2, we can model a recursively enumerable set by an algorithm that lists its elements. The proposition states that the elements can be listed without repetition; but this is optional, since we can store all of the elements already listed and omit duplicates.

We use a recursive bijection \( f \) between the natural numbers \( \mathbb{N} \) and \( \mathbb{N}^* \), the set of finite sequences of elements of \( \mathbb{N} \). We can express any element of \( \mathbb{N}^* \) uniquely in a finite alphabet that consists of the ten digits and the comma symbol. We can then list all of these strings first by length, and then in lexicographic order for each fixed length. We can then let \( f(n) \) be the \( n \)th listed string.

We can now convert the value \( f(n) \) to a finite path \( (x_0, \ldots, x_k) \) in the graph \( G \), in such a way that every finite path is realized. If \( f(n) = (n_0, \ldots, n_k) \), then we let \( x_0 \) be the \( n_0 \)th string in \( \Sigma^* \). For each \( j > 0 \), we let \( x_j \) be the \( n_j \)th neighbor of \( x_{j-1} \). In order to find the \( n_j \)th neighbor of \( x_{j-1} \), we list the elements of the edge set \( G \) until the edge \( (x_{j-1}, x_j) \) arises as the \( j \)th edge from \( x_{j-1} \). There is the technicality that \( x_{j-1} \) might not have an \( n_j \)th neighbor if it only has finitely many neighbors. To avoid this problem, we intersperse trivial edges of the form \((x, x)\) infinitely many times, for every string \( x \in \Sigma^* \), along with the non-trivial edges of \( G \).

Having found every finite path in \( G \), we have thus found every connected pair of vertices \( (x, y) \). Thus, the set of such pairs is recursively enumerable.

### 2.2. Elementary recursive problems

As mentioned after Proposition 2.3, a recursive algorithm might not have any explicit upper bound on its execution time, beyond the tautological bound that running it is one way to calculate how long it runs. This motivates smaller complexity classes that are defined by explicit bounds. The most common notation for a bound on the execution time of an algorithm is asymptotic notation as a function of the input length \( n = |x| \) to a decision problem \( D(x) \). For example, we could ask for a polynomial-time algorithm, by definition one that runs in time \( O(n^k) \) for some fixed \( k \).

We have two reasons to consider a fairly generous “first” bound. First, the recursive class is unfathomably generous; nearly any explicit bound can be considered a huge improvement. Second, the computational complexity of a problem or algorithm depends somewhat on the specific computational model; if we are generous, then we can substantially erase this dependence.

For concreteness, we consider a traditional Turing machine first. By (informal) definition, a Turing machine is a finite-state “head” with an infinite linear memory tape, and deterministic dynamical behavior. We say that an algorithm is elementary recursive if it runs in time

\[
O \left( 2^{2^k} n^k \right)
\]

for some constant \( k \). We call the corresponding complexity class ER. By abuse of terminology, we use ER to refer to both decision problems and function problems, and to numerical bounds. (Note that if \( f(n) \) can be computed in ER, then a running time bound of \( O(f(n)) \) is itself a subclass of ER.)

Without reviewing rigorous definitions, we list some variations in the computational model that do not affect the class ER. We also omit the proof of the following (standard) proposition, as it is peripheral to the main purpose of this article.

**Proposition 2.6.** In any of the following computational models, bounded by elementary recursive computation time except in case 4, the set of decision problems that can be calculated is the same as standard ER.

1. A Turing machine with an infinite tree tape rather than a linear tape; or a random access tape addressable by a separate address tape.
2. A randomized Turing machine whose answers are only probably correct.
3. A quantum Turing machine that can compute in quantum superposition.
4. A Turing machine restricted to an elementary recursive bound on computational space and unrestricted computation time.

**Remark.** An elementary recursive bound is an improvement over another bound that is popular in logic and computer science: primitive recursive. An algorithm is primitive recursive
if it runs in time $O(2^k n)$ for some fixed $k$, where the $k$th operation \( a[k]b \) is defined inductively as follows:
\[
\begin{align*}
    a[1]b &= a + b \\
    a[2]b &= ab \\
    a[3]b &= a^b \\
    a[k+1]b &= a[k](a[k] \cdots (a[k]a) \cdots ).
\end{align*}
\]

The primitive recursive complexity class is PR.

### 2.3. Computable numbers

A computable real number \( r \in \mathbb{R} \) is a real number with a computable sequence of bounding rational intervals. In other words, there is an algorithm that generates rational numbers \( a_n, b_n \in \mathbb{Q} \) such that \( x \in [a_n, b_n] \) and \( b_n - a_n \to 0 \). Many standard algorithms from numerical analysis, including field operations, integration of continuous functions, Newton’s method, etc., have the property that if the input consists of computable numbers, then so does the output. One main limitation of computable real numbers is that inequality tests such as \( a > b \) or \( a \neq b \) are only recursively enumerable, not recursive. In other words, an algorithm can eventually tell us that \( a \) and \( b \) are not equal, and if so which one is greater; but it may be impossible to ever know that they are equal.

Another restricted model of real numbers is \( \hat{\mathbb{Q}} = \mathbb{R} \cap \bar{\mathbb{Q}} \), the real algebraic closure of the rational numbers \( \mathbb{Q} \).

**Theorem 2.7.** There is an encoding of the elements of \( \hat{\mathbb{Q}} \) such that field operations, order relations, and conversion to computable real numbers are all recursive.

One encoding of a real algebraic number \( x \) that can be used to prove Theorem 2.7 is to describe it by a minimal polynomial together with an isolating interval to specify a real root. However, it is just as well to accept the result without worrying about its proof. For example, real algebraic numbers are implemented reliably in Sage [53].

**Theorem 2.8** (Tarski-Seidenberg [44, 47]). It is recursive to determine whether there is a solution to a finite list of polynomial equalities and inequalities with coefficients in \( \hat{\mathbb{Q}} \) in finitely many variables; or to find a solution.

Actually, Tarski and Seidenberg proved the stronger result that it is recursive to decide any assertion over \( \mathbb{R} \) expressed with polynomial relations and first-order quantifiers.

### 3. TRIANGULATIONS OF MANIFOLDS AND MOVES

In this section, we will analyze the form of the input to Theorem 1.1. We will show that given simplicial complexes as input to the homeomorphism problem, we can first confirm that they are 3-manifolds. (It is also easy to confirm that input strings actually represent simplicial complexes, in some convenient data type.) Thus Proposition 2.1 applies: we can view the homeomorphism problem as a non-problem. Actually, Proposition 3.1 below is overkill for this purpose, since it is much harder in dimension \( n = 4 \) than in dimension \( n = 3 \).

We then discuss moves between triangulations of a manifold, mainly to establish Corollary 3.4. In light of Proposition 2.3, Corollary 3.4 is an easy half of Theorem 1.1, one that holds in any dimension \( n \).

**Proposition 3.1.** If \( \tau \) is a finite simplicial complex of dimension \( n \leq 4 \), then it is recursive to determine whether it is a closed PL \( n \)-manifold, and whether or not it is orientable.

**Proof.** The proof is (sort of) by induction on dimension \( n \). The result is trivial if \( n = 0 \), where we need only check that \( \tau \) is a single point. Otherwise, we first check that \( \tau \) is connected, and we must check that the link \( \sigma \) of every vertex is both a closed \((n-1)\)-manifold and a PL \( n \)-sphere. The former condition is the inductive step. The latter condition requires an algorithm to recognize an \((n-1)\)-sphere. If \( \sigma \) is a closed 1-manifold, then it is immediately a 1-sphere, i.e., a circle. If \( \sigma \) is a closed 2-manifold, then we can compute its Euler characteristic. If \( \sigma \) is a closed 3-manifold, then Theorem 1.1 implies that it is recursive to determine if \( \sigma \) is a 3-sphere, although this result was obtained without geometrization by Rubinstein and Thompson (Theorem 7.2) [39, 48].

We can check that \( \tau \) is orientable (and orient it) algorithmically by computing simplicial homology. \( \square \)

The stellar and bistellar subdivision theorems establish that every two triangulations of a compact \( n \)-manifold, in particular a compact \( 3 \)-manifold, are connected by a finite sequence of explicit moves. See the review by Lickorish [25] for a modern treatment and a historical review.

**Theorem 3.2** (Alexander-Newman). If two finite simplicial complexes \( K_1 \) and \( K_2 \) are PL equivalent, then they are connected by a sequence of stellar subdivision moves and their inverses.

Briefly, a stellar move consists of adding a vertex to the interior of some simplex \( \Delta \) of \( K \), and then stellating the link of \( \Delta \).

**Theorem 3.3** (Pachner). If \( K_1 \) and \( K_2 \) are two triangulations of a compact, PL manifold \( M \), then they are connected by bistellar moves.

A bistellar move of a triangulation of an \( n \)-manifold \( M \) consists of a stellation followed an inverse stellation at the same vertex. Equivalently, two triangulations of \( M \) differ by a bistellar move when there is a minimal cobordism between them consisting of a single \((n+1)\)-simplex. In particular, a shellable triangulation of \( M \times I \) yields a sequence of bistellar moves.

Lickorish points out that Newman essentially conjectured and partially proved Theorem 3.3 in an earlier paper, before he and Alexander separately proved Theorem 3.2. Bistellar moves are also called Pachner moves, although arguably they should be called Newman-Pachner moves.

Theorem 3.3 also holds for ideal triangulations of a compact \( 3 \)-manifold with torus boundary components. (Equivalently, a pseudomanifold with singular points with torus links, which are the ideal vertices.)

Theorems 3.2 and 3.3 each have the following corollary.
**Corollary 3.4.** The PL homeomorphism problem for PL $n$-manifolds is in RE.

Actually, there is a simple independent argument for Corollary 3.4. The definition of PL equivalence of two simplicial complexes $K_1$ and $K_2$ is that they have a mutual refinement $K_3$ which consists of a (Euclidean) geometric triangulation of each simplex of $K_1$, and each simplex of $K_2$. This yields a recursive list of moves, since we can take the positions of the vertices of a refinement to be rational numbers. By Proposition 2.5, a recursive list of moves is as good as a finite list of moves for Corollary 3.4; indeed an RE list of moves would already be sufficient.

**Proposition 3.5.** If $K_1$ is a finite simplicial complex with $n_1$ simplices (of arbitrary dimension) and $n_2 ≥ n_1$, then it is recursive to produce a complete list of geometric subdivisions $K_2$ of $K_1$ with $n_2$ simplices.

**Proof.** There are only finitely many simplicial complexes $K_2$ with $n_2$ simplices, and they can be generated recursively. For each candidate for $K_2$, there are only finitely many combinatorial choices for a function from the simplices of $K_2$ to the simplices of $K_1$. For each such choice, we can first check that the simplices of $K_2$ that land in a $k$-simplex $Δ \in K_1$ support a simplicial cycle that represents the fundamental class in $H^k(Δ, ∂Δ)$. We solve for each such cycle for all $Δ$ (where each must be unique if $K_2$ indeed subdivides $K_1$). Then the constraint that each simplex of $K_2$ must be positively oriented in $K_1$ yields a recursive list of moves for Corollary 2.8 to see if there is a solution for those positions.

We conclude this section with the following summary theorem which is variously to Markov, Boone, Adyan, Rabin, and P.S. and S.P. Novikov. (See [38].)

**Theorem 3.6.** The isomorphism problem for finitely presented groups, the PL homeomorphism problem for 4-manifolds, and the recognition of $S^n$ among PL $n$-manifolds for each $n ≥ 5$ are all halting-complete.

It is not known whether either topological or PL recognition of $S^4$ is recursive.

**Remark.** The homeomorphism problem for PL $n$-manifolds in Theorem 3.6, or even recognition of $S^n$, needs to be handled with some care, for several reasons. First, because recognizing whether the input is a PL $n$-manifold is (by Theorem 3.6) an uncomputable promise when $n ≥ 6$. Second, because there are closed manifolds that are PL homeomorphic but not homeomorphic [22]. Third, because there are simplicial complexes that are not PL $n$-manifolds at all, but that are homeomorphic to $S^n$, for each $n ≥ 5$ [7]. The proof of Theorem 3.6 dispenses with all of these concerns as follows. Given an input $x$ to the halting problem $H(x)$ and $n ≥ 4$, there is an algorithm that constructs an $n$-manifold $M(x)$ such that:

1. $M(x)$ is manifestly a closed PL manifold.
2. $M(x)$ is PL homeomorphic to a connected sum of $S^2 \times S^2$ when $n = 4$ or to $S^n$ when $n ≥ 5$, if and only if $M(x)$ is simply connected.
3. $M(x)$ is simply connected if and only if $H(x) = \text{yes}$.

**Remark.** By contrast with Theorem 3.6, the PL homeomorphism problem for simply connected $n$-manifolds with $n ≥ 5$ is recursive [32].

### 4. GEOMETRIZATION IS RECURSIVE

The goal of this section is to prove Theorem 4.4, which says that the geometric decomposition of a 3-manifold $M$ is computable.

#### 4.1. Statement of geometrization

We begin with three results that, together, are one formulation of the geometrization theorem for closed, oriented 3-manifolds.

**Theorem 4.1 (Kneser-Milnor).** Every closed, oriented 3-manifold is a connected sum of prime, closed, oriented 3-manifolds. The summands are unique up to oriented homeomorphism.

**Theorem 4.2 (Jaco-Shalen-Johansson [19, 20]).** A closed, oriented, prime 3-manifold has a minimal collection of incompressible tori, unique up to isotopy and possibly empty, with the property that the complementary regions are either Seifert-fibered or atoroidal.

The decomposition in Theorem 4.2 is called the JSJ decomposition. We can call the tori JSJ tori, and the complementary regions JSJ components. We will use $M$ to denote a general closed, oriented 3-manifold, then $W$ to denote a prime summand of $M$; then $N$ to denote a JSJ component of $W$.

**Theorem 4.3 (Thurston-Hamilton-Perelman).** Suppose that $N$ is an oriented, prime, atoroidal 3-manifold which is either closed or has torus boundary components. Then $N$ is either Seifert-fibered, or its interior $N^*$ has a unique, complete hyperbolic structure with torus cusps.

As everyone knows, Theorem 4.3 was conjectured and partly proven by Thurston [49], then fully proven by Perelman using the Ricci flow program of Hamilton [31]. (Note that Theorem 4.3 implicitly includes the Poincaré conjecture in the Seifert-fibered case.)

**Remark.** Mixing the JSJ decomposition with hyperbolization is a less pure approach than Thurston’s decomposition into geometric components, but we find it convenient for Theorem 1.1. We could recognize spherical and Euclidean components with the same methods as hyperbolic components (Lemma 4.6), while several of the other Thurston geometries induce canonical Seifert fibrations. In fact, every Seifert-fibered 3-manifold or component is geometric. Conversely, every geometric 3-manifold or component is hyperbolic unless it is Seifert-fibered or a Sol manifold (which is cut into a thickened torus in the JSJ decomposition).
4.2. Statement of computational geometrization

Theorem 4.4. If $M$ is a triangulated 3-manifold, then it is recursive to compute its geometric decomposition.

Before proving Theorem 4.4, we state more precisely what is meant by computing its geometric decomposition.

If $N$ is a JSJ component of a summand $W$ of a closed 3-manifold $M$, then we tile it by special polyhedra to describe either a Seifert fibration or a hyperbolic structure. In general, if $X$ is a topological space which is tiled by embedded polyhedra, then we mean in more rigorous terms that $X$ is homeomorphic to a regular CW complex or regular cellulation. We will use the standard fact that every regular CW complex has a barycentric subdivision; see Figure 1.

![Figure 1. A barycentric subdivision of a 2-cell, in this example a square.](image)

**Proposition 4.5.** Every regular CW complex $X$ has a barycentric subdivision which is a simplicial complex.

In light of Proposition 4.5, we can describe a regular cellulation of $N$ by triangulating $N$, and then grouping the simplices to make the cells. We also want to geometricize the cells, which we describe next. In both cases, we call the decorated barycentric subdivision an adapted triangulation.

If $N$ is Seifert fibered, then it is either closed or has vertical torus boundary. We describe the Seifert fibration by a tiling by triangular prisms, such that the Seifert fibers intersect each prism in vertical intervals. $N$ fibers over an orbifold base surface $S$. A triangulation of $S$ lifts to a tiling of $N$ by triangular solid tori; these tori can then be divided into triangular prisms. (We assume that all of the orbifold points of $S$ are vertices of its triangulation, so that the triangular solid tori are all untwisted.) If $N$ has any singular fibers, or if it has a non-trivial Chern number, then its fibration is a non-trivial circle bundle, and the triangular solid tori and prisms are glued to each other with shear transformations. We can realize a unit-slope shear transformation in such a gluing by adding diagonals to the rectangular sides of the prisms, as in Figure 2. By refining the triangulation of $S$, we can assume that all of the shear transformations have unit slope.

If $N$ is hyperbolic, then we choose a triangulation $T^*$ of $N^*$ whose simplices are geometric in the hyperbolic structure. If $N^*$ has cusps, then the cusps are modelled as ideal vertices, along with other vertices that are not ideal; the simplices that subtend these vertices are semi-ideal. We do not allow any of the simplices to be spun, degenerate, or flipped over. It is a standard result that $N^*$ always has a geometric, semi-ideal triangulation. For convenience, we also assume that every tetrahedron has at most one ideal vertex. The corresponding cell in

![Figure 2. A triangular prism with side diagonals (in red); using diagonals to make a shear gluing.](image)

$N$ is then a truncated tetrahedron which is a triangular prism, this time without side diagonals. (See Figure 3.)

![Figure 3. A tetrahedron truncated at one vertex.](image)

If each JSJ component $N$ has an adapted triangulation, then we want to glue them together along their torus boundary components to make a prime summand $W$, and then glue those together to make a general closed, oriented $M$. The second type of gluing is straightforward: If $W_1$ and $W_2$ are two triangulated summands, then we remove open tetrahedron from each summand one and then glue together along the copy of $\partial \Delta^3$ to make $W_1 \# W_2$.

The first type of gluing is more complicated, since the adapted triangulations on the two sides of a JSJ torus $T$ may be significantly mismatched. In this case, if $\tau_1$ and $\tau_2$ are the restrictions of the two triangulations to $T$, we insert a shelled triangulation of $T \times I$ that yields a sequence of bistellar moves to connect $\tau_1$ to $\tau_2$. This sequence of moves certifies that the connecting simplicial complex is homeomorphic to $T \times I$, provided that $\tau_1$ and $\tau_2$ are disjoint as subcomplexes.

We summarize with a top-down description of an adapted triangulation of a general closed, oriented 3-manifold $M$:

1. $M$ has a distinguished collection of disjoint, separating 2-spheres, each triangulated with 4 triangles, that separates it into summands $\{W\}$. (Note that each $W$ is closed; the hole is plugged with a fresh tetrahedron for the remaining steps.)

2. Each $W$ has a distinguished collection of thickened tori $T \times I$ with shelled triangulations. These thickened tori separate $W$ into JSJ components $\{N\}$.

3. The triangulation of each $N$ is marked as the barycentric subdivision of a regular cellulation.
4. If $N$ is Seifert-fibered, then the cells are all triangular prisms parallel to the Seifert fibration, with side diagonals to allow shear.

5. If $N$ is hyperbolic, then each polyhedron is either a tetrahedron, or a triangular prism which is a truncation of a tetrahedron in $N^*$ with one ideal vertex. The triangulation $\tau^*$ of $N^*$ is geometric.

The one 3-manifold whose status is unclear in this description is $M \cong S^3$. It is the unit element in the unique factorization theorem, Theorem 4.1; thus, technically, it has no prime summands at all. Since it is unique, we can assign it any distinguished triangulation, for instance $\partial \Delta^4$, the boundary of a 4-simplex. $S^3$ is also Seifert-fibered and we could ask for an adapted triangulation, but this is moot in context.

4.3. Proof of Theorem 4.4

**Lemma 4.6.** It is recursive to find a geometric triangulation of a hyperbolic 3-manifold $N$ which is either closed or has torus boundary components. Hence, it is in RE to determine if it is hyperbolic.

We give two proofs of Lemma 4.6. Technically, they are proofs of two different results, since they use two different representations of numbers to describe the geometry of $N$. In the first version, we describe the shape of each tetrahedron using algebraic numbers. In the second version, we use floating-point arithmetic.

**Remark.** Manning [26, Thm. 5.2] also proves Lemma 4.6, but as a corollary of a harder result. His Theorem 5.1 shows (without geometrization) that it is recursive to decide whether $N$ is hyperbolic, given the promise that there is some algorithm for the word problem for $\pi_1(N)$. He also uses a single polyhedral fundamental domain to describe the geometry of $N$. Although this differs from a hyperbolic triangulation, which is what we use, the two models seem equivalent in strength.

**Proof of first version of Lemma 4.6.** Suppose that $\tau^*$ is a semi-ideal, geometric triangulation of $N^*$. We can model each tetrahedron $\Delta \in \tau^*$ (non-uniquely) by choosing four vertices in the Poincaré upper half-space model, including one on the boundary if $\Delta$ is semi-ideal. (Note that the ideal vertices of $\tau^*$ are marked in advance.) There is an algebraic formula for each finite edge length $\ell$ and each dihedral angle $\alpha$ of $\Delta$, if these are represented by their exponential values $\exp(\ell)$ and $\exp(i\alpha)$. The main matching condition for $\tau^*$ to be geometric is that if two tetrahedra share a finite edge, then the edge lengths agree; and the total dihedral angle around each edge equals $2\pi$. The first condition is immediately an algebraic condition. The second condition is almost an algebraic condition since the product of the exponentiated angles must be 1; this shows that the total angle is a multiple of $2\pi$, although not which one. We can also bound each ratio $\alpha/\pi$ by a small rational interval to confirm that the total angle is $2\pi$ and not some other multiple; or we can change variables to $\exp(i\alpha/n)$, where $n$ is the number of tetrahedra that meet at the edge. If an edge of $\tau$ is semi-ideal, then the length condition holds automatically.

Since all of the geometric data satisfies algebraic equations, there is a solution to the equations using real algebraic numbers, and complex algebraic numbers with real and imaginary parts. For any fixed triangulation $\tau$, we can thus use Theorem 2.7 to search for a solution and eventually find it, if it exists. We must also search over triangulations using Theorem 3.2 or Theorem 3.3. Since the result is a nested finite search (over triangulations and then candidate geometric structures), we can apply Proposition 2.5. Alternatively, for each triangulation, we can apply Theorem 2.8 to determine if there is a solution, assuming the algebraic change of variables to $\exp(i\alpha/n)$.

**Remark.** If we allowed geometric triangulations with fully ideal edges, then it would not be enough for the sum of the angles around such an edge $e$ to be $2\pi$. Since $e$ goes to itself under hyperbolic translation as well as rotation, gluing together the tetrahedra that contain $e$ could create a non-trivial translational holonomy. The two conditions together, that the total angle is $2\pi$ and the translational holonomy vanishes, are known as a Neumann-Zagier gluing relation [34].

**Remark.** Instead of calculating lengths and angles using positions of vertices in hyperbolic geometry, we can also relate them directly using formulas from hyperbolic and spherical trigonometry.

Since the second proof uses an approximate solution to the gluing equations, we need a criterion to know that an approximate solution is close to an exact one. Given a smooth system of equations $F(x) = 0$, the Newton-Kantorovich theorem establishes a sufficient criterion for Newton’s method to converge from an approximate solution $x_0$ to an exact solution $x_\infty$. Moreover, if $x_\infty$ is an exact solution and $F$ is non-degenerate at $x_\infty$, then the criterion must hold when $x_0$ is a good enough approximation.

Neuberger [33] points out that an ODE analogue of Newton’s method, which is called the continuous Newton’s method, simplifies the Newton-Kantorovich result.

**Theorem 4.7** (Newton-Kantorovich-Neuberger [33]). Let $B_{\epsilon}(x_0) \subset \mathbb{R}^n$ be the open ball of radius $\epsilon$ around $x_0 \in \mathbb{R}^n$, and let

$$F : B_{\epsilon}(x_0) \to \mathbb{R}^n$$

be a $C^2$ smooth function with non-singular Jacobian $DF$. Suppose that

$$||DF(x)^{-1}F(x)|| < \epsilon$$

for all $x \in B_{\epsilon}(x_0)$, where $|| \cdot ||$ is the operator norm. Then there is a unique $x_\infty \in B_{\epsilon}(x_0)$ such that $F(x_\infty) = 0$. Also, given $x_\infty$, (1) eventually holds as $x_0 \to x_\infty$.

**Remark.** The continuous Newton’s method is the ODE formula

$$x(0) = x_0 \quad x'(t) = -DF(x(t))^{-1}F(x) \quad x_\infty = \lim_{t \to \infty} x(t).$$
Although the literature on the continuous Newton’s method is more recent than Thurston’s formulation of geometrization, Kantorovich’s theorem from 1940 also suffices.

If the equation $F(x) = 0$ has a non-singular Jacobian $DF$ in a neighborhood of a solution, as in Theorem 4.7, then the system of equations is also called transverse or first-order rigid. Actually, first order rigidity is the more general condition that $DF$ has maximal rank when

$$U \subseteq \mathbb{R}^n \quad F : U \to \mathbb{R}^m \quad m \geq n,$$

i.e., when there may be more equations than unknowns.

In order to use a Newton-Kantorovich-type criterion, we will need a corollary of the Calabi-Weil rigidity theorem.

**Theorem 4.8 (Calabi-Weil [21, §8.10]).** If $N$ is a closed, hyperbolic 3-manifold, then the induced representation of its fundamental group,

$$\rho : \pi_1(N) \to \text{Isom}(\mathbb{H}^3),$$

is first-order rigid up to conjugacy. (I.e., it is infinitesimally rigid at the level of the first derivative.) The same is true if $N$ is cusped, among representations that are parabolic at the torus cusps.

**Corollary 4.9 (Folklore [15, §1.3]).** If $\tau$ is a geometric triangulation of a closed or cusped hyperbolic 3-manifold $N^*$, then it is first-order rigid except for motion of the non-ideal vertices.

Since Corollary 4.9 is to our knowledge a folklore result, we give a proof in Section 4.4.

**Proof of second version of Lemma 4.6.** We fix the model of each tetrahedron in the upper half space model so that it has exactly six degrees of freedom, or five if one of the vertices is ideal. After ordering the vertices $v_1, v_2, v_3, v_4$, we can put vertex $v_1$ at $(0, 0, 1)$; we can put vertex $v_2$ directly below it (or at $(0, 0, 0)$, allowing it to be the ideal vertex); and we can put $v_3$ at a position of the form $(a, 0, b)$. We approximate the positions of the vertices approximately with rational numbers. We can then approximate the lengths and angles of each tetrahedron in the same form, as well as the first and second derivatives of the lengths and angles as a function of the main variables, the separate positions of the vertices in the ideal models of the tetrahedra.

Suppose that there are $n$ non-ideal vertices. By the implicit function theorem, any exact solution to the gluing equations for the tetrahedra can be perturbed so that some $3n$ of the coordinates are exactly rational. Also by the implicit function theorem, some $3n$ of the angle conditions are implied by the other angle conditions and can be omitted. Finally, the fixed coordinates and omitted angle conditions can be chosen so that the remaining system of constraints, which we can write abstractly as $F(x) = 0$, has a non-singular Jacobian $DF$.

Moreover, the mapping $F$ is real analytic with an explicit formula. Thus, given an approximate solution $x_0$ which is within $\varepsilon$ of a true solution and $\varepsilon$ is small enough, we can majorize $\|DF^{-1}(x)\|$ on the ball $B_\varepsilon(x_0)$ using Taylor series, to confirm equation (1).

As Lemma 4.6 addresses the hyperbolic case of Theorem 4.4, we turn to two lemmas that address the Seifert-fibered case.

**Lemma 4.10.** It is recursive to find an adapted triangulation of a Seifert-fibered manifold $N$ which is either closed or has torus boundary components. Hence, it is in RE to determine if it is Seifert-fibered.

The proof is routine.

**Proof.** We can search through triangulations until we find one that is a barycentric subdivision of a cellulation by triangular prisms. It is then easy to check whether the prisms fit together following the rules in Section 4.2.

Finally, a torus $T$ that has matching Seifert-fibered structure on both sides is not needed and is not a JSJ torus. It is easy to see this case in the proof of Theorem 4.2. The more subtle possibility is that one or both sides might have more than one Seifert fibration. Fortunately this is rare for Seifert-fibered manifolds with boundary. It is addressed by the following result, which we jointly attribute to Seifert [45], Waldhausen [50], Jaco-Shalen [19], and Johannson [20].

**Proposition 4.11 (SWJSJ [16, Ch. VI]).** If $N$ is an orientable, Seifert-fibered 3-manifold with non-empty boundary $\partial N$, then the restriction of the fibration to the boundary is unique unless $N$ is a solid torus $D^2 \times I$, a thickened torus $S^1 \times S^1 \times I$, or a twisted I-bundle over a Klein bottle (a $K^2 \times I$).

**Proof of Theorem 4.4.** We search over triangulations of $M$ using stellar or bistellar moves, and decorations of them, to find an adapted triangulation as described in Section 4.2. A suitable decoration consists of distinguished spheres and thickened tori, and a reverse barycentric subdivision in each JSJ component $N$ to make triangular prisms in the Seifert-fibered case and a combination of once-truncated and ordinary tetrahedra in the hyperbolic case. Within this search, we search for geometric data to describe the hyperbolic structure of each $N$ which is not Seifert-fibered. Since these are nested, infinite searches, we combine them using the RE search algorithm of Proposition 2.5. By the geometrization theorem, we will eventually find a geometric structure that fits the description of Section 4.2.

We examine the JSJ components to verify that all of the spheres and tori are essential and that no two are parallel. We veto the decomposition if it includes a Seifert-fibered solid torus (by checking that the Seifert base is a disk with at most one singular fiber). If all of the components are geometric, and if none are solid tori, then every sphere and torus is essential and no two spheres are parallel. If two distinct tori are parallel, then the component between them is a thickened torus (with an annulus base with no singular fibers); we veto this as well. We also need to veto a Seifert-fibered component homeomorphic to $P^3 \# P^1$, which is the only Seifert-fibered space that is not a prime 3-manifold. It is a twisted circle bundle over a projective plane base with no singular fibers and with vanishing Chern number.

Finally, we need to check that all of the tori are JSJ tori. We need to erase each torus $T$ that has Seifert-fibered components
on both sides with the same induced fibration of $T$; or that could be the same fibration of $T$ if either component has more than one Seifert fibration. The first case is easy to recognize. By Proposition 4.11, we only have to consider two other types of Seifert-fibered components: $S^1 \times S^1 \times I$ and $K^2 \times I$. The first case is only possible if the component $N$ is glued to itself, because we do not allow parallel tori. The torus is needed if and only if the gluing of $N$ makes $W$ a Sol manifold, which is easy to check. In the second case, $K^2 \times I$ only has one torus boundary component, so its refibration does not refer any other tori. Every circle fibration of its boundary extends to a circle fibration of its interior, so we know that we can erase $T$ if we see this component on either side or on both sides.

4.4. Proof of Corollary 4.9

The idea of the proof is that we can convert a first-order deformation of a triangulation of $N^*$ to a deformation of a representation of $\rho$, with the same raw that we can convert a triangulation to $\rho$ in the first place.

Proof. In general, if $\Gamma$ is a discrete group (such as the fundamental group of a topological space) and $G$ is a Lie group, then we can describe a first-order deformation of a homomorphism

$$\rho : \Gamma \to G$$

as a homomorphism

$$\rho' : \Gamma \to G \rtimes \mathfrak{g}.$$ 

Here $\mathfrak{g}$ is the Lie algebra of $G$ viewed as a group under addition, while $G \rtimes \mathfrak{g}$ is the semidirect product in which the non-normal subgroup $G$ acts on the normal subgroup $\mathfrak{g}$ by conjugation. Also, $(\rho, \rho')$ should reduce to $\rho$ under the quotient map

$$\pi : G \rtimes \mathfrak{g} \to G.$$ 

Note that $G \rtimes \mathfrak{g}$ is also the total space of the tangent bundle $TG$. In other words, the extension $\rho'$ is a choice of a tangent vector $\rho'(g) \in T_{\rho(g)}G$ for every $g \in \Gamma$, such that the pairs $(\rho(g), \rho'(g))$ together make a group homomorphism.

Suppose that $\Gamma = \Gamma(X)$ is the fundamental group of a based CW complex $X$. Then we can model $\rho$ (non-uniquely) as a non-commutative cellular cocycle $\alpha \in C^1(X; G)$. Given $\rho$, we can likewise model the extension $\rho'$ (also non-uniquely) as a commutative cocycle $\alpha' \in C^1(X; \mathfrak{g})$, where here $\mathfrak{g}$ is a coefficient system twisted by $\alpha$.

Now let $X = N$, where $N$ has a cellulation $\tau$ that comes from a closed or cusped hyperbolic structure on $N^*$ and a geometric triangulation $\tau^*$. If $N$ is cusped and $\tau^*$ is a semi-ideal triangulation, then we make $\tau$ by truncating the ideal vertices of $\tau^*$. We then want to make a cocycle $\alpha$ from $\gamma$. To do this, we first choose a specific isometry $\tilde{N}^* \cong \mathbb{H}^3$. Then we choose an orthonormal tangent frame at each vertex of $\tau$. Given an edge $e \in \tau$, we let $\alpha(e)$ be the element of $G = \text{Isom}^+(\mathbb{H}^3)$ that takes the tail $\tilde{e}$ of a lift $\tilde{v}$ to the head $\tilde{v}$, and takes the lifted frame of $\tilde{v}$ to the lifted frame of $\tilde{v}$. If $\tau^*$ is cusped, then we require that each truncation edge in $N$ is assigned a parabolic element that fixes the corresponding ideal vertex in $\tau^*$.

In this setting, Theorem 4.8 says that $H^1(N; \mathfrak{g}) = 0$ in the closed case and $H^1(N, \partial N; \mathfrak{g}, p) = 0$ in the cusped case, where $\mathfrak{g}$ is the parabolic Lie subalgebra of $\mathfrak{g}$. The theorem is typically proved using de Rham cohomology rather than cellular cohomology, but these models of cohomology are isomorphic as usual. More explicitly, every 1-cocycle $\alpha' = \delta \beta$, where $\beta$ is a $\mathfrak{g}$-valued 0-cochain on the vertices of $\tau$.

Finally, suppose that $\gamma$ is a first-order deformation of the hyperbolic structure $\gamma$ of $\tau^*$ that satisfies the first derivative of the gluing equations. Then we can lift $\gamma$ to a cocycle $\alpha'$ (non-uniquely) in the same way that $\gamma$ lifts to $\alpha$. Then Theorem 4.8 provides $\beta$, and $\beta$ descends to a first-order motion of the vertices of $\tau^*$ that induces the deformation $\gamma$.

5. HOMEOMORPHISM IS RECURSIVE

We will prove Theorem 1.1 in this section, assuming Theorem 5.2 below. Suppose that $M_1$ and $M_2$ are two closed 3-manifolds given by triangulations. By Theorem 4.4, we know the direct sum decompositions of each one into prime 3-manifolds. These summands can be freely permuted and can only be matched in finitely many ways. If we search over the ways to match them, we then reduce the homeomorphism problem $M_1 \cong M_2$ to the homeomorphism problem $W_1 \cong W_2$ for prime, oriented summands $W_1$ and $W_2$. The only case which is not addressed by this reduction is the one in which $M_1$ or $M_2$ is $S^3$ and therefore has no prime summands. As explained in Section 4.2, we can recognize $S^3$ by any canonical triangulation such as $\partial \Delta^4$.

If $W$ is a prime 3-manifold, then its JSJ decomposition is modelled by a labelled graph $\Gamma$, whose vertices represent JSJ components and whose edges represent connecting tori. (This graph structure inspired the term “graph manifold” for a manifold whose components are all Seifert-fibered [50].) Geometrization shows that the same graph concept is important for all prime 3-manifolds.) Each vertex is labelled by the homeomorphism type of component, which is either Seifert-fibered or hyperbolic.

The labelled graph $\Gamma$ is an invariant of $W$, which at first glance may seem like a complete invariant, provided that the homeomorphism problem for a JSJ component is recursive. However, it is not that simple, because we have to know the allowed permutations of the torus boundary components of a JSJ component $N$, and the allowed homeomorphisms of each torus boundary component. Theorems 5.1 and 5.2 stated here provide this extra information.

Theorem 5.1 (SWJSJ [16, Ch. VI]). Suppose that $N$ is an oriented, Seifert-fibered 3-manifold which is either closed or compact with torus boundary components. Suppose that $N$ is neither a solid torus nor a thickened torus. Then:

1. The oriented homeomorphism type of $N$ is recursive.
2. If $N$ has boundary, then its boundary components can be freely permuted by orientation-preserving homeomorphisms.

3. Suppose that $N$ has boundary, that $N \not\cong K^2 \times I$, and that $f : N \to N$ is an orientation-preserving homeomorphism that sends every boundary torus to itself. The matrices of the mapping class group elements induced by $f$ are all of the form

$$\begin{pmatrix} \lambda & n \\ 0 & \lambda \end{pmatrix},$$

where $\lambda \in \{ \pm 1 \}$.

4. Suppose that $N$ and $f$ are as in claims 3. If the Seifert base of $N$ is orientable, then $\lambda$ is the same for every boundary component, and the case $\lambda = -1$ with $n = 0$ for every boundary component is achievable. If the Seifert base is not orientable, then $n = 0$, and $\lambda$ is unrestricted for every boundary component.

5. If $N \cong K^2 \times I$, then it has a distinguished Seifert fibration whose base is a disk with two singular fibers. Using this fibration, claims 3 and 4 apply.

**Theorem 5.2.** Suppose that $N$ is an oriented, hyperbolic 3-manifold which either closed or cusped. Then the mapping class group of $N$ is its isometry group. It is a finite group and its computation is recursive. If $N_1$ and $N_2$ are two such manifolds, then they are homeomorphic if and only if they are isometric, and recognizing this condition is recursive.

Theorem 5.2 will be proven in the next section. Otherwise, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** As explained at the beginning of this section, it suffices to solve the homeomorphism problem $W_1 \cong W_2$ for prime 3-manifolds $W_1$ and $W_2$.

Let $\Gamma_1$ and $\Gamma_2$ be the decorated graphs that describe their JSJ decompositions. These graphs and their decorations are recursive. If any component is a $K^2 \times I$, then we specifically use the Seifert fibration in case 5. This reduces the homeomorphism problem $W_1 \cong W_2$ to the isomorphism problem $\Gamma_1 \cong \Gamma_2$ for decorated graphs, which by Theorems 5.1 and 5.2 is very nearly a finite search. The only infinite choice listed in Theorem 5.1 is the integer $n$. When $n \neq 0$, it means that we have a homeomorphism of a JSJ component $N$ whose restriction to a boundary torus $T$ has infinite order. If such a group element were available on both sides of $T$, then the two Seifert fibrations would match and $T$ would not be a JSJ torus.

Theorem 5.1 also excludes solid and thickened tori by hypothesis. A JSJ component $N$ cannot be a solid torus. If it is a thickened torus, then it is glued to itself to make a Sol manifold $W$. The homeomorphism problem for Sol manifolds is also routine.

**6. PROOFS OF THEOREM 5.2**

In this section we will give several proofs of Theorem 5.2. Recall that Corollary 3.4 says that the existence of a PL homeomorphism $N_1 \cong N_2$ is in RE; it is also easy to check whether it preserves orientation. So, by Proposition 2.3, it suffices to show that homeomorphism is in coRE, although only one of the proofs will make use of this directly. By a similar argument, finding elements in the mapping class group of a single $N$ is in RE; the remaining task is an algorithm to show that the list is complete.

Recall that if $N$ has boundary, then its interior $N^*$ is cusped and has a semi-ideal triangulation $\tau^*$. In this case, $\tau$ is a cellulation in which semi-ideal tetrahedra are once truncated. We want to geotemrize the truncation that produces $\tau$. We consider a horospheric truncation which is almost but not quite unique, with the following three properties:

1. The horosphere sections lie entirely within the semi-ideal tetrahedra of $\tau^*$, and therefore do not intersect each other.

2. For some common integer $n$, every horospheric torus has area $2^{-n}$.

3. We do not use the smallest value of $n$ that satisfies conditions 1 and 2.

For convenience, we let $N^* = N$ and $\tau^* = \tau$ if $N$ is closed.

Some of the proofs make use of the following lemma.

**Lemma 6.1.** It is recursive to obtain a lower bound in the injectivity radius of $N$ and $\tau$.

**First proof.** Suppose first that $N$ is closed. For each vertex $v \in \tau$, let $\alpha_v$ be the open star of $\tau$ containing $p$. Then the collection $\{ \alpha_v \}$ is a finite open cover of $N$. It follows just from topology that there is some radius $\varepsilon$ such that every ball of radius $\varepsilon$ is contained in some $\alpha_v$. For an explicit calculation, let $\tau'$ be a barycentric subdivision of $\tau$, and for each $v \in \tau$, let $\kappa_v$ be the closed star of $v$ in $\tau'$, then the sets $\kappa_v$ are a closed cover. We can calculate or bound the distance from $\kappa_v$ to $N \setminus \sigma_v$ for some $w \in \tau$ with $\kappa_v \subseteq \alpha_w$. The minimum of all of these distances is thus a lower bound $\varepsilon$ for the injectivity radius.

**Second proof.** In general we use the notation $B(p, r)$ for a hyperbolic ball of radius $r$ centered at $p$.

Let $r$ be the exact injectivity radius of $N$, and let $p$ be a point on a closed geodesic of $N$ of length $2r$. Then $p \in \Delta$ for some cell $\Delta \in \tau$, and we can let $\ell$ be an upper bound of the diameter of $\Delta$. Then in the universal cover

$$\widetilde{N} \subseteq \widetilde{N^*} \cong \mathbb{H}^3,$$

we obtain that at least $1/2r$ lifts of $\Delta$ intersect $B(p, 1)$, and thus at least this many copies of $\Delta$ are contained in $B(p, \ell + 1)$. Thus

$$\frac{1}{2r} \leq \frac{\text{Vol}(B(p, \ell + 1))}{\text{Vol}(\Delta)}.$$
hence
\[ r > \frac{\text{Vol}(\Delta)}{2\text{Vol}(B(p, \ell + 1))}. \]  
(2)

We can calculate an upper bound of this form, if necessary using a lower bound for the numerator and an upper bound for the denominator, for every cell in \( \tau \), since we do not know the position of the shortest geodesic loop in advance.

**Third proof.** This proof is a variation of the second proof using the entire diameter and volume of \( N \). Jørgensen and Thurston proved that the set of possible volumes of \( N^* \) is well-ordered. In particular, there is one of least volume, so there is some constant \( c > 0 \) such that
\[ \text{Vol}(N^*) > c. \]

Our construction of the geometry of \( N \) spares more than half of the volume of \( N^* \), so
\[ \frac{\text{Vol}(N^*)}{2} > \frac{c}{2} = c'. \]

We can obtain an upper bound \( \ell \) on the diameter of all of \( N \) by adding bounds on the diameters of the cells in \( \tau \). Then, we let \( D \) be a convex fundamental domain for \( N \); it has the same volume and diameter at most \( 2\ell \). Thus we obtain an estimate similar to (2), but more robust:
\[ r > \frac{\text{Vol}(D)}{2\text{Vol}(B(p, \ell + 1))} > \frac{c'}{2\text{Vol}(B(p, \ell + 1))}. \]

**Remark.** Without an explicit bound on least-volume closed or cusped hyperbolic manifold, the third proof has the unusual feature of non-constructively proving that an algorithm exists, *i.e.*, without fully stating the algorithm. Meyerhoff [27] established the first lower bound
\[ \text{Vol}(N) \geq \frac{2}{5\pi} \]
in the closed case. In the same paper, he and Jørgensen established
\[ \text{Vol}(N^*) \geq \frac{\sqrt{3}}{4} \implies \text{Vol}(N) \geq \frac{\sqrt{3}}{8} \]
in the cusped case. The exact minimum values are now known [9].

**First proof of Theorem 5.2.** This proof is similar to one given by Scott and Short [43]. We assume geometric triangulations \( \tau^*_1 \) and \( \tau^*_2 \) of \( N^*_1 \) and \( N^*_2 \).

If \( N^*_1 \) and \( N^*_2 \) are homeomorphic and therefore isometric, then we can intersect the tetrahedra of \( \tau^*_1 \) and \( \tau^*_2 \) to make a tiling of \( N_1 \cong N_2 \) by various convex cells with 8 or fewer sides; we can then take a barycentric subdivision to make tetrahedra. We thus obtain a mutual refinement \( \tau_1 \) of \( \tau_1 \) and \( \tau_2 \). If we can bound the complexity of \( \tau_1 \), then we can find it with a finite search or show that it does not exist, rather than using stellar or bistellar moves in both the up and down directions.

Let \( \Delta_1 \in \tau^*_1 \) and \( \Delta_2 \in \tau^*_2 \) be two tetrahedra in the separate triangulations. In the universal cover \( \tilde{N}_1 \), they can only intersect in a single cell with at most 8 sides. In \( N^*_1 \) itself they can intersect many times; however, only as often as different lifts of \( \Delta_1 \) intersect one fixed lift of \( \Delta_2 \). If \( \Delta_1 \) and/or \( \Delta_2 \) are semi-ideal, then their lifts intersect if and only if their truncations do. There is a recursive volume bound on the number of possible intersections by the same argument as the second proof of Lemma 6.1.

Having bounded the necessary complexity of a mutual refinement \( \tau_1 \), we can now search over separate refinements \( \tau_1 \) of \( \tau_1 \) and \( \tau_2 \) of \( \tau_2 \) using Proposition 3.5, and look for an orientating-preserving simplicial isomorphism \( \tau_1 \cong \tau_4 \). The same method can be used to calculate the mapping class group of a single \( N \).

**Second proof.** Suppose that \( X_1 \) and \( X_2 \) are two compact metric spaces, and suppose that for each \( \varepsilon > 0 \) we have a way to make finite \( \varepsilon \)-nets \( S^1 \) and \( S^2 \) for \( X_1 \) and \( X_2 \), and calculate or approximate all distances within \( S^1 \) and within \( S^2 \). If \( X_1 \) and \( X_2 \) are isometric, then there is a function \( f : S^1 \to S^2 \) that changes distances by at most \( 2\varepsilon \). On the other hand, if there is such a function for every \( \varepsilon \), then \( X_1 \) and \( X_2 \) must be isometric.

In our case, we let \( X_4 = N_4 \), where we make sure to use the same truncation area \( 2^{-n} \) to geometrize \( N_1 \) and \( N_2 \) given the geometries of \( N^*_1 \) and \( N^*_2 \). We calculate a common lower bound \( \delta \) on the injectivity radius.

We can choose some convenient coordinates inside each cell \( \Delta \in \tau_1 \). We then have the ability to calculate geodesic segments in \( N_1 \) that are made of geodesic segments in the separate tetrahedra. If \( \Delta \) is truncated, then the geodesic segment might hug the truncation boundary for part of its length, but it still has a finite description. Without more work, we don’t know which of these geodesics are shortest geodesics. However, if a geodesic is shorter than \( \delta \), then it is shortest. Taking \( \delta \gg \varepsilon \to 0 \), we can make \( \varepsilon \)-nets of both \( N_1 \) and \( N_2 \) and look for approximate isometries between these \( \varepsilon \)-nets; it suffices to check distances below the fixed value \( \delta \).

More explicitly, we can use the covering by open stars \( S^r \) in the first proof of Lemma 6.1. There is a \( \delta \) such that if \( d(x, y) < \delta \), then \( x \) and \( y \) and even the connecting short geodesic are all in some open star.

This algorithm does not by itself ever prove that \( N_1 \) and \( N_2 \) are isometric, only that they aren’t. Thus it shows that the homeomorphism problem is in coRE. This is good enough by Proposition 2.3 and Corollary 3.4.

The algorithm also does not by itself determine whether the isometry is orientation-preserving. However, this is very little extra work. Given \( \varepsilon \ll \delta \) and given \( \varepsilon \)-nets \( S_1 \) and \( S_2 \), we can let \( A \) be a set of 4 points inside of a ball of radius \( \delta/2 \) in \( N_1 \) that make an approximate regular tetrahedron. If \( f : S_1 \to S_2 \) is an approximate isometry, then we can check whether \( f \) flips over the points of \( A \). If no orientation-preserving isometry exists, then when \( \varepsilon \) is small enough, either \( f \) will cease to exist or \( A \) will be flipped over.

We can use similar methods to find the mapping class group.
of a single \( N \), since by Mostow rigidity it is also the isometry group of \( N \). We assume that \( N \) has boundary, which is technically short of the full generality of Theorem 5.2, but enough to prove Theorem 1.1. Just as with the method to check whether and approximate \( f \) preserves orientation, we can when \( \varepsilon \) is small enough compute the effect of \( f \) on \( H_1(\partial N) \), which determines which isometry is close to \( f \) (if any).

Third proof. In this proof, we work over the ring \( \hat{\mathbb{Q}} \) of real algebraic numbers rather than using numerical approximations. We assume real algebraic coordinates for \( \mathbb{H}^3 \) and for its isometry group \( \text{Isom}^+(\mathbb{H}^3) \); for example we can take \( \mathbb{H}^3 \) to be the set of positive, unit timelike vectors in \( 3 + 1 \)-dimensional Minkowski geometry, and we can take \( \text{Isom}^+(\mathbb{H}^3) = \text{ISO}(3, 1) \). We again assume that \( N_1 \) and \( N_2 \) are made from \( N_1^* \) and \( N_2^* \) using a common truncation area \( 2^{-n} \).

We assume geometric triangulations \( \tau_1^* \) and \( \tau_2^* \) of \( N_1^* \) and \( N_2^* \) with real algebraic descriptions. Using these triangulations, we can find finite, open coverings of \( N_1 \) and \( N_2 \) by metric balls \( B(p, \varepsilon) \), where each point \( p \) has a real algebraic position and the common radius is (a) also real algebraic, and (b) less than half of the injectivity radius of \( N_1 \) and \( N_2 \). Then we can give each ball the same algebraic coordinates as \( \mathbb{H}^3 \), and we can also calculate the relative position of every pair of balls as some element in \( \text{Isom}^+(\mathbb{H}^3) \). In other words, we obtain atlases of charts for \( N_1 \) and \( N_2 \) using the \( \text{Isom}^+(\mathbb{H}^3) \) pseudogroup. In fact, everything is constructed in the subgroup and subpseudogroup with real algebraic matrix entries.

If there is an isometry between \( N_1 \) and \( N_2 \), then their atlases combine into a larger atlas. There are only finitely many possible patterns of intersection between the balls of \( N_1 \) and the balls of \( N_2 \). For each such pattern, we obtain a finite system of algebraic equalities and inequalities, which says first that the intersection pattern is what is promised, and second that the gluing maps between the atlases are consistent. Theorem 2.8 then says that it is recursive to determine whether this system of equations has a solution. Since we work in the group \( \text{Isom}^+(\mathbb{H}^3) \), we are looking only for orientation-preserving isometries.

7. HOMEOMORPHISM IS IN ER

In this section we will prove Theorem 1.2. The proof generally follows that of Theorem 1.1, with computational improvements from various results that are based on two main other tools. The first tool is normal surface theory, which we can use to find essential spheres and tori and Seifert fibrations. The first part of the proof based on the first tool was previously sketched by Jaco, Letscher, and Rubinstein [17]. The second tool is an ER version of Theorem 2.8 [10], which we can use to bound the complexity of a geometric triangulation of a hyperbolic manifold.

We use the basic fact that a finite composition of ER functions is in ER. In other words, if an algorithm has a bounded number of stages that expand its data by an exponential or otherwise by an ER amount, then it is still in ER.

7.1. Normal surfaces

A central idea of normal surface theory due to Haken is to reduce questions in 3-manifold topology to integer (linear) programming. In full generality, if \( C \subseteq \mathbb{R}^n \) is a rational polytope, then a solution to the integer programming problem defined by \( C \) is a point in \( C \cap \mathbb{Z}^n \). Suppose now that \( C \) is also a proper, closed cone whose apex is \( 0 \). (\( C \) is proper means that \( C \cap -C = \{0\} \).) Then a solution \( v \in C \cap \mathbb{Z}^n \) is fundamental if it is not the sum of two other solutions.

Proposition 7.1. If \( C \) is a rational, closed, proper polytopal cone, then there is an exponential (and therefore ER) bound on the norm of any fundamental solution.

Let \( M \) be a closed, oriented 3-manifold with triangulation \( \tau \). Recall that a normal surface \( S \subseteq M \) intersects each tetrahedron \( \Delta \in \tau \) in 7 types of elementary disks, namely 4 types of triangles and 3 types of quadrilaterals. The surface \( S = S_v \) is given by a vector \( v \in \mathbb{Z}^2_{\leq 0} \) that lists the number of each type of elementary disk. If \( v \) is such a vector, then \( S_v \) is embedded (and uniquely defined) provided that it only uses at most one type of quadrilateral in each tetrahedron. The normal surface equations have a polytopal cone \( C \) of solutions. Then a fundamental surface is one whose vector is a fundamental solution. Sometimes a fundamental surface \( S_v \) which is non-orientable is excluded and replaced by its oriented double cover \( S_{2v} \), but we can allow non-orientable normal surfaces at this state of definitions.

We can represent a normal surface \( S \) by listing all triangles and quadrilaterals in order in each tetrahedron \( \Delta \in \tau \). It is then easy to separate \( S \) into connected components and calculate the topology of each component. This is exponentially inefficient compared to algorithms such as Agol-Hass-Thurston [2], but this is livable for our purposes.

Theorem 7.2 (Rubinstein [39], Thompson [48]). Recognizing the 3-sphere \( S^3 \) is in ER.

The proof of Theorem 7.2 uses a variant known as almost normal surfaces that are allowed one exceptional intersection with a tetrahedron that is either an octagon, or a triangle and a quadrilateral with a connecting annulus. The original papers only claim a recursive algorithm, but the algorithm is based on normal surface theory and an ER bound is immediate. Indeed, Schleimer [41] shows that 3-sphere recognition is in the complexity class NP, which is a much better bound than just ER.

Recall that a Seifert fibered space is small if it is non-Haken (and therefore closed). Small Seifert fibered spaces are a tricky special case that must be handled separately.

Theorem 7.3 (Li [24]). Recognizing small Seifert fibered spaces is in ER.

Li describes his algorithm as merely recursive, but it is clear from each step of his proof that it is in ER. Rubinstein [40] also describes an algorithm to identify small Seifert fibered spaces which could be in ER. However, he appeals to SnapPea, and thus implicitly to Lemma 4.6 and Theorem 5.2, to analyze certain hyperbolic manifolds that arise in his algorithm.
We use Li’s algorithm since it is a self-contained approach using a version of normal surface theory.

The next two steps are to find essential spheres and tori in $M$ to find its prime factorization and then its JSJ decomposition. There are existing algorithms for both decompositions [12, 18, 42], but we appeal to a variation due to Joel Hass and the author.

**Theorem 7.4** ([14]). Let $M$ be a closed, oriented 3-manifold with a triangulation $\tau$. It is in ER to find a maximal collection $\{S\}$ of disjoint normal spheres and projective planes, such that no two are equivalent as normal surfaces. The collection $\{S\}$ includes a maximal collection of essential surfaces. It is then in ER to find a maximal collection $\{T\}$ of disjoint, essential normal tori and Klein bottles in the complement of $\{S\}$, again such that no two are equivalent.

Briefly, Theorem 7.4 uses a generalization of the normal surface equations which we call the disjoint normal surface equations. (They are similar to the crushed triangulation technique defined by Casson [17].) They are equations for a normal surface $S$ which is disjoint from a fixed normal surface $R \subseteq M$. To prove Theorem 7.4, we find each surface $S$ or $T$ sequentially as a fundamental surface, relative to the union of the previous surfaces.

**Theorem 7.5.** It is in ER to find and triangulate the prime summands $\{W\}$ of $M$, to find and triangulate the JSJ components $\{N\}$ within each prime summand $W$, to find their graph $\Gamma$, and to recognize which components $N$ are Seifert fibered and find their fibrations.

**Proof.** We apply Theorem 7.4 to $M$ with its triangulation $\tau$ to obtain a superset of a maximal set of essential spheres. We can ignore the projective planes, since their orientable double covers are automatically present. Then we can cut along all of the spheres (and retriangulate) and cap them to make a multiset summands of $M$. For each non-separating sphere, we create a separate $S^2 \times S^1$ summand. We then use Theorem 7.2 to determine which of these summands are 3-spheres, and discard them; this includes the case where everything is discarded and $M \cong S^3$. What remains is the prime factorization $\{W\}$.

We apply Theorem 7.3 to each summand $W$ to determine if it is small Seifert fibered, and to recognize its topology. In the remainder, we work with each summand $W$ separately and we assume that it is not small Seifert fibered.

The summand $W$ has tori and Klein bottles from the original application of Theorem 7.4 to $M$. We can discard the Klein bottles since their orientable double covers are automatically present. We can then apply Theorem 7.4 once more to find disks of compression for any of the tori, and discard those tori. The remaining tori divide $W$ into a collection $\{N\}$ of complementary regions. We also record the gluing data to make a graph $\Gamma'$ which is similar to the JSJ graph $\Gamma$ defined in Section 5.

The decomposition $\{N\}$ is a refinement of the JSJ decomposition. Each component $N$ is atoroidal: it is either hyperbolic and a JSJ component, or it is Seifert fibered. If $N$ is Seifert fibered and atoroidal, then it has a Haken hierarchy that consists of either one or two essential annuli and then an essential disk. We can apply Theorem 7.4 twice more to find this hierarchy and explicitly calculate a Seifert fibration of $N$. If $N$ does not have an essential annulus, then it is hyperbolic and a JSJ component.

The remaining task is discard tori to discard tori and glue components $N$ until we have the JSJ decomposition. If a component $N$ is a thickened torus $S^1 \times S^1 \times I$, then we discard it unless it is glued to itself. Otherwise, if a torus $T = T_1$ is fibered on each side, then it is routine to check whether the fibrations match. The criterion of Proposition 4.11 is also easy to check.

**7.2. Algebraic bounds**

The topic of this section is an ER version of Lemma 4.6 and Theorem 5.2. Before proving this result itself, we use it to finish the proof of Theorem 1.2.

**Theorem 7.6.** If $N$ has a closed or cusped hyperbolic structure, then it is in ER to find a geometric triangulation. The isomorphism and automorphism problems are also both in ER.

**Proof of Theorem 1.2.** Theorem 1.1 begins with geometric recognition in Theorem 4.4. We can achieve the direct sum decomposition (Theorem 4.1) and JSJ decomposition (Theorem 4.2) using normal surface theory with Theorems 7.2, 7.3, and 7.4; these results also replace Lemma 4.10. We can then replace Lemma 4.6 and Theorem 5.2 with the ER version Theorem 7.6. Then algorithm given in the proof of Theorem 1.1, based in part on the statement of Theorem 5.1, is in ER.

**Theorem 7.7** (Collins, Monk, Vorobiev-Gorjor, Wüthrich [10, Thm. 4]). Suppose that a set $S \subseteq \mathbb{R}^n$ is defined a finite set of polynomial equalities and inequalities over $\mathbb{Q}$. Then it is in ER to calculate a representative finite set $F \in \mathbb{Q}^n$, with one point $p \in F$ in each connected component of $S$.

Theorem 7.7 is of course an ER version of Theorem 2.8. In the statement of the theorem, each element $\alpha \in \mathbb{Q}$ described by its minimal polynomial $a(x) \in \mathbb{Z}[x]$ and an isolating interval $\alpha \in [b, c]$ that contains exactly one root of $a(x)$.

The following result is immediate as a mutual corollary of Theorem 7.7 and the proof of Proposition 3.5.

**Corollary 7.8.** The algorithm in Proposition 3.5 is in ER.

**Lemma 7.9.** If $\alpha \in \overline{\mathbb{Q}}$ is a non-zero complex root of a polynomial $a(x) \in \mathbb{Z}[x]$, then there is an ER upper bound on $|\alpha|$ and $|\alpha^{-1}|$.

**Proof.** Let $n = \deg a$ and write $a(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

We can assume without loss of generality that $a_0 \neq 0$. If $|\alpha| > \sum a_k$, then $|a_n \alpha^n|$ is larger than the total norm of all of the other terms, so by the triangle inequality, $a(\alpha) \neq 0$. This establishes $\sum a_k$ as an upper bound on $|\alpha|$. For the lower
Let $\tau$ be the input triangulation of $N$ as a compact manifold, and let $\tau^*$ be the result of adding a cone to each component of $\partial N$ to make a semi-ideal combinatorial triangulation of $N^*$. The manifold $N^*$ also has a hyperbolic structure which we interpret as a separate manifold. We rename the hyperbolic version of $\tau^*$ as $\tau$. Since $g$ is homotopic to $f$ (or properly homotopic of $N^*$ is not compact), it has (proper) degree 1. The simplices of $g$ may be flat or flipped over, but $g$ has degree 1 since it is homotopic to $f$. Let $\sigma$ be a geometric triangulation of $X$ formed as the barycentric subdivision of the cellulation induced by the self-intersections of $g$. Also let $\kappa = g^{-1}(\sigma)$. Then $\kappa$ is a refinement of the triangulation $\tau^*$, and $g$ is now a simplicial map from $\kappa$ to $\sigma$. See Figure 4. (The figure uses a simplicial refinement of the self-intersections which is simpler than barycentric subdivision; this is not important for the proof.)

Now suppose that we do not know the hyperbolic structure of $N$, only that it must have one because it prime, atoroidal, and acylindrical. If we are given $\kappa$ as a combinatorial refinement of the triangulation $\tau^*$, then we can search for $\sigma$ as a simplicial quotient of $\kappa$, such that we can solve the hyperbolic gluing equations for $\sigma$ to recognize it as a geometric triangulation of a hyperbolic manifold $X$. We obtain a candidate map $g : N^* \to X$. If $g$ has degree 1, and there is also a degree 1 map $h : X \to N^*$, then Mostow rigidity tells us that $g$ and $h$ are both homotopy equivalences and that $X$ and $N^*$ are homeomorphic. (Note that there can be a degree 1 map in one direction between two hyperbolic 3-manifolds that is not a homotopy equivalence [5], even though this cannot happen in the case of hyperbolic surfaces.) We can search for $h$ by the same method of simplicial subdivision that we used to find $g$. This establishes an algorithm to calculate the hyperbolic structure of $N^*$.

We claim that a modified version of this algorithm is in $ER$. We first consider $ER$ candidates for the map $g$. To do this, we make a noncommutative cocycle $\alpha \in \mathcal{C}^1(N; G)$ as in the proof of Corollary 4.9, where

$$G = \text{Isom}^+(\mathbb{H}^3) \cong \text{ISO}(3,1),$$

and with the extra restriction that $\alpha$ is parabolic on each component of $\partial N$. These cocycle equations are algebraic, so Theorem 7.7 guarantees a representative set of solutions. By Mostow or Calabi-Weil rigidity, one of components of the solution space yields a discrete homomorphism

$$\rho : \pi_1(N) \to \text{Isom}^+(\mathbb{H}^3)$$

that describes the hyperbolic geometry of $X$. If we assign some point $p \in \mathbb{H}^3$ to one of the vertices of $\tau$, then in the closed case, its orbit under $\alpha$ is in $ER$ and can be extended on each simplex of $\tau$ to the map $g$. In the cusped case, there are also ideal vertices whose position on the sphere at infinity can be calculated from $\alpha$ as well.

If $N$ has boundary, then we also want a truncated version of $\tau^*$ which is larger than the original $\tau$, and slightly different from the horospheric truncation description in Section 6. If that $\Delta \in \tau^*$ is semi-ideal, then let $p$ be its ideal vertex, let $F$ be the hyperplane containing the face of $\Delta$ opposite to $p$, and let $F'$ be the hypersphere at distance $\log(2)$ from $F$ which is on the same side as $p$. Then we truncate $\Delta$ with $F'$ to make $\Delta'$; or if $\Delta \in \tau^*$ is a non-ideal tetrahedron, we let $\Delta' = \Delta$. We let $X' \subseteq X$ be the union of all $\Delta'$. (In the closed case, we obtain $X' = X$.) $X'$ can have a complicated shape because the truncations are usually mismatched, but we can calculate the positions of its vertices, and it is easy to confirm that it has at least half of the volume of $X$.

Our algorithm does not know which cocycle $\alpha$ gives us a desired $g$ and will not compute this directly. Instead, we can calculate an $ER$ bound for its data complexity, using the complexity bounds in the statement of Theorem 7.7.

In particular, the existence of the map $g$ gives us $ER$ bounds on the parameters used in the third proof of Lemma 6.1. Using Lemma 7.9, the existence of $g$ yields an $ER$ upper bound on the diameter $\ell$ of $X'$ and then a lower bound on its injectivity radius $r$. 

Figure 4. We straighten $f$ to $g$, then subdivide the image and then the domain to make $g$ a simplicial map.

Proof of Theorem 7.6. Let $\tau$ be the input triangulation of $N$ as a compact manifold, and let $\tau^*$ be the result of adding a cone to each component of $\partial N$ to make a semi-ideal combinatorial triangulation of $N^*$. The manifold $N^*$ also has a hyperbolic structure which we interpret as a separate manifold. We rename the hyperbolic version $X$ and assume a homeomorphism

$$f : N^* \to X.$$ 

We fix the vertices of $N^*$ in the map $f$, and straighten all of the tetrahedra, to make a map $g$ that represents $\tau^*$ as a self-intersecting geometric triangulation of $X$. Since $g$ is homotopic to $f$ (or properly homotopic of $N^*$ is not compact), it has (proper) degree 1. The simplices of $g$ may be flat or flipped over, but $g$ has degree 1 since it is homotopic to $f$. Let $\sigma$ be a geometric triangulation of $X$ formed as the barycentric subdivision of the cellulation induced by the self-intersections of $g$. Also let $\kappa = g^{-1}(\sigma)$. Then $\kappa$ is a refinement of the triangulation $\tau^*$, and $g$ is now a simplicial map from $\kappa$ to $\sigma$. See Figure 4. (The figure uses a simplicial refinement of the self-intersections which is simpler than barycentric subdivision; this is not important for the proof.)

Now suppose that we do not know the hyperbolic structure of $N$, only that it must have one because it prime, atoroidal, and acylindrical. If we are given $\kappa$ as a combinatorial refinement of the triangulation $\tau^*$, then we can search for $\sigma$ as a simplicial quotient of $\kappa$, such that we can solve the hyperbolic gluing equations for $\sigma$ to recognize it as a geometric triangulation of a hyperbolic manifold $X$. We obtain a candidate map $g : N^* \to X$. If $g$ has degree 1, and there is also a degree 1 map $h : X \to N^*$, then Mostow rigidity tells us that $g$ and $h$ are both homotopy equivalences and that $X$ and $N^*$ are homeomorphic. (Note that there can be a degree 1 map in one direction between two hyperbolic 3-manifolds that is not a homotopy equivalence [5], even though this cannot happen in the case of hyperbolic surfaces.) We can search for $h$ by the same method of simplicial subdivision that we used to find $g$. This establishes an algorithm to calculate the hyperbolic structure of $N^*$.

We claim that a modified version of this algorithm is in $ER$. We first consider $ER$ candidates for the map $g$. To do this, we make a noncommutative cocycle $\alpha \in \mathcal{C}^1(N; G)$ as in the proof of Corollary 4.9, where

$$G = \text{Isom}^+(\mathbb{H}^3) \cong \text{ISO}(3,1),$$

and with the extra restriction that $\alpha$ is parabolic on each component of $\partial N$. These cocycle equations are algebraic, so Theorem 7.7 guarantees a representative set of solutions. By Mostow or Calabi-Weil rigidity, one of components of the solution space yields a discrete homomorphism

$$\rho : \pi_1(N) \to \text{Isom}^+(\mathbb{H}^3)$$

that describes the hyperbolic geometry of $X$. If we assign some point $p \in \mathbb{H}^3$ to one of the vertices of $\tau$, then in the closed case, its orbit under $\alpha$ is in $ER$ and can be extended on each simplex of $\tau$ to the map $g$. In the cusped case, there are also ideal vertices whose position on the sphere at infinity can be calculated from $\alpha$ as well.

If $N$ has boundary, then we also want a truncated version of $\tau^*$ which is larger than the original $\tau$, and slightly different from the horospheric truncation description in Section 6. If that $\Delta \in \tau^*$ is semi-ideal, then let $p$ be its ideal vertex, let $F$ be the hyperplane containing the face of $\Delta$ opposite to $p$, and let $F'$ be the hypersphere at distance $\log(2)$ from $F$ which is on the same side as $p$. Then we truncate $\Delta$ with $F'$ to make $\Delta'$; or if $\Delta \in \tau^*$ is a non-ideal tetrahedron, we let $\Delta' = \Delta$. We let $X' \subseteq X$ be the union of all $\Delta'$. (In the closed case, we obtain $X' = X$.) $X'$ can have a complicated shape because the truncations are usually mismatched, but we can calculate the positions of its vertices, and it is easy to confirm that it has at least half of the volume of $X$.

Our algorithm does not know which cocycle $\alpha$ gives us a desired $g$ and will not compute this directly. Instead, we can calculate an $ER$ bound for its data complexity, using the complexity bounds in the statement of Theorem 7.7.

In particular, the existence of the map $g$ gives us $ER$ bounds on the parameters used in the third proof of Lemma 6.1. Using Lemma 7.9, the existence of $g$ yields an $ER$ upper bound on the diameter $\ell$ of $X'$ and then a lower bound on its injectivity radius $r$. 

Figure 4. We straighten $f$ to $g$, then subdivide the image and then the domain to make $g$ a simplicial map.
We can now follow the first proof of Theorem 5.2. If $\Delta_1, \Delta_2 \in \tau$ are two tetrahedra, then the intersection complexity of $g(\Delta_1)$ and $g(\Delta_2)$ is no worse than that of $g(\Delta'_1)$ and $g(\Delta'_2)$, and is bounded by an ER function of $\ell$ and $r$. This yields an ER bound on the complexity of the refinements $\sigma$ and $\kappa$. Recall that $\kappa$ is a refinement of $\tau$, which is a slightly modified version of the input description of $N$. Having bounded the complexity of $\kappa$, we can search for it using Corollary 7.8 and solve for $\sigma$ and its geometry. We can also discard $g$ if it does not have degree 1.

Thus far, the algorithm finds an ER collection of candidate maps $g : X^* \to N$ of degree 1, where $N$ varies as well as $g$. At least one of these maps is a homotopy equivalence. Instead of finding an inverse $h$, we can repeat the algorithm to look for degree one maps among the target manifolds $\{N\}$. This induces a transitive relation among these manifolds. If $N$ is chosen at the top of this relation, then the associated map $g : X^* \to N$ must be a homotopy equivalence.

To solve the isomorphism problem, we find geometric triangulations $\sigma_1$ and $\sigma_2$ of the manifolds $N_1$ and $N_2$. We can again follow the first proof of Theorem 5.2, except now with an ER bound on the complexity of $\sigma_1$ and $\sigma_2$, and we can again use Corollary 7.8. The same argument applies for the calculation of the isometry group of a single $N^*$.

8. OPEN PROBLEMS

Theorem 1.2, together with the fact that ER is a fairly generous complexity class, suggests the following conjectures.

**Conjecture 8.1.** If $M$ is a closed, Riemannian 3-manifold, then Ricci flow with surgery on $M$ can be accurately simulated in ER.

In other words, we conjecture that Perelman’s proof of geometrization can be placed in ER.

**Conjecture 8.2.** Every closed, hyperbolic manifold $N$ has a finite-sheeted Haken covering, with an ER bound on the number of sheets.

In other words, we conjecture that the statement of the virtual Haken conjecture, now the theorem of Agol et al [1], can be placed in ER. Maybe the known proof can be as well.

**Conjecture 8.3.** Any two triangulations of a closed 3-manifold $M$ have an ER mutual refinement.

Conjecture 8.3 does not follow from our proof of Theorem 1.2, because the algorithm in Theorem 7.6 only establishes a simplicial homotopy equivalence and then relies on Mostow rigidity. However, the rest of the proof of Theorem 1.2 uses a bounded number of normal surface dissections, which does establish an ER mutual refinement according to the arguments of Mijatović [28, 29]. Also, Conjecture 8.2 and the Haken case of Conjecture 8.3 would together imply the hyperbolic case of Conjecture 8.3, which would then imply the full conjecture. Mijatović [30] also established that any two triangulations of a fiber-free Haken 3-manifold have a primitive recursive mutual refinement.

Cases 3 and 4 of Proposition 2.6 are expected to be false for typical bounds on complexity that are better than ER. Thus, in discussing further improvements to Theorem 1.2, we should consider qualitative complexity classes, such as the famous NP, rather than just bounds on execution time. For one thing, ER is the union of an alternating, nested sequence of time and space complexity classes, as follows:

$$P \subseteq \text{PSPACE} \subseteq \text{EXP} \subseteq \text{EXPSPACE} \subseteq \text{EEXP} \subseteq \cdots.$$ 

Here $P$ is the set of decision problems that can be solved in deterministic polynomial time; PSPACE is solvability in polynomial space with unrestricted (but deterministic) computation time; EXP is deterministic time $\exp(\text{poly}(n))$; etc. The author does not know where a careful version of our proof of Theorem 1.2 would land in this hierarchy.

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