Graph-Based Modeling and Decomposition of Energy Infrastructures

Sungho Shin ∗ Carleton Coffrin ∗∗ Kaarthik Sundar ∗∗ Victor M. Zavala ∗∗∗

∗ University of Wisconsin-Madison, Madison, WI 53706 USA
(e-mail: sungho.shin@wisc.edu; victor.zavala@wisc.edu).
∗∗ Los Alamos National Laboratory, Los Alamos, NM 87545 USA
(e-mail: cjc@lanl.gov; kaarthik@lanl.gov)
∗∗∗ Argonne National Laboratory, Argonne, IL 60439 USA

Abstract: Nonlinear optimization problems are found at the heart of real-time operations of critical infrastructures. These problems are computationally challenging because they embed complex physical models that exhibit space-time dynamics. This work proposes to model these problems as graph-structured optimization problems; such structures are exploited at the modeling level to parallelize function/derivative computations and at the solver level to parallelize linear algebra operations (by using a Schwarz decomposition scheme). The proposed approach is implemented as a general-purpose nonlinear programming solver (that we call MadNLP.jl) that is interfaced to the graph-based modeling package Plasmo.jl. The efficiency of this framework is demonstrated via problems arising in transient gas network optimization and multi-period AC optimal power flow. We show that our framework reduces solution times (compared with off-the-shelf tools) by over 300% (solution times are reduced from 72.36 sec to 23.84 sec for the gas problem and from 515.81 sec to 149.45 sec for the power flow problem).

Keywords: Nonlinear Optimization, Decomposition, Graphs, Energy Systems

1. INTRODUCTION

The real-time operation of modern energy infrastructures requires the solution of large-scale nonlinear programs (NLPs). Important application examples include transient gas network optimization (Sundar and Zlotnik, 2018) and multi-period optimal power flow problems (Geth et al., 2020; Kim and Anitescu, 2020). Achieving real-time solutions for such problems is challenging, as these problems embed complex physical models that require space-time discretization. NLPs arising in this context can easily reach millions of variables and constraints and defy the scope of off-the-shelf solvers. Specifically, scalability bottlenecks are often encountered at the modeling level (function and derivative computations) and at the solver level (computation of the search step).

Large-scale NLPs arising in infrastructures have the key characteristic that they exhibit structures in the form of sparsely connected graphs; we refer to such problems as graph-structured optimization problems (Walther and Bielski, 2019; Shin et al., 2020b; Jalving et al., 2020). Graph-structured problems can be conveniently modeled by using specialized modeling platforms such as Plasmo.jl (Jalving et al., 2019, 2020) and solved by using structure-exploiting optimization solvers such as PIPS-NLP (Chiang et al., 2014). Plasmo.jl is a graph-based modeling platform that enables the modular construction and analysis of highly complex models; this platform also leverages the algebraic modeling capabilities of JuMP.jl (Dunning et al., 2017) and facilitates access to infrastructure modeling tools such as GasModels.jl and PowerModels.jl (Bent et al., 2020; Coffrin et al., 2018). Another key benefit of Plasmo.jl is that it can communicate model structures and this facilitates the implementation of different decomposition strategies such as the alternating direction method of multipliers (Boyd et al., 2011), overlapping Schwarz (Shin et al., 2020a), and parallel interior-point (IP) methods (Chi et al., 2014; Rodriguez et al., 2020).

In this paper, we present a new decomposition framework for graph-structured optimization problems. Our framework uses a restricted additive Schwarz (RAS) decomposition scheme implemented within a filter line-search IP method (Walther and Bielski, 2006). We present a Julia-based implementation of this approach, which we call MadNLP.jl (https://github.com/zavalab/MadNLP.jl). We use our framework to experiment with different decomposition strategies that exploit parallelism at the modeling and solver level. Specifically, we consider a scheme that parallelizes function and derivative computations by exploiting the modular structure of Plasmo.jl. We also consider a scheme that uses RAS (Cai and Sarkis, 1999) to parallelize step computations. RAS has been widely used for the solution of large linear systems arising from discretized partial differential equations (PDEs) (Baly et al., 2019) but we have recently shown that it can also be applied to solve general linear systems arising in graph-structured optimization problems (Shin et al., 2020b; Gerstner et al., 2016). Our computational results indicate that...
our proposed framework can accelerate computations by up to 300% compared with off-the-shelf tools.

The paper is organized as follows: In Section 2 we define the graph-structured problem of interest and discuss how its structure can be exploited at the modeling level. In Section 3 we discuss parallel decomposition schemes and in Section 4 we apply these schemes to transient gas network optimization and multi-period AC optimal power flow problems. Section 5 presents concluding remarks.

2. GRAPH-BASED MODELING

Optimization problems arising in energy infrastructures can be expressed as a graph-structured optimization problem of the form:

\[
\begin{align*}
\min_{x_i \in V} & \sum_{i \in V} f_i(x_i) \\
\text{s.t.} & \quad c_i^f(x_i) = 0, \quad i \in V, \quad (\lambda_i^f) \\
& \quad c_i^l((x_j)_{j \in N_G[i]}) = 0, \quad i \in V, \quad (\lambda_i^l) \\
& \quad x_i \geq 0, \quad i \in V (z_i)
\end{align*}
\]

Here, the undirected graph \( G = (V,E) \) is an ordered pair of the nonempty, strictly ordered node set \( V \) and the edge set \( E \subseteq \{i,j\} \subseteq V : i \neq j \); \( N_G[i] := \{j \in V : \{i,j\} \in E\} \cup \{i\} \) denotes the closed neighborhood of \( i \in V \) on \( G \). For each \( i \in V \), \( x_i \in \mathbb{R}^{n_i} \) is the decision variable; \( f_i(\cdot) \) is the objective function; \( c_i^f(\cdot) \) is the inner equality constraint function; \( c_i^l(\cdot) \) is the linking constraint function; \( \lambda_i^f \in \mathbb{R}^{m_i} \) is the dual variable associated with (1b); \( \lambda_i^l \in \mathbb{R}^{m_i} \) is the dual variable associated with (1c); \( z_i \in \mathbb{R}^{n_i} \) is the dual variable associated with (1d). We note that practical problems impose upper and lower bounds on the variables (e.g., \( x_i^{\ell} \leq x_i \leq x_i^{u} \)) but we use the simplified form (1d) for conciseness of the presentation.

In the context of energy infrastructures the graph \( G \) may intuitively be used as an abstraction of the space-time structure of the problem. Specifically, each node \( i \in V \) is considered as a component (e.g., a subset of buses, generators, storages, and electric lines in power networks or a subset of junctions, compressors, and pipelines in gas networks) located at a particular spatial (geographical) location and at a particular time point. The link constraints (1c) may represent spatial connections (e.g., interconnecting electric lines in power networks or pipelines in gas networks) or temporal connections. However, we will see that the graph \( G \) can also be used as a general abstraction in which each node \( i \in V \) is considered as a variable or constraint of the problem (the graph encodes the general sparsity structure of the problem). The ability to represent the same optimization problem in different forms provides flexibility to identify efficient decomposition strategies.

We define the short-hand notation for (1):

\[
\begin{align*}
\min_{x} & \quad f(x) \\
\text{s.t.} & \quad c(x) = 0 \quad (\lambda) \\
& \quad x \geq 0 \quad (z)
\end{align*}
\]

where \( x := \{x_i\}_{i \in V}, \lambda := \{[\lambda_i^f; \lambda_i^l]\}_{i \in V}, \) \( f(\cdot) := \sum_{i \in V} f_i(\cdot), \) \( c(\cdot) := \{c_i^f(\cdot); c_i^l(\cdot)\}_{i \in V}, \) \( n = \sum_{i \in V} n_i, \) and \( m = \sum_{i \in V} m_i^f + m_i^l. \) Here, \( n_i \) and \( m_i \) are the primal-dual variable dimensions, and \( \{\lambda_i\}_{i \in V} \) denotes the vector concatenation. We use boldface symbols to denote quantities associated with multiple nodes.

Each node \( i \in V \) may contain more than or less than one variable and constraint \((n_i, m_i) \in \mathbb{Z}_{\geq 0}\). Thus, the problem graph \( G = (V,E) \) may be different from the primal-dual coupling graph \( \tilde{G} = (\tilde{V},\tilde{E}) \), where \( \tilde{V} := \mathbb{Z}_{\{1,\ldots,n+m\}}, \) \( \tilde{E} := \{(i,j) : \nabla f_i(x_i,\lambda_i,z_i)[i,j] \neq 0\}. \) Here, \( \nabla (\cdot) \) is the Lagrangian of (2), and we use syntax \( \mathbb{Z}_{[a,b]} := \{a,a+1,\ldots,b\}. \) We can observe that a node \( i \in V \) corresponds to a set of nodes \( U_i \subseteq \mathbb{Z}_{\{1,\ldots,n+m\}} \) in \( \tilde{V} \), which contains multiple variables and constraints. Example graphs \( G \) and \( \tilde{G} \) for a transient gas network problem are depicted in Figure 1 (a detailed problem formulation can be found in Section 4.1). Graph \( G \) contains 24 nodes, each corresponding to a time point in a prediction horizon. Periodicity (over a 24 hours period) is enforced as constraints (this periodicity creates the cycle shape of \( G \)). In this graph, each node embeds the spatial structure of the problem (network and pipelines). Graph \( \tilde{G} \) unfolds the temporal and spatial structure and shows the interconnectivity between all variables and constraints in the problem.

3. GRAPH-BASED DECOMPOSITION

We proceed to describe our IP solver MadNLP.jl; specifically, we describe a new Schwarz decomposition scheme to exploit graph structures within an IP method and we describe its interface to Plasma.jl.

3.1 Interior-Point Method

The IP method implemented in MadNLP.jl finds the solution of (2) by solving a sequence of barrier subproblems:

\[
\min \varphi(x) := f(x) - \mu \log (e^T x) \quad \text{s.t.} \quad c(x) = 0, \quad (3)
\]

with a decreasing sequence for parameter \( \mu \). The KKT conditions for (3) give the nonlinear equations:

\[
\nabla f(x) + A^T \lambda - z = 0; \quad c(x) = 0; \quad X Z e - \mu e = 0, \quad (4)
\]

where \( A := \nabla c(x), \quad X := \text{diag}(z) \), and \( Z := \text{diag}(z) \). A solution of KKT system (4) is obtained by computing primal-dual Newton steps \( d^e \) from:

\[
\begin{align*}
\begin{bmatrix}
W + \Sigma + \delta w I & A^T \\
A & -\delta I
\end{bmatrix}
\begin{bmatrix}
d^w \\
d^e
\end{bmatrix}
= -\left( \nabla \varphi(x) + A^T \lambda \right),
\end{align*}
\]
where \( \delta_w, \delta_c > 0 \) are regularization parameters. The step \( d^* \) computed from (5) is safeguarded by a line-search filter procedure to induce global convergence (Wächter and Biegler, 2006). Typically, the solution of the linear system (5) is the most computationally challenging step in the IP method. This system is typically solved using direct linear solvers based on \( LDL^T \) factorizations (e.g., as those implemented in HSL routines (HSL, 2007)). Decomposition strategies based on Schur complements (Chiang et al., 2014) and iterative strategies (Curtis et al., 2012; Rodriguez et al., 2020) have also been proposed.

The solution of (5) based on a direct block \( LDL^T \) factorization reveals the inertia (the number of positive, zero, negative eigenvalues) of \( M \). This inertia information is crucial in determining the acceptability of the computed step and in triggering the regularization of the linear system. However, such information is not available when using iterative solvers (as that proposed in this work). In MadNLP.jl, we use an inertia-free regularization strategy to determine the acceptability of the step (Chiang and Zavala, 2016). This method performs a simple negative curvature test to trigger regularization.

### 3.2 Restricted Additive Schwarz (RAS)

We propose to use a RAS scheme to solve (5). To explain the mechanics of this method, we need some basic definitions. Consider a partition \( \{V_k\}_{k=1}^K \) of \( V \); we call \( V_k \) non-overlapping subdomains. This partition can be obtained by applying a graph partitioning scheme to \( G \). We then construct a family of overlapping subdomains \( \{V_k^\omega\}_{k=1}^K \) for each \( k = 1, \ldots, K \) (these are constructed by expanding \( V_k \)). The expansion procedure is performed by progressively incorporating adjacent nodes (Shin et al., 2020b) (the size of overlap \( \omega \) represents the expansion level). We observe that:

\[
V_k \subseteq V_k^\omega \subseteq V, \quad k = 1, \ldots, K; \quad \bigcup_{k=1}^K V_k = \bigcup_{k=1}^K V_k^\omega = V,
\]

where \( \sqcup \) denotes disjoint union. With \( \{V_k\}_{k=1}^K \) and \( \{V_k^\omega\}_{k=1}^K \), we define the corresponding index sets in the space of primal-dual variables in \( \mathbb{R}^{n+m} \) as follows:

\[
W_k := \bigcup_{i \in V_k} U_i; \quad W_k^\omega := \bigcup_{i \in V_k^\omega} U_i, \quad k = 1, \ldots, K,
\]

where \( U_i \subseteq W := \mathbb{Z}_{[1,n+m]} \) is the index set of \([x_i; \lambda_i] \) in \([x; \lambda] \). Now we observe that:

\[
W_k \subseteq W_k^\omega \subseteq W, \quad k = 1, \ldots, K; \quad \bigcup_{k=1}^K W_k = \bigcup_{k=1}^K W_k^\omega = W.
\]

We state the RAS scheme for solving (5) as:

\[
d^{(\ell+1)} = d^{(\ell)} + \left( \sum_{k=1}^K \tilde{R}_k M_k^{-1} R_k^\top \right) r^{(\ell)}, \quad \ell = 0, 1, \ldots, (6)
\]

Here, \( \ell \) is the RAS iteration counter, \( r^{(\ell)} := p - M d^{(\ell)} \) is the residual; \( M_k := (W_k^\omega)^{-1} W_k^\omega; \tilde{R}_k := \{\tilde{e}_i^k\}_{i \in W_k^\omega}; \tilde{R}_k = (\tilde{e}_i^k)^\top \in W_k; \) is the \( i \)-th standard basis of \( \mathbb{R}^{n+m} \), and \( \tilde{e}_i^k \) is \( i \) if \( i \in W_k \) and 0 otherwise.

The RAS scheme (6) involves the following steps. We first obtain the residual at the current step \( \ell \). Then, for each overlapping subdomain \( \{V_k^\omega\}_{k=1}^K \), the associated residual is extracted as \( \tilde{R}_k^\top r^{(\ell)} \). The \( k \)-th subsystem is then solved by applying \( M_k^{-1} \) (in MadNLP.jl, a factorization of \( M_k \) is computed with a direct solver and stored, so that the system can be repeatedly solved whenever the new right-hand-side is given). Subsequently, the solution \( M_k^{-1} \tilde{R}_k^\top r^{(\ell)} \) for the \( k \)-th overlapping subdomain is restricted to the non-overlapping subdomain \( V_k \), and then mapped back to the full space by applying \( \tilde{R}_k^\top \) (the indices associated with \( V \setminus V_k \) are set to zero in this step).

Key defining features of RAS are the concepts of overlap and restriction. The overlap allows the dampening of the adverse effect of the truncated domain, and the restriction procedure discards the part of the solution where the adverse truncation effect is strong.

In general, it has been empirically reported that the convergence of the RAS algorithm improves as the size of overlap \( \omega \) becomes large and as the conditioning of \( M \) becomes better (Cai and Saad, 1996). For positive definite \( M \), an exponential relationship between the convergence rate and the size of overlap has been established (Shin et al., 2020b). When \( \omega = 0 \), the RAS scheme reduces to a block-Jacobi scheme (decentralized) while, when \( \omega \) is maximal (\( M_k = M \)), the RAS scheme becomes a direct solution method (centralized). In this sense, RAS provides a bridge between fully decentralized and fully centralized schemes. In the Schwarz submodule of MadNLP.jl, \( \omega \) is set automatically based on the relative size of \( V_k \), and adaptively adjusted whenever a convergence issue occurs.

Algorithm (6) uses a simple static iteration (also called a Richardson iteration), but more sophisticated iterative methods such as the generalized mean residual (GMRES) method can also be used (by treating \( P \) as a preconditioner). Both Richardson and GMRES iterators are implemented in MadNLP.jl.

The construction of partitions for the RAS scheme is illustrated in Figure 2 (for the transient gas network example of Figure 1). Here, by partitioning the problem graph \( G = (V, E) \) (first subfigure), the node set \( V \) is divided into 4 subdomains \( V_1, V_2, V_3, V_4 \) (second subfigure). These subdomains are associated with the primal-dual index sets \( W_1, W_2, W_3, W_4 \) (third subfigure). After applying expansions, the associated blocks \( M_1, M_2, M_3, M_4 \) of \( M \) are identified (last subfigure); these blocks are used by the RAS scheme (6).

### 3.3 Implementation in MadNLP.jl

A schematic of the abstraction layers within MadNLP.jl is shown in Figure 3. The problem to be solved is modeled...
as an OptiGraph object (the core modeling object of Plasmo.jl). The OptiGraph interface is interfaced with multiple JuMP.jl models. The JuMP.jl model objects provide a set of local function oracles (objective, objective gradient, constraint, constraint Jacobian, and Lagrangian hessian). The Plasmo.jl-MadNLP.jl interface collects these local function oracles and creates a set of oracles for the full problem, where the local oracles are evaluated in parallel.

The Solver object of MadNLP.jl is created from the OptiGraph object of Plasmo.jl. The Solver object of MadNLP.jl uses a line-search filter IP method (Wächter and Biegler, 2006) to solve the problem. The step computation is performed by the linear solver specified by the user. The linear solver can be specified either as a direct solver or as the RAS solver. When the RAS solver is chosen, multiple subproblem solver objects are created by using standard direct solvers (e.g., using Ma57 of HSL routines). These subproblem solvers are used for factorization and backsolve for $M_k$ blocks. The RAS scheme (6) exploits multi-thread parallelism available in Julia. After termination of the IP solution procedure, the primal-dual solutions are sent back to the OptiGraph object and Model objects from JuMP.jl so that the user can query the solution via the interface provided by Plasmo.jl and JuMP.jl. See Figure 3 for a comparison with a conventional implementation.

4. CASE STUDIES

The proposed computational framework was tested using transient gas network and multi-period AC power flow problems. We present the problem statements, followed by numerical results and discussion.

4.1 Transient Gas Network

We consider a transient gas network problem (Sundar and Zlotnik, 2018) of the form:

\[
\begin{align*}
\min_{\rho, \alpha, d, v^s, v^c} & \quad \sum_{i \in T} \left( \sum_{j \in C(i)} \gamma P^a_{ijt} + \sum_{e \in E(i)} c_{it} s_{it} - \sum_{d \in D} c_{it} d_{it} \right) \\
\text{s.t.} & \quad \sum_{j \in \mathcal{N}(i)} f^i_{ijt} = \sum_{j \in \mathcal{R}(i)} s_{jt} - \sum_{j \in \mathcal{D}(i)} d_{jt}, i \in \mathcal{N}, t \in \mathcal{T} \quad (7a) \\
\rho^2_{it} - \rho^2_{jt} - \frac{\lambda L}{\rho} v^s_{ijt} &= 0, \quad (i, j) \in \mathcal{P}, t \in \mathcal{T} \quad (7b) \\
L(h^s_{it} + \rho^2_{it}) &= -4 \varphi_{ijt}, \quad (i, j) \in \mathcal{P}, t \in \mathcal{T} \quad (7c) \\
\rho^2_{it} - \rho^2_{jt} &= \frac{\lambda L}{\rho} v^c_{ijt}, \quad (i, j) \in \mathcal{P}, t \in \mathcal{T} \quad (7d) \\
L^2 \rho^2_{it} &= 0, \quad (i, j) \in \mathcal{C}, t \in \mathcal{T} \quad (7e) \\
P_a^i_{ijt} &= \frac{\alpha i_j \rho^2_{it} \rho^2_{jt}}{\rho^2_{it} + \rho^2_{jt}}, \quad (i, j) \in \mathcal{C}, t \in \mathcal{T} \quad (7f) \\
P_a^i_{ijt} - \alpha i_j \rho^2_{it} \rho^2_{jt} &= 0, \quad (i, j) \in \mathcal{C}, t \in \mathcal{T} \quad (7g) \\
\rho^2_{it} &= \frac{\alpha i_j \rho^2_{it} \rho^2_{jt}}{\rho^2_{it} + \rho^2_{jt}}, \quad (i, j) \in \mathcal{C}, t \in \mathcal{T} \quad (7h) \\
\rho^2_{it} &= f^i_{ijt}, \quad (i, j) \in \mathcal{C}, t \in \mathcal{T} \quad (7i)
\end{align*}
\]

where $\rho^2_{it} = \frac{\alpha i_j \rho^2_{it} \rho^2_{jt}}{\rho^2_{it} + \rho^2_{jt}}, \quad P_a^i_{ijt} = W_a A_{ij},$ and $f^i_{ijt} = A_{ij}\varphi_{ijt}^i.$ Here, $\mathcal{N}$ is the set of junctions; $\mathcal{P}$ is the set of pipelines; $\mathcal{C}$ is the set of compressors; $\mathcal{R}$ is the set of receipts; $\mathcal{D}$ is the set of demands; $\mathcal{C}(i)$ is the set of receipts at junction $i \in \mathcal{N}$; $\mathcal{D}(i)$ is the set of demands at junction $i \in \mathcal{N}$; $\mathcal{T}$ is the time index set; $\rho$ is the density; $\varphi^a$ is the average mass flux; $\varphi^-$ is the negative mass flux; $\alpha$ is the compression ratio; $s$ is the supply; $d$ is the demand; $\rho^2$ is the time derivative of density; $P^a$ is the power consumption of compressor; $f$ is the mass flow; $c$ is the gas price; $\gamma$ is the economic factor; $\lambda, L, D, A, \Delta t,$ and $W_a$ are physical parameters. To implicitly enforce the periodicity, we let $\rho^0_{it} = \rho^T_{it},$ where $T$ is the end time index. The gas network under study consists of 2 compressors, 6 junctions (35 junctions after discretization), 4 pipelines (32 pipelines after discretization), 1 receiving points and 5 transfer points, which work either as receipt or delivery. The model is constructed using GasModels.jl (Bent et al., 2020).

4.2 Multi-Period AC Power Flow

We consider a multi-period AC power flow problem with storage (Geth et al., 2020) of the form:

\[
\begin{align*}
\min_{v, \alpha, s^c, s^g} & \quad \sum_{e \in E} \sum_{t \in \mathcal{T}} \left( \sum_{s \in S} P^a_{it} + \sum_{s \in S} c_{it} s_{it} - \sum_{d \in D} c_{it} d_{it} \right) \\
\text{s.t.} & \quad \sum_{j \in \mathcal{N}(i)} f^i_{ijt} = \sum_{j \in \mathcal{R}(i)} s_{jt} - \sum_{j \in \mathcal{D}(i)} d_{jt}, i \in \mathcal{N}, t \in \mathcal{T} \quad (8a) \\
|v^s_{ijt}| & \leq \frac{v^U_{ijt}}{s}, \quad (i, j) \in \mathcal{E}, \ t \in \mathcal{T} \quad (8b) \\
\sum_{k \in \mathcal{G}} s^g_{ikt} - \sum_{k \in \mathcal{L}} s^g_{ikt} + \sum_{k \in \mathcal{S}} s^s_{ikt} &= \sum_{k \in \mathcal{N}_j} s_{ikt}, \quad i \in \mathcal{N}, t \in \mathcal{T} \quad (8c) \\
s_{ikt} &= (Y_{ij} + Y_{ij}^*) |v^s_{ijt}|^2 - Y_{ij} v^c_{ijt} v^s_{ijt}, \quad (i, j) \in \mathcal{E}, t \in \mathcal{T} \quad (8d) \\
\sum_{k \in \mathcal{G}} s^g_{ikt} + (s_{ikt} - s_{ikt}) &= \lambda^g s^c_{ikt} + s^s_{ikt}, \quad k \in \mathcal{S}, t \in \mathcal{T} \quad (8e) \\
|s^g_{ikt}| & \leq \frac{s^g_{ikt}}{s}, 0 \leq s_{ikt} \leq s^s_{ikt}, \quad k \in \mathcal{S}, t \in \mathcal{T} \quad (8f) \\
|s^c_{ikt}| & \leq \frac{s^c_{ikt}}{s}, 0 \leq s^c_{ikt} \leq s^s_{ikt}, \quad k \in \mathcal{S}, t \in \mathcal{T} \quad (8g)
\end{align*}
\]

Here, $\mathcal{G}$ is the set of generators; $\mathcal{N}$ is the set of buses; $\mathcal{E}$ is the set of (directed) branches; $\mathcal{E}^R$ is the set of branches with inverted directions; $\mathcal{S}$ is the set of storage; $T$ is the time index set; $v$ is the voltage; $e \in \mathcal{R}$ is the state of charge; $s$ is the power flow; $s^c \in \mathcal{C}$ is the power generation; $s^g$ is the complex power injected by the storage; $sc$ is the charging rate; $sd$ is the discharging rate; $s_{ckt}$ is the reactive power slack; $\varphi^a, c^a, c^c$ are the generation costs; $s^p$ is the power demand; $Y$ is the admittance; $T$ is the branch complex transformation parameter; $\eta$ is the charging efficiency; $s^\text{loss}$ is the storage energy loss; $\Delta t$ is the time interval. Note that (8) can be reformulated as an NLP with real variables by separately treating the real and imaginary part of the variables and equations (in particular, polar AC power flow formulation is used). The power network under study (a variant of IEEE 14 bus test system) consists of 14 buses, 5 generators, 1 storage, 1 shunt, and 20 branches. The detailed model is constructed with PowerModels.jl (Coffrin et al., 2018).
4.3 Results and Discussion

We compared the proposed method (MadNLP.jl interfaced with Plasmo.jl and Schwarz), with the conventional method (MadNLP.jl interfaced with serial/parallel direct solvers Ma57 (HSL, 2007) or MKL-Pardiso along with non-graph based algebraic modeling language JuMP.jl). The conventional methods are referred to as JuMP-Ma57 and JuMP-PardisoMKL, and the proposed method is referred to as Plasmo-Schwarz/Ma57. Furthermore, the mix of proposed/conventional approaches (JuMP-Schwarz/Ma57, Plasmo-Ma57, and Plasmo-PardisoMKL) were also tested together. For JuMP-Schwarz/Ma57, the graph partitioning tool METIS is used to partition the primal-dual coupling graph $\tilde{G}$ directly. A Richardson scheme is used as an iterator for the RAS scheme. The study was performed by solving the gas (7) and power (8) problems while varying the size of the problems (by increasing the length of the prediction horizon). The code was run on a server computer equipped with 2 CPUs of Intel Xeon CPU E5-2698 v4 running 2.20GHz (20 core for each), and 20 threads are used for the computation. Code to reproduce the results can be found in https://github.com/lanl-ansi/GasModels.jl.

For both problems, we found that the graph-based approach can significantly speed up the solution (see Figure 4). In particular, comparing JuMP-Ma57 and Plasmo-Schwarz/Ma57, Plasmo-Schwarz/Ma57 becomes faster than JuMP-Ma57 when the prediction horizon is longer than or equal to 3 days. Specifically, one can observe that function evaluation is always faster in Plasmo.jl compared to JuMP.jl. This is because the computational saving from function evaluation directly reduces the total solution time (parallelizing the function evaluation itself has no impact on the other part of the algorithm). On the other hand, one can see that the speed-up from parallel linear algebra is only observed when the problem size is sufficiently large (3 days in the gas network and 60 days in power network). This is because the decrease in the problem size also makes the overlap size small (in our implementation, we set the size of overlap using the relative size of the block; so the absolute size of the overlap decreases if the overall problem size becomes small). Because of this, the progress of the RAS scheme (6) may become slow; accordingly, the number of required factorization/backsolve increases. This indicates that RAS is only effective when the problem size is sufficiently large. For the gas problems, the speedup from linear algebra computations was more pronounced, while for the power problem, the speedup from function evaluations was more pronounced. This is because the AC power flow formulation has a large number of nonlinear expressions. By comparing the linear solver time for JuMP-Schwarz/Ma57 and Plasmo-Schwarz/Ma57, we see the advantage of using a graph-based modeling language for obtaining the partitions $\{W_k\}_{k=1}^K$ (recall that for JuMP-Schwarz/Ma57, the Metis graph partitioning routine is directly applied to $\tilde{G}$, while Plasmo-Schwarz/Ma57 uses the user-provided problem graph $G$ is used). One can observe that, in general, the linear solver time is shorter for PlasmoNLP-Schwarz/Ma57. This indicates that the user-provided graph information can be leveraged for obtaining high-quality partitions.

5. CONCLUSIONS AND FUTURE WORK

We have presented a graph-based modeling and decomposition framework for large-scale nonlinear programs arising in energy infrastructures. Here, we introduce a new decomposition paradigm for linear algebra systems within an interior-point method: a restricted additive Schwarz (RAS) scheme. We implement this framework in the Julia package MadNLP.jl and show that the RAS approach accelerates computations (compared to off-the-shelf approaches) by up to a factor of 3. This work focused on applying RAS to conduct temporal decomposition; applying RAS as a spatial decomposition scheme is a future direction of interest. One of the surprising observations was that the RAS scheme is effective at handling instances with a large number of active inequality constraints. We are interested in determining the reasons of this by investigating the convergence properties of the RAS scheme within an interior-point context.

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| Number of Variables | Plasmo-Ma57 | Plasmo-PardisoMKL | Plasmo-Schwarz/Ma57 | JuMP-Ma57 | JuMP-PardisoMKL | JuMP-Schwarz/Ma57 |
|---------------------|-------------|-------------------|---------------------|-----------|----------------|------------------|
| Solution Wall Time (sec) | 10^8        | 10^8              | 10^8                | 10^8      | 10^8           | 10^8             |
| Linear Solver Wall Time (sec) | 10^8        | 10^8              | 10^8                | 10^8      | 10^8           | 10^8             |
| Function Evaluation Wall Time (sec) | 10^8        | 10^8              | 10^8                | 10^8      | 10^8           | 10^8             |

Fig. 4. Solution time (top), linear solver time (middle), function evaluation time (bottom) for transient gas network (left) and multi-period AC optimal power flow (right) problems.

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