Protecting quantum entanglement from amplitude damping

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Abstract

Quantum entanglement is a critical resource for quantum information and quantum computation. However, entanglement of a quantum system is subjected to change due to the interaction with the environment. One typical result of the interaction is the amplitude damping that usually results in the reduction of the entanglement. Here we propose a protocol to protect quantum entanglement from the amplitude damping by applying Hadamard and CNOT gates. As opposed to some recently studied methods, the scheme presented here does not require weak measurement in the reversal process, leading to a faster recovery of entanglement. We propose a possible experimental implementation based on linear optical system.

1. Introduction

A quantum state is subjected to decoherence due to the interaction with the environment. The quantum error correction codes can suppress the decoherence by encoding the logical qubit in multiple physical qubits and performing sufficient measurements and correction operations [1–4]. Another strategy is to rely on the so-called decoherence-free subspace which requires the interaction Hamiltonian to have some appropriate symmetry [5, 6]. Quantum Zeno effect [7, 8] and dynamical decoupling [9] have also been discussed to protect the quantum state.

Amplitude damping is one important type of decoherence which is related to many practical qubit systems [10]. For example, it can happen to a photon qubit in a leaky cavity, or atomic qubit subjected to spontaneous decay, or a superconduction qubit with zero-temperature energy relaxation. Recently, it is shown that weak measurement together with bit flip can recover a quantum state from amplitude damping [11–14]. Another way to restore a qubit state in a weak measurement is using Hadamard and CNOT gates with auxiliary qubits [15]. In this way, no weak measurement is required in the reversal process, and hence the reversal time can be shorter.

Quantum entanglement, which is a critical resource of the quantum information and quantum computation, also decreases due to the amplitude damping. Sun et al first showed that weak measurement together with bit flip can also protect the quantum entanglement [16]. The decoherence can be largely suppressed by uncollapsing the quantum state towards the ground state before the amplitude damping [17]. These ideas were recently implemented in a proof-of-principle experiment [18].

In this paper, we show that, following the approach presented in [15], an arbitrary two-qubit pure state under amplitude damping in a weak measurement can also be probabilistically recovered using Hadamard and CNOT gates with auxiliary qubits. Furthermore, even without weak measurement, quantum entanglement of a two-qubit system under amplitude damping can also be partially protected using our scheme. We also propose a proof-of-principle experiment for this scheme based on a linear optics system. Besides, we can extend our scheme to suppress the decoherence even better by uncollapsing the quantum state of the system towards the ground state.

This paper is organized as follows: in section 2, we briefly introduce the weak measurement and amplitude damping. In section 3, we show that a pure two-qubit state that has undergone a weak measurement can be recovered using Hadamard and CNOT gates with auxiliary qubits. In section 4, we show that quantum entanglement between two qubits that
has undergone amplitude damping can be partially recovered using the same procedure in section 3. An extended scheme to improve the protection is discussed in section 5. In section 6, we discuss the state fidelity. In section 7, we propose a linear optical experiment to implement our scheme. Finally, we summarize the result.

2. Weak measurement and amplitude damping

As opposed to a typical Von Neumann quantum measurement, complete collapse to an eigenstate does not occur in a weak measurement [17]. An example of the weak measurement is the leakage of the field inside a cavity. Suppose that the quantum state of a field in the cavity is a superposition of zero and one photon, i.e., $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. Let us assume that an ideal detector is placed outside the cavity. If the detector registers a click, the quantum state of the cavity is damped.

More generally, an amplitude damping of a single qubit can be described by the following mapping [10]:

$$|0\rangle_s|0\rangle_E \rightarrow |0\rangle_s|0\rangle_E,$$  

$$|1\rangle_s|0\rangle_E \rightarrow \sqrt{\eta}|1\rangle_s|0\rangle_E + \sqrt{1-\eta}|0\rangle_s|1\rangle_E,$$  

where $p \in [0,1]$ is the possibility of decaying the excited state, $q = 1 - p$ and $S\ (E)$ denotes the system (environment). Within the Weisskopf–Wigner approximation, the probability of finding the atom in the excited state decreases exponentially with time and we have $\sqrt{\eta} = e^{-\Gamma t}$.

In a weak measurement, if a detector gets a null-result, we have the following mapping:

$$|0\rangle_s|0\rangle_E \rightarrow |0\rangle_s|0\rangle_E,$$  

$$|1\rangle_s|0\rangle_E \rightarrow \sqrt{\eta}|1\rangle_s|0\rangle_E.$$  

3. Two-qubit state recovery in a weak measurement

In this section, let us consider the situation when we have an arbitrary two-qubit pure state which is given by

$$|\psi\rangle_{in} = \alpha|00\rangle_S + \beta|01\rangle_S + \gamma|10\rangle_S + \delta|11\rangle_S.$$  

When this state undergoes amplitude damping and we get a null-result for the weak measurement, according to the mappings in equations (3) and (4), the system evolves to

$$|\psi\rangle_d = \frac{1}{N_d} (\alpha|00\rangle_S + \beta\sqrt{\eta}|01\rangle_S + \gamma\sqrt{\eta}|10\rangle_S + \delta\eta|11\rangle_S) \otimes (\cos \theta|0\rangle_A + \sin \theta|1\rangle_A)^{\otimes 2}.$$  

where $N_d = \sqrt{|\alpha|^2 + q(|\beta|^2 + |\gamma|^2) + q^2|\delta|^2}$ is the normalization factor.

To recover the damped quantum state back to the initial quantum state, we use the circuit diagram shown in figure 1. Two auxiliary qubits are needed in this scheme. Initially, these two ancillas are both in the $|0\rangle$ state. First we apply a Hadamard gate with angle $\theta$ for each ancilla

$$H_\theta = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right).$$  

The combined system is given by

$$|\psi_1\rangle = \frac{1}{N_d} [\alpha|00\rangle_S + \beta\sqrt{\eta}|01\rangle_S + \gamma\sqrt{\eta}|10\rangle_S + \delta\eta|11\rangle_S] \otimes (\cos \theta|0\rangle_A + \sin \theta|1\rangle_A)^{\otimes 2}.$$  

Then two CNOT gates are separately applied to each pair of the system qubit and the ancilla qubit. The system qubits are the controlled qubits while the ancilla qubits are the target qubits. If $\theta$ is chosen to be $\tan^{-1}(1/\sqrt{\eta})$ or $\tan^{-1}(\exp(\Gamma t))$, the combined state becomes (see appendix A)

$$|\psi_2\rangle = \frac{q}{N_d(1 + q)} [\alpha|00\rangle_S + \beta|01\rangle_S + \gamma|10\rangle_S + \delta|11\rangle_S] \otimes (\cos \theta|0\rangle_A + \sin \theta|1\rangle_A)^{\otimes 2}.$$  

After the CNOT gates, we make a measurement on the ancilla qubits. From equation (9), we can see that if we get the $|00\rangle$ result, the state of the system recovers back to the initial state exactly. The success probability is $P_{00}(q) = q/N_d(1 + q)^2$ which decreases with the decaying probability (see figure 2).
If we get \( |01 \rangle \) or \( |10 \rangle \) for the ancilla qubit, we just repeat the same procedure on one qubit and with \( \theta = \tan^{-1}(1/q) \). For example, if we get \( |01 \rangle \) for the ancilla qubit, the quantum state of the system is \( |\psi\rangle_{\text{out}} = \alpha|00\rangle + \beta q|01\rangle + \gamma|10\rangle + \delta q|11\rangle \). This state can be interpreted as if the first qubit has not decayed while the second qubit has a decay rate of \( 2\Gamma \). In this case, we add another ancilla for the second qubit. By applying a Hadamard gate with \( \theta = \tan^{-1}(1/q) \) for the ancilla and CNOT gate for the ancilla and the second qubit, we obtain the following state:

\[
|\psi_2\rangle = \frac{q}{\sqrt{1 + q^2}} (\alpha|00\rangle + \beta q|01\rangle + \gamma|10\rangle + \delta q|11\rangle) + \frac{1}{\sqrt{1 + q^2}} (\alpha|00\rangle + \beta q^2|01\rangle + \gamma|10\rangle + \delta q^2|11\rangle).
\]

We can see that the state of the system can also be recovered if the state of ancilla qubit is measured to be \( |0\rangle \). The probability in this case is given by \( P_{10}(q) = q^2/(1 + q^2) \). Note that if we get \( |1\rangle \) on the ancilla qubit, the effective decay rate is double and we can repeat the same procedure but with \( \theta = \tan^{-1}(1/q^2) \). Repeating the same procedure again and again we can increase the probability of recovering the quantum state.

If we get the \( |11\rangle \) result, the state of the system is given by the last term in equation (9). Comparing this term with equation (6) we find that the only difference is that the damping coefficient \( \sqrt{q} \) is replaced by \( q \) which means that the decay rate is double. Therefore, by repeating the same procedure but with \( \theta = \tan^{-1}(1/q^2) \), we can still have some probabilities to recover back to the initial state.

The probability of recovering the quantum state versus the damping rate for different repeat times is shown in figure 2 (see appendix A for the calculations). From the figure we can see that the success probability decreases as the damping rate increases. When the quantum state is completely damped \( (p = 1) \), we can never recover it back because the information of the state has been lost. We also can see that the success probability can be significantly increased for the first few repeating times. When we increase the repeating time, we can approach the success probability of double weak measurement \( (q^2) \) which is shown by the dashed line. In practice, we can repeat about three times and we can have significant success probability. After three times the success probability increases by only very small amounts especially when the damping rate is large.

### 4. Two-qubit quantum entanglement protection from amplitude damping

In the previous section, we showed that a two-qubit quantum state in a weak measurement can be recovered probabilistically using the procedure shown in figure 1. In this section, we show that using a similar procedure we can protect two-qubit entanglement when the two-qubit system undergoes a general amplitude damping which is described by equations (1) and (2).

As discussed in the previous section, a general two-qubit pure state is given by \( |\psi_{\text{in}}\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \). The concurrence of this state is \( C = \max \{0, 2|\alpha\delta - \beta\gamma| \} \). We assume that the environment is in the ground state \( |0\rangle \). The evolution of the combined system with amplitude damping is given by

\[
|\psi_d\rangle = \alpha|00\rangle|00\rangle_E + \beta\sqrt{\gamma}|01\rangle|00\rangle_E + \beta\sqrt{\beta}|00\rangle|11\rangle_E + \gamma\sqrt{\alpha}|10\rangle|00\rangle_E + \gamma\sqrt{\alpha}|10\rangle|11\rangle_E + \delta q|11\rangle|00\rangle_E + \delta q|11\rangle|11\rangle_E.
\]

We know that using the procedure shown in figure 1, in this section, we can protect two-qubit entanglement when the two-qubit system undergoes a general amplitude damping which is described by equations (1) and (2).

On tracing out the environment, we obtain the density matrix for the system and from which we can get the damped concurrence [19]

\[
C_d(p) = \max \{0, 2q(|\alpha\delta - \beta\gamma| - |p|\delta^2)\}.
\]

This concurrence is less than the initial value and decreases with an increasing \( p \). This indicates that the entanglement of the system decreases due to the amplitude damping.

To protect the entanglement, we add two ancilla qubits which are initially in the \( |00\rangle \) state and follow the same procedure as in the previous section. We measure the final state of the ancilla qubit. There are four possible outcomes. If the result is \( |00\rangle \), we obtain the density matrix of the system after tracing out the environment to be

\[
\rho_f = \frac{1}{N_f^2} \left( \begin{array}{cccc}
|\alpha|^2 & \beta|\delta| & \beta^*|\alpha| & \beta^*|\delta|
\end{array} \right.
\]

\[
\left. \begin{array}{cccc}
\alpha^*|\beta| & |\gamma|^2 & \beta^*|\gamma| & \gamma^*|\beta|
\end{array} \right.
\]

\[
\left. \begin{array}{cccc}
\gamma^*|\alpha| & \gamma^*|\delta| & |\beta|^2 & \beta^*|\gamma|
\end{array} \right.
\]

\[
\left. \begin{array}{cccc}
\gamma^*|\delta| & \gamma^*|\beta| & \gamma^*|\gamma| & |\delta|^2
\end{array} \right)
\]

\[
\times \left( \begin{array}{cccc}
|\alpha|^2 + p|\beta|^2 + p|\gamma|^2 + p|\delta|^2 & \alpha\beta^* + p\beta^* & \alpha\gamma^* + p\gamma^* & \alpha\delta^* + p\delta^*
\end{array} \right.
\]

\[
\left. \begin{array}{cccc}
\alpha^*\beta + p\gamma^* & |\gamma|^2 + |\beta|^2 & \beta^*\gamma + p\beta^* & \gamma^*\beta
\end{array} \right.
\]

\[
\left. \begin{array}{cccc}
\alpha^*\gamma + p\delta^* & \beta^*\gamma^* & |\gamma|^2 + p|\delta|^2 & \gamma^*\delta
\end{array} \right.
\]

\[
\left. \begin{array}{cccc}
\alpha^*\delta + p\beta^* & \beta^*\gamma^* & |\delta|^2 & |\gamma|^2
\end{array} \right)
\]

where \( N_f^2 = 1 + p(|\beta|^2 + |\gamma|^2 + 2|\delta|^2) + p^2|\delta|^2 \). We note that this result is identical to the result obtained via weak measurement reversal [16]. The probability of getting this result is \( N_f^2 q^2/(1 + q^2) \). The concurrence of the final state is given by

\[
C_f(p) = \max \left\{ 0, \frac{2(|\alpha\delta - \beta\gamma| - |p|\delta^2)}{1 + p(1 + |\delta|^2 - |\alpha|^2 - p^2|\delta|^2)} \right\}.
\]

Comparing the concurrences \( C_d(p) \) and \( C_f(p) \), we find the following features:
applying a Hadamard gate on the ancilla qubit, we apply a CNOT gate on the second qubit and the ancilla qubit. If the ancilla qubit is measured to be in the $|0\rangle$ state, we find that the density matrix of the system after tracing out the environment is the same as equation (13) (see appendix B for calculations). Therefore, even if we do not get the $|00\rangle$ result, an additional procedure can have some probabilities to protect the quantum entanglement.

## 5. Extended scheme

Here we discuss how we can improve the result by extending the scheme in section 3. The extended scheme is shown in figure 4. Before the system qubits undergo amplitude damping, we apply the same quantum circuit as in the recovery part to prepare the system in a more robust quantum state. If the ancilla qubits are measured to be $|00\rangle$, the preparation is successful; otherwise, we should discard the result. The system undergoes amplitude damping after the preparation stage. Finally, we do the same recovery procedure as in section 4 to restore the quantum state and quantum entanglement.

The state of the system after the successful preparation step can be readily obtained from equation (A.3) in appendix A. The only differences are here $q = 1$ and $θ = θ_1$, where $θ_1$ is the rotation angle of the Hadamard gate in the preparation step. If we define $x = \tan^2 θ_1$, we obtain

$$|ψ(p)⟩ = \frac{1}{N_1} (|α00⟩ + β\sqrt{x} |01⟩ + γ\sqrt{x} |10⟩ + δx |11⟩)$$ (16)

where $N_1 = \sqrt{|α|^2 + |β|^2 x + |γ|^2 x^2}$. The success probability is $N_2^2/(1+p^2x^2)$. If $θ_1$ is chosen such that $x$ is less than 1, the system uncouples towards the ground state which has a similar effect as the weak measurement [17, 18]. The ground state is uncoupled to the environment and is less vulnerable to decoherence. Different from the weak measurement scheme [17], here we do not need to wait for the null-result weak measurement.

After preparing the system in the state shown in equation (16), the system undergoes the amplitude damping and the recovery procedure. In the recovery process, we choose the rotation angle of the Hadamard gate such that $xy = 1$ where $y = \tan^2 θ_2$ ($θ_2$ is the rotation angle of the Hadamard gate in the recovery procedure). We measure the state of the ancilla qubits, and if we get the $|00\rangle$ result, the density matrix of the system becomes

$$ρ_f = \frac{1}{N_2} \begin{pmatrix}
|α|^2 & α^*p|β|^2 + p|δ|^2 & αp|β| + p^2|δ|^2 & αp^2|δ|^2 \\
α^*p|β|^2 + p|δ|^2 & |β|^2 & βp|γ| & βp^2|γ| \\
αp|β| + p^2|δ|^2 & βp|γ| & |γ|^2 & γp|δ| \\
αp^2|δ|^2 & βp^2|γ| & γp|δ| & |δ|^2
\end{pmatrix}$$ (17)

where $N_2 = 1 + p^2 (|β|^2 + |γ|^2 + 2|δ|^2) + p^4 x^2 |δ|^2$ is the normalization factor. The success probability for the recovery step is $N_2^2/N_1^2 (1 + y^2)$. The concurrence of the final quantum state is given by

$$C_r(p, x) = \max \left\{ 0, \frac{2(αδ - βγ - px|δ|^2)}{1 + px(1 - |α|^2 + |β|^2 + |δ|^2) + p^2x^2|δ|^2} \right\}$$ (18)
Comparing this concurrence with concurrence in equation (14) and the damped concurrence \(C_d(p)\) in equation (13), we find several new features. (i) When \(x = 1\), the new concurrence returns back to the result in equation (14). This clearly shows that the scheme in section 6 is a special case of the extended scheme discussed in section 3 where there are always some states where the concurrence cannot be improved. (ii) For an arbitrary initial state if \(x\) is small enough, \(C_r(p, x)\) can always be larger than \(C_d(p)\) for arbitrary \(p\) (figures 3(a) and (b)). This is different from the scheme discussed in section 3 where there are always some states where the concurrence cannot be improved. (iii) The smaller the \(x\), the larger the concurrence we can recover (figures 3(a) and (b)). However, the success probability decreases when \(x\) decreases (figures 5(a) and (b)). Therefore, there is a tradeoff between the success probability and the entanglement protection. (iv) If \(x\) is less than 1, both the recovered concurrence and the success probability can be nonzero even if the damped concurrence is zero (figures 3(a) and 5(a)), which never happens in the scheme described in section 3. However, this does not mean that the quantum entanglement can be created by LOCC from a separable state. The reason that happens is because the quantum state damps slower with suitable preparation than without preparation. Without a preparation step the damped quantum state reaches the ESD point when \(p\) is large, but with a preparation step the new state in section 6 is never reached. This shows that the damped quantum state is recovered back to the initial state. However, we should note that in this case the success probability approaches zero (figures 5(a) and (b)).

6. Fidelity

In the previous sections, we show that quantum entanglement can be partially protected in our schemes. Here we show that the quantum state fidelity can also be protected. The fidelity between two quantum states is given by

\[
F(\rho_i, \rho_f) = |\text{Tr}(\sqrt{\rho_i} \sqrt{\rho_f})|^2
\]

where \(\rho_i\) and \(\rho_f\) are the initial and final states, respectively. The second identity is valid when the initial state is pure which is the case in our paper.

For the damped state in equation (11), we can calculate the fidelity between this state and the initial state and it is given by

\[
F_d(p) = (\alpha^2 + \sqrt{\alpha^2 + \gamma^2})(\beta^2 + \gamma^2) + 4p\sqrt{\alpha\beta\gamma \delta} + p(\alpha^2 + \gamma^2)(\beta^2 + \gamma^2) + p^2\alpha^2\delta^2. \tag{20}
\]

For the recovered quantum state by the scheme in section 3 (see equation (13)), the fidelity is

\[
F_r(p) = \frac{1 + 4p\alpha \beta \gamma \delta + p(\alpha^2 + \delta^2)(\beta^2 + \gamma^2) + p^2\alpha^2\delta^2}{1 + p(1 - \alpha^2 + \delta^2) + p^2\delta^2}. \tag{21}
\]

For the recovered quantum state by the extended scheme in section 5 (equation (17)), the fidelity is

\[
F'_r(p) = \frac{1 + 4p\alpha \beta \gamma \delta + p(\alpha^2 + \delta^2)(\beta^2 + \gamma^2) + p^2\alpha^2\delta^2}{1 + p(1 - \alpha^2 + \delta^2) + p^2\delta^2}. \tag{22}
\]

Comparing equations (21) and (22), we find that \(F_r(p)\) is a special case of \(F'_r(p)\) when \(x = 1\). We can also find that when \(x \to 0\), the fidelity \(F_r(p)\) of the extended scheme approaches 1 which means that the quantum state is recovered back to the initial state. Two examples are given in figures 6(a) and (b) where we can see that the fidelity can be controlled by the...
parameter \( x \). One can see that smaller \( x \) gives higher fidelity. Therefore, the fidelity of the quantum state can also be well preserved in the extended scheme. However we should also notice that the success probability decreases with smaller \( x \) (figures 5(a) and (b)).

7. Implementation with linear optics

A possible experimental scheme with a linear optics system is discussed in this section. The experimental setup includes four parts (figure 7): entangled state generation, state preparation, amplitude damping and recovering operations. The setup for entangled state generation and the amplitude damping simulation are the same as that in the weak measurement reversal scheme [16, 18]. We briefly describe these two parts. The polarization-entangled photon pair can be generated by two adjacent type-I crystals and the outcome state is given by

\[
|\psi\rangle = \alpha|HH\rangle + \beta|VV\rangle,
\]

where \( H \) is the horizontal polarization which is denoted as the \( |0\rangle \) state, while \( V \) is the vertical polarization which is denoted as the \( |1\rangle \) state. Here \( \alpha \) and \( \beta \) are two complex numbers and their values can be controlled by a half-wave plate (HWP) and a tilted quarter-wave plate before the crystals. We can also use a HWP to rotate one of the photons or use a type-II phase matching to generate entangled photons with orthogonal polarizations:

\[
|\psi\rangle = \alpha|HV\rangle + \beta|VH\rangle.
\]

The two entangled qubits are then spatially separated and each qubit goes to the state preparation stage. In the preparation part we need to apply CNOT gates on the system qubits and the ancilla qubits. Several methods for CNOT gates based on linear optics systems have been discussed [22–26]. Here we propose to use the scheme proposed by D’Ariano et al where cross-Kerr medium and self-Kerr medium are used [26]. The system qubits act as the control qubit while the ancilla qubits are the target qubits. The ancilla qubits (A1–A4) are photons in the horizontal polarization initialization. We use two HWPs with angle \( \theta_1 \) to rotate the polarizations of A1 and A2. Then their polarizations become a superposition state \( \cos \theta_1|H\rangle + \sin \theta_1|V\rangle \). The polarizers P1 and P2 are horizontal polarized. If we detect one photon for both DA1 and DA2, the state preparation is successful.

After the state preparation, the system qubits undergo amplitude damping which is simulated by a displaced-Sagnac interferometer [16, 18]. The \( H \) photon travels along the solid line (path 0), while the \( V \) photon travels along the dashed line (path 1). There is an HWP with angle \( \theta_2 \) in path 1 which rotates the \( V \) state to a superposition of \( H \) state and \( V \) state, i.e., \( |V\rangle \rightarrow \cos \theta_2|V\rangle + \sin \theta_2|H\rangle \). This is equivalent to an amplitude damping with \( \sqrt{\delta} = \sin \theta_2 \). While the HWPC with angle 0 in path 0 does not rotate the polarization and it is just used to compensate the optical path difference. Then the photon in path 0 and path 1 are combined through another polarization beam splitter.

The recovering part is similar to the preparation part. The only difference is that here the ancilla qubits A3 and A4 are rotated by an angle \( \theta_2 \). For the best recovering, \( \theta_2 \) should be chosen such that \( \tan \theta_1 \cos \theta_2 \tan \theta_2 = 1 \). After the CNOT gates we make a measurement on ancilla qubits A3 and A4 using DA3 and DA4, respectively. If both ancilla qubits are in horizontal polarization, our protection scheme succeeds and the system qubits are very well recovered. To prove the result, we just repeat the same procedure simply by changing the polarizations of polarizers P3 and P5, and we can carry out the quantum state tomography from which we can calculate the concurrence and fidelity.

8. Summary

In this paper, we first show that arbitrary two-qubit pure state under amplitude damping in a weak measurement can be probabilistically recovered using Hadamard and CNOT gates. Then we discuss the situation that a two-qubit system is undergoing amplitude damping but without weak measurement. Using a similar recovering diagram, we show that quantum entanglement can also be partially protected. However, this simple scheme cannot recover some kinds of states. We then propose an extended scheme where we add a step to prepare the system in a more robust state before amplitude damping. In the extended scheme, we can partially recover all kinds of states and the quantum entanglement can
Figure 7. The experimental scheme for protecting quantum entanglement via Hadamard and CNOT gates. The experimental setup includes four parts: entangled state generation, state preparation, amplitude damping and recovering operations. Notations: HWP—half-wave plate, QWP—quarter-wave plate, PBS—polarization beam splitter, CKerr—cross-Kerr medium, SKerr—self-Kerr medium, DA—detector on ancilla qubit, DS—detector on system qubit, HWPC—half-wave plate for compensation, P—polarizer.

be much better recovered. We also show that we can protect the quantum state from sudden death. Finally we propose a linear optical experiment for our protection scheme.

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Appendix A. Quantum state recovery in a weak measurement

When a two-qubit pure state undergoes amplitude damping and a null-result for the weak measurement, the system evolves to equation (6). We add two ancillas which are initially in the |00⟩ state. First we apply a Hadamard gate with angle θ for each ancilla; then the state of the system becomes

\[ |ψ_2⟩ = \frac{α}{N_d}|00⟩_S + \frac{β}{N_d}|01⟩_S + \frac{γ}{N_d}|10⟩_S + \frac{δ}{N_d}|11⟩_S \] \( \otimes \) \( (\cos θ|0⟩_A + \sin θ|0⟩_A)^{⊗2} \)

\[ + \frac{α}{N_d}|00⟩_A + \cos θ \sin θ|01⟩_A \] + \[ + \frac{β}{N_d}|01⟩_A + \cos θ \sin θ|10⟩_A \] + \[ + \frac{γ}{N_d}|10⟩_A + \cos θ \sin θ|11⟩_A \] + \[ + \frac{δ}{N_d}|11⟩_A \] + \[ + \frac{α}{N_d}|00⟩_A + \sin^2 θ|11⟩_A \] + \[ + \frac{β}{N_d}|01⟩_A + \sin θ \cos θ|00⟩_A \] + \[ + \frac{γ}{N_d}|10⟩_A + \sin^2 θ|01⟩_A \] + \[ + \frac{δ}{N_d}|11⟩_A \] + \[ + \frac{α}{N_d}|00⟩_A + \sin^2 θ|11⟩_A \]. \quad (A.1)

Then two CNOT gates are separately applied to each pair of the system qubit and the ancilla qubit. The system qubits are the controlled qubits while the ancillas are the target qubits. We can obtain

\[ |ψ_2⟩ = \frac{α}{N_d}|00⟩_S + \cos θ \sin θ|01⟩_A \] + \[ + \frac{β}{N_d}|01⟩_A + \cos θ \sin θ|10⟩_A \] + \[ + \frac{γ}{N_d}|10⟩_A + \cos θ \sin θ|11⟩_A \] + \[ + \frac{δ}{N_d}|11⟩_A \] + \[ + \frac{α}{N_d}|00⟩_A + \sin θ \cos θ|00⟩_A \] + \[ + \frac{β}{N_d}|01⟩_A + \sin^2 θ|01⟩_A \] + \[ + \frac{γ}{N_d}|10⟩_A + \sin^2 θ|10⟩_A \] + \[ + \frac{δ}{N_d}|11⟩_A \] + \[ + \frac{α}{N_d}|00⟩_A + \sin^2 θ|11⟩_A \].
which is initially in the $P$ state. Only one recovery procedure is given by

$$\psi_{10} = \frac{1}{N_0} (|00\rangle + \beta \sqrt{q} \sin \theta |01\rangle_A) \otimes |00\rangle_A$$ \hspace{1cm} (A.4)$$

where the normalization factor $N_0 = \sqrt{1 + |\beta|^2 + \gamma^2 + \delta^2 q^2}$. We add an ancilla qubit which is initially in the $|0\rangle$ state. After applying the Hadamard gate with angle $\theta$, the combined state becomes

$$|\psi_1\rangle = \cos \theta (|00\rangle S + \beta |01\rangle S + \gamma |10\rangle S) |0\rangle_A$$

$$+ \sin \theta (|00\rangle S + \beta |01\rangle S + \gamma |10\rangle S + \delta |11\rangle S) |1\rangle_A.$$ \hspace{1cm} (A.5)

Applying the CNOT gate on the second qubit and the ancilla, we obtain the state

$$|\psi_2\rangle = \cos \theta (|00\rangle S + \beta |01\rangle S + \gamma |10\rangle S + \delta |11\rangle S) |0\rangle_A$$

$$+ \sin \theta (|00\rangle S + \beta |01\rangle S + \gamma |10\rangle S + \delta |11\rangle S) |1\rangle_A.$$ \hspace{1cm} (A.6)

This state can be rewritten as the state in equation (9) and we have some probability of recovering back to the initial state.

Let us calculate the success probability. First, the probability of the null-result weak measurement for the quantum state in equation (5) is given by $P_0 = N_0^2$. From equation (10), we see that the probabilities of getting $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ results are respectively

$$P_{00}(q) = \frac{q^2}{N_0^2 + (1 + q^2)}; \hspace{1cm} P_{01}(q) = \frac{q}{N_0^2 + (1 + q^2)}; \hspace{1cm} P_{10}(q) = \frac{q}{N_0^2 + (1 + q^2)}; \hspace{1cm} P_{11}(q) = \frac{1}{N_0^2 + (1 + q^2)}.$$ \hspace{1cm} (A.7)

where we neglect the normalization factors in the medium stages because they can always be cancelled out when we calculate the success probability. The success probability for only one recovery procedure is given by

$$P_{N=1} = P_0 P_{00}(q) = \frac{q^2}{(1 + q^2)}.$$ \hspace{1cm} (A.8)

If we get the $|01\rangle$ or $|10\rangle$ result, we can perform an additional recovery procedure and from equation (9) we can see that the success probability is $P_0 P_{01}(q) P_{010}(q)$ where $P_{010}(q) = q^2/(1 + q^2)$. If we get the $|11\rangle$ result, we repeat the recovery procedure and we have the probability $P_0 P_{11}(q) P_{100}(q)$. Thus the success probability for two recovery procedures is given by

$$P_{N=2} = P_0 P_{00}(q) + 2P_0 P_{01}(q) P_{010}(q) + P_0 P_{11}(q) P_{100}(q).$$ \hspace{1cm} (A.9)

Similarly, for three recovery procedures the success probability is given by

$$P_{N=3} = P_0 P_{00}(q) + 2P_0 P_{01}(q) P_{010}(q) + P_0 P_{11}(q) P_{010}(q^2)$$

$$+ 2P_0 P_{11}(q) P_{100}(q^2) + P_0 P_{11}(q) P_{100}(q^4).$$ \hspace{1cm} (A.10)

where $P_{010} = 1/(1 + q^2)$ is the probability of getting the $|1\rangle$ result in equation (9). In principle, we can calculate the success probability for any repeating times. However, in practice we need only about three repeating times because the success probability increases by a very small amount for repeating procedures $N > 3$.

**Appendix B. Quantum entanglement protection without a weak measurement**

The quantum state after the amplitude damping is given by equation (11). We add two ancilla qubits which are initially in the $|0\rangle$ state. After applying the Hadamard and CNOT gates, the state of the system becomes

$$|\psi_2\rangle = \cos^2 \theta (|00\rangle S |00\rangle_E + \beta \sqrt{p} |00\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} |01\rangle S |00\rangle_E + \delta \sqrt{p} |00\rangle S |11\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \cos \theta \sin \theta (|00\rangle S |00\rangle_E + \beta \sqrt{p} |00\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} |01\rangle S |00\rangle_E + \delta \sqrt{p} |00\rangle S |11\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |00\rangle_E$$

$$+ \beta \sqrt{q} |01\rangle S |00\rangle_E + \gamma \sqrt{q} \cos \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \sin \theta (|00\rangle S |00\rangle_E + \beta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_E + \gamma \sqrt{q} \sin \theta |01\rangle S |01\rangle_E$$

$$+ \gamma \sqrt{p} \cos \theta |01\rangle S |00\rangle_E + \delta \sqrt{p} \sin \theta (|00\rangle S |01\rangle_E$$

$$+ \beta \sqrt{q} \cos \theta |01\rangle S |00\rangle_
Measuring the ancilla qubits and if we have the |00⟩_A result and \(\tan \theta = 1/\sqrt{q}\), we can obtain

\[
|\psi\rangle_{\text{out}} = |00⟩_E + \beta \sqrt{p} |00⟩_E |01⟩_E + \gamma \sqrt{p} |00⟩_E |10⟩_E
\]

where we neglect the normalization factor. After tracing out the environment, the density matrix of the system is given by equation (12).

If we get other results we can perform an additional procedure to protect the entanglement. For example, if we get the |01⟩ result, we add an additional qubit for the second qubit. In the recovery procedure, the state evolves as

\[\psi_{\text{r}}(\theta, \phi) = \psi (\theta, \phi) \otimes |0⟩_A\]

for the concurrences (equations (14) and (18)) and fidelities by replacing (B.4). If we choose \(\theta\) such that \(xy = 1\) where \(x = \tan^2 \theta\), we have

\[
|\psi\rangle_{\text{out}} = |00⟩_E + \beta \sqrt{p} |00⟩_E |01⟩_E + \gamma \sqrt{p} |00⟩_E |10⟩_E + \delta \sqrt{p} |00⟩_E |11⟩_E + \delta \sqrt{p} |01⟩_E |00⟩_E + \delta \sqrt{p} |01⟩_E |01⟩_E + \delta \sqrt{p} |01⟩_E |10⟩_E + \delta \sqrt{p} |01⟩_E |11⟩_E + \delta \sqrt{p} |10⟩_E |00⟩_E + \delta \sqrt{p} |10⟩_E |01⟩_E + \delta \sqrt{p} |10⟩_E |10⟩_E + \delta \sqrt{p} |10⟩_E |11⟩_E + \delta \sqrt{p} |11⟩_E |00⟩_E + \delta \sqrt{p} |11⟩_E |01⟩_E + \delta \sqrt{p} |11⟩_E |10⟩_E + \delta \sqrt{p} |11⟩_E |11⟩_E.
\]

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