SIMPLICIAL RESOLUTIONS FOR THE SECOND POWER OF SQUARE-FREE MONOMIAL IDEALS

SUSAN M. COOPER, SABINE EL KHOURY, SARA FARIDI, SARAH MAYES-TANG, SUSAN MOREY, LIANA M. ŞEGA, AND SANDRA SPIROFF

ABSTRACT. Given a square-free monomial ideal $I$, we define a simplicial complex labeled by the generators of $I^2$ which supports a free resolution of $I^2$. As a consequence, we obtain (sharp) upper bounds on the Betti numbers of the second power of any square-free monomial ideal.

1. INTRODUCTION

The question of finding, or even effectively bounding, the Betti numbers of an ideal in a commutative ring is a difficult one. Even more complicated is using the structure of an ideal $I$ to find information about the Betti numbers of its powers $I^r$: predicting something as basic as the minimal number of generators of $I^r$ is a difficult problem.

Taylor’s thesis [10] described a free resolution of any ideal minimally generated by $q$ monomials using the simplicial chain complex of a simplex with $q$ vertices. Taylor’s construction, though often far from minimal, produces a resolution of every monomial ideal $I$. It gives upper bounds $(\binom{q+i}{i+1}) \geq \beta_i(I)$ for the Betti numbers of $I^r$, where $(\binom{q+i}{i+1})$ is the number of $i$-faces of a $q$-simplex. If $I$ is generated by $q$ monomials and $r$ is a positive integer, then the number of generators of $I^r$ generally grows exponentially and as a result, so do the bounds on the Betti numbers of $I^r$ given by Taylor’s resolution.

In this paper, we focus on the case where $r = 2$ and $I$ is a square-free monomial ideal with $q$ generators. In this case, we know that $I^2$ can be generated by at most $\binom{q+1}{2}$ monomials, and hence has a Taylor resolution supported on a simplex with at most $\binom{q+1}{2}$ vertices. The question that we address in this paper is: can we find a subcomplex of this simplex whose simplicial chain complex yields a free resolution of $I^2$? Such a resolution would be closer to minimal than the Taylor resolution.

We answer this question by constructing a simplicial complex on $\binom{q+1}{2}$ vertices which we call $L^2_q$ in honor of the Lyubeznik resolution [9] which was our inspiration. While $L^2_q$ has the same number of vertices as the $\binom{q+1}{2}$-simplex, it is significantly smaller because it has far fewer faces. For a given square-free monomial ideal $I$, we can use further deletions of $L^2_q$ specific to the generators of $I$ to show that $L^2_q$ has an induced subcomplex $L^2_q(I)$ which supports a free resolution of $I^2$. As a result, we find (sharp) upper bounds on the Betti numbers of the second power of any square-free monomial ideal. These bounds are often significantly smaller than the bounds provided by the Taylor resolution (see Section 4).

2010 Mathematics Subject Classification. 13D02; 13F55.

Key words and phrases. powers of ideals; simplicial complex; Betti numbers; free resolutions; monomial ideals.
Section 2 lays out the notation and terminology used in the paper including the construction of simplicial resolutions. In Section 3, we describe the complexes $L^2_q$ (Definition 3.1) and $L^2(I)$ (Definition 3.4) and prove that $L^2(I)$ supports a free resolution of $I^2$ when $I$ is a square-free monomial ideal (Theorem 3.9). Section 4 provides results on the bounds on the Betti numbers that follow from the main results.

This paper is part of a larger project [3] to study resolutions of powers of monomial ideals, which the authors started during the 2019 Banff workshop “Women in Commutative Algebra”.

2. BACKGROUND

Throughout this paper we let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k$. In this section we briefly recall some necessary background about simplicial complexes.

A simplicial complex $\Delta$ over a vertex set $V$ is a set of subsets of $V$ such that if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. An element $\sigma$ of $\Delta$ is called a face and the maximal faces under inclusion are called facets. A simplicial complex can be uniquely determined by its facets, and we use the notation $\Delta = \langle F_0, \ldots, F_q \rangle$ to describe a simplicial complex whose facets are $F_0, \ldots, F_q$.

The dimension of a face $F$ in $\Delta$ is $\dim(F) = |F| - 1$, and the dimension of $\Delta$ is the maximum of the dimensions of its faces.

A simplicial complex with one facet is called a simplex.

If $W \subseteq V$, the subcomplex $\Delta_W = \{ \sigma \in \Delta \mid \sigma \subseteq W \}$ is called the induced subcomplex of $\Delta$ on $W$.

If $\Delta$ is a simplicial complex with vertex $v$, then when we delete $v$ from $\Delta$ we obtain the simplicial complex $\Delta \setminus \{v\} = \{ \sigma \in \Delta \mid v \notin \sigma \}$.

A facet $F$ of $\Delta$ is said to be a leaf if it is the only facet of $\Delta$, or there is a different facet $G$ of $\Delta$, called a joint, such that $F \cap H \subseteq G$ for all facets $H \neq F$. The joint $G$ in this definition is not unique ([4]). A simplicial complex $\Delta$ is a quasi-forest if the facets of $\Delta$ can be ordered as $F_0, \ldots, F_q$ such that for $i = 0, \ldots, q$, the facet $F_i$ is a leaf of the simplicial complex $\langle F_0, \ldots, F_i \rangle$. A connected quasi-forest is called a quasi-tree ([11]).

Example 2.1. The simplicial complex below is a quasi-tree, with leaf order: $F_0, F_1, F_2, F_3$, meaning that each $F_i$ is a leaf of $\langle F_0, \ldots, F_i \rangle$. Note that in this case, the joint of $F_i$ is $F_0$ for every $i \geq 1$.

This complex is in fact $\mathbb{L}_3^3$ as we will see later in Example 3.2.
If $I$ is minimally generated by monomials $m_1, \ldots, m_q$ in $S$, a **minimal free resolution** of $I$ is a (unique up to isomorphism) exact sequence of free $S$-modules

$$0 \to S^{\beta_0} \to S^{\beta_{p-1}} \to \cdots \to S^{\beta_1} \to S^{\beta_0} \to I \to 0$$

where $p \in \mathbb{N}$, $\beta_0 = q$ and for each $i \in \{1, \ldots, p\}$, $\beta_i$ is the smallest possible rank of a free module in the $i$-th spot of any free resolution of $I$. The $\beta_i$, called the **Betti numbers** of $I$, are invariants of the ideal $I$.

Finding ways to describe a free resolution of a given ideal is an open and active area of research. For monomial ideals, combinatorics plays a big role. In her thesis in the 1960’s, Diana Taylor introduced a method of labeling the faces of a simplex $\Delta$ with monomials, and then used this labeling to turn the simplicial chain complex of $\Delta$ into a free resolution of a monomial ideal. This technique has been generalized to other simplicial complexes by Bayer and Sturmfels [2], among others.

More precisely, if $I$ is minimally generated by monomials $m_1, \ldots, m_q$ and $\Delta$ is a simplicial complex on $q$ vertices $v_1, \ldots, v_q$, we label each vertex $v_i$ with the monomial $m_i$, and we label each face of $\Delta$ with the least common multiple of the labels of its vertices. Then, if the labeling of $\Delta$ satisfies certain properties, the simplicial chain complex of $\Delta$ can be “homogenized” using the monomial labels on the faces to give a free resolution of $I$. In this case, we say that $\Delta$ **supports a free resolution** of $I$ and the resulting free resolution is called a simplicial resolution of $I$. Peeva’s book [8] details this method for simplicial as well as other topological resolutions.

**Example 2.2.** Let $I = (x^2, y^2, z^2, xy, xz, yz)$. In the picture below, we label the simplicial complex $\Delta$ in Example 2.1 using the generators of $I$. To make the picture less busy, we have included the labels of the vertices and the facets only.

```
   x^2
 x^2y \quad xy^2z
 xz
 y^2
```

Our main result Theorem 3.9 will prove that, indeed, $\Delta$ does support a free resolution of $I$. This in particular implies that $\beta_i(I)$ is bounded above by the number of $i$-faces of $\Delta$, which is the rank of the $i$-th chain group of $\Delta$. That is,

$$\beta_0(I) \leq 6, \quad \beta_1(I) \leq 9, \quad \beta_2(I) \leq 4, \quad \beta_i(I) = 0 \text{ if } i > 2.$$

We calculate, using Macaulay2 [7], that the actual Betti numbers of $I$ are:

$$\beta_0(I) = 6, \quad \beta_1(I) = 8, \quad \beta_2(I) = 3, \quad \beta_i(I) = 0 \text{ if } i > 2.$$

A major question in the theory of combinatorial resolutions is to determine whether a given simplicial complex supports a free resolution of a given monomial ideal. Taylor proved that a simplex with $q$ vertices always supports a free resolution of an ideal with $q$ generators, or in other words, every monomial ideal has a Taylor resolution. As a result $\binom{q}{i+1}$ (the number of $i$-faces of a simplex with $q$ vertices) is an upper bound for $\beta_i(I)$ if $I$ is any monomial ideal with $q$ generators. We denote the $q$-simplex labeled with the $q$ generators of $I$ by Taylor$(I)$. 
Taylor’s resolution is usually far from minimal. However, if \( I \) is a monomial ideal with a free resolution supported on a (labeled) simplicial complex \( \Delta \), then \( \Delta \) has to be a subcomplex of Taylor(\( I \)). As a result, the question of finding smaller simplicial resolutions of \( I \) turns into a question of finding smaller subcomplexes of Taylor(\( I \)) which support a resolution of \( I \).

One of the best known tools to identify such subcomplexes of the Taylor complex is due to Bayer, Peeva, and Sturmfels \([1]\), and reduces the problem to checking acyclicity of induced subcomplexes. This criterion was adapted in \([5]\) to the class of simplicial trees, and then in \([3]\) to quasi-trees. Theorem 2.3 is this latter adaptation, and will be used in the rest of the paper.

If \( \Delta \) is a subcomplex of Taylor(\( I \)) and \( \mathfrak{m} \) is a monomial in \( S \), let \( \Delta_{\mathfrak{m}} \) be the subcomplex of \( \Delta \) induced on the vertices of \( \Delta \) whose labels divide \( \mathfrak{m} \), and let LCM(\( I \)) denote the set of monomials that are least common multiples of arbitrary subsets of the minimal monomial generating set of \( I \).

**Theorem 2.3 (\([3]\) Criterion for quasi-trees supporting resolutions).** Let \( \Delta \) be a quasi-tree whose vertices are labeled with the monomial generating set of a monomial ideal \( I \) in the polynomial ring \( S \) over a field \( k \). Then \( \Delta \) supports a resolution of \( I \) if and only if for every monomial \( \mathfrak{m} \) in LCM(\( I \)), \( \Delta_{\mathfrak{m}} \) is empty or connected.

If \( I \) is minimally generated by \( q \) monomials, then \( I^2 \) is minimally generated by at most \( \binom{q+1}{2} \) monomials. Our goal in this paper is to find a (smaller) subcomplex of the \( \binom{q+1}{2} \)-simplex which produces a free resolution of \( I^2 \), and only depends on \( q \). The quasi-tree \( \mathbb{L}^2_q \), introduced in the next section, is such a candidate: it has exactly \( \binom{q+1}{2} \) vertices, and for any given ideal \( I \) with \( q \) generators, it has an induced subcomplex \( \mathbb{L}^2(I) \) contained in Taylor(\( I^2 \)) which supports a free resolution of \( I^2 \).

### 3. The quasi-trees \( \mathbb{L}^2_q \) and \( \mathbb{L}^2(I) \)

For an integer \( q \geq 1 \) we now give a description of a simplicial complex \( \mathbb{L}^2_q \), a subcomplex of the \( \binom{q+1}{2} \)-simplex. We will show that if \( I \) is a monomial ideal generated by \( q \) square-free monomials, an induced subcomplex of \( \mathbb{L}^2_q \), which we denote by \( \mathbb{L}^2(I) \), always supports a free resolution of \( I^2 \). The complex \( \mathbb{L}^2_q \) is a far smaller subcomplex of the \( \binom{q+1}{2} \)-simplex, and its construction is motivated by the monomial orderings used to build the Lyubeznik complex \([9]\).

**Definition 3.1.** For an integer \( q \geq 3 \), the simplicial complex \( \mathbb{L}^2_q \) over the vertex set \( \{ \ell_{i,j} : 1 \leq i \leq j \leq q \} \) is defined by its facets as:

\[
\mathbb{L}^2_q = \langle \{ \ell_{i,j} : 1 \leq j \leq q \}_{1 \leq i \leq q}, \quad \{ \ell_{i,j} : 1 \leq i < j \leq q \} \rangle,
\]

where we define \( \ell_{j,i} \) for \( j > i \) by the equality \( \ell_{j,i} = \ell_{i,j} \). For \( q = 1 \) and \( q = 2 \) we use the same construction but note that \( \{ \ell_{i,j} : 1 \leq i < j \leq q \} \) is empty for \( q = 1 \) and is a face but not a facet for \( q = 2 \).

When \( q = 1 \), the ideals \( I \) and \( I^r \) for all \( r \geq 2 \) are principal and \( \mathbb{L}^2_q \) is a point. When \( q = 2 \), the complex \( \mathbb{L}^2_2 \) has only 2 facets, see Example 3.2. Note that \( \mathbb{L}^2_q \) has \( \binom{q+1}{2} \) vertices, which is the number of vertices of the \( \binom{q+1}{2} \)-simplex, and, when \( q > 2 \), it has \( q + 1 \) facets, where one facet has dimension \( \binom{q}{2} - 1 \) and the remaining \( q \) facets have dimension \( q - 1 \).
Example 3.2. The complexes $L_2^2$ and $L_3^2$ are shown on the left and right, respectively.

Proposition 3.3. For $q \geq 1$, $L_q^2$ is a quasi-tree.

Proof. If $q = 1$, then $L_1^2$ is a simplex of dimension 0, and so is a quasi-tree. If $q = 2$, there are only two facets, namely $F_1$ and $F_2$ (as depicted above), and $F_2$ is a leaf of $\langle F_1, F_2 \rangle$ with joint $F_1$, so $L_2^2$ is a quasi-tree. For $q \geq 3$, order the facets of $L_q^2$ by $F_0 = \{ \ell_{i,j} : 1 \leq i < j \leq q \}$, and $F_i = \{ \ell_{i,j} : 1 \leq j \leq q \}$ for $1 \leq i \leq q$. By definition, if $i \neq k$ are nonzero, then $F_i \cap F_k = \{ \ell_{i,k} \} \subseteq F_0$. Thus each $F_i$ is a leaf of $\langle F_0, \ldots, F_i \rangle$ with joint $F_0$, and we are done.

Given a square-free monomial ideal $I$, we now define a labeled induced subcomplex of $L_q^2$, denoted $L_q^2(I)$, which is obtained by deleting vertices from $L_q^2$.

Definition 3.4 ($L_q^2(I)$). For an ideal $I$ minimally generated by the square-free monomials $m_1, \ldots, m_q$, we define $L_q^2(I)$ to be a labeled induced subcomplex of $L_q^2$ formed by the following rules:

1. Label each vertex of $\ell_{i,j}$ of $L_q^2$ with the monomial $m_i m_j$.
2. If for any indices $i, j, u, v \in [q]$ where $[q] = \{1, \ldots, q\}$ with $\{i, j\} \neq \{u, v\}$ we have $m_i m_j | m_u m_v$, then
   - If $m_i m_j = m_u m_v$ and $i = \min\{i, j, u, v\}$, then delete the vertex $\ell_{i,j}$.
   - If $m_i m_j \neq m_u m_v$, then delete the vertex $\ell_{u,v}$.
3. Label each of the remaining faces with the least common multiple of the labels of its vertices.

The remaining labeled subcomplex of $L_q^2$ is called $L_q^2(I)$, and is a subcomplex of $\text{Taylor}(I^2)$.

Remark 3.5.

It follows from Proposition 3.7 below that if $m_i^2$ divides $m_u m_v$ then $u = v = i$, hence the vertices $\ell_{i,i}$ are not deleted in the construction of $L_q^2(I)$.

In Step 2 above, when there is equality, the choice was made to eliminate the vertex $\ell_{i,j}$ with minimum index $i$ so that one has a well-defined definition for $L_q^2(I)$; in fact, one could show that a different choice of elimination would also serve our purposes.

Example 3.6. Let $I = (abe, bc, cdf, ad)$. Setting $m_1 = abe$, $m_2 = bc$ and $m_3 = cdf$, $m_4 = ad$, we first label all vertices of $L_4^2$ with the products $m_i m_j$, but then note that
$m_2m_4 \mid m_1m_3$.

So the (labeled) facets of $\mathbb{L}^2(I)$ are the following five:

| Facet                                      | Dimension |
|--------------------------------------------|-----------|
| $\{m_1^2, m_1m_2, m_1m_4\}$              | 2         |
| $\{m_2^2, m_1m_2, m_2m_3, m_2m_4\}$      | 3         |
| $\{m_3^2, m_2m_3, m_3m_4\}$              | 2         |
| $\{m_4^2, m_1m_4, m_2m_4, m_3m_4\}$      | 3         |
| $\{m_1m_2, m_1m_4, m_2m_3, m_2m_4, m_3m_4\}$ | 4         |

In particular, $\mathbb{L}^2(I)$ is a 4-dimensional complex labeled with the generators of $I^2$.

We now present two preliminary results needed for the proof that when the ideal $I$ is square-free, $\mathbb{L}^2(I)$ supports a free resolution of $I^2$.

**Proposition 3.7.** Let $m_1, \ldots, m_q$ be a minimal square-free monomial generating set for an ideal $I$, let $r$ be a positive integer, and suppose that for some $i \in [q]$ and $1 \leq u_1 \leq \cdots \leq u_r \leq q$,

$$m_i^r \mid m_{u_1} \cdots m_{u_r} \quad \text{or} \quad m_{u_1} \cdots m_{u_r} \mid m_i^r.$$  

Then $u_1 = \cdots = u_r = i$.

**Proof.** If for all, or some, of $j \in [r]$ we have $u_j = i$, then those copies of $m_i$ can be deleted from each side of the division, so one can assume, without loss of generality that $i = 1 < u_1 \leq \cdots \leq u_r \leq q$. Suppose that

$$m_1 = x_1^{a_1} \cdots x_n^{a_n} \quad \text{and} \quad m_{u_j} = x_1^{b_1} \cdots x_n^{b_n},$$

where $a_v, b_v \in \{0, 1\}$ for $j \in [r]$ and $v \in [n]$. It follows that:

- if $m_i^r \mid m_{u_1} \cdots m_{u_r}$, then for every index $v \in [n]$ where $a_v \neq 0$, we have $ra_v = r$ and so $b_1 = \cdots = b_v = 1$. Therefore, we have $m_1 \mid m_{u_j}$ for $j \in [r]$. This is a contradiction since these monomials are minimal generators of $I$.
- if $m_{u_1} \cdots m_{u_r} \mid m_i^r$, then for each nonzero exponent $b_v$ of $m_{u_j}$ we must have $a_v \neq 0$, and so $m_{u_1} \mid m_1$, again a contradiction.

$\square$
Proposition 3.8. Let $I$ be an ideal minimally generated by square-free monomials $m_1, \ldots, m_q$ with $q \geq 2$. Then for every $i \in [q]$ there is a $j \in [q] \setminus \{i\}$ such that
\[ m_u m_v \nmid m_i m_j \text{ for any choice of } u, v \in [q] \setminus \{i, j\}. \]
In particular, $m_i m_j$ is a minimal generator of $I^2$.

Proof. Suppose, by way of contradiction, that there exists $i \in [q]$ such that for every $j \in [q] \setminus \{i\}$ there exist $u, v \in [q] \setminus \{i, j\}$ such that $m_u m_v \mid m_i m_j$.

With $i$ as above, there exist functions $\varphi, \psi: [q] \setminus \{i\} \to [q] \setminus \{i\}$ such that
\[ m_{\varphi(j)} m_{\psi(j)} \mid m_i m_j \quad \text{for all } j \in [q] \setminus \{i\}. \tag{1} \]

For each $k \geq 0$, let $\varphi^k$ denote the composition $\varphi \circ \varphi \circ \cdots \circ \varphi$ ($k$ times). (When $k = 0$, $\varphi^0$ is the identity function.) Let $a \in [q] \setminus \{i\}$. For each $w \geq 1$, set $b_w = \psi(\varphi^{w-1}(a))$. Apply (1) with $j = \varphi^{k-1}(a)$ to get:
\[ m_{\varphi^k(a)} m_{b_k} \mid m_i m_{\varphi^{k-1}(a)} \quad \text{for all } k \geq 1. \]

From this, it is easy to see that
\[ \left( m_{\varphi^k(a)} \cdot \prod_{w=1}^k m_{b_w} \right) \mid \left( m_i m_{\varphi^{k-1}(a)} \cdot \prod_{w=1}^{k-1} m_{b_w} \right) \quad \text{for all } k \geq 2. \]

Inductively, we thus obtain
\[ \left( m_{\varphi^k(a)} \cdot \prod_{w=1}^k m_{b_w} \right) \mid \left( m_i^s m_{\varphi^{k-s}(a)} \cdot \prod_{w=1}^{k-s} m_{b_w} \right) \quad \text{for all } k \geq 2 \text{ and } k > s \geq 1. \tag{2} \]

Assume $\varphi^k(a) = \varphi^{k-s}(a)$ for some $k \geq 2$ and some $s$ with $k > s \geq 1$. After simplifying in (2) we obtain
\[ \left( \prod_{w=k-s+1}^k m_{b_w} \right) \mid m_i^s. \]

For $s = 1$, this implies $m_{b_k} \mid m_i$, but since $b_k \neq i$, this contradicts the minimality of the generating set. If $s > 1$ this is a contradiction according to Proposition 3.7. Therefore, we have shown that the integers $\varphi(a), \varphi^2(a), \ldots$ are distinct. This is a contradiction, since $\varphi^k(a) \in [q] \setminus \{i\}$ for all $k$, and $[q] \setminus \{i\}$ is a finite set. \qed

We are now ready to prove the main result of the paper.

Theorem 3.9 (Main Result). Let $I$ be a square-free monomial ideal. Then $\mathbb{L}^2(I)$ supports a free resolution of $I^2$.

Proof. Suppose $I$ is minimally generated by the square-free monomials $m_1, \ldots, m_q$.

The simplicial complex $\mathbb{L}^2(I)$ is an induced subcomplex of the quasi-tree $\mathbb{L}^2_q$ (Proposition 3.3), and is therefore a quasi-forest itself (see 3.6). Let $V$ denote the set of vertices of $\mathbb{L}^2(I)$. In view of Theorem 3.3 to show that $\mathbb{L}^2(I)$ supports a resolution of $I^2$, we need to show that, for every $m \in \text{LCM}(I^2)$, $\mathbb{L}^2(I)_m$ is connected, where $\mathbb{L}^2(I)_m$ is the induced subcomplex of the complex $\mathbb{L}^2(I)$ on the set $V_m = \{ \ell_{i,j} \in V : m_i m_j \mid m \}$.

Suppose $m \in \text{LCM}(I^2)$. If $q = 1$, then $\mathbb{L}^2(I)_m$ is either empty or a point. If $q = 2$, then $I^2 = (m_1^2, m_1 m_2, m_2^2)$ and $\mathbb{L}^2(I)$, as pictured in Example 3.2 has two facets connected by the vertex $\ell_{1,2}$. If $m \in \{m_1^2, m_2^2\}$, then $\mathbb{L}^2(I)_m$ is a point, and hence
connected. Otherwise, \( m_1 m_2 \mid m \), so the vertex \( \ell_{1,2} \) will be in \( L^2(I)_m \). If either \( \ell_{1,1} \) or \( \ell_{2,2} \) are in \( L^2(I)_m \), they will be connected to \( \ell_{1,2} \). Therefore \( L^2(I)_m \) is connected.

Now assuming \( q \geq 3 \), we use the notation introduced in the proof of Proposition \( \ref{prop:prop1} \) for the facets of \( L^2(I) \), namely \( F_0, \ldots, F_q \). The facets of \( L^2(I)_m \) are the maximal sets among the sets \( F_0 \cap V_m, \ldots, F_q \cap V_m \).

If \( m = m_i^2 \) for some \( i \in [q] \), then Proposition \( \ref{prop:prop2} \) shows that \( L^2(I)_m \) is one point, and hence is connected. Assume now that \( m \neq m_i^2 \) for all \( i \in [q] \), and hence \( F_0 \cap V_m \neq \emptyset \).

To show that \( L^2(I)_m \) is connected, it suffices to show that, for each \( i \in [q] \) such that \( F_i \cap V_m \neq \emptyset \), the intersection between \( F_i \cap V_m \) and \( F_0 \cap V_m \) is nonempty. Note that any vertex in \( F_i \cap V_m \) other than \( \ell_{i,i} \) is also in \( F_0 \cap V_m \). We thus need to show that if \( \ell_{i,i} \in V_m \) for some \( i \in [q] \), then there exists \( b \in [q] \) with \( b \neq i \) such that \( \ell_{i,b} \in V_m \).

Assume \( \ell_{i,i} \in V_m \), hence \( m_i^2 \mid m \). Set
\[
A = \{ j \in [q] : m_j \mid m \}.
\]
Note that \( i \in A \). Since \( m \neq m_i^2 \), we see that \( |A| \geq 2 \). By Proposition \( \ref{prop:prop2} \) applied to the ideal generated by the monomials \( m_j \) with \( j \in A \), there exists \( b \in A \setminus \{i\} \) such that
\[
m_u m_v \text{ does not divide } m_i m_b \text{ for all } u, v \in A \setminus \{i, b\}. \tag{3}
\]

Since \( b \in A \), we have \( m_b \mid m \). We claim that \( m_i m_b \mid m \) as well. Indeed, since \( m_b \) is a square-free monomial, setting \( m = m_i^2 n \), one has
\[
m_b \mid m \Rightarrow m_b \mid m_i^2 n \Rightarrow m_b \mid m_i n \Rightarrow m_i m_b \mid m_i^2 n \Rightarrow m_i m_b \mid m. \tag{4}
\]

In order to conclude \( \ell_{i,b} \in V_m \), we need to show that \( \ell_{i,b} \in V \), that is, \( \ell_{i,b} \) is a vertex of \( L^2(I) \). If \( \ell_{i,b} \notin V \), then we must have \( m_u m_v \mid m_i m_b \) for some \( u, v \in [q] \setminus \{i, b\} \). Since \( m_i m_b \mid m \), we further have \( m_u \mid m \) and \( m_v \mid m \), hence \( u, v \in A \). This contradicts (3) above.

**Remark 3.10.** Given any \( q \geq 2 \), there are square-free monomial ideals \( I \) with \( q \) generators such that \( L^2(I) = L^1_q \) and the resolution supported on \( L^2(I) \) is minimal. The ideal \( I = (xabc, yade, zbdf, wcef) \) is such an example when \( q = 4 \), (see [3]).

### 4. A bound on the Betti numbers of \( I^2 \)

We now consider bounds on the Betti numbers of the second power of a square-free monomial ideal \( I \), as provided by the simplicial complex \( L^2(I) \). Since \( I^2 \) has a free resolution supported on \( L^2(I) \), \( \beta_d(I^2) \) is bounded above by the number of \( d \)-faces of \( L^2(I) \), which itself is bounded above by the number of \( d \)-faces of \( L^2_q \).

It can be seen from the proof of Theorem \( \ref{thm:thm1} \) below that the right-hand term of the inequality \((a)\) below is precisely the number of \( d \)-faces of \( L^2_q \). Note that the bound in \((a)\) depends only on the number of generators \( q \), and not on \( I \) itself. The right-hand term of the inequality \((b)\) below is equal to the number of \( d \)-dimensional faces of \( L^2(I) \), which provides a more precise bound that is dependent on the ideal \( I \).

**Theorem 4.1.** Let \( I \) be a square-free monomial ideal minimally generated by \( q \geq 2 \) monomials. Then for each \( d \geq 0 \) the \( d \)-th Betti number \( \beta_d(I^2) \) satisfies
\[
(a) \quad \beta_d(I^2) \leq \left( \frac{1}{2}(q^2 - q) \right) \frac{1}{d + 1} + q \frac{q - 1}{d}. \]
Furthermore, setting $s$ to be the minimal number of generators of $I^2$ and $t_i$ to be the number of vertices of the form $\ell_{i,j}$ that were deleted from $\mathbb{L}_q^2$ when forming $\mathbb{L}_q^2(I)$, then

$$
(b) \quad \beta_d(I^2) \leq \binom{s-q}{d+1} + \sum_{i=1}^{q} \binom{q-1-t_i}{d}.
$$

By Remark 3.10, the bound in (a) is sharp.

Proof. We begin by proving inequality (b). Theorem 3.9 gives that for each $d \geq 0$, $\beta_d(I^2)$ is bounded above by the number of $d$-dimensional faces of $\mathbb{L}_q^2(I)$. We compute this number next.

The faces of $\mathbb{L}_q^2(I)$ are of two types:

1. Faces that do not contain any vertex of the form $\ell_{i,i}$ for $i \in [q]$.
2. Faces that contain a vertex $\ell_{i,i}$ for some $i \in [q]$, and, as a consequence, all the other vertices have the form $\ell_{i,j}$ with $j \in [q] \setminus \{i\}$.

Let $s$ denote the minimal number of generators of $I^2$ and set $t = \binom{q+1}{2} - s$. Since $\binom{q+1}{2}$ is the number of vertices of $\mathbb{L}_q^2$, the integer $t$ is precisely the number of vertices that are deleted in the construction of $\mathbb{L}_q^2(I)$, as described in Definition 3.4. As noted in Remark 3.5, all the deleted vertices $\ell_{i,j}$ must satisfy $i \neq j$, hence the number of vertices $\ell_{i,j}$ of $\mathbb{L}_q^2(I)$ with $i, j \in [q]$ and $i \neq j$ is $\binom{q}{2} - t$, which is equal to $s - q$.

To construct a $d$-dimensional face of type (1), we need to choose $d + 1$ vertices among the vertices $\ell_{i,j}$ of $\mathbb{L}_q^2(I)$ with $i, j \in [q]$ and $i \neq j$. As noted above, there are $s - q$ such vertices. Thus, the number of $d$-dimensional faces of type (1) is $\binom{s-q}{d+1}$.

Fix $i \in [q]$. To construct a $d$-dimensional face of type (2) that contains $\ell_{i,i}$, we need to choose $d$ vertices among the vertices $\ell_{i,j}$ of $\mathbb{L}_q^2(I)$ that satisfy $j \neq i$. There are $q - 1 - t_i$ such vertices, where $t_i$ denotes the number of vertices $\ell_{i,j}$ of $\mathbb{L}_q^2$ that are deleted in $\mathbb{L}_q^2(I)$. Thus the number of $d$-dimensional faces of type (2) is $\sum_{i=1}^{q} \binom{q-1-t_i}{d}$.

Putting the two computations above together, we have that the number of $d$-dimensional faces of $\mathbb{L}_q^2(I)$ is equal to $\binom{s-q}{d+1} + \sum_{i=1}^{q} \binom{q-1-t_i}{d}$, yielding the inequality (b).

Note that inequality (a) follows from (b) by setting $t_i = 0$ for all $i$ and $s = \binom{q+1}{2}$. In view of our computation above, the right-hand side of inequality (a) is precisely the number of $d$-dimensional faces of $\mathbb{L}_q^2$.

For comparison, the fact that Taylor($I^2$) supports a free resolution of $I^2$ gives an inequality

$$
\beta_d(I^2) \leq \binom{\frac{1}{2}(q^2 + q)}{d+1},
$$

where the binomial on the right side denotes the number of $d$-faces of a $\frac{1}{2}(q^2 + q)$-simplex, which is the largest possible size for Taylor($I^2$).
To get an idea how much Theorem 4.1 improves on this bound, we present the following table, for \( q = 4 \):

| \( d \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) |
|---|---|---|---|---|---|---|---|
| \( d \)-faces of largest possible Taylor(\( I^2 \)) \((\binom{10}{d+1})\) | 10 | 45 | 120 | 210 | 252 | 210 | 120 |
| \( d \)-faces of \( L^2_q \) \( \binom{6}{d+1} + 4\binom{3}{d} \) | 10 | 27 | 32 | 19 | 6 | 1 | 0 |

To put this in context, we examine two specific ideals with 4 generators, and use Macaulay2 to find the Betti numbers of these ideals.

**Example 4.2.** For the ideal \( J = (x, y, z, w) \) Macaulay2 gives the following Betti table for \( J^2 \):

| \( d \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|---|---|---|---|---|---|
| \( \beta_d(J^2) \) | 10 | 20 | 15 | 4 |

These Betti numbers should be compared with the bounds in the table above.

Now let \( I = (abe, bc, cdf, ad) \) be the ideal Example 3.6. The Betti numbers of \( I^2 \) as calculated by Macaulay2 are the following.

| \( d \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) |
|---|---|---|---|---|---|
| \( \beta_d(I^2) \) | 9 | 14 | 6 | 0 |

In this case we should compare these Betti numbers with the bounds given by the Taylor complex with 9 vertices and the bounds given by the Theorem 4.1 \( b \). For the given ideal, we saw that \( L^2(I) \) has 9 vertices, and \( m_2 m_4 \) is an eliminated vertex, hence \( s = 9, t_2 = t_4 = 1, t_1 = t_3 = 0 \) in Theorem 4.1 \( b \). We have:

| \( d \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) |
|---|---|---|---|---|---|---|---|
| \( d \)-faces of Taylor(\( I^2 \)) \( \binom{9}{d+1} \) | 9 | 36 | 84 | 126 | 126 | 84 | 36 |
| \( d \)-faces of \( L^2(I) \) \( \binom{5}{d+1} + 2\binom{3}{d} + 2\binom{2}{d} \) | 9 | 20 | 18 | 7 | 1 | 0 | 0 |

**Acknowledgements.** The bulk of this work was done during the 2019 Banff workshop “Women in Commutative Algebra”. We are grateful to the organizers, the funding agencies (NSF DMS-1934391), and to the Banff International Research Station for their hospitality.

Author Şega and Spiroff were partially supported by grants from the Simons Foundation (#354594, #584932, respectively), and authors Cooper and Faridi were supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).
REFERENCES

1. D. Bayer, I. Peeva, B. Sturmfels, Monomial resolutions, Math. Res. Lett. 5, no. 1-2 (1998) 31–46.
2. D. Bayer, B. Sturmfels, Cellular resolutions of monomial modules, J. Reine Angew. Math. 503 (1998) 123–140.
3. S. M. Cooper, S. El Khoury, S. Faridi, S. Mayes-Tang, S. Morey, L. M. Šega and S. Spiroff, Simplicial resolutions of powers of square free monomial ideals, in preparation.
4. S. Faridi, The facet ideal of a simplicial complex, Manuscripta Math. 109, no. 2 (2014) 159-174.
5. S. Faridi, Monomial resolutions supported by simplicial trees, J. Commut. Algebra 6, no. 3 (2014).
6. S. Faridi, B. Hersey, Resolutions of monomial ideals of projective dimension 1, Comm. Algebra 45, no. 12 (2017) 5453–5464.
7. D.R. Grayson, M.E. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/.
8. I. Peeva, Graded syzygies, Algebra and Applications, 14. Springer-Verlag London, Ltd., London, 2011.
9. G. Lyubeznik, A new explicit finite free resolution of ideals generated by monomials in an $R$-sequence, J. Pure Appl. Algebra 51 (1998) 193-195.
10. D. Taylor, Ideals generated by monomials in an $R$-sequence, Thesis, University of Chicago (1966).
11. X. Zheng, Resolutions of facet ideals, Comm. Algebra 32 no. 6 (2004) 2301-2324.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, 520 MACHRAY HALL, 186 DYSART ROAD, WINNIPEG, MB, CANADA R3T 2N2

Email address: susan.cooper@umanitoba.ca

DEPARTMENT OF MATHEMATICS, AMERICAN UNIVERSITY OF BEIRUT, BLISS HALL 315, P.O. BOX 11-0236, BEIRUT 1107-2020, LEBANON

Email address: se24@aub.edu.lb

DEPARTMENT OF MATHEMATICS & STATISTICS, DALHOUSIE UNIVERSITY, 6316 COBURG RD., PO BOX 15000, HALIFAX, NS, CANADA B3H 4R2

Email address: faridi@dal.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE STREET, ROOM 6290, TORONTO, ON, CANADA M5S 2E4

Email address: smt@math.toronto.edu

DEPARTMENT OF MATHEMATICS, TEXAS STATE UNIVERSITY, 601 UNIVERSITY DR., SAN MARCOS, TX 78666, U.S.A.

Email address: morey@txstate.edu

LIANA M. ŠEGA, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MISSOURI, KANSAS CITY, MO 64110, U.S.A.

Email address: segal@umkc.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSISSIPPI, HUME HALL 335, P.O. BOX 1848, UNIVERSITY, MS 38677 USA

Email address: spiroff@olemiss.edu