Flavor Mixing and CP Violation of Massive Neutrinos

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Abstract

We present an overview of recent progress in the phenomenological study of neutrino masses, lepton flavor mixing and CP violation. We concentrate on the model-independent properties of massive neutrinos, both in vacuum and in matter. Current experimental constraints on the neutrino mass spectrum and the lepton flavor mixing parameters are summarized. The Dirac- and Majorana-like phases of CP violation, which are associated respectively with the long-baseline neutrino oscillations and the neutrinoless double beta decay, are discussed in detail. The seesaw mechanism, the leptogenesis scenario and the strategies to construct lepton mass matrices are briefly described. The features of flavor mixing between one sterile neutrino and three active neutrinos are also explored.
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1 Introduction

Neutrino physics is a peculiar part of flavor physics, because neutrinos belong to super-light flavors in comparison with charged leptons and quarks. There are three central concepts in flavor physics: mass, flavor mixing and CP violation.

- **Mass** represents the inertial energy possessed by a particle when it exists at rest. A massless particle has no way to exist at rest - instead, it must always move at the speed of light. A massive fermion (lepton or quark) must exist in both left-handed and right-handed states, because the field operators responsible for the nonzero mass of a fermion have to be bilinear products of the spinor fields which flip the fermion’s handedness.

- **Flavor mixing** measures the mismatch between flavor eigenstates and mass eigenstates of leptons or quarks, caused by the Higgs interactions. Flavor eigenstates are the members of the weak isospin doublets that transform into each other via the interaction with $W^\pm$ bosons, while mass eigenstates are the states of definite masses created by the interaction with Higgs bosons. If neutrinos were massless, lepton flavor mixing would not exist.

- **CP violation** means that matter and antimatter or a reaction and its CP-conjugated process are distinguishable. It may manifest itself in the weak interactions through nontrivial complex phases residing in the flavor mixing matrix of leptons or quarks. If CPT is invariant, CP violation will give rise to T violation, or vice versa.

The problems of fermion masses, flavor mixing and CP violation are correlated with one another and fundamentally important in particle physics. The study of these problems will ultimately help us understand the nature of matter and the matter-antimatter asymmetry of the universe. Since the early 1960’s, a lot of experimental efforts have been made to measure the parameters of quark mixing and CP violation. The problem of lepton mixing and CP violation was not at the front of experimental particle physics for a long time, however, partly because neutrinos were assumed to be massless in the successful theory of weak and electromagnetic interactions – the standard electroweak model [1]. This situation has dramatically changed since the compelling evidence in favor of atmospheric and solar neutrino oscillations was achieved by Super-Kamiokande (SK) [2], SNO [3], KamLAND [4] and K2K [5] Collaborations in the past five years.

Neutrino oscillation is a quantum phenomenon which can naturally occur if neutrinos are massive and lepton flavors are mixed. Thanks to the elegant SK, SNO, KamLAND and K2K experiments, we are now convinced that the long-standing solar neutrino ($\nu_e$) deficit and the atmospheric neutrino ($\nu_\mu$) anomaly are both due to neutrino oscillations. The study of neutrino masses and lepton flavor mixing is therefore becoming one of the hottest fronts of today’s particle physics and cosmology.

Our present knowledge on the properties of neutrinos comes not only from a number of neutrino oscillation experiments, but also from the direct-mass-search experiments, the neutrinoless double beta decay experiments and the astrophysical or cosmological observations [6]. The following is a partial list of what we have known about neutrino masses and lepton flavor mixing.
• Neutrinos are massive but their masses are tiny. The mass scale of three active neutrinos ($\nu_e$, $\nu_\mu$ and $\nu_\tau$) is expected to be at or below $\mathcal{O}(1)$ eV.

• Two independent mass-squared differences of three neutrinos, which are associated separately with solar and atmospheric neutrino oscillations, are very small and have a strong hierarchy. Typically, $\Delta m^2_{\text{sun}} \sim \mathcal{O}(10^{-5})$ eV$^2$ and $\Delta m^2_{\text{atm}} \sim \mathcal{O}(10^{-3})$ eV$^2$ hold.

• Two lepton flavor mixing angles, which are related separately to solar and atmospheric neutrino oscillations, are much larger than the Cabibbo angle of quark mixing ($\theta_C \approx 13^\circ$). Typically, $\theta_{\text{sun}} \sim 33^\circ$ and $\theta_{\text{atm}} \sim 45^\circ$ hold.

• The other lepton flavor mixing angle, which is relevant to the CHOOZ reactor experiment for neutrino oscillations [7], is very small and even vanishing. The generous upper bound of this angle is $\theta_{\text{chz}} \leq \theta_C$ at present.

It is obvious that the mass spectrum of neutrinos and the mixing pattern of lepton flavors are very different from those of quarks.

However, there exist many open questions about massive neutrinos and lepton flavor mixing. For example,

• Are massive neutrinos Dirac or Majorana particles? If massive neutrinos are Dirac particles, they can be distinguished from their antiparticles. By definition, a Majorana neutrino is identical to its antiparticle. It is possible to identify the Majorana nature of massive neutrinos through the observation of the neutrinoless double beta decay of some even-even nuclei, in which the total lepton number is not conserved.

• How many neutrino species are there? We have known that there are three species of active neutrinos ($\nu_e$, $\nu_\mu$ and $\nu_\tau$), corresponding to three species of charged leptons. It remains unclear whether the light sterile neutrinos, which have been assumed to interpret the controversial LSND data [8] on $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations, are really existent or not.

• Why are neutrino masses so tiny? The fact that the masses of neutrinos are considerably smaller than the masses of charged leptons or quarks is a big puzzle to particle physicists. Although a lot of theoretical models about neutrino masses have been proposed at either low or high energy scales, none of them has proved to be very successful and conceivable.

• What is the absolute scale of neutrino masses? It is very important to know the absolute values of three neutrino masses, because they are fundamental parameters of flavor physics. The mass scale of neutrinos is likely to indicate the energy scale of new physics responsible for the generation of neutrino masses and lepton flavor mixing. Unfortunately, we are only aware of the upper bounds of neutrino masses.

• Why are the mixing angles $\theta_{\text{sun}}$ and $\theta_{\text{atm}}$ so big? The bi-large neutrino mixing pattern is also a mystery to many theorists, because it is “anomalously” different from the familiar tri-small quark mixing pattern. Although the lepton flavor mixing
angles are in general expected to relate to the mass spectra of charged leptons and neutrinos, their specific relations have not convincingly or model-independently been established.

- How small is the mixing angle $\theta_{\text{chz}}$? Current experimental data yield an upper bound $\theta_{\text{chz}} \leq \theta_C$, but the possibility $\theta_{\text{chz}} = 0^\circ$ cannot be excluded. The smallness of $\theta_{\text{chz}}$ requires a good theoretical reason, so does the vanishing of $\theta_{\text{chz}}$. If $\theta_{\text{chz}} = 0^\circ$ held, there would be no chance to observe leptonic CP or T violation in normal neutrino-neutrino and antineutrino-antineutrino oscillations.

- Is there leptonic CP violation? A necessary condition for the existence of CP and T violation in normal neutrino oscillations is $\theta_{\text{chz}} \neq 0$. As CP violation has been discovered in the quark sector, we feel that there is no reason why CP should be conserved in the lepton sector. It is very difficult to detect the effects of CP or T violation in any realistic neutrino oscillation experiments, however.

- Can the leptonic CP-violating phases be determined? If lepton flavor mixing is correlated with CP violation, one has to determine the relevant CP-violating phases through various possible experiments. The neutrinoless double beta decay and long-baseline appearance neutrino oscillations are expected to be sensitive to the Majorana- and Dirac-like phases of CP violation, respectively. To implement such measurements remains a big challenge to experimentalists.

Of course, much more experimental and theoretical efforts are needed to make, in order to answer the important questions listed above. Much more phenomenological attempts are also needed to make, so as to bridge the gap between experimental data and theoretical models.

The purpose of this article is to give an overview of recent progress in the phenomenological study of neutrino masses, lepton flavor mixing and CP violation. We concentrate on the model-independent properties of massive neutrinos, although it is unavoidable to introduce two very attractive ansätze in today’s neutrino physics – the seesaw mechanism and the thermal leptogenesis scenario.

The remaining parts of this article are organized as follows. Section 2 is devoted to a review of the neutrino mass spectrum. First of all, we make a short introduction to the Dirac and Majorana neutrino masses, and to the well-known seesaw mechanism. Then current experimental constraints on neutrino masses, including those from the neutrinoless double beta decay, kinematic measurements, neutrino oscillations and cosmological observations, are briefly summarized. Finally some comments are given on the strategies to construct the phenomenological textures of lepton mass matrices.

In section 3, the model-independent features of lepton flavor mixing are illustrated both in vacuum and in matter. We present a classification of various parametrizations of the $3 \times 3$ lepton flavor mixing matrix, and highlight one of them which is particularly useful for the study of neutrino oscillations. A few concise sum rules for neutrino masses and lepton flavor mixing in matter are derived. Several constant mixing patterns of massive neutrinos are introduced. Finally we point out the differences and similarities between the phenomenon of lepton flavor mixing and that of quark flavor mixing.
Section 4 is devoted to leptonic CP and T violation. The rephasing invariants of CP violation and the unitarity triangles are described both in vacuum and in matter. A few salient features of CP and T violation in long-baseline neutrino oscillations are also discussed. We pay some special interest to the thermal leptogenesis scenario to interpret the observed baryon-antibaryon asymmetry of the universe. Finally we comment on the possible connection or disconnection between the leptonic CP-violating quantities at high and low energy scales.

The conclusion and outlook are presented in section 5.

It is worth remarking that the main body of this article deals with neutrino masses, lepton flavor mixing and CP violation in the scheme of three lepton families. To be complete, the properties of flavor mixing between one sterile neutrino ($\nu_s$) and three active neutrinos ($\nu_e$, $\nu_\mu$, and $\nu_\tau$) are discussed in appendix A.

## 2 Neutrino Mass Spectrum

### 2.1 Theoretical Preliminaries

The standard model of electromagnetic and weak interactions is based on the gauge group $\text{SU}(2)_L \times \text{U}(1)_Y$ [1]. In this framework only the left-handed leptons (and quarks), which transform as $\text{SU}(2)$ doublets, take part in the charged-current weak interactions:

$$-\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} (\nu_e, \nu_\mu, \nu_\tau)_L \gamma^\mu \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_L W^+_{\mu} + \text{h.c.} ,$$

(2.1)

where $(e, \mu, \tau)$ and $(\nu_e, \nu_\mu, \nu_\tau)$ are the flavor eigenstates of charged leptons and neutrinos, respectively. If $\text{SU}(2)_L$ were an exact symmetry, all leptons would be massless and their flavor eigenstates would be physically indistinguishable from their mass eigenstates. In reality, however, this gauge symmetry is badly broken. A spontaneous breakdown of the $\text{SU}(2)_L$ symmetry is realized in the standard model by means of the Higgs mechanism. After the symmetry breaking, the charged leptons (and quarks) acquire their masses through the Yukawa interactions:

$$-\mathcal{L}_l = (e, \mu, \tau)_L M_l \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_R + \text{h.c.} ,$$

(2.2)

where $M_l$ denotes the charged lepton mass matrix and its scale is characterized by the electroweak symmetry breaking scale $v \approx 174 \text{ GeV}$. The vanishing of three neutrino masses follows as a straightforward consequence of the symmetry structure of the standard model, in which only a single Higgs doublet exists and the lepton number conservation is assumed

$\text{I}^1$It is actually the $(B-L)$ symmetry that makes neutrinos exactly massless in the standard model, where $B$ denotes the baryon number and $L$ stands for the total lepton number. The reason is simply that a neutrino $\nu$ and an antineutrino $\bar{\nu}$ have different values of $(B-L)$ [9]. Thus the naive argument for massless neutrinos is valid to all orders in perturbation and non-perturbation theories, if $(B-L)$ is an exact symmetry.
Note that the mass eigenstates of charged leptons can always be chosen to coincide with their flavor eigenstates through an appropriate but physically-irrelevant unitary transformation of the right-handed fields. In such a specific flavor basis the coincidence between mass and flavor eigenstates of neutrinos can also be achieved, if neutrinos are assumed to be exactly massless Weyl particles. Hence there is no lepton flavor mixing within the framework of the standard electroweak model.

2.1.1 Dirac and Majorana Masses

However, the assumption of lepton number conservation or masslessness of neutrinos is not assured by any basic symmetry principle of particle physics. Most reasonable extensions of the standard model (such as the grand unified theories) do allow the absence of lepton number conservation and the existence of nonvanishing neutrino masses. If neutrinos are really massive and their masses are non-degenerate, it will in general be impossible to find a basis of the flavor space in which the coincidence between flavor and mass eigenstates holds both for charged leptons and for neutrinos. In other words, the flavor mixing phenomenon is naturally expected to appear between three charged leptons and three massive neutrinos, just like the flavor mixing between three up-type quarks and three down-type quarks [10].

If neutrinos have nonzero masses, they may be either Dirac or Majorana particles. The field of a massive Dirac neutrino describes four independent states: left-handed and right-handed particle states ($\nu_L$ and $\nu_R$) as well as left-handed and right-handed antiparticle states ($\bar{\nu}_L$ and $\bar{\nu}_R$). Among them the $\nu_L$ and $\bar{\nu}_R$ states, which already exist in the standard model, can take part in weak interactions. The $\nu_R$ and $\bar{\nu}_L$ states need to be introduced into the standard model as necessary ingredients to give the Dirac neutrino a mass, but they should be “sterile” in the sense that they do not take part in the normal weak interactions. A Dirac mass term, which conserves the total lepton number but violates the law of individual lepton flavor conservation, can be written as follows:

$$-\mathcal{L}_D = \overline{(\nu_e, \nu_\mu, \nu_\tau)}_L M_D \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_R + \text{h.c.}, \quad (2.3)$$

where $M_D$ denotes the $3 \times 3$ Dirac neutrino mass matrix. The mass term in (2.3) is quite similar to the mass term of charged leptons in (2.2), hence the scale of $M_D$ should also be characterized by the gauge symmetry breaking scale $v$. In this case the Yukawa coupling constants of three neutrinos must be extremely smaller than those of three charged leptons or six quarks, such that tiny neutrino masses can result. This dramatic difference between the Yukawa couplings of neutrinos and charged leptons (or quarks) is commonly considered to be very unnatural in a sound theory of fermion mass generation, however.

On the other hand, the neutrino $\nu$ may be a Majorana particle, which has only two independent states of the same mass ($\nu_L$ and $\bar{\nu}_R$, or $\nu_R$ and $\bar{\nu}_L$). By definition, a Majorana neutrino is its own antiparticle: $\nu^c \equiv C \bar{\nu}^T = e^{i\Theta} \nu$ [11], where $C$ denotes the charge-conjugation operator and $\Theta$ is an arbitrary real phase. A Majorana mass term, which violates both the law of total lepton number conservation and that of individual
lepton flavor conservation, can be written either as

\[ -\mathcal{L}_{\text{M}(L)} = \frac{1}{2} (\nu_e, \nu_\mu, \nu_\tau)_L M_L \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_R + \text{h.c.} , \]  

(2.4)

or as

\[ -\mathcal{L}_{\text{M}(R)} = \frac{1}{2} (\nu_e^c, \nu_\mu^c, \nu_\tau^c)_L M_R \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_R + \text{h.c.} , \]  

(2.5)

where \( M_L \) and \( M_R \) denote the symmetric 3 \( \times \) 3 mass matrices of left-handed and right-handed Majorana neutrinos, respectively. Note that the mass term \( \mathcal{L}_{\text{M}(L)} \) cannot naturally arise from a simple theory of electroweak interactions which is invariant under the SU(2)\(_L\) \( \times \) U(1)\(_Y\) gauge transformation and has no SU(2)\(_L\) triplet field (like the case in the standard model). It is possible to incorporate the mass term \( \mathcal{L}_{\text{M}(R)} \) into an electroweak theory with SU(2)\(_L\) \( \times \) U(1)\(_Y\) gauge symmetry, because right-handed neutrinos are SU(2)\(_L\) singlets. In general, a neutrino mass Lagrangian may include all of the above-mentioned terms: \( \mathcal{L}_D \), \( \mathcal{L}_{\text{M}(L)} \) and \( \mathcal{L}_{\text{M}(R)} \), in which \( M_D \), \( M_L \) and \( M_R \) are complex mass matrices.

A variety of specific theoretical and phenomenological models have been prescribed in the literature to interpret how lepton masses are generated and why neutrino masses are so tiny. Regardless of the energy scales at which those models are built, the mechanisms responsible for fermion mass generation and flavor mixing can roughly be classified into five different categories [12]: (a) Radiative mechanisms [13]; (b) Texture zeros [14]; (c) Family symmetries [15]; (d) Seesaw mechanisms [16]; and (e) Extra dimensions [17]. Among them, the seesaw mechanisms are particularly natural and interesting. A brief introduction to the seesaw idea will be presented in section 2.1.2. For recent reviews of other interesting models and ans"atze on neutrino masses, we refer the reader to Ref. [12] and Refs. [18]–[21].

### 2.1.2 The Seesaw Mechanism

A simple extension of the standard model is to include one right-handed neutrino in each of three lepton families, while the Lagrangian of electroweak interactions keeps invariant under the SU(2)\(_L\) \( \times \) U(1)\(_Y\) gauge transformation. After spontaneous symmetry breaking, the resultant lepton mass term \( \mathcal{L}_{\text{mass}} \) consists of \( \mathcal{L}_L \) (charged lepton term), \( \mathcal{L}_D \) (Dirac neutrino term) and \( \mathcal{L}_{\text{M}(R)} \) (righ-handed Majorana neutrino term), which have been given in (2.2), (2.3) and (2.5). To be explicit, we have

\[ -\mathcal{L}_{\text{mass}} = -\mathcal{L}_L + \frac{1}{2} (\nu, \nu^c)_L \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \begin{pmatrix} \nu^c \\ \nu \end{pmatrix}_R , \]  

(2.6)

where \( \nu \) denotes the column vector of \( \nu_e \), \( \nu_\mu \) and \( \nu_\tau \) fields; i.e., \( \nu^T \equiv (\nu_e, \nu_\mu, \nu_\tau) \) or \( \nu^c \equiv (\nu_e^c, \nu_\mu^c, \nu_\tau^c)^T \). In obtaining (2.6), we have used the relation \( \overline{\nu}_L M_D \nu_R = (\nu^c)_L M_D^T (\nu^c)_R \) as well as the properties of \( \nu \) and \( \nu^c \). As pointed out in section 2.1.1, the scale of \( M_D \) is characterized by the electroweak symmetry breaking scale \( v \). The scale of \( M_R \) can naturally be much higher than \( v \), because right-handed neutrinos are SU(2)\(_L\) singlets and their corresponding mass term is not subject to the scale of gauge symmetry breaking. It
Figure 2.1: Illustrative plot for the seesaw mass term.

has commonly been expected that the scale of $M_R$ is not far away from the grand unified theory (GUT) scale $\Lambda_{\text{GUT}} \sim 10^{16}$ GeV. In this case, the smallness of left-handed neutrino masses ($m_i \ll v$ for $i = 1, 2, 3$) is essentially attributed to the largeness of right-handed neutrino masses ($M_i \gg v$ for $i = 1, 2, 3$). Such an elegant idea, the so-called seesaw mechanism, was first proposed by Yanagida and by Gell-Mann, Ramond and Slansky in 1979 [16]. Fig. 2.1 illustrates the seesaw mass term given in (2.6).

To show how the seesaw mechanism works, we diagonalize the symmetric $6 \times 6$ neutrino mass matrix in (2.6) by use of the following unitary transformation:

$$
\begin{pmatrix}
V & R \\
S & U
\end{pmatrix}
\begin{pmatrix}
0 & M_D \\
M_D^T & M_R
\end{pmatrix}
\begin{pmatrix}
V^T & R^T \\
S^T & U^T
\end{pmatrix}
= 
\begin{pmatrix}
\bar{M}_\nu & 0 \\
0 & \bar{M}_R
\end{pmatrix},
$$

(2.7)

where $R$, $S$, $U$ and $V$ are the $3 \times 3$ matrices; $\bar{M}_\nu$ and $\bar{M}_R$ denote the $3 \times 3$ diagonal mass matrices with eigenvalues $m_i$ and $M_i$ (for $i = 1, 2, 3$), respectively. It is straightforward to obtain

$$
\begin{align*}
V\bar{M}_{\nu}V^T &= \Delta_V - M_D M_R^{-1} M_D^T, \\
U\bar{M}_R U^T &= \Delta_U + M_R,
\end{align*}
$$

(2.8)

where $\Delta_V \equiv V S^T M_D^T R^* U^T M_R^{-1} U R^t M_D S^* V^T$ and $\Delta_U \equiv M_D^T R^* U^T + U R^t M_D$. Note that both $R$ and $S$ in (2.7) are expected to be of order $M_D / M_R$, as a result of the huge hierarchy between the scales of $M_D$ and $M_R$. Hence $\Delta_V \approx 0$ and $\Delta_U \approx 0$ are excellent approximations for (2.8). We then arrive at $M_R \approx U \bar{M}_R U^T$ for the heavy (right-handed) neutrino mass matrix and the well-known seesaw formula for the light (left-handed) neutrino mass matrix 2:

$$
M_\nu = V \bar{M}_{\nu} V^T \approx -M_D M_R^{-1} M_D^T.
$$

(2.9)

The seesaw mechanism was actually motivated by the SO(10) GUTs [22], which include right-handed neutrinos automatically together with other charged fermions in irreducible 16-dimensional multiplets. In such an attractive framework, $M_D$ can simply be related to the up-type quark mass matrix $M_u$, whose possible textures have well been restricted by current experimental data [12]. If both $M_D = M_u$ and $M_R$ were assumed to be diagonal, $M_\nu$ would also be diagonal and its three mass eigenvalues would be $m_1 = m_u^2 / M_1$, $m_2 = m_2^2 / M_2$ and $m_3 = m_3^2 / M_3$. In this illustrative case, $m_3 \sim 0.1$ eV may simply result from $M_3 \sim 10^{14}$ GeV, a scale not far away from the GUT scale $\Lambda_{\text{GUT}}$.

2Note that the minus sign on the right-hand side of (2.9) can always be absorbed via a redefinition of the approximately unitary matrix $V$. For instance, $V \bar{M}_{\nu} V^T = M_D M_R^{-1} M_D^T$ with $V \equiv i V$ holds.
Without loss of generality, both the charged lepton mass matrix $M_l$ and the heavy Majorana neutrino mass matrix $M_R$ can be arranged to be diagonal and have positive mass eigenvalues. In this specific flavor basis, the approximately unitary matrix $V$ in (2.9) links the mass eigenstates of light neutrinos to their flavor eigenstates; i.e., it is relevant to the phenomenon of lepton flavor mixing at low energies (see section 3.1 for more detailed discussions). A full parametrization of $V$ requires three mixing angles and three CP-violating phases. Thus $M_\nu$ consists of nine free parameters. Note that the Dirac neutrino mass matrix $M_D$ can be parametrized as $M_D = iV\sqrt{M_\nu}O\sqrt{M_R}$ [23], in which $O$ is a complex orthogonal matrix containing three rotation angles and three phase angles. It seems that $M_D$ in this parametrization consists of eighteen parameters: six from $V$, three from $\sqrt{M_\nu}$, six from $O$ and three from $M_R$. However, only fifteen of them (nine real parameters and six CP-violating phases) are independent, due to the seesaw relation. We conclude that the lepton mass matrices $M_l$, $M_D$ and $M_R$ totally have twenty-one free parameters: three from $M_l$, fifteen from $M_D$ and three from $M_R$ in the chosen flavor basis.

It is worth mentioning that (2.9) has been referred to as the Type-I seesaw formula. A somehow similar relation, the so-called Type-II seesaw formula [25], can be derived from the generalized lepton mass term

$$-\mathcal{L}_{\text{mass}} = -\mathcal{L}_l + \frac{1}{2}(\nu, \nu^c)_L \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} \nu^c \\ \nu \end{pmatrix}_R,$$  

(2.10)

where $M_L$ has been defined in (2.4). As $M_L$ must result from a new Yukawa-interaction term which violates the $\text{SU}(2)_L \times U(1)_Y$ gauge symmetry in the standard model, its scale might be much lower than the electroweak symmetry breaking scale $v$. The strong hierarchy between the scales of $M_R$ and $M_L$ or $M_D$ allow us to make some safe approximations in diagonalizing the $6 \times 6$ neutrino mass matrix in (2.10). We find that the heavy (right-handed) neutrino mass matrix remains to take the form $M_R \approx U\sqrt{M_R}U^T$. In contrast, the light (left-handed) neutrino mass matrix is given by

$$M_\nu = \sqrt{M_\nu}V^T \approx M_L - M_DM^{-1}_RM_D^T.$$  

(2.11)

This result is just the Type-II seesaw formula. As emphasized in Ref. [25], the Type-II seesaw relation is a reflection of the left-right symmetry of the theory at high energies. For the phenomenological study of neutrino masses, however, the Type-I seesaw mechanism is more popular and useful because it involves fewer free parameters than the Type-II seesaw mechanism.

Of course, there are some other versions of the seesaw mechanism, which are either more complicated [25] or simpler [26] than what we have discussed above. While the seesaw idea is qualitatively elegant to interpret the smallness of left-handed neutrino masses, it cannot lead to quantitative predictions unless specific textures of neutrino mass matrices (e.g., $M_D$ and $M_R$) are assumed. Many interesting ansätze of lepton mass matrices have so far been proposed and incorporated with the seesaw mechanism (for recent reviews with extensive references, see Ref. [12] and Refs. [18]–[21]), but they are strongly model-dependent and thus beyond the scope of the present review article.

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3The parameter counting is identical in models with and without supersymmetry [24].
Neutrino oscillation is a quantum phenomenon which can naturally happen if neutrinos are massive and lepton flavors are mixed. In a simple two-neutrino mixing scheme, the neutrino flavor eigenstates $\nu_\alpha$ and $\nu_\beta$ are linear combinations of the neutrino mass eigenstates $\nu_a$ and $\nu_b$ [27]:

$$\nu_\alpha = \nu_a \cos \theta + \nu_b \sin \theta$$

$$\nu_\beta = \nu_b \cos \theta - \nu_a \sin \theta,$$

where $\theta$ denotes the flavor mixing angle. Then the probabilities of neutrino oscillations are governed by two characteristic parameters: one of them is the neutrino mass-squared difference $\Delta m^2 \equiv m_b^2 - m_a^2$ and the other is the flavor mixing factor $\sin^2 2\theta$. Corresponding to the “disappearance” and “appearance” neutrino experiments, the survival and conversion probabilities of a neutrino flavor eigenstate $\nu_\alpha$ can explicitly be expressed as

$$P(\nu_\alpha \rightarrow \nu_\alpha) = 1 - \sin^2 2\theta \sin^2 \left(1.27 \frac{\Delta m^2 L}{E}\right),$$

$$P(\nu_\alpha \rightarrow \nu_\beta) = \sin^2 2\theta \sin^2 \left(1.27 \frac{\Delta m^2 L}{E}\right),$$

(2.12)

where $\beta \neq \alpha$, $E$ is the neutrino beam energy (in unit of GeV), and $L$ denotes the distance between the neutrino source and the neutrino detector (in unit of km). A lot of experimental data, including those from solar, atmospheric and reactor neutrino oscillation experiments, have been analyzed by use of (2.12).

(1) The first model-independent evidence for neutrino oscillations was obtained from the Super-Kamiokande (SK) experiment [2] on atmospheric neutrinos, which are produced in the earth’s atmosphere by cosmic rays and are detected in an underground detector. If there were no neutrino oscillation, the atmospheric neutrinos entering and exiting the detector should have a spherical symmetry. In other words, the downward-going and upward-going neutrino fluxes should be equal to each other: $\Phi_\alpha(\theta_z) = \Phi_\alpha(\pi - \theta_z)$ versus the zenith angle $\theta_z$ (for $\alpha = e$ or $\mu$). The SK Collaboration has observed an approximate up-down flux symmetry for the atmospheric $\nu_e$ neutrinos and a significant up-down flux asymmetry for the atmospheric $\nu_\mu$ neutrinos. For instance,

$$\frac{\Phi_\mu(-1 \leq \cos \theta_z \leq -0.2)}{\Phi_\mu(+0.2 \leq \cos \theta_z \leq +1)} = 0.54 \pm 0.04 \neq 1$$

(2.13)

was measured for the multi-GeV $\nu_\mu$ neutrinos, and it is in conflict with the up-down flux symmetry. This result can well be interpreted in the assumption of $\nu_\mu \rightarrow \nu_\tau$ neutrino oscillations. Current SK [2] and CHOOZ [7] data have ruled out the possibility that the atmospheric neutrino anomaly is dominantly attributed to $\nu_\mu \rightarrow \nu_e$ or $\nu_\mu \rightarrow \nu_s$ oscillations, where $\nu_s$ stands for a “sterile” neutrino which does not take part in the normal electroweak interactions (see appendix A for detailed discussions).

The hypothesis of atmospheric neutrino oscillations has recently received very strong support from the K2K [5] long-baseline neutrino experiment, in which the $\nu_\mu$ beam is produced at the KEK accelerator and measured 250 km away at the SK detector. The K2K Collaboration observed a reduction of the $\nu_\mu$ flux and a distortion of the $\nu_\mu$ energy
spectrum, which must take place in the presence of $\nu_\mu \rightarrow \nu_\tau$ oscillations. The possibility that the K2K result is due to a statistical fluctuation instead of neutrino oscillations is found to be less than 1% [5].

It has been shown that the standard $\nu_\mu \rightarrow \nu_\tau$ oscillations provide the best description of the combined SK and K2K data [28], from which $\Delta m^2_{\text{atm}} = (2.6 \pm 0.4) \times 10^{-3} \text{ eV}^2$ and $\sin^2 2\theta_{\text{atm}} = 1.00^{+0.09}_{-0.05}$ are determined at the 1$\sigma$ level. At the 90% confidence level, we have

$$1.65 \times 10^{-3} \text{ eV}^2 \leq \Delta m^2_{\text{atm}} \leq 3.25 \times 10^{-3} \text{ eV}^2$$

and $0.88 \leq \sin^2 2\theta_{\text{atm}} \leq 1.00$.

(2) The long-standing problem associated with solar neutrinos is that the flux of solar $\nu_e$ neutrinos measured in all experiments (e.g., the SK [2] and SNO [3] experiments as well as the earlier Homestake [29], GALLEX-GNO [30] and SAGE [31] experiments) is significantly smaller than that predicted by the standard solar models [32]. The deficit of solar $\nu_e$'s is not the same in different experiments, implying that the relevant physical effect is energy-dependent. It has been hypothesized that the solar neutrino problem is due to the conversion of solar $\nu_e$'s into other active or sterile neutrinos during their travel from the core of the sun to the detectors on the earth. The SNO experiment has model-independently demonstrated that $\nu_e \rightarrow \nu_\mu$ and (or) $\nu_e \rightarrow \nu_\tau$ transitions are dominantly responsible for the solar neutrino deficit.

What the SNO Collaboration has measured is the flux of solar $^8\text{B}$ neutrinos via the charged-current (CC), neutral-current (NC) and elastic-scattering (ES) reactions:

$$\nu_e + d \rightarrow e^- + p + p, \quad \nu_\alpha + d \rightarrow \nu_\alpha + p + n \quad \text{and} \quad \nu_\alpha + e^- \rightarrow \nu_\alpha + e^-,$$

where $\alpha = e, \mu$ or $\tau$. In the presence of flavor conversion, the observed neutrino fluxes in different reactions satisfy

$$\Phi^{\text{CC}} = \Phi_e, \quad \Phi^{\text{NC}} = \Phi_e + \Phi_{\mu\tau}, \quad \Phi^{\text{ES}} = \Phi_e + \frac{\sigma_\mu}{\sigma_e} \Phi_{\mu\tau} \approx \Phi_e + 0.154 \Phi_{\mu\tau},$$

where $\sigma_\mu/\sigma_e \approx 0.154$ is the ratio of elastic $\nu_e$-$e$ and $\nu_\mu$-$\mu$ scattering cross-sections, and $\Phi_{\mu\tau}$ denotes the flux of active non-electron neutrinos. Of course, $\Phi_{\mu\tau} = 0$ or equivalently $\Phi^{\text{CC}} = \Phi^{\text{NC}} = \Phi^{\text{ES}}$ would hold, if there were no flavor conversion. The SNO data [3] yield

$$\Phi_e = (1.76 \pm 0.10) \times 10^6 \text{ cm}^{-2}\text{s}^{-1}, \quad \Phi_{\mu\tau} = (3.41^{+0.66}_{-0.64}) \times 10^6 \text{ cm}^{-2}\text{s}^{-1},$$

a convincing evidence (at the 5.3$\sigma$ level) for the existence of $\nu_\mu$ and $\nu_\tau$ neutrinos in the flux of solar neutrinos onto the earth.

The flavor conversion of solar $\nu_e$ neutrinos is most likely due to neutrino oscillations. In the scheme of two-neutrino oscillations, a global analysis of all available experimental data on solar neutrinos (in particular, those from the SK and SNO measurements) leads to several regions allowed for the parameters $\Delta m^2_{\text{sun}}$ and $\tan^2 \theta_{\text{sun}}$: the SMA (small mixing angle), LMA (large mixing angle) and LOW (low mass-squared difference) regions based on the Mikheev-Smirnov-Wolfenstein (MSW) mechanism [33] as well as the VO (vacuum oscillation) and other possible regions [34], among which the LMA region is most favored. Recently the LMA solution has been singled out as the only acceptable solution to the solar neutrino problem, thanks to the KamLAND [4] experiment.
The KamLAND Collaboration has measured the flux of $\bar{\nu}_e$'s from distant nuclear reactor via the inverse $\beta$-decay reaction $\bar{\nu}_e + p \rightarrow e^+ + n$. The typical baseline of this experiment is 180 km, allowing for a terrestrial test of the LMA solution to the solar neutrino problem. The ratio of the observed inverse $\beta$-decay events to the expected number without $\bar{\nu}_e$ disappearance is $0.611 \pm 0.094$ for $E_\nu > 3.4$ MeV. Such a deficit can naturally be interpreted in the hypothesis of neutrino oscillations, and the parameter space of $(\Delta m^2, \sin^2 2\theta)$ is found to be compatible very well with the LMA region. We are then led to the conclusion that the LMA solution is the only correct solution to the solar neutrino problem. A global analysis of the combined SK, SNO and KamLAND data yields [35]

$$5.9 \times 10^{-5} \text{ eV}^2 \leq \Delta m^2_{\text{sun}} \leq 8.8 \times 10^{-5} \text{ eV}^2$$

(2.17)

and $0.25 \leq \sin^2 \theta_{\text{sun}} \leq 0.40$ at the 90% confidence level.

(3) The purpose of the CHOOZ and Palo Verde reactor experiments [7] is to search for $\bar{\nu}_e \rightarrow \bar{\nu}_e$ disappearance neutrino oscillations in the atmospheric range of $\Delta m^2$. No indication in favor of neutrino oscillations was found from both experiments, leading to a strong constraint on the flavor mixing factor: $\sin^2 2\theta_{\text{chz}} < 0.10$ for $\Delta m^2_{\text{chz}} > 3.5 \times 10^{-3}$ eV$^2$; or $\sin^2 2\theta_{\text{chz}} < 0.18$ for $\Delta m^2_{\text{chz}} > 2.0 \times 10^{-3}$ eV$^2$. The impact of this constraint on the lepton flavor mixing matrix will be discussed in section 3.1.

Let us denote the mass eigenstates of $\nu_e$, $\nu_\mu$ and $\nu_\tau$ neutrinos as $(\nu_1, \nu_2, \nu_3)$, whose eigenvalues are $(m_1, m_2, m_3)$. There are only two independent neutrino mass-squared differences: $\Delta m^2_{21} \equiv m_2^2 - m_1^2$ and $\Delta m^2_{32} \equiv m_3^2 - m_2^2$ or $\Delta m^2_{31} \equiv m_3^2 - m_1^2$. Without loss of generality, we make the identification

$$\Delta m^2_{\text{sun}} = |\Delta m^2_{21}| \ll |\Delta m^2_{32}| = \Delta m^2_{\text{atm}},$$

(2.18)

where the big hierarchy between $\Delta m^2_{\text{sun}}$ and $\Delta m^2_{\text{atm}}$ has been taken into account. Indeed, $\Delta m^2_{\text{sun}} \ll \Delta m^2_{\text{atm}}$ together with $\sin^2 2\theta_{\text{chz}} \ll 1$ implies that solar and atmospheric neutrino oscillations are approximately decoupled. The deficits of solar and atmospheric neutrinos are dominated respectively by $\nu_e \rightarrow \nu_\mu$ and $\nu_\mu \rightarrow \nu_\tau$ transitions. With the help of (2.18), $m_1$ and $m_2$ can be given in terms of $m_3$, $\Delta m^2_{\text{sun}}$ and $\Delta m^2_{\text{atm}}$ [36]:

$$m_1 = \sqrt{m_3^2 + p\Delta m^2_{\text{atm}} + q\Delta m^2_{\text{sun}}},$$

$$m_2 = \sqrt{m_3^2 + p\Delta m^2_{\text{atm}}},$$

(2.19)

where $p = \pm 1$ and $q = \pm 1$ stand for four possible neutrino mass spectra \footnote{The mass convention $m^1_1 < m^2_2 < m^3_3$ has been used in some literature for the redefined neutrino mass eigenstates $\nu'_i$, which are related to the standard neutrino mass eigenstates $\nu_i$ by an orthogonal transformation [37]. We shall not adopt $\nu'_i$ in the present article.}:

$$(p,q) = (-1,-1) : \quad m_1 < m_2 < m_3,$$

$$(p,q) = (-1, +1) : \quad m_1 > m_2 < m_3,$$

$$(p,q) = (+1, -1) : \quad m_1 < m_2 > m_3,$$

$$(p,q) = (+1, +1) : \quad m_1 > m_2 > m_3.$$ 

(2.20)

Current solar neutrino oscillation data favor $q = -1$; i.e., $m_1 < m_2$. The sign of $p$ may be determined from the future long-baseline neutrino oscillation experiments.
Figure 2.2: Illustrative correlation of three neutrino masses for $p = \pm 1$ cases.

Given the best-fit values $\Delta m^2_{\text{sun}} = 7.3 \times 10^{-5}$ eV$^2$ and $\Delta m^2_{\text{atm}} = 2.5 \times 10^{-3}$ eV$^2$, the numerical correlation of $m_1$, $m_2$ and $m_3$ is illustrated in Fig. 2.2 by use of (2.19). We see that three masses become nearly degenerate, if one of them is larger than 0.1 eV (i.e., $m_i > 0.1$ eV). For the $p = -1$ case, the lower bound of $m_3$ is $m_3 \geq \sqrt{\Delta m^2_{\text{atm}} + \Delta m^2_{\text{sun}}}$, which leads to $m_2 \geq \sqrt{\Delta m^2_{\text{sun}}}$. A normal hierarchy $m_1 < m_2 < m_3$ may appear, if $m_1 \sim \mathcal{O}(10^{-2})$ eV or smaller. For the $p = +1$ case, however, $m_1 \approx m_2 > m_3$ always holds, no matter how small $m_3$ is taken (inverted hierarchy).

Finally it is worthwhile to mention the LSND evidence [8] for $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ neutrino oscillations with $\Delta m^2_{\text{LSND}} \sim 1$ eV$^2$ and $\sin^2 2\theta_{\text{LSND}} \sim 10^{-3} - 10^{-2}$. This observation was not confirmed by the KARMEN experiment [38], which is sensitive to most of the LSND parameter space. The disagreement between these two measurements will be resolved by the MiniBOONE experiment [39] at Fermilab in the coming years. Before the LSND result is confirmed on a solid ground, the conservative approach is to set it
aside tentatively and to concentrate on solar and atmospheric neutrino oscillations in the mixing scheme of three lepton families. A natural way to simultaneously accommodate solar, atmospheric and LSND data is to assume the existence of a light sterile neutrino, which may oscillate with three active neutrinos. Such a four-neutrino mixing scheme will be discussed in appendix A.

2.2.2 Cosmological Bounds

Neutrinos play an important role in cosmology and astrophysics [40]. Useful information on neutrino masses can be obtained, for example, from studies of the cosmological relic density and dark matter, the power spectrum in large scale structure surveys, the Lyman $\alpha$ forest, the Big Bang nucleosynthesis, the supernovae, the ultra high energy (UHE) cosmic rays, and the gamma ray bursts. For recent reviews of these interesting topics, we refer the reader to Refs. [40]–[43]. Here we focus our attention on the latest cosmological bound on the sum of light neutrino masses, obtained from the impressive data of the Wilkinson Microwave Anisotropy Probe (WMAP) [44].

According to the Big Bang cosmology, neutrinos were in thermal equilibrium with photons, electrons and positrons in the early universe. When the universe cooled to temperatures of $O(1)$ MeV, neutrinos decoupled from the primordial $e^\pm\gamma$ plasma, leading to the fact that the present-day number density of neutrinos is similar to that of photons. If neutrinos are massive, they contribute to the cosmological matter density [45],

$$\Omega_{\nu}h^2 = \sum_{i} \frac{m_i}{93.5 \text{ eV}},$$

(2.21)

where $\Omega_{\nu}$ denotes the neutrino mass density relative to the critical energy density of the universe, and $h$ is the Hubble constant in units of 100 km/s/Mpc. A tight limit $\Omega_{\nu}h^2 < 7.6 \times 10^{-3}$ (at 95% C.L.) has recently been extracted from the striking WMAP data [44], combined with the additional cosmic microwave background (CMB) data sets (CBI and ACBAR) and the observation of large scale structure from 2dF Galaxy Redshift Survey (2dFGRS) [46]. This impressive bound leads straightforwardly to

$$\sum_{i} m_i < 0.71 \text{ eV},$$

(2.22)

which holds at the same confidence level. Some discussions are in order.

(a) A tight upper bound on $m_i$ can be achieved from (2.22) together with current data on solar and atmospheric neutrino oscillations. To see this point more clearly, we calculate the dependence of $m_1 + m_2 + m_3$ on $m_3$ with the help of (2.19) and (2.22). The numerical results are shown in Fig. 2.3, where the best-fit values $\Delta m^2_{\text{sun}} = 7.3 \times 10^{-5} \text{ eV}^2$ and $\Delta m^2_{\text{atm}} = 2.5 \times 10^{-3} \text{ eV}^2$ have typically been input. Note that only the $q = -1$ or $m_1 < m_2$ case, which is supported by current solar neutrino oscillation data, is taken into account. For the $p = -1$ or $m_2 < m_3$ case, $m_3$ has an interesting lower bound

\(^5\)The exact value of this upper bound depends on other cosmological parameters, as emphasized by Hannestad in Ref. [47], where a somewhat more generous limit $m_1 + m_2 + m_3 < 1.0 \text{ eV}$ (at 95% C.L.) has been obtained for $N_{\nu} = 3$. 

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Figure 2.3: Illustrative dependence of $m_1 + m_2 + m_3$ on $m_3$. The WMAP result sets an upper limit on $m_3$; i.e., $m_3 < 0.24$ eV for both $m_3 > m_2$ and $m_3 < m_2$ cases.

$$m_3 \geq \sqrt{\Delta m^2_{\text{atm}} + \Delta m^2_{\text{sun}}} \approx 0.051 \text{ eV}; \text{ but for the } p = +1 \text{ or } m_2 > m_3 \text{ case, even } m_3 = 0 \text{ is allowed (inverted hierarchy). We see that these two cases become indistinguishable for } m_3 \geq 0.2 \text{ eV, implying the near degeneracy of three neutrino masses. Once the WMAP limit in (2.22) is included, we immediately get } m_3 < 0.24 \text{ eV. As a consequence, we have } m_i < 0.24 \text{ eV for } i = 1, 2 \text{ or 3. Similar results have been obtained in Refs. [47, 48].}$$

(b) The LSND [8] evidence for neutrino oscillations is strongly disfavored, because $\Delta m^2_{\text{LSND}} \sim 1 \text{ eV}^2$ is essentially incompatible with (2.22). However, a marginal agreement between the LSND data and the WMAP data is not impossible in the four-neutrino mixing scheme, if the latter’s limit on neutrino masses is loosened to some extent (e.g., $m_1 + m_2 + m_3 + m_4 < 1.4 \text{ eV for } N_\nu = 4$ [47]). In this sense, it might be premature to claim that the LSND result has been ruled out by the WMAP data [48]. We hope that the MiniBOONE experiment [39] can ultimately confirm or disprove the LSND result.

### 2.2.3 Kinematic Measurements

An essentially model-independent way to determine or constrain neutrino masses is to measure some typical weak decays whose final-state particles include neutrinos. The kinematics of such a process in the case of non-zero neutrino masses is different from that in the case of zero neutrino masses. This provides an experimental opportunity to probe neutrino masses in a direct way. In practice, direct neutrino mass measurements are based on the analysis of the kinematics of charged particles produced together with the neutrino flavor eigenstates $|\nu_\alpha \rangle$ (for $\alpha = e, \mu, \tau$), which are superpositions of the neutrino mass eigenstates $|\nu_i \rangle$ (for $i = 1, 2, 3$): $|\nu_\alpha \rangle = V_{\alpha 1} |\nu_1 \rangle + V_{\alpha 2} |\nu_2 \rangle + V_{\alpha 3} |\nu_3 \rangle$ with $V$ being the $3 \times 3$ lepton flavor mixing matrix. The effective masses of three neutrinos can then be defined as

$$\langle m \rangle_\alpha = \sqrt{\sum_i \left| V_{\alpha i} \right|^2 m_i^2}, \quad (2.23)$$
for \( \alpha = e, \mu \) and \( \tau \). The kinematic limits on \( \langle m \rangle_e, \langle m \rangle_\mu \) and \( \langle m \rangle_\tau \) can respectively be obtained from the tritium beta decay \(^3\text{H} \rightarrow ^3\text{He} + e^- + \nu_e\), the \( \pi^+ \rightarrow \mu^+ + \nu_\mu\) decay and the \( \tau \rightarrow 5\pi + \nu_\tau \) (or \( \tau \rightarrow 3\pi + \nu_\tau \)) decay [6]:

\[
\begin{align*}
\langle m \rangle_e &< 2.2 \text{ eV}, \\
\langle m \rangle_\mu &< 0.19 \text{ MeV}, \\
\langle m \rangle_\tau &< 18.2 \text{ MeV}.
\end{align*}
\] (2.24)

We see that the experimental sensitivity for \( \langle m \rangle_\mu \) is more than four orders of magnitude smaller than that for \( \langle m \rangle_e \), and the experimental sensitivity for \( \langle m \rangle_\tau \) is two orders of magnitude lower than that for \( \langle m \rangle_\mu \).

Note that the unitarity of \( V \) leads straightforwardly to a simple sum rule between \( \langle m \rangle^2_\alpha \) and \( m^2_i \) [49]:

\[
\sum_\alpha \langle m \rangle^2_\alpha = \sum_i m^2_i < \left( \sum_i m_i \right)^2.
\] (2.25)

As the sum of three relativistic neutrino masses has well been constrained by the recent WMAP data [44], \( m_1 + m_2 + m_3 < 0.71 \text{ eV} \) at the 95% confidence level, we are led to

\[
\langle m \rangle_e^2 + \langle m \rangle_\mu^2 + \langle m \rangle_\tau^2 < 0.50 \text{ eV}^2.
\] (2.26)

This generous upper bound implies that \( \langle m \rangle^2_\alpha < 0.50 \text{ eV}^2 \) or \( \langle m \rangle_\alpha < 0.71 \text{ eV} \) holds for \( \alpha = e, \mu \) and \( \tau \). One can see that the cosmological upper bound of \( \langle m \rangle_e \) is about three times smaller than its kinematic upper bound given in (2.24). The former may be accessible in the future KATRIN experiment [50], whose sensitivity is expected to be about 0.3 eV. In contrast, the cosmological upper bound of \( \langle m \rangle_\mu \) is more than five orders of magnitude smaller than its kinematic upper bound given in (2.24), and the upper limit of \( \langle m \rangle_\tau \) set by the WMAP data is more than seven orders of magnitude smaller than its kinematic upper limit. It seems hopeless to improve the sensitivity of the kinematic \( \langle m \rangle_\mu \) and \( \langle m \rangle_\tau \) measurements to the level of 0.71 eV.

If current neutrino oscillation data are taken into account, however, more stringent upper limits can be obtained for the effective neutrino masses \( \langle m \rangle_e, \langle m \rangle_\mu \) and \( \langle m \rangle_\tau \). Substituting (2.19) into (2.23), we get

\[
\langle m \rangle_\alpha = \sqrt{m_3^2 + p (1 - |V_{\alpha 3}|^2) \Delta m^2_{\text{atm}} + q |V_{\alpha 1}|^2 \Delta m^2_{\text{sun}}}.
\] (2.27)

The present experimental data indicate \( |V_{e3}|^2 \approx \sin^2 \theta_{\text{chz}} \ll 1 \) [7] and \( \Delta m^2_{\text{atm}} \ll \Delta m^2_{\text{sun}} \). Furthermore, \( \sin^2 2\theta_{\text{atm}} \approx 1 \) is strongly favored, leading to \( |V_{\mu 3}| \approx \sin \theta_{\text{atm}} \approx 1/\sqrt{2} \) and \( |V_{\tau 3}| \approx \cos \theta_{\text{atm}} \approx 1/\sqrt{2} \) for \( |V_{e3}| \ll 1 \). Therefore, (2.27) can be simplified as

\[
\begin{align*}
\langle m \rangle_e &\approx \sqrt{m_3^2 + p \Delta m^2_{\text{atm}}}, \\
\langle m \rangle_\mu &\approx \sqrt{m_3^2 + \frac{p}{2} \Delta m^2_{\text{atm}}}, \\
\langle m \rangle_\tau &\approx \sqrt{m_3^2 + \frac{p}{2} \Delta m^2_{\text{atm}}}.
\end{align*}
\] (2.28)

We can see that \( \langle m \rangle_\mu \approx \langle m \rangle_\tau \) is a consequence of current neutrino oscillation data. In addition, (2.28) shows that \( \langle m \rangle_e \) is slightly larger than \( \langle m \rangle_\mu \) and \( \langle m \rangle_\tau \) for \( p = +1 \) or
Figure 2.4: Illustrative dependence of $\langle m \rangle_\alpha$ (for $\alpha = e, \mu, \tau$) on $m_3$. Curve a: $\langle m \rangle_e$ with $m_3 < m_2$; Curve b: $\langle m \rangle_\mu \approx \langle m \rangle_\tau$ with $m_3 < m_2$; Curve c: $\langle m \rangle_\mu \approx \langle m \rangle_\tau$ with $m_3 > m_2$; and Curve d: $\langle m \rangle_e$ with $m_3 > m_2$. The WMAP result leads to $\langle m \rangle_\alpha < 0.24$ eV.

$m_2 > m_3$; and it is slightly smaller than $\langle m \rangle_\mu$ and $\langle m \rangle_\tau$ for $p = -1$ or $m_2 < m_3$. The numerical dependence of $\langle m \rangle_e$, $\langle m \rangle_\mu$ and $\langle m \rangle_\tau$ on $m_3$ is illustrated in Fig. 2.4. We find that it is impossible to distinguish between the $m_2 < m_3$ case and the $m_2 > m_3$ case for $m_3 \geq 0.2$ eV. Note that the dependence of $\langle m \rangle_\alpha$ on $m_3$ is similar to the dependence of $m_1 + m_2 + m_3$ on $m_3$ shown in Fig. 2.3. Taking account of the upper limit $m_3 \leq 0.24$ eV obtained in section 2.2.2, we arrive at $\langle m \rangle_\alpha \leq 0.24$ eV for $\alpha = e, \mu$ or $\tau$. This upper bound is suppressed by a factor of three, compared to the upper bound obtained from (2.26) which is independent of the neutrino oscillation data.

The result $\langle m \rangle_\mu \approx \langle m \rangle_\tau < 0.24$ eV implies that there is no hope to kinematically detect the effective masses of muon and tau neutrinos [49]. As the WMAP upper bound is in general valid for a sum of the masses of all possible relativistic neutrinos (no matter whether they are active or sterile), it seems unlikely to loosen the upper limit of $\langle m \rangle_\alpha$ obtained above in the assumption of only active neutrinos. Therefore, the kinematic measurements of $\langle m \rangle_\mu$ and $\langle m \rangle_\tau$ have little chance to reveal the existence of any exotic neutral particles with masses much larger than the light neutrino masses.

2.2.4 Neutrinoless Double-$\beta$ Decay

The neutrinoless double beta decay of some even-even nuclei,

\[ A(Z, N) \rightarrow A(Z + 2, N - 2) + 2e^- , \] (2.29)

can occur through the exchange of a Majorana neutrino between two decaying neutrons inside a nucleus, as illustrated in Fig. 2.5. It would be forbidden, however, if neutrinos were Dirac particles. Thus the neutrinoless double-$\beta$ decay provides us with a unique opportunity to identify the Majorana nature of massive neutrinos.
The rate of the neutrinoless double-$\beta$ decay depends both on an effective neutrino mass term $\langle m \rangle_{ee}$ and on the associated nuclear matrix element. The latter can be calculated, but some uncertainties are involved in the calculations [43].

In the three-neutrino mixing scheme, the effective Majorana mass $\langle m \rangle_{ee}$ is given by

$$\langle m \rangle_{ee} = \left| m_1 V_{e1}^2 + m_2 V_{e2}^2 + m_3 V_{e3}^2 \right|,$$

where $m_i$ (for $i = 1, 2, 3$) denote the physical masses of three neutrinos, and $V_{ei}$ stand for the elements in the first row of the $3 \times 3$ lepton flavor mixing matrix $V$. It is obvious that $\langle m \rangle_{ee} = 0$ would hold, if $m_i = 0$ were taken.

Many experiments have been done to search for the neutrinoless double-$\beta$ decay [51]. Among them, the Heidelberg-Moscow [52] and IGEX [53] $^{76}$Ge experiments have the highest sensitivity and yield the most stringent upper bounds on $\langle m \rangle_{ee}$:

$$\langle m \rangle_{ee} \leq (0.35 - 1.24) \text{ eV} \quad \text{(Heidelberg – Moscow)},$$

$$\langle m \rangle_{ee} \leq (0.33 - 1.35) \text{ eV} \quad \text{(IGEX)},$$

where the uncertainties associated with the nuclear matrix elements have been taken into account. A number of new experiments [51], which may probe $\langle m \rangle_{ee}$ at the level of 10 meV to 50 meV, are in preparation.

While $\langle m \rangle_{ee} \neq 0$ must imply that neutrinos are Majorana particles, $\langle m \rangle_{ee} = 0$ does not necessarily imply that neutrinos are Dirac particles. The reason is simply that the Majorana phases hidden in $V_{ei}$ may lead to significant cancellations on the right-hand side of (2.30), making $\langle m \rangle_{ee}$ vanishing or too small to be detectable [57]. Hence much care has to be taken, if no convincing signal of the neutrinoless double beta decay can be experimentally established: it may imply that (1) the experimental sensitivity is not high enough; (2) the massive neutrinos are Dirac particles; or (3) the vanishing or suppression of $\langle m \rangle_{ee}$ is due to large cancellations induced by the Majorana CP-violating phases. The third possibility has recently been examined [58] from a model-independent point of view and with the help of the latest experimental data. It is found that current neutrino oscillation data do allow $\langle m \rangle_{ee} = 0$ to hold, if the Majorana phases lie in two

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6 After re-analyzing the data from the Heidelberg-Moscow experiment, Klapdor-Kleingrothaus et al. reported the first evidence for the neutrinoless double beta decay [54]. However, their result was strongly criticized by some authors [55]. Future experiments will have sufficiently high sensitivity to clarify the present debates [56], either to confirm or to disprove the alleged result in Ref. [54].
specific regions. To see this point more clearly, we use ρ and σ to denote the Majorana CP-violating phases of \( V \) and take the convention [36]

\[
\arg(V_{e1}) = \rho, \quad \arg(V_{e2}) = \sigma, \quad \arg(V_{e3}) = 0.
\] (2.32)

Of course, ρ and σ have nothing to do with CP and T violation in normal neutrino oscillations. In view of (2.30) and (2.32), we find that \( \langle m \rangle_{ee} = 0 \) requires

\[
\frac{m_1}{m_2} = -\frac{|V_{e3}|^2}{|V_{e1}|^2} \cdot \frac{\sin 2\sigma}{\sin 2\rho}, \quad \frac{m_2}{m_3} = \frac{|V_{e3}|^2}{|V_{e2}|^2} \cdot \frac{\sin 2\rho}{\sin 2(\sigma - \rho)}. \] (2.33)

As \( m_i > 0 \) holds, part of the \((\rho, \sigma)\) parameter space must be excluded. To pin down the whole ranges of \( \rho \) and \( \sigma \) allowed by current neutrino oscillation data under the \( \langle m \rangle_{ee} = 0 \) condition, we utilize (2.18) and (2.33) to calculate the ratio of solar and atmospheric neutrino mass-squared differences:

\[
\frac{\Delta m^2_{\text{sun}}}{\Delta m^2_{\text{atm}}} = \frac{|V_{e3}|^4}{|V_{e1}|^4} \cdot \frac{|V_{e1}|^4 \sin^2 2\rho - |V_{e2}|^4 \sin^2 2\sigma}{|V_{e2}|^4 \sin^2 2(\sigma - \rho) - |V_{e3}|^4 \sin^2 2\rho}. \] (2.34)

Then \( \Delta m^2_{\text{sun}}/\Delta m^2_{\text{atm}} \sim 10^{-2} \) imposes quite strong constraints on the \((\rho, \sigma)\) parameter space, as numerically illustrated in Fig. 2 of Ref. [58]. One can in turn arrive at the allowed ranges of \( m_1/m_2 \) and \( m_2/m_3 \) with the help of (2.33).

We certainly hope that \( \langle m \rangle_{ee} \) is non-zero and measurable. If the parameters of neutrino oscillations are well determined, a measurement of \( \langle m \rangle_{ee} \) will allow us to extract valuable information about the Majorana phases of CP violation and the absolute scale of neutrino masses. Taking account of (2.19) and (2.32), one may rewrite (2.30) as

\[
\langle m \rangle_{ee} = \sqrt{\frac{\sqrt{m_3^2 + p \Delta m^2_{\text{atm}} + q \Delta m^2_{\text{sun}}}}{m^2_3 + p \Delta m^2_{\text{atm}}}} |V_{e1}|^2 e^{2i\rho} + \sqrt{\frac{\sqrt{m_3^2 + p \Delta m^2_{\text{atm}}}}{m^2_3}} |V_{e2}|^2 e^{2i\sigma} + m_3 |V_{e3}|^2, \] (2.35)

where \( |V_{e1}|^2 \) can be expressed in terms of the mixing factors of solar and CHOOZ reactor neutrino oscillations (see section 3.1.1 for details):

\[
|V_{e1}|^2 = \frac{1}{2} \left( \cos^2 \theta_{\text{chz}} + \cos^4 \theta_{\text{chz}} - \sin^2 2\theta_{\text{sun}} \right), \quad |V_{e2}|^2 = \frac{1}{2} \left( \cos^2 \theta_{\text{chz}} - \cos^4 \theta_{\text{chz}} - \sin^2 2\theta_{\text{sun}} \right), \quad |V_{e3}|^2 = \sin^2 \theta_{\text{chz}}. \] (2.36)

We see that \( \langle m \rangle_{ee} \) consists of three unknown parameters: \( m_3, \rho \) and \( \sigma \), which are unable to be determined in any neutrino oscillation experiments. Once \( \Delta m^2_{\text{sun}}, \Delta m^2_{\text{atm}}, \theta_{\text{sun}} \) and \( \theta_{\text{chz}} \) are measured to an acceptable degree of accuracy, one should be able to get a useful constraint on the absolute neutrino mass \( m_3 \) for arbitrary values of \( \rho \) and \( \sigma \) from the observation of \( \langle m \rangle_{ee} \). If the magnitude of \( m_3 \) could roughly be known from
Figure 2.6: Illustrative dependence of $\langle m \rangle_{ee}$ on $m_3$ for the neutrino mass spectrum $m_2 > m_3$ (curves a and b) and the neutrino mass spectrum $m_2 < m_3$ (curves c and d), where we have typically taken $\{\rho, \sigma\} = \{0, 0\}$ or $\{\pi/4, \pi/4\}$ (curves a and c) and $\{\rho, \sigma\} = \{0, \pi/4\}$ or $\{\pi/4, 0\}$ (curves b and d). The region between two dashed lines corresponds to the values of $\langle m \rangle_{ee}$ reported in Ref. [54].

Let us illustrate the dependence of $\langle m \rangle_{ee}$ on $m_3$, $\rho$ and $\sigma$ in a numerical way. For simplicity, we typically take $\Delta m^2_{\text{sun}} = 5 \times 10^{-5}$ eV$^2$, $\Delta m^2_{\text{atm}} = 3 \times 10^{-3}$ eV$^2$, $q = -1$, $\sin^2 2\theta_{\text{sun}} = 0.8$ and $\sin^2 2\theta_{\text{cha}} = 0.05$ in our calculations. We consider four instructive possibilities for the unknown Majorana phases $\rho$ and $\sigma$: (1) $\rho = \sigma = 0$; (2) $\rho = \pi/4$ and $\sigma = 0$; (3) $\rho = 0$ and $\sigma = \pi/4$; and (4) $\rho = \sigma = \pi/4$. The results for $\langle m \rangle_{ee}$ as a function of $m_3$ are shown in Fig. 2.6. We find that it is numerically difficult to distinguish between possibilities (1) and (4), or between possibilities (2) and (3). The reason is simply that the first two terms on the right-hand side of (2.35) dominate over the third term, and they do not cancel each other for the chosen values of $\rho$ and $\sigma$. We also find that the changes of $\langle m \rangle_{ee}$ are rather mild. If reasonable inputs of $\sin^2 2\theta_{\text{sun}}$, $\sin^2 2\theta_{\text{atm}}$ and $\sin^2 2\theta_{\text{cha}}$ are taken, a careful numerical scan shows that the magnitude of $\langle m \rangle_{ee}$ does not undergo any dramatic changes for arbitrary $\rho$ and $\sigma$. Thus a rough but model-independent constraint on the absolute scale of neutrino masses can be obtained from the observation of $\langle m \rangle_{ee}$. Taking account of the alleged experimental region $0.05$ eV $\leq \langle m \rangle_{ee} \leq 0.84$ eV in Ref. [54], one may arrive at the conclusion that $m_3$ is most likely to lie in the range $0.1$ eV $\leq m_3 \leq 1$ eV (see Fig. 2.6). This result implies that $m_1 \approx m_2 \approx m_3$ and $m_1 + m_2 + m_3 \approx 3m_3 \approx (0.3 - 1.5)$ eV hold. Such a sum of three neutrino masses is only partly compatible with the robust WMAP upper bound given in (2.22).

Because of its importance, the neutrinoless double-$\beta$ decay has attracted a lot of very delicate studies. For recent comprehensive analyses of the dependence of $\langle m \rangle_{ee}$ on

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neutrino masses and Majorana phases, we refer the reader to Refs. [43, 55, 59] and references therein.

## 2.3 Lepton Mass Matrices

At low energy scales, the phenomenology of lepton masses and flavor mixing can be formulated in terms of the charged lepton mass matrix $M_l$ and the (effective) neutrino mass matrix $M_\nu$. They totally involve twelve physical parameters:

- three masses of charged leptons ($m_e$, $m_\mu$ and $m_\tau$), which have precisely been measured [6];
- three masses of neutrinos ($m_1$, $m_2$ and $m_3$), whose relative sizes have roughly been known from solar and atmospheric neutrino oscillations;
- three angles of flavor mixing ($\theta_x$, $\theta_y$ and $\theta_z$), whose values have been determined or constrained to an acceptable degree of accuracy from solar, atmospheric and reactor neutrino oscillations;
- three phases of CP violation ($\delta$, $\rho$ and $\sigma$), which are completely unrestricted by current neutrino data.

The long-baseline neutrino oscillation experiments are expected to determine or constrain $\Delta m^2_{21}$, $\Delta m^2_{32}$, $\theta_x$, $\theta_y$, $\theta_z$ and $\delta$ (the Dirac-type CP-violating phase) to a high degree of accuracy in the future. The precision experiments for the tritium beta decay and the neutrinoless double-$\beta$ decay may help determine or constrain the absolute scale of neutrino masses. The possible regions of two Majorana-type CP-violating phases ($\rho$ and $\sigma$) could roughly be constrained by the neutrinoless double-$\beta$ decay. It seems hopeless, at least for the presently conceivable sets of feasible neutrino experiments, to fully determine ($m_1, m_2, m_3$) and ($\rho, \sigma$) [60]. In this case, to establish testable relations between the experimental quantities and the parameters of $M_l$ and $M_\nu$, has to rely on some phenomenological hypotheses.

The specific structures of $M_l$ and $M_\nu$ depend substantially on the flavor basis to be chosen. There are two simple possibilities to choose the flavor basis: (a) $M_l$ is diagonal and has positive eigenvalues; and (b) $M_\nu$ is diagonal and has positive eigenvalues. Let us focus on case (a) in the subsequent discussions, because it has popularly been taken in the phenomenological study of lepton mass matrices.

In the flavor basis where $M_l$ is diagonal (i.e., the flavor eigenstates of charged leptons are identified with their mass eigenstates), the symmetric neutrino mass matrix $M_\nu$ can be diagonalized by a single unitary transformation,

$$V^\dagger M_\nu V^* = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \quad (2.37)$$

---

Case (b) has been considered in Ref. [61], for example, to derive the democratic (nearly bi-maximal) neutrino mixing pattern.
Table 2.1: Nine two-zero textures of the neutrino mass matrix, which are compatible with current neutrino oscillation data. Note that the patterns in the same category (e.g., A₁ and A₂) have very similar phenomenological consequences [63].

| Pattern | By current data | Pattern | By current data |
|---------|-----------------|---------|-----------------|
| A₁ : \( \begin{pmatrix} 0 & 0 & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix} \) favored | A₂ : \( \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix} \) favored |
| B₁ : \( \begin{pmatrix} \times & 0 & \times \\ 0 & \times & \times \\ \times & \times & \times \end{pmatrix} \) favored | B₂ : \( \begin{pmatrix} 0 & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{pmatrix} \) favored |
| B₃ : \( \begin{pmatrix} 0 & 0 & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix} \) favored | B₄ : \( \begin{pmatrix} \times & \times & \times \\ \times & 0 & 0 \\ \times & \times & 0 \end{pmatrix} \) favored |
| C : \( \begin{pmatrix} \times & 0 & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix} \) favored | |
| D₁ : \( \begin{pmatrix} \times & 0 & 0 \\ \times & 0 & \times \\ \times & 0 & \times \end{pmatrix} \) marginally allowed | D₂ : \( \begin{pmatrix} \times & \times & \times \\ \times & 0 & 0 \\ \times & 0 & 0 \end{pmatrix} \) marginally allowed |

The 3 × 3 unitary matrix \( V \) is just the lepton flavor mixing matrix, consisting of three mixing angles and three CP-violating phases. It is straightforward to fix the form of \( M_\nu \) from (2.37),

\[
M_\nu = V \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} V^T,
\]

but extra assumptions have to be made for \( M_\nu \) in order to relate neutrino masses to flavor mixing parameters. The general idea is to assume that some independent matrix elements of \( M_\nu \) are actually dependent upon one another, caused by an underlying (broken) flavor symmetry. In particular, this dependence becomes very simple and transparent, if the relevant matrix elements are exactly equal to zero. Three general categories of \( M_\nu \) with texture zeros have been classified and discussed ¹:

- The three-zero textures of \( M_\nu \). There are totally twenty possible patterns of \( M_\nu \) with three independent vanishing entries, but none of them is allowed by current neutrino oscillation data [62].

- The two-zero textures of \( M_\nu \). There are totally fifteen possible patterns of \( M_\nu \) with two independent vanishing entries. Nine two-zero patterns of \( M_\nu \), as illustrated in Table 2.1, are found to be compatible with current experimental data (but two of them are only marginally allowed [63]).

- The one-zero textures of \( M_\nu \). There are totally six possible patterns of \( M_\nu \) with one vanishing entry, as shown in Table 2.2. It has been found in Ref. [64] that

¹Because \( M_\nu \) is symmetric, a pair of off-diagonal texture zeros of \( M_\nu \) have been counted as one zero.
Table 2.2: Six possible one-zero textures of the neutrino mass matrix. They may agree or disagree with current neutrino oscillation data, if one mass eigenvalue \( m_1 \) or \( m_3 \) vanishes [64].

| Pattern | By current data | Pattern | By current data |
|---------|-----------------|---------|-----------------|
| A :     | \[
\begin{pmatrix}
0 & \times & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & 0 & \\
0 & \times & \times & \\
\end{pmatrix}
\] \( m_1 = 0 \) favored  \\
\( m_3 = 0 \) disfavored | B : \[
\begin{pmatrix}
\times & 0 & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\end{pmatrix}
\] \( m_1 = 0 \) favored  \\
\( m_3 = 0 \) favored |
| C :     | \[
\begin{pmatrix}
\times & \times & 0 & \\
\times & \times & \times & \\
0 & \times & \times & \\
\times & \times & 0 & \\
\times & \times & \times & \\
\end{pmatrix}
\] \( m_1 = 0 \) favored  \\
\( m_3 = 0 \) favored | D : \[
\begin{pmatrix}
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\end{pmatrix}
\] \( m_1 = 0 \) disfavored  \\
\( m_3 = 0 \) favored |
| E :     | \[
\begin{pmatrix}
\times & \times & 0 & \\
\times & \times & \times & \\
\times & \times & 0 & \\
\times & \times & \times & \\
\times & \times & \times & \\
\end{pmatrix}
\] \( m_1 = 0 \) disfavored  \\
\( m_3 = 0 \) disfavored | F : \[
\begin{pmatrix}
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & \times & \\
\times & \times & 0 & \\
\times & \times & \times & \\
\end{pmatrix}
\] \( m_1 = 0 \) disfavored  \\
\( m_3 = 0 \) favored |

Of course, the specific texture zeros of the neutrino mass matrix may not be preserved to all orders or at any energy scales in the unspecified interactions from which neutrino masses are generated. Whether \( m_1 = 0 \) or \( m_3 = 0 \) is stable against radiative corrections should also be taken into account in a concrete theoretical model.

In general, both the charged lepton mass matrix \( M_L \) and the (effective) neutrino mass matrix \( M_\nu \) are not diagonal. A typical ansatz of this nature is the Fritzsch pattern [65] of lepton mass matrices,

\[
M_{L,\nu} = \begin{pmatrix}
0 & C_{L,\nu} & 0 \\
C_{L,\nu} & 0 & B_{L,\nu} \\
0 & B_{L,\nu} & A_{L,\nu}
\end{pmatrix},
\] (2.39)

in which six texture zeros are included. It has been shown in Ref. [66] that such a simple ansatz of lepton mass matrices can naturally predict a normal hierarchy of neutrino masses and a bi-large pattern of lepton flavor mixing. A generalized version of (2.39) with four texture zeros,

\[
M_{L,\nu} = \begin{pmatrix}
0 & C_{L,\nu} & 0 \\
C_{L,\nu} & \tilde{B}_{L,\nu} & B_{L,\nu} \\
0 & B_{L,\nu} & A_{L,\nu}
\end{pmatrix},
\] (2.40)

has carefully been analyzed in Ref. [67]. The motivation to consider the four-zero texture of lepton mass matrices is quite clear. It is well known that the four-zero texture of quark mass matrices is more successful than the six-zero texture of quark mass matrices to interpret the strong hierarchy of quark masses and the smallness of flavor mixing angles [68]. Thus the spirit of lepton-quark similarity motivates us to conjecture that the lepton mass matrices might have the same texture zeros as the quark mass matrices. Such a phenomenological conjecture is indeed reasonable in some specific models of
grand unified theories [69], in which the mass matrices of leptons and quarks are related to each other by a new kind of flavor symmetry. That is why the four-zero texture of lepton mass matrices has been considered as a typical example in a number of interesting model-building works [70].

Taking account of the seesaw mechanism, one may investigate how the texture zeros of $M_D$ and $M_R$ are related to those of $M_\nu$ via (2.9). For example, it has been shown in Ref. [71] that some two-zero textures of $M_\nu$ can easily be reproduced from $M_D$ and $M_R$ with several vanishing entries. Provided both $M_D$ and $M_R$ have texture zeros in their (1,1), (1,3) and (3,1) positions, $M_\nu$ must have the same texture zeros [12]. This seesaw invariance of lepton mass matrices implies that all lepton and quark mass matrices might have a universal texture like (2.40), resulting from a universal (broken) flavor symmetry at the energy scale where the seesaw mechanism works.

There are many other phenomenological ansätze of lepton mass matrices, either at low energy scales or at high energy scales. For recent reviews with extensive references, we refer the reader to Ref. [12], Refs. [18]–[21], and Ref. [69].

3 Lepton Flavor Mixing

3.1 The Flavor Mixing Matrix in Vacuum

The phenomenon of lepton flavor mixing arises from the mismatch between the diagonalization of the charged lepton mass matrix $M_l$ and that of the neutrino mass matrix $M_\nu$ in an arbitrary flavor basis. Without loss of any generality, one can choose to identify the flavor eigenstates of charged leptons with their mass eigenstates. In this specific basis with $M_l$ being diagonal, $M_\nu$ is in general not diagonal. If massive neutrinos are Dirac particles, it is always possible to diagonalize the arbitrary mass matrix $M_\nu$ by a bi-unitary transformation:

$$V^\dagger M_\nu V' = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$  \hspace{1cm} (3.1)

If massive neutrinos are Majorana particles, one may diagonalize the symmetric mass matrix $M_\nu$ by a single unitary transformation:

$$V^\dagger M_\nu V^* = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}.$$ \hspace{1cm} (3.2)

In (3.1) and (3.2), $m_i$ denote the (positive) mass eigenvalues of three active neutrinos. In terms of the mass eigenstates of charged leptons and neutrinos, the Lagrangian of charged-current weak interactions in (2.1) can be rewritten as

$$-\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} \left( \nu_1, \nu_2, \nu_3 \right)_L V^\dagger \gamma^\mu \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_L W^{+\mu}_\mu + \text{h.c.},$$ \hspace{1cm} (3.3)
where the $3 \times 3$ unitary matrix $V$ links the neutrino mass eigenstates ($\nu_1, \nu_2, \nu_3$) to the neutrino flavor eigenstates ($\nu_e, \nu_\mu, \nu_\tau$):

$$
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} =
\begin{pmatrix}
V_{e1} & V_{e2} & V_{e3} \\
V_{\mu1} & V_{\mu2} & V_{\mu3} \\
V_{\tau1} & V_{\tau2} & V_{\tau3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix} .
$$

(3.4)

Obviously $V$ is just the flavor mixing matrix of charged leptons and active neutrinos. The analogue of $V$ in the quark sector is well known as the Cabibbo-Kobayashi-Maskawa (CKM) matrix [10]. In some literature, the lepton mixing matrix $V$ has been referred to as the Maki-Nakagawa-Sakata (MNS) matrix [27] or the Pontecorvo-MNS (PMNS) matrix.

Note that massive neutrinos are commonly believed to be Majorana particles, although there has not been any reliable experimental evidence for that. For simplicity, we use $V$ to denote the lepton flavor mixing matrix in the Majorana case. Then $V$ can always be parametrized as a product of the Dirac-type flavor mixing matrix $U$ and a diagonal phase matrix $P$; i.e., $V = UP$ [72]. The nontrivial CP-violating phases in $P$ characterize the Majorana nature of $V$. It is straightforward to show that the rephasing-invariant quantities $V_{\alpha i}V_{\beta j}V_{\alpha j}^*V_{\beta i}^*$ and $U_{\alpha i}U_{\beta j}U_{\alpha j}^*U_{\beta i}^*$ are identical for arbitrary indices $(\alpha, \beta)$ over $(e, \mu, \tau)$ and $(i, j)$ over $(1, 2, 3)$. This result implies that $V$ and $U$ have the same physical effects in normal neutrino-neutrino and antineutrino-antineutrino oscillations, whose probabilities depend only upon the rephasing-invariant quantities mentioned above. One then arrives at the conclusion that it is impossible to determine the nature of massive neutrinos by studying the phenomena of neutrino oscillations. Only the experiments which probe transitions between left-handed and right-handed neutrino states, like the neutrinoless double beta decay, could tell whether massive neutrinos are of the Majorana type or not.

### 3.1.1 Experimental Constraints

Current experimental constraints on the lepton flavor mixing matrix $V$ come mainly from solar, atmospheric and CHOOZ reactor neutrino oscillations. These three types of neutrino oscillations all belong to the “disappearance” experiments, in which the survival probability of a flavor eigenstate $\nu_\alpha$ is given as

$$
P(\nu_\alpha \rightarrow \nu_\alpha) = 1 - 4 \sum_{i<j} \left( |V_{\alpha i}|^2 |V_{\alpha j}|^2 \sin^2 F_{ji} \right) ,
$$

(3.5)

where $F_{ji} \equiv 1.27 \Delta m^2_{ji} L/E$ with $\Delta m^2_{ji} \equiv m_j^2 - m_i^2$, $L$ stands for the baseline length (in unit of km), and $E$ is the neutrino beam energy (in unit of GeV). Because of CPT invariance, the survival probabilities of neutrinos and antineutrinos in vacuum are equal to each other: $P(\overline{\nu}_\alpha \rightarrow \overline{\nu}_\alpha) = P(\nu_\alpha \rightarrow \nu_\alpha)$. This equality may in general be violated, if neutrinos and antineutrinos propagate in matter [73].

As mentioned in section 2, the solar ($\nu_e \rightarrow \nu_e$) and atmospheric ($\nu_\mu \rightarrow \nu_\mu$) neutrino oscillation data indicate

$$
|\Delta m^2_{21}| = \Delta m^2_{\text{sun}} \ll \Delta m^2_{\text{atm}} = |\Delta m^2_{32}| \approx |\Delta m^2_{31}| .
$$

(3.6)

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In addition, the CHOOZ ($\overline{\nu}_e \rightarrow \overline{\nu}_e$) neutrino oscillation data are obtained at the scale $\Delta m^2_{\text{chz}} \approx |\Delta m^2_{32}|$. To a good degree of accuracy, (3.5) is simplified to

$$P(\nu_e \rightarrow \nu_e) \approx 1 - \sin^2 2\theta_{\text{sun}} \sin^2 \left(1.27 \frac{\Delta m^2_{\text{sun}} L}{E}\right)$$

with

$$\sin^2 2\theta_{\text{sun}} = 4|V_{e1}|^2 |V_{e2}|^2$$

for solar neutrino oscillations;

$$P(\nu_\mu \rightarrow \nu_\mu) \approx 1 - \sin^2 2\theta_{\text{atm}} \sin^2 \left(1.27 \frac{\Delta m^2_{\text{atm}} L}{E}\right)$$

with

$$\sin^2 2\theta_{\text{atm}} = 4|V_{\mu3}|^2 \left(1 - |V_{\mu3}|^2\right)$$

for atmospheric neutrino oscillations; and

$$P(\overline{\nu}_e \rightarrow \overline{\nu}_e) \approx 1 - \sin^2 2\theta_{\text{chz}} \sin^2 \left(1.27 \frac{\Delta m^2_{\text{chz}} L}{E}\right)$$

with

$$\sin^2 2\theta_{\text{chz}} = 4|V_{e3}|^2 \left(1 - |V_{e3}|^2\right)$$

for the CHOOZ neutrino oscillation experiment. In view of the present experimental data from SK [2], SNO [3], KamLAND [4], K2K [5] and CHOOZ [7], we have

$$0.25 \leq \sin^2 \theta_{\text{sun}} \leq 0.4 \quad \text{or} \quad 30.0^\circ \leq \theta_{\text{sun}} \leq 39.2^\circ,$$

$$0.92 < \sin^2 2\theta_{\text{atm}} \leq 1.0 \quad \text{or} \quad 36.8^\circ < \theta_{\text{atm}} < 53.2^\circ,$$

$$0 \leq \sin^2 2\theta_{\text{chz}} < 0.1 \quad \text{or} \quad 0^\circ \leq \theta_{\text{chz}} < 9.2^\circ,$$

at the 90% confidence level. More precise data on $\theta_{\text{sun}}, \theta_{\text{atm}}$ and $\theta_{\text{chz}}$ will be available in the near future.

Taking account of the unitarity of $V$, one may reversely express $|V_{e1}|, |V_{e2}|, |V_{e3}|, |V_{\mu3}|$ and $|V_{\tau3}|$ in terms of $\theta_{\text{sun}}, \theta_{\text{atm}}$ and $\theta_{\text{chz}}$ [36]:

$$|V_{e1}| = \frac{1}{\sqrt{2}} \sqrt{\cos^2 \theta_{\text{chz}} + \sqrt{\cos^4 \theta_{\text{chz}} - \sin^2 2\theta_{\text{sun}}}},$$

$$|V_{e2}| = \frac{1}{\sqrt{2}} \sqrt{\cos^2 \theta_{\text{chz}} - \sqrt{\cos^4 \theta_{\text{chz}} - \sin^2 2\theta_{\text{sun}}}},$$

$$|V_{e3}| = \sin \theta_{\text{chz}},$$

$$|V_{\mu3}| = \sin \theta_{\text{atm}},$$

$$|V_{\tau3}| = \sqrt{\cos^2 \theta_{\text{chz}} - \sin^2 \theta_{\text{atm}}}.$$

The other four matrix elements of $V$ (i.e., $|V_{\mu1}|, |V_{\mu2}|, |V_{\tau1}|$ and $|V_{\tau2}|$) are entirely unrestricted, however, unless one of them or the rephasing invariant of CP violation of $V$ (defined in section 4.1) is measured. A realistic way to get rough but useful constraints
on those four unknown elements is to allow the Dirac phase of CP violation in $V$ to vary between 0 and $\pi$ [74], such that one can find out the maximal and minimal magnitudes of each matrix element. To see this point more clearly, we adopt a simplified parametrization of $V$ in which the Majorana phases of CP violation are omitted [75]:

$$V = \begin{pmatrix}
    c_x c_z & s_x c_z & s_z \\
    -c_x s_y s_z - s_x c_y e^{-i\delta} & s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\
    -c_x c_y s_z + s_x s_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z
\end{pmatrix},$$

(3.15)

where $s_x \equiv \sin \theta_x$, $c_x \equiv \cos \theta_x$, and so on. The Dirac phase $\delta$ affects the magnitudes of $V_{\mu 1}$, $V_{\mu 2}$, $V_{\tau 1}$ and $V_{\tau 2}$, while the Majorana phases of CP violation do not have such effects (see section 3.1.2 for more discussions). Note that three mixing angles ($\theta_x, \theta_y, \theta_z$), which are all arranged to lie in the first quadrant, can be written as

$$\tan \theta_x = \frac{|V_{e 2}|}{|V_{e 1}|}, \quad \tan \theta_y = \frac{|V_{e 3}|}{|V_{\tau 3}|}, \quad \sin \theta_z = |V_{e 3}|.$$

(3.16)

It is then straightforward to obtain

$$|V_{\mu 1}| = \frac{|V_{e 2}||V_{\tau 3}| + |V_{e 1}||V_{e 3}||V_{\mu 3}| e^{i\delta}}{1 - |V_{e 3}|^2},$$

$$|V_{\mu 2}| = \frac{|V_{e 1}||V_{\tau 3}| - |V_{e 2}||V_{e 3}||V_{\mu 3}| e^{i\delta}}{1 - |V_{e 3}|^2},$$

$$|V_{\tau 1}| = \frac{|V_{e 2}||V_{\mu 3}| - |V_{e 1}||V_{e 3}||V_{\tau 3}| e^{i\delta}}{1 - |V_{e 3}|^2},$$

$$|V_{\tau 2}| = \frac{|V_{e 1}||V_{\mu 3}| + |V_{e 2}||V_{e 3}||V_{\tau 3}| e^{i\delta}}{1 - |V_{e 3}|^2}.$$

(3.17)

Varying the Dirac phase $\delta$ from 0 to $\pi$, we are led to the most generous ranges of $|V_{\mu 1}|$, $|V_{\mu 2}|$, $|V_{\tau 1}|$ and $|V_{\tau 2}|$ [76]:

$$\frac{|V_{e 2}||V_{\tau 3}| - |V_{e 1}||V_{e 3}||V_{\mu 3}|}{1 - |V_{e 3}|^2} \leq |V_{\mu 1}| \leq \frac{|V_{e 2}||V_{\tau 3}| + |V_{e 1}||V_{e 3}||V_{\mu 3}|}{1 - |V_{e 3}|^2},$$

$$\frac{|V_{e 1}||V_{\tau 3}| - |V_{e 2}||V_{e 3}||V_{\mu 3}|}{1 - |V_{e 3}|^2} \leq |V_{\mu 2}| \leq \frac{|V_{e 1}||V_{\tau 3}| + |V_{e 2}||V_{e 3}||V_{\mu 3}|}{1 - |V_{e 3}|^2},$$

$$\frac{|V_{e 2}||V_{\mu 3}| - |V_{e 1}||V_{e 3}||V_{\tau 3}|}{1 - |V_{e 3}|^2} \leq |V_{\tau 1}| \leq \frac{|V_{e 2}||V_{\mu 3}| + |V_{e 1}||V_{e 3}||V_{\tau 3}|}{1 - |V_{e 3}|^2},$$

$$\frac{|V_{e 1}||V_{\mu 3}| - |V_{e 2}||V_{e 3}||V_{\tau 3}|}{1 - |V_{e 3}|^2} \leq |V_{\tau 2}| \leq \frac{|V_{e 1}||V_{\mu 3}| + |V_{e 2}||V_{e 3}||V_{\tau 3}|}{1 - |V_{e 3}|^2}.$$

(3.18)

Note that the lower and upper bounds of each matrix element turn to coincide with each other in the limit $|V_{e 3}| \rightarrow 0$. Because of the smallness of $|V_{e 3}|$, the ranges obtained in (3.18) should be quite restrictive. Hence it makes sense to recast the lepton flavor mixing matrix even in the absence of any experimental information on CP violation.

Using the values of $\theta_{sun}$, $\theta_{atm}$ and $\theta_{chz}$ given in (3.13), we calculate $|V_{e 1}|$, $|V_{e 2}|$, $|V_{e 3}|$, $|V_{\mu 3}|$ and $|V_{\tau 3}|$ from (3.14). Then the allowed ranges of $|V_{\mu 1}|$, $|V_{\mu 2}|$, $|V_{\tau 1}|$ and $|V_{\tau 2}|$ can
be found with the help of (3.18). Our numerical results are summarized as [76]

\[
|V| = \begin{pmatrix}
0.70 - 0.87 & 0.50 - 0.69 & < 0.16 \\
0.20 - 0.61 & 0.34 - 0.73 & 0.60 - 0.80 \\
0.21 - 0.63 & 0.36 - 0.74 & 0.58 - 0.80 \\
\end{pmatrix}.
\] (3.19)

This represents our present knowledge on the lepton flavor mixing matrix.

### 3.1.2 Standard Parametrization

Flavor mixing among \( n \) different lepton families is in general described by a \( n \times n \) unitary matrix \( V \), whose number of independent parameters relies on the style of neutrinos. If neutrinos are Dirac particles, \( V \) can be parametrized in terms of \( n(n - 1)/2 \) rotation angles and \( (n - 1)(n - 2)/2 \) phase angles. If neutrinos are Majorana particles, however, a full parametrization of \( V \) requires \( n(n - 1)/2 \) rotation angles and the same number of phase angles \(^9\). The flavor mixing of charged leptons and Dirac neutrinos is completely analogous to that of quarks, for which a number of different parametrizations have been proposed and classified in the literature [78]. In this subsection we classify all possible parametrizations for the flavor mixing of charged leptons and Majorana neutrinos with \( n = 3 \). Regardless of the phase-assignment freedom, we find that there are nine structurally different parametrizations for the \( 3 \times 3 \) lepton flavor mixing matrix \( V \). Although all nine representations are mathematically equivalent, one of them is likely to make the underlying physics of lepton mass generation (and CP violation) more transparent, or is more useful in the analyses of neutrino oscillation data.

The \( 3 \times 3 \) lepton flavor mixing matrix \( V \) defined in (3.4) can be expressed as a product of three unitary matrices \( O_1, O_2 \) and \( O_3 \), which correspond to simple rotations in the complex (1,2), (2,3) and (3,1) planes:

\[
O_1(\theta_1, \alpha_1, \beta_1, \gamma_1) = \begin{pmatrix}
c_1e^{i\alpha_1} & s_1e^{-i\beta_1} & 0 \\
-s_1e^{i\beta_1} & c_1e^{-i\alpha_1} & 0 \\
0 & 0 & e^{i\gamma_1}
\end{pmatrix},
\]

\[
O_2(\theta_2, \alpha_2, \beta_2, \gamma_2) = \begin{pmatrix}
e^{i\gamma_2} & 0 & 0 \\
0 & c_2e^{i\alpha_2} & s_2e^{-i\beta_2} \\
0 & -s_2e^{i\beta_2} & c_2e^{-i\alpha_2}
\end{pmatrix},
\]

\[
O_3(\theta_3, \alpha_3, \beta_3, \gamma_3) = \begin{pmatrix}
c_3e^{i\alpha_3} & 0 & s_3e^{-i\beta_3} \\
0 & e^{i\gamma_3} & 0 \\
-s_3e^{i\beta_3} & 0 & c_3e^{-i\alpha_3}
\end{pmatrix},
\] (3.20)

where \( s_i \equiv \sin \theta_i \) and \( c_i \equiv \cos \theta_i \) (for \( i = 1, 2, 3 \)). Obviously \( O_1O_1^\dagger = O_2O_2^\dagger = O_3O_3^\dagger = 1 \) holds, and any two rotation matrices do not commute with each other. We find twelve different ways to arrange the product of \( O_1, O_2 \) and \( O_3 \), which can cover the whole \( 3 \times 3 \) space and provide a full description of \( V \). Explicitly, six of the twelve different combinations of \( O_i \) belong to the type [79]

\[
V = O_i(\theta_i, \alpha_i, \beta_i, \gamma_i) \otimes O_j(\theta_j, \alpha_j, \beta_j, \gamma_j) \otimes O_k(\theta_k', \alpha_k', \beta_k', \gamma_k')
\] (3.21)

\(^9\)No matter whether neutrinos are Dirac or Majorana particles, the \( n \times n \) flavor mixing matrix has \( (n - 1)^2(n - 2)^2/4 \) rephasing invariants of CP violation [77].
with \( i \neq j \), where the complex rotation matrix \( O_i \) occurs twice; and the other six belong to the type

\[
V = O_i(\theta_i, \alpha_i, \beta_i, \gamma_i) \otimes O_j(\theta_j, \alpha_j, \beta_j, \gamma_j) \otimes O_k(\theta_k, \alpha_k, \beta_k, \gamma_k)
\]  

(3.22)

with \( i \neq j \neq k \), in which the rotations take place in three different complex planes. Note that the products \( O_iO_jO_i \) and \( O_iO_kO_i \) (for \( i \neq k \)) in (3.21) are correlated with each other, if the relevant phase parameters are switched off. Hence only nine of the twelve parametrizations, three from (3.21) and six from (3.22), are structurally different.

In each of the nine distinct parametrizations for \( V \), there apparently exist nine phase parameters. Six of them or their combinations can be absorbed by redefining the arbitrary phases of charged lepton fields and a common phase of neutrino fields. If neutrinos are Dirac particles, one can also redefine the arbitrary phases of Dirac neutrino fields, so as to reduce the number of the remaining phase parameters from three to one. In this case \( V \) contains a single CP-violating phase of the Dirac nature. If neutrinos are Majorana particles, however, there is no freedom to rearrange the relative phases of Majorana neutrino fields. Hence three nontrivial phase parameters are present in the \( 3 \times 3 \) Majorana-like flavor mixing matrix \( V \). Two of the three CP-violating phases in \( V \) can always be factored out as the “pure” Majorana phases through a proper phase assignment of the charged lepton fields, and the remaining one is identified as the “Dirac-like” phase. Both CP- and T-violating effects in normal neutrino oscillations depend only upon the Dirac-like phase.

Of course, different parametrizations of \( V \) are mathematically equivalent, and adopting any of them does not point to physical significance. It is quite likely, however, that one particular parametrization is more useful and transparent than the others in the analyses of data from various neutrino experiments and (or) towards a deeper understanding of the underlying dynamics responsible for lepton mass generation and CP violation. We find that such an interesting parametrization does exist [79]:

\[
V = \begin{pmatrix}
c_x c_z & s_x c_z & s_z \\
-c_x s_y s_z - s_x c_y e^{-i\delta} & -s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\
-c_x c_y s_z + s_x s_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z
\end{pmatrix}
\begin{pmatrix}
e^{i\rho} & 0 & 0 \\
0 & e^{i\sigma} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  

(3.23)

with \( s_x \equiv \sin \theta_x, c_y \equiv \cos \theta_y \), and so on. Without loss of generality, the three mixing angles \((\theta_x, \theta_y, \theta_z)\) can all be arranged to lie in the first quadrant. Arbitrary values between 0 and \( 2\pi \) are allowed for the Dirac CP-violating phase \( \delta \) and the Majorana CP-violating phases \( \rho \) and \( \sigma \).

The parametrization in (3.23) can be regarded as a straightforward generalization of the parametrization in (3.15) to include the Majorana phases of CP violation. Thus the interesting relations obtained in (3.16) remain valid. A remarkable merit of this parametrization is that its three mixing angles \((\theta_x, \theta_y, \theta_z)\) are directly related to the mixing angles of solar, atmospheric and CHOOZ reactor neutrino oscillations:

\[
\theta_x \approx \theta_{\text{sun}}, \quad \theta_y \approx \theta_{\text{atm}}, \quad \theta_z \approx \theta_{\text{chz}},
\]  

(3.24)

which can easily be derived from (3.8), (3.10), (3.12) and (3.23). Other parametrizations of the lepton flavor mixing matrix would not be able to result in such simple relations between the fundamental mixing quantities and the relevant experimental observables.
The CP- and T-violating effects in normal neutrino oscillations are measured by the well-known Jarlskog parameter $J$ [80] (defined in section 4.1), which is proportional to $\sin \delta$ in the following way:

$$J \approx \sin \theta_{\text{sun}} \cos \theta_{\text{sun}} \sin \theta_{\text{atm}} \cos \theta_{\text{atm}} \sin \theta_{\text{che}} \cos^2 \theta_{\text{che}} \sin \delta .$$ (3.25)

To see this point more clearly, we write out the transition probabilities of different neutrino flavor eigenstates in vacuum [12]:

$$P(\nu_\alpha \to \nu_\beta) = -4 \sum_{i<j} \left[ \text{Re} \left( V_{\alpha i} V_{\beta j}^* V_{\alpha j}^* V_{\beta i} \right) \sin^2 F_{ji} \right] - 8 J \prod_{i<j} (\sin F_{ji}) ,$$ (3.26)

where $(\alpha, \beta)$ run over $(e, \mu)$, $(\mu, \tau)$ or $(\tau, e)$; $i$ and $j$ run over 1, 2, 3; and $F_{ji}$ and $\Delta m^2_{ji}$ have been defined below (3.5). In the assumption of CPT invariance, the transition probabilities $P(\nu_\beta \to \nu_\alpha)$ or $P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$ can directly be read off from (3.26) through the replacement $J \to -J$ (i.e., $V \to V^*$). Therefore, the CP-violating asymmetry between $P(\nu_\alpha \to \nu_\beta)$ and $P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$ amounts to the T-violating asymmetry between $P(\nu_\alpha \to \nu_\beta)$ and $P(\nu_\beta \to \nu_\alpha)$:

$$\Delta P = P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$$

$$= P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha)$$

$$= -16 J \sin F_{21} \sin F_{31} \sin F_{32} .$$ (3.27)

The determination of $J$ from $\Delta P$ will allow us to extract the CP-violating phase $\delta$, provided all three mixing angles $(\theta_x, \theta_y, \theta_z)$ have been measured elsewhere. In practical long-baseline neutrino oscillation experiments, however, all these measurable quantities may be contaminated by the terrestrial matter effects. Hence the fundamental parameters of lepton flavor mixing need be disentangled from the matter-corrected ones. Some discussions about the matter effects on lepton flavor mixing and CP violation will be given in sections 3.2 and 4.1.

Regardless of the Majorana phases $\rho$ and $\sigma$, which have nothing to do with normal neutrino oscillations, we have located the Dirac phase $\delta$ in such a way that the matrix elements in the first row and the third column of $V$ are real. As a consequence, the CP-violating phase $\delta$ does not appear in the effective mass term of the neutrinoless double beta decay. Indeed, the latter reads:

$$\langle m \rangle_{ee} = \left| m_1 V_{e1}^2 + m_2 V_{e2}^2 + m_3 V_{e3}^2 \right|$$

$$= \sqrt{a + b \cos 2(\rho - \sigma) + c \cos 2\rho + d \cos 2\sigma} ,$$ (3.28)

where

$$a = m_1^2 c_z^4 + m_2^2 s_z^4 + m_3^2 s_z^4 ,$$

$$b = 2 m_1 m_2 s_z^2 c_z^2 s_z^2 c_z^2 ,$$

$$c = 2 m_1 m_3 s_z c_z^2 s_z^2 s_z^2 c_z^2 ,$$

$$d = 2 m_1 m_3 s_z^2 s_z^2 s_z^2 s_z^2 .$$ (3.29)

It becomes obvious that $\langle m \rangle_{ee}$ is independent of the Dirac phase $\delta$. On the other hand, the CP- or T-violating asymmetry $\Delta P$ in normal neutrino oscillations is independent of
the Majorana phases $\rho$ and $\sigma$. Thus two different types of CP-violating phases can (in principle) be studied in two different types of experiments.

A long-standing and important question is whether the two Majorana phases $\rho$ and $\sigma$ can separately be determined by measuring other possible lepton-number-nonconserving processes, in addition to the neutrinoless double beta decay. While the answer to this question is affirmative in principle, it seems to be negative in practice. The key problem is that those lepton-number-violating processes, in which the Majorana phases can show up, are suppressed in magnitude by an extremely small factor compared to normal weak interactions [79, 81]. Therefore it is extremely difficult, even impossible, to measure or constrain $\rho$ and $\sigma$ in any experiment other than the one associated with the neutrinoless double beta decay.

On the theoretical side, how to predict or calculate those flavor mixing angles and CP-violating phases on a solid dynamical ground is also a big challenge. Phenomenologically, the parametrization in (3.23) is expected to be very useful and convenient, and might even be able to provide some insight into the underlying physics of lepton mass generation. We therefore recommend it to experimentalists and theorists as the standard parametrization of the $3 \times 3$ lepton flavor mixing matrix.

### 3.1.3 Constant Mixing Patterns

Combining (3.13) and (3.24), one can immediately observe the essential feature of lepton flavor mixing: two mixing angles are large and comparable in magnitude ($\theta_x \sim \theta_y \sim 1$), while the third one is very small ($\theta_z \ll 1$). Such a “nearly bi-maximal” flavor mixing pattern has attracted a lot of attention in model building. In particular, a number of constant mixing patterns, which are more or less close to the experimentally-allowed lepton flavor mixing matrix in (3.19), have been proposed. Possible flavor symmetries and their spontaneous or explicit breaking mechanisms hidden in those constant patterns might finally help us pin down the dynamics responsible for neutrino mass generation and lepton flavor mixing. It is therefore worthwhile to have an overview of a few typical constant lepton mixing patterns existing in the literature.

The first example is the so-called “democratic” lepton flavor mixing pattern,

\[
V = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
-\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \\
\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3}
\end{pmatrix},
\]

(3.30)

which was originally obtained by Fritzsch and Xing [61] from the breaking of flavor democracy of the charged lepton mass matrix in the flavor basis where the neutrino mass matrix is diagonal. The predictions of (3.30) for the mixing factors of solar and

\[10\] We only concentrate on those nearly bi-maximal flavor mixing patterns, in which $\theta_x \sim \theta_y >> \theta_z$ holds. Hence the interesting Cabibbo ansatz [82], which has $|V_{\alpha i}| = 1/\sqrt{3}$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) or $\theta_x = \theta_y = 45^\circ$ and $\theta_z \approx 35.3^\circ$, will not be taken into account. Such a “tri-maximal” mixing pattern has completely been ruled out by current experimental data.
Table 3.1: The mixing angles ($\theta_x, \theta_y, \theta_z$) in four constant lepton flavor mixing patterns.

| Mixing Angle | Pattern (3.30) | Pattern (3.31) | Pattern (3.32) | Pattern (3.33) |
|--------------|----------------|----------------|----------------|----------------|
| $\theta_x$   | $45^\circ$     | $45^\circ$     | $35.3^\circ$   | $30^\circ$     |
| $\theta_y$   | $54.7^\circ$   | $45^\circ$     | $45^\circ$     | $45^\circ$     |
| $\theta_z$   | $0^\circ$      | $0^\circ$      | $0^\circ$      | $0^\circ$      |

atmospheric neutrino oscillations, $\sin^2 2\theta_{\text{sun}} = 1$ and $\sin^2 2\theta_{\text{atm}} = 8/9$, are not favored by today’s experimental data. Hence slight modifications of (3.30) have to be made. In Ref. [61] and Ref. [83], some proper perturbative terms have been introduced to break the $S(3)_L \times S(3)_R$ symmetry of the charged lepton mass matrix and the $S(3)$ symmetry of the neutrino mass matrix. The resultant flavor mixing matrix, whose leading term remains to take the form of (3.30), is able to fit current solar, atmospheric and CHOOZ reactor neutrino oscillation data very well.

The second example is the so-called “bi-maximal” neutrino mixing pattern,

$$V = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \quad (3.31)$$

proposed by Barger et al [84] and by Vissani [85]. It predicts $\sin^2 2\theta_{\text{sun}} = \sin^2 2\theta_{\text{atm}} = 1$ for solar and atmospheric neutrino oscillations. As the result $\sin^2 2\theta_{\text{sun}} = 1$ is disfavored by the LMA solution to the solar neutrino problem, slight modifications of (3.31) are also required. A number of possibilities to modify the bi-maximal mixing pattern into nearly bi-maximal mixing patterns have been discussed in the literature [86]. The essential point is to introduce a perturbative term to (3.31), which may naturally be related to the mass ratio(s) of charged leptons or to the Cabibbo angle of quark flavor mixing, such that $\sin^2 2\theta_{\text{sun}}$ deviates from unity to a proper extent.

Another interesting neutrino mixing pattern is

$$V = \begin{pmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \end{pmatrix}. \quad (3.32)$$

It was first conjectured by Wolfenstein long time ago [87] \(^{11}\). Possible (broken) flavor symmetries associated with (3.32) have recently been discussed by several authors [88]. Two straightforward consequences of (3.32) on neutrino oscillations are $\sin^2 2\theta_{\text{sun}} = 8/9$

\(^{11}\)The first and second columns of $V$ were interchanged in the original paper of Wolfenstein [87]. Such an interchange is allowed by current neutrino oscillation data, because $|V_{\text{e}1}|$ and $|V_{\text{e}2}|$ cannot separately be determined from the measurement of $\sin^2 2\theta_{\text{sun}} = 4|V_{\text{e}1}|^2|V_{\text{e}2}|^2$. In (3.14) and (3.32), we have taken a common convention to set $|V_{\text{e}1}| \geq |V_{\text{e}2}|$. 

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and $\sin^2 2\theta_{\text{atm}} = 1$, favored by the present solar and atmospheric neutrino oscillation data. Of course, nonvanishing $V_{e3}$ (or $\theta_z$) and CP-violating phases can be introduced into (3.32), if proper perturbations to the charged lepton and neutrino mass matrices are taken into account.

The last example for constant lepton flavor mixing is given as

$$V = \begin{pmatrix}
\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\
-\frac{\sqrt{2}}{4} & \frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2}
\end{pmatrix}, \quad (3.33)$$

which has recently been discussed by Giunti [89], Peccei [90] and Xing [91]. It predicts $\sin^2 2\theta_{\text{sun}} = 3/4$ and $\sin^2 2\theta_{\text{atm}} = 1$, which are in good agreement with the present solar and atmospheric neutrino oscillation data. Again slight modifications of (3.33) can be done [91], in order to accommodate nonvanishing $V_{e3}$ and leptonic CP violation.

We summarize the values of three flavor mixing angles for each of the four constant patterns in Table 3.1. The similarity and difference between any two patterns can easily be seen. It is worth remarking that a specific constant mixing pattern should be regarded as the leading-order approximation of the “true” lepton flavor mixing matrix, whose mixing angles depend in general on both the ratios of charged lepton masses and those of neutrino masses. Naively, one may make the following speculation:

- Large values of $\theta_x$ and $\theta_y$ could arise from a weak hierarchy or a near degeneracy of the neutrino mass spectrum, as the strong hierarchy of charged lepton masses implies that $m_e/m_\mu$ and $m_\mu/m_\tau$ are unlikely to contribute to $\theta_x$ and $\theta_y$ dominantly.

- Special values of $\theta_x$ and $\theta_y$ might stem from a discrete flavor symmetry of the charged lepton mass matrix or the neutrino mass matrix. Then the contributions of lepton mass ratios to flavor mixing angles, due to flavor symmetry breaking, are only at the perturbative level.

- Vanishing or small $\theta_z$ could be a natural consequence of the explicit textures of lepton mass matrices. It might also be related to the flavor symmetry which gives rise to sizable $\theta_x$ and $\theta_y$.

- Small perturbative effects on a constant flavor mixing pattern can also result from the renormalization-group equations of leptons and quarks, e.g., from the high energy scales to the low energy scales or vice versa [92].

To be more specific, we refer the reader to Ref. [12] and Ref. [21], from which a lot of theoretical models and phenomenological ansätze can be found for the interpretation of possible spectra of lepton masses and possible patterns of lepton flavor mixing.

### 3.2 The Flavor Mixing Matrix in Matter

The effective Hamiltonian responsible for the propagation of neutrinos in vacuum can be written as $\mathcal{H}_{\text{eff}} = \Omega_\nu/(2E)$ in the basis where the flavor eigenstates of charged leptons
are identified with their mass eigenstates. In \( \mathcal{H}_{\text{eff}} \), \( E \gg m_i \) denotes the neutrino beam energy, and \( \Omega_\nu \) is given by

\[
\Omega_\nu = V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^\dagger, \tag{3.34}
\]

in which \( V \) is the 3 \times 3 lepton flavor mixing matrix. Taking account of (3.1) or (3.2), we obtain \( \Omega_\nu = M_\nu M_\nu^\dagger \) [93], no matter whether massive neutrinos are Dirac or Majorana particles. A simple relation can therefore be established between the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \) and the neutrino mass matrix \( M_\nu \). The form of \( \mathcal{H}_{\text{eff}} \) has to be modified, however, in order to describe the propagation of neutrinos in matter.

If neutrinos travel through a normal material medium (e.g., the earth), which consists of electrons but of no muons or taus, they encounter both charged- and neutral-current interactions with electrons. In this case, the effective Hamiltonian responsible for the propagation of neutrinos takes the form \( \tilde{\mathcal{H}}_{\text{eff}} = \tilde{\Omega}_\nu / (2E) \), where [33]

\[
\tilde{\Omega}_\nu = \Omega_\nu + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.35}
\]

The apparent deviation of \( \tilde{\Omega}_\nu \) from \( \Omega_\nu \) arises from the matter-induced effect; namely, \( A = 2\sqrt{2} G_F N_e E \) describes the charged-current contribution to the \( \nu_e e^- \) forward scattering, where \( N_e \) is the background density of electrons and \( E \) stands for the neutrino beam energy. The neutral-current contributions, which are universal for \( \nu_e, \nu_\mu \) and \( \nu_\tau \) neutrinos and can only result in an overall unobservable phase, have been omitted.

Let us concentrate on the long-baseline neutrino oscillation experiments, which are designed to measure the parameters of lepton flavor mixing and CP violation. We shall assume a constant earth density profile (i.e., \( N_e = \text{constant} \)) in section 3.2.1 and section 3.2.3. Such an assumption is rather plausible and close to reality, only if the neutrino beam does not go through the earth’s core [94].

In the chosen flavor basis where \( M_1 \) is diagonal, \( \tilde{\Omega}_\nu \) can be diagonalized by a unitary transformation:

\[
\tilde{\Omega}_\nu = \tilde{V} \begin{pmatrix} \tilde{m}_1^2 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 \\ 0 & 0 & \tilde{m}_3^2 \end{pmatrix} \tilde{V}^\dagger, \tag{3.36}
\]

where \( \tilde{m}_i \) are the effective neutrino masses in matter. The unitary matrix \( \tilde{V} \) is just the matter-modified lepton flavor mixing matrix. Denoting the effective neutrino mass matrix in matter as \( \tilde{M}_\nu \), one may get \( \tilde{\Omega}_\nu = \tilde{M}_\nu \tilde{M}_\nu^\dagger \) similar to \( \Omega_\nu = M_\nu M_\nu^\dagger \). Phenomenologically, it is important to find the analytical relations between \( \tilde{m}_i \) and \( m_i \) as well as the exact relation between \( \tilde{V} \) and \( V \).

### 3.2.1 Generic Analytical Formulas

Combining (3.34), (3.35) and (3.36), we calculate \( \tilde{m}_i \) in terms of \( m_i \) and the matrix elements of \( V \). The results are

\[
\tilde{m}_1^2 = m_1^2 + \frac{1}{3} x - \frac{1}{3} \sqrt{x^2 - 3y} \left[ z + \sqrt{3 (1 - z^2)} \right],
\]

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\[
\tilde{m}_2^2 = m_i^2 + \frac{1}{3} x - \frac{1}{3} \sqrt{x^2 - 3y} \left[ z - \sqrt{3(1 - z^2)} \right],
\]
\[
\tilde{m}_3^2 = m_i^2 + \frac{1}{3} x + \frac{2}{3} z \sqrt{x^2 - 3y},
\]
(3.37)

where \( x, y \) and \( z \) are given by [95]
\[
\begin{align*}
x &= \Delta m_{21}^2 + \Delta m_{31}^2 + A, \\
y &= \Delta m_{21}^2 \Delta m_{31}^2 + A \left[ \Delta m_{21}^2 \left( 1 - \left| V_{e2} \right|^2 \right) + \Delta m_{31}^2 \left( 1 - \left| V_{e3} \right|^2 \right) \right], \\
z &= \cos \left[ \frac{1}{3} \arccos \frac{2x^3 - 9xy + 27A \Delta m_{21}^2 \Delta m_{31}^2 |V_{e1}|^2}{2(x^2 - 3y)^{3/2}} \right] 
\end{align*}
(3.38)
\]

with \( \Delta m_{21}^2 \equiv m_2^2 - m_1^2 \) and \( \Delta m_{31}^2 \equiv m_3^2 - m_1^2 \). Of course, \( \tilde{m}_i^2 = m_i^2 \) can be reproduced from (3.37) when \( A = 0 \) is taken. Only the mass-squared differences \( \Delta \tilde{m}_{21}^2 \equiv \tilde{m}_2^2 - \tilde{m}_1^2 \) and \( \Delta \tilde{m}_{31}^2 \equiv \tilde{m}_3^2 - \tilde{m}_1^2 \), which depend on \( \Delta m_{21}^2, \Delta m_{31}^2, A \) and \( |V_{ei}|^2 \) (for \( i = 1, 2, 3 \)), are relevant to the practical neutrino oscillations in matter.

It is easy to prove that the sum of \( \tilde{m}_i^2 \) is related to that of \( m_i^2 \) through
\[
\sum_{i=1}^{3} \tilde{m}_i^2 = \sum_{i=1}^{3} m_i^2 + A.
(3.39)
\]

Also \( m_i^2, |V_{ei}|^2, A \) and \( \tilde{m}_i^2 \) are correlated with one another through another instructive equation:
\[
A \left( m_i^2 - m_k^2 \right) \left( \tilde{m}_j^2 - m_k^2 \right) |V_{ek}|^2 = \prod_{n=1}^{3} \left( \tilde{m}_n^2 - m_k^2 \right),
(3.40)
\]

where \( (i, j, k) \) run over \( (1, 2, 3) \) with \( i \neq j \neq k \). Assuming \( |\Delta m_{21}^2| \gg |\Delta m_{31}^2| \) and taking \( k = 3 \), one can use (3.40) to derive the approximate analytical result for the correlation between \( \tilde{m}_i^2 \) and \( m_i^2 \) obtained in Ref. [96].

The analytical relationship between the matrix elements of \( \tilde{V} \) and those of \( V \) can also be derived from (3.34), (3.35) and (3.36). After some lengthy but straightforward calculations [97], we arrive at
\[
\tilde{V}_{\alpha i} = \frac{N_i}{D_i} V_{\alpha i} + \frac{A}{D_i} V_{ei} \left[ (\tilde{m}_i^2 - m_j^2) V_{ek} V_{ak} + (\tilde{m}_i^2 - m_k^2) V_{e\alpha} V_{\alpha j} \right],
(3.41)
\]

where \( \alpha \) runs over \( (e, \mu, \tau) \) and \( (i, j, k) \) run over \( (1, 2, 3) \) with \( i \neq j \neq k \), and
\[
\begin{align*}
N_i &= (\tilde{m}_i^2 - m_j^2) (\tilde{m}_i^2 - m_k^2) - A \left[ (\tilde{m}_i^2 - m_j^2) |V_{ek}|^2 + (\tilde{m}_i^2 - m_k^2) |V_{e\alpha}|^2 \right], \\
D_i^2 &= N_i^2 + A^2 |V_{ei}|^2 \left[ (\tilde{m}_i^2 - m_j^2)^2 |V_{ek}|^2 + (\tilde{m}_i^2 - m_k^2)^2 |V_{ej}|^2 \right].
\end{align*}
(3.42)
\]

Obviously, \( A = 0 \) leads to \( \tilde{V}_{\alpha i} = V_{\alpha i} \). This exact and compact formula shows clearly how the flavor mixing matrix in vacuum is corrected by the matter effects. Instructive analytical approximations can be made for (3.41), once the spectrum of neutrino masses is experimentally known or theoretically assumed.

The results obtained above are valid for neutrinos interacting with matter. As for antineutrinos, the corresponding formulas for the effective neutrino masses and flavor
mixing matrix elements in matter can straightforwardly be obtained from (3.37)–(3.42) through the replacements $V \mapsto V^*$ and $A \mapsto -A$.

For illustration, we carry out a numerical analysis of the terrestrial matter effects on lepton flavor mixing and CP violation for a low-energy ($100 \text{ MeV} \leq E \leq 1 \text{ GeV}$) and medium-baseline ($100 \text{ km} \leq L \leq 400 \text{ km}$) neutrino oscillation experiment [93]. In view of current data, we typically take $\Delta m^2_{21} = 5 \times 10^{-5} \text{ eV}^2$, $\Delta m^2_{31} = 3 \times 10^{-3} \text{ eV}^2$, $|V_{\alpha 1}| = 0.816$ and $|V_{\alpha 2}| = 0.571$ have typically been input. In addition, the dependence of the terrestrial matter effect on the neutrino beam energy is given as $A = 2.28 \times 10^{-4} \text{ eV}^2 E/\text{[GeV]}$ [94]. With the help of (3.37) and (3.41), we are then able to compute the ratios $\Delta \tilde{m}_{1i}^2/\Delta m_{1i}^2$ (for $i = 2, 3$) and $|\tilde{\nu}_{\alpha i}|/|\nu_{\alpha i}|$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) as functions of $E$. The relevant numerical results are shown respectively in Fig. 3.1 and Fig. 3.2. We see that the earth-induced matter effect on $\Delta m^2_{21}$ is significant, but that on $\Delta m^2_{31}$ is negligibly small in the chosen range of $E$. The matter effects on $|V_{\mu 3}|$ and $|V_{\tau 3}|$ are negligible in the low-energy neutrino experiment. The smallest matrix element $|V_{e3}|$ is weakly sensitive to the matter effect. In contrast, the other six matrix elements of $V$ are significantly modified by the terrestrial matter effects.

### 3.2.2 Commutators and Sum Rules

If the charged lepton mass matrix $M_l$ and the neutrino mass matrix $M_\nu$ could simultaneously be diagonalized in a given flavor basis, there would be no lepton flavor mixing. A very instructive measure of the mismatch between the diagonalization of $M_l$ and that of $M_\nu$ in vacuum should be the commutators of lepton mass matrices [93], defined as

\[
\begin{align*}
\left[ M_l M_l^\dagger, M_\nu M_\nu^\dagger \right] &\equiv iX, \\
\left[ M_l M_l^\dagger, M_\nu^\dagger M_\nu \right] &\equiv iX', \\
\left[ M_l^\dagger M_l, M_\nu M_\nu^\dagger \right] &\equiv iY, \\
\left[ M_l^\dagger M_l, M_\nu^\dagger M_\nu \right] &\equiv iY'.
\end{align*}
\] (3.43)
Figure 3.2: Ratios $|\tilde{V}_{\alpha i}|/|V_{\alpha i}|$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) changing with the beam energy $E$ (in unit of GeV) for neutrinos ($\nu$) and antineutrinos ($\bar{\nu}$), in which $\Delta m^2_{21} = 5 \times 10^{-5}$ eV$^2$, $\Delta m^2_{31} = 3 \times 10^{-3}$ eV$^2$, $\theta_x \approx 35^\circ$, $\theta_y \approx 40^\circ$, $\theta_z \approx 5^\circ$ and $\delta \approx \pm 90^\circ$ have typically been input.

As for neutrinos propagating in matter, the corresponding commutators of lepton mass matrices are defined as

\begin{align*}
\left[ M_l M_l^\dagger, \bar{M}_\nu \bar{M}_\nu^\dagger \right] & \equiv i \tilde{X}, \\
\left[ M_l M_l^\dagger, \bar{M}_\nu^\dagger \bar{M}_\nu \right] & \equiv i \tilde{X}', \\
\left[ M_l^\dagger M_l, \bar{M}_\nu \bar{M}_\nu^\dagger \right] & \equiv i \tilde{Y}, \\
\left[ M_l^\dagger M_l, \bar{M}_\nu^\dagger \bar{M}_\nu \right] & \equiv i \tilde{Y}'.
\end{align*}  

(3.44)

It is obvious that these commutators must be traceless Hermitian matrices. Without loss of generality, one may again choose to identify the flavor eigenstates of charged leptons with their mass eigenstates. In this specific basis, where $M_l$ takes the diagonal form $D_l \equiv \text{Diag}\{m_e, m_\mu, m_\tau\}$, we obtain $X = Y$, $X' = Y'$ and $\tilde{X} = \tilde{Y}$, $\tilde{X}' = \tilde{Y}'$. 

38
Taking account of (3.1), (3.2), (3.34) and (3.36), we obtain

$$M_{\nu}M_{\nu}^\dagger = VD_{\nu}^2V^\dagger = \Omega_{\nu}$$
$$\bar{M}_{\nu}\bar{M}_{\nu}^\dagger = \bar{V}\bar{D}_{\nu}^2\bar{V}^\dagger = \bar{\Omega}_{\nu}$$

(3.45)

in the chosen flavor basis, where $D_{\nu} \equiv \text{Diag}\{m_1, m_2, m_3\}$ and $\bar{D}_{\nu} \equiv \text{Diag}\{\bar{m}_1, \bar{m}_2, \bar{m}_3\}$. In contrast, there is no direct relationship between $M_{\nu}^\dagger M_{\nu}$ (or $\bar{M}_{\nu}^\dagger \bar{M}_{\nu}$) and $\Omega_{\nu}$ (or $\bar{\Omega}_{\nu}$). Hence we are only interested in the commutators $X$ and $\bar{X}$, whose explicit expressions read as follows:

$$X = i \left( \begin{array}{ccc} 0 & \Delta_{\mu e}Z_{\mu e} & \Delta_{\tau e}Z_{\tau e} \\ \Delta_{e\mu}Z_{e\mu} & 0 & \Delta_{\tau \mu}Z_{\tau \mu} \\ \Delta_{e\tau}Z_{e\tau} & \Delta_{\mu \tau}Z_{\mu \tau} & 0 \end{array} \right),$$

$$\bar{X} = i \left( \begin{array}{ccc} 0 & \Delta_{\mu e}\bar{Z}_{\mu e} & \Delta_{\tau e}\bar{Z}_{\tau e} \\ \Delta_{e\mu}\bar{Z}_{e\mu} & 0 & \Delta_{\tau \mu}\bar{Z}_{\tau \mu} \\ \Delta_{e\tau}\bar{Z}_{e\tau} & \Delta_{\mu \tau}\bar{Z}_{\mu \tau} & 0 \end{array} \right),$$

(3.46)

where $\Delta_{\alpha\beta} \equiv m_\alpha^2 - m_\beta^2$ for $\alpha \neq \beta$ running over $(e, \mu, \tau)$, and

$$Z_{\alpha\beta} \equiv \sum_{i=1}^{3} \left( m_i^2 V_{\alpha i} V_{\beta i}^* \right),$$

$$\bar{Z}_{\alpha\beta} \equiv \sum_{i=1}^{3} \left( \bar{m}_i^2 \bar{V}_{\alpha i} \bar{V}_{\beta i}^* \right).$$

(3.47)

One can see that $\Delta_{\beta\alpha} = -\Delta_{\alpha\beta}$, $Z_{\beta\alpha} = Z_{\alpha\beta}^*$ and $\bar{Z}_{\beta\alpha} = \bar{Z}_{\alpha\beta}^*$ hold. To find out how $\bar{Z}_{\alpha\beta}$ is related to $Z_{\alpha\beta}$, we need to establish the relation between $X$ and $\bar{X}$. With the help of (3.35), it is easy to prove that $X$ and $\bar{X}$ are identical to each other:

$$\bar{X} = i \left[ \bar{\Omega}_{\nu} , D_{\nu}^2 \right] = i \left[ \Omega_{\nu} , D_{\nu}^2 \right] = X.$$

(3.48)

This interesting result, whose validity is independent of the specific flavor basis chosen above, implies that the commutator of lepton mass matrices defined in vacuum is invariant under matter effects. As a straightforward consequence of $\bar{X} = X$, we arrive at $\bar{Z}_{\alpha\beta} = Z_{\alpha\beta}$ from (3.46). Then a set of concise sum rules of neutrino masses emerge [98]:

$$\sum_{i=1}^{3} \left( \bar{m}_i^2 \bar{V}_{\alpha i} \bar{V}_{\beta i}^* \right) = \sum_{i=1}^{3} \left( m_i^2 V_{\alpha i} V_{\beta i}^* \right).$$

(3.49)

Although we have derived these sum rules in the three-neutrino mixing scheme, they may simply be generalized to hold for an arbitrary number of neutrino families. In addition, we emphasize that the useful results obtained above are independent of the profile of matter density (i.e., they are valid everywhere in the sun or in the earth).

It should be noted that $Z_{\alpha\beta}$ and $\bar{Z}_{\alpha\beta}$ are sensitive to a redefinition of the phases of charged lepton fields. The simplest rephasing-invariant equality is $|\bar{Z}_{\alpha\beta}| = |Z_{\alpha\beta}|$, where $(\alpha, \beta)$ may run over $(e, \mu)$, $(\mu, \tau)$ or $(\tau, e)$. Another useful rephasing-invariant relationship is

$$Z_{\epsilon\mu}Z_{\mu\tau}Z_{\tau e} = \bar{Z}_{\epsilon\mu}\bar{Z}_{\mu\tau}\bar{Z}_{\tau e}.$$
As one can see in section 4.1, the imaginary parts of $Z_{e\mu}Z_{\mu\tau}Z_{\tau\mu}$ and $\bar{Z}_{e\mu}\bar{Z}_{\mu\tau}\bar{Z}_{\tau\mu}$ are related respectively to leptonic CP violation in vacuum and that in matter.

It should also be noted that three additional sum rules can similarly be derived from (3.35). We find that $\tilde{\Omega}^\beta_\nu = \Omega^{\alpha \beta}_\nu$ holds, if and only if $(\alpha, \beta) \neq (e, e)$. Thus the following equalities can be obtained:

\[
\begin{align*}
\tilde{\Omega}^\alpha_\nu \tilde{\Omega}^\tau_\nu - \tilde{\Omega}^\mu_\nu \tilde{\Omega}^\tau_\nu &= \Omega^\alpha_\nu \Omega^\tau_\nu - \Omega^\mu_\nu \Omega^\tau_\nu, \\
\tilde{\Omega}^\mu_\nu \tilde{\Omega}^\tau_\nu - \tilde{\Omega}^\mu_\nu \tilde{\Omega}^\tau_\nu &= \Omega^\mu_\nu \Omega^\tau_\nu - \Omega^\mu_\nu \Omega^\tau_\nu, \\
\tilde{\Omega}^\mu \tilde{\Omega}^\tau - \tilde{\Omega}^\mu \tilde{\Omega}^\tau &= \Omega^\mu \Omega^\tau - \Omega^\mu (\Omega^\tau + A).
\end{align*}
\] (3.51)

By use of (3.51) and the unitarity of $V$, one may arrive at

\[
\begin{align*}
\sum_{i \neq j \neq k} \left( m^2_i m^2_j \bar{V}_{ek} V^*_{\mu k} \right) &= \sum_{i \neq j \neq k} \left( m^2_i m^2_j \bar{V}_{ek} V^*_{\mu k} \right), \\
\sum_{i \neq j \neq k} \left( m^2_i m^2_j \bar{V}_{ek} V^*_{\tau k} \right) &= \sum_{i \neq j \neq k} \left( m^2_i m^2_j \bar{V}_{ek} V^*_{\tau k} \right), \\
\sum_{i \neq j \neq k} \left( m^2_i m^2_j \bar{V}_{\mu k} V^*_{\tau k} \right) &= \sum_{i \neq j \neq k} \left[ (m^2_i m^2_j - A_m^2) V_{\mu k} V^*_{\tau k} \right].
\end{align*}
\] (3.52)

These relations, similar to those obtained previously in Ref. [99], imply that $\bar{V}_{\alpha i} V^*_{\beta k}$ (for $\alpha \neq \beta$) can be expressed in terms of $V_{\alpha i} V^*_{\beta k}$, $A m^2_{3j}$ and $A$ [100]. In particular, we are able to get the analytical formula for the sides of three leptonic unitarity triangles in matter (see section 4.2 for the detail).

Let us remark that the results obtained above are only valid for neutrinos propagating in vacuum and in matter. As for antineutrinos, the corresponding formulas can straightforwardly be written out from (3.46)–(3.52) through the replacements $V \rightarrow V^*$ and $A \rightarrow -A$.

### 3.2.3 Effective Mixing Parameters

Although the present non-accelerator neutrino experiments have yielded some impressive constraints on three lepton flavor mixing angles ($\theta_x$, $\theta_y$, and $\theta_z$), a precise determination of them and a measurement of the Dirac-type CP-violating phase $\delta$ have to rely on a new generation of accelerator experiments with very long baselines, including the possible neutrino factories. In such long- or medium-baseline neutrino experiments the terrestrial matter effects, which may deform the oscillating behaviors of neutrinos in vacuum and even fake the genuine CP-violating signals, must be taken into account. It is therefore desirable to figure out how the standard parametrization in (3.23) will be modified by the matter effects, in particular in the case of a constant earth density profile.

The analytical relationship between the elements of the effective flavor mixing matrix $\bar{V}$ in matter and those of the fundamental flavor mixing matrix $V$ in vacuum has been given in (3.41). Adopting the parametrization of $V$ in (3.23), we explicitly obtain

\[
\bar{V}_{\alpha i} = \frac{N_i}{D_i} V_{\alpha i} + \frac{A}{D_i} \sum_k (T_{ak} P_{ki}),
\] (3.53)
where the expressions of $N_i$ and $D_i$ have been given in (3.42), $A$ denotes the matter parameter, $P \equiv \text{Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$ is the diagonal matrix of Majorana phases, and the matter-associated matrix elements $T_{ai}$ read as follows $^{12}$:

$$
T_{e1} = +c_x c_z \left[ (\tilde{m}_1^2 - m_1^2) s_z^2 + (\tilde{m}_1^2 - m_2^2) c_x^2 s_z^2 \right],
$$

$$
T_{e2} = +s_x c_z \left[ (\tilde{m}_2^2 - m_1^2) s_z^2 + (\tilde{m}_1^2 - m_2^2) c_x^2 s_z^2 \right],
$$

$$
T_{e3} = +s_x^2 c_z^2 \left[ (\tilde{m}_3^2 - m_1^2) s_z^2 + (\tilde{m}_1^2 - m_2^2) c_x^2 s_z^2 \right],
$$

$$
T_{\mu 1} = +c_x c_z \left[ (\tilde{m}_1^2 - m_2^2) s_y s_z - (\tilde{m}_1^2 - m_3^2) \left( s_y^2 s_z - s_x c_x c_y e^{-i\delta} \right) \right],
$$

$$
T_{\mu 2} = +s_x c_z \left[ (\tilde{m}_2^2 - m_1^2) s_y s_z - (\tilde{m}_2^2 - m_3^2) \left( c_x^2 s_y s_z + s_x c_x c_y e^{-i\delta} \right) \right],
$$

$$
T_{\mu 3} = -s_x c_z \left[ (\tilde{m}_3^2 - m_1^2) s_y s_z + (\tilde{m}_3^2 - m_2^2) c_x^2 s_y s_z - \Delta m_{21}^2 s_x c_x c_y e^{-i\delta} \right],
$$

$$
T_{\tau 1} = +c_x c_z \left[ (\tilde{m}_1^2 - m_2^2) c_y s_z - (\tilde{m}_1^2 - m_3^2) \left( s_x c_x s_z + s_x c_x c_y e^{-i\delta} \right) \right],
$$

$$
T_{\tau 2} = +s_x c_z \left[ (\tilde{m}_2^2 - m_1^2) c_y s_z - (\tilde{m}_2^2 - m_3^2) \left( c_x^2 c_y s_z - s_x c_x c_y e^{-i\delta} \right) \right],
$$

$$
T_{\tau 3} = -s_x c_z \left[ (\tilde{m}_3^2 - m_1^2) s_x^2 c_y s_z + (\tilde{m}_3^2 - m_2^2) c_x^2 c_y s_z + \Delta m_{21}^2 s_x c_x s_y e^{-i\delta} \right].
$$

Clearly $\tilde{V}_{ai} = V_{ai}$ holds in the case of $A = 0$.

The results given in (3.53) and (3.54) indicate that the diagonal Majorana phase matrix $P$ on the right-hand side of $V$ is not affected by the matter effect. As a natural consequence, the effective flavor mixing matrix $\tilde{V}$ in matter can be parametrized in the same form as $V$:

$$
\tilde{V} = \begin{pmatrix}
\bar{c}_x \bar{s}_x & \bar{s}_x \bar{c}_x & \bar{s}_z \\
-\bar{c}_x \bar{s}_y \bar{s}_z - \bar{s}_x \bar{c}_y e^{-i\delta} & -\bar{s}_x \bar{s}_y \bar{s}_z + \bar{c}_x \bar{c}_y e^{-i\delta} & \bar{s}_z \\
-\bar{c}_x \bar{c}_y \bar{s}_z + \bar{s}_x \bar{s}_y e^{-i\delta} & -\bar{s}_x \bar{c}_y \bar{s}_z - \bar{c}_x \bar{s}_y e^{-i\delta} & \bar{c}_z
\end{pmatrix}
\begin{pmatrix}
e^{i\rho} & 0 & 0 \\
0 & e^{i\sigma} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_x & c_z & s_z \\
\bar{c}_x \bar{c}_z & \bar{s}_x \bar{c}_z & \bar{c}_z \\
\end{pmatrix}
$$

(3.55)

with $\bar{s}_x \equiv \sin \bar{\theta}_x$ and $\bar{c}_x \equiv \cos \bar{\theta}_x$, and so on. It should be noted that the matrix elements $\tilde{V}_{\mu 3}$ and $\tilde{V}_{\tau 3}$ in (3.53) are complex and dependent on the Dirac-type phase $\delta$, as one can see from (3.54). Hence a proper redefinition of the phases for muon and tau fields is needed, in order to make $\tilde{V}_{\mu 3}$ and $\tilde{V}_{\tau 3}$ real in the course from (3.53) to (3.55). The instructive relations between the effective mixing angles in matter ($\bar{\theta}_x, \bar{\theta}_y, \bar{\theta}_z$) and the fundamental mixing angles in vacuum ($\theta_x, \theta_y, \theta_z$) are found to be

$$
\frac{\tan \bar{\theta}_x}{\tan \theta_x} = \frac{N_2 + A \left[ (\tilde{m}_2^2 - m_1^2) s_z^2 + (\tilde{m}_1^2 - m_2^2) c_x^2 s_z^2 \right]}{N_1 + A \left[ (\tilde{m}_1^2 - m_1^2) s_z^2 + (\tilde{m}_2^2 - m_3^2) c_x^2 s_z^2 \right]} D_1 \frac{D_1}{D_2},
$$

$$
\frac{\tan \bar{\theta}_y}{\tan \theta_y} = 1 + \frac{A \Delta m_{21}^2 s_x c_x s_z \cos \delta / (s_y c_y)}{N_3 - A \left[ (\tilde{m}_3^2 - m_1^2) s_z^2 + (\tilde{m}_1^2 - m_2^2) c_x^2 s_z^2 \right] s_z^2},
$$

$$
\frac{\sin \bar{\theta}_z}{\sin \theta_z} = \frac{N_3}{D_3} + \frac{A}{D_3} \left[ (\tilde{m}_3^2 - m_1^2) s_z^2 + (\tilde{m}_1^2 - m_2^2) c_x^2 s_z^2 \right] c_z^2.
$$

(3.56)

12Note that there is a typing error in Eq. (18) of Ref. [101], associated with the sign of $\Delta m_{21}^2 \tilde{s}_1 \tilde{c}_1 \tilde{s}_2 e^{-i\tilde{\delta}}$ in the expression of $T_{\tau 3}$.

13Here we have only presented the next-to-leading order expression for $\tan \bar{\theta}_y / \tan \theta_y$, because the exact result is rather complicated and less instructive. The former works to a high degree of accuracy, and it is identical to the exact result provided that $\theta_y = 45^\circ$ and $\delta = \pm 90^\circ$ hold (i.e., one obtains $\bar{\theta}_y = \theta_y = 45^\circ$ in this special but interesting case).
Figure 3.3: Ratios $\tilde{\theta}_x/\theta_x$, $\tilde{\theta}_y/\theta_y$, $\tilde{\theta}_z/\theta_z$, and $\tilde{\delta}/\delta$ changing with the matter parameter $A$ for neutrinos ($\nu$) and antineutrinos ($\bar{\nu}$), in which $\Delta m_{21}^2 = 5 \times 10^{-5} \text{ eV}^2$, $\Delta m_{31}^2 = 3 \times 10^{-3} \text{ eV}^2$, $\theta_x \approx 35^\circ$, $\theta_y \approx 40^\circ$, $\theta_z \approx 5^\circ$, and $\delta \approx \pm 90^\circ$ have typically been input.

Once the relation between $\tilde{\theta}_y$ and $\theta_y$ is established, one can obtain the relation between the effective CP-violating phase $\tilde{\delta}$ in matter and the genuine CP-violating phase $\delta$ in vacuum by use of the Toshev identity \[102\],

$$\frac{\sin \tilde{\delta}}{\sin \delta} = \frac{\sin 2\theta_y}{\sin 2\tilde{\theta}_y} = \frac{s_y}{s_{\tilde{\theta}_y}} \cdot \frac{c_{\tilde{\theta}_y}}{c_y} .$$ \hspace{1cm} (3.57)

The detailed proof of this elegant identity has been done in Ref. \[103\]. We see that $\tilde{\delta} \approx \delta$ holds as a consequence of $\tilde{\theta}_y \approx \theta_y$ in the leading-order approximation. Hence both $\theta_y$ and $\delta$ are insensitive to the matter effects. Note that the results obtained above are only valid for neutrinos propagating in matter. As for antineutrinos, the corresponding expressions can straightforwardly be obtained from (3.53)–(3.57) through the replacements $\delta \rightarrow -\delta$ and $A \rightarrow -A$. Such formulas should be very useful for the purpose of recasting the fundamental parameters of lepton flavor mixing from the matter-corrected ones \[101\], which can be extracted from a variety of long- and medium-baseline neutrino oscillation experiments in the near future.

To illustrate the dependence of $\tilde{\theta}_x$, $\tilde{\theta}_y$, $\tilde{\theta}_z$, and $\tilde{\delta}$ on the matter effects, we typically take $\Delta m_{21}^2 \approx \Delta m_{\text{sun}}^2 \approx 5 \times 10^{-5} \text{ eV}^2$ and $\Delta m_{31}^2 \approx \Delta m_{\text{atm}}^2 \approx 3 \times 10^{-3} \text{ eV}^2$. In addition,
we adopt \( \theta_x \approx 35^\circ, \theta_y \approx 40^\circ, \theta_z \approx 5^\circ, \) and \( \delta \approx \pm 90^\circ. \) Such a choice for the values of the input parameters is indeed favored by current experimental data on solar and atmospheric neutrino oscillations. With the help of (3.56) and (3.57) as well as the relevant formulas of \( \bar{m}_i^2, N_i \) and \( D_i \) given in section 3.2.1, we are then able to calculate the ratios \( \theta_x/\theta_x, \theta_y/\theta_y, \theta_z/\theta_z, \) and \( \delta/\delta \) as functions of the matter parameter \( A \) in the typical region \( 10^{-7} \text{eV}^2 \leq A \leq 10^{-2} \text{eV}^2, \) which corresponds to the neutrino beam energy \( E \) in the region \( 5 \text{MeV} \leq E \leq 50 \text{GeV} \) for \( N_e \approx 1.5 \text{g/cm}^3 \) [104]. The numerical results are shown in Fig. 3.3. We observe that the matter effects on \( \theta_y \) and \( \delta \) are negligibly small for \( A \leq 5 \times 10^{-2} \text{eV}^2, \) just as indicated by our analytical results. In contrast, \( \theta_z \) is significantly modified when \( A > 10^{-3} \text{eV}^2; \) and \( \theta_x \) is sensitive to the matter effect even for quite small values of \( A. \)

3.3 Comparison with Quark Flavor Mixing

We have seen that the \( 3 \times 3 \) lepton flavor mixing matrix \( V \) is of a bi-large mixing pattern, and most of its matrix elements do not have a strong hierarchy in magnitude. In contrast, the \( 3 \times 3 \) quark flavor mixing matrix \( V_{\text{CKM}} [10] \) is of a tri-small mixing pattern and has a strongly hierarchical structure. Hence \( V_{\text{CKM}} \) can be expanded in powers of a small parameter \( \lambda \approx 0.22, \) the so-called Wolfenstein parameter [105]:

\[
V_{\text{CKM}} \approx \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A\lambda^3 (\rho - i\eta) \\
-\lambda & 1 - \frac{1}{2} \lambda^2 & A\lambda^2 \\
A\lambda^3 (1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix}, \tag{3.58}
\]

where the terms of or below \( \mathcal{O}(\lambda^4) \) have been neglected [106]. Current experimental data yield \( A \approx 0.83, \rho \approx 0.17 \) and \( \eta \approx 0.36 \) [6]. The spirit of lepton-quark similarity motivates us to consider whether the lepton flavor mixing matrix can also be expanded in powers of a Wolfenstein-like parameter, whose magnitude may be somehow larger than \( \lambda. \) We find that it is indeed possible to expand \( V \) in powers of \( \Lambda \equiv |V_{\mu3}| \sim 1/\sqrt{2} \) [91], in view of current neutrino oscillation data.

The parameter \( \Lambda \) measures the strength of flavor mixing in atmospheric neutrino oscillations. The magnitude of flavor mixing in solar neutrino oscillations is then characterized by \( AA^2, \) where \( A \) is a positive coefficient of \( \mathcal{O}(1). \) Because \( |V_{e3}| < 0.16 \) is required by the CHOOZ data [7], we may take \( |V_{e3}| \sim \mathcal{O}(\Lambda^8) \sim 0.06 \) as a typical possibility for \( \Lambda \sim 1/\sqrt{2}. \) Smaller values of \( |V_{e3}| \) are certainly allowed. In some interesting models of lepton flavor mixing [12], \( |V_{e3}| \sim \sqrt{m_e/m_\mu} \sim 0.07 \) is naturally predicted. Hence \( \mathcal{O}(\Lambda^8) \) could be the plausible order of \( |V_{e3}|. \) Let us fix the matrix elements \( V_{e2}, V_{e3} \) and \( V_{\mu3} \) by use of four independent parameters: \( V_{\mu3} = \Lambda, V_{e2} = AA^2 \) and \( V_{e3} = B\Lambda^8 e^{-i\delta}, \) where the positive coefficient \( B \) is of \( \mathcal{O}(1) \) or smaller and \( \delta \) denotes the Dirac phase of leptonic CP violation. Then one may make use of the unitarity of \( V \) to work out the exact analytical expressions for the other six matrix elements. We find that it is more instructive to approximate \( V \) as

\[
V = \begin{pmatrix}
\sqrt{1 - A^2 \Lambda^4} & A\Lambda^2 & BA^8 e^{-i\delta} \\
-A\Lambda^2 \sqrt{1 - \Lambda^4} & \sqrt{(1 - \Lambda^4) (1 - A^2 \Lambda^4)} & \Lambda \\
A\Lambda^3 \left[A - B\Lambda^5 \sqrt{(1 - \Lambda^2) (1 - A^2 \Lambda^4)} e^{i\delta}\right] & -A\Lambda \sqrt{1 - A^2 \Lambda^4} & \sqrt{1 - \Lambda^4}
\end{pmatrix}. \tag{3.59}
\]
In this approximation, the unitary normalization relations of $V$ keep valid to $\mathcal{O}(\Lambda^2) \sim 2\%$. Therefore (3.59) is sufficiently accurate to describe lepton flavor mixing, not only in solar and atmospheric neutrino oscillations, but also in some of the proposed long-baseline neutrino oscillation experiments. As the unitary orthogonality relations of $V$ are valid to $\mathcal{O}(\Lambda^6) \sim 6\%$, the leptonic unitarity triangles and CP violation can also be described by (3.59) to an acceptable degree of accuracy. The off-diagonal asymmetries of the lepton flavor mixing matrix $V$ [107] read as

$$\mathcal{A}_L \equiv \begin{vmatrix} |V_{e2}|^2 - |V_{\mu 1}|^2 & |V_{\mu 3}|^2 - |V_{\tau 2}|^2 & |V_{\tau 4}|^2 - |V_{e3}|^2 \\ V_{\mu 1} & V_{\mu 3} & V_{\tau 4} \\ V_{\mu 3} & V_{\tau 2} & V_{e3} \end{vmatrix}^2 = A^2 \Lambda^6,$$

$$\mathcal{A}_R \equiv \begin{vmatrix} |V_{e2}|^2 - |V_{\mu 3}|^2 & |V_{\mu 1}|^2 - |V_{\tau 2}|^2 & |V_{\tau 3}|^2 - |V_{e1}|^2 \\ V_{\mu 3} & V_{\tau 2} & V_{e1} \\ V_{\mu 1} & V_{\tau 3} & V_{e2} \end{vmatrix}^2 = A^2 \left( A^2 \Lambda^2 - 1 \right).$$

We see that $\mathcal{A}_L > 0$ holds definitely. In comparison, the sign of $\mathcal{A}_R$ cannot be fixed from the present experimental data. It is actually possible to obtain $\mathcal{A}_R = 0$, when $A^2 \Lambda^2 = 1$ is satisfied. In this interesting case, the lepton flavor mixing matrix $V$ is exactly symmetric about its $V_{e3}$-$V_{\mu 2}$-$V_{\tau 1}$ axis [107]. Note that the CKM matrix of quark flavor mixing is approximately symmetric about its $V_{ud}$-$V_{ce}$-$V_{tb}$ axis [108]. This is another difference between the flavor mixing matrices of leptons and quarks.

Note that the Majorana CP-violating phases have been omitted in (3.59). To incorporate $V$ with two Majorana phases $\rho$ and $\sigma$, we simply multiply $V$ on its right-hand side with a pure phase matrix; i.e., $V \rightarrow VQ$ with $Q = \text{Diag}\{e^{i\rho}, e^{i\sigma}, e^{i\theta}\}$. The chosen phase convention of $Q$ is to make the Dirac CP-violating phase $\delta$ not to manifest itself in the effective mass term of the neutrinoless double beta decay:

$$\langle m \rangle_{ee} = |m_1 \left( 1 - A^2 \Lambda^4 \right) e^{2i\rho} + m_2 A^2 \Lambda^4 e^{2i\sigma} + m_3 B^2 \Lambda^6|,$$

where $m_i$ (for $i = 1, 2, 3$) are physical neutrino masses. This result can somehow get simplified, if a specific pattern of the neutrino mass spectrum is assumed. The present experimental upper bound is $\langle m \rangle_{ee} < 0.35$ eV (at the 90% confidence level [52]), from which no constraint on $\rho$ and $\sigma$ can be got.

Now let us establish the direct relations between $(\Lambda, A, B)$ and $(\theta_{\text{atm}}, \theta_{\text{sun}}, \theta_{\text{chz}})$. With the help of (3.14) and (3.59), we find

$$\Lambda = \sin \theta_{\text{atm}},$$

$$A = \frac{\sqrt{\cos^2 \theta_{\text{chz}} - \cos^4 \theta_{\text{chz}} - \sin^2 2\theta_{\text{sun}}}}{\sqrt{2} \sin^2 \theta_{\text{atm}}},$$

$$B = \frac{\sin \theta_{\text{chz}}}{\sin^3 \theta_{\text{atm}}}.\quad (3.62)$$

Once the mixing angles $\theta_{\text{atm}}, \theta_{\text{sun}}$ and $\theta_{\text{chz}}$ are precisely measured, we may use (3.62) to determine the magnitudes of $\Lambda, A$ and $B$. For the purpose of illustration, we typically take $0.25 \leq \sin^2 \theta_{\text{sun}} \leq 0.40$ [35], $\sin^2 2\theta_{\text{atm}} > 0.92$ [2] and $\sin^2 2\theta_{\text{chz}} < 0.1$ [7] to calculate the allowed regions of $\Lambda, A$ and $B$. Then we obtain $0.6 \leq \Lambda \leq 0.8, 0.8 \leq A \leq 1.94$ and $0 \leq B \leq 9.5$. Although the allowed region of $B$ is rather large, we expect that
$B \sim \mathcal{O}(1)$ or a bit smaller is most likely and most suitable for our parametrization. It is also worthwhile to connect ($\Lambda$, $\bar{A}$, $\bar{B}$) to ($\theta_x$, $\theta_y$, $\theta_z$), which are three mixing angles in the standard parametrization of $V$ in (3.23). We find $\sin \theta_x \approx \Lambda^2$, $\sin \theta_y \approx \Lambda$ and $\sin \theta_z = \Lambda^8$. In addition, the Dirac phase of CP violation in the standard parametrization is exactly equal to $\delta$ defined in the present Wolfenstein-like parametrization.

An interesting point is that the effective lepton flavor mixing matrix in matter can similarly be parametrized in terms of four matter-corrected parameters $\tilde{\Lambda}$, $\tilde{\bar{A}}$, $\tilde{\bar{B}}$ and $\tilde{\delta}$:

$$\tilde{V} = \begin{pmatrix} \sqrt{1 - \tilde{A}^2 \tilde{\Lambda}^4} & \tilde{\Lambda} \tilde{\Lambda}^2 & \tilde{B} \Lambda^8 e^{-i \tilde{\delta}} \\ -\tilde{A} \tilde{\Lambda} \sqrt{1 - \tilde{\Lambda}^2} & \sqrt{(1 - \tilde{\Lambda}^2)(1 - \tilde{A}^2 \Lambda^4)} & \tilde{\Lambda} \\ \tilde{\Lambda}^3 \left[ \tilde{\Lambda} - \tilde{B} \Lambda^8 \sqrt{(1 - \tilde{\Lambda}^2)(1 - \tilde{A}^2 \Lambda^4)} e^{i \tilde{\delta}} \right] & -\tilde{\Lambda} \sqrt{1 - \tilde{A}^2 \Lambda^4} & \sqrt{1 - \tilde{\Lambda}^2} \end{pmatrix}, \quad (3.63)$$

where the Majorana phase matrix $Q$ has been omitted for simplicity. There also exist simple relations between the effective mixing angles of $\tilde{V}$ (i.e., $\tilde{\theta}_x$, $\tilde{\theta}_y$ and $\tilde{\theta}_z$) and the corresponding new parameters ($\tilde{\Lambda}$, $\tilde{\bar{A}}$ and $\tilde{\bar{B}}$). It has been shown in section 3.2.3 that $\sin \tilde{\theta}_y \approx \sin \theta_y$ and $\sin \tilde{\delta} \approx \sin \delta$ hold to leading order for a variety of terrestrial long-baseline neutrino oscillation experiments. Therefore, we have $\tilde{\Lambda} \approx \Lambda$ and $\tilde{\delta} \approx \delta$. This result implies that $\Lambda$ and $\delta$ are essentially stable against terrestrial matter effects. Hence the expansion of $\tilde{V}$ in powers of $\tilde{\Lambda} \approx \Lambda$ makes sense. Only $\tilde{\bar{A}}$ and $\tilde{\bar{B}}$ in $V$ are sensitive to the matter-induced corrections. Because of $\tilde{\bar{A}} \propto \sin \tilde{\theta}_x$ and $\tilde{\bar{B}} \propto \sin \tilde{\theta}_z$, two remarkable conclusions can be drawn from Ref. [103] for our new parameters: (a) $\tilde{\bar{A}}/\Lambda$ is suppressed up to the order $\Delta m_{\text{sun}}^2/\Delta m_{\text{atm}}^2$; and (b) $\tilde{\bar{B}}/\bar{B}$ may have the resonant behavior similar to the two-neutrino MSW resonance [33].

Finally we give some speculation on the physical meaning of $\Lambda$. It is well known that the Wolfenstein parameter $\lambda \approx 0.22$ can be related to the ratios of quark masses in the Fritzsch ansatz of quark mass matrices [65] or its modified versions [68]:

$$\lambda \approx \left| \sqrt{\frac{m_u}{m_c}} - e^{i \phi_u} \sqrt{\frac{m_d}{m_s}} \right|, \quad (3.64)$$

where $\phi_u$ denotes the phase difference between the $(1,2)$ elements of up- and down-type quark mass matrices. This relation indicates that the smallness of $\lambda$ is a natural consequence of the strong quark mass hierarchy. Could the largeness of $\Lambda$ be attributed to a relatively weak hierarchy of three neutrino masses? The answer is indeed affirmative for the Fritzsch texture of lepton mass matrices in (2.39), which is compatible with current experimental data on neutrino oscillations if the masses of three neutrinos have a normal but weak hierarchy (typically, $m_1 : m_2 : m_3 \approx 1 : 3 : 10$) [66]. In this phenomenological model, we approximately obtain

$$\Lambda \approx \left| \sqrt{\frac{m_2}{m_3}} - e^{i \phi_\Lambda} \sqrt{\frac{m_\mu}{m_\tau}} \right|, \quad (3.65)$$

where $\phi_\Lambda$ denotes the phase difference between the $(2,3)$ elements of charged lepton and neutrino mass matrices. We find that $\phi_\Lambda \sim \pm 180^\circ$ is practically favored [66], in order to obtain a sufficiently large $\Lambda$. To illustrate, we typically take $m_2/m_3 \sim 0.3$ as well as
\( m_\mu/m_\tau \approx 0.06 \) [6]. Then we arrive at \( \Lambda \sim 0.8 \), a result consistent with our empirical expectation for the order of \( \Lambda \) in (3.59) 14.

4 CP and T Violation

4.1 Rephasing Invariants in Matter

No matter whether massive neutrinos are Dirac or Majorana particles, the effects of CP or T violation in normal neutrino-neutrino and antineutrino-antineutrino oscillations are measured by a universal parameter \( J \) in vacuum or \( \bar{J} \) in matter [80], defined through

\[
\text{Im}(V_{\alpha i}V_{\beta j}^*V_{\alpha j}^*V_{\beta i}^*) = J \sum_\gamma \epsilon_{\alpha \beta \gamma} \sum_k \epsilon_{ijk},
\]

\[
\text{Im}(\bar{V}_{\alpha i}\bar{V}_{\beta j}\bar{V}_{\alpha j}\bar{V}_{\beta i}) = \bar{J} \sum_\gamma \epsilon_{\alpha \beta \gamma} \sum_k \epsilon_{ijk},
\]

(4.1)

where the subscripts \((\alpha, \beta, \gamma)\) and \((i, j, k)\) run respectively over \((e, \mu, \tau)\) and \((1, 2, 3)\). Clearly \( J \) and \( \bar{J} \) are rephasing-invariant; i.e., they are independent of any redefinition of the phases for charged lepton and neutrino fields. Note that \( J \) and \( \bar{J} \) can be expressed in terms of the moduli of four independent matrix elements of \( V \) and \( \bar{V} \), respectively, as follows [12] 15:

\[
J^2 = |V_{\alpha i}|^2|V_{\beta j}|^2|V_{\alpha j}|^2|V_{\beta i}|^2 - \frac{1}{4} \left( 1 + |V_{\alpha i}|^2|V_{\beta j}|^2 + |V_{\alpha j}|^2|V_{\beta i}|^2 \right)^2,
\]

\[
\bar{J}^2 = |\bar{V}_{\alpha i}|^2|\bar{V}_{\beta j}|^2|\bar{V}_{\alpha j}|^2|\bar{V}_{\beta i}|^2 - \frac{1}{4} \left( 1 + |\bar{V}_{\alpha i}|^2|\bar{V}_{\beta j}|^2 + |\bar{V}_{\alpha j}|^2|\bar{V}_{\beta i}|^2 \right)^2,
\]

(4.2)

in which \( \alpha \neq \beta \) running over \((e, \mu, \tau)\) and \( i \neq j \) running over \((1, 2, 3)\). The implication of this result is obvious: the information about leptonic CP violation can in principle be extracted from the measured moduli of the flavor mixing matrix elements.

Now let us establish the relationship between \( \bar{J} \) and \( J \). Note that the imaginary parts of the rephasing-invariant quantities \( Z_{e\mu}\bar{Z}_{e\mu}Z_{\tau e} \) and \( \bar{Z}_{e\mu}\bar{Z}_{e\mu}\bar{Z}_{\tau e} \),

\[
\text{Im}(Z_{e\mu}\bar{Z}_{e\mu}Z_{\tau e}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \left[ m_i^2 m_j^2 m_k^2 \text{Im} \left( V_{ei}V_{\mu j}V_{\tau k}V_{\nu e}V_{\nu \mu}V_{\nu \tau} \right) \right],
\]

\[
\text{Im}(\bar{Z}_{e\mu}\bar{Z}_{e\mu}\bar{Z}_{\tau e}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \left[ \bar{m}_i^2 \bar{m}_j^2 \bar{m}_k^2 \text{Im} \left( \bar{V}_{ei}\bar{V}_{\mu j}\bar{V}_{\tau k}\bar{V}_{\nu e}\bar{V}_{\nu \mu}\bar{V}_{\nu \tau} \right) \right],
\]

(4.3)

14Assuming a somewhat stronger mass hierarchy for three neutrinos, Kaus and Meshkov [109] have proposed a different expansion of the neutrino mixing matrix in terms of \( \Lambda = \sqrt{m_2/m_3} = (\Delta m_{32}^2/\Delta m_{21}^2)^{1/4} \sim 0.37 \). This parameter is associated with \( V_{e2} \) instead of \( V_{e3} \), therefore it is sensitive to the matter effect. In contrast, our parametrization does not rely on the assumption of neutrino mass hierarchy, and its expansion parameter is insensitive to the matter-induced corrections.

15Note that there was a typing error in the original formula of \( J^2 \) given in Ref. [110].
which do not vanish unless leptonic CP and T are good symmetries, amount to each other as a trivial consequence of (3.50). The right-hand side of (4.3) can be expanded in terms of $J$ and $\tilde{J}$. In doing so, one needs to use (4.1) as well as the unitarity conditions of $V$ and $\tilde{V}$ frequently. After some lengthy but straightforward algebraic calculations, we arrive at an elegant relation between the universal CP-violating parameters $J$ in vacuum and $\tilde{J}$ in matter [98]:

$$\tilde{J} \prod_{i<j} (\bar{m}_i^2 - \bar{m}_j^2) = J \prod_{i<j} (m_i^2 - m_j^2). \quad (4.4)$$

This interesting result was first obtained by Naumov [111]. Of course $\tilde{J} = J$ holds if $A = 0$, and $\tilde{J} = 0$ holds if $J = 0$. It is worth mentioning that (4.4) can also be derived from the equality between $\text{Det}(X)$ and $\text{Det}(\tilde{X})$ given in (3.46) [112]. Indeed it is easy to show that

$$\text{Det}(X) = 2J \prod_{\alpha<\beta} (m_\alpha^2 - m_\beta^2) \prod_{i<j} (m_i^2 - m_j^2),$$

$$\text{Det}(\tilde{X}) = 2\tilde{J} \prod_{\alpha<\beta} (m_\alpha^2 - m_\beta^2) \prod_{i<j} (\bar{m}_i^2 - \bar{m}_j^2), \quad (4.5)$$

where the Greek indices run over $(e, \mu, \tau)$; and the Latin indices run over $(1, 2, 3)$. It should be noted that the determinant $\text{Det}(X)$ or $\text{Det}(\tilde{X})$ contains the same information about leptonic CP violation as the universal parameter $J$ or $\tilde{J}$, although their expressions are somehow different [113]. The latter is apparently simpler and more instructive for the description of leptonic CP violation in neutrino oscillations. Taking the standard parametrization of $V$ in (3.23) or $\tilde{V}$ in (3.55), we have

$$J = \sin \theta_x \cos \theta_x \sin \theta_y \cos \theta_y \sin \theta_z \cos^2 \theta_z \sin \delta,$$

$$\tilde{J} = \sin \theta_x \cos \theta_x \sin \tilde{\theta}_y \cos \tilde{\theta}_y \sin \tilde{\theta}_z \cos^2 \tilde{\theta}_z \sin \tilde{\delta}. \quad (4.6)$$

Thus the CP-violating phases $\delta$ and $\tilde{\delta}$ can be related to each other via (4.4) and (4.6).

Once again, the formulas obtained above are valid only for neutrinos propagating in vacuum and in matter. They will become valid for antineutrinos, if the straightforward replacements $V \implies V^*$ and $A \implies -A$ are made.

We remark that the effective CP-violating parameter $\tilde{J}$ depends not only upon the fundamental CP-violating parameter of the lepton flavor mixing matrix ($J$), but also upon the mass-squared differences of neutrinos ($\Delta m_{21}^2$ and $\Delta m_{31}^2$) and the matter-induced effect ($A$). In particular, both the sign and the magnitude of $\tilde{J}$ are dependent on the signs of $\Delta m_{21}^2$ and $\Delta m_{31}^2$. With the help of (3.37) and (3.38), we find that the effective mass-squared differences $\Delta \bar{m}_{21}^2$ and $\Delta \bar{m}_{31}^2$ can keep unchanged in proper arrangements of the signs of $\Delta m_{21}^2$, $\Delta m_{31}^2$ and $A$. A careful analysis of (4.4) leads to the following exact relations [93]:

$$\tilde{J}(+\Delta m_{21}^2, +\Delta m_{31}^2, +A) = -\tilde{J}(-\Delta m_{21}^2, -\Delta m_{31}^2, -A),$$

$$\tilde{J}(+\Delta m_{21}^2, -\Delta m_{31}^2, +A) = -\tilde{J}(-\Delta m_{21}^2, +\Delta m_{31}^2, -A),$$

$$\tilde{J}(+\Delta m_{21}^2, +\Delta m_{31}^2, -A) = -\tilde{J}(-\Delta m_{21}^2, -\Delta m_{31}^2, +A),$$

$$\tilde{J}(+\Delta m_{21}^2, -\Delta m_{31}^2, -A) = -\tilde{J}(-\Delta m_{21}^2, +\Delta m_{31}^2, +A). \quad (4.7)$$
Figure 4.1: The ratio $\tilde{J}/J$ changing with the matter parameter $A$ for neutrinos ($\nu$) and antineutrinos ($\bar{\nu}$), in which $\Delta m^2_{21} = 5 \times 10^{-5}$ eV$^2$, $\Delta m^2_{31} = 3 \times 10^{-3}$ eV$^2$, $\theta_x \approx 35^\circ$, $\theta_y \approx 40^\circ$, $\theta_z \approx 5^\circ$ and $\delta \approx \pm 90^\circ$ have typically been input.

The validity of these relations are independent of both the neutrino beam energy and the baseline length. Note that current experimental data on solar neutrino oscillations favor $\Delta m^2_{21} > 0$, but the sign of $\Delta m^2_{31}$ remains unknown.

It is also worth remarking that the CP- and T-violating effects are measured by the same parameter ($J$ in vacuum or $\tilde{J}$ in matter), as a straightforward consequence of CPT invariance. As for neutrino oscillations, a signal of CP violation comes from the probability asymmetry between $\nu_\alpha \to \nu_\beta$ and $\bar{\nu}_\alpha \to \bar{\nu}_\beta$ transitions; while a signal of T violation is attributed to the probability asymmetry between $\nu_\alpha \to \nu_\beta$ and $\nu_\beta \to \nu_\alpha$ transitions or between $\bar{\nu}_\alpha \to \bar{\nu}_\beta$ and $\bar{\nu}_\beta \to \bar{\nu}_\alpha$ transitions ($\alpha \neq \beta$). The former is roughly associated with the difference between $\tilde{J}(J,A)$ and $\tilde{J}(-J,-A)$, in which the $A$-induced effects essentially add each other; but the latter is roughly associated with the difference between $\tilde{J}(J,A)$ and $\tilde{J}(-J,A)$ or between $\tilde{J}(J,-A)$ and $\tilde{J}(-J,-A)$, in which the $A$-induced effects essentially cancel each other. For this reason, the CP-violating asymmetries in long-baseline neutrino oscillations are rather sensitive to the terrestrial matter effect; while the T-violating asymmetries are almost independent of the terrestrial matter effect (see Refs. [93, 97] for detailed discussions). Measuring T violation is practically more difficult than measuring CP violation, however [114].

To illustrate the dependence of $\tilde{J}$ on matter effects, we typically take $\Delta m^2_{21} = 5 \times 10^{-5}$ eV$^2$ and $\Delta m^2_{31} = 3 \times 10^{-3}$ eV$^2$ as well as $\theta_x \approx 35^\circ$, $\theta_y \approx 40^\circ$, $\theta_z \approx 5^\circ$ and $\delta \approx \pm 90^\circ$ in the standard parametrization of $V$. The result is shown in Fig. 4.1, where $10^{-7}$ eV$^2 \leq A \leq 10^{-2}$ eV$^2$ has been chosen for the matter parameter $A$. One can see that the magnitude of $\tilde{J}$ decreases, when the matter effect becomes significant (e.g., $A \geq 10^{-4}$ eV$^2$). Nevertheless, this feature does not necessarily imply that the CP-violating asymmetries in realistic long-baseline neutrino oscillations would be smaller than their values in vacuum. Very large terrestrial matter effects can significantly modify the frequencies of neutrino oscillations and thus enhance (or suppress) the genuine signals of CP violation.
4.2 Leptonic Unitarity Triangles

A geometric description of the lepton flavor mixing phenomenon is physically intuitive and instructive. The unitarity of the $3 \times 3$ flavor mixing matrix $V$ in vacuum can be expressed by two sets of orthogonality relations and two sets of normalization conditions for its nine matrix elements:

\[ \sum_i (V^*_{\alpha i} V_{\beta i}) = \delta_{\alpha \beta}, \]
\[ \sum_\alpha (V^*_{\alpha i} V_{\alpha j}) = \delta_{ij}, \]  \hspace{1cm} (4.8)

where Greek and Latin indices run over $(e, \mu, \tau)$ and $(1, 2, 3)$, respectively. In the complex plane the six orthogonality relations in (4.8) define six triangles $(\triangle_e, \triangle_\mu, \triangle_\tau)$ and $(\triangle_1, \triangle_2, \triangle_3)$ shown in Fig. 4.2, the so-called unitarity triangles. In general, these six triangles have eighteen different sides and nine different inner (or outer) angles. The unitarity requires that all six triangles have the same area amounting to $J/2$, where $J$ is just the rephasing-invariant measure of CP violation defined in (4.1). If CP were an exact symmetry, $J = 0$ would hold and those unitarity triangles would collapse into lines in the complex plane. Note that the shape and area of each unitarity triangle are irrelevant to the nature of neutrinos; i.e., they are the same for Dirac and Majorana neutrinos.

In addition to $J$, there exist other two characteristic quantities of $V$, resulting from its normalization conditions in (4.8). They are the off-diagonal asymmetries $A_L$ and $A_R$ defined in (3.60). Clearly $A_L = 0$ (or $A_R = 0$) would imply that the flavor mixing matrix
$V$ were symmetric about its $V_{e1} - V_{\mu2} - V_{\tau3}$ (or $V_{e3} - V_{\mu2} - V_{\tau1}$) axis. Geometrically this would correspond to the congruence between two unitarity triangles; i.e.,

$$
\mathcal{A}_L = 0 \implies \triangle_e \cong \triangle_1, \quad \triangle_{\mu} \cong \triangle_2, \quad \triangle_{\tau} \cong \triangle_3;
$$

$$
\mathcal{A}_R = 0 \implies \triangle_e \cong \triangle_3, \quad \triangle_{\mu} \cong \triangle_2, \quad \triangle_{\tau} \cong \triangle_1. \quad (4.9)
$$

Our current knowledge on lepton flavor mixing is rather poor, hence it remains difficult to quantify the magnitude of leptonic CP violation. Nevertheless, we are sure that $\mathcal{A}_L > 0$ definitely holds. The possibility for $\mathcal{A}_R = 0$ cannot be ruled out at present [107]. Useful information about the sides and angles of the leptonic unitarity triangles will be obtained from the medium- and long-baseline neutrino experiments in the near future.

For instance, one might be able to determine three inner angles of the unitarity triangle $\triangle_3$, which are defined as

$$
\alpha_l \equiv \arg \left( \frac{V_{e1}^* V_{e2}}{V_{\mu1}^* V_{\mu2}} \right),
$$

$$
\beta_l \equiv \arg \left( \frac{V_{e1}^* V_{\mu2}}{V_{\mu1}^* V_{\tau2}} \right),
$$

$$
\gamma_l \equiv \arg \left( \frac{V_{\tau1}^* V_{\tau2}}{V_{e1}^* V_{e2}} \right). \quad (4.10)
$$

Then it should be possible to test the self-consistency of our lepton flavor mixing and CP-violating picture (e.g., $\alpha_l + \beta_l + \gamma_l \neq \pi$ would imply the presence of new physics which violates the unitarity of the $3 \times 3$ lepton flavor mixing matrix). The measurement of $\alpha_l$, $\beta_l$ and $\gamma_l$ can in principle be realized in a long-baseline neutrino oscillation experiment. Taking $\Delta m_{21}^2 \approx \Delta m_{32}^2$ and $\Delta m_{31}^2 \approx \Delta m_{23}^2 \approx \pm \Delta m_{\text{atm}}^2$, we simplify (3.26) to [12]

$$
P(\nu_e \to \nu_\mu) = 4|V_{e3}|^2|V_{\mu3}|^2 \sin^2 \theta_{\text{atm}} - 4 \Re \left( V_{e1} V_{\mu2} V_{e2}^* V_{\mu1}^* \right) \sin^2 \theta_{\text{sun}} - 8 J \sin \theta_{\text{sun}} \sin^2 \theta_{\text{atm}},
$$

$$
P(\nu_\mu \to \nu_\tau) = 4|V_{e3}|^2|V_{\mu3}|^2 \sin^2 \theta_{\text{atm}} - 4 \Re \left( V_{\mu1} V_{\tau2} V_{\mu2} V_{\tau1}^* \right) \sin^2 \theta_{\text{sun}} - 8 J \sin \theta_{\text{sun}} \sin^2 \theta_{\text{atm}},
$$

$$
P(\nu_\tau \to \nu_e) = 4|V_{e3}|^2|V_{\mu3}|^2 \sin^2 \theta_{\text{atm}} - 4 \Re \left( V_{\tau1} V_{\mu2} V_{\tau2} V_{e1}^* \right) \sin^2 \theta_{\text{sun}} - 8 J \sin \theta_{\text{sun}} \sin^2 \theta_{\text{atm}}, \quad (4.11)
$$

where $\theta_{\text{atm}} = \theta_{32} \approx \theta_{31}$ and $\theta_{\text{sun}} = \theta_{21}$ defined below (3.5) measure the oscillation frequencies of atmospheric and solar neutrinos, respectively. Provided the baseline satisfies the condition $L \sim E/\Delta m_{\text{sun}}^2$ (i.e., $\theta_{\text{sun}} \sim 1$), the CP-conserving quantities $\Re(V_{\alpha1} V_{\beta2} V_{\gamma3}^*)$ and the CP-violating parameter $J$ could both be determined from (4.11) for changing values of the neutrino beam energy $E$. Then one may get

$$
\tan \alpha_l = - \frac{J}{\Re(V_{e1} V_{\mu2} V_{e2}^* V_{\mu1}^*)},
$$

$$
\tan \beta_l = - \frac{J}{\Re(V_{\mu1} V_{\tau2} V_{\mu2} V_{\tau1}^*)},
$$

$$
\tan \gamma_l = - \frac{J}{\Re(V_{\tau1} V_{\mu2} V_{e2}^* V_{e1})}. \quad (4.12)
$$
in practice, the earth-induced matter effects are likely to fake the genuine CP-violating signals and have to be taken into account. If the neutrino beam energy is low enough (e.g., \( E \sim 0.1 \) to 1 GeV) and the baseline length is not too long (e.g., \( L \sim 100 \) km), the terrestrial matter effects might not be very significant for our purpose [115].

Another interesting unitarity triangle is \( \Delta_\tau \), whose three sides might more easily be measured. To establish \( \Delta_\tau \) needs very precise data, which should be able to show \( |V_{e1}\bar{V}_{\mu1}| + |V_{e3}\bar{V}_{\mu3}| > |V_{e2}\bar{V}_{\mu2}| \) or \( |V_{e2}\bar{V}_{\mu2}| + |V_{e3}\bar{V}_{\mu3}| > |V_{e1}\bar{V}_{\mu1}| \) [116]. Such an accuracy requirement is practically a big challenge, because one side of \( \Delta_\tau \) is much shorter than its other two sides (i.e., \( |V_{e3}\bar{V}_{\mu1}| \sim |V_{e3}\bar{V}_{\mu2}| \gg |V_{e3}\bar{V}_{\mu3}| \), as (3.19) has shown). Note that three sides of the effective unitarity triangle \( \Delta_\tau \) in matter, defined as

\[
\begin{align*}
\tilde{\Delta}_\tau : \quad \tilde{V}_{e1}\bar{V}_{\mu1} + \tilde{V}_{e2}\bar{V}_{\mu2} + \tilde{V}_{e3}\bar{V}_{\mu3} &= 0 , \\
\end{align*}
\]

are possible to be comparable in magnitude. With the help of (3.52), we arrive at

\[
\begin{align*}
\tilde{V}_{e1}\bar{V}_{\mu1} &= \frac{\Delta\tilde{m}_{23}^2 \Delta m_{31}^2}{\Delta m_{31}^2 \Delta m_{32}^2} V_{e1}^\ast V_{\mu1} + \frac{\Delta\tilde{m}_{13}^2 \Delta m_{32}^2}{\Delta m_{32}^2 \Delta m_{31}^2} V_{e2}^\ast V_{\mu2} , \\
\tilde{V}_{e2}\bar{V}_{\mu2} &= \frac{\Delta\tilde{m}_{32}^2 \Delta m_{21}^2}{\Delta m_{21}^2 \Delta m_{32}^2} V_{e2}^\ast V_{\mu2} + \frac{\Delta\tilde{m}_{23}^2 \Delta m_{21}^2}{\Delta m_{21}^2 \Delta m_{32}^2} V_{e3}^\ast V_{\mu3} , \\
\tilde{V}_{e3}\bar{V}_{\mu3} &= \frac{\Delta\tilde{m}_{12}^2 \Delta m_{23}^2}{\Delta m_{23}^2 \Delta m_{13}^2} V_{e3}^\ast V_{\mu3} + \frac{\Delta\tilde{m}_{13}^2 \Delta m_{12}^2}{\Delta m_{12}^2 \Delta m_{13}^2} V_{e1}^\ast V_{\mu1} , \\
\end{align*}
\]

where \( \Delta\tilde{m}_{ij}^2 \equiv m_i^2 - \tilde{m}_j^2 \). The explicit expressions of \( \Delta\tilde{m}_{ij}^2 \) and \( \Delta\tilde{m}_{ij}^2 \), which depend on both \( \Delta m_{ij}^2 \) and \( A \), can be obtained from (3.37). The instructive result in (4.14) clearly shows how three sides of \( \Delta_\tau \) get modified in matter. Of course, one may use (4.14) to evaluate three sides of \( \tilde{\Delta}_\tau \). It is also possible to get a numerical feeling of \( \Delta_\tau \) and \( \tilde{\Delta}_\tau \) directly from Fig. 3.2. For example, we find that three sides of \( \tilde{\Delta}_\tau \) become comparable in magnitude (of order \( |V_{e3}\bar{V}_{\mu3}| \)), when the neutrino beam energy \( E \) is about 1 GeV or somehow larger. In a similar way, we can define and discuss other effective unitarity triangles in matter.

If \( |V_{e3}|, |V_{\mu3}| \) and \( |V_{\tau3}| \) are well determined in the first-round long-baseline neutrino experiments, it should be possible to check one of the three normalization conditions: \( |V_{e3}|^2 + |V_{\mu3}|^2 + |V_{\tau3}|^2 = 1 \). If \( |V_{e1}|, |V_{e2}| \) and \( |V_{e3}| \) are measured to a good degree of accuracy, one can test another normalization condition \( |V_{e1}|^2 + |V_{e2}|^2 + |V_{e3}|^2 = 1 \). At this stage the moduli of \( V_{\mu1}, V_{\mu2}, V_{\tau1} \) and \( V_{\tau2} \) remain unknown. To determine the universal CP-violating parameter \( J \) and to make a full test of the unitarity of \( V \), much more delicate long-baseline neutrino experiments are needed.

### 4.3 Baryon Asymmetry via Leptogenesis

In the universe, the density of baryons compared to that of photons is extremely small:

\[
\eta \equiv n_B/n_s = (2.6 - 6.3) \times 10^{-10} ,
\]

extracted from the Big-Bang nucleosynthesis [6]. This tiny quantity measures the observed matter-antimatter or baryon-antibaryon asymmetry of the universe,

\[
Y_B \equiv \frac{n_B - n_B}{s} \approx \frac{\eta}{7.04} = (3.7 - 8.9) \times 10^{-11} ,
\]

\[\text{Eq. (4.15)}\]
where $s$ denotes the entropy density. To produce a net baryon asymmetry in the standard Big-Bang model, three Sakharov necessary conditions have to be satisfied [117]: (a) baryon number nonconservation, (b) C and CP violation, and (c) a departure from thermal equilibrium. Among a number of interesting and viable baryogenesis mechanisms proposed in the literature [118], Fukugita and Yanagida’s leptogenesis mechanism [119] has recently attracted a lot of attention – due partly to the fact that neutrino physics is entering a flourishing era.

As mentioned in section 2.1.2, a simple extension of the standard model to generate neutrino masses is to include one right-handed neutrino in each of three lepton families, while the Lagrangian of electroweak interactions keeps invariant under the $SU(2)_L \times U(1)_Y$ gauge transformation. In this case, the Yukawa interactions of leptons are described by

$$
-L_Y = \bar{l}_L \phi Y_l e_R + \bar{l}_L \phi Y_\nu \nu_R + \frac{1}{2} \nu_R^\dagger M_{\nu R} \nu_R + \text{h.c.},
$$

where $l_L$ denotes the left-handed lepton doublet, $e_R$ and $\nu_R$ stand respectively for the right-handed charged lepton and Majorana neutrino singlets, and $\phi$ is the Higgs-boson weak isodoublet. The lepton number violation induced by the third term of $L_Y$ allows decays of the heavy (right-handed) Majorana neutrinos $N_i$ (for $i = 1, 2, 3$) to happen; i.e., $N_i \to l + \phi^\dagger$ vs $N_i \to l^c + \phi$, as illustrated in Fig. 4.3. Because each decay mode occurs both at the tree level and at the one-loop level (via the self-energy and vertex corrections), the interference between the tree-level and one-loop decay amplitudes can lead to a CP-violating asymmetry $\varepsilon_i$ between the two CP-conjugated processes [121]:

$$
\varepsilon_i \equiv \frac{\Gamma(N_i \to l + \phi^\dagger) - \Gamma(N_i \to l^c + \phi)}{\Gamma(N_i \to l + \phi^\dagger) + \Gamma(N_i \to l^c + \phi)}
\times \frac{1}{8\pi\left(Y^\dagger_{\nu l} Y_{\nu l}\right)_{ii}} \sum_{j=1}^{3} \left\{ \text{Im} \left[ (Y_{\nu l}^\dagger Y_{l})_{ij} \right] \right\}^2 \left[ f_V \left( \frac{M_j^2}{M_i^2} \right) + f_S \left( \frac{M_j^2}{M_i^2} \right) \right],
$$

where the loop functions $f_V$ (vertex) and $f_S$ (self-energy) are given by

$$
\begin{align*}
    f_V(x) & = \begin{cases} 
        \sqrt{x} \left[ 1 - (1 + x) \ln \frac{1 + x}{x} \right] & \text{(SM)} \\
        -\sqrt{x} \ln \frac{1 + x}{x} & \text{(SUSY)} 
    \end{cases} \\
    f_S(x) & = \begin{cases} 
        \frac{\sqrt{x}}{1 - x} & \text{(SM)} \\
        \frac{1 - x}{2\sqrt{x}} & \text{(SUSY)} 
    \end{cases}
\end{align*}
$$

If $x \gg 1$ holds, we arrive at $f_S \approx 2f_V \approx -1/\sqrt{x}$ (SM) or $f_S \approx 2f_V \approx -2/\sqrt{x}$ (SUSY).

The central idea of baryogenesis via leptogenesis is rather simple [119]. In the early universe, the heavy (right-handed) Majorana neutrinos $N_i$ are present in the primordial

---

16One may also consider a similar extension of the minimal supersymmetric standard model to generate neutrino masses and to resolve the hierarchy problem induced by heavy right-handed Majorana neutrinos [120]. The parameter counting for lepton flavor mixing and leptogenesis is identical in models with and without supersymmetry [24].
Equilibrium decay of heavy Majorana neutrinos $\nu_i$ erases before $CP$-violating asymmetries produced in the decays of $\nu_i$. As shown in Fig. 4.3, the decays of $\nu_i$ are in thermal equilibrium when $Y_i \equiv n_i - n_i^\ast = 106.75$ (SM) or 228.75 (SUSY) is an effective number characterizing the relativistic degrees of freedom which contribute to the entropy $s$, and $d$ accounts for the dilution effects induced by the lepton-number-violating wash-out processes [121]. The lepton asymmetry $Y_L$ is eventually converted into a net baryon asymmetry $Y_B$ via

$$Y_L \equiv \frac{n_L - n_L^\ast}{s} = \frac{d}{g_s} \varepsilon_1 \ .$$  (4.19)
nonperturbative sphaleron processes [122]. The explicit relation between \( Y_L \) (initial) and \( Y_B \) (equilibrium) is given as [123]

\[
Y_B = -c Y_L = -c \frac{d}{g_*} \varepsilon_1 ,
\]

(4.20)

where \( c = (8N_f + 4N_\phi)/(22N_f + 13N_\phi) \) with \( N_f \) being the number of fermion families and \( N_\phi \) being the number of Higgs doublets. Taking \( N_f = 3 \) and \( N_\phi = 1 \) for example, we obtain \( c \approx 1/3 \). Note that the dilution factor \( d \) can be computed by integrating the full set of Boltzmann equations [124]. In the literature [125], some useful analytical approximations for \( d \) have frequently been made.

In the simple mechanism of thermal leptogenesis introduced above, the baryon asymmetry of the universe is attributed to the out-of-equilibrium decay of the lightest heavy (right-handed) Majorana neutrino \( N_1 \). Another interesting scenario of thermal leptogenesis is to consider the lepton-number-violating decays of two heavy Majorana neutrinos, whose masses are approximately degenerate [126]. Because the self-energy contribution to \( \varepsilon_i \) can significantly be enhanced in the case of \( M_i \approx M_j \), it is possible to generate the observed baryon-antibaryon asymmetry \( Y_B \) via the out-of-equilibrium decays of two relatively light and moderately degenerate \( N_i \). Such a scenario could allow the smallest mass of \( N_i \) to be close to or below the maximum reheating temperature of the universe after inflation in the generic supergravity models. Of course, one may abandon the idea of thermal leptogenesis and interpret the baryon asymmetry through the decays of heavy (right-handed) Majorana neutrinos produced non-thermally by the inflaton decay [127]. The non-thermal leptogenesis mechanism seems to be more speculative and less elegant than the thermal leptogenesis mechanism, however. Thus we continue to focus on the simple scenario of thermal leptogenesis in the following.

After spontaneous symmetry breaking, \( \mathcal{L}_Y \) in (4.16) becomes \( \mathcal{L}_{\text{mass}} \) in (2.6), where \( M_l = Y_l \langle \phi \rangle \) for the charged lepton mass matrix and \( M_D = Y_\nu \langle \phi \rangle \) for the Dirac neutrino mass matrix. The scale of \( M_l \) and \( M_D \) is characterized by the gauge symmetry breaking scale \( v \equiv \langle \phi \rangle \approx 174 \text{ GeV} \); but the scale of \( M_R \) may be much higher than \( v \), because right-handed neutrinos are SU(2)_L singlets and their mass term is not subject to the electroweak symmetry breaking. As already shown in section 2.1.2, the light (left-handed) neutrino mass matrix \( M_\nu \) can be given in terms of \( M_D \) and \( M_R \) via the seesaw relation (2.9). Note that lepton flavor mixing at low energy scales stems from a nontrivial mismatch between the diagonalizations of \( M_\nu \) and \( M_l \), while the baryon asymmetry at high energy scales depends on complex \( Y_\nu \) and \( M_R \) in the thermal leptogenesis mechanism. To see the latter more clearly, let us diagonalize the symmetric mass matrix \( M_R \) by a unitary transformation:

\[
U_R^\dagger M_R U_R^* = \begin{pmatrix}
M_1 & 0 & 0 \\
0 & M_2 & 0 \\
0 & 0 & M_3
\end{pmatrix},
\]

(4.21)

where \( M_i \) are the physical masses of \( N_i \). Provided three heavy Majorana neutrinos have a strong mass hierarchy (i.e., \( M_1 \ll M_2 \ll M_3 \)), the CP-violating asymmetry between \( N_1 \rightarrow l + \phi^\dagger \) and \( N_1 \rightarrow l^c + \phi \) decays in (4.17) can be simplified to [128]

\[
\varepsilon_1 \approx -\frac{3}{16\pi v^2} \cdot \frac{M_1}{U_R^T M_D^3 M_D U_R^*} \sum_{j=2}^3 \frac{\text{Im} \left( [U_R^T M_D^3 M_D U_R^*]_{1j} \right)^2}{M_j}.
\]

(4.22)

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In obtaining (4.22), we have taken into account the point that the Dirac neutrino Yukawa coupling matrix $Y_\nu$ takes the form $M_D U_R^* / v$ in the physical basis where $M_R$ is diagonal. One can see that $\varepsilon_1$ depends on the complex phases of $M_D$ and $U_R$. The only possible relationship between $\varepsilon_1$ and the lepton flavor mixing matrix $V$ at low energy scales is due to the seesaw relation (2.9), which links $M_\nu$ to $M_D$ and $M_R$. Therefore we conclude that there is no direct connection between CP violation in heavy Majorana neutrino decays ($\varepsilon_i$) and that in neutrino oscillations ($J$). Such a general conclusion was first drawn by Buchm"uller and Pl"umacher in Ref. [129]. Recently some other authors have carried out more delicate analyses and reached the same conclusion.

Depending on the specific flavor basis that we choose in model building, $\varepsilon_i$ and $V$ can either be completely disconnected or somehow connected. To illustrate, we consider two extreme cases [130]:

- In the flavor basis where $M_\nu$ is diagonal (i.e., lepton flavor mixing and CP violation at low energy scales arise solely from the charged lepton sector [61]), we find that $\varepsilon_1$ has nothing to do with $V$. In this special case, less fine-tuning is expected in building a phenomenological model which can simultaneously interpret the baryon asymmetry of the universe and lepton flavor mixing at low energy scales.

- In the flavor basis where both $M_l$ and $M_R$ are diagonal (i.e., lepton flavor mixing and CP violation at low energy scales arise solely from the neutrino sector), we find that $\varepsilon_1$ can indirectly be linked to $V$ through the seesaw relation in (2.9). In this special case, it is highly nontrivial to build a predictive model or ansatz which can simultaneously interpret the observed baryon asymmetry of the universe and current neutrino oscillation data.

It is worth remarking that the correlation between high-energy physics and low-energy physics may offer us a valuable opportunity to probe the former from the latter, or vice versa [131]. When a specific model is built, however, the textures of $M_R$ and $M_D$ have to be carefully chosen or fine-tuned to guarantee acceptable agreement between the model predictions and the observational or experimental data (see Refs. [128, 132] for example). Many different models or ansätze on leptogenesis and neutrino oscillations have recently been proposed [133], but some of them are rather preliminary and speculative.

An important question is whether the leptogenesis mechanism can experimentally be proved. The answer is unfortunately negative. But baryogenesis via leptogenesis might become a conceivable and acceptable mechanism to interpret the matter-antimatter asymmetry of the universe [134], if (1) the electroweak baryogenesis scenario is ruled out; (2) the Majorana nature of massive neutrinos is verified (e.g., by measuring the neutrinoless double beta decay); and (3) the leptonic CP-violating effect is observed (e.g., in the future long-baseline neutrino oscillation experiments).

5 Concluding Remarks

We have presented an overview of recent progress in the phenomenological study of neutrino masses, lepton flavor mixing and CP violation. Particular attention has been paid to the model-independent properties of massive neutrinos in vacuum and in matter.
With the help of current experimental and observational data, we have obtained some enlightening information about the neutrino mass spectrum. To pin down the absolute mass scale of three active neutrinos, one has to make much more efforts to detect the tritium beta decay and the neutrinoless double beta decay. Precise cosmological data are also expected to play an important role in determining the absolute values of neutrino masses. The relative magnitudes of three neutrino masses can be fixed by means of a variety of long-baseline neutrino oscillation experiments in the coming years.

Different from quark flavor mixing, lepton flavor mixing involves two large mixing angles. Whether the largeness of neutrino mixing angles is associated with a relatively weak hierarchy of neutrino masses remains an open question. We hope that both the smallest angle of lepton flavor mixing and the Dirac phase of CP violation can be measured in the long-baseline neutrino oscillation experiments. It seems hopeless to separately determine two Majorana phases of CP violation from the measurements of the neutrinoless double beta decay and other lepton-number-violating processes. Nevertheless, it is extremely important to realize such measurements in order to identify the Majorana nature of massive neutrinos and to shed light on the nontrivial features of lepton number violation.

To conclude, the robust experimental evidence for neutrino masses and lepton flavor mixing strongly indicates that the standard electroweak model is actually incomplete. This incompleteness motivates us to open a new window to go beyond the standard model. Although there have been a number of unresolved questions associated with massive neutrinos, we are certainly paving the way for satisfactory answers to them. A convincing and predictive theory of massive neutrinos should be achieved in the foreseeable future. Such a theory, which must include new physics of both leptons and quarks at high energy scales, is expected to offer us a deeper insight into the generation of fermion masses, the pattern of flavor mixing and the origin of CP violation.

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A Mixing between Active and Sterile Neutrinos

In addition to the robust evidence for atmospheric and solar neutrino oscillations accumulated from the SK [2], SNO [3], KamLAND [4] and K2K [5] experiments, $\nu_\mu \to \nu_e$ and $\nu_\mu \to \nu_e$ transitions have been observed by the LSND Collaboration [8]. The LSND data can also be interpreted in the assumption of neutrino oscillations, whose mass-squared difference and mixing factor read as

$$\Delta m^2_{\text{LSND}} \sim 1 \text{ eV}^2, \quad \sin^2 2\theta_{\text{LSND}} \sim 10^{-3} - 10^{-2}. \quad (A.1)$$

Because solar, atmospheric and LSND neutrino oscillations involve three distinct mass-squared differences ($\Delta m^2_{\text{sun}} \ll \Delta m^2_{\text{atm}} \ll \Delta m^2_{\text{LSND}}$), a simultaneous interpretation of
them requires the introduction of a light *sterile* neutrino\(^{17}\). There are two typical categories of four-neutrino mixing schemes and their sample mass spectra are shown in Fig. A.1: (a) the \((2+2)\) scheme with two pairs of nearly degenerate massive neutrinos, whose mass-squared gap is characterized by \(\Delta m^2_{\mathrm{LSND}}\); and (b) the \((3+1)\) scheme with a triplet of nearly degenerate massive neutrinos and an isolated massive neutrino, whose mass-squared gap is also characterized by \(\Delta m^2_{\mathrm{LSND}}\). Although each four-neutrino mixing scheme consists of three independent mass-squared differences and six flavor mixing angles, its parameter space can be tightly constrained by current experimental data. The global analyses of all available neutrino oscillation data [137] have shown that the \((2+2)\) mixing scenario is strongly disfavored, while the \((3+1)\) mixing scenario is marginally allowed. However, the recent cosmological upper bound on the sum of all light neutrino masses in (2.22) is too low to be consistent with the LSND result given in (A.1), at least at the 95% confidence level [48]. It is argued by Hannestad [47] and Giunti [48] that the \((3+1)\) mixing scheme may still survive in a very tiny parameter space, if the relevant cosmological data are considered at a higher confidence level. The upcoming MiniBooNE experiment [39] is crucial to firmly confirm or disprove the LSND measurement.

Before a definitely negative conclusion can be drawn from MiniBooNE, we do not think that the LSND data should be completely discarded. In particular, it is worthwhile to study the four-neutrino mixing scenarios in a way without special theoretical biases and (or) empirical assumptions. Starting from such a point of view, we present some model-independent results on the description of four-neutrino mixing and CP violation in this Appendix.

### A.1 Standard Parametrization

Let us begin with a generic SU\((2)_L \times U(1)_Y\) model of electroweak interactions, in which there exist \(n\) charged leptons belonging to isodoublets, \(n\) active neutrinos belonging to isodoublets, and \(n'\) sterile neutrinos belonging to isosinglets. The charged-current weak interactions of leptons are then associated with a rectangular flavor mixing matrix of \(n\)

---

\(^{17}\)A *sterile* particle means that it has vanishing or extremely feeble couplings to the other particles in the standard model. Instead of introducing a light sterile neutrino, a few more far-fetched ideas (such as the violation of CPT symmetry in the neutrino sector [135] and the lepton-number-violating muon decay [136]) have been proposed in the literature to simultaneously interpret the solar, atmospheric and LSND anomalies within the three-neutrino mixing scheme.
rows and \((n+n')\) columns [72]. Without loss of generality, one may choose to identify the flavor eigenstates of charged leptons with their mass eigenstates. In this specific basis, the \(n \times (n+n')\) lepton mixing matrix links the neutrino flavor eigenstates directly to the neutrino mass eigenstates. Although sterile neutrinos do not participate in normal weak interactions, they may oscillate among themselves and with active neutrinos. Once the latter is concerned we are led to a more general \((n+n') \times (n+n')\) lepton flavor mixing matrix [12], defined as \(V\) in the chosen flavor basis. For the mixing of one sterile neutrino \((\nu_s)\) and three active neutrinos \((\nu_e, \nu_\mu, \nu_\tau)\), the explicit form of \(V\) can be written as

\[
\begin{pmatrix}
\nu_s \\
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} = \begin{pmatrix}
V_{s0} & V_{s1} & V_{s2} & V_{s3} \\
V_{e0} & V_{e1} & V_{e2} & V_{e3} \\
V_{\mu0} & V_{\mu1} & V_{\mu2} & V_{\mu3} \\
V_{\tau0} & V_{\tau1} & V_{\tau2} & V_{\tau3}
\end{pmatrix} \begin{pmatrix}
\nu_0 \\
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
\tag{A.2}
\]

where \(\nu_i\) (for \(i = 0, 1, 2, 3\)) denote the mass eigenstates of four neutrinos. If neutrinos are Dirac particles, \(V\) can be parametrized in terms of six mixing angles and three phase angles. If neutrinos are Majorana particles, however, three additional phase angles are required to get a full parametrization of \(V\). We totally need six complex rotation matrices, denoted as \(R_{01}, R_{02}, R_{03}, R_{12}, R_{13}\) and \(R_{23}\), which correspond to simple rotations with angles \(\theta_{ij}\) in the \((0,1), (0,2), (0,3), (1,2), (1,3)\) and \((2,3)\) planes. For simplicity, we assume that each \(R_{ij}\) involves only a single phase angle \(\delta_{ij}\) associated with its \(\sin \theta_{ij}\) term. Explicitly, we have

\[
R_{01}(\theta_{01}, \delta_{01}) = \begin{pmatrix}
c_{01} & \hat{s}_{01}^* & 0 & 0 \\
-\hat{s}_{01} & c_{01} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
R_{02}(\theta_{02}, \delta_{02}) = \begin{pmatrix}
c_{02} & 0 & \hat{s}_{02}^* & 0 \\
0 & 1 & 0 & 0 \\
-\hat{s}_{02} & 0 & c_{02} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
R_{03}(\theta_{03}, \delta_{03}) = \begin{pmatrix}
c_{03} & 0 & 0 & \hat{s}_{03}^* \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\hat{s}_{03} & 0 & 0 & c_{03}
\end{pmatrix},
\]

\[
R_{12}(\theta_{12}, \delta_{12}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & c_{12} & \hat{s}_{12}^* & 0 \\
0 & -\hat{s}_{12} & c_{12} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
R_{13}(\theta_{13}, \delta_{13}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & c_{13} & \hat{s}_{13}^* & 0 \\
0 & 0 & 1 & 0 \\
0 & -\hat{s}_{13} & 0 & c_{13}
\end{pmatrix},
\]

\[
R_{23}(\theta_{23}, \delta_{23}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & c_{23} & \hat{s}_{23}^* \\
0 & 0 & -\hat{s}_{23} & c_{23}
\end{pmatrix}.
\tag{A.3}
\]
where $s_{ij} \equiv \sin \theta_{ij} e^{i\delta_{ij}}$ and $c_{ij} \equiv \cos \theta_{ij}$. There exist numerous different ways to arrange the products of these rotation matrices [78], such that the resultant flavor mixing matrix $V$ covers the whole $4 \times 4$ space. After a proper assignment of the relevant rotation angles, one can find that only sixteen parametrizations of $V$ are structurally distinct: four of them have the $\cos \theta_{ij}$ terms in the $(i, i)$ positions of $V$; and twelve of them have the $\sin \theta_{ij}$ terms in the twelve different $(i, j)$ positions of $V$ (for $i \neq j$).

To avoid unnecessary complication, we shall not write out a lengthy list of the sixteen parametrizations of $V$. Instead we only take an instructive example for illustration. Similar to the representation in (3.23), the $4 \times 4$ neutrino mixing matrix $V$ can be parametrized as $V = R_{23} \otimes R_{13} \otimes R_{03} \otimes R_{12} \otimes R_{02} \otimes R_{01}$ [138]; i.e.,

$$V = \begin{pmatrix}
c_{01}c_{02}c_{03} & c_{02}c_{03}s_{01} & c_{03}s_{02} & s_{03} \\
-c_{01}c_{02}s_{03}s_{13} & -c_{02}s_{01}s_{03}s_{13} & -s_{02}s_{03}s_{13} & c_{03}s_{13} \\
-c_{01}c_{13}s_{02}s_{12} & -c_{13}s_{01}s_{02}s_{12} & +c_{02}c_{13}s_{12} & \\
-c_{12}c_{13}s_{01} & +c_{01}c_{12}c_{13} & \\
-c_{01}c_{02}c_{13}s_{03}s_{23} & -c_{02}c_{13}s_{01}s_{03}s_{23} & -c_{13}s_{02}s_{03}s_{23} & c_{03}s_{13}s_{23} \\
+c_{01}c_{02}s_{13}s_{02}s_{23} & +s_{01}s_{02}s_{12}s_{13}s_{23} & -c_{02}s_{12}s_{13}s_{23} & \\
-c_{01}c_{12}c_{23}s_{02} & -c_{12}c_{23}s_{01}s_{02} & +c_{02}c_{12}c_{23} & \\
+c_{12}s_{01}s_{13}s_{23} & -c_{01}c_{12}c_{13}s_{23} & \\
+c_{23}s_{01}s_{12} & -c_{01}c_{23}s_{12} & \\
-c_{01}c_{02}c_{13}c_{23}s_{03} & -c_{02}c_{13}c_{23}s_{01}s_{03} & -c_{13}c_{23}s_{02}s_{03} & c_{03}c_{13}s_{23} \\
+c_{01}c_{02}s_{13}s_{12}s_{13} & +c_{01}s_{02}s_{12}s_{13}s_{13} & -c_{02}c_{23}s_{12}s_{13} & \\
+c_{01}c_{12}s_{02}s_{23} & +c_{12}s_{01}s_{02}s_{23} & -c_{02}c_{12}s_{23} & \\
+c_{12}c_{23}s_{01}s_{13} & -c_{01}c_{12}c_{23}s_{13} & \\
-s_{01}s_{12}s_{23} & +c_{01}s_{12}s_{23} & \\
\end{pmatrix}.$$ (A.4)

Without loss of generality, the six mixing angles $\theta_{ij}$ can all be arranged to lie in the first quadrant. The six CP-violating phases $\delta_{ij}$ may take arbitrary values between 0 and $2\pi$. Note that $V$ can also be decomposed into a Dirac-like flavor mixing matrix with six rotation angles and three CP-violating phases, multiplied by a diagonal phase matrix with three Majorana phases. In normal neutrino-neutrino and antineutrino-antineutrino oscillations, CP-violating phases of the Majorana nature are unable to be measured.

### A.2 Invariants of CP Violation

No matter whether neutrinos are Dirac or Majorana particles, one may define the Jarlskog rephasing invariants of four-neutrino mixing [80], which govern the CP- and T-violating effects in normal neutrino-neutrino and antineutrino-antineutrino oscillations. To be explicit, we have

$$J_{\alpha\beta}^{ij} \equiv \text{Im} \left( V_{\alpha i} V_{\beta j}^* V_{\alpha j} V_{\beta i}^* \right),$$ (A.5)

where the Greek subscripts run over $(s, e, \mu, \tau)$ and the Latin superscripts run over $(0, 1, 2, 3)$. Of course, $J_{\alpha\beta}^{ii} = J_{\alpha\alpha}^{ii} = 0$ and $J_{\alpha\beta}^{ij} = -J_{\alpha\beta}^{ji} = J_{\alpha\beta}^{\alpha\beta} = J_{\alpha\beta}^{\beta\alpha}$ hold by definition.
The unitarity of $V$ leads to the following correlation equations of $J_{ij}^{a\beta}$:

$$
\sum_i J_{ij}^{a} = \sum_j J_{ij}^{a} = \sum_{\beta} J_{ij}^{a\beta} = \sum_{\alpha} J_{ij}^{a\alpha} = 0 .
$$

(A.6)

Hence there are totally nine independent $J_{ij}^{a\beta}$, whose magnitudes depend only upon three of the six CP-violating phases or their combinations in a specific parametrization of $V$.

We use the standard parametrization of $V$ in (A.4) to calculate $J_{ij}^{a\beta}$. After some lengthy but straightforward calculations, we obtain the exact analytical expressions of nine independent $J_{ij}^{a\beta}$ [140]:

$$
J_{rs}^{02} = c_{01}c_{02}c_{03}c_{12}c_{13}c_{23} s_{02}s_{03}s_{23} \sin \phi_x + c_{01}c_{02}c_{03}c_{12}c_{13}c_{23}s_{01} s_{02}s_{03}s_{13} \sin \phi_y + (c_{12}^2s_{13}^2 - s_{12}^2) c_{01}c_{12}c_{03}c_{13}s_{01}s_{02}s_{03}s_{12}s_{13} \sin(\phi_x - \phi_y) - c_{01}c_{02}c_{03}c_{13}c_{23}s_{01}s_{02}s_{03}s_{23} \sin(\phi_x + \phi_z) + c_{01}c_{02}c_{03}c_{13}c_{23}s_{01}s_{02}s_{03}s_{12}s_{13} \sin(\phi_y - \phi_x) - c_{01}c_{02}c_{03}c_{13}c_{23}s_{01}s_{02}s_{03}s_{12}s_{13} \sin(\phi_x - \phi_y + 2\phi_z) ,
$$

(A.7)

and

$$
J_{rs}^{02} = c_{01}c_{02}c_{03}c_{12}c_{13}s_{01}s_{02}s_{03}s_{13} \sin \phi_y - c_{01}c_{02}c_{03}c_{12}c_{13}s_{01}s_{02}s_{03}s_{12}s_{13} \sin \phi_x ,
$$

$$
J_{rs}^{13} = c_{01}c_{02}c_{03}c_{13}c_{12}s_{01}s_{03}s_{13} \sin \phi_y - c_{01}c_{02}c_{03}c_{13}c_{03}s_{01} s_{02}s_{03}s_{12}s_{13} \sin(\phi_y - \phi_z) ,
$$

(A.8)

as well as

$$
J_{rs}^{12} = - (c_{13}^2s_{02}^2 - s_{13}^2) c_{01}c_{02}c_{03}c_{12}c_{13}c_{23}s_{02}s_{03}s_{23} \sin \phi_x + (c_{02}^2s_{12}^2 - c_{12}^2c_{23}^2 - s_{02}^2s_{03}^2 - s_{12}^2s_{23}^2) c_{01}c_{02}c_{03}c_{12}c_{13}s_{01}s_{03}s_{13} \sin \phi_y + (c_{13}^2s_{02}^2 - c_{13}s_{02}s_{03}s_{23} - c_{13}s_{03}s_{12}s_{23} + s_{02}s_{03}s_{12}s_{23}) c_{01}c_{02}c_{03}c_{12}c_{13}s_{01}s_{02}s_{12} \sin \phi_z + (c_{02}^2s_{12}^2 - c_{03}^2s_{03}^2 - c_{02}s_{03}s_{23} - c_{02}s_{03}s_{12} + s_{02}s_{03}s_{12}s_{23}) c_{01}c_{02}c_{03}c_{13}c_{03}s_{01}s_{02}s_{03}s_{12}s_{13} \sin(\phi_x + \phi_z) + c_{01}c_{02}c_{03}c_{12}c_{13}s_{01}s_{03}s_{13} \sin(\phi_y + \phi_z) - c_{01}c_{02}c_{03}c_{12}c_{13}s_{01}s_{03}s_{23} \sin(\phi_x + \phi_z + 2\phi_z) - c_{01}c_{02}c_{03}c_{13}c_{03}s_{01}s_{02}s_{03}s_{12}s_{13} \sin(\phi_y + \phi_z) - c_{01}c_{02}c_{03}c_{13}c_{03}s_{01}s_{02}s_{03}s_{12}s_{13} \sin(\phi_y + \phi_z + 2\phi_z),
$$

(A.9)
where
\[
\begin{align*}
\phi_x & \equiv \delta_{03} - \delta_{02} - \delta_{23} , \\
\phi_y & \equiv \delta_{03} - \delta_{01} - \delta_{13} , \\
\phi_z & \equiv \delta_{02} - \delta_{01} - \delta_{12} .
\end{align*}
\]

With the help of (A.6), one may easily derive the expressions of all the other rephasing invariants of CP and T violation from (A.7), (A.8) and (A.9). The results obtained above are expected to be very useful for a systematic study of CP- and T-violating effects in the four-neutrino mixing models. The same results are also applicable for the discussion of CP and T violation in the four-quark mixing models [77, 139].

Note that all CP- and T-violating observables in neutrino oscillations must be related linearly to \( J_{\alpha\beta}^{ij} \). To see this point more clearly, we consider that a neutrino \( \nu_\alpha \) converts to another neutrino \( \nu_\beta \) in vacuum. The probability of this conversion is given by
\[
P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4 \sum_{i<j} \left[ \Re \left( V_{\alpha i} V_{\beta j}^* V_{\alpha j}^* V_{\beta i}^* \right) \sin^2 F_{ji} \right] - 2 \sum_{i<j} \left( J_{\alpha\beta}^{ij} \sin 2 F_{ji} \right) ,
\]
where \( F_{ji} \equiv 1.27 \Delta m_{ji}^2 L / E \) with \( \Delta m_{ji}^2 \equiv m_j^2 - m_i^2 \), \( L \) stands for the baseline length (in unit of km), and \( E \) is the neutrino beam energy (in unit of GeV). CPT invariance assures that the transition probabilities \( P(\nu_\beta \to \nu_\alpha) \) and \( P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \) are identical, and they can directly be read off from (A.11) through the replacement \( J_{\alpha\beta}^{ij} \to -J_{\alpha\beta}^{ij} \) (i.e., \( V \to V^* \)). Thus the CP-violating asymmetry between \( P(\nu_\alpha \to \nu_\beta) \) and \( P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \) is equal to the T-violating asymmetry between \( P(\nu_\alpha \to \nu_\beta) \) and \( P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \). The latter can be explicitly and compactly expressed as follows [140]:
\[
\Delta P_{\alpha\beta} \equiv P(\nu_\beta \to \nu_\alpha) - P(\nu_\alpha \to \nu_\beta)
= 16 \left( J_{\alpha\beta}^{12} \sin F_{21} \sin F_{31} \sin F_{32} + J_{\alpha\beta}^{01} \sin F_{10} \sin F_{30} \sin F_{31} \\
+ J_{\alpha\beta}^{02} \sin F_{20} \sin F_{30} \sin F_{32} \right) .
\]

(A.12)

Equivalently, one may obtain
\[
\Delta P_{\alpha\beta} = 16 \left( J_{\alpha\beta}^{23} \sin F_{21} \sin F_{31} \sin F_{32} - J_{\alpha\beta}^{02} \sin F_{10} \sin F_{20} \sin F_{21} \\
- J_{\alpha\beta}^{03} \sin F_{10} \sin F_{30} \sin F_{31} \right) ,
\]

(A.13)
or
\[
\Delta P_{\alpha\beta} = 16 \left( J_{\alpha\beta}^{31} \sin F_{21} \sin F_{31} \sin F_{32} + J_{\alpha\beta}^{01} \sin F_{10} \sin F_{20} \sin F_{21} \\
- J_{\alpha\beta}^{03} \sin F_{20} \sin F_{30} \sin F_{32} \right) .
\]

(A.14)

In getting Eqs. (A.12)–(A.14), the equality
\[
\sin 2 F_{ij} + \sin 2 F_{jk} + \sin 2 F_{ki} = -4 \sin F_{ij} \sin F_{jk} \sin F_{ki}
\]
and (A.6) have been used. Only three of the twelve asymmetries \( \Delta P_{\alpha\beta} \) are independent, and they probe three of the six CP-violating phases (or their combinations) of \( V \). Since only the transitions between active neutrinos can in practice be measured, the useful probability asymmetries for the study of leptonic CP and T violation in long-baseline neutrino oscillation experiments are \( \Delta P_{e\mu}, \Delta P_{\mu\tau} \) and \( \Delta P_{\tau\nu} \).
invariants of CP violation

twelve unitarity quadrangles. As a whole, there are totally thirty-six different topologies among
arise from different orderings of its four sides, and their areas are apparently different



\[ \begin{array}{c}
V_{e_2}V_{e_2}^* \\
V_{s_3}V_{e_1} \\
V_{s_1}V_{e_1} \\
V_{s_1}V_{e_0}^* \\
(\text{a})
\end{array} \quad \begin{array}{c}
V_{s_0}V_{e_0}^* \\
V_{s_1}V_{e_1}^* \\
V_{s_2}V_{e_1} \\
V_{s_3}V_{e_0} \\
(\text{b})
\end{array} \quad \begin{array}{c}
V_{s_0}V_{e_0}^* \\
V_{s_2}V_{e_2}^* \\
V_{s_1}V_{e_2} \\
V_{s_3}V_{e_0} \\
(\text{c})
\end{array} \]

Figure A.2: Three distinct topologies of unitarity quadrangle \( Q_{se} \).

A.3 Leptonic Unitarity Quadrangles

The unitarity of \( V \) implies that there exist twelve orthogonality relations and eight normalization conditions among its sixteen matrix elements. The former corresponds to twelve quadrangles in the complex plane, the so-called unitarity quadrangles. To be explicit, let us write out the twelve orthogonality relations and name their corresponding quadrangles [141]:

\[
\begin{align*}
Q_{se} & : \quad V_{s_0}V_{e_0}^* + V_{s_1}V_{e_1}^* + V_{s_2}V_{e_2}^* + V_{s_3}V_{e_3}^* = 0 , \\
Q_{sj} & : \quad V_{s_0}V_{\mu_0}^* + V_{s_1}V_{\mu_1}^* + V_{s_2}V_{\mu_2}^* + V_{s_3}V_{\mu_3}^* = 0 , \\
Q_{s\tau} & : \quad V_{s_0}V_{\tau_0}^* + V_{s_1}V_{\tau_1}^* + V_{s_2}V_{\tau_2}^* + V_{s_3}V_{\tau_3}^* = 0 , \\
Q_{e\mu} & : \quad V_{e_0}V_{\mu_0}^* + V_{e_1}V_{\mu_1}^* + V_{e_2}V_{\mu_2}^* + V_{e_3}V_{\mu_3}^* = 0 , \\
Q_{e\tau} & : \quad V_{e_0}V_{\tau_0}^* + V_{e_1}V_{\tau_1}^* + V_{e_2}V_{\tau_2}^* + V_{e_3}V_{\tau_3}^* = 0 , \\
Q_{\mu\tau} & : \quad V_{\mu_0}V_{\tau_0}^* + V_{\mu_1}V_{\tau_1}^* + V_{\mu_2}V_{\tau_2}^* + V_{\mu_3}V_{\tau_3}^* = 0 ; \\
& \quad (A.16)
\end{align*}
\]

and

\[
\begin{align*}
Q_{01} & : \quad V_{s_0}V_{s_1}^* + V_{e_0}V_{e_1}^* + V_{s_0}V_{\mu_1}^* + V_{s_0}V_{\tau_1}^* = 0 , \\
Q_{02} & : \quad V_{s_0}V_{s_2}^* + V_{e_0}V_{e_2}^* + V_{s_0}V_{\mu_2}^* + V_{s_0}V_{\tau_2}^* = 0 , \\
Q_{03} & : \quad V_{s_0}V_{s_3}^* + V_{e_0}V_{e_3}^* + V_{s_0}V_{\mu_3}^* + V_{s_0}V_{\tau_3}^* = 0 , \\
Q_{12} & : \quad V_{s_1}V_{s_2}^* + V_{e_1}V_{e_2}^* + V_{s_1}V_{\mu_2}^* + V_{s_1}V_{\tau_2}^* = 0 , \\
Q_{13} & : \quad V_{s_1}V_{s_3}^* + V_{e_1}V_{e_3}^* + V_{s_1}V_{\mu_3}^* + V_{s_1}V_{\tau_3}^* = 0 , \\
Q_{23} & : \quad V_{s_2}V_{s_3}^* + V_{e_2}V_{e_3}^* + V_{s_2}V_{\mu_3}^* + V_{s_2}V_{\tau_3}^* = 0 . \\
& \quad (A.17)
\end{align*}
\]

If six mixing angles and six CP-violating phases of \( V \) are all known, one can plot twelve unitarity quadrangles without ambiguities. Note, however, that each quadrangle has three distinct topologies in the complex plane. For illustration, we take quadrangle \( Q_{se} \) for example and show its three topologies in Fig. A.2, where the sizes and phases of \( V_{s_i}V_{\alpha_i}^* \) (for \( i = 0, 1, 2, 3 \)) have been fixed. One can see that different topologies of quadrangle \( Q_{se} \) arise from different orderings of its four sides, and their areas are apparently different from one another. As a whole, there are totally thirty-six different topologies among twelve unitarity quadrangles.

Now we calculate the areas of all unitarity triangles and relate them to the rephasing invariants of CP violation \( J_{\alpha\beta}^{ij} \). The results for thirty-six different topologies of twelve
unitarity quadrangles are summarized as follows\textsuperscript{18}:

\[
S_{\alpha\beta}^a = \frac{1}{4} \left( J_{10}^{\alpha\beta} + J_{12}^{\alpha\beta} + J_{30}^{\alpha\beta} + J_{32}^{\alpha\beta} \right),
\]

\[
S_{ij}^a = \frac{1}{4} \left( J_{ij}^{\alpha\beta} + J_{ij}^{\beta\mu} + J_{ij}^{\tau\mu} + J_{ij}^{\sigma\tau} \right),
\]

\[
S_{\alpha\beta}^b = \frac{1}{4} \left( J_{10}^{\alpha\beta} + J_{12}^{\alpha\beta} + J_{31}^{\alpha\beta} + J_{32}^{\alpha\beta} \right),
\]

\[
S_{ij}^b = \frac{1}{4} \left( J_{ij}^{\alpha\beta} + J_{ij}^{\beta\mu} + J_{ij}^{\tau\mu} + J_{ij}^{\sigma\tau} \right),
\]

\[
S_{\alpha\beta}^c = \frac{1}{4} \left( J_{10}^{\alpha\beta} + J_{31}^{\alpha\beta} + J_{23}^{\alpha\beta} + J_{02}^{\alpha\beta} \right),
\]

\[
S_{ij}^c = \frac{1}{4} \left( J_{ij}^{\alpha\beta} + J_{ij}^{\beta\mu} + J_{ij}^{\tau\mu} + J_{ij}^{\sigma\tau} \right),
\]

(A.18)

where the subscripts $\alpha\beta = se, s\mu, s\tau, e\mu, e\tau$ or $\mu\tau$, and $ij = 01, 02, 03, 12, 13$ or $23$. Note that the correlation of $J_{ij}^{\alpha\beta}$ allows us to simplify (A.18). Then each $S_{\alpha\beta}^a$ or $S_{ij}^q$ (for $q = a, b, c$) can be expressed as a sum of two independent Jarlskog invariants. Such simplified expressions of $S_{\alpha\beta}^a$ and $S_{ij}^q$ depend on the choice of independent $J_{ij}^{\alpha\beta}$, therefore they may have many different forms. If nine independent Jarlskog invariants are fixed, however, some expressions of $S_{\alpha\beta}^a$ and $S_{ij}^q$ must consist of three $J_{ij}^{\alpha\beta}$. This point will become clear later on.

As $J_{ij}^{\alpha\beta} = -J_{ij}^{\alpha\beta}$ holds by definition, one may easily obtain $S_{\alpha\beta}^q = -S_{\beta\alpha}^q$ and $S_{ij}^q = -S_{ji}^q$ (for $q = a, b, c$). From the sum rule in (A.6), one can also find

\[
\sum_\alpha S_{\alpha\beta}^q = \sum_\beta S_{\alpha\beta}^q = \sum_i S_{ij}^q = \sum_j S_{ij}^q = 0,
\]

(A.19)

where $\alpha$ or $\beta$ runs over $(s, e, \mu, \tau)$, and $i$ or $j$ runs over $(0, 1, 2, 3)$. In addition to (A.19), the following relations can be derived from (A.18):

\[
-S_{se}^a - S_{se}^b - S_{se}^c + S_{se}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{se}^a - S_{se}^a + S_{se}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{se}^a - S_{se}^a - S_{se}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{se}^a - S_{se}^a - S_{se}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{se}^a - S_{se}^a - S_{se}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

\[
-S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a = -S_{se}^a - S_{12}^a - S_{23}^a - S_{30}^a,
\]

(A.20)

\textsuperscript{18}It should be noted that the areas of unitarity quadrangles under discussion are “algebraic areas”, namely, they can be either positive or negative. Of course, it is always possible to take $S_{se}^a = (|J_{se}^{10}| + |J_{se}^{21}| + |J_{se}^{32}| + |J_{se}^{03}|)/4$ or $S_{se}^a = |J_{se}^{10} + J_{se}^{21} + J_{se}^{32} + J_{se}^{03}|/4$, such that $S_{se}^a$ is definitely positive. We find, however, that the language of “algebraic areas” is simpler and more convenient in the description of unitarity quadrangles. In particular, the algebraic area of unitarity quadrangle $Q_{se}$ in the case of topology (b) means an algebraic sum of the areas of its two disassociated triangles, which have opposite signs. Hence both $S_{se}^a = 0$ and $S_{se}^b < 0$ are in general allowed. As for topologies (a) and (c) of $Q_{se}$, $S_{se}^a > 0$ and $S_{se}^c > 0$ are simply a matter of sign or phase convention.
The correlative relations in (A.19) and (A.20) indicate that there are only nine independent $S_{q,\alpha\beta}^q$ and (or) $S_{q,ij}^q$, corresponding to nine independent $J_{ij}^{ij}$. Without loss of generality, let us choose the following nine independent $S_{q,\alpha\beta}^q$ [141]:

\[
S^a_{se} = \frac{1}{2} \left( J_{se}^{02} - J_{se}^{13} - 2J_{se}^{23} \right),
\]

\[
S^a_{st} = \frac{1}{2} \left( J_{st}^{02} + J_{st}^{03} - J_{st}^{23} \right),
\]

\[
S^a_{e\mu} = \frac{1}{2} \left( -J_{e\mu}^{12} - J_{e\mu}^{13} - J_{e\mu}^{23} \right),
\]

\[
S^b_{se} = \frac{1}{2} \left( -J_{se}^{02} - J_{se}^{13} \right),
\]

\[
S^b_{st} = \frac{1}{2} \left( -J_{st}^{02} + J_{st}^{03} + J_{st}^{23} \right),
\]

\[
S^b_{e\mu} = \frac{1}{2} \left( J_{e\mu}^{12} - J_{e\mu}^{13} - J_{e\mu}^{23} \right),
\]

\[
S^c_{se} = \frac{1}{2} \left( J_{se}^{02} - J_{se}^{13} \right),
\]

\[
S^c_{st} = \frac{1}{2} \left( J_{st}^{02} + J_{st}^{03} + J_{st}^{23} \right),
\]

\[
S^c_{e\mu} = \frac{1}{2} \left( -J_{e\mu}^{12} - J_{e\mu}^{13} + J_{e\mu}^{23} \right).
\]

(A.21)

In terms of six flavor mixing angles and three independent phase combinations of $V$, we have expressed the nine independent $J_{ij}^{ij}$ appearing on the right-hand side of (A.21) in section A.2. Then one may directly obtain the explicit expressions of the nine-independent $S_{q,\alpha\beta}^q$ in terms of the same mixing angles and CP-violating phases.

### A.4 Matter Effects on T Violation

The effective Hamiltonians responsible for the propagation of active and sterile neutrinos in vacuum and in matter can respectively be written as [33]

\[
H_{\text{eff}} = \frac{1}{2E} \left( M_{\nu} M_{\nu}^\dagger \right) = \frac{1}{2E} \left( V D_{\nu}^2 V^\dagger \right),
\]

\[
\tilde{H}_{\text{eff}} = \frac{1}{2E} \left( \tilde{M}_{\nu} \tilde{M}_{\nu}^\dagger \right) = \frac{1}{2E} \left( \tilde{V} D_{\nu}^2 \tilde{V}^\dagger \right),
\]

(A.22)

where $D_{\nu} \equiv \text{Diag}\{m_0, m_1, m_2, m_3\}$ with $m_i$ being the fundamental neutrino masses in vacuum, $\tilde{D}_{\nu} \equiv \text{Diag}\{\tilde{m}_0, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3\}$ with $\tilde{m}_i$ being the effective neutrino masses in matter, and $E \gg m_i$ denotes the neutrino beam energy. The deviation of $\tilde{H}_{\text{eff}}$ from $H_{\text{eff}}$ is given by

\[
\Delta H_{\text{eff}} \equiv \tilde{H}_{\text{eff}} - H_{\text{eff}} = \begin{pmatrix}
0' & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(A.23)

where $a = \sqrt{2} G_F N_e$ and $a' = \sqrt{2} G_F N_n/2$ with $N_e$ and $N_n$ being the background densities of electrons and neutrons [142], respectively. In the literature one often assumes
a constant earth density profile (i.e., \( N_e = \text{constant} \) and \( N_n = \text{constant} \)), which is a good approximation for all of the presently-proposed long-baseline neutrino experiments.

Now let us introduce the commutators of \( 4 \times 4 \) lepton mass matrices to describe the mixing of one sterile and three active neutrinos. Without loss of any generality, we continue to work in the afore-chosen flavor basis, where \( M_l \) takes the diagonal form \( D_l = \text{Diag}\{m_s, m_e, m_\mu, m_\tau\} \) with \( m_s = 0 \). Note that we have assumed the \((1, 1)\) element of \( D_l \) to be zero, because there is no counterpart of the sterile neutrino \( \nu_s \) in the charged lepton sector. We shall see later on that our physical results are completely independent of \( m_s \), no matter what value it may take. The commutator of lepton mass matrices in vacuum and that in matter can then be defined as

\[
C \equiv i \left[ M_\nu M_\nu^\dagger, M_l M_l^\dagger \right] = i \left[ V D_\nu^2 V^\dagger, D_l^2 \right],
\]

\[
\tilde{C} \equiv i \left[ \tilde{M}_\nu \tilde{M}_\nu^\dagger, M_l M_l^\dagger \right] = i \left[ \tilde{V} \tilde{D}_\nu^2 \tilde{V}^\dagger, D_l^2 \right].
\]

(A.24)

Obviously \( C \) and \( \tilde{C} \) are traceless Hermitian matrices. In terms of neutrino masses and flavor mixing matrix elements, we obtain the explicit expressions of \( C \) and \( \tilde{C} \) as follows:

\[
C = i \begin{pmatrix}
0 & \Delta_{es} Z_{se} & \Delta_{e\mu} Z_{e\mu} & \Delta_{e\tau} Z_{e\tau} \\
\Delta_{se} Z_{es} & 0 & \Delta_{\mu\mu} Z_{\mu\mu} & \Delta_{\mu\tau} Z_{\mu\tau} \\
\Delta_{e\mu} Z_{e\mu} & \Delta_{\mu\mu} Z_{\mu\mu} & 0 & \Delta_{\tau\mu} Z_{\tau\mu} \\
\Delta_{e\tau} Z_{e\tau} & \Delta_{\mu\tau} Z_{\mu\tau} & \Delta_{\tau\mu} Z_{\tau\mu} & 0
\end{pmatrix},
\]

\[
\tilde{C} = i \begin{pmatrix}
0 & \Delta_{es} \tilde{Z}_{se} & \Delta_{e\mu} \tilde{Z}_{e\mu} & \Delta_{e\tau} \tilde{Z}_{e\tau} \\
\Delta_{se} \tilde{Z}_{es} & 0 & \Delta_{\mu\mu} \tilde{Z}_{\mu\mu} & \Delta_{\mu\tau} \tilde{Z}_{\mu\tau} \\
\Delta_{e\mu} \tilde{Z}_{e\mu} & \Delta_{\mu\mu} \tilde{Z}_{\mu\mu} & 0 & \Delta_{\tau\mu} \tilde{Z}_{\tau\mu} \\
\Delta_{e\tau} \tilde{Z}_{e\tau} & \Delta_{\mu\tau} \tilde{Z}_{\mu\tau} & \Delta_{\tau\mu} \tilde{Z}_{\tau\mu} & 0
\end{pmatrix},
\]

(A.25)

where \( \Delta_{\alpha\beta} \equiv m_{\alpha}^2 - m_{\beta}^2 \) for \( \alpha \neq \beta \) running over \((s, e, \mu, \tau)\), and

\[
Z_{\alpha\beta} \equiv \sum_{i=0}^{3} \left( m_i^2 V_{\alpha i} V_{\beta i}^* \right),
\]

\[
\tilde{Z}_{\alpha\beta} \equiv \sum_{i=0}^{3} \left( \bar{m}_i^2 \bar{V}_{\alpha i} \bar{V}_{\beta i}^* \right).
\]

(A.26)

One can see that \( \Delta_{\beta\alpha} = -\Delta_{\alpha\beta} \), \( Z_{\beta\alpha} = Z_{\alpha\beta}^* \) and \( \tilde{Z}_{\beta\alpha} = \tilde{Z}_{\alpha\beta}^* \) hold. To find out how \( \tilde{Z}_{\alpha\beta} \) is related to \( Z_{\alpha\beta} \), we need to establish the relation between \( \tilde{C} \) and \( C \). Taking account of (A.22) and (A.23), we immediately obtain

\[
\tilde{C} = 2iE \left[ \hat{H}_{\text{eff}}, D_l^2 \right] = C + 2iE \left[ \Delta \hat{H}_{\text{eff}}, D_l^2 \right] = C.
\]

(A.27)

This interesting result indicates that the commutator of lepton mass matrices in vacuum is invariant under terrestrial matter effects. As a straightforward consequence of \( \tilde{C} = C \), we arrive at \( \tilde{Z}_{\alpha\beta} = Z_{\alpha\beta} \) from (A.25); i.e.,

\[
\sum_{i=0}^{3} \left( \bar{m}_i^2 \bar{V}_{\alpha i} \bar{V}_{\beta i}^* \right) = \sum_{i=0}^{3} \left( m_i^2 V_{\alpha i} V_{\beta i}^* \right);
\]

(A.28)
or equivalently
\[ \sum_{i=1}^{3} (\Delta_{i0} V_{\alpha i} \bar{V}_{\beta i}^*) = \sum_{i=1}^{3} (\Delta_{i0} V_{\alpha i} V_{\beta i}^*) , \] (A.29)

where \( \Delta_{i0} \equiv m_i^2 - m_0^2 \) and \( \tilde{\Delta}_{i0} \equiv \bar{m}_i^2 - \bar{m}_0^2 \) for \( i = 1, 2, 3 \). It becomes obvious that the validity of (A.28) or (A.29) has nothing to do with the assumption of \( m_s = 0 \) in the charged lepton sector. Note that the results obtained above are only valid for neutrinos propagating in vacuum and in matter. As for antineutrinos, the corresponding sum rule can straightforwardly be written out from (A.28) or (A.29) through the replacements \( V \mapsto V^*, a \mapsto -a \) and \( a' \mapsto -a' \).

To describe CP or T violation in neutrino oscillations, we consider the rephasing-invariant relationship \( Z_{\alpha \beta} Z_{\beta \gamma} \tilde{Z}_{\gamma \alpha} = Z_{\alpha \beta} Z_{\beta \gamma} Z_{\gamma \alpha} \) for \( \alpha \neq \beta \neq \gamma \) running over \( (s, e, \mu, \tau) \). The imaginary parts of \( Z_{\alpha \beta} Z_{\beta \gamma} Z_{\gamma \alpha} \) and \( \tilde{Z}_{\alpha \beta} \tilde{Z}_{\beta \gamma} \tilde{Z}_{\gamma \alpha} \) read explicitly as
\[ \text{Im}(Z_{\alpha \beta} Z_{\beta \gamma} Z_{\gamma \alpha}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \left[ \Delta_{i0} \Delta_{j0} \Delta_{k0} \text{Im} \left( V_{\alpha i} V_{\beta j} V_{\gamma k} V_{\alpha k} V_{\beta i}^* V_{\gamma j}^* \right) \right] , \] (A.30)

and their equality allows us to derive an interesting relation between the Jarlskog invariant \( J_{ij}^{ij} \) in vacuum and its effective counterpart in matter,
\[ \tilde{J}_{ij}^{ij} \equiv \text{Im} \left( \bar{V}_{\alpha i} \bar{V}_{\beta j} \bar{V}_{\alpha j} \bar{V}_{\beta i}^* \right) , \] (A.31)

where the Greek subscripts run over \( (s, e, \mu, \tau) \) and the Latin superscripts run over \( (0, 1, 2, 3) \). The result is [98]
\[ \tilde{\Delta}_{10} \Delta_{20} \Delta_{30} \sum_{i=1}^{3} \left( J_{ij}^{0i} |V_{\gamma i}|^2 + J_{ij}^{0i} |V_{\alpha i}|^2 + J_{ij}^{0i} |V_{\beta i}|^2 \right) \\
+ \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \left[ \Delta_{i0} \Delta_{j0} \Delta_{k0} \left( \tilde{J}_{ij}^{0i} |\bar{V}_{\gamma i}|^2 + \tilde{J}_{ij}^{0i} |\bar{V}_{\alpha i}|^2 + \tilde{J}_{ij}^{0i} |\bar{V}_{\beta i}|^2 \right) \right] \\
= \Delta_{10} \Delta_{20} \Delta_{30} \sum_{i=1}^{3} \left( J_{ij}^{0i} |V_{\gamma i}|^2 + J_{ij}^{0i} |V_{\alpha i}|^2 + J_{ij}^{0i} |V_{\beta i}|^2 \right) \\
+ \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \left[ \Delta_{i0} \Delta_{j0} \Delta_{k0} \left( J_{ij}^{0i} |V_{\gamma i}|^2 + J_{ij}^{0i} |V_{\alpha i}|^2 + J_{ij}^{0i} |V_{\beta i}|^2 \right) \right] . \] (A.32)

If one “switches off” the mass of the sterile neutrino and its mixing with active neutrinos (i.e., \( a' = 0 \), \( \Delta_{i0} = m_i^2 \), \( \tilde{\Delta}_{i0} = \bar{m}_i^2 \), \( J_{ij}^{0i} = 0 \), and \( \tilde{J}_{ij}^{0i} = 0 \)), then (A.32) is simplified to the elegant Naumov form [111], as shown in (4.4).

The matter-corrected CP-violating parameters \( J_{ij}^{ij} \) can, at least in principle, be determined from the measurement of CP- and T-violating effects in a variety of long-baseline neutrino oscillation experiments. The conversion probability of a neutrino \( \nu_\alpha \) to another neutrino \( \nu_\beta \) in matter is given as
\[ \tilde{P}(\nu_\alpha \rightarrow \nu_\beta) = \delta_{\alpha \beta} - 4 \sum_{i<j} \left[ \text{Re} \left( \bar{V}_{\alpha i} \bar{V}_{\beta j} \bar{V}_{\alpha j} \bar{V}_{\beta i}^* \right) \sin^2 \tilde{F}_{ji} \right] - 2 \sum_{i<j} \left[ J_{ij}^{ij} \sin(2\tilde{F}_{ji}) \right] , \] (A.33)
where $\tilde{F}_{ji} = 1.27\Delta_{ji}L/E$ with $\Delta_{ji} \equiv \tilde{m}_i^2 - \tilde{m}_j^2$. The transition probability $\tilde{P}(\nu_\beta \rightarrow \nu_\alpha)$ can directly be read off from (A.33), if the replacements $\tilde{J}_{ij}^{\alpha\beta} \rightarrow -\tilde{J}_{ij}^{\beta\alpha}$ are made. To obtain the probability $\tilde{P}(\tilde{\nu}_\alpha \rightarrow \tilde{\nu}_\beta)$, however, both the replacements $\tilde{J}_{ij}^{\alpha\beta} \rightarrow -\tilde{J}_{ij}^{\beta\alpha}$ and $(a, a') \rightarrow (-a, -a')$ need be made for (A.33). In this case, $\tilde{P}(\tilde{\nu}_\alpha \rightarrow \tilde{\nu}_\beta)$ is not equal to $\tilde{P}(\nu_\beta \rightarrow \nu_\alpha)$. The difference between $\tilde{P}(\tilde{\nu}_\alpha \rightarrow \tilde{\nu}_\beta)$ and $\tilde{P}(\nu_\beta \rightarrow \nu_\alpha)$ is a false signal of CPT violation, induced actually by the matter effect [73]. Thus the CP-violating asymmetry between $\tilde{P}(\nu_\alpha \rightarrow \nu_\beta)$ and $\tilde{P}(\nu_\beta \rightarrow \nu_\alpha)$ is realistic measured, we are more interested in the T-violating asymmetries. Thus there should not have large cancellation of matter effects in the corresponding CP-violating asymmetries.

\[
\Delta \tilde{P}_{\alpha\beta} \equiv \tilde{P}(\nu_\beta \rightarrow \nu_\alpha) - \tilde{P}(\nu_\alpha \rightarrow \nu_\beta) 
= 16 \left( \tilde{J}_{23}^{23} \sin \tilde{F}_{21} \sin \tilde{F}_{31} \sin \tilde{F}_{32} - \tilde{J}_{13}^{02} \sin \tilde{F}_{10} \sin \tilde{F}_{20} \sin \tilde{F}_{21} 
- \tilde{J}_{23}^{03} \sin \tilde{F}_{10} \sin \tilde{F}_{30} \sin \tilde{F}_{31} \right),
\]

(A.34)

If the hierarchical patterns of neutrino masses and flavor mixing angles are assumed, the expression of $\Delta \tilde{P}_{\alpha\beta}$ may somehow be simplified [143]. Note that only three of the twelve nonvanishing asymmetries $\Delta \tilde{P}_{\alpha\beta}$ are independent, as a consequence of the unitarity of $\tilde{V}$ or the correlation of $\tilde{J}_{ij}^{\alpha\beta}$. Since only the transition probabilities of active neutrinos can be realistically measured, we are more interested in the T-violating asymmetries $\Delta \tilde{P}_{\mu\tau}, \Delta \tilde{P}_{\mu\nu}$ and $\Delta \tilde{P}_{\tau\nu}$. The overall matter contamination residing in $\Delta \tilde{P}_{\alpha\beta}$ is usually to be insignificant. The reason is simply that the terrestrial matter effects in $\tilde{P}(\nu_\alpha \rightarrow \nu_\beta)$ and $\tilde{P}(\nu_\beta \rightarrow \nu_\alpha)$, which both depend on the parameters $(a, a')$, may partly (even essentially) cancel each other in the T-violating asymmetry $\Delta \tilde{P}_{\alpha\beta}$. In contrast, $\tilde{P}(\nu_\alpha \rightarrow \nu_\beta)$ and $\tilde{P}(\tilde{\nu}_\alpha \rightarrow \tilde{\nu}_\beta)$ are associated respectively with $(+a, +a')$ and $(-a, -a')$, thus there should not have large cancellation of matter effects in the corresponding CP-violating asymmetries.

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\(^{19}\)Note that the differences of effective neutrino masses $\tilde{\Delta}_{\alpha\beta} (\text{for } i = 1, 2, 3)$, which must be CP-conserving, keep unchanged for the replacements $\tilde{J}_{ij}^{\alpha\beta} \rightarrow -\tilde{J}_{ij}^{\beta\alpha}$. Therefore the sign flip of $\tilde{J}_{ij}^{\alpha\beta}$ results in that of $\tilde{J}_{ij}^{\alpha\beta}$, as indicated by the sum rules given in (A.32).
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