A Unitary $S$-matrix for 2D Black Hole Formation and Evaporation

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Abstract

We study the black hole information paradox in the context of a two-dimensional toy model given by dilaton gravity coupled to $N$ massless scalar fields. After making the model well-defined by imposing reflecting boundary conditions at a critical value of the dilaton field, we quantize the theory and derive the quantum $S$-matrix for the case that $N=24$. This $S$-matrix is unitary by construction, and we further argue that in the semiclassical regime it describes the formation and subsequent Hawking evaporation of two-dimensional black holes. Finally, we note an interesting correspondence between the dilaton gravity $S$-matrix and that of the $c = 1$ matrix model.
1. Introduction

The discovery that black holes can evaporate by emitting thermal radiation has led to a longstanding controversy about whether or not quantum coherence can be maintained in this process. Hawking’s original calculation [1] suggests that an initial state, describing matter collapsing into a black hole, will eventually evolve into a mixed state describing the thermal radiation emitted by the black hole. The quantum physics of black holes thus seems inherently unpredictable. However, this is clearly an unsatisfactory conclusion, and several attempts have been made to find a description of black hole evaporation in accordance with the rules of quantum mechanics [2, 3], but so far all these attempts have run into serious difficulties. Classically, the loss of information takes place at the singularity of the black hole, which forms a space-like boundary of space-time at which the evolution stops. While it is conceivable that quantum effects may alter this picture by smoothing out the singularity, it is still difficult to see how the information can be retrieved from behind the black hole horizon without a macroscopic violation of causality. Nevertheless, it seems worthwhile to investigate this possibility.

We would like to study some of these issues with the help of a toy model, with the hope that it will capture some of the essential features of the full theory. Recently, there has been considerable interest in 1+1 dimensional dilaton gravity, described by the Lagrangian [6]

\[
S = \frac{1}{2\pi} \int \sqrt{-g} \left[ e^{-2\phi} (R + 4(\nabla \phi)^2 + 4\lambda^2) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right] \quad (1.1)
\]

Here \( \phi \) is the dilaton and \( f_i \) are massless scalar fields. The main virtue of this model is that it is completely soluble, at least classically, while it also possesses classical black hole solutions [4, 5]. A further motivation for studying this model is that, up to a factor of 2 in front of the dilaton kinetic term, the gravitational part of the action (1.1) is identical to the spherically symmetric reduction of the 3+1 dimensional Einstein action.

The information paradox also arises in the context of this toy model, since on the one hand there appears to be no obvious reason why the Lagrangian (1.1) would not describe a well-defined 1+1 dimensional quantum field theory. It even looks exactly soluble and hence it should have some well-defined unitary \( S \)-matrix. On the other hand, semi-classical studies [6]–[10] show that the 1+1 dimensional black holes in this model are unstable and evaporate via the Hawking process. Thus dilaton gravity provides a simple and interesting testing ground in which one can explicitly address the paradox and decide which of the two scenarios is realized.
In this paper we will provide evidence in favor of the first possibility. Namely, we will explicitly construct the quantum theory described by (1.1) in the special case that $N=24$, and after imposing appropriate boundary conditions, derive a well-defined unitary scattering matrix. To explain our approach, we begin in section 2 with a summary of the (semi-)classical properties of the model. The quantization of dilaton gravity and the construction of the $S$-matrix are explained in sections 3 and 4. Finally, in section 5 we discuss some of the properties of the $S$-matrix and clarify its physical interpretation. We will argue that in the semiclassical limit it reproduces the expected semiclassical physics of Hawking radiation. We also point out an interesting correspondence with the scattering equations of the $c=1$ matrix model [14, 15]. Finally, we compare our result with the black hole $S$-matrix proposed by ‘t Hooft [3].

2. The (Semi-)Classical Model.

Dilaton gravity has no local gravitational degrees of freedom: the classical equations of motion of (1.1) uniquely determine the metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ and dilaton field $\phi$ for a given energy-momentum distribution. Explicitly, one can choose a preferred set of coordinates $x^\pm$ such that the classical metric and dilaton are related via

$$ds^2 = e^{2\rho}dx^+dx^-, \quad \rho = \phi. \quad (2.1)$$

In these coordinates the remaining equations of motion read

$$\partial_+ \partial_- e^{-2\phi} = -\lambda^2$$
$$\partial_\pm^2 e^{-2\phi} = T_{\pm\pm} \quad (2.2)$$

where $T_{\pm\pm} = \sum_i \frac{1}{2}(\partial_{\pm} f_i)^2$ is the traceless matter energy-momentum tensor. These equations can be integrated to

$$e^{-2\phi} = M - \lambda^2 x^+ x^- - \int_{-\infty}^{\infty} dy^+ \int_{-\infty}^{\infty} dz^+ T_{++} - \int_{-\infty}^{\infty} dy^- \int_{-\infty}^{\infty} dz^- T_{--} \quad (2.3)$$

If we put the $T_{\pm\pm} = 0$ this reduces to the static black hole solution of mass $M$. The terms involving $T_{\pm\pm}$ represent the classical back reaction of the metric due to the incoming or outgoing matter.
We are interested in the situation where a black hole is formed by incoming matter, but where initially there is no black hole. In this case the geometry in the far past is that of the linear dilaton vacuum

$$ds^2 = -\frac{dx^+dx^-}{\lambda^2x^+x^-}; \quad e^{-2\phi} = -\lambda^2x^+x^-$$  \hspace{1cm} (2.4)

Here the coordinates $x^\pm$ are restricted to a half-line, $x^+ > 0$ and $x^- < 0$; they are related to the standard flat coordinates $r$ and $t$ for which $ds^2 = -dt^2 + dr^2$ via

$$x^\pm = \pm \exp(\lambda(t \pm r))$$  \hspace{1cm} (2.5)

The field $e^{\phi}$ is known to play the role of the coupling constant of the model, and becomes infinite for $r \to -\infty$. It is therefore not appropriate to treat this strong coupling regime as an asymptotic region of space-time, but rather one would need to specify some physically reasonable boundary condition, that makes the model well-defined.

Before we can formulate these boundary conditions, we first need to recall some properties of the semiclassical theory. Namely, it turns out (see [9] and section 4) that in the quantum theory the vacuum carries a negative Casimir energy proportional to the number of scalar fields

$$\langle 0| T_{\pm\pm} |0 \rangle = -\frac{N}{24x^{\pm2}}.$$  \hspace{1cm} (2.6)

The origin of this vacuum energy is that $T_{\pm\pm}$ in equation (2.3) is normal ordered with respect to the ‘Kruskal’ coordinates $x^\pm$, while the asymptotic vacuum is defined in terms of the physical asymptotic coordinates $r$ and $t$. As a consequence, the vacuum solution for the dilaton gravity fields is no longer exactly described by the linear dilaton vacuum (2.4). It is possible, however, to restore this property of the classical theory by introducing a semiclassical dilaton field $\tilde{\phi}$ and metric $\tilde{\rho}$ via

$$e^{-2\phi} = e^{-2\tilde{\phi}} + \frac{N}{12}\tilde{\phi}$$  \hspace{1cm} (2.7)

$$e^{2\rho-2\phi} = e^{2\tilde{\rho}-2\tilde{\phi}}$$

Via this field redefinition one effectively incorporates semiclassical corrections to the equation of motion (2.2), such as the back reaction of the metric due to Hawking radiation

*Note that in the spherically symmetric reduction of the Einstein theory, the line $e^{-2\phi} = 0$ coincides with the origin at $r = 0$ and indeed defines a reflecting boundary.

†In [9] this vacuum energy was taken to be proportional to $N-24$. It will be shown later that (2.6) is the correct value, at least for $N = 24$.  

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In general, however, invariant physical quantities such as an $S$-matrix will not depend on such field redefinitions, and we will therefore continue to work with the original field variables $\phi$ and $\rho$.

The boundary condition we will impose is the same one as proposed by Russo, Susskind and Thorlacius in [9], and is motivated by the following observation. The right-hand side of (2.7) attains a minimum when $e^{-2\tilde{\phi}} = \frac{N}{24}$, and thus only values of $\phi$ larger than a certain critical value $\phi_{cr}$ have a semiclassical interpretation. In other words, the semiclassical theory becomes singular on a certain critical line $x^\pm = x^\pm(\tau)$ at which

$$\phi(x^+, x^-) = \phi_{cr}. \quad (2.8)$$

This critical line is timelike for sufficiently small perturbations around the vacuum, and it is thus natural to identify it with a physical boundary of the 1+1-dimensional space-time at which we can impose reflecting boundary conditions for the $f$-fields. There are many possible choices, but we will for definiteness take the simplest choice and impose Neumann boundary conditions. This choice is coordinate invariant, and implies that there is no net matter energy-momentum flux through the critical line

$$T_{--}(dx^-)^2 = T_{++}(dx^+)^2. \quad (2.9)$$

A natural set of corresponding boundary conditions on the gravitational sector, that are also coordinate invariant, is to demand that the derivative of the dilaton must vanish both along and perpendicular to the boundary

$$\partial_\pm e^{-2\phi}(x^+, x^-) = 0. \quad (2.10)$$

The authors of [9] arrived at an identical set of boundary conditions by imposing the requirement that the semiclassical metric, obtained via (2.7) remains nonsingular at the boundary, and therefore interpreted these conditions as the implementation of cosmic censorship at the critical line $\phi = \phi_{cr}$.

As long as the boundary is time-like, its trajectory is completely determined by the incoming energy flux. By combining equations (2.3) and (2.10), one finds that the boundary trajectory is given by the following elegant equations

$$\lambda^2 x^- = + \int_{x^+}^{\infty} dx^+ T_{++}, \quad (2.11)$$
\[ \lambda^2 x^+ = - \int_{-\infty}^{x^-} dx^- \, T_{--}. \]

Here the second relation follows from the first one by the reflection condition (2.9).

The equations (2.11) will play a central role in the following. As an aside, let us point out that they can in fact be derived from a very simple action principle as follows. Since the boundary trajectory \( x^\pm(\tau) \) represents the only dilaton gravity degree of freedom that couples directly to the matter fields, it should be allowed to eliminate all other gravitational fields from the action via gauge invariance or their equation of motion. If we follow this procedure, we find that the gravitational action can be reduced to a pure boundary term

\[ S_b = \lambda^2 \int d\tau \, x^+ \dot{x}^- \]  

which defines the free dynamics of the critical line. The coupling to matter is described simply by the restricting the integral over \( x^\pm \) in the matter action to the right of the boundary. The equations (2.11) for \( x^\pm \) are then obtained by writing the variation of the matter action as

\[ \delta S_m = \int d\tau \left( T_{++} \delta x^+ + T_{--} \delta x^- \right) \]  

and integrating the resulting equation of motion \( \delta S_b + \delta S_m = 0 \) once with respect to \( \tau \). Thus the model can in a way be thought of as a single quantum mechanical mirror interacting with a free field theory.

When we include the vacuum contribution (2.6) in (2.11) we see that semiclassically there is a low-energy regime for which the boundary stays time-like. In this regime, therefore, there is a well-defined scattering problem that determines the outgoing matter waves from a certain given configuration of incoming matter. However, as soon as the incoming energy density exceeds the Casimir energy of the vacuum, the above semiclassical equations break down and the critical line goes over in a space-like black hole singularity.

This fact appears to be a serious problem in defining the quantum theory, since it is not meaningful to impose reflection conditions on a space-like boundary. Thus at high energies the theory still seems to have the usual problem of information loss. However, near the black-hole singularity quantum fluctuations will be very large and it is not known what new physics may occur here. Hence, to define the model in this regime, we are free to make some assumptions, as long as we do not (drastically) change the known laws of physics in the semiclassical region of space-time. In particular, because the metric loses its classical meaning near the singularity, we are allowed to assume that in the quantum
theory the boundary actually remains time-like. In the following section we work out this idea to construct an $S$-matrix for quantum dilaton gravity.

3. Quantum Dilaton Gravity.

We will now investigate how this semiclassical situation translates to the quantum level. The strategy will be to set up the quantum theory in the low energy regime, where we will adopt the above boundary condition to make the model well-defined. Our aim is to find the quantum mechanical $S$-matrix that describes the scattering of matter off the dynamical boundary. To avoid unnecessary complications due to the conformal anomaly we will restrict our discussion to the critical case, which is dilaton-gravity with 24 scalar fields $f_i$. This special model is by far the simplest, while it still contains all the essential physics. In particular, as will be explained later, it has a semiclassical regime in which black holes are formed and evaporate by emitting Hawking radiation, even though the total conformal anomaly cancels.

We will first describe the quantization of pure dilaton-gravity in the conformal gauge. Later we will combine it with the matter fields (and ghosts) to determine the physical spectrum. In the conformal gauge

$$ds^2 = e^{2\rho}du dv$$

(3.1)

the action of the pure dilaton-gravity theory becomes

$$S = \frac{1}{\pi} \int dudv e^{-2\phi} (2\partial_u \partial_v \rho - 4\partial_u \phi \partial_v \phi + \lambda^2 e^{2\rho}),$$

(3.2)

and the equations of motion can be written as

$$\partial_u \partial_v (\rho - \phi) = 0$$

(3.3)

$$\partial_u \partial_v e^{-2\phi} = \lambda^2 e^{2\rho - 2\phi}.$$

(3.4)

The action (3.2) defines for all values of $\lambda^2$ a conformally invariant field theory. In the quantum theory the modes of the energy-momentum tensor

$$T^a_{uu} = (4\partial_u \rho \partial_u \phi - 2\partial_u^2 \phi) e^{-2\phi}$$

(3.5)
generate a Virasoro algebra with central charge $c = 2$. Furthermore we know that the operator $e^{2\rho - 2\phi}$ representing the cosmological constant must be marginal.

We consider the theory on the Minkowski half-plane with a fixed time-like boundary given by $u = v$. The coordinates $u$ and $v$ differ from the coordinates $x^\pm$ of the previous section by a conformal transformation, that depends on conformal factor $\rho$ and the dilaton field $\phi$. We will now show that in the quantum theory the $x^\pm$-coordinates will appear as a pair of free scalar fields $X^\pm$, and that the complete dilaton gravity can be conveniently reformulated in terms of these fields.

The first equation of motion (3.3) implies that the marginal operator $e^{2\rho - 2\phi}$ factorizes as a product of two chiral components. Therefore, since it has conformal dimension $(1,1)$, we can introduce two chiral scalar fields $X^\pm$ and write

$$e^{2\rho - 2\phi} = \partial_u X^+ \partial_v \tilde{X}^-.$$  \hspace{1cm} (3.6)

Classically the chiral fields $X^+(u)$ and $\tilde{X}^-(v)$ indeed represent the conformal transformation that maps $(u, v)$ on to the coordinates $(x^+, x^-)$ in which $\rho = \phi$. Next we insert this into the second equation of motion (3.4) and the solution after integrating once can be written as

$$\partial_u e^{-2\phi} = \lambda^2 \partial_u X^+ (X^- - \tilde{X}^-)$$
$$\partial_v e^{-2\phi} = -\lambda^2 (X^+ - \tilde{X}^+) \partial_v \tilde{X}^-.$$  \hspace{1cm} (3.7)

Here we introduced two additional chiral fields $X^-(u)$ and $\tilde{X}^+(v)$, that naturally can be combined with $X^+$ and $\tilde{X}^-$ to obtain two ordinary free scalar fields. The boundary conditions on the dilaton gravity fields become very simple in the new variables: the condition that $\partial_u e^{-2\phi} = \partial_v e^{-2\phi} = 0$ at $u = v$ simply translates in to

$$X^\pm(u)|_{u=v} = \tilde{X}^\pm(v)|_{u=v}.$$  \hspace{1cm} (3.8)

Since this boundary condition identifies the left- and right-movers we may from now on drop the tildes and work only with, say, the left-movers $X^\pm(u)$.

Equations (3.6) and (3.7) are not just valid as classical field redefinitions, but with a suitable normal ordering prescription they represent well-defined quantum identifications.

*The following construction in fact follows naturally from the gauge theory formulation of dilaton gravity described in [1].*
of operators. This means that the correlation functions and operator algebra of the dilaton field and the conformal factor are in principle determined by those of $X^\pm$. The operator algebra of the fields $X^\pm$ is most easily obtained by noting that, after substituting the redefinitions into (3.5), the gravitational energy-momentum tensor is given by the familiar free field expression

$$T^g_{uu} = \lambda^2 \partial_u X^+ \partial_u X^-.$$  \hfill (3.9)

From the fact that the fields $X^\pm$ must have the usual scalar operator algebra with the energy-momentum tensor $T^g_{uu}$, it follows that the operators $X^+$ and $X^-$ satisfy the standard free field commutation relations

$$[\partial_u X^+(u_1), X^-(u_2)] = \lambda^{-2} \delta(u_{12}),$$  \hfill (3.10)

with $u_{12} = u_1 - u_2$. Thus we have indeed mapped the pure dilaton gravity theory onto a theory of two free scalar fields. Another method to derive this result is to compute the action $S(X^\pm)$ by substituting (3.6) and (3.7) into (3.2); one obtains the standard free scalar field action.

The correspondence with the semiclassical discussion of the previous section requires that the fields $X^\pm$ are asymptotically identified with the coordinates $x^\pm$ of the linear dilaton vacuum (2.4). This tells us that when $u$ runs from $-\infty$ to $\infty$, $X^+(u)$ should go from 0 to $\infty$ while $X^-(u)$ must go from $-\infty$ to 0 and further that each behaves asymptotically as $e^{\lambda u}$, resp. $e^{-\lambda u}$. Because of these somewhat unconventional asymptotic conditions we can not simply use the standard mode-expansion for the scalar fields $X^\pm$ to construct the dilaton gravity Hilbert space. We find that the only mode-expansion that is consistent with the required asymptotic behaviour is of the form

$$\partial_u X^\pm(u) = \frac{e^{\pm \lambda u}}{\sqrt{2}} + e^{\pm \lambda u} \int d\omega \, x^\pm(\omega) e^{-i\omega u},$$  \hfill (3.11)

where the modes $x^\pm(\omega)$ satisfy

$$[x^+(\omega_1), x^-(\omega_2)] = \lambda^{-2}(\omega + i\lambda) \delta(\omega_1 + \omega_2).$$  \hfill (3.12)

This leads to the following Green function

$$\langle 0|X^+(u_1)X^-(u_2)|0 \rangle = \lambda^{-2} \int_{-\infty}^{\lambda u_{12}} dx \frac{e^x}{x}(1 - \frac{1}{2}x),$$  \hfill (3.13)
where $|0\rangle$ denotes the vacuum state that is annihilated by all modes $x^\pm(\omega)$ with $\omega > 0$. This Green function has the right asymptotic behaviour for large $u_{12}$, while its behaviour at short relative distances has been adjusted such that the conformally normal ordered energy-momentum tensor (3.9) has no vacuum expectation value. This last requirement ensures that the state $|0\rangle$ describes the physical vacuum and fixes the coefficient in front of the first term in (3.11).

In principle, all correlation functions of the original field variables $\rho$ and $\phi$ can now be obtained from the free field correlators of $X^\pm$ via the identifications (3.6) and (3.7). However, some special care is required in regularizing these composite operators. For example, to ensure that the right-hand side of (3.7) correctly behaves as a dimension 1 conformal field, we need to define it as

$$X^- \partial_u X^+ =: X^- \partial_u X^+: -\frac{1}{2\lambda^2} \partial_u \log \partial_u X^+$$

(3.14)

where $::$ denotes usual normal ordering. One should keep in mind, however, that only the expectation values of conformally invariant operators have a precise physical meaning.

4. The S-matrix and the Light-Cone Gauge.

Let us now include the matter fields $f_i$. Similar as for the gravitational fields, the reflecting boundary condition at $u = v$ gives an identification between the left and right moving parts of the fields $f_i$, so we may again work with just the left-movers $f_i(u)$. Because we are working in the conformal gauge we have the usual condition that the sum of the matter and gravitational energy-momentum tensor vanishes. This implies

$$\lambda^2 \partial_u X^+ \partial_u X^- = T^{mn}_{uu}. \hspace{1cm} (4.1)$$

We could impose this condition on physical states in the form of Virasoro constraints, or equivalently, introduce ghosts and demand that physical states and operators are BRST-invariant. Only for the critical theory with 24 scalar fields $f_i$ the BRST-charge $Q$ is nilpotent without the need of adding a one-loop correction to the gravitational energy-momentum tensor.
The reader may have noticed that our formulation of quantum dilaton gravity theory is very similar to critical open string theory, with the matter fields playing the role of the transverse string coordinates while the $X^\pm$ are like the light-cone string coordinates. As we will see momentarily, this correspondence with open string theory proves to be very useful in constructing the $S$-matrix.

A convenient way to describe the physical Hilbert space is to choose the analogue of the light-cone gauge and use the residual conformal symmetry to introduce a physical time coordinate that is defined in terms of either $X^+$ or $X^-$. In fact, for our purpose the light-cone gauge is more than just a convenient choice, but has a precise physical significance: it can be seen that a past observer will identify as the proper time-coordinate along past null infinity the variable

$$\tau_+ = \lambda^{-1} \log(X^+),$$

while a future observer will identify

$$\tau_- = -\lambda^{-1} \log(-X^-)$$

as the proper time-coordinate along future null infinity. These two choices each define an allowed light-cone gauge condition, and each lead to a different description of the same physical Hilbert space. The past observer will use the physical coordinate (4.2) to define creation- and annihilation operators by decomposing the $f_i$ fields in modes as

$$f_i'(\tau_+) = \int \frac{d\omega}{2\pi} \alpha_i(\omega) e^{i\omega\tau_+},$$

with $[\alpha_i(\omega_1), \alpha_j(\omega_2)] = \omega_1 \delta_{ij} \delta(\omega_{12})$. The in-vacuum is annihilated by all $\alpha_i(\omega)$ with $\omega > 0$, while the $\alpha_i(\omega)$ with $\omega < 0$ create the incoming particles. The resulting states are all physical. On the other hand, a future observer, who detects the outgoing particles, will use the physical coordinate (4.3) to write

$$f_i'(\tau_-) = \int \frac{d\omega}{2\pi} \beta_i(\omega) e^{i\omega\tau_-},$$

with $[\beta_i(\omega_1), \beta_j(\omega_2)] = \omega_1 \delta_{ij} \delta(\omega_{12})$, and use these modes to construct the natural out-basis of physical states. Both constructions are the direct analogue of the standard light-cone description of the physical Hilbert space of the open string. In the covariant formalism,
\( \alpha_i(\omega) \) and \( \beta_i(\omega) \) define conformally invariant vertex operators, given by

\[
\alpha_i(\omega) = \int du f'_i(u) (X^+)^{i\omega/\lambda} \tag{4.6}
\]

\[
\beta_i(\omega) = \int du f'_i(u) (-X^-)^{-i\omega/\lambda} \tag{4.7}
\]

They can be compared with the DDF operators [12], which are known to generate the complete physical spectrum. Based on this analogy, it seems a reasonable assumption that also in our case the \( \alpha \) and \( \beta \) oscillators each separately constitute a complete basis of physical operators.

Thus we now arrive at a very simple characterization of the scattering matrix of dilaton gravity. Namely, it is nothing other than the unitary transformation that interchanges the role of \( X^+ \) and \( X^- \) and maps the first light-cone basis of physical states to the second basis. In other words, \( S \) is the intertwiner between the \( \alpha \) and \( \beta \) oscillators

\[
\alpha_i(\omega) \ S = S \ \beta_i(\omega), \tag{4.8}
\]

which, being a canonical transformation, is guaranteed to define a unitary operator. Note that this \( S \)-matrix commutes with the energy operator, so in particular it maps the in-vacuum onto the out-vacuum. Matrix elements of \( S \)

\[
\langle \text{in} | \text{out} \rangle = \langle 0 | \prod_k \alpha_i_k(\omega_k) \prod_l \beta_i_l(\omega_l) | 0 \rangle \tag{4.9}
\]

can thus be written as integrated correlation functions of the covariant vertex operators (4.6)-(4.7) in the free field theory defined by the \( f \) and \( X^\pm \)-fields [11].

To make the relation between the two types of modes more explicit, we can write the formula (4.7) for the outmode \( \beta \) in the light-cone gauge \( u = \tau_+ \) and solve for \( X^- (\tau_+) \) by using the physical state condition (4.1). In this way we can express the right-hand side in terms of the in-modes \( \alpha \). The exact quantum identification of \( X^- (\tau_+) \) can be found by some standard technology of light-cone gauge string theory [13]. We define \( X^- (\tau_+) \) via a fourier mode expansion

\[
X^- (\tau_+) = e^{-\lambda \tau_+} \int \frac{d\omega}{\lambda - i\omega} \hat{X}^- (\omega) e^{i\omega \tau_+} \tag{4.10}
\]

\[
\hat{X}^- (\omega) = \lambda \int du \partial_u X^- (X^+)^{1-i\omega/\lambda}. \tag{4.11}
\]
where the modes $\hat{X}^{-}(\omega)$ are physical operators, provided the composite operator is regularized as in (3.14). Physically, the operator $X^{-}(\tau_{+})$ represents the space-time trajectory of the critical line (2.8). Now, a straightforward calculation shows [13, 11] that the modes $\hat{X}^{-}(\omega)$ satisfy the commutation relations of a Virasoro algebra with central charge $c = 24$

$$\lambda^{2}[\hat{X}^{-}(\omega_{1}), \hat{X}^{-}(\omega_{2})] = (\omega_{1} - \omega_{2})\hat{X}^{-}(\omega_{1} + \omega_{2}) + 2\omega_{1}(\omega_{1}^{2} + \lambda^{2})\delta(\omega_{1} + \omega_{2}).$$

(4.12)

As in critical string theory, this result is sufficient to guarantee that, within the physical Hilbert space, we can indeed solve (4.1) and identify the modes $\hat{X}^{-}(\omega)$ with corresponding modes of the physical energy-momentum tensor

$$\lambda^{2}\hat{X}^{-}(\omega) = L^{in}(\omega) - \lambda^{2}\delta(\omega)$$

(4.13)

$$L^{in}(\omega) = \frac{1}{2}\int d\xi : \alpha_{i}(\xi)\alpha_{i}(\omega - \xi) : .$$

The term $\lambda^{2}\delta(\omega)$ in (4.13) is needed to ensure that the central term in the algebra on both sides has the same form. It represents the constant vacuum contribution (2.6) to the energy density.

Inserting the identification (4.13) into (4.10) and (4.7) in principle gives an expression for the outgoing modes $\beta_{i}(\omega)$ in terms of the ingoing modes. This procedure is the full quantum version of the scattering off of the dynamical boundary $\phi = \phi_{cr}$, described in section 2. Indeed, the above solution (4.13) for $X^{-}$ can be recognized as the mode expansion of the boundary trajectory (2.11), with the vacuum contribution included.

5. Discussion.

In the previous section we outlined a method for obtaining a scattering matrix for quantum dilaton gravity. The construction works for $N = 24$ scalar fields, but we see no fundamental difficulty to generalize it to other values of $N$. In the low energy regime, in which the incoming energy flux stays below the critical value, the $S$-matrix has a clear unambiguous semiclassical interpretation as describing the scattering of $f$-fields off the

*The normal ordering prescription for this expression is uniquely fixed by the covariant definition of the $S$-matrix.
critical line \(2.11\). In terms of the proper asymptotic coordinates \(\tau_{\pm}\), this scattering equation relates the ingoing and outgoing matter modes via

\[
B_i^{\text{in}}(\tau_+) = B_i^{\text{out}}(\tau_-) \tag{5.1}
\]

\[
\tau_- - \tau_+ = -\lambda^{-1} \log[1 - P_\pm(\tau_{\pm})], \tag{5.2}
\]

where we introduced the notation

\[
P_\pm(\tau_{\pm}) = \kappa \int_{\tau_{\pm}}^{\infty} d\sigma e^{\lambda(\tau_{\pm} - \sigma)} T_{\sigma\sigma}^{\text{in}},
\]

\[
P_- (\tau_-) = \kappa \int_{-\infty}^{\tau_-} d\sigma e^{\lambda(\sigma - \tau_-)} T_{\sigma\sigma}^{\text{out}}, \tag{5.3}
\]

with \(\kappa = \frac{24}{N\lambda}\). In the above equations (5.3) the energy-momentum tensors \(T_{\text{in}}\) and \(T_{\text{out}}\) are taken to be normal ordered with respect to the physical vacuum, so the Casimir energy \(2.6\) is explicitly present in (5.2). In the low energy regime

\[
T_{\tau\tau}^{\text{in}} < \frac{N\lambda^2}{24} \tag{5.4}
\]

the boundary line remains time-like and the relation (5.2) between \(\tau_+\) and \(\tau_-\) is a single-valued diffeomorphism of the real line.

The interesting regime of dilaton gravity, where black hole formation and evaporation is expected to take place, is however at high energies. Indeed, when the energy flux exceeds the bound (5.4) the relation (5.2) between \(\tau_+\) and \(\tau_-\) is no longer single-valued, and, as discussed in section 2, this degeneration can be interpreted as the formation of a black hole. In this case the semi-classical scattering equations (5.2) do not give an invertible map from the ingoing to out-going matter waves. This does by no means imply, however, that our quantum construction of the \(S\)-matrix will not be valid in this supercritical regime.

In fact, we have given convincing arguments, based on gauge invariance, why we have a unitary \(S\)-matrix that is defined on the complete physical Hilbert space. To fully establish this fact, however, we need to show that the physical states that we constructed indeed form a complete basis. The close analogy with open string theory should be helpful in this respect, since for this case the corresponding problem was solved long ago, and is known as the no-ghost theorem. A technical difficulty in trying to copy the standard no-ghost theorem appears to be that in our case the energy spectrum is continuous, while the
string spectrum is discrete. This difficulty can however be removed simply by putting the system in a finite box of length $L$. We plan to present further details of this calculation in a future publication [11].

Relation with $C = 1$ Matrix Model.

There is a remarkable correspondence between the present formulation of dilaton gravity and the matrix model of two-dimensional string theory [14]. To explain this relation, let us rewrite the relation (2.11) describing the scattering of the energy-momentum flux off the critical line as follows

$$P_-(\tau) = P_+\left(\tau - \lambda^{-1} \log[1 - P_-(\tau)]\right)$$

(5.5)

with $P_\pm$ defined as in (5.3). The reader familiar with recent developments in two-dimensional string theory will now recognize this equation as the classical scattering equation of tachyons [15]. Namely, in the matrix model, scattering amplitudes in two-dimensional string theory are described in terms of a free fermion field theory [14], in which the string tachyon modes correspond to deformations of the fermi sea. By considering the time evolution of these perturbations, Polchinski derived a classical scattering equation, whose form is exactly identical to (5.5), where $P_\pm$ are identified with the $\tau_\pm$-derivative of the in- and outgoing tachyon field [15]. This suggests therefore an interesting interpretation of the matrix model in which the phase space trajectory of the fermi sea plays the same role as the critical line trajectory $\phi = \phi_{cr}$ in the $X^\pm$ plane.

In both theories, the scattering equation (5.5) defines a canonical transformation. To show this in the case of dilaton gravity, one can use (5.5) to express the fourier modes of the outgoing energy momentum tensor in terms of the incoming field $P_+$ as

$$L_{\text{out}}(\omega) = \int d\tau e^{i\omega\tau} (1 - P_+ (\tau))^{1-i\omega/\lambda}$$

(5.6)

A straightforward calculation [11] then shows that the operator on the right-hand side indeed satisfies the Virasoro algebra, at least at the Poisson bracket level and provided the relation between $\tau_\pm$ is invertible. In the matrix model, on the other hand, $P_\pm$ generate a $U(1)$ current algebra [14, 15]. Hence, while the scattering equations are identical, the canonical structures are different. In both cases, however, the $S$-matrix is characterized as the (unique) unitary quantum operator that implements the canonical transformation.
in the Hilbert space of the theory. In this sense, the dilaton gravity $S$-matrix defines a deformation of that of the $c = 1$ matrix model.

This correspondence between the two models teaches us some important lessons. In particular we learn that even if a classical scattering equation exhibits pathologies of the type discussed above, it can still lead to a well-defined unitary $S$-matrix. In the case of the matrix model this is guaranteed via the mapping to a free fermion theory, but also the bosonic formulation of the quantum theory does apparently not break down even when the classical equations degenerate. This supports our belief that the construction of quantum dilaton gravity described in sections 3 and 4 remains valid in the high energy regime, where black hole formation and evaporation take place.

The Correspondence Principle and Hawking Radiation

In our formulation of quantum dilaton gravity we had to make some assumptions about what happens in the strong coupling regions of space-time. We should make sure, of course, that in making these assumptions we have not inadvertently modified physics in the (semi-)classical regions in an unacceptable way. In other words, we must verify that our quantization procedure satisfies the correspondence principle, in the sense that it reproduces the known semiclassical physics of black holes. In particular, when one sends in a large energy pulse producing a macroscopic black hole, one would like to see that most of the outgoing matter is emitted in the form of approximately thermal radiation.

From equation (5.2) one can see that the criterion that determines if a black hole is macroscopic is whether or not the following condition

$$P_+(\tau_+) < 1$$

is violated for some value of $\tau_+$. If the energy flux exceeds the critical value (5.4) while the above condition remains satisfied, the black hole is in general microscopic and exists only for a rather short time. On the other hand if (5.7) is violated the black hole will be macroscopic and exists for a very long time.

Using the analogy with the moving mirror problem [14], it is now not hard to convince oneself that the outstate will indeed contain a regime describing Hawking radiation. If

* It may be possible, however, to define a suitable large $N$ limit of dilaton gravity in which the correspondence with the $c = 1$ matrix model becomes exact.
we assume that the large incoming energy pulse is concentrated in a small time interval, then for earlier times $\tau_+$ the integral $P_+(\tau_+)$ in (5.3) will be of the form

$$P_+(\tau_+) = p_+ e^{\lambda \tau_+} \quad (5.8)$$

where $p_+ = \kappa \int d\sigma e^{-\lambda \sigma} T_{\sigma \sigma}^{in}$. Thus for values $\tau_+$ smaller than $-\lambda^{-1} \log p_+$, the in- and outgoing modes are related via a well-defined diffeomorphism

$$\tau_- = \tau_+ - \lambda^{-1} \log(1 - p_+ e^{\lambda \tau_+}) \quad (5.9)$$

from the interval $\tau_+ < -\lambda^{-1} \log p_+$ to the real line. The physical effect of precisely this diffeomorphism was analyzed in [17] and we can use their results to conclude that, in the leading semiclassical approximation, the outgoing matter approaches a thermal spectrum for late $\tau_-.$

This indicates that the quantum theory of sections 3 and 4 indeed describes the right physics in the semiclassical limit. It would of course be more interesting to have an explicit form of the $S$-matrix that would manifestly exhibit these features. Also, one would like to understand better how the information that went in the black hole is encoded in the outgoing radiation. An important question, for example, is how long it will take for all the information to come out.

Another interesting question is by what mechanism the information is retrieved from behind the black hole horizon. A closer examination of the model defined in section 3 gives a partial answer to this last question. Namely, it can be seen that, because the boundary is everywhere timelike in the $(u, v)$ plane, it effectively implements the “conveyor belt” scenario by allowing an acausal flow of information along the singularity. Mathematically, this can happen because the conformal mapping $X^-(v)$ or $X^+(u)$ can become non-invertible in the strong coupling region, so that signals, which always propagate causally in the $(u, v)$ plane, may propagate backwards in the physical time defined by $X^\pm.$ Quantum mechanically, this means that the causal structure of space-time becomes fuzzy and indeterminate near the singularity. The key remaining problem is to show that these strong coupling effects have no catastrophic consequences in the semiclassical regime.

Comparison with 't Hooft’s Black Hole $S$-matrix.

An important question is to what extent this approach can be generalized to address the information paradox for four-dimensional black holes. The main new ingredient in that
case is that the fields in addition depend on two angular coordinates. In this connection it is interesting to point out the similarity between our result and ’t Hooft’s black hole S-matrix [3].

The central idea of ’t Hooft’s proposal is that information about the infalling particles is transferred to the outgoing particles via high-energy collisions near the horizon. In particular, he has shown that once the gravitational back-reaction is taken into account, the black hole horizon becomes a dynamical fluctuating surface, described by two light-cone coordinate fields $X^+(u, \theta, \phi)$ and $X^-(v, \theta, \phi)$. The interaction of the horizon with the in- and outgoing energy-momentum is governed by the equation of motion [3, 18]

$$\left(\Delta - 1\right) X^- (\theta, \phi) = \int dx^+ T^+_+ (\theta, \phi)$$

and similar for $X^+(\theta, \phi)$, where $\Delta$ denotes the angular Laplacian. This equation of motion for the horizon is the direct analogue of equation (2.11) determining dynamical boundary, and it actually gives an indication how one could include the angular coordinates in our model.

A further correspondence between our approach and that of [3] is that in both cases the fields $X^\pm$ are canonically conjugate variables, and the S-matrix is essentially the canonical transformation that interchanges the role of $X^+$ and $X^-$. In our theory, however, the variables $X^\pm$ do not describe the position of a permeable horizon, but the trajectory of a reflecting boundary near $r = 0$. In this way we naturally obtain an S-matrix that is defined on the full second quantized Hilbert space of the matter particles, while by analogy with the moving mirror problem we can also explain the origin of Hawking radiation.

Acknowledgements

We would like to thank T. Banks, C. Callan, S. Giddings, A. Jevicki, G. Moore, K. Schoutens, S. Shatashvili, A. Strominger, L. Susskind, L. Thorlacius, and E. Witten for helpful discussions. The research of H.V. is supported by NSF Grant PHY90-21984. and that of E.V. by an Alfred P. Sloan Fellowship, the W.M. Keck Foundation, the Ambrose Monell Foundation, and NSF Grant PHY 91-06210.

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