Evolving Constants of Motion

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Abstract
A critical presentation of Rovelli’s “evolving constants of motion” is given. Previous criticisms by Kuchar concerning the role of factor ordering and the non-existence of observables are dealt with and shown to be unfounded. Kuchar’s criticisms that this approach does not solve the global, multiple choice or Hilbert space problems of time are confirmed, and new insight into why this is so is obtained.

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Rovelli[1] has proposed that one can interpret “timeless” parametrized theories, such as canonical quantum gravity, in terms of “evolving constants of motion.” Kuchař has criticized this approach on a number of grounds, from issues of factor ordering[2] to the existence of observables[3], but, as a means of working with parametrized theories, there is much to recommend it. Some important conclusions can be learned from an independent critical analysis.

In this paper, Rovelli’s approach will be shown to be essentially the Heisenberg picture as formulated in a parametrized theory. The evolving constants of motion are observables which commute with the super-Hamiltonian[4], and they can be constructed from solutions of the Heisenberg equations of motion. One does not have to solve a factor ordering problem to find them. Existence of such solutions and hence of observables is guaranteed, and Kuchař’s doubts[3] to the contrary are unfounded. On the other hand, Rovelli’s program does fail to solve many of the problems of time[2], including the global problem of time, the multiple choice problem and the Hilbert space problem. The reason is that while the solutions to the Heisenberg equations of motion can be found without specifying the Hilbert space structure of a theory and without specifying a foliation of spacetime, to have a complete quantum theory, these must be specified and the observables do not determine them. In particular, self-adjointness of observables is insufficient to determine the inner product.

1 Rovelli’s evolving constants of motion

Rovelli’s approach[1] is perhaps best described in the context of parametrized quantum mechanics. This allows the freedom to be specific in examples without worrying about the particular structure of something like the Wheeler-DeWitt equation. One begins by working in an extended phase space which contains time and its conjugate momentum as well as the spatial phase space variables. I use bold face letters \((\mathbf{q}, \mathbf{p})\) to denote the collection of extended phase space variables, \((q_0, p_0)\) for time and its conjugate momentum, and \((q, p)\) to denote the spatial phase space variables. It is convenient to work as if there were only one spatial degree of freedom, but this is notational. I assume that the extended phase space has the topology \(R^{2n+2}\), where \(n\) is the number of spatial degrees of freedom, and that the extended phase space
variables satisfy the canonical commutation relations: in an obvious index notation,
\[ [q_\mu, p_\nu] = i\delta_{\mu\nu}, \quad [q_\mu, q_\nu] = 0, \quad [p_\mu, p_\nu] = 0. \tag{1} \]

Consider a classical action in parametrized form
\[ S = \int d\lambda \left( p\dot{q} - N(\lambda)H_{cl}(\mathbf{q}, \mathbf{p}) \right), \tag{2} \]
where \( H_{cl}(\mathbf{q}, \mathbf{p}) \) is called the super-Hamiltonian, \( N(\lambda) \) will be called the lapse, and the dot indicates differentiation with respect to the affine parameter \( \lambda \). In non-relativistic physics, the super-Hamiltonian is
\[ H_{cl}(\mathbf{q}, \mathbf{p}) = p_0 + H_{cl}(q, p, q_0). \tag{3} \]
The “super” prefix distinguishes it from the familiar Hamiltonian \( H_{cl}(q, p, q_0) \).

The equations of motion are found by variation. Variation of the lapse gives the super-Hamiltonian constraint
\[ H_{cl}(\mathbf{q}, \mathbf{p}) = 0. \tag{4} \]
This constraint is the signature of a parametrized theory. Varying the extended phase space variables gives the Hamilton equations of motion
\[ \dot{q}_\mu = N(\lambda)\{q_\mu, H_{cl}\}, \quad \dot{p}_\mu = N(\lambda)\{p_\mu, H_{cl}\}. \tag{5} \]

When this system is quantized, one finds the (operator-valued) super-Hamiltonian constraint
\[ \mathcal{H}(\mathbf{q}, \mathbf{p}) = 0. \tag{6} \]
and the Heisenberg equations of motion
\[ \dot{q}_\mu = -iN(\lambda)[q_\mu, \mathcal{H}], \quad \dot{p}_\mu = -iN(\lambda)[p_\mu, \mathcal{H}]. \tag{7} \]
Kuchař’s first objection\[2\] is that there is more than one way to quantize a given classical system to obtain the quantum \( \mathcal{H}(\mathbf{q}, \mathbf{p}) \). This is certainly true, but it is the generic problem one faces when quantizing anything. At the present time, there is no answer to this “multiple choice” problem, except by appeal to experiment. As such, it is a question about choosing between candidate quantum theories and not an obstruction to formulating them. I assume that a factor ordering of the super-Hamiltonian has been given.
The presence of the super-Hamiltonian constraint raises the so-called problem of the frozen formalism\cite{5}. In a theory with (first-class) constraints, observables are defined to be functions which commute (weakly) with all of the constraints. This is necessary because otherwise the action of an observable would take one out of the constraint hypersurface. But here, the super-Hamiltonian would seem to be the generator of time translations, so that if an observable commutes with $\mathcal{H}$, it must be a constant of the motion. Where then are the dynamics one expects of observables? Rovelli\cite{1} introduced the “evolving constants of motion” to resolve the apparent paradox. The multiple conflicting uses of the word “observable” were discussed in \cite{4}, and a simple example was done to show that, properly understood, there is in fact no problem.

The resolution is based on the recognition that observables are members of families of constants of motion parametrized by a label related to time. Given a self-adjoint operator to be measured, it generically has a realization as a distinct observable at each instant of time. Its dynamical evolution is then a reflection of its motion through the family of constants of motion with the passage of time.

To be concrete, consider the elementary example of the parametrized free particle

$$\mathcal{H} = p_0 + \frac{1}{2}p^2 \quad (8)$$

The collection of operators,

$$Q(t) = q + p(t - q_0), \quad (9)$$

parametrized by $t$, are easily verified to be observables

$$[Q(t), \mathcal{H}] = 0. \quad (10)$$

To interpret these observables correctly, one must pass through an unfamiliar step. In the parametrized formalism, $q_0$ is an operator. When it acts on a state defined on a spacelike slice at a fixed instant of time, its eigenvalue is the time associated to the slice on which that state is defined, thus

$$q_0|\psi_1, t_1\rangle = t_1|\psi_1, t_1\rangle. \quad (11)$$

The Hilbert space structure of the theory is still defined at fixed moments of time. One cannot form superpositions of states at different moments of
time—the instants of time define superselection sectors. This means that $q_0$ is not integrated over in the Hilbert space inner product.

While $q_0$ is a self-adjoint operator (its eigenvalues are all real), it is not an “observable” in the practical sense of “something one observes/measures.” One cannot build devices which couple to it directly. Thus, while a coupling like $H_I = cq_0q$ is acceptable mathematically, physically we can only build devices which couple the (by definition, spatial) degrees of freedom of different subsystems. A coupling like $H_I$ may arise indirectly as a consequence of the detailed evolution of some physical degree of freedom, but the true coupling is between spatial degrees of freedom. Finally, and perhaps most importantly, $q_0$ does not commute with the super-Hamiltonian. It is not an observable.

When the observable $Q(t)$ acts on a state, one obtains
\[
\langle \psi_1, t_1 | Q(t) | \psi_1, t_1 \rangle = \langle \psi_1, t_1 | q + p(t - t_1) | \psi_1, t_1 \rangle.
\] (12)

Thus, $Q(t_1) = q$ when acting on states defined on the slice at time $t_1$. It is essential to emphasize that one must give the time of the state that the observable acts on to know its behavior. The operator (9) is not physically meaningful on its own; it acquires meaning in conjunction with the states it acts on. Another way to say this is that an operator does not have physical meaning until one specifies the Hilbert space in which it acts.

Suppose one wishes to compute the expectation value of $q$ at time $t_2$ in terms of states at time $t_1$. One pulls back the operator $q$ from time $t_2$ to time $t_1$ and finds that one has the Heisenberg evolution, based on states at time $t_1$:
\[
\langle \psi_2, t_2 | q | \psi_2, t_2 \rangle = \langle \psi_1, t_1 | e^{-i \frac{p^2}{2} (t_2 - t_1)/2} q e^{i \frac{p^2}{2} (t_2 - t_1)/2} | \psi_1, t_1 \rangle
\] (13)

\[
= \langle \psi_1, t_1 | q + p(t_2 - t_1) | \psi_1, t_1 \rangle.
\]

But, this is just
\[
\langle \psi_2, t_2 | Q(t_2) | \psi_2, t_2 \rangle = \langle \psi_1, t_1 | Q(t_2) | \psi_1, t_1 \rangle.
\] (14)

This confirms what we have been told: $Q(t_2)$ is a constant of the motion; it has the same expectation value on every slice.

Here, we have held the observable fixed and varied the slice. Turning the story around, if we hold the slice and state fixed, say at $t_1$, and vary the
observable, then, as a function of $t$, we have

$$\langle \psi_1, t_1|Q(t)|\psi_1, t_1\rangle = \langle \psi_1, t_1|q + p(t - t_1)|\psi_1, t_1\rangle = \langle \psi_1, t_1|e^{ip^2(t-t_1)/2}q e^{-ip^2(t-t_1)/2}|\psi_1, t_1\rangle.$$  \hspace{1cm} (15)

Thus, $Q(t)$ gives the Heisenberg evolution of $q$ from $t_1$ to $t$ when acting on states at $t_1$! (More generally, $Q(t)$ acting on states at time $\tau$ is the Heisenberg evolution of $q$ from $\tau$ to $t$.) Dynamical evolution is the movement of the self-adjoint operator under observation through the family of constants of motion as time passes.

To be complete, suppose one is in the Heisenberg picture with states defined at time $t_1$, and one wants to follow the Heisenberg evolution of the operator $q + p(t_2 - t_1)$, corresponding to $Q(t_2)$ acting on states at time $t_1$. One can decompose $Q(t_2)$ in terms of observables at time $t_1$ by

$$Q(t_2) = e^{ip^2(t_2-t_1)/2}Q(t_1)e^{-ip^2(t_2-t_1)/2} = Q(t_1) + (t_2 - t_1)P.$$  \hspace{1cm} (16)

(where $P = p$ for all $t$). The Heisenberg evolution of $q + p(t_2 - t_1)$ is then

$$\langle \psi_1, t_1|Q(t) + (t_2 - t_1)P|\psi_1, t_1\rangle.$$  \hspace{1cm} (17)

In the general case of non-relativistic quantum mechanics with time-independent Hamiltonian, the super-Hamiltonian is

$$\mathcal{H} = p_0 + H(q, p).$$  \hspace{1cm} (18)

Any self-adjoint operator $f = f(q, p)$ can be promoted to a family of observables by computing its Heisenberg evolution

$$F(t; q, p, q_0) = e^{iH(t-q_0)} f(q, p) e^{-iH(t-q_0)}.$$  \hspace{1cm} (19)

It is easily verified that

$$[F(t; q, p, q_0), \mathcal{H}] = 0.$$  \hspace{1cm} (20)

The expectation value of the operator $f(q, p)$ at time $t_1$ corresponds to the expectation value of the observable $F(t_1; q, p, q_0)$. Viewed as a function of $t$, $F(t; q, p, q_0)$ acting on states at time $t_1$ is simply the Heisenberg evolution of
$f(q,p)$ from time $t_1$ to $t$. The novel feature of the observable $F(t; q, p, q_0)$ is that its dependence on the operator $q_0$ allows one to shift the fixed initial slice on which the Heisenberg picture is defined. In a sense, some of the freedom of the Schrodinger picture to change slices is incorporated, though dynamics still remains with evolution of the operator. Here, that evolution is seen to be movement through the family of observables as time passes.

Note that $q_0$ cannot be promoted to an observable in this way. From (9), one can construct an operator

$$Q_0 = \frac{1}{p}(Q - q) + q_0$$

which commutes with $\mathcal{H}$—but it is not self-adjoint in the usual inner product.

Consider two of Kuchař’s remarks[3]: a) “one can observe dynamical variables which are not perennial;” b) “perennials are often difficult to observe.” (“Perennial” is synonymous with observable as used here.) Neither of these remarks is truly in conflict with what has just been described. On first consideration, the self-adjoint operator $q$ (at time $t_1$) is measurable[6], and it does not commute with the super-Hamiltonian, so it is not an observable/perennial. Kuchař’s statement is apparently correct. Rovelli takes one small further step. He recognizes that at time $t_1$, $q = Q(t_1)$, that is, it is an instance of an observable. In Rovelli’s scheme, every dynamical variable at an instant is simply the instantaneous form of some observable. This seems a modest step, but it saves one the mental gymnastics of coping with operators which may take one out of the constraint surface, by assuring that they never do.

Kuchař’s second remark is a fact. If one attempts to observe $Q(t_2)$ at time $t_1$, this may be difficult. Fortunately, we don’t experience time in the Heisenberg picture, and there is nothing which says we must try to do so. At each instant of time, there are a collection of observables which are comparatively easy to measure, and these are the ones our attention focuses on. Other observables may be difficult to measure, and our effort to determine them will depend on our interest in them.

This brings us to Kuchař’s key question. He asks, how is $t$ to be observed? In the present context, this question is somewhat misdirected, but it is a very important question in its place and is discussed at length in [4]. Ostensibly, Kuchař wants to argue that to observe change, one must measure something which is not an observable. He argues that one must know $t$ to know when
to measure a particular observable. On the face of it, this seems reasonable, but it puts a false emphasis on $t$ and in doing so misses an essential point.

One does not choose to measure $Q(t)$ at some instant because one knows that instant is labelled by $t$. No, one chooses to measure an operator, say $q$, at the instant which is “now,” and there is an observable associated to this operator—which observable, in an absolute sense, being irrelevant. At a different instant, one again measures $q$, now associated to a different observable, and generally the values obtained are different. Change has been measured. It is true that there is a hidden active agent in the passage of time, but it is enough that it happens. It is inferred only indirectly by the fact that the value of the observable changes.

Alternatively, as a theorist outside of the system, one can select a slice and measure $q$ there. It gives a value associated to one observable. When $q$ is measured again on a different slice, a distinct value associated to another observable is obtained. Change has been measured.

Generally, we want more than to show that observables take different values. We want to coordinate the change in those values. It is not sufficient that time pass; we must mark the passage of time. This cannot be done in the model at hand. It has only one simultaneously measurable observable. Kuchař asserts that $q_0$ is the hand of an ideal Newtonian clock. This is false: $q_0$ is not measurable because it is not a degree of freedom of a physical subsystem.

The traditional approach is to introduce an additional degree of freedom, say with Hamiltonian $H = px$, and call it a clock. This Hamiltonian is used here solely to illustrate a point. As it is not bounded from below, it is unphysical, but consider it temporarily to be a measurable subsystem. (Clocks are critically discussed more fully in [7].) A second family of observables (simultaneously measurable with $Q(t)$) can be constructed

$$X(t) = x + t - q_0.$$  \hspace{1cm} (22)

$X(t)$ is the hand of the clock, and it can be used to coordinate different measurements of $Q(t)$.

Suppose that at some initial instant, arbitrarily labelled $t_1$, the $x$-subsystem is in an eigenstate of $X(t_1)$ with eigenvalue $x_1$. The state of the full system at $t_1$ is then $|\psi_1(q), x_1, t_1\rangle$. The state of the $x$-subsystem at a later time $t_2$ can be deduced from the constancy of the observable $X(t_2)$. 

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Computed at time $t_1$, one finds
\[
\langle \psi_1, x_1, t_1 | X(t_2) | \psi_1, x_1, t_1 \rangle = x_1 + t_2 - t_1
\]
This is what one expects from Heisenberg evolution from $t_1$ to $t_2$ using the $H = p_x$.

By repeatedly measuring $x$ as time passes, one can wait a predetermined $x$-interval between measurements of $Q(t)$. One need never know the dependence of $Q(t)$ on $t$ because one will be able to infer a dependence on $X(t)$ instead. It should be repeated that $X(t)$ refers to the collection of outcomes of measurements of $x$—the label $t$ is largely a convenience to distinguish between different values of the outcomes. Again, behind the whole process, it is assumed that “time passes.” This may be justified either by appeal to experience or by saying that this is the phrase which describes the process of sequentially considering the states associated with different values of $X(t)$. While $X(t)$ can be measured on any slice, it is natural to measure $x$ on some slice and assign $X(t)$ to the value obtained there. Having done so, one may measure $q$ on the same slice and assign $Q(t)$ to that value.

The point here is that the passage of time is an external experiential, if you will, phenomenon. The evolving constants of motion do not explain the passage of time. They show that when one observes measurable quantities like $q$ at a sequence of experienced instants of time $t_k$, these observations can be understood as the measurement of observables $Q(t_k)$ which commute with the super-Hamiltonian $H$. To some, the use of observables which commute with $H$ may seem unnecessary, and they will say one is really measuring $q$ which does not commute with $H$. It is not that one cannot do physics in this way. It is a question of whether it is in keeping with the spirit of other tenets of mathematics and quantum physics where one is ill-disposed to operators which can take one out of the Hilbert space or constraint surface.

### 2 Observables and Integrability

Another of Kuchař’s arguments against Rovelli’s evolving constants of motion concerns the existence of observables that commute with the super-Hamiltonian constraint. Kuchař reminds us of Poincaré’s results on the nonexistence of further integrals of the motion beyond the classical integrals
(energy, center of mass momentum, and angular momentum) for systems like the asymmetric top and the three-body problem. He goes on to argue that gravity itself is unlikely to be integrable (cf. [8]) and therefore there will not be a complete set of observables with which to characterize a system. There is a confusion of terms in this objection, with Kuchař and Rovelli talking past each other.

The problem is with what it means to be a constant of motion. Both Kuchař and Rovelli agree at the outset that an observable which commutes with the super-Hamiltonian is a constant of motion. The difficulty follows as Kuchař begins to speak of what are often referred to as first integrals of the motion while Rovelli speaks of what may be called “second integrals” of the motion, but are more readily recognized as solutions of the equation of motion. When integrating a system of second order differential equations, such as equations of motion, one integration may reduce the order of some equation by one and one obtains a “first integral.” The energy and the momenta of ignorable coordinates are obtained by such a first integration of the Euler-Lagrange equations. These are all first integrals of the motion. A second integration produces a “second integral,” i.e. the solution of the equation.

Since there are existence and uniqueness theorems for the solutions of the equations of motion of dynamical systems, including Einstein’s field equations, the existence of second integrals is not in doubt. Each solution is given uniquely in terms of its initial data, and it is considered a trivial observation that these initial data are constants of motion for that solution. There is a canonical transformation which takes one from any point along a solution back to its initial conditions, so there are canonical transformations which trivialize the motion of the system. The problem with second integrals of the motion is finding them! That is the central challenge of dynamics: solving the equations of motion. This is why first integrals of motion are so valuable. Their existence assists in the integration of the equation of motion.

An alternative definition of a first integral is as a function of the positions and velocities which is constant, \( A(q(t), \dot{q}(t)) = \text{constant} \), along every solution of the equations of motion. It appears that the key feature that distinguishes such an integral from a second integral is that the function \( A \) has no explicit dependence on time, but this is misleading. Kuchař attempts to confuse the distinction between first and second integrals by fixing the energy. The solutions to the equations of motion at fixed energy do not involve
the time, so it seems by this definition any integrals must be first integrals. This is of course absurd. A non-integrable system with time-independent Hamiltonian for which one has the exact time-dependent solution for a given set of initial conditions does not acquire first integrals simply because one fixes the energy nor does the solution cease to exist.

The fixed-energy second integrals are found by inverting the solution of one of the variables for the time and using this to eliminate the time in the remaining solutions of the other variables. It is clear that, when one does this, the expressions for the other variables will involve both initial and final values of the variable which has been used to eliminate time. A true first integral however only depends on initial variables. In an example in the next section, (29) is a first integral while (31) is only a second integral. This is the essential distinction between first and second integrals. In the usual context, it is the time which occurs as a difference of final and initial values, and that is why second integrals are associated with time-dependence.

A system of $n$ degrees of freedom is said to be integrable if it admits $n$ first integrals of the motion in involution. Such a system is equivalent under a time-independent canonical transformation to a system constrained to move geodesically on a regular (flat) torus. It should be emphasized that to say a system is non-integrable does not mean that the solutions to its equations of motion cannot be characterized. It means that they are not equivalent to geodesic motion on a torus. Thus, geodesic motion on a two-dimensional higher genus Riemann surface of constant negative curvature is easily described, for example by circular arcs in the Poincaré disk tesselated by the fundamental domain appropriate to the surface, but the motion is non-integrable.

The evidence is that general relativity is not an integrable theory, and that there are few if any first integrals\[8\]. This is not really surprising. It is in fact a very good thing. I interpret it in the following way. Since general relativity is not integrable, there is not a time-independent canonical transformation to variables in which all the momenta are constant and their conjugate variables evolve linearly in time. This means that the universe cannot be in a superposition of eigenstates of a complete collection of $t$-independent observables (where $t$ is a final value of the time). As a result, time exists. We cannot be trapped in the frozen formalism.

The foundation of Rovelli’s approach is the observation that nevertheless the universe is in a superposition of eigenstates of a complete collection of $t$-
dependent observables. Each of these observables are simply the Heisenberg evolution of some functions of the initial data. If we choose to work in the Heisenberg picture, the universe is frozen in a particular state. At each instant of time, however, we must use a different collection of observables to decompose the state of the universe. This is because we must use a collection of observables evaluated at that moment of time and therefore necessarily a collection of time-independent observables. Since general relativity is not integrable, we cannot use the same set of time-independent observables at every time, so our set must change. In the Heisenberg picture, it is the changing of the set of observables that we use to decompose the state of the universe which reflects the passage of time.

3 The Problems of Time

Returning to the general situation, the challenge in constructing Rovelli’s evolving constants of motion is to solve the Heisenberg equations of motion. Kuchař\(^2\) raises several objections about one’s ability to do this, but the immediate form of his objections rest largely on a misapprehension of the task one faces. Kuchař approaches the problem of construction as one of factor ordering a classical solution of the problem. After emphasizing the difficulty of factor ordering in general, he goes on to ridicule the existence of a time function which he argues is a necessary part of the procedure. He points out the “global problem of time”—that generally there are obstructions to foliating extended phase space with a single time function—and the “multiple choice problem”—that one could use different time functions and one must prove that the quantum theories are equivalent. He argues that Rovelli’s hope to solve the Hilbert space problem, that is to put a Hilbert space structure on the space of solutions to the super-Hamiltonian constraint, is in vain because of the multiple choice problem. He concludes that Rovelli cannot solve any of the problems of time.

The situation is not quite so grim as this, though the conclusions are valid. Kuchař’s version of Rovelli’s approach is a straw man which distorts both the strengths and weaknesses of a careful treatment. The role of the time function is overemphasized, and one especially does not have to factor order a classical solution. One simply needs to solve the Heisenberg equations of motion (7). Doing so may be non-trivial, but existence and uniqueness of the solution
are assured. Perturbative expansions in $\Lambda - \Lambda_0 = \int_{\Lambda_0}^{\Lambda} N(\lambda)d\lambda$ of the solution are easily obtained in the finite dimensional case, and quantum canonical transformation methods are under development to obtain expressions with well-defined factor ordering in more closed form \cite{10}.

van Hove’s theorem may haunt the passage between classical and quantum theory, but it is impotent if one does not bother with the crossing and works always in the quantum realm. The real difficulties are serious enough without adding to them. For example, it is all too true that the field theory case is of a different magnitude of difficulty than finite dimensional examples. The functional nature of $N(\lambda)$ and the consequent notion of “bubble-time” necessarily complicates the production and representation of solutions. It is not an overstatement to say that Rovelli’s approach is in a state of development, but a less polemical critique may clarify the issues and lead to further progress.

To obtain reparameterization-invariant observables, one must eliminate the affine parameter $\Lambda - \Lambda_0$ in the expressions for the solutions, and this raises the first serious obstacle. (A different procedure involving an averaging over $\Lambda - \Lambda_0$ is discussed in \cite{11} and may avoid this difficulty.) To isolate $\Lambda - \Lambda_0$, one must invert the solution of one variable for $\Lambda - \Lambda_0$. One may of course perform canonical transformations on the variables before attempting the inversion. There is unfortunately no non-abelian form of the implicit function theorem of which I’m aware, so it is not clear when this can be done, though some theorem seems likely to be true. Additionally, inversion is generally not unique, because of branch choices or other non-analytic behavior. This means that not every solution of the equations of motion has the same form in terms of the deparametrized variables.

As a simple example, consider the super-Hamiltonian

$$\mathcal{H} = -\frac{1}{2}p_0^2 - aq_0 + \frac{1}{2}p^2 + bq.$$ \hspace{1cm} (24)

Let $Q_\mu, P_\mu$ denote the evolved variables and $q_\mu = q_\mu(\Lambda_0), p_\mu = p_\mu(\Lambda_0)$ denote the initial conditions. The Heisenberg equations of motion are then

$$\dot{Q}_\mu = -i[Q_\mu, \mathcal{H}], \quad \dot{P}_\mu = -i[P_\mu, \mathcal{H}],$$ \hspace{1cm} (25)

where $\mathcal{H}$ is the super-Hamiltonian expressed in terms of the evolved variables $Q_\mu, P_\mu$, and dot means differentiation with respect to $\Lambda$. The solution of
these equations are easily found

\[ P_0 = a(\Lambda - \Lambda_0) + p_0, \]  
\[ Q_0 = -\frac{1}{2}a(\Lambda - \Lambda_0)^2 - p_0(\Lambda - \Lambda_0) + q_0, \]  
\[ P = -b(\Lambda - \Lambda_0) + p, \]  
\[ Q = -\frac{1}{2}b(\Lambda - \Lambda_0)^2 + p(\Lambda - \Lambda_0) + q. \]

Solving for \( \Lambda - \Lambda_0 \) in each of these, one finds the expressions

\[ \Lambda - \Lambda_0 = \frac{P_0 - p_0}{a} \]
\[ = -\frac{P_0}{a} \pm \frac{1}{a}(p_0^2 + 2a(q_0 - Q_0))^{1/2} \]
\[ = -\frac{P - p}{b} \]
\[ = \frac{p}{b} \pm \frac{1}{b}(p^2 + 2b(q - Q))^{1/2} \]

The quadratic equations are solved by shifting \( \Lambda - \Lambda_0 \) to cancel the term linear in \( \Lambda - \Lambda_0 \) and then taking the square root of the remaining expression. When solving for \( \Lambda - \Lambda_0 \), it is assumed to commute with all of the variables, and there are no ordering problems with its coefficients. On the other hand, the evolved variables do not commute with the initial ones; the various commutators can be deduced by taking commutators with the solutions (26). In this example, the forms of \( \Lambda - \Lambda_0 \) are the same as they would be classically (but with appropriate operator ordering of the square-roots).

Equating the first two of (27), one has

\[ P_0 = \pm \left( p_0^2 + 2a(q_0 - Q_0) \right)^{1/2}. \]

Classically it is clear why there are two solutions: for appropriate initial conditions, a particle climbing in a linear potential passes a given point once on its way up and again on the way down. If one uses the second form of (27) to eliminate \( \Lambda - \Lambda_0 \) in either \( P \) or \( Q \), one will also get two solutions. Only if one uses either the first or third forms of (27) will one find unique forms for the deparametrized solutions, and that is because these forms are linearly correlated to \( \Lambda - \Lambda_0 \).
Observables are formed by substituting the form of $\Lambda - \Lambda_0$ obtained from one solution in the solutions for the other variables. Here, one treats the evolved and initial variables as commuting. Each expression consisting of a function of evolved variables equated to a function of initial and evolved variables is then an observable. For example, use the second form of $\Lambda - \Lambda_0$ from (27). Substituting this in the first solution of (26), one has (28) as an observable or alternatively

$$\frac{1}{2}P_0^2 + aQ_0 = \frac{1}{2}P_0^2 + aq_0$$

which one can verify satisfies

$$\left[\frac{1}{2}P_0^2 + aQ_0, \mathcal{H}\right] = 0.$$  \hspace{1cm} (30)

Note that this is a first integral because the initial and final variables are separated.

Substituting the second form of $\Lambda - \Lambda_0$ from (27) in the third solution of (26), one finds

$$P = \frac{bp_0}{a} \pm \frac{b}{a} \left( p_0^2 + 2a(q_0 - Q_0) \right)^{1/2} + p,$$

which can be confirmed to satisfy $[P, \mathcal{H}] = 0$. This is a second integral because $Q_0$ is present on the right-hand side and cannot be separated from the initial variables. As well, the more complicated expression obtained for $Q$ also satisfies $[Q, \mathcal{H}] = 0$. Note that because $P$ and $Q$ depend on $Q_0$, they are actually families of observables parametrized by the value of $Q_0$. ($Q_0$ fulfills the role that $t$ played above.) Further observables can be found by eliminating other forms of $\Lambda - \Lambda_0$ and by taking functional combinations of existing observables. Obviously, not all observables commute nor are they necessarily independent.

The multivaluedness of the expressions for $\Lambda - \Lambda_0$ is a reflection of the global problem of time. When none of the variables has a single-valued expression in terms of $\Lambda - \Lambda_0$, then there is an obstruction to defining a global time function (cf. [12] for one possible way to deal with this). There is an important issue here which deserves closer study, but it is not altogether clear that it is a problem. For example, in the present example, $p_0$ is linearly
correlated to $\Lambda - \Lambda_0$ and would therefore be a good time function. Physics however might dictate that $q_0$ is the time. One has then some sort of time-dependent relativistic super-Hamiltonian and it is not clear why there is a problem if the classical or Heisenberg solutions to the equations of motion are not single-valued in $q_0$. Further work is needed on this point.

Having found a complete set of observables, Rovelli’s next step is to determine the inner product by choosing one in which all of the observables are self-adjoint. If he were successful in this, he would solve the Hilbert space problem; he would succeed in putting a Hilbert space structure on the solutions of the super-Hamiltonian constraint. Kuchař claims he must fail because of a variant on the multiple choice problem: he can choose different time functions to construct his observables, and for each he gets a different set of observables; if one requires that all of these observables be self-adjoint in the same inner product, there are bound to be problems. In the example above, different ways to eliminate $\Lambda - \Lambda_0$ correspond to different choices of Kuchař’s time function. While related to the real issue, this argument may lead one to miss the crucial point.

There is indeed a multiple choice problem which Rovelli has overlooked which defeats this part of his program. So far we have the solutions to the Heisenberg equations of motion, but these solutions are independent of the Hilbert space structure. It has not been emphasized, but the procedure for solving the Heisenberg equations of motion either perturbatively or by canonical transformation\(^\text{[10]}\) involves the algebra of the commutation relations but not the inner product or the states. In the solutions, the spatial and temporal variables are on an equal footing and are indistinguishable. The symmetry between spatial and temporal variables is broken however when one chooses how to make the time-slice through extended configuration space on which one normalizes states. There is nothing except association with physical quantities and their experimental behavior to guide this choice. Simple inspection of the mathematical variables is insufficient. There is a different inner product in which all of the observables are self-adjoint for nearly every choice of slice through the extended configuration space.

As a simple example illustrative of this ambiguity, consider the massless relativistic free particle,

$$\mathcal{H} = -p_0^2 + p^2 = 0.$$ (32)
The solution to the parametrized Heisenberg equations of motion are

\[ Q = 2p(\Lambda - \Lambda_0) + q, \quad Q_0 = -2p_0(\Lambda - \Lambda_0) + q_0, \]  

(33)

with \( P = p \) and \( P_0 = p_0 \) constant. There is clearly symmetry between \( q \) and \( q_0 \) in these equations. Eliminating \( \Lambda - \Lambda_0 \) by solving for it in terms of \( q_0 \) gives

\[ Q(Q_0) = -\frac{p}{p_0}(Q_0 - q_0) + q. \]  

(34)

This system has one physical degree of freedom and requires two observables. They can be chosen to be \( P \) and \( Q \). Requiring that they both be self-adjoint on a hypersurface of constant \( t \) [the eigenvalue of \( q_0 \) as in (11)] in a Hermitian inner product, one would find an inner product. On the other hand, one could solve (34) for \( Q_0(Q) \), and with \( P_0 \), one could find a different inner product integrated over a hypersurface of constant \( q \). The choice of hypersurface distinguishes these from the infinitely many other possibilities, and the correct choice is dictated by physics. The requirement that the observable be self-adjoint is not enough.

In the discussion in [13] of the analogous case of a massive relativistic free particle, it is argued that a unique inner product is found when one requires the boost operator \( Q_0P - QP_0 = q_0p - qp_0 \) (and say \( P \)) be self-adjoint. [This is just another rewriting of the observable (34).] A hidden choice has been made when the inner product takes the form of an integral over a surface at constant \( p_0 \). In the general case of the Wheeler-DeWitt equation, one does not know which variable in the super-Hamiltonian is time, and one does not know how to select the hypersurface on which to define the inner product. The example here makes this obvious. The only \( a \ priori \) justification one has to consider an inner product integrated over slices of constant \( q_0 \) is the historical convention that the variable with the subscript 0 is time. I could however have maliciously mislabelled the variables, and \( q \) might be the physical time. (For completeness, I recall also that it is well known that one cannot use the signature of the operator to help to identify time because the signature of the Wheeler-DeWitt operator is unrelated to time.) Just as in the classical theory, inspection of the solutions of the equations of motion is insufficient to determine which variable is time.

Incidentally, the global problem of time seems to have yet another opportunity to make an appearance. Having chosen one slice on which to define
one’s inner product, it is not clear that this slice can be extended into a foliation. It is difficult to say more without specific examples, but the possibility would seem to exist.

It may have been too optimistic to hope that the Hilbert space problem would be solved by the evolving constants of the motion, but this should not detract from the other real accomplishments of this approach. The Heisenberg picture is brought to the fore, and the solutions of the Heisenberg equations of motion are seen not to require a time function for their definition. They provide a system of observables which commute with the super-Hamiltonian, and through their behavior dynamics can be analyzed. Space and time are on equal footing in these observables. The essential open problem is to define the Hilbert space of states on which these observables act. There is an important interplay between states and time.

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The adjective “measurable” and the verb “to measure” refer to the physical act of determining a value and are used to avoid confusion and redundancy provoked by the words “observable” and “to observe.” They are not meant to invoke a specific interpretation of quantum mechanics and in particular need not imply collapse of the wave function. The relation between various uses of the word observable are discussed in Ref. [4].

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