Probability distributions for one component equations with multiplicative noise.

J.M. Deutsch

University of California
Santa Cruz CA 95064

Abstract

Systems described by equations involving both multiplicative and additive noise are common in nature. Examples include convection of a passive scalar field, polymers in turbulent flow, and noise in dye lasers. In this paper the one component version of this problem is studied. The steady state probability distribution is classified into two different types of behavior. One class has power law tails and the other is of the form of an exponential to a power law. The value of the power law exponent is determined analytically for models having colored gaussian noise. It is found to only depend on the power spectrum of the noise at zero frequency. When non-gaussian noise is considered it is shown that stretched exponential tails are possible. An intuitive understanding of the results is found and makes use of the Lyapunov exponents for these systems.
I. INTRODUCTION

This paper analyzes linear equations containing additive and multiplicative noise. There are many problems in physics which are in this category. The two types of noise are generally uncorrelated with each other. One example is a polymer in a turbulent flow. Analysis of this system yields an equation for bending modes of the polymer with two types of noise \[1,2\]. Thermal noise presents itself in an additive way, while the effects of the random velocity field come in multiplying the coordinates of the polymer, and is hence termed multiplicative. The random flow field, being turbulent, has correlations with long decay times, and are therefore highly colored. Other examples include motion of a passive scalar in a random velocity field \[3,4\], and propagation of light in random media \[5\]. Noise in dye lasers is modeled by multiplicative random equations \[6\]. Most of the applications involve more than one degree of freedom, however the multi-dimensional case is surprisingly similar to the one component case, so it is worth studying it first. The multi-component case is considered in a separate publication \[7\].

Because of its physical applications, it seems worthwhile to attempt to understand general properties of such equations. Drummond \[8\] recently analyzed such a linear one component equation of the type of interest in this paper. He considered the case where both the additive and multiplicative noise where generated by the same random process. He found two types of behavior are possible. In one regime he found that the probability distribution function (PDF) had power law tails, so that high enough moments of the variable of interest did not exist. The other regime had all its moments defined.

In the applications mentioned above, the additive and the multiplicative noise are not correlated with each other which is unlike the cases considered by Drummond \[8\]. Fortunately this problem is also tractable analytically and is the subject of this paper. In this case one can go considerably further in classifying the type of behavior possible, and intuitively how it occurs.

There is a large mathematical literature on the stability of the nth moment of a variable in the absence of additive noise which has been studied by the introduction of Lyapunov functions \[9\]. The stability of moments can be determined if such a function can be found. The method can be extended to additive noise but does not provide a classification of the
kinds of behavior expected for the tails of the distribution.

The classification instead hinges on an important conceptual framework, the Lyapunov exponents for these systems without additive noise, as discussed in section II. All the behavior found here is easily understood in this framework. In this case the PDF does not tend to a time independent limit, however the general scaling form of the distribution has been previously obtained [10,11] and allows one to define a set of Lyapunov exponents $L(q)$. These are not related simply to Lyapunov functions [9]. The $L(q)$'s can then be used to give a more intuitive derivation of the form of the probability distribution's tail in the presence of additive noise discussed in the sections that follow.

The classification into different two regimes is now briefly described. In one regime, it is possible to show that the probability distribution of fields satisfying these types of equations have power law tails. This class of problems occur under a wide variety of conditions. This will be seen first in section IV where a heuristic argument is given to understand when such tails are present and when they should be absent. Then, in section V, the one dimensional version of this problem with gaussian noise is analyzed. In general these power law tails will be present with Gaussian noise, even if it is not short range in time. In this case it is possible to find the exact exponent for the asymptotic probability distribution, for arbitrary time correlations in the multiplicative noise. The exponent depends continuously on the strength of the multiplicative noise in a simple way, only through the power spectrum at zero frequency.

The other type of behavior is where all moments are defined and the PDF has a tail falling faster than any power law. This is shown in section VI to occur for one dimensional models but requires that the multiplicative noise not be Gaussian. In this case the PDF will be of the form of an exponential to a power law, the precise exponent depending on the Lyapunov exponents for high moments. This can result in a stretched exponential distribution.

II. ONE COMPONENT MULTIPLICATIVE EQUATION

A. General equation for the moments

Here we will consider the equation
\[ \frac{dx}{dt} = (-k + f(t))x + \xi(t) \] (1)

This can be interpreted as the equation for a massless particle with friction coefficient \( \nu \) connected to a spring that has a spring coefficient that varies randomly in time as \( k - f(t) \). The spring is perturbed by a random force \( \xi(t) \) representing a heat bath at temperature \( T \).

As a result the \( \xi(t) \) is Gaussian and \( \delta \) function correlated \( \langle \xi(t)\xi(t') \rangle = 2\nu T \delta(t - t') \) \( \mathbb{R}^2 \). \( f(t) \) in general can be considered non-Gaussian with a correlation function

\[ \langle f(t)f(t') \rangle = g(t - t'), \quad \langle f(t) \rangle = 0 \] (2)

Writing \( \tau = (k/\nu)t \), this can be rewritten as

\[ \frac{dx}{d\tau} = -x + \gamma(\tau)x + \eta(\tau) \] (3)

with

\[ \langle \eta(\tau)\eta(\tau') \rangle = \frac{2T}{k} \delta(\tau - \tau') \] (4)

and

\[ \langle \gamma(\tau)\gamma(\tau') \rangle = \frac{1}{k^2} g\left(\frac{\nu}{k}(\tau - \tau')\right). \equiv \sigma(\tau - \tau') \] (5)

### III. DISTRIBUTION IN THE ABSENCE OF ADDITIVE NOISE

Here we will consider eqn. (3) when the additive noise term \( \eta \) is zero. In this case there is not a well defined steady state probability distribution. However it is useful to understand the time dependent form of the distribution for long times. We shall see that in this limit it has a scaling form. The main results found in this section will survive the leap to many components and provide an important conceptual framework for understanding these systems.

First we review why it is that the behavior of the system is characterized by Lyapunov exponents \( \mathbb{R}^2 \) \( \mathbb{R}^1 \)

\[ \langle x(t)^q \rangle \propto e^{L(q)t} \] (6)
where the brackets denote an ensemble average over the noise \( \gamma \). To understand this, it is simplest to dispense with the linear term \(-kx\) in eqn. (3) at the expense of giving \( \gamma(t) \) a nonzero mean \( \langle \gamma \rangle = -1 \). A straightforward way of defining a short range non-gaussian process is to discretize the above eqn. (3), \( t = i\Delta t \) and using Itô discretization, this equation with \( \eta = 0 \) reads

\[
x_{i+1} = x_i(1 + \Delta t\gamma_i)
\]

This defines a multiplicative process. If the different \( \gamma \)'s are all taken to be independent the Lyapunov exponents can be determined by calculating

\[
\langle x_i^q \rangle = \langle (1 + \Delta t\gamma_1)^q \rangle = e^{L(q)t}
\]

Hence the Lyapunov exponents are \( L(q) = (1/\Delta t)\ln(\langle (1 + \Delta t\gamma)^q \rangle) \). In general these are not quadratic and depend on the probability distribution of the \( \gamma_i \)'s. \( L(q) \) should be convex. These multiplicative random processes have been well studied, particularly in recent years in connection with “multifractals”.

Given the moments of \( x \) it is possible to calculate its PDF for long times by performing a steepest descent analysis identical the case of multifractals [10]. It can be checked that the distribution giving such scaling is

\[
\ln P(\ln x) \propto t f((\ln x)/t) + O(\ln t/t)
\]

where

\[
f(\alpha) = L(q) - q\alpha, \quad \alpha = L'(q).
\]

From this \( \langle (\ln x)^2 \rangle \) can be calculated and is proportional to \( t \) for large \( t \), so the the distribution continues to broaden. Therefore \( P(x) \) does not tend towards a time independent, that is steady state, distribution.

Do these results persist when there are correlations between the different \( \gamma_i \)? The answer is that if the correlations are not too long range, these results are still valid. This can be seen by making an analogy with a statistical mechanical model. Denote the random variable \( 1 + \Delta t\gamma_i \) in eqn. (7) by \( \exp(z_i) \). Then given the joint PDF of the variables \( \{z_1, \ldots, z_n\} \)
\[ P\{z_1, \ldots, z_n\} \equiv e^{-V\{z_1, \ldots, z_n\}} \quad (11) \]

\[ \langle x_{nq} \rangle \text{ can be expressed as} \]

\[ \langle x_{nq} \rangle = \int \prod_{i=1}^{n} e^{\sum_{i=1}^{n} (q-1)z_i - V\{z_1, \ldots, z_n\}} dz_1 \ldots dz_n \quad (12) \]

This expression can be thought of as the partition function of a system with \( n \) degrees of freedom. For large \( n \) the free energy of this is extensive if \( V \) is sufficiently short range \[13\]. This means that for large \( n \) \( \langle x_{nq} \rangle \sim \exp(n \Delta t L(q)) \) where \( L(q) \) can be interpreted as the free energy per degree of freedom.

**IV. HEURISTIC EXPLANATION OF POWER LAW TAILS**

Before launching into a detailed mathematical analysis of this problem, it is worthwhile giving a heuristic explanation for the existence of power-law tails and their relationship to the Lyapunov exponents \( L(q) \).

In the absence of additive noise, we just saw that eqn. (3) can be characterized by the different moments of \( x_i \). If we examine the \( q \)th moment \( \langle x_{q}^q \rangle \), it will either exponentially increase or decrease depending on whether \( L(q) \) is positive or negative respectively.

Now examine how the last statement changes in the presence of additive noise. When \( L(q) \) is positive one expects that additive noise will only increase \( \langle x_{q}^q \rangle \) further. When \( L(q) \) is negative \( \langle x_{q}^q \rangle \) cannot be expected to go to zero as even in the absence of any multiplicative force \( \langle x_{q}^q \rangle \) should reach a non-zero steady state value. Therefore we expect that in the limit \( t \rightarrow \infty \) \( \langle x_{q}^q \rangle \) has a non-zero steady state value when \( L(q) \) is negative, and becomes ill defined when \( L(q) \) is positive.

Consider fig. [4]. Here two examples of possible \( L(q) \)'s are illustrated. An important restriction to bear in mind is that \( L(q) \) is a convex function of \( q \). The solid curve represents the case where \( L(q) \) changes sign as a function of \( q \). \( q^* \) represents the point where \( L(q^*) = 0 \). From the above argument, for \( q > q^* \) the moment \( \langle x_{q}^q \rangle \) becomes ill defined in steady state. Below this point the moments are well defined. Such behavior implies that the PDF of \( x \) has a power-law tail \( P(x) \propto x^{-q^*-1} \).

The other case of interest is where \( L(q) \) never crosses through zero for positive \( q \). In this case all moments are defined and \( P(x) \) should have a tail that vanishes more quickly then
any power. In this case, it does not seem possible to obtain the actual form for the tail of $P(x)$ by the heuristic considerations just presented. However in section VI a model will be analyzed where the form of the tail can be completely determined by a knowledge of $L(q)$ for large $q$.

The probability distribution with no additive noise, eqn. (9) can be obtained from eqn. (10) with a knowledge of $L(q)$. With the convex monotonically decreasing $L(q)$ just considered, it can be seen from these equations that $f(\alpha)$ is only defined up to a finite maximum value $\alpha_{max}$. After this point, the probability distribution is zero. This means that $L(q)$’s of this type derive from highly non-gaussian multiplicative noise $\gamma_i$ (cf. eqn. (7)). The probability distribution of $\gamma_i$ must be zero beyond a certain value for this kind of curve, $L(q)$.

V. GAUSSIAN CASE

We will now solve for the averaged moments of $x(\tau)$. Since the noise is stationary, if a convergent moment exists then $\langle x^n(\tau) \rangle = \langle x^n(0) \rangle$. Solving eqn. (3) for $x(0)$ gives

$$x(0) = \int_0^\infty e^{-\int_0^s (1-\gamma(\tau))d\tau} \eta(s)ds$$

(13)

The equation for the 2nth moment averaged over both $\gamma$ and $\eta$ is

$$\langle x^{2n}(0) \rangle = \int_0^\infty \ldots \int_0^\infty \langle e^{-\sum_{i=1}^{2n} \int_0^{s_i} (1-\gamma(\tau))d\tau} \eta(s_1)\eta(s_2)\ldots\eta(s_{2n}) \rangle ds_1 \ldots ds_{2n}$$

(14)

Now we take $\eta$ to be to be gaussian white noise so that Wick’s theorem can be applied to the right hand side to separate out the average into all possible combinations of two point correlation functions. Each term gives the same contribution so one obtains

$$\frac{(2n)!}{2^n n!} \left(\frac{2T}{k}\right)^n \int_0^\infty \ldots \int_0^\infty \langle e^{-\sum_{i=1}^{2n} \int_0^{s_i} (1-\gamma(\tau))d\tau} \delta(s_1 - s_{n+1}) \ldots \delta(s_n - s_{2n}) \rangle ds_1 \ldots ds_{2n}$$

(15)

Now the limits of integration can be restricted to $s_1 > s_2 \ldots > s_n > 0$ at the expense of multiplying by $n!$, as each possible ordering of the $s_i$’s gives an equal contribution. The average with respect to $\gamma$ of the exponential can also be performed. This gives

$$\langle x^{2n}(0) \rangle =$$

$$(2n)! \left(\frac{T}{k}\right)^n \int_0^\infty \int_0^{s_1} \ldots \int_0^{s_{n-1}} e^{-2\sum_{i=1}^{n} s_i} + 2\int \int S(s)\sigma(s-s')S(s')dsds' \rangle ds_1 \ldots ds_n$$

(16)
Here we use the Heaviside function $\theta(s)$ to write

$$S(s) = \sum_{i=1}^{n} \theta(s_i - s). \quad (17)$$

This has the form of a staircase.

### A. White noise correlations

Before we proceed to analyze the case of general correlations in $\gamma$, we examine the case where

$$\sigma(\tau - \tau') = \sigma_o \delta(\tau - \tau'). \quad (18)$$

A differential equation for the time evolution of the probability, $P(x,t)$ can be derived and has the form of a generalized Fokker-Planck equation [14].

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left[ x - \frac{\sigma_o}{2}x + \frac{\partial}{\partial x} \left( \frac{\sigma_o}{2}x^2 + \frac{T}{k} \right) \right] P \quad (19)$$

The steady state solution corresponding to zero particle current, that is, no particles being created or destroyed at the boundaries, is found by setting the expression to the right of the $\frac{\partial}{\partial x}$ equal to 0. Solving this gives

$$P(x) \propto \frac{1}{\left( \frac{2T}{k\sigma_o} + x^2 \right)^{1/2}} \quad (20)$$

A related three dimensional version of this has been derived in the context of polymers in turbulent flow [1]. It shows that the large $x$ behavior of $P(x)$ has a power law tail $P(x) \sim x^{-1-2/\sigma_o}$. We can check that eqn. (16) is correct by substituting eqn. (18) into it. One can then decouple all of the $s_i$ integrations by making the change of variables

$$\Delta_1 \equiv s_2 - s_1, \Delta_2 \equiv s_3 - s_2, \ldots \Delta_{n-1} \equiv s_n - s_{n-1}, \Delta_n \equiv s_n \quad (21)$$

So the right hand side of eqn. (16) becomes

$$\frac{(2n)!}{2^n} \left( \frac{2T}{k} \right)^n \int_0^\infty \ldots \int_0^\infty e^{\sum_{i=1}^{n} (-2\sigma_o \Delta_i + 2\sigma_o i^2 \Delta_i)} d\Delta_1 \ldots d\Delta_n \quad (22)$$

Performing the integration gives
\[ \langle x^{2n} \rangle = \frac{(2n)!}{2^n} \left( \frac{T}{k} \right)^n \prod_{i=1}^{n} \frac{1}{1 - \sigma_\circ_i} \]  \hspace{1cm} (23)

This can be compared with the answer obtained from eqn. (20) by computing moments. The integrals can be done in closed form \cite{15} and give the identical result.

Clearly the precise form of the moments \( \langle x^{2n} \rangle \), and hence \( P(x) \), is dependent on the whole function \( \sigma(s) \). However we shall now see that the large \( x \) behavior of \( P(x) \) only depends on the total area under \( \sigma(s) \).

**B. General Asymptotic Behavior**

To determine the power law exponent of \( P(x) \) for large \( x \), for general correlations \( \sigma(s) \), one locates at which value of \( n \) eqn.(16) becomes divergent. That is if we define \( \alpha \) so that

\[ P(x) \propto x^{-\alpha} \text{ for large } x, \]

then \( \langle x^{2n} \rangle \) becomes divergent when \( 2n - \alpha = -1 \) or \( \alpha = 2n + 1 \).

For example for short range correlations, eqn. (23), becomes divergent when \( n = 1/\sigma_\circ \).

Therefore \( \alpha = 1 + 2/\sigma_\circ \), in agreement with eqn. (20). Therefore the object of this section is to locate the value of \( n \) where eqn. (16) becomes divergent, for general but normalizable \( \sigma(s) \).

To do this we rescale the \( s_i \) in eqn. (17)

\[ s \equiv s_1 p_2 \equiv \frac{s_2}{s_1}, \quad p_3 \equiv \frac{s_3}{s_1}, \ldots, \quad p_n \equiv \frac{s_n}{s_1} \]  \hspace{1cm} (24)

so that

\[ \langle x^{2n} \rangle = (2n)! \left( \frac{T}{k} \right)^n \int_0^{\infty} s^{n-1} F(s) ds F(s) \equiv \int_0^{1} \int_0^{p_3} \ldots \int_0^{p_n} e^{-2s} \int R(p) dp + 2s^2 \int \int R(p) \sigma(s(p-p')) R(p') dp dp' dp_2 \ldots dp_n \]  \hspace{1cm} (25)

\[ R(p) = \theta(1-p) + \sum_{i=2}^{n} \theta(p_i - p) \]

The divergence in \( \langle x^{2n} \rangle \) is therefore controlled by the large \( s \) behavior of \( F(s) \). For large \( s \) the function \( \sigma(s(p-p')) \) becomes very sharply peaked. One can define a limit of functions approaching a \( \delta \) function by

\[ \lim_{s \to \infty} \sigma(s(p-p')) = \frac{\delta(p-p')}{s} \sigma_\circ \]  \hspace{1cm} (26)

\[ \sigma_\circ = \int_{-\infty}^{\infty} \sigma(p) dp \]  \hspace{1cm} (27)
For large $s$, the leading order term in the exponent of the integrand is identical to the short range case

$$-2s\left[\int R(p)dp - \sigma_0 \int \int R(p)\delta(p-p')R(p')dpdp' + O(1/s^P)\right]$$

(28)

Where the exponent $P > 0$ depends on the form of $\sigma(s)$. As $n$ increases, the staircase function $R(p)$ becomes larger. Since the second term is quadratic, and the first term, linear in $R$, for large enough $n$ the expression inside the square brackets becomes positive so that the entire expression will increase linear with $s$, for some values of the $p_i$'s. This leads to a divergence. Lower order corrections in $s$, cannot change this point of divergence as it is determined by the sign of the slope of this expression for large $s$.

Therefore for large $s$, $F(s)$ will only depend on $\sigma_o$, the total area under $\sigma(s)$.

VI. SHORT RANGE NON-GAUSSIAN NOISE

We now relax the restriction that the multiplicative noise be Gaussian, but restrict the analysis to the case of short range correlations. In section II eqn. (3) was discretized. With additive noise included it can also be solved in discretized form, but the results are very similar looking to the continuous case. We can, with little loss of generality use the continuous form of the equation to analyze this problem. With continuous notation the equation for the Lyapunov exponents read

$$\langle e^{q \int_0^t \gamma(\tau)d\tau} \rangle = e^{L(q)t}$$

(29)

Defining $D \equiv 2T/k$ we can still salvage the previous calculation up to eqn. (15), with $(\gamma(\tau) - 1) \rightarrow \gamma(\tau)$, and have

$$\langle x(0)^{2n} \rangle = \frac{(2n)!}{2^n n!} D^n \int_0^\infty \ldots \int_0^\infty \langle e^{2 \sum_{i=1}^n \int_0^{\Delta_i} \gamma(\tau)d\tau} \rangle ds_1 \ldots ds_n.$$  

(30)

Using the change of variables of eqn. (21) and also eqn. (24) defining the Lyapunov exponents, this yields

$$\langle x(0)^{2n} \rangle = \frac{(2n)!}{2^n n!} D^n \int_0^\infty e^{\sum_{i=1}^n L(2i)\Delta_i} d\Delta_1 \ldots d\Delta_n.$$

(31)

Performing the integrations over the $\Delta_i$'s gives the final answer.
The above analysis has assumed that the discretization time step $\Delta t$ is very small so that
$L(q)\Delta t$ is small. Similar expressions to above can be derived for the discretized case without
this restriction on $\Delta t$ which amounts to replacing the last integration over the $\Delta$’s by a
discrete summation. This gives

$$\langle x(0)^{2n} \rangle = (2n)! \left( \frac{D}{2} \right)^n \prod_{i=1}^{n} \frac{1}{-L(2i)}$$

(A. Relation between the distribution and $L(q)$)

By knowing the moments of $x$ one can obtain its PDF. Now that we have an expression
for the moments eqn. (32) as a function of the Lyapunov exponents $L(q)$, we can consider
the problem of how different forms of $L(q)$ affect the probability distribution of $x$. Different
forms of $L(q)$ were shown in fig. 1 and discussed in section IV. The solid line in this figure
represents the case of finite $q^*$, the point beyond which the Lyapunov exponents become
positive. In this case eqn. (32) implies that for $2n > q^*$ the moments are not defined, for $q^*$
an integer. This implies that the p.d.f. has a power law tail $P(x)dx = x^{-q^*-1}$. This is in
agreement with the results found earlier for Gaussian multiplicative noise, and the heuristic
argument of section IV.

The other case to consider is when $L(q)$ is always negative for positive $q$. Consider the
case where the asymptotic form of $L(q)$ for large $q$ is proportional to $-q^\beta$. The restriction
of convexity implies $0 \leq \beta \leq 1$. To find the tail of $P(x)$, one starts by substituting
$L(2i) = -Ki^\beta$ in eqn. (32) and uses Stirlings approximation to simplify the expression.

$$\langle x^q \rangle = e^{q\ln(q-1)-\beta(q/2)(\ln(q/2)-1)+Cq}$$

where $C$ is a constant whose value will not be important in the conclusions to this analysis.
The asymptotic form of $P(x)$ can be obtained by a steepest descent analysis. In general

$$\exp(D(q)) \equiv \langle x^q \rangle = \int P(x)x^q dx$$

Changing variables in the last integration to $u = \ln x$ and performing steepest descent, which
is valid for large $q$, one has $D(q) = f(u(q)) + qu(q)$. Here $\exp(f(\ln x))d(\ln x) = P(x)dx$ and
\( u = u(q) \) is determined by the saddle point equation \( df/du = -q \). We want to solve for \( f(u) \) which can be done by Legendre transform to a different “ensemble” directly borrowing this method from thermodynamics. Then \( f(u) = D(q) - qu \) and \( dD/dq = u \). Carrying out the algebra yields the result

\[
P(x) \propto e^{-Kx^{2/(2-\beta)}} \tag{36}
\]

where \( K \) is a constant. This is stretched exponential behavior. An application of this to experimental systems will be discussed in the following paper. Note that when \( \beta \to 0 \), the tail becomes exponential. The limit where \( L(q) \) approaches a constant for large \( q \), corresponding to \( \beta = 0 \), has qualitatively different behavior for \( f(\alpha) \) defined in eqn. \( \beta \) for finite \( \beta \). In the former case \( f(\alpha) \) goes to negative infinity at a finite value \( \alpha = \alpha_{\text{max}} \), whereas in the latter case \( f(\alpha) \) goes to a finite value at \( \alpha = \alpha_{\text{max}} \).

**VII. CONCLUSIONS**

The PDF for eqn. \( \beta \) has been analyzed in one dimension under a variety of conditions, long range time correlations with gaussian statistics, and nongaussian noise with short range temporal correlations. It has been possible to classify the steady state PDF’s with additive noise by examining the Lyapunov exponents. When the Lyapunov exponents \( L(q) \) pass through zero at finite \( q \), the tails are power law. The exact exponent was worked out for Gaussian multiplicative noise and was found to be independent of the strength of the additive noise. If the Lyapunov exponents remain negative for all positive \( q \) it was argued that the PDF, \( P(x) \) should be of the form \( \log P(x) \propto -x^p \) where \( p \) is a power between 0 and 2.

It is interesting to note that many of the conclusions here hold for coupled many component equations of the same form. This will be seen in the following paper.

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FIGURES

FIG. 1. Examples of two different $L(q)$'s. The solid line represents an $L(q)$ that intersects the horizontal axis at finite $q$. The dashed line represents one that does not.
Fig. 1