Abstract: We develop an approach for investigating geometric properties of Gaussian multiplicative chaos (GMC) in an infinite dimensional set up. The base space is chosen to be the space of continuous functions endowed with Wiener measure, and the random field is a space-time white noise integrated against Brownian paths. In this set up, we show that in any dimension $d \geq 1$ and for any inverse temperature, the GMC-volume of a ball, uniformly around all paths, decays exponentially with an explicit decay rate. The exponential rate reflects the balance between two competing terms, namely the principal eigenvalue of the Dirichlet Laplacian and an energy functional defined over a certain compactification developed earlier in [MV14]. For $d \geq 3$ and high temperature, the underlying Gaussian field is also shown to attain very high values under the GMC -- that is, all paths are “GMC-thick” in this regime. Both statements are natural infinite dimensional extensions of similar behavior captured by $2d$ Liouville quantum gravity and reflect a certain “atypical behavior” of the GMC: while the GMC volume decays exponentially fast uniformly over all paths, the field itself attains atypically large values on all paths when sampled according to the GMC.

It is also shown that, despite the exponential decay of volume for any temperature, for small enough temperature, the normalized overlap of two independent paths tends to follow one of only a finite number of independent paths for most of its allowed time horizon, allowing the GMC probability to accumulate most of its mass along such trajectories.

1. Introduction and the main results.

1.1 Main results: informal description

Consider a centered Gaussian field $\{H(\omega)\}_{\omega \in \mathcal{C}}$ on a complete probability space $(\mathcal{E}, \mathcal{F}, P)$, with the field parametrized by a metric space $\mathcal{C}$ carrying a reference measure $\mu$. For any parameter $\gamma > 0$, the transformed measure

$$M_\gamma(d\omega) = \exp \left\{ \gamma H(\omega) - \frac{1}{2} \gamma^2 \mathbb{E}[H(\omega)^2] \right\} \mu(d\omega), \quad \gamma > 0,$$

on $\mathcal{C}$ is known as a Gaussian multiplicative chaos (GMC) which was constructed by Kahane in a fundamental work [K85]. In a general set up, the random field $H(\cdot)$ might include oscillations, which are however cancelled out after integration w.r.t. suitable test functions -- that is, $\{H(\omega)\}_{\omega \in \mathcal{C}}$ could be interpreted as a distribution. However, these oscillations (when intrinsically present) become highly magnified after exponentiating the field, so that a suitable mollification or a cut-off procedure becomes necessary to interpret the formally defined object (1.1) on a rigorous level. The results of the current

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article are strongly motivated by the ongoing investigations on finite dimensional GMC where the underlying object $H(\cdot)$ is a Gaussian free field in the complex plane leading to a GMC known as the Liouville quantum gravity, a random surface that appears as the scaling limit of random planar maps and exhibits remarkable geometric properties in terms of its multi-fractal spectrum and (non-integer) volume decay exponents, see Section 1.2 for a discussion drawing on analogies to the present results.

We investigate GMC measures in an infinite dimensional setting from a geometric viewpoint for the first time in the present article, whose results can be summarized as follows. The base space $\mathcal{C}_T = C([0, T]; \mathbb{R}^d)$ stands for the metric space of continuous functions $\omega : [0, T] \to \mathbb{R}^d$ equipped with the uniform metric. This space is tacitly endowed with the Wiener measure $\mathbb{P}_x$ corresponding to $\mathbb{R}^d$-valued Brownian path starting at $x \in \mathbb{R}^d$. The Gaussian field $\mathcal{H}_T(\cdot)$ we are interested in is driven by Gaussian space-time white noise $\dot{B}$ (see Section 2.1 for a precise definition) integrated against the Brownian path:

$$\mathcal{H}_T(\omega) = \int_0^T \int_{\mathbb{R}^d} \kappa(\omega_s - y) \dot{B}(s, y) dy ds,$$

so that $E[\mathcal{H}_T(\omega) \mathcal{H}_T(\omega')] = \int_0^T (\kappa \ast \kappa)(\omega_s - \omega'_s) ds$. (1.2)

Here the first integral above is interpreted in the Itô sense (with $\kappa$ being any smooth, spherically symmetric and compactly supported function with $\int_{\mathbb{R}^d} \kappa(x) dx = 1$) and $E$ denotes expectation w.r.t. the noise $\dot{B}$. Thus, the ambient field $\{\mathcal{H}_T(\omega)\}_{\omega \in \mathcal{C}_T}$ is now indexed by Wiener paths $\omega \in \mathcal{C}_T$, and analogous to (1.4), the renormalized GMC probability measure (at cut-off level $T$) is given by

$$\widehat{\mathcal{M}}_{\gamma, T}(d\omega) = \frac{1}{\mathcal{Z}_{\gamma, T}} \exp \left\{ \gamma \mathcal{H}_T(\omega) - \frac{1}{2} \gamma^2 T(\kappa \ast \kappa)(0) \right\} \mathbb{P}_0(d\omega),$$

(1.3)

where $\mathcal{Z}_{\gamma, T}$ is the total mass of the above exponential weight (under $\mathbb{P}_0$) and $\gamma > 0$ is a parameter known as inverse temperature. Our main result, stated formally in Theorem 2.1, quantifies the volume decay exponents of balls in $\mathcal{C}_T$ under $\widehat{\mathcal{M}}_{\gamma, T}$: if $\mathcal{N}_{\gamma, T}(\varphi)$ denotes a neighborhood of radius $r > 0$ around $\varphi \in \mathcal{C}_T$, then for any $d \geq 1$, $r > 0$ and $\gamma > 0$, it holds that

$$\sup_{\varphi \in \mathcal{C}_T} \frac{\mathcal{M}_{\gamma, T}[\mathcal{N}_{\gamma, T}(\varphi)]}{\mathcal{M}_{\gamma, T}(\omega)} \lesssim \exp[-\Theta T] \quad \mathbb{P} - \text{a.s.},$$

(1.4)

as $T \to \infty$, and for some deterministic and explicit constant $\Theta > 0$, which involves two competing terms, namely the principal eigenvalue $\lambda_1 > 0$ of the Dirichlet Laplacian $-\frac{1}{2} \Delta$ (which captures the volume decay rate that under the base measure $\mathbb{P}_0$, i.e., when $\gamma = 0$), and an additional energy functional minimized over (probability measures on) the translation-invariant compactification constructed in [MV14], see (2.6). For $d \geq 3$, for $\gamma > 0$ sufficiently small, the decay rate $\Theta$ simplifies and involves only the principal eigenvalue $\lambda_1$ and a penalization term $\gamma^2 (\kappa \ast \kappa)(0)/2$. A similar exponential lower bound on the GMC probability is also shown to hold pointwise. In the second part of Theorem 2.1 it is shown that

$$\frac{1}{T(\kappa \ast \kappa)(0)} \mathcal{M}_{\gamma, T}(\omega) \mathcal{M}_{\gamma, T}(d\omega) \to \gamma > 0, \quad \mathbb{P} - \text{a.s.} \text{ when } \gamma \in (0, \gamma_1), d \geq 3,$$

(1.5)

i.e., the field $\mathcal{H}_T(\omega)$ attains high values averaged w.r.t. $\mathcal{M}_{\gamma, T}$. Recall that $\text{Var}^{\mathbb{P}}[\mathcal{H}_T] = T(\kappa \ast \kappa)(0)$. The statements (1.4) and (1.5), combined together, confirm the following intuition on the atypical behavior of our GMC measure. Since the former statement implies that the GMC volume decays exponentially uniformly over all paths for any $\gamma > 0$, the only reason that $\mathcal{M}_{\gamma, T}$ could be non-trivial (in the limit $T \to \infty$) is if there are enough paths where the Gaussian field $\mathcal{H}_T$ is atypically large. Thus, it is conceivable that the GMC measure in some sense is “carried by” sufficiently many such thick paths, and it is natural to wonder what their level of thickness could be. The second statement (1.5) in Theorem 2.1 confirms this intuition – when $\gamma > 0$ remains sufficiently small and $d \geq 3$, all
paths are \( \hat{\mathcal{M}}_{\gamma,T} \)-thick, with \( \gamma(\kappa \ast \kappa)(0) \) being the required level of thickness; see [2.7] in Theorem 2.1

Similar results pertinent to GMC measure arising from a multiplicative-noise stochastic heat equation in \( d \geq 3 \) is derived in Corollary 2.2. To complete the picture, we also show that, even though GMC-volume decays exponentially for any temperature \( \gamma > 0 \), once we tune \( \gamma > 0 \) sufficiently large, the normalized overlap of two independent paths, sampled according to \( \hat{\mathcal{M}}_{\gamma,T} \), tends to follow one of only a finite number of independent paths for most of its allowed time horizon, allowing the GMC probability to accumulate most of its mass along such trajectories, see Theorem 2.3 for a precise statement (which holds when \( \gamma \) is at least as large as \( \gamma_1 \); and its complementary phase \( \gamma \in (0, \gamma_1) \) in \( d \geq 3 \) is the regime where all paths are GMC-thick, as underlined by (1.5), see also Remark 2). Let us now explain the background which motivated the current work.

1.2 Motivation: geometry of the Liouville measure.

In a finite dimensional setting, GMC measures share close connection to two-dimensional Liouville quantum gravity which has seen a lot of revived interest in the recent years, see [BP21] for an exposition. In this setting, the GMC measure (or the so-called Liouville measure) appears as the limit \( \mu_{\gamma,\varepsilon}(dx) = \varepsilon^{\Delta/2} \exp(\gamma h_\varepsilon(z)) dz \), with \( h_\varepsilon \) being a suitable approximation (e.g. circle average) of the (log-correlated) two-dimensional Gaussian free field (GFF) with \( \text{Var}(h_\varepsilon) = \log(1/\varepsilon) + O(1) \) and \( dz \) stands for the Lebesgue measure. A rigorous construction of \( \lim_{\varepsilon \to 0} \mu_{\gamma,\varepsilon} \) has been carried out in [K85, RV11, DS11, Ber17] and it is shown that when \( \gamma \in (0, 2) \), \( \mu_{\gamma,\varepsilon} \) converges toward a non-trivial measure \( \mu_\gamma \) which is diffuse and is known as the subcritical GMC; when \( \gamma \geq 2 \), we refer to [DRSV14-I, DRSV14-II, MRV16] for the notion of atomic structure of (super-) critical GMC.

Informally, the Liouville measure is a random surface carrying a Riemannian metric tensor (formally given by the exponential of the GFF) and a parametrization by a domain which preserves the inherent conformal invariance, but distorts the resulting metric and the volume. This distortion also reflects a certain “atypical behavior” of GMC measures: note that the weight \( \exp\{\gamma h_\varepsilon(z) - \frac{1}{2} \gamma^2 \mathbb{E}[h_\varepsilon(z)^2]\} \) vanishes as \( \varepsilon \to 0 \) for each fixed \( z \) (on some domain \( D \subset \mathbb{R}^2 \)), so a non-trivial limiting Liouville measure must be “carried” by sufficiently many thick points \( z \), that is those \( z \) where the field \( h_\varepsilon(z) \) is atypically large. In fact, if \( \gamma \in (0, 2) \) and \( z \in D \) is sampled according to \( \mu_\gamma \), then \( h_\varepsilon(z) \sim \log(1/\gamma) \to 0 \gamma > 0 \). Thus, (1.5) is the infinite-dimensional analogue of the latter assertion (recall that for GFF, \( \text{Var}(h_\varepsilon) = \log(1/\varepsilon) + O(1) \), while for our Gaussian field defined in (1.2), \( \text{Var}^\mathcal{M}(\mathcal{H}_T) = T(\kappa \ast \kappa)(0) \)).

Another prominent aspect of the Liouville measure, which has far reaching consequences, hinges on its multi-fractal spectrum and volume decay exponents on microscopic balls. Indeed, if \( z \) is sampled from the (renormalized) Liouville probability measure \( \mu_\gamma \), the (Liouville) scaling exponent \( \Delta \) of a set \( A \subset \mathbb{R}^2 \) is defined via the volume decay

\[
(P^{\text{GFF}} \otimes \mu_\gamma)[N_\varepsilon(A) \cap A \neq \emptyset] \sim \exp[-(\log 1/\varepsilon)\Delta] \quad \text{as } \varepsilon \downarrow 0,
\]

where \( N_\varepsilon(z) \) is a neighborhood in the Liouville metric [GM19]. Thus, the scaling exponent \( \Delta \) allows one to link the volume of a set in the Euclidean geometry to the same in the Liouville geometry.\footnote{For the Liouville measure \( \mu_\gamma \), it is known that for any \( q \in \mathbb{R} \), \( \mathbb{E}^{\text{GFF}}[\mu_\gamma(B(z, r)^q)] \sim r^\Theta \) where \( B = q(2 + \frac{x^2}{\varepsilon^2}) - \frac{\gamma^2 x^2}{2} \), which is a non-linear function of \( q \), is called the multi-fractal spectrum of the Liouville measure.} From this viewpoint, the first assertion (1.4) in Theorem 2.1 could be interpreted as an infinite dimensional analogue (in an almost sure sense) of (1.6) with the constant \( \Theta > 0 \) in (1.4) linking the volume under the Wiener measure to the same under the GMC measure \( \hat{\mathcal{M}}_{\gamma,T} \).

\footnote{The scaling exponent is a key object that also appears in the celebrated Knizhnik-Polyakov-Zamolodchikov (KPZ) formula [KPZ88] which dictates that if a subset \( A \) which is independent of the GFF, has Euclidean scaling exponent \( x \) (meaning \( P(A \cap B(z, \varepsilon) \neq \emptyset) \sim \varepsilon^x \), with \( B(z, \varepsilon) \) being the Euclidean ball of radius \( \varepsilon \) around \( z \)), then \( A \) has Liouville scaling exponent \( \Delta \); with \( x \) and \( \Delta \) being related by the quadratic relation \( x = \frac{\Delta^2}{2} + (1 - \frac{\Delta^2}{2}) \Delta \), see [RV11, DS11, BGRV16].}
2. Main results.

2.1 Exponential volume decay in GMC-space.

We fix any spatial dimension \( d \geq 1 \), write \( \mathcal{C}_T = C([0,T]; \mathbb{R}^d) \) for the metric space of continuous functions \( \omega : [0,T] \to \mathbb{R}^d \) equipped with the uniform norm
\[
\| \omega \|_{\infty, T} = \sup_{s \in [0,T]} |\omega(s)|, \quad \text{with} \quad \mathcal{N}_r(\omega) = \mathcal{N}_{r,T}(\omega) = \{ \varphi \in \mathcal{C}_T : \| \varphi - \omega \|_{\infty, T} < r \}.
\]
\( \mathcal{C}_T \) is tacitly equipped with the Wiener measure \( \mathbb{P}_x \) corresponding to an \( \mathbb{R}^d \)-valued Brownian motion starting at \( x \in \mathbb{R}^d \). For any \( t > 0 \), \( G_t \) will stand for the \( \sigma \)-algebra generated by the path \( (\omega_s)_{0 \leq s \leq t} \) until time \( t \). Let \( (\mathcal{E}, \mathcal{F}, \mathbb{P}) \) be a complete probability space and \( \hat{B} \) denotes a Gaussian space-time white noise, which is independent of the Brownian path defined above. In other words, if \( \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d) \) denotes the space of rapidly decreasing Schwartz functions, \( \{ \hat{B}(f) \}_{f \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d)} \) is a centered Gaussian process with covariance \( \mathbb{E}[\hat{B}(f) \hat{B}(g)] = \int_0^\infty \int_{\mathbb{R}^d} f(t,x) g(t,x) dx dt \) for \( f, g \in \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^d) \), with \( \mathbb{E} \) denoting expectation w.r.t. \( \mathbb{P} \).

We also fix a nonnegative function \( \kappa \) which is smooth, spherically symmetric and is supported in a ball \( B_{1/2}(0) \) of radius 1/2 around 0 and normalized to have total mass \( \int_{\mathbb{R}^d} \kappa(x) dx = 1 \). For any fixed Brownian path \( \omega \in \mathcal{C}_T \) we define the Itô integral
\[
\mathcal{H}_T(\omega) = \int_0^T \int_{\mathbb{R}^d} \kappa(\omega_s - y) \hat{B}(s,dy)ds, \quad \text{with} \quad \mathbb{E}[\mathcal{H}_T^2(\omega)] = T(\kappa \ast \kappa)(0).
\]
For any \( \gamma > 0 \), the resulting (renormalized) Gaussian multiplicative chaos is a probability measure on the space \( \mathcal{C}_T \) defined as
\[
\tilde{\mathcal{M}}_{\gamma, T}(d\omega) = \frac{1}{Z_{\gamma, T}} \exp \left( \gamma \mathcal{H}_T(\omega) - \frac{\gamma^2 T}{2} (\kappa \ast \kappa)(0) \right) \mathbb{P}_0(d\omega), \quad \text{with} \quad Z_{\gamma, T} = \mathbb{E}_0 [\exp \left( \gamma \mathcal{H}_T(\omega) - \frac{\gamma^2 T}{2} (\kappa \ast \kappa)(0) \right)].
\]

To define the decay rate \( \Theta \) appearing in (1.2), we need some further definitions. We denote by \( \mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^d) \) (resp. \( \mathcal{M}_{\leq 1} \)) the space of probability (resp. sub-probability) measures on \( \mathbb{R}^d \), which acts as an additive group of translations on these spaces. Let \( \tilde{\mathcal{M}}_1 = \mathcal{M}_1 / \sim \) be the quotient space of

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\(^{\dagger \dagger}\)From the viewpoint of directed polymers, results of the article are new, to the best of our knowledge.
$\mathcal{M}_1$ under this action, that is, for any $\mu \in \mathcal{M}_1$, its orbit is defined by $\tilde{\mu} = \{\mu \ast \delta_x : x \in \mathbb{R}^d\} \in \tilde{\mathcal{M}}_1$.

The quotient space $\tilde{\mathcal{M}}_1$ can be embedded in a larger space

$$\tilde{X} := \left\{ \xi : \xi = \{\tilde{\alpha}_i\}_{i \in I}, \alpha_i \in \mathcal{M}_{\leq 1}, \sum_{i \in I} \alpha_i(\mathbb{R}^d) \leq 1 \right\}$$

which consists of all empty, finite or countable collections of orbits from $\tilde{\mathcal{M}}_{\leq 1}$ whose masses add up to at most one. The space $\tilde{X}$ and a metric structure there was introduced in [MV14], and it was shown that under that metric, $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ is densely embedded in $\tilde{X}$ and any sequence in $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ converges along some subsequence to an element $\xi$ of $\tilde{X}$ — that is, $\tilde{X}$ is the compactification of the quotient space $\mathcal{M}_1(\mathbb{R}^d)$, see Section 3.1 for details. On the space $\tilde{X}$ we define an energy functional

$$F_\gamma(\xi) = \frac{1}{2} \sum_{\alpha \in \xi} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\kappa \ast \kappa)(x_1 - x_2) \prod_{j=1}^2 \alpha(dx_j) \quad \forall \xi \in \tilde{X}, \quad \text{and}$$

$$E_{F_\gamma}(\vartheta) = \int_{\tilde{X}} F_\gamma(\xi) \vartheta(d\xi) \quad \vartheta \in \mathcal{M}_1(\tilde{X}).$$

Here $\mathcal{M}_1(\tilde{X})$ denotes the space of probability measures on the space $\tilde{X}$. There is an interesting connection between the structure of the space $\tilde{X}$ and the solution of the variational problem $\sup_{\vartheta \in \mathcal{M}_1(\tilde{X})} E_{F_\gamma}(\vartheta)$: Indeed, there is a non-empty, compact subset $\mathfrak{m}_\gamma \subset \mathcal{M}_1(\tilde{X})$ consisting of the maximizer(s) of the variational problem $\sup_{\vartheta \in \mathfrak{m}_\gamma} E_{F_\gamma}(\vartheta)$, the maximizing set is a singleton $\delta_0 \in \tilde{X}$ for $d \geq 3$ and small enough $\gamma$, and any maximizer assigns positive mass only to those elements of the compactification $\tilde{X}$ whose total mass add up to one, see Proposition 4.5 for details. We are now ready to state our first main result.

**Theorem 2.1** (Scaling exponents of GMC in the Wiener space). Fix any $d \in \mathbb{N}$, $\gamma > 0$ and $r > 0$.

1. **$\mathbb{P}$-almost surely,**

$$\limsup_{T \to \infty} \sup_{\varphi \in \mathcal{G}_T} \frac{1}{T} \log \tilde{\mathcal{H}}_{\gamma,T}[\mathcal{G}_T(\varphi)] \leq -\Theta, \quad \text{where}$$

$$\Theta := \frac{1}{4} \lambda_1(\sqrt{2}r) - \frac{\gamma^2(\kappa \ast \kappa)(0)}{2} + \left[ \frac{1}{2} \sup_{\vartheta \in \mathfrak{m}_\gamma} E_{F_{\gamma}}(\vartheta) - \sup_{\vartheta \in \mathfrak{m}_\gamma} E_{F_{\gamma}}(\vartheta) \right]. \quad \text{(2.5)}$$

Moreover, for any $d \in \mathbb{N}$ and $\gamma > 0$, there exists $r_0 = r_0(d, \gamma)$ such that for $r \in (0, r_0)$, $\Theta > 0$. Conversely, for any $d \in \mathbb{N}$ and $r > 0$, there is $\gamma_c > 0$ such that for $\gamma \in (0, \gamma_c)$, $\Theta > 0$. Finally, for $d \geq 3$ and $\gamma > 0$ sufficiently small,

$$\Theta = \frac{1}{4} \lambda_1(\sqrt{2}r) - \frac{\gamma^2(\kappa \ast \kappa)(0)}{2} > 0.$$  

Finally, for any $d \in \mathbb{N}, \gamma > 0, r > 0$, an exponential lower bound similar to (2.5) also holds pointwise.

2. **Fix $d \geq 3$ and $\gamma > 0$ sufficiently small. Then,**

$$\lim_{T \to \infty} \mathbb{E} \left[ \tilde{\mathcal{H}}_{\gamma,T} \left[ \frac{\mathcal{H}_T(\cdot)}{T} \right] \right] = \gamma(\kappa \ast \kappa)(0) > 0 \quad \mathbb{P} - \text{a.s.} \quad \text{(2.7)}$$

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$^1$Here and in the sequel, $\lambda_1(r)$ will denote for the principal eigenvalue of $-\Delta$ on a ball of radius $r$ around the origin with Dirichlet boundary condition.
Remark 1 (Kahane’s GMC and Theorem 2.1) As remarked earlier, the statements (2.5)-(2.7) imply exponential volume decay, underline the emergence of thick paths and determine the level of their thickness; recall the discussion below (1.5) in Section 1.1. It would be quite intriguing to prove existence of the infinite-volume limit \( \lim_{T \to \infty} \mathcal{M}_{\gamma,T}(\cdot) \) in the full uniform integrability/weak disorder phase and study properties of this limit as done by Kahane [K85] for log-correlated fields. It is reasonable to expect the results of Theorem 2.1 to be helpful in this pursuit. Indeed, the existence of the 2d Liouville measure, already in the \( L^2 \)-phase \( \gamma \in (0, \sqrt{2}) \) requires good estimates for the \( L^2 \)-norm \( \mathbb{E}[|\mu_{\gamma,\varepsilon}(N) - \mu_{\gamma,\varepsilon/2}(N)|^2] \) under the approximating Liouville measure \( \mu_{\gamma,\varepsilon} \) (e.g. defined via the circle average of the GFF), see the construction of Berestycki [Ber17]. Extending the limit to the uniform integrable phase \( \gamma \in (0, 2) \) requires additional information on thick points as one needs to remove points that are thicker than a prescribed level of typical thickness (again see [Ber17]). However, because of the present infinite dimensional set up (and lack of local compactness), additional difficulties are likely to arise, tackling which seems to go beyond the scope of the current article. □

It is also useful to record the following reformulation of the above results concerning a GMC measure arising from the solution of the multiplicative noise stochastic heat equation in \( d \geq 3 \), which can be written as an Itô SDE

\[
du_\varepsilon(t,x) = \frac{1}{2} \Delta u_\varepsilon(t,x) + \gamma(\varepsilon, d) u_\varepsilon(t,x) \hat{B}_\varepsilon(t,x) dt, \quad \text{with} \quad \gamma(\varepsilon, d) = \gamma \varepsilon \frac{d-2}{2},
\]

and with the initial condition is \( u_\varepsilon(0,x) = 1 \). Here \( \hat{B}_\varepsilon(t,x) = (\hat{B} \ast \kappa_\varepsilon)(t,x) = \int_{\mathbb{R}^d} \kappa_\varepsilon(x-y) \hat{B}(t,y) dy \) is a (spatially) mollified noise in \( d \geq 3 \) and \( \kappa_\varepsilon = \varepsilon^{-d} \kappa(x/\varepsilon) \) with \( \kappa \) as before, so that \( \int_{\mathbb{R}^d} \kappa_\varepsilon(x) dx = 1 \) and \( \kappa_\varepsilon(x) dx \Rightarrow \delta_0 \) weakly as probability measures as \( \varepsilon \to 0 \). By Feynman-Kac formula, the solution to (2.8) is given by

\[
u_\varepsilon(t,x) = \mathbb{E}_x \left[ \exp \left\{ \gamma(\varepsilon, d) \int_0^t \int_{\mathbb{R}^d} \kappa_\varepsilon(\omega_s-y) \hat{B}(t-s, dy) ds - \frac{\gamma(\varepsilon, d)^2}{2} (\kappa_\varepsilon \ast \kappa_\varepsilon)(0) \right\} \right].
\]

This solution (resp. the Cole-Hopf solution \( h_\varepsilon := \log u_\varepsilon \) of the Kardar-Parisi-Zhang equation in \( d \geq 3 \)) is directly linked with the total mass \( \mathcal{Z}_{\gamma,T} \) (resp. the free energy \( \log \mathcal{Z}_{\gamma,T} \)) of the GMC measure (2.3), and this link was first observed and used in [MSZ16] (see also [M17, CCM19, CCM19-II, BM19-II, CNN20, LZ20] for further progress). The above representation (2.9) leads to the renormalized GMC probability measure

\[
\mathcal{M}_{\gamma,\varepsilon,t}(d\omega) = \frac{1}{\mathcal{Z}_{\gamma,\varepsilon,t}} \exp \left\{ \gamma \varepsilon \frac{d-2}{2} \int_0^t \int_{\mathbb{R}^d} \kappa_\varepsilon(\omega_s-y) \hat{B}(t-s, dy) ds - \frac{\gamma^2 \varepsilon^{-2} t}{2} (\kappa_\varepsilon \ast \kappa_\varepsilon)(0) \right\} \mathbb{P}_x(d\omega)
\]

with \( \mathcal{Z}_{\gamma,\varepsilon,t} \) being the normalizing constant, as usual. Here is our next result, for which we will also write \( \mathcal{M}_{\gamma,\varepsilon,t} = \mathcal{M}_{\gamma,\varepsilon,t}^{(0)} \).

Corollary 2.2 (Exponential decay of GMC corresponding to SHE in \( d \geq 3 \)). Fix \( d \geq 3 \). Then for any \( \gamma > 0 \) and \( t > 0 \),

\[
\lim_{\varepsilon \downarrow 0} \sup_{\varphi \in \mathcal{C}_t} \varepsilon^2 \log \mathcal{M}_{\gamma,\varepsilon,t}(\mathcal{M}_{\varepsilon,t}(\varphi)) \leq (-\Theta t) \quad \text{in} \ P - \text{probability},
\]

with \( \Theta > 0 \) determined in Theorem 2.7 and with \( \mathcal{M}_{\varepsilon,t}(\varphi) = \{ \omega \in \mathcal{C}_t : \| \omega - \varphi \|_{\infty, t} \leq \varepsilon \} \).

*One such difficulty might already arise from the covariance structure of the underlying field \( \text{Cov}[\mathcal{M}_T(\omega), \mathcal{M}_T(\omega')] = \int_0^T (\kappa \ast \kappa)(\omega_s - \omega'_s) ds \) which depends on the mollification scheme \( \kappa \) of the white-noise field \( \hat{B} \). In contrast, note that \( \text{Var}(h_\varepsilon(z)) = (1/\delta) + R(z, D) \) (with \( R(z, D) \) being the conformal radius of the simply connected domain \( D \) viewed from \( z \)) does not depend on any mollification scheme if we consider the circle average approximation \( h_\varepsilon \) of the 2d GFF \( h \).
Let us now turn to the regime when $\gamma$ is chosen to be large. In this regime, for localization results on the endpoint distributions $Q_{\gamma,t} = \hat{\mathcal{M}}_{\gamma,t}[\omega_t \in \cdot]$ we refer to [CSY03, V07, CC13, BC16, BM19-II]. For the discrete lattice, for log-correlated Gaussian fields (e.g. for $2d$ discrete GFF) it has been shown in [AZ14, AZ15, MRV16] that for large $\gamma$, the normalized covariance of two points sampled from the Gibbs measure is either 0 or 1 and the joint distribution of the Gibbs weights converges in a suitable sense to that of a Poisson-Dirichlet variable and similar results can be found in [BC19] for general Gaussian disordered systems in the lattice setting. The following result extends such statements concerning overlap localization to the current GMC set up.

**Theorem 2.3.** Let $\text{Cov}_T(\omega,\omega') = \frac{1}{T} E[\mathcal{H}_T(\omega)\mathcal{H}_T(\omega')]$ (recall (1.2)). Fix $\gamma > 0$ such that $\gamma$ is a point of differentiability of $\lambda(\gamma) := \inf_{\theta \in \mathbb{R}} \{ \frac{\gamma^2(\kappa \ast \kappa)(0)}{2} - \mathcal{E}_{F,\theta}(0) \}$ (recall (2.1)) and moreover assume that $\lambda'(\gamma) < \gamma(\kappa \ast \kappa)(0)$. Then for every $\varepsilon > 0$ there exist $\delta, T_0 > 0$ and an integer $k \in \mathbb{N}$ and $\omega^{(i)}, ..., \omega^{(k)} \in \mathcal{G}_\infty$ such that

$$P\left[ \mathcal{M}_{\gamma,T}\left( \bigcup_{i=1}^{k} \text{Cov}_T(\omega^{(i)}, \omega^{(k+i)}) \geq \delta \right) \geq 1 - \varepsilon \right] \geq 1 - \varepsilon$$

for all $T \geq T_0$.

**Remark 2** Assuming that $\lambda(\gamma)$ is differentiable at $\gamma$, we always have $\lambda'(\gamma) \leq \gamma(\kappa \ast \kappa)(0)$ and actually for $\gamma < \gamma_1$ (with $\gamma_1$ from Proposition 2.5), we have $\lambda'(\gamma) = \gamma(\kappa \ast \kappa)(0)$ (cf. Proposition 4.1 part (ii)). However, the requirement concerning the strict bound $\lambda'(\gamma) < \gamma(\kappa \ast \kappa)(0)$ in Theorem 2.3 is related to $\gamma$ being large, at least as large as $\gamma_1$. Therefore, if $\gamma$ is sufficiently large such that Theorem 2.3 holds, we necessarily have $\gamma \geq \gamma_1$ — that is, Theorem 2.3 can not hold if $\gamma \in (0, \gamma_1)$ in $d \geq 3$, which is the regime in which $\mathcal{H}_T$ attains large values under $\hat{\mathcal{M}}_{\gamma,T}$, as shown in Theorem 2.1 part (2). \hfill $\square$

### 2.2 Outline of the proofs.

For the convenience of the reader, we briefly describe the main ingredients of the proof. Showing the estimate (2.5) and the identity (2.7) in Theorem 2.1 rely on a large deviation route combined with tools from Malliavin calculus. First, the free energy $\log E_0[e^{\gamma \mathcal{H}_T} \mathbb{I}_A]$ on the path space is decomposed using Itô calculus into a martingale and an integral functional $F(\mu_t) = \int \int V(\cdot - y) \mu_t(dx) \mu_t(dy)$ of the GMC distribution $\mu_t := \mathcal{M}_t[\omega_t \in \cdot]$. The first key observation is that, while $\mu_t$ is itself a probability measure on $\mathbb{R}^d$, the function $F(\cdot)$ depends only on its orbit $\bar{\mu}_t = (\mu_t \ast \delta_x)_{x \in \mathbb{R}^d}$. These orbits are elements of the quotient space $\tilde{\mathcal{M}}_1 = \mathcal{M}_1 / \sim$ of probability measures $\mu \in \mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^d)$ (here $\mu \sim \nu$ if and only if $\nu = \mu \ast \delta_x$ for some $x \in \mathbb{R}^d$). However, since both spaces $\mathcal{M}_1$ as well as its quotient $\tilde{\mathcal{M}}_1$ are non-compact in the (quotient) weak topology, the functional $F$ does not retain any continuity property (even when it is extended to $\tilde{\mathcal{M}}_1$). For the same reasons, a Markovian semigroup evolved by $\mu_t$ (which was constructed in [BC16] for proving localization of discrete directed polymers\footnote{This approach also used the topology of [MV14] but worked with a different (but equivalent) metric which works well in the discrete lattice. Presently in the continuum, we employ the metric directly from [MV14].}) also does not yield an invariant measure in $\tilde{\mathcal{M}}_1$. This is where the construction of [MV14] becomes useful: the quotient $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ can be embedded in the compactified space $\tilde{\mathcal{X}}$, which, in contrast to $\tilde{\mathcal{M}}_1$, is now sufficiently large to contain all limit points of the GMC distribution $\mu_t \in \mathcal{M}_1(\mathbb{R}^d)$ (but still sufficiently small so that the quotient $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ is densely embedded in $\tilde{\mathcal{X}}$\footnote{For example, let $\mu_n$ be a Gaussian mixture $\frac{1}{2} N(n,1) + \frac{1}{2} N(-n,1) + \frac{1}{2} N(0, n)$. While neither $\mu_n$, nor any of its components, has a weak limit in $\mathcal{M}_1(\mathbb{R}^d)$, the sequence $(\mu_n)_n \subset \mathcal{M}_1(\mathbb{R}^d) \hookrightarrow \tilde{\mathcal{X}}$ converges (in the topology of $\tilde{\mathcal{X}}$) along a subsequence to the element $\xi = (\tilde{\alpha}_1, \tilde{\alpha}_1) \in \tilde{\mathcal{X}}$, where $\tilde{\alpha}_1$ is the orbit any Gaussian (i.e., with arbitrary mean) with variance 1.}). By construction, one can then “lift” the above functional $F(\cdot)$ (recall (2.4)) as well
as the Markovian dynamics on this compactification \( \tilde{X} \). The topology of \( \tilde{X} \) (resulting from leveraging the metric directly from [MV14]) then guarantees an invariant measure of the semigroup. Together with the required continuity properties of the aforementioned functionals on \( \tilde{X} \), we then deduce the desired exponential decay of the GMC volume \( \sup_{\varphi \in \mathcal{F}_T} \tilde{M}_T[A \tilde{\kappa}_T(\varphi)] \), with a decay rate which is given by the variational formula on the compactified space \( \tilde{X} \) which always admits minimizer(s) in \( \mathcal{M}_1(\tilde{X}) \) (this minimizer is unique when \( \gamma > 0 \) is small and \( d \geq 3 \) which further simplifies the variational formula). The existence of thick points (2.7) then follow from the above arguments together with some tools from Malliavin calculus. It should be mentioned that the current approach does not rely on sub-additivity arguments which have been previously used as a powerful tool in obtaining deep results about the free energy of directed polymers [CY06]). Here, the uniform exponential decay of the GMC volume (for any temperature \( \gamma \)) results from rather explicit variational formulas involving functionals over (the space of probability measures on) \( \tilde{X} \). From the viewpoint of the discussion in Section 1.2, the decay rates here are the analogues of the scaling exponents of the Liouville measure.

For proving Theorem 2.3 we also transport tools from Malliavin calculus to the current set up: define the (infinite dimensional) Ornstein-Uhlenbeck operator \( \mathcal{L} = -\delta \circ D \) on the abstract Wiener space \( (\mathcal{E}, \mathcal{F}, \mathbb{P}) \) (with \( D \) being the Malliavin derivative and \( \delta \) being “divergence” acting as an adjoint of \( D \)) and deduce a Poincaré inequality \( \text{Var}( \int_0^t \mathcal{L} f_T(\mathbb{B}_r)dr ) \leq 2t \mathbb{E}(\|D f_T(\mathbb{B}_r)\|_{L^2([0,T] \otimes \mathbb{R}^d)})^2 \) for \( f_T(\mathbb{B}, \mathbb{B}) = \frac{1}{\mathbb{p}} \log Z_{\gamma,T} = \frac{1}{\mathbb{p}} \log \mathbb{E}_0[\exp\{\gamma \mathcal{H}_T(\mathbb{B}, \mathbb{B})\}] \). Here \( \mathbb{B}_t(s, x) = e^{-t} \mathbb{B}(s, x) + e^{-t} \eta((e^{2t} - 1)^{-1}s, x) \) is a white noise flow, with \( \eta \) being an independent copy of the space-time white noise \( \mathbb{B} \). Applying Chebyshev’s inequality, we then get a bound \( \mathbb{P}\left[ \int_0^t (\mathcal{L} f_T(\mathbb{B}_r)dr > \gamma \mathbb{p}/2 \right] \leq C(tT^{\mathbb{p}})^{-1} \) for all \( \mathbb{p}, t > 0 \), and together with the assumption that \( \lambda(\gamma) := \lim_{T \to \infty} f_T(\gamma, \mathbb{B}) \) is differentiable, and \( \lambda'(\gamma) < \gamma (\kappa \star \kappa)(0) \), we obtain a localization property showing that for \( t = t(\mathbb{p}, \gamma) \) sufficiently large, \( \lim\inf_{T \to \infty} \mathbb{P}\left[ \frac{1}{t} \int_0^t ds \mathbb{E}_0(\mathbb{B}_s)^{\mathbb{p}} (\kappa \star \kappa)(0) \right] \geq \alpha \geq 1 - \frac{\gamma}{2} \). This builds on a recent technique [BC19] developed for showing localization of general Gaussian disordered systems (on discrete lattices). Together with the fact \( \mathbb{E}_T[|\mathcal{H}_T - T\lambda'(\gamma)|] = o(T) \) (which holds true under the imposed hypotheses) we then have the required concentration of the covariance in Theorem 2.3.

**Organization of the rest of the article:** The rest of the article is organized as follows: In Section 3 we collect the properties of the space \( \tilde{X} \), those of the relevant energy functions defined on \( \tilde{X} \) and \( \mathcal{M}_1(\tilde{X}) \) and their Malliavin derivatives, while the proofs of Theorem 2.1, Theorem 2.3 constitute Section 4.

**Notation:** For convenience, we will adopt the following notation throughout the sequel concerning the GMC measure defined in (2.3).

\[
\begin{align*}
\tilde{M}_T &= \tilde{M}_{\gamma,T}, \\
\mathcal{F}_T &= \mathcal{F}_{\gamma,T} = Z_{\gamma,T} e^{-\frac{\gamma^2}{2} T(\kappa \star \kappa)(0)}, \quad \text{with} \quad Z_T = Z_{\gamma,T} = \mathbb{E}_0[e^{\gamma \mathcal{H}_T}] \\
V &= \kappa \star \kappa.
\end{align*}
\] (2.11)

Also, unless otherwise specified, expectation with respect to the GMC probability measure \( \tilde{\mathcal{M}}_T \) will be written as \( \tilde{\mathbb{E}}_T \). For two independent Brownian motions \( \omega, \omega' \), we write for the product GMC probability

\[
\tilde{\mathcal{M}}_{\omega, \omega'}(d\omega, d\omega') = \frac{1}{\mathcal{L}^2} \exp \left\{ \gamma (\mathcal{H}_T(\omega) + \mathcal{H}_T(\omega')) - \gamma^2 TV(0) \right\} \mathcal{P}_{\gamma,T}(d\omega, d\omega')
\] (2.12)

and expectation with respect to \( \tilde{\mathcal{M}}_{\omega, \omega'} \) will be written as \( \tilde{\mathbb{E}}_{\omega, \omega'} \). Finally, for any \( t > 0 \), \( \mathcal{G}_t \) will stand for the \( \sigma \)-algebra generated by the Brownian path \( (\omega_s)_{0 \leq s \leq t} \). For any \( A \in \mathcal{G}_t \), we will write

\[
Z_t(A) = \mathbb{E}_0[\mathbb{1}_A e^{\gamma \mathcal{H}_T}].
\] (2.13)
3. The space $\tilde{X}$, energy functionals and its Malliavin derivatives.

3.1 The space $\tilde{X}$.

We denote by $\mathcal{M}_1 = \mathcal{M}_1(\mathbb{R}^d)$ (resp., $\mathcal{M}_{\leq 1}$) the space of probability (resp., subprobability) distributions on $\mathbb{R}^d$ and by $\tilde{\mathcal{M}}_1 = \mathcal{M}_1 / \sim$ the quotient space of $\mathcal{M}_1$ under the action of $\mathbb{R}^d$ (as an additive group on $\mathcal{M}_1$), that is, for any $\mu \in \mathcal{M}_1$, its orbit is defined by $\tilde{\mu} = \{\mu \ast \delta_x : x \in \mathbb{R}^d\} \in \tilde{\mathcal{M}}_1$. Then we define

$$\tilde{X} = \left\{ \xi : \xi = \{\tilde{\alpha}_i\}_{i \in I}, \alpha_i \in \mathcal{M}_{\leq 1}, \sum_{i \in I} \alpha_i(\mathbb{R}^d) \leq 1 \right\}$$

(3.1)

to be the space of all empty, finite or countable collections of orbits of subprobability measures with total masses $\leq 1$. Note that the quotient space $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ is embedded in $\tilde{X}$ – that is, for any $\mu \in \mathcal{M}_1(\mathbb{R}^d)$, $\tilde{\mu} \in \tilde{\mathcal{M}}_1(\mathbb{R}^d)$ and the single orbit element $\{\tilde{\mu}\}$ in $\tilde{X}$ belongs to $\tilde{X}$ (in this context, sometimes we will write $\tilde{\mu} \in \tilde{X}$ for $\{\tilde{\mu}\} \in \tilde{X}$).

The space $\tilde{X}$ also comes with a metric structure. If for any $k \geq 2$, $\mathcal{H}_k$ is the space of functions $h : (\mathbb{R}^d)^k \to \mathbb{R}$ which are invariant under rigid translations and which vanish at infinity, we define for any $h \in \mathcal{H} = \bigcup_{k \geq 2} \mathcal{H}_k$, the functionals

$$\mathcal{J}(h, \xi) = \sum_{\tilde{\alpha} \in \xi} \int_{(\mathbb{R}^d)^k} h(x_1, \ldots, x_k) \alpha(dx_1) \cdots \alpha(dx_k).$$

(3.2)

A sequence $\xi_n$ is desired to converge to $\xi$ in the space $\tilde{X}$ if

$$\mathcal{J}(h, \xi_n) \to \mathcal{J}(h, \xi) \quad \forall h \in \mathcal{H}.$$

This leads to the following definition of the metric $D$ on $\tilde{X}$. For any $\xi_1, \xi_2 \in \tilde{X}$, we set

$$D(\xi_1, \xi_2) = \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{1}{1 + \|h_r\|_{\infty}} \left| \mathcal{J}(h_r, \xi_1) - \mathcal{J}(h_r, \xi_2) \right|$$

$$= \sum_{r=1}^{\infty} \frac{1}{2^r} \frac{1}{1 + \|h_r\|_{\infty}} \left| \sum_{\tilde{\alpha} \in \xi_1} \int h_r(x_1, \ldots, x_{k_r}) \prod_{i=1}^{k_r} \alpha(dx_i) \right| - \left| \sum_{\tilde{\alpha} \in \xi_2} \int h_r(x_1, \ldots, x_{k_r}) \prod_{i=1}^{k_r} \alpha(dx_i) \right|.$$

The following result was proved in [MV14] Theorem 3.1-3.2.

**Theorem 3.1.** We have the following properties of the space $\tilde{X}$.

- $D$ is a metric on $\tilde{X}$ and the space $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ is dense in $(\tilde{X}, D)$.
- Any sequence in $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ has a convergent subsequence with a limit point in $\tilde{X}$. Thus, $\tilde{X}$ is the completion and the compactification of the totally bounded metric space $\tilde{\mathcal{M}}_1(\mathbb{R}^d)$ under $D$.
- Let a sequence $(\xi_n)_n$ in $\tilde{X}$ consist of a single orbit $\tilde{\gamma}_n$ and $D(\xi_n, \xi) \to 0$ where $\xi = (\tilde{\alpha}_i)_i \in \tilde{X}$ such that $\alpha_1(\mathbb{R}^d) \geq \alpha_2(\mathbb{R}^d) \geq \ldots$. Then given any $\varepsilon > 0$, we can find $k \in \mathbb{N}$ such that $\sum_{i>1} \alpha_i(\mathbb{R}^d) < \varepsilon$ and we can write $\gamma_n = \sum_{i=1}^{k} \alpha_{n,i} + \beta_n$, such that
  - for any $i = 1, \ldots, k$, there is a sequence $(a_{n,i})_n \subset \mathbb{R}^d$ such that $\alpha_{n,i} \ast \delta_{a_{n,i}} \Rightarrow \alpha_i$ with $\lim_{n \to \infty} \inf_{i \neq j} |a_{n,i} - a_{n,j}| = \infty$.
  - The sequence $\beta_n$ totally disintegrates, meaning for any $r > 0$, $\sup_{x \in \mathbb{R}^d} \beta_n(B_r(x)) \to 0$. 

\[\square\]
3.2 Energy functionals on $\widetilde{X}$.

We now define some key functionals on the space $\widetilde{X}$ which will be quite useful in the present context. First, we set $F_\gamma : \widetilde{X} \to \mathbb{R}$ to be

$$F_\gamma(\xi) = \frac{\gamma^2}{2} \sum_{i \in I} \int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1 - x_2) \prod_{j=1}^2 \alpha_i(dx_j), \quad \xi = (\bar{\alpha}_i)_{i \in I}. \quad (3.3)$$

Because of shift-invariance of the integrand in $(3.3)$, $F_\gamma$ is well-defined on $\widetilde{X}$. Moreover, we have

**Lemma 3.2.** $F_\gamma$ is continuous and non-negative on $\widetilde{X}$, and $F_\gamma(\cdot) \leq \frac{\gamma^2}{2} V(0)$.

**Proof.** For the continuity of $F_\gamma$, we refer to [MV14, Corollary 3.3]. Recall that $V = \kappa \ast \kappa$ and $\kappa$ is rotationally symmetric. Hence, for any $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$, by Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}^d} V(x_1 - x_2) \alpha(dx_1) \alpha(dx_2) = \int_{\mathbb{R}^d} \alpha(dx_1) \alpha(dx_2) \int_{\mathbb{R}^d} dz \kappa(x_1 - z) \kappa(x_2 - z) \leq \int_{\mathbb{R}^d} \alpha(dx_1) \alpha(dx_2) \left[ \int_{\mathbb{R}^d} dz \kappa^2(x_1 - z) \right]^{1/2} \left[ \int_{\mathbb{R}^d} dz \kappa^2(x_2 - z) \right]^{1/2} \leq \alpha(\mathbb{R}^d)^2 \|\kappa\|_2^2. \quad (3.4)$$

Thus, $F_\gamma(\xi) \leq \frac{\gamma^2(\kappa \ast \kappa)(0)}{2} \sum_{i \in I} (\alpha_i(\mathbb{R}^d))^2 \leq \frac{\gamma^2(\kappa \ast \kappa)(0)}{2}$ since for $\xi = (\bar{\alpha}_i)_{i \in I} \in \widetilde{X}$ we have $\sum_{i \in I} \alpha_i(\mathbb{R}^d) \leq 1$. Moreover, since $V = \kappa \ast \kappa$ is non-negative, also $F_\gamma(\cdot) \geq 0$. \hfill $\square$

Next, for any $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$, let

$$\mathcal{G}_t(\alpha) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \alpha(dz) \mathbb{E}_z \left[ \mathbb{1}\{ \omega_t \in dx \} \exp \left\{ \gamma \mathcal{H}_t(\omega) - \frac{\gamma^2}{2} t V(0) \right\} \right]$$

and note that for any $a \in \mathbb{R}^d$ and $t > 0$, $\mathcal{G}_t(\alpha_i) \overset{(d)}{=} \mathcal{G}_t(\alpha_i \ast \delta_a)$. Hence, we may define $\mathcal{G}_t, \mathcal{R}_t : \widetilde{X} \to \mathbb{R}$ as

$$\mathcal{G}_t(\xi) = \sum_{i} \mathcal{G}_t(\alpha_i), \quad \mathcal{R}_t(\xi) = \mathcal{G}_t(\xi) + \mathbb{E}\left[ \mathcal{R}_t - \mathcal{G}_t(\xi) \right] \quad \forall \xi = (\bar{\alpha}_i)_{i \in I} \in \widetilde{X}. \quad (3.5)$$

Next, for any $t > 0$, and for $\xi = (\bar{\alpha}_i)_{i \in I} \in \widetilde{X}$, we set

$$\alpha_i^{(t)}(dx) := \frac{1}{\mathcal{G}_t(\xi)} \int_{\mathbb{R}^d} \alpha_i(dz) \mathbb{E}_z \left[ \mathbb{1}\{ \omega_t \in dx \} \exp \left\{ \gamma \mathcal{H}_t(\omega) - \frac{\gamma^2}{2} t V(0) \right\} \right]$$

$$\xi^{(t)} := (\bar{\alpha}_i^{(t)})_{i \in I} \in \widetilde{X}. \quad (3.6)$$

Recall that $\mathcal{G}_t(\alpha_i) \overset{(d)}{=} \mathcal{G}_t(\alpha_i \ast \delta_a)$. Likewise, we also have $(\alpha_i \ast \delta_a)^{(t)}(dx) \overset{(d)}{=} (\alpha_i^{(t)} \ast \delta_a)(dx)$. Recall that $\mathcal{M}_1(\widetilde{X})$ denotes the space of probability measures on $\widetilde{X}$. For any $\vartheta \in \mathcal{M}_1(\widetilde{X})$, then $\mathcal{G}_t$ further defines a transition kernel

$$\Pi_t(\vartheta, d\xi') = \int_{\widetilde{X}} \pi_t(\xi, d\xi') \vartheta(d\xi) \quad \text{where} \quad \pi_t(\xi, d\xi') = \mathbb{P}[\xi^{(t)} \in d\xi' | \xi] \in \mathcal{M}_1(\widetilde{X}). \quad (3.7)$$

With the above definition, let us record two useful facts.

**Lemma 3.3.** The set

$$\mathbb{m}_\gamma = \{ \vartheta \in \mathcal{M}_1(\widetilde{X}) : \Pi_t \vartheta = \vartheta \text{ for all } t > 0 \} \quad (3.8)$$

of fixed points of $\Pi_t$ is a non-empty, compact subset of $\mathcal{M}_1(\widetilde{X})$. 
Proof. Note that $m_\gamma \neq \emptyset$, because $\delta_0 \in m_\gamma$. Moreover, by the definition of the metric $D$ on $\tilde{X}$ and by the resulting convergence criterion determined by Theorem 3.1, the map $\tilde{X} \ni \xi \mapsto \pi_t(\xi, \cdot)$ is continuous. This property, together with the compactness of $\tilde{X}$ (and therefore also that of $M_1(\tilde{X})$), we have that $M_1(\tilde{X}) \ni \theta \mapsto \Pi_t(\theta, \cdot)$ is continuous too for any $t > 0$. It follows that $m_\gamma$ is a closed subset of the compact metric space $M_1(\tilde{X})$, implying the compactness of $m_\gamma$.

For the next lemma we let $\mathcal{L}(\tilde{X})$ denote the space of all Lipschitz functions $f : \tilde{X} \to \mathbb{R}$ with Lipschitz constant at most 1 and $f(0) = 0$.

**Lemma 3.4.** For $\theta, \theta' \in M_1(\tilde{X})$ let $\mathcal{W}(\theta, \theta') = \sup_{f \in \mathcal{L}(\tilde{X})} \left| \int_{\tilde{X}} f(\xi) \theta(\xi) d\xi - \int_{\tilde{X}} f(\xi) \theta'(\xi) d\xi \right|$ be the Wasserstein metric on $M_1(\tilde{X})$. If $Q_t := \mathcal{M}_{[\omega_t]} \in \mathcal{M}_1(\mathbb{R}^d)$ (so that $Q_t \in \tilde{X}$) and $\nu_T := \frac{1}{T} \int_0^T \delta_{Q_t} dt \in M_1(\tilde{X})$, then for any $s \geq 0$, $\mathbf{P}$-almost surely, $\mathcal{W}(\nu_T, \Pi_s \nu_T) \to 0$ as $T \to \infty$.

*Proof.* Let $f : \tilde{X} \to \mathbb{R}$ be any Lipschitz function vanishing at $\tilde{0} \in \tilde{X}$ and having Lipschitz constant at most 1. Then for any $t \in [0, s)$, $M_{n,t} = \sum_{k=0}^{n} (f(Q_{t+k+s}) - E[f(Q_{t+k+s})|F_{t+k}])$ is an $(F_{t+(n+1)s})_{n \in \mathbb{N}_0}$ martingale (here $F_t$ is the sigma-algebra generated by the noise $\tilde{B}$ up to time $t$). Then by the Burkholder-Davis-Gundy inequality, for some constant $C > 0$, $E[M_{n,t}^2] \leq C(n+1)^2$, and thus, by Jensen’s inequality

$$E\left[\left(\int_0^s f(\tilde{Q}_{t+s}) - E[f(\tilde{Q}_{t+s})|F_t] dt\right)^4\right] \leq Cn^2.$$ 

for some $C = C(s, f) > 0$. Hence $P\left(\left| \int_0^s f(\tilde{Q}_{t+s}) - E[f(\tilde{Q}_{t+s})|F_t] dt \right| \geq (sn)^{4/5} \right)$ is summable and it follows from Borel-Cantelli Lemma that

$$\limsup_{T \to \infty} T^{-4/5} \left| \int_0^s f(\tilde{Q}_{t+s}) - E[f(\tilde{Q}_{t+s})|F_t] dt \right| \leq C(s, f) (3.9)$$

almost surely. Since $f$ has Lipschitz constant at most 1, we also have (choosing $n = \lfloor T/s \rfloor$) $\left| \int_0^s f(\tilde{Q}_{t+s}) - E[f(\tilde{Q}_{t+s})|F_t] dt \right| \leq 2s$. Now for any $s \geq 0$, let $\nu_T^{(s)} = \frac{1}{T} \int_0^T \delta_{Q_{t+s}} dt$, so that $\nu_T = \nu_T^{(0)}$. Then (with $\Pi_s$ as in (3.7)), the above display implies that, for $T > 0$ sufficiently large, we have that almost surely

$$\left| \int_{\tilde{X}} f(\xi) \nu_T^{(s)}(d\xi) - \int_{\tilde{X}} f(\xi) \Pi_s \nu_T(d\xi) \right| \leq C(s, f) T^{-1/5}.$$ 

On the other hand,

$$\left| \int_{\tilde{X}} f(\xi) \nu_T(d\xi) - \int_{\tilde{X}} f(\xi) \nu_T^{(s)}(d\xi) \right| = \frac{1}{T} \int_0^T f(\tilde{Q}_t) dt - \frac{1}{T} \int_s^{T+s} f(\tilde{Q}_t) dt \leq \frac{2s}{T}.$$ 

Combining the last two displays, and using triangle inequality, we obtain that almost surely

$$\limsup_{T \to \infty} T^{1/5} \left| \int_{\tilde{X}} f(\xi) \nu_T(d\xi) - \int_{\tilde{X}} f(\xi) \Pi_s \nu_T(d\xi) \right| \leq C(s, f). (3.10)$$

By the definition of the metric $D$ on $\tilde{X}$, for any $f \in \mathcal{L}(\tilde{X})$, $\sup_{\xi \in \tilde{X}} |f(\xi)| \leq \sup_{\xi \in \tilde{X}} D(\xi, \tilde{0}) \leq 2$ and thus, $\mathcal{L}(\tilde{X})$ forms an equicontinuous family which is closed in the uniform norm. By Ascoli’s theorem, this space is then compact and is also separable. If $\Psi_T(f) := \frac{1}{T} \int_0^T f(\tilde{Q}_t) - E[f(\tilde{Q}_{t+s})|F_t] dt$, then given any $f, g \in \mathcal{L}(\tilde{X})$ with $\|f - g\|_\infty < \delta$, we have $|\Psi_T(f) - \Psi_T(g)| < 2\delta$. Thus $(\Psi_T(\cdot))_{T \geq 0}$ is equicontinuous on the compact metric space $\mathcal{L}(\tilde{X})$, and by (3.10) this family converges pointwise to 0 on a dense subset $(f_n)_n$. Thus, by applying Arzela-Ascoli theorem once more, we obtain that this convergence is uniform, which, together with (3.10) implies the corollary.

$\square$
3.3 Malliavin calculus and free energy derivatives.

We recall some rudimentary facts from Malliavin calculus (see e.g. [N06]) and its consequences for our Hamiltonian $\mathcal{H}_T$. Let $(\mathcal{E}, \mathcal{F}, \mathbb{P})$ be a complete probability space carrying a centered Gaussian process $\{\hat{B}(t)\}_{t \in [0,T] \otimes \mathbb{R}^d}$ with covariance structure $\mathbb{E}[\hat{B}(h)\hat{B}(g)] = \langle h, g \rangle_{L^2([0,T] \otimes \mathbb{R}^d)}$.

For any square integrable random variable $F$ on $(\mathcal{E}, \mathcal{F}, \mathbb{P})$, the Malliavin derivative $DF$ is (when it exists) a random element of $L^2([0,T] \otimes \mathbb{R}^d)$, that can be viewed as a space-time indexed stochastic process $DF = (D_{t,x}F)_{t,x}$. In a particular set up, if

$$F = f(\hat{B}(h_1), \ldots, \hat{B}(h_n))$$

for a $C^\infty$-function $f : \mathbb{R}^n \to \mathbb{R}$, then, the Malliavin derivative is defined as

$$DF = \sum_{i=1}^n \partial_i f(\hat{B}(h_1), \ldots, \hat{B}(h_n))h_i. \quad (3.12)$$

The iterated derivative $D^{(k)}F$ is a random element in the tensor product $L^2([0,T] \otimes \mathbb{R}^d) \otimes \cdots \otimes L^2([0,T] \otimes \mathbb{R}^d)$.

For any $p \geq 1$ and any positive integer $k \geq 1$, note that

$$\|F\|_{k,p} = \left[ \mathbb{E}[|F|^p] + \sum_{i=1}^k \mathbb{E}[\|D^{(i)}F\|_{(L^2)^{\otimes i}}^p] \right]^{1/p}$$

defines a semi-norm, and as the domain of the Malliavin derivative $D$ in $L^p(\mathbb{P})$ is denoted by $\mathbb{D}^{(1,p)}$ in the sense that $\mathbb{D}^{(1,p)}$ is the closure of the class of random variables of the form (3.11) with respect to the norm $\| \cdot \|_{1,p}$. Similarly, $\mathbb{D}^{(k,p)}$ will stand for the completion of the family of smooth random variables with respect to the norm $\| \cdot \|_{k,p}$.

With our space-time white noise $\hat{B}$ in our particular set up, note that for a fixed Brownian path $\omega$, the object $\mathcal{H}_T(\omega, \hat{B}) = \int_0^T \int_{\mathbb{R}^d} \kappa(y - \omega_s) \hat{B}(s,y)dyds$ can be reinterpreted as

$$\hat{B}(h) = \int_0^T \int_{\mathbb{R}^d} h(s,y) \hat{B}(s,y)dyds, \quad \text{with} \quad h(s,y) = \kappa(y - \omega_s) \in L^2([0,T] \otimes \mathbb{R}^d).$$

In particular, if $n = 1$ and $f(x) = x$, then the definition (3.12) dictates that $D\hat{B}(h) = h$ and we have the following implications pertinent to the Malliavin derivative of $\mathcal{H}_T(\omega)$ and the free energy

$$f_T(\hat{B}) = f_T(\gamma, \hat{B}) = \frac{1}{T} \log Z_T = \frac{1}{T} \log \mathbb{E}_0[ e^{\gamma \int_0^T \int_{\mathbb{R}^d} \kappa(y - \omega_s) \hat{B}(s,y)dyds}]. \quad (3.13)$$

Note that

$$T^2 \mathbb{E}[f_T(\hat{B})^2] \leq \mathbb{E}_0[ \mathbb{E}_0[ e^{\gamma \mathcal{H}_T(\omega, \hat{B})} ] + \mathbb{E}_0[ \mathbb{E}_0[ e^{-\gamma \mathcal{H}_T(\omega, \hat{B})} ] = 2e^{\frac{\gamma}{2} TV(0)} < \infty, \quad (3.14)$$

where we used that for any $x > 0$, $(\log(x))^2 \leq x + x^{-1}$ and Jensen’s inequality. Hence, for any $T, \gamma \geq 0$, $f_T(\hat{B}) \in L^2(\mathbb{P})$. Moreover, note that (with $\mathbb{E}_T$ denoting expectation w.r.t. $\mathcal{H}_T$, cf. (2.11)),

$$f_T'(\gamma, \hat{B}) := \frac{\partial}{\partial \gamma} f_T(\gamma, \hat{B}) = \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \mathbb{E}_T[\kappa(y - \omega_s)] \hat{B}(s,y)dyds \quad (3.15)$$

so that, by Itô isometry and Jensen’s inequality,

$$T^2 \mathbb{E}[(f_T'(\gamma, \hat{B}))^2] \leq \int_0^T \int_{\mathbb{R}^d} \mathbb{E}[\mathbb{E}_T[\kappa^2(y - \omega_s)]dyds = T(\kappa \ast \kappa)(0) < \infty. \quad (3.16)$$

Recall (2.12); we will also need the following expressions for the Malliavin derivatives of $\mathcal{H}_T$ and that of the free energy $f_T$:
Lemma 3.5. For any \( t > 0 \) and \( x \in \mathbb{R}^d \),

- \( D_{t,x}[\mathcal{H}_T(\omega)] = \kappa(x - \omega_t) \) and
  \[
  D_{t,x}[f_T(\hat{B})] = \frac{\gamma}{T} \mathbb{E}_T[\kappa(x - \omega_t)].
  \] (3.17)

- Moreover, the second Malliavin derivative of the free energy is given by
  \[
  D^{(2)}_{t,x}[f_T(\hat{B})] = \frac{\gamma^2}{T} \left( \mathbb{E}_T[\kappa(x - \omega_t)^2] - \mathbb{E}_T[\kappa(x - \omega_t)\kappa(x - \omega_t')] \right)
  \] (3.18)

where \( \mathbb{E}_T \) denotes expectation w.r.t. the product GMC measure \( \mathcal{M}_T \) defined w.r.t. two independent Brownian paths.

- For any smooth random variable \( F \) of the form (3.11), we have
  \[
  \mathbb{E}[F \hat{B}(h)] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} h D_{t,x} F dxdt \right].
  \] (3.19)

\textit{Proof.} Note that the first assertion is a consequence of the definition of Malliavin derivative, while (3.17) follows from the chain rule and the fact

\[
D_{t,x}[f_T(\hat{B})] = \frac{\gamma}{T} \mathbb{E}_0[\kappa(x - \omega_t)e^{\gamma \mathcal{H}_T(\omega)}].
\]

Also, the assertion (3.18) follows from

\[
D^{(2)}_{t,x}[f_T(\hat{B})] = \frac{\gamma^2}{T} \mathbb{E}_0[\kappa(x - \omega_t)^2 e^{\gamma \mathcal{H}_T(\omega)}] - \frac{\gamma^2}{T} \mathbb{E}_0[\kappa(x - \omega_t) e^{\gamma \mathcal{H}_T(\omega)}] \left( \frac{\mathbb{E}_0[\kappa(x - \omega_t) e^{\gamma \mathcal{H}_T(\omega)}]}{(\mathbb{E}_0[\kappa(x - \omega_t)e^{\gamma \mathcal{H}_T(\omega)}])^2} \right).
\]

Finally, (3.19) is an easy consequence of integration by parts for Malliavin calculus that asserts that on any Hilbert space \( H \) and \( h \in H \),

\[
\mathbb{E}[F \hat{B}(h)] = \mathbb{E}[(D_{t,x} F, h)_H].
\] (3.20)

\[ \square \]

3.4 Flows on path space and a Poincaré inequality

Using the tools from Malliavin calculus from previous section, the goal of the current section is to prove a Poincaré inequality defined w.r.t. a flow for an infinite dimensional Ornstein-Uhlenbeck (O-U) process.

Let us first recall the definition of an Ornstein-Uhlenbeck operator defined w.r.t. a standard Gaussian measure \( \mu \) in finite dimensions \( \mathbb{R}^n \). Note that if \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable, then its gradient \( \nabla f : \mathbb{R}^n \to \mathbb{R}^n \) defines a vector field and the divergence \( \delta : \mathbb{R}^n \to \mathbb{R} \) can be thought of as an “adjoint” for \( \nabla \) in the Hilbert space \( L^2(\mathbb{R}^n) \), i.e. \( \delta \) acts on a vector field \( v : \mathbb{R}^n \to \mathbb{R}^n \) via the relation \( \mathbb{E}[\nabla f \cdot v] = \mathbb{E}[f \delta v] \). Via this relation we also have, for any \( v = (v^{(1)}, \ldots, v^{(n)}) \) with \( v^{(i)} : \mathbb{R}^n \to \mathbb{R} \) and \( f : \mathbb{R}^n \to \mathbb{R} \) continuously differentiable, by integration by parts

\[
\mathbb{E}[\nabla f \cdot v] = \sum_{i=1}^n \int_{\mathbb{R}^n} \partial_i f(x) v^{(i)}(x) \mu(dx) = \sum_{i=1}^n \int_{\mathbb{R}^n} f(x)(x_i v^{(i)}(x) - \partial_i v^{(i)}(x)) \mu(dx),
\]

and consequently, \( \delta v = \sum_{i=1}^n (x_i v^{(i)} - \partial_i v^{(i)}) \). In particular, the latter identity implies for \( v = f \nabla g : \mathbb{R}^n \to \mathbb{R}^n \)
and any sufficiently smooth \( f, g : \mathbb{R}^n \to \mathbb{R} \), that
\[
\delta(f \nabla g) = \sum_{i=1}^{n} \left[ x_i f(x) \frac{\partial g}{\partial x_i} - \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} + f(x) \frac{\partial^2 g}{\partial x_i^2} \right) \right] = -\nabla f \cdot \nabla g - f(\Delta g - x \cdot \nabla g). \tag{3.21}
\]

Now we can define the Ornstein-Uhlenbeck operator \( \mathcal{L} \) as
\[
\mathcal{L} = -\delta \circ \nabla, \tag{3.22}
\]
and given \( 3.21 \) (for the particular choice \( f = 1 \)), the above definition reduces to
\[
\mathcal{L} g = -\delta(\nabla g) = \Delta g - x \cdot \nabla g. \tag{3.23}
\]

The above finite dimensional setup can be translated to the abstract Gaussian space \((\mathcal{E}, \mathcal{F}, \mathbb{P})\) too by replacing the gradient by the Malliavin derivative \( D \) defined before, while the divergence \( \delta \) acts as an adjoint of \( D \). In other words, for any \( F \in H = \mathbb{D}^{(1,2)} \) and \( u \) in the domain of \( \delta \), we have
\[
\mathbb{E}[\langle DF, u \rangle_H] = \mathbb{E}[\delta(u)]
\]
and the Ornstein-Uhlenbeck operator \( \mathcal{L} \) for \( \mathcal{E} \) then is defined by
\[
\mathcal{L} = -\delta \circ D. \tag{3.24}
\]

Recall the integration by parts formula for Malliavin calculus \( 3.20 \). We apply that formula to the product of the two random variables \( F, G \) of the form \( 3.11 \). Then,
\[
\mathbb{E}[G(DF, h)_H] = -\mathbb{E}[\langle DG, h \rangle_H] + \mathbb{E}[FG\hat{B}(h)].
\]

If \( u \), which is in the domain of \( \delta \) has the form \( u = \sum_{j=1}^{n} F_j h_j \), then by the last display,
\[
\mathbb{E}[\langle DF, u \rangle_H] = \sum_{j=1}^{n} \mathbb{E}[F_j \langle DF, h_j \rangle_H] = \sum_{j=1}^{n} -\mathbb{E}[F(DF_j, h_j)_H] + \mathbb{E}[FF_j \hat{B}(h_j)]
\]
and we conclude that \( \delta(u) = \sum_{j=1}^{n} F_j \hat{B}(h_j) - \langle DF, h \rangle_H \). For the Ornstein-Uhlenbeck operator \( \mathcal{L} \), this means \( \mathcal{L} u = -\delta(Du) = \sum_{j=1}^{n} \langle D^{(2)} F_j, h_j \rangle_H - DF_j \hat{B}(h_j) \). Applying this theory in our setting to the functional \( f_T(\hat{B}) \), the Ornstein-Uhlenbeck operator has the form
\[
(\mathcal{L} f_T)(\hat{B}) = \int_0^T \int_{\mathbb{R}^d} D^{(2)} f_T(\hat{B}) \, dx \, dt - \hat{B}(D f_T(\hat{B})). \tag{3.25}
\]

Let \( \eta \) be space-time white noise which is an independent copy of \( \dot{B} \). That is, \( \{\eta(f)\}_{f \in L^2([0,T] \otimes \mathbb{R}^d)} \) is a centered Gaussian process with covariance \( \mathbb{E}[\eta(f_1)\eta(f_2)] = \langle f_1, f_2 \rangle_{L^2([0,T] \otimes \mathbb{R}^d)} \).

We define the Ornstein Uhlenbeck flow of \( \hat{B} \) at time \( t \geq 0 \) by
\[
\hat{B}_t(s, x) = e^{-t} \dot{B}(s, x) + e^{-t} \eta((e^{2t} - 1)^{-1}s, x) \text{ if } t > 0, \quad \hat{B}_0 = \dot{B}. \tag{3.26}
\]
Recall that \( \lambda \theta^{d/2} \dot{B}(\lambda^2 s, \theta x) \) has the same law as that of \( \dot{B}(s, x) \) for any \( \lambda, \theta > 0 \). Since \( \eta \) is an independent copy of \( \dot{B} \), it follows that for any fixed \( t > 0 \), \( \{\dot{B}_t(s)\}_{s} \) is also a centered Gaussian process with the same covariance structure \( \mathbb{E}[\dot{B}_t(f_1)\dot{B}_t(f_2)] = \langle f_1, f_2 \rangle_{L^2([0,T] \otimes \mathbb{R}^d)} \).

Therefore, we can define
\[
\mathcal{H}_T(\omega, \hat{B}_t) = \int_0^T \int_{\mathbb{R}^d} \kappa(y - \omega_s) \hat{B}_t(s, y) \, dy \, ds, \quad \mathcal{M}_T(\hat{B}_t) = \frac{1}{Z_T(\hat{B}_t)} \exp \{ \gamma \mathcal{H}_T(\omega, \hat{B}_t) \} \mathbb{P}_0(d\omega), \quad Z_T(\hat{B}_t) = \mathbb{E}_0[ \exp \{ \gamma \mathcal{H}_T(\omega, \hat{B}_t) \}]. \tag{3.27}
\]
For expectation $\mathbb{E}^{\mathcal{H}_T}(\hat{B}_t)$ with respect to the probability measure $\mathcal{H}_T(\hat{B}_t)$ we write $\mathcal{E}_T^{\mathcal{H}_T}(\hat{B}_t)$. And if $\omega, \omega'$ are two independent Brownian motions, we write

$$\mathcal{H}_T^\omega(\hat{B}_t) = \frac{1}{Z_T(\hat{B}_t)^2} \exp \left\{ \gamma (\mathcal{H}_T(\omega, \hat{B}_t) + \mathcal{H}_T(\omega', \hat{B}_t)) \right\} \mathbb{P}^\omega_0 (d\omega, d\omega')$$

and for expectation with respect to the probability measure $\mathcal{H}_T^\omega(\hat{B}_t)$ we write $\mathcal{E}_T^{\mathcal{H}_T^\omega}(\hat{B}_t)$.

We also need

**Lemma 3.6.** For any $T, \gamma > 0$,

$$\mathcal{L} f_T(\hat{B}_t) = \gamma^2 V(0) - \frac{\gamma^2}{T} \int_0^T \mathbb{E}^{\mathcal{H}_T^\omega}( [V(\omega_s - \omega'_s)] ds - \gamma f_T(\gamma, \hat{B}_t) \in L^2(\mathbb{P}).$$

(3.28)

for $f_T$ as in (3.13) and $f_T(\gamma, \hat{B}_t) = \frac{\partial}{\partial \gamma} f_T(\gamma, \hat{B}_t)$.

**Proof.** Indeed, recall (3.17) and (3.18) for the first two Malliavin derivatives of $f_T$. Then,

$$\hat{B}_t(Df_T(\hat{B}_t)) = \int_0^T \int_{\mathbb{R}^4} \mathbb{E}^{\mathcal{H}_T^\omega}( [V(\omega_s - \omega'_s)] \hat{B}_t(s, \omega) d\omega ds$$

(3.29)

and with the Ornstein-Uhlenbeck generator, see (3.25), we have (3.28). Moreover, since $0 \leq V(\cdot) \leq V(0)$, by using (3.16), we have $\mathbb{E}[\mathcal{L} f_T(\hat{B}_t)^2] = \mathbb{E}[\mathcal{L} f_T(B)^2] \leq C(\gamma^4 V(0)^2 + \|f'(\gamma, \cdot)\|^2_{L^2(\mathbb{P})}) < \infty$. □

The following Poincaré inequality will be quite useful in the context of proving Theorem 2.23.

**Lemma 3.7.** Let $f_T(\hat{B}_t)$ be the functional defined in (3.13) w.r.t. the flow $\hat{B}_t$ defined in (3.26). Then, for any $T$ and $\gamma$,

$$\text{Var}^\mathbb{P} \left( \frac{1}{t} \int_0^t \mathcal{L} f_T(\hat{B}_t) dr \right) \leq 2 \mathbb{E} \left[ \|Df_T(B)\|^2_{L^2([0, T] \otimes \mathbb{R}^d)} \right].$$

**Proof.** Let Var and Cov stand for variance and covariance w.r.t. $\mathbb{P}$, while $(\mathcal{F}_t)_{t \geq 0}$ stands for the Ornstein-Uhlenbeck semigroup. That is, for $t \geq 0$, and a test function $g \in L^2(\mathbb{P})$ defined on the path space of the white noise so that $\mathcal{L} g \in L^2(\mathbb{P})$,

$$(\mathcal{F}_t g)(\hat{B}) = \mathbb{E}[g(\hat{B}_t)|\hat{B}].$$

Then, for any $0 \leq s \leq t$,

$$\text{Cov}(g(\hat{B}_s), g(\hat{B}_t)) = \text{Cov}(g(\hat{B}_s), \mathbb{E}[g(\hat{B}_t)|\hat{B}_s]) = \text{Cov}(g(\hat{B}_s), \mathcal{F}_{t-s} g(\hat{B}_s)) = \text{Cov}(g(\hat{B}), \mathcal{F}_{t-s} g(\hat{B})).$$

Now let $\{\psi_j\}_{j \geq 0}$ be an orthonormal basis of $L^2(\mathbb{P})$ consisting of eigenfunctions of $\mathcal{L}$, with $\psi_0 \equiv 1$, $\mathcal{L} \psi_0 = \lambda_0 \psi_0 = 0$ and $\mathcal{L} \psi_j = -\lambda_j \psi_j$ with $\lambda_j > 0$ for $j \geq 1$. Then, for $g = \sum_{j \geq 0} a_j \psi_j \in L^2(\mathbb{P})$, we have

$$\mathcal{L} g = -\sum_{j \geq 1} \lambda_j a_j \psi_j, \quad \mathcal{F}_t \mathcal{L} g = -\sum_{j \geq 1} \lambda_j a_j e^{-\lambda_j t} \psi_j.$$

Further, if $g_1 = \sum_{j \geq 0} a_j \psi_j$, $g_2 = \sum_{j \geq 0} b_j \psi_j \in L^2(\mathbb{P})$, then

$$\text{Cov}(g_1(\hat{B}), g_2(\hat{B})) = \sum_{j \geq 1} a_j b_j$$

and if in addition $D^{(2)} g_1, D^{(2)} g_2$ exist, then

$$-\mathbb{E}[g_1(\hat{B}) \mathcal{L} g_2(\hat{B})] = \mathbb{E}[D g_1(\hat{B}) D g_2(\hat{B})].$$

(3.30)
Hence, \( \text{Cov}(\mathcal{L}g(\hat{B}_t), \mathcal{L}g(\hat{B}_t)) = \text{Cov}(\mathcal{L}g(\hat{B}_0), \mathcal{P}_{t-s}\mathcal{L}g(\hat{B}_0)) = \sum_{j \geq 1} \lambda_j^2 a_j^2 e^{-\lambda_j(t-s)} \). Again, if \( g = \sum_{j \geq 0} a_j \psi_j \), then by (3.30),

\[
\mathbf{E}\|Dg(\hat{B})\|^2 = -\mathbf{E}[g(\hat{B})\mathcal{L}g(\hat{B})] = \sum_{j \geq 1} \lambda_j a_j^2.
\]

Thus,

\[
\int_0^t \text{Cov}(\mathcal{L}g(\hat{B}_r), \mathcal{L}g(\hat{B}_r))dr = \sum_{j \geq 1} \int_0^t \lambda_j^2 a_j^2 e^{-\lambda_j(t-r)}dr = \sum_{j \geq 1} \lambda_j a_j^2 (1 - e^{-\lambda_j t}) \leq \mathbf{E}\|Dg(\hat{B})\|^2
\]

and finally

\[
\text{Var}\left( \frac{1}{t} \int_0^t \mathcal{L}g(\hat{B}_r)dr \right) = \frac{1}{t^2} \int_0^t \int_0^t \text{Cov}(\mathcal{L}g(\hat{B}_r), \mathcal{L}g(\hat{B}_s))d\sigma d\sigma \leq \frac{2}{t} \mathbf{E}\|Dg(\hat{B})\|^2.
\]

We now choose \( g(\hat{B}) = f_T(\gamma, \hat{B}) \) and since \( f_T, f_T \in L^2(\mathbf{P}) \), we apply the above bound. \( \square \)

**Corollary 3.8.** For any \( \varepsilon > 0 \) and \( \gamma > 0 \),

\[
\mathbf{P}\left[ \frac{1}{t} \int_0^t (\mathcal{L}f_T)(\hat{B}_r)dr > \frac{\gamma \varepsilon}{2} \right] \leq \frac{8V(0)}{t \varepsilon^2}.
\]

**Proof.** Recall that \( \{\psi_j\}_{j \geq 0} \) is an orthonormal basis of \( L^2(\mathbf{P}) \) consisting of eigenfunctions of \( \mathcal{L} \), with \( \psi_0 \equiv 1 \). Since \( \mathcal{L}f_T(\cdot) = \sum_{j \geq 1} a_j \lambda_j \psi_j(\cdot) \) and \( \psi_j \perp 1 \) for all \( j \geq 1 \), we have \( \mathbf{E}\left[\mathcal{L}f_T(\hat{B}_t)\right] = 0 \). Then by Lemma 3.7 and by Jensen’s inequality,

\[
\text{Var}\left[ \frac{1}{t} \int_0^t (\mathcal{L}f_T)(\hat{B}_r)dr \right] \leq \frac{2}{t} \mathbf{E}\left[ \frac{\gamma^2}{T^2} \int_0^T \int_{\mathbb{R}^d} \mathbf{E}_T[\kappa^2(y - \omega_s)]dyds \right] = \frac{2 \gamma^2 V(0)}{t T}.
\]

Therefore, the claim follows by Chebyshev’s inequality. \( \square \)

4. PROOFS OF MAIN RESULTS: THEOREM 2.1 - THEOREM 2.3

4.1 Proof of Theorem 2.1. \( \blacktriangleright \) Theorem 2.1 will be proved in four steps.

**Step 1:** For any \( x \in \mathbb{R}^d \) and \( A \in \mathcal{G} \) we set (recall the notation from (2.11)-(2.12))

\[
\mathcal{Z}^{(x)}_T(A) := \mathbb{E}_x[\mathbb{I}_A e^{\gamma \mathcal{H}_T(\omega) - \frac{3^2}{2} TV(\omega)}].
\]

**Lemma 4.1.** We set \( M_1^{(1)} = \gamma \int_0^T \int_{\mathbb{R}^d} \mathbf{E}_t[\kappa(y - \omega_t)] \hat{B}(t, y)dy dt \). Let \( Q_t := \mathbf{M}_t[\omega_t \in \cdot] \in \mathcal{M}_1(\mathbb{R}^d) \) so that \( \mathbf{Q}_t \in \mathcal{X} \). Let \( F_\gamma \) and \( \mathcal{F}_\gamma \) be the functionals defined in (3.3) and (3.5), respectively. Then

\[
\frac{1}{T} \log \mathcal{Z}_T = \frac{1}{T} M_1^{(1)} - \frac{1}{T} \int_0^T F_\gamma(\mathbf{Q}_t)dt \quad \text{and} \quad \frac{1}{T} \log(\mathcal{F}_T(\xi)) = \frac{1}{T} M_2^{(2)} - \frac{\gamma^2}{2T} \int_0^T dt \left( \sum_{\alpha_1, \alpha_2 \in \xi} \int_{\mathbb{R}^{2d}} V(x_1 - x_2) \right.
\]

\[
\times \prod_{j=1}^2 \frac{1}{\mathcal{F}_T(\xi)} \int_{\mathbb{R}^d} \alpha_j(dz_j) \mathcal{Z}^{(x_j)}_T(\omega_t^{(i)} \in dz_j),
\]

where \( M_2^{(2)} \) is a mean zero martingale defined below in (4.5).
Proof. Writing $Z_T = \mathbb{E}_0[e^{\gamma \mathcal{H}_T}]$ and applying Itô’s formula to $\log Z_T$ we have
\begin{equation}
\frac{d}{d\log Z_T} = \frac{1}{Z_T} dZ_T - \frac{1}{2Z_T^2} \langle Z_T \rangle
\end{equation}
Again by Itô’s formula,
\begin{equation}
dZ_t = \mathbb{E}_0 \left[ \gamma \int_{\mathbb{R}^d} e^{\gamma \mathcal{H}_t(\omega)} \kappa(y - \omega_t) \tilde{B}(t,y) dy \right] dt + \mathbb{E}_0 \left[ \frac{\gamma^2}{2} \int_{\mathbb{R}^d} e^{\gamma \mathcal{H}_t(\omega)} \kappa(y - \omega_t)^2 dy \right] dt.
\end{equation}
The quadratic variation of $Z_t$ is also given by
\begin{equation}
\langle Z_t \rangle = \mathbb{E}_0 \left[ \gamma \int_{\mathbb{R}^d} e^{\gamma \mathcal{H}_t(\omega)} \kappa(y - \omega_t) \tilde{B}(t,y) dy \right]^2 + \frac{\gamma^2}{2} \mathbb{E}_0 \left[ \int_{\mathbb{R}^d} e^{\gamma \mathcal{H}_t(\omega)} \kappa(y - \omega_t)^2 dy \right] dt
\end{equation}
where $\omega'$ is another Brownian motion independent of $\omega$. Combining (4.2)-(4.4) yields
\begin{equation}
\frac{d}{d\log Z_t} = \gamma \mathbb{E}_t \left[ \int_{\mathbb{R}^d} \kappa(y - \omega_t) \tilde{B}(t,y) dy \right] dt + \frac{\gamma^2}{2} \mathbb{E}_t \left[ \int_{\mathbb{R}^d} \kappa(y - \omega_t)^2 dy \right] dt
\end{equation}
where we recall from (2.12) that $\mathbb{E}_t$ denotes expectation w.r.t. the GMC measure $\tilde{\mathcal{H}}_t$, while $\mathbb{E}_t^\otimes$ stands for the same w.r.t. the product GMC measure $\tilde{\mathcal{H}}_t^\otimes$. Since $\int_{\mathbb{R}^d} \kappa(y - \omega_t)^2 dy = \int_{\mathbb{R}^d} \kappa^2(y) dy = V(0)$, the display above now yields
\begin{equation}
\log Z_T = \gamma \int_0^T \mathbb{E}_t \left[ \int_{\mathbb{R}^d} \kappa(y - \omega_t) \tilde{B}(t,y) dy \right] dt + \frac{\gamma^2}{2} \int_0^T \mathbb{E}_t \left[ V(\omega_t - \omega_t') \right] dt,
\end{equation}
where $\omega, \omega'$ are independent Brownian motions. Consequently,
\begin{equation}
\frac{1}{T} \log Z_T = \frac{\gamma^2}{2} \int_0^T \mathbb{E}_t \left[ \int_{\mathbb{R}^d} \kappa(y - \omega_t) \tilde{B}(t,y) dy \right] dt + \frac{\gamma^2}{2} \int_0^T \mathbb{E}_t \left[ V(\omega_t - \omega_t') \right] dt.
\end{equation}
From the above display the first identity in (4.4) follows. Repeating the Itô computation for $\log(\mathcal{F}_T(\xi))$ also proves the second identity in (4.4) with
\begin{equation}
M^{(2)}_T = \gamma \int_0^T \frac{1}{\mathcal{F}_T(\xi)} \sum_{\alpha \in \xi} \int_{\mathbb{R}^d} \alpha(dz) \mathbb{E}_z \left[ \kappa(y - \omega_s) e^{\gamma \mathcal{H}_t(\omega) - \frac{\gamma^2}{2} TV(0)} \right] \tilde{B}(s,y).
\end{equation}

□

A similar computation as Lemma 4.1 also provides

Lemma 4.2. For any $\gamma > 0$, $\delta > 0$ and as $T \to \infty$, we have $\log Z_T - \mathbb{E}[\log Z_T] = o(T)$.

Proof. For any $s \in [0, T]$ we set
\begin{equation}
X_T = \log Z_T - \mathbb{E}[\log Z_T], \quad X_{T,s} = \mathbb{E}[\log Z_T - \mathbb{E}[\log Z_T] | \mathcal{F}_s],
\end{equation}
where $\mathcal{F}_s$ is the $\sigma$-algebra generated by the noise $\dot{B}$ up to time $s$. We note that $(X_{T,s})_{s \in [0,T]}$ is a martingale and also that

$$X_{T,s} = \mathbb{E} \left[ \gamma \int_0^T \int_{\mathbb{R}^d} \hat{E}[\kappa(y - \omega_t)] \dot{B}(t,y)dydt \right] + \frac{\gamma^2}{2} \int_0^T \mathbb{E} \left[ \hat{E}^s_0[V(\omega_t - \omega_s')] - \hat{E}_t^s[V(\omega_t - \omega_s')] \right] dt \bigg| \mathcal{F}_s \right].$$

The quadratic variation of $X_{T,T}$ is given by

$$\langle X_{T,T} \rangle = \gamma \int_0^T \int_{\mathbb{R}^d} \hat{E}[\kappa(y - \omega_t)] \dot{B}(t,y)dydt = \gamma^2 \int_0^T \int_{\mathbb{R}^d} \left( \hat{E}[\kappa(y - \omega_t)] \right)^2 dydt \leq \gamma^2 TV(0),$$

which we will estimate now as follows: Indeed, again by the martingale property of $(X_{T,s})_{s \in [0,T]}$, we have that for any $a \in \mathbb{R}$, $(\exp \left\{ aX_{T,s} - \frac{a^2}{2} \langle X_{T,s} \rangle \right\})_{s \in [0,T]}$ is also an exponential martingale. Therefore by Chebyshev’s inequality, for any $a,u > 0$,

$$\mathbb{P}(X_T > u) \leq \mathbb{E}[e^{aX_T}] e^{-au} \leq \mathbb{E}[e^{aX_{T,T} - \frac{a^2}{2} \langle X_{T,T} \rangle}] e^{\frac{a^2}{2} \gamma^2 TV(0) - au}.$$

Since $X_{T,0} = 0$, minimizing over $a$ yields

$$\mathbb{P}(X_T > u) \leq \exp \left\{ \min_{a > 0} \left\{ \frac{a^2}{2} \gamma^2 TV(0) - au \right\} \right\} = \exp \left\{ - \frac{u^2}{2\gamma^2 TV(0)} \right\}.$$

Since the same calculations hold by replacing $X_T$ by $-X_T$, it follows that, for any $u > 0$,

$$\mathbb{P}(|\log Z_T - \mathbb{E}[\log Z_T]| > u) \leq 2 \exp \left( - \frac{u^2}{2\gamma^2 TV(0)} \right).$$

In particular, for $\eta \in \left( \frac{1}{2}, 1 \right)$ and for a sequence defined by $T_1 = 1$, and $T_{n+1} = T_n + T^n_\eta$, so that $T_n = n^{\frac{1}{1 - \eta} + o(1)}$ as $n \to \infty$, the upper bound in the last display, combined with Borel-Cantelli lemma, implies that

$$\lim_{n \to \infty} \frac{\log Z_{T_n} - \mathbb{E}[\log Z_{T_n}]}{T_n} = 0, \quad \mathbb{P} \text{ - a.s.} \quad (4.6)$$

In order to strengthen the latter assertion for $T \to \infty$, we apply Lemma 4.1 which implies that

$$\log Z_T = M_T - \frac{\gamma^2}{2} TV(0)$$

where $M_T = \gamma \int_0^T \int_{\mathbb{R}^d} \hat{E}[\kappa(y - \omega_t)] \dot{B}(t,y)dydt$ is a continuous martingale satisfying $\frac{d}{dt} \langle M_T \rangle \leq \gamma^2 V(0)$ for all $T \geq 0$. We now fix a sequence $\varepsilon_n \to 0$ such that $\varepsilon_n^{-1} = n^{o(1)}$. For $n$ large enough, $\gamma^2 V(0) T^n_\eta < \varepsilon_n T_{n+1}$, and by Doob’s inequality, we have

$$\mathbb{P} \left( \sup_{T_n \leq T \leq T_{n+1}} |\log Z_T - \log Z_{T_n} - \mathbb{E}[\log Z_T + \mathbb{E}[\log Z_{T_n}]| > 2 \varepsilon_n T_{n+1} \right)$$

$$\leq \mathbb{P} \left( \sup_{T_n \leq T \leq T_{n+1}} |M_T - M_{T_n}| > \varepsilon_n T_{n+1} \right) \leq (\varepsilon_n T_{n+1})^{-2} \mathbb{E}[\langle M_{T_{n+1}} \rangle - \langle M_{T_n} \rangle],$$

and as $\mathbb{E}[\langle M_{T_{n+1}} \rangle - \langle M_{T_n} \rangle] \leq \gamma^2 V(0) (T_{n+1} - T_n)$, the right-hand side above defines a summable series if we choose $\eta \in (1/2, 1)$ large enough. Together with (4.6), Borel-Cantelli lemma then concludes the proof of the lemma.

\[\square\]

**Step 2:** We will now prove
Lemma 4.3. With $F_{\gamma}$ defined in (3.3), define $\mathcal{E}_{F_{\gamma}} : \mathcal{M}_1(\widetilde{X}) \to \mathbb{R}$ to be

$$\mathcal{E}_{F_{\gamma}}(\vartheta) = \int_{\widetilde{X}} F_{\gamma}(\xi) \vartheta(d\xi).$$  \hfill (4.7)

Then $\mathcal{E}_{F_{\gamma}}$ is continuous on $\mathcal{M}_1(\widetilde{X})$. Moreover, with $\Pi_t(\cdot, \cdot)$ defined in (3.7), we have, for any $T > 0$ and $\xi \in \widetilde{X}$, $\int_0^T \mathcal{E}_{F_{\gamma}}(\Pi_t \delta_\xi) dt \leq - \mathbf{E} \log \mathcal{Z}_T$.

Proof. Recall that $F_{\gamma}$ is continuous on $\widetilde{X}$ and $\widetilde{X}$ is a compact metric space. Thus, $\mathcal{E}_{F_{\gamma}}$ is also continuous on $\mathcal{M}_1(\widetilde{X})$. We will now prove $\int_0^T \mathcal{E}_{F_{\gamma}}(\Pi_t \delta_\xi) dt \leq - \mathbf{E} \log \mathcal{Z}_T$.

Note that by the definition of $\mathcal{E}_{F_{\gamma}}$ and that of $\Pi_t$, we have for any $t$

$$\mathcal{E}_{F_{\gamma}}(\Pi_t \delta_\xi) = \int_{\widetilde{X}} F_{\gamma}(\xi') \Pi_t(\delta_\xi, d\xi') = \int_{\widetilde{X}} F_{\gamma}(\xi') \mathbf{P}[\xi^{(t)} \in d\xi' | \xi] = \mathbf{E}[F_{\gamma}(\xi^{(t)})].$$

On the other hand, $F_{\gamma}(\xi^{(t)}) = \frac{\gamma^2}{2} \sum_{\alpha \in \xi} \int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1 - x_2) \prod_{j=1}^2 \alpha^{(t)}(dx_j)$ and so

$$\mathcal{E}_{F_{\gamma}}(\Pi_t \delta_\xi) = \mathbf{E} \left[ \frac{\gamma^2}{2} \sum_{\alpha \in \xi} \int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1 - x_2) \prod_{j=1}^2 \alpha^{(t)}(dx_j) \right].$$  \hfill (4.8)

Recall that $\mathcal{G}_T(\xi) = \mathcal{G}_T(\xi) + \mathbf{E} \left[ \mathcal{Z}_T - \mathcal{G}_T(\xi) \right]$. We claim that

$$\int_0^T \mathcal{E}_{F_{\gamma}}(\Pi_t \delta_\xi) dt \leq - \mathbf{E} \left[ \log(\mathcal{G}_T(\xi)) \right].$$  \hfill (4.9)

For proving (4.9), we first consider the sum on the right-hand side of (4.8). Since $V, \alpha, \mathcal{Z}_t$ and $\mathcal{G}_t(\xi)$ are nonnegative,

$$\sum_{\alpha \in \xi} \int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1 - x_2) \prod_{j=1}^2 \alpha^{(t)}(dx_j) \leq \sum_{\alpha_1 \in \xi} \sum_{\alpha_2 \in \xi} \int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1 - x_2) \prod_{j=1}^2 \frac{1}{\mathcal{G}_T(\xi)} \alpha_j(dz_j) \mathbb{E}_{z_j} \left[ \mathbb{I}\{\omega_1^{(j)} \in dz_j\} e^{\gamma \mathcal{H}(\omega)} \right],$$

thus by (4.8),

$$\mathcal{E}_{F_{\gamma}}(\Pi_t \delta_\xi) \leq \mathbf{E} \left[ \frac{\gamma^2}{2} \sum_{\alpha_1 \in \xi} \sum_{\alpha_2 \in \xi} \int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1 - x_2) \prod_{j=1}^2 \frac{1}{\mathcal{G}_T(\xi)} \alpha_j(dz_j) \mathbb{E}_{z_j} \left[ \mathbb{I}\{\omega_1^{(j)} \in dz_j\} e^{\gamma \mathcal{H}(\omega)} \right] \right].$$

Claim (4.9) now immediately follows from Lemma 4.1 as $M^{(2)}_T$ in Lemma 4.1 is a martingale that has expectation 0.

For any $\xi = (\tilde{\alpha}_i)_{i \in I} \in \widetilde{X}$, let $\sigma(\xi) = \sum_{i \in I} \alpha_i(\mathbb{R}^d)$ (which is well-defined on $\widetilde{X}$ because $\alpha(\mathbb{R}^d) = (\alpha * \delta_x)(\mathbb{R}^d)$ for any $\alpha \in \mathcal{L}_1$ and $x \in \mathbb{R}^d$) and $\sigma(\cdot) \geq 0$, with identity being true if and only if $\xi = \tilde{0} \in \widetilde{X}$.

We now use (4.9) for those $\xi$ with $\sigma(\xi) > 0$. Using the concavity of the logarithm, we obtain

$$\mathbf{E} \left[ \log(\mathcal{G}_T(\xi)) \right] = \mathbf{E} \left[ \log \left( \frac{\sigma(\xi) \mathcal{G}_T(\xi)}{\sigma(\xi)} + (1 - \sigma(\xi)) \mathcal{Z}_T \right) \right] \geq \sigma(\xi) \mathbf{E} \log \left( \frac{\mathcal{G}_T(\xi)}{\sigma(\xi)} \right) + (1 - \sigma(\xi)) \log \left( \mathcal{Z}_T \right).$$  \hfill (4.10)
As $\int \sigma(\xi)^{-1} \sum_{x} \alpha(x) \, dx = 1$, we can use Jensen’s inequality, so that

$$\log \left( \frac{\mathcal{F}_T(\xi)}{\sigma(\xi)} \right) = \log \left( \int_{\mathbb{R}^d} \left( \frac{\sum_{x} \alpha(x)}{\sigma(\xi)} \right) \mathcal{F}_T[z] \right) \geq \int_{\mathbb{R}^d} \left( \frac{\sum_{x} \alpha(x)}{\sigma(\xi)} \right) \log \mathcal{F}_T[z]$$

and since $\mathcal{F}_T[z] \overset{(d)}{=} \mathcal{F}_T$,

$$\mathbb{E} \log \left( \frac{\mathcal{F}_T(\xi)}{\sigma(\xi)} \right) \geq \int_{\mathbb{R}^d} \left( \frac{\sum_{x} \alpha(x)}{\sigma(\xi)} \right) \mathbb{E} \log \mathcal{F}_T = \mathbb{E} \log \mathcal{F}_T.$$

By using Jensen’s inequality once more, $\log \mathbb{E} \mathcal{F}_T \geq \mathbb{E} \log \mathcal{F}_T$, and both lower bounds, together with (4.9) and (4.10), yield $\int_0^T \mathcal{E}_F(\Pi_t \delta_\xi) \, dt \leq -\mathbb{E}[\log \mathcal{F}_T]$ for any $\xi \in \mathcal{X}$ with $\sigma(\xi) > 0$. The last inequality, when $\sigma(\xi) = 0$, follows immediately by Jensen’s inequality. Indeed, if $\sigma(\xi) = 0$, then $\mathcal{E}_F(\Pi_t \delta_\xi) = 0$ for all $t$ and so $\int_0^T \mathcal{E}_F(\Pi_t \delta_\xi) \, dt = -\mathbb{E}[\log \mathcal{F}_T]$. Thus, the inequality in Lemma 4.3 holds unconditionally.

**Step 3:** Recall (3.7) for definition of $\Pi_t$ and its fixed points $m = \{ \vartheta \in \mathcal{M}_1(\mathcal{X}) : \Pi_t \vartheta = \vartheta \text{ for all } t > 0 \}$. Then the inequality in Lemma 4.3 dictates that, for any $\vartheta \in m_\gamma$,

$$-\frac{1}{T} \mathbb{E}[\log \mathcal{F}_T] \geq \frac{1}{T} \int_0^T \vartheta(\xi) \int_0^T \mathcal{E}_F(\Pi_t \delta_\xi) = \frac{1}{T} \int_0^T \mathcal{E}_F(\Pi_t \vartheta) = \mathcal{E}_F(\vartheta)$$

which proves that $\liminf_{T \to \infty} -\frac{1}{T} \mathbb{E}[\log \mathcal{F}_T] \geq \sup_{\vartheta \in m_\gamma} \mathcal{E}_F(\vartheta)$.

Now Lemma 4.3 implies that, for any $s > 0$, $\mathcal{Y}(\nu_T, m_\gamma, \nu_T) \to 0$ almost surely. Combining this convergence with the fact that $m_\gamma$ is compact, we have the (almost sure) law of large numbers $\mathcal{Y}(\nu_T, m_\gamma, \nu_T) \to 0$ almost surely w.r.t. $\mathbb{P}$. Also note that by Lemma 4.1

$$\frac{1}{T} \mathcal{F}_T = \frac{M^{(1)}}{T} - \frac{1}{T} \int_0^T \mathcal{F}_T(\Pi_t \vartheta)$$

is a martingale with mean zero and quadratic variation given by

$$d\langle M^{(1)} \rangle = \gamma^2 \int_0^T \int_{\mathbb{R}^d} \left( \hat{\mathcal{E}}_t \left[ \kappa(y - \omega_t) \right] \right) \, dy \, dt \leq \gamma^2 \int_0^T \int_{\mathbb{R}^d} \hat{\mathcal{E}}_t \left[ \kappa(y - \omega_t)^2 \right] \, dy \, dt = \gamma^2 T V(0),$$

showing that $\limsup_{T \to \infty} -\frac{1}{T} \log \mathcal{F}_T = \limsup_{T \to \infty} -\frac{1}{T} \int_0^T \mathcal{E}_F(\vartheta) \, dt$ almost surely. Furthermore, by definition of the occupation measures $\nu_T = \frac{1}{T} \int_0^T \mathcal{E}_F(\vartheta) \, dt$ it holds that $\frac{1}{T} \int_0^T \mathcal{E}_F(\vartheta) \, dt = \mathcal{E}_F(\vartheta,T)$. Since $\mathcal{E}_F(\cdot)$ is continuous on $\mathcal{M}_1(\mathcal{X})$, using the almost sure law of large numbers $\mathcal{Y}(\nu_T, m_\gamma, \nu_T) \to 0$, it follows that

$$\limsup_{T \to \infty} -\frac{1}{T} \log \mathcal{F}_T = \limsup_{T \to \infty} \mathcal{E}_F(\nu_T) \leq \sup_{\vartheta \in m_\gamma} \mathcal{E}_F(\vartheta) \quad a.s. \quad (4.11)$$

On the other hand, again by Lemma 4.1 and as $M^{(1)}$ has mean zero, it follows that $-\frac{1}{T} \mathbb{E}[\log \mathcal{F}_T] = \mathbb{E}[\mathcal{E}_F(\nu_T)]$. By Lemma 4.2 both $\mathcal{F}_T$ and $\mathcal{E}_F(\cdot)$ are non-negative and bounded from above by $\gamma^2 T V(0)/2$. Therefore, by reverse Fatou’s lemma and (4.11),

$$\limsup_{T \to \infty} -\frac{1}{T} \mathbb{E}[\log \mathcal{F}_T] = \limsup_{T \to \infty} \mathbb{E}[\mathcal{E}_F(\nu_T)] \leq \mathbb{E} \left[ \limsup_{T \to \infty} \mathcal{E}_F(\nu_T) \right] \leq \sup_{\vartheta \in m_\gamma} \mathcal{E}_F(\vartheta)$$

and therefore $\lim_{T \to \infty} -\frac{1}{T} \mathbb{E}[\log \mathcal{F}_T] = \sup_{\vartheta \in m_\gamma} \mathcal{E}_F(\vartheta)$. Finally, by combining Lemma 4.2 and the previous arguments we have the almost sure statement

$$\lim_{T \to \infty} -\frac{1}{T} \log \mathcal{F}_T = \lim_{T \to \infty} -\frac{1}{T} \mathbb{E}[\log \mathcal{F}_T] = -\sup_{\vartheta \in m_\gamma} \mathcal{E}_F(\vartheta)$$

for any $\vartheta \in m_\gamma$. 

\[ \text{Therefore,} \quad \lim_{T \to \infty} -\frac{1}{T} \log \mathcal{F}_T = \lim_{T \to \infty} -\frac{1}{T} \mathbb{E}[\log \mathcal{F}_T] = -\sup_{\vartheta \in m_\gamma} \mathcal{E}_F(\vartheta), \quad (4.12) \]
Proposition 4.4. Fix $d \in \mathbb{N}$ and $\gamma > 0$. Then the following hold:

(i) It holds $\lambda(\gamma) = \frac{\gamma^2}{2}V(0)$ if $\gamma \leq \gamma_1$ and $\lambda(\gamma) < \frac{\gamma^2}{2}V(0)$ if $\gamma > \gamma_1$.

(ii) Let $f_T(\gamma) = f_T(\gamma, \tilde{B}) = \frac{1}{T} \log Z_{\gamma,T}$ (so that $\lim_{T \to \infty} f_T(\gamma) = \lambda(\gamma)$ almost surely) and assume that $\lambda(\gamma)$ is differentiable at $\gamma$, then $0 \leq \lambda'(\gamma) \leq \gamma V(0)$ and if $\gamma_T = \gamma + o(T)$ as $T \to \infty$, then it holds that

$$\lim_{T \to \infty} f_{\gamma_T}(\gamma_T) = \lambda'(\gamma) \quad \text{a.s. and in } L^1(\mathcal{P})$$

and

$$\lim_{T \to \infty} \hat{E}_{\gamma,T} \left[ \left| T^{-1} \mathcal{H}_T(\omega) - \lambda(\gamma) \right| \right] = 0 \quad \text{a.s. and in } L^1(\mathcal{P}).$$

(iii) Now fix $d \geq 3$. Then $\gamma_1 = \gamma_1(d) > 0$ and for $\gamma \in [0, \gamma_1]$, $\lambda(\gamma) = \frac{\gamma^2}{2}V(0)$. In particular, $\lambda(\gamma)$ is differentiable in $[0, \gamma_1)$ and almost surely

$$\lim_{T \to \infty} T^{-1} \mathcal{H}_T(\omega) = \lambda'(\gamma) = \gamma V(0).$$

We will prove Proposition 4.4 after completing the proof of Theorem 2.1 below, for which we will also need

Proposition 4.5. Fix $d \in \mathbb{N}$ and $\gamma > 0$. Then there exists a non-empty, compact subset $m_\gamma \subset M_1(\tilde{X})$ such that the supremum

$$\sup_{\vartheta \in m_\gamma} \mathcal{E}_{\vartheta}(\cdot) = \sup_{\vartheta \in m_\gamma} \int_{\tilde{X}} F_\vartheta(\xi) \vartheta(d\xi)$$

is attained, and we always have $\sup_{m_\gamma} \mathcal{E}_{\vartheta}(\cdot) \in [0, \gamma^2(\kappa \ast 0)(0)/2]$. Moreover, there exists $\gamma_1 = \gamma_1(d)$ such that $\gamma_1 > 0$ if $d \geq 3$; and if $\gamma \in (0, \gamma_1)$, then $m_\gamma = \{\delta_0\}$ is a singleton consisting of the Dirac measure at $0 \in \tilde{X}$, and consequently in this regime $\sup_{m_\gamma} \mathcal{E}_{\vartheta}(\cdot) = 0$. If $\gamma > \gamma_1$, then $\sup_{m_\gamma} \mathcal{E}_{\vartheta}(\cdot) > 0$. Finally, if $\vartheta \in m_\gamma$ is a maximizer of $\mathcal{E}_{\vartheta}(\cdot)$ and $\vartheta(\xi) > 0$ for $\xi = (\tilde{\alpha}_i)_{i \in I} \in \tilde{X}$, then $\sum_{i \in I} \alpha_i(\mathbb{R}^d) = 1$ (i.e., any maximizer of (4.16) assigns positive mass only to those elements of $\tilde{X}$ whose total mass add up to one).

Proof. The set $m_\gamma \subset M_1(\tilde{X})$ has been defined in Lemma 3.3 which also implies that $m_\gamma$ is non-empty and compact. The fact that the supremum $\sup_{m_\gamma} \mathcal{E}_{\vartheta}(\cdot)$ is attained over $m_\gamma$ is a consequence of the continuity of $\mathcal{E}_{\vartheta}$ (recall Lemma 4.3) on the closed subspace $m_\gamma$ of $M_1(\tilde{X})$ which is compact (because $\tilde{X}$ is compact). Now by Lemma 3.2, $\sup_{m_\gamma} \mathcal{E}_{\vartheta} \leq \gamma^2V(0)/2$ and by definition, $\sup_{m_\gamma} \mathcal{E}_{\vartheta} \geq 0$ and equality holds by Part (i) of Proposition 4.4 if and only if $\gamma \leq \gamma_1$. Moreover, by Part (iii) of Proposition 4.4 $\gamma_1 > 0$ if $d \geq 3$.

It remains to prove that, for $\gamma > \gamma_1$, a maximizer of $m_\gamma$ gives positive probability only to those elements $\xi \in \tilde{X}$ which have total mass 1. We again write $\sigma(\xi) = \sum_{i \in I} \alpha_i(\mathbb{R}^d)$ and using the strict concavity of $x \mapsto \frac{x}{x + 1 - \sigma(\xi)}(x)$ we get a strict inequality $\mathbb{E}[\sigma(\xi^t)] = \mathbb{E}\left[ \frac{\sigma(\xi^t)}{\sigma(\xi^t) + (1 - \sigma(\xi))} \right] < \mathbb{E}\left[ \frac{\sigma(\xi)}{\sigma(\xi) + (1 - \sigma(\xi))} \right] = \sigma(\xi)$, where $\xi^t$ is defined in (3.6). It follows that, for any $t > 0$, $\int \sigma(\xi^t) \vartheta(\vartheta(\xi^t)) \vartheta(d\xi^t) = \int \vartheta(d\xi^t) \mathbb{E}[\sigma(\xi^t)] < \int \vartheta(d\xi^t) \sigma(\xi) \vartheta$ assigns positive mass to $\xi$. But on the other hand we also have $\mathbb{E}_t \vartheta = \vartheta$, since $\vartheta \in m_\gamma$. The resulting contradiction shows that $\vartheta$ assigns positive mass only to those $\xi \in \tilde{X}$ that have total mass 0 or 1.
Next, we note that if $m_\gamma = \{ \delta_0 \}$, then $\sup_{m_\gamma} e_{F_\gamma} = 0$ and by Proposition 4.4 we have $\gamma \leq \gamma_1$. In other words, if $\gamma > \gamma_1$, there exists an element $\delta_0 \neq 0 \in m_\gamma$ which, by our previous remark, satisfies $\theta_0(\mathcal{S}) = 1$ where $\mathcal{S} = \{ \xi \in \tilde{X} : \sigma(\xi) \in \{0,1\} \}$. We also set $\mathcal{A}_1 = \{ \xi \in \tilde{X} : \sigma(\xi) = 1 \}$. It now suffices to show that $\theta_0(\mathcal{A}_1) \in (0,1)$ implies that $\theta_0$ is not a maximizer of $m_\gamma$. Therefore, with $\theta_0(\mathcal{A}_1)$ denoting conditional probability on $\tilde{X}$, we claim that, whenever $\theta_0(\mathcal{A}_1) \in (0,1)$, then $e_{F_\gamma}(\theta_0(\mathcal{A}_1)) > e_{F_\gamma}(\theta_0)$. Indeed, by continuity of $F_\gamma$ we have $F_\gamma(\xi) > 0 = F_\gamma(0)$ for any $\xi \neq 0$. Combining this with the fact that $\theta_0(\mathcal{A}_1) = 1$ shows that, for those $\theta_0$ with $\theta_0(\mathcal{A}_1) \in (0,1)$,

$$e_{F_\gamma}(\theta_0(\mathcal{A}_1)) = \int_{\mathcal{A}_1} F_\gamma(\xi) \theta_0(d\xi) + \frac{1 - \theta_0(\mathcal{A}_1)}{\theta_0(\mathcal{A}_1)} \int_{\mathcal{A}_1} F_\gamma(\xi) \theta_0(d\xi) > \int_{\mathcal{A}_1} F_\gamma(\xi) \theta_0(d\xi) = e_{F_\gamma}(\theta_0).$$

Since $\theta_0 \in m_\gamma \Rightarrow \theta_0(\mathcal{A}_1) \in m_\gamma \supseteq \mathfrak{A}$, the above display implies that the element $\theta_0$ is not a maximizer, which completes the proof of Proposition 4.5.

Let us now conclude

**Proof of Theorem 2.1 (assuming Proposition 4.4):** Recall that $Z_{\gamma, T} := \mathbb{E}_0[e^{\gamma \cdot \mathcal{H}_T}]$ and note that, by Cauchy-Schwarz inequality, for any $\gamma > 0$ and $T > 0$,

$$\sup_{\varphi \in \mathcal{E}_T} \frac{1}{T} \log \mathbb{M}_{\gamma, T}[\mathcal{N}_{\gamma, T}(\varphi)] = \sup_{\varphi \in \mathcal{E}_T} \frac{1}{T} \log \left[ \frac{1}{Z_T} \mathbb{E}_0 \left( e^{\gamma \cdot \mathcal{H}_T} \mathbb{1}_{\mathcal{N}_{\gamma, T}(\varphi)} \right) \right]$$

$$= \sup_{\varphi \in \mathcal{E}_T} \frac{1}{T} \log \left[ \frac{\mathbb{E}_0(e^{\gamma \cdot \mathcal{H}_T})}{\mathbb{E}_0} \mathbb{E}_0 \left( e^{\gamma \cdot \mathcal{H}_T} \mathbb{1}_{\mathcal{N}_{\gamma, T}(\varphi)} \right) \right]$$

$$= -\frac{1}{T} \log \mathbb{E}_0[e^{\gamma \cdot \mathcal{H}_T}] + \sup_{\varphi \in \mathcal{E}_T} \frac{1}{T} \log \mathbb{E}_0 \left( e^{\gamma \cdot \mathcal{H}_T} \mathbb{1}_{\mathcal{N}_{\gamma, T}(\varphi)} \right)$$

$$\leq -\frac{1}{T} \log \mathbb{E}_0[e^{\gamma \cdot \mathcal{H}_T}] + \frac{1}{2T} \log \mathbb{E}_0[e^{\gamma \cdot \mathcal{H}_T}] + \sup_{\varphi \in \mathcal{E}_T} \frac{1}{T} \log P_0[\mathcal{N}_{\gamma, T}(\varphi)]$$

To handle the third term above, note that for any $\varphi \in \mathcal{E}_T$, and with $\omega, \omega'$ denoting two independent Brownian paths,

$$P_0(\omega \in \mathcal{N}_{\gamma, T}(\varphi))^2 = \mathbb{E}_0^2 \left[ \mathbb{1}\{ \omega \in \mathcal{N}_{\gamma, T}(\varphi), \omega' \in \mathcal{N}_{\gamma, T}(\varphi) \} \right].$$

Now if $\omega, \omega' \in \mathcal{N}_{\gamma, T}(\varphi)$, then $\|\omega - \omega'\|_{\infty, T} \leq 2r$. In particular,

$$\mathbb{E}_0^2 \left[ \mathbb{1}\{ \omega \in \mathcal{N}_{\gamma, T}(\varphi), \omega' \in \mathcal{N}_{\gamma, T}(\varphi) \} \right] \leq \mathbb{E}_0^2 \left[ \mathbb{1}\{ \|\omega - \omega'\|_{\infty, T} \leq 2r \} \right] = P_0(\sqrt{2} \|\omega - \omega'\|_{\infty, T} \leq 2r) = P_0(\omega \in \mathcal{N}_{\sqrt{2} \gamma, T}(0))$$

Combining the last two displays, we have,

$$\sup_{\varphi \in \mathcal{E}_T} \log P_0(\omega \in \mathcal{N}_{\gamma, T}(\varphi)) \leq \frac{1}{2} \log P_0(\omega \in \mathcal{N}_{\sqrt{2} \gamma, T}(0)).$$

However, for any $r > 0$, the probability $P_0(\omega \in \mathcal{N}_{\sqrt{2} \gamma, T}(0))$ can be rewritten as $P_0(\tau > T)$, where $\tau$ denotes the first exit time of the standard Brownian motion from the ball $B_{\sqrt{2} \gamma}(0)$, and therefore by the spectral theorem for $-\frac{1}{2} \Delta$ with Dirichlet boundary condition on $B_{\sqrt{2} \gamma}(0)$,

$$\text{(4.18)}$$

To see this, note that $\xi^{(1)} \in \mathcal{S}$ if and only if $\xi \in \mathcal{A}_1$. Thus, for any $A \subseteq \tilde{X}$, we have $\pi_\gamma(\xi, A) = \pi_\gamma(\xi, A \cap \mathcal{A}_1)$ if $\xi \in \mathcal{S}$ and also $\pi_\gamma(\xi, A \cap \mathcal{A}_1) = 0$ if $\xi \notin \mathcal{S}$. The required implication now follows from these two identities, since $P_0(\theta_0(\mathcal{A}_1)) = \frac{1}{\theta_0(\mathcal{A}_1)} \int_{\mathcal{A}_1} \pi_\gamma(\xi, A) \theta_0(d\xi)$ and

$$\int_{\mathcal{A}_1} \pi_\gamma(\xi, A) \theta_0(d\xi) = \int_{\mathcal{A}_1} \pi_\gamma(\xi, A \cap \mathcal{A}_1) \theta_0(d\xi) + \int_{\mathcal{S}^c \cap \mathcal{A}_1} \pi_\gamma(\xi, A \cap \mathcal{A}_1) \theta_0(d\xi) = \Pi_\gamma(\theta_0, A \cap \mathcal{A}_1).$$
we have \( \lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}_0(\omega \in \mathcal{N}_{\sqrt{2}r,T}(0)) = -\lambda_1(\sqrt{2}r) \). Recall that by (4.12), we have \( \mathbb{P} \)-a.s., 
\[ \lim_{T \to \infty} \frac{1}{T} \log Z_{\gamma,T} = \frac{\gamma^2 V(0)}{2} - \sup_{\vartheta \in m_\gamma} \mathcal{E}_{F_\gamma}(\vartheta) \] 
Thus, (4.17)-(4.18) yield, \( \mathbb{P} \)-a.s.,
\[
\begin{align*}
\limsup_{T \to \infty} \sup_{\vartheta \in \mathcal{F}_T} \frac{1}{T} \log \mathcal{M}_{\gamma,T}[\mathcal{N}_{\gamma,T}(\vartheta)] \\
\leq -\frac{1}{4} \lambda_1(\sqrt{2}r) - \lim_{T \to \infty} \frac{1}{T} \log Z_{\gamma,T} + \frac{1}{2} \lim_{T \to \infty} \frac{1}{T} \log Z_{2\gamma,T}
\end{align*}
\]
\[= -\frac{1}{4} \lambda_1(\sqrt{2}r) - \left[ \frac{\gamma^2 V(0)}{2} - \sup_{\vartheta \in m_\gamma} \mathcal{E}_{F_\gamma}(\vartheta) \right] + \frac{1}{2} \left[ \frac{(2\gamma)^2 V(0)}{2} - \sup_{\vartheta \in m_\gamma} \mathcal{E}_{F_{2\gamma}}(\vartheta) \right] = -\Theta, \tag{4.19}
\]
as claimed in (2.5) of Theorem 2.1. By the first part (i) of Proposition 4.4 for any \( d \in \mathbb{N} \) and \( \gamma > 0 \), \( \lambda(\gamma) := \frac{\gamma^2 V(0)}{2} - \sup_{\vartheta \in m_\gamma} \mathcal{E}_{F_\gamma}(\vartheta) \in \left[ 0, \frac{\gamma^2 V(0)}{2} \right] \). Thus, using that \( \lambda(\gamma) \geq 0 \) and \( \lambda(2\gamma) \leq 2\gamma^2 \), we have, for any \( d \in \mathbb{N} \) and \( \gamma > 0 \), \( \Theta \geq \frac{1}{4} \lambda_1(\sqrt{2}r) - \gamma^2 V(0) \). Thus for any \( d \in \mathbb{N} \), if \( \frac{1}{4} \lambda_1(\sqrt{2}r) > \gamma^2 V(0) \) then \( \Theta > 0 \). Since the map \((0,\infty) \ni r \mapsto \lambda_1(r)\) is decreasing, for any \( d \) and \( \gamma \), we find \( r_0 > 0 \) such that \( \Theta > 0 \) for \( r < r_0 \). On the other hand, for given \( d \in \mathbb{N} \) and \( r > 0 \), we find \( r_0 > 0 \) such that \( \Theta \geq \frac{1}{4} \lambda_1(\sqrt{2}r) - \gamma^2 V(0) > 0 \) for any \( \gamma < \gamma_c \). Finally, by part (iii) of Proposition 4.4 for \( d \geq 3 \) and \( \gamma \in [0,\gamma_1/2] \), we have \( \Theta = \frac{1}{4} \lambda_1(\sqrt{2}r) - \gamma^2 V(0) \). Together with a pointwise lower bound similar to (2.5) which will be shown below in Proposition 4.6 the proof of the first part of Theorem 2.1 is thus complete. Also we need to note that the required bound (2.7) in the second part follows directly from part (iii) of Proposition 4.4.

\[\square\]

**Proposition 4.6.** Fix \( d \in \mathbb{N} \), \( r > 0 \). Then there is a constant \( \rho \in (0,\infty) \) such that for any \( \gamma > 0 \) and \( \mathbb{P} \)-a.s.,
\[
\liminf_{T \to \infty} \frac{1}{T} \log \mathcal{M}_{\gamma,T}[\mathcal{N}_{\gamma}(0)] \geq -\left( \lambda_1(\frac{r}{2}) + \rho + \frac{\gamma^2}{2} V(0) + \sup_{\vartheta \in m_\gamma} \mathcal{E}_{F_\gamma}(\vartheta) \right) \tag{4.20}
\]

Together with the previous arguments, the above result will follow from

**Lemma 4.7.** Fix any \( d \in \mathbb{N} \) and any \( r > 0 \). Then there is a constant \( \rho \in (0,\infty) \) and a random variable \( C(\varphi) \) for any \( \varphi \in \mathcal{F}_T \) such that for \( T \) sufficiently large,
\[
\mathbb{P}_0\{\mathcal{N}_{\gamma,T}(\varphi) \in \mathcal{F}_T \} \geq C(\varphi) \exp \left[ - (\lambda_1(\frac{r}{2}) + \rho)T \right] \]

**Proof.** First we write \( H^1_T = \{ f : f(0) = 0, \int_0^T |f'(s)|^2 ds < \infty \} \). For any \( f \in H^1_T \), by the Cameron-Martin theorem we have
\[
\mathbb{P}_0(\omega \in \mathcal{N}_{\gamma/2}(f)) = \int e^{\int_0^T f'(s) \omega ds - \frac{1}{2} \int_0^T |f'(s)|^2 ds} \mathbb{1}\{\omega \in \mathcal{N}_{\gamma/2}(0)\} d\mathbb{P}_0(\omega)
\]
\[= e^{-\frac{1}{2} \int_0^T |f'(s)|^2 ds} \mathbb{E}_0(\mathbb{E}_0 \left[ e^{\int_0^T f'(s) \omega ds} \right] \{\omega \in \mathcal{N}_{\gamma/2}(0)\}) = e^{-\frac{1}{2} \int_0^T |f'(s)|^2 ds} \mathbb{E}_0(\omega \in \mathcal{N}_{\gamma/2}(0)) \mathbb{E}_0 \left[ e^{\int_0^T f'(s) \omega ds} \right] \{\omega \in \mathcal{N}_{\gamma/2}(0)\}. \]

By Jensen’s inequality to the above expectation and also invariance of the set \( \omega \in \mathcal{N}_{\gamma/2}(0) \) with respect to the map \( \omega \mapsto -\omega \), we then have
\[
\mathbb{P}_0(\omega \in \mathcal{N}_{\gamma/2}(f)) > e^{-\frac{1}{2} \int_0^T |f'(s)|^2 ds} \mathbb{P}_0(\omega \in \mathcal{N}_{\gamma/2}(0)). \tag{4.21}
\]

\(^{11}\) It could very well be that a strengthening of the argument above yields that (for \( d \geq 3 \) and \( \gamma \) sufficiently small) the decay rate is simply \( \Theta \approx \frac{1}{4} \lambda_1(\sqrt{2}r) \) (without the penalization term \( \gamma^2 V(0)/2 \) being present). Instead of Cauchy-Schwarz bound above, we could have as well invoked Hölder’s inequality with \( 1/p + 1/q = 1 \) and then optimize over \( p > 1 \) (and \( q > 1 \)). However, the eigenvalue would then carry an extra term \( 1/q \) factor which would bring the decay rate closer to zero when \( q \) gets larger.
We will now handle both the terms on the right-hand side above. Again by the spectral theorem, 
\( \lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}_0(\omega \in \mathcal{N}_{r/2}(0)) = -\lambda_1 < 0 \). For any given \( f \in H^1_T \), let us now handle the term 
\( e^{-\frac{1}{2} \int_0^T |f'(s)|^2 ds} \) in (4.21). For any \( s < t \), with \( s, t \in [0, T] \), let us define the functional
\[
B_{s,t} : \mathcal{C}_T \to \mathbb{R}_+,
\]
\[
B_{s,t}(\varphi) = \inf \left\{ \int_s^t |f'(u)|^2 du : f \in H^1_T, \ f(s) = \varphi(s), f(t) = \varphi(t), \sup_{u \in [s,t]} |\varphi(u) - f(u)| \leq r/2 \right\}.
\]
First remark that, for any \( s < u < t \), we have
\[
B_{s,t} \leq B_{s,u} + B_{u,t}.
\]
That is, the map \( t \mapsto B_{0,t} \) is sub-additive, and therefore by Kingman’s subadditive ergodic theorem we have
\[
\lim_{t \to \infty} t^{-1} B_{0,t} (\cdot) = \rho, \quad \text{a.s.-}\mathbb{P}_0 \quad (4.22)
\]
and the almost sure limit \( \rho \) is deterministic. We need to show that \( \rho \) is finite for which we first note that by Fatou’s lemma, (4.22) implies that \( \rho \leq \mathbb{E}_0(B_{0,1}) \). If we write \( B := B_{0,1} \), then for any \( \varphi \in \mathcal{C}_T \) and for every fixed \( f \in H^1_T \), by change of variables and using the linearity of the relation in the infimum defining \( B(\cdot) \), we have
\[
B(\varphi + f) = B(\varphi) + \int_0^1 |f'(u)|^2 du,
\]
and therefore
\[
\sqrt{B(\varphi + f)} - \sqrt{B(\varphi)} \leq \left( \int_0^1 |f'(u)|^2 du \right)^{1/2},
\]
meaning that the map \( \varphi \mapsto \sqrt{B(\varphi)} \) is Lipschitz. Hence, by Borell’s inequality, \( \sqrt{B} - \text{median}(\sqrt{B}) \) possesses Gaussian tails, implying in particular that
\[
\mathbb{E}_0((\sqrt{B})^2) = \mathbb{E}_0(B) < \infty.
\]
As already remarked, we have \( \rho \leq \mathbb{E}_0(B) \), so that together with the last upper bound we have finiteness of the limit \( \rho \) in (4.22).

Finally, note that for any fixed \( T > 0 \) and \( \varphi \in \mathcal{C}_T \), by lower-semicontinuity of the norm \( H^1_T \ni f \mapsto (\int_0^T |f'(u)|^2 du)^{1/2} \), there exists a (minimizing) function \( f^{(T)} = f^{(T)}(\varphi) \) such that
\[
f^{(T)}(0) = \varphi(0), \quad f^{(T)}(T) = \varphi(T), \quad B_{0,T} = \int_0^T |f^{(T)}(s)|^2 ds.
\]
Then by (4.21),
\[
\mathbb{P}_0^{\odot}(\omega \in \mathcal{N}_r(\varphi) | \varphi) \geq \mathbb{P}_0^{\odot}(\omega \in \mathcal{N}_{r/2}(f^{(T)}(\varphi)) | \varphi) \geq e^{-\frac{1}{2} B_{0,T}} \mathbb{P}_0(\omega \in \mathcal{N}_{r/2}(0))
\]
and by (4.22),
\[
\lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}_0^{\odot}(\omega \in \mathcal{N}_r(\varphi) | \varphi) \geq -\rho/2 + \lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}_0(\omega \in \mathcal{N}_{r/2}(0)).
\]
Combining the finiteness of \( \rho \) together with (4.18) now proves the desired lower bound. \( \square \)

**Remark 3** A uniform exponential lower bound in Proposition 4.6 should follow from a uniform lower bound on the Wiener probability in Lemma 4.7 (with a pre-factor \( C \) which is independent of \( \varphi \) on the right hand side of the bound there).
We now compute the Malliavin derivative on the right-hand side above:

\[ \vartheta(\rho) = \mathbb{E}_0[\mathcal{H}_t(\rho)] \]

We choose \( A_T = \mathcal{N}_r(T) \) (which has strictly positive probability under \( \mathbb{P}_0 \) by Lemma 4.7). Next, using Jensen’s inequality, we note that

\[ \mathbb{E}[\log Z_T(A_T)] = \mathbb{E} \left[ \log \mathbb{E}_0 [\mathcal{E}_T^{A_T}] \right] = \log \mathbb{P}_0(A_T) + \mathbb{E} \left[ \log \mathbb{E}_0 [\mathcal{E}_T^{A_T}] \right] \]

where for the last identity we used that the martingale \( \mathcal{M}_T(\cdot) \) has mean 0 under \( \mathbb{P} \). Combining the last two displays, followed by Lemma 4.7 it holds

\[ \liminf_{T \to \infty} \frac{1}{T} \log \mathbb{E}_0 \left[ \mathcal{K}(\rho) e^{\mathcal{H}_T^r - \frac{2}{2} T \mathbb{E}(V(0))} \right] \geq \liminf_{T \to \infty} \frac{1}{T} \log \mathbb{P}_0(A_T) - \frac{\gamma^2}{2} V(0) \]

with \( \rho = \rho(r) \in (0, \infty) \) defined in (4.22). Hence, \( \mathbb{P} \)-almost surely,

\[ \liminf_{T \to \infty} \frac{1}{T} \log \mathcal{M}_T(\mathcal{K}(0)) \geq - \left( \lambda_1(r/2) + \rho + \frac{\gamma^2}{2} V(0) - \liminf_{T \to \infty} \frac{1}{T} \log Z_T \right) , \]

which proves the required lower bound. \( \square \)

We now owe the reader the

**Proof of Proposition 4.4.** (i) Since \( \sup_{\theta \in \Theta} \mathcal{E}_t(\theta) \geq 0 \) it follows that \( \lambda(\gamma) \leq \frac{\gamma^2}{2} V(0) \) for all \( \gamma \geq 0 \). The fact that \( \gamma \mapsto - \lim_{T \to \infty} \frac{1}{T} \mathbb{E}[\log Z_T] \) is non-decreasing in \([0, \infty)\) and is continuous in \((0, \infty)\) can be shown following the arguments of [CY06] for discrete directed polymers. It follows that \( \lambda(\gamma) = \frac{\gamma^2}{2} V(0) \) if \( \gamma \leq \gamma_1 \) and \( \lambda(\gamma) < \frac{\gamma^2}{2} V(0) \) if \( \gamma > \gamma_1 \).

(ii) Let us now assume that \( \lambda(\gamma) \) is differentiable at \( \gamma \), and note that \( V = \kappa \ast \kappa \) satisfies \( 0 \leq V(\cdot) \leq V(0) \). We will prove the identity

\[ \frac{1}{T} \mathbb{E}[\log Z_T] = \int_0^T r \left( V(t) - \frac{1}{T} \int_0^T \mathbb{E} \left[ \mathcal{E}_{t,T}^0 \left[ V(t) - \mathcal{E}_{t,T}^0 \right] \right] \right) dt \]

below. Assuming this, the statement \( 0 \leq \lambda'(\gamma) \leq V(0) \) follows from the fact that the integrand in (4.23) is bounded by 0 from below and by \( rV(0) \) from above.

We now prove (4.23) using tools from Malliavin calculus and Gaussian integration by parts introduced in the last section. First differentiating \( \mathbb{E}[\log Z_T] \) w.r.t. \( \gamma \) yields

\[ \frac{\partial}{\partial \gamma} \mathbb{E}[\log Z_T] = \mathbb{E} \left[ \mathbb{E}_T \left[ \mathcal{H}_T(\omega) \right] \right] = \mathbb{E}_0 \left[ \int_0^T \int_{\mathbb{R}^d} \kappa(x - \omega_t) e^{\mathcal{H}_T(\omega)} \frac{d}{d\gamma} \mathcal{E}_{t,T}(x) \right] dt \]

From Gaussian integration by parts (cf. (3.19)) we obtain

\[ \mathbb{E}_0 \left[ \int_0^T \int_{\mathbb{R}^d} \kappa(x - \omega_t) \frac{d}{d\gamma} \mathcal{E}_{t,T}(x) \right] = \mathbb{E}_0 \left[ \int_0^T \int_{\mathbb{R}^d} \kappa(x - \omega_t) \frac{d}{d\gamma} \mathcal{E}_{t,T}(x) \right] \]

We now compute the Malliavin derivative on the right-hand side above:

\[ D_{t,x} \frac{e^{\mathcal{H}_T(\omega)}}{Z_{\gamma,T}} = \gamma \kappa(x - \omega_t) \frac{e^{\mathcal{H}_T(\omega)}}{Z_{\gamma,T}} - \gamma \frac{e^{\mathcal{H}_T(\omega)}}{Z_{\gamma,T}} \mathbb{E}_0 \left[ \kappa(x - \omega_t) e^{\mathcal{H}_T(\omega)} \right] \]
Here \( \omega, \omega' \) denote two independent Brownian motions. Thus,

\[
\begin{align*}
\frac{\partial}{\partial \eta} \mathbb{E}[\log Z_{\eta,T}] &= \mathbb{E}_0 \left[ \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \kappa(x - \omega_t) D_{t,x} \frac{e^{\gamma \mathcal{H}_T(\omega)}}{Z_{\eta,T}} \, dx \, dt \right] \right] \\
&= \gamma \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \mathbb{E}_0 \left[ \kappa^2(x - \omega_t) \frac{e^{\gamma \mathcal{H}_T(\omega)}}{Z_{\eta,T}} \right] - \mathbb{E}_0^\otimes \left[ \kappa(x - \omega_t) \kappa(x - \omega'_t) \frac{e^{\gamma (\mathcal{H}_T(\omega) + \mathcal{H}_T(\omega'))}}{Z_{\eta,T}^2} \right] \, dx \, dt \right] \\
&= \gamma T \left( V(0) - \frac{1}{T} \int_0^T \mathbb{E} \left[ \tilde{\mathbb{E}}_{\eta,T}^\otimes [V(\omega_t - \omega'_t)] \right] \, dt \right)
\end{align*}
\]

and the identity (4.23) follows.

We now prove (4.14) and (4.15). The first claim (4.14) is an immediate consequence of the convexity of \( \gamma \mapsto \log Z_{\eta,T} \), which follows readily from Hölder’s inequality. Then the second claim (4.15) can be deduced further from (4.14) using an idea from [PT10] as follows. Recall (3.15) and note that for \( \eta > 0 \),

\[
\tilde{\mathbb{E}}_{\eta,T}^\otimes \left[ \left| \frac{\mathcal{H}_T(\omega)}{T} - f_T^\eta(\gamma) \right| \right] = \tilde{\mathbb{E}}_{\eta,T}^\otimes \left[ \left| \frac{\mathcal{H}_T(\omega)}{T} - \tilde{\mathbb{E}}_{\eta,T}^\otimes \mathcal{H}_T(\omega) \right| \right] \leq \frac{1}{T} \tilde{\mathbb{E}}_{\eta,T}^\otimes \left| \mathcal{H}_T(\omega) - \mathcal{H}_T(\omega') \right|.
\]

Next we can rewrite, using the fundamental theorem of calculus, that for any \( \eta < \gamma \),

\[
\begin{align*}
\int_0^\gamma \tilde{\mathbb{E}}_{\eta,T}^\otimes \left[ \left| \frac{\mathcal{H}_T(\omega)}{T} - \mathcal{H}_T(\omega') \right| \right] \, d\gamma \\
&= (\gamma - \eta) \tilde{\mathbb{E}}_{\eta,T}^\otimes \left[ \left| \frac{\mathcal{H}_T(\omega)}{T} - \tilde{\mathbb{E}}_{\eta,T}^\otimes \mathcal{H}_T(\omega) \right| \right] + \int_0^\gamma \int_0^\gamma \frac{\partial}{\partial \theta} \tilde{\mathbb{E}}_{\eta,T}^\otimes \left[ \left| \frac{\mathcal{H}_T(\omega)}{T} - \tilde{\mathbb{E}}_{\eta,T}^\otimes \mathcal{H}_T(\omega) \right| \right] \, d\theta \, d\gamma
\end{align*}
\]

and by Cauchy-Schwarz inequality, combined with the bound \((a + b)^2 \leq 2a^2 + 2b^2\), we have

\[
\frac{\partial}{\partial \theta} \tilde{\mathbb{E}}_{\eta,T}^\otimes \left[ \left| \frac{\mathcal{H}_T(\omega)}{T} - \mathcal{H}_T(\omega') \right| \right] \leq 4 \text{Var}_{\tilde{\mathbb{E}}_{\eta,T}^\otimes} [\mathcal{H}_T(\cdot)].
\]

Combining the last two displays we have, for \( \gamma > \eta > 0 \),

\[
\begin{align*}
\tilde{\mathbb{E}}_{\eta,T}^\otimes \left[ \left| \frac{\mathcal{H}_T(\omega)}{T} - \mathcal{H}_T(\omega') \right| \right] &\leq \frac{2}{\gamma - \eta} \int_0^\gamma \tilde{\mathbb{E}}_{\eta,T} \left| \frac{\mathcal{H}_T(\omega)}{T} - \tilde{\mathbb{E}}_\theta \mathcal{H}_T(\omega) \right| \, d\theta + 4 \int_0^\gamma \text{Var}_{\tilde{\mathbb{E}}_\theta} \mathcal{H}_T(\cdot) \, d\theta.
\end{align*}
\]

By Jensen’s inequality,

\[
\left( \frac{1}{\gamma - \eta} \int_0^\gamma \tilde{\mathbb{E}}_\theta \left| \frac{\mathcal{H}_T(\omega)}{T} - \tilde{\mathbb{E}}_\theta \mathcal{H}_T(\omega) \right| \, d\theta \right)^2 \leq \frac{1}{\gamma - \eta} \int_0^\gamma \text{Var}_{\tilde{\mathbb{E}}_\theta} \mathcal{H}_T(\cdot) \, d\theta.
\]

Combining the last estimates, we have

\[
\begin{align*}
\tilde{\mathbb{E}}_{\eta,T} \left| \frac{\mathcal{H}_T(\omega)}{T} - f_T^\eta(\gamma) \right| &\leq 2 \sqrt{\frac{\Psi_T(\gamma)}{T(\gamma - \eta)}} + 4 \Psi_T(\gamma), \quad \text{where}
\end{align*}
\]

\[
\Psi_T(\gamma) = \frac{1}{T} \int_0^\gamma \text{Var}_{\tilde{\mathbb{E}}_\theta} \mathcal{H}_T(\cdot) \, d\theta = \int_0^\gamma f_T^\eta(\theta) \, d\theta = f_T^\eta(\gamma) - f_T^\eta(0)
\]

and in the second identity in the above display we used that \( f_T^\eta(\cdot) = \frac{1}{T} \text{Var}_{\tilde{\mathbb{E}}_\theta} \mathcal{H}_T(\cdot) \). It remains to show that the upper bound in (4.24) vanishes in the limit \( T \to \infty \). Recall that we assumed differentiability of \( \eta \mapsto \lambda(\eta) \), and together with convexity of \( f_T(\cdot) \), we have that for any given \( \varepsilon > 0 \), we can choose \( \gamma \) sufficiently close to \( \eta \) such that \( \limsup_{T \to \infty} \Psi_T(\gamma) \leq \varepsilon \) a.s. and in \( L^1 \). Hence,

\[
\lim_{T \to \infty} \tilde{\mathbb{E}}_{\eta,T} \left| \frac{\mathcal{H}_T(\omega)}{T} - f_T^\eta(\gamma) \right| = 0 \quad \text{a.s. and in} \ L^1
\]
and (4.15) follows from (4.14).

(iii) Recall (cf. (3.15)) that we always have the identity \( \hat{E}_{\gamma,T}[\mathcal{H}(\omega)] = \frac{\partial}{\partial T} \log Z_{\gamma,T} \). The goal is to prove that if \( d \geq 3 \) then \( \gamma_1 = \inf\{\gamma: \sup_{m,\gamma} E_{F,\gamma} > 0\} > 0 \). Then, by part (i), for any \( \gamma \in [0, \gamma_1] \) it holds \( \lambda(\gamma) = \frac{\partial}{\partial T} \hat{V}(0) \). Then \( \lambda(\gamma) \) is differentiable for any \( \gamma \in [0, \gamma_1] \) and \( \lim_{T\to\infty} \frac{1}{T} \hat{E}_{\gamma,T}[\mathcal{H}(\omega)] = \frac{\partial}{\partial T} \left( \lim_{T\to\infty} \frac{1}{T} \log Z_{\gamma,T} \right) = \gamma \hat{V}(0) \). It remains to show that \( \gamma_1 > 0 \) for \( d \geq 3 \). It is known that (MSZ16) in dimension \( d \geq 3 \), there exists \( \gamma_0(d) > 0 \), such that \( \mathcal{Z}_{\gamma,T} \) converges almost surely to zero if \( \gamma > \gamma_0 \) and to a non-degenerate, strictly positive random variable if \( \gamma < \gamma_0 \). Furthermore, in any dimension \( d \geq 1 \), the event \( V := \{ \mathcal{Z}_{\gamma,T} \not\to T \to \infty 0 \} \) is a tail event for the process \( t \to \hat{B}(t, \cdot) \), and therefore \( P(V) \in \{0, 1\} \). So if \( \lim_{T\to\infty} \mathcal{Z}_{\gamma,T} > 0 \) almost surely, since for \( x > 0 \), \( -\log(x) < \infty \), \( \lim_{T\to\infty} \frac{1}{T} E[-\log \mathcal{Z}_{\gamma,T}] \leq 0 \). That is, \( \gamma_1 \geq \gamma_0 \), and thus \( \gamma_1 > 0 \).

4.2 Proof of Corollary 2.2. Recall that by Feynman-Kac formula, the solution to (2.8) is given by

\[
u_{\varepsilon}(t, x) = E_x \left\{ \exp \left\{ \gamma(\varepsilon, d) \int_0^t \kappa_\varepsilon(\omega_s - y) \hat{B}(t - s, dy) ds - \frac{\gamma(\varepsilon, d)^2}{2} (\kappa_\varepsilon * \kappa_\varepsilon)(0) \right\} \right\}.
\]

If

\[
\tilde{\mathcal{H}}_{\varepsilon, t}(\omega) = \int_0^t \int_{\mathbb{R}^d} \kappa_\varepsilon(\omega_s - y) \hat{B}(t - s, dy) ds,
\]

the scaling property of the noise \( \hat{B} \) implies that \( \hat{B}(s, dy) ds \) has the same law as that of \( \varepsilon^{d+1} \hat{B}(\varepsilon^{-2}s, d(\varepsilon^{-1}y))d(\varepsilon^{-2}s) \), so that (using \( \kappa_\varepsilon(\cdot) = \varepsilon^{-d} \kappa(\cdot/\varepsilon) \) and changing variables \( \varepsilon^{-2}s \mapsto s \) and \( \varepsilon^{-1}y \mapsto y \)), we obtain from (4.26) that

\[
\tilde{\mathcal{H}}_{\varepsilon, t}(\omega) \overset{d}{=} \varepsilon^{-\frac{d+1}{2}} \int_0^{t/\varepsilon^2} \int_{\mathbb{R}^d} \kappa(\varepsilon^{-1}\omega_{2s} - y) \hat{B}(\varepsilon^{-2}s, dy) ds.
\]

Using now Brownian scaling, we have

\[
E_{\mathbb{P}_x} \left\{ \exp \left\{ \gamma \int_0^{t/\varepsilon^2} \int_{\mathbb{R}^d} \kappa(\varepsilon^{-1}\omega_{2s} - y) \hat{B}(\varepsilon^{-2}s, dy) ds - \frac{\gamma t^2}{2\varepsilon^2} (\kappa * \kappa)(0) \right\} \right\} \overset{d}{=} u_{\varepsilon}(t, x).
\]

Now combining with the invariance of \( \hat{B} \) w.r.t. time reversal, we have the following distributional identity of the processes:

\[
\{ u_{\varepsilon}(t, x) \}_{x \in \mathbb{R}^d} \overset{d}{=} \left\{ \mathcal{Z}_{\gamma, t} \frac{x}{\varepsilon} \right\}_{x \in \mathbb{R}^d},
\]

with \( \mathcal{Z}_{\gamma,T} \) defined in (2.3). The above distributional identity now allows us to use Theorem 2.1 to deduce Corollary 2.2. Indeed, given a fixed \( t > 0 \), we note that Theorem 2.1 implies

\[
\limsup_{T \to \infty} \sup_{\varphi \in \mathcal{C}_T} \frac{1}{T} \log \mathcal{H}_{\gamma,T}[\mathcal{G}_{\varepsilon, t}(\varphi)] \leq -\Theta.
\]

Now Corollary 2.2 is a direct consequence of the above remarks once we set \( \varepsilon = T^{-1/2} \) and use that

\[
P_0^\varphi \left\{ \sup_{0 \leq s \leq t} |\omega_s - \varphi_s| \leq r \varepsilon \right\} = P_0^\varphi \left\{ \sup_{0 \leq s \leq t} |\omega_s - \varphi_s| \leq r \right\}.
\]

(4.29)
4.3 Proof of Theorem 2.3

We will conclude the proof of Theorem 2.3 in this section. Let us first introduce some notation. First recall that $\mathbb{E}_T^\circ$ denotes expectation w.r.t. the product GMC measure $\mathcal{M}_T^{\circ}$ defined in (2.12). Then we set

$$B_\delta = \left\{ \frac{1}{T} \int_0^T \mathbb{E}_T^\circ [V(\omega_s - \omega'_s)]ds \leq \delta \right\}.$$  \hspace{1cm} (4.30)

Also, with $\eta_r(s, x) = e^{-r}\eta(s(e^{2r} - 1), x)$ if $r > 0$ and $\eta_0 = 0$, we set

$$\Phi_T(\omega, \omega', \eta_r) = \frac{1}{Z_T(\eta_r)^2} \exp\{\gamma\mathcal{H}_T(\omega, \eta_r) + \gamma\mathcal{H}_T(\omega', \eta_r)\}$$  \hspace{1cm} (4.31)

and $\hat{Z}_T(\eta_r) = \mathbb{E}_T[e^{\gamma\mathcal{H}_T(\omega, \eta_r)}]$ is the normalization constant so that

$$\mathbb{E}_T[\Phi_T(\cdot, \cdot, \eta_r)] = 1.$$

One important step for the proof of Theorem 2.3 is determined by the following result. Let

$$I_{T,t} = \frac{1}{tT} \int_0^t dr \int_0^T ds \mathbb{E}_T^\circ [V(\omega_s - \omega'_s)\Phi_T(\omega, \omega', \eta_r)]$$  \hspace{1cm} (4.32)

so that the process $(I_{T,t})_{t \geq 0}$ is adapted to the filtration $(\mathcal{H}_t)_{t \geq 0}$ with $\mathcal{H}_t$ being the $\sigma$-algebra generated by $\hat{B}$ and $\eta$ up to time $t$.

**Proposition 4.8.** With the assumption imposed in Theorem 2.3, we have the following assertions.

(a) For any $t, \varepsilon > 0$,

$$\lim_{T \to \infty} \mathbb{P}\left( I_{T,t/T} - \frac{1}{t/T} \int_0^{t/T} \frac{1}{T} \int_0^T \mathbb{E}_T(\eta_r)^\circ [V(\omega_s - \omega'_s)]dsdr \geq \varepsilon \right) = 0.$$

(b) For any $T, \varepsilon_1, \varepsilon_2 > 0$, there exists $\delta' = \delta'(\gamma, t, \varepsilon_1, \varepsilon_2) > 0$ sufficiently small that

$$\mathbb{P}\left( I_{T,t/T} - \frac{1}{t/T} \int_0^{t/T} \mathbb{E}_T^\circ [V(\omega_s - \omega'_s)]ds \geq \varepsilon_1 \bigg| B_\delta \right) \leq \varepsilon_2$$

for all $0 < \delta \leq \delta'$ and $T \geq 0$.

The proof of Proposition 4.8 will be deferred to Appendix A. Assuming the above fact we will conclude the proof of Theorem 2.3.

Let $\varepsilon > 0$ be given. Recall that we assume that $\lambda$ is differentiable and $\lambda'(\gamma) < \gamma V(0)$. If $\rho = \frac{\gamma V(0) - \lambda'(\gamma)}{\gamma}$ and $t$ is large enough, such that $\frac{8}{\varepsilon_1^2} \leq \varepsilon$, then by Corollary 3.8 and (4.14),

$$\liminf_{T \to \infty} \mathbb{P}\left( \rho - \frac{1}{t/T} \int_0^{t/T} \frac{1}{T} \int_0^T \mathbb{E}_T(\eta_r)^\circ [V(\omega_s - \omega'_s)]dsdr \leq \varepsilon \right) \geq 1 - \varepsilon$$

and consequently,

$$\liminf_{T \to \infty} \mathbb{P}\left( \frac{1}{t/T} \int_0^{t/T} \frac{1}{T} \int_0^T \mathbb{E}_T(\eta_r)^\circ [V(\omega_s - \omega'_s)]dsdr \geq \frac{4\rho}{5} \right) \geq 1 - \frac{\varepsilon}{2}.$$  \hspace{1cm} (4.33)
Let \((I_T,t)\) be the process of Proposition 4.8. We define the sets
\[
D = \left\{ \frac{1}{T} \int_0^{t/T} \frac{1}{T} \int_0^T \mathbb{E}^{(\tilde{\omega}_T)}_T [V(\omega_b - \omega'_b)] ds dr \geq \frac{4\rho}{5} \right\}
\]
\[
E = \left\{ \frac{1}{T} \int_0^{t/T} \frac{1}{T} \int_0^T \mathbb{E}^{(\tilde{\omega}_T)}_T [V(\omega_b - \omega'_b)] ds dr \leq \frac{3\rho}{5} \right\}
\]
\[
E_1 = \left\{ I_{T,t/T} - \frac{1}{T} \int_0^{t/T} \frac{1}{T} \int_0^T \mathbb{E}^{(\tilde{\omega}_T)}_T [V(\omega_b - \omega'_b)] ds dr \leq \frac{\rho}{5} \right\}
\]
\[
E_2 = \left\{ I_{T,t/T} - \frac{1}{T} \int_0^{t/T} \mathbb{E}^{(\tilde{\omega}_T)}_T [V(\omega_b - \omega'_b)] ds \leq \frac{\rho}{5} \right\}.
\]
By Proposition 4.8 (a), \(\lim_{T \to \infty} P(E_1) = 1\) and by part (b), we find \(0 < \delta \leq \rho/5\) small enough such that \(P(E_2|B_\delta) \geq 1/2\) for all \(T \geq 0\). Since \(B_\delta \cap E_1 \cap E_2 \subset E\) and since \(D\) and \(E\) are disjoint,
\[
P(B_\delta \cap H_1 \cap H_2) \leq 1 - P(D).
\]
Further,
\[
P(B_\delta \cap E_1 \cap E_2) \geq P(E_1) + P(E_2 \cap B_\delta) - 1 \geq P(E_1) - 1 + \frac{P(B_3)}{2}.
\]
Both inequalities together then yield
\[
P(B_\delta) \leq 2(2 - P(D) - P(E_1)).
\]
Therefore, from (4.33) and \(\lim_{T \to \infty} P(E_1) = 1\) it follows that \(\limsup_{T \to \infty} P(B_\delta) \leq \varepsilon\) and we can deduce that, for every \(\varepsilon > 0\) there exists \(\delta > 0\) sufficiently small such that
\[
\limsup_{T \to \infty} P\left( \frac{1}{T} \int_0^T \mathbb{E}^{(\tilde{\omega}_T)}_T [V(\omega_b - \omega'_b)] ds \leq \delta \right) \leq \varepsilon. \tag{4.34}
\]
Recall that we need to show that for every \(\varepsilon > 0\) there exist \(\delta, T_0 > 0\) and an integer \(k \in \mathbb{N}\) and \(\omega^{(1)}, \ldots, \omega^{(k)} \in \mathcal{C}_\infty\) such that
\[
P\left[ \hat{\mathbb{M}}_{T}^{\otimes} \left( \bigcup_{i=1}^{k} \text{Cov}_T(\omega^{(i)}, \omega^{(k+i)}) \geq \delta \right) \geq 1 - \varepsilon \right] \geq 1 - \varepsilon \tag{4.35}
\]
for all \(T \geq T_0\), where
\[
\text{Cov}_T(\omega, \omega') = \frac{1}{T} \int_0^T (\kappa \ast \kappa)(\omega_s - \omega'_s) ds \quad \text{and}
\]
\[
\text{Cov}_T(\omega') = \mathbb{E}_T(\text{Cov}_T(\omega, \omega')) = \frac{1}{T} \int_0^T \int_{\mathbb{R}^d} \kappa(y - \omega'_s) \mathbb{E}_T[\kappa(y - \omega_s)] dy ds.
\]
Note that (4.34) implies that
\[
\limsup_{T \to \infty} E[\mathbb{E}_T[1_{\mathcal{A}_{T,\delta}}]] \leq \varepsilon, \quad \text{where}
\]
\[
\mathcal{A}_{T,\delta} = \left\{ \omega \in \mathcal{C}_\infty : \int_0^T \int_{\mathbb{R}^d} \kappa(y - \omega_s) \mathbb{E}_T[\kappa(y - \omega'_s)] dy ds \leq \delta T \right\}. \tag{4.36}
\]
Indeed, fix \(k \in \mathbb{N}\), \(\delta > 0\) and set \(\gamma_T = \gamma \sqrt{1 + \frac{k}{T}}\). For \(\eta_1, \ldots, \eta_k\) being independent copies of \(\hat{\mathcal{B}}\), we define
\[
\hat{\mathbb{M}}_{T,k}(d\omega) = \frac{1}{Z_{T,k}} e^{\gamma_T(\mathcal{A}_{T,\delta} + \frac{1}{T} \sum_{i=1}^k \mathcal{A}_{T,\delta}(\omega, \eta_i))} P_0(d\omega)
\]
where as usual, \( Z_{T,k} \) is the normalizing constant and \( \hat{\mathbb{E}}_{T,k} \) stands for expectation with respect to the probability measure \( \hat{\mathcal{M}}_{T,k} \).

We also define

\[
\mathcal{A}_{\delta,k} = \left\{ \omega \in \mathcal{C}_\infty : \int_0^T \int_{\mathbb{R}^d} \kappa(y - \omega_s) \hat{\mathbb{E}}_{T,k} \kappa(y - \omega_s') dy ds \leq \delta T \right\}
\]

\[
\mathcal{B}_{\delta,k} = \left\{ \omega \in \mathcal{C}_\infty : \int_0^T \int_{\mathbb{R}^d} \kappa(y - \omega_s) \hat{\mathbb{E}}_{T,k} \kappa(y - \omega_s') dy ds \leq \delta T \right\}
\]

\[
\mathcal{B}_{\delta,k} = \left\{ \int_0^T \int_{\mathbb{R}^d} \left( \hat{\mathbb{E}}_{T,k} \kappa(y - \omega_s) \right)^2 \leq \delta T \right\}.
\]

Let \( \varepsilon > 0 \). There exist \( K = K(\gamma, \varepsilon) \in \mathbb{N} \) and \( \alpha = \alpha(\gamma, \varepsilon) > 0 \) and \( \delta = \delta(\gamma, \varepsilon) \in (0, 1/2) \) such that, for all \( T \) large enough there exists \( k = k(T) \in \{0, 1, \ldots, K - 1\} \) with

\[
\mathbb{E} \left[ \hat{\mathbb{E}}_{T,k} \left[ \mathbb{1}_{\mathcal{A}_{\delta,k}} \right] \right] \leq \varepsilon/2 \quad \text{and} \quad \mathbb{P}(\mathcal{B}_{\alpha,k}) \leq \varepsilon/2.
\]

The latter is a consequence of (4.34). To show (4.36) (where we have \( k = 0 \)) from the above estimate (where we have \( k = k(T) > 0 \) copies of \( \hat{\mathbb{E}} \)), we can estimate \( \delta \leq \delta^{1-k} \) and

\[
0 \leq \gamma - \gamma \frac{k(T)}{T} \leq \gamma \frac{K(\gamma, \varepsilon)}{T}
\]

and use that \( \hat{\mathbb{E}}_{T,k} [\mathcal{A}_{\delta,k}] \overset{\text{d}}{=} \hat{\mathbb{E}}_{T,k} [\mathcal{A}_{\delta,k}] \), which shows (4.36).

Finally, (4.35) is deduced from (4.36) as follows. Fix \( \varepsilon > 0 \). Then from (4.36), there exist \( \delta > 0 \) small enough and \( T_0 > 0 \) large enough such that \( \mathbb{E} \left[ \hat{\mathbb{E}}_{T} \left[ \mathbb{1}_{\mathcal{A}_{\delta}} \right] \right] \leq \varepsilon^2/2 \) for any \( T \geq T_0 \). Then, from Markov’s inequality it follows

\[
\mathbb{P}(\hat{\mathbb{E}}_{T} \left[ \mathbb{1}_{\mathcal{A}_{\delta}} \right] > \varepsilon/2) \leq \varepsilon.
\]

By Paley-Zygmund inequality**, for any \( i = 1, \ldots, k \), and on the event \( \{ \text{Cov}_T (\omega^{(k+1)}) > 2\delta \} \),

\[
\hat{\mathbb{E}}^\otimes \left[ \mathbb{1} \left\{ \text{Cov}_T (\omega^{(i)}, \omega^{(k+1)}) \geq \delta \right\} \big| \omega^{(k+1)} \right] \geq \hat{\mathbb{E}}^\otimes \left[ \mathbb{1} \left\{ \text{Cov}_T (\omega^{(i)}, \omega^{(k+1)}) \geq 1/2 \text{Cov}_T (\omega^{(k+1)}) \right\} \big| \omega^{(k+1)} \right]
\]

\[
\geq \frac{1}{4} \hat{\mathbb{E}}^\otimes \left[ \text{Cov}_T (\omega^{(k+1)})^2 \big| \omega^{(k+1)} \right] \geq \frac{\delta^2}{V(0)}
\]

Hence,

\[
\hat{\mathbb{E}}^\otimes \left[ \mathbb{1} \left\{ \bigcap_{i=1}^k \{ \text{Cov}_T (\omega^{(i)}, \omega^{(k+1)}) \geq \delta \} \big| \omega^{(k+1)} \right\} \right] \leq \left( 1 - \frac{\delta^2}{V(0)} \right)^k \leq e^{-\frac{\delta^2}{V(0)} k}.
\]

If we choose \( k = \lfloor -\delta^2 V(0) \log(\varepsilon/2) \rfloor \lor 0 \), then

\[
\hat{\mathbb{E}}^\otimes \left[ \mathbb{1} \left\{ \bigcap_{i=1}^k \{ \text{Cov}_T (\omega^{(i)}, \omega^{(k+1)}) \geq \delta \} \right\} \right] \leq \frac{\varepsilon}{2} + \hat{\mathbb{E}}^\otimes \left[ \mathbb{1} \{ \text{Cov}_T (\omega^{(k+1)}) \leq 2\delta \} \right] = \frac{\varepsilon}{2} + \hat{\mathbb{E}}_{T} \left[ \mathbb{1}_{\mathcal{A}_{\delta}} \right]
\]

***For any random variable \( X \geq 0 \), \( \mathbb{P} \left( \frac{X}{\mathbb{E}[X]} \geq \frac{1}{2} \right) \geq \frac{1}{4} \mathbb{E}[X]^2 \).
and furthermore,
\[
P \left( \mathbb{E}_T^\otimes \left\{ \mathbb{I} \left\{ \bigwedge_{i=1}^k \{ \text{Cov}_T(\omega^{(i)}, \omega^{(k+1)}) \geq \delta \} \right\} \right\} \right) \geq 1 - \varepsilon
\]
\[
= P \left( \mathbb{E}_T^\otimes \left\{ \mathbb{I} \left\{ \bigwedge_{i=1}^k \{ \text{Cov}_T(\omega^{(i)}, \omega^{(k+1)}) < \delta \} \right\} \right\} \right) \leq \varepsilon \geq P \left( \mathbb{E}_T^\otimes \left\{ \mathbb{I}_{\mathbb{F}_{T,2s}} \right\} \leq \varepsilon/2 \right) \geq 1 - \varepsilon
\]
where the last inequality is due to (4.37). This completes the proof of (4.35) and therefore that of Theorem 2.3.

\[
\Box
\]

\section*{Appendix A.}

\subsection*{1.1 Proof of Proposition 4.8}

Given the input from previous sections, the arguments needed for the proof of Proposition 4.8 will follow the approach of [BC19] adapted to our setting modulo some modifications. However, in this execution some of the arguments get simplified in our set up, thanks to the estimate

\[
V(x) := (\kappa \ast \kappa)(x) = \int_{\mathbb{R}^d} \kappa(x - y)\kappa(y)dy \leq \left( \int_{\mathbb{R}^d} \kappa^2(x - y)dy \right)^{1/2} \left( \int_{\mathbb{R}^d} \kappa^2(y)dy \right)^{1/2}
\]

\[
= \int_{\mathbb{R}^d} \kappa(y)\kappa(-y)dy = (\kappa \ast \kappa)(0) = V(0)
\]

for which we use that the mollifier \( \kappa(\cdot) \) is a spherically symmetric function around the origin.

Recall that \( \lambda(\gamma) = \lim_{T \to \infty} f_T(\gamma, \cdot) \) = \( \lim_{T \to \infty} \frac{1}{T} \log Z_T \) and for any \( \delta > 0, \gamma > 0 \) and \( t > 0 \), suitable constants \( c(\gamma, t) \) and \( C(\delta) \), define

\[
M_T^2 = c(\gamma, t) \left( C(\delta) \mathbb{E}_T \left[ \left| \lambda' (\gamma) - \frac{\mathcal{H}_V(\omega)}{T} \right| \right] + \delta Z_T^{\frac{12}{13}} \right).
\]

\[\text{Lemma A.1.} \text{ Fix } t > 0 \text{ and } \varepsilon > 0. \text{ Then the following statements hold:}
\]

\begin{itemize}
  \item[(a)] There is \( \delta = \delta(\varepsilon) \) sufficiently small so that
  \[
  \limsup_{T \to \infty} \mathbb{E}[M_T] \leq \varepsilon.
  \]
  \item[(b)] For every \( s \in [0, T] \) and \( r \in [0, t/T] \),
  \[
  \left\| \mathbb{E}_T^\otimes [V(\omega_s - \omega'_s)\Phi_T(\omega, \omega', \eta_r) - \mathbb{E}_T^\otimes \Phi_T(\omega, \omega', \eta_r)] \right\|_{L^1(\mathbb{P})} \leq V(0)\mathbb{E}[M_T].
  \]
  \item[(c)] There exists \( \delta' = \delta'(\gamma, T, \varepsilon) > 0 \) sufficiently small that for every \( s \in [0, T], r \in [0, t/T] \) and \( \delta \in (0, \delta'] \),
  \[
  \left\| \mathbb{E}_T^\otimes [V(\omega_s - \omega'_s)\Phi_T(\omega, \omega', \eta_r) - \mathbb{E}_T^\otimes \Phi_T(\omega, \omega', \eta_r)] \right\|_{L^1(\mathbb{P}, B_\delta)} \leq \varepsilon V(0)\mathbb{P}(B_\delta)
  \]
  where \( L^1(\mathbb{P}, B_\delta) \) is the \( L^1(\mathbb{P}) \) norm defined on the event \( B_\delta \) defined in (4.30).
\end{itemize}

We first conclude the proof of Proposition 4.8 and prove this technical fact afterwards.
Proof of Proposition 4.8 (Assuming Lemma A.1). Let \( t, \varepsilon > 0 \) be fixed. By part (a)-(b) of Lemma A.1 we have \( \limsup_{T \to \infty} E[M_T] \leq \varepsilon^2 \) and
\[
\left\| \mathbb{E}_T \left[ V(\omega_s - \omega'_s)\Phi_T(\omega, \omega', \eta_r) - \mathbb{E}_T \left[ V(\omega_s - \omega'_s) \right] \right] \right\|_{L^1(\mathbb{P})} \leq V(0)E[M_T].
\]
With \( I_{T,t} \) defined in (4.32) and by Lemma A.1 we have
\[
\left\| I_{T,t/T} - \frac{1}{t/T} \int_{t/T}^{t/T+1} t/T \int_0^T \mathbb{E}_T \left[ V(\omega_s - \omega'_s) \right] ds dr \right\|_{L^1(\mathbb{P})} \leq V(0)E[M_T]
\]
and then part (a) of Proposition 4.8 follows from the Markov inequality. The proof of the second part is an identical application of Markov’s inequality and part (c) of Lemma A.1 with the choice \( \varepsilon = \varepsilon_1 \varepsilon_2 \).

Proof of Part (a) and Part (b) of Lemma A.1. We will complete the proof in three main steps. Recall that \( \eta \) is an independent copy of \( \dot{B} \), while
\[
\dot{B}_r(s, \cdot) = e^{-r} \dot{B}(s, \cdot) + e^{-r}\eta(s(e^{2r} - 1)^{-1}, \cdot) \quad \text{if} \ r > 0, \quad B_0 = \dot{B},
\]
which has the same law as that of \( \dot{B} \) and also
\[
\eta_r(s, \cdot) = e^{-r}\eta(s(e^{2r} - 1)^{-1}, \cdot) \quad \text{(d)} \quad \sqrt{1 - e^{-2r}} \eta(s, \cdot), \quad \eta_0 = 0.
\]
We will also use the simple fact that for any \( c \geq 0 \), if \( r \leq t/T \),
\[
T(1 - e^{-cr}) \leq Tc_r \leq ct. \quad \text{(A.2)}
\]

Lemma A.2. Fix \( t, \gamma > 0 \) and \( r \leq t/T \), and denote by
\[
Y_r^{1/2} = \mathbb{E}_T \left[ \exp \left\{ \gamma \left( \mathcal{H}_T(\omega, \dot{B}_r) - \mathcal{H}_T(\omega, \dot{B}) \right) \right\} \right].
\]
Then there is a constant \( c(\gamma, t) \) such that
\[
\sup_T E[Y_r^{-2}] \leq c(\gamma, t).
\]

Proof. By Jensen’s inequality and from the definition of \( Y_r \) we have
\[
E_{\eta}[Y_r^{-2}] \leq \mathbb{E}_T \left[ e^{2\gamma(1-e^{-2r})TV(0)} + \int_0^T \left( V(\omega_s - \omega'_s)ds \right) e^{2\gamma(1-e^{-2r})(\mathcal{H}_T(\omega, \dot{B}) + \mathcal{H}_T(\omega', \dot{B}))} \right] \leq e^{4\gamma(1-e^{-2r})TV(0)} \mathbb{E}_T \left[ e^{2\gamma(1-e^{-2r})(\mathcal{H}_T(\omega, \dot{B}) + \mathcal{H}_T(\omega', \dot{B}))} \right],
\]
by the last display, and an application of (A.2) and the Cauchy-Schwarz inequality, it holds \( E_{\eta}[Y_r^{-2}] \leq c(\gamma, t) E_T [e^{4\gamma(1-e^{-2r})\mathcal{H}_T(\omega, \dot{B})}] \). It is also a straightforward application of Cauchy-Schwarz inequality that for any \( q > 0 \) and \( k = \lfloor \log_2 \frac{T}{qt} \rfloor \) and for any \( T \) large enough such that \( k \geq 1 \),
\[
E_T [e^{\varphi(1-e^{-2r})\mathcal{H}_T(\omega, \dot{B})}] \leq Z_T(\gamma)^{-\frac{1}{2k}} (Z_T(2\gamma)^{\frac{1}{2k}} + 1).
\]
We use the inequality above for \( q = 4 \). Then, if \( T > 0 \) is large enough such that \( k = \lfloor \log_2 \frac{T}{qt} \rfloor \geq 1 \) it holds
\[
E[\eta[Y_r^{-2}]] \leq c(\gamma, t) E \left[ Z_T(\gamma)^{-\frac{1}{2k}} (Z_T(2\gamma)^{\frac{1}{2k}} + 1) \right] \leq c(\gamma, t) \left( e^{\frac{2}{q^2}TV(0)} e^{\frac{2}{q}TV(0)} + e^{\frac{2}{q^2}TV(0)} \right) \leq c(\gamma, t),
\]
where we used for the second inequality Cauchy-Schwarz and Jensen.

\[ \square \]
Lemma A.3. Fix $t, \gamma > 0$ and $r \leq t/T$, and denote by

$$(Y'_r)^{1/2} = \mathbb{E}_T^T \left[ \exp \left\{ \gamma \mathcal{H}_T(\omega, \eta_r) + T(\gamma e^{-r} - 1)\lambda'(\gamma) \right\} \right].$$

Then with $Y_r$ defined in (A.3) and for any $\delta$ we find a constant $C(\delta)$ such that

$$E_\eta[(Y_r - Y'_r)^2] \leq c(\gamma, t) \left( C(\delta) \mathbb{E}_T^T \left[ \left| \lambda'(\gamma) - \frac{\mathcal{H}_T(\omega)}{T} \right| \right] + \delta Z_T^{-\frac{12\gamma}{T}} \right). \tag{A.4}$$

Proof. By Jensen’s inequality and using $V(\omega_s - \omega'_s) \leq V(0)$, followed by an application of (A.2),

$$E_\eta[(Y_r - Y'_r)^2] \leq c(\gamma, t) \mathbb{E}_T^T \left[ \left( e^{\gamma(1-e^{-r}1)}(\mathcal{H}_T(\omega, \hat{B}) + \mathcal{H}_T(\omega', \hat{B}) - 2T\lambda'(\gamma)) - 1 \right)^2 \right]. \tag{A.5}$$

For any $L > 0$ we find a constant such that $(e^{x} - 1)^2 \leq C(L)|x|$ for all $x \leq L$. Also applying (A.2) once more and using the notation $\mathcal{I}(r, T) = (e^{-r} - 1)(\mathcal{H}_T(\omega, \hat{B}) + \mathcal{H}_T(\omega', \hat{B}) - 2T\lambda'(\gamma))$, we have

$$\mathbb{E}_T^T \left[ \left( e^{\gamma(1-e^{-r})1} \mathcal{I}(r, T) - 1 \right)^2 \right] \leq C(L, \gamma, t) \mathbb{E}_T^T \left[ \left| \lambda'(\gamma) - \frac{\mathcal{H}_T(\omega)}{T} \right| \right] + \mathbb{E}_T^T \left[ \left( e^{\gamma(1-e^{-r})1} \mathcal{I}(r, T) - 1 \right)^2 \mathbb{1}\{\gamma \mathcal{I}(r, T) > L\} \right]. \tag{A.6}$$

If $L$ is large enough such that $L \geq 4\gamma T\lambda'(\gamma)$, then

$$\gamma(1-e^{-r})(T\lambda'(\gamma) - \mathcal{H}_T(\omega) + T\lambda'(\gamma) - \mathcal{H}_T(\omega')) = L \geq 4\gamma T\lambda'(\gamma) \geq 4\gamma(1-e^{-r})T\lambda'(\gamma).$$

It follows that

$$-2\gamma(1-e^{-r})(\mathcal{H}_T(\omega) + \mathcal{H}_T(\omega')) > 2\gamma(1-e^{-r})(T\lambda'(\gamma) - \mathcal{H}_T(\omega) - \mathcal{H}_T(\omega')) > L \geq 0$$

and since the left-hand side is positive, we can use (A.2) to get $-\frac{2\gamma L}{T}(\mathcal{H}_T(\omega) + \mathcal{H}_T(\omega')) > L \geq 0$. We may now make use of the indicator. If $T$ is large enough such that $T \geq 6t$, then

$$\mathbb{E}_T^T \left[ \left( e^{\gamma(1-e^{-r})1} \mathcal{I}(r, T) - 1 \right)^2 \mathbb{1}\{\gamma \mathcal{I}(r, T) > L\} \right] \leq e^{-L} Z_T^{-\frac{12\gamma}{T}}. \tag{A.7}$$

Given $\delta > 0$ we simply have to choose $L$ sufficiently large. Indeed, putting (A.3)–(A.7) together proves the claim. \hfill \square

Using very similar arguments as that of the proof of Lemma A.3 and using $V(\cdot) \leq V(0)$ we can show that

Lemma A.4. Fix $t, \gamma > 0$ and $r \leq t/T$, and denote by

$$X_{s, r} = \mathbb{E}_T^T \left[ V(\omega_s - \omega'_s) \exp \left\{ \gamma \left( \mathcal{H}_T(\omega, \hat{B}_r) + \mathcal{H}_T(\omega', \hat{B}_r) - \mathcal{H}_T(\omega, \hat{B}) - \mathcal{H}_T(\omega', \hat{B}) \right) \right\} \right],$$

$$X'_{s, r} = \mathbb{E}_T^T \left[ V(\omega_s - \omega'_s) \exp \left\{ \gamma \left( \mathcal{H}_T(\omega, \eta_r) + \mathcal{H}_T(\omega', \eta_r) + 2T(e^{-r} - 1)\lambda'(\gamma) \right) \right\} \right]. \tag{A.8}$$

Then for any $\delta$ we find a constant $C(\delta)$ such that

$$E_\eta[(X_{s, r} - X'_{s, r})^2] \leq V(0)c(\gamma, t) \left( C(\delta) \mathbb{E}_T^T \left[ \left| \lambda'(\gamma) - \frac{\mathcal{H}_T(\omega)}{T} \right| \right] + \delta Z_T^{-\frac{12\gamma}{T}} \right). \tag{A.9}$$
Recall the definition of $\Phi_T$ from (4.31). Then
\[
\left( \hat{E}_T^{\otimes} \left[ V(\omega_s - \omega'_s) \right] , \hat{E}_T^{\otimes} \left[ V(\omega_s - \omega'_s) \Phi_T(\omega, \omega', \eta_r) \right] \right) \equiv \left( \frac{X_{s,r}}{Y_r} , \frac{X'_{s,r}}{Y'_r} \right)
\]
and as Lemma [A.1] depends only on marginal distributions at fixed $r$, we fix $t, \varepsilon > 0$, $r \leq t/T$ and prove that $\| \frac{X_{s,r}}{Y_r} - \frac{X'_{s,r}}{Y'_r} \|_{L^1(\mathcal{P})} \leq V(0)E[M_T]$. Note that
\[
E_\eta \left[ \left| \frac{X_{s,r}}{Y_r} - \frac{X'_{s,r}}{Y'_r} \right| \right] \leq (E_\eta[|Y_r^{-2}|])^{1/2} (E_\eta[(X_{s,r} - X'_{s,r})^2])^{1/2} + V(0)(E_\eta[|Y_r^{-2}|])^{1/2} (E_\eta[(Y_r - Y'_r)^2])^{1/2}.
\]
Part (a) and Part (b) of Lemma [A.1] follow now from Lemma [A.2]-Lemma [A.4] if we define
\[
M_T = c(\gamma, t) \left( C(\delta) \hat{E}_T \left[ \left| \lambda'(\gamma) - \frac{\mathcal{H}_T(\omega)}{T} \right| \right] + \delta Z_T \right)^{1/2}
\]
choose $T$ large enough, such that, $12t \leq T$ and use (1.15) which yields that $\delta$ can be chosen sufficiently small.

**Proof of Part (c) of Lemma [A.1]** Let us define
\[
X''_{s,r} := \hat{E}_T^{\otimes} \left[ V(\omega_s - \omega'_s)e^{\gamma(\mathcal{H}_T(\omega, \eta_r) + \mathcal{H}_T(\omega', \eta_r))} \right] e^{\gamma(\varepsilon - 1)TV(0)}
\]
\[
Y''_r := \hat{E}_T^{\otimes} \left[ e^{\gamma(\mathcal{H}_T(\omega, \eta_r) + \mathcal{H}_T(\omega', \eta_r))} \right] e^{\gamma(\varepsilon - 1)TV(0)}.
\]
We will show that $\| \frac{X''_{s,r}}{Y''_r} - \hat{E}_T^{\otimes}[V(\omega_s - \omega'_s)] \|_{L^1(\mathcal{P}, B_d)} \leq \mathcal{P}(B_d)eV(0)$. For simplicity, we write $X'' = X''_{s,r}$ and $Y'' = Y''_r$. Also using that $X'' \leq V(0)$, it can be shown that
\[
E_\eta \left[ \frac{X''}{Y''_r} - \hat{E}_T^{\otimes}[V(\omega_s - \omega'_s)] \right] \leq V(0)E_\eta[1 - Y''_r] + E_\eta[X'' - \hat{E}_T^{\otimes}[V(\omega_s - \omega'_s)]].
\]
We first consider the second of the two expectations above:
\[
E_\eta[X'' - \hat{E}_T^{\otimes}[V(\omega_s - \omega'_s)]]
\]
\[
= E_\eta \left[ \int_{\mathbb{R}^d} \left( \hat{E}_T \left[ \kappa(y - \omega_s)e^{\gamma \mathcal{H}_T(\omega, \eta_r) - \frac{\gamma^2}{2}(1-e^{-2\varepsilon})TV(0)} \right] \right) \right]^2 dy \quad (A.11)
\]
\[
\leq \int_{\mathbb{R}^d} E_\eta \left[ \hat{E}_T \left[ \kappa(y - \omega_s)e^{\gamma \mathcal{H}_T(\omega, \eta_r) - \frac{\gamma^2}{2}(1-e^{-2\varepsilon})TV(0)} \right] \right]^2 dy \quad (A.12)
\]
Using $a^2 - b^2 = (a + b)(a - b)$ followed by Cauchy-Schwarz inequality, we obtain
\[
\left( \int_{\mathbb{R}^d} E_\eta \left[ \hat{E}_T \left[ \kappa(y - \omega_s)e^{\gamma \mathcal{H}_T(\omega, \eta_r) - \frac{\gamma^2}{2}(1-e^{-2\varepsilon})TV(0)} \right] \right]^2 dy \right)^2
\]
\[
\leq \int_{\mathbb{R}^d} E_\eta \left[ \hat{E}_T \left[ \kappa(y - \omega_s)e^{\gamma \mathcal{H}_T(\omega, \eta_r) - \frac{\gamma^2}{2}(1-e^{-2\varepsilon})TV(0)} \right] - \hat{E}_T[\kappa(y - \omega_s)] \right]^2 dy
\]
\[
\times \int_{\mathbb{R}^d} E_\eta \left[ \hat{E}_T \left[ \kappa(y - \omega_s)e^{\gamma \mathcal{H}_T(\omega, \eta_r) - \frac{\gamma^2}{2}(1-e^{-2\varepsilon})TV(0)} \right] + \hat{E}_T[\kappa(y - \omega_s)] \right] dy.
\]
For the first factor we note that $E_\eta[\hat{E}_T[\kappa(y - \omega_s)e^{\gamma \mathcal{X}_T(y,\omega_s) - \frac{\gamma^2}{2}(1-e^{-2\gamma})TV(0)]]] = \hat{E}_T[\kappa(y - \omega_s)]$, so that it is bounded above by
\[
\hat{E}_T^\otimes\left[V(\omega_s - \omega'_s)\left(e^{\gamma^2(1-e^{-2\gamma})\int_0^T V(\omega_s - \omega'_s)ds} - 1\right)\right] \leq V(0)\hat{E}_T^\otimes\left[\frac{1}{T}\int_0^T V(\omega_s - \omega'_s)ds\right].
\] (A.13)

For the above exponential we use first inequality (A.2) and then that there exists a constant $c(\gamma,t)$ such that, $e^x - 1 \leq c(\gamma,t)x$ for any $0 \leq x \leq c'(\gamma,t)$. That proves
\[
\hat{E}_T^\otimes\left[e^{\gamma^2(1-e^{-2\gamma})TV(0)}\int_0^T V(\omega_s - \omega'_s)ds\right] \leq c(\gamma,t)\hat{E}_T^\otimes\left[\frac{1}{T}\int_0^T V(\omega_s - \omega'_s)ds\right]
\] (A.14)
The same argumentation for the second factor yields the upper bound $V(0)c(\gamma,t)$. Thus, putting (A.11)-(A.14) together proves
\[
E_\eta|X'' - \hat{E}_T^\otimes[V(\omega_s - \omega'_s)]| \leq V(0)c(\gamma,t)\hat{E}_T^\otimes\left[\frac{1}{T}\int_0^T V(\omega_s - \omega'_s)ds\right].
\]

A similar argumentation also shows that $E_\eta|1 - Y''| \leq c(\gamma,t)\hat{E}_T^\otimes\left[\frac{1}{T}\int_0^T V(\omega_s - \omega'_s)ds\right]$. The last two assertions together with (A.10) then prove
\[
E_\eta\left|\frac{X''}{Y''} - \hat{E}_T^\otimes[V(\omega_s - \omega'_s)]\right| \leq V(0)c(\gamma,t)\hat{E}_T^\otimes\left[\frac{1}{T}\int_0^T V(\omega_s - \omega'_s)ds\right]
\]
and, thus, for given $\varepsilon$ we may choose $\delta$ small enough such that $\left\|\frac{X''}{Y''} - \hat{E}_T^\otimes[V(\omega_s - \omega'_s)]\right\|_{L^1(P,B_\delta)} \leq P(B_\delta)\varepsilon V(0)$.

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References

[AZ14] L.-P. Arguin and O. Zindy. Poisson-Dirichlet statistics for the extremes of a log-correlated Gaussian field. Ann. Appl. Probab. 24 (2014), 1446-1481
[AZ15] L.-P. Arguin and O. Zindy. Poisson-Dirichlet statistics for the extremes of the two-dimensional discrete Gaussian free field. Electron J. Probab., (2015), no. 59 1-15
[BC16] E. Bates and S. Chatterjee. The endpoint distribution of directed polymers. Ann. Probab., arXiv:1612.03443, (2016)
[BC19] E. Bates and S. Chatterjee. Localization in Gaussian disordered systems at low temperature. Ann. Probab. (to appear), arXiv:1906.05502 (2019)
[BP21] N. Berestycki and E. Powell. Gaussian free field, Liouville quantum gravity and Gaussian multiplicative chaos. March 2021, Available at: https://homepage.univie.ac.at/nathanael.berestycki/Articles/master.pdf
[Ber17] N. Berestycki. An elementary approach to Gaussian multiplicative chaos. Electron. Comm. Probab., 22, (2017)
[BGRV16] N. Berestycki, C. Garban, R. Rhodes and V. Vargas. KPZ formula derived from Liouville heat kernel. Journal of the London Mathematical Society, 94, (2016), 186-208

[BPR18] N. Berestycki, E. Powell and G. Ray. A characterisation of the Gaussian free field. Prob. Th. Rel. Fields, 176, 1259-1301 (2020)

[BPR20] N. Berestycki, E. Powell and G. Ray. (1 + \varepsilon)-moment suffice to characterise the GFF. Electron. J. Probab. 26, 1-25 (2021)

[BM19] Y. Bröker and C. Mukherjee. Quenched central limit theorem for the stochastic heat equation in weak disorder. Probability and Analysis in Interacting Physical Systems, (2019), 173-189, arXiv:1710.00631

[BM19-II] Y. Bröker and C. Mukherjee. Localization of the Gaussian multiplicative chaos in the Wiener space and the stochastic heat equation in strong disorder. Ann. Appl. Probab., 29 (2019), 3745-3785, arXiv: 1808.05202

[CCM19] F. Comets, C. Cosco and C. Mukherjee. Renormalizing the Kardar-Parisi-Zhang equation in \(d \geq 3\) in weak disorder. J Stat Phys. 179 713-728 (2020), Available at: arXiv: 1902.04104

[CCM19-II] F. Comets, C. Cosco and C. Mukherjee. Space-time fluctuation of the Kardar-Parisi-Zhang equation in \(d \geq 3\) and the Gaussian free field. Preprint, arXiv: 1905.03200 (2019)

[CC13] F. Comets and M. Cranston. Overlaps and pathwise localization in the Anderson polymer model. Stochastic Processes and their Applications, 123, 2446-2471, (2013).

[CSY03] F. Comets, T. Shiga and N. Yoshida. Directed polymers in a random environment: path localization and strong disorder Bernoulli, 9, 705–728, 2003

[CY06] F. Comets and N. Yoshida, Directed polymers in random environment are diffusive in weak disorder. Ann. Probab. 34, 1746-1770, 2006.

[CNN20] C. Cosco, S. Nakajima and M. Nakashima. Law of large numbers and fluctuations in the sub-critical and \(L^2\) regions for SHE and KPZ equation in dimension \(d \geq 3\). Preprint, arXiv: 2005.12689 (2020)

[DS88] B. Derrida and H. Spohn. Polymers on disordered trees, spin glasses and traveling waves, J. Stat. Phys., 51, 817-840 (1988).

[DRSV14-I] B. Duplantier, R. Rhodes, S. Sheffield and V. Vargas. Critical Gaussian multiplicative chaos: Convergence of the derivative martingale. Ann. Probab., 42, (2014), 1769-1808

[DRSV14-II] B. Duplantier, R. Rhodes, S. Sheffield and V. Vargas. Renormalization of Critical Gaussian Multiplicative Chaos and KPZ Relation. Communications in Mathematical Physics, 330, (2014), pp 283-330

[DS11] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ Invent. Math., 185, 333-393, (2011)

[GM19] E. Gwynne and J. Miller. Existence and uniqueness of the Liouville quantum gravity metric for \(\gamma \in (0, 2)\). Invent. Math, to appear, (2019), arXiv: 1905.00383

[HMP10] X. Hu, J. Miller and Y. Peres. Thick points of the Gaussian free field. Ann. Probab. 38, (2010), 896-926

[K85] J.-P. Kahane, Sur le chaos multiplicatif. Ann. sc. math. Quebec 9, no. 2, 105-150 (1985).

[KPZ88] V. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov. Fractal structure of 2d quantum gravity. Modern Physics Letters A, 3: 819’ 826, 1988.

[LZ20] D. Lygkonis and N. Zygouras. Edwards-Wilkinson fluctuations for the directed polymer in the full \(L^2\)-regime for dimensions \(d \geq 3\). Preprint, arXiv: 2005.12706 (2020)
[MRV16] T. Madaule, R. Rhodes and V. Vargas. Glassy phase and freezing of log-correlated Gaussian potentials. Ann. Appl. Probab., 26, (2016), 643-690

[MS15] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map I: the QLE \((8/3,0)\) metric. Inventiones Mathematicae, to appear, arXiv: 1507.00719, (2015)

[MS16a] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. Preprint, arXiv: 1605.03563, (2016)

[MS16b] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map III: the conformal structure is determined. Preprint, arXiv: 1608.05391, (2016)

[MSZ16] C. Mukherjee, A. Shamov and O. Zeitouni. Weak and strong disorder for the stochastic heat equation and the continuous directed polymer in \(d \geq 3\). Electron. Commun. Probab. 21 Paper No. 61, 12., (2016), arXiv: 1601.01652

[M17] C. Mukherjee. Central limit theorem for Gibbs measures on path spaces including long range and singular interactions and homogenization of the stochastic heat equation. Ann. Appl. Probab., arXiv: 1706.09345, (2017)

[MV14] C. Mukherjee and S.R.S. Varadhan. Brownian occupation measures, compactness and large deviations. Ann. Probab. 44, 3934–3964, (2016), arXiv: 1404.5259

[N06] D. Nualart The Malliavin calculus and related topics (2nd Ed.). Springer-Verlag, Berlin, (2006)

[P10] D. Panchenko. The Ghirlanda-Guerra identities for mixed \(p\)-spin model. Comptes Rendus Mathematique, 348, 189-192 (2010)

[RV10] R. Robert and V. Vargas. Gaussian multiplicative chaos revisited Ann. Probab. 38, 605-631, (2010).

[RV11] R. Rhodes and V. Vargas. KPZ formula for log-infinitely divisible multifractal random measures. ESAIM Probab. Stat., 15, 358-371, 2011

[S14] A. Shamov. On Gaussian multiplicative chaos. Journal of Functional Analysis. 270, 3224-3261, (2016) arXiv:1407.4418 (2014).

[ÜZ00] S.Üstünel and M. Zakai Transformation of measure on Wiener space. Springer-Verlag, Berlin, (2000)

[V07] V. Vargas. Strong localization and macroscopic atoms for directed polymers. Probab. Th. Rel. Fields., 138, 391-410, (2007)