On Pseudo-Convex Partitions of a Planar Point Set

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\textbf{Abstract.} Aichholzer et al. [Graphs and Combinatorics, Vol. 23, 481-507, 2007] introduced the notion of pseudo-convex partitioning of planar point sets and proved that the pseudo-convex partition number $\psi(n)$ satisfies, $\frac{n + 1}{2} \leq \psi(n) \leq \frac{n}{2}$). In this paper we improve the upper bound on $\psi(n)$ to $\left\lceil \frac{3n}{13} \right\rceil$, thus answering a question posed by Aichholzer et al. in the same paper.

\textbf{Keywords.} Convex hull, Discrete geometry, Empty convex polygons, Partition, Pseudo-triangles.

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\section{Introduction}

In 1978 Erdős [5] asked whether for every positive integer $k$, there exists a smallest integer $H(k)$, such that any set of at least $H(k)$ points in the plane, no three on a line, contains $k$ points which lie on the vertices of a convex polygon whose interior contains no points of the set. Such a subset is called an \textit{empty convex $k$-gon} or a $k$-hole. Esther Klein showed $H(4) = 5$ and Harborth [7] proved that $H(5) = 10$. Horton [8] showed that it is possible to construct arbitrarily large set of points without a 7-hole, thereby proving that $H(k)$ does not exist for $k \geq 7$. Recently, after a long wait, the existence of $H(6)$ has been proved by Gerken [6] and independently by Nicolás [12]. Later Valtr [16] gave a simpler version of Gerken’s proof.

Two empty convex polygons are said to be \textit{disjoint} if their convex hulls do not intersect. Let $H(k, \ell)$, $k \leq \ell$ denote the smallest integer such that any set of $H(k, \ell)$ points in the plane, no three on a line, contains both a $k$-hole and a $\ell$-hole which are disjoint. Clearly, $H(3, 3) = 6$ and Horton’s result [8] implies that $H(k, \ell)$ does not exist for all $\ell \geq 7$. Urabe [15] showed that $H(3, 4) = 7$, while Hosono and Urabe [11] showed that $H(4, 4) = 9$. Hosono and Urabe [10] also proved that $H(3, 5) = 10$, $12 \leq H(4, 5) \leq 14$, and $16 \leq H(5, 5) \leq 20$. The results $H(3, 4) = 7$ and $H(4, 5) \leq 14$ were later reconfirmed by Wu and Ding [17]. Recently, using the computer-aided order-type enumeration method, Aichholzer et al. [1] proved that every set of 11 points in the plane, no three on a line, contains either a 6-hole or a 5-hole and a disjoint 4-hole. Recently, this result was proved geometrically by Bhattacharya and Das [2, 3]. Using this Ramsey-type result, Hosono and Urabe [9] proved that $H(4, 5) \leq 13$, which has been consequently tightened to $H(4, 5) = 12$ by Bhattacharya and Das [4]. Hosono and Urabe [9] have also improved the lower bound on $H(5, 5)$ to 17.

The problem of partitioning planer point sets with disjoint holes was first addressed by Urabe [15]. For any set $S$ of points in the plane, denote the \textit{convex hull} of $S$ by $CH(S)$, and cardinality of $S$ by $|S|$. Given a set $S$ of $n$ points in the plane, no three on a line, a \textit{disjoint convex partition} of $S$ is a partition of $S$ into subsets $S_1, S_2, \ldots, S_t$, with $\sum_{i=1}^{t} |S_i| = n$, such that for each $i \in \{1, 2, \ldots, t\}$, $CH(S_i)$ forms a $|S_i|$-gon and $CH(S_i) \cap CH(S_j) = \emptyset$, for any pair of indices $i, j$. Observe that in any disjoint convex partition of $S$, the set $S_i$ forms a
|$S_i$|-hole and the holes formed by the sets $S_i$ and $S_j$ are disjoint for any pair of distinct indices $i, j$.

Let $\kappa(S)$ denote the minimum number of disjoint holes in any disjoint convex partition of $S$. Define $\kappa(n) = \max_S \kappa(S)$, where the maximum is taken over all sets $S$ of $n$ points. $\kappa(n)$ is called the *convex partition number* for all sets $S$ of fixed size $n$, and it is bounded by $\left\lfloor \frac{n+1}{4} \right\rfloor \leq \kappa(n) \leq \left\lceil \frac{n}{2} \right\rceil$. The lower bound was given by Urabe [15] and the upper bound by Hosono and Urabe [11]. The lower bound was later improved to $\left\lfloor \frac{n+1}{4} \right\rfloor$ by Xu and Ding [18].

A pseudo-triangle is a simple polygon with exactly three vertices having interior angles less than $180^\circ$, and is considered to be the natural counterpart of a convex polygon. The recent interest in the properties of pseudo-triangles stems from the study of the combinatorial properties of pseudo-triangulations, which are tessellation of the plane with pseudo-triangles. They provide sparser tessellations than triangulations but retain many of the desirable properties of triangulations. Pseudo-triangulations have received considerable attention in the recent last few years for applications in areas like motion planning, collision detection, ray shooting, rigidity, or visibility (see Rote et al. [13] for a survey of the different properties of pseudo-triangulations and its various applications).

A pseudo-triangle with $k$-vertices is called a $k$-pseudo-triangle. Two empty pseudo-triangles, or a hole and an empty pseudo-triangle are said to be disjoint if their vertex sets as well as their interiors are disjoint. Given a set $S$ of $n$ points in the plane, no three on a line, a *pseudo-convex partition* of $S$ is a partition of $S$ into subsets $S_1, S_2, \ldots S_t$, with $\sum_{i=1}^t |S_i| = n$, such that for each $i \in \{1, 2, \ldots, t\}$, the set $S_i$ forms a $|S_i|$-hole or a $|S_i|$-pseudo-triangle, the holes or pseudo-triangles formed by the sets $S_i$ and $S_j$ are disjoint for any pair of distinct indices $i, j$. If $\psi(S)$ denotes the minimum number of disjoint holes or empty pseudo-triangles in any pseudo convex partition of $S$, then the *pseudo-convex partition number* is defined as $\psi(n) = \max_S \psi(S)$, where the maximum is taken over all sets $S$ of $n$ points.

Recently, Aichholzer et al. [1] introduced the problem of partitioning planer point sets with disjoint holes or empty pseudo-triangles, and introduced the notion of pseudo-convex partitioning in that context. Aichholzer et al. [1] showed that the pseudo-convex partition number $\psi(n)$ satisfies: $\frac{3}{4} \left\lfloor \frac{n}{4} \right\rfloor \leq \psi(n) \leq \left\lceil \frac{n}{4} \right\rceil$. The upper bound follows from the simple observation that every set of 4 points forms either an 4-hole or an empty 4-pseudo-triangle. In their paper, Aichholzer et al. [1] mentioned the possibility of a better upper bound on $\psi(n)$ if $\psi(13) = 3$ holds. In this paper, we answer this question in the affirmative, thus proving that $\psi(n) \leq \left\lceil \frac{3n}{13} \right\rceil$.

2 Notations and Definitions

We first introduce the definitions and notations required for the remaining part of the paper. Let $S$ be a finite set of points in the plane in general position, that is, no three on a line. Denote the *convex hull* of $S$ by $CH(S)$. The boundary vertices of $CH(S)$, and the points of $S$ in the interior of $CH(S)$ are denoted by $V(CH(S))$ and $I(CH(S))$, respectively. A region $R$ in the plane is said to be *empty* in $S$ if $R$ contains no elements of $S$ in its interior. Moreover, for any set $T$, $|T|$ denotes the cardinality of $T$.

By $P := p_1p_2\ldots p_k$ we denote a convex $k$-gon with vertices $\{p_1, p_2, \ldots, p_k\}$ taken in anti-clockwise order. $V(P)$ denotes the set of vertices of $P$ and $I(P)$ the interior of $P$. A set of points $T \subseteq S$ is said to *span* a simple polygon $P$ if $V(P) = T$. 
The \textit{j-th convex layer} of \(S\), denoted by \(L(j, S)\), is the set of points that lie on the boundary of \(CH(S\setminus\bigcup_{i=1}^{j-1} L(i, S))\), where \(L(1, S) = \mathcal{V}(CH(S))\). If \(p, q \in S\) are such that \(pq\) is an edge of the convex hull of the \(j\)-th layer, then the open halfplane bounded by the line \(pq\) and not containing any point of \(S\setminus\bigcup_{i=1}^{j-1} L(i, S)\) will be referred to as the \textit{outer halfplane} induced by the edge \(pq\).

For any three points \(p, q, r \in S\), \(\mathcal{H}(pq, r)\) denotes the open halfplane bounded by the line \(pq\) containing the point \(r\). Similarly, \(\overline{\mathcal{H}}(pq, r)\) is the open halfplane bounded by \(pq\) not containing the point \(r\). Moreover, if \(\angle rpq < \pi\), \(\text{Cone}(rpq)\) denotes the interior of the angular domain \(\angle rpq\). A point \(s \in \text{Cone}(rpq) \cap S\) is called the \textit{nearest angular neighbor} of \(\overline{pq}\) in \(\text{Cone}(rpq)\) if \(\text{Cone}(spq)\) is empty in \(S\). Similarly, for any convex region \(R\) a point \(s \in R \cap S\) is called the \textit{nearest angular neighbor} of \(\overline{pq}\) in \(R\) if \(\text{Cone}(spq) \cap R\) is empty in \(S\). More generally, for any positive integer \(k\), a point \(s \in S\) is called the \textit{k-th angular neighbor} of \(\overline{pq}\) whenever \(\text{Cone}(spq) \cap R\) contains exactly \(k - 1\) points of \(S\) in its interior.

3 Pseudo-Convex Partitioning

Aichholzer et al. [1] showed that \(\psi(n) \leq \left\lceil \frac{n}{4} \right\rceil\). They also mention the possibility of a better upper bound of \(\left\lceil \frac{3n}{13} \right\rceil\) on \(\psi(n)\) by conjecturing that \(\psi(13) = 3\).

In the following theorem we settle this conjecture in the affirmative.

\textbf{Theorem 1.} Every set of 13 points in the plane, in general position, can be partitioned into three sets each of which span either a hole or an empty pseudo-triangle which are mutually disjoint. In other words, \(\psi(13) = 3\).

Theorem 1 immediately establishes a non-trivial upper bound on \(\psi(n)\), which is presented in the following theorem.

\textbf{Theorem 2.} \(\psi(n) \leq \left\lceil \frac{3n}{13} \right\rceil\).

\textit{Proof.} Let \(S\) be a set of \(n\) points in the plane, no three of which are collinear. By a horizontal sweep we can divide the plane into \(\left\lfloor \frac{\psi(n)}{3} \right\rfloor\) disjoint strips, of which \(\left\lceil \frac{\psi(n)}{7} \right\rceil\) contain 13 points each and one remaining strip \(R\), with \(|R| < 13\). The strips having 13 points, can be partitioned into three disjoint holes or empty pseudo-triangles by Theorem 1. Since \(|R| < 13\), at most \(\left\lceil \frac{3|R|}{13} \right\rceil\) disjoint holes or empty pseudo-triangles are needed to partition \(R\), thus proving that \(\psi(n) \leq \left\lceil \frac{3n}{13} \right\rceil\). \(\square\)

4 Proof of Theorem 1

Let \(S\) be a set of 13 points in the plane in general position. A partition of \(S\) into three disjoint subsets \(S_1, S_2, S_3\) is called \textit{admissible} if each \(S_i, i \in \{1, 2, 3\}\), is either empty or it forms an \(|S_i|\)-hole or empty \(|S_i|\)-pseudo-triangle, such that the holes or pseudo-triangles formed by the sets \(S_i\) and \(S_j\) are disjoint for any pair of distinct indices \(i \neq j\). The set \(S\) is said to be \textit{admissible} if there exists an admissible partition of \(S\). To prove Theorem 1 we need to exhibit an admissible partition of \(S\), for all sets of 13 points in the plane, in general position.

Observe that any set of 4 points in the plane always spans a convex quadrilateral or a 4-pseudo-triangle. Hence, we have the following observation.

\textbf{Observation 1} For every integer \(k \geq 1\), we have \(\psi(4k) \leq k\). \(\square\)
Observation 2 $S$ is admissible if some outer halfplane induced by an edge of the second layer contains more than two points of $\mathcal{V}(CH(S))$.

Proof. Suppose some outer halfplane induced by an edge of the second layer contains more than two points of $\mathcal{V}(CH(S))$. This means that there exists two points $p, q \in S$, such that the line segment $pq$ is an edge of $CH(L\{2, S\})$ and $|\mathcal{H}(pq, r) \cap S| \geq 3$, where $r$ is any point of $L\{2, S\}$ different from $p$ and $q$. Then $\{\mathcal{H}(pq, r) \cap S\} \cup \{p, q\}$ spans a $k$-hole, with $k \geq 5$. The remaining eight points of $S$ all lie in the halfplane $\mathcal{H}(pq, r)$. Now, from Observation 1 we have $\psi(8) \leq 2$, which implies that the points in $\mathcal{H}(pq, r) \cap S$ can be partitioned using at most two disjoint holes or empty pseudo-triangles, and the admissibility of $S$ follows. \qed

Fig. 1. Illustration for the proof of (a) Observation 3, (b) Observation 4, and (c) Observation 5.

Observation 3 Let $\mathcal{V}(CH(S)) = \{s_1, s_2, \ldots, s_k\}$, with the vertices taken in the counterclockwise order. If there exists a point $s_i \in \mathcal{V}(CH(S))$, such that $|I(s_{i-1}s_is_{i+1}) \cap S| \geq 2$, then $S$ is admissible, where the indices are taken modulo $k$.

Proof. Suppose $|I(s_{i-1}s_is_{i+1}) \cap S| = a \geq 2$, for some $i = 1, 2, \ldots, k$. Let $p \in S$ be the first angular neighbor of $s_i$ in $\text{Cone}(s_is_{i-1}s_{i+1})$ and $Z = \mathcal{V}(CH(S)\{p, s_i\})$ (Figure 1(a)). Now, $a \geq 2$ implies that $|Z \mathcal{V}(CH(S))| \geq 1$ and $\{Z \mathcal{V}(CH(S))\} \cup \{p, s_{i-1}, s_i, s_{i+1}\}$ forms an empty $k$-pseudo-triangle, with $k \geq 5$. The remaining $13 - k$ points lie in a convex region which can be partitioned using at most two holes or empty pseudo-triangles, since $13 - k \leq 8$, and $\psi(8) \leq 2$ from Observation 1. \qed

Observation 4 $S$ is admissible if there exists three distinct points $s_1, s_2, s_3 \in S$ satisfying the following two conditions:

(A) $|\overline{I}(s_1s_2, s_3) \cap \overline{I}(s_1s_2, s_3)) \cap S| = 0$ and $|\overline{I}(s_2s_3, s_1) \cap S| = 0$.
(B) $|\overline{I}(s_1s_2, s_3) \cap S| \leq 4$ and $|\overline{I}(s_2s_3, s_1) \cap S| \leq 4$.

Proof. Let $|\overline{I}(s_1s_2, s_3) \cap S| = a \leq 4$ and $|\overline{I}(s_2s_3, s_1) \cap S| = b \leq 4$. Then, $|I(s_1s_2s_3) \cap S| = 10 - (a + b)$. Then, there exists a point $\alpha \notin S$ on the line segment $s_2s_3$ such that $|I(s_1s_2\alpha) \cap S| = 5 - a$ and $|I(s_1s_3\alpha) \cap S| = 5 - b$ (see Figure 1(b)). Let $C_\alpha(s_1, s_2) = CH(I(s_1s_2\alpha) \cap S) \cup \{s_1, s_2\}$ and $C_\alpha(s_1, s_3) = CH(I(s_1s_3\alpha) \cap S) \cup \{s_1, s_3\}$. Now, since both $a, b \leq 4$, we have $|\mathcal{V}(C_\alpha(s_1, s_2))| \geq 3$ and $|\mathcal{V}(C_\alpha(s_1, s_3))| \geq 3$. Thus, $S_1 = \mathcal{V}(C_\alpha(s_1, s_2)) \cup \mathcal{V}(C_\alpha(s_1, s_3))$ spans an empty $m$-pseudo-triangle, with $m \geq 5$. Moreover, both $S_2 = I(C_\alpha(s_1, s_2)) \cup \{\overline{I}(s_1s_2, s_3) \cap S\}$, and $S_3 = I(C_\alpha(s_1, s_3)) \cup \{\overline{I}(s_1s_3, s_2) \cap S\}$ lie inside two disjoint convex regions containing at most 4 points each. Therefore, the partition $S = S_1 \cup S_2 \cup S_3$ is admissible. \qed
If there exist three distinct points \( s_1, s_2, s_3 \in S \) satisfying conditions (A) and (B) of Observation 4, then the three points are said to form a heart with the line segment \( s_2 s_3 \) as base and the point \( s_1 \) as pivot. The set \( S \) is admissible if any three of its points form a heart. Observe that if \( s_1, s_2, s_3 \in \mathcal{V}(CH(S)) \) and some edge of the triangle \( s_1 s_2 s_3 \) is also an edge of \( CH(S) \), then condition (A) is automatically satisfied. In such cases, \( s_1, s_2, s_3 \) form a heart whenever condition (B) holds.

Equipped with the above three observations, we now proceed to prove the admissibility of \( S \). The proof of the admissibility of \( S \) is presented in three separate sections. The first section deals with the cases \(|CH(S)| \leq 5\), the second section with the case \(|CH(S)| = 6\), and the third considers the cases \(|CH(S)| \geq 7\).

Let \( \mathcal{V}(CH(S)) = \{s_1, s_2, \ldots, s_k\} \), with the vertices taken in the counter-clockwise order. While indexing a set of points from \( \mathcal{V}(CH(S)) \), we identify indices modulo \( k \).

### 4.1 \(|CH(S)| \leq 5\)

**Lemma 1.** \( S \) is admissible whenever \(|CH(S)| \leq 5\).

**Proof.** Observe that if \(|CH(S)| = 3\), then the admissibility of \( S \) is a direct consequence of Observation 3. Now, we consider the following two cases based on the size of \(|CH(S)|\):

**Case 1:** \(|CH(S)| = 4\). This implies that \(|\mathcal{I}(CH(S))| = 9\). Therefore, either \(|\mathcal{I}(s_2 s_3 s_4) \cap S| \geq 2\) or \(|\mathcal{I}(s_1 s_2 s_4) \cap S| \geq 2\). The admissibility of \( S \) then follows from Observation 3.

**Case 2:** \(|CH(S)| = 5\). Suppose that \(|\mathcal{I}(s_1 s_2 s_3)| = a\), and \(|\mathcal{I}(s_1 s_4)| = b\). If \( a \geq 2\) or \( b \geq 2\), the admissibility of \( S \) is guaranteed from Observation 3. Therefore, assume that both \( a, b \leq 1\). This implies that \(|\mathcal{H}(s_1 s_3 s_4) \cap S| \leq 2\) and \(|\mathcal{H}(s_1 s_4, s_3) \cap S| \leq 2\). Thus, the three points \( s_1, s_3, s_4 \) satisfy Conditions (A) and (B) of Observation 4 and form a heart with \( s_3 s_4 \) as base and \( s_1 \) as pivot. Thus, the admissibility of \( S \) follows. \( \square \)

### 4.2 \(|CH(S)| = 6\)

For any \( i \in \{1, 2, 3\} \), the diagonal \( d := s_1 s_{i+3} \) of the hexagon \( s_1 s_2 s_3 s_4 s_5 s_6 \) is called an \((a, b)\)-splitter of \( CH(S) \), where \( a \leq b \) are integers, if either \(|\mathcal{H}(s_i s_{i+3}, s_{i+1}) \cap \mathcal{I}(CH(S))| = a\) and \(|\mathcal{H}(s_{i+3}, s_{i+1}) \cap \mathcal{I}(CH(S))| = b\), or \(|\mathcal{H}(s_1 s_{i+3}, s_{i+1}) \cap \mathcal{I}(CH(S))| = a\) and \(|\mathcal{H}(s_i s_{i+3}, s_{i+1}) \cap \mathcal{I}(CH(S))| = b\).

We now have the following observation:

**Observation 5** If any one of the three diagonal \( s_2 s_5, s_1 s_4, \) and \( s_3 s_6 \) is not a \((3, 4)\)-splitter of \( CH(S) \), then \( S \) is admissible.

**Proof.** It suffices to prove that \( S \) is admissible whenever \( s_2 s_5 \) is not a \((3, 4)\)-splitter of \( CH(S) \). Suppose the diagonal \( s_2 s_5 \) is a \((a, b)\)-splitter of \( CH(S) \), with \( a \leq 2\), and \( b \geq 5\). Refer to Figure 1(c). W. l. o. g. assume that \(|\mathcal{I}(s_1 s_2 s_5 s_6) \cap S| = a\) and \(|\mathcal{I}(s_2 s_3 s_4 s_5) \cap S| = b\). Now, since \( b \geq 5\), by the pigeon-hole principle, either \(\mathcal{I}(s_2 s_3 s_5)\) or \(\mathcal{I}(s_3 s_4 s_5)\) contains at least 3 points of \( S \). However, if \(|\mathcal{I}(s_3 s_4 s_5) \cap S| \geq 2\), then Observation 3 guarantees the admissibility of \( S \). Therefore, assume that \(|\mathcal{I}(s_3 s_4 s_5) \cap S| = b' \leq 1\). This implies that \(|\mathcal{H}(s_5 s_3, s_2) \cap S| \leq 2\) and \(|\mathcal{H}(s_5 s_2, s_3) \cap S| \leq 4\), and the three points \( s_2, s_3, s_5 \) form a heart with \( s_2 s_3 \) as base and \( s_5 \) as pivot (see Figure 1(c)). The admissibility of \( S \) thus follows from Observation 4. \( \square \)
In light of Observation 5, it suffices to assume that the three diagonals $s_3s_5$, $s_1s_4$, and $s_3s_6$ are $(3,4)$-splitters of $CH(S)$. Consider the partition of the interior of $CH(S)$ by the three diagonals into 7 disjoint regions $R_i$ as shown in Figure 2(a). Let $|R_i|$ denote the number of points of $S$ inside region $R_i$.

Now, we have the following observation:

**Observation 6** If $|R_4| + |R_7| \geq 2$ and $|R_1| \geq 1$, then there exists a point $p \in R_1 \cap S$ such that the three points $p, s_3, s_4$ form a heart with $s_3s_4$ as base and $p$ as pivot.

**Proof.** Let $|R_4| + |R_7| = a \geq 2$. Now, since both the diagonals $s_1s_4$ and $s_3s_6$ are $(3,4)$-splitters of $CH(S)$, we have $|R_2| + |R_3| = b \leq 2$, $|R_5| + |R_6| = b' \leq 2$, and $|R_1| + |R_5| + |R_6| \geq 3$. Let $q_1 \in R_1 \cap S$ be the $(3 - b)$-th angular neighbor of $s_3q_1$ in $R_1$. Let $U_1 = (\text{Cone}(s_1q_3s_4) \setminus I(q_1s_3s_4)) \cap S$ and $V_1 = \mathcal{H}(s_3q_1, s_2) \cap S$, and $\alpha_{11}$ and $\alpha_{12}$ are the points where the rays $s_3q_1$ and $s_5q_1$ intersect the boundary $CH(S)$, respectively (Figure 2(a)). Therefore, $|U_1| \leq 4$ and $|V_1| \leq 4$. Now, if $\text{Cone}(\alpha_{11}q_1\alpha_{12}) \cap S$ is empty, then $q_1 = p$ and the result follows.

Otherwise, suppose that $\text{Cone}(\alpha_{11}q_1\alpha_{12}) \cap S$ is non-empty. Let $q_2 \in R_1 \cap S$ be the nearest angular neighbor of $s_3q_1$ in $\text{Cone}(\alpha_{11}q_1\alpha_{12})$. Let $\alpha_{21}$ and $\alpha_{22}$ be the points where the rays $\overrightarrow{s_3q_2}$ and $\overrightarrow{s_4q_2}$ intersect the boundary $CH(S)$, respectively. Define, $U_2 = (\text{Cone}(s_2q_3s_4) \setminus I(q_2s_3s_4)) \cap S$ and $V_2 = \mathcal{H}(s_3q_2, s_2) \cap S$. Observe that $U_2 \subset U_1$ and $V_2 \subset V_1$, and hence, $|U_2| \leq 4$ and $|V_2| \leq 4$. Therefore, if $\text{Cone}(\alpha_{21}q_2\alpha_{22}) \cap S$ is empty, then $q_2 = p$ is the required point.

If $\text{Cone}(\alpha_{21}q_2\alpha_{22}) \cap S$ is non-empty, we repeat the same procedure again, until we get a point $p = q_i \in R_1 \cap S$ with $|U_i| \leq 4$, $|V_i| \leq 4$, and $|\text{Cone}(\alpha_{i1}q_i\alpha_{i2}) \cap S| = 0$, where $q_i$ is the nearest angular neighbor of $s_3q_i$ in $R_1 \cap S$, $U_i = (\text{Cone}(q_is_3s_4) \setminus I(q_is_3s_4)) \cap S$, $V_i = \mathcal{H}(s_3q_i, s_2) \cap S$, and $\alpha_{i1}$ and $\alpha_{i2}$ are the points where the rays $s_3q_i$ and $s_5q_i$ intersect the boundary $CH(S)$, respectively (see Figure 2(b)).

The admissibility of $S$ when $|CH(S)| = 6$ is now proved by considering the following three cases:

**Case 1:** $|R_4| + |R_7| \geq 2$ and $|R_1| \geq 1$. In this case, Observation 6 guarantees the existence of a point $p \in R_1 \subset S$ such that the three points $p, s_3, s_4$ form a heart with $s_3s_4$ as base and $p$ as pivot (see Figure 2(b)).
Case 2: \(|R_4| + |R_7| \geq 2 \) and \( |R_1| = 0 \). This implies that \( |R_2| + |R_3| \leq 2 \), since \( s_1s_4 \) is a \((3, 4)\)-splitter of \( CH(S) \). Thus, \( |R_1| + |R_2| + |R_3| \leq 2 \), which contradicts the assumption that diagonal \( s_3s_6 \) is a \((3, 4)\)-splitter of \( CH(S) \).

Case 3: \(|R_2| + |R_7| \leq 1 \), \(|R_4| + |R_7| \leq 1 \), and \(|R_6| + |R_7| \leq 1 \). Therefore, \(|R_2| + |R_4| + |R_6| + |R_7| \leq 3 \) which implies that \(|R_1| + |R_3| + |R_5| \geq 4 \). By the pigeon-hole principle, one the three regions \( R_1 \), \( R_3 \), and \( R_5 \) contains at least two points \( S \). W. l. o. g., assume \( |R_1| \geq 2 \). This implies that \(|R_2| + |R_3| \leq 2 \), and hence \(|R_4| + |R_7| \geq 1 \). Combining this with the given inequality we get, \(|R_4| + |R_7| = 1 \), \(|R_2| + |R_3| = 2 \), and \(|R_1| = 2 \). If \( p \in (R_4 \cup R_7) \cap S \), then the three points \( s_1, p, s_6 \) form a heart with \( s_1s_6 \) as base and \( p \) as pivot.

4.3 \( |CH(S)| \geq 7 \)

First, we have the following simple observation.

**Observation 7** \( S \) is admissible whenever \( |CH(S)| \geq 9 \).

**Proof.** Let \( |CH(S)| = k \geq 9 \). This implies that \( |L\{2, S\}| = 13 - k \leq 4 \), and so, there must exist an outer halfplane induced by an edge of \( L\{2, S\} \) containing more than two points of \( S \). The result now follows from Observation 2. \( \square \)

The proof of the admissibility of \( S \) in the remaining cases are presented in the following two lemmas. Hereafter, we shall assume that \( L\{2, S\} = \{p_1, p_2, \ldots, p_m\} \), where the vertices are taken in counter-clockwise order, and the indices are to be identified modulo \( m \).

**Lemma 2.** \( S \) is admissible whenever \( |CH(S)| = 8 \).

**Proof.** If \( |L\{2, S\}| = 3 \) and none of the outer halfplanes induced by the three edges of \( CH(L\{2, S\}) \) contains more than two points of \( V(CH(S)) \), then \( |V(CH(S))| \leq 2 \times 3 < 8 \). This implies that some outer halfplane contains at least three points of \( V(CH(S)) \), and the admissibility of \( S \) is guaranteed from Observation 2.

**Fig. 3.** Illustration of the proof of Lemma 2 (a) \( |L\{2, S\}| = 4 \), (b) \( |L\{2, S\}| = 5 \), and (c) Illustration of the proof of Lemma 3 with \( |L\{2, S\}| = 4 \).

Therefore, \( |L\{2, S\}| \) is either 4 or 5. These two cases are considered separately as follows:
Case 1: \(|L\{2, S\}| = 4\). Let \(L\{3, S\} = \{p\}\) and consider the subdivision of the exterior of the convex quadrilateral \(p_1p_2p_3p_4\) into regions \(R_i\) as shown in Figure 3(a). Note that the regions \(R_i\) and \(R_{i+2}\), for \(i \in \{1, 3, 5, 7\}\), might intersect to produce a region which is not adjacent to the convex hull of the second layer. For the sake of notational convenience we shall assume that this region is a part of the region \(R_i\). This ensures that the regions \(R_i\) are mutually disjoint and by \(|R_i|\) we denote the number of points of \(S\) in region the \(R_i\). Now, since \(|CH(S)| = 8\), Observation 2 implies that we need to verify the admissibility of \(S\) only when every outer halfplane induced by the edges of \(CH(L\{2, S\})\) contains exactly 2 points of \(V(CH(S))\). Therefore, we have \(|R_i| + |R_{i+1}| + |R_{i+2}| = 2\), for \(i \in \{1, 3, 5, 7\}\). Adding these 4 equations and using the fact that \(\sum_{i=1}^{8} |R_i| = 8\) we get, \(|R_1| + |R_3| + |R_5| + |R_7| = 0\). This implies that \(|R_2| = |R_4| = |R_6| = 2\). Let \(\alpha\) be the point of intersection of the diagonals of \(p_1p_2p_3p_4\). W. l. o. g., assume that \(p \in \mathcal{I}(p_2p_3\alpha)\). Observe that the three points \(p, p_2, p_3\) and two points in \(R_4 \cap S\) form a 5-hole. Moreover, the remaining 8 points can be partitioned into two disjoint convex regions with 4 points each, because there exists a point \(\beta\) on the line segment \(p_1p_4\) such that \(|\text{Cone}(\beta p_2) \cap S| = 4\) and \(|\text{Cone}(\beta p_3) \cap S| = 4\) (see Figure 3(a)).

Case 2: \(|L\{2, S\}| = 5\). Consider the partition of the exterior of the empty convex pentagon \(p_1p_2p_3p_4p_5\) into regions \(R_i\)'s as shown in Figure 3(b). Observation 2 implies that \(S\) is admissible unless \(|R_1| + |R_{i+1}| + |R_{i+2}| \leq 2\), for \(i \in \{1, 3, 5, 7\}\). Adding these 5 inequalities and using the fact that \(\sum_{i=1}^{10} |R_i| = 8\) we get, \(|R_1| + |R_3| + |R_5| + |R_7| + |R_9| \leq 2\), that is, \(|R_2| + |R_4| + |R_6| + |R_8| + |R_{10}| \geq 6\). By the pigeon-hole principle, one of these 5 regions must contain exactly two points of \(V(CH(S))\). W. l. o. g., assume that \(|R_2| = 2\). Let \(Z_1 = \{\mathcal{H}(p_3p_4, p_5) \setminus R_7\} \cap \mathcal{V}(CH(S))\) and \(Z_2 = \{\mathcal{H}(p_3p_5, p_2) \setminus R_7\} \cap \mathcal{V}(CH(S))\).

Case 2.1: \(|R_7| \geq 1\). We have \(|R_5| + |R_6| \leq 1\) and \(|R_8| + |R_9| \leq 1\). Therefore, \(Z_1 \cup \{p_4\} = a_1 \leq 4\) and \(|Z_2 \cup \{p_2\}| = a_2 \leq 4\). Now, \(|R_7 \cap S| = 6 - (a_1 + a_2)\) and there exists a point \(s_0 \in R_7 \cap S\) such that \(\text{Cone}(s_0p_3p_1) \cap S\) and \(\text{Cone}(s_0p_3p_5) \cap S\) both contain 4 points and \(\{p_1, p_3, p_5\} \cup (R_2 \cap \mathcal{V}(CH(S)))\) spans a 5-hole. Therefore, \(S\) is admissible.

Case 2.2: \(|R_7| = 0\). Let \(R'_2 = \overline{\mathcal{H}(p_4p_5, p_1)} \cap R_2\). We know that \(|Z_1| \leq 4\), \(|Z_2| \leq 4\), and \(|Z_1| + |Z_2| = 6\). If \(|Z_1| = |Z_4| = 3\), the partition \(S_1 = Z_1 \cup \{p_4\}, S_2 = Z_2 \cup \{p_2\}\), and \(S_3 = (R_2 \cap S) \cup \{p_1, p_3, p_5\}\) is admissible. Otherwise, either \(|Z_1| = 4\) or \(|Z_2| = 4\). W. l. o. g. let \(|Z_1| = 4\). This implies that \(|R_6| = 2\), \(|R_3| + |R_4| = 2\) and \(|R_2| = 0\). Let \(s_i \in S\) be the first angular neighbor of \(\vec{p_1p_2}\) in \(R_6\). Then set \(\{p_1, p_5, s_i\} \cup (R_2 \cap S)\) spans a 5-hole and the admissibility of \(S\) follows.

Lemma 3. \(S\) is admissible whenever \(|CH(S)| = 7\).

Proof. If \(|CH(L\{2, S\})| = 3\), the admissibility of \(S\) follows easily from Observation 2. Therefore, \(4 \leq |L\{2, S\}| \leq 6\). We consider these three cases separately as follows:

Case 1: \(|L\{2, S\}| = 4\). Let \(L\{3, S\} = \{p, q\}\) be such that \(p\) and \(q\) are the first and second angular neighbors of \(\vec{p_3p_5}\) in \(\mathcal{I}(p_1p_2p_3p_4)\). Consider the subdivision of the exterior of the convex quadrilateral \(p_1p_2p_3p_4\) into regions \(R_i\) as shown in Figure 3(c). Again, it follows from Observation 2 that \(|R_i| + |R_{i+1}| + |R_{i+2}| \leq 2\), for \(i \in \{1, 3, 5, 7\}\). Now, using the fact that \(\sum_{i=1}^{8} |R_i| \geq 7\), we can easily conclude that there exists some \(j \in \{2, 4, 6, 8\}\) such that \(|R_j| = |R_{j+2}| = 2\). W. l. o. g., assume \(j = 2\), which implies \(|R_2| = |R_4| = 2\). Observation 2 now implies that \(|R_1| = |R_3| = |R_5| = 0\), which implies that \(|R_7| = |R_8| = k \neq 0\). Let \(s_i \in \mathcal{V}(CH(S))\) be the \(k\)-th angular neighbor of \(\vec{p_1p_2}\) in \(R_7 \cup R_8\), and let \(R'_2 = \overline{\mathcal{H}(p_1s_i, p_3)} \cap R_2\). Observe that \(|R'_2| \neq 0\), since \(p_1 \in L\{2, S\}\) and \(s_i \in \mathcal{V}(CH(S))\).
Case 1.1: $|R_7| + |R_8| = 1$. Let $S_2 = (R_2 \cap S) \cup \{p_1, s_1\}$. If $R_2 \cap S \subset R'_2 \cap S$ then $S_2$ spans a $4$-hole. Otherwise, $|R'_2| = 1$, and $S_2$ spans an empty $4$-pseudo-triangle (see Figure 3(c)). Therefore, the partition of $S$ given by $S_1 = (R_4 \cap S) \cup \{p_2, p_3, p\}$, $S_2$, and $S_3 = (\text{Cone}(p_4p_3) \cap S) \cup \{q\}$.

Case 1.2: $|R_7| + |R_8| = 2$. Again, consider $S_2 = (R_2 \cap S) \cup \{p_1, s_1\}$. Depending on whether $R_2 \cap S$ is empty or non-empty, $S_2$ spans a $4$-hole or empty $4$-pseudo-triangle. Now, if $s_j \in (H(p_3, p_2) \cap R_8) \cap S$, then $(R_4 \cap S) \cup \{p_2, p_3, p, s_j\}$ spans a $6$-hole and the admissible partition of $S$ is immediate. Otherwise, the partition of $S$ given by $S_1 = (R_4 \cap S) \cup \{p_2, p_3, p\}$, $S_2$, and $S_3 = (\text{Cone}(s_iq) \cap S) \cup \{s_i, q\}$ is an admissible partition of $S$, where $\alpha$ is the point of intersection of the lines $\overrightarrow{p_3p}$ and $\overrightarrow{p_1s}$.

Case 2: $|L\{2, S\}| = 5$. Let $L\{3, S\} = \{p\}$ and consider the partition of the exterior of $CH(L\{2, S\})$ into disjoint regions $R_i = \text{Cone}(p_ipp_{i+1}) \setminus \mathcal{I}(p_ipp_{i+1})$, for $i \in \{1, 2, 3, 4, 5\}$. Let $R$ be shaded region inside $CH(L\{2, S\})$ as shown in Figure 4(a).

Case 2.1: $p \in R$. Observe that $\sum_{i=1}^{5} |R_i| = 7$ and by Observation 2 for every $i \in \{1, 2, 3, 4, 5\}$, $|R_i| \leq 2$. Therefore, w. l. o. g. assume $|R_1| = 2$ (Figure 4(a)). If $|R_2| + |R_3| = 2$ and $|R_4| + |R_5| = 3$, then $S_1 = (R_1 \cap S) \cup \{p, p_1, p_2\}$, $S_2 = \text{Cone}(p_1p_4) \cap S$, and $S_3 = (\text{Cone}(p_2p_4) \cap S) \cup \{p_4\}$ is an admissible partition of $S$. Similarly, for $|R_2| + |R_3| = 3$ and $|R_4| + |R_5| = 2$. Otherwise, either $R_2 \cup R_3$ or $R_4 \cup R_5$ has more than 3 points of $S$. W. l. o. g., assume that $|R_2| + |R_3| = 4$. Observation 2 implies that $|R_2| = |R_3| = 2$, and either the partition $S_1 = (R_2 \cap S) \cup \{p, p_2, p_3\}$, $S_2 = \text{Cone}(p_2p_5) \cap S \cup \{p_3\}$ and $S_3 = \text{Cone}(p_3p_5) \cap S$ or the partition $S_1 = (R_2 \cap S) \cup \{p, p_2, p_3\}$, $S_2 = \text{Cone}(p_2p_5) \cap S$ and $S_3 = \text{Cone}(p_3p_5) \cap S \cup \{p_5\}$, is admissible for $S$.

Case 2.2: $p \notin R$. W. l. o. g. let $p \in \mathcal{I}(p_1p_2p_5)$. If $|R_1| + |R_5| \leq 3$, then from arguments similar to Case 1 it follows that $S_2 = ((R_1 \cup R_5) \cap S) \cup \{p_1\}$ spans a $|S_2|$-hole or an empty $|S_2|$-pseudo-triangle. Moreover, $S_1 = ((R_2 \cup R_4) \setminus \mathcal{I}(p_3p_4, p_1)) \cap S \cup \{p, p_2, p_3, p_4, p_5\}$ spans a $|S_1|$-hole. Therefore, the partition of $S$ given by $S_1$, $S_2$, and $S_3 = \mathcal{I}(p_3p_4, p_1) \cap S$ is admissible. Otherwise, $|R_1| + |R_5| = 4$, which implies that $|R_1| = |R_5| = 2$. If $|R_2| \leq 1$, then $S_2 = (R_5 \cap S) \cup \{p, p_1, p_5\}$ contains a $5$-hole and the partition of $S$ given by $S_1 = \text{Cone}(p_1p_3) \cap S$, $S_2$, and $S_3 = \text{Cone}(p_5p_3) \cap S$ is an admissible partition of $S$. Therefore, assume that $|R_2| = 2$. Then there exists a point $\beta \notin S$ on the line segment $p_2p_3$ such that $|\text{Cone}(p_1p_3) \cap S| = |\text{Cone}(p_5p_3) \cap S| = 4$ (see Figure 4(b)), and the partition $S_1 = (R_5 \cap S) \cap \{p, p_1, p_5\}$, $S_2 = \text{Cone}(p_1p_3) \cap S$, and $S_3 = \text{Cone}(p_5p_3) \cap S$ is admissible.
Case 3: $|L\{2, S\}| = 6$. Consider the subdivision of the exterior of $CH(L\{2, S\})$ into 12 regions $R_i$ as shown in Figure 4(c). Note that the regions $R_i$ and $R_{i+2}$, for $i \in \{2, 4, 6, 8, 10, 12\}$, might intersect to produce a region which is not adjacent to the convex hull of the second layer. For the sake of notational convenience we shall assume that this region is a part of the region $R_i$. Observation 2 implies that $S$ is admissible unless $\sum_{i=1}^{12} |R_i| + |R_{i+1}| + |R_{i+2}| \leq 2$, for $i \in \{1, 3, 5, 7, 9, 11\}$. Adding these inequalities and using the fact that $\sum_{i=1}^{12} |R_i| = 7$, we get $|R_1| + |R_3| + |R_5| + |R_7| + |R_9| + |R_{11}| \leq 5$. This implies that for some $i \in \{1, 3, 5, 7, 9, 11\}$, $|R_i| \neq 0$. W. l. o. g. assume that $|R_7| \neq 0$. Let $Z_1 = \mathcal{H}(p_4 p_6, p_5) \cap S$ and $Z_2 = \mathcal{H}(p_1 p_3, p_2) \cap S$. From Observation 2 we know $|Z_1| \leq 5$ and $|Z_2| \leq 5$.

Case 3.1: $|R_7| = 2$. If both $|Z_1|, |Z_2| \leq 4$, then $S_1 = ((R_1 \cup R_7) \cap S) \cup \{p_1, p_3, p_4, p_6\}$, $S_2 = Z_1$ and $S_3 = Z_3$ is an admissible partition of $S$. Next, assume that $|Z_1| = 5$ and $|Z_2| = 2$. Clearly, $|R_6| = |R_8| = 0$ and $|R_9| \neq 0$. If $|R_9| = 2$, then the partition $S_1 = ((R_3 \cup R_9) \cap S) \cup \{p_1, p_2, p_4, p_5\}$, $S_2 = \mathcal{H}(p_2 p_4, p_3) \cap S$, and $S_3 = \mathcal{H}(p_1 p_5, p_6) \cap S$ is admissible. Otherwise, $|R_9| = 1$ and $|\mathcal{H}(p_5 p_6, p_1) \cap S| = 3$. The admissibility of $S$ then follows from Observation 2.

Case 3.2: $|R_7| = 1$ and $|R_i| \leq 1$ for all $i \in \{1, 3, 5, 9, 11\}$. Note that this can be assumed because, by symmetry, the admissibility of $S$ follows by Case 3.1 whenever $|R_i| = 2$ for all $i \in \{1, 3, 5, 9, 11\}$. Now, as before, the admissibility of $S$ is immediate if $|Z_1| = |Z_2| = 4$. Therefore, assume that $|Z_1| = 5$ and $|Z_2| = 3$. The admissibility of $S$ follows from arguments similar to Case 3.1 whenever $|R_8| = 0$. Otherwise, $|R_8| = 1$, and $|Z_1| = 5$. If $|R_9| = 0$, then $|\mathcal{H}(p_5 p_6, p_3) \cap S| = 3$ and the admissibility of $S$ follows from Observation 2. Thus, it suffices to assume that $|R_9| = 1$. Now, if $|R_{11}| = |R_{12}| = 1$, then the partition of $S$ given by $S_1 = ((R_{11} \cup R_3) \cap S) \cup \{p_2, p_5, p_6\}$, $S_2 = \mathcal{H}(p_2 p_6, p_1) \cap S$, and $S_3 = \mathcal{H}(p_3 p_5, p_1) \cap S$ is admissible. Finally, if $|R_{11}| = 0$ and $|R_{12}| = 2$, then $|R_1| = |R_2| = 0$. In this case, the partition of $S$ given by $S_1 = ((R_9 \cup R_3) \cap S) \cup \{p_1, p_2, p_4, p_5\}$, $S_2 = \mathcal{H}(p_1 p_3, p_5) \cap S$, and $S_3 = \mathcal{H}(p_2 p_4, p_6) \cap S$ is admissible. \qed

5 Conclusions

In this paper we prove that every set of 13 points, in general position, can be partitioned into three disjoint regions each of which span an empty convex polygon or an empty pseudo-triangle. This proves that the pseudo-convex partition number $\psi(n) \leq \left\lceil \frac{3n}{13} \right\rceil$, thus answering a question posed by Aichholzer et al. [1].

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