ON SUPPORT VARIETIES AND THE HUMPHREYS CONJECTURE IN TYPE A

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Abstract. Let $G$ be a reductive algebraic group scheme defined over $\mathbb{F}_p$ and let $G_1$ denote the Frobenius kernel of $G$. To each finite-dimensional $G$-module $M$, one can define the support variety $V_{G_1}(M)$, which can be regarded as a $G$-stable closed subvariety of the nilpotent cone. A $G$-module is called a tilting module if it has both good and Weyl filtrations. In 1997, it was conjectured by J.E. Humphreys that when $p \geq h$, the support varieties of the indecomposable tilting modules align with the nilpotent orbits given by the Lusztig bijection. In this paper, we shall verify this conjecture when $G = SL_n$ and $p > n + 1$.

1. Introduction

1.1. Let $G$ be a reductive algebraic group scheme defined over $\mathbb{F}_p$ with Borel subgroup $B$ and maximal torus $T$. Let $\Phi, W, W_p = W \times p\mathbb{Z}\Phi$ and $E = \mathbb{Z}\Phi \otimes \mathbb{Z} \mathbb{R}$ denote the root system, Weyl group, affine Weyl group and Euclidean space respectively. Moreover, let $G_1$ denote the Frobenius kernel and let $k$ be any algebraically closed field of characteristic $p$.

To any finite-dimensional $G_1$-module $M$, one can associate a useful cohomological invariant called the support variety, which is denoted by $V_{G_1}(M)$ (cf. \cite{NPV, Section 2.2} for an overview of the theory). It turns out that support varieties can be identified with subvarieties of the $p$-restricted nilcone

$$\mathcal{N}_1(G) = \{x \in \text{Lie}(G) \mid x^{[p]} = 0\}.$$ 

When $p \geq h$ (the Coxeter number of $G$), one has $\mathcal{N}_1(G) = \mathcal{N}(G)$, where $\mathcal{N}(G)$ is the nilpotent cone (\cite{FP}). So the theory of support varieties establishes a bridge between the cohomology of $G_1$-modules and the geometry of $\mathcal{N}_1(G)$.

When $M$ has the structure of a $G$-module, the support variety $V_{G_1}(M)$ is $G$-stable. It is known that there are only finitely many $G$-orbits in $\mathcal{N} = \mathcal{N}(G)$ (\cite{CM}). Hence, there are only finitely many closed subvarieties of $\mathcal{N}$ which can be realized as the support variety of a $G$-module. A major problem in representation theory has been to determine the support varieties of various types of modules for $G$. Over the years, a number of results have been obtained in this direction (cf. \cite{NPV, DNP, Ha, C}). This paper will be dedicated to computing the support varieties for an important class of $G$-modules, known as the tilting modules (cf. \cite{J} Appendix E for a definition and overview).

1.2. Let $W^+_p$ denote the collection of all minimum length right coset representatives in $W \setminus W_p$. By introducing a certain preorder on $W_p$, one may partition $W_p$ into two sided cells (cf. \cite{Hu}, 7.15). By intersection, one also obtains a partition of $W^+_p$ into right cells (cf. \cite{LX}, Theorem 1.2). Furthermore, there exists the Lusztig bijection, which establishes a correspondence between the right cells of $W^+_p$ and nilpotent orbits (\cite{L}).

For $p \geq h$, it was conjectured by J.E. Humphreys in 1997 that this bijection can be realized by taking the support varieties of the indecomposable tilting modules (cf. \cite{Hu}, Hypothesis 12)]. More precisely, for each $w \in W^+_p$, let $[w] \subset W^+_p$ denote the unique right cell containing $w$ and let $\mathcal{O}_{[w]} \subset \mathcal{N}$ denote the orbit given by the Lusztig bijection. Also, for each $\lambda \in X(T)_+$, let $T(\lambda)$ denote the unique indecomposable tilting module with highest weight $\lambda$ (cf. \cite{J}, Lemma E.3)).
Conjecture 1.2.1. Suppose $p \geq h$, then for each $w \in W_p^+$, $V_{G_1}(T(w \cdot 0)) = \overline{O_{[w]}}$.

1.3. Weight cells. For simplicity, assume now that $G$ is semisimple and simply connected. We will make use of the terminology to be introduced in Section 2. It is well known that there exists a bijection between the elements of $W_p$ (resp. $W_p^+$) and the set of alcoves (resp. dominant alcoves) of $\mathbb{E}$. The space $\mathbb{E}$ is covered by subsets of the form $\tilde{C}$ and $C$ is covered by the intersections $\tilde{C} \cap C$, where $\tilde{C}$ denotes the lower closure of an alcove $C$. Thus, the right cells in $W_p^+$ can be identified with regions in $C$ called weight cells. For each $w \in W_p^+$, they are given by

$$c_{[w]} = \{ \lambda \in \mathcal{C} \mid \lambda \in y \cdot C_0, \ y \in [w] \}.$$ 

In Section 3 it will be shown that if $\lambda, \mu \in \bigcap C \cap X(T)_+$ for some alcove $C$, then $V_{G_1}(T(\lambda)) = V_{G_1}(T(\mu))$. Therefore, the following conjecture is equivalent to Conjecture 1.2.1.

Conjecture 1.3.1 (Humphreys Conjecture). Suppose $p \geq h$ and $\lambda \in c_{[w]} \cap X(T)_+$ for some $w \in W_p^+$, then

$$V_{G_1}(T(\lambda)) = \overline{O_{[w]}}.$$ 

In 1998, Ostrik ([O2, Theorem 6.8]) proved an analogous conjecture for quantum groups of type $A_n$, and in 2006, Bezrukavnikov ([Be, 3.2. Corollary 3]) was able to extend this result to quantum groups of any type. Their results are summarized in Theorem 6.2.2. However, Conjecture 1.3.1 still remains open for all types, and still makes sense when $p \leq h$.

1.4. One of the major obstacles to proving Conjecture 1.3.1 has been the difficulty of determining the weight cells $c_{[w]}$. Although when $G$ is a reductive group of type $A_n$ (i.e., $G = SL_{n+1}(k)$), there is a result due to Shi which gives an explicit description of the weight cells; it is described in Section 1 (cf. [8] for the original result). In fact, progress has already been made in the type $A_n$ situation by Cooper, who first extended Conjecture 1.3.1 by removing the assumption that $p \geq h$. Cooper then made significant progress in verifying this conjecture for small primes, including a complete verification when $p = 2$ (cf. [C, Theorem 7.3.1]).

To be more specific, in the type $A_n$ case, it is well-known that the orbits of $N$ correspond to partitions $\pi \vdash n + 1$ ([CM]). So the Lusztig bijection establishes a correspondence between weight cells and partitions. In Definition 1.2.2 and Remark 1.2.3 these weight cells will be explicitly described by associating a partition, $s(\lambda)$, to each $\lambda \in C$. The main result of this paper is the following theorem, which verifies Conjecture 1.3.1 in type $A_n$ for all $n$, when $p > h = n + 1$.

Theorem 1.4.1. Let $G = SL_{n+1}(k)$ with $p > n + 1$, then for each $\lambda \in X(T)_+$,

$$V_{G_1}(T(\lambda)) = \overline{O_{s(\lambda)}}.$$ 

The proof of Theorem 1.4.1 will begin by showing that $V_{G_1}(T(\lambda)) \subseteq \overline{O_{s(\lambda)'}}$ for each $\lambda \in X(T)_+$, which places an upper bound to the support variety $V_{G_1}(T(\lambda))$. This will require several steps: first in Section 2 some results regarding the alcove geometry associated to the affine Weyl group will be obtained and we will define the weak order on alcoves (see Definition 2.2.1). By recalling a few key identities involving translation functors and wall crossing functors in Section 3, Corollary 3.2.2 will relate this order relation to the ordering of support varieties by inclusion. All of the results in these two sections will hold for arbitrary simple, simply connected groups.

In Section 4 an explicit description of the weight cells in type $A_n$ will be presented. The main result of this section is Proposition 4.3.5 which gives an equivalent characterization of the partitions $s(\lambda)$. Finally, Section 5 will include a proof of Proposition 5.1.7 which establishes the upper bound portion of Theorem 1.4.1 under the slightly relaxed assumption that $p \geq n + 1$.

The remainder of the paper will be dedicated to establishing the lower bound. Section 6 will review the necessary definitions and facts about quantum groups. It will include a result by Andersen, which allows one to “lift” tilting modules over $G$ to tilting modules over an analogous
quantum group (cf. \cite[5.3]{A1}). Some results and conjectures regarding the support varieties of these lifted tilting modules will also be presented, including a complete description of them in type $A$ (see Proposition \cite[5.1.7]{A1}). Section \ref{sec:7} will make use of the fact that in type $A$, every non-dense nilpotent orbit intersects a proper Levi factor (see Lemma \ref{lem:7.1.2}). Proposition \ref{prop:7.2.2} will give the lower bound, which will follow by making direct comparisons to the quantum case when $p > n + 1$. This proposition, along with Proposition \ref{prop:5.1.7} will yield Theorem \ref{thm:1.4.1}

Remark 1.4.2. It is useful to note that the support variety calculations for induced modules in \cite{NPV}, the irreducible modules in \cite{DNP} and the higher sheaf cohomology modules in \cite{Ha} made explicit use of the known character formulas for the corresponding modules, to get the lower bound. However, character formulas for the indecomposable tilting modules, when $G$ is not of type $A_1$, have yet to be determined. In fact, to the author’s best knowledge, there is no known conjecture which predicts the characters of all indecomposable tilting modules for arbitrary semisimple groups (see \cite{LW} and \cite[A2, 3.6: Remark (i)]{A2} for some partial results and conjectures). The lower bound calculation given in Proposition \ref{prop:7.2.2} will only utilize partial information about the characters. Namely, it will use the character formulas given by Soergel in \cite{So1} and \cite{So2} for quantum groups, and the identity \ref{eq:6.3.1}.

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2. Alcove geometry

2.1. In this section, assume that $\Phi$ is an irreducible root system of any type and that $p \geq 1$ is any positive integer. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and $\Phi^+ = \mathbb{N}\Delta \cap \Phi$ denote the basis and the set of positive roots respectively. Let $E = \mathbb{Z}\Phi \otimes \mathbb{R}$ be the Euclidean space. Then $E$ is given the lattice ordering, where for $\lambda, \mu \in E$, $\lambda \leq \mu$ will be taken to mean that $\mu - \lambda \in \mathbb{N}\Phi^+$. Let $\alpha_0 \in \Phi^+$ denote the maximal short root with respect to this ordering. The strong linkage relation also gives an ordering on $E$, where $\lambda \uparrow \mu$ will denote when $\lambda$ is strongly linked to $\mu$ (cf. \cite[II.6]{J}).

The affine Weyl group $W_p = W \times p\mathbb{Z}\Phi$ is a Coxeter group with generators $S = \{s_0, s_1, \ldots, s_n\}$, where $s_0$ denotes the affine reflection and the generators $s_1, \ldots, s_n$ correspond to the basis elements $\alpha_1, \ldots, \alpha_n$ (see Definition \ref{def:2.1.2} below). The group $W_p$ is equipped with the standard length function $\ell : W_p \to \mathbb{N}$, where $\ell(w)$ denotes the length of any reduced expression for $w \in W_p$. Moreover, $W_p$ acts on $E$ by both the linear action and the dot action. As usual, the linear action will be denoted by $\lambda \mapsto w(\lambda)$ and the dot action will be denoted by $\lambda \mapsto w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\rho \in E$ is the half sum of the positive roots.

The group $W_p$ is partially ordered by the Bruhat ordering, which will be denoted by $\leq$. Let $W_p^+$ be the set of minimal length right cosets for the finite Weyl group $W$ in $W_p$, and let

$$C = \{\lambda \in E \mid \langle \lambda + \rho, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta\}$$

denote the dominant chamber of $E$. For each $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$, let

$$H_{\alpha, np} = \{\lambda \in E \mid \langle \lambda + \rho, \alpha \rangle = np\},$$

let $s_{\alpha, np} \in W_p$ denote the affine reflection across $H_{\alpha, np}$ (with respect to the dot action), and let $H = \bigcup_{\alpha \in \Phi^+, n \in \mathbb{Z}} H_{\alpha, np}$. The connected components of $E \setminus H$ are called alcoves. Let $A$ denote the collection of all the alcoves. The collection of dominant alcoves will be denoted by $A^+$; it consists of all the alcoves contained in $C$. The dot action by $W_p$ on $E$ induces a simply transitive action on $A$ by sending $C \mapsto w \cdot C$ for any $C \in A$ and $w \in W_p$. The set $A$ also has an ordering induced by the strong linkage relation on $E$, where $C_1 \uparrow C_2$ if there exists $\lambda_1 \in C_1$ such that $\lambda_1 \uparrow \lambda_2$ for $i = 1, 2$. \section{Alcove geometry}
If $C$ is any alcove, then it is uniquely defined by a set of integers $\{n_\alpha\}_{\alpha \in \Phi^+}$, where

$$C = \{\lambda \in \mathbb{E} \mid (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p, n_\alpha \in \mathbb{Z}, \alpha \in \Phi^+\}.$$ 

The upper closure of $C$ is given by

$$\bar{C} = \{\lambda \in \mathbb{E} \mid (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle \leq n_\alpha p, n_\alpha \in \mathbb{Z}, \alpha \in \Phi^+\}.$$ 

The lower closure of $C$ is defined to be

$$\tilde{C} = \{\lambda \in \mathbb{E} \mid (n_\alpha - 1)p \leq \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p, n_\alpha \in \mathbb{Z}, \alpha \in \Phi^+\}.$$ 

For each $\lambda \in \mathbb{E}$, the unique alcove satisfying $\lambda \in \bar{C}(\lambda)$ is denoted by $C(\lambda)$. The alcove given by $n_\alpha = 1$ for all $\alpha \in \Phi^+$ is called the bottom alcove and will be denoted by $C_0$.

**Remark 2.1.1.** One obtains a bijection between $A$ (resp. $A^+$) and $W_p$ (resp. $W_p^+$) by identifying $w \leftrightarrow w \cdot C_0$. This bijection identifies the strong linkage relation on $A^+$ with the Bruhat order on $W_p^+$, where $w_1 \cdot C_0 \uparrow w_2 \cdot C_0$ if and only if $w_1 \lessdot w_2$ (cf. [1, C.1]).

If $\{n_\alpha\}_{\alpha \in \Phi^+}$ is a set of integers defining an alcove as above, then for any subset $S \subseteq \Phi^+$,

$$F = \left\{ \lambda \in \bar{C} \middle| \begin{array}{ll}
\langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p, & \text{if } \alpha \in S \\
(n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p, & \text{if } \alpha \in \Phi^+ \setminus S
\end{array} \right\}$$

is called a facet, where $F \subset \bar{C}$. We can similarly define the lower closure (resp. upper closure) $\tilde{F} \subset \bar{C}$ (resp. $\bar{F} \subset \bar{C}$). The collection of all facets in $\mathbb{E}$ will be denoted by $\mathcal{F}$, where $\mathcal{F}$ is also acted on by $W_p$ via $F \mapsto w \cdot F$ for $w \in W_p$ and $F \in \mathcal{F}$. Each alcove is a facet in its own right, and thus $A \subseteq \mathcal{F}$.

For any $\lambda \in \mathbb{E}$, denote the unique facet containing $\lambda$ by $F(\lambda)$. Every non-empty facet is of the form $F = F(\lambda)$ for some $\lambda \in \mathbb{E}$. Also, let

$$\text{Stab}_{W_p}(\lambda) = \{w \in W_p \mid w \cdot \lambda = \lambda\}$$

denote the stabilizer subgroup of $\lambda \in \mathbb{E}$. If the facet $F = F(\lambda)$ is given by $S \subseteq \Phi^+$ and $\{n_\alpha\}_{\alpha \in \Phi^+}$ as above, then $\text{Stab}_{W_p}(\lambda)$ is generated by the set of reflections $\{s_{\alpha,n_\alpha p}\}_{\alpha \in S}$.

**Definition 2.1.2.** The walls of $C_0$ are defined to be the hyperplanes: $H_0 = H_{\alpha,p}$ (called the affine wall) and $H_1 = H_{\alpha,0}$ for $\alpha \in \Phi^+$. The elements of $S$ are the reflections across the walls of $C_0$, which are given by $s_\alpha = s_{\alpha,p}$ and $s_i = s_{\alpha,i}$ for $i = 1, \ldots, n$. More generally, for an alcove $C = w \cdot C_0$ with $w \in W_p$, the walls of $C$ are the hyperplanes $w \cdot H_i$ for $i = 0, \ldots, n$.

**Remark 2.1.3.** If $C$ is an alcove defined by the integers $\{n_\alpha\}_{\alpha \in \Phi^+}$, then a hyperplane $H_{\alpha,mp}$ is a wall of $C$ (see Definition 2.1.2), if there is an element $\lambda \in H_{\alpha,mp} \cap \bar{C}$ satisfying $\text{Stab}_{W_p}(\lambda) = \{1, s_{\alpha,mp}\}$. It follows that $m \in \{n_\alpha - 1, n_\alpha\}$, where if $m = n_\alpha$ (resp. $m = n_\alpha - 1$), then $H_{\alpha,mp}$ is called an upper wall (resp. lower wall) of $C$. Moreover, $C \uparrow s_{\alpha,mp} \cdot C$ if and only if $H_{\alpha,mp}$ is an upper wall of $C$.

**Lemma 2.1.4.** For each facet $F = F(\lambda)$,

$$\tilde{F} = \{\mu \in \bar{C} \mid \langle \mu + \rho, \alpha^\vee \rangle = n_\alpha p \text{ for all } \mu \in \text{Stab}_{W_p}(\mu)\}.$$ 

**Proof.** Let $\tilde{F}$ denote the right hand side of the stated identity. Suppose $F$ is given by the data $S \subseteq \Phi^+$ and $\{n_\alpha\}_{\alpha \in \Phi^+}$. In other words, for all $\alpha \notin S$, $(n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p$ and for all $\alpha \in S$, $\langle \lambda + \rho, \alpha^\vee \rangle = n_\alpha p$.

Let us first show $\tilde{F} \subseteq \bar{F}$. By definition $\tilde{F} \subseteq \bar{F}$, so it suffices to show that $\tilde{F}$ doesn’t contain any elements of the form $\mu \in \bar{F}$ satisfying $\langle \mu + \rho, \alpha^\vee \rangle = n_\beta p$ for some $\beta \notin S$. However, if such a $\mu$ exists, then the reflection $s_{\beta,n_\beta p} \in \text{Stab}_{W_p}(\mu)$ and $\lambda \notin s_{\beta,n_\beta p} \cdot \lambda$, since $s_{\beta,n_\beta p} \cdot \lambda = \lambda + m_\beta$, where $m = n_\beta p - \langle \lambda + \rho, \beta^\vee \rangle > 0$. 


To prove $\tilde{F} \supseteq \tilde{F}$, begin by choosing an arbitrary element $\mu \in \tilde{F}$. Let $T \subseteq \Phi^+$ be the collection of all $\beta \in \Phi^+$ satisfying $\langle \mu + \rho, \beta^\vee \rangle = \langle n_\beta - 1 \rangle \rho$. Then the stabilizer of $\mu$ is the subgroup of $W_\rho$ generated by the reflections $s_{\alpha,n_\alpha}$ for $\alpha \in S$ and $s_{\beta,(n_\beta-1)p}$ for $\beta \in T$. However, for all $\alpha \in S$, $s_{\alpha,n_\alpha} \cdot \lambda = \lambda$ and for all $\beta \in T$, $s_{\beta,(n_\beta-1)p} \cdot \lambda = \lambda + m_\beta \leq \lambda$, since $m = (n_\beta - 1)p - \langle \lambda + \rho, \beta^\vee \rangle < 0$. Therefore, $\mu \in \tilde{F}$.

If $F$ is a facet, there exists a unique alcove $C$ such that $F \subseteq \tilde{C}$. Moreover, if $F$ is given by the data $S \subseteq \Phi^+$ and $\{n_\alpha\}_{\alpha \in \Phi^+}$, then $C$ is given by the integers $\{m_\alpha\}_{\alpha \notin S}$, where $m_\alpha = n_\alpha$ if $\alpha \notin S$ and $m_\alpha = n_\alpha + 1$ if $\alpha \in S$. Concretely,

$$C = \left\{ \lambda \in \mathbb{E} \left| \begin{array}{ll}
 n_\alpha p < \langle \lambda + \rho, \alpha^\vee \rangle < (n_\alpha + 1)p, & \text{if } \alpha \in S \\
 (n_\alpha - 1)p < \langle \lambda + \rho, \alpha^\vee \rangle < n_\alpha p, & \text{if } \alpha \notin \Phi^+ \setminus S
\end{array} \right. \right\}.$$ 

2.2. A useful refinement of the Bruhat ordering, called the weak ordering, can be placed on $W_\rho$ (see [Hum 5.9]).

**Definition 2.2.1.** Let $w_1, w_2 \in W_\rho$ be arbitrary and suppose $w_1 = t_1 \cdots t_{m_1}$ is a reduced expression, then $w_1 \leq w_2$ if and only if there is a sequence of elements $w_1 = w_1' \leq w_1' \leq \cdots \leq w_{m_2}' = w_2$,

where $w_i' \in W_\rho$ and $w_i' = t_1 \cdots t_{m_1+i}$ is a reduced expression for $i = 0, \ldots, m_2$. For any two alcoves $C_1, C_2 \in \mathcal{A}$, take $C_1 \leq C_2$ to mean that $w_1 \leq w_2$ for the unique $w_1, w_2 \in W_\rho$ satisfying $w_i \cdot C_0 = C_i$ for $i = 1, 2$. This defines the weak order on $W_\rho$ (respectively $\mathcal{A}$).

The weak ordering restricts to give an order relation on $W_\rho^+$ and $\mathcal{A}^+$.

**Remark 2.2.2.** If $C \in \mathcal{A}$ is of the form $C = w \cdot C_0$ for some $w \in W_\rho$, then for each $i = 0, \ldots, n$ and $s_i \in S$, the alcove $ws_i \cdot C$ is obtained by reflecting $C$ across the wall $w \cdot H_i$ (see Definition 2.1.2). Furthermore, if $w, ws_i \in W_\rho^+$, then $w \leq ws_i$ if and only if $w \cdot H_i$ is an upper wall of $C$ (see Remark 2.1.3). Hence, if $C_1 = w_1 \cdot C_0$ and $C_2 = w_2 \cdot C_0$ are two dominant alcoves, then $C_1 \leq C_2$ if and only if there is a sequence of alcoves $C_1 = C_0 \uparrow \cdots \uparrow C_m = C_2$,

where $C_i = t_1 \cdots t_{m_1+i} \cdot C_0$ for $i = 0, \ldots, m_2$. Thus, by Remark 2.1.3, for each $i = 0, \ldots, m_2 - 1$, $C_{i+1} = s_{\beta_i,n_\beta} \cdot C_i$, where $\beta_i \in \Phi^+$ and $H_{\beta_i,n_\beta}$ is an upper wall of $C_i$.

The following lemma gives an important characterization of the weak order.

**Lemma 2.2.3.** Let $C_1, C_2 \in \mathcal{A}^+$ be two alcoves defined by the nonnegative integers $\{n_\alpha\}_{\alpha \in \Phi^+}$ and $\{m_\alpha\}_{\alpha \in \Phi^+}$ respectively. Then $C_1 \leq C_2$ if and only if $n_\alpha \leq m_\alpha$ for all $\alpha \in \Phi^+$.

**Proof.** Let $C_1 = w_1 \cdot C_0$ and $C_2 = w_2 \cdot C_0$, and suppose $C_1 \leq C_2$. Thus, $w_1 = t_1 \cdots t_{m_1}$ and $w_2 = t_1 \cdots t_{m_1+m_2}$ as in Definition 2.2.1. Let $C_i' = t_1 \cdots t_{m_1+i} \cdot C_0 \in \mathcal{A}^+$ for $i = 0, \ldots, m_2$. For each $i$, the alcove $C_i'$ is defined by the set of integers $\{(n_\alpha)_{i}\}_{\alpha \in \Phi^+}$. By Remark 2.2.2, there exists a root $\beta_i \in \Phi^+$ such that $C_i' = s_{\beta_i,n_\beta} \cdot C_i'$, where $H_{\beta_i,(n_\beta)}$ is an upper wall of $C_i'$. Thus,

$$\{(n_\alpha)_{i+1}\} = \begin{cases} 
(n_\alpha)_i & \text{if } \alpha \neq \beta_i \\
(n_\alpha)_i + 1 & \text{if } \alpha = \beta_i.
\end{cases}$$

It follows that $n_\alpha \leq (n_\alpha)_1 \leq \cdots \leq (n_\alpha)_{m_2} = m_\alpha$ for all $\alpha \in \Phi^+$.

For the converse, perform induction on $d = \sum_{\alpha \in \Phi^+} m_\alpha - n_\alpha$. Observe that since $m_\alpha \geq n_\alpha$ for all $\alpha \in \Phi^+$, then $d$ is equal to the number of hyperplanes separating $C_1$ and $C_2$. This is because the complete set of hyperplanes separating $C_1$ and $C_2$ is given by

$$(2.2.4) \quad \{H_{\alpha,kp} \mid \alpha \in \Phi^+, n_\alpha \leq k \leq m_\alpha - 1\},$$
and has \( d \) elements.

The case where \( d = 0 \) holds because \( C_1 = C_2 \) implies \( C_1 \leq C_2 \). Now for the inductive step, suppose \( d > 1 \). Then \( C_1 \neq C_2 \) since \( d > 0 \), and so there must exist some wall of \( C_1 \) which separates \( C_1 \) and \( C_2 \). If this wall is given by \( H_{\beta, mp} \) for some \( \beta \in \Phi^+ \), then by Remark 2.1.3 \( m \in \{n_\beta - 1, n_\beta\} \), and by (2.2.4), \( n_\beta \leq m \leq n_\beta - 1 \). Thus, \( m = n_\beta \), and hence \( H_{\beta, mp} \) is an upper wall of \( C_1 \). The alcove given by \( C = s_{\beta, mp} \cdot C_1 \in A^+ \) is defined by the integers \( \{r_\alpha\}_{\alpha \in \Phi^+} \), where

\[
r_\alpha = \begin{cases} n_\alpha & \text{if } \alpha \neq \beta \\ n_\alpha + 1 & \text{if } \alpha = \beta. \end{cases}
\]

Thus, \( r_\alpha \leq m_\alpha \) for all \( \alpha \in \Phi^+ \), \( \sum_{\alpha \in \Phi^+} n_\alpha - r_\alpha = d - 1 \) and, by Remark 2.2.2, \( C_1 \leq C \). By the inductive hypothesis, \( C \leq C_2 \), and therefore \( C_1 \leq C_2 \). \( \square \)

By Remark 2.1.1 it follows that if \( C_1 \leq C_2 \), then \( C_1 \uparrow C_2 \). Since by definition, if \( w_1 \leq w_2 \), then \( w_1 \leq w_2 \). However, \( C_1 \uparrow C_2 \) doesn’t generally imply \( C_1 \leq C_2 \) (see the argument in Remark 3.2.3 for a counterexample).

2.3. Stabilizer subroot systems. Each \( \lambda \in \mathbb{E} \) can be associated to a certain subroot system of \( \Phi \).

**Definition 2.3.1.** For each \( \alpha \in \Phi \), let \( d_\alpha = \langle \alpha, \alpha \rangle / \langle \alpha_0, \alpha_0 \rangle \in \{1, 2, 3\} \). For \( \lambda \in \mathbb{E} \), define

\[
\Phi_{\lambda,p} = \{ \alpha \in \Phi \mid d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle \in p\mathbb{Z} \}
\]

to be the stabilizer subroot system of \( \lambda \).

Now \( \Phi_{\lambda,p} \) is a closed subroot system of \( \Phi \) because for any two roots \( \alpha, \beta \in \Phi \), \( d_{\alpha+\beta} \langle \alpha + \beta, \alpha^\vee \rangle = d_\alpha \alpha^\vee + d_\beta \beta^\vee \) (cf. [NPV, 6.2]). Stabilizer systems can also be associated to facets, since if \( F \in \mathcal{F} \) is a facet and \( \lambda, \mu \in \mathbb{F} \), then \( \Phi_{\lambda,p} = \Phi_{\mu,p} \).

**Definition 2.3.2.** For each \( F \in \mathcal{F} \) and any \( \lambda \in \mathbb{F} \), the stabilizer subroot system of \( F \) is given by \( \Phi(F) = \Phi_{\lambda,p} \).

Let \( X \subset \mathbb{E} \) denote the lattice consisting of all \( \lambda \in \mathbb{E} \) satisfying \( \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z} \) for any \( \alpha \in \Phi \), then \( \mathbb{Z}\Phi \subset X \subset \mathbb{E} \). A basis of \( X \) is given by the fundamental weights \( \omega_1, \ldots, \omega_n \in X \), which satisfy \( \langle \omega_j, \alpha^\vee \rangle = \delta_{ij} \). In fact, if \( \Phi \) is the root system associated to a semisimple, simply connected algebraic group \( G \), then \( X = X(T) \) is the weight lattice for \( G \). The extended affine Weyl group is given by \( \widetilde{W}_p = W \ltimes \mathbb{P}X \), where \( \mathbb{P}X \leq \widetilde{W}_p \). The dot action of \( \widetilde{W}_p \) on \( \mathbb{L} \) induces an action on \( \mathcal{F} \), where for each \( F \in \mathcal{F} \)

\[
(2.3.3) \quad \widetilde{W}_p \cdot F = \{ F' \in \mathcal{F} \mid \Phi(F') = w(\Phi(F)) \text{ for some } w \in W \}.
\]

2.4. Lattice points. In this subsection, we assume that \( p \) is a prime number.

**Definition 2.4.1.** A prime \( p \) is said to be good for a root system \( \Phi \) if for any closed subroot system \( \Phi' \subset \Phi \), the quotient \( \mathbb{Z}\Phi / \mathbb{Z}\Phi' \) has no \( p \)-torsion. Equivalently, \( p \) is good unless \( \Phi \) has a component of type \( B_n, C_n, D_n \) and \( p = 2 \); \( \Phi \) has a component of type \( E_6, E_7, F_4 \), \( G_2 \) and \( p = 2, 3 \); or \( \Phi \) has a component of type \( E_8 \) and \( p = 2, 3, 5 \).

**Remark 2.4.2.** It follows that \( p \) is good for \( \Phi \) if and only if \( p \) is good for \( \Phi^\vee \). Also, if \( p \geq h \), then \( p \) is a good prime for \( \Phi \) (and equivalently for \( \Phi^\vee \)).

It will be useful to determine precise conditions under which a facet \( F \) satisfies \( F \cap X \neq \emptyset \), when \( p \) is good (or \( p \geq h \)). Notice that if \( \lambda \in X \), then \( \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z} \) for all \( \alpha \in \Phi \), and thus \( \Phi_{\lambda,p}^\vee \) is also a closed subroot system of \( \Phi^\vee \) (cf. [NPV, 6.2]). Furthermore, since \( p \) is good and \( \mathbb{Z}\Phi^\vee / \mathbb{Z}\Phi_{\lambda,p}^\vee \) contains no \( p \)-torsion, it can verified that

\[
(2.4.3) \quad \Phi_{\lambda,p} = \{ \alpha \in \Phi \mid \langle \lambda + \rho, \alpha^\vee \rangle \in p\mathbb{Z} \}.
\]
Definition 2.4.4. A parabolic subroot system of $\Phi$ is defined to be a subroot system of the form

$$\Phi_I = ZI \cap \Phi,$$

where $I \subseteq \Delta$ is any subset.

The following lemma gives necessary conditions for when $F \cap X \neq \emptyset$.

Lemma 2.4.5. Let $p$ be good and suppose $F \in \mathcal{F}$ satisfies $F \cap X \neq \emptyset$, then $\Phi(F) = w(\Phi_I)$ for some $w \in W$ and some $I \subseteq \Delta$.

Proof. Let $\lambda \in F \cap X$, then $\Phi(F) = \Phi_{\lambda,p}$. By the comments above, $\Phi_{\lambda,p}$ is a closed subroot system of $\Phi^\vee$ and $Z\Phi^\vee / Z\Phi_{\lambda,p}^\vee$ contains no $p$-torsion. If $\alpha \in \Phi$ satisfies $m\alpha^\vee \in \Phi_{\lambda,\alpha}$ for some $m \in Z$, then

$$m \langle \lambda + \rho, \alpha^\vee \rangle = \langle \lambda + \rho, m\alpha^\vee \rangle \in pZ.'$$

Since $Z\Phi^\vee / Z\Phi_{\lambda,p}^\vee$ contains no $p$-torsion, then $p \nmid m$. It follows that $\alpha^\vee \in \Phi_{\lambda,p}^\vee$, and thus $\alpha \in \Phi_{\lambda,p}$. Therefore, $\Phi_{\lambda,p} = \Phi_{\lambda,p} \cap \Phi$ and, by [B, Proposition 24, p. 165], there exists $I \subseteq \Delta$ and $w \in W$ such that $\Phi_{\lambda,p} = w(\Phi_I)$.

When $p \geq h$, necessary and sufficient conditions for when $F \cap X \neq \emptyset$ can be determined.

Proposition 2.4.6. Let $p \geq h$ be prime, then $F \in \mathcal{F}$ satisfies $F \cap X \neq \emptyset$ if and only if $\Phi(F) = w(\Phi_I)$ for some $w \in W$.

Proof. First suppose that $\Phi(F) = w(\Phi_I)$ for some $w \in W$ and $I \subseteq \Delta$. If $\lambda + \rho = \sum_{i \in I} x_i w_i$, then since $p \geq h$, $\Phi_{\lambda,p} = \Phi_I$. By (2.3.3), there exists $x \in \widetilde{W}_p$ such that $F = x \cdot F(\lambda)$. Then $x \cdot \lambda \in F \cap X$.

Since $p$ is good, the converse follows from Lemma 2.4.5.

3. Translation functors and tensor ideals

3.1. In this section, translation functors will be employed to establish some identities relating $\Phi$ to inclusions of thick tensor ideals and support varieties of tilting modules (see the definition below).

Definition 3.1.1. For any reductive group $G$ over a field $k$ of characteristic $p > 0$, let $\mathcal{T} = \mathcal{T}(G)$ denote the full subcategory in the category of rational $G$-modules, consisting of all finite-dimensional tilting modules for $G$.

For each $M \in \mathcal{T}$, the thick tensor ideal generated by $M$ is given by

$$\langle M \rangle = \{ N \in \mathcal{T} \mid N \mid M \otimes L \text{ for some } L \in \mathcal{T} \}.$$

Where $M \mid N$ for $M, N \in \mathcal{T}$, denotes the existence of a decomposition of the form $N = M \oplus T$ for some $T \in \mathcal{T}$.

If $M_1, M_2 \in \mathcal{T}$ satisfy $\langle M_1 \rangle \subseteq \langle M_2 \rangle$, then $V_{G_1}(M_1) \subseteq V_{G_1}(M_2)$. Now recall the definition of translation functors (see also Lemma [6.2.1]).

Definition 3.1.2. Let $M$ be a rational $G$-module. Then for any $\lambda, \mu \in X(T)_+ \cap \overline{C}$, $\lambda_0, \mu_0 \in \overline{C}_0$ with $\lambda_0 \in \overline{W}_p \cdot \lambda$ and $\mu_0 \in \overline{W}_p \cdot \mu$, then $\nu \in X(T)_+$ satisfies $\nu = w(\mu_0 - \lambda_0)$ for some $w \in W$ (cf. [J] II.7: Definition 7.6 and the remarks following Lemma 7.7]).

The following proposition is a clarification of [J, Proposition E.11], which contains a minor mistake in the statement. It provides some information about the behavior of tilting modules under translation functors.
Proposition 3.1.3. Let \( \lambda, \mu \in X(T)_+ \) satisfy \( \mu \in \overline{F(\lambda)} \), then \( T_\mu^\lambda T(\mu) \cong T(\lambda) \) and
\[
T_\mu^\lambda T(\lambda) = [\text{Stab}_{W_p}(\mu) : \text{Stab}_{W_p}(\lambda)] T(\mu).
\]
In particular, \( \langle T(\lambda) \rangle = \langle T(\mu) \rangle \).

Proof. By Lemma 2.1.4, \( \lambda \geq w \cdot \lambda \) for all \( w \in \text{Stab}_{W_p}(\mu) \). The proof of [J Proposition E.11] then implies (3.1.4). \( \square \)

In general, if \( \mu \in \overline{F(\lambda)} \setminus \overline{F(\lambda')} \), then the structure of modules such as \( T_\mu^\lambda T(\lambda) \) or \( T_\mu^\lambda T_\mu^\lambda T(\lambda) \) is much more difficult to describe. However, the following lemma gives some insight into this case.

Proposition 3.1.5. Let \( \lambda, \mu \in X(T)_+ \) and suppose \( \mu \in \overline{F(\lambda)} \). If \( \lambda' \) is the maximal element of \( \text{Stab}_{W_p}(\mu) \cdot \lambda \), then it is the highest weight of the tilting module \( \Theta T(\lambda) = T_\mu^\mu T_\mu^\lambda T(\lambda) \), and thus \( T(\lambda') \mid \Theta T(\lambda) \).

Proof. By definition, \( T(\lambda) \) has a filtration
\[
0 = F_0 \subset \cdots \subset F_m = T(\lambda)
\]
where \( F_i/F_{i-1} \cong V(x_i \cdot \lambda) \) and \( x_i \in W_p \). Now, due to the exactness of the translation functors and [J Proposition II.7.13], \( T_\mu^\lambda T(\lambda) \) has a filtration
\[
0 = F'_0 \subset \cdots \subset F'_m = T_\mu^\lambda T(\lambda),
\]
where \( F'_i/F'_{i-1} = T_\mu^\lambda V(x_i \cdot \mu) = V(x_i \cdot \mu) \). Likewise, \( \Theta T(\lambda) \) has a filtration
\[
0 = F''_0 \subset \cdots \subset F''_m = \Theta T(\lambda),
\]
where \( F''_i/F''_{i-1} = T_\mu^\lambda V(x_i \cdot \mu) \). If \( \text{Stab}_{W_p}(\mu) = \{y_1, \ldots, y_r\} \), then by [J Proposition II.7.13], each \( T_\mu^\lambda V(x_i \cdot \mu) \) has a filtration whose layers are \( V(y_j x_i \cdot \lambda') \) for \( j = 1, \ldots, r \). Thus, \( \Theta T(\lambda') \) has a filtration with layers \( V(y_j x_i \cdot \lambda') \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, r \).

In particular, every layer of the filtration is of the form \( V(z \cdot \lambda') \), where \( z \cdot \mu \leq \mu \). It follows that \( \lambda' \) is a maximal weight with respect to the lattice ordering, and since \( \Theta T(\lambda) \) is a tilting module, it also follows that \( T(\lambda') \mid \Theta T(\lambda) \). \( \square \)

3.2. For the rest of this section, assume \( p \geq h \) and \( G \) is simple and simply connected. Then the ordering \( \leq \) can be directly related to the inclusion ordering of thick tensor ideals.

Lemma 3.2.1. If \( \lambda, \mu \in X(T)_+ \) satisfy \( C(\lambda) \leq C(\mu) \), then
\[
\langle T(\lambda) \rangle \supseteq \langle T(\mu) \rangle.
\]

Proof. By Proposition 3.1.3, we may assume \( \lambda, \mu \in W_p \cdot 0 \), since any two indecomposable tilting modules whose highest weights lie in the lower closure of the same alcove must generate the same thick tensor ideal. Since \( C(\lambda) \leq C(\mu) \), then by Remark 2.2.2, there is a sequence of dominant alcoves
\[
C(\lambda) = C_0' \uparrow C_1' \uparrow \cdots \uparrow C_{m_2}' = C(\mu),
\]
such that for \( i = 0, \ldots, m_2 - 1 \), \( C_{i+1}' = s_{\beta_i,n_p} \cdot C_i' \), where \( \beta_i \in \Phi^+ \) and \( H_{\beta_i,n_p} \) is an upper wall of \( C_i' \).

For \( i = 0, \ldots, m_2 - 1 \), let \( \lambda_{i+1} = s_{\beta_i,n_p} \cdot \lambda_i \), then
\[
\lambda = \lambda_0 \uparrow \lambda_1 \uparrow \cdots \uparrow \lambda_{m_2} = \mu.
\]

Now since \( p \geq h \), then it follows from [J II.6.3 (1)] that for each \( i \) there exists an element \( \nu_i \in C_i' \cap X(T)_+ \) such that \( \text{Stab}_{W_p}(\nu_i) = \{1, s_{\beta_i,n_p}\} \). If \( \Theta_i = T_{\nu_i}^0 T_{\nu_i}^0 \), then by Proposition 3.1.5, \( T(\lambda_{i+1}) \mid \Theta_i T(\lambda_i) \), and thus
\[
\langle T(\lambda_i) \rangle \supseteq \langle \Theta_i T(\lambda_i) \rangle \supseteq \langle T(\lambda_{i+1}) \rangle.
\]
The relationship between thick tensor ideals of tilting modules and their support varieties gives us a useful corollary.

**Corollary 3.2.2.** If \( \lambda, \mu \in X(T)_+ \) satisfy \( C(\lambda) \leq C(\mu) \), then

\[
V_{G_1}(T(\lambda)) \supseteq V_{G_1}(T(\mu)).
\]

**Remark 3.2.3.** It would be tempting to hope \( \lambda \uparrow \mu \) implies \( V_{G_1}(T(\lambda)) \supseteq V_{G_1}(T(\mu)) \), but this is not true in general. For example, take \( G = SL_3(k) \) with \( p \geq 3 \) and let

\[
\lambda + \rho = (p + 1)\omega_1 + (p + 1)\omega_2,
\]

\[
\mu + \rho = s_{\epsilon_1 - \epsilon_2}2p(\lambda + \rho) = (3p - 1)\omega_1 + 2\omega_2
\]

(see Section 4 for notation). Then \( \lambda \uparrow \mu \), since \( \mu - \lambda = (p - 1)(\epsilon_1 - \epsilon_2) \) but \( \langle \mu + \rho, \epsilon_2 - \epsilon_3 \rangle > p \), while \( \langle \lambda + \rho, \epsilon_2 - \epsilon_3 \rangle - p \). Also, \( V_{G_1}(T(\lambda)) = \{0\} \), while \( V_{G_1}(T(\mu)) \neq \{0\} \).

The next lemma relates the support varieties of tilting modules to the support varieties of induced modules.

**Lemma 3.2.4.** Let \( \lambda \in X(T)_+ \), then \( V_{G_1}(T(\lambda)) \subseteq V_{G_1}(H^0(\mu)) \) for any \( \mu \in \overline{C(\lambda)} \cap X(T)_+ \).

**Proof.** Since \( \mu \in \overline{C(\lambda)} \), then by Proposition 3.1.3 \( \langle T(\lambda) \rangle = \langle T(\mu) \rangle \). Furthermore, since \( T(\mu) \) has a good filtration whose layers are of the form \( H^0(w \cdot \mu) \) with \( w \in W_\mu \), then by [NPV, Proposition 6.2.1], \( V_{G_1}(T(\mu)) \subseteq V_{G_1}(H^0(\mu)) \). Therefore,

\[
V_{G_1}(T(\lambda)) = V_{G_1}(T(\mu)) \subseteq V_{G_1}(H^0(\mu)).
\]

We have established the following proposition, which will be a key component in the proof of Theorem 4.1.

**Proposition 3.2.5.** Let \( \lambda, \mu \in X(T)_+ \) satisfy \( C(\lambda) \leq C(\mu) \), then for any \( \nu \in \overline{C(\lambda)} \cap X(T)_+ \)

\[
V_{G_1}(H^0(\nu)) \supseteq V_{G_1}(T(\lambda)) \supseteq V_{G_1}(T(\mu)).
\]

4. **Cell regions in type \( A_n \)**

4.1. For the next two sections, we will assume that \( k \) is an algebraically closed field of characteristic \( p > 0 \) and \( G = SL_{n+1}(k) \). The roots are given by

\[
\Phi = \{ \epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n + 1, i \neq j \},
\]

\[
\Phi^+ = \{ \epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n + 1, i \neq j \},
\]

\[
\Delta = \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_n - \epsilon_{n+1} \},
\]

where \( \epsilon_1, \ldots, \epsilon_{n+1} \) is the standard basis for \( \mathbb{E} \cong \mathbb{R}^{n+1} \). The corresponding fundamental weights are denoted by \( \omega_1, \ldots, \omega_n \in X(T)_+ \). Since \( \Phi \) is simply-laced, it can be normalized so that every root has length \( \sqrt{2} \). Then \( \alpha^\vee = \alpha \) for all \( \alpha \in \Phi \). The Weyl group \( W = \Sigma_{n+1} \) is the group of permutations of \( \{1, \ldots, n + 1\} \), where for any \( w \in \Sigma_{n+1} \) and \( \epsilon_i - \epsilon_j \in \Phi \), define

\[
w(\epsilon_i - \epsilon_j) = \epsilon_{w(i)} - \epsilon_{w(j)}.
\]

In particular, the reflections \( s_{\epsilon_i - \epsilon_j} = (i, j) \in \Sigma_{n+1} \) (in cycle notation).

Let \( \mathcal{P} \) denote the set of all partitions of \( n + 1 \). For each partition \( \pi = (p_1, p_2, \ldots, p_r) \in \mathcal{P} \) with \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 1 \), let \( x_\pi \in \mathcal{N} \) denote the nilpotent matrix which is a direct sum of Jordan blocks of sizes \( p_1, \ldots, p_r \), and let \( O_\pi = G \cdot x_\pi \). The assignment \( \pi \leftrightarrow O_\pi \) gives a bijection between \( \mathcal{P} \) and the set of \( G \)-orbits in \( \mathcal{N} \) ([CM]).
Definition 4.1.1. The set \( \mathcal{P} \) is equipped with the dominance ordering, where if \( \pi = (p_1, p_2, \ldots, p_r) \) and \( \sigma = (q_1, q_2, \ldots, q_s) \), then \( \pi \leq \sigma \) if and only if for \( k = 1, \ldots, n + 1 \),
\[
p_1 + \cdots + p_k \leq q_1 + \cdots + q_k,
\]
where \( p_k = 0 \) (resp. \( q_k = 0 \)) if \( k > r \) (resp. \( k > s \)). Moreover, \( \mathcal{P} \) has an order reversing transposition operation, denoted \( \pi \mapsto \pi' \).

Remark 4.1.2. If \( \pi, \sigma \in \mathcal{P} \), then \( \overline{\mathcal{P}} \subseteq \overline{\mathcal{P}}_\sigma \) if and only if \( \pi \leq \sigma \) (cf. [CM]).

Furthermore, \( \mathcal{P} \) also has a supremum (or least upper bound) operation with respect to \( \leq \).

Definition 4.1.3. Let \( \{\pi_1, \ldots, \pi_t\} \) be any subset of \( \mathcal{P} \), then there exists a least upper bound \( \pi = \sup \{\pi_1, \ldots, \pi_t\} \). To define \( \pi \), set \( \pi_i = (p_{i,1}, \ldots, p_{i,r_i}) \) for \( i = 1, \ldots, t \), then \( \pi = (p_1, \ldots, p_n) \), where \( p_1 = \max \{p_{1,1}, \ldots, p_{t,1}\} \) and for \( i \geq 2 \),
\[
p_1 + \cdots + p_i = \max \{p_{1,1} + \cdots + p_{1,i}, \ldots, p_{t,1} + \cdots + p_{t,i}\}.
\]
It follows from Definition 4.1.1 that \( \pi \) satisfies the least upper bound property.

Example 4.1.4. Suppose \( n + 1 = 6 \), \( \pi_1 = (3, 3) \) and \( \pi_2 = (4, 1, 1) \), then \( \sup \{\pi_1, \pi_2\} = (4, 2) \).

From Definition 2.4.1 recall that every prime \( p \) is good for \( \Phi \). As \( d_\alpha = 1 \) for all \( \alpha \), we have
\[
\Phi_{\lambda,p} = \{\alpha \in \Phi \mid \langle \lambda + \rho, \alpha \rangle \in p\mathbb{Z}\}
\]
for any \( \lambda \in \mathbb{E} \) (see Definition 2.3.1).

Definition 4.1.5. Any subsystem \( \Phi' \subseteq \Phi \) is conjugate to one of type \( A_{p_1-1} \times A_{p_2-1} \times \cdots A_{p_r-1} \), where \( p_1 \geq p_2 \geq \cdots \geq p_r \geq 1 \) and \( A_0 \) denotes the type of the empty root system. Thus, it can be associated to a partition which is given by
\[
\pi(\Phi') = (p_1, p_2, \ldots, p_r) \in \mathcal{P}.
\]
More explicitly, this partition is obtained by choosing a basis \( \Delta(\Phi') \) for \( \Phi' \) and decomposing \( \Delta(\Phi') = \Delta_1 \sqcup \cdots \sqcup \Delta_r \), so that for some \( w \in W \),
\[
w(\Delta(\Phi')) = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_r \subseteq \Delta,
\]
where \( w(\Delta_k) = I_k \), \( \Phi_{I_k} \) is of type \( A_{p_{k-1}} \) and \( p_k = |\Delta_k| + 1 \) for \( k = 1, \ldots, r \) (if \( p_k = 1 \), then \( \Delta_k = I_k = \emptyset \) and \( \Phi_{I_k} = \emptyset \)).

This allows us to associate a partition to each \( \lambda \in \mathbb{E} \) (or equivalently to each \( F \in \mathcal{F} \)).

Definition 4.1.6. For any \( \lambda \in \mathbb{E} \), let \( d(\lambda) = p(\Phi_{\lambda,p}) \).

If \( \lambda \in X(T)_{+} \), then by [NPV] Theorem 6.2.1, \( V_{G_1}(H^0(\lambda)) = \overline{\mathcal{O}(d(\lambda))} \).

4.2. Now we shall give a description of the cell regions in type A which is inspired by the treatment given by Shi in [S] and paraphrased by Cooper in [C] 4.1.

Definition 4.2.1. A subset \( \Psi \subseteq \Phi^+ \) is said to be a positive subroot system if \( \Psi = w(\Phi^+_I) \) for some \( I \subseteq \Delta \) and \( w \in W \). The set \( \Delta(\Psi) = w(I) \) will be referred to as a basis of \( \Psi \) because \( w(I) \) is a basis of the subroot system \( w(\Phi_I) \subseteq \Phi \). We can define \( \pi(\Psi) = \pi(\Phi_I) \) to be the partition associated to \( \Psi \).

A basis of a positive subroot system is also a basis for an actual subroot system of \( \Phi \). Thus, a subset \( \Delta' \subseteq \Phi^+ \) is a basis for a positive subroot system only if for any two distinct elements \( \alpha, \beta \in \Delta' \) with \( \alpha = \epsilon_i - \epsilon_j \) and \( \beta = \epsilon_k - \epsilon_l \), one has \( m_{\alpha,\beta} = \langle \alpha, \beta \rangle \in \{0, -1\} \). The \( m_{\alpha,\beta} \) is 0 case occurs precisely when the indices \( i, j, k, l \) are all distinct, and the \( m_{\alpha,\beta} = -1 \) case occurs precisely when either \( i = l \) or \( j = k \). Conversely, suppose \( \Delta' \subseteq \Phi^+ \) is a basis for a subroot system of \( \Phi \), then \( \Delta' = \Delta_1 \sqcup \cdots \sqcup \Delta_r \) with
\[
\Delta_k = \{\epsilon_{i,k,1} - \epsilon_{i,k,2}, \epsilon_{i,k,2} - \epsilon_{i,k,3}, \ldots, \epsilon_{i,k,p_{k-1}} - \epsilon_{i,k,p_k}\},
\]
where $i_{k,1} < i_{k,2} < \cdots < i_{k,p_k}$ and $p_1 \geq p_2 \geq \cdots \geq p_r \geq 2$. Let $w \in W = \Sigma_{n+1}$, where

$$w(i_{k,j}) = p_1 + p_2 + \cdots + p_k + j$$

for $k = 1, \ldots, r$ and $j = 1, \ldots, p_k$, and

$$w(i) = i$$

if $i \neq i_{k,j}$ for some $k, j$. Now if $I = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_r$, where

$$I_k = \{\epsilon_{p_1+\cdots+p_{k-1}+j} - \epsilon_{(p_1+\cdots+p_{k-1}+j)+1}\}_{j=1,\ldots,p_k-1}$$

for $k = 1, \ldots, r$, then $\Delta' = w^{-1}(I)$. Thus, $\Delta'$ is a basis for the positive subroot system $\Psi = w^{-1}(\Phi_j^\dagger)$ and $\Delta' = \Delta(\Psi)$. Moreover, the partition associated to $\Psi$ is given by

$$\pi(\Psi) = (p_1, \ldots, p_r, 1, \ldots, 1).$$

For example, if $n + 1 = 7$, then the subset $\{\epsilon_1 - \epsilon_6, \epsilon_2 - \epsilon_4, \epsilon_4 - \epsilon_7\} \subseteq \Phi^+$ is a basis for a positive subroot system. However, the subset $\{\epsilon_3 - \epsilon_6, \epsilon_3 - \epsilon_5\} \subseteq \Phi^+$ is not a basis for a positive subroot system.

The following partition can be used to describe both the weight cells and the Lusztig bijection in type $A_n$ (cf. Section 1.3). It was originally defined by Shi (cf. [S]). However, the following formulation of the definition was given by Cooper in [C 4.1].

**Definition 4.2.2.** For each $\lambda \in \mathbb{C}$, set

$$\Gamma_\lambda = \{\alpha \in \Phi^+ \mid \langle \lambda + \rho, \alpha \rangle \geq p\}$$

and define

$$s(\lambda) = \sup \{\pi(\Psi) \mid \Psi \subseteq \Gamma_\lambda \text{ is a positive subroot system}\}.$$

**Remark 4.2.3.** Under the correspondence between $\mathcal{P}$ and the set of nilpotent orbits, each partition $\pi \in \mathcal{P}$ defines a weight cell $c_\pi \subseteq \mathbb{C}$, consisting of all $\lambda \in \mathbb{C}$ satisfying $s(\lambda)^\dagger = \pi$. The bijection $c_\pi \leftrightarrow \mathcal{O}_\pi$ establishes the Lusztig bijection between weight cells and nilpotent orbits.

At this point, all of the notation required to understand the statement of Theorem 1.4.1 has been introduced. In fact, a more general conjecture which places no assumption on $p$ was formulated, and then verified for $p = 2$ by Cooper in [C].

**Conjecture 4.2.4 (Cooper).** For any $\lambda \in X(T)_+$, $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_{\pi}}(\lambda)^\dagger$.

Conjecture 1.2.4 is equivalent to the statement that for $\pi \in \mathcal{P}$ and $\lambda \in c_\pi \cap X(T)_+$, $V_{G_1}(T(\lambda)) = \overline{\mathcal{O}_\pi}$.

### 4.3. Good positive subroot systems

According to Definition 4.2.2, $s(\lambda)$ is calculated by taking the supremum over all partitions of the form $\pi(\Psi)$, where $\Psi \subseteq \Gamma_\lambda$ is a positive subroot system of $\Gamma_\lambda$. However, it will soon be shown that $s(\lambda)$ can also be calculated by taking the supremum of a smaller subset of partitions. Namely, the set of partitions of the form $\pi(\Psi)$, where $\Psi \subseteq \Gamma_\lambda$ is a good positive subroot system (see the following definition).

**Definition 4.3.1.** A positive subroot system $\Psi \subseteq \Phi^+$ is called good, if there are no two elements $\alpha, \beta \in \Delta(\Psi)$ satisfying $\alpha < \beta$.

**Remark 4.3.2.** If $\Psi \neq \emptyset$, then it can be verified that $\Psi$ is good if and only if

$$\Delta(\Psi) = \{\epsilon_{i_1} - \epsilon_{j_1}, \epsilon_{i_2} - \epsilon_{j_2}, \ldots, \epsilon_{i_r} - \epsilon_{j_r}\},$$

where $i_1 < i_2 < \cdots < i_r$ and $j_1 < j_2 < \cdots < j_r$. 
For example, if \( n + 1 = 5 \), then the positive subroot system with basis \( \{ \epsilon_1 - \epsilon_4, \epsilon_2 - \epsilon_3 \} \) is not good since \( \epsilon_2 - \epsilon_3 < \epsilon_1 - \epsilon_4 \). On the other hand, \( \{ \epsilon_1 - \epsilon_4, \epsilon_2 - \epsilon_5 \} \) is the basis for a good positive subroot system.

For any positive subroot system \( \Psi \subseteq \Phi^+ \), let

\[
\Gamma_\Psi = \{ \alpha \in \Phi^+ \mid \alpha \geq \beta \text{ for some } \beta \in \Psi \}.
\]

If \( \Psi \subseteq \Gamma_\lambda \) is a positive subroot system, then \( \Gamma_\Psi \subseteq \Gamma_\lambda \) because for any \( \alpha \in \Gamma_\Psi \), there exists \( \beta \in \Psi \subseteq \Gamma_\lambda \) satisfying \( \alpha \geq \beta \). Thus, since \( \lambda \in \mathcal{C} \),

\[
\langle \lambda + \rho, \alpha \rangle \geq \langle \lambda + \rho, \beta \rangle \geq p.
\]

Moreover, for any good positive subroot system \( \Psi' \subseteq \Gamma_\Psi \), \( \pi(\Psi') \leq s(\lambda) \).

**Lemma 4.3.3.** Let \( \Psi \subseteq \Phi^+ \) be a positive subroot system. Then there exist good positive subroot systems \( \Psi_1, \ldots, \Psi_t \subseteq \Gamma_\Psi \) such that \( \pi(\Psi) \subseteq \sup\{\pi(\Psi_1), \ldots, \pi(\Psi_t)\} \).

**Proof.** For each positive subroot system \( \Psi \subseteq \Phi^+ \), let

\[
M_\Psi = \{\{\alpha, \beta\} \subseteq \Delta(\Psi) \mid \alpha < \beta\}
\]

and \( m_\Psi = |M_\Psi| \). We shall perform induction on \( m = m_\Psi \geq 0 \). The base case, \( m = 0 \), follows from the fact that \( m_\Psi = 0 \) if and only if \( \Psi \) is good. For the inductive step, suppose \( m \geq 1 \) is arbitrary and that for any positive subroot system \( \Psi \subseteq \Phi^+ \) satisfying \( m_\Psi < m \), there exist good positive subroot systems \( \Psi_1, \ldots, \Psi_t \subseteq \Gamma_\Psi \) such that \( \pi(\Psi) \subseteq \sup\{\pi(\Psi_1), \ldots, \pi(\Psi_t)\} \).

Now suppose \( \Psi \subseteq \Phi^+ \) satisfies \( m_\Psi = m \). Let \( \Delta(\Psi) = \Delta_1 \cup \cdots \Delta_r \) and \( \pi(\Psi) = (p_1, \ldots, p_r) \) be as in Definitions 4.3.5 and 4.2.1. For each \( k \) satisfying \( \Delta_k \neq \emptyset \) (i.e., \( p_k \geq 2 \)), write

\[
\Delta_k = \{\epsilon_{i_1,1} - \epsilon_{i_2,2}, \epsilon_{i_3,3} - \epsilon_{i_4,4}, \ldots, \epsilon_{i_{p_k-1},p_k-1} - \epsilon_{i_{p_k},p_k}\},
\]

where \( i_1 < i_2 < \cdots < i_{p_k} \). Since \( m_\Psi \geq 1 \), there exists \( \alpha_1 = \epsilon_{i_{t_1},1} - \epsilon_{i_{t_2},2} \in \Delta_{t_1} \) and \( \alpha_2 = \epsilon_{i_{t_2,2} - \epsilon_{i_{t_3,3}} + 1} \in \Delta_{t_2} \) such that \( \alpha_1 > \alpha_2 \). Our goal is to construct a new positive subroot system \( \Psi' \subseteq \Phi^+ \) by replacing the two “bad” roots \( \alpha_1, \alpha_2 \) with two non-comparable roots \( \beta_1, \beta_2 \) in such a way so that \( \Psi' \subseteq \Gamma_\Psi \) and \( m_{\Psi'} < m_\Psi \).

To do this, let \( \Psi' \subseteq \Gamma_\Psi \) denote the subroot system whose basis, \( \Delta(\Psi') \), is obtained by taking \( \Delta(\Psi) \) and replacing the roots \( \alpha_1, \alpha_2 \) with \( \beta_1 = \epsilon_{i_{t_1,1} - \epsilon_{i_{t_2,2} + 1}} \) and \( \beta_2 = \epsilon_{i_{t_2,2} - \epsilon_{i_{t_1,1} + 1}} \) respectively. The inclusion \( \Psi' \subseteq \Gamma_\Psi \) holds because \( \beta_k \geq \alpha_2 \) for \( k = 1, 2 \) (which implies \( \beta_1, \beta_2 \in \Gamma_\Psi \)). There is a decomposition, \( \Delta(\Psi') = \Delta_1' \cup \cdots \Delta_r' \), where

\[
\Delta_1' = \{\epsilon_{i_{t_1,1} - \epsilon_{i_{t_2,2}}, \ldots, \epsilon_{i_{t_1,1} - \epsilon_{i_{t_2,2}}} + 1, \ldots, \epsilon_{i_{t_2,1} - \epsilon_{i_{t_2,2}}} + 1 - \epsilon_{i_{t_2,1}}\},
\]

\[
\Delta_2' = \{\epsilon_{i_{t_1,1} - \epsilon_{i_{t_2,2}}, \ldots, \epsilon_{i_{t_1,1} - \epsilon_{i_{t_2,2}}} + 1, \ldots, \epsilon_{i_{t_2,1} - \epsilon_{i_{t_1,1}}} + 1 - \epsilon_{i_{t_1,1}}\}
\]

and \( \Delta_k' = \Delta_k \) for \( k \notin \{t_1, t_2\} \). For each \( k \), let \( p'_k = |\Delta'_k| + 1 \). Then \( p'_k = p_k \) for \( k \notin \{t_1, t_2\} \) and \( p'_1 + p'_2 = p_1 + p_2 \) (however, we cannot assume that \( p'_k \geq p'_j \) whenever \( k < j \)). In any case,

\[
\pi(\Psi') = (p'_{\tau(1)}, p'_{\tau(2)}, \ldots, p'_{\tau(r)}),
\]

where \( p'_{\tau(1)} \geq p'_{\tau(2)} \geq \cdots \geq p'_{\tau(r)} \) and \( \tau \) is a permutation of \( \{1, 2, \ldots, r\} \). Also,

\[
p'_1 + \cdots + p'_r \leq p'_{\tau(1)} + \cdots + p'_{\tau(i)}
\]

for \( i = 1, \ldots, r \).

Furthermore, \( m_{\Psi'} < m_\Psi \). To see why this is true, begin by observing that if

\[
\{\alpha, \beta\} \subseteq \Delta(\Psi') \setminus \{\beta_1, \beta_2\} = \Delta(\Psi) \setminus \{\alpha_1, \alpha_2\},
\]

then \( \{\alpha, \beta\} \in M_{\Psi'} \) if and only if \( \{\alpha, \beta\} \in M_\Psi \) By definition, \( \{\beta_1, \beta_2\} \notin M_\Psi \). Thus, it will be sufficient to show that the number of subsets of the form \( \{\alpha, \beta_k\} \in M_{\Psi'} \) is no greater than the number of subsets of the form \( \{\alpha, \alpha_k\} \in M_\Psi \), where \( \alpha \in \Delta(\Psi') \setminus \{\beta_1, \beta_2\} \).
First suppose $\alpha \in \Delta(\Psi')$ satisfies $\{\alpha, \beta_k\} \in M_{\Psi'}$ for both $k = 1, 2$. If $\alpha > \beta_1$ for $k = 1, 2$, then $\alpha > \alpha_1 > \alpha_2$, and hence $\{\alpha, \alpha_k\} \in M_{\Psi}$ for both $k$. Similarly, if $\alpha < \beta_1$ for both $k$, then $\{\alpha, \alpha_k\} \in M_{\Psi}$ for both $k$. If $\alpha > \beta_1$ and $\alpha < \beta_2$, then $\alpha > \alpha_2$ and $\alpha < \alpha_1$ (since $\beta_1 > \alpha_2$ and $\beta_2 < \alpha_1$), and hence $\{\alpha, \alpha_k\} \in M_{\Psi}$ for $k = 1, 2$. Likewise, if $\alpha < \beta_1$ and $\alpha > \beta_2$, then $\{\alpha, \alpha_k\} \in M_{\Psi}$ for both $k$. Suppose now that $\{\alpha, \beta_k\} \in M_{\Psi'}$ only for a single $k$. In this case, if $\alpha < \beta_k$, then $\alpha < \alpha_1$ since $\beta_k < \alpha_1$, and thus $\{\alpha, \alpha_1\} \in M_{\Psi}$. Similarly, if $\alpha > \beta_k$, then $\alpha > \alpha_2$ and $\{\alpha, \alpha_2\} \in M_{\Psi}$. So the number of pairs of the form $\{\alpha, \beta_k\} \in M_{\Psi'}$ is no greater than the number of pairs of the form $\{\alpha, \alpha_k\} \in M_{\Psi}$. Therefore, $m_{\Psi'} \le m_{\Psi} - 1$.

For simplicity, assume $t_1 < t_2$ (the exact same argument will also work when $t_1 > t_2$). Define $\Psi''$ to be the positive subroot system with basis $\Delta(\Psi'') = \Delta(\Psi) \setminus \Delta_{t_2}$. Then $\Pi'' \subseteq \Pi$ and $m_{\Psi''} < m_{\Psi}$, since $M_{\Psi''} \subseteq M_{\Psi} \setminus \{\{\alpha_1, \alpha_2\}\}$. The partition associated to $\Psi''$ is given by
$$
\pi(\Psi'') = (p_1, \ldots, p_{t_2-1}, p_{t_2+1}, \ldots, p_r, 1, \ldots, 1).
$$

Let $\pi = \sup\{\pi(\Psi'), \pi(\Psi'')\}$, and write $\pi = (q_1, \ldots, q_{n+1})$ with $q_1 \ge q_2 \ge \cdots \ge q_{n+1} \ge 0$. By Definition 4.1.3, $\pi(\Psi'') \le \pi$, and hence
$$
p_1 + \cdots + p_i \le q_1 + \cdots + q_i
$$
for $i = 1, \ldots, t_2 - 1$. Moreover, since $p_{t_1} + p_{t_2} = p_1 + p_2$, then
$$
p_1 + \cdots + p_i = p_{t_1} + \cdots + p_{t_1} \le p_{t_1} + \cdots + p_{t_1} \le q_1 + \cdots + q_i
$$
for $i \ge t_2$. Thus, $\pi(\Psi) \le \pi$. Now by the inductive hypothesis, since $m_{\Psi'} < m$ and $m_{\Psi''} < m$, there exist good positive subroot systems $\Psi_1', \ldots, \Psi_r' \subseteq \Pi'$ and $\Psi_1'', \ldots, \Psi_r'' \subseteq \Pi''$ such that $\pi(\Psi') \le \sup\{\pi(\Psi_1'), \ldots, \pi(\Psi_r')\}$ and $\pi(\Psi'') \le \sup\{\pi(\Psi_1''), \ldots, \pi(\Psi_r'')\}$. It follows that
$$
\pi(\Psi) \le \pi \le \sup\{\pi(\Psi_1'), \ldots, \pi(\Psi_r'), \pi(\Psi_1''), \ldots, \pi(\Psi_r'')\},
$$
where $\Psi_1', \ldots, \Psi_r', \Psi_1'', \ldots, \Psi_r'' \subseteq \Pi$ are good positive subroot systems.

For the sake of clarity, a nontrivial example demonstrating the algorithm which was used in the above lemma has been included.

**Example 4.3.4.** Suppose that $n + 1 = 6$ and let $\Delta(\Psi) = \Delta_1 \cup \Delta_2$, where
$$
\Delta_1 = \{\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \epsilon_6\}
$$
$$
\Delta_2 = \{\epsilon_2 - \epsilon_5\}.
$$
Then $m_{\Psi} = 1 = |\{\epsilon_3 - \epsilon_4, \epsilon_2 - \epsilon_5\}|$ and $\pi(\Psi) = (4, 2)$. Let $\alpha_1 = \epsilon_2 - \epsilon_5$ and $\alpha_2 = \epsilon_3 - \epsilon_4$ be as in the above proof, then $\beta_1 = \epsilon_2 - \epsilon_4$ and $\beta_2 = \epsilon_3 - \epsilon_5$. Thus, $\Delta(\Psi_1) = \Delta_1' \cup \Delta_2'$, where
$$
\Delta_1' = \{\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_5\}
$$
$$
\Delta_2' = \{\epsilon_2 - \epsilon_4, \epsilon_4 - \epsilon_6\}.
$$
Then $\Psi_1$ is good, $\Gamma_{\Psi_1} \subseteq \Pi$ and $\pi(\Psi_1) = (3, 3) \le \pi(\Psi)$. Following the algorithm given in the preceding proof, we can also obtain the good subroot system $\Psi_2$ with basis
$$
\Delta(\Psi_2) = \{\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_4 - \epsilon_6\},
$$
by removing $\epsilon_2 - \epsilon_5$ from $\Delta(\Psi)$. Then $\Gamma_{\Psi_2} \subseteq \Pi$ and $\pi(\Psi_2) = (4, 1, 1)$. Finally, observe that $\pi(\Psi) = \sup\{\pi(\Psi_1), \pi(\Psi_2)\}$.

The following proposition gives an equivalent characterization of the partitions $s(\lambda) \in \mathcal{P}$ for $\lambda \in \mathcal{C}$.

**Proposition 4.3.5.** For each $\lambda \in \mathcal{C}$,
$$
s(\lambda) = \sup\{\pi(\Psi) \mid \Psi \subseteq \Gamma_{\lambda} \text{ is a good positive subroot system}\}.
$$
Proof. Let \( \pi = \sup\{\pi(\Psi) \mid \Psi \subseteq \Gamma_\lambda \) is a good positive subroot system}. It follows immediately that \( \pi \leq s(\lambda) \). Conversely, by Lemma 4.3.3 for each positive subroot system \( \Psi \subseteq \Gamma_\lambda \), there exist good positive subroot systems \( \Psi_1, \ldots, \Psi_t \subseteq \Gamma_\Psi \subseteq \Gamma_\lambda \) such that
\[
\pi \geq \sup\{\pi(\Psi_1), \ldots, \pi(\Psi_t) \} \geq \pi(\Psi).
\]
Thus, \( \pi \geq \pi(\Psi) \) for any positive subroot system \( \Psi \subseteq \Gamma_\lambda \), and hence \( \pi \geq s(\lambda). \)

5. THE UPPER BOUND

5.1. In this section, we will prove the upper bound portion of Theorem 1.4.1. A key tool will be the following lemma, which illustrates the importance of good positive subroot systems.

Lemma 5.1.1. If \( \lambda \in \mathcal{C} \) and \( \Psi \subseteq \Gamma_\lambda \subseteq \Phi^+ \) is a good positive subroot system, then there exists an element \( \mu \in \mathcal{C} \) such that \( \Phi_{\mu, p}^+ \supseteq \Psi \) and \( C(\mu) \leq C(\lambda) \).

Proof. Let \( \lambda \in \mathcal{C} \) and let \( \Psi \subseteq \Gamma_\lambda \) be a good positive subroot system. If \( \Psi = \emptyset \), let \( \mu + \rho = (1/n, \ldots, 1/n) \) be given in fundamental basis coordinates. Then \( C(\mu) \) is the bottom alcove, and hence \( C(\mu) \leq C(\lambda) \) since \( C(\lambda) \in \mathcal{A}^+ \). Furthermore, \( \Phi_{\mu, p}^+ \supseteq \Psi = \emptyset \).

For the rest of the proof, we shall assume that \( \Psi \neq \emptyset \). In this case, \( \Delta(\Psi) = \{\alpha_1, \ldots, \alpha_r\} \), where \( \alpha_k = \epsilon_{i_k} - \epsilon_{j_k}, i_1 < i_2 < \cdots < i_r \) and \( j_1 < j_2 < \cdots < j_r \). By performing induction on the rank \( r = |\Delta(\Psi)| \geq 1 \), it will be shown that there exists \( \mu \in \mathcal{C} \) satisfying
\[
(1) \quad \langle \mu + \rho, \alpha_i \rangle = p \text{ for any } \alpha_i \in \Delta(\Psi), \text{ and thus } \Phi_{\mu, p}^+ \supseteq \Psi,
\]
\( \langle \mu + \rho, \epsilon_1 - \epsilon_{j_1-1} \rangle < p \) and \( \langle \mu + \rho, \epsilon_{i_r+1} - \epsilon_{n+1} \rangle < p \),
\( 3) \quad C(\mu) \leq C(\lambda) \).

For each \( \epsilon_i - \epsilon_j \in \Phi^+ \), let \( n_{ij} \geq 1 \) be the unique integer satisfying
\[
(n_{ij} - 1)p \leq \langle \lambda + \rho, \epsilon_i - \epsilon_j \rangle < n_{ij}p.
\]
It is useful to remark that Lemma 2.2.3 implies that (3) will follow if
\[
\langle \mu + \rho, \epsilon_i - \epsilon_j \rangle < n_{ij}p
\]
is satisfied for all \( \epsilon_i - \epsilon_j \in \Phi^+ \).

For the base case, suppose \( r = 1 \), then \( \Delta(\Psi) = \{\epsilon_{i_1} - \epsilon_{j_1}\} \). Now let \( \mu \in \mathcal{C} \) be given by
\[
\mu + \rho = (a_1, \ldots, a_n),
\]
where \( a_k = 1/n \) for \( k \geq i_1 + 1 \), \( a_{i_1} = p - \frac{j_1 - i_1 - 1}{n} \) and, if \( i_1 > 1 \), \( a_k = a \) for \( k \leq i_1 - 1 \) with
\[
0 < a < \frac{1}{(i_1 - 1)n}.
\]
Then \( \mu \) automatically satisfies (1). Moreover,
\[
\langle \mu + \rho, \epsilon_1 - \epsilon_{j_1-1} \rangle = \langle \mu + \rho, \epsilon_1 - \epsilon_{i_1} \rangle + \langle \mu + \rho, \epsilon_{i_1} - \epsilon_{i_1+1} \rangle + \langle \mu + \rho, \epsilon_{i_1+1} - \epsilon_{j_1-1} \rangle
\]
\[
= (i_1 - 1)a + \left( p - \frac{j_1 - i_1 - 1}{n} \right) + \frac{j_1 - i_1 - 2}{n}
\]
\[
= (i_1 - 1)a + p - \frac{1}{n} < p
\]
and
\[
\langle \mu + \rho, \epsilon_{i_1+1} - \epsilon_{n+1} \rangle = \frac{n - i_1 - 1}{n} < p,
\]
so (2) is also satisfied. To show (3), first observe that since (2) holds, then if \( \epsilon_i - \epsilon_j \vDash \epsilon_{i_1} - \epsilon_{j_1} \),
\[
\langle \mu + \rho, \epsilon_i - \epsilon_j \rangle < p \leq n_{ij}p.
\]
On the other hand, since \( \langle \mu + \rho, \epsilon_1 - \epsilon_{n+1} \rangle < p + 1 < 2p \), then if \( \epsilon_i - \epsilon_j \vDash \epsilon_{i_1} - \epsilon_{j_1} \),
\[
\langle \mu + \rho, \epsilon_i - \epsilon_j \rangle < 2p \leq n_{ij}p.
\]
where \( n_{ij} \geq 2 \) because
\[
\langle \lambda + \rho, \epsilon_{i_1} - \epsilon_{j_1} \rangle = \langle \lambda + \rho, \epsilon_i - \epsilon_j \rangle,
\]
and hence \( 2 \leq n_{i_1j_1} \leq n_{ij} \).

For the inductive step, suppose \( r \geq 2 \) and that for any good positive subroot system \( \Psi' \subseteq \Gamma_\lambda \) with \( |\Delta(\Psi')| < r \), there exists \( \nu \in \mathcal{C} \) satisfying conditions (1), (2) and (3). Now let \( \Psi \subseteq \Gamma_\lambda \) be a good positive subroot system with \( \Delta(\Psi) = \{\alpha_1, \ldots, \alpha_r\} \), where \( \alpha_k = \epsilon_{i_k} - \epsilon_{j_k}, i_1 < i_2 < \cdots < i_r \) and \( j_1 < j_2 < \cdots < j_r \). Our goal is to find an element \( \mu \in \mathcal{C} \) corresponding to \( \Psi \) which satisfies all three conditions. Suppose
\[
\mu + \rho = (a_1, \ldots, a_n),
\]
using fundamental basis coordinates. Thus, to determine \( \mu \), it suffices to determine the appropriate coordinates \( a_1, \ldots, a_n \).

Let \( \Psi' \subset \Psi \) be the good positive subroot system with basis \( \Delta(\Psi') = \{\alpha_2, \ldots, \alpha_r\} \), then by the inductive hypothesis, there exists \( \nu \in \mathcal{C} \) corresponding to \( \Psi' \) which is given by
\[
\nu + \rho = (b_1, \ldots, b_n),
\]
and satisfies conditions (1), (2) and (3). We begin by choosing the coordinates \( a_{i_1}, \ldots, a_n \). Set \( a_k = b_k \) for all \( k \geq i_1 + 1 \), and set
\[
a_{i_1} = p - (b_{i_1+1} + \cdots + b_{j_1-1}).
\]
Observe that \( b_1 + \cdots + b_{j_1-1} < p \) because \( \nu \) satisfies (2) and \( j_1 < j_2 \). It follows that \( a_{i_1} > 0 \). Furthermore,
\[
\langle \mu + \rho, \epsilon_{i_1} - \epsilon_{j_1} \rangle = p - (b_{i_1+1} + \cdots + b_{j_1-1}) + b_{i_1+1} + \cdots + b_{j_1-1} = p
\]
and for each \( k \geq 2 \),
\[
\langle \mu + \rho, \epsilon_{i_k} - \epsilon_{j_k} \rangle = \langle \nu + \rho, \epsilon_{i_k} - \epsilon_{j_k} \rangle = p.
\]
Moreover,
\[
\langle \mu + \rho, \epsilon_{i_{r+1}} - \epsilon_{n+1} \rangle = \langle \nu + \rho, \epsilon_{i_{r+1}} - \epsilon_{n+1} \rangle < p.
\]
Thus, if \( a_{i_1}, \ldots, a_n \) are chosen in this way, then \( \mu \) satisfies (1) for any choice of positive real numbers \( a_1, \ldots, a_{i_1-1} \). Additionally, if \( i_1 = 1 \), then \( \mu \) already satisfies (2) because
\[
\langle \mu + \rho, \epsilon_{1} - \epsilon_{j_1-1} \rangle = p - b_{j_1-1} < p.
\]
Also, if \( \epsilon_i - \epsilon_j \in \Phi^+ \) and \( i \geq i_1 + 1 = 2 \), then
\[
\langle \mu + \rho, \epsilon_i - \epsilon_j \rangle = \langle \nu + \rho, \epsilon_i - \epsilon_j \rangle < n_{ij}p,
\]
If \( i = 1 \) and \( j \leq j_1 - 1 \), then
\[
\langle \mu + \rho, \epsilon_1 - \epsilon_j \rangle \leq \langle \mu + \rho, \epsilon_1 - \epsilon_{j_1-1} \rangle < p \leq n_{1j}p,
\]
and if \( j \geq j_1 \), then
\[
\langle \mu + \rho, \epsilon_1 - \epsilon_j \rangle = \langle \mu + \rho, \epsilon_1 - \epsilon_{j_1} \rangle + \langle \mu + \rho, \epsilon_{j_1} - \epsilon_j \rangle
\]
\[
= p + \langle \nu + \rho, \epsilon_{j_1} - \epsilon_j \rangle
\]
\[
< (n_{1j} + 1)p
\]
\[
\leq n_{1j}p.
\]
The inequality \( n_{j_1j} + 1 \leq n_{1j} \) holds because
\[
\langle \lambda + \rho, \epsilon_1 - \epsilon_{j} \rangle \geq p + \langle \lambda + \rho, \epsilon_{j_1} - \epsilon_{j} \rangle \geq p + (n_{j_1j} - 1)p = n_{j_1j}p,
\]
where we set \( n_{j_1j} = 1 \) if \( j_1 = j \). Therefore, the third condition is also satisfied by \( \mu \) when \( i_1 = 1 \), so now assume \( i_1 \geq 2 \).
The problem reduces to finding positive integers \(a_1, \ldots, a_{i_1-1}\) such that (2) and (3) are completely satisfied by \(\mu\). Let us first assume that

\[
a = a_1 = \cdots = a_{i_1-1},
\]

where \(a > 0\). If \(a < b_{j_1-2}/(i_1 - 1)\), then

\[
\langle \mu + \rho, \epsilon_1 - \epsilon_{j_1-1} \rangle = \langle \mu + \rho, \epsilon_1 - \epsilon_{i_1} \rangle + \langle \mu + \rho, \epsilon_1 - \epsilon_{j_1-1} \rangle = (i_1 - 1)a + p - b_{j_1-1} < p,
\]

(5.1.3)

and hence \(\mu\) satisfies (2).

It remains to determine sufficient conditions on \(a\) so that \(\mu\) satisfies (3). If \(i > i_1\), then

\[
\langle \mu + \rho, \epsilon_1 - \epsilon_j \rangle = \langle \nu + \rho, \epsilon_1 - \epsilon_j \rangle < n_{ij}p,
\]

by condition (3) on \(\nu\). Furthermore, if \(j < j_1\), then by (5.1.3),

\[
\langle \mu + \rho, \epsilon_i - \epsilon_j \rangle = (i_1 - 1)a + p - b_{j_1+1} - \cdots - b_{j_1-2} < p \leq n_{ij}p.
\]

Thus, (5.1.2) is met for any \(\epsilon_i - \epsilon_j\) with \(i > i_1\) or \(j < j_1\).

If \(\epsilon_i - \epsilon_j \in \Phi^+\) is such that \(i \leq i_1\) and \(j \geq j_1\), then \(\epsilon_i - \epsilon_j = (\epsilon_i - \epsilon_{j_1}) + (\epsilon_{j_1} - \epsilon_j)\). Now since \(\lambda \in \mathcal{C}\),

\[
\langle \lambda + \rho, \epsilon_i - \epsilon_{j_1} \rangle \geq \langle \lambda + \rho, \epsilon_{i_1} - \epsilon_{j_1} \rangle > p,
\]

and hence,

\[
\langle \lambda + \rho, \epsilon_i - \epsilon_j \rangle \geq p + \langle \lambda + \rho, \epsilon_{j_1} - \epsilon_j \rangle \geq p + (n_{j_1j} - 1)p = n_{jj_1}p.
\]

Thus, \(n_{ij} \geq n_{jj_1} + 1\), where we set \(n_{jj_1} = 1\) if \(j_1 = j\).

Also,

\[
\langle \mu + \rho, \epsilon_i - \epsilon_j \rangle = \langle \mu + \rho, \epsilon_i - \epsilon_{i_1} \rangle + \langle \mu + \rho, \epsilon_{i_1} - \epsilon_{j_1} \rangle + \langle \mu + \rho, \epsilon_{j_1} - \epsilon_j \rangle = (i_1 - i)a + p + \langle \nu + \rho, \epsilon_{j_1} - \epsilon_j \rangle,
\]

since \(\langle \mu + \rho, \epsilon_{j_1} - \epsilon_j \rangle = \langle \nu + \rho, \epsilon_{j_1} - \epsilon_j \rangle\). Hence, if \(a\) is chosen so that for each \(\epsilon_i - \epsilon_j \in \Phi^+\) with \(i \leq i_1\) and \(j \geq j_1\), the inequality

\[
(5.1.4) \quad (i_1 - i)a + \langle \nu + \rho, \epsilon_{j_1} - \epsilon_j \rangle < n_{jj_1}p
\]

holds, then (5.1.2) will hold for any \(\epsilon_i - \epsilon_j\) with \(i \leq i_1\) and \(j \geq j_1\) because

\[
\langle \mu + \rho, \epsilon_i - \epsilon_j \rangle = (i_1 - i)a + \langle \nu + \rho, \epsilon_{j_1} - \epsilon_j \rangle + p < (n_{jj_1} + 1)p \leq n_{ij}p.
\]

Where a solution exists to (5.1.4) for some \(a > 0\), since \(\nu\) satisfies (3) by the inductive hypothesis, and hence \(\langle \nu + \rho, \epsilon_{j_1} - \epsilon_j \rangle < n_{jj_1}p\). Therefore, \(\langle \mu + \rho, \epsilon_i - \epsilon_j \rangle < n_{ij}p\) for all \(\epsilon_i - \epsilon_j \in \Phi^+\) which implies that \(\mu\) satisfies (3).

In summary, we shown that if

\[
\mu + \rho = (a_1, \ldots, a_n),
\]

where \(a_k = b_k\) for \(k \geq i_1\), \(a_{i_1} = p - (b_{i_1+1} + \cdots + b_{j_1-1})\) and \(a_k = a\) for \(k = 1, \ldots, i_1 - 1\) such that \(a > 0\) and satisfies (5.1.3) and (5.1.4), then \(\mu\) satisfies conditions (1), (2) and (3). Therefore, the desired result follows by induction.

\[\square\]

Unfortunately, even when \(p \geq n + 1\) and \(\lambda \in X(T)_+\), Lemma 5.1.4 doesn’t guarantee that for each good positive subroot system \(\Psi \subseteq \Gamma_\lambda\), there exists a weight \(\mu \in X(T)_+\) such that \(\Phi^+_{\mu,p} \supseteq \Psi\) and \(C(\mu) \leq C(\lambda)\). It only ensures the existence of a Euclidean point \(\mu \in \mathcal{C}\) with the desired properties. This issue will be clarified by the following lemma.
Lemma 5.1.5. Let $p \geq n + 1$, then for every non-empty facette $F \subset E$, $F \cap X(T) \neq \emptyset$. Equivalently, every non-empty facette contains a lattice point.

Proof. The statement follows immediately from Proposition 2.3.6 since every subroot system of $\Phi$ is of the form $w(\Phi_I)$ for some $I \subseteq \Delta$ and $w \in W$.

Remark 5.1.6. The conclusion of the preceding lemma generally doesn’t hold for other types. For example, let $\Phi \subset \mathbb{R}^2$ be of type $C_2$ and let $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = 2\epsilon_2$ denote the standard basis. The corresponding fundamental weights are $\omega_1 = \epsilon_1$ and $\omega_2 = \epsilon_1 + \epsilon_2$, so that $p = 2\epsilon_1 + \epsilon_2$.

Consider the facette $F$, consisting of all $\lambda \in \overline{C_0}$ satisfying

$$\langle \lambda + \rho, \alpha_1^\vee \rangle = 0$$
$$\langle \lambda + \rho, \alpha_1^\vee + 2\alpha_2^\vee \rangle = p.$$

If we write $\lambda + \rho = a_1\omega_1 + a_2\omega_2$, then the first equation forces $a_1 = 0$, and so the second equation reduces to $2a_2 = p$. Thus, the only integral solution occurs when $p$ is even and $\lambda + \rho = (p/2)\omega_2$.

Proposition 5.1.7. Let $p \geq n + 1$, then for any $\lambda \in X(T)_+$, $V_{G_1}(T(\lambda)) \subseteq \overline{O_s(\lambda)}$.

Proof. By Lemma 5.1.1 for each good positive subroot system $\Psi \subseteq \Gamma_\lambda$, there exists $\mu \in C$ such that $\Psi \subseteq \Phi_{\mu, p}$ and $C(\mu) \leq C(\lambda)$. Thus, $\pi(\Psi) \leq \pi(\Phi_{\mu, p}) = d(\mu)$. Furthermore, by Lemma 5.1.5 $F(\mu) \cap X(T)_+ \neq \emptyset$, so we may assume that $\mu \in X(T)_+$. Then

$$V_{G_1}(T(\lambda)) \subseteq V_{G_1}(H^0(\mu)) \subseteq \overline{O_{d(\mu)}} \subseteq \overline{O_{\pi(\Psi)}}.$$

Hence, for each good positive subroot system $\Psi \subseteq \Gamma_\lambda$, $V_{G_1}(T(\lambda)) \subseteq \overline{O_{\pi(\Psi)}}$. Therefore, by Proposition 4.3.5, $V_{G_1}(T(\lambda)) \subseteq \overline{O_{s(\lambda)}}$ since $s(\lambda)^I$ is the greatest partition satisfying $s(\lambda)^I \leq \pi(\Psi)^I$ for all good $\Psi \subseteq \Gamma_\lambda$.

6. Quantum groups

6.1. This section will follow the notation and conventions in [BNPP] and [J, Appendix H]. Let $\mathfrak{g} = \mathfrak{g}_C$ denote a finite-dimensional, complex, semisimple Lie algebra and let $G_\mathbb{C}$ denote the split, semisimple, simply connected algebraic group scheme such that $\mathfrak{g} = \text{Lie}(G) \otimes \mathbb{C}$. Denote by $U_q(\mathfrak{g})$, the quantum enveloping algebra with indeterminate $q \in \mathbb{Q}(q)$ and generators $E_\alpha, F_\alpha, K_\alpha$ and $K^{-1}_\alpha$ for $\alpha \in \Delta$, satisfying the quantized Serre relations ([J H.2]).

For $A = \mathbb{Z}[q, q^{-1}]$, let $U_q^A(\mathfrak{g})$ be the Lusztig $A$-form of $U_q(\mathfrak{g})$, which is the $A$-subalgebra generated by the divided powers $E^{(m)}_\alpha, F^{(m)}_\alpha$ and $K^{\pm 1}_\alpha$ (cf. [J H.5]). If $\Gamma$ is an $A$-algebra, set $U_{\Gamma}(\mathfrak{g}) = U^A_q(\mathfrak{g}) \otimes \mathbb{C}$. Finally, for any $\ell > 1$ and any primitive $\ell$th root of unity $\zeta \in \mathbb{C}$, give $\Gamma = \mathbb{C}$ the structure of an $A$-algebra by sending $q \mapsto \zeta$, and write $U_{\zeta}(\mathfrak{g}) = U_{\Gamma}(\mathfrak{g})$.

It is well known that the category of type 1 integrable representations for $U_{\zeta}(\mathfrak{g})$ shares many properties in common with the category of modular representations for $G_k$, where $k$ is an algebraically closed field of characteristic $p > 0$ (cf. [J Appendix H] for an overview). Thus, every $U_{\zeta}(\mathfrak{g})$-module will be assumed to be type 1 and integrable. Suppose $I \subseteq \Delta$ is a subset and $I$ and $p_I$ are the corresponding Levi and (negative) parabolic subalgebras of $\mathfrak{g}$, then one can define Levi and parabolic subalgebras $U_q(I)$ and $U_q(p_I)$ of the quantum enveloping algebra $U_q(\mathfrak{g})$ and, by specialization, the subalgebras $U_{\zeta}(I)$ and $U_{\zeta}(p_I)$ of $U_{\zeta}(\mathfrak{g})$ (cf. [BNPP 2.5]). It is possible to define the induction functor,

$$\text{ind}_{U_{\zeta}(p)}^{U_{\zeta}(\mathfrak{g})}(M) = H^0(U_{\zeta}(\mathfrak{g})/U_{\zeta}(p), M)$$

for any $U_{\zeta}(p)$-module $M$ ([APW 2.4]). When dealing with the Borel subalgebra $U_{\zeta}(\mathfrak{b})$, we will write $H^0_\zeta(M) = \text{ind}_{U_{\zeta}(\mathfrak{b})}^{U_{\zeta}(\mathfrak{g})}(M)$ for any $U_{\zeta}(\mathfrak{b})$-module $M$. 
Let $X$ denote the weight lattice for $\mathfrak{U}_\zeta(\mathfrak{g})$ and let $X^+$ denote the cone of dominant weights. For each $\lambda \in X^+$, $L_\zeta(\lambda) = \text{soc}(H^0_\zeta(\lambda))$ is the corresponding simple highest weight module (cf. [APW Corollary 6.2]). The Weyl modules are defined by $V_\zeta(\lambda) = H^0_\zeta(-w_0\lambda)^*$. The tilting modules are defined in the same way as for algebraic groups. It follows that for each weight $\lambda \in X^+$, there exists a unique indecomposable tilting module $T_\zeta(\lambda)$ for $\mathfrak{U}_\zeta(\mathfrak{g})$ ([J, H.15]). It was proven by Soergel that under some slight restrictions on $\ell$, the formal characters of these modules are determined by certain parabolic Kazhdan-Lusztig polynomials (cf. [So1] and [So2]).

Much work has been done in studying the cohomology of the finite-dimensional Hopf algebra $\mathfrak{u}_\zeta(\mathfrak{g}) \leq \mathfrak{U}_\zeta$, known as the small quantum group ([BNPP 2.2]). For instance, in [GK, Theorem 3] it was shown that when $\ell > h$,

$$H^\text{ev}(\mathfrak{u}_\zeta(\mathfrak{g}), \mathbb{C})_{\text{red}} \cong \mathbb{C}[N].$$

Thus, max-Spec($H^\text{ev}(\mathfrak{u}_\zeta(\mathfrak{g}), \mathbb{C})$) = $N$, where $N \leq \mathfrak{g}$ is the nilpotent cone of $\mathfrak{g}$. To each $\mathfrak{u}_\zeta(\mathfrak{g})$-module $M$, there exists a support variety $V_{\mathfrak{u}_\zeta(\mathfrak{g})}(M) \subseteq N$. If $M$ has the structure of a $\mathfrak{U}_\zeta(\mathfrak{g})$-module, then $V_{\mathfrak{u}_\zeta(\mathfrak{g})}(M)$ is in fact a $G_\zeta$-stable subvariety of $N$ (cf. [BNPP 8.1]).

6.2. Recalling Definition 3.1.1, let $T_\zeta$ denote the full subcategory of all finite-dimensional tilting modules for $\mathfrak{U}_\zeta(\mathfrak{g})$. The thick tensor ideals of $T_\zeta$ have been classified by Ostrik (cf. [O1 Theorem 4.5]). More specifically, it was shown that $\langle T_\zeta(M) \rangle = \langle T_\zeta(\lambda) \rangle$ if and only if $\lambda, \mu \in X^+$ lie in the same weight cell. In further analogy with the algebraic group case, there is a connection between thick tensor ideals and support varieties for quantum tilting modules.

**Lemma 6.2.1.** Let $M, N \in T_\zeta$ be tilting modules with $\langle M \rangle \subseteq \langle N \rangle$, then $V_{\mathfrak{u}_\zeta(\mathfrak{g})}(M) \subseteq V_{\mathfrak{u}_\zeta(\mathfrak{g})}(N)$.

**Proof.** Since $\langle M \rangle \subseteq \langle N \rangle$, then by definition there exists some $L \in T_\zeta$ such that $M \mid N \otimes L$, and hence $V_{\mathfrak{u}_\zeta(\mathfrak{g})}(M) \subseteq V_{\mathfrak{u}_\zeta(\mathfrak{g})}(N)$.

In Section 1.3 it was stated that the varieties $V_{\mathfrak{u}_\zeta(\mathfrak{g})}(T_\zeta(\lambda))$ have been computed for all types when $\ell > h$ by Ostrik and Bezrukavnikov. For convenience, we shall state here what was proven.

**Theorem 6.2.2.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\zeta \in \mathbb{C}$ be a primitive $\ell$th root of unity with $\ell > h$, odd (and not divisible by 3 if $\mathfrak{g}$ has a component of type $G_2$). For each $w \in W^+$, let $c_{[w]} \subseteq \mathbb{C}$ be the corresponding weight cell, and let $\mathcal{O}_{[w]}$ denote the orbit associated to $c_{[w]}$ by the Lusztig bijection. Then if $\lambda \in c_{[w]} \cap X^+$,

$$V_{\mathfrak{u}_\zeta(\mathfrak{g})}(T_\zeta(\lambda)) = \mathcal{O}_{[w]}.$$

For any tilting module $M$ for $G$, it is well known that $M|_{[I_\alpha, I_\alpha]}$ is a tilting module whenever $I \subseteq \Delta$, $I_\alpha$ is a Levi-factor for $G$ and $[I_\alpha, I_\alpha]$ is the derived subgroup of $I_\alpha$ (cf. [J Proposition II.4.24]). An analogous result also holds for quantum groups.

**Proposition 6.2.3.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\zeta \in \mathbb{C}$ be a primitive $\ell$th root of unity where $\ell$ is odd (and not divisible by 3 if $\mathfrak{g}$ has a component of type $G_2$) and is such that $\langle \omega_1 + \rho, \alpha_\ell^\vee \rangle < \ell$ for all fundamental weights $\omega_1, \ldots, \omega_n$. Then for each $I \subseteq \Delta$, $\mathfrak{U}_\zeta([I_\alpha, I_\alpha])$ is the Hopf subalgebra of $\mathfrak{U}_\zeta(\mathfrak{g})$ generated by $E_\alpha^{(m)}$, $F_\alpha^{(n)}$, $K_\alpha^{\pm 1}$ for $\alpha \in I$, and for any $\mathfrak{U}_\zeta(\mathfrak{g})$ tilting module $M$, the restricted module $M|_{\mathfrak{U}_\zeta([I_\alpha, I_\alpha])}$ is a $\mathfrak{U}_\zeta([I_\alpha, I_\alpha])$ tilting module.

**Proof.** Begin by observing that for each fundamental weight $\omega \in X^+$, the restriction $H_\zeta^0(\omega)|_{\mathfrak{U}_\zeta([I_\alpha, I_\alpha])}$ is a tilting module. This is due to the fact that all of the weights of $H_\zeta^0(\omega)|_{\mathfrak{U}_\zeta([I_\alpha, I_\alpha])}$ are $\ell$-minuscule (i.e. they satisfy $\langle \nu + \rho, \alpha_\ell^\vee \rangle < \ell$), and hence the restricted module must be a semisimple tilting module. The proposition follows by adapting the argument in [NT Proposition 3.1] to the quantum setting.

**Remark 6.2.4.** If $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, then the condition that $\langle \omega_i + \rho, \alpha_\ell^\vee \rangle < \ell$ for all fundamental weights $\omega_1, \ldots, \omega_n$, is satisfied precisely when $\ell > n + 1$. 

6.3. Suppose now that $\zeta \in \mathbb{C}$ is a $p^{th}$ root of unity, where $p$ is a prime number, and let $k$ be an algebraically closed field of characteristic $p$. Let $\mathcal{T}$ denote the subcategory of tilting modules for the algebraic group $G = G_k$, and identify $X = X(T)$, where $X$ is the weight lattice for $\mathfrak{U}_\zeta(\mathfrak{g})$. By [AI 5.3], for each tilting module $M$ of $G$, there exists a quantum tilting module, denoted by $M_\zeta$, for $\mathfrak{U}_\zeta(\mathfrak{g})$ satisfying $\text{ch}(M_\zeta) = \text{ch}(M)$. More specifically, if $\lambda \in X(T)_+$ is arbitrary and $M = T(\lambda)$, then

$$ T(\lambda)_\zeta = T_\zeta(\lambda) \oplus \bigoplus_{\mu \neq \lambda} a_\mu T_\zeta(\mu). $$

In particular, since $\text{ch}(T(\lambda)_\zeta) = \text{ch}(T(\lambda))$, then

(6.3.1) \[ \text{ch}(T(\lambda)) = \text{ch}(T_\zeta(\lambda)) + \sum_{\mu \neq \lambda} a_\mu \text{ch}(T_\zeta(\mu)). \]

From the standard properties of the assignment $V_{u_\zeta(\mathfrak{g})}(-)$, one gets $V_{u_\zeta(\mathfrak{g})}(T_\zeta(\lambda)) \subseteq V_{u_\zeta(\mathfrak{g})}(T(\lambda)_\zeta)$ (cf. [O2, Lemma 3.4]). This gives us an immediate corollary to Theorem 6.2.2.

Corollary 6.3.2. Let $G$ be a semisimple, simply connected algebraic group over a field $k$ of characteristic $p > h$, and let $\mathfrak{U}_\zeta(\mathfrak{g})$ be the corresponding quantum group, where $\zeta \in \mathbb{C}$ is a primitive $p^{th}$ root of unity. Then for each $w \in W^+_p$, and $\lambda \in c[w] \cap X(T)_+$,

$$ V_{u_\zeta(\mathfrak{g})}(T(\lambda)_\zeta) \supseteq \overline{C[w]}. $$

6.4. An interesting problem would be to understand how the support varieties for tilting modules of the form $T(\lambda)$ and $T(\lambda)_\zeta$, with $\lambda \in X(T)_+$, are related. It is well known that when the characteristic $p$ is good (in particular if $p > h$), the classification and structure of the $G_C$ orbits on the complex nilpotent cone $N_C = \mathcal{N}(G_C)$ coincide with the $G_k$ orbits on $N_k = \mathcal{N}(G_k)$ (cf. [CM] for the complex case and [P] for the positive characteristic case). This implies that each orbit $O_C$ in $N_C$ uniquely corresponds to an orbit $O_k$ in $N_k$. Moreover, if

$$ O_C = O^{1}_C \cup \ldots \cup O^{m}_C $$

for some orbits $O^{1}_C, \ldots, O^{m}_C$, then

$$ \overline{O_k} = \overline{O^{1}_k} \cup \ldots \cup \overline{O^{m}_k}. $$

It follows that any $G_C$-stable closed subvariety $V_C \subseteq N_C$ uniquely corresponds to a $G_k$-stable closed subvariety $V_k \subseteq N_k$. We now state an interesting conjecture which would realize this correspondence by taking support varieties of tilting modules.

Conjecture 6.4.1. Let $G$ be a semisimple, simply connected algebraic group over a field $k$ of characteristic $p > h$, and let $\mathfrak{U}_\zeta(\mathfrak{g})$ be the corresponding quantum group, where $\zeta \in \mathbb{C}$ is a primitive $p^{th}$ root of unity. Then for any tilting module $M$ for $G_k$, $V_{u_\zeta(\mathfrak{g})}(M_\zeta) = V_C$ if and only if $V_{G_1}(M) = V_k$, where $V_k$ is the unique $G_k$-stable subvariety of $N_k$ corresponding to $V_C$.

Remark 6.4.2. The truth of this conjecture would imply that the correspondence between the $G_C$ and $G_k$-stable closed subvarieties of $N_C$ and $N_k$ described above, can be established by taking support varieties of tilting modules. In fact, if $p > h$, then the conjecture will follow if both Conjecture [1.3.1] holds and an analogous conjecture holds for tilting modules of the form $T(\lambda)_\zeta$, where $\lambda \in X(T)_+$.

The following lemma verifies this conjecture for the trivial orbit closures $\{0\}_C \subseteq N_C$ and $\{0\}_k \subseteq N_k$.

Lemma 6.4.3. Let $G$ be a semisimple, simply connected algebraic group over a field $k$ of characteristic $p > h$ and let $\mathfrak{U}_\zeta(\mathfrak{g})$ be the corresponding quantum group, where $\zeta \in \mathbb{C}$ is a primitive $p^{th}$ root of unity. Then a tilting $G$-module $M$ is $G_1$-projective if and only if $M_\zeta$ is $u_\zeta(\mathfrak{g})$-projective.
Lemma E.8], it follows that \( T(\lambda) \) is \( G_1 \)-projective if and only if \( \langle \lambda, \alpha^\vee \rangle \geq p - 1 \) for all \( \alpha \in \Delta \). The analogous statement also holds in the quantum setting for \( T_\xi(\lambda) \). Since

\[
T(\lambda)_\xi = T(\lambda) \oplus \bigoplus_{\mu \mid \lambda, \mu \neq \lambda} a_\mu T_\xi(\mu),
\]

then \( T(\lambda)_\xi \) is projective if and only if \( a_\mu = 0 \) for any \( \mu \in X(T)_+ \) satisfying \( \langle \mu, \alpha^\vee \rangle < p - 1 \) for some \( \alpha \in \Delta \).

Now observe that \( T(\lambda) \) is \( G_1 \)-projective if and only if \( \lambda = (p - 1)\rho + \nu \), where \( \nu \in X(T)_+ \). Since \( T((p - 1)\rho) = L((p - 1)\rho) \) is a simple tilting module, then \( T_\xi((p - 1)\rho) = T((p - 1)\rho)_\xi \) because both modules are equal to \( L_\xi((p - 1)\rho) \). By highest weight considerations,

\[
T(\lambda)_\xi | T_\xi((p - 1)\rho) \otimes T(\nu)_\xi,
\]

and so \( T(\lambda)_\xi \) is projective if \( T(\lambda) \) is \( G_1 \)-projective. Likewise, if \( T(\lambda)_\xi \) is projective, then \( T_\xi(\lambda) \) is projective, and thus \( \lambda = (p - 1)\rho + \nu \) for some \( \nu \in X(T)_+ \), which implies that \( T(\lambda) \) is \( G_1 \)-projective.

Conjecture[6.4.1] can also be verified for the principal orbit closures: \( N_C \) and \( N_k \).

**Lemma 6.4.4.** Let \( G \) be a semisimple, simply connected algebraic group over a field \( k \) of characteristic \( p > h \) and let \( U_\xi(g) \) be the corresponding quantum group, where \( \xi \in \mathbb{C} \) is a primitive \( p^\text{th} \) root of unity. Then any tilting \( G \)-module \( M \) satisfies \( V_{G_1}(M) = N_k \) if and only if \( V_{U_\xi(g)}(M_\xi) = N_\xi \).

**Proof.** Let \( M = \sum_{\lambda \in X(T)_+} a_\lambda T(\lambda) \) be an arbitrary tilting module for \( G \), then

\[
M_\xi = \sum_{\lambda \in X(T)_+} a_\lambda T(\lambda)_\xi = \sum_{\lambda \in X(T)_+} b_\lambda T_\xi(\lambda).
\]

By using translation identities, it can be deduced that \( V_{G_1}(M) = N_k \) if and only if \( a_\lambda > 0 \) for some \( \lambda \in C_0 \) (cf. [7] Proposition E.11]). By the same argument, \( V_{U_\xi(g)}(M_\xi) = N_\xi \) if and only if \( b_\lambda > 0 \) for some \( \lambda \in C_0 \). On the other hand, by [7] Proposition E.12], it follows that for each \( \lambda \in C_0 \),

\[
a_\lambda = \sum_{w \in W_p^+} (-1)^{\ell(w)} [M : \text{ch } H^0(w \cdot \lambda)].
\]

Moreover, since \( \text{ch } H^0(\mu) = \text{ch } H^0(\mu) \) for any \( \mu \in X(T)_+ \) and since \( \text{ch } M = \text{ch } M_\xi \), then by the same argument,

\[
b_\lambda = \sum_{w \in W_p^+} (-1)^{\ell(w)} [M : \text{ch } H^0(w \cdot \lambda)]
\]

for any \( \lambda \in C_0 \). Thus, \( a_\lambda = b_\lambda \) for each \( \lambda \in C_0 \), and hence \( V_{G_1}(M) = N_k \) if and only if \( V_{U_\xi(g)}(M_\xi) = N_\xi \).

\[
\Box
\]

6.5. **An interesting result in type \( A_n \).** Let \( G = SL_{n+1}(k) \) with \( p > n + 1 \), let \( g = \mathfrak{sl}_{n+1}(\mathbb{C}) \), and let \( \zeta \in \mathbb{C} \) be a primitive \( p^\text{th} \) root of unity. By Corollary 6.3.2, \( V_{U_\xi(g)}(T(\lambda)_\xi) \cong \mathcal{O}_{s(\lambda)^p} \) for any \( \lambda \in X(T)_+ \). It can also be verified that Proposition 3.2.5 holds in the quantum setting. Therefore, the proof of Proposition 5.1.7 may be adapted to the quantum group setting, to yield the following proposition.

**Proposition 6.5.1.** Let \( p > n + 1 \), then for each \( \lambda \in X(T)_+ \), \( V_{U_\xi}(T(\lambda)_\xi) = \mathcal{O}_{s(\lambda)^p} \).

By Remark 6.4.2, this proposition can be combined with Theorem 1.4.1 to prove Conjecture 6.4.1 in the type \( A \) case.
7. The lower bound

7.1. Let \( G = SL_{n+1}(k) \), where \( k \) is an algebraically closed field of characteristic \( p > 0 \), let \( g = \mathfrak{sl}_{n+1}(\mathbb{C}) \) and let \( \zeta \in \mathbb{C} \) be a primitive \( p^{th} \) root of unity. The goal of this section will be to show that for each \( \lambda \in X(T)_+ \),

\[
V_{G_1}(T(\lambda)) \supseteq \overline{O_{s(\lambda)}}.
\]

For any partition \( \pi = (p_1, p_2, \ldots, p_r) \in \mathcal{P} \), we define the subgroup scheme

\[
SL_\pi = SL_{p_1} \times SL_{p_2} \times \cdots \times SL_{p_r} \subseteq SL_{n+1},
\]

and let \( \mathfrak{sl}_\pi = \text{Lie}(SL_\pi) \) denote its Lie algebra. If \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) for \( i = 1, \ldots, n \) and

\[
I_\pi = \{ \alpha_1, \ldots, \alpha_{p_1-1} \} \cup \{ \alpha_{p_1+1}, \ldots, \alpha_{p_1+p_2-1} \} \cup \cdots \cup \{ \alpha_{p_1+\cdots+p_{r-1}+1}, \ldots, \alpha_{p_1+\cdots+p_r-1} \} \subseteq \Delta,
\]

then \( SL_\pi = [L_{I_\pi}, L_{I_\pi}] \) is the derived subgroup of the corresponding Levi factor \( L_{I_\pi} \). For notational simplicity, we will set \( H_\pi = SL_\pi(k) \) and \( \mathfrak{h}_\pi = \mathfrak{sl}_\pi(\mathbb{C}) \).

In Proposition 6.2.3 it was shown that, as in the algebraic setting, there is a natural inclusion of quantum groups \( \mathcal{U}_\zeta(\mathfrak{h}_\pi) \hookrightarrow \mathcal{U}_\zeta(\mathfrak{g}) \) such that tilting modules for \( \mathcal{U}_\zeta(\mathfrak{g}) \) restrict to tilting modules for \( \mathcal{U}_\zeta(\mathfrak{h}_\pi) \). The following lemma is a well known fact about nilpotent orbits in type A (cf. [CM, Theorem 8.2.14]).

**Lemma 7.1.2.** For any partition \( \pi \in \mathcal{P} \), let \( x_\pi \) denote the nilpotent matrix which is a direct sum of Jordan blocks whose sizes are given by the parts of \( \pi \), then \( x_\pi \in \mathfrak{sl}_\pi(k) \) and the orbit \( H_\pi \cdot x_\pi \) is dense in \( \mathcal{N}(H_\pi) \). Moreover, since \( O_\pi = G \cdot x_\pi \), then \( G \cdot \mathcal{N}(H_\pi) = \overline{O_\pi} \).

By the naturality of support varieties, we can identify

\[
V_{(H_\pi)_+}(M|_{H_\pi}) = V_{G_1}(M) \cap \mathfrak{sl}_\pi(k)
\]

for each \( G \)-module \( M \). Under this identification, \( x_\pi \in V_{G_1}(M) \) if and only if \( V_{(H_\pi)_+}(M|_{H_\pi}) = \mathcal{N}(H_\pi) \).

7.2. We now have enough to proceed with a proof of the lower bound. But first, for notational convenience, the following terminology will be introduced.

**Definition 7.2.1.** A module \( M \) is said to have full support, if its support variety is maximal. For instance, if \( M \) is a \( G_1 \)-module with \( p > h \), then \( M \) has full support provided \( V_{G_1}(M) = \mathcal{N}(G) \).

As mentioned in the introduction, the following proposition, along with Proposition 5.1.7 may be combined to give Theorem 1.4.1.

**Proposition 7.2.2.** Let \( p > n + 1 \), then for any \( \lambda \in X(T)_+ \), \( V_{G_1}(T(\lambda)) \supseteq \overline{O_{s(\lambda)}} \).

**Proof.** For any partition \( \pi \in \mathcal{P} \) and \( \lambda \in X(T)_+ \) satisfying \( s(\lambda) = \pi \), it follows from (7.1.3) that this proposition will hold if \( T(\lambda)|_{H_\pi} \) has full support. By Proposition 6.2.3 the module \( (T(\lambda)|_{U_\zeta(\mathfrak{h}_\pi)}) \) is a quantum tilting module, and thus \( (T(\lambda)|_{H_\pi})_{\zeta} = (T(\lambda)|_{U_\zeta(\mathfrak{h}_\pi)})_{\zeta} \). So by Lemma 6.4.4 \( T(\lambda)|_{H_\pi} \) will have full support if and only if \( (T(\lambda)|_{U_\zeta(\mathfrak{h}_\pi)})_{\zeta} \) has full support. Since

\[
T_{\zeta}(\lambda) \mid T(\lambda)_{\zeta} \subseteq V_{U_\zeta(\mathfrak{g})}(T(\lambda)_{\zeta}),
\]

then \( V_{U_\zeta(\mathfrak{g})}(T_{\zeta}(\lambda)) \subseteq V_{U_\zeta(\mathfrak{g})}(T(\lambda)_{\zeta}) \), and hence, by [O1, O2 Lemma 6.4] and [O2 Theorem 6.8], there exist \( \mu, \nu \in X(T)_+ \) such that \( T(\mu) = H^0(\mu), s(\mu) = s(\lambda) \) and \( T_{\zeta}(\mu) \mid T(\lambda)_{\zeta} \otimes T_{\zeta}(\nu) \). Moreover, \( T_{\zeta}(\mu) = T(\mu)_{\zeta} \) and \( T_{\zeta}(\nu) \mid T(\nu)_{\zeta} \) together give

\[
T_{\zeta}(\mu)_{\zeta} \otimes M = (T(\lambda) \otimes T(\nu))_{\zeta}.
\]

for some \( \mathcal{U}_\zeta(\mathfrak{g}) \) tilting module \( M \). Therefore, by Proposition 6.2.3

\[
T(\mu)_{\zeta}|_{U_\zeta(\mathfrak{h}_\pi)} \otimes M|_{U_\zeta(\mathfrak{h}_\pi)} = \mathcal{N}_{\zeta}.
\]
where $N = (T(\lambda) \otimes T(\nu))|_{H_x}$. Since $T(\mu) = H^0(\mu)$, then by [NPV] Theorem 6.2.1, $V_{G_1}(T(\mu)) = \mathcal{O}_\pi$ and so (7.2.2) implies that $V_{(H_x)_1}(T(\mu)|_{H_x}) = N(H_x)$. Also, by (7.2.3) and Lemma 6.4.1 $(T(\mu)|_{H_x})_\zeta = (T(\mu)_\zeta)|_{\mathcal{U}_1(h_x)}$, and hence $N$ and $N_\zeta$ have full support. Thus, $\pi_\mu \in V_{G_1}(T(\lambda) \otimes T(\nu)) \subseteq V_{G_1}(T(\lambda))$, where the inclusion follows from the tensor product identity for support varieties. □

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