Entanglement entropy, horizons and holography

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(Dated: May 1, 2019)

We calculate the entanglement entropy in spaces with horizons, such as Rindler or de Sitter space, using holography. We employ appropriate parametrizations of AdS space in order to obtain a Rindler or static de Sitter boundary metric. The holographic entanglement entropy for the regions enclosed by the horizons can be identified with the standard thermal entropy of these spaces. For this to hold, we define the effective Newton’s constant appropriately and account for the way the AdS space is covered by the parametrizations.

We analyze the relation between entanglement entropy and the thermal entropy in spaces that contain horizons. The framework is provided by the AdS/CFT correspondence [1], which establishes a connection between an asymptotically anti-de Sitter (AdS) bulk space and a conformal field theory (CFT) that can be considered as living on its boundary. The metric on the AdS boundary belongs to a conformal class, a feature that permits the study of the dual CFT on nontrivial backgrounds. We consider parametrizations for which the boundary metric takes the Rindler and static de Sitter form. We focus on the entanglement entropy associated with a CFT confined within a part $A$ of the AdS boundary delimited by an entangling surface $\mathcal{A}$. Our analysis is based on the proposal of refs. [2,3] that the entropy is proportional to the area of an appropriately defined minimal surface (holographic screen) $\gamma_A$ that starts from $\mathcal{A}$ and extends into the bulk.

We consider three parametrizations of $(d+2)$-dimensional AdS space. The first uses global coordinates:

$$ds^2_{d+2} = \frac{R^2}{\cos^2 \chi} \left[ -d\tau^2 + d\chi^2 + \sin^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\Omega^2_{d-1} \right) \right],$$

(1)

with $-\infty < \tau < \infty$, $0 \leq \chi < \pi/2$, $-\pi/2 \leq \theta \leq \pi/2$ for $d > 1$, and $R$ the AdS radius. For $d = 1$, $\theta$ covers the full unit circle: $-3\pi/2 \leq \theta \leq \pi/2$. The second parametrization uses Fefferman-Graham coordinates [4] with a Rindler boundary:

$$ds^2_{d+2} = \frac{R^2}{z^2} \left[ dz^2 - a^2 y^2 d\eta^2 + dy^2 + d\bar{x}_{d-1}^2 \right],$$

(2)

with the boundary located at $z = 0$, $a$ a constant parameter, $-\infty < \eta < \infty$, $0 < y < \infty$ covering the right (R) Rindler wedge and $-\infty < y < 0$ the left (L) wedge. The third parametrization makes use of a de Sitter (dS) slicing of AdS space that results in a metric of the Fefferman-Graham form with a static dS boundary:

$$ds^2_{d+2} = \frac{R^2}{z^2} \left[ dz^2 + \left( 1 - \frac{1}{4} H^2 z^2 \right)^2 \left( -\frac{1}{1 - H^2 \rho^2} dt^2 + \frac{d\rho^2}{1 - H^2 \rho^2} + \rho^2 \, d\Omega^2_{d-1} \right) \right],$$

(3)

where $0 \leq \rho \leq 1/H$ covers the static patch for $d > 1$. There are two such patches in the global geometry, with $\rho = 0$ corresponding to the “North” and “South poles”. For $d = 1$, $\rho$ can be negative and each static patch is covered by $-1/H \leq \rho \leq 1/H$. All the coordinates in the above expressions are taken to be dimensionless, with $R$ the only dimensionful parameter. In particular, $H$ and $a$ are dimensionless. For de Sitter space, the physical Hubble scale is $H/R$.

There are several parametrizations of AdS that correspond to different vacua of the boundary theory [5]. The holographic stress-energy tensor of the dual CFT for the metric [7], computed along the lines of [6], vanishes. This indicates that this form of the metric corresponds to the Minkowski vacuum, for which the Rindler observer is expected to detect a thermal environment. Similarly, the stress-energy tensor for the metric [3] does not contain any singularities on the horizon. Therefore, this metric corresponds to the steady-state vacuum and the static dS observer again detects a thermal background [7]. Thermal properties of de Sitter have been analyzed through holography in [8]. Here we focus on the question of entropy.

We consider first the case of a Rindler boundary. The thermal nature of the vacuum is reflected in the periodicity of the Euclidean “time” coordinate $\eta_E = i\eta$. In order to understand how the parametrization covers the AdS space, we consider the relation between the Euclidean Rindler coordinates and the Euclidean global ones. Similar conclusions can be reached by considering metrics with Lorentzian signature. We use AdS$_3$ as an example, for which the explicit
the cutoff and independent of the width is apparent from the second line of eq. (7) that only the first term in the parenthesis survives in this limit. This term would arise from the entanglement of the right wedge with the left wedge of Rindler space. As the volume. The calculation of the length of the minimal surface for a ($d+1$)-dimensional Rindler boundary is identical to the one in [3] for an infinite belt on a Minkowski boundary. The surface $\eta = 0$ corresponds to Minkowski time $t = 0$. On this surface, the coordinate $y = x > 0$ covers the positive $x$-axis of Minkowski space, and $y = x < 0$ the negative $x$-axis. The two regions are separated by the horizon at $y = 0$. We consider an infinite strip along the $y$-axis with width $l$. The minimal surface extends into the bulk up to a turning point at $z_* = \Gamma \left( \frac{1}{2d} \right) / (2\sqrt{\pi} \Gamma \left( \frac{d+1}{2d} \right)) l$. The entanglement entropy is obtained by dividing the area of the minimal surface by $4G_{d+2}$, with $G_{d+2}$ the Newton’s constant of the bulk space. The entanglement entropy is given by

$$S_A = \frac{2R(R^{d-1}L^{d-1})}{4G_{d+2}} \left( \frac{1}{(d-1)e^{d-1}} - 2^{d-1} \frac{\pi^{d/2}}{\Gamma \left( \frac{d+1}{2} \right)} \frac{1}{(d-1)^{d-1}} \right),$$

with $\epsilon$ a cutoff imposed on the bulk coordinate $z$ as the surface approaches the boundary. For $d = 1$, one must substitute $1/((d-1)e^{d-1})$ with $\log(1/\epsilon)$, and similarly for $l$ in the first line. The term in the parenthesis becomes $\log(l/\epsilon)$. Here $L$ is the large length of the directions perpendicular to the strip, so that $R^{d-1}L^{d-1}$ is the corresponding volume.

We are interested in the limit of eq. (7) when the width of the strip covers the whole positive axis. The entropy would arise from the entanglement of the right wedge with the left wedge of Rindler space. As $z_* \to \infty$ for $l \to \infty$, it is apparent from the second line of eq. (7) that only the first term in the parenthesis survives in this limit. This term is strongly dependent on the cutoff and independent of the width $l$ of the strip. It must be noted, however, that for $d = 1$ the first and second terms combine into $\log(l/\epsilon)$. In the context of our considerations, it is clear that the first term must have a physical meaning. Its nature becomes apparent if we define the effective Newton’s constant for the boundary theory following [2]:

$$G_{d+1} = (d-1)e^{d-1}G_{d+2} \frac{R}{l},$$

with $(d-1)e^{d-1}$ replaced by $1/\log(1/\epsilon)$ for $d = 1$. This definition is natural within an effective theory that implements consistently a cutoff procedure by eliminating the part of AdS space corresponding to $z < \epsilon$. Such a framework is provided, for example, by the Randall-Sundrum (RS) model [10]. If the positive-tension brane in this model is placed
at $z = \epsilon$, the massless graviton is localized arbitrarily close to the boundary, so that the effective Newton’s constant has the parametric dependence of eq. (8). As pointed out in [9], the cutoff dependence of the four-dimensional Newton’s constant is not apparent in [10] because the metric is rescaled by $\epsilon^2$. Our definition differs by a factor of 2 from the one in [9], in which the brane is assumed to bound two mirror AdS regions. On the contrary, we integrate over the AdS region $z > \epsilon$ only once. In a construction along the lines of the RS model, $G_{d+1}$ would be larger by a factor of 2, but the entropy would also be doubled because of the presence of two copies of AdS space. Our definition does not include the factor of 2 and can be viewed as resulting from the regulated action in the context of holographic renormalization [6].

The use of the definition (8) in eq. (7) results in an entanglement entropy $S_R = (R^{d-1}L^{d-1})/(2G_{d+1})$, which is double the expected value of the thermal entropy [12]. This discrepancy can be understood by examining more carefully the limit in which the strip covers the entire right wedge. The strip extends from the horizon at $y = 0$ to a value $y_m$ for which the limit $y_m \rightarrow \infty$ is taken. For any finite, no matter how large, value of $y_m$ the strip is entangled not only with the left wedge, but also with the (infinite domain) beyond $y_m$. As the space is essentially flat, the two contributions are expected to be equal. If one is interested in the entanglement with the left wedge only, the limit $y_m \rightarrow \infty$ must be accompanied by a division by 2 of the computed entanglement entropy. The final result for the Rindler entropy is

$$S_R = \frac{R^{d-1}L^{d-1}}{4G_{d+1}},$$

in agreement with [12].

There is a natural identification of the entanglement entropy with the thermal one, resulting from the fact that the minimal surface $\gamma_A$ corresponding to the right wedge is nothing but the bulk horizon. In the left plot of fig. [1] this is the line starting at the point $(\chi = \pi/2, \theta = -\pi/2)$ and extending towards the point $(\chi = \pi/2, \theta = \pi/2)$. As we pointed out earlier, this second point does not correspond to one in Rindler space and is not a horizon of the boundary metric. Therefore, the (infinite) contribution to the length of the line from the vicinity of $(\chi = \pi/2, \theta = \pi/2)$ should not be taken into account. This interpretation justifies the division by of 2 from a different point of view.

We turn next to the calculation of the entropy of de Sitter space through holographic means. This question has been addressed in the past in the context of the AdS/CFT [9,13,15] or the dS/CFT [16] correspondence. We present here

FIG. 1: The slicing of the Euclidean AdS$_3$ cylinder for a Rindler boundary with $a = 1$ (left) and a static de Sitter boundary with $H = 1$ (right). The Euclidean “time” coordinates $\eta_E$ and $t_E$ are varied between $-\pi/2$ and $\pi/2$. 
an explicit solution that interpolates between the entanglement entropy for regions much smaller than the horizon and the thermal entropy of de Sitter space detected by a static observer. The interpolation is possible through the definition \( [8] \) of the effective Newton’s constant.

In order to make an explicit comparison with the Rindler case we consider how the Euclidean AdS\(_3\) is covered by the coordinates for a Euclidean de Sitter boundary. The explicit relation is

\[
\chi(z, t_E, \rho) = \tan^{-1} \left( \frac{1 - \frac{1}{2} H^2 z^2}{Hz} \sqrt{\cos^2(HT_E) + H^2 \rho^2 \sin^2(HT_E)} \right)
\]

\( (10) \)

\[
\tau_E(z, t_E, \rho) = \tan^{-1} \left( \frac{1 - \frac{1}{2} H^2 z^2}{1 + \frac{1}{4} H^2 z^2} \sin(HT_E) \sqrt{1 - H^2 \rho^2} \right)
\]

\( (11) \)

\[
\theta(z, t_E, \rho) = \tan^{-1} \left( \frac{H \rho}{\sqrt{1 - H^2 \rho^2 \cos(HT_E)}} \right). \quad (12)
\]

In fig. 1 we depict the slicing of the AdS\(_3\) cylinder for \( H = 1 \). Each one of the depicted slices corresponds to a constant value of \( t_E \). The horizontal slice corresponds to \( t_E = 0 \), the vertical one, consisting of two parts, to \( t_E = \pm \pi/2 \), and the other two to \( t_E = \pm \pi/4 \). The intersection of each slice with the boundary corresponds to \( z = 0 \). Each intersection has two symmetric regions, corresponding to the two static patches of the global dS\(_2\). Each one is parametrized by \( \rho \), starting from \( \rho = 1/H \) at the front, at the point \( (\chi = \pi/2, \theta = -\pi/2) \), and finishing at the back, at the point \( (\chi = \pi/2, \theta = \pi/2) \) or \( (\chi = \pi/2, \theta = -3\pi/2) \), for \( \rho = 1/H \). Each slice is covered by lines of constant \( \rho \) and \( z \). These converge to the same point at the center, with \( \chi = 0 \), for \( z \to 2/H \). With respect to entanglement entropy, the minimal surface \( \gamma_A \) for each static wedge corresponds to the intersection of all slices, which connects the two horizons at \( z = 0, \rho = \pm 1/H \). The entanglement occurs between the two static de Sitter patches. The important difference with Rindler space is that the endpoints of the minimal surface correspond to points of dS\(_2\), they are actually the horizons of this space. So, no superfluous factor of 2 is expected. The situation is similar for \( d > 1 \), with the minimal surface \( \gamma_A \) ending on an \((d - 1)\)-dimensional sphere that separates the two hemispheres of the \( t_E = 0 \) slice of dS\(_{d+1}\) [15].

The equivalence between entanglement and thermal entropies is again apparent. The minimal surface is nothing but the bulk horizon of the metric \( [3] \), extending between \( z = 0 \) and \( z = 2/H \) for \( \rho = \pm 1/H \), and between \( \rho = -1/H \) and \( \rho = 1/H \) for \( z = 2/H \). This last part is actually one point, corresponding to \( \chi = 0 \) in global coordinates. For \( d > 1 \) the situation is very similar, but now \( \rho > 0 \) and the cosmological dS horizon is the sphere with \( \rho = 1/H \). The entanglement takes place between degrees of freedom on the two static patches. The holographic images of the patches are separated by the minimal surface starting from the dS horizon on the boundary \( [9] [15] \). This surface acts again as a bulk horizon.

For the calculation of the entanglement entropy for a general dimension \( d \), we use the parametrization of eq. \( [3] \) and consider the interior \( A \) of a spherical entangling surface \( A \) on the boundary. (For \( d = 1 \) we consider a line segment between the two horizons at \( \rho = \pm 1/H \).) The minimal surface \( \gamma_A \) in the bulk can be determined through the minimization of the area

\[
\text{Area}(\gamma_A) = R^d S^{d-1} \int d\rho \rho^{d-1} \frac{\left( 1 - \frac{1}{2} H^2 z^2 \right)^{d-1}}{z^d} \sqrt{\frac{\left( 1 - \frac{1}{4} H^2 z^2 \right)^2}{1 - H^2 \rho^2} + \left( \frac{dz(\rho)}{d\rho} \right)^2}, \quad (13)
\]

with \( S^{d-1} \) the volume of the \((d - 1)\)-dimensional unit sphere. Through the definitions \( \sigma = \sin^{-1}(H \rho), w = 2 \tanh^{-1}(Hz/2) \), the above expression becomes

\[
\text{Area}(\gamma_A) = R^d S^{d-1} \int d\sigma \sin^{d-1}(\sigma) \sinh^d(w) \sqrt{1 + \left( \frac{dw(\sigma)}{d\sigma} \right)^2}. \quad (14)
\]

Minimization of the area results in the differential equation

\[
\tan(\sigma) \tanh(w) w'' + (d - 1) \tanh(w) \left( (w')^3 + w' \right) + d \tan(\sigma) \left( (w')^2 + 1 \right) = 0. \quad (15)
\]

For \( H \to 0 \) and small \( \sigma \), \( w \) we recover the known equation for the minimal surface in the case of a Minkowski boundary. Its solution is \( z(\rho) = \sqrt{\rho_0^2 - \rho^2} \), with \( \rho_0 \) the radius of the entangling surface on the boundary. The expressions for the area of the minimal surface and the corresponding entanglement entropy are given in \( [3] \).

For nonzero \( H \) and a general \( d \), an analytical solution of eq. \( [15] \) in closed form is not possible. However, the features that are important for our analysis become visible in a numerical solution. In fig. 2 we depict the solution
for $d = 3$, $H = 1$ and increasing values of $\rho_0$. It is apparent that as the entangling surface approaches the horizon the minimal surface has a limiting shape. Deep in the bulk it traces the surface with $z = 2/H$, until the parameter $\rho$ reaches the value $\rho = 1/H$. Then it approaches the boundary with constant $\rho$ and diminishing $z$. It is noteworthy that the first part of the surface has vanishing area, as can be seen from eq. (13). The reason is that this part corresponds to a single point in AdS space (the point with global coordinate $\chi = 0$). The total area is dominated by the region near the boundary. For $\rho_0 = 1/H$, the integration of eq. (13) is complicated by the singularity in $\rho$. However, the expression (14) demonstrates that the integral is dominated by the region in which $w \rightarrow Hz \rightarrow 0$, $dw/d\sigma \rightarrow -\infty$ and $\sigma \rightarrow \pi/2$. Cutting off the range of $z$ at $z = \epsilon$, gives

$$\text{Area}(\gamma_A) = R^d S^{d-1} \int_{H\epsilon} \frac{dw}{w^d} = \frac{R^d S^{d-1}}{(d-1)H^{d-1}e^{d-1}}. \tag{16}$$

The entropy becomes

$$S_{dS} = \frac{R^d S^{d-1}}{4G_{d+2}(d-1)H^{d-1}e^{d-1}} = \frac{S^{d-1}}{4G_{d+1}} \left( \frac{R}{H} \right)^{d-1}, \tag{17}$$

in agreement with [17]. The above expressions are also valid for $d = 1$ with $1/((d-1)e^{d-1})$ replaced by $\log(1/\epsilon)$.

In conclusion, the holographic method of Ryu and Takayanagi can reproduce correctly the entropy of spaces with horizons. The entanglement entropy of regions that cover the whole space inside the horizons can be identified with the standard thermal entropy of these spaces. For a precise quantitative agreement, one must define the effective Newton’s constant appropriately and account carefully for the way the AdS space is covered by the coordinates. In this picture the UV divergence of the entanglement entropy is reflected in the suppression of the effective Newton’s constant by the infinite volume near the AdS boundary, an IR effect.

Acknowledgments

We would like to thank E. Kiritsis, I. Papadimitriou, T. Tomaras and N. Toumbas for useful discussions.

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