Dirac System Associated with Hahn Difference Operator

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Abstract
In this paper, we introduce \( q, \omega \)–Dirac system. We investigate the existence and uniqueness of solutions for this system and obtain some spectral properties based on the Hahn difference operator. Also we give two examples, which indicate that asymptotic formulas for eigenvalues.

Keywords
Hahn difference operator · \( q, \omega \)–Dirac system · Eigenvalue problems

Mathematics Subject Classification
Primary 39A13 · 34B09; Secondary 35Q41 · 33D15

1 Introduction and Preliminaries
In [1,2], Hahn introduced the difference operator \( D_{q,\omega} \) which is defined by

\[
D_{q,\omega} f (t) := \begin{cases} 
  \frac{f(qt + \omega) - f(t)}{(qt + \omega) - t}, & t \neq \omega_0, \\
  f'(\omega_0), & t = \omega_0,
\end{cases}
\]

where \( q \in (0, 1) \), \( \omega > 0 \) are fixed and \( \omega_0 := \omega/(1 - q) \). This is valid if \( f \) is differentiable at \( \omega_0 \). In this case, we call \( D_{q,\omega} f \), the \( q, \omega \)-derivative of \( f \). This operator extends the forward difference operator

\[
\Delta_{\omega} f (t) := \frac{f(t + \omega) - f(t)}{(t + \omega) - t},
\]

where \( t \neq \omega_0 \).
where $\omega > 0$ is fixed (see [3–6]) as well as Jackson $q$-difference operator

$$D_q f (t) := \frac{f (qt) - f (t)}{t (q - 1)},$$

where $q \in (0, 1)$ is fixed (see [7–12]).

In [13], the authors gave a rigorous analysis of Hahn’s difference operator and the associated calculus. The existence and uniqueness theorems for general first-order $q, \omega$-initial value problems and the theory of linear Hahn difference equations were studied in [14,15], respectively. Recently, a $q, \omega$-Sturm–Liouville theory has been established in [16], sampling theorems associated with $q, \omega$-Sturm–Liouville problems in the regular setting have been derived in [17] and fractional Hahn calculus have been investigated in [18–20].

In [21], the authors presented the $q$-analog of the one-dimensional Dirac system:

$$\begin{cases} 
- \frac{1}{q} D_{q^{-1}} y_2 + p (x) y_1 = \lambda y_1, \\
D_q y_1 + r (x) y_2 = \lambda y_2, \\
k_{11} y_1 (0) + k_{12} y_2 (0) = 0, \\
k_{21} y_1 (a) + k_{22} y_2 (a q^{-1}) = 0,
\end{cases}$$

where $k_{ij} (i, j = 1, 2)$ are real numbers, $y (x) = \begin{pmatrix} y_1 (x) \\ y_2 (x) \end{pmatrix}$, $0 \leq x \leq a < \infty$. They also gave the existence and uniqueness of the solution and some spectral properties of this system. For the same $q$-Dirac system (1.4)–(1.6), asymptotic formulas for the eigenvalues and the eigenfunctions were investigated in [22] and sampling theory was derived in [23].

In this paper, we introduce a $q, \omega$-version of $q$-Dirac system (1.4)–(1.6). When the $q$-difference operator $D_q$ is replaced by the Hahn difference operator $D_{q, \omega}$, we obtain the following $q, \omega$-Dirac system. Namely, we obtain the system which consists of the $q, \omega$–Dirac equations

$$\begin{cases} 
- \frac{1}{q^1_{q^{-1}}} \frac{1}{q}^{-1} y_2 + p (t) y_1 = \lambda y_1, \\
D_{q, \omega} y_1 + r (t) y_2 = \lambda y_2,
\end{cases}$$

and the boundary conditions

$$\begin{align*}
B_1 (y) & := k_{11} y_1 (\omega_0) + k_{12} y_2 (\omega_0) = 0, \\
B_2 (y) & := k_{21} y_1 (a) + k_{22} y_2 \left( h^{-1} (a) \right) = 0,
\end{align*}$$

where $\omega_0 \leq t \leq a < \infty$, $k_{ij} (i, j = 1, 2)$ are real numbers, $\lambda \in \mathbb{C}$, $p (.)$ and $r (.)$ are real-valued functions defined on $[\omega_0, a]$ and continuous at $\omega_0$, and $h (t)$ is a function defined below. We will establish an existence and uniqueness of the solution.
of the $q$, $\omega$-Dirac equation (1.7). Also we will discuss some spectral properties of the

eigenvalues and the eigenfunctions of this system. Finally, we will give two examples,

which indicate that asymptotic formulas for the eigenvalues.

Let $h(t) := qt + \omega, \ t \in I$; an interval of $\mathbb{R}$ containing $\omega_0$. One can see that

$k$th order iteration of $h(t)$ is given by

$h^k(t) = q^k t + \omega [k]_q, \ t \in I$.

The sequence $h^k(t)$ is uniformly convergent to $\omega_0$ on $I$. Here, $[k]_q$ is defined by

$[k]_q = \frac{1 - q^k}{1 - q}$.

The following relation directly follows from the definition $D_{q,\omega}$

$$
(D_{q,\omega}f)(h^{-1}(t)) = D_{\frac{1}{q}, -\omega} f(t),
$$

where $h^{-1}(t) := (t - \omega)/q, \ t \in I$. The $q, \omega$-type product formula is given by

$$
D_{q,\omega}(fg)(t) = D_{q,\omega}(f(t)) g(t) + f(qt + \omega) D_{q,\omega} g(t).
$$

The $q, \omega$-integral is introduced in [13,24] to be the Jackson–Nörlund sum

$$
\int_a^b f(t) d_{q,\omega} t = \int_{\omega_0}^b f(t) d_{q,\omega} t - \int_{\omega_0}^a f(t) d_{q,\omega} t,
$$

where $\omega_0 < a < b, \ a, b \in I$, and

$$
\int_{\omega_0}^x f(t) d_{q,\omega} t = (x (1 - q) - \omega) \sum_{k=0}^{\infty} q^k f(x q^k + \omega [k]_q), \ x \in I,
$$

provided that the series converges. The fundamental theorem of $q, \omega$-calculus given in [13,24] states that if $f : I \to \mathbb{R}$ is continuous at $\omega_0$, and

$$
F(t) := \int_{\omega_0}^t f(x) d_{q,\omega} x, \ x \in I,
$$

then $F$ is continuous at $\omega_0$. Furthermore, $D_{q,\omega} F(t)$ exists for every $t \in I$ and

$$
D_{q,\omega} F(t) = f(t).
$$
Conversely,

\[
\int_{a}^{b} D_{q,\omega} f(t) \, dq,\omega t = f(b) - f(a), \quad \text{for all } a, b \in I. \tag{1.16}
\]

The \( q,\omega \)-integration by parts for continuous functions \( f, g \) is given in \([13,24]\) by

\[
\int_{a}^{b} f(t) D_{q,\omega} g(t) \, dq,\omega t = f(t) g(t) \bigg|_{a}^{b} - \int_{a}^{b} D_{q,\omega} (f(t)) g(qt + \omega) \, dq,\omega t, \quad a, b \in I. \tag{1.17}
\]

Trigonometric functions of \( q,\omega \)-cosine and sine are defined by

\[
C_{q,\omega}(t, \mu) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (\mu (t (1 - q) - \omega))^{2n}}{(q; q)_{2n}}, \quad t \in \mathbb{C}, \tag{1.18}
\]

\[
S_{q,\omega}(t, \mu) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)} (\mu (t (1 - q) - \omega))^{2n+1}}{(q; q)_{2n+1}}, \quad t \in \mathbb{C}. \tag{1.19}
\]

Here, \((q; q)_k\) is the \( q \)-shifted factorial

\[
(q; q)_k := \begin{cases} 1, & k = 0, \\ \prod_{j=1}^{k} (1 - q^j), & k = 1, 2, \ldots . \end{cases} \tag{1.20}
\]

Furthermore, these functions are a fundamental set of solutions for

\[
-\frac{1}{q} D_{\frac{1}{q},\sqrt{q}} D_{q,\omega} y(t) - \mu^2 y(t) = 0, \tag{1.21}
\]

and satisfy

\[
D_{q,\omega} S_{q,\omega}(t, \mu) = \mu C_{q,\omega}(t, \sqrt{q} \mu), \quad D_{q,\omega} C_{q,\omega}(t, \mu) = -\sqrt{q} \mu S_{q,\omega}(t, \sqrt{q} \mu). \tag{1.22}
\]

Let \( a > 0 \) be fixed and \( L_{q,\omega}^2(\omega_0, a) \) be the set of all complex valued functions defined on \([\omega_0, a]\) for which

\[
\| f( \cdot ) \| = \left( \int_{\omega_0}^{a} |f(t)|^2 \, dq,\omega t \right)^{1/2} < \infty.
\]
The space \( L^2_{q,\omega} (\omega_0, a) \) is a separable Hilbert space with inner product
\[
\langle f, g \rangle := \int_{\omega_0}^{a} f(t) \overline{g(t)} d_{q,\omega} t, \quad f, g \in L^2_{q,\omega} (\omega_0, a),
\]
where \( \overline{z} \) denotes the complex conjugate of \( z \in \mathbb{C} \) (see [16]). Let \( C^2_{q,\omega} (\omega_0, a) \) be the subspace of \( L^2_{q,\omega} (\omega_0, a) \), which consists of all functions \( y(.) \) for which \( y(.) \), \( D_{q,\omega} y(.) \) are continuous at \( \omega_0 \). Let \( H_{q,\omega} \) be the Hilbert space
\[
H_{q,\omega} := \left\{ y(.) = \begin{pmatrix} y_1(.) \\ y_2(.) \end{pmatrix}, \ y_1, y_2 \in C^2_{q,\omega} (\omega_0, a) \right\}.
\]
The inner product of \( H_{q,\omega} \) is defined by
\[
\langle y(.), z(.) \rangle_{H_{q,\omega}} := \int_{\omega_0}^{a} y^\top (t) z(t) d_{q,\omega} t,
\]
where \( \top \) denotes the matrix transpose.

### 2 Fundamental Solutions and Spectral Properties

In this section, we give an existence and uniqueness theorem of the \( q, \omega \)-Dirac equation (1.7).

Let \( y(.) = \begin{pmatrix} y_1(.) \\ y_2(.) \end{pmatrix} \), \( z(.) = \begin{pmatrix} z_1(.) \\ z_2(.) \end{pmatrix} \) \( H_{q,\omega} \). Then, the \( q, \omega \)-Wronskian of \( y(.) \) and \( z(.) \) is defined by
\[
W_{q,\omega} (y, z) (t) := y_1(t) z_2 \left( h^{-1} (t) \right) - z_1(t) y_2 \left( h^{-1} (t) \right).
\]

**Lemma 2.1** The \( q, \omega \)-Wronskian of solutions of the \( q, \omega \)-Dirac equation (1.7) is independent of both \( t \) and \( \lambda \).

**Proof** Let \( y(t, \lambda) = \begin{pmatrix} y_1(t, \lambda) \\ y_2(t, \lambda) \end{pmatrix} \) and \( z(t, \lambda) = \begin{pmatrix} z_1(t, \lambda) \\ z_2(t, \lambda) \end{pmatrix} \) be two solutions of the \( q, \omega \)-Dirac equation (1.7). From (2.1) and (1.11), we have
\[
D_{q,\omega} W_{q,\omega} (y, z) (t, \lambda) = D_{q,\omega} \begin{pmatrix} z_2 \left( h^{-1} (t, \lambda) \right) y_1(t, \lambda) + z_2(t, \lambda) D_{q,\omega} y_1(t, \lambda) \\ -D_{q,\omega} \left( y_2 \left( h^{-1} (t, \lambda) \right) \right) z_1(t, \lambda) - y_2(t, \lambda) D_{q,\omega} z_1(t, \lambda) \end{pmatrix}.
\]

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Since \( y(t, \lambda) \) and \( z(t, \lambda) \) are solutions of the \( q, \omega \)-Dirac equation (1.7) and by using (1.10), we get
\[
D_{q, \omega} W_{q, \omega} (y, z) (t, \lambda) = 0. \tag{2.3}
\]
Therefore, \( W_{q, \omega} (y, z) (t, \lambda) \) is a constant. \( \square \)

**Corollary 2.1** \{\( y(t, \lambda), z(t, \lambda) \)\} forms a fundamental set of solutions of the \( q, \omega \)-Dirac equation if and only if their \( q, \omega \)-Wronskian does not vanish at any point of \([\omega_0, a]\).

**Theorem 2.2** For \( \lambda \in \mathbb{C} \), the \( q, \omega \)-Dirac equation (1.7) has a unique solution
\[
\phi(t, \lambda) = \begin{pmatrix} \phi_1(t, \lambda) \\ \phi_2(t, \lambda) \end{pmatrix}
\]
subject to the initial conditions
\[
\phi_1(\omega_0, \lambda) = c_1, \quad \phi_2(\omega_0, \lambda) = c_2, \tag{2.4}
\]
where \( c_1, c_2 \in \mathbb{C} \).

**Proof** Let
\[
\varphi_1(t, \lambda) = \begin{pmatrix} \varphi_{11}(t, \lambda) \\ \varphi_{12}(t, \lambda) \end{pmatrix} = \begin{pmatrix} C_{q, \omega}(t, \lambda) \\ -\sqrt{q} S_{q, \omega}(t, \sqrt{q} \lambda) \end{pmatrix},
\]
\[
\varphi_2(t, \lambda) = \begin{pmatrix} \varphi_{21}(t, \lambda) \\ \varphi_{22}(t, \lambda) \end{pmatrix} = \begin{pmatrix} S_{q, \omega}(t, \lambda) \\ C_{q, \omega}(t, \sqrt{q} \lambda) \end{pmatrix}. \tag{2.5}
\]

Then, the function
\[
\varphi(t, \lambda) = \begin{pmatrix} \varphi_1(t, \lambda) \\ \varphi_2(t, \lambda) \end{pmatrix} = \begin{pmatrix} c_1 \varphi_{11}(t, \lambda) + c_2 \varphi_{21}(t, \lambda) \\ c_1 \varphi_{12}(t, \lambda) + c_2 \varphi_{22}(t, \lambda) \end{pmatrix}, \tag{2.6}
\]
is a fundamental set of the \( q, \omega \)-Dirac equation (1.7) for \( p(t) = r(t) = 0 \). It is not hard to see that \( W_{q, \omega}(\varphi_1, \varphi_2)(., \lambda) \equiv 1 \).

Define the sequence \( \{\psi_m(., \lambda)\}_{m=1}^{\infty} = \begin{pmatrix} \psi_{m1}(., \lambda) \\ \psi_{m2}(., \lambda) \end{pmatrix} \) of successive approximations by
\( \square \) Springer
\[
\psi_1 (t, \lambda) = \begin{pmatrix} \psi_{11} (t, \lambda) \\ \psi_{12} (t, \lambda) \end{pmatrix} = \begin{pmatrix} c_1 \psi_{11} (t, \lambda) + c_2 \psi_{21} (t, \lambda) \\ c_1 \psi_{12} (t, \lambda) + c_2 \psi_{22} (t, \lambda) \end{pmatrix},
\]
\[
\psi_{m+1} (t, \lambda) = \begin{pmatrix} \psi_{(m+1)1} (t, \lambda) \\ \psi_{(m+1)2} (t, \lambda) \end{pmatrix} = \begin{pmatrix} \psi_{11} (t, \lambda) + q \int_{\omega_0}^{t} \{ \varphi_{21} (t, \lambda) \varphi_{11} (h (s), \lambda) \\
- \varphi_{11} (t, \lambda) \varphi_{21} (h (s), \lambda) \} p (h (s)) \psi_{m1} (h (s), \lambda) \, dq : ws \\\n+ \int_{\omega_0}^{t} \{ \varphi_{21} (t, \lambda) \varphi_{12} (s, \lambda) - \varphi_{11} (t, \lambda) \varphi_{22} (s, \lambda) \} r (s) \psi_{m2} (s, \lambda) \, dq : ws \end{pmatrix},
\]
\[
\psi_{(m+1)2} (t, \lambda) + q \int_{\omega_0}^{t} \{ \varphi_{22} (t, \lambda) \varphi_{11} (h (s), \lambda) \\
- \varphi_{12} (t, \lambda) \varphi_{21} (h (s), \lambda) \} p (h (s)) \psi_{m1} (h (s), \lambda) \, dq : ws \\\n+ \int_{\omega_0}^{t} \{ \varphi_{22} (t, \lambda) \varphi_{12} (s, \lambda) - \varphi_{12} (t, \lambda) \varphi_{22} (s, \lambda) \} r (s) \psi_{m2} (s, \lambda) \, dq : ws \end{pmatrix}
\]
\[
m = 1, 2, 3, \ldots \text{ For a fixed } \lambda \in \mathbb{C}, \text{ there exist positive numbers } B (\lambda), A_1, A_2 \text{ and } A \text{ independent of } t \text{ such that}
\]
\[
|p (t)| \leq A_1, \quad |r (t)| \leq A_2, \quad A = \max \{ A_1, A_2 \},
\]
\[
|\varphi_{ij} (t, \lambda)| \leq \sqrt{\frac{B (\lambda)}{2}} \, i, \quad j = 1, 2, \quad t \in [\omega_0, a].
\]

Let \( K (\lambda) := (|c_1| + |c_2|) \sqrt{\frac{B (\lambda)}{2}} \). Then, from (2.9), \(|\psi_{11} (t, \lambda)| \leq K (\lambda) \) and \(|\psi_{12} (t, \lambda)| \leq K (\lambda) \). Using mathematical induction, we prove that
\[
|\psi_{(m+1)1} (t, \lambda) - \psi_{m1} (t, \lambda)| \leq \prod_{n=1}^{m} (1 + q^n) \, K (\lambda) \, \frac{(AB (\lambda) (t (1 - q) - \omega))^m}{(q : q)_m},
\]
\[
|\psi_{(m+1)2} (t, \lambda) - \psi_{m2} (t, \lambda)| \leq \prod_{n=1}^{m} (1 + q^n) \, K (\lambda) \, \frac{(AB (\lambda) (t (1 - q) - \omega))^m}{(q : q)_m}, \quad m = 1, 2, \ldots.
\]

First, triangle inequalities implies
\[
|\psi_{21} (t, \lambda) - \psi_{11} (t, \lambda)|
\]
\[
\leq q \int_{\omega_0}^{t} \{ \varphi_{21} (t, \lambda) \varphi_{11} (h (s), \lambda) \\
- \varphi_{11} (t, \lambda) \varphi_{21} (h (s), \lambda) \} p (h (s)) \psi_{11} (h (s), \lambda) \, dq : ws \\\n+ \int_{\omega_0}^{t} \{ \varphi_{21} (t, \lambda) \varphi_{12} (s, \lambda) - \varphi_{11} (t, \lambda) \varphi_{22} (s, \lambda) \} r (s) \psi_{12} (s, \lambda) \, dq : ws
\]
\[ \leq q B (\lambda) A K (\lambda) \int_{\omega_0}^{t} d_{q,\omega s} + B (\lambda) A K (\lambda) \int_{\omega_0}^{t} d_{q,\omega s} \]
\[ \leq q A B (\lambda) K (\lambda) \frac{(t (1 - q) - \omega)}{1 - q} + A B (\lambda) K (\lambda) \frac{(t (1 - q) - \omega)}{1 - q} \]
\[ \leq A B (\lambda) K (\lambda) \frac{(t (1 - q) - \omega)}{1 - q} (1 + q) . \]

Assume the correctness of (2.10) for some \( m \). Then,

\[ \left| \psi_{(m+2)1} (t, \lambda) - \psi_{(m+1)1} (t, \lambda) \right| \]
\[ \leq q \left| \int_{\omega_0}^{t} \{ \varphi_{21} (t, \lambda) \varphi_{11} (h (s), \lambda) - \varphi_{11} (t, \lambda) \varphi_{21} (h (s), \lambda) \} p (h (s)) \right| \]
\[ \times \left| (\psi_{(m+1)1} (h (s), \lambda) - \psi_{m1} (h (s), \lambda)) d_{q,\omega s} \right| \]
\[ + \left| \int_{\omega_0}^{t} \{ \varphi_{21} (t, \lambda) \varphi_{12} (s, \lambda) - \varphi_{11} (t, \lambda) \varphi_{22} (s, \lambda) \} r (s) (\psi_{(m+1)2} (s, \lambda) - \psi_{m2} (s, \lambda)) d_{q,\omega s} \right| \]
\[ \leq q B (\lambda) A \int_{\omega_0}^{t} \left| (\psi_{(m+1)1} (h (s), \lambda) - \psi_{m1} (h (s), \lambda)) d_{q,\omega s} \right| \]
\[ + B (\lambda) A \int_{\omega_0}^{t} \left| (\psi_{(m+1)2} (s, \lambda) - \psi_{m2} (s, \lambda)) d_{q,\omega s} \right| \]
\[ \leq q A B (\lambda) K (\lambda) \frac{(AB (\lambda))^m}{(q; q)_m} \prod_{n=1}^{m} (1 + q^n) \int_{\omega_0}^{t} (h (s) (1 - q) - \omega)^m d_{q,\omega s} \]
\[ + A B (\lambda) K (\lambda) \frac{(AB (\lambda))^m}{(q; q)_m} \prod_{n=1}^{m} (1 + q^n) \int_{\omega_0}^{t} (s (1 - q) - \omega)^m d_{q,\omega s} \]
\[ \leq q K (\lambda) \frac{(AB (\lambda))^{m+1}}{(q; q)_m} \prod_{n=1}^{m} (1 + q^n) (t (1 - q) - \omega)^{m+1} \frac{q^m}{1 - q^{m+1}} \]
\[ + K (\lambda) \frac{(AB (\lambda))^{m+1}}{(q; q)_m} \prod_{n=1}^{m} (1 + q^n) (t (1 - q) - \omega)^{m+1} \frac{1}{1 - q^{m+1}} \]
\[ \leq \prod_{n=1}^{m+1} (1 + q^n) K (\lambda) \frac{(AB (\lambda) (t (1 - q) - \omega))^{m+1}}{(q; q)_{m+1}} . \]
Therefore, inequality (2.10) is true for every \( m \in \mathbb{N} \). To guarantee the convergence of the terms on the right side in (2.10), we assume additionally that

\[
|t - \omega_0| < \frac{1}{|AB(\lambda)(1-q)|}, \quad t \in [\omega_0, a].
\]  

(2.11)

Consequently by Weierstrass’ test, the series

\[
\psi_{11}(t, \lambda) + \sum_{m=1}^{\infty} \psi_{(m+1)1}(t, \lambda) - \psi_{m1}(t, \lambda)
\]

\[
\psi_{12}(t, \lambda) + \sum_{m=1}^{\infty} \psi_{(m+1)2}(t, \lambda) - \psi_{m2}(t, \lambda)
\]

(2.12)

converges uniformly in \([\omega_0, a]\). Since the \( m \)-th partial sums of the series is \( \psi_{m+1}(t, \lambda) = \left( \begin{array}{c} \psi_{(m+1)1}(t, \lambda) \\ \psi_{(m+1)2}(t, \lambda) \end{array} \right) \), then \( \psi_{m+1}(., \lambda) \) approaches a function \( \phi(., \lambda) = \left( \begin{array}{c} \phi_1(., \lambda) \\ \phi_2(., \lambda) \end{array} \right) \) uniformly in \([\omega_0, a]\). We can also prove by induction on \( m \) that \( \psi_{mi}(t, \lambda) \) and \( D_{q,\omega}\psi_{mi}(t, \lambda) (i = 1, 2) \) are continuous at \( \omega_0 \), where

\[
D_{q,\omega}\psi_{(m+1)1}(t, \lambda) = c_1D_{q,\omega}\psi_{11}(t, \lambda) + c_2D_{q,\omega}\psi_{21}(t, \lambda)
\]

\[
+ q \left\{ \int_{\omega}^{t} \{ D_{q,\omega}(\psi_{21}(t, \lambda)) \phi_{11}(h(s), \lambda) \\ - D_{q,\omega}(\psi_{11}(t, \lambda)) \phi_{21}(h(s), \lambda) \} p(h(s)) \psi_{m1}(h(s), \lambda) d_{q,\omega}s \right\}
\]

\[
+ \int_{\omega}^{t} \{ D_{q,\omega}(\psi_{21}(t, \lambda)) \phi_{12}(s, \lambda) - D_{q,\omega}(\psi_{11}(t, \lambda)) \phi_{22}(s, \lambda) \} r(s) \psi_{m2}(s, \lambda) d_{q,\omega}s
\]

(2.13)

\[
D_{q,\omega}\psi_{(m+1)2}(t, \lambda) = c_1D_{q,\omega}\psi_{12}(t, \lambda) + c_2D_{q,\omega}\psi_{22}(t, \lambda)
\]

\[
+ q \left\{ \int_{\omega}^{t} \{ D_{q,\omega}(\psi_{22}(t, \lambda)) \phi_{11}(h(s), \lambda) \\ - D_{q,\omega}(\psi_{12}(t, \lambda)) \phi_{21}(h(s), \lambda) \} p(h(s)) \psi_{m1}(h(s), \lambda) d_{q,\omega}s \right\}
\]

\[
+ \int_{\omega}^{t} \{ D_{q,\omega}(\psi_{22}(t, \lambda)) \phi_{12}(s, \lambda) - D_{q,\omega}(\psi_{12}(t, \lambda)) \phi_{22}(s, \lambda) \} r(s) \psi_{m2}(s, \lambda) d_{q,\omega}s
\]

(2.14)

\( m = 1, 2, 3, \ldots \). Therefore, the functions \( \phi_i(., \lambda) \) and \( D_{q,\omega}(\phi_i(., \lambda))) (i = 1, 2) \) are continuous at \( \omega_0 \), i.e., \( \phi(., \lambda) \in C_{q,\omega}^2(\omega_0, a) \). Let \( m \to \infty \) in (2.8), we obtain

\[
\phi(t, \lambda) = \begin{pmatrix} \phi_1(t, \lambda) \\ \phi_2(t, \lambda) \end{pmatrix}
\]
Φ_1 (t, λ) = c_1 ϕ_{11} (t, λ) + c_2 ϕ_{21} (t, λ) + q \int_{ω_0}^{t} \{ ϕ_{21} (t, λ) ϕ_{11} (h (s), λ) - ϕ_{11} (t, λ) ϕ_{21} (h (s), λ) \} p (h (s)) φ_1 (h (s), λ) d_{q,ωs} + \int_{ω_0}^{t} \{ ϕ_{21} (t, λ) ϕ_{12} (s, λ) - ϕ_{11} (t, λ) ϕ_{22} (s, λ) \} r (s) φ_2 (s, λ) d_{q,ωs} \tag{2.15}

Φ_2 (t, λ) = c_1 ϕ_{12} (t, λ) + c_2 ϕ_{22} (t, λ) + q \int_{ω_0}^{t} \{ ϕ_{22} (t, λ) ϕ_{11} (h (s), λ) - ϕ_{12} (t, λ) ϕ_{22} (h (s), λ) \} p (h (s)) φ_1 (h (s), λ) d_{q,ωs} + \int_{ω_0}^{t} \{ ϕ_{22} (t, λ) ϕ_{12} (s, λ) - ϕ_{12} (t, λ) ϕ_{22} (s, λ) \} r (s) φ_2 (s, λ) d_{q,ωs} \tag{2.16}

Now we prove that Φ (., λ) solves the q, ω-Dirac equation (1.7) subject to (2.4). It is not hard to see that Φ_1 (ω_0, λ) = c_1 and Φ_2 (ω_0, λ) = c_2. From (2.15) and using (1.11) and (1.15), we have

\[ D_{q,ω} φ_1 (t, λ) = c_1 D_{q,ω} ϕ_{11} (t, λ) + c_2 D_{q,ω} ϕ_{21} (t, λ) + q \int_{ω_0}^{t} \{ D_{q,ω} (ϕ_{21} (t, λ)) ϕ_{11} (h (s), λ) - D_{q,ω} (ϕ_{11} (t, λ)) ϕ_{21} (h (s), λ) \} p (h (s)) φ_1 (h (s), λ) d_{q,ωs} + \int_{ω_0}^{t} \{ D_{q,ω} (ϕ_{21} (t, λ)) ϕ_{12} (s, λ) - D_{q,ω} (ϕ_{12} (t, λ)) ϕ_{22} (s, λ) \} r (s) φ_2 (s, λ) d_{q,ωs} + \{ ϕ_{21} (h (t), λ) ϕ_{12} (t, λ) - ϕ_{11} (h (t), λ) ϕ_{22} (t, λ) \} r (t) φ_2 (t, λ) \tag{2.17} \]

Since the function (2.6) is a fundamental solution set of the q, ω-Dirac equation (1.7) for p (t) = 0 ≡ r (t) and from Corollary 2.2, we get

\[ D_{q,ω} φ_1 (t, λ) = λ ϕ_2 (t, λ) - r (t) φ_2 (t, λ) \tag{2.18} \]

The validity of the other equation in the q, ω-Dirac equation (1.7) is proved similarly.

For the uniqueness assume Y (t, λ) = \begin{pmatrix} Y_1 (t, λ) \\ Y_2 (t, λ) \end{pmatrix} and Z (t, λ) = \begin{pmatrix} Z_1 (t, λ) \\ Z_2 (t, λ) \end{pmatrix} are two solutions of the q, ω-Dirac equation (1.7) together with (2.4). Applying the q, ω-
integration to the \( q, \omega \)-Dirac equation (1.7) yields

\[
Y_1(t, \lambda) = c_1 + \int_{\omega_0}^{t} (\lambda - r(s)) Y_2(s, \lambda) \, dq, \omega \, ds, \quad (2.19)
\]

\[
Y_2(t, \lambda) = c_2 + q \int_{\omega_0}^{t} (p(h(s)) - \lambda) Y_1(h(s), \lambda) \, dq, \omega \, ds, \quad (2.20)
\]

and

\[
Z_1(t, \lambda) = c_1 + \int_{\omega_0}^{t} (\lambda - r(s)) Z_2(s, \lambda) \, dq, \omega \, ds, \quad (2.21)
\]

\[
Z_2(t, \lambda) = c_2 + q \int_{\omega_0}^{t} (p(h(s)) - \lambda) Z_1(h(s), \lambda) \, dq, \omega \, ds. \quad (2.22)
\]

Since \( Y(t, \lambda), Z(t, \lambda), p(t) \) and \( r(t) \) are continuous at \( \omega_0 \), then exist positive numbers \( N_{\lambda,t}, M_{\lambda,t} \) such that

\[
\sup_{n \in \mathbb{N}} \left| Y_i(h^n(t), \lambda) \right| = N_{i\lambda,t}, \quad \sup_{n \in \mathbb{N}} \left| Z_i(h^n(t), \lambda) \right| = \tilde{N}_{i\lambda,t},
\]

\[
N_{\lambda,t} = \max \{ N_{i\lambda,t}, \tilde{N}_{i\lambda,t} \}, \quad i = 1, 2, \quad (2.23)
\]

\[
\sup_{n \in \mathbb{N}} \left| \lambda - r(h^n(t)) \right| = M_{1\lambda,t}, \quad \sup_{n \in \mathbb{N}} \left| \lambda - p(h^n(t)) \right| = M_{2\lambda,t},
\]

\[
M_{\lambda,t} = \max \{ M_{1\lambda,t}, M_{2\lambda,t} \}. \quad (2.24)
\]

We can prove by mathematical induction on \( k \) that

\[
|Y_1(t, \lambda) - Z_1(t, \lambda)| \leq 2q^k \left[ \frac{2^k}{k!} \right] M_{\lambda,t}^k N_{\lambda,t} \frac{(t(1-q) - \omega)^k}{(q; q)_k}, \quad (2.25)
\]

\[
|Y_2(t, \lambda) - Z_2(t, \lambda)| \leq 2q^k \left[ \frac{2^k+4k+(-1)^{k+1}+1}{k!} \right] M_{\lambda,t}^k N_{\lambda,t} \frac{(t(1-q) - \omega)^k}{(q; q)_k}, \quad (2.26)
\]

\( k \in \mathbb{N}, \ t \in [\omega_0, a] \). Indeed, if (2.25) holds at \( k \in \mathbb{N} \), then from (2.19) and (2.21) we have

\[
|Y_1(t, \lambda) - Z_1(t, \lambda)| \leq M_{\lambda,t} 2q^k \left[ \frac{2^k+4k+(-1)^{k+1}+1}{k!} \right] M_{\lambda,t}^k N_{\lambda,t} \int_{\omega_0}^{t} (s(1-q) - \omega)^k \, dq, \omega \, ds.
\]
Similarly, if (2.26) holds at \( k \in \mathbb{N} \), then from (2.20) and (2.22) we have

\[
|Y_2(t, \lambda) - Z_2(t, \lambda)|
\leq q M_{A,t} 2q \left| \frac{M_{A,t} N_{A,t}}{(q; q)_k} \right| \int_{a_0}^{t} (h(s) - \omega)^k d_{q,\omega} s
\]

\[
= 2q q \left| \frac{t^2}{q} \right| M_{A,t} N_{A,t} (q; q)_k \sum_{n=1}^{\infty} (t - \omega)^n q^{(n+1)k} (t - \omega)^k
\]

\[
= 2q k^{2+4k+(-1)^{k+1}+1} \left| \frac{t^2}{q} \right| M_{A,t} N_{A,t} (t - \omega)^k (q; q)_{k+1}^{k+1}.
\]

Hence, (2.25) and (2.26) hold true at \( k + 1 \). Consequently, (2.25) and (2.26) are true for all \( k \in \mathbb{N} \) because from (2.23) it is satisfied at \( k = 0 \). Since

\[
\lim_{k \to \infty} 2q \left| \frac{t^2}{q} \right| M_{A,t} N_{A,t} (t - \omega)^k (q; q)_k = 0,
\]

and

\[
\lim_{k \to \infty} 2q \left| \frac{t^2}{q} \right| M_{A,t} N_{A,t} (t - \omega)^k (q; q)_k = 0,
\]

then \( Y_1(t, \lambda) = Z_1(t, \lambda) \) and \( Y_2(t, \lambda) = Z_2(t, \lambda) \) for all \( t \in [\omega_0, a] \), i.e., \( Y(t, \lambda) = Z(t, \lambda) \). This proves the uniqueness. \( \square \)

**Lemma 2.3** If \( \lambda_1 \) and \( \lambda_2 \) are two different eigenvalues of the \( q, \omega \)-Dirac system (1.7)–(1.9), then the corresponding eigenfunctions \( y(t, \lambda_1) \) and \( z(t, \lambda_2) \) are orthogonal, i.e.,

\[
\int_{\omega_0}^{a} y^\top(t, \lambda_1) z(t, \lambda_2) d_{q,\omega} t = 0
\]

\[
\int_{\omega_0}^{a} y^\top(t, \lambda_1) z(t, \lambda_2) d_{q,\omega} t = 0,
\]

where \( y^\top = (y_1, y_2) \).

**Proof** Since \( y(t, \lambda_1) \) and \( z(t, \lambda_2) \) are solutions of the \( q, \omega \)-Dirac system (1.7)–(1.9)

\[
\left\lbrace \begin{array}{l}
-\frac{1}{q} D_{q,\omega} y_2(t, \lambda_1) + \{ p(t) - \lambda_1 \} y_1(t, \lambda_1) = 0, \\
D_{q,\omega} y_1(t, \lambda_1) + \{ r(t) - \lambda_1 \} y_2(t, \lambda_1) = 0,
\end{array} \right.
\]

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and
\[
\begin{align*}
\begin{cases}
- \frac{1}{q} \frac{D_1}{D_{0^+}} \omega \ z_2 (t, \lambda_2) + \{ p (t) - \lambda_2 \} \ z_1 (t, \lambda_2) = 0,
D_{q, \omega} z_1 (t, \lambda_2) + \{ r (t) - \lambda_2 \} \ z_2 (t, \lambda_2) = 0.
\end{cases}
\end{align*}
\]

Multiplying by \( z_1, z_2, -y_1 \) and \(-y_2 \), respectively, and adding together we have
\[
\begin{align*}
- \frac{1}{q} \frac{D_1}{D_{0^+}} \omega \ (y_2 (t, \lambda_1)) \ z_1 (t, \lambda_2) + D_{q, \omega} (y_1 (t, \lambda_1)) \ z_2 (t, \lambda_2) \\
+ \frac{1}{q} \frac{D_1}{D_{0^+}} \omega \ (z_2 (t, \lambda_2)) \ y_1 (t, \lambda_1) - D_{q, \omega} (z_1 (t, \lambda_2)) \ y_2 (t, \lambda_1) \\
= \lambda_1 \ (y_1 (t, \lambda_1)) \ z_1 (t, \lambda_2) + y_2 (t, \lambda_1) \ z_2 (t, \lambda_2) - \lambda_2 \ (y_1 (t, \lambda_1)) \ z_1 (t, \lambda_2) + y_2 (t, \lambda_1) \ z_2 (t, \lambda_2) .
\end{align*}
\]

Using (1.10) and (1.11), we obtain
\[
\begin{align*}
D_{q, \omega} \left\{ y_1 (t, \lambda_1) \ z_2 (h^{-1} (t), \lambda_2) - y_2 (h^{-1} (t), \lambda_1) \ z_1 (t, \lambda_2) \right\} \\
= (\lambda_1 - \lambda_2) \ (y_1 (t, \lambda_1)) \ z_1 (t, \lambda_2) + y_2 (t, \lambda_1) \ z_2 (t, \lambda_2) .
\end{align*}
\]

Applying the \( q, \omega \)-integration to (2.33) yields
\[
\begin{align*}
(\lambda_1 - \lambda_2) \int_{\omega_0}^a \ y_1 (t, \lambda_1) \ z_1 (t, \lambda_2) + y_2 (t, \lambda_1) \ z_2 (t, \lambda_2) \ d_{q, \omega} t \\
= \left\{ y_1 (t, \lambda_1) \ z_2 (h^{-1} (t), \lambda_2) - y_2 (h^{-1} (t), \lambda_1) \ z_1 (t, \lambda_2) \right\} \bigg|_{\omega_0}^a .
\end{align*}
\]

It follows from the boundary conditions (1.8) and (1.9) the right-hand side vanishes. It is concluded that
\[
(\lambda_1 - \lambda_2) \int_{\omega_0}^a \ y^T (t, \lambda_1) \ z (t, \lambda_2) \ d_{q, \omega} t = 0 .
\]

The lemma is thus proved, since \( \lambda_1 \neq \lambda_2 \).

\begin{lemma}
The eigenvalues of the \( q, \omega \)-Dirac system (1.7)–(1.9) are real.
\end{lemma}

\begin{proof}
Assume the contrary that \( \lambda_0 \) is a nonreal eigenvalue of the \( q, \omega \)-system (1.7)–(1.9). Let \( y (t, \lambda_0) \) be a corresponding (nontrivial) eigenfunction. \( \overline{\lambda_0} \) is also an eigenvalue, corresponding to the eigenfunction \( \overline{y} (t, \lambda_0) \). Since \( \lambda_0 \neq \overline{\lambda_0} \) by the previous lemma
\[
\int_{\omega_0}^a \left\{ |y_1 (t, \lambda_0)|^2 + |y_2 (t, \lambda_0)|^2 \right\} \ d_{q, \omega} t = 0 .
\]
\end{lemma}
Hence \( y(t, \lambda_0) = 0 \) and this is a contradiction. Consequently, \( \lambda_0 \) must be real. \( \square \)

**Lemma 2.5** The eigenvalues of the \( q, \omega \)-Dirac system (1.7)–(1.9) are simple.

**Proof** Let \( \phi(t, \lambda) = \begin{pmatrix} \phi_1(t, \lambda) \\ \phi_2(t, \lambda) \end{pmatrix} \) be a solution of the \( q, \omega \)-Dirac equation (1.7) together with
\[
\phi_1(\omega_0, \lambda) = k_{12}, \quad \phi_2(\omega_0, \lambda) = -k_{11}.
\]

It is obvious that \( \phi(t, \lambda) \) satisfies the boundary condition (1.8). To find the eigenvalues of the \( q, \omega \)-Dirac system (1.7)–(1.9) we have to insert this function into the boundary condition (1.9) and find the roots of the obtained equation. So, putting the function \( \phi(t, \lambda) \) into the boundary condition (1.9) we get the following characteristic function
\[
\Delta(\lambda) = k_{21}\phi_1(a, \lambda) + k_{22}\phi_2(h^{-1}(a), \lambda).
\]

Then, \( \frac{d\Delta(\lambda)}{d\lambda} = k_{21}\frac{\partial\phi_1(a, \lambda)}{\partial\lambda} + k_{22}\frac{\partial\phi_2(h^{-1}(a), \lambda)}{\partial\lambda} \). Let \( \lambda_0 \) be a double eigenvalue, and \( \phi^0(t, \lambda_0) \) one of the corresponding eigenfunctions. Then, the conditions \( \Delta(\lambda_0) = 0, \frac{d\Delta(\lambda_0)}{d\lambda} = 0 \) should be fulfilled simultaneously, i.e.,
\[
\begin{align*}
&k_{21}\phi_1^0(a, \lambda_0) + k_{22}\phi_2^0(h^{-1}(a), \lambda_0) = 0, \\
&k_{21}\frac{\partial\phi_1^0(a, \lambda_0)}{\partial\lambda} + k_{22}\frac{\partial\phi_2^0(h^{-1}(a), \lambda_0)}{\partial\lambda} = 0.
\end{align*}
\]

Since \( k_{21} \) and \( k_{22} \) can not vanish simultaneously, it follows from that
\[
\phi_1^0(a, \lambda_0) \frac{\partial\phi_2^0(h^{-1}(a), \lambda_0)}{\partial\lambda} - \phi_2^0(h^{-1}(a), \lambda_0) \frac{\partial\phi_1^0(a, \lambda_0)}{\partial\lambda} = 0.
\]

Now, differentiating the \( q, \omega \)-Dirac equation (1.7) with respect to \( \lambda \), we obtain
\[
\begin{align*}
&\begin{cases}
-D_1^q \frac{\partial y_2(t, \lambda)}{\partial\lambda} + (p(t) - \lambda) \frac{\partial y_1(t, \lambda)}{\partial\lambda} = y_1(t, \lambda), \\
D_{q, \omega} \left( \frac{\partial y_1(t, \lambda)}{\partial\lambda} \right) + (r(t) - \lambda) \frac{\partial y_2(t, \lambda)}{\partial\lambda} = y_2(t, \lambda).
\end{cases}
\end{align*}
\]

Multiplying the \( q, \omega \)-Dirac equation (1.7) and (2.42) by \( \frac{\partial y_1(t, \lambda)}{\partial\lambda}, \frac{\partial y_2(t, \lambda)}{\partial\lambda}, -y_1(t, \lambda) \) and \( -y_2(t, \lambda) \), respectively, adding them together and applying the \( q, \omega \)-integration, we obtain
\[
\begin{align*}
&\begin{cases}
y_2(h^{-1}(t), \lambda) \frac{\partial y_1(t, \lambda)}{\partial\lambda} - y_1(t, \lambda) \frac{\partial y_2(h^{-1}(t), \lambda)}{\partial\lambda} \bigg|_{t=a}^a \\
\quad - \int_{\omega_0}^a \left\{ (y_1(t, \lambda))^2 + (y_2(t, \lambda))^2 \right\} d_{q, \omega} t.
\end{cases}
\end{align*}
\]
Putting $\lambda = \lambda_0$, taking account that
\[ \frac{\partial \phi_1^0(t, \lambda_0)}{\partial \lambda} \bigg|_{t=\omega_0} = \frac{\partial \phi_2^0(t, \lambda_0)}{\partial \lambda} \bigg|_{t=\omega_0} = 0, \]
and using the equality (2.41), we obtain
\[
\int_{\omega_0}^{a} \left\{ \left( \phi_1^0(t, \lambda_0) \right)^2 + \left( \phi_2^0(t, \lambda_0) \right)^2 \right\} dq = 0.
\]
Hence $\phi_1^0(t, \lambda_0) = \phi_2^0(t, \lambda_0) = 0$, which is impossible. Consequently $\lambda_0$ must be a simple eigenvalue.

\section*{3 Examples}

\textbf{Remark 3.1} (see \cite{16,17}) The $q$, $\omega$-sine and cosine functions defined by (1.18) and (1.19) have real and simple zeros $\{\pm g_n\}_{n=1}^{\infty}$, $\{\pm j_n\}_{n=1}^{\infty}$, respectively,
\[
g_n = \omega_0 + q^{-n} (1 - q)^{-1} (1 + O(q^n)), \quad j_n = \omega_0 + q^{-n+1/2} (1 - q)^{-1} (1 + O(q^n)), \quad n \geq 1.
\]

\textbf{Example 3.2} Consider the $q$, $\omega$-Dirac system (1.7)–(1.9) in which $p(t) = r(t) = 0$:
\[
\begin{cases}
-\frac{1}{q} D_q y_2 = \lambda y_1, \\
D_{q,\omega} y_1 = \lambda y_2,
\end{cases}
\]
(3.1)
\[
y_1(\omega_0) = 0, \\
y_2 \left(h^{-1}(\pi)\right) = 0.
\]
(3.2) (3.3)
It is easy to see that a solution of (3.1) and (3.2) is given by
\[
\phi^T(t, \lambda) = \left( -S_{q,\omega}(t, \lambda), \quad -C_{q,\omega}(t, \sqrt{q} \lambda) \right).
\]
By substituting this solution in (3.3), we obtain $\Delta(\lambda) = C_{q,\omega} \left(h^{-1}(\pi), \sqrt{q} \lambda\right)$. By the previous Remark 3.1, the eigenvalues are
\[
\lambda_n = \frac{q^{-n+1}}{(1-q)(\pi-\omega_0)} \left(1 + O(q^n)\right), \quad n = 1, 2, \ldots
\]
(3.4)
These approximate the eigenvalues of the $q$-Dirac system as $\omega \to 0^+$, see Example 1 in \cite{23}.

\textbf{Example 3.3} Consider the $q$, $\omega$-Dirac system (3.1) together with the following boundary conditions
\[
y_2(\omega_0) = 0,
\]
(3.5)
In this case

\[ \phi^T(t, \lambda) = (C_{q, \omega}(t, \lambda), -\sqrt{q} S_{q, \omega}(t, \sqrt{q} \lambda)) \]

Since \( \Delta(\lambda) = \sqrt{q} S_{q, \omega}(h^{-1}(\pi), \sqrt{q} \lambda) \), then the eigenvalues are given by

\[ \lambda_n = \frac{q^{-n+1/2}}{(1-q) (\pi - \omega_0)} (1 + O(q^n)), \quad n = 1, 2, \ldots \]  

(3.7)

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