Para-Krawtchouk polynomials on a bi-lattice and a quantum spin chain with perfect state transfer

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Abstract
Analogues of Krawtchouk polynomials defined on a bi-lattice are introduced. They are shown to provide a (novel) spin chain with perfect transfer. Their characterization, as well as their connection to the quadratic Hahn algebra, is given.

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1. Introduction

Progress in exactly solving physical models is often tied to advances in the theory of orthogonal polynomials and special functions (OPSF). We here present one such instance. In looking for spin chains that admit perfect state transfer (PST), we have found a new remarkable family of OPs defined on a bi-lattice (i.e. a lattice formed from the union of two regular lattices). We here report on these findings.

The transfer of quantum state from one location to another is of clear importance in quantum information. It is at the root of exchanges between individuals or between components of an eventual quantum computer. It is also fundamental in the development of quantum algorithms. Spin chains, i.e. one-dimensional coupled systems of spins, are being used in the design of quantum wires that will effect this transport [1]. These models have the merit of avoiding the need of external control as the dynamics of the chain is responsible for the transmission. An issue of interest is the efficiency of these wires. Ideally, one would wish for the transfer to be perfect, in other words, that the probability be 1 of finding the input state as output at some time. It has been shown that this can be achieved in various cases by properly engineering the couplings between the spins [1, 3].

The simplest systems in which this can be realized are XX spin chains with nearest-neighbor interactions and inhomogeneous couplings. Many aspects of PST are revealed by focusing on the one-excitation dynamics. Under this restriction, the XX-Hamiltonian takes the
form of a Jacobi or a tridiagonal Hermitian matrix $J$. Such matrices are intimately connected to OPs as they entail three-term recurrence relations. Moreover, the conditions for PST to occur can be expressed as conditions on their eigenvalues. As a matter of fact, PST requires that the differences between neighboring eigenvalues of $J$ be odd integers (up to a scale factor). The matrix $J$ contains the complete information on the couplings of the full Hamiltonian. Determining $XX$ spin chains with PST, therefore, amounts to an inverse spectral problem, i.e. to the problem of constructing Jacobi matrices $J$ that have given sets of eigenvalues satisfying the PST conditions. Algorithms have been developed to that end by exploiting the underlying connection with the theory of OPs.

It is of practical interest to identify spin chains with PST that admit exact solutions in terms of special functions. This obviously allows an analytical study of the dynamics. The interest is doubled if the examination of spin chains with PST leads to the full characterization of OPs that had eluded attention. This paper is in that vein. We purported to determine the $XX$ spin chains with PST corresponding to one-excitation spectra of bi-lattice form. In doing so, we found a new family of OPs that we have called para-Krawtchouk polynomials. We have studied their properties and found that they share many of the features of classical OPs.

The rest of the paper is organized as follows. In section 2, we review the relation between $XX$ spin chains with PST and OPs. In section 3, we present a new chain of that type which is amenable to an analytic treatment. It corresponds to a one-excitation energy spectrum of bi-lattice form and is valid for a chain where the number $N + 1$ of sites is even. The coupling coefficients are the recurrence parameters for the para-Krawtchouk polynomials. These turn out to form a one-parameter generalization of the Krawtchouk polynomials which are related to the PST model first presented in [1]. Sections 4 and 5 are devoted to the characterization of the para-Krawtchouk polynomials. In section 4, an explicit expression is given in terms of the complementary Bannai–Ito polynomials which arose in recent studies of Dunkl or $-1$ polynomials. In section 5, their relations with the quadratic Hahn algebra is established. In section 6, equipped with the appropriate mathematical tools developed in sections 4 and 5, we appeal to the Christoffel transform to obtain in the case of an odd number $N + 1$ of states, the chain with PST which is again associated with a bi-lattice spectrum.

2. OPs and quantum spin chains with PST

Consider the $XX$ spin chain with Hamiltonian

$$H = \frac{1}{2} \sum_{l=0}^{N-1} J_{l+1}(\sigma_l^x \sigma_{l+1}^x + \sigma_l^y \sigma_{l+1}^y) + \frac{1}{2} \sum_{l=0}^{N} B_l(\sigma_l^z + 1),$$  \quad (2.1)$$

where $J_l > 0$ are the constants coupling the sites $l - 1$ and $l$ and $B_l$ are the strengths of the magnetic field at the sites $l$ ($l = 0, 1, \ldots, N$). This operator acts on $\mathbb{C}^{2^{N+1}}$. The symbols $\sigma_l^x$, $\sigma_l^y$, $\sigma_l^z$ stand for the Pauli matrices, which act as follows on the standard basis $\{|0\rangle, |1\rangle\}$ of $\mathbb{C}^2$:

$$\sigma^x|1\rangle = |0\rangle, \quad \sigma^y|1\rangle = i|0\rangle, \quad \sigma^z|1\rangle = |1\rangle,$$

$$\sigma^x|0\rangle = |1\rangle, \quad \sigma^y|0\rangle = -i|1\rangle, \quad \sigma^z|0\rangle = -|0\rangle.$$  \quad (2.2, 2.3)

The index on these symbols indicate on which $\mathbb{C}^2$ factor they act. It is straightforward to see that the $z$-projection of the total spin is conserved:

$$\left[ H, \frac{1}{2} \sum_{l=0}^{N} (\sigma_l^z + 1) \right] = 0.$$  \quad (2.4)
This implies that the eigenstates of $H$ split into subspaces labeled by the number of spins over the chain that are in state $|\uparrow\rangle$.

Most state transfer properties can be obtained by focusing on the one-excitation subspace. Assume that the register is initially prepared in the state $|0\rangle^{\otimes(N+1)} = |0, 0, \ldots, 0\rangle$. Introduce the unknown state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

onto the first state (labeled by $l = 0$). We wish to recuperate the state $|\psi\rangle$ at the last site ($l = N$) after some time. The component $\alpha|0\rangle$ is automatically found at the last site because $|0\rangle^{\otimes(N+1)}$ is an eigenstate of $H$. In examining the transfer of the arbitrary state $|\psi\rangle$, it therefore suffices to consider how $|1\rangle|0\rangle^{\otimes N}$ can possibly evolve into $|0\rangle^{\otimes N}|1\rangle$ and that only requires looking at the one-excitation subspace. A natural basis for that subspace, equivalent to $\mathbb{C}^{N+1}$, is $|e_n\rangle = (0, 0, \ldots, 1, \ldots, 0)$, where the only 1 (spin up) occupies the $n$th position. In that basis, the restriction $J$ of $H$ to the one-excitation subspace is given by the following $(N+1) \times (N+1)$ Jacobi matrix:

$$J = \begin{pmatrix} B_0 & J_1 & 0 & \cdots & 0 \\ J_1 & B_1 & J_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_N & B_N \end{pmatrix}.$$ 

Its action on the basis vectors $|e_n\rangle$ reads

$$J|e_n\rangle = J_{n+1}|e_{n+1}\rangle + B_n|e_n\rangle + J_n|e_{n-1}\rangle.$$ 

Note also that the conditions

$$J_0 = J_{N+1} = 0$$

are assumed.

Let $x_s$, $s = 0, 1, \ldots, N$ be the eigenvalues of the matrix $J$. They are all real and non-degenerate. Moreover, they are labeled in increasing order, i.e. $x_0 < x_1 < x_2 < \cdots < x_N$.

To the Jacobi matrix $J$, one can associate the monic OPs $P_n(x)$ defined by the three-term recurrence relation

$$P_{n+1}(x) + B_n P_n(x) + U_n P_{n-1}(x) = x P_n(x), \quad n = 0, 1, \ldots, N,$$

where $U_n = J^2_n > 0$ and $P_{-1} = 0$, $P_0(x) = 1$.

$P_{N+1}(x)$ is the characteristic polynomial of the spectral points $x_s$:

$$P_{N+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_N).$$

The polynomials $P_n(x)$ satisfy the orthogonality relation

$$\sum_{j=0}^{N} P_n(x_j) P_m(x_j) w_j = h_n \delta_{nm},$$

where

$$h_n = U_1, U_2, \ldots, U_N.$$ 

The discrete weights $w_j > 0$ are uniquely determined by the recurrence coefficients $B_n$ and $U_n$. 

3
The PST condition means that there exists a time $T$ such that
\[ e^{iTJ}|e_0\rangle = e^{i\phi}|e_N\rangle, \]
(2.11)
where $\phi$ is a real number. In other words, after some time $T$, the initial state $|e_0\rangle$ evolves into the state $|e_N\rangle$ (up to inessential phase factor $e^{i\phi}$).

It is well known that the PST property is equivalent to the following two conditions [3].

(i) The eigenvalues $x_s$ satisfy
\[ x_{s+1} - x_s = \frac{\pi}{T} M_s, \]
(2.12)
where $M_s$ are positive odd numbers.

(ii) The matrix $J$ is mirror-symmetric $RJR = J$, where the matrix $R$ (reflection matrix) is
\[
R = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}.
\]

Property (ii) is equivalent to either two of the following properties [12]:

(ii') the weights $w_s$ (up to a normalization) are given by the expression
\[ w_s = \frac{1}{|P_{n+1}'(x_s)|} > 0; \]
(2.13)
(ii'') the polynomial $P_N(x)$ satisfies
\[ P_N(x_s) = A(-1)^s, \]
(2.14)
with some constant $A$ not depending on $s$.

Properties (ii') and (ii'') prove convenient from a constructive point of view. Indeed, a simple algorithm allows us to reconstruct polynomials $P_n(x)$ and Jacobi matrix $J$ simply from a given spectrum $x_s$ verifying condition (i). As a matter of fact, the polynomials $P_{n+1}(x)$ and $P_n(x)$ can be determined explicitly using property (ii'); the polynomials $P_n(x)$, $n = N - 1, N - 2, \ldots, 1$, and the matrix $J$ are then obtained iteratively with the help of the Euclidean division algorithm.

Note that the construction ensures in all cases that the polynomials are positive definite, and that the recurrence coefficients always generate a matrix $J$ with mirror symmetry.

From a mathematical perspective, interesting new families of explicit OPs can be generated from spectral datasets only, through this connection with chains admitting PST. A key step is in the choice of the grids. The simplest case is that of a uniform grid (i.e. $x_s$ is an arithmetic progression in $s$); in this instance, formula (2.13) gives the binomial distribution to which the Krawtchouk polynomials correspond. We thus recover the well-known example described in [1].

Other choices of grids $x_s$ with simple invariance properties (i.e. the geometric progression, two uniform grids separated by a gap etc) were considered in [12, 13].

They generate examples of OPs $P_n(x)$ and of corresponding Jacobi matrices $J$ with the PST property. Such explicit examples may also be useful from a practical point of view. Indeed, as shown in [12], starting from any explicit Hamiltonian $H$ with the PST property, one can generate many other Hamiltonians with the same PST property. This procedure called ‘spectral surgery’, in [12], involves the removal of spectral points and is equivalent to Darboux
transformations of the Hamiltonian $H$. Such manipulations can generate spin chains with more suitable properties (say, with a smoother behavior of the coefficients of the matrix $J$) than the initial spin chain.

3. Finite bi-lattice and para-Krawtchouk polynomials

Consider a finite bi-lattice of eigenvalues

$$x_s = s + \frac{1}{2}(\gamma - 1)(1 - (-1)^s), \quad s = 0, 1, \ldots, N,$$

where $N$ is odd and the parameter $\gamma$ satisfies the restrictions

$$0 < \gamma < 2.$$

For $\gamma \neq 1$, we have two uniform sublattices that are intermeshed. One sublattice corresponds to even $s$: $x_{2s} = 2s$.

The other sublattice corresponds to odd $s$:

$$x_{2s+1} = 2s + \gamma.$$

When $\gamma = 1$, we obtain the uniform lattice $x_s = s$. The PST condition for uniform lattice leads to the Krawtchouk polynomials [1]. The OPs $P_n(x)$ corresponding to the weights (2.13) with $x_s$ given by (3.1) will be called para-Krawtchouk polynomials. This term is justified by the observation that the spectrum (3.1) coincides (when $N = \infty$) with the spectrum of the para-bosonic oscillator [8].

Direct calculation with the help of formula (2.13) yields two different expressions for the weights $w_s$ depending on whether $s$ is even or odd:

$$w_{2s} = \frac{2^{-N}(1 + \gamma/2)_{2J}}{(2J)_{2s}} \frac{(-J)_s(-\gamma/2 - J)_s}{s!(1 - \gamma/2)_s}, \quad s = 0, 1, \ldots, J,$$

and

$$w_{2s+1} = \frac{2^{-N}(1 - \gamma/2)_{2J}}{(2J)_{2s+1}} \frac{(-J)_s(\gamma/2 - J)_s}{s!(1 + \gamma/2)_s}, \quad s = 0, 1, \ldots, J,$$

where $J = (N - 1)/2$ is a positive integer (recall that $N$ is odd).

The weights are normalized in a standard way:

$$\sum_{s=0}^{N} w_s = 1. \quad (3.6)$$

Equation (3.6) can be verified by observing that

$$\sum_{s=0}^{J} w_{2s} = \frac{2^{-N}(1 + \gamma/2)_{2J}}{(2J)_{2J}} 2F_1\left( -J, -\gamma/2 - J; 1 \right). \quad (3.7)$$

The hypergeometric function in (3.7) can be calculated by using the Chu–Vandermonde identity

$$2F_1\left( -J, b; 1 \right) = \frac{(c - b)_J}{(c)_J}, \quad (3.8)$$
which is valid for arbitrary \( b, c \) and for positive integers \( J \). Hence,

\[
\sum_{j=0}^{J} w_{2j} = \frac{2^{-N}(J + 1)_j}{(1/2)_j} = 1/2.
\]

Similarly,

\[
\sum_{j=0}^{J} w_{2j+1} = \frac{2^{-N}(J + 1)_j}{(1/2)_j} = 1/2
\]

and we obtain the desired property \( (3.6) \).

Note that the weights \( w_j \) resemble the corresponding weights for the Meixner polynomials on bi-lattices \( [7] \). But in contrast to these examples of OPs on bi-lattices that have been considered thus far \( [7] \), the para-Krawtchouk polynomials have explicit recurrence coefficients \( J_n \) and \( B_n \):

\[
B_n = \frac{N - 1 + \gamma}{2}, \quad U_n = J_n^2 = \frac{n(N + 1 - n)((N + 1 - 2n)^2 - \gamma^2)}{4(N - 2n)(N - 2n + 2)}.
\]

There is an even more convenient representation for the recurrence coefficients

\[
U_n = A_{n-1}C_n, \quad B_n = -A_n - C_n,
\]

where

\[
A_n = \frac{(N - n)(N - 1 - 2n + \gamma)}{2(2n - N)}, \quad C_n = \frac{n(N + 1 - 2n - \gamma)}{2(2n - N)}.
\]

The para-Krawtchouk polynomials satisfy the difference equation

\[
E(x)P_n(x + 2) + F(x)P_n(x - 2) - (E(x) + F(x))P_n(x) = 2n(n - N)P_n(x),
\]

where

\[
E(x) = \frac{(x - N + 1)(x - N + 1 - \gamma)}{2}, \quad F(x) = \frac{x(x - \gamma)}{2}.
\]

Surprisingly, equation \( (3.14) \) puts the para-Krawtchouk polynomials in the category of classical OPs on uniform grids \( [6] \). Note, however, that the translation \( x \rightarrow x + 2 \) corresponds to the shift \( x_i \rightarrow x_{i+2} \). This means that with respect to the grid \( (3.1) \) equation \( (3.14) \) is of fourth order.

\( F(x) \) vanishes at the two points \( x = 0 \) and \( x = \gamma \) which correspond to the smallest points of the two sublattices. Similarly, \( E(x) \) vanishes at the two points \( x = N - 1 \) and \( x = N - 1 + \gamma \) which coincide with the largest points of the two sublattices.

When \( \gamma = 1 \), we obtain the classical symmetric Krawtchouk polynomials with recurrence coefficients \( [4] \)

\[
B_n = \frac{N}{2}, \quad U_n = \frac{n(N + 1 - n)}{4}.
\]

Equation \( (3.14) \) becomes in this case the square of the eigenvalue equation for the Krawtchouk polynomials \( [4] \).

Returning to the PST property, consider the affine transformation of the spectral points \( (3.1) \)

\[
\tilde{x}_i = \alpha x_i + \beta,
\]

with arbitrary real parameters \( \alpha, \beta \). The corresponding monic polynomials \( \tilde{P}_n(x) \) will satisfy the recurrence relation

\[
\tilde{P}_{n+1}(x) + \tilde{B}_n \tilde{P}_n(x) + \tilde{U}_n \tilde{P}_{n-1}(x) = x \tilde{P}_n(x),
\]

where

\[
\tilde{B}_n = \alpha B_n + \beta, \quad \tilde{U}_n = \alpha^2 U_n.
\]
Using appropriate values for the parameters $\alpha$ and $\beta$, we can always achieve condition (i) for the PST iff

$$\gamma = \frac{M_1}{M_2},$$  \hspace{1cm} (3.19)

where $M_1$ and $M_2$ are positive co-prime integers and $M_1$ is odd.

4. Relation with the complementary Bannai–Ito polynomials

A breakthrough [11] in the theory of OPs theory has been realized recently with the discovery of classical OPs that are eigenfunctions of continuous or discrete Dunkl operators defined using reflections. These OPs are often referred to as $-1$ polynomials since they arise through $q \to -1$ limit of $q$-OPs. Remarkably, this has allowed for a complete characterization of the four-parameter Bannai–Ito polynomials in a discrete variable. A significant feature of these classes of OPs is that their spectra $\sigma_z$ depend asymmetrically on the parity of the labeling index as in the case of the para-Krawtchouk polynomials.

We shall now provide an explicit expression for these para-Krawtchouk polynomials in terms of the complementary Bannai–Ito polynomials [11]. These are the kernel polynomials of the Bannai–Ito polynomials; i.e. they are obtained from the latter through a Christoffel transform. These complementary Bannai–Ito polynomials depend on four parameters $r_1$, $r_2$, $\rho_1$, $\rho_2$ and are defined through the three-term recurrence relation

$$W_{n+1}(x) + (-1)^n \rho_2 W_n(x) + v_n W_{n-1}(x) = x W_n(x), \hspace{1cm} (4.1)$$

with

$$v_{2n} = -\frac{n(n + \rho_1 - r_1 + 1/2)(n + \rho_1 - r_2 + 1/2)(n - r_1 - r_2)}{(2n + 1 + g)(2n + g)},$$

$$v_{2n+1} = -\frac{(n + g + 1)(n + \rho_1 + \rho_2 + 1)(n + \rho_2 - r_1 + 1/2)(n + \rho_2 - r_2 + 1/2)}{(2n + 1 + g)(2n + g + 2)},$$

where we denote

$$g = \rho_1 + \rho_2 - r_1 - r_2.$$  \hspace{1cm} (4.2)

They can be expressed in terms of the Racah polynomials as follows [11]:

$$W_{2n}(x) = \kappa_n^{(1)} 3F_3 \left( \begin{array}{c} -n, n + g + 1, \rho_2 + x, \rho_2 - x \\ \rho_1 + \rho_2 + 1, \rho_2 - r_1 + 1/2, \rho_2 - r_2 + 1/2 \end{array} ; 1 \right), \hspace{1cm} (4.3)$$

and

$$W_{2n+1}(x) = \kappa_n^{(2)} (x - \rho_2) 3F_3 \left( \begin{array}{c} -n, n + g + 2, \rho_2 + 1 + x, \rho_2 + 1 - x \\ \rho_1 + \rho_2 + 2, \rho_2 - r_1 + 3/2, \rho_2 - r_2 + 3/2 \end{array} ; 1 \right), \hspace{1cm} (4.4)$$

where the normalization coefficients $\kappa_n^{(1,2)}$ are needed to ensure that polynomials $W_n(x)$ are monic.

Putting

$$r_2 = \rho_2 = 0, \hspace{1cm} r_1 = \frac{N + 1 + \gamma}{4}, \hspace{1cm} \rho_1 = \frac{\gamma - N - 3}{4},$$

the recurrence relation (4.1) then reads

$$W_{n+1}(x) + \frac{n(N + 1 - n)((N + 1 - 2n)^2 - \gamma^2)}{16(N - 2n)(N + 2 - 2n)} W_{n-1}(x) = x W_n(x). \hspace{1cm} (4.6)$$

Comparing (4.6) with the recurrence relation for the para-Krawtchouk polynomials we conclude that

$$P_n(x) = 2^n W_n(x/2 - (N - 1 + \gamma)/4), \hspace{1cm} (4.7)$$
i.e. that the para-Krawtchouk polynomials can be expressed in terms of the complementary Bannai–Ito polynomials with a shifted argument.

Consider the polynomial \( P_{N+1}(x) \). By the assumption that \( N = 2M + 1 \) is odd, and hence by (4.3), we have

\[
P_{N+1}(x) = \text{const } {}_3F_2 \left( \begin{array}{c} -M - 1, x/2 - M/2 - \gamma/4, -x/2 + M/2 + \gamma/4 \\ \gamma - 2M/4, -\gamma + 2M/4 \\ \end{array} ; 1 \right).
\]

(4.8)

On the one hand, the hypergeometric function \( {}_3F_2(1) \) in (4.8) can be simplified using the Pfaff–Saalschütz summation formula [4]. This yields the factorization

\[
P_{N+1}(x) = \text{const } \left( \frac{\gamma - x}{2} \right)^{M+1} \left( \frac{x - 2M}{2} \right)^{M+1},
\]

(4.9)

where \( (x)_n = x(x - 1) \cdots (x + n - 1) \) is the shifted factorial (Pochhammer symbol).

On the other hand, \( P_{N+1}(x) \) is the characteristic polynomial (2.9). Comparing (4.9) and (2.9) we arrive at the explicit expression (3.1) for the spectral points.

5. Algebraic interpretation: Hahn algebra

It is possible to relate the para-Krawtchouk polynomials to the quadratic Hahn algebra [2, 6, 14].

Introduce two operators \( X \) and \( Y \) on the space of polynomials \( f(x) \): let \( X \) be the operator of multiplication by the argument \( x \)

\[
Xf(x) = xf(x)
\]

and \( Y \) be the difference operator

\[
Yf(x) = E(x)f(x + 2) + F(x)f(x - 2) - (E(x) + F(x))f(x),
\]

(5.2)

where the functions \( E(x) \) and \( F(x) \) are given by (3.15). Note that the operator \( Y \) is the operator on lhs of equation (3.14) satisfied by the para-Krawtchouk polynomials.

Introduce also a third operator \( Z \) which is the commutator of \( X \) and \( Y \)

\[
Z = [X, Y] = XY - YX.
\]

(5.3)

It is then easily verified that the commutators \([Y, Z]\) and \([Z, X]\) are quadratic expressions in terms of the operators \( X, Y, Z \):

\[
[Y, Z] = -4[X, Y] + C_1X + GY + M_1,
\]

\[
[Z, X] = -4X^2 + C_2Y + GX + M_2,
\]

(5.4)

where \([X, Y]\) stands for the anticommutator and

\[
C_1 = 4(1 - N^2), \quad C_2 = -4, \quad G = 4(N + \gamma - 1),
\]

\[
M_1 = 2(N + \gamma - 1)(N^2 - 1), \quad M_2 = 2(1 - N)(N + \gamma - 1).
\]

(5.5)

The polynomial algebra with three generators satisfying (5.3) and (5.4) as defining relations is known as the Hahn algebra [2].

It has for the Casimir operator

\[
Q = Z^2 - 4X^2 + (C_1 + 16)X^2 + C_2Y^2 + G[X, Y] + (2M_1 - 4G)X + 2M_2Y,
\]

(5.6)

which commutes with the operators \( X, Y, Z \): \([Q, X] = [Q, Y] = [Q, Z] = 0\).

In the realization (5.1) and (5.2), \( Q \) reduces to a constant

\[
Qf(x) = qf(x),
\]

(5.7)
where
\[ q = (N - 1)(N + \gamma - 1)(N^2 - 2N + \gamma(N + 3) - 7). \] (5.8)

The Hahn algebra admits a basis \( \pi_n \), \( n = 0, 1, \ldots, N \), where the operator \( Y \) is diagonal
\[ Y \pi_n = \lambda_n \pi_n, \] (5.9)

whereas the operator \( X \) is 3-diagonal:
\[ X \pi_n = a_{n+1} \pi_{n+1} + B_n \pi_n + a_n \pi_{n-1}. \] (5.10)

The coefficients \( a_n, B_n \) and \( \lambda_n \) can be calculated from the representations of the Hahn algebra [2] with the initial conditions \( a_0 = a_{N+1} = 0 \):
\[ \lambda_n = 2n(n - N). \] (5.11)

The coefficients \( a_n \) and \( B_n \) coincide with the recurrence coefficients (3.11) for the para-Krawtchouk polynomials.

This gives a simple algebraic interpretation of the para-Krawtchouk polynomials. It is obvious that the basis vectors \( \pi_n \) are given by the para-Krawtchouk polynomials in the realization (5.1) and (5.2):
\[ \pi_n = P_n(x). \] (5.12)

It is worth noting that the para-Krawtchouk polynomials can be obtained from a periodic reduction (with period 4) of the Darboux transformations of finite Jacobi matrices. Indeed, as already noted in [9], such periodic reductions will lead to the Hahn algebra and to the appearance of non-trivial sublattices for the spectral points. Only ordinary Hahn polynomials were analyzed from this point of view in [9]. The para-Krawtchouk polynomials provide a new non-trivial example of such periodic closures.

6. The case of odd number of the eigenvalues

So far, we considered the case of an even number \( N + 1 \) of eigenvalues \( x_s \), \( s = 0, 1, \ldots, N \) with \( N = 1, 3, 5, \ldots \). Using the Christoffel transform, it is possible to obtain the para-Krawtchouk polynomials corresponding to an odd number of eigenvalues \( x_s \), \( s = 0, 1, \ldots, N - 1 \). Indeed, consider the polynomials
\[ \tilde{P}_n(x) = \frac{P_{n+1}(x) - A_n P_n(x)}{x - x_N}, \] (6.1)

where \( A_n \) are given by (3.13). It is not difficult to check that
\[ A_n = \frac{P_{n+1}(x_N)}{P_n(x_N)}, \]

and hence formula (6.1) defines new OPs \( \tilde{P}_n(x) \) obtained by the Christoffel transform [10] of the polynomials \( P_n(x) \). The polynomials \( \tilde{P}_n(x) \) satisfy the recurrence relation
\[ \tilde{P}_{n+1}(x) + \tilde{B}_n \tilde{P}_n(x) + \tilde{U}_n \tilde{P}_{n-1}(x) = x \tilde{P}_n(x) \] (6.2)

with the coefficients [12]
\[ \tilde{U}_n = U_n \frac{A_n}{A_{n-1}}, \quad \tilde{B}_n = B_{n+1} + A_{n+1} - A_n. \] (6.3)

A simple calculation yields
\[ \tilde{U}_n = \frac{n(N - n)((2n - N)^2 - (\gamma - 1)^2)}{(2n - N)^2} \] (6.4)
and
\[ \tilde{B}_n = \frac{N + \gamma}{2} - 1 + \frac{(\gamma - 1)N}{4} \left( \frac{1}{2n - N} - \frac{1}{2n + 2 - N} \right). \]
\[ (6.5) \]

The Jacobi matrix \( \tilde{J} \) with the recurrence coefficients \( \tilde{U}_n \) and \( \tilde{B}_n \) has size \( N \times N \) and its eigenvalues \( \tilde{x}_s \) coincide with the eigenvalues of the matrix \( J \) apart from the final value \( x_N \):
\[ \tilde{x}_s = s + \frac{1}{2}(\gamma - 1)(1 - (-1)^s), \quad s = 0, 1, \ldots, N - 1. \]
\[ (6.6) \]

Thus, the polynomials \( \tilde{P}_n(x) \) defined by (6.2), (6.4), (6.5) have properties similar to those of the para-Krawtchouk polynomials. Clearly, the Jacobi matrix \( \tilde{J} \) generates a spin chain with the PST property under the restriction (3.19). Indeed, Christoffel transforms like (6.1) preserve the PST property and can be used to construct new examples of spin chains with PST (see [12] for details).

7. Conclusions

In summary, we have presented an analytically solvable XX spin chain model with inhomogeneous nearest-neighbor couplings that effect perfect state transfer. It was obtained by exploiting the intimate connection between inverse spectral problems for Jacobi matrices and orthogonal polynomial (OP) theory. We posited a spectrum of bi-lattice form for one excitation and identified the corresponding coupling constants of the Hamiltonian. The associated OPs were seen to form a remarkable family that had hitherto not been identified. They have been called para-Krawtchouk polynomials as their spectral properties are analogous to that of the para-Bose oscillator. We have presented their characterization and observed that they enjoy properties similar to those of classical OPs. We have further shown that they are related to the quadratic Hahn algebra.

We have thus provided an example of fruitful cross-fertilization between the study of systems with PST and the theory of OPs. We trust further results can be reaped from this connection and in view of their nice features, we believe that the para-Krawtchouk polynomials will find more applications now that they have been discovered and characterized.

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