LOWER BOUND OF RIESZ TRANSFORM KERNELS REVISITED AND COMMUTATORS ON STRATIFIED LIE GROUPS

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Abstract. Let $G$ be a stratified Lie group and $\{X_j\}_{1 \leq j \leq n}$ a basis for the left-invariant vector fields of degree one on $G$. Let $\Delta = \sum_{j=1}^n X_j^2$ be the sub-Laplacian on $G$ and the $j^{th}$ Riesz transform on $G$ is defined by $R_j := X_j(-\Delta)^{-\frac{1}{2}}$, $1 \leq j \leq n$. In this paper we give a new version of the lower bound of the kernels of Riesz transform $R_j$ and then establish the Bloom-type two weight estimates as well as a number of endpoint characterisations for the commutators of the Riesz transforms and BMO functions, including the $L^\log L(G)$ to weak $L^1(G)$, $H^1(G)$ to $L^1(G)$ and $L^\infty(G)$ to BMO($G$). Moreover, we also study the behaviour of the Riesz transform kernel on a special case of stratified Lie group: the Heisenberg group, and then we obtain the weak type (1, 1) characterisations for the Riesz commutators.

1. Introduction and statement of main results

The Calderón-Zygmund theory of singular integrals has a central role in modern harmonic analysis with extensive applications to other fields such as partial differential equations and complex analysis. A prototype of singular integral on the real line is the Hilbert transform which is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. At the end-point $p = 1$, the Hilbert transform is bounded from $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ and is bounded from the Hardy space $H^1(\mathbb{R})$ into $L^1(\mathbb{R})$ while at the end-point $p = \infty$, the Hilbert transform is bounded from $L^\infty(\mathbb{R})$ to the BMO space BMO$(\mathbb{R})$. Further study on singular integrals and the related partial differential equations leads to the commutator $[b, H]$ of the Hilbert transform $H$ and a BMO function $b$ defined by

$$[b, H]f(x) = b(x)Hf(x) - H(bf)(x)$$

for suitable functions $f$ (introduced by A.P. Calderón [4]). It is well-known that the commutator of the Hilbert transform has the following properties:

a) $\|[b, H]\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \approx \|b\|_{\text{BMO}(\mathbb{R})}$ ([7]);

b) $\|b, H\|_{L^p_0(\mathbb{R}) \to L^p_0(\mathbb{R})} \approx \|b\|_{\text{BMO}_c(\mathbb{R})}$ with $\mu, \lambda \in A_p$ for $1 < p < \infty$ and $\nu = (\frac{1}{\mu})^{\frac{1}{p}}$, where $A_p$ denotes the Muckenhoupt weights ([2], [15]);

c) $[b, H]$ is bounded from $L \log^2 L(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$ if and only if $b \in \text{BMO}(\mathbb{R})$ ([24], [13], [1]);

d) $[b, H]$ is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ if and only if $b$ equals a constant almost everywhere ([12]);

e) $[b, H]$ is bounded from $L^\infty_c(\mathbb{R})$ to BMO$(\mathbb{R})$ if and only if $b$ equals a constant almost everywhere ([12]);

f) $[b, H]$ is of weak type (1, 1) if and only if $b$ is in $L^\infty(\mathbb{R})$ ([1]).

For more details, we refer to the references listed above. We also point out that there are quite a number of recent results on the characterisations of commutators in the above forms for singular integrals in different settings, see for example [10], [22], [9], [21], [26], [24], [15], [19], [13], [8], [1].
Inspired by these classical results above, it is natural to ask whether these results hold on the Heisenberg group \( \mathbb{H}^n \). Note that in several complex variables, the Heisenberg group \( \mathbb{H}^n \) is the boundary of the Siegel upper half space, whose roles are holomorphically equivalent to the unit sphere and the unit ball in \( \mathbb{C}^n \). And hence, the role of the Riesz transform on \( \mathbb{H}^n \) is similar to the role of Hilbert transform on the real line.

The first four authors in [9] established an analogous results of a) above in the more general setting of stratified Lie groups by studying the behaviour of the Riesz transform kernels and obtaining a lower bound on the kernel. Note that the Riesz transform kernel does not have an explicit representation, and hence the key difficulty is to obtain a suitable version of kernel lower bounds. To overcome this, in [9] they studied and made good use of the group structures and the dilations related to the stratified condition, and then established a first version of the kernel lower bound for Riesz transforms.

However, it is not clear whether the kernel lower bound for Riesz transforms introduced in [9] is enough to further study analogous results of b), c), d) and e) on stratified Lie groups.

The aim of this paper is three fold. First, we provide a better understanding of the Riesz transform kernel behaviour on stratified Lie groups, which is stronger than the lower bound obtained in [9] and is new in the literature. Second, we apply these kernel lower bounds to study the endpoint characterisations of boundedness of commutators, i.e., we establish similar versions of b), c), d) and e) for the Riesz commutators (but with some improvement) to study the endpoint characterisations of boundedness of commutators, i.e., we establish obtained in [9] and is new in the literature. Second, we apply these kernel lower bounds. To overcome this, in [9] they studied and made good use of the group structures and the dilations related to the stratified condition, and then established a first version of the kernel lower bound for Riesz transforms.

We now recall the BMO space on \( G \), which is the dual space of \( H^1(G) \) [11, Chapter 5], defined as

\[
\text{BMO}(G) := \{ b \in L^1_{\text{loc}}(G) : \| b \|_{\text{BMO}(G)} < \infty \},
\]

where

\[
\| b \|_{\text{BMO}(G)} := \sup_B \frac{1}{|B|} \int_B |b(g) - b_B| dg,
\]

and \( b_B := \frac{1}{|B|} \int_B b(g) \, dg \), where \( B \) denotes the ball on \( G \) defined via a homogeneous norm \( \rho \). We also mention that the weighted BMO space \( \text{BMO}_\nu(G) \) is defined via replacing the measure \( |B| \) in (1.1) by \( \nu(B) \). See Section 2 for details.

The first part of main results of this paper is to establish an enhanced version of the lower bound of the Riesz transform kernel on stratified Lie groups and then study the properties of commutators, including the following:

b') \( \| [b, R_j] \|_{L^p_{\nu}(G) \to L^q_{\nu}(G)} \approx \| b \|_{\text{BMO}_\nu(G)} \) with \( \mu, \lambda \in A_p(G) \) for \( 1 < p < \infty \) and \( \nu = \left( \frac{\mu}{\lambda} \right)^{1/r} \);

c') \( [b, R_j] \) is bounded from \( L^{p \log^+ L}(G) \) to \( L^{1,\infty}(G) \) if and only if \( b \in \text{BMO}(G) \);

d') \( [b, R_j] \) is bounded from \( H^1(G) \) to \( L^1(G) \) if and only if \( b \) equals a constant almost everywhere;
e’') \([b,R_j]\) is bounded from \(L^\infty_c(G)\) to \(\text{BMO}(G)\) if and only if \(b\) equals a constant almost everywhere.

The second part of the main results of this paper is to consider the characterisation of the endpoint boundedness of the commutator with respect to the weak type \((1,1)\), i.e., we aim to study:

f’’ \([b,R_j]\) is of weak type \((1,1)\) if and only if \(b\) is in \(L^\infty(G)\).

However, it is not clear whether one can establish this result on general stratified Lie groups \(G\). Here we study the characterisation of boundedness of commutator from \(L^1\) to \(L^{1,\infty}\) by focusing on the special case of stratified Lie group: the Heisenberg group \(\mathbb{H}^n\), which can be identified with \(\mathbb{C}^n \times \mathbb{R}\) with the group structure (for all the notation on \(\mathbb{H}^n\) we refer to Section 2).

To be more specific, we begin by recalling a recent result by the first four authors [9], where they studied the behaviour of the kernel \(K_j\) of Riesz transform \(R_j\) on \(G\) and obtained that: \(K_j \neq 0\) in \(G \setminus \{0\}\). Then, based on this behaviour they further obtained the following version of kernel bounds which implies an analogues of a) for the commutator of \(R_j\) and the BMO space on \(G\).

**Theorem A** ([9]). Fix \(j = 1, \ldots, n\). There exist \(0 < \varepsilon_0 \leq 1\) and \(C > 0\) such that for any \(0 < \eta < \varepsilon_0\) and for all \(g \in G\) and \(r > 0\), we can find \(g_\ast = g_\ast(j,g,r) \in G\) satisfying

\[
\rho(g, g_\ast) = r, \quad |K_j(g_1, g_2)| \geq Cr^{-Q}, \quad \forall g_1 \in B(g, \eta r), \ g_2 \in B(g_\ast, \eta r).
\]

To obtain b’’, c’’, d’’ and e’’, we point out that the result in Theorem A may not be enough. To achieve this, we need an enhanced version of the kernel lower bound. Thus, the main results of this paper are three fold. First, we establish an enhanced version of the kernel lower bound of the Riesz transform \(R_j\) on \(G\) via constructing a type of “twisted truncated sector” on \(G\), which is of independent interest and will be useful in studying other problems.

**Theorem 1.1.** Suppose that \(G\) is a stratified Lie group with homogeneous dimension \(Q\) and that \(j \in \{1,2,\ldots,n\}\). There exist a large positive constant \(r_o\) and a positive constant \(C\) such that for every \(g \in G\) there exists a “twisted truncated sector” \(G \subset G\) such that \(\inf_{g' \in G} \rho(g, g') = r_o\) and that for every \(g_1 \in B(g,1)\) and \(g_2 \in G\), we have

\[
|K_j(g_1, g_2)| \geq C \rho(g_1, g_2)^{-Q}, \quad |K_j(g_2, g_1)| \geq C \rho(g_1, g_2)^{-Q},
\]

and all \(K_j(g_1, g_2)\) as well as all \(K_j(g_2, g_1)\) have the same sign.

Moreover, this “twisted truncated sector” \(G\) is regular, in the sense that \(|G| = \infty\) and that for any \(R > 2r_o\),

\[
|B(g,R) \cap G| \approx R^Q,
\]

where the implicit constants are independent of \(g\) and \(R\).

Here we point out that the set \(G\) that we constructed in Theorem 1.1 above is a connected open set spreading out to infinity, which plays the role of the “truncated sector centered at a fixed point” in the Euclidean setting. The shape of \(G\) here may not be the same as the usual sector since the norm \(\rho\) (or the Carnot–Carathéodory metric \(d\) on \(G\) is different from the standard Euclidean metric. However, such a kind of twisted sector always exists.

Second, we establish the Bloom-type two weight estimates for the commutators \([b,R_j]\).

**Theorem 1.2.** Let \(\mu,\lambda \in A_\rho(G), \ 1 < p < \infty\). Further set \(\nu = (\frac{\mu}{\lambda})^\frac{1}{p}\). Suppose \(j \in \{1, \ldots, n\}\). Then

\begin{enumerate}
  \item if \(b \in \text{BMO}_\nu(G)\), then
    \[
    \|\|b, R_j\|f\|_{L^p(G)} \lesssim \|b\|_{\text{BMO}_\nu(G)} \cdot \|\mu\|_{A_\rho(G)} \|\mathbf{1}_{A_\rho(G)}\|^\text{max}(1,\frac{1}{p}) \|f\|_{L^p(G)}.
    \]
\end{enumerate}
Theorem 1.4. Suppose that
\[ L \] is bounded from \( L^p_0(\mathcal{G}) \) to \( L^q_\chi(\mathcal{G}) \), then \( b \in \text{BMO}_\nu(\mathcal{G}) \) with
\[
\|b\|_{\text{BMO}_\nu(\mathcal{G})} \lesssim \| [b, R_j] \|_{L^p_0(\mathcal{G}) \to L^q_\chi(\mathcal{G})}.
\]

For the proof of part (i) in the above theorem, we point out that it follows directly from [19] (see also [15], where they neglected the sharp constant argument) with only minor changes since for this part we only need to use the upper bound of the Riesz transform, which satisfies the standard size and smoothness condition of Calderón–Zygmund type. For part (ii), we use the idea and techniques originated from [19] and then adapt it to our setting according to the lower bound of Riesz kernel obtained in Theorem 1.1.

Next, we establish the endpoint estimates of the commutators \([b, R_j]\).

Theorem 1.3. Suppose that \( \mathcal{G} \) is a stratified Lie group and that \( j = 1, 2, \ldots, n \). Then given a function \( b \in L^1_{\text{loc}}(\mathcal{G}) \), \( b \in \text{BMO}(\mathcal{G}) \) if and only if \( [b, R_j] \) is bounded from \( L \log^+ L(\mathcal{G}) \) to weak \( L^1(\mathcal{G}) \), i.e., there exists a constant \( \theta_b \) such that for all \( \lambda > 0 \) and for all \( f \in L \log^+ L(\mathcal{G}) \),
\[
\{ g \in \mathcal{G} : \| [b, R_j](f)(g) \| > \lambda \} \leq \theta_b \int_{\mathcal{G}} \left| \frac{f(g)}{\lambda} \right| \left( 1 + \log^+ \left( \frac{|f(g)|}{\lambda} \right) \right) dg.
\]

The proof of this theorem is twofold, the necessity and sufficiency. For the sufficiency, we point out that it follows directly from [24] with only minor changes since for this part we only need to use the upper bound of the Riesz transform, which satisfies the standard size and smoothness condition of Calderón–Zygmund type. For the necessary part, we use the idea of Uchiyama [27] and the technique that has been further explored and studied in [13] and [1]. To be more specific, we write \( \| [b, R_j](f)(g) \| \geq |R_j(bf)(g)| - |b(g)||R_j(f)(g)| \), and then by choosing a suitable function \( f \) that is closely related to \( b \) and with cancellation condition and by making good use of the lower bound in Theorem 1.1, we show that \( |R_j(bf)(g)| \) is the main term and \( |b(g)||R_j(f)(g)| \) acts as the “error” term due to the cancellation of \( f \) and the smoothness condition of the kernel of \( R_j \). Together with the boundedness of \([b, R_j]\), we show that \( b \) is in \( \text{BMO}(\mathcal{G}) \).

We also point out that this endpoint characterisation in Theorem 1.3 above is sharp since following the method in [24] in the Euclidean setting and using the lower bound in Theorem 1.1, it is easy to construct a function \( b \in \text{BMO}(\mathcal{G}) \) such that \([b, R_j]\) fails to be weak type (1, 1).

Third, besides the weak type (1, 1), it is natural to study the endpoint estimates like \( H^1(\mathcal{G}) \) to \( L^1(\mathcal{G}) \) and the \( L^\infty(\mathcal{G}) \) to \( \text{BMO}(\mathcal{G}) \). We also have the following characterisations. Denote by \( L^\infty_c(\mathcal{G}) \) the subspace of \( L^\infty(\mathcal{G}) \) of compactly supported functions.

Theorem 1.4. Suppose that \( \mathcal{G} \) is a stratified Lie group, \( b \in \text{BMO}(\mathcal{G}) \) and \( j \in \{1, 2, \ldots, n\} \). Then the following statements are equivalent:

(i) \([b, R_j]\) is bounded from \( H^1(\mathcal{G}) \) to \( L^1(\mathcal{G}) \);
(ii) \([b, R_j]\) is bounded from \( L^\infty_c(\mathcal{G}) \) to \( \text{BMO}(\mathcal{G}) \);
(iii) \( b \) equals a constant almost everywhere.

The proof of this theorem follows from the strategy of the classical results [14]. We make use of the Riesz transform kernel lower bound in Theorem 1.1 and adapt the idea in [14] to this setting with necessary changes.

Next, we turn to the study of the behaviour of the kernel of the Riesz transform on \( \mathbb{H}^n \).

Theorem 1.5. Suppose \( j = 1, 2, \ldots, 2n \), and \( R_j := X_j(-\Delta_{\mathbb{H}^n})^{-\frac{1}{2}} \), is the \( j \)th Riesz transform on \( \mathbb{H}^n \). Let \( K_j(g), g \in \mathbb{H}^n \), be the kernel of \( R_j \). Then we have
\[
K_j(g) \neq 0, \quad \text{a.e. } g \in \mathbb{H}^n.
\]
This is an obvious fact for classical Hilbert transforms and Riesz transforms. However, it is not known before for Riesz transforms on Heisenberg groups. This is still open on general stratified Lie groups.

Next, concerning the commutator \([b, R_j]\) to be of weak type \((1, 1)\), we have the following characterisation.

**Theorem 1.6.** Suppose \(j = 1, 2, \ldots, 2n\), and \(R_j := X_j(-\Delta_H)^{-\frac{1}{2}}\), is the \(j\)th Riesz transform on \(\mathbb{H}^n\). Then given a function \(b \in L^1_\text{loc}(\mathbb{H}^n)\), \([b, R_j]\) is of weak type \((1, 1)\) if and only if \(b \in L^\infty(\mathbb{H}^n)\).

The proof of this theorem follows from the result in Theorem 1.5 and from the idea and approach in [1] for Hilbert transform.

This paper is organised as follows. In Section 2 we recall the necessary preliminaries on stratified Lie groups \(G\). In Section 3 we give a deep study of the behaviour of the Riesz transform kernels and the kernel lower bounds, and then prove Theorem 1.1. In Section 4, by using the kernel lower bound that we established and the original idea in [19], we prove Theorem 1.2. In Section 5, by using the kernel lower bound that we establish, we prove Theorem 1.3, the characterisation of BMO(\(G\)) via the endpoint \((L \log L)^\pm L^1\) estimates of \([b, R_j]\). In Section 6 we prove Theorem 1.4 and in the last section we show the behaviour of the Riesz transform kernel on \(\mathbb{H}^n\) (Theorem 1.5) and then give the characterisation of boundedness of Riesz commutator from \(L^1\) to \(L^1, \infty\) (Theorem 1.6).

**2. Preliminaries on stratified Lie groups \(G\)**

Recall that a connected, simply connected nilpotent Lie group \(G\) is said to be stratified if its left-invariant Lie algebra \(g\) (assumed real and of finite dimension) admits a direct sum decomposition

\[
g = \bigoplus_{i=1}^k V_i, \quad [V_1, V_i] = V_{i+1}, \quad \text{for } i \leq k - 1 \text{ and } [V_1, V_k] = 0;
\]

\(k\) is called the step of the group \(G\).

One identifies \(g\) and \(G\) via the exponential map

\[
\exp : g \to G,
\]

which is a diffeomorphism.

We fix once and for all a (bi-invariant) Haar measure \(dx\) on \(G\) (which is just the lift of Lebesgue measure on \(g\) via \(\exp\)).

There is a natural family of dilations on \(g\) defined for \(r > 0\) as follows:

\[
\delta_r \left( \sum_{i=1}^k v_i \right) = \sum_{i=1}^k r^i v_i, \quad \text{with } v_i \in V_i.
\]

We choose once and for all a basis \(\{X_1, \cdots, X_n\}\) of \(V_1\) and consider the sub-Laplacian \(\Delta = \sum_{j=1}^n X_j^2\). Observe that \(X_j (1 \leq j \leq n)\) is homogeneous of degree 1 with respect to the dilations, and \(\Delta\) of degree 2 in the sense that:

\[
X_j (f \circ \delta_r) = r (X_j f) \circ \delta_r, \quad 1 \leq j \leq n, \quad r > 0, \quad f \in C^1,
\]

\[
\delta_r \circ \Delta \circ \delta_r = r^2 \Delta, \quad \forall r > 0.
\]

For \(i = 1, \cdots, k\), let \(n_i = \dim V_i\) and \(m_i = n_1 + \cdots + n_i\) and \(m_0 = 0\), clearly, \(n_1 = n\). Set \(N = m_k\). Two important families of diffeomorphisms of \(G\) are the translations and dilations of \(G\). For any \(g \in G\), the (left) translation \(\tau_g : G \to G\) is defined as

\[
\tau_g(g') = g \circ g'.
\]
The dilations on \( g \) allow the definition of dilation on \( G \), which we still denote by \( \delta_r \). For any \( \lambda > 0 \), the dilation \( \delta_\lambda : G \to G \), is defined as
\[
(2.1) \quad \delta_\lambda(g) = \delta_\lambda(x_1, x_2, \ldots, x_N) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_N} x_N),
\]
where \( \alpha_j = i \) whenever \( m_j-1 < j \leq m_i, \ i = 1, \ldots, k \). Therefore, \( 1 = \alpha_1 = \cdots = \alpha_{n-1} < \alpha_{n+1} = 2 = \cdots = \alpha_n = k \). For any set \( E \subset G \), denote by \( \tau_g(E) = \{g \circ g' : g' \in E\} \) and \( \delta_r(E) = \{\delta_r(g) : g \in E\} \).

Let \( Q \) denote the homogeneous dimension of \( G \), namely,
\[
(2.2) \quad Q = \sum_{i=1}^{k} i \dim V_i.
\]

And let \( p_h (h > 0) \) denote the heat kernel (that is, the integral kernel of \( e^{h\Delta} \)) on \( G \). For convenience, we set \( p_h(g) = p_h(0,g) \) (that is, in this note, for a convolution operator, we will identify the integral kernel with the convolution kernel) and \( p(g) = p_1(g) \).

Recall that (c.f. for example [11])
\[
(2.3) \quad p_h(g) = h^{-\frac{Q}{2}} p(\delta_{\sqrt{h}}(g)), \quad \forall h > 0, \ g \in G.
\]

The kernel of the \( j \)-th Riesz transform \( X_j(-\Delta)^{\frac{-1}{2}} \) \((1 \leq j \leq n)\) is written simply as \( K_j(g,g') = K_j(g^{-1} \circ g) \). It is well-known that
\[
(2.4) \quad K_j \in C^\infty(G \setminus \{0\}), \quad K_j(\delta_r(g)) = r^{-Q} K_j(g), \quad \forall g \neq 0, \ r > 0, \ 1 \leq j \leq n,
\]
which also can be explained by (2.3) and the fact that
\[
K_j(g) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} h^{-\frac{Q}{2}} X_j p_h(g) \, dh = \frac{1}{\sqrt{\pi}} \int_0^{\infty} h^{-\frac{Q}{2}-1} (X_j p)(\delta_{\sqrt{h}}(g)) \, dh.
\]

Next we recall the homogeneous norm \( \rho \) (see for example [11]) on \( G \) which is defined to be a continuous function \( g \to \rho(g) \) from \( G \) to \([0,\infty)\), which is \( C^\infty \) on \( G \setminus \{0\} \) and satisfies
(a) \( \rho(g^{-1}) = \rho(g) \);
(b) \( \rho(\delta_r(g)) = r \rho(g) \) for all \( g \in G \) and \( r > 0 \);
(c) \( \rho(g) = 0 \) if and only if \( g = 0 \).

For the existence (also the construction) of the homogeneous norm \( \rho \) on \( G \), we refer to [11] Chapter 1, Section A. For convenience, we set
\[
\rho(g,g') = \rho(g^{-1} \circ g') = \rho(g^{-1} \circ g'), \quad \forall g, g' \in G.
\]

Recall that this defines a quasi-distance in the sense of Coifman–Weiss, in fact, we have the following improved pseudo-triangle inequality, there exists a constant \( C_p \geq 1 \) such that (see [3])
\[
(2.5) \quad |\rho(g_1,g_2) - \rho(g_1,g_3)| \leq C_p \rho(g_2,g_3) \quad \forall g_1, g_2, g_3 \in G.
\]

We now denote by \( d \) the Carnot–Carathéodory metric associated to \( \{X_j\}_{1 \leq j \leq n} \), which is equivalent to \( \rho \) in the sense that there exists constants \( C_1, C_2 > 0 \) such that for every \( g_1, g_2 \in G \) (see [3]),
\[
(2.6) \quad C_1 \rho(g_1,g_2) \leq d(g_1,g_2) \leq C_2 \rho(g_1,g_2).
\]

We point out that the Carnot–Carathéodory metric \( d \) even on the most special stratified Lie group, the Heisenberg group, is not smooth on \( G \setminus \{0\} \).

In the sequel, to avoid confusing notation, for \( g \in G \) and \( r > 0 \), \( B(g,r) \) denotes the open ball defined by \( \rho \).
For the Folland–Stein BMO space $\text{BMO}(G)$, note that we have an equivalent norm, defined by

$$\|b\|_{\text{BMO}(G)}' = \sup_B \inf_c \frac{1}{|B|} \int_B |b(x) - c|dx.$$  

For a ball $B$, the infimum above is attained and the constants where this happens can be found among the median values.

**Definition 2.1.** A median value of a function $b$ over a ball $B$ will be any real number $m_b(B)$ that satisfies simultaneously

$$|\{x \in B : b(x) < m_b(B)\}| \geq \frac{1}{2} |B|$$

and

$$|\{x \in B : b(x) > m_b(B)\}| \geq \frac{1}{2} |B|.$$  

Following the standard proof in [26, p.199], we can see that the constant $c$ in the definition of $\|b\|_{\text{BMO}(G)}'$ can be chosen to be a median value of $b$.

We now recall that given a weight $\nu$, the weighted BMO space $\text{BMO}_\nu(G)$ is defined as $\text{BMO}_\nu(G) := \{b \in L^1_{\text{loc}}(G) : \|b\|_{\text{BMO}_\nu(G)} < \infty\}$, where

$$\|b\|_{\text{BMO}_\nu(G)} := \sup_B \nu(B) \int_B |b(x) - b_B|dx.$$  

We also recall the definition of a Muckenhoupt $A_p$ weight. Let $w(x)$ be a nonnegative locally integrable function on $G$. For $1 < p < \infty$, we say $w$ is an $A_p$ weight, written $w \in A_p$, if

$$[w]_{A_p} := \sup_B \left( \frac{1}{|B|} \int_B w^p(x)dx \right)^{1/p} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w(x)} \right)^{1/(p-1)}dx \right)^{p-1} < \infty.$$  

Here the supremum is taken over all balls $B \subset G$. The quantity $[w]_{A_p}$ is called the $A_p$ constant of $w$.

Given $w \in A_p$ with $1 < p < \infty$, the space $L^p_w(G)$ is defined as the set of $w$-measurable functions with $\|f\|_{L^p_w(G)} := \left( \int_G |f(x)|^p w(x)dx \right)^{1/p} < \infty$.

We also recall the definition for Heisenberg group. Recall that $\mathbb{H}^n$ is the Lie group with underlying manifold $\mathbb{C}^n \times \mathbb{R} = \{[z,t] : z \in \mathbb{C}^n, t \in \mathbb{R}\}$ and multiplication law

$$[z,t] \circ [z',t'] = [z_1, \cdots, z_n, t] \circ [z'_1, \cdots, z'_n, t']$$

$$:= [z_1 + z'_1, \cdots, z_n + z'_n, t + t' + 2\text{Im}(\sum_{j=1}^n z_j \bar{z}'_j)].$$  

The identity of $\mathbb{H}^n$ is the origin and the inverse is given by $[z,t]^{-1} = [-z,-t]$. Hereafter, we agree to identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and to use the following notation to denote the points of $\mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$: $g = [z, t] \equiv [x, y, t] = [x_1, \cdots, x_n, y_1, \cdots, y_n, t]$ with $z = [z_1, \cdots, z_n]$, $z_j = x_j + iy_j$ and $x_j, y_j, t \in \mathbb{R}$ for $j = 1, \ldots, n$. Then, the composition law $\circ$ can be explicitly written as

$$g \circ g' = [x, y, t] \circ [x', y', t'] = [x + x', y + y', t + t' + 2(y, x') - 2(x, y')],$$  

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.

The Lie algebra of the left invariant vector fields of $\mathbb{H}^n$ is generated by (here and in the following, we shall identify vector fields as the associated first order differential operators)

(2.7) $$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$
for \( j = 1, \ldots, n \). We now denote \( X_{n+j} = Y_j \) for \( j = 1, \ldots, n \). Then the sub-Laplacian on \( \mathbb{H}^n \) is \( \Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2 \).

The Heisenberg distance derived form the the Korányi norm (which is also the standard homogeneous norm on \( \mathbb{H}^n \))

\[
d_K(g) = (\|z\|^4 + t^2)^{\frac{1}{4}}, g = (z,t) \in \mathbb{H}^n,
\]

is given by

\[
d_K(g,g') = d_K(g^{-1} \circ g) = d_K(g^{-1} \circ g'), \quad \forall g, g' \in \mathbb{H}^n.
\]

It is also a quasi-distance and there exists a constant \( C_K \geq 1 \) such that

\[
d_K(g_1, g_2) \leq C_K (d_K(g_1, g') + d_K(g', g_2)), \quad \forall g_1, g_2, g' \in \mathbb{H}^n.
\]

**Notation:** In what follows, \( C, C' \), etc. will denote various constants which depend only on the triple \( (\mathcal{G}, \rho, \{X_j\}_{1 \leq j \leq n}) \). By \( A \lesssim B \), we shall mean \( A \leq CB \) with such a \( C \), and \( A \sim B \) stands for \( A \leq CB \) and \( B \leq CA \).

3. **Revisit of the lower bound for kernel of Riesz transform \( R_j := X_j(-\Delta)^{-\frac{1}{2}} \) and proof of Theorem 1.1**

In this section, we study a suitable version of the lower bound for the kernel of the Riesz transform \( R_j := X_j(-\Delta)^{-\frac{1}{2}}, j = 1, \ldots, n \), on stratified Lie group \( \mathcal{G} \). Here we make good use of the dilation structure on \( \mathcal{G} \). It is not clear whether one can obtain similar lower bounds for the Riesz kernel on a general nilpotent Lie groups which is not stratified.

To begin with, we first recall that by (2.6) and the classical estimates for heat kernel and its derivations on stratified groups (see for example [28]), it is well-known that

\[
|K_j(g, g')| + \rho(g, g') \sum_{i=1}^n (|X_{i,g} K_j(g, g')| + |X_{i,g'} K_j(g, g')|)
\]

\[
\lesssim \rho(g, g')^{-Q}, \quad 1 \leq j \leq n, \ g \neq g',
\]

where \( X_{i,g} \) denotes the derivation with respect to \( g \).

In [9], the first four authors showed the following properties for the Riesz kernel \( K_j \).

**Lemma 3.1** ([9]). For all \( 1 \leq j \leq n \), we have \( K_j \neq 0 \) in \( \mathcal{G} \setminus \{0\} \).

Now we prove Theorem 1.1.

**Proof of Theorem 1.1.** For any fixed \( j \in \{1, \ldots, n\} \), by Lemma 3.1 and the scaling property of \( K_j \) (c.f. (2.3)), there exists \( \tilde{g}_j \) satisfying

\[
\rho(\tilde{g}_j) = 1 \quad \text{and} \quad K_j(\tilde{g}_j^{-1}) \neq 0.
\]

Since \( K_j \) is a \( C^\infty \) function on \( \mathcal{G}\setminus\{0\} \), there exists \( 0 < \epsilon < 1 \) such that

\[
K_j(g') \neq 0 \quad \text{and} \quad |K_j(g')| > \frac{1}{2}|K_j(\tilde{g}_j^{-1})|
\]

for all \( g' \in B(\tilde{g}_j^{-1}, 4C_\rho \epsilon) \), where \( C_\rho > 1 \) is the constant in (2.5). To be more specific, we have that for all \( g' \in B(\tilde{g}_j^{-1}, 4C_\rho \epsilon) \), the values \( K_j(g') \) and \( K_j(\tilde{g}_j^{-1}) \) have the same sign.

Take a large \( r_o \) with \( r_o > \frac{1}{\epsilon} \), we now set

\[
G_\epsilon := \bigcup_{r \geq \frac{r_o}{2\epsilon}} B(\tilde{g}_j, \epsilon),
\]
Moreover, we have
\[ \alpha := \min_{g \in B(\tilde{g}, \epsilon)} \rho(g) \geq 1 - C_{\rho} \epsilon \geq 1 - \frac{1}{100} \]
and
\[ \beta := \max_{g \in B(\tilde{g}, \epsilon)} \rho(g) \leq 1 + C_{\rho} \epsilon \leq 1 + \frac{1}{100}. \]

Now for any \( g \in G \), we let
\[ G := \tau_g(\tau_g(G_e)). \]
It is clear that
\[ \inf_{g' \in G} \rho(g, g') = r_o. \]
We now show that the set \( G \) defined as above satisfies all the required conditions.

First, we point out that for any \( g_2 \in G \) and for any \( g_1 \in B(g, 1) \), following similar calculation and estimates in [9, Theorem 1.1], we obtain that
\[ |K_j(g_1, g_2)| \geq \frac{1}{2} r^{-Q}|K_j(\tilde{g}_j^{-1})|, \quad |K_j(g_2, g_1)| \geq \frac{1}{2} r^{-Q}|K_j(\tilde{g}_j^{-1})|. \]
Moreover, all the \( K_j(g_1, g_2) \) as well as all the \( K_j(g_2, g_1) \), \( g_1, g_2 \in B(g, 1) \), have the same sign. We denote
\[ C := \min_{1 \leq j \leq n} \frac{1}{2} |K_j(\tilde{g}_j^{-1})|, \]
then \( (1.3) \) holds with the constant \( C \) defined above.

We now only show that for any \( R > 2r_o \), \( (1.4) \) holds, since the other properties are obvious. In fact, we note that
\[ B(g, R) \supset B(g, R) \cap G = \tau_g(B(0, R) \cap G_e). \]
Moreover, we have
\[ B(0, R) \cap G_e \supset \bigcup_{\frac{1}{100} \leq r \leq \frac{1}{100} R} \delta_r(B(\tilde{g}_j, \epsilon)) \supset \delta_{\frac{1}{100} R}(B(\tilde{g}_j, \epsilon)). \]
As a consequence, we have
\[ |B(0, 1)| R^Q = |B(g, R)| \geq |B(0, R) \cap G| \geq \left| \delta_{\frac{1}{100} R}(B(\tilde{g}_j, \epsilon)) \right| \geq C_r R^Q, \]
which shows that \( (1.4) \) holds. This completes the proof of Theorem 1.1. \( \square \)

By using \( r_o r \) to replace \( r_o \) in the above proof, we can also get the similar result for any ball \( B(g, r) \).

**Corollary 3.2.** Suppose that \( G \) is a stratified Lie group with homogeneous dimension \( Q \) and that \( j \in \{1, 2, \ldots, n\} \). There exist a large positive constant \( r_o \) and a positive constant \( C \) depending on \( K_j \) such that for every \( g \in G \) there exists a set \( G \subset G \) such that \( \inf_{g' \in G} \rho(g, g') = r_o r \) and that for every \( g_1 \in B(g, r) \) and \( g_2 \in G \), we have
\[ |K_j(g_1, g_2)| \geq C \rho(g_1, g_2)^{-Q}, \quad |K_j(g_2, g_1)| \geq C \rho(g_1, g_2)^{-Q}, \]
all \( K_j(g_1, g_2) \) as well as all \( K_j(g_2, g_1) \) have the same sign.
Moreover, the set $G$ is regular, in the sense that $|G| = \infty$ and that for any $R > 2r_0$, $B(g, R) \cap G \approx R^Q$, where the implicit constants are independent of $g$ and $R$.

4. Two weight estimates for commutator $[b, R_j]$ and the proof of Theorem 1.2

To begin with, we point out that by considering the stratified Lie group $G$ as a space of homogeneous type in the sense of Coifman and Weiss with the metric $\rho$, we have a system $\mathcal{D}(G)$ of dyadic cubes on $G$. We refer to the original construction from [5] and the refinement from [16]. See also [18, Section 2] for a summary.

Next we also recall that there exist adjacent systems of dyadic cubes on $G$, denoted by $\mathcal{D}^1(G), \ldots, \mathcal{D}^T(G)$, such that for each ball $B \subset G$, there exist $t \in \{1, 2, \ldots, T\}$ and $S \in \mathcal{D}^t(G)$ satisfying

$$B \subset S \subset CB,$$

where $C$ is an absolute constant independent of $t$ and $S$, and $CB$ denotes the ball with the same center as $B$ and radius $C$ times that of $B$. See [16] and [18, Section 2.4] for more details. Associated to each systems of dyadic cubes, one has the dyadic BMO space as follows. A dyadic weighted BMO space associated with the system $\mathcal{D}^t(G)$ is defined as $\text{BMO}_{\nu, \mathcal{D}^t(G)} := \{ b \in L^1_{\text{loc}}(G) : \| b \|_{\text{BMO}_{\nu, \mathcal{D}^t(G)}} < \infty \}$, where $\| b \|_{\text{BMO}_{\nu, \mathcal{D}^t(G)}} := \sup_{S \in \mathcal{D}^t(G)} \frac{1}{|S\nu(S)|} \int_S |b(g) - bs\nu dg|^\nu$.

Then according to the dyadic structure theorem studied in [16, 18], one has

$$\text{BMO}_{\nu}(G) = \bigcap_{t=1}^T \text{BMO}_{\nu, \mathcal{D}^t(G)}(G).$$

Thus, to verify a function $b$ is in $\text{BMO}_{\nu}(G)$, it suffices to verify it belongs to each weighted dyadic BMO space $\text{BMO}_{\nu, \mathcal{D}^t(G)}(G)$.

Given a dyadic cube $S \in \mathcal{D}^t(G)$ with $t = 1, \ldots, T$, and a measurable function $f$ on $G$, we define the local mean oscillation of $f$ on $S$ by

$$w_\lambda(f; S) = \inf_{c \in \mathbb{R}} \left((f - c)_S\right)^{\ast}(\lambda|S|), \quad 0 < \lambda < 1,$$

where $f^{\ast}$ denotes the non-increasing rearrangement of $f$.

With these notation and dyadic structure theorem above, following the same proof in [19, Lemma 2.1], we also obtain that for any weight $\nu \in A_2(G)$, we have

$$\| b \|_{\text{BMO}_{\nu}(G)} \leq C \sum_{t=1}^T \| b \|_{\text{BMO}_{\nu, \mathcal{D}^t(G)}} \leq C \sum_{t=1}^T \sup_{S \in \mathcal{D}^t(G)} w_\lambda(b; S) \frac{|S|}{\nu(S)}, \quad 0 < \lambda \leq \frac{1}{2^{Q+2}},$$

where $C$ depends on $\nu$.

Next, we point out that from the construction of dyadic system $\mathcal{D}^t(G)$ as in [5, 16], for every dyadic cube $S \in \mathcal{D}^t(G)$ in level $l$, there exist two balls $B_1$ and $B_2$ with radius $r_1$ and $r_2$, respectively, such that

$$B_1 \subset S \subset B_2$$

and that $C_1 r_1 \leq 2^{-l} \leq C_1 r_1$ and $C_2 r_2 \leq 2^{-l} \leq C_2 r_2$, where the constants $C_1, C_1, C_2$ and $C_2$ are independent of $r_1, r_2$ and $l$.

To prove Theorem 1.2, we first need to establish the following result.

**Proposition 4.1.** Suppose that $G$ is a stratified Lie group with homogeneous dimension $Q$ as defined in Section 2, $b \in L^1_{\text{loc}}(G)$ and that $K_j$ is the kernel of the $j$th Riesz transform on $G$, $j = 1, 2, \ldots, n$. Let $r_\alpha$ be the constant in Corollary 3.2. Then for any $k_0 > r_\alpha$ and for
any dyadic cube \( S \in D(G) \), there exist measurable sets \( E \subset S \) and \( F \subset k_0B_1 \) with \( B_1 \) the ball in \([1,2]\), such that

1. \(|E \times F| \sim |S|^2|,\)
2. \( w \frac{1}{2^{Q+2}}(b; S) \leq |b(g) - b(g')|, \quad \forall (g, g') \in E \times F,\)
3. \( K_j(g, g') \) and \( b(g) - b(g') \) do not change sign for any \((g, g') \in E \times F,\)
4. \( |K_j(g, g')| \geq C \rho(g, g')^{-Q} \) for any \((g, g') \in E \times F.\)

**Proof.** From the fact \([4.2]\), for any dyadic cube \( S \in D^t(G) \) with \( t = 1, 2, \ldots, T \), we now consider the ball \( B_2 \) containing \( S \) with radius comparable to the side-length of \( S \). For simplicity, we denote it by \( B = B(x_0, r) \).

From Corollary \([3.2]\) we have that there exists a large positive constant \( r_0 \) such that for this \( B(x_0, r) \), there exists a set \( G \subset G \) such that \( \inf_{g' \in G} \rho(g, g') = r_o \) and \( K_j(g, g') \) do not change sign for any \((g, g') \in B(x_0, r) \times G \). Moreover, \( (4) \) holds.

Now to show the other three properties, we need to consider an appropriate subset of \( G \). We define it as follows. For any \( k_0 > r_o \), we let \( F_{k_0} := k_0B \cap G \). Then by using Corollary \([3.2]\) again we have that

\[
(4.3) \quad |F_{k_0}| \sim k_0^Q \sim k_0^Q |B|.
\]

From the definition of \( w \frac{1}{2^{Q+2}}(b; S) \) we see that there exists a subset \( E \subset S \) with \(|E| = \frac{1}{2^{Q+2}}|S| \) such that for any \( g \in G \),

\[
(4.4) \quad w \frac{1}{2^{Q+2}}(b; S) \leq |b(g) - m_b(F_{k_0})|.
\]

Next, we show that there exist \( E \subset E \) and \( F \subset F_{k_0} \) such that \( |E| = \frac{1}{2^{Q+2}}|S|, |F| = \frac{1}{2}|F_{k_0}| \) and that

\[
(4.5) \quad |b(g) - m_b(F_{k_0})| \leq |b(g) - b(g')|, \quad \forall (g, g') \in E \times F
\]

and moreover, \( b(g) - b(g') \) does not change sign in \( E \times F. \)

To see this, we let

\[
E_1 = \{g \in E : b(g) \geq m_b(F_{k_0})\}, \quad E_2 = \{g \in E : b(g) \leq m_b(F_{k_0})\};
\]

\[
F_1 = \{g' \in F_{k_0} : b(g') \geq m_b(F_{k_0})\}, \quad F_2 = \{g' \in F_{k_0} : b(g') \leq m_b(F_{k_0})\}.
\]

Then we have

\[
|F_1| \geq \frac{1}{2}|F_{k_0}|, \quad |F_2| \geq \frac{1}{2}|F_{k_0}|,
\]

and there exists \( i \in \{1, 2\} \) such that \(|E_i| \geq \frac{1}{2}|E|\). Without loss of generality we assume \(|E_1| \geq \frac{1}{2}|E|\). Hence, there exist \( E \subset E_1 \) and \( F \subset F_1 \) such that

\[
|E| = \frac{1}{2}|E| \quad \text{and} \quad |F| = \frac{1}{2}|F_{k_0}|.
\]

Thus, for any \((g, g') \in E \times F\), we have

\[
|b(g) - m_b(F_{k_0})| = b(g) - m_b(F_{k_0}) \leq b(g) - b(g'),
\]

which implies that \([4.5]\) holds and that \( b(g) - b(g') \) does not change sign in \( E \times F. \)

As a consequence, we get that \((2) \) and \((3) \) hold. Next, from \([4.3]\), we get that

\[
|E \times F| = |E| \times |F| = \frac{1}{4}|E| \cdot |F| = \frac{1}{2^{Q+3}}|S| \cdot |F_{k_0}| \sim k_0^Q |S|^2,
\]

which shows that \((1) \) holds.

The proof of Proposition \([4.1]\) is complete. \(\square\)
we have
\[ g,g \]
\[ K \]
From property (3) in Proposition 4.1, we get that
\[ \text{From H"older's inequality, we further have} \]
\[ (4.2), \text{we obtain that} \]
\[ \text{for all} \ (b(g) - b(g')) \]
\[ \text{Proof of Theorem 1.2.} \]
\[ \text{Note that it suffices to prove (ii), since (i) follows from [15] or [19] using the size and smoothness of Riesz transform kernel only.} \]
\[ \text{To prove (ii), based on (4.1), it suffices to show that there exists a positive constant } C \]
\[ \text{such that for all dyadic cubes } S \in \mathcal{D}(g), \]
\[ (4.6) \]
\[ w_{\frac{1}{p^*}}(b; S) \leq C \frac{\nu(S)}{|S|} \| [b, R_j] \|_{L^p(g) \rightarrow L^p(g)}. \]
\[ \text{To see this, we first note that property (2) in Proposition 4.1 implies that} \]
\[ w_{\frac{1}{p^*}}(b; S)|E \times F| \leq \int_{E \times F} |b(g) - b(g')| \ dg dg'. \]
\[ \text{From this, using property (4) in Proposition 4.1 and the fact that } \rho(g, g') \leq \overline{C}(k_0 + 1) \text{diam}(S) \]
\[ \text{for all } (g, g') \in E \times F \text{ with the constant } \overline{C} \text{ depending only on } C_2 \text{ and } C_2 \text{ in the inclusion} \]
\[ \text{we obtain that} \]
\[ w_{\frac{1}{p^*}}(b; S)|E \times F| \leq C|S| \int_{E \times F} |b(g) - b(g')| \frac{1}{\rho(g, g')} \ dg dg'. \]
\[ \text{From property (3) in Proposition 4.1 we get that } K_j(g, g') \text{ and } b(g) - b(g') \text{ do not change sign for any} \ (g, g') \in E \times F. \]
\[ \text{Hence, taking into account the property (1) in Proposition 4.1 we have} \]
\[ w_{\frac{1}{p^*}}(b; S) \leq C \frac{1}{|S|} \int_{E \times F} |b(g) - b(g')| |K_j(g, g')| \ dg dg' \]
\[ = \frac{1}{|S|} \int_{E \times F} (b(g) - b(g')) K_j(g, g') \ dg dg' \]
\[ \leq C \frac{1}{|S|} \int_E [b, R_j] (\chi_F)(g) \ dg. \]
\[ \text{From H"older's inequality, we further have} \]
\[ w_{\frac{1}{p^*}}(b; S) \leq C \frac{1}{|S|} \left( \int_E \| [b, R_j] (\chi_F)(g) \|^p \lambda(g) \ dg \right)^{\frac{1}{p}} \left( \int_S \lambda^{-\frac{1-p}{p-1}}(g) \ dg \right)^{1-\frac{1}{p}} \]
\[ \leq C \frac{1}{|S|} \mu(F)^{\frac{1}{p}} \left( \int_S \lambda^{-\frac{1-p}{p-1}}(g) \ dg \right)^{1-\frac{1}{p}} \| [b, R_j] \|_{L^p(g) \rightarrow L^p(g)} \]
\[ \leq C \frac{1}{|S|} \mu(S)^{\frac{1}{p}} \left( \int_S \lambda^{-\frac{1-p}{p-1}}(g) \ dg \right)^{1-\frac{1}{p}} \| [b, R_j] \|_{L^p(g) \rightarrow L^p(g)} \]
\[ \leq C \frac{\nu(S)}{|S|} \| [b, R_j] \|_{L^p(g) \rightarrow L^p(g)}, \]
\[ \text{where the last inequality follows from the definition of the weight } \nu. \] This proves (4.6). The proof of Theorem 1.2 is complete. \[ \square \]

5. **Endpoint characterisation of \( \text{BMO}(\mathcal{G}) \) via the \( L \log^+ L \rightarrow L^{1,\infty} \) boundedness of the commutator \([b, R_j]\) and the proof of Theorem 1.3**

**Proof of Theorem 1.3.** For the sufficient part, we point out that it follows directly from [24] with only minor changes, since the whole proof can be adapted from Euclidean space to stratified Lie groups and the key conditions for the operator \( T \) in [24] are the upper bound of size and smoothness properties of the kernel; all of which we have in the setting at hand.
Thus, we have

\[ M(b, B) = \inf_c \frac{1}{|B|} \int_B |b(g) - c| \, dg \]

and we now prove

\[ \sup_B M(b, B) \leq C(Q, \theta_k). \tag{5.1} \]

We claim that it suffices to prove (5.1) for the ball \( B(0, 1) \). To see this, for a measurable function \( f \) on \( \mathcal{G} \), \( g_0 \in \mathcal{G} \) and \( r > 0 \), we define the translation and dilation of \( f \) by

\[ \tau_{g_0}(f)(g) := f(g_0 \circ g), \quad \delta_r(f)(g) := f(\delta_r(g)). \]

Remark that the Riesz transforms are translation-invariant and satisfy the scaling property, namely, for any \( g \in \mathcal{G} \), \( r > 0 \) and for every \( j = 1, 2, \ldots, n \),

\[ \tau_g(R_j(f)) = R_j(\tau_g(f)), \quad \delta_r(R_j(f)) = R_j(\delta_r(f)). \]

So we have

\[ [\tau_{g_0} \circ \delta_r(b), R_j](f) = \tau_{g_0} \circ \delta_r\left([b, R_j]\left(\delta_1 \circ \tau_{g_0}^{-1}(f)\right)\right). \]

Thus, it suffices to prove (5.1) for the ball \( B(0, 1) \).

Let

\[ M := \frac{1}{|B(0, 1)|} \int_{B(0, 1)} |b(g) - m_b(B(0, 1))| \, dg, \]

where \( m_b(B(0, 1)) \) is the median of \( b \) over \( B(0, 1) \) as in Definition 2.1. Since

\[ [b - m_b(B(0, 1)), R_j] = [b, R_j], \]

without loss of generality, we may assume that \( m_b(B(0, 1)) = 0 \). This means that we can find disjoint subsets \( E_1, E_2 \subset B(0, 1) \) such that

\[ E_1 \supset \{ g \in B(0, 1) : b(g) < 0 \}, \quad E_2 \supset \{ g \in B(0, 1) : b(g) > 0 \}, \]

and \( |E_1| = |E_2| = \frac{1}{2} |B(0, 1)| \).

Define \( \varphi(g) = \chi_{E_2}(g) - \chi_{E_1}(g) \). Then \( \varphi \) satisfies \( \text{supp} \varphi \subset B(0, 1) \),

\[ \|\varphi\|_{L^\infty(G)} = 1, \quad \varphi(g)b(g) \geq 0, \quad \int_{B(0, 1)} \varphi(g) \, dg = 0, \]

and

\[ \frac{1}{|B(0, 1)|} \int_{B(0, 1)} \varphi(g) b(g) \, dg = M. \]

In the following, for \( i = 1, \ldots, 10 \), \( A_i \) denotes a positive constant depending only on \( K_j, Q, C_g \) in (2.3) and \( A_i, 1 \leq i < i \). For the ball \( B(0, 1) \), take the set \( G \) as in Theorem 1.1. For \( g \in G \),

\[ \left| [b, R_j] \varphi(g) \right| = |b(g) R_j(\varphi)(g) - R_j(b \varphi)(g)| \geq |R_j(b \varphi)(g)| - |b(g)||R_j(\varphi)(g)|. \]

We estimate these two terms separately. By Theorem 1.1

\[ |R_j(b \varphi)(g)| = \int_{B(0, 1)} |K_j(g, g')| |b(g')| \, dg' \geq A_1 M \rho(g)^{-Q}. \]

For the second term, since \( \int_{B(0, 1)} \varphi(g) \, dg = 0 \), by (3.1), we have

\[ |R_j(\varphi)(g)| \leq \int_{B(0, 1)} |K_j(g, g') - K_j(g, 0)| |\varphi(g')| \, dg' \leq A_2 \rho(g)^{-Q-1}. \]

Thus, we have

\[ \left| [b, R_j] \varphi(g) \right| \geq A_1 M \rho(g)^{-Q} - A_2 |b(g)| \rho(g)^{-Q-1}. \]
Let
\[ F := \left\{ g \in G : |b(g)| > \frac{A_1}{2A_2} M \rho(g) \text{ and } \rho(g) < M^{\frac{1}{Q}} \right\}, \]
then by Theorem 1.1 we have
\[
\left| \left\{ g \in \mathcal{G} : |[b, R_j] \varphi(g)| > \frac{A_1}{2} \right\} \right| \geq |(G \setminus F) \cap B(0, M^{\frac{1}{Q}})|
\]
(5.2)
where the last inequality follows by taking
\[ M > (2r_0)^Q. \]
By assumption, we also have
\[
\left| \left\{ g \in \mathcal{G} : |[b, R_j] \varphi(g)| > \frac{A_1}{2} \right\} \right| \leq \theta_b \int_{B(0,1)} \frac{2|\varphi(g)|}{A_1} \left( 1 + \log^+ \left( \frac{2|\varphi(g)|}{A_1} \right) \right) \, dg
\]
(5.3)
\[
= \frac{2\theta_b}{A_1} \left( 1 + \log^+ \left( \frac{2}{A_1} \right) \right).
\]
Then (5.2) and (5.3) imply that
\[
|F| \geq A_3 M - |F|,
\]
by assuming that \( M > \frac{4\theta_b}{A_1 \log^+ \left( \frac{2}{A_1} \right)}. \)

Let \( \psi(g) := \text{sgn}(b(g)) \chi_{\mathcal{F}}(g) \), then for \( g \in B(0,1), \)
\[
|b, R_j| \psi(g)| \geq |b(R_j(\psi)(g)) - R_j(b\psi)(g)| \geq |R_j(b\psi)(g)| - |b(g)||R_j(\psi)(g)|.
\]
From Theorem 1.1, we have
\[
|R_j(b\psi)(g)| = \int_{G} K_j(g, g') b(g') \psi(g') \, dg' = \int_{F} |K_j(g, g')| |b(g')| \, dg'
\]
\[
\geq A_4 \int_{F} \rho(g')^{-Q} |b(g')| \, dg' \geq \frac{A_4 A_1}{2A_2} \int_{F} M \rho(g')^{-Q + 1} \, dg'
\]
\[
\geq \frac{A_4 A_1}{2A_2} M^{\frac{1}{Q}} |F|
\]
\[
\geq A_5 M^{1+\frac{1}{Q}}.
\]
For the second term, by (3.1), for \( g \in B(0,1), \) we have
\[
|R_j(\psi)(g)| \leq \int_{F} |K_j(g, g')| |\psi(g')| \, dg' \leq A_6 \int_{F} \rho(g')^{-Q} \, dg' \leq A_7 \log M.
\]
Therefore, for \( g \in B(0,1), \) we have
\[
|b, R_j| \psi(g)| \geq A_5 M^{1+\frac{1}{Q}} - A_7 |b(g)| \log M.
\]
By our assumption on \( R_j, \) we have
\[
\left| \left\{ g \in \mathcal{G} : |b, R_j| \psi(g)| > \frac{A_5}{2} M^{1+\frac{1}{Q}} \right\} \right| \leq \theta_b \int_{G} \frac{2|\psi(g)|}{A_5 M^{1+\frac{1}{Q}}} \left( 1 + \log^+ \left( \frac{2|\psi(g)|}{A_5 M^{1+\frac{1}{Q}}} \right) \right) \, dg
\]
\[
\leq A_8 \theta_b M^{-\frac{1}{Q}},
\]
where the last inequality follows by taking \( M \) large enough \( (M > (\frac{2}{A_5})^{\frac{Q}{Q+1}}). \)
On the other hand,
\[
A_8 \theta_b M^{-\frac{1}{Q}} \geq \left| \left\{ g \in B(0,1) : |b, R_j| \psi(g)| > \frac{A_5}{2} M^{1+\frac{1}{Q}} \right\} \right|
\]
where the last inequality comes from the fact that \( M^{-\frac{1}{d}} \log M < Q e^{-1} \) whenever \( M > e^Q \).

Therefore, we have
\[
M \leq \left( \frac{A_8}{A_{10}} \right)^Q \theta_b^Q.
\]

Summing up the above estimates, we can obtain that
\[
M \leq \max \left\{ (2r_o)^Q, \frac{4\theta_b}{A_1A_3} \left( 1 + \log^+ \left( \frac{2}{A_5} \right) \right), \left( \frac{2}{A_5} \right)^{-\frac{1}{d+1}}, e^Q, \left( \frac{\theta_b A_8}{A_{10}} \right)^Q \right\} =: C(Q, \theta_b).
\]

The proof of Theorem 6.3 is complete. \( \square \)

6. ENDPOINT CHARACTERISATION OF COMMUTATOR \([b, R_j]\) VIA \( H^1(\mathcal{G})\) AND \( \text{BMO}(\mathcal{G})\) AND PROOF OF THEOREM 1.4

Recall that the Hardy space \( H^1(\mathcal{G}) \) can be characterized by the atomic decomposition \([\text{III}]\).

**Definition 6.1.** The space \( H^1(\mathcal{G}) \) is the set of functions of the form \( f = \sum_{j=1}^\infty \lambda_j a_j \) with \( \{\lambda_j\} \in \ell^1 \) and \( a_j \) a \((1,q)\) atom, \(1 < q \leq \infty\), meaning that it is supported on a ball \( B \subset \mathcal{G} \), has mean value zero \( \int_B a(g)dg = 0 \) and has a size condition \( \|a\|_{L^q(\mathcal{G})} \leq |B|^\frac{1}{q-1} \). The norm of \( H^1(\mathcal{G}) \) is defined by:
\[
\|f\|_{H^1(\mathcal{G})} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : \{\lambda_j\} \in \ell^1, f = \sum_{j=1}^\infty \lambda_j a_j, a_j \text{ a } (1,q)\text{-atom} \right\}
\]
with the infimum taken over all possible representations of \( f \) via atomic decompositions.

We point out that for any \( q \in (1, \infty] \), the definitions of \( H^1(\mathcal{G}) \) via these \((1,q)\)-atoms are equivalent.

**Proposition 6.2.** Suppose that \( \mathcal{G} \) is a stratified Lie group, \( b \in \text{BMO}(\mathcal{G}) \) and \( j \in \{1,2,\ldots,n\} \). Then the following statements are equivalent:

(i) \([b, R_j]\) is bounded from \( H^1(\mathcal{G}) \) to \( L^1(\mathcal{G}) \);

(ii) \( b \) satisfies the following condition: for any \((1,p)\)-atom \( a \) with \( 1 < p < \infty \) supported in a ball \( B \) and \( \bar{g} \in B \),
\[
\left( \int_{(r_o B)^c} \left| K_j(g,\bar{g}) \right| dg \right) \left| \int_B b(g')a(g')dg' \right| \leq C,
\]
where \( r_o \) is the one in Theorem 1.1.

**Proof.** Note that \( b \in \text{BMO}(\mathcal{G}) \). We have that \([b, R_j]\) is bounded on \( L^p(\mathcal{G}) \) for \( 1 < p < \infty \). Assume that \( a \) is a \((1,p)\)-atom which is supported in some ball \( B \), then by \([9] \) Theorem 1.2, we can see that \([b, R_j](a)\) makes sense and belongs to \( L^p(\mathcal{G}) \).

For any \( g \in \mathcal{G} \), we can write
\[
[b, R_j](a)(g) = \chi_{r_o B}(g)[b, R_j](a)(g) + \chi_{(r_o B)^c}(g) (b(g) - b_B) R_j(a)(g)
\]
\[
- \chi_{(r_o B)^c}(g) \int_{\mathcal{G}} (K_j(g,g') - K_j(g,\bar{g})) (b(g') - b_B) a(g')dg'
\]
\[ -\chi_{(r_oB)^c}(g) \int \mathcal{K}_j(g, \tilde{g}) (b(g') - b_B) a(g')dg' =: I_1(g) + I_2(g) + I_3(g, \tilde{g}) + I_4(g, \tilde{g}), \]

where \( \tilde{g} \) is any point in \( B \).

For \( I_1 \), by Hölder’s inequality, \[8\] Theorem 1.2 and the definition of atom, we have

\[
\|I_1\|_{L^1(\mathcal{G})} \leq |r_oB|^{1 - \frac{1}{p}} \left( \int_{r_oB} |[b, R_j](a)(g)|^p dg \right)^{\frac{1}{p}} \leq C|B|^{1 - \frac{1}{p}} \|a\|_{L^p(\mathcal{G})} \leq C|B|^{\frac{1}{p} - 1} = C,
\]

for any \( 1 < p < \infty \).

By the method of choosing \( r_o \) in Theorem 1.1, we can assume \( r_o = 2^\gamma \) for some \( \gamma > 1 \) and \( \gamma \in \mathbb{N} \).

For the term \( I_2 \), since \( a \) has mean value zero, by (3.1), we have

\[
\|I_2\|_{L^1(\mathcal{G})} \leq \int_{(r_oB)^c} \left| b(g) - b_B \right| \int_B \left| (\mathcal{K}_j(g, g') - \mathcal{K}_j(g, \tilde{g})) a(g')dg' \right| dg 
\leq \sum_{l=\gamma+1}^{\infty} \int_{2^{l-1}B} \int_{2^lB} \left| b(g) - b_B \right| \left( \int_B \left| \frac{\rho(g', \tilde{g})}{\rho(g, \tilde{g})}\right|^p dg' \right)^{\frac{1}{p}} \|a\|_{L^p(\mathcal{G})} dg
\leq C \sum_{l=\gamma+1}^{\infty} 2^{-l} \frac{1}{|2^lB|} \int_{2^lB} \left| b(g) - b_B \right| dg,
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \). By noting that \( |b_{2^lB} - b_{2^{l-1}B}| \leq 2Q \|b\|_{\text{BMO}(\mathcal{G})} \), we get that

\[
\|I_2\|_{L^1(\mathcal{G})} \leq C \sum_{l=\gamma+1}^{\infty} 2^{-l} 2Q \|b\|_{\text{BMO}(\mathcal{G})} = C.
\]

For the term \( I_3 \), by using (3.1), Hölder’s inequality and the \( L^p \) norm of the atom \( a \), we have

\[
\|I_3\|_{L^1(\mathcal{G})} \leq \sum_{l=\gamma+1}^{\infty} \int_{(2^{l-1}B)\setminus 2^lB} \int_B \frac{d(g', \tilde{g})}{d(g, \tilde{g})^{q+1}} \left| b(g') - b_B \right| \left| a(g') \right| dg' dg
\leq C \sum_{l=\gamma+1}^{\infty} 2^{-l} \int_B \left| b(g') - b_B \right| \left| a(g') \right| dg' 
\leq C \sum_{l=\gamma+1}^{\infty} 2^{-l} \frac{1}{|B|} \int_B \left| b(g') - b_B \right|^{p'} dg' \left| a(g') \right| \frac{1}{p'}
\leq C \sum_{l=\gamma+1}^{\infty} 2^{-l} \|b\|_{\text{BMO}(\mathcal{G})} 
\leq C,
\]

where the fourth inequality follows from the John–Nirenberg inequality for \( \text{BMO} \) space.

For the term \( I_4 \), by the mean value zero property of \( a \), we have

\[
\|I_4\|_{L^1(\mathcal{G})} = \int_{(r_oB)^c} \int_B \mathcal{K}_j(g, \tilde{g})(b(g') - b_B) a(g')dg' dg 
\]
Therefore,\[ \text{(6.9)} \]
\[ \left( \int_{(\tau, B)^c} |K_j(g, \tilde{g})| dg \right) \left( \int_B b(g')a(g') dg' \right). \]

From \text{(6.2)}, \text{(6.3)}, \text{(6.4)} and \text{(6.5)}, we can see that, for any \((1, p)\)-atom \(a\),
\[ |||b, R_j|||_{L^1(G)} \leq C \]
if and only if \(||I_4||_{L^1(G)} \leq C\). Then Proposition \text{6.2} follows from \text{(6.6)}. \(\Box\)

**Proposition 6.3.** Suppose that \(G\) is a stratified Lie group, \(b \in \text{BMO}(G)\) and \(j \in \{1, 2, \ldots, n\}\). Then the following statements are equivalent:

(i) \([b, R_j]\) is bounded from \(L^\infty_c(G)\) to \(\text{BMO}(G)\);

(ii) \(b\) satisfies the following condition: For any ball \(B\), any \(\tilde{g} \in B\) and \(f \in L^\infty_c(G)\),
\[ \left( \frac{1}{|B|} \int_B |b^g - b_B| dg \right) \left( \int_{(\tau, B)^c} K_j(\tilde{g}, g') f(g') dg' \right) \leq C ||f||_{L^\infty(G)}, \]
where \(\tau_o\) is the one in Theorem \text{1.1}.

**Proof.** Let \(f\) be a bounded function with compact support, then \(f \in L^p(G)\). Since \(b \in \text{BMO}(G)\), by \text{[9] Theorem 1.2} we can see that \([b, R_j]f \in L^p(G)\) for any \(1 < p < \infty\). Thus, \([b, R_j]f\) is a locally integrable function. Fix any ball \(B \subset G\), we can write
\[ f = f\chi_{r_o B} + f\chi_{(r_o B)^c} =: f_1 + f_2, \]
where \(r_o\) is the constant in Theorem \text{1.1}. Then for \(g \in B\), and for any \(\tilde{g} \in B\), we have
\[ [b, R_j](f)(g) - ([b, R_j]f)_B = [b, R_j](f_1)(g) - ([b, R_j]f_1)_B + (b^g - b_B)(R_j(f_2)(g) - R_j(f_2)(\tilde{g})) \]
\[ - \frac{1}{|B|} \int_B (b^g - b_B) (R_j(f_2)(g') - R_j(f_2)(\tilde{g})) dg' \]
\[ + \frac{1}{|B|} \int_B (R_j((b - b_B)f_2)(g') - R_j((b - b_B)f_2)(\tilde{g})) dg' \]
\[ + (b^g - b_B)R_j(f_2)(\tilde{g}). \]

Set
\[ J_1(g) = [b, R_j](f_1)(g), \]
\[ J_2(g, \tilde{g}) = (b^g - b_B)(R_j(f_2)(g) - R_j(f_2)(\tilde{g})), \]
\[ J_3(g', \tilde{g}) = R_j((b - b_B)f_2)(g') - R_j((b - b_B)f_2)(\tilde{g}), \]
\[ J_4(g, \tilde{g}) = (b^g - b_B)R_j(f_2)(\tilde{g}). \]

Then we have
\[ \text{(6.8)} \]
\[ [b, R_j](f)(g) - ([b, R_j]f)_B = J_1(g) - (J_1)_B + J_2(g, \tilde{g}) - (J_2(\cdot, \tilde{g}))_B \]
\[ + (J_3(\cdot, g))_B + J_4(g, \tilde{g}). \]

Therefore,
\[ \text{(6.9)} \]
\[ \frac{1}{|B|} \int_B ||[b, R_j](f)(g) - ([b, R_j]f)_B|| dg \]
\[ \leq \frac{2}{|B|} \int_B |J_1(g)| dg + \frac{2}{|B|} \int_B |J_2(g, \tilde{g})| dg + \frac{1}{|B|} \int_B |(J_3(\cdot, g))_B| dg \]
\[ + \frac{1}{|B|} \int_B |J_4(g, \tilde{g})| dg \]
\[ =: 2L_1 + 2L_2(\tilde{g}) + L_3 + L_4(\tilde{g}). \]
For the term $L_1$, since $b \in \text{BMO}(\mathcal{G})$, by Hölder’s inequality and [9, Theorem 1.2], for any $1 < p < \infty$, we have
\[
L_1 \leq \left( \frac{1}{|B|} \int_B |b, R_j|(f_1)(g) \right)^p \frac{1}{p} \tag{6.10}
\]
\[
\leq C|B|^{-\frac{1}{p}} \|b\|_{\text{BMO}(\mathcal{G})} \left( \int_{r_0B} |f_1(g)|^p \right)^{\frac{1}{p}} \leq C(r_0) \|b\|_{\text{BMO}(\mathcal{G})} \|f\|_{L^\infty(\mathcal{G})}.
\]

For any $\tilde{g} \in B$, we have
\[
L_2(\tilde{g}) \leq \frac{1}{|B|} \int_B |b(g) - b_B||R_j(f_2)(g) - R_j(f_2)(\tilde{g})| \, dg.
\]
By (3.1), we can obtain
\[
|R_j(f_2)(g) - R_j(f_2)(\tilde{g})| \leq \|f\|_{L^\infty(\mathcal{G})} \sum_{l=\gamma+1}^\infty \int_{2^lB \setminus 2^{l-1}B} |K_j(g, g') - K_j(\tilde{g}, g')| \, dg'
\]
\[
\leq C\|f\|_{L^\infty(\mathcal{G})} \sum_{l=\gamma+1}^\infty 2^{-l} \frac{1}{|2^lB|} \int_{2^lB} |b(g)| \, dg'
\]
\[
\leq C\|f\|_{L^\infty(\mathcal{G})}.
\] Therefore, for any $\tilde{g} \in B$,
\[
L_2(\tilde{g}) \leq C\|f\|_{L^\infty(\mathcal{G})} \frac{1}{|B|} \int_B |b(g) - b_B| \, dg \leq C\|b\|_{\text{BMO}(\mathcal{G})} \|f\|_{L^\infty(\mathcal{G})}. \tag{6.11}
\]

For the term $L_3$, we use (3.1) again, then for any $g, g' \in B$, we have
\[
|J_3(g', g)| \leq C\|f\|_{L^\infty(\mathcal{G})} \sum_{l=\gamma+1}^\infty 2^{-l} \frac{1}{|2^lB|} \int_{2^lB} |b(g_1) - b_B| \, dg_1
\]
\[
\leq C\|b\|_{\text{BMO}(\mathcal{G})} \|f\|_{L^\infty(\mathcal{G})}.
\] Thus we have
\[
L_3 \leq C\|b\|_{\text{BMO}(\mathcal{G})} \|f\|_{L^\infty(\mathcal{G})}. \tag{6.12}
\]

From (6.8), we have
\[
|J_4(g, \tilde{g})| \leq |[b, R_j](f)(g) - ([b, R_j]f)_B| + |J_1(g)| + |(J_1)_B| + |J_2(g, \tilde{g})|
\]
\[
+ \|(J_2(\cdot, \tilde{g}))_B\| + |(J_3(\cdot, g))_B|,
\]
which means
\[
L_4(\tilde{g}) \leq \frac{1}{|B|} \int_B |[b, R_j](f)(g) - ([b, R_j]f)_B| \, dg + 2L_1 + 2L_2(\tilde{g}) + L_3. \tag{6.13}
\]

By (6.9), (6.13) and (6.11)-(6.12), we can see that $[b, R_j]$ is bounded from $L^\infty_c(\mathcal{G})$ to BMO(\mathcal{G}) if and only if $L_4(\tilde{g}) \leq C\|f\|_{L^\infty(\mathcal{G})}$ for any $\tilde{g} \in B$, i.e.,
\[
C\|f\|_{L^\infty(\mathcal{G})} \geq \frac{1}{|B|} \int_B |J_4(g, \tilde{g})| \, dg
\]
\[
= \frac{1}{|B|} \int_B |(b(g) - b_B)R_j(f_2)(\tilde{g})| \, dg
\]
\[
= \left( \frac{1}{|B|} \int_B |b(g) - b_B| \, dg \right) \int_{(r_0B)^c} K_j(\tilde{g}, g') f(g') \, dg'.
\]
This proves the proposition.

Proof of Theorem 1.4. From Proposition 6.2 and Proposition 6.3, we can see that it is sufficient to show both (6.1) and (6.7) are equivalent to the condition (iii).

For the equivalence of (6.1) and (iii). If $b$ equals a constant almost everywhere, then for any atom $a$ supported in some ball $B = B(g_0, r)$ and for any $\tilde{g} \in B$,

$$\left( \int_{(r,B)^c} |K_j(g, \tilde{g})| \, dg \right) \left( \int_B b(g')a(g') \, dg' \right) \leq C \left( \int_{(r,B)^c} |K_j(g, \tilde{g})| \, dg \right) \left( \int_B a(g') \, dg' \right) = 0,$$

due to the mean value zero property of atom.

Conversely, assume that (6.1) holds. Let $G$ be the set in Corollary 3.2 then $\inf_{g \in G'} \rho(g_0, g') = r_0r$, $|G| = \infty$ and for $g \in G$, $\tilde{g} \in B$, we have

$$|K_j(g, \tilde{g})| \geq C \rho(g, g_0)^{-Q}.$$ 

Therefore

$$C \geq \left( \int_{(r,B)^c} |K_j(g, \tilde{g})| \, dg \right) \left( \int_B b(g')a(g') \, dg' \right) \geq C \left( \int_G \rho(g, g_0)^{-Q} \, dg \right) \left( \int_B b(g')a(g') \, dg' \right),$$

where $a$ is any $(1,p)$-atom supported in $B$ with $1 < p < \infty$. This is impossible unless

(6.14) $$\int_B b(g')a(g') \, dg' = 0$$

for every ball $B$ and any $(1,p)$-atom supported in $B$ with $1 < p < \infty$. We recall a result from [23]: one can define the space $H^1_{\text{fin}}(G)$ as the set of all finite linear combinations of $(1,p)$-atoms, which is endowed with the natural norm

$$\|f\|_{H^1_{\text{fin}}(G)} = \inf \left\{ \left( \sum_{j=1}^N |\lambda_j|^p \right)^{\frac{1}{p}} : f = \sum_{j=1}^N \lambda_j a_j, \text{ a}_j (1,p) - \text{atoms, } N \in \mathbb{N} \right\}.$$

Note that $H^1_{\text{fin}}(G)$ is dense in $H^1(G)$. Moreover, from [23] Proposition 2], we know that the two norms $\| \cdot \|_{H^1_{\text{fin}}(G)}$ and $\| \cdot \|_{H^1(G)}$ are equivalent on $H^1_{\text{fin}}(G)$. Hence, if (6.14) holds for every $(1,p)$-atom, then we obtain that $b$ is a zero linear functional on $H^1_{\text{fin}}(G)$, and hence extends to a zero linear functional on $H^1(G)$. This shows that $b$ is in $\text{BMO}(G)$ with

$$\|b\|_{\text{BMO}(G)} = 0.$$

Thus, $b$ equals a constant almost everywhere.

For the equivalence of (6.7) and (iii). It is easy to see that if $b$ equals a constant almost everywhere, then (6.7) holds. Conversely, take $f_N(g) = \chi_{G \cap B(g_0, N)}(g)$, $N \in \mathbb{N}$, in (6.7) to obtain

$$C \geq \left( \frac{1}{|B|} \int_B |b(g) - b_B| \, dg \right) \left( \int_{G \cap B(g_0, N)} K_j(\tilde{g}, g') \, dg' \right) = \left( \frac{1}{|B|} \int_B |b(g) - b_B| \, dg \right) \left( \int_{G \cap B(g_0, N)} |K_j(\tilde{g}, g')| \, dg' \right) \geq C_1 \left( \frac{1}{|B|} \int_B |b(g) - b_B| \, dg \right) \left( \int_{G \cap B(g_0, N)} \rho(g', g_0)^{-Q} \, dg' \right) = C_2 \log N \left( \frac{1}{|B|} \int_B |b(g) - b_B| \, dg \right),$$

\[\square\]
for all $N \in \mathbb{N}$ large enough. Letting $N$ go to infinity we have $b(g) = b_B$ a.e. in $B$, and hence $b$ must be constant almost everywhere.

7. Endpoint characterisation of commutator $[b, R_j]$ on Heisenberg groups $\mathbb{H}^n$ and proof of Theorem 1.6

Proof of Theorem 1.6. We handle the Riesz transform kernel by using the idea in the proof of [20] Proposition 3.1.

Recall that (see for example [17] and [12]) the explicit expression of heat kernel on the Heisenberg group $\mathbb{H}^n$ is as follows: for $g = (z, t) \in \mathbb{H}^n$,

$$p_h(g) = \frac{1}{2(4\pi h)^{n+1}} \int_{\mathbb{R}} \exp\left(\frac{\lambda}{4h} (it - \|z\|^2 \coth \lambda)\right) \left(\frac{\lambda}{\sinh \lambda}\right)^n d\lambda,$$

where $\|z\| = \sum_{j=1}^n \|z_j\|^2$.

For any $g = (z, t) \in \mathbb{H}^n$, by using the explicit expression of the heat kernel above and by Fubini's theorem, we have

$$(-\Delta_{\mathbb{H}^n})^{-\frac{1}{2}}(g) = C \int_0^{+\infty} h^{-\frac{1}{2}} p_h(g) dh$$

$$= C' \int_0^{+\infty} h^{-n-\frac{3}{2}} \exp\left(\frac{\lambda}{4h} \left(it - \|z\|^2 \coth \lambda\right)\right) dh \left(\frac{\lambda}{\sinh \lambda}\right)^n d\lambda$$

$$= C'' \int_0^{+\infty} (\|z\|^2 \lambda \coth \lambda - i\lambda t)^{-n-\frac{1}{2}} \left(\frac{\lambda}{\sinh \lambda}\right)^n d\lambda.$$

Then by (2.7), for $j = 1, \cdots, n$, we can obtain

$$X_j(-\Delta_{\mathbb{H}^n})^{-\frac{1}{2}}(g) = C \left(\frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}\right) \int_{\mathbb{R}} (\|z\|^2 \lambda \coth \lambda - i\lambda t)^{-n-\frac{1}{2}} \left(\frac{\lambda}{\sinh \lambda}\right)^n d\lambda$$

$$= C(-2n - 1) \left[x_j \int_{\mathbb{R}} (\|z\|^2 \lambda \coth \lambda - i\lambda t)^{-n-\frac{1}{2}} \left(\frac{\lambda}{\sinh \lambda}\right)^{n+1} \cosh \lambda d\lambda \right.$$

$$- iy_j \int_{\mathbb{R}} (\|z\|^2 \lambda \coth \lambda - i\lambda t)^{-n-\frac{1}{2}} \left(\frac{\lambda}{\sinh \lambda}\right)^n d\lambda].$$

Observe that

$$\|z\|^2 \lambda \coth \lambda - i\lambda t = \frac{\lambda}{\sinh \lambda} d_K^2(g) \left(\frac{\|z\|^2}{d_K^2(g)} \cosh \lambda - i \frac{t}{d_K^2(g)} \sinh \lambda\right)$$

$$= \frac{\lambda}{\sinh \lambda} d_K^2(g) \cosh(\lambda - i\phi),$$

where

$$-\frac{\pi}{2} \leq \phi = \phi(\|z\|, t) \leq \frac{\pi}{2}, \quad e^{i\phi} = d_K^2(g)(\|z\|^2 + it),$$

and $d_K(g)$ is the Korányi norm as defined in (2.8). Therefore,

$$X_j(-\Delta_{\mathbb{H}^n})^{-\frac{1}{2}}(g) = C d_K^{Q-1}(g) \left[x_j \int_{\mathbb{R}} \left(\frac{\lambda}{\sinh \lambda}\right)^{-\frac{1}{2}} \cosh(\cosh(\lambda - i\phi))^{-\frac{n+1}{2}} d\lambda$$

$$- iy_j \int_{\mathbb{R}} \left(\frac{\lambda}{\sinh \lambda}\right)^{-\frac{1}{2}} \lambda(\cosh(\lambda - i\phi))^{-\frac{n+1}{2}} d\lambda].$$

Then by the Cauchy integral theorem, we have

$$X_j(-\Delta_{\mathbb{H}^n})^{-\frac{1}{2}}(g) = C d_K^{Q-1}(g) F_j(g),$$

where

$$F_j(g) = x_j \int_{\mathbb{R}} \left[\frac{\sinh(\lambda + i\phi)}{\lambda + i\phi}\right]^\frac{1}{2} \cosh(\lambda + i\phi)(\cosh(\lambda))^{-\frac{n+1}{2}} d\lambda$$
Similarly,
\[ X_{n+j}(-\Delta_{H^n})^{-\frac{1}{2}}(g) = C d_{-K}^{-1}(g) H_j(g), \]
where
\[ H_j(g) = y_j \int_{\mathbb{R}} \left[ \frac{\sinh(\lambda + i\phi)}{\lambda + i\phi} \right]^\frac{3}{2} \cosh(\lambda + i\phi)(\cosh \lambda)^{-\frac{3}{2}} d\lambda + ix_j \int_{\mathbb{R}} \left[ \frac{\sinh(\lambda + i\phi)}{\lambda + i\phi} \right]^\frac{3}{2} (\cosh \lambda)^{-\frac{3}{2}} d\lambda. \]

Let
\[ A_n(w) = \int_{\mathbb{R}} \left[ \frac{\sinh(\lambda + w)}{\lambda + w} \right]^\frac{3}{2} \cosh(\lambda + w)(\cosh \lambda)^{-\frac{3}{2}} d\lambda, \quad w \in \mathbb{C}, \]
\[ B_n(w) = \int_{\mathbb{R}} (\lambda + w) \left[ \frac{\sinh(\lambda + w)}{\lambda + w} \right]^\frac{3}{2} (\cosh \lambda)^{-\frac{3}{2}} d\lambda, \quad w \in \mathbb{C}. \]

Then
\[ F_j(g) = x_j A_n(i\phi) - iy_j B_n(i\phi), \quad H_j(g) = y_j A_n(i\phi) + ix_j B_n(i\phi). \]

Notice that \( A_n(w) \) and \( B_n(w) \) are analytic in some domain on \( \mathbb{C} \), which contains the segment \([-\frac{\pi}{2}, \frac{\pi}{2}]\) of the imaginary axis, and \( A_n(0) \neq 0, B_n(0) = 0 \). Thus, \( A_n(i\phi) \) has at most a finite number of zero points on \([-\frac{\pi}{2}, \frac{\pi}{2}]\), i.e., there exist \( \{ \phi_\ell \}_{\ell=1}^N \subset [-\frac{\pi}{2}, \frac{\pi}{2}] \) such that \( A_n(i\phi_\ell) = 0 \). By noting that \( \phi = \phi(\|z\|^2, t) \), we see that \( \{ \phi_\ell \}_{\ell=1}^N \) corresponds to a set \( \mathcal{H}_N \) in \( \mathbb{H}^n \) with
\[ \mathcal{H}_N := \{(z,t) \in \mathbb{H}^n : \phi_\ell \neq \phi(\|z\|^2, t), \ell = 1, \ldots, N\}, \]
which has measure zero.

Therefore, for any fixed \( \phi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \backslash \{ \phi_\ell \}_{\ell=1}^N \), when we fixed \( |z_j|^2 = x_j^2 + y_j^2 \) with \( x_j \cdot y_j \neq 0 \), there are at most two \( z_j \) satisfying \( F_j(g) = 0 \) (or \( H_j(g) = 0 \)). Consequently, the measure of the set of \( g \) satisfying \( F_j(g) = 0 \) (or \( H_j(g) = 0 \)) is zero. This completes the proof. \( \square \)

Proof of Theorem [10]. We first prove the sufficient part. Suppose \( j \in \{1, \ldots, 2n\} \) and \( b \in L^\infty(\mathbb{H}^n) \) with \( ||b||_{L^\infty(\mathbb{H}^n)} \neq 0 \). For \( f \in L^1(\mathbb{H}^n) \), and for any \( \lambda > 0 \), we have
\begin{align*}
|\{ g \in \mathbb{H}^n : |b R_j(f)(g)| > \lambda \}| &\leq |\{ g \in \mathbb{H}^n : |b(g) R_j(f)(g)| > \lambda/2 \}| + |\{ g \in \mathbb{H}^n : |R_j(bf)(g)| > \lambda/2 \}| \\
&\leq C ||b||_{L^\infty(\mathbb{H}^n)} \frac{||f||_{L^1(\mathbb{H}^n)}}{\lambda},
\end{align*}
which shows that \( b, R_j \) is of weak type \((1,1)\), where the last inequality follows from the fact that \( R_j \) is of weak type \((1,1)\).

For the necessity part. Suppose that \( b \in L^1_{loc}(\mathbb{H}^n) \), then \( b \) is finite almost everywhere and almost every point is a Lebesgue point of \( b \).

Let \( f = \frac{1}{|B(0,1)|} \chi_{B(0,1)} \). For every \( \epsilon > 0 \), set \( f_\epsilon(g) = \frac{1}{\epsilon^2} f(\delta_{\epsilon,-1}(g)) \) and \( f^\epsilon_\epsilon(g) = f_\epsilon(g^{-1} \circ g) \).

Fix any Lebesgue point \( g' \) of \( b \), since \( K_j \in C^\infty(\mathbb{H}^n \setminus \{0\}) \), for any \( g \neq g' \), we have
\begin{align*}
\lim_{\epsilon \to 0} \left| b, R_j(f^\epsilon_\epsilon)(g) \right| &\leq \lim_{\epsilon \to 0} \left| P.V. \int_{\mathbb{H}^n} K_j(g, \tilde{g})(b(g) - b(\tilde{g}))f_\epsilon(g^{-1} \circ \tilde{g})d\tilde{g} \right| \\
&= \lim_{\epsilon \to 0} \frac{1}{|B(g', \epsilon)|} \left| P.V. \int_{|B(g', \epsilon)|} K_j(g, \tilde{g})(b(g) - b(\tilde{g}))d\tilde{g} \right| \\
&= |K_j(g, g')| \left| b(g) - b(g') \right|.
\end{align*}
Thus,

\[(7.1) \quad |\{ g \in \mathbb{H}^n \setminus \{g'\} : |K_j(g, g')| \cdot |b(g) - b(g')| > \lambda\}| \leq \frac{||[b, R_j]]||_{L^1(\mathbb{H}^n) \to L^{1,\infty}(\mathbb{H}^n)}}{\lambda}.
\]

By Theorem 1.5, we can see that \(K_j(g) \neq 0\) almost everywhere on \(S(0, 1)\). Fix small \(\varepsilon > 0\) and take \(\Gamma\) to be a compact subset of \(S(0, 1)\) such that \(K_j(g) \neq 0\) on \(\Gamma\) and \(\sigma(S(0, 1) \setminus \Gamma) < \varepsilon\), where \(\sigma\) is the Radon measure on \(S(0, 1)\). Let \(C_K = \inf\{|K_j(g)| : g \in \Gamma\}\), since \(K_j \in C^\infty(\mathbb{H}^n \setminus \{0\})\), \(j = 1, \cdots, 2n\), we have \(C_K > 0\).

Set

\[S_\Gamma(g') = \left\{ g \in \mathbb{H}^n : \delta_{d_K(g,g')}^{-1}(g^{-1} \circ g) \in \Gamma \right\},\]

\[\Lambda_\lambda(g') = \left\{ g \in \mathbb{H}^n : \delta_{d_K(g,g')}^{-1}(g^{-1} \circ g) \in \Gamma, \frac{|b(g) - b(g')|}{d_K(g,g')^Q} > \lambda \right\}.
\]

Then for any \(r > 0\), we have

\[(7.2) \quad |B(0, r) \setminus S_\Gamma(0)| < \varepsilon \frac{r^Q}{Q}.
\]

By the homogeneous property of \(K_j\) (2.3) and (7.1), we have

\[|\Lambda_\lambda(g')| \leq \left\{ g \in \mathbb{H}^n : \delta_{d_K(g,g')}^{-1}(g^{-1} \circ g) \in \Gamma, |b(g) - b(g')| |K_j(g, g')| > C_K \lambda \right\} \leq \frac{1}{C_K \lambda} ||[b, R_j]]||_{L^1(\mathbb{H}^n) \to L^{1,\infty}(\mathbb{H}^n)}.
\]

Since \([b - c, R_j] = [b, R_j]\), \(j = 1, \cdots, 2n\), for any constant \(c\) and \(b\) is finite almost everywhere, we may assume \(b(0) = 0\), then we have

\[(7.3) \quad |\Lambda_\lambda(0)| = \left\{ g \in \mathbb{H}^n : \delta_{d_K(g,g')}^{-1}(g) \in \Gamma, \frac{|b(g)|}{d_K(g,g')^Q} > \frac{1}{2^Q} \right\},
\]

\[|\Lambda_\lambda(0)| \leq \frac{1}{C_K \lambda} ||[b, R_j]]||_{L^1(\mathbb{H}^n) \to L^{1,\infty}(\mathbb{H}^n)}.
\]

Let \(g' \neq 0\), \(g \in B(g', \frac{1}{2}d_K(g,g')|b(g')|^{1/Q}) \cap S_\Gamma(g')\) and \(g \notin \Lambda_{d_K(g,g')^{-Q}}(g')\), then

\[|b(g)| \geq |b(g')| \frac{|b(g) - b(g')|}{d_K(g,g')^Q} d_K(g,g')^Q \geq \left(1 - \frac{1}{2^Q}\right) |b(g')|
\]

for almost every \(g' \in \mathbb{H}^n\). Then by (7.3), we have

\[I_{g', \Gamma} := \left\{ g \in B\left(g', \frac{1}{2}d_K(g,g')|b(g')|^{1/Q}\right) \cap S_\Gamma(g') \cap S_\Gamma(0) \setminus \Lambda_{d_K(g,g')^{-Q}}(g') : \frac{|b(g')|}{d_K(g,g')^Q} > \frac{1}{C_K \lambda} \right\}
\]

\[\leq \left\{ g \in S_\Gamma(0) : \frac{|b(g)|}{d_K(g,g')^Q} > \frac{1}{C_K \lambda} \right\}
\]

\[\leq \frac{C_K \lambda}{C_K \lambda} ||[b, R_j]]||_{L^1(\mathbb{H}^n) \to L^{1,\infty}(\mathbb{H}^n)}.
\]

where \(C_{d_K}\) is the constant in (2.9).

Suppose that \(|b(g')| \geq 2^Q\), then for any \(g \in B(g', \frac{1}{2}d_K(g,g')|b(g')|^{1/Q})\), by (2.9),

\[d_K(g) \leq C_{d_K}(d_K(g,g') + d_K(g',0)) \leq C_{d_K}\left(\frac{1}{2}d_K(g')|b(g')|^{1/Q} + d_K(g')\right) \leq C_{d_K}d_K(g')|b(g')|^{1/Q}.
\]
That is,
\[ B(g', \frac{1}{2} d_K(g') |b(g')|^{\frac{1}{2}}) \subset B(0, C_{d_K} d_K(g') |b(g')|^{\frac{1}{2}}). \]

Therefore,
\[ I_{g', \Gamma} \geq \left| B\left(g', \frac{1}{2} d_K(g') |b(g')|^{\frac{1}{2}}\right) \cap S_{\Gamma}(g') \right| - \left| B(0, C_{d_K} d_K(g') |b(g')|^{\frac{1}{2}}) \setminus S_{\Gamma}(0) \right| - \left| \Lambda_{d_K}(g')^{-Q}(g') \right|. \]

Observe that, by (7.5), we have
\[
\left| B\left(g', \frac{1}{2} d_K(g') |b(g')|^{\frac{1}{2}}\right) \cap S_{\Gamma}(g') \right| = \left| B\left(0, \frac{1}{2} d_K(g') |b(g')|^{\frac{1}{2}}\right) \cap S_{\Gamma}(0) \right| - \left| B\left(0, \frac{1}{2} d_K(g') |b(g')|^{\frac{1}{2}}\right) \cap S_{\Gamma}(0)^c \right| > \frac{1}{Q^2} (\omega_Q - \varepsilon) d_K(g')^{Q|b(g')|},
\]

where \( \omega_Q \) is the Radon measure of \( S(0, 1) \), and
\[
\left| B\left(0, C_{d_K} d_K(g') |b(g')|^{\frac{1}{2}}\right) \setminus S_{\Gamma}(0) \right| + \left| \Lambda_{d_K}(g')^{-Q}(g') \right| < \frac{\varepsilon}{Q} C_{d_K} d_K(g')^{Q|b(g')|} + \frac{1}{C_{K_j}} d_K(g')^{Q||b, R_j||_{L^1(\mathbb{H}^n)}} L^1(\mathbb{H}^n) \to L^{1, \infty}(\mathbb{H}^n).
\]

Consequently, we obtain
\[
I_{g', \Gamma} > \frac{1}{Q} C_{d_K} d_K(g')^{Q|b(g')|} \left( \frac{\omega_Q}{2Q} - \frac{1 + C_{d_K}^{Q} 2Q}{2Q} \varepsilon \right) - \frac{1}{C_{K_j}} d_K(g')^{Q||b, R_j||_{L^1(\mathbb{H}^n)}} L^1(\mathbb{H}^n) \to L^{1, \infty}(\mathbb{H}^n).
\]

Now take \( \varepsilon = \frac{\omega_Q}{2(1 + C_{d_K}^{Q} 2Q)} \), we have
\[
I_{g', \Gamma} > \frac{\omega_Q}{2Q + 1} d_K(g')^{Q|b(g')|} - \frac{1}{C_{K_j}} d_K(g')^{Q||b, R_j||_{L^1(\mathbb{H}^n)}} L^1(\mathbb{H}^n) \to L^{1, \infty}(\mathbb{H}^n).
\]

Now combining the inequalities (7.4) and (7.5), we obtain that
\[
|b(g')| < \frac{2Q + 1}{C_{K_j} \omega_Q} \left( 1 + \frac{2Q C_{d_K}^{Q}}{2Q - 1} \right) ||b, R_j||_{L^1(\mathbb{H}^n)} L^1(\mathbb{H}^n) \to L^{1, \infty}(\mathbb{H}^n).
\]

To sum up, for almost all \( g' \in \mathbb{H}^n \),
\[
|b(g')| \leq \max \left\{ 2Q, \frac{2Q + 1}{C_{K_j} \omega_Q} \left( 1 + \frac{2Q C_{d_K}^{Q}}{2Q - 1} \right) ||b, R_j||_{L^1(\mathbb{H}^n)} L^1(\mathbb{H}^n) \to L^{1, \infty}(\mathbb{H}^n) \right\}.
\]

This completes the proof of Theorem 1.6. \( \square \)

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