COMPLEMENTS ON LOG SURFACES

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Abstract. More strong version of the main inductive theorem about the complements on surfaces is proved and the models of exceptional log del Pezzo surfaces with $\delta = 0$ are constructed.

Introduction

The theory of complements on algebraic varieties has been created by V. V. Shokurov in the papers [15], [16]. It is a powerful tool for studying algebraic varieties, extremal contractions and singularities. Roughly speaking, the complement is a "good" divisor in the multiple anticanonical linear system. The advantage of this theory is that the concept of complement is an invariant in Log Minimal Model Program. Moreover a complement has an inductive property, this means that the complement finding problem for an $n$-dimensional variety is reduced to the same one for an $(n-1)$-dimensional variety. See the papers [16], [13], [12] with reference to the theory of complements on the high-dimensional varieties. For example, the application of this theory for the three-dimensional varieties is given in the papers [5], [6], [10], [11].

Thus, in order to study effectively the three-dimensional contractions and singularities it is important to classify the log del Pezzo surfaces completely. The last open two-dimensional problem (in the framework of the theory of complements) is the classification of exceptional log del Pezzo surfaces. The exceptional log del Pezzo surfaces $(S, D)$ are divided into three types: $\delta(S, D) = 0$, $\delta(S, D) = 1$ and $\delta(S, D) = 2$, where

$$\delta(S, D) = \# \{ E \mid E \text{ is a divisor with a discrepancy } a(E, D) \leq -\frac{6}{7} \}.$$

The cases $\delta(S, D) = 1$ and $\delta(S, D) = 2$ were classified in the papers [9], [16]. To study the remaining case $\delta(S, D) = 0$ the theory of complements on surfaces must be applied in more wide set of coefficients.

This work was done with the partial support of the Russian Foundation for Basic Research (grant no. 02-01-00441), the Leading Scientific Schools (grant no. 00-15-96085) and INTAS-OPEN (grant no. 2000#269.)
Therefore it will be considered when all coefficients of a boundary $D$ are greater then or equal to $1/2$.

One of the main results of this paper given in §2 is more strong version of the main inductive theorem about the complements on surfaces. Using this inductive theorem we construct the models of exceptional log del Pezzo surfaces with $\delta = 0$ in §4 (see definition 4.15). In §3 we give the classification of non-rational exceptional log surfaces. Also, one type of exceptional log del Pezzo surfaces with $\delta = 0$ is described completely in §4 (see theorem 4.3).

I am grateful to Professor Yu.G. Prokhorov for valuable remarks.

1. Preliminary facts and results

All varieties are algebraic and are assumed to be defined over $\mathbb{C}$, the complex number field. The main definitions, notations and notions used in the paper are given in [3], [13].

**Definition 1.1.** Put $\Phi_{\text{sm}} = \{1 - 1/m \mid m \in \mathbb{Z}_{>0} \cup \{\infty\}\}$ and $\Phi_m = \Phi_{\text{sm}} \cup [6/7, 1]$. A coefficient $d$ is called standard if $d \in \Phi_{\text{sm}}$.

Put $\mathbb{Z}/(n) = \{k/n \mid k \in \mathbb{Z}_{>0}\}$.

**Definition 1.2.** For fix $n \in \mathbb{N}$ put

$$\mathcal{P}_n = \{a \mid 0 \leq a \leq 1, \quad (n+1)a > na\}.$$ 

It is clear that

$$\mathcal{P}_n = \{0\} \cup \left[\frac{1}{n+1}, \frac{1}{n}\right] \cup \left[\frac{2}{n+1}, \frac{2}{n}\right] \cup \ldots \cup \left[\frac{k}{n+1}, \frac{k}{n}\right] \cup \ldots \cup \left[\frac{n}{n+1}, 1\right].$$

**Definition 1.3.** Let us define the set $\Phi^{[a,b]} = (\Phi_{\text{sm}} \cap [0, b)) \cup [a, b)$. Put $\Phi_i = \Phi^{[a,b]}$, where $a = \frac{i-1}{i}$ and $b = \frac{i}{i+1}$.

**Definition 1.4.** The pair $(X, D = \sum d_k D_k)$ is called a pair of type $\Phi_i$ if the three following conditions are satisfied:

1. $d_k \in \Phi_i$ for all $k$;
2. there exists $j$ such that $d_j > \frac{i-1}{i}$;
3. the divisor $K_X + D$ is $1$-log terminal.

**Definition 1.5.** Let $(X/Z \ni P, D)$ be a pair, where $D$ is a subboundary. Then a $\mathbb{Q}$-complement of $K_X + D$ is a log divisor $K_X + D'$ such that $D' \geq D$, $K_X + D'$ is log canonical and $n(K_X + D') \sim 0$ for some $n \in \mathbb{N}$.  

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Definition 1.6. Let $X$ be a normal variety and let $D = S + B$ be a subboundary on $X$ such that $B$ and $S$ have no common components, $S$ is an effective integral divisor and $\mathcal{B} \leq 0$. Then we say that the divisor $K_X + D$ is $n$-complementary if there is a $\mathbb{Q}$-divisor $D^+$ such that

(1) $n(K_X + D^+) \sim 0$ (in particular, $nD^+$ is an integral divisor);
(2) the divisor $K_X + D^+$ is log canonical;
(3) $nD^+ \geq nS + \mathcal{B} (n + 1)B$.

In this situation the $n$-complement of $K_X + D$ is $K_X + D^+$. The divisor $D^+$ is called an $n$-complement too.

Let $X$ be a semi-smooth variety in codimension 1. Then the divisor $K_X + D$ is $n$-semicomplementary if there is a $\mathbb{Q}$-divisor $D^+$ satisfying previous conditions (1), (3) and next condition (2'):

(2') the divisor $K_X + D^+$ is semi log canonical.

Proposition 1.7. [15, example 5.2], [1] theorem 19.4] Let $X$ be a semi-smooth, connected, complete curve. Let $D$ be a boundary on $X$ contained in the smooth part of $X$. Assume that $-(K_X + D)$ is a nef divisor on every component of $X$. Then

(1) the divisor $K_X + D$ is $n$-semicomplementary for $n \in \{1, 2, 3, 4, 6\}$;
(2) if the divisor $K_X + D$ is not 1- and 2-semicomplementary then $X \cong \mathbb{P}^1$ and $\mathcal{B}D = \mathcal{B}D^+ = 0$.

The following statements show the invariance of complements with respect to the log minimal model program and their inductive properties.

Proposition 1.8. [15, lemma 5.4] Let $f: X \to Z$ be a birational contraction of varieties and let $D$ be a subboundary on $X$. If the divisor $K_X + D$ is $n$-complementary then the divisor $K_Z + f(D)$ is also $n$-complementary.

Proposition 1.9. [16, lemma 4.4] Let $f: X \to Z$ be a birational contraction of varieties and let $D$ be a subboundary on $X$. Assume that

(1) the divisor $K_X + D$ is $f$-nef;
(2) the coefficient of every non-exceptional component of $D$ meeting $\text{Exc } f$ belongs to $\mathbb{P}^n$;
(3) the divisor $K_Z + f(D)$ is $n$-complementary.

Then the divisor $K_X + D$ is also $n$-complementary.

For the two-dimensional varieties we have more strong theorem about the inductive property of complements then for the high-dimensional varieties [18, proposition 4.4.1].
Theorem 1.10. \cite{prop4.4.3} Let \((X/Z \ni P, D = S + B)\) be a log surface with the following properties:

1. the divisor \(K_X + D\) is divisorial log terminal;
2. the divisor \(-(K_X + D)\) is nef and big over \(Z\);
3. \(S = \Delta \cap f^{-1}(P) \neq 0\) in the neighborhood of \(f^{-1}(P)\).

Assume that near \(f^{-1}(P) \cap S\) there exists an \(n\)-semicomplement \(K_S + \text{Diff}_S(B)\) of \(K_S + \text{Diff}_S(B)\). Then near \(f^{-1}(P)\) there exists an \(n\)-complement \(K_X + S + B^+\) of \(K_X + S + B\) such that \(\text{Diff}_S(B^+) = \text{Diff}_S(B^+)\).

Definition 1.11. Let \((X/Z \ni P, D)\) be a contraction of varieties, where \(D\) is a boundary. In the case when \(\dim Z \neq 0\) the contraction is said to be \textit{exceptional} if for every \(Q\)-complement \(D'\) there is at most one divisor \(E\) (not necessarily exceptional) such that \(a(E, D') = -1\). In the case when \(\dim Z = 0\) the log variety is said to be \textit{exceptional} if the pair \((X, D')\) is kawamata log terminal for every \(Q\)-complement \(D'\).

Definition 1.12. Let \((X, D)\) be an exceptional log variety. Define

\[
\delta(X, D) = \# \left\{ E \mid E \text{ is an exceptional or non-exceptional divisor with a discrepancy } a(E, D) \leq -\frac{6}{7} \right\}.
\]

Lemma 1.13. Let \((X \ni P, \alpha C + B)\) be a germ of two-dimensional log terminal pair, where \((X \ni P)\) is a non-cyclic singularity, \(C\) is a curve, \(B \geq 0\) and \(\alpha \geq 0\). Then

1. the divisor \(K_X + \alpha C + B\) is not \((1 - \alpha)\)-log terminal;
2. the divisor \(K_X + \alpha C + B\) is strictly \((1 - \alpha)\)-log canonical if and only if \((X \ni P, \alpha C + B) \cong_{an} (\mathbb{C}^2 \ni 0, \alpha \{xy = 0\}) / \mathbb{D}_n\), where \(\mathbb{D}_n \subset SL_2(\mathbb{C})\) is a dihedral subgroup.

\textbf{Proof.} For some number \(0 < c \leq 1\) the pair \((X, cC + B)\) is log canonical, but not purely log terminal \cite{th2.1.2}. Let \(f : (Y, E) \to (X \ni P)\) be an inductive blow-up of this log pair \cite{th1.9, prop3.1.4}. Then the divisor \(K_Y\) is \(f\)-nef since \(f\) is the blow-up of the central vertex of minimal resolution graph \cite{6}. Therefore we have

\[
a(E, \alpha C + B) = -\frac{\alpha}{c} + (1 - \frac{\alpha}{c}) \cdot a(E, B) \leq -\frac{\alpha}{c} \leq -\alpha.
\]

The equality holds if and only if \(c = 1, B = 0\) and \(a(E, 0) = 0\). By the classification of two-dimensional log terminal singularities we obtain the required statement (for example, see \cite{th2.1.2}). \hfill \qed
Lemma 1.14. Let \((X \ni P, \alpha C + B)\) be a germ of two-dimensional log terminal pair, where \((X \ni P)\) is a cyclic singularity, \(C\) is a curve, \(B = \sum b_iB_i \geq 0\) and \(\alpha \geq 0\). Assume that the pair \((X, C + B)\) is not purely log terminal and \(b_i \geq \frac{1}{2}\) for all \(i\). Then

1. the divisor \(K_X + \alpha C + B\) is not \((1 - \alpha)\)-log terminal;
2. the divisor \(K_X + \alpha C + B\) is strictly \((1 - \alpha)\)-log canonical if and only if:

   a. \((X \ni P, \alpha C + B) \cong_{an} (\mathbb{C}^2 \ni 0, \alpha\{xy = 0\})/\mathbb{Z}_n(n - 1, 1);\)
   b. \((X \ni P, \alpha C + B) \cong_{an} (\mathbb{C}^2 \ni 0, \alpha\{x^2 + y^2 = 0\})/\mathbb{Z}_4(3, 1);\)
   c. \((X \ni P, \alpha C + B) \cong_{an} (\mathbb{C}^2 \ni 0, \alpha\{x^2 + y^4 = 0\})/\mathbb{Z}_2(1, 1),\)
   where \(\alpha \leq 2/3;\)
   d. \((X \ni P, \alpha C + B) \cong_{an} (\mathbb{C}^2 \ni 0, \frac{1}{2}\{x = 0\} + \frac{1}{2}\{x + y^3 = 0\})/\mathbb{Z}_2(1, 1).\)

Proof. Assume that \(C\) is a reducible curve or \(B\) consists of at least two divisors. Let \(\psi : \tilde{X} \to X\) be a minimal resolution and \(\Gamma\) be its graph. The proper transforms of \(C\) and \(B\) are denoted by \(\tilde{C}\) and \(\tilde{B}\). The curve of \(\Gamma\) intersecting \(\tilde{C}\) is denoted by \(\tilde{E}\). Let us contract all other curves of \(\Gamma\). We obtain a blow-up \(f : (Y, E) \to (X \ni P)\). Write

\[ K_Y + aE + B_Y + C_Y = f^*(K_X + C + B).\]

Since

\[ 0 \leq -2 + \deg \text{Diff}_E(0) + (B_Y + C_Y) \cdot E = (1 - a)E^2 \]

then \(a \geq 1\). Hence

\[ a(E, \alpha C + B) = -\alpha a + (1 - \alpha)a(E, B) \leq -\alpha a \leq -\alpha.\]

The equality holds if and only if \(a = 1, B = 0\) and \(a(E, 0) = 0\). By the classification of two-dimensional log terminal singularities we obtain subcase (a), or subcase (d) considered below (in this situation \(B = 0\)).

Assume that \(C\) is an irreducible curve and \(B\) consists of at most one divisor. If \((X \ni P) \not\cong_{an} (\mathbb{C}^2 \ni 0)/\mathbb{Z}_n(1, 1)\) then arguing as above we can find the curve \(E\) such that inequality (1) holds. If we have an equality in (2) then we obtain subcase (b).

Let \((X \ni P) \cong_{an} (\mathbb{C}^2 \ni 0)/\mathbb{Z}_n(1, 1)\). Put \(E = \text{Exc} \psi\). If \((\tilde{C} + \tilde{B}) \cdot E \geq 2\) then lemma is proved by the same arguments. If we have an equality in (2) then we obtain subcase (c). Therefore we may assume that

\[(X \ni P, C) \cong_{an} (\mathbb{C}^2 \ni 0, \{x = 0\})/\mathbb{Z}_n(1, 1), B = b_1B_1\text{ and } \tilde{C} \cap \tilde{B}_1 \cap E = P.\]

Take an usual blow-up at the point \(P\). Then

\[ a(E', \alpha C + B) = -\alpha(1 + \frac{1}{n}) - b_1(1 + \frac{1}{n}) + \frac{2}{n}, \]
where $E'$ is a corresponding exceptional curve. It can easily be checked that $a(E', \alpha C + B) > -\alpha$ if and only if $n = 2$, $\alpha < 1/2$ and we have $a(E, \alpha C + B) \leq -\alpha/2 - 1/4 < -\alpha$. Moreover $a(E', \alpha C + B) = -\alpha$ if and only if $n = 2$, $\alpha = b_1 = \frac{1}{2}$; that is, we obtain subcase (d) (in this situation $B \neq 0$).

**Proposition 1.15.** Let the pair $(X \ni P, \alpha C + B)$ be of type $\Phi_i$, where $i = 2, 3, 4, 5, 6$ and $\alpha \geq \frac{1}{i - 1}$. Then one of the following possibilities holds.

1. $(X \ni P, \alpha C + B)_\text{an}(C^2 \ni 0, \alpha \{x = 0\} + \beta \{x + y^k = 0\})$, where $i = 2$, $k \geq 2$, $\alpha > 1/2$, $\beta > 1/2$, $\alpha + \beta < 1 + \frac{1}{2k}$.
2. $(X \ni P, \alpha C + B)_\text{an}(C^2 \ni 0, \alpha \{x = 0\} + \frac{1}{2}\{x + y^k = 0\})$, where $k \geq 2$. If $k = 2$ then $i$ is arbitrary. If $k = 3$ then $i = 2, 3$. If $k \geq 4$ then $i = 2$ and $\alpha < \frac{1}{2} + \frac{1}{2k}$.
3. $(X \ni P, \alpha C + B)_\text{an}(C^2 \ni 0, \alpha \{x = 0\} + \frac{1}{2}\{x + y^k = 0\})$, where $k \geq 2$. If $k = 2$ then $i$ is arbitrary. If $k = 3$ then $i = 2, 3, 4$. If $k \geq 4$ then $i = 2$ and $\alpha < \frac{1}{2} + \frac{1}{4k/2}$.
4. $(X \ni P, \alpha C + B)_\text{an}(C^2 \ni 0, \alpha \{x = 0\} + b_1\{y = 0\})/\mathbb{Z}_n(q, 1)$, where $(n, q) = 1$ and $\frac{\alpha - 1 + b_1}{1 - \alpha} < q \leq n$ (the case $n = q = 1$ is possible).

**Proof.** By lemma 1.13 $(X \ni P)$ is a cyclic singularity or a smooth point. The condition that $K_X + D$ is $\frac{1}{i}$-log terminal divisor and the form of the coefficients of a divisor $D$ are principal in the proposition proof.

Assume that the divisor $K_X + C + B$ is purely log terminal. If $(X \ni P)$ is a smooth point then $(X \ni P, \alpha C + B)_\text{an}(C^2 \ni 0, \alpha \{x = 0\} + b_1\{y = 0\})$. By the same argument as in the proof of proposition 1.9 we obtain case 4).

Assume that the divisor $K_X + C + B$ is not purely log terminal. Then by lemma 1.13 $(X \ni P)$ is a smooth point. If the divisor $K_X + C + B$ is divisorial log terminal then we obtain case 3) with $k = 2$. Suppose that the divisor $K_X + C + B$ is not divisorial log terminal. Then there are two possibilities for a divisor $B$.

Let $B = b_1B_1 \neq 0$. Then $C$ is a smooth curve and $B_1$ is tangent to $C$ at the point $P$. Therefore we obtain cases 1) and 2).

Let $B = 0$. Then $C$ is a singular curve and we obtain case 3) with $k \geq 3$. 

**Remark 1.16.** The case $\Phi_m = \bigcup_{i \geq 1} \Phi_i$ is developed in proposition 1.9 [2].
2. Main inductive theorem on surfaces

Next theorem 2.1 is more strong version of the main inductive theorem about the complements on surfaces [16, theorem 2.3].

**Theorem 2.1.** Let \((S, D = \sum d_i D_i)\) be a projective log surface with the following properties:

1. the divisor \(K_S + D\) is log canonical, but not kawamata log terminal;
2. the divisor \(-(K_S + D)\) is nef;
3. there exists a \(\mathbb{Q}\)-complement of \(K_S + D\);
4. \(d_i \geq \frac{1}{2}\) for all \(i\).

Then there is 1-, 2-, 3-, 4- or 6-complement of \(K_S + D\) which is not kawamata log terminal, except the cases from example 2.3.

Besides, if there is an infinite number of divisors \(E\) with a discrepancy \(a(E, D) = -1\) then there is 1- or 2-complement of \(K_S + D\) which is not kawamata log terminal.

**Proof.** In many cases this theorem is true without condition (4). When proving this theorem we follow the paper [16]. The cases using condition (4) are considered in details.

Applying a minimal log terminal modification [13, definition 3.1.3] we may assume that the pair \((S, D)\) is divisorial log terminal. Put \(C = D \not\equiv 0\) and \(B = \{D\}\). We have three cases depending on the numerical dimension of a divisor \(-(K_S + C + B)\).

**Case I.** Assume that \(-(K_S + C + B)\) is a big divisor. Then all required statements immediately follow by proposition 1.7 and theorem 1.10. Let us remark that condition (4) on a boundary \(D\) is unnecessary in this case.

Before discussing two remaining cases let us make more precise the structure of a log surface \((S, D)\).

Let \(S\) be a non-rational surface. Then our theorem is proved in [16, theorem 2.3], [13, theorem 8.2.1]. Moreover there exists 1- or 2-complement which is not kawamata log terminal and there are at most two divisors \(E\) with a discrepancy \(a(E, D) = -1\). Let us remark that condition (4) on a boundary \(D\) is also unnecessary in this case.

Let \(C\) be not the chain of rational curves. Then our theorem is also true without condition (4) on the coefficients of a boundary \(D\) [16, theorem 2.3].

Thus we may assume that \(S\) is a rational surface and \(C\) is a chain of rational curves.

**Case II.** Assume that \(K_S + C + B \not\equiv 0\) and \(-(K_S + C + B)\) is not a big divisor. By proposition 2.5 [16] we can assume that the divisor.
$-(K_S + C + B)$ is semi-ample. Let $\nu: S \to Z \cong \mathbb{P}^1$ be the morphism given by a linear system $|-m(K_S + C + B)|$, where $m \gg 0$. The next lemma is basic to construct the complements.

**Lemma 2.2.** [16, lemma 2.20, lemma 2.21] Let $(S, D)$ be a projective log surface with a structure of fibration onto a curve $f: S \to Z$, where $D$ is a boundary. Let $C = \cup D \neq 0$ and $B = \{D\}$. Assume that the following conditions are satisfied:

1. there exists a section $C_1 \subset C$ of $f$;
2. the divisor $K_C + \text{Diff}_C(B)$ is $n$-semicomplementary;
3. the divisor $K_S + C + (n+1)B/\mathbb{N}$ is numerically trivial on a general fiber;
4. the divisor $-(K_S + C + B)$ is nef;
5. the divisor $K_S + C + B$ is log terminal in some (analytic) neighborhood of a divisor $C$.

Then the divisor $K_S + C + B$ is $n$-complementary. Moreover conditions (1) and (3) can be replaced by condition (1'):

1. there exists a multi-section $C_1 \subset C$ of $f$ and $S$ is a rational surface.

There are three possibilities for $C$.

A). Let $C_1 \subset C$ be a multi-section of $\nu$. Then the required statements don’t depend on condition (4) on the coefficients of $D$ and follow by lemma 2.2 and proposition 1.7.

B). Let $C$ has the unique section $C_1$ of $\nu$. Lemma 2.2 cannot be applied if and only if there is a horizontal component $B_i$ of $B$ with a coefficient $b_i \in \mathbb{Z}/(n+1)$. On the other hand, if there is a horizontal component $B_j$ of $B$ with a coefficient $b_j \in \mathbb{Z}/(n+1)$, then we consider the divisor $K_S + C + B - \varepsilon B_j$, where $0 < \varepsilon \ll 1$. It has the same $n$-complements as the divisor $K_S + C + B$ (see definition 1.0). Since $-(K_S + C + B) + \varepsilon B_j$ is a nef and big divisor then our theorem is reduced to case I. Therefore we assume that all horizontal components of $B$ have the coefficients from the set $\mathbb{Z}/(n+1)$.

Assume that $C$ is a reducible divisor. Then the divisor $K_C + \text{Diff}_C(B)$ is 1- or 2-semicomplementary by proposition 1.7.

If it is 1-semicomplementary then $B_{\text{hor}} = \frac{1}{2}B_1 + \frac{1}{2}B_2$ or $B_{\text{hor}} = \frac{1}{2}B_1$. Hence the divisor $K_S + C + B$ is 2-semicomplementary.

If it is 2-semicomplementary then we have a contradiction with condition (4).

Assume that $C = C_1$. Then we have $n = 1, 3$ by condition (4). If $n = 1$ then there is a 2-complement of $K_S + C + B$ as before. Consider
the case \( n = 3 \). Then \( B_{\text{hor}} = \frac{1}{3}B_1 + \frac{1}{2}B_2 \) or \( B_{\text{hor}} = \frac{1}{2}B_1 \). The divisor \( K_S + C + B \) doesn't have 1-, 2-, 3-, 4- and 6-complement if and only if the divisor \( K_C + \text{Diff}_C(B) \) doesn't have 1-, 2-, 4- and 6-complement, that is, (after simple calculations)

\[(*) \quad (C, \text{Diff}_C(B)) = (\mathbb{P}^1, (3/5 + \varepsilon_1)P_1 + (2/3 + \varepsilon_2)P_2 + (5/7 + \varepsilon_3)P_3),\]

where \( \varepsilon_1 + \varepsilon_2 + \varepsilon_3 < \frac{2}{105} \) and \( \varepsilon_i \geq 0 \) for all \( i \). By lemma 2.2 the divisor \( K_S + C + B \) is 12-complementary (the index 12 is not always a minimal one).

Let \( \nu: S \xrightarrow{\psi_1} S' \xrightarrow{\psi_2} S \xrightarrow{\phi} Z \) be a contraction of all curves in the fibres of \( \nu \) (with the help of log minimal model program) with \( (K_S + C + \cup_{i} 13B_i \cup 12) \cdot E > 0 \). Since \( (K_S + C + \cup_{i} 13B_i \cup 12) \cdot C = 0 \) then \( \psi_1 \) doesn't contract the curves intersecting \( C \). We get that the divisor \( (K_{S'} + C' + \cup_{i} 13B'_i \cup 12) \) is nef and in particular, it is nef over \( Z \), where \( C' \) and \( B' \) are the images of \( C \) and \( B \). The cone of \( S/Z \) is polyhedral and generated by contractible extremal curves \([16\text{ proposition } 2.5]\). Let \( \nu: S \xrightarrow{\psi_1} S' \xrightarrow{\psi_2} S \xrightarrow{\phi} Z \) be a contraction of all curves not intersecting \( C' \) in the fibres of \( \nu \). Then \( \rho(S/Z) = 1 \). The pair \( (C, \text{Diff}_C(B)) \) is the same one as in \([1]\), where \( C \) and \( B \) are the images of \( C' \) and \( B' \). Note that either \( \text{Diff}_C(0) = 0 \), or \( \text{Diff}_C(0) = \frac{2}{3}P_2 \) and \( \varepsilon_2 = 0 \).

1). Consider the case \( \text{Diff}_C(0) = 0 \). Since \( C \cdot B_{\text{hor}} = 0 \) then \( C^2 \leq 0 \). Hence, the linear system \( |C + mf| \) gives a birational morphism \( \psi: S \to F_k \), where \( f \) is a general fiber and \( m \gg 0 \) \([10\text{ proposition } 1.10]\). We obtain that \( \overline{S} \cong F_k \).

2). Consider the case \( \text{Diff}_C(0) = \frac{2}{3}P_2 \) and \( \varepsilon_2 = 0 \). Let \( \phi: S' \to S \) be the blow-up with the unique exceptional curve at the point \( P_2 \) such that \( \text{Sing} S' \cap C' = \emptyset \) and the divisor \( K_{S'} \) is \( \phi \)-nef. Put \( f_2 = \text{Exc} \phi \). By the same argument as in the previous case the linear system \( |C + mf| \) gives a birational morphism \( \psi: S \to F_k \). Let \( E = \text{Exc} \psi, \hat{D} = \psi(C') + \cup_{i} 13\psi(B_i) \cup 12 \) and \( \psi(E) = P \). Then either

\[
(\mathbb{F}_k \ni P, \hat{D}) \cong_{\text{an}} (C^2 + (y = 0) + (x^2 + y^2) = 0),
\]

or

\[
(\mathbb{F}_k \ni P, \hat{D}) \cong_{\text{an}} (C^2 + (y = 0) + (x^2 + y^2) = 0).
\]

Since \( a(E, \hat{D}) = 0 \) and \( f_2^2 < -1 \) then \( \psi \) is a weighted blow-up with weights \((1,3)\) or \((2,3)\) (cf. \([12\text{ lemma } 5.5]\)). In the second case if we take a blow-up with weights \((2,3)\) then the following condition must be satisfied: \( t \geq 2 \). The result is summarized in the next example.
Example 2.3. 1). Let  
\[(F, D) = (F, E_\infty + (1/2)E_1 + (1/2)E_2 + (7/12)f_1 + (2/3)f_2 + (3/4)f_3),\]
where \(E_\infty\) is a minimal section, \(E_i\) is a zero section, \(f_i\) is a fiber. Let \(h: S \to \mathbb{F}_k\) be a birational contraction:
\[
K_S + \tilde{E}_\infty + (1/2)\tilde{E}_1 + (1/2)\tilde{E}_2 + (7/12)\tilde{f}_1 + (2/3)\tilde{f}_2 + (3/4)\tilde{f}_3 + 
\sum a_i\Delta_i = h^*(K_{\mathbb{F}_k} + D^+) \equiv 0.
\]
Assume that \(a_i \in \{0\} \cup [1/2, 1) \cup \mathbb{Z}/\{12\}\), \(h(\Delta_i) \not\in E_\infty\) for all \(i\). Since the pair \((\mathbb{F}_k, D^+)\) is kawamata log terminal outside \(E_\infty\) then there is only finite number of such surfaces \(S\) by lemma 3.1.9 [13]. The log surface
\[
(S, D) = (S, \tilde{E}_\infty + (1/2)\tilde{E}_1 + (1/2)\tilde{E}_2 + (3/5 + \varepsilon_1)\tilde{f}_1 + (2/3 + \varepsilon_2)\tilde{f}_2 + 
(5/7 + \varepsilon_3)\tilde{f}_3 + 
\sum a_i\Delta_i)
\]
satisfies the condition of theorem 2.1 where \(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leq \frac{2}{105}\) and \(\varepsilon_i \geq 0\) for all \(i\). If \(5/7 + \varepsilon_3 < 8/11\) then we have 10-complement of \(K_S + D\). If \(5/7 + \varepsilon_3 \geq 8/11\) then we have 12-complement. Note also that sometimes we can easily change the coefficients \(a_i\), and we can contract \(\tilde{E}_\infty\) if \(k > 0\).

2). Let \(\overline{S} \to \mathbb{P}^1\) be an extremal generically \(\mathbb{P}^1\)-fibration, that is, \(\rho(\overline{S}/\mathbb{P}^1) = 1\). Assume that \(\text{Sing} \overline{S} \subset f_2\), where the fiber \(f_2\) is shown in one of the following figures.

![Fig. 1](image1.png)

![Fig. 2](image2.png)
Let us consider the minimal resolution of a surface $S$ and contract all $(-1)$ curves not intersecting the proper transform of $E_\infty$. We obtain $\mathbb{F}_k$. On the ruled surface $\mathbb{F}_k$ the image of $E_\infty$ is a minimal section, the images of $B_1$ and $B_2$ from figures 1 and 2 are the sections, the image of $B_1$ from figure 3 is a 2-multi-section.

Consider the log surface 
\[(\mathcal{S}, D^+) = (\mathcal{S}, E_\infty + B + (7/12)f_1 + (3/4)f_3),\]

where $B = \frac{1}{2}B_1 + \frac{1}{2}B_2$ (in the case of figure 1 or 2) or $B = \frac{1}{2}B_1$ (in the case of figure 3). Arguing as above in the previous point of example we can construct the birational morphisms $h: S \to \mathcal{S}$. It is clear that the same statements take place about the structure of $h$ and the complements of $K_S + D$.

C). Let $C$ be in a fiber of $\nu$. Put $P = \nu(C)$ and $f = \nu^{-1}(P)$. The case, where the general fiber is an elliptic curve is considered in III, B). Therefore we assume that the general fiber of $\nu$ is a rational curve. The divisor $K_C + \text{Diff}_C(B)$ has an $n$-semicomplement of minimal index, where $n \in \{1, 2, 3, 4, 6\}$ (see proposition 1.7). Assume that there is a horizontal component $B_i$ with a coefficient $b_i \not\in \mathbb{Z}/(n + 1)$. Then considering the divisor $K_S + C + B - \varepsilon B_i$ we reduce our problem to case I. Therefore we assume that all horizontal components of $B$ have the coefficients from the set $\mathbb{Z}/(n + 1)$. Let us show that this possibility is impossible.

Let $\nu: S \xrightarrow{\psi} S' \to Z$ be a contraction of all curves in the fiber $f$ not lying in $C$. Put $C' = \psi(C)$, $B' = \psi(B)$. Since $K_S + C + B \equiv 0$ over $Z$ then $(C, \text{Diff}_C(B)) = (C', \text{Diff}_{C'}(B'))$.

Assume that $C$ is a reducible curve. Then $n = 1, 2$. If $n = 1$ then all coefficients of horizontal components of $B$ are equal to 1/2 by condition (4). Since the divisor $K_{S'} + C' + B'$ is divisorial log terminal and numerical trivial over $Z$ then there is a divisor in $\text{Diff}_{C'}(B')$ with a coefficient 1/2. A contradiction with $n = 1$. If $n = 2$ then we have same contradiction.

The case, where $C$ is an irreducible curve, is considered similarly.

**Case III.** Assume that $K_S + C + B \equiv 0$. 

![Diagram](image-url)
Definition 2.4. Let $D$ be a $\mathbb{Q}$-divisor on a projective variety $X$. Define the numerical dimension of a divisor $D$:

$$ \nu_{\text{num}}(D) = \max \{ \nu_{\text{num}}(D') \mid D' \geq 0, \text{Supp } D' \subseteq \text{Supp } D \}.$$

The linear dimension (Iitaka dimension) $\kappa(D)$ is defined similarly.

We have $\kappa(B) = \nu_{\text{num}}(B)$ \cite[proposition 2.12]{16}. Let us consider all possibilities of $\nu_{\text{num}}(B)$ case by case.

A). Let $\nu_{\text{num}}(B) = 2$. The cone $\overline{\text{NE}}(S)$ is polyhedral and generated by contractible extremal curves since there exists a divisor $B'$ (Supp $B' \subseteq$ Supp $B$) such that the divisor $-(K_S + C + B) + \varepsilon B'$ is nef and big \cite[prop. 2.5]{16}. Let the divisor $K_C + \text{Diff}_C(B)$ be $n$-semicomplementary. Let us contract all exceptional curves $E$ with $(K_S + C + \lfloor (n+1)B \rfloor/n) \cdot E > 0$. We obtain either $\nu: S \xrightarrow{\psi} S' \xrightarrow{\nu'} Z \cong \mathbb{P}^1$ and $(K_{S'} + C' + \lfloor (n+1)B' \rfloor/n) \cdot f > 0$, where $f$ is a general fiber of $\nu'$, or $\psi: S \to S'$ and $-(K_{S'} + C' + \lfloor (n+1)B' \rfloor/n)$ is a nef divisor. By the construction none component of $C$ is contracted by $\psi$, and $C'$ doesn’t lie in the fibres of $\nu'$ in the first case. Therefore $C$ has the horizontal components of $\nu$ in the first case. Moreover, since $\nu_{\text{num}}(B) = 2$ then we have $B_{\text{hor}} = \frac{1}{2}B_1 + \frac{1}{2}B_2$ or $B_{\text{hor}} = \frac{1}{2}B_1$ by condition (4). Thus the first case is reduced to case II, B). This new possibility was included in example 2.3.

In the second case the divisor $K_{S'} + C' + \lfloor (n+1)B' \rfloor/n$ is $n$-complementary without condition (4) on the coefficients of a boundary $B$ \cite{16} (here it is essential that the cone $\overline{\text{NE}}(S')$ is polyhedral and generated by contractible extremal curves). By proposition 1.9 the divisor $K_S + C + B$ is $n$-complementary.

B). Let $\nu_{\text{num}}(B) = 1$. Then for some divisor $B'$ (Supp $B' \subseteq$ Supp $B$) the linear system $|B'|$ gives a fibration $\nu: S \to Y$ and the divisor $B$ lies in the fibres. If some component of $C$ is a section then the general fiber is $\mathbb{P}^1$. Hence there is a multi-section $C_1 \subseteq C$. By lemma 2.2 our theorem is proved. Therefore we may assume that $C$ lies in the fibres and the general fiber is an elliptic curve. Arguing as above, we contract all curves $E \not\subseteq$ Supp $C$ such that $(K_S + C + \lfloor (n+1)B \rfloor/n) \cdot E \geq 0$, where $n$ is the semicomplementary index of $K_S + \text{Diff}_C(B)$. As a result we obtain $\nu: S \xrightarrow{\phi} S' \xrightarrow{\nu'} Y$, where all fibers of $\nu'$ are irreducible, except the fiber consisting of $C'$. Let $\psi: S' \to S''$ be the contraction of components of $C'$. We get the model with $\rho(S'') = 2$. The cone of surface $S''$ has two extremal rays: a fiber of $\nu'$, a (multi-)section $E$. If $(K_{S''} + C'' + \lfloor (n+1)B'' \rfloor/n) \cdot E \leq 0$ then the divisor $-(K_{S''} + C'' + \lfloor (n+1)B'' \rfloor/n)$ is nef. In this case the theorem is proved in \cite{16} without condition (4) on the coefficients of a boundary $B$. If $(K_{S''} +
$C'' + \land (n+1)B'' \cdot E > 0$ then we have a fibration $\theta: S \to Z \cong \mathbb{P}^1$ and $C$ contains a section of $\theta$. The horizontal part of divisor $B$ for $\theta$ is $B_{\text{hor}} = \frac{1}{2}B_1 + \frac{1}{2}B_2$ or $B_{\text{hor}} = \frac{1}{2}B_1$ by condition (4). Since $K_S + C + B \equiv 0$ over $Y$ and $K_{S''} + C'' \equiv 0$ over $Y$ then $\text{Diff}_C(B) = \text{Diff}_{C''}(0)$. Hence the coefficients of $\text{Diff}_C(B)$ are standard. In particular, if $n = 1, 3$ then we have $\text{Diff}_C(B) = P_1 + P_2$ and $\text{Diff}_C(B) = \frac{2}{3}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3$ respectively. Therefore, if $n = 1, 2, 3, 4, 6$ then the divisor $K_S + C + B$ is 2-, 2-, 6-, 4-, 6-complementary respectively by lemma 2.2.

C). Let $\nu_{\text{num}}(B) = 0$, that is, the divisor $B$ is contracted. Then our theorem is true without condition (4) on the coefficients of a boundary $D$ [16].

Remark 2.5. Condition (3) of theorem 2.1 can be replaced by one of the following more strong conditions (see [16, proposition 2.5]):

(3’) the divisor $-(K_S + D)$ is big;

(3”) the cone $\overline{\text{NE}}(S)$ is polyhedral and generated by contractible extremal curves;

(3”) the divisor $-(K_S + D)$ is semi-ample;

(3”) there exists a contraction $\nu: S \to Z$ with the following property:

if $(K + D) \cdot E = 0$ then $E \subset \text{Exc} \nu$.

In example 2.3 the log surfaces $(S, D)$ satisfy the conditions enumerated, of course except condition (3”).

The next corollary is very important for the applications.

Corollary 2.6. Let $(S, D = \sum d_iD_i)$ be a projective log surface. Assume that

(1) the divisor $K_S + D$ is kawamata log terminal;

(2) the divisor $-(K_S + D)$ is nef;

(3) there exists a $\mathbb{Q}$-complement of $K_S + D$;

(4) $d_i \geq \frac{1}{2}$ for all $i$;

(5) there exists an effective $\mathbb{Q}$-divisor $D' \geq D$ such that the divisor $-(K_S + D')$ is nef and the pair $(S, D')$ is not kawamata log terminal.

Then there is 1-, 2-, 3-, 4- or 6-complement of $K_S + D$ which is not kawamata log terminal, except the cases appearing in example 2.3.

Besides, if there is an infinite number of divisors $E$ with a discrepancy $a(E, D) = -1$ and the pair $(S, D')$ is log canonical then there is 1- or 2-complement of $K_S + D$ which is not kawamata log terminal.

Proof. Replacing the divisor $D'$ with suitable $D + \lambda(D' - D)$, where $\lambda > 0$ we may assume that the divisor $K_S + D'$ is log canonical but not kawamata log terminal.
At first let us prove that there is a $\mathbb{Q}$-complement of $K_S + D'$. If the divisor $-(K_S + D)$ is big then the cone $\text{NE}(S)$ is polyhedral and generated by contractible extremal curves and we obtain the required statement [16, proposition 2.5]. Therefore it can be assumed that the linear system $|-m(K_S + D)|$ gives a fibration $\nu: S \to Z$, where $m \gg 0$. Adding the required number of general fibres of $\nu$ to the divisor $K_S + D'$ we have our statement.

Let $f: \tilde{S} \to S$ be a minimal log terminal modification of the pair $(S, D')$ [13, definition 3.1.3]. We have

$$K_{\tilde{S}} + \sum E_i + \cup \tilde{D}' \cup \{\tilde{D}'\} = f^*(K_S + D'),$$

where $\tilde{D}'$ is a proper transform of $D'$. Put

$$\tilde{D} = \sum E_i + \cup \tilde{D}' \cup \sum_{i: \text{Supp } \tilde{D}_i \not\subset \text{Supp } \{\tilde{D}'\}} d_i \tilde{D}_i,$$

where $\tilde{D}_i$ is a proper transform of $D_i$. Thus, the statement of corollary must be proved for the divisor $K_{\tilde{S}} + \tilde{D}$. If the divisor $-(K_{\tilde{S}} + \tilde{D})$ is nef then it is nothing to be proved by theorem 2.1. Therefore it can be assumed that the divisor $-(K_{\tilde{S}} + \tilde{D})$ is not nef. A $\mathbb{Q}$-complement of a divisor $K_{\tilde{S}} + \tilde{D}$ is denoted by $\tilde{\Theta}$. We can assume that $\cup \tilde{\Theta} \cup = \cup \tilde{D} \cup$.

Let us prove that we can contract all exceptional curves $E$ such that $(K_{\tilde{S}} + \tilde{D}) \cdot E > 0$ on every step.

A). Assume that $\nu_{\text{num}}(\{\tilde{\Theta}\}) = 2$. Then arguing as in the proof of theorem 2.1 (case III,A) the cone $\text{NE}(\tilde{S})$ is polyhedral and generated by contractible extremal curves. Q.E.D.

B). Assume that $\nu_{\text{num}}(\{\tilde{\Theta}\}) = 1$. By proposition 2.12 [16] for some divisor $\tilde{\Theta}'$ (Supp $\tilde{\Theta}' \subset$ Supp$\{\tilde{\Theta}\}$) the linear system $|\tilde{\Theta}'|$ gives a fibration $\nu: \tilde{S} \to Z$ and a divisor $\{\tilde{\Theta}\}$ lies in the fibres of $\nu$. If $(K_{\tilde{S}} + \tilde{D}) \cdot E > 0$ then a curve $E$ lies in the fibres of $\nu$. Therefore it can be contracted.

C). Assume that $\nu_{\text{num}}(\{\tilde{\Theta}\}) = 0$. Then a divisor $\{\tilde{\Theta}\}$ is contracted by the definition.

Thus we get a birational morphism $\phi: \tilde{S} \to \overline{S}$. It is clear that $\phi$ doesn’t contract the components of $\cup \tilde{D} \cup$, and the curve contracted intersects some component $\tilde{\Theta}_1$ of $\cup \tilde{D} \cup$ on every step. Put $\overline{D} = \phi(\tilde{D})$.

It remains to prove that an $n$-complement $\overline{D}^+$ of $K_{\overline{S}} + \overline{D}$ induces an $n$-complement of $K_{\tilde{S}} + \tilde{D}$ ($n = 1, 2, 3, 4$ or 6). Put

$$K_{\tilde{S}} + \tilde{D}^+ = \phi^*(K_{\overline{S}} + \overline{D}^+).$$
We must prove that

\[ \langle \widetilde{D} \rangle + \frac{\langle n+1 \rangle \{ \widetilde{D} \}}{n} \leq \langle \widetilde{D}^+ \rangle + \{ \widetilde{D}^+ \}. \]

By the above this requirement is enough to check in the case, when \( \phi \) is a contraction of the unique curve \( E \). Let \( P = \phi(E) \). By the classification of two-dimensional log terminal pairs \([13] \text{ theorem 2.1.2}\) and by condition (4) we conclude that there are at most one divisor of \( \{ \widetilde{D} \} \) passing through the point \( P \) and \( (S \ni P) \) is a cyclic singularity. Let \( d_1 \) be a divisor passing through the point \( P \). If the coefficient of divisor \( \overline{D}_1 \) is more then \( d_1 \) then we consider it instead of \( d_1 \). Since the divisor \( K_{\overline{D}_j} + \text{Diff}_{\overline{D}_j}(\{ \widetilde{D} \}) \) is \( n \)-semicomplementary there are the following cases (the case \( n = 1 \) is obvious).

1. \( (S \ni P, \overline{\mathcal{S}}_1 + d_1 \overline{D}_1) \cong (\mathbb{C}^2 \ni 0, \{ x = 0 \} + d_1 \{ y = 0 \})/\mathbb{Z}_2(1,1) \) and \( n = 4 \). Then by proposition \([13] \text{ requirement (**) must be checked for } d_1 \in (1/2, \frac{3}{2}) \).

2. \( (S \ni P, \overline{\mathcal{S}}_1 + d_1 \overline{D}_1) \cong (\mathbb{C}^2 \ni 0, \{ x = 0 \} + d_1 \{ y = 0 \})/\mathbb{Z}_2(1,1) \) and \( n = 6 \). Then by proposition \([13] \text{ requirement (**) must be checked for } d_1 \in ( \frac{2}{3}, \frac{5}{6} ) \).

3. \( (S \ni P, \overline{\mathcal{S}}_1 + d_1 \overline{D}_1) \cong (\mathbb{C}^2 \ni 0, \{ x = 0 \} + d_1 \{ y = 0 \})/\mathbb{Z}_3(1,1) \) and \( n = 6 \). Then by proposition \([13] \text{ requirement (**) must be checked for } d_1 \in (1/2, \frac{3}{4}) \).

4. \( (S \ni P, \overline{\mathcal{S}}_1 + d_1 \overline{D}_1) \cong (\mathbb{C}^2 \ni 0, \{ x = 0 \} + d_1 \{ y = 0 \}) \).

Requirement (***) is equivalent to the following one:

\[ -\frac{\langle n+1 \rangle a(E, \overline{\mathcal{S}}_1 + d_1 \overline{D}_1)}{n} \leq -a(E, \overline{\mathcal{S}}_1 + \frac{\langle n+1 \rangle d_1}{n} \overline{D}_1). \]

Since \( a(E, \overline{\mathcal{S}}_1 + d_1 \overline{D}_1) \leq -1/2 \) then \( \phi \) is a toric blow-up. Requirement (***) in cases 1), 2) and 3) is checked directly. In case 4) the weights of weighted blow-up \( \phi \) are denoted by \( (\alpha, \beta) \). Then either \( (\alpha, \beta) = (\alpha, 1) \) and \( d_1 \geq 1/2 \), or \( (\alpha, \beta) = (\alpha, 2) \), \( d_1 \geq 3/4 \) and \( n = 4 \), or \( (\alpha, \beta) = (\alpha, 3) \), \( d_1 \geq 5/6 \) and \( n = 6 \). Now requirement (***) is also fulfilled by direct calculation.

**Corollary 2.7.** Under the notation of corollary 2.6 let us decline condition (4) on a boundary \( D \). Then there is \( n \)-complement of \( K_S + D \) which is not Kawamata log terminal, where \( n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 35, 36, 40, 41, 42, 43, 56 or 57.

Besides, if there is an infinite number of divisors \( E \) with a discrepancy \( a(E, D) = -1 \) and the pair \((S, D')\) is log canonical then there is 1-, 2- or 6-complement of \( K_S + D \) which is not Kawamata log terminal.
Proof. By the proof of corollary 2.6 it follows that the divisor $K_S + D'$ has a $\mathbb{Q}$-complement. Therefore we obtain our statement by theorem 2.3 [16]. □

3. Exceptional non-rational log surfaces

Theorem 3.1. (cf. [13] proposition 9.2.2]) Let $(S, D = \sum d_i D_i)$ be a projective log surface, where $D$ is a boundary. Assume that the following conditions are satisfied:

1. there exists a $\mathbb{Q}$-complement $\Theta = \sum \theta_i \Theta_i$ of $K_S + D$;
2. the surface $S$ is non-rational and the pair $(S, D)$ is exceptional;
3. we have $\Theta \neq 0$ or $S$ has a non Du Val singularity.

Then there is $2$-, $3$-, $4$- or $6$-complement of $K_S + D$. Besides, one of the following cases takes place.

1. $S \cong C \times \mathbb{P}^1$, where $C$ is an elliptic curve, $D_i$ are the sections of corresponding $\mathbb{P}^1$-bundle.
2. $S \cong \mathbb{P}_C(E)$, where $E$ is an indecomposable vector bundle of degree $1$ on an elliptic curve $C$. Up to multiplication by an invertible sheaf, $E$ is a nontrivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C(0) \to 0.$$ 

Then $D_i \sim 2E - f^*(O + t_i)$ or $D_i \sim 4E - 2f^*O$, where $E$ is a section of $f: S \to C$ and $t_i$ is an element of order $2$ in Pic$(C)$.

Proof. Let $\phi: \tilde{S} \to S$ be a minimal resolution. Then $K_S + \Theta = \phi^*(K_S + \Theta) \equiv 0$. Condition (3) implies $\Theta \neq 0$, that is, $\kappa(\tilde{S}) = -\infty$. Let $S_{\text{min}}$ be a minimal model of $\tilde{S}$. By the condition $S_{\text{min}}$ is a minimal ruled surface over a curve $C$ with $p_a(C) \geq 1$. The image of divisor $\Theta$ on $S_{\text{min}}$ is denoted by $\Theta$. If there is an irreducible curve $E$ with $E^2 < 0$ on $S_{\text{min}}$ then

$$(K_{S_{\text{min}}} + E) \cdot E = \left( - \sum_{i, \Theta_i \neq E} \theta_i \Theta_i \right) \cdot E + \varepsilon E^2 < 0,$$

where $\varepsilon > 0$. Hence $p_a(E) = 0$ and we have a contradiction with $p_a(C) \geq 1$.

Since $0 \geq 8 - 8p_a(C) = K_{S_{\text{min}}}^2 = \Theta^2 \geq 0$ then $p_a(C) = 1$, $\Theta_i^2 = \Theta_i \cdot \Theta_j = 0$ for all $i, j$. Since $\Theta_i < 1$ for all $i$ then the pair $(S_{\text{min}}, \Theta)$ is terminal. Therefore $\mathbb{P}_C(E) \cong S_{\text{min}} \cong \tilde{S} \cong S$, where deg $E \geq 0$. By chapter 5 [14] and by examples 1.1, 2.1 [16] we obtain the remaining statements. □

Remark 3.2. [16] example 2.1] In case 2) of theorem 3.1 the linear system $|4E - 2f^*O|$ gives a structure of elliptic fibration with three
degenerate (double) fibres, which are linear equivalent to $2E - f^*(O + t_i)$.

**Corollary 3.3.** Under the conditions of theorem 3.1 we have $\delta(S, D) = 0$.

4. **Construction of models of log del Pezzo surfaces with $\delta = 0$**

The classification of exceptional log surfaces with $\delta = 1, 2$ was given in the papers [9], [16]. The exceptional non-rational log surfaces were completely classified in theorem 3.1. Thus it remains to study the last remaining case—the exceptional rational log surfaces with $\delta = 0$.

**Definition 4.1.** The pair $(S, D)$ is called a log del Pezzo surface, where $D$ is a boundary, if the following conditions are satisfied:

1. the divisor $-(K_S + D)$ is nef;
2. the divisor $K_S + D$ is log canonical;
3. there exists a $\mathbb{Q}$-complement of $K_S + D$.

Let us consider the limiting case.

**Example 4.2.** [1, remark 1.2], [17, examples 4.2, 5.3] 1). Let $S = C \times C$, where $C = \mathbb{C}/(\mathbb{Z} + \varepsilon_3 \mathbb{Z})$ is an elliptic curve and $\varepsilon_3$ is a primitive root of unity of order 3. The group $\mathbb{Z}_3$ acts on the curve $C$ by the multiplication on $\varepsilon_3$. Then $S = S/\mathbb{Z}_3$ is a surface with $3K_S \sim 0$, $\rho(S) = 4$, and Sing $S$ consists of nine singularities $\frac{1}{3}(1, 1)$.

2). Let the surface $S = J(C)$ be the jacobian of hyperelliptic curve $C$: $y^2 = x^5 - 1$ of genus 2. The group $\mathbb{Z}_5$ is generated by the automorphism $(x, y) \mapsto (\varepsilon_5 x, y)$ of curve $C$, where $\varepsilon_5$ is a primitive root of unity of order 5. Then $S = S/\mathbb{Z}_5$ is a surface with $5K_S \sim 0$, $\rho(S) = 2$, and Sing $S$ consists of five singularities $\frac{1}{5}(2, 1)$.

3). Let us consider three irreducible curves $E_1 \sim \mathcal{O}_{\mathbb{P}^2}(1)$, $E_2 \sim E_3 \sim \mathcal{O}_{\mathbb{P}^2}(4)$ on $\mathbb{P}^2$. The curve $E_2$ and the curve $E_3$ has three ordinary double points $Q_5, Q_6, Q_7$ and $Q_8, Q_9, Q_{10}$ respectively. The line $E_1$ intersects the curves $E_2$ and $E_3$ at the points $Q_1, Q_2, Q_3$ and $Q_4$. The curve $E_2$ intersects $E_3$ at the points $Q_1, \ldots, Q_{10}$. Let us take the usual blow-ups of $\mathbb{P}^2$ at the points $Q_1, \ldots, Q_{10}$ and contract the proper transforms of the curves $E_1, E_2, E_3$. We get a surface $\widetilde{S}$ with $3K_{\widetilde{S}} \sim 0$, $\rho(\widetilde{S}) = 8$, and Sing $\widetilde{S}$ consists of three singularities $\frac{1}{3}(1, 1)$. When contracting $(-2)$ curves on $\widetilde{S}$ we get a surface $S$ with $3K_S \sim 0$ and with three non Du Val singularities $\frac{1}{3}(1, 1)$.

**Theorem 4.3.** Let $S$ be a rational exceptional log del Pezzo surface $(D = 0)$. Assume that $a(E, 0) > -1/2$ for all $E$ and there is no a
\( \mathbb{Q} \)-complement \( \Theta = \sum \frac{1}{2} \Theta_i \) of \( K_S \). Then the surface \( S \) is of example 4.2.

**Proof.** The surface \( S \) must have a non Du Val singularity otherwise \( h^0(S, \mathcal{O}_S(-K_S)) \geq 1 \).

**Lemma 4.4.** Let \( (X \ni P) \) be a two-dimensional non Du Val singularity. Assume that

\[
M = \min \{ a(E, 0) \mid E \text{ is an exceptional divisor} \} > -\frac{1}{2},
\]

Then \( (X \ni P) \cong an(\mathbb{C}^2 \ni 0)/\mathbb{Z}_{2n+1}(n,1) \), where \( n \geq 1 \). In particular, \( M = -\frac{n}{2n+1} \leq -\frac{1}{3} \).

**Proof.** If \( (X \ni P) \) is a non-cyclic singularity then the blow-up of central vertex of minimal resolution graph gives a discrepancy \( \leq -\frac{1}{2} \). Therefore \( (X \ni P) \) is a cyclic singularity. Let \( \tilde{X} \rightarrow X \) be a blow-up with the unique exceptional curve \( E \) such that its self-intersection index \( k \) on the minimal resolution of \( (X \ni P) \) is at most \(-3\). Then

\[
-\frac{1}{2} < a(E, 0) = -1 + \frac{-2 + \text{Diff}_E(0)}{E^2} = -1 + \frac{-2 + \frac{m_1 - 1}{m_1} + \frac{m_2 - 1}{m_2}}{-k + \frac{n_1}{m_2} + \frac{n_2}{m_2}}.
\]

Hence \( (m_1, q_1) = (1, 0), (m_2, q_2) = (m_2, m_2 - 1), k = 3 \). \( \square \)

Let \( P_1, \ldots, P_r \) be non Du Val singularities of \( S \) of types \( \frac{1}{2m_1 + 1}(n_1,1), \ldots, \frac{1}{2n_r + 1}(n_r,1) \) respectively. Let \( f: \tilde{S} \rightarrow S \) be a minimal resolution. Then \( K_{\tilde{S}} + \Delta = f^* K_S \). By lemma 4.4

\[
h^2(\tilde{S}, \mathcal{O}_{\tilde{S}}(-2K_{\tilde{S}} - \psi 3\Delta_\psi)) = h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3K_{\tilde{S}} + \psi 3\Delta_\psi)) = 0,
\]

except the case \( n_1 = \ldots = n_r = 1 \) and \( K_S \equiv 0 \). Let us determine the remaining possibilities of \( n_1, \ldots, n_r \). By Riemann-Roch theorem and Noether’s formula we have the next system

\[
\begin{cases}
0 = h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(-2K_{\tilde{S}} - \psi 3\Delta_\psi)) = 3K_{\tilde{S}}^2 + r + 1 - 3 \cdot \sum_{i=1}^r \frac{n_i}{2n_i + 1} + h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(-2K_{\tilde{S}} - \psi 3\Delta_\psi)) \\
K_{\tilde{S}}^2 - \sum_{i=1}^r \frac{n_i}{2n_i + 1} + \rho(S) + n_1 + \ldots + n_r \leq 10.
\end{cases}
\]

**Lemma 4.5.** [2, corollary 9.2] Let \( X \) be a rational surface with Kawamata log terminal singularities and with \( \rho(X) = 1 \). Then

\[
\sum_{P \in \text{Sing } X} \frac{m_P - 1}{m_P} \leq 3,
\]

where \( m_P \) is the order of the local fundamental group \( \pi_1(U_P \setminus \{P\}) \) (\( U_P \) is a sufficiently small neighborhood of \( P \)).
Taking into account $K_S^2 \geq 0$ and lemma 4.5 we obtain $K_S^2 = 0$, $\rho(S) = 2$, $n_1 = \ldots = n_5 = 2$, $\text{Sing } S = \{P_1, \ldots, P_5\}$ by the system. Moreover, we have $K_S \equiv 0$. Indeed, let $K_S \not\equiv 0$. If there is a curve $E$ with $E^2 < 0$ on $S$ then we can contract it and obtain a contradiction with lemma 4.5. Therefore we have generically $\mathbb{P}^1$-fibration $S \to Z$, but it is impossible by classification of such fibrations [13, theorem 7.1.12].

Let $\widetilde{S} \to S$ be a canonical cover. There are two cases [1, theorem C].

A). Let $\widetilde{S}$ be an abelian surface. Then theorem C [1] implies that $S$ is a surface from example 4.2 (points 1) or 2)) and $n_1 = \ldots = n_9 = 1$ or $n_1 = \ldots = n_5 = 2$ respectively.

B). Let $\widetilde{S}$ be a $K_3$-surface. Then by theorem 5.1 [17] we get that $n_1 = n_2 = n_3 = 1$. It is clear that one of the minimal models of $\widetilde{S}$ is $S_{\text{min}} \cong \mathbb{P}^2$. Let $\varphi: \widetilde{S} \to S_{\text{min}}$ be a corresponding birational morphism. Put $\Delta = \varphi(\varnothing) = 1/3(\varphi(E_1) + \varphi(E_2) + \varphi(E_3))$, where $E_i$ is an exceptional curve over the point $P_i$. Let $\varphi$ contracts a curve $E$ different from $E_i$ for all $i$. Then the pair $(S_{\text{min}} \ni \Phi, \Delta)$ is canonical, where $\Phi = \varphi(E)$. It is easy to prove that $(S_{\text{min}} \ni \Phi, \Delta)$ is analytically isomorphic either $(\mathbb{C}^2 \ni 0, 1/3\{x^3 + y^3 = 0\})$, where $l = 3, 4$, or $(\mathbb{C}^2 \ni 0, 1/5\{x^3 + xy^2 = 0\})$, or $(\mathbb{C}^2 \ni 0, 1/7\{x^2y + y^4 = 0\})$.

**Lemma 4.6.** There exists a surface $S'$ such that $\psi(E_i)$ is (+1) nonsingular rational curve for some $i$, where $\varphi: S \to S'$ is a birational morphism.

**Proof.** If $\Delta = 1/3E_j$ then $p_a(\Phi_j) = 28$. The curve $\Phi_j$ must have two singular points of multiplicity 4 and eight singular points of multiplicity 3, we get a contradiction. Therefore there exists a curve $\Phi_j$ such that $(\mathcal{O}_{\mathbb{P}^2}(5) - E_j)$ is a nef divisor. Take a resolution of the curve $\Phi_j$ singularities. We obtain a curve with a self-intersection index $\geq +1$. Q.E.D.

We can assume that the linear system $|\psi(E_i)|$ gives a birational morphism $S' \to S_{\text{min}}$ [9] proposition 1.10) and $\Delta = 1/3E_1 + 1/3E_2 + 1/3E_3$.

Sorting out all variants of $E_1, E_2, E_3$ on $S_{\text{min}} \cong \mathbb{P}^2$ the reader will easily prove that there are four $(-1)$ curves on $\widetilde{S}$ such that every curve intersects all $E_i$ and they are mutually disjoint. Let us contract them $S \to S'$. We obtain a surface $S'$ from case 3) of example 4.2.

**4.7.** The classification of log del Pezzo surfaces is very important to study the three-dimensional extremal contractions and singularities, because there is an induction from a (local) three-dimensional contraction to a two-dimensional log variety [12, 16, 11]. Let us remark that in the result of induction we obtain the log surfaces $(S, D)$ such that
the divisor \(-(K_S + D)\) is nef, big and the coefficients of \(D\) are standard. In order to get an effective classification, the exceptional log del Pezzo surfaces with \(\delta = 1, 2\) are considered in more wide set of coefficients – \(\Phi_{\text{sm}}\) [16], [9]. In this case the big condition is replaced, for instance, on the requirement of existence of \(Q\)-complement of \(K_S + D\). The later allows to give the classification of log Enriques surfaces with \(\delta \geq 1, 2\). Therefore, in the case \(\delta = 0\) the set \(\Phi_{\text{sm}}\) will be extended to \(\Phi_i\). Now the main goal is to construct the models of \((S, D)\) with Picard number 1 or 2.

**Definition 4.8.** Let \((S, D)\) be an exceptional log del Pezzo surface with \(\delta(S, D) = 0\), where \(S\) is a rational surface. Then the pair \((S, D)\) of type \(\Phi_i\) is called an exceptional log del Pezzo surface of type \(\Phi_i\), where \(i = 2, 3, 4, 5, 6\).

**4.9.** Let \(i = 3, 4, 5, 6\). Put

\[
\hat{D} = \sum_{d_k \geq \frac{1}{i-1}} c(D_k)D_k + \sum_{d_k < \frac{1}{i-1}} d_kD_k,
\]

where \(c(D_k) = c(S, D - d_kD_k; D_k)\) is a log canonical threshold of a divisor \(D_k\) for the pair \((S, D - d_kD_k)\). By proposition [1.15] the divisor \(K_S + \hat{D}\) is log canonical. By corollary [2.6] the divisor \(-K_S - \hat{D}\) is nef. Assume that \(\rho(S) \geq 3\). Then there exists an exceptional curve \(E\) with \((K_S + \hat{D}) \cdot E > 0\) (see the proof of theorem 4.1 from the paper [16]). Let \(\varphi: S \to S'\) be a contraction of \(E\). In contrast to the case \(\delta(S, D) \geq 1\) the birational morphism \(\varphi\) can contract the curve from \(D\) with a coefficient \(> \frac{i-1}{i}\).

**Lemma 4.10.** Let \(D = \alpha E + D^o\), \(\alpha \geq \frac{i-1}{i}\) and \(P = \varphi(E)\). Then we have one of the following cases.

1. \((S' \ni P, \varphi(D)) \cong_{\text{an}} \mathbb{C}^2 \ni 0, \frac{1}{2}\{x = 0\} + (\frac{3}{1} + \varepsilon)\{x + y^3 = 0\}\), where \(\varepsilon < \frac{1}{10}\) and \(i = 4\). A morphism \(\varphi\) is a weighted blow-up with weights \(3, 1\).
2. \((S' \ni P, \varphi(D)) \cong_{\text{an}} \mathbb{C}^2 \ni 0, (\frac{2}{3} + \varepsilon_1)\{x = 0\} + (\frac{3}{2} + \varepsilon_2)\{x + y^2 = 0\}\), where \(\varepsilon_1 + \varepsilon_2 < \frac{5}{12}\) and \(i = 4\). A morphism \(\varphi\) is a weighted blow-up with weights \(2, 1\).
3. \((S' \ni P, \varphi(D)) \cong_{\text{an}} \mathbb{C}^2 \ni 0, \frac{1}{2}\{x = 0\} + (\frac{3}{2} + \varepsilon)\{x + y^l = 0\}\), where \(l = 3, 4\), \(i = 3\) and \(\varepsilon < \frac{3}{12} - \frac{1}{l}\). A morphism \(\varphi\) is a weighted blow-up with weights \((l, 1)\).
4. \((S' \ni P, \varphi(D)) \cong_{\text{an}} \mathbb{C}^2 \ni 0, (\frac{i-1}{i} + \varepsilon)\{x^2 + y^3 = 0\}\), where \(i = 5, 4, 3\), \(\varepsilon < \frac{1}{180}\) if \(i = 5\), \(\varepsilon < \frac{1}{20}\) if \(i = 4\), \(\varepsilon < \frac{1}{12}\) if \(i = 3\). A morphism \(\varphi\) is a weighted blow-up with weights \((3, 2)\).
Proof. Note that $E$ must intersect a curve from $D^\circ$ with a coefficient $\geq \frac{i-1}{i}$. Let us consider the case when $i = 3$ and there exists a point on $E$ with $k = 3$ from point 2) of proposition 1.15. Then

$$(K_S + D) \cdot E > (K_S + E + D^\circ) \cdot E = -2 + \deg \text{Diff}_E(0) + D^\circ \cdot E \geq -2 + \frac{3}{2} + \frac{2}{3} > 0.$$  

A contradiction. Therefore $E$ has a coefficient 1 in $\hat{D}$. There are three possibilities.

1). Assume that $\deg \text{Diff}_E(0) = 0$. Then $(S' \ni P) \cong (\mathbb{C}^2 \ni 0)/\mathbb{Z}_n(1,1)$. Since $(K_S + \hat{D}) \cdot E > 0$ and $(K_S + D) \cdot E \leq 0$ then simple calculations show that the possibility is not realized.

2). Assume that $\deg \text{Diff}_E(0) = \frac{k-1}{k}P$, where $k \geq 2$. For example, consider the case $i = 3$. Since $\deg \text{Diff}_E(D^\circ) < 2$ and $\deg \text{Diff}_E(\hat{D} - E) > 2$ then $k = 2, 3, 4, 5$ and $P$ is a non-singular point of $S'$. Moreover we have

1) $\text{Diff}_E(D^\circ) = \frac{1}{2}P + (\frac{2}{3} + \varepsilon_1)P_1 + (\frac{2}{3} + \varepsilon_2)P_2$ or

2) $\text{Diff}_E(D^\circ) = \frac{k-1}{k}P + \frac{1}{2}P_1 + (\frac{2}{3} + \varepsilon)P_2$, where $k = 3, 4, 5$.

It remains to check that $-a(E, \varphi(D)) < \frac{i}{i+1}$.

3). Assume that $\deg \text{Diff}_E(0) = \frac{k_1-1}{k_1}P_1 + \frac{k_2-1}{k_2}P_2$, where $k_1, k_2 \geq 2$. Since $\deg \text{Diff}_E(D^\circ) < 2$ then it can be assumed that $k_1 = 2$. By direct calculations we obtain case (4).

□

4.11. If $i = 4, 5, 6$ then we repeat the procedure for $S'$ described above. If $i = 4$ then the case, when there is a point on an exceptional curve from case (1) of lemma 4.10 is impossible by the same argument as case 2) of proposition 1.15 with $k = 3$ (see the proof of lemma 4.10). As a result we get a surface $\mathcal{S}$ with $\rho(\mathcal{S}) = 1$, or a surface $\mathcal{S}$ with $\rho(\mathcal{S}) = 2$ and with structure of generically $\mathbb{P}^1$-fibration.

Now let $i = 3$. Put $D' = \sum d'_k D'_k = \varphi(D)$. Let us repeat the procedure described above. If there is no a point on $D'_k$ from case (2) of lemma 4.10 the divisor $\hat{D}'$ is defined as the divisor $\hat{D}$. Otherwise, put $c(D'_k) = 3/4$.

Let $\varphi': S' \to S''$ be a contraction of a curve $E'$ from $D'$ with a coefficient $\geq 2/3$. Two new cases can be appeared.

1). The case, when there is a point on $E'$ case (3) of lemma 4.10 is similarly impossible.

2). Assume that there is a point on $E'$ from case (2) of lemma 4.10
Then $\text{Diff}_{E'}(\hat{D'} - \frac{3}{4}E') \neq \frac{3}{2}Q$, where $Q$ is a point of $E'$. Indeed, otherwise we have
\[
\begin{cases}
(K_{S'} + \hat{D'}) \cdot E' = (K_{S'} + \frac{3}{2}E' + 3/4D'_2) \cdot E' = -\frac{1}{2} - \frac{1}{4}E'^2 > 0 \\
(K_{S'} + D') \cdot E' = (K_{S'} + (\frac{2}{3} + \varepsilon_1)E' + (\frac{2}{3} + \varepsilon_2)D'_2) \cdot E' = \\
= -\frac{2}{3} + 2\varepsilon_2 - (\frac{1}{3} - \varepsilon_1)E'^2 \leq 0.
\end{cases}
\]

Since $\varepsilon_1 + \varepsilon_2 < \frac{1}{24}$ then this system of inequalities is contradictorily. The simple calculations show that the following new case is possible only.

(I) \((S'' \ni P', \varphi'(D')) \cong_{an} (\mathbb{C}^2 \ni 0, (\frac{2}{3} + \varepsilon)\{x^2 + y^5 = 0\})\),

where $\varepsilon < \frac{1}{120}$, $P' = \varphi'(E')$ and $\varphi'$ is a weighted blow-up with weights $(2, 1)$. Repeating the procedure we obtain the surface $S$ described above.

4.12. Let $i = 2$. Now the main problem is how much to increase the divisor $D = \sum d_l D_l$ up to $\hat{D} = \sum c_l D_l$, where $c_l \geq d_l$ for all $l$. After this increase the divisor $K_S + \hat{D}$ must be log canonical, but not kawamata log terminal. As before there is an exceptional curve $E$ with $(K_S + D) \cdot E > 0$ under the condition $\rho(S) \geq 3$. The corresponding morphism is denoted by $\varphi : S \to S'$.

Let us describe the construction of $\hat{D}$. Let $D_t$ be a non-singular curve. If there is a point on $D_t$ from point 1) or 2) of proposition 4.15 with $k = 3$ then we put $c_t = \frac{2}{3}$.

Consider the remaining cases. If there is a point on $D_t$ from point 1) or 2) of proposition 4.15 with $k = 2$ then we put $c_t = \frac{3}{4}$.

In the remaining cases we arbitrarily increase the other coefficients $d_l$ up to maximal possible values.

Lemma 4.13. Let $D = \alpha E + D^o$, $\alpha \geq \frac{1}{2}$ and $P = \varphi(E)$. Then we have one of the following cases.

(1) \((S' \ni P, \varphi(D)) \cong_{an} (\mathbb{C}^2 \ni 0, (\frac{1}{2} + \varepsilon)\{x^3 + y^4 = 0\})\), where $\varepsilon < \frac{1}{24}$.

A morphism $\varphi$ is a weighted blow-up with weights $(1, 1)$.\]
Proof. Since the divisor $S + D$ is $\frac{1}{k}$-log terminal then the singularities of $S$ lying on $E$ are Du Val singularities of type $\mathbb{A}_n$ (see lemma 1.15). For the same reason there is at most one singular point on a curve $E$.

Assume that $E$ has the point of tangency of multiplicity 3 with a curve $D_1$ from $D^o$. Since $(K_S + D) \cdot E \leq 0$ then $\text{Diff}_E(D^o - d_1D_1) = 0$ and we obtain case (1).

If $E$ has the point of tangency of multiplicity 3 with a curve $D_1$ from $D^o$ then arguing as above in lemma 1.10 we obtain cases (2) and (3).

For the remaining possibility we have cases (4) and (5). \qed

4.14. Let us repeat the procedure described above. As a result of multiple procedure repetition two new singularities can appear similarly to point 4.11.

(II) $(\mathbb{C}^2 \ni 0, (\frac{1}{2} + \varepsilon_1)\{y = 0\} + (\frac{1}{2} + \varepsilon_2)\{x = 0\} + (\frac{1}{2} + \varepsilon_3)\{x + y^k = 0\})$, where $\varepsilon_1 + k(\varepsilon_2 + \varepsilon_3) < \frac{1}{6}$, and

(III) $(\mathbb{C}^2 \ni 0, (\frac{1}{2} + \varepsilon_1)\{y = 0\} + (\frac{1}{2} + \varepsilon_2)\{x^2 + y^{2k+1} = 0\})$, where $\varepsilon_1 + 2k\varepsilon_2 < \frac{1}{6}$.

The results above-mentioned allow to define the model of exceptional log del Pezzo surface of type $\Phi_i$ (cf. [10, §5]).

Definition 4.15. Let $(S, D)$ be an exceptional log del Pezzo surface of type $\Phi_i$ ($i = 2, 3, 4, 5, 6$) except the following points (if $i = 6$ then there are no exceptions).

(1) If $i = 5$ then see point (4) of lemma 4.10
(2) If $i = 4$ then see points (1), (4) of lemma 4.10
(3) If \( i = 3 \) then see points (2), (3), (4) of lemma 4.10 and case (I)
of point 4.11.

(4) If \( i = 2 \) then see points (1), (3), (5) of lemma 4.13. It is possible
with another restrictions on the values of \( \varepsilon, \varepsilon_1, \varepsilon_2 \). Also see cases
(II) and (III) of point 4.14.

Then the pair \((S,D)\) is called a model of type \( \Phi_i \) if one of the following
two conditions is satisfied.

A). \( \rho(S) = 1 \).

B). \( \rho(S) = 2 \), the cone \( \text{NE}(S) \) is generated by two extremal rays
\( R_1 \) and \( R_2 \). The ray \( R_1 \) gives generically \( \mathbb{P}^1 \)-fibration. If the
ray \( R_2 \) gives a birational contraction of a curve \( E \) then \( E \) is a
component of divisor \( D \) with a coefficient \( \geq i - 1 \).

Remark 4.16. Let us remark that in the model of type \( \Phi_i \) definition
the condition, that the divisor \( K_S + D \) is \( \frac{1}{i} \)-log terminal, is not fulfilled
at the non-singular points of surface \( S \) only.

4.17. The very important problem is to classify the models of type
\( \Phi_i \). When the model classification of type \( \Phi_2 \) is finished it is remained
to describe the exceptional surfaces \( S \) such that the divisor \( K_S \) is \( \frac{1}{2} \)-
log terminal and there exists a 2-complement \( \Theta = \sum \frac{1}{2} \Theta_i \) of \( K_S \) (see
theorem 4.3). This completes the classification of exceptional log del
Pezzo surfaces (see [9], [16]) and allows to describe log Enriques surfaces
completely (see [8], [7]).

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