On Kerr-Schild Symmetries and Conservation Laws in General Relativity

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Abstract

In the present work, the spin-coefficient formalism of Newman and Penrose is used to formulate geometric constraints for the existence of Kerr-Schild groups, i.e. continuous groups of generalized Kerr-Schild transformations. In addition, by characterizing the geometric structure of the deformed Einstein tensor of the generalized Kerr-Schild class, restrictions are imposed on the existence of apparent conservation laws in generic spacetimes, which are defined via considering special Kerr-Schild currents whose associated Kerr-Schild vector fields coincide with timelike Killing vector fields of pairs of stationary background geometries. The feasibility of the derived conditions is demonstrated by considering concrete, suitably simple models of generalized Kerr-Schild spacetimes.

Introduction

The generalized Kerr-Schild class is a remarkably geometrically rich class of spacetimes which plays an important role in general relativity. This is mainly due to the fact that it provides a large number of comparatively simple geometric models which are known to be of great importance for gravitational physics.

Defined in relation to a fixed geometric background characterized by a so-called seed metric, the spacetimes belonging to this particular class have a very remarkable physical property: their associated mixed Einstein tensor (and thus also the mixed deformed Ricci) is linear in the only free scalar parameter of the theory, the so-called profile function of geometry.

This is the main reason why the associated geometric framework has so far proved so successful in finding exact solutions to Einstein’s equations, in particular to the vacuum field equations, the Einstein-Maxwell equations and the field equations for perfect fluid and null radiation fields [4, 11, 12, 13, 14, 15, 23].

As it turns out, however, the generalized Kerr-Schild framework is not only of great use for the pursuit of exact solutions to the field equations of general
relativity, but also has a wide range of applicability in gravitational physics owing to its versatility.

This versatility is reflected, among other things, by the fact that there are specific types of generalized Kerr-Schild deformations which, as first shown by Coll, Hildebrandt, Senovilla [3], form continuous groups of metric transformations, so-called Kerr-Schild groups, which have the general structure of either finite or possibly even infinite (local) Lie algebras.

These groups, whose geometric structure proves to be even richer than that of isometries or conformal symmetries, are generated by associated so-called Kerr-Schild vector fields, which must be determined with respect to the deformation null direction required for the definition of the generalized Kerr-Schild class. Somewhat surprisingly, it typically turns out in this context that for each Kerr-Schild group there is one and only one infinitesimal Kerr-Schild generator, which is because, on the one hand, generic Kerr-Schild geometries have no symmetries and, on the other hand, the only class of vector fields capable of simultaneously generating several Kerr-Schild groups (with different associated null geodesic deformation vector fields) is exactly the class of Killing vector fields, which, however, is trivial for spacetimes without symmetries.

This, however, shows that Kerr-Schild groups may exist on spacetimes with a comparatively low degree of symmetry or even no symmetries, which comes from the fact that these spacetimes often have generalized, so-called causal symmetries, which are associated with the existence of special Lie algebras and corresponding diffeomorphisms that respect the causal structure of spacetime in that they represent transformations that allow one to map causal future-directed vectors onto causal future-directed vectors. As it turns out in this context, the causal symmetries mentioned contain a subclass of so-called pure causal symmetries [6, 7], which contains exactly those continuous (local) groups of generalized Kerr-Schild transformations alias Kerr-Schild groups that are dealt with in the present work. The existence of Kerr-Schild symmetries therefore refers indirectly to the existence of causal symmetries, which in turn highlights the importance of Kerr-Schild groups for a variety of potential applications in gravitational physics.

However, as it turns out, there arise two very specific questions in this context, namely, on the one hand, under which conditions Kerr-Schild groups exist in generic geometric settings and, on the other hand, whether or not there are corresponding Kerr-Schild currents and associated apparent conservation laws on these spacetimes, which are compatible with the standard local conservation laws of general relativity.

Using the spin-coefficient framework of Newman and Penrose [18, 19], the given work intends to answer both of these questions by formulating geometric constraints for the existence of the said algebraic structures and physical laws. The latter is achieved by expanding the deformed part of the Einstein tensor of the generalized Kerr-Schild class into spin-coefficients, which entails an unambiguous characterization of the structure of the same tensor for any geometric background. In this context, it then turns out that a very specific choice must be made for the associated null tetrad, which contains the null
geodesic vector field of the Kerr-Schild group, in order to fulfill the associated spin-coefficient relations and therefore to satisfy the said geometric constraints. This is demonstrated by considering some concrete but simple models of generalized Kerr-Schild geometries, which include the important Reissner-Nordström and Bonnor-Vaidya family of spacetimes, for which the relevant spin-coefficients all have exactly the right structure and thus guarantee the existence of a Kerr-Schild group. Furthermore, it turns out in this context that - under certain restrictions on the Einstein tensor of the background geometry - the existence of the said group gives rise to an apparent conservation law associated with a class of Kerr-Schild currents, of which a single representative can be specified in such a way that it coincides with a timelike Killing vector field of the background spacetime in the case that the selfsame spacetime has a sufficient degree of symmetry. The physical significance of this particular case is discussed in the later stages of this work.

**Kerr-Schild Groups and Spin-Coefficients**

Considering a pair of spacetimes \((\mathcal{M}, g)\) and \((\tilde{\mathcal{M}}, \tilde{g})\), it is a well-known possibility that the corresponding metrics \(g_{ab}\) and \(\tilde{g}_{ab}\) are linked by a generalized Kerr-Schild relation of the form

\[
\tilde{g}_{ab} = g_{ab} + f l^a l^b,
\]

where, based on the fact that the null geodesic vector field \(l^a\) can always be completed to a normalized null geodesic frame \((l^a, k^a, m^a, \bar{m}^a)\) whose components satisfy \(-k_a l^a = m_a \bar{m}^a = 1\), the so-called seed metric \(g_{ab}\) can be split up in the 2+2-form

\[
g_{ab} = -2l_{(a} k_{b)} + 2m_{(a} \bar{m}_{b)}
\]

with respect to a null co-tetrad field \((-k_a, -l_a, \bar{m}_a, m_a)\) and the Kerr-Schild metric \(\tilde{g}_{ab}\) can be decomposed in a very similar way in the form

\[
\tilde{g}_{ab} = -2(\tilde{k}_{a} \tilde{l}_{b}) + 2m_{(a} \bar{m}_{b)}
\]

with respect to a given null co-tetrad field \((-\tilde{k}_a, -\tilde{l}_a, \bar{m}_a, m_a)\), provided that \(\tilde{k}_a = k_a - \frac{2}{f} l_a\).

Accordingly, due to the fact that \(l^a\) is lightlike, it is easily found in this context that the inverse Kerr-Schild metric is given by

\[
\tilde{g}^{ab} = g^{ab} - f l^a l^b
\]

and that, furthermore, the inverse seed metric can be decomposed in the 2+2-form

\[
g^{ab} = -2l^{(a} k^{b)} + 2m^{(a} \bar{m}^{b)}
\]
with respect to a null tetrad field \((l^a, k^a, m^a, \bar{m}^a)\) and that the inverse Kerr-Schild metric \(\tilde{g}^{ab}\) can be decomposed in the 2+2-form
\[
\tilde{g}^{ab} = -2l^a k^b + 2m^a \bar{m}^b,
\]
provided that \(\tilde{k}^a = k^a + \frac{f}{2} l^a\).

As a direct consequence, using relations (1) and (2), the affine connection
\[
C^a_{bc} = \frac{1}{2} \nabla_b (f l^a c) + \frac{1}{2} \nabla_c (f l^a b) - \frac{1}{2} \nabla^a (f l^b c) + \frac{1}{2} f D f l^a l^b l^c
\]
(7)
can be defined, which relates the pair of covariant derivatives \(\nabla_a\) and \(\tilde{\nabla}_a\), from which it can be concluded that
\[
C^a_{ba} = C^a_{bc} l^b = C^a_{bc} l^c = C^a_{bc} l^a = 0,
\]
(8)
and
\[
\tilde{D} l^a = D l^a = 0, \quad \tilde{\nabla}_a l^a = \nabla_a l^a, \quad \tilde{\nabla}_{[a} l_{b]} = \nabla_{[a} l_{b]}
\]
(9)
applies in the present context; whereas it has to be noted that \(l_b = \tilde{g}_{ab} l^b = g_{ab} l^b\).

This allows one to calculate the Riemann tensor
\[
\tilde{R}^a_{bcd} = R^a_{bcd} + E^a_{bcd}
\]
(10)
where \(E^a_{bcd} = 2 \nabla_c C^a_{db} + 2 C^a_{dc} C^c_{db} \) holds by definition. Moreover, using the definition \(E^a_b = 2(g^{ad} - f l^a l^d)(\nabla_{[a} C^m_{db]} + 2 C^m_{[a} C^m_{b]d})\), the corresponding Ricci tensor can be calculated by contracting indices, which leads to the result
\[
\tilde{R}^a_b = R^a_b + E^a_b - \frac{1}{2} f R^a_c l^c l^b - \frac{1}{2} f R^a_b l^c l^c.
\]
(11)
Similarly, the corresponding Ricci scalar can be calculated in the next step, which reads
\[
\tilde{R} = R + E - f R^a_c l^c l_a,
\]
(12)
where \(E = \delta^b_a E^a_b\) holds by definition.

Hence, using (11) and (12), the Einstein tensor of the generalized Kerr-Schild spacetime \((\tilde{M}, \tilde{g})\)
\[
\tilde{G}^a_b = \tilde{R}^a_b - \frac{1}{2} \delta^a_b \tilde{R}
\]
(13)
can be determined, which by definition has the form
\[
\tilde{G}^a_b = G^a_b + \rho^a_b
\]
(14)
when it is decomposed with respect to the background metric \(g_{ab}\). The corresponding expression for the deformed part of the Einstein tensor of the generalized Kerr-Schild class then reads
\[ \rho^a_b = E^a_b - \frac{1}{2} f R^a_c l^c l_b - \frac{1}{2} f R^c_b l_c l^a - \frac{1}{2} \delta^a_b (E - f R^d_c l^d) \]  

(15)

but alternatively it can also be brought into the much simpler form

\[ \rho^a_b = -\frac{1}{2} f R^a_c l^c l_b - \frac{1}{2} f R^c_b l_c l^a + \frac{1}{2} \delta^a_b (f R^d_c l_d l^c - \nabla_d \nabla^c (f l^d l_c)) + \]  

\[ \frac{1}{2} (\nabla_c \nabla^a (f l^c l_b) + \nabla^c \nabla_b (f l^c l^a) - \nabla_c \nabla^c (f l^a l_b)). \]  

(16)

What is remarkable about this result is that the given mixed deformed Einstein tensor is obviously linear in the profile function \( f \); an instance that holds neither with regard to the deformed Einstein tensor \( \tilde{G}^{ab} \) with lowered indices nor with respect to its counterpart \( \tilde{G}^{ab} \) with raised indices.

Therefore, based on the fact that the above applies to any pair of generalized 'Kerr-Schild related metrics', i.e. to any pair of metrics related by a generalized Kerr-Schild ansatz, it can be concluded that generalized Kerr-Schild deformations are prime examples of linear metric deformations, making them particularly attractive for seeking new solutions to Einstein’s field equations.

As it turns out, however, the generalized Kerr-Schild framework is not only of great use for the search for exact solutions to the field equations of general relativity, but also has attractive physical properties that are worth investigating for various reasons. This is reflected, among other things, by the fact that the large class of Kerr-Schild metric deformations contains an interesting subclass, namely the class of generalized Kerr-Schild transformations; a class that was first specified by Coll, Hildebrandt and Senovilla via considering special classes of diffeomorphisms \( \phi : M \to M \) defined with respect to a given background spacetime \((M, g)\) under the assumption that there exists a function \( f : M \to \mathbb{R} \), usually called the profile function of the geometry, and a smooth null vector field \( l^a \) such that \( \phi^* g_{ab} = g_{ab} + f l_a l_b \), provided that \( l_a = g_{ab} l_b \) applies in the present context. The said relation can also be understood as resulting in a 'new' metric of generalized Kerr-Schild type (1), whereas, however, a necessary prerequisite for this to hold is the validity of the conditions

\[ L_\xi g_{ab} = f l_a l_b, \quad L_\xi l_a = g l_a, \]  

(17)

according to which \( L_\xi \) denotes the Lie derivative with respect to a vector field \( \xi^a \), generally referred to as so-called Kerr-Schild vector field of the geometry. This is because the validity of the system of equations (10) guarantees that the Kerr-Schild form (1) is stable under Lie derivatives of any order.

Consequently, provided that there exist functions \( f_s : M \to \mathbb{R} \) and \( g_s : M \to \mathbb{R} \) such that \( \phi^*_s g_{ab} = g_{ab} + f_s l_a l_b \) and \( \phi^*_s l_a = g_s l_a \) for all \( s \in I \subseteq \mathbb{R} \), the existence of an associated one-parameter family \( \{ \phi_s \} \) of Kerr-Schild transformations ensures the existence of a local Lie algebra with group structure \( \phi_s \phi_t = \phi_{s+t} \),
where, in this context, the Kerr-Schild vector field $\xi^a$ satisfying relation (17) is the infinitesimal generator of the corresponding Kerr-Schild group. It should be noted, however, that the algebraic structures associated with this type of vector field differ considerably from those associated with ‘ordinary’ Killing or conformal vector fields, which can be deduced, among other things, from the fact that the set of all Kerr-Schild vector fields for a given metric does not have the structure of a vector space. Instead, the said set is defined as a union of local Lie algebras, usually called Kerr-Schild algebras, which are determined with respect to a congruence of integral curves of the null geodesic deformation vector field of the generalized Kerr-Schild class. As it turns out, the corresponding Kerr-Schild algebras are generally finite-dimensional, but may as well be infinite-dimensional in special (degenerate) cases, in which the Kerr-Schild vector field leaves the integral curves of the deformation null direction invariant.

Surprisingly, as was found by García-Parrado and Senovilla [6, 7], the corresponding Kerr-Schild groups and associated symmetries occur as a special case of so-called pure causal symmetries, which, as already indicated in the introduction, are metric deformations mapping future-directed causal vectors onto future-directed causal vectors. These are specified by the relations $L_\xi g_{ab} = \alpha g_{ab} + \beta S_{ab}$ and $L_\xi S_{ab} = \alpha S_{ab} + \beta g_{ab}$ for the metric $g_{ab}$ and a tensor field $S_{ab}$, which reduce to the form $L_\xi g_{ab} = \alpha g_{ab} + \beta l_a l_b$, $L_\xi l_a = \gamma l_a$ in the degenerate case relevant to the present work. In this vein, it turns out that causal symmetries are causal maps which, contrary to the very similar case of causally decreasing diffeomorphisms described in [8, 9], are strictly associated with the existence of groups or rather monoids of transformations, which is because the inverse of a causal symmetry is generally not a causal symmetry. To be more precise, the set of causal symmetries is a submonoid of the diffeomorphism group of a given Lorentzian manifold, or of one of its subgroups, which is interesting in that submonoids and their generalizations, generally known as semigroups, have been studied by mathematicians for a long time, so that there is a multitude of methods for treating these special algebraic structures far beyond the scope of their natural field of application in general relativity.

However, since the generality of (17) gives the impression that Kerr-Schild groups and the associated causal symmetries could be defined, in principle, in relation to a generic geometric background spacetime, the question now arises under which precise geometric circumstances and for which particular class of background spacetimes the groups and symmetries mentioned exist in practice, and furthermore whether or not there are any conserved quantities associated with their existence.

As a basis for answering these questions, the remaining part of the present section will use the spin-coefficient framework of Newman and Penrose [18, 19] to formulate geometric constraints linked to the existence of Kerr-Schild groups and associated symmetries. For this purpose, using the decomposition $\xi^a = L^a + Kk^a + Mm^a + M\bar{m}^a$, the conditions (17) (in combination with the additional
condition \( L \xi = g \xi \) can be used to deduce the set of scalar relations

\[
DL + D'K = (\gamma + \bar{\gamma})K + (\bar{\tau} - \pi)M + (\tau - \bar{\pi})\bar{M}, \quad (18)
\]

\[
DK = 0, \quad \delta K = (\bar{\alpha} + \beta - \tau)K + (\bar{\rho} - \rho)M, \quad (19)
\]

\[
\delta L - D'M = -(\bar{\alpha} + \beta + \tau)L + (\mu + \bar{\xi} - \gamma)M - \lambda \bar{M}, \quad (20)
\]

\[
\delta K - DM = -(\bar{\alpha} + \beta + \bar{\pi})K + (\bar{\rho} + \rho - \epsilon)M + \sigma \bar{M}, \quad (21)
\]

\[
\delta' M + \delta \bar{M} = (\rho + \bar{\rho})L - (\mu + \mu)K + (\alpha - \beta)M + (\bar{\alpha} - \beta)\bar{M}, \quad (22)
\]

\[
DM = -(\tau + \bar{\pi})K - (\rho + \bar{\epsilon} - \epsilon)M - \sigma \bar{M}, \quad (23)
\]

\[
\delta M = -\sigma L - \lambda K + (\beta - \bar{\alpha})M, \quad (24)
\]

whereas it has to be noted that the geodetics of the null deformation direction is used here as a precondition in the sense that \( \epsilon + \bar{\epsilon} = \kappa = 0 \) is concluded from the validity of \( DL^a = (\epsilon + \bar{\epsilon})l^a - \kappa m^a - \kappa \bar{m}^a = 0 \).

In view of these findings, one immediately obtains - through the use of (19) in combination with the identity \( \delta DK - D\delta K = -\sigma \delta' K - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta K \) - the geometric constraints

\[
D(\bar{\alpha} + \beta - \tau) = \sigma(\alpha + \bar{\beta} - \bar{\tau}) + (\bar{\alpha} + \beta - \tau)(\bar{\rho} - \rho - \epsilon) + (\tau + \bar{\pi})(\rho - \bar{\rho}), \quad (25)
\]

\[
D(\rho - \bar{\rho}) = \rho^2 - \bar{\rho}^2, \quad (26)
\]

which can be met by requiring that the deformation null direction give rise to a foliation of the Lorentzian manifold \( M \) in lightlike hypersurfaces, which is the case if and only if

\[
\nabla_{[a} l_{b]} = \nabla_{[a} l_{b]} = 0. \quad (27)
\]

From this, however, it can be concluded that \( \epsilon = \bar{\epsilon} = \bar{\alpha} + \beta - \tau = \rho - \bar{\rho} = 0 \).

To proceed, since there is no obvious way to decouple the resulting set of first order differential relations (18 - 24), one may stick to the considerably simpler special case in which \( \xi^a = L^a + K^a \). As a result, one obtains the much simpler set of relations

\[
DL + D'K = (\gamma + \bar{\gamma})K, \quad DK = 0, \quad \delta L = -2\tau L, \quad (28)
\]

\[
(\rho + \bar{\rho})L = (\mu + \bar{\mu})K, \quad \sigma L = -\lambda K, \quad \tau = -\bar{\pi}, \quad (29)
\]

which, as can be seen, cannot only be decoupled, but also allows the direct calculation of the unknowns \( L \) and \( K \), respectively.

Using now in this context the identities \( \delta D'K - D'\delta K = D\delta L - \delta DL + 2\tau DL - \sigma \delta' L - \bar{\rho} \delta L = 0 \) and \( \delta' \delta L - \delta L = (\mu - \bar{\mu})DL - (\bar{\alpha} - \beta)\delta' L + (\alpha - \beta)\delta L \), the additional geometric constraints

\[
\delta(\gamma + \bar{\gamma}) = 0, \quad D\tau = \bar{\rho} \tau + \sigma \bar{\tau}, \quad (30)
\]

\[
\delta \bar{\tau} - \delta' \tau = (\alpha - \beta)\bar{\tau} - (\bar{\alpha} - \beta)\tau \quad (31)
\]

result, whereas it has to be noted that, since \( DL \propto K \) can be deduced from (29), there must either additionally hold \( \mu = \bar{\mu} \) or \( D(\rho + \bar{\rho})(\mu + \bar{\mu}) = D(\mu + \bar{\mu})(\rho + \bar{\rho}) \).
and $D\sigma \lambda = D\lambda \sigma$. In this context, however, the first requirement proves to be much simpler, which is not least because its validity can always be guaranteed by simply requiring that the complex null vectors of the corresponding null tetrad should be hypersurface-forming, so that $L_m m^a \propto n^a$, $\tilde{m}^a$ should be valid in the present context.

Consequently, provided that the spin-coefficients listed and the functions $L$ and $K$ can be chosen in such a way that geometric constraints $(25 – 31)$ are satisfied, the existence of a Kerr-Schild group is guaranteed. However, given the specific nature of the constraints mentioned, it becomes clear that they cannot be met in relation to generic background fields, but only in relation to special classes of background spacetimes.

In order to find interesting candidates for such background spacetimes, in relation to which the listed geometrical boundary conditions can be fulfilled, it makes sense to consider only those that can be foliated by lightlike hypersurfaces. For such spacetimes, it is usually possible to introduce null Gaussian coordinates [5, 16], in which the line element of spacetime takes the form

$$ds^2 = - H dv^2 + 2 W_b dx^b dv + q_{bc} dx^b dx^c,$$

(32)

where $H = H(v, r, x^2, x^3)$, $W_b = W_b(v, r, x^2, x^3)$ and $q_{bc} = q_{bc}(v, r, x^2, x^3)$ applies by definition in the present context. For simplicity’s sake, one may even stick to the (still very general) class of so-called Robinson-Trautman spacetimes, which is the class of all spacetimes with line element (32) that can be foliated by a collection of null hypersurfaces that are orthogonal to a geodesic, shear-free and expanding congruence of null curves.

Of all the spacetimes that have such a geometric structure, the simplest candidates are those belonging to the so-called Kundt class of spacetimes, which are spacetimes that permit a null geodesic congruence with vanishing expansion, shear and twist, so that $\epsilon = \kappa = \rho = \sigma = 0$. Obviously, the geometric constraints related to this class of spacetimes take a very simple form, as can easily be concluded from the fact that $l^a$ represents a Killing vector field for a subclass of these types of geometries. However, on closer inspection, it turns out that the requirement that the null geodesic congruence is expansion-free leads to the exclusion of very interesting types of spacetimes, so that the given class of spacetimes turns out to be nothing more than a valuable special case of the more general class of Robinson-Trautman spacetimes.

Another important special case of the given setting arises if the 'standard Robinson-Trautman form' of the above line element (32) with $W_b = 0$ is made. This case, already found by Coll, Hildebrant and Senovilla in their pioneering work on Kerr-Schild symmetries, represents the class of all spacetimes of the Kerr-Schild class (defined in relation to a flat background metric) for which Kerr-Schild groups exist. This includes an interesting subclass, commonly known as the Bonnor-Vaidya family of spacetimes, for which the next section will demonstrate that geometric constraints $(25 – 31)$ are all met simultaneously.

Ultimately, as it commonly turns out in this context, the existence of Kerr-Schild symmetries is accompanied by the existence of a class of conserved Kerr-
Schild currents (provided that a single additional geometric constraint is additionally imposed), as shown in the following section on the basis of a decomposition of the deformed Einstein tensor of the generalized Kerr-Schild class in spin-coefficient form, which is universally valid regardless of the problem under consideration. As will be further clarified, this definition permits the specification of special classes of pure causal symmetries, which shall be as *phantom symmetries*, which could play an important role for the characterization of ‘asymptotically conserved’ quantities in generic spacetimes (i.e. of quantities that are only conserved at early and late times) in the future.

**Conserved generalized Kerr-Schild Currents and Phantom Symmetries**

A very interesting consequence of the existence of a Kerr-Schild group is the possibility of formulating apparent (that is, purely formal) conservation laws that lead to actual conservation laws in the case that the generalized Kerr-Schild spacetime under consideration exhibits a suitably high degree of symmetry at early and late times.

For the purpose of finding such conservation laws, the stress-energy tensor may be contracted with a Kerr-Schild vector to construct a current \( j^a = T^a_{\xi b} \xi^b \), which turns out to be subject to the relation

\[
\nabla_a j^a = \nabla_a (T^a_{\xi b} \xi^b) = 0,
\]

provided that \( \Phi_{00} = 0 \). The class of spacetimes on which Kerr-Schild groups exist is therefore the class of generalized Kerr-Schild spacetimes on which the null dominant energy condition is not only fulfilled, but on which the corresponding inequality is exactly saturated. More precisely, it is the class of generalized Kerr-Schild spacetimes in relation to which both \( \Phi_{00} = 0 \) and the geometric constraints (25 – 31) are met.

This is also true if the background spacetime itself is a generalized Kerr-Schild spacetime, in which case the given relation reads

\[
\tilde{\nabla}_a \tilde{j}^a = \tilde{\nabla}_a (\tilde{T}^a_{\xi b} \xi^b) = 0,
\]

where \( \tilde{T}^a_b \) is subject to the deformed field equations

\[
\tilde{G}^a_b = G^a_b + \rho^a_b = 8\pi \tilde{T}^a_b,
\]

which are defined in full accordance with relations (14) and (16), in such a manner that the background field equations

\[
G^a_b = 8\pi T^a_b
\]

remain valid.
A necessary and sufficient condition for the validity of (34) is $\Phi_{00} = 0$, as can be concluded from the fact that $\rho^a_b l^a_b = 0$. This can be seen as follows: Using the decomposition relations

$$\nabla_a l_b = -(\gamma + \bar{\gamma}) l^a_b + \bar{\tau} a m_b + \tau a \bar{m}_b + (\alpha + \beta)m_a b +$$

$$+ (\bar{\alpha} + \beta)\bar{m}_a b - \bar{\sigma} m_a b - \bar{\tau} m_a b - \bar{\rho} m_a b - \rho m_a b,$$

(37)

$$\nabla_a k_b = (\gamma + \bar{\gamma}) k^a b - \pi k^a m_b - \bar{\pi} k^a b - \nu l^a m_b - \bar{\nu} l^a b - (\alpha + \beta)m_a k_b -$$

$$- (\bar{\alpha} + \beta)\bar{m}_a k_b + \lambda m_a m_b + \lambda \bar{m}_a b + \mu m_a m_b + \mu \bar{m}_a b,$$

(38)

$$\nabla_a m_b = (\gamma - \bar{\gamma}) l^a m_b - \bar{\nu} l^a b + \tau l^a k_b + (\epsilon - \bar{\epsilon}) k^a m_b - \bar{\tau} k^a l_b +$$

$$+ (\alpha - \bar{\alpha}) m_a m_b + (\beta - \bar{\beta}) m_a m_b + \bar{\mu} m_a m_b - \rho m_a b + \lambda \bar{m}_a b - \sigma \bar{m}_a b,$$

(39)

one finds (after a lengthy computation) that the deformed part $\rho^a_b$ of the mixed Einstein tensor $\bar{G}^a_b$ can be decomposed in the form

$$\rho^a_b = \mathfrak{G}_1 l^a b + \mathfrak{G}_2 (l^a k_b + k^a l_b) + \mathfrak{G}_3 (l^a m_b + m^a l_b) +$$

$$+ \mathfrak{G}_4 (\bar{m}^a m_b + m^a \bar{m}_b) + \mathfrak{G}_5 m^a m_b + \mathfrak{G}_5 \bar{m}^a \bar{m}_b,$$

(40)

where the individual components - with respect to the setting fixed in the previous section - take the form

$$\mathfrak{G}_1 = -\frac{1}{2} D^2 f - \bar{\tau} \delta f - \tau \delta f + \rho D f - (\gamma + \bar{\gamma} - \mu) D f -$$

$$- 2 |\tau|^2 f + 2 (\gamma + \bar{\gamma}) \rho f - (\Psi_2 + \bar{\Psi}_2 - 4 \Pi) f,$$

$$\mathfrak{G}_2 = -\rho (D f - D \rho f + 2 \rho^2 f),$$

$$\mathfrak{G}_3 = \frac{1}{2} D \delta f + \tau D f - \frac{1}{2} \rho \delta f + \frac{1}{2} \sigma \delta f + D \tau f - \tau \rho f + \bar{\tau} \sigma f - \Phi_{01} f,$$

$$\mathfrak{G}_4 = -\frac{1}{2} D^2 f + \rho D f + \Phi_{00} f,$$

$$\mathfrak{G}_5 = -\sigma D f - D \sigma f + 2 \bar{\rho} \sigma,$$

provided that, according to the usual conventions, the definitions $\Pi = \rho \frac{\rho}{\rho^2}$ and $D^2 = D^a D_a = \delta \delta' + \beta \delta^2 + (\beta - \alpha) \delta + (\bar{\beta} - \bar{\alpha}) \delta'$ are used and the condition $\mu = \bar{\mu}$ is met in the present context.

Consequently, using the spin-coefficient framework presented in [13], it becomes clear that $\nabla_a \bar{T}^a_b = 0$ in combination with relation (8) implies that

$$\nabla_a \tau^a_b = 0$$

(41)

holds whenever a generalized Kerr-Schild spacetime with an associated Kerr-Schild group is given and not only the conditions $L_{\xi \gamma a b} = \bar{f} l_a b$, $L_{\xi \gamma a} = \bar{g} l_a$, $L_{\xi \gamma a b} = f l_a b$, $L_{\xi \gamma a} = g l_a$, but also the geometric constraints $\Phi_{00} = \Phi_{00} = 0$ and

$$\tau \Phi_{01} + \tau \bar{\Phi}_{10} = \sigma \Phi_{01} + \rho \bar{\Phi}_{10} = \sigma \bar{\Phi}_{10} + \rho \Phi_{01} = 0$$

(42)
are met, from which it can be concluded that \( \sigma = \bar{\sigma} \) and \( \sigma \bar{\tau} = \rho \tau \) must apply in the given context.

Given this particular case, it becomes clear that for these types of generalized Kerr-Schild spacetimes there must hold

\[
\tilde{\nabla}_a \tilde{j}^a = \nabla_a j^a = 0. \tag{43}
\]

In consequence, however, considering the special case in which the background spacetime belongs to the Kerr-Schild class, it even follows that

\[
\partial_a \tau_b^a = 0, \tag{44}
\]

which implies on the basis of the validity of (8) that (42) reduces to the form

\[
\tilde{\nabla}_a \tilde{j}^a = \partial_a j^a = 0. \tag{45}
\]

Next, by applying Gauss’ theorem to equation (34), where it shall be assumed that the boundary of the integration region is a spacelike hypersurface \( \Sigma \), the expression

\[
\tilde{Q} = \int_{\Sigma} j_a d\tilde{\Sigma}^a \tag{46}
\]

can be deduced, which is defined with respect to the hypersurface element \( d\tilde{\Sigma}^a = \sqrt{\tilde{h}} \tilde{n}^a d^3x \).

Accordingly, as it now turns out in this context, the quantity obtained represents a purely formal one without any real physical significance; at least unless \( j^a \)
represents an actual mass-energy current, in which case (45) represents an exact physical conservation law. However, this is the case if and only if the Kerr-Schild vector field coincides with a timelike Killing vector field of the Kerr-Schild geometry, so that the obtained quantity \( \tilde{Q} \) associated with an energy-momentum tensor \( \tilde{j}_a \tilde{n}^a = \tilde{T}_{ab} \xi^a \xi^b \) can be interpreted as the mass-energy measured by a stationary observer evolving along the tangent lines of \( \xi^a \). In this particular case, then, the flow generated by \( \xi^a \) can be used to determine a collection of hypersurfaces \( \{ \Sigma_t \} \), which provide a foliation of spacetime in spacelike hypersurfaces, where \( t \) is the Killing parameter along the integral curves of \( \xi^a \), which is subject to the relation \( (\xi \nabla) t = 1 \). The problem, however, is that not every generalized Kerr-Schild spacetime has a sufficient degree of symmetry, so that (43) is in general not an actual physical conservation law, but rather only a purely formal one, which makes its physical interpretation much more difficult.

In this context, however, the following observation may prove useful in practice: Since it is always possible to consider the special case in which the Kerr-Schild metric listed in relation (1) becomes time-independent as direct result of the fact that suitable conditions are imposed on the geometry of spacetime, it can be assumed that in such a case there is a non-vanishing, smooth, timelike Killing vector field \( \xi^a \), which allows one to define the conserved quantity

\[
\tilde{Q}_0 = \int_{\Sigma} j_a^{(0)} d\tilde{\Sigma}^a_{(0)}, \tag{47}
\]
where the zero in the parentheses indicates that the respective vector and covector fields do not depend on the Killing time parameter \( t \).

Consequently, it can be concluded that, by specifying suitable conditions for a geometric transition from a given generic Kerr-Schild metric to an associated stationary Kerr-Schild metric at early and late times of the evolution of spacetime, the purely formal expression (46) can always be traced back to the conserved quantity (47). This can be achieved by requiring that the profile function of the geometry becomes time-independent at early and late stages of the evolution of spacetime, so that \( \lim_{t \to \pm \infty} f \to f_0 \) applies in the present context, where \( f_0 \) is independent of the Killing parameter \( t \), and by further assuming that \( \lim_{t \to \pm \infty} \partial_t f \) and \( \lim_{t \to \pm \infty} \partial^2_t f \) no longer depend on \( t \), so that spacetime reaches stationary geometric configurations at early and late times, which in turn has the consequence that the limit \( \lim_{t \to \pm \infty} \tilde{Q} \to \tilde{Q}_0 \pm \) leads to quantity \( \tilde{Q}_0 \pm \) that are constant in time. In order to clarify this, a special case of the given setting shall now be discussed. More precisely, the concrete example of Bonnor-Vaidya spacetime shall be considered, which is a spacetime whose associated metric can be read off from the line element

\[
ds^2 = -(1 - \frac{2Mr - e^2}{r^2}) dv^2 + 2dvdr + v^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{48}
\]

in Eddington-Finkelstein coordinates, where \( M = M(v) \) and \( e = e(v) \) apply by definition. As can be seen, the metric of this spacetime can be cast in Kerr-Schild form (1) by using the definition \( l_a = -dv_a \). The energy-momentum tensor of the corresponding solution of the Einstein-Maxwell equations, which may be assumed to describe the absorption of a charged null fluid by a non-rotating charged black hole, is given by the expression

\[
T_{ab} = \frac{1}{4\pi r^2} (\dot{M} - \frac{e\dot{e}}{r}) dv_a dv_b. \tag{49}
\]

By performing a 2+2-decomposition of the corresponding metric and its inverse, one finds that the associated coordinate vector fields can be combined to a null tetrad of the form

\[
l^a = -\partial^a_r,
\]

\[
k^a = \partial^a_v + \left( \frac{v^2 - 2Mr + e^2}{2v^2} \right) \partial^a_r,
\]

\[
m^a = \frac{1}{\sqrt{2r}} (\partial^a_\theta + \frac{i}{\sin \theta} \partial^a_\phi),
\]

\[
m^\ast = \frac{1}{\sqrt{2r}} (\partial^\ast_\theta - \frac{i}{\sin \theta} \partial^\ast_\phi). \tag{50}
\]
According to this particular choice, the spin-coefficients take the form

$$
\epsilon = \tau = \pi = \kappa = \sigma = \nu = \lambda = 0, \quad \beta = -\alpha = \frac{\cot \theta}{2\sqrt{2r}},
$$

\begin{align*}
\gamma &= -\frac{1}{r^2} (M - \frac{e^2}{r}), \\
\rho &= 1, \\
\mu &= \frac{r^2 - 2Mr + e^2}{2r^3}, \\
\phi_1 &= \frac{e}{\sqrt{2}r^2}, \\
\Psi_2 &= -\frac{1}{r^3} (M - \frac{e^2}{r}), \\
\phi_0 &= \phi_2 = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0,
\end{align*}

from which it can be concluded that the relevant spin-coefficients all have exactly the right structure and thus guarantee the existence of a Kerr-Schild group.

A reasonable choice for the Kerr-Schild vector field of the spacetime is

$$
\xi_a = \partial_a v,
$$

so that one has $K = 1$ and $L = \frac{r^2 - 2Mr + e^2}{2r^3}$. This is due to the fact that the given vector field is both a Kerr-Schild vector field in relation to the deformed Kerr-Schild metric and Killing vector field in relation to the flat Minkowskian background. Accordingly, by considering the case in which

$$
\lim_{v \to \pm \infty} f \to \frac{2M_0^+ - e_0^+}{r^2} \quad \text{and} \quad \lim_{v \to \pm \infty} \partial_v f \to \frac{2(M_0^+ - e_0^+)}{r^2},
$$

it holds, where the functions $\lim_{v \to \pm \infty} M = M_0^\pm$, $\lim_{v \to \pm \infty} M_0^\pm$ and $\lim_{v \to \pm \infty} e = e_0^\pm$ and $\lim_{v \to \pm \infty} \dot{e} = \dot{e}_0^\pm$ such that $M_0^\pm$, $e_0^+$ and $\dot{M}_0^\pm$, $\dot{e}_0^\pm$ are independent of $v$, it is guaranteed that Kerr-Schild vector field actually represents a Killing vector field at early and late stages and that therefore the limit $\lim_{v \to \pm \infty} \tilde{Q} \to \tilde{Q}_0^\pm$ leads as required to quantities $\tilde{Q}_0^\pm = \int \frac{M_0^+ r^2 - e_0^+ \dot{e}_0^+}{r^2 - 2M_0^+ r + e_0^+} dr$ that are constant in time. A choice of physical interest in this regard is the case in which an initially given Reissner-Nordström spacetime changes its geometric structure due to the accretion of charged null particles; a case that occurs naturally if the conditions $M_0^+ > M_0^-$ and $e_0^+ < e_0^-$ are imposed in the present context. The result of this is that the associated Kerr-Schild vector field turns out to be a timelike Killing vector field at early and late stages of the evolution of the gravitating system under consideration, so that there are physical charges which are conserved in relation to the stationary initial and final geometric configurations of spacetime, whereas the associated type of causal symmetry shall be called phantom symmetry in order to indicate that it is not a real symmetry, but an apparent symmetry, which arises as a result of imposing suitable conditions on the geometry of spacetime at early and times.

Accordingly, it can be concluded that in the case of generic spacetimes, which do not exhibit any symmetries and thus do not lead to exact conservation laws, this form of generalized symmetry is the best possible substitute on the basis of which physical laws can be formulated. This applies not only to Kerr-Schild symmetries or even more general causal symmetries, but to all classes of metric deformations; may they be linear or fully nonlinear.

As is argued in [10], the formulation of corresponding conditions and the specification of associated stationary 'four-geometric initial data' allows the specification of a special class of generic spacetimes, the consideration of which could be useful not only in the given Kerr-Schild case, but in general. It is to
be expected that future studies on this subject will emphasize this aspect of the theory even more.

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