FAMILIES OF (3,3)-SPLIT JACOBIANS

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Abstract. We compute all the “special cases” of (3,3)-split Jacobians and we give an explicit parametrization of the Igusa–Clebsch invariants of (3,3)-split Jacobians isogenous to a product of two elliptic curves from the Hesse pencil.

1. Introduction

If $C$ is a curve of genus two, equipped with an optimal covering $\phi: C \to E$ of degree $n$ of an elliptic curve then there exists an elliptic curve $E'$ such that the Jacobian $\text{Jac}(C)$ is isogenous to $E \times E'$ via an isogeny of degree $n^2$. Such a Jacobian is said to be $(n,n)$-split. Classical examples for $n = 2$ and $n = 3$ are attributed to Legendre, Jacobi and Bolza. For modern work on the topic for $n \in \{2, 3, 4, 5\}$, see [1], [5], [6], [9], [10], [14], [15].

The problem of finding the curve $E'$ (or its isomorphism class), given the map $\phi$, was considered in [9]. Explicit examples are given in [9] and [15] when $n = 3$.

This paper deals with the case $n = 3$. In the first two sections, we review the known results and offer minor corrections. In the remaining sections, we consider a similar problem – given two elliptic curves $E$ and $E'$, find the isomorphism class of the genus-2 curve $C$, such that $\text{Jac}(C) \sim E \times E'$, if it exists. When dealing with both of these problems, one is not only interested in concrete examples, but also in parametrizations of infinite families. In Section 2 we give a parametrization of the Igusa–Clebsch invariants of $\text{Jac}(C)$ in terms of two parameters that define a pair of elliptic curves from the Hesse pencil. We hope that this approach can be generalized for various $n \geq 4$.

Notations and definitions. Throughout the paper, $K$ denotes a field of characteristic $\text{char}(K) \neq 2$ and $\overline{K}$ denotes an algebraic closure of $K$. Unless otherwise specified, varieties are projective and curves are smooth. If $C$ is a curve of genus two, defined over $K$, we say that its Jacobian $\text{Jac}(C)$ is split if it is isogenous over $K$ to a product $E \times E'$ of two elliptic curves. Given a commutative ring $R$ and polynomials $F, G \in R[x]$, we shall denote by $\text{Res}_x(F,G)$ the resultant of $F$ and $G$ and we shall denote by $\text{Disc}_x(F)$ the discriminant of $F$.

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2. A CURVE OF GENUS TWO COVERING A CURVE OF GENUS ONE

Let $C$ be a curve of genus two, defined over $K$, and equipped with a covering $\phi : C \to E$ of degree $n$, where $E$ is a curve of genus one and $n$ is coprime to $\text{char}(K)$. Recall that $C$ is hyperelliptic since the linear system defined by its canonical divisor $K_C$, defines a 2-to-1 map to $\mathbb{P}^1$, by Riemann–Roch. Let $\iota$ denote the hyperelliptic involution on $C$, so that $C/\iota \cong \mathbb{P}^1$. For a Weierstraß point $W \in C(\overline{K})$, one can define the Abel–Jacobi map $\alpha : C \hookrightarrow \text{Jac}(C)$, given by $P \mapsto [P - W]$, and embed $C$ into its Jacobian. One also has a corresponding isomorphism $E \cong \text{Jac}(E)$, given by $P \mapsto [P - \phi(W)]$. The morphism $\phi : C \to E$ induces a group morphism $\phi^* : \text{Jac}(C) \to E$ so that $\phi^* \circ \alpha = \phi$. The group morphism $\phi^*$ commutes with $[-1]$ and therefore induces a map $\text{Jac}(C)/[-1] \to E/[-1]$. Thus we obtain the following commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\phi} & E \\
\downarrow{\pi_C} & & \downarrow{\pi_E} \\
C/\iota & \xrightarrow{f} & E/[-1]
\end{array}
$$

Here $\pi_C$ and $\pi_E$ denote the canonical maps.

**Remark 2.1.** The Abel–Jacobi map $\alpha$ is in general only defined over a quadratic extension of $K$, but this issue is easily resolved when $n$ is odd, as we shall explain below.

**Assumption 2.2.** From now on, we will assume that the covering map $\phi : C \to E$ is **optimal**, which is to say that it does not factor through a non-trivial isogeny.

2.1. **Ramification analysis.** Kuhn [9] analysed the ramification of the map $f$. We recall the main results. The map $\pi_C$ has six geometric ramification points, whereas $\pi_E$ has four, by Riemann–Hurwitz. These are, of course, the points fixed by $\iota$. Let $W_1, \ldots, W_6$ denote the ramification points of $\pi_C$ and let $T_1, \ldots, T_4$ denote the ramification points of $\pi_E$. Moreover, let $w_1, \ldots, w_6$ and $t_1, \ldots, t_4$ denote their respective images under the corresponding canonical maps $\pi_C$ and $\pi_E$. It is clear from the above that $\phi(\{W_i\}) \subseteq \{T_j\}$ and $f(\{w_i\}) \subseteq \{t_j\}$.

By Riemann–Hurwitz, we have that $\phi$ has ramification degree 2, meaning that $\phi$ either doubly ramifies at two distinct points or it has one triple ramification point. We distinguish two cases – either this ramification occurs above some $T_j$ or it does not. These are referred to as the “special” case and the “general” case, respectively. As $\iota$ acts on the ramification divisor $R_\phi$ of $\phi$, if there are two distinct ramification points, they cannot lie above two distinct $T_j$. In the generic case, the map $\pi_E \circ \phi = f \circ \pi_C$ ramifies at $4n$ double points that lie above the $T_j$. Since $\pi_C$ ramifies at six double points, we have that $f$ ramifies at $(4n - 6)/2 = 2n - 3$ double points above the $t_j$, none of which is any of the $w_i$. By Riemann–Hurwitz, $f$ has ramification degree $2n - 2$, which means that there is one more doubly ramified point that does not lie above the $t_j$. In the special case, all of the ramification lies above
the $t_j$. Since $f$ is a finite map between smooth varieties, it is flat and every fibre has $n = \deg f$ geometric points, counting with multiplicities.

**Lemma 2.3.** With the notations as above, for every $i \in \{1,2,\ldots,6\}$ the divisor $f^*(\sum_{j=1}^4 t_j)$ contains $w_i$ with odd multiplicity and any other points with even multiplicity.

*Proof.* See Lemma in §1 of [9]. □

**Lemma 2.4.** Let $C$ be a curve of genus two and let $\phi_1: C \to E_1$ be an optimal covering of an elliptic curve $(E_1,O_1)$ with $\deg \phi_1 = n$ coprime to $\char(K)$. Then, possibly after extending the base field, there exists an elliptic curve $(E_2,O_2)$, an optimal covering $\phi_2: C \to E_2$, and an isogeny $\varphi: E_1 \times E_2 \to \Jac(C)$ such that:

1. $\deg \phi_2 = n$;
2. $\varphi = \phi_1^* + \phi_2^*$;
3. $\deg \varphi = n^2$;
4. $\Ker(\varphi) \cong E_1[n] \cong E_2[n]$.

*Proof.* The covering map $\phi_1$ induces an embedding $\phi_1^* : E_1 \hookrightarrow \Jac(C)$, with respect to an isomorphism $E_1 \cong \Jac(E_1)$. The elliptic curve $E_2$ is given as $\Ker(\phi_2) \subset \Jac(C)$, which is connected because $\phi$ is optimal. Let $\alpha : C \hookrightarrow \Jac(C)$ be an embedding, not necessarily defined over $K$. Recalling that Jacobians are (canonically) self-dual, let $\eta : \Jac(C) \to E_2$ denote the map dual to the inclusion $E_2 \hookrightarrow \Jac(C)$. The covering map $\phi_2 : C \to E_2$ is then obtained as the composition $\eta \circ \alpha$. The isogeny $\varphi : E_1 \times E_2 \to \Jac(C)$ is given by $\varphi = \phi_1^* + \phi_2^*$ and its kernel is the image of $E_i[n]$ under the embedding $\phi_i^* : E_i \hookrightarrow \Jac(C)$ for both $i \in \{1,2\}$. For details, see Lemma in §2 of [9] or Lemma 1.6 in [4]. □

**Definition 2.5.** With the assumptions of Lemma 2.4 the Jacobian $\Jac(C)$ is said to be $(n,n)$-split and the curves $E_1$ and $E_2$ are said to be *glued along their $n$-torsion*.

**Remark 2.6.** The constructions in Lemma 2.4 depend on the choice of the embedding $\alpha : C \to \Jac(C)$, which need not be $K$-rational.

### 2.2. Optimal coverings of odd degree

Let $\phi_1 : C \to E_1$ be an optimal covering of odd degree $n$. Let $\pi_i : E_i \to \mathbb{P}^1$ be the canonical maps, let $T_j$ be the geometric ramification points of $\pi_1$ and let $t_j = \pi_1(T_j)$. It follows from Lemma 2.3 that there is a unique ramification point of $\pi_1$, say $T_4$, such that exactly three of the $W_i$ map to it under $\phi_1$. Moreover, there is exactly one $W_i$ above each point in $\{T_1,T_2,T_3\}$. We index the points so that $W_1, W_2, W_3$ lie above $T_4$. It follows that the divisors $W_1 + W_2 + W_3$ and $W_4 + W_5 + W_6$ are $K$-rational and that the point $T_4$ is $K$-rational. Analogous statements hold for the points $w_i$ and $t_j$. Thus we conclude that the curve $C$ admits an affine plane model $y^2 = P(x)Q(x)$, where $P(x), Q(x) \in K[x]$ are cubics whose roots are $\{w_1, w_2, w_3\}$ and $\{w_4, w_5, w_6\}$, respectively. Since the class of the canonical divisor $K_C \sim 2W_i$ is $K$-rational, so is the class of the divisor $W_1 - W_2 + W_3$. Since the latter is a divisor of degree one and invariant under
the action of the hyperelliptic involution, it follows that \( \phi_1 \) induces a canonical \( K \)-rational embedding \( C \hookrightarrow \text{Jac}(C) \), given by

\[
P \mapsto [P - W_1 + W_2 - W_3],
\]

which is compatible with the isomorphism \( E_1 \cong \text{Jac}(E_1) \), given by \( P \mapsto [P - T_4] \), and the involutions. We summarize with the following lemma.

**Lemma 2.7.** If \( n \) is odd then there is a canonical and \( K \)-rational choice for \( E_2 \) and the associated morphisms.

**Definition 2.8.** If \( n \) is odd, we will implicitly assume the embedding (2.2) and we will say that \( E_2, \phi_2, \) and \( f_2 \) are complementary to \( E_1, \phi_1, \) and \( f_1 \), respectively.

**Lemma 2.9.** In the case of optimal coverings of odd degree \( n \), the roles of the divisors \( w_1 + w_2 + w_3 \) and \( w_4 + w_5 + w_6 \) are exchanged between the complementary maps \( f_1 \) and \( f_2 \). We have \( f_{1*}(w_1 + w_2 + w_3) = 3\pi_{1*}(O_1) \) and \( f_{2*}(w_4 + w_5 + w_6) = 3\pi_{2*}(O_2) \), and hence also \( f_{1*}(w_4 + w_5 + w_6) = \pi_{1*}(E_1[2] \setminus \{O_1\}) \) and \( f_{2*}(w_1 + w_2 + w_3) = \pi_{2*}(E_2[2] \setminus \{O_2\}) \), where the identity element \( O_1 \in E_1(K) \) is given by \( T_4 \).

**Proof.** See §4 of [9] or Theorem 1.7 in [4]. \( \square \)

### 3. Covering Maps of Degree Three

From now on, we will fix \( n = 3 \) and focus exclusively on that case. We therefore assume that \( \text{char}(K) \not\in \{2, 3\} \). It follows from Lemma 2.3 and its preceding paragraph that the ramification of the map \( f_1: C \to E_1 \) is restricted to one of two possibilities. This ultimately leads to a parametrization of the \( K \)-isomorphism invariants of the curves involved. We will deal with the generic case first.

**3.1. The generic case.** The results in this subsection appear in [9]. Let \( \phi_1: C \to E_1 \) be an optimal covering of degree 3. In the generic case, the map \( f_1 \) is doubly ramified at a \( K \)-rational point that is not in any of the fibres \( f_1^*(t_j) \). Let us denote the image of this point under \( f_1 \) by \( t_0 \). Since \( t_0 \) and \( t_4 \) are \( K \)-rational and we are interested in isomorphism classes, we may and do assume that \( t_0 = 0, t_4 = \infty, \) and \( f_1^*(0) = 2 \cdot 0 + \infty \). In summary, we assume that the ramification of \( f_1 \) is as depicted in Fig. 3.1, where the unramified points above \( t_1, \ldots, t_4 \) are the \( w_i \).

![Figure 3.1. Ramification of the map \( f_1 \) in the generic case.](image-url)
In other words, we assume, without loss of generality, that 

\[ f_1(x) = \frac{x^2}{P(x)}, \]

where \( P(x) = x^3 + ax^2 + bx + c \in K[x] \) has roots \( w_1, w_2, w_3 \) over \( \overline{K} \). The \( w_i \) are pairwise distinct and none of them equals zero. This can be expressed as

\[
\text{Res}_x(x, P(x)) = c \neq 0, \\
\text{Disc}_x(P(x)) = a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2 \neq 0.
\]

The pullback of \( t_1 + t_2 + t_3 \) corresponds to the roots of \( D(x)^2Q(x) \), where \( D(x) \) and \( Q(x) \) are cubics in \( K[x] \). Moreover, the roots of \( D(x) \) are the ramification points distinct from 0, and the roots of \( Q(x) \) are \( w_1, w_2, w_3 \). Since

\[
\frac{df_1(x)}{dx} = -\frac{x(x^3 - bx - 2c)}{P(x)^2}
\]

and the roots of the numerator correspond precisely to the doubly ramified points of \( f_1 \), we can take \( D(x) = x^3 - bx - 2c \). The ramification points are again pairwise distinct so we have

\[
\text{Disc}_x(D(x)) = 4(b^3 - 27c^2) \neq 0. \tag{3.1}
\]

From this we can calculate the nonic \( D(x)^2Q(x) \) whose roots correspond to the divisor \( f_1^* \circ f_1^* \circ (d_1 + d_2 + d_3) \), where the \( d_i \) are the roots of \( D(x) \). In particular, we have the following equality, up to multiplication by constants,

\[
\text{Res}_y(x^2P(y) - y^2P(x), D(y)) = D(x)^2Q(x).
\]

This resultant is easily found to equal

\[
c(x^3 - bx - 2c)^2(4cx^3 + b^2x^2 + 2bcx + c^2),
\]

whence we can take \( Q(x) = 4cx^3 + b^2x^2 + 2bcx + c^2 \). It follows that, up to quadratic twists, the curve \( C \) admits an affine plane model given by

\[
y^2 = P(x)Q(x) = (x^3 + ax^2 + bx + c)(4cx^3 + b^2x^2 + 2bcx + c^2).
\]

By Lemma 2.9 we may assume

\[
f_2(x) = \frac{(x + d)^2(x + e)}{4cx^3 + b^2x^2 + 2bcx + c^2}
\]

for some \( d, e \in K \), up to multiplication by a constant. To determine \( d \) and \( e \), we apply the procedure used to obtain \( Q(x) \) from \( f_1 \) to the map \( f_2 \). In doing so, we must ultimately obtain a cubic polynomial \( R(x) \) that is a multiple of \( P(x) \), by Lemma 2.9. Working over the field \( K(a, b, c, d, e) \), we compute the polynomial \( R(x) \) and perform Euclidean division on \( P(x) \) and \( R(x) \). By the argument above, the remainder must equal zero and we obtain three polynomial equations over the ring \( K[a, b, c, d, e] \). More details can be found in [4], where the computations are performed over \( K(a, b, c)[d, e] \). The approach here is slightly different. In the ring \( K[z, a, b, c, d, e] \), let \( I \) denote the ideal generated by

\[
1 - z \text{Disc}_x(P) \text{Disc}_x(Q) \text{Res}_x(x + d, x + e) \text{Res}_x(x, P) \text{Res}_x(x + e, Q) \text{Res}_x(x + d, Q)
\]
and by the coefficients of the remainder obtained by dividing \( R(x) \) and \( P(x) \). Eliminating the variable \( z \) and computing the primary decomposition of the corresponding elimination ideal gives us, among others, the following two equations:

\[
\begin{align*}
bd - 3c &= 0, \\
acd - 4ace + b^2c - bc + 3cde &= 0.
\end{align*}
\]

We therefore take

\[
f_2(x) = \frac{(bx + 3c)^2((b^3 - 4abc + 9c^2)x + b^2c - 3ac^2)}{4cx^3 + b^2x^2 + 2bcx + c^2}, \tag{3.2}
\]

which also covers the cases when \( \infty \) is a zero of \( f_2 \). Now one can determine the modular invariants of \( E_1 \) and \( E_2 \). An affine plane model for \( E_1 \) can be determined by requiring that the set of branch points of the canonical map \( \pi_1 \) is \( \{ t_1, t_2, t_3, \infty \} \), i.e. \( \infty \) and the image under \( f_1 \) of the three roots of \( Q(x) \). Likewise, an affine plane model for \( E_2 \) can be determined by requiring that \( \pi_2 \) ramifies above \( \infty \) and the image under \( f_2 \) of the three roots of \( P(x) \). The corresponding cubics can be obtained from resultants \( \text{Res}_y(xP(y) - y^2, Q(y)) \) and \( \text{Res}_y(xQ(y) - (y + d)^2(y + e), P(y)) \) and are omitted here. The \( j \)-invariants of the two elliptic curves can then be obtained from the cubics by a direct calculation. We obtain the following two expressions, which appear in §6 of [9]:

\[
\begin{align*}
j(E_1) &= \frac{16(a^2b^4 + 12b^5 - 126ab^3c + 216a^2bc^2 + 405b^2c^2 - 972ac^3)^3}{(b^3 - 27c^2)^3(a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2)^2}, \\
j(E_2) &= \frac{256(a^2 - 3b)^3}{a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2}.
\end{align*}
\]

### 3.2. The special cases.

In this subsection we will deal with the cases in which one or both of the maps \( f_i \) are special, i.e. there is a triple ramification point above the branch locus of \( \pi_i \).

#### 3.2.1. First map special.

Suppose that \( f_1 \) is special and \( f_2 \) is not. By passing to a quadratic extension of \( K \) if necessary and applying a suitable automorphism of \( \mathbf{P}^1 \), we may and do assume that \( f_1 \) has a triple zero at 0 and a simple pole at \( \infty \), so that

\[
\begin{align*}
f_1(x) &= \frac{x^3}{x^2 + ax + b}, \\
f_2(x) &= \frac{(x + c)^2(x + d)}{Q(x)}.
\end{align*}
\]

The argument used in the previous section gives \( Q(x) = (a^2 - 4b)x^3 - 2abx^2 - 3b^2x \) and therefore \( C \) admits an affine plane model given by

\[
y^2 = x((a^2 - 4b)x^2 - 2abx - 3b^2)(x^2 + ax + b).
\]

As before, we impose the generic ramification on \( f_2 \) and use Lemma 2.9. The difference in this case is that \( R(x) \) is a priori a cubic so we must identify its leading
coefficient with zero and then find the remainder by dividing by $P(x)$. This ultimately gives

\begin{align*}
ac - 3b = 0, \\
3ad - 3b + c^2 - 4cd = 0.
\end{align*}

It follows that we can take

\[ f_2(x) = \frac{(ax + 3b)^2(a(a^2 - 4b)x + b(a^2 - 3b))}{x((a^2 - 4b)x^2 - 2abx - 3b^2)}. \]

Simple resultant computations finally give models of $E_1$ and $E_2$ from which we obtain

\begin{align*}
j(E_1) &= \frac{16(16a^6 - 144a^4b + 405a^2b^2 - 324b^3)^3}{729b^4(a^2 - 3b)^3(a^2 - 4b)^2}, \quad j(E_2) = \frac{256(a^2 - 3b)^3}{b^2(a^2 - 4b)}. \end{align*}

A simpler parametrization can be obtained by reversing the roles of the maps. On p. 29 of [4] one can find the parametrizations obtained without assuming that $\infty$ is a pole of $f_1$.

3.2.2. Second map special. Suppose that $f_2$ is special and $f_1$ is not. This is analogous to the general case and we may and do assume that

\begin{align*}
f_1(x) &= \frac{x^2}{x^3 + ax^2 + bx + c}, \quad f_2(x) = \frac{(x + d)^3}{4cx^3 + b^2x^2 + 2bcx + c^2}.
\end{align*}

Following the same argument, we obtain:

\begin{align*}
bd - 3c &= 0, \\
-2b + 3ad - 3d^2 &= 0,
\end{align*}

whence we can take

\begin{align*}
f_1(x) &= \frac{x^2}{(bx + 3c)(9cx^2 + 2b^2x + 3bc)} \quad f_2(x) = \frac{(bx + 3c)^3}{4cx^3 + b^2x^2 + 2bcx + c^2}.
\end{align*}

and the curve $C$ admits an affine plane model given by

\[ y^2 = (bx + 3c)(9cx^2 + 2b^2x + 3bc)(4cx^3 + b^2x^2 + 2bcx + c^2). \]

The $j$-invariants of $E_1$ and $E_2$ are then easily found to be

\begin{align*}
j(E_1) &= \frac{64b^3}{c^2}, \quad j(E_2) = \frac{64(4b^3 - 27c^2)^3}{729b^3c^4}. \quad (3.3)
\end{align*}

Note that in all cases one must assume that the numerator and the denominator of $f_i$ both have distinct roots and no common factors with one another. These conditions can be expressed as the non-vanishing of the corresponding resultants, omitted here.
3.2.3. Both maps special. Finally, suppose that both $f_1$ and $f_2$ are special. Without loss of generality, we assume

$$f_1(x) = \frac{x^3}{x^2 + ax + b},$$

where $b \neq 0$ and $a^2 - 4b \neq 0$. We will consider two cases. First suppose that $\infty$ is a triple zero of $f_2$, meaning that $f_2$ is of the form $1/Q(x)$. By the usual arguments, we find

$$Q(x) = (a^2 - 4b)x^3 - 2abx^2 - 3b^2 x.$$  

Applying the same argument to $f_2$, we conclude that $x^2 + ax + b$ must be divisible by

$$3(a^2 - 4b)x^2 + 4ab(a^2 - 4b)x - 16b^2(a^2 - 3b).$$

The remainder obtained by dividing these two polynomials is

$$-a(3a^2 - 8b)(a^2 - 4b)x - a^2 b(3a^2 - 8b).$$

Given that $a^2 - 4b \neq 0$ and $b \neq 0$, the remainder is identically zero if and only if $a = 0$ or $b = 3a^2/8$. For $a = 0$, we obtain

$$f_1(x) = \frac{x^3}{x^2 + b}, \quad f_2(x) = \frac{1}{4x^3 + 3bx}, \quad j(E_1) = j(E_2) = 1728.$$  

For $b = 3a^2/8$, we obtain

$$f_1(x) = \frac{x^3}{8x^2 + 8ax + 3a^2}, \quad f_2(x) = \frac{1}{32x^3 + 48ax^2 + 27a^2 x},$$

$$j(E_1) = j(E_2) = \frac{-873722816}{59049} = -2 \frac{6 \cdot 239^3}{3^{10}}.$$  

Now suppose instead that, up to multiplication by a constant, we have

$$f_2(x) = \frac{(x + c)^3}{(a^2 - 4b)x^3 - 2abx^2 - 3b^2 x}.$$  

The usual argument ultimately yields the following equations

$$ac - b - c^2 = 0,$$

$$8a^2 b - 28b^2 + 2bc^2 - 3c^4 = 0,$$

$$8a^3 - 36ab + 22bc - 11c^3.$$  

Let $u \in \mathbb{K}$ be such that $u^2 = -2$. We again have two cases:

$$b = \frac{2a^2}{9}, \quad c = \frac{2a}{3}$$

and

$$b = \frac{2a^2(1 \pm u)}{(3 \pm u)^2}, \quad c = \frac{2a}{3 \pm u}.$$  

The first case corresponds to $j(E_1) = j(E_2) = 1728$, while the second case corresponds to $j(E_1) = j(E_2) = -873722816/59049$. 
This shows that there are exactly two pairs of isomorphism classes of $E_1$ and $E_2$ such that both covering maps $C \to E_i$ of degree three have “special” ramification, correcting [9] (where it is claimed that there is exactly one) and [15] (where it is claimed that there are exactly nine).

**Remark 3.1.** Suppose that a curve of genus two covers an elliptic curve of $j$-invariant $J_1$ “generically” 3-to-1 and that it covers the complementary curve, of $j$-invariant $J_2$, “specially” 3-to-1. Then it is readily verified, using the parametrizations above, that

$$F(J_1, J_2) := J_1^3 - 1296J_1^2 - 729J_1J_2 + 559872J_1 - 80621568 = 0,$$

which is an equation describing a singular affine curve of genus zero. The nine $j$-invariant pairs given in [15] are obtained as the intersection of the curves $F(J_1, J_2) = 0$ and $F(J_2, J_1) = 0$. However, this does not correspond to the $j$-invariant pairs of curves whose 3-to-1 coverings by a genus-2 curve are both special as the two curves are obtained under the assumption that, for each pair, one covering is always generic. For example, the point $(1728, 1728)$ is on both curves. To see that it is on the first curve, we can take $(b, c) = (3t^2, -t^3)$ in (3.3), for $t \neq 0$. However, these parameters do not define a hyperelliptic curve because $t$ is a multiple root of both $P(x)$ and $Q(x)$ in this case.

### 4. Gluing two elliptic curves

In the previous section we started with an optimal covering $C \to E_1$ of degree 3 and constructed the complementary curve $E_2$. In this section, and the one following it, we adopt a different approach. We start with two elliptic curves $E_1, E_2$ and an isomorphism $E_1[3] \cong E_2[3]$. We wish to construct a curve of genus two, whose Jacobian is isogenous to $E_1 \times E_2$. This approach can be found in [6]. First we recall some definitions and classical results.

Let $A$ be an abelian variety over $K$ and let $\lambda: A \to A^\vee$ be a polarization. Suppose that $m \in \mathbb{Z}$ is coprime to $\text{char}(K)$ and such that $\text{Ker}(\lambda) \subset A[m]$. Let

$$e_m: A[m](\overline{K}) \times A^\vee[m](\overline{K}) \to \mu_m$$

denote the Weil pairing. Then we can associate to $\lambda$ a skew-symmetric pairing

$$e_\lambda: \text{Ker}(\lambda) \times \text{Ker}(\lambda) \to \mu_m$$

that is defined for any pair $(P, Q)$ of geometric points as $e_\lambda(P, Q) = e_m(P, \lambda(Q))$, where $R$ is such that $[m]R = Q$. This does not depend on $R$ or $m$ (see §16 in [12]).

**Lemma 4.1** (Mumford). Let $\varphi: A \to B$ be an isogeny whose degree is coprime to $\text{char}(K)$ and let $\lambda: A \to A^\vee$ be a polarization induced by a line bundle $\mathcal{L}$. Then the following are equivalent:

1. There exists a line bundle $\mathcal{M}$ on $B$ such that $\mathcal{L} = \varphi^*(\mathcal{M})$, inducing a polarization $\lambda': B \to B^\vee$,
2. $\text{Ker}(\varphi) \subset \text{Ker}(\lambda)$ and $e_\lambda$ is trivial on $\text{Ker}(\varphi) \times \text{Ker}(\varphi)$.

**Proof.** See Proposition 16.8 in [12] or Theorem 2 and its Corollary in §23 in [13]. □
Corollary 4.2. Let \( \phi_1 : C \to E_1 \) be an optimal covering of an elliptic curve by a curve of genus two, such that \( \deg \phi_1 = n \) is coprime to \( \text{char}(K) \), and let \( E_2 \) be the complementary elliptic curve. Let \( \alpha : E_1[n] \sim \to E_2[n] \) be the induced canonical isomorphism (with respect to an embedding of \( C \)). Then \( \alpha \) inverts the Weil pairing, i.e.

\[
e_n(P, Q) = e_n(\alpha(P), \alpha(Q))^{-1}
\]

for any \( P, Q \in E_1[n](\overline{K}) \).

In other words, Lemma 4.1 provides a criterion for deciding when a polarization “descends” through an isogeny. In view of the lemma, our starting data are two elliptic curves \( E_1, E_2 \) and an isomorphism \( \alpha : E_1[n] \sim \to E_2[n] \) that is anti-symplectic with respect to the Weil pairing, i.e. \( \alpha \) satisfies (4.1) for any \( P, Q \in E_1[n](\overline{K}) \), where \( n \) is coprime to \( \text{char}(K) \).

We assume the usual principal polarization on \( E_1 \times E_2 \), given by the divisor

\[
\Theta = \{O_1\} \times E_2 + E_1 \times \{O_2\}.
\]

Let \( \Gamma_\alpha \subset (E_1 \times E_2)[n] \) denote the graph of \( \alpha \) and let

\[
\varphi : E_1 \times E_2 \to (E_1 \times E_2)/\Gamma_\alpha =: J
\]

be the canonical map, which is clearly an isogeny.

Lemma 4.3. The isogeny \( \varphi : E_1 \times E_2 \to J \) induces a principal polarization of \( J \).

Proof. See p. 156 in [6]. \( \square \)

Lemma 4.4. Suppose that \( n \) is odd. Then there exists a unique effective divisor \( C \) on \( J \) such that \( D = \varphi^*(C) \) is linearly equivalent to \( n\Theta \) and fixed by \( [-1]_{E_1 \times E_2} \). The divisor \( C \) is fixed by \( [-1]_J \) and principally polarizes \( J \).

Proof. See Proposition 1.1 and Corollary 1.2 in [6]. \( \square \)

It is a well known theorem of Weil (Satz 2 in [17]) that over \( \overline{K} \) any principally polarized abelian surface is either a Jacobian or a product of two elliptic curves (with the usual polarizations). Therefore the question of whether or not \( J \) is a Jacobian reduces to the question of whether or not the divisor \( C \) is geometrically irreducible.

Lemma 4.5. The divisor \( C \) is geometrically irreducible if and only if the divisor \( D \) is geometrically irreducible.

Proof. See Proposition 1.3 in [6]. \( \square \)

Proposition 4.6. Let \( n \geq 3 \) be an odd integer, let \( E_1 \) and \( E_2 \) be elliptic curves, let \( \Theta := E_1 \times \{O_2\} + \{O_1\} \times E_2 \), and let \( \alpha : E_1[n] \to E_2[n] \) be an anti-symplectic isomorphism. Let \( D \) be the unique divisor on \( E_1 \times E_2 \) that is linearly equivalent to \( n\Theta \), invariant under the translations by points of \( \Gamma_\alpha \), and invariant under \( [-1]_{E_1 \times E_2} \). Then \( (E_1 \times E_2)/\Gamma_\alpha \) is not a Jacobian if and only if \( D \) contains a 2-torsion point of \( E_1 \times E_2 \) that is not a point of order two on \( E_1 \times \{O_2\} \) or a point of order two on \( \{O_1\} \times E_2 \).
Proof. As before, let $J$ and $C$ respectively denote the images of $E_1 \times E_2$ and $D$ under the isogeny $\varphi : E_1 \times E_2 \to (E_1 \times E_2)/\Gamma_\alpha$. The divisor $C$ is either a curve of genus two or a sum of two elliptic curves that meet in a rational 2-torsion point. Since $[-1]_J$ induces an involution $\iota$ on $C$, we conclude that $C(K)$ contains exactly six points fixed by $\iota$ if and only if it is reducible and that it contains exactly seven points fixed by $\iota$ if and only if it is reducible. Since $n$ is odd, the restriction of $\varphi$ to the 2-torsion is an isomorphism and there is exactly one geometric point of $(E_1 \times E_2)[2]$ above each point of $C(K)$ that is fixed by $\iota$. Therefore $D(K)$ contains at most seven 2-torsion points. On the other hand, $D(K)$ contains at least the order-2 points of $E_1 \times \{O_2\}$ and $\{O_1\} \times E_2$ and the claim follows. \hfill \qed

Remark 4.7. There exist examples with $C$ reducible. Let $\gamma : E_1 \to E_2$ be an isogeny of two elliptic curves, of degree $n - 1$. Let $\alpha : E_1[n] \sim \to E_2[n]$ be the anti-symplectic isomorphism that is the restriction of $\gamma$ to the $n$-torsion and let $\Gamma_\alpha$ denote its graph. Then the map

$$
\phi : E_1 \times E_2 \to E_1 \times E_2, \quad (P, Q) \mapsto (nP, \gamma(Q))
$$

is an isogeny with kernel $\text{Ker}(\phi) = \Gamma_\alpha$ and therefore $J := (E_1 \times E_2)/\Gamma_\alpha \cong E_1 \times E_2$.

5. Gluing two elliptic curves along the 3-torsion

In this section we will deal with the case $n = 3$, given two elliptic curves from the Hesse pencil. We begin by fixing additional assumptions.

Assumption 5.1. The field $K$ is of characteristic $\text{char}(K) \neq 3$ and it contains a primitive third root of unity.

From now on, let $w \in K$ be such that $1 + w + w^2 = 0$.

5.1. Prerequisites. The one-dimensional family of curves given by

$$
E_a : x^3 + y^3 + z^3 + 3axyz = 0, \quad (5.1)
$$

is called the Hesse pencil\footnote{The 3 does not usually appear in the definition. We added it for simplicity of the formulas.}. With the exception of $a^3 = -1$, each $a \in K$ defines an elliptic curve $E_a$. We choose $[1 : -1 : 0]$ to be the identity element. The group morphisms are given in the Appendix. We shall denote the set of elliptic curves in the Hesse pencil by $\mathcal{H}$.

The $j$-invariant of $E_a$ is

$$
j(E_a) = -\frac{27a^3(a^3 - 8)^3}{(a^3 + 1)^3} \quad (5.2)
$$

and $j : \mathcal{H} \to \mathbb{P}^1$ is 12-to-1, except above $j = 0$ and $j = 1728$. Every element of

$$
\left\{a, aw, aw^2, \frac{-a + 2}{a + 1}, \frac{-a + 2}{w}, \frac{-a + 2w}{a + w}, \frac{-a + 2w^2}{a + w^2}, \frac{-wa + 2}{a + w}, \frac{-wa^2 + 2}{a + w^2}, \frac{-a + 2w}{a + w}, \frac{wa}{a + w^2} \right\}
$$

defines the same isomorphism class.
The 3-torsion subgroup of every elliptic curve in $\mathcal{H}$ is fully $K$-rational and is given by $xyz = 0$. Therefore the same nine points in $\mathbb{P}^2$ are the 3-torsion points of every element of $\mathcal{H}$. Moreover, the Hesse pencil is exactly the family of all cubics passing through these nine points.

Let $S = [-1 : 0 : 1]$ and $T = [-w : 1 : 0]$. These two points generate $E[3]$ for every $E \in \mathcal{H}$. From now on, we are going to fix an isomorphism $\eta: E[3] \isomto (\mathbb{Z}/3\mathbb{Z})^2$ for every elliptic curve $E$ in $\mathcal{H}$, given by $S \mapsto (1, 0)$ and $T \mapsto (0, 1)$.

Let $E$ be an elliptic curve in the Hesse pencil. The Weil pairing on $E[3]$ is completely determined by the value $e_3(S, T)$ and we can easily calculate

$$e_3(S, T) = w.$$ 

For example, by using III §8 in [16], we find that $e_3(S, T) = g(P + T)/g(P)$, where

$$g = \frac{x^2z + y^2x + z^2y}{xyz} \in K(E)$$

and $P \in E(K) \setminus (E[3] \cup (P + E[3]))$. It follows that the Weil pairing on $E[3]$ is given by

$$e_3(P, Q) = w^{\det(\eta(P), \eta(Q))}$$

and we can interpret it as the determinant map

$$\det: \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}.$$ 

Since $\text{Aut}(\mathbb{Z}/3\mathbb{Z}) \cong \text{GL}_2(\mathbb{Z}/3\mathbb{Z})$ is a group of order 48, an anti-symplectic isomorphism $E_1[3] \isomto E_2[3]$ corresponds to one of the 24 elements of the coset $[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]\text{SL}_2(\mathbb{Z}/3\mathbb{Z})$. However, since each isomorphism can be composed with $[-1]$, we are left with 12 distinct cases at most.

Before we deal with the general case, we consider a concrete example in which gluing two elliptic curves does not give a Jacobian.

**Example 5.2.** Let $a = -(1 + 2t^3)/(3t^2)$ for some $t \in K$, such that

$$t(t^3 - 1)(8t^3 + 1) \neq 0.$$ 

Then the elliptic curve

$$E_a: x^3 + y^3 + z^3 - \frac{1 + 2t^3}{t^2}xyz = 0$$

has a rational point $[t : t : 1]$ of order two. Let $b = (1 - 4t^3)/(3t)$. Then the map

$$\gamma: E_a \to E_b, \quad [x : y : z] \mapsto [f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z)],$$

where

$$f_1 = x(-2t^2y^2 - t^2xy + t^2x^2 - yz + 2t^3xz + tz^2),$$

$$f_2 = y(-2t^2x^2 - t^2xy + t^2y^2 - xz + 2t^3yz + tz^2),$$

$$f_3 = tz(x + y + t)(x + y - 2tz),$$

may be an anti-symplectic isomorphism.
is an isogeny whose kernel is the cyclic group of order two that is generated by the point \([t : t : 1]\). Restricting \(\gamma\) to the 3-torsion, we obtain the isomorphism \(\alpha : E_a[3] \to E_b[3]\) that corresponds to \(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\) \(\in \text{GL}_2(\mathbb{Z}/3\mathbb{Z})\). We have that \(J := (E_a \times E_b)/\Gamma_\alpha\) is isomorphic to \(E_a \times E_b\) (recall Remark 4.7). We note that \(a\) and \(b\) satisfy
\[
3a^2b^2 + a^3 + b^3 - 3ab + 2 = 0, \tag{5.3}
\]
describing a singular affine curve of genus zero.

We now consider the isomorphism from Example 5.2 in full generality. From now on, we fix \(\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\). Let \(E_1\) and \(E_2\) be two elliptic curves in \(\mathcal{H}\), corresponding to parameters \(a\) and \(b\), respectively. Let \(A\) and \(G\) respectively denote the images of \(E_1 \times E_2\) and \(\Gamma_\alpha\) in \(\mathbb{P}^8\) under the Segre embedding
\[
\sigma : ([x_1 : y_1 : z_1], [x_2 : y_2 : z_2]) \mapsto [x_1x_2 : y_1y_2 : x_1z_2 : y_1z_2 : y_1x_2 : y_1z_2 : z_1x_2 : z_1y_2 : z_1z_2].
\]
The identity element of \(A\) is \(O_A = [1 : -1 : 0 : -1 : 1 : 0 : 0 : 0 : 0]\) and the inversion morphism \([-1]_A\) is given by
\[
[X_1 : X_2 : \cdots : X_9] \mapsto [X_5 : X_4 : X_6 : X_2 : X_1 : X_3 : X_8 : X_7 : X_9]. \tag{5.4}
\]

### 5.2. The computations.

In this subsection we go over the steps that will finally lead to the Igusa–Clebsch invariants of the curve \(C\) of genus two, such that \(\text{Jac}(C)\) is isogenous to \(E_1 \times E_2\), if it exists. Many technical details will be omitted, but appear in the Appendix.

The Igusa–Clebsch invariants are the invariants \(A', B', C', D'\) defined on p. 319 of [11]. The Igusa invariants are the invariants \(J_2, J_4, J_6, J_8, J_{10}\) defined on p. 324 of [11]. By the corresponding absolute invariants we mean the values
\[
\begin{align*}
\dot{j}_1 &= J_2^5 J_{10}, & \dot{j}_2 &= J_2^3 J_4 J_{10}, & \dot{j}_3 &= J_2^J_6 J_{10}.
\end{align*}
\]

We start by computing the ideal \(I = I(A)\) that defines \(A\) as a variety in \(\mathbb{P}^8\). This is a straightforward computation of an elimination ideal and is omitted.

**Lemma 5.3.** Let \(W_1\) and \(W_2\) denote the set of geometric points of order two on \(\sigma(E_1 \times \{O_2\})\) and the set of points of order two on \(\sigma(\{O_1\} \times E_2)\), respectively. Then any hyperplane section on \(A\) that is invariant under \([-1]_A\) contains either \(W_1 \cup W_2\) or its complement in \(A[2](\overline{K})\).

**Proof.** The two eigenspaces of (5.4) are respectively generated by the sets
\[
\begin{align*}
S_1 &= \{X_1 + X_5, X_2 + X_4, X_3 + X_6, X_7 + X_8, X_9\}, \\
S_2 &= \{X_1 - X_5, X_2 - X_4, X_3 - X_6, X_7 - X_8\}. \tag{5.5}
\end{align*}
\]
By adding the corresponding linear forms from (5.5) to \(I\), we find that \(A[2](\overline{K})\) consists of six points that are in the zero locus of the ideal generated by \(S_1\) and ten points that are in the zero locus of the ideal generated by \(S_2\). Since any linear form that is fixed by \([-1]_A\) is a linear combination of the elements of exactly one of these two sets, we are done. \(\square\)
A fact crucial to our approach is that the translations by the points of \( A[3] \) are linear. In fact, they can be extended to automorphisms of \( \mathbb{P}^8 \). This is a consequence of the fact that \( A \) is embedded in \( \mathbb{P}^8 \) via the global sections of \( \mathcal{L}(D) \), where \( D \) is a divisor linearly equivalent to \( 3\Theta \) (cf. [13]). It can also be shown directly, using the addition formulas. In particular, the group of translations by the points in \( G \) is generated by the following two automorphisms:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & w^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & w^2 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & w & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & w & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

From this we can immediately find the nine divisors on \( A \) that map to \( C \) under \( \phi \). They are the hyperplane sections defined by the following linear forms:

\[
L_1 = X_1 + X_5 + X_9, \quad L_6 = w^2X_3 + wX_4 + X_8,
\]

\[
L_2 = wX_1 + w^2X_5 + X_9, \quad L_7 = wX_2 + w^2X_6 + X_7,
\]

\[
L_3 = w^2X_1 + wX_5 + X_9, \quad L_8 = wX_3 + w^2X_4 + X_8,
\]

\[
L_4 = X_3 + X_4 + X_8, \quad L_9 = w^2X_2 + wX_6 + X_7.
\]

We note that the divisor \( D \), that is invariant under \([−1] \), is \( Z(L_1) \cap A \) and that it does not contain \( O_A \). Now we can use Lemma 5.3 and compute the scheme that is the intersection of \( D \) and the nine points of \( A[2](\overline{K}) \) that are not 2-torsion points on \( E_1 \times \{O_2\} \) or \( \{O_1\} \times E_1 \). Taking the corresponding ideal in the ring \( K[X_1, \ldots, X_9, a, b] \) and eliminating the \( X_i \) gives

\[
3a^2b^2 + a^3 + b^3 - 3ab + 2 = 0 \quad (5.6)
\]

Note that this matches (5.3) from Example [5.2]. Thus we have obtained the following.

**Proposition 5.4.** The quotient \( J \) is a Jacobian if and only if (5.6) holds.

Let \( \varphi: A \rightarrow A/G = J \) denote the isogeny. The quotient \( J \) can be found as the quotient of the variety \( A \) under the group action of \( G \), where \( G \) acts by point translation (see Lecture 10 in [8], for example). Note that since \( \varphi^* \) is injective and \( \dim_K(\mathcal{L}(nC)) = n^2 \), we have that the subspace of \( G \)-invariants of \( \mathcal{L}(nD) \) is \( n^2 \)-dimensional. In particular, we need to find nine linearly independent polynomials in the ring \( K[X_1, \ldots, X_9] \) that generate the ring \( K[X_1, \ldots, X_9]^G \) of \( G \)-invariants, in order to obtain an explicit embedding of \( J \) into \( \mathbb{P}^8 \). We may take the nine invariants \( L_i^3 \) for this purpose. We remark that computing \( J \) is not feasible, unsurprisingly.
From now on, let us assume that $3a^2b^2 + a^3 + b^3 - 3ab + 2 \neq 0$ so that $C$ is irreducible. It follows (see [7]) that the global sections of $\mathcal{L}(2C)$ define the map $\kappa: A \to \mathbb{K}$, where $\mathbb{K} = A/[-1] \subset \mathbb{P}^3$ is the Kummer surface of $A$. Then $\kappa(D)$ is a conic in $\mathbb{P}^3$ and the image under $\kappa$ of the 2-torsion points that lie on $D$ gives six pairwise distinct (geometric) points on $\kappa(D)$ that are the branch locus of the canonical 2-to-1 map $C \to \kappa(D)$. By finding a rational point on the conic $\kappa(D)$, we obtain an isomorphism $\kappa(D) \cong \mathbb{P}^1$ and the image of the six branch points will give us a sextic that defines a plane model of $C$. From this model one can directly compute the absolute invariants. To accomplish this, we may take the following four $G$-invariants to define $\kappa$:

\[
\begin{align*}
X_1^2 + X_5^2 + X_9^2, \\
X_2X_4 + X_3X_7 + X_6X_8, \\
X_2X_5 + X_4X_6 + X_7X_8, \\
X_2X_8 + X_3X_6 + X_4X_7.
\end{align*}
\]

Note that all of this is done for concrete values $a, b \in \mathbb{K}$. An alternative approach is to compute the curve $C = \varphi(D)$ directly, compute the canonical divisor $K_C$, and then find the image in $\mathbb{P}^1$ of the six points of $J[2](\mathbb{K})$ that lie on $C$, under the canonical map defined by $|K_C|$. However, this turns out to be significantly slower in practice than the Kummer surface approach.

We make an important observation. The absolute invariants of $C$, as functions of the parameters $a$ and $b$, will have certain symmetries. For example, the abelian surface $E_1 \times E_2$ is isomorphic to $E_2 \times E_1$ and the isomorphism (which is just a permutation of the coordinates) leaves $G$ intact so that $(E_1 \times E_2)/\Gamma_\alpha$ and $(E_2 \times E_1)/\Gamma_\alpha$ will give the same absolute invariants. Similarly, the same invariants are obtained for $(aw, bw^2)$ and $(aw^2, bw)$.

**Remark 5.5.** Recall that for each curve in $\mathcal{H}$ there are eleven other curves in $\mathcal{H}$ that are isomorphic to it, with the exception of the usual two isomorphism classes. It is a natural question to ask which of the 144 possible isomorphic pairs $E_1 \times E_2$ result in the same isomorphism class of $C$ when modding out $G$. It turns out that the pairs are grouped into twelve sets of twelve pairs and for each set every pair results in the same isomorphism class of $C$. Moreover, all of the remaining eleven choices of $\alpha$ can be reduced to the case we are considering by applying an isomorphism to a suitable product of two elliptic curves in $\mathcal{H}$. Our choice of $\alpha$ is particularly suitable for computations because of the simplicity of the equation defining $D$ and the symmetries mentioned above.

To obtain the absolute invariants of $C$ as functions of $(a, b)$, the first thing we do is make several degree estimates. For example, we can take $a$ and $b$ to be two large consecutive primes or $a \in \{0, 1\}$ and $b$ a large prime and get estimates for the degrees of various monomials that appear in the invariants. Then we notice that the so-called discriminant $J_{10}$, that appears in the denominators, is going to be zero for choices of $(a, b)$ that either do not define a pair of elliptic curves or do not define
a quotient $J$ that is a Jacobian. By factoring the invariants obtained for a choice of various $a, b \in \mathbb{Z}$ and combining this information with the degree estimates, we conclude that, up to a constant, $J_{10}$ equals

$$(a^3 + 1)(b^3 + 1)(3a^2b^2 + a^3 + b^3 - 3ab + 2)^{12}. \quad (5.7)$$

To obtain the numerators, we use interpolation. We compute the absolute invariants of $C$ for many choices of $(a, b)$ and multiply by $(5.7)$ in each case. From the inherent symmetries mentioned above, we conclude that the numerators are linear combinations of monomials $a^mb^n$, where $m \equiv n \pmod{3}$. This significantly reduces the number of non-zero coefficients and makes the computation reasonably fast. Using our empirical bounds on the degrees and an empirical bound on the size of the coefficients, we can interpolate over the fields $F_p$ for a suitable set of primes $p$ and then lift the results using the Chinese remainder theorem. Then it is easy to convert the Igusa invariants to the Igusa–Clebsch invariants, following the formulas in [11]. We summarize our results in the following proposition.

**Proposition 5.6.** Let $E_1$ and $E_2$ be two elliptic curves, respectively given by the Hesse models $x^3 + y^3 + z^3 + 3axyz = 0$ and $x^3 + y^3 + z^3 + 3bxyz = 0$, with identity $O_i = [1 : -1 : 0]$, for $a, b \in K$ such that $(a^3 + 1)(b^3 + 1)(3a^2b^2 + a^3 + b^3 - 3ab + 2) \neq 0$. Let $\alpha : E_1[3] \rightrightarrows E_2[3]$ be the isomorphism given by $[-1 : 0 : 1] \mapsto [-1 : 0 : 1]$ and $[-w : 1 : 0] \mapsto [-1 : w : 0]$ and let $\Gamma_\alpha$ be the graph of $\alpha$. Then the abelian surface $(E_1 \times E_2)/\Gamma_\alpha$ is isomorphic to the Jacobian of a curve of genus two, whose Igusa–Clebsch invariants are as follows:

$$I_2 = 72(9a^6b^6 - 30(a^7b^4 + a^4b^7) - 88a^5b^5 + a^8b^2 + a^2b^8 + 54(a^6b^3 + a^3b^6) + 65a^4b^4 - 32(a^7b + ab^7) - 104(a^5b^2 + a^2b^5) + 40(a^6 + b^6) + 44a^3b^3 + 100(a^4b + ab^4) - 68a^2b^2 + 16(a^3 + b^3) + 112ab - 20),$$

$$I_4 = 36(3a^2b^2 + a^3 + b^3 - 3ab + 2)^4(9a^4b^4 + 240a^3b^5 + 8(a^4b + ab^4) + 240a^2b^2 + 160(a^3 + b^3) + 256ab + 320)$$

$$I_6 = 72(3a^2b^2 + a^3 + b^3 - 3ab + 2)^4(729a^{10}b^{10} - 3402(a^{11}b^8 + a^8b^{11}) + 30456a^9b^9 + 81(a^{12}b^6 + a^6b^{12}) - 70794(a^{10}b^7 + a^7b^{10}) - 201555a^8b^8 - 2160(a^{11}b^5 + a^5b^{11}) + 60(a^{12}b^3 + a^3b^{12}) + 106560(a^9b^6 + a^6b^9) - 148932a^7b^7 - 121608(a^{10}b^4 + a^4b^{10}) + 480(a^{11}b^2 + a^2b^{11}) - 358740(a^8b^5 + a^5b^8) - 8(a^{12} + b^{12}) + 156928(a^9b^3 + a^3b^9) + 336444a^6b^6 - 50160(a^{10}b + ab^{10}) + 81072(a^7b^4 + a^4b^7) - 462906a^5b^5 - 167112(a^8b^2 + a^2b^8) + 84224(a^9 + b^9) + 455568(a^6b^3 + a^3b^6) + 761040a^4b^4 + 181152(a^7b + ab^7) - 93600(a^5b^2 + a^2b^5) + 219552(a^6 + b^6) + 383424a^3b^3 + 564480(a^4b + ab^4) + 88512a^2b^2 + 74624(a^3 + b^3) + 314112ab - 55040),$$
\[ I_{10} = 36864(a^3 + 1)(b^3 + 1)(3a^2b^2 + a^3 + b^3 - 3ab + 2)^{12}. \]

When \( K \) is a number field or a finite field of characteristic \( \text{char}(K) > 5 \), it is possible to construct a genus-2 curve over \( K \) with given Igusa–Clebsch invariants (see [3] and [11]). When \( K \) is a number field, the recent work of Bruin, Sijsling, and Zotine [2] allows one to verify numerically over \( \mathbb{C} \) that a curve obtained from Igusa–Clebsch invariants of the form above indeed has a \((3,3)\)-split Jacobian.

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**Appendix**

Given a smooth projective plane curve over \( K \) of the form
\[ x^3 + y^3 + z^3 + \lambda xyz = 0 \]  
(A.1)
for \( \lambda \in K \), we give it the structure of an elliptic curve as follows. The identity element is \( O = [1 : -1 : 0] \). The inversion morphism is given by
\[ [x : y : z] \mapsto [y : x : z]. \]

The addition morphism is given by
\[ ([x_1 : y_1 : z_1], [x_2 : y_2 : z_2]) \mapsto [x_2y_1^2y_2x_1 - x_1y_2^2y_1z_1 : x_1y_2^2z_1 - x_1y_2z_1 : x_2y_1^2z_2 - x_1y_1z_2]. \]

The point duplication morphism is given by
\[ [x : y : z] \mapsto [y(x^3 - z^3) : x(z^3 - y^3) : z(y^3 - x^3)]. \]

One can easily obtain the formulas for the corresponding morphisms on an abelian surface that is the product of two curves of the form (A.1), embedded in \( \mathbb{P}^8 \) via the Segre embedding.

**A.1. The Magma code.** The following code contains the equations defining the abelian surface \( A \), the divisor \( D \), and subschemes of \( A[2] \).

```magma
R<x> := PolynomialRing(Rationals());
K<ω> := NumberField(1+x+x^2);
L<a,b> := FunctionField(K,2);
P8<X1,X2,X3,X4,X5,X6,X7,X8,X9> := ProjectiveSpace(L,8);

s := 3*a;
t := 3*b;
// E1: x^3 + y^3 + z^3 + s*x*y*z = 0
// E2: x^3 + y^3 + z^3 + t*x*y*z = 0
```
// equations defining the surface A
A := Scheme(P8, [
    X1^3 + X2^3 + X3^3 + s*X1*X2*X3,
    X1^2*X2 + X4^2*X5 + X7^2*X8 + t*X1*X4*X8,
    X1*X2^2 + X4*X5^2 + X7*X8^2 + t*X1*X5*X8,
    X2^3 + X5^3 + X8^3 + t*X2*X5*X8,
    X1^2*X3 + X4^2*X6 + X7^2*X9 + t*X1*X4*X9,
    X1*X2*X3 + X4*X5*X6 + X7*X8*X9 + t*X1*X5*X9,
    X2^2*X3 + X5^2*X6 + X8^2*X9 + t*X2*X5*X9,
    X1*X3^2 + X4*X6^2 + X7*X9^2 + t*X1*X6*X9,
    X2*X3^2 + X5*X6^2 + X8*X9^2 + t*X2*X6*X9,
    X3^3 + X6^3 + X9^3 + t*X3*X6*X9,
    X1^2*X4 + X2^2*X5 + X3^2*X6 + s*X1*X2*X6,
    X1*X4^2 + X2*X5^2 + X3*X6^2 + s*X1*X5*X6,
    X4^3 + X5^3 + X6^3 + s*X4*X5*X6,
    X1^2*X7 + X2^2*X8 + X3^2*X9 + s*X1*X2*X9,
    X1*X4*X7 + X2*X5*X8 + X3*X6*X9 + s*X1*X5*X9,
    X4^2*X7 + X5^2*X8 + X6^2*X9 + s*X4*X5*X9,
    X1*X7^2 + X2*X8^2 + X3*X9^2 + s*X1*X8*X9,
    X4*X7^2 + X5*X8^2 + X6*X9^2 + s*X4*X8*X9,
    X7^3 + X8^3 + X9^3 + s*X7*X8*X9,
    X2*X4 - X1*X5,
    X3*X4 - X1*X6,
    X3*X5 - X2*X6,
    X2*X7 - X1*X8,
    X3*X7 - X1*X9,
    X5*X7 - X4*X8,
    X6*X7 - X4*X9,
    X3*X8 - X2*X9,
    X6*X8 - X5*X9
]);

// the identity on A
A20 := Scheme(P8, [ X1 - X5, X2 + X5, X3, X4 + X5, X6, X7, X8, X9 ]);

// the three (geometric) points of order two on E1 (in A)
A21 := Scheme(P8, [ X5^3 + 1/2*s*X5^2*X6 + 1/2*X6^3, X1 + X5, X2 + X5, X3 + X5 , X4 - X5, X7, X8, X9 ]);

// the three (geometric) points of order two on E2 (in A)
A22 := Scheme(P8, [ X5^3 + 1/2*t*X5^2*X8 + 1/2*X8^3, X1 + X5, X2 - X5, X3, X4 + X5, X6, X7 + X8, X9 ]);

// the remaining nine (geometric) points of A[2]
A23 := Scheme(P8, [ X5^3 + -1/4*s*t*X5^2*X9 + -1/4*X6^2*X9 + -1/4*s*X8^2*X9 + -1/4*X9^3, X5^2*X6 + 1/2*t*X5^2*X9 + 1/2*X8^2*X9,]
X5*X6^2 + 1/2*t*X5*X6*X9 + 1/2*X8*X9^2,
X6^3 + 1/2*t*X6^2*X9 + 1/2*X9^3,
X5^2*X8 + 1/2*s*X5^2*X9 + 1/2*X6^2*X9,
X5*X8^2 + 1/2*s*X5*X8*X9 + 1/2*X6*X9^2,
X8^3 + 1/2*s*X8^2*X9 + 1/2*X9^3,
X6*X8 - X5*X9,
X1 - X5, X2 - X5, X3 - X6, X4 - X5, X7 - X8
]);

// the divisor on A that is invariant under the action of G and [-1]
D := Scheme(P8, [ X1 + X5 + X9 ]) meet A;
A20 subset D;
Union(A21,A22) subset D;

The following code leads to (5.6).

R<T,X1,X2,X3,X4,X5,X6,X7,X8,X9,a,b> := PolynomialRing(Rationals(),12);
s := 3*a;
t := 3*b;
I := ideal < R | [
  X5^3 + -1/4*s*t*X5^2*X9 + -1/4*t*X6^2*X9 + -1/4*s*X8^2*X9 + -1/4*X9^3,
  X5^2*X6 + 1/2*t*X5^2*X9 + 1/2*X8^2*X9,
  X5*X6^2 + 1/2*t*X5*X6*X9 + 1/2*X8*X9^2,
  X6^3 + 1/2*t*X6^2*X9 + 1/2*X9^3,
  X5^2*X8 + 1/2*s*X5^2*X9 + 1/2*X6^2*X9,
  X5*X8^2 + 1/2*s*X5*X8*X9 + 1/2*X6*X9^2,
  X8^3 + 1/2*s*X8^2*X9 + 1/2*X9^3,
  X6*X8 - X5*X9,
  X1 - X5, X2 - X5, X3 - X6, X4 - X5, X7 - X8,
  X9*T - 1, X1 + X5 + X9
]>;
GroebnerBasis(EliminationIdeal(I,10))[1];

Given a bound on the degrees, the following code can be used to verify the Igusa–Clebsch invariants in Proposition 5.6, by verifying that the formulas hold for sufficiently many pairs (a, b).

// dehomogenize a homogeneous polynomial
function Dehomogenize(f)
  S := Parent(f);
  R := PolynomialRing(BaseRing(S));
p1 := hom< S -> R | [ R.1, 1 ] >(f);
p2 := hom< S -> R | [ 1, R.1 ] >(f);
if Degree(p1) ge Degree(p2) then
  return p1;
else
  return p2;
end if;
end function;

S<y> := PolynomialRing(Rationals());
K<ω> := NumberField(1+y+y^2);
R<x> := PolynomialRing(K);
P8<X1,X2,X3,X4,X5,X6,X7,X8,X9> := ProjectiveSpace(K,8);
P2<X,Y,Z> := ProjectiveSpace(K,2);
P1<u,v> := ProjectiveSpace(K,1);
P3<y1,y2,y3,y4> := ProjectiveSpace(K,3);
WP := WeightedProjectiveSpace(K, [2,4,6,10]);

// the Kummer quotient map
q := map< P8 -> P3 | [ X1^2 + X5^2 + X9^2, X2*X4 + X3*X7 + X6*X8, X2*X3 + X4*X6 + X7*X8, X2*X8 + X3*X6 + X4*X7 ]>;

// our formulas for the Igusa-Clebsch invariants, obtained by interpolation
function MyInvariants(a,b)
  return [72*(-20 + 16*a^3 + 40*a^6 + 112*a*b + 100*a^4*b - 32*a^7*b - 68*a^2*b^2 - 104*a^5*b^2 + a^8*b^2 + 44*a^3*b^3 + 54*a^6*b^3 + 100*a*b^4 + 65*a^4*b^4 - 32*a^7*b^4 - 68*a^2*b^5 - 104*a^5*b^5 + 30*a^7*b^6 - 32*a^2*b^7 - 30*a^4*b^7 + a^2*b^8),
           36*(2 + a^3 - 3*a*b + 3*a^2*b^2 + b^3)^4*(320 + 160*a^3 + 256*a*b + 8*a^4*b + 240*a^2*b^2 + 160*a*b^3 + 240*a^3*b^3 + 8*a*b^4 + 9*a^4*b^4),
           72*(2 + a^3 - 3*a*b + 3*a^2*b^2 + b^3)^4*(-55040 + 74624*a^3 + 219552*a^6 + 84224*a^9 - 8*a^12 + 314112*a*b + 564480*a^4*b + 181152*a^7*b - 50160*a^10*b + 88512*a^2*b^2 - 93600*a^5*b^2 - 167112*a^8*b^2 + 480*a^11*b^2 + 74624*a^3*b^3 + 383424*a^3*b^3 + 455568*a^6*b^3 + 156928*a^9*b^3 + 60*a^12*b^3 + 64*a^12*b^3 + 761040*a^4*b^4 + 81072*a^7*b^4 - 121608*a^10*b^4 - 93600*a^2*b^5 - 462096*a^5*b^5 - 358740*a^8*b^5 - 2160*a^11*b^5 + 219552*b^6 + 455568*a^3*b^6 + 336444*a^6*b^6 + 106560*a^9*b^6 + 81*a^12*b^6 + 181152*a^4*b^7 + 81072*a^4*b^7 - 148932*a^7*b^7 - 70794*a^10*b^7 - 167112*a^2*b^8 - 358740*a^5*b^8 - 201555*a^8*b^8 - 3402*a^11*b^8 + 884224*a^9*b^9 + 156928*a^3*b^9 + 106560*a^6*b^9 + 30456*a^9*b^9 - 50160*a*b^10 - 121608*a^4*b^10 - 70794*a^7*b^10 + 729*a^10*b^10 + 480*a^2*b^11 - 2160*a^5*b^11 - 3402*a^8*b^11 - 8*b^12 + 60*a^2*b^12 + 81*a^6*b^12)];
36864*(1 + a^3)*(1 + b^3)*(2 + a^3 - 3*a*b + 3*a^2*b^2 + b^3)^12;
end function;

function VerifyInvariants(a,b)
    if (1 + a^3)*(1 + b^3)*(2 + a^3 - 3*a*b + 3*a^2*b^2 + b^3) eq 0 then
        print("ERROR: The quotient is not a Jacobian!");
        return false;
    end if;
    s := 3*a;
    t := 3*b;
    A := Scheme(P8, [X1^3 + X2^3 + X3^3 + s*X1*X2*X3,
        // many equations are omitted here! (see above for the full list)
        X6*X8 - X5*X9]);

    D := Scheme(P8, [ X1 + X5 + X9 ]) meet A;
    A21 := Scheme(P8, [ X5^3 + 1/2*s*X5^2*X6 + 1/2*X6^3, X1 + X5, X2 + X5, X3 + X6, X4 - X5, X7, X8, X9 ]);
    A22 := Scheme(P8, [ X5^3 + 1/2*t*X5^2*X8 + 1/2*X8^3, X1 + X5, X2 - X5, X3, X4 + X5, X6, X7 + X8, X9 ]);

    // H is a conic that lies in Y1 + 2*Y2 = 0; we project to IP^2
    // B is the image of the order-2 points of E1 and E2 (embedded in A)
    proj := map< P3 -> P2 | [ Y2, Y3, Y4 ] >;
    H := proj(q(D));
    B := proj(q(Union(A21,A22)));

    // we find the inverse of the parametrization IP^1 -> H
    paramStrings := Split(Sprint(Parametrization(Conic(Curve(H)))),"\n");
    coord1 := eval paramStrings[#paramStrings-1];
    coord2 := eval paramStrings[#paramStrings];
    paraminv := map< P2 -> P1 | [ coord1, coord2 ] >;

    // we have the branch locus of the canonical map C -> IP^1 and we can
    // recover C (up to quadratic twists)
    branch := paraminv(B);
    p := Basis(Ideal(branch))[1];
    poly := Dehomogenize(p);

    // we compare the actual Igusa-Clebsch invariants with our formula
    return WP!(IgusaClebschInvariants(HyperellipticCurve(poly))) eq WP!(MyInvariants(a,b));
end function;
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