Gauge Field Strength Tensor
from the Overlap Dirac Operator

K.F. Liu, A. Alexandru, and I. Horváth
Dept. of Physics and Astronomy, University of Kentucky, Lexington, KY 40506

Abstract

We derive the classical continuum limit of the operator $\text{tr}_s \sigma_{\mu\nu} D^{ov}(x, x)$ with $D^{ov}$ being the overlap Dirac operator and show that it corresponds to the gauge field strength tensor $F_{\mu\nu}(x)$. 
1 Introduction

Lattice gauge operators are usually constructed explicitly from link variables. For example, the Wilson gauge action uses the product of gauge links at the boundary of the square plaquette. Similarly, the gauge field strength tensor can be defined through suitable combinations of such elementary plaquettes. To improve scaling behavior of the action and other gauge operators, rectangular and more sophisticated loops have been included \[1, 2, 3, 4\]. Furthermore, it was shown that operators with smeared gauge links, being less ultralocal, are effective in filtering out ultraviolet fluctuations and improving the efficiency of numerical calculations \[5, 6, 7, 8\].

A different approach is realized in constructing topological charge densities via Ginsparg-Wilson Dirac operators \[9\]. It uses matrix elements of the Dirac operator as a starting point. By virtue of the fact that this type of Dirac operators are inevitably non-ultralocal \[10\], they entail a sum of all gauge loops and are thus automatically smeared. While technically involved, this approach has the advantage that Ginsparg-Wilson operators incorporate an exact lattice chiral symmetry \[11\]. This leads to an index theorem on the lattice \[9\].

The overlap Dirac operator \[12\] offers a concrete example which satisfies Ginsparg–Wilson relation \[13\]. Its compact form makes it amenable to analytic as well as numerical calculations. The associated topological charge density was shown to have the correct classical continuum limit via weak-coupling expansion \[14\] and by direct calculations \[15, 16, 17\]. It is with this topological density operator that the subdimensional long-range structure has been discovered \[18, 19, 20\] and confirmed \[21\] in QCD, as well as in 2-D CP(N-1) models \[22\]. It is also with this operator that the required negativity of the topological density correlator is borne out clearly with only a handful of configurations \[23\]. Whereas, detecting this negativity to the same precision using conventional operators, such as those used in the glueball calculation \[24\], would require much larger statistics. Based on these observations, it was suggested by one of the authors that the condition of chiral symmetry plays a relevant role in efficient suppression of the ultraviolet noise \[25\], and that all gauge operators can be constructed from the chirally symmetric Dirac operator \[26\]. This way, one can also have a formulation of lattice QCD where the gauge action, the \(\theta\) term and the fermion action are all expressed in terms of the lattice Dirac operator \[26\].

In the present work, we will concentrate on the gauge field strength tensor. It was suggested that the classical limit of the tensor component of the overlap operator, \(\text{tr}_s \sigma_{\mu\nu} D^0v(x, x)\), is proportional to the gauge field strength tensor \(F_{\mu\nu}(x)\) \[26, 27\]. We shall explicitly calculate it and show that this is indeed the case. In view of the fact the chirally symmetric Dirac operators are non-ultralocal \[10\], the derivation is somewhat non-trivial. Since the overlap Dirac operator is expected to be local on gauge configurations of interest \[28, 29\], our result implies that the quantum operator constructed this way represents a valid definition of gauge field strength tensor in
lattice QCD. This could provide a practical tool for evaluating gluonic observables. A preliminary version of the calculation was given in Ref. [30].

2 Formulation

Our goal in this paper is to show that if we discretize a classical gauge field $A_\mu(x)$ on a hypercubic lattice with (classical) lattice spacing $a$, and consider the overlap Dirac operator $D^{\sigma\nu}$ on such background, then

$$\text{tr}_s \sigma_{\mu\nu} D^{\sigma\nu}(x, x) \propto a^2 F_{\mu\nu}(x) \quad \text{for} \quad a \rightarrow 0 \quad (1)$$

where “$\text{tr}_s$” denotes the spinor trace. By classical SU(3) gauge configuration we mean any configuration of the gauge field that is smooth (differentiable arbitrarily many times) almost everywhere. In the above equation we have implicitly assumed that $x$ is not a singular point of $A_\mu(x)$. It was also implicitly assumed that point $x$ is a lattice grid point of a superimposed lattice for arbitrary lattice spacing $a$. This is technically most easily (and without loss of generality) achieved if we focus on the point $x = 0$ with the origin of the lattice coordinate system aligned with the Cartesian coordinate system of $R^4$. Also, to avoid an extensive discussion of technical issues associated with the transcription of the field with singularities onto the hypercubic lattice, we will implicitly assume in the following that the field is smooth everywhere. This is sensible since due to the locality of the overlap operator, the result is expected to be valid for an arbitrary non-singular space–time point $x$. For a discussion relevant to this point, the reader is referred to Ref. [15].

In the convention that we will use, the continuum gauge potential $A_\mu(x)$ is the vector field of traceless Hermitian matrices $^2$ and the corresponding field–strength tensor is

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i [A_\mu(x), A_\nu(x)]. \quad (2)$$

With the covariant derivative defined as

$$D_\mu \phi(x) = (\partial_\mu + iA_\mu(x)) \phi(x), \quad (3)$$

one has

$$[D_\mu, D_\nu] \phi(x) = iF_{\mu\nu}(x) \phi(x). \quad (4)$$

\footnotetext[1]{Note that we use the convention that the real–valued arguments of lattice quantities (such as $D(x, x)$ on LHS of Eq. (1)) are given in parenthesis, while the integer–valued lattice coordinates are written as subscripts (such as $D_{n,n}$).

\footnotetext[2]{Note that this differs from conventions of Ref. [26], where anti–Hermitian gauge potentials were used instead. The equations below can be obtained from equations of Ref. [26] via substitutions $A_\mu(x) \rightarrow iA_\mu(x)$, $F_{\mu\nu}(x) \rightarrow iF_{\mu\nu}(x)$. The value of constant $c^T$ in Eq. (9) is the same in both conventions.}
The transcription of $A_\mu(x)$ to the hypercubic lattice with integer coordinates $n \equiv (n_1, n_2, n_3, n_4)$ is accomplished in a standard manner. If $a$ is the classical lattice spacing, we associate the lattice site $n$ with the space–time point $x = an$, and the lattice link variable $U_{n,\mu}$ is defined as

$$U_{n,\mu}(a) \equiv \exp(iaA_\mu(an))$$

The overlap Dirac operator $D_{ov}$ is given by [12]

$$D_{ov} = \rho \left( 1 + X \frac{1}{\sqrt{X^\dagger X}} \right); \quad X = \mathcal{D} - R - \rho + 4r$$

where $-\rho, \rho \in (0, 2r)$, is the negative mass parameter and

$$D_\mu = \frac{1}{2}[U_\mu S_\mu - S_\mu^\dagger U_\mu^\dagger], \quad R = \frac{r}{2} \sum_\mu [U_\mu S_\mu + S_\mu^\dagger U_\mu^\dagger]$$

with

$$(S_\mu)_{m,n} \equiv \delta_{m,n-\bar{\mu}} \quad (U_\mu)_{m,n} \equiv U_{m,\mu} \delta_{m,n}$$

We shall take the Euclidean $\gamma$–matrices to be Hermitian, i.e. $\gamma_\mu^\dagger = \gamma_\mu$ and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$.

With the above defining relations, we shall proceed to show the following in an explicit calculation:

If $A_\mu(x)$ is a smooth SU(3) gauge potential on $R^4$, and $U(a)$ is the transcription of this field to the hypercubic lattice with classical lattice spacing $a$, then

$$\text{tr}_s \sigma_{\mu\nu} D_{ov}^{0,0}(U(a)) = c^T a^2 F_{\mu\nu}(0) + \mathcal{O}(a^3),$$

where $D_{ov}^{0,0}$ is the matrix element of the overlap operator at $(m,n) = (0,0)$. The non-zero constant $c^T = c^T(\rho)$ is independent of $A_\mu(x)$, and $\sigma_{\mu\nu} \equiv \frac{1}{2i}[\gamma_\mu, \gamma_\nu]$.

3 Calculation

To proceed with the calculation, we shall assume that $\text{tr}_s \sigma_{\mu\nu} D_{ov}^{0,0}$ has a Taylor expansion in $a$ and we will compute the leading contributions.

To evaluate the diagonal element $D_{ov}^{0,n}$, we introduce the momentum variable in the following way [15, 17]

$$D_{ov}^{0,n} = \sum_m D_{ov}^{m,n} \delta_{n,m} = \sum_m D_{ov}^{m,n} \int_{-\pi}^\pi \frac{d^4k}{(2\pi)^4} e^{-ik(n-m)}$$

$$= \int_{-\pi}^\pi \frac{d^4k}{(2\pi)^4} e^{-ikn} \sum_m D_{ov}^{m,n} e^{ikm}.\quad (11)$$
This will allow us to evaluate the inverse square root in Eq. (6). Next, we define the diagonal matrices \( K(k) \) as

\[
(K(k))_{n,m} \equiv e^{ikn} \delta_{n,m}.
\]  

(12)

These matrices are unitary: \( K^\dagger(k) = K^{-1}(k) = K(-k) \). If we now introduce the vector \( 1 \) such that \( 1_n = 1 \), a vector with all entries set to 1, we can rewrite the above expression as

\[
D_{ov}^{n,n} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left( K^{-1}(k)D_{ov}(k)1 \right)_n.
\]  

(13)

To calculate \( K^{-1}(k)D_{ov}(k) \), we assume that we can express \( \frac{1}{\sqrt{X^\dagger X}} \) as a power series in \( X^\dagger X \). Then

\[
K^{-1}(k)D_{ov}(k) = \rho \left( 1 - Y \frac{1}{\sqrt{Y^\dagger Y}} \right),
\]  

(14)

where

\[
Y = -\bar{X} = -K^{-1}(k)XK(k) = M + \bar{R} - \bar{D} - i\bar{\beta},
\]  

(15)

and

\[
\bar{D}_\mu = \frac{1}{2} \left( e^{ik\mu} (U_\mu S_\mu - 1) - e^{-ik\mu} (S_\mu U_\mu^\dagger - 1) \right),
\]

\[
\bar{R} = \frac{r}{2} \sum_\mu \left( e^{ik\mu} (U_\mu S_\mu - 1) + e^{-ik\mu} (S_\mu U_\mu^\dagger - 1) \right),
\]

\[
M = \rho + r \sum_\lambda (c_\lambda - 1),
\]  

(16)

where \( s_\mu = \sin k_\mu, \quad c_\mu = \cos k_\mu \).

Eqs. (10), (13), and (14), lead to

\[
\text{tr}_s \sigma_{\mu\nu} D_{0,0}^{ov} = -\rho \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \text{tr}_s \sigma_{\mu\nu} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} \right)_0.
\]  

(17)

### 3.1 Computational strategy

As we mentioned before, we assume that \( \text{tr}_s \sigma_{\mu\nu} D_{0,0}^{ov} \) has a Taylor expansion in \( a \). We will compute the leading contributions by taking derivatives with respect to \( a \) and then evaluating the limit \( a \to 0 \). We assume that all the matrices and matrix products are well defined and that we can take derivatives in the usual fashion. The non-trivial part of the calculation is taking the derivative of \( \frac{1}{\sqrt{Y^\dagger Y}} \) with respect to \( a \).

For this we will write

\[
\frac{1}{\sqrt{Y^\dagger Y}} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \frac{1}{\sigma^2 + Y^\dagger Y},
\]  

(18)
under the condition that $Y^\dagger Y$ does not have zero eigenvalues, which is satisfied for the case of classical gauge fields sufficiently close to the continuum limit \cite{28}. Noting that $(M^{-1})' = -M^{-1}M'M^{-1}$ for a matrix $M$, we have

\[
\begin{align*}
\frac{d}{da} \sqrt{Y^\dagger Y} &= -\frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \frac{1}{\sigma^2 + Y^\dagger Y} \left( \frac{d}{da} Y^\dagger Y \right) \frac{1}{\sigma^2 + Y^\dagger Y} \\
\frac{d^2}{da^2} \sqrt{Y^\dagger Y} &= \frac{2}{\pi} \int_{-\infty}^{\infty} d\sigma \frac{1}{\sigma^2 + Y^\dagger Y} \left( \frac{d}{da} Y^\dagger Y \right) \frac{1}{\sigma^2 + Y^\dagger Y} \\
&\quad - \frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \frac{1}{\sigma^2 + Y^\dagger Y} \left( \frac{d^2}{da^2} Y^\dagger Y \right) \frac{1}{\sigma^2 + Y^\dagger Y}.
\end{align*}
\]

(19)

We see that the problem is reduced to computing derivatives of $Y$ with respect to $a$. The only matrices that depend on $a$ are the link matrices $U_\mu$. Since we are only interested in the limit $a \to 0$ and since we will only calculate the contributions up to order $a^2$, we only need the following limits

\[
\begin{align*}
\lim_{a \to 0} U_\mu &= 1, \\
\lim_{a \to 0} \frac{d}{da} U_\mu &= iA_\mu(0), \\
\lim_{a \to 0} \frac{d^2}{da^2} U_\mu &= -A_\mu(0)1 + 2iN\nabla A_\mu(0).
\end{align*}
\]

(20)

where

\[
(N\nabla A_\mu(0))_{m,n} \equiv \sum_{\alpha} n_\alpha \partial_\alpha A_\mu(0) \delta_{m,n},
\]

(21)

with $n_\alpha$ being the component of the 4-vector $n$. We see that in the limit $a \to 0$ both $U_\mu$ and $\frac{d}{da} U_\mu$ reduce to an identity matrix in the space-time coordinates, but they are not necessarily diagonal in color space. However, the second derivative $N\nabla A_\mu(0)$ has a term that is different: this matrix is still diagonal in the space-time index (since it is the derivative of a diagonal matrix) but the diagonal elements are not equal.

To compute $(Y \sqrt{Y^\dagger Y} 1)_0$ we will need to justify several relations.

- **Relation 1:**

\[
Y_0^\dagger Y_0 1 = z 1,
\]

(22)

where

\[
\begin{align*}
Y_0 &= \lim_{a \to 0} Y = M + \tilde{R}_0 + \tilde{D}_0 + i\beta, \\
\tilde{D}_{0,\mu} &= \lim_{a \to 0} \tilde{D}_\mu = \frac{1}{2} \left( e^{ik_\mu} (S_\mu - 1) - e^{-ik_\mu} (S_\mu^\dagger - 1) \right), \\
\tilde{R}_0 &= \lim_{a \to 0} \tilde{R} = \frac{r}{2} \sum_{\mu} \left( e^{ik_\mu} (S_\mu - 1) + e^{-ik_\mu} (S_\mu^\dagger - 1) \right),
\end{align*}
\]

(23)
and
\[ z = \sum_{\mu} s^2_\mu + M^2, \]  \hspace{1cm} (25)
is a number.

To show Eq. (22), we write
\[ Y^\dagger_0 Y_0 = ((M + \tilde{R}_0) + (\tilde{P}_0 + i\beta)) ((M + \tilde{R}_0) - (\tilde{P}_0 + i\beta)) \]
\[ = (M + \tilde{R}_0)^2 - (\tilde{P}_0 + i\beta)^2 + [\tilde{P}_0 + i\beta, M + \tilde{R}_0]. \]  \hspace{1cm} (26)
The commutator is zero since \([S_\mu, S_\nu] = 0\) and thus we have
\[ Y^\dagger_0 Y_0 = M^2 + \tilde{R}^2_0 + 2M\tilde{R}_0 + \beta^2 - \tilde{P}^2_0 - i\{\beta, \tilde{P}_0\}. \]  \hspace{1cm} (27)
It is easy to see that \(\tilde{R}_0 1 = 0\) and \(\tilde{P}_0 1 = 0\) since the “derivative” like term, \(S_\mu - 1\), vanishes when acting on \(1\). This “derivative” term comes from the continuum limit of \((U_\mu S_\mu - 1)\) which becomes \((S_\mu - 1)\) as \(a \to 0\). We thus have
\[ Y^\dagger_0 Y_0 1 = (M^2 + \beta^2) 1 = z 1. \]  \hspace{1cm} (28)

**Relation 2:**
\[ \lim_{a \to 0} \frac{1}{\sigma^2 + Y^\dagger_0 Y_0} 1 = \frac{1}{\sigma^2 + z} 1, \]  \hspace{1cm} (29)
This can be straightforwardly shown if we expand \(\frac{1}{\sigma^2 + Y^\dagger_0 Y_0}\) as a power series in \(Y^\dagger_0 Y_0\) and apply Relation 1 in Eq. (22) successively.

**Relation 3:**
\[ \frac{1}{\sigma^2 + Y^\dagger_0 Y_0} N \nabla A_\mu(0) 1 = \frac{1}{\sigma^2 + z} N \nabla A_\mu(0) 1 - \frac{1}{(\sigma^2 + z)^2} \Delta N \nabla A_\mu(0) 1, \]  \hspace{1cm} (30)
where
\[ \Delta = 2M\tilde{R}_0 - i\{\beta, \tilde{P}_0\} = 2M\tilde{R}_0 - 2i \sum_{\mu} s_\mu (\tilde{D}_\mu)_0. \]  \hspace{1cm} (31)
To compute the second order derivative in Eq. (19) we will need this relation when the matrix \(\frac{1}{\sigma^2 + Y^\dagger_0 Y_0}\) acts on non-constant vectors of the form \(N \nabla A_\mu(0) 1\) in Eq. (20).

To show this we write \(Y^\dagger_0 Y_0\) as
\[ Y^\dagger_0 Y_0 = z + \Delta_2 (\tilde{R}_0^2 - \tilde{P}_0^2 + 2M\tilde{R}_0 - i\{\beta, \tilde{P}_0\}), \]  \hspace{1cm} (32)
where \(\Delta_2\) and \(\Delta\) are unrelated. Using the fact that
\[ (S_\nu - 1) N \nabla A_\mu(0) 1 = ((N + \nu) \nabla A_\mu(0) - N \nabla A_\mu(0)) 1 \]
\[ = \nu \nabla A_\mu(0) 1 = \partial_\nu A_\mu(0) 1, \]  \hspace{1cm} (33)
which is a constant vector, we can easily see that \( \Delta_2 N \nabla A_\mu (0) \mathbf{1} = 0 \) and \( \Delta \Delta N \nabla A_\mu (0) \mathbf{1} = 0 \). This is because these terms include double “derivatives” like \((S_\mu - 1)(S_\nu - 1)\) and since the first “derivative” produces a constant vector, the second “derivative” acting on it makes it vanish. Now that we have \( Y_0^\dagger Y_0 N \nabla A_\mu \mathbf{1} = (z + \Delta) N \nabla A_\mu \mathbf{1} \) and \( \Delta \Delta N \nabla A_\mu (0) \mathbf{1} = 0 \), we can prove by induction that

\[
(Y_0^\dagger Y_0)^k N \nabla A_\mu \mathbf{1} = z^k N \nabla A_\mu \mathbf{1} + k z^{k-1} \Delta N \nabla A_\mu \mathbf{1}.
\]

(34)

From a series expansion \( P(Y_0^\dagger Y_0) = \frac{1}{\sigma^2 + Y_0^\dagger Y_0} \) and Eq. (34), we see that

\[
P(Y_0^\dagger Y_0) N \nabla A_\mu \mathbf{1} = P(z) N \nabla A_\mu \mathbf{1} + P'(z) \Delta N \nabla A_\mu \mathbf{1}.
\]

(35)

Since \( P(z) = \frac{1}{\sigma^2 + z} \) and \( P'(z) = -\frac{1}{(\sigma^2 + z)^2} \), Eqs. (34) and (35) lead to Relation 3 in Eq. (30).

As we mentioned before, we assume that our function admits a Taylor expansion. In this case, we can write

\[
\text{tr}_s \sigma_{\mu \nu} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} \mathbf{1} \right)_0 = \lim_{a \to 0} \text{tr}_s \sigma_{\mu \nu} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} \mathbf{1} \right)_0 + a \lim_{a \to 0} \text{tr}_s \sigma_{\mu \nu} \frac{d}{da} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} \mathbf{1} \right)_0 + \frac{1}{2} a^2 \lim_{a \to 0} \text{tr}_s \sigma_{\mu \nu} \frac{d^2}{da^2} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} \mathbf{1} \right)_0 + O(a^3).
\]

(36)

We will now proceed to carry out our calculation order by order.

### 3.2 Calculation details

#### 3.2.1 Order 0

We need to compute

\[
\lim_{a \to 0} \text{tr}_s \sigma_{\mu \nu} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} \mathbf{1} \right)_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \text{tr}_s \sigma_{\mu \nu} \left( Y_0 \frac{1}{\sigma^2 + Y_0^\dagger Y_0} \mathbf{1} \right)_0,
\]

(37)

where the relevant term is \( \text{tr}_s \sigma_{\mu \nu} Y_0 \frac{1}{\sigma^2 + Y_0^\dagger Y_0} \mathbf{1} = \frac{1}{\sigma^2 + z} \text{tr}_s \sigma_{\mu \nu} Y_0 \mathbf{1} \). Now

\[
Y_0 \mathbf{1} = (M + \tilde{R}_0 - i \tilde{\beta} - \tilde{\mathcal{D}}_0) \mathbf{1} = (M - i \tilde{\beta}) \mathbf{1},
\]

(38)

due to the fact \( \tilde{R}_0 \mathbf{1} = \tilde{\mathcal{D}}_0 \mathbf{1} = 0 \). Since \( Y_0 \mathbf{1} \) has only scalar and vector spinor components and no tensor component, it leads to \( \text{tr}_s \sigma_{\mu \nu} Y_0 \mathbf{1} = 0 \) and the zeroth order contribution is thus zero, i.e:

\[
\lim_{a \to 0} \text{tr}_s \sigma_{\mu \nu} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} \mathbf{1} \right)_0 = 0.
\]

(39)
3.2.2 Order 1

We need to compute

\[
\lim_{a \to 0} \text{tr}_s \sigma_{\mu \nu} \frac{d}{da} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} 1 \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \text{tr}_s \sigma_{\mu \nu} \left( \frac{Y_0'}{\sigma^2 + Y_0 Y_0} 1 \right).
\]

(40)

The first term contribution vanishes since

\[
Y_0' \frac{1}{\sigma^2 + Y_0 Y_0} 1 = \frac{1}{\sigma^2 + z} Y_0' 1 = \frac{1}{\sigma^2 + z} (\tilde{R}_0' - \tilde{D}_0') 1,
\]

(41)

and then we are left again with only scalar and vector spinor components. Thus

\[
\text{tr}_s \sigma_{\mu \nu} Y_0' \frac{1}{\sigma^2 + Y_0 Y_0} 1 = 0.
\]

To evaluate the second term we need to compute \((Y^\dagger Y)_0' 1 = (2M\tilde{R}_0' - 2i \sum_\alpha s_\alpha \tilde{D}_0') 1\) using the following identities

\[
\tilde{R}_0' = \frac{r}{2} \sum_\mu i A_\mu(0) \left( e^{i k_\mu S_\mu} - e^{-i k_\mu S_\mu} \right),
\]

(42)

\[
\tilde{D}_0' = \frac{i}{2} A_\mu(0) \left( e^{i k_\mu S_\mu} + e^{-i k_\mu S_\mu} \right).
\]

(43)

We have then \(\tilde{R}_0' 1 = -r \sum \alpha s_\alpha A_\alpha(0) 1\) and \(\tilde{D}_0' 1 = ic_\alpha A_\alpha(0) 1\) and

\[
(Y^\dagger Y)_0' 1 = 2 \sum_\alpha s_\alpha A_\alpha(0) (-Mr + c_\alpha) 1,
\]

(44)

a constant vector. We have then

\[
Y_0' \frac{1}{\sigma^2 + Y_0 Y_0} (Y^\dagger Y)_0' \frac{1}{\sigma^2 + Y_0 Y_0} 1 = \frac{1}{\sigma^2 + z} Y_0 \frac{1}{\sigma^2 + Y_0 Y_0} (Y^\dagger Y)_0' 1
= \frac{1}{\sigma^2 + z} 2 \sum_\alpha s_\alpha A_\alpha(0) (-Mr + c_\alpha) Y_0 \frac{1}{\sigma^2 + Y_0 Y_0} 1
= \left( \frac{1}{\sigma^2 + z} \right)^2 2 \sum_\alpha s_\alpha A_\alpha(0) (-Mr + c_\alpha) Y_0 1
= \left( \frac{1}{\sigma^2 + z} \right)^2 2 \sum_\alpha s_\alpha A_\alpha(0) (-Mr + c_\alpha) (M - i\beta) 1.
\]

(45)

We see that we only have scalar and vector spinor components and this term vanishes too after taking the spinor trace. Thus the first order contribution vanishes.

\[
\lim_{a \to 0} \text{tr}_s \sigma_{\mu \nu} \frac{d}{da} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} 1 \right) = 0.
\]

(46)
3.2.3 Order 2

The derivation of the second order contribution is somewhat involved, but employs the same steps as above. The details of the calculations are presented in Appendix A for the perusal of interested readers. The main result is that the second order contribution is not zero and is given by

$$\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \lim_{a \to 0} \text{tr}_s \sigma_{\mu\nu} \frac{d^2}{da^2} \left( Y \frac{1}{\sqrt{YY}} \right)_0$$

$$= - \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} 4(M\mu c_\nu + r s_\nu^2 c_\nu + r s_\mu c_\mu)$$

$$\frac{z^{3/2}}{2} F_{\mu\nu}(0). \quad (47)$$

Together with the (null) results from the zeroth and first orders in Eqs. (39) and (46), the final result is

$$\text{tr}_s \sigma_{\mu\nu} D_{0,0}^{on} = a^2 F_{\mu\nu}(0) \rho \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} 2(M\mu c_\nu + r s_\nu^2 c_\nu + r s_\mu c_\mu)$$

$$\frac{z^{3/2}}{2} + O(a^3). \quad (48)$$

Comparing with Eq. (49), we find

$$c^T(\rho) = \rho \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} 2(M\mu c_\nu + r s_\nu^2 c_\nu + r s_\mu c_\mu)$$

$$\frac{z^{3/2}}{2}. \quad (49)$$

With $r = 1$ and $\rho = 1.368$ (which corresponds to $\kappa = 0.19$ in the Wilson Dirac operator), we find $c^T = 0.11157$.\(^3\)

4 Conclusions

We have shown in an explicit calculation that, for the overlap Dirac operator, the classical continuum limit of $\text{tr}_s \sigma_{\mu\nu} D_{0,0}^{on}(x, x)$ is proportional to the gauge field strength tensor $F_{\mu\nu}(x)$.

Based on the experience of studying the QCD vacuum structure with the topological charge density defined from the overlap operator, it is found that one can obtain clear signals with only a handful of gauge configurations\(^1\),\(^2\),\(^3\),\(^4\),\(^5\),\(^6\). This

\(^3\)We should remark that $D_\mu$ and $R$ in Eq. (7) can be written as $D_\mu(x) = \frac{1}{2}[U_\mu(x)e^{\alpha\beta\mu} - e^{-\alpha\beta\mu}U_\mu(x)]$ and $R(x) = \frac{r}{2} \sum_\mu [U_\mu(x)e^{\alpha\beta\mu} + e^{-\alpha\beta\mu}U_\mu(x)]$, such as defined in Ref. [17]. Upon taking derivatives with respect to $a$ in $e^{\pm\alpha\beta\mu}$ and $U_\mu$ in Eq. (39), the results in Eqs. (39), (40), (41), and (48) were obtained [30]. However, due to the fact that $\sigma_{\mu\nu}$ in Ref. [30] is defined with an opposite sign from the one used here which is $\sigma_{\mu\nu} \equiv \frac{1}{2i}[\gamma_\mu, \gamma_\nu]$, the result in Ref. [30] is negative of that in Eq. (48).
is presumably due to the non-ultralocal nature of the overlap operator which serves as an efficient filter of the ultraviolet fluctuations \cite{25, 26}. It is worthwhile then to study whether other operators defined in a similar fashion share this property. For example, it would be interesting to see if the calculation of glueball masses, glue momentum and angular momentum in the nucleon, etc. can benefit from employing the overlap-based definition of the field strength tensor

\[ \mathcal{O}(x) \equiv \frac{1}{c^T} \text{tr}_s \sigma_{\mu\nu} D^{ov}(x, x) \tag{50} \]

which is properly normalized. We should point out that the value of the constant \( c^T = c^T(\rho) \) depends on the mass parameter \( \rho \) used to define the overlap operator. We will study this \( \rho \)-dependence in detail elsewhere \cite{31}.

Finally, we wish to mention that, for the purposes of studying QCD vacuum structure it is useful to be able to expand gauge observables in low-lying Dirac eigenmodes. Indeed, such expansions in the case of overlap-based topological density proved to be useful in studying the low-energy behavior of the topological vacuum structure \cite{32, 19}. Thus, one rationale for defining all gauge operators in terms of Dirac kernels is the fact that it allows such an expansion for an arbitrary operator \cite{26}. We note that for the purpose of eigenmode expansion, the expression for the field strength tensor in terms of the squared lattice Dirac operators was also considered in Ref. \cite{33}. 

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Appendix A

We need to compute

$$\lim_{a \to 0} \text{tr}_s \sigma_{\mu \nu} \frac{d^2}{da^2} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} I \right)_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \text{tr}_s \sigma_{\mu \nu} \left( Y''^\dagger \frac{1}{\sigma^2 + Y'' Y_0} I \right)_0$$

$$- \frac{2}{\pi} \int_{-\infty}^{\infty} d\sigma \text{tr}_s \sigma_{\mu \nu} \left( Y' \frac{1}{\sigma^2 + Y' Y_0} (Y^\dagger Y)_0 \frac{1}{\sigma^2 + Y_0 Y_0} I \right)_0$$

$$+ \frac{2}{\pi} \int_{-\infty}^{\infty} d\sigma \text{tr}_s \sigma_{\mu \nu} \left( Y_0 \left[ \frac{1}{\sigma^2 + Y_0 Y_0} (Y^\dagger Y)_0 \right]^2 \frac{1}{\sigma^2 + Y_0 Y_0} I \right)_0$$

$$- \frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \text{tr}_s \sigma_{\mu \nu} \left( Y_0 \frac{1}{\sigma^2 + Y_0 Y_0} (Y^\dagger Y)_0 \frac{1}{\sigma^2 + Y_0 Y_0} I \right)_0 \tag{51}$$

The first term in Eq. (51) turns out to be zero, since the expression

$$Y''_0 \frac{1}{\sigma^2 + Y'' Y_0} I = \frac{1}{\sigma^2 + z} Y''_0 I = \frac{1}{\sigma^2 + z} (\tilde{R}''_0 - \tilde{D}''_0) I, \tag{52}$$

only has scalar and vector spinor components and, as a result, it vanishes upon taking the the spinor trace in Eq. (51). The second term in Eq. (51) also vanishes, because

$$Y' \frac{1}{\sigma^2 + Y' Y_0} (Y^\dagger Y)_0 \frac{1}{\sigma^2 + Y_0 Y_0} I = \left( \frac{1}{\sigma^2 + z} \right)^2 2 \sum_\alpha s_\alpha A_\alpha(0) (\frac{\mu}{\gamma}) Y_0' I \tag{53}$$

also has only scalar and vector components. The third term in Eq. (51) vanishes for the same reason, since

$$Y_0 \left[ \frac{1}{\sigma^2 + Y_0 Y_0} (Y^\dagger Y)_0 \right]^2 \frac{1}{\sigma^2 + Y_0 Y_0} I = \left( \frac{1}{\sigma^2 + z} \right)^3 \left( 2 \sum_\alpha s_\alpha A_\alpha(0) (\frac{\mu}{\gamma}) \right)^2 Y_0 I \tag{54}$$

has only spinor and vector components.

The only non-zero contributions come from the fourth term in Eq. (51). To compute its contribution, we need to evaluate

$$(Y^\dagger Y)'_0 I = (Y_0 Y_0'' + Y_0'' Y_0 + 2 Y_0'' Y_0'') I. \tag{55}$$

We will compute each term separately. For the first term, we have

$$Y''_0 = \tilde{R}''_0 - \tilde{D}''_0 = -\frac{1}{2} \sum_\mu \left( e^{ik_\mu} (\gamma_\mu - r) U''_{\mu 0} S_\mu - e^{-ik_\mu} (\gamma_\mu + r) S^\dagger_\mu U''_{\mu 0} \right), \tag{56}$$

where $U''_{\mu 0} = -A_\mu(0)^2 + 2 i N \nabla A_\mu(0)$. In this case, $Y''_0 I$ will have a constant vector part

$$(Y''_0 I)_c = \sum_\mu \left( A_\mu(0)^2 (i s_\mu \gamma_\mu - r c_\mu) + i e^{-ik_\mu} (\gamma_\mu + r) \partial_\mu A_\mu(0) \right) I, \tag{57}$$
and a non-constant part

\[ (Y''_0 1)_{nc} = -2i \sum_{\mu} N \nabla A_\mu(0)(c_\mu \gamma_\mu - i rs_\mu)1. \]  \hfill (58)

In considering the first term in Eq. (55), we note that the derivative terms in \( Y^t_0 \), i.e. \( \tilde{R}^t_0 + \tilde{P}^t_0 \), vanish when acting on constant vectors as shown in Eq. (38). Furthermore, the derivatives acting on the non-constant part produce constant vectors as in Eq. (33). As a result, we get a constant term and a non-constant term

\[ (Y''_0 Y'_{0} 1)_c = (M + i \beta) \sum_{\mu} \left[ A_\mu(0)^2(i s_\mu \gamma_\mu - r c_\mu) + ie^{-ik_\mu}(\gamma_\mu + r) \partial_\mu A_\mu \right] 1 \]
\[ -2i \sum_{\mu,\nu}(c_\mu \gamma_\mu + i rs_\mu)(c_\nu \gamma_\nu - i rs_\nu) \partial_\mu A_\nu(0)1, \]
\[ (Y''_0 Y'_{0} 1)_{nc} = -2i(M + i \beta) \sum_{\mu} N \nabla A_\mu(0)(c_\mu \gamma_\mu - i rs_\mu)1. \]  \hfill (59)

From Eq. (56), we have

\[ Y''_0 = \tilde{R}''_0 + \tilde{P}''_0 = \frac{1}{2} \sum_{\mu} \left( e^{ik_\mu}(\gamma_\mu + r)U''_{\mu0}S_\mu - e^{-ik_\mu}(\gamma_\mu - r)S^\dagger_{\mu0}U''_{\mu0} \right). \]  \hfill (60)

Consequently, the second term, \( Y''_0 Y_0^t 1 \), gives

\[ (Y''_0 Y_0^t 1)_c = -\sum_{\mu} \left[ A_\mu(0)^2(i s_\mu \gamma_\mu + r c_\mu) + ie^{-ik_\mu}(\gamma_\mu - r) \partial_\mu A_\mu(0) \right] (M - i \beta)1, \]
\[ (Y''_0 Y_0^t 1)_{nc} = 2i \sum_{\mu} N \nabla A_\mu(0)(c_\mu \gamma_\mu + i rs_\mu)(M - i \beta)1. \]  \hfill (61)

The last term to evaluate is \( 2Y''_0 Y_0^t 1 \). We note that since this term only involves first derivatives it will only produce a constant vector. Using

\[ Y'_0 = -\frac{1}{2} \sum_{\mu} iA_\mu(0) \left( e^{ik_\mu}(\gamma_\mu - r)S_\mu + e^{-ik_\mu}(\gamma_\mu + r)S^\dagger_{\mu} \right), \]  \hfill (62)

we get

\[ 2Y''_0 Y_0^t 1 = 2 \sum_{\mu,\nu} A_\mu(0)A_\nu(0)(c_\mu \gamma_\mu + i rs_\mu)(c_\nu \gamma_\nu - i rs_\nu)1. \]  \hfill (63)

Putting all the contributions from Eqs. (59), (61), and (63) together, we obtain

\[ ((Y''Y)^t_0 1)_c = \sum_{\mu} \left[ A_\mu(0)^2(-2Mr c_\mu - 2s^2_\mu) + ie^{-ik_\mu} \partial_\mu A_\mu(0)(2Mr + 2is_\mu) \right] 1 \]
\[ + 2 \sum_{\mu,\nu}(A_\mu(0)A_\nu(0) - i \partial_\mu A_\nu(0))(c_\mu \gamma_\mu + i rs_\mu)(c_\nu \gamma_\nu - i rs_\nu)1, \]  \hfill (64)

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and
\[(Y^\dagger Y)^{\mu}_{\nu}1)_{nc} = 4 \sum_{\mu} N \nabla A_{\mu}(0)s_{\mu}(c_{\mu} - Mr)1.\] (65)

We now return to evaluating the last term of the second order contribution in Eq. (51)
\[Y_0 \frac{1}{\sigma^2 + Y_0^2 Y_0} (Y^\dagger Y)^{\mu}_{\nu}0 = \frac{1}{\sigma^2 + z} Y_0 (Y^\dagger Y)^{\mu}_{\nu}0 \]
\[= \left(\frac{1}{\sigma^2 + z}\right)^2 Y_0((Y^\dagger Y)^{\mu}_{\nu}0c) + \frac{1}{\sigma^2 + z} Y_0 (Y^\dagger Y)^{\mu}_{\nu}0((Y^\dagger Y)^{\mu}_{\nu}01)_{nc}.\] (66)

From Eq. (30), we find for the non-constant term contribution
\[Y_0 \frac{1}{\sigma^2 + Y_0^2 Y_0} ((Y^\dagger Y)^{\mu}_{\nu}01)_{nc} = \frac{1}{\sigma^2 + z} Y_0((Y^\dagger Y)^{\mu}_{\nu}01)_{nc} - \left(\frac{1}{\sigma^2 + z}\right)^2 Y_0\Delta((Y^\dagger Y)^{\mu}_{\nu}01)_{nc}.\] (67)

It is clear from Eq. (65) that the non-constant term \((Y^\dagger Y)^{\mu}_{\nu}01)_{nc}\) is a scalar. Furthermore, from Eq. (31), we see that \(\Delta\) is a scalar too. Since \(Y_0\) has only scalar and vector components, the terms in Eq. (67) above have the same spinor structure. Thus, after taking the spinor trace, all these terms vanish.

Putting together the above results, we have
\[\lim_{a \to 0} \text{tr}_s \sigma_{\mu\nu} \frac{d^2}{da^2} \left(Y \frac{1}{\sqrt{Y^\dagger Y}}1\right)_0 = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\sigma \left(\frac{1}{\sigma^2 + z}\right)^2 \text{tr}_s \sigma_{\mu\nu} \left(Y_0((Y^\dagger Y)^{\mu}_{\nu}01)_{c}\right)_0.\] (68)

One can perform the integration over \(\sigma\), since \(Y_0((Y^\dagger Y)^{\mu}_{\nu}01)_{c}\) has no \(\sigma\) dependence.

We then have
\[\lim_{a \to 0} \text{tr}_s \sigma_{\mu\nu} \frac{d^2}{da^2} \left(Y \frac{1}{\sqrt{Y^\dagger Y}}1\right)_0 = -\frac{1}{2z^{3/2}} \text{tr}_s \sigma_{\mu\nu} \left(Y_0((Y^\dagger Y)^{\mu}_{\nu}01)_{c}\right)_0.\] (69)

Since \((Y^\dagger Y)^{\mu}_{\nu}01)_{c}\) is a constant vector, when \(Y_0\) acts from the left, the derivative terms in \(Y_0\) vanish and, as a result, we have \(Y_0((Y^\dagger Y)^{\mu}_{\nu}01)_{c} = (M - i\beta)((Y^\dagger Y)^{\mu}_{\nu}01)_{c}\). Since the first term in Eq. (64) is a scalar, its contribution vanishes after taking the trace. We have then
\[\text{tr}_s \sigma_{\mu\nu} Y_0((Y^\dagger Y)^{\mu}_{\nu}01)_{c}\]
\[= 2\text{tr}_s \sigma_{\mu\nu}(M - i\beta) \sum_{\alpha,\beta} (A_{\alpha}(0)A_{\beta}(0) - i\partial_{\alpha}A_{\beta}(0))(c_{\alpha}\gamma_{\alpha} + i\epsilon_{\alpha})(c_{\beta}\gamma_{\beta} - i\epsilon_{\beta})1\]
\[= 2\text{tr}_s \sigma_{\mu\nu} \sum_{\alpha,\beta} (A_{\alpha}(0)A_{\beta}(0) - i\partial_{\alpha}A_{\beta}(0))(Mc_{\beta}\gamma_{\beta} + r\epsilon_{\beta}c_{\alpha}\delta_{\beta} - \epsilon_{\beta}c_{\alpha}\delta_{\beta}1.\]

(70)
To finish our calculation, we will use the fact that $z$ and $M$ are even functions of $k_\mu$ and thus any integral over $k_\mu$ that involves an odd power of $s_\mu$ will vanish. For the spinor trace we use the relation $\text{tr}\,\sigma_{\mu\nu}\gamma_\alpha\gamma_\beta = 4i(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})$ and we finally obtain

$$
\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \lim_{a \to 0} \text{tr}\,\sigma_{\mu\nu} \frac{d^2}{da^2} \left( Y \frac{1}{\sqrt{Y^\dagger Y}} \right) _0 = -\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{4(Mc_\mu c_\nu + rs_\mu^2 c_\nu + rs_\nu^2 c_\mu)}{z^{3/2}} F_{\mu\nu}(0). \quad (71)
$$

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