ON ABSOLUTELY CONTINUOUS CURVES OF PROBABILITIES ON THE LINE

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(Communicated by Enrico Valdinoci)

Abstract. In recent collaborative work we studied existence and uniqueness of a Lagrangian description for absolutely continuous curves in spaces of Borel probabilities on the real line with finite moments of given order. Of course, a measurable velocity driving the evolution in Eulerian coordinates is necessary to define the Eulerian and Lagrangian descriptions of fluid flow; here we prove that in this case it is also sufficient for a Lagrangian description. More precisely, we argue that the existence of the integrable velocity along an absolutely continuous curve in the set of Borel probabilities on the line is enough to produce a canonical Lagrangian description for the curve; this is given by the family of optimal maps between the uniform distribution on the unit interval and the measures on the curve. Moreover, we identify a necessary and sufficient condition on said family of optimal maps which ensures that the measurable velocity along the curve exists.

1. Introduction. Over the last decade, absolutely continuous curves in the Wasserstein space have been showed to provide the right setting for studying the evolution of a probability measure in many important situations. Velocities along such curves provide the connection to PDE, and a series of authors have exploited this fact for various equations of Hamiltonian or gradient flow type [2], [7], [12] etc. Interestingly, a purely metric definition of absolutely continuous curves of probabilities \([0, T] \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d)\) implies under some conditions [3] the existence of a velocity field \(v\) such that

\[
\partial_t \mu + \nabla \cdot (v \mu) = 0 \quad \text{in the sense of distributions in } (0, T) \times \mathbb{R}^d. \quad (CE)
\]

In applications it is first established that \(v\) exists, then it is proceeded to proving that it has the expected form \(v = A(\mu)\), where \(A\) is an operator chosen such that the evolutionary PDE under scrutiny can be written as

\[
\partial_t \mu + \nabla \cdot [A(\mu) \mu] = 0 \quad \text{as distributions.}
\]

In this paper we will be mainly preoccupied with the existence of an integrable (in some precise sense) velocity and its Lagrangian flow for a given absolutely continuous curve in the Wasserstein space over the real line. The one-dimensional case is an important first step in understanding the general case; however, it has been

2010 Mathematics Subject Classification. 34A12, 34A34, 35A02, 35A24, 35F20, 35Q35, 60E05, 76A02.

Key words and phrases. Continuity equation, Lagrangian flow, optimal transport, Wasserstein metric, Wasserstein space.
already established [16] that similar conclusions in higher dimensions hold under more severe restrictions on the probabilities on the curve.

In a recent collaborative paper [4] we studied the uniqueness of the Lagrangian flow associated to the curve, i.e. the flow of its velocity. However, we left open the existence issue for said velocity; as shown in [4] and recalled below, for absolutely continuous curves of Borel probabilities in the 1-Wasserstein space (i.e. $p = 1$) the existence of the measurable velocity is not automatic (as it is when $p > 1$). This is another factor that motivates our interest in investigating the existence of the velocity associated to a given curve.

1.1. Preliminaries. Let us recall the definition of and some basic facts about absolutely continuous curves in the Wasserstein space. If $1 \leq p < \infty$ and $d \geq 1$ is an integer, we denote by $\mathcal{P}_p(\mathbb{R}^d)$ the $p$-Wasserstein space on $\mathbb{R}^d$, i.e. the set of Borel probabilities on $\mathbb{R}^d$ with finite $p$-moment and endowed with the $p$-Wasserstein metric

$$W_p(\mu, \nu) := \inf \left\{ \left( \mathbb{E}[|X - Y|^p] \right)^{\frac{1}{p}} \mid \text{law}(X) = \mu, \text{law}(Y) = \nu \right\}.$$ 

Also, in this paper $\mathcal{P}_p^{ac}(\mathbb{R}^d)$ stands for the set of all Borel probability measures $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ which are absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^d$. Let us begin by recalling the definition of $AC^q(0, T; \mathcal{P}_p(\mathbb{R}^d))$ (see [3]):

**Definition 1.1.** If $1 \leq p < \infty$ and $1 \leq q \leq \infty$, a path $[0, T] \ni t \mapsto \mu_t \in \mathcal{P}_p(\mathbb{R}^d)$ is said to lie in $AC^q(0, T; \mathcal{P}_p(\mathbb{R}^d))$ provided that there exists $\beta \in L^q(0, T)$ such that

$$W_p(\mu_s, \mu_t) \leq \int_s^t \beta(\tau) d\tau \text{ for all } 0 \leq s \leq t \leq T.$$

It is proved in [3] that if $1 < p < \infty$, then any such curve admits a Borel velocity $v : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ of minimal norm in the sense that:

$$(\mu, v) \text{ satisfies } (CE) \text{ in the sense of distributions in } (0, T) \times \mathbb{R}^d \quad (1.1)$$

$$v(t, \cdot) \in L^p(\mu(t, \cdot); \mathbb{R}^d) \text{ for a.e. } t \in (0, T) \text{ and } \quad (0, T) \ni t \mapsto \|v(t, \cdot)\|_{L^p(\mu(t, \cdot); \mathbb{R}^d)} \in L^q(0, T); \quad (1.2)$$

$$\int_0^T \|v(t, \cdot)\|_{L^p(\mu(t, \cdot); \mathbb{R}^d)} dt \text{ is minimal among all velocities satisfying } (1.1) \text{ and } (1.2). \quad (1.3)$$

Moreover, it is showed in [3] that this “velocity of minimal norm” is unique, in the sense that if $v_1$ and $v_2$ satisfy (1), (2), (3) above for a.e. $t \in (0, T)$, then $v_1(t, \cdot) \equiv v_2(t, \cdot) \mu_t$ a.e.

Note that $AC^q(0, T; \mathcal{P}_p(\mathbb{R}^d)) \subset AC^1(0, T; \mathcal{P}_1(\mathbb{R}^d))$ for all $1 \leq p < \infty$ and $1 \leq q \leq \infty$.

**The one-dimensional case.** In [4] the following theorem is proved for $d = 1$; it states that on the line there is at most one integrable (in the sense specified below) velocity, the minimality condition on its norm being redundant. As customary, $\mathbb{R}^1$ is denoted simply by $\mathbb{R}$.

**Theorem 1.2.** Consider a path $\mu \in AC^1(0, T; \mathcal{P}_1(\mathbb{R}))$ for some $0 < T < \infty$. Then there exists at most one Borel velocity $v$ for $\mu$ such that $v \in L^1(\mu)$ (as a function of both time and space). More precisely, if $v_1, \ v_2 : (0, T) \times \mathbb{R} \to \mathbb{R}$ are Borel maps such that $v_i \in L^1(\mu)$ for $i = 1, 2$, and such that both $(\mu, v_1)$ and $(\mu, v_2)$ satisfy (CE) in
the sense of distributions, then for Lebesgue a.e. \( t \in (0, T) \) we have \( v_1(t, \cdot) \equiv v_2(t, \cdot) \) in the \( \mu(t, \cdot) \)-a.e. sense.

The following is an obvious consequence.

**Corollary 1.3.** Let \( 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \) and \( \mu \in AC^q(0, T; \mathcal{P}_p(\mathbb{R})) \) be given. Then, there exists at most one Borel map \( v : (0, T) \times \mathbb{R} \to \mathbb{R} \) such that \( (\mu, v) \) satisfies (1.1) (with \( d = 1 \)) and

\[
v \in L^1(\mu), \text{ i.e. } \int_0^T \int_{\mathbb{R}} |v(t, y)| \mu(t, dy)dt < \infty. \tag{1.4}
\]

This uniqueness result enables us to make the following definition:

**Definition 1.4.** Let \( 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \) and \( \mu \in AC^q(0, T; \mathcal{P}_p(\mathbb{R})) \) be given. If it exists, the Borel map \( v : (0, T) \times \mathbb{R} \to \mathbb{R} \) such that (1.1) and (1.4) are satisfied is called the \( L^1 \)-velocity associated to \( \mu \).

Let

\[
\chi := \mathcal{L}^1|_{(0,1)}, \text{ i.e. the one-dimensional Lebesgue measure restricted to } (0, 1).
\]

It has been known since Fréchet [9] that any Borel map \( S \) defined on (in our case) \( I := (0,1) \) can be monotonically rearranged over \( I \), i.e. there exists a nondecreasing map \( M : I \to \mathbb{R} \) such that \( \chi(S^{-1}(B)) = \chi(M^{-1}(B)) \) for all Borel sets \( B \subset \mathbb{R} \).

In other words, there exists a nondecreasing function \( M \) such that the Lebesgue measure of the pre-images of any Borel set through \( S \) and \( M \) coincide. This function satisfies

\[
M(x) = \inf \{ y \in \mathbb{R} : F(y) > x \} \text{ for all } x \in (0, 1),
\]

where

\[
F(y) := \mu((\infty, y]) \tag{1.5}
\]

is the cumulative distribution function of the Borel probability measure \( \mu := S_\# \chi \), i.e. \( M \) is the generalized inverse of \( F \). Here we introduced the notation \( S_\# \mu = \nu \) to mean that

\[
\nu(B) = \mu(S^{-1}(B)) \text{ for all Borel sets } B \subset \mathbb{R}.
\]

It also turns out that \( F \) is the generalized inverse of \( M \).

Consider the problem

\[
\partial_t X(t, x) = v(t, X(t, x)), \quad \text{under } X(0, x) = X_0(x), \quad x \in I, \tag{Flow}
\]

where \( X_0 : I \to \mathbb{R}, \ v : (0, T) \times \mathbb{R} \to \mathbb{R} \) are given functions. Note that the solution \( X \) is written as a function of two variables in order to account not only for the time-variable but also for the initial value prescribed for \( X \). If \( X_0 \equiv \text{Id} \) in \( I \) and the solution exists and is unique for all \( x \in I \), loosely speaking, the function \( X \) is called the classical flow of \( v \). The terminology comes primarily from Fluid Dynamics: if \( v \) stands for the velocity of fluid flow, then \( X(t, x) \) accounts for the position at time \( t \) of the fluid particle that was initially \( (t = 0) \) at position \( x \) (if \( X_0 \equiv \text{Id} \)) or, more generally, \( X_0(x) \).

**Definition 1.5 (Flow of a Borel map).** Let \( v : [0, T] \times \mathbb{R} \to \mathbb{R} \) and \( X_0 : I \to \mathbb{R} \) be Borel functions. We say that \( X : [0, T] \times I \to \mathbb{R} \) is a flow map for \( v \) if

(i) the map \( [0, T] \ni t \to X(t, x) \) is absolutely continuous for a.e. \( x \in I \);

(ii) \( \partial_t X(t, x) = v(t, X(t, x)) \) for a.e. \( x \in I \).

Furthermore, we say that \( X : [0, T] \times I \to \mathbb{R} \) is a flow map for \( v \) starting at \( X_0 \) if, beside (i), (ii), the following is satisfied:

(iii) \( X(0, x) = X_0(x) \) for a.e. \( x \in I \).
This equation is the basis for the Lagrangian description of fluid flow (where the trajectory of a particle is observed through time). It is worth mentioning that (Flow) and (CE) are closely connected to the Transport Equation
\[ \partial_t F + v \cdot \partial_y F = 0 \text{ in } (0,T) \times \mathbb{R}. \]  

Indeed, the cumulative distribution functions \( F(t, \cdot) \) of \( \rho(t, \cdot) \) formally satisfy (Trans) if \((\rho, v)\) satisfies (CE). In order to clarify these connections, note that in the regular case where \( \rho(t, \cdot) \ll L^1 \) and is smooth as a function of both variables, we can differentiate \( F(t, M(t,x)) = x \) with respect to \( t \) and get
\[ \partial_t F(t, M(t,x)) + \partial_y M(t, x) \partial_y F(t, M(t,x)) = 0, \]
which, in light of \( M \) satisfying (Flow) yields the pointwise version of (Trans). Of course, (CE) can be obtained by differentiating (Trans) with respect to \( y \). However, such regularity is not expected in general. We will do with much less, including giving up on the assumption \( \rho(t, \cdot) \ll L^1 \).

If \( p = 1 \), then even when \( d = 1 \) and \( \rho(t, \cdot) \in P^1_{ac}(\mathbb{R}) \) for all \( t \in (0,T) \), a velocity \( v \) satisfying (1.1) and (1.4) may not exist. The examples below appear in [4]; the points they make are central to the main pursuit of this paper (existence of an integrable velocity for a given curve), therefore we carry them over in full detail.

**Example 1.6.** Let \( M : (0,1) \times (0,1) \to \mathbb{R} \) be the family of optimal maps given by:
\[ M(t,x) = \begin{cases} x & \text{if } x \in [0,1-t), \\ 1+x & \text{if } x \in [1-t,1] \end{cases} \]
for all \( t \in (0,1) \). Also, \( M(0,x) = x \) and \( M(1,x) = 1+x \) for all \( x \in [0,1] \). Then we can easily compute the curve \( \rho(t,\cdot) = M(t,\cdot)\#\chi \) to obtain \( \rho(t,\cdot) = \chi_{[0,1-t]} + \chi_{[2-t,2]} \), which shows that \( \rho(t,\cdot) \in P^1_{ac}(\mathbb{R}) \) for all \( p \geq 1 \) and for all \( t \in (0,1) \). However, for all \( 0 \leq s \leq t \leq 1 \)
\[ W_p(\rho(s,\cdot), \rho(t,\cdot)) = \left( \int_0^1 |M(s,x) - M(t,x)|^p \, dx \right)^{\frac{1}{p}} = |t-s|^{\frac{1}{p}}, \]
and this is bounded by \( \int_0^1 \beta(t) d\tau \) for some \( \beta \in L^1(0,1) \) if and only if \( p = 1 \) (in which case we may take \( \beta \equiv 1 \in L^\infty(0,1) \)). Thus, \( \rho \in AC^{\infty}(0,1; P^1_{ac}(\mathbb{R})) \) but \( \rho \notin AC^q(0,1; P_p(\mathbb{R})) \) for any \( 1 < p < \infty \) and any \( 1 \leq q \leq \infty \).

Next, assume that the \( L^1 \)-velocity \( v \) associated to \( \rho \) exists. Then, for all \( \varphi \in C^1_c(\mathbb{R}) \), the function \( t \mapsto \int_\mathbb{R} \varphi(y) \rho(t,dy) \) is absolutely continuous on \([0,1]\) and
\[ \frac{d}{dt} \int_\mathbb{R} \varphi(y) \rho(t,dy) = \int_\mathbb{R} v(t,y) \varphi'(y) \rho(t,dy) \text{ at a.e. } t \in (0,1), \]
i.e.
\[ \varphi(2-t) - \varphi(1-t) = \int_0^{1-t} v(t,y) \varphi'(y) \, dy + \int_{1-t}^2 v(t,y) \varphi'(y) \, dy \]
for a.e. \( t \in [0,1] \). Take \( \varphi \in C^1_c(\mathbb{R}) \) such that \( \varphi \equiv 1 \) on \([0,5/8]\) and \( \varphi \equiv 2 \) on \([11/8,2]\). Then the above equality must be satisfied, in particular, at a.e. \( t \in (3/8,5/8) \); this yields \( 1 = 0 \), a contradiction. In conclusion, we have produced an example of a curve lying in the “most regular” subset of the \( AC^q(0,1; P^1_{ac}(\mathbb{R})) \) families of curves (namely, \( AC^{\infty}(0,1; P^1_{ac}(\mathbb{R})) \)), and yet whose \( L^1 \)-velocity does not exist.
Example 1.7. Let \( f(z) = (1 - \ln z)^{-1} \), so that \( f'(z) = z^{-1} (1 - \ln z)^{-2} \) for \( z \in (0, 1) \). Then \( f \in L^\infty(0, 1) \) and \( f' \in L^p(0, 1) \) if and only if \( p = 1 \), so that \( f \in W^{1,1}(0, 1) \) but \( f \notin W^{1,p}(0, 1) \) for any \( p > 1 \). Finally, set \( g(x) := \min \{-f'(x), -e/4\} \) and note that \( g \) is continuous on \((0, 1]\), increasing on \((0, e^{-1}]\) and constant on \([e^{-1}, 1]\). Just like \( f' \), we have \( g \in L^p(0, 1) \) if and only if \( p = 1 \). Theorem 2.2 answers Question 2 in the affirmative, while Corollary 2.8 assures a positive answer to Question 1. See also M the graphs of \( f \) and \( g \) and their \( L^1 \)-velocity exists.

Remark 1.8. We know from [3] that \( p > 1 \) guarantees the existence of the integrable velocity \( v \). Examples 1.6 and 1.7 show that in the case \( p = 1 \) things are not clear; the first example shows a curve in \( AC^\infty(0, 1; P_1(\mathbb{R})) \) for which \( v \) does not exist, while the second shows a curve that belongs to \( AC^q(0, 1; P_1(\mathbb{R})) \) if and only if \( p = q = 1 \), all the measures on it have point masses, and yet the velocity exists.

1.2. Main results. It is important to note that the curve \( \rho \) from Example 1.6 also exhibits an irregular family of optimal maps \( M \) such that \( M(t, \cdot) \# \chi = \rho(t, \cdot) \), i.e. \( M \notin W^{1,1}(0, 1; L^1(I)) \). In [10] it was proved that \( p = 2 \) implies \( M \in W^{1,2}(0, T; L^2(I)) \) and \( M(t, x) = v(t, M(t, x)) \) for a.e. \((t, x) \in (0, T) \times I\). The proof of this fact can readily be extended to the case \( p > 1 \), \( q > 1 \). That leaves the cases \( p = 1 \), \( q \geq 1 \) and \( p \geq 1 \), \( q = 1 \) open to discussion. As a partial result, we proved in [4] that if \( v \) exists for a curve in \( AC^1(0, 1; P_1(\mathbb{R})) \), then \( M \in W^{1,1}(0, T; L^1(I)) \) and the above “ODE” holds.

The following natural questions arise:

Question 1. If \( v \) exists for some \( \rho \in AC^1(0, 1; P_1(\mathbb{R})) \), does this imply \( M \in W^{1,1}(0, T; L^1(I)) \)? If so, do \( M \) and \( v \) satisfy \((\text{Flow})\)?

Question 2. Is the converse of the above statement true, i.e. if \( M \in W^{1,1}(0, T; L^1(I)) \), is it true that \( v \) exists? Again, will \( M \) and \( v \) satisfy \((\text{Flow})\) then?

Note that if the first question is settled in the affirmative, we can combine that with the existence of \( v \) in the case \( p > 1 \) to conclude that for \( p > 1 \), \( q \geq 1 \) we have that \( M \in W^{1,1}(0, T; L^1(I)) \) (in fact, \( M \in W^{1,q}(0, T; L^p(I)) \)) and \((\text{Flow})\) is satisfied. Furthermore, if both questions above are answered affirmatively, then, in light of Example 1.6, we have a complete understanding of the situation. Namely, if \( p > 1 \), then \( M \in W^{1,1}(0, T; L^1(I)) \), \( v \) exists and \((\text{Flow})\) is satisfied; if \( p = 1 \), then \( M \in W^{1,1}(0, T; L^1(I)) \) if and only if \( v \) exists (in which case \((\text{Flow})\) holds!).

Main results: The next section provides all the answers to the above questions: Theorem 2.2 answers Question 2 in the affirmative, while Theorem 2.6 provides a positive answer to Question 1. See also Corollary 2.8 and Corollary 3.2 for summaries of a more general flavor.

In Section 3 the multi-dimensional case (for which a less precise version of the first part of Question 2 above is answered in the affirmative) and some applications are discussed, whereas Section 4 brings some open questions to the reader’s attention.
2. *AC^q(0, T; \mathcal{P}_p(\mathbb{R})) curves and their integrable velocities.* The following Borel measurability lemma will be useful.

**Lemma 2.1.** Let $\rho \in AC^1(0, T; \mathcal{P}_1(\mathbb{R}^d))$ for some $T > 0$ and some positive integer $d$. Then the set $\mathcal{A}$ of all $(t, y) \in [0, T] \times \mathbb{R}^d$ such that $y$ is an atom for $\rho(t, \cdot)$ is Borel.

**Proof.** Denote by $C_+^c$ the nonnegative cone of $C_c(\mathbb{R}^d)$. Consider, for every positive integer $m$ and every $\xi \in C_+^c$, the set

$$\mathcal{A}_m^\xi := \left\{ (t, y) \in [0, T] \times \mathbb{R}^d : \int_{\mathbb{R}^d} \xi(z) \rho(t, dz) \geq \frac{1}{m} \xi(y) \right\}.$$

Note that the absolute continuity in time of the left hand side of the above inequality and the continuity in $y$ of the right hand side imply that the difference is a continuous function of $(t, y)$. Therefore, $\mathcal{A}_m^\xi$ is the nonnegative set of a continuous function, which makes it a closed subset of $[0, T] \times \mathbb{R}^d$. Thus, $\mathcal{A}_m := \left\{ (t, y) \in [0, T] \times \mathbb{R}^d : \int_{\mathbb{R}^d} \xi(z) \rho(t, dz) \geq \frac{1}{m} \xi(y) \text{ for all } \xi \in C_+^c \right\}$ is closed as well, by being an arbitrary intersection of closed sets. But $\mathcal{A} = \bigcup_{m \geq 1} \mathcal{A}_m$, so the proof is concluded. □

The result below is a great generalization of Proposition 4.2 in [10]. As mentioned earlier, one can easily generalize this proposition to deal with the case $p > 1$, $q > 1$. Indeed, the key to the approach in [10] was the fact

$$\lim_{h \to 0} \int_0^T \left\| \frac{M(t + h, \cdot) - M(t, \cdot)}{h} - \dot{M}(t, \cdot) \right\|_{L^q(\mathcal{I})}^q dt = 0$$

for $p = q = 2$, which can easily be extended to $p > 1, q > 1$. However, in order to settle the most general case ($p = q = 1$), we pursue a dramatically different proof.

**Theorem 2.2.** Let $M \in W^{1,1}(0, T; L^1(\mathcal{I}))$ be such that $M(t, \cdot)$ is nondecreasing and right-continuous for all $t \in [0, T]$, set $\rho(t, \cdot) := M(t, \cdot) \# \chi$ and let $F(t, \cdot)$ denote its cumulative distribution function (see (1.5)) for all $t \in [0, T]$. Define $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ by

$$v(t, y) := \int_{F(t, y)}^{F(t, y-)} \dot{M}(t, z) dz,$$  \hspace{1cm} (2.1)

where the average above is taken to be $\dot{M}(t, F(t, y))$ if $F(t, \cdot)$ is continuous at $y \in \mathbb{R}$. Then:

(i) $\rho \in AC^1(0, T; \mathcal{P}_1(\mathbb{R}))$;

(ii) There exists a Borel set $\mathcal{T} \subset [0, T]$ of full Lebesgue measure such that $v$ coincides with a Borel function on $\mathcal{T} \times \mathbb{R}$;

(iii) $M(t, x) = v(t, M(t, x))$ for $L^2$-a.e. $(t, x) \in [0, T] \times \mathcal{I}$;

(iv) $v$ is the integrable velocity of $\rho$ in the sense of Definition 1.4.

**Proof.** (i) That $\rho \in AC^1(0, T; \mathcal{P}_1(\mathbb{R}))$ follows from

$$W_1(\rho(s, \cdot), \rho(t, \cdot)) = \left\| M(s, \cdot) - M(t, \cdot) \right\|_{L^1(\mathcal{I})} \leq \int_s^t \left\| \dot{M}(\tau, \cdot) \right\|_{L^1(\mathcal{I})} d\tau \text{ for all } 0 \leq s \leq t \leq T,$$

which shows that we may choose $\beta(t) = \left\| \dot{M}(t, \cdot) \right\|_{L^1(\mathcal{I})}$ in Definition 1.1.
(ii) Next let us show that $v$ from (2.1) is a Borel map (after, possibly, redefining it on a negligible set). Since $M, \dot{M} \in L^1((0, T) \times I)$, they both admit Borel representatives; so, by possibly redefining them on negligible sets, we may assume they are Borel maps. Consider a sequence $\{\varphi_n\} \subset C_c((0, T) \times I)$ which converges to $\dot{M}$ in $L^1((0, T) \times I)$ and set

$$\phi_n(t, x) := \int_0^x \varphi_n(t, z) \, dz, \quad \Phi(t, x) = \liminf_{n \to \infty} \phi_n(t, x) \text{ for all } (t, x) \in (0, T) \times I.$$ 

Clearly, $\Phi$ is Borel as $\liminf$ of a sequence of continuous functions. Thus, if to $\dot{v}$ it on a negligible set). Since

$$\{\varphi_n\} \subset C_c((0, T) \times I)$$

they are Borel maps. Consider a sequence

$$\{\text{representatives; so, by possibly redefining them on negligible sets, we may assume}
$$

$$(\text{U is a Borel map. Let} \quad L \quad \text{in order to conclude that a Borel velocity field for the flow}
$$

$$\text{we wish to identify this velocity as} \quad v \quad \text{in}
$$

$$\quad \text{that} \quad \dot{M} \quad \text{in}
$$

$$\quad \text{implies it is defined at all points in}
$$

$$\quad \text{we infer}
$$

$$\quad \text{for all} \quad t \quad \text{and all}
$$

$$\quad \text{Thus,} \quad w(t, y) = v(t, y) \quad \text{for all} \quad t \quad \text{and all}
$$

$$\quad \text{Let} \quad \mathcal{S} \subset (0, T) \times I \quad \text{be the set of all the Lebesgue points of} \quad M; \quad \text{since} \quad \dot{M} \quad \text{in} \quad L^1((0, T) \times I) \quad \text{we know that} \quad \mathcal{L}^2(\mathcal{S}) = T = \mathcal{L}^2((0, T) \times I). \quad \text{According to Theorem} \quad 2.2 \quad [16], \quad \text{it is enough to show that}
$$

$$\quad \dot{M}(t, x_1) = \dot{M}(t, x_2) \quad \text{for some} \quad (t, x_1), \quad (t, x_2) \in \mathcal{S} \quad \text{implies} \quad \dot{M}(t, x_1) = \dot{M}(t, x_2)
$$

(2.2)

in order to conclude that a Borel velocity field for the flow $M$ exists; however, we wish to identify this velocity as $v$ given by (2.1). Now, to prove (2.2), first note that it is trivially true if $x_1 = x_2$. Assume next, without loss of generality, that $0 < x_1 < x_2 < 1$. Since $M(t, \cdot)$ is nondecreasing and right-continuous (which implies it is defined at all points in $I$), we infer $M(t, x) = M(t, x_1) = M(t, x_2)$ for all $x_1 < x < x_2$. In particular, if we set $\delta := (x_2 - x_1)/2 > 0$, we have that

$$\quad M(t, x_1 + z) = M(t, x_2 - z) \quad \text{for all} \quad 0 \leq z < \delta. \quad (2.3)$$

Since $(t, x_1), \quad (t, x_2)$ are Lebesgue points for $\dot{M}$, we have that

$$\lim_{h \to 0} \frac{1}{h^2} \int_t^{t+h} \int_{x_1}^{x_1+|h|} |\dot{M}(s, z) - \dot{M}(t, x_1)| \, dz \, ds = 0,$$

$$\lim_{h \to 0} \frac{1}{h^2} \int_t^{t+h} \int_{x_2}^{x_2-|h|} |\dot{M}(s, z) - \dot{M}(t, x_2)| \, dz \, ds = 0. \quad \text{Consequently, by using the fact that} \quad M(\cdot, z) \in W^{1,1}(0, T) \quad \text{for a.e.} \quad z \in I \quad \text{with} \quad \dot{M}(\cdot, z) \quad \text{as its a.e. derivative, we get}
$$

$$\dot{M}(t, x_1) = \lim_{h \to 0} \frac{1}{h^2} \int_t^{t+h} \int_{x_1}^{x_1+|h|} \dot{M}(s, z) \, dz \, ds$$

$$= \lim_{h \to 0} \frac{1}{|h|} \int_0^{|h|} \frac{M(t+h, x_1+z) - M(t, x_1+z)}{h} \, dz$$

$$= \dot{M}(t, x_1) = \dot{M}(t, x_2).$$
and
\[ \dot{M}(t, x_2) = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \int_{x_2-|h|}^{x_2} \dot{M}(s, z) \, dz \, ds \]
\[ = \lim_{h \to 0} \frac{1}{h} \int_0^{h} \frac{M(t + h, x_2 - z) - M(t, x_2 - z)}{h} \, dz. \]

Thus, since $|h|$ will ultimately drop below $\delta$, we use (2.3) to deduce
\[ \dot{M}(t, x_2) = \dot{M}(t, x_1) = \lim_{h \to 0} \frac{1}{h} \int_0^{h} \frac{M(t + h, x_2 - z) - M(t + h, x_1 + z)}{h} \, dz. \]

For all $z$ such that $0 \leq z \leq |h| < \delta$ we also have $x_1 + z < x_2 - z$, which means the numerator of the integrand above is nonnegative for sufficiently small $|h|$ (due to $M(t, \cdot)$ being nondecreasing for all $t \in [0, T]$). Since $h$ tends to zero from either the left or right, the limit must be zero, i.e. $\dot{M}(t, x_1) = \dot{M}(t, x_2)$.

Let $t \in (\text{proj}_{0, T} \mathcal{S}) \cap \mathcal{T}$. If $(t, M(t, x)) \in \mathcal{B}$ for some $x \in I$ such that $(t, x) \in \mathcal{S}$, then $F(t, M(t, x)) = x$ and so $w(t, M(t, x)) = v(t, M(t, x)) = \dot{M}(t, x)$. If, on the other hand, $(t, M(t, x)) \in \mathcal{A}$, then
\[ x_1 := F(t, M(t, x) -) \leq x < F(t, M(t, x)) =: x_r \]
and $M(t, x) = M(t, z)$ for all $z \in [x_1, x_r)$. Since $t \in \text{proj}_{0, T} \mathcal{S}$, almost all points $z \in [x_1, x_r)$ satisfy that $(t, z) \in \mathcal{S}$. By (2.2) we deduce $\dot{M}(t, z) = \dot{M}(t, x)$ for all $z \in [x_1, x_r)$ for which $(t, z) \in \mathcal{S}$, and since this is true for almost all $z \in [x_1, x_r)$, we have that
\[ w(t, M(t, x)) = v(t, M(t, x)) = \int_{x_1}^{x_r} \dot{M}(t, z) \, dz = \int_{x_1}^{x_r} \dot{M}(t, x) \, dz = \dot{M}(t, x). \]

(iv) For $\zeta \in C^1_c(\mathbb{R})$ we have that $[0, T] \ni t \mapsto \zeta(M(t, x))$ lies in $W^{1,1}(0, T)$ for a.e. $x \in I$. With all the integrability conditions satisfied, we get that
\[ [0, T] \ni t \mapsto \int_I \zeta(M(t, x)) \, dx = \int_\mathbb{R} \zeta(y) \rho(t, dy) \]
also belongs to $W^{1,1}(0, T)$ and, by (iii) above,
\[ \frac{d}{dt} \int_\mathbb{R} \zeta(y) \rho(t, dy) = \int_I \zeta'(M(t, x)) v(t, M(t, x)) \, dx = \int_\mathbb{R} \zeta'(y) v(t, y) \rho(t, dy), \]
which is equivalent to the fact that $\rho$ and $v$ satisfy the continuity equation in the sense of distributions. Finally,
\[ \int_0^T \|v(t, \cdot)\|_{L^1(\rho(t, \cdot))} \, dt = \int_0^T \|v(t, M(t, \cdot))\|_{L^1(I)} \, dt = \|\dot{M}\|_{L^1([0, T] \times I)}, \]
which shows that $v \in L^1(\rho)$.

\[ \Box \]

Remark 2.3. Due to the uniqueness of the $L^1$-velocity $v$ (when it exists) for a given $\rho \in AC^1(0; T; \mathcal{P}_1(\mathbb{R}))$, every time we refer to a Lagrangian flow map $X$ for $v$ under the constraint $X_{t \# \chi} = \rho_t$, we may simply call it a Lagrangian flow map associated to $\rho$.

Let us denote by $\mathcal{P}_p^r(\mathbb{R})$ the set of all Borel probability measures $\mu$ on $\mathbb{R}$ with finite $p$-th moments (i.e. $\int_\mathbb{R} |y|^p \mu(t, dy) < +\infty$) which do not have atoms. The proposition below was proved in [4] (Theorem 2.11) under the stronger assumption that $\rho(t, \cdot) \ll \mathcal{L}^1$ for all $t \in [0, T]$. This is, therefore, a generalization.
Proposition 2.4. Let $\rho \in C^{n}(0,T;P_{p}^{r}(\mathbb{R}))$ for some $1 \leq p < \infty$ and $1 \leq q \leq \infty$. If $p = 1$, assume also that the $L^{1}$-velocity of $\rho$ exists. Denote by $M(t,\cdot)$ the optimal map pushing forward $\chi$ to $\rho(t,\cdot)$. Then $M \in W^{1,q}(0,T;L^{p}(I))$ and it is a flow map associated to $\rho$.

Proof. Since

$$
\|M(t,\cdot)\|_{L^{p}(I)}^{p} = \int_{\mathbb{R}} |y|^{p} \rho(t,dy) = W_{p}^{p}(\rho(t,\cdot),\delta_{0})
$$

$$
\leq 2^{p-1} \left( W_{p}^{p}(\rho(t,\cdot),\rho_{0}) + \int_{\mathbb{R}} |y|^{p} \rho_{0}(dy) \right)
$$

$$
\leq 2^{p-1} \left[ \left( \int_{0}^{T} \|v(t,\cdot)\|_{L^{p}(\rho(t,\cdot))} dt \right)^{p} + \int_{\mathbb{R}} |y|^{p} \rho_{0}(dy) \right] < \infty \quad \text{for all } t \in [0,T]
$$

and

$$
\int_{0}^{T} \left( \int_{I} |v(t,M(t,x))|^{p} dx \right)^{q/p} dt = \int_{0}^{T} \|v\|_{L^{p,q}(\rho,\cdot)}^{q} dt < \infty,
$$

it suffices to prove that $v(t,M(t,x))$ is the distributional time-derivative of $M$ to obtain the desired thesis.

Consider a standard mollifier $\eta^{\varepsilon}(y) = \eta(y/\varepsilon)/\varepsilon$ for $0 < \varepsilon < 1$ and let

$$
\rho^{\varepsilon}(t,\cdot) = \eta^{\varepsilon} * \rho(t,\cdot), \quad E^{\varepsilon}(t,\cdot) = \eta^{\varepsilon} * [v(t,\cdot) \rho(t,\cdot)].
$$

Here, $\eta \in C_{c}^{\infty}(\mathbb{R})$ is supported in $[-1,1]$, nonnegative, even and $\int \eta = 1$. Thus, for fixed $y \in \mathbb{R}$, $z \mapsto \eta^{\varepsilon}(z-y)$ is smooth and supported in $[y-\varepsilon,y+\varepsilon]$. So, it can be used as a test function in (CE) to deduce that

$$
[0,T] \ni t \mapsto \int_{\mathbb{R}} \eta^{\varepsilon}(z-y) \rho(t,dy) = \rho^{\varepsilon}(t,y)
$$

is absolutely continuous and

$$
\partial_{t} \rho^{\varepsilon}(t,y) = \int_{\mathbb{R}} \partial_{t} \eta^{\varepsilon}(z-y) v(t,z) \rho(t,dy)
$$

$$
= -\int_{\mathbb{R}} \eta^{\varepsilon}(y-z) v(t,z) \rho(t,dy) = -\partial_{y} E^{\varepsilon}(t,y)
$$

for a.e. $t \in [0,T]$. It is straightforward that $\rho^{\varepsilon}(t,\cdot)$ and $E^{\varepsilon}(t,\cdot)$ converge narrowly to $\rho(t,\cdot)$ and $v(t,\cdot)\rho(t,\cdot)$, respectively. Now let $F^{\varepsilon}(t,\cdot)$ and $F(t,\cdot)$ be the cumulative distribution functions of $\rho^{\varepsilon}(t,\cdot)$ and $\rho(t,\cdot)$, respectively. By repeated applications of Fubini’s theorem, we note

$$
F^{\varepsilon}(t,y) = \int_{y}^{\infty} \int_{\mathbb{R}} \eta^{\varepsilon}(z-u) \rho(t,du) \, dz
$$

$$
= \int_{\mathbb{R}} \int_{-\infty}^{\infty} 1_{(-\infty,y]}(z) \eta^{\varepsilon}(z-u) \rho(t,du) \, dz
$$

$$
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} 1_{(-\infty,y-u]}(z-u) \eta^{\varepsilon}(z-u) \, dz \right) \rho(t,du)
$$

$$
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} 1_{(-\infty,y-u]}(z) \eta^{\varepsilon}(z) \, dz \right) \rho(t,du)
$$

$$
= \int_{\mathbb{R}} \left[ \eta^{\varepsilon} * F(t,\cdot) \right](y),
$$
i.e. $F^\varepsilon(t, \cdot) = \eta^\varepsilon \ast F(t, \cdot)$ for all $\varepsilon > 0$ and all $t \in [0, T]$. Since $\rho(t, \cdot)$ is atom-free, $F(t, \cdot)$ is continuous on $\mathbb{R}$, which implies $F^\varepsilon(t, \cdot)$ converges uniformly on compact sets to $F(t, \cdot)$ as $\varepsilon \to 0^+$. Also, $F^\varepsilon(t, \cdot)$ is smooth with $\partial_y F^\varepsilon(t, y) = \rho'(t, y)$ for all $t$ and $y$.

Since $\rho \in AC^1(0, T; \mathcal{P}_1(\mathbb{R}))$, we deduce that
\[ \int_{\mathbb{R}} |y| \rho(t, dy) \leq C < \infty \quad \text{for all } t. \]

We also have that $t \mapsto F^\varepsilon(t, y)$ is absolutely continuous for a.e. $y \in \mathbb{R}$, with $\partial_t F^\varepsilon(t, y) = -E^\varepsilon(t, y)$. Indeed, this comes as a consequence of $\partial_t \rho^\varepsilon(t, y) = -\partial_y E^\varepsilon(t, y)$. In order to prevent some integrability issues, we also introduce a cut-off function in $y$, namely, $\xi_k \in C^1_c(\mathbb{R})$ such that $\xi_k(y) = 1$ if $|y| \leq k$, $\xi_k(y) = 0$ if $|y| \geq 3k$ and $|\xi_k(y)| \leq \min \{ 2|y|, k + 1 \}$, $|\xi'_k(y)| \leq 1$ for all $y \in \mathbb{R}$. Let $\varphi \in C^1_c(\mathbb{R})$ and note that
\[ [0, T] \ni t \mapsto \xi_k(y) \varphi(F^\varepsilon(t, y)) \rho^\varepsilon(t, y) \]
is also absolutely continuous for a.e. $y \in \mathbb{R}$, with
\[
\frac{\partial}{\partial t} \left[ \xi_k(\varphi \circ F^\varepsilon) \rho^\varepsilon \right] =
\xi_k(y) \varphi'(F^\varepsilon(t, y)) \partial_t F^\varepsilon(t, y) \rho^\varepsilon(t, y) + \xi_k(y) \varphi(F^\varepsilon(t, y)) \partial_t \rho^\varepsilon(t, y)
- \left[ \xi_k(y) \varphi'(F^\varepsilon(t, y)) E^\varepsilon(t, y) \rho^\varepsilon(t, y) + \xi_k(y) \varphi(F^\varepsilon(t, y)) \partial_y E^\varepsilon(t, y) \right],
\]
i.e., for any $\zeta \in C^1_c(0, T)$, we have:
\[
\int_0^T \zeta(t) \xi_k(y) \varphi(F^\varepsilon(t, y)) \rho^\varepsilon(t, y) dt
= \int_0^T \zeta(t) \left[ \xi_k(y) \varphi'(F^\varepsilon(t, y)) E^\varepsilon(t, y) \rho^\varepsilon(t, y) + \xi_k(y) \varphi(F^\varepsilon(t, y)) \partial_y E^\varepsilon(t, y) \right] dt \quad \text{for a.e. } y \in \mathbb{R}. \tag{2.4}
\]

We would like to integrate the above equality in $y$ over $\mathbb{R}$, then integrate by parts over $\mathbb{R}$ the last term in the right hand side; for this we need to show both sides are integrable over $\mathbb{R}$. First,
\[
\left| \int_0^T \zeta(t) \xi_k(y) \varphi(F^\varepsilon(t, y)) \rho^\varepsilon(t, y) dt \right| \leq 2 \| \zeta \|_{\infty} \| \varphi \|_{\infty} \int_0^T |y| \rho^\varepsilon(t, y) dt,
\]
and we continue by noticing that
\[
\int_{\mathbb{R}} |y| \rho^\varepsilon(t, y) = \int_{\mathbb{R}} |y| \int_{\mathbb{R}} \eta^\varepsilon(y - z) \rho(t, dz) dy
= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |y| \eta^\varepsilon(y - z) dy \right) \rho(t, dz)
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |z| + |y - z| \eta^\varepsilon(y - z) dy \right) \rho(t, dz)
= \int_{\mathbb{R}} |z| \rho(t, dz) + \int_{\mathbb{R}} \rho(t, dz) \int_{\mathbb{R}} |y| \eta^\varepsilon(y) dy
\leq \int_{\mathbb{R}} |z| \rho(t, dz) + C \varepsilon, \quad \text{where } C := \int_{-1}^1 |y| \eta(y) dy.
\]
Thus, \( \int_{\mathbb{R}} |\xi_k(y)| \rho^\varepsilon(t, y) \, dy \) is bounded by a finite constant which is independent of \( t \) and \( 0 < \varepsilon < 1 \). So, \((t, y) \mapsto \xi_k(y) \varphi(F^\varepsilon(t, y)) \rho^\varepsilon(t, y)\) is in \( L^\infty((0, T); L^1(\mathbb{R})) \), with bounds independent of \( \varepsilon \in (0, 1) \) and \( k \). As for the right hand side of (2.4), we see that:

\[
|E^\varepsilon(t, y)| \leq \int_{\mathbb{R}} \eta^\varepsilon(y - z) |v(t, z)| \rho(t, dz) \leq \frac{1}{\varepsilon} \max_R |\eta(t, \cdot)|_{L^1(\rho(t, \cdot))}.
\]

Thus,

\[
\int_{\mathbb{R}} |\xi_k(y) \varphi(F^\varepsilon(t, y)) E^\varepsilon(t, y) \rho^\varepsilon(t, y)| \, dy \leq \frac{C}{\varepsilon} \|v(t, \cdot)\|_{L^1(\rho(t, \cdot))} \|\varphi\|_{\infty},
\]

where we absorbed the uniform bound on \( \int_{\mathbb{R}} |\xi_k(y)| \rho^\varepsilon(t, y) \, dy \) proved above in the constant \( C \).

Note that the right hand side of this inequality lies in \( L^1(0, T) \), so

\[(t, y) \mapsto \xi_k(y) \varphi'(F^\varepsilon(t, y)) E^\varepsilon(t, y) \rho^\varepsilon(t, y)\]

is in \( L^1((0, T) \times \mathbb{R}) \) (even though, in this case, the bound may be of order \( \varepsilon^{-1} \)).

Finally, the last term in (2.4) is \( \xi_k(y) \varphi(F^\varepsilon(t, y)) \partial_y E^\varepsilon(t, y) \) and it satisfies

\[
\int_{\mathbb{R}} \left| \xi_k(y) \varphi(F^\varepsilon(t, y)) \partial_y E^\varepsilon(t, y) \right| \, dy \leq (k + 1) \|\varphi\|_{\infty} \int_{\mathbb{R}} \left| (\eta^\varepsilon)'(y - z) \right| |v(t, z)| \rho(t, dz) \, dy
\]

\[
= (k + 1) \|\varphi\|_{\infty} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |(\eta^\varepsilon)'(y - z)| \, dy \right) |v(t, z)| \rho(t, dz)
\]

\[
= \varepsilon^{-1} (k + 1) \|\varphi\|_{\infty} \|\eta^\varepsilon\|_{L^1(\mathbb{R})} \|v(t, \cdot)\|_{L^1(\rho(t, \cdot))},
\]

so it lies in \( L^1((0, T) \times \mathbb{R}) \) as well (note that it was this estimate for which the cut-off function \( \xi_k \) was introduced in the first place). Since \([0, T] \ni t \mapsto \|v(t, \cdot)\|_{L^1(\rho(t, \cdot))}\) lies in \( L^1(0, T) \), we deduce that (2.4) can be integrated with respect to \( y \) over \( \mathbb{R} \) and Fubini’s Theorem may be applied to discover, after a spatial integration by parts of the last term in the right hand side (which leads to the cancelation of the first term in the right hand side), that

\[
\int_0^T \zeta(t) \int_{\mathbb{R}} \xi_k(y) \varphi(F^\varepsilon(t, y)) \rho^\varepsilon(t, y) \, dy \, dt
\]

\[
= - \int_0^T \zeta(t) \int_{\mathbb{R}} \xi_k(y) \varphi(F^\varepsilon(t, y)) E^\varepsilon(t, y) \, dy \, dt,
\]

with integrands in \( L^1((0, T) \times \mathbb{R}) \).

Next, we let \( \varepsilon \to 0^+ \) and use the uniform convergence of \( F^\varepsilon(t, \cdot) \) to \( F(t, \cdot) \) on compact sets, and the narrow convergence of \( \rho^\varepsilon(t, \cdot) \) to \( \rho(t, \cdot) \), along with that of \( E^\varepsilon(t, \cdot) \) to \( v(t, \cdot) \rho(t, \cdot) \) to infer that for each \( t \in [0, T] \) we have (since \( \xi_k \) is continuous and compactly supported)

\[
U^\varepsilon(t) := \int_{\mathbb{R}} \xi_k(y) \varphi(F^\varepsilon(t, y)) \rho^\varepsilon(t, y) \, dy \xrightarrow{\varepsilon \to 0^+} \int_{\mathbb{R}} \xi_k(y) \varphi(F(t, y)) \rho(t, dy)
\]

and

\[
V^\varepsilon(t) := \int_{\mathbb{R}} \xi'_k(y) \varphi(F^\varepsilon(t, y)) E^\varepsilon(t, y) \, dy \xrightarrow{\varepsilon \to 0^+} \int_{\mathbb{R}} \xi'_k(y) \varphi(F(t, y)) v(t, y) \rho(t, dy)
\]
By some well-known convolution properties, we have
\[ |U^\varepsilon(t)| \leq (k + 1)\|\varphi\|_\infty \text{ and } |V^\varepsilon(t)| \leq \|\varphi\|_\infty \|v(t, \cdot)\|_{L^1(\rho(t, \cdot))}.\]

Next we let \( \varepsilon \to 0^+ \) in (2.5) and use Dominated Convergence over \([0, T]\) to get
\[
\int_0^T \zeta(t) \int_{\mathbb{R}} \xi_k(y) \varphi(F(t, y)) \rho(t, dy) \, dt = - \int_0^T \zeta(t) \int_{\mathbb{R}} \xi_k(y) \varphi(F(t, y)) v(t, y) \rho(t, dy) \, dt. \tag{2.6}
\]

Note that
\[
\left| \int_{\mathbb{R}} [\xi_k(y) - y] \varphi(F(t, y)) \rho(t, dy) \right| \leq \|\varphi\|_\infty \int_{\{|y| \geq k\}} |\xi_k(y) - y| \rho(t, dy) \\
\leq 3 \|\varphi\|_\infty \int_{\{|y| \geq k\}} |y| \rho(t, dy)
\]
and
\[
\left| \int_{\mathbb{R}} [\xi_k'(y) - 1] \varphi(F(t, y)) v(t, y) \rho(t, dy) \right| \\
\leq \|\varphi\|_\infty \int_{\{|y| \geq k\}} |\xi_k'(y) - 1| |v(t, y)| \rho(t, dy) \\
\leq 2 \|\varphi\|_\infty \int_{\{|y| \geq k\}} |v(t, y)| \rho(t, dy).
\]

But
\[
\lim_{k \to \infty} \int_{\{|y| \geq k\}} |y| \rho(t, dy) = 0 = \lim_{k \to \infty} \int_{\{|y| \geq k\}} |v(t, y)| \rho(t, dy)
\]
and also
\[
\int_{\{|y| \geq k\}} |y| \rho(t, dy) \leq \int_{\mathbb{R}} |y| \rho(t, dy), \quad \int_{\{|y| \geq k\}} |v(t, y)| \rho(t, dy) \leq \int_{\mathbb{R}} |v(t, y)| \rho(t, dy)
\]
for all integers \( k \geq 1 \) and a.e. \( t \in [0, T] \). Since both right hand sides of the above inequalities lie in \( L^1(0, T) \), we use Dominated Convergence on \([0, T]\) to get from (2.6) that
\[
\int_0^T \zeta(t) \int_{\mathbb{R}} y \varphi(F(t, y)) \rho(t, dy) \, dt = - \int_0^T \zeta(t) \int_{\mathbb{R}} \varphi(F(t, y)) v(t, y) \rho(t, dy) \, dt. \tag{2.7}
\]

We now use the fact that
\[
\rho(t, \cdot) \text{ is non-atomic implies } F(t, M(t, x)) = x \text{ for a.e. } x \in I, \tag{2.8}
\]
and that \( M_{t\#} \chi = \rho_t \) to conclude:
\[
\int_0^T \zeta(t) \int_I \varphi(x) M(t, x) \, dx \, dt = - \int_0^T \zeta(t) \int_I v(t, M(t, x)) \varphi(x) \, dx \, dt
\]
for all \( \zeta \in C^1_c(0, T) \), \( \varphi \in C^1_c(I) \). Thus, the distributional time-derivative of \( M(t, x) \) is \( v(t, M(t, x)) \). Of course, the last displayed equality and the uniform \( L^q - L^p \) bounds obtained in the first paragraph of this proof also imply that for a.e. \( x \in I \) the function \( t \mapsto M(t, x) \) is absolutely continuous on \([0, T]\) and its a.e. time derivative is \( v(\cdot, M(\cdot, x)) \), so \( M \) is as in Definition 1.5. \( \square \)
Now let \( \rho(t, \cdot) := M(t, \cdot) \# \chi \). By Theorem 2.8 [4], we deduce \( \rho \in AC^1(0, T; P_1(\mathbb{R})) \) and \( v \) from the theorem above is its integrable velocity. In order to prove the converse we begin with a generalization of Lemma 8.1.10 [3].

**Lemma 2.5.** Let \( \mu \) be a Borel probability on \( \mathbb{R}^d \), \( E \) be a \( \mathbb{R}^n \)-valued measure on \( \mathbb{R}^d \) with finite total variation and absolutely continuous with respect to \( \mu \). Let \( \phi : \mathbb{R}^d \to [0, \infty] \) be convex and lower semicontinuous. Then

\[
\int_{\mathbb{R}^d} \phi \left( \frac{\partial * E(y)}{\partial * \mu(y)} \right) \partial * \mu(y) \, dy \leq \int_{\mathbb{R}^d} \phi \left( \frac{dE}{d\mu}(y) \right) \mu(dy)
\]

(2.9)

for any smooth and positive convolution kernel \( \partial : \mathbb{R}^n \to \mathbb{R} \).

**Proof.** Define the convex function \( \Phi \) by \( \Phi(z, s) = s \phi(z/s) \) if \( s > 0 \), \( \Phi(0, 0) = 0 \) and \( \Phi \equiv \infty \) elsewhere. Note that \( \Phi : \mathbb{R}^{n+1} \to [0, \infty] \) is convex, lower semicontinuous and positively 1-homogeneous (i.e. \( \Phi(\lambda z, \lambda s) = \lambda \Phi(z, s) \) for all \( \lambda > 0 \)). For each \( y \in \mathbb{R}^d \) we define, for all Borel sets \( A \subset \mathbb{R}^d \), the measure

\[
\mu_y(A) := \int_A \partial(y - z) \, \mu(dz).
\]

These measures are all finite and we set \( \tilde{\mu}_y(A) = \mu_y(A)/\mu_y(\mathbb{R}^d) \) to rescale them into probability measures. For any Borel map \( U : \mathbb{R}^d \to \mathbb{R}^{m+1} \) we apply Jensen’s inequality to the convex map \( \Phi \) and the probability measures \( U \# \tilde{\mu}_y \) to get

\[
\Phi \left( \int_{\mathbb{R}^d} U(z) \tilde{\mu}_y(dz) \right) \leq \int_{\mathbb{R}^d} \Phi(U(z)) \tilde{\mu}_y(dz).
\]

By the positive 1-homogeneity of \( \Phi \) we deduce

\[
\Phi \left( \int_{\mathbb{R}^d} U(z) \, \mu_y(dz) \right) \leq \int_{\mathbb{R}^d} \Phi(U(z)) \, \mu_y(dz) \quad \text{for all } y \in \mathbb{R}^d.
\]

Let \( U := (dE/d\mu, 1) \) and note that the above inequality yields

\[
\frac{\partial * E(y)}{\partial * \mu(y)} = \Phi \left( \int_{\mathbb{R}^d} \frac{dE}{d\mu}(z) \partial(y - z) \, \mu(dz), \int_{\mathbb{R}^d} \partial(y - z) \, \mu(dz) \right)
\]

\[
\leq \int_{\mathbb{R}^d} \Phi \left( \frac{dE}{d\mu}(z), 1 \right) \partial(y - z) \, \mu(dz)
\]

\[
= \int_{\mathbb{R}^d} \frac{dE}{d\mu}(z) \partial(y - z) \, \mu(dz).
\]

Finally, we integrate with respective to \( y \) over \( \mathbb{R}^d \) to finish the proof. \( \square \)

The plan is to use the above lemma in order to control the integrability of the mollified objects from the proof of the theorem below.

**Theorem 2.6.** Let \( \rho \in AC^1(0, T; P_1(\mathbb{R})) \) for which the integrable velocity \( v \) exists and let \( M(t, \cdot) \) be the right-continuous optimal map pushing \( \chi \) forward to \( \rho(t, \cdot) \) for all \( t \in [0, T] \). Then \( M \in W^{1,1}(0, T; L^1(I)) \).

**Proof.** Our strategy now is to mollify the measures \( \rho(t, \cdot) \), apply Proposition 2.4 to the curves consisting of the mollified measures, then recover the general case in the limit. To achieve that, consider a mollification kernel \( \eta \) as in the proof of Proposition 2.4, except that we now assume \( \eta > 0 \) on \( \mathbb{R} \). Set \( v^\varepsilon(t, \cdot) := E^\varepsilon(t, \cdot)/\rho^\varepsilon(t, \cdot) \), which is well-defined since \( \rho^\varepsilon(t, \cdot) > 0 \) on \( \mathbb{R} \). It is easy to check that \( v^\varepsilon \) is the integrable velocity of the curve \( \rho^\varepsilon \in AC^1(0, T; P_1^m(\mathbb{R})) \). Furthermore, it is straight-forward to
Indeed, let \( \zeta \in C(\mathbb{R}) \) for which there exists a positive number \( C \) such that \(|\zeta(y)| \leq C(1 + |z|)\) for all \( z \in \mathbb{R} \). Then,

\[
\int_{\mathbb{R}} \zeta(y) \rho^\varepsilon(t,y) \, dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \eta^\varepsilon(z-y) \zeta(y) \, dy \right) \rho(t,dz) = \int_{\mathbb{R}} \eta^\varepsilon * \zeta(z) \rho(t,dz).
\]

We infer the desired convergence by noting that \( \eta^\varepsilon * \zeta \) converges to \( \zeta \) locally uniformly, while near \( \pm \infty \) the integrals involved are small due to \(|\eta^\varepsilon * \zeta(z)| \leq C'(1+|z|)\) and the finiteness of the first moment of \( \rho(t,\cdot) \).

Since \( M(t,\cdot) \# \chi = \rho(t,\cdot) \) implies

\[
\int_0^T \int_I |v(t,M(t,x))| \, dx \, dt = \int_0^T \|v(t,\cdot)\|_{L^1(\mu(\cdot,\cdot))} \, dt < \infty,
\]

we have that \( f(t,x) := v(t,M(t,x)) \) belongs to \( L^1((0,T) \times I) \). By the de la Vallée-Poussin Lemma, there exists an even, strictly convex, \( C^1 \) function \( \phi : \mathbb{R} \to [0,\infty) \) with \( \phi(0) = 0 \) and super linear growth at infinity such that

\[
\int_0^T \int_I \phi(f(t,x)) \, dx \, dt < \infty.
\]

Thus,

\[
\int_{\mathbb{R}} \phi(v(t,y)) \rho(t,dy) = \int_I \phi(v(t,M(t,x))) \, dx < \infty \text{ for a.e. } t \in (0,T).
\]

We now apply Lemma 2.5 to \( \mu := \rho(t,\cdot) \), \( E := v(t,\cdot) \rho(t,\cdot) \), the convex function \( \phi \) and the convolution kernel \( \vartheta := \eta^\varepsilon \) to conclude

\[
\int_{\mathbb{R}} \phi(v^\varepsilon(t,y)) \rho^\varepsilon(t,dy) \leq \int_{\mathbb{R}} \phi(v(t,y)) \rho(t,dy) \text{ for a.e. } t \in (0,T),
\]

which implies

\[
\int_0^T \int_{\mathbb{R}} \phi(v^\varepsilon(t,y)) \rho^\varepsilon(t,dy) \, dt \leq \int_0^T \int_{\mathbb{R}} \phi(v(t,y)) \rho(t,dy) \, dt \text{ for all } \varepsilon > 0.
\]

This is equivalent to

\[
\int_0^T \int_{\mathbb{R}} \phi(v^\varepsilon(t,M^\varepsilon(t,x))) \, dx \, dt \leq \int_0^T \int_{\mathbb{R}} \phi(v(t,M(t,x))) \, dx \, dt < \infty,
\]

(2.11)

where \( M^\varepsilon \) is the family of optimal maps between \( \chi \) and the measures of \( \rho^\varepsilon \). Since \( \rho^\varepsilon \in AC^1(0,T;\mathcal{P}_1^r(\mathbb{R})) \) with integrable velocity \( v^\varepsilon \), we deduce from Proposition 2.4 that \( M^\varepsilon \in W^{1,1}(0,T;L^1(I)) \) and \( M^\varepsilon(t,x) = v^\varepsilon(t,M^\varepsilon(t,x)) \) for a.e. \((t,x) \in (0,T) \times I \). Therefore, the last displayed inequality implies

\[
\sup_{\varepsilon > 0} \int_0^T \int_I \phi(M^\varepsilon(t,x)) \, dx \, dt < \infty,
\]

which, due to the super-linear growth of \( \phi \), implies that the family of functions \( \{M^\varepsilon\}_{\varepsilon > 0} \) is sequentially weakly relatively compact in \( L^1((0,T) \times I) \). Therefore, it is enough to show that some sequence \( \{M^{\varepsilon_n}\}_n \subset \{M^\varepsilon\}_{\varepsilon > 0} \) converges weakly in \( L^1((0,T) \times I) \) to \( M \) as \( \varepsilon_n \to 0^+ \) in order to conclude that \( M \in W^{1,1}(0,T;L^1(I)) \).
This follows easily from the convergence of \( \rho^\varepsilon(t, \cdot) \) to \( \rho(t, \cdot) \) in the 1-Wasserstein distance (2.10), i.e. for all \( t \in [0, T] \) we have
\[
\| M^\varepsilon(t, \cdot) - M(t, \cdot) \|_{L^1(I)} = W_1(\rho^\varepsilon(t, \cdot), \rho(t, \cdot)) \to 0 \quad \text{as} \quad \varepsilon \to 0^+.
\]
The equiboundedness (both in \( t \) and \( \varepsilon \)) of the first moments of \( \rho^\varepsilon(t, \cdot) \) and \( \rho(t, \cdot) \) enables us to use Dominated Converges to infer that, in fact,
\[
\lim_{\varepsilon \to 0^+} \| M^\varepsilon - M \|_{L^1((0,T) \times I)} = 0. \tag{2.12}
\]
This concludes the proof. \( \square \)

**Remark 2.7.** 1. The previous two theorems show that the integrable velocity \( v \) of a curve \( \rho \in AC^1(0, T; \mathcal{P}_1(\mathbb{R})) \) exists if and only if the corresponding family of optimal maps \( M \) belongs to \( W^{1,1}(0, T; L^1(I)) \). In this case we have \( M(t, x) = v(t, M(t, x)) \) for \( \mathcal{L}^2 \)-a.e. \((t, x) \in (0, T) \times I \).

2. Referring to the proof of Theorem 2.6: note that since the weak \( L^1 \)-limit of any convergent sequence extracted from \( \{v^\varepsilon(\cdot, M\varepsilon(\cdot, \cdot))\} \}_{\varepsilon > 0} \) for \( \varepsilon_n \to 0^+ \) must, according to Theorem 2.2, be \( v(\cdot, M(\cdot, \cdot)) \), we deduce that
\[
v^\varepsilon(\cdot, M\varepsilon(\cdot, \cdot)) \rightharpoonup v(\cdot, M(\cdot, \cdot)) \quad \text{weakly in} \quad L^1((0,T) \times I) \quad \text{as} \quad \varepsilon \to 0^+. \tag{2.13}
\]
In fact, in light of (2.11) and (2.13), we conclude (see, e.g. [6], [18])
\[
v^\varepsilon(\cdot, M\varepsilon(\cdot, \cdot)) \rightrightarrows v(\cdot, M(\cdot, \cdot)) \quad \text{strongly in} \quad L^1((0,T) \times I) \quad \text{as} \quad \varepsilon \to 0^+. \tag{2.14}
\]
Thus, the Lagrangian description of \( \rho \) consisting of optimal maps is strongly (in \( L^1 \)) stable up to first order (i.e. both \( M \) and \( \dot{M} \)) with respect to mollification.

Now let \( X_0 : I \to \mathbb{R} \) be a Borel map, and denote \( \rho_0 := X_0 \# \chi \). If \( M_0 : I \to \mathbb{R} \) is the optimal map pushing \( \chi \) forward to \( \rho_0 \), then there exists a Borel measure preserving map \( g_0 \) (i.e. \( g_0 \# \chi = \chi \)) such that \( X_0 = M_0 \circ g_0 \) (see [15]). This is called a **polar decomposition** [5] of \( X_0 \). According to [15], if \( d = 1 \) then the polar factorization holds for any Borel map \( X_0 \); no non-degeneracy restriction on the measure \( \rho_0 \) is necessary. The proposition below is a generalization of Corollary 2.12 in [4], in the sense that now \( \rho(t, \cdot) \) (for \( t = 0 \) and at later times) may have atoms.

**Corollary 2.8.** Let \( \rho \in AC^q(0, T; \mathcal{P}_q(\mathbb{R})) \) for some \( 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \). If \( p = 1 \) assume also that the \( L^1 \)-velocity of \( \rho \) exists. Let \( X_0 : I \to \mathbb{R} \) be any Borel map such that \( X_0 \# \chi = \rho_0 \), and let \( X_0 = M_0 \circ g_0 \) be a polar factorization of \( X_0 \) (\( M_0 \) is monotone nondecreasing and \( g_0 \# \chi = \chi \)). Then there exists a flow map \( X \in W^{1,q}(0, T; L^p(I)) \) associated to \( \rho \) that starts at \( X_0 \). More precisely, \( X \) can be chosen such that \( X(t, \cdot) = M(t, \cdot) \circ g_0 \) for all \( t \in [0, T] \), where \( M \) is the family of optimal maps associated to \( \chi \) and \( \rho \).

**Proof.** There exists a Borel set \( A \subset I \) such that \( \chi(A) = 1 \) and
\[
M(t, z) = M_0(z) + \int_0^t v(s, M(s, z))ds \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad z \in A.
\]
But \( 1 = \chi(A) = \chi(g_0^{-1}(A)) \) due to \( g_0 \# \chi = \chi \), so \( g_0(x) \in A \) for \( \mathcal{L}_1 \)-a.e. \( x \in I \). Thus, \( M(t, g_0(x)) \)
\[
= M_0(g_0(x)) + \int_0^t v(s, M(s, g_0(x)))ds \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad \mathcal{L}_1 \text{-a.e.} \quad x \in A.
\]
Thus, \( X_t := M_t \circ g_0 \) satisfies **(Flow)** with \( X(0, \cdot) = X_0 \). \( \square \)
Corollary 2.9. Let \( \rho \in AC^1(0, T; P_1(\mathbb{R})) \). Then \((\text{Trans})\) is satisfied for some \( v \in L^1(\rho) \) if and only if \( v \) is the integrable velocity associated to \( \rho \).

**Proof.** The converse implication is trivial, as it only takes differentiating \((\text{Trans})\) with respect to \( y \) in the sense of distributions. As far as the direct implication is concerned, we will use the same approximations for \( \rho \) as in the proof of Proposition 2.4 and let us recall from said proof that \( \partial_t F^\varepsilon(t, y) = -E^\varepsilon(t, y) \), which implies

\[
\int_0^T \int_{\mathbb{R}} \partial_t \varphi(t, y) F^\varepsilon(t, y) \, dy \, dt = \int_0^T \int_{\mathbb{R}} E^\varepsilon(t, y) \varphi(t, y) \, dy \, dt
\]

for all \( \varphi \in C^1_c((0, T) \times \mathbb{R}) \). But if we set \( F(t, y) := \rho(t, (\infty, y)) \) (the cumulative distribution function of \( \rho(t, \cdot) \)), then \((2.12)\) also means (see, e.g., [17])

\[
\|F^\varepsilon - F\|_{L^1((0, T) \times \mathbb{R})} = \int_0^T W_1(\rho^\varepsilon(t, \cdot), \rho(t, \cdot)) \, dt \to 0 \text{ as } \varepsilon \to 0^+,
\]

so we can use the fact that \( E^\varepsilon \) converges narrowly to \( v \rho \) (first as functions of \( y \) for fixed \( t \) [3], then in both variables by Dominated Convergence due to \( \|E^\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|v(t, \cdot)\|_{L^1(Y(t, \cdot))} \) for a.e. \( t \in (0, T) [3] \)) to pass to the limit and get the distributional form of \((\text{Trans})\).

\[\Box\]

3. Applications.

3.1. Monotone rearrangements of maps in \( W^{1,1}(0, T; L^1(I)) \). Let \( Q \) denote the open unit cube in \( \mathbb{R}^d \).

**Theorem 3.1.** Let \( X \in W^{1,1}(0, T; L^1(Q; \mathbb{R}^d)) \) and denote \( \rho(t, \cdot) := X(t, \cdot)_# L^d | Q \) for \( L^1 \)-a.e. \( t \in [0, T] \). Then \( \rho \in AC^1(0, T; P_1(\mathbb{R}^d)) \) and it admits an integrable velocity.

**Proof.** We consider the Borel map \( Y : [0, T] \times Q \to [0, T] \times \mathbb{R}^d \) given by \( Y(t, x) := (t, X(t, x)) \). Since \( X \in L^1((0, T) \times Q; \mathbb{R}^d) \), let \( \mu^\pm \) be the Borel vector measures of finite total variation whose densities with respect to the Lebesgue measure \( L^{1+d} \) are \( X^\pm \), respectively. Let \( \nu^\pm = Y_# \mu^\pm \). Take \( B \subset [0, T] \times \mathbb{R}^d \) a Borel set such that \( \rho(B) = 0 \); since \( X(t, \cdot)_# \chi = \rho(t, \cdot) \) for \( L^1 \)-a.e. \( t \in [0, T] \), we have

\[
\int_0^T \int_Q 1_B \circ Y(t, x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^d} 1_B(t, y) \rho(t, dy) \, dt = 0,
\]

which implies \( 1_B \circ Y(t, x) = 0 \) for \( L^{1+d} \)-a.e. \( (t, x) \in [0, T] \times Q \). Thus,

\[
0 = \int_0^T \int_Q 1_B \circ Y(t, x) \dot{X}^\pm(t, x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^d} 1_B(t, y) \nu^\pm(dt, dy),
\]

which means (due to the arbitrariness of \( B \) as a negligible set with respect to \( \rho \)) \( \nu^\pm \ll \rho \). By the Radon-Nikodym Theorem, we get nonnegative Borel measurable functions \( v^\pm : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) such that

\[
\int_0^T \int_Q \zeta(t, X(t, x)) \cdot \dot{X}^\pm(t, x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^d} \zeta(t, y) \cdot v^\pm(t, y) \rho(t, dy) \, dt \quad (3.1)
\]
for all $\zeta \in C_b([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$. Since $\tilde{X} \in L^1((0, T) \times Q; \mathbb{R}^d)$, we deduce $v := v^+ - v^- \in L^1(\rho; \mathbb{R}^d)$ and
\[
\int_0^T \int_Q \zeta(t, X(t, x)) \cdot \tilde{X}(t, x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^d} \zeta(t, y) \cdot v(t, dy) \, dt
\]
for all $\zeta \in C_b([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$. In particular, if $\zeta \equiv \nabla \varphi$ for some arbitrary $\varphi \in C^1_c((0, T) \times \mathbb{R}^d)$, we see that the above equality is equivalent to
\[
\int_0^T \int_{\mathbb{R}^d} \{\partial_t \varphi(t, y) + v(t, y) \cdot \nabla \varphi(t, y)\} \rho(t, dy) \, dt = 0 \text{ for all } \varphi \in C^1_c((0, T) \times \mathbb{R}^d),
\]
which, along with the integrability of $v$ with respect to $\rho$, yields the desired conclusion.

Three important consequences arise, the first in view of Theorem 2.6:

**Corollary 3.2.** The integrable velocity $v$ of a curve $\rho \in AC^1(0, T; \mathcal{P}_1(\mathbb{R}))$ exists if and only if there exists a family of maps $X \in W^{1,1}(0, T; L^1(I))$ such that $X(t, \cdot) \# \chi = \rho(t, \cdot)$ for $\mathcal{L}^1$-a.e. $t \in [0, T]$. In this case, the time-dependent family of monotone rearrangements $M(t, \cdot)$ of $X(t, \cdot)$ satisfies $M \in W^{1,1}(0, T; L^1(I))$ and
\[
M(t, x) = v(t, M(t, x)) \text{ for } \mathcal{L}^2 \text{-a.e. } (t, x) \in (0, T) \times I.
\]

The second is just a reformulation of a subset of the statements above; however, in this form it displays a much stronger conclusion than Loeper’s from [13] in the case $d = 1$.

**Corollary 3.3.** If $X \in W^{1,q}(0, T; L^p(I))$ for some $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then the time-dependent family of monotone rearrangements $M(t, \cdot)$ of $X(t, \cdot)$ also satisfies $M \in W^{1,q}(0, T; L^p(I))$.

Finally, since the monotone rearrangement of any measure preserving map is the identity, we can use Corollary 3.2 to conclude:

**Corollary 3.4.** If $g \in W^{1,1}(0, T; L^1(I))$ is a family of measure-preserving maps (i.e. $g(t, \cdot) \# \chi = \chi$ for all $t \in [0, T]$) and $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a Borel map such that $\dot{g}(t, x) = v(t, g(t, x))$ a.e. in $[0, T] \times I$, then $g$ is time-independent and $v \equiv 0$.

### 3.2. Action minimization and Hamilton-Jacobi equations in the Wasserstein space

On the “tangent bundle” to $\mathcal{P}_p(\mathbb{R})$, namely
\[
\bigcup_{\mu \in \mathcal{P}_p(\mathbb{R})} \{\mu\} \times L^p(\mu)
\]
we define a Lagrangian map
\[
\tilde{L}(\mu, \xi) = \Psi(\|\xi\|_{L^p(\mu)}) - \mathcal{F}(\mu),
\]
where $\Psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function.

On the set of curves in $AC^0(0, T; \mathcal{P}_p(\mathbb{R}))$ which admit an integrable velocity we define the action
\[
\mathcal{A}[T; \rho] = \int_0^T \tilde{L}(\rho(t, \cdot), v(t, \cdot)) \, dt
\]
under the assumption
\[
\Psi(s) \leq C(1 + s^q) \text{ for all } s \geq 0.
\]
Define
\[
L(X, V) = \Psi(\|V\|_{L^p(X_\# \chi)}) - \mathcal{F}(X)
\]
for all $X \in L^p(I)$ and $V \in L^p(X_\# \chi)$, where $F(X) := \tilde{F}(X_\# \chi)$. We now know that the existence of the integrable velocity renders $\rho \in AC^1(0, T; P_p(\mathbb{R}))$ equivalent to $M \in W^{1,q}(0, T; L^p(I))$; in fact, by Corollary 3.2 there is a one-to-one and onto correspondence between curves in $AC^1(0, T; P_p(\mathbb{R}))$ with integrable velocities and $W^{1,q}(0, T; L^p(I))$, where the subscript $\uparrow$ indicates that maps $M$ in this set satisfy $M(t, \cdot)$ is nondecreasing for all $t \in [0, T]$. Let $X \in W^{1,q}(0, T; L^p(I))$ and $\rho(t, \cdot) = X(t, \cdot)_\# \chi$. Denote by $M(t, \cdot)$ the monotone rearrangement of $X(t, \cdot)$. Then $[X(s, \cdot), X(t, \cdot)]_\# \chi$ is a transport plan between $\rho(s, \cdot)$ and $\rho(t, \cdot)$, which implies

$$W_p(\rho(s, \cdot), \rho(t, \cdot)) \leq \|X(s, \cdot) - X(t, \cdot)\|_{L^p(I)}$$

for all $s, t \in I$. Since

$$\|X(s, \cdot) - X(t, \cdot)\|_{L^p(I)} \leq \left| \int_s^t \|\dot{X}(\tau, \cdot)\|_{L^p(I)} d\tau \right|$$

for all $s, t \in I$, we deduce

$$\|\dot{M}(t, \cdot)\|_{L^p(I)} \leq \|\dot{X}(t, \cdot)\|_{L^p(I)}$$

for a.e. $t \in [0, T]$. It follows that

$$A[T; \rho] = \int_0^T L(M(t, \cdot), \dot{M}(t, \cdot)) dt \leq \int_0^T L(X(t, \cdot), \dot{X}(t, \cdot)) dt.$$

Thus, the constrained problem of minimizing $A[T; \cdot]$ over $W^{1,q}_+(0, T; L^p(I))$ is equivalent to the (unconstrained) problem of minimizing $A[T; \cdot]$ over $W^{1,q}(0, T; L^p(I))$. Of course, one can take $U_0 : L^p(I) \to \mathbb{R}$ rearrangement invariant (meaning $U_0(X)$ only depends on the measure $X_\# \chi$, not on $X$ itself), fix $M_0 \in L^p(I)$ monotone non-decreasing, $g_0 : I \to I$ measure preserving such that $X_0 = M_0 \circ g_0$ and go further to solving for the value function

$$U(t, M) := \sup \left\{ U_0(Y(t, \cdot)) - \int_0^t L(Y(s, \cdot), \dot{Y}(s, \cdot)) ds : Y \in W^{1,q}(0, T; L^p(I)), Y(0, \cdot) = M_0 \right\}$$

(3.2)

in $W^{1,q}_+(0, T; L^p(I))$. If $M(t, \cdot)$ is a maximizer above, then $M(t, \cdot) \circ g_0$ is a maximizer for

$$\sup \left\{ U_0(X(t, \cdot)) - \int_0^t L(X(s, \cdot), \dot{X}(s, \cdot)) ds : X \in W^{1,q}(0, T; L^p(I)), X(0, \cdot) = X_0 \right\}$$

(3.3)

over $W^{1,q}(0, T; L^p(I))$. These observations help to considerably simplify the analysis of Hamilton-Jacobi equations in the Wasserstein space performed in [11]. Indeed, we see that the constrained (because the maps $M(t, \cdot)$ should be nondecreasing) problem (3.2) is equivalent to the non-constrained one (where the monotonicity constraint on $M(t, \cdot)$ is dropped).

4. Open problems. Of course, Theorem 3.1 only shows that some integrable velocity $v$ (it may not be unique if $d > 1$) exists if a map $X$ exists as indicated. It does not provide a Lagrangian description of the curve as a flow of $v$, i.e. it does not ensure the existence of a map $Y \in W^{1,1}(0, T; L^1(Q; \mathbb{R}^d))$ such that $Y(t, \cdot)_\# \mathcal{L}^d|_Q = X(t, \cdot)_\# \mathcal{L}^d|_Q$ and $\dot{Y}(t, x) = v(t, Y(t, x))$ for $\mathcal{L}^{1+d}$-a.e $(t, x) \in (0, T) \times Q$. According to Theorem 2.2 and Theorem 2.6, if $d = 1$ all this is achieved by taking $Y(t, \cdot) \equiv M(t, \cdot)$ the monotone rearrangement of $X(t, \cdot)$.
Leaving aside the Lagrangian description, if $d > 1$ it is not even clear whether the existence of an integrable velocity $v$ along a curve $\rho \in AC^1(0,T;P_1(\mathbb{R}^d))$ ensures the existence of a map $Y \in W^{1,1}(0,T;L^1(Q;\mathbb{R}^d))$ such that $Y(t,\cdot)\#L^d|_{Q}=\rho(t,\cdot)$ for all $t \in [0,T]$. However, since Lemma 2.5 applies to arbitrary dimensions, a careful inspection of the proof of Theorem 2.6 reveals that the flow maps $X$ associated to the regularized velocities $v^\varepsilon$ have subsequences which converge weakly in $W^{1,1}(0,T;L^1(Q;\mathbb{R}^d))$ to some $X \in W^{1,1}(0,T;L^1(Q;\mathbb{R}^d))$. This weak convergence is insufficient to conclude that $X(t,\cdot)\#L^d|_{Q}=\rho(t,\cdot)$. We do not expect we can do better than that if $d > 1$; however, this may work under some restrictions on $\rho$ (similar to the ones imposed in [16]). We plan to address the multi-dimensional case in the near future.

Acknowledgments. Adrian Tudorascu acknowledges the partial support provided by the NSF Grant DMS–16 00 272, and the Simons Foundation award #246063. The author would like to thank the anonymous referee for the suggestions to improve the presentation.

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Received June 2018; revised March 2019.

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