SOME EXAMPLES OF CONTINUOUS IMAGES OF RADON-NIKODÝM COMPACT SPACES

ALEXANDER D. ARVANITAKIS, ANTONIO AVILÉS

Abstract. We provide a characterization of continuous images of Radon-Nikodým compacta lying in a product of real lines and model on it a method for constructing natural examples of such continuous images.

1. Introduction

This note contains some ideas to address the well known open problem whether the class of Radon-Nikodým (or briefly RN) compact spaces is stable under continuous images. In particular, we provide a method to construct a number of concrete compact spaces which seem to be natural candidates to be counterexamples, but we are unable to decide whether they are Radon-Nikodým compact or not. We think that the understanding of these examples could be a useful tool towards the solution of the problem.

2. Basic definitions and notations

We denote by \( \mathbb{N} = \mathbb{N}^\infty \) the set of all sequences of natural numbers, and we consider that \( 0 \in \mathbb{N} \). If \( s \in \mathbb{N}^{<\infty} \) is a finite sequence of natural numbers and \( \sigma \in \mathbb{N} \), then \( s < \sigma \) means that \( s \) is an initial segment of \( \sigma \). We call \( \mathcal{N}_s = \{ \sigma \in \mathbb{N} : s < \sigma \} \).

If \( K \) is a topological space, a map \( d : K \times K \to \mathbb{R} \) is said to \( \varepsilon \)-fragment \( K \) if for every (closed) set \( L \subset K \) there exists a nonempty relative open subset \( U \) of \( L \) of \( d \)-diameter less than \( \varepsilon \), that is, \( \sup\{d(x, y) : x, y \in U\} < \varepsilon \). If \( d \) \( \varepsilon \)-fragments \( K \) for every \( \varepsilon \) we shall say that \( d \) fragments \( K \).

Given a bounded family \( \Delta \) of continuous functions over \( K \) we shall denote by \( d_\Delta \) the uniform pseudometric over \( \Delta \), that is \( d_\Delta(x, y) = \sup\{|f(x) - f(y)| : f \in \Delta\} \). When we view a compactum lying as a subset of a product of real lines, \( K \subset \mathbb{R}^\Gamma \), we shall identify every element \( \gamma \in \Gamma \) with the continuous function on \( K \) given by projection onto the \( \gamma \)-th coordinate. Thus, for a given \( \Delta \subset \Gamma \), \( d_\Delta(x, y) = \sup\{|x_\gamma - y_\gamma| : \gamma \in \Delta\} \).

Originally, RN compact spaces are defined as weak* compact subsets of dual Banach spaces with the Radon-Nikodým property. From an intrinsic topological point of view, they can be characterized as those compact spaces \( K \) for which there exists a lower semicontinuous metric \( d : K \times K \to \mathbb{R} \) which fragments \( K \). The

2000 Mathematics Subject Classification. 46B26, 54G12.

Key words and phrases. Radon-Nikodým compact.

The second author was supported by a Marie Curie Intra-European Fellowship MCEIF-CT2006-038768 and research projects MTM2005-08379 (MEC and FEDER) and Séneca 00690/P1/04.
class of quasi Radon-Nikodým (qRN) compact spaces is a superclass of the class
of RN compacta which is stable under continuous images. According to the defi-
nition of the first author [1], a compact space is qRN compact if there exists a lower
semicontinuous map $f : K \times K \to \mathbb{R}^+$ which is nonzero out of the diagonal and
which fragments $K$. This is actually equivalent to other definitions given by Fabian,
Heiler and Matoušková [6] and Reznichenko, as it is shown in [9] and [2]. We refer
to the recent survey of Fabian [5] on this subject, as well as other articles containing
different approaches or partial results concerning the problem of continuous images
as [10], [7] or [3].

In this note, we will work with compact spaces viewed as closed subsets of Ty-
chonoff cubes $[-1, 1]^\Gamma$. We present below the two known characterizations of both
RN and qRN compact spaces in terms of these embeddings. Theorem 1 can be
found in [8]. With respect to Theorem 2, the equivalence of (2) and (3) is proven
in [6], and the equivalence of (1) with the others in [2].

**Theorem 1.** A compact space $K$ is RN compact if and only if there exists an em-
bedding of $K$ into a product of intervals $K \subseteq [-1, 1]^\Gamma$ such that $d_{\Gamma}$ fragments $K$.

**Theorem 2.** For a compact space $K$ the following are equivalent:

1. $K$ is quasi RN compact.
2. There exists an embedding of $K$ into a product of intervals $K \subseteq [-1, 1]^\Gamma$
   and a map $u : \Gamma \to \mathbb{N}$ such that for every finite sequence $s$ of naturals of
   length $n > 0$ the pseudometric $d_{u^{-1}(\mathbb{N})} \frac{1}{2^n}$ fragments $K$.
3. For any embedding of $K$ into a product of intervals $K \subseteq [-1, 1]^\Gamma$ there
   exists a map $u : \Gamma \to \mathbb{N}$ such that for every finite sequence $s$ of naturals of
   length $n > 0$ the pseudometric $d_{u^{-1}(\mathbb{N})} \frac{1}{2^n}$ fragments $K$.

The picture is that we have three classes of compact sets, RN compacta, their
continuous images (ciRN) and quasi RN (qRN) compacta,

$$RN \subseteq ciRN \subset qRN$$

and we do not know at all whether any of the inclusions is strict.

A key conceptual difference between RN and qRN is that there is no analogue
of condition (3) in Theorem 2 for RN, referring to any embedding of $K$ into a cube.
This makes the class qRN easier to handle than RN in many aspects: in order
to check that some space $K$ is qRN compact we take any embedding of $K$ and we try
to see whether there is function $u : \Gamma \to \mathbb{N}$ fulfilling the mentioned condition (3).
However, if we want to check that a certain space is RN compact we must find an
appropriate embedding, probably different from the obvious ones. Such a difficulty
is found, of course, in the problem of the continuous image.

3. A characterization of continuous images of RN compacta

The main result of this note, and the inspiration for the announced concrete
examples, will be Theorem 5 where we give a similar characterization as those
appearing in Theorems 1 and 2 for the intermediate class ciRN of the continuous images of Radon-Nikodym compacta.

**Definition 3.** A family $\Delta \subset C(K, [-1, 1])$ of continuous functions over $K$ is said to be a Namioka family if the pseudometric $d_{\Delta}$ fragments $K$.

Using this concept, Theorem 1 can be restated saying that a compact space is RN if and only if there exists a Namioka family $\Delta \subset C(K)$ which separates the points of $K$. We shall need the fact that if a Namioka family exists, then indeed it can be chosen to be much bigger than simply a separating family.

**Lemma 4.** Let $K$ be an RN compactum. Then there exists a Namioka family $\Delta$ such that $C(K) = \bigcup_{n \in \mathbb{N}} n\Delta^{\|\cdot\|}$.

**Proof:** We consider $\Delta_0$ some Namioka family over $K$ and then we define $\Delta$ to be the set of all $f \in C(K)$ such that $|f(x) - f(y)| \leq d_{\Delta_0}(x, y)$ for all $x, y \in K$. Clearly $d_{\Delta} = d_{\Delta_0}$ so $\Delta$ is a Namioka family. On the other hand, $\bigcup_{n \in \mathbb{N}} n\Delta$ is a linear lattice of functions which separates the points of $K$, so by the Stone-Weierstrass Theorem, it is uniformly dense in $C(K)$. $\square$

**Theorem 5.** For a compact space $K \subset [a, b]^\Gamma$ the following are equivalent:

1. $K$ is the continuous image of an RN compactum.
2. There exists a function $u : \Gamma \rightarrow \mathbb{N}$ and a family of compact sets $K_s \subset [a, b]^\Gamma$ for $s \in \mathbb{N}^{<\omega}$ such that:
   a. $d_{\Gamma}^{-1}(\mathbb{N}_s)$ fragments $K_s$ for every $s$, and
   b. If $\text{length}(s) = n > 0$, then for every $x \in K$ there exists $y \in K_s$ such that $d_{\Gamma}^{-1}(\mathbb{N}_s)(x, y) \leq \frac{1}{2^n}$.

Of course, the use of the numbers $\frac{1}{2^n}$ is inessential, we could have used any numbers $\varepsilon_n$ converging to 0. We give an equivalent reformulation of Theorem 5 which will be more suitable for the purpose of making the proof more transparent, though it will be the previous statement which we will be more relevant in further discussion.

**Theorem 5(b).** For a compact space $K \subset [a, b]^\Gamma$ the following are equivalent:

1. $K$ is a continuous image of an RN compactum.
2. For every $\varepsilon > 0$ there exists a countable decomposition $\Gamma = \bigcup_m \Gamma_m$ and compact sets $K_m^\varepsilon \subset [a, b]^\Gamma$ such that:
   a. $K_m^\varepsilon$ is fragmented by $d_{\Gamma_m}$
   b. For every $m \in \mathbb{N}$ and $x \in K$ there is $y \in K_m^\varepsilon$ such that $d_{\Gamma_m}(x, y) \leq \varepsilon$.

Before passing to the proof, we state a lemma from [1] that we will often use.
Lemma 6. Let \( f : Q \to S \) be a continuous surjection between compact spaces, and \( d : Q \times Q \to \mathbb{R} \) a lower semicontinuous map that fragments \( Q \). Then the map \( \hat{d}(x,y) = \inf \{d(u,v) : f(u) = x, f(v) = y\} \) fragments \( S \).

Proof: Theorem \( \text{[5]} \) is easily seen to be equivalent to Theorem \( \text{[5]}(\text{b}) \), we leave this to the reader. We shall prove Theorem \( \text{[5]}(\text{b}) \). Without loss of generality, we suppose that \([a, b] = [-2, 2]\).

\((1) \Rightarrow (2)\): Let \( L \) be RN compact and \( \pi : L \to K \subset [-2, 2]^{\Gamma} \) a continuous surjection. We take a Namioka family \( \Delta \) on \( L \) as in Lemma \( \text{[4]} \) so that we view \( L \subset [-1, 1]^{\Delta} \). We fix \( \varepsilon \in (0, 1) \). For every \( \gamma \in \Gamma \) we have a continuous function on \( L \) given by \( z \mapsto \pi(z)_{\gamma} \), so since \( C(L) = \bigcup_{m \in \mathbb{N}} m\Delta^{\|} \) we have that:

\[
\text{for every } \gamma \in \Gamma \text{ there exists } m(\gamma) \in \mathbb{N} \text{ and } \delta(\gamma) \in \Delta \text{ such for every } z \in L, |\pi(z)_{\gamma} - m(\gamma)z_{\delta(\gamma)}| \leq \varepsilon.
\]

We set
\[
\Gamma_{m} = \{ \gamma : m(\gamma) = m \}
\]
\[
K_{m} = \{ x \in [-2, 2]^\Gamma : \exists z \in L : mz_{\delta(\gamma)} = x, \forall \gamma \in \Gamma_{m} \}
\]

Each \( K_{m} \) is compact, because it is the continuous image of \( L \) under the map \( z \mapsto (mz_{\delta(\gamma)})_{\gamma \in \Gamma} \).

We check condition (b). If \( x \in K \), then \( x = \pi(z) \) for some \( z \in L \), and we know that \( |x_{\gamma} - m(\gamma)z_{\delta(\gamma)}| \leq \varepsilon \) for every \( \gamma \in \Gamma \). We define \( y \) to be an element of \([-2, 2]^{\Gamma} \) such that \( y_{\gamma} = mz_{\delta(\gamma)} \) for \( \gamma \in \Gamma_{m} \). Then \( y \in K_{m} \) and \( d_{\Gamma_{m}}(x, y) \leq \varepsilon \).

For condition (a), We consider \( K_{m} \subset [-2, 2]^{\Gamma_{m}} \) the projection of \( K_{m} \) to the coordinates of \( \Gamma_{m} \). We prove that \( d_{\Gamma_{m}} \) fragments \( K_{m} \) (this is equivalent to say that \( d_{\Gamma_{m}} \) fragments \( K_{m} \)). We have a continuous surjection \( \phi : L \to K_{m} \) given by \( \phi(z) = (mz_{\delta(\gamma)})_{\gamma \in \Gamma_{m}} \). Observe that \( d_{\Gamma_{m}}(\phi(z), \phi(z')) = m\sup_{\gamma \in \Gamma_{m}} |z_{\delta(\gamma)} - z'_{\delta(\gamma)}| \leq md_{\Delta}(z, z') \), so
\[
d_{\Gamma_{m}}(x, x') \leq m \inf \{d_{\Delta}(z, z') : \phi(z) = x, \phi(z') = x'\}.
\]

Since \( d_{\Delta} \) fragments \( L \), the conclusion follows from Lemma \( \text{[6]} \).

\((2) \Rightarrow (1)\): Let us call \( \varepsilon_{n} = 2^{-n} \). Without loss of generality, we assume that \( \Gamma_{m} \cap \Gamma_{m'} = \emptyset \) for \( m \neq m' \). For \( \gamma \in \Gamma \) and \( n \in \mathbb{N} \), call \( m_{n}(\gamma) \) to be the only \( m \) such that \( \gamma \in \Gamma_{m_{n}} \).

We consider again \( K_{m} \subset [a, b]^{\Gamma_{m}} \) the projection of \( K_{m} \) to the coordinates \( \Gamma_{m} \). The metric \( d_{\Gamma_{m}} \) fragments \( K_{m} \) so this is an RN compactum.

Let \( L \subset \prod_{n \in \mathbb{N}} \prod_{m \in \mathbb{N}} K_{m}^{\gamma} \) be the set consisting of all \( x \) for which there exists \( g(x) \in [a, b]^{\Gamma} \) such that
\[
d_{\Gamma_{m}}(g(x), p_{mn}(x)) \leq \varepsilon_{n}
\]
where \( p_{mn}(x) \in K^\varepsilon_m \) is the coordinate of \( x \) in the factor \( K^\varepsilon_m \).

Notice that this element \( g(x) \) is uniquely determined by \( x \), since \( p_{mn}(\gamma_n(x)) \rightarrow g(x) \) as \( n \rightarrow \infty \). The space \( L \) is compact and the map \( g : L \rightarrow [a, b]^{\mathcal{N}} \) is continuous (these two facts are easily checked considering net convergence). Moreover, \( g(L) \supseteq K \) (this follows from part (b) of condition (2) in the theorem). Since the class RN is closed under the operations of taking countable products and closed subspaces, \( L \) is RN compact and \( K \) is a continuous image of an RN compactum. □

4. A WAY TO CONSTRUCT CONTINUOUS IMAGES OF RN COMPACTA

We think now of a continuous image of an RN compactum as a compact space \( K \) that satisfies part (2) of Theorem 5. For simplicity, we shall assume that \([a, b] = [0, 1], \mathcal{N} = \mathcal{N} \) and \( u : \mathcal{N} \rightarrow \mathcal{N} \) is just the identity map. We can think of Theorem 5 as giving a constructive process to produce continuous images of RN compacta in the following way:

**Step 1.** Begin with a family \( \{K_s : s \in \mathbb{N}^{<\omega}\} \) of compact subsets of \([0, 1]^{\mathcal{N}}\) such that \( d_{\mathcal{N}s} \) fragments \( K_s \).

**Step 2.** For every \( s \) of length \( n \), consider the \( \frac{1}{2^n} \mathcal{N}s \)-enlargement of \( K_s \),
\[
[K_s] = \{x \in [0, 1]^{\mathcal{N}} : \exists y \in K_s \ d_{\mathcal{N}s}(x, y) \leq \frac{1}{2^n}\},
\]
and some compact set \( L_s \subset [K_s] \).

**Step 3.** Finally, \( K = \bigcap_{s \in \mathbb{N}^{<\omega}} L_s \) is a continuous image of an RN compactum.

This would be the general procedure but still it does not seem that much constructive: How can we get compacta \( K_s \) for step 1? And how can we get the sets \( L_s \) of step 2? Well, some canonical choices can help us in this task:

For step 1 we can begin with an RN compactum \( L \subset [0, 1]^{\mathcal{N}} \) which is fragmented by \( d_{\mathcal{N}} \) and make \( K_s = L \) for all \( s \). We know plenty of concrete examples of such objects as we shall describe later. For step 2, simply observe that the sets \( [K_s] \) described above are themselves compact, so we can take \( L_s = [K_s] \), in our case
\[
L_s = \{x \in [0, 1]^{\mathcal{N}} : \exists y \in L : d_{\mathcal{N}s}(x, y) \leq \frac{1}{2^n}\}.
\]

After this we shall call the resulting ciRN compactum \( \bar{L} = \bigcap_{s \in \mathbb{N}^{<\omega}} L_s \subset [0, 1]^{\mathcal{N}} \). Notice that it only depends on the RN compactum \( L \subset [0, 1]^{\mathcal{N}} \) that we took as starting point of our construction. It follows from the proof of Theorem 5 that \( \bar{L} \) is indeed a continuous image of some closed subset of the countable power \( L^{\mathcal{N}} \).

A trivial example of compact space \( L \subset [0, 1]^{\mathcal{N}} \) which is fragmented by \( d_{\mathcal{N}} \) is a scattered compactum. Recall that a compactum is scattered if every nonempty subset contains an isolated point. Scattered compacta are indeed fragmented by
any metric, even the discrete metric. Scattered compacta, being totally disconnected, are typically found as compact subsets of the Cantor cube $L \subset \{0, 1\}^\mathbb{N}$ and in this case, it follows immediately that also $\tilde{L} \subset \{0, 1\}^\mathbb{N}$ is totally disconnected. It is a known fact that continuous images of RN compacta (indeed all $q$RN compacta) which are totally disconnected are RN compacta. Thus, scattered compacta seem not to be good starting points for our procedure if we are looking for candidates to be counterexamples to the problem of the continuous images.

As we mentioned in the previous paragraph, the problem of the continuous images has a positive solution in the case when the image is totally disconnected. So we focus rather on connected compacta, where the problem becomes harder. We shall obtain our connected RN compactum by taking the convex hull of a scattered compactum:

**Proposition 7.** Let $S \subset \{0, 1\}^\mathbb{N}$ be a scattered compactum and let $\overline{\sigma}(S) \subset \{0, 1\}^\mathbb{N}$ be the closure of its convex hull in $[0, 1]^\mathbb{N}$, then $d_N$ fragments $\overline{\sigma}(S)$.

We notice that it is not known in general whether the closed convex hull of an RN compact space is again RN compact (when such an operation makes sense and produces a compact set). This is indeed a particular instance of the problem of the continuous images, cf. [8].

Proof: Let $P(S)$ denote the space of regular Borel probability measures on $S$ endowed with the weak$^*$ topology. Recall that every continuous function $f : S \rightarrow [0, 1]$ induces a continuous function $\hat{f} : P(S) \rightarrow [0, 1]$ given by $\hat{f}(\mu) = \int f(d\mu)$. Applying this fact to every coordinate function over $S$, we find a natural continuous function $g : P(S) \rightarrow [0, 1]^\mathbb{N}$ whose image is precisely $g(P(S)) = \overline{\sigma}(S)$. To get convinced about this latter fact, notice that $g(\delta_s) = s$ for every $s \in S$ ($\delta_s$ denotes the corresponding Dirac measure) and $g$ commutes with convex linear combinations, and recall the well know fact that $P(S) = \overline{\sigma}\{\delta_s : s \in S\}$. Now, we can view $P(S)$ as weak$^*$ compact subset of $C(S)^*$ where $C(S)$ is the space of continuous functions over $S$. Since $S$ is scattered, $C(S)$ is an Asplund space and this implies that any weak$^*$ compact subset of $C(S)^*$, and in particular $P(S)$, is fragmented by the norm of $C(S)^*$ (cf. [4]). We observe the following: for every $x, y \in P(S)$,

$$d_N(g(\mu), g(\nu)) \leq \|\mu - \nu\| = \sup\{|h(\mu) - h(\nu)| : h \in C(S), \|h\| \leq 1\}.$$ 

Hence, the pseudometric $d_N(g(\mu), g(\nu))$ fragments $P(S)$ as well, and making use of Lemma 6 we conclude that $d_N$ fragments $g(P(S)) = \overline{\sigma}(S)$. □

The map $g$ appearing in the proof of Proposition 7 is one-to-one provided that the linear span of the coordinate functions is dense in $C(S)$, and in that case we would have that $\overline{\sigma}(S) = P(S)$. This will happen if for instance the coordinate functions form an algebra of functions over $S$, by the Stone-Weierstrass theorem.

Summarizing, the ciRN compacta that we are proposing are those obtained in the following way:
Step 0. Begin with a scattered compactum $S \subset \{0, 1\}^N$.

Step 1. Consider its closed convex hull $L(S) = \overline{\text{co}}(S) \subset [0, 1]^N$.

Step 2. Finally, take the ciRN compactum $\hat{L}(S) = \bigcap_{s \in N^{<\omega}} L(S)_s = \bigcap_{s \in N^{<\omega}} \{x \in [0, 1]^N : \exists y \in L(S) \ d_N(x, y) \leq \frac{1}{2^n}\}$, where $n$ is the length of $s$.

Problem 8. Let $S \subset \{0, 1\}^N$ be a compact scattered space. Is the space $\hat{L}(S)$ an RN compact?

There are a couple of cases when we know that the answer to this question is positive. One case occurs if the weight of $S$ is less than $b[2]$. This will not interfere if we take $S$ to be of weight the continuum, or at least weight $b$, which on the other hand is the natural choice. The other case is when $S$ is Eberlein compact, which implies that $\hat{L}(S)$ is Eberlein compact as well. We do not know much more out of these two cases, and those which can be obtained by mixing the up.

5. The space of almost increasing functions

We promised concrete examples and we are going to describe a very concrete one in this section. The only variable on which the space $\hat{L}(S)$ depends is on the choice of some scattered compact $S \subset \{0, 1\}^N$, which by the remarks above we should take care not to be Eberlein compact. One natural example of such is an ordinal interval $[0, \alpha]$ endowed with the order topology. The way of viewing a scattered compactum of this type as a subset of $\{0, 1\}^N$ is to fix a well order $\prec$ on $N$ and to declare

$$S = \{x \in \{0, 1\}^N : \forall i \prec j \ x_i \leq x_j\}$$

It is easy to check that the closed convex hull of this is nothing else than

$$\overline{\text{co}}(S) = \{x \in [0, 1]^N : \forall i \prec j \ x_i \leq x_j\}$$

It requires a little bit more work to realize that $\hat{L}(S)$ has also a nice description as the set of “almost increasing functions”. We consider the following distance defined on $N$: $d(\sigma, \tau) = \frac{1}{2^{\min(|\tau|, |\sigma|) - 1}}$.

Theorem 9. In this case, we can describe the space $\hat{L}(S)$ as follows:

$$\hat{L}(S) = \{x \in [0, 1]^N : \forall \sigma \prec \tau \ x_\sigma \leq x_\tau + d(\sigma, \tau)\}$$

Proof: It is enough to observe that for $x \in [0, 1]^N$ and $\varepsilon > 0$, the following two conditions are equivalent:

1. For every $\sigma \prec \tau$, $x_\sigma \leq x_\tau + 2\varepsilon$.
2. There exists $y \in L(S)$ such that $|x_\sigma - y_\sigma| \leq \varepsilon$ for all $\sigma$.

Namely, if condition (1) holds, we can define $y_\sigma = \inf\{1, x_\tau + \varepsilon : \tau \geq \sigma\}$. □
6. RN Quotients of a qRN Compactum

It is not only that we do not know whether every qRN compactum is RN, moreover we do not know the answer to such a question as whether every qRN compactum has an RN quotient of the same weight. It is shown in [2] that every qRN compactum is a subspace of a product of $d$ many RN compact spaces, and also that every qRN compactum of weight less than $b$ is RN compact ($d$ and $b$ denote the domination and bounding cardinal numbers, following the notation in [11]). As corollary one gets:

**Proposition 10.** Let $K$ be a qRN compactum of weight $\kappa$.

1. If $\kappa > d$, then $K$ has an RN quotient of weight $\lambda$ for every $\lambda < \kappa$.
2. If $cf(\kappa) > d$, then $K$ has an RN quotient of weight $\kappa$.
3. Every quotient of $K$ of weight less than $b$ is RN compact.

Observe that the previous statements give no information about a compact space $K$ of weight $c$ under CH. We do not know whether the space of almost increasing functions is RN compact, but at least we can show the following.

**Theorem 11.** Let $K \subset [0,1]^N$ be the compact space of almost increasing functions associated to a well order of $\mathcal{N}$.

1. For every cardinal $\lambda < c$, $K$ has an RN quotient of weight $\lambda$.
2. If $c$ is a regular cardinal, then $K$ has an RN quotient of weight $c$.
3. If the well order $(\prec)$ is chosen so that every infinite $(\prec)$-interval is dense in the Baire space $\mathcal{N}$, then $K$ has an RN quotient of weight $c$.

Before entering the proof, we introduce some auxiliary definitions and results. For $\sigma \in \mathbb{N}^\mathbb{N}$, we call a $\sigma$-good sequence a $\prec$-increasing sequence $\tau^* : \tau^1 \prec \tau^2 \prec \cdots$ such that $d(\sigma, \tau^n) \leq 2^{-n}-2$. For $\tau^i$, $\sigma^i$ two sequences in $\mathcal{N}$ we say $\tau^* \prec \sigma^*$ if $\tau_i^* \prec \sigma_i$ for all $i, j$. A family $Z$ of such sequences is called separated if for any $\zeta^*, \xi^* \in Z$, either $\zeta^* \prec \xi^*$ or $\xi^* \prec \zeta^*$, so that $(\prec)$ is a well order on $Z$.

Given a $(\prec)$-increasing sequence $\tau^*$ in $\mathcal{N}$ we produce a continuous function $\phi[\tau^*] : K \rightarrow [0,1]$ as follows. First, we consider the map $\Phi : [0,1]^{[0,1]} \rightarrow [0,1]^{[0,1]}$ which consists in associating to each sequence $(x^n)$ a sequence $(y^n)$ with the property that $|y^n - y^{n+1}| \leq 2^{-n}$, that we construct recursively: given $y^n$, we choose $y^{n+1}$ to be closest number to $x^{n+1}$ that satisfies $|y^n - y^{n+1}| \leq 2^{-n}$; in a formula:

$$\Phi((x^k)_{k \in \mathbb{N}})_{n+1} = y^{n+1} = x^{n+1} + \text{sign}(y^n - x^{n+1}) \cdot \max(|y^n - x^{n+1}|, 2^{-n}, 0).$$

Notice that $\Phi$ is continuous because this recursive formula is continuous on $x_{n+1}$ and $y_n$, so $y^{n+1}$ depends continuously on $x_1, \ldots, x_{n+1}$. For $x \in K \subset [0,1]^\mathcal{N}$, we also call $\Phi[\tau^*](x) = \Phi((x_{\tau^n})_{n \in \mathbb{N}})$. Clearly, the image of $\Phi$ consists of convergent sequences so one can define

$$\phi[\tau^*](x) = \lim_{n \in \mathbb{N}} \Phi((x_{\tau^n})_{n \in \mathbb{N}})$$

The function $\phi[\tau^*]$ is continuous on $K$. The reason is that for every $m$, the map $x \mapsto \Phi[\tau^*](x)_m$ is continuous, and $\phi[\tau^*](x)$ is the uniform limit of this sequence of
continuous functions, \(|\Phi[\tau^*](x)_n - \Phi[\tau^*](x)_{n+1}| \leq 2^{-n}\).

**Lemma 12.** Let \(\sigma \in \mathcal{N}\) and let \(Z\) be a separated family of \(\sigma\)-good sequences. Then, \(\mathcal{F} = \{\phi[\tau^*] : \zeta^* \in Z\}\) is a Namioka family.

Proof: Given \(m \in \mathbb{N}\) and \(L\) a closed subset of \(K\) we will try to find a nonempty open set \(V\) of \(L\) of uniform diameter less than or equal to \(2^{3-m}\) for the uniform metric associated to the family \(\mathcal{F}\).

For every \(n \in \mathbb{N}\), \(\sigma|n = (\sigma_0, \ldots, \sigma_{n-1})\), and we call \(d_n = d_{\mathcal{N}\sigma|n}\) the corresponding pseudometric. Since \(d_n \frac{1}{2^n}\)-fragments \(K\) we can find a nonempty open set \(U \subset L\) with \(d_n\)-diameter less than or equal to \(\frac{1}{2^n}\), for every \(n \leq m\). On the other hand, \(\zeta^n \in \mathcal{N}\sigma|n\) for every \(\zeta \in Z\) because \(d(\zeta^n, \sigma) \leq 2^{-n-1}\). This means that for every \(\zeta \in Z\) and every \(n \leq m\), the set \(\{x_{\zeta^n} : x \in U\}\) lies in an interval, say \(I_{\zeta^n}\), of diameter less than or equal to \(2^{-n}\).

**Remark A:** We note that, if it would be the case that \(\Phi[\tau^*](x)_n \in I_{\zeta^n}\) for every \(x \in U\) and every \(\zeta \in Z\) then we would be done, because this would imply that, for every \(x \in U\), \(\phi[\tau^*](x)\) lies in an interval of diameter at most \(\frac{1}{2^n}\), namely \(I_{\zeta^n} + \frac{1}{2^n}\).

If the the condition expressed in the remark above fails, we begin a procedure of reducing \(U\) and changing the intervals \(I_n\) as follows. If it is not the case it means that there exists \(x \in U\) and some \(\zeta\) such that \(\Phi[\tau^*](x)_m \notin I_{\zeta^n}\). We choose \(\zeta^*_m\) to be the \((-\infty)\)-minimum \(\zeta^*\) to have this property and fix the corresponding \(x \in U\). In particular we will have that \(\Phi[\tau^*](x)_n \neq x_{\zeta^n}\). By the definition of \(\Phi\), if this happens it is because there existed some \(j < n\) such that \(|x_{\zeta^j} - x_{\zeta^j+1}| > 2^{-j}\). Actually, since \(x\) is almost increasing and \(d(\zeta^j, \zeta^{j+1}) \leq 2^{-j-1}\) (for \(\zeta^*\) is \(\sigma\)-good), it must be the case that \(x_{\zeta^j} + 2^{-j} < x_{\zeta^{j+1}}\).

We define

\[
U_1 = \{y \in U : |y_{\zeta^n} - x_{\zeta^n}| < 2^{-m-2} \text{ and } |y_{\zeta^j} - x_{\zeta^j}| < 2^{-m-5} \text{ and } |y_{\zeta^{j+1}} - x_{\zeta^{j+1}}| < 2^{-m-5}\}
\]

**Claim B:** For every \(y \in U_1\) and every \(\xi^* \succ \zeta^*\), we have that \(x_{\zeta^j} + 2^{-j-1} < y_{\zeta^j}\).

Proof of the claim:

\[x_{\zeta^j} + 2^{-j} < x_{\zeta^{j+1}} \leq y_{\zeta^{j+1}} + 2^{-m-5} \leq y_{\zeta^j} + d(\zeta^{j+1}, \zeta^j) + 2^{-m-5} \leq y_{\zeta^j} + 2^{-j-1}\]

Notice that for every \(y \in U_1\):

- If \(\xi^* < \zeta^*\), \(\Phi[\xi^*](y)_m \in I_m\),
- \(|\Phi[\xi^*](y)_m - \Phi[\tau^*](x)_m| < 2^{-m-2}\).
Therefore, if in addition $\Phi[\zeta^*](y)_m \in I_\mathcal{L}$ for every $\xi^* \succ \zeta^*$, the proof would be finished, on the same grounds as in the above Remark A. If not, we must repeat the procedure of reduction and pass to a new further open set $U_2$, and look now at the minimum $\zeta^*_2 \succ \zeta^*$ such that $\Phi[\zeta^*_2](x)_n \notin I_{\mathcal{L}}$, etc. In order to conclude, we observe that the reduction procedure cannot be repeated infinitely many times, so that for some $k \in \mathbb{N}$, an open set $U_k$ of small $\mathcal{F}$ diameter will be found. The reason for the impossibility of infinite repetitions is precisely the inequality of Claim B. There are only finitely many possible values for the natural number $j \leq m$, so for some $j$, after infinitely many steps we would have a sequence $x^{(1)}_\zeta < x^{(2)}_\zeta < \cdots$ of numbers in $[0,1]$ with $x^{(n)}_\zeta + 2^{-j-1} < x^{(n+1)}_\zeta$. This is a contradiction which finishes the proof of the lemma. □

**Lemma 13.** Let $\sigma \in \mathcal{N}$, let $Z$ be a separated family of $\sigma$-good sequences of cardinality $\kappa$, and $\mathcal{F} = \{\phi[\zeta^*] : \zeta \in \mathcal{Z}\}$. Consider $L$ the quotient space of $K$ induced by the equivalence relation $(x \sim y \iff \forall f \in \mathcal{F} f(x) = f(y))$. Then $L$ is an RN compactum of weight $\kappa$.

Proof: It is clear that $L$ is RN compact since $\mathcal{F}$ is a Namioka family. The point is to prove that the weight of $L$ is not less than $\kappa$. For this we shall show that actually $L$ contains a copy of the compact space $P[0,\kappa]$ of probability measures on the ordinal interval $[0,\kappa]$. Remember that this space can be identified with

$$P[0,\kappa] = \{f : [0,\kappa] \rightarrow [0,1] : \alpha \prec \beta \Rightarrow f(\alpha) \leq f(\beta)\}.$$  

Let $Z = \{\zeta^*(\alpha) : \alpha \prec \kappa\}$ be an $(\prec)$-increasing enumeration of $Z$. To every $f \in P[0,\kappa]$, we associate the function $\psi(f) \in K$ defined as

- $\psi(f)(\sigma) = 0$ if $\sigma \prec \zeta^1(0)$
- $\psi(f)(\sigma) = f(\alpha)$ if $\zeta^1(\alpha) \prec \sigma \prec \zeta^1(\alpha + 1)$
- $\psi(f)(\sigma) = 1$ if $\sigma \succ \zeta^1$ for all $\zeta \in Z$.

Then $\psi : P[0,\kappa] \rightarrow K$ is a one-to-one continuous map. The key fact is that after composing with the quotient map onto $L$,

$$\psi : P[0,\kappa] \rightarrow K \rightarrow L$$

the function is still one-to-one. The reason is that $\phi[\zeta^*(\alpha)](\psi(f)) = f(\alpha)$, for every $\alpha \prec \kappa$ and every $f \in P[0,\kappa]$. □

Proof of Theorem [1] In the case of item (3), it is clear that, for any $\sigma \in \mathcal{N}$, we can find a separated $\sigma$-good family of size $\kappa$, since actually we can find a $\sigma$-good sequence inside any infinite $(\prec)$-interval. For (1) and (3), we shall show that for every cardinal $\lambda \leq \kappa$ there is a separated family of $\sigma$-good sequences of cardinality $\text{cf}(\lambda)$. Let $\nu : \mathcal{N} \rightarrow \text{Ord}$ be the function which associates to every $\sigma \in \mathcal{N}$ the ordinal which indicates its position in the well order $(\prec)$. Let us pick a point

$$\sigma \in \bigcap_{\alpha(\tau) < \lambda} c\mathcal{N}\{\xi : o(\tau) \prec o(\xi) \prec \lambda\},$$

where $c\mathcal{N}(\cdot)$ indicates the closure in the Baire space of irrationals $\mathcal{N}$ with its usual metric topology. Such a point $\sigma$ exists because $\mathcal{N}$ is hereditarily Lindelof and we
suppose that \( cf(\lambda) > \omega \). It is clear that we can find a separated family \( Z \) of \( \sigma \)-good sequences of size \( cf(\lambda) \), all of them below the ordinal position \( \lambda \).

**References**

[1] A. D. Arvanitakis: Some remarks on Radon-Nikodým compact spaces. Fund. Math. **172** (2002), no. 1, 41–60. Zbl 1012.46021

[2] A. Avilés: Radon-Nikodým compact spaces of low weight and Banach spaces. Studia Math. **166** (2005), no. 1, 71–82. Zbl 1086.46014

[3] A. Avilés: Linearly ordered Radon-Nikodým compact spaces. Topology Appl. 154 (2007) 404–409. Zbl 1109.54022

[4] M. Fabian: Gâteaux differentiability of convex functions and topology. Weak Asplund spaces. Canadian Mathematical Society Series of Monographs and Advanced Texts. New York. Zbl 0883.46011

[5] M. Fabian: Overclasses of the class of Radon-Nikodm compact spaces. Methods in Banach space theory. Proceedings of the V conference on Banach spaces, Cáceres, Spain, September 13–18, 2004. Cambridge: Cambridge University Press. London Mathematical Society Lecture Note Series **337**, 197-214 (2006). Zbl pre05183835

[6] M. Fabian, M. Heissler, E. Matoušková: Remarks on continuous images of Radon-Nikodm compacta. Commentat. Math. Univ. Carol. **39** (1998), no.1, 59-69. Zbl 0937.46015

[7] M. Iancu, S. Watson: On continuous images of Radon-Nikodým compact spaces through the metric characterization. Topol. Proc. **26** (2001-2002), no. 2, 677-693. Zbl 1083.54006

[8] I. Namioka: Radon-Nikodým compact spaces and fragmentability. Mathematika **34** (1987), no. 2, 258–281. Zbl 0654.46017

[9] I. Namioka: On generalizations of Radon-Nikodým compact spaces. Proceedings of the 16th Summer Conference on General Topology and its Applications (New York). Topology Proc. **26** (2001/02), no. 2, 741–750. Zbl 1083.54012

[10] J. Orihuela, W. Schachermayer, M. Valdivia: Every Radon-Nikodým Corson compact space is Eberlein compact. Studia Math. **98** (1991), 157-174. Zbl 0771.46015

[11] E. K. van Douwen: The integers and topology. Handbook of set-theoretic topology, 111–167, North-Holland, Amsterdam, 1984. Zbl 0561.54004

**National Technical University of Athens, Department of Mathematics, Athens 15780, Greece**

*E-mail address: aarva@math.ntua.gr*

**Université de Paris VII, Equipe de Logique Mathématique, UFR de Mathématiques, 2 Place Jussieu, 75251 Paris, France**

*E-mail address: avileslo@um.es, aviles@logique.jussieu.fr*