A REMARK ON A 3-FOLD CONSTRUCTED BY COLLIOT-THÉLÈNE AND VOISIN

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ABSTRACT. A classical question asks whether the Abel-Jacobi map is universal among all regular homomorphisms. In this paper, we prove that we can construct a 4-fold which gives the negative answer in codimension 3 if the generalized Bloch conjecture holds for a 3-fold constructed by Colliot-Thélène and Voisin in the context of the study of the defect of the integral Hodge conjecture in degree 4.

1. INTRODUCTION

For a smooth complex projective variety $X$, we denote by $CH^p(X)$ the $p$-th Chow group of $X$ and by $A^p(X)$ (resp. $CH^p(X)_\text{hom}$) the subgroup of cycle classes algebraically equivalent to zero (resp. homologous to zero). A homomorphism $\phi: A^p(X) \to A$ to an Abelian variety $A$ is called regular if for any smooth connected projective variety $S$ with a base point $s_0$ and for any codimension $p$ cycle $\Gamma$ on $S \times X$, the composition

$$S \to A^p(X) \to A, s \mapsto \phi(\Gamma_s(s-s_0))$$

is a morphism of algebraic varieties. An important example of such homomorphisms is given as follows. We consider the Abel-Jacobi map $CH^p(X)_\text{hom} \to J^p(X)$, where $J^p(X) = H^{2p-1}(X, \mathbb{Z})/(H^{2p-1}(X, \mathbb{Z}(p)) + F^pH^{2p-1}(X, \mathbb{Z}))$ is the $p$-th Griffith intermediate Jacobian. Then the image $J^p_a(X) \subset J^p(X)$ of $A^p(X)$ by the Abel-Jacobi map is an Abelian variety, and the induced map

$$\psi^p: A^p(X) \to J^p_a(X),$$

which we also call Abel-Jacobi, is regular [7,8]. A classical question [13, Section 7] asks whether the Abel-Jacobi map $\psi^p: A^p(X) \to J^p_a(X)$ is universal among all regular homomorphisms $\phi: A^p(X) \to A$, that is, whether every such $\phi$ factors through $\psi^p$. It is true for $p = 1$ by the theory of Picard variety, and for $p = \dim X$ by the theory of Albanese variety. It is also true for $p = 2$, which is proved by Murre [11,12] using the Merkurjev-Suslin theorem [10]. To the author’s knowledge, this question is open for $3 \leq p \leq \dim X - 1$.

In this paper, we prove that we can construct a 4-fold which gives the negative answer in codimension 3 if the generalized Bloch conjecture holds for a 3-fold constructed by Colliot-Thélène and Voisin in the context of the study of the defect of the integral Hodge conjecture in degree 4. They ask whether $Z^4(X)$ is trivial for a 3-fold $X$ such that the Chow group $CH_0(X)$ of 0-cycles is supported on a proper closed subset [6, Section 6]. Then they construct an example which conjecturally gives the negative answer as follows [6, Subsection 5.7]. Let $G = Z/5$. We fix a
generator $g \in G$, then $G$ acts on $\mathbb{P}^3 = \text{Proj} \mathbb{C}[X,Y]$ and $\mathbb{P}^3 = \text{Proj} \mathbb{C}[X_0, \ldots, X_3]$ by
\[ g^*X = X, \quad g^*Y = \zeta Y, \quad g^*X_i = \zeta^i X_i \quad (i = 0, \ldots, 3), \]
where $\zeta$ is a primitive $5$-th root of unity. Let $H \subset \mathbb{P}^1 \times \mathbb{P}^3$ be the hypersurface defined by a very general $G$-invariant polynomial of type $(3,4)$, and $X$ be a resolution of $H/G$. Then $X$ satisfies the following two properties \cite[Proposition 5.7]{Walker}:

(i) $H^p(X, \mathcal{O}_X) = 0$ for all $p > 0$;
(ii) there is a class $\alpha \in H^4(X, \mathbb{Z}(2)) = Hdg^4(X, \mathbb{Z})$ such that $\alpha \cdot F = 5$, but $C \cdot F$ is even for any curves $C \subset X$, where $F$ is the class of fibers of the morphism $X \to \mathbb{P}^1$ induced by the first projection $H \to \mathbb{P}^1$.

The property (i) implies $CH_0(X) = \mathbb{Z}$ if the generalized Bloch conjecture \cite[Conjecture 11.23]{ColliotThelene} holds for $X$. The property (ii) implies more than $Z^4(X) \neq 0$. We have $\text{Coker} (H^4(X, \mathbb{Z}(2))_{\text{tors}} \to Z^4(X)) \neq 0$, where a non-zero element is given by the image of $\alpha$. Our main theorem is:

**Theorem 1.1.** Let $X$ be a smooth projective variety such that

(i) $CH_0(X)$ is supported on a $2$-dimensional closed subset;
(ii) $\text{Coker} (H^4(X, \mathbb{Z}(2))_{\text{tors}} \to Z^4(X)) \neq 0$.

Then the Abel-Jacobi map $\psi^3 : A^3(X \times E) \to J^3_{\alpha}(X \times E)$ is not universal for some smooth elliptic curve $E$.

**Corollary 1.2.** Let $X$ be the $3$-fold constructed by Colliot-Thélène and Voisin. Then the Abel-Jacobi map $\psi^3 : A^3(X \times E) \to J^3_{\alpha}(X \times E)$ is not universal for some smooth elliptic curve $E$ if the generalized Bloch conjecture holds for $X$.

This paper is organized as follows. In Section 2, we explain a factorization of the Abel-Jacobi map $\psi^p$ due to Walker. In Section 3, we study regular homomorphisms on the torsion subgroup $A^p(X)_{\text{tors}}$. In Section 4, we prove a proposition on the defect of the integral Hodge conjecture and the torsion subgroup of the Abel-Jacobi kernel. In Section 5, we prove Theorem 1.1.

Throughout this paper, the base field is the field of complex numbers $\mathbb{C}$.

**Notation.** For a smooth projective variety $X$, we denote by $N^iH^j(X, \mathbb{Z}(k))$ the coniveau filtration on $H^j(X, \mathbb{Z}(k))$. Recall that it is defined as
\[ N^iH^j(X, \mathbb{Z}(k)) = \text{Ker} \left( H^j(X, \mathbb{Z}(k)) \to \lim_{Z \subset X} H^j(X - Z, \mathbb{Z}(k)) \right), \]
where $Z \subset X$ runs through all codimension $\geq i$ closed subsets of $X$. For an Abelian group $G$ and a prime number $l$, we denote by $G_{l, \text{tors}}$ the subgroup of $l$-primary torsion elements of $G$.

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2. **A factorization of the Abel-Jacobi map $\psi^p$ due to Walker**

As an application of the theory of the Lawson homology and the morphic cohomology, the following theorem is proved by Walker:
Lemma 3.1. Let $\mathcal{H}$ be a non-zero. A map, and $\tilde{\psi}$ is universal.

First we prove the existence of a universal regular homomorphism. The map $\tilde{\psi}$ restricted to an isomorphism $\tilde{\psi}: A^p(X)\rightarrow B_{\text{tors}}$ is a surjection regular homomorphism.

Therefore the Abel-Jacobi map $\psi^p$ is not universal if the kernel $\text{Ker}(\psi^p) = \text{Coker}(\mathcal{H}(X, \mathbb{Z}(p)))_{\text{tors}} \rightarrow (\mathcal{H}(X, \mathbb{Z}(p))/\mathcal{N}^p H^{2p-1}(X, \mathbb{Z}(p)))_{\text{tors}}$ is non-zero.

3. Regular homomorphisms on the torsion subgroup $A^p(X)_{\text{tors}}$

Lemma 3.1. Let $X$ be a smooth projective variety and $\phi: A^p(X)\rightarrow A$ be a surjective regular homomorphism. Assume that the restriction $\phi_{\text{tors}}: A^p(X)_{\text{tors}}\rightarrow A_{\text{tors}}$ is an isomorphism. Then $\phi$ is universal.

Proof. First we prove the existence of a universal regular homomorphism $\phi_0: A^p(X)\rightarrow A_0$. By Saito’s criterion [15, Theorem 2.2] (see also [12, Proposition 2.1]), it’s enough to prove $\dim B \leq \dim A$ for any surjective regular homomorphism $\psi: A^p(X)\rightarrow B$. Such a homomorphism $\psi$ is restricted to a surjection $\psi_{\text{tors}}: A^p(X)_{\text{tors}}\rightarrow B_{\text{tors}}$. Indeed, by [15, Proposition 1.2] (see also [12, Lemma 1.6.2]), there exists an Abelian variety $C$ and $\Gamma \in CH^p(C \times X)$ such that the composition $C \rightarrow A^p(X)\rightarrow B, s\mapsto \psi(\Gamma_c((s-s_0))$ is an isogeny, and the restriction $C_{\text{tors}}\rightarrow B_{\text{tors}}$ is a surjection. By assumption, we have $A^p(X)_{\text{tors}} \cong A_{\text{tors}}$. Then we have

$$\dim B = \frac{1}{2} \text{co-rank } B_{\text{tors}} \leq \frac{1}{2} \text{co-rank } A_{\text{tors}} = \dim A.$$

The existence follows.

The map $\phi_0$ is surjective since the image of a regular homomorphism is an Abelian variety [12, Lemma 1.6.2]. The induced map $A_0 \rightarrow A$ is surjective and restricted to an isomorphism $(A_0)_{\text{tors}} \cong A_{\text{tors}}$, therefore it is an isomorphism. The proof is done.

We review the Bloch-Ogus theory on the coniveau spectral sequence [3]. For a smooth projective variety $X$, let $\mathcal{H}^q(X, \mathbb{Z}(r))$ be the Zariski sheaf on $X$ associated to the presheaf $U \mapsto H^q(U, \mathbb{Z}(r))$. Then the $E_2$ term of the coniveau spectral sequence is given by

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(X, \mathbb{Z}(r))) \Rightarrow N^* H^{p+q}(X, \mathbb{Z}(r)),$$

and we have $E_2^{p,q} = 0$ if $p > q$ [3, Corollary 6.3]. We also have $E_2^{p,q} = 0$ if $(p, q) \notin [0, \dim X] \times [0, \dim X]$.

Let $f^p: H^{p-1}(X, \mathcal{H}^p(X, \mathbb{Z}(p))) \rightarrow H^{2p-1}(X, \mathbb{Z}(p))$ be the edge homomorphism.

**Lemma 3.2.** There is a short exact sequence:

$$0 \rightarrow H^{p-1}(X, \mathcal{K}_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l \rightarrow \text{Ker}(f^p \otimes \mathbb{Q}_l/\mathbb{Z}_l: H^{p-1}(X, \mathcal{H}^p(X, \mathbb{Z}(p))) \otimes \mathbb{Q}_l/\mathbb{Z}_l \rightarrow N^{p-1} H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l) \rightarrow \text{Ker}(\tilde{\psi}^p_{\text{tors}}: A^p(X)_{\text{tors}} \rightarrow J(N^{p-1} H^{2p-1}(X, \mathbb{Z}(p)))_{\text{tors}}) \rightarrow 0$$
for any smooth projective variety $X$ and any prime number $l$, where $K_p$ is the Zariski sheaf on $X$ associated to the Quillen $K$-theory.

**Proof.** We use the Bloch map $\lambda^p: CH^p(X)_{l\text{-tors}} \rightarrow H^{2p-1}(X, \mathbb{Q}_l/(p))$ (see also [5]). By the construction of the Bloch map and [9, Theorem 5.1], we have a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H^{p-1}(X, K_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \rightarrow & H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}(p))) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \rightarrow & A^p(X)_{l\text{-tors}} & \rightarrow & 0 \\
& & \downarrow f^p \otimes \mathbb{Q}_l/\mathbb{Z}_l & & \downarrow \lambda^p & & & & \\
0 & \rightarrow & H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \rightarrow & H^{2p-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(p)) & & & & \\
\end{array}
$$

We prove that it induces another commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H^{p-1}(X, K_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \rightarrow & H^{p-1}(X, \mathcal{H}^p(\mathbb{Z}(p))) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \rightarrow & A^p(X)_{l\text{-tors}} & \rightarrow & 0 \\
& & \downarrow f^p \otimes \mathbb{Q}_l/\mathbb{Z}_l & & \downarrow \tilde{\lambda}^p & & & & \\
0 & \rightarrow & N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \rightarrow & N^{p-1}H^{2p-1}(X, \mathbb{Q}_l/\mathbb{Z}_l(p)) & \rightarrow & 0
\end{array}
$$

It is enough to prove the image of $H^{p-1}(X, K_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ in $N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is zero. It follows by observing that $H^{p-1}(X, K_p) \otimes \mathbb{Q}_l/\mathbb{Z}_l$ is divisible and

$$
\begin{align*}
\text{Ker} \left( N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \rightarrow H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \right) \\
= \text{Coker} \left( H^{2p-1}(X, \mathbb{Z}(p))_{l\text{-tors}} \rightarrow (H^{2p-1}(X, \mathbb{Z}(p))/N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))_{l\text{-tors}} \right)
\end{align*}
$$

is finite. We prove that $\tilde{\lambda}^p$ coincides with the restriction $\tilde{\psi}^p|_{l\text{-tors}}$. In commutative triangles

$$
\begin{array}{ccc}
N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \rightarrow & A^p(X)_{l\text{-tors}} \\
\lambda^p (\text{resp. } \tilde{\lambda}^p|_{l\text{-tors}}) & & \lambda^p (\text{resp. } \tilde{\psi}^p|_{l\text{-tors}})
\end{array}
$$

$\tilde{\lambda}^p$ (resp. $\tilde{\psi}^p|_{l\text{-tors}}$) is the unique lift of $\lambda^p$ (resp. $\psi^p|_{l\text{-tors}}$) since $A^p(X)_{l\text{-tors}}$ is divisible [3 Lemma 7.10]. Therefore it’s enough to prove that $\tilde{\lambda}^p$ coincides with $\tilde{\psi}^p|_{l\text{-tors}}$. It follows from [2 Proposition 3.7]. The proof is done by the snake lemma.

We prove an analogue of the Roitman theorem [14, Theorem 3.1]:

**Proposition 3.3.** Let $X$ be a smooth projective variety such that $CH_0(X)$ is supported on a 3-dimensional closed subset. Let $p \in \{3, \dim X - 1\}$. Then the restriction

$$
\tilde{\psi}^p|_{l\text{-tors}}: A^p(X)_{l\text{-tors}} \rightarrow J(N^{p-1}H^{2p-1}(X, \mathbb{Z}(p)))_{l\text{-tors}}
$$

is an isomorphism. Therefore $\tilde{\psi}^p$ is universal.

**Remark 3.4.** There is a 4-fold $X$ such that $A^3(X)$ has infinite $l$-torsions for some prime $l$ [10]. Therefore the assumption on $CH_0(X)$ is essential.

**Proof.** The second statement follows from the first one by Lemma 3.1.

We prove that $\tilde{\psi}^3|_{l\text{-tors}}$ is an isomorphism. By Lemma 3.2 it’s enough to prove that

$$
\text{Ker}(f^3) = \text{Im} \left( H^0(X, \mathcal{H}^4(\mathbb{Z}(3))) \rightarrow H^2(X, \mathcal{H}^3(\mathbb{Z}(3))) \right)
$$

is torsion. The group $H^0(X, \mathcal{H}^4(\mathbb{Z}(3)))$ is torsion by [6 Proposition 3.3 (i)] (it is actually zero as a consequence of the Bloch-Kato conjecture, see [6 Theorem 3.1]), so it follows.
Let $d = \dim X$. We prove that $\widetilde{\psi}^{d-1}|_{\text{tors}}$ is an isomorphism. By Lemma 5.2, it’s enough to prove that

$$\text{Ker}(f^{d-1}) = \text{Im}(H^{d-4}(X, \mathcal{H}^d(\mathbb{Z}(d-1)))) \to H^{d-2}(X, \mathcal{H}^{d-1}(\mathbb{Z}(d-1)))$$

is torsion. The group $H^{d-4}(X, \mathcal{H}^d(\mathbb{Z}(d-1)))$ is torsion by [6, Proposition 3.3 (ii)], so it follows.

4. A proposition on the defect of the integral Hodge conjecture and the torsion subgroup of the Abel–Jacobi kernel

For a smooth projective variety $X$, let $Z^{2p}(X) = Hdg^{2p}(X, \mathbb{Z})/H^{2p}_{\text{alg}}(X, \mathbb{Z}(p))$ be the defect of the integral Hodge conjecture in degree $2p$. Inspired by the work of Soulé and Voisin [17], we prove:

**Proposition 4.1.** Let $X$ be a smooth projective variety such that $\text{Ker}(H^{2p}(X, \mathbb{Z}(p))_{\text{tors}} \to Z^{2p}(X)_{\text{tors}}) \neq 0$. Then there exists a smooth elliptic curve $E$ such that $\text{Ker}(\psi^{p+1}|_{\text{tors}}: A^{p+1}(X \times E)_{\text{tors}} \to J^{p+1}_q(X \times E)_{\text{tors}}) \neq 0$.

**Remark 4.2.** The assumption of Proposition 4.1 for $p = 2$ is satisfied by Kollar’s example [11, p.134, Lemma] (see also [17, Section 2]). It is a very general hypersurface in $\mathbb{P}^4$ of degree $l^3$ for a prime number $l \geq 5$. When it contains a certain smooth degree $l$ curve, the same conclusion follows from [17, Theorem 4]. The details are given in [17, Section 4].

**Proof.** We have the following exact sequence:

$$0 \to \text{Coker}(H^{2p}(X, \mathbb{Z}(p))_{\text{tors}} \to Z^{2p}(X)_{\text{tors}}) \to H^{2p}_{\text{alg}}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}/\mathbb{Z} \to Hdg^{2p}(X, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z}.$$ 

Let $\alpha \in \text{Coker}(H^{2p}(X, \mathbb{Z}(p))_{\text{tors}} \to Z^{2p}(X)_{\text{tors}})$ be a non-trivial element, and we use the same notation for its image in $H^{2p}_{\text{alg}}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}/\mathbb{Z}$. Let $\tilde{\alpha} \in CH^p(X) \otimes \mathbb{Q}/\mathbb{Z}$ be an element which maps to $\alpha$ via the surjection

$$c^p \otimes \mathbb{Q}/\mathbb{Z}: CH^p(X) \otimes \mathbb{Q}/\mathbb{Z} \to H^{2p}_{\text{alg}}(X, \mathbb{Z}(p)) \otimes \mathbb{Q}/\mathbb{Z}.$$ 

Let $k \subset \mathbb{C}$ be an algebraically closed field such that $\text{tr.deg}_k k < \infty$ and both $X$ and $\tilde{\alpha}$ are defined over $k$. Let $E$ be a smooth elliptic curve such that $j(E) \notin k$. We fix one component $\mathbb{Q}/\mathbb{Z}$ of $CH^1(E)_{\text{tors}} = (\mathbb{Q}/\mathbb{Z})^2$, and we identify $\tilde{\alpha}$ with an element in $CH^p(X) \otimes CH^1(E)_{\text{tors}}$. By the Schoen theorem [16, Theorem 0.2], the image of $\tilde{\alpha}$ by the exterior product map

$$CH^p(X) \otimes CH^1(E)_{\text{tors}} \to CH^{p+1}(X \times E)$$

is non-zero. Let $\beta \in CH^{p+1}(X \times E)$ be the image. Then $\beta \in A^{p+1}(X \times E)_{\text{tors}}$. We prove $\beta \in \text{Ker}(\psi^{p+1}: A^{p+1}(X \times E) \to J^{p+1}_q(X \times E))$. It’s enough to prove that $\beta$ is in the kernel of the cycle class map of the Deligne cohomology:

$$c^{p+1}_{D}: CH^{p+1}(X \times E) \to H^{2p+2}_{D}(X \times E, \mathbb{Z}(p+1)).$$

The composition of $c^{p+1}_{D}$ with the exterior product map factors through

$$c^p_{D} \otimes c^1_{D}: CH^p(X) \otimes CH^1(E)_{\text{tors}} \to H^2_{D}(X, \mathbb{Z}(p)) \otimes H^2_{D}(E, \mathbb{Z}(1))_{\text{tors}}.$$ 

Now it’s enough to prove that $\tilde{\alpha}$ is in the kernel of this map. Since we have an isomorphism

$$H^{2p}_{D}(X, \mathbb{Z}(p)) \otimes H^2_{D}(E, \mathbb{Z}(1))_{\text{tors}} \cong Hdg^{2p}(X, \mathbb{Z}) \otimes H^2_{D}(E, \mathbb{Z}(1))_{\text{tors}},$$

the proof is done by the choice of $\tilde{\alpha}$. 

$\square$
5. Proof of Theorem 1.1

Proof of the Theorem 1.1. For any smooth projective curve $E$, the group $CH_0(X \times E)$ is supported on a 3-dimensional closed subset. By Proposition 5.3, we have

$$\text{Ker}(\pi^3) = \text{Ker}(\psi^3|_{\text{tors}} : A^3(X \times E)_{\text{tors}} \to J^3_0(X \times E)_{\text{tors}}).$$

By Proposition 4.1 together with the fact that $Z^4(X) = Z^4(X)_{\text{tors}}$ (Theorem 1 (iv)), there exists a smooth elliptic curve $E$ such that

$$\text{Ker}(\psi^3|_{\text{tors}} : A^3(X \times E)_{\text{tors}} \to J^3_0(X \times E)_{\text{tors}}) \neq 0.$$

We have $\text{Ker}(\pi^3) \neq 0$, therefore the Abel-Jacobi map $\psi^3 : A^4(X \times E) \to J^3(X \times E)$ is not universal. \hfill \Box

A “homology counterpart” of Theorem 1.1 can be proved by a similar strategy.

Theorem 5.1. Let $X$ be a smooth projective variety such that

(i) $CH_0(X)$ is supported on a 2-dimensional closed subset;

(ii) $\text{Coker}(H^2_0(X, \mathbb{Z}(1))_{\text{tors}} \to \mathbb{Z}_2(X)) \neq 0$.

Then the Abel-Jacobi map $\psi_1 : A_1(X \times E) \to J_1(X \times E)_{\text{tors}}$ is not universal for some smooth elliptic curve $E$.

References

[1] Ballico, E., Catanese, F., Ciliberto, C (eds.): Classification of irregular varieties, Lecture Notes in Mathematics, 1515, Springer-Verlag, Berlin, 1992.
[2] Bloch, S.: Torsion algebraic cycles and a theorem of Roitman. Compositio Math. 39 (1979), no. 1, 107–127.
[3] Bloch, S., Ogus, A.: Gersten’s conjecture and the homology of schemes, Ann. Sci. École Norm. Sup. (4) 7 (1974), 181–201 (1975).
[4] Bloch, S., Srinivas, V.: Remarks on correspondences and algebraic cycles, Amer. J. Math. 105 (1983), no. 5, 1235–1253.
[5] Colliot-Thélène, J.-L., Voisin, C.: Cohomologie non ramifiée et conjecture de Hodge entière, Duke Math. J. 161 (2012), no. 5, 735–801.
[6] Griffiths, P. A.: Periods of integrals on algebraic manifolds. I. Local study of the period mapping, Amer. J. Math. 90 (1968), 805–865.
[7] Lieberman, D.: Intermediate Jacobians, in Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer-School in Math.), 125–139, Wolters-Noordhoff, Groningen.
[8] Murre, J. P.: Applications of algebraic $K$-theory to the theory of algebraic cycles, in Algebraic geometry, Sitges (Barcelona), 1983, 216–261, Lecture Notes in Math., 1124, Springer, Berlin.
[9] Murre, J. P.: Abel-Jacobi equivalence versus incidence equivalence for algebraic cycles of codimension two, Topology 24 (1985), no. 3, 361–367.
[10] Roitman, A. A.: The torsion of the group of 0-cycles modulo rational equivalence, Ann. of Math. (2) 111 (1980), no. 3, 553–569.
[11] Saito, H.: Abel varieties attached to cycles of intermediate dimension, Nagoya Math. J. 75 (1979), 95–119.
[12] Schoen, C.: On certain exterior product maps of Chow groups, Math. Res. Lett. 7 (2000), no. 2-3, 177–194.
[13] Soulé, C., Voisin, C.: Torsion cohomology classes and algebraic cycles on complex projective manifolds, Adv. Math. 198 (2005), no. 1, 107–127.
[14] Voisin, C.: Hodge theory and complex algebraic geometry. II, translated from the French by Leila Schneps, reprint of the 2003 English edition, Cambridge Studies in Advanced Mathematics, 77, Cambridge University Press, Cambridge, 2007.
[15] Walker, M. E.: The morphic Abel-Jacobi map, Compos. Math. 143 (2007), no. 4, 909–944.
