Turning Weight Multiplicities into Brauer Characters

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August 2018

Abstract

We describe two methods for computing $p$-modular Brauer character tables for groups of Lie type $G(p^f)$ in defining characteristic $p$, assuming that the ordinary character table of $G(p^f)$ is known, and the weight multiplicities of the corresponding algebraic group $G$ are known for $p$-restricted highest weights.

1 Introduction

Let $G(q)$ be finite group of Lie type arising from a connected reductive algebraic group $G$ of simply-connected type over an algebraic closure $\overline{\mathbb{F}}_q$ of characteristic $p$.

The irreducible representations of $G(q)$ over $\overline{\mathbb{F}}_p$ (called defining characteristic representations) are restrictions of irreducible representations of the algebraic group $G$. The irreducible representations of $G$ over $\overline{\mathbb{F}}_p$ can be described in terms of weight multiplicities.

If the weight multiplicities for $G$ are explicitly known for sufficiently many irreducible representations (parameterized by $p$-restricted highest weights) a lot of information about the irreducible representations of $G(q)$ for small powers $q$ of $p$ can be computed. For example the degrees of all irreducible representations of $G(q)$ or the composition factors of tensor products of irreducible representations.

But this description does not yield a relation of the irreducible representations of $G(q)$ in characteristic $p$ with the ordinary (complex) representations of $G(q)$. This relation is provided by the $p$-modular Brauer character table of $G(q)$, equivalently by the $p$-modular decomposition matrix of $G(q)$ or the (ordinary) characters of the $p$-modular projective indecomposable modules.

In this note we describe two ways to compute from weight multiplicities of a representation of $G$ the corresponding Brauer character of $G(q)$.

As an application of these methods we compute some character tables for the modular ATLAS project [JLPW95, W+18], which has the aim to provide all Brauer character tables for all primes for the groups whose ordinary character table is mentioned in the ATLAS [CCN+85]. More precisely, we compute the full $p$-modular Brauer character tables for the groups (in ATLAS notation) $F_4(2)$,
$2^2 . O_8^+(3), O_8^- (3), O_{10}^+(2), O_{10}^-(2)$ and $S_{10}(2)$ where $p$ is the defining characteristic. We also find partial tables for $E_6(2)$ and $3^2 E_6(2)$.

One possible application of these tables is the induction of Brauer and projective characters to (sporadic) overgroups.

In Section 2 we describe the groups we consider here in more detail and explain how we represent them for computations.

In Section 3 we give an overview of weight multiplicities encoded in dominant characters, and in particular we sketch in 3.3 how to compute with such dominant characters (this may be of independent interest).

Section 4 describes a first method to compute Brauer characters which uses explicit small degree representations of $G(q)$ and computations with dominant characters.

In Section 5 we show how to compute a parameterization of conjugacy classes of elements of $p'$-order in $G(q)$ (without an explicit representation of the group).

Section 6 describes a second method to compute Brauer characters which uses the weight multiplicities directly as well as some invariants of conjugacy classes.

In all sections we illustrate the theoretical descriptions with the example of $G$ of type $D_4$ and $G(q) \in \{\text{Spin}_{8}^{+}(3), \text{Spin}_{8}^{-}(3)\}$.

Acknowledgements. I thank Klaus Lux for convincing me that for some applications it is not enough to know the irreducible representations of $G(q)$ in defining characteristic only in terms of weight multiplicities. This motivated the preparation of these notes. And I thank Thomas Breuer for answering some questions about Brauer character tables, for helping with the computation of some ordinary character tables with Magma, and for distributing the new character tables described here with the GAP-package CTblLib.

## 2 Setup and Notation

### 2.1 Root Data

We consider connected reductive groups $G$ over an algebraic closure $\overline{F}_p$ of a finite field of characteristic $p$ and their finite subgroups $G(q) = G^F$ of fixed points under a Frobenius morphism $F$.

If $G$ is of rank $r$ and $T \leq G$ is a maximal torus of $G$ there is an associated root datum $(X, \Phi, Y, \Phi^\vee)$, see [Spr98, 7.4, 9.6]. Here $X \cong \mathbb{Z}^r$ is the character group $\text{Hom}(T, \overline{F}_p)$ and $Y \cong \mathbb{Z}^r$ is the cocharacter group $\text{Hom}(\overline{F}_p, T)$ of the torus $T$, and $\Phi \subset X$ is the set of roots and $\Phi^\vee \subset Y$ is the set of coroots of $G$ with respect to $T$.

If $F$ is a Frobenius morphism of $G$, we assume that $F(T) = T$ and in that case $F$ naturally induces $\mathbb{Z}$-linear maps on $X$ and $Y$ which are of the form $q \cdot F_0$ where $F_0$ is of finite order permuting the roots and $q$ is a power of $p$, see [Car85, 1.17–1.18]. We write $G(q) = G^F$ for the finite group of $F$-fixed points to indicate the parameter $q$ determined by $F$. (We exclude here more general $F$, leading for
example to Suzuki and Ree groups, to keep some statements simpler; except for some remarks in the proof of 3.4.1.)

The algebraic group $G$ is determined up to isomorphism by the root datum and the field $\mathbb{F}_p$. The finite group $G(q)$ of $F$-fixed points is determined up to isomorphism by the root datum, $F_0$ and the number $q$.

For computations we use the setup from the CHEVIE [GHL+96] software package. See also [BL13, Section 2] for more details.

Let $l$ be the semisimple rank of $G$. We encode the root datum by two matrices $(A, A^\vee)$ in $\mathbb{Z}^{l \times r}$. There is a natural pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$ and we choose dual $\mathbb{Z}$-bases of $X$ and $Y$ with respect to that pairing. We write a set of simple roots with respect to the basis of $X$ and put the coefficients into the rows of $A$, similarly we write the corresponding simple coroots with respect to the basis of $Y$ and put the coefficients into the rows of $A^\vee$.

If $\alpha_i$ is a simple root and $\alpha_i^\vee$ the corresponding simple coroot ($1 \leq i \leq l$), we define

$$s_i : X \to X, x \mapsto x - \langle x, \alpha_i^\vee \rangle \alpha_i,$$

$$s_i^\vee : Y \to Y, y \mapsto y - \langle \alpha_i, y \rangle \alpha_i^\vee.$$

The $s_i$, $1 \leq i \leq l$, are a set of Coxeter generators of the Weyl group $W$ of $G$ (and the same is true for the $s_i^\vee$, $1 \leq i \leq l$). Using the $i$-th rows of $A$ and $A^\vee$ we can write down $s_i$ as matrix acting on $X$ with respect to our basis of $X$. The transposed of that matrix represents the action of $s_i^\vee$ on $Y$.

The set of all roots $\Phi$ can be found as the orbits of the simple roots under the action of $W$.

If $F_0 : X \to X$ is a $\mathbb{Z}$-linear map of finite order which fixes the set of roots and whose dual map $F_0^\vee : Y \to Y$ fixes the set of coroots, then there is for each power $q$ of $p$ a Frobenius map $F$ on $G$ which induces $qF_0$ on $X$.

### 2.2 Simply-Connected Groups

From now we assume that $G$ is a semisimple group of simply connected type. In our setup this means that $r = l$ and that the coroots span the lattice $Y$. Therefore, we choose a set of simple coroots $\alpha_1^\vee, \ldots, \alpha_l^\vee$ as $\mathbb{Z}$-basis of $Y$. The corresponding dual basis $\omega_1, \ldots, \omega_l$ of $X$ is called the set of fundamental weights. The matrix $A^\vee$ is the identity matrix of size $l$ and the matrix $A$ is the transposed of the Cartan matrix of the root system of $G$ (the $l \times l$ matrix with $(i,j)$-entry $\langle \alpha_j, \alpha_i^\vee \rangle$).

The set $X_+ = \{ \sum_i a_i \omega_i \mid a_i \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq i \leq l \}$ is called the set of dominant weights and for a real number $b$ we call $X_b = \{ \sum_i a_i \omega_i \in X_+ \mid a_i < b \text{ for } 1 \leq i \leq l \}$ the set of $b$-restricted weights. The $W$-orbit of each $\lambda \in X$ contains a unique dominant weight.

There is a partial order on $X$ defined by: $\lambda, \mu \in X$, then $\lambda \geq \mu$ if and only if $\lambda - \mu = \sum_i b_i \alpha_i$ with $b_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq l$. 

3
2.3 Example

We consider the root system of type $D_4$. Its Cartan matrix is

$$
\begin{pmatrix}
2 & 0 & -1 & 0 \\
0 & 2 & -1 & 0 \\
-1 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{pmatrix}
$$

and can be encoded in the Dynkin diagram

```
1 2
3   4
```

We encode the root datum of the simply connected algebraic groups of type $D_4$ by the pair of matrices

$$(A, A^\vee) = \left( \begin{pmatrix}
2 & 0 & -1 & 0 \\
0 & 2 & -1 & 0 \\
-1 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \right).$$

Then the generating reflections $s_1, \ldots, s_4$ of the Weyl group $W$ act on $X$ by

$$
\begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

and $s_1^{\vee}, \ldots, s_4^{\vee}$ act on $Y$ by the transposed of the printed matrices.

For each characteristic $p$ these data determine the corresponding reductive algebraic group $G$ which is isomorphic to the Spin group $\text{Spin}_8(\mathbb{F}_p)$.

If we choose as $F_0$ the identity matrix, then for each power $q$ of $p$ the corresponding Frobenius morphism of $G$ yields a group of fixed points $G(q) = \text{Spin}_8^+(q)$.

The $F_0$ that permutes the first two simple roots (and coroots) yields the finite groups $G(q) = \text{Spin}_8^{-}(q)$, and from $F_0$ permuting the simple roots 1, 2, 4 cyclically we find the Steinberg triality groups $G(q) = ^3D_4(q)$.

3 Weight Multiplicities

We use the setup from 2.2. So $G$ is a connected reductive group of simply-connected type over $\mathbb{F}_p$, with a Frobenius morphism $F$, maximal torus $T = F(T)$, and finite group of fixed points $G(q) = G^F$.

3.1 Irreducible Representations

We recall a few standard facts about irreducible representations of $G$ and $G(q)$ in their defining characteristic $p$.

If $M$ is a rational module for $G$ then $M$ considered as $T$-module splits into a direct sum of non-zero subspaces $M_\mu$ which are common eigenspaces for all
\( t \in T \). Here \( \mu \in X \) is the homomorphism \( \mu : T \to \mathbb{F}_p^X \) which maps each \( t \in T \) to its eigenvalue \( \mu(t) \) on \( M_\mu \). The set \( \{ \mu \in X \mid M_\mu \neq 0 \} \) is called the weights of \( M \), it is a union of \( W \)-orbits on \( X \) and when \( \mu, \nu \in X \) are in the same \( W \)-orbit, then \( M_\mu \) and \( M_\nu \) have the same dimension.

If \( M \) is irreducible then it has a unique highest weight \( \lambda \in X_+ \) with \( \lambda \geq \mu \) for all weights of \( M \). Chevalley [Jan03, II 2.7] showed that such an \( M \) is determined up to isomorphism by its highest weight, we denote this module \( L(\lambda) \). And for each dominant \( \lambda \in X_+ \), there is such an irreducible module \( L(\lambda) \).

Steinberg [Ste68, 13.1] showed that for fixed characteristic \( p \) all \( L(\lambda) \) can be described in terms of a finite number of them. For this write an arbitrary dominant \( \lambda \in X_+ \) as a finite linear combination \( \lambda = \sum_{i=0}^k p^i \lambda_i \) where all \( \lambda_i \in X_\mu \) are \( p \)-restricted (write the entries of \( \lambda \) in base \( p \)). Steinberg’s tensor product theorem says that

\[
L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)^{[i]} \otimes \cdots \otimes L(\lambda_k)^{[k]},
\]

where \( L(\lambda)^{[i]} \) is the \( i \)-th Frobenius twist of \( L(\lambda) \), \( G \) acts on this module by the composition of its action on \( L(\lambda) \) and the field automorphism \( c \mapsto c^{p^i} \) of \( \mathbb{F}_p \).

Steinberg [Ste68, 13.3] also showed that the restrictions of all \( \{ L(\lambda) \mid \lambda \in X_\mu \} \) (the \( L(\lambda) \) for \( q \)-restricted weights) to \( G(q) \) yield a set of representatives of all isomorphism classes of irreducible representations of \( G(q) \) over \( \mathbb{F}_p \).

### 3.2 Characters by Weight Multiplicities

A lot of information about a finite dimensional rational \( G \)-module \( M \) is encoded in the list of weights \( \mu \) of \( M \) and the dimensions of the weight spaces \( M_\mu \). Using that the set of weights consists of \( W \)-orbits which each contain a unique dominant weight, we get an efficient description of these data by a dominant character, which only lists the dominant weights of \( M \) and the dimensions of the corresponding weight spaces. If \( \mu \in X \) is a weight of \( M \) we call \( m_\mu(M) = \dim(M_\mu) \) the multiplicity of the weight \( \mu \) in \( M \).

The orbit lengths of weights are not difficult to compute because for a weight \( \sum_{i=1}^t a_i \omega_i \) its stabilizer in \( W \) is the parabolic subgroup \( \langle s_i \mid a_i = 0 \rangle \).

We can find all weights in a \( W \)-orbit by a standard orbit calculation using the matrices for the generators \( s_1, \ldots, s_t \) of \( W \) given in 2.1.

Here is an example. Let \( G \) be of type \( D_4 \), \( p = 3 \), and \( \lambda = \omega_2 + 2\omega_4 \). We denote elements of \( X \) by their coefficient vectors with respect to the basis \( \omega_1, \ldots, \omega_4 \) of fundamental weights as in Example 2.3. In the following table we show the dominant character of \( L(\lambda) \) and the lengths of the \( W \)-orbits of the mentioned dominant weights.

| \( \mu \)  | \( m_\mu \) | \( |\mu|^W | \) |
|-----------|-----------|----------|
| (0, 1, 0, 2) | 1         | 32       |
| (0, 1, 1, 0) | 1         | 48       |
| (1, 0, 0, 1) | 3         | 32       |
| (0, 1, 0, 0) | 6         | 8        |
In our example, we can use the $s_i$ from 2.3 (matrices acting on $X$), the orbit of $(0, 1, 0, 0)$ consists of the weights $(0, 1, 0, 0)$, $(0, -1, 1, 0)$, $(1, 0, -1, 1)$, $(-1, 0, 0, 1)$, $(1, 0, 0, -1)$, $(-1, 0, 1, -1)$, $(0, 1, -1, 0)$, $(0, -1, 0, 0)$ and the corresponding weight spaces of $L((0, 1, 0, 2))$ have dimension 6.

3.3 Computing with Dominant Characters

Given dominant characters of rational $G$-modules $M$ and $M'$ we can

(a) compute the total dimension, $\dim M = \sum_{\mu \in X_+} m_{\mu}(M) \cdot |\mu^W|$, 

(b) compute the dominant character of the $i$-th Frobenius twist $M[i]$, for this just multiply all weights by $p^i$, 

(c) compute the character of $M \otimes M'$, using that for $v \in M_\mu$ and $v' \in M'_{\mu'}$, the vector $v \otimes v'$ is in the weight space $(M \otimes M')_{\mu + \mu'}$ because $\mu(t)\mu'(t) = (\mu + \mu')(t)$ for $t \in T$; to compute the weight multiplicities of $M_\mu \otimes M'_{\mu'}$, one needs to enumerate at least one of the orbits $\mu^W$ or $\mu'^W$, 

(d) find the composition factors of $M$ provided the dominant characters of those are available, for this enumerate the dominant weights of $M$ with respect to a total ordering which refines the partial ordering $>$ on $X$, then the first weight $\lambda$ and its multiplicity $m$ shows the multiplicity of $L(\lambda)$ as composition factor of $M$, subtract $m$ times the dominant character of $L(\lambda)$ and proceed recursively.

Now assume that we know for a fixed $p$ the dominant characters of $L(\lambda)$ for all $p$-restricted $\lambda \in X_p$. Fix the map $F_0 : X \to X$ and let $q$ be a power of $p$.

Then we can

(e) compute the dimensions of all irreducible representations of $G(q)$ over $\mathbb{F}_p$, using the Steinberg tensor product theorem for all $q$-restricted weights (only the dimensions of the $L(\lambda)$ for $p$-restricted $\lambda$ are needed), 

(f) compute the dominant characters of $L(\lambda)$ for all $q$-restricted weights, using the tensor product theorem and (b) and (c), 

(g) find the composition factors of the restriction of $L(\lambda)$ to $G(q)$ for arbitrary $\lambda \in X$, for this note that $L(\lambda)$ and $L(\lambda(qF_0))$ restrict to the same representation of $G(q)$, so decompose an arbitrary $\lambda \in X$ as $\sum_{i \geq 0} q^i \lambda_i F_0^i$ with all $\lambda_i \in X_q$ being $q$-restricted to get $L(\lambda) |_{G(q)} = \bigotimes_{i \geq 0} L(\lambda_i) |_{G(q)}$, use (h) and this point (g) recursively, 

(h) decompose tensor products of irreducible representations of $G(q)$ over $\mathbb{F}_p$, use (c), (d) and (g) for factors not corresponding to $q$-restricted weights.
We continue the example of \( G \) in type \( D_4 \), \( p = 3 \) from 3.2. Let \( \lambda = \omega_2 + 2\omega_4 \) and \( \lambda' = 2\omega_4 \). The dominant character of \( L(\lambda) \otimes L(\lambda') \) is

\[
\mu \quad m_\mu \quad |\mu^W| \quad (\text{cont.}) \quad (\text{cont.})
\begin{align*}
(0, 1, 0, 4) & \quad 1 \quad 32 & \quad (1, 0, 0, 3) & \quad 6 \quad 32 & \quad (0, 1, 0, 2) & \quad 24 \quad 32 \\
(0, 1, 1, 2) & \quad 2 \quad 96 & \quad (1, 0, 1, 1) & \quad 11 \quad 96 & \quad (0, 1, 1, 0) & \quad 34 \quad 48 \\
(0, 1, 2, 0) & \quad 3 \quad 48 & \quad (2, 1, 0, 0) & \quad 15 \quad 32 & \quad (1, 0, 0, 1) & \quad 63 \quad 32 \\
(1, 2, 0, 1) & \quad 4 \quad 96 & \quad (0, 3, 0, 0) & \quad 6 \quad 8 & \quad (0, 1, 0, 0) & \quad 112 \quad 8 \\
\end{align*}
\]

So, for the algebraic group \( G \) the tensor product contains one composition factor isomorphic to \( L((0, 1, 0, 4)) = L((0, 1, 0, 1)) \otimes L((0, 0, 0, 1)) \). This composition factor restricted to \( \text{Spin}_8^+(3) \) or to \( \text{Spin}_8^-(3) \) is the tensor product of the restrictions of \( L((0, 1, 0, 1)) \) and \( L((0, 0, 0, 1)) \) to those groups, while the restriction to \( ^3D_4(3) \) is the tensor product of the restrictions of \( L((0, 1, 0, 1)) \) and \( L((0, 1, 0, 0)) \) (because \( F_0 \) permutes \( \omega_1 \rightarrow \omega_2 \rightarrow \omega_4 \rightarrow \omega_1 \)) to the finite group. A similar computation shows that these tensor products for the finite groups have five composition factors (corresponding to 3-restricted weights) in each case.

### 3.4 Some Dominant Characters

**Theorem 3.4.1** Let \( p \) and \( G \) as in one row of the following table

\[
\begin{array}{c|c|c}
p & \text{Lie type of } G & \text{group name} \\
\hline
3 & D_4 & \text{Spin}_8(\mathbb{F}_p) \\
2 & B_4 & \text{Spin}_9(\mathbb{F}_p) \\
2 & F_4 & F_4(\mathbb{F}_p) \\
2, 3 & A_5 & \text{SL}_6(\mathbb{F}_p) \\
2 & C_5 & \text{Sp}_{10}(\mathbb{F}_p) \\
2 & D_5 & \text{Spin}_{10}(\mathbb{F}_p) \\
\end{array}
\]

Then the dominant characters of the irreducible rational representations \( L(\lambda) \) of \( G \) are known for all \( p \)-restricted highest weights \( \lambda \).

If \( G \) is simply connected of type \( E_6 \) and \( p = 2 \) then we know the dominant characters of 44 irreducible \( L(\lambda) \) with 2-restricted \( \lambda \) (there are 64 such \( \lambda \)).

**Proof.** These characters were computed with the strategy and programs described in [Lüb01]. The characters are available on the web page [Lüb18]. (Because of the size of these data we do not reproduce them within this article.) The computations involved several weeks of CPU time.

We remark that the case \( F_4 \) and \( p = 2 \) is particularly easy. In this case \( G \) has an exceptional automorphism \( \tilde{F} \) whose square is the Frobenius morphism which acts as 2 \( \text{Id} \) on \( X \).

If we number the simple roots and the fundamental weights according to the Dynkin diagram \( 1 \overset{2}{\longrightarrow} \overset{3}{\longrightarrow} \overset{4}{\longrightarrow} \), then \( \tilde{F} \) maps \( \omega_1, \omega_2, \omega_3, \omega_4 \) to \( 2\omega_4, 2\omega_3, \omega_2, \omega_1 \),
respectively. The Steinberg tensor product theorem also holds for $\tilde{F}$: Let

$$M = \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1)\}$$

and write an arbitrary dominant $\lambda \in X^+$ as $\lambda = \sum_{i=0}^{k} \lambda_i \tilde{F}^i$ where all $\lambda_i \in M$. Then

$$L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)[1] \otimes \cdots \otimes L(\lambda_k)[k]$$

where now $[i]$ denotes the $i$-th twist with $\tilde{F}$, see [Ste68, 11.2]. It is easy to compute the dominant characters of $L(\lambda)$ for $\lambda \in M$. This was already done in [Vel70].

4 Brauer Table from Small Representation and Tensoring

Let $G$, $p$ and $G(q)$ be as in previous sections. In this section we describe a first method to find the Brauer character table of a finite group $G(q)$ in defining characteristic $p$.

We recall that Brauer characters are defined relative to an embedding of the multiplicative groups $\mathbb{F}_p^\times \hookrightarrow \mathbb{C}^\times$. There is a convention how to choose this map which is used in the Modular Atlas [JLPW95] and in the CTblLib library of character tables [Bre13] distributed with GAP [GAP18] (based on the notion of Conway polynomials). To compute the Brauer character value for a group element given by its representing matrix over a finite extension of $\mathbb{F}_p$ one computes its eigenvalues in $\mathbb{F}_p$, lifts them to $\mathbb{C}$ via the mentioned embedding, and sums up the images. GAP provides a function for this computation. Brauer character values are only computed on elements of $p'$-order.

To start, we assume that we have an explicit faithful matrix representation of $G(q)$ over a finite extension of $\mathbb{F}_p$. For the groups considered in this article these are available from GAP commands like $SL(6, 3)$ or $Sp(10, 2)$ or from the ATLAS of group representations [Wil], accessible in GAP via the AtlasRep package [WPN+11].

We also assume that

- we know the weight multiplicities of $G$ for all $L(\lambda)$ with $p$-restricted $\lambda$,
- we can compute representatives of the conjugacy classes of $G(q)$ in the given representation,
- we can identify the composition factors of this representation in terms of their labels by highest weights,
- and one of:
  - the ordinary character table of $G(q)$ is known and we can identify the conjugacy classes in the given representation with those in the character table, or
we can compute the character table of $G(q)$ (in this case the identification of the conjugacy classes with those of a known table is automatic and essentially unique).

Sometimes we may need a tool, called the MeatAxe [Par84]. Given representing matrices of a set of generators of a group over a finite field, it can find representing matrices of the generators for each (absolutely irreducible) composition factor.

Given representing matrices of a set of group generators for two representations over the same field it is easy to compute representing matrices for the tensor product representation (Kronecker product of matrices). If we want to compute Brauer character values for group elements in several representations we just add representatives of all classes of $p'$-elements to our set of group generators.

The table of Brauer characters is now computed as follows.

1. If the given representation of $G(q)$ is not absolutely irreducible compute the composition factors with the MeatAxe.

2. Compute the Brauer character of the trivial representation, and of the composition factors we have found. (Recall that we know the corresponding highest weights.)

3. Using weight multiplicities we determine the composition factors of tensor products of representations for which we already know the corresponding Brauer characters, using the method sketched in 3.3.

3a) If there is a tensor product which contains only one composition factor for which we do not yet know the Brauer character, then we can compute the Brauer character of this composition factor by tensoring and subtracting known Brauer characters.

3b) Otherwise, we use a tensor product which has one or several composition factors which are easy to label (e.g., via their degrees), compute representing matrices for this tensor product, use the MeatAxe to find the composition factors, and compute the Brauer characters of new composition factors as above.

4. If not all Brauer characters are found go back to step (3).

We do not call this description an algorithm, because it is not clear that we will always be able to identify the label of new composition factors found with the MeatAxe. And there can be a practical problem if we need to apply the MeatAxe to representations of dimensions exceeding a few thousands.

But in practice this method worked very well in cases we have tried.
4.1 Example \( G(q) = \text{Spin}_{-8}(3) \)

In this case we can find a 16-dimensional representation of \( G(q) \) over \( \mathbb{F}_3 \) and we can compute class representatives and the ordinary character table with Magma [BCP97].

The MeatAxe yields two 8-dimensional absolutely irreducible composition factors over \( \mathbb{F}_9 \).

We have a description of \( G(q) \) in terms of a root datum as in 2.3 and we know the weight multiplicities of \( G \) for all 3-restricted weights as stated in 3.4.1.

Using weight multiplicities we compute (writing \( \chi_\lambda \) for the character of \( L(\lambda) \)):

\[
\chi_{(1,0,0,0)} \otimes \chi_{(0,1,0,0)} = \chi_{(1,1,0,0)} + \chi_{(0,0,0,1)}
\]

\[
\chi_{(1,0,0,0)} \otimes \chi_{(1,0,0,0)} = \chi_{(2,0,0,0)} + \chi_{(0,0,1,0)} + \chi_{(0,0,0,0)}
\]

\( G \) has three 8-dimensional irreducible representations corresponding to the highest weights \( \lambda \in \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 1)\} \). Using the MeatAxe we find the third irreducible of degree 8 as composition factor of the tensor product of the given ones. And we find the 28-dimensional module \( L((0, 0, 1, 0)) \) from the tensor product of an 8-dimensional module with itself.

The three 8-dimensional representations are permuted by the first Frobenius twist as described by \( F_0 \) (see 3.3(g)). Applying the corresponding Galois automorphism to the Brauer character values, we find that the two representations we started with are swapped. So, we started with \( L((1, 0, 0, 0)) \) and \( L((0, 1, 0, 0)) \) (in any order) and the third 8-dimensional module is \( L((0, 0, 0, 1)) \).

At this stage we know the Brauer characters of \( L(\lambda) \) for all fundamental weights \( \lambda \). It turns out that from here we can always find a tensor product of known Brauer characters which contains only one constituent whose Brauer character is not yet known:

\[
\chi_{(0,0,0,2)} = \chi_{(0,0,0,1)} \otimes \chi_{(0,0,0,1)} - \chi_{(0,0,1,0)} - \chi_{(0,0,0,0)}
\]

\[
\chi_{(0,1,0,1)} = \chi_{(0,0,0,1)} \otimes \chi_{(0,1,0,0)} - \chi_{(1,0,0,0)}
\]

\[
\chi_{(2,1,0,2)} = \chi_{(1,1,0,2)} \otimes \chi_{(1,0,0,0)} - \chi_{(0,1,1,2)} - 2\chi_{(1,2,0,1)} - 2\chi_{(1,0,0,3)}
\]

\[
-3\chi_{(1,0,1,1)} - 2\chi_{(2,1,0,0)} - 4\chi_{(0,1,0,2)} - 2\chi_{(0,1,0,0)}
\]

and so on. That is, we can find all remaining Brauer characters just by simple computations with characters.

5 Semisimple Classes

In this section we find representatives of semisimple conjugacy classes without an explicit representation of \( G(q) \). Let \( T \subseteq G, F, F_0, F(T) = T, G(q) \) as in previous sections.
We use the following facts, see [Car85, 3.7, 3.1, 3.2, 3.5].

Each semisimple conjugacy of $G$ intersects $T$ and the intersection is a single $W$-orbit of $T$.

A semisimple conjugacy class of $G$ intersects $G(q)$ if and only if the class is $F$-stable. In that case the intersection is a single $G(q)$-conjugacy class.

The maximal torus can be recovered from the root datum as $T \cong Y \otimes_{\mathbb{Z}} \mathbb{F}_p^\times$.

Writing $Q^\times_p$ for the additive group of rational numbers with denominator not divisible by $p$ there is an isomorphism $\mathbb{F}_p^\times \cong (Q^\times_p / \mathbb{Z})^+$. There is also an explicit choice of such an isomorphism in terms of Conway polynomials, such that the composition of $(Q^\times_p / \mathbb{Z})^+ \to \mathbb{C}^\times, \xi \mapsto \exp(2\pi i r / s)$, yields the embedding $\mathbb{F}_p^\times \hookrightarrow \mathbb{C}^\times$ mentioned in the beginning of Section 4.

The centralizer in $G$ of an element $t \in T$ is also a reductive group, parametrized by the same lattices $X, Y$ as $G$ and the subset of the roots $\alpha \in \Phi$ with $\alpha(t) = 0 \in Q^\times_p / \mathbb{Z}$ and the corresponding coroots. If $w \in W$ with $w(F(t)) = t$ then the Frobenius morphism on the centralizer is described by the matrix $F_0w$.

Combining this and using the chosen $\mathbb{Z}$-basis of $Y$ we can identify $T$ with the additive group $(Q^\times_p / \mathbb{Z})^l$. The action of $w \in W$ on $Y$ extends to an action on $(Q^\times_p / \mathbb{Z})^l, w(t) = tw$ (formal matrix multiplication). The same holds for the Frobenius action, $F(t) = t(qF_0)$. Evaluating a weight $\lambda \in X$ on $t \in T = (Q^\times_p / \mathbb{Z})^l$ is also done by a matrix product $\lambda(t) = \lambda t^{tr}$.

This leads to the following algorithm to determine the semisimple conjugacy classes of $G(q)$ by finding representatives of the $F$-stable $W$-orbits of $T$.

1. Determine a set of representatives $F_0w \in F_0W$ under conjugation of $W$.
   (Or, representatives $w$ of the $F_0$-conjugacy classes of $W$.)

2. For each $F_0w$ found in (1) find all solutions of the equation $t(qF_0w - id) = 0 \in (Q^\times_p / \mathbb{Z})^l$.

3. For each element $t$ found in (2) compute its $W$-orbit in $(Q^\times_p / \mathbb{Z})^l$ and take the (lexicographically) minimal element as representative.

4. For each representative $t$ from (3) compute the roots $\alpha \in \Phi$ with $\alpha(t) = 0 \in (Q^\times_p / \mathbb{Z})$, and a $w \in W$ with $w(F(t)) = t$.

Note that we may find some orbits/classes several times during the algorithm.

We remark that in practice we have used programs which parametrize semisimple classes generically for $G(q)$ for all prime powers $q$ and specialized to the specific small $q$ considered here. But the much more elementary approach sketched above is sufficient for the application in this paper.

In addition to the representatives of semisimple classes we need the following information:

**Power maps.** For small positive integers $k$ we want to know for each semisimple class of an element $s \in G(q)$ the class of $s^k$. If the class of $s$ is represented by
an element \( t \in (\mathbb{Q}_p/\mathbb{Z})^l \) as above, we compute \( kt \) and the lexicographically minimal element of its \( W \)-orbit.

**Multiplication with central elements.** For each semisimple class of an element \( s \in G(q) \) and each element \( z \in Z(G(q)) \) we want to know the class of \( sz \) (this is well defined). If the class is represented by \( t \in (\mathbb{Q}_p/\mathbb{Z})^l \) and \( z \) is represented by \( c \in (\mathbb{Q}_p/\mathbb{Z})^l \) we compute the lexicographically minimal element in the \( W \)-orbit of \( t + c \).

### 5.1 Example

We use again the example \( G(q) = \text{Spin}_9^-(3) \) and the setup from 2.3. The matrix of \( F_0 \) is the permutation matrix of the transposition \((1, 2)\). The computations can be done with the basic functions of the CHEVIE package [GHL+96].

Let \( w = s_1^\vee s_3^\vee s_5^\vee s_7^\vee s_9^\vee s_1^\vee s_3^\vee s_5^\vee s_7^\vee s_1^\vee \) as matrix acting on \( Y \). We have to consider the equation

\[
t(qF_0^{tr}w - \text{id}) = tM = 0 \in (\mathbb{Q}_p/\mathbb{Z})^l,
\]

where \( M = \begin{pmatrix}
2 & 3 & 3 & 0 \\
0 & -1 & -3 & 0 \\
0 & -3 & -1 & 0 \\
-3 & 0 & -3 & -4
\end{pmatrix}.
\]

With the Smith normal form algorithm we find unimodular matrices \( L, R \in \mathbb{Z}^{4 \times 4} \) with \( LMR = \text{diag}(0, 0, 8, 8) \). So, the solutions of the original equation are

\[(0, 0, i/8, j/8) \cdot L \quad \text{with} \quad 0 \leq i, j < 8.
\]

Considering the specific solution \( t' = (0, 0, 1/4, 1/4) L = (1/4, 1/4, 1/2, 1/2) \) its \( W \)-orbit contains 8 torus elements, the minimal representative in that orbit is \( t = (1/4, 1/4, 0, 0) \). The roots \( \{ \alpha \in \Phi \mid t'\alpha = 0 \} \) form a subsystem of the root system of \( G \) of type \( A_3 \). This yields the root datum of the centralizer \( C = C_G(t') \).

The Frobenius action on this centralizer is described by the matrix \( F_0 w \) (the transposed of \( w \) described the action of the same element of \( W \) on \( X \)). With CHEVIE we see that the centralizer \( C \) is of type \( A_3(q) + T(q + 1) \) (Dynkin diagram of type \( A_3 \) with trivial Frobenius action and a central torus \( Z^0 \) with \( |(Z^0)^F| = q + 1 \)).

For \( q = 3 \) we find the centralizer order 48522240.

We see that \( t' \) has order 4 and we can identify the classes of \( kt' \) for \( k = 2, 3 \) and so all power maps for this class.

The center of \( G(q) \) is of order 2 and the non-trivial element in the center is \( c = (1/2, 1/2, 0, 0) \). The element \( t' + c \) is in the same \( W \)-orbit as \( t' \).

### 6 Brauer Table from Weight Multiplicities

Let \( G, T, p, F, G(q) \) as in the previous sections. We fix \( q \) and assume that
• we know the weight multiplicities of $G$ for all $L(\lambda)$ with $p$-restricted $\lambda$,

• we know the ordinary character table of $G(q)$ (abstractly, that is without a labelling of the conjugacy classes by representatives in a concrete group), including the power maps of the classes,

• we have the representatives of semisimple classes of $G(q)$ in form of torus elements as explained in Section 5.

With this information it is easy to compute the values of Brauer characters as functions on the given representatives of semisimple classes. Let for a fixed $p$-restricted dominant weight $\lambda$ the weight multiplicities of $L(\lambda)$ be given as a dominant character, and let $t \in (\mathbb{Q}_p/\mathbb{Z})^d$ be a representative of a semisimple class. If $\mu$ is a dominant weight of $L(\lambda)$ with multiplicity $m_\mu$ then for each $\mu'$ in the $W$-orbit of $\mu$ the element $t$ has $m_\mu$ times the eigenvalue $a = \mu't^{tr} \in \mathbb{Q}_p/\mathbb{Z}$ on the weight space $L(\lambda)_{\mu'}$. We can lift these eigenvalues to $\exp(2\pi ia) \in \mathbb{C}^\times$ and add them all up over all weights of $L(\lambda)$ to find the Brauer character value of $t$ on $L(\lambda)$.

To be able to relate the Brauer characters found so far with ordinary characters we need to identify the conjugacy classes described by representatives $t \in T$ with the classes of $p'$-elements in the given ordinary character table of $G(q)$. This map is usually not unique and can be difficult to determine.

Recall that our class representatives in $T$ found as described in Section 5 come with the following information:

• element order,

• centralizer order,

• power maps,

• permutation of classes by multiplication with central elements.

All of this information is also contained in the abstract character tables as they come from the ATLAS [CCN+85] or the GAP character table library CTblLib [Bre13] (the permutation from multiplication with central elements can be computed from so called class multiplication coefficients).

Of course, the identification of classes we are looking for must be compatible with these data.

For character tables in GAP we can compute the group of table automorphisms. These consist of permutations of the conjugacy classes, compatible with power maps, which leave the set of irreducible characters invariant. We want to find the identifications of classes modulo these table automorphisms.

It is not a priori clear that modulo table automorphisms there is only one such identification. And we do not have a practical algorithm which will find all possible identifications compatible with our data (of course, by brute force one can try all
possibilities and check compatibility, but that is not practical). Therefore, each case needs some ad hoc procedure. We mention some typical arguments in the example below.

6.1 Example $G(q) = \text{Spin}_8^-$ (3)

We consider again the group $G(q) = \text{Spin}_8^-$ (3). We have found representatives $t \in T$ of the $3^4 = 81$ semisimple classes of $G(q)$ together with the data mentioned above. The character table of $G(q)$ is available in GAP under the name "2.08- (3)". The table has 640 table automorphisms.

The identifications of the two center elements are clear. There are for example 10 classes of elements of order 82 and the squares of their representative are in 10 classes of elements of order 41 (the 41st powers are the non-trivial element in the center). It turns out that there is a table automorphism which permutes the 10 classes of elements of order 82 cyclically and their 2nd powers accordingly and fixes all other classes. Furthermore, the 7th power map also permutes the 10 classes of 82-elements cyclically. This shows that we can identify one class of an 82-element arbitrarily and then the identification of all other classes of 82-elements and 41-elements is determined from the 7th and 2nd power maps.

A similar argument works for 8 classes of 104-elements and their powers. The choice of the identification of one of these classes can be made independently from the choice for the 82-elements because the common powers of these elements are only the center elements.

There are 4 classes of 40-elements with centralizer of order 160. Their 5th powers yield 8-elements which are also powers of 104-elements and so are already identified. This leaves only one possibility for identifying these 4 classes.

There are 3 classes of 8-elements with centralizer order 64. Only one of these classes has elements whose 2nd power has centralizer order 69120 which fixes the identification of that class. For the other two classes there are two possible identifications which are both compatible with our data.

We can proceed with this type of arguments until the number of possible identifications becomes reasonably small: two choices for one class of 8-elements as mentioned, four choices each for one class of 40-elements and for one class of 56-elements, and two choices for another class of 40-elements.

At this stage we just try out any of the 64 combinations of choices. In each case the full identification of classes then follows from the power maps. For each of these indentifications we construct the hypothetical Brauer character table and compute the corresponding decomposition matrix (that is express the restrictions of the ordinary characters to $p'$-classes as linear combinations of the Brauer characters). The entries of a decomposition matrix must be non-negative integers. This conditions rules out 60 of the 64 possibilities because these yield some non integer or negative coefficients. Two pairs of the remaining four possible identifications
differ only by a table automorphism, so that modulo table automorphisms we are left with two possible identifications.

Further investigation shows that from one of the two possible Brauer character tables we get the other via a Galois automorphism which raises complex roots of unity to their $647$th power. This shows that both tables are correct with respect to some choice of $p$-modular system, or with respect to some choice of the identification $\mathbb{F}_p^\times \cong \mathbb{Q}_p'/\mathbb{Z}$.

6.2 Some new Brauer character tables

Using the techniques described in this paper we get the following contribution to the Modular Atlas Project.

Theorem 6.2.1 The $p$-modular Brauer character tables and their fusion into the corresponding ordinary character table are known for the following cases:

| $p$ | Lie type | group name | ATLAS name | name in GAP |
|-----|----------|------------|------------|-------------|
| 2   | $F_4$    | $F_4(2)$   | $F_4(2)$   | $F_4(2)$    |
| 3   | $D_4$    | Spin$_{+}^+(3)$ | $2^2.O_8^+(3)$ | $2^2.08^+(3)$ |
| 3   | $D_4$    | Spin$_{-}^-(3)$ | $2.O_8^-(3)$ | $2.08^-(-3)$ |
| 2   | $D_5$    | Spin$_{+}^{10}(2)$ | $O_{10}^+(2)$ | $O_{10}+(2)$ |
| 2   | $D_5$    | Spin$_{-}^{10}(2)$ | $O_{10}^-(2)$ | $O_{10}^-(2)$ |
| 2   | $C_5$    | Sp$_{10}^{(2)}$ | $S_{10}(2)$ | $S_{10}(2)$ |

Furthermore, partial 2-modular Brauer character tables are known for the groups $E_6(2)$ and $^2E_6(2)_{sc}$ (ATLAS names $E_6(2)$ and $3.2E_6(2)$, GAP names $E_6(2)$ and $3.2E_6(2)$). In these cases 44 of the 64 irreducible Brauer characters are known. (We cannot get decomposition numbers from this partial information.)

These character tables are too big to be printed in this article. They will be available in future versions of the GAP character table library CTblLib [Bre13].

Actually, not all ordinary character tables mentioned above are printed in the ATLAS [CCN+85] (but they are available in GAP): for $2^2.O_8^+(3)$ only the table of the simple quotient is printed, tables of $2.O_8^-(3)$ or its simple quotient are not printed, the availability of the table of $S_{10}(2)$ is only mentioned in the Improvements to the ATLAS [JLPW95, App. 2]. For $3.2E_6(2)$ only the table of the simple quotient was printed, the table of the extension was computed by the author [Lüb17]. The table of $E_6(2)$ was not printed in the ATLAS. Actually, while trying to identify semisimple classes of this table we discovered an error in that ordinary table. It turned out that Bill Unger had recently recomputed that table with Magma and also discovered the error. A corrected ordinary table of $E_6(2)$ will also become available with future version of the character table library CTblLib.
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