Parametric down-conversion can produce photons that are entangled both in polarization and in space. Here we show how the spatial entanglement can be used to purify the polarization entanglement using only linear optical elements. Spatial entanglement as an additional resource leads to a substantial improvement in entanglement output compared to a previous scheme. Interestingly, in the present context the thermal character of down-conversion sources can be turned into an advantage. Our scheme is realizable with current technology.

Entanglement is an essential resource for quantum communication. It inevitably becomes degraded when the entangled particles propagate away from each other. Entanglement purification is therefore essential for the implementation of quantum communication over all but very modest distances. Entanglement purification describes methods to generate close to maximally entangled pairs out of a larger number of less perfectly entangled pairs. This is usually perceived as a problem, because it means that some quantum information protocols that would work for single-pair sources fail for PDC sources. Here we will show that not only is entanglement purification with linear optics still possible for PDC sources, but their characteristics can even be turned into an advantage. The main reason for this is that in PDC both polarization and spatial entanglement can be produced naturally, and the spatial entanglement can be used as an additional resource.

Fig. 1 shows the type of source that we have in mind. A pump pulse coming from below traverses a non-linear crystal where it can produce correlated pairs of photons into the modes $a_1$ and $b_1$. After the crystal it is reflected and traverses the crystal a second time, now producing correlated pairs into the spatial modes $a_2$ and $b_2$. The photon pairs can additionally be entangled in polarization. It is experimentally possible to fix the distance between the crystal and the mirror such that the phase between the first and second possibility to create photon pairs is stable and equal to a multiple of $2\pi$.

Then the situation is approximately described by the Hamiltonian

$$
H = \gamma K^+ + \gamma^* K^- = \gamma(a_{1H}^\dagger b_{1V}^\dagger + a_{1V}^\dagger b_{1H}^\dagger + a_{2H}^\dagger b_{2V}^\dagger + a_{2V}^\dagger b_{2H}^\dagger) + h.c.,
$$

where $H$ and $V$ denote vertical and horizontal polarization and we have defined $K^+ = a_{1H}^\dagger b_{1V}^\dagger + a_{1V}^\dagger b_{1H}^\dagger + a_{2H}^\dagger b_{2V}^\dagger + a_{2V}^\dagger b_{2H}^\dagger$. Considering the single-pair state created by this source, $K^+|0\rangle$ (we will disregard normalization where it is not essential), one sees that it creates photon pairs that are entangled both in polarization and in the spatial modes. In a different notation, the state $K^+|0\rangle$ could be written as $(11 + 22)(VV + HH)$, so there are two qubits on each side, one represented by the polarization modes and one by the spatial modes, and both the polarization and the spatial qubits are in a singlet state. The total entanglement content is therefore two “ebits”.

The four-photon state produced by this source is given by

$$
(K^+)^2|0\rangle = (a_{1H}^\dagger b_{1H}^\dagger + a_{1V}^\dagger b_{1V}^\dagger + a_{2H}^\dagger b_{2H}^\dagger + a_{2V}^\dagger b_{2V}^\dagger)^2|0\rangle = ((a_{1H}^\dagger b_{1H}^\dagger + a_{1V}^\dagger b_{1V}^\dagger)^2 + (a_{2H}^\dagger b_{2H}^\dagger + a_{2V}^\dagger b_{2V}^\dagger)^2
\] + 2(a_{1H}^\dagger b_{1H}^\dagger + a_{1V}^\dagger b_{1V}^\dagger)(a_{2H}^\dagger b_{2H}^\dagger + a_{2V}^\dagger b_{2V}^\dagger))|0\rangle
$$

One sees that this state contains a component where there is one photon in each spatial mode $a_1$, $a_2$, $b_1$ and $b_2$. But it also has components of comparable magnitude with two photons each in the upper modes $a_1$ and
where two and four photons have been produced. In both the source to Alice and Bob. In the text we analyze the cases will be imperfect when the photons have travelled from in polarization. In practice, spatial and polarization ent angle- with a fixed phase between these two possibilities, leading t o both lower spatial modes. For four photons, one selects those cases where the photons are both in the upper or both in the glement. On each side the two spatial modes are combined on a

$|a_1 a_2 b_1 b_2 \rangle$ consists of a non-linear crystal pumped by a laser pulse. The pump pulse is reflected from a mirror such that it traverses the crystal twice. Photons can be created in pairs both into the upper modes $a_1$ and $b_1$ and into the lower modes $a_2$ and $b_2$, with a fixed phase between these two possibilities, leading to spatial entanglement. The photons are additionally entangled in polarization. In practice, spatial and polarization entanglement will be imperfect when the photons have travelled from the source to Alice and Bob. In the text we analyze the cases where two and four photons have been produced. In both cases the setup shown serves to purify the polarization entanglement. On each side the two spatial modes are combined on polarizing beam splitters. For two photons, one selects those cases where the photons are both in the upper or both in the lower spatial modes. For four photons, one selects those cases where there is one photon in each output mode. Then both the pair of photons in modes $a_1$ and $b_1$ and the pair in $a_2$ and $b_2$ have higher polarization entanglement than before.

$|a_1 a_2 b_1 b_2 \rangle$, or two photons each in the lower modes $a_2$ and $b_2$. This shows the quasi-thermal nature of down-conversion: given two PDC sources, the probability that each emits a pair is of the same order of magnitude as the probability that one of them emits four photons and the other one doesn’t emit any photons at all. As mentioned above, this is usually perceived as a problem. However, the state contains a lot of potentially useful entanglement. Expanding one easily shows that it is a maximally entangled state in $10 \times 10$ dimensions and thus contains $2^{\log 10} = 3.32$ ebits, i.e. significantly more than two separate polarization-entangled pairs.

So far we have been talking about the ideal case of perfect polarization and spatial entanglement. In practice neither of them will be perfect. For example, the photons traveling from the source to Alice’s station may suffer depolarization, consisting of both bit-flip and phase errors, which reduces the polarization entanglement. The spatial entanglement is affected if the phase between the two possibilities for creating photons is not exactly stable. However, the probability for bit-flip errors in the spatial modes is extremely low, cross-talk between the two spatial modes on each side can be easily avoided e.g. by having two separate optical fibers.

Fig. 1 shows the basic setup for our purification scheme. The two spatial modes on each side are combined on polarizing beam splitters (PBS). This resembles the setup in [1], but with a different source. A PBS transmits horizontally polarized photons and reflects vertically polarized ones. In the language of modes this corresponds to the transformations $a_{1H} \rightarrow a_{2H}, a_{1V} \rightarrow a_{1V}, a_{2H} \rightarrow a_{1H}, a_{2V} \rightarrow a_{2V}$, and analogously for the modes on Bob’s side. Here we have denoted the spatial modes behind the PBS by the same names as the original spatial modes.

The basic reason why the setup of fig. 1 performs entanglement purification is the following. On the one hand, the PBS ensure that photons of different polarization that are originally in the same spatial mode end up in different spatial modes. On the other hand, photons are always created into corresponding pairs of spatial modes. As a consequence, selecting certain distributions of photons over the spatial modes allows one to get rid of the cases where a bit-flip error has occurred in polarization, cf. [9]. Phase errors can be purified in a second step, by first transforming them into bit-flip errors [10]. This leads to universal purification protocols.

Let us first illustrate the purification effect of fig. 1 for the simplest case, where only a single photon pair has been produced by the source. In the ideal case the state is therefore given by $K^+(0) = (a_{1H}^+ b_{1H}^† + a_{1V}^+ b_{1V}^† + a_{2H}^+ b_{2H}^† + a_{2V}^+ b_{2V}^†) \langle 0 \rangle$. One sees that this state is not changed by the action of the two PBS, so also after the PBS the photons will be in the upper modes $a_1$ and $b_1$, or in the lower modes $a_2$ and $b_2$. But suppose that a bit-flip error in polarization has occurred on the way to Alice’s station, e.g. exchanging $a_{1V}$ and $a_{1H}$. Then after the PBS one of the photons will be in an upper mode and the other one in a lower mode. Therefore by selecting only those events where both photons are up (one in $a_1$ and one in $b_1$) or both are down (one in $a_2$ and one in $b_2$), one can purify away all bit-flip errors.

Several remarks are in order. First it is worth noting that for the case of a single photon pair the above setup is actually a realization of the purification scheme proposed in [7], which uses CNOT operations on each side. The PBS is an implementation of the CNOT operation between a spatial-mode and a polarization qubit, since the spatial mode is flipped or not flipped as a function of the polarization. This implies that the above scheme also works if the original spatial entanglement is not perfect. The more efficient scheme of [10] can also be realized in this way.

Second, the PBS transform spatial entanglement into polarization entanglement. To see this, consider the amplitude for finding the two photons in modes $a_1$ and $b_1$ after the PBS. There are two ways of reaching this final state. Either the photons can have come from the two upper modes, then they must have been reflected by the PBS and thus must both be vertically polarized, or they came from the lower modes, then they must have been...
transmitted and thus be horizontally polarized. If there is a fixed phase between these two possibilities, i.e. if there was original spatial entanglement, then one has a polarization-entangled state.

Third, we have stated that a purified polarization-entangled pair is produced in the pairs of modes \(a_1 - b_1\) or \(a_2 - b_2\). With present technology, these good cases can only be selected a posteriori. For purification schemes involving several steps this means that one will sometimes run the second step although the first step did not actually produce a pair. Avoiding this kind of inefficiency would require a method for non-destructive detection of photons.

The above method for the purification of single photon pairs is interesting in its own right because of its great simplicity. The main experimental requirements are phase stability of the setup and good overlap of the photon wavepackets on the two PBS. Without good overlap, the polarization of the photons behind the PBS could be inferred from their temporal characteristics, which means that the polarization entanglement would be affected.

Let us now turn to the case where four photons are produced by the source in fig. 1, subsequently referred to as “four-photon case”. Ideally one would have the state \(\ket{2}\). As before, this state is unchanged by the action of the PBS, subsequently called the “four-mode cases”. For the ideal state, this projects onto

\[
(a_{1H}^\dagger b_{1H}^\dagger + a_{1V}^\dagger b_{1V}^\dagger)(a_{2H}^\dagger b_{2H}^\dagger + a_{2V}^\dagger b_{2V}^\dagger)\ket{0},
\]

(3)
a state of two independent polarization-entangled pairs, one in the upper and one in the lower modes. Note that the other terms in \(\ket{3}\) have all four photons in the upper modes or all four photons in the lower modes, so they do not lead to even threefold coincidences. Again, to arrive at the state \(\ket{3}\), spatial entanglement has been transformed into polarization entanglement by the two PBS. Consider the two photons in the upper modes: they can have been both reflected by the PBS if they are vertically polarized, or both transmitted if they are horizontally polarized. Again the polarization entanglement arises from the fixed phase between these two possibilities.

To understand why the setup has a purifying effect, suppose again that a single bit-flip error occurs in one of the spatial modes, e.g. mode \(a_1\). Then the affected photon is diverted by the PBS from the path that it would take in the error-less case, and thus there cannot be a photon in each of the four output modes, but there will be only a three-fold coincidence. Furthermore recall that in the ideal case there are no threefold coincidences, so diverting a single photon does not turn the cases thrown away in the ideal case into four-fold coincident cases. Therefore by selecting the four-mode cases one can indeed purify away single bit-flip errors. To study the actual magnitude of the purification effect, this simple argument has to be supplemented by a detailed calculation, cf. below.

It is important to note that in the present protocol both the upper and the lower pair of photons can be used. This is in contrast to the scheme of \(\ket{4}\), where only one of the output pairs was useful. The reason for this substantial improvement in entanglement output is that we have succeeded in using spatial entanglement as an additional resource. In contrast to usual purification schemes, where entanglement is concentrated into a single pair, while the other pair is discarded, here the spatial entanglement is concentrated into polarization entanglement, and in the four-mode case all photons are kept. However, photons are indeed discarded, since a four-mode event occurs only in about 40 percent of the four-photon cases (the exact value depending on the particular four-photon state).

It is interesting to compare the present protocol to the performance of the PBS scheme for single-pair sources without spatial entanglement \(\ket{3}\), i.e. for an initial state approximately corresponding to \(\ket{4}\), but with imperfect polarization entanglement. Selecting four-mode cases behind the PBS, which happen with a probability of 50 percent, projects \(\ket{3}\) onto

\[
(a_{1H}^\dagger a_{2H}^\dagger b_{1H}^\dagger b_{2H}^\dagger + a_{1V}^\dagger a_{2V}^\dagger b_{1V}^\dagger b_{2V}^\dagger)\ket{0},
\]

(4)
which has 1 ebit of entanglement between Alice and Bob. This is an upper bound for the real case where the original polarization entanglement is less than perfect. Thus for true single-pair sources the protocol outputs one purified entangled pair in 50 percent of the four-photon cases, where the entanglement fidelities before and after purification are related in the same way as for the protocol of \(\ket{4}\), cf. our discussion in \(\ket{4}\). The well-known S-shaped curve describing this relationship is plotted in figure 2. Here we have defined the entanglement fidelity \(F\) of a mixed state \(\rho\) as \(F = \langle \psi | \rho | \psi \rangle\), where \(|\psi\rangle = \frac{1}{\sqrt{2}}(\ket{H}\ket{H} + \ket{V}\ket{V})\) is the desired maximally entangled pure state.

On the other hand, for our present scheme that uses spatial entanglement the probability of a four-mode case behind the PBS is approximately 0.4, and every four-mode case corresponds to two purified pairs, where the relationship between initial and final polarization entanglement fidelities is also plotted in figure 2. Below we will describe in more detail how these curves were obtained. One sees that in the ideal case the new curve is always above the curve of \(\ket{3}\), while for reasonably good spatial entanglement the new curves are still substantially above the ideal curve of \(\ket{4}\) over a wide range of initial fidelities. Let us stress again that moreover there are two output pairs instead of one. It is particularly remarkable that there is no lower threshold for purification, in contrast to the previous schemes of \(\ket{3}\) and \(\ket{4}\), which had fidelity
thresholds of $F = 1/2$, as can be seen in figure 2. Also this feature is understandable because of the presence of spatial entanglement, which is converted into polarization entanglement by the PBS.

The above comparison to the case of single-pair sources shows clearly that for the present protocol the thermal character of the down-conversion source is not destructive, but actually helpful, provided that there is a stable phase between the two possible photon creation events. Parts of the final four-mode four-photon amplitude come from cases where two photons were created into the same spatial mode on both sides. These contributions account for cases where two photons were created into the same spatial mode, e.g. $a_{1H}$, affects the density matrix elements in the following way:

\[
|0\rangle\langle 0| \rightarrow |0\rangle\langle 0|
\]

\[
|1_H,0_V\rangle|1_H,0_V\rangle \rightarrow \frac{1}{2}(|1_H,0_V\rangle|1_H,0_V\rangle + |0_H,1_V\rangle|0_H,1_V\rangle)
\]

\[
|2_H,0_V\rangle|2_H,0_V\rangle \rightarrow \frac{1}{4}(|2_H,0_V\rangle|2_H,0_V\rangle + |1_H,1_V\rangle|1_H,1_V\rangle + |0_H,2_V\rangle|0_H,2_V\rangle),
\]

and analogously for all other diagonal matrix elements, while all off-diagonal elements are transformed into zero. Here we have defined $|1_H,0_V\rangle = a_{1H}^\dagger|0\rangle$, etc.

We define the partially depolarizing channel $C_s$ as the application of the fully depolarizing channel with probability $1 - s$, while the system remains undisturbed with probability $s$. In our calculations we consider states created by the source (5), which is characterized by $r$ and $\phi$, and then apply depolarizing channels $C_s$ in the spatial modes $a_1$ and $a_2$, which could e.g. correspond to a situation where the distance from the source to Alice is much larger than the distance from the source to Bob. The polarization entanglement fidelity of individual photon pairs created in this way is $\frac{1}{2}+s$ which is the fidelity before purification defining the x-axis in fig. 2. Fig. 2 shows that effective purification can be achieved for realistic values of $r$ and $\phi$. The final fidelity that can be achieved is determined by the quality of the spatial entanglement.

A first experimental realization of the present scheme is under way. The scheme is scalable in principle. Several sources of the type of fig. 1 can produce photons in parallel, which can then be fed into stacked arrays of polarizing beam splitters. Phase stability of the whole setup has to be achieved. The methods of the present work can be adapted to the case of energy-time entanglement (3), which allows to go to longer distances. True long-distance quantum communication protocols will probably also require the capability of storing photons in order to overcome the problem of photon loss. A protocol combining photons and atomic ensembles has recently been proposed (14).

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