Distance-regular graphs, the subconstituent algebra, and the $Q$-polynomial property

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Abstract

This survey paper contains a tutorial introduction to distance-regular graphs, with an emphasis on the subconstituent algebra and the $Q$-polynomial property.

Keywords. Distance-regular graph; subconstituent algebra; $Q$-polynomial; tridiagonal pair.

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1 Introduction

This survey paper contains a tutorial introduction to distance-regular graphs, with an emphasis on the subconstituent algebra and the $Q$-polynomial property. Our treatment is roughly based on the unpublished lecture notes [61], along with two more recent versions [80]. The treatment is not comprehensive; instead we restrict out attention to those topics that seem most important. A proof is given for every main result in the paper. We do not assume any prior knowledge about distance-regular graphs. We intend the present paper to complement the excellent recent works [3, 19].

A hypercube is an elementary example of a $Q$-polynomial distance-regular graph. The subconstituent algebra of a hypercube is described in [27]. We advise the beginning reader to treat the hypercubes as a running example, using [27] as a guide.

As we go along, we will encounter a linear-algebraic object called a tridiagonal pair. As a warmup, we now define a tridiagonal pair.

Let $F$ denote a field. Let $V$ denote a nonzero, finite-dimensional vector space over $F$. An $F$-linear map $A : V \to V$ is called diagonalizable whenever $V$ is spanned by the eigenspaces of $A$.

**Definition 1.1.** (See [33, Definition 1.1].) A tridiagonal pair on $V$ is an ordered pair of $F$-linear maps $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

(i) Each of $A, A^*$ is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad (0 \leq i \leq d),$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of $A^*$ such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad (0 \leq i \leq \delta),$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace $W$ of $V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

**Note 1.2.** According to a common notational convention, $A^*$ denotes the conjugate transpose of $A$. We are not using this convention. In a tridiagonal pair $A, A^*$ the maps $A$ and $A^*$ are arbitrary subject to (i)–(iv) above.

Referring to Definition [11] by [33, Lemma 4.5] the integers $d$ and $\delta$ from (1), (2) are equal; we call this common value the diameter of the pair.

The concept of a tridiagonal pair was formally introduced in [33]. However, the concept shows up earlier [67–69] in connection with the subconstituent algebra $T$ of a $Q$-polynomial.
distance-regular graph. As we will see, for such a graph every irreducible $T$-module gives a tridiagonal pair in a natural way.

We refer the reader to [3, 32, 36, 37, 53] for background and historical remarks about tridiagonal pairs.

We will encounter tridiagonal pairs of the following sort.

**Definition 1.3.** (See [71, Definition 1.1].) A Leonard pair on $V$ is a tridiagonal pair $A, A^*$ on $V$ such that the eigenspaces $\{V_i\}_{i=0}^{D}$ and $\{V_i^*\}_{i=0}^{D}$ all have dimension one.

We refer the reader to [3, 32, 36, 37, 53] for background and historical remarks about tridiagonal pairs.

### 2 Distance-regular graphs

We now turn our attention to distance-regular graphs. After a brief review of the basic definitions, we will describe the Bose-Mesner algebra, the dual Bose-Mesner algebra, and the subconstituent algebra. For more information we refer the reader to [2, 19, 27, 67, 69].

Let $\mathbb{R}$ denote the field of real numbers. Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices with rows and columns indexed by $X$ and all entries in $\mathbb{R}$. Let $I \in \text{Mat}_X(\mathbb{R})$ denote the identity matrix. Let $V = \mathbb{R}^X$ denote the vector space over $\mathbb{R}$ consisting of the column vectors with coordinates indexed by $X$ and all entries in $\mathbb{R}$. The algebra $\text{Mat}_X(\mathbb{R})$ acts on $V$ by left multiplication. We call $V$ the standard module. We endow $V$ with a bilinear form $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^t v$ for $u, v \in V$, where $t$ denotes transpose. This bilinear form is symmetric. For $u \in V$ we abbreviate $\|u\|^2 = \langle u, u \rangle$. Note that $\|u\|^2 \geq 0$, with equality if and only if $u = 0$. For $u, v \in V$ and $B \in \text{Mat}_X(\mathbb{R})$ we have $\langle Bu, v \rangle = \langle u, B^t v \rangle$. For all $y \in X$, define a vector $\hat{y} \in V$ that has $y$-coordinate 1 and all other coordinates 0. The vectors $\{\hat{y}\}_{y \in X}$ form an orthonormal basis for $V$. For later use, define a matrix $J \in \text{Mat}_X(\mathbb{R})$ that has all entries 1.

Let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$ and edge set $\mathcal{R}$. Vertices $y, z$ are adjacent whenever $y, z$ form an edge. Let $\partial$ denote the path-length distance function for $\Gamma$, and define $D = \max\{\partial(y, z) | y, z \in X\}$. We call $D$ the diameter of $\Gamma$. For $y \in X$ and an integer $i \geq 0$ define $\Gamma_i(y) = \{z \in X | \partial(y, z) = i\}$. We abbreviate $\Gamma(y) = \Gamma_1(y)$. For an integer $k \geq 0$ we say that $\Gamma$ is regular with valency $k$ whenever $|\Gamma_i(y)| = k$ for all $y \in X$. We say that $\Gamma$ is distance-regular whenever for all integers $h, i, j$ ($0 \leq h, i, j \leq D$) and for all vertices $y, z \in X$ with $\partial(y, z) = h$, the number $p_{h, i, j} = |\Gamma_i(y) \cap \Gamma_j(z)|$ is independent of $y$ and $z$. The $p_{h, i, j}$ are called the intersection numbers of $\Gamma$. From now until the end of Section 19, we assume that $\Gamma$ is distance-regular with $D \geq 3$. By construction $p_{h, i, j} = p_{h, i, j}^\Gamma$ for $0 \leq h, i, j \leq D$. We abbreviate

$$c_i = p_{1, i-1}^\Gamma (1 \leq i \leq D), \quad a_i = p_{1, i}^\Gamma (0 \leq i \leq D), \quad b_i = p_{1, i+1}^\Gamma (0 \leq i \leq D - 1).$$

Note that $a_0 = 0$ and $c_1 = 1$. Moreover

$$c_i > 0 \quad (1 \leq i \leq D), \quad b_i > 0 \quad (0 \leq i \leq D - 1).$$
The graph $\Gamma$ is regular with valency $k = b_0$. Moreover,

$$c_i + a_i + b_i = k \quad (0 \leq i \leq D),$$

where $c_0 = 0$ and $b_D = 0$. For $0 \leq i \leq D$ define $k_i = p_{b,i}^\delta$ and note that $k_i = |\Gamma_i(y)|$ for all $y \in X$. We have $k_0 = 1$ and $k_1 = k$. By a routine counting argument, $k_{i-1}b_{i-1} = k_ic_i$ for $1 \leq i \leq D$. Consequently

$$k_i = \frac{b_0b_1 \cdots b_{i-1}}{c_1c_2 \cdots c_i} \quad (0 \leq i \leq D). \quad (3)$$

By the triangle inequality, the following hold for $0 \leq h, i, j \leq D$:

(i) $p_{h,j}^0 = 0$ if one of $h, i, j$ is greater than the sum of the other two;

(ii) $p_{h,j}^0 \neq 0$ if one of $h, i, j$ is equal to the sum of the other two.

The following results are verified by routine counting arguments:

$$p_{0,j}^h = \delta_{h,j} \quad (0 \leq h, j \leq D); \quad p_{i,0}^h = \delta_{h,i} \quad (0 \leq h, i \leq D);$$

$$p_{i,j}^0 = \delta_{i,j}k_i \quad (0 \leq i, j \leq D); \quad \sum_{i=0}^D p_{i,j}^h = k_j \quad (0 \leq h, j \leq D).$$

We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq D$ define a matrix $A_i \in \text{Mat}_X(\mathbb{R})$ with $(y, z)$-entry

$$(A_i)_{y,z} = \begin{cases} 1, & \text{if } \delta(y, z) = i; \\ 0, & \text{if } \delta(y, z) \neq i \end{cases} \quad (y, z \in X).$$

We call $A_i$ the $i^{th}$ distance matrix of $\Gamma$. For $y \in X$ we have $A_i\hat{y} = \sum_{z \in \Gamma_i(y)} \hat{z}$. We abbreviate $A = A_1$ and call this the adjacency matrix of $\Gamma$. We observe (i) $A_0 = I$; (ii) $\sum_{i=0}^D A_i = J$; (iii) $A_i = A_i^t = A_i$ ($0 \leq i \leq D$); (iv) $A_iA_j = \sum_{h=0}^D p_{i,j}^h A_h$ ($0 \leq i, j \leq D$). Consequently the matrices \{A_i\}_{i=0}^D form a basis for a commutative subalgebra $M$ of $\text{Mat}_X(\mathbb{R})$, called the Bose-Mesner algebra of $\Gamma$. The distance matrices are symmetric and mutually commute. Therefore they can be simultaneously diagonalized. Consequently $M$ has a second basis \{E_i\}_{i=0}^D such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^D E_i = I$; (iii) $E_i^t = E_i$ ($0 \leq i \leq D$); (iv) $E_iE_j = \delta_{i,j}E_i$ ($0 \leq i, j \leq D$).

We call \{E_i\}_{i=0}^D the primitive idempotents of $\Gamma$. The primitive idempotent $E_0$ is called trivial. We have

$$V = \sum_{i=0}^D E_iV \quad \text{(orthogonal direct sum)}.$$ 

For $0 \leq i \leq D$ the subspace $E_iV$ is a common eigenspace of $M$. Note that

$$E_0V = \mathbb{R}1, \quad 1 = \sum_{y \in X} \hat{y}.$$
For $0 \leq i \leq D$ let $m_i$ denote the dimension of $E_i V$. We have $m_i = \text{tr}(E_i)$, where $\text{tr}$ denotes trace. Note that $m_0 = 1$.

We recall the dual Bose-Mesner algebras of $\Gamma$. For the rest of this section, fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{R})$ with $(y, y)$-entry

$$(E_i^*)_{y,y} = \begin{cases} 1, & \text{if } \partial(x, y) = i; \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X). \quad (4)$$

We call $E_i^*$ the $i^{th}$ dual primitive idempotent of $\Gamma$ with respect to $x$ [67, p. 378]. For $y \in X$ we have $E_i^* \hat{y} = \hat{y}$ if $\partial(x, y) = i$, and $E_i^* \hat{y} = 0$ if $\partial(x, y) \neq i$. We observe (i) $\sum_{i=0}^D E_i^* = I$; (ii) $E_i^* = E_i^*$ $(0 \leq i \leq D)$; (iii) $E_i^* E_j^* = \delta_{i,j} E_i^*$ $(0 \leq i, j \leq D)$. By these facts $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{R})$. We call $M^*$ the dual Bose-Mesner algebra of $\Gamma$ with respect to $x$ [67, p. 378].

We recall the subconstituents of $\Gamma$ with respect to $x$. From (4) we find

$$E_i^* V = \text{Span}\{\hat{y} | y \in \Gamma_i(x)\} \quad (0 \leq i \leq D). \quad (5)$$

By (5) and since $\{\hat{y}\}_{y \in X}$ is an orthonormal basis for $V$, we find

$$V = \sum_{i=0}^D E_i^* V \quad \text{(orthogonal direct sum)}.$$

For $0 \leq i \leq D$ the subspace $E_i^* V$ is a common eigenspace for $M^*$. Observe that the dimension of $E_i^* V$ is equal to $k_i$. Also $\text{tr}(E_i^*) = k_i$. We call $E_i^* V$ the $i^{th}$ subconstituent of $\Gamma$ with respect to $x$. Note that $E_0^* V = \mathbb{R}\hat{x}$. Also note that

$$A E_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \quad (0 \leq i \leq D),$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$.

We recall the subconstituent algebra of $\Gamma$ with respect to $x$. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{R})$ generated by $M$ and $M^*$. Observe that $T$ has finite dimension. The algebra $T$ is closed under the transpose map, because $M$ and $M^*$ are closed under the transpose map. We call $T$ the subconstituent algebra (or Terwilliger algebra) of $\Gamma$ with respect to $x$ [67, Definition 3.3]. See [13, 14, 18, 25, 27, 30, 63, 67, 69] for more information on the subconstituent algebra.

We recall the $T$-modules. By a $T$-module we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. A $T$-module $W$ is called irreducible whenever $W \neq 0$ and $W$ does not contain a $T$-module besides 0 and $W$. Let $W$ denote a $T$-module and let $W'$ denote a $T$-module contained in $W$. Then the orthogonal complement of $W'$ in $W$ is a $T$-module. It follows that each $T$-module is an orthogonal direct sum of irreducible $T$-modules. In particular, $V$ is an orthogonal direct sum of irreducible $T$-modules.

The following relations are verified by matrix multiplication:

$$|X| E_0^* E_0 E_0^* = E_0^*, \quad |X| E_0 E_0^* E_0 = E_0.$$
3 Some polynomials

Throughout this section $\Gamma = (X, \mathcal{R})$ denotes a distance-regular graph with diameter $D \geq 3$. We will discuss two sequences of polynomials that are associated with the distance matrices of $\Gamma$.

**Lemma 3.1.** We have

\[
AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \quad (1 \leq i \leq D-1), \\
AA_D = b_{D-1}A_{D-1} + a_DA_D.
\]

**Proof.** This is $A_iA_j = \sum_{h=0}^{D} p_{i,j}^h A_h$ with $j = 1$. \qed

Let $\lambda$ denote an indeterminate. Let $\mathbb{R}[\lambda]$ denote the $\mathbb{R}$-algebra of polynomials in $\lambda$ that have all coefficients in $\mathbb{R}$.

**Definition 3.2.** We define some polynomials $\{v_i\}_{i=0}^{D+1}$ in $\mathbb{R}[\lambda]$ such that

\[
v_0 = 1, \quad v_1 = \lambda, \quad \lambda v_i = b_{i-1}v_{i-1} + a_i v_i + c_{i+1} v_{i+1} \quad (1 \leq i \leq D),
\]

where $c_{D+1} = 1$.

**Lemma 3.3.** The following (i)–(iv) hold:

(i) $\deg v_i = i \quad (0 \leq i \leq D + 1)$;

(ii) the coefficient of $\lambda^i$ in $v_i$ is $(c_1 c_2 \cdots c_i)^{-1} \quad (0 \leq i \leq D + 1)$;

(iii) $v_i(A) = A_i \quad (0 \leq i \leq D)$;

(iv) $v_{D+1}(A) = 0$.

**Proof.** (i), (ii) By Definition 3.2.

(iii), (iv) Compare Lemma 3.1 and Definition 3.2. \qed

**Corollary 3.4.** The following hold:

(i) the algebra $M$ is generated by $A$;

(ii) the minimal polynomial of $A$ is $c_1 c_2 \cdots c_D v_{D+1}$.

**Proof.** By Lemma 3.3 and since $\{A_i\}_{i=0}^{D}$ is a basis for $M$. \qed

Next we consider the eigenvalues of $A$. Since $\{E_i\}_{i=0}^{D}$ form a basis for $M$, there exist real numbers $\{\theta_i\}_{i=0}^{D}$ such that

\[
A = \sum_{i=0}^{D} \theta_i E_i.
\]
Lemma 3.5. The following (i)–(iii) hold:

(i) the polynomial $v_{D+1}$ has $D + 1$ mutually distinct roots $\{\theta_i\}_{i=0}^D$;

(ii) the eigenspaces of $A$ are $\{E_iV\}_{i=0}^D$;

(iii) for $0 \leq i \leq D$, $\theta_i$ is the eigenvalue of $A$ for $E_iV$.

Proof. (i) The roots of $v_{D+1}$ are mutually distinct by Corollary 3.4(ii) and since $A$ is diagonalizable. These roots are $\{\theta_i\}_{i=0}^D$ by (6).

(ii), (iii) By (6).

Definition 3.6. For $0 \leq i \leq D$ we call $\theta_i$ the $i$th eigenvalue of $\Gamma$ (with respect to the given ordering of the primitive idempotents).

For convenience we adjust the normalization of the polynomials $v_i$.

Definition 3.7. Define the polynomial

$$u_i = \frac{v_i}{k_i} \quad (0 \leq i \leq D).$$

Lemma 3.8. We have

$$u_0 = 1, \quad u_1 = k^{-1}\lambda,$$

$$\lambda u_i = c_iu_{i-1} + a_iu_i + b_{i+1}u_{i+1} \quad (1 \leq i \leq D - 1),$$

$$\lambda u_D - c_Du_D - a_DU_D = k^{-1}v_{D+1}.$$

Proof. Evaluate the recurrence in Definition 3.2 using $v_i = k_i u_i$ $(0 \leq i \leq D)$ and (3).

Recall that $\{A_i\}_{i=0}^D$ and $\{E_i\}_{i=0}^D$ are bases for the vector space $M$. Next we describe how these bases are related.

Lemma 3.9. For $0 \leq j \leq D$ we have

(i) $A_j = \sum_{i=0}^D v_j(\theta_i)E_i$;

(ii) $E_j = |X|^{-1}m_j \sum_{i=0}^D u_i(\theta_j)A_i$.

Proof. (i) We have

$$A_j = v_j(A) = v_j(A) \sum_{i=0}^D E_i = \sum_{i=0}^D v_j(\theta_i)E_i.$$

(ii) Define $S = |X|^{-1}m_j \sum_{i=0}^D u_i(\theta_j)A_i$. We show that $E_j = S$. Expanding $AS$ using Lemma 3.1, we routinely obtain $AS = \theta_j S$. By this and since $S \in M$, we obtain $S = \alpha E_j$ for some $\alpha \in \mathbb{R}$. In the equation $S = \alpha E_j$, take the trace of each side. We have $\text{tr}(S) = m_j$, because $\text{tr}(A_i) = \delta_{0,\ell}|X|$ for $0 \leq \ell \leq D$. We have $\text{tr}(E_j) = m_j$. By these comments $\alpha = 1$, so $E_j = S$. 

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Lemma 3.10. For $0 \leq i, j \leq D$,

$$E_i A_j = v_j(\theta_i) E_i = A_j E_i.$$  

Proof. To verify this equation, eliminate $A_j$ using Lemma 3.9(i) and simplify the result. \qed

Lemma 3.11. For $0 \leq i \leq D$ we have

(i) $v_i(\theta_0) = k_i$;

(ii) $u_i(\theta_0) = 1$.

Proof. (i) The matrix $A_i$ has constant row sum $k_i$. Therefore $A_i J = k_i J$. We have $E_0 = |X|^{-1} J$, so $A_i E_0 = k_i E_0$. By Lemma 3.10 $A_i E_0 = v_i(\theta_0) E_0$. By these comments $v_i(\theta_0) = k_i$.

(ii) By (i) and $v_i = k_i u_i$. \qed

It is often said that the polynomials $\{u_i\}_{i=0}^D$ and $\{v_i\}_{i=0}^D$ are orthogonal [2, p. 201]. Our next goal is to explain what this means. We will bring in a bilinear form, and explain what is orthogonal to what.

We endow the vector space $\text{Mat}_X(\mathbb{R})$ with a bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle B, C \rangle = \text{tr}(BC^t) \quad B, C \in \text{Mat}_X(\mathbb{R}).$$

This bilinear form is symmetric. For $B \in \text{Mat}_X(\mathbb{R})$ we abbreviate $\|B\|^2 = \langle B, B \rangle$. Note that $\|B\|^2 \geq 0$, with equality if and only if $B = 0$.

Lemma 3.12. For $G, H, K \in \text{Mat}_X(\mathbb{R})$ we have

$$\langle GH, K \rangle = \langle H, G^t K \rangle = \langle G, KH^t \rangle.$$  

Proof. Use $\text{tr}(BC) = \text{tr}(CB)$. \qed

Lemma 3.13. For $0 \leq i, j \leq D$ we have

(i) $\langle E_i, E_j \rangle = \delta_{i,j} m_i$;

(ii) $\langle A_i, A_j \rangle = \delta_{i,j} k_i |X|$;

(iii) $\langle A_i, E_j \rangle = v_i(\theta_j) m_j$.

Proof. (i) We have

$$\langle E_i, E_j \rangle = \text{tr}(E_i E_j^t) = \text{tr}(E_i E_j) = \delta_{i,j} \text{tr}(E_i) = \delta_{i,j} m_i.$$ 

(ii) We have

$$\langle A_i, A_j \rangle = \text{tr}(A_i A_j^t) = \text{tr}(A_i A_j) = \sum_{h=0}^D p_{i,j}^h \text{tr}(A_h) = p_{i,j}^0 |X| = \delta_{i,j} k_i |X|.$$ 

(iii) Eliminate $A_i$ using Lemma 3.9(i), and evaluate the result using (i) above. \qed
Proposition 3.14. We have

\[ \sum_{\ell=0}^{D} u_{\ell}(\theta_i)u_{\ell}(\theta_j)k_{\ell} = \delta_{i,j}m_{i}^{-1}|X| \quad (0 \leq i, j \leq D), \]

\[ \sum_{\ell=0}^{D} u_{\ell}(\theta_i)u_{\ell}(\theta_j)m_{\ell} = \delta_{i,j}k_{i}^{-1}|X| \quad (0 \leq i, j \leq D). \]

Proof. The first equation comes from \( \langle E_i, E_j \rangle = \delta_{i,j}m_{i} \). In this equation, eliminate \( E_i \) and \( E_j \) using Lemma 3.9(ii), and evaluate the result using Lemma 3.13(ii). The second equation comes from \( \langle A_i, A_j \rangle = \delta_{i,j}k_{i}|X| \). In this equation, eliminate \( A_i \) and \( A_j \) using Lemma 3.9(i), and evaluate the result using Lemma 3.13(i).

Proposition 3.15. We have

\[ \sum_{\ell=0}^{D} v_{\ell}(\theta_i)v_{\ell}(\theta_j)k_{\ell}^{-1} = \delta_{i,j}m_{i}^{-1}|X| \quad (0 \leq i, j \leq D), \]

\[ \sum_{\ell=0}^{D} v_{\ell}(\theta_i)v_{\ell}(\theta_j)m_{\ell} = \delta_{i,j}k_{i}|X| \quad (0 \leq i, j \leq D). \]

Proof. Combine Definition 3.7 and Proposition 3.14.

Note 3.16. The relations in Proposition 3.14 and Proposition 3.15 are called the orthogonality relations for the polynomials \( \{u_i\}_{i=0}^{D} \) and \( \{v_i\}_{i=0}^{D} \), respectively.

Our next goal is to give some formulas for the intersection numbers \( p_{i,j}^h \).

Lemma 3.17. For \( 0 \leq h, i, j \leq D \),

\[ p_{i,j}^h = |X|^{-1}k_h^{-1}\langle A_i A_j, A_h \rangle = |X|^{-1}k_h^{-1}\langle A_h, A_i A_j \rangle. \]

Proof. To verify these equations, expand \( A_i A_j \) using \( A_i A_j = \sum_{\ell=0}^{D} p_{i,j,\ell}A_{\ell} \), and evaluate the results using Lemma 3.13(ii).

Lemma 3.18. For \( 0 \leq h, i, j \leq D \),

\[ k_h p_{i,j}^h = k_i p_{j,h}^i = k_j p_{h,i}^j = |X|^{-1}\text{tr}(A_h A_i A_j). \]

Proof. Routine application of Lemma 3.17.

Proposition 3.19. For \( 0 \leq h, i, j \leq D \),

\[ p_{i,j}^h = |X|^{-1}k_i k_j \sum_{\ell=0}^{D} u_{\ell}(\theta_i)u_{\ell}(\theta_j)u_{\ell}(\theta_h)m_{\ell}. \]

Proof. In the equation \( p_{i,j}^h = |X|^{-1}k_h^{-1}\langle A_i A_j, A_h \rangle \), eliminate \( A_h, A_i, A_j \) using Lemma 3.9(i), and evaluate the result using Lemma 3.13(i).
4 The geometry of the eigenspaces

Throughout this section $\Gamma = (X, \mathcal{R})$ denotes a distance-regular graph with diameter $D \geq 3$. Recall the standard module $V$ and the adjacency matrix $A$. Recall that for $0 \leq j \leq D$ the subspace $E_j V$ is an eigenspace of $A$ with eigenvalue $\theta_j$. This eigenspace is spanned by the vectors $\{E_j \hat{w} | w \in X\}$. Note that for $y, z \in X$ the following scalars are equal:

$$
\langle E_j \hat{y}, E_j \hat{z} \rangle, \quad (y, z)\text{-entry of } E_j,
$$

$$
\langle \hat{y}, E_j \hat{z} \rangle, \quad y\text{-coordinate of } E_j \hat{z},
$$

$$
\langle E_j \hat{y}, \hat{z} \rangle, \quad z\text{-coordinate of } E_j \hat{y}.
$$

Next, we have some comments of a geometric nature.

Lemma 4.1. For $0 \leq i, j \leq D$ and $y, z \in X$ with $\partial(y, z) = i$,

(i) $\langle E_j \hat{y}, E_j \hat{z} \rangle = |X|^{-1}m_j u_i(\theta_j)$;

(ii) $\|E_j \hat{y}\|^2 = \|E_j \hat{z}\|^2 = |X|^{-1}m_j$;

(iii) $u_i(\theta_j) = \frac{\langle E_j \hat{y}, E_j \hat{z} \rangle}{\|E_j \hat{y}\| \|E_j \hat{z}\|}$;

(iv) $u_i(\theta_j)$ is the cosine of the angle between $E_j \hat{y}$ and $E_j \hat{z}$.

Proof. (i) The $(y, z)$-entry of $E_j$ is found in Lemma 3.9(ii).

(ii) Set $y = z$ and $i = 0$ in part (i).

(iii) Combine (i), (ii).

(iv) By (iii) and trigonometry.

Corollary 4.2. We have

$$-1 \leq u_i(\theta_j) \leq 1 \quad (0 \leq i, j \leq D).$$

Proof. By Lemma 4.1(iv) and trigonometry.

Corollary 4.3. For $0 \leq i, j \leq D$ the following are equivalent:

(i) $u_i(\theta_j) = 1$;

(ii) $E_j \hat{y} = E_j \hat{z}$ for all $y, z \in X$ at $\partial(y, z) = i$;

(iii) there exists $y, z \in X$ such that $\partial(y, z) = i$ and $E_j \hat{y} = E_j \hat{z}$.

Proof. By Lemma 4.1(iv) and trigonometry.

Corollary 4.4. For $0 \leq i, j \leq D$ the following are equivalent:

(i) $u_i(\theta_j) = -1$;

(ii) $E_j \hat{y} = -E_j \hat{z}$ for all $y, z \in X$ at $\partial(y, z) = i$;
(iii) there exists $y, z \in X$ such that $\vartheta(y, z) = i$ and $E_j \hat{y} = -E_j \hat{z}$.

**Proof.** By Lemma 4.1(iv) and trigonometry. □

The following reformulation of Corollary 4.3 will be useful.

**Lemma 4.5.** For $0 \leq j \leq D$ the following are equivalent:

(i) $u_i(\theta_j) \neq 1$ for $1 \leq i \leq D$;

(ii) the vectors $\{E_j \hat{y}|y \in X\}$ are mutually distinct.

Assume that (i), (ii) hold. Then $j \neq 0$.

**Proof.** The equivalence of (i), (ii) follows from Corollary 4.3. The last assertion is from Lemma 3.11(ii). □

**Definition 4.6.** For $0 \leq j \leq D$, we call $E_j$ nondegenerate whenever the equivalent conditions (i), (ii) hold in Lemma 4.5. In this case $j \neq 0$.

The following definition is motivated by Lemma 4.1(iv).

**Definition 4.7.** For $0 \leq j \leq D$, we call the sequence $\{u_i(\theta_j)\}_{i=0}^D$ the cosine sequence of $E_j$ (or $\theta_j$).

**Lemma 4.8.** (See [4, Section 4.1.B].) For a real number $\theta$ and a sequence of real numbers $\{\sigma_i\}_{i=0}^D$, the following are equivalent:

(i) $\theta$ is an eigenvalue of $\Gamma$ with cosine sequence $\{\sigma_i\}_{i=0}^D$;

(ii) $\sigma_0 = 1$, $\sigma_1 = k^{-1}\theta$, and

$$c_i\sigma_{i-1} + a_i\sigma_i + b_i\sigma_{i+1} = \theta\sigma_i \quad (1 \leq i \leq D - 1),$$

$$c_D\sigma_{D-1} + a_D\sigma_D = \theta\sigma_D.$$

**Proof.** Use Lemma 3.5(i) and Lemma 3.8. □

5 The Krein parameters and the dual distance matrices

Our next topic is the Krein parameters. Throughout this section $\Gamma = (X, \mathcal{R})$ denotes a distance-regular graph with diameter $D \geq 3$.

For $B, C \in \text{Mat}_X(\mathbb{R})$ define the matrix $B \circ C \in \text{Mat}_X(\mathbb{R})$ with entries

$$(B \circ C)_{y,z} = B_{y,z}C_{y,z} \quad (y, z \in X).$$

The operation $\circ$ is called entrywise multiplication, or Schur multiplication, or Hadamard multiplication. We have $A_i \circ A_j = \delta_{i,j}A_i$ for $0 \leq i, j \leq D$. Recall that $\{A_i\}_{i=0}^D$ is a basis for
the Bose-Mesner algebra \( M \). By these comments, \( M \) is closed under \( \circ \). Recall that \( \{ E_i \}_{i=0}^D \) is a basis for \( M \). Therefore, there exist real numbers \( q_{i,j}^h \) (\( 0 \leq h, i, j \leq D \)) such that

\[
E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{i,j}^h E_h \quad (0 \leq i, j \leq D). \tag{8}
\]

The \( q_{i,j}^h \) are called the *Krein parameters* of \( \Gamma \). By construction \( q_{i,j}^h = q_{j,i}^h \) for \( 0 \leq h, i, j \leq D \). Shortly we will show that \( q_{i,j}^h \geq 0 \) for \( 0 \leq h, i, j \leq D \).

In order to avoid dealing directly with entrywise multiplication, we bring in a certain map \( p \). For the rest of this section, fix a vertex \( x \in X \).

**Definition 5.1.** For \( B \in \text{Mat}_X(\mathbb{R}) \) let \( B^p \) denote the diagonal matrix in \( \text{Mat}_X(\mathbb{R}) \) with \((y,y)\)-entry

\[
(B^p)_{y,y} = B_{x,y} \quad (y \in X).
\]

**Lemma 5.2.** For \( B, C \in \text{Mat}_X(\mathbb{R}) \),

\[
(B \circ C)^p = B^p C^p.
\]

**Proof.** By Definition 5.1.

**Lemma 5.3.** We have

(i) \( A_i^p = E_i^* \) (\( 0 \leq i \leq D \));

(ii) \( I^p = E_0^* \);

(iii) \( J^p = I \).

**Proof.** This is routinely checked using Definition 5.1.

Recall the dual Bose-Mesner algebra \( M^* = M^*(x) \).

**Lemma 5.4.** The restriction \( p|_M : M \to M^* \) is an isomorphism of vector spaces.

**Proof.** The map \( p \) is \( \mathbb{R} \)-linear, and sends the basis \( \{ A_i \}_{i=0}^D \) of \( M \) to the basis \( \{ E_i^* \}_{i=0}^D \) of \( M^* \).

We caution the reader that the map in Lemma 5.4 is not an algebra isomorphism in general.

**Lemma 5.5.** For \( B, C \in M \),

\[
\langle B^p, C^p \rangle = |X|^{-1} \langle B, C \rangle. \tag{9}
\]

**Proof.** Using Definition 5.1 one finds that each side of (9) is equal to the \((x,x)\)-entry of \( B C^t \).

We mention some useful facts about the map \( p \).
Lemma 5.6. For $B \in M$ we have

(i) $E_0 E_0^* B = E_0 B^p$;
(ii) $E_0^* E_0 B^p = |X|^{-1} E_0^* B$;
(iii) $B E_0^* E_0 = B^p E_0$;
(iv) $B^p E_0 E_0^* = |X|^{-1} B E_0^*$.

Proof. (i) Recall that $E_0 = |X|^{-1/2} J$. Recall that for $E_0^*$, the $(x,x)$-entry is 1 and all other entries are 0. For $y, z \in X$ we compare the $(y,z)$-entry of each side of the given equation. We have

$$(E_0 E_0^* B)_{y,z} = (E_0)_{y,x} B_{x,z} = |X|^{-1} B_{x,z} = (E_0)_{y,z} (B^p)_{z,z} = (E_0 B^p)_{y,z}.$$ 

(ii) In the equation (i), multiply each side on the left by $E_0^*$ and evaluate the result using $|X| E_0^* E_0 = E_0^*$.

(iii), (iv) Take the transpose of each side in (i), (ii) above. 

Definition 5.7. For $0 \leq i \leq D$, define the matrix $A_i^* = A_i^*(x)$ by

$$A_i^* = |X|(E_i)^p.$$ 

Thus $A_i^*$ is diagonal with $(y,y)$-entry

$$(A_i^*)_{y,y} = |X|(E_i)_{x,y} \quad (y \in X). \quad (10)$$

We call $A_i^*$ the $i^{th}$ dual distance matrix of $\Gamma$ with respect to $x$.

Lemma 5.8. With the above notation,

(i) the matrices $\{A_i^*\}_{i=0}^D$ form a basis for $M^*$;
(ii) for $0 \leq i \leq D$ and $y \in X$,

$$A_i^* \hat{y} = m_i u_j(\theta_i) \hat{y}, \quad j = \partial(x,y).$$

Proof. (i) By Lemma 5.4 and since $\{E_i\}_{i=0}^D$ form a basis for $M$.

(ii) By (10) and since $|X|(E_i)_{x,y} = m_i u_j(\theta_i)$. 

Lemma 5.9. The following (i)–(iv) hold:

(i) $A_0^* = I$;
(ii) $\sum_{i=0}^D A_i^* = |X| E_0^*$;
(iii) $(A_i^*)^t = A_i^*$ \quad ($0 \leq i \leq D$);
(iv) $A_i^* A_j^* = \sum_{h=0}^D q^h_{i,j} A_h^* \quad (0 \leq i, j \leq D)$. 

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Proof. (i) Use $A^*_0 = |X|E^*_0$ and $E_0 = |X|^{-1}J$.
(ii) Apply $p$ to each side of $\sum_{i=0}^{D} E_i = I$.
(iii) $A^*_i$ is diagonal.
(iv) Apply $p$ to each side of (8), and evaluate the result using Lemma 5.2 along with Definition 5.7.

We have seen that $\{E^*_i\}^{D}_{i=0}$ and $\{A^*_i\}^{D}_{i=0}$ are bases for the vector space $M^*$. Next we describe how these bases are related.

**Lemma 5.10.** For $0 \leq j \leq D$ we have

(i) $A^*_j = m_j \sum_{i=0}^{D} u_i(\theta_j) E^*_i$;
(ii) $E^*_j = |X|^{-1} \sum_{i=0}^{D} v_j(\theta_i) A^*_i$.

**Proof.** In Lemma 3.9 we displayed some equations that show how $\{E^*_i\}^{D}_{i=0}$ and $\{A^*_i\}^{D}_{i=0}$ are related. Apply $p$ to each side of these equations.

**Lemma 5.11.** For $0 \leq i, j \leq D$,

$$E^*_i A^*_j = m_j u_i(\theta_j) E^*_i = A^*_j E^*_i.$$ 

**Proof.** To verify this equation, eliminate $A^*_j$ using Lemma 5.10(i) and simplify the result.

**Lemma 5.12.** For $0 \leq i, j \leq D$ we have

(i) $\langle E^*_i, E^*_j \rangle = \delta_{i,j} k_i$;
(ii) $\langle A^*_i, A^*_j \rangle = \delta_{i,j} m_i |X|$;
(iii) $\langle A^*_i, E^*_j \rangle = m_i v_j(\theta_i)$.

**Proof.** (i) Routine.
(ii) Use Lemma 5.10 and Definition 5.7.
(iii) In the given equation, eliminate $E^*_j$ using Lemma 5.10(ii), and simplify the result using (ii) above.

In the next few lemmas, we describe the Krein parameters in various ways.

**Lemma 5.13.** For $0 \leq h, i, j \leq D$,

$$q_{i,j}^h = |X|^{-1} m_h^{-1} \langle A^*_i A^*_j, A^*_h \rangle = |X|^{-1} m_h^{-1} \langle A^*_h, A^*_i A^*_j \rangle.$$ 

**Proof.** To verify these equations, expand $A^*_i A^*_j$ using $A^*_i A^*_j = \sum_{\ell=0}^{D} q_{i,j}^{\ell} A^*_\ell$, and evaluate the results using Lemma 5.12(ii).

**Lemma 5.14.** For $0 \leq h, i, j \leq D$,

$$m_{i,j}^h = m_i q_{j,h}^i = m_j q_{i,h}^j = |X|^{-1} \text{tr}(A^*_h A^*_i A^*_j).$$ 

**Proof.** Routine application of Lemma 5.13.
Lemma 5.15. The following (i)–(iv) hold.

(i) \( q_{0,j}^h = \delta_{h,j} \) (0 \( \leq h, j \leq D \));

(ii) \( q_{i,0}^h = \delta_{h,i} \) (0 \( \leq h, i \leq D \));

(iii) \( q_{i,j}^0 = \delta_{i,j}m_i \) (0 \( \leq i, j \leq D \));

(iv) \( \sum_{i=0}^{D} q_{i,j}^h = m_j \) (0 \( \leq h, j \leq D \)).

Proof. (i) By Lemmas 5.12, 5.13 we obtain
\[
q_{0,j}^h = |X|^{-1}m_h^{-1}\langle A_0^*A_j^*, A_h^* \rangle = |X|^{-1}m_h^{-1}\langle A_j^*, A_h^* \rangle = \delta_{h,j}.
\]

(ii) By (i) and \( q_{i,0}^h = q_{0,i}^h \).

(iii) By (i) and Lemma 5.14.

(iv) We have
\[
\sum_{i=0}^{D} q_{i,j}^h = |X|^{-1}m_h^{-1}\sum_{i=0}^{D} \langle A_i^*A_j^*, A_h^* \rangle = m_h^{-1}\langle E_0^*A_j^*, A_h^* \rangle = m_h^{-1}m_j\langle E_0^*, A_h^* \rangle = m_j.
\]

Proposition 5.16. For 0 \( \leq h, i, j \leq D \),
\[
q_{i,j}^h = |X|^{-1}m_h^{-1}\sum_{\ell=0}^{D} u_\ell(\theta_i)u_\ell(\theta_j)u_\ell(\theta_h)k_\ell.
\]

Proof. In the equation \( q_{i,j}^h = |X|^{-1}m_h^{-1}\langle A_i^*A_j^*, A_h^* \rangle \), eliminate \( A_h^*, A_i^*, A_j^* \) using Lemma 5.10(i), and evaluate the result using Lemma 5.12(i). \( \square \)

6 Reduction rules

Throughout this section \( \Gamma = (X, R) \) denotes a distance-regular graph with diameter \( D \geq 3 \). Fix \( x \in X \) and write \( T = T(x) \). We will display a number of relations involving \( E_0 \) and \( E_0^* \). These relations are informally known as reduction rules; see [25, Section 7], [54, Sections 9, 11, 13].

Lemma 6.1. For 0 \( \leq i \leq D \),
\[
E_0E_0^*A_i = E_0E_i^*, \quad E_0^*E_0E_i^* = |X|^{-1}E_0^*A_i,
\]
\[
A_iE_0^*E_0 = E_i^*E_0, \quad E_i^*E_0E_0^* = |X|^{-1}A_iE_0^*.
\]

Proof. Apply Lemma 5.6 with \( B = A_i \). \( \square \)

Lemma 6.2. For 0 \( \leq i \leq D \),
\[
E_0^*E_0A_i^* = E_0^*E_i, \quad E_0E_0^*E_i = |X|^{-1}E_0A_i^*,
\]
\[
A_i^*E_0E_0^* = E_iE_0^*, \quad E_iE_0^*E_0 = |X|^{-1}A_i^*E_0.
\]
Proof. Apply Lemma [5.6] with $B = E_i$. 

**Lemma 6.3.** For $0 \leq i, j \leq D$ we have

(i) $E_0 A_i^* E_j = \delta_{i,j} E_0 A_i^*$;

(ii) $E_0 E_i^* E_j = |X|^{-1} k_i u_i(\theta_j) E_0 A_j^*$;

(iii) $E_0 A_i^* A_j = k_j u_j(\theta_i) E_0 A_i^*$;

(iv) $E_0 E_i^* A_j = \sum_{h=0}^{D} p_{i,j}^h E_0 E_h^*$.

**Proof.** (i) Observe

$$E_0 A_i^* E_j = |X| E_0 E_i^* E_j = \delta_{i,j} |X| E_0 E_i^* = \delta_{i,j} E_0 A_i^*.$$ 

(ii) Observe

$$E_0 E_i^* E_j = E_0 E_i^* A_j = k_i u_i(\theta_j) E_0 E_i^* E_j = |X|^{-1} k_i u_i(\theta_j) E_0 A_j^*.$$ 

(iii) Observe

$$E_0 A_i^* A_j = |X| E_0 E_i^* A_j = |X| k_j u_j(\theta_i) E_0 E_i^* A_j = k_j u_j(\theta_i) E_0 A_i^*.$$ 

(iv) Observe

$$E_0 E_i^* A_j = E_0 E_i^* A_j = \sum_{h=0}^{D} p_{i,j}^h E_0 E_h^*.$$ 

**Lemma 6.4.** For $0 \leq i, j \leq D$ we have

(i) $E_0^* A_i E_j^* = \delta_{i,j} E_0^* A_i$;

(ii) $E_0^* E_i^* E_j^* = |X|^{-1} m_i u_i(\theta_j) E_0^* A_j^*$;

(iii) $E_0^* A_i A_j^* = m_j u_i(\theta_j) E_0^* A_i$;

(iv) $E_0^* E_i A_j^* = \sum_{h=0}^{D} p_{i,j}^h E_0^* E_h.$

**Proof.** Similar to the proof of Lemma 6.3.

**Lemma 6.5.** For $0 \leq i, j \leq D$ we have

(i) $E_j A_i^* E_0 = \delta_{i,j} A_i^* E_0$;

(ii) $E_j E_i^* E_0 = |X|^{-1} k_i u_i(\theta_j) A_i^* E_0$;

(iii) $A_j A_i^* E_0 = k_j u_j(\theta_i) A_i^* E_0$;

(iv) $A_j E_i^* E_0 = \sum_{h=0}^{D} p_{i,j}^h E_h^* E_0.$
Proof. Take the transpose of everything in Lemma 6.3.

Lemma 6.6. For $0 \leq i, j \leq D$ we have

(i) $E^*_i A_i E^*_0 = \delta_{i,j} A_i E^*_0$;

(ii) $E^*_j E_i E^*_0 = |X|^{-1} m_i u_j(\theta_i) A_j E^*_0$;

(iii) $A^*_j A_i E^*_0 = m_j u_i(\theta_j) A_i E^*_0$;

(iv) $A^*_j E_i E^*_0 = \sum_{h=0}^{D} q^h_{i,j} E_h E^*_0$.

Proof. Take the transpose of everything in Lemma 6.4.

Lemma 6.7. For $0 \leq i \leq D$ we have

(i) $E_0 E^*_i E_0 = |X|^{-1} k_i E_0$;

(ii) $E^*_0 E_i E^*_0 = |X|^{-1} m_i E^*_0$.

Proof. (i) Observe

$$E_0 E^*_i E_0 = E_0 E^*_0 A_i E_0 = k_i E_0 E^*_0 E_0 = |X|^{-1} k_i E_0.$$ 

(ii) Observe

$$E^*_0 E_i E^*_0 = E^*_0 E_0 A^*_i E^*_0 = m_i E^*_0 E_0 E^*_0 = |X|^{-1} m_i E^*_0.$$ 

Lemma 6.8. For $0 \leq i, j \leq D$ we have

(i) $A_i E^*_0 A_j = |X| E^*_i E_0 E^*_j$;

(ii) $E_i E^*_0 A_j = A^*_i E_0 E^*_j$;

(iii) $A^*_i E^*_0 E_j = E^*_i E_0 A^*_j$;

(iv) $E_i E^*_0 E_j = |X|^{-1} A^*_i E_0 A^*_j$.

Proof. (i) Observe

$$A_i E^*_0 A_j = |X| A_i E^*_0 E_0 E^*_j = |X| E^*_i E_0 E^*_j.$$ 

(ii) Observe

$$E_i E^*_0 A_j = |X| E_i E^*_0 E_0 E^*_j = A^*_i E_0 E^*_j.$$ 

(iii) Observe

$$A_i E^*_0 E_j = A_i E^*_0 E_0 A^*_j = E^*_i E_0 A^*_j.$$ 

(iv) Observe

$$E_i E^*_0 E_j = E_i E^*_0 E_0 A^*_j = |X|^{-1} A^*_i E_0 A^*_j.$$
Corollary 6.9. $ME_0^*M$ and $M^*E_0M^*$ span the same subspace of $\text{Mat}_X(\mathbb{R})$.

**Proof.** By Lemma 6.8. \hfill \square

Lemma 6.10. (See [25, Proposition 11.1].) We have

$$\sum_{i=0}^{D} k_i^{-1} E_i^* E_0^* E_i^* = \sum_{j=0}^{D} m_j^{-1} E_j^* E_0^* E_j.$$  \hfill (11)

**Proof.** Observe

$$\sum_{i=0}^{D} k_i^{-1} E_i^* E_0^* E_i^* = |X|^{-1} \sum_{i=0}^{D} k_i^{-1} A_i E_0^* A_i$$

$$= |X|^{-1} \sum_{i=0}^{D} k_i^{-1} \left( \sum_{r=0}^{D} v_i(\theta_r) E_r \right) E_0^* \left( \sum_{s=0}^{D} v_i(\theta_s) E_s \right)$$

$$= |X|^{-1} \sum_{r=0}^{D} \sum_{s=0}^{D} E_r E_0^* E_s \left( \sum_{i=0}^{D} k_i^{-1} v_i(\theta_r) v_i(\theta_s) \right)$$

$$= |X|^{-1} \sum_{r=0}^{D} \sum_{s=0}^{D} E_r E_0^* E_s \left( \delta_{r,s} m_r^{-1} |X| \right)$$

$$= \sum_{j=0}^{D} m_j^{-1} E_j^* E_0^* E_j.$$ \hfill \square

Definition 6.11. Define the matrix $e_0 = e_0(x)$ to be $|X|$ times the common value of the matrices (11).

Referring to Definition 6.11, the matrix $e_0$ is symmetric. Moreover $e_0$ is contained in the common span of $ME_0^*M$ and $M^*E_0M^*$. In the next result we describe another property of $e_0$.

Let $Z(T)$ denote the center of $T$.

Proposition 6.12. (See [25, Corollary 11.3, Proposition 11.4].) For the matrix $e_0$ from Definition 6.11, we have

(i) $e_0 \in Z(T)$;

(ii) $E_0 e_0 = E_0$;

(iii) $E_0^* e_0 = E_0^*$;

(iv) $e_0^2 = e_0$. 

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Proof. (i) Using $e_0 = |X| \sum_{i=0}^{D} k_i^{-1} E_i^* E_i^*$ we see that $e_0$ commutes with $E^*_\ell$ for $0 \leq \ell \leq D$. Therefore $e_0$ commutes with everything in $M^*$. Using $e_0 = |X| \sum_{j=0}^{D} m_j^{-1} E_j E_j^*$ we see that $e_0$ commutes with $E_\ell$ for $0 \leq \ell \leq D$. Therefore $e_0$ commutes with everything in $M$. The result follows since $T$ is generated by $M, M^*$.

(ii) Observe

\[ E_0 e_0 = E_0 |X| \sum_{j=0}^{D} m_j^{-1} E_j E_j^* E_0 = |X| E_0 E_0^* E_0 = E_0. \]

(iii) Observe

\[ E_0^* e_0 = E_0^* |X| \sum_{i=0}^{D} k_i^{-1} E_i^* E_i^* E_0 = |X| E_0^* E_0 E_0^* = E_0^*. \]

(iv) Observe

\[ (I - e_0)e_0 \in (I - e_0) \text{Span}(ME_0^* M) = \text{Span}(M(I - e_0) E_0^* M) = \text{Span}(M^0 M) = 0. \]

We will say more about $e_0$ in the next section.

We finish this section with a comment.

**Lemma 6.13.** For $B \in M$ we have the logical implications

\[ B = 0 \iff BE_0^* = 0 \iff E_0^* B = 0. \]

Moreover for $C \in M^*$ we have the logical implications

\[ C = 0 \iff CE_0 = 0 \iff E_0 C = 0. \]

**Proof.** By Lemma 6.1 the following are nonzero for $0 \leq i \leq D$:

\[ E_i E_0^*, \quad E_0^* E_i, \quad E_i^* E_0, \quad E_0 E_i^*. \]

The result is a routine consequence of this.

\[ \square \]

## 7 The primary $T$-module

Throughout this section $\Gamma = (X, R)$ denotes a distance-regular graph with diameter $D \geq 3$. Fix $x \in X$ and write $T = T(x)$. Our next goal is to describe the primary $T$-module [13, Section 5], [25, Sections 8, 9], [54, 67, Lemma 3.6]. Recall the vector $1 = \sum_{y \in X} \hat{y}$. For $0 \leq i \leq D$ define the vector $1_i = \sum_{y \in \Gamma_i(x)} \hat{y}$. Observe that

\[ A_i \hat{x} = 1_i = E_i^* 1 \quad (0 \leq i \leq D). \]

Consequently

\[ ME_0^* V = M^* E_0 V. \]
Lemma 7.1. The vector space $M^*E_0^*V = M^*E_0V$ is an irreducible $T$-module.

Proof. Define $\mathcal{V} = M^*E_0^*V = M^*E_0V$. We have $MV \subseteq \mathcal{V}$ since $\mathcal{V} = M^*E_0^*V$. We have $M^*V \subseteq \mathcal{V}$ since $\mathcal{V} = M^*E_0V$. Therefore $T\mathcal{V} \subseteq \mathcal{V}$, so $\mathcal{V}$ is a $T$-module. We show that the $T$-module $\mathcal{V}$ is irreducible. The standard $T$-module $\hat{x}$ is a direct sum of irreducible $T$-modules. There exists an irreducible $T$-module that is not orthogonal to $\hat{x}$. This $T$-module contains $\hat{x}$, so it contains $M\hat{x} = \mathcal{V}$. This $T$-module must equal $\mathcal{V}$ by irreducibility. \hfill \Box

Definition 7.2. Define $\mathcal{V} = M^*E_0^*V = M^*E_0V$. The $T$-module $\mathcal{V}$ is called primary.

Lemma 7.3. For $0 \leq i \leq D$ we have

$$A_i^*1 = |X|E_i\hat{x}. \quad (13)$$

Proof. Both vectors in (13) have $y$-coordinate $|X|(E_i)_{x,y}$ for $y \in X$. \hfill \Box

Definition 7.4. For $0 \leq i \leq D$ let $1^*_i$ denote the common vector in (13).

We clarify the definitions. Note that $1_0 = \hat{x}$ and $1^*_0 = 1$. Moreover

$$1^*_0 = \sum_{i=0}^{D} 1_i, \quad 1_0 = |X|^{-1} \sum_{i=0}^{D} 1^*_i.$$ 

The following result is routinely verified.

Lemma 7.5. For the primary $T$-module $\mathcal{V}$,

(i) $1_i$ is a basis for $E_i^*\mathcal{V}$ ($0 \leq i \leq D$);

(ii) $\{1_i\}_{i=0}^{D}$ is a basis for $\mathcal{V}$;

(iii) $1^*_i$ is a basis for $E_i\mathcal{V}$ ($0 \leq i \leq D$);

(iv) $\{1^*_i\}_{i=0}^{D}$ is a basis for $\mathcal{V}$.

Next we explain how the primary $T$-module $\mathcal{V}$ is related to the matrix $e_0$ from Definition 6.11. Let $\mathcal{V}^\perp$ denote the orthogonal complement of $\mathcal{V}$ in $V$. We have an orthogonal direct sum $V = \mathcal{V} + \mathcal{V}^\perp$.

Lemma 7.6. With the above notation,

(i) $(e_0 - I)\mathcal{V} = 0$;

(ii) $e_0\mathcal{V}^\perp = 0$.

In other words, $e_0$ acts on $V$ as the orthogonal projection $V \to \mathcal{V}$.
Proof. (i) Observe
\[(e_0 - I)V = (e_0 - I)M E_0^* V = M(e_0 - I)E_0^* V = 0.\]

(ii) Note that \(V^\perp\) is a \(T\)-module, so \(e_0 V^\perp \subseteq V^\perp\). Also,
\[e_0 V^\perp \subseteq e_0 V \subseteq \text{Span}(ME_0^* MV) \subseteq ME_0^* V = V.\]
Therefore,
\[e_0 V^\perp \subseteq V^\perp \cap V = 0.\]

We saw in Lemma 7.5 that \(\{1_i\}_{i=0}^D\) and \(\{1_i^*\}_{i=0}^D\) are bases for the primary \(T\)-module \(V\). Next we describe how these bases are related.

Lemma 7.7. For \(0 \leq j \leq D\) we have

(i) \(1_j = |X|^{-1} D_j \sum_{i=0}^D u_j(\theta_i) 1_i^*;\)

(ii) \(1_j^* = m_j \sum_{i=0}^D u_i(\theta_j) 1_i.\)

Proof. (i) Observe
\[1_j = A_j \hat{x} = k_j \sum_{i=0}^D u_j(\theta_i) E_i \hat{x} = |X|^{-1} k_j \sum_{i=0}^D u_j(\theta_i) 1_i^*.\]

(ii) Observe
\[1_j^* = |X| E_j \hat{x} = m_j \sum_{i=0}^D u_i(\theta_j) A_i \hat{x} = m_j \sum_{i=0}^D u_i(\theta_j) 1_i.\]

Next we describe how the algebra \(T\) acts on the bases \(\{1_i\}_{i=0}^D\) and \(\{1_i^*\}_{i=0}^D\).

Lemma 7.8. For \(0 \leq i, j \leq D\) we have

(i) \(E_i^* 1_j = \delta_{i,j} 1_j;\)

(ii) \(A_i^* 1_j = m_i u_j(\theta_i) 1_j;\)

(iii) \(E_i 1_j = |X|^{-1} m_i k_j u_j(\theta_i) \sum_{h=0}^D u_h(\theta_i) 1_h;\)

(iv) \(A_i 1_j = \sum_{h=0}^D p_{i,j}^h 1_h.\)

Proof. These are routinely checked using the reduction rules in Lemmas 6.5, 6.6 along with Lemma 7.7.

Lemma 7.9. For \(0 \leq i, j \leq D\) we have
(i) \( E_i 1^*_j = \delta_{i,j} 1^*_j \);
(ii) \( A_i 1^*_j = k_i u_i(\theta_j) 1^*_j \);
(iii) \( E^*_i 1^*_j = |X|^{-1} k_i m_j u_i(\theta_j) \sum_{h=0}^D u_i(\theta_h) 1^*_h \);
(iv) \( A^*_i 1^*_j = \sum_{h=0}^D q^h i,j \).

Proof. These are routinely checked using the reduction rules in Lemmas 6.5, 6.6 along with Lemma 7.7.

Next we bring in the bilinear form.

Lemma 7.10. For \( 0 \leq i, j \leq D \) we have

(i) \( \langle 1_i, 1_j \rangle = \delta_{i,j} k_i \);
(ii) \( \langle 1^*_i, 1^*_j \rangle = \delta_{i,j} |X| m_i \);
(iii) \( \langle 1_i, 1^*_j \rangle = k_i m_j u_i(\theta_j) \).

Proof. (i) Routine.
(ii) Observe
\[
\langle 1^*_i, 1^*_j \rangle = |X|^2 \langle E_i \hat{x}, E_j \hat{x} \rangle = |X|^2 \langle \hat{x}, E_i E_j \hat{x} \rangle = \delta_{i,j} |X|^2 \langle \hat{x}, E_i \hat{x} \rangle = \delta_{i,j} |X| m_i.
\]
(iii) Observe
\[
\langle 1_i, 1^*_j \rangle = |X| \langle A_i \hat{x}, E_j \hat{x} \rangle = |X| \langle \hat{x}, A_i E_j \hat{x} \rangle = |X| u_i(\theta_j) \langle \hat{x}, E_j \hat{x} \rangle = k_i m_j u_i(\theta_j).
\]

8 The Krein condition and the triple product relations

The Krein condition states that the Krein parameters are nonnegative. In this section we will prove the Krein condition, and also derive some relations called the triple product relations.

Throughout this section \( \Gamma = (X, \mathcal{R}) \) denotes a distance-regular graph with diameter \( D \geq 3 \). Fix \( x \in X \) and write \( T = T(x) \).

In the following result, the second item is a variation on [5, Proposition 5.1].

Lemma 8.1. (See [5, Proposition 5.1], [24, Lemmas 3.1, 4.1].) For \( 0 \leq h, i, j, r, s, t \leq D \),

(i) \( \langle E^*_h A_i E^*_j, E^*_r A_s E^*_t \rangle = \delta_{h,r} \delta_{i,s} \delta_{j,t} k_h p^h i,j \);
(ii) \( \langle E_h A^*_i E_j, E_r A^*_s E_t \rangle = \delta_{h,r} \delta_{i,s} \delta_{j,t} m_h q^h i,j \).
Proof. (i) Using $\text{tr}(BC) = \text{tr}(CB)$,
\[
\langle E_h^*A_iE_j^*, E_r^*A_sE_t^* \rangle = \text{tr}(E_h^*A_iE_j^*(E_r^*A_sE_t^*)^t)
\]
\[
= \text{tr}(E_h^*A_iE_j^*E_r^*A_sE_t^*)
\]
\[
= \delta_{h,r}\delta_{j,t}\text{tr}(E_h^*A_iE_j^*A_s)
\]
and
\[
\text{tr}(E_h^*A_iE_j^*A_s) = \sum_{y \in X} \sum_{z \in X} (E_h^*)_{y,g}(A_i)_{y,z}(E_j^*)_{z,z}(A_s)_{z,y}
\]
\[
= \sum_{y \in X} \sum_{z \in X} (E_h^*)_{y,g}(A_i \circ A_s)_{y,z}(E_j^*)_{z,z}
\]
\[
= \delta_{i,s} \sum_{y \in X} \sum_{z \in X} (E_h^*)_{y,g}(A_i)_{y,z}(E_j^*)_{z,z}
\]
\[
= \delta_{i,s} \sum_{y \in \Gamma_i(x), \quad z \in \Gamma_j(x), \quad \partial(y,z)=i} 1
\]
\[
= \delta_{i,s} h_i^h p_{i,j}.
\]

(ii) We have
\[
\langle E_h^*A_iE_j, E_r^*A_sE_t \rangle = \text{tr}(E_h^*A_iE_j(E_r^*A_sE_t)^t)
\]
\[
= \text{tr}(E_h^*A_iE_jA_r^*E_t^*E_r)
\]
\[
= \delta_{h,r}\delta_{j,t}\text{tr}(E_h^*E_j^*A_s)
\]
and
\[
\text{tr}(E_h^*A_iE_jA_s) = \sum_{y \in X} \sum_{z \in X} (E_h^*)_{y,z}(A_i^*)_{z,z}(E_j)_{z,y}(A_s)_{y,y}
\]
\[
= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_h^*)_{y,z}(E_i)_{z,z}(E_j)_{z,y}(E_s)_{x,y}
\]
\[
= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_s)_{x,y}(E_h \circ E_j)_{y,z}(E_i)_{z,x}
\]
\[
= |X|^2 \left( (x,x)\text{-entry of } E_s(E_h \circ E_j)E_i \right)
\]
\[
= |X| \text{tr}(E_s(E_h \circ E_j)E_i)
\]
\[
= |X| \text{tr}((E_h \circ E_j)E_i E_s)
\]
\[
= \delta_{i,s} |X| \text{tr}((E_h \circ E_j)E_i)
\]
\[
= \delta_{i,s} \sum_{\ell=0}^D q_{h,j}^\ell \text{tr}(E_i E_i)
\]
\[
= \delta_{i,s} q_{h,j}^i \text{tr}(E_i)
\]
\[
= \delta_{i,s} q_{h,j}^i m_i
\]
\[
= \delta_{i,s} m_h q_{i,j}^h.
\]
Corollary 8.2. For $0 \leq h, i, j \leq D$ we have

\begin{enumerate}[(i)]
\item $\|E_h^* A_i E_j^*\|^2 = k_{h}^{h} p_{h,i,j}^{h}$;
\item $\|E_h A_i^* E_j\|^2 = m_{h} q_{h,i,j}^{h}$.
\end{enumerate}

Proof. Set $r = h$, $s = i$, $t = j$ in Lemma 8.1.

The following result is called the Krein condition. See [2, p. 69] for a discussion of the history.

Theorem 8.3. We have $q_{h,i,j}^{h} \geq 0$ for $0 \leq h, i, j \leq D$.

Proof. By Corollary 8.2 (ii) and since $\|B\|^2 \geq 0$ for all $B \in \text{Mat}_X(\mathbb{R})$.

Theorem 8.4. (See [67, Lemma 3.2].) For $0 \leq h, i, j \leq D$ we have

\begin{enumerate}[(i)]
\item $E_h^* A_i E_j^* = 0$ if and only if $p_{h,i,j}^{h} = 0$;
\item $E_h A_i^* E_j = 0$ if and only if $q_{h,i,j}^{h} = 0$.
\end{enumerate}

Proof. By Corollary 8.2 and since $\|B\|^2 = 0$ implies $B = 0$ for all $B \in \text{Mat}_X(\mathbb{R})$.

The relations in Theorem 8.4 are called the triple product relations.

We bring in some notation. For subspaces $R, S$ of $\text{Mat}_X(\mathbb{R})$, define

$$RS = \text{Span}\{rs | r \in R, s \in S\}.$$  

Theorem 8.5. (See [60, Section 7].) With the above notation,

\begin{enumerate}[(i)]
\item the vector space $M^* M M^*$ has an orthogonal basis

$$\{E_h^* A_i E_j^* | 0 \leq h, i, j \leq D, p_{h,i,j}^{h} \neq 0\};$$
\item the vector space $M M^* M$ has an orthogonal basis

$$\{E_h A_i^* E_j | 0 \leq h, i, j \leq D, q_{h,i,j}^{h} \neq 0\}.$$  

\end{enumerate}

Proof. By Lemma 8.1 and Theorem 8.4.

We mention a consequence of Theorem 8.4.

Proposition 8.6. For $0 \leq i, j \leq D$ we have

$$A_i E_j^* V \subseteq \sum_{0 \leq h \leq D} E_h^* V, \quad A_i^* E_j V \subseteq \sum_{0 \leq h \leq D} E_h V. \quad (14)$$
Proof. Concerning the containment on the left in (14),

\[ A_i E_j^* V = IA_i E_j^* V = \sum_{h=0}^{D} E_h^* A_i E_j^* V = \sum_{0 \leq h \leq D \atop p_{h,i,j}^i \neq 0} E_h^* A_i E_j^* V \subseteq \sum_{0 \leq h \leq D \atop p_{h,i,j}^i \neq 0} E_h^* V. \]

The containment on the right in (14) is similarly obtained. \qed

We finish this section with some comments about the primary \( T \)-module.

Lemma 8.7. For the primary \( T \)-module \( V \) and \( 0 \leq h, i, j \leq D \),

(i) \( E_h^* A_i E_j^* = 0 \) on \( V \) if and only if \( p_{h,i,j}^i = 0 \);

(ii) \( E_h A_i^* E_j = 0 \) on \( V \) if and only if \( q_{h,i,j}^i = 0 \).

Proof. Use parts (i), (iv) of Lemmas 7.8, 7.9 \qed

Lemma 8.8. For \( 0 \leq i, j \leq D \) the following holds on the primary \( T \)-module \( V \):

\[ E_i^* A_j E_i^* = p_{i,j}^i E_i^* \]

\[ E_i A_j^* E_i = q_{i,j}^i E_i. \]

Proof. Use parts (i), (iv) of Lemmas 7.8, 7.9 \qed

9 The function algebra and the Norton algebra

Throughout this section \( \Gamma = (X, R) \) denotes a distance-regular graph with diameter \( D \geq 3 \). In Theorem 8.4 we saw that each vanishing Krein parameter gives a triple product relation. In this section we consider the vanishing Krein parameters from another point of view, due to Cameron, Goethals, and Seidel [5, Proposition 5.1]. We will also briefly mention Norton algebras.

Recall the basis \( \{ \hat{y} \}_{y \in X} \) for the standard module \( V \).

Definition 9.1. We turn the vector space \( V \) into a commutative, associative, \( R \)-algebra with multiplication \( \circ \) defined as follows:

\[ \hat{y} \circ \hat{z} = \delta_{y,z} \hat{y} \quad y, z \in X. \] (15)

The algebra \( V \) is isomorphic to the algebra of functions \( X \to \mathbb{R} \). Motivated by this, we call the algebra \( V \) the function algebra.

In order to illustrate the multiplication \( \circ \), let \( v, w \in V \) and write

\[ v = \sum_{y \in X} v_y \hat{y}, \quad w = \sum_{y \in X} w_y \hat{y} \quad v_y, w_y \in \mathbb{R}. \]

Then

\[ v \circ w = \sum_{y \in X} v_y w_y \hat{y}. \]
Lemma 9.2. For the function algebra $V$, the multiplicative identity is $1 = \sum_{y \in X} \hat{y}$.

Proof. Routine. \hfill \Box

For the rest of this section, fix $x \in X$ and write $T = T(x)$.

Lemma 9.3. For $v \in V$ and $0 \leq i \leq D$,
\begin{equation}
A_i^* v = |X| E_i \hat{x} \circ v.
\end{equation}

Proof. Write $v = \sum_{y \in X} v_y \hat{y}$. Pick $y \in X$. The $y$-coordinate of $A_i^* v$ is
\[(A_i^* v)_y = (A_i^*)_{y,y} v_y = |X| (E_i)_{x,y} v_y.
\]
The $y$-coordinate of $E_i \hat{x} \circ v$ is
\[(E_i \hat{x} \circ v)_y = (E_i)_{y,y} v_y = (E_i)_{x,y} v_y.
\]
The result follows. \hfill \Box

We bring in some notation. For subspaces $R, S$ of $V$ define
\[R \circ S = \text{Span}\{r \circ s | r \in R, s \in S\}.
\]

Theorem 9.4. (See [5, Proposition 5.1].) For $0 \leq i, j \leq D$,
\begin{equation}
E_i V \circ E_j V = \sum_{0 \leq h \leq D} q_{i,j}^h E_h V.
\end{equation}

Proof. We first establish the inclusion $\subseteq$. By construction $E_i V = \text{Span}\{E_i \hat{y} | y \in X\}$. We show that for $y \in X$,
\[E_i \hat{y} \circ E_j V \subseteq \sum_{0 \leq h \leq D} q_{i,j}^h E_h V.
\]
Since our base vertex $x$ is arbitrary, we may assume without loss of generality that $x = y$. By Proposition 8.6 and Lemma 9.3,
\[E_i \hat{x} \circ E_j V = A_i^* E_j V \subseteq \sum_{0 \leq h \leq D} q_{i,j}^h E_h V.
\]
Next we establish the inclusion $\supseteq$. For $0 \leq h \leq D$ such that $q_{i,j}^h \neq 0$, we show that
$E_i V \circ E_j V \supseteq E_h V$. We have

\[
E_i V \circ E_j V = \text{Span}\{E_i \hat{y} \circ E_j \hat{z}|y, z \in X\}
\supseteq \text{Span}\{E_i \hat{y} \circ E_j \hat{y}|y \in X\}
= \text{Span}\{(E_i \circ E_j)\hat{y}|y \in X\}
= (E_i \circ E_j)V
\supseteq (E_i \circ E_j)E_h V
= \left(|X|^{-1} \sum_{\ell=0}^{D} q_{i,j}^\ell E_\ell \right)E_h V
= |X|^{-1} q_{i,j}^h E_h V
= E_h V.
\]

Next we briefly review the Norton algebra.

**Lemma 9.5.** (See [5, Proposition 5.2].) For $0 \leq j \leq D$ we endow $E_j V$ with a binary operation $\star$ as follows:

\[
u \star v = E_j (u \circ v) \quad u, v \in E_j V.
\]

Then for $u, v, w \in E_j V$ and $\alpha \in \mathbb{R}$,

(i) $u \star v = v \star u$;

(ii) $u \star (v + w) = u \star v + u \star w$;

(iii) $(\alpha u) \star v = \alpha (u \star v)$.

**Proof.** This is routinely checked. \hfill \square

Referring to Lemma 9.5, the vector space $E_j V$ together with the operation $\star$ is called the $j^{th}$ Norton algebra for $\Gamma$; see [5]. This algebra is commutative and nonassociative. It has no multiplicative identity in general. See [31,46,50,77] for recent results on the Norton algebra.

## 10 The function algebra and nondegenerate primitive idempotents

Throughout this section $\Gamma = (X, \mathcal{R})$ denotes a distance-regular graph with diameter $D \geq 3$. Recall the function algebra $V$ from Definition 9.1. We will show that a primitive idempotent $E$ of $\Gamma$ is nondegenerate if and only if the eigenspace $EV$ generates $V$ in the function algebra.

For a subspace $U \subseteq V$ we describe the subalgebra of $V$ generated by $U$. This subalgebra contains $1$ by Lemma 9.2. To see what else is in the subalgebra, we define a binary relation on $X$ called $U$-equivalence.
Definition 10.1. Vertices $y, z$ in $X$ are said to be $U$-equivalent whenever for all $u \in U$, the $y$-coordinate of $u$ is equal to the $z$-coordinate of $u$. Observe that $U$-equivalence is an equivalence relation.

Definition 10.2. For a subset $Y \subseteq X$ define $\hat{Y} = \sum_{y \in Y} \hat{y}$.

Lemma 10.3. For a subspace $U \subseteq V$ the following are equal:

(i) the subalgebra of the function algebra $V$ generated by $U$;
(ii) $\text{Span}\{\hat{Y} \mid Y \text{ is a } U\text{-equivalence class}\}$.

Proof. (i) $\subseteq$ (ii): Note that $\text{Span}\{\hat{Y} \mid Y \text{ is a } U\text{-equivalence class}\}$ is a subalgebra of $V$ that contains $U$.

(i) $\supseteq$ (ii): Let $Y$ denote a $U$-equivalence class. We show that $\hat{Y}$ is contained in the subalgebra of $V$ generated by $U$. List the $U$-equivalence classes $Y = Y_0, Y_1, \ldots, Y_n$. For $u \in U$ write

$$u = \sum_{i=0}^{n} \alpha_i(u) \hat{Y}_i \quad \alpha_i(u) \in \mathbb{R}.$$ 

For $1 \leq i \leq n$ there exists $u_i \in U$ such that $\alpha_0(u_i) \neq \alpha_i(u_i)$. We have

$$\hat{Y} = \prod_{i=1}^{n} \frac{u_i - \alpha_i(u_i)1}{\alpha_0(u_i) - \alpha_i(u_i)},$$

where the product is with respect to $\circ$. Therefore $\hat{Y}$ is contained in the subalgebra of $V$ generated by $U$.

Corollary 10.4. For a subspace $U \subseteq V$ the following are equivalent:

(i) $U$ generates the function algebra $V$;
(ii) each $U$-equivalence class has cardinality one.

Proof. By Lemma 10.3.

We have been discussing a subspace $U$ of $V$. Next we consider the special case in which $U = EV$, where $E$ is a primitive idempotent of $\Gamma$. Let us compute the $EV$-equivalence classes. We have $EV = \text{Span}\{E\hat{w} \mid w \in X\}$. Above Lemma 4.1 we saw that for $y, z \in X$,$$
y\text{-coordinate of } E\hat{z} = z\text{-coordinate of } E\hat{y}. \quad (19)$$

Lemma 10.5. Let $E$ denote a primitive idempotent of $\Gamma$. Then for $y, z \in X$ the following are equivalent:

(i) $y, z$ are in the same $EV$-equivalence class;
(ii) $E\hat{y} = E\hat{z}$.
Proof. Condition (i) holds, if and only if the \(y\)-coordinate of \(E\hat{w}\) is equal to the \(z\)-coordinate of \(E\hat{w}\) for all \(w \in X\). Condition (ii) holds, if and only if the \(w\)-coordinate of \(E\hat{y}\) is equal to the \(w\)-coordinate of \(E\hat{z}\) for all \(w \in X\). By these comments and (19), the conditions (i), (ii) are equivalent.

**Theorem 10.6.** Let \(E\) denote a primitive idempotent of \(\Gamma\). Then the following are equivalent:

(i) \(EV\) generates the function algebra \(V\);

(ii) \(E\) is nondegenerate.

Proof. By Corollary 10.4 and Lemma 10.5 we have the logical implications

\[
EV \text{ generates the function algebra } V \\
\Leftrightarrow \text{ each } EV\text{-equivalence class has cardinality one} \\
\Leftrightarrow \{E\hat{y}\}_{y \in X} \text{ are mutually distinct} \\
\Leftrightarrow E \text{ is nondegenerate}.
\]

\[\square\]

11 **The \(Q\)-polynomial property and Askey-Wilson duality**

In this section we discuss the \(Q\)-polynomial property and its connection to Askey-Wilson duality.

Throughout this section \(\Gamma = (X, R)\) denotes a distance-regular graph with diameter \(D \geq 3\). Recall the primitive idempotents \(\{E_i\}_{i=0}^D\) of \(\Gamma\).

**Definition 11.1.** The ordering \(\{E_i\}_{i=0}^D\) is called \(Q\)-polynomial whenever the following hold for \(0 \leq h, i, j \leq D\):

(i) \(q^h_{i,j} = 0\) if one of \(h, i, j\) is greater than the sum of the other two;

(ii) \(q^h_{i,j} \neq 0\) if one of \(h, i, j\) is equal to the sum of the other two.

**Definition 11.2.** We say that \(\Gamma\) is \(Q\)-polynomial whenever there exists at least one \(Q\)-polynomial ordering of the primitive idempotents.

For the rest of this section, we assume that the ordering \(\{E_i\}_{i=0}^D\) is \(Q\)-polynomial. Define

\[
c^*_i = q^i_{1,i-1} \quad (1 \leq i \leq D), \quad a^*_i = q^i_{1,i} \quad (0 \leq i \leq D), \quad b^*_i = q^i_{1,i+1} \quad (0 \leq i \leq D - 1).
\]

Note that \(a^*_0 = 0\) and \(c^*_1 = 1\). Moreover

\[
c^*_i > 0 \quad (1 \leq i \leq D), \quad b^*_i > 0 \quad (0 \leq i \leq D - 1).
\]
From Lemma 5.15(iv) we obtain
\[ c_i^* + a_i^* + b_i^* = m_1 \quad (0 \leq i \leq D), \]
where \( c_0^* = 0 \) and \( b_D^* = 0 \). By Lemma 5.14 we have \( m_i c_i^* = m_{i-1} b_{i-1}^* \) for \( 1 \leq i \leq D \). Consequently
\[ m_i = \frac{b_i^* b_{i-1}^* \cdots b_1^*}{c_i^* c_{i-1}^* \cdots c_1^*} \quad (0 \leq i \leq D). \tag{20} \]

For the rest of this section, fix \( x \in X \) and write \( T = T(x) \). Recall the bases \( \{ A_i^* \}_{i=0}^D \) and \( \{ E_i^* \}_{i=0}^D \) for \( M^* \). We abbreviate \( A_i^* = A_{i,x}^* \) and call this the dual adjacency matrix (with respect to \( x \) and the given \( Q \)-polynomial structure).

Our next goal is to show that \( A_i^* \) is a polynomial of degree \( i \) in \( A^* \) for \( 0 \leq i \leq D \).

**Lemma 11.3.** We have
\[
A^* A_i^* = b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^* \quad (1 \leq i \leq D - 1),
\]
\[
A^* A_D^* = b_D^* A_{D-1}^* + a_D^* A_D^*.
\]

**Proof.** This is \( A_i^* A_j^* = \sum_{h=0}^{D} q_{i,j,h}^* A_h^* \) with \( j = 1 \). \( \square \)

**Definition 11.4.** We define some polynomials \( \{ v_i^* \}_{i=0}^{D+1} \) in \( \mathbb{R}[\lambda] \) such that
\[
v_0^* = 1, \quad v_1^* = \lambda, \quad \lambda v_i^* = b_{i-1}^* v_{i-1}^* + a_i^* v_i^* + c_{i+1}^* v_{i+1}^* \quad (1 \leq i \leq D),
\]
where \( c_{D+1}^* = 1 \).

**Lemma 11.5.** The following (i)–(iv) hold:

(i) \( \deg v_i^* = i \) \( (0 \leq i \leq D + 1) \);

(ii) the coefficient of \( \lambda^i \) in \( v_i^* \) is \( (c_1^* c_2^* \cdots c_i^*)^{-1} \) \( (0 \leq i \leq D + 1) \);

(iii) \( v_i^*(A^*) = A_i^* \) \( (0 \leq i \leq D) \);

(iv) \( v_{D+1}^*(A^*) = 0 \).

**Proof.** (i), (ii) By Definition 11.4
(iii), (iv) Compare Lemma 11.3 and Definition 11.4 \( \square \)

**Corollary 11.6.** The following hold:

(i) the algebra \( M^* \) is generated by \( A^* \);

(ii) the minimal polynomial of \( A^* \) is \( c_1^* c_2^* \cdots c_D v_{D+1}^* \).

**Proof.** By Lemma 11.5 and since \( \{ A_i^* \}_{i=0}^D \) is a basis for \( M^* \). \( \square \)
Next we consider the eigenvalues of $A^*$. By Lemma 5.10(i),

$$A^* = m_1 \sum_{i=0}^{D} u_i(\theta_1) E_i^*.$$  

Abbreviate

$$\theta_i^* = m_1 u_i(\theta_1) \quad (0 \leq i \leq D), \quad (21)$$

so that

$$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*.$$  

(22)

**Lemma 11.7.** The following (i)--(iii) hold:

(i) the polynomial $v_{D+1}^*$ has $D + 1$ mutually distinct roots $\{\theta_i^*\}_{i=0}^{D};$

(ii) the eigenspaces of $A^*$ are $\{E_i^* V\}_{i=0}^{D};$

(iii) for $0 \leq i \leq D$, $\theta_i^*$ is the eigenvalue of $A^*$ for $E_i^* V$.

**Proof.** (i) The roots of $v_{D+1}^*$ are mutually distinct by Corollary 11.6(ii) and since $A^*$ is diagonal. These roots are $\{\theta_i^*\}_{i=0}^{D}$ by (22).

(ii), (iii) By (22). $\Box$

**Definition 11.8.** For $0 \leq i \leq D$ we call $\theta_i^*$ the $i^{th}$ dual eigenvalue of $\Gamma$ (with respect to the given $Q$-polynomial structure).

For convenience we adjust the normalization of the polynomials $v_i^*$.

**Definition 11.9.** Define the polynomial

$$u_i^* = \frac{v_i^*}{m_i} \quad (0 \leq i \leq D).$$  

(23)

**Lemma 11.10.** We have

$$u_0^* = 1, \quad u_1^* = m_1^{-1} \lambda, \quad \lambda u_i^* = c_i^* u_{i-1}^* + a_i^* u_i^* + b_i^* u_{i+1}^* \quad (1 \leq i \leq D - 1), \quad \lambda u_D^* - c_D^* u_{D-1}^* - a_D^* u_D^* = m_D^{-1} v_{D+1}^*.$$  

**Proof.** Evaluate the recurrence in Definition 11.4 using $v_i^* = m_i u_i^* (0 \leq i \leq D)$ and (20). $\Box$

We just defined the polynomials $\{u_i^*\}_{i=0}^{D}$. Next we explain how the polynomials $\{u_i\}_{i=0}^{D}$ and $\{u_i^*\}_{i=0}^{D}$ are related.

**Theorem 11.11.** (See [20], p. 14.) We have

$$u_i(\theta_j) = u_j^*(\theta_i^*) \quad (0 \leq i, j \leq D).$$  

(24)
Proof. Using Lemma 5.11 and Lemma 11.5(iii),
\[ u_i(\theta_j)E_i^* = m_j^{-1}A_j^* E_i^* = m_j^{-1}v_j^*(A^*)E_i^* = u_j^*(A^*)E_i^* = u_j^*(\theta_i^*)E_i^*. \]
The result follows. ∎

We will comment on Theorem 11.11 shortly.

Lemma 11.12. For \(0 \leq i \leq D\) we have

(i) \(v_i^*(\theta_0^*) = m_i^*;\)

(ii) \(u_i^*(\theta_0^*) = 1.\)

Proof. By Theorem 11.11 we obtain \(u_i^*(\theta_0^*) = u_0(\theta_i) = 1\), giving (ii). By Definition 11.9 we get (i). ∎

In Propositions 3.14 and 3.15 we obtained some orthogonality relations for the polynomials \(\{u_i\}_{i=0}^D\) and \(\{v_i\}_{i=0}^D\). Next we give the analogous orthogonality relations for the polynomials \(\{u_i^*\}_{i=0}^D\) and \(\{v_i^*\}_{i=0}^D\).

Proposition 11.13. We have

\[
\sum_{\ell=0}^D u_i^*(\theta_\ell^*)u_j^*(\theta_\ell^*)k_\ell = \delta_{i,j}m_i^{-1}|X| \quad (0 \leq i, j \leq D),
\]

\[
\sum_{\ell=0}^D u_i^*(\theta_\ell^*)v_j^*(\theta_\ell^*)m_\ell = \delta_{i,j}k_i^{-1}|X| \quad (0 \leq i, j \leq D).
\]

Proof. Combine Proposition 3.14 and Theorem 11.11. ∎

Proposition 11.14. We have

\[
\sum_{\ell=0}^D v_i^*(\theta_\ell^*)v_j^*(\theta_\ell^*)k_\ell = \delta_{i,j}m_i |X| \quad (0 \leq i, j \leq D),
\]

\[
\sum_{\ell=0}^D v_i^*(\theta_\ell^*)v_j^*(\theta_\ell^*)m_\ell^{-1} = \delta_{i,j}k_i^{-1} |X| \quad (0 \leq i, j \leq D).
\]

Proof. Combine Definition 11.9 and Proposition 11.13. ∎

Equation (24) is called Askey-Wilson duality \[73\] or Delsarte duality \[45\]. In \[45\] D. Leonard classifies the orthogonal polynomial sequences that satisfy Askey-Wilson duality. See \[2\, p. 260\] for a more comprehensive treatment. The classification shows that the orthogonal polynomial sequences that satisfy Askey-Wilson duality belong to the terminating branch of the Askey scheme; this branch consists of the \(q\)-Racah polynomials \[1\] along with their limiting cases \[39\]. The theory of Leonard pairs \[29\, 71\, 73\, 74\, 76\, 78\] provides a modern approach to Askey-Wilson duality.
12 The function algebra characterization of the $Q$-polynomial property

In this section we characterize the $Q$-polynomial property in terms of the function algebra.

Throughout this section $\Gamma = (X, \mathcal{R})$ denotes a distance-regular graph with diameter $D \geq 3$. Recall the function algebra $V$ and the primitive idempotents $\{E_i\}_{i=0}^D$. Recall that

$$E_0V = \mathbb{R}1, \quad 1 = \sum_{y \in X} \hat{y}.$$ 

We remind the reader that for subspaces $R, S$ of $V$,

$$R \circ S = \text{Span}\{r \circ s | r \in R, s \in S\}.$$  

For an integer $n \geq 0$ define

$$R^{\circ n} = R \circ R \circ \cdots \circ R \quad (n \text{ copies}).$$  

(25)

We interpret $R^{\circ 0} = \mathbb{R}1$.

The following result appears in [61, Lecture 23]. It is also mentioned in [19, p. 30].

**Theorem 12.1.** The following are equivalent:

(i) the ordering $\{E_i\}_{i=0}^D$ is $Q$-polynomial;

(ii) $E_1$ is nondegenerate and

$$E_1V \circ E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V \quad (0 \leq i \leq D),$$  

(26)

where $E_{-1} = 0$ and $E_{D+1} = 0$;

(iii) for $0 \leq i \leq D$,

$$\sum_{\ell=0}^i E_{\ell}V = \sum_{\ell=0}^i (E_1V)^{\circ \ell}. \quad (27)$$

**Proof.** (i) $\Rightarrow$ (ii) For the given $Q$-polynomial structure, the dual eigenvalues are $\theta_i^* = m_{1i}u_i(\theta_1) \ (0 \leq i \leq D)$. The scalars $\{\theta_i^*\}_{i=0}^D$ are mutually distinct, so $\theta_i^* \neq \theta_0^*$ for $1 \leq i \leq D$. Therefore $E_1$ is nondegenerate. By Theorem 9.4 we obtain

$$E_1V \circ E_iV = \sum_{0 \leq h \leq D \atop q_{1,i}^h \neq 0} E_hV \quad (0 \leq i \leq D).$$

The ordering $\{E_i\}_{i=0}^D$ is $Q$-polynomial, so $q_{1,i}^h = 0$ if $|h - i| > 1 \ (0 \leq h, i \leq D)$. By these comments we obtain (26).

(ii) $\Rightarrow$ (iii) For $0 \leq i \leq D$ define $P_i = \sum_{\ell=0}^i (E_1V)^{\circ \ell}$. By Theorem 9.4 there exists a subset
$S_i \subseteq \{0, 1, \ldots, D\}$ such that $P_i = \sum_{k \in S_i} E_k V$. We show that $S_i = \{0, 1, \ldots, i\}$ for $0 \leq i \leq D$. By construction $S_0 = \{0\}$ and $S_1 = \{0, 1\}$. Using (29) we obtain $S_i \subseteq \{0, 1, \ldots, i\}$ for $0 \leq i \leq D$. By construction $S_{i-1} \subseteq S_i$ for $1 \leq i \leq D$. For $1 \leq i \leq D$ we have $S_{i-1} \neq S_i$; otherwise $P_{i-1} = P_i$, which forces $i \geq 2$ and $E_1 V \circ P_{i-1} \subseteq P_{i-1}$, which forces $P_{i-1} = V$ by Theorem 10.6, which forces $S_{i-1} = \{0, 1, \ldots, D\}$, which contradicts $S_{i-1} \subseteq \{0, 1, \ldots, i-1\} \subseteq \{0, 1, \ldots, D-1\}$. By these comments $S_i = \{0, 1, \ldots, i\}$ for $0 \leq i \leq D$.

(iii) $\Rightarrow$ (i) Let $0 \leq i \leq D$ and $0 \leq j \leq D - i$. We show that $q_{i,j}^{i+j} \neq 0$ and $q_{i,j}^h = 0$ for $i + j < h \leq D$. By (27),

$$(E_0 V + E_1 V + \cdots + E_i V) \circ (E_0 V + E_1 V + \cdots + E_j V) = E_0 V + E_1 V + \cdots + E_{i+j} V.$$  

By this and Theorem 9.4 we obtain $q_{i,j}^h = 0$ for $i + j < h \leq D$. Also by Theorem 9.4 there exists $0 \leq r \leq i$ and $0 \leq s \leq j$ such that $q_{r,s}^{i+j} \neq 0$. By our above comments $i + j \leq r + s$. By construction $0 \leq r \leq i$ and $0 \leq s \leq j$, so $r = i$ and $s = j$. Therefore $q_{i,j}^{i+j} \neq 0$. 

### 13 Irreducible $T$-modules and tridiagonal pairs

Throughout this section, $\Gamma = (X, \mathcal{R})$ denotes a distance-regular graph with diameter $D \geq 3$. We assume that $\Gamma$ is $Q$-polynomial with respect to the ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents. Fix $x \in X$ and write $T = T(x)$. We will show that $A, A^*$ act on each irreducible $T$-module as a tridiagonal pair.

**Lemma 13.1.** The algebra $T$ is generated by $A, A^*$.

**Proof.** The algebra $T$ is generated by $M$ and $M^*$. Moreover $M$ is generated by $A$, and $M^*$ is generated by $A^*$. 

We review a few points:

- the eigenspaces of $A$ are $\{E_i V\}_{i=0}^D$;
- for $0 \leq i \leq D$, $\theta_i$ is the eigenvalue of $A$ for $E_i V$;
- the eigenspaces of $A^*$ are $\{E_i^* V\}_{i=0}^D$;
- for $0 \leq i \leq D$, $\theta_i^*$ is the eigenvalue of $A^*$ for $E_i^* V$;
- for $0 \leq i \leq D$ we have

$$AE_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V,$$

where $E_{-1}^* = 0$ and $E_D^* = 0$.

**Lemma 13.2.** For $0 \leq i \leq D$ we have

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V,$$

where $E_{-1} = 0$ and $E_D = 0$. 

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Proof. By the containment on the right in \([14]\), together with Definition \([11,1]\). □

In Section 2 we mentioned that the standard module \(V\) is an orthogonal direct sum of irreducible \(T\)-modules. Let \(W\) denote an irreducible \(T\)-module. Observe that \(W\) is an orthogonal direct sum of the nonzero subspaces among \(\{E_i^*W\}_{i=0}^D\). Similarly \(W\) is an orthogonal direct sum of the nonzero subspaces among \(\{E_iW\}_{i=0}^D\).

**Lemma 13.3.** Let \(W\) denote an irreducible \(T\)-module. Then for \(0 \leq i \leq D\) we have

\[
AE_i^*W \subseteq E_{i-1}^*W + E_i^*W + E_{i+1}^*W,
\]

\[
A^*E_iW \subseteq E_{i-1}W + E_iW + E_{i+1}W.
\]

Proof. By Lemma \([13,2]\) and the comment above it. □

Let \(W\) denote an irreducible \(T\)-module. By the endpoint of \(W\) we mean \(\min\{i|0 \leq i \leq D, E_i^*W \neq 0\}\). By the diameter of \(W\) we mean \(|\{i|0 \leq i \leq D, E_i^*W \neq 0\}| - 1\). By the dual endpoint of \(W\) we mean \(\min\{i|0 \leq i \leq D, E_iW \neq 0\}\). By the dual diameter of \(W\) we mean \(|\{i|0 \leq i \leq D, E_iW \neq 0\}| - 1\).

**Lemma 13.4.** \([67, \text{Lemma 3.4, Lemma 3.9}]\) Let \(W\) denote an irreducible \(T\)-module with endpoint \(\rho\) and diameter \(d\). Then \(\rho, d\) are nonnegative integers such that \(\rho + d \leq D\). Moreover the following (i), (ii) hold:

\[\text{(i) } E_i^*W \neq 0 \text{ if and only if } \rho \leq i \leq \rho + d \quad (0 \leq i \leq D);\]

\[\text{(ii) } W = \sum_{i=\rho}^{\rho+d} E_i^*W \quad \text{(orthogonal direct sum)}.\]

Proof. (i) By construction \(E_\rho^*W \neq 0\) and \(E_i^*W = 0\) for \(0 \leq i < \rho\). Suppose there exists an integer \(i\) \((\rho < i \leq \rho + d)\) such that \(E_i^*W = 0\). Define \(W' = E_\rho^*W + E_{\rho+1}^*W + \cdots + E_{\rho+d}^*W\). By construction \(0 = W' \subseteq W\). Also by construction, \(A^*W' \subseteq W'\). By Lemma \([13,3]\) and \(E_i^*W = 0\) we obtain \(AW' \subseteq W'\). By these comments \(W'\) is a \(T\)-module. We have \(W = W'\) since the \(T\)-module \(W\) is irreducible. This contradicts the fact that \(d\) is the diameter of \(W\). We conclude that \(E_i^*W \neq 0\) for \(\rho \leq i \leq \rho + d\). By the definition of the diameter \(d\) we have \(E_i^*W = 0\) for \(\rho + d < i \leq D\).

(ii) By (i) and the comments above Lemma \([13,3]\). □

**Lemma 13.5.** \([67, \text{Lemma 3.4, Lemma 3.12}]\) Let \(W\) denote an irreducible \(T\)-module with dual endpoint \(\tau\) and dual diameter \(\delta\). Then \(\tau, \delta\) are nonnegative integers such that \(\tau + \delta \leq D\). Moreover the following (i), (ii) hold:

\[\text{(i) } E_iW \neq 0 \text{ if and only if } \tau \leq i \leq \tau + \delta \quad (0 \leq i \leq D);\]

\[\text{(ii) } W = \sum_{i=\tau}^{\tau+\delta} E_iW \quad \text{(orthogonal direct sum)}.\]

Proof. Similar to the proof of Lemma \([13,4]\). □

**Theorem 13.6.** (See \([67, \text{Lemmas 3.9, 3.12}]\).) The pair \(A, A^*\) acts on each irreducible \(T\)-module as a tridiagonal pair.

Proof. By Definition \([1,1]\) and Lemmas \([13,1, 13,3, 13,4, 13,5]\). □

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Corollary 13.7. Let $W$ denote an irreducible $T$-module. The the diameter of $W$ is equal to the dual diameter of $W$.

Proof. By the comment below Note 1.2.

It is an ongoing project to describe the irreducible $T$-modules for a $Q$-polynomial distance-regular graph $\Gamma$. Comprehensive treatments can be found in [3, 32, 36, 37, 67–69, 78]. In addition, there are papers about the thin condition [10,12,21]; $\Gamma$ being bipartite [6,13,14,11]; $\Gamma$ being almost-bipartite [9,12]; $\Gamma$ being dual bipartite [22]; $\Gamma$ being almost dual bipartite [23]; $\Gamma$ being 2-homogeneous [15,17,18,53]; $\Gamma$ being tight [56]; $\Gamma$ being a hypercube [27]; $\Gamma$ being a Doob graph [63]; $\Gamma$ being a Johnson graph [49,62]; $\Gamma$ being a Grassmann graph [48]; $\Gamma$ being a dual polar graph [84]; $\Gamma$ having a spin model in the Bose-Mesner algebra [16,52]. Some miscellaneous topics about irreducible $T$-modules can be found in [26,34,35,40,43,44,55,59,60,75,83].

14 Recurrent sequences

In this section we have some comments about finite sequences that satisfy a 3-term recurrence. In later sections we will apply these comments to $Q$-polynomial distance-regular graphs.

Throughout this section fix an integer $D \geq 3$, and let $\{\theta_i\}_{i=0}^D$ denote scalars in $\mathbb{R}$.

Definition 14.1. Let $\beta, \gamma, \varrho$ denote scalars in $\mathbb{R}$.

(i) The sequence $\{\theta_i\}_{i=0}^D$ is said to be recurrent whenever $\theta_{i-1} \neq \theta_i$ for $2 \leq i \leq D-1$, and

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}$$

is independent of $i$ for $2 \leq i \leq D-1$.

(ii) The sequence $\{\theta_i\}_{i=0}^D$ is said to be $\beta$-recurrent whenever

$$\theta_{i-2} - (\beta + 1)\theta_{i-1} + (\beta + 1)\theta_i - \theta_{i+1}$$

is zero for $2 \leq i \leq D-1$.

(iii) The sequence $\{\theta_i\}_{i=0}^D$ is said to be $(\beta, \gamma)$-recurrent whenever

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma$$

for $1 \leq i \leq D-1$.

(iv) The sequence $\{\theta_i\}_{i=0}^D$ is said to be $(\beta, \gamma, \varrho)$-recurrent whenever

$$\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) = \varrho$$

for $1 \leq i \leq D$. 

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Lemma 14.2. The following are equivalent:

(i) the sequence \( \{\theta_i\}_{i=0}^{D} \) is recurrent;

(ii) the scalars \( \theta_{i-1} \neq \theta_i \) for \( 2 \leq i \leq D - 1 \), and there exists \( \beta \in \mathbb{R} \) such that \( \{\theta_i\}_{i=0}^{D} \) is \( \beta \)-recurrent.

Suppose (i), (ii) hold. Then the common value of (28) is equal to \( \beta + 1 \).

Proof. Routine.

Lemma 14.3. For \( \beta \in \mathbb{R} \) the following are equivalent:

(i) the sequence \( \{\theta_i\}_{i=0}^{D} \) is \( \beta \)-recurrent;

(ii) there exists \( \gamma \in \mathbb{R} \) such that \( \{\theta_i\}_{i=0}^{D} \) is \( (\beta, \gamma) \)-recurrent.

Proof. Routine.

Lemma 14.4. The following (i), (ii) hold for all \( \beta, \gamma \in \mathbb{R} \).

(i) Suppose \( \{\theta_i\}_{i=0}^{D} \) is \( (\beta, \gamma) \)-recurrent. Then there exists \( \varrho \in \mathbb{R} \) such that \( \{\theta_i\}_{i=0}^{D} \) is \( (\beta, \gamma, \varrho) \)-recurrent.

(ii) Suppose \( \{\theta_i\}_{i=0}^{D} \) is \( (\beta, \gamma, \varrho) \)-recurrent, and that \( \theta_{i-1} \neq \theta_{i+1} \) for \( 1 \leq i \leq D - 1 \). Then \( \{\theta_i\}_{i=0}^{D} \) is \( (\beta, \gamma) \)-recurrent.

Proof. Let \( p_i \) denote the expression on the left in (31), and observe

\[
p_i - p_{i+1} = (\theta_{i-1} - \theta_{i+1})(\theta_{i-1} - \beta \theta_i + \theta_{i+1} - \gamma)
\]

for \( 1 \leq i \leq D - 1 \). Assertions (i), (ii) are both routine consequences of this.

The following result is handy.

Lemma 14.5. Let \( \beta, \gamma, \varrho \in \mathbb{R} \) and assume that \( \{\theta_i\}_{i=0}^{D} \) is \( (\beta, \gamma, \varrho) \)-recurrent. Then

\[
(2 - \beta)\theta_i^2 - 2\gamma \theta_i - \varrho = (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \quad (0 \leq i \leq D),
\]

(32)

where \( \theta_{-1} \) and \( \theta_{D+1} \) are defined by (30) at \( i = 0 \) and \( i = D \).

Proof. To verify (32) for \( 1 \leq i \leq D \), eliminate \( \theta_{i+1} \) using (30), and evaluate the result using (31). To verify (32) for \( 0 \leq i \leq D - 1 \), eliminate \( \theta_{i-1} \) using (30), and evaluate the result using (31).


## 15 The tridiagonal relations

Throughout this section \( \Gamma = (X, \mathcal{R}) \) denotes a distance-regular graph with diameter \( D \geq 3 \). Let \( \{E_i\}_{i=0}^D \) denote a \( Q \)-polynomial ordering of the primitive idempotents of \( \Gamma \). Fix \( x \in X \) and write \( T = T(x) \). We will show that \( A, A^* \) satisfy a pair of relations called the tridiagonal relations. We will also obtain a recurrence satisfied by the eigenvalues \( \{\theta_i\}_{i=0}^D \) and the dual eigenvalues \( \{\theta_i^*\}_{i=0}^D \). We now state our two main results.

**Theorem 15.1.** (See [69, Lemma 5.4].) With the above notation, there exist real numbers \( \beta, \gamma, \gamma^*, \varrho, \varrho^* \) such that

\[
0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*],
\]

\[
0 = [A^*, A^*2A - \beta A^*AA^* + AA^*2 - \gamma^*(A^*A + AA^*) - \varrho^*A].
\]

**Definition 15.2.** The relations (33), (34) are called the tridiagonal relations; see [72].

**Theorem 15.3.** (See [45, Proposition 3] and [2, Theorem 5.1].) With the above notation, the scalars

\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}
\]

are equal and independent of \( i \) for \( 2 \leq i \leq D - 1 \).

We will prove Theorems 15.1, 15.3 shortly.

**Lemma 15.4.** For \( 0 \leq i, j, r \leq D \) we have

(i) \( E_i^*A^rE_j^* = \begin{cases} 0 & \text{if } r < |i - j|; \\ \neq 0 & \text{if } r = |i - j|, \end{cases} \)

(ii) \( E_iA^rE_j = \begin{cases} 0 & \text{if } r < |i - j|; \\ \neq 0 & \text{if } r = |i - j|. \end{cases} \)

**Proof.** (i) By Theorem 8.4(i), \( E_i^*A^rE_j^* = 0 \) if and only if \( p_{r,i}^0 = 0 \). The matrix \( A^r \) is a polynomial in \( A \) with degree exactly \( r \). The scalar \( p_{r,i}^0 \) is zero if \( r < |i - j| \) and nonzero if \( r = |i - j| \). The result follows.

(ii) Similar to the proof of (i).

We are going to prove a sequence of results. Each result has a ‘dual’ obtained by interchanging \( A, A^* \) and \( E_i, E_i^* \) for \( 0 \leq i \leq D \). We will not state each dual explicitly.

**Lemma 15.5.** For \( 0 \leq i, j, r, s \leq D \) we have

\[
E_i^*A^rA^sE_j^* = \begin{cases} \theta_{j+s}^*E_i^*A^rA^sE_j^* & \text{if } r + s = i - j; \\ \theta_{j-s}^*E_i^*A^rA^sE_j^* & \text{if } r + s = j - i; \\ 0 & \text{if } r + s < |i - j|. \end{cases}
\]
Proof. Using \( I = \sum_{h=0}^{D} E_h^* \) and \( A^* = \sum_{h=0}^{D} \theta_h^* E_h^* \) we find
\[
E_i^* A^* I A^* E_j^* = \sum_{h=0}^{D} E_i^* A^* E_h^* E_j^*, \quad E_i^* A^* A^* E_j^* = \sum_{h=0}^{D} \theta_h^* E_i^* A^* E_h^* E_j^*.
\]
In the above sums, evaluate each summand using Lemma 15.4(i). The result follows.

Lemma 15.6. For \( 0 \leq i \leq D - 1 \),
\[
E_i A^* E_{i+1} - E_{i+1} A^* E_i = (E_0 + E_1 + \cdots + E_i) A^* - A^* (E_0 + E_1 + \cdots + E_i).
\]

Proof. Recall the convention \( E_{-1} = 0 \). For \( 0 \leq j \leq D - 1 \) we have
\[
E_j A^* = E_j A^* (E_0 + E_1 + \cdots + E_D) = E_j A^* (E_{j-1} + E_j + E_{j+1}),
\]
and similarly
\[
A^* E_j = (E_{j-1} + E_j + E_{j+1}) A^* E_j.
\]
Sum (35), (36) over \( j = 0, 1, \ldots, i \). Take the difference between the two sums.

Recall the Bose-Mesner algebra \( M \) of \( \Gamma \).

Lemma 15.7. We have
\[
\text{Span}\{RA^* S - SA^* R | R, S \in M\} = \{YA^* - A^* Y | Y \in M\}.
\]

Proof. Recall that \( \{E_i\}_{i=0}^{D} \) is a basis for \( M \). Note that \( \{E_0 + \cdots + E_i\}_{i=0}^{D} \) is a basis for \( M \). Observe that
\[
\text{Span}\{RA^* S - SA^* R | R, S \in M\}
= \text{Span}\{E_i A^* E_j - E_j A^* E_i | 0 \leq i, j \leq D\}
= \text{Span}\{E_i A^* E_{i+1} - E_{i+1} A^* E_i | 0 \leq i \leq D - 1\}
= \text{Span}\{(E_0 + \cdots + E_i) A^* - A^* (E_0 + \cdots + E_i) | 0 \leq i \leq D - 1\}
= \text{Span}\{(E_0 + \cdots + E_i) A^* - A^* (E_0 + \cdots + E_i) | 0 \leq i \leq D\}
= \{YA^* - A^* Y | Y \in M\}.
\]

Let \( \lambda, \mu \) denote commuting indeterminates. We define some polynomials \( P(\lambda, \mu) \) and \( P^*(\lambda, \mu) \) as follows. Given real numbers \( \beta, \gamma, \varrho \) define
\[
P(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma (\lambda + \mu) - \varrho.
\]
Given real numbers \( \beta, \gamma^*, \varrho^* \) define
\[
P^*(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma^* (\lambda + \mu) - \varrho^*.
\]

Lemma 15.8. For real numbers \( \beta, \gamma, \varrho \) the following are equivalent:

\[
\]
Proof. Let $C$ denote the expression on the right in (i). We have

$$C = (E_0 + \cdots + E_D)C(E_0 + \cdots + E_D) = \sum_{i=0}^{D} \sum_{j=0}^{D} E_iCE_j.$$ 

For $0 \leq i, j \leq D$ use $E_iA = \theta_iE_i$ and $AE_j = \theta_jE_j$ to get

$$E_iCE_j = E_iA^*E_j(\theta_i - \theta_j)P(\theta_i, \theta_j).$$

(i) $\Rightarrow$ (ii): For $1 \leq i \leq D$ we show $P(\theta_{i-1}, \theta_i) = 0$. We have $C = 0$, so

$$0 = E_{i-1}CE_i = E_{i-1}A^*E_i(\theta_{i-1} - \theta_i)P(\theta_{i-1}, \theta_i).$$

We have $E_{i-1}A^*E_i \neq 0$ and $\theta_{i-1} - \theta_i \neq 0$, so $P(\theta_{i-1}, \theta_i) = 0$.

(ii) $\Rightarrow$ (i): We show that $C = 0$. The polynomial $P(\lambda, \mu)$ is symmetric, so $P(\theta_i, \theta_{i-1}) = 0$ for $1 \leq i \leq D$. To show that $C = 0$, it suffices to show that $E_iCE_j = 0$ for $0 \leq i, j \leq D$. Let $i, j$ be given. We evaluate $E_iCE_j$ using (39). If $|i - j| > 1$ then $E_iA^*E_j = 0$, so $E_iCE_j = 0$. If $|i - j| = 1$ then $P(\theta_i, \theta_j) = 0$, so $E_iCE_j = 0$. If $i = j$ then $\theta_i - \theta_j = 0$, so $E_iCE_j = 0$. In all cases $E_iCE_j = 0$, so $C = 0$.

We are now ready to prove Theorem 15.1.

Proof of Theorem 15.1. By Lemma 15.7 (with $R = A^2$ and $S = A$), there exists $Z \in M$ such that

$$A^2A^*A - AA^*A^2 = ZA^* - A^*Z.$$  

The matrices $\{A^i\}_{i=0}^{D}$ form a basis for $M$, so there exists a polynomial $f$ with degree at most $D$ such that $Z = f(A)$. Let $d$ denote the degree of $f$. We show that $d = 3$. First assume that $d > 3$. Multiply each term in (40) on the left by $E_d^*$ and the right by $E_0^*$. Evaluate the result using Lemmas 15.3 15.5 to get

$$0 = c(\theta_0^* - \theta_d^*)E_d^*A^dE_0^*,$$

where $c$ is the leading coefficient of $f$. By construction $c \neq 0$. We have $\theta_0^* - \theta_d^* \neq 0$ since $d \neq 0$. Also $E_d^*A^dE_0^* \neq 0$ by Lemma 15.3. Therefore (41) gives a contradiction. Next assume that $d < 3$. Multiply each term in (40) on the left by $E_3^*$ and on the right by $E_0^*$. Evaluate the result using Lemmas 15.3 15.5 to get

$$0 = (\theta_1^* - \theta_2^*)E_3^*A^3E_0^*.$$

We have $\theta_1^* - \theta_2^* \neq 0$. Also $E_3^*A^3E_0^* \neq 0$ by Lemma 15.3. Therefore (42) gives a contradiction.

We have shown that $d = 3$. Define $\beta = c^{-1} - 1$. Divide both sides of (40) by $c$, and evaluate the result using $d = 3$ and $c^{-1} = \beta + 1$. We find that there exist $\gamma, \varrho \in \mathbb{R}$ such that

$$(\beta + 1)(A^2A^*A - AA^*A^2) = A^3A^* - A^*A^3 - \gamma(A^2A^* - A^*A^2) - \varrho(AA^* - A^*A).$$
In this equation, we rearrange the terms to get

$$0 = [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \varrho A^*]. \quad (43)$$

This is the first tridiagonal relation. To get the second tridiagonal relation, pick an integer $i$ ($2 \leq i \leq D - 1$). Multiply each term in (43) on the left by $E^*_i - 2$ and on the right by $E^*_i + 1$. Simplify the result using Lemma 15.3 to get

$$0 = E^*_i - 2 A^3 E^*_i + (\beta + 1) \theta^*_i - (\beta + 1) \theta^*_i + (\beta + 1) \theta^*_i - \theta^*_i + 1). \quad (44)$$

We have $E^*_i - 2 A^3 E^*_i \neq 0$ by Lemma 15.4 so the coefficient in (44) must be zero. Therefore the sequence $\{\theta^*_i\}_{i=0}^{D}$ is $\beta$-recurrent. By Lemma 14.4 there exists $\gamma^* \in \mathbb{R}$ such that $\{\theta^*_i\}_{i=0}^{D}$ is $(\beta, \gamma^*)$-recurrent. By Lemma 14.4 there exists $\varrho^* \in \mathbb{R}$ such that $\{\theta^*_i\}_{i=0}^{D}$ is $(\beta, \gamma^*, \varrho^*)$-recurrent. Consequently $P^*(\theta^*_i, \theta^*_i) = 0$ for $1 \leq i \leq D$. Now $\beta, \gamma^*, \varrho^*$ satisfy (34) by Lemma 15.8.

Proposition 15.9. (See [72, Theorem 4.3].) We refer to the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ from Theorem 15.1.

(i) The expressions

$$\frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are both equal to $\beta + 1$ for $2 \leq i \leq D - 1$;

(ii) $\gamma = \theta_{i-1} - \beta \theta_i + \theta_{i+1}$ ($1 \leq i \leq D - 1$);

(iii) $\gamma^* = \theta^*_{i-1} - \beta \theta^*_i + \theta^*_{i+1}$ ($1 \leq i \leq D - 1$);

(iv) $\varrho^* = \theta^2_{i-1} - \beta \theta_{i-1} \theta_i + \theta^2_i - \gamma (\theta_{i-1} + \theta_i)$ ($1 \leq i \leq D$);

(v) $\varrho^* = \theta^2_{i-1} - \beta \theta^*_{i-1} \theta^*_i + \theta^2_i - \gamma^*(\theta^*_{i-1} + \theta^*_i)$ ($1 \leq i \leq D$).

Proof. We start with item (iv).

(iv) By (33) and Lemma 15.8
(v) By (34) and Lemma 15.8
(ii) By Lemma 14.4 and (iv) above.
(iii) By Lemma 14.4 and (v) above.
(i) The sequence $\{\theta_i\}_{i=0}^{D}$ is $(\beta, \gamma)$-recurrent by (ii), so $\{\theta_i\}_{i=0}^{D}$ is $\beta$-recurrent. Similarly $\{\theta^*_i\}_{i=0}^{D}$ is $\beta$-recurrent. The result follows.

Theorem 15.3 is immediate from Proposition 15.9(i).

Remark 15.10. The tridiagonal relations (33), (34) are the defining relations for the tridiagonal algebra $[72, \text{Definition 3.9}].$ Special cases of the tridiagonal algebra include the universal enveloping algebra of the Onsager Lie algebra $O$ [76, Remark 34.5], the $q$-Onsager algebra $O_q$ [79], and the positive part $U_q^+$ [76, Remark 34.7] of the $q$-deformed enveloping algebra $U_q(\mathfrak{sl}_2)$ [11].

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16 The primary $T$-module and the Askey-Wilson relations

Throughout this section $\Gamma = (X, R)$ denotes a distance-regular graph with diameter $D \geq 3$. Fix $x \in X$ and write $T = T(x)$. In Section 7 we described the primary $T$-module. In this section, we give more information under the assumption that $\Gamma$ is $Q$-polynomial.

Throughout this section, we assume that $\Gamma$ is $Q$-polynomial with respect to the ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents.

**Lemma 16.1.** For the primary $T$-module $V$ the following hold.

(i) With respect to the basis $\{1_i\}_{i=0}^D$ the matrices representing $A$ and $A^*$ are

$$A : \begin{pmatrix} a_0 & b_0 \\ c_1 & a_1 & b_1 \\ & c_2 & \ddots & \ddots \\ 0 & & \cdots & b_{D-1} \\ & & & c_D \\ \end{pmatrix}, \quad A^* : \text{diag}(\theta_0, \theta_1, \ldots, \theta_D).$$

(ii) With respect to the basis $\{1_i^*\}_{i=0}^D$ the matrices representing $A$ and $A^*$ are

$$A : \text{diag}(\theta_0, \theta_1, \ldots, \theta_D), \quad A^* : \begin{pmatrix} a_0^* & b_0^* \\ c_1^* & a_1^* & b_1^* \\ & c_2^* & \ddots & \ddots \\ 0 & & \cdots & b_{D-1}^* \\ & & & c_D^* \\ \end{pmatrix}.$$  

**Proof.** By parts (ii), (iv) of Lemmas 7.8, 7.9.

**Lemma 16.2.** The pair $A, A^*$ acts on the primary $T$-module $V$ as a Leonard pair.

**Proof.** By Definition 1.3 and Lemma 16.1.

In Section 15 we saw that $A, A^*$ satisfy the tridiagonal relations. In the literature, there is another pair of relations called the Askey-Wilson relations \[85\], that resemble the tridiagonal relations but are more elementary. It is shown in \[81\] Theorem 1.5] that any Leonard pair satisfies the Askey-Wilson relations. This and Lemma 16.2 imply that $A, A^*$ satisfy the Askey-Wilson relations on the primary $T$-module $V$. We will show directly, that on the primary $T$-module $V$ the $A, A^*$ satisfy the Askey-Wilson relations.

We have a comment about Lemma 15.4. In that lemma there are some inequalities. We will need the fact that these inequalities hold on the primary $T$-module $V$.

**Lemma 16.3.** For $0 \leq i, j \leq D$ and $r = |i - j|$ the following hold on the primary $T$-module $V$:

$$E_i^* A^* E_j^* \neq 0, \quad E_i A^* E_j \neq 0.$$
We now display the Askey-Wilson relations. Recall the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ from Theorem 15.1. The following result is a variation on [81, Theorem 1.5].

**Theorem 16.4.** (See [81, Theorem 1.5].) There exist real numbers $\omega, \eta, \eta^*$ such that the following hold on the primary $T$-module $V$:

\[
\begin{align*}
A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* &= \gamma^* A^2 + \omega A + \eta I, \quad (45) \\
A^2A - \beta A^*AA^* + AA^2 - \gamma^*(A^*A + AA^*) - \varrho^* A &= \gamma A^*A^2 + \omega^* A + \eta^* I. \quad (46)
\end{align*}
\]

**Proof.** Let $L$ denote the expression on the left in (45). By the tridiagonal relation (33), we see that $[A, L] = 0$ on $V$. The restriction of $A$ to $V$ is diagonalizable and has all eigenspaces of dimension one. Recall from linear algebra that for a diagonalizable linear transformation $T$ of dimension one. Recall from linear algebra that for a diagonalizable linear transformation $\sigma$ that has all eigenspaces of dimension one, any linear transformation that commutes with $\sigma$ must be a polynomial in $\sigma$. Therefore there exists a polynomial $f$ such that $L = f(A)$ on $V$. The minimal polynomial of $A$ on $V$ has degree $D + 1$, so we may choose $f$ such that its degree is at most $D$. Let $d$ denote the degree of $f$. We show that $d \leq 2$. Suppose that $d \geq 3$. In the equation $L = f(A)$, multiply each term on the left by $E_d^*$ and on the right by $E_0^*$. We have $E_d^* L E_0^* = 0$ by Lemma 15.3, so $0 = E_d^* f(A) E_0^*$ on $V$. By Lemma 15.4 we have $E_d^* f(A) E_0^* = \alpha E_d^* A^d E_0^*$, where $\alpha$ is the leading coefficient of $f$. By construction $\alpha \neq 0$. By Lemma 16.3 we have $E_d^* A^d E_0^* \neq 0$ on $V$. This is a contradiction, so $d \leq 2$. There exist real numbers $\varepsilon, \omega, \eta$ such that $L = \varepsilon A^2 + \omega A + \eta I$ on $V$. We show that $\gamma^* = \varepsilon$. Note that on $V$,

\[
\gamma^* E_2^* A^2 E_0^* = (\theta_0^* - \beta \theta_1^* + \theta_2^*) E_2^* A^2 E_0^* = E_2^* L E_0^* = E_2^* (\varepsilon A^2 + \omega A + \eta I) E_0^* = \varepsilon E_2^* A^2 E_0^*.
\]

By Lemma 16.3 we have $E_2^* A^2 E_0^* \neq 0$ on $V$, so $\gamma^* = \varepsilon$. We have shown that (45) holds on $V$. Interchanging the roles of $A, A^*$ in the argument so far, we see that there exist real numbers $\omega^*, \eta^*$ such that on $V$,

\[
A^2A - \beta A^*AA^* + AA^2 - \gamma^*(A^*A + AA^*) - \varrho^* A = \gamma A^*A^2 + \omega^* A^* + \eta^* I. \quad (47)
\]

We show that $\omega = \omega^*$. Take the commutator of (45) with $A^*$. This shows that on $V$,

\[
\begin{align*}
A^2 A^* - \beta A^* AA^* + \beta A^* AA^* A - A^* A^2 - \gamma(AA^* - A^2 A) &= \gamma^*(A^2 A^* - A^* A^2) + \omega(AA^* - A^* A).
\end{align*}
\]

Next, take the commutator of (47) with $A$. This shows that on $V$,

\[
A^2 A^2 - \beta A^* AA^* A + \beta AA^* AA^* - A^2 A^2 - \gamma^*(A^* A^2 - A^2 A) &= \gamma(A^2 A - AA^2) + \omega^*(A^* A - AA^*).
\]

Adding the above two equations, we find that on $V$,

\[0 = (\omega - \omega^*)(AA^* - A^* A).\]

We have $AA^* \neq A^* A$ on $V$, because the $T$-module $V$ is irreducible. Therefore $\omega = \omega^*$. We have shown that (46) holds on $V$. \qed
Definition 16.5. (See [85].) The relations (45), (46) are called the Askey-Wilson relations.

Next we consider how to compute the scalars $\omega, \eta, \eta^*$ from Theorem 16.4. To facilitate this computation, we bring in some notation. Recall from Proposition 15.9 that
\begin{align*}
\gamma &= \theta_{i+1} - \beta \theta_i + \theta_i + 1 \quad (1 \leq i \leq D - 1), \quad (48) \\
\gamma^* &= \theta_{i+1} - \beta \theta_i^* + \theta_i^* + 1 \quad (1 \leq i \leq D - 1). \quad (49)
\end{align*}

Definition 16.6. Define the real numbers
\[
\theta_{-1}, \quad \theta_{D+1}, \quad \theta_{-1}^*, \quad \theta_{D+1}^*
\]
such that (48), (49) hold at $i = 0$ and $i = D$.

Recall the polynomials $P, P^*$ from (37), (38).

Lemma 16.7. The following hold for $0 \leq i \leq D$:
\begin{enumerate}
  \item[(i)] $P(\theta_i, \theta_i) = (\theta_i - \theta_{i+1})(\theta_i - \theta_{i+2})$;
  \item[(ii)] $P^*(\theta_i^*, \theta_i^*) = (\theta_i^* - \theta_{i+1}^*)(\theta_i^* - \theta_{i+2}^*)$.
\end{enumerate}

Proof. (i) By Lemma 14.5 and since $\{\theta_i\}_{i=0}^D$ is $(\beta, \gamma, \varrho)$-recurrent.

(ii) Similar to the proof of (i). \hfill \Box

Proposition 16.8. (See [81, Theorem 5.3].) With the above notation, we have
\begin{enumerate}
  \item[(i)] $\omega = a_i^* (\theta_i - \theta_{i+1}) + a_i^* (\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_{i+1} + \theta_i) \quad (1 \leq i \leq D)$,
  \item[(ii)] $\omega = a_i (\theta_i^* - \theta_{i+1}^*) + a_{i-1} (\theta_{i-1}^* - \theta_{i-2}^*) - \gamma(\theta_{i+1}^* + \theta_i^*) \quad (1 \leq i \leq D)$,
  \item[(iii)] $\eta = a_i^* (\theta_i - \theta_{i+1})(\theta_i - \theta_{i+1}) - \omega \theta_i - \gamma^* \theta_i^2 \quad (0 \leq i \leq D)$,
  \item[(iv)] $\eta^* = a_i (\theta_i^* - \theta_{i+1}^*)(\theta_i^* - \theta_{i+1}^*) - \omega \theta_i^* - \gamma \theta_i^2 \quad (0 \leq i \leq D)$.
\end{enumerate}

Proof. We start with item (iii).

(iii) In the Askey-Wilson relation (45), multiply each term on the left by $E_i$ and on the right by $E_i^*$. This shows that on $V$,
\[
E_i A^* E_i P(\theta_i, \theta_i) = E_i (\gamma^* \theta_i^2 + \omega \theta_i + \eta).
\]

By Lemma 8.8 we have $E_i A^* E_i = a_i^* E_i$ on $V$. By Lemma 7.5(iii), $E_i \neq 0$ on $V$. By these comments,
\[
a_i^* P(\theta_i, \theta_i) = \gamma^* \theta_i^2 + \omega \theta_i + \eta.
\]

Solve this equation for $\eta$, and evaluate the result using Lemma 16.7(i).

(iv) Similar to the proof of (iii).

(i) Subtract (iii) (at $i$) from (iii) (at $i - 1$).

(ii) Similar to the proof of (i).

\hfill \Box
The Pascasio characterization of the $Q$-polynomial property

In [57] Pascasio characterized the $Q$-polynomial distance-regular graphs using the dual eigenvalues $\theta_i^*$ and the intersection numbers $a_i$. In this section we give a proof of her result that uses some ideas of Hanson [28].

Throughout this section $\Gamma = (X, R)$ denotes a distance-regular graph with diameter $D \geq 3$.

**Definition 17.1.** Let $E$ denote a nontrivial primitive idempotent of $\Gamma$. We say that $\Gamma$ is $Q$-polynomial with respect to $E$ whenever there exists a $Q$-polynomial ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of $\Gamma$ such that $E = E_1$.

**Theorem 17.2.** (See [57, Theorem 1.2].) Let $E = |X|^{-1} \sum_{i=0}^D \theta_i^* A_i$ denote a nontrivial primitive idempotent of $\Gamma$. Then $\Gamma$ is $Q$-polynomial with respect to $E$ if and only if the following conditions hold:

(i) $\theta_i^* \neq \theta_0^* \ (1 \leq i \leq D)$;

(ii) there exist real numbers $\beta, \gamma^*$ such that

$$\theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* = \gamma^* \quad (1 \leq i \leq D - 1); \tag{50}$$

(iii) there exist real numbers $\gamma, \omega, \eta^*$ such that

$$a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) = \gamma \theta_i^{*2} + \omega \theta_i^* + \eta^* \quad (0 \leq i \leq D),$$

where $\theta_{-1}^*, \theta_{D+1}^*$ are defined such that (50) holds at $i = 0$ and $i = D$.

We will prove Theorem 17.2 shortly. In the meantime, let $\{E_i\}_{i=0}^D$ denote an ordering of the primitive idempotents of $\Gamma$, and abbreviate $E = E_1$. Let $x \in X$ and write $T = T(x)$, $A^* = A_1^*$.

**Definition 17.3.** With the above notation, define a graph $\Delta_E$ with vertex set $\{0, 1, \ldots, D\}$ such that for $0 \leq i, j \leq D$, the vertices $i, j$ are adjacent whenever $i \neq j$ and $q_{i,j}^1 \neq 0$.

**Lemma 17.4.** For the graph $\Delta_E$ in Definition 17.3, the vertex 0 is adjacent to vertex 1 and no other vertex.

*Proof.* We have $q_{0,j}^1 = \delta_{1,j}$ for $0 \leq j \leq D$. \qed

**Lemma 17.5.** For distinct vertices $i, j$ of $\Delta_E$ the following are equivalent:

(i) $i, j$ are adjacent in $\Delta_E$;

(ii) $E_i A^* E_j \neq 0$;

(iii) $E_i A^* E_j \neq 0$ on the primary $T$-module $V$.

*Proof.* By the triple product relations and Lemma 8.7. \qed
Lemma 17.6. For the graph \( \Delta_E \) in Definition 17.3, assume that \( E \) is nondegenerate. Then \( \Delta_E \) is connected.

Proof. Let \( S \subseteq \{0, 1, \ldots, D\} \) denote the connected component of \( \Delta_E \) that contains 0, 1. We show that \( S = \{0, 1, \ldots, D\} \). Define \( U = \sum_{i \in S} E_i V \). By construction \( E_0 V \subseteq U \) and \( EV \subseteq U \). By Lemma 5.14 and Theorem 9.4, the following holds for \( 0 \leq i \leq D \):

\[
EV \circ E_i V = \sum_{0 \leq h \leq D \atop q_{i,h} = 0} E_h V.
\]

Therefore \( EV \circ U \subseteq U \). We assume that \( E \) is nondegenerate, so the function algebra \( V \) is generated by \( EV \). By these comments \( U = V \), so \( S = \{0, 1, \ldots, D\} \).

Proof of Theorem 17.2. First we assume that \( \Gamma \) is \( Q \)-polynomial with respect to \( E \). We saw earlier that \( E \) satisfies (i)–(iii). Next we assume that \( E \) satisfies (i)–(iii), and show that \( \Gamma \) is \( Q \)-polynomial with respect to \( E \). Fix \( x \in X \) and write \( T = T(x) \). Abbreviate \( E^* = E_1^* \) and \( A^* = A_1^* \). For the time being, let \( \{E_i\}_{i=2}^{D} \) denote any ordering of the remaining nontrivial primitive idempotents of \( \Gamma \). For \( 0 \leq i \leq D \) let \( \theta_i \) denote the eigenvalue of \( \Gamma \) for \( E_i \). Consider the graph \( \Delta_E \) from Definition 17.3. By Lemma 17.4, in \( \Delta_E \) the vertex 0 is adjacent to vertex 1 and no other vertices. Note that \( E \) is nondegenerate by condition (i) in the theorem statement, so the graph \( \Delta_E \) is connected in view of Lemma 17.6. We will show that \( \Delta_E \) is a path. By (50) the sequence \( \{\theta_i^*\}_{i=0}^{D} \) is \( (\beta, \gamma^*) \)-recurrent. By Lemma 14.4 there exists \( \gamma^* \in \mathbb{R} \) such that \( \{\theta_i^*\}_{i=0}^{D} \) is \( (\beta, \gamma^*, \gamma^*) \)-recurrent. By this and Lemma 14.5 we obtain

\[
P^*(\theta_i^* , \theta_i^*) = (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) \quad (0 \leq i \leq D),
\]

where the polynomial \( P^* \) is from (38).

Claim 1. On the primary \( T \)-module \( V \),

\[
A^2 - \beta A^* AA^* + AA^* - \gamma^* (A^* A + AA^*) - \gamma^* A = \gamma A^* + \omega A^* + \eta^* I.
\]

Proof of Claim 1. Let \( \mathcal{L} \) denote the left-hand side of (51). On the \( T \)-module \( V \),

\[
\mathcal{L} = \sum_{i=0}^{D} \sum_{j=0}^{D} E_i^* \mathcal{L} E_j^* = \sum_{i=0}^{D} \sum_{j=0}^{D} E_i^* A E_j^* P^*(\theta_i^*, \theta_j^*)
\]

\[
= \sum_{i=0}^{D} E_i^* A E_i^* P^*(\theta_i^*, \theta_i^*)
\]

\[
= \sum_{i=0}^{D} E_i^* A E_i^* P^*(\theta_i^*, \theta_i^*)
\]

\[
= \sum_{i=0}^{D} E_i^* a_i (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*)
\]

\[
= \sum_{i=0}^{D} E_i^* (\gamma \theta_i^2 + \omega \theta_i^* + \eta^*)
\]

\[
= \gamma A^* + \omega A^* + \eta^* I.
\]

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We have shown (51).

Claim 2. Let \( i, j \) denote vertices in \( \Delta_E \) that are at distance \( \partial(i, j) = 2 \). Assume that there exists a unique vertex \( h \) in \( \Delta_E \) that is adjacent to both \( i \) and \( j \). Then \( \gamma = \theta_i - \beta \theta_h + \theta_j \).

Proof of Claim 2. In the equation (51), multiply each term on the left by \( E_i \) and on the right by \( E_j \). Simplify the result using Lemma \ref{lem:17.5}. To aid this simplification, note that

\[
E_i A^2 E_j = E_i A^* \left( \sum_{r=0}^{D} E_r \right) A^* E_j = E_i A^* E_h A^* E_j
\]

and

\[
E_i A^* A A^* E_j = E_i A^* \left( \sum_{r=0}^{D} \theta_r E_r \right) A^* E_j = \theta_h E_i A^* E_h A^* E_j.
\]

By these comments, the following holds on the \( T \)-module \( V \):

\[
0 = E_i A^* E_h A^* E_j (\theta_i - \beta \theta_h + \theta_j - \gamma).
\]

We show that \( E_i A^* E_h A^* E_j \neq 0 \) on \( V \). By Lemma \ref{lem:17.5} and the construction, \( E_i A^* E_h \) and \( E_h A^* E_j \) are nonzero on \( V \). The dimension of \( E_h V \) is one, so \( E_h A^* E_j V = E_h V \). By these comments \( E_i A^* E_h A^* E_j V = E_i A^* E_h V \neq 0 \). We have shown that \( E_i A^* E_h A^* E_j \neq 0 \) on \( V \), so \( \gamma = \theta_i - \beta \theta_h + \theta_j \). Claim 2 is proved.

We can now easily show that \( \Delta_E \) is a path. Since \( \Delta_E \) is connected, and since vertex 0 is adjacent only to vertex 1, it suffices to show that each vertex in \( \Delta_E \) is adjacent to at most two other vertices in \( \Delta_E \). Suppose there exists a vertex \( i \) of \( \Delta_E \) that is adjacent to at least three vertices in \( \Delta_E \). Of all such vertices, pick \( i \) such that \( \partial(0, i) \) is minimal. Without loss of generality, we may assume that the vertices of \( \Delta_E \) are labelled such that \( \partial(0, i) = i \), and vertices \( 0, 1, 2, \ldots, i \) form a path in \( \Delta_E \). By construction \( i \geq 1 \). By assumption, there exist distinct vertices \( j, j' \) in \( \Delta_E \) that are adjacent to \( i \) and not equal to \( i - 1 \). By construction, \( \partial(i-1, j) = 2 \) and \( i \) is the unique vertex in \( \Delta_E \) that is adjacent to both \( i - 1, j \). By Claim 2, \( \gamma = \theta_{i-1} - \beta \theta_i + \theta_j \). Repeating the argument with \( j \) replaced by \( j' \), we obtain \( \gamma = \theta_{i-1} - \beta \theta_i + \theta_{j'} \). By these comments \( \theta_j = \theta_{j'} \) for a contradiction. We conclude that \( \Delta_E \) is a path. Relabelling \( \{E_j\}_{j=0}^{D} \) if necessary, we may assume without loss of generality that vertices \( i - 1 \) and \( i \) are adjacent in \( \Delta_E \) for \( 1 \leq i \leq D \). The ordering \( \{E_i\}_{i=0}^{D} \) is \( Q \)-polynomial by Theorem \ref{thm:12.1} because item (ii) of that theorem is satisfied by \( E = E_1 \). By these comments and Definition \ref{def:17.4} the graph \( \Gamma \) is \( Q \)-polynomial with respect to \( E \). □

Note 17.7. Referring to Theorem \ref{thm:17.2} assume that \( \Gamma \) is \( Q \)-polynomial with respect to \( E \). For this \( Q \)-polynomial structure the eigenvalue sequence \( \{\theta_i\}_{i=0}^{D} \) is obtained as follows:

1. \( \theta_0 \) is the valency \( k \) of \( \Gamma \);
2. \( \theta_1 = k \theta_1^* / \theta_0^* \) by Lemma \ref{lem:17.8};
3. \( \theta_2, \theta_3, \ldots, \theta_D \) are recursively found using

\[
\theta_{i-1} - \beta \theta_i + \theta_{i+1} = \gamma \quad (1 \leq i \leq D - 1),
\]

where \( \beta, \gamma \) are from Theorem \ref{thm:17.2}.

Note 17.8. A variation on Theorem \ref{thm:17.2} is given in \ref{thm:38}.
18 Distance-regular graphs with classical parameters

In [4] Section 6.1 Brouwer, Cohen, and Neumaier introduce a type of distance-regular graph, said to have classical parameters. In [4] Section 8.4 they show that these graphs are $Q$-polynomial. In this section we give a short proof of this fact, using the Pascasio characterization from Theorem 17.2.

Throughout this section $\Gamma = (X, \mathcal{R})$ denotes a distance-regular graph with diameter $D \geq 3$. Let $k$ denote the valency of $\Gamma$.

We now recall what it means for $\Gamma$ to have classical parameters. We will use the following notation. For a nonzero integer $b$ define

$$\left[ \begin{array}{c} i \\ 1 \end{array} \right] = \left[ \begin{array}{c} i \\ 1 \end{array} \right]_b = 1 + b + b^2 + \cdots + b^{i-1}.$$  

**Definition 18.1.** (See [4] p. 193.) The graph $\Gamma$ has **classical parameters** $(D, b, \alpha, \sigma)$ whenever the intersection numbers satisfy

$$c_i = \left[ \begin{array}{c} i \\ 1 \end{array} \right] \left( 1 + \alpha \left[ \begin{array}{c} i - 1 \\ 1 \end{array} \right] \right) \quad (0 \leq i \leq D),$$

$$b_i = \left( \left[ \begin{array}{c} D \\ 1 \end{array} \right] - \left[ \begin{array}{c} i \\ 1 \end{array} \right] \right) \left( \sigma - \alpha \left[ \begin{array}{c} i \\ 1 \end{array} \right] \right) \quad (0 \leq i \leq D).$$

**Theorem 18.2.** (See [4] Section 8.4.) Assume that $\Gamma$ has classical parameters $(D, b, \alpha, \sigma)$. Then the following (i)–(iv) hold.

(i) $\theta = \frac{b_1}{b} - 1$ is an eigenvalue of $\Gamma$.

(ii) Let $E = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$ denote the associated primitive idempotent. Then

$$\frac{\theta_i^*}{\theta_0^*} = 1 + \left( \frac{\theta}{k} - 1 \right) \left[ \begin{array}{c} i \\ 1 \end{array} \right] b^{1-i} \quad (0 \leq i \leq D).$$

(iii) $\theta \neq k$.

(iv) $\Gamma$ is $Q$-polynomial with respect to $E$.

**Proof.** (i), (ii) Apply Lemma 4.8 with $\theta = \frac{b_1}{b} - 1$ and

$$\sigma_i = 1 + \left( \frac{\theta}{k} - 1 \right) \left[ \begin{array}{c} i \\ 1 \end{array} \right] b^{1-i} \quad (0 \leq i \leq D).$$

(iii) Suppose $\theta = k$. We have $b_1 = b(k + 1)$, so $b > 0$. We have $b \geq 1$ since $b$ is an integer. Therefore $b_1 \geq k + 1$, a contradiction. We have shown that $\theta \neq k$.  

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(iv) The conditions of Theorem 17.2 are satisfied using $\beta = b + b^{-1}$ and

$$
\gamma = \frac{\alpha(b^2 + 1) + \sigma(b - 1) + 1 - b}{b},
$$

$$
\gamma^* = \theta_0^* \frac{\alpha(b^2 - b) + \sigma(b - 1) + b^2 - b}{kb},
$$

$$
\omega = \Psi(\theta_1^* - \theta_0^*) - 2\gamma \theta_0^*,
$$

$$
\eta^* = \gamma \theta_0^{*2} - \Psi \theta_0^*(\theta_1^* - \theta_0^*),
$$

where

$$
\Psi = 1 - \sigma - \frac{\alpha}{b}\left[\frac{D + 1}{1}\right].
$$

Lemma 18.3. (See [4, Corollary 8.4.2].) Assume that $\Gamma$ has classical parameters $(D, b, \alpha, \sigma)$. For the $Q$-polynomial structure in Theorem 18.2, the eigenvalue sequence is

$$
\theta_i = \frac{b_i}{b^i} - \left[\frac{i}{1}\right] \quad (0 \leq i \leq D).
$$

Proof. Routine calculation using Note 17.7.

19 The balanced set characterization of the $Q$-polynomial property

In this section we give a characterization of the $Q$-polynomial property, known as the balanced set condition. The result is given in Theorem 19.2 below. The result first appeared in [66]. More recent versions can be found in [3, 4, 61, 70].

Throughout this section $\Gamma = (X, R)$ denotes a distance-regular graph with diameter $D \geq 3$. Recall the valency $k$ of $\Gamma$.

Lemma 19.1. Fix $x \in X$ and write $T = T(x)$. Then for $0 \leq i, j, \ell \leq D$ and $y, z \in X$ the $(y, z)$-entry of $A_i A_j^* A_j$ is equal to

$$
|X| \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} \langle E_{\ell} \hat{x}, E_{\ell} \hat{w} \rangle.
$$

Proof. We have

$$
(A_i A_j^* A_j)_{y, z} = \sum_{w \in X} (A_i)_{y, w} (A_j^*)_{w, w} (A_j)_{w, z} = \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} (A_j^*)_{w, w}
$$

$$
= |X| \sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} \langle E_{\ell} \hat{x}, E_{\ell} \hat{w} \rangle.
$$

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For the rest of this section, let $E$ denote a nontrivial primitive idempotent of $\Gamma$, and write $E = |X|^{-1} \sum_{n=0}^{D} \theta_n^* A_n$. Recall that $E$ is nondegenerate if and only if $\theta_n^* \neq \theta_0^*$ for $1 \leq n \leq D$. Also recall that for $y, z \in X$,

$$\langle E\hat{y}, E\hat{z} \rangle = |X|^{-1} \theta_n^*, \quad n = \partial(y, z). \quad (52)$$

**Theorem 19.2.** (See [66, Theorem 1.1].) With the above notation, the following (i)–(iii) are equivalent:

(i) $\Gamma$ is $Q$-polynomial with respect to $E$;

(ii) $E$ is nondegenerate and for all $0 \leq i, j \leq D$ and all distinct $y, z \in X$,

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w} - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} E\hat{w} = p_i^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} (E\hat{y} - E\hat{z}),$$

where $h = \partial(y, z)$;

(iii) $E$ is nondegenerate and for all $y, z \in X$,

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w} - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} E\hat{w} \in \text{Span}(E\hat{y} - E\hat{z}).$$

**Proof.** Fix $x \in X$ and write $T = T(x)$. Write $E = E_1$ and $A^* = A_1^*$.

(i) $\Rightarrow$ (ii) $E$ is nondegenerate since $\{\theta_n^*\}_{n=0}^D$ are mutually distinct. Recall the Bose-Mesner algebra $M$. By Lemma [15.7],

$$\text{Span}\{RA^* S - SA^* R | R, S \in M\} = \{YA^* - A^*Y | Y \in M\}.$$  

Taking $R = A_i$ and $S = A_j$, we obtain

$$A_i A^* A_j - A_j A^* A_i = \sum_{n=1}^{D} r_{i,j}^n (A^* A_n - A_n A^*) \quad (53)$$

for some scalars $\{r_{i,j}^n\}_{n=1}^D$ in $\mathbb{R}$. We show that

$$r_{i,j}^h = p_{i,j}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} \quad (1 \leq h \leq D). \quad (54)$$

Let $h$ be given, and pick $z \in X$ such that $\partial(x, z) = h$. We compute the $(x, z)$-entry of each term in (53). We do this using Lemma [19] with $\ell = 1$ and $y = x$ along with (52). A brief calculation yields

$$p_{i,j}^h (\theta_i^* - \theta_j^*) = r_{i,j}^h (\theta_0^* - \theta_h^*),$$

and (54) follows. Pick distinct $y, z \in X$ and write $h = \partial(y, z)$. We show that

$$\sum_{w \in \Gamma_i(y) \cap \Gamma_j(z)} E\hat{w} - \sum_{w \in \Gamma_j(y) \cap \Gamma_i(z)} E\hat{w} = p_{i,j}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} (E\hat{y} - E\hat{z}). \quad (55)$$
Since the base vertex \( x \) is arbitrary, without loss of generality it suffices to show that in (55), the left-hand side minus the right-hand side is orthogonal to \( E \hat{x} \). This orthogonality is routinely obtained from (53) and (54) along with Lemma 19.1 (with \( \ell = 1 \)).

(ii) \( \Rightarrow \) (iii). Clear.

(iii) \( \Rightarrow \) (i). We assume that \( E \) is nondegenerate, so \( \theta^*_n \neq \theta^*_0 \) for \( 1 \leq n \leq d \).

Claim 1. Pick an integer \( h \) (\( 1 \leq h \leq D \)) and \( y, z \in X \) such that \( \partial(y, z) = h \). Then

\[
\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E \hat{w} - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} E \hat{w} = r^h_{1,2}(E \hat{y} - E \hat{z}),
\]

where

\[
r^h_{1,2} = p^h_{1,2} \frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_h}.
\]

Proof of Claim 1. By assumption there exists \( \alpha \in \mathbb{R} \) such that

\[
\sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E \hat{w} - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} E \hat{w} = \alpha(E \hat{y} - E \hat{z}).
\]

For each term in the above equation, take the inner product with \( E \hat{y} \) using (52). A brief calculation yields

\[
p^h_{1,2}(\theta^*_1 - \theta^*_2) = \alpha(\theta^*_0 - \theta^*_h).
\]

Therefore

\[
\alpha = p^h_{1,2} \frac{\theta^*_1 - \theta^*_2}{\theta^*_0 - \theta^*_h},
\]

and Claim 1 is proved.

Claim 2. We have

\[
AA^*A_2 - A_2A^*A = \sum_{n=1}^{D} r^n_{1,2}(A^*A_n - A_nA^*).
\]

Proof of Claim 2. For \( y, z \in X \) we compute the \((y, z)\)-entry of the left-hand side of (57) minus the right-hand side of (57). We do this computation using Lemma 19.1 (with \( \ell = 1 \)). For \( y = z \) the \((y, z)\)-entry is zero. For \( y \neq z \) the \((y, z)\)-entry is equal to \(|X|\) times

\[
\left\langle E \hat{x}, \sum_{w \in \Gamma(y) \cap \Gamma_2(z)} E \hat{w} - \sum_{w \in \Gamma_2(y) \cap \Gamma(z)} E \hat{w} - r^h_{1,2}(E \hat{y} - E \hat{z}) \right\rangle,
\]

where \( h = \partial(y, z) \). The above scalar is zero by Claim 1. Claim 2 is proved.

Conceivably \( \theta^*_1 = \theta^*_2 \). In this case \( r^h_{1,2} = 0 \) for \( 1 \leq h \leq D \). So by Claim 2,\( AA^*A_2 = A_2A^*A \).

In this equation we eliminate \( A_2 \) using \( A_2 = (A^2 - a_1A - kI)/c_2 \) and get

\[
A^2A^*A - AA^*A^2 = k(A^*A - AA^*).
\]
We will return to this equation shortly.

Claim 3. Assume that \( \theta_1^* \neq \theta_2^* \). Then there exist scalars \( \beta, \gamma, \varrho \in \mathbb{R} \) such that

\[
0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*].
\]  

(59)

Proof of Claim 3. Referring to (56), the scalar \( p_{i,2}^h \) is zero if \( h > 3 \) and nonzero if \( h = 3 \). Therefore \( r_{i,2}^h \) is zero if \( h > 3 \) and nonzero if \( h = 3 \). The matrices \( A_2 \) and \( A_3 \) appear in (57). Recall that \( A_2 \) and \( A_3 \) are polynomials in \( A \) that have degrees 2 and 3, respectively. Evaluating (57) using this fact, we obtain

\[
A^3A^* - A^*A^3 \in \text{Span}(A^2A^*A - AA^*A^2, A^2AA^* - A^*A^2, AA^* - A^*A).
\]

Therefore there exist \( \beta, \gamma, \varrho \in \mathbb{R} \) such that

\[
A^3A^* - A^*A^3 = (\beta + 1)(A^2A^*A - AA^*A^2) + \gamma(A^2AA^* - A^*A^2) + \varrho(AA^* - A^*A).
\]

In this equation we rearrange the terms to obtain (59). Claim 3 is proved.

Recall our notation \( E = E_1 \). For the time being, let \( \{E_i\}_{i=2}^D \) denote any ordering of the remaining nontrivial primitive idempotents of \( \Gamma \). For \( 0 \leq i \leq D \) let \( \theta_i \) denote the eigenvalue of \( \Gamma \) for \( E_i \). Recall the graph \( \Delta_E \) from Definition 17.3. The graph \( \Delta_E \) is connected since \( E \) is nondegenerate. Recall that in \( \Delta_E \), vertex 0 is adjacent to vertex 1 and no other vertex. We will show that \( \Delta_E \) is a path. To do this, it suffices to show that each vertex \( i \) in \( \Delta_E \) is adjacent to at most 2 vertices in \( \Delta_E \).

Claim 4. For distinct vertices \( i, j \) in \( \Delta_E \) that are adjacent,

(i) if \( \theta_1^* = \theta_2^* \) then \( \theta_i \theta_j = -k \);

(ii) if \( \theta_1^* \neq \theta_2^* \) then \( P(\theta_i, \theta_j) = 0 \), where we recall

\[
P(\lambda, \mu) = \lambda^2 - \beta \lambda \mu + \mu^2 - \gamma(\lambda + \mu) - \varrho.
\]

Proof of Claim 4. First assume that \( \theta_1^* = \theta_2^* \). Then (58) holds. In (58), multiply each term on the left by \( E_i \) and on the right by \( E_j \). Simplify the result to get

\[
0 = E_iA^*E_j(\theta_i - \theta_j)(\theta_i \theta_j + k).
\]

We have \( E_iA^*E_j \neq 0 \) since \( i, j \) are adjacent in \( \Delta_E \). The scalar \( \theta_i - \theta_j \) is nonzero since \( i \neq j \). Therefore \( \theta_i \theta_j + k = 0 \) so \( \theta_i \theta_j = -k \). Next assume that \( \theta_1^* = \theta_2^* \). Then (59) holds. In (59), multiply each term on the left by \( E_i \) and the right by \( E_j \). Simplify the result to get

\[
0 = E_iA^*E_j(\theta_i - \theta_j)P(\theta_i, \theta_j).
\]

We have \( E_iA^*E_j \neq 0 \) since \( i, j \) are adjacent in \( \Delta_E \). The scalar \( \theta_i - \theta_j \) is nonzero since \( i \neq j \). Therefore \( P(\theta_i, \theta_j) = 0 \).

Claim 5. We have \( \theta_1^* \neq \theta_2^* \).

Proof of Claim 5. Suppose that \( \theta_1^* = \theta_2^* \). By Claim 4 and since vertex 0 is adjacent to vertex 1, we have \( \theta_0 \theta_1 = -k \). We have \( \theta_0 = k \) so \( \theta_1 = -1 \). The graph \( \Delta_E \) is connected, so vertex 1 is adjacent to some nonzero vertex \( j \). By Claim 4 we have \( \theta_1 \theta_j = -k \). By this and \( \theta_1 = -1 \),
we obtain $\theta_j = k$. This implies $j = 0$, for a contradiction. Claim 5 is proved.

Claim 6. Each vertex $i$ in $\Delta_E$ is adjacent at most two vertices in $\Delta_E$.

Proof of Claim 6. By Claims 4, 5 we see that for each vertex $j$ in $\Delta_E$ that is adjacent vertex $i$, the eigenvalue $\theta_j$ is a root of the polynomial

$$P(\theta_i, \mu) = \theta_i^2 - \beta \theta_i \mu + \mu^2 - \gamma (\theta_i + \mu) - \varphi.$$ 

This polynomial is quadratic in $\mu$, so it has at most two distinct roots. Claim 6 is proved.

We have shown that the graph $\Delta_E$ is a path. Consequently the graph $\Gamma$ is $Q$-polynomial with respect to $E$. \hfill \Box

The balanced set condition has subtle combinatorial implications; see [7, 8, 12, 47, 51, 65, 69].

20 Directions for future research

In this section, we extend the $Q$-polynomial property to graphs that are not necessarily distance-regular.

Throughout this section, let $\Gamma = (X, \mathcal{R})$ denote a finite, undirected, connected graph, without loops or multiple edges, with diameter $D \geq 1$. We do not assume that $\Gamma$ is distance-regular. Let $A$ denote the adjacency matrix of $\Gamma$.

**Definition 20.1.** An ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of $A$ is called $Q$-polynomial whenever

$$\sum_{\ell=0}^{i} V_\ell = \sum_{\ell=0}^{i} (V_1)^{\otimes \ell} \quad (0 \leq i \leq d).$$

(60)

We are using the notation (25).

**Definition 20.2.** The graph $\Gamma$ is said to be $Q$-polynomial whenever there exists at least one $Q$-polynomial ordering of the eigenspaces of $A$.

**Lemma 20.3.** Assume that $\Gamma$ is $Q$-polynomial. Then $\Gamma$ is regular.

**Proof.** Let $\{V_i\}_{i=0}^d$ denote a $Q$-polynomial ordering of the eigenspaces of $A$. Setting $i = 0$ in (60), we obtain $V_0 = \mathbb{R}1$. Therefore, the vector $1$ is an eigenvector for $A$. Consequently $\Gamma$ is regular. \hfill \Box

Assume for the moment that $\Gamma$ is $Q$-polynomial. We do not expect that $\Gamma$ is distance-regular. However, we do expect that the following conjecture is true. Let $M$ denote the subalgebra of $\text{Mat}_{X}(\mathbb{R})$ generated by $A$.

**Conjecture 20.4.** If $\Gamma$ is $Q$-polynomial, then $B \circ C \in M$ for all $B, C \in M$.

**Note 20.5.** Conjecture 20.4 asserts that if $\Gamma$ is $Q$-polynomial then $M$ is the Bose-Mesner algebra of a symmetric association scheme [4, Section 2.2].
For the rest of this section, fix \( x \in X \). Define \( D(x) = \max\{\partial(x, y) | y \in X\} \). For \( 0 \leq i \leq D(x) \) define the matrix \( E^*_i = E^*_i(x) \) as in line (4). Note that \( \{E^*_i\}_{i=0}^{D(x)} \) form a basis for a commutative subalgebra \( M^* = M^*(x) \) of \( \text{Mat}_X(\mathbb{R}) \).

**Definition 20.6.** Let \( \{V_i\}_{i=0}^d \) denote an ordering of the eigenspaces of \( A \). A matrix \( A^* \in \text{Mat}_X(\mathbb{R}) \) is called a dual adjacency matrix (with respect to \( x \) and \( \{V_i\}_{i=0}^d \)) whenever:

(i) \( A^* \) generates \( M^* \);

(ii) for \( 0 \leq i \leq d \) we have

\[
A^*V_i \subseteq V_{i-1} + V_i + V_{i+1},
\]

where \( V_{-1} = 0 \) and \( V_{d+1} = 0 \).

**Definition 20.7.** An ordering \( \{V_i\}_{i=0}^d \) of the eigenspaces of \( A \) is called \( Q \)-polynomial with respect to \( x \) whenever there exists a dual adjacency matrix with respect to \( x \) and \( \{V_i\}_{i=0}^d \).

**Definition 20.8.** We say that \( \Gamma \) is \( Q \)-polynomial with respect to \( x \) whenever there exists an ordering of the eigenspaces of \( A \) that is \( Q \)-polynomial with respect to \( x \).

Another generalization we could adopt, is to allow the adjacency matrix \( A \) to be weighted. A weighted adjacency matrix is obtained from the classical adjacency matrix by replacing each entry 1 by a nonzero real scalar. The only requirement on the scalars is that the weighted adjacency matrix is diagonalizable. A weighted adjacency matrix is used in [18].

**Problem 20.9.** Investigate how the above variations on the \( Q \)-polynomial property are related.

**Remark 20.10.** In [58] Sho Suda introduced the \( Q \)-polynomial property for coherent configurations. A coherent configuration is a combinatorial object more general than a graph.

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