SYMPLECTIC RESOLUTION OF ORBIFOLDS WITH HOMOGENEOUS ISOTROPY

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Abstract. We construct the symplectic resolution of a symplectic orbifold whose isotropy locus consists of disjoint submanifolds with homogeneous isotropy, that is, all its points have the same isotropy groups.

1. Introduction

An orbifold is a space which is locally modelled on balls of $\mathbb{R}^n$ quotient by a finite group. These have been very useful in many geometrical contexts [18]. In the setting of symplectic geometry, symplectic orbifolds have been introduced mainly as a way to construct symplectic manifolds by resolving their singularities. The problem of resolution of singularities and blow-up in the symplectic setting was posed by Gromov in [10]. Few years later, the symplectic blow-up was rigorously defined by McDuff [14] and it was used to construct a simply-connected symplectic manifold with no Kähler structure.

McCarthy and Wolfson developed in [12] a symplectic resolution for isolated singularities of orbifolds in dimension 4. Later on, Cavalcanti, Fernández and the first author gave a method of performing symplectic resolution of orbifold isolated singularities in all dimensions [4]. This was used in [7] to give the first example of a simply-connected symplectic 8-manifold which is non-formal, as the resolution of a suitable symplectic 8-orbifold. This manifold was proved to have also a complex structure in [3].

Niederkrüger and Pasquotto [16, 17] provided a method for resolving symplectic orbifold singularities via symplectic reduction, which can be used for some classes of symplectic singularities, including cyclic orbifold singularities, even if these are not isolated. Recently, Chen [5] has detailed a method of resolving arbitrary symplectic 4-orbifolds, using the fact that the singular points of the underlying space have to be isolated in dimension 4. The novelty is that there can be also surfaces of non-trivial isotropy, and the symplectic orbifold form has to be modified on these surfaces also. In this dimension, the work of the authors with Tralle [15] also serve to resolve symplectic 4-orbifolds whose isotropy set is of codimension 2. In such case the orbifold is topologically a manifold (the isotropy points are non-singular), so the question only amounts to change the orbifold symplectic form into a smooth symplectic form.

Key words and phrases. Orbifold, Symplectic, Isotropy, Resolution.

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Bazzoni, Fernández and the first author [1] have given the first construction of a symplectic resolution of an orbifold of dimension 6 with isotropy sets of dimension 0 and 2, although the construction is ad hoc for the particular example at hand as it satisfies that the normal bundle to the 2-dimensional isotropy set is trivial. This was used to give the first example of a simply-connected non-Kähler manifold which is simultaneously complex and symplectic.

In this paper we give a procedure to resolve a wider type of singularities in a symplectic orbifold $X$ of arbitrary dimension $2n$. We are able to develop such resolution for orbifolds $X$ whose isotropy set is composed of disjoint submanifolds $D_i$ so that each of the $D_i$ have the same isotropy groups at all its points. We call this $D_i$ a homogeneous isotropy set and such orbifold $X$ an HI-orbifold. This allows the existence of positive dimensional submanifolds composed of singular points. The singular points of the topological underlying space are not isolated, hence new techniques are required in order to perform the resolution. We are able to endow the normal bundle to $D_i$ with a nice structure in which to effectively perform fiberwise the algebraic resolution of singularities of [6], and then glue these local resolutions into a resolution $\tilde{X}$ of $X$.

The general strategy is to endow the normal bundle $\nu_D$ of any homogeneous isotropy submanifold $D \subset X$ with the structure of an orbifold bundle with structure group $U(k)$, where $2k$ is the codimension of $D$. The singularities of $X$ at the points of $D$ are quotient singularities in the fibers $F = \mathbb{C}^k/\Gamma$ of $\nu_D$, where $\Gamma$ is the isotropy group of $D$. The usual resolution of singularities for algebraic geometry allows to resolve each of the fibers $F$ of $\nu_D$ separately. However, we need this resolution to glue nicely when we change trivializations. For this we need an improvement of the classical theorem of resolution of singularities by Hironaka [11]. This improvement is the constructive resolution of singularities by Encinas and Villamayor [6], which is compatible with group actions. Using their result we are able to construct the resolution $\tilde{\nu}_D$ of $X$ near $D$ as a smooth manifold.

The resolution $\tilde{\nu}_D$ has the structure of a fiber bundle over $D$, with fiber the resolution $\tilde{F}$ of $F = \mathbb{C}^k/\Gamma$. Both base $D$ and fiber $\tilde{F}$ of the total space $\tilde{\nu}_D$ are symplectic, but this does not imply directly that $\tilde{\nu}_D$ admits a symplectic form. First we need to prove that there is no cohomological obstruction for this, which amounts to finding a cohomology class on the total space $\tilde{\nu}_D$ that restricts to the cohomology class of the symplectic form of the fiber. Secondly, we have to develop a globalization procedure for symplectic fiber bundles with non-compact symplectic fiber. The final step is to glue the symplectic form on $\tilde{\nu}_D$ with the original symplectic form of $X \setminus D$.

The main result is:

**Theorem 1.** Let $(X, \omega)$ be a symplectic orbifold with isotropy set consisting of disjoint homogeneous isotropy subsets. Then there exists a symplectic manifold $(\tilde{X}, \tilde{\omega})$ and a smooth map $b : (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ which is a symplectomorphism outside an arbitrarily small neighborhood of the isotropy set of $X$.

We conclude the paper with some examples in which Theorem 1 applies.
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2. Orbifolds

We start by giving the basic definitions and results of symplectic orbifolds that we will need later.

Definition 2. An \( n \)-dimensional (differentiable) orbifold is a Hausdorff and second-countable space \( X \) endowed with an atlas \{\( (U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha) \}\}, where \{\( V_\alpha \)\} is an open covering of \( X \), \( U_\alpha \subset \mathbb{R}^n \), \( \Gamma_\alpha < \text{Diff}(U_\alpha) \) is a finite group acting by diffeomorphisms, and \( \phi_\alpha : U_\alpha \to V_\alpha \subset X \) is a \( \Gamma_\alpha \)-invariant map which induces a homeomorphism \( U_\alpha/\Gamma_\alpha \cong V_\alpha \).

There is a condition of compatibility of charts for intersections. For each point \( x \in V_\alpha \cap V_\beta \) there is some \( V_\delta \subset V_\alpha \cap V_\beta \) with \( x \in V_\delta \) so that there are group monomorphisms \( \rho_{\delta\alpha} : \Gamma_\delta \hookrightarrow \Gamma_\alpha \), \( \rho_{\delta\beta} : \Gamma_\delta \hookrightarrow \Gamma_\beta \), and open embeddings \( \iota_{\delta\alpha} : U_\delta \to U_\alpha \), \( \iota_{\delta\beta} : U_\delta \to U_\beta \), which satisfy \( \iota_{\delta\alpha} (\gamma(x)) = \rho_{\delta\alpha}(\gamma)(\iota_{\delta\alpha}(x)) \) and \( \iota_{\delta\beta} (\gamma(x)) = \rho_{\delta\beta}(\gamma)(\iota_{\delta\beta}(x)) \), for all \( \gamma \in \Gamma_\delta \).

For an orbifold \( X \), a change of charts is the map
\[
\psi_{\alpha\beta}^\delta = \iota_{\delta\beta} \circ \iota_{\delta\alpha}^{-1} : \iota_{\delta\alpha}(U_\delta) \subset U_\alpha \to \iota_{\delta\beta}(U_\delta) \subset U_\beta.
\]
So the change of charts between the chart \( U_\alpha \) and \( U_\beta \) depends on the inclusion of a third chart \( U_\delta \). This dependence is up to the action of an element in \( \Gamma_\delta \). In general, we abuse notation and write \( \psi_{\alpha\beta} \) for any change of chart between \( U_\alpha \) and \( U_\beta \).

For any point \( x \in X \), by taking \( U \) a small enough neighbourhood we can arrange always a chart \( U \subset \mathbb{R}^n \), \( U/\Gamma \cong V \) so that the group \( \Gamma \) acting on \( U \) leaves the point \( x \) fixed, i.e. \( \gamma(x) = x \) for all \( \gamma \in \Gamma \). In this case, we call \( \Gamma \) the isotropy group at \( x \), and we denote it by \( \Gamma_x \).

We call \( x \in X \) a smooth point if a neighbourhood of \( x \) is homeomorphic to a ball in \( \mathbb{R}^n \), and singular otherwise. We call \( x \in X \) a regular point if the isotropy group \( \Gamma_x = \{ \text{Id} \} \) is trivial, and we call it an isotropy point if it is not regular. Clearly a regular point is smooth, but not conversely. We say that an orbifold \( X \) is smooth if all its points are smooth. This is equivalent to \( X \) being a topological manifold. Finally, let us denote by \( \Sigma \) the set of isotropy points of an orbifold \( X \).

Proposition 3. Every orbifold \( X \) has an atlas \{\( (U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha) \}\} where the isotropy groups \( \Gamma_\alpha < O(n) \).

Proof. Let \( \phi : U \to V \cong U/\Gamma \) be a small orbifold chart around a point \( x \in X \), with \( \Gamma \) acting on \( U \subset \mathbb{R}^n \) by diffeomorphisms. We can suppose that the point \( x = \phi(0) \) and that all elements of \( \Gamma \) fix 0. We consider the standard metric \( g_{\text{std}} \) on \( U \) and take \( g := \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* g_{\text{std}} \). Then \( g \) is a Riemannian metric on \( U \) and it is \( \Gamma \)-invariant.
We consider now the exponential map for the metric $g$, $\exp_0 : T_0 U = \mathbb{R}^n \to U$. Since any $\gamma \in \Gamma$ acts by isometries, we have $\exp_0 \circ d_0 \gamma(v) = \gamma \circ \exp_0(v)$ for all $v \in \mathbb{R}^n$. Take $\epsilon > 0$ small enough so that $\exp_0 : B_\epsilon(0) \to U' = \exp_0(B_\epsilon(0)) \subset U$ is a diffeomorphism. Then we have a chart $\phi' = \phi \circ \exp_0 : B_\epsilon(0) \to V' = \phi(U')$ and the group $\Gamma$ acts on $B_\epsilon(0)$ via $\gamma \mapsto d_0 \gamma$. Moreover, $d_0 \gamma$ are isometries with respect to the metric $g$ at the point 0, i.e. $g|_0$. If we take an orthonormal basis of $\mathbb{R}^n$ with respect to $g|_0$, then $\Gamma < O(n)$. □

**Proposition 4.** Let $X$ be an orbifold, and let $\Sigma$ be its isotropy subset. For every conjugacy class of finite subgroup $H < O(n)$, we can define the set

$$
\Sigma_H = \{ x \in X | \Gamma_x \cong H \}.
$$

Then the closure $\overline{\Sigma}_H$ is an orbifold, and $\Sigma_H = \overline{\Sigma}_H \setminus \bigcup_{H < H'} \Sigma_{H'}$ is a smooth manifold.

**Proof.** Let $x_0 \in \Sigma$ be an isotropy point and take a local chart $(U, V, \phi, \Gamma)$ near $x$ with $\Gamma < O(n)$. Let $\Gamma = \{ \gamma_1 = \text{Id}, \gamma_2, \ldots, \gamma_N \}$ and consider the linear subspaces $L_i = \ker(\gamma_i - \text{Id}) \subset \mathbb{R}^n$, for $1 \leq i \leq N$. For every subgroup $H < \Gamma$, we define $L_H = \bigcap_{\gamma_i \in H} L_i \subset \mathbb{R}^n$. This gives a finite collection of subspaces, which are stratified, in the sense that $H' < H$ implies that $L_H \supset L_{H'}$. For given $H < \Gamma$, let $L_H^0 = L_H \setminus \bigcup_{H' > H} L_{H'}$. If $L_H^0$ is not empty, then a point $x \in L_H^0$ satisfies that its isotropy is exactly $H$. So $\Sigma_H \cap V = \phi(L_H^0 \cap U)$. Clearly $\overline{\Sigma}_H = \phi(L_H \cap U)$, hence it is an orbifold with chart $(U \cap L_H, V \cap \overline{\Sigma}_H, \phi, \Gamma/\langle H \rangle)$. Note that for any conjugate $\tilde{H} = \gamma H \gamma^{-1}$, $L_H = \gamma L_H$ and $\phi(L_H \cap U) = \phi(L_{\tilde{H}} \cap U)$, and the converse also holds. Take the minimal normal subgroup $\langle H \rangle$ containing $H$. Then $\Gamma/\langle H \rangle$ acts on $L_H$. □

An orbifold function $f : X \to \mathbb{R}$ is a continuous function such that $f \circ \phi_\alpha : U_\alpha \to \mathbb{R}$ is smooth for every $\alpha$. Note that this is equivalent to giving smooth functions $f_\alpha$ on $U_\alpha$ which are $\Gamma_\alpha$-equivariant and which agree under the changes of charts. An orbifold partition of unity subordinated to the open cover $\{ V_\alpha \}$ of $X$ consists of orbifold functions $\rho_\alpha : X \to [0, 1]$ such that the support of $\rho_\alpha$ lies inside $V_\alpha$ and the sum $\sum_\alpha \rho_\alpha \equiv 1$ on $X$.

**Proposition 5.** Let $X$ be an $n$-orbifold. For any sufficiently refined locally finite open cover $\{ V_\alpha \}$ of $X$ there exists an orbifold partition of unity subordinated to $\{ V_\alpha \}$.

**Proof.** Take an open cover $\{ V_\alpha \}$ of $X$ formed by coordinate patches $V_\alpha \cong B_{3\epsilon}(0)/\Gamma_\alpha$ with $\Gamma_\alpha < O(n)$ and so that $V_\alpha' \cong B_{\epsilon}(0)/\Gamma_\alpha$ is also an open cover of $X$. We can suppose that $V_\alpha$ is locally finite. Take $f : \mathbb{R}^n \to \mathbb{R}$ be a radial bump function so that $f \equiv 0$ on $B_{3\epsilon}(0) \setminus B_{2\epsilon}(0)$ and $f \equiv 1$ on $B_{\epsilon}(0)$. Since $f$ is a radial function and $\Gamma_\alpha < O(n)$, it descends to the quotient and gives a continuous function $f_\alpha : V_\alpha \to \mathbb{R}$ which can be extended by zero to all $X$ so we write $f_\alpha : X \to \mathbb{R}$. The sum $\sum_\beta f_\beta(x) > 0$ at all points of $X$ because the sets $V_\alpha'$ form a cover of $X$. We define $\rho_\alpha = f_\alpha/\sum_\beta f_\beta$, and thus $\sum_\alpha \rho_\alpha \equiv 1$ on $X$. □
Let $X$ be an orbifold with atlas $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$. An orbifold tensor on $X$ is a collection of tensors $T_\alpha$ on each $U_\alpha$ which are $\Gamma_\alpha$-invariant, and which agree under the changes of charts. In particular, we have the set of orbifold differential forms $\Omega^p_{\text{orb}}(X)$, orbifold Riemannian metrics $g$, and orbifold almost complex structures $J$. The exterior differential, covariant derivatives, Lie bracket, Nijenhuis tensor, etc, are defined in the usual fashion.

**Proposition 6.** Let $X$ be an orbifold. There exists an orbifold Riemannian metric $g$ on $X$.

**Proof.** Let us consider an atlas $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$ where the isotropy groups $\Gamma_\alpha \subset O(n)$, whose existence is proved in Proposition 3. Consider the standard metric $g_\alpha$ on $U_\alpha$ which is in particular $\Gamma_\alpha$-invariant. Take a differentiable partition of unity $\rho_\alpha$ subordinated to $\{V_\alpha\}$, given by Proposition 5. Define $g = \sum_\alpha \rho_\alpha g_\alpha$. This is an orbifold tensor on $X$, as $g_\alpha$ are orbifold tensors and $\rho_\alpha$ orbifold functions. It is an orbifold Riemannian metric by the usual convexity argument. □

An orbifold $X$ is orientable if all $\Gamma_\alpha$ acts by orientation preserving diffeomorphisms and all embeddings $i_\delta$ in Definition 2 preserve orientation. In this case we have an atlas with all $\Gamma_\alpha < SO(n)$ and all changes of charts preserving orientation. This is equivalent to the existence of a globally non-zero orbifold form of degree $2n$, called a volume form.

Given an orbifold $X$, the orbifold forms $(\Omega^p_{\text{orb}}(X), d)$ define the orbifold De Rham cohomology algebra, and its cohomology is denoted $H_{\text{orb}}^*(X)$. This is isomorphic to the usual singular cohomology with real coefficients [4],

$$H_{\text{orb}}^*(X) \cong H^*(X, \mathbb{R}). \quad (1)$$

### 3. Symplectic Orbifolds

**Definition 7.** A symplectic orbifold $(X, \omega)$ is an orbifold $X$ equipped with an orbifold 2-form $\omega \in \Omega^2_{\text{orb}}(X)$ such that $d\omega = 0$ and $\omega^n > 0$, where $2n = \dim X$. In particular, it is oriented.

An almost Kähler orbifold $(X, J, \omega)$ consists of an orbifold $X$, and orbifold almost complex structure $J$ and an orbifold symplectic form $\omega$ such that $g(u, v) = \omega(u, Ju)$ defines an orbifold Riemannian metric with $g(Ju, Jv) = g(u, v)$.

A Kähler orbifold is an almost Kähler orbifold satisfying the integrability condition that the Nijenhuis tensor $N_J = 0$. This is equivalent to requiring that the changes of charts are biholomorphisms of open sets of $\mathbb{C}^n$.

**Proposition 8.** Let $(X, \omega)$ be a symplectic orbifold. Then $(X, \omega)$ admits an almost Kähler orbifold structure $(X, \omega, J, g)$.

**Proof.** Consider an auxiliary orbifold Riemannian metric $g_0$ on $X$. We define the orbifold endomorphism $A \in \text{End}(TX)$ by the requirement $g_0(u, Av) = \omega(u, v)$. The adjoint of $A$ with respect to $g$ is the orbifold endomorphism $A^* \in \text{End}(TX)$ such that $g_0(u, A^*v) = g_0(Au, v)$. We have that $A^* = -A$ since $g_0(u, A^*v) = g_0(Au, v) = \omega(u, Ju) = -\omega(u, u) = 0$. This implies that $g_0(Au, v) = \omega(u, Jv) = 0$. Therefore, $A$ is an almost Kähler structure.
\[ g_0(v, Au) = \omega(v, u) = -\omega(u, v) = -g_0(u, Av) = g_0(u, -Av). \] The orbifold endomorphism \( B = AA^* = -A^2 \) is symmetric and positive. Indeed \( g_0(u, Bu) = g_0(A^*u, A^*u) > 0 \) for \( u \neq 0 \), and \( g_0(u, Bv) = g_0(A^*u, A^*v) = g_0(A^*v, A^*u) = g_0(v, Bu) \).

Let us see that \( B \) admits a square root \( \sqrt{B} \in \text{End}(TX) \), which is an orbifold endomorphism. On every chart \( \phi : U \to V = U/\Gamma \), \( B \) is given by a matrix valued function \( B(x) \) on \( U \) which is \( \Gamma \)-equivariant. At every \( x \in U \), it has positive eigenvalues and diagonalises, so we can define \( \sqrt{B} \) locally as the matrix which has the same eigenvectors as \( B \) with eigenvalues the (positive) square root of the eigenvalues of \( B \). We have to see that \( \sqrt{B} \) is \( \Gamma \)-equivariant. We take a real constant \( \mu > 0 \) so that \( \|\mu B - \text{Id}\| < 1 \), in some operator norm, so we have

\[
\sqrt{\mu \sqrt{B}} = \sqrt{\mu B} = \text{Id} + \frac{1}{2} \mu B - \frac{1}{8} \mu^2 B^2 + \frac{1}{16} \mu^3 B^3 + \ldots
\]

by the usual power series expansion of the square root. This yields the formula \( \sqrt{B} = \frac{1}{\sqrt{\mu}}(\text{Id} + \frac{1}{2} \mu B + \ldots) \). As \( \Gamma \) commutes with \( B \), we have that it also commutes with \( \sqrt{B} \).

Now define \( J = - (\sqrt{B})^{-1} A \), which is an orbifold endomorphism. As \( \sqrt{B} = \sqrt{-A^2} \) commutes with \( A \) by the power series expansion, its inverse \( (\sqrt{B})^{-1} \) also commutes with \( A \), and hence \( J \) commutes with both \( \sqrt{B} \) and \( A \). Also \( J^2 = B^{-1} A^2 = (-A^2)^{-1} A^2 = -\text{Id} \), so \( J \) is an orbifold almost complex structure. As \( J^* = A^* \sqrt{B}^* = -A \sqrt{B} = -J \), we have that \( g(u, v) = \omega(u, Jv) \) is a symmetric bilinear orbifold tensor. Moreover

\[
g(u, v) = \omega(u, Jv) = g_0(u, AJv) = g_0(u, (\sqrt{AA^*})^{-1} AA^* v) = g_0(u, \sqrt{AA^*} v),
\]

which implies that \( g \) is positive definite, and hence an orbifold Riemannian metric. Finally, \( J \) is compatible with \( \omega \) since \( \omega(Ju, Jv) = g(Ju, AJv) = g(J^*Ju, Av) = g(u, Av) = \omega(u, v) \). So \((X, \omega, g, J)\) is an almost Kähler orbifold. \( \square \)

In the case of symplectic orbifolds, the structure of the isotropy set given in Proposition 4 can be improved.

**Corollary 9.** The isotropy set \( \Sigma \) of \((X, \omega)\) consists of immersed symplectic suborbifolds \( \Sigma_H \). Moreover, if we endow \( X \) with an almost Kähler orbifold structure \((\omega, J, g)\), then the \( \Sigma_H \) are almost Kähler suborbifolds.

**Proof.** Put any almost Kähler structure \((\omega, J, g)\) on \( X \) as provided by Proposition 8. Fix a chart \((U, V, \phi, \Gamma)\) with \( \Gamma < O(n) \), and \( U \subset \mathbb{R}^{2n} \) a neighborhood of \( 0 \). As \( J \) is an orbifold almost complex structure, \( \Gamma \) preserves \( J \), in particular \( d_0 \gamma \circ J_0 = J_0 \circ d_0 \gamma \) for all \( \gamma \in \Gamma \). As \( \gamma \) is linear, we have that \( d_0 \gamma = \gamma \), hence \( \gamma \) preserves the complex structure of \( \mathbb{C}^n = (\mathbb{R}^{2n}, J_0) \). This means that \( \Gamma \subset \text{GL}(n, \mathbb{C}) \cap O(2n) = U(n) \).

As proved in Proposition 4, the isotropy set \( \Sigma \cap V = 0 \cap L_H \), for some subgroups \( H < \Gamma \). As \( L_H = \bigcap_{\gamma \in H} L_\gamma \), where \( L_\gamma = \text{ker}(\gamma - \text{Id}) \), and \( \gamma \) are complex endomorphisms, we have that \( L_H \) is a complex linear subspace of \( \mathbb{C}^n \). This proves that \( J_0 \) leaves invariant \( T_0 \Sigma_H = L_H \), the (orbifold) tangent
space of $\Sigma_H$ at the origin. This happens at every point, hence $\Sigma_H$ is an almost Kähler orbifold. In particular, it is a symplectic suborbifold of $(X, \omega)$. \hfill $\Box$

The following result is a Darboux theorem for symplectic orbifolds.

**Proposition 10.** Let $(X, \omega)$ be a symplectic orbifold and $x_0 \in X$. There exists an orbifold chart $(U, V, \phi, \Gamma)$ around $x_0$ with local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ such that the symplectic form has the expression $\omega = \sum dx_i \wedge dy_i$, and $\Gamma < U(n)$ is a subgroup of the unitary group.

**Proof.** Take an initial orbifold chart $(U, V, \psi, \Gamma)$ with $\Gamma < U(n)$ and $x_0 = \psi(0)$, possible by Corollary 9. Consider the evaluation of $\omega$ at the origin $\omega|_{x_0}$. We take a basis of $\mathbb{R}^{2n}$ such that $\omega|_{x_0}$ has standard form, that is $\omega|_{x_0} = \sum dx_i \wedge dy_i$. Let $\omega_0$ be the symplectic form with constant coefficients which equals to $\omega|_{x_0}$. Since $U$ is contractible we have that $\omega - \omega_0 = d\mu$, for some $\mu \in \Omega^1(V)$. We can suppose that $\mu$ is $\Gamma$-invariant, since otherwise we put $\bar{\mu} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* \mu$ and $\bar{\mu}$ also satisfies

$$d\bar{\mu} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* d\mu = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^*(\omega - \omega_0) = \omega - \omega_0.$$

We can further suppose that $\mu|_{x_0} = 0$ vanishes as a 1-form, since otherwise we put $\bar{\mu} = \mu - \mu|_{x_0}$ which also satisfies $d\bar{\mu} = \omega - \omega_0$ and $\bar{\mu}$ is $\Gamma$-equivariant.

Now we apply Moser trick. Consider $\omega_t = t\omega + (1 - t)\omega_0 = \omega_0 + t d\mu$. Consider a vector field $X_t$ such that $i_{X_t} \omega_t = -\mu$. Let us call $\varphi_t$ the flow of the vector field $X_t$ at time $t$, which satisfies $\frac{d}{dt} |_{t=s} \varphi_t(x) = X_t |_{\varphi_t(x)}$ for each $x \in U$. Then for each $s$,

$$\frac{d}{dt} |_{t=s} \varphi_t^* \omega_t = \frac{d}{dt} |_{t=s} \varphi_t^* \omega_s + \varphi_t^* \left( \frac{d}{dt} |_{t=s} \omega_t \right) = \varphi_t^* (\mathcal{L}_{X_s} \omega_s) + \varphi_t^* (d\mu)$$

$$= \varphi_t^* (d(i_{X_s} \omega_s) + i_{X_s} d\omega_s) + \varphi_t^* (d\mu) = -\varphi_t^* (d\mu) + \varphi_t^* (d\mu) = 0,$$

using Cartan formula for the Lie derivative $\mathcal{L}_X = dX + i_X d$. This implies that $\omega_t = \varphi_t^* \omega_0 = \varphi_t^* \omega_0 = \varphi_t^* \omega$. The change of coordinates is then given by the diffeomorphism $\varphi := \varphi_1$ which is defined in some neighborhood of $0 \in U$. Recall that, since $\mu$ vanishes at $0 \in U$, $\varphi_t(0) = 0$ for all $t$, so $\varphi(0) = 0$. Finally, as $\mu$ and $\omega_t$ are $\Gamma$-equivariant, and $i_{X_t} \omega_t = -\mu$, we have that the vector fields $X_t$ are $\Gamma$-equivariant. Therefore the flow $\varphi_t$ are $\Gamma$-equivariant diffeomorphisms, and so $\varphi$ is $\Gamma$-equivariant. Summarising, we have a diffeomorphism $\varphi : U' \to U$ between two neighborhoods of $0$ and $\varphi^* \omega = \omega_0$ is a constant symplectic form on $U'$. Moreover, since $\varphi \gamma \varphi^{-1} = \gamma$ for all $\gamma \in \Gamma$, the $\Gamma$-action induced by $\varphi$ on $U'$ is the same as on $U$. The sought orbifold chart is $(U', V, \psi \circ \varphi, \Gamma)$. \hfill $\Box$

**Corollary 11.** Let $(X, \omega)$ be a symplectic orbifold. Then $(X, \omega)$ admits a Darboux orbifold atlas, i.e. an atlas $\{(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)\}$ where all the isotropy groups $\Gamma_\alpha < U(n)$ and the expression in coordinates of $\omega$ on each $U_\alpha \subset \mathbb{R}^{2n}$ is the canonical form of $\mathbb{R}^{2n}$, i.e. $\omega|_{U_\alpha} = \sum dx_j \wedge dy_j = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$.

Moreover, if $\Sigma_H \subset X$ is an isotropy suborbifold of codimension $2k$, we can arrange that for each open set $V_\alpha$ which intersects $\Sigma_H$, the intersection $\Sigma_H \cap V_\alpha$ is given by $\{z_1 = 0, \ldots, z_k = 0\} \subset U_\alpha$. 

Proof. By Proposition 10, there is a Darboux atlas as required. Let us see that it can be adapted to the submanifold $D$. For each chart $(U_\alpha, V_\alpha, \phi_\alpha, \Gamma_\alpha)$ intersecting $D$, $D \cap V_\alpha = \phi_\alpha(L_H \cap U_\alpha)$, where $L_H \subset \mathbb{C}^n$ is a complex linear subspace, being the fixed subset of $\Gamma_\alpha$. We can then take a unitary basis of $\mathbb{C}^n$ so that $L_H = \{z_1 = 0, \ldots, z_k = 0\}$, and clearly the symplectic form is again $\omega_0$ since $U(n) < \text{Sp}(2n, \mathbb{R})$. □

4. Tubular neighbourhood of the isotropy set

From now on we shall restrict to the case where the isotropy locus $\Sigma$ is already a smooth submanifold.

Definition 12. We say that an isotropy subset $\Sigma_H$ is homogeneous if $\Sigma_H = \Sigma_H$. That is, all its points have isotropy equal to $H$.

By Proposition 4, if $\Sigma_H$ is homogeneous, then it is a submanifold.

Definition 13. We say that an orbifold $X$ is HI (abbreviature for homogeneous isotropy) if all its isotropy subsets are homogeneous.

From now on we shall work exclusively with an HI orbifold $X$.

Lemma 14. If $\Sigma_H$ is an homogeneous isotropy set, then it is isolated, that is, no other isotropy set intersects it. Moreover, around any point $x_0 \in \Sigma_H$ we have a chart $(U, V, \phi, H)$, where $U \cong U' \times U''$, $U' \subset \mathbb{R}^d$, $U'' \subset \mathbb{R}^{n-d}$, $H < O(n-d)$, where $d$ is the dimension of $\Sigma_H$, $V \cong U' \times (U''/H)$, and $\Sigma_H$ corresponds to $U' \times \{0\}$.

If $(X, \omega)$ is a symplectic orbifold of dimension $2n$ and $\Sigma_H$ is an homogeneous isotropy set of dimension $2d$, then for every $x \in \Sigma_H$ there is a Darboux chart $(U, V, \phi, H)$ around $x$, where $U \cong U' \times U''$, $U' \subset \mathbb{C}^d$, $U'' \subset \mathbb{C}^{n-d}$, $H < O(n-d)$, $V \cong U' \times (U''/H)$, and $\Sigma_H$ corresponds to $U' \times \{0\}$.

Proof. We have $\Sigma_H \cap V = \phi(L_H \cap U)$. The linear subspace $L_H$ is $d$-dimensional, so we can write $\mathbb{R}^n = L_H \oplus (L_H)^\perp$. Note that $\Gamma$ fixes $L_H$, so it acts on $(L_H)^\perp \cong \mathbb{R}^{n-k}$. Moreover $\Gamma = H$. The result follows.

The statement for symplectic orbifolds follows analogously using Corollary 11.

To understand the structure around an homogeneous isotropy subset, let us introduce the notion of orbifold bundle, as bundle of orbifolds over a manifold. For a space with a geometric structure $(M, G)$ we understand a smooth manifold $M$ with a Lie group $G$ acting on $M$. We call $G$ the automorphism group of the structure and write $G = \text{Aut}(M)$.

Definition 15. Let $M$ be a space with some geometric structure and let $\Gamma < \text{Aut}(M)$ be a finite subgroup of automorphisms of $M$, and let $B$ be a smooth manifold. An orbifold bundle $E$ with fiber $F = M/\Gamma$ and base space $B$ consists of an orbifold $E$ endowed with an open cover $\{V_\alpha\}$ and with orbifold charts $\phi_\alpha : U_\alpha \times M \to V_\alpha$ so that:
(1) The groups $\Gamma_\alpha < \text{Aut}(M)$ act on $U_\alpha \times M$ as $\gamma(x,m) = (x,\gamma m)$ for all $\gamma \in \Gamma_\alpha$.

(2) All the groups $\Gamma_\alpha$ are all conjugated to $\Gamma$ by some automorphism of $M$, so all the quotients $M/\Gamma_\alpha$ are isomorphic to $F = M/\Gamma$.

(3) The changes of charts of this atlas of $E$ are maps of the form

$$\varphi_{\alpha\beta} : \iota_{\delta\alpha}(U_\delta) \times M \rightarrow \iota_{\delta\beta}(U_\delta) \times M, (x,m) \rightarrow (\psi_{\alpha\beta}(x),A_{\alpha\beta}(x)m),$$

with $A_{\alpha\beta}: \iota_{\delta\alpha}(U_\delta) \rightarrow \text{Aut}(M)$ is a smooth map taking values in the group of automorphisms of $M$.

Note that from the definition of orbifold, the maps $A_{\alpha\beta}$ are compatible with the actions of the local groups $\Gamma_\alpha$ and $\Gamma_{\beta}$ in the sense that $A_{\alpha\beta}(x)\gamma m = \rho_{\alpha\beta}(\gamma)A_{\alpha\beta}(x)m$ for all $\gamma \in \Gamma_\alpha$, where $\rho_{\alpha\beta} = \rho_{\delta\beta} \circ \rho_{\delta\alpha}^{-1} : \Gamma_\alpha \rightarrow \Gamma_{\beta}$ are all group isomorphisms. Note that it must be $\rho_{\alpha\beta}(\gamma) = A_{\alpha\beta}(x)\gamma A_{\alpha\beta}(x)^{-1}$, so (2) in Definition 15 is automatic.

An orbifold bundle satisfies that $E$ is topologically a fiber bundle of the form $F = M/\Gamma \rightarrow E \rightarrow B$. The transition functions are induced by $A_{\alpha\beta}$ on $M/\Gamma$.

A vector orbifold bundle corresponds to the case where $M$ is a (real or complex) vector space and $\text{Aut}(M)$ is a subgroup of the group of linear maps of $M$.

Now let $(X,\omega)$ be an HI symplectic orbifold, and let $D = \Sigma_H$ be an homogeneous isotropy set of dimension $2d$. Let $2k = 2n - 2d$ be the codimension of $D$. The orbifold tangent space $TX$ is given in local charts $(U,V,\phi,\Gamma)$ by $T_xU$ with the action of $\Gamma_x < \text{GL}(T_xU)$ induced by $d_x\gamma$, for $\gamma \in \Gamma$ acting on $U$. If $x \in D$, then $T_xU$ is a symplectic vector space and $T_xD$ is the fix set of $\Gamma_x$. The symplectic orthogonal $(T_xD)_{\perp}\omega \cong \mathbb{R}^{2k}$ has the action induced by $\Gamma_x$, and we define the orbifold normal space as $$\nu_{D,x} = (T_xD)_{\perp}\omega/\Gamma_x.$$  The normal bundle $\nu_D$ is the union of all $\nu_{D,x}$, for $x \in D$.

**Proposition 16.** Let $(X,\omega)$ be an HI symplectic orbifold, and let $D \subset X$ be an isotropy submanifold. Then the normal bundle $\nu_D$ admits the structure of a symplectic orbifold vector bundle over $D$.

**Proof.** We take a collection of symplectic charts $(U_\alpha,V_\alpha,\phi_\alpha,\Gamma_\alpha)$ adapted to $D$, given by Corollary 11. Denote $2d = \text{dim } D$ and let $2k = 2n - 2d$ be the codimension of $D$. Then $U_\alpha \cong U'_\alpha \times U''_\alpha$, where $U'_\alpha \subset \mathbb{C}^d$, $U''_\alpha \subset \mathbb{C}^k$, $\Gamma_\alpha < \text{U}(k)$, and $V_\alpha \cong U'_\alpha \times (U''_\alpha/\Gamma_\alpha)$. Then $\phi : U'_\alpha \rightarrow V'_\alpha$ is a diffeomorphism, and $\{V'_\alpha\}$ is a covering by charts of $D$.

For any $p \in U'_\alpha \subset D$, the tangent space $T_pD \cong \mathbb{C}^d \times \{0\}$ and $(T_pD)_{\perp}\omega \cong \{0\} \times \mathbb{C}^k$. Therefore $\nu_D|U'_\alpha \cong U'_\alpha \times (\mathbb{C}^k/\Gamma_\alpha)$, where $\nu_D|U'_\alpha$ denotes the collection of normal spaces to points $p \in U'_\alpha$. Then there is an orbifold chart $$U'_\alpha \times \mathbb{C}^k \rightarrow \nu_D|U'_\alpha,$$

where $\Gamma_\alpha$ acts on $\mathbb{C}^k$ by the inclusion $\Gamma_\alpha < \text{U}(k)$. The fiber is $M = \mathbb{C}^k$ with $\text{Aut}(M) = \text{U}(k)$. Let us see that the orbifold changes of charts satisfy (3) in
Definition 15. By Definition 2, the change of charts for $U_{\alpha}$ and $U_{\beta}$ is given by a map
\[ \psi_{\alpha\beta} : \tau_{\delta}\alpha(U'_{\delta} \times U''_{\delta}) \to \tau_{\delta}\beta(U'_{\delta} \times U''_{\delta}), \quad \psi_{\alpha\beta}(x, y) = (\psi'_{\alpha\beta}(x, y), \psi''_{\alpha\beta}(x, y)). \]
The group homomorphisms $\rho_{\delta}\alpha : \Gamma_{\delta} \to \Gamma_{\alpha}$ and $\rho_{\delta}\beta : \Gamma_{\delta} \to \Gamma_{\beta}$ are isomorphisms (since all points have the same isotropy), so the map $\rho_{\alpha\beta} = \rho_{\delta}\beta \circ \rho_{\delta}\alpha^{-1} : \Gamma_{\alpha} \to \Gamma_{\beta}$ is an isomorphism. The map $\psi_{\alpha\beta}$ satisfies $\psi_{\alpha\beta}(x, \gamma y) = \rho_{\alpha\beta}(\gamma)(\psi_{\alpha\beta}(x, y))$, i.e.
\[ \psi''_{\alpha\beta}(x, \gamma y) = \rho_{\alpha\beta}(\gamma)\psi''_{\alpha\beta}(x, y), \tag{2} \]
for $\gamma \in \Gamma_{\alpha}$. Take a point $x = (x, 0) \in U'_{\alpha} \subset U_{\alpha}$. The map at the tangent space $T_{x}X$ is given by $(d\psi_{\alpha\beta})(x, 0)$. Therefore the induced map on $(T_{x}D)^{\perp} = \{0\} \times \mathbb{C}^{k}$ is given by the differential in the direction of $y$, which is
\[ A_{\alpha\beta}(x) = \frac{\partial \psi''_{\alpha\beta}}{\partial y}_{|_{(x, 0)}}. \]
By differentiating (2), we have $A_{\alpha\beta}(x)\gamma m = \rho_{\alpha\beta}(\gamma)A_{\alpha\beta}(x)m$, for $m \in \mathbb{C}^{k}$. Note that $A_{\alpha\beta}(x) \in \text{Sp}(2k, \mathbb{R})$, since $\psi_{\alpha\beta}$ are symplectomorphisms. We consider the geometric space $M = \mathbb{C}^{k}$ with group $\text{Aut}(M) = \text{Sp}(2k, \mathbb{R})$. This completes the proof. \hfill \Box

Proposition 17 (Tubular neighbourhood for orbifolds). Let $X$ be an orbifold and $D \subset X$ an homogeneous isotropy submanifold. Then there exists a tubular neighborhood of $D$ in $X$ which is diffeomorphic (as orbifolds) to a neighborhood of the zero section of the orbifold normal bundle $\nu_{D}$.

Proof. Consider an orbifold Riemannian metric $g$ for $X$. We use the exponential map associated to the metric to find the desired diffeomorphism. Take the normal bundle $\nu_{D} = \{(x, u) | u \in (T_{x, (0)}D)^{\perp}\}$ and let $D = D \times \{0\} \subset \nu_{D}$ be the zero section. Define $\exp : \nu_{D} \to U/\Gamma \subset X$ by $\exp([x, u]) = [\alpha_{|(x, (0), u)}(1)]$, where $\alpha_{|(x, (0), u)}$ is the geodesic from $(x, 0) \in U$ with direction $u$. The brackets stand for the equivalence classes modulo the local isotropy groups. We have to see that the map $\exp$ is defined locally in each orbifold chart, $\exp : \nu_{D}|_{U'} \to U/\Gamma = U' \times (U''/\Gamma)$, and it is $\Gamma$-equivariant. The isotropy groups $\Gamma$ act by isometries on the orbifold charts and hence commute with the exponential map, so $\exp(x, \gamma u) = \gamma(\exp(x, u))$ for $\gamma \in \Gamma$. There are open sets $U, V$ with $D \subset U \subset \nu_{D}, D \subset V \subset M$, so that $\exp : U \to V$ is defined. As $\exp$ is the identity on $D$, it yields an orbifold diffeomorphism $\exp : U \to V$ for small open sets. \hfill \Box

Now let $(X, \omega)$ be a symplectic orbifold with an homogeneous isotropy submanifold $D \subset X$. Let $2d$ be the dimension of $D$ and $2k = 2n - 2d$ its codimension. Then we take $(\omega, g, J)$ any orbifold almost Kähler structure for $(X, \omega)$. For $x_{0} \in D$, we take an orbifold Darboux chart $(U, V, \phi, \Gamma)$ adapted to $D$, with $\Gamma < \text{U}(k)$. So the lifting of $D$ to $U$ is given by $\{z_{d+1} = 0, \ldots, z_{n} = 0\}$. By compatibility of $g$ and $\omega$, we have $(T_{x_{0}}D)^{\perp} = (T_{x_{0}}D)^{\perp}_{g}$, and it has the structure of a $J$-complex subspace of $T_{x_{0}}U = \mathbb{C}^{n}$, and it is given by $(T_{x_{0}}D)^{\perp} = \{z_{1} = 0, \ldots, z_{d} = 0\}$. The action of $\Gamma$ on $U$ is given by $\gamma(x, y) = (x, \gamma y)$ for $x = (z_{1}, \ldots, z_{d}) \in \mathbb{C}^{d}$ and $y = (z_{d+1}, \ldots, z_{n}) \in \mathbb{C}^{k}$. 

Under the diffeomorphism $F : U \to V$ provided by Proposition 17, where $U$ is a neighbourhood of the zero section $D \subset \nu_D$ and $V$ is a neighbourhood of $D \subset X$, we can consider the pull-back of $\omega$ to $U$, which we will call $\tilde{\omega}$ again. So $\omega \in \Omega^2_{orb}(U)$ is a symplectic orbifold form.

**Proposition 18.** Let $(X, \omega)$ a symplectic orbifold and $D$ a homogeneous isotropy submanifold. The bundle $\nu_D$ admits a closed 2-form $\tilde{\omega}$ such that:

- $\tilde{\omega}$ and $\omega$ coincide along the zero section $D \subset \nu_D$, in particular $\tilde{\omega}$ is symplectic on an open set $U$ with $D \subset U \subset \nu_D$.
- Restricted to any fiber $F_x = \nu_{D,x} = (T_x D)^\perp / \Gamma_x$, the form $\tilde{\omega}|_{F_x}$ is constant on the vector space $(T_x D)^\perp$.

**Proof.** We consider a local trivialization of $\nu_D$, given by a chart $\phi : U_\alpha \times \mathbb{C}^k \to \nu_D|_{U_\alpha}$, with group $\Gamma_\alpha < U(k)$. Consider the form $\omega_x := \omega(x,0)|_{(T_x D)^\perp}$, which is a $\Gamma_\alpha$-equivariant symplectic 2-form on the vector space $(T_x D)^\perp$. Write $\omega_x = \sum b_{ij}(x)dy_i \wedge dy_j$ and let $\beta = d(\sum b_{ij}(x)y_i dy_j)$. Then $\beta$ is closed and satisfies $\beta|_{F_x} = \omega_x$ for every $x \in D$. Averaging over $\Gamma_\alpha$, we have a $\Gamma_\alpha$-invariant form $\tilde{\beta}$ satisfying the same conditions. Now consider $\omega'_\alpha = \pi^*(\omega|_D) + \tilde{\beta}$. This is $\Gamma_\alpha$-invariant, $(\omega'_\alpha)(x,0) = \omega(x,0)$ for all $x \in U_\alpha$ and it is constant on fibers. Clearly $\omega'_\alpha = \pi^*(\omega|_D) + d\eta_\alpha$, for some $\eta_\alpha \in \Omega^1(U_\alpha \times \mathbb{C}^k)$. Note that the 2-forms $d\eta_\alpha$ restrict to 0 on $U_\alpha \times \{0\}$ and restrict to $\omega_x$ on every fiber $F_x$ over a point $x \in U_\alpha$. The forms $\eta_\alpha$ can be supposed invariant by averaging over $\Gamma_\alpha$.

Take any smooth orbifold partition of unity $\rho_\alpha$ subordinated to the cover $U_\alpha$ of $D$. Consider the form

$$\tilde{\omega} = \pi^*(\omega|_D) + \sum_\alpha d((\pi^*\rho_\alpha)\eta_\alpha).$$

Note that $\tilde{\omega}$ is invariant by the local groups since all objects involved in its definition are. Restricting to a fiber $F_x$, we have $\tilde{\omega}|_{F_x} = \sum d(\rho_\alpha(x)\eta_\alpha) = \sum \rho_\alpha(x)\omega_x = \omega_x$. For $(x,0) \in \nu_D$, we have from the expression $\tilde{\omega} = \pi^*(\omega|_D) + \sum d(\pi^*\rho_\alpha) \wedge \eta_\alpha + \sum(\pi^*\rho_\alpha) d\eta_\alpha$ and the fact that $\eta_\alpha$ vanishes at $(x,0)$, that $\tilde{\omega}(x,0) = \omega(x,0)$. In particular, $\tilde{\omega}$ is non-degenerate at every point $(x,0)$ in the zero section, which implies that $\tilde{\omega}$ is also non-degenerate in some open neighborhood $U$ of the zero section in $\nu_D$. Since $\tilde{\omega}$ is closed, it is symplectic on $U$.

The next result is the orbifold version of the tubular neighbourhood theorem for symplectic submanifolds.

**Proposition 19** (Symplectic tubular neighborhood for orbifolds). Let $(X, \omega)$ be a symplectic orbifold and let $D \subset X$ be an homogeneous isotropy submanifold. Let $U \subset \nu_D$ be a neighborhood of $D$ in the orbifold normal bundle $\nu_D$ and suppose that $(U, \tilde{\omega})$ is a symplectic manifold such that the symplectic form $\tilde{\omega}$ satisfies that $\tilde{\omega}_x$ and $\omega_x$ coincide on $T_x X$ for all points $x \in D$. Then there are open sets $U', V'$ with $D \subset U' \subset U \subset \nu_D$ and $D \subset V' \subset X$ and an orbifold symplectomorphism $\varphi : (U', \tilde{\omega}) \to (V', \omega)$ so that $\varphi|_D = \text{Id}_D$. 
Proof. The proof is similar to the equivariant Darboux theorem (Proposition 10). Take first any orbifold diffeomorphism \( h : U \subset \nu_D \to V \subset X \) such that \( h|_D = \text{Id}_D \) by Proposition 17 (maybe reducing \( U \) if necessary). Let us call \( i : D \to \nu_D \) the inclusion of \( D \) as the zero section, and let \( \omega_0 = \tilde{\omega}, \omega_1 = h^*(\omega) \), so that \( \omega_0 \) and \( \omega_1 \) are two symplectic forms on \( U \subset \nu_D \) such that \( i^*(\omega_1 - \omega_0) = 0 \).

By (1), the orbifold De Rham cohomology \( H^2_{\text{orb}}(\nu_D) \cong H^2(\nu_D) \). Hence the inclusion \( i : D \to \nu_D \) induces an isomorphism \( i^* : H^2_{\text{orb}}(\nu_D) \to H^2(D) \). So there exists an orbifold one form \( \mu \in \Omega^1_{\text{orb}}(V) \) such that \( d\mu = \omega_1 - \omega_0 \). We can suppose that the restriction \( i^*\mu \) of \( \mu \) to the zero section vanishes. Indeed, if not then we would consider the form \( \tilde{\mu} = \mu - \pi^*i^*\mu \) which also satisfies \( d\tilde{\mu} = d\mu - \pi^*i^*(\omega_1 - \omega_0) = d\mu = \omega_1 - \omega_0 \), and \( i^*\tilde{\mu} = i^*\mu + i^*\pi^*i^*\mu = i^*\mu - i^*\mu = 0 \).

Consider the form \( \omega_t = t\omega_1 + (1 - t)\omega_0 = \omega_0 + t d\mu \), for \( 0 \leq t \leq 1 \). Since \( i^*\omega_0 = i^*\omega_1 = i^*\omega_1 \) is symplectic on the zero section \( D \), we can suppose, reducing \( U \) if necessary, that \( \omega_0 \) is symplectic on some neighborhood, which we call \( U \) again, of the zero section \( D \) of \( \nu_D \). The equation \( i_X\omega_t = -\mu \) admits a unique solution \( X_t \), which is a vector field on \( V \). Since \( i^*\mu = 0 \), it follows that \( X_t|_{x} = 0 \) for every \( x \in D \subset \nu_D \). Now consider the flow \( \varphi_t \) of the family of vector fields \( X_t \). There is some \( U' \subset U \) such that \( \varphi_t : U' \to U \) for all \( t \in [0, 1] \). Moreover \( \varphi_0 = \text{Id}_U \), and \( \varphi|_D = \text{Id}_D \). We compute

\[
\frac{d}{dt} \bigg|_{t=s} \varphi_t^*\omega_t = \varphi_t^*\left( \mathcal{L}_{X_t}\omega_s \right) + \varphi_t^*(d\mu) \\
= \varphi_t^*(d(i_X\omega_s) - i_Xd\omega_s) + \varphi_t^*(d\mu) = -\varphi_t^*(d\mu) + \varphi_t^*(d\mu) = 0.
\]

This implies that \( \omega_0 = \varphi_0^*\omega_0 = \varphi_1^*\omega_1 \). So \( \varphi_1 : (U', \tilde{\omega}) \to (U, h^*(\omega)) \) is a symplectomorphism. It remains to see that \( \varphi \) is \( \Gamma_\alpha \)-equivariant by all the local isotropy groups \( \Gamma_\alpha \). Fix a chart of \( \nu_D \) and suppose that the group \( \Gamma \) acts on this chart. As \( \omega_t \) and \( \mu \) are \( \Gamma \)-equivariant, we have that \( X_t \) are \( \Gamma \)-equivariant. This implies that the diffeomorphisms \( \varphi_t \) are \( \Gamma \)-equivariant.

Given \( \varphi = \varphi_1 \) as above, take the composition \( \psi = h \circ \varphi : (U', \tilde{\omega}) \to (V, \omega) \), which is our desired orbifold symplectomorphism of \( U' \) onto \( V' = \psi(U') \subset V \).

\[\text{Proposition 20.} \quad \text{Let} \ (X, \omega, g, J) \ \text{be an almost Kähler orbifold and} \ D \ \text{a homogeneous isotropy submanifold. An open neighborhood} \ V \ \subset \nu_D \ \text{of the zero section} \ D = D \times \{0\} \subset \nu_D \ \text{admits an orbifold almost Kähler structure} \ (\tilde{\omega}, \tilde{g}, \tilde{J}) \ \text{such that:}
\]

- For a point \((x, 0)\) in the zero-section we have that, under the natural splitting \( T_{(x,0)}(\nu_D) = T_xD \times (T_xD)^\perp \), the restriction of \((\tilde{\omega}, \tilde{g}, \tilde{J})\) to \( T_xD \) and \((T_xD)^\perp\) coincides with \((\omega, g, J)\).
- The tensors \( \tilde{\omega}, \tilde{g} \) and \( \tilde{J} \) are constant along the fibers \( F_x = \nu_{D,x} \), for \( x \in D \).

\[\text{Proof.} \quad \text{We take the symplectic structure} \ \tilde{\omega} \ \text{provided by Proposition 18. Let us define first an auxiliary metric} \ g' \ \text{on} \ V \ \subset \nu_D. \ \text{We define} \ g' \ \text{so that} \ g \ \text{on} \ T_xD \ \text{and on} \ (T_xD)^\perp \ \text{for} \ x \ \in \ D. \ \text{On the fiber} \ F_x = \nu_{D,x} = ((T_xD)^\perp)/\Gamma_x, \ \text{the tensors} \ g_{x|_{(T_xD)^\perp}} \ \text{and} \ J_{x|_{(T_xD)^\perp}} \ \text{are} \ \Gamma_x \text{-equivariant, so we can define constant}
\]
tensors on $F_x$, which vary smoothly for $x \in D$. Define $g'_x$ equal to $g_x|_{(T_x,D)^⊥}$ at any point $y \in F_x$.

Now we extend $g'$ to a Riemannian metric on $V \subset \nu_D$. This is done as follows. For $(x,u) \in V \subset \nu_D$, with $u \neq 0$, we consider the splitting $T_{(x,u)\nu_D} = T_{(x,u)F_x} \oplus (T_{(x,u)F_x})^{⊥ω}$. We define $g'$ by making these subspaces orthogonal so that $g'$ restricted to $(T_{(x,u)F_x})^{⊥ω}$ is $π^*(g|_{T_xD})$ under the isomorphism $π_x : (T_{(x,u)F_x})^{⊥ω} \to T_xD$. The metric $g'$ may not be equivariant, so we make it equivariant by averaging and then we use the method of the proof of Proposition 8 to modify $g'$ into an orbifold Riemannian metric $\tilde{g}$ such that $\tilde{g}(u,v) = \tilde{ω}(u,\tilde{J}v)$ defines an orbifold almost-Kähler structure $\tilde{J}$. Note that the tensor $A$ defined by $g'(u,Av) = \tilde{ω}(u,v)$ satisfies that $A = J$ at the points of $D \subset \nu_D$, as desired. For $(x,u) \in F_x$, the definition $g'(u,Av) = \tilde{ω}(u,v)$ and the fact that $T_{(x,u)F_x} \oplus (T_{(x,u)F_x})^{⊥ω}$ is at the same time the Riemannian orthogonal decomposition, implies that $A$ equals $J_x|_{T_xD}^⊥$ restricted to $T_{(x,u)F_x}$. So $\tilde{J}$ is constant along $F_x$. This concludes the proof.

To proceed further, we will use the natural retraction of [13, Prop. 2.2.4],

$$r : \text{Sp}(2k, \mathbb{R}) \rightarrow U(k), \quad r(A) = A(A^tA)^{-1/2} \quad (4)$$

We note that there is a group $Γ \subset U(k)$ and an isomorphism $ρ : Γ \rightarrow Γ' \subset U(k)$, such that $A$ is $Γ$-equivariant, in the sense that $A \circ γ = ρ(γ) \circ A$, then $r(A)$ is also $Γ$-equivariant.

**Lemma 21.** Let $A, C \in U(k)$ and $B \in \text{Sp}(2k, \mathbb{R})$ such that $A = B^{-1}CB$. Then $A = r(B)^{-1}Cr(B)$.

**Proof.** The fact $B \in \text{Sp}(2k, \mathbb{R})$ means that $B^tJ_0B = J_0$, where $J_0$ is the matrix of the standard complex structure. So $B^t = -J_0B^{-1}J_0$, $A^tA = C^tC = \text{Id}$, $AJ_0 = J_0A$ and $CJ_0 = J_0C$. Then

$$(B^tB)A = -J_0B^{-1}J_0BA = -J_0B^{-1}J_0CB = -J_0B^{-1}CJ_0B$$

$$= -J_0AB^{-1}J_0B = -AJ_0B^{-1}J_0B = A(B^tB).$$

This means that $A$ commutes with $B^tB$. Therefore $A$ commutes with $(B^tB)^{1/2}$ as well. Hence $r(B)^{-1}Cr(B) = (B^tB)^{1/2}B^{-1}CB(B^tB)^{-1/2} = (B^tB)^{1/2}A(B^tB)^{-1/2} = A$, as required.

**Proposition 22.** The normal orbifold bundle $\nu_D$ admits an atlas such that the transition functions $A_{αβ}$ are $U(k)$-valued. In the terminology of Definition 15, the structure group of $\nu_D$ reduces to $U(k)$.

**Proof.** By Propositions 16 and 20, the normal orbifold bundle $\nu_D$ admits an almost Kähler structure $(ω, J, g)$ which is constant along the fibers, and it also admits the structure of a $\text{Sp}(2k, \mathbb{R})$-orbifold bundle. Call $h$ the hermitian metric associated with $(ω, J, g)$. Take an atlas $\{(U_α × \mathbb{C}^k, Γ_α, ω_0)\}_{α \in I}$ of $\nu_D$ so that $Γ_α < U(k)$, $ω_0$ the standard symplectic form in $\mathbb{C}^k$, and the transition functions are $A_{αβ} : U_α ∩ U_β \rightarrow \text{Sp}(2k, \mathbb{R})$. 


Fix a chart $U_\alpha \times \mathbb{C}^k$ and call $(x, y)$ the corresponding coordinates. The hermitian metric $h$ induces a linear hermitian metric $h_x$ on each fiber $\{x\} \times \mathbb{C}^k$ varying smoothly with $x \in U_\alpha$. Using a $h_x$-unitary frame, this is determined by a matrix $C_\alpha(x) \in \text{Sp}(2k, \mathbb{R})$. The orbifold almost Kähler structure in the chart is given by tensors $(\omega_0, J_\alpha, g_\alpha)$, which are $\Gamma_\alpha$-equivariant. If we introduce new coordinates $(x, \tilde{y}) = (x, C_\alpha(x)y)$ then the orbifold almost Kähler structure is given by the standard tensors $(\omega_0, J_\alpha, g_\alpha)$ defining the complex structure and metric in $\mathbb{C}^k$, but the action is given by the varying group $\Gamma^x_\alpha = C_\alpha(x)\Gamma_\alpha C_\alpha(x)^{-1}$. Clearly $\Gamma^x_\alpha < U(k)$ because it preserves the hermitian structure $(\omega_0, J_\alpha, g_\alpha)$. The group $\Gamma^x_\alpha$ acts on the fiber $\{x\} \times \mathbb{C}^k$ and vary with the point $x \in U_\alpha$, so the action is not linear on the chart $U_\alpha \times \mathbb{C}^k$. On the other hand, in the coordinates $(x, \tilde{y})$ the transition functions of the bundle are $U(k)$-valued as we want.

Now define new coordinates $(x, y') = (x, r(C_\alpha(x))^{-1}y)$ where $r$ is the retraction (4). The hermitian metric in the new coordinates is the standard metric of $\mathbb{C}^k$ because it was so in the coordinates $(x, \tilde{y})$ and $r(C_\alpha(x))^{-1} \in U(k)$. So the orbifold almost Kähler structure in the coordinates $(x, y')$ is given by $(\omega_0, J_\alpha, g_\alpha)$. However, the isotropy group is the group $\Gamma_\alpha < U(k)$ that we began with. Indeed, $\Gamma_\alpha = C_\alpha(x)^{-1}\Gamma^x_\alpha C_\alpha(x)$ implies, by Lemma 21, that $\Gamma_\alpha = r(C_\alpha(x))^{-1}\Gamma^x_\alpha r(C_\alpha(x))$. Carrying out this procedure for each coordinate patch, the corresponding transition functions are in $U(k)$, whereas the isotropy is given by the groups $\Gamma_\alpha < U(k)$. □

**Corollary 23.** If $D \subset X$ is a connected homogeneous isotropy submanifold, then the normal bundle admits an atlas $\{U_\alpha \times \mathbb{C}^k\}$ with the transition functions $A_{\alpha\beta} : U_\alpha \cap U_\beta \to U(k)$ and with the group $\Gamma$ fixed. Actually, the image of $A_{\alpha\beta}$ lies in the normalizer of $\Gamma < U(k)$, i.e. in the subgroup of $U(k)$ given by $N_{U(k)}(\Gamma) = \{A \in U(k) | A^\Gamma A^{-1} = \Gamma\}$.

**Remark 24.** Therefore, if an homogeneous isotropy submanifold $D \subset X$ has an isotropy group $\Gamma < U(k)$ with finite normalizer, then its normal bundle $\nu_D$ has constant transition functions $A_{\alpha\beta}$ and hence the Chern class $c_1(\nu_D) = 0$.

## 5. Resolution of the normal bundle

In this section we will use the previous nice structure of the normal bundle $\nu_D$ of an HI-submanifold $D \subset X$ of a symplectic orbifold $X$, to construct a symplectic resolution of $\nu_D$.

By Corollary 23, we fix an atlas $\{U_\alpha \times \mathbb{C}^k\}$ with $\Gamma < U(k)$ acting on the fiber, and with the transition functions $A_{\alpha\beta} : U_\alpha \cap U_\beta \to N_{U(k)}(\Gamma)$. The group $G = N_{U(k)}(\Gamma)$ is a closed Lie subgroup of $U(k)$ since $\Gamma$ is finite. In particular $G$ is compact, and acts on $\mathbb{C}^k/\Gamma$ by matrix multiplication. Recall that $F_x \cong \mathbb{C}^k/\Gamma$ is a singular complex variety, hence it admits a constructive algebraic resolution, see [6] and [20]. This resolution has the property that any algebraic action on the singular variety admits a unique lifting to the resolution.

**Theorem 25 ([20, Prop. 7.6.2]).** Let $X \subset W$ be a subscheme of finite type of a smooth scheme $W$, with $X$ reduced, and $\theta \in \text{Aut}(W)$ an algebraic automorphism of
Let \( b : \bar{X} \to X \) be the constructive resolution of singularities. Then \( \theta : X \to X \) lifts uniquely to an isomorphism \( \bar{\theta} : \bar{X} \to \bar{X} \) of the constructive resolution of singularities \( \bar{X} \) of \( X \) such that \( b \circ \bar{\theta} = \theta \circ b \).

Note that the uniqueness of the lifting follows immediately from the existence because any two liftings have to coincide in the Zariski open set where \( b : \bar{X} \to X \) is an isomorphism.

The compact group \( G = N_{U(k)}(\Gamma) < U(k) \) has a complexification \( G^c < \text{GL}(k, \mathbb{C}) \) which is an algebraic group. We claim that \( G^c < N_{\text{GL}(k, \mathbb{C})}(\Gamma) \). The normalizer \( N_{\text{GL}(k, \mathbb{C})}(\Gamma) < \text{GL}(k, \mathbb{C}) \) is a complex Lie group that contains \( G \), hence it contains \( G^c \), which is its Zariski closure. Thus the group \( G^c \) acts naturally on \( F = \mathbb{C}^k / \Gamma \) by matrix multiplication, i.e. \( A \cdot [u] = [Au] \) for \( A \in G^c \). Here the bracket stands for the equivalence class of \( u \in \mathbb{C}^k \) in the quotient \( \mathbb{C}^k / \Gamma \). For \( A \in G^c \), this is well defined because if \( [u] = [u'] \) then there exists \( \gamma \in \Gamma \) with \( u = \gamma u' \) and hence \( Au = A\gamma u' = \gamma'Au' \) for some \( \gamma' \in \Gamma \), since \( A \in N_{\text{GL}(k, \mathbb{C})}(\Gamma) \).

**Proposition 26.** The fiber \( F = \mathbb{C}^k / \Gamma \) and its constructive resolution \( \bar{F} \) are quasi-projective varieties.

**Proof.** Since \( \Gamma < U(k) \) is a finite group, the quotient \( F = \mathbb{C}^k / \Gamma \) is an affine variety, i.e. there is an embedding \( \iota : F \to \mathbb{C}^N \) for some \( N \in \mathbb{N} \). Indeed \( \mathbb{C}[x_1, \ldots, x_k] / \mathbb{C}[x_1, \ldots, x_k] \), the \( \mathbb{C} \)-algebra of polynomials invariant by the action of \( \Gamma \), is a finitely generated \( \mathbb{C} \)-algebra, say \( \mathbb{C}[x_1, \ldots, x_k] / \mathbb{C}[f_1, \ldots, f_N] \) for some \( f_j \in \mathbb{C}[x_1, \ldots, x_k] \). Defining \( \iota : \mathbb{C}^k / \Gamma \to \mathbb{C}^N, \iota([(x_1, \ldots, x_k)]) = (f_1(x), \ldots, f_N(x)) \), we have an embedding of \( F \) into \( \mathbb{C}^N \). This proves that \( F \) is an affine variety, hence it is quasi-projective. We can use the model \( \iota(F) \subset \mathbb{C}^N \) to perform the resolution of singularities. The resolution \( \bar{F} \) of \( \iota(F) \) is obtained via a finite numbers of blow-ups starting from \( \mathbb{C}^N \) so \( \bar{F} \) is quasi-projective. \( \square \)

Select an embedding \( \iota : F = \mathbb{C}^k / \Gamma \to \mathbb{C}^N \) as in Proposition 26. Let \( \bar{F} \) be the constructive resolution of the algebraic variety \( \iota(F) \subset \mathbb{C}^N \). The action \( G^c \times F \to F \), \((g, y) \mapsto gy\), is an algebraic map. There is a well-defined map \( G^c \times \bar{F} \to G^c \times F \), \((g, y) \mapsto (g, g \cdot y) \), by Theorem 25. This is a bijection between smooth algebraic varieties, and it is holomorphic on the Zariski dense open subset \( G^c \times \bar{F} \setminus G^c \times Z \), where \( Z \) is the exceptional locus. In particular it is continuous. Therefore it is holomorphic everywhere, hence algebraic. This implies that the map \( G^c \to \text{Aut}(\bar{F}) \) is holomorphic, in particular the map \( G \to \text{Aut}(\bar{F}) \) is smooth.

Let \( b : \bar{F} \to F \) be the blow-up map, and denote by \( Z = b^{-1}(0) \) the exceptional divisor. For the bundle \( \nu_D \), each transition matrix \( A_{\alpha\beta}(x) \in G < U(k) \) has a corresponding unique lifting \( B_{\alpha\beta}(x) : \bar{F} \to \bar{F} \) which satisfies \( b(B_{\alpha\beta}(x)y) = A_{\alpha\beta}(x)(b(y)) \), for each \( y \in \bar{F} \), i.e. \( b \circ B_{\alpha\beta}(x) = A_{\alpha\beta}(x) \circ b \). The maps \( B_{\alpha\beta}(x) \) depend smoothly on \( x \). This is because \( A_{\alpha\beta}(x) \) depend smoothly on \( x \), and the map \( G^c \to \text{Aut}(\bar{F}) \) is holomorphic.
Proposition 27. The maps $B_{\alpha\beta}(x)$ for $x \in U_\alpha \cap U_\beta$ are the transition functions of a smooth fiber bundle $\tilde{\nu}_D \to D$ with $\tilde{F}$ as fiber.

There is a map $b : \tilde{\nu}_D \to \nu_D$ which is a diffeomorphism outside the subbundle $E \to D$ whose fiber is the exceptional locus $Z \subset \tilde{F}$.

Proof. We only need to check the cocycle condition. In a triple intersection we know that $A_{\alpha\beta} \circ A_{\delta\alpha} \circ A_{\beta\delta} = \text{Id}_F$. Since lifting respects composition and the identity lifts to the identity, we have that $B_{\alpha\beta} \circ B_{\delta\alpha} \circ B_{\beta\delta} = \text{Id}_F$, as required. □

We call $b$ the blow-up map, because it is induced on each fiber by the blow-up map $b : \tilde{F} \to F$.

The next step consists on constructing a symplectic form on the resolution $\tilde{F}$ of the complex variety $F = \mathbb{C}^k/\Gamma$, with $\Gamma < U(k)$ as above. Here, $F \cong F_x$ is diffeomorphic to the orbifold normal space $(T_xD^\perp)/\Gamma$ of the HI-submanifold $D \subset X$. Since $D$ does not intersect any other isotropy submanifold of the orbifold $X$, we see that $0 \in \mathbb{C}^k$ is the only fixed point of the action of the group $\Gamma < U(k)$. Hence the singular locus of $F$ reduces to the point $[0] \in F = \mathbb{C}^k/\Gamma$. The exceptional locus is $Z = b^{-1}([0]) \subset \tilde{F}$, and consists of a finite union of irreducible components $Z_j$ which are divisors intersecting transversally.

Proposition 28. The resolution $\tilde{F}$ of $F = \mathbb{C}^k/\Gamma$ admits a Kähler structure $(\omega_F, J_F, g_F)$ which is invariant by the action of $G = N_{U(k)}(\Gamma)$ on $\tilde{F}$.

Proof. By Proposition 26, $\tilde{F}$ is a quasi-projective variety, so it is a complex submanifold of some $CP^N$ for $N$ high enough. Consider $(CP^N, \omega_{FS}, J, g_{FS})$ the standard Kähler structure on $CP^N$, where $\omega_{FS}$ is the Fubini-Study Kähler form. The restriction of $(\omega_{FS}, J, g_{FS})$ to $\tilde{F}$ defines a Kähler structure $(\omega_1, J_\tilde{F}, g_1)$ on $\tilde{F}$, where $J_\tilde{F}$ is the given complex structure on $\tilde{F}$.

The complex structure $J_\tilde{F}$ is preserved by the transition functions $B_{\alpha\beta}(x)$ because they act on $\tilde{F}$ as biholomorphisms. But the symplectic structure $\omega_1$ may not be preserved, so we need to make an average. As $G$ is compact, we put on $G$ any right-invariant Riemannian metric and call $\mu$ the measure induced by this metric. Let

$$\omega_\tilde{F} = \frac{1}{\mu(G)} \int_G h^* \omega_1 d\mu(h) \in \Omega^2(\tilde{F}).$$

We claim that $\omega_\tilde{F}$ is a symplectic form invariant by the action of $G$ on $\tilde{F}$. For the invariance, take $g \in G$ and compute

$$g^* \omega_\tilde{F} = \frac{1}{\mu(G)} \int_G g^*(h^* \omega_1) d\mu(h) = \frac{1}{\mu(G)} \int_G (hg)^*(\omega_1) d\mu(h) = \frac{1}{\mu(G)} \int_G k^* \omega_1 d\mu(k) = \omega_\tilde{F},$$

where we have made the change of variables $hg = k$, and $d\mu(h) = d\mu(k)$ since translations are isometries. The closeness is clear as $d\omega_\tilde{F} = \frac{1}{\mu(G)} \int_G d(h^* \omega_1) d\mu(h) = \omega_{FS}$.
Proof. For any choice of \( \varepsilon > 0 \) we consider 
\[ U = U^Z = b^{-1}(B_{\varepsilon}(0)/\Gamma) \] 
and 
\[ V = V^Z = b^{-1}(B_{\varepsilon}(0)/\Gamma), \] 
where the balls are taken with respect to the metric \( g_F \) on \( \mathbb{C}^k \). Consider also 
\[ W = b^{-1}(\{ \frac{1}{2} \varepsilon < |z| < \frac{3}{2} \varepsilon \}/\Gamma) \] 
so that \( U^Z \setminus V^Z \subset W \).

As the map \( b : \tilde{F} \to F \) is a diffeomorphism outside \( Z \subset \tilde{F} \), we see that 
\( W \) is homotopy equivalent to a lens-space \( S^{2k-1}/\Gamma \subset F = \mathbb{C}^k/\Gamma \). In particular, 
\( H^2(W, \mathbb{R}) = 0 \), so we have 
\[ \omega_{\tilde{F}} - b^*(\omega_F) = d\eta, \] 
for some \( \eta \in \Omega^1(W) \). Take \( \rho : \tilde{F} \to \mathbb{R} \) a bump-function so that \( \rho = 1 \) on \( V \) and 
\( \rho = 0 \) on \( \tilde{F} \setminus U \). Define the form 
\[ \Omega_{\delta} = b^*(\omega_F) + \delta d(\rho\eta), \] 
for \( \delta > 0 \). Note that \( \Omega_{\delta} = (1-\delta)b^*(\omega_F) + \delta \omega_{\tilde{F}} \) in \( W \cap V \), and 
\( \Omega_{\delta} = b^*(\omega_F) \) in \( W \setminus U \). This shows that \( \Omega_{\delta} \) can be extended to all \( \tilde{F} \) so that \( \Omega_{\delta} \) equals \( (1-\delta)b^*(\omega_F) + \delta \omega_{\tilde{F}} \) on \( V \) and \( b^*(\omega_F) \) on \( \tilde{F} \setminus U \). Moreover \( \Omega_{\delta} \) is obviously closed.

We need to show that \( \Omega_{\delta} \) is non-degenerate for an adequate choice of \( \delta \). We already know that \( \Omega_{\delta} \) is non-degenerate except for the set \( U \setminus V \), on which \( \Omega_{\delta} \) has the form 
\[ \Omega_{\delta} = b^*(\omega_F) + \delta d(\rho\eta) \] 
for \( \delta > 0 \). Since \( b^*(\omega_F) \) is non-degenerate on \( U \setminus V \), by choosing \( \delta \) small enough, the form \( \Omega_{\delta} \) will be non-degenerate on \( U \setminus V \). Note that 
\[ \Omega_{\delta} = (1-\delta)b^*(\omega_F) + \delta \omega_{\tilde{F}} \] 
on \( V \), both \( (\tilde{F}, g_{\tilde{F}}, J_{\tilde{F}}, \omega_{\tilde{F}}) \) and \( (F, g_F, J_F, \omega_F) \) are Kähler,
and \( b \) is a biholomorphism. From this we see that for a tangent vector \( u \) at a point in \( V \) we have 
\[ \Omega_{\delta}(u, -J_{\tilde{F}}u) = (1-\delta)(b^*g_F)(u, u) + \delta g_{\tilde{F}}(u, u) > 0 \] 
if \( u \neq 0 \). This also shows that \( \Omega_{\delta} \) is \( J_{\tilde{F}} \)-tame in \( V \), hence 
\( (V, \Omega_{\delta}, J_{\tilde{F}}, (1-\delta)b^*g_F + \delta g_{\tilde{F}}) \) defines a Kähler structure on \( V \).

It remains to see the invariance under the structure group \( G = N_{U(k)}(\Gamma) \) of \( E \).
Recall that the average over the compact Lie group \( G \) of a form \( \alpha \) is given by
\frac{1}{\mu(\alpha)} \int_G h^\ast \alpha d\mu(h), \text{ where } \mu \text{ is the measure induced by any right-invariant metric on } G. \text{ The average operator is a linear projection onto the vector subspace of } G\text{-invariant forms. The form } \eta \text{ can be chosen to be } G\text{-invariant by averaging over } G \text{ in the equation (5). The bump function } \rho \text{ can also be chosen } G\text{-invariant. It suffices to take } \rho = b^\ast \rho_0 \text{ with } \rho_0 \text{ a bump-function on } \mathbb{C}^k \text{ which is radial with respect to the metric } g_F. \text{ Indeed, since } G \text{ acts on } F \text{ by unitary matrices and } \rho_0 \text{ is radial, } \rho_0 \text{ is a } G\text{-invariant function on } F. \text{ Since the actions of } G \text{ on } \tilde{F} \text{ and } F \text{ commute with } b, \text{ for } h \in G \text{ we have } h^\ast b^\ast \rho_0 = (b \circ h)^\ast \rho_0 = b^\ast \rho_0, \text{ proving the invariance of } \rho = b^\ast \rho_0. \quad \square

The proposition above shows that we can construct a symplectic form } \Omega_{\tilde{F}} \text{ on the fiber } \tilde{F} \text{ of } \tilde{\nu}_D. \text{ Now we will globalize the construction to obtain a Kähler form in some small neighborhood of the exceptional locus } E \text{ of } \tilde{\nu}_D. \text{ Note that we have a blow-up map}

\begin{align*}
b : \tilde{\nu}_D &\to \nu_D,
\end{align*}

such that \( b^{-1}(F_x) = \tilde{F}_x \). Let \( 0_x \) be the origin of the fiber \( F_x \), \( Z_x = b^{-1}(0_x) \cong Z \) \text{ the exceptional divisor. We denote } D \subset \nu_D \text{ the zero section, and } E = b^{-1}(D) \text{ the exceptional locus of the blow-up. Then } b : E \to D \text{ is a fibre bundle, whose fiber is } Z_x \text{ at every } x \in D.

\textbf{Remark 30. The question of whether a bundle with symplectic fibers over a symplectic base space admits a symplectic form defined on the total space of the bundle is not entirely trivial and there are some topological obstructions [9]. For instance, consider the Hopf fibration } S^1 \to S^3 \to S^3 \text{ and multiply by } S^1 \text{ to get a torus bundle } S^1 \times S^1 \to S^3 \times S^1 \to S^2 \text{. Both base and fiber are symplectic, however the total space has trivial second cohomology so it is not symplectic.}

The first thing that we need is to find a cohomology class } [\eta] \text{ on the manifold } \tilde{\nu}_D \text{ that restricts to the cohomology class } [\Omega_{\tilde{F}}] \text{ of the symplectic form of the fiber } \tilde{F}, \text{ constructed in Proposition 29. If we do this, the cohomological obstructions of Remark 30 vanish and we will be able to construct a symplectic form on all of } E \text{ (cf. Proposition 35).}

\textbf{Proposition 31. The homology } H_{2k-2}(\tilde{F}) \text{ of } \tilde{F} \text{ is freely generated by the exceptional divisors } Z_j, j = 1, \ldots, l \text{ (the irreducible components of } Z \subset \tilde{F}). \text{ In other words } H_{2k-2}(\tilde{F}) = \bigoplus_{j=1}^l \mathbb{Z}\langle Z_j \rangle.

\textbf{Proof.} \text{ The exceptional locus } Z \text{ of the constructive resolution of singularities of [6] is a tree of exceptional divisors } Z_j \text{ with normal crossings. These are smooth complex submanifolds of dimension } k - 1, \text{ hence } (2k - 2)\text{-dimensional smooth real manifolds, so } H_{2k-2}(Z_j) = \mathbb{Z}\langle Z_j \rangle. \text{ Now, } Z_i \cap Z_j \text{ for } i \neq j \text{ is of complex dimension}
\[ H_{2k-2}(Z) = H_{2k-2}(Z/(\cup_{i \neq j}(Z_i \cap Z_j))) = H_{2k-2} \left( \bigvee_{j=1}^{l} Z_j/((\cup_{i \neq j}(Z_i \cap Z_j)) \right) \]
\[ \cong \bigoplus_{j=1}^{l} H_{2k-2}(Z_j) = \bigoplus_{j=1}^{l} \mathbb{Z}(Z_j). \]

There is a deformation retract from \( \tilde{F} \) to \( Z \) induced by lifting the radial vector field \( r \frac{\partial}{\partial r} \) from \( F = \mathbb{C}_k^\infty/\Gamma \) to \( b : \tilde{F} \to F \). Therefore \( H_{2k-2}(\tilde{F}) = H_{2k-2}(Z) = \bigoplus_{j=1}^{l} \mathbb{Z}(Z_j) \), as required.

This proposition means that in the bundle \( \tilde{F} \to \tilde{\nu}_D \to D \) there is a canonical unordered basis for \( H_{2k-2}(\tilde{F}) \) at the level of chains, namely the set of exceptional divisors. Note that for each ordering of the exceptional divisors \( Z_j \), we have a basis of \( H_{2k-2}(\tilde{F}) \), but the transition functions \( B_{\alpha \beta}(x) : \tilde{F} \to \tilde{F} \) induce a permutation on this basis, so it is the (unordered) set \( \{Z_1, \ldots, Z_l\} \) what is preserved. This property of the bundle \( \tilde{\nu}_D \) will be crucial to construct a symplectic form on the total space \( \tilde{\nu}_D \).

Poincaré duality for \( \tilde{F} \) gives an isomorphism
\[ PD : H^2_c(\tilde{F}, \mathbb{R}) \cong H_{2k-2}(F, \mathbb{R}). \]
Note that \( H^2_c(\tilde{F}, \mathbb{R}) \cong H^2(\tilde{F}, \mathbb{R}) \). Consider the radial function \( r : \tilde{F} \to [0, \infty) \), and introduce the sets \( A_R = \{y \in \tilde{F} | r(y) \leq R\} \subset \tilde{F} \), for each \( R > 0 \). Then
\[ H^2_c(\tilde{F}, \mathbb{R}) \cong H^2_c(A_R, \mathbb{R}) \cong H^2(\overline{A}_R, \partial A_R, \mathbb{R}) \cong H^2(\overline{A}_R, \mathbb{R}) \cong H^2(\tilde{F}, \mathbb{R}), \]

since \( \partial A_R \cong S^{2k-1}/\Gamma \) has \( H^2(\partial A_R, \mathbb{R}) = 0 \).

6. SYMPLECTIC FORM ON THE RESOLUTION OF THE NORMAL BUNDLE

Now we construct a global symplectic form on \( \tilde{\nu}_D \) which coincides with \( \Omega_{\tilde{F}} \) on every fiber. The construction will provide a symplectic form on a neighbourhood of the exceptional locus \( E \subset \tilde{\nu}_D \). First we deal with the cohomological obstruction mentioned in Remark 30.

**Proposition 32.** Let \( \tilde{F} \to \tilde{\nu}_D \to D \) be as before, with \( (\tilde{F}, \Omega_{\tilde{F}}) \) the symplectic structure on \( \tilde{F} \). There exists a cohomology class \( a \in H^2(\tilde{\nu}_D, \mathbb{R}) \) whose restriction to each fiber is \([\Omega_{\tilde{F}}]\).

**Proof.** Consider the atlas of the bundle \( \tilde{\nu}_D \) consisting of charts \( \phi_\alpha : U_\alpha \times \tilde{F} \to V_\alpha \subset \tilde{\nu}_D \), and with changes of trivializations \( B_{\alpha \beta} : U_\alpha \cap U_\beta \to \text{Sympl}(\tilde{F}, \Omega_{\tilde{F}}) \). We refine the open cover given by the \( U_\alpha \subset D \) in such a way that there exists a smooth map \( T_\alpha : [0, 1]^{2n-2k} \to U_\alpha \) with image \( Q_\alpha \subset U_\alpha \), so that the simplices \( Q_\alpha \) form a triangulation of \( D \). As \( D \) is compact and symplectic, it is an oriented manifold of dimension \( 2n - 2k \). Let \([D]\) \( \in H_{2n-2k}(D) \) denote its fundamental class, which can be defined by the chain \( \sum_\alpha Q_\alpha \in C_{2n-2k}(D) \).
On the other hand, consider the cohomology class \([\Omega_{\tilde{F}}] \in H^2(\tilde{F}, \mathbb{R})\). By Poincaré duality, 
\(H^2(\tilde{F}, \mathbb{R}) \cong H^2_c(\tilde{F}, \mathbb{R}) \cong H_{2k-2}(\tilde{F}, \mathbb{R})\).
Choose a basis \(\{Z_1, \ldots, Z_l\}\) of exceptional divisors of \(H_{2k-2}(\tilde{F})\). There exists unique real numbers \(a_i \in \mathbb{R}\) so that 
\(\text{PD}[\Omega_{\tilde{F}}] = \sum_{i=1}^{l} a_i[Z_i]\). For each trivialization \(\phi_\alpha : U_\alpha \times \tilde{F} \to \tilde{V}_\alpha \subset \tilde{\nu}_D\), consider the chain 
\[A_\alpha = \sum_{i=1}^{l} a_i \phi_\alpha(Q_\alpha \times Z_i) \in C_{2n-2}(\tilde{\nu}_D).\]
We claim that the chain \(A = \sum A_\alpha\) is closed, so it defines a homology class 
\([A] \in H_{2n-2}(\tilde{\nu}_D)\). Certainly, 
\[\partial A = \sum \partial A_\alpha = \sum \sum_{i} a_i \phi_\alpha(\partial Q_\alpha \times Z_i).\] 
If \(x \in \partial Q_\alpha \cap \partial Q_\beta \subset U_\alpha \cap U_\beta\), the transition function \(g = B_{\alpha\beta}(x) : \tilde{F} \to \tilde{F}\) is a symplectomorphism of \((\tilde{F}, \Omega_{\tilde{F}})\), hence it preserves the homology class \(\text{PD}(\Omega_{\tilde{F}})\) = 
\(\sum_{i=1}^{l} a_i[Z_i]\). On the other hand, \(g\) permutes the exceptional divisors \(Z_i\). But if 
\(g(Z_{i_1}) = Z_{i_2}\) then the corresponding coefficients in \([\Omega_{\tilde{F}}]\) are the same, i.e. \(a_{i_1} = a_{i_2}\). This follows from the equality 
\(\text{PD}(\Omega_{\tilde{F}}) = \sum_{i=1}^{l} a_i[Z_i] = (g)_* \text{PD}(\Omega_{\tilde{F}}) = \sum_{i=1}^{l} a_i[g(Z_i)]\), by looking on both sides at the coefficient of \([Z_{i_2}]\). Therefore, if 
\(g(Z_{i_1}) = Z_{i_2}\) then 
\[a_{i_1} \phi_\alpha(T \times Z_{i_1}) + a_{i_2} \phi_\beta(T \times Z_{i_2}) = 0 \in C_{2n-3}(\tilde{\nu}_D),\] 
where \(T \subset \partial Q_\alpha \cap \partial Q_\beta\) is a \((2n-3)\)-simplex that is common to the boundary of both \(Q_\alpha\) and \(Q_\beta\). Note that we are taking into account that the orientations of \(T\) induced by \(Q_\alpha\) and \(Q_\beta\) are opposite. Plugging (7) into (6), we get that \(\partial A = 0\).

Hence \(A \in H_{2n-2}(\tilde{\nu}_D)\) determines via Poincaré duality a unique \(a = [\eta] \in H^2(\tilde{\nu}_D, \mathbb{R})\) so that \(\text{PD}(a) = A\). The relation between \(a = [\eta]\) and \(A\) is given by the equality 
\(\int_{\tilde{\nu}_D} \eta \wedge \beta = \int_A \beta\), for all \([\beta] \in H^{2n-2}(\tilde{\nu}_D)\). To see that the cohomology class \([\eta]\) restricts to \([\Omega_{\tilde{F}}]\) over each fiber \(\tilde{F}\), we need to check that 
\(\int_{\tilde{F}} \eta \wedge \gamma = \int_{\tilde{F}} \Omega_{\tilde{F}} \wedge \gamma\) for all \([\gamma] \in H^{2k-2}(\tilde{F})\). For this, take any \(x \in D\) with fiber \(\tilde{F}_x \subset \tilde{\nu}_D\), and some \(Q_\alpha\) containing \(x\). Take any \([\gamma] \in H^{2k-2}(\tilde{F}_x)\). Consider a bump \(2(n-k)\)-form \(\nu \in \Omega^{2n-2k}(D)\) with support contained in \(Q_\alpha\) and \(\int_D \nu = 1\). Then \(\pi^*\nu\) has support in \(Q_\alpha \times \tilde{F}\), and so 
\[\int_{\tilde{F}_x} \eta \wedge \gamma = \int_{Q_\alpha \times \tilde{F}} \eta \wedge \gamma \wedge \pi^*\nu = \int_{\tilde{\nu}_D} \eta \wedge \gamma \wedge \pi^*\nu = \int_A \gamma \wedge \pi^*\nu\] 
\[= \int_{A \cap (Q_\alpha \times \tilde{F})} \gamma \wedge \pi^*\nu = \sum_i a_i \int_{Q_\alpha \times Z_i} \gamma \wedge \pi^*\nu = \sum_i a_i \int_{Z_i} \gamma = \int_{\tilde{F}_x} \Omega_{\tilde{F}} \wedge \gamma.\] 
\[\square\]

In [19] it is given a construction of a symplectic form on the total space of a fiber bundle with symplectic base and compact symplectic fibers, once we know the existence of a cohomology class that restricts to the cohomology class of the
symplectic form on the fibers. We have to do a slight extension to a case with non-compact symplectic fiber. We start with a lemma.

**Lemma 33.** Let \((B,g)\) be a compact Riemannian manifold and let \(\omega\) be a symplectic form in \(B\). There exists a constant \(m > 0\) which satisfies the following. For each \(x \in B\) and \(u \in T_xB\), there exists \(v \in T_xB\) so that \(\omega(u,v) \geq m|u||v|\).

**Proof.** Let \(S(TB)\) be the unit sphere bundle of \(B\), and consider the function \(s : S(TB) \rightarrow \mathbb{R}, s(x,u) = \max_{v \in S(T_xB)} \omega(u,v)\). This is a continuous function, which is strictly positive since \(\omega\) is symplectic. It follows that \(s\) attains a minimum \(m\) on the compact set \(S(TB)\), so for all \(x \in B\) and for all \(u \in T_xB\) with \(|u| = 1\) there exists \(v \in T_xB\) with \(|v| = 1\) so that \(\omega(u,v) \geq m\). This implies the required assertion.  

**Definition 34.** Let \(B\) be a compact manifold, and \((N,\omega_N)\) a (possibly non-compact) symplectic manifold with a proper height function \(H : N \rightarrow [0,\infty)\). A proper symplectic bundle is a fiber bundle \(N \rightarrow M \rightarrow B\) such that the transition functions take values in \(\text{Sympl}(N,\omega_N,H) = \{f : N \rightarrow M | f^*\omega_N = \omega_N, H \circ f = H\}\).

If \(N \rightarrow M \rightarrow B\) is a proper symplectic bundle, then the height function \(H\) defines a smooth proper function \(H_M : M \rightarrow [0,\infty)\). For \(R > 0\), we introduce the sets \(M_R = H_M^{-1}([0,R]) \subset M\) and \(N_R = H^{-1}([0,R]) \subset N\). Then \(N_R\) and \(M_R\) are compact and \(N_R \rightarrow M_R \rightarrow B\) is a fibre bundle. If \(R > 0\) is a regular value of \(H\), then \((N_R,\omega_R)\) is a symplectic manifold with boundary, so \(N_R \rightarrow M_R \rightarrow B\) is a compact symplectic bundle.

**Proposition 35.** Let \(N \rightarrow M \rightarrow B\) be a proper symplectic bundle, where the base space \((B,\omega_B)\) is a compact symplectic manifold, \((N,\omega_N)\) is a symplectic manifold with height function \(H : N \rightarrow [0,\infty)\). Suppose that there exists a cohomology class \(e \in H^2(M,\mathbb{R})\) which restricts to \([\omega_N]\) on every fiber. Fix \(R > 0\). Then there exists a closed 2-form \(\omega_M \in \Omega^2(M)\) which is non-degenerate on \(M_R \subset M\), so that \(\omega_M\) restricts to \(\omega_N\) on every fiber \(N_x = \pi^{-1}(x) \subset M\).

**Proof.** Take \(e = [\eta]\) with \(\eta \in \Omega^2(E)\) a representative of the class \(e\). Take \(U_\alpha\) a good cover of \(B\) so that \(\phi_\alpha : U_\alpha \times N \rightarrow V_\alpha \subset M\) are trivialisations of the bundle \(M\), and the transition functions \(g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{Sympl}(N,\omega_N,H)\). On each trivialisation the (locally defined) vertical projection \(q_\alpha : U_\alpha \times N \rightarrow N\) induces an isomorphism in cohomology, hence \((\phi_\alpha^{-1})^*q_\alpha^*\omega_N - \eta|_{V_\alpha} = d\theta_\alpha\) for some 1-form \(\theta_\alpha \in \Omega^1(V_\alpha)\). Take a partition of unity \(\rho_\alpha\) subordinated to the open cover \(U_\alpha\) of \(B\) and define

\[
\omega_M = K\pi^*(\omega_B) + \eta + \sum_\alpha d((\pi^*\rho_\alpha)\theta_\alpha),
\]

for a real number \(K > 0\) to be chosen later. We claim that \(\omega_M\) is symplectic in \(M_R \subset M\) if \(K > 0\) is large enough. The form \(\omega_M\) is clearly closed. We rewrite it
\[
\omega_M = K \pi^* \omega_B + \eta + \sum_\alpha ( \pi^* d \rho_\alpha ) \wedge \theta_\alpha + \sum_\alpha ( \pi^* \rho_\alpha ) \wedge ( ( \phi_\alpha^{-1} )^* q_\alpha^* \omega_N - \eta )
\]

\[
= K \pi^* \omega_B + \sum_\alpha ( \pi^* d \rho_\alpha ) \wedge \theta_\alpha + \sum_\alpha ( \pi^* \rho_\alpha ) ( \phi_\alpha^{-1} )^* q_\alpha^* \omega_N.
\]

On a fiber \( N_x = \pi^{-1}(x) \), we have

\[
(\omega_M)|_{N_x} = \sum_\alpha \rho_\alpha(x) ( \phi_\alpha^{-1} )^* q_\alpha^* \omega_N = \sum_\alpha \rho_\alpha(x) \omega_N = \omega_N,
\]

since all \( \phi_\alpha : \{x\} \times N \to N_x \) are symplectomorphims. We are using here that the transition functions of the bundle are symplectomorphisms of \( (N, \omega_N) \).

To see that \( \omega_M \) is non-degenerate on \( M_R \), take a vector \( u \in T_y M \) and let us see that there exists another vector \( v \) such that \( \omega_M(u, v) \neq 0 \). We fix some (any) background metrics on \( M \) and \( D \) and assume that \( |u| = 1 \). If \( u \in T_y N_{\pi(y)} \) lies in the tangent space to the fiber, then it is clear since \( \omega_M|_{N_{\pi(y)}} = \omega_N \) is symplectic. Since being non-degenerate is an open condition, there is an open set \( G \subset S(TM_R) \) containing all tangent spaces to the fibers \( T_y N_{\pi(y)} \), for \( y \in M_R \), with the property that \( \omega_M \) is non-degenerate on \( G \). The set \( G \) can be taken of the form \( G = \{ (y, w) \in S(TM_R) \mid \text{dist}(w, T_y N_{\pi(y)}) < \delta \} \), for some \( \delta > 0 \). As \( M_R \) is compact, we can take a uniform \( \delta \) for all points \( y \in M_R \).

Take now \( u \in S(T_y M_R) \setminus G \). There exists a constant \( \delta_1 > 0 \) so that \( |\pi_*(u)| \geq \delta_1 \). By Lemma 33, there exists a constant \( m_1 > 0 \) (independent of \( u \)) and a vector \( w \in T_y M \) (depending on \( u \)) so that \( \pi^* \omega_B(u, w) \geq m_1 |\pi_*(u)| |\pi_*(w)| \). By compactness, there are constants \( C_1, C_2 > 0 \) so that the map \( \pi_* : (T_y N_{\pi(y)})^\perp \to T_{\pi(y)} B \) satisfies that

\[
C_1 |v| \leq |\pi_*(v)| \leq C_2 |v|,
\]

for all \( y \in M_R \). Choosing \( w \in (T_y N_{\pi(y)})^\perp \) and unitary, we have that \( |\pi_*(w)| \geq C_1 \). Finally call \( \mu \) the second term in (8) so that \( \omega_M = K \pi^* \omega_B + \mu \). Then

\[
\omega_M(u, w) = K \pi^* \omega_B(u, w) + \mu(u, w)
\]

\[
\geq Km_1 |\pi_*(u)| |\pi_*(w)| - m_2 |u||w| \geq Km_1 \delta_1 C_1 - m_2,
\]

where \( m_2 \) is a constant which bounds \( \mu \) on \( M_R \). The above constants are valid for all \( y \in M_R \) and for all \( u \not\in G \) with \( |u| = 1 \). It is enough to take \( K \geq \frac{m_2}{m_1 \delta_1 C_1} + 1 \) to get that the form \( \omega_M \) is non-degenerate on \( M_R \).

Applying Proposition 35 to the symplectic bundle \( \tilde{F} \to \tilde{\nu}_D \to D \) with symplectic fiber \((\tilde{F}, \Omega_{\tilde{F}})\) and height function given by \( H(y) = |b(y)| \) for \( y \in \tilde{F} = \mathbb{C}^k / \Gamma \), we have the following.

**Theorem 36.** The bundle \( \tilde{F} \to \tilde{\nu}_D \to B \) admits closed 2-form \( \omega_{\tilde{\nu}_D} \) so that:

- The restriction of \( \omega_{\tilde{\nu}_D} \) to each fiber \( \tilde{F}_x \) coincides with \( \Omega_{\tilde{F}_x} \).
- If \( E \subset \tilde{\nu}_D \) is the exceptional locus, then the form \( \omega_{\tilde{\nu}_D} \) is non-degenerate on a neighborhood \( U^E \) of \( E \) in \( \tilde{\nu}_D \).
The form $\omega_{\tilde{\nu}_D}$ has the local expression

$$\omega_{\tilde{\nu}_D} = K\pi^*\omega_D + \sum_a d\pi^*(\rho_a) \land \eta_a + \sum_a (\pi^*(\rho_a)(\phi^{-1}_a)^*\Omega_F),$$

for some $K > 0$ large enough, a finite atlas of symplectic-bundle charts $\phi_a : U_a \times \mathcal{F} \to V_a \subset \tilde{\nu}_D$, some 1-forms $\eta_a$, and a partition of unity $\rho_a$ subordinated to the cover $U_a$ of $D$.

7. Gluing the symplectic form

Finally, we glue the symplectic form $\omega_{\tilde{\nu}_D}$ constructed in Theorem 36 with the symplectic form of the symplectic orbifold $(X, \omega)$. Recall some notations of the previous sections. We have a symplectic fiber bundle $\pi : \tilde{\nu}_D \to D$ with fiber $\mathcal{F}$, the exceptional divisor $E \subset \tilde{\nu}_D$ is a fiber sub-bundle $\pi : E \to D$ with fiber $Z$, and the blow-up map is denoted $b : \tilde{\nu}_D \to \nu_D$. Recall that by Proposition 18, the space $\nu_D$ admits a closed orbifold 2-form $\tilde{\omega}$, which is symplectic on a neighborhood of the zero section. There is also a radial function $\tilde{H}(y) = |b(y)|$, for $y \in \tilde{\nu}_D$, where $b(y) \in F_{\pi(y)} \cong \mathbb{C}^k/\Gamma$ and $|b(y)|$ is its norm in $\mathbb{C}^k$. We denote $U_R = \{ y \in \tilde{\nu}_D | \tilde{H}(y) < R \}$ for $R > 0$. We fix a neighbourhood $W = U_{R_0} \subset \tilde{\nu}_D$ of the exceptional locus, such that $\omega_{\tilde{\nu}_D}$ is symplectic on $W$, as provided by Theorem 36.

**Proposition 37.** For $\varepsilon > 0$ small enough there exists a symplectic form $\Omega_W$ on $W$ so that $\Omega_W = (1-\varepsilon)b^*(\tilde{\omega}) + \varepsilon\frac{1}{K}\omega_{\tilde{\nu}_D}$ on some small neighborhood $U_\delta \subset W$ of $E$ in $\tilde{\nu}_D$, and $\Omega_W = b^*(\tilde{\omega})$ outside of some larger neighborhood $U_{\delta'} \subset W$, $0 < \delta < \delta' < R_0$.

**Proof.** By construction $\omega_{\tilde{\nu}_D} = K\pi^*(\omega_D) + \eta + \sum_a d\pi^*(\rho_a)\theta_a$, where the form $\eta$ is a representative of the Poincaré dual of the homology class given by the cycle $A = \sum_a \sum_l a_lQ_a \times Z_l$. In particular we can take $\eta$ to be very close to a Dirac delta around the cycle $A$, hence we can suppose that the support of $\eta$ is contained in a small neighborhood of $E$, say $U_{\delta/2}$, for $0 < \delta < R_0$. By the construction of $\tilde{\omega}$ in Proposition 18, we have $b^*(\tilde{\omega}) = \pi^*(\omega_D) + d\pi^*(\sum_a (\pi^*(\rho_a)\theta_a))$, for some 1-forms $\eta_a$. On the other hand, outside of the support of $\eta$, the form $\frac{1}{K}\omega_{\tilde{\nu}_D} = \pi^*(\omega_D) + d\left(\frac{1}{K}\sum_a (\pi^*(\rho_a)\theta_a)\right)$. This implies that $b^*(\tilde{\omega})$ and $\frac{1}{K}\omega_{\tilde{\nu}_D}$ define the same cohomology class outside $U_\delta$. So there exists a 1-form $\gamma$ such that $\frac{1}{K}\omega_{\tilde{\nu}_D} - b^*(\tilde{\omega}) = d\gamma$ on $W \setminus U_{\delta/2}$.

Define $\Omega_W = b^*(\tilde{\omega}) + \varepsilon d(\rho\gamma)$, with $\rho : E \to [0, 1]$ a bump function so that $\rho \equiv 1$ on some $U_\delta$ and $\rho \equiv 0$ outside some $U_{\delta'}$ with $\delta < \delta' < R_0$. The form $\Omega_W$ satisfies

$$\Omega_W = b^*(\tilde{\omega}) + \varepsilon d\gamma = b^*(\tilde{\omega}) + \varepsilon \left(\frac{1}{K}\omega_{\tilde{\nu}_D} - b^*(\tilde{\omega})\right) = (1-\varepsilon)b^*(\tilde{\omega}) + \varepsilon\frac{1}{K}\omega_{\tilde{\nu}_D},$$

(11) on $U_\delta \setminus U_{\delta/2}$. We extend $\Omega_W$ with the same formula to all of $U_\delta$. Also $\Omega_W = b^*(\tilde{\omega})$ on $W \setminus U_{\delta'}$.

It remains to see that $\Omega_W$ is symplectic on $W$ if we choose $\varepsilon > 0$ small enough. This is clear on $W \setminus U_{\delta'}$. On $U_{\delta'} \setminus U_\delta$, we have $\Omega_W = b^*(\tilde{\omega}) + \varepsilon d(\rho\gamma)$, where
\( b^*(\tilde{\omega}) \) is non-degenerate. As this is a compact set, making \( \varepsilon > 0 \) small we can assure that \( \Omega_W \) is symplectic there. Finally, take \( y \in U_\varepsilon \), then \( T_y\tilde{\nu}_D \cong T_y\tilde{\pi}_{\pi(y)} \times T_{\pi(y)}D \) by splitting (non-canonically) into vertical directions and projecting onto \( D \). The form \( b^*(\tilde{\omega}) \) vanish on the vertical directions, whereas \( \omega_{\tilde{\nu}_D} \) is symplectic over \( T_y\tilde{\pi}_{\pi(y)} \), hence for \( u \in T_y\tilde{\pi}_{\pi(y)} \) there is some \( v \) such that \( \Omega_W(u,v) \neq 0 \). The same happens for vectors in a neighbourhood of \( S(T_y\tilde{\pi}_{\pi(y)}) \). Finally, for unitary vectors \( u \in T_y\tilde{\nu}_D \) such that \( |\pi_*(u)| \geq \delta_1 \) (using some background metrics), we have \( |b^*(\tilde{\omega})(u,v)| \geq m_1\delta_1C_1 \) for some constant \( m_1 > 0 \) provided by Lemma 33, and some constant \( C_1 > 0 \) provided by (9), and a suitable unitary vector \( w \). We can bound \( |\omega_{\tilde{\nu}_D}(u,w)| \leq m_2 \), so for \( \varepsilon > 0 \) small enough, we have that the expression (11) implies that \( |\Omega_W(u,w)| > 0 \). This completes the proof. \( \square \)

Take the form \( \Omega_W \) constructed in the Proposition 37. It is symplectic on some neighborhood \( W \) of \( E \subset \tilde{\nu}_D \). By Proposition 19, there are neighborhoods \( U \subset \nu_D \) and \( V \subset X \) of \( D \) and a symplectomorphism \( \varphi : (U,\tilde{\omega}) \to (V,\omega) \). By shrinking we can arrange that \( \varphi \) be defined on larger open sets. Consider the open set \( \tilde{U} = b^{-1}(U) \subset \tilde{\nu}_D \), which we assume contained in \( W \).

We define
\[
\tilde{X} = W \cup_f (X \setminus \tilde{U}_\varepsilon),
\]
where \( \tilde{U}_\varepsilon = \varphi(b(U_\varepsilon)) \) is a tubular neighborhood of \( D \subset X \) of radius \( \varepsilon > 0 \). This is chosen with \( \varepsilon > \varepsilon' \), given in Proposition 37. The gluing map is \( f = \varphi \circ b : W \setminus U_\varepsilon \to V \subset X \), whose image is some open set \( V \subset \mathcal{V} \). Note that \( V \subset X \) is the result of removing a tubular neighborhood of \( D \subset X \) from a larger tubular neighborhood, i.e. \( V \) is a fiber bundle over \( D \) with fiber \( (\varepsilon,R_0) \times S^{2k-1}/\Gamma \). Since \( f^*(\omega) = b^*\varphi^*\omega = b^*\tilde{\omega} = \Omega_W \), we see that \( f \) is a symplectomorphism. Hence \( \tilde{X} \) is a symplectic manifold. We have proved the following.

**Theorem 38.** Let \((X,\omega)\) be a symplectic orbifold such that all its isotropy set consists of homogeneous disjoint embedded submanifolds in the sense of definition 13. There exists a symplectic manifold \((\tilde{X},\tilde{\omega})\) and a smooth map \( b : (\tilde{X},\tilde{\omega}) \to (X,\omega) \) which is a symplectomorphism outside an arbitrarily small neighborhood of the isotropy points.

**Remark 39.** If the isotropy submanifold \( D \subset X \) is such that its normal tangent spaces \( F = C^k/\Gamma \) are not singular spaces (for instance, when \( D \) has codimension 2 in \( X \)), then the constructive resolution has \( \tilde{F} = F \) and \( E = D \). In this case Theorem 38 serves to obtain a smooth symplectic form on \( X \) from an orbifold symplectic form. This construction appears in [15].

8. **Examples**

In this section, we want to give some examples where we can apply Theorem 1.

**Example 1. A symplectic divisor.** Let \((X,\omega)\) be a symplectic orbifold of dimension \( 2n \) such that the isotropy locus \( D \subset X \) is a divisor, that is, \( \dim D = 2n - 2 \), and the isotropy is given by \( \Gamma = \mathbb{Z}_k = \langle g \rangle \) acting on the normal space \( C \) by
Let $(\tilde{X}, \tilde{\omega})$ be a smooth symplectic manifold with symplectic form $\tilde{\omega}$ with some prescribed isotropy group (in [15] the dimension of the orbifold is 4, but the result holds for arbitrary dimension).

**Example 2. A product.** Let $(M, \omega_1)$ be a symplectic orbifold with isolated orbifold singularities. By [4], we have a symplectic resolution $b : (\tilde{M}, \tilde{\omega}_1) \to (M, \omega_1)$. Let $(N, \omega_2)$ be a smooth symplectic manifold. Then $(X = M \times N, \omega_1 + \omega_2)$ is a symplectic orbifold with homogeneous isotropy sets. Actually, if $x \in M$ is a singular point of $M$, then $D = \{x\} \times N$ is an isotropy submanifold of $X$. The map $b : (\tilde{M} \times N, \tilde{\omega}_1 + \omega_2) \to (M \times N, \omega_1 + \omega_2)$ is a symplectic resolution, agreeing with Theorem 1. In this case, the symplectic normal bundle to $D$ is trivial.

**Example 3. Symplectic bundle over an orbifold.** Let $(F, \omega_F)$ be a symplectic manifold, $(B, \omega_B)$ a symplectic orbifold with isolated singularities, and let $F \to M \xrightarrow{\pi} B$ be a smooth bundle, where $(M, \omega)$ is a symplectic orbifold such that $(F_x, \omega|_{F_x})$ is symplectomorphic to $(F, \omega_F)$, for all fibers $F_x = \pi^{-1}(x)$, $x \in B$ (that is, $M$ is a symplectic orbifold over an orbifold symplectic base). For a small orbifold chart $(U, V, \varphi, \Gamma)$ of $B$, we have $\pi^{-1}(V) \cong V \times F \cong (U/\Gamma) \times F = (U \times F)/\Gamma$, where $\Gamma$ acts on the first factor. As we are assuming that $B$ has isolated singularities, the isotropy sets are $F_x$, where $x \in B$ is a singularity of $B$. Theorem 1 guarantees the existence of a symplectic resolution of $M$.

Actually, the resolution is given as follows. Take a resolution $b : (\tilde{B}, \tilde{\omega}_B) \to (B, \omega_B)$ provided by [4], and take the pull-back of the bundle $F \to \tilde{M} \xrightarrow{\tilde{\pi}} \tilde{B}$. Then for every singular point $x \in B$ with orbifold chart $(U, V, \varphi, \Gamma)$, we glue the symplectic form $\tilde{\omega}_B \times \omega_F$ on $\tilde{\pi}^{-1}(V) \cong \tilde{V} \times F$ to $\omega_M$ along the complement of a neighbourhood of $F_x$. Theorem 1 does the job without having to care about the details.

**Example 4. Mapping torus.** Let $(M, \omega_M)$ be a compact symplectic orbifold with isolated singularities. Let $f : M \to M$ be an orbifold symplectomorphism and consider the mapping torus $M_f = (M \times [0,1])/\sim$ with $\sim$ with $(x, 0) \sim (f(x), 1)$. Let $t$ be the coordinate of $[0,1]$ and consider a circle $S^1$ with coordinate $\theta$. Then $X = M_f \times S^1$ is a symplectic orbifold with symplectic form $\omega = \omega_M + dt \wedge d\theta$. The isotropy sets are $2$-tori. Take a singular point $x \in M$ and let $x_0 = x, x_1 = f(x_0), x_2 = f^2(x_0), \ldots$ be the orbit of $x$. As all of them are singular points and there are finitely many of them in $M$, there is some $n > 0$ such that $x_n = x_0$. 


and we take the minimum of such $n$. Consider the circle $C_x$ given by the image of \{x_0, \ldots, x_{n-1}\} \times [0, 1] in $M_f$, which is a $n : 1$ covering of $[0, 1]/ \sim = S^1$. Then $D = C_x \times S^1$ is an isotropy set of $X = M_f \times S^1$. Theorem 1 gives a symplectic resolution of $X$. This can be constructed alternatively by taking the symplectic resolution $b : \tilde{M} \to M$ of $M$ given by \cite{4}. If we arrange to do it in an equivariant way around the singular points, then we may lift $f$ to a symplectomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$ of the resolved manifold, and $\tilde{X} = \tilde{M}_j \times S^1$ is a symplectic resolution of $X$.

**Example 5. An example with non-trivial normal bundle.** Take a standard 6-torus $T^6 = \mathbb{R}^6/\mathbb{Z}^6$ with the standard symplectic form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$, and consider the maps

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, x_2, -x_3, -x_4, -x_5, -x_6),$$

$$g(x_1, x_2, x_3, x_4, x_5, x_6) = f(x_1 + \frac{1}{2} x_2, x_3, x_4, -x_5, -x_6).$$

Then $X = T^6/(f, g)$ is a symplectic orbifold. The isotropy locus are the subsets $S_a = \{(x_1, x_2, a_3, a_4, a_5, a_6) | (x_1, x_2) \in \mathbb{R}^2\}$, for $a = (a_3, a_4, a_5, a_6) \in \{0, 1/2\}^4$. Each of them is isomorphic to $\mathbb{R}^2/\langle (1/2, 0), (0, 1) \rangle$. The normal structure is $F = \mathbb{C}^2/\mathbb{Z}_2$, with action $(z_1, z_2) \sim (-z_1, -z_2)$. The normal bundle is the quotient of the trivial bundle $T^2 \times F \to T^2$ over $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, by the map $g$, hence it is non-trivial (although it is trivializable).

**Example 6. Resolving the quotient of a symplectic nilmanifold.** To give an explicit example of a resolution, we shall take a symplectic 6-nilmanifold from \cite{2} and perform a suitable quotient to get a symplectic 6-orbifold with homogeneous isotropy. For instance we take the nilmanifold corresponding to the Lie algebra $L_{6,10}$ of Table 2 in \cite{2}, which is symplectic since it appears in Table 3 of \cite{2}. Take the group of $(7 \times 7)$-matrices given by the matrices

$$
\begin{pmatrix}
1 & x_2 & x_1 & x_4 & x_1x_2 & x_5 & x_6 \\
0 & 1 & 0 & -x_1 & x_1 & x_1^2/2 & x_3 \\
0 & 0 & 1 & 0 & x_2 & -x_4 & x_2^2/2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & x_1 & x_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

where $x_i \in \mathbb{R}$, for any $i = 1, \ldots, 6$. Then, a global system of coordinate functions $\{x_1, \ldots, x_6\}$ for $G$ is given by $x_i(a) = x_i$, with $i = 1, \ldots, 6$. Note that if a matrix $A \in G$ has coordinates $a_i$, then the change of coordinates of $a \in G$ by the left
translation $L_A$ are given by

$$
L_A^*(x_1) = x_1 + a_1, \quad L_A^*(x_2) = x_2 + a_2, \\
L_A^*(x_3) = x_3 + a_1x_2 + a_3, \quad L_A^*(x_4) = x_4 - a_2x_1 + a_4, \\
L_A^*(x_5) = x_5 + \frac{1}{2}a_2x_1^2 - a_1x_4 + a_1a_2x_1 + a_5, \\
L_A^*(x_6) = x_6 + \frac{1}{2}a_1x_2^2 + a_2x_3 + a_1a_2x_2 + a_6.
$$

A standard calculation shows that a basis for the left invariant 1-forms on $G$ consists of

$$
\{dx_1, dx_2, dx_3 - x_1dx_2, dx_4 + x_2dx_1, dx_5 + x_1dx_4, dx_6 - x_2dx_3\}.
$$

Let $\Gamma$ be the discrete subgroup of $G$ consisting of matrices with entries $(x_1, x_2, \ldots, x_6) \in (2\mathbb{Z})^2 \times \mathbb{Z}^4$, that is $x_1$ are integer numbers and $x_1, x_2$ are even. It is easy to see that $\Gamma$ is a subgroup of $G$. So the quotient space of right cosets $M = \Gamma \setminus G$ is a compact 6-manifold. Hence the 1-forms

$$
e_1 = dx_1, e_2 = -dx_2, e_3 = dx_3 - x_1dx_2 - dx_4 - x_2dx_1 = d(x_3 - x_4 - x_1x_2), \\
e_4 = dx_4 + x_2dx_1, e_5 = dx_5 + x_1dx_4, e_6 = dx_6 - x_2dx_3
$$
satisfy

$$
de_1 = de_2 = de_3 = 0, de_4 = e_1e_2, de_5 = e_1e_4, de_6 = e_2e_3 + e_2e_4.
$$

This coincides with $L_{6,10}$ in Table 2 in [2]. The symplectic form of $M$ is $\omega = e_1e_6 + e_2e_5 - e_3e_4$ (see Table 3 in [2]).

Now we consider the map $\varphi(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, -x_2, -x_3, -x_4, -x_5, x_6)$. This is given in terms of the matrices as $\varphi(A) = PAP$, where $P$ is the diagonal matrix $P = \text{diag}(1, 1, 1, -1, -1, -1)$. Note that for $N \in \Gamma$, $PNAP = (PNP)(PAP)$. As $\varphi(\Gamma) = \Gamma$, we see that $\varphi$ descends to $M = \Gamma \setminus G$. This is clearly a symplectomorphism with $\varphi^2 = \text{Id}$, hence

$$
X = M/\langle \varphi \rangle
$$
is a symplectic orbifold. The isotropy locus is formed by the sets

$$
S_b = \{(x_1, b_2, b_3, b_4 - b_2x_1, b_5 + \frac{1}{2}b_2x_1^2, b_6)(x_1, x_6) \in \mathbb{R}^2\},
$$
for $b = (b_2, b_3, b_4, b_5) \in \{0, 1\} \times \{0, 1/2\}^3$. This is a collection of 16 tori, each of them of homogeneous isotropy $\mathbb{C}^2/\mathbb{Z}_2$. This is computed computed solving the equation $\varphi(x) = Ax$ for some $A \in \Gamma$, which translates to $x_1 = L_A^*(x_1), -x_i = L_A^*(x_i)$ for $2 \leq i \leq 5$ and $x_6 = L_A^*(x_6)$.

The above manifold $M$ is a circle bundle (with coordinate $x_6$) over a mapping torus (with coordinate $x_1$) of a 4-torus (with coordinates $x_2, x_3, x_4, x_5$). Then we take a quotient of $T^4$ by $\mathbb{Z}_2$ acting as $\pm \text{Id}$. So this fits with Example 4 above.

Let us compute the Betti numbers of the resolution $\tilde{X}$ of $X$. The Betti numbers of $M$ appear in Table 2 of [2] and are $b_1(M) = 3, b_2(M) = 5, b_3(M) = 6$. Easily
we get that $H^1(M) = \langle e_1, e_2, e_3 \rangle$ and $H^2(M) = \langle e_2 e_3, e_1 e_5, e_1 e_3, e_2 e_6, e_3 e_6 + e_4 e_6 \rangle$. Taking the invariant part by the action of $\varphi$, we have

$$H^1(X) = \langle e_1 \rangle, \quad H^2(X) = \langle e_2 e_3 \rangle,$$

so $b_1(X) = 1$ and $b_2(X) = 1$. By Poincaré duality, $b_4(X) = b_5(X) = 1$. Now $\chi(X) = 0$ since $\chi(M) = 0$ and the ramification locus are $T^2$ which have $\chi(T^2) = 0$. Therefore $b_3(X) = 2$.

The resolution process changes $F = \mathbb{C}^2/\mathbb{Z}_2$ by the single blow-up at the origin $\tilde{F}$, which has exceptional divisor $Z = \mathbb{C}P^1$ with $Z^2 = -2$. Then each exceptional locus increases by 1 the second Betti number $b_2$ (cf. the computations of cohomology in [8]). Therefore $b_1(\tilde{X}) = 1, b_2(\tilde{X}) = 1 + 16 = 17$. By Poincaré duality, $b_4(\tilde{X}) = 17, b_5(\tilde{X}) = 1$. Again $\chi(\tilde{X}) = 0$, since the exceptional divisors are $\mathbb{C}P^1$-bundles over $T^2$ and hence they have $\chi(E) = 0$. So $b_3(\tilde{X}) = 34$.

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