Analytical Solutions of ARL for SAR(p)_L Model on a Modified EWMA Chart

Piyatida Phanthuna¹, Yupaporn Areepong²*

¹Faculty of Science and Technology, Rajamangala University of Technology Phra Nakhon, 10800, Thailand
²Faculty of Applied Science, King Mongkut’s University of Technology North Bangkok, 10800, Thailand

Received June 7, 2021; Revised July 30, 2021; Accepted August 22, 2021

Abstract  A modified exponentially weighted moving average (EWMA) scheme expanded from an EWMA chart is an instrument for immediate detection on a small shifted size. The objective of this research is to suggest the average run length (ARL) with the explicit formula on a modified EWMA control chart for observations of a seasonal autoregressive model of order p th (SAR(p)_L) with exponential residual. A numerical integral equation method is brought to approximate ARL for checking an accuracy of explicit formulas. The results of two methods show that their ARL solutions are close and the percentage of the absolute relative change (ARC) is obtained to less than 0.002. Furthermore, the modified EWMA chart with the SAR(p)_L model is tested to shift detection when the parameters c and λ are changed. In addition, the modified EWMA control chart is compared to performance with the EWMA scheme and such that their results encourage the modified EWMA chart for a small shift. Finally, this explicit formula can be applied to various real-world data. For example, two data about information and communication technology are used for the validation and the capability of our techniques.

Keywords  Explicit Formula, Seasonal Autoregressive, Average Run Length

1. Introduction

A control chart is one of the instruments for Statistical Process Control (SPC). The ordinary control charts are well known such as a Shewhart control chart [1], an exponentially weighted moving average (EWMA) chart [2], a cumulative sum (CUSUM) scheme [3]. A Shewhart chart is the basis of a control chart that detected a large shift on 3-sigma control limits speedily. Next, EWMA and CUSUM control charts are developed to a small shift detection appropriately. For the EWMA control chart, many recent works of literature were found to this chart usage, for example, Li et al. [4] introduced the EWMA control chart to invent the transient patterns of motivation and potential of specifying seasonal faster motivation. Moreover, Nawaz et al. [5] presented the EWMA control chart by integrating multiscale principal component analysis for improving the monitoring efficiently and detecting an online multiscale mistake. Recently, Hu and Liu [6] detected the positive shifts of zero-inflated poisson models by using a weighted score test statistic on an upper-sided exponentially weighted moving average control chart.

Meanwhile, the EWMA control chart is improved to a performance by many researchers. One of them is the modified EWMA control chart which was originally presented by Patel and Divecha [7] and developed by Khan et al. [8]. The modified EWMA statistic is expanded by adding a multiple of a previous shift term and a constant c for an abrupt detection of autocorrelated observations. The modified EWMA control chart was continuously studied in various literature [9-11].

For the ability comparison of control charts, one of the well-known measurements is the average run length (ARL) [12] presented by using numerous calculations such as a
Markov chain [13], a Monte Carlo simulation [14], a numerical integral equation (NIE) method [15], and an explicit formula such that the later one can be found to exact solutions. For example, Petcharat [16] suggested the explicit formula of the ARL for the SAR(p)_L process in the case of an exponential error on the EWMA chart. Next, Sukparungsee and Areepong [17] presented ARL solutions by using the explicit formula on an EWMA control chart of the AR(p) model for white noise with exponential distribution. Recently, Phanthuna et al. [18] introduced the modified EWMA control chart on the AR(1) process in the case of exponential white noise by using the explicit formula for solving the ARL.

In this research, the modified EWMA chart of the SAR(p)_L model with exponential white noise is proposed. A SAR(p)_L model is an autoregressive model added to consider a seasonal component in time series analysis. A seasonal autoregressive model can be applied to many fields such as engineering [19], environment [20] and communication [21].

Nowadays, global communication has rapidly and continuously entered into the digital age. Therefore, information and communication technology plays a crucial role as a tool for accessing information in a digital platform such that the use of the internet is essential for supporting many areas such as educations, businesses and commerces, entertainment, etc. For this case study, the ARL explicit formula is applied to the percentages of internet users by news and business website categories for commerces, entertainment, etc. For this case study, the ARL explicit formula is applied to real data of time series. Discussion and Conclusion are described in sections 4 and 5, respectively.

Finally, the future research is suggested in section 6.

2. Materials and Methods

This section suggests designing a modified EWMA statistic together with observations of a SAR(p)_L model. Next, explicit formulas of the ARL are derived and compared with the NIE method.

2.1. The Modified EWMA Scheme with a SAR(p)_L Model

Patel and Divecha [7] and Khan et al. [8] expanded a modified exponentially weighted moving average (EWMA) control chart originated from the classical EWMA scheme. For a random variable sequence, \( X_t \) is an observation of a SAR(p)_L model with average \( \mu \) and variance \( \sigma^2 \), where \( t \) is a positive integer.

SAR(p)_L model:

\[
X_t = \eta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + \epsilon_t,
\]

where \( \eta \) is a reasonable constant, \( \phi_i \) is an autoregressive coefficient \( |\phi_i| < 1 \), at \( i = 1, 2, \ldots, p, L \) is a seasonal period and \( \epsilon_t \) defined white noise sequences with the exponential distribution as \( \epsilon_t \sim \text{Exp}(\beta) \).

Modified EWMA statistic:

\[
Y_t = (1-\lambda) Y_{t-1} + \lambda X_t + c (X_t - X_{t-1}),
\]

where \( \lambda \) is an exponential smoothing parameter with \( 0 < \lambda < 1 \), \( c \) is a constant, the mean of \( Y_t \) is \( \mu \) such that \( Y_0 = \mu = X_0 \) is an initial value and the asymptotic variance of \( Y_t \) is \( (\lambda + 2\lambda c + 2c^2)\sigma^2 / (2 - \lambda) \). In addition, an EWMA statistics is found in (2), when \( c = 0 \).

Control bounds of a modified EWMA chart:

\[
\mu \pm W_t \sqrt{\frac{\lambda + 2\lambda c + 2c^2}{2 - \lambda} \sigma^2},
\]

where \( W_t \) is suitable to control width limits, and \( \lambda \) is an exponential smoothing parameter. In similar ways, the control bounds of a classical EWMA chart can be looked for (3), as \( c = 0 \).

If Equation (1) of a SAR(p)_L model is substituted into (2), then the modified EWMA statistic with a SAR(p)_L model can be written as:

\[
Y_t = (1-\lambda) Y_{t-1} + \lambda X_t + c (X_t - X_{t-1})
\]

\[
Y_t = (1-\lambda) Y_{t-1} + \lambda \left( \eta + \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon_t \right)
\]

\[
+ c \left( \eta + \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon_t - X_{t-1} \right)
\]

\[
Y_t = (1-\lambda) Y_{t-1} + (c+\lambda) \sum_{i=1}^{p} \phi_i X_{t-i} - cX_{t-1}
\]

\[
+ (c+\lambda) \epsilon_t + (c+\lambda) \eta.
\]

For detection of an out-of-control process, the corresponding stopping time on SAR(p)_L model of the modified EWMA control chart, where \( Y_t = u \) is the initial value, \( a \) and \( b \) are lower and upper control limits, is defined as:

\[
r_{a,b} = \inf \{ t > 0; Y_t < a \cup Y_t > b \}.
\]

On the other hand, \( Y_t \) is in an in-control process can be reorganized in the error term \( \epsilon_t \) as:

\[
a < Y_t < b
\]
Therefore, 

$$a < \left( (1 - \lambda)u + (c + \lambda) \sum_{i=1}^{p} \phi_i X_{i,t-\lambda} \right) - cX_0 + (c + \lambda) \varepsilon_t + (c + \lambda) \eta \right) \right) < b$$ 

$$\left( \frac{a - (1 - \lambda)u + cX_0}{(c + \lambda)} \right) - \left( \frac{b - (1 - \lambda)u + cX_0}{(c + \lambda)} \right) < \varepsilon_t < \left( \frac{-\sum_{i=1}^{p} \phi_i X_{i,t-\lambda} - \eta}{(c + \lambda)} \right) - \left( \frac{-\sum_{i=1}^{p} \phi_i X_{i,t-\lambda} - \eta}{(c + \lambda)} \right).$$  \(6\) 

An initial value of the ARL for the SAR(p)\(_L\) model on the modified EWMA control chart is determined as:

$$ARL(u) = E_u(\tau_{u,\lambda}).$$  \(7\) 

### 2.2. Analytical Solutions of ARL 

For the modified EWMA control chart of the SAR(p)\(_L\) model, ARL is solved by using the method of the Fredholm integral equation of the second kind [22]. ARL\(_u\) recalled for an initial ARL can be described as:

$$ARL(u) = 1 + \int_a^b \left[ (1 - \lambda)u - cX_0 + (c + \lambda) \sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + (c + \lambda)k + (c + \lambda) \right] f(k)dk.$$ 

After that, the integral variable is changed for an easier way such that

$$z = (1 - \lambda)u - cX_0 + (c + \lambda) \sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + (c + \lambda)k + (c + \lambda),$$

then ARL\(_u\) is rearranged as:

$$ARL(u) = 1 + \frac{1}{c + \lambda} \int_a^b ARL(z)f\left( \frac{z - (1 - \lambda)u + cX_0}{(c + \lambda)} - \sum_{i=1}^{p} \phi_i X_{i,t-\lambda} - \eta \right) dz.$$  \(8\)

where \(f(z) = \exp(-z / \beta) / \beta.\) Therefore,

$$ARL(u) = 1 + \frac{(1 - \lambda)u - cX_0}{\beta(1 + \lambda)} \cdot \sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta \cdot \int_a^b ARL(z) \cdot e^{-\frac{z}{\beta(1 + \lambda)}} dz.$$  \(9\)

In the next step, the integral equation in (9) is proved by using Banach’s fixed point theorem [23] for checking the persistence and uniqueness of ARL solutions on the SAR(p)\(_L\) model of the modified EWMA scheme such that this procedure is presented in Appendix.

#### 2.2.1. Explicit Formula

After checking a unique solution of ARL, Equation (9) can be converted by setting new variables as:

$$IE = \int_a^b ARL(z) \cdot e^{-\frac{z}{\beta(1 + \lambda)}} dz$$

and

$$D(u) = e^{\frac{(1 - \lambda)u - cX_0}{\beta(1 + \lambda)}} \cdot \sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta.$$ 

then this equation can be rewritten as:

$$ARL(u) = 1 + \frac{D(u)}{\beta(1 + \lambda)} \cdot IE.$$  \(10\)

Next step, IE is discussed and ARL\(_z\) is replaced by (10) as follows:

$$IE = \int_a^b ARL(z) \cdot e^{-\frac{z}{\beta(1 + \lambda)}} dz$$

$$IE = \int_a^b \left[ 1 + \frac{D(u)}{\beta(1 + \lambda)} \cdot IE \right] \cdot e^{-\frac{z}{\beta(1 + \lambda)}} dz$$

$$IE = \int_a^b e^{\frac{(1 - \lambda)u - cX_0}{\beta(1 + \lambda)}} + \int_a^b D(z) \cdot IE \cdot e^{-\frac{z}{\beta(1 + \lambda)}} dz$$

$$IE = -\beta(1 + \lambda) \left[ e^{\frac{-b}{\beta(1 + \lambda)}} - e^{\frac{-a}{\beta(1 + \lambda)}} \right]$$

$$-IE \cdot e^{\frac{-b}{\beta(1 + \lambda)}} \cdot e^{\frac{-cX_0}{\beta(1 + \lambda)}} \cdot e^{\frac{-\sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta}{\beta}} \cdot e^{\frac{-\sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta}{\beta}} - e^{\frac{-a}{\beta(1 + \lambda)}}$$

$$IE = \frac{-\beta(1 + \lambda) \left[ e^{\frac{-b}{\beta(1 + \lambda)}} - e^{\frac{-a}{\beta(1 + \lambda)}} \right]}{1 + e^{\frac{-b}{\beta(1 + \lambda)}} \cdot e^{\frac{-cX_0}{\beta(1 + \lambda)}} \cdot e^{\frac{-\sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta}{\beta}} \cdot e^{\frac{-\sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta}{\beta}} - e^{\frac{-a}{\beta(1 + \lambda)}}} \cdot e^{\frac{-b}{\beta(1 + \lambda)}} \cdot e^{\frac{-cX_0}{\beta(1 + \lambda)}} \cdot e^{\frac{-\sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta}{\beta}} \cdot e^{\frac{-\sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta}{\beta}} \cdot e^{\frac{-a}{\beta(1 + \lambda)}}$$

Thus, the IE of (11) is substituted into (10), then ARL\(_u\) can be rearranged as:

$$ARL(u) = 1 + \frac{(1 - \lambda)u - cX_0}{\beta(1 + \lambda)} \cdot \sum_{i=1}^{p} \phi_i X_{i,t-\lambda} + \eta \cdot \int_a^b ARL(z) \cdot e^{-\frac{z}{\beta(1 + \lambda)}} dz.$$  \(11\)
Finally, Equation (12) is the explicit formula to solve the ARL on the modified EWMA control chart of the SAR(p)L model.

\[ ARL_\text{u}(u) = 1 - \frac{\sum_{j=0}^{n} d_j \cdot e^{\lambda (z_j - (1 - \lambda)u + cX_n)}}{\lambda \cdot e^{\lambda cX_n}} \]  \hspace{1cm} (12)

where \( \beta_i > \beta_0 \) and \( \delta \) is the size of mean shift for an out-of-control process such that this ARL situation is called to be \( ARL_1 \).

ARL solutions of the NIE method and the explicit formula are compared on the modified EWMA chart for observations of the SAR(p)L model with the absolute relative change (ARC) [24] computed as:

\[ \text{ARC}\% = \left| \frac{ARL(u) - ARL_{c}(u)}{ARL(u)} \right| \times 100\%. \]  \hspace{1cm} (14)

Moreover, each control chart can be compared to perform by measuring the relative mean index (RMI) [25] defined as:

\[ \text{RMI} = \frac{1}{n} \sum_{i=1}^{n} \frac{ARL_{c}(i) - ARL_{s}(i)}{ARL_{s}(i)}, \]  \hspace{1cm} (15)

where \( ARL_{c}(i) \) is the ARL of row \( i \) on the tested control chart, \( ARL_{s}(i) \) is the lowest ARL of row \( i \) from all the control charts such that a control chart is more effective if the RMI value is lower.

### 3.1. Experimental Results

For this section, a simulation of the in-control process is typically given \( ARL_0 = 370 \) such that the initial parameters are set \( u = 1, X_0, X_1, ..., X_{10}, \beta_0 = 1 \). For the out-of-control process, \( \beta_i \) is computed by determining mean shift sizes (\( \delta \)) to be 0.01, 0.02, 0.03, 0.05, 0.10, 0.15, 0.2, 0.3, 0.5, 1.0, 1.5 and 2.0. Since the lower bound \( a \) is studied on the exponential distribution of \( \epsilon_i \) which is in the interval \([0, \infty)\), the upper bound \( b \) is found by using the least \( a \) to be 0.

In Tables 1 and 2, the \( ARL_0 \) of the modified EWMA control chart at \( c = 1 \) is computed by using two techniques to be the explicit formula and the NIE method with various \( \lambda \) and \( \phi \) for the SAR(1,12) model and the SAR(2,12) model, respectively. The results of two methods are compared with the ARC for checking the precision of solutions such that all of them are of little value, less than 0.002, correspondingly.
Table 2. Comparison of $ARL_0 = 370$ from two techniques for the SAR(2)$_{12}$ model on the modified EWMA control chart at $c = 1$ and $\lambda = 0.05$

| $\lambda$ | $\phi_1$ | $\phi_2$ | $b$ | Explicit | NIE | ARC (%) |
|----------|----------|----------|-----|----------|-----|---------|
| 0.05     | 0.1      | 0.2      | 1.90196 | 370.10454 | 370.10418 | 0.0000973 |
|          | 0.2      | 0.3      | 1.54352 | 370.14372 | 370.14349 | 0.0000621 |
|          | 0.3      | 0.5      | 1.13179 | 370.39688 | 370.39677 | 0.0000297 |
| 0.10     | 0.1      | 0.2      | 1.99495 | 370.33503 | 370.33417 | 0.0001378 |
|          | 0.2      | 0.3      | 1.60479 | 370.01020 | 370.00969 | 0.0002322 |
|          | 0.3      | 0.5      | 1.16523 | 370.37571 | 370.37547 | 0.0000648 |
| 0.20     | 0.1      | 0.2      | 2.20547 | 370.21995 | 370.21686 | 0.0008346 |
|          | 0.2      | 0.3      | 1.74013 | 370.21995 | 370.21686 | 0.0008346 |
|          | 0.3      | 0.5      | 1.23788 | 370.00551 | 370.00472 | 0.0002135 |

The capability of control charts can be compared by calculating the $ARL$ in Table 3 and Fig. 1A for the SAR(1)$_{12}$ model and Table 4 and Fig. 1B for the SAR(2)$_{12}$ model. The results of the modified EWMA control charts for all $c$ can be effectively detected to be faster than the EWMA scheme for a small shift size. Correspondingly, the RMI of the modified EWMA control charts is lower than the EWMA chart. Moreover, the modified EWMA chart is more performance when $c$ is increased.

Table 3. ARL Comparison among the original and the modified EWMA control charts adjusted various $c$ for the SAR(1)$_{12}$ model at $\lambda = 0.05$, $\phi = 0.05$, $ARL_0 = 370$

| $\delta$ | EWMA | Modified EWMA | $c = 1$ | $c = 3$ | $c = 10$ | $c = 50$ |
|----------|------|----------------|--------|--------|--------|--------|
| 0.00     | 370  | 370            | 370    | 370    | 370    | 370    |
| 0.01     | 300.784 | 108.536 | 70.895 | 59.901 | 56.414 |
| 0.02     | 245.347 | 63.667 | 39.624 | 33.085 | 31.052 |
| 0.03     | 200.934 | 45.087 | 27.696 | 23.085 | 21.661 |
| 0.05     | 136.365 | 28.527 | 17.515 | 14.646 | 13.764 |
| 0.10     | 55.244 | 14.984 | 9.472 | 8.038 | 7.597 |
| 0.15     | 24.489 | 10.252 | 6.708 | 5.777 | 5.489 |
| 0.2      | 11.887 | 7.854 | 5.310 | 4.633 | 4.423 |
| 0.3      | 3.779 | 5.448 | 3.899 | 3.477 | 3.345 |
| 0.5      | 1.308 | 3.544 | 2.761 | 2.538 | 2.467 |
| 1.0      | 1.008 | 2.173 | 1.897 | 1.813 | 1.785 |
| 1.5      | 1.000 | 1.745 | 1.605 | 1.560 | 1.546 |
| 2.0      | 1.000 | 1.541 | 1.457 | 1.430 | 1.421 |
| RMI      | 3.074 | 0.886 | 0.383 | 0.245 | 0.202 |

Table 4. ARL Comparison among the original and the modified EWMA control charts adjusted various $c$ for the SAR(2)$_{12}$ model at $\lambda = 0.20$, $\phi_1 = 0.3$, $\phi_2 = 0.5$, $ARL_0 = 370$

| $\delta$ | EWMA | Modified EWMA | $c = 1$ | $c = 3$ | $c = 10$ | $c = 50$ |
|----------|------|----------------|--------|--------|--------|--------|
| 0.00     | 370  | 370            | 370    | 370    | 370    | 370    |
| 0.01     | 300.045 | 101.396 | 79.656 | 71.983 | 69.383 |
| 0.02     | 258.348 | 58.944 | 45.053 | 40.361 | 38.791 |
| 0.03     | 221.575 | 41.648 | 31.611 | 28.274 | 27.162 |
| 0.05     | 168.950 | 26.368 | 20.031 | 17.945 | 17.252 |
| 0.10     | 98.256 | 13.956 | 10.811 | 9.774 | 9.430 |
| 0.15     | 64.225 | 9.630 | 7.627 | 6.962 | 6.740 |
| 0.2      | 45.113 | 7.436 | 6.013 | 5.536 | 5.376 |
| 0.3      | 25.563 | 5.230 | 4.382 | 4.094 | 3.996 |
| 0.5      | 11.395 | 3.470 | 3.064 | 2.921 | 2.873 |
| 1.0      | 3.776 | 2.179 | 2.059 | 2.016 | 2.001 |
| 1.5      | 2.251 | 1.764 | 1.718 | 1.701 | 1.695 |
| 2.0      | 1.720 | 1.563 | 1.545 | 1.538 | 1.536 |
| RMI      | 4.620 | 0.308 | 0.096 | 0.024 | 0.002 |
For Fig. 2A and Fig. 2B, three different $\lambda$, 0.05, 0.10 and 0.20, are compared to $ARL_1$ of the modified EWMA chart with $c = 1$ such that higher $\lambda$ can be detected to shift faster than on observations of the SAR(1)_{12} model and the SAR(2)_{12} model, respectively.
3.2. Application for Real Data

Two practical datasets are used in this case study which are the percentages of internet users by news and business website categories in Thailand [26]. First, the autocorrelation of the observations is assessed by using the Box-Jenkins technique to determine the fitting of forecast time series data models. Next, applying the t-statistic is proved that the datasets are suitable for a \( \text{SAR}(p)_1 \) model. Researchers verified that the white noise is followed by an exponential distribution.

Dataset 1 is the percentage of internet users by news category collected monthly from January 2015 to December 2020. This dataset proves an autocorrelated time series suitable for the \( \text{SAR}(1)_1 \) model. This \( \text{SAR}(1)_1 \) model can be written as follows:

\[
X_t = 22.42 + 0.767 X_{t-1} + \epsilon_t, \quad \text{where} \quad \epsilon_t \sim \text{Exp}(4.27).
\]

Dataset 2 is the percentage of internet users by business category collected monthly from January 2017 to

---

**Figure 2.** ARL comparison of various \( \lambda \) for the \( \text{SAR}(1)_1 \) model in Part A and the \( \text{SAR}(2)_1 \) model in Part B
December 2020. The SAR(2) model is suitable for this dataset. This SAR(2) model can be found as:

\[ X_t = 0.619X_{t-1} + 0.358X_{t-2} + \epsilon_t, \]

where \( \epsilon_t \sim \text{Exp}(2.19) \).

For Dataset 1, the ARL of control charts is evaluated in Table 5 and Fig. 3A for the SAR(1) model. Accordingly, Dataset 2 of the SAR(2) model is presented to the ARL results on control charts in Table 6 and Fig. 3B. The results of both datasets similarly appear to simulated data such that the modified EWMA control charts are found to have higher performance than the EWMA chart when a shift detection is of small size and these charts adjusted \( c \) enlarge.

**Table 5.** ARL Comparison of control charts for observations of the percentage of internet users by news category on the SAR(1) model. Accordingly, \( \lambda = 0.05, ARL_0 = 370 \)

| \( \delta \) | EWMA | Modified EWMA |
|---|---|---|
| | | \( c = 1 \) | \( c = 3 \) | \( c = 10 \) | \( c = 50 \) |
| 0.00 | 370 | 370 | 370 | 370 | 370 |
| 0.01 | 286.443 | 36.378 | 21.709 | 17.934 | 16.780 |
| 0.02 | 222.705 | 19.063 | 11.411 | 9.485 | 8.900 |
| 0.03 | 174.030 | 12.900 | 7.846 | 6.582 | 6.197 |
| 0.05 | 107.867 | 7.835 | 4.953 | 4.231 | 4.012 |
| 0.10 | 35.537 | 4.007 | 2.777 | 2.465 | 2.370 |
| 0.15 | 13.290 | 2.765 | 2.067 | 1.887 | 1.832 |
| 0.2 | 5.758 | 2.1714 | 1.724 | 1.606 | 1.570 |
| 0.3 | 1.883 | 1.623 | 1.400 | 1.340 | 1.321 |
| 0.5 | 1.059 | 1.254 | 1.174 | 1.151 | 1.144 |
| 1.0 | 1.001 | 1.065 | 1.049 | 1.044 | 1.043 |
| 1.5 | 1.000 | 1.028 | 1.022 | 1.021 | 1.020 |
| 2.0 | 1.000 | 1.015 | 1.013 | 1.012 | 1.012 |
| RMI | 8.954 | 0.496 | 0.133 | 0.040 | 0.012 |

**Table 6.** ARL Comparison of control charts for observations of the percentage of internet users by business category on the SAR(2) model \( \lambda = 0.20, ARL_0 = 370 \)

| \( \delta \) | EWMA | Modified EWMA |
|---|---|---|
| | | \( c = 1 \) | \( c = 3 \) | \( c = 10 \) | \( c = 50 \) |
| 0.00 | 370 | 370 | 370 | 370 | 370 |
| 0.01 | 304.601 | 97.284 | 75.617 | 67.994 | 65.406 |
| 0.02 | 256.187 | 56.196 | 42.534 | 37.927 | 36.384 |
| 0.03 | 219.077 | 39.610 | 29.791 | 26.528 | 25.440 |
| 0.05 | 166.301 | 25.030 | 18.859 | 16.828 | 16.152 |
| 0.10 | 96.045 | 13.237 | 10.187 | 9.181 | 8.846 |
| 0.15 | 62.513 | 9.139 | 7.199 | 6.554 | 6.339 |
| 0.2 | 43.779 | 7.063 | 5.685 | 5.223 | 5.068 |
| 0.3 | 24.705 | 4.976 | 4.158 | 3.878 | 3.784 |
| 0.5 | 10.967 | 3.315 | 2.923 | 2.785 | 2.738 |
| 1.0 | 3.636 | 2.100 | 1.984 | 1.941 | 1.926 |
| 1.5 | 2.181 | 1.711 | 1.665 | 1.648 | 1.642 |
| 2.0 | 1.677 | 1.522 | 1.504 | 1.497 | 1.495 |
| RMI | 4.833 | 0.319 | 0.099 | 0.025 | 0 |
Figure 3. ARL comparison of control charts for real data of the SAR(1)_{12} model in Part A and the SAR(2)_{12} model in Part B.

For the modified EWMA control chart with \( c = 1 \), ARL solutions when \( \lambda \) adapts, 0.05, 0.10 and 0.20, are shown in Fig. 4A and Fig. 4B for observations of Datasets 1 and 2, respectively. They increase \( \lambda \) to make a shift detection quick.
4. Discussion

For the proposed explicit formula, their $ARL_0$ and $ARL_1$ results are closed to the NIE method such that the percentage of the absolute relative change (ARC) is obtained to lower than 0.002. Moreover, this explicit formula of ARL is used for comparing the EWMA and the modified EWMA control charts adjusted $c$ such that the modified EWMA control charts get less ARL and RMI value than the EWMA chart for a small shift and a high $c$, then they summarized that modified EWMA control charts can be detected more quickly. Afterward, the modified EWMA control chart is experimented to vary $\lambda$ such that the results are presented to better efficacy for higher $\lambda$. In addition, this explicit formula can be applied with real data such as the percentages of internet users by news and business website categories in Thailand, then these results are agreeable to simulated situations. For this research, the explicit formula can be used under the conditions of the SAR(p)L model in the case of exponential white noise.
5. Conclusions

In this research, the modified EWMA control chart is presented for observations of the SAR(p) L model with exponential residual. The explicit formula of the ARL is derived to measure the efficiency of this control chart and compared to the NIE method for checking the exactness of this explicit formula by using the ARL and ARC solutions such that the results of two methods are not much different. Furthermore, the modified EWMA control charts adjusted are compared to the EWMA by using the ARL and RMI calculations. Finally, this process can be applied for observing real-life situations.

6. Future Research

For future studies, we will develop the explicit formula of the ARL on this control chart for other models and construct explicit formulas for modern control charts. Other techniques will be suggested for calculating the ARL such as the Markov chain approach, Monte Carlo simulation, and Martingale approach.

Appendix

For proving Banach’s fixed point theorem, we suppose that \( f : [a, b] \to [a, b] \) is a contraction mapping for all \( u \in [a, b] \) on the complete metric space \( (C([a, b]), \| \cdot \|_{c}) \) such that \( C([a, b]) \) is a set of all continuous functions on the compact interval \([a, b]\). Next step, \( ARL_1, ARL_2 \) are given to be a solution to (9) for all \( ARL_1, ARL_2 \in C([a, b]) \) such that \( \| f(ARL_1) - f(ARL_2) \|_c \leq r \| ARL_1 - ARL_2 \|_c \) is proved as follows:

\[
\| f(ARL_1) - f(ARL_2) \|_c = \sup_{u \in [a, b]} \| ARL_1(u) - ARL_2(u) \|_c \leq \sup_{u \in [a, b]} \left( \frac{(1-x)u}{e^{\beta(c+\lambda)}}, \frac{-\zeta X_c}{e^{\beta(c+\lambda)}}, \frac{\sum \phi X_{c,u+\zeta}}{e^{\beta(c+\lambda)}}, \frac{1}{e^{\beta(c+\lambda)}} \right) \| ARL_1 - ARL_2 \|_c.
\]

References

[1] W. A. Shewhart. Economic Control of Quality of Manufactured Product, Martino Fine Books, 2015.
[2] W. S. Roberts. Control chart tests based on geometric moving averages. Technometrics, Vol.1, No.3, 239-250, 1959.
[3] E. S. Page. Continuous inspection schemes, Biometrika, Vol.41, 100-115, 1954.
[4] H. Li, Q. Xu, Y. He, X. Fan, S. Li. Modeling and predicting reservoir landslide displacement with deep belief network and EWMA control charts: a case study in Three Gorges Reservoir, Landslides, Vol.17, 693-707, 2020.
[5] M. Nawaz, A. S. Maulud, H. Zabiri, S. A. A. Taqvi, A. Idris. Improved process monitoring using the CUSUM and EWMA-based multiscale PCA fault detection framework, Chinese Journal of Chemical Engineering, Vol.29, 253-265, 2021.
[6] Q. Hu, L. Liu. Weighted Score test based EWMA control charts for Zero-Inflated Poisson Models, Computers & Industrial Engineering, Vol.152, 106966, 2021.
[7] A. K. Patel, J. Divecha. Modified exponentially weighted moving average (EWMA) control chart for an analytical process data, Journal of Chemical Engineering and Materials Science, Vol.2, 12-20, 2011.
[8] N. Khan, M. Aslam, C. H. Jun. Design of a control chart using a modified EWMA statistic, Quality and Reliability Engineering International, Vol.33, No.5, 1095-1104, 2017.
Analytical Solutions of ARL for SAR(p) Model on a Modified EWMA Chart

[9] A. Saghir, M. Aslam, A. Faraz, L. Ahmad, C. Heuchenne. Monitoring process variation using modified EWMA. Quality and Reliability Engineering International, Vol. 36, No.1, 328-339, 2020.

[10] A. Saghir, L. Ahmad, M. Aslam. Modified EWMA control chart for transformed gamma data, Communications in Statistics - Simulation and Computation, 1-14, 2019. DOI: 10.1080/03610918.2019.1619762.

[11] P. Phanthuna, Y. Areepong, S. Sukparungsee. Detection capability of the modified EWMA chart for the trend stationary AR(1) model, Thailand Statistician, Vol.19, No.1, 70-81, 2021.

[12] M. O. A. Abu-Shawiesh, M. Riaz, Q. Khalique. MTSD-TCC: A robust alternative to Tukey's Control Chart (TCC) based on the Modified Trimmed Standard Deviation (MTSD), Mathematics and Statistics, Vol. 8, No.3, 262-277, 2020. DOI: 10.13189/ms.2020.080304.

[13] M. B. C. Khoo, P. Castagliola, J. Y. Liew, W. L. Teoh, P. E. Maravelakis. A study on EWMA charts with runs rules - the Markov chain approach, Communications in Statistics - Theory Methods, Vol.45, No.14, 4156-4180, 2016.

[14] M. I. Flury, M. B. Quaglino. Multivariate EWMA control chart with highly asymmetric gamma distributions, Quality Technology and Quantitative Management, Vol.15, No.2, 230-252, 2018.

[15] P. Phanthuna, Y. Areepong, S. Sukparungsee. Numerical integral equation methods of average run length on modified EWMA control chart for exponential AR(1) process, Proceedings of the International MultiConference of Engineers and Computer Scientists, Vol.2, 845-847, 2018.

[16] K. Petcharat. An analytical solution of ARL of EWMA procedure for SAR(P) process with exponential white noise, Far East Journal of Mathematical Sciences, Vol.98, No.7, 831-843, 2015.

[17] S. Sukparungsee, Y. Areepong. An explicit analytical solution of the average run length of an exponentially weighted moving average control chart using an autoregressive model, Chiang Mai Journal of Science, Vol.44, No.3, 1172-1179, 2017.

[18] P. Phanthuna, Y. Areepong, S. Sukparungsee. Exact run length evaluation on a two-sided modified exponentially weighted moving average chart for monitoring process mean, Computer Modeling in Engineering and Sciences, Vol.127, No.1, 23-41, 2021.

[19] G. Lowry, F. U. Bianeyin, N. Shah. Seasonal autoregressive modelling of water and fuel consumptions in buildings, Applied Energy, Vol.84, No.5, 542-552, 2007.

[20] J. Weiss, P. Bernardara, M. Andreewsky, M. Benoit. Seasonal autoregressive modeling of a skew storm surge series, Ocean Modelling, Vol.47, 41-54, 2012.

[21] A. A. Amin, M. A. Ismail. Gibbs sampling for double seasonal autoregressive models, Communications for Statistical Applications and Methods, Vol.22, No.6, 557-573, 2015.

[22] M. Almousa. Adomian decomposition method with modified Bernstein polynomials for solving nonlinear Fredholm and Volterra integral equations, Mathematics and Statistics, Vol.8, No.3, 278-285, 2020. DOI: 10.13189/ms.2020.080305.

[23] S. Shukla, S. Balasubramanian, M. A. Pavlović. Generalized Banach fixed point theorem, Bulletin of the Malaysian Mathematical Sciences Society, Vol.39, 1529-1539, 2016.

[24] Q. T. Nguyen, K. P. Tran, P. Castagliola, G. Celano, S. Lardjane. One-sided synthetic control charts for monitoring the multivariate coefficient of variation, Journal of Statistical Computation and Simulation, Vol.89, No.10, 1841-1862, 2019.

[25] A. Tang, P. Castagliola, J. Sun, X. Hu. Optimal design of the adaptive EWMA chart for the mean based on median run length and expected median run length, Quality Technology and Quantitative Management, Vol.16, 439-458, 2018.

[26] Internet Innovation Research Center Co. Ltd., “TrueHits Statistics”, Truehits, http://truehits.net/monthly/ (accessed May. 1, 2021).