ON FIBERS OF ALGEBRAIC INVARIANT MOMENT MAPS

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Abstract. In this paper we study some properties of fibers of the invariant moment map for a Hamiltonian action of a reductive group on an affine symplectic variety. We prove that all fibers have equal dimension. Further, under some additional restrictions, we show that the quotients of fibers are irreducible normal schemes.

Contents

1. Introduction 1
2. Preliminaries 4
2.1. Hamiltonian actions 4
2.2. Conical Hamiltonian varieties 7
2.3. Local structure of Hamiltonian actions 7
2.4. Some results concerning $\psi_{G,X}, C_{G,X}$ 10
3. Dimensions of fibers 12
3.1. A Hamiltonian version of the Luna-Richardson theorem 13
3.2. A stratification of a fiber of $\psi_{G,X}//G$ 14
3.3. The proof of Theorem 3.1 15
4. Some results concerning Weyl groups 17
4.1. Some technical propositions 17
4.2. The structure of Weyl groups of affine Hamiltonian varieties 21
4.3. Examples of computation of Weyl groups 25
5. Fibers of $\hat{\psi}_{G,X}$ and untwisted varieties 28
5.1. Reducedness of fibers of $\hat{\psi}_{G,X}$ 28
5.2. Proof of Theorem 1.4 31
5.3. Some classes of untwisted varieties 33
5.4. Some counterexamples 35
6. Some open problems 36
7. Notation and conventions 36
References 38

1. Introduction

Let $K$ be a connected compact Lie group acting on a symplectic real manifold $M$ by symplectomorphisms. Suppose there exists a moment map $\mu : M \to \mathfrak{k}^*$ (see, for instance, [GS] for the definition of moment maps). It is an important problem in symplectic geometry...
to study properties of $\mu$. In fact, usually one studies not the map $\mu$ itself, but some coarser map, which we call the \textit{invariant moment map}. It is constructed as follows. One chooses a Weyl chamber $C \subset \mathfrak{t}^*$. The inclusion $C \hookrightarrow \mathfrak{t}^*$ induces a homeomorphism $C \cong \mathfrak{t}^*/K$ of topological spaces. By definition, the invariant moment map $\psi$ is the composition of $\mu : M \rightarrow \mathfrak{t}^*$ and the quotient map $\mathfrak{t}^* \rightarrow C$. It turns out that the map $\psi$ has the following amazing properties provided $M$ is compact:

(a) The image of $\psi$ is a convex polytope in $C$.
(b) All fibers of $\psi$ are connected.
(c) $\psi$ is an open map onto its image.

(a) and (b) were proved by Kirwan in [Ki], (c) is due to Knop, [Kn5]. Since $\mu$ is $K$-equivariant, one can extract some information about the image of $\mu$ from (a). From (b) one derives that all fibers of $\mu$ are connected. Hamiltonian $K$-manifolds satisfying (a)-(c) were called \textit{convex} in [Kn5]. In fact, all interesting classes of Hamiltonian manifolds (compact manifolds, Stein complex manifolds, cotangent bundles) are convex, see [Kn5] for details.

An algebraic analog of the category of smooth manifolds with an action of a compact Lie group is the category of smooth \textit{affine} varieties acted on by a reductive algebraic group. Similarly to the case of compact groups one can define the notion of a Hamiltonian action of a reductive group, see Subsection 2.1. It is an interesting problem to understand:

(1) what are algebraic analogs of properties (a)-(c)?
(2) what varieties satisfy these properties?

The study of these two questions was initiated by Knop in the early 90's (see the details below).

In the sequel all groups and varieties are defined over $\mathbb{C}$. First of all, we need to define the invariant moment map in the algebraic category. Let $X$ be a symplectic algebraic variety and $G$ a reductive algebraic group acting on $X$ in a Hamiltonian way. Fix a moment map $\mu_{G,X} : X \rightarrow \mathfrak{g}^*$ for this action. In the sequel it will be convenient to identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of a nondegenerate invariant symmetric form of $\mathfrak{g}$ and consider $\mu_{G,X}$ as a morphism $X \rightarrow \mathfrak{g}$. By the invariant moment map for $X$ we mean the morphism $\psi_{G,X} := \pi_{G,\mathfrak{g}} \circ \mu_{G,X}$, where $\pi_{G,\mathfrak{g}}$ denotes the quotient morphism $\mathfrak{g} \rightarrow \mathfrak{g}/G$ for the adjoint action $G : \mathfrak{g}$. Note that the relation between $\mu_{G,X}$ and $\psi_{G,X}$ is more loose than in the case of compact groups. For example, one cannot determine $\mathrm{im} \mu_{G,X}$ by $\mathrm{im} \psi_{G,X}$.

It turns out that the morphism $\psi_{G,X}$ does have some good properties.

**Theorem 1.1.** The morphism $\psi_{G,X}$ is equidimensional (i.e., all irreducible components of nonempty fibers have the same dimensions equal, obviously, to $\dim X - \dim \mathrm{im} \psi_{G,X}$).

In fact, a more precise result holds, see Theorem 3.1.

However, $\psi_{G,X}$ does not seem to have other good properties. For example, even its general fiber may be disconnected, see [Kn4], Introduction. Therefore one needs to modify the morphism $\psi_{G,X}$.

To this end we introduce a kind of Stein factorization of $\psi_{G,X}$. Namely, let $A$ denote the integral closure of the subalgebra $\psi_{G,X}^*(\mathbb{C}[\mathfrak{g}])$ in $\mathbb{C}[X]^G$. Set $C_{G,X} := \mathrm{Spec}(A)$. There are a natural $G$-invariant morphism $\widetilde{\psi}_{G,X} : X \rightarrow C_{G,X}$ and a finite morphism $\tau_{G,X} : C_{G,X} \rightarrow \mathfrak{g}/G$ such that $\tau_{G,X} \circ \widetilde{\psi}_{G,X} = \psi_{G,X}$. Note that at least the general fibers of $\widetilde{\psi}_{G,X} : X \rightarrow C_{G,X}$ are connected whenever $G$ is connected. The idea to replace $\psi_{G,X}$ with $\widetilde{\psi}_{G,X}$ is due to F. Knop, see [Kn1].
In [Kn4] Knop proved that any fiber of \( \tilde{\psi}_{G,X} \) is connected provided \( X \) is the cotangent bundle of some smooth irreducible (not necessarily affine) \( G \)-variety. On the other hand, he constructed an example of a four-dimensional affine Hamiltonian \( \mathbb{C}^4 \)-variety \( X \) such that \( \tilde{\psi}_{G,X} \) has a disconnected fiber.

On the other hand, Theorems 1.2.5, 1.2.7 from [Lo2] describe the image of \( \tilde{\psi}_{G,X} \). This description is particularly easy when \( X \) satisfies some additional conditions that can be described as a presence of a grading on \( \mathbb{C}[X] \) compatible with the structure of a Hamiltonian variety.

**Definition 1.2.** An affine Hamiltonian \( G \)-variety \( X \) equipped with an action \( \mathbb{C}^\times : X \) commuting with the action of \( G \) is said to be *conical* if the following two conditions are fulfilled

- (Con1) The morphism \( \mathbb{C}^\times \times X//G \to X//G, (t, \pi_{G,X}(x)) \mapsto \pi_{G,X}(tx) \), can be extended to a morphism \( \mathbb{C} \times X//G \to X//G \).
- (Con2) There exists a positive integer \( k \) (called the *degree* of \( X \)) such that \( t^*\omega = t^{-k}\omega \) and \( \mu_{G,X}(tx) = t^k\mu_{G,X}(x) \) for all \( t \in \mathbb{C}^\times, x \in X \). Here \( \omega \) denotes the symplectic form on \( X \) and \( t^*\omega \) is the push-forward of \( \omega \) under the automorphism of \( X \) induced by \( t \).

For example, a symplectic \( G \)-module and the cotangent bundle of a smooth affine \( G \)-variety are conical.

If \( X \) is conical, then \( C_{G,X} \) is a quotient of a vector space by a finite group and \( \tilde{\psi}_{G,X} \) is surjective, see [Lo2], Theorem 1.2.7. More precisely, there is a subspace \( a \subset \mathfrak{g} \) (called the Cartan space of \( X \)) and a subgroup \( W \subset N_G(a)/\mathbb{Z}_G(a) \) (the Weyl group) such that \( C_{G,X} \cong a/W \) and the finite morphism \( \tau_{G,X} : C_{G,X} \to \mathfrak{g}/G \) is induced by the embedding \( a \hookrightarrow \mathfrak{g} \). So the subspace \( a \subset \mathfrak{g} \) and the group \( W \) encode the difference between \( \tilde{\psi}_{G,X} \) and \( \psi_{G,X} \). This description partially generalizes Knop’s results for cotangent bundles and symplectic vector spaces ([Kn1],[Kn7]).

We have no examples of conical Hamiltonian \( G \)-varieties, where \( \tilde{\psi}_{G,X} \) has a disconnected fiber. We conjecture that in this case all fibers of \( \tilde{\psi}_{G,X} \) are connected and, more precisely, that \( X \) enjoys the following property:

- (Irr) Any fiber of \( \tilde{\psi}_{G,X}//G : X//G \to C_{G,X} \) is irreducible.

We are able to prove (Irr) only under another restriction on \( X \).

**Definition 1.3.** An affine Hamiltonian \( G \)-variety \( X \) is said to be *untwisted* if

- (Utw1) \( C_{G,X} \) is smooth.
- (Utw2) The morphism \( \tilde{\psi}_{G,X} \) is smooth in codimension 1 (that is, the complement to the set of smooth points of \( \tilde{\psi}_{G,X} \) in \( X \) has codimension at least 2).

**Theorem 1.4.** Let \( G \) be connected and \( X \) a conical Hamiltonian \( G \)-variety.

1. If \( X \) is untwisted, then any fiber of \( \tilde{\psi}_{G,X}//G \) is a normal Cohen-Macaulay scheme.
2. If \( X \) satisfies (Utw1) and all fibers of \( \tilde{\psi}_{G,X}//G \) are normal (as schemes), then \( X \) satisfies (Irr).
3. Suppose \( X \) is algebraically simply connected. If \( X \) satisfies (Irr), then \( X \) is untwisted.

The term ”untwisted” is partially justified by Remark 5.6.

We recall that a smooth irreducible variety \( X \) is called *algebraically simply connected* if a finite étale morphism \( \varphi : Y \to X \) is an isomorphism whenever \( Y \) is irreducible.

Note that a fiber of \( \tilde{\psi}_{G,X}//G \) can be thought as an algebraic analog of a Marsden-Weinstein reduction, [MW].
Now let us describe some classes of conical untwisted Hamiltonian $G$-varieties. Knop showed in [Kn3] that the cotangent bundle of any smooth irreducible affine variety is untwisted. In the present paper we give alternative proofs of this result and prove that a symplectic $G$-module is untwisted.

Let us briefly describe the content of the paper. In Section 2 we recall some known results concerning Hamiltonian actions in the algebraic setting. Section 3 is devoted to the proof of Theorem 1.1 (in fact, of a more precise statement). In Section 4 we prove some results concerning the Weyl groups of Hamiltonian actions (see above). These results are used in the proof of Theorem 1.4. Besides, they play a crucial role in the computation of Weyl groups and root lattices of affine $G$-varieties, the former is done in the preprint [Lo4]. Section 5 is devoted to the proof of Theorem 1.4. We also present there some classes of untwisted varieties. In Section 6 we discuss some open problems related to the subject of the paper. Finally, Section 7 contains conventions and the list of notation we use. In the beginning of Sections 2-5 their content is described in more detail.

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2. Preliminaries

In this section $G$ is a reductive algebraic group and $X$ is a smooth variety equipped with a regular symplectic form $\omega$ and an action of $G$ by symplectomorphisms.

In Subsection 2.1 we recall the definition of a Hamiltonian action and give some examples. Subsection 2.2 is devoted to conical Hamiltonian varieties introduced in [Lo2]. In Subsection 2.3 we study a local structure of Hamiltonian actions. At first, we recall the theory of cross-sections of Hamiltonian actions (Proposition 2.19) tracing back to Guillemin-Sternberg, [GS]. Next, in this subsection we recall the symplectic slice theorem from [Lo3]. These two results are key ingredients of most proofs in this paper. Finally, in Subsection 2.4 we recall some results from [Lo2], [Lo5]. The most important ones are Propositions 2.31, 2.32.

2.1. Hamiltonian actions. Let $U$ be an open subset of $X$ and $f$ a regular function on $U$. The skew-gradient $v(f)$ of $f$ is, by definition, the regular vector field on $U$ given by the equality

$$\omega_x(v(f), \eta) = \langle dx f, \eta \rangle, x \in U, \eta \in T_x X.$$  

For $f, g \in \mathbb{C}[U]$ one defines their Poisson bracket $\{f, g\} \in \mathbb{C}[U]$ by

$$\{f, g\} = \omega(v(f), v(g)).$$

Clearly, $\{f, g\} = L_{v(f)} g$, where $L$ denotes the Lie derivative.

To any element $\xi \in \mathfrak{g}$ one associates the velocity vector field $\xi_*$. Suppose there is a linear map $\mathfrak{g} \to \mathbb{C}[X], \xi \mapsto H\xi$, satisfying the following two conditions:

(H1) The map $\xi \mapsto H\xi$ is $G$-equivariant.

(H2) $v(H\xi) = \xi_*$. 

Definition 2.1. The action $G : X$ equipped with a linear map $\xi \mapsto H\xi$ satisfying (H1),(H2) is said to be Hamiltonian and $X$ is called a Hamiltonian $G$-variety.
Remark 2.2. Very often the definition of a Hamiltonian action is given in a slightly different way. Namely, for a connected group $G$ condition (H1) is replaced by the condition $\{H_\xi, H_\eta\} = H_{[\xi, \eta]}$. However, these two conditions are equivalent provided (H2) is fulfilled. Note also that one can consider Hamiltonian actions on arbitrary Poisson varieties, see, for example, [Lo2].

For a Hamiltonian action $G : X \to \mathfrak{g}^*$ we define the morphism $\mu_{G,X} : X \to \mathfrak{g}^*$ by the formula

$$\langle \mu_{G,X}(x), \xi \rangle = H_\xi(x), \xi \in \mathfrak{g}, x \in X.$$

This morphism is called the moment map of the Hamiltonian $G$-variety $X$.

Conditions (H1),(H2) are equivalent, respectively, to

1. $\mu_{G,X}$ is $G$-equivariant.
2. $\langle d_x \mu_{G,X}(v), \xi \rangle = \omega_x(\xi_v, v)$, for all $x \in X, v \in T_x X, \xi \in \mathfrak{g}$.

Here and below we write $\xi_v$ instead of $\xi_x v$.

Any two maps $\mu_{G,X} : X \to \mathfrak{g}^*$ satisfying conditions (M1),(M2) differ by an element of $\mathfrak{g}^{*G}$. Moreover, $H_{[\xi, \eta]} = \{H_\xi, H_\eta\} = \omega(\xi_\ast, \eta_\ast)$ (see, for example, [GS],[V2]). Conversely, for any $\eta \in \mathfrak{g}^{*G}$ there exists the unique Hamiltonian $G$-variety $X_\eta$ coinciding with $X$ as a symplectic $G$-variety and such that $\mu_{G,X_\eta} = \mu_{G,X} + \eta$.

Let us choose some effective $G$-module $V$ and put $(\xi, \eta) = \text{tr}_{V}(\xi \eta)$ for $\xi, \eta \in \mathfrak{g}$. The form $(\cdot, \cdot)$ is $G$-invariant, symmetric and its restriction to the Lie algebra of any reductive subgroup of $G$ is nondegenerate. Using this form, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$. In particular, we may consider $\mu_{G,X}$ as a morphism from $X$ to $\mathfrak{g}$.

Let us now give some examples of Hamiltonian $G$-varieties.

Example 2.3 (Cotangent bundles). Let $X_0$ be a smooth $G$-variety, $X := T^*X_0$ the cotangent bundle of $X_0$. $X$ is a symplectic algebraic variety (the symplectic form is presented, for example, in [GS],[V2]). The action of $G$ on $X$ is Hamiltonian. The moment map is given by $\langle \mu_{G,X}((y, \alpha)), \xi \rangle = \langle \alpha, \xi_y \rangle$. Here $y \in X_0, \alpha \in T_y^*X_0, \xi \in \mathfrak{g}$.

Example 2.4 (Symplectic vector spaces). Let $V$ be a vector space equipped with a non-degenerate skew-symmetric bilinear form $\omega$. Then $V$ is a symplectic variety. Let $G$ act on $V$ by linear symplectomorphisms. Then the action $G : V$ is Hamiltonian. The moment map $\mu_{G,V}$ is given by $\langle \mu_{G,V}(v), \xi \rangle = \frac{1}{2} \omega(\xi_v, v), \xi \in \mathfrak{g}, v \in V$.

Example 2.5 (Model varieties). This example generalizes the previous one. Let $H$ be a reductive subgroup of $G$, $\eta \in \mathfrak{g}^H$, $V$ a symplectic $H$-module. Put $U = (\mathfrak{g}(\eta)/H)^\ast$. Let us equip the homogeneous vector bundle $X = G \ast_H (U \oplus V)$ with a certain closed 2-form. Let $\eta_n, \eta_s$ denote nilpotent and semisimple parts of $\eta$, respectively. If $\eta_n \neq 0$, choose an $\mathfrak{sl}_2$-triple $(\eta_n, h, f)$ in $\mathfrak{g}(\eta_n)^H$ (where $h$ is semisimple and $f$ is nilpotent). If $\eta_n = 0$, we set $h = f = 0$. The $H$-module $U$ can be identified with $\mathfrak{g}(f) \cap \mathfrak{h}^\perp$. Fix a point $x = [1, (u, v)] \in X$. The tangent space $T_x X$ is naturally identified with $\mathfrak{h}^\perp \oplus U \oplus V$, where $U \oplus V$ is the tangent space to the fiber of the projection $G \ast_H (U \oplus V) \to G/H$ and the embedding $\mathfrak{h}^\perp \to T_x X$ is given by $\xi \mapsto \xi_z$. Put

$$\omega_x(u_1 + v_1 + \xi_1, u_2 + v_2 + \xi_2) = \omega_V(v_1, v_2) + \langle \xi_1, u_2 \rangle - (\xi_2, u_1) + \langle \eta + u + \mu_{H,V}(v), [\xi_1, \xi_2] \rangle,$$

$$u_1, u_2 \in U, v_1, v_2 \in V, \xi_1, \xi_2 \in \mathfrak{h}^\perp.$$

The corresponding map $\omega : U \oplus V \to \wedge^2 (\mathfrak{h}^\perp \oplus U \oplus V)^\ast$ is $H$-equivariant. Thus $\omega$ can be extended to the unique $G$-invariant 2-form on $X$, which is denoted also by $\omega$. It turns out that $\omega$ is closed and nondegenerate in any point of the zero section $G/H$, [Lo3], assertion 1 of Proposition 1. If $\eta$ is nilpotent, then $\omega$ is nondegenerate on the whole variety $X$. In
the general case the subset $X_r = \{ x \in G *_H (U \oplus V) | \omega_x \text{ is nondegenerate in } x \}$ is affine. The action $G : X_r$ is Hamiltonian. The moment map is given by (see [Lo3], assertion 3 of Proposition 1)

$$\mu_{G,X_r}([g,(u,v)]) = \text{Ad}(g)(\eta + u + \mu_{H,V}(v)).$$

We denote the Hamiltonian variety $X_r$ by $M_G(H, \eta, V)$ and call it a model variety.

**Remark 2.6.** The Hamiltonian structure on $M_G(H, \eta, V)$ depends on the choice of an $\mathfrak{sl}_2$-triple $(\eta_n, h, f)$ in $\mathfrak{sl}_2(\eta_s)^H$ (if $\eta_n \neq 0$). However, Hamiltonian varieties corresponding to different choices of $h, f$ are isomorphic (see Remark 1 from [Lo3]). In the sequel we say that $(\eta_n, h, f)$ is an $\mathfrak{sl}_2$-triple generating $M_G(H, \eta, V)$.

**Remark 2.7.** For $\eta_0 \in \mathfrak{g}^G$ the Hamiltonian $G$-varieties $M_G(H, \eta + \eta_0, V)$, $M_G(H, \eta, V)_{\eta_0}$ are naturally identified. They even coincide as subsets in $G *_H (U \oplus V)$.

Now we consider two constructions with Hamiltonian varieties.

**Example 2.8** (Restriction to a subgroup). Let $H$ be a reductive subgroup of $G$ and $X$ a Hamiltonian $G$-variety. Then $X$ is a Hamiltonian $H$-variety with the moment map $\mu_{H,X} = p \circ \mu_{G,X}$. Here $p$ denotes the restriction map $\mathfrak{g}^* \to \mathfrak{h}^*$.

**Example 2.9** (Products). Suppose $X_1, X_2$ are Hamiltonian $G$-varieties. Being the product of symplectic varieties, the variety $X_1 \times X_2$ has a natural symplectic structure. The action $G : X_1 \times X_2$ is Hamiltonian. The moment map is given by the formula $\mu_{G,X_1 \times X_2}(x_1, x_2) = \mu_{G,X_1}(x_1) + \mu_{G,X_2}(x_2)$ for $x_1 \in X_1, x_2 \in X_2$.

**Remark 2.10.** It follows directly from the construction of a model variety that if $(H, \eta, V)$ is the same as in Example 2.5 and $V_0$ is a trivial symplectic $H$-module, then the Hamiltonian $G$-varieties $M_G(H, \eta, V \oplus V_0) \cong M_G(H, \eta, V) \times V_0$ are isomorphic (the action $G : V_0$ is assumed to be trivial).

Now we define some important numerical invariants of an irreducible Hamiltonian $G$-variety $X$. For an action of $G$ on an algebraic variety $Y$ we denote by $m_G(Y)$ the maximal dimension of a $G$-orbit on $Y$. The number $m_G(X) - m_G(\text{im } \mu_{G,X})$ is called the defect of $X$ and is denoted by $\text{def}_G(X)$. The number $\text{dim } X - \text{def}_G(X) - m_G(X)$ is called the corank of $X$ and is denoted by $\text{cork}_G(X)$. Equivalently, $\text{cork}_G(X) = \text{tr. deg } C(X)^G - \text{def}_G(X)$. An irreducible Hamiltonian $G$-variety $X$ such that $\text{cork}_G(X) = 0$ is called coisotropic.

It follows from the standard properties of the moment map (see, for example, [GS], [V2]) that the defect and the corank of $X$ coincide, respectively, with $\dim \ker \omega|_{\mathfrak{g}(x)} \cap \text{rk } \omega|_{(\mathfrak{g}(x))}^\perp$ for a point $x \in X$ in general position. Further, the following statement holds, see [Lo2], Proposition 3.1.7.

**Lemma 2.11.** $\dim C_{G,X} = \dim \text{im } \psi_{G,X} = \text{def}_G(X)$.

**Definition 2.12.** Let $X_1, X_2$ be Hamiltonian $G$-varieties. A morphism $\varphi : X_1 \to X_2$ is called Hamiltonian if it is an étale $G$-equivariant symplectomorphism intertwining the moment maps.

Note that a Hamiltonian morphism $\varphi : X_1 \to X_2$ induces the unique morphism $\varphi_0 : C_{G,X_1} \to C_{G,X_2}$ such that $\tilde{\psi}_{G,X_2} \circ \varphi = \varphi_0 \circ \tilde{\psi}_{G,X_1}$.

**Remark 2.13.** One can similarly define Hamiltonian actions on complex analytic manifolds. The definitions of the corank and the defect can be extended to this case without any noticeable modifications.
2.2. Conical Hamiltonian varieties. The definition of a conical Hamiltonian variety was given in Introduction, Definition 1.2.

Example 2.14 (Cotangent bundles). Let $X_0, X$ be as in Example 2.3. The variety $X$ is a vector bundle over $X_0$. The action $\mathbb{C}^\times : X$ by fiberwise multiplication turns $X$ into a conical variety of degree 1.

Example 2.15 (Symplectic vector spaces). The symplectic $G$-module $V$ equipped with the action $\mathbb{C}^\times : V$ given by $(t, v) \mapsto tv$ is conical of degree 2.

Example 2.16 (Model varieties). This example generalizes the previous one. Let $H, \eta, V$ be as in Example 2.5 and $X = M_G(H, \eta, V)$. Suppose that $\eta$ is nilpotent. Here we define an action $\mathbb{C}^\times : X$ turning $X$ into a conical Hamiltonian variety of degree 2. Let $(\eta, h, f)$ be the $\mathfrak{sl}_2$-triple in $\mathfrak{g}^H$ generating $X$. As a $G$-variety, $X = G* _H(U \oplus V)$, where $U = \mathfrak{j}_0(f) \cap h^\perp$. Note that $h$ is an image of a coroot under an embedding of Lie algebras. In particular, there exists a one-parameter subgroup $\gamma : \mathbb{C}^\times \to G$ with $\frac{d}{dt}|_{t=0} \gamma = h$. Since $[h, [h, f]] = 0, [h, f] = -2f$, we see that $\gamma(t)(h^\perp) = h^\perp, \gamma(t)(U) = U$. Define a morphism $\mathbb{C}^\times \times X \to X$ by formula

$$ (t, [g, (u, v)]) \mapsto [g\gamma(t), t^2 \gamma(t)^{-1}u, tv], t \in \mathbb{C}^\times, g \in G, u \in U, v \in V. $$

One checks directly that the morphism (2.1) is well-defined and determines an action of $\mathbb{C}^\times$ on $X$ commuting with the action of $G$. Let us check that $X$ with this action is a conical Hamiltonian variety. The action of $\mathbb{C}^\times$ on $X//G$ coincides with that induced by the action $\mathbb{C}^\times : X$ given by

$$ (t, [g, (u, v)]) \mapsto [g, t^2\gamma(t)^{-1}u, tv]. $$

The eigenvalues of $\text{ad}(h)$ on $\mathfrak{j}_0(f)$ are not positive. Thus the morphism (2.2) can be extended to a morphism $\mathbb{C} \times X \to X$. This yields (Con1). (Con2) for $k = 2$ is verified directly using the construction of Example 2.5.

Remark 2.17. Let $X$ be as in the previous example. The action $\mathbb{C}^\times : X$ induces a non-negative grading on $\mathbb{C}[X]^G$. In the notation of the previous example $\mathbb{C}[X]^G \cong \mathbb{C}[U \oplus V]^H$. The grading on $\mathbb{C}[U \oplus V]^H$ is induced from the following grading on $\mathbb{C}[U \oplus V]$:

all elements of $V^* \subset \mathbb{C}[U \oplus V]$ have degree 1. The $H$-module $U^*$ is naturally identified with $\mathfrak{j}_0(\eta) \cap h^\perp$. Put $\mathfrak{g}_i = \{\xi \in \mathfrak{g} | [h, \xi] = i\xi\}$. All elements of $\mathfrak{j}_0(\eta) \cap h^\perp \cap \mathfrak{g}_i$ have degree $i + 2$.

Lemma 2.18 ([Lo2], Lemma 3.3.6). Let $X$ be a conical Hamiltonian $G$-variety of degree $k$. Then

1. $0 \in \text{im} \psi_{G,X}$.
2. Assume that $X$ is irreducible and normal. Then the subalgebra $\mathbb{C}[C_{G,X}] \subset \mathbb{C}[X]^G$ is $\mathbb{C}^\times$-stable. The morphisms $\psi_{G,X} : X \to C_{G,X}, \tau_{G,X} : C_{G,X} \to \mathfrak{g}/G$ are $\mathbb{C}^\times$-equivariant, where the action $\mathbb{C}^\times : \mathfrak{g}/G$ is induced from the action $\mathbb{C}^\times : \mathfrak{g}$ given by $(t, x) \mapsto t^k x, t \in \mathbb{C}^\times, x \in \mathfrak{g}$.

3. Under the assumptions of assertion 2, there is the unique point $\lambda_0 \in C_{G,X}$ such that $\tau_{G,X}(\lambda_0) = 0$. For any point $\lambda \in C_{G,X}$ the limit $\lim_{t \to 0} t \lambda$ exists and is equal to $\lambda_0$.

2.3. Local structure of Hamiltonian actions. Firstly, we review the algebraic variant of the Guillemin-Sternberg local cross-section theory, see [Kn3], Section 5, [Lo2], Subsection 5.1. Let $L$ be a Levi subgroup of $G$ and $\mathfrak{l}$ the corresponding Lie algebra. Put $\mathfrak{l}^{pr} = \{\xi \in \mathfrak{l} | \mathfrak{j}_0(\xi) \subset \mathfrak{l}\}$. 
Proposition 2.19 ([Kn3], Theorem 5.4 and [Lo2], Corollary 5.1.3, Propositions 5.1.2, 5.1.4, 5.1.7). Let \( x \in X, l = \im (\mu_{G,X}(x)_s), Y = \mu_{G,X}^{-1}(pr_l) \). Then

1. \( T_yX = \im y \oplus T_yY \) is a skew-orthogonal direct sum for any \( y \in Y \). In particular, \( Y \) is a smooth subvariety of \( X \) and the restriction of \( \omega \) to \( Y \) is nondegenerate. Thus \( Y \) is equipped with a symplectic structure.
2. The action \( N_G(L) : Y \) is Hamiltonian with the moment map \( \mu_{G,X} \).\( |Y \).
3. The natural morphism \( G*_{N_G(L)} Y \rightarrow X \) is étale. Its image is saturated.
4. If \( x \) is in general position, then the natural morphism \( G*_{N_G(L)} Y \rightarrow X \) is an open embedding and \( N_G(L) \) permutes the connected components of \( Y \) transitively.

A subset \( Z^0 \) of a \( G \)-variety \( Z \) is said to be saturated if there exist a \( G \)-invariant morphism \( \varphi : Z \rightarrow Z_0 \) and a subset \( Z^0_0 \subset Z_0 \) such that \( Z^0 = \varphi^{-1}(Z^0_0) \).

Definition 2.20. An irreducible (=connected) component of \( \mu_{G,X}^{-1}(pr_l) \) equipped with the structure of a Hamiltonian \( L \)-variety obtained by restriction of the Hamiltonian structure from \( \mu_{G,X}^{-1}(pr_l) \) is called an \( L \)-cross-section of \( X \).

Definition 2.21. The Levi subgroup \( L = Z_G(\mu_{G,X}(x)_s), \) where \( x \in X \) is in general position, is said to be the principal centralizer of \( X \).

Note that the principal centralizer is determined uniquely up to \( G \)-conjugacy.

Lemma 2.22. Let \( L \) be the principal centralizer and \( X_L \) an \( L \)-cross-section of \( X \). Then the following conditions are equivalent:

1. \( m_G(X) = \dim G \).
2. \( \def_G(X) = \rk G \).
3. \( \im \mu_{G,X} = \mathfrak{g} \).
4. \( L \) is a maximal torus in \( G \) and \( \def_L(X_L) = \def_L(X_L) = \rk G \).
5. The stabilizer in general position for the action \( G : X \) is finite.

Under these conditions, \( \cork G(X) = \dim X - \dim G - \rk G \).

Proof. The equivalence of conditions (1)-(4) was proved in [Lo5], Lemma 4.5. The equality for \( \cork G(X) \) follows from (1) and (2). It is well-known that (5) is equivalent to (1).

Lemma 2.23. Let \( L \) be the principal centralizer and \( X_L \) an \( L \)-cross-section of \( X \). Suppose that the stabilizer in general position \( L_0 \) for the action \( L : X_L \) is reductive and that \( 0 \in \im \psi_{G,X}. \) Then \( \im \mu_{G,X} = \im (I \cap I_L) \).

Proof. From Proposition 2.19 it follows that \( \im \mu_{G,X} = G\im \mu_{L,X_L} \). Since \( 0 \in \im \psi_{G,X}, \) we see that \( 0 \in \im \mu_{L,X_L} \). By Theorem 4.1.1 from [Lo2], \( (L, L) \subset L_0 \). Therefore \( \im \mu_{L,X_L} = I \cap I_L \).

Now we turn to the problem of describing the structure of an affine Hamiltonian \( G \)-variety in some neighborhood of a point with closed \( G \)-orbit. A neighborhood is taken with respect to the complex topology (in the sequel we call such neighborhoods analytical).

At first, we define some invariants of the triple \( (G, X, x) \). Put \( H = G_x, \eta = \mu_{G,X}(x) \). The subgroup \( H \subset G \) is reductive and \( \eta \in \mathfrak{g}^H \). Put \( V = (\mathfrak{g}_x)^{\perp}/(\mathfrak{g}_x \cap \mathfrak{g}_x^{\perp}) \). This is a symplectic \( H \)-module. We say that \( (H, \eta, V) \) is the determining triple of \( X \) at \( x \). For example, the determining triple of \( X = M_G(H, \eta, V) \) at \( x = [1, (0, 0)] \) is \( (H, \eta, V) \), see [Lo3], assertion 4 of Proposition 1.

As the name suggests, a determining triple should determine the structure of the Hamiltonian \( G \)-variety \( X \) near \( x \). In fact, a slightly stronger claim holds.
Definition 2.24. Let \( X_1, X_2 \) be affine Hamiltonian \( G \)-varieties, \( x_1 \in X_1, x_2 \in X_2 \) be points with closed \( G \)-orbits. The pairs \((X_1, x_1), (X_2, x_2)\) are called \textit{analytically equivalent}, if there are saturated open analytical neighborhoods \( O_1, O_2 \) of \( x_1 \in X_1, x_2 \in X_2 \), respectively, that are isomorphic as complex-analytical Hamiltonian \( G \)-manifolds.

Remark 2.25. An open saturated analytical neighborhood in \( X \) is the inverse image of an open analytical neighborhood in \( X//G \) under \( \pi_{G,X} \). See, for example, [Lo3], Lemma 5.

Proposition 2.26 (Symplectic slice theorem, [Lo3]). Let \( X \) be an affine Hamiltonian \( G \)-variety, \( x \in X \) a point with closed \( G \)-orbit, \((H, \eta, V)\) the determining triple of \( X \) at \( x \). Then the pair \((X, x)\) is analytically equivalent to the pair \((M_G(H, \eta, V), [1, (0, 0)])\).

Now we prove two lemmas, which will be used in Subsection 4.1.

We have two approaches to the local study of affine Hamiltonian varieties: the cross-sections theory and the symplectic slice theorem. Let us establish a connection between them.

Lemma 2.27. Let \( x \in X \) be a point with closed \( G \)-orbit and \((H, \eta, V)\) the determining triple of \( X \) at \( x \). Put \( M = Z_G(\eta_x) \). Denote by \( X_M \) the unique \( M \)-cross-section of \( X \) containing \( x \). Then the following assertions hold

1. \( Mx \) is closed in \( X_M \) and \((H, \eta, V)\) is the determining triple of \( X_M \) at \( x \).
2. There exists an affine saturated open (with respect to Zariski topology) neighborhood \( X_M^0 \subset X_M \) of \( x \) such that the following conditions are satisfied:
   (a) the natural morphism \( X_M^0//M \to X//G, \pi_{M,X_M}(z) \mapsto \pi_{G,X}(z) \) is étale;
   (b) for any \( z \in X_M^0 \) the orbit \( Mz \) is closed in \( X_M^0 \) (equivalently, in \( X_M \)) iff \( Gz \) is closed in \( X \).

Proof. The morphism \( \varphi : G *_M X_M \to X, [g, x] \mapsto gx \), is étale (assertion 3 of Proposition 2.19). Since \( Gx \) is closed in \( X \), we see that \( G[1, x] \) is closed in \( G *_M X_M \), equivalently, \( Mx \) is closed in \( X_M \). Since \( G_z \subset Z_G(\mu_{G,X}(z)) \subset Z_G(\mu_{G,X}(z)_s) = M \), we have \( G_z = M_z \) for \( z \in X_M \). By construction of \( \mu_{M,X_M} : \mu_{M,X_M}(z) = \mu_{G,X}(z) \). Assertion 1 will follow if we check that the \( H \)-modules \( g_* x^\perp/(g_* x^\perp \cap g_* x) \) and \( m_* x^\perp/(m_* x^\perp \cap m_* x) \) are isomorphic. Here the skew-orthogonal complement to \( g_* x \) (resp., to \( m_* x \)) is taken in \( T_x X \) (resp., in \( T_x X_M \)). The existence of an isomorphism stems from \( g_* x = m_* x \perp + m_* x \) and assertion 1 of Proposition 2.19.

By the above, the orbits \( G[1, x], Gx \) are closed and isomorphic via \( \varphi \). It follows from Luna’s fundamental lemma, [Lu], that for some open affine neighborhood \( U \) of the point \( \pi_{M,X_M}(x) \) in \( X_M//M \cong (G *_M X_M)//G \) the morphism \( \varphi//G : U \to X//G \) is étale and

\[
\pi_{G,G*_M X_M}^{-1}(U) \cong U \times_X G X \tag{2.3}
\]

Clearly, \( \pi_{G,G*_M X_M}^{-1}(U) \cong G *_M \pi_{G,X_M}^{-1}(U) \). Thanks to (2.3), we see that for all \( z \in X_M^0 := \pi_{M,X_M}^{-1}(U) \) the orbit \( G[1, z] \) is closed in \( G *_M \pi_{M,X_M}^{-1}(U) \) iff \( Gz \) is closed in \( X \).

The next lemma studies the behavior of determining triples under replacing \( G \) with some connected subgroup \( G^1 \subset G \) containing \((G, G)\).

Lemma 2.28. Let \( x \in X \) be a point with closed \( G \)-orbit and \((H, \eta, V)\) the determining triple of \( X \) at \( x \). Then \( G^1 x \) is closed in \( X \) and the determining triple of the Hamiltonian \( G^1 \)-variety \( X \) at \( x \) has the form \((H \cap G^1, \eta_0, V \oplus V_0)\), where \( V_0 \) is a trivial \( H \cap G^1 \)-module and \( \eta_0 \) is the projection of \( \eta \) to \( g^1 \).
Proof. Since $G^1$ is a normal subgroup of $G$, we see that all $G^1$-orbits in $Gx$ have the same dimension whence closed. Obviously, $G^1_x = G^1 \cap H, \mu_{G^1, X}(x) = \eta_0$. Clearly, $g^1_x \subset g_x$ and $g_* x \subset g_1^1 x \subset g_1 x$. Therefore we have a natural embedding $g_* x \subset g_1^1 x \subset g_1 x$ and a natural projection $g_* x / (g_* x \cap g_1^1 x) \rightarrow g_1^1 x / (g_1^1 x \cap g_1 x)$ and a natural projection $g_1^1 x / (g_1^1 x \cap g_1 x) \rightarrow g_1 x / (g_1 x \cap g_1 x)$. The cokernel of the former is a quotient of the $H \cap G^1$-module $g_1^1 x / g_1 x \subsetneq (g_1 x / g_1 x)^*$, while the kernel of the latter is a submodule in $g_1 x / g_1 x$. Since $g_1 x / g_1 x$ is a trivial $H \cap G^1$-module, we are done. □

2.4. Some results concerning $\tilde{\psi}_{G,X}, C_{G,X}$. Let us, at first, define two important invariants of a Hamiltonian variety: its Cartan space and Weyl group. The proofs of the facts below concerning these invariants can be found in [Lo2], Subsection 5.2.

Let $L$ be the principal centralizer and $X_L$ an $L$-cross-section of a Hamiltonian $G$-variety $X$. It turns out that $\mu_{Z(L), X_L}$ is an affine subspace in $\mathfrak{z}(l)$. We denote this affine subspace by $a^{(X_L)}_{G,X}$ and call it the Cartan space of $X$. It intersects the Lie algebra of the inefficiency kernel for the action $Z(L) : X_L$ in the unique point (by the inefficiency kernel of a group action $\Gamma : Y$ we mean the kernel of the corresponding homomorphism $\Gamma \rightarrow \text{Aut}(Y)$). Taking this point as the origin in $a^{(X_L)}_{G,X}$ we may (and will) consider $a^{(X_L)}_{G,X}$ as a vector space.

The group $N_G(L, X_L)$ acts linearly on $a^{(X_L)}_{G,X}$. We denote the image of $N_G(L, X_L)$ in $\text{GL}(a^{(X_L)}_{G,X})$ by $W^{(X_L)}_{G,X}$ and call it the Weyl group of $X$. If $G$ is connected, then $W^{(X_L)}_{G,X}$ is naturally identified with $N_G(L, X_L)/L$.

Note that, in a suitable sense, the pair $(a^{(X_L)}_{G,X}, W^{(X_L)}_{G,X})$ does not depend up to $G$-conjugacy from the choice of $L, X_L$. When a particular choice of $L, X_L$ does not matter, we write $a^{(G,X)}_{G,X}$ for $a^{(X_L)}_{G,X}$ and $W^{(G,X)}_{G,X}$ for $W^{(X_L)}_{G,X}$.

Note that $\mu_{L, X_L} \subset a^{(X_L)}_{G,X} \hookrightarrow l//L$. There is the unique $G$-invariant morphism $\tilde{\psi}_{G,X} : X \rightarrow a^{(G,X)}_{G,X}/W^{(G,X)}_{G,X}$ coinciding with $\psi_{N_G(L, X_L), X_L}$ on $X_L$. The morphism $\psi_{G,X} : X \rightarrow g//G$ is the composition of $\tilde{\psi}_{G,X}$ and the finite morphism $\tau^1_{G,X} : a^{(G,X)}_{G,X}/W^{(G,X)}_{G,X} \rightarrow g//G$ induced by the embedding $a^{(G,X)}_{G,X} \hookrightarrow g$. So $\tilde{\psi}_{G,X}$ factors through $\psi_{G,X}$ and the respective morphism $\tau^2_{G,X} : C_{G,X} \rightarrow a^{(G,X)}_{G,X}/W^{(G,X)}_{G,X}$ is finite and dominant.

**Lemma 2.29.** Assume, in addition, that $X$ is conical of degree $k$. Then $a^{(X_L)}_{G,X}$ is a vector subspace of $g$ so one can equip $a^{(X_L)}_{G,X}$ with the action of $\mathbb{C}^\times$ given by $(t, \xi) \mapsto t^k \xi$. Let us equip $a^{(G,X)}_{G,X}/W^{(G,X)}_{G,X}$ with the induced action. Then the morphisms $\tau^1_{G,X}, \tau^2_{G,X}$ are $\mathbb{C}^\times$-equivariant.

**Proof.** Note that $X_L$ is $\mathbb{C}^\times$-stable and the morphism $\mu_{L, X_L} : X_L \rightarrow l$ is $\mathbb{C}^\times$-equivariant (here $\mathbb{C}^\times$ acts on $l$ by $(t, \xi) \mapsto t^k \xi$). Now everything follows directly from the definitions of $a^{(X_L)}_{G,X}$ and the morphisms $\tau^1_{G,X}, \tau^2_{G,X}$. □

Now we want to describe the behavior of $\tilde{\psi}_{G,X}$ under some simple modifications of the pair $(G, X)$. To do this we need to recall some results obtained in [Lo5]. The proofs of these results are mostly straightforward.

Let $X, L, X_L$ be such as above. Let $M$ be a Levi subgroup of $G$ containing $L, G^1$ a connected subgroup of $G$ containing $(G, G), L^1 := G^1 \cap L, G_1, \ldots, G_k$ be all simple normal subgroups of $G$, so that $G = Z(G)^0 G_1 \ldots G_k$ is the decomposition into the locally direct product. Finally, let $X'$ be another affine irreducible Hamiltonian $G$-variety and $\varphi : X \rightarrow X'$ a generically finite dominant $G$-equivariant morphism such that $\mu_{G, X'} \circ \varphi = \mu_{G, X}$. 

By Lemma 6.9 from [Lo5], \( a^{(X_L)}_{G,X} = a^{(X_L)}_{G^1,X} \) \( W^{(X_L)}_{G,X} \) is a normal subgroup of \( W^{(X_L)}_{G,X} \).

Suppose \( G \) is connected. Recall, [Lo5], Lemmas 4.6.6.10, that there exists the unique \( M \)-cross-section \( X_M \) of \( X \) containing \( X_L \) and \( a^{(X_L)}_{M,X_M} = a^{(X_L)}_{G,X} L_{W,X_M} = W^{(X_L)}_{G,X} \cap M/L \).

By Lemma 4.6 from [Lo5], \( L \) is the principal centralizer of \( X' \) and there exists the unique \( L \)-cross-section \( X'_L \) of \( X' \) such that \( \varphi(X_L) \subset X'_L \). Further, by Lemma 6.11 from [Lo5], \( a^{(X_L)}_{G,X} a^{(X_L)}_{G^1,X} \) \( W^{(X_L)}_{G,X} \subset W^{(X_L)}_{G,X'} \).

Suppose, in addition, that \( 0 \in \im \psi_{G,X} \). Recall, [Lo5], Lemma 4.6, that \( L' \) is the principal centralizer and \( X_L \) is an \( L' \)-cross-section of the Hamiltonian \( G' \)-variety \( X \). Further, by [Lo5], Lemma 6.13, \( a^{(X_L)}_{G,X} \cap g^1 \subset a^{(X_L)}_{G^1,X} \) the groups \( W^{(X_L)}_{G,X} \) \( W^{(X_L)}_{G,X} \) are naturally identified, and the orthogonal projection \( g \to g' \) induces the \( W^{(X_L)}_{G,X} \)-equivariant epimorphism \( a^{(X_L)}_{G,X} \to a^{(X_L)}_{G^1,X} \).

Finally, suppose \( X \) satisfies the equivalent conditions of Lemma 2.22. Put \( T = L, T_i = L \cap G_i \). Recall, [Lo5], Lemma 4.6, that \( T_i \) is the principal centralizer of the Hamiltonian \( G_i \)-variety \( X \) and there is the unique \( T_i \)-cross-section \( X_{T_i} \) of \( X \) containing \( (\prod_{j \neq i} G_j)_{X_T} \). Further, Lemma 6.14 from [Lo5] implies that \( a^{(X_{T_i})}_{G_i,X} = t_i, W^{(X_{T_i})}_{G_i,X} \subset \prod_{i=1}^k W^{(X_{T_i})}_{G_i,X} \) and the projection of \( W^{(X_{T_i})}_{G_i,X} \) to \( GL(t_i) \) coincides with \( W^{(X_{T_i})}_{G_i,X} \).

**Lemma 2.30.** Let \( G, X, X_L, M, G^1, L_1, G_1, \ldots, G_k, X', \varphi, X_M, X'_L, T, T_i, X_{T_i} \) be as above.

1. \( \hat{\psi}_{G,X} \) is the composition of \( \hat{\psi}_{G^1,X} \) and the natural morphism of quotients \( a^{(X_L)}_{G^1,X}/W^{(X_L)}_{G^1,X} \to a^{(X_L)}_{G,X}/W^{(X_L)}_{G,X} \) induced by the inclusion \( W^{(X_L)}_{G^1,X} \subset W^{(X_L)}_{G,X} \).

2. Suppose \( G \) is connected. Then the following diagram is commutative.

\[
\begin{array}{ccc}
X_M & \xrightarrow{\hat{\psi}_{M,X_M}} & a^{(X)}_{M,X_M}/W^{(X)}_{M,X_M} \\
\downarrow & & \downarrow \tau_{M,X_M}^i \\
X & \xrightarrow{\hat{\psi}_{G,X}} & a^{(X)}_{G,X}/W^{(X)}_{G,X} \\
& & \downarrow \tau_{G,X}^i \\
& & g\!/G
\end{array}
\]

Here the morphism \( X_M \to X \) is the inclusion, the morphism \( a^{(X)}_{M,X_M}/W^{(X)}_{M,X_M} \to a^{(X)}_{G,X}/W^{(X)}_{G,X} \) is given by \( W^{(X)}_{M,X_M} \xi \to W^{(X)}_{G,X} \xi \), and the morphism \( m\!/M \to g\!/G \) is induced by the restriction of functions from \( g \) to \( m \).

3. The following diagram is commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{\hat{\psi}_{G,X}} & X' \\
\downarrow & & \downarrow \\
a^{(X)}_{G,X}/W^{(X)}_{G,X} & \xrightarrow{\hat{\psi}_{G,X}} & a^{(X)}_{G,X}/W^{(X)}_{G,X'}
\end{array}
\]

4. Suppose \( G \) is connected and \( 0 \in \im \hat{\psi}_{G,X} \). Then the following diagram is commutative.

\[
\begin{array}{ccc}
X & \xrightarrow{\hat{\psi}_{G,X}} & X' \\
\downarrow & & \downarrow \\
a^{(X)}_{G,X}/W^{(X)}_{G,X} & \xrightarrow{\hat{\psi}_{G,X}} & a^{(X)}_{G^1,X}/W^{(X)}_{G^1,X}
\end{array}
\]
(5) Suppose $G$ is connected and $X$ satisfies the equivalent conditions of Lemma 2.22. Then the following diagram, where the map $a_{G,X}^{(i)}/W_{G,X}^{(i)} \rightarrow a_{G_i,X}^{(i)}/W_{G_i,X}^{(i)}$ is induced by the natural epimorphism $g_i \rightarrow g_i$, is commutative.

\[
\begin{array}{ccc}
X & \rightarrow & a_{G_i,X}^{(i)}/W_{G_i,X}^{(i)} \\
\downarrow & & \downarrow \\
a_{G,X}^{(i)}/W_{G,X}^{(i)} & \rightarrow & a_{G,X}^{(i)}/W_{G,X}^{(i)}
\end{array}
\]

Proof. The proofs of assertions 1,3,4 follow directly from the definition of $\hat{\psi}_{G,X}$. Let us prove assertion 2. The commutativity of the right square of the diagram follows directly from the definition of $\tau_{G,X}^{(i)}$. To prove the commutativity of the left square we note that both morphisms $X_M \rightarrow a_{G,X}^{(i)}/W_{G,X}^{(i)}$ from the diagram are $M$-invariant and their restrictions to $X_L$ coincide with $\hat{\psi}_{N_G(L,X_L),X_L}$. To complete the proof it remains to recall that $MX_L$ is dense in $X_M$.

We proceed to assertion 5. The morphism $\hat{\psi}_{G_i,X}|_{X_i}$ is $Z(G)\prod_{j\neq i} G_j$-invariant. It follows that $\hat{\psi}_{G_i,X}$ is $G$-invariant. It remains to note that the restrictions of both morphisms $X \rightarrow a_{G_i,X}^{(i)}/W_{G_i,X}^{(i)}$ coincide on $X_T$. □

Proposition 2.31. The morphism $\hat{\psi}_{G,X}/G : X//G \rightarrow a_{G,X}^{(i)}/W_{G,X}^{(i)}$ is equidimensional and open. Further, for any closed subvariety $Y \subset \im \hat{\psi}_{G,X}$ and any irreducible component $Z$ of $(\hat{\psi}_{G,X}/G)^{-1}(Y)$ the subset $(\hat{\psi}_{G,X}/G)(Z)$ is dense in $Y$.

Proof. Note that $a_{G,X}^{(i)}/W_{G,X}^{(i)}$ is a normal variety of dimension $\text{def}_G(X)$. Thanks to Theorem 1.2.3 from [Lo2], $\hat{\psi}_{G,X}/G$ is equidimensional. The openness stems from [Ch], Proposition 3 in Section 5.5. The last assertion of the proposition is an easy corollary of the fact that $\hat{\psi}_{G,X}/G$ is equidimensional. □

Proposition 2.32 ([Lo2], Theorem 1.2.7). Suppose $X$ is conical. Then $C_{G,X} \cong a_{G,X}^{(i)}/W_{G,X}^{(i)}$ and $\tilde{\psi}_{G,X} = \hat{\psi}_{G,X}$. Further, the algebra $\mathbb{C}[C_{G,X}]$ coincides with the intersection of $\mathbb{C}[X]$ and the Poisson center of $\mathbb{C}(X)^G$.

3. Dimensions of fibers

Throughout the section $G$ is a connected reductive group and $X$ is a Hamiltonian $G$-variety with symplectic form $\omega$.

In Subsection 3.1 we prove a variant of the Luna-Richardson restriction theorem ([LR]) for Hamiltonian varieties. This allows us to reduce a general affine Hamiltonian $G$-variety to one satisfying the equivalent conditions of Lemma 2.22.

Subsection 3.2 deals with a stratification of fibers of the morphism $\psi_{G,X}/G : X//G \rightarrow g//G$. A stratum consists of the images of all points with closed $G$-orbit and the same determining triple. The main results of the subsection are the proof that any stratum is smooth and the formula for the dimensions of the strata (Proposition 3.5).

The main part of this section is Subsection 3.3. There we prove the following result that strengthens Theorem 1.1.
Theorem 3.1. The morphisms $\psi_{G,X}, \hat{\psi}_{G,X}, \check{\psi}_{G,X}$ are equidimensional. The morphisms $\hat{\psi}_{G,X}, \check{\psi}_{G,X}$ are open. For any closed irreducible subvariety $Y \subset \text{im} \hat{\psi}_{G,X}$ and any irreducible component $\hat{Y} \subset \hat{\psi}_{G,X}^{-1}(Y)$ the subvariety $\pi_{G,X}(\hat{Y}) \subset X//G$ is an irreducible component of $(\hat{\psi}_{G,X}//G)^{-1}(Y)$.

The proof uses the stratification introduced in Subsection 3.2 and the estimate on dimensions of fibers of $\pi_{G,X}$ obtained in Proposition 3.7.

3.1. A Hamiltonian version of the Luna-Richardson theorem. Let $H$ be a reductive subgroup of $G$. The subvariety $X^H \subset X$ is smooth (see [PV], Subsection 6.5) and $N_G(H)$-stable. Let us equip $X^H$ with a structure of a Hamiltonian $N_G(H)$-variety.

Proposition 3.2. (1) $\omega|_{X^H}$ is nondegenerate, thus $X^H$ is equipped with the symplectic structure.

(2) The action $N_G(H) : X^H$ is Hamiltonian with the moment map $\mu_{N_G(H),X^H} = \mu_{G,X}|_{X^H}$.

Proof. For a symplectic vector space $V$ and a reductive subgroup $H \subset \text{Sp}(V)$ the $H$-modules $V^H$ and $V/(V^H)^\perp$ are isomorphic. Thus $\omega|_{V^H}$ is nondegenerate. Since $T_x(X^H) = (T_xX)^H$, see [PV], Subsection 6.5, we see that $\omega|_{X^H}$ is nondegenerate.

Note that the Lie algebra of $N_G(H)$ coincides with $\mathfrak{g}^H + \mathfrak{h}$. Since $\mu_{G,X}$ is $G$-equivariant, we have $\mu_{G,X}(X^H) \subset \mathfrak{g}^H$. Clearly, $\mu_{G,X}|_{X^H}$ is $N_G(H)$-equivariant. It remains to check that

$$v(H_\xi|_{X^H})_x = \xi_x$$

for all $\xi \in \mathfrak{g}^H + \mathfrak{h}, x \in X^H$. Obviously, $v(H_\xi)_x = \xi_x = 0$ for all $\xi \in \mathfrak{h}, x \in X^H$. Thus (3.1) holds for $\xi \in \mathfrak{h}$. Now let $\xi \in \mathfrak{g}^H$. Then $H_\xi \in \mathbb{C}[X]^H$, and $v(H_\xi)_x$ is an $H$-invariant vector for $x \in X^H$. It follows from the construction of the symplectic form on $X^H$ that $v(H_\xi)_x = v(H_\xi|_{X^H})_x$.

Now we will apply the previous construction to a special choice of $H$.

Let $L$ be the principal centralizer of $X$ and $X_L$ an $L$-cross-section. By Corollary 4.2.3 from [Lo2], the restriction of $\pi_{(L,L),X_L} : X_L \to X_L//(L,L)$ to $X^{(L,L)}_L \subset X_L$ is an isomorphism. Denote by $L_0$ the unit component of the inefficiency kernel of the action $L : X_L//(L,L) \cong X^{(L,L)}_L$. It follows from Theorem 4.2.1, [Lo2], that $L_0 = (L,L)/T_0$, where $T_0$ is the unit component of the inefficiency kernel for the action $Z(L) : X_L$. Let $X_0$ be the unique connected component of $X^{L_0}$ containing $X^{(L,L)}_L$. Put $\tilde{G}_0 = N_G(L_0, X_0)$ (the stabilizer of $X_0$ under the action of $N_G(L_0)$), $G_0 = \tilde{G}_0/L_0$. We identify $\mathfrak{g}_0$ with $\mathfrak{g}^{L_0} \cap \mathfrak{l}_L^0$. It follows from Proposition 3.2 that the action $\tilde{G}_0 : X_0$ is Hamiltonian with moment map $\mu_{G,X}|_{X_0}$. By Remark 3.1.2 from [Lo2], the action $G_0 : X_0$ is Hamiltonian with the moment map $\mu_{G_0,X_0} := p \circ \mu_{\tilde{G}_0,X_0}$, where $p$ denotes the natural projection $\tilde{G}_0 \to G_0$.

The following proposition is what we mean by a "Hamiltonian version of the Luna-Richardson theorem".

Proposition 3.3. In the notation introduced above the following statements hold.

(1) The morphism $X_0//G_0 \to X//G$ induced by the restriction of functions is an isomorphism.

(2) $\text{md}_{G_0}(X_0) = \dim G_0$, $\text{def}_{G_0}(X_0) = \text{def}_G(X), \text{cork}_G(X) = \text{cork}_{G_0}(X_0)$.

(3) $L/L_0$ is the principal centralizer of $X_0$. The subvariety $X^{L_0}_L$ is dense in the unique $L/L_0$-cross-section $X_{0L}$. The subvariety $X^{L_0}_L$ of $X_0$, $a_{G_0,X_0}^{(X_L)} = a_{G,X}^{(X_{0L})} - \xi_0$, where $\xi_0 \in L_0 \cap a_{G,X}^{(X_L)}$, is open.

$W_{G_0,X_0}^{(X_{0L})} = W_{G,X}^{(X_L)}$. 

ON FIBERS OF ALGEBRAIC INVARIANT MOMENT MAPS 13
(4) \( \hat{\psi}_{G,X}|_{X_0} = \hat{\psi}_{G_0,X_0} \).

In the proof we will use some notions of the theory of algebraic transformation groups. Let \( Y \) be an irreducible affine variety acted on by a reductive group \( H \). It is known, see [PV], Theorem 7.12, that there exists an open subset \( Y_0 \subset Y/\!/H \) such that for any \( y \in Y_0 \) the closed orbit in \( \pi_{H,Y}^{-1}(y) \) is isomorphic to \( H/C \), where \( C \) is a reductive subgroup of \( H \).

**Definition 3.4.** Such a subgroup \( C \) (determined uniquely up to \( H \)-conjugacy) is called the principal isotropy subgroup for the action \( H : Y \).

The action \( H : Y \) is called stable if its general orbit is closed and locally free if \( m_H(Y) = \dim H \).

**Proof of Proposition 3.3.** The action \( Z(L)^0 : X_I^{L,L} \cong X_L//((L,L) \cong \), is stable ([Lo2], Proposition 4.5.1). Thus \( L_0 \) is the unit component of the principal isotropy subgroup for the action \( L : X_L \). Since the natural morphism \( G}*L \to X \) is étale and its image is saturated, we see that the group \( L_0 \) is the unit component of the principal isotropy subgroup for the action \( G : X \) and that the morphism \( X_0//G_0 \to X//G \) is dominant. By the Luna-Richardson theorem ([LR]), the morphism \( X_0//G_0 \to X//G \) is an isomorphism and the action of \( G_0 \) on \( X_0 \) is locally free. The latter yields \( \text{def} G_0(X_0) = \text{rk} G_0 = \text{rk} G - \text{rk} L_0 = \text{deg} G(X) \). By Theorem 1.2.9 from [Lo2], \( C(X)^G = \text{Quot}(C[X]^G) \), \( C(X_0)^{G_0} = \text{Quot}(C[X_0]^{G_0}) \). So

\[
\text{cork}_G(X) = \text{tr. deg } C(X)^G - \text{def} G(X) = \text{tr. deg } C(X_0)^{G_0} - \text{def} G_0(X_0) = \text{cork}_{G_0}(X_0).
\]

We proceed to assertion 3. Since \( m_{G_0}(X_0) = \dim G_0 \), the maximal torus \( L/L_0 \subset G_0 \) is the principal centralizer of \( X_0 \) (see Lemma 2.22) and \( a_{G_0,X_0} = 1 \cap l_0 = a_{G,X} - \xi_0 \) for any \( L/L_0 \)-cross-section \( X_{0L} \) of \( X_0 \). The natural morphism \( X_{L}///L \to X//G \) is dominant and quasifinite, therefore so is the natural morphism \( (X_L^{L_0})//((L/L_0) \to X_0//G_0 \). It follows from [Lo2], Theorem 1.2.9, that the actions \( L/L_0 : X_{L_0}/G_0 : X_0 \) are stable. It follows that \( \dim X_{L_0} = \dim (X_{L_0})//((L/L_0) + \dim L/L_0 = \dim X_0//G_0 + \dim L/L_0 = \dim X_0 - \dim G_0 + \dim L/L_0 \). Since \( \mu_{G,X}(X_L^{L_0}) \subset \mu_{G,X}(X_L) \cap g_0 \subset \text{pr} \cap g_0 \subset (1/l_0)_{\text{pr}} \) (the last subset is taken w.r.t. the Lie algebra \( g_0 \)), we see that \( X_{L_0} \) lies in the unique \( L/L_0 \)-cross-section \( X_{0L} \) of \( X_0 \). Comparing the dimensions, we see that \( X_{L_0} \) is dense in \( X_{0L} \). The equality for the Weyl groups stems from \( N_G(L,X_L)/L_0 \subset G_0 \), \( N_G(L,X_{L_0}) = N_G(L,X_L) \).

Finally, both morphisms in assertion 4 are \( G_0 \)-invariant and their restrictions to \( X_{L_0} \) are equal to the restriction of \( \hat{\psi}_{N_G(L,X_L),X_L} \).

3.2. **A stratification of a fiber of** \( \psi_{G,X}//G \). In this subsection we introduce a stratification of fibers of the morphism \( \psi_{G,X}//G : X//G \to \mathfrak{g}//G \). We consider fibers of \( \psi_{G,X}//G \) as algebraic varieties. Namely, let \( \eta \in \mathfrak{g} \), \( H \) be a reductive subgroup of \( G_\eta \) and \( V \) a symplectic \( H \)-module. We put

\[
S_{G,X}(H,\eta,V) = \{ \pi_{G,X}(x)|Gx \text{ is closed}, (H,\eta,V) \text{ is the determining triple of } X \text{ at } x \}.
\]

Clearly, \( S_{G,X}(H_1,\eta_1,V_1) = S_{G,X}(H_2,\eta_2,V_2) \) iff there is \( g \in G \) and a linear isomorphism \( \iota : V_1 \to V_2 \) such that \( \text{Ad}(g)\eta_1 = \eta_2, gH_1g^{-1} = H_2 \) and \( (ghg^{-1})\iota(v) = \iota(hv) \) for all \( h \in H_1 \).

The main result of this subsection is the following

**Proposition 3.5.** Let \( X,G,H,\eta,V \) be as above, \( \lambda = \pi_{G,\mathfrak{g}}(\eta) \). Then \( S_{G,X}(H,\eta,V) \) is a locally-closed smooth subvariety of pure codimension \( \text{cork}_G(X) - \dim V^H \) in \( (\psi_{G,X}//G)^{-1}(\lambda) \).
Proof. First, we show that $S_{G,X}(H, \eta, V)$ is a locally-closed subvariety of $X//G$. Denote by $Y$ the set of all points $x \in X$ such that $Gx$ is closed, $G_x = H$, and $T_x X/\mathfrak{g}_x x \cong V \oplus (\mathfrak{g}_\eta/\mathfrak{h})^*$. It follows from the Luna slice theorem applied to any point of $Y$ that $Y$ is a locally-closed subvariety in $X$. Therefore $Y_\eta = Y \cap \mu^{-1}_{G,X}(\text{Ad}(G)\eta)$ is a locally closed subvariety of $X$. Since all orbits in $Y_\eta$ are closed in $X$, we see that $Y_\eta$ is an open saturated subvariety of $Y$. Thus $S_{G,X}(H, \eta, V) = \pi_{G,X}(Y_\eta)$ is open in $Y//G$.

Applying Proposition 2.26, we reduce the codimension and smoothness claims to the case $X = M_G(H, \eta, V)$. Put $\mathfrak{s} = \mathfrak{z}(\eta)$. Choose an $\mathfrak{sl}_2$-triple $(\eta_1, h, f)$ in $\mathfrak{h}^*$ generating $M_G(H, \eta, V)$. Denote by $U$ the $H$-module $\mathfrak{z}(\eta) \cap \mathfrak{h}^*$.

Lemma 3.6. In the above notation $\eta$ is an isolated point of $(\eta + \mathfrak{z}(f)) \cap \text{Ad}(G)\eta$.

Proof of Lemma 3.6. Note that $T_\eta(\eta + \mathfrak{z}(f)) = \mathfrak{z}(f), T_\eta\text{Ad}(G)\eta = [\mathfrak{g}, \eta]$. It is enough to show $\mathfrak{z}(f) \cap [\mathfrak{g}, \eta] = \{0\}$. The equality $\mathfrak{s} = \mathfrak{z}(\eta_1)$ yields $[\mathfrak{g}, \eta] = [\mathfrak{s}^+, \eta] + [\mathfrak{s}, \eta] = \mathfrak{s}^+ \oplus [\mathfrak{s}, \eta_1]$. Thanks to the representation theory of $\mathfrak{sl}_2$, $[\mathfrak{s}, \eta_1] \cap \mathfrak{z}(f) = 0$ whence the required equality. □

In virtue of Remark 2.10, it is enough to assume that $V^H = \{0\}$. Put $x := [1, (0, 0)]$. Everything will follow if we check that $\pi_{G,X}(x)$ is an isolated point in $S_{G,X}(H, \eta, V)$. Indeed, by Proposition 2.31, $	ext{cork}_{G}(X) = \dim X//G - \text{def}_{G}(X) = \dim_{\pi_{G,X}(x)}(\psi_{G,X//G})^{-1}(\lambda)$.

There exists a neighborhood $O'$ of $\eta$ in $\eta + \mathfrak{z}(f)$ such that $O' \cap \text{Ad}(G)\eta = \eta$. Replacing $O'$ with $HO'$, if necessary, we may assume that $O'$ is $H$-stable. Set $O := \{[g, (u, v)] \in M_G(H, \eta, V)|\eta + u + \mu_{H,V}(v) \in O'\}$. By definition, $O$ is an open $G$-subvariety of $X$ containing $x$. It is enough to show that any point $x_1 \in O$ with closed $G$-orbit and the determining triple $(H, \eta, V)$ is $G$-conjugate to $x$. Assume the converse. Put $x_1 = [g, (u, v)], u \in U, v \in V, (u, v) \neq 0$. Recall that $\mu_{G,X}(x_1) = \text{Ad}(g)(\eta + u + \mu_{H,V}(v))$. Since $\mu_{G,X}(x_1) = \eta$, Lemma 3.6 implies that $u + \mu_{H,V}(v) = 0$. Since $U \cap \mathfrak{h} = \{0\}$, we have $u = 0$. The subgroup $H_v \subset H$ is conjugate to $H$ in $G$. Thus $v \in V^H = \{0\}$. Contradiction. □

3.3. The proof of Theorem 3.1. At first, we obtain an estimate for the dimension of a fiber of $\pi_{G,X}$.

Proposition 3.7. The dimension of any fiber of $\pi_{G,X} : X \rightarrow X//G$ does not exceed $\dim X - \text{def}_{G}(X) - \frac{\text{cork}_{G}(X)}{2}$.

Proof. The proof is carried out in two steps. Firstly, we consider the case when $X$ satisfies the equivalent conditions of Lemma 2.22 and then deduce the general case from this one.

Step 1. Suppose $X$ satisfies the equivalent conditions of Lemma 2.22. Then

$$\text{def}_{G}(X) + \frac{\text{cork}_{G}(X)}{2} = \frac{\dim X - \dim G + \text{rk } G}{2}.$$

Let $y \in X//G, x$ be a point from the unique closed $G$-orbit in $\pi_{G,X}^{-1}(y), H = G_x, \eta = \mu_{G,X}(x), U = (\mathfrak{z}(\eta)/\mathfrak{h})^*, V = (\mathfrak{g}_x x)/(\mathfrak{g}_x x \cap (\mathfrak{g}_x)^\perp)$. The $H$-modules $U \oplus V$ and $T_x X/\mathfrak{g}_x x$ are isomorphic.

Using the Luna slice theorem, we see that it is enough to check

$$\dim \pi_{H,U\oplus V}^{-1}(0) \leq \dim U + \dim V - \frac{\dim X - \dim G + \text{rk } G}{2}$$

Lemma 3.8 ([Sch2], Proposition 2.10). Let $H$ be a reductive group, $T_H$ a maximal torus of $H$, and $V$ a self-dual $H$-module. Then

$$\dim \pi_{H,V}^{-1}(0) \leq \frac{1}{2}(\dim V - \dim V^{T_H} + \dim H - \dim T_H).$$
Lemma 3.9. $U \oplus V$ is a self-dual $H$-module.

Proof of Lemma 3.9. Note that the $H$-modules $U \oplus V$ and $T_x X/\mathfrak{g}_x$ are isomorphic. The module $T_x X$ is symplectic, while the module $\mathfrak{g}_x \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{h}^+$ is orthogonal. Hence both these modules are self-dual. Therefore the quotient module $U \oplus V$ is self-dual too. \hfill \square

We see that the $H$-module $U \oplus V$ satisfies the assumptions of Lemma 3.8. Let $T_H$ be a maximal torus of $H$. Let us show that $\dim U_T H \geq \mathrm{rk} \mathfrak{g} - \mathrm{rk} \mathfrak{h}$. Since $\dim \mathfrak{h}^+ = \mathrm{rk} \mathfrak{h}$, it is enough to show that $\dim \mathfrak{u}(\xi) T_H \geq \mathrm{rk} \mathfrak{g}$ for any $\xi \in \mathfrak{g}^+$. It is enough to check the last inequality for $\xi \in \mathfrak{g}^+$ in general position. But in this case $\xi$ is semisimple. Thence $\mathfrak{u}(\xi)$ is a Levi subalgebra of $\mathfrak{g}$ and everything is clear.

By Lemma 3.8, we have the following inequalities

$$
\dim \pi_{H, U \oplus V}^{-1}(0) \leq \frac{1}{2} (\dim U + \dim V - \dim U_T H - \dim V_T H + \dim \mathfrak{h} - \mathrm{rk} \mathfrak{h})
$$

(3.3)

$$
\leq \frac{1}{2} (\dim U + \dim V - (\mathrm{rk} \mathfrak{g} - \mathrm{rk} \mathfrak{h}) + \dim \mathfrak{h} - \mathrm{rk} \mathfrak{h}).
$$

One may check directly that the last expression in (3.3) coincides with the r.h.s of (3.2).

Step 2. Now we consider the general case. Let $X_0, G_0$ be as in Subsection 3.1.

By Proposition 3.3, $\cork_G(X) = \cork_{G_0}(X_0), \mathrm{def}_G(X) = \mathrm{def}_{G_0}(X_0)$. The proposition will follow if we show that

$$
\mathrm{codim}_X \pi_{G, X}^{-1}(y) \geq \mathrm{codim}_{X_0} \pi_{G_0, X_0}^{-1}(y),
$$

for any $y \in X//G$. It follows from Proposition 3.3 that $X_0//G_0 \cong X//G, \pi_{G_0, X_0}^{-1}(y) = \pi_{G, X}^{-1}(y) \cap X_0$. Now (3.4) stems from the following general fact of Algebraic geometry:

$$
\dim \pi_{G, X}^{-1}(y) \cap Z \geq \dim \pi_{G, X}^{-1}(y) \cap X - \dim X
$$

for any subvarieties $Y, Z$ of an irreducible variety $X$ and $x \in Y \cap Z$ provided $X$ is smooth. \hfill \square

Proof of Theorem 3.1. Clearly, $\tilde{\psi}_{G, X}, \psi_{G, X}$ are equidimensional provided $\tilde{\psi}_{G, X}$ is. As we mentioned above, any equidimensional morphism to a normal variety is open.

To prove the theorem it remains to check that for all $\lambda \in \mathfrak{g}//G$ and any irreducible component $Z$ of $\psi_{G, X}^{-1}(\lambda)$ the equality $\dim \pi_{G, X}(Z) = \dim X//G - \mathrm{def}_G(X)$ and the inequality $\dim Z \leq \dim X - \mathrm{def}_G(X)$ take place (the opposite inequality holds automatically, since $\mathrm{def}_G(X) = \dim \mathrm{im} \psi_{G, X}^{-1}$). The former equality will imply

$$
\dim \pi_{G, X}(Z) = \dim X//G - \mathrm{def}_G(X) + \dim Y
$$

(3.5)

for an irreducible component $Z$ of $\tilde{\psi}_{G, X}^{-1}(Y)$, where $Y \subset \mathrm{im} \tilde{\psi}_{G, X}$ is an arbitrary closed irreducible subvariety (recall that, by Proposition 2.31, $\mathrm{im} \tilde{\psi}_{G, X} = \mathrm{im}(\tilde{\psi}_{G, X}//G)$ is an open subvariety in $a^{(c)}(\psi_{G, X}^{-1}(Y))$. Thanks to Proposition 2.31, (3.5) holds if $\pi_{G, X}(Z)$ is an irreducible component in $\tilde{\psi}_{G, X}^{-1}(Y)$.

Choose a subvariety $S_{G, X}(H, \eta, V) \subset (\psi_{G, X}//G)^{-1}(\lambda)$ (see Subsection 3.2) such that $\pi_{G, X}(Z) \cap S_{G, X}(H, \eta, V)$ is dense (and so, in virtue of Proposition 3.5, open) in $\pi_{G, X}(Z)$. Further, choose a point $x \in Z \cap \pi_{G, X}^{-1}(S_{G, X}(H, \eta, V))$ with closed $G$-orbit. Applying Proposition 2.26 to $x$, we may replace $X$ with $M_G(H, \eta, V)$. Thanks to Remark 2.10, we may assume that $V^H = 0$. From Proposition 3.5 it follows that $\pi_{G, X}(Z)$ is a point. By Proposition 3.7, $\dim Z \leq \dim X - \mathrm{def}_G(X) - \frac{1}{2} \mathrm{cork}_G(X)$. It follows that $\cork_G(X) = 0, \dim(\psi_{G, X}//G)^{-1}(\lambda) = 0, \dim Z = \dim X - \mathrm{def}_G(X)$. This verifies the claim in the beginning of the previous paragraph and completes the proof. \hfill \square
Corollary 3.10. For any $\lambda \in \text{im} \psi_{G,X}$ and any irreducible component $Z$ of $\psi_{G,X}^{-1}(\lambda)$ there exists an open subset $Z_0 \subset Z//G$ such that $Z_0$ is smooth (as a variety), $\text{codim}_{Z//G}(Z//G) \setminus Z_0 \geq 2$, and for any $z \in Z_0$ and any point $x \in \pi_{G,X}^{-1}(z)$ with closed $G$-orbit the following condition holds:

(*) $M_G(H, \eta, V/V^H)$ is coisotropic, where $(H, \eta, V)$ is the determining triple of $X$ at $x$.

Moreover, $M_G(H, \eta, V/V^H)$ does not depend (up to an isomorphism) on the choice of $z$.

Proof. (*) is equivalent to $\text{cork}_G(X) = \text{cork}_G(M_G(H, \eta, V)) = \dim V^H$. It follows from Theorem 3.1 that $Z$ maps dominantly whence, by the standard properties of quotient morphisms, surjectively onto some irreducible component of $(\psi_{G,X}/G)^{-1}(\lambda)$. The required claims follow now from Proposition 3.5.

Corollary 3.11. Let $Y$ be a closed irreducible subvariety in $\text{im} \hat{\psi}_{G,X}$. Then $\hat{\psi}_{G,X}^{-1}(\tilde{Y}) = Y$ for any irreducible component $\tilde{Y}$ of $\hat{\psi}_{G,X}^{-1}(Y)$.

Proof. According to Theorem 3.1, $\pi_{G,X}(\tilde{Y})$ is an irreducible component of $(\hat{\psi}_{G,X}/G)^{-1}(Y) \subset X//G$. It remains to apply Proposition 2.31.

Corollary 3.12. A simply connected affine conical Hamiltonian $G$-variety satisfies (Utw1).

Proof. Thanks to Proposition 2.32, $\tau_{G,X} : C_{G,X} \to a_{G,X}^{(\cdot)}/W_{G,X}^{(\cdot)}$ is an isomorphism. By Theorem 3.1, the morphism $\tilde{\psi}_{G,X} : X \to C_{G,X}$ is equidimensional. Since $G$ is connected, the subalgebra $\mathbb{C}[X]^G$ is integrally closed in $\mathbb{C}[X]$. Thus $\mathbb{C}[C_{G,X}]$ is integrally closed in $\mathbb{C}[X]$. In other words, a general fiber of $\tilde{\psi}_{G,X}$ is connected. Summarizing, we see that $\tilde{\psi}_{G,X}$ is an equidimensional morphism with a connected general fiber from a simply connected variety $X$ to $C_{G,X} \cong a_{G,X}^{(\cdot)}/W_{G,X}^{(\cdot)}$. The proof of the proposition is based on an idea of Panyushev [Pa] and is completely analogous to that given in [Kn7], Theorem 7.2.

4. Some results concerning Weyl groups

Throughout the section $G, X, \omega$ have the same meaning as in the previous section.

In this section we study the structure of the Weyl group $W_{G,X}^{(\cdot)}$. Subsection 4.1 contains three technical propositions, which play a crucial role in the subsequent exposition. Propositions 4.1, 4.3 allow one to reduce the study of an arbitrary affine Hamiltonian $G$-variety to the study of a coisotropic conical model variety. Proposition 4.6 describes the behavior of Weyl groups under this reduction.

Using results of Subsection 4.1, in Subsection 4.2 we establish some properties of Weyl groups of varieties satisfying the equivalent conditions of Lemma 2.22. In particular, we get some restrictions on varieties with a ”small” Weyl group (Proposition 4.9, Corollary 4.14) and show that a Weyl group cannot be ”too small” (Corollary 4.16). As a consequence of Corollary 4.16 we get some explicit restrictions on Weyl groups for simple $G$ of types $A - E$ in Proposition 4.17, Corollary 4.19.

Finally, in Subsection 4.3 we compute the Weyl groups of linear actions of simple groups satisfying some additional restrictions. This computation will be used in Subsection 5.3 to check that any symplectic $G$-module is an untwisted Hamiltonian variety.

4.1. Some technical propositions.
Proposition 4.1. Let $L$ be the principal centralizer and $X_L$ an $L$-cross-section of $X$, $\xi \in a_{G,X}^{(XL)}$, $\alpha = \pi_{W_{G,X}^{(XL)}}^{a_{G,X}^{(XL)}}(\xi)$, $M = Z_G(\xi)$. Suppose $\alpha \in \text{im} \psi_{G,X}$. Choose an irreducible component $Z$ of $\psi_{G,X}^{-1}(\alpha)$. Then there exists $x \in X$ possessing the following properties:

(a) $x \in Z$.
(b) $\mu_{G,X}(x) \in \mathfrak{z}(\mathfrak{m}) \cap \mathfrak{m}^{pr}$.
(c) A unique $M$-cross-section $X_M$ of $X$ containing $x$ contains $X_L$ and $\hat{\psi}_{M,X_M}(x) = \pi_{W_{M,X_M}^{(XL)}}^{a_{M,X_M}^{(XL)}}(\xi)$.
(d) $Gx$ is closed in $X$.
(e) Let $(H, \eta, V)$ be the determining triple of $X$ (or, equivalently, of $X_M$) at $x$ and $\hat{G}$ be a connected subgroup of $M$ containing $(M, M)H^{\circ}$. The orbit $\hat{G}x$ is closed in $X_M$ and the Hamiltonian $\hat{G}$-variety $\hat{X} := M_G(H \cap \hat{G}, \eta, V/V^{H})$ is coisotropic.

Remark 4.2. If $X$ satisfies the equivalent conditions of Lemma 2.22, then so does the Hamiltonian $\hat{G}$-variety $\hat{X}$. This stems easily from Proposition 2.26.

Proof of Proposition 4.1. Choose a point $z \in Z$ with closed $G$-orbit. Let us show that $gz$ satisfies (b),(c) for some $g \in G$. Put $M_1 = Z_G(\mu_{G,X}(z))$. Since $\pi_{G,g}(\mu_{G,X}(z)_s) = \pi_{G,g}(\xi)$, we have $M_1 \sim_G M$. Let $X_{M_1}$ be an $M_1$-cross-section of $X$ containing $z$, $L_1$ be the principal centralizer and $X_{L_1}$ an $L_1$-cross-section of $X_{M_1}$. Replacing $z$ with $gz$ for an appropriate element $g \in G$, we may assume that $L_1 = L, X_{L_1} = X_L$. Next, replacing $z$ with $mz$ for some $m \in M_1$, one obtains $\mu_{G,X}(z)_s \in a_{M_1,M_1}^{(XL)} = a_{G,X}^{(XL)}$. By the commutative diagram of assertion 2 of Lemma 2.30, for some $n \in N_G(L, X_L)$ the following equality holds

\begin{equation}
\hat{\psi}_{M_1,X_{M_1}}(z) = \pi_{W_{M_1,X_{M_1}}^{(XL)}}^{a_{M_1,X_{M_1}}^{(XL)}}(n\xi).
\end{equation}

Note that $\psi_{M_1,X_{M_1}}(z) \in \mathfrak{z}(\mathfrak{m}_1) \hookrightarrow \mathfrak{m}_1/M_1$. From (4.1) it follows that $\pi_{M_1,M_1}(n\xi) \in \mathfrak{z}(\mathfrak{m}_1) \hookrightarrow \mathfrak{m}_1/M_1$ whence $n\xi \in \mathfrak{z}(\mathfrak{m}_1)$. On the other hand, $n\xi \in \mathfrak{z}(\text{Ad}(n)\mathfrak{m}) \cap (\text{Ad}(n)\mathfrak{m})^{pr}$ and so $\mathfrak{m}_1 \subset \text{Ad}(n)\mathfrak{m}$. We have seen above that $M_1 \sim_G M$ whence $M_1 = nMn^{-1}$. Replacing $z$ with $n^{-1}z$, we get the point $z$ satisfying (a)-(c). Put $\alpha' = \pi_{W_{M,M}^{(XL)}}^{a_{M}^{(XL)}}(\xi)$.

According to Lemma 2.27, there exists an open affine $M$-saturated subvariety $X^0_M \subset X_M$ containing $z$ such that for any $x \in X^0_M$ the orbit $Gx$ is closed in $X$ iff $Mx$ is closed in $X_M$. Further, by Lemma 2.28, $\hat{G}x \subset X_M$ is closed whenever $Mx$ is closed.

From assertion 2 of Lemma 2.30, Theorem 3.1 and the fact that the natural morphism $G * X_M \to X$ is étale we get $\dim Z \cap X^0_M = \dim \hat{\psi}_{M,M}^{-1}(\alpha')$. Hence there is an irreducible component $Z'$ of $\hat{\psi}_{M,M}^{-1}(\alpha')$ containing $z$ and contained in $Z \cap X^0_M$. By Corollary 3.10, there is an open subset $Y^0 \subset \pi_{M,X^0}(Z')$ such that any point $x \in \pi_{M,X^0}^{-1}(Y^0)$ with closed $M$-orbit satisfies (a)-(d) and (e) for $\hat{G} = M$. When $\hat{G} \neq M$, there is a covering $T_0 \times \hat{G} \to M$ and a finite Hamiltonian morphism $T^*(T_0) \times \hat{G} \to M_H(H \cap \hat{G}, \eta, V/V^H)$, where $T_0$ is a torus. Since $H^\circ \subset \hat{G}$, we are done.

Proposition 4.3. Let $X, L, X_L$ be as in Proposition 4.1, $T_0$ denote the unit component of the inefficiency kernel of the action $Z(L)^\circ : X_L, \xi_0 \in a_{G,X}^{(XL)}, M = Z_G(\xi_0)$. Suppose $0 \in \text{im} \psi_{G,X}$. Put $\mathfrak{z} := \mathfrak{z}(\mathfrak{m}) \cap a_{G,X}^{(XL)}, \mathfrak{Z} := \pi_{W_{G,X}^{(XL)}}^{a_{G,X}^{(XL)}}(\mathfrak{z})$. Choose an irreducible component $\hat{Z}$ of $\psi_{G,X}^{-1}(\mathfrak{Z})$. 

\[\]
Let $\xi \in \mathfrak{z}$ be a point in general position. Then there is a component $Z$ of $\psi_{G,X}^{-1}(\pi_{W_{G,X}^{(c)}(x)}(\xi))$ lying in $\tilde{Z}$ and a point $x \in Z$ satisfying the conditions (b)-(e) of Proposition 4.1 and

\begin{enumerate}[(f)]
\item $G_x^0 \subset (M, M) T_0$.
\end{enumerate}

**Remark 4.4.** Under the assumptions of Proposition 4.3 one may assume that $\tilde{G}$ defined in (d) coincides with $(M, M) T_0$. If $X$ satisfies the equivalent conditions of Lemma 2.22, then one can take $(M, M)$ for $\tilde{G}$.

**Proof of Proposition 4.3.** The morphism $\psi_{G,X}$ is open, Theorem 3.1. So $Z, \tilde{Z}$ do exist. Choose a point $z \in Z$ satisfying conditions (a)-(e) and such that $\pi_{G,X}(\tilde{Z})$ is the only component of $(\psi_{G,X}/G)^{-1}(Z)$ (see Theorem 3.1) containing $\pi_{G,X}(z)$.

Let $X_M$ be as in (c). By the choice of $z$, any irreducible component $\tilde{Z}'$ of $\psi_{M,X_M}^{-1}(\mathfrak{z})$ containing $z$ is contained in $\tilde{Z} \cap X_M$, compare with the proof of Proposition 4.1. As in that proof, there is an open subset $Y^0 \subset \pi_{M,X_M}(\tilde{Z}')$ such that any $x \in \pi_{M,X_M}^{-1}(Y^0)$ with closed $M$-orbit satisfies conditions (a)-(e) (for appropriate $\xi$).

It remains to prove that $M_x^0 \subset (M, M) T_0$ for a general point $x \in \tilde{Z}'$ with closed $M$-orbit. Recall (see the discussion preceding Proposition 3.3) that $L_0 := (L, L) T_0$ is the unit component of the principal isotropy group for the action $M : X_M$. Let $C$ denote the principal isotropy subgroup for the action $M : \tilde{Z}'$, so $L_0 \subset C$. By the definition of $C$, there exists an irreducible component $X_1$ of $X_M^C$ such that $\pi_{M,X_M}(X_1 \cap \tilde{Z}')$ is dense in $\pi_{M,X_M}(\tilde{Z}')$.

By Lemma 3.2, the action $N_0(C, X_1) : X_1$ is Hamiltonian with moment map $\mu_{N_0(C,X_1), X_1} = \mu_{M,X_M}|_{X_1}$. Since $0 \in \text{im} \psi_{G,X}$, we get $0 \in \psi_{M,X_M}(\tilde{Z}')$, equivalently, $\mu_{M,X_M}(X_1)$ contains a nilpotent element. Since $C$ acts trivially on $X_1$, we get

\[ \mu_{M,X_M}(X_1) \subset \mathfrak{m}^C \cap (\xi + c^1) \]

for any $\xi \in \text{im} \mu_{M,X_M}(X_1)$. Since there is a nilpotent element in $\mu_{M,X_M}(X_1)$, we see that the r.h.s. of (4.2) coincides with $\mathfrak{m}^C \cap c^1$. For brevity, put $\mathfrak{s} = \mathfrak{m}^C \cap c^1$. This is an ideal in $\mathfrak{m}^C$.

Choose $x \in \tilde{Z}' \cap X_1$ and put $\eta = \mu_{M,X_M}(x)$. Then $\eta_\mathfrak{s} \in \mathfrak{z}$ and $(\eta_\mathfrak{s} - \xi + \eta_\mathfrak{n}) \in \mathfrak{s}$. Clearly, $c^C \subset \mathfrak{z}(\mathfrak{m}^C)$. Thus $[\eta_\mathfrak{s} - \xi, \eta_\mathfrak{n}] = 0$ whence $\eta_\mathfrak{n} = \xi = (\eta - \xi)_s \in \mathfrak{s}$ and

\[ \mu_{M,X_M}(x)_s \in \mathfrak{z} \cap c^1, \forall x \in \tilde{Z}' \cap X_1. \]

**Lemma 4.5.** $\mathfrak{m} = \mathfrak{z} + \mathfrak{t}_0 + [\mathfrak{m}, \mathfrak{m}]$.

**Proof.** It is enough to check that

\[ t = \mathfrak{z} + \mathfrak{t}_0 + t_1 := t \cap [\mathfrak{m}, \mathfrak{m}], \]

where $t$ denotes a Cartan subalgebra of $t$. Recall that

\[ t = \mathfrak{z}(\mathfrak{m}) \oplus t_1, \]

\[ \mathfrak{z} = \mathfrak{z}(\mathfrak{m}) \cap a_{G,X}^{(c)} = \mathfrak{z}(\mathfrak{m}) \cap (\mathfrak{l}(t) \cap t_0^1) = \mathfrak{z}(\mathfrak{m}) \cap t_0^1. \]

Since $\mathfrak{z}(\mathfrak{m}), t_1, t_0$ are the Lie algebras of algebraic groups, we see that $(\cdot, \cdot)$ is nondegenerate on $\mathfrak{z}(\mathfrak{m}), t_1, t_0, \mathfrak{z}$. To prove (4.4) it is enough to note that $t_0 + t_1 = \mathfrak{z}$.

If $\mathfrak{c} \not\subset [\mathfrak{m}, \mathfrak{m}] + t_0$, then, thanks to Lemma 4.5, the r.h.s. of (4.3) is a proper subspace in $\mathfrak{z}$. Hence $\psi_{M,X_M}(\tilde{Z}') = \psi_{M,X_M}(\tilde{Z}' \cap X_1)$ is not dense in $\mathfrak{z}$. Since $\mathfrak{z} \cap \text{im} \psi_{M,X_M}$ is an open subset in $\mathfrak{z}$, we get a contradiction with Corollary 3.11.
Proposition 4.6. Let \( X, L, X_L, M, X_M, \hat{G} \) be as in Proposition 4.1, \( \hat{L} = L \cap \hat{G} \). Let \( x \in X \) satisfy conditions (a)-(d) of Proposition 4.1 (for some \( Z \)) and \( \hat{X} \) be the model variety constructed by \( x \) as in (e). Then \( \hat{L} \) is the principal centralizer of \( \hat{X} \) and there is an \( \hat{L} \)-cross-section \( \hat{X}_L \) of \( \hat{X} \) such that \( a^{(\hat{X}_L)}_{\hat{G}, \hat{X}} \) is a \( W^{(X_L)}_{G,X} \cap M/L \)-stable subspace of \( a^{(X_L)}_{G,X} \) and \( W^{(\hat{X}_L)}_{G,\hat{X}} \) lies in the image of \( W^{(X_L)}_{G,X} \cap M/L \) in \( GL(a^{(\hat{X}_L)}_{\hat{G}, \hat{X}}) \).

Proof. Recall, see Lemma 2.30, that \( \hat{L} \) is the principal centralizer and \( X_L \) is an \( \hat{L} \)-cross-section of the Hamiltonian \( \hat{G} \)-variety \( X_M \). Let \( (H, \eta, V) \) denote the determining triple of \( X \) at \( x \). Thanks to Lemmas 2.27,2.28, \( (H \cap \hat{G}, \eta, V/V^H \oplus V_0) \) is the determining triple of the Hamiltonian \( \hat{G} \)-variety \( X_M \) at \( x \), where \( V_0 \) is a trivial \( H \cap \hat{G} \)-module. Put \( \hat{X}' := M_G(H \cap \hat{G}, \eta, V/V^H \oplus V_0) \cong \hat{X} \times V_0. \) It is enough to prove the analogue of the assertion of the proposition for \( \hat{X}' \).

By Proposition 2.26, there is a \( \hat{G} \)-saturated analytical open neighborhood \( O \) of \([1, (0, 0)]\) in \( \hat{X}' \), that is isomorphic (as a Hamiltonian \( \hat{G} \)-manifold) to a saturated analytical neighborhood of \( x \) in \( (X_M)^{\eta} \). One may assume additionally that \( O \) is connected. By [Lo3], Lemma 5, \( O_1 := \pi_{\hat{G}, \hat{X'}}(O) \) is an open neighborhood of \( \pi_{\hat{G}, \hat{X'}}([1, (0, 0)]) \) in \( \hat{X}'//\hat{G} \). Further, according to Example 2.16, \( \hat{X}' \) is a conical Hamiltonian variety. Replacing \( O \) with a smaller neighborhood, we may assume that \( t.O \subset O \) for \( 0 \leq t \leq 1 \). Note that \( \hat{L} \) is the principal centralizer of the Hamiltonian \( \hat{G} \)-variety \( \hat{X}' \). Since \( \hat{G}X_L \) is an open subvariety of \( X_M \) (in Zariski topology), we have \( X_L \cap O \neq \emptyset \). Choose an \( \hat{L} \)-cross-section \( \hat{X}'_L \) of \( \hat{X}' \) such that some connected component of \( X_L \cap O \) is contained in \( \hat{X}'_L \cap O \).

Lemma 4.7. The manifold \( \hat{X}'_L \cap O \) is connected.

Proof of Lemma 4.7. Let \( (\eta, h, f) \) be an \( \mathfrak{s}_2 \)-triple in \( \hat{g}^{H \cap \hat{G}} \) generating the model variety \( \hat{X}' \). Note that the action \( \mathbb{C}^\times : \hat{X}' \) preserves \( \hat{X}'_L \). Let \( Y^0, Y^1 \) be two distinct connected components of \( \hat{X}'_L \cap O \), \( y^i \in Y^i, i = 0, 1 \), and \( y^0, 0 \leq t \leq 1 \), a continuous curve connecting \( y^0, y^1 \) in \( \hat{X}'_L \). There is a positive real \( \tau < 1 \) such that \( \tau y^i \in O \) for all \( t, 0 \leq t \leq 1 \). Finally, note that \( \tau_1 y^i \in Y^i \) for all real \( \tau_1 \) such that \( \tau \leq \tau_1 \leq 1 \) and \( i = 0, 1 \). Therefore \( t \mapsto \tau y^i \) is a continuous curve in \( \hat{X}'_L \cap O \) connecting points from \( Y^0, Y^1 \). Contradiction. \( \Box \)

Now we can complete the proof of the proposition. One easily deduces from Proposition 2.26 that \( a^{(\hat{X}_L)}_{\hat{G}, \hat{X}} = a^{(X_L)}_{\hat{G}, X_M} \). The equalities \( W^{(X_L)}_{G,X_M} = W^{(X_L)}_{M,X_M} = W^{(X_L)}_{G,X} \cap M/L \) hold, see the discussion preceding Lemma 2.30. By Lemma 4.7, \( N^L_G(\hat{L}, X'_L) = N^L_G(\hat{L}, X'_L \cap O) \). It remains to recall that \( X'_L \cap O \hookrightarrow X_L \) whence \( N^L_G(\hat{L}, X'_L \cap O) \subset N^L_G(\hat{L}, X_L) \). \( \Box \)

Remark 4.8. We use the notation of Proposition 4.1. Let \( \hat{X}' \), \( \hat{X}'_L \), \( O \) be as in the proof of Proposition 4.6. It can be checked using the definitions of the morphisms \( \psi \), that the following diagram is commutative
is the projection along \( V \) of analytical manifolds. The morphism easily sees that the orthogonal projection \( a \) of \( W \) is an translation by the projection of \( a \). The structure of Weyl groups of affine Hamiltonian varieties.

4.2. The structure of Weyl groups of affine Hamiltonian varieties. In this subsection \( G \) is a connected reductive group, \( T \) is a maximal torus of \( G \), \( X \) is a conical affine Hamiltonian \( G \)-variety satisfying the equivalent conditions of Lemma 2.22, and \( X_T \) is a \( T \)-cross-section of \( X \). The goal of this subsection is to obtain some information about \( W^{(i)}_{G,X} \) and some restrictions on a Hamiltonian \( G \)-variety \( X \) with a given Weyl group. All results are based on Propositions 4.1,4.2,4.3,4.6. These propositions allow one to reduce the study of \( W^{(i)}_{G,X} \) to the case when \( G \) is semisimple and \( X \) is a model Hamiltonian variety \( M_G(H, \eta, V) \) such that \( \text{cork}_G(X) = 0 \) and \( \eta \) is nilpotent. First of all, we need to find out when the Weyl group of the last variety is trivial.

Proposition 4.9. Let \( G \) be a connected reductive group, \( H \) a reductive subgroup, \( \eta \) an \( H \)-invariant nilpotent element of \( g \), and \( V \) a symplectic \( H \)-module. Suppose \( X := M_G(H, \eta, V) \) satisfies the equivalent conditions of Lemma 2.22 and \( \text{cork}_G(X) = 0 \).

1. If \( W^{(i)}_{G,X} = \{1\} \), then

\[ \begin{align*}
\text{(*) } \eta = 0, (G, G) &\subset H, (G, G) \cong G_1 \times \ldots \times G_k \text{ for some } k, \text{ where } G_i \cong \text{SL}_2. \\
\text{Moreover, the } G \text{-modules } V/V^{(G,G)} \text{ and } V_1 \oplus V_2 \oplus \ldots \oplus V_k \text{ are isomorphic, where } V_i \text{ is the direct sum of two copies of the two-dimensional irreducible } G_i \text{-module.}
\end{align*} \]

2. Conversely, if \( G \) is semisimple, and \( X \) satisfies \( (*) \), then \( W^{(i)}_{G,X} = \{1\} \).

Proof. Suppose, at first, that \( G \) is semisimple. Let us prove the first assertion.

Since \( \text{cork}_G(X) = 0 \), we see that the field \( \mathbb{C}(X)^G \) is Poisson commutative, compare with [V2], Section 2.3. It follows from Proposition 2.32 that \( \hat{\psi}_{G,X}/G : X/G \to a^{(i)}_{G,X}/W^{(i)}_{G,X} \) is an isomorphism. By Lemma 2.29, \( \mathbb{C}[X]^G \) and \( \mathbb{C}[a^{(i)}_{G,X}/W^{(i)}_{G,X}] \) are isomorphic as graded algebras (where all elements of \( a^{(i)}_{G,X} \) are supposed to have degree 2, the grading on \( \mathbb{C}[X]^G \)
is described in Remark 2.17). Thus the equality $W_{G,X}^{(v)} = \{1\}$ is equivalent to the condition that $\mathbb{C}[X]^{G}$ is generated by elements of degree 2.

The morphism $M_{G}(H^\circ, \eta, V) \to M_{G}(H, \eta, V), \{g, (u, v)\} \mapsto \{g, (u, v)\}$, satisfies the assumptions of assertion 4 of Lemma 2.30. Thus $W_{G,M_{G}(H^\circ, \eta, V)}^{(v)} = \{1\}$ and we may assume that $H$ is connected.

Put $U = (\mathfrak{g}(\eta) \cap \mathfrak{h})^\perp$. Let us equip the algebra $\mathbb{C}[U \oplus \bigwedge^{2} \mathfrak{g}]$ with the grading described in Remark 2.17. The algebra $\mathbb{C}[U \oplus \bigwedge^{2} \mathfrak{g}]$ is generated by elements of degree 2. Recall that $\eta \in U^\ast$ has degree 4. Therefore $\eta = 0$. Any element from $U^\ast \cong \mathfrak{h}^\perp$ has degree 2. Therefore $U/H \cong U^H$, equivalently, $\mathbb{C}[U/U^H]^H = \mathbb{C}$. But the $H$-module $U/U^H$ is orthogonal (that is, possesses a nondegenerate $H$-invariant symmetric form) because $U$ is orthogonal. Hence $U = U^H$. Equivalently, $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^\perp$. In particular, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. By Lemma 2.22,

$$\dim X = \dim \mathfrak{g} + \text{rk} \mathfrak{g}.$$  

But $\dim X = 2 \dim \mathfrak{g} - 2 \dim \mathfrak{h} + \dim V$. Note that $m_{H}(V) = \dim H$, for $m_{G}(X) = \dim G$ and $\mathfrak{g}/\mathfrak{h}$ is a trivial $\mathfrak{h}$-module. Since $V$ is a symplectic $H$-module, we have

$$\dim V = \dim \mathfrak{h} + \text{rk} \mathfrak{h} + \text{cork}_H(V).$$

From (4.5), (4.6) it follows that

$$\dim \mathfrak{g} + \text{rk} \mathfrak{g} = \dim X = 2 \dim G/H + \dim V \geq 2 \dim \mathfrak{g} - \dim \mathfrak{h} + \text{rk} \mathfrak{h}.$$  

We deduce from (4.7) that $\dim \mathfrak{g} - \text{rk} \mathfrak{g} \leq \dim \mathfrak{h} - \text{rk} \mathfrak{h}$. Since $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, the last inequality is equivalent to $\mathfrak{g} = \mathfrak{h}$.

Let $G = G_{1} \ldots G_{k}$ be the decomposition into the locally direct product of simple subgroups. By the discussion before Lemma 2.30, $W_{G_{i}, V}^{(v)} = \{1\}$. By Propositions 4.1, 4.3, there exists a point $x \in \psi_{G_{i}, V}^{-1}(0)$ satisfying the conditions (a)-(f) of those propositions (with $G, X$ replaced with $G_{i}, V$). Let $(H_{0}, \eta_{0}, V_{0})$ be the determining triple of the $G_{i}$-variety $V$ at $x$.

Thanks to Proposition 4.6, $W_{G_{i}, M_{G_{i}}(H_{0}, \eta_{0}, V_{0})}^{(v)} = \{1\}$. By assertion 1, $\eta_{0} = 0, H_{0} = G_{i}$ whence $V_{0} = V$. Further, $V/V^{G_{i}}$ is a coisotropic $G_{i}$-module of dimension $\dim \mathfrak{g}_{i} + \text{rk} \mathfrak{g}_{i}$. Using the classification of coisotropic modules obtained in [Lo1],[Kn6], we get $G_{i} = \text{SL}_{2}$. Since $W_{G_{i}, V/V^{G_{i}}} = \{1\}$, we see that $\mathbb{C}[V/V^{G_{i}}]$ is generated by an element of degree 2. One easily deduces from this that $V/V^{G_{i}}$ is isomorphic to the direct sum of two copies of the irreducible 2-dimensional $\text{SL}_{2}$-module.

Since $V/V^{G_{i}} \cong V^{G_{i}} \bigwedge^{2}$ is a symplectic $G_{i}$-module and $\text{Sp}(V/V^{G_{i}})_{G_{i}}$ is a torus, the group $\prod_{j \neq i} G_{j}$ acts trivially on $V/V^{G_{i}}$. Note that $(\bigoplus_{i=1}^{k} V^{G_{i}})^{\bigwedge^{2}} = \bigcap_{i=1}^{k} V^{G_{i}} = V^{G} = 0$. The last equality holds because $\text{cork}_{G}(V) = 0$. To complete the proof of assertion 1 note that $\prod_{i=1}^{k} G_{i}$ acts on $V$ effectively. It follows that the natural epimorphism $\prod_{i=1}^{k} G_{i} \to G$ is an isomorphism.

Now suppose that $X$ is of the form indicated in (*). It is enough to check the equality $W_{G,X}^{(v)} = \{1\}$ for $k = 1$. Here the equality follows from the observation that $\mathbb{C}[X]^{G}$ is generated by an element of degree 2.

We proceed to the case when $G$ is not necessarily semisimple. Let $x$ be a point of $X$ satisfying conditions (a)-(f) of Propositions 4.1,4.6 for $M = G, \hat{G} = (G,G)$ and $\hat{X}$ be the model variety constructed by $\hat{G}, x$ in (e). By Proposition 4.6, $W_{G, \hat{X}}^{(v)} = \{1\}$. Therefore

$$(G,G) = \prod_{i=1}^{k} G_{i}, \hat{X} = \bigoplus_{i=1}^{k} V_{i}.$$  

Since any stabilizer of a point with closed $G$-orbit is
conjugate to a subgroup of \( H \), we have \((G, G) \subset H, \eta = 0\). By the above, there is a point \( x \in V^{(G, G)} \to X \) such that \( V/\mathfrak{g}_x x \cong \bigoplus_{i=1}^{k} V_i \). This observation completes the proof. \( \square \)

Now we are going to obtain a sufficient condition for \( W^{(s)}_{G, X} \) to intersect any subgroup of \( W(g) \) conjugate to a certain fixed subgroup. To state the corresponding assertion we need some definitions.

**Definition 4.10.** A subset \( A \subset \Delta(g) \) is called **completely perpendicular** if the following two conditions take place:

1. \( (\alpha, \beta) = 0 \) for any \( \alpha, \beta \in A \).
2. \( \text{Span}_x(A) \cap \Delta(g) = A \cup -A \).

For example, any one-element subset of \( \Delta(g) \) is completely perpendicular.

**Definition 4.11.** A pair \((\mathfrak{h}, V)\), where \( \mathfrak{h} \) is a reductive subalgebra of \( \mathfrak{g} \) and \( V \) is an \( \mathfrak{h} \)-module, is said to be a \( \mathfrak{g} \)-stratum. Two \( \mathfrak{g} \)-strata \((\mathfrak{h}_1, V_1), (\mathfrak{h}_2, V_2)\) are called **equivalent** if there exist \( g \in G \) and a linear isomorphism \( \varphi : V_1/\mathfrak{h}_1 \to V_2/\mathfrak{h}_2 \) such that \( \text{Ad}(g)\mathfrak{h}_1 = \mathfrak{h}_2 \) and \( \text{Ad}(g)\xi \varphi(v_1) = \varphi(\xi v_1) \) for all \( \xi \in \mathfrak{h}_1, v_1 \in V_1/\mathfrak{h}_1 \).

**Definition 4.12.** Let \( Y \) be a smooth affine variety and \( y \in Y \) a point with closed \( G \)-orbit. The pair \((\mathfrak{g}_y, T_y Y/\mathfrak{g}_y, y)\) is called the \( \mathfrak{g} \)-stratum of \( y \). We say that \((\mathfrak{h}, V)\) is a \( \mathfrak{g} \)-stratum of \( Y \) if \((\mathfrak{h}, V)\) is equivalent to a \( \mathfrak{g} \)-stratum of a point of \( Y \). In this case we write \((\mathfrak{h}, V) \sim_{\mathfrak{g}} Y \).

**Remark 4.13.** Let us justify the terminology. Pairs \((\mathfrak{h}, V)\) do define some stratification of \( Y/G \) by varieties with quotient singularities. Besides, analogous objects were called ”strata” in [Sch3], where the term is borrowed from.

Let \( A \) be a nonempty completely perpendicular subset of \( \Delta(g) \). By \( S^{(A)} \) we denote the \( \mathfrak{g} \)-stratum \((\mathfrak{g}^{(A)}; \sum_{\alpha \in A} V^\alpha)\), where \( V^\alpha \) is, by definition, the direct sum of two copies of the two-dimensional irreducible \( \mathfrak{g}^{(A)}/\mathfrak{g}^{(A\backslash\{\alpha\})} \)-module.

**Corollary 4.14.** If \( W^{(s)}_{G, X} \cap W(\mathfrak{g}^{(A)}) = \{1\} \), then \( S^{(A)} \sim_{\mathfrak{g}} X \).

**Proof.** Put \( M = Z_G(\bigcap_{\alpha \in A} \ker \alpha) \). We remark that \( G^{(A)} = (M, M) \). Choose a point \( x \in X \) satisfying conditions (a)-(f) of Propositions 4.1, 4.3 for general \( \xi \in \mathfrak{z}(m) \). Let \((H, \eta, V)\) be the determining triple of \( X \) at \( x \) and \( \overset{\sim}{X} = M_{G^{(A)}(H \cap G^{(A)}, \eta, V/V^H)} \). By Proposition 4.6, \( W^{(s)}_{G^{(A)}, \overset{\sim}{X}} = \{1\} \). Using Proposition 4.9, we see that \( S^{(A)} \) is equivalent to the \( \mathfrak{g} \)-stratum of \( x \). \( \square \)

Now we obtain some restriction on \( W^{(s)}_{G, X} \), namely, we check that \( W^{(s)}_{G, X} \) is large in the sense of the following definition.

**Definition 4.15.** A subgroup \( \Gamma \subset W(g) \) is said to be **large** if for any two roots \( \alpha, \beta \in \Delta(g) \) such that \( \beta \neq \pm \alpha, (\alpha, \beta) \neq 0 \) there exists \( \gamma \in \mathbb{R}\alpha + \mathbb{R}\beta \) with \( s_{\gamma} \in \Gamma \).

**Corollary 4.16.** The subgroup \( W^{(s)}_{G, X} \subset W(g) \) is large.

**Proof.** Assume the converse. Choose \( \alpha, \beta \in \Delta(g) \) such that \( \beta \neq \pm \alpha, (\alpha, \beta) \neq 0 \) but \( s_{\gamma} \notin W^{(s)}_{G, X} \) for all \( \gamma \in \Delta(g) \cap (\mathbb{R}\alpha + \mathbb{R}\beta) \). Put \( M = Z_G(\ker \alpha \cap \ker \beta) \). Note that \( (M, M) = G^{(\alpha, \beta)} \). Let \( x \in X \) satisfy conditions (a)-(f) of Propositions 4.1, 4.3 for general \( \xi \in \mathfrak{z}(m) \). Let \((H, \eta, V)\) be the determining triple of \( X \) at \( x \). Put \( \overset{\sim}{X} := M_{G^{(\alpha, \beta)}(H \cap G^{(\alpha, \beta)}, \eta, V/V^H)} \). It follows from Proposition 4.6 that \( W^{(s)}_{G^{(\alpha, \beta)}, \overset{\sim}{X}} \) contains no reflection. Let \( \overset{\sim}{G} \) denote the simply
connected covering of $G^{(\alpha, \beta)}$. It is a simple simply connected group of rank 2. Further, denote by $\tilde{H}$ the connected normal subgroup of $\tilde{G}$ corresponding to $\frak{h}$. Put $\tilde{X} = \frak{M}_G(\tilde{H}, \eta_\frak{h}, V/V^H)$. It is a coisotropic variety. There is a natural morphism $\tilde{X} \to \tilde{X}$ satisfying the assumptions of the fourth assertion of Lemma 2.30. Therefore the group $W^{(\cdot)}_{\tilde{G}, \tilde{X}}$ contains no reflection. On the other hand, by Corollary 3.12, the group $W^{(\cdot)}_{\tilde{G}, \tilde{X}}$ is generated by reflections. Therefore $W^{(\cdot)}_{\tilde{G}, \tilde{X}} = \{1\}$. By Proposition 4.9, $\tilde{G}$ is isomorphic to the direct product of several copies of SL$_2$. Since $\tilde{G}$ is simple and of rank 2, this is absurd. 

Now let us describe large subgroups of $W(g)$ for simple groups $G$ of types $A-E$.

Firstly, we consider the situation when $g$ is simple and has type $A, D, E$, in other words, when all elements of $\Delta(g)$ are of the same length.

Recall the classification of maximal proper root subsystems in $\Delta(g)$ (see [D]). We fix a system $\alpha_1, \ldots, \alpha_r \in \Delta(g)$ of simple roots. Let $\alpha_0$ be the minimal (=lowest) root and $n_1, \ldots, n_r$ (uniquely determined) nonnegative integers satisfying $\alpha_0 + n_1\alpha_1 + \ldots + n_r\alpha_r = 0$. A proper root subsystem $\Delta_0 \subset \Delta(g)$ is maximal iff it is $W(g)$-conjugate to one of the following root subsystems.

(a) $\text{Span}_Z(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r) \cap \Delta(g)$ for $n_i = 1$.

(b) $\text{Span}_Z(\alpha_0, \alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r) \cap \Delta(g)$ for prime $n_i$.

The number $n_i$ depends only on $\Delta_0$. We will call this number the characteristic of $\Delta_0$.

For a proper subgroup $\Gamma \subset W(g)$ let $\Delta_\Gamma$ denote the set of all $\alpha \in \Delta(g)$ such that $s_\alpha \in \Gamma$.

**Proposition 4.17.** Let $g$ be a simple Lie algebra of type $A, D, E$, $\text{rk} \ g > 1$, and $\Gamma$ a proper subgroup in $W(g)$. Then $\Gamma$ is large iff $\Delta_\Gamma$ is a maximal proper root subsystem in $\Delta(g)$ of characteristic 1 or 2.

**Lemma 4.18.** Let $g$ be a simple Lie algebra of type $A, D, E$. Then $\Delta_\Gamma$ is a root subsystem in $\Delta(g)$ for any subgroup $\Gamma \subset W(g)$.

*Proof.* Let $\alpha, \beta \in \Delta(g)$. Since all roots of $\Delta(g)$ are of the same length, we see that $\alpha + \beta \in \Delta(g)$, (resp., $\alpha - \beta \in \Delta(g)$) iff $(\alpha, \beta) < 0$, (resp., $(\alpha, \beta) > 0$).

We need to check that $\alpha \in \Delta_\Gamma$ implies $-\alpha \in \Delta_\Gamma$ and that $\alpha, \beta \in \Delta_\Gamma$, $\alpha + \beta \in \Delta_\Gamma$ imply $\alpha + \beta \in \Delta_\Gamma$. The first implication follows directly from the definition of $\Delta_\Gamma$. To prove the second one we note that $\alpha + \beta = s_\alpha \beta$ whenever $\alpha, \beta, \alpha + \beta \in \Delta(g)$, while $s_{s_\alpha \beta} = s_\alpha s_\beta s_\alpha \in \Gamma$ provided $\alpha, \beta \in \Delta_\Gamma$. 

*Proof of Proposition 4.17.* The subgroup $\Gamma \subset W(g)$ is large iff

(A) $\{\alpha, \beta, \alpha + \beta\} \cap \Delta_\Gamma \neq \emptyset$ for all $\alpha, \beta \in \Delta(g)$ such that $\alpha + \beta \in \Delta(g)$.

One checks directly that a maximal root subsystem $\Delta_\Gamma \subset \Delta(g)$ of characteristic 1 or 2 satisfies (A).

Now let $\Delta_\Gamma$ be a root subsystem of $\Delta(g)$ satisfying (A). At first, assume that $\Delta_\Gamma$ is not maximal. Let $\Delta_1$ be a maximal proper root subsystem of $\Delta(g)$ containing $\Delta_\Gamma$. Choose $\alpha \in \Delta_1 \setminus \Delta_\Gamma$. We see that $\alpha + \beta \notin \Delta(g)$ for all $\beta \notin \Delta_1$. Otherwise $\{\alpha, \beta, \alpha + \beta\} \cap \Delta_\Gamma = \emptyset$. Analogously, $\alpha - \beta \notin \Delta(g)$. Therefore $\alpha \perp \Delta \setminus \Delta_1$. Since the root system $\Delta$ is irreducible, there is $\gamma \in \Delta$ such that $(\alpha, \gamma) \neq 0$, $\gamma \notin \Delta \setminus \Delta_1$. By the above, any such $\gamma$ necessarily lies in $\Delta_\Gamma$. Without loss of generality, we may assume that $(\alpha, \gamma) = -1$ whence $\alpha + \gamma \in \Delta$. Then, automatically, $\alpha + \gamma \in \Delta_1 \setminus \Delta_\Gamma$. It follows that $\alpha + \gamma \perp \Delta \setminus \Delta_1$, which contradicts the choice of $\gamma$. 


It remains to show that the characteristic of $\Delta_\Gamma$ is less than 3. Assume that $\Delta_\Gamma = \text{Span}_\mathbb{Z}\{\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_r\} \cap \Delta(g)$, where $n_i > 2$. Let $\pi^{\vee}_i$ denote the dual fundamental weight corresponding to $\alpha_i$. The subset $\Delta_\Gamma \subset \Delta(g)$ coincides with the set of all $\alpha \in \Delta$ such that $n_i$ divides $\pi^{\vee}_i(\alpha)$. So it is enough to check that there are $\alpha, \beta \in \Delta(g)$ such that $\langle \pi^{\vee}_i, \alpha \rangle = \langle \pi^{\vee}_i, \beta \rangle = 1$, and $\alpha + \beta \in \Delta(g)$. There is $\gamma \in \Delta(g)$ with $\langle \pi^{\vee}_i, \gamma \rangle = 2$. Choose such an element $\gamma = \sum_{j=1}^r m_j \alpha_j$ such that $\sum m_j$ is minimal possible. One sets $\alpha := \alpha_i, \beta := \gamma - \alpha_i \in \Delta(g)$.

\begin{proof}

For $g$ of type $A_l$ or $D_l$ the required assertion stems directly from Proposition 4.17. Suppose $g \simeq \mathfrak{sp}_{2l}, l \geq 2$. If $l = 2$, then $\Gamma \subset W(g)$ is large if $\Delta_\Gamma \neq \emptyset$. All nonempty subsets $\Delta_\Gamma \subset \Delta(\mathfrak{sp}_4)$ do appear in Table 4.1. Now suppose $l > 2$. Let $\Delta_0$ denote the subset of all short roots in $\Delta(g)$ and $W_0$ the subgroup of $W(g)$ generated by $s_\alpha, \alpha \in \Delta_0$. Note that $W_0$ is the Weyl group of the root system $D_l$. By the definition of a large subgroup, the subgroup $\Gamma_0$ generated by $s_\alpha, \alpha \in \Delta_\Gamma \cap \Delta_0$, is large in $W_0$. If $\Delta_0 \cap \Delta_\Gamma$ is of type (a) (see Table 4.1), then $\Gamma$ is large in $W(g)$ iff $\Delta_\Gamma$ contains a long root. If $\Delta_0 \cap \Delta_\Gamma$ is of type (b) or $\Delta_0 \subset \Delta_\Gamma$, then $\Gamma_0$ is large in $W(g)$. Since $\Gamma \subset N_{W(g)}(\Gamma_0)$, we see that large subgroups in $W(g)$ are precisely those presented in Table 4.1.

The proof for $g \simeq \mathfrak{so}_{2l+1}, l \geq 2$, follows easily from the duality between the root systems $B_l, C_l$.

\end{proof}

4.3. Examples of computation of Weyl groups. In this subsection we classify pairs $(G, V)$, where $G$ is a simple algebraic group, and $V$ is a symplectic $G$-module such that $\text{def}_G(V) = \text{rk}_G W^{(i)}_{G, V} \neq W(g)$. The computation for $V \simeq U \oplus U^*$ (and, more generally, $X = T^*(G * H V)$) is made in [Lo4], Section 5, so here we consider only the case $X \neq U^* \oplus U$.

**Lemma 4.20.** Let $G$ be a simple group, $X := M_G(H, \eta, V)$, where $\eta$ is nilpotent, satisfy the equivalent conditions of Lemma 2.22. If $s_\alpha \not\in W^{(i)}_{G, X}$ for some $\alpha \in \Delta(g)$, then there exist a
indicated in column 4 are determined uniquely up to
and Littelmann, [KL]. All such symplectic modules
theorem, we see that there is a prime

\[ \frac{\text{tr}_{U \oplus V}(h^2)}{\text{tr}_h(h^2)} = 1 - \frac{4}{\text{tr}_{h}(h^2)}. \]

Here \( U := \mathfrak{z}_g(\eta)/\mathfrak{h} \) and \( h \) is a coroot in \( \mathfrak{s} \).

**Proof.** By Corollary 4.14, \( S^{(\alpha)} \sim_\mathfrak{g} X \). Equivalently, there is a subalgebra \( \mathfrak{s} \subset \mathfrak{h} \) such that \( \mathfrak{s} \sim G \mathfrak{g}^{(\alpha)} \) and \( (\mathfrak{s}, \mathbb{C}^2 \oplus \mathbb{C}^2) \sim_\mathfrak{h} U \oplus V \). The last condition implies that the \( \mathfrak{s} \)-modules \( \mathfrak{h}/\mathfrak{s} \oplus (\mathbb{C}^2)^{\oplus 2} \) and \( U \oplus V \) differ by a trivial summand. Comparing the traces of \( h^2 \) on these two modules, we get the claim. \( \square \)

Here is the main result of this subsection.

**Proposition 4.21.** Let \( G \) be a simple group and \( V \) a symplectic \( G \)-module satisfying the equivalent conditions of Lemma 2.22 such that \( V \not\cong U \oplus U^* \) for any \( G \)-module \( U \). Then \( W_{G,V}^{(i)} \neq W(\mathfrak{g}) \) iff \( V \) is contained in Table 4.2. The group \( W_{G,V}^{(i)} \) is presented in the forth column of the table.

| N | \( \mathfrak{g} \) | \( V \) | \( W_{G,V}^{(i)} \) |
|---|---|---|---|
| 1 | \( \mathfrak{sl}_6 \) | \( V = V(\pi_3) \oplus V(\pi_1)^{\oplus 2} \oplus V(\pi_5)^{\oplus 2} \) | \( A_1 \times A_3 \) |
| 2 | \( \mathfrak{sp}_4 \) | \( V = V(\pi_1) \oplus V(\pi_2)^{\oplus 2} \) | \( C_1 \times C_1 \) |
| 3 | \( \mathfrak{sp}_6 \) | \( V = V(\pi_3) \oplus V(\pi_1)^{\oplus 2} \) | \( C_1 \times C_2 \) |
| 4 | \( \mathfrak{so}_{11} \) | \( V = V(\pi_5) \oplus V(\pi_1)^{\oplus 4} \) | \( B_1 \times B_4 \) |
| 5 | \( \mathfrak{so}_{13} \) | \( V = V(\pi_6) \oplus V(\pi_1)^{\oplus 2} \) | \( B_2 \times B_4 \) |

In the fourth column we indicate the type of a root subsystem in \( \Delta(\mathfrak{g}) \) such that the reflections corresponding to its roots generate \( W_{G,V}^{(1)} \). By \( B_1 \) (resp., \( C_1 \)) we mean a root subsystem containing two opposite short (resp., long) roots in \( B_n \) (resp., \( C_n \)). Root subsystems indicated in column 4 are determined uniquely up to \( W(\mathfrak{g}) \)-conjugacy.

**Proof of Proposition 4.21.** By Corollary 4.14, \( S^{(\alpha)} \sim_\mathfrak{g} V \) for some \( \alpha \in \Delta(\mathfrak{g}) \). There is an \( \text{SL}_2 \)-stable prime divisor \( D' \) on \( \mathbb{C}^2 \oplus \mathbb{C}^2 \) such that \( m_{\text{SL}_2}(D') = 2 \). Applying the Luna slice theorem, we see that there is a prime \( G \)-stable divisor \( D \) on \( V \) such that \( m_G(D) < \dim G \). All \( G \)-modules \( V \) with \( m_G(V) = \dim G \) possessing such a divisor \( D \) were classified by Knop and Littelmann, [KL]. All such symplectic modules \( V \) such that \( V \not\cong U \oplus U^* \) are presented in Table 4.2. Let us show that for these modules the inequality \( W_{G,V}^{(i)} \neq W(\mathfrak{g}) \) does hold.

**Case 1.** \( \mathfrak{g} = \mathfrak{sl}_6, V = V(\pi_3) \oplus V(\pi_1)^{\oplus 2} \oplus V(\pi_5)^{\oplus 2} \). We can consider \( V \) as a symplectic \( \widetilde{G} := \text{SL}_6 \times \text{SL}_2 \)-module, where \( \text{SL}_2 \) acts on \( V(\pi_1)^{\oplus 2} \oplus V(\pi_5)^{\oplus 2} \) as on \( \mathbb{C}^2 \oplus (V(\pi_1) \oplus V(\pi_5)) \). This module has a finite stabilizer in general position and is coisotropic, see [Kn6],[Lo1]. The Weyl group \( W_{G,V}^{(i)} \) was computed in [Kn6], Table 12, it corresponds to the root system \( A_1 \times A_1 \times A_3 \). By the discussion preceding Lemma 2.30, \( W_{G,V}^{(i)} = W_{\text{SL}_2}^{(i)} \times W_{G,V}^{(i)} \). It follows that \( W_{G,V}^{(i)} = A_1 \times A_3 \).

**Case 2.** \( \mathfrak{g} = \mathfrak{sp}_4, V = V(\pi_1) \oplus V(\pi_2)^{\oplus 2} \). We can consider \( V \) as a symplectic \( \widetilde{G} := \text{Sp}_4 \times \mathbb{C}^\times \)-module. Here \( \mathbb{C}^\times \) acts trivially on \( V(\pi_1) \) and as \( \text{SO}_2 \) on \( V(\pi_2)^{\oplus 2} \cong \mathbb{C}^2 \otimes V(\pi_2) \). Again, this module is coisotropic and has a finite stabilizer in general position. Using tables obtained
in [Kn6], we see that $W^{(t)}_{G,V} \cong A_1 \oplus A_1$. But $W_{G,V}^{(t)} = W^{(t)}_{G,V}$, see assertion 3 of Lemma 2.30. Using Lemma 4.20, we see that $s_\alpha \in W^{(t)}_{G,V}$ for all long roots $\alpha$.

**Case 3. $g = \mathfrak{sp}_6, V = V(\pi_3) \oplus V(\pi_1)^{\otimes 2}$.** One argues exactly as in the previous case.

Before proceeding to the remaining two cases let us make some remarks.

Firstly, $s_\alpha \in W^{(t)}_{G,V}$ for all short roots $\alpha$. One checks this using Lemma 4.20 (the fraction in the l.h.s. of (4.8) (the index of the $G$-module $V$) can be computed using Table 1 of [AEV]).

Since $s_\alpha \in W^{(t)}_{G,V}$ for any short root $\alpha$, it follows from Corollary 4.16, Proposition 4.17 that $W^{(t)}_{G,V}$ is either the whole Weyl group $W(g)$ or is maximal among all proper subgroups generated by reflections. The latter holds iff $\mathbb{C}[C_{G,V}] \cong \mathfrak{g}(\mathbb{C}[V]^G)$ (the center of the Poisson algebra $\mathbb{C}[V]^G$) contains two linearly independent elements of degree 4.

**Case 4. $g = \mathfrak{so}_{11}, V = V(\pi_5) \oplus V(\pi_1)^{\oplus 4}$.**

Note that $\operatorname{Sp}(V)^G \cong \mathfrak{sp}_4$. We consider the $\mathbb{Z}^2$-grading on $\mathbb{C}[V]$ induced by the degrees with respect to $V(\pi_5)$ and $V(\pi_1)^{\oplus 4}$. By [Sch2], $\mathbb{C}[V]^G$ is freely generated by a 21-dimensional subspace $U \subset \mathbb{C}[V]^G$ such that

1. $U$ is an $\mathfrak{sp}_4$-submodule in $\mathbb{C}[V]^G$.
2. There is the decomposition $U = U_1 \oplus U_2 \oplus U_3 \oplus U_4$, where $U_1 \cong S^2 \mathbb{C}^4, U_2 \cong \wedge^2 \mathbb{C}^4, U_3 \cong \mathbb{C}^4, U_4 \cong \mathbb{C}^4$ (isomorphisms of $\mathfrak{sp}_4$-modules).
3. $U_i, i = 1, 4$, is homogeneous with respect to the $\mathbb{Z}^2$-grading on $\mathbb{C}[V]$. The degrees of $U_1, U_2, U_3, U_4$ are $(2, 0), (2, 2), (1, 2), (0, 4)$, respectively.

Let $P_1, P_2$ denote the Poisson bivectors on $V(\pi_5)$ and $V(\pi_1)^{\oplus 4}$ respectively. Now let $f_1, f_2$ be homogeneous elements of $\mathbb{C}[V]$ of bidegrees, say $(d_1, d_1'), (d_2, d_2')$. Then $\{f_1, f_2\} = (P_1, df_1 \wedge df_2) + (P_2, df_1 \wedge df_2)$. The bidegrees of the first and the second summand are, respectively, $(d_1 + d_2 - 2, d_1' + d_2'), (d_1 + d_2, d_1' + d_2' - 2)$. For instance, if $f_1 \in \mathbb{C}[V(\pi_5)]$, then $\{f_1, f_2\}$ is homogeneous of bidegree $(d_1 + d_2 - 2, d_2')$.

Let us check that $U_4 \subset 3(\mathbb{C}[V]^G)$. Since $V(\pi_5)$ and $V(\pi_1)^{\oplus 4}$ are skew-orthogonal, we get $\{U_4, U_1\} = 0$. Suppose $\{U_4, U_2\} \neq \{0\}$. Since $U_4 \subset \mathbb{C}[V]^G \times \mathfrak{sp}_4$, we see that $\{U_4, U_2\}$ is isomorphic (as an $\mathfrak{sp}_4$-module) to a submodule of $\wedge^2 \mathbb{C}^4$. But $\{U_4, U_2\}$ consists of homogeneous elements of degree $(2, 4)$ whence $\{U_4, U_2\} \subset U_1 U_4 + U_2^2$. Both summands are isomorphic to $S^2 \mathbb{C}^4$. This contradicts $\{U_4, U_2\} \neq \{0\}$. Finally, the degree of $\{U_3, U_4\}$ equals $(1, 4)$ whence $\{U_3, U_4\} = 0$.

Let $q$ denote a homogeneous element in $\mathbb{C}[g]^G$ corresponding to an invariant nondegenerate form. Then $\mu^*_{G,V}(q)$ is a homogeneous element of $\mathbb{C}[V]^G$ of degree 4. It remains to check that $\mu^*_{G,V}(q) \notin U_4$. One checks easily that if $\mu_{G,V(\pi_1)^{\oplus 4}}$ contains an element $\xi$ such that $(\xi, \xi) \neq 0$. If $v \in V(\pi_1)^{\oplus 4} \hookrightarrow V$ is such that $\mu_{G,V}(v) = \xi$, then $q(\mu_{G,V}(v)) \neq 0, f(v) = 0$ for any $f \in U_4$. The last equality holds because $U_4 \subset \mathbb{C}[V(\pi_5)]$.

By Corollary 4.19, $W^{(t)}_{G,V}$ corresponds either to $B_1 \oplus B_4$ or to $B_2 \oplus B_3$. Thanks to Corollary 4.14, it remains to prove that $S^4 \not\twoheadrightarrow g V$, where $A = \{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4\}$. If $S^4 \twoheadrightarrow g V$, then

\begin{equation}
\dim V^{g(A)} + \dim g - \dim (g(A)) = \dim V - 8.
\end{equation}

But $\dim V(\pi_5)^{g(A)} = 3, \dim V(\pi_5)^{b(A)} = 8$ (recall that the weight system of $V(\pi_5)$ consists of all weights of the form $\pm \frac{1}{2}(\epsilon_1 \pm \ldots \pm \epsilon_5)$ without multiplicities; $V(\pi_5)_{\lambda} \subset V(\pi_5)^{b(A)}$ iff $(\lambda, \epsilon_1 - \epsilon_2) = (\lambda, \epsilon_3 - \epsilon_4) = 0$, $\dim g(A) = 15$. So (4.9) does not hold.

**Case 5. $g = \mathfrak{so}_{13}, V = V(\pi_6) \oplus V(\pi_1)^{\otimes 2}$.** By [Sch2], the algebra $\mathbb{C}[V]^G$ is freely generated by 12 elements $f(2, 0, 0), f(1, 1, 0), f(0, 2, 0), f(0, 0, 4), f(0, 0, 8), f(1, 0, 4), f(0, 1, 4), f(2, 0, 4), f(1, 1, 4), f(0, 2, 4), f(0, 1, 2), f(1, 1, 6), f(0, 2, 0)$.
where the lower index indicates the grading with respect to the decomposition $V = V(\pi_1) \oplus V(\pi_1) \oplus V(\pi_6)$. Note that $\mathrm{Sp}(V)^G \cong \mathrm{SL}_2$. The elements $f_{(0,0,4)}, f_{(0,0,8)}, f_{(1,1,2)}, f_{(1,1,6)}$ are $\mathrm{SL}_2$-invariant, $\mathrm{Span}_C(f_{(2,0,0)}, f_{(1,1,0)}, f_{(0,2,0)}), \mathrm{Span}_C(f_{(2,0,4)}, f_{(1,1,4)}, f_{(0,2,4)}) \cong S^2 \mathbb{C}^2$, $\mathrm{Span}(f_{(1,0,4)}, f_{(0,1,4)}) \cong \mathbb{C}^2$.

Analogously to the previous case (i.e., using the grading and the $\mathrm{SL}_2$-module structure of $\mathbb{C}[V]^G$, we check that $f_{(0,0,4)}, f_{(0,0,8)} \in \mathfrak{z}(\mathbb{C}[V]^G)$. Let $\mu_1, \mu_2, \mu$ denote the moment maps for the actions $G : V(\pi_6), G : V(\pi_1)^{\otimes 2}, V$. Clearly, $\mu = \mu_1 + \mu_2$. Further, put $f_2(\xi) = \mathrm{tr}(\xi^2), f_4(\xi) = \mathrm{tr}(\xi^4), \xi \in \mathfrak{g}$ (the traces are taken in the tautological $\mathfrak{so}_{13}$-module). We have shown that $\mu_1^*(f_2), \mu_1^*(f_4) \in \mathfrak{z}(\mathbb{C}[V]^G)$. On the other hand, $\mu^*(f_2), \mu^*(f_4) \in \mathfrak{z}(\mathbb{C}[V]^G)$. Let us check that $\mu_1^*(f_2), \mu_1^*(f_4), \mu^*(f_2), \mu^*(f_4)$ are algebraically independent. Analogously to the previous case one checks that $\mu_1^*(f_2), \mu^*(f_2)$ are independent. It remains to check that the equality
\begin{equation}
(4.10) \quad a \mathrm{tr}(\xi_i^4) + b \mathrm{tr}(\xi_1 + \xi_2)^4 + c \mathrm{tr}(\xi_i^2)^2 + d \mathrm{tr}(\xi_i^2) \mathrm{tr}(\xi_1 + \xi_2)^2 + e \mathrm{tr}(\xi_i + \xi_2)^2)^2 = 0,
\end{equation}
implies $a = b = 0$. The isotropy subalgebras in general positions for $V(\pi_6), V(\pi_1)^{\otimes 2}$ are $\mathfrak{g}^{\alpha_1, \alpha_2, \alpha_4, \alpha_5}, \mathfrak{g}^{\alpha_2, \alpha_3, \alpha_6}$, respectively. By Lemma 2.23, $\mathrm{im} \mu_1 = G \mathrm{Span}_C(\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_4 + \varepsilon_5 + \varepsilon_6)$, $\mathrm{im} \mu_2 = G \mathrm{Span}_C(\varepsilon_1)$. Since $\mathrm{tr}(\xi_i^4)$ and $\mathrm{tr}(\xi_i^2)^2$ are not proportional, we get $a + b = 0$. Writing down the terms of (4.10) of bidegree $(3,1)$ with respect to $(\xi_1, \xi_2)$, we get
\[
4b \mathrm{tr}(\xi_1^2 \xi_2) + 2d \mathrm{tr}(\xi_1^2) \mathrm{tr}(\xi_1 \xi_2) + 4e \mathrm{tr}(\xi_1^2) \mathrm{tr}(\xi_1 \xi_2) = 0.
\]
Putting $\xi_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + i(\varepsilon_4 + \varepsilon_5 + \varepsilon_6), \xi_2 = \varepsilon_1$, we see that $b = 0$.

To prove that the group $W_{G,V}^{(1)}$ has the form indicated in Table 4.2 it is enough to check that $S^{(A)} \not\subset \mathfrak{g} V$ for $A = \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4, \varepsilon_5 - \varepsilon_6\}$. This is done analogously to the previous case.

\section{5. Fibers of $\widehat{\psi}_{G,X}$ and untwisted varieties}

Throughout this section $G, X$ are as in the previous one.

The goal of this section is to prove Theorem 1.4 and establish some examples of untwisted varieties. Subsection 5.1 contains some technical results used in the proof of Theorem 1.4. The proof itself is given in Subsection 5.2. In Subsection 5.3 we describe some classes of untwisted Hamiltonian varieties. We state a result by Knop that the cotangent bundle of an affine variety is untwisted and show that any symplectic module is untwisted. Finally, in Subsection 5.4 we give two counterexamples: of a Hamiltonian variety not satisfying (Irr) and of a conical coisotropic model variety not satisfying (Utw2). The former counterexample is due to F. Knop.

\subsection{5.1. Reducedness of fibers of $\widehat{\psi}_{G,X}$}

\textbf{Proposition 5.1.} Let $\xi \in \mathfrak{a}_{G,X}^{(l)}$ be such that $(W_{G,X}^{(l)})(\xi) = \{1\}$. Then the fiber $\widehat{\psi}_{G,X}^{-1}(\pi_{W_{G,X}^{(l)} \mathfrak{a}_{G,X}^{(l)}}(\xi))$ is reduced.

\textbf{Proof.} We preserve the notation of Proposition 4.1 and Remark 4.8 and put $\widehat{G} = M$. The image of $\xi$ in $\mathfrak{a}_{G,X}^{(l)} \cap W_{G,X}^{(l)}$ is a smooth point. Thanks to Theorem 3.1, the schematic fiber in interest is a local complete intersection. So to verify that this fiber is reduced it is enough to prove that it is generically reduced (see, for example, [E], Propositions 18.13,18.15). In other words, we need to show that $\widehat{\psi}_{G,X}$ is smooth at any point $x \in X$ satisfying conditions (a)-(e)
of Proposition 4.1 for any irreducible component $Z$. By Remark 4.8, $W^{(i)}_{M,X,M} = \{1\}$. Using the commutative diagram of Remark 4.8, we see that it is enough to prove the proposition in the case when $\xi = 0$ and $X = M_G(H, \eta, V)$, where $\eta$ is nilpotent, $\text{cork}_G(X) = 0$, and $W^{(i)}_{G,X} = \{1\}$. In this case $a_{G,X}^{(X)} / W^{(i)}_{G,X} \cong X / G$. Let $X_L, L, G_0, X_0, L_0$ be as in Proposition 3.3. An element of $N_G(L_0)$ acting trivially on $a_{G,X}^{(X)}$ lies in $Z_G(a_{G,X}^{(X)}) = L$. Thus (see the discussion preceding Lemma 2.30) $W^{(i)}_{G_0,X_0} / W^{(i)}_{G_0,X_0} \cong G_0 / G_0$. So $G_0$ is connected. There is a point $y \in X_0$ of the form $[g, 0]$, we may assume that $g = e$. It follows directly from Example 2.5 that $X_0 \cong M_G(N_H(L_0)/L_0, \eta, V^{L_0})$. Note that $\eta$ is a nilpotent element of $g^{L_0}$ so $\eta \in g^{L_0} \cap t_0^{L_0} \cong g_0$. By assertion 1 of Proposition 3.3, it is enough to check that the fiber $\pi_{G_0,X_0}(0)$ is generically reduced. Replacing $(G, X)$ with $(G_0, X_0)$ we may assume, in addition, that $X$ satisfies the equivalent conditions of Lemma 2.22. It follows that $X$ satisfies condition (*) of Proposition 4.9.

We may replace $G$ with a covering and assume that $G = T_0 \times H^c$, where $T_0$ is a torus. Further, by assertion 4 of Lemma 2.30, $W^{(i)}_{G,X} = \{1\}$, where $\tilde{X} := M_G(H^c, \eta, V)$. So we may replace $X$ with $\tilde{X}$ and assume that $H$ is connected. Since in this case $X = T^*(T_0) \times V$, we reduce to the case $H = G, X = V$. Changing $G$ by a covering again, we may assume that $G \cong (G, X) \times Z$, where $Z$ is a torus.

Recall that $a_{G,V} \cong V//G$. The required claim will follow if we show that the zero fibers of the morphisms $\pi_{(G,G),V}, \pi_{Z,V//G}^{(G)}$ are reduced. For the former morphism this stems easily from the decomposition $V \cong V^{(G,G)} \oplus \bigoplus_{i=1}^k V_i$. Put, for brevity, $U_1 := V^{(G,G)}, U_2 := \bigoplus_{i=1}^k V_i$. Note that $\text{Sp}(U_2^{(G,G)})$ is a torus of dimension $k$ acting trivially on $U_2//G$. So it remains to prove that $\pi_{Z,U_1}(0)$ are reduced. Since $\text{cork}_G(V) = 0$, we have $\dim V = 4k + 2 \dim Z$. It follows that $\dim U_2 = 2 \dim Z$. Further, by the above, $k + \dim Z = \dim V//G = \dim U_1//Z + \dim U_2//G$, whence $\dim U_1//Z = \dim Z$. Since $C(U_1)^Z = \text{Quot}(C[U_1]^Z)$ (see [Lo2], Theorem 1.2.9, for the proof in the general case), it follows that $Z$ acts on $U_1$ locally effectively. Thus the weight system of $Z$ in $U_1$ coincides with $\lambda_1, \ldots, \lambda_r, -\lambda_1, \ldots, -\lambda_r$, where $\lambda_1, \ldots, \lambda_r$ form a basis in $j^*$. Now the claim is easy. \hfill \Box

**Proposition 5.2.** Suppose $0 \in \im \psi_{G,X}$, so that $0 \in a_{G,X}^{(X)}$. Let $s \in W^{(i)}_{G,X}$ be a reflection. Put $\Gamma_s := (a_{G,X}^{(X)})^s$ (the fixed point hyperplane of $s$), $D_s = \pi_{W^{(i)}_{G,X} a_{G,X}^{(X)}}(\Gamma_s)$. Let $\tilde{Z}$ be an irreducible component of $\psi_{G,X}^{(X)}(D_s)$. Let $\xi$ be a general point in $\Gamma_s$, $M := Z_G(\xi)$, and $x \in \tilde{Z}$ satisfy conditions (a)-(f) of Propositions 4.1, 4.3. In the notation of those propositions put $\tilde{G} = T_0(M, M)$ and let $\tilde{X}$ be as in (e) of Proposition 4.1. Then the multiplicity of $\tilde{Z}$ in $\psi_{G,X}^{(X)}(D_s)$ equals 1 or 2 and the latter holds iff $W^{(i)}_{G,X} = \{1\}$.

**Proof.** We use the notation of Propositions 4.1, 4.3 and Remark 4.8. From the choice of $M$ it follows that $s \in W^{(X)}_{G,X,M}$. Put $\Gamma_s' = (a_{G,X}^{(X)})^s$, and let $D_s'$ be the image of $\Gamma_s'$ in $a_{G,X}^{(X)} / W^{(X)}_{G,X}$. Let $\tilde{Z}_M$ be an irreducible component of $\tilde{Z} \cap X_M$ containing $x$. Also $\tilde{Z}_M$ is an irreducible component of $\psi_{G,X}^{(X)}(\pi_{W^{(X)}_{G,X} a_{G,X}^{(X)}}(\Gamma_s'))$. Let $\tilde{Z}'$ denote an irreducible component of $\psi_{G,X}^{(X)}(D_s')$ containing a connected component of $O \cap \tilde{Z}_M$, where $O$ is as in the proof of Proposition 4.6. Clearly, $\tilde{Z}' = \tilde{Z} \times VH$ for some subvariety $\tilde{Z} \subset \tilde{X}$, which is an irreducible
component of $\psi_{G,X}^{-1}(D_s')$. From the commutative diagram of Remark 4.8 one deduces that precisely one of the following possibilities holds:

1. $W^{(\cdot)}_{G,X} \cong \mathbb{Z}_2$. The multiplicity of $\tilde{Z}$ in $\psi_{G,X}^{-1}(D_s)$ equals the multiplicity of $\tilde{Z}$ in $\psi_{G,X}^{-1}(D_s')$.

2. $W^{(\cdot)}_{G,X} = \{1\}$. By Proposition 5.1, the multiplicity of $\tilde{Z}$ in $\psi_{G,X}^{-1}(D_s')$ is one. Further, the multiplicity of $\tilde{Z}$ in $\psi_{G,X}^{-1}(D_s)$ is 2.

It remains to consider the first possibility. Note that, by definition of $M$, one gets $\Gamma_s \subset \mathfrak{z}(m)$ whence $\Gamma_s' \subset \mathfrak{z}(\mathfrak{g})$. Since $W^{(\cdot)}_{G,X} \neq \{1\}$, it follows that $\dim \mathfrak{a}^{(\cdot)}_{G,X} \cap [\mathfrak{g}, \mathfrak{g}] = 1$. Further, note that $T_0$ acts trivially on $\tilde{X}_L$. Since $\mathfrak{a}^{(\cdot)}_{G,X} = \mathfrak{z}(\xi) \cap t_0$ and $t_0$ projects surjectively to $\mathfrak{z}(\mathfrak{g})$ (recall that $\mathfrak{g} = t_0 + [m, m]$), we see that $\mathfrak{a}^{(\cdot)}_{G,X} \cap \mathfrak{z}(\mathfrak{g}) = \{0\}$. Therefore $\mathfrak{a}^{(\cdot)}_{G,X} \subset [\mathfrak{g}, \mathfrak{g}], \dim \mathfrak{a}^{(\cdot)}_{G,X} = 1, Z(\mathfrak{g}) = 1, Z(\mathfrak{g}) \circ \subset T_0$. From the last inclusion it follows that $Z(\mathfrak{g}) \circ$ acts trivially on $\tilde{X}$.

Replacing $(G, X)$ with $((\tilde{G}, \tilde{G}), \tilde{X})$, we reduce the problem to the proof of the following claim.

(**) Suppose that $X = M_G(H, \eta, V)$, $G$ is semisimple, $\eta$ is nilpotent, cork$_G(X) = 0$, $\dim \mathfrak{a}^{(\cdot)}_{G,X} = 1, W^{(\cdot)}_{G,X} = \{1, s\},$ where $s$ is a reflection. Then the fiber $\psi_{G,X}^{-1}(0)$ is reduced.

As in the proof of Proposition 5.1, we see that $\psi_{G,X} = \pi_{G,X}$. So it is enough to check that $\pi_{H, U \oplus V}^{-1}(0)$ is reduced, where $U = (\mathfrak{z}(\eta)/\mathfrak{h})^\ast$. Note that $\mathbb{C}[U \oplus V]^H$ is generated by an element of degree 4. Recall that $\eta \in (\ast)^H \subset \mathbb{C}[U \oplus V]^H$ is of degree 4. So if $\eta \neq 0$, we are done. If $\mathfrak{g} \neq \mathfrak{h}$, then there is an element $q \in \mathbb{C}[\mathfrak{g}/\mathfrak{h}]^H$ corresponding to an $H$-invariant quadratic form on $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{h}^\perp$. It has degree 4. Any fiber of $q$ is reduced, since $\dim \mathfrak{g}/\mathfrak{h} > 1$. Therefore it remains to consider the case $H = G, X = V$. The reducedness of fibers in this case follows from the observation that a homogeneous generator of $\mathbb{C}[V]^G$ is irreducible. □

**Proposition 5.3.** Again, we keep the notation of Proposition 4.1 and Remark 4.8. Suppose $X$ is untwisted and $0 \in \im \psi_{G,X}$. Let $x$ be a point satisfying conditions (a)-(d) of Proposition 4.1 for a point $\xi_0 \in \mathfrak{a}^{(X_L)}_{G,X}$. Then the Hamiltonian $\tilde{G}$-variety $\tilde{X}$ is untwisted and $W^{(X_L)}_{G,\tilde{X}} = (W^{(X_L)}_{G,X})_{\xi_0}$.

**Proof.** At first, we consider the case when $\tilde{G} = M$. Let $s$ be a reflection lying in $(W^{(X_L)}_{G,X})_{\xi_0}$. Let $\Gamma_s, D_s$ have the same meaning as in Proposition 5.2 and $D_s'$ denote the image of $\Gamma_s$ in $\mathfrak{a}^{(X_L)}_{M,\tilde{X}}/W^{(X_L)}_{M,\tilde{X}}$ (we write $\tilde{X}_L$ instead of $\tilde{X}_L$ because $L = \tilde{L}$). By Proposition 5.1, it is enough to show that $\psi_{M,\tilde{X}}^{-1}(D_s')$ is reduced. Note that the last scheme is non-empty because $\psi_{M,\tilde{X}}(x) \in D_s'$. Choose a component $\tilde{Z}$ of $\psi_{M,\tilde{X}}^{-1}(D_s')$. Note that $\psi_{G,X}(\tilde{Z} \cap O) \subset D_s$. Choose a component $Z$ of $\psi_{G,X}^{-1}(D_s)$ containing a connected component of $\tilde{Z} \cap O$. Since $\tilde{G} = M$, we see that the morphism $\mathfrak{a}^{(X_L)}_{M,X_M}/W^{(X_L)}_{M,X_M} \to \mathfrak{a}^{(X_L)}_{G,X_M}/W^{(X_L)}_{G,X_M}$ from the commutative diagram of Remark 4.8 can be inverted. So one can consider the morphism $\hat{\psi} : \tilde{X} \to \mathfrak{a}^{(X_L)}_{G,X}/W^{(X_L)}_{G,X}$ from this diagram. It follows from the diagram that the multiplicities of $\tilde{Z}$ in $\hat{\psi}_{G,X}^{-1}(D_s)$ and of $Z$ in $\psi_{G,X}^{-1}(D_s)$ coincide. By (Utw2) the latter is 1. It follows that the multiplicity of $\tilde{Z}$
There is a natural Hamiltonian morphism $\rho$. From Proposition 2.32 it follows that the Weyl groups of $M$ scheme whence Cohen-Macaulay ([E], Proposition 18.13).

Recall that $\phi$. Proof of assertion 1.

Changing $M$ by a covering, we may assume that $M = \hat{G} \times T_0$, where $T_0$ is a torus. There is a natural Hamiltonian morphism $\rho: T^*(T_0) \times \hat{X} \rightarrow M_M(H, \eta, V)$. By Lemma 5.4, $T^*(T_0) \times \hat{X}$ is untwisted and the natural morphism $C_{M, T^*(T_0) \times \hat{X}} \rightarrow C_{M, M_M(H, \eta, V)}$ is étale. From Proposition 2.32 it follows that the Weyl groups of $M_M(H, \eta, V)$ and $T^*(T_0) \times \hat{X}$ coincide. This implies all required claims.

5.2. Proof of Theorem 1.4.

Lemma 5.5. If $X$ is an affine Hamiltonian variety satisfying (Utw1), then all schematic fibers of $\tilde{\psi}_{G,X}/G$ are Cohen-Macaulay.

Proof. By the Hochster-Roberts theorem (see, for instance, [PV], Theorem 3.19), $X//G$ is Cohen-Macaulay. Since $C_{G,X}$ is smooth and $\psi_{G,X}/G$ is equidimensional (from Proposition 2.31), we see that any fiber of $\tilde{\psi}_{G,X}/G$ is a locally complete intersection in a Cohen-Macaulay scheme whence Cohen-Macaulay ([E], Proposition 18.13).

Proof of assertion 1. Recall that $C_{G,X} \cong \mathfrak{a}_{G,X}/W_{G,X}^{(c)}$, Proposition 2.32. Thanks to Lemma 5.5, it remains to prove that any fiber of $\tilde{\psi}_{G,X}/G$ is smooth in codimension 1. Since $X$ is conical (of degree, say, $k$), there are actions $\mathbb{C}^x : X//G, \mathfrak{a}_{G,X}/W_{G,X}^{(c)}$ such that the former extends to a morphism $\mathbb{C} \times X//G : \rightarrow X//G$, the latter is induced by $\mathbb{C}^x \times \mathfrak{a}_{G,X} \rightarrow \mathfrak{a}_{G,X}, (t, v) \mapsto t^kv$, and the morphism $\tilde{\psi}_{G,X}/G$ is $\mathbb{C}^x$-equivariant. Applying a standard argument, we see that it is enough to prove that $\tilde{\psi}_{G,X}/G^{-1}(0)$ is smooth in codimension 1.

Let us use the notation of Corollary 3.10. Put $\lambda := 0$ and choose $z \in Z_0$ and $x \in \pi_{G,X}^{-1}(z)$ with closed $G$-orbit. Put $\hat{X} := M(G(H, \eta, V/V^H)$. By Proposition 5.3, $\mathfrak{a}_{G,X}/W_{G,X}^{(c)} \cong \mathfrak{a}_{G,X}/W_{G,X}^{(c)}$. Since $\text{cork}_{G}(\hat{X}) = 0$, we have $\hat{X}/G \cong \mathfrak{a}_{G,X}/W_{G,X}^{(c)}$. Taking quotients in the commutative diagram of Remark 4.8, we get the following commutative diagram

$$
V^H \times \hat{X}/G \xleftarrow{\text{pr}_2} O/G \xrightarrow{} X/G \xrightarrow{\tilde{\psi}_{G,X}/G}
$$
It follows that $\hat{\psi}_{G,X}/G$ is smooth at $z$. Since one may take an arbitrary point of $Z_0$ for $z$ we are done by Proposition 3.3.

\textbf{Proof of assertion 2.} Choose a nonzero point $\xi \in a^{(i)}_{G,X}$ and put $a_0 := \mathbb{C}\xi$. Further, put $Y := (X//G) \times_{a^{(i)}_{G,X}/W^{(i)}_{G,X}} a^{(i)}_{G,X}$, $Y_0 := (X//G) \times_{a^{(i)}_{G,X}/W^{(i)}_{G,X}} a_0$ (here $a_0$ maps to $a^{(i)}_{G,X}/W^{(i)}_{G,X}$ via the composition $a_0 \to a^{(i)}_{G,X} \to a^{(i)}_{G,X}/W^{(i)}_{G,X}$).

Let us check that $Y$ is normal (as a scheme) and Cohen-Macaulay. Indeed, the morphism $a^{(i)}_{G,X} \to a^{(i)}_{G,X}/W^{(i)}_{G,X}$ is flat, since $X$ satisfies (Utw1). Therefore the morphism $Y \to X//G$ is flat. But, as we have already remarked, $X//G$ is Cohen-Macaulay. By Corollary 18.17 from [E], $Y$ is Cohen-Macaulay. Note that $X//G$ is smooth in codimension 1 over $a^{(i)}_{G,X}/W^{(i)}_{G,X}$. Hence $Y$ is smooth in codimension 1 over $a^{(i)}_{G,X}$ hence normal.

Again, being a complete intersection in a Cohen-Macaulay variety, $Y_0$ is Cohen-Macaulay. Similarly to the previous paragraph, $Y_0$ is normal.

Let us show that $\mathbb{C}[a_0]$ is integrally closed in $\mathbb{C}[Y_0]$. Let $\tilde{a}_0$ denote the spectrum of the integral closure of $\mathbb{C}[a_0]$ in $\mathbb{C}[Y]$. There is an action of $\mathbb{C}^\times$ on $Y_0$ lifted from $\mathbb{C}^\times : X//G$. The morphism $Y_0 \to a_0$ is $\mathbb{C}^\times$-equivariant. Therefore there is an action $\mathbb{C}^\times : \tilde{a}_0$ contracting $\tilde{a}_0$ to the unique point over $0 \in a_0$. It follows that $\tilde{a}_0 \cong \mathbb{C}$. Since the zero fiber of the morphism $Y \to a_0$ is normal, we see that the morphism $\tilde{a}_0 \to a_0$ is étale in 0. From the $\mathbb{C}^\times$-equivariance it follows that it is an isomorphism.

Thus a general fiber of the morphism $Y_0 \to a_0$ is irreducible. Thanks to the presence of $\mathbb{C}^\times$-action, the same is true for any fiber but the zero one. It follows easily from (Con1),(Con2) that $(\hat{\psi}_{G,X}/G)^{-1}(0)$ is connected. Since $(\hat{\psi}_{G,X}/G)^{-1}(0)$ is normal, it is irreducible. □

\textbf{Proof of assertion 3.} By Proposition 2.32, $X$ satisfies (Utw1). Assume that $X$ does not satisfy (Utw2). By Proposition 5.1, there is $s \in W^{(i)}_{G,X}$ such that some irreducible component $\tilde{Z} \subset \hat{\psi}^{-1}_{G,X}(D_s)$ (where, as above, $D_s$ denotes the image of $(a^{(i)}_{G,X})^s$ in $a^{(i)}_{G,X}/W^{(i)}_{G,X}$) is of multiplicity 2. Put $Y = \pi_{G,X}(\tilde{Z})$. By Proposition 2.31, $Y$ is an irreducible component of $(\hat{\psi}_{G,X}/G)^{-1}(D_s)$. It follows from (Irr) that $Y = (\hat{\psi}_{G,X}/G)^{-1}(D_s)$. Thanks to Theorem 3.1, the set of closed orbits of any two components $\tilde{Z}_1, \tilde{Z}_2 \subset \hat{\psi}^{-1}_{G,X}(D_s)$ is the same. By Proposition 5.2, the multiplicity of any component $\tilde{Z}_1$ in $\hat{\psi}^{-1}_{G,X}(D_s)$ is 2. Let $f \in \mathbb{C}[a^{(i)}_{G,X}/W^{(i)}_{G,X}]$ be such that $(f) = D_s$. Let us remark that $f$ is not a square in $\mathbb{C}(X)$. Assume the converse, let $f = f_1^2$, $f_1 \in \mathbb{C}(X)$. Then $f_1 \in \mathbb{C}[C_{G,X}] = \mathbb{C}[a^{(i)}_{G,X}/W^{(i)}_{G,X}]$ which is absurd.

Put $\tilde{A} := \mathbb{C}[X][t]/(t^2 - f)$. There is a natural morphism of schemes $\rho : \text{Spec}(\tilde{A}) \to X$. This morphism is unramified over $X \setminus \hat{\psi}^{-1}_{G,X}(D_s)$. Note also that the group $\mathbb{Z}_2$ acts on $\text{Spec}(\tilde{A})$ so that $\rho$ is the quotient for this action. Hence the restriction of $\rho$ to any irreducible component of $\text{Spec}(\tilde{A})$ is dominant. Since $f$ is not a square in $\mathbb{C}[X]$, we see that $\text{Spec}(\tilde{A})$ is irreducible. Recall that $(f) = 2D$ for some $D$. It follows that $\rho$ is unramified over $\hat{\psi}^{-1}_{G,X}(D_s)$. So $\rho$ is étale in codimension 1. Let $\tilde{X}$ denote the normalization of $\text{Spec}(\tilde{A})$ and $\tilde{\rho}$ is the natural morphism $\tilde{X} \to X$. Since $X$ is smooth, we see that $\tilde{\rho}$ is also étale in codimension 1. Besides, $\tilde{\rho}$ is finite. Applying the Zariski-Nagata theorem to $\tilde{\rho}$, we obtain that $\tilde{\rho}$ is étale. However, by our assumptions, $X$ is simply connected whence $\tilde{\rho}$ is an isomorphism. It follows that the image of $t$ in $\mathbb{C}[\tilde{X}]$ lies in $\mathbb{C}(X)$. This contradicts the condition that $f$ is not a square in $\mathbb{C}(X)$.

□
Remark 5.6. In fact, assertion 3 can be generalized to non simply connected varieties. Namely, suppose $X$ satisfies (Irr). Then there exists an untwisted conical Hamiltonian $G$-variety $\tilde{X}$ and a free action of a finite group $\Gamma$ on $\tilde{X}$ by Hamiltonian automorphisms (see Definition 2.12) such that $X \cong \tilde{X}/\Gamma$ and $\pi_{\Gamma,X} : \tilde{X} \to X$ is a Hamiltonian morphism. The proof of this claim is similar to that of assertion 3.

5.3. Some classes of untwisted varieties.

Proposition 5.7. Let $X$ be coisotropic, simply connected and conical (e.g. a symplectic vector space). Then $X$ is untwisted.

Proof. Thanks to Proposition 2.32, $X$ satisfies (Utw1). Furthermore, $X$ obviously satisfies (Irr). Applying assertion 3 of Theorem 1.4, we complete the proof. □

Theorem 5.8. Let $X_0$ be a smooth irreducible affine $G$-variety. Then $X := T^*X_0$ is an untwisted Hamiltonian $G$-variety.

Proof. (Utw1) is checked in Satz 6.6 of [Kn1]. (Utw2) follows from [Kn3], Corollary 7.6. □

As the preprint [Kn3] is not published, below we present alternative proofs of Theorem 5.8.

Theorem 5.9. Let $V$ be a symplectic $G$-module. Then $V$ is an untwisted Hamiltonian $G$-variety.

We will prove this theorem after some auxiliary considerations.

Proposition 5.10. Let $X$ be a conical Hamiltonian $G$-variety and $G_0, X_0$ be such as in the discussion preceding Proposition 3.3. If the Hamiltonian $G_0$-variety $X_0$ satisfies (Irr), then so does $X$.

Proof. The action $\mathbb{C}^\times : X$ preserves $X_0$ and so gives rise to the structure of a conical Hamiltonian $G_0$-variety on $X_0$. By Proposition 3.3, the following diagram, where the horizontal arrows are quotient morphisms for the actions $G_0/G_0^\circ$ on $X_0/G_0^\circ$, $a_{G,X}^{(\cdot)}/W_{G_0^\circ,X_0}^{(\cdot)}$, is commutative.

\[
\begin{array}{ccc}
X_0/G_0^\circ & \longrightarrow & X/G \\
\downarrow \widehat{\psi}_{G_0,X_0} & & \downarrow \widehat{\psi}_{G,X} \\
a_{G,\cdot}^{(\cdot)}/W_{G_0^\circ,X_0}^{(\cdot)} & \longrightarrow & a_{G,\cdot}^{(\cdot)}/W_{G,\cdot}^{(\cdot)}
\end{array}
\]

Choose $\xi \in a_{G,\cdot}^{(\cdot)}/W_{G,\cdot}^{(\cdot)}$ and a point $\xi' \in a_{G_0,X_0}^{(\cdot)}/W_{G_0^\circ,X_0}^{(\cdot)}$ mapping to $\xi$. By the previous commutative diagram, $(\widehat{\psi}_{G,X}/G)^{-1}(\xi')$ is the quotient of $\widehat{\psi}_{G_0,X_0}/G_0^\circ(\xi')$ by some finite group. In particular, $(\widehat{\psi}_{G,X}/G)^{-1}(\xi)$ is irreducible. □

Proposition 5.11. Let $X$ be an irreducible conical affine Hamiltonian $G$-variety such that $\dim a_{G,\cdot}^{(\cdot)} = \text{rk} G$. Let $G = Z(G)^1 G_1 \dot{\ldots} G_k$ be the decomposition of $G$ into the locally direct product of simple normal subgroups and the unit component of the center. If $X$ satisfies (Utw1) and is untwisted as a Hamiltonian $G_i$-variety for any $i$, then $X$ is untwisted as a Hamiltonian $G$-variety. Conversely, if $X$ is untwisted as a Hamiltonian $G$-variety, then so is it as a Hamiltonian $G_i$-variety.
\textbf{Proof.} By Proposition 5.1, if $\hat{\psi}_{G,X}^{-1}(D)$ is not reduced for some divisor $D$ of $a^{(i)}_{G,X}/W^{(i)}_{G,X}$, then (in the notation of Proposition 5.2) $D = D_s$ for some reflection $s \in W^{(i)}_{G,X}$. By assertion 5 of Lemma 2.30, if $s$ is a reflection in $W^{(i)}_{G,X}$, then $s$ is also a reflection in $W^{(i)}_{G,X}$ for some $i$ and vice versa. If $W^{(i)}_{G,X}$ is generated by reflections, then $\hat{\psi}_{G,X}$ is identified with the product of the morphisms $\hat{\psi}_{G_i,X_i}$. Now the proof is straightforward. \hfill \Box

\textbf{Proof of Theorem 5.9.} Applying Proposition 5.10, we reduce the proof to the case when $V$ satisfies the equivalent conditions of Lemma 2.22. Further, thanks to Proposition 5.11, we may (and will) assume that $G$ is simple.

Suppose $V$ is not untwisted. By Proposition 2.32, $\hat{\psi}_{G,V}$ is not smooth in codimension 1. By Proposition 5.2, in the notation of that proposition, for some $s \in W^{(i)}_{G,V}$ there is a point $x \in \hat{\psi}_{G,V}^{-1}(D_s)$ satisfying conditions (a)-(f) with $W^{(i)}_{G,X} = \{1\}$. From Remark 4.4 it follows that one can take $(M,M)$ for $\hat{G}$. Note that $\hat{g} \cong \mathfrak{sl}_2$. By Proposition 4.9, $g_x = \mathfrak{sl}_2, V/(g_x + V^0) \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ (here $\mathbb{C}^2$ denotes the irreducible two-dimensional $\mathfrak{sl}_2$-module). In the proof of Proposition 4.21 we have seen that

(A) all modules $V$ containing such a point $x$ are presented in Table 4.2;

(B) if $\alpha \in \Delta(\mathfrak{g})$ is such that $S^{(\alpha)} \rightarrow \mathfrak{g} V$, then $s_{\alpha w} \not\in W^{(i)}_{G,V}$ for some $w \in W(\mathfrak{g})$.

Let us choose a point $x \in \hat{\psi}_{G,V}^{-1}(0)$ satisfying conditions (a)-(e) of Proposition 4.1. Let $\hat{X}$ be the corresponding model $G$-variety. Let us check that $(\hat{\psi}_{G,V}/G)^{-1}(0)$ is irreducible. Clearly, $(\hat{\psi}_{G,V}/G)^{-1}(0)$ is connected. From Proposition 3.5 it follows that $(\hat{\psi}_{G,V}/G)^{-1}(0)$ is smooth in codimension 2 (as a variety). Applying the Hartshorne connectedness theorem, we see that $(\hat{\psi}_{G,V}/G)^{-1}(0)$ is irreducible.

Therefore $\hat{X}$ does not depend on the choice of $x$. If $S^{(\alpha)} \rightarrow \mathfrak{g} \hat{X}$ for some $\alpha \in \Delta(\mathfrak{g})$, then $S^{(\alpha)} \rightarrow V$. By Proposition 4.6, $W^{(i)}_{G,\hat{X}}$ is $W(\mathfrak{g})$-conjugate to a subgroup in $W^{(i)}_{G,V}$. Let us check that these two groups are, in fact, $W(\mathfrak{g})$-conjugate. By (B) and Corollary 4.14, if $s_{\alpha w} \in W^{(i)}_{G,V}$ for all $w \in W(\mathfrak{g})$, the the same inclusions hold for $\hat{X}$. But both $W^{(i)}_{G,V}, W^{(i)}_{G,\hat{X}}$ are contained in Table 4.1. Therefore they are $W(\mathfrak{g})$-conjugate.

It follows from the commutative diagram of Remark 4.8 that the fiber $(\hat{\psi}_{G,V}/G)^{-1}(0)$ is smooth in $x$. By Proposition 3.3, this fiber is smooth in codimension 1. Proceeding as in the proof of assertion 1 of Theorem 1.4, we see that any fiber of $\hat{\psi}_{G,V}/G$ is normal. Applying assertions 2,3 of Theorem 1.4, we see that $V$ is untwisted. \hfill \Box

\textbf{First alternative proof of Theorem 5.8.} Suppose that (Utw2) does not hold. Let $Y$ be a prime divisor in $X$ consisting of singular points of $\hat{\psi}_{G,X}$.

\textit{Step 1}. Note that $Y$ is $\mathbb{C}^*$-stable. Therefore there is a point $x \in X_0 \cap Y$ with closed $G$-orbit. Set $H := G_x, V_0 := T_x X_0/\mathfrak{g}_x, X'_0 := G \ast H V_0, X' := T^* X_0$. As we noted in [Lo3], $X' \cong M_G(H,0, V_0 + V^0)$. By [Kn1], Satz 6.5, $W^{(i)}_{G,X}$ is conjugate to $W^{(i)}_{G,X'}$. Using Remark 4.8, we see that $X'$ does not satisfy (Utw2). So we may assume that $X'_0 \cong G \ast H V_0$. Also [Kn1], Satz 6.5, implies that $T^* \hat{X}_0$, where $\hat{X}_0 := G \ast H V_0$, is not untwisted. So we may assume that $X$ is simply connected.

\textit{Step 2}. Analogously to the proof of Theorem 5.9, we may assume that $m_G(X) = \dim G$ and $G$ is simple. In [Lo4], Section 5 (see especially Lemma 5.4.1), we checked that condition
(B) of the proof of Theorem 5.9 (with $V$ replaced by $X$) holds. Now we can proceed as in that proof. Note that [Lo4] uses results of Section 4 but not of Section 5.

Let us also present a proof that does not use the classification results of [Lo4].

Second alternative proof of Theorem 5.8. Again, we may assume that $X$ is simply connected. Recall the notation $Y$ from the previous proof.

Set $D := \psi_{G,X}(Y)$. Let us check that any irreducible component of $\tilde{\psi}_{G,X}^{-1}(D)$ is of multiplicity 2. We may assume that $Y \neq \tilde{\psi}_{G,X}^{-1}(D)$. It follows from the connectedness theorem of [Kn4], that $\tilde{\psi}_{G,X}^{-1}(z)$ is connected for any $z \in D$. Applying the Hartshorne connectedness theorem to the Cohen-Macaulay scheme $(\tilde{\psi}_{G,X}/G)^{-1}(D)$ and using the Knop connectedness theorem, we may assume that (probably, after replacing $Y$ with another component of multiplicity 2) there is an irreducible component $Y_1$ of $\tilde{\psi}_{G,X}^{-1}(D)$ such that

- $\operatorname{codim}_{Y/G}(Y_1//G) \cap (Y//G) = 1$,
- $\tilde{\psi}_{G,X}(Y_1 \cap Y)$ is dense in $D$.
- $Y_1$ is of multiplicity 1.

Choose a general point $\alpha \in D$. It follows from Proposition 3.5 that there is a point $x \in Y \cap Y_1$ satisfying the conditions (a)-(f) of Propositions 4.1, 4.3. By Proposition 5.2, $Y_1$ has multiplicity 2, contradiction.

So any component of $\tilde{\psi}_{G,X}^{-1}(D)$ has multiplicity 2. Recall that by our assumptions $X$ is simply connected. Arguing as in the last paragraph of the proof of assertion 3 of Theorem 1.4, we get a contradiction.

5.4. Some counterexamples. The following example of a Hamiltonian variety not satisfying (Irr) is due to F. Knop.

Example 5.12. Put $G = \mathbb{C}^\times, X = \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C} \times \mathbb{C}$. Choose coordinates $x_1, \ldots, x_4$ on $X$ corresponding to the above decomposition. Define the action $G : X$ by $t(x_1, x_2, x_3, x_4) = (tx_1, t^{-1}x_2, x_3, x_4)$. Put $\alpha := (x_1^2 - x_1^{-1}x_3^2)dx_1 + x_4dx_3$. Clearly, $\alpha$ is $G$-invariant. Put

$$\omega := -d\alpha = 2x_1x_2dx_1 \wedge dx_2 - 2x_1^{-1}x_3x_1 \wedge dx_3 - dx_3 \wedge dx_4.$$ 

One checks directly that $\omega$ is nondegenerate. The action $G : X$ is Hamiltonian with $\mu_{G,X}(x) = \langle \alpha, \frac{\partial}{\partial x} \rangle_x = x_1^2x_2^2 - x_3^2$. It is clear that $\mu_{G,X}^{-1}(a)$ is irreducible whenever $a \neq \{0\}$. It follows that $C_{G,X} \cong \mathfrak{g}$. On the other hand, $\mu_{G,X}^{-1}(0)$ has two connected components.

We remark that the Hamiltonian variety in Example 5.12 is the smallest one in the sense that both group and variety have the smallest possible dimensions.

Now let us present an example of a coisotropic conical model variety $X = M_G(H, \eta, V)$ that is twisted. An example, where the group $W_{G,X}^{(\cdot)}$ is not generated by reflections, can be found in [Lo2], Subsection 5.10. In the following example $W_{G,X}^{(\cdot)}$ is generated by reflections but (Utw2) does not hold. Note that this example is very similar to that from [Lo2].

Example 5.13. Put $G = \mathrm{SL}_2 \times \mathbb{C}^\times, X := M_G(\mathbb{Z}_2 \times \mathrm{SL}_2, 0, \mathbb{C}^2 \oplus \mathbb{C}^2)$, where $\mathbb{C}^2$ denotes the two-dimensional irreducible $\mathrm{SL}_2$-module with the symplectic form given by $(u, v) \mapsto \det(u, v)$ and the nontrivial element $\sigma \in \mathbb{Z}_2 \subset \mathbb{C}^\times$ acts on $\mathbb{C}^2 \oplus \mathbb{C}^2$ as follows: $\sigma(v_1, v_2) = (v_2, -v_1)$.

One easily checks that $W_{G,X}^{(\cdot)} = N_G(T)/T \cong \mathbb{Z}_2$ and (Utw2) does not hold.
6. SOME OPEN PROBLEMS

Firstly, we state two conjectures concerning property (Irr). Below $G$ is a connected reductive group.

**Conjecture 6.1.** Any conical irreducible Hamiltonian $G$-variety $X$ satisfies (Irr).

The following conjecture is a weaker version of the first one.

**Conjecture 6.2.** $X = M_G(H, \eta, V)$, where $\eta$ is nilpotent, satisfies (Irr).

In virtue of the local cross-section and symplectic slice theorems (Propositions 2.19, 2.26) one can deduce from Conjecture 6.2 that any fiber of $\psi_{G,X}$ is normal (as a variety).

Unlike the first conjecture, the second one can be reduced to some case-by-case consideration. Let us sketch the scheme of this reduction.

At first, one reduces the problem to the case when $X$ satisfies the equivalent conditions of Lemma 2.22 and then to the case when $X$ is algebraically simply connected. Here one should check that $X$ satisfies (Utw2). This will follow if one verifies the following assertion:

\[(*)\] for any $\alpha \in \Delta(\mathfrak{g})$ such that $S(\alpha) \sim_{\mathfrak{g}} X$ there is $w \in W(\mathfrak{g})$ such that $s_{w\alpha} \not\in W_{G,X}^{(1)}$.

Finally, it is enough to check $(*)$ only for some special quadruples $(G, H, \eta, V)$. By analogy with Section 7 of [Lo4], we call such quadruples quasiessential. By definition, a quadruple $(G, H, \eta, V)$ is quasiessential if $M_G(H, \eta, V)$ satisfies the equivalent conditions of Lemma 2.22 and for any ideal $h_1 \subset h$ there is $\alpha \in \Delta(\mathfrak{g})$ such that $S(\alpha) \sim_{\mathfrak{g}} M_G(H, \eta, V)$ but $S(\alpha) \not\sim_{\mathfrak{g}} M_G(H_1, \eta, V)$, where $H_1$ is a subgroup of $H$ corresponding to $h_1$. It is not very difficult to show that if $(G, H, \eta, V)$ is quasiessential, then $G$ is simple and $H$ is semisimple.

The next conjecture strengthens assertion 1 of Theorem 1.4.

**Conjecture 6.3.** Let $X = M_G(H, \eta, V)$, where $\eta$ is nilpotent, be untwisted. Then any fiber $Y$ of $\tilde{\psi}_{G,X}/G$ has symplectic singularities. This means that there is a resolution of singularities $\tilde{Y} \to Y$ such that the symplectic form on the smooth part of $Y$ is extended to some regular form on $\tilde{Y}$.

Finally, we would like to propose a conjecture giving an estimate on dimensions of fibers of $\mu_{G,X}$.

**Conjecture 6.4.** Let $X$ be an irreducible affine Hamiltonian $G$-variety. Then $\dim \mu_{G,X}^{-1}(\eta) \leq \dim X - (m_G(X) + \text{def}_G(X) + \dim G\eta)/2$.

If $X$ is the cotangent bundle of a (not necessarily affine) $G$-variety $X_0$ this conjecture follows from Vinberg’s theorem on the modality of the action of a Borel subgroup of $G$ on $X_0$, see [V1].

7. NOTATION AND CONVENTIONS

For an algebraic group denoted by a capital Latin letter we denote its Lie algebra by the corresponding small German letter. For roots and weights of semisimple Lie algebras we use the notation of [OV].
~_G_ the equivalence relation induced by an action of a group \( G \).

\( C_{G,X} \) the spectrum of the integral closure of \( \psi^*_G(\mathbb{C}[g]^G) \) in \( \mathbb{C}[X]^G \).

cork\(_G\)(\( X \)) the corank of a Hamiltonian \( G \)-variety \( X \).
def\(_G\)(\( X \)) the defect of a Hamiltonian \( G \)-variety \( X \).
\( e_\alpha \) a nonzero element of the root subspace \( g^\alpha \subset g \).
\( (f) \) the zero divisor of a rational function \( f \).
\( (G,G) \) (resp., \( [g,h] \)) the commutant of a group \( G \) (resp., of a Lie algebra \( g \)).
\( G^0 \) the connected component of unit of an algebraic group \( G \).
\( G*_HV \) the equivalence class of \((g,v)\) in \( G*_HV \).
\( Gx \) the stabilizer of \( x \in X \) under an action \( G : X \).
\( g^0 \) the root subspace of \( g \) corresponding to a root \( \alpha \).
\( g^{(A)} \) (resp., \( G^{(A)} \)) the subalgebra \( g \) generated by \( g^\alpha, \alpha \in A \cup -A \) (resp., the corresponding connected subgroup of \( G \)).
\( m_G(X) \) \( = \max_{x \in X} \dim Gx \).
\( N_G(H), (\text{resp., } N_G(h), n_g(h)) \) the normalizer of an algebraic subgroup \( H \) in an algebraic group \( G \) (resp., of a subalgebra \( h \subset g \) in \( G \), of a subalgebra \( h \subset g \) in a Lie algebra \( g \)).
\( N_G(H,Y) \) the stabilizer of \( Y \) under the action of \( N_G(H) \).
\( \text{Quot}(A) \) the fraction field of \( A \).
\( \text{rk}(G) \) the rank of an algebraic group \( G \).
\( s_\alpha \) the reflection in a Euclidian space corresponding to a vector \( \alpha \).
\( \text{Span}_F(A) \) the linear span of a subset \( A \) of a module over a field \( F \).
\( \text{tr.deg} A \) the transcendence degree of an algebra \( A \).
\( U^\perp \) the skew-orthogonal complement to a subspace \( U \subset V \) of a symplectic vector space \( V \).
\( V^g = \{v \in V | gu = 0\} \), where \( g \) is a Lie algebra and \( V \) is a \( g \)-module.
\( V(\lambda) \) the irreducible module of the highest weight \( \lambda \) over a reductive algebraic group or a reductive Lie algebra.
\( W(g) \) the Weyl group of a reductive Lie algebra \( g \).
\( X^G \) the fixed point set for an action \( G : X \).
\( X//G \) the categorical quotient for an action \( G : X \), where \( G \) is a reductive group and \( X \) is an affine \( G \)-variety.
\( Z(G), (\text{resp., } Z_g(h)) \) the center of an algebraic group \( G \) (resp., a Lie algebra \( g \)).
\( Z_G(H), (\text{resp., } Z_G(h), z_g(h)) \) the centralizer of a subgroup \( H \) in an algebraic group \( G \) (resp., of a subalgebra \( h \subset g \) in \( G \), of a subalgebra \( h \subset g \) in a Lie algebra \( g \)).
\( \alpha^\vee \) the dual root of \( \alpha \in \Delta(g) \).
\( \Delta(g) \) the root system of a reductive Lie algebra \( g \).
\( \mu_{G,X} \) the moment map for a Hamiltonian \( G \)-variety \( X \).
\( \xi_s, (\text{resp., } \xi_n) \) the semisimple (the nilpotent) part of an element \( \xi \) of an algebraic Lie algebra.
\( \xi_s \) the velocity vector field associated with \( \xi \in g \).
the system of simple roots for a reductive Lie algebra \( g \).

the (categorical) quotient morphism \( X \to X//G \).

the morphism of (categorical) quotients induced by a \( G \)-equivariant morphism \( \varphi \).

the homomorphism \( \mathbb{C}[X_2] \to \mathbb{C}[X_1] \) induced by a morphism \( \varphi : X_1 \to X_2 \).

the natural morphism \( X \to C_{G,X} \).

the natural morphism \( X \to a_{G,X}^{(1)}/W_{G,X}^{(1)} \).

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