BULK UNIVERSALITY FOR NON-HERMITIAN RANDOM MATRICES

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Abstract. We prove the non-Hermitian analogue of the celebrated Wigner-Dyson-Mehta bulk universality phenomenon, i.e. we show that in the bulk the local eigenvalue statistics of a large random matrix with independent, identically distributed centred entries are universal, in particular they asymptotically coincide with those of the Ginibre ensemble in the corresponding symmetry class. The analogous result in the edge regime was proven recently in [12].

1. Introduction

Consider a large $n \times n$ matrix $X$ with independent, identically distributed (i.i.d.) centred entries with variance $n^{-1}$. According to the circular law [5, 22, 6, 25], the spectrum of $X$ converges to the unit disk in the complex plane with uniform spectral density. The typical distance between nearby eigenvalues is $n^{-1/2}$. We consider the eigenvalue point process after rescaling it by a factor of $n^{1/2}$ around a fixed point $z_0 \in \mathbb{C}$, $|z_0| < 1$. In case of the Ginibre ensemble, i.e. if the entries of $X$ are Gaussian, all correlation functions of this rescaled point process can be computed explicitly, for both the real and the complex case, in the $n \to \infty$ limit, see Remark 2.4. Beyond the Gaussian case no explicit formulas are available, but the outstanding conjecture asserts that the local eigenvalue statistics are given by exactly the same formulas for essentially any distribution of the matrix elements. In this paper we prove this conjecture in the bulk regime. The analogous result at the edge of the spectrum, $|z_0| = 1$, has been obtained recently in [12] relying on supersymmetric methods to obtain a lower tail estimate for the lowest singular value of $X$ [13]. Prior to our works, these universality conjectures have only been proven under the restriction that the first four moments of the common distribution of the matrix elements (almost) match the first four moments of the standard Gaussian [27]. Matching the second moment amounts to a simple rescaling, but the requirement of matching any higher moments was an artefact of the proof. In the current work we remove this condition.

Local spectral universality questions have been motivated by Eugene Wigner’s pioneering idea to model spectral statistics of complex quantum systems by those of simple random matrix ensembles that respect the basic symmetries but otherwise may not resemble at all to the initial quantum Hamiltonian. The original Wigner-Dyson-Mehta (WDM) conjecture [24] concerned Hermitian random matrix ensembles, most prominently the Wigner ensemble that is characterized by i.i.d. entries (up to the Hermitian symmetry). Since the resolution of the WDM conjecture about ten years ago via the three-step strategy (see [17, 18] for an overview of the major steps and references), in the recent years many local spectral universality results have been obtained for random matrix ensembles.

\textit{Key words and phrases.} Ginibre ensemble, Girko’s formula, Bulk universality.

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of increasing generality. However, apart from [27] and [12] all results have been restricted to Hermitian ensembles.

The main reason why the three-step strategy has not yet been extended beyond the Hermitian case is the lack of a good analogue of the celebrated Dyson Brownian Motion (DBM), a system of stochastic differential equations for the eigenvalues under a natural matrix flow. The DBM is the essential core of the three-step strategy; its fast convergence to local equilibrium is the ultimate mechanism behind universality. This dynamical approach is extremely robust since it not only detects universality but also induces it. Unfortunately, the non-Hermitian analogue of the DBM [9, Appendix A] involves overlaps of eigenvectors as well, making the rigorous analysis extremely complicated and currently beyond reach.

In the current work, similarly to our edge universality proof [12], we circumvent the non-Hermitian DBM. As standard in non-Hermitian spectral analysis, we use Girko’s formula [22] in the form given in [27] that expresses linear eigenvalue statistics of $X$ in terms of resolvents of a family of $2n \times 2n$ Hermitian matrices

$$H^z := \begin{pmatrix} 0 & X - z \\ X^* - \overline{z} & 0 \end{pmatrix}$$

parametrized by $z \in \mathbb{C}$. This formula asserts that

$$\sum_{\sigma \in \text{Spec}(X)} f(\sigma) = -\frac{1}{4\pi} \int_{\mathcal{C}} \Delta f(z) \int_0^\infty \text{Tr} G^z(i\eta) d\eta dz$$

for any smooth, compactly supported test function $f$, where $G^z(w) := (H^z - w)^{-1}$ is the resolvent of $H^z$. The key point is that we are back to the Hermitian world and all tools and results developed for Hermitian ensembles in the last years are available.

Utilizing Girko’s formula requires a very good understanding of the resolvent of $H^z$ along the imaginary axis for all $\eta > 0$. A posteriori, our proof shows that only the regime $\eta \sim n^{-1}$ is relevant for the local eigenvalue statistics of $X$, but a priori we need to control all scales. On very small scales $\eta \ll n^{-1}$, there are no eigenvalues, hence $\text{Tr} G^z$ is negligible. On the scale $\eta \sim n^{-1}$ we rely on the result from [11] on the universality of the few small singular values of $X - z$. Above this microscopic scale, i.e. for $\eta \gg n^{-1}$, the resolvent becomes deterministic and it obeys a type of law of large numbers, called the local law in this context. However, the typical error term in the local law is of order $1/(\eta n)$ and it is not sufficient to guarantee that this regime has a negligible contribution to (1.2). On the other hand, we need to control $\text{Tr} G^z$ only in distribution, hence we can use further cancellations within Green function comparison arguments. We remark that ideas based solely on local laws were sufficient for the edge proof in [12], no Hermitian universality result was needed.

Apart from relying on [11], for the bulk, we need to use two additional ideas. First, for larger $\eta$, the dependence of the resolvent $G^z(i\eta)$ on $z$ is weaker, so we may use the cancellation originating from $\int_{\mathcal{C}} \Delta f(z) dz = 0$. Second, we use a stochastic flow applied to the resolvent which leads to the stochastic advection equation studied by several authors recently [23, 8, 28, 29, 19, 3]. The key mechanism in these works is that controlling the time dependent resolvent along the characteristics of the Burgers equation one may relate the original resolvent $G^z(i\eta)$ at a small $\eta$ to a modified resolvent $\tilde{G}^z(i\eta)$ at a much larger $\tilde{\eta} \gg \eta$, which can then be more efficiently bounded by the local law. In fact, the $\eta \gg n^{-1}$ regime is further split into two parts: for $\eta \gg n^{-1/2}$ we can use a Green function comparison argument along an Ornstein-Uhlenbeck (OU) flow up to infinite time to connect the original matrix directly with a Ginibre matrix since in this regime the error in the local law is small. For $n^{-1} \ll \eta \ll n^{-1/2}$ we first have to use the stochastic advection equation for short time to increase $\eta$ above $n^{-1/2}$ and then we can use again the OU flow up to infinite time.
Notations and conventions. We introduce some notations we use throughout the paper. For any $k \in \mathbb{N}$ we use the notation $[k] := \{1, \ldots, k\}$. We write $\mathbb{D}$ for the unit disk, $\mathbb{H}$ for the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} | \Im z > 0\}$, and for any $z \in \mathbb{C}$ we use the notation $dz := \frac{1}{2}(dz \wedge d\overline{z})$ for the two dimensional volume form on $\mathbb{C}$. For any $2n \times 2n$ matrix $A$ we use the notation $(A) := (2n)^{-1} \text{Tr} A$ to denote the normalized trace of $A$. For positive quantities $f, g$ we write $f \lesssim g$ and $f \sim g$ if $f \leq C g$ or $c g \leq f \leq C g$, respectively, for some constants $c, C > 0$ which depends only on the constants appearing in (2.1). For non-negative functions $f(B_1, B_2), g(B_1, B_2)$ we use the notation $f \leq_B g$ if there exist constants $C(B_1)$ such that $f(B_1, B_2) \leq C(B_1) g(B_1, B_2)$ for all $B_1, B_2$. We denote vectors by bold-faced lower case Roman letters $\mathbf{x}, \mathbf{y} \in \mathbb{C}^k$, for some $k \in \mathbb{N}$. Vector and matrix norms, $\|x\|$ and $\|A\|$, indicate the usual Euclidean norm and the corresponding induced matrix norm. Moreover, for a vector $\mathbf{x} \in \mathbb{C}^k$, we use the notation $d\mathbf{x} := dx_1 \ldots dx_k$.

We will use the concept of “with very high probability” meaning that for any fixed $D > 0$ the probability of the event is bigger than $1 - n^{-D}$ if $n \geq n_0(D)$. Moreover, we use the convention that $\xi > 0$ denotes an arbitrary small constant.

We use the convention that quantities without tilde refer to a general matrix with i.i.d. entries, whilst any quantity with tilde refers to the Ginibre ensemble, e.g. we use $X, \{\sigma_i\}_{i=1}^n$ to denote a non-Hermitian matrix with i.i.d. entries and its eigenvalues, respectively, and $\tilde{X}, \{\tilde{\sigma}_i\}_{i=1}^n$ to denote their Ginibre counterparts.

2. Model and main results

We consider real or complex i.i.d. matrices $X$, i.e. matrices whose entries are independent and identically distributed as $x_{ab} \overset{d}{=} n^{-1/2} \chi$ for a (real or complex) random variable $\chi$. We require that the random variable $\chi$ satisfies the following two assumptions.

Assumption 2.1. We assume that $\mathbb{E}\chi = 0$ and $\mathbb{E}|\chi|^2 = 1$. In the complex case we also assume $\mathbb{E}\chi^2 = 0$ (this holds, for example, if $\Re \chi$ and $\Im \chi$ are i.i.d.). In addition, we assume the existence of high moments, i.e. that there exist constants $C_p > 0$ for each $p \in \mathbb{N}$, such that

\begin{equation}
\mathbb{E}|\chi|^p \leq C_p.
\end{equation}

Assumption 2.2. There exist $\alpha, \beta > 0$ such that the probability density $g : \mathbb{R} \to [0, \infty)$ of the random variable $\chi$ satisfies

\begin{equation}
g \in L^{1+\alpha}(\mathbb{R}), \quad \|g\|_{1+\alpha} \leq n^{\beta},
\end{equation}

where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ in the real and complex case, respectively.

Remark 2.3. We remark that we assume (2.2) only to control the probability that the smallest singular value of $X - z$ is in a very small regime close to zero, say $[0, n^{-l}]$ for some large $l > 0$, in Lemma 3.3. The assumptions in (2.2) are not used anywhere else in the paper.

We denote the eigenvalues of $X$ by $\sigma_1, \ldots, \sigma_n \in \mathbb{C}$, and define the $k$-point correlation function $p_k^{(n)}$ of $X$ implicitly as

\begin{equation}
\int_{\mathbb{C}^k} F(z_1, \ldots, z_k)p_k^{(n)}(z_1, \ldots, z_k) dz_1 \ldots dz_k = \left(\begin{array}{c} n \\ k \end{array}\right) E \sum_{i_1, \ldots, i_k} F(\sigma_{i_1}, \ldots, \sigma_{i_k}),
\end{equation}

for any smooth compactly supported test function $F : \mathbb{C}^k \to \mathbb{C}$, with $i_j \in \{1, \ldots, n\}$ for $j \in \{1, \ldots, k\}$ all distinct. For the important special case when $\chi$ follows a standard real or complex Gaussian distribution, we denote the $k$-point function of the Ginibre matrix $X$ by $p_k^{(n, \text{Gin}(F))}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. The circular law implies that the 1-point function converges

\[
\lim_{n \to \infty} p_1^{(n)}(z) = \frac{1}{\pi} \mathbb{1}(z \in \mathbb{D}) = \frac{1}{\pi} \mathbb{1}(|z| \leq 1)
\]
to the uniform distribution on the unit disk. On the scale $n^{-1/2}$ of individual eigenvalues the scaling limit of the $k$-point function has been explicitly computed in the case of complex and real Ginibre matrices, $X \sim \text{Gin}(\mathbb{R}), \text{Gin}(\mathbb{C})$, i.e. for any fixed $z_1, \ldots, z_k \in \mathbb{C}$ there exist scaling limits $p^{(\infty)}_{z_1, \ldots, z_k} = p^{(\infty, \text{Gin}(\mathbb{F}))}_{z_1, \ldots, z_k}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ such that

$$
\lim_{n \to \infty} p^{(\infty, \text{Gin}(\mathbb{F}))}_{n} (z_1 + \frac{w_1}{n^{1/2}}, \ldots, z_k + \frac{w_k}{n^{1/2}}) = p^{(\infty, \text{Gin}(\mathbb{F}))}_{z_1, \ldots, z_k} (w_1, \ldots, w_k).
$$

**Remark 2.4.** The $k$-point correlation function $p^{(\infty, \text{Gin}(\mathbb{F}))}_{z_1, \ldots, z_k}$ of the Ginibre ensemble in both the complex and real cases $\mathbb{F} = \mathbb{C}, \mathbb{R}$ is explicitly known; see [21] and [24] for the complex case, and [7, 14, 20] for the real case, where the appearance of $n^{-1/2}$ real eigenvalues causes a singularity in the density. In the complex case $p^{(\infty, \text{Gin}(\mathbb{C}))}_{z_1, \ldots, z_k}$ is determinantal, i.e. for any $w_1, \ldots, w_k \in \mathbb{C}$ it holds

$$
p^{(\infty, \text{Gin}(\mathbb{C}))}_{z_1, \ldots, z_k} (w_1, \ldots, w_k) = \det \left( K^{(\infty, \text{Gin}(\mathbb{C}))}_{z_1, z_j} (w_i, w_j) \right)_{1 \leq i, j \leq k}
$$

where for any complex numbers $z_1, z_2, w_1, w_2$ the kernel $K^{(\infty, \text{Gin}(\mathbb{C}))}_{z_1, z_2} (w_1, w_2)$ is defined by

(i) For $z_1 \neq z_2$, $K^{(\infty, \text{Gin}(\mathbb{C}))}_{z_1, z_2} (w_1, w_2) = 0$.

(ii) For $z_1 = z_2$ and $|z_1| > 1$, $K^{(\infty, \text{Gin}(\mathbb{C}))}_{z_1, z_2} (w_1, w_2) = 0$.

(iii) For $z_1 = z_2$ and $|z_1| < 1$,

$$
K^{(\infty, \text{Gin}(\mathbb{C}))}_{z_1, z_2} (w_1, w_2) = \frac{1}{\pi} e^{-\frac{|w_1|}{2} - \frac{|w_2|}{2} + w_1 \overline{w_2}},
$$

(iv) For $z_1 = z_2$ and $|z_1| = 1$,

$$
K^{(\infty, \text{Gin}(\mathbb{C}))}_{z_1, z_2} (w_1, w_2) = \frac{1}{2\pi} \left[ 1 + \text{erf} \left( -\sqrt{2} (z_1 \overline{w_2} + w_1 z_2) \right) \right] e^{-\frac{|w_1|}{2} - \frac{|w_2|}{2} + w_1 \overline{w_2}},
$$

where

$$
\text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_{\gamma_z} e^{-t^2} dt,
$$

for any $z \in \mathbb{C}$, with $\gamma_z$ any contour from $0$ to $z$.

For the corresponding much more involved formulas for $p^{(\infty, \text{Gin}(\mathbb{R}))}_{z_1, \ldots, z_k}$ we refer the reader to [7].

Our main result is the universality of $p^{(\infty, \text{Gin}(\mathbb{R}, \mathbb{C}))}_{z_1, \ldots, z_k}$ in the bulk. In particular we show, that the bulk-scaling limit of $p^{(\infty)}_{z_1, \ldots, z_k}$ agrees with the known scaling limit of the corresponding real or complex Ginibre ensemble.

**Theorem 2.5 (Bulk universality).** Let $X$ be an i.i.d. $n \times n$ matrix with real or complex entries that satisfy Assumption 2.1 and 2.2. Then, for any fixed integer $k \geq 1$, and complex spectral parameters $z_1, \ldots, z_k$ such that $|z_j| < 1$, $j = 1, \ldots, k$, and for any compactly supported smooth function $F : \mathbb{C}^k \to \mathbb{C}$, we have the bound

$$
\int_{\mathbb{C}^k} F(w) \left[ p^{(n)}_{k} \left( z + \frac{w}{\sqrt{n}} \right) - p^{(\infty, \text{Gin}(\mathbb{F}))}_{k} (w) \right] dw = O \left( \sqrt{n} \right),
$$

where the constant in $O(\cdot)$ may depend on $k$ and $F$, and $c > 0$ is a small constant depending on $k$ and the $C^\infty$-norm of $F$.

**Proof strategy.** For the proof of Theorem 2.5 we study the Hermitization $H^z$ of $X - z$ defined by

$$
H^z := \begin{pmatrix} 0 & X - z \\ \overline{X^* - z} & 0 \end{pmatrix}.
$$

Similarly, we define $\tilde{H}^z$ replacing $X$ by $\tilde{X}$. In the following by $G^z, \tilde{G}^z$ we denote the resolvent of $H^z$ and $\tilde{H}^z$, respectively, i.e. $G^z(w) := (H^z - w)^{-1}$ and $\tilde{G}^z(w) := (\tilde{H}^z - w)^{-1}$, with $w \in \mathbb{H}$. Note that the block structure of $H^z$ induces a symmetric spectrum around
0, i.e. $\lambda_i^2 = -\lambda_i^2$ for $i = 1, \ldots, n$, where $\{\lambda_i\}_{i=1}^n$ denote the eigenvalues of $H^z$ ordered as $\lambda_{i,n} \leq \ldots \lambda_{i,1} \leq 0 \leq \lambda_{i,2} \leq \ldots \lambda_{i,n}$. Note that the positive eigenvalues $\{\lambda_i^2\}_{i=1}^n$ of $H^z$ are exactly the singular values of $X - z$.

In the limit $n \to +\infty$ the resolvent of $H^z$ becomes deterministic and its limit can be found by solving the scalar equation

$$(2.6) \quad - \frac{1}{m^z} = w + \tilde{m}^z - \frac{|z|^2}{w + \tilde{m}^z}, \quad \tilde{m}^z(w) \in \mathbb{H}, \quad w \in \mathbb{H},$$

which is a special case of the matrix Dyson equation (MDE), see e.g. [2]. On the imaginary axis $\tilde{m}^z(i\eta) = i\tilde{m}^z(i\eta)$. Then for $\eta > 0$ we define

$$u = u^z(i\eta) := \frac{\Im \tilde{m}^z(i\eta)}{\eta + m^z(i\eta)}, \quad M = M^z(i\eta) := \begin{pmatrix} \tilde{m}^z(i\eta)I & -zu(\eta)I \\ -zu(\eta)I & \tilde{m}^z(i\eta)I \end{pmatrix},$$

where $I$ is the $n \times n$ identity matrix. Moreover,

$$(2.7) \quad u^z(i\eta) \lesssim 1, \quad \|M^z(i\eta)\| \lesssim 1, \quad \|(M^z)^*(i\eta)\| \lesssim 1$$

hold uniformly in $z$ as long as $|z| \leq 1 - \tau$ for some fixed $\tau > 0$. Define

$$\rho^x(x) := \frac{1}{\pi} \Im \tilde{m}^z(x + i0)$$

to be the self consistent density of states (scDos) of $H^z$. By [4, Proposition 3.2] we have the bound $\Im \tilde{m}^z(i\eta) \sim |1 - |z|^2|^{1/2} + \eta^{1/3} \lesssim 1$ for $|z| < 1$, also implying that $\rho^x(0) \geq c_\tau$ uniformly for $|z| \leq 1 - \tau$, for some positive constant $c_\tau$, depending only on $\tau$. The bound on the derivative (2.7) and analyticity implies that $\tilde{m}^z(w)$ is also regular in $\Re w$, in particular

$$(2.8) \quad \rho^x(x) \geq c_\tau$$

uniformly for $|z| \leq 1 - \tau$ and $|x| \leq c_\tau'$ with some positive constants $c_\tau'$ and $c_\tau''$ depending only on $\tau$. In particular, if $z$ is inside the bulk of the circular law, then an entire neighborhood of 0 is in the bulk of the spectrum of $H^z$. We may sometimes drop the $z$-dependence of $\tilde{m}^z$, $M^z$, etc. in the notation.

The main inputs for the proof of Theorem 2.5 are the following three propositions. The first one is the optimal local law for $G^z$ in Proposition 2.6. The averaged local law in (2.10) and the entry-wise local law (choosing $x$ and $y$ being the coordinate vectors in (2.9)) have been proven in [4, Theorem 5.2]. We defer the proof, using the averaged and the entry-wise local law as an input, of the isotropic local law in (2.9) to Appendix A.

**Proposition 2.6** (Local law for $G^z$ on the imaginary axis). Let $X$ be an i.i.d. $n \times n$ matrix, whose entries satisfy Assumption 2.1 and 2.2, and let $H^z$ as in (2.5). Then for any deterministic vectors $x, y$ and matrix $R$, and any $\xi > 0$, $\tau > 0$ we have the bound

$$(2.9) \quad |\langle x, (G^z(i\eta) - M^z(i\eta))y \rangle| \leq n^\xi \|x\| \|y\| \left( \frac{1}{\sqrt{n\eta}} + \frac{1}{n\eta} \right)$$

$$(2.10) \quad |\langle R(G^z(i\eta) - M^z(i\eta)) \rangle| \leq \frac{n^\xi \|R\|}{n\eta},$$

uniformly in $|z| \leq 1 - \tau$ and $\eta > 0$ with very high probability, as long as $n$ is sufficiently large, $n \geq n_0$, where $n_0$ is uniform in $z$, it depends only on $\tau, \xi$ and the control parameters in Assumption 2.1 and 2.2.

The second ingredient in order to prove universality for the small singular values of $X - z$ and $\tilde{X} - z$ is the averaged local law in Proposition 2.7 (which immediately follows by [10, Theorem 3.4] using that $2\sqrt{n} \text{Tr}(X - z)^*(X - z) - w)^{-1} = \text{Tr}(H^z - \sqrt{w})^{-1}$ for $\Re w, \Im w > 0$, choosing the branch of $\sqrt{w}$ with $3\sqrt{w} > 0$) that holds true not only along the imaginary axis but also for all spectral parameters $w$ with $|\Re w| \leq C$, for some fixed constant $C > 0$. 

| User-Supplied Notes |
Proposition 2.7 (Averaged bulk law for $G^z$). Fix $\tau > 0$ and consider $z \in \mathbb{C}$, with $|z| \leq 1 - \tau$. Let $X$ be an i.i.d. $n \times n$ matrix, whose entries satisfy Assumption 2.1 and 2.2, and let $\{\lambda_i^z\}_{i=1}^n$ be the singular values of $X - z$. Then for any $\xi, \epsilon > 0$ we have the bound

\begin{equation}
\label{2.11}
|\langle G^z(\mathrm{i}n) - M^z(\mathrm{i}n) \rangle| = \left| \frac{1}{2n} \sum_{i=1}^n \left[ \frac{1}{\lambda_i^z - w} - \frac{1}{\lambda_i^z + w} \right] - \tilde{m}^z(w) \right| \leq \frac{n^\xi}{n^{3\epsilon} w},
\end{equation}

uniformly in $|z| \leq 1 - \tau$ and $w \in \{w \in \mathbb{C} : |\Re w| \leq C, \Im w \geq n^{-1-\epsilon}\}$, for some fixed $C > 0$, with very high probability, as long as $n$ is sufficiently large, $n \geq n_0$, where $n_0$ is uniform in $z$, it depends only on $\tau, \xi, \epsilon, C$ and the control parameters in Assumption 2.1 and 2.2.

We define the quantiles $\gamma_i^z$ of the scDos $\rho^z(x)$ by

\begin{equation}
\label{2.12}
\frac{i}{n} = \int_0^{\gamma_i^z} \rho^z(x) \, dx, \quad 1 \leq i \leq n.
\end{equation}

Using standard arguments, from Proposition 2.7 and (2.8) we conclude the following rigidity bound:

Corollary 2.8 (Singular values rigidity). Fix $\tau > 0$. Let $X$ be an $n \times n$ matrix with i.i.d. entries, and let $\lambda_i^z$ be the singular values of $X - z$, and let $\gamma_i^z$ be the quantiles defined in (2.12), then

\begin{equation}
\label{2.13}
|\lambda_i^z - \gamma_i^z| \leq \frac{n^\xi}{n}, \quad 1 \leq i \leq cn,
\end{equation}

for some small fixed constant $c = c_\tau > 0$, uniformly in $|z| \leq 1 - \tau$, with very high probability for any $\xi > 0$ and for $n \geq n_0$.

The Hermitized matrix $H^z$ is related to the eigenvalues $\sigma_i$ of $X$ via Girko’s Hermitization formula (1.2) in the form:

\begin{equation}
\label{2.14}
\frac{1}{n} \sum_{i=1}^n f_{\sigma_0}(\sigma_i) = \frac{1}{4\pi n} \int_\mathbb{C} \Delta f_{\sigma_0}(z) \log |\det H^z| \, dz = -\frac{1}{4\pi n} \int_\mathbb{C} \Delta f_{\sigma_0}(z) \int_{-\infty}^{+\infty} \Im \text{Tr} G^z(\mathrm{i}n) \, d\eta \, dz,
\end{equation}

for rescaled test functions $f_{\sigma_0}(z) := n f (\sqrt{n}(z - z_0))$ around a fixed reference point $z_0 \in \mathbb{C}$, where $f : \mathbb{C} \to \mathbb{C}$ is smooth and compactly supported.

As third ingredient, in order to control the eigenvalues of $H^z$ on the critical scale $\sim n^{-1}$ in Girko’s formula in (2.14), we use the universality for the smallest $n^\omega$ eigenvalues, for some small fixed $\omega > 0$, stated in Proposition 2.9. Define $V_1 = X - z$ and $V_2 = \tilde{X} - z$, then Proposition 2.7 implies that $V_1, V_2$ satisfy the properties in [11, Definition 3.1]. Hence, applying [11, Theorem 3.2] with initial conditions $V_1$ and $V_2$, choosing the same coupling for the DBM flows of the singular of $V_1, V_2$ and of the comparison Ginibre ensemble, we conclude the following proposition.

Proposition 2.9. Fix $\tau > 0$ and consider $z \in \mathbb{C}$, with $|z| \leq 1 - \tau$. Let $X$ be an $n \times n$ matrix with real or complex i.i.d. entries and $\tilde{X}$ be an $n \times n$ real or complex Ginibre matrix. For any $t \geq 0$, denote by $\{\lambda_i^z(t)\}_{i=1}^n$ and $\{\mu_i^z(t)\}_{i=1}^n$ the singular values of the matrices $X - z + n^{-1/2}B_1^{(1)}$ and $\tilde{X} - z + n^{-1/2}B_1^{(2)}$, respectively, with $B_1^{(1)}$ and $B_1^{(2)}$ matrix valued standard real or complex Brownian motions. Fix $\omega_0 > 0$, then there exist constants $\omega, \omega_0 > 0$ and a coupling of the processes $\{\lambda_i^z(t)\}_{i=1}^n, \{\mu_i^z(t)\}_{i=1}^n$ such that

\begin{equation}
\label{2.15}
|\lambda_i^z(t_\omega) - \mu_i^z(t_\omega)| \leq n^{-1-\omega}, \quad 1 \leq i \leq n^\omega,
\end{equation}

with very high probability, where $t_\omega := n^{-1+\omega_0}$, if $n$ is sufficiently large, $n \geq n_0$, where $n_0$ is uniform in $z$, it depends only on $\tau$ and the control parameters in Assumption 2.1 and 2.2.
To avoid misunderstanding, we point out some imprecisions in the statement of [11, Theorem 3.2], the formulation above already remedied them. First, the authors apparently forgot to match the density of eigenvalues at 0 for the two initial matrices \( V \) and \( W \) (using their notation). Second, \( \nu_i \)'s in their Definition 3.1 are undefined, but from the context they are the eigenvalues of \( V \). Finally either a multiplicative constant is missing in the main result [11, (3.6)] or the statement holds only for sufficiently large \( N \geq N_0 \). In both cases, as it follows from their proof, the hidden constant is uniform in the control parameters in [11, Definition 3.1].

Finally, to control the very small \( \eta \ll n^{-1} \) regime in (2.14) we need a lower tail estimate on the lowest singular value of \( X-z \):

**Corollary 2.10** (Tail estimate for \( \lambda_1 \)). Fix \( \tau > 0 \) and \( \delta > 0 \). Consider \( z \in \mathbb{C} \), with \( |z| \leq 1 - \tau \). Then there exists a constant \( c > 0 \) such that

\[
\mathbb{P} \left( \lambda_1^z \leq n^{-1-\delta} \right) \leq n^{-c\delta}
\]

uniformly in \( |z| \leq 1 - \tau \), where \( \lambda_1 \) denotes the smallest singular value of \( X-z \).

This result follows from [26, Theorem 3.2] with the choice of \( A = \delta, \gamma = \frac{1}{2} + \delta \). Alternatively, one can also prove (2.16) by noting that this bound for the Gaussian case, i.e. for the smallest eigenvalue \( \mu_1 \) of \( \tilde{H} \) has been proven in [13, Eq. (4a)]. Next, Proposition 2.9 guarantees that the same bound holds for \( \lambda_1^\xi(t_\gamma) \). Finally, the small time \( t_\gamma \) can be removed by a simple Green function comparison theorem.

The rest of the paper contains the proof of the main result, Theorem 2.5, and it is organised as follows: In Section 3 we give some \textit{a priori} bounds and estimate the \( \eta \)-regimes in Girko’s formula in (2.14) that are either “very small” (\( \eta \ll n^{-1} \)) or “very large” (\( \eta \gg n^{1+\epsilon} \)) by using the local law in Proposition 2.6 and the bound for the smallest singular value of \( X-z \) in Corollary 2.10; these are fairly standard steps. In Section 4 we handle the important microscopic regime \( \eta \in [n^{-1-\epsilon}, n^{-1+\epsilon}] \) by using Proposition 2.9 and we show how to separate the contributions of this regime in Girko’s formula from the mesoscopic scales \( \eta \geq n^{-1+\epsilon} \). Section 5 is devoted to the mesoscopic scales by using a long time Green function comparison argument, combined with the stochastic advection flow. In Appendix A we give the proof of the isotropic version of the local law, Proposition 2.6.

**3. Preliminary reductions**

In order to resolve the eigenvalues on their natural scale, we define the rescaled test function

\[
f_{\eta_0}(z) := n f(\sqrt{n}(z-z_0)),
\]

for any fixed \( |z_0| < 1 \) and \( z \in \mathbb{C} \). From now on we fix the scales

\[
\eta_0 := n^{-1-\delta_0}, \quad \eta_1 := n^{-1+\delta_1}, \quad \eta_2 := n^{-1/2+\delta_2},
\]

for some fixed \( \delta_0, \delta_1, \delta_2 > 0 \) whose interrelations will be given later, but the reader can keep in mind that \( \delta_0 \leq \frac{1}{100} \delta_1 \leq \frac{1}{1000} \delta_2 \).

To prove bulk universality in Theorem 2.5, by inclusion-exclusion principle, it is enough to prove the following proposition for any fixed \( k \in \mathbb{N} \).

**Proposition 3.1.** Let \( k \in \mathbb{N} \) and fix \( z_1, \ldots, z_k \) such that \( |z_j| < 1 \) for all \( j \in [k] \), and let \( f^{(1)}, \ldots, f^{(n)} \) be smooth compactly supported test functions. Let \( X \) satisfy Assumptions 2.1 and 2.2. Denote the eigenvalues of \( X \) and of a complex Ginibre matrix \( \tilde{X} \) by \( \{\sigma_i\}_{i=1}^n \) and \( \{\tilde{\sigma}_i\}_{i=1}^n \), respectively. Then we have the bound

\[
\mathbb{E} \left[ \prod_{j=1}^k \left( \frac{1}{n} \sum_{i=1}^n f^{(j)}(\sigma_i) - \frac{1}{\pi} \int \gamma dz \right) \right] = \mathcal{O} \left( n^{-c(k)} \right),
\]
for some small constant $c(k) > 0$, where the implicit multiplicative constant in $\mathcal{O}(\cdot)$ depends on the norms $\|\Delta f^{(j)}\|_1$, $j = 1, \ldots, k$.

For any $j \in [k]$ we split the $\eta$-integration in Girko’s formula (2.14) as follows:

\[(3.4) \quad \frac{1}{n} \sum_{i=1}^{n} f^{(j)}_{s_j}(\sigma_i) - \frac{1}{\pi} \int_{\mathbb{D}} f^{(j)}_{s_j}(z) \, dz = \frac{1}{4\pi n} \int_{\mathbb{C}} \Delta f^{(j)}_{s_j}(z) \log |\det(H^s - i\tau)| \, dz
\]

\[\quad - \frac{1}{2\pi} \int_{\mathbb{C}} \Delta f^{(j)}_{s_j}(z) \int_{\eta_0}^{\eta_1} [(3G^s(\eta)) - \Im \tilde{m}^s(\eta)] \, d\eta \, dz
\]

\[\quad - \frac{1}{2\pi} \int_{\mathbb{C}} \Delta f^{(j)}_{s_j}(z) \int_{\eta_0}^{\eta_1} [(3G^s(\eta)) - \Im \tilde{m}^s(\eta)] \, d\eta \, dz
\]

\[\quad - \frac{1}{2\pi} \int_{\mathbb{C}} \Delta f^{(j)}_{s_j}(z) \int_{\eta_0}^{\eta_1} [(3G^s(\eta)) - \Im \tilde{m}^s(\eta)] \, d\eta \, dz
\]

\[\quad + \frac{1}{2\pi} \int_{\mathbb{C}} \Delta f^{(j)}_{s_j}(z) \int_{T}^{\eta_2} \left( \Im \tilde{m}^s(\eta) - \frac{1}{\eta + 1} \right) \, d\eta \, dz
\]

\[\quad =: I_{1}^{(j)} + I_{2}^{(j)} + I_{3}^{(j)} + I_{4}^{(j)} + I_{5}^{(j)} + I_{6}^{(j)},
\]

with $\eta_0, \eta_1, \eta_2$ defined in (3.2) and $T$ is very large, say $T = n^{100}$. Similarly, we define $\tilde{I}_{l}^{(j)}$ for the Ginibre ensemble, with $l = 1, \ldots, 6$. We split (3.4) in this way, since the different regimes will be treated using different techniques. In particular, $I_{2}^{(j)}, I_{3}^{(j)}$ will be estimated using universality for the eigenvalues close to 0 of the Hermitised matrix $H^s$ which easily follows from [11, Theorem 3.2]. For the term $I_{4}^{(j)}$ we use the stochastic advection flow for $d(G^s)$ in the form introduced in [23] in the bulk regime (see also [1] in the edge regime). We estimate $I_{5}^{(j)}$ via a Green’s function comparison (GFT) approach similar to [12, Lemma 4.3], whilst the integrals $I_{1}^{(j)}$ and $I_{6}^{(j)}$ will be estimated by easy direct computations.

Using the local law for $H^s$ in Proposition 2.6 to bound $I_{2}^{(j)}, I_{3}^{(j)}, I_{4}^{(j)}, I_{5}^{(j)}$, and the bounds proven in [4, Proof of Theorem 2.5] adapting the proof of [6, Lemma 4.12], for $I_{4}^{(j)}$ and $I_{6}^{(j)}$, we immediately conclude the following a priori bounds:

**Lemma 3.2.** For any $j \in [k]$ the bounds

\[(3.5) \quad \left| I_{1}^{(j)} \right| \leq \frac{n^{1+\xi} \| \Delta f^{(j)} \|_1}{T^2}, \quad \left| I_{2}^{(j)} + I_{3}^{(j)} + I_{4}^{(j)} + I_{5}^{(j)} + I_{6}^{(j)} \right| \leq n^\xi \| \Delta f^{(j)} \|_1, \quad \left| I_{6}^{(j)} \right| \leq \frac{n \| \Delta f^{(j)} \|_1}{T},
\]

hold with very high probability for any $\xi > 0$. The same bounds hold for $\tilde{I}_{l}^{(j)}$.

Moreover, by Corollary 2.10 and [4, Proposition 5.7] it follows that for $I_{2}^{(j)}$ we have an improved bound which holds true only in expectation:

**Lemma 3.3.** For any $j \in [k]$ we have that

\[\mathbb{E} \left| I_{2}^{(j)} \right| \leq n^{-c \delta},\]

for some small $c > 0$. The same bound holds for $\tilde{I}_{2}^{(j)}$ as well.

We remark that the only place throughout the paper when Assumption 2.2 is used is to ensure that [4, Proposition 5.7] is applicable to bound a very small regime close to zero, say $[0, n^{-l}]$ for some large $l > 0$, in $I_{2}^{(j)}$.

By (3.4), Lemma 3.2 and Lemma 3.3 we easily conclude the following proposition.
Proposition 3.4. For any \( k \in \mathbb{N} \) we have

\[
\mathbb{E} \left[ \prod_{j=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} f_{ij}^{(j)}(\sigma_i) - \frac{1}{n} \int_{D} f_{ij}^{(j)}(z)dz \right) - \prod_{j=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} f_{ij}^{(j)}(\tilde{\sigma}_i) - \frac{1}{n} \int_{D} f_{ij}^{(j)}(z)dz \right) \right] = \mathbb{E} \left[ \prod_{j=1}^{k} \left( I_{3j}^{(j)} + I_{4j}^{(j)} + I_{5j}^{(j)} \right) - \prod_{j=1}^{k} \left( \tilde{I}_{3j}^{(j)} + \tilde{I}_{4j}^{(j)} + \tilde{I}_{5j}^{(j)} \right) \right] + \mathcal{O} \left( n^{-c(k)} \right),
\]

for a small constant \( c(k) > 0 \).

We conclude this section with the lemma below which immediately follows by [4, Eq. (4.2)] and the fact that \( \int_{\mathbb{C}} \Delta f(z)dz = 0 \). We omit the elementary proof.

Lemma 3.5. Fix \( \tau > 0 \). Let \( \tilde{m}^z \) be the solution of (2.6), and let \( \gamma_i^z \), with \( 1 \leq i \leq n \) be the quantiles defined in (2.12), then the following holds true

\[
|\partial^k z m^z| \lesssim 1, \quad |\partial^k z \gamma_i^z| \lesssim \gamma_i^z^z,
\]

for any \( k \in \mathbb{N} \), uniformly for any \( |z| \leq 1 - \tau \). Moreover, for any smooth compactly supported test function \( f : \mathbb{C} \rightarrow \mathbb{C} \) and \( |z_0| \leq 1 - \tau \) we have that

\[
\left| \int \Delta f_{m_0}(z) \partial z m^z dz \right| \lesssim 1.
\]

All implicit constants depend only on \( \tau \), hence the estimates are uniform in \( z \) as long as \( |z| \leq 1 - \tau \).

Note that \( \int_{\mathbb{C}} |\Delta f_{m_0}| \) in absolute value would be naively bounded by \( n \).

4. Bulk universality for non-Hermitian random matrices

The main result of this section is the proof of Theorem 2.5. In particular, we prove Proposition 4.1, which, combined with Proposition 3.4, immediately concludes Theorem 2.5. We always assume that \( |z| \leq 1 - \tau \) with some fixed \( \tau > 0 \) and we will not carry the \( \tau \)-dependence. The implicit constants in all estimates below depend may depend only on \( \tau \), but they are uniform in \( z \).

Proposition 4.1. For any \( k \in \mathbb{N} \) we have that

\[
\mathbb{E} \left[ \prod_{j=1}^{k} \left( I_{3j}^{(j)} + I_{4j}^{(j)} + I_{5j}^{(j)} \right) - \prod_{j=1}^{k} \left( \tilde{I}_{3j}^{(j)} + \tilde{I}_{4j}^{(j)} + \tilde{I}_{5j}^{(j)} \right) \right] = \mathcal{O} \left( n^{-c(k)} \right),
\]

for some small constant \( c(k) > 0 \).

The main inputs for the proof of Proposition 4.1 are the following two lemmata. In Lemma 4.2 we prove that products of \( I_{3k}^{(j)} \) and products of \( \tilde{I}_{3k}^{(j)} \) are close in expectation using a long time GFT, whose proof is postponed to Section 5.

Lemma 4.2. For any \( k \in \mathbb{N} \) we have that

\[
\mathbb{E} \left[ \prod_{j=1}^{k} \left( I_{4j}^{(j)} + I_{5j}^{(j)} \right) - \prod_{j=1}^{k} \left( \tilde{I}_{4j}^{(j)} + \tilde{I}_{5j}^{(j)} \right) \right] = \mathcal{O} \left( n^{-c(k)} \right),
\]

for a small constant \( c(k) > 0 \).

Then, using the universality result in Proposition 2.9 as an input, in the following lemma we prove that \( I_{5j}^{(j)} - \tilde{I}_{5j}^{(j)} \) is small with very high probability. The proof of Lemma 4.3 is postponed to the end of this section.
Lemma 4.3. Let $k \in \mathbb{N}$, then for any non-empty subset $\mathcal{J} \subset [k]$, $\mathcal{J} \neq \emptyset$, we have

\begin{equation}
E \left[ \prod_{i_1 \in \mathcal{J}} I_3^{(i_1)} - \prod_{i_1 \in \mathcal{J}} \widetilde{I}_3^{(i_1)} \right] \times \prod_{i_2 \in \mathcal{J}^c} (I_4^{(i_2)} + I_5^{(i_2)}) = O \left( n^{-c(k)} \right),
\end{equation}

for a small constant $c(k) > 0$.

Combining Lemma 4.2 and Lemma 4.3 we prove Proposition 4.1.

Proof of Proposition 4.1. In order to apply Lemma 4.2, we write

\begin{equation}
\prod_{j=1}^{k}(I_3^{(j)} + I_4^{(j)} + I_5^{(j)}) = \sum_{\mathcal{J} \subset [k]} \prod_{i_1 \in \mathcal{J}} I_3^{(i_1)} \times \prod_{i_2 \in \mathcal{J}^c} (I_4^{(i_2)} + I_5^{(i_2)}).
\end{equation}

Similarly, we rewrite the product of $\tilde{I}_3^{(j)} + \tilde{I}_4^{(j)} + \tilde{I}_5^{(j)}$. Then, by linearity of the expectation, it is enough to prove that

\begin{equation}
E \left[ \prod_{i_1 \in \mathcal{J}} I_3^{(i_1)} \times \prod_{i_2 \in \mathcal{J}^c} (I_4^{(i_2)} + I_5^{(i_2)}) - \prod_{i_1 \in \mathcal{J}} \tilde{I}_3^{(i_1)} \times \prod_{i_2 \in \mathcal{J}^c} (\tilde{I}_4^{(i_2)} + \tilde{I}_5^{(i_2)}) \right] = O \left( n^{-c(k)} \right),
\end{equation}

for each subset $\mathcal{J} \subset [k]$. If $\mathcal{J} = \emptyset$ then (4.3) follows by Lemma 4.2, hence in the following we assume that $|\mathcal{J}| \geq 1$. We can write

\begin{align*}
E \left[ \prod_{i_1 \in \mathcal{J}} I_3^{(i_1)} \times \prod_{i_2 \in \mathcal{J}^c} (I_4^{(i_2)} + I_5^{(i_2)}) - \prod_{i_1 \in \mathcal{J}} \tilde{I}_3^{(i_1)} \times \prod_{i_2 \in \mathcal{J}^c} (\tilde{I}_4^{(i_2)} + \tilde{I}_5^{(i_2)}) \right] &= E \left[ \prod_{i_1 \in \mathcal{J}} I_3^{(i_1)} - \prod_{i_1 \in \mathcal{J}} \tilde{I}_3^{(i_1)} \right] \times E \left[ \prod_{i_2 \in \mathcal{J}^c} (I_4^{(i_2)} + I_5^{(i_2)}) - \prod_{i_2 \in \mathcal{J}^c} (\tilde{I}_4^{(i_2)} + \tilde{I}_5^{(i_2)}) \right] \\
&- E \left[ \prod_{i_1 \in \mathcal{J}} I_3^{(i_1)} - \prod_{i_1 \in \mathcal{J}} \tilde{I}_3^{(i_1)} \right] \times E \left[ \prod_{i_2 \in \mathcal{J}^c} (I_4^{(i_2)} + I_5^{(i_2)}) \right] \\
&+ E \left[ \prod_{i_2 \in \mathcal{J}^c} (I_4^{(i_2)} + I_5^{(i_2)}) \right] \times E \left[ \prod_{i_2 \in \mathcal{J}^c} (\tilde{I}_4^{(i_2)} + \tilde{I}_5^{(i_2)}) - \prod_{i_2 \in \mathcal{J}^c} (\tilde{I}_4^{(i_2)} + \tilde{I}_5^{(i_2)}) \right].
\end{align*}

Note that the expectation is decoupled in the two last terms; this holds for their first summands since $I_3^{(i)}$ and $\tilde{I}_3^{(i)}$ are independent, while the second summands cancel each other. Finally, using the high probability bounds of Lemma 3.2 and Lemma 4.2, Lemma 4.3 in the above equality we conclude (4.3). This concludes the proof of Proposition 4.1 and so of Theorem 2.5. \qed

Finally we conclude this section with the proof of Lemma 4.3.

Proof of Lemma 4.3. Consider the Ornstein-Uhlenbeck (OU) flow defined by

\begin{equation}
dX_t = -\frac{1}{2} X_t dt + \frac{dB_t}{\sqrt{n}}, \quad X_0 = X,
\end{equation}

where $B_t$ is a real or complex matrix valued Brownian motion, i.e. $B_t \in \mathbb{R}^{n \times n}$ or $B_t \in \mathbb{C}^{n \times n}$, accordingly with $X$ being real or complex, where $(b_{k})_{ab}$ in the real case, and $\sqrt{2}\mathbb{N}[(b_{k})_{ab}], \sqrt{2}\mathbb{C}[(b_{k})_{ab}]$ in the complex case are independent standard real Brownian motions for $a, b \in [n]$. The integrals $I_3^{(j)}(t)$ are defined as in (3.4) replacing $H^z$ by $H_t$, with $H_t = H_t^z$ being the Hermitization of $X_t$ - $z$. We define $\tilde{I}_3^{(j)}(t)$ exactly in the same way using another OU flow $d\tilde{X}_t$ with initial condition $\tilde{X}_0 = \tilde{X}$ and with an independent driving white noise.
Similarly to [12, Proof of Lemma 4.3], applying Green’s Function Comparison Theorem (GFT) and using (3.7) of Lemma 3.5, first along the $dX_t$ flow and then along the $d\tilde{X}_t$ flow, we conclude that

$$
(4.5) \quad \mathbb{E} \left[ \prod_{i_1 \in J} I_3^{(i_1)}(t_1) - \prod_{i_2 \in J} \tilde{I}_3^{(i_2)}(t_1) \times \prod_{i_2 \in J} (I_4^{(i_2)}(t_1) + I_5^{(i_2)}(t_1)) \right] = \mathcal{O} \left( \frac{t_1}{\sqrt{n\eta_0}} \right),
$$

with $t_1 := n^{-1+\omega_1}$, for some small fixed $\omega_1 > 0$, and $\eta_0$ being defined in (3.2). In this estimate no cancellation in the first factors between $\prod I_3^{(i_1)}$ and $\prod \tilde{I}_3^{(i_2)}$ was used, the bound (4.5) holds also for each factor separately. Note that compared with [12, Proof of Lemma 4.3] the GFT used here is in the bulk of the scDos of $H^\pm$ and not at the cusp. This accounts for the different $n$ and $\eta_0$ exponents in the error term (4.5) and in [12, Eq. (36)].

We omit the details of the proof of (4.5) since it is very similar to the more involved and delicate long time GFT argument that will be presented in Section 5 to prove Lemma 4.2. To conclude the proof of Lemma 4.3 it is therefore enough to prove that

$$
(4.6) \quad \mathbb{E} \left| I_3^{(i_1)}(t_1) - \tilde{I}_3^{(i_1)}(t_1) \right| = \mathcal{O} \left( n^{-c} \right)
$$

since (4.6) clearly implies (4.2) using linearity of the expectation and the high probability a priori bounds in Lemma 3.2 together with $\mathcal{J} \neq \emptyset$.

In the remainder of the proof we prove (4.6). We will apply Proposition 2.9, so we first construct a matrix $X_{t_1}$ such that $X_{t_1} := X_{t_1} + \sqrt{ct_1}U$, for a Ginibre matrix $U$ independent of $X_{t_1}$ and for some constant $c > 0$, and consider the flow

$$
(4.7) \quad d\tilde{X}_t = \frac{d\hat{B}_t}{\sqrt{n}}, \quad \tilde{X}_0 = X_{t_1}^#,
$$

where $\hat{B}_t$ denotes a standard matrix valued real or complex Brownian motion. In particular, the solution of (4.7) is such that after a time $t = ct_1$ we have $\tilde{X}_{ct_1} = X_{t_1}$, with $X_{t_1}$ the solution of (4.4). We repeat the same construction for the process with initial condition $\tilde{X}_{t_1}$. In order to prove (4.6), we need to compare the singular values of $X_t$ with its Gaussian counterparts.

The flow $\tilde{X}_t$ defined in (4.7) and its tilde counterpart induce the following flows for the singular values $\{\lambda_i(t)\}_{i=1}^n$, $\{\mu_i(t)\}_{i=1}^n$ of $\tilde{X}_t - z$ and $\tilde{X}_t - z$: (see, e.g. [16, Eq. (5.8)], [11, Eq. (3.7)])

$$
(4.8) \quad dr_i(t) = \sqrt{\frac{2}{\beta n}} db_i + \frac{1}{n} \sum_{j \neq i} \frac{1}{r_i(t) - r_j(t)} + \frac{1}{r_i(t) + r_j(t)} \right] dt + \beta r_i(t) dt, \quad 1 \leq i \leq n,
$$

with initial data $\{r_i(0)\}_{i=1}^n = \{\lambda_i(0)\}_{i=1}^n$ or $\{r_i(0)\}_{i=1}^n = \{\mu_i(0)\}_{i=1}^n$, where $\lambda_i(0) = \lambda_i^*(0)$ and $\mu_i(0) = \mu_i^*(0)$ are the singular values of $\tilde{X}_0 - z$ and $\tilde{X}_0 - z$, respectively, and $\{b_i, i = 1, 2, \ldots, n\}$ is a collection of independent standard real or complex Brownian motions. As usual, the parameter $\beta = 1, 2$ for real and complex ensembles, respectively.

The main statement of Proposition 2.9 is that one may couple the processes $\{\lambda_i(t)\}_{i=1}^n$, $\{\mu_i(t)\}_{i=1}^n$ in such a way that the difference between them is small after a short time. In fact, along the proof of Proposition 2.9 in [11] this coupling was realized by choosing the same stochastic differential $db$ in the flows for $\lambda_i(t)$ and $\mu_i(t)$. From now on we assume this coupling between them. In this way, we can use (2.15) to estimate the difference $|\lambda_i(ct_1) - \mu_i(ct_1)|$, for $1 \leq i \leq n^\omega$, with some exponent $0 < \omega \leq 10 \omega$ that we will choose shortly. For larger indices, $n^\omega \leq i \leq c'n$, we use the bulk rigidity bound from Corollary 2.8
to control $|\lambda_i(c t_1) - \mu_i(c t_1)|$, and finally we use the trivial $O(1)$ bound on this difference for $i \geq c' n$.

Combining all these estimates on $|\lambda_i(c t_1) - \mu_i(c t_1)|$, we conclude that

$$E \left| I_3^{(1)}(t_1) - \bar{I}_3^{(1)}(t_1) \right| \lesssim \int |\Delta f_{x,n}(z)| \int_{n_0}^{n_1} \sum_{i=1}^{n} \frac{\eta}{\lambda_i(c t_1)^2 + \eta^2} - \frac{\eta}{\mu_i(c t_1)^2 + \eta^2} \, d\eta dz$$

$$\lesssim \int_{n_0}^{n_1} \left( \sum_{i=1}^{n} + \sum_{i=c' n}^{n} \right) \left( \frac{\eta |\lambda_i(c t_1) - \mu_i(c t_1)| |\lambda_i(c t_1) + \mu_i(c t_1)|}{\lambda_i(c t_1)^2 + \eta^2} \right) \, d\eta$$

$$\lesssim \frac{n^2}{n^{1+\omega} \eta_0} + \frac{n^{1+\delta_2} \eta_2}{n^2} + n^2.$$  

The constant $c' > 0$ is chosen sufficiently small so that $\gamma_i$ is still in the bulk and thus (2.13) holds (we also used (2.8)). We choose the exponents such that $\overline{\sigma} + \delta_0 \leq 10 \omega_n$ and $\delta_1 \leq 10 \overline{\sigma}$, to guarantee that the r.h.s. in (4.9) is bounded by $n^{-c(k)}$, for some constant $c(k) > 0$. Thus (4.9) implies (4.6) and so combining it with (4.5) concludes the proof of Lemma 4.3. 

\[ \square \]

5. Green function comparison theorem (GFT) for long times

This section is devoted to the proof of Lemma 4.2. The proof has two main ingredients. First, we use the stochastic advection flow in the form presented in [23] to approximate the resolvent in $I_4^{(1)}$, which lives on scales $\eta \in [\eta_1, \eta_2]$, with another resolvent whose spectral parameter has an imaginary part proportional to $\eta_2$. This quantity, called $J_4^{(1)}$ later in (5.28), lives on scales of order $\eta_2$. Combining it with $J_5^{(1)}$ that already lives on scales above $\eta_2$, we define $L^{(1)} := J_4^{(1)} + J_5^{(1)}$ that lives on scales above $\eta_2$, see (5.29) later. Second, we conclude that products of $L^{(1)}$ at the larger spectral parameter $\eta_2$ are close to their Ginibre counterparts using a long time GFT similarly to [12], concluding the proof of Lemma 4.2. The first step is necessary since the long time GFT is affordable only for resolvents with spectral parameters with a relatively large imaginary part.

From now on we often drop the superscript $z$ to simplify the notation, i.e. we write $m = m^z := (G^z) = \frac{i}{2\pi} \text{Tr} G^z$, $\lambda_i = \lambda_i^z$, etc. However, we will put back the superscript $z$-integration in the definition of $I_4^{(1)}$ plays an important role in the argument. Note that the spectrum of $H^z$ is symmetric around zero, hence its eigenvalues come in pairs, $\lambda_i$ and $-\lambda_i$. Due to this symmetry we can write $m(w)$ as

$$m(w) = \frac{1}{2n} \text{Tr} G^z(w) = \frac{1}{2n} \sum_{i=1}^{n} \left[ \frac{1}{\lambda_i - w} - \frac{1}{\lambda_i + w} \right].$$

Before proceeding, we recall the definition of $I_4^{(1)}$ and $I_5^{(1)}$:

$$I_4^{(1)}(X) = -\frac{1}{2\pi} \int_{\mathbb{C}} \Delta f_{x,1}(z) \int_{\eta_1}^{\eta_2} \left[ 3m(i\eta) - 3\tilde{m}(i\eta) \right] \, d\eta dz,$$

$$I_5^{(1)}(X) = -\frac{1}{2\pi} \int_{\mathbb{C}} \Delta f_{x,1}(z) \int_{\eta_2}^{T} \left[ 3m(i\eta) - 3\tilde{m}(i\eta) \right] \, d\eta dz,$$

with $\eta_1 = n^{-1+\delta_1}$, $\eta_2 = n^{-1/2+\delta_2}$ defined in (3.2). We added the argument $X$ to stress that we view $I_4^{(1)}$ and $I_5^{(1)}$ as functions of the random matrix $X$ via the Stieltjes transform $m$ of the eigenvalues of the Hermitization of $X - z$ from (5.1).

Along the proof of Lemma 4.2 we define three stochastic matrix flows to relate $X$ and $\bar{X}$. Before the actual proof we introduce these flows to guide the reader.
(i) \( \hat{X}_t \) is an Ornstein-Uhlenbeck (OU) flow with initial condition \( \hat{X}_0 := X \), see (5.5). At time \( t_2 \) we can write \( \hat{X}_{t_2} \overset{d}{=} X_{t_2} + \sqrt{ct_2}U_1 \) with some i.i.d. random matrix \( X_{t_2} \), independent Ginibre matrix \( U_1 \) and constant \( c \) close to 1.

(ii) \( X_t \) is a Dyson Brownian Motion (DBM) flow with initial condition \( X_0 := \hat{X}_{t_2} \), see (5.11), i.e. \( X_t \overset{d}{=} X_0 + \sqrt{t}U_3 \) with a Ginibre matrix \( U_3 \). In particular, \( X_{ct_2} \overset{d}{=} \hat{X}_{t_2} \).

(iii) \( X'_t \) is another OU flow with initial condition \( X'_0 = X_0 \), see (5.33).

With these notations, along the proof we will justify the following chain of approximations for the appropriate choice of time \( t_2 \overset{d}{=} \eta_2 \):

\[
\mathbb{E} \prod_j \left( I_4^{(j)} + I_2^{(j)} \right) (X) \overset{(1)}{=} \mathbb{E} \prod_j \left( I_4^{(j)} + I_2^{(j)} \right) (\hat{X}_{t_4}) \overset{(2)}{=} \mathbb{E} \prod_j \left( I_4^{(j)} + I_2^{(j)} \right) (X_{ct_2}) \quad \overset{(3)}{=} \mathbb{E} \prod_j \left( J_4^{(j)}(X_0, 0) + I_2^{(j)}(X_{ct_2}) \right) \quad \overset{(4)}{=} \mathbb{E} \prod_j L_0^{(j)}(X_0, X_0 + \sqrt{ct_2}U_3) \\
\overset{(5)}{=} \mathbb{E} \prod_j L_{\infty}^{(j)}(X'_n, X'_n + \sqrt{ct_2}U_3) - \int_0^\infty \mathbb{E} \frac{d}{dt} \prod_j L_t^{(j)}(X'_t, X'_t + \sqrt{ct_2}U_3) dt \\
\overset{(6)}{=} \mathbb{E} \prod_j L_{\infty}^{(j)}(X'_n, X'_n + \sqrt{ct_2}U_3).
\]

Here \( J_4 \), given in (5.28), is a version of \( I_4 \) along the flow \( X'_t \), and \( L_t^{(j)} \) is defined by

\[
L_t^{(j)}(X'_t, X'_t + \sqrt{ct_2}U_3) := J_4^{(j)}(X'_t, t) + I_2^{(j)}(X'_t + \sqrt{ct_2}U_3).
\]

Now we informally explain the key mechanism in each step in (5.3). Step (1) in (5.3) will be a simple GFT argument given in Lemma 5.1. Step (2) is from construction since \( X_{ct_2} \overset{d}{=} \hat{X}_{t_2} \). Step (3) is obtained from the stochastic advection flow along the DBM process \( X_t \) and \( J_4 \) is the endpoint of \( I_4 \) in this flow. This step was motivated by [23, 1]. The error is estimated in Lemma 5.2 and it even holds in high probability. Step (4) is just a definition and Step (5) is just integrating back the time derivative of \( \prod L_t^{(j)} \) along the OU flow \( X'_t \). Finally in Step (6) we show that this derivative term is negligible, this will be the main part of the proof of Lemma 5.4. After all these steps, we arrive at the last expression in (5.3) that contains \( X'_n \) which is a purely Gaussian matrix since the OU-flow \( X'_t \) has the Ginibre ensemble as its large time limit. So the dependence on the distribution of the original matrix \( X \) is eliminated. Now we may repeat the same argument starting from the Ginibre matrix \( \hat{X} \) instead of \( X \), we will arrive to the same object at the end of (5.3). This will prove Lemma 4.2.

After this summary, we start the actual proof. From now on, we focus on the general matrix \( X \) as a starting point, keeping in mind that the same procedure will be done for its Ginibre counterpart \( \hat{X} \). Using the Ornstein-Uhlenbeck (OU) flow, in Lemma 5.1 we first prove that we can add a small Gaussian component to \( H^2 \) at the price of a small error. We define the flow

\[
d\hat{X}_t = -\frac{1}{2} \hat{X}_t dt + \frac{d\hat{B}_t}{\sqrt{n}}, \quad \hat{X}_0 = X,
\]

where \( \hat{B}_t \) is a matrix valued standard real or complex Brownian motion. In particular, \( \hat{X}_t \) has a Gaussian component with variance \( 1 - e^{-t} \), i.e. slightly smaller than \( t \). We then construct a matrix \( \hat{X}_t \) such that

\[
\hat{X}_t \overset{d}{=} \hat{X}_t + \sqrt{ct}U_1,
\]

where \( U_1 \) is a Ginibre matrix independent of \( \hat{X}_t \) and \( c > 0 \) is a constant slightly smaller than 1. Note that the entries of \( \hat{X}_t \) have zero expectation and variance \( c(t)n^{-1} \), with
function comparison (GFT), used to prove Lemma (5.8)
\[ \hat{Z} \]
\[ \text{with} \]
\[ I \]
the definition of \( \eta \) the evolution of the DBM flow \( (I_{4}^{(j)} + I_{5}^{(j)})(X) \) with \( (I_{4}^{(j)} + I_{5}^{(j)})(\hat{X}) \) as stated in Lemma 5.1. We omit its proof since it is a simplified version of the long time Green’s function comparison (GFT), used to prove Lemma 5.3.

**Lemma 5.1.** Let \( \eta_{1}, \eta_{2} \) be defined in (3.2) and fix \( t_{2} := \eta_{2} = n^{-1/2+\delta_{2}} \), then we have
\[ \mathbb{E} \left[ \prod_{j=1}^{k} \left( I_{4}^{(j)} + I_{5}^{(j)} \right)(X) - \prod_{j=1}^{k} \left( I_{4}^{(j)} + I_{5}^{(j)} \right)(\hat{X}) \right] = O \left( \frac{t_{2}}{\sqrt{m\eta_{1}}} \right), \]
for a small constant \( c(k) > 0 \).

In order to ensure that the error term in (5.10) is of order \( n^{-c} \), we choose \( \delta_{1} \geq 10\delta_{2} \) in the definition of \( \eta_{1} \).

From now on fix \( t_{2} := \eta_{2} \) and consider only \( \hat{X}_{t_{2}} \). In order to estimate \( I_{4}^{(j)}(\hat{X}_{t_{2}}) \), we consider the following Dyson Brownian Motion (DBM):
\[ dX_{t} = \frac{dB_{t}}{\sqrt{n}}, \quad X_{0} = \hat{X}_{t_{2}}, \]
where \( dB_{t} \) is a standard real or complex matrix Brownian motion. Note that \( X_{t} \overset{d}{=} X_{0} + \sqrt{n}U_{3} \), with \( U_{3} \) a Ginibre matrix independent of \( X_{0} \). We run the flow (5.11) for a time \( t = ct_{2} \), then, by computing the variances of the added Gaussian components, we find that
\[ X_{ct_{2}} \overset{d}{=} \hat{X}_{t_{2}}, \quad H_{ct_{2}} \overset{d}{=} \hat{H}_{t_{2}}, \]
with \( \hat{H}_{t_{2}}, H_{ct_{2}} \) being the Hermitization of \( \hat{X}_{t_{2}} - z \) and \( X_{ct_{2}} - z \), respectively. In other words, this ensemble can be viewed in two different ways, once as a time-\( t_{2} \) evolution of the OU flow starting from \( X \) and once as a time-\( ct_{2} \) evolution of the DBM flow (5.11) starting from \( \hat{X}_{t_{2}} \).

In contrast with (5.5), the flow defined in (5.11) does not preserve the second moment of \( X_{t} \), but the associated flow for the eigenvalues of the Hermitization of \( X_{t} - z \) in (5.14) is much easier to handle than the one for the eigenvalues of the OU matrix flow \( H_{t} \) which involves eigenvectors as well. Exactly as in (4.8), the matrix flow defined in (5.11) induces the flow for the eigenvalues of the Hermitised matrix
\[ H_{t} = H_{t}^{c} := \begin{pmatrix} 0 & X_{t} - z \\ X_{t}^{*} - z & 0 \end{pmatrix}, \]
defined by
\begin{equation}
(5.14) \quad d\lambda_i(t) = \sqrt{\frac{2}{\beta n}} \, db_i + \frac{1}{n} \sum_{j \neq i} \left[ \frac{1}{\lambda_i(t) - \lambda_j(t)} + \frac{1}{\lambda_i(t) + \lambda_j(t)} \right] \, dt + \frac{\beta - 1}{\beta n \lambda_i(t)} \, dt, \quad 1 \leq i \leq n,
\end{equation}

with initial data \( \{\lambda_i(0)\}_{i=1}^n = \{\lambda_i^{(t_2)}\}_{i=1}^n \), where \( \{\lambda_i^{(t_2)}\}_{i=1}^n \) are the singular values of \( X_{t_2} - z \). In (5.14) \( \beta = 1, 2 \) corresponds to the real and complex case, respectively, and \( \{b_i\} \) is a collection of independent real Brownian motions.

Similarly to (5.1) we define the empirical Stieltjes transform of \( H_t \) as
\begin{equation}
(5.15) \quad m_t^i(w) = m_t(w) := \frac{1}{2n} \sum_{i=1}^n \left[ \frac{1}{\lambda_i(t) - w} - \frac{1}{\lambda_i(t) + w} \right], \quad w \in \mathbb{H}.
\end{equation}

By (5.14)-(5.15) and Ito’s formula it follows that
\begin{equation}
(5.16) \quad dm_t(w) = \sqrt{\frac{2}{\beta n}} \, db_i(t) + 2 \frac{\beta - 1}{2\beta n^2} \sum_{i=1}^n \lambda_i(t)(\lambda_i(t) - w)^2 \, dt + m_t(w) \partial_w m_t(w) \, dt
\end{equation}

Moreover, we define the deterministic function \( \tilde{m}_t = \tilde{m}_t(w) \) (still implicitly depending on \( z \)) as the solution of the complex Burgers equation
\begin{equation}
(5.17) \quad \partial_t \tilde{m}_t = -\tilde{m}_t \partial_w \tilde{m}_t,
\end{equation}

with initial data \( \tilde{m}_0 = \tilde{m}_0(w) \), where \( \tilde{m}_0 = \tilde{m}_0^{(t_2)} \) is the Stieltjes transform of the scDos of \( H_0 = H_0^{(t_2)} \) as defined in (2.6). Note that by the definition of (5.5) and (5.11) it follows that
\( m_{ct_2}(w) \xrightarrow{d} m(w) \) and \( \tilde{m}_{ct_2}(w) = \tilde{m}(w) \), for any \( w \in \mathbb{C} \). The characteristics \( t \mapsto w_t \) of the flow in (5.17) are defined by the ODE
\begin{equation}
(5.18) \quad \frac{d}{dt} w_t = -\tilde{m}_t(w_t)
\end{equation}

with some initial condition \( w_0 \). We will choose the initial condition on the imaginary axis, \( \Re w_{t=0} = 0 \). Note that (5.18) implies \( \Re w_t = 0 \) for any \( t \geq 0 \) if initially \( \Re w_{t=0} = 0 \), since \( \Re \tilde{m}_t(\eta) = 0 \) for any \( \eta > 0 \) by symmetry around 0 of the scDos of \( H_t \). Thus we can write \( w_t = i\eta_t \) with some positive \( \eta_t \). Moreover, since \( t_2 = \eta_2 \), by (5.18) it follows that \( \eta_t \sim c_{t_2} + \eta - t_2 \), for any \( 0 \leq t \leq c_{t_2} \) and \( \eta_1 \leq \eta \leq \eta_2 \). In particular, note that the imaginary part of the characteristics initially is \( \eta_{t=0} = \eta_2 + \eta \) and at the final time it is \( \eta_{t=t_2} = \eta \).

We now follow the evolution of \( m_t(\eta) - \tilde{m}_t(i\eta) \) along the DBM flow \( X_t \). For any \( \eta_1 \leq \eta \leq \eta_2 \), integrating in time \( d(m_t(\eta) - \tilde{m}_t(i\eta)) \) from 0 to \( c_{t_2} \), by easy computations, using (5.16)-(5.18), it follows that
\begin{equation}
(5.19) \quad m(\eta_2) - \tilde{m}(i\eta) \xrightarrow{d} m_{ct_2}(\eta) - \tilde{m}_{ct_2}(i\eta) = m_0(\eta_{t=0}) - \tilde{m}_0(i\eta_{t=0}) + \int_0^{c_{t_2}} [A_1(t) \, dt + dA_2(t)],
\end{equation}

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where 
\[ A_1(t) := (m_2(i\eta) - \tilde{m}_2(i\eta)) \partial_\eta m_1(i\eta), \]
\[ dA_2(t) := \sqrt{\frac{1}{2\beta n^3}} \sum_{i=1}^{n} \left[ \frac{1}{(\lambda_i(t) + i\eta)^2} - \frac{1}{(\lambda_i(t) - i\eta)^2} \right] dB_i(t) \]
\[ + \frac{2 - \beta}{2\beta n^2} \sum_{i=1}^{n} \left[ \frac{1}{(\lambda_i(t) - i\eta)^3} - \frac{1}{(\lambda_i(t) + i\eta)^3} \right] dt \]
\[ + \frac{1}{2n^2} \sum_{i=1}^{n} \left[ (\lambda_i(t) + i\eta)(\lambda_i(t) - i\eta)^2 - \frac{1}{(\lambda_i(t) + i\eta)^2} \right] dt \]
\[ + \beta \frac{1}{2\beta n} \sum_{i=1}^{n} \left[ \frac{1}{\lambda_i(t)(\lambda_i(t) - i\eta)^2} - \frac{1}{\lambda_i(t)(\lambda_i(t) + i\eta)^2} \right] dt. \tag{5.20} \]

Note that the last two lines of \( dA_2(t) \) can be combined into one term as
\[ \frac{\eta(n\beta - 4)}{2\beta n^2} \sum_{i=1}^{n} \left( \frac{1}{(\lambda_i(t) - i\eta)^2(\lambda_i(t) + i\eta)^2} - \frac{1}{(\lambda_i(t) - i\eta)^4} \right) = \frac{\eta(3\beta - 2)}{\beta n^2} \sum_{i=1}^{n} \frac{1}{(\lambda_i(t) - i\eta)^2}. \tag{5.21} \]

In Lemma 5.2, we prove that the contribution to \( f_4^{(j)} \) (see (5.2)) of the second term in the r.h.s. of (5.19) is smaller than \( n^{-1} \) with very high probability.

**Lemma 5.2.** Let \( \eta_1, \eta_2 \) be defined in (3.2), and let \( A_1(t), dA_2(t) \), with \( 0 \leq t \leq \eta_2 \), be defined in (5.20), then for any \( j \in [k] \) we have
\[ \left| \frac{1}{2\pi} \int_{C} \Delta f_4^{(j)}(z) \int_{\eta_1}^{\eta} \int_{0}^{\eta_2} \Im [A_1(t) + dA_2(t)] \, dt \, d\eta \, dz \right| \leq \frac{\eta^{\xi}}{n^{\xi}}, \tag{5.22} \]
for any \( \xi > 0 \) with very high probability.

The bound (5.22) applied in (5.19) shows that, with high probability, the difference between
\[ m_{\eta_1\eta_2}(i\eta) - \tilde{m}_{\eta_1\eta_2}(i\eta) \quad \text{and} \quad m_{0}(i\eta = 0) - \tilde{m}_{0}(i\eta = 0) \tag{5.23} \]
is negligible for our purposes. The left quantity in (5.23) is a function of \( H_{\eta_2} \) and involves resolvents at scale \( \eta_1 = \eta \) which can be as small as \( \eta_1 \), while the right quantity contains resolvents of the Hermitian of the matrix \( X_0 - z = X_{\eta_2} - z \) at a scale \( \eta_1 = \eta \) and \( \eta \sim \eta_2 \). Since the distribution of \( X_{\eta_1} \) and \( X_{\eta_2} \) coincide by (5.12), and \( (I_4^{(j)} + I_5^{(j)})(X_{\eta_2}) \) is a function of \( X_{\eta_2} \), the relation (5.23) allows us to compute the expectation of products of \( (I_4^{(j)} + I_5^{(j)})(X_{\eta_2}) \) in terms of similar quantities involving resolvents at much larger scales. On this scale, we will use a long time GFT in Lemma 5.4, to compare directly with Ginibre. We repeat the same construction starting with the Ginibre matrix \( \hat{X} \) instead of \( X \). Combining all these facts will yield the following lemma.

**Lemma 5.3.** Let \( \eta_1, \eta_2 \) be defined in (3.2) and fix \( t_2 := \eta_2 \sim n^{-1/2+\delta_2} \). Then we have
\[ \mathbb{E} \left[ \prod_{j=1}^{k} (I_4^{(j)} + I_5^{(j)})(\hat{X}_{t_2}) - \prod_{j=1}^{k} (I_4^{(j)} + I_5^{(j)})(\hat{X}_{t_2}) \right] = \mathcal{O} \left( \frac{n^{\xi}}{\sqrt{n\eta_2}} \right). \tag{5.24} \]

Combining Lemma 5.3 with Lemma 5.1 applied to both \( \prod(I_4 + I_5)(X) \) and its Gaussian counterpart \( \prod(I_4 + I_5)(\hat{X}) \) we conclude the proof of Lemma 4.2.

**Proof of Lemma 5.2 and Lemma 5.3.**
Proof of Lemma 5.2. By (5.20) and (5.21) it follows that to prove (5.22) we have to bound the following four terms

\begin{equation}
(5.25) \quad n^{-2} \sum_{i=1}^{n} \eta_i \frac{i \lambda_i(t) - i \eta_i}{(\lambda_i(t) - i \eta_i)^2}, \quad \frac{1}{n^{3/2}} \sum_{i=1}^{n} \frac{d\beta_i(t)}{(\lambda_i(t) \pm i \eta_i)^2}, \quad n^{-2} \sum_{i=1}^{n} \frac{1}{(\lambda_i(t) \pm i \eta_i)^3}, \quad (m_i(i \eta_i) - m_i(i \eta_i)) \partial \omega_m_i(i \eta_i).
\end{equation}

Recall that \( \eta_i \sim cn + \eta - t \), with \( 0 \leq t \leq ct \) and that \( t_2 = \eta_2 \) is defined in (3.2). Using that for an analytic function \( h(w) \) we have \( \partial h = -i \partial \bar{h} \) with \( \eta = \bar{\omega} \), and using the local law in Proposition 2.6, we bound the contribution to (5.22) of the last term in (5.25):

\[ \int_C \Delta f_{z_j}(z) \int_{\eta_1}^{\eta_2} \int_{\eta_1}^{\eta_2} \left( m(z \eta_i) - \tilde{m}(z \eta_i) \right) \partial \omega \tilde{m}(z \eta_i) \ dt \ d\eta \ dz \]

\[ = -\frac{i}{2} \int_C \Delta f_{z_j}(z) \int_{\eta_1}^{\eta_2} \int_{\eta_1}^{\eta_2} \partial \eta \left( m(z \eta_i) - \tilde{m}(z \eta_i) \right)^2 \ d\eta \ dz + O(\eta_2 \log \eta_1) \]

\[ = -\frac{i}{2} \int_C \Delta f_{z_j}(z) \int_{\eta_1}^{\eta_2} \left( m(z \eta_i) - \tilde{m}(z \eta_i) \right)^2 \ d\eta \ dz + O\left( n^\xi \eta_2 \log \eta_1 \right) = O\left( \frac{n^\xi}{\eta_1^2} \right), \]

with very high probability for any \( \xi > 0 \). Here we plugged in \( -\partial \omega \tilde{m}(z \eta_i) + \partial \omega \tilde{m}(z \eta_i) \), used that \( |\partial \omega \tilde{m}| \lesssim 1 \) in the error term, and performed the \( \eta \)-integration. We also used that \( \int_C |\Delta f_{z_j}(z)| \lesssim n \).

Now we turn to the estimate of all the terms in the first line of (5.25). They all follow the same lines, so in the remaining part of the proof we consider only the term with the stochastic differential that is the most delicate to handle. The other two terms are left to the reader. Note that the rigidity of the \( \lambda_i(t) \) along all the flow is ensured by Proposition 2.7 and Corollary 2.8. We omit the \( t \)-dependence from \( \lambda_i(t) \) since it plays no role, but we put back the superscript \( z \) since now the \( z \)-dependence will be crucial. We start estimating the stochastic term (with the choice of the minus sign in \( \pm \)) as follows

\[ \int_C \Delta f_{z_j}(z) \int_{\eta_1}^{\eta_2} \int_{\eta_1}^{\eta_2} \frac{1}{n^{3/2}} \sum_{i=1}^{n} \left[ \frac{1}{(\lambda_i(t) - i \eta_i)^2} - \frac{1}{(\eta_i - i \eta_i)^2} \right] \ d\beta_i(t) \ d\eta \ dz \]

\[ + \int_C \Delta f_{z_j}(z) \int_{\eta_1}^{\eta_2} \int_{\eta_1}^{\eta_2} \frac{1}{n^{3/2}} \sum_{i=1}^{n} \left[ \frac{1}{(\eta_i - i \eta_i)^2} - \frac{1}{(\gamma_i - i \eta_i)^2} \right] \ d\beta_i(t) \ d\eta \ dz =: E_1 + E_2. \]

Note that the we could subtract \( (\gamma_i - i \eta_i)^{-2} \) since it is independent of \( z \) hence its \( z \)-integral it is equal to zero by \( \int_C \Delta f_{z_j}(z) = 0 \). We start estimating \( E_1 \). In particular, we write \( E_1 \) as

\[ \int_C \Delta f_{z_j}(z) \int_{\eta_1}^{\eta_2} \int_{\eta_1}^{\eta_2} \frac{1}{n^{3/2}} \sum_{i=1}^{n} \left( \frac{\gamma_i^2 - \lambda_i^2}{(\lambda_i^2 - i \eta_i)^2} - \frac{\lambda_i^2}{(\eta_i - i \eta_i)^2} \right) \ d\beta_i(t) \ d\eta \ dz, \]

and estimate the quadratic variation of the inner \( d\beta_i(t) \)-integral as

\[ \frac{1}{n^5} \int_{\eta_1}^{\eta_2} \sum_{i=1}^{n} \left| \gamma_i - \lambda_i \right|^2 \left( |\gamma_i^2 - \lambda_i^2| |\gamma_i - \lambda_i|^2 + \eta_i^2 \right) \ dt \lesssim \frac{1}{n^5} \int_{\eta_1}^{\eta_2} \frac{n^\xi}{\eta_i^2} \ dt \lesssim \frac{n^\xi}{\eta_1^{3/2}}, \]

with very high probability for any \( \xi > 0 \). Here we used the rigidity bound in (2.13) for indices \( i \leq cn \) and a trivial bound for larger indices. In the final integration we used \( \eta_i \sim cn + \eta - t \).

Hence, by Schwarz inequality we can bound the quadratic variation of \( E_1 \) by

\[ \left( n \int_{\eta_1}^{\eta_2} \frac{n^\xi}{n^2 \eta_i^2} \ d\eta \right)^2 \lesssim \frac{n^\xi}{(\eta_1)^2}. \]
with very high probability. Hence, by the Burkholder-Davis-Gundy (BDG) inequality we conclude that

\[ |E_1| \lesssim n^2 (\eta_1)^{-1} \]

with very high probability.

To conclude the estimate of the stochastic term we are left with the estimate of \( E_2 \). Note that by (2.6) and (2.12) it follows that \( \gamma^z_i = \hat{\gamma}^z_i \), i.e. \( \hat{\gamma}^z_i \) depends only on \( u = |z| \) but not on the phase of \( z \). Hence, using (3.6) of Lemma 3.5 we estimate \( E_2 \) as follows

\[
E_2 = \int_C \Delta f^{(j)}_z(z) \int_{|z|}^{n_2} \frac{1}{n_3^{3/2}} \sum_{i=1}^{n} \left[ -\frac{2}{\gamma_i^z - \gamma_i^z} + \frac{3(\gamma_i^z - x)}{(x - i \eta)^2} \right] \, db_i(t) d\eta \, dz
\]

\[
= \int_C \Delta f^{(j)}_z(z) \int_{|z|}^{n_2} \frac{1}{n_3^{3/2}} \sum_{i=1}^{n} \left[ -\frac{f(|z|)}{2 \eta |z|} \left( \hat{\gamma}^z_i - \eta \right) \right] \, db_i(t) d\eta \, dz.
\]

Here we used second order Taylor expansion of the functions \( u \mapsto \hat{\gamma}^z_i, u = |z| \), and we used that terms linear in \( z \) vanish after \( z \)-integration against \( \Delta f^{(j)}_z \). Then we estimate the quadratic variation of the inner \( db_i(t) \)-integral as

\[
\frac{C}{n^3} \int_{|z|}^{n_2} \frac{1}{n_3^{3/2}} \sum_{i=1}^{n} \left[ \left( \frac{\hat{\gamma}^z_i}{\gamma_i^z - \eta} \right)^4 \right] \, d\eta \lesssim \frac{1}{n^3} \int_{|z|}^{n_2} \sum_{i=1}^{n} \left[ \frac{1}{\gamma_i^z + \hat{\gamma}^z_i - \eta} \right] \lesssim \frac{1}{n^3} \eta^2
\]

in the regime where \( |z - z| \leq n^{-1/2} \) using that \( f \) is compactly supported, hence \( \Delta f^{(j)}_z(z) = 0 \) outside of this regime. Finally, similarly to \( E_1 \), by the BDG inequality we conclude that

\[ |E_2| \leq n^{-1 + \xi}, \]

with very high probability for any \( \xi > 0 \). Combining (5.26) and (5.27) we conclude the proof of Lemma 5.2.

**Proof of Lemma 5.3.** Using (5.12) we can compare \( \mathbb{E} \prod (I_4 + I_5)(X_{ctz}) \) with its Gaussian counterpart instead of \( \mathbb{E} \prod (I_4 + I_5)(\hat{X}_{tz}) \). Define

\[
J^{(j)}_4(X, t) := -\frac{1}{\sqrt{2\pi}} \int_C \Delta f^{(j)}_z(z) \int_{|z|}^{n_2} \left[ 3 \eta t(i(\eta t^2 + n_2)) - 3 \eta \bar{t} i(\eta t^2 + n_2) \right] \, d\eta \, dz,
\]

the arguments indicate that \( J_4 \) is a function of \( X'_i \) and additionally also of \( t \) via \( \eta \), and

\[
L^{(j)}_k := L^{(j)}_k(X'_i, X'_i + \sqrt{ct2U_3}) := J^{(j)}_4(X', t) + J^{(j)}_4(X'_i, t) + I^{(j)}_k(X'_i, t, U_3).
\]

Note that, similarly to \( L^{(j)}_0 \), the new quantity \( J^{(j)}_4 \) involves resolvents whose spectral parameter is at least \( n_2 \), unlike \( L^{(j)}_k(\hat{X}_{tz}) \) with resolvents starting from scale \( \eta_1 \), see (5.9).

The price we pay for this convenience is that the two terms in \( L^{(j)}_k \) contain resolvents of slightly different matrices, \( X_t \) and \( X_t + \sqrt{ct2U_3} \). As a consequence of (5.12) and (5.23) from Lemma 5.2 it follows that

\[
\mathbb{E} \left[ \prod_{j=1}^{k} (I^{(j)}_4 + I^{(j)}_5)(X_{ctz}) - \prod_{j=1}^{k} L^{(j)}_0(X_{ctz}) \right] = \mathcal{O} \left( n^{-c(k)} \right),
\]

for some small constant \( c(k) > 0 \). Recall that

\[
X_{ctz} \overset{d}{=} X_0 + \sqrt{ct2U_3}.
\]

We can repeat all these arguments starting from the Ginibre matrix and define \( \hat{L}^{(j)}_0 \), the Ginibre counterpart of \( L^{(j)}_0 \). Thus the bound (5.30) also holds if we replace \( I^{(j)}_4, L^{(j)}_0 \) by \( \hat{I}^{(j)}_4 \) and \( \hat{L}^{(j)}_0 \). Hence Lemma 5.3 is proven modulo that we can compare \( \prod L^{(j)}_0 \) with \( \prod \hat{L}^{(j)}_0 \) which is done in the following Lemma 5.4. \( \square \)
Lemma 5.4. With the definition (5.29) we have

\[
\mathbb{E} \left[ \prod_{j=1}^{k} L_0^{(j)} - \prod_{j=1}^{k} \tilde{L}_0^{(j)} \right] = \mathcal{O} \left( \frac{1}{\sqrt{n\eta_2}} \right).
\]

Proof. Recall that the variance of the matrix elements of \( Z \) is not exactly \( 1/n \), it is \( c(t_2)/n \), where \( c(t_2) = 1 - c_2 \) with some constant \( c \) close to 1. In order to prove (5.32) we consider the OU flow

\[
dX'_t = -\frac{1}{2}X'_t dt + \sqrt{c(t_2)} dB'_t, \quad X'_0 = X_0,
\]

with \( X_0 \) defined in (5.11) and \( B'_t \) a matrix valued standard real or complex Brownian motion, i.e. \( B'_t \in \mathbb{R}^{n \times n} \) or \( B'_t \in \mathbb{C}^{n \times n} \), accordingly with \( X_0 \) being real or complex, where their matrix elements, \( (B'_t)_{ab} \) in the real case, and \( \sqrt{2\mathbb{R}}[(B'_t)_{ab}], \sqrt{2\mathbb{I}}[(B'_t)_{ab}] \) in the complex case, are independent standard real Brownian motions for \( a, b \in [n] \).

By (5.33) it follows that \( \overline{X}_t \cong e^{-t/2} X_0 + \sqrt{c(t_2)} \sqrt{1 - e^{-t}} U_4 \), where \( U_4 \) is a real or complex Ginibre matrix independent of \( X_0 \). Note that the first two moments of \( X'_t \) are preserved along (5.33), i.e. the entries of \( \overline{X}_t \) have zero expectations and variance \( c(t_2)n^{-1} \). Note that (5.33) induces a flow for the Hermitisation \( H'_t \) of \( X'_t - z \), similarly to (5.7).

In order to prove (5.32) we define the observable

\[
Z_t := \prod_{j=1}^{k} L_t^{(j)},
\]

where \( L_t^{(j)} = L_t^{(j)}(X'_t, X'_t + \sqrt{c(t_2)} Z) \) is defined in (5.28). Indeed,

\[
\mathbb{E} \left[ \prod_{j=1}^{k} L_0^{(j)} - \prod_{j=1}^{k} \tilde{L}_0^{(j)} \right] = \mathbb{E}[Z_0 - Z_\infty],
\]

since the first two moments of \( H_0 \), and so by (5.31) the first two moments of \( H_{t=t_2} \), do not change along the flow \( H'_t \), and the flow relaxes to its equilibrium which is the Ginibre ensemble, hence \( Z_\infty \stackrel{d}{=} Z_t \) holds for any \( t \geq 0 \).

In the remainder of this section we bound

\[
|\mathbb{E}[Z_0 - Z_\infty]| \leq \int_{0}^{+\infty} \left| \mathbb{E} \frac{dZ_t}{dt} \right| dt.
\]

Define the matrix \( W'_t \) by

\[
H'_t = W'_t + \begin{pmatrix} 0 & -zI \\ -zI & 0 \end{pmatrix},
\]

with \( I \) the \( n \times n \) identity. In particular the entries \( w_{ab}(t) \) of \( W'_t \) have zero expectation and variance \( c(t_2)n^{-1} \). To make our notation easier we omit the prime for the entries of \( W'_t \).

Let \( \chi' \) be such that \( w_{ab}(0) \cong \chi' n^{-1/2} \) for any \( a, b \in [n] \), hence \( \chi' \) has zero expectation and variance \( c(t_2) \), since the entries of the initial condition \( X_0 \) in (5.33) have variance \( c(t_2) \). Then the flow (5.33) induces a flow \( d\chi'_t = -\chi'_t dt/2 + c(t_2) dB'_t \) on the entry distribution \( \chi' \) with solution

\[
\chi'_t = e^{-t/2} \chi' + \sqrt{c(t_2)} \int_{0}^{t} e^{-(t-s)/2} dB'_s, \quad \text{i.e.} \quad \chi'_t \cong e^{-t/2} \chi' + \sqrt{c(t_2)} \sqrt{1 - e^{-t}} g,
\]

where \( g \sim \mathcal{N}(0, 1) \) is a standard real or complex Gaussian, independent of \( \chi' \), with \( \mathbb{E}g^2 = 0 \) in the complex case. By linearity of cumulants we find

\[
\kappa_{i,j}(\chi'_t) = e^{-(i+j)t/2} \kappa_{i,j}(\chi') + \begin{cases} c(t_2)(1 - e^{-t}) \kappa_{i,j}(g), & i + j = 2 \\ 0, & \text{else}, \end{cases}
\]
where \( \kappa_{i,j}(x) \) denotes the joint cumulant of \( i \) copies of \( x \) and \( j \) copies of its conjugate \( \overline{x} \), in particular \( \kappa_{2,0}(x) = \kappa_{0,2}(x) = \kappa_{1,1}(x) = 1 \) for \( x = \chi', g \) in the real case, and \( \kappa_{0,2}(x) = \kappa_{2,0}(x) = 0 \neq \kappa_{1,1}(x) = 1 \) for \( x = \chi', g \) in the complex case.

To estimate the derivative of \( Z_t \) in (5.36), we compute it by Ito’s formula using (5.33):

\[
E \frac{dZ_t}{dt} = E \left[ -\frac{1}{2} \sum_\alpha w_\alpha(t) \partial_\alpha Z_t + \frac{1}{2} \sum_{\alpha,\beta} \kappa_1(\alpha,\beta) \partial_\alpha \partial_\beta Z_t \right],
\]

where \( \alpha, \beta \in [2n]^2 \) are double indices, \( w_\alpha(t) \) are the entries of \( W'_t \), and

\[
\kappa_1(\alpha, \beta, \ldots, \beta_j) := \kappa(w_\alpha(t), w_\beta(t), \ldots)
\]
denotes the joint cumulant of \( w_\alpha, w_\beta, \ldots \), and \( \partial_\alpha := \partial_{w_\alpha} \). By (5.38) and the independence of \( \chi' \) and \( g \) it follows that \( \kappa_1(\alpha,\beta) = \kappa_0(\alpha,\beta) \) for all \( \alpha, \beta \) and

\[
\kappa_j(\alpha,\beta_1,\ldots,\beta_j) = \begin{cases} e^{-\frac{t+1}{2} n - \frac{j+1}{2} \kappa_j(\chi')} & \text{if } \alpha \notin [n]^2 \cup [n+1, 2n]^2 \text{, } \beta_i \in \{ \alpha, \alpha' \} \forall i \in [j] \\ 0 & \text{otherwise,} \end{cases}
\]

for \( j > 1 \), where for a double index \( \alpha = (a,b) \), we use the notation \( \alpha' := (b,a) \), and \( l,k \) with \( l+k = j+1 \) denote the number of double indices among \( \alpha, \beta_1, \ldots, \beta_j \) which correspond to the upper-right, or respectively lower-left corner of the matrix \( W'_t \). In the sequel the value of \( \kappa_j(\chi') \) is of no importance, but we note that Assumption 2.1 ensures the bound \( |\kappa_{k,l}(\chi')| \leq C^{k+l} < +\infty \) for any \( k,l \).

To compute the right hand side of (5.39), we will use the cumulant expansion that holds for any smooth function \( g \) of a collection of random variables \( w = \{w_\alpha\} \):

\[
E w_\alpha g(w) = \sum_{m=0}^K \sum_{\beta_1,\ldots,\beta_m \in [2n]^2} \frac{\kappa(\alpha,\beta_1,\ldots,\beta_m)}{m!} \mathbb{E} \partial_{\beta_1} \ldots \partial_{\beta_m} g(w) + \Omega(K,g),
\]

where the error term \( \Omega(K,g) \) goes to zero as the expansion order \( K \) goes to infinity. In our application the error is negligible for, say, \( K = 100 \) since with each derivative we gain an additional factor of \( n^{-1/2} \) and due to the independence (5.41) the sums of any order have effectively only \( n^d \) terms. Applying (5.42) to (5.39) with \( g = \partial_{\alpha} Z_t \), the first order term is zero due to the assumption \( E w_\alpha(t) = 0 \), and the second order term cancels.

Therefore, in the rest of this section we separately estimate the third-, fourth-, and higher order terms from (5.42) in the formula

\[
||E[Z_0 - Z_\infty]| | \leq \int_0^{+\infty} \left| \mathbb{E} \left[ -\frac{1}{2} \sum_\alpha w_\alpha(t) \partial_\alpha Z_t + \frac{1}{2} \sum_{\alpha,\beta} \kappa_1(\alpha,\beta) \partial_\alpha \partial_\beta Z_t \right] \right| dt.
\]

All estimates are performed for fixed \( t \) and then we will integrate the answer in time. Note that the convergence of the \( t \)-integral is easily ensured by the exponentially decaying factor \( e^{-ct} \) from the cumulant decay (see (5.41)), so the time parameter does not play an important role. In the following we always omit the \( t \)-dependence of the resolvent, i.e. we use the notation \( G = G_t := (H'_t - w)^{-1} \), with \( w \in \mathbb{H} \).

Since \( \partial_\alpha \)-derivatives of \( Z_t \) consist of products of resolvents, naively, one would like to use the local law to all these terms. This simple strategy indeed works for terms of order five and higher, but not for the third and fourth order terms. For example, using the entry-wise local of Proposition 2.6 the \( j \)-th order terms are bounded by \( e^{-t(j+1)/2} n^{j+1+\epsilon} \), for any \( j \geq 3 \). This is clearly not affordable for \( j = 3,4 \) since we get an error term \( n^{3/2+\epsilon} \) or \( n^6 \) instead of the \( O(n^{-\epsilon}) \) needed to prove (5.32). In order to improve the naive estimate for the few explicit third and fourth order terms, we will use the following two mechanisms:
and the fact that which gives a better estimate for averages of Green functions. Concretely, 

\begin{equation}
\int \Delta f_{\alpha}(z) \Im \bar{m}^\tau dz = O(1)
\end{equation}

uniformly in \( |z_0| \leq 1 - \tau \). Naively, with absolute value inside, the integral in (5.44) would be bounded only by \( n \). Similar idea was already used in the proof of Lemma 5.2.

(ii) Instead of the entrywise local law, we sometimes use the isotropic local law in Proposition 2.6 which gives a better estimate for averages of Green functions. Concretely, for small \( \eta \), we will use

\begin{equation}
\sum_a G_{ab}(i\eta) = \langle 1, G(i\eta)c_a \rangle \lesssim \eta^{-1/2}, \quad \sum_{ab} G_{ab}(i\eta) = \langle 3, G(i\eta)\mathbb{1} \rangle \lesssim \sqrt{\eta} \eta^{-1/2},
\end{equation}

where \( \mathbb{1} \) denotes the 2n-dimensional vector whose entries are all 1’s and \( c_a \in \mathbb{R}^{2n} \) denote the coordinate vector. Note that using only the entry-wise local law in (5.45) we would only get a bound \( n\eta^{1/2} \eta^{-1/2} \) and \( n^{-3/2} \eta^{-1/2} \), respectively for the first and the second sum in (5.45). Hence, using the isotropic local law we gain a factor \( n^{-1/2} \) for each sum that we can perform. In order to exploit the improvement in (5.45) we will often use that \( G_{aa} = (G_{aa} - \bar{m}) + \bar{m} \), since the term \( \bar{m} \) does not depend on the summation index and \( |G_{aa} - \bar{m}| \) is smaller than \( G_{aa} \) by the entry-wise local law in Proposition 2.6.

Finally, before starting with the actual estimates we make a further simplification to make our presentation clearer. Note that \( L_l^{(j)} \) defined in (5.29) is a sum of two terms: the first one is a function of the matrix \( X'_l \), while second one is a function of the matrix \( X'_l + \sqrt{c_2} i T U_3 \). In order to make our exposition cleaner in the following part of the proof we assume that both terms in the definition of \( L_l^{(j)} \) depend on the same matrix, say \( X'_l \). Within the proof of Green function comparison arguments this is not a restriction, since resolvents of both types behave in the same way under derivatives with respect to matrix elements of \( X'_l \). It only saves us from carrying an additional notation to distinguish between the two types of resolvent.

Since \( L_l^{(j)} = L_l^{(j)} \), defined in (5.29), consists of the sum of two terms, we have to distinguish when the derivative \( \partial_b \) hits the first or the second term. Since in the following we will estimate terms of order three or higher, as an example, we consider how the third derivative \( \partial_b^3 \) acts on the two terms in the definition of \( L_l^{(j)} \):

\begin{equation}
Q_1 := \int_C \Delta f_{s_j}^{(j)}(z) \int_{\eta_2}^{\eta_T} \frac{1}{n} \sum_{abc} G_{ca}(i\eta)G_{ba}(i\eta)G_{bc}(i\eta) \, d\eta \, dz,
\end{equation}

\begin{equation}
Q_2 := \int_C \Delta f_{s_j}^{(j)}(z) \int_{\eta_1}^{\eta_T} \frac{1}{n} \sum_{abc} G_{ca}(i(c\eta_2 + \eta))G_{ba}(i(c\eta_2 + \eta))G_{bc}(i(c\eta_2 + \eta)) \, d\eta \, dz.
\end{equation}

Using the entry-wise local law in Proposition 2.6 and the fact that \( \int_C |\Delta f_{s_j}^{(j)}| \lesssim n \) we have the following bounds

\[ |Q_1| \lesssim \int_{\eta_2}^{\eta_T} \frac{n^{1+\xi}}{\eta^2} d\eta \lesssim \frac{n^{1+\xi}}{\eta_2^2}, \quad |Q_2| \lesssim \int_{\eta_1}^{\eta_T} \frac{n^{1+\xi}}{(\eta_2 + \eta)^2} d\eta \lesssim \frac{n^{1+\xi}}{\eta_2^2}, \]

with very high probability for any \( \xi > 0 \). Note that the main contribution to the integral in \( Q_1 \) comes from the small \( \eta \)-regime around \( \eta_2 \) giving a bound \( n\eta_2^{-2} \). On the other hand, even if the main contribution to \( Q_2 \) comes from the regime \( \eta \sim \eta_1 \), we estimate \( |Q_2| \) by \( n\eta_2^{-1} \), since \( \eta_2 + \eta \sim \eta_2 \) for any \( \eta \in [\eta_1, \eta_2] \). Hence, since the bounds for terms of the form \( Q_1 \) or \( Q_2 \) are of the same order, to make the presentation of the estimates in the remainder of this section cleaner, from now on we will only consider the terms of the form \( Q_1 \).
Order three terms. For the third order, when computing $\partial_\alpha \partial_\beta_1 \partial_\beta_2 Z$, through the Leibniz rule we have to consider all possible assignments of derivatives $\partial_\alpha$, $\partial_\beta_1$, $\partial_\beta_2$ to the factors $L^{(1)}$, \ldots, $L^{(k)}$. Since the particular functions $f^{(j)}$ and complex parameters $z_j$ play no role in the argument, there is no loss in generality in considering only the assignments

$$
\begin{align*}
\left(\partial_{\alpha_1 \beta_1} L^{(1)}\right) \prod_{j>1} L^{(j)}, & \quad \left(\partial_{\alpha_1} L^{(1)}\right) \left(\partial_{\beta_2} L^{(2)}\right) \prod_{j>2} L^{(j)}, \\
\left(\partial_{\alpha} L^{(1)}\right) \left(\partial_{\beta_1} L^{(2)}\right) \left(\partial_{\beta_2} L^{(3)}\right) & \prod_{j>3} L^{(j)}.
\end{align*}
$$

(5.47)

The cumulants in the expansion of (5.43) are non zero only if $\beta_j \in \{\alpha, \alpha'\}$ for any $j \geq 1$, with $\alpha = (a, b)$ and $\alpha' = (b, a)$. Note that $a \neq b$ since for $a = b$ the cumulants $\kappa_2(\alpha, \beta_1, \ldots)$ vanish. By Lemma 3.2 it follows that $|L^{(j)}| \leq n^{\xi}$ with very high probability, hence we can bound the products that do not have derivatives in (5.47) by $n^{k\xi}$. In the following we will omit the spectral parameter of the resolvent $G$, since it will always be $\eta$, and often use that

$$\int_C |\Delta f^{(j)}(z)| \, dz \lesssim n$$

to handle the final $z$ integral.

In order to prove Lemma 5.4 we plug (5.47) in (5.43) and estimate them term by term. We start with the estimate of the last term in (5.47):

$$
n^{-3/2} e^{-3t/2} \sum_{a, b \neq 0} \left( \frac{1}{n} \sum_{c} \int_C \Delta f^{(1)}_{z_1} \int_{\eta_2}^{T} G_{ca} G_{bc} \, d\eta d\eta \right)^3 \right| \lesssim n^{-3/2} e^{-3t/2} \sum_{a, b \neq 0} |G_{ba}(\eta_2) - G_{ba}(iT)|^3 \lesssim e^{-3t/2} n^{\xi} \frac{n^{\xi}}{m^2} \quad (5.48)
$$

where $T = n^{100}$ and in the first line we performed the $c$-summation and used the resolvent identity $\partial_\eta G = iG^2$ to explicitly perform the $\eta$-integration. In the bound (5.48) we used that $a \neq b + n \pmod{2n}$ to apply the entry-wise local law. On the other hand, if $a = b + n \pmod{2n}$ then the l.h.s. of (5.48) is trivially bounded by $n^{-1/2+\xi} e^{-3t/2}$.

Next, we bound the first term in (5.47) as follows:

$$
n^{-3/2} e^{-3t/2} \sum_{a, b \neq 0} \left( \frac{1}{n} \sum_{c} \int_C \Delta f^{(1)}_{z_1} \int_{\eta_2}^{T} G_{ca} G_{ba} G_{bc} \, d\eta d\eta \right) \lesssim n^{-3/2} e^{-3t/2} \sum_{a, b \neq 0} \int_{\eta_2}^{T} \hat{m} G_{ca}(G_{bb} - \hat{m}) G_{bc} \, d\eta \lesssim e^{-3t/2} n^{1/2+\xi} \int_{\eta_2}^{T} \frac{1}{n \eta^2} \, d\eta + e^{-3t/2} n^{3/2+\xi} \int_{\eta_2}^{T} \frac{1}{(n \eta)^2} \, d\eta \lesssim e^{-3t/2} n^{1/2+\xi} \int_{\eta_2}^{T} \frac{1}{n \eta^2} \, d\eta + e^{-3t/2} n^{3/2+\xi} \int_{\eta_2}^{T} \frac{1}{(n \eta)^2} \, d\eta \lesssim e^{-3t/2} \frac{n^{\xi} \log n}{\sqrt{n}} + n^{\xi} e^{-3t/2} \frac{n^{\xi}}{\sqrt{n \eta^2}} + n^{\xi} e^{-3t/2} \frac{n^{\xi}}{\sqrt{n \eta^2}} \quad (5.49)
$$

Note that to estimate the first two terms in the r.h.s. of (5.49) we used the isotropic improvement (5.45) respectively once and twice. Moreover, we assumed that $c \neq a, a + n$ and $c \neq b, b + n \pmod{2n}$, if this is not the case then the l.h.s. of (5.49) is bounded by $n^{-1/2+\xi} e^{-3t/2}$ using once the improvement (5.45). Finally, the proof of the bound
Similarly to the estimate of the third order terms we can bound the products above without derivatives by $n^{k\xi}$. As in the previous section we plug the terms (5.50) in (5.43) and estimate them one by one. Exactly as in (5.48), it follows that the last term in (5.50) is bounded by $n^{k\xi}(nT)^{-2}e^{-2t}$.

Next, adding and subtracting $\hat{m}$ to estimate the term $G_{bb}$ below, we bound the third term in (5.50) by:

$$e^{-3t/2}(\sqrt{nT})^{-1} \text{ for the last remaining term in (5.47) is left to the reader, since it follows using the same ideas as the bounds (5.48)-(5.49).}$$

**Order four terms.** For the fourth-order Leibniz rule we have to consider the assignments

$$n \text{ of the estimate for (5.51).}$$

In the first term in the r.h.s. of (5.51) we performed the $c$-summation. The second term in the r.h.s. of (5.51), using entry-wise local law, is bounded by

$$e^{-2t} \left| \sum_c \int_{\eta_2}^{\eta_1} G_{ca}(G_{bb} - m)G_{ac} d\eta \right|^2 \lesssim e^{-2t} \left| \sum_c \int_{\eta_2}^{\eta_1} G_{ca}(G_{bb} - m)G_{ac} d\eta \right|^2 \lesssim \frac{n^{k\xi}}{nT^2}.$$ (5.52)

In (5.52) we assumed that $c \neq a, a + n \mod 2n$, if this is not the case then the l.h.s. of (5.52) is trivially bounded by $e^{-2t}n^{-1/2+\frac{\xi}{2}}$. In order to bound the first term in the r.h.s. of (5.51), instead, we need to use the improvement (5.44). Indeed, using the resolvent identity $\partial_\eta G = iG^2$, performing integration by parts, and using the local law for $G_{aa} - \hat{m}$, we have

$$\frac{1}{n} \int_{\gamma_1} \Delta f_{z_1}(c)(G^2)_{aa} \hat{m} d\eta = \frac{1}{n} \int_{\gamma_1} \Delta f_{z_1} \left( G_{aa} \hat{m} \right)_{\eta_2}^{\eta_1} - \int_{\eta_2}^{\eta_1} G_{aa} \partial_\eta \hat{m} d\eta \right) d\eta = \frac{1}{n} \int_{\gamma_1} \Delta f_{z_1} \left( \hat{m} \right)_{\eta_2}^{\eta_1} - \int_{\eta_2}^{\eta_1} \hat{m} \partial_\eta \hat{m} d\eta \right) d\eta + O \left( \frac{n^{k\xi}}{\sqrt{n}} \right).$$ (5.53)

where in the last equality we used that the first term in the r.h.s. of (5.53) is smaller than $n^{-1}$ by (5.44). Combining (5.52) and (5.53) we conclude that the l.h.s. of (5.51) is bounded by $e^{-2t}[n^{k\xi}(nT)^{-1} + n^{-1/2+\xi}]$. Following exactly the same kind of computations of the estimate for (5.51), using (5.44) and in addition the isotropic improvement (5.45) we conclude that the first term in (5.50) is bounded by $e^{-2t}[n^{k\xi}(nT)^{-1} + n^{-1/2+\xi}]$ as well. The proof that the two remaining terms in (5.50) are bounded by $e^{-2t}[n^{k\xi}(nT)^{-1} + n^{-1/2+\xi}]$ is easier. It is left to the reader.
Higher order terms. For terms of order at least five, we can always use an entry-wise local law for the first and last $G$ with respect to the trace index, gaining a factor $n^{-1+\xi}$, after integration, respect to the trivial bound $n^5$. We simply bound all the other terms by $n^5$. Hence, for the terms of order $l$, with $l \geq 5$ we get a bound $n^{5-(l-4)/2}e^{-l/2}$.

Combining all the estimates on order three, four and higher order derivatives, integrating in $t$ from 0 to $+\infty$, and using (5.35)-(5.36), we complete the proof of Lemma 5.4. □

Appendix A. Isotropic local law for $H^z$

Proof of Proposition 2.6. All along the proof we drop the $z$-dependence of $H$, $G$ and $M$. The averaged local law (2.10) and the entry-wise local law (i.e. (2.9) for $x, y$ being coordinate vectors) have been proven in [4, Theorem 5.2] for $\eta \gg n^{-1}$ even for inhomogeneous variance profile for $X$. The isotropic local law, i.e. (2.9) for any vectors, can be proven from the averaged local law by considering the identity (we omitted the $z$ superscript)

\[(A.1) \quad G - M = -MD + M[S[G - M]M + M[S(G - M)(G - M)],\]

where

\[D := (H - \mathbb{E}H)G + S[G],\]

and the linear map $S$ on any $2n \times 2n$ matrix $R$ is given by

\[S[R] := \left(\begin{array}{cc}
\frac{1}{n} \sum_{i=n+1}^{2n} R_{ii} & \mathbf{0} \\
\mathbf{0} & \left(\frac{1}{n} \sum_{i=1}^{n} R_{ii}\right) \mathbf{I}
\end{array}\right) \in \mathbb{C}^{2n \times 2n}.
\]

In particular, the average local law guarantees that

\[(A.2) \quad \|S[G(i\eta) - M(i\eta)]\| \leq \frac{n^{\xi}}{n\eta}\]

with very high probability. The block matrix $H = H^z$ is a very special case of the general ensembles with arbitrary expectation and possible correlations studied in [15]. According to [15, Theorem 4.1], the error term $D$ in isotropic sense (i.e. tested against deterministic vectors) can be estimated in the convenient norms for any $p \geq 1$

\[\|D\|_p := \sup \{\mathbb{E}(\langle x, Dy\rangle)^p : \|x\| = \|y\| = 1\},\]

by

\[(A.3) \quad \|D\|_p \leq n^{\xi} \sqrt{\frac{\|S\|_q}{n\eta}} \left(1 + \|G\|_q\right)^C \left(1 + \frac{\|G\|_q}{n^{1/2}}\right)^{Cp}, \quad q := Cp^4,
\]

with some large constant $C$. We stress that, unlike most results in [15], the estimates from [15, Theorem 4.1] do not require any flatness condition on $S$, see [15, Assumption E]. This is essential since the matrix $H^z - \mathbb{E}H^z$ with large zero blocks does not satisfy flatness.

If we ignore the last (quadratic in $G - M$) term in (A.1) and assume that $\|G\|_q$ is bounded, then the boundedness of $M$ from (2.7), the consequence of the average law (A.2), and the estimate (A.3) applied for arbitrary large $p$ would give $\|G - M\|_p \leq n^{\xi}/\sqrt{m\eta}$, which immediately shows (2.9) after a Markov inequality. Intuitively, for $\eta \gg n^{-1}$ the quadratic term is negligible and the bound $\|G - M\|_p \leq n^{\xi}/\sqrt{m\eta}$ for any $p$ together with the boundedness of $M$ implies the boundedness of $\|G\|_p$ for any $p$. To make this reasoning rigorous we run the usual bootstrap argument from [15, Section 5.3] to break the apparent circularity by starting from a large $\eta$ and gradually reducing it. Since this standard argument has been presented many times, we omit the details.

So far we proved the isotropic local law for $\eta \gg n^{-1}$. Following the easy argument in [12, Appendix A], we can easily extend all these estimates to any small $\eta > 0$. This proves Proposition 2.6. □
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