UNIFICATION OF THE SOLUBLE TWO-DIMENSIONAL VECTOR COUPLING MODELS

by

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Abstract

The general theory of a massless fermion coupled to a massive vector meson in two dimensions is formulated and solved to obtain the complete set of Green’s functions. Both vector and axial vector couplings are included. In addition to the boson mass and the two coupling constants, a coefficient which denotes a particular current definition is required for a unique specification of the model. The resulting four parameter theory and its solution are shown to reduce in appropriate limits to all the known soluble models, including in particular the Schwinger model and its axial vector variant.
The fact that the theory of a massless fermion with a current-current coupling in one space and one time dimension can be solved exactly was discovered by Thirring [1] some years ago. The model subsequently was solved by Johnson [2] who realized that an essential ingredient had to be a very precise definition of the current operator. He adopted a procedure in which the current is realized as an average of spacelike and timelike limits of the product of two field operators. This averaging process was motivated by the need to obtain a covariant result, but introduced the somewhat undesirable feature of a timelike limit which does not fit comfortably into a Cauchy initial value formulation.

The most general solution of this model was obtained by the author [3] using an extension of Schwinger’s gauge invariant definition of the current $j^\mu$. Specifically, one writes in the case of a charged fermion $\psi(x)$ coupled to an external field $A_\mu$ [4]

$$j^\mu(x) = \lim_{x \rightarrow x'} \frac{1}{2} \psi(x) q \alpha^\mu \exp \left[ i q \int_{x'}^{x} dx' \left( \xi A^\mu - \eta \gamma^5 \epsilon_{\mu\nu} A^\nu \right) \right] \psi(x')$$

where the Dirac matrices $\alpha^0$ and $\alpha^1$ are conveniently taken to be the unit matrix and the Pauli matrix $\sigma_3$, respectively, and the limit is taken from a spacelike direction. The parameters $\xi$ and $\eta$ are required by Lorentz invariance to satisfy the constraint

$$\xi + \eta = 1 .$$

For reasons of symmetry it is convenient throughout this paper to keep both of these parameters in the general formulation since $\xi = 1$ ($\eta = 1$) corresponds to vector (axial vector) conservation [5]. In particular the Johnson solution can be seen to coincide with the choice $\xi = \eta = \frac{1}{2}$.

Perhaps the most well known of the two-dimensional models is the Schwinger model [6] which is simply QED for a massless fermion. Although its greatest success was the confounding of the conventional view that gauge invariance implied zero mass, it continues to be studied in widely varied applications. An extension of the Schwinger model to the massive vector meson case was made by Sommerfield [7] using Johnson’s current definition and by Brown [8] who employed Schwinger’s $\xi = 1$ limit of Eq. (1). The author [9] showed that the results of refs. 7 and 8 are obtained as the $\xi = \frac{1}{2}$ and $\xi = 1$ limits respectively of a formulation in which Eq. (1) is used as the current definition.
A model in which only a single component of the fermion field was coupled via the current operator to a massive vector meson was subsequently proposed and solved by the author [10]. This model (hereafter SCM) has the same Green’s functions as one which was subsequently proposed by Jackiw and Rajaraman [11]. The latter formulation has come to be known as the chiral Schwinger model. However, as has been pointed out [12], the name is somewhat unfortunate since the equations of motion are inconsistent with zero bare mass for the photon even though the Green’s functions are a consistent set by virtue of the equivalence to the SCM.

More recently an attempt has been made [13] to generalize the SCM by allowing an arbitrary admixture of vector and axial vector coupling. This could in principle provide an interpolation between the Schwinger model and the SCM. However, it fails to accommodate a variation of the Schwinger model in which the coupling is through the axial vector current as well as the known $\xi$ dependence of the Thirring model and the vector meson model of refs. 7-9. In this paper the task of finding the most general formulation of the soluble two dimensional theories is carried out and its solution obtained. It reduces in all the appropriate limits to the known soluble models.

One begins with the Lagrangian
\[
\mathcal{L} = \frac{i}{2} \psi^\alpha \partial_\mu \psi + \frac{1}{4} G^\mu\nu G_{\mu\nu} - \frac{1}{2} G^\mu\nu (\partial_\mu B_\nu - \partial_\nu B_\mu) - \frac{1}{2} \mu_0^2 B_\mu B_\mu + e j^\mu B_\mu + j^\mu A_\mu + J^\mu B_\mu
\]
where $A_\mu$ and $B_\mu$ are external sources and $e$ is the axial vector coupling constant. The most general current allowed by Lorentz invariance for $e = 0$ is
\[
j^\mu(x) = \lim_{x \to x'} \frac{1}{2} \psi(x) \alpha^\mu q(1 + r \gamma_5) \exp \left[ i q \int_{x'}^x dx'' (\xi A_\mu - \eta \gamma_5 \epsilon^\mu\nu A_\nu + r \eta \gamma_5 A_\mu - r \xi \epsilon^\mu\nu A_\nu) \right] \psi(x') \tag{2}
\]
where $\gamma_5 = \alpha^1$. By careful application of the action principle and functional differentiation techniques a complete solution of the interacting theory can be obtained.

It is easy to see that the first quantity which must be computed is the current correlation function $D^{\mu\nu}$. From general considerations one infers that its Fourier transform can
be written as
\[ D^{\mu
u}(p) = (g^{\mu\alpha} + r^{\mu\alpha}) (g^{\nu\beta} + r^{\nu\beta}) \left[ D_1 \epsilon_\alpha \epsilon_\beta \epsilon_\sigma \epsilon_\tau p^\sigma p^\tau \\
+ D_2 p_\alpha p_\beta + D_3 (p_\alpha \epsilon_\beta \epsilon_\sigma p^\sigma + p_\beta \epsilon_\alpha \epsilon_\sigma p^\sigma) \right] . \] (3)

From ref. 3 it is found that for \( e = 0 \), \( D_1 = \xi/\pi p^2 \), \( D_2 = \eta/\pi p^2 \), and \( D_3 = 0 \). The vacuum-to-vacuum transition amplitude \( <0\sigma_1|0\sigma_2> \) (\( \sigma_1 \) and \( \sigma_2 \) specify distinct spacelike surfaces) can be put in the form
\[
<0\sigma_1|0\sigma_2> = <0\sigma_1|0\sigma_2>_{A=J=0} \exp \left[ \frac{i}{2} \int J_\mu(x) G^{\mu\nu}(x-x') J_\nu(x') dx dx' \right] \\
\exp \left[ \frac{i}{2} \int A_\mu(x) D^{\mu\nu}(x-x') A_\nu(x') dx dx' \right] \\
\exp \left[ i \int J_\mu(x) M^{\mu\nu}(x-x') A_\nu(x') dx dx' \right].
\]

Using the result
\[
<0\sigma_1|0\sigma_2> = \exp \left[ -ie \int dx \frac{\delta^2}{\delta J^\mu(x) \delta A_\mu(x)} \right] <0\sigma_1|0\sigma_2>_{e=0}
\]
where
\[
<0\sigma_1|0\sigma_2>_{e=0} = \exp \left[ \frac{i}{2} \int J_\mu(x) G^{\mu\nu}_0 (x-x') J_\nu(x') dx dx' \right] \\
\exp \left[ \frac{i}{2} \int A_\mu(x) D^{\mu\nu}_{e=0}(x-x') A_\nu(x') dx dx' \right]
\]
and
\[
G^{\mu\nu}_0(p) = \left( g^{\mu\nu} + \frac{p^{\mu} p^{\nu}}{\mu_0^2} \right) \frac{1}{p^2 + \mu_0^2} \] (4)

one obtains the formal result
\[
D = D_{e=0} \left[ 1 - e^2 G_0 D_{e=0} \right]^{-1} . \] (5)

In this equation (5) all Lorentz indices have been suppressed so that obtaining the actual solution of (5) is much more involved in the general case than might otherwise be expected.

A practical approach to this problem consists of writing
\[
D_i(p) = \sum_0^{\infty} D_i^{(n)} \quad i = 1, 2, 3
\]
and using Eqs. (3-5) to obtain $D_i^{(n+1)}$ in terms of $D_i^{(n)}$. Not surprisingly, in the case of $D_3$ the symmetry in $\mu \leftrightarrow \nu$ which characterizes the exact result (3) is not manifest in this approach so long as $D_1^{(n)}$ and $D_2^{(n)}$ are unconstrained. Thus one simplifies the calculation by imposing that symmetry in each order, thereby leading to an expression for $D_3^{(n)}$ in terms of $D_1^{(n)}$ and $D_2^{(n)}$. The consequence of this is that one obtains a two dimensional matrix relation (rather than a three dimensional one) of the form

$$D^{(n+1)} = KD^{(n)}$$

where $D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$ and

$$K = \left[ \begin{array}{cc}
(r^2 - 1) + \frac{p^2}{\mu_0^2} (\xi r^2 - \eta) \\
\xi(1 - r^2)^2 \left( 1 + \eta \frac{p^2}{\mu_0^2} \right) + 2\frac{p^2}{\mu_0^2} (r^2 - 1) \xi^2 + \frac{p^4}{\mu_0^4} \xi^2 & -\frac{r^2 p^4 \xi^2}{\mu_0^2} \\
\frac{p^4}{\mu_0^4} & -\eta(1 - r^2)^2 \left( 1 + \xi \frac{p^2}{\mu_0^2} \right) - 2\eta^2 (1 - r^2) \frac{p^2}{\mu_0^2} - \frac{p^4 \eta^2}{\mu_0^2} \end{array} \right]$$

The solution is thus

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = (1 - K)^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \frac{1}{\pi p^2} .$$

Very considerable algebra allows one to solve this formal equation. As a first step it is found that

$$\det(1 - K) = \left[ 1 - \frac{e^2}{\pi \mu_0^2} (\xi r^2 + \eta) \right] \frac{p^2 + \mu^2}{p^2 + \mu_0^2}$$

where $\mu$ the physical renormalized mass of the theory is given by

$$\mu^2 = \mu_0^2 \frac{[1 + \frac{\xi e^2}{\pi \mu_0^2} (1 - r^2)][1 - \frac{\eta e^2}{\pi \mu_0^2} (1 - r^2)]}{1 - \frac{e^2}{\pi \mu_0^2} (\xi r^2 + \eta)} .$$

It is encouraging to note that for $r = 0$ one obtains

$$\mu^2 = \mu_0^2 + \xi e^2 / \pi$$

appropriate to the vector meson model [9]; $r = \mu_0 = \eta = 0$ gives

$$\mu^2 = e^2 / \pi$$
of the Schwinger model; \( r = \pm 1 \) gives
\[
\mu^2 = \mu_0^2 \left[ 1 - \frac{e^2}{\pi \mu_0^2} \right]^{-1}
\]
as in the SCM (after allowing for a trivial difference in the definition of \( e \)); and for \( e \to 0, \ r \to \infty, \ er \to e, \ \xi = 0 \) gives
\[
\mu^2 = e^2/\pi
\]
as it must for the axial vector Schwinger model. The result (6) allows one finally to obtain
\[
D_1(p) = \frac{\xi}{\pi C_\xi} \left[ D(p) + \frac{e^2 \xi}{\pi \mu_0^2} \frac{C_\eta}{1 - \frac{e^2}{\pi \mu_0^2} (\xi r^2 + \eta)} \Delta(p) \right]
\]
\[
D_2(p) = \frac{\eta}{\pi C_\eta} \left[ D(p) + \frac{e^2 \eta}{\pi \mu_0^2} \frac{C_\xi}{1 - \frac{e^2}{\pi \mu_0^2} (\xi r^2 + \eta)} \Delta(p) \right]
\]
\[
D_3(p) = -\frac{e^2 \xi \eta r}{\pi^2 \mu_0^2} \frac{1}{1 - \frac{e^2}{\pi \mu_0^2} (\xi r^2 + \eta)} \Delta(p)
\]
where
\[
C_\xi = 1 + \frac{\xi e^2}{\pi \mu_0^2} (1 - r^2)
\]
\[
C_\eta = 1 - \frac{\eta e^2}{\pi \mu_0^2} (1 - r^2)
\]
\[
D(p) = 1/p^2
\]
and
\[
\Delta(p) = 1/(p^2 + \mu^2)
\]
One immediate result which follows from \( D_{\mu\nu}(p) \) is the equal time commutator of the charge density \( j^0(x) \) with the current density \( j^1(x) \). It has the form
\[
[j^0(x), j^1(x')] = -\frac{i}{\pi} \partial_t \delta(x - x') \frac{1}{C_\xi C_\eta} \left\{ 1 + r^2 + \frac{e^2}{\pi \mu_0^2} \left( \xi C_\eta + \eta C_\xi \right)^2 + r^2 \left( \xi C_\eta + \eta C_\xi \right)^2 \right\}
\]
The vector meson propagator can now be computed in terms of \( D_{\mu\nu} \) according to
\[
G_{\mu\nu} = G_0^{\mu\nu} + e^2 G_0^{\mu\alpha} D_{\alpha\beta} G_0^{\beta\nu}
\]
The result is

\[ G^{\mu\nu}(p) = \left( g^{\mu\nu} + \frac{p^\mu p^\nu}{\mu^2} \right) \Delta + \frac{e^2}{\pi \mu_0^2} \frac{1}{C_\xi C_\eta \mu_0^2} \left\{ \left[ r(p^\mu \epsilon^\nu p_\alpha + p^\nu \epsilon^\mu p_\alpha) + (1 + r^2)p^\mu p^\nu \right] (D - \Delta) + p^\mu p^\nu \frac{1 + r^2 - \frac{e^2}{\pi \mu_0^2} (\eta + \xi r^4)}{1 - \frac{e^2}{\pi \mu_0^2} (\xi r^2 + \eta)} \Delta \right\} . \]

Finally, the calculation of the bosonic sector is completed by means of

\[ M^{\mu\nu} = eG^{\mu\alpha}_0 D_{\alpha\beta} g^{\beta\nu} \]

which yields

\[ M^{\mu\nu}(p) = \frac{e}{\pi \mu_0^2} \frac{1}{\mu^2} \left\{ \frac{1}{1 - \frac{e^2}{\pi \mu_0^2} (\xi r^2 + \eta)} p^\mu \left[ (\xi r^2 + \eta) p^\nu + r\epsilon^{\nu\alpha} p_\alpha \right] \Delta + \left[ \epsilon^{\mu\alpha} p_\alpha \epsilon^{\nu\beta} p_\beta \left( \frac{\xi}{C_\xi} + \frac{\eta r^2}{C_\eta} \right) + p^\mu p^\nu \left( \frac{\eta}{C_\eta} + \frac{\xi r^2}{C_\xi} \right) \right] (D - \Delta) \right\} . \]

The fermionic sectors of the model can now be obtained in a fairly straightforward fashion. The 2n-point functions \( G(x_1, \ldots, x_{2n}) \) are calculated from

\[ <0\sigma_1|0\sigma_2 > G(x_1, \ldots, x_{2n}) = \exp \left[ -ie \int \frac{\delta^2}{\delta A^\mu(x) \delta J_\mu(x)} \right] G_{e=0}(x_1, \ldots, x_{2n}) \]

using [14]

\[ G_{e=0}(x_1, \ldots, x_{2n}) = \exp \left[ i \sum_i q_i \int G_0(x - x_i)(1 + r\gamma_5)\alpha^\mu(x) A_\mu(x) dx \right] \]

where \( G_0(x) \) is defined by

\[ \alpha^\mu \frac{1}{i} \partial_\mu G_0(x) = \delta(x) . \]

It is convenient to cast this into the form

\[ G(x_1, \ldots, x_{2n}) = \exp \left[ i \sum_i q_i \int A^\mu(x) N_\mu(x - x_i) dx \right] \exp \left[ i \sum_i q_i \int J^\mu(x) M_\mu(x - x_i) dx \right] G_{0,0,0}(x_1, \ldots, x_{2n}) . \]
Tedious calculation yields

\[-iN_\mu(p) = \left\{ p_\mu \left[ \frac{1}{C_\eta} + r\gamma_5 \frac{1}{C_\xi} \right] - \epsilon_{\mu\alpha} p^\alpha \gamma_5 \left[ \frac{1}{C_\xi} + r\gamma_5 \frac{1}{C_\eta} \right] \right\} D(p) \]

\[+ \frac{e^2}{\pi \mu_0^2} \left\{ \frac{1}{1 - \frac{e^2}{\pi \mu_0^2}(\xi r^2 + \eta)} \left[ r^2 p_\mu \frac{1}{C_\eta} + r p_\mu \gamma_5 \frac{1}{C_\xi} - r^2 \epsilon_{\mu\alpha} p^\alpha \gamma_5 \right] + \epsilon_{\mu\alpha} p^\alpha r \right\} \Delta(p) \]

\[ - \epsilon_{\mu\alpha} p^\alpha r \left[ - \epsilon_{\mu\alpha} p^\alpha \gamma_5 \left[ \frac{\xi}{C_\xi} - r\gamma_5 \frac{\eta}{C_\eta} \right] (1 - r^2) \right] \Delta(p) \]

and

\[M_\mu = \frac{ie}{\mu_0^2} \left\{ \left[ p_\mu \left( \frac{1}{C_\eta} + r\gamma_5 \frac{1}{C_\xi} \right) - \epsilon_{\mu\alpha} p^\alpha \gamma_5 \left( \frac{1}{C_\xi} + r\gamma_5 \frac{1}{C_\eta} \right) \right] D(p) \]

\[+ \left[ \epsilon_{\mu\alpha} p^\alpha \gamma_5 \left( \frac{1}{C_\xi} + r\gamma_5 \frac{1}{C_\eta} \right) + \frac{e^2}{\pi \mu_0^2} \frac{1}{1 - \frac{e^2}{\pi \mu_0^2}(r^2 + \eta)} r p_\mu \left( \frac{r}{C_\eta} + \gamma_5 \frac{1}{C_\xi} \right) \right] \Delta(p) \].

The solution of the model is completed with the calculation of \(G_{0,0,e}(x_1, \ldots x_{2n})\). In this case one finds

\[G_{0,0,e}(x_1, \ldots x_{2n}) = \exp \left\{ \frac{ie^2}{\mu_0^2} \sum_{i,j} q_i q_j \left[ (1 - r^2) \left( \frac{1}{C_\eta} - \gamma_5 \frac{1}{C_\xi} \right) D(x_i - x_j) \right. \right. \]

\[\left. + \left. \frac{1}{1 - \frac{e^2}{\pi \mu_0^2}(\xi r^2 + \eta)} r^2 C_\xi C_\eta + \frac{C_\eta}{C_\xi} \gamma_5 + r(\gamma_5 + \gamma_5) \right) \Delta(x_i - x_j) \left\} \right\} G_0(x_1, \ldots x_{2n}) ,

a special case of which gives the two point function

\[G(x) = \exp \left\{ -i \pi \left( \frac{e^2}{\mu_0^2} \right)^2 (1 - r^2) \left( \frac{1}{C_\xi} \right) \right. \]

\[\left. - \frac{1}{\mu_0^2} \frac{1}{1 - \frac{e^2}{\pi \mu_0^2}(r^2 + \eta)} \left( r^2 C_\xi C_\eta + C_\eta \frac{C_\eta}{C_\xi} + 2r\gamma_5 \right) \right\} G_0(x) .

The solution derived here agrees in all particulars with results which have been obtained for the Thirring model (\(e, \mu_0 \to \infty, e/\mu_0 \) finite), the vector meson model, the Schwinger model (vector and axial vector) and the single component model. It should also be noted that it coincides with the model of ref. 13 provided that \(\xi = 1\) (i.e., in the case of the original Schwinger definition of the current).

Before concluding it is of interest to state the limitations which must be placed on the results obtained. Clearly, there exists another (though somewhat less interesting) case
in which the current is coupled to the derivative of a scalar field. That set of models can be handled by identical techniques provided that the free vector meson propagator is replaced by $p_{\mu}p_{\nu}/(p^2 + \mu^2)$. Are there any other models which fall outside the scope of this work? The answer is certainly yes! An example of such a model is one in which the left and right chiral projections of the current operator are each coupled to an independent external vector potential. If one subsequently couples the sum of the two chiral currents to a single vector meson with equal coefficients, there is no limit in which such a system can reduce to the Schwinger model. This is despite the fact that the Lagrangians of the two systems formally appear to be the same. Thus the model considered in this paper includes all the known soluble models but also provides a valuable guide in showing that there must yet exist a wider class of soluble models which are distinct from these. Details of the model presented here and the additional inequivalent extensions which it suggests will be provided in subsequent publications.

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2. K. Johnson, Nuovo Cimento 20, 773 (1961).
3. C.R. Hagen, Nuovo Cimento 51B, 169 (1967).
4. In order to simplify the final form of the fermionic Green’s functions one uses here a charge matrix
   \[ q = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]
   which acts in a two dimensional charge space of the Hermitian field \( \psi(x) \).
5. It is well to point out here that a significant fraction of the subsequent literature in this subfield chooses to ascribe the effect of the \( \xi \) parameter to the freedom available in the regularization of a certain logarithmically divergent integral. Although the more physical definition of Eq. (1) is the choice of this paper, identical results can be obtained using either approach.
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12. C.R. Hagen, Phys. Rev. Lett. 55, 2223 (1985).
13. A. Bassetto, L. Grignolo, and P. Zanca, Phys. Rev. 50D, 1077 (1994).
14. In dealing with the fermionic sectors of the model a matrix with a subscript should be taken to mean that it acts on that particular index of the \( 2n \) point function.