SINGULARITY CATEGORIES OF GORENSTEIN MONOMIAL ALGEBRAS

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Abstract. In this paper, we study the singularity category $D_{sg}(\text{mod} A)$ and the $\mathbb{Z}$-graded singularity category $D_{sg}(\text{mod}^\mathbb{Z} A)$ of a Gorenstein monomial algebra $A$. Firstly, for a positively graded 1-Gorenstein algebra, we prove that its $\mathbb{Z}$-graded singularity category admits a silting object. Secondly, for $A = kQ/I$ being a 1-Gorenstein monomial algebra, which is viewed as a $\mathbb{Z}$-graded algebra by setting each arrow to be degree one, we prove that $D_{sg}(\text{mod}^\mathbb{Z} A)$ has a tilting object. In particular, $D_{sg}(\text{mod}^\mathbb{Z} A)$ is triangle equivalent to the derived category of a hereditary algebra $H$ which is of finite representation type. Finally, we give a characterization of 1-Gorenstein monomial algebras $A$, and describe their singularity categories by using the triangulated orbit categories of type $A$.

1. Introduction

The singularity category of an algebra is defined to be the Verdier quotient of the bounded derived category with respect to the thick subcategory formed by complexes isomorphic to bounded complexes of finitely generated projective modules [16], see also [30]. Recently, Orlov’s global version [52] attracted a lot of interest in algebraic geometry and theoretical physics. In particular, the singularity category measures the homological singularity of an algebra [30]: the algebra has finite global dimension if and only if its singularity category is trivial.

Gorenstein (also called Iwanaga-Gorenstein) algebra $A$, where by definition $A$ has finite injective dimensions both as a left and a right $A$-module, is inspired from commutative ring theory. For a Gorenstein algebra $A$, the injective dimensions of $A$ are equal both as a left and a right $A$-module. When these dimensions are less or equal to $d$ for a positive integer $d$, we call $A$ a $d$-Gorenstein algebra. A fundamental result of Buchweitz [16] and Happel [30] states that for a Gorenstein algebra $A$, the singularity category is triangle equivalent to the stable category of Gorenstein projective (also called (maximal) Cohen-Macaulay) $A$-modules, which generalizes Rickard’s result [51] on self-injective algebras. It is worth noting that Gorenstein algebras, especially 1-Gorenstein algebras are playing an important role in representation theory of finite dimensional algebras, which include some important classes of algebras, e.g. the cluster-tilted algebras [14, 15], 2-CY-tilted algebras [38, 53], or more general the endomorphism algebras of cluster tilting objects in triangulated categories [45], the class of 1-Gorenstein algebras defined by Geiss, Leclerc and Schröer via quivers with relations associated with symmetrizable Cartan matrices [25].

In general, it is difficult to describe the singularity categories and the Gorenstein projective modules. Many people are trying to describe them for some special classes of algebras, see e.g. [18, 19, 20, 21, 32, 33, 41, 42, 43, 47, 48, 49, 55, 58, 59, 62]. In this paper, we focus on describing the (graded) singularity categories of Gorenstein monomial algebras. For a monomial algebra $A$, the authors in [20] gave an explicit classification of indecomposable Gorenstein projective
we use this result to describe the singularity categories of quiver representations over local rings $A_i$, where $(A_{i \geq 0})$ is the graded algebra of monomial modules, then $T$ is a tilting object, see Theorem 4.5. In particular, Gorenstein monomial algebras, we prove that it is true, i.e. their graded singularity categories are defined as follows. For a $Z$-graded $A$-module $X$, $X_{\geq i}$ is a $Z$-graded sub $A$-module of $X$ defined by

$$(X_{\geq i})_j := \begin{cases} 
0 & \text{if } j < i \\
X_j & \text{if } j \geq i,
\end{cases}$$

and $X_{\leq i}$ is a $Z$-graded factor $A$-module $X/X_{\geq i+1}$ of $X$.

Now we define a $Z$-graded $A$-module by

$$T := \bigoplus_{i \geq 0} A(i)_{\leq 0},$$

where $(i)$ is the grade shift functor. We prove that for any positively graded Gorenstein algebra $A = \bigoplus_{i \geq 0} A_i$ such that $\text{gl.dim} A_0 < \infty$, if $T = \bigoplus_{i \geq 0} A(i)_{\leq 0}$ is a Gorenstein projective $A$-module, then $T$ is tilting object in $D_{sg}(\text{mod}^Z A)$, see Proposition 3.3. It is worth noting that we use this result to describe the singularity categories of quiver representations over local rings $k[X]/(X^k)$, $k > 1$, see [17].

A natural question is that when does $D_{sg}(\text{mod}^Z A)$ admit a tilting object for a positively graded algebra $A = \bigoplus_{i \geq 0} A_i$. In this case, $A_0$ should satisfy that $\text{gl.dim} A_0 < \infty$, see Lemma 4.4. We refer to [31, 32, 33, 11, 12, 43, 47, 51, 60, 62] for recent results related to this question. For Gorenstein monomial algebras, we prove that it is true, i.e. their graded singularity categories admit tilting objects, see Theorem 4.5. In particular, $D_{sg}(\text{mod}^Z A)$ is triangle equivalent to $D^b(H)$, where $H$ is a hereditary algebra of finite representation type since $A$ is CM-finite, see Proposition 4.9. As corollaries, for $A$ being a Gorenstein quadratic monomial algebra (in particular, gentle algebra), then there exists a hereditary algebra $B = kQ$ with $Q$ a disjoint union of finitely many simply-laced quivers of type $A_1$ such that $D_{sg}(\text{mod}^Z A) \simeq D^b(\text{mod} B)$; for $A$ being a Gorenstein Nakayama algebra, then there exists a hereditary algebra $B = kQ$ with $Q$ a disjoint union of finitely many simply-laced quivers of type $A$ such that $D_{sg}(\text{mod}^Z A) \simeq D^b(\text{mod} B)$.

At last, we consider the $1$-Gorenstein monomial algebra, which is a special kind of monomial algebras. First, we character $1$-Gorenstein monomial algebra $A = kQ/I$ by the minimal generators of the two-sided ideal $I$, see Theorem 5.3, which generalised [22, Proposition 3.1] for gentle algebras. Second, there exists a hereditary algebra $B = kQ^B$ with $Q^B$ a disjoint union of finitely many simply-laced quivers of type $A$ such that $D_{sg}(\text{mod}^Z A) \simeq D^b(\text{mod} B)$, see Theorem 5.9.
Third, we describe explicitly its ordinary singularity category $D_{sg}(\text{mod} A)$ by using the triangulated orbit categories of type $A$ in the sense of [37], see Theorem 6.16. In fact, in order to prove this result, we consider a kind of gluing algebras, which is defined by T. Brüstle in [12], and we call them Brüstle’s gluing algebras. We prove that singularity categories, and also Gorenstein properties are invariant under Brüstle’s gluing process, see Theorem 6.9 and Proposition 6.11.

The paper is organized as follows. In Section 2, we collect some materials on positively graded algebras, Gorenstein algebras, (graded) Gorenstein projective modules and (graded) singularity categories. In Section 3, we prove that there is a silting object in $D_{sg}(\text{mod}^Z A)$ for a 1-Gorenstein positively graded algebra $A = \bigoplus_{i \geq 0} A_i$ if $\text{gl.dim} A_0 < \infty$. In Section 4, we prove that $D_{sg}(\text{mod}^Z A)$ admits a tilting object for $A$ a Gorenstein monomial algebra. In Section 5, we characterize the 1-Gorenstein monomial algebras. In Section 6, we describe the singularity categories for 1-Gorenstein monomial algebras.

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After finishing a first version of this paper appeared in arXiv, we were informed kindly from Professor Osamu Iyama and Professor Kota Yamaura that H. Minamoto, Y. Kimura and K. Yamaura \[30\] has also obtained the same result with Theorem 5.3. We are deeply indebted to them for this and for their helpful comments!

2. Preliminaries

Throughout this paper $k$ is an algebraically closed field and algebras are finite-dimensional $k$-algebras unless specified. We denote by $D$ the $k$-dual, i.e. $D(-) = \text{Hom}_k(-, k)$.

Let $A$ be a $k$-algebra. We denote by mod $A$ the category of finitely generated (left) modules, by proj $A$ the category of finitely generated projective $A$-modules.

For an additive category $A$, we use Ind $A$ to denote the set of all non-isomorphic indecomposable objects in $A$.

Let $\mathcal{A}$ be an abelian category, $\mathcal{X}$ a full additive subcategory of $\mathcal{A}$. Let $M \in \mathcal{A}$ be an object. A right $\mathcal{X}$-approximation of $M$ is a morphism $f : X \to M$ such that $X \in \mathcal{X}$ and any morphism $X' \to M$ from an object $X' \in \mathcal{X}$ factors through $f$. Dually one has the notion of left $\mathcal{X}$-approximation. The subcategory $\mathcal{X} \subseteq \mathcal{A}$ is said to be contravariantly finite (resp. covariantly finite) provided that each object in $\mathcal{A}$ has a right (resp. left) $\mathcal{X}$-approximation. The subcategory $\mathcal{S} \subseteq \mathcal{A}$ is said to be functorially finite provided it is both contravariantly finite and covariantly finite.

2.1. Positively graded algebras. Let $A$ be a $(\mathbb{Z})$-graded algebra, i.e., $A = \bigoplus_{i \in \mathbb{Z}} A_i$, where $A_i$ is the degree $i$ part of $A$. A $(\mathbb{Z})$-graded $A$-module $X$ is of form $\bigoplus_{i \in \mathbb{Z}} X_i$, where $X_i$ is the degree $i$ part of $X$. The category $\text{mod}^Z A$ of (finitely generated) $\mathbb{Z}$-graded $A$-modules is defined as follows.

- The objects are graded $A$-modules,
- For graded $A$-modules $X$ and $Y$, the morphism space from $X$ to $Y$ in $\text{mod}^Z A$ is defined by

$$\text{Hom}_{\text{mod}^Z A}(X, Y) := \text{Hom}_A(X, Y)_0 := \{f \in \text{Hom}_A(X, Y) | f(X_i) \subseteq Y_i \text{ for any } i \in G\}.$$  

We denote by proj$^Z A$ the full subcategory of $\text{mod}^Z A$ consisting of projective objects.
For $i \in \mathbb{Z}$, we denote by $(i) : \text{mod}^\mathbb{Z} A \to \text{mod}^\mathbb{Z} A$ the grade shift functor. Then for any two graded $A$-modules $X, Y$,

(1) \[ \text{Hom}_A(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_A(X, Y(i))_0. \]

Denote by $J_A$ the Jacobson radical of $A$.

**Proposition 2.1** ([62]). Assume that $J_A = J_{A_0} \oplus \bigoplus_{i \in \mathbb{Z} \setminus \{0\}} A_i$. We take a set $\overline{PT}$ of idempotents of $A_0$ such that $\{eA_0 | e \in \overline{PT}\}$ is a complete list of indecomposable projective $A_0$-modules. Then the following assertions hold.

(a) Any complete set of orthogonal primitive idempotents of $A_0$ is that of $A$.

(b) A complete list of simple objects in $\text{mod}^\mathbb{Z} A$ is given by

\[ \{S(i) | i \in \mathbb{Z}, S \text{ is a simple } A_0\text{-module}\}. \]

(c) A complete list of indecomposable projective objects in $\text{mod}^\mathbb{Z} A$ is given by

\[ \{eA(i) | i \in \mathbb{Z}, e \in \overline{PT}\}. \]

(d) A complete list of indecomposable injective objects in $\text{mod}^\mathbb{Z} A$ is given by

\[ \{D(Ae)(i) | i \in \mathbb{Z}, e \in \overline{PT}\}. \]

Let $A$ be a positively graded algebra, i.e., $A = \bigoplus_{i \geq 0} A_i$. Note that for a positively graded algebra $A$, the equation

\[ J_A = J_{A_0} \oplus \bigoplus_{i \neq 0} A_i \]

always holds. So $A$ satisfies the assumption of Proposition 2.1, see [62, Proposition 2.18].

2.2. **Gorenstein algebra and Singularity category.** Let $A$ be an algebra. A complex

\[ P^\bullet : \cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to \cdots \]

of finitely generated projective $A$-modules is said to be **totally acyclic** provided it is acyclic and the Hom complex $\text{Hom}_A(P^\bullet, A)$ is also acyclic [5]. An $A$-module $M$ is said to be (finitely generated) **Gorenstein projective** provided that there is a totally acyclic complex $P^\bullet$ of projective $A$-modules such that $M \cong \text{Ker} d^0$ [23]. We denote by $\text{Gproj} A$ the full subcategory of $\text{mod} A$ consisting of Gorenstein projective modules.

The following lemma follows from the definition of Gorenstein projective module easily.

**Lemma 2.2.** Let $A$ be an algebra. Then

(i) \[ \text{Gproj} A = \{ M \in \text{mod} A | \exists \text{ an exact sequence } 0 \to M \to P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots , \]

\[ \text{with } P^i \in \text{proj} A, \text{ker } d^i \in \mathbb{Z}_A, \forall i \geq 0 \}. \]

(ii) If $M$ is Gorenstein projective, then $\text{Ext}^1_A(M, L) = 0$, $\forall i > 0$, for all $L$ of finite projective dimension or of finite injective dimension.

(iii) If $P^\bullet$ is a totally acyclic complex, then all $\text{Im } d^i$ are Gorenstein projective; and any truncations

\[ \cdots \to P^i \to \text{Im } d^i \to 0, \to \text{Im } d^i \to P^{i+1} \to \cdots \]

and

\[ 0 \to \text{Im } d^i \to P^{i+1} \to \cdots \to P^j \to \text{Im } d^j \to 0, i < j \]

are $\text{Hom}_A(-, \text{proj } A)$-exact.
Definition 2.3. An algebra $A$ is called a Gorenstein (or Iwanaga-Gorenstein) algebra if $A$ satisfies $\text{inj. dim } A < \infty$ and $\text{inj. dim } A_A < \infty$.

Then a $k$-algebra $A$ is Gorenstein if and only if $\text{inj. dim } A_A < \infty$ and $\text{proj. dim } D(A) < \infty$. Observe that for a Gorenstein algebra $A$, we have $\text{inj. dim } A = \text{inj. dim } A_A$, see [7, Lemma 6.9] and [30, Lemma 1.2], the common value is denoted by $\text{G. dim } A$. If $\text{G. dim } A \leq d$, we say that $A$ is a $d$-Gorenstein algebra.

Definition 2.4. Let $A$ be a Gorenstein algebra. A finitely generated $A$-module $M$ is called (maximal) Cohen-Macaulay if

$$\text{Ext}^i_A(M, A) = 0 \text{ for } i \neq 0.$$ 

The full subcategory of Cohen-Macaulay modules in $\text{mod } A$ is denoted by $\text{CM}(A)$. Let $\mathcal{X}$ be a subcategory of $\text{mod } A$. Then $\perp \mathcal{X} := \{ M | \text{Ext}^i(M, X) = 0, \text{ for all } X \in \mathcal{X}, i \geq 1 \}$. Dually, we can define $\mathcal{X} \perp$.

From Theorem 2.5, it is easy to see that for a Gorenstein algebra, the concept of Gorenstein projective modules coincides with that of Cohen-Macaulay modules.

Theorem 2.5 ([8], [23]). Let $A$ be an algebra and let $d \geq 0$. Then the following statements are equivalent:

(a) the algebra $A$ is $d$-Gorenstein;

(b) $\text{Gproj}(A) = \Omega^d(\text{mod } A)$.

In this case, a module $G$ is Gorenstein projective if and only if there is an exact sequence $0 \to G \to P^0 \to P^1 \to \cdots$ with each $P^i$ projective.

For a module $M$ take a short exact sequence $0 \to \Omega(M) \to P \to M \to 0$ with $P$ projective. The module $\Omega(M)$ is called a syzygy module of $M$. Note that syzygy modules of $M$ are not uniquely determined, while they are naturally isomorphic to each other in $\text{mod } A$.

Theorem 2.6 ([6], see also [10]). Let $A$ be a $d$-Gorenstein algebra. Then

(i) $\mathcal{P}^{< \infty}(\text{mod } A) = \mathcal{P}^{\leq d}(\text{mod } A)$;

(ii) $\text{Gproj } A$ and $\mathcal{P}^{\leq d}(\text{mod } A)$ are functorially finite in $\text{mod } A$;

(iii) For any $M \in \text{mod } A$, there exist the following exact sequences

$$0 \to Y_M \to X_M \to M \to 0,$$

$$0 \to M \to Y^M \to X^M \to 0,$$

such that $X_M, X^N \in \text{Gproj } A$, $Y^M \in \mathcal{P}^{\leq d-1}(\text{mod } A)$, $Y_M \in \mathcal{P}^{\leq d}(\text{mod } A)$.

In fact, for a $d$-Gorenstein algebra $A$, $(\text{Gproj } A, \mathcal{P}^{\leq d}(\text{mod } A))$ is a (complete) cotorsion pair in $\text{mod } A$. Here the notion of cotorsion pair is defined in [57], for complete cotorsion pair, see [23], [26].

Similarly, for graded algebras, one can define the notions of graded Gorenstein algebras, graded Gorenstein projective modules. We denote by $\text{Gproj}^Z A$ the full subcategory of $\text{mod } Z A$ formed by all $Z$-graded Gorenstein projective modules.

For a graded algebra $A$, every (finitely generated) projective modules and injective modules are gradable, see [27, Corollary 3.4]. Let $F: \text{mod } Z A \to \text{mod } A$ be the forgetful functor, then for any $M \in \text{mod } Z A$, $M$ is graded projective (resp. graded injective) if and only if $F(M)$ is projective (resp. injective) as $A$-module, see [28, Proposition 1.3, Proposition 1.4]. In fact, one
has \( \text{grinj. dim}_A M = \text{inj. dim}_A M \) for any graded \( A \)-module \( M \), see also [16], where \( \text{grinj. dim}_A M \) denotes the injective dimension in \( \text{mod}^\mathbb{Z} A \), which is called the \textit{graded injective dimension} of \( M \). Thus the graded algebra \( A \) is graded Gorenstein if and only if it is Gorenstein as an ungraded ring. Certainly, for a finite-dimensional positively graded algebra \( A = \bigoplus_{i \geq 0} A_i \), which we mainly focus on, \( A \) is \( d \)-graded Gorenstein if and only if \( A \) is \( d \)-Gorenstein. So we do not distinguish them in the following.

Let \( A \) be a graded Gorenstein algebra. A graded \( A \)-module \( M \) is called \textit{graded (maximal) Cohen-Macaulay} if \( M \) satisfies that

\[
\text{Ext}^i_{\text{mod}^\mathbb{Z} A}(M, A(j)) = 0 \quad \text{for} \quad i \neq 0, j \in \mathbb{Z}.
\]

We denote by \( \text{CM}^\mathbb{Z}(A) \) the full subcategory of \( \text{mod}^\mathbb{Z} A \) formed by all \( \mathbb{Z} \)-graded Cohen-Macaulay modules. Since for arbitrary graded algebra, every \textit{graded projective module} is a direct summand of a \textit{graded free module}, we get that \( \text{CM}^\mathbb{Z}(A) = \perp \text{proj}^\mathbb{Z}(A) \). It is worth noting that the graded version of Theorem 2.5 and Theorem 2.6 are also valid. In particular, \( \text{Gproj}^\mathbb{Z} A = \text{CM}^\mathbb{Z}(A) = \perp \text{proj}^\mathbb{Z}(A) \)

**Lemma 2.7.** Let \( A = \bigoplus_{i \geq 0} A_i \) be a graded algebra. Then the forgetful functor \( F : \text{mod}^\mathbb{Z} A \to \text{mod} A \) induces a functor from \( \text{Gproj}^\mathbb{Z} A \) to \( \text{Gproj} A \), which is also denoted by \( F \).

**Proof.** Since \( F \) is exact and maps graded projective modules to projective modules, we only need to prove that \( F \) induces a functor from \( \perp \text{proj}^\mathbb{Z}(A) \) to \( \perp \text{proj} A \).

For any \( M, N \in \text{mod}^\mathbb{Z} A \), take a projective resolution of \( M \):

\[
\cdots \xrightarrow{f_{i+1}} P_i \xrightarrow{f_i} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0
\]

in \( \text{mod}^\mathbb{Z} A \) with \( P_i \) graded projective. Then for any \( j \in \mathbb{Z} \), we get an exact sequence:

\[
\text{Hom}_{\text{mod}^\mathbb{Z} A}(P_0, N(j)) \to \text{Hom}_{\text{mod}^\mathbb{Z} A}(\ker f_0, N(j)) \to \text{Ext}^1_{\text{mod}^\mathbb{Z} A}(M, N(j)) \to 0.
\]

Proposition 2.1 implies that \( P_i \) is projective in \( \text{mod} A \) for any \( i \), we also get an exact sequence:

\[
\text{Hom}_{A}(P_0, N(j)) \to \text{Hom}_{A}(\ker f_0, N(j)) \to \text{Ext}^1_{A}(M, N(j)) \to 0.
\]

From (1), we get that \( \text{Hom}_{A}(P_0, N(j)) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\text{mod}^\mathbb{Z} A}(P_0, N(j)) \), and \( \text{Hom}_{A}(\ker f_0, N(j)) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\text{mod}^\mathbb{Z} A}(\ker f_0, N(j)) \), and then

\[
\text{Ext}^1_{A}(M, N) \cong \bigoplus_{j \in \mathbb{Z}} \text{Ext}^1_{\text{mod}^\mathbb{Z} A}(M, N(j)).
\]

By induction, one can check that

\[
\text{Ext}^i_{A}(M, N) \cong \bigoplus_{j \in \mathbb{Z}} \text{Ext}^i_{\text{mod}^\mathbb{Z} A}(M, N(j)), \forall i.
\]

From the above, our desired result follows directly. \( \square \)

**Definition 2.8 ([16]).** Let \( A = \bigoplus_{i \geq 0} A_i \) be a graded algebra. The singularity category is defined to be the Verdier localization

\[
D_{sg}(\text{mod} A) := D^b(\text{mod} A)/K^b(\text{proj} A),
\]

the \((\mathbb{Z}_-)\)-graded singularity category is

\[
D_{sg}(\text{mod}^\mathbb{Z} A) := D^b(\text{mod}^\mathbb{Z} A)/K^b(\text{proj}^\mathbb{Z} A),
\]

We denote by \( \pi : D^b(\text{mod} A) \to D_{sg}(\text{mod} A) \) and \( \pi^\mathbb{Z} : D^b(\text{mod}^\mathbb{Z} A) \to D_{sg}(\text{mod}^\mathbb{Z} A) \) the localization functor.
Theorem 2.9 (Buchweitz’s Theorem, see also [40] for a more general version). Let $A$ be a graded algebra. Then

(i) $\text{Gproj} A$ and $\text{Gproj}^Z A$ are Frobenius categories with the projective modules and $\mathbb{Z}$-graded projective modules as the projective-injective objects respectively.

(ii) There is an exact embedding $\Phi : \text{Gproj} A \rightarrow D_{sg}(\text{mod} A)$ given by $\Phi(M) = M$, where the second $M$ is the corresponding stalk complex at degree 0, and $\Phi$ is an equivalence if and only if $A$ is Gorenstein.

(iii) There is an exact embedding $\Phi^Z : \text{Gproj}^Z A \rightarrow D_{sg}(\text{mod}^Z A)$ given by $\Phi^Z(M) = M$, where the second $M$ is the corresponding stalk complex at degree 0, and $\Phi^Z$ is an equivalence if and only if $A$ is (graded) Gorenstein.

3. Existence of silting objects in $D_{sg}(\text{mod}^Z A)$ for 1-Gorenstein positively graded algebras

Let $\mathcal{T}$ be a triangulated category. An object $T \in \mathcal{T}$ is called tilting if it satisfies the following conditions.

(a) $\text{Hom}_\mathcal{T}(T, T[i]) = 0$ for any $i \neq 0$.

(b) $T = \text{thick}_\mathcal{T} T$.

Recall that an object $T$ of $\mathcal{T}$ is called a silting subcategory [39, 11] if it generates $\mathcal{T}$ and $\text{Hom}_\mathcal{T}(T, T[m]) = 0$ for $m > 0$.

Let $\mathcal{T}$ be an algebraic triangulated Krull-Schmidt category. If $\mathcal{T}$ has a tilting object $T$, then there exists a triangle equivalence $\mathcal{T} \simeq K^b(\text{proj End}_\mathcal{T}(T))$, see [36].

In this section, we always assume that $A$ is a positively graded Gorenstein algebra.

Similar to [62, Theorem 3.1], we get the following lemma.

Lemma 3.1. Let $A$ be a positively graded $d$-Gorenstein algebra. If $D_{sg}(\text{mod}^Z A)$ has tilting objects, then $A_0$ has finite global dimension.

Proof. We assume that $A$ is a $d$-Gorenstein algebra.

Suppose for a contradiction that $D_{sg}(\text{mod}^Z A)$ has a tilting object $T$ and $\text{gl.dim} A_0 = \infty$. Since $A$ is positively graded, for any $A_0$-module $X$, the degree 0 part of a projective resolution of $X$ in $\text{mod}^Z A$ gives a projective resolution of $X$ in $\text{mod} A_0$. So we have

$$\text{Ext}^i_{\text{mod}^Z A}(A_0/J_{A_0}, A_0/J_{A_0}) = \text{Ext}^i_{A_0}(A_0/J_{A_0}, A_0/J_{A_0})$$

for any $i > 0$.

Next since $\text{gl.dim} A_0 = \infty$, the $A_0$-module $A_0/J_{A_0}$ has infinite projective dimension. Thus we have

$$\text{Ext}^i_{A_0}(A_0/J_{A_0}, A_0/J_{A_0}) \neq 0$$

for any $i > 0$.

Denote by $\Omega$ the syzygy translation of $\text{mod}^Z A$. Since $A$ is a $d$-Gorenstein algebra, from the graded version of Theorem 2.5, we get that $\Omega^j(A_0/J_{A_0})$ is a graded Gorenstein projective $A$-module for any $j \geq d$. Let

$$\cdots \rightarrow Q_d \xrightarrow{f_d} Q_{d-1} \xrightarrow{f_{d-1}} \cdots \xrightarrow{f_1} Q_0 \xrightarrow{f_0} A_0/J_{A_0} \rightarrow 0$$

be a projective resolution of $A_0/J_{A_0}$ as graded $A$-module. Let $\Sigma$ be the shift functor of $D^b(\text{mod}^Z A)$, and $(1)$ be the shift functor of $D_{sg}(\text{mod}^Z A)$ which is induced by $\Sigma$. It is not hard to see that $A_0/J_{A_0} \simeq \Omega^j(A_0/J_{A_0})(j)$ for any $j \geq 0$ in $D_{sg}(\text{mod}^Z A)$. So for any $i$, we get
that

\[ \text{Hom}_{D_{sg}(\text{mod}^{Z}A)}(A_0/J_{A_0}, (A_0/J_{A_0})(i)) \]
\[ = \text{Hom}_{D_{sg}(\text{mod}^{Z}A)}(\Omega^d(A_0/J_{A_0})(d + i), \Omega^d(A_0/J_{A_0})(d + i)) \]
\[ = \text{Hom}_{D_{sg}(\text{mod}^{Z}A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})(i)). \]

Since \( \text{Gproj}^{Z}(A) \simeq D_{sg}(\text{mod}^{Z}A) \), and \( \text{Gproj}^{Z}(A) \) is a full subcategory of \( \text{mod}^{Z}(A) \), we get that

\[ \text{Hom}_{D_{sg}(\text{mod}^{Z}A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})(i)) \]
\[ = \text{Hom}_{\text{Gproj}^{Z}(A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})(i)) \]
\[ = \text{Hom}_{\text{mod}^{Z}(A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})(i)), \]

where [1] is the shift functor of \( \text{Gproj}^{Z}(A) \) given by the cosyzygy functor. Since \( \Omega^d(A_0/J_{A_0}) \) is a graded Gorenstein projective \( A \)-module, there exists an exact sequence

\[ 0 \to \Omega^d(A_0/J_{A_0}) \xrightarrow{g_0} P_0 \xrightarrow{g_1} P_1 \xrightarrow{g_2} \cdots \to P_i \to \cdots \]

with \( P_i \) graded projective and \( \text{Coker}(g_i) \) graded Gorenstein projective for any \( i \geq 0 \). Since

\[ \text{Ext}^i_{\text{mod}^{Z}(A)}(\Omega^d(A_0/J_{A_0}), P) = 0 \]

for any projective module \( P \) and \( i > 0 \), we get that

\[ \text{Hom}_{\text{mod}^{Z}(A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})(i)) \]
\[ = \text{Ext}^i_{\text{mod}^{Z}(A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})). \]

Similarly, for any \( i > d \),

\[ \text{Ext}^i_{\text{mod}^{Z}(A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})) \]
\[ = \text{Ext}^{i-d}_{\text{mod}^{Z}(A)}(\Omega^d(A_0/J_{A_0}), A_0/J_{A_0}) \]
\[ = \text{Ext}^{i-d+1}_{\text{mod}^{Z}(A)}(\Omega^{d-1}(A_0/J_{A_0}), A_0/J_{A_0}) \]
\[ = \text{Ext}^i_{\text{mod}^{Z}(A)}(A_0/J_{A_0}, A_0/J_{A_0}). \]

So

\[ \text{Hom}_{D_{sg}(\text{mod}^{Z}A)}(A_0/J_{A_0}, (A_0/J_{A_0})(i)) \]
\[ = \text{Hom}_{D_{sg}(\text{mod}^{Z}A)}(\Omega^d(A_0/J_{A_0})(d + i), \Omega^d(A_0/J_{A_0})(d + i)) \]
\[ = \text{Hom}_{D_{sg}(\text{mod}^{Z}A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})(i)) \]
\[ = \text{Hom}_{\text{Gproj}^{Z}(A)}(\Omega^d(A_0/J_{A_0}), \Omega^d(A_0/J_{A_0})(i)) \]
\[ = \text{Ext}^i_{\text{mod}^{Z}(A)}(A_0/J_{A_0}, A_0/J_{A_0}) \]
\[ = \text{Ext}^i_{A_0}(A_0/J_{A_0}, A_0/J_{A_0}) \]
\[ \neq 0 \]

for any \( i > d \).

However, if \( D_{sg}(\text{mod}^{Z}A) \) has a tilting object, then \( \text{Hom}_{T}(X, Y(i)) = 0 \) holds for any \( X, Y \in T \) and \( |i| >> 0 \), which is a contradiction. □
Recall that the truncation functors

\((-)_{\geq i} : \text{mod}^\mathbb{Z} A \to \text{mod}^\mathbb{Z} A, \quad (-)_{\leq i} : \text{mod}^\mathbb{Z} A \to \text{mod}^\mathbb{Z} A\)

are defined as follows. For a \(\mathbb{Z}\)-graded \(A\)-module \(X\), \(X_{\geq i}\) is a \(\mathbb{Z}\)-graded sub \(A\)-module of \(X\) defined by

\[
(X_{\geq i})_j := \begin{cases} 0 & \text{if } j < i \\ X_j & \text{if } j \geq i,
\end{cases}
\]

and \(X_{\leq i}\) is a \(\mathbb{Z}\)-graded factor \(A\)-module \(X/X_{\geq i+1}\) of \(X\).

We define a \(\mathbb{Z}\)-graded \(A\)-module by

\[
T := \bigoplus_{i \geq 0} A(i)_{\leq 0}.
\]

Since \(A(i)_{\leq 0} = A(i)\) is zero in \(D_{sg}(\text{mod}^\mathbb{Z} A)\) for sufficiently large \(i\), we can regard \(T\) as an object in \(D_{sg}(\text{mod}^\mathbb{Z} A)\).

**Lemma 3.2.** Let \(A\) be a positively graded Gorenstein algebra. If \(A_0\) has finite global dimension, then we have \(D_{sg}(\text{mod}^\mathbb{Z} A) = \text{thick} T\), where \(T = \bigoplus_{i \geq 0} A(i)_{\leq 0}\).

**Proof.** The proof is based on that of [62, Lemma 3.5].

For any object \(N\) in \(D_{sg}(\text{mod}^\mathbb{Z} A)\), there exists a graded Gorenstein projective \(A\)-module such that it is isomorphic to \(N\) in \(D_{sg}(\text{mod}^\mathbb{Z} A)\). So it is enough to show that all the graded \(A\)-modules are in the subcategory \(\text{thick} T\).

We regard \(A_0\) as the natural \(\mathbb{Z}\)-graded factor \(A\)-module, i.e. \(A_0(0) = A(0)_{\leq 0}\). Any object in \(\text{mod}^\mathbb{Z} (A)\) has a finite filtration by simple objects in \(\text{mod}^\mathbb{Z} (A)\) which are given by simple \(A_0\)-modules concentrated in some degree by Proposition 2.21. Every short exact sequence in \(\text{mod}^\mathbb{Z} A\) gives a triangle in \(D^b(\text{mod}^\mathbb{Z} A)\), and then a triangle in \(D_{sg}(\text{mod}^\mathbb{Z} A)\), so we only need to check that \(S(i)\) is in \(\text{thick} T\) for any simple \(A_0\)-module \(S\), and any \(i \in \mathbb{Z}\). Since \(A_0\) has finite global dimension, it is enough to show that \(A_0(i)\) is in \(\text{thick} T\) for any \(i \in \mathbb{Z}\). We divide the proof into two parts.

(i) We show \(A_0(i)\) is in \(\text{thick} T\) for any \(i \geq 0\) by induction on \(i\). Obviously we have \(A_0(0) = A(0)_{\leq 0}\) is in \(\text{thick} T\). We assume \(A_0(0), \ldots, A_0(i - 1)\) is in \(\text{thick} T\). This implies that \(S(0), \ldots, S(i - 1)\) is in \(\text{thick} T\) for any simple \(A_0\)-module \(S\) since \(A_0\) has finite global dimension, and for any graded \(A\)-module \(N\), if \(N_j = 0\) for any \(j > 0\) and \(j < 1 - i\) then \(N \in \text{thick} T\).

There exists an exact sequence

\[
0 \to (A(i)_{\leq 0})_{\geq 1 - i} \to A(i)_{\leq 0} \to A_0(i) \to 0,
\]

in \(\text{mod}^\mathbb{Z} A\). By the inductive hypothesis, we have \((A(i)_{\leq 0})_{\geq 1 - i}\) is in \(\text{thick} T\). Thus we have \(A_0(i)\) is in \(\text{thick} T\).

(ii) We show that \(A_0(-i)\) is in \(\text{thick} T\) for any \(i \geq 1\). We assume \(A_0(-j)\) is in \(\text{thick} T\) for any \(j < i\). We put \(n := \text{proj. dim}_{A_0^{\text{op}}} D(A_0) + 1\). Then there exists an exact sequence

\[
0 \to X \to Q^{n-1} \to \cdots \to Q^1 \to Q^0 \to D(A_0) \to 0
\]

in \(\text{mod}^\mathbb{Z} A^{\text{op}}\) such that \(Q^j\) is a \(\mathbb{Z}\)-graded projective \(A\)-modules for \(0 \leq j \leq n - 1\), and \(X_{\leq 0} = 0\). Then there exists an exact sequence

\[
0 \to A_0 \to D(Q^0) \to D(Q^1) \to \cdots \to D(Q^{n-1}) \to D(X) \to 0
\]

in \(\text{mod}^\mathbb{Z} A\) such that \(D(Q^j)\) is a \(\mathbb{Z}\)-graded projective \(A\)-module for \(0 \leq j \leq n - 1\), and \((D(X))_{\geq 0} = 0\). We have \(A_0(-i) = D(X)(-i)(-n)\) in \(D_{sg}(\text{mod}^\mathbb{Z} A)\). Since \((D(X)(-i))_{\geq 1} = 0\), we have \(D(X)(-i)\) is in \(\text{thick} T\) by the inductive hypothesis. Thus we have \(A_0(-i)\) is in \(\text{thick} T\). □
Theorem 3.3. Let $A$ be a positively graded 1-Gorenstein algebra. If $A_0$ has finite global dimension, then $D_{sg}(\mod^Z A)$ admits a silting object.

Proof. We prove it by constructing a silting object in $D_{sg}(\mod^Z A)$ which is motivated by \cite{62}.

Let $T := \bigoplus_{i \geq 0} A(i)_{<0}$, which is viewed as an object in $D_{sg}(\mod^Z A)$. We take a minimal projective resolution

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow A(i) \rightarrow A(i)_{<0} \rightarrow 0$$

of $A(i)_{<0}$ in $\mod^Z A$. Since $A$ is positively graded, we have $(P_j)_{<0} = 0$ for $j > 0$. Thus $\langle \Omega^j T \rangle_{<0} = 0$ holds for any $j > 0$. Since $T = T_{\leq 0}$, we have $\Hom_{\mod^Z A}(T, \Omega^j T) = 0 = \Hom_{\mod^Z A}(\Omega^j T, T)$ for any $j > 0$. Let $0 \rightarrow \Omega(T) \xrightarrow{f_0} P \xrightarrow{f_1} T \rightarrow 0$ be the exact sequence.

Since $A$ is 1-Gorenstein, $\Omega T \in \Gproj^Z(A)$. Similar to the proof of Lemma 3.3, we get that for any $i > 1$,

$$\Hom_{\Gproj^Z(A)}(\Omega T, \Omega T[i]) = \Ext^i_{\mod^Z A}(\Omega T, \Omega T) = \Ext^i_{\mod^Z A}(T, T) = \Ext^{i-1}_{\mod^Z A}(\Omega T, T) = \Ext^1_{\mod^Z A}(\Omega^{-1} T, T) = \Hom_{\mod^Z A}(\Omega^j T, T) = 0.$$

For $i = 1$, $\Hom_{\Gproj^Z(A)}(\Omega T, \Omega T[1]) = \Ext^1_{\mod^Z A}(\Omega T, \Omega T) = \Hom_{\mod^Z A}(\Omega T, T) = 0$. Therefore, $\Hom_{\Gproj^Z(A)}(\Omega T, \Omega T[i]) = 0$ for any $i > 0$.

Together with Lemma 3.2 we get that $\Omega T$ is a silting object in $D_{sg}(\mod^Z A)$. Since $T \cong \Omega T(1)$ in $D_{sg}(\mod^Z A)$, we also get that $T$ is a silting object in $D_{sg}(\mod^Z A)$. \hfill \Box

The following proposition is also useful, recently, we use it to consider the graded singularity categories of quiver representations over local rings, see \cite{17}.

Proposition 3.4. Let $A$ be a positively graded Gorenstein algebra such that $A_0$ has finite global dimension. If $T = \bigoplus_{i \geq 0} A(i)_{\leq 0}$ is a graded Gorenstein projective, then $T$ is a tilting object in $D_{sg}(\mod^Z A)$. In particular, in this case, $\End_{\Gproj^Z(A)}(T)$ is of finite global dimension.

Proof. Since $A$ is a Gorenstein algebra, we get that $\Gproj^Z(A)$ is equivalent to $D_{sg}(\mod^Z A)$. Since $T$ is Gorenstein projective,

$$\Hom_{\Gproj^Z(A)}(T, T(i)) = \Hom_{\Gproj^Z(A)}(T, \Omega^{-1} T) = \Hom_{\Gproj^Z(A)}(\Omega^j T, T)$$

for any $i \in \mathbb{Z}$. From the proof of Theorem 3.3, we get that

$$\Hom_{\mod^Z A}(T, \Omega^j T) = 0 = \Hom_{\mod^Z A}(\Omega^j T, T)$$

for any $j > 0$. So $\Hom_{\Gproj^Z(A)}(T, T(i)) = 0$ for any $i \neq 0$, together with Lemma 3.2 we get that $T$ is a tilting object in $D_{sg}(\mod^Z A)$. For the proof of that $\End_{\Gproj^Z(A)}(T)$ has finite global dimension, it is same to that of \cite{62} Theorem 3.7, we omit it here. \hfill \Box

Example 3.5. Let $A$ be a cluster-tilted algebra of type $E_6$ with its quiver as Fig. 1 shows.
is finitely generated, and is isomorphic to

\[ \text{End}_A(I) \cong \text{End}_B(I) \text{ if } I \in \text{Gproj}_A \]

then so does \( \Gamma \).

Let \( \Gamma \) be the endomorphism of \( T \) such that

\[ \Gamma(0) = 0 \]

It is easy to check that \( T = \bigoplus_{i \geq 0} A(i) \leq 0 \cong B \) is a tilting object in \( D_{sg}(\text{mod} \mathbb{Z} A) \).

(b) Let \( B_2 \) be the algebra where its quiver is as Fig. 3 shows with \( \alpha_3 \beta_1 \alpha_2 = 0, \beta_2 \alpha_1 \gamma = 0 \). Then \( B_2 \) is a tilting algebra, and \( A \cong B_2 \times \text{Ext}^2_{B_2}(DB_2, B_2) \). So \( A \) is a positively graded algebra with \( A_0 = B_2 \) and \( A_1 = \text{Ext}^2_{B_2}(DB_2, B_2) \). It is easy to see that \( T = \bigoplus_{i \geq 0} A(i) \leq 0 \cong B_1 \) is a tilting object in \( D_{sg}(\text{mod} \mathbb{Z} A) \). Then \( A \) is a positively graded algebra with \( A_0 = B_2 \) and \( A_1 = \text{Ext}^2_{B_2}(DB_2, B_2) \).

It is easy to check that \( B_2 \) is a tilting object in \( D_{sg}(\text{mod} \mathbb{Z} A) \).

However, in general case, \( T \) is not a tilting object in \( D_{sg}(\text{mod} \mathbb{Z} A) \).

**Example 3.6.** Let \( A = KQ/I \) be the cluster-tilted algebra of type \( \mathbb{D}_6 \) with \( Q \) as the following figure shows. Then \( A \) is a positively graded algebra by setting \( \alpha_4, \beta_1, \beta_2 \) to be degree one, and other arrows to be degree zero. Let \( T = \bigoplus_{i \geq 0} A(i) \leq 0 \). It is easy to see that \( T \) is a silting object, which is not a tilting object, in \( D_{sg}(\text{mod} \mathbb{Z} A) \).

\[ \alpha_4 \beta_2 \]

\[ \alpha_1 \alpha_3 \gamma_1 \beta_2 \]

\[ \beta_1 \]

\[ \gamma_2 \]

\[ \text{Fig. 4. A cluster-tilted algebra of type } \mathbb{D}_6. \]

At the end of this section, we consider the case when \( T = \bigoplus_{i \geq 0} A(i) \leq 0 \) is a Gorenstein projective \( A \)-module. Let \( \Gamma \) be the endomorphism of \( T \) in \( \text{Gproj}^Z_A \).

**Theorem 3.7** ([62]). Take a decomposition \( T = \bigoplus P \in \text{Mod}^Z_A \) where \( P \) is a direct sum of all indecomposable non-projective direct summand of \( T \). Then

1. \( P \) is finitely generated, and is isomorphic to \( T \) in \( \text{Gproj}^Z_A \).
2. There exists an algebra isomorphism \( \Gamma \cong \text{End}_A(P) \), and if \( A_0 \) has finite global dimension, then so does \( \Gamma \).
Similar to [62, Section 3.2], we take a positive integer \( l \) such that \( A = A_{\leq l} \). Let \( U := \bigoplus_{i=0}^{l-1} A(i) \leq 0 \). Then there exists an algebra isomorphism

\[
\begin{pmatrix}
A_0 & A_1 & \cdots & A_{l-2} & A_{l-1} \\
A_0 & \cdots & A_{l-3} & A_{l-2} \\
\vdots & \vdots \\
A_0 & A_1 \\
A_0
\end{pmatrix}
\]

If we decompose \( U = T \oplus P' \) for some projective direct summand of \( U \), then

\[
\text{End}_A(U)_0 \simeq \left( \text{End}_A(P')_0 \quad \text{Hom}_A(T, P')_0 \right).
\]

Then \( \Gamma \) is a quotient algebra of \( \text{End}_A(U)_0 \).

Finally, we give an example of cluster-tilted algebras. Recall that cluster categories of weighted projective lines are by definition of the form \( \mathcal{C}_X = \mathbb{D}^b(\text{coh}(X))/\langle \tau^{-1}[1] \rangle \), where \( \text{coh}(X) \) is the category of coherent sheaves on a weighted projective line, see [9]. \( \mathcal{C}_X \) is a triangulated 2-Calabi-Yau category with cluster structure, see [37, 9]. The endomorphism algebras of cluster tilting objects in \( \mathcal{C}_X \) are also called cluster-tilted algebras. It is well known that an algebra \( A \) is a cluster-tilted algebra if and only if \( A = B \rtimes \text{Ext}^2_B(DB, B) \) for some quasi-tilted algebra \( B \), see [2, 63, 64]. In this way, \( A \) can be viewed as a positively graded algebra with \( A_0 = B \), and \( A_1 = \text{Ext}^2_B(DB, B) \).

Let \( A = KQ/I \) be a quotient of the path algebra of the following quiver \( Q \) modulo the ideal \( I \) generated by the elements described below (in \( Q \) the arm with arrows \( x_i \) contains \( p_i \) (\( p_i \geq 2, 1 \leq i \leq 3 \)) arrows):

![Diagram of a quiver](image)

\[
I : x_1^{p_1} + x_2^{p_2} + x_3^{p_3} \eta x_i^{j-1}
\]

for \( i = 1, 2, 3, j = 1, \ldots, p_i \).

Fig. 5. Cluster-tilted algebra of a canonical algebra with weights \( (p_1, p_2, p_3) \).

From [9], we know that \( A = KQ/I \) is a cluster-tilted algebra of a canonical algebra with weights \( (p_1, p_2, p_3) \). Then \( A \) is naturally a positively graded algebra with \( A_0 \) the corresponding canonical algebra.

Let \( \Delta(q_1, q_2, q_3) \ (q_i \geq 0) \) be the quiver as the following diagram shows (in \( \Delta(q_1, q_2, q_3) \) the arm with arrows \( x_i \) contains \( q_i \) arrows)
Let $B$ be the canonical algebra with weights $(p_1, p_2, p_3)$, i.e. its quiver is constructed from that in Fig. 5 by removing the arrow $\eta$, and its relation is $x_1^{p_1} + x_2^{p_2} + x_3^{p_3}$.

**Proposition 3.8.** Let $A$ be the cluster-tilted algebra of a canonical algebra with weights $(p_1, p_2, p_3)$ as Fig. 5 shows. Then $D_{sg}(\text{mod}^Z A)$ is triangle equivalent to $D^b(K\Delta(p_1 - 2, p_2 - 2, p_3 - 2))$.

**Proof.** From [32, Corollary 4.10], we get that the corresponding canonical algebra $A_0 = B$ is a tilting object in $D_{sg}(\text{mod}^Z A)$. We give a different proof here.

It is easy to see that the corresponding canonical algebra $A_0 = B$ is a (graded) Gorenstein projective $A$-module when viewing as a graded $A$-module concentrated in degree zero. So from Proposition 3.4, we get that $\text{Gproj}_A$ as Fig. 5 shows, where

$$\xymatrix{ \cdots & x_1 & x_1 & x_1 & x_2 & x_2 & \cdots & x_2 }$$

Fig. 6. $\Delta(q_1,q_2,q_3)$.

Recall that an algebra $A$ is called to be CM-finite if there are only finitely many indecomposable Gorenstein projective modules over $A$.

**Corollary 3.9.** Let $A$ be the cluster-tilted algebra of a canonical algebra with weights $(p_1, p_2, p_3)$ as Fig. 5 shows, where $p_1 \leq p_2 \leq p_3$. Then $A$ is CM-finite if and only if

$$(p_1, p_2, p_3) \in \{(2, p, q), (3, 3, r), (3, 4, 4), (3, 4, 5), (3, 4, 6)\}|p, q \geq 2, r \geq 3\}.$$

**Proof.** First, from Proposition 3.8 we get that $\text{Gproj}_A$ is triangle equivalent to $D^b(\text{mod} H)$ for some hereditary algebra. For $\text{Gproj}_A$, it is a 3-Calabi-Yau triangulated category, see [38, Theorem 3.3]. In fact, [32, Theorem 4.15] shows that $\text{Gproj}_A$ is triangulated equivalent to $D^b(\text{mod} H)/\tau^{-1}[2]$, where $\tau$ is the Auslander-Reiten functor of $D^b(\text{mod} H)$, and $[1]$ is the shift functor.

So $A$ is CM-finite if and only if $H$ is of finite representation type, and then the result follows immediately.

4. **Realising the graded singularity categories of Gorenstein monomial algebras as derived categories**

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver. A path $p$ of length $n$ in $Q$ is a sequence $p = \alpha_n \cdots \alpha_2 \alpha_1$ of arrows satisfying $s(\alpha_i) = t(\alpha_{i-1})$ for $2 \leq i \leq n$, its starting vertex is $s(p) = s(\alpha_1)$ and its
Theorem 4.3.\...for nonzero paths such that $q \in S$ and $p = qp'$ for some nontrivial path $p'$. Dually one defines a right-minimal path in $S$. A path $p$ in $S$ is minimal in $S$ if there is no proper subpath $q$ of $p$ inside $S$.

Let $kQ$ be the path algebra of $Q$. An admissible ideal $I$ of $kQ$ is monomial if it is generated by some paths of length at least two. In this case, the quotient algebra $A = kQ/I$ is called a monomial algebra.

Let $A = kQ/I$ be a monomial algebra as above. Let $F$ be the set formed by all the minimal paths among the paths in $I$. A path is said to be a nonzero path in $A$ provided that $p$ does not belong to $I$, or equivalently, $p$ does not contain a subpath in $F$. Then the set of nonzero paths forms a $k$-basis of $A$.

For a nonzero path $p$, we consider the left ideal $Ap$ and the right ideal $pA$. Note that $Ap$ has a basis given by all nonzero paths $q$ such that $q = q'p$ for some path $q'$, and similar for $pA$. For a nonzero nontrivial path $p$, we define $L(p)$ to be the set of right-minimal paths in the set \{nonzero paths $q|s(q) = t(p)$ and $qp = 0$\} and $R(p)$ the set of left-minimal paths in the set \{nonzero paths $q|t(q) = s(p)$ and $pq = 0$\}.

**Definition 4.1** (16). Let $A = kQ/I$ be a monomial algebra as above. We call a pair $(p,q)$ of nonzero paths in $A$ perfect provided that the following conditions are satisfied:

(P1) both of the nonzero paths $p,q$ are nontrivial satisfying $s(p) = t(q)$ and $pq = 0$ in $A$;

(P2) if $pq' = 0$ for a nonzero path $q'$ with $t(q') = s(p)$, then $q' = q''p$ for some path $q''$; in other words, $R(p) = \{q\}$;

(P3) if $p'q = 0$ for a nonzero path $p'$ with $s(p') = t(q)$, then $p' = p''p$ for some path $p''$; in other words, $L(q) = \{p\}$.

From (16) (20), we have the following exact sequence of left $A$-modules

$$0 \rightarrow Ap \xrightarrow{inc} A_{\pi(q)} \xrightarrow{\pi_i} Aq \rightarrow 0.$$ \hspace{1cm} \text{(4)}

In particular, $\Omega(Aq) \simeq Ap$.

**Definition 4.2** (20). Let $A = kQ/I$ be a monomial algebra. We call a nonzero path $p$ in $A$ a perfect path, if there exists a sequence $p = p_1, p_2, \ldots, p_n, p_{n+1} = p$ of nonzero paths such that $(p_i, p_{i+1})$ are perfect pairs for all $1 \leq i \leq n$. If the given nonzero paths $p_i$ are pairwise distinct, we refer to the sequence $p = p_1, p_2, \ldots, p_n, p_{n+1} = p$ as a relation-cycle for $p$.

A perfect path has a unique relation-cycle. Let $n \geq 1$. By a basic $(n-)\text{cycle } Z_n$, we mean a quiver consisting of $n$ vertices and $n$ arrows which form an oriented cycle.

The following theorem characterizes the indecomposable Gorenstein projective modules for a monomial algebra clearly.

**Theorem 4.3** (20). Let $A$ be a monomial algebra. Then there is a bijection

$$\{\text{perfect paths in } A\} \xleftarrow{1:1} \text{Ind Gproj} A$$

sending a perfect path $p$ to the $A$-module $Ap$.\hspace{1cm} \text{(5)}
For a monomial algebra $A$, by [1] Chapter III.2, Lemma 2.4, Lemma 2.6], its indecomposable projective modules (also Gorenstein projective modules) and injective modules are as Fig. 7 and Fig. 8 show respectively.

For a monomial algebra $A = kQ/I$, it is easy to see that $A$ is a positively graded algebra by setting each arrow degree one. In the following, we always consider $A$ to be positively graded in this way.

**Lemma 4.4.** Let $A = kQ/I$ be a monomial algebra. Then the forgetful functor $F : \text{Gproj}^{\mathbb{Z}} A \to \text{Gproj} A$ is dense. In particular, for any indecomposable graded Gorenstein projective module $X$, we have that $\text{top}(X)$ is simple.

**Proof.** From Fig. 7, it is easy to see that any graded Gorenstein projective module is gradable, and $F : \text{Gproj}^{\mathbb{Z}} A \to \text{Gproj} A$ is dense. For the last sentence, obviously, $X = A p$ for some perfect path $p$, and then $\text{top}(X)$ is isomorphic to the simple module corresponding to the vertex $t(p)$.

Lemma 4.4 implies that $\text{Ind} \text{Gproj}^{\mathbb{Z}} A = \bigcup_{i \in \mathbb{Z}} (\text{Ind} \text{Gproj} A)(i)$ by viewing each indecomposable $X \in \text{Gproj} A$ as a graded Gorenstein projective module with $\text{top}(X)$ degree zero.

**Theorem 4.5.** Let $A = kQ/I$ be a Gorenstein monomial algebra. Then $D_{sg}(\text{mod}^{\mathbb{Z}} A)$ has a tilting object.

**Proof.** We prove it by constructing a tilting object in $D_{sg}(\text{mod}^{\mathbb{Z}} A)$ similar to Theorem 3.3. We take a positive integer $l$ such that $A \leq l$ and define a $\mathbb{Z}$-graded $A$-module by

$$T := \bigoplus_{0 \leq i \leq l} A(i)_{\leq 0}. $$

Since $A(i)_{\leq 0} = A(i)$ is zero in $D_{sg}(\text{mod}^{\mathbb{Z}} A)$ for sufficiently large $i$, we have that $\bigoplus_{i \geq 0} A(i)_{\leq 0}$ is an object in $D_{sg}(\text{mod}^{\mathbb{Z}} A)$. In this way, $T \cong \bigoplus_{i \geq 0} A(i)_{\leq 0}$ in $D_{sg}(\text{mod}^{\mathbb{Z}} A)$, and then $\text{thick}(T) = D_{sg}(\text{mod}^{\mathbb{Z}} A)$ by Lemma 5.2 since $A_0$ is semisimple.

We take a minimal projective resolution

$$\cdots \to P_2 \to P_1 \to A(i) \to A(i)_{\leq 0} \to 0$$

of $A(i)_{\leq 0}$ in $\text{mod} A$. Since $A$ is positively graded, we have $(P_j)_{\leq 0} = 0$ for $j > 0$. Thus $(\Omega^n T)_{\leq 0} = 0$ holds for any $j > 0$. In particular, $\text{top}(\Omega^n T)$ is homogeneous of degree one.

Since $A$ is Gorenstein, for $T$, from Theorem 2.1 we get that there is an exact sequence

$$0 \to A_T \xrightarrow{\alpha} N_T \xrightarrow{\beta} T \to 0$$

where $N_T$ is graded Gorenstein projective $A$-module, and $A_T$ is graded $A$-module of finite projective dimension. Let $P_{A_T}$ be the projective cover of $A_T$. Then we get the following
commutative diagram with each row and each column short exact:

\[
\begin{array}{cccc}
L & \rightarrow & P_{AT} & \rightarrow & A_T \\
\downarrow c & & \downarrow & & \downarrow a \\
M & \rightarrow & P_{AT} \oplus (\bigoplus_{0\leq i \leq l} A(i)) & \rightarrow & N_T \\
\downarrow d & & \downarrow & & \downarrow b \\
\Omega T & \rightarrow & \bigoplus_{0\leq i \leq l} A(i) & \rightarrow & T.
\end{array}
\]

Then \( L \) is of finite projective dimension, and \( M \) is Gorenstein projective. In particular, \( \Omega T \cong M \) in \( D_{sg}(\text{mod}^Z A) \). Assume that \( M = \bigoplus_{i=1}^m M_i \) with \( M_i \) indecomposable for \( 1 \leq i \leq m \). Then \( M_i \) is Gorenstein projective, \( \text{proj.dim}_A M_i = 0 \) or \( \infty \) for each \( i \). Without losing of generality, we assume that \( M_i \) is non-projective for \( 1 \leq i \leq t \), and \( M_i \) is projective for \( t + 1 \leq i \leq m \). In this way, \( d \) is \( (d_1, \ldots, d_m) : M = \bigoplus_{i=1}^m M_i \rightarrow \Omega(T) \) with \( d_i : M_i \rightarrow \Omega(T) \). Then \( d_i \neq 0 \) for \( 1 \leq i \leq t \).

In fact, if \( d_i = 0 \), then from the short exact sequence

\[ 0 \rightarrow L \rightarrow M \rightarrow \Omega(T) \rightarrow 0, \]

we get that \( M_i \) is a direct summand of \( L \) which implies that \( M_i \) has finite projective dimension, and then it is projective, a contradiction.

From the definition of \( T \), easily, \( \text{top}(\Omega(T)) \) is homogeneous of degree one, and \( (\Omega(T))_{\geq 1} = \Omega(T) \). Lemma 3.3 shows that \( \text{top}(M_i) \) is simple for \( 1 \leq i \leq m \), and then \( \text{deg}(\text{top}(M_i)) \geq 1 \) for \( 1 \leq i \leq t \) since there is a nonzero morphism \( d_i : M_i \rightarrow \Omega(T) \). Furthermore, since \( A \) is positively graded, we have that \( (\Omega^j(M))_{\geq 1} = \Omega^j(M) \) for any \( j \geq 0 \). Here \( \Omega^j(M) = M \) for \( j = 0 \).

Similarly, assume that \( N_T = \bigoplus_{i=1}^p N_i \), where \( N_i \) is indecomposable for \( 1 \leq i \leq n \), \( N_i \) is non-projective for \( 1 \leq i \leq s \); \( N_i \) is projective for \( s + 1 \leq i \leq n \). Then \( \text{deg}(\text{top}(N_i)) \leq 0 \) for \( 1 \leq i \leq s \) since \( (T)_{\leq 0} = T \). In fact, \( s = t \) by \( \Omega(N_T) \cong M \) in \( \text{Gproj}^Z A \).

Denote by \( L \) the shift functor in \( D_{sg}(\text{mod}^Z A) \). Then for any \( p > 0 \)

\[
\text{Hom}_{D_{sg}(\text{mod}^Z A)}(T, T(-p)) = \text{Hom}_{D_{sg}(\text{mod}^Z A)}(N_T, N_T(-p))
\]

\[
= \text{Hom}_{\text{Gproj}^Z A}(N_T, \Omega^p(N_T))
\]

\[
= \text{Hom}_{\text{Gproj}^Z A}(\bigoplus_{i=1}^s N_i, \Omega^{p-1}(\bigoplus_{i=1}^s M_i)).
\]

For each \( 1 \leq i \leq s \), since \( \text{deg}(\text{top}(N_i)) \leq 0 \), and \( (\Omega^{p-1}(\bigoplus_{j=1}^s M_j))_{\geq 1} = \Omega^{p-1}(\bigoplus_{j=1}^s M_j) \), it is easy to see that \( \text{Hom}_{\text{mod}^Z A}(N_i, \Omega^{p-1}(\bigoplus_{j=1}^s M_j)) = 0 \), which yields that

\[
\text{Hom}_{\text{Gproj}^Z A}(\bigoplus_{i=1}^s N_i, \Omega^{p-1}(\bigoplus_{i=1}^s M_i)) = 0.
\]

So \( \text{Hom}_{D_{sg}(\text{mod}^Z A)}(T, T(-p)) = 0 \).

On the other hand,

\[
\text{Hom}_{D_{sg}(\text{mod}^Z A)}(T, T(p)) = \text{Hom}_{D_{sg}(\text{mod}^Z A)}(T(-p), T)
\]

\[
= \text{Hom}_{D_{sg}(\text{mod}^Z A)}(N_T(-p), N_T)
\]

\[
= \text{Hom}_{\text{Gproj}^Z A}(\Omega^p(N_T), N_T)
\]

\[
= \text{Hom}_{\text{Gproj}^Z A}(\Omega^{p-1}(\bigoplus_{i=1}^s M_i), \bigoplus_{i=1}^s N_i).
\]

For any morphism \( f : \Omega^{p-1}(\bigoplus_{i=1}^s M_i) \rightarrow \bigoplus_{i=1}^s N_i \) in \( \text{mod}^Z A \), since

\[
(\Omega^{p-1}(\bigoplus_{j=1}^s M_j))_{\geq 1} = \Omega^{p-1}(\bigoplus_{j=1}^s M_j),
\]
we get that \((\text{Im } f)_{\geq 1} = \text{Im } f\). The morphism \(b : N_T \to T\) induces a morphism \(b' = (b_1, \ldots, b_s) : \oplus_{i=1}^s N_i \to T\). Since \((T)_{\leq 0} = T\), we get that \(b'f = 0\). Extend \(f\) to be a morphism

\[
\begin{pmatrix}
 f \\
 0
\end{pmatrix} : \Omega^{p-1}(\oplus_{i=1}^s M_i) \to (\oplus_{i=1}^s N_i) \bigoplus (\oplus_{i=s+1}^n N_i) = N_T.
\]

It is easy to see that \(\begin{pmatrix}
 f \\
 0
\end{pmatrix}\) factors through \(a : A_T \to N_T\), and then it also factors through \(e : P_{AT} \to A_T\) by Lemma 2.2 since \(\Omega^{p-1}(\oplus_{i=1}^s M_i)\) is Gorenstein projective and \(L\) is of finite projective dimension. Then we get that \(f = 0\) in \(\text{Gproj}^Z A\). Since \(f\) is arbitrary, we get that

\[
\text{Hom}_{D_{sg}(\text{mod}^Z A)}(T, T(p)) \cong \text{Hom}_{\text{Gproj}^Z A}(\Omega^{p-1}(\oplus_{i=1}^s M_i), \oplus_{i=1}^n N_i) = 0.
\]

To sum up, \(T\) is a tilting object in \(D_{sg}(\text{mod}^Z A)\). \(\square\)

In the following, we prove that \(D_{sg}(\text{mod}^Z A)\) is triangle equivalent to \(D^b(\text{mod } H)\), where \(H\) is a hereditary algebra of finite representation type. Before that, we give some lemmas.

**Lemma 4.6.** Let \(A = kQ/I\) be a monomial algebra. Then \(A\) is CM-finite. In particular, \(\text{Gproj}^Z A\) has finitely many indecomposable objects up to the shift of degree.

**Proof.** Since \(A\) is a finite-dimensional algebra over \(k\), there are only finitely many perfect paths in \(A\). From [11] in Theorem 1.3 we get that there are only finitely many indecomposable Gorenstein projective \(A\)-modules.

The forgetful functor \(F : \text{Gproj}^Z A \to \text{Gproj} A\) in Lemma 1.4 shows that \(\text{Gproj}^Z A\) has finitely many indecomposable objects up to the shift of degree. \(\square\)

Recall that a hereditary path algebra \(kQ\) is of finite representation type if every connected component of \(Q\) is of Dynkin type.

**Lemma 4.7** (see e.g. [11, 61]). For a finite-dimensional algebra \(B\) over \(k\), if \(D^b(\text{mod } B)\) has finitely many indecomposable objects up to the shift of complexes, then there exists a hereditary algebra \(H\) of finite representation type such that \(D^b(\text{mod } B) \simeq D^b(\text{mod } H)\).

**Lemma 4.8.** Let \(B\) be a finite-dimensional algebra over \(k\), if \(K^b(\text{proj } B)\) has finitely many indecomposable objects up to the shift of complexes, then \(B\) is of finite global dimension. Furthermore, in this case, \(D^b(\text{mod } B)\) also has finitely many indecomposable objects up to the shift of complexes.

**Proof.** Suppose for a contradiction that \(B\) is of infinite global dimension. Since \(\text{gl. dim } B = \max\{\text{proj. dim}_B S | S\text{ is a simple module}\}\), there exists a simple module \(S\) with infinite projective dimension. Let

\[
\cdots \to P_n \to \cdots P_1 \to P_0 \to S \to 0
\]

be a minimal projective resolution of \(S\). Then for any \(n \geq 0\), the complex \(P_n \to \cdots P_1 \to P_0\) is an indecomposable object in \(K^b(\text{proj } B)\). Since \(S\) has infinite projective dimension, we get that \(P_i \neq 0\) for any \(i \geq 0\), and then the complexes \(P_n \to \cdots P_1 \to P_0\) are pairwise non-isomorphic even up to the shift of complexes. So we get that there are infinitely many indecomposable objects up to the shift of complexes, a contradiction. \(\square\)

**Proposition 4.9.** Let \(A\) be a Gorenstein monomial algebra. Then there exists a hereditary algebra \(H\) of finite representation type such that \(\text{Gproj}^Z A \simeq D^b(\text{mod } H)\).
Proof. Theorem 4.5 shows that there is a tilting object $T$ in $\text{Gproj}^Z A$. Without losing of generality, we assume that $T$ is basic. Denote by $B = \text{End}_{\text{Gproj}^Z A}(T, T)^{op}$. Then $\text{Gproj}^Z A \simeq K^b(\text{proj} B)$.

For any indecomposable object in $\text{Gproj}^Z A$, there exists a perfect path $p$ such that it is isomorphic to $Ap$ up to some certain shift of degree. Definition 1.2 shows that there exists a sequence

$$p = p_1, p_2, \ldots, p_n, p_{n+1} = p$$

of nonzero paths such that $(p_i, p_{i+1})$ are perfect pairs for all $1 \leq i \leq n$. Then $Ap_i$ is Gorenstein projective for $1 \leq i \leq n$, and $\Omega(Ap_i)$ is isomorphic to $Ap_{i+1}(m_{i+1})$ by (4) for some $M_{i+1}$, and then $\Omega^n p (Ap) \cong Ap(m_p)$ for some certain $m_p$ in $\text{Gproj}^Z A$. Obviously, $m_p > 0$.

For any indecomposable object $X$ in $\text{Gproj}^Z A$, $X \cong Ap(i)$ for some perfect path $q$ and $i \in \mathbb{Z}$. There exists an integer $0 \leq j < m_q$ satisfying $i \equiv j \pmod{m_q}$, and then $X$ is isomorphic to $Ap(j)$ up to the syzygy functor $\Omega$. Let $m = \max \{m_p | p$ is a perfect path in $A\}$. Then $m$ is a finite number since there are only finitely many perfect paths in $A$. By viewing each indecomposable $X$ in $\text{Gproj} A$ as a graded Gorenstein projective module with top($X$) degree zero, let $X = \bigcup_{0 \leq i < m} \text{Ind}_{\text{Gproj} A}(i)$. Then $X$ is a finite set, and for any indecomposable object $X$ in $\text{Gproj}^Z A$, there exists an object $Y \in X$ such that $X$ is isomorphic to $Y$ up to the syzygy functor $\Omega$. So there are only finitely many indecomposable objects in $\text{Gproj}^Z A$ up to the syzygy functor $\Omega$.

For the triangle equivalence $\text{Gproj}^Z A \simeq K^b(\text{proj} B)$, the shift of complexes in $K^b(\text{proj} B)$ corresponds to the syzygy functor $\Omega$ in $\text{Gproj}^Z A$, so there are only finitely many indecomposable objects in $K^b(\text{proj} B)$ up to the shift of complexes. Lemma 4.5 implies $K^b(\text{proj} B) \simeq D^b(\text{mod} B)$, together with Lemma 4.8 we get that there exists a hereditary algebra $H$ of finite representation type such that $\text{Gproj}^Z A \simeq D^b(\text{mod} H)$. \[\square\]

Recall that a monomial algebra $A = kQ/I$ is called to be quadratic monomial provided that the ideal $I$ is generated by paths of length two.

**Corollary 4.10.** Let $A$ be a Gorenstein quadratic monomial algebra. Then there exists a hereditary algebra $B = kQ$ with $Q$ a disjoint union of finitely many simply-laced quivers of type $A_1$ such that $\text{Gproj}^Z A \simeq D^b(\text{mod} B)$.

**Proof.** We take a positive integer $l$ such that $A = A_{\leq l}$ and define a $\mathbb{Z}$-graded $A$-module by

$$T := \bigoplus_{0 \leq i \leq l} A(i)_{\leq 0}.$$ 

From the proof of Theorem 4.5, we get that $N_T$ is a tilting object in $\text{Gproj}^Z A$, where $N_T$ satisfies that there is an exact sequence

$$0 \to A_T \overset{a}{\to} N_T \overset{b}{\to} T \to 0$$

where $N_T$ is a graded Gorenstein projective $A$-module, and $A_T$ is a graded $A$-module of finite projective dimension. Without losing generality, we assume that $N_T$ is basic, and $N_T = Ap_1(r_1) \oplus \cdots \oplus Ap_n(r_n)$. By Theorem 5.7, we get that $\text{Gproj} A \simeq \mathcal{T}_d_1 \times \mathcal{T}_d_2 \times \cdots \times \mathcal{T}_d_m$, where $\mathcal{T}_d_i = D^b(k)/[d_i]$ is the triangulated orbit category in the sense of [37]. So $\text{Hom}_{\text{Gproj} A}(Ap_i, Ap_j) = 0$ for any $i \neq j$. Since $\text{Hom}_{\text{Gproj} A}(Ap_i(m_i), Ap_j(m_j))$ is a subset of $\text{Hom}_{\text{Gproj} A}(Ap_i, Ap_j)$, we get that there is no nonzero morphism from $Ap_i(m_i)$ to $Ap_j(m_j)$ in $\text{Gproj}^Z A$, and then

$$\text{End}_{\text{Gproj}^Z A}(N_T, N_T) \cong K \times K \times \cdots \times K.$$
Therefore, there exists a hereditary algebra \( B = kQ \) with \( Q \) a disjoint union of finitely many simply-laced quivers of type \( A_1 \) such that \( Gproj^Z A \simeq D^b(\text{mod } B) \).

**Corollary 4.11.** Let \( A \) be a Gorenstein Nakayama algebra. Then there exists a hereditary algebra \( B = kQ \) with \( Q \) a disjoint union of finitely many simply-laced quivers of type \( A \) such that \( Gproj^Z A \simeq D^b(\text{mod } B) \).

**Proof.** From [55, Proposition 1], we get that \( Gproj A \) is triangle equivalent to the stable category of a self-injective Nakayama algebra, which is also triangle equivalent to the triangulated orbit category \( D^b(\text{mod } H)/\Sigma^n (n \geq 1) \), where \( H \) is a hereditary algebra of type \( A \) and \( \Sigma \) is the shift functor of \( D^b(\text{mod } H) \). Proposition 4.5 shows that \( Gproj^Z A \) is triangle equivalent to a hereditary algebra of finite representation type.

Suppose for a contradiction that there exists a connected component \( \mathcal{C} \) of \( Gproj^Z A \) such that it is equivalent to the derived category of a hereditary algebra of type \( \mathbb{D} \) or \( \mathbb{E} \), then there exists an almost split triangle
\[
L \to M_1 \oplus M_2 \oplus M_3 \to N \to \Sigma L,
\]
for some nonzero indecomposable objects \( L, M_1, M_2, M_3, N \in Gproj^Z A \). So there exists a projective module \( P \) such that \( L \to M_1 \oplus M_2 \oplus M_3 \oplus P \to N \) is an almost split sequence. Since the forgetful functor \( F : Gproj^Z A \to Gproj A \) is dense, it is easy to see that it preserves almost split sequences, and then \( F(L) \to F(M_1) \oplus F(M_2) \oplus F(M_3) \oplus F(P) \to F(N) \) is an almost split sequence in \( Gproj A \), which implies that \( F(L) \to F(M_1) \oplus F(M_2) \oplus F(M_3) \to F(N) \to \Sigma F(L) \) is an almost split triangle in \( Gproj A \), giving a contradiction to that \( Gproj A \) is triangle equivalent to the stable category of a self-injective Nakayama algebra. So every connected component of \( Gproj^Z A \) is equivalent to the derived category of a hereditary algebra of type \( A \). \( \square \)

Note that C. M. Ringel gives a description of \( Gproj A \) for any Nakayama algebras [55], recently, D. Shen gives a characterization of Gorenstein Nakayama algebras [59].

Recall that \( A \) is a self-injective Nakayama algebra if and only if \( A = k \) or there exists an oriented cycle \( Z_n \) with the vertex set \( \{1, 2, \ldots, n\} \) and the arrow set \( \{\alpha_1, \ldots, \alpha_n\} \), where \( s(\alpha_i) = i \) and \( t(\alpha_i) = i + 1 \), such that \( A \) is isomorphic to \( kZ_n/J^d \) for some \( d \geq 2 \), where \( J \) denotes the two-sided ideal of \( kZ_n \) generated by arrows.

The following remark is well known.

**Remark 4.12.** Let \( A = kQ/I \) be a monomial algebra, where \( Q \) is connected. Then \( A \) is self-injective if and only if \( A \) is a self-injective Nakayama algebra.

**Proof.** We only need to prove the necessary part. Suppose \( A = kQ/I \) is a self-injective monomial algebra. For any vertex \( i \in Q \), its corresponding indecomposable projective module \( P_i \) is as Fig. 1 shows, and its corresponding indecomposable projective module \( I_i \) is as Fig. 2 shows. Since \( A \) is self-injective, \( P_i \) is an indecomposable injective module as Fig. 2 shows, which implies that \( P_i \) (and also \( I_i \)) is a string module with its string
\[
\cdot \to \cdot \to \cdots \to \cdot \to \cdot.
\]
If \( A \neq k \), then for any vertex \( i \), there is at most one arrow starting from \( i \), and at most one arrow ending to \( i \). Since \( Q \) is connected and \( A \) is self-injective which is not isomorphic to \( k \), we get that there is no sink vertex and no source vertex in \( Q \), so for any vertex \( i \), there is only one arrow starting from \( i \), and only one arrow ending to \( i \), and then \( Q \cong Z_n \) for some \( n \). So \( A \cong kZ_n/I \) which is a Nakayama algebra, and then \( A \) is a self-injective Nakayama algebra. \( \square \)

**Remark 4.13.** For a monomial Gorenstein algebra \( A = kQ/I \), there are many ways to make it to be a positively graded algebra, not only by setting each arrow to be degree one. For any positively grading on \( A \) such that its zero part \( A_0 \) satisfying \( \text{gl.dim } A_0 < \infty \), then \( Gproj^Z A \)
admits a tilting object by the same construction in Theorem 4.3. Furthermore, all the results in Section 4 hold in this situation. In other words, the results do not depend on the grading.

5. Characterization of 1-Gorenstein monomial algebras

In this section, we give a characterization of 1-Gorenstein monomial algebras $kQ/I$ by minimal paths in $I$, which is a generalization of the characterization of 1-Gorenstein gentle algebras, see [22] Proposition 3.1].

First, we fix some notations. Let $A = kQ/I$ be an algebra. For any vertex $i$ in $Q$, we denote by $P_i$ the corresponding indecomposable projective module and $S_i$ the corresponding simple module.

**Lemma 5.1.** Let $A = kQ/I$ be a monomial algebra. Let $F$ be the set formed by all the minimal paths among the paths in $I$. If the nontrivial paths $p,q$ satisfies that $pq \in F$, and $p$ is not a perfect path, then $A$ is not 1-Gorenstein.

**Proof.** Denote by $p = \alpha_i \cdots \alpha_1$ with $\alpha_i \in Q$ for all $1 \leq i \leq n$. Suppose for a contradiction that $A$ is 1-Gorenstein. For the indecomposable projective module $P_{t(q)}$, which has a basis formed by nonzero paths starting at $s(q)$, it has a submodule $M$ with a basis $\{p'|p'q \text{ is a nonzero path in } A\}$. Obviously, $\text{top}(M) = S_{t(q)}$, and then $M$ is indecomposable. Furthermore, since $A$ is 1-Gorenstein, the Gorenstein projective modules coincide with the torsionless modules, and then $M$ is an indecomposable Gorenstein projective module. It is easy to see that the nonzero path $p \notin M$, and then $M$ is not projective. Theorem 4.3 implies that there exists a perfect path $q'$ such that $M = Ap'$. Let $(p',q')$ be the perfect pair. Obviously, $(p',q') = t(q') = t(q)$. Since $p \notin M$, we get that $pq' = 0$ and then $p = p'/p'$ for some nonzero path $p'$. On the other hand, since $pq = \alpha_n \cdots \alpha_1 q \in F$, which implies $\alpha_{n-1} \cdots \alpha_1 q$(equals to $q$ if $n = 1$) is nonzero, and then $\alpha_{n-1} \cdots \alpha_1$ is in $M$. So $\alpha_{n-1} \cdots \alpha_1 q'$ is nonzero. Together with that $pq' = \alpha_n \alpha_{n-1} \cdots \alpha_1 q'$ is zero, we get that $p' = p$. So $(p,q')$ is a perfect pair. Since $q'$ is a perfect path, we get that so is $p$, giving a contradiction. So $A$ is not 1-Gorenstein.

**Lemma 5.2.** Let $A = kQ/I$ be a monomial algebra. Let $F$ be the set formed by all the minimal paths among the paths in $I$. Assume that for any nonzero path $p$ with the property that there exists a nonzero path $q$ such that $pq \in F$, then $p$ is a perfect path. Then $Ap$ is an indecomposable Gorenstein projective $A$-module for any nonzero path $p$.

**Proof.** First, $Ap$ is indecomposable for any nonzero path $p$. If $Ap$ is not projective, i.e., $Ap$ is not isomorphic to $P_{t(p)}$, then there exists a nonzero path $q'$ such that $q'p \in I$. Then $L(p)$ is nonempty, let $q$ be a path in $L(p)$. Then obviously, $R(q)$ is nonempty. Since $qp \in I$, there exists $p' \in R(q)$ such that $p = p'p''$. We claim that $qp' \in F$. In fact, since $qp' \in I$ and $p',q$ are nonzero, there exist nonzero paths $q_1,q_2,p_1,p_2$ such that $q = q_2q_1,p' = p_1p_2$ and $q_1p_1 \in F$. Then $qp_1 = q_2q_1p_1 \in I$, which implies that there exists a path $p_1' \in R(q)$ such that $p_1 = p_1'p_1''$ for some nonzero path $p_1''$. On the other hand, $p' = p_1p_2 = p_1'p_1''p_2$, which is also in $R(q)$. So $p_1''$ and $p_2$ are trivial paths, and then $p' = p_1$. Similarly, from $q_1p = q_1p_1p_2p'' \in I$ and $q = q_2q_1 \in L(p)$, we get that $q_2$ is trivial, and then $q = q_1$. Therefore, $qp' \in F$.

From the hypothesis, we get that $(q,p')$ is a perfect pair, and $p'$ is a perfect path. So $Ap'$ is a non-projective Gorenstein projective module. We claim that $Ap'$ is isomorphic to $Ap'$ as $A$-modules. Recall that $Ap$ (resp. $Ap'$) has a basis $S$ (resp. $S'$) given by all nonzero paths $q'$ such that $q'p \notin I$ (resp. $q'p' \notin I$). Since $p = p'p''$, it is easy to see that $S \subseteq S'$. Conversely, suppose for a contradiction that there exists a nonzero path $q'$ satisfying that $q'p' \notin I$, and $q'p \notin I$. Without losing generality, assume that $q' \in L(p)$. Similar to the above, we can get that there exists a nonzero path $p_3$ such that $qp_3 \in F$ and $p = p_3p_4$ for some nonzero path $p_4$. 

Together with \( p = p'p'' \), we get that either \( l(p') \geq l(p_3) \) or \( l(p') \leq l(p_3) \). If \( l(p') \geq l(p_3) \), then \( p' = p_3p_5 \) for some nonzero path \( p_5 \). So \( q'p' = q'p_3p_5 \in I \), giving a contradiction. If \( l(p') \leq l(p_3) \), then \( p_3 = p'p_6 \) for some nonzero path \( p_6 \), which yields that \( qp_3 = qp'p_6 \in I \). From \( q'p_3 \in F \), we get that \( (q', p_3) \) is a perfect pair, and then there exists a path \( q_1 \) such that \( q = q_1q' \). On the other hand, \( q', q \in L(p) \), which implies that \( q_1 \) is trivial and \( q = q' \). By both of \( (q, p') \) and \( (q', p_3) \) are perfect pairs, we get that \( p' = p_3 \). Then \( q'p' = qp' \in F \), giving a contradiction to that \( q'p' \notin I \). Therefore, \( Ap \) is isomorphic \( Ap' \) as \( A \)-modules, and then \( Ap \) is a non-projective Gorenstein projective module.

To sum up, \( Ap \) is an indecomposable Gorenstein projective \( A \)-module for any nonzero path \( p \).

From Fig. 7, it is easy to get the following lemma, which is a special case of [17, Theorem 2.2].

**Lemma 5.3 ([17]).** Let \( A = kQ/I \) be a monomial algebra. Then for any semisimple module \( M \), the first syzygy module of \( M \) is isomorphic to a direct sum \( \oplus Ap_{\lambda(p)} \), where \( p \) runs over all the nonzero paths in \( A \) and each \( \lambda(p) \) is some index set.

**Theorem 5.4.** Let \( A = kQ/I \) be a monomial algebra. Let \( F \) be the set formed by all the minimal paths among the paths in \( I \). Then \( A \) is 1-Gorenstein if and only if for any nonzero path \( p \) with the property that there exists a nonzero path \( q \) such that \( pq \in F \), we have that \( p \) is a perfect path.

*Proof.* It follows from Lemma 5.3 that if \( A \) is 1-Gorenstein, then for any nonzero path \( p \) with the property that there exists a nonzero path \( q \) such that \( pq \in F \), we have that \( p \) is a perfect path.

Conversely, by Theorem 2.5 we only need to check that \( \Omega(\text{mod } A) = \text{Gproj } A \).

First, Lemma 5.3 and Lemma 5.2 implies that for any semisimple module, its first syzygy is Gorenstein projective. For any finite-dimensional module \( M \), by induction, we assume that the first syzygy of \( \text{rad}(M) \) is Gorenstein projective. Then there are exact sequences

\[
0 \to \text{rad}(M) \to M \to \text{top}(M) \to 0, \quad 0 \to N_2 \to P_2 \to \text{rad}(M) \to 0,
\]

and \( 0 \to N_1 \to P_1 \to \text{top}(M) \to 0 \) with \( N_1, N_2 \) Gorenstein projective, \( P_1, P_2 \) projective. So we get the following commutative diagram with each row and column short exact:

\[
\begin{array}{ccc}
N_2 & \to & P_2 \to \text{rad}(M) \\
& \downarrow & \downarrow \\
& & N_1 \oplus P_1 \to P_1 \to \text{top}(M).
\end{array}
\]

Since \( \text{Gproj }(A) \) is closed under taking extensions, by the short exact sequence in the first column, we get that \( N \in \text{Gproj}(A) \), and then \( \Omega(M) \) is Gorenstein projective. So \( \Omega(\text{mod } A) \subseteq \text{Gproj}(A) \).

On the other hand, it is obvious that \( \text{Gproj}(A) \subseteq \Omega(\text{mod } A) \), and then \( \text{Gproj}(A) = \Omega(\text{mod } A) \). So Theorem 2.5 shows that \( A \) is 1-Gorenstein.

As a special class of monomial algebras, gentle algebras have some nice properties.

**Definition 5.5 ([5]).** The pair \((Q, I)\) is called gentle if it satisfies the following conditions.

- Each vertex of \( Q \) is starting point of at most two arrows, and end point of at most two arrows.
• For each arrow $\alpha$ in $Q$ there is at most one arrow $\beta$ such that $\alpha \beta \notin I$, and at most one arrow $\gamma$ such that $\gamma \alpha \notin I$.
• The set $I$ is generated by zero-relations of length 2.
• For each arrow $\alpha$ in $Q$ there is at most one arrow $\beta$ with $t(\beta) = s(\alpha)$ such that $\alpha \beta \in I$, and at most one arrow $\gamma$ with $s(\gamma) = t(\alpha)$ such that $\gamma \alpha \in I$.

A finite dimensional algebra $A$ is called gentle, if it has a presentation as $A = KQ/\langle I \rangle$ where $(Q, I)$ is gentle. For a gentle algebra $\Lambda = KQ/\langle I \rangle$, we denote by $\mathcal{C}(\Lambda)$ the set of equivalence classes (with respect to cyclic permutation) of repetition-free cyclic paths $\alpha_1 \ldots \alpha_n$ in $Q$ such that $\alpha_i \alpha_{i+1} \in I$ for all $i$, where we set $n+1 = 1$. Then Theorem 5.4 yields the following corollary immediately.

Corollary 5.6 (22). Let $\Lambda = KQ/\langle I \rangle$ be a finite dimensional gentle algebra. Then $\Lambda$ is 1-Gorenstein if and only if for any arrows $\alpha, \beta$ in $Q$ satisfying $s(\beta) = t(\alpha)$ and $\beta \alpha \in I$, there exists $c \in \mathcal{C}(\Lambda)$ such that $\alpha, \beta \in c$.

The following result gives a description of the elements in $\mathbf{F}$ for a 1-Gorenstein monomial algebra.

Proposition 5.7. Let $A = kQ/I$ be a 1-Gorenstein monomial algebra. Let $\mathbf{F}$ be the set formed by all the minimal paths among the paths in $I$. Then for any sequence

$$p_{n+1} = p_1, p_1, \ldots, p_2, p_1$$

of nonzero paths such that $(p_{i+1}, p_i)$ are perfect pairs for all $1 \leq i \leq n$, where $p_1 = \alpha_1 \cdots \alpha_1$, $p_2 = \alpha_1 + p_2 \cdots \alpha_1 + 1$, $p_n = \alpha \sum_{i=1}^{n-1} \alpha_i + \alpha_{n-1} + 1$ and $r_i = l(p_i)$ for all $1 \leq i \leq n$, we have $r_1 + r_2 = r_2 + r_3 = \cdots = r_n + r_1$. Furthermore, $\alpha_j + r - 1 \cdots \alpha_j \in \mathbf{F}$ for any $j \geq 1$, where $r = r_1 + r_2$. Here we set $\sum_{i=1}^{n-1} r_i + 1 = 1$.

Proof. Obviously, $p_{i+1} p_i \in \mathbf{F}$ for all $1 \leq i \leq n$. Without losing generality, we assume that $p_2 p_1$ is (one of) the longest path in $\{p_{i+1} p_i | 1 \leq i \leq n\}$. Then $p_2 p_1 = \alpha_1 + p_2 \cdots \alpha_1 + 1 \alpha_1 \cdots \alpha_1 \in \mathbf{F}$. So $(\alpha_1 + p_2 \alpha_1 + \cdots \alpha_1, \alpha_1)$ is a perfect pair. Then $\alpha_1 + p_2 \cdots \alpha_1 + 1 \alpha_1 \cdots \alpha_1$ is perfect path, which implies that there exists a nonzero path $q_1$ such that $(q_1, \alpha_1 + p_2 \cdots \alpha_1 + 1 \alpha_1 \cdots \alpha_1)$ is a perfect pair. Since $p_3 \alpha_1 + p_2 \cdots \alpha_1 + 1 \alpha_1 \cdots \alpha_1 + 2 = p_3 \alpha_2 \cdots \alpha_1 + 2 \in I$, and then $p_2 = q_1 q_2$ for some path $q_1$. So $q_1 = \alpha_1 + p_2 + p_3 \cdots \alpha_1 + 1 \alpha_1 \cdots \alpha_1$ for some $1 \leq s_1 \leq r_3$. Similarly, $\alpha_1 + p_2 + p_3 \cdots \alpha_1 + 1 \alpha_1 \cdots \alpha_1 + 3$ is also a perfect path. Then there exists a nonzero path $q_2$ such that $(q_2, \alpha_1 + p_2 + p_3 \cdots \alpha_1 + 1 \alpha_1 \cdots \alpha_1 + 3)$ is a perfect pair. We can also get that $p_3 = q_2 q_2$, and then $p_2 = \alpha_1 + p_2 + p_3 + \cdots \alpha_1 + 1 \alpha_1 + 3$ for some $1 \leq s_2 \leq r_2$. Since $(q_1, \alpha_1 + p_2 + p_3 \cdots \alpha_1 + 1 \alpha_1 \cdots \alpha_1)$ is a perfect pair, we get that $q_1 \alpha_1 + p_2 + p_3 \cdots \alpha_1 + 3 \notin I$, and then $1 \leq s_1 < s_2 \leq r_3$. Inductively, there exists $1 < s_1 < s_2 \cdots < s_r \leq s_3$ such that $(\alpha_1 + p_2 + p_3 \cdots \alpha_1 + 1 \alpha_1 + 3)$ is a perfect pair. Then $\alpha_1 + p_2 + p_3 \cdots \alpha_1 + 1 \alpha_1 + 3 \cdots \alpha_1 + 1 \alpha_1 + 3 \alpha_1 + r_1 + 1 \alpha_1 + r_1 + 1 \alpha_1 + r_1 + 1 \alpha_1 + r_1 + 1 \alpha_1$ is a perfect pair.

Corollary 5.8. Let $A = kQ/I$ be a Nakayama algebra with $Q$ an oriented cycle. If $A$ is 1-Gorenstein, then $A$ is self-injective.

Proof. Assume that $Q = Z_n$ for some $b \geq 1$ with $\alpha_i : i \to i + 1$ being the arrow. Recall that $\mathbf{F}$ is the set formed by all the minimal paths among the paths in $I$. Let $q \in \mathbf{F}$ be one of the longest path. Denote by $m = l(q)$. Then Proposition 5.4.1 shows every path of length $m$ is in $\mathbf{F}$, and then easily $I = J^m$, where $J$ is the two-sided ideal generated by arrows. So $A = kZ_n/J^m$ is self-injective. □
In the following, we give a characterization of $\text{Gproj}^Z_A$ for 1-Gorenstein monomial algebra $A = kQ/I$.

**Theorem 5.9.** Let $A = kQ/I$ be a 1-Gorenstein monomial algebra. Then there exists a hereditary algebra $B = kQ^B$ with $Q^B$ a disjoint union of finitely many simply-laced quivers of type $\Delta$ such that $\text{Gproj}^Z_A \cong D^b(\text{mod} \ B^{op})$.

**Proof.** Similar to proof of Theorem 4.5, we take a positive integer $l$ such that $A = A_{\leq l}$ and define a $Z$-graded $A$-module by

$$T := \bigoplus_{0 \leq i \leq l} A(i)_{\leq 0}.$$  

Then $T$ is a tilting object in $D_{sg}(\text{mod}^Z A)$. Let $PT \to T$ be the minimal graded projective cover. Then $0 \to \Omega(T) \to PT \to T \to 0$ is a short exact sequence, which implies that $\Omega(T)$ is a (graded) Gorenstein projective module since $A$ is 1-Gorenstein. So $\Omega(T)$ is a tilting object in $\text{Gproj}^Z_A$. Note that $\text{top}(\Omega(T))$ is homogeneous of degree 1. So every indecomposable summand of $\Omega(T)$ is of form $Ap(−1)$ for some perfect path $p$. Here $Ap$ is viewed as a graded module such that $\text{top}(Ap)$ is degree zero.

Without losing generality, we assume that $\Omega(T)$ is basic, and then $\Omega(T) \cong \oplus_{i=1}^n Ap_i(−1)$ for some pairwise different perfect paths $p_1, \ldots, p_n$. Let $B = \text{End}_{\text{Gproj}^Z_A}(\Omega(T)(1))$. Assume that $B = kQ^B/I^B$, where $(Q^B, I^B)$ is a bound quiver. Recall that the vertex set of $Q^B$ is $\{Ap_1, \ldots, Ap_n\}$, and the arrow set of $Q^B$ from $Ap_i$ to $Ap_j$ is formed by the irreducible morphisms from $Ap_i$ to $Ap_j$ in the subcategory of $\text{Gproj}^Z_A$ formed by $\text{add} \Omega(T)$.

For any nonzero morphism $f \in \text{Hom}_{\text{mod}^Z A}(Ap_i, Ap_j)$, since every arrow is of degree one, and $f$ preserves grades, we get that $\text{top}(Ap_i) = S_{t(p_i)} = \text{top}(Ap_j) = S_{t(p_j)}$, which implies that $t(p_i) = t(p_j)$, and $f$ maps $\text{top}(Ap_i)$ to $\text{top}(Ap_j)$ which is nonzero, then $f$ is an epimorphism. In particular, $\dim_k \text{Hom}_{\text{mod}^Z A}(Ap_i, Ap_j) \leq 1$ for any $i, j$. From these, we also get that $Q^B$ is an acyclic quiver.

For any nonzero morphism $f \in \text{Hom}_{\text{mod}^Z A}(Ap_i, Ap_j)$, if $f$ is zero in $\text{Gproj}^Z_A$, then $f$ factors through the projective cover $P_{t(p_i)} \to Ap_j$ in $\text{mod}^Z A$, which is of form $f = \beta \alpha$, where $\beta : Ap_i \to P_j(p_j)$. Since $t(p_j) = t(p_i)$, we get that there is nonzero morphism $\beta : Ap_i \to P_j(p_j)$, similar to the above, we get that $\beta$ is an isomorphism, and then $Ap_i$ is projective, a contradiction. So $f \neq 0$ in $\text{Gproj}^Z_A$. Therefore, the arrow set of $Q^B$ from $Ap_i$ to $Ap_j$ is formed by the irreducible morphisms from $Ap_i$ to $Ap_j$ in $\text{add} \Omega(T)$.

We define a partial order for the set $\{Ap_1, \ldots, Ap_n\}$ as follows. We set $Ap_i \leq Ap_j$ if and only if $p_i, p_j$ have the same ending point, and $p_j = p_ip_{i,j}$ for some path $p_{i,j}$. One can check that it is well-defined. Obviously, we can assume that the partial order is $Ap_1 \leq \cdots \leq Ap_m$, $Ap_{m+1} \leq \cdots \leq Ap_{m+2}$, $\ldots$, $Ap_{m+1+1} \leq \cdots \leq Ap_{m+n}$. In the following, we prove that there is an irreducible morphism from $Ap_i$ to $Ap_j$ in $\text{add} \Omega(T)$ if and only if $i = j + 1$ and $Ap_i \leq Ap_{i+1}$.

Denote by $(p_i, p_i)$ the perfect pair for any $1 \leq i \leq n$. If there is a nonzero morphism $f$ from $Ap_i$ to $Ap_j$, then $t(p_i) = t(p_j)$ and $f$ is surjective. By noting that $Ap_i$ has a basis given by all nonzero paths $q$ such that $q = q'p_i$ for some path $q'$, we get that $q_i$ is not in $Ap_i$. Since $f$ is surjective, we get that $q_i$ is not in $Ap_i$ and then $q_ip_j = 0$. By (P3) of Definition 4.1, we get that $p_j = p_ip_{i,j}$ for some path $p_{i,j}$, and then $Ap_i \leq Ap_j$. Conversely, if $Ap_i \leq Ap_j$, then $p_j = p_ip_{i,j}$, and there is a nonzero morphism $f_{ij} : Ap_i \to Ap_j$ induced by the multiplication of $p_{i,j}$ for any $i, j$. In particular, $f_{ij} = f_{i−1,j} \cdots f_{i+1,i,2}f_{i,i+1}$, and $\text{Hom}_{\text{Gproj}^Z_A}(Ap_i, Ap_j)$ is spanned by $f_{ij}$ if $Ap_i \leq Ap_j$. So if there is an irreducible morphism $f : Ap_i \to Ap_j$ in $\text{add} \Omega(T)$, then $j = i + 1$ and $Ap_i \leq Ap_j$. Conversely, one can check that $f_{i,i+1} : Ap_i \to Ap_{i+1}$ is irreducible in $\text{add} \Omega(T)$.
when \( A_{p_i} \leq A_{p_{i+1}} \) since any morphism from \( A_{p_i} \) to \( A_{p_j} \) with \( j \neq i \) factors through \( f_{i,i+1} \). Then

\[
Q^B = \bigcup_{l=1}^{r} (A_{p_{m_{l-1}+1}} \to A_{p_{m_{l-2}+1}} \to \cdots \to A_{p_{m_{l}}}).
\]

Here \( m_0 = 0 \). Since each irreducible morphism \( f_{i,i+1} : A_{p_i} \to A_{p_{i+1}} \) is surjective, we get that \( I^B = 0 \). Therefore, there exists a hereditary algebra \( B = kQ^B \) with \( Q^B \) a disjoint union of finitely many simply-laced quivers of type \( A \) such that \( \text{Gproj}^B A \simeq D^b(\text{mod }B)^{\text{op}} \).

\[
\square
\]

6. Singularity categories of 1-Gorenstein monomial algebras

Finally, we characterize singularity categories for 1-Gorenstein monomial algebras. Before that, we give a precise definition of the \textit{gluing of algebras}, which is defined in \cite{12}. After that, we prove that singularity category is an invariant under taking this kind of gluing. Finally, we use it to characterize the singularity categories of 1-Gorenstein monomial algebras.

For any vertex \( i \) in any quiver, by abusing notations, we always use \( e_i \) to denote the corresponding \textit{primitive idempotent}.

6.1. Gluing of algebras. Let \( S = kQ/I \) be a finite-dimensional quiver algebra (not necessarily monomial), where \( Q = (Q_0, Q_1, s, t) \). We do not assume \( Q \) to be connected. Let \( E \) be an \textit{involution} on the set of vertices of \( Q \). The following procedure associates a new quiver \( Q(E) \) to \( Q \) by gluing together each pair \( x \neq E(x) \) to one vertex. Precisely, for each \( x \in Q_0 \), we define \( \bar{x} = \{x, E(x)\} \). The quiver \( Q(E) = (Q(E)_0, Q(E)_1, s(E), t(E)) \) is then defined as follows:

- \( Q(E)_0 = \{\bar{x} : x \in Q_0\} \),
- \( Q(E)_1 = Q_1 \),
- \( s(E)(\alpha) = s(\alpha) \) and \( t(E)(\alpha) = t(\alpha) \) for any \( \alpha \in Q_1 \).

From the definition it follows that any path in \( Q(E) \) is also a path in \( Q(E) \), Hence we can regard \( I \) as a subset of \( kQ(E) \). Let \( I(E) \) be the ideal of \( kQ(E) \) that is generated by \( I \) and set \( S_E = kQ(E)/I(E) \), which is called the \textit{Brüstle’s gluing algebra of \( S \) by gluing the vertices along \( E \).}

In the following, we always assume that \( S_E \) is finite-dimensional.

Lemma 6.1. Keep the notations as above. Then \( S_E \) is finite-dimensional if and only if there is no nonzero paths \( p_m, \ldots, p_1 \) in \( S \) such that \( t(p_i) = E(s(p_{i+1})) \) and \( t(p_i) \neq E(t(p_i)) \) for any \( i \in \mathbb{Z}/m\mathbb{Z} \). In particular, if \( S_E \) is finite-dimensional, then there is no nonzero path from \( x \) to \( E(x) \) for any \( x \neq E(x) \) in \( S \).

Proof. Suppose for a contradiction that there exist no nonzero paths \( p_m, \ldots, p_1 \) in \( S \) such that \( t(p_i) = E(s(p_{i+1})) \) and \( t(p_i) \neq E(t(p_i)) \) for any \( i \in \mathbb{Z}/m\mathbb{Z} \), then \( (p_m \cdots p_1)^l \) is a path in \( Q(E) \) for any \( l > 0 \). From the definition of \( I(E) \), it is easy to see that \( (p_m \cdots p_1)^l \) is nonzero in \( S_E \), which implies that \( S_E \) is infinite-dimensional, giving a contradiction.

Conversely, we assume that \( (x_1, E(x_1)), \ldots, (x_n, E(x_n)) \) are the pairs of vertices with \( x_i \neq E(x_i) \). For \( 1 \leq j \leq n \), define the involution \( E_j \) on the set of vertices of \( Q \) such that \( E_j(x_i) = E(x_i) \) for any \( 1 \leq i < j \), and \( E(x) = x \) otherwise. It is easy to see that \( E_n = E \). Then we get a series of algebras \( S, S_{E_1}, \ldots, S_{E_n} = S_E \). We prove that all of these algebras are finite-dimensional.

Denote by \( N_0 \) the length of the longest path in \( Q \) which is nonzero in \( S \). For any nonzero path \( q \) in \( S_{E_1} \), if \( q \) is path in \( S \), then it is also nonzero in \( S \), and then the length of \( q \) is less than \( N_0 \). Otherwise, \( q = q_r \cdots q_1 \), where \( t(q_i), s(q_{i+1}) \in \{x_1, E(x_1)\} \) for any \( 1 \leq i < r \), and \( q_1, \ldots, q_r \) are nonzero paths in \( S \). Note that \( t(q_i) \neq E(s(q_i)) \) for any \( 2 \leq i < r \) by the assumption. So \( t(q_i) = E(s(q_i)) \in \{x_1, E(x_1)\} \). If \( r \geq 4 \), then \( p_2 = q_3, p_1 = q_2 \) satisfy that \( t(p_i) = E(s(p_{i+1})) \)
and \( t(p_i) \neq E(t(p_i)) \) for any \( i \in \mathbb{Z}/2\mathbb{Z} \), contradicts. So \( r < 4 \), which implies that the length of \( q \) is less than \( 3N_0 \), and then \( S_{E_1} \) is finite-dimensional.

By induction, we assume that \( S_{E_{n-1}} \) is finite-dimensional. Similar to the above, one can check that \( S_E = S_{E_n} \) is finite-dimensional.

For \( S = kQ/I \), and \( E \) is an involution on the set of vertices of \( Q \), we define another new quiver \( \tilde{Q} \) from \( Q \) by adding an arrow \( \alpha_{(x, E(x))} \) between \( x \) and \( E(x) \) (in either direction) if \( x \neq E(x) \). Then \( I \) can be viewed as a subset of \( k\tilde{Q} \). Let \( \tilde{I} \) be the ideal of \( k\tilde{Q} \) that is generated by \( I \) and \( \tilde{S} = k\tilde{Q}/\tilde{I} \). It is easy to see that if \( S_E \) is finite-dimensional, then so is \( \tilde{S} \).

**Example 6.2.** Let \( S = kQ/I \) be the quiver algebra, where \( Q \) is the quiver as Fig. 9 shows, and \( I \) is generated by \( \alpha_{i+2}\alpha_{i+1}\alpha_i \), for any \( 1 \in \mathbb{Z}/6 \). Let \( E \) be the involution on the set of vertices which maps \( 3 \) to \( 6 \), and if \( i \) to \( i \) otherwise. Then \( S_E \) is the algebra \( kQ(E)/I_E \) with \( Q(E) \) as Fig. 10 shows, and \( I_E \) is generated by \( \alpha_{i+2}\alpha_{i+1}\alpha_i \), for any \( 1 \in \mathbb{Z}/6 \). \( \tilde{S} = k\tilde{Q}/\tilde{I} \) is the algebra with \( \tilde{Q} \) as Fig. 11 shows, and \( \tilde{I} \) is also generated by \( \alpha_{i+2}\alpha_{i+1}\alpha_i \), for any \( 1 \in \mathbb{Z}/6 \).

![Fig. 9. The quiver Q in Example 6.2](image)

![Fig. 10. The quiver Q(E) in Example 6.2](image)

![Fig. 11. The quiver \( \tilde{Q} \) in Example 6.2](image)

Keep the notations as above. Similar to [12], we define a functor \( \Phi : \text{mod} \ S_E \to \text{mod} \ \tilde{S} \) as follows. For any finite-dimensional representation \( M = (M_i, \phi_\alpha)_{\alpha \in Q, \alpha \in E, \alpha} \) of \( (Q(E), I(E)) \), we define \( \Phi(M) = (N_j, \psi_\beta)_{\beta \in Q, \beta \in \tilde{Q}} \) by \( N_j = M_i \) if \( j = i \) and \( \psi_\beta = \phi_\beta \) for any \( \beta \in Q(E)_i = Q_i \), and identity for any newly added arrow \( \alpha_{(x, E(x))} \). It is easy to see that \( \Phi \) is an exact fully faithful functor, which induces that \( \text{mod} \ S_E \) is equivalent to the full subcategory of \( \text{mod} \ \tilde{S} \) consisting of those modules \( (N_j, \psi_\beta)_{\beta \in Q, \beta \in \tilde{Q}} \) with \( \psi_\beta \) bijective whenever \( \beta \) is any newly added arrow \( \alpha_{(x, E(x))} \).

In the following, we prove that \( D_{sg}(\tilde{S}), D_{sg}(S_E) \) and \( D_{sg}(S) \) are triangulated equivalent. For any two algebras \( A \) and \( B \), if \( D_{sg}(A) \) and \( D_{sg}(B) \) are triangle equivalent, then we call \( A \) and \( B \) to be singularity equivalent, similar to the definition of derived equivalent.

**Lemma 6.3.** Keep the notations as above. Assume that there is only one pair of vertices \( (x, E(x)) \) such that \( x \neq E(x) \), and the added arrow for \( \tilde{Q} \) is \( \gamma = \alpha_{(x, E(x))} : x \to E(x) \). Then for any vertex \( i \neq E(x) \) in \( Q_0 \), there are short exact sequences in \( \text{mod} \ \tilde{S} \):

\[
\begin{align*}
(7) \quad 0 & \to (\tilde{S}(e_{E(x)})_{\tilde{E}_i})^{\oplus t_i} \to \tilde{S}e_i \oplus (\tilde{S}e_x)^{\oplus t_i} \to \Phi(S_Ee_i) \to 0, \\
(8) \quad 0 & \to \tilde{S}e_i \to \Phi(S_Ee_i) \to B_z^{\oplus t_i} \to 0,
\end{align*}
\]

and
for some $t_i$, where $B_x$ is the cokernel of the natural injective morphism $f_\gamma : \mathcal{S}e_{E(x)} \to \mathcal{S}e_x$ induced by the arrow $\gamma$.

Proof. Since $S_E$ is finite-dimensional, from Lemma 6.1 there is no nonzero path from $x$ to $E(x)$ or from $E(x)$ to $x$ in $S$.

In the following, we use the structure of projective modules and the definition of $\Phi$ to prove our desired results. In order to describe the structure of projective modules, for simplicity, we assume that there is only one arrow $\alpha_1$ ending to $x$, only one arrow $\alpha_2$ starting at $x$, only one arrow $\alpha_3$ ending to $E(x)$, and only one arrow $\alpha_4$ starting at $E(x)$. Then the structure of the indecomposable projective module $S_{Ee_1}$ is as Fig. 12 shows without losing of generality. The structure of $\Phi(S_{Ee_1})$ is as Fig. 13 shows. So we get that there exists a short exact sequence in mod $\mathcal{S}$:

$$0 \to \mathcal{S}e_i \to \Phi(S_{Ee_1}) \to B_x^{\oplus t_i} \to 0.$$ 

In fact, $t_i$ is the number of the arrows $\alpha_3$ in Fig. 12.

It is worth noting that $\Phi(S_{Ee_2}) = \mathcal{S}e_x$ by Fig. 13 since there is no nonzero path from $x$ to $E(x)$ in $\mathcal{S}$.

By the definition, there is a short exact sequence $0 \to \mathcal{S}e_{E(x)} \xrightarrow{f} \mathcal{S}e_x \to B_x \to 0$, and then we get the following pull-back diagram:

$$
\begin{array}{c}
\mathcal{S}e_i \\
\downarrow \\
\mathcal{S}e_i 
\end{array} 
\begin{array}{c}
\xrightarrow{(\mathcal{S}e_{E(x)})^{\oplus t_i}} \\
\downarrow \\
(\mathcal{S}e_x)^{\oplus t_i} 
\end{array} 
\begin{array}{c}
\mathcal{S}e_i \\
\downarrow \\
\Phi(S_{Ee_1}) 
\end{array} 
\begin{array}{c}
\xrightarrow{B_x^{\oplus t_i}} \\
\end{array}
$$

So the short exact sequence (7) follows immediately.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig12}
\caption{The structure of indecomposable projective module $S_{Ee_1}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig13}
\caption{The structure of $\Phi(S_{Ee_1})$.}
\end{figure}

Lemma 6.4. Keep the notations as above. Assume that there is only one pair of vertices $(x, E(x))$ such that $x \neq E(x)$, and the added arrow for $\overline{Q}$ is $\gamma = \alpha_{(x,E(x))} : x \to E(x)$. Let $B_x$ be the cokernel of the natural injective morphism $f_\gamma : \mathcal{S}e_{E(x)} \to \mathcal{S}e_x$ induced by the arrow $\gamma$. Then

$$\text{Hom}_\mathcal{S}(B_x, \Phi(M)) = 0 = \text{Ext}_\mathcal{S}^i(B_x, \Phi(M))$$

for any $i \geq 1$ and $M \in \text{mod } S_E$. In particular, $\text{End}_\mathcal{S}(B_x) \cong k$, i.e. $B_x$ is a brick.
Proof. For any morphism \( f : B_x \to \Phi(M) \), by restriction to the full subquiver formed by the added arrow \( \gamma \), it is easy to see that \( f|_{\text{top}(B_x)} = 0 \) and then \( f = 0 \).

From the definition of \( \Phi \), and the structure of \( \bar{S}e_x \) and \( \bar{S}e_{E(x)} \), we get that for any morphism \( g : \bar{S}e_{E(x)} \to \Phi(M) \), there exists a morphism \( h : \bar{S}e_x \to \Phi(M) \) such that \( g = hf_\gamma \). Applying \( \text{Hom}_S(-, \Phi(M)) \) to

\[
0 \to \bar{S}e_{E(x)} \xrightarrow{f_x} \bar{S}e_x \to B_x \to 0,
\]

we get that \( \text{Ext}_S^1(B_x, \Phi(M)) = 0 \) for any \( i \geq 1 \).

From the structure of \( B_x \), it is easy to see that any proper submodule of \( B_x \) is in \( \text{Im} \Phi \), so \( \text{End}_S(B_x) \cong k \) by the above. \( \square \)

Lemma 6.5. Keep the notations as above. Assume that there is only one pair of vertices \((x, E(x))\) such that \( x \neq E(x) \), and the added arrow for \( \bar{Q} \) is \( \gamma = \alpha_{(x, E(x))}: x \to E(x) \). Then the derived functor of \( \Phi : D^b(\text{mod } S_E) \to D^b(\text{mod } \bar{S}) \) is fully faithful.

Proof. Since \( \Phi : \text{mod } S_E \to \text{mod } \bar{S} \) is exact, its derived functor is \( D^b(\Phi) : D^b(\text{mod } S_E) \to D^b(\text{mod } \bar{S}) \). For any \( M, N \in \text{mod } S_E \), take a projective resolution of \( M \)

\[
\cdots \xrightarrow{f^{n+1}} M^n \xrightarrow{f^n} \cdots \xrightarrow{f^1} P^0 \xrightarrow{f^0} M \to 0.
\]

Denote by \( M^i = \text{Im } f^i \) for any \( i \geq 0 \). Note that \( M^i \) is a \( i \)-th syzygy of \( M \), and \( M^0 = M \).

First, we check that \( \Phi \) induces that \( \text{Ext}_S^1(\Phi(M), \Phi(N)) \cong \text{Ext}_{S_E}^1(M, N) \). In fact, we get a short exact sequence

\[
0 \to \Phi(M^1) \to \Phi(U^0) \xrightarrow{\Phi(f^0)} \Phi(M) \to 0.
\]

From (8), we get that there is a short exact sequence

\[
0 \to Q^0 \to \Phi(U^0) \to B_x \to 0.
\]

By applying \( \text{Hom}_S(-, \Phi(N)) \) to the above sequence, from Lemma 6.3, we get that

\[
\text{Ext}_S^1(\Phi(U^0), \Phi(N)) = 0, \text{ for any } i > 0.
\]

From (8), we get that following exact sequence

\[
\text{Hom}_S(\Phi(U^0), \Phi(N)) \to \text{Hom}_S(\Phi(M^1), \Phi(N)) \to \text{Ext}_S^1(\Phi(M), \Phi(N)) \to 0.
\]

Since \( \Phi \) is fully faithful, it is easy to see that \( \text{Ext}_S^1(\Phi(M), \Phi(N)) \cong \text{Ext}_{S_E}^1(M, N) \).

For \( i > 1 \), by induction, we get that \( \Phi \) induces that \( \text{Ext}_{S_E}^i(M^1, N) \cong \text{Ext}_{S_E}^{i-1}(M, N) \), and then

\[
\text{Ext}_{S_E}^i(\Phi(M), \Phi(N)) \cong \text{Ext}_{S_E}^{i-1}(\Phi(M^1), \Phi(N)) \cong \text{Ext}_{S_E}^{i-1}(M^1, N) \cong \text{Ext}_{S_E}^i(M, N).
\]

To sum up, we get that \( D^b(\Phi) \) induces

\[
\text{Hom}_{D^b(S)}(D^b(\Phi)(M^\bullet), D^b(\Phi)(N^\bullet)) = \text{Hom}_{D^b(S_E)}(M^\bullet, N^\bullet)
\]

for any stalk complexes \( M^\bullet, N^\bullet \). By induction on the length of complexes, and using the “five Lemma”, one can check that \( D^b(\Phi) \) is fully faithful. \( \square \)

By abusing notations, we use \( \Phi \) to denote its derived functor \( D^b(\Phi) \) in the following.

Lemma 6.6 (14). Let \( \mathcal{T} \) be a triangulated category with two full triangulated subcategories \( \mathcal{T}' \) and \( S \). Set \( S' = S \cap \mathcal{T}' \). Then the natural inclusion \( \mathcal{T}' \hookrightarrow \mathcal{T} \) induces an exact functor \( J : \mathcal{T}'/S' \to \mathcal{T}/S \). Suppose that either

(a) every morphism from an object in \( S \) to an object in \( \mathcal{T}' \) factors through some object in \( S' \), or

(b) every morphism from an object in \( \mathcal{T}' \) to an object in \( S \) factors through some object in \( S' \).
(b) every morphism from an object in $T'$ to an object in $S$ factors through some object in $S'$.

Then the induced functor $J : T'/S' \to T/S$ is fully faithful.

**Proposition 6.7.** Keep the notations as above. Assume that there is only one pair of vertices $(x, E(x))$ such that $x \neq E(x)$, and the added arrow for $\tilde{Q}$ is $\gamma = \alpha_{(x,E(x))} : x \to E(x)$. Then the exact functor $\Phi : \text{mod } S_E \to \text{mod } \tilde{S}$ induces a triangle equivalence $\tilde{\Phi} : D_{sg}(S_E) \to D_{sg}(\tilde{S})$.

**Proof.** From [7], we get that $\text{proj} \dim S(M) < \infty$ if $\text{proj} \dim S_E M < \infty$ for any finitely generated $S_E$-module $M$ since $\Phi$ is exact. So $\Phi(K^b(\text{proj } S_E)) \subseteq K^b(\text{proj } \tilde{S})$, which implies that $\Phi$ induces an exact functor $\tilde{\Phi} : D_{sg}(S_E) \to D_{sg}(\tilde{S})$. In the following, we use Lemma 6.6 to prove that $\tilde{\Phi}$ is fully faithful.

First, we check that $\Phi(K^b(\text{proj } S_E)) = K^b(\text{proj } \tilde{S}) \cap \Phi(D^b(\text{mod } S_E))$. From the above, we get that $\Phi(K^b(\text{proj } S_E)) \subseteq K^b(\text{proj } \tilde{S}) \cap \Phi(D^b(\text{mod } S_E))$. Conversely, for any object in $K^b(\text{proj } \tilde{S}) \cap \Phi(D^b(\text{mod } S_E))$, it is of form

$$\Phi(X^\bullet) = \cdots \xrightarrow{\Phi(d^{-1})} \Phi(X^1) \xrightarrow{\Phi(d^1)} \Phi(X^{i+1}) \xrightarrow{\Phi(d^{i+1})} \Phi(X^{i+2}) \xrightarrow{\Phi(d^{i+2})} \cdots$$

for some bounded complex $X^\bullet = \cdots \xrightarrow{d^{-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \xrightarrow{d^{i+2}} \cdots$.

Since $\Phi(X^\bullet) \in K^b(\text{proj } \tilde{S})$, from [35, 4.1, Lemma, b]), we get that there is an epimorphism of complexes $f^\bullet : Q^\bullet \to \Phi(X^\bullet)$ such that $f^\bullet$ is a quasi-isomorphism, where $Q^\bullet = \cdots \xrightarrow{e^{-1}} Q^i \xrightarrow{e^i} Q^{i+1} \xrightarrow{e^{i+1}} Q^{i+2} \xrightarrow{e^{i+2}} \cdots$ is a bounded complex with $Q^i \in \text{proj } \tilde{S}$ for all $i$. Here $f^\bullet = (f^i : Q^i \to \Phi(X^i))_i$. For every $Q^i$, by [35], there exists $P^i \in \text{proj } S_E$ such that

$$(10) \quad 0 \to Q^i \xrightarrow{g^i} \Phi(P^i) \to B_x^{\oplus s_i} \to 0$$

is exact for some integer $s_i$. Lemma 6.4 implies that $f^i$ factors through $g^i$ for any $i$, and then there exists $h^i : P^i \to X^i$ such that $\Phi(h^i)g^i = f^i$ for any $i$ since $\Phi$ is fully faithful.

Similar to the above, by applying $\text{Hom}_{\tilde{S}}(-, \Phi(P^{i+1}))$ to the short exact sequence (10), we get that there exists $P^i : P^i \to P^{i+1}$ such that the following diagram is commutative for each $i$ since $\Phi$ is fully faithful:

$$
\begin{array}{ccc}
\Phi(P^i) & \xrightarrow{\Phi(p^i)} & \Phi(P^{i+1}) \\
\downarrow{g^i} & & \downarrow{g^{i+1}} \\
Q^i & \xrightarrow{e^i} & Q^{i+1} 
\end{array}
$$

It follows from $\Phi(p^{i+1}p^i)g^i = g^{i+2}e^{i+1}e^i = 0$ that $\Phi(p^{i+1}p^i)$ factors through $B_x^{\oplus s_i}$, which implies that $\Phi(p^{i+1}p^i) = 0$ since $\text{Hom}_{\tilde{S}}(B_x, \Phi(P^{i+1})) = 0$. So $p^{i+1}p^i = 0$ for any $i$, which implies that $P^\bullet = (P^i, p^i)_i$ is a complex in $K^b(\text{proj } S_E)$. In particular $g^\bullet = (g^i : Q^i \to \Phi(P^i))$ is a morphism of complexes.

In the following, we check that $h^{i+1}p^i = d'h^i$ for each $i$. In fact,

$$\Phi(h^{i+1}p^i)g^i = \Phi(h^{i+1})g^{i+1}e^i = f^{i+1}e^i = \Phi(d^i)f^i = \Phi(d'h^i)g^i.$$
In the following, we check that $h^\bullet$ is a quasi-isomorphism. Since $\Phi(h^\bullet)g^\bullet = \Phi(f^\bullet)$, and $f^\bullet$ is a quasi-isomorphism, we only need to check that $g^\bullet$ is a quasi-isomorphism since $\Phi$ is fully faithful and exact.

Let $B^\bullet = (B^i, u_i)_i$ be the cokernel of $g^\bullet$. Then $B^i = B_x^{\oplus s_i}$. We get a short exact sequence of complexes $0 \to Q^\bullet \xrightarrow{\Phi(P^\bullet)} \Phi(P^\bullet) \to B^\bullet \to 0$, and then a long exact sequence of cohomological groups:

$$\cdots \to H^i(Q^\bullet) \to H^i(\Phi(P^\bullet)) \to H^i(B^\bullet) \to H^{i+1}(Q^\bullet) \to \cdots.$$

From Lemma 6.3, we get that $\text{End}_S(B_x) = k$. So $H^i(B^\bullet) = B_x^{\oplus t_i}$ for some $t_i$. On the other hand, $H^{i+1}(Q^\bullet) \cong H^{i+1}(\Phi(X^\bullet)) = \Phi(H^{i+1}(X^\bullet))$, so $\text{Hom}_S(H^{i}(B^\bullet), H^{i+1}(Q^\bullet)) = 0$, and then $H^{i}(B^\bullet)$ is the cokernel of $H^{i}(Q^\bullet) \to H^{i}(\Phi(P^\bullet))$. Noting that both of $H^{i}(Q^\bullet)$ and $H^{i}(\Phi(P^\bullet))$ are in the image of $\Phi$, we get that $H^{i}(B^\bullet)$ is also in the image of $\Phi$ since $\Phi$ is fully faithful and exact. Together with $H^{i}(B^\bullet) = B_x^{\oplus t_i}$, we get that $H^{i}(B^\bullet) = 0$, and then $B^\bullet$ is an acyclic complex. So $g^\bullet$ is a quasi-isomorphism. Then we get that $h^\bullet$ is a quasi-isomorphism, which yields that $X^\bullet \in K^b(\text{proj} \ S_E)$ since $P^\bullet \in K^b(\text{proj} \ S_E)$.

To sum up, we get that $\Phi(K^b(\text{proj} \ S_E)) = K^b(\text{proj} \ S) \cap \Phi(D^b(\text{mod} \ S_E))$.

Second, we prove that $\Phi(K^b(\text{proj} \ S_E)) \subseteq D^b(\text{mod} \ S_E)$ and $K^b(\text{proj} \ S) \subseteq D^b(\text{mod} \ S)$ satisfy Lemma 6.6 (a). In fact, for any $Q^\bullet = (Q^i, e^i)_i \in K^b(\text{proj} \ S)$, $X^\bullet = (X^i, d^i)_i \in D^b(\text{mod} \ S_E)$, and any morphism $f^\bullet = (f^i)_i : Q^\bullet \to \Phi(X^\bullet)$, from [10], we get a short exact sequence for each $Q^i$:

$$0 \to Q^i \xrightarrow{g^i} \Phi(P^i) \to B_x^{\oplus s_i} \to 0$$

where $P^i \in \text{proj} \ S_E$. Similar to the above, we get that there exists $p^i : P^i \to P^{i+1}$ such that the following diagram is commutative for each $i$:

$$\begin{array}{ccc}
\Phi(P^i) & \xrightarrow{\Phi(p^i)} & \Phi(P^{i+1})\\
\downarrow{g^i} & & \downarrow{g^{i+1}}\\
Q^i & \xrightarrow{e^i} & Q^{i+1},
\end{array}$$

and $P^\bullet = (P^i, p^i)_i$ is a complex in $K^b(\text{proj} \ S_E)$. Furthermore, similar to the above, we can also get that $f^i : Q^i \to \Phi(X^i)$ factors through $g^i$ as $f^i = \Phi(h^i)g^i$ for some morphism $h^i : P^i \to X^i$ for each $i$, and $h^\bullet = (h^i : P^i \to X^i)_i : P^\bullet \to X^\bullet$ is a morphism of complexes. So we get that $f^\bullet = \Phi(h^\bullet)g^\bullet$ factors through $\Phi(P^\bullet)$ which is in $\Phi(K^b(\text{proj} \ S_E))$. So $\Phi(K^b(\text{proj} \ S_E)) \subseteq D^b(\text{mod} \ S_E)$ and $K^b(\text{proj} \ S) \subseteq D^b(\text{mod} \ S)$ satisfy Lemma 6.6 (a), and then the induced functor $\tilde{\Phi} : D_Sg(S_E) \to D_Sg(S)$ is fully faithful.

Finally, we check that $\Phi$ is dense.

For any object in $D^b(S)$, we can assume that it is of form $(Q^i, d^i)_i \in K^{-b}(\text{proj} \ S_E)$ by the equivalence $D^b(S_E) \simeq K^{-b}(\text{proj} \ S_E)$. Similar to the above, we get that there is a short exact sequences of bounded above complexes: $0 \to Q^\bullet \xrightarrow{\Phi(P^\bullet)} \Phi(P^\bullet) \to B^\bullet \to 0$, where $B^i = B_x^{\oplus s_i}$. For $i$ sufficiently small, $H^i(Q^\bullet) = 0$, which implies that $H^i(\Phi(P^\bullet)) \cong H^i(B^\bullet)$ for $i$ sufficiently small. Since $H^i(\Phi(P^\bullet)) \cong \Phi(H^i(P^\bullet))$ and $H^i(B^\bullet) = B_x^{\oplus t_i}$ for some $t_i$, we get that both of them are zero for $i$ sufficiently small. Then $\Phi(P^\bullet), B^\bullet \in K^{-b}(\text{proj} \ S_E)$ since $\text{proj} \dim_S \Phi(P^i) \leq 1, \text{proj} \dim_S B_x \leq 1$ for any $i$. For $B^\bullet = (B^i, b^i)_i$, since $\text{Hom}_S(S, B_x) = k$, we get that $\text{Im} b^i \in \text{add} B_x$. Denote by $n$ the number satisfying $H^i(B^\bullet) = 0$ for any $i \leq n$. Then there is a
Lemma 6.10. Keep the notations as above. Assume that there is only one pair of vertices.

\[ \cdots \rightarrow B^{n-2} \overset{b^{n-2}}{\rightarrow} B^{n-1} \rightarrow \text{Im} b^{n-1} \rightarrow 0 \rightarrow 0 \rightarrow \cdots \]

\[ \cdots \rightarrow B^{n-2} \overset{b^{n-2}}{\rightarrow} B^{n-1} \overset{b^{n-1}}{\rightarrow} B^n \overset{b^{n+1}}{\rightarrow} B^{n+1} \overset{b^{n+1}}{\rightarrow} B^{n+2} \rightarrow \cdots \]

Since the complex in the first row is acyclic, and \( \text{Im} b^{n-1}, B^n / \text{Im} b^{n-1} \in \text{add} B_\varepsilon \), we get that \( B^\bullet \) is in \( K^b(\text{proj} \ S) \) since it is isomorphic to the complex in the third row in \( D^b(\text{mod} \ S) \). From \( 0 \rightarrow Q^\bullet \overset{\Phi(P^\bullet)}{\rightarrow} B^\bullet \rightarrow 0 \), we get that \( Q^\bullet \cong \Phi(P^\bullet) \) in \( D_{sg}(\bar{S}) \), and then \( \bar{\Phi} \) is dense.

Therefore, \( \bar{\Phi} \) is a triangulated equivalence.

In the following, we prove that \( D_{sg}(\bar{S}) \) and \( D_{sg}(S) \) are triangulated equivalent. Note that \( S \) is a subalgebra of \( \bar{S} \), and then \( S \) can be viewed as a left (also right) \( S \)-module. In particular, \( \bar{S} \) is projective as a left (also right) \( S \)-module. Then there is an adjoint triple \((j_*, j^*, \bar{j})\), where \( j_* = \bar{S} \otimes_S - : \text{mod} \ S \rightarrow \text{mod} \bar{S}, j^* = s\bar{S} \otimes_{\bar{S}} - : \text{mod} \bar{S} \rightarrow \text{mod} S \), and \( \bar{j} = \text{Hom}_S(s\bar{S}, -) : \text{mod} \bar{S} \rightarrow \text{mod} S \). Both of these three functors are exact, and they induce triangulated functors on bounded derived categories, which are denoted by the same notations respectively.

Lemma 6.8. Keep the notations as above. Assume that there is only one pair of vertices \((x, E(x))\) such that \( x \neq E(x) \), and the added arrow for \( Q \) is \( \gamma = \alpha_{(x, E(x))} : x \rightarrow E(x) \). Then \( D_{sg}(S) \) and \( D_{sg}(\bar{S}) \) are triangulated equivalent.

Proof. Since \( \bar{S} \) is projective as a left (also right) \( S \)-module, we get that both of \( j_*, j^* \) preserve projective modules. Together with that they are exact, we get that they induces triangulated functors on singularity categories, which are denoted by \( j_* \) and \( j^* \) respectively. By induction on the length of modules, one can get that the adjunction \( \text{Id} \rightarrow j^* j_* \) induces that \( M \) is a direct summand of \( j^* j_* (M) \), and \( j^* j_* (M) \cong M \oplus \text{dim}_S M \) for any \( S \)-module \( M = (M_\alpha, \phi_\alpha)_{\alpha \in Q_\alpha} \). In particular, we get that \( j^* j_* \cong \text{Id}_{D_{sg}(\bar{S})} \) since \( \text{mod} \bar{S} \) is a generator of \( D_{sg}(\bar{S}) \). Therefore, \( j_* \) is fully faithful.

So we get the following result by combining Proposition 6.7 and Lemma 6.8.

Theorem 6.9. Let \( S = kQ/I \) be a finite-dimensional algebra, and \( E \) an involution on the set of vertices of \( Q \). Assume that the algebra \( S_E \) constructed from \( S \) by gluing the vertices along \( E \) is finite-dimensional. Then \( S \) and \( S_E \) are singularity equivalence.

Proof. We assume that \((x_1, E(x_1)), \ldots, (x_n, E(x_n))\) are the pairs of vertices with \( x_i \neq E(x_i) \). For \( 1 \leq j \leq n \), define the involution \( E_j \) on the set of vertices of \( Q \) such that \( E_j(x_i) = E(x_i) \) for any \( 1 \leq i \leq j \), and \( E(x) = x \) otherwise. It is easy to see that \( E_n = E \). Then we get a series of algebras \( S, S_{E_1}, \ldots, S_{E_n} = S_E \). Easily, \( S_{E_{i+1}} \) can be viewed as a Brüstle’s gluing algebra of \( S_E \) by gluing \( x_{i+1} \) to \( E(x_{i+1}) \) for any \( 0 \leq i < n \). Combining Proposition 6.7 and Lemma 6.8 we get that \( D_{sg}(S_{E_i}) \cong D_{sg}(S_{E_{i+1}}) \) for any \( 0 \leq i < n \), which implies that \( D_{sg}(S) \cong D_{sg}(S_E) \).

At the end of this subsection, we consider the Gorenstein property of \( S \) and \( S_E \).

Lemma 6.10. Keep the notations as above. Assume that there is only one pair of vertices \((x, E(x))\) such that \( x \neq E(x) \), and the added arrow for \( Q \) is \( \gamma = \alpha_{(x, E(x))} : x \rightarrow E(x) \). Then \( S \) is Gorenstein if and only if \( \bar{S} \) is Gorenstein.
Proof. If $\bar{S}$ is Gorenstein, then for any indecomposable injective $\bar{S}$-module $D(\bar{S})e_i$, we get that there is an exact sequence which is a minimal projective resolution of $D(\bar{S})e_i$:

$$0 \to P^m \to P^{m-1} \to \cdots \to P^0 \to D(\bar{S})e_i \to 0.$$ 

By applying the restriction functor $j^*$ to the above exact sequence, we get that the following exact sequence:

$$0 \to j^*(P^m) \to j^*(P^{m-1}) \to \cdots \to j^*(P^0) \to j^*(D(\bar{S})e_i) \cong D(S)e_i \oplus (D(S)e_{E(x)})^{\oplus s_i} \to 0.$$ 

Then we get that $\text{proj.dim}_S(D(S)e_i) < \infty$ since $j^*$ preserves projective modules. Similarly, one can check that $\text{inj.dim}_S(Se_i) < \infty$ since $j^*$ also preserves injective modules, and $j^*(\bar{S}e_i) \cong S e_i \oplus (S e_{E(x)})^{\oplus r_i}$ for some $r_i$. Therefore, $\bar{S}$ is Gorenstein.

Conversely, for $S$ is Gorenstein, it is easy to see that $\bar{S}$ is a quotient algebra of $\bar{S}$, and then there is an exact functor $i_* : \text{mod} S \to \text{mod} \bar{S}$. Easily, $j^*i_* \simeq \text{Id}$. For any $S$-module $M = (M_i, M_{ij})_{i \in Q_0, \alpha \in Q_1}$, one can check that there is a morphism $f_M : j_*(M) \to j_*(M)$ with $\text{Im}(f_M) = i_*(M)$. Obviously, there is an exact sequence:

$$0 \to j_*(S e_x) \to j_*(M) \xrightarrow{f_M} (D(\bar{S})e_x) \to 0.$$ 

For any projective $S$-module $U$, from (11), we have that $\text{proj.dim}_S(i_*(U)) \leq 1$ since $j_*(U)$ is projective. Similarly, for any injective $S$-module $V$, we have that $\text{inj.dim}_S(i_*(V)) \leq 1$.

Since $S_E$ is finite-dimensional, there is no nonzero paths from $x$ to $\bar{E}(x)$, and no nonzero paths from $E(x)$ to $x$ in $S$, which yields that $\bar{S}e_x \cong i_*(Se_{E(x)})$ and $D(\bar{S})e_x \cong i_*(D(S)e_x)$. Since $S$ is Gorenstein, $\text{proj.dim}_S(D(S)e_x) < \infty$. By applying $i_*$ to the minimal projective resolution of $D(S)e_x$, one can get that $\text{proj.dim}_S(D(\bar{S})e_x) = \text{proj.dim}_S(i_*(D(S)e_x)) < \infty$ since $i_*$ is exact and $\text{proj.dim}_S(i_*(U)) \leq 1$ for any projective $S$-module $U$. Similarly, we can prove that $\text{inj.dim}_S(\bar{S}e_x) < \infty$.

For any indecomposable injective $\bar{S}$-module $D(\bar{S})e_i$, $i \in Q_0$, then $D(\bar{S})e_i = j_*(D(S)e_i)$. Since $S$ is Gorenstein, $\text{proj.dim}_S(D(S)e_i) < \infty$. By applying $j_*$ to the minimal projective resolution of $D(S)e_i$, we get that $\text{proj.dim}_S(j_*(D(S)e_i)) < \infty$. From (11), it is easy to see that $\text{proj.dim}_S(D(\bar{S})e_i) = \text{proj.dim}_S(j_*(D(S)e_i)) < \infty$ if $\text{proj.dim}_S(D(S)e_x) < \infty$. Similarly, for any indecomposable projective $\bar{S}$-module $\bar{S}e_i$, we can get that $\text{inj.dim}_S(\bar{S}e_i) < \infty$. Therefore, $\bar{S}$ is Gorenstein.

\begin{proposition}
Let $S = kQ/I$ be a finite-dimensional algebra, and $E$ an involution on the set of vertices of $Q$. Assume that the algebra $S_E$ constructed from $S$ by gluing the vertices along $E$ is finite-dimensional. Then $S_E$ is Gorenstein if and only if $S$ is Gorenstein.
\end{proposition}

Proof. Similar to the proof of Theorem 6.9 we only need to prove for the case that there is only one pair of vertices $(x, E(x))$ such that $x \neq E(x)$, and the added arrow for $\bar{Q}$ is $\gamma = \alpha(x, E(x)) : x \to E(x)$.

If $S$ is Gorenstein, from Lemma 6.10, we get that $\bar{S}$ is Gorenstein. For any indecomposable injective $S_E$-module $D(S_E)e_i$, $i \neq x$, the dual of (17) implies that there is a short exact sequence in $\text{mod} \bar{S}$:

$$0 \to \Phi(D(S_E)e_i) \to D(\bar{S})e_i \oplus (D(\bar{S})e_{E(x)})^{\oplus s_i} \to (D(\bar{S})e_x)^{\oplus s_i} \to 0,$$

for some $s_i$.

So $\text{inj.dim}_S(\Phi(D(S_E)e_i)) \leq 1$, together with $\bar{S}$ is Gorenstein, we get that $\text{proj.dim}_S(\Phi(D(S_E)e_i)) < \infty$, for any $i$.

From the proof of Proposition 6.7, we get that $\Phi(K^b(\text{proj} S_E)) = K^b(\text{proj} \bar{S}) \cap \Phi(D^b(\text{mod} S_E))$, $\Phi(D(S_E)e_1) \in K^b(\text{proj} \bar{S}) \cap \Phi(D^b(\text{mod} S_E)) = \Phi(K^b(\text{proj} S_E))$, which implies that there is a bounded complex $P^*$ in $K^b(\text{proj} S_E)$, such that $\Phi(P^*) \cong \Phi(D(S_E)e_1)$ in $D^b(\text{mod} \bar{S})$. Since
exists a nonzero (nontrivial) path \( p \) since \((p)\) contradicts Lemma 6.12.

By considering the opposite algebras, one can prove that \( \text{inj. dim}_{S_E} S_E c_i < \infty \) dually for any \( i \). Then \( S_E \) is Gorenstein.

Conversely, if \( S_E \) is Gorenstein, there are exact functors \( \Phi : \text{mod } S_E \to \text{mod } \bar{S} \), and \( j' : \text{mod } \bar{S} \to \text{mod } S \), combining them, we get an exact functor \( j' \Phi : \text{mod } S_E \to \text{mod } S \). From the definitions, we get that \( j' \Phi \) preserves projective modules and injective modules. In particular, for \( i \neq E(x) \), we get that \( j' \Phi(S_E c_i) = S_E e_i \oplus (S_SE_{E(x)})^{\oplus \ell_i} \), for some \( t_i \), and \( j' \Phi(S_E e_x) = S_E x \oplus S_E E_{E(x)} \). Dually, for \( i \neq x \), \( j' \Phi(D(S_E c_i) = D(S) e_i \oplus (D(S) e_x)^{\oplus \ell_i} \) for some \( s_{x} \), and \( j' \Phi(D(S_E) e_x) = D(S) e_{x} \oplus D(S) E_{E(x)} \). Since \( S_E \) is Gorenstein, for any \( i \neq E(x) \), it is easy to see that \( \text{inj. dim}_{S} j' \Phi(S_E c_i) < \infty \), and so \( \text{proj. dim}_{S} S_E c_i < \infty \). For \( i = E(x) \), from \( j' \Phi(S_E e_x) = S_E e_{x} \oplus S_E E_{E(x)} \), we can also get that \( \text{inj. dim}_{S} S_E E_{E(x)} < \infty \). Dually, one can get that
\[
\text{proj. dim}_{S} D(S) e_x < \infty
\]
for any \( i \in Q_0 \). Then \( S \) is Gorenstein. \( \square \)

6.2. Singularity categories of 1-Gorenstein monomial algebras. In this subsection, we describe the stable categories of Gorenstein projective modules (i.e. singularity categories) for 1-Gorenstein monomial algebras.

For a 1-Gorenstein monomial algebra \( A = kQ/I \), we denote by \( C(A) \) the set of equivalence classes (with respect to cyclic permutation) of repetition-free cyclic paths \( c = \alpha_1 \cdots \alpha_n \in Q \) such that there is an integer \( r \) such that any oriented path \( \beta_r \cdots \beta_1 \in \mathbb{F} \), where \( \beta_i \in \{\alpha_1, \ldots, \alpha_n\} \). In this case, \( n \) is called the length of \( c \), \( r \) is called the length of relations of \( c \). Proposition 5.7 implies that for any perfect path \( p \), there is one and only one \( c \in C(A) \) such that \( p \) is along \( c \), i.e. \( p \) is in the full subquiver generated by \( c \). In fact, we have the following lemma.

**Lemma 6.12.** Let \( A = kQ/I \) be a 1-Gorenstein monomial algebra. Then for any arrow \( \alpha \), there is at most one oriented cycle \( c \in C(A) \) such that \( \alpha \) is in \( c \).

**Proof.** Suppose for a contradiction that there are two non-equivalent oriented cycles \( c_1, c_2 \in C(A) \) such that \( \alpha \) is in both of them. It is easy to see that there is an arrow \( \beta \) such that \( c_1 = \alpha_n \cdots \alpha_1 \beta \) and \( c_2 = \beta_m \cdots \beta_1 \beta \) such that \( \alpha_1 \neq \beta_1 \). By definition, there is some paths \( p_1 \) along \( c_1 \), \( p_2 \) along \( c_2 \) such that \( p_1 \alpha_1 \beta \in \mathbb{F} \) and \( p_2 \beta_1 \beta \in \mathbb{F} \). From Theorem 5.4, we get that \( (p_1 \alpha_1 \beta) \) and \( (p_2 \beta_1 \beta) \) are perfect pairs, which gives a contradiction. \( \square \)

**Lemma 6.13.** Let \( A = kQ/I \) be a 1-Gorenstein monomial algebra. For any two perfect paths \( p_1, q_1 \), if \( p_1, q_1 \) are along two non-equivalent oriented cycle \( c_1, c_2 \in C(A) \), then \( \text{Hom}_A(Ap_1, Aq_1) = 0 \).

**Proof.** Let \( (p_2, p_1) \) and \( (q_2, q_1) \) be the perfect pairs. For any morphism \( f : Ap_1 \to Aq_1 \), since the top of every indecomposable Gorenstein projective module is simple, if \( f \) maps \( \text{top}(Ap_1) \) into \( \text{top}(Aq_1) \), then \( f \) is surjective. In particular, \( t(p_1) = t(q_1) \). By noting that \( Ap \) has a basis given by all nonzero paths \( q \) such that \( q = q' p \) for some path \( q' \), it is easy to see that \( p_2 q_1 \in I \) since \( p_2 p_1 \in I \) and \( f \) is surjective. Then there exists a nonzero path \( q_1' \) such that \( q_1 = p_1 q_1' \) since \( (p_2, p_1) \) is a perfect pair. Then there are two non-equivalent cycles \( c_1, c_2 \) containing \( p_1 \), a contradiction to Lemma 5.12.

If \( \text{Im}(f) \subseteq \text{rad}(Ap_2) \), then there exists a nonzero path \( p' \) from \( t(q_1) \) to \( t(p_1) \), and \( p' q_1 \) is nonzero. Since \( p_2 p_1 \in I \) and \( f \) is a morphism, we get that \( p_2 p' q_1 \in I \), and then there exists a path \( q_2' \) such that \( q_2 q_2' = p_2 p' \) since \( (q_2, q_1) \) is a perfect pair. Since \( p' q_1 \) is nonzero, then there exists a nonzero (nontrivial) path \( p'' \) such that \( q_2 = p'' p' \), \( p_2 = d_2 p'' \). So the nontrivial path \( p'' \) is along \( c_1 \) and \( c_2 \), giving a contradiction to Lemma 6.12. \( \square \)
Lemma 6.14. Keep the notations as above. Let $A = kQ/I$ be a 1-Gorenstein monomial algebra. Then

$$D_{sg}(A) \simeq \coprod_{c \in C(A)} D_{sg}(S_c).$$

Proof. Denote by $\iota = \iota_c : S_c \to A$ the natural embedding. Then $A$ is left (and also right) $S_c$-module. So we get an adjoint pair $(\iota_*, \iota^*)$, where $(\iota_* = A \otimes_{S_c} - : \text{mod } S_c \to \text{mod } A$, and $\iota^* : \text{mod } A \to \text{mod } S_c$ is the restriction functor. First, we check that $A$ is projective as left (and also right) $S_c$-module. Let $C(A) = \{c_1, \ldots, c_n\}$. Let $I_c$ be the restriction of $I$ to the subquiver $c$ for any $c \in C(A)$. Then $I_c$ is a two sided ideal of $kc$ and $S_c = kc/I_c$ by viewing $c$ as a quiver. For any arrow $\alpha$, Lemma 6.12 shows that there is at most one $c \in C(A)$ such that $\alpha$ is in $c$, together with Theorem 5.4, we get that $I = \langle I_c : c \in C(A) \rangle$. For any nonzero paths $p_1 = \alpha_\iota \cdots \alpha_1$, and $p_2 = \beta_\iota \cdots \beta_1$ with $t(\beta_j) = s(\alpha_i)$, where $\alpha_i, \beta_j \in Q_0$ for any $i, j$, if $p_1$ is in the subquiver $c$, and $\beta_j \notin c$ for some $c \in C(A)$, then the combination $p_1p_2$ is not zero; similarly, if $p_2$ is in the subquiver $c$ and $\alpha_1 \notin c$ for some $c \in C(A)$, then the combination $p_1p_2$ is not zero. From the structure of projective modules characterized in Chapter III.2, Lemma 2.4, we get that $A$ is projective as left $S_c$-module. By considering the dual quiver, similarly, one can get that $A$ is projective as right $S_c$-module. So $\iota$ and $\iota^*$ are exact functors and preserve projective modules. So they induce exact functors between $D_{sg}(S_c)$ and $D_{sg}(A)$, which are denoted by $\overline{\iota}_*$ and $\overline{\iota}^*$ respectively.

One can check that the adjunction $\text{Id} \to \iota^* \iota_*$ induces that $\iota^* \iota_*(M) \cong M \oplus P_M$ for some projective $S_c$-module $P_M$. So $\overline{\iota}^* \overline{\iota}_* \cong \text{Id} D_{sg}(S_c)$ and then $\overline{\iota}_*$ is fully faithful. For any perfect path $p$ along $c$, by definition, we get that $\iota(S_c p) = Ap$. Lemma 6.13 implies that

$$\text{Hom}_{D_{sg}(\text{mod } A)}((\overline{\iota}_{c})_*(D_{sg}(\text{mod } S_c)), (\overline{\iota}_c)^*(D_{sg}(\text{mod } S_c))) = 0$$

for any $c \neq c' \in C(A)$. So there is a fully faithful functor

$$F = \coprod_{c \in C(A)} (\overline{\iota}_c)_* : \coprod_{c \in C(A)} D_{sg}(S_c) \to D_{sg}(A).$$

Since $A$ is 1-Gorenstein, by Theorem 5.4, we get that $S_c$ is also 1-Gorenstein. Then Theorem 4.3 and Theorem 4.4 yield that $F$ is dense.

Lemma 6.15. Keep the notations as above. Let $A = kQ/I$ be a 1-Gorenstein monomial algebra. Then for any $c \in C(A)$,

$$D_{sg}(S_c) \simeq D^b(\mathcal{A}_{r-1})/[\tau^n],$$

where $n = l(c)$, $r$ is the length of relations for $c$, $D^b(\mathcal{A}_{r-1})$ is the derived category of Dynkin type $\mathcal{A}_{r-1}$ with $\tau$ the Auslander-Reiten functor, and $D^b(\mathcal{A}_{r-1})/[\tau^n]$ denotes the triangulated orbit category in the sense of [37].

Proof. Let $Q(n)$ be a basic $n$-cycle, that is an oriented cycle of $n$ vertices, $S = kQ(n)/J^r$, where $J$ is the ideal of $kQ(n)$ generated by arrows of $Q(n)$. Then $S$ is a self-injective Nakayama algebra with $\text{mod } S \simeq D^b(\mathcal{A}_{r-1})/[\tau^n]$.

For $c$, there is a series of quivers $Q^0 = Q(n), Q^1, \ldots, Q^m$, and involutions $E_0, E_1, \ldots, E_m$ on the sets of vertices of $Q^0, Q^1, \ldots, Q^m$ respectively, such that $Q^{i+1} = Q^i(E_i)$ for $1 \leq i < m$, and $Q^m(E_m) = c$. Denote by $S_i = Q^i(E_i)/I(E_i)$ for $0 \leq i \leq m$. It is easy to see that $I(E_i)$ is...
generated by paths of length \(r\) for each \(i\), so \(S_m = S_c\). Then Theorem 6.9 implies that \(S, S_1, \ldots, S_m = S_c\) are singularity equivalent. So
\[
D_{sg}(S_c) \simeq D_{sg}(S) \cong \text{mod} S \simeq D^b(A_{r-1})/\left[\tau^n\right].
\]
\[\square\]

By combining Lemma 6.14 and Lemma 6.15, we get the following result immediately.

**Theorem 6.16.** Let \(A\) be a 1-Gorenstein monomial algebra, and \(C(A) = \{c_1, \ldots, c_n\}\). Then
\[
\text{Gproj} A \simeq D_{sg}(A) \cong \bigoplus_{c \in C(A)} D^b(A_{r_c-1})/\left[\tau^{n_c}\right],
\]
where \(n_c = l(c)\), \(r_c\) is the length of relations for \(c\), \(D^b(A_{r_c-1})\) is the derived category of Dynkin type \(A_{r_c-1}\) with \(\tau\) the Auslander-Reiten functor, and \(D^b(A_{r_c-1})/\left[\tau^{n_c}\right]\) denotes the triangulated orbit category in the sense of [37].

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