Wormholes with a space- and time-dependent equation of state

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This paper discusses various wormhole solutions in a spacetime with an equation of state that is both space and time dependent. The solutions obtained are exact and generalize earlier results on wormholes supported by phantom energy.

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I. INTRODUCTION

Traversable wormholes, whose possible existence was first conjectured by Morris and Thorne in 1988 [1], are actually shortcuts that could in principle be used for traveling to remote parts of our Universe or to different universes altogether. The meticulous analysis in [1] has shown that such wormholes can only be held open by the use of exotic matter. Such matter violates the weak energy condition (WEC), which requires the stress-energy tensor $T_{\alpha\beta}$ to obey $T_{\alpha\beta}u^\alpha u^\beta \geq 0$ for all time-like vectors and, by continuity, all null vectors. For example, given the radial outgoing null vector $(1,1,0,0)$, we obtain

$$T_{\alpha\beta}u^\alpha u^\beta = \rho + p \geq 0.$$  

(Recall that $T_{ii} = \rho$ and $T_{rr} = p$ in the orthonormal frame of reference.) So if the WEC is violated, we have $\rho + p < 0$. Interest in traversable wormholes has increased in recent years due to an unexpected connection, the discovery that our Universe is undergoing an accelerated expansion [2, 3]. This acceleration, caused by a negative pressure dark energy, implies that $\ddot{a} > 0$ in the Friedmann equation $\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)$. (Our units are taken to be those in which $G = c = 1$.) The equation of state is $p = -K\rho$, $K > 1/3$, and $\rho > 0$. While the condition $K > 1/3$ is required for an accelerated expansion, larger values for $K$ are also of interest. For example, $K = 1$ corresponds to a cosmological constant.

Of particular importance for us is the case $K > 1$, referred to as phantom energy. For this case we have $\rho + p < 0$, in violation of the weak energy condition. As noted realier, this condition is the primary prerequisite for the existence of traversable wormholes. (Strictly speaking, the notion of dark or phantom energy applies only to a homogeneous distribution of matter in the Universe, while wormhole spacetimes are necessarily inhomogeneous. However, the extension to spherically symmetric inhomogeneous spacetimes has been carried out [4].)

In a recent paper, F. Rahaman et al. [5] discussed wormhole solutions that depend on a variable equation of state, i. e., $\frac{\ddot{a}}{a} = -m(r)$, where $m(r) > 1$ for all $r$. The variable $r$ refers to the radial coordinate in the line element

$$ds^2 = -e^{2f(r)}dt^2 + \frac{1}{1 - b(r)/r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2); \quad (1)$$

in other words, $m = m(r)$ is independent of direction. It is shown that given a specific shape function $b = b(r)$, it is possible to determine $m = m(r)$ and vice versa. It is also assumed that $f'(r) \equiv 0$, referred to as the “zero tidal-force solution” in Ref. [3]. An earlier study [6] assumed that the equation of state is a function of time. In this paper we will assume that $m = m(r,t)$ is a continuous function of both $r$ and $t$. As in Ref. [3], however, we retain the assumption that the function values are independent of direction. For reasons that will become apparent later, we also assume that the change in $t$ is very gradual.

The main goal in this paper is to show that the time-dependent metric describes a slowly evolving wormhole structure without assigning specific functions to $b$ and $m$. Moreover, the function $f$ in line element (1) need not be a constant.

Evolving wormhole geometries are also discussed in Refs. [7] and [8].

II. THE PROBLEM

Normally, one would begin with the general line element

$$ds^2 = -e^{2\gamma(r)}dt^2 + e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2)$$

In view of line element (1),

$$e^{2\alpha(r)} = \frac{1}{1 - b(r)/r^2}.$$  

As already noted, $b = b(r)$ is the shape function; $b(r_0) = r_0$, where $r = r_0$ is the radius of the throat. As a result,

$$\lim_{r \to r_0^+} \alpha(r) = +\infty.$$  

Recall that $\gamma(r)$ is referred to as the redshift function. This function must be finite everywhere to avoid an event horizon.
Returning to the function \( m = m(r, t) \), since, for any fixed \( t \), the function is invariant under rotation, we need a time-dependent metric with the same property \[1\]:

\[
ds^2 = -e^{2\gamma(t,r)} dt^2 + e^{2\alpha(r,t)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

(3)

This metric describes an evolving wormhole.

Observe that the shape function is now given by

\[
b(r, t) = r(1 - e^{-2\alpha(r,t)}).
\]

(4)

To obtain a traversable wormhole, the shape function must obey the usual flare-out conditions at the throat, modified to accommodate the time dependence:

\[
b(r_0, t) = r_0 \quad \text{and} \quad \frac{\partial}{\partial r} b(r_0, t) < 1 \quad \text{for all } t.
\]

Another requirement is asymptotic flatness: \( b(r, t)/r \to 0 \) as \( r \to \infty \).

The components of the Einstein tensor in the orthonormal frame are available from Ref. \[2\]:

\[
G_{tt} = 2 \frac{r}{e^{2\alpha(r,t)}} \frac{\partial}{\partial r} \alpha(r, t) + \frac{1}{r^2} (1 - e^{-2\alpha(r,t)}),
\]

(5)

\[
G_{rr} = 2 \frac{r}{e^{2\alpha(r,t)}} \frac{\partial}{\partial r} \gamma(r, t) - \frac{1}{r^2} (1 - e^{-2\alpha(r,t)}),
\]

(6)

\[
G_{\theta\theta} = G_{\phi\phi} = -e^{2\gamma(r,t)} \left[ \frac{\partial^2}{\partial t^2} \alpha(r, t) + \left( \frac{\partial}{\partial t} \alpha(r, t) \right)^2 \right] - e^{-2\alpha(r,t)} \left[ -\frac{\partial^2}{\partial r^2} \gamma(r, t) + \frac{\partial}{\partial r} \gamma(r, t) \left( \frac{\partial}{\partial r} \alpha(r, t) - \frac{\partial}{\partial t} \alpha(r, t) \right)^2 \right] - \frac{1}{r^2} e^{-2\alpha(r,t)} \left( -\frac{\partial}{\partial r} \gamma(r, t) + \frac{\partial}{\partial t} \alpha(r, t) \right). \]

(7)

Recall that from the Einstein field equations in the orthonormal frame, \( G_{\alpha\beta} = 8\pi T_{\alpha\beta} \), the components of the Einstein tensor are proportional to the components of the stress-energy tensor. In particular, \( T_{tt} = T_{rr} = \frac{1}{8\pi} G_{tt} \), with \( \pm f \) is interpreted as the energy flux in the outward radial direction \[10\]. The WEC now becomes \( \rho + p \pm 2f \geq 0 \).

So if the WEC is violated, then

\[
\frac{1}{8\pi} \left[ \frac{2}{r} e^{-2\alpha(r,t)} \left( \frac{\partial}{\partial r} \alpha(r, t) + \frac{\partial}{\partial t} \alpha(r, t) \right) \right. \\
\left. \pm \frac{4}{r} e^{-\gamma(r,t)} e^{-\alpha(r,t)} \frac{\partial}{\partial t} \alpha(r, t) \right] < 0. \]

(9)

It will be seen in the next section that \( \alpha(r, t) \) depends directly on \( m(r, t) \). So if \( m(r, t) \) changes slowly enough with respect to time, then the last term on the left-hand side of inequality \[9\] becomes negligible, ensuring that the WEC will always be violated.

From the Einstein field equations \( G_{\alpha\beta} = 8\pi T_{\alpha\beta} \) and the equation of state \( p = -m(r, t) \rho \), we have \( G_{tt} = 8\pi \rho \) and \( G_{rr} = 8\pi [-m(r, t) \rho] \). Using Eqs. \[5\] and \[6\], we obtain the following system of equations:

\[
G_{tt} = 8\pi T_{tt} = 2 \frac{r}{e^{2\alpha(r,t)}} \frac{\partial}{\partial r} \alpha(r, t) \\
+ \frac{1}{r^2} \left( 1 - e^{-2\alpha(r,t)} \right),
\]

\[
G_{rr} = 8\pi T_{rr} = 8\pi [-m(r, t) \rho] \\
= 2 \frac{r}{e^{2\alpha(r,t)}} \frac{\partial}{\partial r} \gamma(r, t) - \frac{1}{r^2} \left( 1 - e^{-2\alpha(r,t)} \right).
\]

(10)

After substituting and rearranging terms, we have

\[
m(r, t) \frac{\partial}{\partial r} \alpha(r, t) \\
= -\frac{\partial}{\partial r} \gamma(r, t) - \frac{1}{2r} \left( e^{2\alpha(r,t)} - 1 \right) [m(r, t) - 1]. \]

(10)

### III. THE REDSHIFT FUNCTION

In this section we solve Eq. \[10\] by letting the redshift function take on a specific form. One obvious choice is \( \gamma(r, t) \equiv \text{constant} \), so that \( \frac{\partial}{\partial r} \gamma(r, t) \equiv 0 \); the other is \( \frac{\partial}{\partial r} \gamma(r, t) = \frac{m(r,t)-1}{r} \), allowing the solution of Eq. \[10\] by separation of variables.

If \( \frac{\partial}{\partial r} \gamma(r, t) \equiv 0 \), then Eq. \[10\] becomes

\[
\frac{\partial}{\partial r} \alpha(r, t) = -\frac{1}{2r} \left( e^{2\alpha(r,t)} - 1 \right) \left( 1 - \frac{1}{m(r, t)} \right).
\]

Rewritten as

\[
\frac{2}{e^{2\alpha(r,t)} - 1} \frac{\partial}{\partial r} \alpha(r, t) = -\frac{1}{r} \left( 1 - \frac{1}{m(r, t)} \right),
\]

one recognizes the form \( \int \frac{dr}{e^{\alpha(r,t)} - 1} = \ln (e^{\alpha} - 1) + u \). The result is

\[
\ln \left( e^{2\alpha(r,t)} - 1 \right) - 2\alpha(r, t) = -\ln r + \int_e^{r} \frac{dr'}{r'm(r', t)}.
\]

Next, we solve for \( e^{2\alpha(r,t)} \):

\[
e^{2\alpha(r,t)} = \left[ 1 - e^{\int_e^{r} \frac{dr'}{r'm(r', t)}} \right]^{-1}.
\]

(11)

Evidently,

\[
b(r, t) = e^{\int_e^{r} \frac{dr'}{r'm(r', t)}}.
\]

(12)
At \( r = r_0 \), we have
\[
e^{-r_0 \frac{dr}{r_0}} = 1, \tag{13}
\]
which implies that \( b(r_0, t) = r_0 \) for all \( t \). Also,
\[
\frac{\partial}{\partial r} b(r_0, t) = e^{r_0 \frac{dr_0}{r_0}} \frac{1}{rm(r_0, t)} \bigg|_{r=r_0} = r_0 \frac{1}{r_0 m(r_0, t)} = \frac{1}{m(r_0, t)} < 1.
\]
So the flare-out conditions are met without any additional assumptions on either \( b \) or \( m \). Since \( m = m(r, t) \) is continuous, the constant \( c \) can be uniquely determined from Eq. (13):
\[
\int_{r_0}^{r_0} \frac{dr}{rm(r, t)} = \ln r_0
\]
for any fixed \( t \). (So \( c \) is actually a function of \( t \).)
The line element is now seen to be
\[
ds^2 = -dt^2 + \left[ 1 - e^{\frac{2r}{m(r, t)}} \right]^{-1} \, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \tag{14}
\]
As an illustration, in the special case \( m(r, t) \equiv K \), we obtain \( c = r_0^{1-K} \). The result is
\[
e^{2\alpha(r)} = \frac{1}{1 - \left( \frac{c}{r} \right)^{1-1/K}},
\]
which is Lobo’s solution [11].

Remark: Returning to inequality (9), since \( \alpha(r, t) \) depends on \( m(r, t) \), \( (\partial/\partial t) \alpha(r, t) \) is small only if \( (\partial/\partial t) m(r, t) \) is small, explaining our earlier requirement that \( m(r, t) \) change only gradually with respect to time.

For the other choice of \( \gamma(r, t) \),
\[
\frac{\partial}{\partial r} \gamma(r, t) = \frac{m(r, t) - 1}{2r},
\]
Eq. (10) becomes
\[
m(r, t) \frac{\partial}{\partial r} \alpha(r, t)
= \frac{m(r, t) - 1}{2r} - \frac{1}{2r} \left( e^{2\alpha(r,t)} - 1 \right) \left( m(r, t) - 1 \right)
= \frac{1}{2r} \left( m(r, t) - 1 \right) \left( -1 - e^{2\alpha(r,t)} + 1 \right),
\]
or
\[
2 \frac{\partial}{\partial r} \alpha(r, t) = e^{2\alpha(r,t)} = \frac{1}{r} \left( 1 - \frac{1}{m(r, t)} \right).
\]
It follows that
\[
e^{-2\alpha(r,t)} = \int_{c_1}^{r} \frac{1}{r'} \left( 1 - \frac{1}{m(r', t)} \right) \, dr'.
\]
So by Eq. (11),
\[
b(r, t) = r \left( 1 - e^{-2\alpha(r,t)} \right)
= r \left[ 1 - \int_{c_1}^{r} \frac{1}{r'} \left( 1 - \frac{1}{m(r', t)} \right) \, dr' \right].
\]
It now becomes apparent that \( c = r_0 \) for all \( t \), since we must have \( b(r_0, t) = r_0 \). Thus
\[
b(r, t) = r \left[ 1 - \int_{r_0}^{r} \frac{1}{r'} \left( 1 - \frac{1}{m(r', t)} \right) \, dr' \right], \tag{15}
\]
while the line element becomes
\[
ds^2 = -e^{2\gamma(r,t)} dt^2 + \int_{r_0}^{r} \frac{dr}{r} \left( 1 - \frac{1}{m(r,t)} \right) \, dr' + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2). \tag{16}
\]
Once again,
\[
\frac{\partial}{\partial r} b(r_0, t) = \frac{1}{m(r_0, t)} < 1 \quad \text{for all } t.
\]
Returning to the redshift function, we have up to this point
\[
\gamma(r, t) = \int_{c_1}^{r} \frac{m(r', t) - 1}{2r'} \, dr'. \tag{17}
\]
The constant \( c_1 \) will be obtained from the junction conditions, described below.

Based on previous studies involving static wormholes supported by phantom energy [11] [12] [13], our spacetime is not likely to be asymptotically flat. The wormhole material will therefore have to be cut off at some \( r = a \) and joined to an external Schwarzschild spacetime. Moreover, in Ref. [13], \( b = b(r) \) actually attains a maximum value at some \( r = a \), which then becomes a natural place at which to perform the junction. Accordingly, we will assume that for any fixed \( t \), \( b(r, t) \) has a maximum value at some \( r = a \). (While other values could be chosen, the choice suggested here yields a particularly elegant solution.) So we proceed by determining the critical value:
\[
\frac{\partial}{\partial r} b(r, t) = 1 - \int_{r_0}^{r} \frac{1}{r'} \left( 1 - \frac{1}{m(r', t)} \right) \, dr' + r \left[ \frac{1}{r} - \frac{1}{m(r, t)} \right] = 0.
\]
By assumption, equality holds for some \( r = a \). As a consequence,
\[
\int_{r_0}^{a} \frac{1}{r} \left( 1 - \frac{1}{m(r, t)} \right) \, dr = \frac{1}{m(a, t)}. \tag{18}
\]
As noted in Ref. [11], to match our interior solution to the exterior Schwarzschild solution

\[ ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

at \( r = a \) (t fixed) requires continuity of the metric. Because of the assumption of spherical symmetry, the components \( g_{\theta\theta} \) and \( \theta\phi \) are already continuous [11]. As a result, the continuity requirement has to be imposed only on the remaining components. For every \( t \)

\[ g_{ii(\text{int})}(a) = g_{i(\text{ext})}(a) \quad \text{and} \quad g_{ij(\text{int})}(a) = g_{ij(\text{ext})}(a) \]

for the interior and exterior components, respectively. These conditions now imply that (for every \( t \))

\[ \gamma_{\text{int}}(a) = \gamma_{\text{ext}}(a) \quad \text{and} \quad b_{\text{int}}(a) = b_{\text{ext}}(a). \]

Hence

\[ e^{2\alpha(a,t)} = \frac{1}{1 - \frac{(b(a,t)}{a}} = \frac{1}{1 - \frac{2M}{a}}. \]

We now see that the total mass of the wormhole for \( r \leq a \) is given by \( M = \frac{a}{2} b(a,t) \) for any fixed \( t \). So by Eqs. (15) and (18)

\[ M = \frac{1}{2} a \left(1 - \frac{1}{m(a,t)}\right). \]

By Eq. (17),

\[ e^{2\gamma(a,t)} = e^{\int_{r_0}^{r} \frac{m(r,t) - 1}{r} \, dr} = 1 - \frac{2M}{a} \]

\[ = 1 - \frac{2}{a} \left(1 - \frac{1}{m(a,t)}\right) \]

or

\[ e^{2\gamma(a,t)} = \frac{1}{m(a,t)}. \quad (19) \]

Eq. (19) can now be used to determine \( c_1 = c_1(t) \), thereby completing the line element, Eq. (18).

As an illustration, if \( m(r,t) \equiv K \), then \( a = r_0 e^{1/(K-1)} \) [from Eq. (15)] and \( c_1 = r_0 (Ke)^{1/(K-1)} \).

IV. ADDITIONAL SOLUTIONS

As noted in Ref. [13], to obtain additional exact solutions, \( \gamma \) must depend directly on \( \alpha \). The corresponding condition for the time-dependent case is

\[ \frac{\partial}{\partial t} \gamma(r,t) = F[\alpha(r,t)] \frac{\partial}{\partial r} \alpha(r,t) \]

for some elementary function \( F \). Since these cases are just extensions of the cases discussed in Ref. [13], we will merely summarize the results.

If

\[ F[\alpha(r,t)] = - \frac{m(r,t)}{e^{2\alpha(r,t)}}, \]

then Eq. (10) becomes

\[ m(r,t) \frac{\partial}{\partial r} \alpha(r,t) \]

\[ = m(r,t) \frac{\partial}{\partial r} \alpha(r,t) \]

\[ = \frac{1}{2r} \left( e^{2\alpha(r,t)} - 1 \right) \left(m(r,t) - 1\right]. \]

The solution is

\[ e^{2\alpha(r)} = \int_{r_0}^{r} \frac{1}{1 - \frac{1}{m(r',t)}} \, dr'. \]

(If \( m(r,t) \equiv K \), the solution reduces to

\[ e^{2\alpha(r)} = \frac{1}{\ln \left(\frac{r}{r_0}\right)} \]

discussed in Ref. [13].) Also,

\[ b(r,t) = r \left[1 - \int_{r_0}^{r} \frac{1}{1 - \frac{1}{m(r',t)}} \, dr'\right]. \]

As in the previous cases,

\[ b(r_0, t) = r_0 \quad \text{and} \quad \frac{\partial}{\partial r} b(r_0, t) = \frac{1}{m(r_0, t)} < 1 \]

for all \( t \).

The determination of the redshift function and the junction to an exterior Schwarzschild solution follows along the lines discussed in Sec. III.

Another solution comes from

\[ F[\alpha(r,t)] = - \frac{2m(r,t)}{e^{2\alpha(r,t)} + 1}. \]

Substituting in Eq. (10) and simplifying, we get

\[ \frac{\partial}{\partial r} \alpha(r,t) \]

\[ = \frac{2}{2r} \left( e^{2\alpha(r,t)} - 1 \right) \left(m(r,t) - 1\right] \]

The solution is

\[ e^{2\alpha(r)} = \left[e^{\int_{r_0}^{r} \left(1 - \frac{1}{m(r',t)}\right) \, dr'} - 1\right]^{-1}. \]

(If \( m(r,t) \equiv K \), this reduces to

\[ e^{2\alpha(r)} = \left(\frac{1}{r_0}\right)^{1-1/K} - 1 \]

also discussed in Ref. [13].) Here

\[ b(r,t) = r \left(1 - e^{-2\alpha(r,t)}\right) \]

\[ = r \left(2 - e^{\int_{r_0}^{r} \left(1 - \frac{1}{m(r',t)}\right) \, dr'}\right). \]

Once again,

\[ b(r_0, t) = r_0 \quad \text{and} \quad \frac{\partial}{\partial r} b(r_0, t) = \frac{1}{m(r_0, t)} < 1 \]

for all \( t \).
V. SUMMARY

This paper discusses several exact solutions of the Einstein field equations describing traversable wormholes supported by a generalized form of phantom energy: the equation of state is given by $p/\rho = -m(r,t)$, $m(r,t) > 1$. The function $m = m(r,t)$ is a continuous function of $r$ and $t$, invariant under rotation, slowly evolving in time.

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