On spectral and orbital stability for the Klein–Gordon equation coupled to an anharmonic oscillator

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Abstract

We obtain explicit characterization of spectral and orbital stability of solitary wave solutions to the \( U(1) \)-invariant Klein–Gordon equation in one spatial dimension coupled to an anharmonic oscillator. We also give the complete analysis of the spectrum of the linearization at a solitary wave.

1 Introduction

In the present article we study the spectral stability of the \( U(1) \)-invariant Klein–Gordon equation on a line with a concentrated nonlinearity:

\[
\ddot{\psi}(x,t) = \partial_x^2 \psi(x,t) - m^2 \psi(x,t) + \delta(x) a(|\psi(0,t)|^2) \psi(0,t), \quad \psi(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}, \quad (1.1)
\]

where \( m > 0 \). Above, \( a(\cdot) \) is a real-valued differentiable function, so that the model is \( U(1) \)-invariant. The equation is understood in the sense of distributions. Physically, equation (1.1) describes the Klein–Gordon field coupled to a nonlinear oscillator located at \( x = 0 \), with \( a(|\psi|^2) \psi \) being the oscillator force. Equation (1.1) admits finite energy solutions of the form \( \phi(x, \omega) e^{-i \omega t} \), \( \omega \in \mathbb{R} \), called solitary waves.

The solitary waves form a two-dimensional solitary manifold in the Hilbert space of finite energy states of the system.

We will determine conditions for orbital and spectral stability of solitary waves, formulating the results in terms of values of \( a \) and \( a' \) evaluated at \(|\phi(0,\omega)|^2|\); see Theorem 3.6 below. Then, in Theorem 4.5, we give the complete description of the spectrum of the linearization at a solitary wave.

Equation (1.1) was proposed to model electron’s transitions between Bohr’s quantum orbits, which is one of the fundamental problems of Quantum Mechanics. More precisely, (1.1) models the interaction of the electron (represented by the anharmonic oscillator) with the electromagnetic radiation (represented by the Klein–Gordon field). The global attraction of any finite energy solution to the set of all solitary waves in local energy norms, which was established in [KK07], can be interpreted as the relaxation of a perturbed electron to the Bohr orbit, where it no longer loses the energy via the radiation, in the complete agreement with Bohr’s postulate on quantum jumps.

We pursue further properties of the model (1.1) since it is a convenient playground for establishing asymptotic stability results, similarly to [BKKS08, KKS12], allowing one to explicitly check all the spectral properties of the linearized equation. In the present article, we will obtain the spectral and
orbital stability results. Moreover, we obtain the complete description of the spectrum of the linearized equation: these results will be needed for the subsequent proof of asymptotic stability. We point out that many pieces of the spectral analysis which we develop in the present article can be carried over essentially verbatim to models similar to (1.1), such as the models where the concentrated nonlinearity is substituted by its regularized versions, such as the self-interaction based on the mean field \[\text{[KK09]}\].

We mention that the local and global well-posedness of (1.1) has already been proved in \[\text{[KK07]}\]. In \[\text{[Kop09, Kop10]}\], asymptotic stability of solitary waves was obtained for discrete Schrödinger and Klein-Gordon equations. Let us also mention that related results on local well-posedness, orbital stability, and linear instability of solitary waves in the nonlinear Klein–Gordon equation in the external \[\delta\]-function potential were obtained in \[\text{[CGOR19]}\].

The paper is organized as follows. The model and its solitary wave solutions are described in Section 2. The linearization at a solitary wave is carried out in Section 3, where the standard properties of the linearized operator are obtained. The detailed structure of the spectrum of the linearized operator is derived in Section 4 (see Theorem 4.5).

## 2 The model

We define \(\Psi(x,t) = \begin{bmatrix} \psi(x,t) \\ \pi(x,t) \end{bmatrix} \in \mathbb{C}^2\) and rewrite (1.1) in the vector form:

\[
\dot{\Psi}(t) = \begin{bmatrix} 0 & 1 \\ m^2 & 0 \end{bmatrix} \Psi(t) + \delta(x) \begin{bmatrix} 0 \\ a(|\psi|^2)\psi \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi \\ \pi \end{bmatrix} \in \mathbb{C}^2, \quad \Psi|_{t=0} = \Psi_0 := \begin{bmatrix} \psi_0 \\ \pi_0 \end{bmatrix}. \tag{2.1}
\]

The oscillator force \(a(|\psi|^2)\psi\) admits a real-valued potential,

\[
a(|\psi|^2)\psi = -\nabla \text{Re} \psi \cdot \text{Im} U(\psi), \quad \psi \in \mathbb{R}^2, \quad U \in C^2(\mathbb{R}^2), \tag{2.2}
\]

where \(U(\psi) = u(|\psi|^2)\), with \(u(\tau) = \frac{1}{2} \int_0^\tau a(s) \, ds\). Then (2.1) is formally a Hamiltonian system with the Hamiltonian functional

\[
\mathcal{H}(\Psi) = \frac{1}{2} \int_{\mathbb{R}} \left( |\pi|^2 + |\partial_x \psi|^2 + m^2 |\psi|^2 \right) \, dx + U(\psi(0)), \quad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}), \tag{2.3}
\]

which is conserved for sufficiently regular finite energy solutions. Equation (1.1) is \(U(1)\)-invariant: if \(\psi(x,t)\) is a solution, then so is \(e^{i\theta}\psi(x,t)\) for any \(\theta \in \mathbb{R}\). The Nöther theorem implies the charge conservation: the value of the functional

\[
Q(\Psi) = -\frac{1}{2} \text{Im} \int_{\mathbb{R}} \left( \overline{\psi(x)} \pi(x) - \overline{\pi(x)} \psi(x) \right) \, dx, \quad \Psi = \begin{bmatrix} \psi(x) \\ \pi(x) \end{bmatrix}, \tag{2.4}
\]

is conserved for solutions to (1.1).

The local and global existence result for the Cauchy problem (2.1) proved in \[\text{[KK07, Theorem 2.1]}\]:

**Theorem 2.1.** Assume that the potential is represented by \(U(\psi) = u(|\psi|^2)\) with \(u \in C^2(\mathbb{R})\) and that \(U\) satisfies the inequality

\[
U(\psi) \geq A - B|\psi|^2 \quad \text{for} \quad \psi \in \mathbb{C}, \quad \text{where} \quad A \in \mathbb{R}, \quad 0 \leq B < m.
\]

Then:

1. For every \(\Psi_0 \in \mathcal{E} := H^1(\mathbb{R}) \oplus L^2(\mathbb{R})\) the Cauchy problem (2.1) has a unique solution \(\Psi \in C_0(\mathbb{R}, \mathcal{E})\).

2. The energy of \(\Psi\) is conserved: for all \(t \in \mathbb{R}\), \(\mathcal{H}(\Psi(t)) = \mathcal{H}(\Psi(0))\).
3. There exists \( \Lambda(\Psi_0) > 0 \) such that \( \sup_{t \in \mathbb{R}} \| \Psi(t) \|_\mathcal{C} \leq \Lambda(\Psi_0) < \infty \).

The main subject of this paper is the analysis of the spectral and orbital stability of “quantum stationary states”, or solitary waves [GSS87], which are finite energy solutions of the form

\[
\Psi(x, t) = \Phi(x)e^{-i\omega t}, \quad \omega \in \mathbb{R}, \quad \Phi(x) = \begin{bmatrix} \phi(x) \\ -i\omega \phi(x) \end{bmatrix}, \quad \phi \in H^1(\mathbb{R}). \tag{2.5}
\]

Above, \( H^1(\mathbb{R}) \) is the Sobolev space of complex-valued measurable functions satisfying \( \int_\mathbb{R} (|\partial_x \psi|^2 + |\psi|^2) \, dx < \infty \). By (2.1), the frequency \( \omega \in \mathbb{R} \) and the amplitude \( \phi(x) \) solve the following nonlinear eigenvalue problem:

\[
\omega^2 \phi(x) = -\partial_x^2 \phi(x) + m^2 \phi(x) - \delta(x)a(|\phi(0)|^2)\phi(x), \quad x \in \mathbb{R}. \tag{2.6}
\]

**Definition 2.2.** The solitary manifold \( S \) is the set of all amplitudes \( \Phi(\cdot, \omega) = \begin{bmatrix} \phi(\cdot, \omega) \\ -i\omega \phi(\cdot, \omega) \end{bmatrix} \), where \( \phi(x, \omega)e^{-i\omega t} \) is a nonzero solitary wave solution to (1.1), with all possible \( \omega \in \mathbb{R} \).

**Lemma 2.3.** The set \( S \) is given by

\[
S = \left\{ \Phi(x, \omega)e^{i\theta} = \begin{bmatrix} \phi(x, \omega) \\ -i\omega \phi(x, \omega) \end{bmatrix} e^{i\theta}, \quad \phi(x, \omega) = Ce^{-\kappa|x|}, \quad \theta \in [0, 2\pi] \right\},
\]

where \( |\omega| < m, \kappa = \sqrt{m^2 - \omega^2} > 0 \), and \( C > 0 \) satisfies the relation

\[
a(C^2) = 2\kappa. \tag{2.7}
\]

If \( a'(C^2) \neq 0 \), then \( C \) is locally a \( C^1 \)-function of \( \omega \).

**Proof.** Equation (2.6) implies that \( \partial_x^2 \phi(x, \omega) = (m^2 - \omega^2)\phi(x, \omega), \quad x \neq 0 \), and hence \( \phi(x, \omega) = C \pm e^{-\kappa_\pm|x|} \) for \( \pm x > 0 \), where \( \kappa_\pm \) satisfy \( \kappa_\pm^2 = m^2 - \omega^2 \). Since \( \phi(\cdot, \omega) \in H^1(\mathbb{R}) \), we conclude that \( \kappa_\pm > 0, |\omega| < m \), and that \( \kappa_\pm = \sqrt{m^2 - \omega^2} > 0 \). Since the function \( \phi(x, \omega) \) is continuous at \( x = 0 \), one has \( C_- = C_+ = C \). Thus, the solitary waves are solutions of the form

\[
\phi(x, \omega) = Ce^{-\kappa |x|}, \quad \kappa = \sqrt{m^2 - \omega^2} > 0, \quad C \neq 0. \tag{2.8}
\]

For our convenience, we assume that \( C > 0 \), and write solitary wave solutions in the form

\[
\phi(x, \omega) = e^{i\theta} Ce^{-\kappa |x|}, \quad \kappa = \sqrt{m^2 - \omega^2} > 0, \quad C > 0, \quad \theta \in \mathbb{R} \mod 2\pi. \tag{2.9}
\]

The algebraic equation satisfied by the constant \( C > 0 \) is obtained by collecting coefficients at \( \delta(x) \) in (2.6):

\[
0 = \partial_x \phi(0+, \omega) - \partial_x \phi(0-, \omega) + a(|\phi(0, \omega)|^2)\phi(0, \omega). \tag{2.10}
\]

This implies that \( 0 = -2\kappa C + a(C^2)C \), hence \( 2\kappa = a(C^2) \).

We note that there is the following relation:

\[
-2\omega = \frac{d}{d\omega} \kappa^2 = \frac{d}{d\omega} \frac{a(C^2)^2}{4} = a(C^2)a'(C^2)C \frac{dC}{d\omega}, \tag{2.11}
\]

which shows that \( C \) is locally a \( C^1 \)-function of \( \omega \) as long as \( a'(C^2) \neq 0 \).\( \Box \)
3 Linearization at a solitary wave

Let us linearize the nonlinear Klein–Gordon (2.1) at a solitary wave 
\[ e^{-i\omega t+i\theta} \Phi(x,\omega) = e^{-i\omega t+i\theta} \begin{bmatrix} \phi(x,\omega) \\ -i\omega \phi(x,\omega) \end{bmatrix}, \]
where \( \omega \in (-m, m), \theta \in \mathbb{R} \mod 2\pi, \) and \( \phi(x,\omega) = C(\omega)e^{-\kappa|x|}, \) with \( \kappa = \sqrt{m^2 - \omega^2} > 0 \) and \( C(\omega) > 0 \) (see Lemma 2.3). Substituting 
\[ \Psi(x,t) = e^{-i\omega t+i\theta}(\Phi(x,\omega) + \mathcal{X}(x,t)), \quad \mathcal{X}(x,t) = \begin{bmatrix} \mathcal{X}_1(x,t) \\ \mathcal{X}_2(x,t) \end{bmatrix} \in \mathbb{C}^2, \quad (3.1) \]
into (2.1), we obtain:
\[ -i\omega(\Phi + \mathcal{X}) + \dot{\mathcal{X}} = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{bmatrix} (\Phi + \mathcal{X}) + \delta(x) \begin{bmatrix} 0 \\ a(|C + \mathcal{X}_1|^2)(C + \mathcal{X}_1) \end{bmatrix}. \]
Equality \( \Phi_2 = -i\omega \Phi_1 \) and equation (2.6) lead to 
\[ \dot{\mathcal{X}} = \begin{bmatrix} i\omega \\ \partial_x^2 - m^2 \end{bmatrix} \mathcal{X}(x,t) + \delta(x) \begin{bmatrix} 0 \\ a(|C + \mathcal{X}_1|^2)(C + \mathcal{X}_1) - a(|C|^2)C \end{bmatrix}. \quad (3.2) \]
The first order part of (3.2) is given by 
\[ \dot{\mathcal{X}}(x,t) = \begin{bmatrix} i\omega \\ \partial_x^2 - m^2 \end{bmatrix} \mathcal{X}(x,t) + \delta(x) \begin{bmatrix} 0 \\ a(C^2)\mathcal{X}_1(0,t) + C^2 a'(C^2) 2 \text{Re } \mathcal{X}_1(0,t) \end{bmatrix}. \quad (3.3) \]
In the case \( a'(C^2) \neq 0, \) the operator in the right-hand side is \( \mathbb{R} \)-linear but not \( \mathbb{C} \)-linear; to study its spectrum, it is convenient to rewrite (3.3) in the real form. For \( \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \in \mathbb{C}^2, \) we denote 
\[ \Psi = \begin{bmatrix} \text{Re } \psi_1 \\ \text{Im } \psi_1 \\ \text{Re } \partial_t \psi_1 \\ \text{Im } \partial_t \psi_1 \end{bmatrix} \in \mathbb{R}^4. \] The solitary wave \( \Phi(x,\omega)e^{-i\omega t} \in \mathbb{C}^2 \) is then represented by \( e^{i\omega t}\Phi(x,\omega) \in \mathbb{R}^4, \)
where 
\[ \Phi(x,\omega) = \begin{bmatrix} \phi(x,\omega) \\ 0 \\ 0 \\ -\omega \phi(x,\omega) \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \]
We identify the perturbation \( \mathcal{X} \in \mathbb{C}^2 \) with the real vector \( X = \begin{bmatrix} \text{Re } \mathcal{X}_1 \\ \text{Im } \mathcal{X}_1 \\ \text{Re } \mathcal{X}_2 \\ \text{Im } \mathcal{X}_2 \end{bmatrix} \in \mathbb{R}^4. \) Then (3.3) becomes 
\[ \dot{X}(x,t) = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & \partial_x^2 - m^2 & \partial_x^2 - m^2 & \omega \end{bmatrix} X(x,t) + \delta(x) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (3.4) \]
where (cf. (2.7)) 
\[ \alpha := a(C^2) = 2\kappa > 0, \quad \kappa := C^2 a'(C^2)/a(C^2). \quad (3.5) \]
This gives a system which is \( \mathbb{C} \)-linear.
Remark 3.1. We note that the above definition of $\kappa$ is compatible with the pure power case $a(\tau) = \tau^\kappa$, $\kappa > 0$, $\tau \geq 0$, when $a'(\nabla^2) = \kappa a(\nabla^2)$.  

Let us denote

$$L_\kappa(\omega) = -\partial_x^2 + m^2 - \alpha(1 + 2\kappa)\delta(x) - \omega^2. \quad (3.6)$$

Remark 3.2. The domains of all operators which we consider require a careful definition. For example, $A = -\partial_x^2 + c\delta(x)$, $c \in \mathbb{R}$, is defined as $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with the domain 

$$\mathcal{D}(A) = \{ \psi \in H^2(\mathbb{R}, \mathbb{C}) \cup H^2(\mathbb{R}, \mathbb{C}), \quad \partial_x\psi(0+) - \partial_x\psi(0-) = c\psi(0) \},$$

where it is selfadjoint.

Lemma 3.3 (Spectrum of $L_\kappa$). Let $\omega \in (-m, m)$, $\kappa \in \mathbb{R}$. The operator $L_\kappa(\omega)$ is selfadjoint and satisfies 

$$\sigma_{\text{ess}}(L_\kappa(\omega)) = [\kappa^2, +\infty), \quad \sigma_{\text{p}}(L_\kappa(\omega)) = \emptyset, \quad \kappa \leq -1/2, \quad \Lambda_\kappa(\omega) := -4\kappa^2(\kappa + \kappa^2), \quad \kappa > -1/2,$$

where $\kappa = \sqrt{m^2 - \omega^2}$. The eigenvalue $\Lambda_\kappa(\omega)$ is simple.

Proof. Solving the equation 

$$(-\partial_x^2 - \alpha(1 + 2\kappa)\delta(x) + \kappa^2)\psi = \Lambda\psi, \quad \psi \in L^2(\mathbb{R}),$$

we find that $\Lambda$ has to satisfy $\Lambda < \kappa^2$ and that 

$$\psi(x) = Ce^{-|x|\sqrt{\kappa^2 - \Lambda}}, \quad x \in \mathbb{R}, \quad C \neq 0,$$

while $\Lambda$ is obtained from the jump condition at $x = 0$:

$$\alpha(1 + 2\kappa) = 2\sqrt{\kappa^2 - \Lambda}. \quad (3.7)$$

This shows that there could only be an eigenvalue if $\kappa > -1/2$. Taking into account that $\alpha = 2\kappa$, one arrives at $\kappa^2(1 + 2\kappa)^2 = \kappa^2 - \Lambda$, so $\Lambda = -4\kappa^2(\kappa + \kappa^2)$. In particular, setting $\kappa = 0$, one obtains the point spectrum $\{0\}$ of $L_0$; by (2.6), the corresponding eigenvector is $\phi$. 

In terms of operators (3.6), the system (3.4) reads as

$$\dot{X}(x, t) = A(\omega, \kappa)X(x, t), \quad A(\omega, \kappa) := \begin{bmatrix} 0 & -\omega & 1 & 0 \\
\omega & 0 & 0 & 1 \\
-L_\kappa(\omega) - \omega^2 & 0 & 0 & -\omega \\
0 & -L_0(\omega) - \omega^2 & \omega & 0 \end{bmatrix}. \quad (3.8)$$

Theorem 2.1 generalizes to equation (3.8): for every initial function $X(x, 0) = X_0 \in \mathcal{E}$, the equation admits a unique solution $X(x, t) \in C_b(\mathbb{R}, \mathcal{E})$. Denote

$$\Sigma = \begin{bmatrix} 0 & I_2 \\
-I_2 & 0 \end{bmatrix}, \quad H(\omega, \kappa) = \begin{bmatrix} L_\kappa + \omega^2 & 0 & 0 & \omega \\
0 & L_0 + \omega^2 & -\omega & 0 \\
0 & -\omega & 1 & 0 \\
\omega & 0 & 0 & 1 \end{bmatrix};$$

then the operator $A(\omega, \kappa)$ from (3.8) corresponding to a linearization at the solitary wave is factored into

$$A(\omega, \kappa) = \Sigma H(\omega, \kappa).$$
Denote
\[ \mathbf{G}_1 = \mathbf{G}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_2 = \mathbf{G}_2^{-1} = \begin{bmatrix} I_2 & 0 \\ 0 & \sigma_2 \end{bmatrix} \] (3.9)
and consider the conjugated version of \( \mathbf{H}(\omega, \kappa) \):
\[
\tilde{\mathbf{H}}(\omega, \kappa) = \mathbf{G}_2 \mathbf{G}_1 \mathbf{H}(\omega, \kappa) \mathbf{G}_1^{-1} \mathbf{G}_2^{-1} = \begin{bmatrix} L_\kappa + \omega^2 & \omega & 0 & 0 \\ \omega & 1 & 0 & 0 \\ 0 & 0 & L_0 + \omega^2 & \omega \\ 0 & 0 & \omega & 1 \end{bmatrix} =: \begin{bmatrix} H_\kappa(\omega) & 0 \\ 0 & H_0(\omega) \end{bmatrix}. \] (3.10)

We start with the spectrum of \( H_\kappa(\omega) \) by Lemma 3.4.

**Lemma 3.4.** For any \( \kappa \in \mathbb{R} \), the essential spectrum of \( H_\kappa(\omega) \) is given by

1. \(
\sigma_{ess}(H_\kappa(\omega)) = \begin{cases} 
[m^2, +\infty), & \omega = 0, \\
[c^-(\omega), 1] \cup [c^+(\omega), +\infty), & \omega \in (-m, m) \setminus \{0\},
\end{cases}
\)

where
\[
c^\pm(\omega) = \frac{m^2 + 1 \pm \sqrt{(m^2 - 1)^2 + 4\omega^2}}{2}.
\]

We note that \( 0 < c^-(\omega) \leq \min\{1, m^2\}, c^+(\omega) \geq \max\{1, m^2\} \).

2. \(
\sigma_p(H_\kappa(\omega)) = \begin{cases} 
\emptyset, & \kappa \leq -1/2, \ \omega \in (-m, m) \setminus \{0\}, \\
\{1\}, & \kappa \leq -1/2, \ \omega = 0, \\
\{\lambda^\pm_\kappa(\omega)\}, & \kappa > -1/2, \ \omega \in (-m, m),
\end{cases}
\)

with the eigenvalues \( \lambda^\pm_\kappa(\omega) \) given by
\[
\lambda^\pm_\kappa(\omega) = \left( \Lambda_\kappa + \omega^2 + 1 \pm \sqrt{(\Lambda_\kappa + \omega^2 + 1)^2 - 4\lambda_\kappa^2} \right)/2, \] (3.12)

with \( \Lambda_\kappa(\omega) = -4\omega^2(\kappa + \kappa^2) \) (when \( \kappa > -1/2 \)) from Lemma 3.3. In particular, \( 0 \in \sigma_p(H_\kappa(\omega)) \) if and only if \( \kappa = 0 \), and
\[
\sigma_p(H_0(\omega)) = \{0\} \cup \{\omega^2 + 1\},
\]

with the corresponding eigenvectors
\[
\psi_0(x) = \begin{bmatrix} \phi(x) \\ \omega \phi(x) \end{bmatrix}, \quad \psi_{\omega^2+1}(x) = \begin{bmatrix} \omega \phi(x) \\ \phi(x) \end{bmatrix}.
\]

**Proof.** Let us compute the essential spectrum of \( H_\kappa(\omega) \). Since \( \delta(x) \) is a relatively compact perturbation of the Laplacian in one dimension, by the Weyl theorem, when computing the essential spectrum of \( H_\kappa(\omega) \), we can replace \( L_\kappa \) in (3.11) by \( L_\kappa^{(0)} = -\partial_x^2 + m^2 - \omega^2 \). Then the values of the essential spectrum are those \( \lambda \in \mathbb{C} \) such that
\[
\det \begin{bmatrix} \xi^2 + \omega^2 - \lambda & \omega \\ \omega & 1 - \lambda \end{bmatrix} = (1 - \lambda) \left( \xi^2 + m^2 - \lambda - \frac{\omega^2}{1 - \lambda} \right) = 0 \quad \text{for some } \xi \in \mathbb{R}. \] (3.13)
Thus, either $\lambda = 1$, or, expressing $\xi^2$ in terms of $\lambda$, we arrive at the inequality
\[
-m^2 + \lambda + \frac{\omega^2}{1 - \lambda} = \frac{(\lambda - m^2)(1 - \lambda) + \omega^2}{1 - \lambda} = \frac{\lambda^2 - \lambda(\lambda^2 + 1) + m^2 - \omega^2}{\lambda - 1} \geq 0.
\] (3.14)

There are two roots $\lambda = c^\pm(\omega)$ of the numerator and root $\lambda = 1$ of the denominator; these roots satisfy the inequalities
\[
0 < c^-(\omega) < 1 < c^+(\omega), \quad \omega \in (-m, m) \setminus \{0\}.
\]
(We note that when $\omega = 0$, one has $c^-(0) = 1$ if $m^2 \geq 1$ and $c^+(0) = 1$ if $m^2 \leq 1$.) We conclude that the essential spectrum (the set where (3.13) is satisfied) is given by the intervals
\[
\lambda \in [c^-(\omega), 1] \cup [c^+(\omega), +\infty).
\]

Let us now study the point spectrum of $H_\kappa(\omega)$. First let us consider when $\lambda = 1$ is an eigenvalue of $H_\kappa(\omega)$. One can see from (3.11) that this is only possible when $\omega = 0$ (with eigenfunctions of the form $\left[\begin{array}{c} 0 \\ u_2(x) \end{array}\right]$, $u_2 \in L^2(\mathbb{R})$); in this case, the rest of the point spectrum comes from Lemma 3.3 (($\Lambda_\kappa(0) = -4m^2(\kappa + \kappa^2)$ in the case $\kappa > -1/2$), agreeing with $\Lambda^-_\kappa(0) = 1$ and $\Lambda^+_\kappa(0) = \Lambda_\kappa(0)$ from (3.12).

Now we consider the case $\lambda \neq 1$. The operator
\[
H_\kappa(\omega) - \lambda I_2 = \left[\begin{array}{cc} L_\kappa + \omega^2 - \lambda & \omega \\ \omega & 1 - \lambda \end{array}\right]
\] (3.15)
has zero eigenvalue if and only if so does the Schur complement of $1 - \lambda$, which is given by
\[
L_\kappa(\omega) + \omega^2 - \lambda - \frac{\omega^2}{1 - \lambda} = L_\kappa(\omega) - \lambda - \frac{\lambda \omega^2}{1 - \lambda}.
\]
Thus, $\lambda \neq 1$ is an eigenvalue of $H_\kappa(\omega)$ if and only if
\[
\Lambda = \lambda + \frac{\lambda \omega^2}{1 - \lambda}
\] (3.16)
is an eigenvalue of $L_\kappa(\omega)$; see Figure 1. Now the proof follows from Lemma 3.3, leading to the expressions (3.12). Let us point out that the eigenfunctions corresponding to $\lambda \neq 1$ are given by (cf. (3.7))
\[
u^\pm(x) = \left[\begin{array}{c} 1 - \lambda \pm \frac{\lambda \omega^2}{1 - \lambda} \\ -\omega \end{array}\right] e^{-|x|\sqrt{\omega^2 - \lambda}}.
\] (3.17)

Lemma 3.4 gives the spectrum of the selfadjoint operator $H(\omega, \kappa) : L^2(\mathbb{R}, \mathbb{C}^4) \to L^2(\mathbb{R}, \mathbb{C}^4)$:

**Corollary 3.5.** For any $\kappa \in \mathbb{R}$, the essential spectrum of $H(\omega, \kappa)$ is given by
\[
\sigma_{\text{ess}}(H(\omega, \kappa)) = [c^-(\omega), 1] \cup [c^+(\omega), +\infty).
\]
For $\kappa < 0$, one has $H(\omega, \kappa) \geq 0$, with eigenvalue $\lambda = 0$ of multiplicity one. For $\kappa = 0$, one has $H(\omega, 0) \geq 0$, with eigenvalue $\lambda = 0$ of multiplicity two. For $\kappa > 0$ (when $\Lambda_\kappa(\omega)$ from Lemma 3.3 is negative), the operator $H(\omega, \kappa)$ has exactly one simple negative eigenvalue $\lambda_{\kappa}^-$ given by (3.12).

The following theorem gives the orbital stability result for solitary wave solutions to (1.1).
Theorem 3.6. Assume that there are solitary wave solutions for \( \omega \in (\omega_1, \omega_2) \subset (-m, m) \). The solitary wave \( \phi(x, \omega)e^{-i\omega t} \) is orbitally stable if and only if

\[
\kappa < \frac{\omega^2}{m^2}.
\]

Above, \( \kappa := |\phi(0, \omega)|^2 a'(|\phi(0, \omega)|^2)/a(|\phi(0, \omega)|^2) \) could be negative (see (3.5)).

Let us recall that for nonzero solitary waves \( a(|\phi(0, \omega)|^2) \neq 0 \).

Proof. The Grillakis–Shatah–Strauss theory \([GSS87]\) applies when either \( H \geq 0 \), with \( \lambda = 0 \) a simple eigenvalue with the corresponding eigenvector \( J\Phi \), satisfying the assumptions of \([GSS87], \text{Theorem 1}\). (when \( \kappa < 0 \)), or \( H \) has a negative spectrum consisting of one simple eigenvalue, has its kernel spanned by \( J\Phi \), while the rest of its spectrum is positive and separated away from \( z = 0 \), satisfying the assumptions of \([GSS87], \text{Theorem 2}\). In the case \( \kappa = 0 \), the operator \( A_0(\omega) = \begin{bmatrix} i\omega & \omega^2 \\ -L_0(\omega) & 1 + i\omega \end{bmatrix} \) in the right-hand side of (3.3) is \( \mathbb{C} \)-linear. Moreover, it can be represented as \( A_0(\omega) = J\tilde{H}_0(\omega) \), where

\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \tilde{H}_0(\omega) = \begin{bmatrix} L_0(\omega) + \omega^2 & -i\omega \\ i\omega & 1 \end{bmatrix}.
\]

The operator \( \tilde{H}_0(\omega) \) has the same spectral properties as the operator \( H_0(\omega) \). Namely,

\[
\sigma_{\text{ess}}(\tilde{H}_0(\omega)) = [c^-(\omega), 1] \cup [c^+(\omega), +\infty), \quad \sigma_p(\tilde{H}_0(\omega)) = \{0\} \cup \{\omega^2 + 1\},
\]

with eigenvalue \( \lambda = 0 \) being simple. Hence, \([GSS87], \text{Theorems 1}\) applies, showing that the solitary wave \( \phi(x, \omega)e^{-i\omega t} \) is orbitally stable.

The instability in the case \( \kappa > \omega^2/m^2 \) follows from \([GSS87, \text{Theorem 3}]\); in the critical case, the instability follows from the Jordan block structure of the zero eigenvalue; see e.g. \([CP03]\).

The spectrum of the linearization at a solitary wave also follows from the arguments in \([Kol73]\) and \([GSS87]\); we consider it next.

Lemma 3.7 (Spectrum of \( A(\omega, \kappa) \): basic properties).

1. If \( \lambda \) belongs to \( \sigma_p(A(\omega, \kappa)) \), then so do \( \bar{\lambda} \), \(-\lambda\), and \(-\bar{\lambda}\).
2. \( \sigma_{\text{ess}}(A(\omega, \kappa)) = i(\mathbb{R} \setminus (-m + |\omega|, m - |\omega|)) \).
3. \( \sigma_p(A(\omega, \kappa)) \subset \mathbb{R} \cup i\mathbb{R} \). There is a pair of a positive and a negative eigenvalues of \( A(\omega, \kappa) \) if and only if \( \kappa > 0 \) and \( |\omega| < \min\{m, \Omega_\kappa\} \), with
\[
\Omega_\kappa := m\sqrt{\kappa}.
\]  

(3.18)

4. If \( \omega \in (-m, m) \) and \( \kappa \notin \{0, \omega^2/m^2\} \), then the zero eigenvalue of \( A(\omega, \kappa) \) is of geometric multiplicity 1 and of algebraic multiplicity 2.

5. If \( \omega \in (-m, m) \setminus \{0\} \) and \( \kappa = \omega^2/m^2 \), then the zero eigenvalue of \( A(\omega, \kappa) \) is of geometric multiplicity 1 and of algebraic multiplicity 4.

6. If \( \omega \in (-m, m) \setminus \{0\} \) and \( \kappa = 0 \), then the zero eigenvalue of \( A(\omega, \kappa) \) is of geometric multiplicity 2 and of algebraic multiplicity 2.

7. If \( \omega = 0 \) and \( \kappa = 0 \), then eigenvalue \( z = 0 \) of \( A(\omega, \kappa) \) is of geometric multiplicity 2 and of algebraic multiplicity 4.

Proof. We follow the arguments of Kolokolov [Kol73] and Grillakis–Shatah–Strauss [GSS87]. For Part (1), one notices that \( A(\omega, \kappa) \) has real coefficients (hence \( \lambda \) is also an eigenvalue) and then that \( A(\omega, \kappa)^* = -H(\omega, \kappa)\Sigma \), which is conjugate to \( -\Sigma H(\omega, \kappa) \) (hence \( -\lambda \) is also an eigenvalue). Part (2) follows by fixing \( \lambda \in i\mathbb{R} \) with \( |\lambda| \geq m - |\omega| \) and considering the Weyl sequences supported away from \( x = 0 \).

To prove Part (3), it is convenient to write \( A(\omega, \kappa) \) from (3.8) in the form similar to the nonlinear Schrödinger theory [Kol73]. We consider the conjugated versions of \( \Sigma \) and \( H(\omega, \kappa) \), with
\[
\tilde{H}(\omega, \kappa) = G_2 G_1 H(\omega, \kappa) G_1^{-1} G_2^{-1}
\]
from (3.10) and with
\[
\tilde{\Sigma} = G_2 G_1 \Sigma G_1^{-1} G_2^{-1} = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}.
\]  

(3.19)

Now we are in the framework of A. Kolokolov [Kol73]. Consider the eigenvalue problem for \( \tilde{\Sigma} \tilde{H}(\omega, \kappa) \):
\[
\tilde{\Sigma} \tilde{H}(\omega, \kappa) \psi = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} H_\kappa(\omega) & 0 \\ 0 & H_0(\omega) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}, \quad \psi = \begin{bmatrix} u \\ v \end{bmatrix} \in L^2(\mathbb{R}, \mathbb{C}^4).
\]  

(3.20)

First, we notice that if \( \kappa \leq 0 \), then \( \tilde{H}(\omega, \kappa) \) is nonnegative and selfadjoint, hence one can extract the square root (also nonnegative and selfadjoint); therefore,
\[
\sigma_d(\tilde{\Sigma} \tilde{H}) \setminus \{0\} = \sigma_d(\tilde{H}^{1/2} \tilde{\Sigma} \tilde{H}^{1/2}) \setminus \{0\} \subset i\mathbb{R}
\]
since \( \tilde{H}^{1/2} \tilde{\Sigma} \tilde{H}^{1/2} \) is antiselfadjoint.

From now on, we assume that \( \kappa > 0 \). Eliminating \( v \), we get
\[
-\sigma_1 H_0(\omega) \sigma_1 H_\kappa(\omega) u = \lambda^2 u, \quad u \in L^2(\mathbb{R}, \mathbb{C}^2).
\]

For \( \lambda \neq 0 \), one can see that \( u \) is orthogonal to
\[
\ker(\sigma_1 H_0(\omega) \sigma_1) = \ker \left( \begin{bmatrix} 1 & \omega \\ \omega & L_0 + \omega^2 \end{bmatrix} \right) = \begin{bmatrix} -\omega \phi \\ \phi \end{bmatrix}
\]  

(3.21)

(cf. (2.6)), hence we can write
\[
H_\kappa(\omega) u = -\lambda^2 (\sigma_1 H_0(\omega) \sigma_1)^{-1} u = -\lambda^2 \sigma_1 H_0(\omega)^{-1} \sigma_1 u.
\]
Coupling this relation with $u$ and taking into account that $\langle u, (\sigma_1 H_0(\omega) \sigma_1)^{-1} u \rangle > 0$, we see that $\lambda^2 \in \mathbb{R}$. To find whether $-\lambda^2$ can be negative, one considers the minimization problem

$$
\mu = \inf \left\{ \langle u, H_\kappa(\omega) u \rangle : \langle u, u \rangle = 1, \ u \in (\ker(\sigma_1 H_0(\omega) \sigma_1))^\perp = \left[ -\omega \phi \right] \phi \right\},
$$

which implies that $u$ satisfies

$$
H_\kappa(\omega) u = \mu u + \nu \left[ -\omega \phi \right] \phi,
$$

with $\mu, \nu \in \mathbb{R}$ the Lagrange multipliers. We note that if $\mu \leq 0$, then $\nu \neq 0$, or else one would have $\mu = \lambda^-_\kappa$ (the only negative eigenvalue of $H_\kappa(\omega)$), which is not possible since $u^-(x)$ from (3.17) corresponding to eigenvalue $\lambda^-_\kappa$ is not orthogonal to $\ker(\sigma_1 H_0(\omega) \sigma_1)$: one has

$$
(u^-)^* \left[ -\omega \phi \phi \right] = \left[ 1 - \lambda^-_\kappa \right] \left[ -\omega \right] e^{-|x|^2/\kappa} e^{-|x|} = -(2 - \lambda^-_\kappa) e^{-|x|^2/\kappa} e^{-|x|},
$$

where $2 - \lambda^-_\kappa$ is strictly positive. So, $\nu \neq 0$ (or else $\mu$ would have to be positive); then one concludes from (3.22) that $\mu > \lambda^-_\kappa$. We rewrite (3.23) as

$$
(H_\kappa(\omega) - \mu) u = \nu \left[ -\omega \phi \right] \phi, \quad u = \nu(H_\kappa(\omega) - \mu)^{-1} \left[ -\omega \phi \right] \phi.
$$

The sign of $\mu$ could be found from the condition that $u$ is orthogonal to $\ker(\sigma_1 H_0(\omega) \sigma_1)$:

$$
\left\langle \left[ -\omega \phi \phi \right], (H_\kappa(\omega) - z)^{-1} \left[ -\omega \phi \phi \right] \right\rangle = 0.
$$

We consider

$$
h(z) = \left\langle \left[ -\omega \phi \phi \right], (H_\kappa(\omega) - z)^{-1} \left[ -\omega \phi \phi \right] \right\rangle, \quad z \in (\lambda^-_\kappa, \kappa^-) \subset \rho(H_\kappa(\omega)).
$$

Since $h(z)$ is monotonically increasing on $\rho(H_\kappa(\omega))$, the sign of $\mu$ is opposite to the sign of $h(0)$, which is given by

$$
h(0) = \left\langle \left[ -\omega \phi \phi \right], H_\kappa(\omega)^{-1} \left[ -\omega \phi \phi \right] \right\rangle = \left\langle \left[ -\omega \phi \phi \right], \left[ -\omega \phi \phi \right] + \left[ \omega \partial_\omega \phi + \phi \right] \right\rangle = \partial_\omega \left( \omega \| \phi \|^2 \right).
$$

Above, we used the relation

$$
H_\kappa(\omega) \left[ -\omega \phi \phi \right] = \left[ \frac{L_\omega + \omega^2}{\omega} \right] \left[ -\omega \phi \phi \right] = \left[ -\omega \phi \phi \right],
$$

which in turn follows from taking the $\omega$-derivative of (2.6) (after we substitute $\phi(x, \omega)$), which yields

$$
L_\omega \partial_\omega \phi = 2 \omega \phi.
$$

We conclude from (3.24) that there is $\lambda^2 > 0$ (hence, there is a pair of a positive and a negative eigenvalues) if and only if $h(0) = \partial_\omega \left( \omega \| \phi \|^2 \right) > 0$.

Let us prove Parts (4) and (5). Given $\omega \in (-m, m)$, let us compute the geometric multiplicity of $\lambda = 0$. We consider $\hat{\mathbf{A}} = \hat{\Sigma} \hat{H}(\omega, \kappa)$, with $\hat{\Sigma}$ from (3.19) and $\hat{H}(\omega, \kappa)$ from (3.10). Since $\kappa \neq 0$, by Lemma 3.4 the geometric multiplicity of $\lambda = 0$ equals 1, with

$$
\ker(\hat{H}(\omega, \kappa)) = \text{Span} \{ \phi(x) e_3 - \omega \phi(x) e_4 \},
$$
with \{e_i\}_{1 \leq i \leq 4} the standard basis in \mathbb{C}^4.

Now let us compute the algebraic multiplicity of \lambda = 0. By (3.21) and (3.25) we have

\[
\Sigma \mathbf{H}(\omega, \kappa) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \phi & \omega \partial_{w} \phi + \phi & 0 \\
0 & -\omega \phi & 0 & 0 \\
-\omega \phi & 0 & 0 & 0
\end{bmatrix} = -i \begin{bmatrix}
0 & \sigma_1 & H_\kappa(\omega) & 0 \\
\sigma_1 & 0 & 0 & H_0(\omega) \\
H_\kappa(\omega) & 0 & \phi & -\omega \phi \\
H_0(\omega) & 0 & \phi & -\omega \phi
\end{bmatrix} = 0,
\]

thus \lambda = 0 is an eigenvalue of \Sigma \mathbf{H}(\omega, \kappa) of multiplicity at least two. To be able to extend this Jordan chain, solving

\[
\Sigma \mathbf{H}(\omega, \kappa) \Xi = \begin{bmatrix}
-\partial_{\omega} \phi \\
\omega \partial_{\omega} \phi + \phi \\
0 \\
0
\end{bmatrix}, \quad \Xi \in \mathbb{C}^4,
\]

we need to make sure that the right-hand side is orthogonal to the kernel of the adjoint of the operator in the left-hand side,

\[
\ker (\mathbf{H}(\omega, \kappa) \Sigma) = \text{Span}\left\{ \begin{bmatrix}
-\omega \phi \\
\phi \\
0 \\
0
\end{bmatrix} \right\};
\]

thus, the condition to have a Jordan block of a larger size is

\[
0 = \left\langle \begin{bmatrix}
-\omega \phi \\
\phi \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-\partial_{\omega} \phi \\
\omega \partial_{\omega} \phi + \phi \\
0 \\
0
\end{bmatrix} \right\rangle = \partial_{\omega} \left( \omega \| \phi \|^2 \right). \quad (3.29)
\]

Let us compute explicitly the right-hand side in (3.29). Since \| \phi(\omega) \|^2 = \frac{C(\omega)^2}{\kappa} \) (see Lemma 2.3), one computes:

\[
\frac{d}{d\omega}(\omega \| \phi(\omega) \|^2) = \frac{C^2}{\kappa} + \omega \left( \frac{2C}{\kappa} \frac{dC}{d\omega} + \frac{\omega C^2}{\kappa^3} \right) = \frac{1}{\kappa^2} \left( m^2 C^2 - \frac{\omega^2 a(C^2)}{d'(C^2)} \right) = \frac{C^2}{\kappa^2} \left( m^2 - \frac{\omega^2}{\kappa} \right), \quad (3.30)
\]

where \( C = C(\omega); \) to express \( dC/d\omega, \) we used (2.11). Thus, if \( \kappa \neq 0, \partial(\omega \| \phi(\omega) \|^2) = 0 \) if and only if \( \kappa = \omega^2/m^2 \) (we will consider the case \( \kappa = 0 \) separately).

If \( \kappa \neq \omega^2/m^2, \kappa \neq 0, \) since the condition (3.29) is not satisfied, equation (3.28) has no \( L^2 \) solutions hence the algebraic multiplicity is exactly two. If \( \kappa = \omega^2/m^2 > 0, \) then equation (3.28) has an \( L^2 \)-solution: the algebraic multiplicity jumps by at least one. This means that an eigenvalue arrives at \( z = 0; \) by the symmetry of \( \sigma(A(\omega, \kappa)) \) with respect to \( \mathbb{R} \) and \( i\mathbb{R}, \) there is also the opposite sign eigenvalue arriving at \( z = 0, \) thus the algebraic multiplicity of \( z = 0 \) jumps by two. Standard considerations (see e.g. [CP03]) show that for \( \omega \neq \pm \Omega_{\kappa} \) the algebraic multiplicity of zero eigenvalue cannot be more than four. Indeed, let us show that the equation

\[
(\Sigma \mathbf{H})^4 \Gamma = \phi(x) e_3 - \omega \phi(x) e_4, \quad \Gamma \in L^2(\mathbb{R}, \mathbb{C}^4),
\]
has no solutions. If the algebraic multiplicity of \( \lambda = 0 \) is at least four (hence \( \kappa = \omega^2 / m^2 \)) so that there are \( \Xi, \Theta \in L^2(\mathbb{R}, \mathbb{C}^4) \) such that

\[
(\Sigma \mathbf{H}) \Xi = -\partial_\omega \phi e_1 + (\omega \partial_\omega \phi + \phi) e_2, \quad (\Sigma \tilde{\mathbf{H}}) \Theta = \Xi, \tag{3.31}
\]

where, as one can readily see from the block form of \( \Sigma \mathbf{H} \) (see (3.20)), one can decompose \( \Xi(x) = \xi_3(x)e_3 + \xi_4(x)e_4 \), then, trying to solve

\[
(\Sigma \tilde{\mathbf{H}})^2 \Gamma = \Xi, \quad \Gamma \in L^2(\mathbb{R}, \mathbb{C}^4),
\]

we need to make sure that \( \Xi \) is orthogonal to the kernel of \((\Sigma \tilde{\mathbf{H}})^2 \) which contains in particular \( \Sigma((-\partial_\omega \phi e_1 + \omega \partial_\omega \phi + \phi) e_2) = \Sigma(\Sigma \mathbf{H}) \Xi = -\mathbf{H} \Xi \). We arrive at the following necessary condition:

\[
\langle \Xi, \tilde{\mathbf{H}}(\omega, \kappa) \Xi \rangle = 0. \tag{3.32}
\]

Taking into account that \( \tilde{\mathbf{H}}(\omega, \kappa) = \begin{bmatrix} H_\kappa(\omega) & 0 \\ 0 & H_0(\omega) \end{bmatrix} \) is semi-positive-definite on vectors of the form \( \xi_3(x)e_3 + \xi_4(x)e_4 \), with the one-dimensional kernel (this follows from Lemma 3.4), while by (3.31) the function \( \Xi \) is not in this kernel, we conclude that the left-hand side is strictly positive, hence the condition (3.32) can not be satisfied.

For Part (6), we consider the case \( \omega \neq 0, \kappa = 0 \). By Lemma 3.4, the geometric multiplicity of \( \lambda = 0 \) equals 2, with

\[
\ker(\tilde{\mathbf{H}}(\omega, 0)) = \text{Span} \{ \phi(x)e_1 - \omega \phi(x)e_2, \phi(x)e_3 - \omega \phi(x)e_4 \}.
\]

To have a Jordan block, we would need to solve

\[
\Sigma \mathbf{H}(\omega, 0) \phi(x)e_4 = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} H_0(\omega) & 0 \\ 0 & H_0(\omega) \end{bmatrix} \Xi = -i \phi(x)e_1. \tag{3.33}
\]

Since \( \omega \neq 0 \), the right-hand side of (3.33) is not orthogonal to

\[
\ker((\Sigma \mathbf{H}(\omega, 0))^*) = \ker(\tilde{\mathbf{H}}(\omega, 0)\Sigma) \ni \begin{bmatrix} -\omega \phi(x) \\ \phi(x) \\ 0 \\ 0 \end{bmatrix},
\]

hence (3.33) has no solutions.

Finally, for Part (7), we consider the case \( \omega = 0, \kappa = 0 \). In this case, the geometric multiplicity of \( \lambda = 0 \) equals 2, with

\[
\ker(\tilde{\mathbf{H}}(0, 0)) = \text{Span} \{ \phi(x)e_1, \phi(x)e_3 \}.
\]

The Jordan block corresponding to \( \phi(x)e_1 \) is of size at least two since

\[
\Sigma \mathbf{H}(0, 0) \phi(x)e_4 = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} H_0(0) & 0 \\ 0 & H_0(0) \end{bmatrix} \phi(x)e_4 = -i \phi(x)e_1.
\]

Now we notice that \( \phi(x)e_4 \) is not orthogonal to \( \ker(\tilde{\mathbf{H}}(0, 0)\Sigma) \) which contains \( \phi(x)e_4 \); therefore, the corresponding Jordan block is of size exactly two. Similarly, there is a Jordan block of size exactly two corresponding to \( \phi(x)e_3 \).

This completes the proof of Lemma 3.7. \[ \square \]

**Remark 3.8.** By Lemma 3.7 (3), there is a positive eigenvalue of \( A(\omega, \kappa) \) if and only if \( \omega^2 < \min\{m^2, m^2 \kappa \} \). We see from (3.30) that in this case one has \( \partial_\omega(\omega \| \phi(\omega) \|)^2 > 0 \). Since the charge of the Klein–Gordon field is given by \( Q(\phi(\omega)) = \omega \| \phi(\omega) \|^2 \) (see (2.4)), the above is in agreement with the Kolokolov stability condition

\[
\partial_\omega Q(\phi(\omega)) < 0 \tag{3.34}
\]

derived in [Kol73] in the context of the nonlinear Schrödinger equation.
4 The spectrum of the linearization operator

Now we are going to perform a complete analysis of the spectrum of the linearization operator. For convenience, we consider the linearization operator $A(\omega, \kappa)$ from (3.8) using $\kappa > 0$ as a parameter. Recall that

$$A(\omega, \kappa) = \begin{bmatrix}
0 & -\omega & 1 & 0 \\
\omega & 0 & 0 & 1 \\
0 & 0 & 0 & -\omega \\
\partial_x^2 - m^2 + (1 + 2\kappa)\alpha \delta(x) & 0 & 0 & \omega \\
\partial_x^2 - m^2 + \alpha \delta(x) & \omega & 0 & 0
\end{bmatrix}, \quad \alpha = 2\sqrt{m^2 - \omega^2}. \quad (4.1)$$

For $x \neq 0$, substituting $\Psi(x) = w e^{-\nu|x|}$ with some $\nu \in \mathbb{C}$, $\text{Re}\nu \geq 0$, into the equation $(A(\omega, \kappa) - \lambda)\Psi = 0$, we get

$$\begin{bmatrix}
-\lambda & -\omega & 1 & 0 \\
\omega & -\lambda & 0 & 1 \\
\nu^2 - m^2 & 0 & -\lambda & -\omega \\
0 & \nu^2 - m^2 & \omega & -\lambda
\end{bmatrix}w = 0, \quad \nu \in \mathbb{C}, \quad w \in \mathbb{C}^4, \quad w \neq 0, \quad (4.2)$$

which, via the Schur complement idea, is equivalent to

$$\begin{bmatrix}
-\lambda & -\omega & 1 & 0 \\
\omega & -\lambda & 0 & 1 \\
S(\omega, \lambda, \nu) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}w = 0, \quad w \in \mathbb{C}^4, \quad w \neq 0, \quad (4.3)$$

with $S(\omega, \lambda, \nu) \in \text{End}(\mathbb{C}^2)$ the Schur complement of the top right block $I_2$:

$$S(\omega, \lambda, \nu) = \begin{bmatrix}
\nu^2 - m^2 & 0 \\
0 & \nu^2 - m^2
\end{bmatrix} - \begin{bmatrix}
-\lambda & -\omega \\
\omega & -\lambda
\end{bmatrix}^2 = \begin{bmatrix}
\nu^2 - m^2 + \omega^2 - \lambda^2 & -2\lambda \omega \\
2\lambda \omega & \nu^2 - m^2 + \omega^2 - \lambda^2
\end{bmatrix}. \quad (4.4)$$

The condition to have nonzero solution $w \in \mathbb{C}^4$ to (4.3) is equivalent to

$$\det S(\omega, \lambda, \nu) = (m^2 - \omega^2 + \lambda^2 - \nu^2)^2 + 4\lambda^2 \omega^2 = 0.$$

This gives

$$m^2 - \omega^2 + \lambda^2 - \nu^2 = -2i\lambda \omega, \quad m^2 - \omega^2 + \lambda^2 - \nu^2 = 2i\lambda \omega,$$

$$m^2 - (\omega - i\lambda)^2 = \nu^2, \quad m^2 - (\omega + i\lambda)^2 = \nu^2,$$

allowing one to express $\nu \in \mathbb{C}$ in terms of $\omega$ and $\lambda$:

$$\nu = \sqrt{m^2 - (\omega \pm i\lambda)^2}.$$

We choose the cuts in the complex plane $\lambda$ from the branching points to infinity:

$$C_+ := (-i\infty, -i(m - \omega)] \cup [i(m + \omega, i\infty),$$

$$C_- := (-i\infty, -i(m + \omega)] \cup [i(m - \omega, i\infty),$$

defining

$$\nu_\pm(\omega, \lambda) = \sqrt{m^2 - (\omega \pm i\lambda)^2}, \quad (4.5)$$

with

$$\text{Re}\nu_\pm(\omega, \lambda) > 0, \quad \lambda \in \mathbb{C} \setminus C_\pm. \quad (4.6)$$
The zero eigenvalue of the Schur complement $S$ from (4.4) corresponds to two eigenvectors $u_{\pm} \in \mathbb{C}^2$, depending on the choice $\nu = \nu_{\pm}$; these eigenvectors are given by

$$
\begin{bmatrix}
2\lambda \omega \\
-m^2 + \omega^2 + \nu_{\pm}^2 - \lambda^2
\end{bmatrix} = \begin{bmatrix}
2\lambda \omega \\
-m^2 + \omega^2 + (m^2 - \omega^2 + \lambda^2 - 2i\lambda \omega) - \lambda^2
\end{bmatrix} = 2\lambda \omega \begin{bmatrix}
1 \\
1 + i
\end{bmatrix},
$$

so we can use $u_{\pm} = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}$. By (4.3), the corresponding vector from the null space of $A(\omega, \kappa) - \lambda I$ is then $w_{\pm} = \begin{bmatrix} u_{\pm} \\ v_{\pm} \end{bmatrix} \in \mathbb{C}^4$, with $v_{\pm} = -\begin{bmatrix} -\lambda & -\omega \\
\omega & -\lambda \end{bmatrix} u_{\pm} = \begin{bmatrix} \lambda + i\omega \\
-\omega + i\lambda \end{bmatrix} \in \mathbb{C}^2$. Thus, one has

$$
w_{\pm} = \begin{bmatrix} 1 + i
\lambda \mp i\omega
\omega \pm i\lambda
\end{bmatrix} \in \mathbb{C}^4.
$$

Therefore, an eigenfunction corresponding to the eigenvalue $\lambda \in \mathbb{C}$ is of the form

$$
\Psi(x, \omega, \lambda) = A w_{\pm} e^{-\nu_{\pm}|x|} + B w_{-\nu_{\pm}} e^{-\nu_{-}|x|} = A \begin{bmatrix} 1 \\
l - i\omega \\
\lambda + i\omega
\end{bmatrix} e^{-\nu_{\pm}|x|} + B \begin{bmatrix} 1 \\
-1 \\
\lambda + i\omega
\end{bmatrix} e^{-\nu_{-}|x|}, \tag{4.7}
$$

with $A, B \in \mathbb{C}$ not simultaneously zeros and $\text{Re} \nu_{\pm} > 0$ (when the corresponding coefficient is nonzero). The values of $A$ and $B$ are obtained from the jump conditions at $x = 0$: substituting $\Psi$ into $(A(\omega, \kappa) - \lambda I)\Psi = 0$ and collecting the terms with $\delta$-function gives:

$$
\begin{cases}
(-2\nu_{\pm} + \alpha(1 + 2\kappa))A + (-2\nu_{-} + \alpha(1 + 2\kappa))B = 0, \\
(-2\nu_{\pm} + \alpha)iA + (-2\nu_{-} + \alpha)(-iB) = 0.
\end{cases} \tag{4.8}
$$

The condition to have $A, B \in \mathbb{C}$ not simultaneously zeros,

$$
\det \begin{bmatrix}
-2\nu_{\pm} + \alpha(1 + 2\kappa) & -2\nu_{-} + \alpha(1 + 2\kappa) \\
-2\nu_{\pm} + \alpha & 2\nu_{-} - \alpha
\end{bmatrix} = 0,
$$

takes the form $-8\nu_{\pm}\nu_{-} + 4(\nu_{\pm} + \nu_{-})\alpha(1 + \kappa) - 2\alpha^2(1 + 2\kappa) = 0$, which we rewrite as

$$
D_{\omega, \kappa}(\lambda) = 0 \tag{4.9}
$$

with

$$
D_{\omega, \kappa}(\lambda) = \alpha^2(1 + \kappa)^2 - 2(\nu_{\pm} + \nu_{-})\alpha(1 + \kappa) + 4\nu_{\pm}\nu_{-} - \alpha^2\kappa^2, \tag{4.10}
$$

with $\alpha = 2\sqrt{m^2 - \omega^2}$ from (3.5) and $\nu_{\pm}(\omega, \lambda) = \sqrt{m^2 - (\omega \pm i\lambda)^2}$ from (4.5). The function $D_{\omega, \kappa}(\lambda)$ is analytic in $\lambda$ in $\mathbb{C} \setminus (C_- \cup C_+)$, since there are two possible values for each of the square roots in the definition (4.5) of $\nu_{\pm}$, $D_{\omega, \kappa}(\lambda)$ can be continued analytically through the cuts $C_-$ and $C_+$ to an analytic function on the four-sheet cover of $\mathbb{C}$, which we also denote by $D_{\omega, \kappa}(\lambda)$. We call the sheet defined by conditions (4.6) the physical sheet of $D_{\omega, \kappa}(\lambda)$.

### 4.1 Embedded eigenvalues and virtual levels

We start by studying embedded eigenvalues and virtual levels of the operator $A(\omega, \kappa)$. Before formulating our results, let us mention that a virtual level (also known as a threshold resonance) can be defined as a limit point of an eigenvalue family which corresponds to values of a perturbation parameter in an interval when this limit point no longer corresponds to a square-integrable eigenfunction. The virtual levels usually occur at thresholds of the essential spectrum (the endpoints of the essential spectrum or the points where the continuous spectrum changes its multiplicity). For more on the phenomenon of virtual levels, see e.g. [JK79, JN01, Yaf10, GN20, EGT19].
Lemma 4.1 (Embedded eigenvalues and virtual levels of $A(\omega, \kappa)$).

1. There are embedded eigenvalues $\lambda = \pm 2\omega i$ if and only if $\kappa = 0$ and $|\omega| \geq m/3$.

2. For any $\kappa \in \mathbb{R}$ and $\omega \in (-m, m) \setminus 0$ there are no virtual levels at the embedded thresholds $\lambda = \pm i(m + |\omega|)$.

3. There are virtual levels at $\lambda = \pm i(m - |\omega|)$ if and only if $\kappa \in \left[ -\frac{1}{2}, \frac{1}{\sqrt{2}} \right)$ and $\omega = \pm T_\kappa$, where

\[ T_\kappa = \frac{m (1 + 2\kappa)^2}{3 + 4\kappa}; \tag{4.11} \]

in particular, $\lambda = \pm im$ are virtual levels if and only if $\kappa = -1/2$ and $\omega = 0$. By Part 1, these are genuine virtual levels (with non-$L^2$ eigenfunction) if and only if $\kappa \neq 0$.

Proof. Because of the symmetry with respect to the sign of $\omega$, we will assume that $0 \leq \omega < m$.

For Part (1), it suffices to consider $\lambda = i\Lambda$ with $\Lambda \geq m - \omega$. In this case

\[ \nu_\pm = \sqrt{m^2 - (\omega + \Lambda)^2} \in i\mathbb{R}; \]

therefore, in the expression for the corresponding eigenfunction (4.7) one would need to take $B = 0$.

There are two cases to consider:

- If $\kappa \neq 0$, the system (4.8) shows that one would also have $A = 0$; hence, $\lambda$ cannot be an eigenvalue;

- If $\kappa = 0$ and $\omega \in [m/3, m)$, there is an embedded eigenvalue $\lambda = 2\omega i$ since the system (4.8) is satisfied for $B = 0$ and any $A \in \mathbb{C}$. (Note that in this case $\Lambda = 2\omega \geq m - \omega$ and $\nu_+ = \kappa = \alpha/2$.)

For Part (2), it is enough to consider $\omega \in (0, m)$, $\lambda = i(m + \omega)$. At this value of $\lambda$, one has $\nu_+ = 0$,

\[ \nu_{\pm} = \sqrt{m^2 - (m + 2\omega)^2} \in i\mathbb{R} \setminus \{0\}. \]

Thus,

\[ D_{\omega, \kappa}(i(m + \omega)) = \alpha^2 (1 + 2\kappa) - 2\nu_{\pm} \alpha (1 + \kappa) \]

can not be zero since $\alpha > 0$. In the case $\omega = 0$, one has $\nu_{\pm} = 0$, and then $D_{\omega, \kappa}(im) = 0$ if and only if $\kappa = -1/2$.

For Part (3), we need to consider $\omega \in (0, m)$, $\lambda = i(m - \omega)$. In this case

\[ \nu_- = 0, \quad \nu_+ = \sqrt{m^2 - (2\omega - m)^2} = 2\sqrt{\omega(m - \omega)} > 0 \]

by (4.6). We need to solve:

\[ 0 = D_{\omega, \kappa}(i(m - \omega)) = \alpha^2 (1 + 2\kappa) - 2\nu_{\pm} \alpha (1 + \kappa). \]

The equation has no solution for $\kappa = -1$. In the case $\kappa \neq -1$, it is equivalent to

\[ \sqrt{m^2 - \omega^2} \frac{1 + 2\kappa}{1 + \kappa} = 2\sqrt{\omega(m - \omega)}; \]

which leads to

\[ \omega = T_\kappa = m \frac{(1 + 2\kappa)^2}{3 + 4\kappa} = m \left( \kappa + \frac{1}{4} + \frac{1}{4(3 + 4\kappa)} \right). \tag{4.12} \]

We note that $0 \leq T_\kappa < m$ as long as $\kappa \in [-1/2, 1/\sqrt{2})$.\]
4.2 Discrete spectrum

Now we will study the discrete spectrum of the operator \((A(\omega, \kappa))\). Recall that for any \(\kappa \in \mathbb{R}\) and \(\omega \in (-m, m)\) the operator has zero eigenvalue of of multiplicity 2. We start with the special cases \(\kappa = 0\) and \(\omega = 0\).

**Lemma 4.2** (Eigenvalues of \(A(\omega, 0), \omega \neq 0\)).

\[
\sigma_p(A(\omega, 0)) = \{0, \pm 2\omega i\} \quad 0 < |\omega| < m,
\]

with eigenvalues \(\lambda = \pm 2\omega i\) being simple. The only eigenvalue of \(A(0, 0)\) is \(\lambda = 0\) of algebraic multiplicity 4 and geometric multiplicity 2.

**Proof.** Considering for which values \(\lambda\) the matrix

\[
A(\omega, 0) - \lambda I_4 = \begin{bmatrix}
-\lambda & -\omega & 1 & 0 \\
\omega & 0 - \lambda & 0 & 1 \\
-L_0(\omega) - \omega^2 & 0 & -\lambda & -\omega \\
0 & -L_0(\omega) - \omega^2 & \omega & -\lambda
\end{bmatrix}
\] (4.13)

has eigenvalue zero reduces to studying this question for the Schur complement of the top right block \(I_2\),

\[
S = \begin{bmatrix}
-L_0(\omega) - \omega^2 & 0 \\
0 & -L_0(\omega) - \omega^2
\end{bmatrix} - \begin{bmatrix}
-\lambda & -\omega \\
\omega & -\lambda
\end{bmatrix}^2 = \begin{bmatrix}
-L_0(\omega) - \lambda^2 & -2\lambda \omega \\
2\lambda \omega & -L_0(\omega) - \lambda^2
\end{bmatrix}.
\] (4.14)

By Lemma 3.3, \(L_0(\omega)\) has the only eigenvalue 0; we conclude that \(\lambda\) can be obtained from the equation

\[
\det \begin{bmatrix}
-\lambda^2 & -2\lambda \omega \\
2\lambda \omega & -\lambda^2
\end{bmatrix} = 0,
\]

which gives \(\lambda = 0\) and \(\lambda = \pm 2\omega i\). Recall that the eigenvalues \(\pm 2\omega i\) are embedded into the essential spectrum if \(|\omega| \geq m/3\). In the case \(\omega = 0\), the geometric multiplicity could be obtained from substituting \(\lambda = 0, \omega = 0,\) and \(\kappa = 0\) into (4.13) and taking into account that \(A_0(\omega) = 0\) is a simple eigenvalue of \(L_0(\omega)\) (Lemma 3.3). \(\Box\)

**Lemma 4.3** (Eigenvalues of \(A(0, \kappa)\)).

\[
\sigma_p(A(0, \kappa)) = \begin{cases}
0, & \kappa \leq -1/2, \\
\pm 2m \sqrt{\kappa(1 + \kappa)}, & \kappa > -1/2.
\end{cases}
\]

The eigenvalues \(\lambda = \pm 2m \sqrt{\kappa(1 + \kappa)}\) are simple (real if \(\kappa > 0\) and purely imaginary if \(-1/2 < \kappa < 0\)).

**Proof.** In the case \(\omega = 0\), one has \(\nu_\pm = \sqrt{m^2 + \lambda^2}\) and \(\alpha = 2m\). Then

\[
D_{0,\kappa}(\lambda) = 4m^2(1 + 2\kappa) - 8m(1 + \kappa)\sqrt{m^2 + \lambda^2} + 4(m^2 + \lambda^2)
\]

\[
= 8m^2(1 + \kappa) + 4\lambda^2 - 8m(1 + \kappa)\sqrt{m^2 + \lambda^2}.
\]

In the case \(\kappa = -1\), \(D_{0,\kappa}(\lambda)\) has the only root \(\lambda = 0\) of multiplicity 2. For \(\kappa \neq -1\), \(D_{0,\kappa}(\lambda) = 0\) if and only if

\[
2m^2 + \frac{\lambda^2}{1 + \kappa} > 0 \quad \text{and} \quad \left(2m^2 + \frac{\lambda^2}{1 + \kappa}\right)^2 = 4m^2(m^2 + \lambda^2).
\] (4.15)

The second condition gives \(\lambda^2 = 0\) and \(\frac{\lambda^2}{1 + \kappa} = 4m^2 \kappa\). Using the first condition of (4.15), we conclude that \(\kappa > -1/2\), and then \(\lambda = 2m \sqrt{\kappa(1 + \kappa)}\). \(\Box\)
Denote

\[ K_\omega = \frac{\sqrt{m^2 - (2|\omega| - m)^2} - \kappa}{2\kappa - \sqrt{m^2 - (2|\omega| - m)^2}} = \frac{2\sqrt{|\omega|}}{2 - 2\sqrt{|\omega|/m}} - 1, \quad \omega \in (-m, m). \quad (4.16) \]

We note that

\[ K_\omega \leq 0 \text{ for } \omega \leq m/3, \quad \text{and} \quad K_\omega \geq 0 \text{ for } \omega \geq m/3. \quad (4.17) \]

We note also that for \( \omega \in [0, m) \) the function \( \omega \mapsto K_\omega \) is the inverse to the function \( \kappa \mapsto T_\kappa \) defined in (4.11).

**Lemma 4.4.** For \( \omega \in (-m, m) \) and \( \kappa \in (-\infty, 0) \cup (0, K_\omega) \), \( \sigma_p(A(\omega, \kappa)) = \{0\} \).

**Proof.** For \( \omega = 0 \), the statement follows from Lemma 4.3; because of the symmetry with respect to the sign of \( \omega \), it suffices to consider \( \omega \in (0, m) \). Due to Lemma 3.7 (3), it suffices to prove that

\[ \sigma_p(A(\omega, \kappa)) \cap i\mathbb{R} = \{0\}, \quad \kappa \in (-\infty, 0) \cup (0, K_\omega), \quad \omega \in (0, m). \]

It suffices to prove that

\[ D_{\omega, \kappa}(\lambda) \neq 0, \quad \lambda \in i\mathbb{R} \setminus \{0\}, \quad \kappa \in (-\infty, 0) \cup (0, K_\omega), \quad \omega \in (0, m), \quad (4.18) \]

with \( D_{\omega, \kappa}(\lambda) \) defined in (4.10). We rewrite \( D_{\omega, \kappa}(\lambda) \) as

\[ D_{\omega, \kappa}(\lambda) = (\alpha - 2\nu_+)(\alpha - 2\nu_-) + 2\kappa \alpha (\alpha - \nu_+ - \nu_-). \quad (4.19) \]

Note that \( \lambda = \pm 2\omega i \) are not roots of \( D_{\omega, \kappa}(\lambda) \neq 0 \) for \( \omega > 0 \) and \( \kappa \neq 0 \). Indeed, for \( \lambda = 2\omega i \) one has \( \alpha - 2\nu_+ = 0 \), and then

\[ D_{\omega, \kappa}(2\omega i) = \alpha \kappa (\alpha - 2\nu_-) = \alpha \kappa \left(2\sqrt{m^2 - \omega^2} - 2\sqrt{m^2 - 9\omega^2}\right) \neq 0; \]

the case \( \lambda = -2\omega i \) is treated similarly.

For \( \lambda \neq \{0, \pm 2\omega i\} \), we may rewrite (4.19) as

\[ D_{\omega, \kappa}(\lambda) = (\alpha - 2\nu_+)(\alpha - 2\nu_-) \left(1 + \kappa \alpha \left(\frac{1}{\alpha - 2\nu_+} + \frac{1}{\alpha - 2\nu_-}\right)\right). \]

It remains to prove that for \( \omega \in (0, m) \) and \( \kappa \in (-\infty, 0) \cup (0, K_\omega) \), the equation

\[ 1 + \kappa \alpha \left(\frac{1}{\alpha - 2\nu_+} + \frac{1}{\alpha - 2\nu_-}\right) = 0 \]

has no solutions \( \lambda \in i\mathbb{R} \setminus \{0, \pm 2\omega i\} \). Denoting \( \Lambda = i\lambda \), we rewrite the above equation as

\[ 1 + \kappa \alpha Q(\Lambda) = 0, \quad (4.20) \]

where

\[ Q(\Lambda) = \frac{1}{\sqrt{m^2 - \omega^2} - \sqrt{m^2 - (\omega + \Lambda)^2}} + \frac{1}{\sqrt{m^2 - \omega^2} - \sqrt{m^2 - (\omega - \Lambda)^2}} \]

\[ = \frac{1}{\Lambda} \left[ \frac{\kappa + \sqrt{\kappa^2 - \lambda^2 - 2\omega\Lambda}}{\Lambda + 2\omega} + \frac{\kappa + \sqrt{\kappa^2 - \Lambda^2 + 2\omega\Lambda}}{\Lambda - 2\omega} \right]. \quad (4.21) \]

Due to (4.17) it suffices to prove that there are no solutions to (4.20) in the following domains:

(1) \( 0 < \omega < m/3, \quad \kappa < T_\kappa(\omega) < 0, \quad 2\omega < \Lambda < m - \omega; \)
(2) $0 < \omega < m/3$, $\kappa < T_\kappa(\omega) < 0$, $0 < \Lambda < 2\omega$;
(3) $m/3 < \omega < m$, $0 < \kappa < T_\kappa(\omega)$, $0 < \Lambda < m - \omega$.

(1) Note that $Q(\Lambda) > 0$ for $2\omega < \Lambda < m - \omega$, and

$$Q'(\Lambda) = -\sum_{\pm} \frac{\Lambda \pm \omega}{\sqrt{m^2 - (\Lambda \pm \omega)^2} \left(\kappa - \sqrt{m^2 - (\Lambda \pm \omega)^2}\right)^2} < 0.$$ 

Hence,

$$\inf_{(2\omega, m-\omega)} Q(\Lambda) = Q(m - \omega) = \frac{1}{\kappa} + \frac{1}{\kappa - \sqrt{m^2 - (2\omega - m)^2}} = \frac{1}{\kappa} \frac{1}{\sqrt{m^2 - \omega^2 - 2\sqrt{\omega(m - \omega)}}}$$

$$= \frac{1}{\kappa} \frac{1}{\sqrt{m^2 + 2\omega - \omega^2}} = \frac{1}{\kappa} \frac{2}{\sqrt{m^2 - \omega^2 - 2\sqrt{\omega(m - \omega)}}} = \frac{1}{\kappa} |K_\omega|^{-1} > 0. \quad (4.22)$$

and

$$|\kappa| \kappa Q(\Lambda) > |K_\omega| \kappa Q(m - \omega) = 1.$$ 

Therefore (4.20) has no solutions in the first domain.

(2) Note that $0 < \Lambda < \min\{2\omega, m-\omega\}$ in the second and the third domains. Hence $|\Lambda(\Lambda + 2\omega)| < \kappa^2$, and the following expansion holds:

$$\sqrt{m^2 - (\omega \pm \Lambda)^2} = \sqrt{\kappa^2 - \Lambda(\Lambda + 2\omega)} = \kappa \left(1 - \frac{\Lambda(\Lambda + 2\omega)}{2\kappa^2} - \sum_{n=2}^{\infty} \frac{(2n - 3)!!\Lambda^{n-1}(\Lambda + 2\omega)^{n-1}}{n!2^n \kappa^{2n}} \right).$$

The above leads to

$$\frac{\kappa + \sqrt{m^2 - (\omega \pm \Lambda)^2}}{\Lambda(\Lambda + 2\omega)} = \frac{2\kappa}{\Lambda(\Lambda + 2\omega)} - \frac{1}{2\kappa} - \sum_{n=2}^{\infty} \frac{(2n - 3)!!\Lambda^{n-1}(\Lambda + 2\omega)^{n-1}}{n!2^n \kappa^{2n-1}},$$

and then we derive from (4.21):

$$Q(\Lambda) = \frac{\kappa + \sqrt{m^2 - (\omega + \Lambda)^2}}{\Lambda(\Lambda + 2\omega)} + \frac{\kappa + \sqrt{m^2 - (\omega - \Lambda)^2}}{\Lambda(\Lambda - 2\omega)}$$

$$= -\frac{1}{\kappa} - \frac{4\kappa}{4\omega^2 - \Lambda^2} - \sum_{n=1}^{\infty} \frac{(2n - 1)!!\Lambda^n(\Lambda + 2\omega)^n + (\Lambda - 2\omega)^n}{(n + 1)!2^{n+1} \kappa^{2n+1}} < 0 \quad (4.23)$$

since each summand in $\sum_{n=1}^{\infty}$ is positive for $0 < \Lambda < 2\omega$. This immediately implies that (4.20) has no solution in the second domain.

(3) In the third domain $m - \omega < 2\omega$. Let us show that $Q(\Lambda)$ decreases monotonically on $(0, m - \omega)$. Indeed, (4.23) implies

$$Q'(\Lambda) = -\frac{8\kappa}{(4\omega^2 - \Lambda^2)} - \sum_{n=1}^{\infty} \frac{(2n - 1)!!n\Lambda^{n-1}(\Lambda + 2\omega)^n + (\Lambda - 2\omega)^n}{(n + 1)!2^{n+1} \kappa^{2n+1}}$$

$$- \sum_{n=1}^{\infty} \frac{(2n - 1)!!n\Lambda^n(\Lambda + 2\omega)^{n-1} + (\Lambda - 2\omega)^{n-1}}{(n + 1)!2^{n+1} \kappa^{2n+1}} < 0.$$

Hence, similarly to (4.22),

$$\sup_{(0, m-\omega)} |Q(\Lambda)| = |Q(m - \omega)| = \frac{1}{\sqrt{m^2 - (2\omega - m)^2}} - \frac{1}{\kappa} = \frac{1}{\kappa} |K_\omega|^{-1},$$

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and
\[ \kappa \Re |Q(\Lambda)| < K_\omega \Re |Q(m - \omega)| = 1. \]

Hence (4.20) has no solutions in this domain. \(\square\)

Now we collect all the facts about the eigenvalues obtained above.

**Theorem 4.5** (Eigenvalues of \(A(\omega, \kappa)\)).

1. For \(\kappa = \omega^2/m^2\), one has \(\sigma_p(A(\omega, \kappa)) = \{0\}\) of algebraic multiplicity 4; its geometric multiplicity is 1 if \(\omega \neq 0\) and 2 if \(\omega = 0\);
2. For \(\omega \neq 0, \kappa = 0\), one has \(\sigma_p(A(\omega, 0)) = \{0, \pm 2\omega i\}\) (\(\pm 2\omega i \in \sigma_{\text{ess}}(A(\omega, \kappa))\) when \(|\omega| \geq m/3\));
3. For \(\kappa > \omega^2/m^2\), one has \(\sigma_p(A(\omega, \kappa)) = \{0, \pm \lambda\}\) with some \(\lambda > 0\);
4. For \(\kappa \in (K_\omega, \omega^2/m^2)\), one has \(\sigma_p(A(\omega, \kappa)) = \{0, \pm i\Lambda\}, \Lambda > 0\);
5. \(\kappa \leq K_\omega, \kappa \neq 0\), one has \(\sigma_p(A(\omega, \kappa)) = \{0\}\).

![Figure 2: Location of simple eigenvalues ±\(\lambda\) for different values of parameters \(\kappa \in \mathbb{R}\) and \(\omega \in (-m, m)\). The values of \(\omega\) on the Kolokolov curve \(\Omega_\kappa\) (\(\kappa = \omega^2/m^2\)) correspond to collision of two simple eigenvalues at \(\lambda = 0\) as indicated by the Kolokolov condition. The values of \(\omega\) on the virtual level curve \(T_\kappa\) (blue curves for \(\kappa \in (-1/2, 1/\sqrt{2})\)) correspond to virtual levels at thresholds \(\pm i(m - |\omega|)\). The regions between the Kolokolov and virtual level lines (dots on the plot) correspond to two simple purely imaginary eigenvalues \(\pm i\Lambda\) in the spectral gap. The values \(|\omega| \geq m/3\) at \(\kappa = 0\) correspond to embedded eigenvalues \(\pm 2\omega i\). Besides eigenvalues mentioned on this plot, there is always eigenvalue \(\lambda = 0\).](image)

**Proof.** Part (1) in the case \(\omega \neq 0\) follows from Lemma 3.7 (5) and from Lemma 4.2 in the case \(\omega = 0\). Moreover, in the case \(\omega \neq 0\), one can arrive at the same conclusion from considering the function \(D_{\omega, \kappa}(\lambda)\). We have:

\[
D_{\omega, \kappa}(\lambda) = (\alpha - 2\nu_+)(\alpha - 2\nu_-)(1 + \kappa \Re Q(i\lambda)).
\]
We notice that
\[ \alpha - 2\nu_\pm = 2(\kappa - \sqrt{\kappa^2 + \lambda \pm 2i\lambda}) = O(\lambda), \quad \lambda \to 0. \]

Moreover, by (4.23),
\[ Q(i\lambda) = -\frac{1}{\kappa} - \frac{\kappa}{\omega^2(1 + \frac{\lambda}{\kappa})} + O(\lambda^2) = -\frac{1}{\kappa} - \frac{\kappa}{\omega^2} + O(\lambda^2) = -\frac{m^2}{\kappa\omega^2} + O(\lambda^2), \quad |\lambda| < 2|\omega|, \]
therefore,
\[ 1 + \kappa\omega Q(i\lambda) = 1 - \frac{\kappa m^2}{\omega^2} + O(\lambda^2) = \left\{ \begin{array}{ll}
O(1), & \omega \neq \Omega_\kappa, \\
O(\lambda^2), & \omega = \Omega_\kappa, \quad |\lambda| \leq 2|\omega|.
\end{array} \right. \]

It follows that if \( \omega = \Omega_\kappa \), then \( \lambda = 0 \) is a root of (4.24) of order four.

Part (2) follows from Lemma 4.2, and Part (3) follows from Lemma 3.7 (3). Let us prove Part (4).

By Part (1), for \( \omega = \Omega_\kappa \), \( \sigma_p(A(\omega, \kappa)) = \{0\} \) of algebraic multiplicity 4. For \( \omega \geq \Omega_\kappa \), the two nonzero eigenvalues start moving away from the origin, becoming purely real for \( \omega < \Omega_\kappa \) and purely imaginary for \( \omega > \Omega_\kappa \). These two purely imaginary eigenvalues hit the threshold points \( \pm i(m - |\omega|) \) at some values \( \pm T_\kappa \), with \( \Omega_\kappa < T_\kappa < m \), becoming virtual levels (see Lemma 4.1), since by Part (1), there are no eigenvalues at this threshold. It remains to note that \( \omega = T_\kappa \) corresponds to \( \kappa = K_\omega \). Indeed, solving (4.12), we get
\[ \kappa = K_\omega = \frac{1}{2} \left( \sqrt{\frac{\omega^2}{m^2} + \frac{\omega}{m}} - 1 \right) = \frac{2\left(\frac{|\omega|}{m+|\omega|}\right)^{1/2} - 1}{2 - 2\left(\frac{|\omega|}{m+|\omega|}\right)^{1/2}}. \]
The condition for \( T_\kappa \) to be smaller than \( m \) (so that the root of \( D_{\omega,\kappa}(\lambda) \) indeed arrives at \( i(m - |\omega|) \) when \( \omega = \pm T_\kappa \in (-m, m) \)) is \( (1 + 2\kappa)^2 < 3 + 4\kappa \), which gives the requirement \( \kappa \in (-\frac{1}{2}, \sqrt{2}) \).

Part (5) follows from Lemma 4.4. The absence of nonzero roots for \( \kappa < K_\omega \), \( \kappa \neq 0 \), is explained by the fact that that when we passing the thresholds \( \kappa = K_\omega \) the nonzero root \( \lambda(\omega, \kappa) \) of \( D_{\omega,\kappa}(\lambda) \) moves onto the unphysical sheet of the Riemann surface of \( D_{\omega,\kappa}(\lambda) \). Recall that in the case \( \kappa = 0 \), \( |\omega| \geq m/3 \) the point spectrum also contains embedded eigenvalues \( \pm 2\omega i \) by Lemma 4.11).

This completes the proof of Theorem 4.5. \( \square \)

A Appendix: Reducing \( D_{\omega,\kappa}(\lambda) = 0 \) to a cubic with explicit solution

For the completeness, we provide an explicit solution to the equation \( D_{\omega,\kappa}(\lambda) = 0 \), with \( D_{\omega,\kappa}(\lambda) \) from (4.10), which we can cast as a cubic equation. Albeit explicit, this solution does not yield the spectral properties of \( A(\omega, \kappa) \) as readily as the methods employed in Section 4; yet this solution enables, for example, the analysis of large \( \kappa \) asymptotics of roots of \( D_{\omega,\kappa}(\lambda) \) on physical and unphysical sheets of the Riemann surface of \( D_{\omega,\kappa}(\lambda) \). Denote \( \Sigma = \nu_+ + \nu_- \), with \( \nu_\pm = \nu_\pm(\omega, \lambda) \) from (4.5); one has
\[ -2\nu_+\nu_- = -\Sigma^2 + (\nu_+^2 + \nu_-^2) = -\Sigma^2 + (2m^2 - 2\omega^2 + 2\lambda^2) = -\Sigma^2 + 2\lambda^2 + \alpha^2/2. \]
Then (4.9) takes the form
\[ \alpha^2(1 + \kappa)^2 - 2\alpha(1 + \kappa)\Sigma - 2(-\Sigma^2 + 2\lambda^2 + \alpha^2/2) - \alpha^2\kappa^2 = 0 \]
and then \( \Sigma^2 - \alpha(1 + \kappa)\Sigma + (\alpha^2\kappa - 2\lambda^2) = 0 \). Equation (A.1) implies
\[ \Sigma = \alpha(1 + \kappa) \pm \sqrt{\alpha^2(1 - \kappa)^2 + 8\lambda^2}/2. \]
Further, from (A.1) and (A.2),
\[
-2\nu_+\nu_- = -\Sigma^2 + 2\lambda^2 + \alpha^2/2 = \alpha^2(1 + \kappa)^2/2 - \alpha^2\kappa^2/2 - \alpha(1 + \kappa)\Sigma
\]
\[
= \frac{\alpha^2(1 + \kappa)^2}{2} - \alpha^2\kappa^2/2 - \alpha(1 + \kappa)\left(\frac{(1 + \kappa) \pm \sqrt{\alpha^2(1 - \kappa)^2 + 8\lambda^2}}{2}\right)
\]
\[
= -\frac{\alpha^2\kappa^2}{2} \pm \alpha(1 + \kappa)\sqrt{\alpha^2(1 - \kappa)^2 + 8\lambda^2},
\]
so
\[
\nu_+^2\nu_-^2 = \frac{\alpha^4\kappa^4}{16} + \alpha^2(1 + \kappa)^2/16 - \alpha^2(1 - \kappa)^2 + 8\lambda^2) \pm \frac{\alpha^3(1 + \kappa)^2\kappa^2}{8} \sqrt{\alpha^2(1 - \kappa)^2 + 8\lambda^2}. \tag{A.3}
\]
On the other side, the definition of \(\nu_{\pm}\) (4.5) implies that
\[
\nu_+^2\nu_-^2 = ((\omega + i\lambda)^2 - m^2)((\omega - i\lambda)^2 - m^2) = ((\omega - m)^2 + \lambda^2)((\omega + m)^2 + \lambda^2)
\]
\[
= (\omega^2 - m^2)^2 + \lambda^4 + 2\lambda^2(\omega^2 + m^2) = \frac{\alpha^4}{16} + \lambda^4 + 4\lambda^2m^2 - \frac{\lambda^2\kappa^2}{2}. \tag{A.4}
\]
From (A.3) and (A.4), simplifying and denoting \(x = \lambda^2\), we obtain:
\[
x^2 + 4xm^2 - x\alpha^2 - \frac{\alpha^4\kappa^4}{8} + \frac{\alpha^4\kappa^2}{8} - \frac{x\alpha^2\kappa^2}{2} - x\alpha^2\kappa = \pm \frac{\alpha^3(1 + \kappa)^2\kappa^2}{8} \sqrt{\alpha^2(1 - \kappa)^2 + 8x}. \tag{A.5}
\]
Defining
\[
c(\omega, \kappa) = 4m^2 - \alpha^2 - \alpha^2\kappa - \alpha^2\kappa^2/2 \tag{A.6}
\]
and squaring (A.5) yields
\[
\left(\frac{x^2 + cx + \frac{\alpha^4\kappa^2(1 - \kappa)^2}{8}}{4}\right)^2 = \frac{\alpha^8(1 - \kappa)^2\kappa^4}{64} + \frac{\alpha^6(1 + \kappa)^2\kappa^4 x}{8}.
\]
There is a root \(x = 0\). To find nonzero roots, we simplify the above and cancel \(x\), arriving at
\[
x^3 + 2cx^2 + x\left[c^2 + \frac{\alpha^4\kappa^2(1 - \kappa^2)}{4}\right] + c^2\frac{\alpha^4\kappa^2(1 - \kappa^2)}{4} - \frac{\alpha^6(1 + \kappa)^2\kappa^4}{8} = 0. \tag{A.7}
\]
Writing
\[
x = y - 2c/3, \tag{A.8}
\]
we reduce the cubic equation (A.7) to the form
\[
y^3 + py + q = 0 \tag{A.9}
\]
with
\[
p = -\frac{c^2}{3} + \frac{\alpha^4\kappa^2(1 - \kappa^2)}{4}, \quad q = -\frac{2c^3}{27} + \frac{\alpha^4\kappa^2(1 - \kappa^2)}{12} - \frac{\alpha^6(1 + \kappa)^2\kappa^4}{8}. \tag{A.10}
\]
If the discriminant
\[
\Delta(\omega, \kappa) = -4p^3 - 27q^2 \tag{A.11}
\]
with \(c = c(\omega, \kappa)\) from (A.6), is negative, then there is exactly one real root \(y \in \mathbb{R}\) of equation (A.9) (and two complex conjugate roots with nonzero imaginary part). The real root is given by
\[
y_1 = \left( -\frac{q}{2} + \frac{(-\Delta)^{1/2}}{108} \right)^{1/3} + \left( -\frac{q}{2} - \frac{(-\Delta)^{1/2}}{108} \right)^{1/3}, \text{ and then (A.8) yields}
\]
\[
x_1 = \left( -\frac{q}{2} + \frac{(-\Delta)^{1/2}}{108} \right)^{1/3} + \left( -\frac{q}{2} - \frac{(-\Delta)^{1/2}}{108} \right)^{1/3} - \frac{2c}{3}. \tag{A.12}
\]
Let us mention that roots of (A.9) with nonzero imaginary part correspond to roots \( \lambda = \pm \sqrt{y - 2c^3/3} \) of \( D_{\omega, \kappa}(\lambda) \) with nonzero real and imaginary parts. By Lemma 3.7 (3), these roots do not correspond to eigenvalues of \( A(\omega, \kappa) \).

We are going to show that the discriminant \( \Delta(\omega, \kappa) \) is negative either if \( |\omega| < m \) is sufficiently close to \( m \) or if \( |\kappa| \) is sufficiently large, so that there is exactly one real solution to (A.9) and therefore exactly one pair of roots to \( D_{\omega, \kappa}(\lambda) = 0 \) which are located on the Riemann sheet corresponding to eigenvalues of \( A(\omega, \kappa) \) (that is, when the real parts of \( \nu_+ \) and \( \nu_- \) from (4.5) are positive).

**Lemma A.1.**

1. There is \( \kappa_0 > 0 \) such that \( \Delta(\omega, \kappa) < 0 \) for \( |\kappa| > \kappa_0 \).

2. For each \( \kappa \in \mathbb{R} \setminus \{-1, 0\} \), there is \( \omega_0 \in (0, m) \) such that \( \Delta(\omega, \kappa) < 0 \) for \( |\omega| \in (\omega_0, m) \).

**Proof.** Using (A.10), one computes:

\[
\Delta(\omega, \kappa) = -4p^3 - 27q^2 = -\frac{a^6\kappa^4(1 + \kappa)^2}{2} \left( c^3 - \frac{c^2\alpha^2(1 - \kappa)^2}{8} - \frac{9\alpha^4\kappa^2(1 - \kappa^2)}{8} + \frac{\alpha^6\kappa^2(1 - \kappa^2)(1 - \kappa^2)}{8} + \frac{27\alpha^6(1 + \kappa)^2\kappa^4}{32} \right).
\]

Since \( c = 4m^2 - \alpha^2(1 + \kappa + \kappa^2/2) \), one concludes that, for each fixed \( \kappa \in \mathbb{R} \) different from \(-1\) and \(0\), \( \Delta(\omega, \kappa) < 0 \) if \( \alpha = 2\sqrt{m^2 - \omega^2} \) is sufficiently small (that is, if \( |\omega| \) is sufficiently close to \( m \)).

Alternatively, one can keep the highest order powers of \( \kappa \), substituting \( c \sim -\alpha^2 \kappa^2/2 \) and getting

\[
\Delta(\omega, \kappa) \sim -\alpha^6 \kappa^6 \left( -\frac{4\alpha^6 \kappa^6}{16} + \frac{\alpha^6 \kappa^6}{16} - \frac{2\alpha^2 \kappa^2}{16} + \frac{2\alpha^6 \kappa^6}{64} \right) = -\frac{16\alpha^6 \kappa^6}{8},
\]

showing that \( \Delta(\omega, \kappa) \) is strictly negative for \( |\kappa| \) large enough. \(\square\)

**Remark A.2.** Let us note that for \( |\kappa| \to \infty \), one has:

\[
c \sim -\frac{\alpha^2 \kappa^2}{2}, \quad (-\Delta)^{1/2} \sim \frac{\alpha^6 \kappa^6}{2\sqrt{2}}, \quad q \sim -\frac{2}{27}c^3 - \frac{\alpha^6 \kappa^6}{8} \sim -\frac{29}{27} \frac{\alpha^6 \kappa^6}{8},
\]

and (A.12) yields

\[
x_1 \sim \left( \frac{\alpha^6 \kappa^6}{8} + \frac{1}{108} \frac{\alpha^6 \kappa^6}{2\sqrt{2}} \right)^{1/3} + \left( \frac{\alpha^6 \kappa^6}{8} - \frac{1}{108} \frac{\alpha^6 \kappa^6}{2\sqrt{2}} \right)^{1/3} + \frac{2\alpha^2 \kappa^2}{3} \sim c\alpha^2 \kappa^2,
\]

with some \( c > 1 \), showing that \( \lambda_{1,2} = \pm \sqrt{x_1} \) are real. For \( \kappa > 0 \) these values \( \lambda_{1,2} \) are exactly the two real nonzero eigenvalues of \( A(\omega, \kappa) \) which appear in Theorem 4.5 (3). For \( \kappa < 0 \), since \( A(\omega, \kappa) \) has no real eigenvalues (Lemma 3.7 (3)), \( \lambda_{1,2} \) correspond to resonances of \( A(\omega, \kappa) \) (that is, to zeros of (4.10) on one of the unphysical Riemann sheets of \( D_{\omega, \kappa}(\lambda) \)).

The expression (A.12) can be used to analyze zero eigenvalues of \( A(\omega, \kappa) \).

**Lemma A.3.** \( x_1 = 0 \) in the following cases: \( \kappa = -1, \kappa = 0, \) and \( \kappa = \omega^2/m^2 \).

**Proof.** Since \( x_1 = 0 \) corresponds to \( y_1 = 2c/3 \) (see (A.8)), we need to solve

\[
(2c/3)^3 + (2c/3)p + q = 0.
\]

Substituting \( c \) from (A.6), \( p, q \) from (A.10), and simplifying, one arrives at

\[
\left( 4m^2 - (1 + \kappa + \kappa^2/2)\alpha^2 \right) \frac{\alpha^4 \kappa^2(1 - \kappa^2)}{4} - \frac{\alpha^6(1 + \kappa)^2\kappa^4}{8} = 0.
\]

Canceling common factors (corresponding to solutions \( \kappa = -1 \) and \( \kappa = 0 \)), one obtains \( \omega^2 = km^2 \). \(\square\)

**Remark A.4.** The value \( \kappa = \omega^2/m^2 \) is in agreement with the value of \( \Omega_\kappa \) from (3.18). By Lemma 3.7, we know that only the cases \( \kappa = 0 \) and \( \kappa = \omega^2/m^2 \) correspond to higher algebraic multiplicity of eigenvalue \( \lambda = 0 \), while the case \( \kappa = -1 \) has to correspond to resonances: one can see from (4.10) that in this case the values \( \lambda = 0 \) are located on the Riemann sheet of \( D_{\omega, \kappa}(\lambda) \) characterized by \( \nu_+ < 0 \).
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