NORMAL CR STRUCTURES ON COMPACT 3-MANIFOLDS

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Abstract. We study normal CR compact manifolds in dimension 3. For a choice of a CR Reeb vector field, we associate a Sasakian metric on them, and we classify those metrics. As a consequence, the underlying manifolds are topologically finite quotiens of $S^3$ or of a circle bundle over a Riemann surface of positive genus. In the latter case, we prove that their CR automorphisms group is a finite extension of $S^1$, and we classify the normal CR structures on these manifolds.

1. Introduction

Analogs of complex manifolds in odd dimensions, pseudo-conformal CR manifolds are particular contact manifolds, with a complex structure on the corresponding distribution of hyperplanes, satisfying an integrability condition (see Section 2). Contrary to complex geometry, CR geometry is locally determined by a finite system of local invariants (like in the cases of conformal or projective structures), \[17\], \[9\], \[16\]. Therefore the space of locally non-isomorphic CR structures is a space with infinitely many parameters.

In this paper, we focus our attention on normal CR manifolds, which admit global Reeb vector fields preserving the CR structure, in particular their CR automorphisms group has dimension at least 1. Our main result is that, for a compact normal CR 3-manifold, which is topologically not a quotient of $S^3$, this CR automorphisms group is a finite extension of a circle, thus the Reeb vector field is unique up to a constant (Section 4, Theorem 2). This, together with the classification of Sasakian compact 3-manifolds (see Section 3), allows us to obtain the classification of normal CR structures on these manifolds (Section 4, Corollary 4).

The question of classifying compact CR manifolds has first been solved in situations with a high order of local symmetry: the classification of flat compact CR manifolds, where the local CR automorphism group is $PSU(n+1,1)$ (if the manifold has dimension $2n+1$), is due to E. Cartan \[3\] and to D. Burns and S. Shnider \[2\]; in dimension 3, homogeneous, simply-connected, CR manifolds are either flat or (3-dimensional) Lie groups, and have been classified by E. Cartan \[3\] (see also \[3\]). In this case, there is no intermediate symmetry because E. Cartan has showed that a homogeneous CR manifold whose CR automorphism group has dimension greater than 3 is automatically flat.

In dimension 3, the normal CR structures are always deformations of a flat one (Theorem 1, see also \[1\]), and the key point is that, for a CR Reeb vector field $T$, they admit compatible Sasakian metrics, for which $T$ is
Killing (see Section 3 for details); these metrics are closely related to locally conformally Kähler metrics with parallel Lee form, natural analogs of Kähler structures on non-symplectic complex manifolds [11].

Topologically, every compact normal CR (or Sasakian) 3-manifold is a Seifert fibration (Proposition 5, see also [8], [7] and [1]), but it turns out that the Sasakian structures themselves can be explicitly described on these manifolds: Theorem 1, Proposition 5 (these are extended versions of the results announced in [1]); In particular, if a compact Sasakian 3-manifold is not covered by $S^3$ (or non-spherical), its Sasakian structure is always regular, i.e. it is a (finite quotient of a) circle bundle over a Riemann surface and its metric arises from a Kaluza-Klein construction (Corollary 2), which leads to an elementary description of any Sasakian metric on these manifolds, either directly, or as deformations of a CR flat one.

Although the classification of Sasakian structures on compact 3-manifolds is complete in all cases (the explicit description in the case of finite quotients of $S^3$ is more elaborate, but still possible, see Section 3), the question of classifying the associated CR structures (which are normal) is more subtle. It is solved in Section 4 for non-spherical manifolds, and is still open for $S^3$.

Acknowledgements. The author is grateful to P. Gauduchon for his constant support during the last few years.

2. CR geometry on 3-manifolds and Sasakian structures

Let $M$ be a $2n + 1$-dimensional manifold. In all generality, a CR structure on $M$ is a field of complex structures on a field of hyperplanes of $M$. But, as this concept is inspired by the structure of a real hypersurface in a complex manifold, one generally searches for CR structures satisfying the conditions described below:

Let $Q$ be a field of hyperplanes in a $2n + 1$-dimensional real manifold $M$, then the Levi form $L^Q : Q \times Q \to TM/Q$ is defined by $L(X, Y) := [X, Y]$ mod $Q$. Let $J \in \text{End}(Q)$, $J^2 = -\mathbb{I}$ be an almost-complex structure on $Q$. We say that the Levi form is non-degenerate iff $L^Q(X, \cdot) \in \text{Hom}(Q, TM/Q)$ is non-zero for any non-zero $X \in Q$, and that it is $J$-invariant iff $L^Q(J \cdot, J \cdot) = L^Q(\cdot, \cdot)$. If the Levi form is $J$-invariant, the Nijenhuis tensor of $J$ is defined as a linear map $N : \Lambda^2 Q \to Q$, by:

$$4N(X, Y) = [JX, JY] - J[JX, Y]^Q - J[X, JY]^Q - [X, Y],$$

where $Z^Q$ denotes the component in $Q$ of the vector $Z$ — by means of a non-canonic linear projection —; the Nijenhuis tensor is independent of this projection.

Convention We call a tensor any multi-linear object defined on subspaces or/and quotients of $TM$ (namely $Q$ and $TM/Q$). For example, $L^Q : Q \times Q \to TM/Q$ is a tensor.

Definition 1. Let $M$ be a odd-dimensional connected real manifold. A CR structure on $M$ is a field of hyperplanes $Q$, with an almost complex structure $J \in \text{End}(Q)$, such that the Levi form $L^Q$ is $J$-invariant. The CR structure
J on M is called formally integrable if the Nijenhuis tensor vanishes identically; it is called pseudo-conformal if the Levi form is non-degenerate, and pseudo-convex if $L^Q(J\cdot,\cdot)$ is a positive definite (symmetric) 2-form on Q.

Some authors consider only (formally) integrable CR structures; this is because only these can arise as real hypersurfaces in a complex manifold (in which case we call them integrable). Note that, in the $C^\infty$ case, the vanishing of the Nijenhuis tensor does not necessarily imply integrability (the analog of the Newlander-Nirenberg theorem holds only for analytic CR manifolds) [13], [14].

We consider only pseudo-conformal CR structures: pseudo-conformal geometry is, like projective or conformal geometry, a semisimple G-structure (or a parabolic geometry) [16], [17], in particular they admit a unique Cartan connection, whose curvature (see below) locally characterizes the geometry of a pseudo-conformal manifold; another consequence of this is that the group of diffeomorphisms of M preserving a given pseudo-conformal structure is a Lie group [9].

If $(M,Q,J)$ is a pseudo-conformal manifold, then $Q$ is a contact structure on $M$, i.e. $Q$ is (locally) the kernel of a 1-form $\eta : TM \to L$ with values in a real line bundle, such that $\eta \wedge d\eta$ does not vanish. We consider only orientable manifolds $M$, such that $L$ is also orientable (hence topologically trivial), and then $\eta \wedge d\eta$ is a volume form on $M$. Obviously, to each contact form $\eta$ we can uniquely associate a Reeb vector field $T$ such that $\eta(T) = 1$ and $d\eta(T,\cdot) = 0$ everywhere (in particular, the Lie derivative of $\eta$ along $T$ vanishes: $\mathcal{L}_T \eta = 0$).

The hyperplane $Q$ admits a natural conformal- (pseudo-) Hermitian structure, represented by the (pseudo-)Hermitian symmetric forms $h := -\frac{1}{2}d\eta(J\cdot,\cdot)$ on $Q$, for any contact form $\eta$ (the factor $\frac{1}{2}$ is useful when we consider Sasakian metrics — see below).

The choice of a contact form $\eta$ (and of its Reeb vector field $T$) yields the Tanaka-Webster connection $\nabla$, defined as follows [18], [20]:

1. $T$, $Q$, and $J$ are parallel with respect to $\nabla$;
2. If $\tau(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ is the torsion of $\nabla$, then $\tau(X,Y) = d\eta(X,Y)T, \forall X,Y \in Q$;
3. If $\tilde{\tau}(X) := \tau(T,X), X \in Q$, then $\tilde{\tau} : Q \to Q$ is $J$-anti-invariant ($\tilde{\tau}(JX) = -J\tilde{\tau}(X)$).

We remark that this connection preserves the CR structure, and it has minimal torsion. $\tilde{\tau}$ cannot vanish unless $T$ is a CR Reeb vector field, i.e. the diffeomorphism group generated by $T$ on $M$ (that already preserves $Q$, as $T$ is a Reeb vector field) preserves $J$, i.e. $\mathcal{L}_T J = 0$.

**Definition 2.** The pseudo-conformal structure of $M$ is called normal iff it admits a CR Reeb vector field.

In particular, the dimension of the Lie group of pseudo-conformal automorphisms of a normal CR manifold is at least 1.

If $T'$ is another Reeb vector field on $M$, then the corresponding contact form $\eta'$ is equal to $f^{-1}\eta$, for a positive function $f : M \to \mathbb{R}$, and

$$T' = fT + X_f,$$
where \( X_f \in Q \) is determined by the fact that \( \mathcal{L}_T \eta' = 0 \), thus

\[
d\eta(X_f, \cdot) = -df(X) \iff X_f = -\frac{1}{2}(df|_Q \circ J)^\sharp = \frac{1}{2}J(df|_Q)^\sharp,
\]

where the “rising of indices” \( df^\sharp \) is made with respect to the Hermitian metric \( h \) on \( Q \). Then \( \mathcal{L}_T J = 0 \) iff \( [T', JX] - J[T', X] = 0, \forall X \in Q \), thus

\[-\nabla_J XT' - d\eta(X_f, JX)T + J(\nabla_X T' + d\eta(X_f, X)T) = 0,\]

where \( \nabla \) is the Tanaka-Webster connection corresponding to \( \eta \). We get then

\[-J\nabla_JX(df|_Q)^\sharp - \nabla_X(df|_Q)^\sharp = 0,\]

which leads to (see also [3]):

**Proposition 1.** If \( f \) is a function relating \( 2 \) CR Reeb vector fields \( T \) and \( T' = fT + X_f, X_f \in Q \), then the Hessian of \( f \), restricted to \( Q \) (defined by \( \text{Hess}^Q f(X, Y) := X.Y.f - \nabla_XY.f \)), is \( J \)-invariant \((\text{with respect to the Tanaka-Webster connection} \nabla \text{ on} \ M \text{ induced by the contact form} \ \eta)\):

\[\text{Hess}^Q f(X, Y) = \text{Hess}^Q f(JX, JY), \forall X, Y \in Q.\]

In particular, if \( \dim M = 3 \), the above condition means that \( \text{Hess}^Q f \) is a scalar multiple of \( h \).

### 2.1. Sasakian geometry.

A Sasakian structure on an odd-dimensional manifold \( M \) is a Riemannian metric \( g \) on \( M \) and a unitary Killing vector field \( T \) such that \( \nabla_T T = 0 \) and \( \nabla \bullet T : Q \to Q \) \((\text{where} \ Q := T^\perp)\) is an almost complex structure \( J \) \((\text{compatible with the metric as it is an anti-symmetric endomorphism})\). It is easy to see that \( T \) preserves \( \nabla T \), as it is Killing, so a Sasakian structure on \( M \) is a special case of a normal (pseudo-convex) CR structure on \( M \) \((\text{we remark that} \ T, \text{followed by a basis of} \ Q, \text{oriented by} \ J, \text{yield to an orientation of} \ M)\).

Actually, if \( \dim M = 3 \), then any CR Reeb vector field \( T \) of a normal CR structure on \( M \) yields a unique Sasakian structure:

**Proposition 2.** If \( T \) is a CR Reeb vector field on the \( 3 \)-manifold \( M \), then there is a unique Sasakian metric \( g \) on \( M \) for which \( T \) or \( -T \) is the corresponding Killing vector field.

**Proof.** As \( T \) preserves \( Q \) and \( J \), it \((\text{and all its multiples by a constant})\) preserves the Riemannian metric defined by

\[
g := \eta^2 - \frac{1}{2}d\eta(J\bullet, \cdot),
\]

\((\text{we replace, if necessary,} \ T \text{ with} -T, \ \eta \text{ with} -\eta, \text{ such that} \ g \text{ is positive definite}) \text{ with respect to which} \ T \text{ is Killing and} \ \nabla_T T = 0. \ \nabla T \text{ is then identified to a} \ J \text{-invariant, anti-symmetric endomorphism of} \ Q, \text{ thus equal to} \ fJ, \text{ for a function} \ f. \text{ But}
\]

\[d\eta(X, Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = -2g(fJX, Y), \forall X, Y \in Q,
\]

as \( \eta = g(T, \cdot) \), thus \( f \equiv 1 \) \( \Box \)

**Remark.** For a general, pseudo-conformal manifold \( M^{2n+1} \), let \( (p, q) \), \( p \le q \), \( p + q = n \), be the signature of the Hermitian form \( h \). Then the flat model of the pseudo-conformal geometry of signature \( (p, q) \) is the homogeneous
space $PSU(p + 1, q + 1)/H^{p,q}$, where $H^{p,q}$ is the isotropy subgroup of the point $[1 : 0 : \ldots : 0] \in \mathbb{C}^{p+1}$, for the action of $PSU(p + 1, q + 1)$ on the real hypersurface $M^{p,q}$ of $\mathbb{C}^{p+1}$ defined by the equation

$$x_1\bar{x}_1 + \ldots + x_p\bar{x}_p - x_{p+1}\bar{x}_{p+1} - \ldots - x_{p+q}\bar{x}_{p+q} + x_{n+1} = 0.$$ 

It turns out that it exists a canonical $H^{p,q}$-bundle $P$ over any pseudo-conformal manifold $M^{2n+1}$ (whose Hermitian structure on $Q$ has signature $(p,q)$), and a canonical Cartan connection $\omega : TP \to psu(p+1,q+1)$ (where the latter is the Lie algebra of the above mentioned group) [17], [4]. Its curvature measures the obstruction to the construction of a local diffeomorphism $\Xi$ would induce a group structure on the universal covering of $P$, locally isomorphic to $PSU(p + 1, q + 1)$, and it locally determines the pseudo-conformal structure, see Tanaka [17]; see also [18], [4]; see [9] for a general theory of Cartan connections, and [16] for a general theory of simple graded Lie algebras and $G$-structures.

The curvature of the Cartan connection is identified, if $n > 1$, to the pseudo-conformal tensor of Chern and Moser [4], which is a component of the curvature of any Tanaka-Webster connection [17], [20]. It is the equivalent of the Weyl tensor in conformal geometry [17]. In dimension 3, i.e. $n = 1$, this tensor vanishes identically and the curvature of the Cartan connection is determined by another tensor (see the convention above), called the Tanaka curvature, see [17], page 187, Theorem 12.3.

In the case when the Tanaka-Webster connection $\nabla$ corresponds to a positive Reeb vector field $T$ (i.e. $h(X, X) = -\frac{1}{2}\eta(JX, X) > 0$, $\forall X \in Q$, $X \neq 0$), we denote by $k$ the sectional curvature of the plane $Q$: $k := h(R(X, JX) X, JX)$, for $X \in Q$, $h(X, X) = 1$. The Tanaka curvature is then defined as the tensor $\Phi : S^2Q \to TM/Q$ that satisfies:

1. $\Phi$ is trace-free, i.e. $\Phi(X, X) + \Phi(JX, JX) = 0$;
2. $\Phi(X, X)(T) = -kh(\tilde{\tau}(X), X) + 2h(\nabla_{JX}\tilde{\tau})(X), JX) - \frac{1}{2}\nabla_X \nabla_{JX} - \nabla_{XJX} + \nabla_{JX} \nabla_X - \nabla_{JXJX} k - 4h(\nabla_{\tilde{\tau}}(X), X),$

for any $X \in Q$; the Laplacian $\Delta$ is defined as

$$\Delta \sigma := (\nabla_Y \nabla_{JY} - \nabla_{YJ} \nabla_Y - \nabla_{JY} \nabla_{JY} - \nabla_{JY}(JY)) \sigma,$$

for a unitary $Y \in Q$, and a tensor $\sigma \in \text{End}(Q)$. $\Phi$ is independent of the Reeb field $T$, and of the associated connection $\nabla$ [17].

**Remark.** The Tanaka curvature is the analog of the Cotton-York tensor from conformal geometry; however, the terms contained in the expression of $\Phi$ change by terms involving up to 4th order derivatives of $f$, for a change of the Tanaka-Webster connection determined by $\eta' = f^{-1}\eta$. The invariance, in dimension 3, of the Tanaka curvature is, thus, a highly non-trivial fact.

We denote by $\Box_X$ the second order differential operator on functions $f : M \to \mathbb{R}$, acting as

$$\Box_X f := X.JX.f + JX.X.f - \nabla_X.JX.f - \nabla_{JX}X.f,$$

and it is obvious that $\Box_X$ depends quadratically on $X \in Q$. Then, if $T$ is a positive $CR$ Reeb vector field, then the above expression for $\Phi$ becomes a
lot simpler:
\[
\Phi(X, X)(T) = -\frac{1}{2} \Box_X k, \ \forall X \in Q.
\]

**Proposition 3.** The sectional curvature of the plane \( Q \) in \( M \), with respect to the Sasakian structure induced by \( T \), is equal to \(-2k - 3\).

**Proof.** We denote by \( \nabla \), resp. \( \nabla^0 \), the Tanaka-Webster connection, resp. the Levi-Civita (Sasakian) connection associated to the positive CR Reeb vector field \( T \). We have
\[
\begin{align*}
\nabla_T T - \nabla^0_T T &= 0; \\
\nabla_X T - \nabla^0_X T &= -JX, \ \forall X \in Q; \\
\nabla_T X - \nabla^0_T X &= -JX, \ \forall X \in Q; \\
\nabla_X Y - \nabla^0_X Y &= \frac{1}{2} T d\eta(X, Y) = T g(JX, Y), \ \forall X, Y \in Q.
\end{align*}
\]
The claimed result now follows from a straightforward computation. \( \square \)

On the other hand, if we define the operator \( \Box^0_X \) as in (3), by replacing \( \nabla \) with \( \nabla^0 \), we have
\[
\Box_X f = \Box^0_X f, \ \forall f : M \to \mathbb{R}, \ \forall X \in Q.
\]

### 2.2. Regular Sasakian structures.

A Sasakian structure on a compact 3-manifold is called regular if the Reeb vector field \( T \) is induced by a free circle action on \( M \). In that case, there is a \( S^1 \) fibration \( M \xrightarrow{\pi} \Sigma \) of \( M \) over a Riemann surface \( \Sigma \), \( T \) is tangent to the fibers, and \( Q \) is a connection in the principal bundle \( M \to \Sigma \), of connection form \( i\eta : TM \to i\mathbb{R} \) (where \( i\mathbb{R} \) is the Lie algebra of \( S^1 \simeq U(1) \)), and of curvature form \( i d\eta \). On the other hand, \( T \) is a Killing vector field, thus the metric \( g \) on \( M \) induces a Riemannian metric \( g^0 \) on \( \Sigma \), compatible with the induced complex structure \( J \); the Kähler form on \( \Sigma \) is then \( \omega = g^0(J\cdot, \cdot) \), thus \( \pi^* \omega = \frac{1}{2} d\eta \). Given such a metric on \( \Sigma \) and a connection, the metric constructed as above (the horizontal space of the connection is defined to be orthogonal to the vertical — this procedure is usually called a [Kaluza-Klein construction](#)) on \( M \) is Sasakian.

**Remark.** It turns out then that the Chern class of the \( S^1 \)-bundle \( M \to \Sigma \) is always positive: this is because we chose \( T \) to be positive (see [1], Section 3, for a detailed explanation); in particular, we obtain a positive Chern class for the Hopf fibration \( S^3 \to S^2 \), apparently contradictory to the negative Chern class of the tautological bundle \( \mathcal{O}(-1) \) on \( \mathbb{C}P^1 \) \((\mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^1)\); this is because the canonical metric on \( S^3 \) is a Sasakian structure with the opposite orientation.

**Remark.** All normal CR compact 3-manifolds are covered by circle bundles over a Riemann surface [1], [8], [1], see also next Section, and, if this circle bundle is not the Hopf fibration \( S^3 \to S^2 \), then all Sasakian structures are, up to a finite quotient, regular [1], see also next Section). If \( M \) is covered by \( S^3 \), then any Sasakian structure on \( M \) is a deformation of a regular one [1], see next Section. Therefore the study of these particular Sasakian structures is essential to the understanding of compact normal CR 3-manifolds.

As a direct consequence to Proposition [3], we have, in this case:
Corollary 1. Let $M$ be a compact regular Sasakian 3-manifold, thus $M \xrightarrow{\pi} \Sigma$ is a $S^1$-bundle over a Riemann surface $\Sigma$. Then $-k$ is equal to the (Gaussian) curvature of $\Sigma$.

The reason for this is that, if $\nabla^\Sigma$ is the Levi-Civita connection on $\Sigma$, $\nabla$ is the Tanaka-Webster connection on $M$, and $\tilde{X}$ denotes the horizontal lift (in $Q$) of a vector $X \in T\Sigma$, we have:

$$\tilde{\nabla}^\Sigma_X Y = \nabla_{\tilde{X}} \tilde{Y}, \forall X, Y \in T\Sigma.$$ 

We also get, from the above equality, that $\Box_X f = \Box^\Sigma_X f$, for a function $f$ constant on the fibers of $\pi$ (where $\Box^\Sigma_X$ is defined by a relation analogous to (3)), so the Tanaka curvature (4) has a particularly simple expression in this case:

Proposition 4. Let $M$ be a compact regular Sasakian 3-manifold, fibered over a Riemann surface $\Sigma$ of positive genus. Then the CR structure of $M$ is flat iff $\Sigma$ has constant curvature.

Proof. We have to prove that, if $k$ is the Gauss curvature of the Riemann surface $\Sigma$, then it satisfies $\Box_X k = 0, \forall X \in T\Sigma$ iff $k$ is constant (we omit the indices referring to $\Sigma$, as we only use the metric, the Levi-Civita connection and the operator $\Box_X$ on $\Sigma$ in this proof). First we prove the following fact: if $f : \Sigma \to \mathbb{R}$, then

$$\Box_X f = 0, \forall X \in T\Sigma \iff J(df)^2 \text{ is Killing.} (6)$$

We need to prove that $\nabla J(df)^2$ is anti-symmetric, thus it is enough to check that $g(\nabla_X J(df)^2, X) = 0, \forall X \in T\Sigma$, but this is equal to

$$-X.JX.f + \nabla X JX.f = \frac{1}{2}(\Box_X f + (X.JX - JX.X - \nabla X JX + \nabla JX X).f) = \frac{1}{2} \Box_X f.$$

If the genus of $\Sigma$ is greater that 1, it admits no non-zero Killing vector field. If $\Sigma$ is a torus, non-zero vector fields vanish nowhere, but $J(df)^2$ should vanish in the critical points of $f$ (e.g. the maximum points).

Remark. The only possibility to find a non-constant function $f : \Sigma \to \mathbb{R}$ such that $\Box_X f = 0, \forall X$ is that $\Sigma$ is a sphere admitting a isometric $S^1$ action. Then $f$ is constant on the orbits of this action and has as only critical points the poles (0-dimensional orbits) of this action.

3. Vaisman metrics on compact complex surfaces and Sasakian structures on compact 3-manifolds

Definition 3. A compact CR 3-manifold $M$ is called primary if its fundamental group $\pi_1(M)$ contains no non-trivial finite subgroups.

This notion arises from the geometry of complex surfaces (see below). All circle bundles over a Riemann surface (regular Sasakian manifolds) are primary; this is no longer the case if we factor them by a finite group of orientation-preserving bundle automorphisms, unless it acts either trivially, or with no fixed points, on the basis.

We recall that a Riemannian product of a Sasakian manifold with a circle is a Vaisman metric (or locally conformally Kähler metric with parallel Lee
form) on the resulting complex manifold (which is usually called a \textit{generalized Hopf manifold}) [19], [8], see also [3]. Starting from a compact 3-manifold, we obtain thus a compact complex surface, on which the Vaisman metrics are classified in [3]:

**Proposition 5.** Let \((M, g, T)\) be a Sasakian compact 3-manifold; then \(M \times S^1\) is one of the following complex surfaces, and \(T\) is (up to a constant) the following holomorphic vector field:

1. \(M \times S^1\) is a properly elliptic surface, admitting two holomorphic circle actions: the first one is given by the factor \(S^1\), and the second one is infinitesimally induced by \(T\) (whose orbits in \(M\) are, therefore, closed). If \(M\) is primary, then the Vaisman metric on \(M \times S^1\) is regular (i.e. it is obtained by a Kaluza-Klein construction on an elliptic fiber bundle — see above);
2. \(M \times S^1\) is a Kodaira surface, admitting two holomorphic circle actions; all the other conclusions above still hold;
3. \(M \times S^1\) is a Hopf surface of class 1, given by the contraction \(g \in \text{End}(\mathbb{C}^2)\), \(g(x, y) := (\alpha x, \beta y)\), with \(\alpha, \beta \in \mathbb{R}\), \(0 < \alpha \leq \beta < 1\), and the Reeb vector field \(T\) is induced by the field \(\text{i log } \alpha x \partial_x + \text{i log } \beta y \partial_y\) on \(\mathbb{C}^2\). If \(M\) is primary, then \(M \times S^1\) is a primary Hopf surface (of class 1) and \(M \simeq S^3\) (in particular \(\pi_1(M) = 0\)).

We recall that a primary properly elliptic surface is a non-flat elliptic bundle over a Riemann surface of genus \(g > 1\), a primary Kodaira surface is a non-flat elliptic bundle over an elliptic curve, and a primary Hopf surface is a quotient of \(\mathbb{C}^2 \setminus 0\) by the infinite cyclic group generated by a contraction \(g\). Non-primary (or secondary) surfaces above considered are finite quotients of primary ones.

**Proof.** The almost-complex structure \(J^s\) on \(M \times S^1\) is defined as follows: \(J^s|_Q := J = \nabla T\), and \(J^s(V) := T\), \(V\) being the unitary, oriented, generator of \(T S^1\). The product metric is, then, given by its Kähler form

\[
\omega := \frac{1}{2} d\eta - \eta \wedge (\eta \circ J^s).
\]

It is easy to prove that \(J^s\) is integrable and that

\[
d\omega = -2(\eta \circ J^s) \wedge \omega,
\]

such that the Lee form of the Hermitian metric \(\omega\), is, by definition, \(\eta \circ J^s\), thus parallel [15], [1], [1].

The \textit{Lee vector field} \(V\) (the metric dual of the Lee form) is a holomorphic vector field [13], and it is automatically given (up to a positive constant) by the complex geometry of the surface \(M \times S^1\) ([1]), which can only be of the three kinds enumerated in the Proposition. In the first two cases, it follows from the descriptions of Sections 3 and 4 of [1], see also Theorem 2 of the same paper, that \(J V = T\) always has closed orbits. In the same cases, \(V\) and \(T\) generate the tangent space of the fibers of the elliptic fibration, and the Riemannian product situation can only occur if generic fibers are biholomorphically \((S^1\text{, can}) \times (S^1, l\cdot \text{can})\) (the first factor corresponds to the orbits of \(V\), and the second (of unknown length \(l\)) to the orbits of \(T\)). If \(M\) is primary, then \(\pi_1(M \times S^1)\) contains no non-trivial finite subgroup, hence...
$M \times S^1$ is a principal elliptic bundle over a complex curve of positive genus (see [2]), as any Vaisman metric is regular on these surfaces, (1), Theorem 2, it immediately follows that the Sasakian metric on $M$ is regular (in the primary case).

If $M \times S^1$ is a Hopf surface, then it is necessarily of class 1 (see [1]), i.e. it is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by a group $H$ generated by a normal finite subgroup $H$ and by a holomorphic contraction $g$ of $\mathbb{C}^2$ as in the Proposition, in general with $\alpha, \beta \in \mathbb{C}, 0 < |\alpha| \leq |\beta| < 1)$. We first need the orbits of $V$ to be closed: as $V := \log \alpha x \partial_x + \log \beta y \partial_y$, see [1], Proposition 8, we obtain that $\alpha = |\alpha| \varepsilon_1$ and $\beta = |\beta| \varepsilon_2$, where $\varepsilon^k = 1 \Leftrightarrow k = np, p \in \mathbb{Z}$, for $j = 1, 2$.

Then it is clear that $M$ is a finite quotient of $S^3$. In the primary case, $M \times S^1$ is the primary Hopf surface $(\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$, where the action of $\mathbb{Z}$ is generated by $g$. As the generic orbits of $V$ cross the orthogonal hypersurfaces (tangent to $V^\perp$) $n$ times, it follows that $n = 1$ and $M \simeq S^3$. As a Hopf surface is a finite quotient of a primary one, we get $n = 1$ in general, as claimed. □

**Corollary 2.** In the cases 1. and 2. from Proposition 3, a primary normal CR 3-manifold is topologically a circle bundle over a Riemann surface of positive genus. Moreover, the orbits of any CR Reeb vector field are the fibers of such a fibration, and therefore have all the same length.

We claim that all Sasakian structures on a given 3-manifold are deformations of a standard one; in the following definitions (see also [1]) we currently consider the Sasakian structure $(M, g, T)$ to be given by its CR structure $J$, together with the contact form $\eta$ and Reeb vector field $T$:

**Definition 4.** (i) A standard Sasakian structure is, up to finite quotient, a regular structure such that the basis has a metric of constant curvature;

(ii) A first type deformation of a Sasakian structure $(M, g, T)$ is a Sasakian structure $(M, g', T')$, where $T' = fT + X_f, X_f \in Q$ is another CR Reeb vector field associated to the same CR structure $(M, Q, J)$. (see also previous Section);

(iii) A second type deformation of a Sasakian structure $(M, \eta, T)$ is defined by the (deformed) contact form $\eta'$, with the same CR Reeb vector field $T$, and such that:

\begin{equation}
\eta' := \eta + d\sigma \circ J_0, \quad Q' := \ker \eta', \quad J' := J_0|Q',
\end{equation}

where $J_0|Q := J$ and $J_0(T) := 0$ is an extension $J_0 \in \text{End}(TM)$ of $J; \sigma$ is a (small enough) function on $M$, such that $d\sigma(T) = 0$ and $\eta'$ is a contact form.

(iv) A 0-type deformation of a Sasakian structure consist in multiplying $T$ with a positive constant, and keeping the CR structure fixed.

After a first type deformation, the normal oriented vector of $Q$ becomes $T'$, and the metric on $Q$ changes as

\begin{equation}
g'|Q = f^{-1}g|Q.
\end{equation}
Remark. If \((M, g, T)\) is regular, then a second type deformation consist in a conformal change of the metric on the basis \(\Sigma\), and a subsequent change of connection in the \(S^1\)-bundle \(M \to \Sigma\). We can always obtain, by this procedure, a metric of constant curvature on \(\Sigma\), and the corresponding Sasakian metric on \(M\) is determined only by the choice of a connection of fixed curvature, i.e. of an element of the affine space modeled on \(H^1(\Sigma, \mathbb{R})\). Standard primary Sasakian structures are then determined by this latter choice and a complex (conformal) structure on \(\Sigma\); both these choices are unique if \(\Sigma \simeq S^2\).

On any Hopf surface \(C^2 \setminus \{0\}/\mathbb{Z}_g \rtimes H\) of class 1, a Vaisman metric has been constructed in \([6]\), see also \([1]\), such that it induces a first type deformation of the round Sasakian structure on \(S^3\) (the unique standard one). We also know \([1]\) that, for a given surface, two Vaisman metrics are second type deformations of each other (and they induce second type deformations on the corresponding Sasakian manifolds). We have then \([1]\):

**Theorem 1.** Any Sasakian structure on a compact 3-manifold \(M\) is a deformation of a standard one; a second type deformation if \(M\) is a Seifert fibration not recovered by \(S^3\), a composition a first type and of a second type deformation if \(M\) is a finite quotient of \(S^3\). Moreover, a standard Sasakian structure on \(S^3\) is unique up to a global rescaling (0-type deformation).

4. Normal CR structures on non-spherical Seifert fibrations

Let \(M\) be a a Seifert fibration over a 2-dimensional orbifold \(\Sigma\), of non-zero genus. Then \(M\) is a finite quotient of a circle bundle over a Riemann surface of positive genus. If \(M\) is primary, then it is a circle bundle over a Riemann surface \(\Sigma\) which is not a sphere, and we have seen, in the previous Section, that Sasakian structures on such manifolds are regular, i.e. given by a Riemannian metric \(g\) on \(\Sigma\) and a connection in the \(S^1\)-bundle \(M \to \Sigma\), see also \([4]\).

Let us consider now \(M\) as a normal CR manifold; we ask then if it admits more than one Sasakian structure associated to it. In other words: Can a Sasakian structure on \(M\) admit CR infinitesimal automorphisms other than the Killing field \(T\)?

The answer is negative; more precisely:

**Theorem 2.** A Sasakian structure on a non-spherical Seifert fibration does not admit non-trivial first type deformations. Equivalently, the connected component of the CR automorphism group of a normal CR structure on \(M\) is isomorphic to \(S^1\).

**Proof.** Consider a Sasakian structure on \(M\), with the usual notations. (see Section \([3]\)) We suppose, with no loss of generality, that \(M\) is primary, thus \(M \to B\) is a \(S^1\)-bundle over a complex curve \(\Sigma\), such that the Reeb vector field \(T\) is tangent to the fibers, which all are of equal length \(l_T\) (and we can suppose \(l_T = 1\)).

We want to prove that any function \(f\) satisfying \(\Box_X f = 0, \forall X \in Q\) is constant. The function \(If : \Sigma \to \mathbb{R}\), defined by

\[
If(x) := \int_{M_x} f \eta,
\]

is constant.
where $M_x := \pi^{-1}(x)$ is a fiber of the projection $M \to \Sigma$, satisfies $\Box^\Sigma_X IF = 0$, thus, from (8), it is constant (recall that $\Sigma$ is of positive genus). If $f > 0$ (which we can assume by adding a constant to the bounded function $f$), then $T' = fT + X_f$ (see (6)) is another $CR$ Reeb vector field on $M$, that thus circular orbits (see Section 3) of an equal length $l^{T'}$ (as $(M, Q, J, T')$ is a primary Sasakian manifold, it is regular, see Section 3). Suppose $IF \equiv 1$. We can also compute

$$I'f^{-1}(x') := \int_{M'_x} f^{-1} \eta',$$

where $M'_x$ is the orbit associated to $x'$ (a point in the orbit space $\Sigma'$ of $T'$ — for topological reasons, the projection $M \to \Sigma'$ is still a principal $S^1$-fibration). $I'f^{-1}$ is also a constant, and we would like to relate it to $IF$, and to the length $l^{T'}$ of the orbits of $T'$; the reason for that is the following Lemma:

**Lemma 1.** a). If two $CR$ Reeb vector fields coincide on a common orbit, they coincide everywhere.

b). Suppose that $T$ and $T'$ are two arbitrary $CR$ Reeb vector fields, and that the length of the orbits of each of them (measured in the corresponding metrics) is equal to 1. If $IF \equiv 1$ and $I'f^{-1} \equiv 1$, then $T \equiv T'$ everywhere.

**Remark.** The point a). is a particular case of b).; it is explicitly formulated as it will often be used throughout the proof of Theorem 3.

**Proof.** The volume form of the original Sasakian metric is $\lambda := \frac{1}{2} \eta \wedge d\eta$, for $\eta = g(T, \cdot)$, and the volume form of the deformed metric is $\lambda' := \frac{1}{2} \eta' \wedge d\eta'$, where $\eta' = f^{-1} \eta$ (see (10)). We denote by $\nu$ the volume of $\Sigma$, equal (as the fibers of the $S^1$-bundle $M \to \Sigma$ are of length $1$) to the volume of $(M, g, T)$. The volume $\nu'$ of $(M, g', T')$ is then equal to

$$\nu' = \int_M f^{-2} \lambda.$$

On the other hand, from Hölder’s inequality, we have

$$\left( \int_M f^{-2} \lambda \right) \left( \int_M f \lambda \right)^2 \geq \left( \int_M \lambda \right)^3 = \nu^3. \tag{11}$$

But the integral of $f$ on $M$ is equal to the integral of $IF$ on $\Sigma$, thus to $\nu$. We get then

$$\nu' \geq \nu. \tag{12}$$

We have thus:

$$\text{(} IF \equiv 1 \text{ and } l^T = 1 \text{)} \Rightarrow \nu' \geq \nu. \tag{13}$$

In each of the claims a). or b)., both $T$ and $T'$ have orbits of length $1$. On the other hand, we also have $IF \equiv 1 \equiv I'f^{-1}$, thus we can apply the implication (13) for $f^{-1}$, too (starting from $(M, g', T')$):

$$\text{(} IF^{-1} \equiv 1 \text{ and } l^{T'} = 1 \text{)} \Rightarrow \nu \geq \nu'.$$

We have thus equality in (14), which implies that $f$ is constant.
It is known that the group $G$ of $CR$ automorphisms of $M$ is a Lie group \[9, 17\]. We consider the connected subgroup $G^f$ associated to the Lie subalgebra generated by $T$ and $T'$. This group acts on $M$ by $CR$ automorphisms, and its orbits are connected. We consider the decomposition of $M$ in orbits of $G^f$; they are of three kinds, according to their dimension:

1. circles $M_{x_i}$, $i \in I \subset B$;
2. immersed surfaces $S \subset M$;
3. open orbits.

Suppose $S$ is a $G^f$-orbit of dimension 2 in $M$; it contains all the circles $M_{x_i}$ that intersect it, and it is immersed, hence it projects, via the bundle projection $M \to \Sigma$, onto an immersed connected curve $C \subset \Sigma$. There are two cases:

1. $C$ is an open segment;
2. $C$ is a circle.

In both cases, we define $X^C$ to be a unitary continuous vector field in $TC$, and $X^S$ to be its horizontal lift to $M$; we have then $X_f = T' - T = kX^S$, and, as $X_f = \frac{1}{2}J(df|_Q)^2$ from \[\text{(1)}\], we get

\[df(X^S) \equiv 0,\] (14)

thus $f$ is constant on the orbits of $X^S$.

If $C$ is a circle, then it has a finite length $l_C$. Assume that $f$ is not constant on $S$; then $f$ has a regular value $s_0$. Fix a point $y_0 \in M_{x_0}$, such that $f(x_0) = s_0$, and fix some other points $y_s \in M_{x_0}$ close to it, such that $f(y_s) = s$ are still regular values of $f$ on $S$. All these numbers $s$ close to $s_0$ are images of compact immersed curves, all diffeomorphic and horizontal (i.e. tangent to $X^S$). Consider $C_s$ to be their connected component containing the points $y_s \in M_{x_0}$; then $C_s$ are precisely the orbits of $X^S$ starting from $y_s$; they all have thus equal length (which is an integer multiple of $l_C$, as they all cover $C$).

Note that this length is measured by the Riemannian metric $g$, and that for $g'$ the lengths need not be equal any longer: Indeed, the curves $C_s$ are still the same, but the unitary vector fields on them, for $g'$, are $\sqrt{f}X^S$ (see Section \[\text{3}\]), and $f$ is precisely non-constant in a neighbourhood of $y_0$.

On the other hand, the group $G^f$, the orbit $S$ and the circles $C_s$ can also be considered starting from the Sasakian metric $(M, g', T')$, in which case an analog reasoning yields that they have equal length, in the metric $g'$ (the regular points of $f$ are also regular points for $f^{-1}$), which leads to a contradiction.

We obtain thus $f \equiv 1$ on $S$. We can suppose that $T'$ is sufficiently close to $T$ (by replacing $f$, if necessary, with $\varepsilon f + 1 - \varepsilon$, for $\varepsilon > 0$ small), such that the orbits of $T$ and the ones of $T'$ are homotopic circles in the torus $S$. We can even suppose that one particular orbit $M'_{x_0}$ of $T'$ lies in a small tubular neighbourhood of an orbit $M_{x_0}$ of $T$. The projection of this neighbourhood onto $M_{x_0}$ induces then a diffeomorphism $\phi : M'_{x_0} \to M_{x_0}$, such that $\phi^* \eta = \eta'$.
(because \( f \equiv 1 \) on \( S \)). Thus

\[
l^{T'} = \int_{M'_{s_0}} \eta' = \int_{M_{s_0}} \eta = l^T = 1,
\]

thus we can apply the point b) in Lemma [1]. This implies that \( f \equiv 1 \) (resp. \( \varepsilon f + 1 - \varepsilon \equiv 1 \), which is the same thing).

The same reasoning can be applied if the projection \( C \) of \( S \) in \( \Sigma \) is a segment of finite length for the metric induced by \( T \) (and also for the metric induced by \( T' \), if we suppose \( T \) and \( T' \) to be sufficiently close to each other).

Suppose now \( C \) is an immersed open curve in \( \Sigma \) of infinite length. We have

\[
T' = fT + Xf = fT + rX^S,
\]

where \( r : S \to \mathbb{R} \) is a function, equal to \(-1/2df(JX^S)\) [1]. In the following lines, \( X \) stands for \( X^S \); it is unitary and contained in \( TS \cap Q \):

\[
X.JX.f - \nabla_X JX.f - (JX.X.f - \nabla_JX.X.f) = 2T.f = 2f',
\]

\[
X.JX.f - \nabla_X JX.f + (JX.X.f - \nabla_JX.X.f) = \Box_X f = 0,
\]

thus, by summation, we get \( X.r - \nabla_X JX.f = f' \), but \( \nabla_X JX \perp JX \) and it still lies in \( Q \), hence it is collinear with \( X \); but we know from (13) that \( X.f \equiv 0 \). We have thus

\[
X.r = f'.
\]

We recall that \( f \), thus \( f' \), too, are constant on the orbits of \( X = X^S \), which are curves of infinite length. If \( f' \neq 0 \) on such an orbit, the function \( r \) satisfying the equation above is not bounded, but this is impossible as \( M \) is compact.

Thus \( f \) is constant on \( S \). The rest of the argument used for the case when \( C \) had finite length can also be applied here.

We have thus proven that \( G^f \) cannot admit any 2-dimensional orbits. We are going to prove now that the number of 1-dimensional orbits (which are vertical circles) is finite.

Suppose we have an infinite number of such orbits; then we can extract a sequence of points \( a_n \to a \in \Sigma, \ n \to \infty \), such that \( M_{a_n}, \ n \in \mathbb{N} \), are orbits of \( G^f \). Then so is \( M_a \), as the union of all 1-dimensional orbits in \( M \) is closed. We want to prove that \( f \) is constant on \( M_a \).

The normal bundle of \( M_a \) in \( M \) is also the restriction to \( M_a \) of the plane bundle \( Q \), and will still be denoted by \( Q \). It is a complex line bundle, and its metric depends on the CR Reeb vector field in \( g^f \) that induces a Sasakian metric on \( M \). Consider two such vector fields \( T \) and \( T' = fT + X_f \) (where \( X_f \equiv 0 \) on \( M_a \)), and fix two arbitrary different points \( x_1, x_2 \in M_a \). The \( S^1 \) (integral) actions corresponding to \( T \), resp. \( T' \) on \( M \) induce two different diffeomorphisms \( \psi \), resp. \( \psi' : S_1 \to S_2 \), where \( S_i \) are contractible surfaces in \( M \), locally defined around \( x_i \), tangent to \( Q_{x_i} \) at \( x_i \) and transverse to the orbits of \( G^f \). The choice of such surfaces is not essential, as we are interested in the differential at \( x_i \) of \( \psi \), resp. \( \psi' \).
Consider $\Psi := (\psi')^{-1} \circ \psi$, which is a diffeomorphism of a neighbourhood $U$ of $x_1$ in $S_1$ into $S_1$. If we denote by $b_n$ the intersections of $M_{a_n}$ with $U$, we obviously have

$$\Psi(b_n) = b_n, \ n \in \mathbb{N}.$$  

**Lemma 2.** If $\Psi : U \rightarrow \mathbb{R}^k$ is a diffeomorphism from a neighbourhood of 0 in $\mathbb{R}^k$ into $\mathbb{R}^k$ that has a sequence, converging to 0, of fixed points, then the differential $(d\Psi)_0$ has at least one eigenvector corresponding to the eigenvalue 1.

**Proof.** By subtracting the inclusion $1_U$ of $U$ in $\mathbb{R}^k$, we get a function $\Psi - 1_U$ which has a sequence, converging to 0, of zeros. Then the kernel of its differential at 0 is non-trivial.

From (15) and the previous Lemma, we conclude that it exists a non-zero $Y \in \mathbb{Q} \times S_1$ such that

$$\frac{d\psi}{dx_1}(Y) = \frac{d\psi'}{dx_1}(Y).$$  

But these differentials are equal to the differentials of the $S^1$ (integral) actions induced by $T$, resp. $T'$, on $M$, and these actions preserve the complex structure of $Q$. Then $JY$ satisfies (15) as well, so the two differentials $\frac{d\psi}{dx_1}$ and $\frac{d\psi'}{dx_1}$ coincide. On the other hand, the $S^1$ action induced by $T$ preserves the corresponding metric $g$ on $Q$, and hence so does $\frac{d\psi}{dx_1}$. The same holds for $(d\psi')_{x_1}$ and $g' = fg$, so we get $f(x_1) = f(x_2)$. As $x_1, x_2$ were arbitrarily chosen, $f$ is constant on $M_{a_1}$, thus everywhere (Lemma 1, a).

So the only remaining situation is when $M$ is a union of open orbits and a finite number of circular orbits of $G^f$; as $M$ is connected, there needs to be only one open orbit $U$ (dense in $M$). We study now the structure of the Lie algebra $g^f$. We will suppose that $G^f$ acts effectively on $M$.

Every element $V \in g^f$ can be written as

$$V := f^V T + X_V,$$

where $f^V : M \rightarrow \mathbb{R}$ is a function, and $X_V := X_{f^V}$. Because $If^V$ is a constant, we get a linear homomorphism

$$I : g^f \rightarrow \mathbb{R}$$

induced by the integral of the functions $f^V$ along the fibers of $T$. The kernel of this homomorphism is a hyperplane $H \subset g^f$, and it contains all the brackets $[T, V]$, for any $V \in g^f$; indeed, the function $f^{[T, V]}$ is precisely the derivative along $T$ of $f^V$, denoted by $f'^V$, hence its integral on the orbits of $T$ vanishes.

**Lemma 3.** 1. The bracket with $T$, $\text{ad}_T \in \text{End}(g^f)$, induces an automorphism of $H$ (still denoted by $\text{ad}_T$), which is $\mathbb{C}$-diagonalizable, and whose eigenvalues are pure imaginary (hence non-zero);

2. If $V$ is the real part of an eigenvector of $\text{ad}_T$, then the bracket $[V, \text{ad}_T V]$ is a non-zero multiple of $T$.

**Proof.** 1. Because all orbits of $T$ have length 1, it means that the exponential of $T$, $\exp T \in G^f$, is contained in the isotropy subgroup of any point in the open orbit $U$, but the intersection of all these isotropy groups is
trivial, as \( G^f \) acts effectively on \( M \). In particular, \( \text{Ad}_{\exp T} = \exp(\text{ad}_T) \) acts trivially on \( g^f \), so the exponential of the endomorphism \( \text{ad}_T \in \text{End}(H) \) is the identity. It follows that its eigenvalues are imaginary (integer multiples of \( 2\pi i \)), and that its Jordan decomposition reduces to the diagonal part.

On the other hand, we know from Proposition \[5\] that the only \( CR \) Reeb vector fields commuting with \( T \) are multiples of \( T \); therefore, if \([T, V] = 0\) for \( V \in H \), then \( V = 0 \) and \( \text{ad}_T \) is non-singular, hence all its eigenvalues are non-zero.

**Remark.** It follows that \( \dim H \) is always even, and that \( \text{ad}_T \) is the (commutative) product of a complex structure \( J \) on \( H \) with a diagonal matrix with real eigenvalues.

2. If \( V \) is the real part of an eigenvector of \( \text{ad}_T \), then \( \text{ad}_T^2(V) \) is a multiple of \( V \). On the other hand,

\[
[T, [V, \text{ad}_T V]] = [\text{ad}_T V, \text{ad}_T V] + [V, \text{ad}_T^2 V] = 0,
\]

so \([V, \text{ad}_T V]\) commutes with \( T \), hence it is collinear to it (see Proposition \[4\]).

Consider the case when the function \( f = f^V \) corresponds to the real part \( V \) of an eigenvector of \( \text{ad}_T \). Then \( G^f \) has dimension 3, as its Lie algebra is generated by \( T, V \), and \( \text{ad}_T V \). We will obtain a contradiction, hence the Theorem will follow.

Denote by \( f' \) the function associated to \( V' = \text{ad}T(V) \); we have

\[
V = fT + X_f, \quad V' = f'T + X_{f'}; \quad V' = [T, V], \quad X_{f'} = [T, X_f].
\]

We also have

\[
[T, V'] = -aV, \quad a = -4\pi^2 l^2, \quad l \in \mathbb{N}^*,
\]

(where \( \pm 2\pi il \) are eigenvalues of \( \text{ad}_T \)) hence \( f \), restricted to any orbit of \( T \), is a solution of the differential equation \( f'' = -af \), in particular it is a sinusoid function:

\[
f(s) = k_x \sin(2\pi ls),
\]

for \( s \) an arc length parameter (for the Sasakian metric induced by \( T \)) on the fiber \( M_x \), and its only critical points are the maximum and the minimum.

Let us compute, from (17), the bracket \([V, V']\):

\[
[fT + X_f, f'T + X_{f'}] = (f'f'' - f'^2)T + fX_{f'} - df(X_{f'})T - f'X_f + df'(X_f)T + [X_f, X_{f'}].
\]

As this has to be a constant multiple of \( T, kT \), it follows that, on a circular orbit of \( G^f \) (where \( df = df' = 0 \)), we have

\[
f f'' - f'^2 = k,
\]

independently on the circular orbit. But, from (19), \( f f'' - f'^2 = -4\pi^2 l^2 k_x^2 \), where \( l \) is a global constant, and \( k_x \) is the amplitude of the sinusoid \( f|_{M_x} \).

The only critical points of \( f \) are then its maximums and its minimums, obtained only on the circular orbits, with the values \( \pm k_x \); indeed, on the
open orbit $U, V = fT + X_f$ has to be linearly independent of $T$, thus $df|_Q \neq 0$.

The function $f$ has the following properties:

1. it has only a finite set of critical points;
2. any of these (isolated) points is either a maximum or a minimum.

Then, after deforming $f$ if necessary (in order to get a function with non-singular Hessian at these critical points), we obtain a Morse function $\varphi : M \to \mathbb{R}$, with a finite set of $2lm$ critical points (where $m$ is the number of circular orbits of $G^f$), which are local maximums or local minimums; The topology of $M$ is thus obtained by gluing $lm$ points to $lm$ 2-cells, which implies, as $M$ is connected, that $l = m = 1$ and $M$ is homeomorphic to $S^3$, which contradicts our hypothesis.

Corollary 3. On a compact, normal CR manifold $M$, the only solutions of the equation

$$\Box_X f = 0, \forall X \in Q$$

are the constants.

Corollary 4. A compact, normal CR manifold $M$, admits a unique Sasaki-Kähler structure associated to it. For $M$ the total space of a circle bundle over a Riemann surface $\Sigma$ of positive genus, an isomorphism class of normal CR structures on $M$ is determined by an isometry class of Riemannian metrics on $\Sigma$, of unitary volume, together with a choice of an element in the affine space modeled on $H^1(\Sigma, \mathbb{R})$.

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