On the complete integrability of the discrete Nahm equations

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Abstract
The discrete Nahm equations, a system of matrix valued difference equations, arose in the work of Braam and Austin on half-integral mass hyperbolic monopoles.

We show that the discrete Nahm equations are completely integrable in a natural sense: to any solution we can associate a spectral curve and a holomorphic line-bundle over the spectral curve, such that the discrete-time DN evolution corresponds to walking in the Jacobian of the spectral curve in a straight line through the line-bundle with steps of a fixed size. Some of the implications for hyperbolic monopoles are also discussed.

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1 Introduction

This paper is concerned with two closely related stories: one about the complete integrability of a discrete-time system of nonlinear matrix equations (the discrete Nahm or DN system), the other having to do with $SU_2$-monopoles on hyperbolic three-space $H^3$. The link between these two stories is given by the Braam–Austin version of the ADHMN construction, which is a correspondence between hyperbolic monopoles (of integral or half-integral mass) and certain solutions of the DN system [3].

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We shall show, using methods very close to those of [8], that the DN system is completely integrable in a natural sense: to any solution we can associate a spectral curve $S$ and a holomorphic line-bundle $\mathcal{L} \to S$, such that the discrete-time DN evolution corresponds to walking in the Jacobian of $S$ in a straight line through $\mathcal{L}$ with steps of a fixed size. The main novelty in this is that $S$ lies in $\mathbb{P}_1 \times \mathbb{P}_1$ rather than in the total space of $\mathcal{O}(d) \to \mathbb{P}_1$. It turns out that the geometry of $\mathbb{P}_1 \times \mathbb{P}_1$ gives rise in an entirely natural way to a discrete-time system. At the technical level, the new geometric set-up means that it is necessary to develop a number of modifications of the modern theory of algebraically integrable systems (by which we mean the body of knowledge that is surveyed, for example, in [9]).

This account is accessible to readers with no knowledge of (or interest in) hyperbolic monopoles. On the other hand the origin of the DN system in the theory of hyperbolic monopoles provided us with essential insights in this work, and is probably the main reason for its interest. Therefore we have also described how the particular solutions of the DN systems that are linked by Braam and Austin to hyperbolic monopoles arise within our general framework. This leads in particular to constraints on spectral curves of hyperbolic monopoles analogous to those previously known in the euclidean case.

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### 1.1 Nahm equations

The Nahm equations comprise the following non-linear system of ordinary differential equations:

$$\frac{dT_1}{dz} = [T_2, T_3], \quad \frac{dT_2}{dz} = [T_3, T_1], \quad \frac{dT_3}{dz} = [T_1, T_2]$$

(1.1)

where the $T_i$ are functions of the real variable $z$, with values in the complex, skew-hermitian $k \times k$ matrices. They form a completely integrable system which reduces, when $k = 2$, to the Euler top equations. In particular there is a Lax formulation

$$\frac{dA}{dz} = [A, A_+]$$

(1.2)

where

$$A := A(\zeta) = (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2, \quad A_+ := A_+(\zeta) = -iT_3 + (T_1 - iT_2)\zeta.$$  

(1.3)

The complete integrability is obtained from this by setting up the eigenvalue problem

$$A(\zeta)f(\eta, \zeta) = \eta f(\eta, \zeta).$$

(1.4)

Then if $A(\zeta)$ evolves according to (1.2) and

$$\frac{df}{dz} + A_+ f = 0,$$

(1.5)
the eigenvalue $\eta$ remains constant. In particular the equation
\[ \det(\eta - A(\zeta)) = 0 \tag{1.6} \]
must be independent of $z$, so the coefficients of this equation are a set (in fact a complete set) of conserved quantities for the system (1.4).

For some purposes (and in particular to allow an easy comparison with the discrete Nahm system to be introduced below) it is useful to reformulate these equations slightly. First one introduces a ‘gauged’ version by adding a further skew-hermitian matrix-valued function $T_0$, and writing
\[ \frac{dT_1}{dz} - [T_0, T_1] = [T_2, T_3], \quad \frac{dT_2}{dz} - [T_0, T_2] = [T_3, T_1], \quad \frac{dT_3}{dz} - [T_0, T_3] = [T_1, T_2]. \tag{1.7} \]
If one regards $d/dz - T_0$ as a connection, then (1.7), modulo gauge equivalence, is equivalent to the original system (1.1) (modulo conjugation by constant matrices). To be quite explicit, the gauge group here is the space of smooth maps $g(z)$ into $U(k)$ and
\[ g(T_0, T_i) = (gT_0g^{-1} - (dg/dz)g^{-1}, gT_ig^{-1}). \tag{1.8} \]
Now, following Donaldson [5], we introduce the ‘complex variables’ $\sigma = T_0 + iT_1$, $\tau = T_2 + iT_3$. Then (1.7) becomes
\[ \frac{d\tau}{dz} = [\sigma, \tau], \quad \frac{d\tau^*}{dz} = -[\sigma^*, \tau^*]; \tag{1.9} \]
and
\[ \frac{d}{dz}(\sigma + \sigma^*) = [\sigma, \sigma^*] + [\tau, \tau^*]. \tag{1.10} \]

### 1.2 Discrete Nahm equations

In [3] Braam and Austin found a discrete version of (1.1). The relation of this system to the theory of monopoles on hyperbolic space will be described in §1.4. For the moment, let us just write it down:
\[ \beta_i\gamma_{i+1} = \gamma_{i+1}\beta_{i+2}, \quad \beta_i^*\gamma_{i+1}^* = \gamma_{i+1}^*\beta_i^* \tag{1.11} \]
and
\[ \gamma_{i-1}^*\gamma_i - \gamma_{i+1}\gamma_i^* + \beta_i^*\beta_i = 0. \tag{1.12} \]
Here the discrete variable $i$ runs over $I = \{a, a+2, \ldots, b\} \subset 2\mathbb{Z}$ and the $\beta$’s and $\gamma$’s are $k \times k$ complex matrices, with the $\gamma$’s invertible. There is a gauge group $G$ which consists of sequences $(g_i)$ of unitary matrices, acting as follows:
\[ \beta_i \mapsto g_i\beta_i g_i^{-1}, \quad \gamma_i \mapsto g_i\gamma_i g_{i+1}^{-1}. \tag{1.13} \]
This system of equations was also supplemented by a boundary condition which we shall consider later.
The equations have a formal similarity to the standard Nahm system, with (1.3) and (1.10) resembling, respectively (1.11) and (1.12). One aspect of this is that the latter really are a discretization of the former. To see this, rescale $I$ by multiplying by $h$ (which is to be thought of as small and positive). Given $\sigma$ and $V$, really are a discretization of the former. To see this, rescale $I$ by multiplying by $h$ (which is to be thought of as small and positive). Given $\sigma$ and $\tau$, set

$$\gamma_{i+1}^* = \frac{1}{2h} + \sigma(h(i + 1)), \quad \beta_i^* = \tau(hi). \quad (1.14)$$

Then we have

$$\beta_{i+2}^* \gamma_{i+1}^* - \gamma_{i+1}^* \beta_i^* = \left[ \frac{d\tau}{dz} - [\sigma, \tau] \right]_{z=ih} + O(h) \quad (1.15)$$

and

$$\gamma_{i-1}^* \gamma_i^* - \gamma_i^* \gamma_{i+1}^* + [\beta_i^*, \beta_i] = -\left[ \frac{d\sigma^*}{dz} + [\sigma, \sigma^*] - [\tau, \tau^*] \right]_{z=ih} + O(h), \quad (1.16)$$

so that (1.11) and (1.12) are satisfied to lowest order in $h$ by virtue of (1.3) and (1.10).

Note further that it is reasonable to think of the Braam–Austin equations as the evolution equations of a discrete-time system. For given $\gamma_{i-1}$ and $\beta_i$, we solve (1.12) for $\gamma_{i+1}$ and then (1.11) determines $\beta_{i+2}$. This procedure gives a unique evolution (up to gauge) provided that the quantity $\gamma_{i-1}^* \gamma_i^* + [\beta_i^*, \beta_i]$ is positive-definite. If this fails, then the evolution cannot be continued beyond this point.

We remark also that there is a natural way to fix the gauge by taking $\gamma_{i+1} > 0$ to be the (positive) square root of $\gamma_{i-1}^* \gamma_i^* + [\beta_i^*, \beta_i]$, at every step of the evolution: in other words, we take the $\gamma_i$ to be self-adjoint. Comparing with (1.14), we see that this corresponds to the gauge $T_0 = 0$ and so to the original form (1.3) of the Nahm equations.

### 1.3 Statement of results

The main purpose of this paper is to explain that the Braam–Austin system also shares a more profound property with the standard Nahm equations, their complete integrability. In order to state our results more precisely we must give a minor reformulation the Braam–Austin equations.

First of all let us complexify the system, replacing $\beta$ by $-A$, $\beta^*$ by $D$, $\gamma^*$ by $P^+$ and $\gamma$ by $-P^-$. (The choice of signs is for later convenience only.) We replace the index set $I$ by a set $Z = \{r_0, r_0 + 1, \ldots, r_1 - 1, r_1\}$ of consecutive integers ($r_0 \geq -\infty, r_1 \leq +\infty$).

We assume given a complex $k$-dimensional vector space $V_r$ attached to each $r \in Z$ and naturally interpret $A_r$ and $D_r$ as endomorphisms of $V_r$. By contrast $P^+$ and $P^-$ map adjacent vector spaces to each other and we shall choose the numbering so that $P_r^+$ maps $V_r$ to $V_{r+1}$, while $P_r^-$ maps $V_r$ to $V_{r-1}$.

Now by discrete Nahm data at $r \in Z$, we mean a triple $(A_r, B_r, D_r)$ of endomorphisms of $V_r$. Given discrete Nahm data at adjacent points $r$ and $r + 1$ in $Z$ and maps $P_r^+ : V_r \rightarrow V_{r+1},\ P_r^- : V_{r+1} \rightarrow V_r$, we say that the discrete Nahm (DN) equations are satisfied on $[r, r + 1]$ if the following hold:

$$P_{r+1}^- A_{r+1} - A_r P_r^- = 0,\ P_r^+ D_r - D_{r+1} P_r^- = 0 \quad (1.17)$$
and

\[ B_r = P_{r+1}^r P_r^+ + A_r D_r, \quad B_{r+1} = P_r^+ P_{r+1}^- + D_{r+1} A_{r+1}. \] (1.18)

Furthermore we shall say that the DN equations are satisfied on \( Z \) if for every pair of adjacent points \( r, r + 1 \) in \( Z \), the DN equations are satisfied on \([r, r + 1]\). It is clear that (1.17) corresponds to (1.11) and that if the second of (1.18) holds with \( r \) replaced by \( r - 1 \), then we have at \( r \)

\[ B_r = P_{r+1}^r P_r^+ + A_r D_r, \quad B_{r+1} = P_r^+ P_{r+1}^- + D_{r+1} A_{r+1}. \]

which yields (1.12). Thus (1.17) and (1.18) provide a reformulation of the Braam–Austin system except at the end-points of \( Z \). As in (1.13) there is a natural gauge freedom given by the action of \( g_r \in \text{GL}(V_r) \) where \( g_r \) acts by conjugation on the triple \((A_r, B_r, D_r)\) and by \( P_r^{\pm} \mapsto g_{r+1} P_r^{\pm} g_r^{-1} \).

As well as taking care of the end-points, the introduction of \( B \) allows us to define the spectral curve \( S \) of DN data. Given \((A_r, B_r, D_r)\) consider

\[ S_r = \{ \det(\eta \zeta A_r + \eta B_r + \zeta + D_r) = 0 \}. \] (1.19)

This defines an algebraic curve in \( \mathbb{C}^2 \) which has a natural compactification in \( \mathbb{P}_1 \times \mathbb{P}_1 \) and which is gauge-independent. The data also define a holomorphic line-bundle \( L_r \) over \( S_r \) as the cokernel of the multiplication map

\[ \mathbb{C}^k \otimes \mathcal{O}(-1,-1) \xrightarrow{M_r(\eta,\zeta)} \mathbb{C}^k \otimes \mathcal{O} \]

where \( M_r \) is the matrix in (1.19). Our main results are as follows:

**Theorem 1.1.** Given a solution \((A, B, D, P^{\pm})\) of the DN system (1.17) and (1.18) in \( Z \), we have \( S_r = S_{r'} \) and \( L_r = L_r \otimes L^{r'-r} \) for all \( r, r' \in Z \).

Thus the spectral curve is constant for the DN evolution and that evolution corresponds to walking in a straight line on the Jacobian of \( S \), with steps of fixed size, corresponding to the line-bundle \( L = \mathcal{O}(1,-1) \) over \( \mathbb{P}_1 \times \mathbb{P}_1 \).

The converse of Theorem 1.1 is as follows:

**Theorem 1.2.** Let \( S \) be a smooth curve of bidegree \((k, k)\) in \( \mathbb{P}_1 \times \mathbb{P}_1 \) and let \( L \) be a regular holomorphic line-bundle over \( S \). Then there is canonically associated to \((S, L)\) a solution \((A, B, D, P^{\pm})\) of the DN equations over \( Z = \{r_0, \ldots, r_1\} \), such that the spectral curve of the solution is \( S \) and \( L_r \otimes L^{-1} \) is an integral power of \( L \), for every \( r \) in \( Z \). The set \( Z \) is determined by the condition: \( r \in Z \) if and only if for all integers \( m \) between 0 and \( r \) (inclusive), \( L \otimes L^m \) is regular.

The term ‘regular’ is defined below (Definition 2.1); the set of regular elements is a dense open subset of the Jacobian of \( S \).

Combining these two Theorems we obtain a further result about the evolution from initial data of the DN system.

1Strictly we should assume that \( S_r \) is smooth here.
Theorem 1.3. Let \((a,b,d)\) be a triple of \(k \times k\) matrices such that
\[
\{ \det(\eta \zeta a + \eta b + \zeta + d) = 0 \}
\]
is smooth in \(\mathbb{P}_1 \times \mathbb{P}_1\) and let \(\mathcal{L} \to S\) be defined as above. Then there exists a unique solution \((A,B,D,P^\pm)\) of the DN system on \(Z\) such that \(Z\) contains \(0\) and \((A_0,B_0,D_0)\) is gauge-equivalent to \((a,b,d)\). Moreover \(Z = \{0\}\) iff both \(\mathcal{L} \otimes \mathcal{L}\) and \(\mathcal{L} \otimes \mathcal{L}^{-1}\) fail to be regular.

These theorems will be proved in §§2–3 below. For greater clarity we describe in §2 the (1:1) correspondence between triples \((A,B,D)\) (modulo conjugation) and pairs \((S,\mathcal{L})\) where \(S \subset \mathbb{P}_1 \times \mathbb{P}_1\) and \(\mathcal{L}\) is a regular line-bundle on \(S\). With this established, we show that the DN evolution corresponds to straight-line motion on the Jacobian in §3.

Of course the correspondence between triples of matrices (or more generally matrix polynomials) and pairs \((S,\mathcal{L})\) is a fundamental part of the modern theory of completely integrable systems, but in that setting the curve \(S\) is naturally embedded in the total space of \(\mathcal{O}(d) \to \mathbb{P}_1\) (for some positive integer \(d\)). The present work may be viewed as an attempt to understand what aspects of this theory change when \(\mathcal{O}(d)\) is replaced by \(\mathbb{P}_1 \times \mathbb{P}_1\). From this point of view it seems natural to ask whether the DN system is part of a discrete integrable hierarchy and whether other interesting families of discrete integrable systems arise by generalizations of the present construction.

1.4 Relation to monopoles and instantons

This subsection outlines how these results were motivated by, and bear on hyperbolic monopoles. The reader interested only in the complete integrability of the discrete Nahm system may skip it.

When supplemented boundary conditions, there is a correspondence, the Nahm transform, between solutions on \(0 < z < 2\) of (1.1) and solutions of the euclidean Bogomolny equations
\[
\nabla_1 \Phi = [\nabla_2, \nabla_3], \quad \nabla_2 \Phi = [\nabla_3, \nabla_1], \quad \nabla_3 \Phi = [\nabla_1, \nabla_2],
\]
where \(\nabla_j = \partial_j + A_j\) are the components of a unitary connection on \(\mathbb{R}^3\) and \(\Phi\), the Higgs field, is a section of the adjoint bundle. On the other hand there is also a twistor correspondence for these equations, yielding an algebraic curve \(S \subset T\mathbb{P}_1\) which determines the monopole. From the work of Hitchin and Murray \([7]\) one knows that the curve (1.6) determined by the Nahm data which corresponds to a monopole, coincides with \(S\).

The discrete Nahm system arose in the work of Braam and Austin on hyperbolic monopoles. These are solutions of the Bogomolny equations on hyperbolic space \(H^3\), subject to certain boundary conditions \([1]\). These boundary conditions yield two numerical invariants for each solution, the magnetic charge \(k\), a positive integer, and the mass \(p\), a positive real number. When \(p\) is an integer of half-integer, the moduli space of hyperbolic monopoles of mass \(p\) and charge \(k\) can be identified with a moduli space of circle-invariant instantons on \(S^4\) of topological charge (instanton number) \(2pk\). By
decomposing the ADHM description of such instantons under the action of the circle, Braam and Austin proved that there is a (1:1) correspondence between

(i) Solutions of the $k \times k$ DN system in $\{1,2,\ldots,2p+1\}$, with boundary condition that $B_1 - D_1 A_1$ is of rank 1, plus reality conditions, and

(ii) hyperbolic monopoles of charge $k$ and mass $p$.

(Actually Braam–Austin considered mainly the case where $2p$ is odd, and considered their system to be defined on the set $\{1-2p,3-2p,\ldots,-2,0,2,\ldots,2p-1\}$.)

On the other hand, it is also known that the monopole is determined by a spectral curve in $\mathbb{P}_1 \times \mathbb{P}_1$. This suggested to us that the Braam–Austin system should be integrable in terms of the geometry of such a curve. Indeed a previous calculation of the first author gave the equation of the spectral curve of the monopole in terms of the corresponding Braam–Austin data in the form

$$\det(\eta \zeta \beta_i - \eta (\gamma_{i-1}^* \gamma_{i-1} + \beta_i^* \beta_i) + \zeta - \beta_i^*) = 0.$$ 

The reader will recognize this as the equation (1.19) that we have already used to associate a spectral curve to DN data. It follows that given a hyperbolic monopole, the associated spectral curve $S_{\text{mon}}$, say, agrees with the spectral curve $S_{\text{DN}}$ of the corresponding solution of the Braam–Austin equations. After Theorems 1.1 and 1.2, it follows that there exists a holomorphic line-bundle $L$ over $S_{\text{mon}}$ which gives rise to the corresponding solution of the Braam–Austin equations. In fact we have the following

**Theorem 1.4.** Let $(A, \Phi)$ be an $SU_2$-monopole on $H^3$ with charge $k$ and mass $p \in \frac{1}{2} \mathbb{Z}_{>0}$. Assume the spectral curve of $(A, \Phi)$ is smooth. Then the solution of the DN system given by Theorem 1.2, applied to $(S_{\text{mon}}, \mathcal{O}(k -1, 0))$ coincides with the solution associated by Braam and Austin to $(A, \Phi)$.

There is a slight abuse of notation here in that $\mathcal{O}(k - 1, 0)$ is not regular in the sense of Definition 2.1. However, as we shall see, $L^r(k - 1, 0)$ is regular for $r = 1, \ldots, 2p$, and the Theorem states that the corresponding solution of DN on $\{1, \ldots, 2p\}$ agrees with the one obtained from the Braam–Austin correspondence. This proves one of the main parts of the following

**Conjecture 1.5.** Let $S$ be a curve of bidegree $(k,k)$ in $\mathbb{P}_1 \times \mathbb{P}_1$. Then $S$ is the spectral curve of a hyperbolic monopole of charge $k$ and mass $p \in \frac{1}{2} \mathbb{Z}_{>0}$ iff

(o) $S$ does not intersect the anti-diagonal;

(i) $S$ has no multiple components;

(ii) $S$ is real, $L_{|S}^{2p+k}$ is holomorphically trivial;

(iii) $L^{p+1/2}(k -1, 0)_{|S}$ has a real structure;

(iv) $H^0(S, L^r(k -2, 0)) = 0$ for $r = 1, 2, \ldots, 2p + 1$. 

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To interpret the word ‘real’ here, recall that when \( \mathbb{P}_1 \times \mathbb{P}_1 \) is viewed as the twistor space of \( H^3 \), it is equipped with a natural real structure \( \sigma : (p, q) \mapsto (\sigma_0(q), \sigma_0(p)) \), where \( \sigma_0 \) is the antipodal map. The subgroup of \( SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) that commutes with \( \sigma \) is an ‘antidiagonal’ copy of \( SL_2(\mathbb{C}) \) which corresponds to the isometry group of \( H^3 \). This has two orbits on \( \mathbb{P}_1 \times \mathbb{P}_1 \): the anti-diagonal \( \bar{\triangle} \), which is the set of all pairs \( (p, \sigma_0(p)) \) in \( Q \), and its complement. Condition (o) is equivalent to \( S \) being a compact subset of \( Q - \bar{\triangle} \). These conditions should be compared with those for the euclidean monopole in [8, p. 146].

Of these conditions, (o) and (ii) are known from [2, 11]. What is new here is that (iii) and (iv) are proved to be necessary conditions, (iv) being a restatement of the regularity of \( L^r(k-1, 0) \). Condition (iii) follows from the reality conditions of Braam and Austin, and is equivalent to the existence of a canonical real structure switching the summands in

\[
H^0(S, L^{p+1/2+t}(k-1, 0)) \oplus H^0(S, L^{p+1/2-t}(k-1, 0)).
\]

It is hoped that the methods of this paper might be refined to prove that (i)–(iv) are indeed sufficient conditions; for this one would need to show that the solution of the DN system given by Theorem 1.2 (applied to \( (S, O(k-1, 0)) \)) satisfies the boundary conditions and reality conditions written down by Braam and Austin.

A further extension of the present work might uncover the Nahm description of non-integral hyperbolic monopoles. Since such monopoles still have spectral curves in \( \mathbb{P}_1 \times \mathbb{P}_1 \), it is tempting to believe that one of the corresponding solutions of the DN equations, now defined, presumably, on an infinite set \( Z \), should provide such a Nahm description.

Two final remarks. First, the continuum limit can be seen as the limit as the curvature of \( H^3 \) goes to 0. In terms of the twistor spaces, this corresponds to the singular limit of a family of embedded quadrics in \( \mathbb{P}_3 \). Second, since the hyperbolic monopoles described by the Braam-Austin system correspond to \( S^1 \)-invariant instantons over \( S^4 \), the present work provides, in principle at least, some non-trivial solutions to the ADHM equations solutions associated with algebraic curves.

## 2 DN triples and spectral curves

### 2.1 Notation

From now on, we write \( Q = \mathbb{P}_1 \times \mathbb{P}_1 \), with homogeneous coordinates \( [w_0, w_1], [z_0, z_1] \) on the two factors. It is sometimes helpful to think of \( Q \) as \( \mathbb{P}(E^+) \times \mathbb{P}(E^-) \), where \( E^\pm \) are complex symplectic vector spaces of dimension 2, not canonically isomorphic. The symplectic form in \( E^\pm \) will be denoted \( \langle \cdot, \cdot \rangle \).

In particular, then, if \( O(a, b) = p_1^*O(a) \otimes p_2^*O(b) \), there are canonical isomorphisms

\[
H^0(Q, O(1, 1)) = E^+ \otimes E^-, \quad H^0(Q, O(a, b)) = S^a E^+ \otimes S^b E^-
\]

These are a system of quadratic matrix equations.

The natural action of \( SL_2(\mathbb{C}) \) on \( Q \) entails that \( E^+ \) and \( E^- \) should be the two inequivalent two-dimensional irreducible representations of \( SL_2(\mathbb{C}) \); this corresponds to the anti-diagonal embedding mentioned above.

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provided \( a \) and \( b \) are non-negative. (Here \( S^m \) denotes \( m \)-th symmetric power.) We shall usually denote by \( L \) the line-bundle \( O(1, -1) \).

The first of (2.1) leads to an evaluation map \( E^+ \otimes E^- \otimes O \to O(1, 1) \) whose kernel \( K \) will be very important in what follows. In addition to the defining exact sequence

\[
0 \to K \to E^+ \otimes E^- \otimes O \to O(1, 1) \to 0, \tag{2.2}
\]

(where the second map is given by \( g_1 \otimes g_2 \mapsto \langle g_1, w \rangle \langle g_2, z \rangle \)) we have an exact sequence

\[
0 \to O(-1, -1) \to E^+ \otimes O(0, -1) \oplus E^- \otimes O(-1, 0) \to K \to 0. \tag{2.3}
\]

Here the maps are \( f \mapsto (f \otimes w, f \otimes z) \) and \( (g_1, g_2) \mapsto g_1 \otimes z - g_2 \otimes w \), \( w \) and \( z \) standing for the tautological sections of \( E^+(1, 0) \) and \( E^-(0, 1) \) respectively.

If a local section of \( K \) is represented in the form \( g_1 \otimes z - g_2 \otimes w \) then \( \langle g_1, w \rangle \) and \( \langle g_2, z \rangle \) are local sections of \( L \) and \( L^{-1} \) respectively. Since each of these vanish if \( (g_1, g_2) \) is in the image of \( O(-1, -1) \) in (2.3), we obtain sheaf maps \( K \to L^\pm \) and hence exact sequences

\[
0 \to E^- \otimes O(-1, 0) \to K \to L \to 0 \tag{2.4}
\]

and

\[
0 \to E^+ \otimes O(0, -1) \to K \to L^{-1} \to 0. \tag{2.5}
\]

These four exact sequences will be much used below.

\subsection*{2.2 DN maps and spectral curves}

In order to give an invariant statement of the correspondence between matrix data and holomorphic line-bundles over algebraic curves, we introduce the notion of a DN map of charge \( k \). This is just an injective complex-linear map \( \alpha : \mathbb{C}^k \to \mathbb{C}^k \otimes E^+ \otimes E^- \). Choosing a basis in \( E^+ \otimes E^- \) we may consider \( \alpha \) as a list \((\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11})\) of maps \( \mathbb{C}^k \to \mathbb{C}^k \). It will turn out to be natural to regard the two copies of \( \mathbb{C}^k \) as different, so that the natural notion of equivalence is given by the action of \( GL_k \times GL_k \) on \( \alpha \), \( (g_1, g_2)\alpha = g_1\alpha g_2^{-1} \). In particular any injective linear map \( \alpha : U \to V \otimes E^+ \otimes E^- \), where \( \dim U = \dim V = k \) gives rise to a DN map, by choosing bases in \( U \) and \( V \). The freedom in choosing these bases corresponds precisely to the \( GL_k \times GL_k \)-action just mentioned. In our application, \( \alpha_{10} \) is invertible and may be used to identify the two copies of \( \mathbb{C}^k \). Then the DN map takes the form \((A, B, 1, D)\) and \( A, B \) and \( D \) will be identified with the matrix data in \( \S 1.3 \).

The spectral curve \( S(\alpha) \) associated to a DN map \( \alpha \) is defined as follows. Identifying \( E^+ \) with the space of sections of \( O(1, 0) \) and \( E^- \) with the space of sections of \( O(0, 1) \), we may think of \( \alpha \) as an element of \( H^0(Q, \mathbb{C}^k \otimes \mathbb{C}^k \otimes O(1, 1)) \). Because \( \alpha \) is assumed injective, taking the determinant, we obtain \( 0 \neq \det \alpha \in H^0(Q, O(k, k)) \). Then we put

\[
S(\alpha) := \{ \det \alpha = 0 \}. \tag{2.6}
\]

Thus \( S(\alpha) \) is an algebraic curve of bidegree \((k, k)\) in \( Q \). In addition, we define a sheaf \( \mathcal{L}(\alpha) \) by the exactness of

\[
0 \to \mathbb{C}^k \otimes O(-1, -1) \overset{\alpha}{\to} \mathbb{C}^k \to \mathcal{L}(\alpha) \to 0. \tag{2.7}
\]
We shall refer to \((S(\alpha), \mathcal{L}(\alpha))\) as the spectral data determined by \(\alpha\). It is clear that the spectral data depends only on the \(GL_k \times GL_k\)-equivalence class of \(\alpha\) and that the support of \(\mathcal{L}(\alpha)\) is contained in \(S(\alpha)\).

Twisting (2.3) by \(\mathcal{O}(0,-1)\) and \(\mathcal{O}(-1,0)\) and taking the corresponding long exact sequences, we note that

\[
H^i(\mathcal{L}(\alpha)(-1,0)) = 0, \quad H^i(\mathcal{L}(\alpha)(0,-1)) = 0 \text{ for all } i.
\]

Since the genus of \(S\) is \((k-1)^2\), if \(E\) is a bundle over \(Q\) of rank \(n\) and bidegree \((a,b)\),

\[
\text{ind}(E) := \dim H^0(S,E) - \dim H^1(S,E) = k(a+b) - nk(k-2)
\]

(Riemann–Roch). In particular, the above vanishing of cohomology implies that the degree of \(\mathcal{L}(\alpha)(-1,0)\) is \(k(k-2)\); so the degree of \(\mathcal{L}\) is \(k(k-1)\). Accordingly, let \(J = J(S)\) denote the set of holomorphic line-bundles on \(S\) of degree \(k(k-1)\).

**Definition 2.1.** The element \(\mathcal{L} \in J\) is called regular iff both \(\mathcal{L}(-1,0)\) and \(\mathcal{L}(0,-1)\) are in the complement of the \(\vartheta\)-divisor; i.e. if and only if

\[
H^0(S, \mathcal{L}(-1,0)) = H^1(S, \mathcal{L}(-1,0)) = 0, \quad H^0(S, \mathcal{L}(0,-1)) = H^1(S, \mathcal{L}(0,-1)) = 0.
\]

The set of regular elements of \(J\) is denoted by \(J^{\text{reg}}\).

The ‘kinematic’ part of our construction now has the following statement:

**Theorem 2.2.** Let \(S\) be a smooth curve in \(Q\) of bidegree \((k,k)\). Then there is a natural bijection between \(J(S)^{\text{reg}}\) and the set

\[
\{ \alpha : \mathbb{C}^k \to \mathbb{C}^k \otimes E^+ \otimes E^- | S(\alpha) = S \}/GL_k \times GL_k.
\]

**Proof** In one direction this bijection is the map which assigns to \(\alpha\) the spectral data \((S(\alpha), \mathcal{L}(\alpha))\). We shall show that \(\mathcal{L}(\alpha)\) is a holomorphic line-bundle on \(S\), i.e. a locally free sheaf of \(\mathcal{O}_S\)-modules of rank 1. To prove that \(\mathcal{L}(\alpha)\) is a sheaf of \(\mathcal{O}_S\) modules it is necessary and sufficient to show that the ideal \(\mathcal{I} \subset \mathcal{O}\) defined by \(\det \alpha\) annihilates \(\mathcal{L}(\alpha)\).

In other words, if \(v \in \mathbb{C}^k\) then \((\det \alpha)v\) is in the image of \(\alpha\). But by definition, the matrix \(\beta\) of cofactors of \(\alpha\) satisfies \(\alpha \beta = \beta \alpha = \det \alpha \cdot \text{Id}\). Hence \(\alpha\) carries \(\beta v\) to \((\det \alpha)v\), as required.

We now use the assumption that \(S\) is smooth to prove that \(\mathcal{L}(\alpha)\) has rank 1. For suppose not. Then there is a point \((\eta_0, \zeta_0)\) on \(S\) such that the nullity \(n\) of \(\alpha(\eta_0, \zeta_0)\) is at least 2. By replacing \(\eta\) by \(\eta - \eta_0\) and \(\zeta\) by \(\zeta - \zeta_0\) we may suppose this point is \((0,0)\). With such a choice of coordinates, \(\alpha\) takes the form

\[
\alpha(\eta, \zeta) = \eta \zeta \alpha_{00} + \eta \alpha_{01} + \zeta \alpha_{10} + \alpha_{11}
\]

where the nullity of \(\alpha_{11}\) is equal to \(n\). Choosing an appropriate basis of \(V\), we may suppose that the first two rows of \(\alpha_{11}\) are identically zero. But now when we expand \(\det \alpha(\eta, \zeta) = a \eta + b \zeta + \ldots\) in ascending powers of \((\eta, \zeta)\), we have \(a = b = 0\). For each
term in the expansion of the determinant contains an entry from the first row of $\alpha(\eta, \zeta)$ and an entry from the second row, so each term has a factor of $\eta^2$, $\eta\zeta$, or $\zeta^2$. Hence the curve is singular at $(0,0)$, contradiction.

Finally let us show that $L(\alpha)$ is locally free. This is a local question, so we may once again assume that we are working near the point $(0,0) \in S$. Choose the basis in $\mathbb{C}^k$ so that the first row of $\alpha_{11}$ in (2.10) is identically zero, the remaining rows being linearly independent. Then obviously $L(\alpha_0)$ is generated by the first basis vector $e_1$, but by continuity, the same is true of $L(\alpha)_x$ for all $x \in S$ sufficiently close to 0. In other words there exists an open set $U$ of $S$ containing 0 such that multiplication by $e_1$ followed by projection to $L(\alpha)$ gives an isomorphism $O_S|U \cong L(\alpha)|U,$ as required for $L(\alpha)$ to be locally free. Since we have already seen that $L(\alpha)(-1,0)$ and $L(\alpha)(0,-1)$ have no cohomology this completes the map from the set of DN maps to $J^{reg}$.

To go in the other direction, we show how to construct a DN map $\alpha(L)$, given a pair $(S,L)$ with $L$ in $J^{reg}$. Given such $L$, we may consider the sheaf cohomology groups

$$U(L) := H^0(S,K \otimes L)\text{ and } V(L) := H^0(S,L).$$

By definition of $K$ there is a natural map $\alpha(L) : U \to V \otimes E^+ \otimes E^-$. The next three lemmas are devoted to showing that $\alpha(L)$ is a DN map with spectral curve equal to $S$. Namely we establish in turn that $U(L)$ and $V(L)$ are of the correct dimension $k$, that $\alpha(L)$ is injective, and that $S = S(\alpha(L))$.

**Lemma 2.3.** If $L \in J(S)^{reg}$ then $H^0(S,L)$, $H^0(S,L \otimes L)$ and $H^0(S,L \otimes L^{-1})$ are $k$-dimensional.

**Proof** Let $C^+$ be a generator of $Q$ in the linear system of $O(1,0)$. Then we have the structure sequence

$$0 \to O(-1,0) \to O \to O_{C^+} \to 0. \tag{2.11}$$

Since $S$ is smooth, we may choose $C^+$ so that $S \cap C^+$ consists of $k$ distinct points. Then we obtain

$$0 \to O_S(-1,0) \to O_S \to O_{S \cap C^+} \to 0 \tag{2.12}$$

and the latter is a skyscraper sheaf, supported at the $k$ points of $S \cap C^+$. Tensor (2.12) with $L$ and take global sections, to obtain

$$\cdots \to H^0(S,L(-1,0)) \to H^0(S,L) \to \mathbb{C}^k \to H^1(S,L(-1,0)) \to \cdots.$$ 

Hence if $L$ is regular, $H^0(S,L)$ is $k$-dimensional, by evaluation of sections on a generically chosen generator $C^+$. The same argument works for $H^0(S,L \otimes L)$, for $L \otimes L(-1,0) = L(0,-1)$ has no cohomology.

Similarly, evaluation of sections on a generic generator $C^-$ in the linear system of $O(0,1)$ shows that $H^0(S,L)$ and $H^0(S,L \otimes L^{-1})$ are both $k$-dimensional. QED

**Lemma 2.4.** If $L$ is regular, then $H^0(S,K \otimes L)$ is $k$-dimensional, and the sequence

$$0 \to H^0(S,K \otimes L) \xrightarrow{\alpha} H^0(S,L) \otimes E^+ \otimes E^- \xrightarrow{m} H^0(S,L(1,1)) \to 0$$

is exact. Here $m$ is the multiplication map which arises by identifying $E^+ \otimes E^-$ with $H^0(Q,O(1,1))$. 

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Proof It is plain that the sequence in question arises by tensoring \((2.2)\) with \(\mathcal{L}\) and taking global sections. Thus \(m\) is surjective if \(H^1(S, K \otimes \mathcal{L})\) vanishes. Tensoring \((2.3)\) with \(\mathcal{L}\) and taking the long exact sequence yields a surjective map

\[
E^+ \otimes H^0(S, \mathcal{L}(0, -1)) \oplus E^- \otimes H^0(S, \mathcal{L}(-1, 0)) \to H^1(S, K \otimes \mathcal{L}),
\]

so that if \(\mathcal{L}\) is regular, \(H^1(S, K \otimes \mathcal{L}) = 0\).

On the other hand, from \((2.2)\), \(K\) is of rank 3 and bidegree \((-1, -1)\), \(K(a, b)\) has bidegree \((3a - 1, 3b - 1)\) and from \((2.8)\),

\[
\text{ind}(K(a, b)) = k(3a + 3b - 2) - 3k(k - 2) = k(3(a + b - k) + 4).
\]

Hence \(\text{ind}(K \otimes \mathcal{L}) = k\) and so \(H^0(S, K \otimes \mathcal{L})\) is \(k\)-dimensional. QED

To summarise: given a DN-map \(\alpha\) we can construct a pair \((S(\alpha), \mathcal{L}(\alpha))\) consisting of a smooth curve \(S\) in \(Q\) of bi-degree \((k, k)\) and a regular line bundle \(\mathcal{L} \to S\); conversely from such a pair \((S, \mathcal{L})\) we can construct a DN-map \(\alpha(\mathcal{L})\). Obviously we would like these two constructions to invert each other. We will show this by showing that if we start with a pair \((S, \mathcal{L})\) then the spectral curve and line bundle, say \((S', \mathcal{L}')\) constructed from the DN-map of \(\alpha(\mathcal{L})\) is isomorphic to \((S, \mathcal{L})\).

Consider the sequence

\[
H^0(S, K \otimes \mathcal{L}) \otimes \mathcal{O}_Q(-1, -1) \xrightarrow{M} H^0(S, \mathcal{L}) \otimes \mathcal{O}_Q \xrightarrow{\text{ev}} \mathcal{O}_S(\mathcal{L})
\]

Then \(S'\) is defined by \(\det(M) = 0\) and \(\mathcal{L}'\) is the cokernel of \(M\).

Clearly \(\text{ev} \circ M = 0\), moreover the proof of Lemma \((2.3)\) shows that for generic points \(z\) of \(S\) we can always find a section \(\psi\) of \(\mathcal{L}\) with \(\psi(z) \neq 0\). At these points the kernel of \(\text{ev}\) has at least co-dimension 1 and thus \(\det(M)\) vanishes. From this we conclude that \(\det(M) = 0\) on \(S\) so that \(S \subset S'\) but both are smooth curves of the same bi-degree so they must be equal. The cokernel of \(M\) is now a line bundle over \(S = S'\). Because \(\text{ev}\) vanishes on the image of \(M\) it induces a map from \(\text{coker}M = \mathcal{L}'\) to \(\mathcal{L}\). As both of these are line bundles of the same degree this map is a holomorphic section of the trivial bundle \(\mathcal{L}' \otimes \mathcal{L}^*\), and hence either the zero section or everywhere non-vanishing. However this section is non-vanishing at points of \(S\) for which there is a non-vanishing section of \(\mathcal{L}\). As we have already argued this happens generically on \(S\) and thus \(\mathcal{L}\) and \(\mathcal{L}'\) are isomorphic.

3 DN evolution and motion on the Jacobian

We turn now to an explanation of the claim that the discrete-time evolution of the DN equations corresponds to straight-line motion on \(J(S)\). More precisely we shall prove Theorems \([1.1]\) and \([1.2]\) here.

3.1 Solutions of the DN system from an algebraic curve

In this section we shall prove Theorem \([1.2]\). So we assume given a smooth curve \(S\) of bi-degree \((k, k)\) in \(Q\) and a regular line bundle \(\mathcal{L}\) over \(S\). By moving \(S\) by an element of
SL₂(ℂ), we may suppose that the point with coordinates ((0 : 1), (1 : 0)) does not lie on S.

According to Theorem 2.2, these data give a DN map

\[ \alpha = \alpha_{00} w_0 z_0 + \alpha_{01} w_0 z_1 + \alpha_{10} w_1 z_0 + \alpha_{11} w_1 z_1 \]

such that each of the \( \alpha_{ij} \) maps \( U(\mathcal{L}) = H^0(S, \mathcal{K} \otimes \mathcal{L}) \) into \( V(\mathcal{L}) = H^0(S, \mathcal{L}) \). Evaluating at \(((0 : 1), (1 : 0))\) gives the element \( \alpha_{01} \); since this is not on S, it follows that \( \alpha_{01} \) is an isomorphism. Thus we may break the symmetry of the problem by using \( \alpha_{01} \) to identify \( U(\mathcal{L}) \) with \( V(\mathcal{L}) \). Having done so, the DN map takes the form

\[ M(w, z) = w_0 z_0 A + w_0 z_1 B + w_1 z_0 + w_1 z_1 D \]

where \( A, B \) and \( D \) are endomorphisms of a \( k \)-dimensional vector space. These will be identified with the DN data of the same name that were introduced in §3.

In order to complete the definition of DN data, we must define the operators \( P^\pm \). These arise directly from the geometry of \( \mathbb{P}_1 \times \mathbb{P}_1 \) from the basic exact sequences \( (2.4) \) and \( (2.5) \) as follows. After tensoring with \( \mathcal{L} \), we obtain from the corresponding long exact sequence,

\[ \cdots E^- \otimes H^0(S, \mathcal{L}(-1, 0)) \rightarrow U(\mathcal{L}) \overset{\phi^+}{\rightarrow} V(\mathcal{L} \otimes \mathcal{L}) \rightarrow E^- \otimes H^1(S, \mathcal{L}(-1, 0)) \cdots \quad (3.1) \]

and

\[ \cdots E^+ \otimes H^0(S, \mathcal{L}(0, -1)) \rightarrow U(\mathcal{L}) \overset{\phi^-}{\rightarrow} V(\mathcal{L} \otimes L^{-1}) \rightarrow E^+ \otimes H^1(S, \mathcal{L}(0, -1)) \cdots . \quad (3.2) \]

Since \( \mathcal{L} \) is assumed regular, \( \phi^\pm \) are isomorphisms. On the other hand we have just identified \( U(\mathcal{L}) \) with \( V(\mathcal{L}) \), so \( \phi^\pm \) gives rise to an isomorphism \( P^\pm : V(\mathcal{L}) \rightarrow V(\mathcal{L} \otimes L^{\pm 1}) \).

Now suppose that \( \mathcal{L} \) and \( \mathcal{L} \otimes L \) are both regular. Then the construction we have just described yields \( k \)-dimensional vector spaces \( V_0 = V(\mathcal{L}), V_1 = V(\mathcal{L} \otimes L) \), maps \( M_0, M_1 \), where

\[ M_r(\eta, \zeta) = \eta \zeta A_r + \eta B_r + \zeta + D_r \quad (3.3) \]

and operators \( P_0^+: V_0 \rightarrow V_1, P_1^- : V_1 \rightarrow V_0 \). (Here \( \eta = w_0/w_1, \zeta = z_0/z_1 \) are being used to make the formulae more readable.) Unravelling the definitions, we obtain the following formulae relating these maps:

\[ M_r(\eta, \zeta) s_r(\eta, \zeta) = 0, \quad (3.4) \]

\[ (\eta A_0 + 1) s_0(\eta, \zeta) = P_0^+ s_0(\eta, \zeta), \quad (3.5) \]

\[ (\eta B_0 + D_0) s_0(\eta, \zeta) = -\zeta P_0^+ s_0(\eta, \zeta), \quad (3.6) \]

\[ (\zeta A_1 + B_1) s_1(\eta, \zeta) = P_1^- s_1(\eta, \zeta), \quad (3.7) \]

\[ (\zeta + D_1) s_1(\eta, \zeta) = -\eta P_1^- s_1(\eta, \zeta), \quad (3.8) \]

for all \( s_r \in V_r \) and \( (\eta, \zeta) \in S \)

The equations have been written at such length to emphasize that here \( (\eta, \zeta) \) are not independent parameters; they live on \( S \).
Proposition 3.1. The data defined by equations (3.3–3.8) satisfy the DN equations on [0, 1]:

\[
P_0^+ D_0 = D_1 P_0^+, \quad P_1^- P_0^+ + A_0 D_0 = B_0, \quad P_1^- A_1 = A_0 P_1^-, \quad P_1^- P_0^+ + D_1 A_1 = B_1.
\]

Proof: Since \( P_0^+ \) is a map \( V_0 \rightarrow V_1 \), the definition of \( M_1 \) implies that

\[
[\eta \zeta A_1 P_0^+ + \eta B_1 P_0^+ + \zeta P_0^+ + D_1 P_0^+] s_0(\eta, \zeta) = 0. \tag{3.9}
\]

Use (3.7) to combine the first two terms, and (3.5) in the third to obtain

\[
[\eta P_1^- P_0^+ - \eta B_0 - D_0 + D_1 P_0^+] s_0(\eta, \zeta) = 0. \tag{3.10}
\]

Now write

\[
DP^+ = [D, P^+] + P^+ D = [D, P^+] + (1 + \eta A_0) D_0 \tag{3.11}
\]

making use of (3.3). Substituting this into (3.10) we obtain, finally

\[
[\eta (P_1^- P_0^+ - B_0 + A_0 D_0) + [D, P^+]] s_0(\eta, \zeta) = 0. \tag{3.12}
\]

One derives similarly from \( M_0 P_1^- = 0 \) a linear combination of the other two equations. The proof is now completed with the aid of the lemma below which says that we can conclude from (3.12) that the two terms must vanish separately. QED

Lemma 3.2. Suppose the relation \((Q_0 \eta + Q_1) s(\eta, \zeta)\) holds for matrices \(Q_0\) and \(Q_1\) and all \((\eta, \zeta) \in S, s \in V(\mathcal{L})\). Then \(Q_0 = 0, Q_1 = 0\).

Proof If not there exist \(a \in \mathbb{C}, s \in V(\mathcal{L})\), such that \((Q_0 a + Q_1) s \neq 0\), where we may suppose \(a\) has the property that the intersection of \(S\) with \(\{\eta = a\}\) consists of \(k\) distinct points \((a, b_1), \ldots, (a, b_k)\). From the given relation, we have

\[
(Q_0 a + Q_1) s(a, b_j) = 0 \quad j = 1, \ldots, k.
\]

Because \(\mathcal{L}\) is regular, \(V(\mathcal{L})\) is identified with \(\mathbb{C}^k\) by evaluation at these points (Lemma 2.3). This contradicts the assumption that \((Q_0 a + Q_1) s \neq 0\). QED

3.2 From the DN equations to spectral data

We now turn to the proof of Theorem 1.1. The basic idea here has already been described in §1.3 and at greater length in Theorem 2.2. What remains to be proved is that if we have a solution of the DN system in \([0, 1]\) (say), then the two spectral curves \(S_0\) and \(S_1\) coincide, and that the two line-bundles \(\mathcal{L}_0\) and \(\mathcal{L}_1\) satisfy \(\mathcal{L}_1 = \mathcal{L}_0 \otimes L\).

For this a Lax formulation of the DN system is needed; for this we are indebted to Richard Ward [12], who noted that

\[
\hat{W}^+ = P^+ - \lambda A
\]

and

\[
\hat{W}^- = P^- + \lambda^{-1} D
\]
form a Lax pair for the discrete Nahm equations. In order to interpret these formulae it is essential to think in terms of ‘discrete gauge theory’, as follows.

Given \( Z = \{r_0, \ldots, r_1\} \) we may think of the vector spaces \( V_r \) as forming a vector bundle \( V \) over the discrete space \( Z \). The \( A, B, D \) become sections of the corresponding bundle of endomorphisms, while \( P^\pm \) are the discrete analogue of connections (more precisely, of parallel transport operators). Denote by \( \Gamma(V) \) the space of sections of \( V \); this is just the set of sequences \( f \) with \( f_r \in V_r \) for all \( r \). Then the formulae for \( \hat{W}^\pm \) make sense as operators on \( \Gamma(V) \). Specifically, if \( f \in \Gamma(V) \), then

\[
(\hat{W}^+ f)_r = P^+_{r-1} f_{r-1} - \lambda A_r f_r.
\]

Ward’s observation is that the condition \([\hat{W}^+, \hat{W}^-] = 0\), for all values of \( \lambda \), is equivalent to the DN equations.

To recover \( S \), we follow standard practice and ask for simultaneous eigensections for \( \hat{W}^\pm \). The commutativity of \( \hat{W}^+ \) and \( \hat{W}^- \) means that \( \hat{W}^+ \) acts on any eigenspace of \( \hat{W}^- \). The conditions that \( \hat{W}^\pm \) have simultaneous eigensections defines an algebraic curve; in the right coordinates, this curve is given precisely by \( \det M(\eta, \zeta) = 0 \).

In order to put this plan into action, we shall replace the above operators by

\[
W^+ = P^+ - \eta A - 1
\]

and

\[
W^- = \eta P^- + \zeta + D
\]

and study the conditions on \((\eta, \zeta)\) under which there exists a section \( f \) which satisfies

\[
W^+ f = 0 \quad W^- f = 0.
\]

Of course Ward’s original operators are recovered by deleting 1 from the definition of \( W^+ \) and \( \zeta \) from the definition of \( W^- \). This particular modification is motivated by the definitions (3.5) and (3.8) of \( P^\pm \) above. For future reference note that in homogeneous coordinates, the Ward operators become

\[
W^+ = z_1 P^+ - w_0 A - w_1, \quad W^- = w_0 P^- + z_0 + z_1 D. \quad (3.13)
\]

Let \( K^\pm = K^\pm(\eta, \zeta) \) be the space of \( W^\pm \)-parallel sections of \( V \). If the rank of \( V \) is \( k \) then since a parallel section is determined by its value at any point, \( K^\pm \) is a \( k \)-dimensional complex vector space and can be identified with any one of the \( V_r \) (by evaluation). We shall look for the condition on \((\eta, \zeta)\) that makes the intersection \( K^+ \cap K^- \) inside \( \Gamma(V) \) non-trivial.

**Theorem 3.3.** Given data \( A, D, P^\pm \) satisfying the discrete Nahm equations, let

\[
B = P^+ P^- + DA = P^- P^+ + AD
\]

and consider

\[
M = \eta \zeta A + \eta B + \zeta + D.
\]

Then the condition \( \det M = 0 \), viewed as an equation for \((\eta, \zeta)\), is independent of \( r \) and is equivalent to the condition that \( K^-(\eta, \zeta) \cap K^+(\eta, \zeta) \neq 0 \).
Proof: Using the above formulae,

\[ M = \eta \zeta A + \eta (P^-P^+ + AD) + \zeta + D = \eta \zeta A + \eta P^-(W^+ + \eta A + 1) + \eta AD + \zeta + D \]

\[ = \eta P^-W^+ + (\eta P^- + \zeta + D) + \eta A(\eta P^- + \zeta + D) \]

where we have used \([P^-, A] = 0\). Recognizing \(W^-\) in the second and third terms of this we obtain

\[ M = \eta P^-W^+ + (\eta A + 1)W^- = \eta P^-W^+ + P^+W^- - W^+W^- . \]

Because \([W^+, W^-] = 0\), this also yields

\[ M = \eta P^-W^+ + P^+W^- - W^-W^+ \]

In particular we see that

\[ M|K^- = \eta P^-W^+, \quad M|K^+ = P^+W^- \]

It follows that \(K^+ \cap K^- \neq 0\) iff \(\det M = 0\). Since the first condition is independent of \(r\), it follows that the condition \(\det M = 0\) is also. The key point is that the operator \(M\) on \(\Gamma(V)\) is of order zero in the sense that its value at \(V_r\) depends only upon \(f_r\). QED

In order to derive the relation between \(L_1\) and \(L_0\), it is convenient to dualize. Then given \(g^t_r \in V^*_r\), we may consider

\[ [g^tW^+]_0 = g^t_1P^+_0 - g^t_0(\eta A_0 + 1) \quad (3.14) \]

and

\[ [g^tW^-]_1 = \eta g^t_0P^-_1 + g^t_1(\zeta + D_1) . \quad (3.15) \]

**Proposition 3.4.** Let \((A, B, D, P_{\text{prim}})\) satisfy the DN equations in \([0, 1]\). Let \(g^t_r \in V^*_r\) \((r = 0, 1)\) satisfy \([g^tW^-]_1 = 0\). Then

(a) If \(g^t_1M_1 = 0\) we have also \([g^tW^+]_0 = 0\) and \(g^t_0M_0 = 0\)

(b) If \(g^t_0M_0 = 0\) we have also \([g^tW^+]_0 = 0\) and \(g^t_1M_1 = 1\)

Before giving the proof note that dualizing the sequence which defines \(L_r\), we obtain on \(S\)

\[ 0 \rightarrow L^*_r \rightarrow V^*_r \overset{M_r}{\rightarrow} V^*_r(1, 1) \]

and that according to this lemma \(W^-\) defines an identification \(L^*_0 \otimes L^{-1} \rightarrow L^*_1\) (recall the homogeneity \(\mathbf{3.13}\) of \(W^-\)). Hence this result completes the proof of Theorem \(\mathbf{1.1}\).

**Proof of Proposition** We shall only do part (a), since part (b) is very similar. Since \(B_1 = (P^+_0P^-_1) + D_1A_1\), we have

\[ g^t_1M_1 = g^t_1P^+(\eta P^-) + g^t_1(\zeta + D_1)(1 + \eta A_1) . \quad (3.16) \]
But we are given
\[ g_1^t(\zeta + D_1) = -\eta g_0^t P_1^- \] (3.17)
so inserting this in (3.16),
\[ g_1^t M_1 = -g_1^t P_0^+(\eta P_1^-) + g_0^t (1 + \eta A_0) \eta P_1^- , \]
where we have used the DN equation \( A_0 P_1^- = P_1^- A_1 \). But the RHS is now equal to
\(-\eta [g^t W^+]_0 P_1^- \). This proves the first part of (a).

We now have the following three equations:
\[ g_1^t(\eta A_1 + \eta B_1 + \zeta + D_1) = 0 \] (3.18)
\[ g_0^t(\eta P_1^-) = -g_1^t(\zeta + D_1) \] (3.19)
\[ g_1^t P_0^+ = g_0^t(\eta A_0 + 1) \] (3.20)
\[ g_1^t P_0^+ = g_0^t(\eta A_0 + 1) \] (3.21)

From the first of these, there exists a vector \( h \) such that
\[ h^t(1, -\eta) = g_1^t(\zeta A_1 + B_1, \zeta + D_1) \]
and comparing with the second, \( h^t = g_0^t P_1^- \), so
\[ g_0^t P_1^- = g_1^t(\zeta A_1 + B_1). \]

Now expand \( g_0^t M_0 \) using \( B_0 = P^- P^+ + A_0 D_0 \)
\[ g_0^t M_0 = g_0^t \eta P_1^- P_0^+ + g_0^t (1 + \eta A_0)(\zeta + D_0) = \eta g_1^t(\zeta A_1 + B_1) P^+ - g_1^t P_0^+ (\zeta + D_0) \] (3.22)

Since \( P_0^+ D_0 = D_1 P_0^+ \), we may rearrange this to obtain
\[ g_0^t M_0 = g_1^t M_1 P^+ , \]
completing the proof of (a). QED

4 Application to hyperbolic monopoles

We now want to relate the general integration of the DN system to the particular solutions that correspond to hyperbolic monopoles, as discussed in §1.4 and prove Theorem 1.4.

We start by assuming that \( S \) is the spectral curve of a hyperbolic monopole of charge \( k \) and mass \( p \in \frac{1}{2} \mathbb{Z} \). We wish to prove first that \( L^r(k - 1, 0) \) is regular for \( r = 1, \ldots , 2p \), i.e. the vanishing theorem
\[ V_r = H^0(S, L^r(k - 2, 0) = 0 \text{ if } r = 1, 2, \ldots , 2p + 1. \]

The first step in the proof is to interpret each of these groups in terms of the cohomology of the bundle \( E \to Q \) that corresponds, by twistor theory, to the hyperbolic monopole. This starts from the description of \( E \) in terms of \( S \), by the two exact sequences
\[ 0 \to L^{-p}(-k, 0) \to E \to L^p(k, 0) \to 0 \] (4.1)
and

\[ 0 \to L^p(0, -k) \to E \to L^{-p}(0, k) \to 0 \]  \hspace{1cm} (4.2)

related by the real structure. The obvious composites \( L^{-p}(-k, 0) \to L^{-p}(0, k) \) and \( L^p(0, -k) \to L^p(k, 0) \) are both given by multiplication by \( F \), the equation of the spectral curve \( S \). It follows that \( L^{2p+k} \) is holomorphically trivial over \( S \).

From the structure sequence

\[ 0 \to \mathcal{O}(-k; -k) \to \mathcal{O} \to \mathcal{O}_S \to 0 \]  \hspace{1cm} (4.3)

we obtain the exact sequence

\[ \to H^0(Q, L^r(k - 2, 0)) \to H^0(S, L^r(k - 2, 0)) \xrightarrow{\delta} H^1(Q, L^r(-2, -k)) \to \]

and for \( r \) in the given range, \( H^0(Q, L^r(k - 2, 0)) = 0 \), so that \( \delta \) is injective. Now tensor (4.2) with \( L^{-p}(-2, 0) \) to get

\[ 0 \to L^r(-2, -k) \to EL^{-p}(-2, 0) \to L^{-2p}(-2, k) \to 0. \]  \hspace{1cm} (4.4)

From this we obtain an exact sequence

\[ H^0(Q, L^r(-2p)(-2, k)) \to H^1(Q, L^r(-2, -k)) \to H^1(Q, EL^{-p}(-2, 0)) \to \]

and again, for \( r = 1, \ldots, 2p + 1 \), the first space vanishes. It follows that the composite map

\[ H^0(S, L^r(k - 2, 0)) \to H^1(Q, EL^{-p}(-2, 0)) \]

is injective. Next we claim that the right-hand side is a summand in the group \( H^1(\mathbb{P}_3, \tilde{E}(-2)) \) where \( \tilde{E} \) is the bundle which represents the \( S^1 \)-invariant instanton corresponding to our given hyperbolic monopole. The non-trivial point in this is the following. There is a map \( \mathbb{P}^0_3 = \mathbb{P}_3 - L^+ \cup L^- \to Q \) where \( L^\pm \) are projective lines. Then \( \tilde{E}|_{\mathbb{P}^0_3} = \pi^*(E) \), but when \( p \) is integral or half-integral, this extends to a holomorphic bundle over the whole of \( \mathbb{P}_3 \), corresponding to the extension of the \( S^1 \)-invariant instanton to all of \( S^4 \).

In this situation, one always has \( H^1(\mathbb{P}_3, \tilde{E}(-2)) = 0 \) \([1]\), so if the above claim is true, then we obtain the vanishing of \( V_r \) that we require. Since \( \pi^* \) gives a map

\[ H^1(Q, EL^{-p}(-2, 0)) \to H^1(\mathbb{P}^0_3, \tilde{E}(-2)) \]

the main thing is to show that everything in the image of this map extends to \( \mathbb{P}_3 \). For this we must go into the description of \( E \) and \( \tilde{E} \) in more detail, and describe in particular the extension of \( \pi^*(E) \) to \( \mathbb{P}_3 \).

### 4.1 Background on \( E \) and \( \tilde{E} \)

Let us begin with the description of the map \( \pi \). In terms of homogeneous coordinates \((z_0, \ldots, z_3)\) in \( \mathbb{P}_3 \), the \( \mathbb{C}^\times \)-action is given by

\[(z_0, z_1, z_2, z_3) \mapsto (\lambda^{1/2}z_0, \lambda^{-1/2}z_1, \lambda^{1/2}z_2, \lambda^{-1/2}z_3),\]

where \( \lambda \) is a non-zero complex number. The bundle \( E \) is the \( \mathbb{C}^\times \)-equivariant bundle associated to the representation of \( \mathbb{C}^\times \) on \( \mathbb{C}^3 \) given by multiplication. The bundle \( \tilde{E} \) is the extension of \( E \) to \( \mathbb{P}_3 \) obtained by identifying the fibers above \( z_0 = 0 \) with the fibers above \( z_0 = 1 \).
the fixed set consists of

\[ L^+ = \{ z_0 = z_2 = 0 \} \quad \text{and} \quad L^- = \{ z_1 = z_3 = 0 \} \]

The map \( \pi \) is the corresponding quotient map and gives affine coordinates \( \eta = z_0/z_2, -1/\zeta = z_1/z_3 \) in \( Q \). In these coordinates, the anti-diagonal has the equation \( 1 + \eta\zeta = 0 \).

Now \( \pi^*O(a, b) \) is isomorphic to \( O(a + b) \) over \( \mathbb{P}^3_3 \); the different possible values of \( a \) and \( b \) are distinguished on \( \mathbb{P}^3_3 \) by the different possible lifts of the \( \mathbb{C}^* \)-action. Indeed, the lifted action on \( \pi^*O(a, b) \) is given by

\[ \pi^*f(\lambda^{1/2}z_0, \lambda^{-1/2}z_1, \lambda^{1/2}z_2, \lambda^{-1/2}z_3) = \lambda^{(a-b)/2} \pi^*f(z_0, z_1, z_2, z_3). \]

Combining this with the overall homogeneity of \( f \), we have found a way to represent the pull-back of a local section \( f \) of \( O(a, b) \) on \( \mathbb{P}^3_3 \); as a function satisfying

\[ \pi^*f(\lambda z_0, z_1, z_2, z_3) = \lambda^a \pi^*f(z_0, z_1, z_2, z_3) \]

and

\[ \pi^*f(z_0, \lambda z_1, z_2, \lambda z_3) = \lambda^b \pi^*f(z_0, z_1, z_2, z_3) \]

In particular, such a section extends smoothly through \( L^+ \) if \( a \geq 0 \) and through \( L^- \) if \( b \geq 0 \). Using this idea, we can see how \( \pi^*E \) extends to \( \mathbb{P}^3_3 \) as follows.

The extension class defining (4.1) is defined by taking a trivialization \( s \) of \( L^{2p+k} \) and mapping it (via (4.3) \( \otimes L^{2p+k} \)) into \( H^1(L^{2p+k}(-k, k)) \). Explicitly, we may take \( s_+ \) to be smooth section of \( L^{2p+k} \) supported near to \( S \), such that \( s_+ \neq 0 \) on \( S \), but \( \overline{\partial}s_+ = 0 \) on \( S \). Then the extension class is defined by \( \theta_+ = \overline{\partial}s_+/F \). Similarly, (4.2) is represented by \( \theta_- = \overline{\partial}s_-/F \), where \( s_- \) is a section of \( L^{-2p-k} \) with the same properties as \( s_+ \). It follows that the pull-back of \( E \) may be identified with the smooth bundle \( C^\infty(p, -p-k) \oplus C^\infty(-p, p+k) \) with the twisted \( \overline{\partial} \)-operator

\[ \overline{\partial}_+(u, v) = (\overline{\partial}u + \theta_+v, \overline{\partial}v) \]

and equally with \( C^\infty(-p - k, p) \oplus C^\infty(p + k, -p) \) with the twisted \( \overline{\partial} \)-operator

\[ \overline{\partial}_-(u, v) = (\overline{\partial}u + \theta_-v, \overline{\partial}v). \]

Now \( \theta_+ \) is of bidegree \((2p, -2p-2k)\) while \( \theta_- \) is of bidegree \((-2p-2k, 2p)\) so the former extends through \( L^+ \) and the latter through \( L^- \). It follows that \( \pi^*E \) extends to \( \mathbb{P}^3_3 \), as required.

With this understood, we can try to prove our claim. \( V_r \) maps into \( H^1(Q, \mathcal{O}(r - 2, -k - r)) \) which vanishes if \( r = 1 \) and extends through \( L^+ \) if \( r \geq 2 \). On the other hand,

\[ H^0(S, L^r(k - 2, 0)) = H^0(S, \mathcal{O}(r - 2p - 2, 2p + k - r)) \rightarrow H^1(Q, \mathcal{O}(r - 2p - 2 - k, 2p - r)) \]

injectively for \( r = 1, 2, \ldots, 2p + 1 \) and again this vanishes identically if \( r = 2p + 1 \) and extends through \( L^- \) if \( r \leq 2p \). Hence each \( v \in V_r \) gives rise to an element of \( H^1(EL^r-p(-2, 0)) \) which extends to \( \mathbb{P}^3_3 \) on being pulled back. The claim now follows from the vanishing theorem of [1].
4.2 Boundary conditions

To finish the proof of Theorem 1.4 we need to use the boundary conditions satisfied by the \((A, B, D)\) coming from a monopole \([3]\). There are actually two of these which are interchanged by the real structure but for our purposes it is enough to know that \(B_1 - D_1 A_1\) is rank 1. Letting \(X = B_1 - D_1 A_1\) we have

\[
M_1(\eta, \zeta) = (\zeta + D_1)(A_1 \eta + 1) + X \eta.
\]

We can use this to factorise

\[
M_1 : \mathcal{O}^k(-1, -1) \to \mathcal{O}^k
\]

into \(M_0(\eta, \zeta) = G(\zeta) \circ F(\eta)\) where

\[
F(\eta) : \mathcal{O}^k(-1, -1) \to \mathcal{O}^{k+1}(0, -1)
\]

and

\[
G(\zeta) : \mathcal{O}^{k+1}(0, -1) \to \mathcal{O}^k
\]

as follows. We identify the image of \(X\) with \(\mathcal{O}\) and then define \(F(v)\) by \(F(v) = (A_0(v)\eta + v, X(v)\eta)\). We define \(G\) by \(G(v, w) = \zeta v + D_0(v) + w\) where we identify \(\mathcal{O}\) with the image of \(X\) which is inside \(\mathcal{O}^k\).

It follows from \([2]\) that the spectral curve of a monopole does not intersect the anti-diagonal. In particular it cannot contain a generator of the quadric. We claim that this implies that \(F\) is injective and \(G\) is surjective. To see this note that if \(F(\eta)\) is not injective for some \(\eta\) then \(M_0(\eta, \zeta) = G(\zeta) F(\eta)\) is not injective for that \(\eta\) and all \(\zeta\) so \(\det(M_0)\) would vanish on a generator which is not possible. Similarly \(G(\zeta)\) must be onto for all \(\zeta\).

Let \(J\) be the kernel of \(G\) and \(V\) the image of \(F\). Then from the discussion in the previous paragraph the only way that \(\det(M_0)\) can vanish is when \(J \subset V\). Hence the spectral curve is given precisely by the condition \(J \subset V\). Moreover the cokernel of \(M_0\) is the cokernel of \(G\) and hence we have

\[
0 \to J \to V \to \mathcal{O}^k \to \text{coker}(M_0) \to 0
\]

as an exact sequence of bundles over the spectral curve. So we have

\[
\text{coker}(M_0) = \det(V)^* \otimes J.
\]

But we also have that \(V = \mathcal{O}^k(-1, -1)\), as \(F\) is injective, so \(\det(V) = \mathcal{O}(-k, -k)\). Moreover we have

\[
0 \to K \to \mathcal{O}^{k+1}(0, -1) \to \mathcal{O}^k
\]

and hence

\[
K = \det(\mathcal{O}^{k+1}(0, -1)) = \mathcal{O}(0, -k - 1).
\]

Finally

\[
\text{coker}(M_0) = \mathcal{O}(k, -1).
\]

Now applying Theorem 1.3 proves Theorem 1.4.
5 Concluding Remarks

We have given a rather complete account of the ‘discrete linearization’ of the discrete Nahm equations on the Jacobian of algebraic curves in $\mathbb{P}_1 \times \mathbb{P}_1$. We have also shown that a solution of these equations, corresponding by Braam–Austin to a hyperbolic monopole, arises by a canonical application of our construction, the algebraic curve in this case being the spectral curve of the monopole.

Apart from Conjecture 1.5, various questions remain. In one direction, it would be of interest to compare our results with other approaches to discretizations of integrable systems such as [10]. One could also ask for an elaboration of the method to yield explicit (e.g. in terms of $\vartheta$-functions) solutions of the DN system. This would presumably entail an appropriate analogue of the methods developed in [6]. In this connection we note that in [12] the general solution of $k = 2$ is written down in terms of elliptic functions (though the boundary conditions are not considered in detail there). A special case, is a solution in trigonometric functions, corresponding to the axially symmetric hyperbolic monopole with $k = 2$, is given by the following.

Example 5.1. Pick $p > 0$, let $\phi = \pi/(2p + 2)$ and

$$S_p = \{(\eta - e^{i\phi}\zeta)(\eta - e^{-i\phi}\zeta) = 0\}.$$  

Then $S_p$ is a real reducible curve in $Q$ and the restriction of $L^{2p+2}$ to $S$ is holomorphically trivial. Applying our construction, with $L_1 = \mathcal{O}(2, -1)$ as in Theorem 1.4, we obtain the solution

$$A_r = \begin{pmatrix} 0 & -s/s_{r+1} \\ 0 & 0 \end{pmatrix}, \quad B_r = \begin{pmatrix} -s_{r+1}/s_r & 0 \\ 0 & -s_r/s_{r+1} \end{pmatrix}, \quad D_j = \begin{pmatrix} 0 & 0 \\ s/s_r & 0 \end{pmatrix},$$

with

$$P_r^+ = \begin{pmatrix} 1 & 0 \\ 0 & s_r/s_{r+1} \end{pmatrix}, \quad P_r^- = \begin{pmatrix} -s_{r+1}/s_r & 0 \\ 0 & -1 \end{pmatrix},$$

where $s = \sin \phi$ and $s_k = \sin k\phi$. The solution satisfies the Braam-Austin boundary condition at $r = 1$, and the corresponding one at $r = 2p$ if this is an integer. In this case, the DN equations come down to the trigonometric identity

$$\sin a\phi \sin(a + 2)\phi + \sin^2 \phi = \sin^2(a + 1)\phi.$$ 

The solution is also real ($A_r = -D_r^*$, $B_r = B_r^*$, $P_r^+ = -[P_{r+1}^-]^*$) with respect to the hermitian inner product $g_r$ on $V_r = H^0(S, L^r(1, 0))$ given by the matrix

$$g_r = \begin{pmatrix} s_{r+1} & 0 \\ 0 & s_r \end{pmatrix}.$$ 

Note that these formulae are algebraic in the coefficients of $S_p$ as is to be expected.

We shall now make some remarks which fit our construction in a more general framework. First of all, in §§2–3 we have closely followed [8] and have used ‘elementary’ arguments throughout. However, the representation of $\mathcal{L}$ over $S$ as a cokernel is a special
Now from (2.3) and the assumption that $L \delta$:

the description of the bundle $\overline{\mathcal{E}}$ the DN equations; we claim that these are essentially the linear operators of the monad $d$ that the composite $\bigoplus$ computed by the Beilinson spectral sequence with $[3, \S 3]$. The assumption that $L$ is regular gives that $\langle w, w' \rangle$ and $\langle z, z' \rangle$, $w, z, w', z'$ being the obvious homogeneous coordinates on the two factors.

Then $p_{1*}(\mathcal{L}' \otimes \mathcal{O}_\Delta)$ is isomorphic to $\mathcal{L}$. On the other hand, this direct image is also computed by the Beilinson spectral sequence

$$E_1^{ij} = R^j p_{1*}(\mathcal{R}_i \otimes \mathcal{L}').$$

The assumption that $\mathcal{L}$ is regular gives that $E_1^{-1,j} = 0$ and it is also easily checked that $E_1^{-2,0} = E_1^{0,1} = 0$. Hence we obtain $E_1 = E_2$ and an exact sequence

$$0 \to H^1(\mathcal{L}(-1, -1) \otimes \mathcal{O}(-1, -1)) \xrightarrow{d_2} H^0(\mathcal{L}) \otimes \mathcal{O} \longrightarrow \mathcal{L} \to 0.$$ 

Now from (2.3) and the assumption that $\mathcal{L}$ is regular, the connecting homomorphism $\delta : H^0(S, \mathcal{K} \otimes \mathcal{L}) \to H^1(\mathcal{L}(-1, -1))$ is an isomorphism. Finally it can be checked that that the composite $d_2 \circ \delta$ agrees with $\alpha(\mathcal{L})$.

Our last remarks concern the operators $W^\pm$ of $\S 3.2$ that gave a ‘Lax representation’ for the DN equations; we claim that these are essentially the linear operators of the monad description of the bundle $\mathcal{E}$ over $\mathbb{P}_3$. Indeed a straightforward but tedious comparison with $[3, \S 3]$, shows that their monad (3.1) can be naturally interpreted as the sequence

$$\bigoplus_j V_{2j} \otimes L^{-j}(-\frac{1}{2}, -\frac{1}{2}) \xrightarrow{(W^+, W^-)} \bigoplus_j V_{2j} \otimes (L^\frac{1}{2} - j \oplus L^{\frac{1}{2} - j}) \xrightarrow{(-W^-, W^+)} \bigoplus_j V_{2j} \otimes L^{-j}(\frac{1}{2}, \frac{1}{2})$$

down on $Q$. Here we have labelled the vector spaces as in Braam and Austin because it is more symmetrical, and $W^\pm$ are as in (3.13). The main point we want to make is that the basic condition that the monad maps form a complex now becomes the integrability condition $[W^+, W^-] = 0$. It seems very likely that with a little further work one should be able to obtain a canonical identification of $V_{2j}$ here with the space $H^0(S, L^{j+k+\frac{1}{2}}(k, -1, 0))$, thereby giving another proof of Theorem 1.4, but we shall not pursue this here. We remark also that in $[3]$ the Beilinson spectral sequence is applied to give monad descriptions of stable bundles over Hirzebruch surfaces and in particular over $Q$. Those monads are different from the one above, and it would be interesting to clarify the relation between them.
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