Algebra versus analysis in statistical mechanics and quantum field theory

Barry M. McCoy
Institute for Theoretical Physics
State University of New York
Stony Brook, N.Y. 11794

Abstract

I contrast the profound differences in the ways in which algebra and analysis are used in physics. In particular I discuss the fascinating phenomenon that theoretical physicists devote almost all their efforts to algebraic problems even though all problems of experimental interest require some methods of analysis.

1 Introduction

When I first started research in physics I was certain that the mathematics used by physics was analysis. After all, physics was all phrased in the language of partial differential equations and integrals. We spoke of Fourier series and complex analysis and we all learned about norms in Hilbert spaces. Algebra, if it was considered at all, was concerned with the classification of finite groups and the determination of which polynomial equations could be solved in terms of radicals. It was considered to be a subject of great intellectual interest but not much practical importance.

But when I reflect on the many decades which have passed since those days of innocence it becomes clear that I have never genuinely used analysis at all. I have never really cared if the norm in some space was $L^2$ or $L^1$ and I have certainly never used the fact that there exist functions which are everywhere continuous but nowhere differentiable. In the end all the integrals I ever did were algebraic and all the special functions I ever used turned out to have group theory interpretations.

Moreover in the years since I was a graduate student there has been an explosion of knowledge about many body problems (in either statistical mechanics or quantum field theory) which can be explicitly solved and in the end the “solvability” of these problems depends on the fact that they have an algebraic structure. Not coincidently this explosion of knowledge about solvable statistical mechanical problems is totally connected with the explosive growth in our knowledge of algebra in the last 35 years.

The impact of algebra in physics has been so complete that it now can be said that modern physics has almost completely abandoned analysis and is now totally dominated by algebra.

\footnote{e-mail mccoy@insti.physics.sunysb.edu}
On the other hand there has very recently been a new development in statistical mechanics which does not seem to fit well into this algebraic framework. This development is the discovery by Orrick, Nickel, Guttmann, and Perk [1]-[2] that the magnetic susceptibility of the two dimensional Ising model has a natural boundary in the complex temperature plane. I believe that this phenomenon of the natural boundary is not related to algebra but instead owes its existence to analysis.

However, the boundary between algebra and analysis is nebulous and vague and the purpose of this lecture is to clarify the concepts in a way which makes contact with the way in which physicists actually treat the subjects. This will be done by contrasting the following 8 properties and then using these properties to discuss the new phenomena found for the Ising susceptibility and, more generally, how the addition of a magnetic field to the two dimensional Ising model may turn an algebraic problem into a problem in analysis.

| Algebra                        | Analysis                                      |
|-------------------------------|-----------------------------------------------|
| 1. Only finite processes allowed | Infinite processes are needed                  |
| 2. Continuable complex functions | Functions of real variables                  |
| 3. Integrable systems         | Perturbation theory                           |
| 4. Solvable models            | Series expansions                             |
| 5. S-matrix theory            | Field theory                                  |
| 6. Overdetermined holonomic systems | Unstable equations (small denominators)     |
| 7. Computable in polynomial time | Not computable in polynomial time          |
| 8. Simple fixed points        | Multiple length scales                        |
| 9. Ising model at \( H = 0 \) | Ising model at \( H \neq 0 \)               |

However it is possible that not all mathematicians and physicists will be in agreement with some or all of what I have to say and to further explain my intentions I will conclude this introduction by quoting from the introduction to “Penguin Island” written by Anatole France who discusses corresponding problems in the writing of history.

“What is the good, my dear sir, of giving yourself so much trouble, why compose a history when all you need to do is to copy the best-known ones in the usual way? If you have a fresh view or an original idea, if you present men and things from an unexpected point of view, you will surprise the reader. And the reader does not like being surprised. He never looks in a history for anything but the stupidities that he knows already. If you try to instruct him you only humiliate him and make him angry. So do not try to enlighten him; he will only cry out that you insult his beliefs.

“Historians copy from one another. Thus they spare themselves trouble and avoid the appearance of presumption. Imitate them and do not be original. An original historian is the object of distrust, contempt, and loathing from everybody.

“Do you imagine, sir,” added he, “that I should be respected and honored as I am if I had put innovations into my historical works? And what are innovations? They are impertinences.”
2 Finite versus infinite processes

The first distinction between algebra and analysis which we meet in our education is that analysis always seems to involve limits and infinite processes while algebra never involves limits and uses only finite operations. But even this most elementary distinction can be misleading and requires at least a passing discussion.

We of course define derivatives as

\[
\frac{df(x)}{dx} = \lim_{\Delta \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

and because a limiting process is used we call this analysis. But in fact in most elementary courses we actually only consider functions such as

\[x^n, \sin x, e^x, \ln x.\]

Thus for example we have

\[
\frac{dx^2}{dx} = \lim_{\Delta \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x
\]

and even though we have written limit signs in almost every step the entire computation is nothing but algebra. Indeed from this point of view most calculus problems are nothing more than an exercise in algebraic bookkeeping and this is, in fact, how most students treat their first course in calculus.

We actually do not see much true analysis until we prove that there exist functions which are everywhere continuous but nowhere differentiable, but this theorem is much too hard for a beginning course.

We similarly make serious oversimplifications when we first introduce Riemann integrals as limits of sums. This appears to be a concept in analysis but again in practice almost our entire attention is restricted to integrands which have algebraic or isolated essential singularities and thus we are usually dealing with a branch of algebraic geometry without actually admitting it. Furthermore the most serious parts of analysis as used in integration arise only later when the concepts of Lebesgue measure and Lebesgue integration are introduced.

On the other hand in the last 50 years we have learned that algebras can be infinite as is seen in following brief history of the subject

| Year | Theorist(s) | Description |
|------|-------------|-------------|
| 1870's | Lie | Finite Continuous Groups |
| 1944 | Onsager | Infinite loop algebra |
| 1968 | Kac/Moody | Affine Lie Algebra |
| 1985 | Drinfeld/Jimbo | Quantum finite and affine Lie algebra |

Here, for such objects as affine Lie algebras which have infinite dimension, the finiteness can be seen for example in the fact that they can be described by a finite number of generators and relations. In fact one of the first things done in the theory of these infinite dimensional algebras is to prove that the definition in terms of the finite number of generators and relations and the definition in terms of an infinite number of modes are in fact equivalent. This serves as a very nice example that something which might be thought of as part of analysis is in fact a part of algebra.
3 Complex versus real variables

Closely connected with the distinction between finite and infinite is the distinction between real and complex variables and here also great confusion sets in because of the way we teach calculus. Beginning students know nothing of complex variables and thus we have no choice but to consider all variables to be real. But because we only use algebraic (or trigonometric) functions all of the functions we actually use can be continued to complex variables. The upshot is that we are actually using analytic functions of a complex variable without ever admitting it.

This confusion goes much beyond the elementary calculus courses. Consider the power series

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]  

which converges in the circle \(|z| < R\). This function \(f(z)\) is analytic inside this circle. In general this is the end of the story and the function can never be continued outside the circle \([9]\). The most familiar example of such a function with a natural boundary is the theta constant

\[ \theta_3(0; q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \]  

which has a natural boundary at \(|q| = 1\). More generally we may consider series of the form

\[ \sum_k a_{n_k} z^{n_k} \quad \text{where} \quad \lim_{k \to \infty} k/n_k = 0. \]

These series are called “lacunary series” and they all have natural boundaries. Another example is the series \([9]\)

\[ \sum_{n=0}^{\infty} \pm c_n z^n \]  

where the signs \(\pm\) are chosen as independent random variables. Among these series there is at least one which is noncontinuable and if \(|c_n|^{1/n} \to 1\) the continuable power series form at most a countable subset. Hille remarks (page 88 of vol 2 of ref.\([9]\)) that “this fact lends some substance to the assertion that is commonly made that a power series “normally” is noncontinuable.”

Nevertheless even though generically analytic functions have natural boundaries most analytic functions used by physicists do not have any natural boundaries at all and can be analytically continued to multi (possibly infinite) sheeted Riemann surfaces. These Riemann surfaces almost by definition have an algebraic structure and are often related to the representation theory of groups. Furthermore the theta functions which do have natural boundaries are at the foundation of algebraic geometry. Thus in practice physicists’ use of complex analysis is limited to algebra.

Physicists love complex variables and in particular we love functions which can be analytically continued. Indeed the very notion of solving a problem is taken to mean that the answer can be expressed as some special functions in complex variable theory. We have long ago given away real variables to the mathematicians, applied mathematicians, computer scientists, and people who forecast the weather.
4 Classical integrability versus canonical perturbation theory

The meaning and import of these two preceding distinctions between algebra and analysis is made very concrete in classical mechanics in the distinction between an integrable system and a perturbation expansion.

**Definition of Classical Integrability (Liouville 1836)**

A classical system with $2N$ degrees of freedom and a Hamiltonian $H$ is said to be integrable if there are $N$ independent operators $J_i$ ($i = 1, \cdots, N$) with $J_1 = H$ such that

$$\{J_i, J_k\} = 0$$

where $\{A, B\}$ denotes the Poisson Bracket. (Here when $i$ or $k = 1$ means that $J_k$ is a constant of the motion. The remaining equations in (8) say that these constants are all compatible (in involution) with each other.)

These systems may all be solved in closed form by canonical transformations and have four important properties

1. These systems have no chaos
2. The dependence on the initial conditions is smooth
3. The equations of the orbits may be continued into the complex plane
4. There is sufficient analyticity in $t$ as $t \to \infty$ to determine the asymptotic behavior of the orbit from (slightly) imprecise initial data.

If Liouville’s condition fails to hold then properties 1-4 no longer hold. However if a system is in some sense “close” to an integrable system we often study it by setting up a “perturbation” theory in some parameter $\lambda$ such that at $\lambda = 0$ the system is integrable. These classical perturbation theories have been studied for almost 2 centuries and they all have the property that they never converge because of resonances due to “small denominators.” The key to understanding these nonconvergent expansions is the famous theorem of Kolmogorov, Arnold and Moser (KAM) which dates from the mid 1950’s → ’60’s [120 years after Liouville]. This theorem loosely states that even though commuting constants of the motion do not exist the system still has a finite measure of orbits (the KAM tori) which behave as if they did have all the constants of the integrable system. However there are also a finite measure of chaotic orbits and these orbits are infinitesimally close to the KAM tori. When the “perturbation” parameter is made large enough it can happen that these KAM tori disappear altogether.

5 Commuting transfer matrices versus series expansions

In the last 30 years this classical notion of Liouville Integrability has been extended to lattice statistical mechanics and quantum systems by means of the notion of the
commuting transfer matrix which was first introduced into physics with the work of Baxter [11] on the 8 vertex model.

A transfer matrix for a statistical system defined on a (say) square lattice with nearest neighbor interactions builds up the full partition function, which is defined as

\[ Z = \sum_{\text{all states}} e^{-E/kT} \]  

(9)

where \( E \) is the interaction energy of the system, by adding one row of interactions at a time. The matrix \( T \) is thus

\[ T_{\{\sigma'\},\{\sigma\}} = e^{-\mathcal{E}(\{\sigma'\},\{\sigma\})/kT} \]  

(10)

where \( \{\sigma'\} \) and \( \{\sigma\} \) denote the configurations of the variables in 2 adjacent rows and \( \mathcal{E}(\{\sigma'\},\{\sigma\}) \) is that part of the interaction energy which depends only on these two configurations. This full partition function (9) is now given as

\[ Z = \text{Tr}T(u)^{L_h} \]  

(11)

where \( L_h \) is the number of rows of the lattice.

As an example we can consider a square lattice where variables \( \sigma = \pm 1 \) are on the vertical links and variables \( \mu = \pm \) are on the horizontal links. The transfer matrix which adds one row to the lattice is then

\[ T(\{\sigma'\},\{\sigma\}) = \sum_{\mu_1,\ldots,\mu_N} w(\sigma'_1,\sigma_1|\mu_1,\mu_2)w(\sigma'_2,\sigma_2|\mu_2,\mu_3)\cdots w(\sigma'_N,\sigma_N|\mu_N,\mu_1). \]  

(12)

**Definition**

A statistical system with a transfer matrix \( T \) depending on a parameter \( u \) is said to be integrable if

\[ [T(u),T(u')] = 0 \]  

(13)

Typically these families of transfer matrices have a value of \( u \) (which may be taken to be zero for example) such that \( T(0) \) is either the identity or a shift operator. Here we can expand

\[ T(u) = T(0)[1 + uH + O(u^2)] \]  

(14)

where \( H \) will be of the form

\[ H = \sum_{j=1}^{N} H_{j,j+1} \]  

(15)

where \( H_{j,j+1} \) depends only of the variables on sites \( j \) and \( j + 1 \) and \( N \) is the number of sites in the row. If we now consider \( n^{th} \) derivatives of \( T(u) \) with respect to \( u \) we see that from (13) we obtain the same condition as the classical Liouville condition (8) except that the Poisson brackets are replaced by commutators.

The condition (13) on the transfer matrix is a global condition which depends on all the spins in the row. To be fulfilled it is sufficient for a local condition on the interaction energies to hold. This local condition is referred to as a star-triangle or Yang–Baxter
equation. For the case where the transfer matrix is given by (12) this star triangle equation is

$$\sum_{\gamma,\mu,\nu} w(\alpha, \gamma|\mu, \mu')w'(\gamma, \beta|\nu, \nu')w''(\mu', \nu'|\nu'', \mu', \mu'') = \sum_{\gamma,\mu,\nu,\mu'} w''(\mu, \nu'|\nu'', \mu', \mu'')w(\alpha, \gamma|\mu, \mu')w'(\gamma, \beta|\nu, \nu')$$

(16)

where the spectral variable in $w, w'$ and $w''$ is $u, u', u''$ respectively. Whenever (13) and the corresponding star-triangle holds it has been possible to exactly compute the eigenvalues of $T(u)$ by algebraic means. Furthermore for most of these models the order parameters can be computed.

The most famous of these models is the Ising model in zero magnetic field in two dimensions. This model has a variable $\sigma_{i,j} = \pm 1$ at each site $i, j$ of a lattice with $L_v$ rows and $L_h$ columns which interact with an interaction energy

$$E = -\sum_{i,j} (E_v \sigma_{i,j} \sigma_{i+1,j} + E_h \sigma_{i,j} \sigma_{i,j+1})$$

(17)

The maximum eigenvalue of the transfer matrix was computed by Onsager [3] and from this it is found that in the thermodynamic limit the free energy per site is (with $N = L_v L_h$)

$$f = -kT \lim_{N \to \infty} \frac{1}{N} \ln Z_N$$

$$= -kT \left( \ln2 + \frac{1}{2} (2\pi)^{-2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \ln[\cosh 2E_v/kT \cosh 2E_h/kT$$

$$- \sinh 2E_h/kT \cos \theta_1 - \sinh 2E_v/kT \cos \theta_2]) \right)$$

(18)

This free energy has a singularity of the form

$$|T - T_c|^2 \ln|T - T_c|$$

(19)

at a temperature $T_c$ defined from

$$\sinh 2E_v/kT_c \sinh 2E_h/kT_c = 1.$$ 

(20)

On the finite lattice the partition function has no singularities but instead has zeros in the complex $T$ plane which as $N \to \infty$ lie on the curve

$$\cosh 2E_v/kT \cosh 2E_h/kT - \sinh 2E_h/kT \cos \theta_1 - \sinh 2E_v/kT \cos \theta_2$$

(21)

where $0 \leq \theta_1, \theta_2 \leq 2\pi$. In the limit $N \to \infty$ this curve of zeroes becomes the logarithmic singularity and branch cut of (19).

The spontaneous magnetization (order parameter) of the Ising model is [14] (for $T < T_c$)

$$M = (1 - k^2)^{1/8}$$

(22)

where

$$k = (\sinh 2E_v/kT \sinh 2E_h/kT)^{-1}.$$ 

(23)
It should be remarked that the free energy depends on $E_v/kT$ and $E_h/kT$ separately while $M$ depends only on the single variable $k$. The variable $u$ of the commuting transfer matrix relation (13) is the variable which parametrises the curve in the plane of $E_v/kT$, $E_h/kT$ on which the $k$ of (23) is constant. This curve may be expressed in terms of Jacobi elliptic functions where $k$ is the modulus and $u$ is the variable.

These models with commuting transfer matrices play the role in statistical mechanics and quantum spin chains which the Liouville integrable models play in classical mechanics. The search for solutions of (13) has led to the invention of quantum groups and has been the physics behind the many discoveries made in algebra in the last 30 years.

On the other hand most problems in statistical mechanics are not of this integrable form and these models have been studied for over 60 years by means of a variety of series expansions. The first of these expansions is the expansion of the pressure in terms of the density as set up by Mayer [12] in the 30’s. In the 50’s the study of high and low temperatures series expansion was initiated [13] and work on these expansions continues to the present day. These series expansions play the role analogous to canonical perturbation theory in classical mechanics. What is lacking in this analogy of commuting transfer matrices and series expansions with classical mechanics is that we do not know for the statistical systems what plays the role of the KAM theorem.

6 S-matrix versus field theory

In 1967 Yang [15] studied the S matrix for scattering of particles in one space and one time dimension which interact with a delta function potential and found that 1) the 2 body S matrix satisfies a set of overdetermined equations and 2) that all n-body S matrix elements are determined from the 2 body S matrix. Remarkably enough this consistency equation for the 2 body S matrix is exactly the star-triangle equation of Baxter [11] with the parameter $u$ replaced by the momentum $p$. It is for this reason that equation (16) is often referred to as the Yang–Baxter equation.

Just as solutions of the star triangle equation may be taken as the starting point for the study of integrable systems in statistical mechanics so the Yang–Baxter equation may be taken as the starting point for the study of solvable S matrix theories in 1 + 1 dimensions. This is the point of view taken by Zamolodchikov [16] in 1981 in his initial study of the theory of factorizing S matrices and since then a large number of these factorizing S matrix theories have been found. These factorizing S-matrices in 1 + 1 dimensions realize the ideas of the S-matrix bootstrap which originated in work in particle physics [17] in the early 60’s. There is obviously a close match (if not a 1-1 correspondence) between statistical systems (in 2 dimensions) with commuting transfer matrices and quantum S matrix systems in 1 + 1 dimensions.

In contrast with these factorizable S matrices is the concept of Lagrangian field theory. There are many approaches to this subject but at least at the computational level the subject is usually treated in perturbation theory in terms of Feynman diagrams. These Feynman diagrams plays the role in field theory which canonical perturbation theory plays in classical mechanics. Just as in classical mechanics the perturbation theories do not converge. However, in contrast with classical mechanics the analogue of the KAM theorem is not known,
Many (if not most) of the S matrices which satisfy the Yang Baxter equation may be interpreted as the S matrix for scattering in some suitable Lagrangian field theory. Thus, for example it is said that the sine Gordon Field theory has an exactly known S matrix. On the other hand this identification is made in a curious fashion because in field theory it is the off mass shell Greens function which is the object of fundamental interest and with the single exception of the Ising model these off mass shell greens functions are not known for any of the models with factorizable S matrices and the agreement of the S matrix with the Lagrangian is made by arguments such as saying that they have the same symmetry properties and that they agree to the one loop level.

There is a folk theorem which says that in theories with factorizable S-matrices the semi-classical treatment of the corresponding field theory is exact.

These integrable statistical systems and S matrix theories play the role in statistical mechanics and field theory that Liouville integrable systems play in classical mechanics and their relation to general systems may be summarized as follows.

| Integrable                          | Perturbation                        |
|-------------------------------------|-------------------------------------|
| Liouville’s Condition               | Canonical Orbit Perturbation theory |
| Commuting transfer matrices         | Mayer graph expansions              |
| S-matrix theory                     | High temperature expansions         |
|                                     | Feynman Diagrams                     |

7 Holonomic and D finite systems

Definition of holonomic

A system of partial differential equations is said to be holonomic \[18\] or maximally overdetermined if only a finite number of initial conditions (instead of a function) need to be specified to uniquely determine the solution.

This notion of holonomic is the multivariable generalization of the notion of D-finiteness.

Definition of D-finite

A function \(f(z)\) is said to be D(ifferentially) finite \[19\] if there exists an integer \(k\) and polynomials \(P_0(z), \ldots, P_k(z)\) such that \(P_0(z) \neq 0\) and

\[
0 = P_0(z)f(z) + P_1(z)f'(z) + \cdots + P_k(z)f^{(k)}(z) \tag{24}
\]

In 1977 Kashiwara and Kawai \[18\] proved the remarkable theorem that every Feynman diagram is a holonomic function of the external momentum. However, even though this is true for each diagram in the perturbation expansion the complete sum of all the diagrams is not in general holonomic.

On the other hand there are field theories where the entire Greens function is still holonomic. This is illustrated by the field theory based on the Ising model and is the reason that the series of papers of Sato, Miwa, and Jimbo \[20\] on the Ising model is
entitled “Holonomic Quantum Field Theory.” It is strongly suspected that all of the
field theories which have factorizable S-matrices are holonomic field theories. These field
theories then are clearly analogous to the integrable systems of classical mechanics and
the fact that the full Greens function does not share the property of being holonomic
with the individual Feynman diagrams seems to be related to the fact that in classical
mechanics the canonical perturbation theory does not converge and that a finite fraction
of the orbits will not lie on $N$ dimensional KAM tori.

Furthermore there is an important connection between the concepts of D-finiteness
and the absence of natural boundaries. In particular let

$$F(x, y) = \sum_{n \geq 0} y^n H_n(x)$$

be a D-finite series in $y$ with rational coefficients. For $n \geq 0$ let $S_n$ be the set of poles of
$H_n(x)$ and let $S = \cup_n S_n$. Then the set $S$ has only a finite number of accumulation points
and thus has no natural boundaries [21]. This strongly suggests that natural boundaries
(if they arise at all) will not be present in any finite order or perturbation theory but
will only occur in the full sum.

8 Polynomial versus NP complete systems

Thus far I have contrasted algebra and analysis by using concepts familiar to most
physicists and mathematicians. But an even broader comparison of the two subjects
can be made if we extend our point of view to embrace computer science. Here an
important concept is NP completeness.

The determination of the time it takes to run an algorithm on a computer is obviously
of great importance. In particular if we have a problem (such as the computation of
a partition function or free energy per site) which depends on a number $N$ of “input
parameters” it is of great practical importance how the running time of the algorithm
increases with $N$. If the running time increases as a power of $N$ the problem is said to
be solvable in polynomial time. For a lattice statistical mechanical system such as the
Ising model (17) the number $N$ is the number of lattice points in the system.

Unhappily there are very few problems which physicists are interested in which have
been shown to be solvable in polynomial time. What almost always happens is that the
running time increases exponentially (or worse) as the size of the system increases. This
unfortunate fact is known to every physicist whoever tried to do a numerical computation
on a large system.

Computer scientists deal all the time with difficult problems for which no polynomial-
time algorithm has been found, despite decades of attempts; being immodest, they
strongly (and probably rightly) suspect that no polynomial-time algorithm for these
problems exists. A famous example is the Traveling Salesman Problem: given $N$ cities on
the plane, determine whether there exists a tour of total length $\leq L$. The naive approach
requires checking $N!$ possible tours; and while better algorithms have been found, the $N!$
has never been reduced to $N^p$. Note, however, that if someone claims to exhibit a tour
of length $\leq L$, this claim can trivially be checked in polynomial (in fact linear) time.
A problem for which a purported solution is checkable in polynomial time is said to
belong to the family NP. (The letters stand for Nondeterministic Polynomial. The idea is that if a purported solution is polynomial-time checkable, then the original problem is solvable in polynomial time by a “nondeterministic Turing machine” that could test all possible solutions in parallel.) Now, among the problems lying in NP, computer scientists distinguish a subclass of problems called \textit{NP-complete}: these are the most difficult problems in the class NP, in the sense that if one NP-complete problem were solvable in polynomial time, then all problems in NP would be. And literally hundreds of problems — the Traveling Salesman, graph \(q\)-colorability, . . . — have been proven rigorously to be NP-complete. So, either all of these problems are polynomial-time solvable (P=NP), or none of them are (P \(\neq\) NP). Computer scientists strongly suspect that the latter is true, and their Holy Grail is to prove it \[22, 23\].

On the other hand when a physicist refers to “solving a problem” he/she almost always means that the problem has been reduced to an algebraic problem in the sense I have discussed above. This algebraic structure almost certainly will allow the problem to be solved in polynomial time.

There is obviously a gap between the computer science concepts of polynomial time versus NP complete and the physicist’s concept of integrable versus nonintegrable. In particular it is not clear that there is a one-to-one match between the term NP complete and the term nonintegrable. Nevertheless some physics problems have become so well known that they have attracted the attention of computer scientists and in the past 15 years Ising spin glasses \[24\] and the monomer dimer problem \[25\] have been shown to be NP complete. The most recent of these studies is by Istrail \[26\] in a paper (accompanied by a press release) entitled \textit{Statistical Mechanics, Three Dimensionality and NP-Completeness 1. Universality of intractability for the partition function of the Ising model across non-planar lattices.} This paper suggests (but to the best of my reading does not actually say) that

\begin{itemize}
  \item \textbf{(Suggestion)} The free energy per site of the three dimensional Ising model on a homogeneous lattice is NP complete.
\end{itemize}

Istrail interprets his suggestion to imply that it is impossible to “solve” the three dimensional Ising model.

This may or may not be considered surprising depending on your point of view. For over 50 years people have repeatedly tried and failed to find a solution to the three dimensional Ising model which more or less generalizes Onsager’s result \[18\]. In addition people have searched for and failed to find families of commuting transfer matrices which include the three dimensional Ising model. All of this has led people to suppose that a “solution” does not exist. The suggestion of Istrail fits nicely into this body of experience.

But what one would like is a more constructive discussion of the problem. For example one would like to know if by looking at the high temperature series expansion of the free energy there is any way to see some difference between the 2 and 3 dimensional model. The most standard high temperature series expansion is to consider the isotropic case with only one interaction energy \(E = (E_v = E_h \text{ in two dimensions})\) and use \(v = \tanh E/kT\) as the expansion variable. Then in dimension \(d\) the (exponential of the) free
energy can be written in terms of $v$ as

$$e^{-f/kT} = 2 \cosh^d E/kT \left(1 + \sum_{n=2}^{\infty} a_{2n} v^{2n}\right).$$

(26)

In two dimensions the coefficients $a_n$ may be quickly obtained from the exact solution (18) to as large an $n$ as desired. For three dimensions the best results available [30] only go up to $N = 26$

| order $n$ | the coefficients $a_n$ |
|-----------|------------------------|
| 4         | 3                      |
| 6         | 22                     |
| 8         | 192                    |
| 10        | 2,046                  |
| 12        | 24,853                 |
| 14        | 329,334                |
| 16        | 4,649,601              |
| 18        | 68,884,356             |
| 20        | 1,059,830,112          |
| 22        | 16,809,862,992         |
| 24        | 273,374,177,222        |
| 26        | 4,539,862,959,852      |

A finite number of terms from a power series expansion will only define a polynomial and thus in strict principle we can never learn anything about singularities in the free energy from the 12 coefficients given above without further assumptions. But as physicists we “do the best we can” and accordingly high temperature series expansions for free energies (and specific heats) are inevitably analyzed by assuming that there is a critical value of $v_c$ (at the radius of convergence of the full infinite series (26) such that at $v_c$ there is a singularity of the form

$$f \sim A(v_c - v)^{2-\alpha} \quad \text{or} \quad A(v_c - v)^{2\ln|v_c - v|}$$

(27)

which generalizes (19). But even for the two dimensional case the expression (27) is not the exact answer so several further terms which are analytic at $v_c$ and possibly a few further confluent singularities at $v_c$ are assumed to exist and the resulting form is fitted to the series expansion by some technique such as differential Padé approximants.

Ultimately this procedure of extracting a critical exponent from the high temperature series data depends on the form of the singularities we have assumed to fit the data and can be thought of as a nondeterministic step. But in section 4 we pointed out that most power series do not lead to such simple algebraic (or logarithmic) functions and that in general natural boundaries are to be expected. Thus generically the assumptions made to analyze a finite number of terms in high temperature series expansions can fail to be correct for the full infinite series. To the extent that it is felt that the three dimensional Ising model is “computationally intractable” and NP complete so it would seem to become increasingly unlikely that the specific heat should be described by such a very special and nongeneric form as (27).
I hasten to point out that there is no known useful necessary and sufficient condition for the coefficients of a power series expansion which tell if a natural boundary is present. Moreover it is not expected that the first 12 terms would even come close to approximating this criterion even if a criterion were known. Nevertheless there is a genuine problem here which demands to be studied.

9 Simple fixed points

The free energy, spontaneous magnetization and correlation functions of the two dimensional Ising model at \( H = 0 \) are all exactly known and the model has a phase transition at the critical temperature \( T_c \). Moreover the properties at this phase transition are all in qualitative agreement with the properties of real phase transitions as seen in experiments even though for these real systems we are unable to compute the phase transition behavior theoretically. Therefore starting in the mid '60's a very successful phenomenology of phase transitions was developed and converted into a physical theory in the 70's which says in effect the the real world behaves like the two dimensional Ising model at \( H = 0 \).

The crucial ingredient in this description is the notion of a single unstable fixed point which plays a key role in the theory of the renormalization group \[27\]-\[29\].

The correlation functions of the 2-dimensional Ising model at \( H = 0 \) have the property that if \( T \neq T_c \) the correlations (on a lattice of unit spacing) decay at large separations as

\[
< \sigma_{0,0} \sigma_{\vec{R}} > \sim M^2 + A(\theta)|\vec{R}|^{-p} e^{-|\vec{R}|/\xi(T,\theta)} \tag{28}
\]

where \( \theta \) is the angle which \( \vec{R} \) makes with the \( x \) axis and \( p = 1/2 \) if \( T > T_c \) and 2 is \( T < T_c \). If the system is isotropic (\( E_v = E_h \)) and if \( T \to T_c \) the dependence on \( \theta \) disappears and so we will suppress it in our further discussion. The quantity \( \xi(T) \) is called the correlation length and it has the property that as \( T \to T_c \)

\[
\xi(T) \sim |T - T_c|^{-\nu} \text{ with } \nu = 1 \tag{29}
\]

At \( T_c \) the two point function of the Ising model behaves for large \( |\vec{R}| \) as

\[
< \sigma_{0,0} \sigma_{\vec{R}} >_{T=T_c} \sim \frac{C}{|\vec{R}|^{d-2+\eta}} \text{ with } \eta = 1/4 \tag{30}
\]

Definition of scaling limit

The scaling limit is the limit in which

\[
|\vec{R}| \to \infty, \quad T \to T_c \tag{31}
\]

with \( |\vec{R}|/\xi(T) = |\vec{R}| |T - T_c|^{\nu} = r \) fixed \( \tag{32} \)

Definition of Scaled two point function

The scaled two point function (for \( T \) either above or below \( T_c \)) is defined as

\[
G_{\pm}(r) = \lim_{M \to -2} M^2 < \sigma_{0,0} \sigma_{\vec{R}} > \tag{33}
\]

13
where
\[ M_\pm = (1 - k^{\pm 2})^{1/8} \]  
(34)
(and we note that \( M_- \) is the spontaneous magnetization). These scaled two point functions [31] exist and are nonzero for the 2 dimensional Ising model at \( H = 0 \).

**Definition of simple fixed point scaling**

If the behavior of \( G_\pm (r) \) as \( r \to 0 \) is
\[ G_\pm (r) \sim C' \frac{r^{d-2+\eta}}{r^{d-2+\eta}} \]  
(35)
where (1) the \( \eta \) of (30) and (33) are the same and (2) the constant \( C' \) is obtained from \( C \) by using the definition (33) then the system is said to have simple fixed point scaling.

These two defining properties of simple fixed point scaling have been verified for the two dimensional Ising model. The equality of the exponents \( \eta \) was demonstrated [32] in 1977 and the equality of the constants was demonstrated [33] in 1991. But these computations rely completely on all of the special properties of the Ising model which lead to the Painlevé representation of the scaled two point function [31]. There are no other integrable models for which the corresponding computations have been yet carried out. Therefore it is totally correct to say that the two dimensional Ising model in zero magnetic field is the only system for which simple fixed point scaling has ever been proven to hold.

It is, however, usually assumed that this simple fixed point scaling will hold for all integrable models coming from a family of commuting transfer matrices. Moreover it is assumed much more generally that this scaling holds in the general theory of critical phenomena [27]–[29] and in particular it is assumed to hold for the three dimensional Ising model and for real fluids such as noble gases and CO\(_2\).

Indeed this assumption of single fixed point scaling is much older than the theory of critical phenomena. It is used in the renormalization theory in quantum field theory in the work of Gell–Mann and Low [31] on short distance behavior in QED and was later carried forward by Callan [32] and Symanzik [33] in what is now called the Callen-Symanzik equation. If this simple fixed point assumption fails then many of the results which are obtained from the renormalization group will break down.

Because of this very widespread use of results which follow from single fixed point scaling it is of great importance find an appropriate justification for it. Thus it is somewhat disturbing that for the Ising model this scaling has only been proven in ref. [32] and [33] by using very strong integrability properties. These integrability properties are certainly algebraic in the sense I have discussed above and therefore it is believable that eventually we will be able to prove analogous results for other integrable statistical mechanical models.

But if the existence of single fixed point scaling in integrable models does indeed require the use of an algebraic structure then this very fact casts doubt on whether they can hold for a non integrable system such as the three dimensional Ising model, a simple fluid, or an interacting quantum field theory such as QED.

It would seem quite possible that for non integrable systems the exponent part of the scaling law could hold while the equality of the constants could fail. Even worse it
could happen that as \( r \to 0 \) the Green’s function could behave as

\[
G(r) \sim \frac{\ln^a r}{r^{d-2+\eta}} \quad \text{or} \quad \sim \ln^a r \ln \ln \frac{b r}{r^{d-2+\eta}}
\]

where each logarithm effectively introduces another scale. Since such logarithmic “shock wave” layers are known to happen [32] in the Korteweg de Vries equation which is an integrable partial differential equation it would seem perfectly possible that multiple scales could exist in a non integrable system.

We are thus lead to the following

**Major Question**

Does single fixed point scaling hold for all systems or does it only hold for the integrable (algebraic) systems?

### 10 The Ising model for \( H \neq 0 \)

Now that I have explored in detail the many different ways in which analysis and algebra are used in physics I may return to the topic which originated this discussion in the first place; The study of the susceptibility of the two dimensional Ising model at \( H = 0 \) made in references [1]-[2].

The magnetic susceptibility is the derivative of the magnetization and is expressed in terms of the two point correlation function as follows:

\[
kT \chi = kT \frac{\partial M(H)}{\partial H} \bigg|_{H=0} = \sum_{j,k} \left( < \sigma_{0,0} \sigma_{j,k} > - M^2 \right) \quad (38)
\]

where by the interaction of the Ising model with the magnetic field \( H \) we mean the addition of the term \(-H \sum_{j,k} \sigma_{j,k}\) to the interaction energy (17).

The susceptibility is different for \( T \) above and below \( T_c \) and in ref.[31] it is shown that

\[
kT \chi_+(T) = k(1 - k^{-2})^{1/4} \sum_{l=0}^{\infty} \tilde{\chi}^{(2l+1)}(n)
\]

\[
kT \chi_-(T) = (1 - k^2)^{1/4} \sum_{l=1}^{\infty} \tilde{\chi}^{(2l)}(n)
\]

where \( \tilde{\chi}^{(n)} \) is the sum over \((j, k)\) of the \( n \) particle contribution to the two point function.

For finite \( n \) each term \( \tilde{\chi}^{(n)} \) in (39) is a holonomic function of \( T \). But in ref. [1] and [2] compelling evidence (just short of a proof) is given which indicates that the full sum \( \chi_+(T) \) is not a holonomic function but has a natural boundary on the same curve (21) on which the partition function on the finite lattice has zeros. This natural boundary is a new phenomenon.
But once it is accepted that the susceptibility at $H = 0$ has a natural boundary then it is to be expected that for all $H \neq 0$ the two dimensional Ising model can have a natural boundary on the curve where the partition function on the finite lattice has zeroes. This is in agreement with the intuition suggested by ref. [25], which proved that the two–dimensional monomer dimer problem is NP complete, because the Ising model in a magnetic field can be reduced to this monomer dimer problem.

We thus see that there are sound reasons for believing that the Ising model in a magnetic field is a problem in analysis and not of algebra. Therefore I pose the question:

**Does simple fixed point scaling fail for the two dimensional Ising model with $H \neq 0$?**

The situation, however, is more subtle than this preceding discussion indicates because in 1989 Zamolodchikov [37] showed that in the scaling limit there is a continuum S-matrix model in the same universality class as the Ising model at $T = T_c$, $H > 0$ which is integrable and this S matrix model has subsequently been shown to arise from an integrable lattice model [38] (which however is not the lattice Ising model at $T = T_c$, $H > 0$.) The relation between these two models, one of which is integrable and the other which is not, is not understood.

### 11 Conclusions

I have now completed the survey of eight properties which can be used to characterize the difference between algebra and analysis in statistical mechanics (and quantum field theory) and have given one concrete example where at least some of these differences can be seen to occur. Several of the questions raised about the possible breakdown of scaling contradict beliefs long held by many researchers in both statistical mechanics and quantum field theory and may be classified as impertinences in the sense contemplated by Anatole France. But impertinent as some of my distinctions between algebra and analysis are, there are no known theorems which prove them to be incorrect.

Moreover when taken together the survey presented here is very disturbing because almost all our theoretical computations and intuition come from algebraic problems whereas all real problems of experimental interest will not possess these algebraic structures. This leads to the nagging suspicion that there must be phenomena which are left out of our phenomenological picture of critical phenomena and quantum field theory.

One property of nature which manifestly is left out of all solvable problems in statistical mechanics and quantum field theory is the phenomenon of particle production and the converse phenomenon of decay of an unstable particle. This is very relevant to Zamolodchikov’s model [16] because it has a mass spectrum of 8 particles of which 5 are above the 2 particle decay threshold. Therefore when the model is perturbed away from $T = T_c$ these 5 particles will decay (at least using perturbation theory [40]).

But what is meant by an unstable particle in a mathematical sense? One common definition is to continue the 2 point function as a function of momentum through the 2 particle cut onto what is called the second sheet and to identify an unstable particle as a pole on this second sheet. But in deriving the analytic structure of the two point function for the Ising model we make an expansion in terms of the complete set of eigenvectors.
and eigenvalues of the transfer matrix. For the Ising model these eigenvalues are very
smooth for a finite size system and they go over smoothly to the usual kinematic cuts
in the thermodynamic limit. But once the system becomes non integrable it is known
from computer studies of the eigenvalues that what looks like randomness creeps into
the eigenvalues due to “avoided crossings.” For the same reason that “noisy” coefficients
in a power series expansion could generate natural boundaries it would seem that as a
function of $k$ the two point function could also have a natural boundary and cannot be
continued onto a second sheet. If this is the case the definition of unstable particle as a
pole on the second sheet cannot be used.

What is needed is some way to quantify the influence of a nonintegrable perturbation
on an integrable statistical mechanical system. In other words at the very least

**We need a statistical mechanical analogue of the KAM theorem.**

One possible version of such a theorem would be to prove that in the scaling limit
the intuitive ideas of the renormalization group [28]-[29] are exact. For example it would
seem not unreasonable to expect that the addition of bonds to the Ising model which
destroy the planarity of the lattice (such as next nearest neighbor interactions) will have
absolutely no effect on the scaling function even though the arguments of Istrail [26]
would lead one to believe that the system is NP complete.

The fact is that 25 years after the formulation of the renormalization group no such
rigorous theorem has been proven even though such a theorem is sorely needed if we are
to understand how algebra and analysis are related in the study of physical systems. In
the end it is ultimately unacceptable to ignore the fact that algebraic systems are only
a set of measure zero in the space of all systems. No matter how much we physicists
love algebra we cannot ignore the fact that analysis exists.

**Acknowledgments**

I am very pleased to acknowledge many useful discussions with A. Guttmann, W.
Orrick and A. Sokal. This work is supported in part by NSF grant DMR0073058.

**References**

[1] B. Nickel, On the singularity structure of the 2D Ising model susceptibility. J. Phys.
A 32 (1999) 3889: Addendum to “On the singularity structure of the 2D Ising model
susceptibility, J. Phys. A 33 (2000) 1693.

[2] W.P. Orrick, B. Nickel, A.J. Guttmann, and J.H.H. Perk, The susceptibility of the
square Ising model: new developments, J. Stat. Phys. (in press).

[3] L. Onsager, Crystal statistics I. A two dimensional model with an order disorder
transition, Phys. Rev. 65 (1944) 117.

[4] V. Kac, Simple irreducible graded Lie algebras of finite growth, Izvestia, AN USSR
(sér. mat.) 32 (1968) 1923-1967.
[5] R.V. Moody, A New class of Lie algebras, J. Algebra 10 (1968) 211-230.

[6] V.G. Drinfel’d, Hopf algebras and the quantum Yang-Baxter equation, Soviet Math. Doklady 32 (1985) 254-258.

[7] M. Jimbo, A q–difference analogue of U(q) and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985) 63-69.

[8] V. Kac, Infinite dimensional Lie Algebras, third ed. (Cambridge, 1990).

[9] E. Hille, Analytic Function Theory. Vol. 1 (Ginn and Co. 1959), 132-136; Vol. 2 (Ginn and Co. 1962), 87-92.

[10] A.M. Kolmogorov, The conservation of quasi–periodic motion for small changes in the Hamiltonian function. Dokl. Akad. Nauk. USSR 98 (1954) 527-530.

V.I. Arnold, Small denominators and problems of stability of motion in classical and celestial mechanics, Russian Mat. Surveys 18, 6 (1963) 85-191; Proof of a theorem of A.N. Kolmogorov on the invariance of quasi–periodic motions under small perturbations of the Hamiltonian, Russian Math. Surveys, 18, 5(1963) 9-36.

J. Moser. Lectures on Hamiltonian systems, Mem. Am. Math. Soc. 81 (1968) 1-60.

[11] R.J. Baxter, Eight-vertex model in lattice statistics, Phys. Rev. Letts. 26 (1971) 832-3; Partition function of the eight-vertex lattice model, Ann. Phys. 70 (1972) 193-228.

[12] J.E. Mayer and M.D. Mayer, Statistical Mechanics (Wiley, 1940) Chapter 13, 277-284.

[13] C. Domb, On the theory of cooperative phenomena in crystals, Adv. Phys. 9 (1960) 149-244; 245-361.

[14] C.N. Yang, The spontaneous magnetization of the two dimensional Ising model, Phys. Rev. 85 (1952) 808.

[15] C.N. Yang, Some Exact results for the many-body problem in one dimension with repulsive delta-function interactions, Phys. Rev. Letts. 19 (1967) 1312-1314; S matrix for the One-dimensional N-body problem with repulsive or attractive δ– function interaction, Phys. Rev. 168 (1968) 1920-1923.

[16] A.B. Zamolodchikov and Al. B Zamolodchikov, Factorized S-matrices in two dimensions as exact solutions of certain relativistic quantum field theories, Ann. Phys. 120 (1979) 253-291.

[17] G.F. Chew, S-Matrix theory of strong interactions, (Benjamin, Reading 1961).

[18] M. Kashiwara and Y. Kawai, Holonomic systems of linear differential equations and Feynman integrals, Pub. RIMS 12, Suppl. (1977) 131-140.
[19] L. Lipshitz, D-finite power series, J. Alg. 122 (1989) 353; A Holonomic systems approach to special function identities, J. Comp. and Appl. Math. 32 (1990) 321-368.

[20] M. Sato, T. Miwa, and M. Jimbo, Studies on holonomic quantum fields I-V, Pub. RIMS, 14, (1978) 223; 15 91979) 201; 15 (1979) 577; 15 (1979) 871; 16 (1980) 531.

[21] Due to M. Bousquet–Mélou, unpublished. Quoted in ref. 2.

[22] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness (Freeman, San Francisco, 1979).

[23] See also the article by S. Cook in the website www.claymath.org/prize-problems/p-vs-np.

[24] F. Barahona, On the computational complexity of Ising spin glass models, J. Phys. A 15 (1982) 3241–3253.

[25] M. Jerrum, Two-dimensional monomer-dimer systems are computationally intractable, J. Stat. Phys. 48 (1987) 121

[26] S. Istrail, Statistical Mechanics, three dimensionality and NP-completeness, Preprint; see also quotes in the article by B. Cipra, Statistical physicists phase out a dream, Science vol. 288 (2000) 1561.

[27] L.P. Kadanoff, Physics 2 (1966) 263.

[28] K.G. Wilson and J. Kogut, The renormalization group and the $\epsilon$ expansion, Phys. Repts. 12C (1974) 75–199.

[29] M.E. Fisher, The renormalization group theory of critical behavior, Rev. Mod. Phys. 46 (1974) 597–616.

[30] A.J. Guttmann and I.G. Enting, Series studies of the Potts model: I. the simple cubic Ising model, J. Phys. A 26 (1993) 807–821; The high-temperature specific heat exponent of the 3D ising model, J. Phys. A 27 (1994) 8007–8010.

[31] T.T. Wu, B.M. McCoy, C.A. Tracy and E. Barouch, Spin–spin correlation functions for the two–dimensional Ising model: exact theory in the scaling region, Phys. Rev. B13 (1976) 315.

[32] B.M. McCoy, C.A. Tracy and T.T. Wu, Painlevé equations of the third kind, J. Math. Phys. 18 (1977) 1058.

[33] C.A. Tracy, Asymptotics of $\tau$-function arising in the two-dimensional Ising model, Comm. Math. Phys. 142 (1991) 297.

[34] M. Gell–Mann and F.E. Low, Quantum electrodynamics at short distances, Phys. Rev. 95 (1954) 1300.
[35] K. Symanzik, Small distance behavior in field theory and power counting, Comm. Math. Phys. 18 (1970) 227.

[36] C.G. Callen, Broken scale invariance in scalar field theory, Phys. Rev. D2 (1970) 1541.

[37] A.B. Zamolodchikov, Integrable field theory from conformal field theory, Adv. Stud. in Pure Math. 19 (1989) 641; Integrals of motion and the S-matrix of the (scaled) $T = T_c$ Ising model with a magnetic field, Int. J. Mod. Phys. A4 (1989) 4235.

[38] S.O. Warnaar, B. Nienhuis and K.A. Seaton, New construction of solvable lattice models including an Ising model in a field, Phys. Rev. Letts. 69 (1992) 710; Int. J. Mod. Phys. B7 (1993) 3727.

[39] M.J. Ablowitz and H. Segur, Asymptotic solutions of the Korteweg–de Vries equation, Stud. Appl. Math. 57 (1977) 13–44.

[40] G. Mussardo, Off-critical statistical models: Factorized scattering theories and bootstrap program, Phys. Reports 218 (1992) 215.