The First Ascent into the Incidence Algebra of the Fibonacci Cobweb Poset

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Abstract

The explicit formulas for Möbius function and some other important elements of the incidence algebra are delivered. For that to do one uses Kwaśniewski’s construction of his Fibonacci cobweb poset in the plane grid coordinate system.

1 Fibonacci cobweb poset

The Fibonacci cobweb poset \( P \) has been invented by A.K. Kwaśniewski in [1, 2, 3] for the purpose of finding combinatorial interpretation of fibonomial coefficients and eventually their recurrence relation. At first the partially ordered set \( P \) (Fibonacci cobweb poset) was there defined via Hasse diagram as follows: \( P \) looks like famous rabbits growth tree but it has the specific cobweb in addition, i.e. it consists of levels labeled by Fibonacci numbers (the \( n \)-th level consist of \( F_n \) elements). Every element of \( n \)-th level \( (n \geq 0) \) is in partial order relation with every element of the \( (n+1) \)-th level but it’s not with any element from the level in which he lies \((n\text{-th level})\) except from it.

In [1] A. K. Kwaśniewski defined cobweb poset \( P \) as infinite labeled digraph oriented upwards as follows: Let us label vertices of \( P \) by pairs of coordinates: \( \langle i, j \rangle \in \mathbb{N}_0 \times \mathbb{N}_0 \), where the second coordinate is the number of level in which the element of \( P \) lies (here it is the \( j \)-th level) and the first one is the number of this element in his level (from left to the right), here \( i \).
Following \[\text{[1]}\] we shall refer to \(\Phi_s\) as to the set of vertices (elements) of the \(s\)-th level, i.e.:

\[\Phi_s = \{(j, s), \quad 1 \leq j \leq F_s\}, \quad s \in \mathbb{N} \cup \{0\}.\]

For example \(\Phi_0 = \{(1, 0)\}, \quad \Phi_1 = \{(1, 1)\}, \quad \Phi_2 = \{(1, 2)\}, \quad \Phi_3 = \{(1, 3), (2, 3)\}, \quad \Phi_4 = \{(1, 4), (2, 4), (3, 4)\}, \quad \Phi_5 = \{(1, 5), (2, 5), (3, 5), (4, 5), (5, 5)\} \ldots\)

Then \(P\) is a labeled graph \(P = (V, E)\) where

\[V = \bigcup_{p \geq 0} \Phi_p, \quad E = \{\langle j, p \rangle, \langle q, p + 1 \rangle \}, \quad 1 \leq j \leq F_p, \quad 1 \leq q \leq F_{p+1}.\]

We can now define the partial order relation on \(P\) as follows: let \(x = (s, t), y = (u, v)\) be elements of cobweb poset \(P\). Then

\[(x \leq y) \iff (t < v) \lor (t = v \land s = u).\]

### 2 The Incidence Algebra \(I(P)\)

Let us recall that one defines the incidence algebra of a locally finite partially ordered set \(P\) as follows (see \([\text{II} \text{ I}]\)):

\[I(P) = \{f : P \times P \longrightarrow \mathbb{R}; \quad f(x, y) = 0 \quad \text{unless} \quad x \leq y\}.\]

The sum of two such functions \(f\) and \(g\) and multiplication by scalar are defined as usual. The product \(h = f * g\) is defined as follows:

\[h(x, y) = (f * g)(x, y) = \sum_{z \in P: x \leq z \leq y} f(x, z) \cdot g(z, y).\]

It is immediately verified that this is an associative algebra over the field of reals (or any associative ring) \(\mathbb{R}\).

The incidence algebra has an identity element \(\delta(x, y)\), the Kronecker delta. Also the zeta function of \(P\) defined for any poset by:

\[\zeta(x, y) = \begin{cases} 1 & \text{for } x \leq y \\ 0 & \text{otherwise} \end{cases}\]

is an element of \(I(P)\). For the Fibonacci cobweb poset \(\zeta\) was expressed by \(\delta\) in \([\text{II}]\) from where quote the result:

\[\zeta = \zeta_1 - \zeta_0 \quad (1)\]
where for \( x, y \in \mathbb{N}_0 \):

\[ \zeta_1(x, y) = \sum_{k=0}^{\infty} \delta(x + k, y) \]  

(2)

\[ \zeta_0(x, y) = \sum_{k \geq 0} \sum_{s \geq 0} \sum_{r=1}^{F_s - k - 1} \delta(k + F_s + r, y). \]  

(3)

Using coordinates more convenient formula was given by the present author in [6]:

\[ \zeta(x, y) = \zeta(\langle s, t \rangle, \langle u, v \rangle) = \delta(s, u)\delta(t, v) + \sum_{k=1}^{\infty} \delta(t + k, v). \]  

(4)

3 The Möbius function and Möbius inversion formula on \( P \)

The knowledge of \( \zeta \) enables us to construct other typical elements of incidence algebra of \( P \). The one of them is Möbius function indispensable in numerous inversion type formulas of countless applications [3, 5]. Of course the \( \zeta \) function of a locally finite partially ordered set is invertible in incidence algebra and its inversion is the famous Möbius function \( \mu \) i.e.:

\[ \zeta * \mu = \mu * \zeta = \delta. \]

The Möbius function \( \mu \) of Fibonacci cobweb poset \( P \) was presented for the first time by the present author in [7]. It was recovered just by the use of the recurrence formula for Möbius function of locally finite partially ordered set \( I(P) \) (see [4]):

\[
\begin{cases}
\mu(x, x) = 1 & \text{for all } x \in P \\
\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)
\end{cases}
\]  

(5)

The grid coordinates system definition of \( P \) allowed the present author to derive an explicit formula for Möbius function of cobweb poset \( P \), [6]. Namely for \( x = \langle s, t \rangle, \ y = \langle u, v \rangle, \ 1 \leq s \leq F_t, \ 1 \leq u \leq F_v, \ t, v \in \mathbb{N}_0 \) we
have

\[ \mu(x, y) = \mu((s, t), (u, v)) = \delta(t, v)\delta(s, u) - \delta(t + 1, v) + \sum_{k=2}^{\infty} \delta(t + k, v)(-1)^k \prod_{i=t+1}^{v-1} (F_i - 1) \]

(6)

where \( \delta \) is the usual Kronecker delta

\[ \delta(x, y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \]

The formula (6) enables us to formulate the following theorem (see [4]):

**Theorem 3.1. (Möbius Inversion Formula for \( P \))**

Let \( f(x) = f((s, t)) \) be a real valued function, defined for \( x = (s, t) \) ranging in cobweb poset \( P \). Let an element \( p = (p_1, p_2) \) exist with the property that \( f(x) = 0 \) unless \( x \geq p \).

Suppose that

\[ g(x) = \sum_{\{y \in P: y \leq x\}} f(y). \]

Then

\[ f(x) = \sum_{\{y \in P: y \leq x\}} g(y)\mu(y, x). \]

Hence using coordinates of \( x, y \) in \( P \) i.e. \( x = (s, t), y = (u, v) \) if

\[ g((s, t)) = \sum_{v=0}^{t-1} \sum_{u=1}^{F_v} (f((u, v)) + f((s, t))) \]

then we have

\[ f((s, t)) = \sum_{v \geq 0} \sum_{u=1}^{F_v} g((u, v))\mu((s, t), (u, v)) = \]

\[ = \sum_{v \geq 0} \sum_{u=1}^{F_v} g((u, v)) \left[ \delta(v, t)\delta(u, s) - \delta(v + 1, t) + \sum_{k=2}^{\infty} \delta(v + k, t)(-1)^k \prod_{i=v+1}^{t-1} (F_i - 1) \right]. \]

(7)
4 Examples of Other Elements of I(P)

The knowledge of $\zeta$ enables one to construct the typical elements of incidence algebra perfectly suitable for calculating number of chains, of maximal chains etc. in finite sub-posets of $P$. We shall then now deliver some of them in explicit form.

(1) The function $\zeta^2 = \zeta \ast \zeta$ counts the number of elements in the segment $[x, y]$ (where $x = (s, t), y = (u, v)$), i.e.:

$\zeta^2(x, y) = (\zeta \ast \zeta)(x, y) = \sum_{x \leq z \leq y} \zeta(x, z) \cdot \zeta(z, y) = \sum_{x \leq z \leq y} 1 = \text{card } [x, y]
$

Therefore for $x, y \in P$ as above, we have:

$\text{card } [x, y] = \left( \sum_{i=t+1}^{v-1} \frac{F_i}{\sum_{j=1}^{i}} \right) + 2$

(2) For any incidence algebra the function $\eta$ is defined as follows:

$\eta(x, y) = (\zeta - \delta)(x, y) = \begin{cases} 1 & x < y \\ 0 & \text{otherwise} \end{cases}$

Hence $\eta^k(x, y), \ (k \in \mathbb{N})$ counts the number of chains of length $k$, (with $(k + 1)$ elements) from $x$ to $y$.

The corresponding function for $x, y$ being elements of Fibonacci cob-web poset $P$, $(x = (s, t), y = (u, v))$ is then given by formula:

$\eta((s, t), (u, v)) = \sum_{k=1}^{\infty} \delta(t + k, v)$

(3) Now let

$C(x, y) = (2\delta - \zeta)(x, y) = \begin{cases} 1 & x = y \\ -1 & x < y \\ 0 & \text{otherwise} \end{cases}$
For elements of $\mathbf{P}$ we have:

$$C(\langle s, t \rangle, \langle u, v \rangle) = \delta(t, v)\delta(s, u) - \sum_{k=1}^{\infty} \delta(t + k, v)$$

The inverse function $C^{-1}(x, y)$ counts the number of all chains from $x$ to $y$. From the recurrence formula one infers that

$$\left\{ \begin{array}{l}
C^{-1}(x, x) = \frac{1}{\chi(x, x)} \\
C^{-1}(x, y) = -\frac{1}{\chi(x, x)} \sum_{x < z \leq y} C(x, z) \cdot C^{-1}(z, y)
\end{array} \right.$$  

(4) For any incidence algebra the function $\chi$ is defined as follows:

$$\chi(x, y) = \begin{cases} 
1 & y \text{ covers } x \\
0 & \text{otherwise}.
\end{cases}$$

Therefore $\chi^k(x, y), \ (k \in \mathbb{N})$ counts the number of maximal chains of length $k$, (with $(k + 1)$ elements) from $x$ to $y$.

The corresponding function for $x, y$ being elements of Fibonacci cobweb poset $\mathbf{P} \ (x = \langle s, t \rangle, y = \langle u, v \rangle)$ is expressed by formula:

$$\chi(\langle s, t \rangle, \langle u, v \rangle) = \delta(t + 1, v).$$

Finally let

$$M(x, y) = (\delta - \chi)(x, y) = \begin{cases} 
1 & x = y \\
-1 & y \text{ covers } x \\
0 & \text{otherwise}
\end{cases}$$

For elements of $\mathbf{P}$ we have:

$$M(\langle s, t \rangle, \langle u, v \rangle) = \delta(t, v)\delta(s, u) - \delta(t + 1, v).$$

Then the inverse function of $M$:

$$M^{-1} = \frac{\delta}{\delta - \chi} = \delta + \chi + \chi^2 + \chi^3 + \ldots$$

counts the number of all maximal chains from $x$ to $y$. 

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Closing Remark
As the poset $P$ is not of binomial type $\mathbb{N}$ the further study of the $I(P)$ algebra looks promising as far as obstacles are concerned.

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