Bourgain–Brezis–Mironescu–Maz’ya–Shaposhnikova limit formulae for fractional Sobolev spaces via interpolation and extrapolation

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Abstract
The real interpolation spaces between $L^p(\mathbb{R}^n)$ and $\dot{H}^{r,p}(\mathbb{R}^n)$ (resp. $H^{r,p}(\mathbb{R}^n)$), $r > 0$, are characterized in terms of fractional moduli of smoothness, and the underlying seminorms are shown to be “the correct” fractional generalization of the classical Gagliardo seminorms. This is confirmed by the fact that, using the new spaces combined with interpolation and extrapolation methods, we are able to extend the Bourgain–Brezis–Mironescu–Maz’ya–Shaposhnikova limit formulae, as well as the Bourgain–Brezis–Mironescu convergence theorem, to fractional Sobolev spaces. On the other hand, we disprove a conjecture of Brazke et al. (Bourgain–Brezis–Mironescu convergence via Triebel–Lizorkin spaces. https://arxiv.org/abs/2109.04159) suggesting fractional convergence results given in terms of classical Gagliardo seminorms. We also solve a problem proposed in Brazke et al. (Bourgain–Brezis–Mironescu convergence via Triebel–Lizorkin spaces. https://arxiv.org/abs/2109.04159) concerning sharp forms of the fractional Sobolev embedding.

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1 Preamble

1.1 The Bourgain–Brezis–Mironescu–Maz’ya–Shaposhnikova limit formulae

In their celebrated paper [3], Bourgain–Brezis–Mironescu studied limits of Gagliardo semi-norms defined by

\[ \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}, \quad s \in (0, 1), \quad p \in (1, \infty), \]

and showed that while

\[ \lim_{s \to 1^-} \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} < \infty \iff f \text{ is constant}, \]

the Gagliardo seminorms can be normalized so that the \( L^p \) norm of the gradient can be recovered: For \( f \in W^{1,p}(\mathbb{R}^n) \), it holds

\[ \lim_{s \to 1^-} (1 - s)^{\frac{1}{p}} \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = C(p, n) \|\nabla f\|_{L^p(\mathbb{R}^n)}, \]

where \( C(p, n) \) is a constant that depends only on \( p \) and \( n \), and can be computed explicitly.

Later, Maz’ya–Shaposhnikova [24] used a different normalization to deal with the case where the (smoothness) parameter \( s \) tends to zero, and proved that

\[ \lim_{s \to 0^+} s^{\frac{1}{p}} \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} = c(p, n) \|f\|_{L^p(\mathbb{R}^n)}. \]

These results have sparked intense research interest and have been used and extended in many different directions (for recent related material, applications and references cf. [5–7, 10, 12, 15, 34]). The original proofs were somewhat complicated, and the process of finding the correct normalizations involved ad-hoc considerations. For example, the correct normalization for (1.4) was found using the sharp form of the Sobolev embedding and, conversely, the limit result (1.3) suggested the sharp Fractional Sobolev inequality obtained in [4].

In [20, 25], it was observed that these limiting results can be naturally understood in the more general context of scales of interpolation spaces. Given a pair of compatible spaces \((X_0, X_1)\), interpolation methods (cf. [2]) give rise to scales of spaces, in particular, the scale \((X_0, X_1)_{s,q}, \ s \in (0, 1), \ q \in [1, \infty]\), produced by the real method, will play an important role in this paper. We colloquially refer to \( X_0 \) and \( X_1 \) as “end point spaces”, and we think that, as the parameter \( s \) varies from 0 to 1, we are moving from one end point space in the scale, to the other. In this setting, the limit theorems take the form of a continuity result that can be informally interpreted as saying that “with appropriate normalizations we can recover the end point spaces” (cf. Theorem 4.7),

\[ \lim_{s \to 0^+} c_{s,q} \|f\|_{(X_0, X_1)_{s,q}} = \sup_{t > 0} K(t, f; X_0, X_1); \]

\[ \lim_{s \to 1^-} c_{s,q} \|f\|_{(X_0, X_1)_{s,q}} = \sup_{t > 0} \frac{K(t, f; X_0, X_1)}{t}. \]

1 We assume that \( p > 1 \), since, as in [5], our main focus will be the fractional Sobolev spaces defined via \((-\Delta)^{\frac{s}{2}}\). The case \( p = 1 \) requires a separate treatment.

2 See Sect. 4 for background information on interpolation and extrapolation.
For many pairs of spaces \((X_0, X_1)\), we actually have estimates: For \(f \in X_0 \cap X_1\),
\[
\sup_{t > 0} K(t, f; X_0, X_1) \approx \|f\|_{X_0}, \quad \sup_{t > 0} \frac{K(t, f; X_0, X_1)}{t} \approx \|f\|_{X_1}.
\] (1.6)

Furthermore, if the pair is “exact” then one can replace the equivalences signs in (1.6) by equalities\(^4\) (cf. [20, 25]).

From the interpolation point of view, (1.3) and (1.4) are not separate limiting results, but part of the recovery of end point norms of a particular interpolation scale. Moreover, we note that the limiting formulae (1.5) are somewhat stronger, in the sense that we are free to choose the index \(q\), which indeed, is not dependent on the original pair of spaces, as it is the case in (1.3) or (1.4), where \(q = p\).

In fact, the normalization constants produced by interpolation are “universal”, in the sense that they are independent of the originating pair of spaces and have an explicit formula\(^5\)
\[
c_s,q = \begin{cases} 
(s(1-s)q)^{1/2}, & s \in (0, 1), \quad q \in [1, \infty), \\
1, & q = \infty.
\end{cases}
\] (1.7)

The correct formulae for the normalizations originate from considerations arising in the extrapolation theory of [16], where one needs to work with continuous scales, and must pay close attention to the constants that appear in the estimates. It turns out that normalized interpolation norms have certain crucial monotonicity properties that will play a role in what follows. In particular, it was shown in [16, (3.2), p. 19] that, if \(q \leq r\), then
\[
c_{s,r} \|f\|_{(X_0, X_1)_{s,r}} \leq c_{s,q} \|f\|_{(X_0, X_1)_{s,q}}.
\]

Furthermore, the usual reiteration formulae of interpolation theory, as exemplified by Holmstedt’s reiteration formula (cf. [1, Theorem 2.1, p. 307], [16, (3.15), p. 33], [19, Lemmas 2.1–2.3, pp. 61–63], [20], etc.) can be sharpened, when the norms have been normalized according to (1.7).

Combining (1.5) and the fact that
\[
(L^p(\mathbb{R}^n), \dot{W}^1_p(\mathbb{R}^n))_{s,p} = \dot{W}^{s,p}(\mathbb{R}^n),
\] (1.8)
with hidden equivalence constants independent of \(s\), we can derive versions of (1.3) and (1.4). The proof of (1.8) hinges upon the known computation of the underlying \(K\)-functional in terms of a \(p\)–modulus of continuity (cf. [1, Theorem 4.12, p. 339] and the references therein),
\[
K\left(u, f; L^p(\mathbb{R}^n), \dot{W}^1_p(\mathbb{R}^n)\right) := \inf_{f = f_0 + f_1} \left\{ \|f_0\|_{L^p(\mathbb{R}^n)} + u \|\nabla f_1\|_{L^p(\mathbb{R}^n)} \right\}
\approx \sup_{|h| \leq u} \|\Delta h f\|_{L^p(\mathbb{R}^n)},
\]

\(^3\) Throughout the paper, the symbol \(f \lesssim g\) indicates the existence of a constant \(c > 0\), which is independent of all essential parameters, such that \(f \leq c g\). The symbol \(f \approx g\) means that \(f \lesssim g\) and \(g \lesssim f\).

\(^4\) From this point of view, the constants that appear in (1.3) and (1.4) are connected to the computation of the underlying \(K\)-functionals.

\(^5\) There is a method to define and compute the normalizations in order insure the continuity of the interpolation scales. In particular, the normalizations for other methods of interpolation are also known. For example, for the complex method of Calderón, \(c_s = 1\); for the \(J\)–method, \(c_{s,q} = \left(1 - s \right)q^{-1/q'}\), where \(1/q + 1/q' = 1\). In the case of quasi-Banach spaces some adjustments are necessary (cf. [16, p. 35] for more details).
where
\[ \Delta_h^1 f(x) = \Delta_h f(x) = f(x + h) - f(x), \quad x, h \in \mathbb{R}^n. \]

The higher order differences \( \Delta_h^k, k \in \mathbb{N} \), are defined inductively,
\[ \Delta_h^{k+1} = \Delta_h \Delta_h^k, \]
and we can write
\[ \Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + (k - j)h). \tag{1.9} \]

The corresponding \( K \)-functionals for higher order Sobolev spaces \( \dot{W}_p^k(\mathbb{R}^n) \) are given by (cf. [1, Theorem 4.12, p. 339], [20], and the references therein)
\[ K(u^k, f; L_p(\mathbb{R}^n), \dot{W}_p^k(\mathbb{R}^n)) \approx \sup \| \Delta_h^k f \|_{L_p(\mathbb{R}^n)} \tag{1.10} \]
\[ K(u^k, f; L_p(\mathbb{R}^n), W_p^k(\mathbb{R}^n)) \approx \min \{1, u^k \} \| f \|_{L_p(\mathbb{R}^n)} + \sup \| \Delta_h^k f \|_{L_p(\mathbb{R}^n)}. \tag{1.11} \]

Combining (1.10) and (1.11) with (1.5), we can prove Bourgain–Brezis–Mironescu–Maz’ya–Shaposhnikova higher order limit results in a unified fashion\(^6\) (cf. [20]).

Very recently, Brazke et al. [5], using heavy Harmonic Analysis machinery, obtained a new approach to (1.3) and suggested a possible extension to fractional Sobolev spaces, through the use of Triebel–Lizorkin spaces. The paper [5] contains a number of interesting new inequalities and poses a number of intriguing open problems. It was their paper that provided the initial impetus for the present work. However, in order to solve some of the main questions asked in [5] we will rely on interpolation and extrapolation methods. Interestingly, as we hope to show, interpolation theory will provide us with the crucial spaces, and the appropriate scalings, to resolve the outstanding issues.

The first problem proposed in [5, p. 5] that we shall discuss here is the extension of (1.3) to the fractional case. The interpolation method provides the strategy, although to implement it requires the introduction of new spaces as we now explain.

Let \( t > 0 \), and consider the fractional Sobolev spaces via the fractional Laplacian\(^7\)
\[ \dot{H}^{1, p}(\mathbb{R}^n) = \{ f : \| f \|_{\dot{H}^{1, p}(\mathbb{R}^n)} = \| (-\Delta)^{\frac{1}{2}} f \|_{L_p(\mathbb{R}^n)} < \infty \}, \]
\[ H^{1, p}(\mathbb{R}^n) = \{ f : \| f \|_{H^{1, p}(\mathbb{R}^n)} = \| f \|_{L_p(\mathbb{R}^n)} + \| \nabla f \|_{L_p(\mathbb{R}^n)} < \infty \}, \]
and the corresponding interpolation pairs \( (L_p(\mathbb{R}^n), \dot{H}^{1, p}(\mathbb{R}^n)), (L_p(\mathbb{R}^n), H^{1, p}(\mathbb{R}^n)) \). As a consequence of \( L_p \)-boundedness of the Riesz transforms,
\[ \| (-\Delta)^{\frac{1}{2}} f \|_{L_p(\mathbb{R}^n)} \approx \| \nabla f \|_{L_p(\mathbb{R}^n)}, \quad p \in (1, \infty), \tag{1.12} \]
and thus \( \dot{H}^{1, p}(\mathbb{R}^n) = \dot{W}_p^1(\mathbb{R}^n) \) and \( H^{1, p}(\mathbb{R}^n) = W_p^1(\mathbb{R}^n) \).

In view of the previous discussion the program we shall follow should be clear. Indeed, a direct application of (1.5) yields the following extension of the Bourgain–Brezis–Mironescu–

\(^6\) The proof of property (1.6) related to \( (L_p(\mathbb{R}^n), \dot{W}_p^k(\mathbb{R}^n)) \) can be seen, e.g., in [27, Proposition 3, p. 139].

\(^7\) \( \dot{H}^{1, p}(\mathbb{R}^n) \) is also known as the space of Riesz potentials, while \( H^{1, p}(\mathbb{R}^n) \) is the space of Bessel potentials (cf. [27]).
Maz’ya–Shaposhnikova limits: Let \( t > 0, s \in (0, 1), p > 1 \), then for \( f \in H^{t-p}(\mathbb{R}^n) \),

\[
\begin{align*}
\lim_{s \to 1^-} {(1 - s)^{\frac{1}{p}}} \| f \|_{(L^p(\mathbb{R}^n), H^{t-p}(\mathbb{R}^n))_{s,p}} & \approx \| (-\Delta)^{\frac{1}{p}} f \|_{L^p(\mathbb{R}^n)}, \\
\lim_{s \to 0^+} s^{\frac{1}{p}} \| f \|_{(L^p(\mathbb{R}^n), H^{t-p}(\mathbb{R}^n))_{s,p}} & \approx \| f \|_{L^p(\mathbb{R}^n)},
\end{align*}
\tag{1.13}
\]

with a corresponding result for the inhomogeneous pair \( (L^p(\mathbb{R}^n), H^{t-p}(\mathbb{R}^n)) \):

\[
\begin{align*}
\lim_{s \to 1^-} {(1 - s)^{\frac{1}{p}}} \| f \|_{(L^p(\mathbb{R}^n), H^{t-p}(\mathbb{R}^n))_{s,p}} & \approx \| f \|_{H^{t-p}(\mathbb{R}^n)}, \\
\lim_{s \to 0^+} s^{\frac{1}{p}} \| f \|_{(L^p(\mathbb{R}^n), H^{t-p}(\mathbb{R}^n))_{s,p}} & \approx \| f \|_{L^p(\mathbb{R}^n)}. \tag{1.14}
\end{align*}
\]

As we noted before, the case \( t = 1 \) in (1.13) (resp. (1.14)) corresponds to (1.3) and (1.4), while integer values can be seen to correspond to the higher order limit theorems of [20]. However, to obtain meaningful statements for arbitrary \( t > 0 \), we are required to find explicit descriptions of the indicated interpolation spaces, and control the corresponding interpolation norms within constants independent of the parameter \( s \). The difficulty here is due to the fact that the nonlocal operator \((-\Delta)^{\frac{1}{p}}\) is involved. We will overcome these obstructions using a tool from Approximation Theory: Fractional differences. Fractional differences give an appropriate extension of (1.9) and lead to a natural extension of (1.10) and (1.11).

We now introduce fractional differences and a class of seminorms that extends the classical Gagliardo seminorms (1.1) to the fractional case.

### 1.2 Interpolation and fractional differences: the Butzer seminorms

Butzer and his collaborators (cf. [8, 9] and the references therein) apparently introduced the idea of modifying (1.9) in order to define fractional differences of any order \( t > 0 \) so that a related fractional differentiation approach can be developed. For \( t > 0 \), we let \( \Delta_h^t \) denote the Butzer operator\(^8\) given by

\[
\Delta_h^t f(x) = \sum_{j=0}^{\infty} (-1)^j \binom{t}{j} f(x + (t - j)h), \tag{1.15}
\]

where by definition

\[
\binom{t}{j} = \frac{t(t-1)\ldots(t-j+1)}{j!}, \quad \binom{t}{0} = 1.
\]

Obviously if \( t = k \in \mathbb{N} \), then \( \binom{t}{j} = 0 \) for all \( j \geq k+1 \), and (1.15) coincides with the classical differences (1.9). Remarkably, the \( K \)-functionals, we are seeking to compute in relation with (1.13)–(1.14), can be formally obtained\(^9\) by replacing \( \Delta_h^t \) by \( \Delta_h^0 \) on the right hand side of

---

\(^8\) The concept of fractional differences was already used by Butzer et al. [8] to introduce fractional order moduli of smoothness, which has recently become a powerful tool in approximation theory, e.g., in the study of sharp inequalities between moduli of smoothness (Jackson–Marchaud–Ulyanov inequalities), cf. [21, 22, 29, 30] and the references therein.

\(^9\) For the details of the computation we refer to [21], with [8, 33] as a forerunners.
(1.10) and (1.11), to obtain
\[
K(u^t, f; L^p(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n)) \approx \sup_{|h| \leq u} \| \Delta_h^t f \|_{L^p(\mathbb{R}^n)},
\]
\[
K(u^t, f; L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n)) \approx \min\{1, u^t\} \| f \|_{L^p(\mathbb{R}^n)} + \sup_{|h| \leq u} \| \Delta_h^t f \|_{L^p(\mathbb{R}^n)}.
\]

Armed with the formulae for the \(K\)-functionals we can give a complete description of the interpolation spaces \((L^p, \dot{H}^{t,p}(\mathbb{R}^n))_{s,q}\) (resp. \((L^p, H^{t,p}(\mathbb{R}^n))_{s,q}\)) (cf. Lemma 4.8 below) in terms of the following (semi)norms.

**Definition 1.1** Let\(^{10}\) \(1 \leq p \leq \infty, 0 < s < t\), then we define the Butzer seminorms, as follows\(^{11}\)
\[
\| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t} := \left\{ \begin{array}{ll}
\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\Delta_h^t f(x)|^p}{|h|^{p+n}} \, dx \, dh \right)^{\frac{1}{p}}, & p < \infty \\
\sup_{(x,h)} \frac{|\Delta_h^t f(x)|}{|h|^p}, & p = \infty.
\end{array} \right. (1.16)
\]
We let\(^{12}\)
\[
\| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t} := (s(t - s)p)^{-\frac{1}{p}} \| f \|_{L^p(\mathbb{R}^n)} + \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t}. (1.17)
\]
Observe that when \(t = 1\) we recover the classical Gagliardo seminorms (cf. [1.1])
\[
\| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), 1} = \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \quad s \in (0, 1). (1.18)
\]
Furthermore, if \(s < t < 1\) then \(\| \cdot \|_{\dot{W}^{s,p}(\mathbb{R}^n), t}\) defines an equivalent semi-norm on \(\dot{W}^{s,p}(\mathbb{R}^n)\), i.e.,
\[
\| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t} \approx \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)}. (1.19)
\]
Note, however, that the constants of equivalence blow-up, as \(s\) approaches \(t\) (cf. Lemma 4.10 below):
\[
\frac{1}{(t - s) \max\{p, 2\}} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)} \lesssim \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t} \lesssim \frac{1}{(t - s) \min\{p, 2\}} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)}. (1.20)
\]

By direct computation we characterize the interpolation norms (cf. Lemma 4.8 below): Suppose that \(1 < p < \infty, 0 < s < 1\) and \(t > 0\) then
\[
\| f \|_{(L^p(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n))_{s,p}} \approx \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t},
\]
with hidden constants of equivalence independent of \(s\) (but depending on \(n, p, \) and \(t\)).

In particular, to derive the fractional Bourgain–Brezis–Mironescu–Mazʻya–Shaposhnikova limit formulae, we simply replace the interpolation seminorms \(\| \cdot \|_{(L^p, \dot{H}^{t,p}(\mathbb{R}^n))_{s,p}}\) by \(\| \cdot \|_{\dot{W}^{s,p}(\mathbb{R}^n), t}\) in (1.13) (resp., replace \(\| \cdot \|_{(L^p, H^{t,p}(\mathbb{R}^n))_{s,p}}\) with \(\| \cdot \|_{\dot{W}^{s,p}(\mathbb{R}^n), t} + (s(1 -

\(^{10}\) In this paper we shall be interested in the case \(1 < p < \infty\).

\(^{11}\) Note also that (1.16) enables us to introduce Sobolev spaces of any order \(s > 0\) (not necessarily \(s < 1\) like in (2.1)), via higher order differences. For instance, one can introduce for \(s \in (0, 2)\),
\[
\| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), 2} := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x + 2h) - 2f(x + h) + f(x)|^p}{|h|^{p+N}} \, dx \, dh \right)^{\frac{1}{p}}.
\]

\(^{12}\) The appearance of the prefactor \((s(t - s)p)^{\frac{1}{p}}\) in front of \(\| f \|_{L^p(\mathbb{R}^n)}\) is dictated by standard normalizations in interpolation theory. This will become clear later, cf. Lemma 4.8.
\[ s) \frac{1}{p} \| f \|_{L^p(\mathbb{R}^n)} \) in (1.14)). For example, the general fractional homogeneous version reads now

\[
\begin{align*}
\lim_{s \to t^-} (t - s)^{\frac{1}{p}} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n),t} & \approx \| (-\Delta)^{\frac{s}{2}} f \|_{L^p(\mathbb{R}^n)}, \\
\lim_{s \to 0^+} s^{\frac{1}{p}} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n),t} & \approx \| f \|_{L^p(\mathbb{R}^n)}. 
\end{align*}
\]

(1.21)

This answers completely the question raised in [5, p. 5] asking for a possible fractional extension of (1.3) [see also (1.12)]. Note that (1.21) and (1.20) show that we cannot use the classical Gagliardo seminorms for this purpose.

Further evidence of the important rôle of the Butzer seminorms in our project is provided by the following result, which answers another question raised in [5].

1.3 The fractional Bourgain–Breizis–Mironescu convergence theorem via Butzer seminorms

We consider the extension to fractional Sobolev spaces of the Bourgain–Breizis–Mironescu convergence theorem [3],

**Theorem 1.2** Let \( p \in (1, \infty) \), assume that \( f_k \in \mathcal{S}(\mathbb{R}^n) \), \( k \in \mathbb{N} \), and

\[ f_k \rightharpoonup f \ \text{weakly in} \ L^p(\mathbb{R}^n) \ \text{as} \ k \to \infty. \]

Let \( \{s_k\}_{k \in \mathbb{N}} \subset (0, 1) \) be such that \( s_k \uparrow 1 \) and, moreover, assume that

\[ \Lambda := \sup_{k \in \mathbb{N}} \left( \| f_k \|_{L^p(\mathbb{R}^n)} + (1 - s_k)^{\frac{1}{p}} \| f_k \|_{\dot{W}^{s_k,p}(\mathbb{R}^n)} \right) < \infty. \]

Then \( f \in W^1_p(\mathbb{R}^n) \) and there exists \( C = C(p, n) > 0 \) such that

\[ \| f \|_{L^p(\mathbb{R}^n)} + \| \nabla f \|_{L^p(\mathbb{R}^n)} \leq C \Lambda. \]

In fact, \( f_k \to f \) strongly in \( L^p_{\text{loc}}(\mathbb{R}^n) \).

In [5] the authors propose a plan of attack for a possible fractional extension of Theorem 1.2. The idea is based on the use of the Triebel-Lizorkin spaces \( \dot{F}^s_p,q(\mathbb{R}^n) \) (we refer to Sect. 2 for the definitions). Indeed, based on this idea, a new proof of Theorem 1.2 was presented in [5], using the following sharp Sobolev-type inequality:

**Theorem 1.3** Let \( s \in (0, 1), p \in (1, \infty) \) and \( \Lambda > 1 \). Assume \( 1 - s \leq \frac{1}{2 \Lambda} \) and let \( 1 - \bar{r} = \Lambda(1 - s) \). Then there exists \( C \), which depends only on \( n, p \) and \( \Lambda \), such that

\[ \| f \|_{\dot{F}^s_{p,r}(\mathbb{R}^n)} \leq C \left( \| f \|_{L^p(\mathbb{R}^n)} + (1 - s)^{\frac{1}{p}} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)} \right), \quad r \in [0, \bar{r}]. \]

With this result at their disposal the proof of Theorem 1.2 given in [5] now proceeds by combining Theorem 1.3 (taking limits as \( r \to 1^- \)), with (1.12) and the Littewood–Paley estimate

\[ \| f \|_{\dot{F}^s_{p,1}(\mathbb{R}^n)} \approx \| (-\Delta)^{\frac{s}{2}} f \|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \]

and then applying the Rellich–Kondrachov theorem.

In view of this, the following conjecture was formulated in [5].
Open Problem 1.4 [5, Question 1.11] Let $p \in (1, \infty)$, assume that
\[ f_k \rightharpoonup f \quad \text{weakly in} \quad L^p(\mathbb{R}^n) \quad \text{as} \quad k \to \infty. \]

Let $t \in (0, 1)$ and $(s_k)_{k \in \mathbb{N}} \subset (0, t)$ such that $s_k \uparrow t$ and assume that
\[ \Lambda := \sup_{k \in \mathbb{N}} \left( \| f_k \|_{L^p(\mathbb{R}^n)} + (t - s_k)^{\frac{1}{p}} \| f_k \|_{W^{s_k,p}(\mathbb{R}^n)} \right) < \infty. \]

Then $f \in H^{1,p}(\mathbb{R}^n)$ and there exists $C = C(p, n, t) > 0$ such that
\[ \lim \limsup_{t \uparrow t} \| (-\Delta)^{\frac{1}{2}} f_k \|_{L^p(\mathbb{R}^n)} \leq C \Lambda. \]

Note that in the case $t = 1$, the conjectured result is exactly Theorem 1.2. However, as we have indicated above, the standard Gagliardo seminorms are “too far” from the “exact” interpolation seminorms $\| \cdot \|_{W^{s_k,p}(\mathbb{R}^n), t}$ (cf. [1.16]) for $0 < t < 1$. We use this insight to give a counterexample to Open Problem 1.4 (cf. Sect. 3, Proposition 3.1 for the precise statement, and Sect. 5 for a proof).

The problem thus remains: What spaces should we use to formulate and prove a fractional extension of Theorem 1.2? Once again interpolation, via the Butzer seminorms, comes to our rescue and we are essentially able to prove that the conjectured result in Open Problem 1.4 is true if in its statement we replace “Gagliardo seminorms” by “Butzer seminorms” (cf. Sect. 3, Theorem 3.2 for the precise statement, and Sect. 5 for a complete proof).

1.4 Sharp Sobolev inequalities

Another problem proposed in [5] deals with Sobolev type inequalities. Let $0 < s < t < 1$, an elementary version of Sobolev’s inequality asserts that
\[ \| f \|_{W^{s,2}(\mathbb{R}^n)} \leq \gamma_{s,t} \left( \| f \|_{L^2(\mathbb{R}^n)} + \| f \|_{W^{t,2}(\mathbb{R}^n)} \right). \] (1.22)

Here $\gamma_{s,t}$ is a positive constant which depends, in particular, on the smoothness parameters $s$ and $t$. Furthermore, at least formally, (1.22) is still valid in the limiting values $s = 0$ and $t = 1$. According to (1.2), (1.3) and (1.4), the validity of the previous assertion should be reflected in the behavior of the equivalence constant $\gamma_{s,t}$ that appears in (1.22), in terms of the blow-ups $\min\{t, 1-t\}^{\frac{1}{2}}$ and $\min\{s, 1-s\}^{\frac{1}{2}}$. Indeed, the following result was obtained in [5, Corollary 1.3].

Theorem 1.5 Let $0 < s < t < 1$ and $f \in S(\mathbb{R}^n)$. Then there exists $C = C(n) > 0$, such that
\[ \min\{s, 1-s\}^{\frac{1}{2}} \| f \|_{W^{s,2}(\mathbb{R}^n)} \leq C \left( \| f \|_{L^2(\mathbb{R}^n)} + \min\{t, 1-t\}^{\frac{1}{2}} \| f \|_{W^{t,2}(\mathbb{R}^n)} \right). \]

It is well known that the $L^p$-counterpart of (1.22) is also true, i.e.,
\[ \| f \|_{W^{s,p}(\mathbb{R}^n)} \leq C_{s,t} \left( \| f \|_{L^p(\mathbb{R}^n)} + \| f \|_{W^{t,p}(\mathbb{R}^n)} \right) \]
provided that $0 < s < t < 1$ and $1 < p < \infty$. Accordingly, a similar result, in the spirit of Theorem 1.5, was expected. However, the set of harmonic analysis techniques developed in [5] does not work outside the Hilbert setting given by $p = 2$. This leads to the following question, explicitly raised in [5]:
Open Problem 1.6 Let \( p \in (1, \infty), 0 < \theta < s < t < 1 \) and \( f \in S(\mathbb{R}^n) \). Does there exist \( C = C(n, p, \theta) > 0 \), such that

\[
\min\{s, 1 - s\}^{1/p} \| f \|_{\dot{W}^{s, p}(\mathbb{R}^n)} \leq C \left( \| f \|_{L^p(\mathbb{R}^n)} + \min\{t, 1 - t\}^{1/p} \| f \|_{\dot{W}^{t, p}(\mathbb{R}^n)} \right) \tag{1.23}
\]

We provide a positive answer to this problem. Indeed, in Sect. 3, Theorem 3.9, we state a somewhat stronger result which is then proved in Sect. 5.

We now turn to explain the organization of the paper. The brief Sect. 2 contains the basic background and notation concerning the function spaces we shall consider; Sect. 3 and (1.4).

\section{2 Background on function spaces}

Let \( s \in (0, 1) \) and \( p \in (1, \infty) \), the (fractional) Sobolev space \( \dot{W}^{s, p}(\mathbb{R}^n) \) is equipped with the Gagliardo seminorm

\[
\| f \|_{\dot{W}^{s, p}(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{sp+n}} \, dx \, dy \right)^{1/p} .
\tag{2.1}
\]

The Riesz potential space \( \dot{H}^{s, p}(\mathbb{R}^n) \) is endowed with

\[
\| f \|_{\dot{H}^{s, p}(\mathbb{R}^n)} := \| (-\Delta)^{s/2} f \|_{L^p(\mathbb{R}^n)} .
\]

The spaces \( \dot{H}^{s, p}(\mathbb{R}^n) \) make sense for any \( s \in \mathbb{R} \). In particular, \( \dot{H}^{k, p}(\mathbb{R}^n) \), \( k \in \mathbb{N} \), coincides with the classical Sobolev space \( W^{k, p}(\mathbb{R}^n) \) and

\[
\| f \|_{\dot{H}^{k, p}(\mathbb{R}^n)} \approx \| f \|_{\dot{W}^{k, p}(\mathbb{R}^n)} := \| \nabla^k f \|_{L^p(\mathbb{R}^n)}, \quad p \in (1, \infty) .
\tag{2.2}
\]

We let \( H^{s, p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \dot{H}^{s, p}(\mathbb{R}^n) \) (the space of Bessel potentials) and \( \| f \|_{H^{s, p}(\mathbb{R}^n)} = \| f \|_{L^p(\mathbb{R}^n)} + \| (-\Delta)^{s/2} f \|_{L^p(\mathbb{R}^n)} \).

Let \( S(\mathbb{R}^n) \) be the Schwartz space of all complex-valued, rapidly decreasing and infinitely differentiable functions on \( \mathbb{R}^n \), and let \( S'(\mathbb{R}^n) \) be the dual space of all tempered distributions. Let

\[
\dot{S}(\mathbb{R}^n) = \{ \varphi \in S(\mathbb{R}^n) : (D^\alpha \varphi)(0) = 0 \text{ for } \alpha \in \mathbb{N}_0^n \},
\]

considered as a topological subspace of \( S(\mathbb{R}^n) \). As usual, the dual space \( \dot{S}'(\mathbb{R}^n) \) can be understood within \( S'(\mathbb{R}^n) \) by identifying elements whose difference is a polynomial.

The (homogeneous) Triebel–Lizorkin space \( \dot{F}^{s,q}_{p,q}(\mathbb{R}^n) \), \( s \in \mathbb{R}, p \in (0, \infty), q \in (0, \infty], \) is formed by all \( f \in \dot{S}'(\mathbb{R}^n) \) such that

\[
\| f \|_{\dot{F}^{s,q}_{p,q}(\mathbb{R}^n)} := \left\| \left( \sum_{j=-\infty}^{\infty} 2^{jsq} |\Delta_j f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty ,
\]
with the obvious modification if \( q = \infty \). Here, \( \{\Delta_j f : j \in \mathbb{Z}\} \) is the standard dyadic Littlewood–Paley decomposition of \( f \). The inhomogeneous counterparts, \( F^s_{p,q}(\mathbb{R}^n) \), can be introduced similarly within the dual pairing \( (\dot{S}(\mathbb{R}^n), \dot{S}'(\mathbb{R}^n)) \); we refer to the monograph [32] for further details.

Interchanging the roles of \( L^p(\mathbb{R}^n) \) and \( \ell^q(\mathbb{Z}) \) in the definition of \( \dot{F}^s_{p,q}(\mathbb{R}^n) \) we arrive at the Besov spaces. Specifically, the (homogeneous) Besov space \( \dot{B}^s_{p,q}(\mathbb{R}^n) \), \( s \in \mathbb{R}, p, q \in (0, \infty] \), is formed by all \( f \in \dot{S}'(\mathbb{R}^n) \) such that
\[
\| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \left( \sum_{j=-\infty}^{\infty} 2^{jsq} \| \Delta_j f \|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} < \infty,
\]
with the obvious modifications if \( q = \infty \).

We shall use the following standard notations. Given a quasi-Banach space \( X \) and \( \lambda > 0 \), we will denote by \( \lambda X \) the space endowed with \( \| f \|_{\lambda X} = \lambda \| f \|_X \), \( f \in X \).

Let \( Y \) be a quasi-Banach space and let \( \beta > 0 \). We will use the notation \( \lambda X \hookrightarrow \beta Y \) to indicate that there exists a positive constant \( C \), independent of \( \lambda \) and \( \beta \), such that
\[
\beta \| f \|_Y \leq C \| f \|_X, \quad f \in X.
\]
In particular, if \( \lambda = \beta = 1 \) then we simply write \( X \hookrightarrow Y \). By \( X = Y \) we mean that \( X \hookrightarrow Y \) and \( Y \hookrightarrow X \).

We collect some well-known relations between the function spaces described above (cf. [2, 27, 32]).

**Lemma 2.1** (i) Let \( s \in \mathbb{R}, p \in (0, \infty) \) and \( q \in (0, \infty] \). Then
\[
\dot{B}^s_{p,\min\{p,q\}}(\mathbb{R}^n) \hookrightarrow \dot{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p,\max\{p,q\}}(\mathbb{R}^n).
\]
In particular,
\[
\dot{F}^s_{p,p}(\mathbb{R}^n) = \dot{B}^s_{p,p}(\mathbb{R}^n).
\]
(ii) Let \( s \in (0, 1) \) and \( p \in (1, \infty) \). Then
\[
\dot{W}^{s,p}(\mathbb{R}^n) = \dot{B}^s_{p,p}(\mathbb{R}^n).
\]
(iii) Let \( s \in \mathbb{R} \) and \( p \in (1, \infty) \). Then
\[
\dot{F}^s_{p,2}(\mathbb{R}^n) = \dot{H}^{s,p}(\mathbb{R}^n).
\]

### 3 Main results

In this section we provide a list of statements of the main results we have obtained. The proofs will be given in Sect. 5.

We start outlining some details of a counterexample that provides a negative answer to Open Problem 1.4. To avoid technicalities, we switch temporarily from \( \mathbb{R}^n \) to the unit circle \( \mathbb{T} \).\(^{13}\) However, our construction can be easily modified to deal with \( \mathbb{R}^n \) and \( p > \frac{2n}{n+1} \); in this connection see Remark 5.1 below.

\(^{13}\) Spaces of periodic functions are defined similarly as their analogues on \( \mathbb{R}^n \), simply replacing \( L^p(\mathbb{R}^n) \) by \( L^p(\mathbb{T}) \).
Proposition 3.1 Let $p \in (1, \infty)$ and $t \in (0, 1)$. Let $f$ be formally associated to a Fourier series as follows,

$$f(x) \sim \sum_{\nu=1}^{\infty} \nu^{-t-1+\frac{1}{p}} \cos(\nu x), \quad x \in \mathbb{T}. \quad (3.1)$$

Then, $f \in L^p(\mathbb{T})$ but $f \notin H^{1,p}(\mathbb{T})$. Furthermore, given any sequence $(s_k)_{k \in \mathbb{N}} \subset (0, t)$ with $s_k \uparrow t$,

$$\sup_{k \in \mathbb{N}} (t - s_k)^{\frac{1}{p}} \|f\|_{\tilde{W}^{s,p}(\mathbb{T})} < \infty.$$

In fact, we are able to go far beyond, and we show that a correct formulation of Open Problem 1.4 is obtained by means of replacing the classical seminorms $\| \cdot \|_{W^{s,p}(\mathbb{R}^n)}$ by the Butzer seminorms $\| \cdot \|_{\tilde{W}^{s,p}(\mathbb{R}^n), t}$ (cf. [1.16] and [1.19]).

Theorem 3.2 Let $p \in (1, \infty)$, assume that

$$f_k \rightharpoonup f \text{ weakly in } L^p(\mathbb{R}^n) \text{ as } k \to \infty.$$

Let $t \in (0, 1)$ and $(s_k)_{k \in \mathbb{N}} \subset (0, t)$ be such that $s_k \uparrow t$. Assume that

$$\Lambda := \sup_{k \in \mathbb{N}} \left( \|f_k\|_{L^p(\mathbb{R}^n)} + (t - s_k)^{\frac{1}{p}} \|f_k\|_{\tilde{W}^{s_k,p}(\mathbb{R}^n), t} \right) < \infty.$$

Then $f \in H^{1,p}(\mathbb{R}^n)$ and there exists $C = C(n, p, t) > 0$ such that

$$\lim_{k \to \infty} \sup_{t \uparrow \bar{t}} \|(-\Delta)^{\frac{1}{2}} f_k\|_{L^p(\mathbb{R}^n)} \leq C \Lambda.$$

To prove Theorem 3.2 we need an extension of Theorem 1.3 formulated in terms of the Butzer seminorms $\| \cdot \|_{\tilde{W}^{s,p}(\mathbb{R}^n), t}$. It turns out that our method is flexible enough to incorporate both, the case $s \to 0+$, as well as to deal with inhomogeneous Triebel–Lizorkin spaces.

Theorem 3.3 Let $0 < s < t \leq 1$, $p \in (1, \infty)$ and $\Lambda > 1$.

1. Assume $t - s \leq \frac{1}{2\Lambda}$ and let $t - \tilde{r} = \Lambda (t - s)$. Then, there exists $C$, which depends only on $n$, $p$, $t$ and $\Lambda$, such that

$$\|f\|_{\tilde{F}^{s,p}_{p,2}(\mathbb{R}^n)} \leq C \left( \|f\|_{L^p(\mathbb{R}^n)} + (t - s)^{\frac{1}{p}} \|f\|_{\tilde{W}^{s,p}(\mathbb{R}^n), t} \right). \quad (3.2)$$

There is a corresponding result for homogeneous Triebel–Lizorkin spaces,

$$\|f\|_{\tilde{F}^{s,p}_{p,2}(\mathbb{R}^n)} \leq C \left( \|f\|_{L^p(\mathbb{R}^n)} + (t - s)^{\frac{1}{p}} \|f\|_{\tilde{W}^{s,p}(\mathbb{R}^n), t} \right), \quad r \in [0, \tilde{r}]. \quad (3.3)$$

2. Assume $s < \frac{1}{2\Lambda}$ and let $\tilde{r} = \frac{1}{\Lambda} s$. Then there exists $C$, which depends only on $n$, $p$, $t$ and $\Lambda$, such that

$$\|f\|_{\tilde{F}^{s,p}_{p,2}(\mathbb{R}^n)} \leq C \left( \|f\|_{L^p(\mathbb{R}^n)} + s^{\frac{1}{p}} \|f\|_{\tilde{W}^{s,p}(\mathbb{R}^n), t} \right). \quad (3.4)$$

Likewise, for homogeneous Triebel–Lizorkin spaces we have

$$\|f\|_{\tilde{F}^{s,p}_{p,2}(\mathbb{R}^n)} \leq C \left( \|f\|_{L^p(\mathbb{R}^n)} + s^{\frac{1}{p}} \|f\|_{\tilde{W}^{s,p}(\mathbb{R}^n), t} \right), \quad r \in [0, \tilde{r}]. \quad (3.5)$$

We apply our method to provide an extension of (1.3)–(1.4) in terms of the fractional Laplacian and Butzer seminorms.
Theorem 3.4 Let \( t \in (0, 1) \) and \( p \in (1, \infty) \). Assume \( f \in H^{t,p}(\mathbb{R}^n) \). Then
\[
\lim_{s \to t^+} (t - s)^{\frac{1}{p}} \| f \|_{W^{t,p}(\mathbb{R}^n), t} \approx \| (-\Delta)^{\frac{1}{2}} f \|_{L^p(\mathbb{R}^n)} \tag{3.6}
\]
and
\[
\lim_{s \to 0^+} s^{\frac{1}{p}} \| f \|_{W^{t,p}(\mathbb{R}^n), t} \approx \| f \|_{L^p(\mathbb{R}^n)}. \tag{3.7}
\]
The corresponding results for inhomogeneous spaces also hold true:
\[
\lim_{s \to t^-} (t - s)^{\frac{1}{p}} \| f \|_{W^{t,p}(\mathbb{R}^n), t} \approx \| f \|_{H^{t,p}(\mathbb{R}^n)} \tag{3.8}
\]
and
\[
\lim_{s \to 0^+} s^{\frac{1}{p}} \| f \|_{W^{t,p}(\mathbb{R}^n), t} \approx \| f \|_{L^p(\mathbb{R}^n)}. \tag{3.9}
\]

Remark 3.5 Observe that (3.6) and (3.7) with \( t = 1 \) give back, up to equivalence constants, the classical formulae (1.3) and (1.4).

As indicated by Theorems 3.2 and 3.4, the classical seminorms \( \| \cdot \|_{W^{t,p}(\mathbb{R}^n)} \) are not the optimal choices for dealing with Sobolev-type inequalities involving \( H^{t,p}(\mathbb{R}^n) \). This phenomenon is illustrated in the following result.

Theorem 3.6 Let \( 1 < p < \infty \) and \( 0 < r < t < 1 \). Then there exists \( C = C(n, p, r, t) > 0 \), such that
\[
\| f \|_{W^{t,p}(\mathbb{R}^n)} \leq C((s - r)(t - s))^{\frac{1}{\max[1,2]}} \left( \frac{1}{(s - r)^{\frac{1}{p}}} \| f \|_{L^p(\mathbb{R}^n)} + \frac{1}{(t - s)^{\frac{1}{p}}} \| (-\Delta)^{\frac{1}{2}} f \|_{L^p(\mathbb{R}^n)} \right), \tag{3.10}
\]
for every \( s \in (r, t) \). In the limiting cases, \( r = 0 \) and \( t = 1 \), we have
\[
\| f \|_{W^{t,p}(\mathbb{R}^n)} \leq C(t - s)^{\frac{1}{\max[1,2]}} \left( \frac{1}{(r - s)^{\frac{1}{p}}} \| f \|_{L^p(\mathbb{R}^n)} + \frac{1}{(t - s)^{\frac{1}{p}}} \| (-\Delta)^{\frac{1}{2}} f \|_{L^p(\mathbb{R}^n)} \right) \tag{3.11}
\]
for \( 0 < t < 1 \) and
\[
\| f \|_{W^{t,p}(\mathbb{R}^n)} \leq C(s - r)^{\frac{1}{\max[1,2]}} \left( \frac{1}{(s - r)^{\frac{1}{p}}} \| (-\Delta)^{\frac{1}{2}} f \|_{L^p(\mathbb{R}^n)} + \frac{1}{(1 - s)^{\frac{1}{p}}} \| \nabla f \|_{L^p(\mathbb{R}^n)} \right) \tag{3.12}
\]
for \( 0 < r < 1 \), respectively.

As an immediate consequence of (3.11) and (3.12), we obtain

Corollary 3.7 Let \( 0 < r < t < 1 \). Then
\[
\sup_{s \in [r, t]} (t - s)^{\frac{1}{p}} - \frac{1}{\max[1,2]} \| f \|_{W^{t,p}(\mathbb{R}^n)} \leq C \left( \| f \|_{L^p(\mathbb{R}^n)} + \| (-\Delta)^{\frac{1}{2}} f \|_{L^p(\mathbb{R}^n)} \right) \tag{3.13}
\]
and
\[
\sup_{s \in (r, t]} (s - r)^{\frac{1}{p}} - \frac{1}{\max[1,2]} \| f \|_{W^{t,p}(\mathbb{R}^n)} \leq C \left( \| (-\Delta)^{\frac{1}{2}} f \|_{L^p(\mathbb{R}^n)} + \| \nabla f \|_{L^p(\mathbb{R}^n)} \right). \tag{3.14}
\]
Remark 3.8 Theorem 3.6 is a considerable improvement of Theorem 1.6 in [5]: Let \( 1 < p < \infty \) and \( 0 \leq r < t \leq 1 \). Then there exists \( C = C(n, p) > 0 \) such that

\[
\| f \|_{\dot{W}_r^{s.p}(\mathbb{R}^n)} \leq C \left( \frac{1}{(s-r)^{\frac{1}{p}}} \| f \|_{\dot{F}_p^{s,2}(\mathbb{R}^n)} + \frac{1}{(t-s)^{\frac{1}{p}}} \| f \|_{\dot{F}_p^{t,2}(\mathbb{R}^n)} \right), \quad s \in (r,t).
\]

(3.15)

The estimates provided in Theorem 3.6 sharpen (3.15) due to the appearance of one of the additional prefactors \((s-r)(t-s)^{\frac{1}{p}}\), \((t-s)^{\frac{1}{p}}\) and \((s-r)^{\frac{1}{p}}\). Note that the constant \( C \) in (3.15) does not depend on \( r \) and \( t \), but this is not the case in (3.10). The reason behind this additional flexibility in (3.15) is explained by its non-optimality.

On the other hand, (3.13) sharpens the following inequality obtained in Corollary 1.7 in [5]: Let \( 1 < p < \infty \) and \( 0 < r < t \leq 1 \). Then there exists \( C = C(n, p, r) > 0 \) such that

\[
\sup_{s \in [r,t]} (t-s)^{\frac{1}{p}} \| f \|_{\dot{W}_r^{s.p}(\mathbb{R}^n)} \leq C \left( \| f \|_{L^p(\mathbb{R}^n)} + \| (-\Delta)^{\frac{1}{2}} f \|_{L^p(\mathbb{R}^n)} \right).
\]

(3.16)

In particular, this inequality with \( t = 1 \) gives one of the estimates in the classical Bourgain–Brezis–Mironescu formula (1.3). However, (3.16) is not optimal if \( t \in (0,1) \). In fact, as was already observed in [5, Remark 1.8], the inequality (3.16) is only useful with \( p < 2 \). Specifically, if \( p \geq 2 \) and \( 0 < t < 1 \), it is well known that

\[
\sup_{s \in [r,t]} \| f \|_{\dot{W}_r^{s.p}(\mathbb{R}^n)} \leq C \left( \| f \|_{L^p(\mathbb{R}^n)} + \| (-\Delta)^{\frac{1}{2}} f \|_{L^p(\mathbb{R}^n)} \right).
\]

(3.17)

This obstruction can be overcome via the inequality (3.13). Specifically, if \( p \in (1,2) \) then the prefactor \((t-s)^{\frac{1}{p}}\) on the left-hand side of (3.16) is now improved by \((t-s)^{\frac{1}{p}-\frac{1}{2}}\) in (3.13). On the other hand, if \( p \in [2,\infty) \), then (3.13) coincides with the optimal estimate (3.17). On the other hand, inequality (3.14), with \( p < 2 \), seems to be new.

We give a positive answer to Open Problem 1.6. In fact, we obtain a stronger version by means of removing the \( \theta \)-dependence from the constant \( C \), and working with inhomogeneous norms on both sides of (1.23). Furthermore, our method works with the more general family of seminorms formed by \( \| \cdot \|_{\dot{W}_r^{s.p}(\mathbb{R}^n)} \) (cf. [1.16]) for arbitrary order of smoothness \( \alpha > 0 \).

Theorem 3.9 Let \( p \in (1,\infty) \), \( \alpha > 0 \), and \( 0 < s < t < \alpha \). Then there exists a positive constant \( C \), independent of \( s \) and \( t \), such that

\[
\| f \|_{L^p(\mathbb{R}^n)} + \min\{s, \alpha - s\}^{\frac{1}{p}} \| f \|_{\dot{W}_r^{s.p}(\mathbb{R}^n)} \leq C \left( \| f \|_{L^p(\mathbb{R}^n)} + \min\{t, \alpha - t\}^{\frac{1}{p}} \| f \|_{\dot{W}_r^{s.p}(\mathbb{R}^n)} \right).
\]

In particular, if \( \alpha = 1 \) then (cf. [1.18])

\[
\| f \|_{L^p(\mathbb{R}^n)} + \min\{s, 1-s\}^{\frac{1}{p}} \| f \|_{\dot{W}_r^{s.p}(\mathbb{R}^n)} \leq C \left( \| f \|_{L^p(\mathbb{R}^n)} + \min\{t, 1-t\}^{\frac{1}{p}} \| f \|_{\dot{W}_r^{t.p}(\mathbb{R}^n)} \right)
\]

for all \( 0 < s < t < 1 \).

4 Interpolation/extrapolation: a primer

The goal of this section is to present our interpolation/extrapolation based methodology. These methods will be applied in Sect. 5 to prove all the results that were stated in Sect. 3.
4.1 Revisiting some classical interpolation formulae

All the results contained in this subsection are well known. However, for the purposes of this paper, it is important to establish these results with sharp control on the dependance with respect to the parameters involved. For the convenience of the reader, who may not be familiar with interpolation theory, we give a self-contained exposition, since it is hard to find the material organized in a manner that fits our needs in this paper.

Let \((A_0, A_1)\) be a quasi-semi-normed pair. The \emph{real interpolation space} \((A_0, A_1)_{\theta, q}\), \(\theta \in (0, 1)\), \(q \in (0, \infty]\), is formed by all the elements \(f \in A_0 + A_1\), such that

\[
\| f \|_{(A_0, A_1)_{\theta, q}} := \left( \int_0^\infty (t^{-\theta} K(t, f; A_0, A_1))^q \frac{dt}{t} \right)^{1/q} < \infty, \tag{4.1}
\]

where \(K(t, f; A_0, A_1)\) denotes the \(K\)-functional for the pair \((A_0, A_1)\), defined for \(t > 0\), by

\[
K(t, f; A_0, A_1) := \inf_{f = f_0 + f_1} \left( \| f_0 \|_{A_0} + t \| f_1 \|_{A_1} \right).
\]

We shall simply write \(K(t, f)\) when the underlying pair of spaces \((A_0, A_1)\) is understood from the context. We refer to the standard references on interpolation theory \([1, 2, 31]\). Sometimes it is more convenient to work with the modified \(K\)-functional given by

\[
K_p(t, f; A_0, A_1) := K_p(t, f) = \inf_{f = f_0 + f_1} \left( \| f_0 \|_{A_0}^p + t^p \| f_1 \|_{A_1}^p \right)^{\frac{1}{p}}
\]

for \(p \in (0, \infty)\). Clearly \(K(t, f) \approx K_p(t, f)\) with equivalence constants depending only on \(p\). Furthermore, it is plain that

\[
(A_0, A_1)_{\theta, q} = (A_1, A_0)_{1-\theta, q} \tag{4.2}
\]

with equality of norms.

Let \(A\) be a Banach space and \(p \in (0, \infty)\). The vector-valued Lebesgue spaces \(L^p(\mathbb{R}^n; A)\) are formed by all strongly measurable functions \(f : \mathbb{R}^n \rightarrow A\) such that

\[
\| f \|_{L^p(\mathbb{R}^n; A)} := \left( \int_{\mathbb{R}^n} \| f(x) \|_A^p \, dx \right)^{1/p} < \infty.
\]

The next result is well known.

**Lemma 4.1** Let \((A_0, A_1)\) be a pair of Banach spaces and let \(\theta \in (0, 1)\), \(p \in (0, \infty)\). Then

\[
(L^p(\mathbb{R}^n; A_0), L^p(\mathbb{R}^n; A_1))_{\theta, p} = L^p(\mathbb{R}^n; (A_0, A_1)_{\theta, p})
\]

and

\[
\| f \|_{(L^p(\mathbb{R}^n; A_0), L^p(\mathbb{R}^n; A_1))_{\theta, p}} \approx \| f \|_{L^p(\mathbb{R}^n; (A_0, A_1)_{\theta, p})}
\]

where the hidden equivalence constants depend only on \(p\).

**Proof** It is plain that

\[
K(t, f; L^p(\mathbb{R}^n; A_0), L^p(\mathbb{R}^n; A_1))^p \approx \int_{\mathbb{R}^n} K_p(t, f(x); A_0, A_1)^p \, dx.
\]
Therefore using Fubini’s theorem,
\[
\|f\|_{L^p(\mathbb{R}^n; A_0), L^p(\mathbb{R}^n; A_1)_{\theta,p}} \approx \int_{0}^{\infty} t^{-\theta p} \int_{\mathbb{R}^n} K(t, f(x); A_0, A_1)^p \, dx \, \frac{dt}{t}
\]
\[
\approx \int_{\mathbb{R}^n} \|f(x)\|_{(A_0, A_1)_{\theta,p}}^p \, dx.
\]

\[
\square
\]

For \( s \in \mathbb{R} \) and \( p \in (0, \infty) \), the space \( \ell^s_p(\mathbb{Z}) \) is formed by all the scalar-valued sequences \( \xi = (\xi_j)_{j \in \mathbb{Z}} \), such that
\[
\|\xi\|_{\ell^s_p(\mathbb{Z})} := \left( \sum_{j = -\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p} < \infty.
\]

Similarly, we can define the spaces \( \ell^s_p(\mathbb{N}_0) \) where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

**Lemma 4.2** Let \( -\infty < s_0 < s_1 < \infty, \theta \in (0, 1), s = (1-\theta)s_0 + \theta s_1 \), and let \( p, q \in (0, \infty) \). Then
\[
\left( \ell^{s_0}_q(\mathbb{Z}), \ell^{s_1}_q(\mathbb{Z}) \right)_{\theta,p} = \ell^s_p(\mathbb{Z}),
\]
with
\[
\left( \frac{1}{\theta \min p,q} + \frac{1}{(1-\theta) \min p,q} \right) \|\xi\|_{\ell^s_p(\mathbb{Z})} \leq \|\xi\|_{\ell^{s_0}_q(\mathbb{Z}), \ell^{s_1}_q(\mathbb{Z})_{\theta,p}} \leq \left( \frac{1}{\theta \min p,q} + \frac{1}{(1-\theta) \min p,q} \right) \|\xi\|_{\ell^s_p(\mathbb{Z})}
\]
uniformly w.r.t. \( \theta \). The corresponding result also holds true for \( \mathbb{N}_0 \)-indexed sequences.

**Proof** It is well known and easy to see that
\[
K_q(t, \xi; \ell^{s_0}_q(\mathbb{Z}), \ell^{s_1}_q(\mathbb{Z})) = \left( \sum_{j = -\infty}^{\infty} [\min\{2^{js_0}, 2^{js_1}\} |\xi_j|]^{q'} \right)^{1/q'}.
\]

By the monotonicity properties of the \( K \)-functional, the following estimates hold uniformly w.r.t. \( \theta \in (0, 1) \),
\[
\|\xi\|_{\ell^{s_0}_q(\mathbb{Z}), \ell^{s_1}_q(\mathbb{Z})_{\theta,p}} \approx \left( \sum_{j = -\infty}^{\infty} 2^{l(\theta s_1 - s_0)} p K(2^{-l(s_1 - s_0)}, \xi; \ell^{s_0}_q(\mathbb{Z}), \ell^{s_1}_q(\mathbb{Z}))^p \right)^{1/p}
\]
\[
\approx \left( \sum_{l = -\infty}^{\infty} 2^{l(\theta s_1 - s_0)p} \left( \sum_{j = -\infty}^{\infty} [2^{js_0} \min\{1, 2^{(j-l)(s_1 - s_0)}\} |\xi_j|]^{q'} \right)^{p/q'} \right)^{1/p}
\]
\[
\approx A + B,
\]
where
\[
A := \left( \sum_{l = -\infty}^{\infty} 2^{l(\theta - 1)(s_1 - s_0)p} \left( \sum_{j = -\infty}^{\infty} 2^{js_1 q} |\xi_j|^q \right)^{p/q} \right)^{1/p}
\]
and
\[
B := \left( \sum_{l=-\infty}^{\infty} 2^{l(\theta(s_1-s_0))p} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^q \right)^{p/q} \right)^{1/p}.
\]

We consider now two possible cases. Assume first \( p \geq q \). Therefore
\[
A \geq \left( \sum_{l=-\infty}^{\infty} 2^{l(\theta-1)(s_1-s_0)p} \sum_{j=-\infty}^{l} 2^{jsp} |\xi_j|^p \right)^{1/p} = \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \sum_{l=j}^{\infty} 2^{l(\theta-1)(s_1-s_0)p} \right)^{1/p} \approx \frac{1}{(1-\theta)^{1/p}} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p}.
\]

By the sharp version of Hardy’s inequality (see, e.g., [28, p. 196]),
\[
A = \left( \sum_{l=-\infty}^{\infty} \left( 2^{l(\theta-1)(s_1-s_0)q} \sum_{j=-\infty}^{l} 2^{jsp} |\xi_j|^q \right)^{p/q} \right)^{1/p} \lesssim \frac{1}{(1-\theta)^{1/q}} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p}.
\]

Hence the term \( A \) can be estimated by
\[
\frac{1}{(1-\theta)^{1/p}} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p} \lesssim A \lesssim \frac{1}{(1-\theta)^{1/q}} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p}.
\]

Furthermore, similar estimates also hold for \( B \), namely,
\[
\frac{1}{\theta^{1/p}} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p} \lesssim B \lesssim \frac{1}{\theta^{1/q}} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p}.
\]

Combining (4.3), (4.4) and (4.5), we arrive at the desired result under \( p \geq q \).

Suppose now \( p < q \). Then
\[
A \leq \left( \sum_{l=-\infty}^{\infty} 2^{l(\theta-1)(s_1-s_0)p} \sum_{j=-\infty}^{l} 2^{jsp} |\xi_j|^p \right)^{1/p} \approx \frac{1}{(1-\theta)^{1/p}} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p}.
\]

On the other hand, by a sharp version of Copson’s inequality (see, e.g., [23]) we find that
\[
A \gtrsim \frac{1}{(1-\theta)^{1/q}} \left( \sum_{j=-\infty}^{\infty} 2^{jsp} |\xi_j|^p \right)^{1/p}.
\]
Hence
\[
\left(1 - \theta\right)^{1/p} \left(\sum_{j=-\infty}^{\infty} 2^{j/p} |\xi_j|^p \right)^{1/p} \lesssim A \lesssim \frac{1}{\left(1 - \theta\right)^{1/p}} \left(\sum_{j=-\infty}^{\infty} 2^{j/p} |\xi_j|^p \right)^{1/p},
\]
with hidden constants of equivalence uniform w.r.t. $\theta \in (0, 1)$. The term $B$ can be estimated similarly. Combining estimates yields the desired result for $p < q$. \qed

We close this subsection by recalling the well-known characterization of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ as an interpolation space between $L^p(\mathbb{R}^n)$ and $\dot{W}_p^k(\mathbb{R}^n)$; see, e.g., [1, Chapter 5, Corollary 4.13, p. 341] and [2, Theorem 6.3.1, p. 147].

**Lemma 4.3** Let $s \in (0, 1)$, $k \in \mathbb{N}$ and $p \in (1, \infty)$. Then\(^{14}\)
\[
(L^p(\mathbb{R}^n), \dot{W}_p^k(\mathbb{R}^n))_{s,q} = \dot{B}^s_{p,q}(\mathbb{R}^n).
\]

### 4.2 Extrapolation theory

Generally speaking, the extrapolation theory of Jawerth and Milman [16] develops methods to recover the endpoint spaces of a given parametrized scale of quasi-Banach spaces $\{A_\theta : \theta \in (0, 1)\}$. In particular, it contains a careful analysis of the asymptotic behavior of the related quasi-norms $\| \cdot \|_{A_\theta}$ as $\theta \to 0^+$ or $\theta \to 1^-$. In [16] it is shown that $(\theta(1 - \theta)p)^{1/p}$ is the appropriate normalization factor for the real interpolation method $(A_0, A_1)_{\theta,p}$ (cf. [4.1]). This assertion is well-illustrated by the following results.

**Lemma 4.4** Let $p \in (0, \infty)$. Then there exists $C > 0$, which depends only on $p$, such that, for every $\theta \in (0, 1)$ and $f \in A_0 \cap A_1$,
\[
\|f\|_{(A_0, A_1)_{\theta,p}} \leq C \left(\frac{1}{\theta^{1/p}} \|f\|_{A_0} + \frac{1}{(1 - \theta)^{1/p}} \|f\|_{A_1}\right).
\]

**Proof** Since $K(t, f) \leq \min\{\|f\|_{A_0}, t \|f\|_{A_1}\}$, it follows that
\[
\|f\|_{(A_0, A_1)_{\theta,p}}^p = \int_0^1 t^{-\theta p} K(t, f)^p \frac{dt}{t} + \int_1^\infty t^{-(1 - \theta)p} K(t, f)^p \frac{dt}{t} \\
\leq \|f\|_{A_0}^p \int_0^1 t^{-(1 - \theta)p} \frac{dt}{t} + \|f\|_{A_1}^p \int_1^\infty t^{-\theta p} \frac{dt}{t} \\
= \|f\|_{A_0}^p \frac{1}{(1 - \theta)p} + \|f\|_{A_1}^p \frac{1}{\theta p}.
\]

**Lemma 4.5** (i) Assume $\theta \in (0, 1)$ and $0 < p \leq q < \infty$. Then
\[
(\theta(1 - \theta)p)^{1/p} (A_0, A_1)_{\theta,p} \hookrightarrow (\theta(1 - \theta)q)^{1/q} (A_0, A_1)_{\theta,q}.
\]

(ii) Assume further that $(A_0, A_1)$ is ordered (i.e., $A_1 \hookrightarrow A_0$). Let $0 < \theta < \eta < 1$ and let $p, q \in (0, \infty)$. Then, the norm of the embedding\(^{15}\)
\[
(A_0, A_1)_{\eta,p} \hookrightarrow (A_0, A_1)_{\theta,q}
\]

---

\(^{14}\) The limiting values $p = 1, \infty$ are also admissible.

\(^{15}\) The assumption that the couple $(A_0, A_1)$ is ordered is not restrictive, i.e., a corresponding embedding result still holds for general couples $(A_0, A_1)$. However, this embedding is simplified when dealing with ordered couples, which will be the only case of interest in this paper.
does not exceed

\[
\begin{cases}
\frac{(\eta(1-\eta))^{1/p}}{q^{1/q}} \quad & \text{if } p \leq q, \\
\frac{1}{(\eta-\theta)^{1/q-1/p}} + \frac{\eta^{1/p}(1-\eta)^{1/p}}{q^{1/q}} \quad & \text{if } p > q,
\end{cases}
\]

uniformly w.r.t. \(\eta\) and \(\theta\).

**Proof** The formula that was stated in (i) can be found in \([16, (3.2), p. 19]\). Concerning (ii) with \(p = q\), i.e.,

\[(\eta(1-\eta))^{1/p}(A_0, A_1, p, q) \rightarrow (\theta(1-\theta))^{1/p}(A_0, A_1, \theta, p), \tag{4.6}\]

we refer to \([19, Corollary 2.1]\). Furthermore the case \(p \leq q\) in (ii) can be obtained as a direct consequence of (i) and (4.6).

Next we concentrate on the case \(p > q\) in (ii). Since \(A_1 \hookrightarrow A_0\), it is plain to see that \(K(t, f) \approx \|f\|_{A_0}\) for \(t > 1\). Then

\[
\|f\|_{(A_0, A_1)_{p,q}} \approx \left(\int_0^1 (t^{-\theta}K(t, f))^q \frac{dt}{t}\right)^{1/q} + \frac{1}{\theta^{1/q}} \|f\|_{A_0}. \tag{4.7}
\]

On the one hand, applying Hölder’s inequality, we can estimate the integral given on the right-hand side of (4.7) by

\[
\left(\int_0^1 (t^{-\theta}K(t, f))^q \frac{dt}{t}\right)^{1/q} \lesssim \frac{1}{(\eta-\theta)^{1/q-1/p}} \left(\int_0^1 (t^{-\eta}K(t, f))^p \frac{dt}{t}\right)^{1/p}. \tag{4.8}
\]

On the other hand, using monotonicity properties of the \(K\)-functional, the second term in the right-hand side of (4.7) can be dominated as follows

\[
\|f\|_{A_0} \lesssim \min\{\eta, 1-\eta\}^{1/p} \left(\int_0^\infty (t^{-\eta}K(t, f))^p \frac{dt}{t}\right)^{1/p}. \tag{4.9}
\]

Inserting (4.8) and (4.9) into (4.7), we get

\[
\|f\|_{(A_0, A_1)_{p,q}} \lesssim \left(\frac{1}{(\eta-\theta)^{1/q-1/p}} + \frac{\min\{\eta, 1-\eta\}^{1/p}}{\theta^{1/q}}\right)\|f\|_{(A_0, A_1)_{p,q}}.
\]

\(\square\)

The previous result tells us that the standard interpolation norm \(\|\cdot\|_{(A_0, A_1)_{p,q}}\) given in (4.1) can be renormalized by \((\theta(1-\theta))^{1/p}\|\cdot\|_{(A_0, A_1)_{p,q}}\) so that sharp estimates w.r.t. the interpolation parameter \(\theta\) can now be achieved. Another important example of this phenomenon occurs in reiteration formulas. Indeed, we recall the following sharp versions of reiteration properties for the real method.

**Lemma 4.6** \([19, Theorems 2.11 and 2.12]\) and \([20, Theorem 3]\) Let \(s_0, s_1, \theta \in (0, 1)\) and \(p, q \in (0, \infty)\).

(i) Let \(s = (1-\theta)s_0 + \theta s_1\). The following embeddings hold uniformly w.r.t. \(\theta\)

\[
(\theta(1-\theta))^{-1/\min\{p,q\}}(A_0, A_1)_{s,p} \hookrightarrow ((A_0, A_1)_{s_0,q}, (A_0, A_1)_{s_1,q})_{\theta,p} \hookrightarrow (\theta(1-\theta))^{-1/\max\{p,q\}}(A_0, A_1)_{s,p}.
\]
The following embeddings hold uniformly w.r.t. $s$ and $\theta$

$$s_1^{1/p - 1/q}[(1 - s_1)^{-1/q} + (1 - \theta)^{-1/\min\{p, q\}}](A_0, A_1)_{\theta s_1, p}$$

$$\hookrightarrow (A_0, (A_0, A_1)_{s_1, q})_{\theta, p} \hookrightarrow s_1^{1/p} (1 - \theta)^{-1/\max\{p, q\}}(A_0, A_1)_{\theta s_1, p}.$$

A key role in our arguments is played by the continuity properties of general interpolation scales obtained in [25] (see also [20]). In particular, we will make use of the following result for the real interpolation method.

**Theorem 4.7** Let $p > 0$ and $f \in A_0 \cap A_1$. Then

$$\lim_{\theta \to 1^{-}} (\theta (1 - \theta) p)^{\frac{1}{p}} \|f\|_{(A_0, A_1)_{\theta, p}} = \sup_{t > 0} \frac{K(t, f; A_0, A_1)}{t}$$

(4.10)

and

$$\lim_{\theta \to 0^{+}} (\theta (1 - \theta) p)^{\frac{1}{p}} \|f\|_{(A_0, A_1)_{\theta, p}} = \sup_{t > 0} K(t, f; A_0, A_1).$$

(4.11)

### 4.3 Butzer seminorms via interpolation

A key result in our approach is that the family of semi-norms $\| \cdot \|_{\dot{W}^{s, p}(\mathbb{R}^n), t}$ introduced in (1.16) can be generated (with sharp constants) via interpolation of the classical pair $(L^p(\mathbb{R}^n), H^{t, p}(\mathbb{R}^n))$. This is the assertion contained in the following

**Lemma 4.8** Let $\theta \in (0, 1), t \in (0, \infty)$ and $p \in (1, \infty)$. Then

$$\|f\|_{(L^p(\mathbb{R}^n), \dot{H}^{t, p}(\mathbb{R}^n))_{\theta, p}} \approx \|f\|_{\dot{W}^{\theta, p}(\mathbb{R}^n), t}$$

(4.12)

and

$$\|f\|_{(L^p(\mathbb{R}^n), H^{t, p}(\mathbb{R}^n))_{\theta, p}} \approx \|f\|_{W^{\theta, p}(\mathbb{R}^n), t}$$

(4.13)

where hidden equivalence constants are independent of $\theta$ (but depending on $n$, $p$ and $t$).

**Remark 4.9** Letting $t = 1$ in the previous result (cf. [2.2] and [1.18]), we have

$$\|f\|_{(L^p(\mathbb{R}^n), \dot{W}^{1, p}(\mathbb{R}^n))_{\theta, p}} \approx \|f\|_{\dot{W}^{\theta, p}(\mathbb{R}^n)}$$

(4.14)

and, cf. [1.16],

$$\|f\|_{(L^p(\mathbb{R}^n), W^{1, p}(\mathbb{R}^n))_{\theta, p}} \approx \frac{1}{(\theta (1 - \theta) p)^{\frac{1}{p}}} \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{W^{\theta, p}(\mathbb{R}^n)}.$$  

In particular, (4.14) was already shown in [25, Lemma 1].

**Proof of Lemma 4.8** We shall employ the characterizations of the corresponding $K$-functional given by

$$K(u^t, f; L^p(\mathbb{R}^n), \dot{H}^{t, p}(\mathbb{R}^n)) \approx \frac{1}{u^n} \int_{|h| < u} \|A_h^t f\|_{L^p(\mathbb{R}^n)} dh \approx \sup_{|h| < u} \|A_h^t f\|_{L^p(\mathbb{R}^n)}$$

(4.15)

for $u > 0$; this characterization is well known in the classical case $t = k \in \mathbb{N}$ (see, e.g., [1, Chapter 5, (4.42), p. 341] and [2, Theorem 6.7.3]), for the general case $t \in (0, \infty)$ we refer to [21, (1.10)] (with [8, 33] as a forerunners).
Inserting the first equivalence in (4.15) into the definition of interpolation space (cf. [4.1]), we have

\[
\|f\|_{(L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n))_{\theta,p}} \approx \left( \int_0^\infty u^{-\theta t p - np} \left( \int_{|h| < u} \| \Delta_h f \|_{L^p(\mathbb{R}^n)} \, dh \right)^p \frac{du}{u} \right)^{1/p}.
\]

Consequently, the desired result (4.12) would be established once we are able to prove (cf. [1.16])

\[
\|f\|_{\tilde{W}^{t,p}(\mathbb{R}^n), t} \approx \left( \int_0^\infty u^{-\theta t p - np} \left( \int_{|h| < u} \| \Delta_h f \|_{L^p(\mathbb{R}^n)} \, dh \right)^p \frac{du}{u} \right)^{1/p} \quad (4.16)
\]

uniformly w.r.t. \( \theta \in (0, 1) \).

To deal with the estimate \( \lesssim \) in (4.16), we apply Hölder’s inequality and change the order of integration so that

\[
\int_0^\infty u^{-\theta t p - np} \left( \int_{|h| < u} \| \Delta_h f \|_{L^p(\mathbb{R}^n)} \, dh \right) \frac{du}{u} \lesssim \int_0^\infty u^{-\theta t p - n} \int_{|h| < u} \| \Delta_h f \|_{L^p(\mathbb{R}^n)} \, dh \frac{du}{u}
\]

\[
\approx \int_{|h| \leq 1} |h|^{-\theta t p - n} \| \Delta_h f \|_{L^p(\mathbb{R}^n)} \, dh
\]

\[
= \|f\|_{\tilde{W}^{t,p}(\mathbb{R}^n), t}.
\]

Concerning the estimate \( \gtrsim \) in (4.16), we use basic monotonicity properties and the second estimate given in (4.15) to get

\[
\|f\|_{\tilde{W}^{t,p}(\mathbb{R}^n), t} = \int_{\mathbb{R}^n} \frac{\| \Delta_h f \|_{L^p(\mathbb{R}^n)}^p}{|h|^{|\theta t p + n|}} \, dh
\]

\[
= \sum_{j = -\infty}^\infty \int_{2^{j-1} < |h| \leq 2^j} \frac{\| \Delta_h f \|_{L^p(\mathbb{R}^n)}^p}{|h|^{|\theta t p + n|}} \, dh
\]

\[
\approx \sum_{j = -\infty}^\infty 2^{-j \theta t p} \sup_{|h| \leq 2^j} \| \Delta_h f \|_{L^p(\mathbb{R}^n)}^p
\]

\[
\approx \sum_{j = -\infty}^\infty 2^{-j \theta t p} \left( \frac{1}{2^{jn \theta t p}} \int_{|h| < 2^j} \| \Delta_h f \|_{L^p(\mathbb{R}^n)} \, dh \right)^p.
\]

Next we focus on (4.13). The \( K \)-functional for the pair \((L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n))\) can be estimated as (see [33, (4.2)])

\[
K(u, f; L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n)) \approx \min\{1, u\} \| f \|_{L^p(\mathbb{R}^n)} + \sup_{|h| \leq u^{-\theta}} \| \Delta_h f \|_{L^p(\mathbb{R}^n)}
\]

for \( u > 0 \) and \( f \in L^p(\mathbb{R}^n) \). Therefore, by (4.15),

\[
K(u, f; L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n)) \approx \min\{1, u\} \| f \|_{L^p(\mathbb{R}^n)} + K(u, f; L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n)).
\]

By elementary computations we find

\[
\| f \|_{(L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n))_{\theta,p}} \approx \left( \int_0^\infty u^{-\theta p \min\{1, u\}^p} \frac{du}{u} \right)^{1/p} \| f \|_{L^p(\mathbb{R}^n)} + \| f \|_{(L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n))_{\theta,p}}
\]

\[
= \frac{1}{\theta(1 - \theta)^p} \| f \|_{L^p(\mathbb{R}^n)} + \| f \|_{(L^p(\mathbb{R}^n), H^{t,p}(\mathbb{R}^n))_{\theta,p}}.
\]
Therefore, by (4.12),

\[ \| f \|^p_{L^p(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)} \approx \frac{1}{\theta(1-\theta)p} \| f \|^p_{L^p(\mathbb{R}^n)} + \| f \|^p_{\dot{W}^{s,p}(\mathbb{R}^n), t}. \]

\[ \square \]

The following result shows that the seminorm \( \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t} \) is equivalent to the classical Gagliardo seminorm \( \| f \|_{W^{s,p}(\mathbb{R}^n)} \), but the constants of equivalence blow-up as \( s \to t^- \).

**Lemma 4.10** Let \( 0 < s < t < 1 \) and \( 1 < p < \infty \). Then

\[ \frac{1}{(t-s)_{\max[p,2]}} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t} \lesssim \| f \|_{W^{s,p}(\mathbb{R}^n), t} \lesssim \frac{1}{(t-s)_{\min[p,2]}} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \tag{4.17} \]

where the hidden equivalence constants are independent of \( s \).

**Proof** Recall the relationships between \( \dot{H}^{t,p}(\mathbb{R}^n) \) and \( \dot{B}^{t,q}_{p,q}(\mathbb{R}^n) \) (cf. Lemma 2.1),

\[ \dot{B}^{t}_{p,\min[p,2]}(\mathbb{R}^n) \hookrightarrow \dot{H}^{t,p}(\mathbb{R}^n) \hookrightarrow \dot{B}^{t}_{p,\max(p,2)}(\mathbb{R}^n), \]

therefore, by interpolation, the following embeddings hold uniformly w.r.t. \( \theta \in (0, 1) \)

\[ (L^p(\mathbb{R}^n), \dot{B}^{t}_{p,\min[p,2]}(\mathbb{R}^n))_{\theta,p} \hookrightarrow (L^p(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n))_{\theta,p} \hookrightarrow (L^p(\mathbb{R}^n), \dot{B}^{t}_{p,\max(p,2)}(\mathbb{R}^n))_{\theta,p}. \tag{4.18} \]

Furthermore, by Lemma 4.3, we can rewrite

\[ (L^p(\mathbb{R}^n), \dot{B}^{t}_{p,q}(\mathbb{R}^n))_{\theta,p} = (L^p(\mathbb{R}^n), (L^p(\mathbb{R}^n), \dot{W}^{1,s,p}(\mathbb{R}^n))_{r,q})_{\theta,p}, \quad q \in (0, \infty), \]

with related equivalence constants independent of \( \theta \). Combining this and Lemma 4.6(ii), we obtain the embeddings

\[ (1-\theta)^{-1/\min[p,q]}(L^p(\mathbb{R}^n), \dot{W}^{1,s,p}(\mathbb{R}^n))_{\theta,t,p} \hookrightarrow (L^p(\mathbb{R}^n), \dot{B}^{t}_{p,q}(\mathbb{R}^n))_{\theta,p} \hookrightarrow (1-\theta)^{-1/\max[p,q]}(L^p(\mathbb{R}^n), \dot{W}^{1,s,p}(\mathbb{R}^n))_{\theta,t,p}. \]

In light of Lemma 4.8 [see also (4.14)], the previous embeddings turn out to be equivalent to

\[ (1-\theta)^{-1/\max[p,q]} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)} \lesssim \| f \|_{(L^p(\mathbb{R}^n), \dot{B}^{t}_{p,q}(\mathbb{R}^n))_{\theta,p}} \lesssim (1-\theta)^{-1/\min[p,q]} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)}, \tag{4.19} \]

Combining (4.18) and (4.19), we find

\[ (1-\theta)^{-1/\max[p,2]} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)} \lesssim \| f \|_{(L^p(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n))_{\theta,p}} \lesssim (1-\theta)^{-1/\min[p,2]} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)}. \]

Equivalently (cf. [4.12])

\[ (1-\theta)^{-1/\max[p,2]} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)} \lesssim \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t} \lesssim (1-\theta)^{-1/\min[p,2]} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)} \]

which, after the change of variables \( \theta \leftrightarrow \frac{s}{t} \), yields the desired result (4.17). \( \square \)
5 Proofs of the main results

5.1 Proof of Proposition 3.1

We now recall some results that we need from the theory of Fourier series with monotone coefficients.

Let $T$ be a unit circle. Suppose that the Fourier series of $f \in L^1(T)$ is given by

$$f(x) \sim \sum_{\nu=1}^{\infty} c_{\nu} \cos(\nu x), \quad x \in T,$$

with

$$c_{\nu} \geq c_{\nu+1} \geq \cdots \geq 0 \quad \text{and} \quad c_{\nu} \to 0. \quad (5.2)$$

A well-known theorem of Hardy–Littlewood (see, e.g., [35, p. 129, Vol. II]) asserts that $f \in L^p(T)$ if and only if

$$\sum_{\nu=1}^{\infty} \nu^{p-2} c_{\nu}^p < \infty. \quad (5.3)$$

Furthermore

$$\|f\|_{L^p(T)}^p \approx \sum_{\nu=1}^{\infty} \nu^{p-2} c_{\nu}^p. \quad (5.4)$$

This result has been further extended in [11] in order to deal with other spaces of smooth functions (namely, Besov spaces and Sobolev spaces), as well as more general classes of Fourier series with monotone-type coefficients (the so called, general monotone class). In particular, it was shown in [11, Theorem 4.25] that if $f$ is given by (5.1) with coefficients satisfying (5.2) then it belongs to $H^{t,p}(T)$, $t \in \mathbb{R}$, $p \in (1, \infty)$, if and only if

$$\sum_{\nu=1}^{\infty} \nu^{t(p-1)+p-1} c_{\nu}^p < \infty. \quad (5.5)$$

Moreover,

$$\|f\|_{H^{t,p}(T)}^p \approx \sum_{\nu=1}^{\infty} \nu^{t(p-1)+p-1} c_{\nu}^p. \quad (5.6)$$

Let $t > 0$ and assume $c_{\nu} = \nu^{-t-1+\frac{1}{p}}$, $\nu \in \mathbb{N}$. By virtue of (5.3) and (5.4)

$$\|f\|_{L^p(T)}^p \approx \sum_{\nu=1}^{\infty} \nu^{p-2} \nu^{-tp-p+1} = \sum_{\nu=1}^{\infty} \nu^{-tp-1} < \infty \quad (5.5)$$

and

$$\|f\|_{H^{t,p}(T)}^p \approx \sum_{\nu=1}^{\infty} \nu^{tp+p-2} \nu^{-tp-p+1} = \sum_{\nu=1}^{\infty} \nu^{-1} = \infty,$$

respectively. Consequently, the Fourier series $f$ defined by (5.1) belongs to $L^p(T)$ but not to $H^{t,p}(T)$. 
To conclude, it remains to show that
\[ \| f \|_{\dot{W}^{s_k, p}(\mathbb{T})} \leq C (t - s_k)^{-\frac{1}{p}} \]
uniformly w.r.t. \( k \in \mathbb{N} \). To proceed, we argue as follows. In view of (4.14),
\[ \| f \|_{\dot{W}^{s_k, p}(\mathbb{T})} \approx \left( \int_0^\infty u^{-s_k p} K(u, f; L^p(\mathbb{T}), \dot{W}^{l_1}_p(\mathbb{T}))^p \frac{du}{u} \right)^{\frac{1}{p}}, \]
with hidden constants of equivalence independent of \( k \). Next we estimate \( K(u, f; L^p(\mathbb{T}), \dot{W}^{l_1}_p(\mathbb{T})) \).

For \( u > 0 \), the classical moduli of smoothness of \( f \in L^p(\mathbb{T}) \) is defined by
\[ \omega(f, u)_p := \sup_{|h| \leq u} \| \Delta_h f \|_{L^p(\mathbb{T})}. \]
Assume further that \( f \) satisfies (5.1) and (5.2). Then it is known that \( \omega(f, u)_p \) can be estimated in terms of the Fourier coefficients of \( f \). Namely,
\[ \omega(f, u)_p \approx \left( \sum_{\nu=1}^\infty \min\{1, \nu u\} \nu v^{p-2} c_v \right)^{\frac{1}{p}}, \quad u \in (0, 1); \quad \tag{5.9} \]
(cf. [29, Theorem 6.2] and [14, Theorem 6.1] where this is actually done even in the more general context of general monotone Fourier coefficients.)

Specializing (5.9) to \( f \) given by (5.1) (with \( c_v = v^{-t-1+\frac{1}{p}} \)), yields
\[ \omega(f, l^{-1})_p \approx l^{-1} \left( \sum_{\nu=1}^l \nu (l-\nu t - 1) \nu^{p-1} \right)^{\frac{1}{p}} + \left( \sum_{\nu=1}^\infty \nu v^{p-1} \right)^{\frac{1}{p}} \approx \frac{1}{(t(1-t))^{\frac{1}{p}}} l^{-t} \]
for every \( l \in \mathbb{N} \) and \( t \in (0, 1) \). According to (4.15) (with \( t = 1 \)) and (5.8), the previous estimate is equivalent to
\[ K(l^{-1}, f; L^p(\mathbb{T}), \dot{W}^{l_1}_p(\mathbb{T})) \approx \frac{1}{(t(1-t))^{\frac{1}{p}}} l^{-t}. \]
Furthermore, we have the trivial estimate
\[ K(u, f; L^p(\mathbb{T}), \dot{W}^{l_1}_p(\mathbb{T})) \leq \| f \|_{L^p(\mathbb{T})}. \]

Putting together (5.7), (5.10), (5.11), applying monotonicity properties of \( K \)-functionals, and taking into account that \( s_k \uparrow t \), we obtain
\[ \| f \|_{\dot{W}^{s_k, p}(\mathbb{T})} \leq \int_0^1 u^{-s_k p} K(u, f; L^p(\mathbb{T}), \dot{W}^{l_1}_p(\mathbb{T}))^p \frac{du}{u} + \frac{1}{s_k} \| f \|_{L^p(\mathbb{T})} \]
\[ \approx \sum_{l=1}^\infty l^{p-1} K(l^{-1}, f; L^p(\mathbb{T}), \dot{W}^{l_1}_p(\mathbb{T}))^p + \| f \|_{L^p(\mathbb{T})} \]
\[ \approx \sum_{l=1}^\infty l^{-(t-s_k)p-1} + \| f \|_{L^p(\mathbb{T})} \]
\[ \approx \frac{1}{t-s_k} + \| f \|_{L^p(\mathbb{T})} \]
\[ \leq \frac{1}{t-s_k}. \]
This concludes the proof of (5.6). \( \square \)

**Remark 5.1** Similar ideas can be used to establish the counterpart of Proposition 3.1 in \( \mathbb{R}^n \).

We will only sketch the proof. Consider the monotone function

\[
F_0(u) = \begin{cases} 
  u^{-t-n+\frac{\alpha}{p}} & \text{if } u > 1, \\
  1 & \text{if } u \in (0, 1], 
\end{cases}
\]

and define the radial function \( f(x) = f_0(|x|), x \in \mathbb{R}^n \), where \( f_0 \) is the inverse Fourier–Hankel transform of \( F_0 \), i.e.,

\[
f_0(u) = \frac{2}{\Gamma\left(\frac{\alpha}{2}\right)(2\sqrt{\pi})^n} \int_0^\infty F_0(\xi) j_{n/2-1}(u\xi) \xi^{n-1} d\xi,
\]

where \( j_\alpha(u) = \Gamma(\alpha + 1)u^{-\alpha}/(\Gamma\alpha) \) is the normalized Bessel function \( j_\alpha(0) = 1, \) \( \alpha \geq -1/2, \) and \( C_\alpha \) is the classical Bessel function of the first kind of order \( \alpha \).

Applying the analogue of Hardy–Littlewood theorem for \( L^p(\mathbb{R}^n), \) \( p > \frac{2n}{n+1}, \) (cf. [13, Theorem 1] and [14, (4.10)]), we have

\[
\|f\|_{L^p(\mathbb{R}^n)} \approx \int_0^\infty u^{n-1}F_0(u)^p du = \int_0^1 u^{n-1} du + \int_1^\infty u^{-t-1} du < \infty,
\]

which gives the Euclidean setting counterpart of (5.5). On the other hand, using the counterpart of (5.4) for \( H^{\frac{p}{p},p}(\mathbb{R}^n) \), which may be found in [11, Theorem 4.8], we have

\[
\|f\|_{H^{\frac{p}{p},p}(\mathbb{R}^n)} \approx \int_0^1 u^{n-1}F_0^p(u) du + \int_1^\infty \frac{u^{p+n-1}F_0^p(u)}{u} du = \int_0^1 u^{n-1} du + \int_1^\infty \frac{du}{u} = \infty.
\]

The analogue of (5.9) for the moduli of smoothness \( \omega(f, u)_p = \sup_{\|\| \leq u} \|\Delta_h f\|_{L^p(\mathbb{R}^n)} \) (cf. [5.8]) was obtained in [14, Corollary 4.1 and (7.6)], namely,

\[
\omega(f, u)_p \approx u^p \int_0^{1/u} \xi^{p+n-1}F_0^p(\xi) d\xi + \int_0^1 \xi^{n-1}F_0^p(\xi) d\xi.
\]

In particular (since \( t \in (0, 1) \)),

\[
\omega(f, u)_p \approx u^t \quad \text{for} \quad u \in (0, 1).
\]

Following now line by line the arguments given in the proof of Proposition 3.1, one can show that there exists \( C > 0 \) such that

\[
\|f\|_{W^{k,p}(\mathbb{R}^n)} \leq C(t - s_k)^{-\frac{1}{p}} \quad \text{for every} \quad k \in \mathbb{N}.
\]

### 5.2 Proof of Theorem 3.3

We give a unified proof in the inhomogeneous setting (i.e., (3.2) and (3.4)). It follows from Lemma 4.8 that

\[
(s(t - s))^{\frac{1}{p}} \|f\|_{(L^p(\mathbb{R}^n), H^{\frac{p}{p},p}(\mathbb{R}^n))_{s,t}} \approx \|f\|_{L^p(\mathbb{R}^n)} + (s(t - s))^{\frac{1}{p}} \|f\|_{W^{k,p}(\mathbb{R}^n),_{s,t}}.
\]
Consequently, both (3.2) and (3.4) will be established if we are able to show that
\[
\|f\|_{F_{p,2}^N(\mathbb{R}^n)} \lesssim (s(t - s))^{\frac{1}{p}} \|f\|_{(L^p(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n))^\frac{1}{p}}. \tag{5.12}
\]
To prove this inequality, we invoke the method of retractions (cf. [31]) which shows that the interpolation space \((L^p(\mathbb{R}^n), H^{1,p}(\mathbb{R}^n))\) can be isomorphically identified (with constants of equivalence independent of \(s\)) with \((L^p(\mathbb{R}^n; L^2(\mathbb{N}_0)), L^p(\mathbb{R}^n; L^2(\mathbb{N}_0)))\). In view of the interpolation formula given in Lemma 4.1,
\[
\|f\|_{(L^p(\mathbb{R}^n; L^2(\mathbb{N}_0)), L^p(\mathbb{R}^n; L^2(\mathbb{N}_0)))^\frac{1}{p}} \approx \|f\|_{L^p(\mathbb{R}^n; (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0)))^\frac{1}{p}},
\]
which reduces\(^{16}\) (5.12) to the following sharp embedding at the level of sequence spaces
\[
(s(t - s))^{\frac{1}{p}} (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0))^{\frac{1}{p}} \hookrightarrow L^2(\mathbb{N}_0). \tag{5.13}
\]
Since (cf. Lemma 4.2)
\[
(\tilde{r}(t - \tilde{r}))^{\frac{1}{2}} (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0))^{\frac{1}{2}} = L^2(\mathbb{N}_0),
\]
the embedding (5.13) turns out to be equivalent to
\[
(s(t - s))^{\frac{1}{p}} (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0))^{\frac{1}{p}} \hookrightarrow (\tilde{r}(t - \tilde{r}))^{\frac{1}{2}} (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0))^{\frac{1}{2}}. \tag{5.14}
\]
It remains to show the validity of (5.14). To do this, we distinguish two possible cases. Suppose first that \(p \leq 2\), then by Lemma 4.5(ii) (note that \(\tilde{r} < s\))
\[
(s(t - s))^{\frac{1}{p}} (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0))^{\frac{1}{p}} \hookrightarrow (\tilde{r}(t - \tilde{r}))^{\frac{1}{2}} (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0))^{\frac{1}{2}}.
\]
Suppose now that \(p > 2\). Applying again Lemma 4.5(ii), we have
\[
\left(\frac{1}{(s - \tilde{r})^{1/2 - 1/p}} + \frac{s^{1/p}(t - s)^{1/p}}{\tilde{r}^{1/2}}\right) (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0))^{\frac{1}{p}} \hookrightarrow (L^2(\mathbb{N}_0), L^2(\mathbb{N}_0))^{\frac{1}{2}}. \tag{5.15}
\]
Furthermore, it is easy to check that under the assumptions (1) & (2) in Theorem 3.3, the constant in (5.15) behaves like
\[
\frac{1}{(s - \tilde{r})^{1/2 - 1/p}} + \frac{s^{1/p}(t - s)^{1/p}}{\tilde{r}^{1/2}} \approx (s(t - s))^{\frac{1}{p}} (\tilde{r}(t - \tilde{r}))^{-\frac{1}{2}},
\]
which yields the desired embedding (5.14).

For the homogeneous setting (i.e., (3.3) and (3.5)), observe that it is enough to deal with \(r = \tilde{r}\), since for \(r \in (0, \tilde{r})\) we have
\[
\|f\|_{F^r_{p,2}(\mathbb{R}^n)} \leq \|f\|_{F^0_{p,2}(\mathbb{R}^n)} + \|f\|_{F^\tilde{r}_{p,2}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\tilde{F}^\tilde{r}_{p,2}(\mathbb{R}^n)}.
\]
Henceforth, we concentrate on (3.3) and (3.5) with \(r = \tilde{r}\). The methodology for the inhomogeneous setting given above (i.e., (3.2) and (3.4)) yields (cf. [5.12])
\[
\|f\|_{F^r_{p,2}(\mathbb{R}^n)} \lesssim (s(t - s))^{\frac{1}{p}} \|f\|_{(L^p(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n))^{\frac{1}{p}}}. \tag{5.16}
\]
\(^{16}\) Here we must take into account that \(F^r_{p,2}(\mathbb{R}^n)\) can be identified with the vector-valued space \(L^p(\mathbb{R}^n; L^2(\mathbb{N}_0))\).
Indeed, in this case the analogue of (5.13) reads as
\[
(s(t - s))^{\frac{1}{p}} (\ell_2(\mathbb{Z}), \ell_2^2(\mathbb{Z}))_{\tilde{r}, p} \hookrightarrow \ell_2^2(\mathbb{Z}),
\]  
(5.17)
where
\[
\tilde{r} = \begin{cases} 
  t - \Lambda(t - s), & \text{if } s \to t^-, \\
  \frac{1}{\Lambda}s, & \text{if } s \to 0^+.
\end{cases}
\]

Here the couple \((\ell_2(\mathbb{Z}), \ell_2^2(\mathbb{Z}))\) is not ordered and we cannot invoke Lemma 4.5(ii) directly to deal with (5.17). However, this difficulty can be overcome as follows. We write \(\mathbb{Z} = \mathbb{N}_0 \cup \mathbb{N}_-\) where \(\mathbb{N}_- := \{-j : j \in \mathbb{N}\}\). Since \(\ell_2^1(\mathbb{N}_0) \hookrightarrow \ell_2(\mathbb{N}_0)\) and \(\ell_2^2(\mathbb{N}_-) \hookrightarrow \ell_2^2(\mathbb{N}_-)\), it follows from (5.13) that
\[
(s(t - s))^{\frac{1}{p}} (\ell_2(\mathbb{N}_0), \ell_2^2(\mathbb{N}_0))_{\tilde{r}, p} \hookrightarrow \ell_2^2(\mathbb{N}_0)
\]  
(5.18)
and, by (4.2), and a simple change of variables,
\[
(s(t - s))^{\frac{1}{p}} (\ell_2(\mathbb{N}_-), \ell_2^2(\mathbb{N}_-))_{\tilde{r}, p} \hookrightarrow \ell_2^2(\mathbb{N}_-).
\]  
(5.19)
Let \(\xi = (\xi_j)_{j \in \mathbb{Z}}\) and consider the related sequences \(\xi^1 = (\xi_j)_{j \in \mathbb{N}_0}\) and \(\xi^2 = (\xi_j)_{j \in \mathbb{N}_-}\). By triangle inequality and (5.18), (5.19), we derive
\[
\|\xi\|_{\ell_2^2(\mathbb{Z})} \leq \|\xi^1\|_{\ell_2^1(\mathbb{N}_0)} + \|\xi^2\|_{\ell_2^2(\mathbb{N}_-)} \\
\leq C(s(t - s))^{\frac{1}{p}} \left(\|\xi^1\|_{(\ell_2(\mathbb{N}_0), \ell_2^1(\mathbb{N}_0))_{\tilde{r}, p}} + \|\xi^2\|_{(\ell_2(\mathbb{N}_-), \ell_2^2(\mathbb{N}_-))_{\tilde{r}, p}}\right) \\
\leq 2C(s(t - s))^{\frac{1}{p}} \|\xi\|_{(\ell_2(\mathbb{Z}), \ell_2^2(\mathbb{Z}))_{\tilde{r}, p}},
\]
where the last step follows immediately from the interpolation property applied to the canonical projections \(\xi \mapsto \xi^i, i = 1, 2\). The proof of (5.17) is complete establishing that (5.16) holds. Now the rest of the proof follows line by line the arguments provided for the inhomogeneous case. \(\square\)

5.3 Proof of Theorem 3.2

Without loss of generality, we may assume that \(t - s_k \leq \frac{1}{4}\). According to (3.3) there is \(C = C(n, p, t) > 0\) such that, for every \(r \leq t - 2(t - s_k)\),
\[
\|f_k\|_{\ell^r_{p, 2}(\mathbb{R}^n)} \leq C \left(\|f_k\|_{L^p(\mathbb{R}^n)} + (t - s_k)^{\frac{1}{p}} \|f_k\|_{W^{s_k, p}(\mathbb{R}^n), t}\right) \leq C \Lambda.
\]
Taking limits as \(k \to \infty\), and noting that \(\lim_{k \to \infty} t - 2(t - s_k) = t\), the previous estimate yields
\[
\limsup_{k \to \infty} \left(\|f_k\|_{L^p(\mathbb{R}^n)} + \|f_k\|_{\ell^r_{p, 2}(\mathbb{R}^n)}\right) \lesssim \Lambda \quad \text{for all} \quad r \in (0, t).
\]
Therefore, one can apply [5, Lemma 2.6] to derive \(f \in H^{1, p}(\mathbb{R}^n)\) and
\[
\|(-\Delta)^{\frac{1}{2}} f\|_{L^p(\mathbb{R}^n)} \lesssim \Lambda.
\]
\(\square\)
Remark 5.2 It may be instructive to revisit the counterexample provided in Proposition 3.1 to show the important role played by the seminorms $\| \cdot \|_{\dot{W}^{s,p}(\mathbb{T}),t}$ in Theorem 3.2. Consider the Fourier series $f$ defined by (3.1). It was shown in Proposition 3.1 that the condition
\[
\sup_{k \in \mathbb{N}} (t - s_k)^{\frac{1}{p}} \| f \|_{\dot{W}^{s_k,p}(\mathbb{T}),t} < \infty
\]
is satisfied but $f \notin H^{t,p}(\mathbb{T})$. Next we check that this example does not contradict Theorem 3.2 since
\[
\limsup_{k \to \infty} (t - s_k)^{\frac{1}{p}} \| f \|_{\dot{W}^{s_k,p}(\mathbb{T}),t} = \infty. \quad (5.20)
\]
Indeed, applying [29, Theorem 6.2] one can estimate, for every $l \in \mathbb{N}$,
\[
K(t^{-l}, f; L^p(\mathbb{T}), \dot{H}^{t,p}(\mathbb{T})) \approx l^{-l} \left( \sum_{v=1}^{l} v^{p} p - 2 (p - 1) \right)^{1/p} + \left( \sum_{v=l}^{\infty} v^{p} p - 2 (p - 1) \right)^{1/p} \\
\approx l^{-l} (1 + \log l)^{1/p}
\]
and thus, by Lemma 4.8,
\[
\| f \|^{p}_{\dot{W}^{s_k,p}(\mathbb{T}),t} \approx \| f \|^{p}_{(L^p(\mathbb{T}), \dot{H}^{t,p}(\mathbb{T}))} \\
\geq \sum_{l=0}^{\infty} 2^{l s_k} p K(t^{-l}, f; L^p(\mathbb{T}), \dot{H}^{t,p}(\mathbb{T}))^p \\
\approx \sum_{l=0}^{\infty} 2^{-l(t-s_k)p} (1 + l) \\
\geq (t - s_k)^{-1} \sum_{l=\left\lfloor \frac{1}{t-s_k} \right\rfloor}^{\infty} 2^{-l(t-s_k)p} \\
\approx \frac{(t - s_k)^{-1}}{1 - 2^{-l(t-s_k)p}} \approx (t - s_k)^{-2}.
\]
Hence
\[
(t - s_k)^{1/p} \| f \|_{\dot{W}^{s_k,p}(\mathbb{T}),t} \geq (t - s_k)^{-1/p}
\]
which yields (5.20) (since $s_k \uparrow t$).

5.4 Proof of Theorem 3.4

Let $(A_0, A_1) = (L^p(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n))$. Recall that for this couple for $f \in H^{t,p}(\mathbb{R}^n)$, (cf. [33, Corollary 10, p. 75], [21, Sect. 1.3, Property 1])
\[
\sup_{t>0} K(t, f; L^p(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n)) \approx \| (\Delta)^{\frac{t}{2}} f \|_{L^p(\mathbb{R}^n)}
\]
and
\[
\sup_{t>0} K(t, f; L^p(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n)) \approx \| f \|_{L^p(\mathbb{R}^n)}.
\]
Therefore (3.6) [respectively, (3.7)] is an immediate consequence of Lemma 4.8 and (4.10) [respectively, (4.11)].

Concerning the inhomogeneous setting [i.e., (3.8) and (3.9)], the proofs are immediate consequences of their homogeneous counterparts [i.e., (3.6) and (3.7)] applied to (1.17).

5.5 Proof of Theorem 3.6

Assume $0 < r < t < 1$. Applying Lemma 4.4 to the Triebel–Lizorkin pair $(A_0, A_1) = (\dot{F}_{p,2}^r(\mathbb{R}^n), \dot{F}_{p,2}^t(\mathbb{R}^n))$, one has

$$\|f\|_{(\dot{F}_{p,2}^r(\mathbb{R}^n), \dot{F}_{p,2}^t(\mathbb{R}^n))_{\theta,p}} \lesssim \frac{1}{\theta^{1/p}} \|f\|_{\dot{F}_{p,2}^t(\mathbb{R}^n)} + \frac{1}{(1 - \theta)^{1/p}} \|f\|_{\dot{F}_{p,2}^r(\mathbb{R}^n)}$$

for every $\theta \in (0, 1)$. Since (cf. Lemma 2.1(i))

$$\dot{F}_{p,2}^r(\mathbb{R}^n) \hookrightarrow \dot{B}_{p,\max[p,2]}^r(\mathbb{R}^n) \quad \text{and} \quad \dot{F}_{p,2}^t(\mathbb{R}^n) \hookrightarrow \dot{B}_{p,\max[p,2]}^t(\mathbb{R}^n),$$

the previous inequality implies

$$\|f\|_{(\dot{B}_{p,\max[p,2]}^r(\mathbb{R}^n), \dot{B}_{p,\max[p,2]}^t(\mathbb{R}^n))_{\theta,p}} \lesssim \frac{1}{\theta^{1/p}} \|f\|_{\dot{B}_{p,\max[p,2]}^r(\mathbb{R}^n)} + \frac{1}{(1 - \theta)^{1/p}} \|f\|_{\dot{B}_{p,\max[p,2]}^t(\mathbb{R}^n)}.$$

Next we compute the interpolation norm given in the left-hand side of (5.21). Indeed, since $r, t \in (0, 1)$, we can invoke Lemma 4.3 to state

$$\dot{B}_{p,\max[p,2]}^r(\mathbb{R}^n) = (L^p(\mathbb{R}^n), \dot{W}_p^1(\mathbb{R}^n))_{r,\max[p,2]}$$

and

$$\dot{B}_{p,\max[p,2]}^t(\mathbb{R}^n) = (L^p(\mathbb{R}^n), \dot{W}_p^1(\mathbb{R}^n))_{t,\max[p,2]}.$$

Therefore, in light of Lemmas 4.6(i) and (4.14), we get, uniformly w.r.t. $\theta$,

$$\|f\|_{(\dot{B}_{p,\max[p,2]}^r(\mathbb{R}^n), \dot{B}_{p,\max[p,2]}^t(\mathbb{R}^n))_{\theta,p}} \approx \|f\|_{(L^p(\mathbb{R}^n), \dot{W}_p^1(\mathbb{R}^n))_{r,\max[p,2]}} \|f\|_{(L^p(\mathbb{R}^n), \dot{W}_p^1(\mathbb{R}^n))_{t,\max[p,2]}}$$

$$\lesssim (\theta(1 - \theta))^{-1/\max[p,2]} \|f\|_{(L^p(\mathbb{R}^n), \dot{W}_p^1(\mathbb{R}^n))_{(1 - \theta)r + \theta t,p}}$$

$$\approx (\theta(1 - \theta))^{-1/\max[p,2]} \|f\|_{\dot{W}^{(1 - \theta)r + \theta t,p}(\mathbb{R}^n)}.$$

Combining this and (5.21), we arrive at

$$\frac{1}{(\theta(1 - \theta))^{1/\max[p,2]}} \|f\|_{\dot{W}^{(1 - \theta)r + \theta t,p}(\mathbb{R}^n)} \lesssim \frac{1}{\theta^{1/p}} \|f\|_{\dot{F}_{p,2}^t(\mathbb{R}^n)} + \frac{1}{(1 - \theta)^{1/p}} \|f\|_{\dot{F}_{p,2}^r(\mathbb{R}^n)}.$$

(5.22)

Given $s \in (r, t)$, we choose $\theta \in (0, 1)$ such that $s = (1 - \theta)r + \theta t$ and make the corresponding change of variables in (5.22), i.e.,

$$\frac{1}{((s - r)(t - s))^{1/\max[p,2]}} \|f\|_{\dot{W}^{s,p}(\mathbb{R}^n)} \lesssim \frac{1}{(s - r)^{1/p}} \|f\|_{\dot{F}_{p,2}^t(\mathbb{R}^n)} + \frac{1}{(t - s)^{1/p}} \|f\|_{\dot{F}_{p,2}^r(\mathbb{R}^n)}.$$

The proofs in the limiting cases $r = 0$ and $t = 1$ [i.e., (3.11) and (3.12) respectively] are easier and we omit further details. □
5.6 Proof of Theorem 3.9

Since $H^{s,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$, it follows from Lemma 4.5(ii) that

$$\left( s(\alpha - s) \right)^{1/p} \| f \|_{L^p(\mathbb{R}^n)} \| f \|_{H^{s,p}(\mathbb{R}^n)}^{\frac{1}{p}} \lesssim (t(\alpha - t))^{1/p} \| f \|_{L^p(\mathbb{R}^n)} \| f \|_{H^{s,p}(\mathbb{R}^n)}^{\frac{1}{p}}.$$  

Combining with (4.13) yields

$$\| f \|_{L^p(\mathbb{R}^n)} + (s(\alpha - s))^{1/p} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)} \approx (s(1 - s))^{1/p} \| f \|_{L^p(\mathbb{R}^n)} \| f \|_{H^{s,p}(\mathbb{R}^n)}^{\frac{1}{p}} \lesssim (t(1 - t))^{1/p} \| f \|_{L^p(\mathbb{R}^n)} + (t(\alpha - t))^{1/p} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)}.$$  

The desired result now follows since $s(\alpha - s) \approx \min\{s, \alpha - s\}$. □

Appendix A: Sharp versions of fractional Sobolev inequalities

For $p \in (0, \infty)$ and $q \in (0, \infty)$, the Lorentz space $L^{p,q}(\mathbb{R}^n)$ is formed by all measurable functions $f$ defined on $\mathbb{R}^n$ such that

$$\| f \|_{L^{p,q}(\mathbb{R}^n)} := \left( \int_0^\infty \left( u^\frac{1}{p} f^*(u) \right)^q du \right)^{\frac{1}{q}} < \infty$$

(with the usual modification if $q = \infty$). As usual, $f^*$ denotes the non-increasing rearrangement of $f$. By $f^{**}$ we denote the maximal function given by $f^{**}(u) := \frac{1}{u} \int_0^u f^*(v) dv$. We refer to [1, 2] for detailed accounts on Lorentz spaces.

Let $0 < s < 1$, $1 \leq p < \frac{n}{s}$ and $p^* = \frac{np}{n - sp}$. The Bourgain–Brezis–Mironescu–Maz’ya–Shasposhnikova formula (cf. [4, 24]) claims that there exists $C = C(n, p) > 0$ such that

$$\| f \|_{L^{p,s,p}(\mathbb{R}^n)} \leq C \frac{s(1 - s)}{(n - sp)p} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n)}; \quad (5.23)$$

see also [20]. This inequality can be considered as a sharp version of the classical Sobolev embedding

$$\dot{W}^1_p(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-sp}}(\mathbb{R}^n), \quad 1 \leq p < n. \quad (5.24)$$

Indeed, (5.24) follows from (5.23) by taking limits as $s \to 1^-$ (cf. [1.3]).

Let $t \in (0, 1)$ be fixed. Note that the formula (5.23) does not provide any insight if $s \to t^-$. So the natural question here is: can we obtain an analogue of (5.23) for fractional smoothness $t$? Or equivalently, what is the fractional counterpart of (5.23) related to the classical embedding

$$\dot{H}^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-sp}}(\mathbb{R}^n), \quad 1 < p < \frac{n}{t}? \quad (5.25)$$

The answer is given again in terms of the Butzer seminorms $\| \cdot \|_{\dot{W}^{s,p}(\mathbb{R}^n), t}$ (cf. [1.16]).

Theorem 5.3  Let $0 < s < t \leq 1$, $1 < p < \frac{n}{s}$ and $p^* = \frac{np}{n - sp}$. Assume $f \in W^{s,p}(\mathbb{R}^n)$. Then there exists $C = C(n, p, t) > 0$ such that

$$\| f \|_{L^{p,s,p}(\mathbb{R}^n)} \leq C \frac{s(t - s)}{(n - sp)p} \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t}; \quad (5.26)$$

In particular, if $t = 1$ then one recovers (5.23).
Remark 5.4 Let $t > 0$ and $1 < p < \frac{n}{t}$. Taking limits as $s \to t^-$ in (5.26) and applying Theorem 3.4, we recover (5.25).

Proof of Theorem 5.3 Case 1: Assume $t < \frac{n}{p}$ and $s \in (0, t)$. Under these assumptions, the Hardy–Littlewood–Sobolev theorem (cf. [26, p. 139]) asserts

$$
\hat{H}^{t, p} (\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-tp}} (\mathbb{R}^n).
$$

If we interpolate this embedding with the trivial one $L^p (\mathbb{R}^n) \hookrightarrow L^p (\mathbb{R}^n)$, we arrive at

$$
\left\| f \right\|_{(L^p (\mathbb{R}^n), L^{\frac{np}{n-tp}} (\mathbb{R}^n))_{\frac{1}{p}, p}} \lesssim \left\| f \right\|_{(L^p (\mathbb{R}^n), \hat{H}^{t, p} (\mathbb{R}^n))_{\frac{1}{p}, p}} \tag{5.27}
$$

uniformly with respect to $s \in (0, t)$. In light of Lemma 4.8,

$$
\left\| f \right\|_{(L^p (\mathbb{R}^n), \hat{H}^{t, p} (\mathbb{R}^n))_{\frac{1}{p}, p}} \approx \left\| f \right\|_{\dot{W}^{t, p} (\mathbb{R}^n), t}. \tag{5.28}
$$

Next we show that

$$
\left\| f \right\|_{L^{p^*} (\mathbb{R}^n)} \approx (s(t - s))^{\frac{1}{p}} \left\| f \right\|_{(L^p (\mathbb{R}^n), L^{\frac{np}{n-tp}} (\mathbb{R}^n))_{\frac{1}{p}, p}}. \tag{5.29}
$$

Indeed, since $L^{\frac{np}{n-tp}} (\mathbb{R}^n) = (L^p (\mathbb{R}^n), L^\infty (\mathbb{R}^n))_{\frac{np}{p}, p}$ (see, e.g., [2, Theorem 5.2.1, p. 109]), we can write

$$(L^p (\mathbb{R}^n), L^{\frac{np}{n-tp}} (\mathbb{R}^n))_{\frac{1}{p}, p} = (L^p (\mathbb{R}^n), (L^p (\mathbb{R}^n), L^\infty (\mathbb{R}^n))_{\frac{np}{p}, p})_{\frac{1}{p}, p},$$

with hidden constants of equivalence independent of $s$. Invoking now the sharp reiteration formula given in Lemma 4.6(ii), we get

$$(L^p (\mathbb{R}^n), L^{\frac{np}{n-tp}} (\mathbb{R}^n))_{\frac{1}{p}, p} = (t - s)^{-\frac{1}{p}} (L^p (\mathbb{R}^n), L^\infty (\mathbb{R}^n))_{\frac{sp}{p}, p}. \tag{5.30}$$

Furthermore, using the well-known characterization of the $K$-functional (cf. [2, Theorem 5.2.1, p. 109])

$$
K(u, f; L^p (\mathbb{R}^n), L^\infty (\mathbb{R}^n)) \approx \left( \int_0^u (f^*(v))^p \, dv \right)^{\frac{1}{p}}
$$

and applying Fubini’s theorem, we obtain

$$
\left\| f \right\|_{(L^p (\mathbb{R}^n), L^\infty (\mathbb{R}^n))_{\frac{np}{p}, p}} = \int_0^\infty (u - \frac{sp}{p}) K(u, f; L^p (\mathbb{R}^n), L^\infty (\mathbb{R}^n))^{\frac{1}{p}} \frac{du}{u}
$$

$$
\approx \int_0^\infty u^{\frac{np}{p}} \int_0^u (f^*(v))^p \, dv \frac{du}{u}
$$

$$
= \int_0^\infty (f^*(v))^p \int_0^\infty u^{\frac{np}{p}} \frac{du}{u} \, dv
$$

$$
= \frac{n}{sp} \left\| f \right\|_{L^{p^*} (\mathbb{R}^n)}.\tag{5.31}
$$

Inserting this into (5.30), we arrive at the desired estimate (5.29).

Combining now (5.27)–(5.29) we achieve (5.26).

Case 2: Assume that either $t > \frac{n}{p}$ and $s \in \left(0, \frac{n}{p}\right)$.

\[\text{Proof details here.}\]
We will make use of the well-known embedding (see, e.g., [27, p. 164])

\[ \dot{H}^{\frac{n}{p}}(\mathbb{R}^n) \hookrightarrow \text{BMO}(\mathbb{R}^n) \]

where BMO(\mathbb{R}^n) is the space of bounded mean oscillation functions of John–Nirenberg [18] endowed with the seminorm

\[ \|f\|_{\text{BMO}(\mathbb{R}^n)} := \|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \]

and \( \hat{f} \) is the sharp maximal function of \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), i.e.,

\[ \hat{f}(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy, \quad x \in \mathbb{R}^n, \]

where the supremum runs over all cubes \( Q \) in \( \mathbb{R}^n \) with sides parallel to the axes of coordinates and \( f_Q = \frac{1}{|Q|} \int_Q f \).

In this case, the analogue of (5.27) reads as

\[ \|f\|_{(L^p(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))_{\frac{n}{p}, p}} \lesssim \|f\|_{(L^p(\mathbb{R}^n), \dot{H}^{\frac{n}{p}}(\mathbb{R}^n))_{\frac{n}{p}, p}} \]

for all \( s \in (0, \frac{n}{p}) \).

We make the following claim, uniformly with respect to \( s \in (0, \frac{n}{p}) \),

\[ (L^p(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))_{\frac{n}{p}, p} \hookrightarrow s^{-\frac{1}{p}}(n - sp)L^{p^*}(\mathbb{R}^n). \]

Assuming momentarily the validity of this claim, then (5.26) follows easily from (5.31) and (5.28) (with \( t = \frac{n}{p} \)).

It remains to show (5.32). To proceed with, we will make use of the known estimate (cf. [17, Corollary 3.3])

\[ K(u, f; L^p(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n)) \approx \left( \int_0^u \left( f^{#*}(v) \right)^p \, dv \right)^{\frac{1}{p}}. \]

Accordingly, by Fubini’s theorem,

\[ \|f\|_{(L^p(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n))_{\frac{n}{p}, p}}^p = \int_0^\infty \left( u^{-\frac{n}{p}} K(u, f; L^p(\mathbb{R}^n), \text{BMO}(\mathbb{R}^n)) \right)^p \frac{du}{u} \\
\approx \int_0^\infty u^{-\frac{n}{p}} \int_0^u (f^{#*}(v))^p \, dv \, \frac{du}{u} \\
= \frac{n}{sp} \int_0^\infty \frac{1}{v^{\frac{n}{p}}} \frac{f^{#*}(v)}{v} \, dv. \]

We can compare \( f^# \) and \( f^{**} \) via the following weak-type estimate (cf. [1, Proposition 8.10, p. 398])

\[ (f - f_\infty)^{**}(u) \lesssim \int_u^\infty f^{#*}(v) \, \frac{dv}{v} \]
where \( f_\infty := \lim_{|Q| \to \infty} \frac{1}{|Q|} \int_Q f \). Note that \( f_\infty = 0 \) since \( f \in L^p(\mathbb{R}^n) \). Accordingly, by Hardy’s inequality (see, e.g., [28, p. 196]),

\[
\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq \left( \int_0^\infty [u^{\frac{1}{p^*}} f^{*\star}(u)]^p \frac{du}{u} \right)^{\frac{1}{p}} 
\leq \left( \int_0^\infty \left( u^{\frac{1}{p^*}} \int_0^\infty f^{*\star}(v) \frac{dv}{v} \right)^p \frac{du}{u} \right)^{\frac{1}{p}} 
\leq p^* \left( \int_0^\infty u^{\frac{p}{p^*}} (f^{*\star}(u))^p \frac{du}{u} \right)^{\frac{1}{p}} 
\approx s \frac{1}{p} p^* \|f\|_{(L^p(\mathbb{R}^n), BMO(\mathbb{R}^n))_{\frac{p}{p^*}}},
\]

where we have used (5.33) in the last step. The proof of (5.32) is complete.

**CASE 3:** Assume \( t = \frac{n}{p} \) and \( s \to t^- \). Let \( t_0 \in (0, t) \) be fixed and choose \( \theta \in (0, 1) \) such that \( s = (1-\theta)t_0 + \theta t \). Then there exists a constant \( C > 0 \), which is independent of \( s \), such that

\[
\|f\|_{((L^p(\mathbb{R}^n), \dot{H}^{t_0,p}(\mathbb{R}^n)))_{\frac{n}{p}, \min \{p, 2\}}} \leq C \|f\|_{W^{s,p}(\mathbb{R}^n), t} \tag{5.34}
\]

Indeed, since (see, e.g., [2, Sect. 6.4, pp. 149–153])

\[
\dot{B}_{p, \min \{p, 2\}}^{t_0}(\mathbb{R}^n) = (L^p(\mathbb{R}^n), \dot{H}^{t_0,p}(\mathbb{R}^n))_{\frac{n}{p}, \min \{p, 2\}},
\]

and

\[
\dot{B}_{p, \min \{p, 2\}}^{t_0}(\mathbb{R}^n) \hookrightarrow \dot{H}^{t_0,p}(\mathbb{R}^n),
\]

we have

\[
((L^p(\mathbb{R}^n), \dot{H}^{t_0,p}(\mathbb{R}^n))_{\frac{n}{p}, \min \{p, 2\}}, \dot{H}^{t_0,p}(\mathbb{R}^n))_{\theta, p} \hookrightarrow ((L^p(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n))_{\frac{n}{p}, \min \{p, 2\}}, \dot{H}^{t,p}(\mathbb{R}^n))_{\theta, p} \tag{5.35}
\]

with related embedding constant independent of \( s \). Furthermore, in virtue of the sharp version of the reiteration formula given in Lemma 4.6(ii) and (4.2) (and taking into account that \( \theta \to 1^- \)), the following holds

\[
(L^p(\mathbb{R}^n), \dot{H}^{t_0,p}(\mathbb{R}^n))_{1-(1-\theta)(1-\frac{n}{p}), \theta} \hookrightarrow ((L^p(\mathbb{R}^n), \dot{H}^{t_0,p}(\mathbb{R}^n))_{\frac{n}{p}, \min \{p, 2\}}, \dot{H}^{t_0,p}(\mathbb{R}^n))_{\theta, p}
\]

uniformly with respect to \( \theta \), or equivalently, by Lemma 4.8,

\[
\|f\|_{((L^p(\mathbb{R}^n), \dot{H}^{t_0,p}(\mathbb{R}^n))_{\frac{n}{p}, \min \{p, 2\}}, \dot{H}^{t_0,p}(\mathbb{R}^n))_{\theta, p}} \lesssim \|f\|_{W^{s,p}(\mathbb{R}^n), t}. \tag{5.36}
\]

Putting together (5.35) and (5.36), the estimate (5.34) is achieved.

By the well-known fact (see, e.g., [32, Theorem 1(i), Sect. 5.2.3, p. 242]) that \( -\Delta \frac{n}{2} \) acts as an isomorphism from \( \dot{H}^{t_0,p}(\mathbb{R}^n) \) onto \( L^p(\mathbb{R}^n) \) and from \( \dot{H}^{t,p}(\mathbb{R}^n) \) onto \( \dot{H}^{-t_0,p}(\mathbb{R}^n) \), we easily derive that

\[
K(u, f; \dot{H}^{t_0,p}(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n)) \approx K(u, (-\Delta)^{\frac{n}{2}} f; L^p(\mathbb{R}^n), \dot{H}^{-t_0,p}(\mathbb{R}^n))
\]

and thus

\[
\|f\|_{((\dot{H}^{t_0,p}(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n))_{\theta, p}} \approx \|(-\Delta)^{\frac{n}{2}} f\|_{(L^p(\mathbb{R}^n), \dot{H}^{-t_0,p}(\mathbb{R}^n))_{\theta, p}}
\]

where the hidden constants are independent of \( \theta \). According to Lemma 4.8, the last estimate can be equivalently rewritten as

\[
\|f\|_{((\dot{H}^{t_0,p}(\mathbb{R}^n), \dot{H}^{t,p}(\mathbb{R}^n))_{\theta, p}} \approx \|(-\Delta)^{\frac{n}{2}} f\|_{W^{t_0(1-\theta), p}(\mathbb{R}^n), t-t_0}. \tag{5.37}
\]
Since $0 < \theta (t - t_0) < t - t_0 < t = \frac{n}{p}$, we can apply Case 1 so that (take into account that $f \in W^{3, p}(\mathbb{R}^n)$ and, in particular, $f \in H^{0, p}(\mathbb{R}^n)$ since $t_0 < s$)

$$\| (\Delta)^{\frac{q}{2}} f \|_{L^{\frac{np}{n-s-t_0}}(\mathbb{R}^n)} \lesssim (1 - \theta)^{\frac{1}{p}} \| (\Delta)^{\frac{q}{2}} f \|_{W^{q(t-t_0), p}(\mathbb{R}^n), t-t_0}. \quad (5.38)$$

As usual, we denote by $I_{\alpha}$, $\alpha \in (0, n)$, the Riesz potential operator

$$(I_{\alpha} f)(x) := c_{n, \alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n.$$ 

By well-known mapping properties of $I_{\alpha}$ (cf. [27, Chapter V, Sect. 1.2])

$$I_{t_0} : L^{\frac{np}{n-s-t_0}}(\mathbb{R}^n) \to L^{p^*(u)}(\mathbb{R}^n). \quad (5.39)$$

However, our method requires to control the norm of this operator with respect to $s$. Accordingly, to make the exposition self-contained, we provide below a detailed proof of (5.39) using standard techniques. It follows from O’Neil’s inequality [26, Lemma 1.5] that

$$(I_{t_0} f)^*(u) \lesssim \int_0^\infty f^{**}(v) v^{\frac{t_0}{n}} - 1 \, dv$$

and thus, by Hardy’s inequalities (cf. [28, p. 196]),

$$\| I_{t_0} f \|_{L^{p^*(u)}(\mathbb{R}^n)} \lesssim \left( \int_0^\infty u^{\frac{p}{p^*}} \left( \int_0^\infty f^{**}(v) v^{\frac{t_0}{n}} - 1 \, dv \right)^p \, du \right)^{\frac{1}{p}}$$

$$\lesssim p^* \left( \int_0^\infty \left( u^{\frac{n-(s-t_0)p}{np}} f^{**}(u) \right)^p \, du \right)^{\frac{1}{p}}$$

$$\lesssim \frac{p^*}{1 - \frac{1}{p} + \frac{s-t_0}{n}} \left( \int_0^\infty \left( u^{\frac{n-(s-t_0)p}{np}} f^*(u) \right)^p \, du \right)^{\frac{1}{p}}$$

$$\approx \frac{1}{n - sp} \| f \|_{L^{\frac{np}{n-(s-t_0)p}}(\mathbb{R}^n)}$$

where we have used $s \to t^- = \left( \frac{n}{p} \right)^- \text{in the last step. This proves (cf. [5.39])}$

$$\| I_{t_0} \|_{L^{\frac{np}{n-(s-t_0)p}}(\mathbb{R}^n) \to L^{p^*(u)}(\mathbb{R}^n)} \lesssim (n - sp)^{-1} \quad (5.40)$$

uniformly with respect to $s \to t^-$. 

Since $I_{t_0} (\Delta)^{\frac{q}{2}} f = f$, in light of (5.40), (5.38), (5.37) and (5.34) we get

$$\| f \|_{L^{p^*(u)}(\mathbb{R}^n)} \lesssim (n - sp)^{-1} \| (\Delta)^{\frac{q}{2}} f \|_{L^{\frac{np}{n-(s-t_0)p}}(\mathbb{R}^n)}$$

$$\lesssim (n - sp)^{-\frac{1}{p}} \| (\Delta)^{\frac{q}{2}} f \|_{W^{q(t-t_0), p}(\mathbb{R}^n), t-t_0}$$

$$\approx (n - sp)^{-\frac{1}{p}} \| f \|_{(H^{0, p}(\mathbb{R}^n), H^{0, p}(\mathbb{R}^n))_{\phi, p}}$$

$$\lesssim (n - sp)^{-\frac{1}{p}} \| f \|_{W^{s, p}(\mathbb{R}^n), t}.$$ 

The proof is finished. \qed

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As an immediate consequence of Theorem 5.3 and the well-known inequality between Lorentz spaces (cf. [28, p. 192])
\[
\left(\frac{q}{p}\right)^{\frac{1}{q}} \| f \|_{L^{p,q}(\mathbb{R}^n)} \leq \left(\frac{r}{p}\right)^{\frac{1}{r}} \| f \|_{L^{p,r}(\mathbb{R}^n)}, \quad r < q,
\]
we obtain the following

**Corollary 5.5** Let \( 0 < s < t \leq 1, \ 1 < p \leq \frac{n}{s} \) and \( p^* = \frac{np}{n-sp} \). Assume \( f \in W^{s,p}(\mathbb{R}^n) \). Then there exists \( C = C(n, p, t) > 0 \) such that
\[
\| f \|_{L^{p^*}(\mathbb{R}^n)} \leq C (t-s)(n-sp)^{1-p} \| f \|_{W^{s,p}(\mathbb{R}^n), t}.
\]

**Appendix B: Sharp relationships between Sobolev–Triebel–Lizorkin spaces**

The methodology proposed in [5] relies on sharp comparison estimates between \( \dot{W}^{s,p}(\mathbb{R}^n) \), \( \dot{F}^{s}_{p,2}(\mathbb{R}^n) \) and \( \dot{F}^{s}_{p,p}(\mathbb{R}^n) \) using tools from harmonic analysis. We now provide an approach to these results relying on the interpolation techniques presented in Sect. 4.

Let \( s \in (0, 1) \) and \( p \in (1, \infty) \). Recall that (cf. Lemma 2.1)
\[
\dot{W}^{s,p}(\mathbb{R}^n) = \dot{F}^{s}_{p,p}(\mathbb{R}^n), \quad (5.41)
\]
\[
\dot{F}^{s}_{p,2}(\mathbb{R}^n) \hookrightarrow \dot{W}^{s,p}(\mathbb{R}^n), \quad p \geq 2,
\]
and
\[
\dot{W}^{s,p}(\mathbb{R}^n) \hookrightarrow \dot{F}^{s}_{p,2}(\mathbb{R}^n), \quad p \leq 2.
\]

The sharp behavior of the related equivalence constants in these embeddings plays a key role in the Bourgain–Brezis–Mironescu method. In particular, note that the family of semi-norms \( \{ \| \cdot \|_{\dot{F}^{s}_{p,p}(\mathbb{R}^n)} : s \in (0, 1) \} \) converges to \( \| \cdot \|_{\dot{F}^{s}_{p,p}(\mathbb{R}^n)} \) as \( s \rightarrow 1^- \), however this is not the case for \( \{ \| \cdot \|_{\dot{W}^{s,p}(\mathbb{R}^n)} : s \in (0, 1) \} \) (cf. [1.2]). In particular, the hidden equivalence constants in (5.41) must exhibit a certain blow-up as \( s \) approaches the limiting values 0 and 1. The precise answer is contained in the following

**Theorem 5.6** Let \( 0 < s < t < \infty \) and \( p \in (1, \infty) \). Then there exists a positive constant \( C \), which is independent of \( s \), such that

(a) \[
C^{-1}\left(\frac{1}{s^{\max\{p,2\}}} + \frac{1}{(t-s)^{\max\{p,2\}}}\right) \| f \|_{\dot{F}^{s}_{p,p}(\mathbb{R}^n)} \leq \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t}
\]
\[
\leq C \left(\frac{1}{s^{\min\{p,2\}}} + \frac{1}{(t-s)^{\min\{p,2\}}}\right) \| f \|_{\dot{F}^{s}_{p,p}(\mathbb{R}^n)}, \quad (5.42)
\]

(b) \[
C\left(\frac{1}{s} + \frac{1}{(t-s)^{\frac{1}{p}}}\right) \| f \|_{\dot{F}^{s}_{p,2}(\mathbb{R}^n)} \leq \| f \|_{\dot{W}^{s,p}(\mathbb{R}^n), t}, \quad p \leq 2.
\]
\[(c) \quad \|f\|_{\mathcal{W}^s_p(\mathbb{R}^n), t} \leq C \left( \frac{1}{s} + \frac{1}{(t-s)} \right) \|f\|_{\tilde{F}^s_{p,2}(\mathbb{R}^n)}, \quad p \geq 2.\]

**Remark 5.7** The previous result can be understood as a generalization to higher order smoothness of [5, Theorems 1.2 and 1.5] which corresponds to the classical case \( t = 1, \) cf. [1.18].

**Proof of Theorem 5.6** (a): By Lemma 4.8, for every \( s \in (0, t), \)
\[
\|f\|_{\mathcal{W}^s_p(\mathbb{R}^n), t} \approx \|f\|_{(L^p(\mathbb{R}^n), \dot{H}^s_{-p}(\mathbb{R}^n))_{\frac{t}{s}, p}}. \tag{5.43}
\]
The interpolation space given in the right-hand side of the previous estimate can be explicitly computed via the retraction method. Specifically, \((L^p(\mathbb{R}^n), \dot{H}^s_{-p}(\mathbb{R}^n))\) can be identified with the vector-valued pair \((L^p(\mathbb{R}^n; \ell^2_2(\mathbb{Z})), \ell^p(\mathbb{R}^n; \ell^l_2(\mathbb{Z})))\) and thus, by Lemma 4.1,
\[
\| (f_j) \|_{(L^p(\mathbb{R}^n; \ell^2_2(\mathbb{Z})), L^p(\mathbb{R}^n; \ell^l_2(\mathbb{Z})))_{\frac{t}{s}, p}} \approx \| (f_j) \|_{L^p(\mathbb{R}^n; (\ell^2_2(\mathbb{Z}), \ell^l_2(\mathbb{Z})))_{\frac{t}{s}, p}}. \tag{5.44}
\]
Furthermore, according to Lemma 4.2,
\[
\left( \frac{1}{s \max\{p, 2\}} + \frac{1}{(t-s) \max\{p, 2\}} \right) \|\xi\|_{\ell^2_2(\mathbb{Z})} \lesssim \|\xi\|_{(\ell^2_2(\mathbb{Z}), \ell^l_2(\mathbb{Z}))_{\frac{t}{s}, p}} \lesssim \left( \frac{1}{s \min\{p, 2\}} + \frac{1}{(t-s) \min\{p, 2\}} \right) \|\xi\|_{\ell^l_2(\mathbb{Z})}. \tag{5.45}
\]
As a combination of (5.44) and (5.45),
\[
\left( \frac{1}{s \max\{p, 2\}} + \frac{1}{(t-s) \max\{p, 2\}} \right) \| (f_j) \|_{L^p(\mathbb{R}^n; \ell^l_2(\mathbb{Z}))} \lesssim \| (f_j) \|_{(L^p(\mathbb{R}^n; \ell^2_2(\mathbb{Z})), L^p(\mathbb{R}^n; \ell^l_2(\mathbb{Z})))_{\frac{t}{s}, p}} \lesssim \left( \frac{1}{s \min\{p, 2\}} + \frac{1}{(t-s) \min\{p, 2\}} \right) \| (f_j) \|_{L^p(\mathbb{R}^n; \ell^l_2(\mathbb{Z}))}. \tag{5.46}
\]
By the well-known fact that \( \tilde{F}^s_{p,2}(\mathbb{R}^n) \) is a retract of \( L^p(\mathbb{R}^n; \ell^l_2(\mathbb{Z})) \), we can put together (5.46) and (5.43) to arrive at (5.42).

(b): Let \( p \leq 2 \). Applying Lemmas 4.2 and 4.5 (i),
\[
\|\xi\|_{\ell^2_2(\mathbb{Z})} \approx (s(t-s))^\frac{1}{2} \|\xi\|_{(\ell^2_2(\mathbb{Z}), \ell^l_2(\mathbb{Z}))_{\frac{t}{s}, 2}} \lesssim (s(t-s))^\frac{1}{2} \|\xi\|_{(\ell^2_2(\mathbb{Z}), \ell^l_2(\mathbb{Z}))_{\frac{t}{s}, p}}. \tag{5.47}
\]
Taking into account that \( \tilde{F}^s_{p,2}(\mathbb{R}^n) \) is a retract of \( L^p(\mathbb{R}^n; \ell^l_2(\mathbb{Z})) \), it follows now from (5.43), (5.44) and (5.47) that
\[
(s(t-s))^{-\frac{1}{p}} \| f \|_{\tilde{F}^s_{p,2}(\mathbb{R}^n)} \lesssim \| f \|_{\mathcal{W}^s_p(\mathbb{R}^n), t}.
\]

The proof of (c) follows similar ideas as those given in (b). \( \square \)

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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