Action of a finite quantum group on the algebra of complex $N \times N$ matrices

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Abstract

Using the fact that the algebra $\mathcal{M} \cong M_N(\mathbb{C})$ of $N \times N$ complex matrices can be considered as a reduced quantum plane, and that it is a module algebra for a finite dimensional Hopf algebra quotient $\mathcal{H}$ of $U_q sl(2)$ when $q$ is a root of unity, we reduce this algebra $\mathcal{M}$ of matrices (assuming $N$ odd) into indecomposable modules for $\mathcal{H}$. We also show how the same finite dimensional quantum group acts on the space of generalized differential forms defined as the reduced Wess Zumino complex associated with the algebra $\mathcal{M}$.

Keywords: quantum groups, differential calculus, gauge theories, non commutative geometry.

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1 Introduction

When $q$ is a root of unity ($q^N = 1$), the quantized enveloping algebra $U_q sl(2, \mathbb{C})$ possesses interesting quotients that are finite dimensional Hopf algebras. The structure of the left regular representation of such an algebra was investigated in [3] and the pairing with its dual in [2]. We call $\mathcal{H}$ the Hopf algebra quotient of $U_q sl(2, \mathbb{C})$ defined by the relations $K^N = 1, X_\pm^N = 0$ (we shall define the generators $K, X_\pm$ in a later section), and $\mathcal{F}$ its dual. It was shown\footnote{Warning: the authors of [3] actually consider a Hopf algebra quotient defined by $K^{2N} = 1, X_\pm^N = 0$, so that their algebra is, in a sense, twice bigger than ours.} in [3] that the non semi-simple algebras $\mathcal{H}$ is isomorphic with the direct sum of a complex matrix algebra and of several copies of suitably defined matrix algebras with coefficients in the ring $Gr(2)$ of Grassmann numbers with two generators. The explicit structure (for all values of $N$) of those algebras, including the expression of generators themselves, in terms of matrices with coefficients in $\mathbb{C}$ or $Gr(2)$, was obtained by [7]. Using these results, the representation theory of $\mathcal{H}$, for the case $N = 3$, was presented in [4].

Following this work, the authors of [1], studied the action of $\mathcal{H}$ (case $N = 3$) on the algebra of complex matrices $M_3(\mathbb{C})$. In the letter [8], a reduced Wess-Zumino complex $\Omega_{\text{WZ}}(\mathcal{M})$ was introduced, thus providing a differential calculus bicovariant with respect to the action of the quantum group $\mathcal{H}$ on the algebra $M_3(\mathbb{C})$ of complex matrices. This differential algebra (that could be used to generalize gauge field theory models on an auxiliary smooth manifold) was also analysed in terms of representation theory of $\mathcal{H}$ in the same letter. In particular, it was shown that $M_3(\mathbb{C})$ itself can be reduced into the direct sum of three indecomposable representations of $\mathcal{H}$. A general discussion of several other properties of the dually paired Hopf algebras $\mathcal{F}$ and $\mathcal{H}$ (scalar products, star structures, twisted derivations etc. ) can also be found there, as well as in the article [9]. Other properties of $SL_q(2, \mathbb{C})$ at third (or fourth) root of unity should also be discussed in [11].

In the present paper, after recalling several basic definitions and properties, we show that the algebra of usual $N \times N$ complex matrices (assuming $N$ odd) decomposes, under the action of the quantum group $\mathcal{H}$, into a direct sum of $N$ indecomposable representations of dimension $N$ that we call $N_p$. Every indecomposable module of this type contains an invariant irreducible subspace of dimension $p$. In particular, $N_N$ itself is irreducible. We also show how these representations appear as particular subrepresentations of the projective indecomposable modules of $\mathcal{H}$.

Finally we shall give the action of generators of $\mathcal{H}$ on the elements of the reduced Wess-Zumino complex.

A short discussion about the use of those structures in physics is given at the end.

2 The dually paired finite dimensional quantum groups $\mathcal{F}$ and $\mathcal{H}$

2.1 $\mathcal{M} \doteq M_N(\mathbb{C})$ as a finite dimensional quantum plane

The algebra of $N \times N$ matrices can be generated by two elements $x$ and $y$ with relations:

$$xy = qyx \quad \text{and} \quad x^N = y^N = 1,$$

(1)
where $q$ denotes a $N$-th root of unity ($q \neq 1$) and $\mathbb{I}$ denotes the unit matrix. Explicitly, $x$ and $y$ can be taken as the following matrices:

$$
\begin{align*}
x &= \begin{pmatrix}
1 & q^{-1} \\
q^{-2} & \ddots \\
& \ddots & q^{-(N-1)} \\
\end{pmatrix} \\
y &= \begin{pmatrix}
0 & \vdots \\
\vdots & \mathbb{I}_{N-1} \\
\vdots & \vdots \\
1 & 0 & \cdots & 0
\end{pmatrix}
\end{align*}
$$

(2)

This result can be found in [6].

**Warning**: for technical reasons, we shall assume in all this paper that $N$ is odd.

### 2.2 The dually paired quantum groups $\mathcal{F}$ and $\mathcal{H}$

#### 2.2.1 The quantum group $\mathcal{F}$

We now consider a free associative algebra generated by four *a priori* non-commuting symbols $a, b, c, d$ and define the following “change of variables” (notice that the symbol $\otimes$ does not denote the usual tensor product since it involves also a matrix multiplication):

$$
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \otimes \begin{pmatrix}
x \\
y
\end{pmatrix},
$$

(3)

and

$$
\begin{pmatrix}
\tilde{x} \\
\tilde{y}
\end{pmatrix} = \begin{pmatrix}
x \\
y
\end{pmatrix} \otimes \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
$$

(4)

One then imposes that quantities $x', y'$ (and $\tilde{x}, \tilde{y}$) obtained by the previous matrix equalities should satisfy the same relations as $x$ and $y$ (the multiplication of these elements make perfect sense within the tensor product of the corresponding algebras). One obtains in this way the relations:

$$
\begin{align*}
qca &= ac & qdb &= bd \\
qba &= ab & qdc &= cd \\
qb & = bc & ad - da &= (q - q^{-1})bc
\end{align*}
$$

(5)

together with

$$
\begin{align*}
a^N &= \mathbb{I}, & b^N &= 0 \\
c^N &= 0, & d^N &= \mathbb{I}
\end{align*}
$$

(6)

One also take the central element $D = da - q^{-1}bc = ad - qbc$ to be equal to $\mathbb{I}$. The algebra defined by $a, b, c, d$ and the above set of relations will be called $\mathcal{F}$. It is clearly a quotient of $Fun(SL_q(2, \mathbb{C}))$, the algebra of polynomial functions on the quantum group $SL_q(2, \mathbb{C})$. Since $a^N = \mathbb{I}$, multiplying the relation $ad = \mathbb{I} + qbc$ from the left by $a^{N-1}$ leads to

$$
d = a^{N-1}(\mathbb{I} + qbc)
$$

(7)

so that $d$ can be eliminated. The algebra $\mathcal{F}$ can therefore be *linearly* generated—as a vector space—by the elements $a^\alpha b^\beta c^\gamma$ where indices $\alpha, \beta, \gamma$ run in the set $\{0, 1, \cdots, N - 1\}$. We see that $\mathcal{F}$ is a *finite dimensional* associative algebra, whose dimension is

$$
\dim(\mathcal{F}) = N^3.
$$

$\mathcal{F}$ is not only an associative algebra but a Hopf algebra (cf. previously given references). The coproduct of generators can be read directly from the above $2 \times 2$ matrix with entries $a, b, c, d$, for instance, $\Delta a = a \otimes a + b \otimes c$, etc. .

This quantum group, by construction, *coacts* on $\mathcal{M}$. 

2.2.2 The quantum group $H$ acts on $M$

Multiplication and comultiplication being interchanged by duality, it is clear that the dual $H$ of $F$ is also a quantum group (of the same dimensionality). For compatibility reasons with previous references, we choose the linear basis $K^\alpha X^\beta_+ X^\gamma_-$ in $H$, where $K, X_+ \text{ and } X_- \text{}$ are defined by duality as follows:

\[
\begin{align*}
\langle K, a \rangle &= q \langle K, b \rangle = 0 \\
\langle K, c \rangle &= 1 \\
\langle K, d \rangle &= q^{-1}
\end{align*}
\]

(8)

From multiplication and comultiplication in $F$, one gets:

**Multiplication:**

\[
\begin{align*}
KX_+ &= qX_+ K \\
[X_+, X_-] &= \frac{1}{(q-q^{-1})} (K - K^{-1}) \\
K^N &= 1 \\
X^N_+ &= X^N_- = 0.
\end{align*}
\]

(9)

**Comultiplication:** The comultiplication is an algebra morphism, i.e., $\Delta(XY) = \Delta X \Delta Y$.

It is given by

\[
\begin{align*}
\Delta X_+ &= X_+ \otimes \mathbb{1} + K \otimes X_+ \quad (10)
\end{align*}
\]

\[
\begin{align*}
\Delta X_- &= X_- \otimes K^{-1} + \mathbb{1} \otimes X_- \\
\Delta K &= K \otimes K \\
\Delta K^{-1} &= K^{-1} \otimes K^{-1}.
\end{align*}
\]

There is also an antipode and a counit but we shall not need them in the sequel.

The quantum group $H$ acts on itself (by left or right multiplication), it acts also on its dual $F$ from the left or from the right, as follows:

\[
\langle y, h_L[f] \rangle = \langle yh, f \rangle \quad \langle y, h_R[f] \rangle = \langle hy, f \rangle
\]

(11)

where $y, h \in H$, $f \in F$. These actions can also be expressed as:

\[
\begin{align*}
h_L[f] &= \langle \text{id} \otimes h, \Delta f \rangle_{(2)} \\
h_R[f] &= \langle \text{id} \otimes h, \Delta f \rangle_{(1)}
\end{align*}
\]

(12)

where the notation $\langle \cdot, \cdot \rangle_{(1)}$ means that we only pair the first term in the tensor product (resp. the second).

Finally, $H$ also acts on the reduced quantum plane $M$ (the algebra of $N \times N$ matrices) since its dual $F$ coacts on it. There are again two possibilities, left or right, but we shall use the left action.

The left action of $H$ on $M$ is generally defined as follows. If we denote the right coaction of $F$ on $M$ as:

\[
\delta_R[m] = m_{(1)} \otimes f_{(2)}
\]

(13)

then:

\[
h^L[m] = m_{(1)} \langle h, f_{(2)} \rangle \\
h \in H.
\]

(14)

The action of generators of $H$ on generators of $M$ is given by the following table.

| Left | $K$ | $X_+$ | $X_-$ |
|------|-----|-------|-------|
| $x$  | $qx$ | 0     | $y$   |
| $y$  | $q^{-1}y$ | $x$ | 0      |
Using the coproduct, one finds the expression of these generators on an arbitrary element of \( \mathcal{M} \)

\[
K^L[x^r y^s] = q^{r-s} x^r y^s
\]
\[
X^L_+[x^r y^s] = q^r (1 + q^{-2} + \cdots + q^{-2(s-1)}) x^{r+1} y^{s-1}
\]
\[
= q^r \left( \frac{1 - q^{-2s}}{1 - q^{-2}} \right) x^{r+1} y^{s-1}
\]
\[
X^L_-[x^r y^s] = q^s (1 + q^{-2} + \cdots + q^{-2(r-1)}) x^{r-1} y^{s+1}
\]
\[
= q^s \left( \frac{1 - q^{-2r}}{1 - q^{-2}} \right) x^{r-1} y^{s+1}
\]

(16)

with \( 1 < r, s < N \).

Remember that \( \mathcal{M} \) itself is not a quantum group, but a module (a representation space) for the quantum group \( \mathcal{H} \). However, taking \( h \in \mathcal{H}, m_1, m_2 \in \mathcal{M} \) and writing \( \Delta h = \sum h_1 \otimes h_2 \), one can check the following supplementary compatibility condition \( h[m_1m_2] = \sum h_1[m_1]h_2[m_2] \) between the module structure of \( \mathcal{M} \), the algebra structure of \( \mathcal{M} \) and the coproduct in \( \mathcal{H} \). This makes \( \mathcal{M} \) a module algebra over \( \mathcal{H} \).

In order to reduce this module into indecomposable modules, it is necessary to know at least part of the representation theory of \( \mathcal{H} \). This is recalled in the next subsection.

### 2.3 Representation theory of \( \mathcal{H} \)

As already stated in the introduction, using a result by [3], the explicit structure (for all values of \( N \)) of those algebras, including the expression of generators \( X_\pm, K \) themselves, in terms of matrices with coefficients in \( \mathbb{C} \) or in the Grassmann algebra \( Gr(2) \) with two generators \( \theta_1, \theta_2 \), was obtained by [7]. We shall not need the general theory but only the following fact: when \( N \) is odd, \( \mathcal{H} \) is isomorphic with the direct sum

\[
\mathcal{H} = M_N \oplus (M_{N-1}|1(\Lambda^2))_0 \oplus (M_{N-2}|2(\Lambda^2))_0 \oplus \cdots \oplus (M_{N+1}|N-1(\Lambda^2))_0
\]

(17)

where:

- \( M_N \) is a \( N \times N \) complex matrix
- An element of the \( (M_{N-1}|1(\Lambda^2))_0 \) block (space that we shall just call \( M_{N-1|1} \)) is of the following form :

\[
\begin{pmatrix}
\bullet & \bullet & \cdots & \bullet & \bullet \\
\bullet & \bullet & \cdots & \bullet & \bullet \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\bullet & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \cdots & \circ & \circ
\end{pmatrix}
\]

(18)

We have introduced the following notation:

\( \bullet \) is an even element of the ring \( Gr(2) \) of Grassmann numbers with two generators, i.e., of the kind :

\[
\bullet = \alpha + \beta \theta_1 \theta_2, \quad \alpha, \beta \in \mathbb{C}
\]

\( \circ \) is an odd element of the ring \( Gr(2) \) of Grassmann numbers with two generators, i.e., of the kind :

\[
\circ = \gamma \theta_1 + \delta \theta_2, \quad \gamma, \delta \in \mathbb{C}
\]

\(^2\)Remember that \( \theta_1^2 = \theta_2^2 = 0 \) and that \( \theta_1 \theta_2 = -\theta_2 \theta_1 \)
- An element of the $M_{N-2/2}$ block is of the kind:

\[
\begin{pmatrix}
\bullet & \cdots & \bullet & \circ & \circ \\
\bullet & \cdots & \bullet & \circ & \circ \\
\vdots & \vdots & \vdots & \vdots & \\
\bullet & \cdots & \bullet & \circ & \circ \\
\circ & \cdots & \circ & \bullet & \circ
\end{pmatrix}
\] (19)

- etc.

Notice that $H$ is not a semi-simple algebra: its Jacobson radical $J$ is obtained by selecting in equation [17] the matrices with elements proportional to Grassmann variables. The quotient $H/J$ is then semi-simple... but no longer Hopf!

Projective indecomposable modules (PIM's, also called principal modules) for $H$ are directly given by the columns of the previous matrices.

- From the $M_{N}$ block, one obtains $N$ equivalent irreducible representations of dimension $N$ that we shall denote $N_{irr}$.
- From the $M_{N-p}$ block (assume $p < N - p$), one obtains
  - $(N - p)$ equivalent indecomposable projective modules of dimension $2N$ that we shall denote $P_{N-p}$ with elements of the kind

\[
\begin{pmatrix}
\bullet & \cdots & \bullet & \circ & \circ \\
\circ & \cdots & \circ & \bullet & \circ
\end{pmatrix}
\] (20)

- $p$ equivalent indecomposable projective modules (also of dimension $2N$) that we shall denote $P_{p}$ with elements of the kind

\[
\begin{pmatrix}
\circ & \cdots & \circ & \bullet & \bullet \\
\bullet & \cdots & \bullet & \circ & \circ
\end{pmatrix}
\] (21)

Other submodules can be found by restricting the range of parameters appearing in the columns defining the PIM's and imposing stability under multiplication by elements of $H$. In this way, one can determine, for each PIM the lattice of its submodules. For a given PIM of dimension $2N$ (with the exception of $N_{irr}$), one finds totally ordered sublattices (displayed below) with exactly three non trivial terms: the radical (here, it is the biggest non trivial submodule of a given PIM), the socle (here it is the smallest non trivial submodule), and one "intermediate" submodule of dimension exactly equal to $N$. However the definition of this last submodule (up to equivalence) depends on the choice of an arbitrary complex parameter $\lambda$, so that we have a chain of inclusions for every such parameter. The collection of all these sublattices fully determines the lattice structure of submodules of a given principal module.

Since we have two types of principal modules (besides the irreducible ones, $N_{irr}$), we obtain explicitly the following two types of chains of inclusions:

- First type: submodules of $P_{N-p}$.

\[
0 \subset N - p \subset N_{N-p} \subset N + p \subset 2N = P_{N-p}
\]

where \(\subset\) represent inclusion.

An element of the submodule $N_{N-p}$ (which has dimension $N$) is of the kind:

\[
N_{N-p} = \left( \beta_1 \theta_1 \theta_2 \ \beta_2 \theta_1 \theta_2 \ \cdots \ \beta_{N-p} \theta_1 \theta_2 \ \cdots \ \gamma_1 \theta_\lambda \ \cdots \ \gamma_p \theta_\lambda \right)
\] (22)

\[\text{[3] Here we label the submodules by underlining their dimension.}\]
where: \[ \theta_\lambda = \lambda_1 \theta_1 + \lambda_2 \theta_2 \quad \lambda = \lambda_1/\lambda_2 \in \mathbb{C} P^1 \]

Notice that the submodule \( N_{N-p} \) itself is the direct sum of an invariant sub-module of dimension \((N-p)\), and a vector subspace of dimension \(p\). We shall denote this as follows:

\[ N_{N-p} = N - p \in p \]

with

\[ N - p = \begin{pmatrix} \beta_1 \theta_1 \theta_2 & \beta_2 \theta_1 \theta_2 & \cdots & \beta_{N-p} \theta_1 \theta_2 & 0 & \cdots & 0 \end{pmatrix} \quad (23) \]

- Second type: submodules of \( P_p \).

An element of the submodule \( N_p \) (which has dimension \(N\)) is of the kind:

\[ N_p = \begin{pmatrix} \gamma_1 \theta_1 \gamma_2 \theta_2 & \cdots & \gamma_{N-p} \theta \lambda_{N-p} & \beta_1 \theta_1 \theta_2 & \beta_2 \theta_1 \theta_2 & \cdots & \beta_p \theta_1 \theta_2 \end{pmatrix} \quad (24) \]

Notice that the submodule \( N_p \) itself is the direct sum of an invariant sub-module of dimension \(p\), and a vector subspace of dimension \(N-p\)

\[ N_p = p \in (N-p) \]

with

\[ p = \begin{pmatrix} 0 & 0 & \cdots & 0 & \beta_1 \theta_1 \theta_2 & \beta_2 \theta_1 \theta_2 & \cdots & \beta_p \theta_1 \theta_2 \end{pmatrix} \quad (25) \]

Notice that the quotient of a PIM by its own radical defines an irreducible representation for the quantum group \( \mathcal{H} \). The irreducible representations for \( \mathcal{H} \) are therefore of dimensions 1, 2, 3, \ldots, \(N\). Warning: the projective cover of a given indecomposable module appearing in one of the above sublattices is not necessarily equal to the principal module that appears as the maximum element of the same sublattice.

We have already noticed that each PIM contains indecomposable submodules \( N_p \) of dimension exactly equal to \(N\), each such submodule containing itself one invariant irreducible subspace of dimension \(p\) (the precise definition involves the choice of a parameter \(\lambda\)). As we shall see, these submodules of dimension \(N\) are exactly those that appear in the decomposition of the algebra of complex \(N \times N\) matrices into representations of \( \mathcal{H} \).

As an example, let us explicitly describe the case \(N = 5\). Then, \( \dim \mathcal{H} = 5^3 = 125 \).

We can write \(5 = 5 + 0 = 4 + 1 = 3 + 2\), so that \( \mathcal{H} = M_5 \oplus (M_{4\times1}(\mathbb{L}^2))_0 \oplus (M_{3\times2}(\mathbb{L}^2))_0 \) and we have five principal modules, one is irreducible (5\(_{irr}\), of dimension 5), the others (\(P_4, P_1, P_3, P_2\)) are projective indecomposable and have the same dimension 10. One can also write \( \mathcal{H} = 5 \mathbf{5}_{irr} + 4 P_4 + 1 P_1 + 3 P_3 + 2 P_2 \).

The lattices of submodules read:

\[
\begin{align*}
0 & \hookrightarrow 4 \hookrightarrow 5_1 \hookrightarrow 6 \hookrightarrow 10' = P_4 \\
0 & \hookrightarrow 1 \hookrightarrow 5_1 \hookrightarrow 9 \hookrightarrow 10'' = P_1 \\
0 & \hookrightarrow 3 \hookrightarrow 5_1 \hookrightarrow 7 \hookrightarrow 10''' = P_3 \\
0 & \hookrightarrow 2 \hookrightarrow 5_2 \hookrightarrow 8 \hookrightarrow 10'''' = P_2
\end{align*}
\]

Besides the five-dimensional irreducible representation which is itself a PIM, there are four others: \(4_{irr} = P_4/\mathbf{5}, 1_{irr} = P_1/\mathbf{5}, 3_{irr} = P_3/\mathbf{2}\) and \(2_{irr} = P_2/\mathbf{2}\).
3 Reduction of the algebra $\mathcal{M} = M_N(\mathbb{C})$, $N$ odd, into indecomposable representations of $\mathcal{H}$

We shall now focus our attention on the case of $N \times N$ complex matrices, in the case where $N$ is odd, and show how this familiar algebra can be reconstructed as a sum of representations for the finite dimensional quantum group $\mathcal{H}$.

A vectorial basis of this algebra is given by matrices

$$\left\{ x^r y^s \right\}, \quad r, s \in \{0, 1, \cdots, N - 1\}$$

where $x$ and $y$ are the particular generators already defined in section 2. The left action of generators of $\mathcal{H}$ on an arbitrary element of $\mathcal{M}$ is given by:

$$K[x^r y^s] = q^{(r-s)x^r y^s}$$

$$X_+[x^r y^s] = q^{r - \frac{2s}{1 - q^2}} x^{r+1} y^{s-1}$$

$$X_- [x^r y^s] = q^{r - \frac{2s}{1 - q^2}} x^{r-1} y^{s+1}$$

The generator $K$ always acts as an automorphism, for this reason, in order to study the invariant subspaces of $\mathcal{M}$ under the left action of $\mathcal{H}$, we shall only have to consider the action of $X_+$ et $X_-$.

Forgetting numerical factors, the action de $X_+$ and of $X_-$ on a given element of $\mathcal{M}$ can be written as follows:

$$x^{r+1} y^{s-1} \overset{X_+}{\longrightarrow} x^r y^s \overset{X_+}{\longrightarrow} x^{r-1} y^{s+1}$$

It is then easy to decompose the space of $N \times N$ matrices into invariant subspaces for this action.

• Starting from the element $x^{N-1}$, we have:

$$\begin{array}{c}
0 \\
X_+ \\
X_- \\
x^{N-1} \\
x^{N-2} y \cdots \cdots y^{N-2} \overset{X_+}{\longrightarrow} x^{N-3} y \cdots \cdots y^{N-3} \overset{X_+}{\longrightarrow} y^{N-2}
\end{array}$$

We obtain in this way an irreducible subspace of dimension $N$, denoted $N_{irr}$. A base of this subspace is given by:

$$\left\{ x^r y^s \right\} \quad \text{avec} \quad (r + s) = N - 1$$

• Starting from the element $x^{N-2}$, we obtain the following diagram:
We obtain again an invariant subspace of the same dimension $N$, denoted $N_{N-1}$. A base of this subspace is given by:

\[ \{x^r y^s\} \quad \text{avec} \quad (r + s) = N - 2 \quad \text{[modulo $N$]} \]

This space is the direct sum of an invariant subspace of dimension $N - 1$ (hence the notation), a basis of which being given by

\[ \{x^r y^s\} \quad \text{avec} \quad (r + s) = N - 2, \]

and a non-invariant vector subspace of dimension 1 generated by $\{x^{N-1}y^{N-1}\}$.

- The process can be repeated, up to:

\[ \begin{array}{c}
\vdots \\
0 \\
\end{array} \]

We obtain in this last case a subspace of still the same dimension $N$, denoted $N_1$ with a base given by:

\[ \{x^r y^s\} \quad \text{avec} \quad (r + s) = 0 \quad \text{[modulo $N$]} \]

It is the direct sum of an invariant vector subspace of dimension 1, generated by $I$, and a non-invariant vector subspace of dimension $N - 1$.

To conclude, we see that, under the left action of $\mathcal{H}$, the algebra of $N \times N$ matrices can be decomposed into a direct sum of invariant subspaces of dimension $N$, according to

- $N_N = N_{\text{irr}}$ : irreducible
- $N_{N-1}$ : reducible indecomposable, with an invariant subspace of dimension $N - 1$.
- $N_{N-2}$ : reducible indecomposable, with an invariant subspace of dimension $N - 2$.
- $\vdots$
- $N_1$ : reducible indecomposable, with an invariant subspace of dimension 1.

These representations of dimension $N$ coincide exactly with those also called $N_p$ (or $N_{N-p}$) in the previous section. Using these notations, the algebra of matrices $N \times N$ can be written

\[ \mathcal{M} = N_N \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_{N-1} \quad (26) \]

with:

\[
\begin{align*}
N_N &= N_{\text{irr}} \\
N_{N-1} &= (N - 1) \in 1 \\
N_{N-2} &= (N - 2) \in 2 \\
&\vdots \\
N_1 &= 1 \in (N - 1)
\end{align*}
\]

Continuing the example given at the end of section 2 we see that

\[ M(5, \mathbb{C}) = \mathbf{5} \oplus \mathbf{2}_1 \oplus \mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{5}_1 \]
4 Action of $\mathcal{H}$ on the reduced Wess-Zumino complex $\Omega_{WZ}(\mathcal{M})$

4.1 The reduced Wess-Zumino complex $\Omega_{WZ}(\mathcal{M})$

The Wess-Zumino complex was constructed in [5] as the unique (up to a redefinition of $q \rightarrow q^{-1}$) quadratic differential algebra on the quantum plane, bicovariant with respect to the action of the quantum group $SL_q(2, \mathbb{C})$. The commutation relations between the $N \times N$ matrices $x, y$ and their differentials $dx, dy$, and between the differentials themselves are given by:

\[
\begin{align*}
xy &= qyx \\
x \, dx &= q^2 dx \, x \\
y \, dx &= q \, dx \, y \\
x \, dy &= q^2 \, dy \, x \\
x^2 &= 0 \\
y \, dx + q^2 dy \, dx &= 0
\end{align*}
\]

The reduced quantum plane itself is not a quadratic algebra, since it contains relations like $x^N = y^N = 1$, but Leibniz rule, together with the fact that $N$ is a root of unity imply that such compatibility relations like $dx^N = dy^N = 0$ are automatically satisfied [8][9], indeed:

\[
\begin{align*}
d(x^N) &= d(x^{N-1})x + x^{N-1}dx = \ldots = (1 + q + q^2 + \ldots + q^{N-1})x^{N-1}dx = 0 \\
d(y^N) &= d(y^{N-1})y + y^{N-1}dy = \ldots = (1 + q + q^2 + \ldots + q^{N-1})y^{N-1}dy = 0
\end{align*}
\]

We call “reduced Wess-Zumino complex” $\Omega_{WZ}(\mathcal{M})$ the quotient of the differential algebra of Wess-Zumino by the corresponding differential ideal. Note that $\dim(\Omega_{WZ}^0(\mathcal{M})) = N^2$, $\dim(\Omega_{WZ}^1(\mathcal{M})) = 2N^2$ and $\dim(\Omega_{WZ}^2(\mathcal{M})) = N^2$. Therefore, $\dim(\Omega_{WZ}(\mathcal{M})) = 4N^2$.

4.2 Remarks

Here is a list of questions that can be asked about $\Omega_{WZ}(\mathcal{M})$.

- Study the action of $\mathcal{H}$.
- Decompose this differential algebra into representations of $\mathcal{H}$
- Study the cohomology of $d$
- Extend the star operation(s) of $\mathcal{M}$ to star (or graded star) operations of the differential algebra
- Study algebraic connections in $\Omega_{WZ}(\mathcal{M})$ and their curvature
- etc.

These questions are studied in [8][9], mostly with the particular choice $N = 3$. Here we shall only give explicitly the action of $\mathcal{H}$ on this differential algebra, for an arbitrary $N$.

4.3 Action of $\mathcal{H}$

We already gave in [4] the left action of generators of $\mathcal{H}$ on an arbitrary element of $\mathcal{M}$. The left action of the same elements on generators $dx, dy$ of the Manin dual $\mathcal{M}'$ of $\mathcal{M}$ (although the later is not quadratic, we can use this terminology !) is given by:

\[
\begin{align*}
K^L[dx] &= qdx \\
X^L_+[dx] &= 0 \\
X^L_-[dx] &= dy \\
K^L[dy] &= q^{-1}dy \\
X^L_+[dy] &= dx \\
X^L_-[dy] &= 0
\end{align*}
\]
Using these two pieces of information, we now find the action of generators of \( \mathcal{H} \) on arbitrary elements of \( \Omega_{WZ}(\mathcal{M}) \) thanks to the coproduct:

Let us write: \( \Delta(h) = h^{(1)} \otimes h^{(2)} \) whenever \( h \in \mathcal{H} \), then, for \( m_1 \in \mathcal{M} \) and \( m_2 \in \mathcal{M}' \), we have:

\[
h^{L}[m_1 m_2] = h^{L}_{(1)}[m_1] \otimes h^{L}_{(2)}[m_2]
\]

The left action of generators of \( \mathcal{H} \) on arbitrary elements of \( \Omega_{WZ}(\mathcal{M}) \) is then given by:

\[
\begin{align*}
K[x^r y^s dx] &= q^{r+1-s} x^r y^s dx \\
K[x^r y^s dy] &= q^{r-s-1} x^r y^s dy \\
X_+[x^r y^s dx] &= q^s \frac{1 q^{-2s}}{1-q^{-2}} x^{r+1} y^{s-1} dx \\
X_+[x^r y^s dy] &= q^s \frac{1 q^{-2s}}{1-q^{-2}} x^{r+1} y^{s-1} dy + q^{r-s} x^r y^s dx \\
X_-[x^r y^s dx] &= q^{s-1} \frac{1 q^{-2r}}{1-q^{-2}} x^{r-1} y^{s+1} dx + x^r y^s dy \\
X_-[x^r y^s dy] &= q^{s+1} \frac{1 q^{-2r}}{1-q^{-2}} x^{r-1} y^{s+1} dx
\end{align*}
\]

5 Comments

The usual algebra \( \mathcal{M} \) of \( N \times N \) complex matrices (or the group of its unitary elements) is very often used in various fields of physics. The observation that it is a module algebra for a finite dimensional non commutative, non cocommutative (and non semi-simple) quantum group, and that it can be correspondingly decomposed into the sum of \( N \) indecomposable representations (of dimension \( N \)) came up — for us — as a surprise. One could then be tempted to speak of “hidden symmetry”, whenever \( \mathcal{M} \) plays a role in the description of some physical process, but we do not know yet the physical meaning of such a “symmetry”, and its interpretation should certainly depend upon the particular situation at hand. The above properties nevertheless suggest that, in several branches of physics, it may be worth to study the appearance and meaning of “symmetries” associated with this quantum group \( \mathcal{H} \) (which is a kind of “fat” version of the algebra of the group \( \mathbb{Z}_N \)).

Notice that representations of \( \mathcal{H} \) are also, \textit{a priori}, particular representations of \( U_q sl(2, \mathbb{C}) \) when \( q \) is a root of unity (representations for which \( K^N = 1 \), and \( X^N = 0 \)). Such representations appear in several examples of conformal field theories, and also in the study of abelian anyons (particles obeying a one-dimensional statistics associated with the braid group, or with a Hecke algebra). It may be simpler and conceptually more appropriate to discuss such problems in terms of \( \mathcal{H} \) than with the infinite dimensional Hopf algebra \( U_q sl(2, \mathbb{C}) \).

One of the purposes of the letter [8] was to construct generalized differential forms as elements of the tensor product \( \Xi \) of the space of usual (De Rham) forms on a (usual) space-time and of the differential algebra defined by the reduced Wess-Zumino complex. One obtains naturally on \( \Xi \) an action of a Hopf algebra which is the product of \( \mathcal{H} \), times the envelopping algebra of the Lie algebra of the Lorentz group. \( \mathcal{H} \) appears therefore as a discrete (but neither commutative nor cocommutative) analogue of the Lorentz group.

Finally, the group of unitary elements of the quotient of \( \mathcal{H} \) by its radical, when \( N \) is odd, is isomorphic with

\[
U(N) \times (U(N - 1) \times U(1)) \times (U(N - 2) \times U(2)) \times (U(N - 3) \times U(3)) \times \ldots
\]

For \( N = 3 \), one obtains \( U(3) \times U(2) \times U(1) \). After appropriate identification of several \( U(1) \) factors, one recognizes the gauge group describing of the standard model of elementary
particles. This last observation was made in [10] where it was suggested that \( \mathcal{H} \) could play the role of a Lorentz group for the “internal space” and that it would be tempting to devise a generalized gauge theory using this finite quantum group as a basic ingredient. This last comment can also be related to the construction described in [8].

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