ALEXANDROV CURVATURE OF CONVEX HYPERSURFACES
IN HILBERT SPACE

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Abstract. It is shown that convex hypersurfaces in Hilbert spaces have non-
negative Alexandrov curvature. This extends an earlier result of Buyalo for
convex hypersurfaces in Riemannian manifolds of finite dimension.

1. Introduction

In this paper, the following result is established:

Theorem 1. If $C$ is an open set in a Hilbert space $H$ and $\bar{C}$ is locally convex, then
$\partial C$ is a nonnegatively curved Alexandrov space under the induced length metric.

Questions of this sort go back to [2], where Alexandrov defined Alexandrov
curvature and showed that it characterizes boundaries of locally convex bodies
in $\mathbb{R}^3$. This was generalized by Buyalo to the case of locally convex sets of full
dimension in a Riemannian manifold in [3]. If the ambient manifold has a positive
lower bound $\kappa$ on sectional curvature, it has also been shown in [1] that the convex
boundary has Alexandrov curvature $\geq \kappa$.

The proof of Theorem 1 relies on approximating $\partial C$ by smooth manifolds, where
the connection between curvature and convexity is well understood. Due to the pos-
sibly infinite dimension of $H$, we cannot smooth by integrating over $H$ against a
mollifier. As currently known smoothing operators for infinite dimensional spaces
do not preserve convexity, we proceed by integrating over a suitably chosen finite
dimensional subspace. Lemma 7 shows this can be done in such a way that the
curvature of $\partial C$ is controlled by the curvature of smooth, finite-dimensional ap-
proximating manifolds. A similar approximation of infinite-dimensional curvature
by finite dimensional curvature is outlined in [4].

2. Basic definitions

We begin by defining curvature in the sense of Alexandrov. There are several
equivalent definitions, and we will find it most convenient to work with comparison
angles.

Definition 2. For three points $x, y, z$ in a metric space $(X, d)$, the comparison
angle $\tilde{\angle}xyz$ is defined as

$$\tilde{\angle}xyz = \arccos \frac{d^2(x, y) - d^2(x, z) + d^2(y, z)}{2d(x, y)d(y, z)}.$$ 

Recall that $(X, d)$ is called a length space if the distance between any two points
equals the infimum of the lengths of paths between them.
Furthermore, for each rectifiable curve $\sigma$ in $X$, there exists a polygonal path approximating smooth graphs. This enables us to control the Alexandrov curvature in the form

$$\hat{\zeta}bac + \hat{\zeta}cap + \hat{\zeta}pab \leq 2\pi$$

for any quadruple $(a; b, c, p)$ of distinct points in $U_x$. In this case, $X$ is called a nonnegatively curved Alexandrov space.

If $X$ is a Riemannian manifold, then nonnegative Alexandrov curvature is equivalent to nonnegative sectional curvature.

It will also be helpful to fix notation for polygonal paths.

**Definition 4.** For two points $p, q$ in a vector space $V$, $\sigma_{pq} : [0, 1] \to V$ denotes the constant speed linear path:

$$\sigma_{pq}(t) = (1 - t)p + tq.$$

**Definition 5.** A path $\tau : [0, 1] \to V$ is called a polygonal path if it can be written in the form

$$\tau(t) = \sum_{i=1}^{k-1} \sigma_{p_ip_{i+1}}(kt - i)1_{[i/k, (i+1)/k]}(t)$$

for some set of points $p_1, \ldots, p_k \in V$. Here $1_A$ denotes the characteristic function of the set $A$.

### 3. Approximation by smooth manifolds

In this section, we prove two technical lemmas which allow us to approximate $C^{1,1}$ convex functions $f$ on a Hilbert space by convex functions that are smooth on a finite-dimensional linear subspace. This enables us to control the Alexandrov curvature of graph $f$, the graph of $f$ in $H \times \mathbb{R}$, via the sectional curvature of the approximating smooth graphs.

**Lemma 6.** Let $f : V \to (X, d)$ be a $\lambda$-bi-Lipschitz map from a Banach space $V$ onto a metric space $(X, d)$. For any rectifiable curve $\sigma : [0, 1] \to X$ and any $\varepsilon > 0$, there exists a polygonal path $\tau : [0, 1] \to V$ such that $f \circ \tau(0) = \sigma(0)$, $f \circ \tau(1) = \sigma(1)$, and $\forall t \in [0, 1], |\sigma(t) - f \circ \tau(t)| < \varepsilon$ and $|l(\sigma) - l(f \circ \tau)| < \varepsilon$.

**Proof.** For each rectifiable curve $\sigma_0 : [0, 1] \to X$ and $\varepsilon > 0$, define

$$B_1^2(\sigma_0) = \{ \sigma : [0, 1] \to X; \forall t \in [0, 1], d(\sigma_0(t), \sigma(t)) < \varepsilon, |l(\sigma_0) - l(\sigma)| < \varepsilon \}.$$

For each rectifiable curve $\sigma_0 : [0, 1] \to V$ and $\varepsilon > 0$, define

$$B_2^2(\sigma_0) = \{ \sigma : [0, 1] \to V; \forall t \in [0, 1], |\sigma_0(t) - \sigma(t)| < \varepsilon, |l(\sigma_0) - l(\sigma)| < \varepsilon \}.$$

Fix a rectifiable curve $\sigma_0 : [0, 1] \to V$ and $\varepsilon > 0$. For any $\sigma \in B_2^2(\sigma_0)$, for all $t \in [0, 1]$,

$$|\sigma_0(t) - \sigma(t)| < \varepsilon \implies |f \circ \sigma_0(t) - f \circ \sigma(t)| < \lambda \varepsilon.$$

Furthermore,

$$|l(f \circ \sigma_0) - l(f \circ \sigma)| \leq |l(\sigma_0) - l(\sigma)| + |l(f \circ \sigma_0) - l(\sigma)| + |l(f \circ \sigma) - l(\sigma)|$$

$$\leq \varepsilon + |l(f \circ \sigma_0) + l(\sigma)| + |l(f \circ \sigma) + l(\sigma)|$$

$$\leq \varepsilon + \lambda |l(\sigma_0) + l(\sigma)| + \lambda l(\sigma) + l(\sigma)$$

$$\leq \varepsilon + (\lambda + 1)l(\sigma_0) + (\lambda + 1)(l(\sigma_0) + \varepsilon)$$

$$\leq 2(\lambda + 1)(\varepsilon + l(\sigma_0)).$$
So for \( \varepsilon' = 2(\lambda + 1)(\varepsilon + l(\sigma_0)) \),

\[
B^2_\varepsilon(\sigma_0) \subset f^{-1}(B^1_{\varepsilon'}(f \circ \sigma_0))
\]

By a similar argument, for any rectifiable curve \( \sigma_0 : [0, 1] \to X \) and \( \varepsilon > 0 \),

\[
f^{-1}(B^1_{\varepsilon'}(\sigma_0)) \subset B^2_\varepsilon(f^{-1} \circ \sigma_0),
\]

for \( \varepsilon' = 2(\lambda + 1)(\varepsilon + l(\sigma_0)) \). Thus the \( B^2 \)'s and \( f^{-1}(B^1) \)'s determine equivalent topologies on the space of rectifiable curves \( \sigma : [0, 1] \to V \). Polygonal paths are dense under the \( B^2 \)-topology, so they are dense under the \( f^{-1}(B^1) \)-topology. \( \square \)

**Lemma 7.** Let \( f : \Omega \to \mathbb{R} \) be a \( C^{1,1} \) convex function, where \( \Omega \) is a domain in a Hilbert space \( H \). For any \( x_0 \in \Omega \), there exists \( R > 0 \) such that \( Y \), the graph of \( f \) over \( B_R(x_0) \), satisfies the quadruple condition

\[
Z_{bac} + \tilde{Z}_{cap} + \tilde{Z}_{pab} \leq 2\pi
\]

for any quadruple \( (a; b, c, p) \) of distinct points, under the induced length metric \( d \) from \( X \times \mathbb{R} \).

**Proof.** \( f \) is convex, hence Lipschitz continuous for some Lipschitz constant \( L \geq 1 \). Let \( \tilde{f} : \Omega \to f(\Omega) \subset \text{graph}_f \) be defined by \( \tilde{f}(x) = (x, f(x)) \), and note that \( \tilde{f} \) is \( \sqrt{1 + L^2} \)-bi-Lipschitz. Choose \( R > 0 \) such that \( B_{3R}(x_0) \subset \Omega \). Suppose that \( (a; b, c, p) \) is a quadruple of distinct points such that

\[
\tilde{Z}_{bac} + \tilde{Z}_{cap} + \tilde{Z}_{pab} = 2\pi + \varepsilon_0 > 2\pi,
\]

where \( (a; b, c, p) = (\tilde{f}(a'); \tilde{f}(b'), \tilde{f}(c'), \tilde{f}(p')) \) and \( a', b', c', p' \in B_R(x_0) \). The comparison angles vary continuously in the intrinsic distances, so there exists \( \varepsilon > 0 \) such that if \( (A; B, C, D) \) is a quadruple of points in some other metric space \( (X_1, d_1) \) with

\[
|d(a, b) - d_1(A, B)| < \varepsilon, \quad |d(a, c) - d_1(A, C)| < \varepsilon, \quad |d(a, p) - d_1(A, P)| < \varepsilon,
\]

\[
|d(b, c) - d_1(B, C)| < \varepsilon, \quad |d(b, p) - d_1(B, P)| < \varepsilon, \quad |d(c, p) - d_1(C, P)| < \varepsilon,
\]

then

\[
\tilde{Z}_{BAC} + \tilde{Z}_{CAP} + \tilde{Z}_{PAB} = 2\pi + (\varepsilon_0/2) > 2\pi.
\]

By Lemma 6 we may approximate \( d(a, b) \) by the length of the image under \( \tilde{f} \) of a polygonal path \( \tau_1 \) determined by points \( a' = q_1, q_2, \ldots, q_{k_1-1}, b' = q_{k_1} \in B_{2R}(x_0) \) such that

\[
d(a, b) + (\varepsilon/3) \geq \sum_{i=1}^{k_1-1} l(\tilde{f} \circ \sigma_{q_i, q_{i+1}}) = l(\tilde{f} \circ \tau_1) \geq d(a, b).
\]

Similarly, we may approximate \( d(a, c) \) by the image under \( \tilde{f} \) of a polygonal path determined by points \( a' = q_{k_1+1}, q_{k_1+2}, \ldots, c' = q_{k_2} \in B_{2R}(x_0) \) such that

\[
d(a, c) + (\varepsilon/3) \geq \sum_{i=k_1+1}^{k_2-1} l(\tilde{f} \circ \sigma_{q_i, q_{i+1}}) \geq d(a, c).
\]

Continue in this manner choosing \( q_{k_2+1}, q_{k_2+2}, \ldots, q_m, \ldots, q_{k_0} \) to approximate the remaining four intrinsic distances.

The \( k_0 + 1 \) points \( q_1, \ldots, q_{k_0}, x_0 \) lie in a \( k_0 \)-dimensional subspace of \( H \), which we will identify as \( \mathbb{R}^n, \ n = k_0 \). Let \( \varphi_\delta : \mathbb{R}^n \to \mathbb{R} \) be the standard \( C^\infty \) mollifier supported on the \( \delta \)-ball, and define \( f_\delta : B_{5R/2}(x_0) \to \mathbb{R} \) by \( f_\delta = f * \varphi_\delta \), where the
convolution occurs in the \( \mathbb{R}^n \)-variables and \( \delta < R/2 \). Let \( \hat{f}_\delta(x) = (x, f_\delta(x)) \). As \( f \) is assumed to be convex and \( C^{1,1} \), it is easy to check the following properties:

1. \( f_\delta|_{B_{2R}(x_0) \cap \mathbb{R}^n} \) is \( C^\infty \).
2. \( f_\delta|_{B_{2L}(x_0) \cap \mathbb{R}^n} \) is \( L \)-Lipschitz.
3. \( f_\delta \to f \) pointwise as \( \delta \to 0 \).
4. On \( \mathbb{R}^n \cap \overline{B}_{2L}(x_0) \), \( \nabla_{\mathbb{R}^n} f_\delta \to \nabla f \) uniformly as \( \delta \to 0 \).
5. For every rectifiable curve \( \sigma : [0,1] \to B_{2R}(x_0) \), \( l(\hat{f}_\delta \circ \sigma) \to l(f \circ \sigma) \). This convergence is uniform on sets \( \{ \sigma : [0,1] \to B_{2R}(x_0) ; l(\sigma) < C \} \) with \( C \in \mathbb{R} \).
6. \( f_\delta \) is convex.

Let \( Y_\delta \) denote the graph of \( f_\delta \) over \( B_{2R}(x_0) \) with metric \( d_\delta \) induced by \( H \times \mathbb{R} \), and let \( Y_{\delta,n} \) denote the graph of \( f_\delta \) over \( B_{2R}(x_0) \cap \mathbb{R}^n \) with metric \( d_{\delta,n} \) induced by \( \mathbb{R}^n \times \mathbb{R} \). Note that \( f_\delta|_{B_{2R}(x_0) \cap \mathbb{R}^n} \) is a \( C^\infty \) convex function over a domain in \( \mathbb{R}^n \), so \( Y_{\delta,n} \) is a Riemannian manifold of nonnegative sectional curvature. In particular, it satisfies the quadruple condition. We will obtain a contradiction by showing

\[
|d(a,b) - d_{\delta,n}(\hat{f}_\delta(a'), \hat{f}_\delta(b'))| < \varepsilon, \quad |d(a,c) - d_{\delta,n}(\hat{f}_\delta(a'), \hat{f}_\delta(c'))| < \varepsilon,
\]

\[
|d(a,p) - d_{\delta,n}(\hat{f}_\delta(a'), \hat{f}_\delta(p'))| < \varepsilon, \quad |d(b,c) - d_{\delta,n}(\hat{f}_\delta(b'), \hat{f}_\delta(c'))| < \varepsilon,
\]

\[
|d(b,p) - d_{\delta,n}(\hat{f}_\delta(b'), \hat{f}_\delta(p'))| < \varepsilon, \quad |d(c,p) - d_{\delta,n}(\hat{f}_\delta(c'), \hat{f}_\delta(p'))| < \varepsilon.
\]

Let \( C = d(a,b) + d(a,c) + \cdots + d(c,p) + \varepsilon \). Choosing \( \delta_0 \) small with respect to \( C \), we have for all \( \delta < \delta_0 \),

\[
\tau \in \{ \sigma : [0,1] \to B_{2R}(x_0) ; l(\sigma) < C \} \implies l(\hat{f}_\delta \circ \tau) - l(f \circ \tau) \leq \varepsilon/3.
\]

Recall that \( \tau_1 \) is the polygonal path determined by \( q_1, \ldots, q_{k_1} \),

\[
l(\tau_1) = l(f \circ \tau_1) = \sum_{i=1}^{k_1-1} l(f \circ \sigma_{q_i,q_{i+1}}) \leq d(a,b) + (\varepsilon/3) < C,
\]

so \( l(\hat{f}_\delta \circ \tau_1) \leq l(f \circ \tau_1) + (\varepsilon/3) \) for \( \delta < \delta_0 \). \( \hat{f}_\delta \circ \tau_1 : [0,1] \to Y_{\delta,n} \) is a path from \( \hat{f}_\delta(a') \) to \( \hat{f}_\delta(b') \), so

\[
d_{\delta,n}(\hat{f}_\delta(a'), \hat{f}_\delta(b')) \leq l(\hat{f}_\delta \circ \tau_1) \leq l(f \circ \tau_1) + (\varepsilon/3) \leq d(a,b) + (2\varepsilon/3).
\]

Applying Lemma \[1\] again, choose \( \tau_2 : [0,1] \to B_{2R}(x_0) \cap \mathbb{R}^n \) such that

\[
d_{\delta,n}(\hat{f}_\delta(a'), \hat{f}_\delta(b')) \geq l(\hat{f}_\delta \circ \tau_2) - (\varepsilon/6).
\]

Note that

\[
l(\tau_2) \leq l(f \circ \tau_2) \leq d_{\delta,n}(\hat{f}_\delta(a'), \hat{f}_\delta(b')) + (\varepsilon/6) \leq d(a,b) + (5\varepsilon/6) < C.
\]

For \( \delta < \delta_0 \),

\[
l(\hat{f}_\delta \circ \tau_2) \geq l(\hat{f} \circ \tau_2) - (\varepsilon/3),
\]

so

\[
d_{\delta,n}(\hat{f}_\delta(a'), \hat{f}_\delta(b')) > l(\hat{f} \circ \tau_2) - \varepsilon \geq d(a,b) - \varepsilon.
\]

The remaining inequalities follow in a similar manner, for the same choice of \( C \) and \( \delta_0 \). So for \( \delta < \delta_0 \), the quadruple \( (\hat{f}_\delta(a'), \hat{f}_\delta(b'), \hat{f}_\delta(c'), \hat{f}_\delta(p')) \) violates the quadruple condition in the Riemannian manifold of nonnegative sectional curvature \( Y_{\delta,n} \). Therefore our original assumption is false and \( Y \) satisfies the quadruple condition. \[\square\]
4. Proof of Theorem

Proof of Theorem 1. We must prove the quadruple condition holds in a neighborhood of every $x_0 \in \partial C$. Let $C' = B_{2\rho}(x_0) \cap C$, where $\rho$ is chosen small enough to make $C'$ convex. Note that the intrinsic balls of radius $\rho$ about $x_0$ are the same for $C$ and $C'$. Choose a point $y \in C'$, and $r \in (0, \rho/2)$ such that $B_{2r}(y) \subset C'$. Let $H'$ be the hyperplane through $x_0$ with normal vector $y - x_0$. For any $x \in H' \cap B_{2r}(x_0)$, let $L_x$ be the line through $x$ spanned by $y - x_0$. $L_x \cap C'$ is convex and $C'$ is open and bounded, so $L_x \cap C'$ is a bounded interval. $x + (y - x_0) \in L_x \cap C'$, so $L_x \cap C' \neq \emptyset$.

Considering $y - x_0$ as the upward direction, let $f(x)$ denote the $\mathbb{R}$-coordinate of the bottom endpoint of $L_x \cap C'$ in $H' \times \mathbb{R}$. $f : H' \cap B_{2r}(x_0) \to \mathbb{R}$ is then a convex function, as the epigraph is convex. Furthermore, the graph of $f$ is a neighborhood of $x_0$ in $\partial C'$, and thus also in $\partial C$ since $2r < \rho$.

$f$ is convex, hence Lipschitz continuous for some Lipschitz constant $L \geq 1$. As shown in [5], for all small enough $\varepsilon > 0$, the inf-sup-convolution

$$g_{\varepsilon}(x) = \inf_{z \in H' \cap B_{2r}(x_0)} \sup_{y \in H' \cap B_{2r}(x_0)} \left[ f(y) - \frac{\|y - z\|^2_H}{2\varepsilon} + \frac{\|x - z\|^2_H}{\varepsilon} \right]$$

is a $C^{1,1}$ convex function on $H' \cap B_r(x_0)$. $g_\varepsilon$ is $L$-Lipschitz, and $g_\varepsilon \to f$ uniformly on $H' \cap B_r(x_0)$.

By Lemma 7, the graph of $g_\varepsilon$ over $H' \cap B_R(x_0)$ satisfies the quadruple condition for $R = r/3$. The graph of $f$ over $H' \cap B_R(x_0)$ then satisfies the quadruple condition by continuity. □

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