Computing Hecke Operators for Arithmetic Subgroups of General Linear Groups

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Abstract

We present an algorithm to compute the Hecke operators on the equivariant cohomology of an arithmetic subgroup Γ of the general linear group GL_n. This includes GL_n over a number field or a finite-dimensional division algebra. As coefficients, we may use any finite-dimensional local coefficient system. Unlike earlier methods, the algorithm works for the cohomology H^i in all degrees i. It starts from the well-rounded retract ˜W, a Γ-invariant cell complex which computes the cohomology [1]. It extends ˜W to a new well-tempered complex ˜W^+ of one higher real dimension, using a real parameter called the temperament. The algorithm has been coded up for SL_n(Z) for n = 2, 3, 4, and it performs successfully in tests for various Γ.

1 Introduction

1.1 The 2 × 2 Case

We begin by describing our Hecke operator algorithm in the simplest case. Let E be the space of real symmetric matrices Z = [a b]

b c . The group G = GL_2(R) acts on E by Z → gZg^t. The action preserves the cone of positive definite matrices X ⊆ E. We mod out in X by the positive scalar multiples of the identity, the homotheties. Then the action (when det g > 0) is equivalent

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to the well-known action of $\text{SL}_2(\mathbb{R})$ on the upper half-plane $\mathbb{H} \subset \mathbb{C}$ by linear fractional transformations. $X$ is called the Klein model of $\mathbb{H}$. The action on $\mathbb{H}$ has the advantage of being complex-analytic, but the Klein model has a different advantage: $X$ has a linear structure inherited from $\mathbb{H}$.

A matrix $Z \in X$ defines a positive definite quadratic form on $\mathbb{R}^2$, which we use to measure the lengths of vectors in the standard lattice $L_0 = \mathbb{Z}^2$. Divide $Z$ by a homothety so the shortest non-zero vector in $L_0$ has length 1. We say $Z$ is well rounded if two linearly independent vectors in $Z^2$ have length 1. Figure 1 shows the $Z = 1$ level curves of five forms, which are well rounded because $(1, 0), (0, 1)$ have $Z = 1$. The figure suggests there is a continuous path, a pencil of well-rounded forms, connecting the first and last pictures.

To draw pictures in $X$, we divide $Z$ by a different homothety to make its trace 1. The image of $X$ is then an open circular disc in the $ab$-plane. The well-rounded forms form a tree $\tilde{W}$ in $X$, shown on the left in Figure 2. The vertical edge in the center of the tree is the pencil in Figure 1.

We can also understand $Z$ in terms of lattices. $X = G/K$, where $K$ is the subgroup of orthogonal matrices. $Z = gg^t$ for some $g \in G$. The rows of $g$ are a basis of $\mathbb{R}^2$, and the $\mathbb{Z}$-linear combinations of this basis are a lattice $L = L_0g$. Let $\Gamma_0 = \text{GL}_2(\mathbb{Z})$, the integer matrices of determinant $\pm 1$. Then $Y = \Gamma_0 \backslash G$ is the space of lattices, and $Y/K = \Gamma_0 \backslash X$. The tree $\tilde{W}$ is $\Gamma_0$-equivariant and is a strong deformation retract of $X$. It is called the well-rounded retract.

Let $m$ be a positive integer. The Hecke correspondence $T_m$ is the one-to-many map that carries a lattice $L$ to its sublattices of index $m$. Figure 3 shows the three sublattices for $\ell = 2$.

Let $\Gamma \subseteq \Gamma_0$ be a subgroup of finite index. The correspondence $T_m$ acts on the cohomology $H^1(\Gamma \backslash X)$, and this action, also denoted $T_m$, is a Hecke operator. By work of Eichler and Shimura $[23]$, the cohomology can be un-
Figure 2: The well-rounded retract for $\text{GL}_2(\mathbb{Z})$, and its translate by $T_2$.

Figure 3: A lattice and its three sublattices of index two.
understood in terms of modular forms (Eisenstein series and cusp forms) on $\Gamma \backslash \mathfrak{H}$. The Hecke operators act on modular forms, and their eigenvalues give important arithmetic information.

We would like to compute the Hecke operators by how they act on the tree $\tilde{W}$. Unfortunately, the Hecke correspondences do not carry the tree to itself. $T_2$ moves it within the disc, as shown on the right in Figure 2. (The $T_m$ are one-to-many on $\Gamma_0 \backslash X$ but are defined by a one-to-one map on $X$; see Section 4.1 for the definition.) The deformation retraction $X \to \tilde{W}$ will carry the right-hand tree back to the left-hand tree, but when $m$ is large this is hard to use for computation. One edge may retract back to a path through many edges, starting and ending with fractions of edges.

This paper introduces the well-tempered complex $\tilde{W}^+$ for $T_m$. For simplicity, let $m$ be a prime $\ell$ for the rest of the Introduction. The well-tempered complex is a regular cell complex of dimension $\dim \tilde{W} + 1$. It is a fibration $\tilde{W}^+ \to [1, \ell^2]$, where the coordinate $\tau$ in the base is called the temperament. The fiber over $\tau = 1$ is on the left side of Figure 2, with the fiber over $\ell^2$ on the right. The fibers are all trees, and they continuously deform as $\tau$ varies. We show there are a finite number of critical temperaments where the cell structure abruptly changes. Figure 4 shows an example for $T_2$, around $\tau = 2$ in the base $[1, 4]$. All the fibers are 3-valent trees—every vertex is on three edges—except in the middle picture, where for a moment some vertices are on four edges.

A deformation retract of $\Gamma \backslash X$ is $\Gamma \backslash \tilde{W}_1$, the finite graph which is the left side of Figure 2 mod $\Gamma$. We compute $H^1(\Gamma \backslash \tilde{W}_\tau)$ as $\tau$ varies from 1 to $\ell^2$. At the critical temperaments, the cohomology does not change, but...
the cell structure does, and we keep track of it. When $\tau$ reaches the last temperament $\ell^2$, we have the information we need to compute $T_\ell$ on the cohomology. The algorithm is explained in more detail in Section 6.

1.2 The General Case

For $\Gamma \subseteq \text{GL}_2(\mathbb{Z})$, there is already a well-known method for computing Hecke operators, modular symbols \cite{19} \cite{26}. It works on the top degree of cohomology, $H^1$. The contribution of our paper is to give a Hecke operator algorithm on the cohomology $H^i(\Gamma \backslash X; \rho)$ for all $i = 0, 1, 2, \ldots$. This holds for $\text{GL}_n$ for all $n \geq 2$. Our algorithm works for all local coefficient systems $\rho$.

The general setting of this paper is the algebraic group $G$ where $G(\mathbb{Q})$ is the general linear group of a division algebra $D$ (possibly commutative) of finite dimension over $\mathbb{Q}$. When $D = \mathbb{Q}$, we have $G(\mathbb{Z}) = \Gamma_0 = \text{GL}_n(\mathbb{Z})$, the case described in the Introduction so far (for $n = 2$). In \cite{1}, Avner Ash constructed the well-rounded retract in this general setting of division algebras. In Section 2, we set up the rest of the paper by reviewing notation and facts from \cite{1}. The retract depends on a family of weights; in Section 3, we show that varying the weights over an interval of temperaments gives a fibration.

For any of the $G$ in this paper, the definition of a Hecke correspondence starts from a sublattice $M_0 \subset L_0$ of finite index. (In our $2 \times 2$ example for $T_\ell$, the $M_0$ is any one of the index-$\ell$ sublattices of $\mathbb{Z}^2$.) The form $Z$ generalizes to a Hermitian form $\bar{Z}$ which defines a Hermitian metric on $L_0$. The main idea of the well-tempered complex is to lie about the lengths of vectors $x \in L_0 - M_0$. We use a family of weights where the weighted length of $x$ is $\tau$ times the length of $x$ if $x \in L_0 - M_0$, but the weighted length of $x$ equals the length of $x$ if $x \in M_0$. When $\tau = 1$, there are no lies, and we obtain the usual well-rounded retract. When $\tau$ is large, vectors $x \in L_0 - M_0$ cannot have small weighted length: all the vectors of weighted length 1 in a well-rounded lattice will be in $M_0$. For $\tau$ between the extremes, the critical temperaments occur when the changing weights make the cell structure vary. This is the topic of Section 4.

In Section 5, we show that we can compute the Hecke operators with the well-tempered complex. In Section 6, we indicate how the different fibers of the well-tempered complex are computed in practice. This involves a weighted generalization of the Voronoi polyhedron, the Hecketope.

The algorithm has been coded up for $\text{SL}_n(\mathbb{Z})$ for $n = 2, 3, 4$ by the first
author. It performs successfully for congruence subgroups of small levels \( N \).
In the near future, we hope to test it for \( \text{GL}_2 \) when \( D \) is a quadratic number field or another small Galois extension of \( \mathbb{Q} \).

Our algorithm for Hecke operators has two parts, one-time work and every-time work. The one-time work is to compute the well-tempered complex for a given \( \Gamma_0, L_0, \) and Hecke correspondence \( T \). The every-time work is to compute the Hecke operators for \( T \) on \( H^i(\Gamma_0 \backslash X, \rho) \) for various \( i \) and \( \rho \). To compute the cohomology of subgroups \( \Gamma' \), like the classical \( \Gamma_0(N) \) and \( \Gamma(N) \) of level \( N \), we simply do the every-time work for the coinduced representation \( \text{Coind}^{\Gamma_0}_{\Gamma} \rho \). Other Hecke-operator algorithms, including the classical algorithms with modular symbols, require one to do the equivalent of the one-time work and the every-time work every time. Even for a single Hecke correspondence, it is possible to run our algorithm as a parallel computation (see Theorem 5).

Our current implementation is a proof of concept written in Sage \[27\]. In terms of speed, it is not yet competitive with Gunnells’ algorithm in \[18\] \[5\]. Plans for the future include translating parts of the implementation to C++ and parallelizing it.

1.3 Related Work

The Ash-Rudolph algorithm generalizes the modular symbol algorithm \[9\]. For any \( n \), it computes Hecke operators on \( H^i(\Gamma \backslash X; \rho) \) for \( \Gamma \subseteq \text{SL}_n(\mathbb{Z}) \). It works only in the top degree, \( i = \text{vcd} \Gamma \). Ash and his collaborators have many papers that study cohomology using this algorithm, starting with \[3\].

Gunnells extended the Ash-Rudolph algorithm to the next lower degree, where \( i = \text{vcd} \Gamma - 1 \). This works for \( \Gamma \subseteq \text{SL}_n(\mathbb{Z}) \) for all \( n \) \[18\]. In a series of papers including \[5\] \[6\] \[7\] \[4\], Ash, Gunnells, and the first author have computed Hecke operators on \( H^5(\Gamma_0(N) \backslash X; \rho) \), which is in the cuspidal range of cohomology, for a variety of levels \( N \) and coefficient systems \( \rho \).

From the Hecke eigenvalues, these papers find Galois representations that are conjecturally attached to the cohomology.

For other computations of Hecke operators on the cohomology of subgroups of \( \text{SL} \) or \( \text{GL} \) over a number field, see for example \[21\] \[13\] \[29\] \[2\] \[28\] \[22\] \[12\] \[24\] \[8\] \[17\].
1.4 Acknowledgments

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We thank Tony Bahri for suggesting the name well tempered. See also [14]. Soli Deo Gloria.

2 Background on the Well-Rounded Retract

We summarize notation and facts from [1]. We reverse left and right, and rows and columns, relative to [1] because we prefer to have our arithmetic groups acting on the left on cell complexes.

Let $D$ be a division algebra (possibly commutative) of finite dimension over $\mathbb{Q}$. Let $S = D \otimes_{\mathbb{Q}} \mathbb{R}$. Let $G = \text{GL}_n(S)$. The space of row vectors $S^n$ is an $\mathbb{R}$-vector space, a left $S$-module, and a right $G$-module.

A $\mathbb{Z}$-lattice in $D^n$ is a finitely generated subgroup of row vectors that contains a $\mathbb{Q}$-basis of $D^n$. A $\mathbb{Z}$-lattice in $S^n$ is a finitely generated subgroup that contains an $\mathbb{R}$-basis. Let $L_0 \subset D^n$ be a $\mathbb{Z}$-lattice. The subset $A$ of $D$ that stabilizes $L_0$ is an order in $D$. The subgroup $\Gamma_0$ of $G$ that stabilizes $L_0$ is an arithmetic group. We fix $L_0$, and hence $A$ and $\Gamma_0$, once and for all.

Fix an arithmetic subgroup $\Gamma \subseteq \Gamma_0$. Let $Y = \Gamma \backslash G$. If $\Gamma = \Gamma_0$, then $Y$ is the space of $\mathbb{Z}$-lattices in $S^n$ stable under $A$ and isomorphic to $L_0$ as $A$-module, and these are the lattices of the form $L_0g$ for $g \in G$. If $\Gamma$ is a congruence subgroup, then the elements of $Y$ are lattices with a level structure. In [1], only $\Gamma_0$ is used, but passing to $\Gamma$ brings no real changes.

We choose a particular positive definite inner product $\langle \cdot, \cdot \rangle$ on the $\mathbb{R}$-vector space $S^n$; we will present the definition in Section 3.2. The $K \subset G$ which preserves $\langle \cdot, \cdot \rangle$ is a maximal compact subgroup. $X = G/K$ is a symmetric space. The virtual cohomological dimension $\text{vcd} \Gamma$ is $\dim(Y/K) - n$.

Let $\mathcal{P}(D^n)$ be the set of lines (rank-one left $D$-submodules) in $D^n$. The quotient $\mathcal{P}(D^n)/\Gamma$ is finite. A set of weights for $\Gamma$ is a function $\varphi : \mathcal{P}(D^n)/\Gamma \to \mathbb{R}_+$. Such a $\varphi$ defines a set of weights for $L_0$, a $\Gamma$-invariant function $L_0 - \{0\} \to \mathbb{R}_+$ also denoted $\varphi$, by $\varphi(x) = \varphi(Dx)$. We always normalize $\varphi$ by dividing through by a positive real number so that the maximum value in its image is 1. For $L = L_0g$, a set of weights $\varphi$ for $L_0$ defines $\varphi^L : L - \{0\} \to \mathbb{R}_+$, a set of weights for $L$, by $\varphi^L(xg) = \varphi(x)$ for $x \in L_0 - \{0\}$.
Here are the main definitions and the main theorem of \[\Pi\].

**Definition 1.** Let \( L = L_0g \in Y \). The *arithmetic minimum* is

\[
m(L) = \min\{\varphi^L(x)\langle x,x \rangle \mid x \in L - \{0\}\}.
\]

The *minimal vectors* are

\[
M(L) = \{x \in L \mid \varphi^L(x)\langle x,x \rangle = m(L)\}.
\]

We say \( L \) is *well rounded* if \( SM(L) = S^n \). The set of well-rounded lattices in \( Y \) with minimum 1 is denoted \( \hat{W} \).

The functions \( m \) and \( M \) are \( K \)-invariant, because \( \langle , \rangle \) is. Hence \( \hat{W} \) is \( K \)-invariant.

**Theorem 1** (\[\Pi\] Thm. 2.11). \( W = \hat{W}/K \) is a strong deformation retract of \( Y/K \). It is compact and of dimension \( \text{vcd } \Gamma \). The universal cover \( \hat{W} \) of \( W \) is a locally finite regular cell complex in \( X \) on which \( \Gamma \) acts cell-wise with finite stabilizers of cells. This cell structure has a natural barycentric subdivision which descends to a finite cell complex structure on \( W \).

**Definition 2.** \( W = \hat{W}/K \) is the *well-rounded retract*.

3 Varying the Weights

3.1 Statement of Results

In this paper, we start with the trivial bundle \( Y \times I \) over an interval \( I \). The value \( \tau \in I \) is the *temperament*. \( G \) acts on \( Y \times I \) by acting on each fiber \( Y \times \{\tau\} \) in the obvious way. There is a bundle isomorphism \( (Y \times I)/K \cong (Y/K) \times I \) with fibers \( Y/K \).

A *one-parameter family of weights* for \( L_0 \) is a map \( \varphi_\tau : (L_0 - \{0\}) \times I \to \mathbb{R}^+ \) which is a \( \Gamma \)-invariant set of weights for any given \( \tau \), and such that \( \varphi_\tau(x) \) is real analytic in \( \tau \) for any given \( x \). We divide through by a positive real scalar, which depends continuously on \( \tau \), so that the maximum of \( \varphi_\tau \) is 1.

\[\footnote{Strictly speaking, this is a ramified cover, because certain points of \( W \) have finite stabilizer subgroups in \( \Gamma_0 \). The barycentric subdivision in the last sentence of the theorem produces a triangulation which is compatible with the ramified covering map.}\]
for all \( \tau \). Why we use real analytic functions will appear during the proof of Theorem 2.

A one-parameter family of weights \( \varphi_\tau \) determines \( \varphi^L_\tau, m_\tau(L), M_\tau(L), \hat{W}_\tau \), and \( W_\tau = \hat{W}_\tau/K \) for any given \( \tau \). By Theorem 1 there is a strong deformation retraction \( R_\tau(t) \) of the fiber over \( \tau \) onto \( W_\tau \). This means \( R_\tau(t) : ((Y \times \{\tau\})/K) \times [0, 1] \to (Y \times \{\tau\})/K \) is continuous, \( R_\tau(0) \) is the identity on the fiber \( (Y \times \{\tau\})/K \), the image of \( R_\tau(1) \) is \( W_\tau \), and \( R_\tau \) fixes \( W_\tau \) pointwise for all \( t \). Our next theorem is that \( R_\tau(t) \) is also continuous in \( \tau \).

**Theorem 2.** \( R_\tau(t) \) is a continuous function \((Y \times I)/K) \times [0, 1] \to (Y \times I)/K\).

**Corollary.** \( \{(w \times \tau)/K \mid \tau \in I, w \in \hat{W}_\tau\} \) is a strong deformation retract of \((Y \times I)/K\). It has dimension \( \text{vcd} \Gamma \). It is compact if \( I \) is compact. The map from the retract to \( I \) is a fibration.

The proof of Theorem 2 is an extension of the proof of Theorem 1. In Section 3.2, we summarize notation and facts from the proof of Theorem 1 in [1]. In Section 3.3 we prove Theorem 2.

**3.2 The Well-Rounded Retraction**

First, we recall the definition of the inner product \( \langle , \rangle \). Let \( v \) run through the archimedean places of the center of \( D \). Let \( D_v \) be the corresponding completion; we identify \( D_v \) with an algebra of square matrices over \( \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \), as appropriate. For \( x \in D_v \), let \( x^* \) be the conjugate transpose, where \textit{conjugate} means the identity on \( \mathbb{R} \) and the usual conjugate on \( \mathbb{C} \) or \( \mathbb{H} \). By \( \mathbf{x} \) we denote a row vector. Define the positive definite inner product \( \langle , \rangle_v \) on \( D_v^n \) by \( \langle x, y \rangle_v = \sum_{i=1}^{n} \text{Trace}_{D_v}(x^*_iy_i) \). We have \( S = \prod_v D_v \) and \( G = \prod_v G_v \), where \( G_v = \mathbb{GL}_n(D_v) \). Define \( \langle , \rangle \) on \( S^n = \prod_v D_v^n \) by \( \langle (x_v), (y_v) \rangle = \sum_v \langle x_v, y_v \rangle_v \).

The left action of \( S \) on \( S^n \) is stable under adjoint of \( \mathbb{R} \)-linear maps. \( K \) is the subgroup of \( G \) preserving \( \langle , \rangle \). Observe that \( K = \prod_v K_v \), where \( K_v \) is the maximal compact subgroup of \( G_v \) preserving \( \langle , \rangle_v \). Also, \( X = \prod_v X_v \) where \( X_v = G_v/K_v \).

Throughout this Section 3.2 \( \tau \in I \) is fixed. For \( j = 1, \ldots, n \), define

\[
Y^{(j)}_\tau = \{ L \in Y_\tau \mid \dim_D(D M_\tau(L)) \geq j \text{ and } m_\tau(L) = 1 \}.
\]

Set \( Y^{(0)}_\tau = Y \). Then \( Y^{(0)}_\tau \supset Y^{(1)}_\tau \supset Y^{(2)}_\tau \supset \cdots \) is a nested sequence of subspaces in the fiber \( Y \) over \( \tau \). By definition, \( \hat{W}_\tau = Y^{(n)}_\tau \). There is a
$K$-equivariant strong deformation retraction $r^{(j)}_\tau(t)$ from $Y^{(j-1)}_\tau$ to $Y^{(j)}_\tau$ for $t \in [0,1]$, whose definition we give below at (1). To obtain $R_\tau(t)$, we string together the $r^{(j)}_\tau(t)$ for $j = 1, \ldots, n$, rescaling the $n$ consecutive $t$s into $[0,1]$, while modding out by $K$. We write $r^{(j)}_\tau(t)$ as $r^{(j)}_\tau(L, t)$ when we need to specify $L$.

The action of the positive real homotheties gives $r^{(1)}_\tau$. Suppose $j \geq 2$ and $L \in Y^{(j-1)}_\tau$. We now define $r^{(j)}_\tau(L, t)$. If $L$ is in $Y^{(j)}_\tau$ already, set $r^{(j)}_\tau(L, t) = L$ for all $t$. Otherwise, let $P = S \cdot M_\tau(L)$, an $S$-submodule of $S^n$. Let $Q$ be the orthogonal complement of $P$ in $S^n$ with respect to $\langle , \rangle$. By [1, Lemma 2.6], $Q$ is also an $S$-submodule, $S^n = P \oplus Q$, and if $\lambda$ is a non-zero real number then the map $F_\lambda(p + q) = p + \lambda q$ (for $p \in P$, $q \in Q$) is given by right multiplication by some matrix $F_\lambda \in G$. The heart of the argument is to prove that $\mu_\tau(L) = \sup\{\lambda \mid m_\tau(L F_\lambda) < 1\}$ exists and satisfies $0 < \mu_\tau(L) < 1$, with $m_\tau(L \cdot F_\lambda) = 1$ at $\lambda = \mu_\tau(L)$. (Intuitively, we shrink $\lambda$ down from 1 through positive values until $L F_\lambda$ gains at least one new independent minimal vector; this new vector appears at $\lambda = \mu_\tau(L)$.) Define

$$r^{(j)}_\tau(L, t) = \begin{cases} L \cdot F_{(1 + (\mu_\tau(L) - 1) t)} & \text{if } L \in Y^{(j-1)}_\tau - Y^{(j)}_\tau \\ L & \text{if } L \in Y^{(j)}_\tau. \end{cases} \quad (1)$$

This is $K$-invariant, and [1] proves it is continuous in $L$ and $t$.

### 3.3 Proof of Theorem 2

Let $Y^{(j)} \subseteq Y \times I$ be the union of the $Y^{(j)}_\tau$ for $\tau \in I$. Let $L \in Y^{(j-1)}$. We must show $r^{(j)}_{\tau'}(L, t)$ is continuous as $\tau' \in I$ varies in a neighborhood of $\tau$. We will show it is continuous for both $\tau' \in (\tau - \varepsilon, \tau]$ and $\tau' \in [\tau, \tau + \varepsilon)$ for small $\varepsilon > 0$. We will call either $(\tau - \varepsilon, \tau]$ or $[\tau, \tau + \varepsilon)$ a half-interval around $\tau$.

A punctured half-interval is a half-interval around $\tau$ with $\tau$ removed.

As above, the action of the positive real homotheties gives the retraction $r^{(1)}$. This is continuous as a function of $\tau$ because $\varphi_\tau$ is continuous in $\tau$ and all the weights are positive. From now on, let $j \geq 2$.

The dimension $\text{dim}_D(DM_\tau(L))$ may or may not be constant on small half-intervals. If there is no small half-interval on a given side on which it is constant, it nevertheless varies upper semi-continuously: the dimension has a certain value at $\tau$, and has a constant but smaller value on a small punctured half-interval. The reason for the upper-semicontinuity is that $\text{dim}_S(SM_\tau(L))$
is the number of linearly independent vectors $x \in L$ with
\[ \varphi^{L}(x) \langle x, x \rangle - m_{\tau}(L) = 0. \] (2)

For a given $x$, (2) is an upper semi-continuous condition, because $\varphi^{L}(x)$ is a real analytic positive function of $\tau$, $m_{\tau}(L)$ is a min of real analytic functions of $\tau$, and $\langle x, x \rangle$ is a positive constant independent of $\tau$. If the left side of (2) is $\neq 0$, then it will remain $\neq 0$ on a small half-interval around $\tau$, by continuity. But if it is 0, then—by analyticity—it will either remain 0 on a small half-interval around $\tau$, or it will be $\neq 0$ throughout a small punctured half-interval around $\tau$.

Showing $r_{\tau}^{(j)}(L, t)$ is continuous in $\tau'$ breaks into two cases. In the first case, $\dim_{S} P$ is constant for $\tau'$ in a small half-interval around $\tau$. This is equivalent to saying $L \in Y_{\tau'}^{(j-1)}$ for all $\tau'$ in the neighborhood. Then $P$ and $Q$ vary continuously with $\tau'$ as $S$-submodules of $S^{n}$, hence so does $LF_{\lambda}$. The minimum $m_{\tau'}$ varies continuously with $\tau'$, so $\mu_{\tau'}(L)$ varies continuously with $\tau'$ and remains bounded away from 1 on the half-interval. Hence $r_{\tau}^{(j)}(L, t)$ varies continuously with $\tau'$ throughout the half-interval.

In the second case, $\dim_{S}(SM_{\tau'}(L))$ is upper semi-continuous. Here $L \notin Y_{\tau'}^{(j-1)}$ for $\tau'$ on a small punctured half-interval around $\tau$; rather, there is a $j' < j$ so that $L \in Y_{\tau'}^{(j'-1)}$ for $\tau'$ in that punctured half-interval. The retraction $r^{(j)}$ will not be operating on $L$ when $\tau' \neq \tau$, because $L \notin Y_{\tau'}^{(j-1)}$; rather, it will be operating on the image $L'$ of $L$ under $r^{(j')}, r^{(j'+1)}, \ldots, r^{(j-1)}$. We must show that $L'$ is arbitrarily close to $L$ when $\tau'$ is arbitrarily close to $\tau$. Consider the function $\mu$ used in (1) above. When $\tau'$ is very close to $\tau$, the new minimal vectors that will appear when $\lambda = \mu_{\tau'}(L)$ are a subset of those that are already in $M_{\tau}(L)$. This means that $\mu_{\tau'}(L) \to 1$ as $\tau' \to \tau$. In (1), this means $L F_{(1+(\mu_{\tau'}(L)-1))t}$ converges to $L$ as $\tau' \to \tau$. The convergence is uniform as $L$ and $t$ vary in small compact neighborhoods. This proves the continuity of $r_{\tau}^{(j)}(L, t)$.

4 The Well-Tempered Complex

4.1 Hecke Correspondences

We review Hecke correspondences, following [23 §3.1 and p. 76]. Subgroups $\Gamma_{1}$ and $\Gamma_{2}$ of $G$ are commensurable if $\Gamma_{1} \cap \Gamma_{2}$ has finite index in
both \( \Gamma_1 \) and \( \Gamma_2 \). Commensurability is an equivalence relation. The \textit{commensurator} of \( \Gamma_1 \) is \( \hat{\Gamma}_1 = \{ a \in G \mid a^{-1} \Gamma_1 a \text{ is commensurable with } \Gamma_1 \} \). If \( \Gamma_1 \) and \( \Gamma_2 \) are commensurable, then \( \hat{\Gamma}_1 = \hat{\Gamma}_2 \).

Define \( \Delta = \{ a \in G \mid L_0 a \subseteq L_0 \} \). This is a sub-semigroup of \( G \) containing \( \Gamma_0 \). If \( a \in \Delta \), then \( M_0 = L_0 a \) is a sublattice of \( L_0 \) of finite index. Thus \( \Gamma_0 \subseteq \Delta \subseteq \hat{\Gamma}_0 \). The arithmetic group \( \Gamma = \Gamma_0 \cap a^{-1} \Gamma_0 a \) is the common stabilizer in \( G \) of \( L_0 \) and \( M_0 \). One calls \((\Gamma_0, \Delta)\) a \textit{Hecke pair}.

A point in \( \Gamma_0 \setminus X \) has the form \( \Gamma_0 gK \) with \( g \in G \). Define two maps \( \begin{array}{c} \Gamma \setminus X \\ \Gamma_0 \setminus X \end{array} \xrightarrow{p} \xrightarrow{q} \Gamma_0 \setminus X \) (3) by \( p : \Gamma gK \mapsto \Gamma_0 gK \) and \( q : \Gamma gK \mapsto \Gamma_0 agK \). The map \( q \) is well defined: \( (\forall \gamma \in \Gamma) \Gamma_0 ag = \Gamma_0 a \gamma g \) is true if and only if \( \Gamma_0 = \Gamma_0 a \gamma a^{-1} \), and the latter is implied by \( a \gamma a^{-1} \in a(\Gamma_0 \cap a^{-1} \Gamma_0 a)^{-1} = a \Gamma_0 a^{-1} \cap \Gamma_0 \subseteq \Gamma_0 \).

The \textit{Hecke correspondence} \( T_a \) is the one-to-many map \( \Gamma_0 \setminus X \to \Gamma_0 \setminus X \) given by \( T_a = q \circ p^{-1} \). It sends one point of \( \Gamma_0 \setminus X \) to \([\Gamma_0 : \Gamma]\) points of \( \Gamma_0 \setminus X \), counting multiplicities.

The \textit{Hecke algebra} for the Hecke pair \((\Gamma_0, \Delta)\) is the free abelian group on the set of correspondences \( T_a \) for \( a \in \Delta \), with multiplication defined by the composition of the correspondences. This is equivalent to the traditional definition as the algebra of double cosets \( \Gamma_0 a \Gamma_0 \) for \( a \in \Delta \) [23, p. 54]. The Hecke algebra is commutative.

4.1.1 Examples

Let \( D = \mathbb{Q} \), and let \( L_0 \) be the standard lattice \( \mathbb{Z}^n \). Then \( \Gamma_0 = \text{GL}_n(\mathbb{Z}) \). All lines in \( \mathbb{P}^n(\mathbb{Q}) \) are equivalent mod \( \Gamma_0 \), so the constant function \( \varphi = 1 \) is the only set of weights for \( \Gamma_0 \). Here \( \Delta \) is the semigroup of all \( n \times n \) matrices with entries in \( \mathbb{Z} \) and non-zero determinant. \( \hat{\Gamma}_0 = \text{GL}_n(\mathbb{Q}) \). For a prime \( \ell \in \mathbb{Z} \) and \( k \in \{1, \ldots, n\} \), define \( T_{\ell,k} = T_a \) for \( a = \text{diag}(1, \ldots, 1, \ell, \ldots, \ell) \).

The Hecke algebra is generated by the \( T_{\ell,k} \) for all primes \( \ell \) and \( k \in \{1, \ldots, n\} \) [23 §3.2].
When \( D \) is an algebraic number field of class number one, the Hecke algebra is defined in a similar way. The ring of integers \( \mathcal{O}_D \) replaces \( \mathbb{Z} \).

When \( D \) is a number field of class number greater than one, what we have defined is a subalgebra of the full Hecke algebra, namely the subalgebra consisting of those correspondences that come from the principal ideal classes. The full Hecke algebra is defined over the adeles; see, for example, [16, p. 47].

4.2 Definition of the Well-Tempered Complex

Fix a set of weights \( \varphi \) for \( \Gamma_0 \). The lattice \( L_0 \), and \( \varphi \), determine the well-rounded retract for \( \Gamma_0 \) as in Section 2. Fix \( a \in \Delta \), and let \( \Gamma = \Gamma_0 \cap a^{-1} \Gamma_0 a \) as in Section 4.1. The well-tempered complex \( W^+ \) will be determined by \( L_0 \), \( \varphi \), and \( a \), and will naturally have an action of \( \Gamma \).

We impose an additional hypothesis on \( \varphi \) making \( \varphi \) compatible with \( a \):

\[
\varphi(xa) = \varphi(x) \quad \text{for all } x \in L_0 - \{0\}. \tag{4}
\]

The hypothesis is satisfied, for instance, if \( \varphi \) is the constant function 1.

Let \( M_0 = L_0 a \). For the rest of the paper, we extend the set of weights \( \varphi \) for \( \Gamma_0 \) to a set of weights \( \varphi_\tau \) for \( \Gamma \) in a particular way.

**Definition 3.** For \( x \in L_0 - \{0\} \) and \( \tau \geq 1 \), define

\[
\varphi_\tau(x) = \begin{cases} 
\varphi(x) & \text{if } x \in M_0 - \{0\}, \\
\tau^2 \varphi(x) & \text{if } x \notin M_0.
\end{cases}
\]

**Remark.** The idea here comes from \( m(L) \) in Definition 1. The weighted squared length of a vector \( x \in L \) is \( \varphi^2(x) \langle x, x \rangle \). The squared length \( \langle x, x \rangle \) scales by \( c^2 \) when we multiply \( x \) by \( c \in \mathbb{R} \). By multiplying the weight by \( \tau^2 \) when \( x \notin M_0 \), we mimic the effect of scaling the length of \( x \) linearly by \( \tau \). We pretend \( x \notin M_0 \) gets longer by \( \tau \), linearly. When \( x \in M_0 \), we do not pretend to change the length.

Choose \( \tau_0 > 1 \), and let \( I = [1, \tau_0] \). The well-tempered complex depends on \( \tau_0 \), but we will see that the complexes for two different \( \tau_0 \) are isomorphic when \( \tau_0 \) is sufficiently large. In Theorem 4 below, we will derive a particular \( \tau_0 > 1 \) for computing Hecke operators.

**Definition 4.** The well-tempered complex \( W^+ \) for \( L_0 \), \( \varphi \), and \( a \) is the image of \( (Y \times [1, \tau_0])/K \) under the retraction \( R_\tau(t) \) of Theorem 2 where \( \varphi \) satisfies (4) and where \( \varphi_\tau \) is as in Definition 3.
4.3 Cells in the Well-Tempered Complex

**Theorem 3.** The universal cover \( \tilde{W}^+ \) of the well-tempered complex \( W^+ \) is a locally finite regular cell complex on which \( \Gamma \) acts cell-wise with finite stabilizers of cells. This cell structure has a natural barycentric subdivision which descends to a finite cell complex structure on \( W^+ \).

**Proof.** We summarize notation and facts from the proof of Theorem 1 in [1], at first with \( \tau \) fixed. For any ring \( R \), let \( M_n(R) \) be the ring of \( n \times n \) matrices over \( R \), and \(( \ldots )^t \) the transpose. For \( Z_v \in M_n(D_v) \), let \( Z_v^* \) be the result of applying \( * \) to each matrix entry of \( Z_v \). We say \( Z_v \in M_n(D_v) \) is Hermitian if \( Z_v^t = Z_v^* \). Let \( E_v \) be the \( R \)-vector space of Hermitian matrices in \( M_n(D_v) \).

\( Z_v \) is positive definite if Trace\( D_v x Z_v x^* > 0 \) for all \( x \in D_v^n - \{0\} \). Identify \( X_v = G_v/K_v \) with the open cone of positive definite matrices in \( E_v \), via \( gK_v \leftrightarrow g(g^*)^t \) for \( g \in G_v \). Let \( E = \prod_v E_v \), an \( R \)-vector space, and identify points \( X = \prod_v X_v \) with products \( Z = \prod_v Z_v \) of positive definite matrices. If \( x \in L_0 \) and \( Z = g(g^*)^t \) for \( g \in G \), the length \( \langle xg, xg \rangle \) is denoted \( Z[x] \).

As mentioned in the Introduction, \( E \) puts a system of linear coordinates on \( X \). If \( x \in L_0 \) is fixed, then \( Z[x] \) is an \( R \)-linear function of \( Z \). The cells in our complexes will be defined using this linear structure.

The definitions of \( m(L) \), \( M(L) \), and well rounded from Definition 1 are equivalent to the following for \( Z \in X \):

\[
m_\tau(Z) = \min\{\varphi_\tau(x)Z[x] \mid x \in L_0 - \{0\}\} \tag{5}
\]

\[
M_\tau(Z) = \{x \in L_0 \mid \varphi_\tau(x)Z[x] = m_\tau(Z)\}
\]

\( Z \) is well rounded if \( S M_\tau(Z) = S^n \).

If \( \tilde{W}_\tau \) is the set of well-rounded \( Z \) in \( X \) with minimum \( m_\tau(Z) = 1 \), then \( \tilde{W}_\tau \) is the inverse image of \( W_\tau \) under the covering map \( X \to Y/K \); this is the universal cover of \( W_\tau \).

A point \( Z \in \tilde{W}_\tau \) satisfies the linear equations and inequalities

\[
\varphi_\tau(x)Z[x] = 1 \quad \text{if} \quad x \in M_\tau(Z),
\]

\[
\varphi_\tau(x)Z[x] \geq 1 \quad \text{if} \quad x \in L_0 - \{0\}.
\]

For each finite subset \( M \subseteq L_0 \), we define \( \sigma_\tau(M) \) to be the set of all \( Z \in X \) such that

\[
\varphi_\tau(x)Z[x] = 1 \quad \text{for} \quad x \in M \tag{6}
\]
\[ \varphi_\tau(x)Z[x] > 1 \quad \text{for } x \in L_0 - (M \cup \{0\}). \]

This is a system of linear equations and inequalities in the linear coordinates of \( E \). (Intuitively, it is a linear programming problem in \( E \).) Either \( \sigma_\tau(M) \) is empty, or it is a convex polyhedron, which is topologically a closed cell. When the cell is non-empty, we call \( M \) the set of minimal vectors for \( \sigma_\tau(M) \).

[1] shows that the non-empty \( \sigma_\tau(M) \) give \( \tilde{W}_\tau \) the structure of a locally finite regular cell complex on which \( \Gamma_0 \) acts cell-wise with finite stabilizers of cells. It shows that the cell structure has a natural barycentric subdivision which descends to a finite cell complex structure on \( W_\tau \). An element \( \gamma \in \Gamma_0 \) acts by \( \gamma \sigma_\tau(M) = \sigma_\tau(M\gamma^{-1}) \).

We now consider how these constructions vary with \( \tau \). For each finite subset \( M \subseteq L_0 \), extend (6) to the following system of equations and inequalities in \( Z \) and \( \tau \):

\[
\begin{align*}
\varphi_\tau(x)Z[x] &= 1 & \text{for } x \in M, \\
\varphi_\tau(x)Z[x] &> 1 & \text{for } x \in L_0 - (M \cup \{0\}), \\
1 &\leq \tau \leq \tau_0.
\end{align*}
\]

By Definition 3, this is equivalent to

\[
\begin{align*}
\varphi(x)Z[x] &= 1 & \text{for } x \in M \cap M_0, \\
\varphi(x)Z[x] &= \frac{1}{\tau^2} & \text{for } x \in M - M_0, \\
\varphi(x)Z[x] &> 1 & \text{for } x \in M_0 - (M \cup \{0\}), \\
\varphi(x)Z[x] &> \frac{1}{\tau^2} & \text{for } x \in L_0 - (M_0 \cup M), \\
1 &\leq \tau \leq \tau_0.
\end{align*}
\]

We now add one more variable to our system of linear coordinates. Replace \( E \) with the real vector space \( E \times \mathbb{R} \), where the coordinate on the second factor is \( u \). Rewrite (7) on \( E \times \mathbb{R} \) by setting

\[ u = \frac{1}{\tau^2}. \]

In the first four equations of (7), \( u \) occurs (if at all) only to the first power. The last equation bounds \( u \) between two constants, \( 1/\tau_0^2 \) and 1. Thus the equations and inequalities (7) are still linear in the coordinates of \( Z \) and \( u \),
still a linear programming problem. Exactly as in [1], the solution set for \( M \) is either empty or is a convex polyhedron \( \sigma(M) \), which is topologically a closed cell. Taken over all \( M \), the \( \sigma(M) \) give \( \tilde{W}^+ \) the structure of a regular cell complex on which \( \Gamma \) acts cell-wise with finite stabilizers of cells. The cell structure has a natural barycentric subdivision which descends to a cell complex structure on \( W^+ \).

We must show there are only finitely many cells \( \sigma \) up to the action of \( \Gamma \). Since a convex polyhedron is the convex hull of its vertices, it suffices to show there are only finitely many zero-dimensional cells (vertices) \( \sigma \) in \( \tilde{W}^+ \) up to the action of \( \Gamma \). Fix a \( \mathbb{Z} \)-basis of \( L_0 \). Given \( b \in \mathbb{R} \), define the box for \( b \) to be the set of vectors in \( L_0 \) whose coordinates with respect to the basis all have absolute value \( \leq b \). Let \( \Gamma' \) be the intersection of the conjugates of \( \Gamma \) in \( \Gamma_0 \); this is a normal subgroup of \( \Gamma_0 \) of finite index. By the finiteness for the ordinary well-rounded retract \( \tilde{W}_1 \) mod \( \Gamma' \), there is a \( b_0 \) so that every vertex of \( \tilde{W}_1 \) has, modulo \( \Gamma' \), the form \( \sigma_1(M) \) for an \( M \) contained in the box for \( b_0 \). Then every vertex that is a solution to (7) with \( 1 \leq \tau \leq \tau_0 \) has, modulo \( \Gamma' \), the form \( \sigma(M) \) for an \( M \) contained in the box for \( b_0 \tau_0 \). The box has only finitely many subsets. \( \square \)

**Notation.** For any \( \tau \geq 1 \), we write \( u = 1/\tau^2 \) for the rest of the paper. We write \( W_u = W_{1/\tau^2} \), and similarly for \( \tilde{W}_u \).

### 4.4 Refining the Well-Tempered Complex

In \( \tilde{W}_\tau \) for a given \( \tau \), the non-empty cells \( \sigma_\tau(M) \) are in one-to-one correspondence with their sets of minimal vectors \( M \). In \( \tilde{W}^+ \), the cells \( \sigma \) are also indexed by a finite amount of combinatorial data. For computation in \( \tilde{W}^+ \), it will be more convenient to take a certain refinement of the cell structure. In the refinement, the combinatorial data in \( L_0 \) will be kept separate from the combinatorial data along the \( \tau \)-axis.

Since a non-empty \( \sigma \) is a closed convex polyhedron, its projection onto the \( u \)-coordinate is a compact subinterval of \([1/\tau_0^2, 1]\), possibly a single point. Consider the set \( \mathcal{U} \) of left and right endpoints of these intervals, taken over all \( \sigma \). Because the cells are convex, the endpoints are the images of certain vertices of the \( \sigma_\tau \)s. By the finiteness in Theorem 3, \( \mathcal{U} \) is a finite subset of \([1/\tau_0^2, 1]\). We denote its elements

\[
1/\tau_0^2 = u^{(0)} < u^{(1)} < \cdots < u^{(i_r)} = 1.
\] (8)
The \( u^{(i)} \), and their corresponding \( \tau^{(i)} = 1/\sqrt{u^{(i)}} \), are called the **critical temperaments**. We will always refine the cells \( \sigma \) of Theorem 3 by cutting them into closed pieces along the hyperplanes \( u = u^{(i)} \) for \( i = 0, \ldots, i_r \). Each non-empty cell of the refinement is indexed by a pair. The pair is \((M, [u^{(i-1)}, u^{(i)})]\) if the projection of the cell to the \( u \)-coordinate is \([u^{(i-1)}, u^{(i)})\]. The pair is \((M, [u^{(i)}, u^{(i)})]\) if the projection is \([u^{(i)}]\). We will write \([u, u']\) as shorthand for both \([u^{(i-1)}, u^{(i)})\] and \([u^{(i)}, u^{(i)})\].

### 4.5 The First and Last Temperaments

Since \( M_0 \) is a sublattice of finite index in \( L_0 \), there is a positive integer \( N \) so that \( NL_0 \subseteq M_0 \subseteq L_0 \). Let \( \xi \) be the maximum value of \( \varphi \) on \( x \in M_0 - \{0\} \). Let \( \eta \) be the minimum value of \( \varphi \) on \( x \in L_0 - M_0 \).

**Theorem 4.** Let \( L_0 \), \( \varphi \), and \( a \) be as in Theorem 3. Define

\[
\tau_0 = \sqrt{\xi/\eta N}.
\]

Let \( \tau \geq \tau_0 \). Then the map \( X \to X \) given by \( gK \mapsto a^{-1}gK \) descends mod \( \Gamma \) to give a cell-preserving homeomorphism from the well-rounded retract \( W_1 \) over 1 to the well-rounded retract \( W_\tau \) over \( \tau \). If a cell over \( \tau = 1 \) is \( \sigma_1(Q) \) with index set \( Q \subseteq L_0 - \{0\} \), then the cell that corresponds to \( \sigma_1(Q) \) under the homeomorphism has index set \( Qa \).

We call the endpoints of \([1, \tau_0]\) the **first** and **last** temperaments, respectively. For example, if \( D = \mathbb{Q} \) and \( T_a = T_{\ell,k} \) as in Section 4.1.1 then \( \tau_0 = \ell \).

We turn to the proof of Theorem 4.

**Lemma.** With notation as in Theorem 3, let \( \tau \geq \tau_0 \). Let \( L = L_0g \) be a well-rounded lattice in the fiber over \( \tau \). Then every minimal vector \( x \in M_\tau(L) \) is in \( M_0g \).

**Proof of Lemma.** First suppose \( \tau > \tau_0 \). Assume by way of contradiction that \( x \in M_\tau(L) \) is a minimal vector with \( x \notin M_0g \). Note that \(Nx \in M_0g \). The weighted length of \( x \) is \( \varphi_\tau(x)\langle x, x \rangle \geq \tau^2\eta\langle x, x \rangle \). The weighted length of \( Nx \) is \( \varphi_\tau(Nx)\langle Nx, Nx \rangle \geq \xi N^2\langle x, x \rangle \). Since \( x \) was minimal, we have \( \tau^2\eta \leq \xi N^2 \), or \( \tau \leq \sqrt{\xi/\eta}N \), contradicting \( \tau > \tau_0 \).

When \( \tau = \tau_0 \), the argument needs to be modified because there may be ties. \( \tau^2\eta = \xi N^2 \) may occur for an \( x \) with \( x \notin M_0g \) but \( Nx \in M_0g \). This says that two of the conditions in (7), those for \( x \) and \( Nx \), are identical and
redundant in the fiber over this particular $\tau_0$. We simply drop $x$ from the minimal vector set and keep $Nx$.

Proof of Theorem 4. Let $g \in G$ be such that $L_0gK \in W_\tau$. We use the notation $Z = g(g^*)^t$ of the proof of Theorem 3. To say $L_0gK$ is well rounded over $\tau$ is to say that there is a finite subset $M \subset L_0 - \{0\}$ so that

\[
\varphi_\tau(x)Z[x] = 1 \quad \text{for } x \in M
\]

\[
\varphi_\tau(x)Z[x] > 1 \quad \text{for } x \in L_0 - (M \cup \{0\}).
\]

By the Lemma, $M$ is a subset of $M_0 - \{0\}$; every $x \in M$ has the form $x = ya$ for some $y \in L_0 - \{0\}$. Let $Q$ be the set of these $y$, so that $M = Qa$. Let $Z_1 = aZ(a^*)^t$. Then

\[
\varphi_\tau(x)Z_1[y] = 1 \quad \text{for } y \in Q
\]

\[
\varphi_\tau(x)Z_1[y] > 1 \quad \text{for } y \in L_0 - (Q \cup \{0\})
\]

with $x = ya$. Since $x \in M_0$, Definition 3 says $\varphi_\tau(x)$ is independent of $\tau$, hence equals $\varphi_1(x)$. By the hypothesis (4), $\varphi_1(x) = \varphi_1(y)$. Thus

\[
\varphi_1(y)Z_1[y] = 1 \quad \text{for } x \in Q
\]

\[
\varphi_1(y)Z_1[y] > 1 \quad \text{for } x \in L_0 - (Q \cup \{0\}).
\]

In other words, $Z_1$ is well rounded in the fiber over 1.

It is easy to see this map is a homeomorphism between the well-rounded retracts in the two fibers. Since the map from the fiber over $\tau$ to the fiber over 1 was multiplication by $a$, the map from the fiber over 1 to the fiber over $\tau$ is multiplication by $a^{-1}$. The map on sets of minimal vectors from the fiber over 1 to the fiber over $\tau$ is $Q \mapsto Qa$. Therefore, $Z_1$ is well rounded in the fiber over $\tau$.

Corollary. For any two $\tau_1, \tau_2 \geq \tau_0$, the intersections of the well-tempered complex with the fibers over $\tau_1$ and $\tau_2$ are equal to each other as subsets of $Y/K$, independently of $\tau$.

Proof. By Theorem 4, both are equal to $a^{-1}$ times the fiber over $\tau = 1$. □

5 Computing Hecke Operators

Let the Hecke pair $(\Gamma_0, \Delta)$ be as above. Let $\rho$ be any left $\mathbb{Z}\Delta$-module. There is a natural left action of the Hecke algebra on cohomology groups for $\Gamma_0$. 

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with coefficients in $\rho \ [10, \S 1.1]$. For $a \in \Delta$, the action of the Hecke correspondence $T_a$ on the cohomology is called the Hecke operator, and it will also be denoted $T_a$. In this section, we will give an algorithm that uses the well-tempered complex to compute $T_a$.

5.1 Equivariant Cohomology

First, we specify the cohomology theory we will use. This is the equivariant cohomology $H^*_\Gamma_0(X; \rho)$, the cohomology of the group $\Gamma_0$ with coefficients in the complex of cochains on $X$ with coefficients in $\rho \ [11, VII.7]$.

In this paragraph, say a prime is “bad” if it divides the order of a finite subgroup of $\Gamma_0$. Since $\Gamma_0$ is arithmetic, it has at most finitely many finite subgroups up to conjugacy. Hence there are at most finitely many bad primes. Suppose the module underlying the coefficient system $\rho$ is a vector space over a field whose characteristic is not divisible by the bad primes; this includes characteristic zero. Then the equivariant cohomology $H^*_\Gamma_0(X; \rho)$ is canonically isomorphic to the cohomology $H^*(\Gamma_0 \setminus X; \rho)$ of the space $\Gamma_0 \setminus X$ with coefficients in $\rho$, as well as to the group cohomology $H^*(\Gamma_0; \rho)$.

To compute equivariant cohomology, we may replace $X$ with any acyclic cell complex on which $\Gamma_0$ acts. The symmetric space $X = G/K$ is contractible. By Theorem 1, the well-rounded retract $\widetilde{W}$ is a strong deformation retract of $X$, hence contractible, hence acyclic. Thus $H^*_\Gamma_0(X; \rho) = H^*_\Gamma_0(\widetilde{W}; \rho)$. Again by Theorem 1, the fiber $\widetilde{W}_\tau$ of the well-tempered complex $\widetilde{W}^+$ over any $\tau$ is a strong deformation retract of $X$, hence acyclic. This holds in particular for the fibers $\widetilde{W}_\tau$ over the critical temperaments $\tau^{(i)}$. Denote the portion of $\widetilde{W}^+$ with $u$ between two consecutive critical temperaments, $u \in [u^{(i-1)}, u^{(i)}]$, by $\widetilde{W}_{[u^{(i-1)}, u^{(i)}]}$. By Theorem 2, $\widetilde{W}_{[u^{(i-1)}, u^{(i)}]}$ is a strong deformation retract of $X \times [u^{(i-1)}, u^{(i)}]$, hence is acyclic. Any of the $\widetilde{W}_{u^{(i)}}$ or $\widetilde{W}_{[u^{(i-1)}, u^{(i)}]}$ can be used to compute the equivariant cohomology of $\Gamma$. We can compute it in finite terms because the complexes have only finitely many $\Gamma$-orbits of cells. $\widetilde{W}_{[u^{(i-1)}, u^{(i)}]}$ has dimension one higher than the vcd, but its cohomology in degree vcd+1 will be zero.
5.2 Definition of the Hecke Operator

Apply the equivariant cohomology functor to the diagram (3):

\[
\begin{array}{ccc}
H^*_\Gamma(X; \rho) & \xrightarrow{q^*} & H^*_\Gamma_0(X; \rho) \\
p_* & & p_*
\end{array}
\]

The Hecke operator \( T_a \) on \( H^*_\Gamma_0(X; \rho) \) is by definition \( p_*q^* \).

The map \( q^* : H^*_\Gamma_0(X; \rho) \to H^*_\Gamma(X; \rho) \) is the natural pullback map for \( q \). The map \( p_* : H^*_\Gamma(X; \rho) \to H^*_\Gamma_0(X; \rho) \) is the transfer map \([11, III.9]\) for \( p \), which is defined because \( \Gamma = \Gamma_0 \cap a^{-1}\Gamma_0 a \) has finite index in \( \Gamma_0 \).

5.3 Computing Hecke Operators

We now describe how to use (3) with the well-tempered complex to compute the Hecke operators in practice.

First, we compute \( p_* \). We use \( \tau = 1 \), the first temperament, when working with \( p \). The retracts \( \tilde{W} \) and \( \tilde{W}_1 \) are equal by definition. \( \Gamma_0 \) acts on \( \tilde{W} \), and the smaller group \( \Gamma \) acts on \( \tilde{W}_1 \). Computing the transfer map is straightforward. (In practice it is tricky to get the orientation questions correct. This is true for all the cells, and especially for the cells with non-trivial stabilizer subgroups. This comment applies to all the computations in this paper.)

Next, we compute \( q^* \). The pullback map is natural on cohomology, but we must account for the factor of \( a \) in the definition of \( q \). The key is to use the last temperament \( \tau_0 \) when working with \( q \). We compute \( H^*_\Gamma(X; \rho) \) as \( H^*_\Gamma(\tilde{W}_{\tau_0}; \rho) \). By Theorem 4, there is a homeomorphism of cell complexes \( \tilde{W}_{\tau_0} \to \tilde{W}_1 \), from the last temperament to the first, given by multiplication by \( a \). As we saw for \( p \), \( \tilde{W}_1 \) equals \( \tilde{W} \). Thus there is a cellular map which enables us to compute \( q^* : H^*_\Gamma_0(\tilde{W}; \rho) \to H^*_\Gamma(\tilde{W}_{\tau_0}; \rho) \).

During the computation, the cells are represented by the combinatorial data \((M, [u, u'])\) of Section 4.4. By Theorem 4, the map in the last paragraph involves dividing \( M \) by \( a \); the map sends \((M, [u, u'])\) to \((Ma^{-1}, [u, u'])\).
index sets $M$ are subsets of the sublattice $M_0 = L_0a$. This is why $Ma^{-1}$ will be a subset of $L_0$ (will be “integral”).

Computing only $p_*$ and $q^*$ does not give us the Hecke operator. The map of Theorem 4 involves dividing or multiplying by $a$. It is not a $\Gamma$-map, because $a \in \Delta$ but $a \notin \Gamma$ in general. For this reason, we cannot directly map $H_\Gamma^*(\tilde{W}_0; \rho)$ to $H_\Gamma^*(\tilde{W}_1; \rho)$.

To overcome this last difficulty, we use the whole well-tempered complex to define a chain of morphisms and quasi-isomorphisms. For $i = 1, \ldots, i_r$, in the portion $\tilde{W}_{u(i-1), u(i)}$ over the fibers $u \in [u(i-1), u(i)]$, define the closed inclusions of the fibers on the left and right sides:

$$\tilde{W}_{u(i-1)} \xleftarrow{l(i-1)} \tilde{W}_{u(i-1), u(i)} \xrightarrow{r(i)} \tilde{W}_{u(i)}$$

By Theorems 2 and 3, we can compute the pullbacks $(l(i-1))^*$ and the pushforwards $(r(i))^*$ on $H_\Gamma^*(\ldots; \rho)$. The pullback is a naturally defined cellular map. The pushforward $(r(i))^*$ is a quasi-isomorphism, the inverse of the pullback $(r(i))^*$; we compute the pullback at the cochain level using the cellular map, then invert the map on cohomology.

We summarize our algorithm as a theorem.

**Theorem 5.** With notation as above, the Hecke operator $T_a$ on equivariant cohomology (9) may be computed in finite terms as the composition

$$p_*l(0)^* l(1)^* r_* l(2)^* \ldots l(i-1)^* r_* l(i)^* q^*$$

Note that the maps $p_*$, $q^*$, and $l(i)^*$ and $r_* (i)$ for the various $i$ can be computed in parallel to speed up the computation.

### 5.4 Cohomology of Subgroups

Let $\Gamma' \subseteq \Gamma_0$ be an arithmetic subgroup. We wish to compute Hecke operators on the equivariant cohomology $H_\Gamma^*(X; \rho)$ for any $\Gamma'$. By Shapiro’s Lemma [11, III.6.2], $H_\Gamma^*(X; \rho) \cong H_{\Gamma_0}^*(X; \text{Coind}_{\Gamma_0}^\Gamma \rho)$. We use Theorem 5 to compute the latter.

### 6 Computing the Well-Tempered Complex

We now describe practical algorithms for finding a list of representatives of the $\Gamma$-orbits of cells in the well-tempered complex.
In broad terms, there are two kinds of steps. First, we take an individual $\tau$ and find all the cells in $W_\tau$ up to $\Gamma$-equivalence. Second, we determine the interval of $\tau'$ around the given $\tau$ where the cell structure does not change. The endpoints of this interval are two consecutive members of the set of critical temperaments. We also carry out the first step at the critical temperaments themselves. We repeat the first and second steps, starting at different $\tau$, until the entire interval $[1, \tau_0]$ has been covered.

As we said in the Introduction, our algorithm for Hecke operators has two parts, one-time work and every-time work. The one-time work is to compute $\tilde{W}^+$ for a given $L_0$, set of weights $\phi$, and $a \in \Delta$. The every-time work is to pick a $\rho$ and compute the Hecke operators on $H^*_{\Gamma_0}(X; \rho)$. As in Section 5.4, computing Hecke operators for a range of different levels $N$ comes under the heading of every-time work.

6.1 Finding the Cells for One Fiber

In Section 4.3 we introduced the space $E_v$ of $n \times n$ Hermitian matrices over $D_v$. Put the positive definite inner product $\langle\langle A, B \rangle\rangle_v = \sum_{i,j=1}^n A_{ij}B_{ij}^*$ on $E_v$. Then $\langle\langle A, B \rangle\rangle_v = \sum_v \text{Trace}_{D_v}^R \langle\langle A, B \rangle\rangle_v$ is a positive definite inner product on $E = \prod_v E_v$. For a row vector $x \in L_0$, let $\psi = \psi(x)$ be the matrix whose component $\psi_v$ at the $v$-th place has $i,j$ entry equal to $(x_v)^*i(x_v)j$. Let $Z = g(g^*)^t.$ The next proposition generalizes Voronoi’s original work on perfect forms [30, §15].

**Proposition.** With $x$, $\psi$, and $Z$ as above, $Z[x] = \langle\langle Z_v, \psi \rangle\rangle$.

**Proof.** For elements in the image of the embedding $D \hookrightarrow D_v$, we drop the subscript $v$ for simplicity.

\[
Z[x] = \langle xg, xg \rangle \\
= \sum_v \sum_{i=1}^n \text{Trace}_{D_v}^R ((xg)_i^*(xg)_i) \\
= \sum_v \text{Trace}_{D_v}^R \left( \sum_{i=1}^n (xg)_i^*(xg)_i \right) \\
= \sum_v \text{Trace}_{D_v}^R \left( \sum_{i=1}^n \left( \sum_{j=1}^n x_j g_{ji} \right)^* \left( \sum_{k=1}^n x_k g_{ki} \right) \right)
\]
\[
= \sum_v \text{Trace}_R^D \left( \sum_{i,j,k} g_{ji}^* x_j^* x_k g_{ki} \right)
\]
\[
= \sum_v \text{Trace}_R^D \left( \sum_{i,j,k} g_{ki} g_{ji}^* x_j^* x_k \right)
\]
\[
= \sum_v \text{Trace}_R^D \left( \sum_{j,k} \left( \sum_i g_{ki} g_{ji}^* \right) x_j^* x_k \right)
\]
\[
= \sum_v \text{Trace}_R^D \left( \sum_{j,k} (Z_v)_{kj} (x_k^* x_j)^* \right)
\]
\[
= \sum_v \text{Trace}_R^D \langle \langle Z_v, \psi \rangle \rangle_v
\]
\[
= \langle \langle Z_v, \psi \rangle \rangle
\]

Now let the notation be as in Theorem 4.

\[
\varphi_\tau(x) Z[x] = \begin{cases} 
\varphi(x) Z[x] & \text{if } x \in M_0 - \{0\}, \\
\tau^2 \varphi(x) Z[x] & \text{if } x \notin M_0.
\end{cases}
\]

Define

\[
\psi_\tau(x) = \begin{cases} 
\psi & \text{if } x \in M_0 - \{0\}, \\
\tau^2 \psi & \text{if } x \notin M_0.
\end{cases}
\]

**Definition 5.** For a given \( \tau \in [1, \tau_0] \), the Hecketope at \( \tau \) is the intersection of \( X \) with the convex hull \( H_\tau \) in \( E \) of the points \( \psi_\tau(x) \) for all \( x \in L_0 - \{0\} \).

The *Voronoi polyhedron* is the Hecketope at \( \tau = 1 \) [26, §A.3.4].

We must explain why we took the intersection with \( X \) in the definition. The points \( \psi_\tau(x) \) lie on the boundary of the cone \( X \) in \( E \). They are not in \( X \) (assuming \( n \geq 2 \)), because their rank over \( D \) is one; they are positive semidefinite, not positive definite. If \( F \) is a face of \( H_\tau \) and \( \text{int} F \) is the relative interior of \( F \), then either \( \text{int} F \) lies in \( X \) or is disjoint from \( X \). The Hecketope contains only the \( \text{int} F \) that lie in \( X \).

We remark that \( H_\tau \) is not compact, because it has infinitely many vertices \( \psi_\tau \). It is not locally finite at the faces where \( \text{int} F \) does not lie in \( X \).

Examining Formulas (7) gives the
Corollary. There is a one-to-one, inclusion-reversing correspondence between the faces of the Hecketope at $\tau$ and the faces of the fiber $\widetilde{W}_\tau$ of the well-tempered complex over $\tau$.

For instance, a facet (codimension-1 face) of the Hecketope corresponds to a vertex (dimension 0) of the well-rounded retract. When $D = \mathbb{Q}$, the polyhedral cones in the Voronoi tiling of $X$ as in [30] are exactly the cones spanned by the faces of the Voronoi polyhedron.

If $F$ is a face of $\mathcal{H}_\tau$ for which $\text{int } F$ is disjoint from $X$, then Formulas (7) determine a set of points which are not well rounded. That is why we exclude such $\text{int } F$ from the Hecketope.

6.2 Finding Representative Cells modulo $\Gamma$

For a given $\tau$, we need an algorithm to find a complete set of representatives of the cells of $\tilde{W}_\tau$ modulo $\Gamma$. By the last Corollary, it is equivalent to provide an algorithm to find a complete set of representatives of the faces of the Hecketope $\mathcal{H}_\tau \cap X$ modulo $\Gamma$. We describe two such algorithms.

6.2.1 Contiguous Forms

The first algorithm generalizes Voronoi’s own method. Find a facet $F_0$ of the Hecketope at which to start. Enumerate all the facets $F_{01}, F_{02}, \ldots$ of this facet (these have codimension two in the Hecketope). Each $F_{0j}$ is the border between exactly two facets of the Hecketope; the first is $F_0$, and the second is a new facet contiguous to $F_0$. Voronoi gave an algorithm for finding the contiguous facet when $D = \mathbb{Q}$ [30, §§22–23]; a modern formulation, with ideas for implementing it on computers, appears in [20, §7.8]. Enumerate all the facets contiguous to $F_0$ across $F_{01}, F_{02}, \ldots$. Continue this process recursively. When the algorithm computes a new facet, check whether it is $\Gamma$-equivalent to one we have already found. The algorithm stops when all the new facets one finds are $\Gamma$-equivalent to old facets. Once we have a list of finitely many facets, it is straightforward to enumerate their faces in all the smaller dimensions and classify them up to $\Gamma$-equivalence.

Specialized, highly optimized algorithms have been designed for particular $D$ and $n$. Examples include [25] and [15].
6.2.2 A Global Method

At present, our code uses a second method. We begin with an intuitive description. We choose a finite subset $B \subset L_0 - \{0\}$, roughly the lattice points in a ball around the origin. We take the convex hull $H'$ of $\{\psi_\tau(x) \mid x \in B\}$ in $E$. $H' \cap X$ is a chunk of the Hecketope $H \cap X$ near the origin of $E$. The chunk has been truncated far from the origin (since $H'$ is compact, while $H$ is not). Near the region where the truncation occurs, $H'$ will have faces which are not faces of $H$.

Let us make precise how we choose $B$. Choose a reasonably good lattice basis of $L_0$ as $\mathbb{Z}$-lattice. (If $D = \mathbb{Q}$ and $L_0 = \mathbb{Z}^n$, choose the standard basis.) Put on $L_0$ the usual 2-norm for the coordinates on that basis. Choose a parameter $c$, the vertex count, which is the approximate number of points we want to have in $B$. Using the classical formula for the volume of a ball in $\dim R S^n$ dimensions, choose an $r_1$ so that the ball of radius $r_1$ contains about $1.25c$ vectors of $L_0 - \{0\}$. (We always count $x$ and $-x$ as a single vector. Similarly, if $u \in D$ is a unit with norm of absolute value 1 in each $D_v$, we count $x$ and $xu$ as a single vector. The extra 0.25 is a margin of error.) Let $L_1$ be the list of these $x \in L_0$ within radius $r_1$. Choose $r_2$ so that the ball of radius $r_2$ contains about $1.25c$ vectors of $M_0 - \{0\}$. Let $L_2$ be the list of these $x \in M_0$. If $x \in L_1 \cap L_2$, remove $x$ from $L_1$. For each $x \in L_1 \cup L_2$, compute $\psi_\tau(x)$. Give $\psi_\tau(x)$ a score; we have experimented with different scores, but currently we use $\sum z^*z$ for $z$ running down the diagonal of the matrix $\psi_\tau(x)$. Let $B$ contain the best $c$ vectors $x \in L_1 \cup L_2$, where “best” means the scores are the least. If the scores are tied, take a few more vectors than $c$ until the tie is broken. The main point of this algorithm is that $\psi_\tau(x)$ is scaled by a factor of $\tau^2$ when $x \in L_1$, but not when $x \in L_2$. When $\tau$ is near 1, $B$ contains the shortest $c$ vectors in $L_0 - \{0\}$. But when $\tau$ is near $\tau_0$, almost all the vectors in $B$ are in $M_0 - \{0\}$. For mid-range $\tau$, $B$ contains a mixture.

Once $B$ is chosen, we compute the convex hull $H'$ of $\{\psi_\tau(x) \mid x \in B\}$ in $E$ using Sage's class Polyhedron over $\mathbb{Q}$ [27]. The Polyhedron class has a method which enumerates all the faces of $H'$ in every dimension. We keep those faces whose relative interior lies in $X$, discarding those where it does not. We use [27] to check whether the face is really a face of $H$, and we discard it if not. The faces that remain form a set $\tilde{W}_\tau$. We find representatives of the $\Gamma$-orbits in $\tilde{W}_\tau$ by a straightforward search.

If we choose $c$ too small, we observe that the set $\tilde{W}_\tau$ will not be a closed subcomplex. Some small-dimensional faces of $H$ will be computed incor-
rectly in \( \mathcal{H}' \) near where \( \mathcal{H} \) is truncated. Because of the inclusion-reversing correspondence in the last Corollary, this means some large-dimensional faces will be missing in \( \tilde{W}_\tau \); for example, two \( k \)-cells may be missing their common \((k-1)\)-dimensional face. However, when \( c \) is large enough, we do get a closed cell complex equal to \( W_\tau \).

As we have said, computing the Hecketopes is one-time work. We take the position that one-time work can be relatively slow, as long as the every-time work is fast. Still, choosing a large \( c \) makes the one-time work very slow, and it generates large intermediate files until we have finished modding out by \( \Gamma \). A plan for our future work on the Hecketopes is to replace the global method with the method of contiguous forms.

### 6.3 Finding Consecutive Critical Temperaments

In this section, let \( F \) be a given facet of the Hecketope at \( \tau \). Dually, \( F \) is a vertex of the retract \( \tilde{W}_\tau \) over \( \tau \). As in \([7]\), \( F \) is characterized by a set \( M \) of minimal vectors in \( L_0 - \{0\} \). Because \( F \) is a vertex of the retract, there is a unique \( Z \) satisfying \([7]\) for \( M \) and \( \tau \). As before, change variables to \( u \) by \( u = 1/\tau^2 \).

Our next goal is to find the largest interval \([u_1, u_2]\) containing a given \( u \) so that, for all \( u' \in (u_1, u_2) \), \([7]\) still has a unique solution at \( u' \) for our chosen \( F \) and \( M \). At the endpoints \( u' = u_1 \) or \( u' = u_2 \), the solution to \([7]\) could be unique for a proper superset of \( M \) (in other words, some new independent minimal vectors could appear).

To find \([u_1, u_2]\), we observe that \([7]\), for a given \( M \), defines a pencil of Hermitian forms in the variable \( u' \). The left-hand side of the equations is a linear function of the entries of \( Z \). The right-hand sides are either 1 or \( u' \), depending on whether or not \( x \in M_0 \). Using symbolic algebra in Sage, we find the solution \( Z = Z(u') \) of this pencil. The entries of the matrix \( Z(u') \) are linear polynomials in \( u' \).

The coefficients in this pencil are in \( D \) (are “rational”), since all the \( x \in L_0 \). Thus the symbolic computations we are about to do are based on rational arithmetic. For simplicity, in the rest of this section, we describe our implementation of the algorithm for \( D = \mathbb{Q} \), where \( S = \mathbb{R} \). If \( D \) is a more general number field or division algebra, the algorithm would need modifications, such as using complex interval arithmetic when we use real interval arithmetic below.

It may be that \( Z(u') \) only has a solution at the one point \( u' = u \). This
happens if and only if we are at a critical temperament. If it happens, we stop and report that the largest interval \([u_1, u_2]\) is \([u, u]\). Otherwise, we continue searching for a larger interval.

The algorithm we are about to describe is like Voronoi’s original contiguous form algorithm [30, §§22–23] [20, §7.8], but adapted to the variation of the temperament. We find \(\det Z(u')\) symbolically as a polynomial \(f(u')\). Replacing \(f\) by its gcd with its derivative, we guarantee that \(f\) has no repeated roots. Using real interval arithmetic (Sage’s RIF), we find approximations to the real roots of \(f\). Our \(u\) itself is not a root, since \(Z\) is positive definite there. The real roots closest to \(u\) on either side are where \(Z\) become a degenerate Hermitian form. The interval \([u_1, u_2]\) must be properly contained in the interval cut out by the real roots, because of positive definiteness. We now describe how to find \(u_1\) (the left side); finding \(u_2\) is similar. We find \(u_1\) using a bisection argument. Let \(u_r\) be the real root of \(f\) closest to \(u\) on the left. Start with \(u'\) halfway between \(u_r\) and \(u\). If the equations in (7) have no new minimal vectors beyond those in \(M\), move \(u'\) further to the left, halfway between its current position and \(u_r\). After enough bisections, we must find some new minimal vectors, because we are getting closer and closer to where \(\det Z(u') = 0\), the boundary of \(X\). Once we find a \(u'\) where the set \(M'\) of minimal vectors in the equations in (7) is a proper superset of \(M\), we are essentially done. The finite number of vectors in \(M' - M\) give finitely many conditions; the rightmost of those conditions determines \(u_1\).

6.4 Finding the Well-Tempered Complex

We complete Section 6, giving a practical algorithm for finding a list of representatives of the \(\Gamma\)-orbits of cells in the well-tempered complex \(\tilde{W}^+\). The input is \(L_0\), a set of weights \(\varphi\), and \(a \in \Delta\). We begin by finding the Hecketope at \(u^{[0]} = 1\). This finds the fiber of \(\tilde{W}^+\) over \(u^{[0]} = 1\) (the well-rounded retract of \([\Pi]\)), together with the smallest possible \(u^{[1]}\) so that the cell structure does not change for \(u' \in \{(u^{[1]}, u^{[0]}]\). At the critical temperament \(u = u^{[1]}\), the cell structure has changed, and we compute the fiber over it. Next, we take a \(u' < u^{[1]}\) but close to \(u^{[1]}\), and we compute the fiber over \(u'\) together with the interval around \(u'\). If \(u'\) was close enough, then the right-hand endpoint of the interval will indeed be \(u^{[1]}\) (if not, we pick \(u'\) even closer to \(u^{[1]}\) and start over). The interval around \(u'\) where the cell structure does not change is then \((u^{[2]}, u^{[1]}]\). Here \(u^{[2]}\) is the next critical temperament. We continue in this way. We stop when the left-hand endpoint of the interval is
$1/\tau_0^2$. We have generated the points $u^{[0]}, \ldots, u^{[i_r]}$ in right-to-left order, but we renumber them as $u^{(i)}$ in left-to-right order to match (8).

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