RANDOM PROCESSES AND CENTRAL LIMIT THEOREM
IN BESOV SPACES

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\textbf{Abstract.}

We study sufficient conditions for the belonging of random process to certain
Besov space and for the Central Limit Theorem (CLT) in these spaces.

We investigate also the non-asymptotic tail behavior of normed sums of centered
random independent variables (vectors) with values in these spaces.

Main apparatus is the theory of mixed (anisotropic) Lebesgue-Riesz spaces, in
particular so-called permutation inequality.

Key words and phrases: ordinary and generalized Besov spaces, Central Lim-
it Theorem (CLT) in Banach spaces, mixed (anisotropic) Lebesgue-Riesz spaces,
norms, Orlicz spaces, Monte-Carlo method, Prokhorov’s condition, characteristi-
cal functional, Rosenthal constants and inequalities, metric entropy and entropy
integrals, Grand Lebesgue spaces, exponential upper tail estimates, triangle and
Minkowsky inequalities, permutation inequalities, moments.

2000 Mathematics Subject Classification. Primary 37B30, 33K55; Secondary
34A34, 65M20, 42B25.

1 Notations. Statement of problem.

Let $T = \{t\} = [0, 1]$ be ordinary unit closed numerical segment, $f : T \rightarrow R$ be a
numerical measurable function, which may be extended on the whole line $R$ by zero
value: $f(t) := 0$, $t \notin T$. The classical Lebesgue-Riesz norm $|f|_p = |f|_{p,T}$ of such a
function is defined as follows:

$$|f|_p = |f|_{p,T} \overset{def}{=} \left( \int_T |f(t)|^p \, dt \right)^{1/p}, \ p \ge 1; \quad (1.1)$$

$$L_p = L_{p,T} \overset{def}{=} \{ f, \ f : T \rightarrow R, \ |f|_{p,T} < \infty \}. \quad (1.1a)$$

Denote
\[ S_h[f](t) = S[f](t, h) := f(t + h) - f(t), \quad h \in [-1, 1], \quad t \in [0, 1]; \]
and denote also by \( \Delta [f, \delta]_p, \delta \in [0, 1] \) the \( L_p \) module of continuity of the function \( f \) from the space \( L_p \):

\[
\Delta [f, \delta]_p \overset{df}{=} \sup_{|h| \leq \delta} |S_h[f] |_p. \quad (1.2)
\]

It is known that

\[ f \in L_{p,T} \Rightarrow \lim_{\delta \to 0^+} \Delta [f, \delta]_p = 0. \]

We reserve the notation \( \omega (\cdot) \) for the elements of probability space, which should appear further.

Recall that the so-called ordinary Besov space \( B^p_{\alpha,s} \) on the functions defined on considered set \( T = [0, 1] \) may be defined in particular as a space of all measurable function having a finite norm

\[
||f||_{B^p_{\alpha,s}} \overset{df}{=} |f|_p + \left\{ \int_0^1 [\delta^{-\alpha} \Delta [f, \delta]_p]^s \frac{d\delta}{\delta} \right\}^{1/s} = |f|_p + ||f||_{B^{\alpha_p}_{\alpha,s}}, \quad p, s = \text{const}, \geq 1, \quad \alpha = \text{const}, \quad (1.3)
\]

where

\[
||f||_{B^{\alpha_p}_{\alpha,s}} \overset{df}{=} \left\{ \int_0^1 [\delta^{-\alpha} \Delta [f, \delta]_p]^s \frac{d\delta}{\delta} \right\}^{1/s}.
\]

The detail investigation of these spaces may be found in the classical monographs \([4],[5],[6],[58],[71]\); see also articles \([13],[17]\).

It is offered and considered, in particular, in the article \([13]\) the following generalization of these spaces. Namely, denote

\[
\Delta_q [f, \delta]_p \overset{df}{=} \left[ \int_{-\delta}^{\delta} |S_h[f] q_p dh \right]^{1/q} = \left[ \int_{-\delta}^{\delta} |S[f](t, h)| q_p dh \right]^{1/q}, \quad q \geq 1, \quad (1.4)
\]
so that \( \Delta_{\infty} [f, \delta]_p = \Delta [f, \delta]_p \), with evident generalization on the case when \( p = \infty \) or on both the cases \( p = \infty \) and \( q = \infty \).

The so-called generalized Besov space \( B^{p,q}_{\alpha,s} \) on the functions defined on the considered set \( T = [0, 1] \) may be defined in particular as a space of all measurable function having a finite norm

\[
||f||_{B^{p,q}_{\alpha,s}} \overset{df}{=} |f|_p + \left\{ \int_0^1 [\delta^{-\alpha} \Delta_q [f, \delta]_p]^s \frac{d\delta}{\delta} \right\}^{1/s} = |f|_p + ||f||_{B^{\alpha_p,q}_{\alpha,s}}, \quad p, s = \text{const}, \geq 1, \quad \alpha = \text{const}, \quad (1.5)
\]
where the semi-norm \( ||f||_{B^{\alpha_p,q}_{\alpha,s}} \) is defined as follows
Another approach to the definition of (more generalized) Besov’s spaces may be found in articles [21], [36].

Our goal in this article is deducing of sufficient conditions for the belonging of almost all path of random process to certain Besov space and finding the sufficient conditions for the Central Limit Theorem (CLT) in these spaces.

We intend to investigate also the non-asymptotical tail estimated, for instance, exponential decreasing, for the distribution for the norm of normed sums of centered independent random processes with values in these spaces.

The offered here results are formulated in the very simple and natural terms generated only by the source random process: metric distance between its values and so on.

The paper is organized as follows. In the next section we consider the case of generalized Besov spaces. The third section contains the Grand Lebesgue Spaces norm estimation for Besov norm of random processes. The case of ordinary Besov spaces is investigated in fourths section.

The fifth section is devoted to the Central Limit Theorem in generalized Besov spaces. The Central Limit Theorem (CLT) in the classical Besov’s spaces is content of the next section.

We deduce in the seventh section the non-asymptotical estimates, in particular, exponential ones for the norm of normed sums of independent random processes with paths in Besov spaces.

As ordinary, the last section contains the concluding remarks, namely, some remarks about possible generalization on the multivariate case.

In detail: let $\xi = \xi(t) = \xi(t, \omega)$, $t \in T = [0,1]$ be numerical bi-measurable valued random process defined apart from the set $T$ on some probability space $(\Omega = \{\omega\}, B, P)$. Question: under what conditions on the $\xi(t)$ almost all its trajectories belong to certain Besov space

$$\mathbb{P} \left( \xi(\cdot) \in B_{a,s}^{p} \right) = 1$$

or more generally satisfies the Central Limit Theorem (CLT) in these spaces.

This problem for the general separable Banach space $B$ instead the Besov’s space $B_{a,s}^{p}$ can be regarded as a classic, see, e.g. [11], [32], [43]. The case namely of Besov space is considered in the articles [9], [12], [13], [30], [72].

The applications of the CLT in the Banach spaces in the Monte-Carlo method and in statistics is described in the articles [16], [22], [47], [48].

We intend to generalize obtained therein results, for instance, on the case of random processes with exponential decreasing tails of distributions.
Let us to pay attention that the expression for $\Delta_q[f, \delta]_p$ may be written as follows. Denote $T(\delta) = [-\delta, \delta]$, so that $T = T(1)$; then

$$\Delta_q[f, \delta]_p = |S(\cdot, \cdot)|_{T; q, T(\delta)}, \quad (1.7)$$

where the right-hand side of an equality (1.7) is a nothing more than a so-called mixed, or equally anisotropic Lebesgue-Riesz norm, see [3], [5], chapter 2.

Further,

$$||f||_{B^{a,p,q}_{\alpha,s}} = |\Delta_q[f, \delta]_p|_{s, T, \mu}, \quad (1.8)$$

where the measure $\mu$ is follow

$$\mu(\delta) = \delta^{-1-\alpha}d\delta, \quad \delta \in (0, 1). \quad (1.9)$$

Therefore, the value $||f||_{B^{a,p,q}_{\alpha,s}}$ may be represented also through a three dimensional mixed norm.

We recall here for readers convenience the definition and used for us simple properties of the so-called mixed (anisotropic) Lebesgue (Lebesgue-Riesz) spaces, which appeared in the famous article of Benedek A. and Panzone R. [3]. More detail information about this spaces with described applications see in the books of Besov O.V., Ilin V.P., Nikolskii S.M. [5], chapter 1,2; Leoni G. [34], chapter 11; Lieb E., Loss M. [35], chapter 6.

Let $X_k, A_k, \mu_k$, $k = 1, 2, \ldots, l$ be measurable spaces with sigma-finite separable non-trivial measures $\mu_k$. The separability denotes that the metric space $A_k$ relative the distance

$$\rho_k(D_1, D_2) = \mu_k(D_1 \Delta D_2) = \mu_k(D_1 \setminus D_2) + \mu_k(D_2 \setminus D_1)$$

is separable.

Let also $p = (p_1, p_2, \ldots, p_l)$ be $l$-dimensional numerical vector such that $1 \leq p_j < \infty$.

The anisotropic (mixed) Lebesgue-Riesz space $L_\vec{p}$ consists on all the totally measurable real valued function $f = f(x_1, x_2, \ldots, x_l) = f(\vec{x})$:

$$f : \otimes_{k=1}^l X_k \rightarrow R$$

with finite norm $|f|_{\vec{p}} =$

$$|f|_{p_1, p_2, \ldots, p_l} = |f|_{p_1, X_1; p_2, X_2; \ldots, p_l, X_l} = |f|_{p_1, X_1, \mu_1; p_2, X_2, \mu_2; \ldots, p_l, X_l, \mu_l} \overset{def}{=}$$

$$\left(\int_{X_1} \mu_1(dx_1) \left(\int_{X_{l-1}} \mu_{l-1}(dx_{l-1}) \ldots \left(\int_{X_1} |f(\vec{x})|^{p_1} \mu_1(dx_1)\right)^{p_2/p_1}\ldots\right)^{1/p_l}\right)^{1/p_l}. \quad (1.10)$$

In particular, for the r.v. $\xi$

$$|\xi|_p = [\mathbb{E}|\xi|^p]^{1/p}, \quad p \geq 1.$$
Note that in general case $|f|_{p_1,p_2} \neq |f|_{p_2,p_1}$, but $|f|_{p,p} = |f|_p$.

Observe also that if $f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2)$ (condition of factorization), then $|f|_{p_1,p_2} = |g_1|_{p_1} \cdot |g_2|_{p_2}$ (formula of factorization).

Note that under conditions of separability of the measures $\{\mu_k\}$ these spaces are also separable Banach spaces.

These spaces appear in the Theory of Approximation, Functional Analysis, theory of Partial Differential Equations, theory of Random Processes etc.

Let for example $l = 2$; we agree to rewrite for clarity the expression for $|f|_{p_1,p_2}$ as follows:

$$|f|_{p_1,p_2} := |f|_{p_1,X_1; p_2,X_2} = |f|_{p_1,X_1,\mu_1; p_2,X_2,\mu_2}.$$  

Analogously,

$$|f|_{p_1,p_2,p_3} = |f|_{p_1,X_1; p_2,X_2; p_3,X_3} = |f|_{p_1,X_1,\mu_1; p_2,X_2,\mu_2; p_3,X_3,\mu_3}.$$  

Let us give an example. Let $\eta = \eta(x, \omega)$ be bi-measurable random field, $(X = \{x\}, A, \mu)$ be measurable space, $p = \text{const} \in [1, \infty)$. As long as the expectation $\mathbf{E}$ is also an integral, we deduce

$$\mathbf{E}|\eta(\cdot, \cdot)|^p_{p,X} = \mathbf{E} \int_X |\eta(x, \cdot)|^p \mu(dx) = \int_X \mathbf{E}|\eta(x, \cdot)|^p \mu(dx) \left[ \int_\Omega |\eta(x, \omega)|^p \mathbf{P}(d\omega) \right];$$

$$[\mathbf{E}|\eta|^m_{p,X}]^{1/m} = \left[ \left\{ \int_X |\eta(x)|^p \mu(dx) \right\}^{m/p} \right]^{1/m} = \int_\Omega \mathbf{P}(d\omega) \left\{ \int_X |\eta(x)|^p \mu(dx) \right\}^{m/p}^{1/m} = |\eta(\cdot, \cdot)|_{p,X; m,\Omega}^m, \quad p,m \geq 1. \quad (1.11)$$

We will use also the so-called permutation inequality in the terminology of an article [1]; see also [5], chapter 1, p. 24-26. Indeed, let $(Z, B, \mu)$ be another measurable space and $\phi : (\tilde{X}, Z) = \tilde{X} \otimes Z \rightarrow R$ be measurable function. In what follows $\tilde{X} = \otimes_{k=1}^K X_k$. Let also

$$r = \text{const} \geq \tilde{p} \overset{\text{def}}{=} \max_j p_j.$$  

It is true the following inequality (in our notations):

$$|\phi|_{p,\tilde{X},r,Z} \leq |\phi|_{r,Z; \tilde{p},\tilde{X}}. \quad (1.12)$$

We put in what follows $Z = \Omega$, $\mu = \mathbf{P}$.
2 The case of generalized Besov spaces.

Let us introduce some new notations.

\[ V[f](t, z, \delta) := S[f](t, z \cdot \delta), \quad t, z, \delta \in T. \]  

Further, let \( m = \text{const} \geq 1 \); the Pisier’s natural distance \( d_m(t, s), \ t, s \in T \) on the set \( T \) of order \( m \), \( m = \text{const} \geq 1 \) generated by our random process \( \xi(t) \) is defined as follows

\[ d_m(t, s) \overset{\text{def}}{=} |\xi(s) - \xi(t)|_{m, \Omega, P} = [\mathbb{E}|\xi(s) - \xi(t)|^m]^{1/m}, \]  

if there exists (for given value \( m \)) and is finite, see [60], [61].

Denote also

\[ \sigma_m(h) = \sup_{t \in T} d_m(t + h, t) = \sup_{t \in T} [\mathbb{E}|\xi(t + h) - \xi(t)|^m]^{1/m}. \]  

Introduce a new measure on at the same set \( T = [0, 1] \)

\[ \nu(A) = \nu_{\alpha, q, s}(A) = \int_A \delta^\gamma \ d\delta, \]  

where

\[ \gamma = \gamma(\alpha, q, s) = \frac{s}{q} - \alpha s - 1. \]  

**Theorem 2.1.** Suppose that for some value \( m \geq \max(p, q, s) \)

\[ |V[\xi]|_{m, \Omega, P; p, T; q, T; s, T, \nu} < \infty. \]  

Then

\[ \mathbb{P} \left( \xi(\cdot) \in B^{\alpha, p, q}_{\alpha, s} \right) = 1 \]  

and moreover

\[ |\|\xi\|_{B^{\alpha, p, q}_{\alpha, s}}|_{m, \Omega, P} \leq |V[\xi]|_{m, \Omega, P; p, T; q, T; s, T, \nu}. \]  

**Proof.** The expression for \( \Delta_q[\xi, \delta]_p \) may be rewritten as follows

\[ \Delta_q[\xi, \delta]_p = \left[ \int_{-\delta}^\delta |S[\xi](t, h)|^q_{p, T} \ dh \right]^{1/q} = \delta^{1/q} \left[ \int_{-1}^1 |S[\xi](t, \delta z)|^q_{p} \ dz \right]^{1/q} = \delta^{1/q} \left[ \int_{-1}^1 |V[\xi](t, \delta z)|^q_{p} \ dz \right]^{1/q} = \delta^{1/q} |V(t, \cdot, \cdot)|_{p, T; q, T}. \]
Therefore
\[ \| \xi \| B^{o,p,q}_{\alpha,s} = |V[\xi](\cdot, \cdot, \cdot)|_{p,T; q,T; s,\nu,T} \]
and correspondingly
\[ \| \xi \| B^{o,p,q}_{\alpha,s} \big|_{m,\Omega,P} = |V[\xi](\cdot, \cdot, \cdot)|_{p,T; q,T; s,\nu,T; m,\Omega,P}. \] (2.8)

Since \( m \geq \max(p, q, s) \), we can use the permutation inequality (1.12):
\[ \| \xi \| B^{o,p,q}_{\alpha,s} \big|_{m,\Omega,P} \leq |V[\xi](\cdot, \cdot, \cdot)|_{m,\Omega,P; p,T; q,T; s,\nu,T} < \infty. \] (2.9)
Thus,
\[ \xi(\cdot) \in B^{o,p,q}_{\alpha,s} \] (2.10)
with probability one, Q.E.D.

**Remark 2.1. Simplification.**

We intend to convince the reader on the simplicity of conditions of theorem 2.1. Namely, the expression for the right-hand side of (2.6) and (2.7a) may be rewritten as follows.
\[ |V[\xi]|_{m,\Omega,P; p,T; q,T; s,\nu,T} = |d_m(t + z\delta, t)|_{p,T; q,T; s,\nu,T}. \] (2.11)
Note that the Pisier’s natural distance \( d_m(\cdot, \cdot) \) from (2.2) and therefore in (2.11) may be relatively easy calculated or estimated in the majority of practical cases.

**Remark 2.2.**

It is reasonable to choose as a capacity of the value \( m \) in the theorem 2.1 its minimal value
\[ m := m_0 \overset{\text{def}}{=} \max(p, q, s). \]
Correspondingly
\[ \left( \| \xi \| B^{o,p,q}_{\alpha,s} \right)_{m_0,\Omega,P} \leq |V[\xi]|_{m_0,\Omega,P; p,T; q,T; s,\nu,T}. \] (2.12)

But for exponential non-asymptotical estimations for the tail of distribution of the (random) norm
\[ \tau := \| \xi \| B^{o,p,q}_{\alpha,s} \]
the estimation (2.7a) is very convenient.

**Remark 2.3.**

If in addition to the conditions of theorem 2.1 for some value \( k \geq p \), for instance for the value \( k = p \)
\[ |\xi(\cdot, \cdot)|_{k, P, \Omega; p, X} < \infty, \quad (2.13) \]

then

\[ \xi(\cdot) \in B_{\alpha, s}^{p, q} \quad (2.14) \]

with probability one and moreover

\[
\left| |\xi| B_{\alpha, s}^{p, q, \min(m, k), \Omega, P} \right| \leq V[\xi] \left| m, \Omega, \mathbb{P}; p, T; q, T; s, T, \nu \right| + |\xi(\cdot, \cdot)|_{k, P, \Omega; p, T}. \quad (2.14a)
\]

Indeed, it follows from the condition (2.13) that \( \xi(\cdot) \in L_p(T) \) a.e., see e.g. [49].

Let us choose for simplicity in (2.14a) \( k = m \geq \max(p, q, s) \), then we obtain for arbitrary such a values \( m \)

\[
\left( |\xi| B_{\alpha, s}^{p, q, m, \Omega, P} \right) \leq V[\xi] \left| m, \Omega, \mathbb{P}; p, T; q, T; s, T, \nu \right| + |\xi(\cdot, \cdot)|_{m, P, \Omega; p, T}. \quad (2.14b)
\]
Remark 2.4.

An example. Suppose in addition to the conditions of theorem 2.1

\[ d_m(t + h, t) \leq C_m h^\beta, \quad \beta = \beta(m) = \text{const} \in (0, 1]. \]

If

\[ \beta + 1/q > \alpha, \quad m \geq \max(p, q, s), \]

then

\[ \xi(\cdot) \in B_{\alpha,s}^{p,q} \]

with probability one and moreover

\[ \|\xi\|_{B_{\alpha,s}^{p,q}} \leq C_m s^{-1/s} (\beta - \alpha + 1/q)^{-1/s}. \quad (2.15) \]

Remark 2.5.

The case when \( p = q = s = \infty \) correspondent to the so-called Hölder space, see, e.g. [63] - [66].

3 Grand Lebesgue Spaces norm estimation for Besov norm of random processes.

We recall in the beginning of this section briefly the definition and some simple properties of the so-called Grand Lebesgue spaces; more detail investigation of these spaces see in [15], [26], [29], [33], [43], [44]; see also reference therein.

Recently appear the so-called Grand Lebesgue Spaces \( GLS = G(\psi) = G\psi = G(\psi; B), \quad 1 < B \leq \infty \), spaces consisting on all the random variables (measurable functions) \( f : \Omega \rightarrow R \) with finite norms

\[ \|f\|_{G(\psi)} \overset{def}{=} \sup_{p \in (1, B)} |f|_p / \psi(p). \quad (3.1) \]

Here \( \psi(\cdot) \) is some continuous positive on the open interval \((1, B)\) function such that

\[ \inf_{p \in (1, B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p > B. \]

We will denote

\[ \text{supp}(\psi) \overset{def}{=} (1, B) = \{ p : \psi(p) < \infty \}. \]

The set of all \( \psi \) functions with support \( \text{supp}(\psi) = (1, B) \) will be denoted by \( \Psi(B) = \Psi(1, B) \).

These spaces are complete Banach spaces and moreover rearrangement invariant, see [4], and are used, for example, in the theory of probability [29], [43], [44]; theory
of Partial Differential Equations [15], [26]; Functional Analysis [15], [26], [33], [44];
theory of Fourier series, theory of martingales, mathematical statistics, theory of
approximation etc.

Notice that in the case when $\psi(\cdot) \in \Psi(\infty)$ and a function $p \to p \cdot \log \psi(p)$ is
convex, then the correspondent space $G\psi(\infty)$ coincides with some exponential Orlicz
space.

Conversely, if $B < \infty$, then the space $G\psi(B)$ does not coincides with the classical
rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

Suppose $\|f\|_{G\psi} \in (0, \infty)$; then

$$T_f(u) \overset{def}{=} \max(\mathbf{P}(f > u), \mathbf{P}(f < -u)) \leq \exp \left[ -\tilde{\psi}^*(\ln u) \right], \; u > e.$$ 

where $\tilde{\psi}(p) = p \cdot \ln \psi(p)$, and

$$g^*(y) = \sup_{p} (p|y| - g(p))$$

is Young-Fenchel, or Legendre transform of the function $g = g(p)$.

The last relations implies that in the case when $\|f\|_{G\psi} \in (0, \infty)$ the function $f(\cdot)$ obeys the exponential decreasing tail of distribution or equally belongs to some exponential Orlicz’s space.

In detail: this $G\psi$ space coincides with the exponential Orlicz’s space relative
the Young function

$$N(z) = \exp \left( \tilde{\psi}^*(\ln |z|) \right) - 1, \; |z| > e,$$

see [29].

Denote for instance

$$\psi_l(p) = p^{1/l}, \; l = \text{const} > 0.$$ 

The case $\psi(p) = \psi_2(p) = \sqrt{p} = p^{1/2}$ correspondent to the so-called subgaussian
space

$$\text{Sub(}\Omega) \overset{def}{=} G\psi_2.$$ 

It is known that there exists absolute constants $C_1, C_2 \in (0, \infty)$ such that any centered (mean zero) r.v. $\eta$ belongs to this space $\text{Sub(}\Omega)$ with positive finite norm $\|\eta\|_{G\psi_2} = \sigma^2 \in (0, \infty)$ iff

$$\forall \lambda \in \mathbb{R} \Rightarrow \mathbf{E} \exp(\lambda \eta) \leq \exp(0.5 \; C_1^2 \; \sigma^2 \; \lambda^2) \quad (3.2)$$

or equally

$$T_f(u) \leq \exp(-C_2 u^2), \; u \geq 0. \quad (3.2a)$$

In the more general case, i.e. when $\|f\|_{G\psi_l} = 1$, then

$$T_f(u) \leq \exp \left( -C(l) \; u^l \right), \; u \geq 0. \quad (3.2b)$$
Thus, the theory of Grand Lebesgue spaces allows us to obtain in particular the exponential estimates for tails of distributions for random variables, or equally estimate the norm of functions in exponential Orlicz spaces.

**Remark 3.1** If we introduce the discontinuous function

\[ \psi(r)(p) = 1, \ p = r; \quad \psi(r)(p) = \infty, \ p \neq r, \ p, r \in (1, B) \]

and define formally \( C/\infty = 0, \ C = \text{const} \in \mathbb{R}^1 \), then the norm in the space \( G(\psi_r) \) coincides with the \( L_r \) norm:

\[ ||f||G(\psi_r) = |f|_r. \]

Therefore, the Grand Lebesgue Spaces are the direct generalization not only of the exponential Orlicz’s spaces, but also the classical Lebesgue-Riesz spaces \( L_r \).

Let us return to the formulated above problem. We will apply the mentioned in the second section one-dimensional degree of freedom \( m \geq \max(p, q, s) \).

Denote

\[ \nu(m) = |V[\xi]|_{m, \Omega, P; p, T; q, T; s, T, \nu} + |\xi(\cdot, \cdot)|_{m, \Omega, P; p, T}. \quad (3.3) \]

**Proposition 3.1.** Suppose in addition to the conditions of theorem 2.1 that for some value \( B > \max(p, q, s) \Rightarrow \nu(B) < \infty \). It follows immediately from the assertion of theorem 2.1 (2.14a) that

\[ \left( ||\xi||B^{p,q}_{\alpha,s} \right) G\nu \leq 1. \quad (3.4) \]

As a consequence:

\[ T_{||\xi||B^{p,q}_{\alpha,s}}(u) \leq \exp \left[ -\nu^*(\ln u) \right], \ u > e. \quad (3.4a) \]

Let us consider some examples.

**Example 3.1.** Suppose under the conditions (and notations) of the remark (2.4) \( C_m \leq m^{1/l} \), \( l = \text{const} > 0 \). It follows from the inequality (2.15)

\[ \left\| \left\| \xi \right\|B^{p,q}_{\alpha,s} \right\| G\psi_l \leq s^{-1/s} (\beta - \alpha + 1/q)^{-1/s}, \ \beta > \alpha - 1/q. \quad (3.5) \]

In the case when in addition \( \xi(t) \) is centered Gaussian process, for instance, is ordinary Brownian motion: \( \xi(t) = w(t) \), we conclude \( l = 2 \) and following

\[ \left\| \left\| \xi \right\|B^{p,q}_{\alpha,s} \right\| \text{Sub}(\Omega) \leq s^{-1/s} (\beta - \alpha + 1/q)^{-1/s}. \quad (3.5a) \]

Evidently, for the Brownian motion \( \beta = 1/2 \).

**Example 3.2.** It may happen that in the inequality (2.15) \( C_m = \text{const} = s^{1/s} \) (for simplicity) and for the values \( m \geq m_o := \max(p, q, s) \)

\[ \beta = \beta(m) = \alpha - \frac{1}{q} + m^{-\gamma}, \ \gamma = \text{const} > 0. \quad (3.6) \]

We get substituting into (2.15)
at that the case of the values \( m \in [1, m_0] \) may be considered by means of Lyapunov’s inequality.

The estimation (3.7) may be rewritten as follows.

\[
\left\| \frac{\delta m}{\delta \Omega} \right\|_{B_{m,s}^{\omega,p,q}} \leq m^{\gamma/s}
\]

and thus

\[
T_{\|\xi\|_{B_{m,s}^{\omega,p,q}}} \leq \exp \left( -C(s/\gamma) u^{s/\gamma} \right), \ C(s/\gamma) \in (0, \infty), \ u > 0.
\]

**Example 3.3.** Assume that for some constants \( C \in (0, \infty), \ \theta \geq 2 \)

\[
d_m(t, t + h) \leq C h^{\theta - m}, \ \theta - 1 \leq m < \theta.
\]

Denote

\[
\tilde{\theta} = \theta - \alpha + \frac{1}{q}
\]

and let still \( \tilde{\theta} > 1 \).

We observe \[
\left\| \frac{\delta m}{\delta \Omega} \right\|_{B_{m,s}^{\omega,p,q}} \leq \infty, \ \psi(\theta) < \infty, \ \text{where}
\]

\[
\psi(\theta)(m) = (\tilde{\theta} - m)^{-1/s}, \ 1 \leq m < \tilde{\theta},
\]

therefore

\[
T_{\|\xi\|_{B_{m,s}^{\omega,p,q}}} \leq C_{1}(s, p, q, \theta, \alpha) u^{-\tilde{\theta}} (\ln u)^{\tilde{\theta}/s}, \ u > e^2.
\]

We used some estimations from an article [50].

### 4 The case of ordinary Besov spaces.

This case is more complicated. Note first of all

\[
\Delta[f]_p \overset{def}{=} \sup_{|h| \leq \delta} |S_h[f]|_{p,T} =
\]

\[
\sup_{|z| \leq 1} |\xi(t + z\delta) - \xi(t)|_{p,T} = \sup_{|z| \leq 1} \theta(z),
\]

where

\[
\theta(z) = \theta(z, \delta) = |\xi(t + z\delta) - \xi(t)|_{p,T},
\]

meaning that \( t \in T = [0, 1] \).

Let as before \( m \geq \max(p, q, s) \) and denote
\[
\mu(\delta) = \mu_m(\delta) := \sup_{|z| \leq 1} |\theta(z, \delta)|_{m, \Omega, P; p, T}; \quad (4.2)
\]

and introduce the following distance (more exactly, semi-distance) on the set \([-1, 1] : \]
\[
\rho(z_1, z_2) = \rho_m(z_1, z_2) := \sup_{\delta \in (0, 1)} \left[ \left| \frac{\theta(z_1, \delta) - \theta(z_2, \delta)}{\mu_m(\delta)} \right| \right]. \quad (4.3)
\]

The finiteness of this distance for certain segment \(m \in [\max(p, q, s), B], B = \text{const} \in (\max(p, q, s), \infty) \) will be presumed.

We intend to apply the Orlicz’s spaces norm tail estimates for the distribution of maximum (supremum) of random fields, based on the so-called entropy technique, see e.g. [14], [11], [32], [29], [43], chapter 3, sections 3.4 - 3.6, [61] etc.

Some preliminary notations. Denote by \(H(T, d, \epsilon) = H(d, \epsilon)\) the metric entropy of the set \(T = [0, 1]\) at the point \(\epsilon > 0\) relative certain distance function \(d = d(z_1, z_2)\) \(z_1, z_2 \in T\), i.e. the natural logarithm of the minimal number of the closed balls with radii \(\epsilon, \epsilon > 0\) in the distance \(d(\cdot, \cdot)\) which cover all the set \(T\). Put also
\[
N(T, d, \epsilon) = N(d, \epsilon) = \exp H(T, d, \epsilon) = \exp H(d, \epsilon);
\]
and denote for brevity
\[
N_m(\epsilon) := N(T, \rho_m, \epsilon); \quad H_m(\epsilon) := \ln N(T, \rho_m, \epsilon). \quad (4.4)
\]

Introduce following G.Pisier [60], [61]; see also [43], chapter 3, section 3.17 the variables
\[
V(m) \quad \text{def} = 9 \int_0^{D_m} N_m^{1/m}(\epsilon) \, d\epsilon, \quad (4.5)
\]
where
\[
D_m = \text{diam}(T, \rho_m) := \sup_{z_1, z_2 \in T} \rho_m(z_1, z_2) \leq 2. \quad (4.5a)
\]

Define the following \(\psi -\) function \(\beta = \beta(m) : \]
\[
\beta(m) := V(m) \cdot |\mu_m(\cdot)|_{s, T, \nu}, \quad \text{it is meaning } \delta \in T, \text{ and suppose the finiteness of these function for at last one value } m > \max(p, q, s) : \]
\[
\exists L > \max(p, q, s) \Rightarrow \beta(L) < \infty. \quad (4.6)
\]

**Theorem 4.1.** Let the condition (4.6) be satisfied. Then almost all the trajectories of the random process \(\xi(t)\) belong to the space \(B_{\alpha, s}^{p, q}\) and herewith
\[
||| \xi |||_{B_{\alpha, s}^{p, q}} \cdot G\beta \leq 1, \quad (4.7)
\]
with correspondent tail estimation.

**Proof.** Let \(m \in (\max(p, q, s), L)\). Let us consider the normed random process (fields)
\[
\zeta_m(z, \delta) := \frac{\theta(z, \delta)}{\mu_m(\delta)}.
\]

We observe using again the permutation inequality
\[
| \zeta_m(z, \delta) |_{m, \Omega, \mathbf{P}} \leq 1,
\]
\[
| \zeta_m(z_1, \delta) - \zeta_m(z_2, \delta) |_{m, \Omega, \mathbf{P}} \leq \rho_m(z_1, z_2).
\]
Since the so-called entropy integral (4.5) convergent, one can apply the main result of the article [61]; see also [43], chapter 3, section 3.17:
\[
\sup_z | \zeta_m(z, \delta) |_{m, \Omega, \mathbf{P}} \leq V(m),
\]
or equally
\[
\sup_z | \theta(z, \delta) |_{m, \Omega, \mathbf{P}} \leq V(m) \cdot \mu_m(\delta),
\]
which is equivalent to the estimate (4.7).

Example 4.1. Let in (4.7) \(m\) be a fixed number \(m = r \in (\max(p, q, s), L)\). As long as the ordinary Lebesgue spaces are the particular case of Grand Lebesgue spaces, we propose from the assertion and conditions of theorem 4.1 the following \(L_r(\Omega)\) estimation
\[
|\|\| \xi \|\|_{B^{o,p}_{\alpha,s}} \leq 1.
\]

Example 4.2. Let in addition to the conditions of theorem 4.1 in (4.7) \(L = \infty\) and \(\beta(m) \leq m^{1/l}, \ l = \text{const} > 0\). Then
\[
|\|\| \xi \|\|_{B^{o,p}_{\alpha,s}} \leq G \psi \leq 1,
\]
with correspondent exponential tail estimation
\[
T|\|\|_{B^{o,p}_{\alpha,s}}(u) \leq e^{-C(\psi)u}, \ u \geq 1.
\]

We intend now to offer some important generalization of theorem 4.1. Namely, let \(\lambda_m = \lambda_m(\delta), \ \delta \in [0, 1]\) be some family of non-negative continuous functions such that
\[
\lim_{\delta \to 0+} \frac{\mu_m(\delta)}{\lambda_m(\delta)} = 0, \ \mu_m(\delta) \leq \lambda_m(\delta), \ \lambda_m(0+) = \lambda_m(0) = 0.
\]

For instance, \(\lambda_m\) may be the \(L_m(\Omega)\) deterministic component of a factorable module of continuity for the r.p. \(\xi(t)\), see in detail a preprint [51].

Let \(m \in (\max(p, q, s), L)\), and let us consider the normalized random process
\[
\tau_m(z, \delta) := \frac{\theta(z, \delta)}{\lambda_m(\delta)}.
\]
Introduce an another (bounded) distance on the set $T$:

$$r(z_1, z_2) = r_m(z_1, z_2) := \sup_{\delta \in (0, 1)} \left[ \frac{|\theta(z_1, \delta) - \theta(z_2, \delta)|_{m, \Omega; p, T}}{\lambda_m(\delta)} \right]. \tag{4.12}$$

Introduce also again following G. Pisier [60], [61] the variables

$$V_r(m) = V_{r, \tau}(m) \overset{def}{=} 9 \int_0^{D_{m,r}} N^{1/m} (T, r, \epsilon) \, d\epsilon, \tag{4.12a}$$

$$D_{m,r} = \text{diam}(T, \rho_m) := \sup_{z_1, z_2 \in T} r_m(z_1, z_2) \leq 2. \tag{4.12a}$$

Define the following new $\psi$ - function $\beta_r = \beta_r(m)$:

$$\beta_r(m) := V_r(m) \cdot |\lambda_m(\cdot)|_{s, T, \nu},$$

it is meaning $\delta \in T$, and suppose the finiteness of these function for at last one value $m > \max(p, q, s)$:

$$\exists L_r > \max(p, q, s) \Rightarrow \beta_r(L_r) < \infty. \tag{4.13}$$

We observe analogously to the proof of theorem 4.1

**Theorem 4.1a.** Let the condition (4.13) be satisfied. Then almost all the trajectories of the random process $\xi(t)$ belong to the space $B_{\alpha, s}^{o,p}$ and wherein

$$\left\| \left\| \xi \right\|_{B_{\alpha, s}^{o,p}} \right\| G\beta_r \leq 1, \tag{4.14}$$

with correspondent tail estimation.

**Example 4.3.** Let $\xi(t) = w(t)$, $0 \leq t \leq 1$ be the ordinary Brownian motion (Wiener’s process). It is well known that one can take

$$\lambda_m(\delta) = \sqrt{m} \cdot \sqrt{\delta \cdot |\ln \delta|}, \ 0 \leq \delta \leq 1/e. \tag{4.15}$$

and $L_r = \infty$.

We get from the proposition (4.14) of theorem 4.1a that if $\alpha < (2s)^{-1}$, then almost everywhere

$$w(\cdot) \in B_{\alpha, s}^{o,p}$$

and moreover

$$P \left( \left\| w(\cdot) \right\|_{B_{\alpha, s}^{o,p}} > u \right) \leq \exp \left( -C(\alpha, p, s) \cdot u^2 \right), \ u \geq 1. \tag{4.16}$$
5 Central limit theorem in generalized Besov spaces.

1. Let \((B, || \cdot ||_B)\) be certain separable Banach space built on the real valued functions defined on our set \(T\) and \(\{\xi_j\} = \{\xi_j(t)\}, \ t \in T, \ \xi_1(t) = \xi(t), \ j = 1, 2, \ldots\) be a sequence of centered in the weak sense: \(E(\xi_i, b) = 0 \ \forall b \in B^*\) or equally \(E\xi_j(t) = 0, \ t \in T\) independent identical distributed (i.; i.d.) random variables (r.v.) (or equally random vectors, with at the same abbreviation r.v.) defined on some non-trivial probability space \((\Omega = \{\omega\}, F, P)\) with values in the space \(B\). Denote

\[ S_n = S_n(t) = S(n) = S(n, t) = n^{-1/2} \sum_{j=1}^{n} \xi_j(t), \ n = 1, 2, \ldots. \]

If we suppose that the r.v. \(\xi\) has a weak second moment:

\[ \forall b \in B^* \Rightarrow (Rb, b) := E(\xi, b)^2 < \infty, \]

then the characteristic functional (more exactly, the sequence of characteristic functionals)

\[ \phi_{S(n)}(b) := E e^{i \langle S(n), b \rangle} \]

of \(S(n)\) converges as \(n \to \infty\) to the characteristic functional of (weak, in general case) Gaussian r.v. \(S = S(\infty)\) with parameters \((0, R)\):

\[ \lim_{n \to \infty} \phi_{S(n)}(b) = e^{-0.5 \langle Rb, b \rangle}. \]

Symbolically: \(S \sim N(0, R)\) or \(\text{Law}(S) = N(0, R)\). The operator \(R = RS\) is called the covariation operator, or variance of the r.v. \(S:\)

\[ R = \text{Var}(S); \]

note that \(R = \text{Var}(\xi)\).

We recall the classical definition of the CLT in the space \(B\).

(We will investigate in the sequel the case when the space \(B\) is our Besov’s space \(B = B^{\alpha,p,q}_{\alpha,s}.\))

**Definition 5.1.** We will say as ordinary that the mean zero r.p. \(\xi(t)\) or equally the sequence \(\{\xi_j\}, \ \xi(t) = \xi_1(t)\) satisfies the CLT in the space \(B\), write: \(\{\xi_j\} \in \text{CLT} = \text{CLT}(B)\) or simple: \(\xi \in \text{CLT}(B)\), if the limiting Gaussian r.v. \(S\) belongs to the space \(B\) with probability one: \(P(S \in B) = 1\) and the sequence of distributions \(\text{Law}(S(n))\) converges weakly as \(n \to \infty\), i.e. in the Prokhorov - Skorokhod sense, to the distribution of the r.v. \(S = S(\infty)\):

\[ \lim_{n \to \infty} \text{Law}(S(n)) = \text{Law}(S). \] (5.1)

The equality (5.1) implies that for any continuous functional \(F : B \to R\)

\[ \lim_{n \to \infty} P(F(S(n)) < x) = P(F(S) < x) \] (5.2)
for all positive values \( x \).

In particular,

\[
\lim_{n \to \infty} P(||S(n)||B < x) = P(||S||B < x), \ x > 0.
\]

2. The problem of describing of necessary (sufficient) conditions for the infinite-dimensional CLT in Banach space \( B \) has a long history; see, for instance, the monographs [2], [11], [20], [32], [43] and articles [19], [18], [73]; see also reference therein.

The applications of considered theorem in statistics and method Monte-Carlo see, e.g. in [16], [48], [52], [53].

3. The cornerstone of this problem is to establish the weak compactness of the distributions generated in the space \( B \) by the sequence \( \{S(n)\} \):

\[
u_n(D) = P(S(n) \in D),
\]

where \( D \) is Borelian set in \( B \); see [62]; [7], [8].

4. We will apply the famous Rosenthal’s constants and inequality, see the classical work of H.P.Rosenthal [67]; see also [25], [28], [56], [59] etc.

Let \( p = \text{const} \geq 1 \), \( \{\zeta_k\} \) be a sequence of numerical centered, i.; i.d. r.v. with finite \( p \)-th moment \( |\zeta|_p < \infty \). The following constants, more precisely, functions on \( p \), are called constants of Rosenthal-Dharmadhikari-Jogdeo-Johnson-Schechtman-Zinn-Latala-Ibragimov-Pinelis-Sharachmedov-Talagrand-Utev...:

\[
K_R(p) \overset{\text{def}}{=} \sup_{n \geq 1} \sup_{\{\zeta_k\}} \left[ \frac{n^{-1/2} \sum_{k=1}^{n} \zeta_k}{|\zeta_1|_p} \right].
\]

We will use the following ultimate up to an error value \( 0.5 \cdot 10^{-5} \) estimate for \( K_R(p) \), see [56] and reference therein:

\[
K_R(p) \leq \frac{C_R}{\epsilon \cdot \log p}, \quad C_R = \text{const} := 1.77638.
\]

Note that for the symmetrical distributed r.v. \( \zeta_k \) the constant \( C_R \) may be reduced up to a value 1.53572 and that both the boundaries are exact.

5. We retain here all the notations (and conditions) of the second section, for instance, the Pisier’s notations [60], [61]

\[
V(m) \overset{\text{def}}{=} 9 \int_0^{P_m} N_m^{1/m}(\epsilon) \ d\epsilon, \quad (5.3)
\]

\[
D_m = \text{diam}(T, \rho_m) := \sup_{z_1, z_2 \in T} \rho_m(z_1, z_2) \leq 2.
\]

\[
\beta(m) := V(m) \cdot |\mu_m(\cdot)|_{s,T,\nu}, \quad (5.4)
\]

and add some news:

\[
\tilde{\beta}(m) := K_R(m) \cdot V(m) \cdot |\mu_m(\cdot)|_{s,T,\nu}, \quad (5.4a)
\]
it is meaning \( \delta \in T \), and suppose the finiteness of these function for at last one value \( m > m' \overset{\text{def}}{=} \max(2, p, q, s) \):

\[
\exists L' > \max(2, p, q, s) \Rightarrow \tilde{\beta}(L') < \infty. \tag{5.5}
\]

6. **Theorem 5.1.** Suppose that for some value \( m \geq \max(2, p, q, s) \)

\[
| V[\xi] |_{m, \Omega, P; p, T; q, T; s, T, \nu} < \infty. \tag{5.6}
\]

Then

\[
\xi(\cdot) \in CLT \left[ B_{\alpha, s}^{p, q} \right] \tag{5.7}
\]

and moreover

\[
\sup_n \left\| S_n \right\| B_{\alpha, s}^{p, q} \bigg|_{m, \Omega, P} \leq \tilde{\beta}(m) \tag{5.8}
\]

or equally

\[
\sup_n \left\| S_n \right\| B_{\alpha, s}^{p, q} \bigg| \left[ 1 + \beta \right] \leq 1. \tag{5.8a}
\]

7. **Proof.**

We deduce using Rosenthal’s inequality

\[
\overline{d}_m(t, s) \leq K_R(m) \cdot d_m(t, s),
\]

since here \( m \geq 2 \).

We can apply the proposition (2.7a) of theorem 2.1

\[
\sup_n \left\| S_n(\cdot) \right\| B_{\alpha, s}^{p, q} \bigg|_{m, \Omega, P} \leq K_R(m) \cdot |d_m(t + z\delta, t)|_{p, T; q, T; s, T, \nu} < \infty.
\]

The right-hand of the last estimate meaning that \( t \in T, z \in T, \delta \in T \).

8. As long as the Banach space \( B_{\alpha, s}^{p, q} \) is separable and the function \( y \to |y|^m, m \geq 1 \) satisfies the well known \( \Delta_2 \) condition, there exists a linear compact operator \( U : B_{\alpha, s}^{p, q} \to B_{\alpha, s}^{p, q} \), which dependent only on the distribution \( \text{Law}(\xi), \xi = \xi_1 \), such that

\[
P \left( U^{-1} \xi \in B_{\alpha, s}^{p, q} \right) = 1 \tag{5.9}
\]

and moreover

\[
E\|U^{-1}\xi\|^m B_{\alpha, s}^{p, q} < \infty, \tag{5.10}
\]

[44]; see also [10], [55].

9. Let us consider the sequence of the r.v. in the space \( B_{\alpha, s}^{p, q} \) : \( \eta_k(x) = U^{-1}[\xi_k](x) \); it is also a sequence of i., i.d. r.v. in the space \( B_{\alpha, s}^{p, q} \), and we can apply the inequality (5.8) taking the value \( m \) :
\[
\sup_n \mathbb{E}[|U^{-1}[S_n]|^m B^{\alpha,p,q}_{\alpha,s}] \leq K^m R(p) \mathbb{E}[|U^{-1}[\xi]|^m B^{\alpha,p,q}_{\alpha,s}] = C_m(p) < \infty. \tag{5.11}
\]

We get using Tchebychev’s inequality
\[
\sup_n \mathbb{P}(||U^{-1}[S_n]|B^{\alpha,p,q}_{\alpha,s} > Z) \leq C_m(p)/Z^m < \epsilon, \tag{5.12}
\]
for sufficiently greatest values \( Z = Z(\epsilon), \epsilon \in (0,1) \).

Denote by \( W = W(Z) \) the set
\[
W = \{ f : f \in B^{\alpha,p,q}_{\alpha,s}, ||U^{-1}[f]|B^{\alpha,p,q}_{\alpha,s} \leq Z \}. \tag{5.13}
\]
Since the operator \( U \) is compact, the set \( W = W(Z) \) is compact set in the space \( B^{\alpha,p,q}_{\alpha,s} \). It follows from the inequality (5.12) that
\[
\sup_n \mathbb{P}(S(n) \notin W(Z)) \leq \epsilon.
\]

Thus, the sequence \( \{S_n\} \) satisfies the famous Prokhorov’s criterion [62] for weak compactness of the family of distributions in the separable metric spaces.

This completes the proof of theorem 5.1.

6 Central limit theorem in the classical Besov spaces.

Denote
\[
S_n(t) = n^{-1/2} \sum_{i=1}^{n} \xi_i(t), \quad t \in T = [0,1],
\]
where the random processes \( \xi_i(t) \) are independent copies of the centered (mean zero) r.p. \( \xi(t) \) belonging to the space \( B^{\alpha,p}_{\alpha,s} \) with probability one. For instance, \( \xi(\cdot) \) may satisfy the conditions of theorem 4.1. However, the conditions of offered further theorem 6.1 ”absorb” ones in theorem 4.1.

Define as above
\[
\theta_n(z) = \theta_n(z,\delta) = |S_n(t + z\delta) - S_n(t)|_{p,T}, \tag{6.0}
\]
meaning that \( t \in T = [0,1] \).

We assume in the sequel
\[
m \geq \max(2, p, q, s) \tag{6.1}
\]
and denote
\[
\bar{\mu}(\delta) = \bar{\mu}_m(\delta) := \sup_{|z| \leq 1} |\theta_n(z,\delta)|_{m,\Omega,p; p,T}; \tag{6.2}
\]
It follows from the Rosenthal’s inequality

\[
\]
\[ \overline{p}_m(\delta) \leq K_R(m) \cdot \mu_m(\delta). \]  

(6.3)

Introduce also the following distance (more exactly, semi-distance) on the set 

\[ \overline{p}(z_1, z_2) = \overline{p}_m(z_1, z_2) := \sup_n \sup_{\delta \in (0,1)} \left[ \frac{\theta_n(z_1, \delta) - \theta_n(z_2, \delta)}{\overline{p}_m(\delta)} \right]. \]  

(6.4)

Evidently,

\[ \overline{p}_m(z_1, z_2) \leq K_R(m) \cdot \rho_m(z_1, z_2). \]  

(6.5)

The finiteness of the distance \( \rho_m(z_1, z_2) \) for certain segment \( m \in [\max(2, p, q, s), B) \), \( B = \text{const} \in (\max(2, p, q, s), \infty) \) will be presumed.

Introduce again following G. Pisier [60], [61]; see also [43], chapter 3, section 3.17 the variables

\[ V(m) \overset{\text{def}}{=} 9 \int_0^{\overline{D}_m} N(T, \overline{p}_m, \epsilon) \, d\epsilon, \]  

(6.6)

where

\[ \overline{D}_m = \text{diam}(T, \overline{p}_m) := \sup_{z_1, z_2 \in T} \overline{p}_m(z_1, z_2) < \infty. \]  

(6.7)

Define the following \( \psi - \) function \( \overline{\beta} = \overline{\beta}(m) : \)

\[ \overline{\beta}(m) := V(m) \cdot |\overline{p}_m(\cdot)|_{s, T, \nu}, \]

it is meaning \( \delta \in T \); and suppose the finiteness of these function for at last one value \( m > \max(2, p, q, s) \) :

\[ \exists L > \max(2, p, q, s) \Rightarrow \overline{\beta}(L) < \infty. \]  

(6.8)

**Theorem 6.1.** Let the condition (6.8) be satisfied. Then the (centered) r.p. \( \xi(t) \) satisfies the CLT in the space \( B^{\alpha, p}_{a, s} \) and moreover

\[ \sup_n \left\| \left\| S_n(\cdot) \right\|_{B^{\alpha, p}_{a, s}} \right\| G\overline{\beta} \leq 1, \]  

(6.9)

with correspondent tail estimation.

**Proof.** Let \( m \in (\max(2, p, q, s), L) \). We use the propositions (2.7); (2.7a) of theorem 2.1 applied for the sequence random processes \( S_n(t) \); note that all our estimations are uniformly in \( n \). This dives us the estimate (6.9).

The remainder part of proof theorem 6.1 is completely analogous to one in the theorem 5.1.

**Example 6.1.** Let in (6.1) (and further) \( m \) be a fixed number \( m = r \in (\max(2, p, q, s), L) \). As long as the ordinary Lebesgue spaces are the particular case of Grand Lebesgue spaces, we propose from the assertion and conditions of theorems
4.1 and 6.1 that the r.p. \( \xi(t) \) satisfies the CLT in the space \( B_{\alpha,s}^{\alpha,p} \) and there holds the following \( L_r(\Omega) \) estimation

\[
\sup_n \left\| S_n(\cdot) \right\|_{B_{\alpha,s}^{\alpha,p}} < \infty. \tag{6.10}
\]

**Example 6.2.** Let in addition to the conditions of theorem 6.1 \( L = \infty \) and \( \overline{\beta}(m) \leq m^{1/l} \), \( l = \text{const} > 0 \). Then the r.p. \( \xi(t) \) satisfies again the CLT in the space \( B_{\alpha,s}^{\alpha,p} \) and moreover

\[
\sup_n \left\| S_n(\cdot) \right\|_{B_{\alpha,s}^{\alpha,p}} \leq 1, \tag{6.11}
\]

with correspondent exponential tail estimation

\[
\sup_n T \left\| G_{\psi_{L/(l+1)}} \right\|_{B_{\alpha,s}^{\alpha,p}} (u) \leq e^{-C_2(l)u^{l/(l+1)}}, \quad u \geq 1. \tag{6.11a}
\]

It is no hard to generalize this example on the case which was considered in (4.10)-(4.11), theorem 4.1a. Indeed, let \( \lambda_m = \lambda_m(\delta) \), \( \delta \in [0,1] \) be some family of non-negative continuous functions such that

\[
\lim_{\delta \to 0^+} \frac{\mu_m(\delta)}{\lambda_m(\delta)} = 0, \quad \mu_m(\delta) \leq \lambda_m(\delta), \quad \lambda_m(0^+) = \lambda_m(0) = 0. \tag{6.12}
\]

Define the following new \( \psi - \) function \( \overline{\beta}_r = \overline{\beta}_r(m) \):

\[
\overline{\beta}_r(m) := K_R(m) \cdot V_r(m) \cdot |\lambda_m(\cdot)|_{s,T,v},
\]

it is meaning \( \delta \in T \), and suppose the finiteness of these function for at last one value \( \overline{m} > \overline{m_0} := \max(2,p,q,s) \):

\[
\exists \overline{L}_r > \overline{m_0} = \max(2,p,q,s) \Rightarrow \overline{\beta}_r(\overline{L}_r) < \infty. \tag{6.13}
\]

We observe analogously to the proof of theorems 4.1a and 6.1

**Theorem 6.1a.** Let the condition (6.13) be satisfied. Then the (centered) r.p. \( \xi(t) \) satisfies the CLT in the space \( B_{\alpha,s}^{\alpha,p} \) and moreover

\[
\sup_n \left\| S_n(\cdot) \right\|_{B_{\alpha,s}^{\alpha,p}} \left\| G_{\overline{\beta}_r} \right\| \leq 1, \tag{6.14}
\]

with correspondent tail estimation.

### 7 Non-asymptotical estimates.

We intend to obtain in this section the non-asymptotical estimates for the probability
\[ \mathcal{P}(u) \overset{\text{def}}{=} \sup_n \mathbb{P}\left( \|S_n(\cdot)\|_{B_{\alpha,s}^{o,p,q}} > u \right), \quad u \geq e, \quad (7.1) \]

for example the exponential decreasing type bounds.

Define a new function

\[ \kappa(m) := K_R(m) \cdot |d_m(t + z\delta, t)|_{p,T; q,T; s,T,\nu}, \quad m > m_0 := \max(2, p, q, s) \quad (7.2) \]

and suppose its finiteness for the values \( m \in (m_0, L') \), where \( L' \leq \infty \), i.e. one can \( L' = \infty \).

We apply the inequality (5.8):

\[ \sup_n \left| \|S_n(\cdot)\|_{B_{\alpha,s}^{o,p,q}} \right|_{m,\Omega,p} \leq \kappa(m), \quad m < L'. \quad (7.3) \]

**Proposition 7.1.**

Since \( \tau \overset{\text{def}}{=} \|S_n(\cdot)\|_{B_{\alpha,s}^{o,p,q}} \in G\kappa \) and moreover \( ||\tau||_{G\kappa} = 1 \), we conclude

\[ T_\tau(u) \leq \exp \left( -\tilde{\kappa}^*(\ln u) \right), \quad u > e. \quad (7.4) \]

where (we recall) \( \tilde{\psi}(p) = p \cdot \ln \psi(p) \), and

\[ g^*(y) = \sup_p (p|y| - g(p)) \]

is Young-Fenchel, or Legendre transform of the function \( g = g(p) \).

**Example 7.1.**

Suppose under the conditions of remark 2.4. and hence under the conditions of theorem 2.1

\[ d_m(t + h, t) \leq C_m h^\beta, \quad \beta = \text{const} \in (0, 1], \quad (7.5) \]

If as before \( C_m \leq C \cdot m^{1/l}, \quad l = \text{const} > 0, \)

\[ \beta + 1/q > \alpha, \quad m \geq \max(2, p, q, s), \]

then the r.p. \( \xi(\cdot) \) satisfies the CLT in the space \( B_{\alpha,s}^{o,p,q} \) and herewith

\[ \sup_n T_{\|S_n\|_{B_{\alpha,s}^{o,p,q}}} (u) \leq c_1(l; \alpha, \beta, p, q, s) s^{-1/s} \times \]

\[ (\beta - \alpha + 1/q)^{-1/s} \exp \left\{ -c_2(l; \alpha, \beta, p, q, s) u^{l/(l+1)} \right\}, \quad u \geq e, \quad (7.6) \]

\( c_1, c_2 \in (0, \infty) \).

**Example 7.2.**

This time the variables \( m, \ C_m \) can be considered as finite positive constants such that \( m \geq \max(2, p, q, s) \). Let again \( \beta + 1/q > \alpha \). We deduce on the basis of proposition 7.1
\[
\sup_n T_{\|S_n\|B_{\alpha,s}^{p,q}}(u) \leq c_3(m; \alpha, \beta, p, q, s) s^{-1/s} \times \\
(\beta - \alpha + 1/q)^{-1/s} u^{-m}, \ u \geq 2.
\] (7.7)

**Proposition 7.2.**

The correspondent result for the classical Besov’s spaces contains in theorem 6.1. In detail: Let all the condition (6.8) be satisfied. Then the (centered) r.p. \( \xi(t) \) satisfies the CLT in the space \( B_{\alpha,s}^{p,q} \) and moreover

\[
\sup_n \mathbb{P} \left( \|S_n(\cdot)\|B_{\alpha,s}^{p,q} > u \right) \leq \exp \left( -\frac{\beta}{\ln u} \right), \ u > e.
\] (7.8)

8 Concluding remarks. Multivariate case.

**A.** The possible generalizations may be undertaken in two directions: multidimensional "time" \( t \) and multivariate increment \( h \), see detail definitions and investigations of these Besov’s spaces in [5], [31], [37], [68].

By our opinion, these generalizations are not hard for the point of view of offered here problems.

**B.** The considered above Central Limit Theorem in Besov’s spaces may be grounded also for (strong) stationary sequences \( \{\xi_i\} \), superstrong mixingales, martingales etc., see [39] [40], [41], [42], [46], [57].

In these articles was obtained in particular the analogs of Rosenthal’s inequalities for stationary sequences, superstrong mixingales and martingales.

For example, the "Rosenthal's-Osekowski's" constant \( K_{RO,s}(p) \) for the centered martingales, more exactly, for the mean zero identical distributed martingale differences \( \{\zeta_k\}, \ k = 1, 2, \ldots \) (relative arbitrary filtration), i.e.

\[
K_{RO,s}(p) \overset{\text{def}}{=} \sup_n \sup_{\{\zeta_k\}} \left[ \frac{n^{-1/2} \sum_{k=1}^n |\zeta_k|}{|\zeta_1|} \right],
\]

may be estimated as follows

\[
K_{RO,s}(p) \leq 15.7858 \cdot \frac{p}{\ln p}, \ p \geq 2,
\]

see [42], [57].

For the centered strictly stationary sequence of the r.v. \( \{\zeta_k\}, \ k = 1, 2, \ldots \) satisfying the so-called superstrong mixing condition with coefficient \( \beta(k) \) the analogous constant may be named as a constant of Rosenthal-Nachapetyan \( K_{RN}(p) \) [39] and may be estimated as follows

\[
K_{RN}(p) \leq C_p \cdot p^p \cdot \left[ \sum_{k=1}^\infty \beta(k)(k+1)^{(p-2)/2} \right]^{1/p}.
\]

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See also [43], chapter 2, section 2.9.

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