SURGERY PRESENTATIONS OF
COLOURED KNOTS AND OF THEIR COVERING LINKS

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Abstract. We consider knots equipped with a representation of their knot groups onto a dihedral group $D_{2n}$ (where $n$ is odd). To each such knot there corresponds a closed 3–manifold, the (irregular) dihedral branched covering space, with the branching set over the knot forming a link in it. We report a variety of results relating to the problem of passing from the initial data of a $D_{2n}$-coloured knot to a surgery presentation of the corresponding branched covering space and covering link. In particular, we describe effective algorithms for constructing such presentations. A by-product of these investigations is a proof of the conjecture that two $D_{2n}$-coloured knots are related by a sequence of surgeries along $\pm 1$–framed unknots in the kernel of the representation if and only if they have the same coloured untying invariant (a $\mathbb{Z}_n$-valued algebraic invariant of $D_{2n}$-coloured knots).

1. Introduction

The starting point for this work was the authors’ desire to explore the quantum topology of covering spaces as a means of acquiring a deeper understanding of how quantum invariants actually encode topological information. Recent results in the case of cyclic covering spaces (see e.g. [9, 10]) suggest the existence of such a theory.

Having understood the cyclic case, the natural next step is to consider the branched dihedral covering spaces. These spaces have long played an important role in knot theory, dating back to Reidemeister’s use of the linking matrix of a knot’s dihedral covering link to distinguish knots with the same Alexander polynomial ([19], see also e.g. [17]). More recently they have also been used in investigations of knot concordance (e.g. [7]). In addition, branched dihedral covers are useful in 3–manifold topology: for example, it turns out that every 3–manifold is a 3–fold branched dihedral covering space over some knot (see e.g. [2 Theorem 11.11]).

Quantum invariants for 3–manifolds are typically constructed using surgery presentations. To investigate the quantum topology of covering spaces, then, it seems we need a combinatorial theory of surgery presentations of covering spaces.

The cyclic case is well-known. Recall that there is a famous trick for obtaining surgery presentations of $n$–fold cyclic covers for any natural number $n$ (see e.g. [20 Chapter 6D]). We wish to generalize this trick to dihedral covers, so we’ll begin by reviewing how it goes.
One first performs crossing changes to untie the knot by introducing ±1-framed unknots along which surgery is carried out. The unknots are chosen to have linking number zero with the knot. After this step, we have a surgery presentation of the given knot as a ±1-framed link L lying in the complement of an unknot U, where each component of L has linking number zero modulo n with U. For the purpose of generalization, this last condition can be restated: every component of L lies in the kernel of the mod n linking homomorphism \( \text{Link}_n: H_1(\mathbb{S}^3 - N(U)) \to \mathbb{Z}_n \).

Because this condition is satisfied the construction of a cyclic cover can now be completed by lifting L to the n–fold cyclic cover of \( \mathbb{S}^3 \) branched over U, which is of course again \( \mathbb{S}^3 \).

We would like analogous procedures for classes of covering spaces corresponding to other groups, in particular to the dihedral groups. The key feature which permitted construction in the cyclic case was the existence of a knot (the unknot) which every other knot could be transformed into via surgeries in the kernel of the mod n linking homomorphism, and whose branched cyclic cover could be constructed explicitly.

To discuss how this generalizes it’s worth introducing a few definitions.

**Definition 1** (\( G \)-coloured knots). For a finite group G and a closed orientable 3–manifold \( M \), define a \( G \)-coloured knot in \( M \) to be a pair \((K, \rho)\) of an oriented knot \( K \subset M \) and a surjective representation \( \rho: \pi_1(M - N(K)) \to G \). Unless otherwise specified it will be assumed that \( M = \mathbb{S}^3 \).

**Definition 2** (Surgery in \( \text{ker} \rho \)). Let \( (K, \rho) \) be a \( G \)-coloured knot in a 3–manifold \( M \). If \( L \subset M - K \) is an integer–framed link each of whose components is specified by a curve lying in \( \text{ker} \rho \) then we can perform surgery along \( L \) to obtain a new \( G \)-coloured knot \( (K', \rho') \) in a 3–manifold \( M' \), as follows:

- Remove tubular neighbourhoods \( N(L_i) \) of the components \( L_i \) of \( L \), and reattach them to \( M - \bigcup N(L_i) \) so as to match the meridional discs to the framing curves.
- To specify the induced representation \( \rho' \), we must state the value it takes for an arbitrary curve \( \gamma \) in \( M' - K' \). Homotope \( \gamma \) into \( M - \bigcup N(L_i) \), then evaluate it in the restriction of \( \rho \). This value is well-determined because the components of \( L \) lie in the kernel of \( \rho \).

Such surgery is called surgery in \( \text{ker} \rho \).

**Definition 3** (Complete set of base-knots). A complete set of base-knots\(^1\) for a group G is a set \( \Psi \) of \( G \)-coloured knots \( (K_i, \rho_i) \) in 3–manifolds \( M_i \), such that any \( G \)-coloured knot \( (K, \rho) \) in \( \mathbb{S}^3 \) can be obtained from some \( (K_i, \rho_i) \in \Psi \) by surgery in \( \text{ker} \rho_i \).

To generalize the procedure from the cyclic case to some other group G, we must find a complete set of base-knots whose desired covering spaces (and covering links) we know how to construct explicitly, and into whose covering spaces we know how to lift surgery presentations for any \( G \)-coloured knot.

This paper deals with the case when G is the dihedral group \( D_{2n} \) with \( n \) any odd integer— the group of permutations of the vertices of a regular polygon with

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\(^1\)The term base-knot imitates base-point.
n sides. Its presentation is

\[ D_{2n} := \left\{ t, s \mid t^2 = s^n = 1, \; tst = s^{-1} \right\}. \]

As permutations on the set of vertices of the regular polygon, these generators correspond to

\[ t = \left( \begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ 1 & n & n-1 & \ldots & 3 & 2 \end{array} \right) \]

and

\[ s = \left( \begin{array}{cccccc} 1 & 2 & 3 & \ldots & n-1 & n \\ 2 & 3 & 4 & \ldots & n & 1 \end{array} \right). \]

Elements in \( D_{2n} \) of the form \( s^a \) are called rotations, and elements of the form \( ts^a \) are called reflections. The cyclic group of rotations \( C_n := \langle s \rangle \) is a normal subgroup in \( D_{2n} \).

We’ll present a \( D_{2n} \)–colouring \( \rho \) of a knot \( K \subset S^3 \) by labeling every arc of a knot diagram for \( K \) by the image under \( \rho \) of the corresponding Wirtinger generator. More generally, we can present a \( D_{2n} \)–colouring of a knot \( K \) in a closed 3–manifold \( M \) by a diagram of a link \( L \cup K_1 \) in \( S^3 \) where:

- \( L \) is integer–framed, and surgery along \( L \) turns \( (S^3, K_1) \) into \( (M, K) \).
- Every arc of the diagram is labeled by an element of \( D_{2n} \).
- Wirtinger relations are satisfied.
- When the framing curve of any component of \( L \) is expressed as a product of Wirtinger generators of \( \pi_1 \left( S^3 - (L \cup K_1) \right) \), the product of the corresponding labels is \( 1 \in D_{2n} \).

Our goal in this paper is to give a combinatorial procedure for constructing surgery presentations of the irregular dihedral branched covering space corresponding to some \( D_{2n} \)–coloured knot, together with the covering link it contains. In Section 2 we’ll recall exactly what these phrases refer to.

Roughly speaking, we’ll describe two approaches to this problem, corresponding to two different complete sets of base-knots for \( D_{2n} \). The sets of base-knots will be introduced shortly. The construction of their corresponding dihedral covering spaces, covering links, and how to lift surgery presentations in the complement of the base-knot, will be discussed in detail in Sections 3 and 4.

**The untying approach.** This first approach begins with exactly the same procedure for untying knots as is used when constructing surgery presentations of cyclic covers. It may be viewed as an adaptation of that approach to the case of \( D_{2n} \).

**Theorem 1.** Consider the following diagram, which depicts a \( D_{2n} \)–coloured unknot in the 3–manifold that results from surgery on the disjoint \( kn \)–framed unknot (recall that this is the \( (kn, 1) \)–lens-space).

\[ s \]

\[ \text{framing} = kn \]

\[ t \]

\[ U \]

Note that this is a different convention from the one normally used for a Fox \( n \)–colouring of a knot, in which an arc which we would label by \( ts^a \) is labeled simply by \( a \) (see [6]).
The set of these $D_{2n}$-coloured knots for $k = 0, 1, \ldots, n-1$ is a complete set of base-knots for $D_{2n}$.

It follows from the previous theorem that every $D_{2n}$-coloured knot $(K, \rho)$ has a presentation of the following form, for some $0 \leq k < n$:

$$\text{framing} = kn$$

Here, the thick lines "-" denote parallel strands, $T$ is some tangle, and every component of the framed link which results lies in $\ker \rho$ and has linking number zero with $U$.

As an example, see the surgery presentation for a $D_{14}$-coloured $5_2$ knot given in Figure 1.

**Theorem 2.** A surgery presentation for the irregular dihedral branched covering space $M$ determined by the $D_{2n}$-coloured knot $(K, \rho)$, and for the covering link $\tilde{K}$ of $K$ sitting inside $M$ is as shown in Figure 2. In that figure, a small zero near an introduced surgery component means it has zero framing and $\tilde{U}_1 \cup \cdots \cup \tilde{U}_{n+1}$ is the covering link of $U$, becoming $\tilde{K}$ after the surgeries are performed.

**Band projection approach.** This approach is based on a choice of band projection for a Seifert surface $F$ of the $D_{2n}$-coloured knot. To a basis for $H_1(F)$ there corresponds a Seifert matrix and a colouring vector (to be defined in Section 4.1.1). The colouring vector determines the $D_{2n}$-colouring of the knot. The heart of this approach will be realizing algebraic operations on the Seifert matrix and colouring vector by sliding bands and performing $\pm 1$-framed surgeries on unknots in $\ker \rho$.

While this method seems to be less efficient in practice, it is a stronger theoretical result because it arises from an equivalence relation on $D_{2n}$-coloured knots in $S^3$ whose corresponding equivalence classes can be detected with a certain algebraic invariant: the coloured untying invariant.

**Definition 4.** We say that two $D_{2n}$-coloured knots $(K_1, \rho_1)$ and $(K_2, \rho_2)$ in $S^3$ are $\rho$-equivalent if one can be obtained from the other by a sequence of surgeries on $\pm 1$-framed unknots in $\ker \rho$.

We alert the reader that we are restricting to surgeries along $\pm 1$-framed unknots, so that this is an equivalence relation on $D_{2n}$-coloured knots in $S^3$. 

![Figure 1. A surgery presentation for a $D_{14}$-coloured $5_2$ knot.](image)
This equivalence can be defined as an equivalence relation on coloured knot diagrams without reference to surgery in the following way:

\[ \left\{ \begin{array}{c}
g_1 \ g_2 \\
\vdots \\
g_r
\end{array} \right\} \iff \left\{ \begin{array}{c}
g_1 \ g_2 \\
\vdots \\
g_r
\end{array} \right\} \quad \text{with } \prod_{i=1}^{r} g_i = 1 \in D_{2n}. \]

**Theorem 3.** Any \( D_{2n} \)-coloured knot \((K, \rho)\) is \( \rho \)-equivalent to one of the \( D_{2n} \)-coloured knots of Figure 3 for \( k = 0, 1, \ldots, n - 1 \). This implies that this set of knots (the pretzel knots \( p(2kn + 1, 1, -n) \) for \( k = 0, 1, \ldots, n - 1 \) with the specified colouring) is a complete set of base-knots for \( D_{2n} \).

Because too much extra notation would need to be introduced at this point in order to explain it, an explicit construction of the irregular branched dihedral
covering spaces corresponding to the set of base-knots of Figure 3 is pushed off to Section 4.4.

Having just defined a new equivalence relation, several questions immediately arise. How many equivalence classes are there? Can they be detected with algebraic information?

What we would really like is a theorem characterizing these classes in terms of a readily computable algebraic invariant. There are many prototypes for this in the recent literature. One example is the result of Murakami–Nakanishi [15], which is closely related to results of Matveev [13], which characterizes $\Delta$–equivalence classes of links in terms of their linking matrices. Another is the result of Habiro [11] classifying knots, all of whose finite-type invariants up to a certain degree are equal, via surgery along tree claspers. Yet another is the work of Naik–Stanford [16] which links $S$–equivalence classes of knots to double-delta moves. The influence of this point of view on recent research should be clear.

In [14, Section 6] a non-trivial function from $D_{2n}$–coloured knots to $\mathbb{Z}_n$ was defined. Its value for a $D_{2n}$-coloured knot in $S^3$, in terms of a Seifert matrix $S$ and a vector $\vec{w}$ which determines the $D_{2n}$-colouring $\rho$, is given by the formula:

$$cu(K,\rho) = \frac{2(\vec{w}^T \cdot S \cdot \vec{w})}{n} \mod n.$$ 

The value $cu(K,\rho) \in \mathbb{Z}_n$ was called the \textit{coloured untying invariant} of $(K,\rho)$. It was proven there that $cu$ is invariant under surgery in $\ker \rho$, and so, in particular, is constant function on $\rho$–equivalence classes (see also [12]). It was also shown there that every possible value is realized by some $D_{2n}$-coloured knot. These facts imply that the number of $\rho$–equivalence classes is at least $n$. On the other hand, because the complete sets of base-knots in Theorem 1 and in Theorem 3 both have cardinality $n$, it follows that $n$ is also an upper bound for the number of $\rho$–equivalence classes. Thus a by-product of our constructions is:

**Corollary 4.** Two $D_{2n}$-coloured knots have the same coloured untying invariant if and only if they are $\rho$–equivalent. In particular, the number of $\rho$–equivalence classes of $D_{2n}$-coloured knots is $n$.

For $n$ prime this was [14, Conjecture 1], where it was proved for $n = 3$ and for $n = 5$. This conjecture was also the subject of [12], where bordism theory was used to put an upper bound of $2n$ on the number of $\rho$–equivalence classes.

**The view from here.** As stated at the beginning of the introduction, our motivation is to develop a theory of quantum topology for dihedral covering spaces and covering links. How to proceed? Many tantalizing hints can be found in the literature.

One possible route would be to generalize recent results in the cyclic case [8] [9] due to Garoufalidis and Kricker. The results culminate in a universal formula for the LMO invariant of a cyclic branched cover in terms of the rational lift of the loop expansion of the Kontsevich invariant [10]. This rational lift may be viewed as a version of the Kontsevich invariant coloured by the canonical representation $\pi_1 \left( \frac{S^3 - N(K)}{24} \right) \to \mathbb{Z}$.

Using the surgery presentations in this paper, one should be able to obtain a version of these constructions where the colouring group is $D_{2n}$ instead of $\mathbb{Z}$. Taking the ‘1–loop part’ will give an analogue to the Alexander polynomial. The
2–loop part should determine the Casson(-Walker–Lescop) invariant for an irregular dihedral covering space $M$ (which can be any 3-manifold).

Another clue for the shape of such a theory is a mysterious formula for the Rohlin invariant of a dihedral branched covering space that was discovered in the seventies by Cappell and Shaneson [3, 4]. Recall that the Rohlin invariant is the mod 2 reduction of the Casson–Walker invariant, which is the unique finite type invariant of degree 1.

The theory of knot concordance has long been a blind spot for ‘traditional’ quantum topology. The classical invariants which access this type of information are typically constructed from systems of covering spaces. One of our longer term goals is to develop sufficient technology to make contact with these constructions.

**Odds and ends.** The paper concludes in Section 5 with a variety of odds and ends which are immediate corollaries of the constructions in the previous sections. First, the choice of a complete set of base-knots in Theorem 3, which was made after trial and error, is of course not the only one possible. Some other choices are also worth mentioning. By choosing the twist knots in Figure 4 as a complete set of base-knots, we can prove that the surgery link in Theorem 1 can be chosen to have linking number zero with the component labeled by $s$. There $m = 1 - \frac{(n+1)^2}{2}$ if $\frac{n+1}{2}$ is even, while if $\frac{n+1}{2}$ is odd then $m = 2 - \frac{n^2+1}{2}$. Choosing the torus knots of Figure 5 gives a picture that is easy to lift (see [14] for the $n = 3$ and $n = 5$ cases) but is not a natural end-point for our algorithms. Using it we can prove that a 3–manifold with $D_{2n}$-symmetry has a surgery presentation with $D_{2n}$-symmetry, extending a ‘visualization’ result of Przytycki and Sokolov [18] and of Sakuma [21]. Finally, we may choose a complete set of base-knots which differ only by the choice of their $D_{2n}$-colouring, as shown in Figure 6.

Although the methods in this paper are elementary, the results appear to be new. Swenton [23], and independently Yamada [24] for $n = 3$, give quite different algorithms for translating from dihedral covering presentations to surgery presentations, ‘forgetting’ the knot.

**Some further problems.**

- Explore the relationship between the untying approach and the band projection approach. In particular, how can one calculate the coloured untying invariant of a $D_{2n}$-coloured knot in a 3-manifold other than $S^3$?
- Explore the possibility of using Goeritz surfaces instead of Seifert surfaces in the band projection approach, giving the torus knots as a complete set of base knots directly.
- Find minimal complete sets of base-knots for groups other than the dihedral group. Use these to find presentations for other classes of covering spaces.
- Extend the results of this paper to $D_{2n}$-coloured algebraically split links (the extension to boundary links is straightforward).

2. Dihedral branched covering spaces

In this section we recall the way in which $(K, \rho)$ presents a closed 3–manifold $M$.

We first recall how a *monodromy representation* characterizes an (unbranched) covering space. Let $pr: \tilde{X} \to X$ be an $n$–fold (unbranched) covering space of a
closed 3–manifold $X$ with basepoint $\ast$. An oriented loop $\ell \subset X$ based at $\ast$ lifts to a collection of distinct loops $\ell_1, \ldots, \ell_n$ each starting and ending at one of the $n$ preimages $*_1, \ldots, *_n$ of $\ast$ in $\tilde{X}$. Sending the initial point of each of these paths to its endpoint gives a permutation of $*_1, \ldots, *_n$, inducing a representation

$$\pi_1(X, \ast) \to \text{Sym} \left( \text{pr}^{-1}(\ast) \right)$$

which is unique up to relabeling lifts of the basepoint. Choosing a different basepoint in $X$ modifies the representation via some bijection $\text{pr}^{-1}(\ast) \simeq \text{pr}^{-1}(\ast')$.

The theory of covering spaces tells us that two covering spaces are equivalent (that is, homeomorphic by a homeomorphism respecting the covering map) if and only if their monodromy representations are the same (after some relabeling). Thus we can specify a covering space by giving a representation $\pi_1(X, \ast) \to \text{Sym} \left( \text{pr}^{-1}(\ast) \right)$.

From an intuitive cut-and-paste point of view it is natural to present a covering space by means of its monodromy. This allows one to construct it by cutting the base space into cells, taking the appropriate number of copies of each cell, and gluing them together according to the representation. The following example, which plays a part in the proof of Theorem\textsuperscript{2} is a good illustration of this.
Example. Consider a genus two handlebody equipped with base point and a representation $\rho$ from its fundamental group onto $D_{14}$. To construct the covering space whose monodromy group is given by this representation, we begin by cutting the handlebody into a cell:

Now we take seven copies of this cell, and glue them together according to $\rho$.

We now construct $M$. Begin from a $D_{2n}$-coloured knot $(K, \rho)$, and consider the $n$–sheeted (unbranched) covering space $\tilde{X}$ of the knot complement $X$ of $K$ (the closure in $S^3$ of the complement of a tubular neighbourhood $N(K)$ of $K$) with monodromy given by $\rho$, where $D_{2n}$ is thought of as a subgroup of $\text{Sym}(*)_{1,\ldots,n}$.

Consider the boundary of this covering space. What is it? Well, the $D_{2n}$-colouring $\rho$ sends a meridian to a reflection, and the longitude may be chosen so that it is sent to 1. It follows that the boundary of $\tilde{X}$ is a collection of $\frac{n+1}{2}$ tori—$\frac{n+1}{2}$ two–sheeted coverings and 1 one–sheeted covering of the boundary torus $\partial N(K)$ of $X$. Glue $\frac{n+1}{2}$ solid tori into these boundary components, longitude to longitude, such that a meridional disc is glued into some lift of a power of the meridian downstairs.

This is the desired space $M$: the branched dihedral covering space of $S^3$ associated to the $D_{2n}$-coloured knot $(K, \rho)$. The cores of the glued-in tori, with orientations induced by the orientation of $K$, form the covering link $\tilde{K}$.

Remark. It is more usual to specify a covering space by a conjugacy class of subgroups of $\pi_1(X)$ (corresponding to the image of $\pi_1(\tilde{X})$ under the projection). If a covering space is determined by a monodromy representation then the corresponding class of subgroups is given by taking the stabilizer of a chosen element in $\text{Sym}(*)_{1,\ldots,n}$.
Remark. The 3–manifold $M$ is usually referred to as the *irregular* branched dihedral covering space associated to $(K, \rho)$ (as we referred to it in the introduction), because it corresponds to the preimage under $\rho$ of $(t)$ which is not a normal subgroup of $D_{2n}$.

3. Untying approach

This approach consists of two steps. The first is to obtain a surgery presentation of a $D_{2n}$-coloured knot $(K, \rho)$ in the complement of an unknot in a lens–space (one of the base-knots of Theorem 1). Such a presentation is called a *separated dihedral surgery presentation* of $(K, \rho)$. The second step is to lift the separated dihedral surgery presentation to a surgery presentation of the dihedral branched covering space and of the covering link.

3.1. Obtaining a separated dihedral surgery presentation. The construction consists of three steps: use surgery to untie the knot, perform handleslides to concentrate the non-trivial labels onto a single surgery component, and finish with another round of surgery to untie that surgery component. We’ll also describe some moves which put the labels and surgery curves in the resulting diagram in a standard form.

3.1.1. Untying the knot. We can untie any knot $K$ by crossing changes, realized by surgery on $\pm 1$–framed unknots which have linking number zero with $K$. This allows us to present $K$ as a $\pm 1$–framed link $L$ in the complement of a standard unknot $U \subset S^3$, such that surgery on $L$ recovers $K \subset S^3$. In the following section we generalize this procedure to $D_{2n}$-coloured knots.

Let us begin by reminding ourselves that the arcs of a knot in $S^3$ are all coloured by reflections (elements of the form $ts^a \in D_{2n}$). This follows from the Wirtinger relations. Near a crossing where the over-crossing arc is labeled $g_1$, the under-crossing arcs must be labeled $g_2$ and either $g_1^{-1}g_2g_1$ or $g_1g_2g_1^{-1}$ for some $g_2 \in D_{2n}$. If any arc is labeled by a rotation then all arcs in the knot diagram would be labeled by rotations (because $C_n \triangleleft D_{2n}$) which would contradict surjectivity of the $D_{2n}$-colouring $\rho$.

When performing surgery, the colours of the arcs of the introduced surgery component are induced as follows:

**Lemma 5.** Let $g_1$ and $g_2$ be elements in $D_{2n}$. The local moves depicted below induce colours on the added surgery components as shown. (The two strands “being twisted” can be from the knot or from surgery components.)

\[
\begin{align*}
\begin{array}{c}
g_1 \\
g_2
\end{array} & \quad \begin{array}{c}
g_2^{-1}g_1g_2 \\
g_2^{-1}g_1g_2g_1^{-1}g_2
\end{array} \quad \iff \quad 
\begin{array}{c}
g_1 \\
g_2
\end{array} & \quad \begin{array}{c}
g_1 \\
g_2
\end{array}
\end{align*}
\]
Proof. The precise claim is that there is a PL-homeomorphism $h$ between the two spaces, taking the knot in one space onto the knot in the other space, such that the pulled-back representation of the knot group is as shown. The homeomorphism $h$ is to cut $S^3 - T$ along a disc spanning $T$ a tubular neighbourhood of the introduced surgery component, do a $2\pi$ twist in the appropriate direction, then reglue the disc and the solid torus.

The label on an arc of the right-hand diagram is determined by the image under $h$ of a path representing the appropriate element of the fundamental group. For example, we obtained the label $g_1^{-1}g_2$ in the first local move by finding the image under $h$ of the Wirtinger generator corresponding to the appropriate meridian of the introduced surgery curve:

We’ll use this lemma to untie $D_{2n}$-coloured knots. In that case we’ll have $g_1 = ts^a$ and $g_2 = ts^b$ for some $a, b \in \mathbb{Z}_n$, so that $g_1g_2^{-1} = s^{a+b}$, and the typical move will look like:

If some meridian of $K$ maps to a reflection $ts^a$, then because $C_n$ is a normal subgroup of $D_{2n}$, all meridians of $K$ map to that same reflection $ts^a$. Arcs in $L$ are labeled by rotations, and conjugation by a reflection $ts^a$ of a rotation $s^b$ maps it to $s^{-b}$. Therefore for any component $C_i$ of $L$ there exists $j \in \mathbb{Z}_n$ such that all arcs of $C_i$ are labeled either $s^j$ or $s^{-j}$.

Because $\rho$ is surjective, $s$ is generated by labels of the arcs of the knot $K$. Thus there exists an element in $\pi_1 \left( S^3 - N(K) \right)$ which is represented by a curve $C$ which has a meridian which maps to $s$. Because $s$ is a rotation, this curve passes under an
even number of arcs of $K$, and we may choose such a curve to have linking number zero with $K$ because the labels on $K$’s arcs all have order 2. Perform surgery on $C$, and set $C_1 := C$. Now untie the resulting knot by crossing changes, realized by $\pm 1$-framed surgeries along unknots which have linking number zero with the knot. We obtain surgery presentation $L$ for $K$ in the complement of the unknot $U$, for which an arc of $C_1$ is labeled $s$, and so all arcs of $C_1$ are labeled either $s$ or $s^{-1}$. We have shown the following lemma:

**Lemma 6.** The surgery presentation $L$ may be chosen such that the arcs of $C_1$ are labeled $s$ and $s^{-1}$.

We call $C_1$ the distinguished surgery component.

3.1.2. **Handleslides.** A surgery component whose arcs are all labeled 1 (the identity in $D_{2n}$) is said to be in $\ker \rho$. The second step of the construction is to perform handleslides so as to arrange that every surgery component except for one distinguished component is in $\ker \rho$. The following lemma tells us how labels transform under handleslides.

**Lemma 7.** Two diagrams that differ by one of the moves shown below present equivalent $D_{2n}$-coloured knots. (The displayed components are surgery components.)

**Proof.** When two diagrams are related by a handleslide, the corresponding spaces are related by a PL-homeomorphism which is the identity outside a genus two handlebody containing the two involved components and the ‘path’ of the slide.

To observe how labels transform: pick a curve representing some arc in the right-hand diagram, isotope the curve so that it lies outside the genus two handlebody corresponding to a handle slide which will take us to the left-hand diagram, then read off what that curve maps to in the left-hand diagram.

For example, the label $s^{b-a}$, above can be obtained as shown below:
Note further that once we know what the label on one of the arcs of a component is, the labels on all of its arcs are determined by the fact that they must induce a well-defined representation onto $D_{2n}$.

By Lemma 7 we may repeatedly perform handleslides until all of the surgery components are in ker $\rho$ except $C_1$ (the distinguished surgery component). By Lemma 8 we may assume $C_1$ has an arc labeled $s$. For each $1 < i \leq \mu$ let $a_i \in \mathbb{Z}_n$ be an element such that some arc of $C_i$ is labeled $s^{a_i}$. The effect of sliding $C_1$ over $C_i$ is to replace $a_i$ by $a_i - 1$ (or to $a_i + 1$ depending on which version of the handleslide is used). Thus sliding $C_1$ over $C_i$ repeatedly $a_i$ times (or $-a_i$ times) kills the labels of the arcs of $C_i$. Repeat for all $i = 2, \ldots, \mu$.

Remark. Readers who try some examples will find that this second step can add significant complexity to the construction. However things are not so bad when $n = 3$. The reason is that there will only ever be a single handleslide required to kill the label on a surgery component, because $1 + 2 = 0 \text{ mod } 3$ or $1 - 1 = 0 \text{ mod } 3$. Note further that in this situation the surgery components will remain framed unknots after the handleslides.

3.1.3. Putting the presentation into a standard form. After the first two steps we have a diagram where:

- The knot $K$ has been untied and is in its standard position $U$.
- There are a number of surgery components, each of which has linking number zero with the knot.
- Every surgery component, except one, is in ker $\rho$, i.e. has all of its arcs labeled $1 \in D_{2n}$.
- The remaining component $C_1$ has each of its arcs labeled either $s$ or $s^{-1}$.

The final step is to add extra surgery components so that the two component sublink $U \cup C_1$ becomes a standard two component unlink. We will require that the surgery components introduced to make this happen are in ker $\rho$.

In the neighbourhood of a crossing in $C_1$, either all arcs will be labeled $s$, in which case we can reverse the crossing by:

(3.3)

or the crossing will have one incident arc labeled $s$ and another incident arc labeled $s^{-1}$, which can be dealt with by:
Thus, we can reverse any crossing on $C_1$ by surgery in $\ker \rho$, untying $C_1$ and unlinking it from $U$.

Notice that the framing of the distinguished component must end up a multiple of $n$. This is because labels induce a well-defined representation of $\pi_1(S^3 - N(K))$ onto $D_{2n}$, so contractible curves map to 1. The framing curve of every surgery component (in particular, the distinguished component) bounds a disc in the corresponding torus being glued in and is thus contractible. Since the distinguished component is labeled $s$ and is disjoint from the knot (so its longitude maps to 1), its framing must vanish modulo $n$.

It is possible to introduce extra surgery components into the presentation which will change that framing by $n^2$. It follows that $k$ may be chosen so that $0 \leq k < n$. To do this, coil the distinguished surgery component into $n$ parallel strands:

Then add a $\pm 1$–framed surgery component. (Choose $+1$ to increase framing by $n^2$, and $-1$ to decrease framing by $n^2$.)

The distinguished surgery component may now be tied in a knot, but we can untie it using surgery in $\ker \rho$, as shown in Equation (3.3). Observe that such moves do not change the framing of the distinguished surgery component because the linking number of the introduced surgery components with $C_1$ is zero.

All arcs of $U$ are labeled by some reflection $ts^a \in D_{2n}$, but by the ambient isotopy of Figure 7 we may conjugate this label by $s$, so that the label on the arcs of $U$ becomes $ts^{a-2}$. Repeating $\frac{a}{2}$ mod $n$ times, we obtain a presentation for $(K, \rho)$ in which all arcs of $U$ are labeled $t$.

To summarize:

**Proposition 8.** Any $D_{2n}$-coloured knot $(K, \rho)$ has an separated dihedral surgery presentation, i.e. it has an surgery presentation $L = C_1 \cup \cdots \cup C_\mu$ such that:
The distinguished surgery component $C_1$ has all its arcs labeled $s$ and has framing $kn$ with $0 \leq k < n$.

- All the other components $C_2, \ldots, C_\mu$ are in $\ker \rho$.
- All arcs of $U$ are labeled $t$.
- $C_1 \cup U$ is the standard 2-component unlink.

In Section 5.1.1 we will additionally show that a separated dihedral surgery presentation may be chosen such that the components $C_2, \ldots, C_\mu$ in $\ker \rho$ all have linking number zero with the distinguished component $C_1$.

3.1.4. Example: The $D_{14}$-coloured $5_2$ knot. As an example, let’s see how we obtain a separated dihedral surgery presentation of a $D_{14}$-coloured $5_2$ knot.
3.2. Constructing the cover. Take a separated dihedral surgery presentation of some $D_{2n}$-coloured knot $(K, \rho)$. It consists of a framed link $L$ in a genus two handlebody $H$, embedded into a link in the way shown in Figure 8. Our goal in this section is to lift this picture to a surgery presentation of $M$, the $n$-fold dihedral covering space of $S^3$ branched over the knot $K$ whose monodromy is given by $\rho$.

Our starting point is Figure 9 which tells us how to use the separated dihedral surgery presentation to construct the knot complement $X := S^3 - N(K)$. This is achieved by doing surgery on $L$, attaching 2-handles to the curves $A$ and $B$, and finishing by attaching a ball to the resulting $S^2$ boundary component.

The knot complement $X$ comes equipped with a representation $\rho: \pi_1 (X) \rightarrow D_{2n}$ determined by the labels $s$ and $t$.

On the boundary of $H$ we have also marked the meridian $m$ and a choice of longitude $l$ of $K$. This data will be referred to below as the peripheral markings.
We recover $S^3$ with the knot $K$ embedded in it by gluing a solid torus $N(K)$, displayed in Figure 10 into the boundary of $X$ (a torus), so as to match up the curves $m$ and $l$.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure10.png}
\caption{The solid torus $N(K)$ with embedded knot $K$.}
\end{figure}

With these preliminaries in hand, we can now describe the construction of $M$. The first step is to construct $\tilde{X}_\rho$, which is defined to be the (unbranched) covering space of $X$ whose monodromy is specified by $\rho$. The following steps construct $\tilde{X}_\rho$.

1. Take $\tilde{H}_\rho$, the $n$–fold covering space of $H$ whose monodromy is specified by $\rho$. Lift the surgery link in $H$ to $\tilde{H}_\rho$ and do surgery on that link.
2. Lift $A$ and $B$, the 2–handle attaching circles, to systems of curves $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ on $\tilde{H}_\rho$.
3. Attach 2–handles to these systems of curves.
4. Attach a ball to each of the $n$ resulting $S^2$ boundary components.

Figure 12 shows how the attaching circles and peripheral markings lift to $\tilde{H}_\rho$, in the special case that $n = 7$. The general case is clear from this picture.

Consider now the boundary of $\tilde{X}_\rho$, the space we have just constructed. Inspecting Figure 12 we observe that it consists of $\frac{n+1}{2}$ tori:

$$\partial \left( \tilde{X}_\rho \right) = T_1 \sqcup T_2 \sqcup \ldots \sqcup T_{\frac{n+1}{2}}.$$

The torus $T_1$ is marked as shown in Figure 11 on the left. Under the restriction of the covering map $\tilde{X}_\rho \to X$ to this boundary component, $T_1$ is a one–sheeted covering of $\partial N(K)$. The other tori, $T_i$ where $i$ runs from 2 to $\frac{n+1}{2}$, are marked as shown in Figure 11 on the right. These tori give two–sheeted coverings of $\partial N(K)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure11.png}
\caption{The torus $T_1$ (on the left) and a torus $T_i$ for some $2 \leq i \leq \frac{n+1}{2}$ (on the right), together with their markings.}
\end{figure}

The branched irregular dihedral covering space $M$, together with the covering link $\left\{ \tilde{K}_i \right\}_{i=1}^{\frac{n+1}{2}}$, is obtained from $\tilde{X}_\rho$ by:
Figure 12. The lifts of the attaching circles and peripheral markings to \( \tilde{H}_\rho \), in the case that \( n = 7 \).

(1) Gluing a copy of \( N(K) \) into \( T_1 \) so as to match \( m \) to \( m_1 \) and \( l \) to \( l_1 \).

(2) For each \( i \) such that \( 2 \leq i \leq \frac{n+1}{2} \), gluing a copy of \( N(K) \) into \( T_i \) so as to match \( m \) to the curve \( m_im_{n-i+2} \), and \( l \) to either \( l_i \) or \( l_{n-i+2} \).

This completes the construction of \( M \).

Our task is to turn the construction we have just detailed into a surgery presentation for \( M \). Consider the sequence below, where the index \( i \) runs from 2 to \( \frac{n+1}{2} \), and \( j = n - i + 2 \).
The first move is to slide the attaching circle $A_i$ over the attaching circle $A_j$. Before we do that, we’ll get the longitude marking $l_i$ out of the way by sliding it over $A_j$ first. Next we attach a 2–handle to $A_i$. Observe that the result can be embedded in $S^3$, and that the torus $T_i$ is now embedded in this diagram. Glue a copy of $N(K)$ into $T_i$ in the required way (matching $m$ to $m_i m_j$). In a similar way we can immediately attach a 2–handle to $A_1$ and glue a copy of $N(K)$ into $T_1$. These are the three steps which we carried out in the sequence above.

After the above sequence, if we attach 2–handles to to the circles $A_{n+2}$ through $A_n$ and another 2–handle to $B_1$, then the boundary of the space is a copy of $S^2$ (it is connected and of genus 0). Call this boundary $Y$. A 2–sphere $S^2$ can only bound a ball (Schönflies Theorem) so plugging $Y$ with a 3–ball right away is the same as attaching 2–handles to $B_2, \ldots, B_n \subset Y$ and then plugging what is left of the boundary with 3–balls. In other words, we can discard $B_2, \ldots, B_n$ without changing the result.

In the same way, we can add extra attaching circles for 2–handles into $Y$ without changing the result. Let’s then attach 2–handles into $Y$ to cut the complement in $S^3$ of the handlebody into solid tori, in the way indicated in Figure 14. The attaching circles of the extra 2–handles are labeled $E_1, \ldots, E_{n+3}$ in the figure.

We are done. The space constructed is in the complement of a $\frac{n+1}{2}$ component unlink in the three–sphere, and attaching the remaining 2–handles and balls is equivalent to doing surgery on that unlink, in precisely the way detailed in Theorem 2.

To illustrate with an example, the surgery presentation for the dihedral branched covering space and covering link for the $D_{14}$-coloured $5_2$ knot considered in Section 3.1.4 is as given in Figure 13.

\[\text{framing} = -7\]

\[\text{Figure 13. A surgery presentation for the dihedral covering space and covering link of the } D_{14}\text{-coloured } 5_2\text{ knot of Section 3.1.4.}\]
4. Band projection approach

In this approach we obtain a surgery presentation of a $D_{2n}$-coloured knot $(K, \rho)$ as a link $L$ consisting of $\pm 1$-framed unknotted surgery components in $\ker \rho$ which live in the complement of an element in a complete set of base-knots in $S^3$. We then lift this presentation to a surgery presentation of the branched dihedral cover $M$. A by-product of this approach is a proof of a conjecture that two $D_{2n}$-coloured knots are $\rho$-equivalent if and only if they have the same coloured untying invariant.

4.1. Tools.

4.1.1. The surface data. Take a $D_{2n}$-coloured knot $(K, \rho)$ and choose $F$ a Seifert surface for $K$. Let $x_1, \ldots, x_{2g}$ be a basis of $H_1(F)$ and let $(\xi_1, \ldots, \xi_{2g})$ be the associated basis for $H_1(S^3 - F)$ uniquely characterized by the condition that $\text{Link}(x_i, \xi_j) = \delta_{ij}$ (see e.g. [2, Definition 13.2]). A curve representing $\xi_i$ under-crosses an even number of arcs of the knot diagram, therefore the representative $\tilde{\xi}_i$ of $\xi_i$ in $\pi_1(S^3 - K)$ is mapped by $\rho$ to the product of an even number of reflections, i.e. a rotation $s^a \in C_n \subset D_{2n}$. This image is independent of which representative $\tilde{\xi}_i$ of $\xi_i$ we chose because $C_n$ is commutative. Let the colouring vector of $(K, \rho)$ associated to the basis $\{x_1, \ldots, x_{2g}\}$ of $H_1(F)$ be the vector of these images

$$\vec{v} := (v_1, \ldots, v_{2g})^T := (\rho(\xi_1), \ldots, \rho(\xi_{2g}))^T \in (C_n)^{2g}.$$ 

The colouring vector determines $\rho$ restricted to $\pi_1(S^3 - F)$, and so, via the HNN construction over the Seifert surface, determines $\rho$ up to an inner automorphism of
$D_{2n}$ (the details of the construction are recalled in the proof of Lemma 9 below). Actually, a small trick shows that every such inner automorphism is realized by an isotopy of the diagram, so the colouring vector determines $\rho$ uniquely [14, Proof of Lemma 4].

Let $\tau^\pm$ denote the pushoff from $F$ in the direction of its positive (negative) normal (as determined by the orientation of the knot), and let $S = (\text{Link}(\tau^- x_i, x_j))_{1 \leq i,j \leq 2g}$ be a Seifert matrix for $K$ with respect to $\{x_1, \ldots, x_{2g}\}$.

We call the pair $(S, \vec{v})$ the surface data for the $D_{2n}$-coloured knot $(K, \rho)$ corresponding to a choice of Seifert surface $F$ and a choice of basis for $H_1(F)$. The surface data satisfies the following property:

Lemma 9. Let $\vec{w} := (w_1, \ldots, w_{2g})^T \in \mathbb{Z}^{2g}$ be a vector of integers satisfying $v_i = s^{w_i}$. For any vector of integers $\vec{z} := (z_1, \ldots, z_{2g})^T \in \mathbb{Z}^{2g}$ we have

$$\vec{z}^T \cdot (S + S^T) \cdot \vec{w} \equiv 0 \mod n,$$

and in particular

$$\vec{w}^T \cdot S \cdot \vec{w} \equiv 0 \mod n.$$

Proof. The proof is essentially the same as [4, proof of Proposition 1.1]. Because $(\xi_1, \ldots, \xi_{2g})$ is a basis for $H_1(S^3 - F)$ and because $C_n$ is abelian, the colouring vector $\vec{v}$ determines the map $\bar{\rho} : \pi_1(S^3 - F) \to C_n$ induced by $\rho$ via the condition $\bar{\rho}(y) = s^{\text{Link}(y, \alpha)}$, where

$$\alpha := \sum_{i=1}^{2g} w_i \cdot x_i.$$

The extension of $\pi_1(S^3 - F)$ to $\pi_1(S^3 - K)$ is given by adding a generator $m$ corresponding to a choice of meridian of $K$, modulo the relation

$$m \cdot \tau^+ z \cdot m^{-1} = \tau^- z$$

for all $z \in \pi_1(F)$, corresponding to the fact that the path $m \cdot z \cdot m^{-1} z^{-1}$ is contractible in $\pi_1(S^3 - K)$ (the HNN construction).

Because we know that $\bar{\rho}$ extends to $\rho$ and that $\rho(m)$ is a reflection, it follows that

$$\rho(\tau^- z) = \rho(m \cdot \tau^+ z \cdot m^{-1}) = \rho(m) \cdot \rho(\tau^+ z) \cdot \rho(m) = \rho(-\tau^+ z)$$

Therefore

$$\text{Link}(\tau^+ z, \alpha) = \text{Link}(-\tau^- z, \alpha) \mod n$$

The term on the right equals $-\text{Link}(\tau^+ \alpha, z)$. Therefore

$$z \cdot (L_S + L_{ST}) \cdot \alpha \equiv 0 \mod n$$

where $L_S$ and $L_{ST}$ are the linking pairings of $S$ and of $ST$ correspondingly in $S^3$.

Setting $z = \sum_{i=1}^{2g} z_i x_i$ gives

$$\vec{z}^T \cdot (S + S^T) \cdot \vec{w} \equiv 0 \mod n$$

and setting $z = \alpha$ gives

$$\vec{w}^T \cdot (S + S^T) \cdot \vec{w} = 2\vec{w}^T \cdot S \cdot \vec{w} \equiv 0 \mod n.$$

\qed
Remark. The vectors $\vec{w}$ and $\vec{w} \mod n$ are called the $p$–colouring vector in [12] and in [14] respectively. When $\alpha$ is represented by a simple closed curve, that curve is called a $mod p$ characteristic knot of $(K, \rho)$ in [4].

4.1.2. The coloured untying invariant. In [14, Section 6] it was shown that the following expression

$$cu(K, \rho) = \frac{2(\vec{w}^T \cdot S \cdot \vec{w})}{n} \mod n$$

depends neither on the choice of Seifert surface $F$ nor on the choice of basis for $H_1(F)$. Hence it is an invariant of $D_{2n}$-coloured knots. It is also shown that this is a non-trivial $\mathbb{Z}_n$–valued invariant of $D_{2n}$-coloured knots in $S^3$ which is constant on $\rho$–equivalence classes. A homological version of this invariant seems to provide a generalization to $D_{2n}$-coloured knots in more general 3–manifolds [14, 12].

The culmination of this section is to show that two knots are $\rho$–equivalent if and only if they have the same untying invariant.

4.1.3. Band Projection. Any knot has a band projection (see for instance [2, Proposition 8.2]). This is a projection of the following form:

![Figure 15. A band projection of a knot.](image)

Pairs of bands $B_{2i-1}$ and $B_{2i}$ for $i = 1, \ldots, g$ will be called twin bands. We may choose a band projection such that that knot is oriented as shown in the figure.

A knot in band projection comes equipped with a canonical choice of a Seifert surface $F$ and a choice of basis for $H_1(F)$: let $x_1, \ldots, x_{2g}$ be elements of $H_1(F)$ such that for each $1 \leq i \leq 2g$ the class $x_i$ is represented by a curve in $F$ which threads once through the band $B_i$, with orientations as determined by Figure 15. Recall that the associated basis $\xi_1, \ldots, \xi_{2g}$ for $H_1(S^3 - F)$ is determined by the condition $\text{Link}(x_i, \xi_j) = \delta_{ij}$. In this case the class $\xi_i$ is represented by the appropriately oriented boundary of a small disc which the band intersects the interior of transversely as shown in Figure 15.

The surface data of a knot in band projection refers to the Seifert matrix and colouring vector for this canonical choice of basis.

4.1.4. Band slides. At the heart of this approach are moves which allow us to realize algebraic manipulations of the surface data by ambient isotopies which modify the choice of band projection of a fixed $D_{2n}$-coloured knot.

We say that some band projection is obtained from another by doing a band slide of band $B_{2i-1}$ counterclockwise over band $B_{2i}$ if it is obtained by the following sequence of ambient isotopies:
Similarly we can slide $B_{2i-1}$ clockwise over $B_{2i}$, and we can slide $B_{2i}$ over $B_{2i-1}$ both clockwise and counterclockwise.

These moves fix $F$ but change the choice of basis for $H_1(F)$, and so will change the surface data. The effect on the choice of basis is:

- Sliding $B_{2i-1}$ counterclockwise (respectively clockwise) over $B_{2i}$:
  \[(x_1, \ldots, x_{2i-1}, x_{2i}, \ldots, x_{2g}) \mapsto (x_1, \ldots, x_{2i-1}, x_{2i} \pm x_{2i-1}, \ldots, x_{2g})\]

- Sliding $B_{2i}$ counterclockwise (respectively clockwise) over $B_{2i-1}$:
  \[(x_1, \ldots, x_{2i-1}, x_{2i}, \ldots, x_{2g}) \mapsto (x_1, \ldots, x_{2i-1} \pm x_{2i}, x_{2i}, \ldots, x_{2g})\]

And the corresponding effect on the colouring vector is as follows:

- Sliding $B_{2i-1}$ counterclockwise (respectively clockwise) over $B_{2i}$:
  \[(v_1, \ldots, v_{2i-1}, v_{2i}, \ldots, v_{2g}) \mapsto (v_1, \ldots, v_{2i-1}, v_{2i} \cdot v_{2i-1}^{\mp 1}, \ldots, v_{2g})\]

- Sliding $B_{2i}$ counterclockwise (respectively clockwise) over $B_{2i-1}$:
  \[(v_1, \ldots, v_{2i-1}, v_{2i}, \ldots, v_{2g}) \mapsto (v_1, \ldots, v_{2i-1} \cdot v_{2i}^{\pm 1}, v_{2i}, \ldots, v_{2g})\]

The corresponding effects on the Seifert matrix are $S \mapsto \left( P_{(2i-1,2i)}^\pm \right)^T (B_{(2i-1,2i)}^\pm)^T$ and $S \mapsto \left( P_{(2i,2i-1)}^\pm \right)^T (B_{(2i,2i-1)}^\pm)^T$ for $P_{j,k}^\pm := I \pm E_{j,k}$.

**Example.** Let $(K, \rho)$ be a $D_{2n}$-coloured genus one knot for which

\[
(S, \vec{v}) = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix},
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
\]

with respect to a given basis of $H_1(F)$. The effect of band sliding $B_1$ over $B_2$ counterclockwise is as follows:

\[
(S, \vec{v}) \mapsto \begin{pmatrix}
  a_{11} + a_{12} + a_{21} + a_{22} \\
  a_{21} + a_{22}
\end{pmatrix},
\begin{pmatrix}
  v_1 \\
  v_2 \cdot v_1^{-1}
\end{pmatrix}
\]

The following two lemmas are crucial in this approach. They show how much freedom band slides give us to engineer the colouring vector.

**Lemma 10.** For twin bands $B_{2i-1}$ and $B_{2i}$ for which either $v_{2i-1}$ or $v_{2i}$ generates $C_n$, band slides allow us to transform the pair $(v_{2i-1}, v_{2i})$ to any other pair $(v'_{2i-1}, v'_{2i})$ for which either $v'_{2i-1}$ or $v'_{2i}$ generates $C_n$.

**Proof.** Assume without the limitation of generality that $v_{2i-1}$ generates $C_n$. Then by sliding $B_{2i-1}$ over $B_{2i}$ an appropriate number of times, we can transform $v_{2i}$ into a generator of $C_n$ (in fact into any element of $C_n$). We can therefore assume that both $v_{2i-1}$ and $v_{2i}$ generate $C_n$. Symmetrically, we can assume that both $v'_{2i-1}$ and $v'_{2i}$ generate $C_n$. Slide $B_{2i-1}$ over $B_{2i}$ until the corresponding entry in the colouring vector becomes $v'_{2i}$, and then sliding $B_{2i}$ over $B_{2i-1}$ until the corresponding entry in the colouring vector becomes $v'_{2i-1}$.

**Lemma 11.** For any pair of twin bands $B_{2i-1}$ and $B_{2i}$, by band slides we can obtain a band projection which induces a colouring vector such that either $v_{2i-1}$ vanishes, or $v_{2i}$ vanishes, as desired.
Proof. Equip \( C_n \) with a total ordering, and for each \( i = 1, \ldots, g \) slide \( B_{2i} \) over \( B_{2i-1} \) if \( v_{2i-1} \geq v_{2i} \), or \( B_{2i-1} \) over \( B_{2i} \) otherwise. We obtain a pair \((v_{2i-1}, v_{2i})\) which is smaller than \((v_{2i-1}, v_{2i})\) in the lexical ordering induced by the ordering on \( C_n \). Repeat until we kill \( v_{2i} \) (in which case we’re finished) or \( v_{2i-1} \).

Now that we have obtained a colouring vector with \( v_{2i-1} = 1 \), if we want a colouring vector with \( v_{2i} = 1 \), exchange the \( 2i \)’th and the \((2i - 1)\)’st entries in the colouring vector by a sequence of band slides corresponding to the following operations on entries of the colouring vector:

\[
(1, v_{2i}) \rightarrow (v_{2i}, v_{2i}) \rightarrow (v_{2i}, 1)
\]

Analogously, if \( v_{2i} = 1 \) and we want \( v_{2i-1} \) to vanish, we can reverse the sequence of band slides above. \(\square\)

4.2. Reduction of Genus. The goal of this section is to show that any \( D_{2n} \)-coloured knot \((K, \rho)\) of genus \( g \) is \( \rho \)-equivalent to a \( D_{2n} \)-coloured knot of genus 1.

The proof consists of three steps. We first show that for any \((K, \rho)\) we may choose a band projection such that the induced colouring vector has its first entry equal to \( s \). The second step is to arrange every other entry to be 1. Having prepared such a band projection the final step is to reduce genus by \( \rho \)-equivalences.

4.2.1. Step 1: Engineer a band projection such that \( v_1 = s \). If \( n \) is prime, engineering a band projection such that \( v_1 = s \) is straightforward (Lemma 10), and one may proceed directly to Step 2. If \( n \) is composite, however, a more involved argument may be required.

Our strategy is to construct the desired band projection directly, by finding an appropriate cut system. For the purpose of this section’s discussion we’ll formalize a few terms.

Definition 5. A cut on some Seifert surface for some knot \( K \) is a simple non-separating oriented curve lying on the surface whose two boundary points lie on \( K \).

Definition 6. Consider some cut \( C \) on some Seifert surface \( F \). The ring around \( C \) is a particular simple closed oriented curve in \( S^3 - F \), constructed in the following way. Seifert surfaces are bi-collared, so we may thicken \( F \) in \( S^3 \) to \( F \times [-1, 1] \). The original surface \( F \) is regarded as occupying the 0–slice of this cylinder. Let the boundary points of \( C \) be \( C_0 \) and \( C_1 \) (so that \( C \) runs from \( C_0 \) to \( C_1 \)). The ring around \( C \) is now the loop which starts at \( C_0 \times \{1\} \), follows the curve \( C \times 1 \) to \( C_1 \times \{1\} \), loops around \( K \) to \( C_1 \times \{-1\} \) via the path \( \gamma_1 \) shown in Figure 10 returns along \( C \times \{-1\} \) to \( C_0 \times \{-1\} \), then loops back around \( K \) to its starting point using the obvious path \( \gamma_0 \).

So given a cut on a Seifert surface, we may take the ring around it, which now evaluates in the representation \( \rho \) to give a well-defined element of \( C_n \).

The constructions which resolve this step can now be described by the following two lemmas.

Lemma 12. Consider a \( D_{2n} \)-coloured knot \((K, \rho)\), and a Seifert surface \( F \) for \( K \). If there exists a cut \( C \) on the surface whose corresponding ring evaluates to \( s \), then the knot has a band projection whose corresponding colouring vector has its first entry, \( v_1 \), equal to \( s \).
Lemma 13. Every Seifert surface $F$ of a $D_{2n}$-coloured knot $(K,\rho)$ has a cut on it whose corresponding ring evaluates to $s$.

We’ll explain the proofs of these lemmas in turn.

Proof of Lemma 12. This proof is essentially a re-reading of the standard manipulations that show that every Seifert surface has a band projection (see e.g. Chapter 6).

A system of cuts on $F$, $C_1$ through $C_{2g}$, is called a cut system if when we remove the bands coming from the regular neighbourhoods of the cuts, we are left with a disc. If we have a cut system on $F$, then the disc that remains after we remove the bands from it has its boundary marked with $2g$ pairs of intervals, corresponding to the two sides that are created when an arc is cut open. Label these intervals using $B_1$ through $B_{2g}$, say, depending on which cut an interval came from. (So, in particular, each label will appear twice.) If we have chosen our cuts so that these labels appear in the usual “product of commutators” order, then an ambient isotopy which takes this disc into a standard unknotted disc position will carry the original Seifert surface into standard band-projection position. Furthermore, that ambient isotopy will also carry the rings around the cuts to the rings around the “standard” cuts of a knot in band projection (see Figure 17), which are the usual $\xi_i$’s.

So our only task is to show that any given cut $C_1$ may be completed to a cut system, $C_1$ through $C_{2g}$, marking the disc in the desired “product of commutators” order. This is a standard manipulation.
Proof of Lemma 13. Begin with any band projection of the given $D_{2n}$-coloured knot. Using Lemma 11 kill even numbered entries in the colouring vector by band slides.

Next we’ll introduce the collection of cuts amongst which we’ll find our desired cut. To every vector $(a_1, \ldots, a_g) \in \mathbb{Z}^g$ associate a cut in the way illustrated by Figure 18.

![Figure 18](image)

**Figure 18.** The cut for $g = 2$ and $(a_1, a_2) = (3, -2)$

We claim that we can choose the vector $(a_1, \ldots, a_g)$ so that the ring around the corresponding cut evaluates under $\rho$ to $s$.

Our next task, then, is to determine how the image under $\rho$ of the ring around one of these cuts depends on the given vector. Well, observe that this ring is homologous in $H_1(S^3 - F)$ to

$$\sum_{i=1}^g \left( a_i (\tau^+ x_{2i-1} - \tau^- x_{2i-1}) + (\tau^- x_{2i} - \tau^+ x_{2i}) \right),$$

where, recall, $\tau^\pm x$ denotes the push-off from the Seifert surface of a curve $x$ in the positive (resp. negative) direction. Furthermore, note for all $i$ that $\tau^+ x_{2i-1} - \tau^- x_{2i-1}$ is homologous to $\xi_{2i}$ and $\tau^- x_{2i} - \tau^+ x_{2i}$ is homologous to $\xi_{2i-1}$. Thus the ring around the cut corresponding to the vector $(a_1, \ldots, a_g)$ evaluates under $\rho$ to

$$(v_2)^{a_1} (v_4)^{a_2} \cdots (v_{2g})^{a_g}.$$

To finish the proof we ask: can we choose the vector $(a_1, \ldots, a_g)$ so that this expression evaluates to $s$? The answer is yes, because we assumed that $\rho$ was surjective. (Here are some quick details: Because $\rho$ is surjective, there will be some curve $\psi$ in the complement of $K$ mapping to $s$. Note that it will have to link $K$ an even number of times. So we can write $\psi$ as some product

$$\gamma^k_1 \psi_1 \gamma^k_2 \psi_2 \cdots \gamma^k_j \psi_j,$$

where $\gamma$ is some fixed loop based at the base-point $\star$ which intersects the Seifert surface exactly once, in the positive direction, where each $\psi_i$ is a loop based at $\star$ in the complement of the Seifert surface, and where $\sum_i k_i = 0$.

Each of these factors $\psi_i$ is mapped under $\rho$ to the element of $\mathcal{C}_n$ given by the formula:

$$\rho(\psi_i) = (v_2)^{\text{Link}(\psi_i, x_2)} (v_4)^{\text{Link}(\psi_i, x_4)} \cdots (v_{2g})^{\text{Link}(\psi_i, x_{2g})}.$$

So $\rho$ of the above product of curves, which equals $s$ by the choice of $\psi$, gives the desired expression for $s$.)

□
4.2.2. **Step 2: Kill** \(v_i\) for \(i > 1\). First, kill \(v_{2i}\) for \(i = 1, \ldots, g\) by Lemma [11](note that this leaves \(v_1\) untouched because \(s\) generates \(C_n\)). If \(v_3 = s^a\) and if \(a > 0\), first exchange \(v_1\) and \(v_2\) by band slides using Lemma [10](then slide bands as follows:

\[\begin{array}{c}
\text{1} \quad s \quad s^a \\
\text{1} \quad s \quad 1 \\
\end{array}\]

The second arrow is obtained by sliding both attaching segments of \(B_3\) and the left attaching segment of \(B_4\) all the way around the knot counterclockwise. This does not effect the colouring vector because \(v_4 = 1\).

Repeat the above steps \(n - a\) times. After this step (if we switch back \(v_1\) and \(v_2\)), \(v_3\) which has been killed while the rest of the colouring vector has been unchanged. Now slide \(B_3\) and \(B_4\) over \(B_5\) and \(B_6\):

\[\begin{array}{c}
\text{and repeat the sequence of slides which we used to kill } v_3 \text{ in order to kill } v_5. \end{array}\]

Repeat all steps above to kill \(v_{2i+1}\) for all \(i = 1, 2, \ldots, g - 1\), and the colouring vector becomes \(\vec{v} = (s, 1, \ldots, 1)^T\) as required.

4.2.3. **Step 3: Surgery to reduce genus.** Now that we have a band projection with respect to which \(\vec{v} = (s, 1, \ldots, 1)^T\), we can reduce genus by surgery. Assume that \(g > 1\). By surgery we trivialize bands \(B_i\) for \(i > 2\), starting from the right.

If \(B_i\) links with \(B_{2g}\) for some \(i < 2g\), we may isotopy \(B_i\) to make sure it passes first over and then under \(B_{2g}\) with respect to the orientation of \(x_{2g}\):

\[\begin{array}{c}
\text{Denote by } \bar{C} \text{ and by } \bar{D} \text{ collective linkage of other bands with } B_{2g} \text{ and } B_{2g-1} \text{ correspondingly. Consider Figure [19] } \end{array}\]
Denoting by $\rho(\delta_i)$ the $\rho$–image of the Wirtinger generator corresponding to $\delta_i$, we have

$$v_1 = \rho(\delta_{4g} \cdot \delta_{4g-1}) = \rho(\delta_{4g-3} \cdot \delta_{4g-2}) = 1$$

$$v_2 = \rho(\delta_{4g-1} \cdot \delta_{4g-2}) = \rho(\delta_{4g-4} \cdot \delta_{4g-3}) = 1.$$ 

Therefore $\rho(\delta_{4g}) = \rho(\delta_{4g-1}) = \rho(\delta_{4g-2}) = \rho(\delta_{4g-3}) = \rho(\delta_{4g-4})$. We also know that conjugation of $\rho(\delta_1)$ by the $\rho$–image of all the arcs in $\bar{C}$ which cross over $B_{2g}$ gives $\rho(\delta_{1g-3})$, which is equal to $\rho(\delta_{4g})$. In other words, conjugation of $\rho(\delta_{4g})$, which equals $ts^a$ for some $a \in \mathbb{Z}_n$, by the $\rho$–image of a $+1$–framed component $C'_1$ which loops once around $\bar{C}$ equals $\rho(\delta_{1g})$. The component $C'_1$ is in $\pi_1(S^3 - F)$ (let’s allow ourselves to confuse curves with the homotopy classes which they represent) so $\rho(C'_1) = s^b$ for some $b \in \mathbb{Z}_n$ and therefore

$$ts^a = s^{-b} \cdot ts^a \cdot s^b = ts^{a+b}$$

which is possible only if $b = 0$. Thus $C'_1$ is in ker $\rho$, and a $+1$–framed component $C_1$ which loops once around $\bar{C}$ and once around $B_{2g}$ is also in ker $\rho$ (see Figure 19). By performing surgery along $C_1$ we may unlink $\bar{C}$ and $B_{2g}$. Thus after this step $B_{2g}$ won’t be linked with any other bands.

Next, using the fact that $v_{2g} = 1$, untwist $B_{2g}$ by surgery on $\pm 1$–framed components which ring $B_{2g}$:

$$ts^a \quad \Leftrightarrow \quad ts^a$$

(4.3)

Finally, if $B_{2g}$ is knotted then it can be untied by surgery in ker $\rho$.

$$ts^a \quad \Leftrightarrow \quad ts^a$$

(4.4)
The resulting diagram represents a $D_{2n}$-coloured knot of lower genus that the one we began with, as $B_{2g}$ and $B_{2g-1}$ unravel and vanish (see Figure 20).

![Figure 20](image)

**Figure 20.** After untwisting $B_{2g}$, untying it, and unlinking everything from it.

### 4.3. Genus one knots

In the last section we saw that any $D_{2n}$-coloured knot $(K, \rho)$ is $\rho$-equivalent to a $D_{2n}$-coloured genus one knot $(K', \rho')$. Let $(K', \rho')$ be given in band projection, with respect to which it has surface data:

\[
(S, \vec{v}) := \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{12} + 1 & a_{22} \end{array} \right), \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right)
\]

(4.5)

We may change any crossing between a band and itself by $\rho$-equivalence (Equation 4.4). We may thus take $B_1$ and $B_2$ to be unknotted. Since two-component homotopy links are uniquely characterized by their linking number (i.e. $\pi_1(N(x_1)) \simeq \mathbb{Z}$), we may view $(S, \vec{v})$ as giving rise to a unique $D_{2n}$-coloured knot. From now on we shall do so, and $B_1$ and $B_2$ will always be assumed to be unknotted.

By Lemma 10 we may take $(v_1, v_2) = (s, 1)$, fixing the colouring vector (surjectivity of $\rho$ implies non-vanishing of the colouring vector). Now that $v_2 = 1$ we may add or subtract twists from $B_2$ by $\rho$-equivalence as in 4.3. Therefore we may set $a_{22}$ to any integer we please. Let’s set it to $\frac{1-n}{2}$.

Since $\vec{w}^T \cdot S \cdot \vec{w} = 0 \mod n$ (Lemma 9 recalling that $\vec{w} = (w_1, \ldots, w_{2g})$ with $v_i = s^{w_i}$), we also know that $a_{11} = 0 \mod n$. We may add or subtract $n^2$ full twists in $B_1$ by surgery on a unit–framed component which rings $n$ times around $B_1$, thus setting $a_{11} = kn$ for some $0 \leq k \leq n-1$. This is illustrated below in the case $n = 3$ (with the number of full twists indicated near the bands), where the second surgery is there to keep $B_1$ unknotted.
Next, we may add or subtract $n$ from $a_{12}$ by $\pm 1$–framed surgery on a component $C$ which has no self-intersections in the projection to the plane which gives the band projection of $K'$, and for which $\text{Link}(C, B_1) = n$ and $\text{Link}(C, B_2) = 1$ ($C$ is in $\ker \rho$). The surgery is illustrated below in the case $n = 3$:

Cancel the twists added in $B_1$ and $B_2$ by surgery on a unit–framed component which rings $n$ times around $B_1$ as in (4.6) and surgery on a unit–framed component which rings once times around $B_2$ as in (4.3).

Thus we may choose $0 \leq a_{12} < n$. Because for $\vec{x} = (0, 1)^T$ we have $\vec{x}^T \cdot S \cdot \vec{w} = 0 \mod n$ (Lemma 9) and because also $a_{12} + 1 = a_{21}$, this implies that we may choose $a_{12} = \frac{n-1}{2}$.

We are almost there— we have shown that any $D_{2n}$-coloured knot is equivalent to a $D_{2n}$-coloured knot for which there exists a coloured Seifert matrix of the form:

\[(4.8) \quad (S, \vec{v}) = \left( \begin{pmatrix} kn & n-1 \\ \frac{n+1}{2} & \frac{n}{2} \end{pmatrix}, \begin{pmatrix} s \\ 1 \end{pmatrix} \right) \]

with $k = 0, \ldots, n - 1$.

To simplify one step further, to get the easiest knot to lift that we can, perform one additional band slide:

\[(S, \vec{v}) \mapsto \left( \begin{pmatrix} kn + n+1 \\ \frac{n+1}{2} & \frac{n}{2} \end{pmatrix}, \begin{pmatrix} s \\ 1 \end{pmatrix} \right) \]

The surface data $(S, \vec{v})$ uniquely determines a genus one $D_{2n}$-coloured knot if we assume unknotted bands (as we do). We denote this knot $(B_k, \rho_k)$ (see Figure 21).
where the thick line "−−−−−−−" denotes an arbitrary number of strands). It is the pretzel knot \( p(2kn + 1, -1, -n) \) with a certain \( D_{2n} \)-colouring. In this section we have shown that any knot is \( \rho \)-equivalent to such a knot for some \( k = 0, \ldots, n - 1 \), proving Theorem 3.

### 4.4. Constructing the cover.
In the previous section we proved that for any \( D_{2n} \)-coloured knot \((K, \rho)\) with coloured untying invariant \( 0 \leq k < n \) there exists a \( \pm 1 \)-framed link \( L \) in the complement of the \( D_{2n} \)-coloured pretzel knot \((B_k, \rho_k)\) of Figure 21 whose components are unknotted and in \( \ker \rho \), and surgery along which recovers \((K, \rho)\). The goal of this section is to construct the branched dihedral covering space and covering link corresponding to this data, and to lift the surgery information to this cover.

#### 4.4.1. Language and Notation.
Coordinates in \( \mathbb{R}^3 \subset S^3 \) will be employed to explicitly describe configurations of objects in 3-space. Denote by \( \Sigma \subset \mathbb{R}^2 \) the surface arising from a sufficiently large disc when small discs centred at the points \((-2, 0), (-1, 0), (1, 0) \) and \( (2, 0) \) are removed. The surgery link \( L \) will lie inside \( \Sigma \times [0, 1] \subset \mathbb{R}^3 \). We will think of \( L \) as being the closure of a \( \pm 1 \) framed tangle \( T \) such that diagram of \( L \) arising from the projection onto \( \Sigma \times \{0\} \) is as pictured in Figure 22.

The knot \( B_k \) over which we’ll be taking a branched dihedral cover can be assumed to live in \( \mathbb{R}^3 - (\Sigma \times [0, 1]) \), as pictured in Figure 22 (using the convention that the coordinate \( x_2 \) increases into the page).

The \( D_{2n} \)-colouring \( \rho \) induces a representation from \( \pi_1 (\Sigma \times [0, 1]) \) into \( D_{2n} \), which we shall also call \( \rho \) by abuse of notation. To describe this representation, choose a base point for \( \Sigma \times [0, 1] \) lying on the surface \( \Sigma \times \{0\} \), and specify the images of generators for with respect to this basepoint as shown in Figure 23. The two ‘outer’ generators map to \( ts \), while the two ‘inner’ generators map to \( t \).
4.4.2. Constructing $\Sigma \times [0,1]$. We now have language and notation which is sufficiently explicit to describe the construction of $\Sigma \times [0,1]$, the dihedral cover of $\Sigma \times [0,1]$ with respect to $\rho$, and its embedding in the branched dihedral covering over $(B_k, \rho_k)$, which is $S^3$ because $B_k$ is a 2–bridge knot (see e.g. [1]).
We build $\Sigma \times [0, 1]$ embedded in $\mathbb{R}^2 \times [0, 1] \subset \mathbb{R}^3$ by slotting together copies of $\Sigma \times [0, 1]$ cut open along planes, as shown in Figure 25. These are our “lego blocks”, which we can bend, stretch, and shrink. To construct $\Sigma \times [0, 1]$ (embedded in $\mathbb{R}^3$) we take $n$ blocks denoted $X_1, \ldots, X_n$ (copies of the cut-open $\Sigma \times [0, 1]$ of Figure 25) and slot them together in the usual way: always matching an $A$ to an $A'$, and so on, and using the representation to decide which copy of the 3-cell one passes to when crossing a cut.

To see that combinatorially the blocks end up lined up in a line, consider the graph with $n$ vertices labeled $1, \ldots, n$ and an arc connecting vertices $i$ and $j$ if and only if $X_i$ and $X_j$ are incident in $\Sigma \times [0, 1]$, i.e. if and only if we slot $X_i$ and $X_j$ together, which is if and only if $t(i) = j$ or $ts(i) = j$ where $D_{2n}$ is acting on $1, \ldots, n$ by symmetries of the regular $n$-gon (remember that when crossing a cut labeled $t$ we are going to be crossing from $X_i$ to $X_{t(i)}$, and similarly for $ts$). When $n = 7$ the graph is given in Figure 26. Because $t(1) = 1$ and $ts(\frac{n+3}{2}) = \frac{n+3}{2}$, the graph will consist of two loops and a path from $1$ to $\frac{n+3}{2}$.

Now that we know what $\Sigma \times [0, 1]$ looks like combinatorially, we describe its embedding in $\mathbb{R}^3$ which we will use in the presentation of the final result. For this purpose it is useful to notice that the construction of $\Sigma \times [0, 1]$ defines a permutation...
$\tau$ of $1, \ldots, n$, taking $i$ (representing $X_i$) to the position of $X_i$ on the path from 1 to $\frac{1}{\gamma}$ (one plus its distance on the graph from the vertex labeled 1). Thus for Figure 26:

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 6 & 7 & 5 & 3 \end{pmatrix}$$

Now, for each $i = 1, \ldots, n$, if $\tau(i)$ is even, bend the arms of the dumbbell (Figure 25) down and place the block $X_i$ in $\mathbb{R}^3$ the position shown in Figure 27. If $\tau(i)$ is even, bend the arms up and place the result in the position shown in Figure 28. The reader can observe that the resulting identifications are exactly those determined by the representation. Finish the construction by gluing up the four remaining pairs of cuts— the cuts next to each other around the points $(-n-1,0,0)$, $(-1,0,0)$, $(1,0,0)$ and $(n+1,0,0)$.

For example, the result for $n = 3$ is displayed in Figure 29.
4.5. The branching set. We began this section with a framed link in $\Sigma \times [0, 1]$ in the complement of a coloured knot $B_k$. Since $B_k$ happens to be a 2–bridge knot, its dihedral covering space is $S^3$ with an $\frac{n+1}{2}$–component covering link embedded in it (see e.g. [1]). It remains for us to describe this link, and show how $\Sigma \times [0, 1]$ embeds into its complement.

To present the result we need to introduce some additional notation. The result will use certain braids on $2(n+1)$ strands. The strands of the braids will be indexed by the set

$I_n = \{-n-1 \leq i \leq n+1, i \in \mathbb{Z}/\{0\}\}.$

The coordinate $x_3$ will be the vertical coordinate of the braid, and the projections of the endpoints of the strands to the $(x_1, x_2)$–plane will be the points $\{(x, 0), x \in I_n\}$. (Note that these are precisely the coordinates of the ‘holes’ in the construction we just gave of $\widetilde{\Sigma} \times [0, 1]$.)

Let $i < j$ be indices from $I_n$. Let $X[i, j]$ denote the braid you get by putting a clockwise half-twist into the group of strands starting with the strand at position $i$, up to the strand at position $j$. For example, if $n = 4$, then $X[-2, 2]$ denotes the braid shown in Figure 30.

We can now state the result.

**Theorem 14.** Take the construction given earlier of $\Sigma \times [0, 1]$ as a subset of $\mathbb{R}^3$. The branching set over $B_k$ lies in its complement as shown in Figure 31, where $B$ denotes the braid:

$$X[-n, n] \cdot X[-n+1, n-1] \cdots X[-2, 2] \cdot X[-1, 1].$$

5. Odds and Ends

In this section we consider several corollaries to the constructions given in the previous sections. In Section 5.1 we list some different choices of complete sets
Figure 30. The half-twist $\mathcal{X}[-2, 2]$.

5.1. Different choices for a complete set of base-knots. Our choice of $(B_k, \rho_k)$ as a complete set of base-knots was made because we have an explicit algorithm to reduce any $D_{2n}$-coloured knot to one of them by surgery, and because in addition we know how to explicitly find their branched dihedral covering spaces, covering links, and the lifts of the surgery presentations. This set was found by trial and
error. Other complete sets of base-knots are possible of course, and some of these have advantages over \((B_k, \rho_k)\).

Our starting point is a genus one knot with unknotted bands and with the surface data given by Equation 4.8 repeated here for the reader’s convenience.

\[
(S, \vec{v}) = \left( \frac{kn}{n+1}, \frac{n-1}{2n}, \frac{n+1}{2}, 1, s \right)
\]

5.1.1. Linking number zero with the distinguished component. In this section we prove that we may choose a separated dihedral surgery presentation such that the curves in \(\ker \rho\) all have linking number zero with the distinguished surgery component. First perform the band slide we did in order to obtain \((B_k, \rho_k)\):

\[
(S, \vec{v}) = \left( \frac{kn}{n+1}, \frac{n-1}{2n}, \frac{n+1}{2}, 1, s \right) \mapsto \left( \frac{kn + n + 1}{n+1}, 0, \frac{1}{1-n}, s^{-1} \right)
\]

Perform \(\frac{n+1}{2}\) additional surgeries between the bands:

\[
(S, \vec{v}) \mapsto \left( \frac{(k+1)n + 1}{n+1}, \frac{n+1}{2}, s^{-1} \right)
\]

Slide \(B_2\) over \(B_1\) repeatedly \(\frac{n+1}{2}\) times:

\[
(S, \vec{v}) \mapsto \left( \frac{(k+1)n + m + 1}{1}, 0, s^{-1} \right)
\]

where \(k' = 0, \ldots, n-1\) and \(m = \frac{n+1}{2} - 2 \sum_{i=1}^{n+1} i\). If \(\frac{n+1}{2}\) is even, then \(m = -\frac{(n+1)^2}{2}\), while if \(\frac{n+1}{2}\) is odd then \(m = 1 - \frac{n+1}{2}\). This is the twist knot with \((k+1)n + m + 1\) twists. Untie this knot by a single surgery as shown in Figure 32 where we redefine \(k' := k+1\) and \(m' := m+1\). Put this into a separated dihedral surgery presentation by untying the distinguished surgery component by surgery in \(\ker \rho\). We obtain a separated dihedral surgery presentation where all surgery components in \(\ker \rho\) have linking number zero not only with the knot, but also with the distinguished surgery component.

Figure 32. Untying the twist knot.
5.1.2. Torus knot presentation. By constructing complete sets of base knots with cardinality $n$ in previous sections, we proved Corollary 4 which states that two knots are $\rho$-equivalent if and only if they have the same coloured untying invariant. As calculated in [14] (see also [12]), the left-hand $((2k+1)n, 2)$–torus knots of Figure 33 are examples of $D_{2n}$-coloured knots with coloured untying invariant $k = 1, \ldots, n$. Thus we have:

**Corollary 15.** The knots depicted in Figure 33 (the $((2k+1)n, 2)$–torus knots with the given colouring for $k = 1, \ldots, n$) comprise a complete set of base-knots for $D_{2n}$.

The surgery presentation of the branched dihedral covering and of the covering link which this picture gives is:

![Diagram](image)

with the thick line denoting $n + 1$ parallel strands and with

![Diagram](image)

being the lift of the covering link, slotted into the lift of the torus knot at the dotted line, where the strands of the covering link of the torus knot thread up out of the page through the holes indicated.

5.1.3. One knot, different colourings. We can choose a complete set of base-knots as a fixed knot $K$ whose colouring varies. Use [4.2] to kill $a_{22}$ and then for $a_{11} = kn$
slide $B_2$ over $B_1$ counterclockwise repeatedly $k$ times. We obtain:

\[(S, \vec{v}) \mapsto \begin{pmatrix} 0 & \frac{n-1}{2} \\ \frac{n+1}{2} & 0 \end{pmatrix}, \begin{pmatrix} s \\ s^k \end{pmatrix} \]

Since the coloured Seifert matrix uniquely characterizes a $D_{2n}$-coloured knot modulo $\rho$-equivalence, this gives a minimal complete set of base-knots, as in Figure \[6\].

5.2. Visualizing dihedral actions on manifolds. The following section deals with an observation due to Makoto Sakuma, that Corollary 15 implies a visualization theorem for $D_{2n}$ actions on manifolds. We summarize his argument, essentially contained in \[21\].

Let $D_{2n}$ act on a closed oriented connected 3–manifold $M$ via orientation preserving diffeomorphisms $f := (f_t, f_s)$ where $f_t^n = f_s^n = 1$, and $f_t f_s f_t = f_s^{n-1}$. Actually the assumption that $f(t)$ and $f(s)$ are smooth may be replaced by the weaker assumption that they be locally linear \[21, \text{Remark 2.3}\]. Viewing the 3–sphere as a one point compactification of $\mathbb{R}^3$, the claim is then that $M$ has a surgery presentation $L \subset S^3$ such that $L$ is invariant under $\frac{2\pi}{n}$ rotation around the $Z$–axis and under $\pi$ rotation around the $X$–axis as a framed link.

The proof is by taking the quotient smooth orbifold $\mathcal{O} := M/D_{2n}$ (see \[5, \text{Section 2.1}\]), with singular set $\Sigma$. So $\text{pr} : M \to \mathcal{O}$ is a $2n$–fold regular dihedral covering space (see \[20\]) with monodromy given by a representation $\psi : \pi_1(\mathcal{O} - \Sigma) \to D_{2n}$ induced by the action of $f$. The idea is to construct a surgery link $\mathcal{L}$ to make the following diagram commute:

\[\begin{array}{ccc}
M & \xrightarrow{\text{surg}(\mathcal{L})} & S^3 \\
\downarrow \text{pr}_\psi & & \downarrow \text{pr}_{\psi_1} \\
\Sigma \subset \mathcal{O} & \xrightarrow{\text{surg}(\mathcal{L})} & S^3 \supset \mathcal{L} \cup t((2k+1)n, 2)
\end{array}\]

where $\text{surg}(-)$ performs surgery by its argument (note that this is not a map), and $\Sigma$ and $t((2k+1)n, 2)$ are the covering loci. The lifted link $\mathcal{L}$ will then have the required dihedral symmetry by construction, inherited from the dihedral symmetry of $t((2k+1)n, 2)$ lying symmetrically along a torus.

The link $\mathcal{L}$ is constructed as the combination of two framed links $\mathcal{L}_1 \cup \mathcal{L}_2$ such that

1. The sublink $\mathcal{L}_1$ is in $\ker \rho$, its components are $\pm 1$–framed and are unknotted, and $\text{surg}(\mathcal{L}_1) : S^3 \to S^3$ takes $t((2k+1)n, 2)$ to some $D_{2n}$-coloured knot $(K', \rho')$.

2. For the sublink $\mathcal{L}_2$, the procedure $\text{surg}(\mathcal{L}_2) : S^3 \to \mathcal{O}$ takes $(K', \rho')$ to $(\Sigma, \psi)$.

The sublink $\mathcal{L}_1$ is given to us by Corollary \[15\] while $\mathcal{L}_2$ may be constructed in complete analogy with \[21\] Pages 383–384 and Section 4.

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