SECOND DERIVATIVE ESTIMATES FOR UNIFORMLY ELLIPTIC OPERATORS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we obtain a uniform \(W^{2,\varepsilon}\)-estimate of solutions to the fully nonlinear uniformly elliptic equations on Riemannian manifolds with a lower bound of sectional curvature using the ABP method.

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1. Introduction

We study regularity estimates for solutions to a class of the fully nonlinear uniformly elliptic equations

\[ F(D^2 u, x) = f \quad \text{in } B_R(z_0) \subset M. \]

on a complete Riemannian manifold \(M\), where the operator \(F\) satisfies the hypothesis \(H1\). Under the assumption that sectional curvature of \(M\) is nonnegative, the Krylov-Safonov Harnack estimate \([KS]\) was initiated by Cabré in his paper \([Ca]\), where a priori global Harnack inequality for linear elliptic equations was established by obtaining the Aleksandrov-Bakelman-Pucci (ABP) estimate on \(M\). Later, Kim \([K]\) improved Cabré’s result removing the sectional curvature assumption and imposing the certain conditions on the squared distance function. Recently, Wang and Zhang \([WZ]\) proved a version of the ABP estimate on \(M\) with a lower bound of Ricci curvature, and hence a locally uniform Harnack inequality for nonlinear elliptic operators on \(M\) provided that the sectional curvature is bounded from below. A priori Harnack estimates have been extended in \([KL]\) for viscosity solutions using the regularization of Jensens sup-convolution on Riemannian manifolds. The Hölder continuity is obtained as an immediate consequence of the Harnack inequality. In \([KKL, KL]\), the parabolic Harnack inequality and the ABP-Krylov-Tso type estimate were established in the Riemannian setting.

In this paper, we investigate a uniform \(W^{2,\varepsilon}\)-regularity (for some \(\varepsilon > 0\)) of solutions to \([1]\) on Riemannian manifolds with a lower bound of sectional curvature. In the Euclidean
space, a uniform $W^{2,\varepsilon}$-estimate (for some $\varepsilon > 0$) for linear, nondivergent elliptic operators with measurable coefficients was first discovered by Lin [1]. It is known that for any $p \geq 1$, a uniform $W^{2,p}$-estimate for uniformly elliptic equations with measurable coefficients is not valid; see [PT], [U]. In [C] [C, Chapter 7], Caffarelli dealt with $W^{2,\varepsilon}$-estimates for fully nonlinear elliptic operators, where the ABP estimate is a keystone in the proof together with the Calderón-Zygmund technique. The ABP estimate proved by Aleksandrov, Bakelman, and Pucci in sixties has played a crucial role in the Krylov-Safonov theory for nondivergent elliptic equations with measurable coefficients, and in the development of the regularity theory for fully nonlinear equations.

Making use of the ABP type estimate on Riemannian manifolds, we follow Caffarelli’s approach to extend $W^{2,\varepsilon}$-estimates for fully nonlinear elliptic operators on Riemannian manifolds under the assumption that sectional curvature is bounded from below. It can be checked that the a straightforward adaptation of the Euclidean method yields the $W^{2,\varepsilon}$-estimate on Hadamard manifolds which are complete and simply-connected Riemannian manifolds with nonpositive sectional curvature everywhere. In general, it is not applicable directly due to the existence of the cut locus. Indeed, it is difficult to use the squared distance functions as global test functions as in the Euclidean case. To proceed with the ABP method, we introduce the notion of the special contact set in Definition 3.5 and squared distance functions. With the help of the Calderón-Zygmund technique, the notion of the special contact set enables us to employ an iterative procedure using the ABP type estimate in Proposition 3.14. Therefore we deduce a (locally) uniform $W^{2,\varepsilon}$-estimate for a class of solutions to the fully nonlinear uniformly elliptic equations in Definition 2.8 which includes the solutions to (1).

Theorem 1.1 ($W^{2,\varepsilon}$-estimate). Let $M$ be a complete Riemannian manifold with the sectional curvature bounded from below by $-\kappa$ for $\kappa \geq 0$. Let $0 < R \leq R_0$ and $x_0 \in M$ and $f \in L^{2\eta}(B_{2R}(x_0))$ for $\eta := 1 + \log_2 \cosh(4\sqrt{\kappa}R_0)$. There exist uniform constant $\varepsilon > 0$ and $C > 0$ such that if a smooth function $u$ belongs to $S^r(\Lambda, \Lambda, f)$ in $B_{2R}(x_0)$, then we have that $u \in W^{2,\varepsilon}(B_R(x_0))$ with the estimate

$$\left( \frac{1}{\text{Vol}(B_R(x_0))} \int_{B_R(x_0)} |u|^{\varepsilon} + |\nabla u|^{\varepsilon} + |\nabla^2 u|^{\varepsilon} \right)^{\frac{1}{\varepsilon}} \leq C \left( \|u\|_{L^{2\eta}(B_{2R}(x_0))} + \left( \frac{1}{\text{Vol}(B_{2R}(x_0))} \int_{B_{2R}(x_0)} |R^2 f|^{\eta} \right)^{\frac{1}{\eta}} \right),$$

where $\varepsilon > 0$ and $C > 0$ depend only on $n, \Lambda, \Lambda$, and $\sqrt{\kappa}R_0$, and we denote $\int_Q f := \frac{1}{\text{Vol}(Q)} \int_Q f \text{ dVol}$.

When a Riemannian manifold has nonnegative sectional curvature, i.e., $\kappa = 0$, the $W^{2,\varepsilon}$-estimate is global, and depends only on dimension $n$, and the ellipticity constants $\Lambda$, and $\Lambda$.

2. Preliminaries

Throughout this paper, let $(M, g)$ be a smooth, complete Riemannian manifold of dimension $n$, where $g$ is the Riemannian metric. A Riemannian metric defines a scalar product and a norm on each tangent space, i.e., $\langle X, Y \rangle_x := g_*(X, Y)$ and $|X|^2_x := \langle X, X \rangle_x$ for $X, Y \in T_xM$, where $T_xM$ is the tangent space at $x \in M$. Let $d(\cdot, \cdot)$ be the Riemannian distance on $M$. For a given point $y \in M$, $d_*(x)$ denotes the distance to $x$ from $y$, i.e., $d_*(x) := d(x, y)$. A Riemannian manifold is equipped with the Riemannian measure $\text{Vol} = \text{Vol}_g$ on $M$ which is denoted by $|\cdot|$ for simplicity.
For a smooth function \( u : M \to \mathbb{R} \), the gradient \( \nabla u \) of \( u \) is defined by
\[
\langle \nabla u, X \rangle := du(X)
\]
for any vector field \( X \) on \( M \), where \( du : T M \to \mathbb{R} \) is the differential of \( u \). The Hessian \( D^2u \) of \( u \) is defined as
\[
D^2u(X, Y) := \langle \nabla_X \nabla u, Y \rangle,
\]
for any vector fields \( X, Y \) on \( M \), where \( \nabla \) denotes the Riemannian connection of \( M \). We observe that the Hessian \( D^2u \) is a symmetric 2-tensor over \( T \mathbb{M} \), and \( D^2u(x, y) \) at \( x \in M \) depends only on the values \( x, y \) at \( x \), and \( u \) in a small neighborhood of \( x \). By the metric, the Hessian of \( u \) at \( x \) is canonically identified with a symmetric endomorphism of \( T_x \mathbb{M} \):
\[
D^2u_x(X) := \nabla_X \nabla u_x, \quad \forall X \in T_x \mathbb{M}.
\]
We will write \( D^2u(x)(X, Y) = \langle D^2u(x) \cdot X, Y \rangle \) for \( X \in T_x \mathbb{M} \). In terms of local coordinates \( (x^i) \) of \( M \), the components of \( D^2u \) are written by
\[
(D^2u)_{ij} := \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial u}{\partial x^k},
\]
where \( \{\Gamma^k_{ij}\} \) are the Christoffel symbols of the Riemannian connection \( \nabla \) of \( M \). Here and in what follows, we adopt the Einstein summation convention. In [H] (see [Au]), the norm of the Hessian of \( u \), \( |D^2u| \) is defined in local coordinates by
\[
|D^2u|^2 := g^{ir} g^{js} (D^2u)_{ij} (D^2u)_{rs}.
\]
Denote by \( R \) the Riemannian curvature tensor defined as
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z
\]
for any vector fields \( X, Y, Z \) on \( M \). For two linearly independent vectors \( X, Y \in T_x \mathbb{M} \), the sectional curvature of the plane generated by \( X \) and \( Y \) is defined as
\[
\sec(X, Y) := \frac{\langle R(X, Y)X, Y \rangle}{|X|^2 |Y|^2 - \langle X, Y \rangle^2}.
\]
The Ricci curvature tensor denoted by \( \mathrm{Ric} \) is defined as follows: for a unit vector \( X \in T_x \mathbb{M} \) and an orthonormal basis \( \{X, e_2, \cdots, e_n\} \) of \( T_x \mathbb{M} \),
\[
\mathrm{Ric}(X, X) := \sum_{j=2}^n \sec(X, e_j).
\]
1. \( \mathrm{Ric} \geq \kappa \) on \( M \) \( (\kappa \in \mathbb{R}) \) stands for \( \mathrm{Ric}_x \geq \kappa g_x \) for all \( x \in M \). We refer to [D Le] for Riemannian geometry.

Assuming the Ricci curvature to be bounded from below, Bishop-Gromov’s volume comparison theorem says that the volume of balls does not increase faster than the volume of balls in the model space (see [V] for instance). In particular, the volume comparison implies the following (locally uniform) volume doubling property.

**Theorem 2.1** (Bishop-Gromov). Assume that \( \mathrm{Ric} \geq -(n-1)\kappa \) on \( M \) for \( \kappa \geq 0 \). For any \( 0 < r < R \), we have
\[
\frac{\mathrm{Vol}(B_{2r}(z))}{\mathrm{Vol}(B_r(z))} \leq 2^n \cosh^{-1}(2\sqrt{\kappa} R) =: \mathcal{D},
\]
where \( \mathcal{D} \) is the so-called doubling constant.
One can check that the doubling property \( \Box \) yields that for any \( 0 < r < R < R_0 \),
\[
\frac{\text{Vol}(B_r(z))}{\text{Vol}(B_{r}(z))} \leq \mathcal{D} \left( \frac{R}{r} \right)^n,
\]
for \( \mathcal{D} := 2^n \cosh^{-n} \left( 2 \sqrt{\kappa} R_0 \right) \). According to the volume comparison, it is easy to prove the following lemma. Below and hereafter, we denote \( \mathcal{D} \).

**Lemma 2.2.** Assume that for any \( z \in M \) and \( 0 < r < 2R_0 \), there exists a doubling constant \( \mathcal{D} > 0 \) such that
\[
\text{Vol}(B_{2r}(z)) \leq \mathcal{D} \text{Vol} (B_r(z)).
\]
Then we have that for any \( B_s(y) \subset B_r(z) \) with \( 0 < r < R < R_0 \),
\[
\left\{ \int_{B_s(y)} |\rho^2 f|^{\eta} \right\}^{\frac{1}{\eta}} \leq 2 \left\{ \int_{B_s(z)} |\rho^2 f|^{\eta} \right\}^{\frac{1}{\eta}}; \quad \eta := \frac{1}{n} \log_2 \mathcal{D}.
\]
In particular, if the sectional curvature of \( M \) is bounded from below by \( -\kappa (\kappa \geq 0) \), then (3) holds for \( \eta := 1 + \log_2 \cosh(4 \sqrt{\kappa} R_0) \).

A Hessian bound for the squared distance function is the following lemma which is proved in [CMS, Lemma 3.12] making use of the formula for the second variation of energy provided that the sectional curvature is bounded from below.

**Lemma 2.3.** Let \( x, y \in M \). If \( \text{Sec} \geq -\kappa (\kappa \geq 0) \) along a minimizing geodesic joining \( x \) to \( y \), then for any unit vector \( X \in T_x M \),
\[
\limsup_{r \to 0} \frac{d^2 \left( \exp_x rX \right) + d^2 \left( \exp_x -rX \right) - 2d^2 (x)}{r^2} \leq 2 \mathcal{H} \left( \sqrt{d_\gamma (x)} \right),
\]
where \( \mathcal{H}(t) := t \coth(t) \) for \( t \geq 0 \).

It is not difficult to obtain the following corollary modifying the proof of [CMS, Lemma 3.12] with the help of the monotonicity of a composed function \( \psi \).

**Corollary 2.4.** Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a smooth, even function such that \( \psi'(s) \geq 0 \) for \( s \in [0, +\infty) \). Let \( x, y \in M \). If \( \text{Sec} \geq -\kappa (\kappa \geq 0) \) along a minimizing geodesic joining \( x \) to \( y \), then for any unit vector \( X \in T_x M \),
\[
\limsup_{r \to 0} \frac{\psi \left( d^2 \left( \exp_x rX \right) \right) + \psi \left( d^2 \left( \exp_x -rX \right) \right) - 2\psi \left( d^2 (x) \right)}{r^2} \leq 2 \sqrt{d_\gamma (x)} \coth \left( \sqrt{d_\gamma (x)} \right) \psi' \left( d^2 (x) \right) + 4d_\gamma^2 (x) \psi'' \left( d^2 (x) \right).
\]

According to [CMS, Proposition 2.5], the cut locus of \( y \in M \) is characterized as the set of points at which the squared distance function \( d^2 \) is not smooth. We state it as a lemma which says that the semi-convexity of the squared distance functions fails at the cut locus.

**Lemma 2.5.** Let \( x, y \in M \). If \( x \in \text{Cut}(y) \), then there is a unit vector \( X \in T_x M \) such that
\[
\liminf_{r \to 0} \frac{d^2 \left( \exp_x rX \right) + d^2 \left( \exp_x -rX \right) - 2d^2 (x)}{r^2} = -\infty.
\]

Let \( M \) and \( N \) be Riemannian manifolds of dimension \( n \) and \( \phi : M \to N \) be smooth. The Jacobian of \( \phi \) is defined as
\[
\text{Jac} \phi(x) := |\det \phi(x)| \quad \text{for } x \in M.
\]
Now we state the area formula, which can be proved by using the area formula in Euclidean space and a partition of unity: For a Lipschitz continuous function $\phi : M \to M$, and a measurable set $E \subset M$, we have

$$\int_E \text{Jac} \phi \, d\text{Vol} = \int_M \mathcal{H}^0 \left( E \cap \phi^{-1}(y) \right) \, d\text{Vol}(y),$$

where $\mathcal{H}^0$ is the counting measure.

Now, we present a standard theorem called the weak type $(1, 1)$ estimate in the classical harmonic analysis, which will be used in the proof of our key estimate in Proposition\textsuperscript{[5,13]} 3.14 The proof relies on the volume doubling property and Vitali’s covering lemma; see [38] Chapter 1 for details.

**Lemma 2.6** (Weak type $(1, 1)$). Assume that $\text{Sec} \geq -\kappa$ for $\kappa \geq 0$ and let $x_0 \in M$ and $0 < R \leq R_0$. For an integrable function $f$ in $B_R(x_0)$, the Hardy-Littlewood maximal operator $m_{B_R(x_0)}$ over $B_R(x_0)$ is defined as

$$m_{B_R(x_0)}(f)(z) := \sup_{z \in B_R(x_0)} \int_{B_R(x)} |f| \, d\text{Vol}.$$

Then there exists a uniform constant $C_1 := 2D^{1+\log_2 5}$ for the doubling constant $D := 2^n \cosh^{-1}\left(2\sqrt{R_0}\right)$ such that

$$\left| \{z \in B_R(x_0) : m_{B_R(x_0)}(f)(z) \geq h\} \right| \leq C_1 \frac{\|f\|_{L^1(B_R(x_0))}}{h} \quad \forall h > 0.$$

For the rest of this section, we recall the concept of the uniformly elliptic operators. Let $\text{Sym} TM$ be the bundle of symmetric 2-tensors over $M$. An operator $F : \text{Sym} TM \times M \to \mathbb{R}$ is said to be uniformly elliptic with the so-called ellipticity constants $0 < \lambda \leq \Lambda$, if we have

(H1) $\lambda \text{trace}(P_x) \leq F(S_x + P_x, x) - F(S_x, x) \leq \Lambda \text{trace}(P_x), \quad \forall x \in M$

for any $S \in \text{Sym} TM$, and positive semi-definite $P \in \text{Sym} TM$. As extremal cases of the uniformly elliptic operators, Pucci’s operators are defined as follows: for any $x \in M$, and $S_x \in \text{Sym} TM_x$

$$M^+_{\lambda, \Lambda}(S_x) := \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$

$$M^-_{\lambda, \Lambda}(S_x) := \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $e_i = e_i(S_x)$ are the eigenvalues of $S_x$. We will usually drop the subscripts $\lambda$ and $\Lambda$, and write $M^\pm$. When $\lambda = \Lambda = 1$, $M^\pm$ simply coincide with the trace operator, that is, $M^\pm(D^2 u) = \Delta u$. We observe that (H1) is equivalent to the following: for any $S, P \in \text{Sym} TM$,

$$M^+(P_x) \leq F(S_x + P_x, x) - F(S_x, x) \leq M^-(P_x) \quad \forall x \in M.$$

The following lemma is concerned with basic properties of the Pucci operators; see \textsuperscript{[CC]} Chapter 2 for details.

**Lemma 2.7.** Let $\text{Sym}(n)$ denote the set of $n \times n$ symmetric matrices. For $S, P \in \text{Sym}(n)$, the followings hold:

(i) $M^+(S) = \sup_{A \in \mathcal{A}_{\lambda, \Lambda}} \text{trace}(AS)$, and $M^-(S) = \inf_{A \in \mathcal{A}_{\lambda, \Lambda}} \text{trace}(AS)$,

where $\mathcal{A}_{\lambda, \Lambda}$ consists of positive definite symmetric matrices in $\text{Sym}(n)$, whose eigenvalues lie in $[\lambda, \Lambda]$. 

(ii) $M^-(S) = -M^+(S)$.

(iii) $M^+(tS) = tM^+(S)$ for any $t \geq 0$.

(iv) $M^-(S + P) \leq M^-(S) + M^+(P) \leq M^+(S) + M^+(P)$.

In order to study a uniform $W^{2,\infty}$-regularity for a class of uniformly elliptic equations such as \([1]\), we introduce a more general class of solutions to the uniformly elliptic equations by using the Pucci operators as in \([CC]\) Chapter 2. We notice that the solution to the fully nonlinear elliptic equation \([1]\) belongs to the class $S^* (\lambda, \Lambda, f - F(0, \cdot))$ below.

**Definition 2.8.** Let $\Omega \subset M$ be open, and let $0 < \lambda \leq \Lambda$. We define a class of supersolutions $\mathcal{S}(\lambda, \Lambda, f)$ by the set of $u \in C^2(\Omega)$ satisfying

$$M^-(D^2u) \leq f \quad \text{a.e. in } \Omega.$$ 

Similarly, a class of subsolutions $\mathcal{S}(\lambda, \Lambda, f)$ is defined as the set of $u \in C^2(\Omega)$ such that

$$M^+(D^2u) \geq f \quad \text{a.e. in } \Omega.$$ 

We also define

$$S^* (\lambda, \Lambda, f) := \mathcal{S}(\lambda, \Lambda, f) \cap \mathcal{S}(\lambda, \Lambda, -|f|).$$

We write shortly $\mathcal{S}(f), S(f), \text{and } S^*(f)$ for $\mathcal{S}(\lambda, \Lambda, f), \mathcal{S}(\lambda, \Lambda, f)$, and $S^* (\lambda, \Lambda, f)$, respectively.

3. **Uniform $W^{2,\infty}$-estimate for elliptic operators**

3.1. **ABP type estimate.** We recall the ABP type estimate on Riemannian manifolds established by Cabrè \([Ca]\) that plays an important role in the proof of the Harnack inequality and $W^{2,\infty}$-estimate for fully nonlinear elliptic operators in nondivergence form. A major difficulty in proving the ABP estimate on manifolds is that non-constant affine functions cannot be generalized to an intrinsic notion on general manifolds. Cabrè \([Ca]\) Lemma 4.1] replaced affine functions by quadratic functions which are squared distance functions in order to show the ABP type estimate. On the basis of this idea, Wang and Zhang \([WZ]\) introduced the contact set defined as follows; see \([CC]\) \([S]\) \([W]\) for the Euclidean case.

**Definition 3.1** (Contact set). Let $\Omega$ be a bounded, open set in $M$ and let $u \in C(\Omega)$. For a given $h > 0$ and a compact set $E \subset M$, the contact set associated with $u$ of opening $h$ with vertex set $E$ is defined by

$$\mathcal{G}_h(u; E; \Omega) := \left\{ x \in \Omega : \inf_{y \in E} \left\{ u + \frac{h}{2} d^2(x) \right\} = u(x) + \frac{h}{2} d^2(x) \quad \text{for some } y \in E \right\}.$$

**Remark 3.2.** We recall the $c$-convexification of $u \in C(\Omega)$ for a given cost function $c(x, y)$, defined as

$$u^{cc}(x) := \sup_{y \in E \in \Omega} \left\{ u(z) + c(z, y) - c(x, y) \right\} \quad \forall x \in \Omega.$$ 

It is easy to check that $u \geq u^{cc}$ and $u^{cc}$ is continuous in $\Omega$ if $c$ is continuous. One can also prove that for $c(x, y) = \frac{h}{2} d^2(x, y), \quad \mathcal{G}_{\frac{h}{2}}(u; E; \Omega) = \{ x \in \Omega : u(x) = u^{cc}(x) \}$.

We refer to \([V]\) Chapter 5] for more details about $c$-convex functions.

To prove the ABP type estimate, Jacobian of the normal map $\phi$ on the contact set below, which corresponds to the image of the gradient mapping in the Euclidean space, was computed explicitly by Cabrè. The following lemma is an improved estimate by Wang and Zhang \([WZ]\) Theorem 1.2] using a standard theory of Jacobi fields.
Then we have that For \( z \)
\[ \phi(x) := \exp_x \nabla u(x). \]
For a given point \( x \in \Omega \), assume that \([0, 1] \ni t \mapsto \exp_x t \nabla u(x) \) is a unique minimizing geodesic joining \( x \) to \( \phi(x) \). Then we have
\[ \text{Jac} \phi(x) \leq \mathcal{J}^n \left( \sqrt{k} |\nabla u(x)| \right) \left\{ \mathcal{H} \left( \sqrt{k} |\nabla u(x)| \right) + \frac{\Delta u(x)}{n} \right\}^n, \]
where \( \mathcal{H}(\tau) = \tau \coth(\tau) \), and \( \mathcal{J}(\tau) = \sinh(\tau)/\tau \) for \( \tau \geq 0 \).

Using the Jacobian estimate of the normal map in Lemma 3.3, we have the following ABP type estimate in \([C\alpha, \text{Lemma 4.1}] \) and \([WZ, \text{Theorem 1.2}] \).

**Lemma 3.3.** Assume that \( \text{Ric} \geq -(n-1)\kappa \) for \( \kappa \geq 0 \). Let \( u \) be a smooth function in \( \Omega \subset M \). Define the normal map \( \phi : \Omega \to M \) as
\[ \phi(x) := \exp_x \nabla u(x). \]
For a given point \( x \in \Omega \), assume that \([0, 1] \ni t \mapsto \exp_x t \nabla u(x) \) is a unique minimizing geodesic joining \( x \) to \( \phi(x) \). Then we have
\[ \text{Jac} \phi(x) \leq \mathcal{J}^n \left( \sqrt{k} |\nabla u(x)| \right) \left\{ \mathcal{H} \left( \sqrt{k} |\nabla u(x)| \right) + \frac{\Delta u(x)}{n} \right\}^n, \]
where \( \mathcal{H}(\tau) = \tau \coth(\tau) \), and \( \mathcal{J}(\tau) = \sinh(\tau)/\tau \) for \( \tau \geq 0 \).

Using the Jacobian estimate of the normal map in Lemma 3.3, we have the following ABP type estimate in \([C\alpha, \text{Lemma 4.1}] \) and \([WZ, \text{Theorem 1.2}] \).

**Lemma 3.4 (ABP type estimate \([C\alpha, WZ]\))**. Assume that \( \text{Ric} \geq -(n-1)\kappa \) on \( M \) for \( \kappa \geq 0 \). For \( x_0 \in M \) and \( 0 < r \leq R \), assume that \( \partial B_r(x_0) \subset B_R(x_0) \). Let \( u \) be a smooth function on \( B_R(x_0) \) such that
\[ u \geq 0 \quad \text{on} \quad B_R(x_0) \setminus B_5(x_0) \quad \text{and} \quad \inf_{B_5(x_0)} u \leq 1. \]
Then we have that
\[ |B_5(x_0)| \leq \int_{\mathcal{G}_5(u \setminus B_5(x_0))} \mathcal{J}^n \left( \sqrt{k} |\nabla u| \right) \left\{ \mathcal{H} \left( \sqrt{k} |\nabla u| \right) + \frac{\Delta u}{n} \right\}^n, \]
where \( \mathcal{G}_5(u) := \mathcal{G}_5 \left( u; B_7(x_0); B_R(x_0) \right) \).

Making use of the ABP type estimate and well-understood barrier functions below, we will investigate in Lemma 3.4 the measure of the contact set at which the second derivatives of the supersolution have a uniform lower bound. First, we recall from \([C\alpha]\) the barrier function on Riemannian manifolds and its scale invariant properties.

**Lemma 3.5.** Assume that \( \text{Sec} \geq -\kappa \) for \( \kappa \geq 0 \). For \( x_0 \in M \) and \( 0 < r \leq R \), assume that \( \partial B_r(x_0) \subset B_R(x_0) \). For a given \( \delta \in (0,1) \), there exists a continuous function \( \psi_\delta = \psi_\delta(\cdot; r; x_0) \) in \( M \) defined by
\[ \psi_\delta(\cdot; r; x_0) := \psi_\delta \left( \frac{d^2}{r^2} \right) \]
for a smooth, even function \( \psi_\delta : \mathbb{R} \to \mathbb{R} \) with \( \psi_\delta' > 0 \) on \((0, +\infty)\), and \( \psi_\delta \) satisfies the following:
(i) \( \psi_\delta \geq 0 \) in \( M \setminus B_5(x_0) \),
(ii) \( \psi_\delta \leq 0 \) in \( B_2(x_0) \),
(iii) \( r^2 M (\Delta^2 \psi_\delta) + (n+1) \Lambda \mathcal{H}(2 \sqrt{R} \kappa) \leq 0 \) in \((B_7(x_0) \setminus B_6(x_0)) \setminus \text{Cut}(x_0) \),
(iv) \( r^2 \Delta \psi_\delta < C_\delta \text{Id in } B_R(x_0) \setminus \text{Cut}(x_0) \),
(v) \( \frac{1}{R} \sqrt{R} |\nabla \psi_\delta| < C_\delta \) in \( B_R(x_0) \setminus \text{Cut}(x_0) \),
(vi) \( \psi_\delta \geq -C_\delta \) in \( M \),
where \( \mathcal{H}(\tau) := \tau \coth(\tau) \) for \( \tau \geq 0 \). Here, the constant \( C_\delta > 0 \) depends only on \( \delta, n, \lambda, \Lambda \), and \( \mathcal{H}(2 \sqrt{R} \kappa) \) (independent of \( r \) and \( x_0 \)).

**Proof.** We give a sketch of the proof; see \([C\alpha, \text{Lemma 5.5}] \) and \([WZ, \text{Lemma 4.2}] \) for details. Fix \( 0 < \delta < 1 \) and consider
\[ \psi_\delta(s) := \left( \frac{3^2}{5^4} \right)^{\alpha \delta} - \left( \frac{s}{3^2} \right)^{\alpha \delta} \quad \text{for } s \geq \delta^2, \]
for a positive constant $\alpha$ to be chosen later, which will depend only on $\delta,n,\lambda,\Lambda$, and $\mathcal{H}(2\sqrt{R_0})$. After fixing a large constant $\alpha > 0$, we will extend $\psi_\delta$ smoothly in $\mathbb{R}$ to be an even function and to satisfy that $\psi' > 0$ in $(0, +\infty)$. Now we define

$$v_\delta(\cdot; r; z_0) := \psi_\delta\left(\frac{d_{z_0}}{r^2}\right) \text{ in } M,$$

where $d_{z_0}$ is the distance function to $z_0$. It is clear that (i), (vi) and (ii) hold for $\alpha \geq 1$.

In order to check (iii), we recall that a closed set $\text{Cut}(z_0)$ has measure zero and that

$$M^+ \left(\frac{d^2_{z_0}(x)}{r^2}\right) \leq 2nA \mathcal{H}\left(\sqrt{d_{z_0}(x)}\right) \leq nA \mathcal{H}(2\sqrt{R_0}) \quad \forall x \in B_{7r}(z_0) \setminus \text{Cut}(z_0),$$

from Lemma 2.3. As in the proof of [Ca, Lemma 5.5] and [WZ, Lemma 4.2], we can select $\alpha > 0$ sufficiently large so that (ii), (iii) hold, where $\alpha > 0$ depends only on $\delta, n, \lambda, \Lambda$, and $\mathcal{H}(2\sqrt{R_0})$. Lastly, for a fixed $\alpha > 0$, it is not hard to check (iv) and (v) using Corollary 2.4, since $\psi'_\delta(s)$ is bounded in $[0, +\infty)$, and $\psi'_\delta(s) \leq 0$ for $s \geq 1$. □

**Remark 3.6.** In Lemma 3.5, (iv), we have obtained an upper bound of $r^2D^2v_\delta(\cdot; r; z_0)$ in $B_{8r}(x_0) \setminus \text{Cut}(z_0)$. According to Corollary 2.4 we deduce that for any $x \in B_{8r}(x_0)$ and any unit vector $X \in T_xM$,

$$\limsup_{r \to 0} \frac{1}{r} \left| v_\delta(\exp_t tX; r; z_0) + v_\delta(\exp_{-t} tX; r; z_0) - 2v_\delta(x; r; z_0) \right| \leq \frac{C_\delta}{r^2},$$

where $C_\delta$ is the same constant as in Lemma 3.5.

Since the barrier function in Lemma 3.5 are not smooth on the cut locus of the center point, we need the following technical lemma to apply the ABP type estimate to a sum of smooth function and the scale invariant barrier function directly in Lemma 3.8.

**Lemma 3.7.** Assume that $\text{Sec} \geq -\kappa$ for $\kappa \geq 0$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $B_{7r}(z_0) \subset B_{8r}(x_0)$. Let $u$ be a smooth function on $B_{8r}(x_0)$, and let

$$w := u + h \sum_{j=1}^{k} r_j^2v_\delta(\cdot; r_j; z_j) + h \sum_{j=1}^{k} \frac{1}{2} d_{z_j}^2 + v_\delta(\cdot; r; z_0)$$

for $k \in \mathbb{N} \cup \{0\}$, where $h > 0$, $r_j > 0$, $B_{7r_j}(z_j) \subset B_{8r}(x_0)$, $y_j \in \overline{B_{7r_j}(z_j)}$ for $j = 1, \ldots, k$, and $v_\delta(\cdot; r; z)$ is the barrier function with respect to $r$ and $z$ as in Lemma 3.5. Assume that $x \in \overline{B_{7r}(z_0); B_{8r}(x_0)}$, that is, for some $y_0 \in B_{8r}(z_0)$,

$$\inf_{B_{8r}(x_0)} \left( w + \frac{1}{2r^2} d_{y_0}^2 \right) = w(x) + \frac{1}{2r^2} d_{y_0}^2(x).$$

Then we have the following:

(i) $x \notin \bigcup_{j=0}^{k} \text{Cut}(z_j) \cup \bigcup_{j=0}^{k} \text{Cut}(y_j)$

(ii) $w$ is smooth at $x$, and satisfies that $r^2|\nabla w(x)| = d_{y_0}(x) < 2R$ and

$$-\mathcal{H}(2\sqrt{R_0}) I \leq r^2 D^2 w(x) \leq r^2 D^2 u(x) + r^2 h k \left( C_\delta \mathcal{H}(2\sqrt{R_0}) I + C_\delta I \right)$$

where $C_\delta > 0$ is the constant as in Lemma 3.5.

**Proof.** Once (i) is proved, (ii) easily follows from Lemma 2.3 and Lemma 3.5. Note that $\mathcal{H}$ is nondecreasing in $[0, +\infty)$. So it suffices to show that $x \notin \bigcup_{j=0}^{k} \text{Cut}(z_j) \cup \bigcup_{j=0}^{k} \text{Cut}(y_j)$.

We will only prove that $x \notin \text{Cut}(z_0)$ since the proofs for the other cases are similar.
Suppose to the contrary that \( x \in \text{Cut}(z_0) \). Since \( w - w(x) \) lies above \(-\frac{1}{2rz^2}d_{y_0}^2 + \frac{1}{2rz^2}d_{j}^2(x)\), we take the second difference quotient and use Lemma 2.3 to have that for any unit vector \( X \in T_xM \),
\[
\left( r^2 D^2 u \cdot X, X \right)_x + r^2 \liminf_{t \to 0} \frac{1}{t^2} \left[ v_\delta \left( \exp_x tX; r; z_0 \right) + v_\delta \left( \exp_x -tX; r; z_0 \right) - 2v_\delta \left( x; r; z_0 \right) \right] \\
+ hr^2 \sum_{j=1}^k r_j \limsup_{t \to 0} \frac{1}{t^2} \left[ v_\delta \left( \exp_x tX; r_j; z_j \right) + v_\delta \left( \exp_x -tX; r_j; z_j \right) - 2v_\delta \left( x; r_j; z_j \right) \right] \\
+ hr^2 \sum_{j=1}^k \limsup_{t \to 0} \frac{1}{t^2} \left[ d_{y_j}^2 \left( \exp_x tX \right) + d_{y_j}^2 \left( \exp_x -tX \right) - 2d_{y_j}^2(x) \right] \\
\geq - \liminf_{t \to 0} \frac{1}{2r^2} \left[ d_{y_0}^2 \left( \exp_x tX \right) + d_{y_0}^2 \left( \exp_x -tX \right) - 2d_{y_0}^2(x) \right] \\
\geq - \mathcal{H} \left( \sqrt{d(x,y_0)} \right) \geq \mathcal{H} \left( 2\sqrt{d} \right).
\]

Combined with Remark 3.6 and Lemma 2.3, this implies that for any unit vector \( X \in T_xM \),
\[
r^2 \liminf_{t \to 0} \frac{1}{t^2} \left[ v_\delta \left( \exp_x tX; r; z_0 \right) + v_\delta \left( \exp_x -tX; r; z_0 \right) - 2v_\delta \left( x; r; z_0 \right) \right] \geq - C > -\infty.
\]

Using strict monotonicity of \( \psi_\delta \) in Lemma 3.5, we find a positive constant \( c_0 > 0 \) such that for small \( |t| \in (0, 1) \)
\[
\frac{v_\delta \left( \exp_x tX; r; z_0 \right) - v_\delta \left( x; r; z_0 \right)}{d_{y_0}^2 \left( \exp_x tX \right) - d_{y_0}^2(x)} \leq 2r^2 \psi_\delta \left( \frac{d_{y_0}^2(x)}{r^2} \right) =: c_0,
\]
where we notice that \( x \neq z_0 \) and hence \( \psi_\delta \left( \frac{d_{y_0}^2(x)}{r^2} \right) \) is positive since we assume \( x \in \text{Cut}(z_0) \). Thus we deduce that for any unit vector \( X \in T_xM \),
\[
\liminf_{t \to 0} \frac{d_{y_0}^2 \left( \exp_x tX \right) + d_{y_0}^2 \left( \exp_x -tX \right) - 2d_{y_0}^2(x)}{t^2} \geq \frac{1}{c_0} \liminf_{t \to 0} \frac{v_\delta \left( \exp_x tX; r; z_0 \right) + v_\delta \left( \exp_x -tX; r; z_0 \right) - 2v_\delta \left( x; r; z_0 \right)}{t^2} \geq - \frac{C}{c_0 r^2} > -\infty,
\]
which contradicts to the assumption that \( x \in \text{Cut}(z_0) \) from Lemma 2.3. Therefore, \( x \) is not a cut point of \( z_0 \), which finishes the proof.

In the following, we obtain the measure estimate of the contact set that consists of points, where \( u + v_\delta \) has a global tangent concave paraboloid from below. [Ca] Lemma 5.1, [Kr] Lemma 3.1 and [WZ] Proposition 4.1 dealt with estimates of the measure of the level sets of the solution \( u \) to establish pointwise estimates; the Harnack estimate and the weak Harnack inequality. In order to study the bound of the second derivatives of the solution, we keep the barrier function in the estimate of Lemma 3.5. This is the first step to estimate the distribution function of \( |D^2 u| \), the norm of the Hessian.

**Lemma 3.8.** Assume that \( \text{Sec} \geq -\kappa \) for \( \kappa \geq 0 \). For \( z_0, x_0 \in M \) and \( 0 < r \leq R \leq R_0 \), assume that \( B_{2r}(z_0) \subset B_R(x_0) \). Let \( u \) be a smooth function on \( \overline{B_R(x_0)} \) such that \( u \in \mathcal{S}(\Lambda, \lambda, f) \) in \( B_{r_1}(z_0) \).

\[
u \geq 0 \quad \text{on} \quad B_R(x_0) \setminus B_{5r}(z_0) \quad \text{and} \quad \inf_{B_{r}(z_0)} u \leq 1.
\]
For a given \( \delta \in (0, 1) \), there exist uniform constants \( \epsilon_0 > 0 \) and \( \mu_0 > 0 \) depending only on \( \delta, n, \lambda, \Lambda \) and \( \sqrt{r} R_0 \), such that if
\[
\left( \frac{1}{|B_{r \delta}(z_0)|} \int_{G_{r \delta}(u + v_\delta(\cdot; r; z_0)) \cap B_{r \delta}(z_0)} |r^2 f^{++}| \right)^{\frac{1}{2}} \leq \epsilon_0; \quad \eta := 1 + \log_2 \cosh(4 \sqrt{r} R_0),
\]
then
\[
\left| \frac{\partial}{\partial x} (u + v_\delta(\cdot; r; z_0)) \cap B_{r \delta}(z_0) \right| \geq \mu_0 > 0,
\]
where \( G_{r \delta}(u + v_\delta(\cdot; r; z_0)) := G_{r \delta}(u + v_\delta(\cdot; r; z_0); \overline{B_{r \delta}(z_0)}; B_{r \delta}(x_0)) \) and \( v_\delta(\cdot; r; z_0) \) is as in Lemma 3.3.

**Proof.** Let \( \Omega := B_{r \delta}(x_0), E := \overline{B_{r \delta}(z_0)} \), and
\[
w := u + v_\delta(\cdot; r; z_0).
\]
For \( x \in G_{r \delta}(w) := G_{r \delta}(w; E; \Omega) \), there exists \( y \in E \) such that
\[
\inf_{B_{r \delta}(x_0)} \left( w + \frac{1}{2r^2} d^2 \right) = w(x) + \frac{1}{2r^2} d^2(x).
\]
According to Lemma 3.7, one can check that \( x \notin \text{Cut}(z_0) \cup \text{Cut}(y) \),
\[
\nabla w(x) = -r^2 d(x, y) \nabla d(y, x), \quad D^2 \left( w + \frac{1}{2r^2} d^2 \right)(x) \geq 0, \quad \text{and} \quad y = \exp_\gamma r^2 \nabla w(x) \notin \text{Cut}(x).
\]
Now, consider the smooth function \( \phi : B_{r \delta}(x_0) \setminus \text{Cut}(z_0) \rightarrow M \) defined by
\[
\phi(x) := \exp_\gamma r^2 \nabla w(x).
\]
We use (4) and the properties (i),(ii) of \( v_\delta(\cdot; r; z_0) \) in Lemma 3.3 to deduce that
\[
B_{r \delta}(z_0) \subset \phi \left( G_{r \delta}(w) \cap B_{r \delta}(x_0) \right),
\]
where we refer to the proof of [Ca, Lemma 4.1] for details. From Remark 3.2 and Lemma 3.7, we observe that \( G_{r \delta}(w) \) is measurable and
\[
G_{r \delta}(w) \subset \left\{ x \in B_{r \delta}(x_0) \setminus \text{Cut}(z_0) : -C I < r^2 D^2 w(x) < C I, \ r^2 |\nabla w(x)| < 2 R_0 \right\} =: \Omega_0
\]
for some \( C > 0 \), where \( \phi \) is Lipschitz in a bounded, open set \( \Omega_0 \). Note that a closed set \( \text{Cut}(z_0) \) has measure zero. Now we apply the area formula to obtain
\[
|B_\varepsilon(z_0)| \leq \left| \frac{\partial}{\partial x} \left( \phi \left( G_{r \delta}(w) \cap B_{r \delta}(z_0) \right) \right) \right| \leq \int_{G_{r \delta}(w) \cap B_{r \delta}(z_0)} \text{Jac} \phi(x) d \text{Vol}(x).
\]
By making use of Lemma 3.3 and Lemma 3.7, we have that for \( x \in G_{r \delta}(w) \),
\[
\text{Jac} \phi(x) \leq \mathcal{H}^n \left( \sqrt{r} \nabla w \right) \left\{ \mathcal{H} \left( \sqrt{r} \nabla w \right) + \frac{r^2 \Delta w}{n} \right\}^n
\]
\[
\leq \mathcal{H}^n \left( 2 \sqrt{r} R_0 \right) \left\{ \mathcal{H} \left( 2 \sqrt{r} R_0 \right) + \frac{1}{\lambda} \left\{ \mathcal{M}^* \left( r^2 D^2 w \right) + n \Lambda \mathcal{H} \left( 2 \sqrt{r} R_0 \right) \right\} \right\}^n
\]
\[
\leq \frac{\mathcal{H}^n \left( 2 \sqrt{r} R_0 \right)}{\lambda^n} \left\{ \frac{r^2}{\lambda} \mathcal{M}^* \left( D^2 w \right) + (n + 1) \Lambda \mathcal{H} \left( 2 \sqrt{r} R_0 \right) \right\}^n
\]
\[
\leq \frac{\mathcal{H}^n \left( 2 \sqrt{r} R_0 \right)}{\lambda^n} \left\{ r^2 f^+ + r^2 \mathcal{M}^* \left( D^2 v_\delta(\cdot; r; z_0) \right) + (n + 1) \Lambda \mathcal{H} \left( 2 \sqrt{r} R_0 \right) \right\}^n,
\]
where we recall that $\mathcal{S}(r)$ and $\mathcal{H}(r)$ are nondecreasing for $r \geq 0$. From Lemma [3.5] we notice that

$$r^2 M^+ \left( D^2 v_\delta(\cdot; r; z_0) \right) + (n + 1) \Delta \mathcal{H} \left( 2 \sqrt{r} R_0 \right) \leq \left\{ n \Lambda C_\delta + (n + 1) \Lambda \mathcal{H} \left( 2 \sqrt{r} R_0 \right) \right\} \chi_{B_{R}(z_0)}$$

in $B_{7r}(z_0) \setminus \text{Cut}(z_0)$ for $C_\delta > 0$ as in Lemma [3.5] where $\chi_{B_{R}(z_0)}$ stands for the characteristic function. Combined with (4) and (6), this provides that

$$\left( \frac{|B_{r}(z_0)|}{|B_{7r}(z_0)|} \right)^{\frac{1}{r}} \leq \left( \frac{1}{|B_{7r}(z_0)|} \int_{G_{r}(w) \cap B_{7r}(z_0)} \text{Jac} \phi(x) \ d \text{Vol}(x) \right)^{\frac{1}{r}} \leq \tilde{C} \left( \frac{1}{|B_{7r}(z_0)|} \int_{G_{r}(w) \cap B_{7r}(z_0)} \bigl| r^2 f^\eta \bigr|^p \right)^{\frac{1}{p}} + \tilde{C} \left( \frac{|G_{r}(w) \cap B_{6r}(z_0)|}{|B_{7r}(z_0)|} \right)^{\frac{1}{p}}$$

for a uniform constant $\tilde{C} > 0$ depending only on $\delta, n, \Lambda, \text{ and } \sqrt{r} R_0$. Using Bishop-Gromov’s Theorem [2.1] we have that

$$\frac{|G_{r}(w) \cap B_{6r}(z_0)|}{|B_{7r}(z_0)|} \geq \frac{1}{C} \left( \frac{1}{|D|} \left( \frac{1}{7} \right)^{\log_2 D} \right) \geq 2 \mu_\delta$$

for $D := 2^n \cosh^{-1}(2 \sqrt{r} R_0)$. By selecting $\epsilon_\delta := \mu_\delta^{\frac{1}{p}}$, we conclude that

$$\frac{|G_{r}(w) \cap B_{6r}(z_0)|}{|B_{7r}(z_0)|} \geq \mu_\delta.$$ 

\[\square\]

**Corollary 3.9.** Assume that Sec $\geq -\kappa$ for $\kappa \geq 0$. For $z_0, x_0 \in M$ and $0 < r \leq R \leq R_0$, assume that $B_{7r}(z_0) \subset B_R(x_0)$. Let $u$ be a smooth function on $\overline{B_R}(x_0)$ and let

$$\tilde{u} := u + h \sum_{j=1}^k r_j v_\delta(\cdot; r_j; z_0) + h \sum_{j=1}^k \frac{1}{2} d_j,$$

for $k \in \mathbb{N}$, where $h > 0$, $r_j > 0$, $B_{7r_j}(z_j) \subset B_R(x_0), y_j \in \overline{B_{7r_j}(z_j)}$ for $j = 1, \ldots, k$. Assume that

$$\tilde{u} \geq 0 \quad \text{on } B_R(x_0) \setminus B_{5r}(z_0), \quad \inf_{B_{5r}(z_0)} \tilde{u} \leq 1,$$

and $\tilde{u}$ satisfies $\mathcal{M} \left( D^2 \tilde{u} \right) \leq f$ a.e. in $B_{7r}(z_0) \setminus \left( \bigcup_{j=1}^k \text{Cut}(y_j) \cup \bigcup_{j=1}^k \text{Cut}(y_j) \right)$ with

$$\left( \int_{B_{7r}(z_0)} \bigl| r^2 f^p \bigr| \right)^{\frac{1}{p}} \leq \epsilon_\delta.$$

Then we have

$$\frac{|G_{r}(\tilde{u} + v_\delta(\cdot; r; z_0)) \cap B_{6r}(z_0)|}{|B_{7r}(z_0)|} \geq \mu_\delta > 0,$$

where $G_{r}(\tilde{u} + v_\delta(\cdot; r; z_0)) := G_{r}(\tilde{u} + v_\delta(\cdot; r; z_0); \overline{B_{7r}(z_0)}; B_R(x_0))$ and the uniform constants $\epsilon_\delta, \mu_\delta > 0$ are as in Lemma [3.8].

**Proof.** Let $\tilde{w} := \tilde{u} + v_\delta(\cdot; r; z_0)$. From Lemma [3.7] we observe that

$$G_{r}(\tilde{w}) := G_{r}(\tilde{w}; \overline{B_{7r}(z_0)}; B_R(x_0)) \subset \Omega_0,$$
where
\[
\tilde{\Omega}_0 := \left\{ x \in B_R(x_0) \setminus \left( \bigcup_{j=0}^k \text{Cut}(z_j) \cup \bigcup_{j=1}^k \text{Cut}(y_j) \right) : -C1 < r^2D_0^2\tilde{w}(x) < C1, \quad r^2|\nabla\tilde{w}(x)| < 2R_0 \right\}
\]
for some \( C > 0 \). We note that \( G_{\tilde{w}}(\bar{B}) \) is measurable according to Remark 3.2 and that a closed set \( \bigcup_{j=0}^k \text{Cut}(z_j) \cup \bigcup_{j=1}^k \text{Cut}(y_j) \) has measure zero. Thus \( \tilde{w} \) is smooth in \( \tilde{\Omega}_0 \) and the function \( \tilde{\phi} : \tilde{\Omega}_0 \to M \) defined as
\[
\tilde{\phi}(x) := \exp_x r^2\nabla\tilde{w}(x),
\]
is Lipschitz continuous in \( \tilde{\Omega}_0 \). The remaining part of the proof is the same as the proof of Lemma 3.3. \( \square \)

3.2. Calderón-Zygmund Technique. We quote this subsection from [Ca, Section 6] to introduce the Calderón-Zygmund technique which is one of main tools for the proof of uniform \( L^p \)-estimates of the Hessian of solutions to uniformly elliptic equations. We first present the Christ decomposition [Ch], which generalizes the Euclidean dyadic decomposition for so-called “spaces of homogeneous type” (see Theorem 3.10). In harmonic analysis, a metric measure space \( \mathcal{X} = (X, d, \nu) \) is called a space of homogeneous type when a nonnegative Borel measure \( \nu \) satisfies the doubling property with a doubling constant \( D > 0 \):
\[
\nu(B_{2r}(x)) \leq D \nu(B_r(x)) < +\infty \quad \forall x \in X, \quad r > 0.
\]
A Riemannian manifold with nonnegative Ricci curvature has the volume doubling property with the doubling constant \( D = 2^a \). When a Riemannian manifold \( M \) has a negative lower bound of the Ricci curvature, the Riemannian measure of \( M \) has a locally uniform doubling property; see Bishop-Gromov’s Theorem [Bl]. As a matter of fact, one can see that the following Christ decomposition is valid for the metric measure space equipped with a local doubling measure.

**Theorem 3.10** (Christ). Assume that the Ricci curvature of \( M \) is bounded from below. There exist a countable collection \( \{ \mathcal{Q}^{k,\alpha} : k \in \mathbb{Z}, \alpha \in I_k \} \) of open subsets of \( M \) and positive uniform constants \( \delta_0 \in (0, 1), c_1 \) and \( c_2 \) (with \( 2c_1 \leq c_2 \) ) such that

(i) \( |M \setminus \bigcup_{\alpha \in I_k} \mathcal{Q}^{k,\alpha} | = 0 \) for \( k \in \mathbb{Z} \),
(ii) if \( l \leq k, \alpha \in I_k, \) and \( \beta \in I_l \), then either \( \mathcal{Q}^{k,\alpha} \subset \mathcal{Q}^{l,\beta} \) or \( \mathcal{Q}^{k,\alpha} \cap \mathcal{Q}^{l,\beta} = \emptyset \).
(iii) for any \( \alpha \in I_k \) and any \( l < k \), there is a unique \( \beta \in I_l \) such that \( \mathcal{Q}^{k,\alpha} \subset \mathcal{Q}^{l,\beta} \),
(iv) \( \text{diam}(\mathcal{Q}^{k,\alpha}) \leq c_2 \delta_0^{k} \)
(v) any \( \mathcal{Q}^{k,\alpha} \) contains some ball \( B_{c_1\delta_0}(x^{k,\alpha}) \).

The open set \( \mathcal{Q}^{k,\alpha} \) in Theorem 3.10 is called a dyadic cube of generation \( k \) on \( M \). The property (iii) asserts that for any \( \alpha \in I_k \), there is a unique \( \beta \in I_{k-1} \) such that \( \mathcal{Q}^{k,\alpha} \subset \mathcal{Q}^{k-1,\beta} \).
We call \( \mathcal{Q}^{k-1,\beta} \) the predecessor of \( \mathcal{Q}^{k,\alpha} \) which is denoted by \( \mathcal{Q}^{k,\alpha} \) for simplicity.

For the rest of the paper, we fix some small numbers;
\[
\delta := \frac{2c_1}{c_2} \delta_0 \in (0, \delta_0), \quad \text{and} \quad \delta_1 := \frac{\delta_0(1 - \delta_0)}{2} \in \left( 0, \frac{\delta_0}{2} \right),
\]
where \( \delta_0 \in (0, 1), c_1 \) and \( c_2 \) are the constants in Theorem 3.10. For a given \( R > 0 \), we define \( k_R \in \mathbb{N} \) to satisfy
\[
c_2 \delta_0^{k-1} < R \leq c_2 \delta_0^{k-2}.
\]
The number \( k_R \) means that a dyadic cube of generation \( k_R \) is comparable to a ball of radius \( R \). The following technical lemma is quoted from [Lemma 6.5], which deals with the relation between dyadic cubes and comparable balls.

**Lemma 3.11.** Let \( x_0 \in M \) and \( R > 0 \).

(i) If \( Q \) is a dyadic cube of generation \( k \) such that

\[
  k \geq k_R \quad \text{and} \quad Q \subset B_{2R}(x_0),
\]

then there exist \( \overline{\tau} \in Q \) and \( \overline{r}_k \in (0, R) \) such that

\[
  B_{\overline{\tau}}(\overline{\tau}) \subset Q \subset \overline{Q} \subset B_{\overline{\tau}}(\overline{\tau}) \subset B_{7R}(x_0)
\]

and

\[
  B_{5R}(x_0) \subset B_{7R}(\overline{\tau}).
\]

In fact, for \( k \geq k_R \), the radius \( \overline{r}_k \) is defined by

\[
  \overline{r}_k := \frac{c_1}{\delta} k^\delta = \frac{1}{2} c_2^\delta k^{\delta}.
\]

(ii) If \( Q \) is a dyadic cube of generation \( k_R \) and \( d(x_0, Q) \leq \delta_1 R \), then \( Q \subset B_{2R}(x_0) \) and hence (8) and (9) hold for some \( \overline{\tau} \in Q \) and \( \overline{r}_k = \overline{r}_k = \frac{1}{2} c_2^\delta k^{\delta} \in \left\{ \frac{\delta R}{4}, \frac{R}{4} \right\} \). Moreover,

\[
  B_{\delta_1 R}(x_0) \subset B_{\overline{r}_k}(\overline{\tau}).
\]

(iii) There exists at least one dyadic cube \( Q \) of generation \( k_R \) such that \( d(x_0, Q) \leq \delta_1 R \).

Using the Calderón-Zygmund technique, Lemma 3.12 follows from Theorem 3.10; the proof can be found in [Lemma 6.3].

**Lemma 3.12.** Let \( Q_1 \subset M \) be a dyadic cube, \( A \subset B \subset Q_1 \) be measurable sets, and \( \sigma \in (0, 1) \) satisfying

(i) \( |A| \leq \sigma|Q_1| \) and

(ii) if \( Q \subset Q_1 \) is a dyadic cube satisfying \( |A \cap Q| > \sigma|Q| \), then \( \overline{Q} \subset B \).

Then we have \( |A| \leq \sigma|B| \).

### 3.3. Proof of \( W^{2,\xi} \)-estimate.

This subsection is devoted to the proof of a (locally) uniform \( W^{2,\xi} \)-estimate for uniformly elliptic operators. Instead of the contact sets in Definition 3.1, we introduce the special contact set so as to proceed with the ABP method using Lemma 3.12 in Proposition 3.14 with the help of the Calderón-Zygmund technique. For the special contact set, we make use of sums of the barrier functions and the squared distance functions as global test functions on a Riemannian manifold since the scale invariant barrier functions are well-understood in Lemma 3.5. With the choice of \( \delta \in (0, 1) \) in (7), the barrier function \( v_\delta(\cdot; r; z) \) in Lemma 3.5 will be denoted by \( v(\cdot; r; z) \) below and hereafter.

**Definition 3.13** (Special contact set). Let \( \Omega \) be a bounded, open set in \( M \). For \( k \in \mathbb{N} \), let \( Q_k(\Omega) \) be the set of global test functions on \( \Omega \) defined as

\[
  Q_k(\Omega) := \left\{ \sum_{j=1}^k r_j^2 v(\cdot; r_j; z_j) + \sum_{j=1}^k d_j^2 : \; r_j > 0, B_{\gamma_j}(z_j) \subset \Omega, y_j \in \overline{B}_{\gamma_j}(z_j), \forall j = 1, \ldots, k \right\}.
\]

For \( u \in C(\Omega) \), and \( k \in \mathbb{N} \), define the special contact set \( \mathcal{C}^k(u; \Omega) \) associated with \( u \) of degree \( k \) over \( \Omega \) by

\[
  \mathcal{C}^k(u; \Omega) := \left\{ x \in \Omega : \inf_{Q_l} (u + q) = u(x) + q(x) \quad \text{for some} \; q \in Q_l(\Omega) \quad \text{with} \; 1 \leq l \leq k \right\}.
\]
We also define

\[ \mathcal{F}^k(u; \Omega) := \mathcal{F}^k(u; \Omega) \cap \mathcal{F}^k(-u; \Omega). \]

Now, we obtain the following power decay estimate of the measure of the special contact set with respect to the degree, using Lemma 3.8.

**Proposition 3.14.** Assume that Sec \( \geq -\kappa \) for \( \kappa \geq 0 \), and let \( x_0 \in M \) and \( 0 < 7R \leq R_0 \). Let \( u \) be a smooth function in \( B_{7R}(x_0) \) such that

\[ |u|_{L^\infty(B_{3R}(x_0))} \leq 1/2, \]

and \( u \in \Sigma(\lambda, \Lambda, f) \) on \( B_{7R}(x_0) \) with

\[ \left( \int_{B_{7R}(x_0)} |R^2 f + |\eta|^m| \right)^{\frac{1}{m}} \leq \epsilon, \quad \eta := 1 + \log_2 \cosh(4\sqrt{R_0}). \]

Let \( Q_1 \) be a dyadic cube of generation \( k_R \) such that \( d(x_0, Q_1) \leq \delta_1 R \), and let \( \tilde{r}_k \in \left[ \frac{6R}{2}, \frac{R}{2} \right) \) be the radius in Lemma 3.11 (ii). Then we have

\[ \left( \frac{Q_1 \setminus \mathcal{F}^k(\tilde{r}_k; B_{7R}(x_0))}{|Q_1|} \right) \leq C \left( 1 - \frac{\mu}{2} \right) \quad \forall i = 1, 2, \cdots, \]

where the uniform constants \( K \in \mathbb{N} \) and \( C \geq 1 \) depend only on \( n \), \( \lambda \), \( \Lambda \), and \( \sqrt{R_0} \), and the constants \( \epsilon := \epsilon_\delta, \mu := \mu_\delta \) are as in Lemma 3.8 with \( \Omega \).

**Proof.** (i) First, we prove

\[ \left( \frac{Q_1 \setminus \mathcal{F}^k(\tilde{r}_k; B_{7R}(x_0))}{|Q_1|} \right) \leq 1 - \mu. \]

In fact, from Lemma 3.11 we find \( \tilde{z}_{k_6} \in Q_1 \) and \( \tilde{r}_{k_6} \in \left[ \frac{6R}{2}, \frac{R}{2} \right) \) for \( k = k_R \) such that

\[ B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6}) \subset Q_1 \subset \overline{Q_1} \subset B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6}) \subset B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6}) \subset B_{7R}(x_0). \]

We notice that \( 0 \leq u + \frac{1}{2} \leq 1 \) in \( B_{7R}(x_0) \) and recall from Lemma 2.2 that

\[ \left( \int_{B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6})} |\tilde{r}_{k_6}^2 f + |\eta|^m| \right)^{\frac{1}{m}} \leq 2 \left( \int_{B_{7R}(x_0)} |R^2 f + |\eta|^m| \right)^{\frac{1}{m}} \leq \epsilon. \]

Thus we apply Lemma 3.8 to \( u + \frac{1}{2} \) in order to obtain

\[ \frac{\mathcal{G}^k_u \left( u + \frac{1}{2} + v(\cdot; \tilde{r}_{k_6}; \tilde{z}_{k_6}); B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6}); B_{7R}(x_0) \right) \cap B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6})}{|B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6})|} \geq \mu. \]

Since we have

\[ \mathcal{G}^k_u \left( u + \frac{1}{2} + v(\cdot; \tilde{r}_{k_6}; \tilde{z}_{k_6}); B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6}); B_{7R}(x_0) \right) \subset B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6}); B_{7R}(x_0), \]

we deduce that

\[ \frac{\mathcal{F}^k(\tilde{r}_{k_6}; B_{7R}(x_0)) \cap Q_1}{|Q_1|} \geq \frac{\mathcal{F}^k(\tilde{r}_{k_6}; B_{7R}(x_0)) \cap B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6})}{|B_{\tilde{r}_{k_6}}(\tilde{z}_{k_6})|} \geq \mu, \]

which implies (11).

(ii) For \( i \in \mathbb{N} \), we define

\[ A := Q_1 \setminus \mathcal{F}^k(\tilde{r}_{k_6}; B_{7R}(x_0)), \]
and
\[ B := (Q_1 \setminus Q^{K-1}(\tau_{K_0} u; B_{7R}(x_0))) \cup \left\{ z \in Q_1 : m_{B_{7R}(x_0)}(R^2 f^+ \eta) (z) > K^{i-1} \right\}, \]
where \( m_{B_{7R}(x_0)} \) is the maximal operator over \( B_{7R}(x_0) \) in Lemma 2.6. It is clear that \( A \subset B \subset Q_1 \) and \( |A| \leq (1 - \mu)|Q_1| \) according to (11). Now we claim that
\[ |A| \leq (1 - \mu)|B|, \]
for a large number \( K \in \mathbb{N} \) to be chosen later. Using Lemma 3.12 it suffices to show that if \( Q \subset Q_1 \) is a dyadic cube of generation \( k > k_0 \) such that \( |A \cap Q| > (1 - \mu)|Q| \), then \( Q \subset B \). Suppose to the contrary that \( Q \not\subset B \), let \( \tilde{z} \) be a point in \( Q \setminus B \subset Q_1 \). From Lemma 3.11 again, we find \( z_0 \in Q \) and \( \tau_k \in (0, R) \) such that
\[ \tilde{z} \in Q \cap Q^{K-1}(\tau_{K_0} u; B_{7R}(x_0)) \cap \left\{ z \in Q_1 : m_{B_{7R}(x_0)}(R^2 f^+ \eta)(z) \leq K^{i-1} \right\} \]
\[ \subset B_{\tau_j}(z_0) \cap Q^{K-1}(\tau_{K_0} u; B_{7R}(x_0)) \cap \left\{ z \in Q_1 : m_{B_{7R}(x_0)}(R^2 f^+ \eta)(z) \leq K^{i-1} \right\}, \]
and \( B_{\tau_j}(z_0) \subset Q \subset Q \setminus B_{\tau_j}(z_0) \subset B_{\tau_j}(z_0) \subset B_{7R}(x_0) \). From the definition of \( Q^{K-1}(\tau_{K_0} u; B_{7R}(x_0)) \), we consider
\[ \tilde{u} := \tau_{K_0}^2 u + q \quad \text{in } B_{7R}(x_0) \]
for some \( q \in Q_1(B_{7R}(x_0)) \) with \( 1 \leq l \leq K^{i-1} \), satisfying
\[ \inf_{B_{3R}(x_0)} \tilde{u} = \tilde{u}(\tilde{z}). \]
Indeed, we can write
\[ \tilde{u} := \tau_{K_0}^2 u + \sum_{j=1}^{l} r_j^2 v(\cdot; r_j; z_j) + \sum_{j=1}^{l} \frac{1}{2} d_{z_j}^2 \quad \text{in } B_{7R}(x_0), \]
where \( r_j > 0, B_{r_j}(z_j) \subset B_{7R}(x_0) \) and \( y_j \in B_{r_j}(z_j) \) for \( j = 1, \ldots, l \). For large constants \( K > K_0 > 1 \), define
\[ \tilde{w} := \frac{1}{r_j^{2} K_0^{i-1}} \tilde{u} - \frac{1}{r_j^{2} K_0^{i-1}} \tilde{u}(\tilde{z}), \]
which is nonnegative in \( B_{7R}(x_0) \), and vanishes at \( \tilde{z} \in B_{\tau_j}(z_0) \). Recall from Lemma 3.5 that
\[ r_j^2 D^2 v(\cdot; r_j; z_j) \leq C \quad \text{in } B_{7R}(x_0) \setminus \text{Cut}(z_j) \quad \forall j = 1, 2, \ldots, l \]
for a uniform constant \( C > 0 \), depending only on \( n, \lambda, \Lambda, \mathcal{H} \left( 2 \sqrt{\kappa} R_0 \right) \), since \( B_{r_j}(z_j) \subset B_{7R}(x_0) \) with \( 7R < R_0 \). This combined with Lemma 2.3 implies that
\[ \mathcal{M}(D^2 \tilde{w}) \leq \frac{1}{r_j^2 K_0^{i-1}} \tau_{K_0}^2 f^+ + \frac{1}{r_j^2 K_0^{i-1}} \sum_{j=1}^{l} r_j^2 \mathcal{M}^+ \left( D^2 v(\cdot; r_j; z_j) \right) + \sum_{j=1}^{l} \mathcal{M}^+ \left( D^2 d_{z_j}^2 / 2 \right) \]
\[ \leq \frac{1}{r_j^2 K_0^{i-1}} \tau_{K_0}^2 f^+ + \frac{1}{r_j^2 K_0^{i-1}} K^{i-1} n \Lambda \left( C + \mathcal{H} \left( 2 \sqrt{\kappa} R_0 \right) \right) \]
\[ =: \tilde{f} \quad \text{in } B_{7R}(x_0) \setminus \left( \bigcup_{j=1}^{l} \text{Cut}(z_j) \cup \bigcup_{j=1}^{l} \text{Cut}(y_j) \right). \]
By choosing $K_0 \in \mathbb{N}$ sufficiently large, we deduce that
\[
\left( \int_{B_{\overline{B}_{\sigma}(z_0)}} \overline{v}_k^{\ell} \right)^{\frac{1}{\ell}} \leq \frac{1}{K_0 K^{l-1}} \left( \int_{B_{\overline{B}_{\sigma}(z_0)}} |R^2 f^{+ \mu}| \right)^{\frac{1}{\mu}} + \frac{1}{K_0} n \Lambda \left\{ C + \mathcal{H}^c \left( 2 \sqrt{r} R_{K_0} \right) \right\}
\leq \frac{1}{K_0 K^{l-1}} \left( \int_{B_{\overline{B}_{\sigma}(z_0)}} |R^2 f^{+ \mu}| \right)^{\frac{1}{\mu}} + \epsilon
\leq \frac{1}{K_0 K^{l-1}} K^{l-1} + \frac{1}{K_0} + \frac{\epsilon}{2} \leq \epsilon
\]
since $\tilde{z} \in B_{\overline{B}_{\sigma}(z_0)} \cap \left\{ z \in \mathcal{Q}_1 : m_{B_{\overline{B}_{\sigma}(z_0)}} \left( |R^2 f^{+ \mu}| \right) \left( z \right) \leq K^{(l-1)\mu} \right\}$. Thus, we apply Corollary 3.9 to $\tilde{w}$ in $B_{\overline{B}_{\sigma}(z_0)}$ to obtain that
\[
\frac{\mathcal{G}_{\mathcal{Q}_1} \left( \tilde{w} + v(\overline{v}; \mathcal{T}_k; z_0); B_{\overline{B}_{\sigma}(z_0)} \right)}{|B_{\overline{B}_{\sigma}(z_0)}|} \geq \mu.
\]
Now we claim that
\[
\mathcal{G}_{\mathcal{Q}_1} \left( \tilde{w} + v(\overline{v}; \mathcal{T}_k; z_0); B_{\overline{B}_{\sigma}(z_0)} \right) \subset \mathcal{G}^{K} \left( \mathcal{T}_k^{j}; \mu; B_{\sigma}(x_0) \right)
\]
for a large uniform constant $K \in \mathbb{N}$. In fact, let $x \in \mathcal{G}_{\mathcal{Q}_1} \left( \tilde{w} + v(\overline{v}; \mathcal{T}_k; z_0); B_{\overline{B}_{\sigma}(z_0)} \right)$. Then we have
\[
\inf_{B_{\mathcal{Q}_1}(x_0)} \left\{ \tilde{w} + v(\overline{v}; \mathcal{T}_k; z_0) + \frac{1}{2\mathcal{T}_k^2} \right\} = \tilde{w}(x) + v(x; \mathcal{T}_k; z_0) + \frac{1}{2\mathcal{T}_k^2} d_{y_0}^2(x)
\]
for some $y_0 \in B_{\overline{B}_{\sigma}(z_0)}$, that is,
\[
\left\{ \mathcal{T}_k^{j}; \mathcal{Q}_1 \right\} \cap \left\{ \mathcal{Q}_1 \right\} \geq \left| \frac{\mathcal{G}^{K} \left( \mathcal{T}_k^{j}; \mu; B_{\sigma}(x_0) \right)}{|B_{\overline{B}_{\sigma}(z_0)}|} \right| \geq \frac{\mathcal{G}^{K} \left( \mathcal{T}_k^{j}; \mu; B_{\sigma}(x_0) \right)}{|B_{\overline{B}_{\sigma}(z_0)}|} \geq \mu,
\]
which means that $|A \cap \mathcal{Q}| < (1 - \mu) |Q|$. This contradicts to the assumption $|A \cap \mathcal{Q}| > (1 - \mu) |Q|$. Therefore, we have proved (13) according to Lemma 3.12.

(ii) Let
\[
\alpha_i = \frac{|Q_1 \setminus \mathcal{G}^{K} \left( \mathcal{T}_k^{j}; \mu; B_{\sigma}(x_0) \right)|}{|Q_1|}, \quad \text{and} \quad \beta_i = \frac{|\mathcal{Q}_1 : m_{B_{\mathcal{Q}_1}(x_0)} \left( |R^2 f^{+ \mu}| \right) \left( z \right) > K^{(l-1)\mu} \right|}{|Q_1|}.
\]
From (ii), we have
\[
\alpha_i \leq (1 - \mu) (\alpha_{i-1} + \beta_{i-1}) \quad \forall i \in \mathbb{N}
\]
which implies that
\[
\alpha_i \leq (1 - \mu)^i \alpha_0 + \sum_{j=0}^{i-1} (1 - \mu)^{i-j} \beta_j \leq (1 - \mu)^i + \sum_{j=0}^{i-1} (1 - \mu)^{i-j} \beta_j.
\]
In order to prove (16), we claim that for any $(18)$

Gromov’s Theorem 2.1. Hence we have that

\[
R \in B(x_0)
\]

the norm of the Hessian of the solution

which is bounded by a uniform constant depending only on $\Lambda$.

Therefore we select $\varepsilon < \Lambda$ sufficiently large so that

\[
\alpha_i \leq (1 + Ci) (1 - \mu)^i \quad \forall i \in \mathbb{N},
\]

which implies (16). \hfill $\Box$

Proposition 3.14 yields a power decay estimate (16) of the distribution function of $|D^2 u|$, the norm of the Hessian of the solution $u$ to the uniformly elliptic equation.

**Lemma 3.15** (Decay estimate). Assume that $\text{Sec} \geq -\kappa$ for $\kappa \geq 0$. Let $x_0 \in M$ and $0 < 7R \leq R_0$. Let $u$ be a smooth function in $B_{7R}(x_0)$ such that $||u||_{L^\infty(B_{7R}(x_0))} \leq 1/2$, and $u \in S^s(\lambda, \Lambda, f)$ in $B_{7R}(x_0)$ with

\[
\left( \int_{B_{7R}(x_0)} |R^2 f|^\eta \right)^{\frac{1}{\eta}} \leq \varepsilon/2; \quad \eta := 1 + \log_2 \cosh(4 \sqrt{kR_0}).
\]

Let $Q_1$ be a dyadic cube of generation $k_R$ such that $d(x_0, Q_1) \leq \delta_1 R$. Then we have

\[
\left| \left\{ |R^2 |D^2 u| \geq h \right\} \cap Q_1 \right| \leq C \varepsilon^{\varepsilon_0} |Q_1| \quad \forall h > 0
\]

for uniform constants $\varepsilon_0 \in (0, 1)$ and $C > 0$ depending only on $n, \lambda, \Lambda, \sqrt{kR_0}$, and hence

\[
\left( \int_{Q_1} |R^2 |D^2 u| \right)^{\frac{1}{\eta}} \leq C,
\]

where $\varepsilon \in (0, 1)$ and $C > 0$ are uniform constants.

**Proof.** Applying Proposition 3.14 to $\pm u$, we have

\[
\left| \left\{ \left| R^K |D^2 u| \right|^i \right\} \cap Q_1 \right| \leq C \left(1 - \mu \right)^i \quad \forall i = 1, 2, \ldots.
\]

In order to prove (16), we claim that for any $i \in \mathbb{N} \cup \{0\}$

\[
\left| \left\{ \left| R^K |D^2 u| \right|^i \right\} \subseteq \left\{ \left| R^K |D^2 u| \right| \leq CK^i \right\} \right|
\]

where $C > 0$ is a uniform constant depending only on $n, \lambda, \Lambda, \sqrt{kR_0}$. In fact, fix $i \in \mathbb{N} \cup \{0\}$, and $x \in \left| R^K |D^2 u| \right| \subseteq \left\{ \left| R^K |D^2 u| \right| \leq C K^i \right\}$.

Then there exist $q_1 \in Q_{l_1} (B_{7R}(x_0))$, and $q_2 \in Q_{l_2} (B_{7R}(x_0))$ for $1 \leq l_1, l_2 \leq K^i$ such that

\[
-q_1 + q_2 \leq \tau_{l_1} |D^2 u| - \tau_{l_2} |D^2 u| \leq q_2 - q_1 \quad \text{in} \quad B_{7R}(x_0).
\]
From the definition of of $Q_{i_1}(B_{7r}(x_0))$, we can write

\begin{equation}
-q_1 + q_1(x) := - \sum_{j=1}^{l_i} r_j^2 v(\cdot; r_j; z_j) - \sum_{j=1}^{l_i} \frac{1}{2} d_j^2 + \sum_{j=1}^{l_i} r_j^2 v(x; r_j; z_j) + \sum_{j=1}^{l_i} \frac{1}{2} d_j^2(x)
\leq \tau_{k_0}^2 u - \tau_{k_0}^2 u(x) \quad \text{in } B_{7r}(x_0)
\end{equation}

for some $r_j > 0$, $B_{7r_j}(z_j) \subset B_{7r}(x_0)$ and $y_j \in \overline{B}_{7r_j}(z_j)$ $(j = 1, 2, \cdots, l_i)$. For a unit vector $X \in T_x M$, we have

\[
\liminf_{t \to 0} \frac{\tau_{k_0}^2 u(x) + \tau_{k_0}^2 u(x) - 2 \tau_{k_0}^2 u(x)}{t^2} \\
\geq - \sum_{j=1}^{l_i} r_j^2 \limsup_{t \to 0} \frac{1}{t} \left\{ v(\exp_x tX; r_j; z_j) + v(\exp_x -tX; r_j; z_j) - 2v(x; r_j; z_j) \right\} \\
- \sum_{j=1}^{l_i} \limsup_{t \to 0} \frac{1}{2t^2} \left\{ d_{r_j}^2(\exp_x tX) + d_{r_j}^2(\exp_x -tX) - 2d_{r_j}^2\right\}.
\]

We recall from Remark 3.6 and Lemma 2.3 to obtain that

\[
\liminf_{t \to 0} \frac{\tau_{k_0}^2 u(x) + \tau_{k_0}^2 u(x) - 2 \tau_{k_0}^2 u(x)}{t^2} \geq -C_{1} \geq -CK^i,
\]

for a uniform constant $C > 0$. Thus we deduce that for any unit vector $X \in T_x M$,

\[
\tau_{k_0}^2 \left\langle D^2u(x) \cdot X, X \right\rangle \geq -CK^i.
\]

According to our argument above using (19), we obtain (18) since $\tau_{k_0} \in \left(0, \frac{\epsilon}{2}\right]$.

\[
\left| \left\{ \left| R^2 D^2u \right| > CK^i \right\} \cap Q_1 \right| \leq \left| Q_1 \setminus \mathcal{K} (\tau_{k_0}^2 u; B_{7r}(x_0)) \right| \leq C \left(1 - \frac{\mu}{2}\right)^{i+1} |Q_1|.
\]

which implies (16).

Finally, let $\epsilon := \epsilon_0/2$, we have

\[
\int_{Q_1} \left( R^2 |D^2u| \right)^2 d \text{Vol} = \frac{1}{|Q_1|} \left\{ \int_{Q_1 \cap \left| R^2 |D^2u| \leq 1 \right\}} + \int_{Q_1 \cap \left| R^2 |D^2u| > 1 \right\}} \right\} \left( R^2 |D^2u| \right)^2 d \text{Vol} \\
\leq 1 + \frac{1}{|Q_1|} \int_0^{\infty} h^{\epsilon-2} \left| \left\{ \left| R^2 D^2u \right| \geq h \right\} \cap Q_1 \right| dh \\
= 1 + \frac{\epsilon_0}{2} \int_0^{\infty} h^{\epsilon_0/2-2}Ch^{-\epsilon_0}dh,
\]

where the last quantity is bound by a uniform constant. This completes the proof. \(\square\)

**Corollary 3.16.** Under the same assumption as Lemma 3.15 we have

\[
\left( \int_{Q_1} |R \nabla u|^2 \right)^{1/2} \leq C,
\]

where $\epsilon \in (0, 1)$ and $C > 0$ are uniform constants.

**Proof.** It suffices to show that for any $i \in \mathbb{N} \cup \{0\}$

\begin{equation}
\mathcal{K} (\tau_{k_0}^2 u; B_{7r}(x_0)) \subset \left\{ x \in B_{7r}(x_0) : R |\nabla u(x)| \leq CK^i \right\},
\end{equation}

since the corollary follows from the same argument as in Lemma 3.15. Fix $i \in \mathbb{N} \cup \{0\}$, and $x \in \mathcal{K} (\tau_{k_0}^2 u; B_{7r}(x_0))$. Then there exist $q_1 \in Q_{i_1}(B_{7r}(x_0))$, and $q_2 \in Q_{i_1}(B_{7r}(x_0))$.
for $1 \leq l_1, l_2 \leq K'$ satisfying (19). We recall from Lemma 2.3 and Lemma 5.5 that for any $B_{7R}(z) \subset B_{7R}(x_0)$ and $y \in \overline{B_{7R}(z)}$,

\[
\frac{1}{R} d^2 \nabla y \mid_{\mathcal{C}(y)} = \frac{2}{R} d_y < 28 \quad \text{in} \ B_{7R}(x_0) \setminus \text{Cut}(y),
\]

\[
\frac{1}{7R} r^2 |\nabla v(\cdot; r, z)| < C \quad \text{in} \ B_{7R}(x_0) \setminus \text{Cut}(z)
\]

for a uniform constant $C > 0$ depending only on $n, \lambda, \Lambda$, and $\sqrt{\kappa} R_0$. According to Lemma 3.7 and (20), we see that

\[
\mathcal{C}(x) \cap \bigcup_{j=1}^{l_1} \text{Cut}(z_j) \cup \bigcup_{j=1}^{l_1} \text{Cut}(y_j),
\]

and hence

\[
\frac{\delta^2_0}{4} R \| \nabla u(x) \| \leq \frac{7^2}{R} \| \nabla u(x) \| \leq C l_1 \leq C K^i
\]

since $\delta_0 \in \left(\frac{\delta R_0}{2}, \frac{\delta R_0}{4}\right]$. Therefore, it follows that

\[
R \| \nabla u(x) \| \leq C K^i \quad \forall x \in \mathcal{G}^i \left(\mathcal{C}_k u; B_{7R}(x_0)\right),
\]

which completes the proof of (21). \(\square\)

**Proof of Theorem 1.1** According to [WZ, Theorem 1.5], we have the weak Harnack inequality which provides a uniform $L^\infty$-estimate of $u$. By passing from cubes to balls with the help of Bishop-Gromov’s Theorem 2.1, we conclude a uniform $W^{2,\infty}$-estimate; refer to Remark 8.3 and Theorem 8.1 of [Ca] for details. \(\square\)

**References**

[Au] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer, 1998.

[Ca] X. Cabré, *Nondivergent elliptic equations on manifolds with nonnegative curvature*, Comm. Pure Appl. Math. 50 (1997), 623-665.

[C] L.A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. Math. 130 (1989), 189-213.

[CC] L. A. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations*, American Mathematical Soc., 1995.

[Ch] M. Christ, *A T(b) theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. 60/61 (1990), 601-628.

[CMS] D. Cordero-Erausquin, R. J. McCann and M. Schmuckenschläger, *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*, Invent. Math. 146 (2001), 219-257.

[D] M. P. do Carmo, *Riemannian geometry*, Springer, 1992.

[H] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*. American Mathematical Soc., 1999.

[K] S. Kim, *Harnack inequality for nondivergent elliptic operators on Riemannian manifolds*, Pacific J. Math. 213 (2004), 281-293.

[KKL] S. Kim, S. Kim and K.-A. Lee, *Harnack inequality for nondivergent parabolic operators on Riemannian manifolds*, Calc. Var. 49 (2014), 669-706.

[KL] S. Kim and K.-A. Lee, *Parabolic Harnack inequality of viscosity solutions on Riemannian manifolds*, arXiv:1304.7351.

[KS] N. V. Krylov and M. V. Safonov, *A property of the solutions of parabolic equations with measurable coefficients* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), 161-175; Math. USSR Izvestija 16 (1981), 151-164 (English).

[Le] J. M. Lee, *Riemannian manifolds: an introduction to curvature*, Springer, 1997.

[L] F.-H. Lin, *Second derivative $L^2$-estimates for elliptic equations of nondivergent type*, Proceedings of the American Mathematical Society 96 (1986), 447-451.

[PT] C. Pucci and G. Talenti, *Elliptic (second-order) partial differential equations with measurable coefficients and approximating integral equations*, Adv. in Math. 19 (1976), 48-105.
[S] O. Savin, Small perturbation solutions for elliptic equations, Comm. Partial Differential Equations 32 (2007), 557-578.

[St] E. M. Stein, Singular integrals and differentiability properties of functions. Princeton university press, 1970.

[U] N. N. Ural’ceva, Impossibility of $W^{2,q}_2$-bounds for multi-dimensional elliptic equations with discontinuous coefficients, Zap. Naun. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 5 (1967), 250-254 (Russian).

[V] C. Villani, Optimal transport: old and new, Springer, 2008.

[W] L. Wang, On the regularity theory of fully nonlinear parabolic equations. I, Comm. Pure Appl. Math. 45 (1992), 27-76.

[WZ] Y. Wang and X. Zhang, An Alexandroff-Bakelman-Pucci estimate on Riemannian manifolds, Adv. Math 232 (2013), 499-512.

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