PARABOLIC TYPE EQUATIONS AND MARKOV STOCHASTIC PROCESSES ON ADELES

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Abstract. In this paper we study the Cauchy problem for new classes of parabolic type pseudodifferential equations over the rings of finite adeles and adeles. We show that the adelic topology is metrizable and give an explicit metric. We find explicit representations of the fundamental solutions (the heat kernels). These fundamental solutions are transition functions of Markov processes which are adelic analogues of the Archimedean Brownian motion. We show that the Cauchy problems for these equations are well-posed and find explicit representations of the evolution semigroup and formulas for the solutions of homogeneous and non-homogeneous equations.

1. Introduction

During the last twenty years the interest on stochastic models on $p$-adics and adeles has been increasing mainly because these models are convenient for describing phenomena whose space of states display a hierarchical structure. All these developments have been motivated by a conjecture in statistical physics asserting that the non exponential relaxation of several models describing complex systems, such as glasses and proteins, is a consequence of a hierarchical structure of the state space which can in turn be put in connection with $p$-adic structures. The pioneering work of Avetisov et al. on $p$-adic techniques for describing spontaneous symmetry breaking in the models of spin glasses and relaxation processes in complex systems gives a very strong motivation for developing a theory of parabolic type pseudодifferential equations and their corresponding stochastic processes on $p$-adics and adeles, see [1], [4], [5], [6], [7], [8], [9], [10], [11], [12], [17], and references therein], [22], [25], [26], [27] and references therein], [28], [29], [35], [39], [44], [46], and references therein], [50], [51], among others.

Another two motivations for studying pseudodifferential equations on adeles are the following. In [24] Haran established a connection between explicit formulas for the Riemann zeta function and adelic pseudodifferential operators, see also [14]. In [31] Manin posed the conjecture that the physical space is adelic, which can be considered as an extension of the Volovich conjecture on the non Archimedean nature of the physical space at the Planck scale [43], [47], [45]. This conjecture conducts naturally to consider models involving partial differential equations on adelic
spaces. Some preliminary results such as studying pseudodifferential operators are presented in [16] and references therein], [18], [33], [36].

In this article we work exclusively with complex valued functions on adeles, this due to the fact that most of the physical models that motivate our theory require ‘real valued probabilities’. However, recently new models of complex systems using ‘p-adic valued probabilities’ have emerged, see e.g. [31], [32]. The use of complex-valued functions allow us to take advantage of the classical harmonic and functional analysis. However, the classical derivative is not defined for complex-valued functions on adeles, implying the consideration of pseudodifferential operators. For the sake of simplicity we formulated all our results for finite adeles and adeles on \( \mathbb{Q} \), however all the results are still valid if the field of rational numbers \( \mathbb{Q} \) is replaced by an algebraic number field, or by the function field of an algebraic curve over a finite field. We study the Cauchy problem for parabolic type pseudodifferential equations over the rings of finite adeles and adeles involving a natural generalization of the Taibleson operator [2], [39]. The considered pseudodifferential operator is not a straightforward generalization of the Taibleson operator and is natural only from the point of view of its connection with the adelic topology and the Fourier transform. By so far we are unaware of any similar results. Other adelic pseudodifferential operators with different symbols have been studied in [24], [33], [30], [36].

The article is organized as follows. In Section 2 we summarize some well-known results on p-adic and adelic analysis. In Sections 3, 4 we introduce metric structures on the rings of finite adeles and adeles, see Propositions 3.3, 4.1. These metric structures induce the adelic topology and are naturally connected with the Fourier transform. In addition, they allow us to use classical results on Markov processes, see e.g. [20]. We compute the Fourier transform of radial functions defined on the ring of finite adeles, see Theorem 3.10, and we introduce adelic analogues of the Taibleson operators and Lizorkin spaces of the second kind and prove some basic properties of them. In Section 5 we study the heat kernels on the ring of finite adeles, see Definition 5.2 and Theorem 5.6. We give an ‘explicit formula’ for the heat kernel as a series involving Chebyshev type functions, i.e. products of powers of primes, some arithmetic operators and exponential functions depending on \( t \), see Proposition 5.3. We require the Prime Number Theorem to establish the existence of the adelic heat kernels, see Proposition 5.1. In Section 6 we show that the adelic heat kernels are the transition functions of Markov processes, see Theorem 6.3. In Sections 8, 9 we study the heat kernels on the ring of adeles, see Definition 8.1 and Theorem 8.2 and show that the heat kernels are the transition functions of Markov processes, see Theorem 8.3. In Sections 7, 10 we study Cauchy problems for parabolic type equations involving adelic versions of the Taibleson operator. We show that these problems are well-posed and find explicit formulas for the solutions of homogeneous and non-homogeneous equations, see Proposition 7.3, Theorems 7.5, 7.8, 7.9, Proposition 10.3 and Theorems 10.4, 10.5.

Finally we hope that this article will raise interest on studying pseudodifferential equations and stochastic processes on adeles. We are still at the beginning to develop a complete theory, there are many open problems and questions, among them, we propose the study of adelic Schrödinger equations and their connection with Feynman and Feynman-Kac integrals.
2. Preliminaries

In this section we fix the notation and collect some basic results on \( p \)-adic and adelic analysis that we will use through the article. For a detailed exposition on \( p \)-adic and adelic analysis the reader may consult [2], [23], [27], [38], [43], [46].

2.1. Adeles on \( \mathbb{Q} \). Let \( p \) be a fixed prime number, and let \( x \) be a nonzero rational number. Then \( x \) may be represented uniquely as \( x = p^k \frac{a}{b} \) with \( p \nmid ab \) and \( k \in \mathbb{Z} \). The function

\[
|x|_p := \begin{cases} 
  p^{-k} & \text{if } x \neq 0, \\
  0 & \text{if } x = 0 
\end{cases}
\]

gives rise to a non-Archimedean absolute value on \( \mathbb{Q} \). The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is defined as the completion of \( \mathbb{Q} \) with respect to the distance induced by \( | \cdot |_p \).

Any non-zero \( p \)-adic number \( x_p \) has a unique representation of the form

\[
(2.1) \quad x_p = p^\gamma \sum_{i=0}^{\infty} a_i p^i,
\]

where \( \gamma = \gamma(x_p) \in \mathbb{Z} \), \( a_i \in \{ 0, 1, \ldots, p - 1 \} \), \( a_0 \neq 0 \). Series (2.1) converges in the \( p \)-adic absolute value. The integer \( \gamma \) is called the \( p \)-adic order of \( x_p \), and it will be denoted as \( \text{ord}_p(x_p) \), with \( \text{ord}_p(0) := +\infty \). Note that \( |x_p|_p = p^{-\text{ord}_p(x_p)} \). With the topology induced by \( | \cdot |_p \), \( \mathbb{Q}_p \) is a locally compact topological field. The unit ball \( \mathbb{Z}_p \) of \( \mathbb{Q}_p \) is a compact topological ring. Let \( dx_p \) denote the Haar measure of the topological group \( (\mathbb{Q}_p, +) \) normalized by the condition \( \text{vol}(\mathbb{Z}_p) = 1 \). For a detailed presentation of the integration theory on \( \mathbb{Q}_p \) see [23], [46].

Along the article, the variables \( p, q \) will denote ‘primes’, including ‘the infinite prime’, denoted by \( \infty \). To each prime \( p \) corresponds an absolute value \( | \cdot |_p \) on \( \mathbb{Q} \), with \( | \cdot |_{\infty} \) corresponding to the usual Euclidean norm. In addition, \( \mathbb{Q}_p \) denotes the completion of \( \mathbb{Q} \) with respect to \( | \cdot |_p \), note that \( \mathbb{Q}_\infty = \mathbb{R} \).

The ring of adeles of \( \mathbb{Q} \), denoted \( \mathbb{A} \), is defined by

\[
\mathbb{A} = \{ (x_\infty, x_2, x_3, \ldots) : x_p \in \mathbb{Q}_p, \text{ and } x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \}.
\]

Alternatively, we can define \( \mathbb{A} \) as the restricted product of the \( \mathbb{Q}_p \) with respect to the \( \mathbb{Z}_p \). The componentwise addition and multiplication give to \( \mathbb{A} \) a ring structure.

Furthermore, \( \mathbb{A} \) can be made into a locally compact topological ring by taking as a base for the topology, certainly the restricted product topology, all the sets of the form \( U \times \prod_{p \notin S} \mathbb{Z}_p \), where \( S \) is any finite set of primes containing \( \infty \), and \( U \) is any open subset in \( \prod_{p \in S} \mathbb{Q}_p \).

The restricted product topology is not equal to the product topology. However, the following relation holds. Take \( S \) as before and consider the group

\[
G_S = \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p.
\]

Then, the product topology on \( G_S \) is identical to the one induced by the restricted product topology on \( G_S \), thus \( G_S \) is a locally compact subgroup of \( \mathbb{A} \), and the locally compact topological group \( (\mathbb{A}, +) \) has a Haar measure, denoted \( dx_\mathbb{A} \), which coincides on \( G_S \) with the product measure \( \prod_p dx_p \), where \( dx_\infty \) is the Lebesgue measure of \( \mathbb{R} \). We also note that any set of the form

\[
(2.2) \quad \prod_{p \in S} p^{b_p} \mathbb{Z}_p \times \prod_{p \notin S} \mathbb{Z}_p,
\]
where $l_p$ are arbitrary integers, is a compact subset of $\mathbb{A}$.

The ring of finite adeles over $\mathbb{Q}$, denoted $\mathbb{A}_f$, is defined by

$$\mathbb{A}_f = \{(x_2, x_3, \ldots) : x_p \in \mathbb{Q}_p, \text{ and } x_p \in \mathbb{Z}_p \text{ for all but finitely many } p\}.$$  

From now on, we consider $\mathbb{A}_f$ as a topological ring with respect to the restricted product topology. Then $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. Since $(\mathbb{A}_f, +)$ is a locally compact topological group, it has a Haar measure, denoted $dx_{\mathbb{A}_f}$, which agrees with the product measure $\prod_{p<\infty} dx_p$ on open subgroups of type $\prod_{p \leq N} \mathbb{Q}_p \times \prod_{p > N} \mathbb{Z}_p$, with $N \in \mathbb{N}$.

Furthermore $dx_{\mathbb{A}} = dx_{\infty} dx_{\mathbb{A}_f}$. For a detailed presentation of the integration theory on $\mathbb{A}$ and $\mathbb{A}_f$ see [23, Chapter 1], see also [38], [49].

In this article we work exclusively with complex valued functions on adeles. Having complex valued functions defined on a locally compact topological group, we have the notion of continuous function and may use the functional spaces $L^\rho(\mathbb{A}_f)$ and $L^\rho(\mathbb{A})$, $\rho \geq 1$ defined in the standard way.

For studying solutions of parabolic equations we need notations for several spaces of functions which depend on time and adelic (space) variables. We denote by:

(i) $C(I, X)$ the space of continuous functions $u$ on a time interval $I$ with values in $X$;
(ii) $C^1(I, X)$ the space of continuously differentiable functions $u$ on a time interval $I$ such that $u' \in X$;
(iii) $L^1(I, X)$ the space of measurable functions $u$ on $I$ with values in $X$ such that $\|u\|$ is integrable;
(iv) $W^{1,1}(I, X)$ the space of measurable functions $u$ on $I$ with values in $X$ such that $u' \in L^1(I, X)$.

### 2.2. Fourier transform on adeles

Let $p$ be a finite prime, and let $\chi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be the additive character defined by

$$\chi_p(x_p) = \exp(-2\pi i \{x_p\}),$$

where

$$\{x_p\} := \begin{cases} \sum_{i=-k}^{-1} a_i p^i & \text{if } x = \sum_{i=-k}^{\infty} a_i p^i \text{ with } k > 0 \text{ and } 0 \leq a_i \leq p-1, \\ 0 & \text{otherwise.} \end{cases}$$

A function $f_p : \mathbb{Q}_p \to \mathbb{C}$ which is locally constant with compact support is called a *Bruhat-Schwartz function*. The space of such functions is denoted as $\mathcal{S}(\mathbb{Q}_p)$. Note that in $p$-adic analysis the space $\mathcal{S}(\mathbb{Q}_p)$ coincides with the space of test functions $\mathcal{D}(\mathbb{Q}_p)$, see [16] for details. For $f_p \in \mathcal{S}(\mathbb{Q}_p)$, its Fourier transform $\hat{f}_p$ is defined by

$$\hat{f}_p(\xi_p) = \int_{\mathbb{Q}_p} \chi_p(-x_p \xi_p) f_p(x_p) dx_p.$$  

The Fourier transform induces a linear isomorphism of $\mathcal{S}(\mathbb{Q}_p)$ onto $\mathcal{S}(\mathbb{Q}_p)$ satisfying $\hat{\hat{f}}_p(\xi_p) = f_p(-\xi_p)$.
In the case \( p = \infty \) the additive character is defined by \( \chi_{\infty} (x_\infty) := \exp (2\pi ix_\infty) \).

Let \( \mathcal{S}(\mathbb{R}) \) denote the Schwartz space. The Fourier transform of \( f_\infty \in \mathcal{S}(\mathbb{R}) \), denoted \( \hat{f}_\infty \), is defined by

\[
\hat{f}_\infty (\xi_\infty) = \int_\mathbb{R} \chi_{\infty} (-x_\infty \xi_\infty) f_\infty (x_\infty) \, dx_\infty.
\]

The Fourier transform induces a linear isomorphism of \( \mathcal{S}(\mathbb{Q}_\infty) \) onto \( \mathcal{S}(\mathbb{Q}_\infty) \) satisfying \( \hat{\hat{f}}_\infty (\xi_\infty) = f_\infty (-\xi_\infty) \).

The additive adelic character \( \chi : \mathbb{A} \to \mathbb{C} \) is defined by

\[
\chi (x) = \prod_p \chi_p (x_p) \quad \text{for } x = (x_\infty, x_2, x_3, \ldots).
\]

An adelic function is said to be Bruhat-Schwartz if it can be expressed as a finite linear combination, with complex coefficients, of factorizable functions \( f = \prod_{p \leq \infty} f_p \), where \( f_p \) satisfies the following conditions: (A1) \( f_\infty \in \mathcal{S}(\mathbb{R}) \); (A2) \( f_p \in \mathcal{S}(\mathbb{Q}_p) \) for \( p < \infty \); (A3) \( f_p \) is the characteristic function of \( \mathbb{Z}_p \) for all but finitely many \( p < \infty \). The adelic space of Bruhat-Schwartz functions is denoted as \( \mathcal{S}(\mathbb{A}) \). The space of Bruhat-Schwartz functions \( \mathcal{S}(\mathbb{A}_f) \) is defined in a similar form except that only conditions A2 and A3 are required.

The Fourier transform of a factorizable adelic Bruhat-Schwartz function is defined by

\[
\hat{f} (\xi) = \prod_{p \leq \infty} \int_{\mathbb{Q}_p} f_p (x_p) \chi (-x_p \xi_p) \, dx_p.
\]

This definition may be extended to arbitrary adelic Bruhat-Schwartz functions by linearity. The Fourier transform gives a linear isomorphism of \( \mathcal{S}(\mathbb{A}) \) to \( \mathcal{S}(\mathbb{A}) \) satisfying \( \hat{\hat{f}} (\xi) = f (-\xi) \). Analogous definitions and results hold for the Fourier transform on \( \mathbb{A}_f \). The Fourier transform may be extended to the space \( L^2(\mathbb{A}) \) (or to \( L^2(\mathbb{A}_f) \)), where it is a unitary operator and the Steklov–Parseval equality holds.

We will also use the notation \( \mathcal{F}\varphi \) for the Fourier transform and \( \mathcal{F}^{-1}\varphi \) for the inverse Fourier transform. We used as a main reference for this section [23, Chapter 1], see also [33], [38], [49].

Since \( \mathbb{A} \) (resp. \( \mathbb{A}_f \)) is a locally compact topological group, a convolution operation between functions is also defined on \( \mathcal{S}(\mathbb{A}) \) and \( L^2(\mathbb{A}) \) (resp. \( \mathcal{S}(\mathbb{A}_f) \) and \( L^2(\mathbb{A}_f) \)). It is connected with the Fourier transform in the usual way, see e.g. [40] for details.

3. Metric structures, Distributions and Pseudodifferential Operators on \( \mathbb{A}_f \)

3.1. A structure of complete metric space for the finite adeles. In the previous section the restricted product topology on adeles was described. By far authors are unaware of any article introducing metric on adeles producing the same topology. We show that for the finite adeles the topology is metrizable and present a non-Archimedean metric on \( \mathbb{A}_f \). Moreover, in this metric each ball is a compact set and the Fourier transform of a radial function is again a radial function. Hence, despite of the complicated form of the presented metric we believe that it is natural for the ring of finite adeles.
Consider the following two functions:

\begin{equation}
\|x\|_1 := \max_p |x_p|, \quad x \in \mathbb{A}_f,
\end{equation}

and

\begin{equation}
\|x\|_0 := \max_p \frac{|x_p|}{p}, \quad x \in \mathbb{A}_f.
\end{equation}

Both functions are well defined and may be used to introduce a metric on $\mathbb{A}_f$. However, the topology induced by the metric $\|x - y\|_1$ does not coincide with the restricted product topology which may be easily seen from the following example. The sequence of adeles $x^{(k)} := (0, 0, \ldots, 0, 1, 0, \ldots)$, $k \in \mathbb{N}$ converges to 0 in the restricted product topology, but does not converge in the metric generated by $\|\cdot\|_1$. The metric $\|x - y\|_0$ induces the same topology as the restricted product topology, however it does not satisfy the above mentioned properties. For instance, with respect to this metric only balls of radiuses less than 1 are compact, and the Fourier transform of a radial function is not necessary a radial function. We left checking of these statements to reader, all required proofs may be obtained similarly to the proofs in this article.

To overcome the mentioned problems we define a function

\begin{equation}
\|x\| := \begin{cases} 
\|x\|_0 & \text{if } x \in \prod_p \mathbb{Z}_p, \\
\|x\|_1 & \text{if } x \notin \prod_p \mathbb{Z}_p,
\end{cases}
\end{equation}

for arbitrary $x \in \mathbb{A}_f$. Note that $\|x\|_0 \leq \|x\| \leq \|x\|_1$ for any $x \in \mathbb{A}_f$. We introduce the function (our metric)

\begin{equation}
\rho(x, y) := \|x - y\|, \quad x, y \in \mathbb{A}_f.
\end{equation}

The function $\|\cdot\|$ can be also represented as

$$\|x\| = \max_p p^{-[\text{ord}_p(x_p)]}, \quad x \in \mathbb{A}_f \setminus \{0\},$$

where

\begin{equation}
[[t]] := \begin{cases} [t] & \text{if } t \geq 0 \\
[t] + 1 & \text{if } t < 0,
\end{cases}
\end{equation}

here $[\cdot]$ denotes the integer part function.

**Remark 3.1.** The range of values of the function $\rho$ coincides with the set $\{0\} \cup \{p^j : p \text{ is prime, } j \in \mathbb{Z} \setminus \{0\}\}$.

**Remark 3.2.** It may seem odd that the proposed metric does not attain the value 1. It is possible to define another metric, using instead of $\|\cdot\|_0$ in \eqref{3.3} the function $(\|\cdot\|_0)_+$, see \eqref{3.11} for the definition of $\cdot^+$ operator. Due to Bertrand’s postulate the generated metrics are equivalent. In some cases like in Corollary \ref{3.13} such change simplifies formulas. However, calculations with this metric become more complicated. For this reason we do not use it in this article.

**Proposition 3.3.** The restricted product topology on $\mathbb{A}_f$ is metrizable, the metric is given by \eqref{3.4}. Furthermore, $(\mathbb{A}_f, \rho)$ is a complete non-Archimedean metric space.
Proof. The fact that \( \rho(x, y) \) is a non-Archimedean metric is a consequence of the fact that

\[
\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad x, y \in \mathbb{A}_f,
\]

which can be checked easily case by case.

We now show that \((\mathbb{A}_f, \rho)\) is a complete metric space. Let \( x^{(n)} = (x_p^{(n)})_p \) be a Cauchy sequence in \( \mathbb{A}_f \) with respect to \( \rho \). Since we have coordinate-wise convergence, we may define \( \bar{x}_p := \lim_{n \to \infty} x_p^{(n)} \) in \( \mathbb{Q}_p \) and \( \bar{x} := (\bar{x}_p)_p \). We assert that \( \bar{x} \in \mathbb{A}_f \). Indeed, \( \rho(x^{(n)}, x^{(m)}) < 1 \) for all \( n, m \geq M_0 \), hence \( \rho(x^{(n)}, x^{(m)}) = \|x^{(n)} - x^{(m)}\|_0 \). Due to the properties of \( p \)-adic absolute value it follows from \( |x_p - y_p|_p < 1 \) that \( |x_p - y_p|_p \leq 1 \). Therefore \( |x_p^{(n)} - x_p^{(m)}|_p \leq 1 \) for all \( p \) and \( n_0, m \geq M_0 \). Then \( |x_p^{(n)} - \bar{x}_p|_p \leq 1 \) for all \( p \) and \( n_0 \geq M_0 \). Since \( (\bar{x}_p)_p \in \mathbb{A}_f \), there exist a constant \( N' \geq N \) such that \( x_p^{(n_0)} \in \mathbb{Z}_p \) for \( p \geq N \). Then also \( \bar{x}_p \in \mathbb{Z}_p \) for \( p \geq N \). To show that \( \lim_{n \to \infty} \rho(x^{(n)}, x) = 0 \), consider arbitrary \( \epsilon > 0 \) and take an integer \( N' \geq N \) such that \( 1/N' < \epsilon \). Since \( |x_p^{(n)} - \bar{x}_p|_p \leq 1 \) for all \( p \) and \( n \geq M_0 \), and \( x_p^{(n)} \to \bar{x}_p \) for any \( p \), we have for \( n \) big enough

\[
\rho(x^{(n)}, \bar{x}) = \max\left\{ \max_{p < N'} |x_p^{(n)} - \bar{x}_p|_p, \max_{p \geq N'} |x_p^{(n)} - \bar{x}_p|_p \right\} \leq \max\left\{ \max_{p < N'} \frac{|x_p^{(n)} - \bar{x}_p|_p}{p}, \max_{p \geq N'} \frac{1}{N'} \right\} \leq \max\left\{ \max_{p < N'} \frac{|x_p^{(n)} - \bar{x}_p|_p}{p}, \epsilon \right\} = \epsilon.
\]

Let \( \tau_{\mathbb{A}_f} \) denote the restricted product topology on \( \mathbb{A}_f \), and let \( \tau_{\rho} \) denote the topology induced by \( \rho \) on \( \mathbb{A}_f \). We want to show that \( \tau_{\mathbb{A}_f} = \tau_{\rho} \). Set \( U := \prod_p \mathbb{Z}_p \). Then the family

\[ x + yU, \quad x \in \mathbb{A}_f, \quad y \in \mathbb{A}_f \setminus \{0\}, \]

is a base for \( \tau_{\mathbb{A}_f} \). Note that \((\mathbb{A}_f, +, \cdot)\) is a topological ring with respect to \( \tau_{\rho} \), and that the set \( U \) coincides with \( \{x \in \mathbb{A}_f : \rho(0, x) \leq \frac{1}{2}\} \) which is open in \( \tau_{\rho} \) due to the non-Archimedean nature of the metric. Then \( x + yU \in \tau_{\rho} \) for any \( x \in \mathbb{A}_f \), \( y \in \mathbb{A}_f \setminus \{0\} \), i.e. \( \tau_{\mathbb{A}_f} \subset \tau_{\rho} \). We now show that \( \tau_{\rho} \subset \tau_{\mathbb{A}_f} \). The family of balls

\[
B_{\epsilon}(x^{(0)}) = \left\{ x \in \mathbb{A}_f : \rho(x^{(0)}, x) \leq \epsilon \right\}, \quad x^{(0)} = (x_p^{(0)})_p \in \mathbb{A}_f
\]

is a base for \( \tau_{\rho} \). We have

\[
B_{\epsilon}(x^{(0)}) = \prod_p \left( x_p^{(0)} + p^{-\alpha_p(\epsilon)} \mathbb{Z}_p \right),
\]

where \( \alpha_p(\epsilon) = \lceil \log_2 p \epsilon \rceil \), here the function \( \lceil \cdot \rceil \) is defined by (3.5). Note that for \( p \) big enough \( \alpha_p(\epsilon) = 0 \) and \( x_p^{(0)} = 0 \) in \( \mathbb{Z}_p \). Therefore by (3.7) we have \( B_{\epsilon}(x^{(0)}) \in \tau_{\mathbb{A}_f} \).

**Corollary 3.4.** \( B_{\epsilon}(x^{(0)}) \) is a compact subset for any \( \epsilon > 0 \).

**Proof.** By (3.7), \( B_{\epsilon}(x^{(0)}) \) is a translation of a compact subset \( \prod_p p^{-\alpha_p(\epsilon)} \mathbb{Z}_p \), cf. (2.2).

**Remark 3.5.** The following properties of the space \((\mathbb{A}_f, \rho)\) hold.
(i) \((\mathbb{A}_f, \rho)\) is \(\sigma\)-compact space. Indeed, consider
\[K_N := \prod_{p \leq N} p^{-Np} \times \prod_{p > N} \mathbb{Z}_p \text{ for } N \in \mathbb{N}.\]
Then \(K_N\) is a compact subgroup with respect to \(\tau_{\mathbb{A}_f}\), see e.g. [23] Section 5.1 and \(\mathbb{A}_f = \bigcup_N K_N\).

(ii) \((\mathbb{A}_f, \rho)\) is second-countable topological space. Indeed, by applying twice the Weak Approximation Theorem, see e.g. [23, Theorem 1.4.4], one gets that \(\beta + \alpha \prod_p \mathbb{Z}_p, \beta \in \mathbb{Q}, \alpha \in \mathbb{Q} \setminus \{0\}\) is a countable base for the topology of \(\mathbb{A}_f\).

(iii) \((\mathbb{A}_f, \rho)\) is a semi-compact space, i.e. a locally compact Hausdorff space with a countable base.

Metric \(\rho\) allows us to introduce an adelic ball (given by (3.6)) and an adelic sphere, given by
\[(3.8) \quad S_r(x^{(0)}) = \left\{ x \in \mathbb{A}_f : \rho(x^{(0)}, x) = r \right\}, \quad x^{(0)} = (x_p^{(0)})_p \in \mathbb{A}_f.\]
Note that by Remark 3.1 the radius \(r\) of the adelic sphere may possess only values equal to any non-zero integer power of prime number. We now introduce some notations and compute volumes of adelic balls and adelic spheres.

Given a positive real number \(x\), we define
\[(3.9) \quad \Phi(x) = \prod_p p^{[\log p x]},\]
where \([\cdot]\) is defined by (3.5), i.e. for \(x \geq 1\) we take a product over all prime numbers each taken in the largest power \(\alpha_p\) such that \(p^{\alpha_p} \leq x\) and for \(x < 1\) we take a product over all prime numbers each taken in the largest power \(\alpha_p\) such that \(p^{\alpha_p} \leq px\), see also (3.7). Note that only a finite number of terms in this product differs from 1 and that the function \(\Phi(x)\) is non-decreasing, right-continuous and piecewise constant. Then \(\Phi(x) = 1\) if \(1/2 \leq x < 2\). If \(x \geq 2\), \(\Phi(x)\) coincides with the exponential of the second Chebyshev function \(\psi(x) = \sum_{p \leq x} \log p \ln p = \sum_{p \leq x} \ln p\), where the last sum is taken over all powers of prime numbers not exceeding \(x\).

It is easy to check using (3.9), (3.5) and properties of the entire part function that for any prime number \(p\) and any \(j \in \mathbb{Z} \setminus \{0\}\),
\[(3.10) \quad \Phi(p^{-j}) = -\frac{p}{\Phi(p^j)}.\]

**Definition 3.6.** For \(n \in \mathbb{R}, n > 0\) we define the next and previous non-zero power of a prime operators as
\[(3.11) \quad n_+ = \min \left\{ p^\beta : n < p^\beta, \ p \text{ prime}, \ \beta \in \mathbb{Z} \setminus \{0\} \right\},\]
\[(3.12) \quad n_- = \max \left\{ p^\beta : p^\beta < n, \ p \text{ prime}, \ \beta \in \mathbb{Z} \setminus \{0\} \right\}.\]

It is easy to see that the following relations hold for any number \(n = p^j\), where \(p\) is a prime and \(j \in \mathbb{Z} \setminus \{0\}\)
\[(3.13) \quad (n_-)_+ = n, \quad (n_+)^{-1} = (n^-)_-,\]
\[(n_+)_- = n, \quad (n^-)^{-1} = (n^+)_+,\]
\[\Phi\left((p^j)_-\right) = \frac{\Phi(p^j)}{p}.\]
By using the operators \((\cdot)_-\) and \((\cdot)_+\) we can completely order the set of non-zero powers of primes. This total order will be very relevant in the next sections. To simplify notations we will write \(p^j_-\) instead of \((p^j)_-\) and \(p^j_+\) instead of \((p^j)_+\).

**Lemma 3.7.** (i) The adelic ball \(B_r := B_r(0)\) is a compact subset and its volume is given by
\[
\text{vol}(B_r) = \Phi(r).
\]
(ii) The adelic sphere \(S_r := S_r(0)\) is a compact subset and its volume is given by
\[
\text{vol}(S_r) = \Phi(r) - \Phi(r_-).
\]

**Proof.** The compactness of \(B_r(0)\) was established in Corollary 3.4. Since \(S_r(0)\) is a closed subset of \(A_f\) and \(S_r(0) \subset B_r(0)\) we conclude that \(S_r(0)\) is compact. The formulas for volumes follows immediately from (3.7), (3.9) and (3.5).

3.2. The Fourier transform of radial functions.

**Definition 3.8.** A function \(f : A_f \to \mathbb{C}\) is said to be radial if its restriction to any sphere \(S_r, r > 0\), is a constant function, i.e. \(f|_{S_r} = f_r \in \mathbb{C}, r > 0\).

By abuse of notation we will denote a radial function \(f\) in the form \(f(\|\xi\|)\).

**Lemma 3.9.** Let \(f : A_f \to \mathbb{C}\) be an integrable function. Then the following assertions hold:

(i) \[
\int_{A_f} f(\xi) d\xi_{A_f} = \sum_{p^m, m \in \mathbb{Z}\setminus\{0\}} \int_{S_{p^m}} f(\xi) d\xi_{S_{p^m}}.
\]
In the particular case in which \(f\) is a radial function this formula takes the form
\[
\int_{A_f} f(\xi) d\xi_{A_f} = \sum_{p^m, m \in \mathbb{Z}\setminus\{0\}} f(p^m) \text{vol}(S_{p^m}).
\]

(ii) Take \(A^{(i)} = \bigsqcup_{m \in J} S_{p^m} \subset A_f\), where \(J\) is a (countable) subset of \(\mathbb{Z}\setminus\{0\}\), then
\[
\int_{A_f} f(\xi) 1_{A^{(i)}}(\xi) d\xi_{A_f} = \sum_{p^m, m \in J} \int_{S_{p^m}} f(\xi) d\xi_{S_{p^m}}.
\]
In the particular case in which \(f\) is a radial function this formula takes the form
\[
\int_{A_f} f(\xi) 1_{A^{(i)}}(\xi) d\xi_{A_f} = \sum_{p^m, m \in J} f(p^m) \text{vol}(S_{p^m}).
\]

(iii) Assume that \(A_f = \bigsqcup_{i \in \mathbb{N}} A^{(i)}\) with each \(A^{(i)}\) is a disjoint union of spheres, then
\[
\int_{A_f} f(\xi) d\xi_{A_f} = \sum_{i \in \mathbb{N}} \int_{A^{(i)}} f(\xi) d\xi_{A^{(i)}}.
\]

**Proof.** The proof follows by general techniques in measure theory, the compactness of the adelic balls and spheres, see Lemma 3.7 and the characterization of the adelic integrals for positive functions given in [23, p. 21].

To simplify notations, throughout this subsection the expressions \(\|0\|^{-1}\) and \(|0|_p^{-1}\) in the inequalities mean \(\infty\). The following theorem describes the Fourier transform of a radial function.
Claim 1. The proofs are given later.

Claim 2. Otherwise.

\[ f(x) := (F_{\mathbb{C}}^{-1}x,f)(x) = \sum_{q^{i}<\|x\|^{-1}} \Phi(q^{i}) (f(q^{i}) - f(q^{i}_{+})) \quad \text{for any } x \in \mathbb{A}_{f}, \]

where \( q^{i} \) runs through all non-zero powers of prime numbers; the functions \( \|x\| \), \( \Phi(x) \) and \( q^{i}_{+} \) are defined by (3.3), (3.4) and (3.5).

Remark 3.11. It follows from (3.3) that the Fourier transform of a radial function is again a radial function.

Proof. We represent the ring of finite adeles \( \mathbb{A}_{f} \) as a disjoint union of the following sets

\[ \mathbb{A}_{f} = \{0\} \cup \bigcup_{q} \mathbb{A}^{(0,q)} \cup \bigcup_{q} \mathbb{A}^{(1,q)}, \]

where

\[ \mathbb{A}^{(0,q)} := \bigcup_{j<0} S_{q^{j}} = \{ \xi \in \mathbb{A}_{f} : 0 < \|\xi\| < 1, \|\xi_{j}\| = \frac{|\xi_{j}|_{q}}{q}, \frac{|\xi|_{p}}{p} < \frac{|\xi_{j}|_{q}}{q} \text{ for } p \neq q \}, \]

\[ \mathbb{A}^{(1,q)} := \bigcup_{j>0} S_{q^{j}} = \{ \xi \in \mathbb{A}_{f} : \|\xi\| > 1, \|\xi\| = |\xi_{j}|_{q}, |\xi_{p}|_{p} < |\xi_{j}|_{q} \text{ for } p \neq q \}. \]

Note that on the sets \( \mathbb{A}^{(0,q)} \) we have \( \|\xi\| = \|\xi\|_{0} \) and on the sets \( \mathbb{A}^{(1,q)} \) we have \( \|\xi\| = \|\xi\|_{1} \). Then \( \hat{f}(x) = \sum_{q} \hat{f}^{(0,q)}(x) + \sum_{q} \hat{f}^{(1,q)}(x) \), where

\[ \hat{f}^{(k,q)}(x) := \int_{\mathbb{A}^{(k,q)}} \chi(\xi \cdot x) f(\|\xi\|) \, d\xi_{\mathbb{A}_{f}}, \quad k = 0, 1, q \text{ is a prime}. \]

We set

\[ \beta_{q} := \beta_{q}(x) = -[\log_{q} \|x\|] \]

with convention that \( \beta_{q}(0) = +\infty \). We also set \( \delta(t) = 1 \) if \( t = 0 \) and \( \delta(t) = 0 \) otherwise.

To simplify the proof, we first present the final formulas for the functions \( \hat{f}^{(k,q)}(x) \), the proofs are given later.

Claim 1.

(A) \[ \sum_{q} \hat{f}^{(1,q)}(x) = 0 \quad \text{if } \|x\| > 1, \]

(B) \[ \sum_{q} \hat{f}^{(1,q)}(x) = \sum_{q^{i}<\|x\|^{-1}} \left\{ \left(1 - \frac{1}{q}\right) \sum_{j=1}^{\beta_{q}(x)-1} f(q^{j}) \Phi(q^{j}) \right\}
\]

\[ - \sum_{q} \frac{1}{q} f(\|x\|^{-1}) \Phi(\|x\|^{-1}) \delta\left(\frac{|\xi_{j}|_{q}}{q} - |\xi_{j}|_{q} \right) \quad \text{if } \|x\| < 1. \]

Claim 2.

(C) \[ \sum_{q} \hat{f}^{(0,q)}(x) = \sum_{q} \left\{ \left(1 - \frac{1}{q}\right) \sum_{j=-\infty}^{\|x\|^{-1}} f(q^{j}) \Phi(q^{j}) \right\} \quad \text{if } \|x\| < 1, \]
\[
\sum_q f^{(0,q)}(x) = \sum_q \left\{ \left(1 - \frac{1}{q} \right) \sum_{j=-\infty}^{\beta_q(x)-1} f(q^j) \Phi(q^j) \right\} \\
- \sum_q \frac{1}{q} f(\|x\|^{-1}) \Phi(\|x\|^{-1}) \delta(\|x_q| - \|x\|) \quad \text{if } \|x\| > 1.
\]

(D)

Combining (A), (B), (C), (D) we obtain
\[
\hat{f}(x) = \sum_q \left\{ \left(1 - \frac{1}{q} \right) \sum_{j=\beta_q(x)+1}^{\beta_q(x)-1} f(q^j) \Phi(q^j) \right\} \\
- \sum_{q,j} \frac{1}{q} f(\|x\|^{-1}) \Phi(\|x\|^{-1}) \delta(q^j - \|x\|).
\]

(3.16)

Note that the last sum over \(q\) and \(j\) involving the function \(\delta\) means that we take the only term corresponding to the prime number \(q\) such that \(\|x\| = q^j\) for some \(j \in \mathbb{Z} \setminus \{0\}\).

Now the proof of the theorem may be finished as follows. Since \(f \in L^1(\mathbb{A}_f)\) and \(\text{vol}(S_{\mu_f}) = \text{vol}(\{\xi \in \mathbb{A}_f : \|\xi\| = p^j\}) = \Phi(p^j) - \Phi(p^j_+),\) see Lemma 3.7, the series \(\sum_q \Phi(q^j) - \Phi(q^j_+)\) is convergent. Because of the inequality \(\Phi(q^j) - \Phi(q^j_+) \geq \frac{1}{2} \Phi(q^j)\) the series \(\sum_q \Phi(q^j) |f(q^j)|\) converges as well, hence we may arbitrary reorder the terms in (3.16).

By the properties of the entire part function, the following inequalities hold for the function \(\beta_q\):
\[
q^j \leq q^{-\log_q \|x\|-1} < q^{-\log_q \|x\|} = \|x\|^{-1}, \quad j < \beta_q, \ j \in \mathbb{Z}, \\
q^j \geq q^{-\log_q \|x\|} \geq q^{-\log_q \|x\|} = \|x\|^{-1}, \quad j \geq \beta_q, \ j \in \mathbb{Z},
\]
where the equality in the second inequality is possible only when \(\|x\|\) is a power of \(q\). Suppose in (3.16), that \(\|x\| = p^k\) for some prime number \(p\) and integer \(k \neq 0\). It follows from inequalities (3.17), (3.18) that the formula (3.16) may be written as
\[
\hat{f}(x) = \sum'_{q^j < \|x\|^{-1}} \left(1 - \frac{1}{q} \right) f(q^j) \Phi(q^j) - \frac{1}{p} f(\|x\|^{-1}) \Phi(\|x\|^{-1}) \\
- \sum'_{q^j < \|x\|^{-1}} f(q^j) \Phi(q^j) - \sum'_{q^j < \|x\|^{-1}} f(q^j) \Phi(q^j_+) = f(\|x\|^{-1}) \Phi(\|x\|^{-1}) \\
= \sum'_{q^j < \|x\|^{-1}} f(q^j) \Phi(q^j) - \sum'_{q^j < \|x\|^{-1}} f(q^j) \Phi(q^j) = \sum'_{q^j < \|x\|^{-1}} \Phi(q^j) (f(q^j) - f(q^j_+)),
\]
where we have used (3.11)–(3.13) and \(\sum'\) means that the value \(j = 0\) is omitted in the summation.

The checking of the formula (3.14) in the case \(\|x\| = 0\) is left to the reader. \(\square\)

**Proof of Claim 3.** We assume that \(x \neq 0\). The case \(x = 0\) may be checked directly. With the use of (2.3), Lemma 3.9 and the fact that \(1_{S_{\mu_f}}(x)\) is a factorizable function, we may write \(f^{(1,q)}\) as
\[
\hat{f}^{(1,q)}(x) = \int \chi_q(x_q \xi_0) f(|\xi_q|_{\|q\|}) \left\{ \prod_{p \neq q} \int \chi_p(x_p \xi_p) d\xi_p \right\} d\xi_q.
\]

**Proof of Claim 4.** We assume that \(x \neq 0\). The case \(x = 0\) may be checked directly.
Denote by $\alpha_p(\xi_q)$ the largest integer satisfying $p^{\alpha_p(\xi_q)} \leq |\xi_q|_q$ (i.e., $\alpha_p(\xi_q) = \lfloor \log_p |\xi_q|_q \rfloor = \lfloor \log_p |\xi_q|_q \rfloor$). Note that the equality $p^{\alpha_p(\xi_q)} = |\xi_q|_q$ is impossible for $|\xi_q|_q > 1$ and $p \neq q$, hence $p^{\alpha_p(\xi_q)} < |\xi_q|_q$. Recall that

$$
\int_{|\xi|_p \leq p^{\alpha_p(\xi_q)}} x_p (x_p \xi_q) d\xi_p = \begin{cases} p^{\alpha_p(\xi_q)} & \text{if } |x_p|_p \leq p^{-\alpha_p(\xi_q)}, \\ 0 & \text{if } |x_p|_p \geq p^{-\alpha_p(\xi_q)}+1, \end{cases}
$$

and since $p^{\alpha_p(\xi_q)} < |\xi_q|_q < p^{\alpha_p(\xi_q)+1}$, we have

$$
\int_{|\xi|_p \leq p^{\alpha_p(\xi_q)}} x_p (x_p \xi_q) d\xi_p = \begin{cases} p^{\alpha_p(\xi_q)} & \text{if } |x_p|_p < p|\xi_q|^{-1}, \\ 0 & \text{if } |x_p|_p > p|\xi_q|^{-1}, \end{cases}
$$

which implies

$$
\prod_{p \neq q} \int_{|\xi|_p < |\xi_q|} x_p (x_p \xi_q) d\xi_p = \left( \prod_{p \neq q} p^{\alpha_p(\xi_q)} \right) \mathbf{1}_B(\xi_q) = \frac{\Phi(|\xi_q|_q)}{|\xi_q|_q} \mathbf{1}_B(\xi_q),
$$

where $\mathbf{1}_B(\xi_q)$ is the characteristic function of the set

$$
B := \left\{ \xi_q \in \mathbb{Q}_q : \max_{p \neq q} \frac{|x_p|_p}{|p|_p} < |\xi_q|_q^{-1} \right\}.
$$

Therefore

$$
\tilde{f}^{(1,q)}(x) = \int_{q \leq |\xi|_q < \left( \max_{p \neq q} \frac{|x_p|_p}{|p|_p} \right)^{-1}} x_q (x_q \xi_q) f(\xi_q) \Phi(|\xi_q|_q) d\xi_q.
$$

Note that it follows from (3.20) that $\tilde{f}^{(1,q)}(x) = 0$ if $\max_{p \neq q} \frac{|x_p|_p}{|p|_p} \geq \frac{1}{q}$.

Set $\gamma_q$ to be the largest integer satisfying $q^{\gamma_q} < \left( \max_{p \neq q} \frac{|x_p|_p}{|p|_p} \right)^{-1}$, then

$$
\tilde{f}^{(1,q)}(x) = \sum_{j=1}^{\gamma_q} \frac{f(q^{j}) \Phi(q^{j})}{q^{j}} \int_{|\xi|_q = q^{j}} x_q (x_q \xi_q) d\xi_q.
$$

We recall that

$$
\int_{|\xi|_q = q^{j}} x_q (x_q \xi_q) d\xi_q = \begin{cases} q^{j} (1 - q^{-1}) & \text{if } |x_q|_q \leq q^{-j}, \\ -q^{j-1} & \text{if } |x_q|_q = q^{-j+1}, \\ 0 & \text{if } |x_q|_q \geq q^{-j+2}. \end{cases}
$$

Note that the integral (3.22) is non-zero when $|\xi_q|_q \leq \frac{q}{|\xi|_q} = \left( \frac{|x|_q}{|p|_q} \right)^{-1}$. Since for $|\xi_q|_q > 1$ the equality $|\xi_q|_q = \left( \max_{p \neq q} \frac{|x_p|_p}{|p|_p} \right)^{-1}$ is impossible, the last inequality may be combined with $|\xi_q|_q < \left( \max_{p \neq q} \frac{|x_p|_p}{|p|_p} \right)^{-1}$ into the inequality $|\xi_q|_q \leq \left( \max_{p \neq q} \frac{|x_p|_p}{|p|_p} \right)^{-1} - ||x||_0^{-1}$, cf. (3.1). Then it follows from (3.20)–(3.22) that

$$
\tilde{f}^{(1,q)}(x) = \int_{q \leq |\xi|_q \leq ||x||_0^{-1}} x_q (x_q \xi_q) f(|\xi_q|_q) \Phi(|\xi_q|_q) d\xi_q.
$$

Note that $\tilde{f}^{(1,q)}(x) = 0$ if $||x||_0^{-1} < q$ and that

$$
\{ x \in A_f : ||x||_0^{-1} \geq 2 \} = \{ x \in A_f : \max_{p} \frac{|x|_p}{|p|} \leq \frac{1}{2} \} = \prod_p \mathbb{Z}_p,
$$
hence for any \( x \) outside of \( \prod_p \mathbb{Z}_p \), the sum \( \sum_q \hat{f}^{(1,q)}(x) \) vanishes. Therefore we have non-zero terms in \( \sum_q \hat{f}^{(1,q)}(x) \) only if \( \|x\|_0 < 1 \). In such case \( \|x\|_0 = \|x\| \). We recall definition (3.15) of \( \beta_q \) and inequalities (3.17), (3.18). Then it follows from (3.21), (3.23) that

\[
\hat{f}^{(1,q)}(x) = \left( 1 - \frac{1}{q} \right) \sum_{j=1}^{\beta_q-1} f(q^j) \Phi(q^j) \quad \text{if} \quad \frac{|x|_q}{q} < \|x\|
\]

and

\[
\hat{f}^{(1,q)}(x) = \left( 1 - \frac{1}{q} \right) \sum_{j=1}^{\beta_q-1} f(q^j) \Phi(q^j) - \frac{1}{q} f(\|x\|^{-1}) \Phi(\|x\|^{-1}) \quad \text{if} \quad \frac{|x|_q}{q} = \|x\|.
\]

By combining the formulas (3.24) - (3.25) we obtain formula (3.6).

**Proof of Claim 2.** We assume that \( x \neq 0 \). The case \( x = 0 \) may be checked directly. The required calculations are mostly similar to the previous ones, however there are some subtle variations. We have \( \|\xi\| = q^{-1} |\xi|_q \) and

\[
\hat{f}^{(0,q)}(x) = \int_{q^{-1}|\xi|_q < 1} \chi_q(x_q \xi) f(q^{-1} |\xi|_q) \left\{ \prod_{p \neq q} p^{-1} |\xi|_p q^{-1} |\xi|_q \right\} \chi_p(x_p \xi_p) d\xi_p.
\]

Let \( \alpha_p(\xi) \) denote the largest power \( p^\alpha_p(\xi) \) satisfying \( p^{-1} p^\alpha_p(\xi) < q^{-1} |\xi|_q \) which is equal to \( 1 + \lfloor \log_p q^{-1} |\xi|_q \rfloor \), and since \( q^{-1} |\xi|_q < 1 \), the last quantity is equal to \( \lfloor \log_q q^{-1} |\xi|_q \rfloor \), cf. (3.5). Hence similarly to (3.19) with the use of (3.9) we obtain

\[
\prod_{p \neq q} \int_{p^{-1} |\xi|_p q^{-1} |\xi|_q} \chi_p(x_p \xi_p) d\xi_p = \left( \prod_{p \neq q} p^\alpha_p(\xi) \right) \Phi(q^{-1} |\xi|_q) = \frac{\Phi(q^{-1} |\xi|_q)}{q^{\lfloor \log_q q^{-1} |\xi|_q \rfloor}} \mathbf{1}_B(\xi_q),
\]

where \( \mathbf{1}_B(\xi_q) \) is the characteristic function of the set

\[
B := \{ \xi_q \in \mathbb{Q}_q : |\xi|_q < q \left( \max_{p \neq q} |x_p|_p \right)^{-1} \}.
\]

Since \( |\xi|_q \) is a power of \( q \) and \( q^{-1} |\xi|_q < 1 \), we have \( \lfloor \log_q q^{-1} |\xi|_q \rfloor = \log_q |\xi|_q \) and \( q^{\lfloor \log_q q^{-1} |\xi|_q \rfloor} = q^{\log_q |\xi|_q} = |\xi|_q \). Since \( q^{-1} |\xi|_q < 1 \) is equivalent to \( |\xi|_q \leq 1 \), we obtain

\[
\hat{f}^{(0,q)}(x) = \int_{|\xi|_q \leq 1} \chi_q(x_q \xi) f(q^{-1} |\xi|_q) \Phi(q^{-1} |\xi|_q) |\xi|_q d\xi_q.
\]

It follows from (3.22) that the last integral is non-zero when \( |x|_q \leq q |\xi|_q^{-1} \), which may be combined with (3.26) into \( \max_p |x_p|_p \leq q |\xi|_q^{-1} \) or, equivalently, into \( |\xi|_q \leq q (\max_p |x_p|_p)^{-1} \). Hence the domain of integration in the last integral is

\[
|\xi|_q \leq \min\left\{ 1, \frac{q}{\max_p |x_p|_p} \right\},
\]

and we have to consider two cases.
of $\beta$ similarly to (3.24) and (3.25) we have
determine possible values of $f$ and inequalities (3.17), (3.18). As a result, from (3.27), (3.28) and (3.22)
we obtain

$$
\hat{f}^{(0,q)}(x) = \left(1 - \frac{1}{q}\right) \sum_{j=-\infty}^{0} f(q^{j+1}) \Phi(q^{j+1}) - \left(1 - \frac{1}{q}\right) \sum_{j=-\infty}^{-1} f(q^j) \Phi(q^j).
$$

Case 2: \( \|x\| > 1 \), i.e. \( \max_p |x_p|_p = \|x\| > 1 \). In this case it is sufficient to
determine possible values of $|\xi|_q$ from the inequality \( |\xi|_q \leq \frac{\|x\|}{q} \), because they satisfy the
inequality \( |\xi|_q \leq 1 \) even in the case \( \|x\| < q \). Recall definition (3.15)
of $\beta_q$ and inequalities (3.17), (3.18). As a result, from (3.27), (3.28) and (3.22)
similarly to (3.24) and (3.25) we have

$$
\hat{f}^{(0,q)}(x) = \left(1 - \frac{1}{q}\right) \sum_{j=-\infty}^{\beta_q} f(q^{j+1}) \Phi(q^{j+1})
= \left(1 - \frac{1}{q}\right) \sum_{j=-\infty}^{\beta_q-1} f(q^j) \Phi(q^j) \quad \text{if } |x_q|_q < \|x\|
$$

and

$$
\hat{f}^{(0,q)}(x) = \left(1 - \frac{1}{q}\right) \sum_{j=-\infty}^{\beta_q-1} f(q^j) \Phi(q^j) - \frac{1}{q} f(\|x\|^{-1}) \Phi(\|x\|^{-1}) \quad \text{if } |x_q|_q = \|x\|.
$$

The last two equalities give us formulas (C)–(D).

As a corollary from Theorem 3.10 we derive a sufficient condition for the Fourier
transform of a radial function to be non-negative.

**Corollary 3.12.** Let $f$ be a real-valued non-increasing radial function, i.e. \( f = f(\|\xi\|) \) and \( f(\xi) \geq f(\zeta) \) for any \( \xi, \zeta \in \mathcal{A}_f \) satisfying \( \|\xi\| \leq \|\zeta\| \). Then

$$
\hat{f}(x) := (\mathcal{F}_{\xi \rightarrow x} f)(x) \geq 0 \quad \text{for any } x \in \mathcal{A}_f.
$$

On the base of Theorem 3.10 we may compute the Fourier transforms of characteristic functions of balls and spheres.

**Corollary 3.13.** Let $f$ be a characteristic function of a ball, i.e. \( f = 1_{B_r}(x) \),
\( r \in \{p^i : p \text{ is prime}, j \in \mathbb{Z} \setminus \{0\}\} \). Then

$$
\hat{f}(\xi) = \Phi(r) 1_{B_R}(\xi), \quad R = (r^{-1})_-. 
$$

Let $g$ be a characteristic function of a sphere, i.e. \( g = 1_{S_r}(x) \), \( r \in \{p^i : p \text{ is prime}, j \in \mathbb{Z} \setminus \{0\}\} \). Then

$$
\hat{g}(\xi) = \Phi(r) 1_{B_R}(\xi) - \Phi(r_-) 1_{B_{R+}}(\xi), \quad R = (r^{-1})_-.
$$

**Proof.** As it follows from (3.14) we have at most one non-zero term equal to $\Phi(r)$
in the Fourier transform of the ball $B_r$, and this term is present if and only if
\( r < \|x\|^{-1} \) which is equivalent to $\|x\| < r^{-1}$ or \( \|x\| \leq (r^{-1})_- \).

The second statement follows from the presentation $S_r = B_r \setminus B_{r^-}$ and the
properties of the operators \('_-' and '++': \((r^-)_-\) = \((r^{-1})_\) = r^{-1} = R_. \)
3.3. Distributions on $\mathbb{A}_f$. In this subsection we consider $\mathbb{A}_f$ as the complete metric space $(\mathbb{A}_f, \rho)$. As it was previously mentioned in Subsection 2.2 the space $S(\mathbb{A}_f)$ of Bruhat-Schwartz functions consists of finite linear combinations of factorizable functions $f = \prod_p f_p$, where a finite number of the functions $f_p$ are in $S(\mathbb{Q}_p)$ and the rest of the functions are the characteristic functions of the sets $\mathbb{Z}_p$, i.e. $f = \prod_{p \leq N} f_p \times \prod_{p > N} \Omega_p(|x_p|_p)$. For the sake of simplicity, from now on we will use test function to mean Bruhat-Schwartz function. The spaces $S(\mathbb{Q}_p)$ consist of compactly supported locally constant functions. We show that the same property characterizes the space $S(\mathbb{A}_f)$. Despite a similar result was already proved in [33], the adelic metric $\rho$ allows us to introduce the notion of ‘parameter of constancy’ for functions in $S(\mathbb{A}_f)$ and to give a construction of a topology for $S(\mathbb{A}_f)$ in a similar way as for the spaces $S(\mathbb{Q}_p)$.

Definition 3.14. We say that a function $f$ is locally constant if for any $x \in \mathbb{A}_f$ there exists a constant $\ell(x) > 0$ such that $f(x + y) = f(x)$ for any $y \in B_{\ell(x)}(0)$.

The same reasoning as in the $p$-adic case, see e.g. [16] or [2], shows that for a compactly supported function $f$ the same constant $\ell$ may be chosen for all points $x \in \mathbb{A}_f$.

Definition 3.15. Let $f$ be a non-zero compactly supported function. We define the parameter of constancy $\ell$ of $f$ as the largest non-zero integer power of a prime number such that

\[(3.29) \quad f(x + y) = f(x) \quad \text{for any } x \in \mathbb{A}_f, \quad y \in B_{\ell}(0).\]

By definition we set the parameter of constancy of function 0 to be equal $+\infty$.

Lemma 3.16. The function $f \in S(\mathbb{A}_f)$ if and only if it is locally constant with compact support.

Proof. The statement is trivial for $f \equiv 0$. Suppose $f \in S(\mathbb{A}_f) \setminus \{0\}$, and $f = \sum_{m=1}^{M} f^{(m)}(x)$, where each function $f^{(m)}$ is factorizable, $f^{(m)} = \prod_{p \leq N, m} f_p^{(m)} \times \prod_{p > N} \Omega_p(|x_p|_p)$. Since each $f^{(m)}$ is compactly supported, so is $f$.

Let $f_p^{(m)}$ denote the parameter of constancy of the function $f_p^{(m)}$, i.e. $f_p^{(m)}(x_p + y_p) = f_p^{(m)}(x_p)$ for all $y_p$ such that $|y_p|_p \leq f_p^{(m)}$. Note that our definition of the parameter of constancy on $\mathbb{Q}_p$ is different from the one presented in [40]. Such change of definition is justified by necessity to make the parameter of constancy independent on $p$. Consider

$$\ell = \min \left\{ \frac{1}{2}, \min_{p,m} \frac{f_p^{(m)}}{p} \right\}.$$ 

Since we have only finite number of parameters $f_p^{(m)}$, $\ell > 0$. It is easy to check that (3.29) holds with this parameter $\ell$, i.e. the function $f$ is locally constant.

Suppose now that $f$ is a locally constant function with compact support. Let $K = \text{supp } f$. Since $f$ is locally constant, for each $x \in K$ there exists a ball $B_{\ell(x)}(x)$, which is an open set, such that $f$ is constant on $B_{\ell(x)}(x)$. Then there exists a finite number of these balls, say $B_{r_1}(x_1), \ldots, B_{r_n}(x_n)$, covering $K$. Since the metric is non-Archimedean we may assume that these balls are disjoint. Therefore

\[(3.30) \quad f(x) = f(x_1) \cdot 1_{B_{r_1}(x_1)}(x) + \cdots + f(x_n) \cdot 1_{B_{r_n}(x_n)}(x),\]

where each characteristic function $1_{B_{r_i}(x_i)}(x)$ is factorizable, cf. (3.7). \qed
Remark 3.17. Let $\mathcal{P}(A_f)$ denote the set of parameters of constancy of functions from $\mathcal{S}(A_f)$. Then
\[
\mathcal{P}(A_f) = \{ l \in \mathbb{Q} : l = p^m, \ p \text{ is a prime, } m \in \mathbb{Z} \setminus \{0\} \} \cup \{+\infty\}.
\]

By considering the characteristic functions of the adelic balls $B_r$ we verify that every number in $\mathcal{P}$ is an admissible parameter of constancy. $\mathcal{P}(A_f)$ is a countable and totally ordered set.

We define by $\mathcal{S}'_{R}(A_f)$ the subspace of test functions with supports contained in the adelic ball $B_R$ and parameters of constancy $\geq l$. Then the following embedding holds: $\mathcal{S}'_{R}(A_f) \subset \mathcal{S}'_{R'}(A_f)$ whenever $R \leq R'$, $l \geq l'$. As in the $p$-adic setting, see e.g. [2, 43, 46], we define the convergence in $\mathcal{S}(A_f)$ in the following way: $f_k \to 0$, $k \to \infty$ in $\mathcal{S}(A_f)$ if and only if

(i) $f_k \in \mathcal{S}'_{R}(A_f)$ where $R$ and $l$ do not depend on $k$;

(ii) $f_k \to 0$ uniformly as $k \to \infty$.

With this notion of convergence $\mathcal{S}(A_f)$ becomes a complete topological vector space. In addition,
\[
\mathcal{S}_R(A_f) = \lim \inf_{l \to 0} \mathcal{S}'_{R}(A_f), \quad \mathcal{S}(A_f) = \lim \inf_{R \to \infty} \mathcal{S}_R(A_f).
\]

Note that the second inductive limit makes sense because $\mathcal{P}(A_f)$ is totally ordered.

The following proposition shows that the spaces $\mathcal{S}'_{R}(A_f)$ possess similar properties to their $p$-adic analogues.

Proposition 3.18. For arbitrary $l \leq R$ the space $\mathcal{S}'_{R}(A_f)$ is non-trivial and finite dimensional, its dimension is equal to $\Phi(R)/\Phi(l)$, with a basis given by the characteristic functions of disjoint balls $B_l(x^{(n)}) \subset B_R$. If $f \in \mathcal{S}'_{R}(A_f)$ then $\hat{f} = \mathcal{F}_{x \to \xi} f \in \mathcal{S}'_{(1/R)}(A_f)$. Moreover, $\mathcal{F} \mathcal{S}'_{R}(A_f) = \mathcal{S}'_{(1/R)}(A_f)$.

Proof. Note that $B_R$ is a finite disjoint union of balls of type $B_l(x_i)$ and the number of such balls is $\text{vol}(B_R)/\text{vol}(B_l)$. The first statement follows from this observation by [3, 30].

For the second part it is enough to consider the Fourier transform of the characteristic function of a ball $B_l(x^{(n)}) \subset B_R$. We obtain from Corollary 3.13
\[
\hat{1}_{B_l(x^{(n)})}(\xi) = \int_{A_f} \chi(-x \cdot \xi) 1_{B_l}(x - x^{(n)}) \, dx_{A_f} = \chi(-x^{(n)} \cdot \xi) 1_{B_l}(\xi) = \chi(-x^{(n)} \cdot \xi) \Phi(l) 1_{B_{(1/l)}^-}(\xi),
\]
hence the Fourier transform is supported in the ball $B_{(1/l)}^-$. Since $x^{(n)} \in B_R$, for any $y \in B_{(1/R)}^-$ we have $\|x^{(n)} \cdot y\| < 1$ hence $\chi(x^{(n)} \cdot y) = 1$ and the Fourier transform of a ball $B_l(x^{(n)})$ is locally constant with the parameter of constancy $\geq (1/R)^-$. The last part follows from the observation that $(\{(n^{-1})_{-1}\}) = (n_{-})_{-} = n$ for any non-zero power of a prime. \hfill \Box

Proposition 3.19. (i) Let $K$ be a compact subset of $\mathcal{S}(A_f)$. The space of test functions $\mathcal{S}(A_f)$ is dense in the space $C(K)$ of continuous functions on $K$. (ii) The space of test functions $\mathcal{S}(A_f)$ is dense in $L^q(A_f)$ for $1 \leq q < \infty$.

Proof. The proof follows the classical pattern, see e.g. [2, 43, 46]. \hfill \Box
Denote by \( S'(\mathbb{A}_f) \) the \( \mathbb{C} \)-vector space of all (complex-valued) linear continuous functionals on \( S(\mathbb{A}_f) \). This space is the space of Bruhat-Schwartz distributions on \( \mathbb{A}_f \). For the sake of simplicity we will use distribution instead of Bruhat-Schwartz distribution. We equip \( S'(\mathbb{A}_f) \) with the weak topology. The following proposition allows to simplify checking that a functional belongs to the space \( S'(\mathbb{A}_f) \) stating that every linear functional on \( S(\mathbb{A}_f) \) is continuous.

**Proposition 3.20.** (i) \( S'(\mathbb{A}_f) \) is the \( \mathbb{C} \)-vector space of all (complex-valued) linear functionals on \( S(\mathbb{A}_f) \). (ii) \( S'(\mathbb{A}_f) \) is complete.

**Proof.** Due to Proposition 3.18 the proof of this proposition is completely similar to the proof given for the analogous statement in the \( p \)-adic case, see e.g. [2], [46]. □

3.4. Pseudodifferential operators and the Lizorkin space on \( \mathbb{A}_f \). As it was mentioned in the introduction, the classical derivative cannot be defined for complex-valued functions on adeles. Instead we consider pseudodifferential operators. The function \( \| \cdot \| \) which generates the metric \( \rho \) allows us to introduce a natural generalization of the Taibleson operator \( D^\gamma_{\mathbb{A}_f} := D^\gamma, \gamma > 0 \), defined on \( S(\mathbb{A}_f) \) by

\[
(D^\gamma f)(x) = \mathcal{F}_{\xi \to x}^{-1}(\|\xi\|^\gamma \mathcal{F}_{\xi \to x}f), \quad f \in S(\mathbb{A}_f).
\]

**Lemma 3.21.** With the above notation,

\[
D^\gamma : S(\mathbb{A}_f) \to C(\mathbb{A}_f) \cap L^2(\mathbb{A}_f).
\]

**Proof.** Since \( \mathcal{F}_{\xi \to x}f \) may be represented as a linear combination of functions of type \( 1_{B_r(\xi_0)}(\xi) \), it is sufficient to consider the case \( \mathcal{F}_{\xi \to x}f = 1_{B_r(\xi_0)}(\xi) \). If \( 0 \notin B_r(\xi_0) \) then \( \|\xi\|^\gamma 1_{B_r(\xi_0)}(\xi) \in S(\mathbb{A}_f) \) because \( \|\xi\|^\gamma \) is locally constant outside of the origin, and hence

\[
\mathcal{F}_{\xi \to x}^{-1}(\|\xi\|^\gamma 1_{B_r(\xi_0)}(\xi)) \in S(\mathbb{A}_f) \subset L^2(\mathbb{A}_f).
\]

If \( 0 \in B_r(\xi_0) \) then \( B_r(\xi_0) = B_r(0) \) and \( \|\xi\|^\gamma \leq r^\gamma \) on the \( B_r(0) \), hence \( \|\xi\|^\gamma 1_{B_r(0)}(\xi) \in L^1(\mathbb{A}_f) \cap L^2(\mathbb{A}_f) \). Thus \( \mathcal{F}_{\xi \to x}^{-1}(\|\xi\|^\gamma 1_{B_r(\xi_0)}(\xi)) \in C(\mathbb{A}_f) \cap L^2(\mathbb{A}_f) \). □

The space \( S(\mathbb{A}_f) \) is not invariant under the action of the operator \( D^\gamma \). To overcome such an inconvenience, we introduce the following space

\[
\mathcal{L}_0(\mathbb{A}_f) := \mathcal{L}_0 = \{ f \in S(\mathbb{A}_f) : \hat{f}(0) = 0 \}.
\]

The space \( \mathcal{L}_0 \) can be equipped with the topology of the space \( S(\mathbb{A}_f) \), which makes \( \mathcal{L}_0 \) a complete space. Note that

\[
\mathcal{L}_0 = \mathcal{F}\{ h \in S(\mathbb{A}_f) : h(0) = 0 \}.
\]

This space is an adelic analogue of the Lizorkin space of the second kind. We refer the reader to [1] for the theory of the \( p \)-adic Lizorkin spaces. Recently in [30] an adelic version of the Lizorkin space of the first kind was introduced.

**Lemma 3.22.** With the above notation the following assertions hold:

(i) \( D^\gamma \mathcal{L}_0 = \mathcal{L}_0 \) for \( \gamma > 0 \).
(ii) \( f \in \mathcal{L}_0 \) if and only if \( f \in S \) and \( \int_{\mathbb{A}_f} f(x) \, dx_{\mathbb{A}_f} = 0 \).
(iii) \( \mathcal{L}_0 \) is dense in \( S \) with respect to the \( L^2 \)-norm.
(iv) \( \mathcal{L}_0 \) is dense in \( L^2(\mathbb{A}_f) \).
Proof. (i) Take $f \in L_0$, then $\|\xi\|^{\gamma} \hat{f}(\xi) \in \mathcal{S}(\mathcal{A}_f)$ because $\hat{f}$ is equal to 0 in some neighborhood of 0 and $\|\xi\|^{\gamma}$ is locally constant outside of the origin. Therefore $D^\gamma f \in L_0$, i.e. $D^\gamma L_0 \subseteq \mathcal{L}_0$. The converse inclusion follows from the fact that $\hat{\mathcal{A}} \in \mathcal{S}(\mathcal{A}_f)$ for any $h \in \mathcal{L}_0$.

(ii) The statement follows from $\hat{f}(0) = \int_{A_f} f(x) \, dx_{A_f}$ which is the consequence of $\mathcal{F}\mathcal{S}(\mathcal{A}_f) = \mathcal{S}(\mathcal{A}_f)$.

(iii) By (3.32) and the fact that the Fourier transform preserves the $L^2$ norm, it is sufficient to show that $1_{B_{\gamma}}(x)$ can be arbitrarily closely approximated by functions from $L_0$ in the $L^2$ norm. As such approximating functions we may use $1_{B_{\gamma}} - 1_{B_{\gamma-m}}(x)$ for $m$ big enough.

(iv) The statement follows from (iii) since $\mathcal{S}(\mathcal{A}_f)$ is dense $L^2(\mathcal{A}_f)$, see Proposition 3.19.

We will use the notation $\mathcal{D}(D^\gamma)$ to denote the domain of the operator $D^\gamma$. We refer reader to [37] for notions of essentially self-adjoint operators and to [3] and [21] for the definition of strongly continuous ($C_0$) semigroups and related notions.

Since $L_0(\mathcal{A}_f) \subseteq L^2(\mathcal{A}_f)$, we may consider the operator $D^\gamma$ as an operator acting on $L^2(\mathcal{A}_f)$. It is easy to see that the operator $D^\gamma$ with the domain $\mathcal{D}(D^\gamma) = L_0(\mathcal{A}_f)$ is symmetric. Moreover, similarly to the proof of Lemma 3.22 (i) we may check that $(D^\gamma \pm i)L_0(\mathcal{A}_f) = L_0(\mathcal{A}_f)$, i.e. the ranges of the operators $D^\gamma \pm i$ are dense in $L^2(\mathcal{A}_f)$, hence the operator $D^\gamma$ is essentially self-adjoint, see [37, Corollary to Theorem VIII.3] for details. The following description of the self-adjoint closure holds.

**Lemma 3.23.** The closure of the operator $D^\gamma$, $\gamma > 0$ (let us denote it by $D^\gamma$ again) with domain

$$\mathcal{D}(D^\gamma) := \left\{ f \in L^2(\mathcal{A}_f) : \|\xi\|^{\gamma} \hat{f} \in L^2(\mathcal{A}_f) \right\}$$

is a self-adjoint operator. Moreover, the following assertions hold:

(i) $D^\gamma$ is a positive operator;

(ii) $D^\gamma$ is $m$-accretive, i.e. $-D^\gamma$ is an $m$-dissipative operator;

(iii) the spectrum $\sigma(D^\gamma) = \left\{ p^{\gamma j} : p \text{ is a prime, } j \in \mathbb{Z} \setminus \{0\} \right\} \cup \{0\}$;

(iv) $-D^\gamma$ is the infinitesimal generator of a contraction $C_0$ semigroup $(T(t))_{t \geq 0}$.

Moreover, the semigroup $(T(t))_{t \geq 0}$ is bounded holomorphic (or analytic) with angle $\pi/2$.

Proof. (i) It follows from the Steklov–Parseval equality that for any $f \in L^2(\mathcal{A}_f)$

$$\langle D^\gamma f, f \rangle = \langle \|\xi\|^{\gamma} \mathcal{F} f, \mathcal{F} f \rangle = \int_{A_f} \|\xi\|^{\gamma} |\mathcal{F} f|^2 d\xi_{A_f} \geq 0.$$

(ii), (iv) The result follows from the well-known corollary from the Lumer-Phillips theorem, see e.g. [21, Chapter 2, Section 3] or [13]. For the property of the semigroup of being holomorphic, see e.g. [3, 3.7] or [21, Chapter 2, Section 4.7].

(iii) Since $D^\gamma$ is self-adjoint and positive, $\sigma(D^\gamma) \subset [0, \infty)$. Consider the eigenvalue problem $D^\gamma f = \lambda f$, $f \in \mathcal{D}(D^\gamma)$, $\lambda > 0$. By applying the Fourier transform we obtain the equivalent equation

$$\langle \|\xi\|^{\gamma} - \lambda \rangle \hat{f} = 0.$$
If $\lambda = p^{\gamma j}$ for some prime $p$ and $j \in \mathbb{Z} \setminus \{0\}$ then the inverse Fourier transform of the characteristic function of $S_p = \{ \xi \in \mathbb{A}_f : ||\xi|| = p^j \}$ is a solution of (3.34). If $\lambda \not\in \{ p^{\gamma j} : p$ is a prime, $j \in \mathbb{Z} \setminus \{0\} \}$, then the functions $|1/||\xi||^{\gamma} - \lambda|$ and $||\xi||^{\gamma} - \lambda|$ are bounded, hence the equation $D^\gamma f - \lambda f = h$ is uniquely solvable for any $h \in L^2(\mathbb{A}_f)$ and $\lambda \in \rho(D^\gamma)$. The point 0 belongs to $\sigma(D^\gamma)$ as a limit point.

The representation of the generated semigroup $\{ T(t) \}_{t \geq 0}$ is presented in detail in Theorem 7.5.

4. Metric structures, Distributions and Pseudodifferential Operators on $\mathbb{A}$

4.1. A structure of complete metric space for the adeles. We recall that $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. Then any $x \in \mathbb{A}$ can be written uniquely as $x = (x_\infty, x_f) \in \mathbb{R} \times \mathbb{A}_f = \mathbb{A}$.

Set for $x, y \in \mathbb{A}$

$$\rho_\mathbb{A}(x, y) := |x_\infty - y_\infty|_\infty + \rho(x_f, y_f),$$

where $\rho(x, y)$ was defined in (3.4). Then $(\mathbb{A}, \rho_\mathbb{A})$ is a complete metric space, see Proposition 3.3. Note that $\rho_\mathbb{A}(x, y)$ is topologically equivalent to

$$\tilde{\rho}(x, y) := \max \{|x_\infty - y_\infty|_\infty, \rho(x, y)\}, \quad x, y \in \mathbb{A},$$

which induces on $\mathbb{A}$ the product topology. The topology of the restricted product on $\mathbb{A}$ is equal to the product topology on $\mathbb{R} \times \mathbb{A}_f$, where $\mathbb{R}$ is equipped with the usual topology and $\mathbb{A}_f$ with the restricted product topology. Hence the following result holds.

**Proposition 4.1.** The restricted product topology on $\mathbb{A}$ is metrizable, a metric is given by $\rho_\mathbb{A}$. Furthermore, $(\mathbb{A}, \rho_\mathbb{A})$ is a complete metric space and $(\mathbb{A}, \rho_\mathbb{A})$ as a topological space is homeomorphic to $(\mathbb{R}, |\cdot|_\infty) \times (\mathbb{A}_f, \rho)$.

**Remark 4.2.** $(\mathbb{A}, \rho_\mathbb{A})$ is a second-countable topological space. Indeed, $(B_{\infty}^{(i)} \times B_f^{(j)})_{i,j \in \mathbb{N}}$ is a countable base, where $(B_{\infty}^{(i)})_{i \in \mathbb{N}}$ is a countable base of $(\mathbb{R}, |\cdot|_\infty)$ and $(B_f^{(j)})_{j \in \mathbb{N}}$ is a countable base of $(\mathbb{A}_f, \rho)$, see Remark 3.2(ii). Therefore $(\mathbb{A}, \rho_\mathbb{A})$ is a semi-compact space.

4.2. Distributions on $\mathbb{A}$. The space of Bruhat-Schwartz functions, denoted $S(\mathbb{A})$, consists of finite linear combinations of functions of type $h(x) = h_\infty(x_\infty) h_f(x_f)$ with $h_\infty \in S(\mathbb{R})$, Schwartz space on $\mathbb{R}$, and $h_f \in S(\mathbb{A}_f)$. The space $S(\mathbb{A})$ is dense in $L^p(\mathbb{A}, dx_\mathbb{A})$ for $1 \leq p < +\infty$, see e.g. [18, Theorem 2.9]. The space of distributions on $S(\mathbb{A})$ is the strong dual space of $S(\mathbb{A})$.

4.3. Pseudodifferential operators and the Lizorkin space on $\mathbb{A}$. We consider the pseudodifferential operator $D^\beta_\mathbb{R} =: D^\beta$, $\beta > 0$ on $S(\mathbb{R})$ defined by

$$\left( D^\beta h \right)(x_\infty) = F_{\xi_\infty \to x_\infty}^{-1} \left( |\xi_\infty|^\beta |F_{x_\infty \to x}|^|\cdot|^\beta h \right), \quad h \in S(\mathbb{R}).$$

Recall that the operator $D^\beta$ is the real Riesz fractional operator and represents a fractional power of the Laplacian, see e.g. [41] §8, [42] §25.

We introduce the pseudodifferential operator $D^{\alpha, \beta}_\mathbb{A} =: D^{\alpha, \beta}$, $\alpha, \beta > 0$ on $S(\mathbb{A})$ defined by

$$\left( D^{\alpha, \beta} h \right)(x) = F_{\xi \to x}^{-1} \left( \left( |\xi_\infty|^\beta + |\xi_f|^\alpha \right) F_{x \to x} h \right), \quad h \in S(\mathbb{A}).$$
Lemma 4.3. With the above notation
\[ D^{\alpha, \beta} : \mathcal{S}(\mathbb{A}) \to C(\mathbb{A}, \mathbb{C}) \cap L^2(\mathbb{A}) \]

Proof. It is sufficient to consider a factorizable function \( h = h_\infty h_f \), \( h_\infty \in \mathcal{S}(\mathbb{R}) \), \( h_f \in \mathcal{S}(\mathbb{A}_f) \). Since \( \hat{h}(\xi) = \hat{h}_\infty(\xi_\infty) \hat{h}_f(\xi_f) \),
\[ (D^{\alpha, \beta}h)(x) = h_f(x_f) \left( D^\beta h_\infty \right)(x_\infty) + h_\infty(x_\infty) \left( D^\alpha h_f \right)(x_f). \]

Note that \( D^\beta h_\infty \in L^2(\mathbb{R}) \cap C(\mathbb{R}, \mathbb{C}) \), \( D^\alpha h_f \in L^2(\mathbb{A}_f) \cap C(\mathbb{A}_f, \mathbb{C}) \), cf. Lemma 3.21 and since \( dx_h = dx_\infty dx_f \), we conclude \( h_f D^\beta h_\infty, h_\infty D^\alpha h_f \in C(\mathbb{A}, \mathbb{C}) \cap L^2(\mathbb{A}) \). \( \square \)

The space \( \mathcal{S}(\mathbb{A}) \) is not invariant under the action of the operator \( D^{\alpha, \beta} \). To overcome such an inconvenience, we introduce an adelic version of the Lizorkin space of the second kind. First we recall that the real Lizorkin space of test functions, see e.g. [41, §2] or [42, §25], is defined by
\[ \mathcal{L}_0(\mathbb{R}) = \left\{ f_\infty \in \mathcal{S}(\mathbb{R}) : \int_\mathbb{R} x_n^\infty f_\infty(x_\infty) \, dx_\infty = 0, \text{ for } n \in \mathbb{N} \right\}. \]

The real Lizorkin space can be equipped with the topology of the space \( \mathcal{S}(\mathbb{R}) \), which makes \( \mathcal{L}_0(\mathbb{R}) \) a complete space. The real Lizorkin space is invariant with respect to \( D^\beta \), is dense in \( L^p(\mathbb{R}) \), \( 1 < p < \infty \), and admits the following characterization: \( f_\infty \in \mathcal{L}_0(\mathbb{R}) \) if and only if \( f_\infty \in \mathcal{S}(\mathbb{R}) \) and
\[ \frac{d^n}{dx_\infty^n} \mathcal{F} f(\xi_\infty) \bigg|_{\xi_\infty = 0} = 0, \text{ for } n \in \mathbb{N}. \]

We introduce an adelic Lizorkin space of the second kind \( \mathcal{L}_0 := \mathcal{L}_0(\mathbb{A}) \) as
\[ \mathcal{L}_0(\mathbb{A}) = \mathcal{L}_0(\mathbb{R}) \otimes \mathcal{L}_0(\mathbb{A}_f). \]

The space \( \mathcal{L}_0(\mathbb{A}) \) consists of finite linear combinations of factorizable functions \( h(x) = h_\infty(x_\infty) h_f(x_f) \) with \( h_\infty \in \mathcal{L}_0(\mathbb{R}) \), \( h_f \in \mathcal{L}_0(\mathbb{A}_f) \). Note that \( \mathcal{L}_0(\mathbb{A}) \) is a subspace of \( \mathcal{S}(\mathbb{A}) \) and it may be equipped with the topology of \( \mathcal{S}(\mathbb{A}) \).

Lemma 4.4. With the above notation the following assertions hold:
(i) \( D^{\alpha, \beta} \mathcal{L}_0 = \mathcal{L}_0 \) for \( \alpha, \beta > 0 \);
(ii) \( \mathcal{L}_0 \) is dense in \( \mathcal{L}^2(\mathbb{A}) \).

Proof. (i) It is sufficient to consider a factorizable function \( h = h_\infty h_f, h_\infty \in \mathcal{L}_0(\mathbb{R}), h_f \in \mathcal{L}_0(\mathbb{A}_f) \). Since \( D^\alpha h_f \in \mathcal{L}_0(\mathbb{A}_f) \) and \( D^\beta h_\infty \in \mathcal{L}_0(\mathbb{R}) \), see Lemma 3.22 and [41, (9.1)], we conclude from (4.3) that \( D^{\alpha, \beta} \mathcal{L}_0(\mathbb{A}) \subset \mathcal{L}_0(\mathbb{A}) \).

Conversely, take \( h \in \mathcal{L}_0(\mathbb{A}) \). We want to show that the equation \( D^{\alpha, \beta} g = h \) has a solution \( g \in \mathcal{L}_0(\mathbb{A}) \). We may assume without loss of generality that \( \hat{h} = h_\infty \hat{h}_f \), \( h_\infty \in \mathcal{L}_0(\mathbb{R}), h_f \in \mathcal{L}_0(\mathbb{A}_f) \). Applying the Fourier transform we obtain
\[ \hat{g}(\xi) = \frac{\hat{h}_\infty(\xi_\infty)\hat{h}_f(\xi_f)}{|\xi_\infty|^\beta + ||\xi_f||^\alpha}. \]

Since \( \hat{h}_f \in \mathcal{L}_0(\mathbb{A}_f) \), it follows from (3.30) that
\[ \hat{h}_f(\xi_f) = \sum_{i=1}^N c_i 1_{B_h(\xi_f)}(\xi_f), \]
where the balls $B_R(\xi_i)$ are disjoint and $0 \notin B_R(\xi_i)$ for $i = 1, \ldots, N$. It follows from non-Archimedean property that the function $\|\xi_f\|$ is constant on each of the balls $B_R(\xi_i)$, hence we may rewrite (4.4) as
\[
\hat{g}(\xi) = \sum_{i=1}^{N} c_i \hat{h}_\infty(\xi_{\infty}) 1_{B_R(\xi_i)}(\xi_f),
\]
where $d_i := \|\xi_i\|^{\alpha} > 0$ are constants. It may be easily checked that the functions $c_i \hat{h}_\infty(\xi_{\infty})$ are Fourier transforms of real Lizorkin functions and the functions $1_{B_R(\xi_i)}(\xi_f)$ are Fourier transforms of Lizorkin functions on $\mathbb{A}_f$, thus $g \in L_0(\mathbb{A})$.

(ii) Since $L_0(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, see [41, Thm. 3.2], and $L_0(\mathbb{A}_f)$ is dense in $L^2(\mathbb{A}_f)$ by Lemma 3.22 the tensor product $L_0(\mathbb{A}) = L_0(\mathbb{R}) \otimes L_0(\mathbb{A}_f)$ is dense in the tensor product $L^2(\mathbb{R}) \otimes L^2(\mathbb{A}_f)$ which is isomorphic to the space $L^2(\mathbb{A})$, see e.g. [37, Theorem II.10].

Similarly to Subsection 3.4 we may consider the operator $D^{\alpha, \beta}$ as an operator acting on $L^2(\mathbb{A})$. It is easy to see that the operator $D^{\alpha, \beta}$ with the domain $\mathcal{D}(D^{\alpha, \beta}) = \mathcal{S}(\mathbb{A})$ is symmetric. Moreover, similarly to the proof of Lemma 4.3 (i) we may check that $(D^{\alpha, \beta} \pm i)L_0(\mathbb{A}) = L_0(\mathbb{A})$, i.e. the ranges of the operators $D^{\alpha, \beta} \pm i$ are dense in $L^2(\mathbb{A}_f)$, hence the operator $D^{\alpha, \beta}$ is essentially self-adjoint. The following description of the closure holds.

**Lemma 4.5.** The closure of the operator $D^{\alpha, \beta}$, $\alpha, \beta > 0$ (let us denote it by $D^{\alpha, \beta}$ again) with domain
\[
\mathcal{D} (D^{\alpha, \beta}) := \left\{ f \in L^2(\mathbb{A}) : (|\xi_{\infty}|^{\beta} + \|\xi\|^{\alpha}) \hat{f} \in L^2(\mathbb{A}) \right\}
\]
is a self-adjoint operator. Moreover, the following assertions hold:

(i) $D^{\alpha, \beta} \geq 0$;

(ii) $D^{\alpha, \beta}$ is $m$-accretive, i.e. $-D^{\alpha, \beta}$ is an $m$-dissipative operator;

(iii) the spectrum $\sigma(D^{\alpha, \beta}) = [0, \infty)$;

(iv) $-D^{\alpha, \beta}$ is the infinitesimal generator of a contraction $C_0$ semigroup $\{T_{\alpha, \beta}(t)\}_{t \geq 0}$.

Moreover, the semigroup $\{T_{\alpha, \beta}(t)\}_{t \geq 0}$ is bounded holomorphic (or analytic) with angle $\pi/2$.

5. The Adelic Heat Kernel on $\mathbb{A}_f$

In this section we introduce the adelic heat kernel on $\mathbb{A}_f$ as the inverse Fourier transform of $e^{-t|y|^{\alpha}}$ with $y \in \mathbb{A}_f$, $|y|$ defined by (5.3), $\alpha > 1$ and $t > 0$.

In Sections 5.1 5.4 and 7 we work only with finite adeles, for this reason in the variables we omit the subindex ‘$f$’.

**Proposition 5.1.** Consider the function $|y|^{\beta} e^{-t|y|^{\alpha}}$ for fixed $t > 0$, $\beta \geq 0$ and $\alpha > 1$. Then
\[
|y|^{\beta} e^{-t|y|^{\alpha}} \in L^0(\mathbb{A}_f, dy_{\mathbb{A}_f})
\]
for any $1 \leq q < +\infty$.

**Proof.** It is sufficient to show that for any $t > 0$ and $\beta \geq 0$
\[
I(t) := \int_{\mathbb{A}_f} |y|^{\beta} e^{-t|y|^{\alpha}} dy_{\mathbb{A}_f} < +\infty.
\]
According to Lemmas 3.9 and 3.7
\[
\int_{\mathbb{A}_f} \|y\|^\beta e^{-\frac{1}{2}y^*y} \, dy_{\mathbb{A}_f} = \sum_{p^m, m \neq 0} p^{m\beta} e^{-tp^m} (\Phi(p^m) - \Phi(p^m))
\]
thus we have to prove the convergence of the latter series. We consider two cases: 
m < 0 and m > 0.

If m < 0, then \(p^{m\beta} e^{-tp^m} \leq 1\) and
\[
S_-(t) := \sum_{p^m, m < 0} p^{m\beta} e^{-tp^m} (\Phi(p^m) - \Phi(p^m))
\leq \sum_{p^m, m < 0} (\Phi(p^m) - \Phi(p^m)) = \Phi(1/2).
\]

Let m > 0. We have
\[
S_+(t) := \sum_{p^m, m > 0} p^{m\beta} e^{-tp^m} (\Phi(p^m) - \Phi(p^m)) \leq \sum_{p^m, m > 0} p^{m\beta} e^{-tp^m} \Phi(p^m).
\]
We recall that the Prime Number Theorem is equivalent to
\[
\ln \Phi(x) = \psi(x) \sim x, \quad x \to \infty,
\]
see e.g. [15], hence there exists a constant C such that
\[
S_+(t) \leq \sum_p \sum_{m=1}^{\infty} p^{m\beta} e^{-tp^m} \leq Cp^m.
\]

We want to show the existence of a positive constant \(M = M(\beta)\) such that
\[
p^{m\beta} e^{-tp^m} + Cp^m \leq Mp^{-1-m} \quad \text{for all } m \geq 1 \text{ and prime } p,
\]
or equivalently that \(p^{1+m(\beta+1)} e^{-tp^m} + Cp^m \leq M\). Since \(p^{1+m(\beta+1)} \leq e^{(\beta+1)p^m}\) for all \(m \geq 1\) and \(p \geq 2\), consider \(e^{(C+\beta+1)p^m - tp^m}\). This expression is less than or equal to 1 when \(p^m \geq \left(\frac{C+\beta+1}{t}\right)^{\frac{1}{\beta}}\) and hence there exist only a finite number of pairs \((p,m)\) for which it can be greater than 1, so the announced constant exists. Therefore
\[
S_+(t) \leq M \sum_p \sum_{m=1}^{\infty} p^{-1-m} = 2M \sum_{p} p^{-2} < +\infty. \qedhere
\]

**Definition 5.2.** We define the adelic heat kernel on \(\mathbb{A}_f\) as
\[
Z(x, t; \alpha) := Z(x, t) = \int_{\mathbb{A}_f} \chi(\xi \cdot x) e^{-t\|\xi\|^\alpha} \, d\xi_{\mathbb{A}_f}, \quad x \in \mathbb{A}_f, \quad t > 0, \quad \alpha > 1.
\]

By Proposition 5.1 the integral is convergent. When considering \(Z(x, t)\) as a function of \(x\) for \(t\) fixed we will write \(Z_t(x)\). By applying Theorem 3.10 to the function \(e^{-t\|\xi\|^\alpha}\) we obtain the following result.

**Proposition 5.3.** The following representation holds for the heat kernel:
\[
Z(x, t) = \sum_{q^j < \|x\|^{-1}, j \neq 0} \Phi(q^j) \left(e^{-tq^j\alpha} - e^{-t(q^i)^\alpha}\right) \quad \text{for } t > 0, \quad x \in \mathbb{A}_f,
\]
where \(q^j\) runs through all non-zero powers of prime numbers; functions \(\|x\|, \Phi(x)\) and \(q^i\) are defined by (5.3), (5.9) and (5.11). For \(x = 0\) the expression \(\|0\|^{-1}\) in the representation means \(\infty\).
Lemma 5.4. The following estimate holds for the heat kernel:

\[(5.3)\quad Z(x, t) \leq 2t\|x\|^{-\alpha} \Phi(\|x\|^{-1}), \quad x \in \mathbb{A}_f \setminus \{0\}, \ t > 0.\]

Proof. From the inequality \[1 - e^{-x} \leq x \text{ valid for } x \geq 0 \]
we obtain \[e^{-tq^\alpha} - e^{-t(q^\alpha)} \leq 1 - e^{-t(q^\alpha)} \leq t(q^\alpha)^\alpha.\]

Then with the use of the inequality \[\frac{1}{t} \Phi(q^\alpha) \leq \Phi(q^\alpha) - \Phi(q^\alpha)\]
we have

\[
Z(x, t) = \sum_{q^\alpha < \|x\|^{-1}} \Phi(q^\alpha) \left(e^{-tq^\alpha} - e^{-t(q^\alpha)}\right) \leq t \sum_{q^\alpha < \|x\|^{-1}} \Phi(q^\alpha) (q^\alpha)^\alpha \\
\leq 2t\|x\|^{-\alpha} \sum_{q^\alpha < \|x\|^{-1}} \left(\Phi(q^\alpha) - \Phi(q^\alpha)\right) = 2t\|x\|^{-\alpha} \Phi(\|x\|^{-1}).
\]

Corollary 5.5. With the above notation the following assertions hold:

(i) \[Z(x, t) \geq 0 \text{ for } t > 0;\]
(ii) \[\lim_{t \to 0^+} Z(x, t) = 0 \text{ for any } x \in \mathbb{A}_f \setminus \{0\};\]
(iii) For any \(\varepsilon > 0\) there exists a constant \(C = C(\varepsilon)\) such that for any \(t > 0\)

\[(5.4)\quad \int_{\|x\| > \varepsilon} Z_t(y) dy_{A_f} \leq Ct < +\infty.\]

Proof. The statements (i) and (ii) immediately follows from formulas \[(5.2)\] and \[(5.3),\]
respectively.

(iii) By Proposition \[(5.3),\] Lemma \[(5.9)\] and \[(5.3),\]
we have

\[
\int_{\|x\| > \varepsilon} Z_t(y) dy_{A_f} = \sum_{p^\alpha > \varepsilon} \text{vol}(S^\alpha) \cdot \sum_{q^\alpha < p^{-k}} \Phi(q^\alpha) \left(e^{-tq^\alpha} - e^{-t(q^\alpha)}\right) \\
\leq \sum_{p^\alpha > \varepsilon} \Phi(p^\alpha) \cdot 2tp^{-k\alpha} \Phi(p^{-k}) = 2t \sum_{p^\alpha > \varepsilon} p^{-k\alpha} < +\infty,
\]
where we have used \[(3.10)\] and \[(3.13).\]

Theorem 5.6. The adelic heat kernel on \(\mathbb{A}_f\) satisfies the following:

(i) \(Z(x, t) \geq 0\) for any \(t > 0;\)
(ii) \(\int_{\mathbb{A}_f} Z_t(x) dx_{A_f} = 1\) for any \(t > 0;\)
(iii) \(Z_t(x) \in L^1(\mathbb{A}_f)\) for any \(t > 0;\)
(iv) \(Z_t(x) * Z_t(x) = Z_{t+t'}(x)\) for any \(t, t' > 0;\)
(v) \(\lim_{t \to 0^+} Z_t(x) = \delta(x) \text{ in } S'(\mathbb{A}_f);\)
(vi) \(Z_t(x)\) is a uniformly continuous function for any fixed \(t > 0;\)
(vii) \(Z(x, t)\) is uniformly continuous in \(t\), i.e. \(Z(x, t) \in C((0, \infty), C(\mathbb{A}_f))\) or \(\lim_{t \to t} \max_{x \in \mathbb{A}_f} |Z(x, t) - Z(x, t')| = 0\) for any \(t > 0.\)

Proof. (i) It follows from Corollary \[(5.5)\]
(ii) For any \(t > 0\) the function \(e^{-t\|\xi\|^\alpha}\) is continuous at \(\xi = 0\) and by Proposition \[(5.3)\] we have \(e^{-t\|\xi\|^\alpha} \in L^1(\mathbb{A}_f) \cap L^2(\mathbb{A}_f).\) Then \(Z_t(x) \in C(\mathbb{A}_f, \mathbb{R}) \cap L^2(\mathbb{A}_f).\) Now the statement follows from the inversion formula for the Fourier transform on \(\mathbb{A}_f.\)
(iii) The statement follows from (i) and (ii).
(iv) By the previous property \(Z_t(x) \in L^1(\mathbb{A}_f)\) for any \(t > 0.\) Then

\[
Z_t(x) * Z_t(x) = \mathcal{F}_{\xi \to x}^{-1} \left(e^{-t\|\xi\|^\alpha} - e^{-t\|\xi\|^\alpha}\right) = \mathcal{F}_{\xi \to x}^{-1} \left(e^{-\|\xi\|^\alpha} - e^{-\|\xi\|^\alpha}\right) = Z_{t+t'}(x).
\]
Lemma 6.2. With the above notation the following assertions hold:

The result follows from Theorem 5.6, see [20, Sec. 2.1] for further details.

Proof. Let $A_x, \rho$ be the complete non-Archimedean metric space and use the terminology, notation and results of [20] Chapters Two, Three. Let $B$ denote the $\sigma$-algebra of the Borel sets of $A_x$. Then $(A_x, B, d\mu_x)$ is a measure space. Let $1_B (x)$ denote the characteristic function of a set $B \in B$.

We assume along this section that $\alpha > 1$ and set

$$ p(t, x, y) := Z(x - y, t) \quad \text{for } t > 0, x, y \in A_x, $$

and

$$ P(t, x, B) := \begin{cases} \int_B p(t, x, y) \, d\mu_x, & \text{for } t > 0, x \in A_x, B \in B \\ 1_B (x), & \text{for } t = 0. \end{cases} $$

Lemma 6.1. With the above notation the following assertions hold:

(i) $p(t, x, y)$ is a normal transition density;

(ii) $P(t, x, B)$ is a normal transition function.

Proof. The result follows from Theorem 5.6, see [20] Sec. 2.1 for further details.

Lemma 6.2. The transition function $P(t, y, B)$ satisfies the following two conditions:

(i) for each $u \geq 0$ and a compact $B$

$$ \lim_{x \to \infty} \sup_{t \leq u} P(t, x, B) = 0; \quad \text{[Condition L(B)]} $$
(ii) for each $\epsilon > 0$ and a compact $B$

$$\lim_{t \to 0^+} \sup_{x \in B} P(t, x, A_f \setminus B_{\epsilon}(x)) = 0. \quad \text{[Condition M(B)]}$$

**Proof.** Since $B$ is a compact, $\text{dist}(x, B) = d(x) \to \infty$ as $x \to \infty$. Since the function $\Phi(x)$ is non-decreasing, we obtain from (5.33) that $Z(x-y, t) \leq 2u(d(x))^{-\alpha} \Phi((d(x))^{-1})$ for any $y \in B$ and $t \leq u$. Hence $P(t, x, B) \leq 2u(d(x))^{-\alpha} \cdot \Phi((d(x))^{-1}) \cdot \text{vol}(B) \to 0$ as $x \to \infty$.

To verify Condition M(B) we proceed as follows: for $y \in A_f \setminus B_{\epsilon}(x)$ we have $\|x-y\| > \epsilon$. The statement follows from (5.3):

$$P(t, x, A_f \setminus B_{\epsilon}(x)) \leq C(\epsilon)t \to 0, \quad t \to 0^+. \quad \square$$

**Theorem 6.3.** $Z(x, t)$ is the transition density of a time- and space homogenous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.

**Proof.** The result follows from [20, Theorem 3.6], Remark 6.2 (ii) and Lemmas 6.1, 6.3.

**Remark 6.4.** The more strict version of Condition M(B) which is sufficient for the continuity of a Markov process, namely, that for each $\epsilon > 0$ and a compact $B$

$$\lim_{t \to 0^+} \frac{1}{t} \sup_{x \in B} P(t, x, A_f \setminus B_{\epsilon}(x)) = 0, \quad \text{[Condition N(B)]}$$

does not hold for the function $Z(x, t)$. This may be easily seen if we take $\epsilon = 1/4$. In such case by Proposition 6.3 and Lemma 6.1 we have

$$\int_{A_f \setminus B_{1/4}(x)} Z_t(x-y) \, dy_{A_f} \geq \int_{S_{1/3}(x)} Z_t(x-y) \, dy_{A_f}$$

$$= \text{vol} S_{1/3}(x) \cdot \sum_{q' < 3, j \neq 0} \Phi(q') \left( e^{-tq'^{\alpha} - e^{-t(q_j^+)^{\alpha}}} \right) \geq \frac{1}{3} \left( e^{-2\alpha t} - e^{-3\alpha t} \right),$$

hence

$$\lim_{t \to 0^+} \frac{1}{t} \sup_{x \in B} P(t, x, A_f \setminus B_{1/4}(x)) \geq \frac{3^\alpha - 2^\alpha}{3} \neq 0.$$

7. Cauchy Problem for Parabolic Type Equations on $A_f$

Consider the following Cauchy problem

$$\begin{cases}
\frac{\partial u(x, t)}{\partial t} + D^\alpha u(x, t) = 0, & x \in A_f, \ t \in [0, +\infty), \\
u(x, 0) = u_0(x), & u_0(x) \in D(D^\alpha),
\end{cases} \tag{7.1}$$

where $\alpha > 1$, $D^\alpha$ is the pseudodifferential operator defined by (5.31) with the domain given by (3.33) and $u : A_f \times [0, +\infty) \to \mathbb{C}$ is an unknown function.

We say that a function $u(x, t)$ is a solution of (7.1) if $u \in C([0, +\infty), D(D^\alpha)) \cap C^1([0, +\infty), L^2(A_f))$ and $u$ satisfies equation (7.1) for all $t \geq 0$.

We understand the notions of continuity in $t$, differentiability in $t$ and equalities in the $L^2(A_f)$ sense, as it is customary in the semigroup theory. More precisely, we say that a function $u(x, t)$ is continuous in $t$ at $t_0$ if $\lim_{t \to t_0} \|u(x, t) - u(x, t_0)\|_{L^2(A_f)} = 0$; the function $u_t(x, t)$ is the time derivative of function $u(x, t)$ at
Lemma 7.1. Since \( Z \) with Theorem 5.6 (v). We define the function notations, set \( u \) the initial value from \( t \) defined and belongs to \( \mathcal{A} \). Therefore Cauchy problem (7.1) is well-posed, i.e. it is uniquely solvable with the solution continuously dependent on the initial data, and its solution is given by \( u(x, t) = T(t)u_0(x), \quad t \geq 0 \), see e.g. \( 3, 13, 24 \). However the general theory does not give an explicit formula for the semigroup \( (T(t))_{t \geq 0} \). We show that the operator \( T(t) \) for \( t > 0 \) coincides with the operator of convolution with the heat kernel \( Z_t \ast \cdot \). In order to prove this, we first construct a solution of Cauchy problem (7.1) with the initial value from \( \mathcal{S}(\mathcal{A}) \) without using the semigroup theory. Then we extend the result to all initial values from \( \mathcal{D}(\mathcal{D}^\alpha) \), see Proposition 7.3 and Theorem 7.5.

We show in Theorem \( 7.8 \) that in the case \( u_0 \in \mathcal{L}_0(\mathcal{A}) \), the function \( u(x, t) \) is the solution of Cauchy problem (7.1) in a stricter sense, i.e. \( u(x, t) \in C^1([0, \infty), \mathcal{L}_0(\mathcal{A})) \) and all limits and equalities are understood pointwise.

7.1. Homogeneous equations with initial values in \( \mathcal{S}(\mathcal{A}_T) \). We first consider Cauchy problem (7.1) with the initial value from the space \( \mathcal{S}(\mathcal{A}_T) \). To simplify notations, set \( \mathcal{Z}_0 \ast u_0 = (Z_t \ast u_0)_{t \to 0} := u_0 \). Note that such definition is consistent with Theorem 5.6 (v). We define the function (7.2) \( u(x, t) = Z_t(x) \ast u_0(x), \quad t \geq 0 \).

Since \( Z_t(x) \in \mathcal{L}^1(\mathcal{A}_T) \) for \( t > 0 \) and \( u_0(x) \in \mathcal{S}(\mathcal{A}_T) \subset \mathcal{L}^\infty(\mathcal{A}_T) \), the convolution exists and is a continuous function, see \( 10 \) Theorem 1.1.6.

Lemma 7.1. Let \( u_0 \in \mathcal{S}(\mathcal{A}_T) \) and \( u(x, t), \ t \geq 0 \) is defined by (7.2). Then \( u(x, t) \) is continuously differentiable in time for \( t \geq 0 \) and the derivative is given by (7.3) \( \frac{\partial u}{\partial t}(x, t) = -\mathcal{F}^{-1}_{\xi \to x}(\|\xi\|^\alpha e^{-t\|\xi\|^\alpha} \cdot 1_{B_R}(\xi) \ast u_0(x), \) where \( 1_{B_R}(\cdot) \) is the characteristic function of the ball \( B_R, \ R = (1/\ell)_- \) and \( \ell \) is the parameter of constancy of the function \( u_0 \), see \( 8 \) Theorem 2.49 and Proposition 3.18.

Proof. Let \( h_t(x) \) be a function defined by the right-hand side of (7.3). Since \( \|\xi\|^\alpha e^{-t\|\xi\|^\alpha} \cdot 1_{B_R}(\xi) \in \mathcal{L}^1(\mathcal{A}_T) \cap \mathcal{L}^2(\mathcal{A}_T) \) for any \( t \geq 0 \), the function \( h_t(x) \) is well-defined and belongs to \( C(\mathcal{A}_T) \cap \mathcal{L}^2(\mathcal{A}_T) \).

Let \( t_0 \geq 0 \). Consider a limit

\[
\lim_{t \to t_0} \frac{u(x, t) - u(x, t_0)}{t - t_0} - h_t(x, t_0) = \lim_{t \to t_0} \frac{e^{-t\|\xi\|^\alpha} - e^{-t_0\|\xi\|^\alpha}}{t - t_0} \widehat{u_0}(\xi) + \|\xi\|^\alpha e^{-t_0\|\xi\|^\alpha} 1_{B_R}(\xi) \cdot \widehat{u_0}(\xi) \]

\[
= \lim_{t \to t_0} \frac{(e^{-t\|\xi\|^\alpha} - e^{-t_0\|\xi\|^\alpha})}{t - t_0} + \|\xi\|^\alpha e^{-t_0\|\xi\|^\alpha} 1_{B_R}(\xi) \cdot \widehat{u_0}(\xi) \]

where we have applied Steklov-Parseval equality and the fact that \( \operatorname{supp} \widehat{u_0} \subset B_R \) which follows from Proposition 3.18. By applying the Mean-Value Theorem twice we obtain

\[
e^{-t\|\xi\|^\alpha} - e^{-t_0\|\xi\|^\alpha} + \|\xi\|^\alpha e^{-t_0\|\xi\|^\alpha} = (t' - t_0)\|\xi\|^{2\alpha} e^{-t''\|\xi\|^\alpha}.
\]
where \( t' = t'(\|\xi\|) \) is a point between \( t_0 \) and \( t \) and \( t'' = t''(\|\xi\|) \) is a point between \( t_0 \) and \( t' \) (and thus between \( t_0 \) and \( t \)). Hence

\[
\left\| \frac{e^{-t}\|\xi\|^\alpha - e^{-t_0}\|\xi\|^\alpha}{t - t_0} + \|\xi\|^\alpha e^{-t_0}\|\xi\|^\alpha \right\|_{L^2(\mathbb{A}_f)} \leq |t - t_0|R^{2\alpha}\|\hat{u}_0(\xi)\|_{L^2(\mathbb{A}_f)} \to 0, \quad t \to t_0,
\]

i.e. \( h_t(x) \) is the time derivative of the function \( u(x,t) \) for any \( t \geq 0 \).

The proof of the continuous differentiability in time of \( u(x,t) \) follows from the time continuity of \( h_t(x) \) which can be checked similarly.

**Lemma 7.2.** Let \( u_0 \in \mathcal{S}(\mathbb{A}_f) \) and \( u(x,t), t \geq 0 \) is defined by (7.2). Then \( u(x,t) \in D(D^\alpha) \) for any \( t \geq 0 \) and

\[
D^\alpha u(x,t) = F_{\xi \to x}^{-1}\left(\|\xi\|^\alpha e^{-t}\|\xi\|^\alpha \right) F_{\xi \to x}(\mathbb{1}_{B_R}(\xi)) \cdot u_0(\xi)
\]

where \( \mathbb{1}_{B_R}(\cdot) \) is the characteristic function of the ball \( B_R \), \( R = (1/e) \) and \( e \) is the parameter of constancy of the function \( u_0 \), see (5.20) and Proposition 5.15.

**Proof.** Note that \( \hat{u}_0 \in \mathcal{S}(\mathbb{A}_f) \) which implies that \( e^{-t}\|\xi\|^\alpha \cdot \hat{u}_0(\xi) \in L^1(\mathbb{A}_f) \cap L^2(\mathbb{A}_f) \) and \( \|\xi\|^\alpha e^{-t}\|\xi\|^\alpha \cdot \hat{u}_0(\xi) \in L^1(\mathbb{A}_f) \cap L^2(\mathbb{A}_f) \) for any \( t \geq 0 \). Hence we may calculate \( D^\alpha u(x,t) \) by formula (3.31). For \( t > 0 \) we obtain

\[
D^\alpha u(x,t) = F_{\xi \to x}^{-1}\left(\|\xi\|^\alpha e^{-t}\|\xi\|^\alpha \right) F_{\xi \to x}(\mathbb{1}_{B_R}(\xi)) \cdot u_0(\xi)
\]

where we have used the fact that \( \text{supp} \hat{u}_0 \subset B_R \).

For \( t = 0 \) we obtain

\[
D^\alpha u(x,0) = D^\alpha u_0(x) = F_{\xi \to x}^{-1}\left(\|\xi\|^\alpha \cdot \hat{u}_0(\xi)\right) = \mathcal{F}_{\xi \to x}(\mathbb{1}_{B_R}(\xi)) = F_{\xi \to x}^{-1}\left(\|\xi\|^\alpha e^{-0}\|\xi\|^\alpha \right) \mathbb{1}_{B_R}(\xi) \cdot u_0(\xi).
\]

As an immediate consequence from Lemmas 7.1 and 7.2 we obtain

**Proposition 7.3.** Let the function \( u_0 \in \mathcal{S}(\mathbb{A}_f) \). Then the function \( u(x,t) \) defined by (7.2) is a solution of Cauchy problem (7.1).

### 7.2. Homogeneous equations with initial values in \( L^2(\mathbb{A}_f) \)

Consider the operator \( T(t), t \geq 0 \) of convolution with the heat kernel, i.e.

\[
T(t)u = Z_t \ast u.
\]

Since \( Z_t \in L^2(\mathbb{A}_f) \), the convolution \( Z_t \ast u \) is a continuous function of \( x \) for \( t > 0 \) and any \( u \in L^2(\mathbb{A}_f) \), see [10] Theorem 1.1.6].

**Lemma 7.4.** The operator \( T(t) : L^2(\mathbb{A}_f) \to L^2(\mathbb{A}_f) \) is bounded.

**Proof.** Consider a function \( u \in L^2(\mathbb{A}_f) \). Since \( Z_t \in L^1(\mathbb{A}_f) \), see Theorem 5.6 (iii), by the Young inequality and Theorem 5.6 (ii)

\[
\|Z_t \ast u\|_{L^2} \leq \|Z_t\|_{L^1} \cdot \|u\|_{L^2} = \|u\|_{L^2}.
\]

Hence \( T(t)u = Z_t \ast u \in L^2(\mathbb{A}_f) \) and \( \|T(t)\| \leq 1 \).

**Theorem 7.5.** Let \( \alpha > 1 \). Then the following assertions hold.

(i) The operator \( -D^\alpha \) generates a \( C_0 \) semigroup \( (T(t))_{t \geq 0} \). The operator \( T(t) \) coincides for each \( t \geq 0 \) with the operator \( T(t) \) given by (7.5).
(ii) Cauchy problem (7.1) is well-posed and its solution is given by \(u(x,t) = Z_t * u_0, \ t \geq 0\).

**Proof.** According to Lemma 3.23, the operator \(-D^\alpha\) generates a \(C_0\) semigroup \((T(t))_{t \geq 0}\). Hence Cauchy problem (7.1) is well-posed, see e.g. [13, Theorem 3.1.1]. By Proposition 7.3, \((T(t))_{t \geq 0}\) is holomorphic, Cauchy problem (7.1) is well-posed, see e.g. [13, Theorem 3.1.1], [21, Ch. 2, Proposition 6.2].

**Remark 7.6.** Since the semigroup \((T(t))_{t \geq 0}\) is holomorphic, Cauchy problem (7.1) possesses smoothing effect, see e.g. [3, Corollary 3.7.2]. More precisely, consider Cauchy problem (7.1) with weaker requirement on the initial value, namely, let \(u_0 \in L^2(A_f)\). Then there exists a unique function

\[u(x,t) \in C([0,\infty), L^2(A_f)) \cap C([0,\infty), D(D^\alpha)) \cap C^\infty((0,\infty), L^2(A_f))\]

satisfying the equation for \(t > 0\) and satisfying the initial condition. That is, this weaker Cauchy problem is solvable for arbitrary initial data and the solution is infinitely differentiable in \(t > 0\).

### 7.3. Homogeneous equations with initial values in \(L_0(A_f)\)

We now consider the Cauchy problem

\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} + D^\alpha u(x,t) = 0, & x \in A_f, \ t \in [0,\infty), \\
u(x,0) = u_0(x), & u_0(x) \in L_0(A_f),
\end{cases}
\]

with the initial value from the space \(L_0(A_f)\) and the pseudodifferential operator \(D^\alpha\) with the smaller domain \(D(D^\alpha) = L_0(A_f)\).

We say that a function \(u(x,t)\) is a classical solution of (7.6) if \(u \in C^1([0,\infty), L_0(A_f))\) and \(u\) satisfies equation (7.6) for all \(t \geq 0\), with the understanding that all the involved limits are taken in the topology of \(L_0(A_f)\).

**Lemma 7.7.** Let \(u_0 \in L_0(A_f)\) and the function \(u(x,t)\) is defined by (7.2). Then \(u(x,t) \in L_0(A_f)\) for any \(t > 0\).

**Proof.** Since \(e^{-t\|\xi\|^\alpha}\) is locally constant outside of the origin, the function \(h_t(\xi) = e^{-t\|\xi\|^\alpha} \cdot \tilde{u}_0(\xi) \in S(A_f)\) with \(h_t(0) = 0\). Then \(F_{\xi \rightarrow x}^{-1}(e^{-t\|\xi\|^\alpha} \cdot \tilde{u}_0(\xi)) = Z_t(x) * u_0(x) \in L_0(A_f)\) for \(t > 0\).

**Theorem 7.8.** Let the function \(u_0 \in L_0(A_f)\). Then the function \(u(x,t)\) defined by (7.2) is the classical solution of Cauchy problem (7.6).

**Proof.** By Lemma 7.7, the function \(u(x,t)\) is correctly defined and \(u(x,t) \in D(D^\alpha) = L_0(A_f)\) for all \(t \geq 0\).

We assert that there exist constants \(\ell\) and \(R\) not dependent on \(t\) such that \(u(x,t) \in S^\ell_R\) and \(D^\alpha u(x,t) \in S^{\ell R}_R\) for all \(t \geq 0\). Consider the function \(h_t(\xi) = e^{-t\|\xi\|^\alpha} \cdot \tilde{u}_0(\xi), \ t \geq 0\). Since \(u_0 \in L_0\), the function \(e^{-t\|\xi\|^\alpha}\) is locally constant on the support of \(\tilde{u}_0\). Moreover, the parameter of constancy of \(e^{-t\|\xi\|^\alpha}\) on the support of \(\tilde{u}_0\) does not depend on \(t\). Hence there exist parameters \(\ell'\) and \(R'\) such that \(h_t \in S^{\ell'}_{R'}\) for any \(t \geq 0\). By Proposition 3.18 we have \((F_{\xi \rightarrow x}^{-1}(h_t)(x) = u(x,t) \in S^{(1/R')}_{(1/\ell')}\) for any \(t \geq 0\). Similar proof works for the function \(g_t(\xi) := F_{\xi \rightarrow x}(D^\alpha u(x,t)) = \|\xi\|^\alpha e^{-t\|\xi\|^\alpha} \cdot \tilde{u}_0(\xi)\).
We recall that for finite dimensional spaces, the uniform convergence is equivalent to the $L^2$-convergence. Since $\mathcal{S}_d$ is a finite dimensional space, cf. Proposition 3.18 by applying Lemmas 5.21 and 7.22 we have $\frac{\partial u}{\partial t}(x, t) \in \mathcal{L}_0(\mathbb{A}_f)$, $u(x, t)$ is a solution of Cauchy problem (7.6) and $u \in C^1([0, \infty), \mathcal{L}_0(\mathbb{A}_f))$. □

7.4. Non homogeneous equations. Consider the following Cauchy problem

\begin{equation}
\begin{cases}
\frac{\partial u(x, t)}{\partial t} + D^\alpha u(x, t) = f(x, t), & x \in \mathbb{A}_f, \ t \in [0, T], \ T > 0, \\
u(x, 0) = u_0(x), & u_0(x) \in D(D^\alpha).
\end{cases}
\end{equation}

We say that a function $u(x, t)$ is a solution of (7.7) if $u \in C([0, T], D(D^\alpha)) \cap C^1([0, T], L^2(\mathbb{A}_f))$ and if $u$ satisfies equation (7.7) for $t \in [0, T]$.

**Theorem 7.9.** Let $\alpha > 1$ and let $f \in C([0, T], L^2(\mathbb{A}_f))$. Assume that at least one of the following conditions is satisfied:

(i) $f \in L^1([0, T], D(D^\alpha))$;
(ii) $f \in W^{1,1}([0, T], L^2(\mathbb{A}_f))$.

Then Cauchy problem (7.7) has a unique solution given by

$$u(x, t) = \int_{\mathbb{A}_f} Z(x - y, t) u_0(y) dy_{\mathbb{A}_f} + \int_0^t \left\{ \int_{\mathbb{A}_f} Z(x - y, t - \tau) f(y, \tau) dy_{\mathbb{A}_f} \right\} d\tau.$$

**Proof.** With the use of Theorem 7.6 the proof follows from well-known results of the semigroup theory, see e.g. [3, Proposition 3.1.16], [13, Proposition 4.1.6]. □

8. The Adelic Heat Kernel on $\mathbb{A}$

We recall that the Archimedean heat kernel is defined as

$$Z(x_\infty, t; \beta) = \int_{\mathbb{R}} \chi_\infty(\xi_\infty x_\infty) e^{-t|\xi_\infty|_\infty^\beta} d\xi_\infty, \quad t > 0, \ \beta \in (0, 2].$$

This heat kernel is a solution of the pseudodifferential equation

$$\frac{\partial u(x_\infty, t)}{\partial t} + F_{\xi_\infty \to x_\infty}^{-1} \left| \xi_\infty \right|^\beta F_{x_\infty \to \xi_\infty}^{-1} u(x_\infty, t) \right).$$

For a more detailed discussion of the Archimedean heat kernel and its properties the reader may consult [19, Section 2] and references therein.

From now on we will denote heat kernel (5.1) as $Z(x_f, t; \alpha)$.

**Definition 8.1.** For fixed $\alpha > 1$, $\beta \in (0, 2]$ we define the heat kernel on $\mathbb{A}$ as

$$Z(x, t; \alpha, \beta) := \int_{\mathbb{A}_f} \chi(-\xi \cdot x) e^{-t(|\xi\|_\infty^\alpha + \|\xi_f\|^\alpha)} d\xi_{\mathbb{A}_f}, \quad x \in \mathbb{A}_f, \ t > 0.$$

Since $e^{-t|\xi\|_\infty^\alpha} \in L^1(\mathbb{R}, d\xi_\infty)$, $e^{-t\|\xi_f\|^\alpha} \in L^1(\mathbb{A}_f, d\xi_{\mathbb{A}_f})$, cf. [19, Property 2.2] and Proposition 5.1, and $d\xi_{\mathbb{A}_f} = d\xi_\infty d\xi_{\mathbb{A}_f}$, we have

\begin{equation}
Z(x, t; \alpha, \beta) = F^{-1}(e^{-t|\xi\|_\infty^\alpha}) F^{-1}(e^{-t\|\xi_f\|^\alpha}) = Z(x_\infty, t; \beta) Z(x_f, t; \alpha).
\end{equation}

For $t > 0$ fixed, we use the notation $Z_t(x; \alpha, \beta)$ instead of $Z(x, t; \alpha, \beta)$.

**Theorem 8.2.** The adelic heat kernel on $\mathbb{A}$ possesses the following properties

(i) $Z(x, t; \alpha, \beta) \geq 0$ for any $t > 0$;
(ii) $\int_{\mathbb{A}_f} Z(x, t; \alpha, \beta) d\xi_{\mathbb{A}_f} = 1$ for any $t > 0$;
(iii) $Z_t(x; \alpha, \beta) \in L^1(\mathbb{A})$ for any $t > 0$;
we suppose that $\alpha > 0$ and for $x, y \in A$

\[ Z_t(x; \alpha, \beta) = Z_{t+\epsilon}(x; \alpha, \beta) \text{ for any } t, t' > 0; \]

\[ \lim_{\epsilon \to 0^+} Z_t(x; \alpha, \beta) = \delta(x) \text{ in } S'(A); \]

\[ Z_t(x; \alpha, \beta) \text{ is uniformly continuous for any fixed } t > 0; \]

\[ Z(x; t; \alpha, \beta) \text{ is uniformly continuous in } t, \text{ i.e. } Z(x; t; \alpha, \beta) \in C((0, \infty), C(A)) \]

\[ \lim_{t \to t'} \max_{x \in A} |Z(x; t; \alpha, \beta) - Z(x; t'; \alpha, \beta)| = 0 \text{ for any } t > 0. \]

**Proof.** The statement follows from (8.1) and the corresponding properties for $Z(x, \infty; t; \beta)$ and $Z(x, f; t; \alpha)$, see [19, Section 2] and Theorem 5.6. \[ \square \]

9. **Markov Processes on $A$**

Let $B(A)$ denote the $\sigma$-algebra of the Borel sets of $(A, \rho_B)$. Along this section we suppose that $\alpha > 1$ and $\beta \in (0, 2]$ are fixed parameters. We set

\[ p(t, x; y; \alpha, \beta) := Z(x-y, t; \alpha, \beta) \quad \text{for } t > 0, x, y \in A. \]

Note that

\[ p(t, x; y; \alpha, \beta) = Z(x, \infty; y, t; \beta) Z(f-y, t; \alpha) =: p(t, x; \infty; \beta) p(t, f-y; \alpha), \]

where $p(t, x; \infty; \beta) = Z(x, \infty; y, t; \beta)$ and $p(t, f-y; \alpha) := p(t, f-y; \alpha) = Z(f-y, t; \alpha)$. We also define for $x, y, \in R$ and $B \in B(R)$

\[ P(t, x, \infty; B; \beta) := \begin{cases} \int_B p(t, x, \infty; y; \beta) dy = 0 & \text{for } t > 0, \\ 1_B(x) & \text{for } t = 0 \end{cases} \]

and for $x, y \in A$ and $B \in B(A)$

\[ P(t, x, B; x, \alpha, \beta) := \begin{cases} \int_B p(t, x, \infty; \beta) p(t, f-y; \alpha) dy \rho_B(x) = 0 & \text{for } t > 0, \\ 1_B(x) & \text{for } t = 0 \end{cases} \]

**Lemma 9.1.** With the above notation the following assertions hold:

(i) $p(t, x; y; \alpha, \beta)$ is a normal transition density;

(ii) $P(t, x, B; \alpha, \beta)$ is a normal transition function.

**Proof.** The statement follows from the corresponding properties for the functions $p(t, x, \infty; \beta)$ and $p(t, f-y, \alpha)$, see Lemma 6.1. \[ \square \]

**Lemma 9.2.** The transition function $P(t, x, B; \alpha, \beta)$ satisfies the following two conditions:

(i) for each $u \geq 0$ and a compact $B$

\[ \lim_{t \to \infty} \sup_{t \leq u} P(t, x, B; \alpha, \beta) = 0; \quad [\text{Condition } L(B)] \]

(ii) for each $\epsilon > 0$ and a compact $B$

\[ \lim_{t \to 0^+} \sup_{t \in B} P(t, x, A \setminus B_\epsilon(x); \alpha, \beta) = 0, \quad [\text{Condition } M(B)] \]

where $B_\epsilon(x) := \{ y \in A : \rho_B(x, y) < \epsilon \}$. 

Proof: (i) Note that there exist compact subsets $K_\infty \subset \mathbb{R}$ and $K_f \subset \mathbb{A}_f$ such that $\mathbb{B} \subset K_\infty \times K_f$. Then
\[
P(t, x, \mathbb{B}; \alpha, \beta) \leq P(t, x, \infty; \beta)P(t, x, K_f).
\]
Since $\rho_{\alpha} (0, x) \to \infty$ we have either $\rho (0, x_f) \to \infty$ or $|x_\infty| \to \infty$. Therefore it is sufficient to show that
(9.1) \[
\lim_{x_f \to \infty} \sup_{t \leq u} P(t, x_f, K_f) = 0
\]
and
(9.2) \[
\lim_{x_\infty \to \infty} \sup_{t \leq u} P(t, x_\infty, K_\infty; \beta) = 0.
\]
The equality (9.1) follows from Lemma 6.2. By [19 (2.2)]
(9.3) \[
Z(t, x_\infty; \beta) \leq \frac{C t^\beta}{t^\beta + x_\infty^2} \quad \text{for } t > 0, x_\infty \in \mathbb{R}.
\]
Then
\[
P(t, x_\infty, K_\infty; \beta) = \int_{K_\infty} Z(t, x_\infty - y_\infty; \beta) dy_\infty \leq C t^\beta \int_{K_\infty} \frac{1}{t^\beta + (x_\infty - y_\infty)^2} dy_\infty.
\]
As $x_\infty \to \infty$ we have $\mathrm{dist}(x_\infty, K_\infty) \to \infty$ and $|x_\infty - y_\infty| \geq \mathrm{dist}(x_\infty, K_\infty)$ for any $y_\infty \in K_\infty$ and
\[
\frac{1}{t^\beta + (x_\infty - y_\infty)^2} \leq \frac{1}{\mathrm{dist}^2(x_\infty; K_\infty)}.
\]
Hence
\[
\lim_{x_\infty \to \infty} \sup_{t \leq u} P(t, x_\infty, K_\infty; \beta) \leq \lim_{x_\infty \to \infty} C u^\beta \int_{K_\infty} \frac{1}{\mathrm{dist}^2(x_\infty; K_\infty)} dy_\infty = 0.
\]
(ii) Since $\hat{\mathbb{B}}_\epsilon (x) \supseteq \hat{\mathbb{B}}_\beta (x_\infty) \times \mathbb{B}_\beta (x_f)$, where
\[
\hat{\mathbb{B}}_\beta (x_\infty) = \{ y_\infty \in \mathbb{R} : |x_\infty - y_\infty| < 2 \beta \}
\]
and $\mathbb{B}_\beta (x_f)$ is given by [3.3], we have $\mathbb{A} \setminus \hat{\mathbb{B}}_\epsilon (x) \subseteq \mathbb{A} \setminus \hat{\mathbb{B}}_\beta (x_\infty) \times \mathbb{B}_\beta (x_f)$ and with the use of [19 (2.1)] and Theorem 5.6 (ii) we obtain
\[
P(t, x, \mathbb{A} \setminus \hat{\mathbb{B}}_\beta (x); \alpha, \beta) \leq \left( \int_{|x_\infty - y_\infty| \geq \frac{2}{\beta}} p(t, x_\infty, y_\infty; \beta) \, dy_\infty \right) + \left( \int_{p(x_f, y_f) > \frac{2}{\beta}} p(t, x_f, y_f) \, dy_{\hat{A}_f} \right)
\]
\[
\leq P(t, x_\infty, \mathbb{R} \setminus \hat{\mathbb{B}}_\beta (x); \beta) + P(t, x_f, \mathbb{A} \setminus \hat{\mathbb{B}}_\beta (x_f)).
\]
Now the result follows from Lemma 6.2 and the inequality
\[
P(t, x_\infty, \mathbb{R} \setminus \hat{\mathbb{B}}_\beta (x_f)) = \int_{|y_f| \geq \frac{2}{\beta}} Z(t, y_\infty; \beta) \, dy_\infty \leq C \int_{|y_\infty| \geq \frac{2}{\beta}} \frac{t^\beta}{t^\beta + y_\infty^2} \, dy_\infty
\]
\[
= C \int_{|z_\infty| \geq \frac{2}{\beta} t^{1/\beta}} \frac{1}{1 + z_\infty^2} \, dz_\infty \to 0 \quad \text{as } t \to +0.
\]
Theorem 9.3. \( Z(x, t; \alpha, \beta) \) with \( \alpha > 1 \) and \( \beta \in (0, 2] \) is the transition density of a time- and space homogenous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.

Proof. The result follows from [20, Theorem 3.6], Remark [4.2 (ii) and Lemmas 0.1 and 0.2] \( \square \)

10. Cauchy problem for parabolic type equations on \( \mathbb{A} \)

In this section we study Cauchy problems for parabolic type equations on \( \mathbb{A} \) and present analogues of the results of Subsections 7.1, 7.2 and 7.4.

10.1. Homogeneous equations. Consider the following Cauchy problem

\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} + D^{\alpha, \beta} u(x, t) = 0, & x \in \mathbb{A}, \ t \in [0, +\infty), \\
u(x, 0) = u_0(x), & u_0(x) \in D(D^{\alpha, \beta}),
\end{cases}
\]

where \( \alpha > 1, \beta \in (0, 2] \), \( D^{\alpha, \beta} \) is the pseudodifferential operator defined by (1.2) with the domain given by (1.3) and \( u : \mathbb{A} \times [0, +\infty) \rightarrow \mathbb{C} \) is an unknown function.

We say that a function \( u(x, t) \) is a solution of (10.1), if \( u \in C([0, \infty), D(D^{\alpha, \beta})) \cap C^1([0, \infty), L^2(\mathbb{A})) \) and if \( u \) satisfies equation (10.1) for all \( t \geq 0 \).

As in Section 7 we understand the notions of continuity, differentiability and equalities in the sense of \( L^2(\mathbb{A}) \).

We first consider Cauchy problem (10.1) with the initial value from \( S(\mathbb{A}) \). We define the function (10.2) \( u(x, t) := u(x, t; \alpha, \beta) = Z_t(x; \alpha, \beta) * u_0(x) = Z_t(x) * u_0(x), \ t \geq 0 \),

where \( Z_0 * u_0 = (Z_0 * u_0) \mid_{t=0} := u_0 \). Note that such definition is consistent with Theorem 8.2 (v). Since \( Z_t(x) \in L^1(\mathbb{A}) \) for \( t > 0 \) and \( u_0(x) \in S(\mathbb{A}) \subset L^\infty(\mathbb{A}) \), the convolution exists and is a continuous function, see Theorem 8.2 (ii), [40, Theorem 1.1.6].

Lemma 10.1. Let \( u_0 \in S(\mathbb{A}) \) and \( u \) is defined by (10.2). Then \( u \in C([0, \infty), D(D^{\alpha, \beta})) \) and

\[
D^{\alpha, \beta} u = \mathcal{F}_{\xi=0}^{-1} \left( (|\xi|_\infty^\beta + ||\xi||^\alpha) e^{-t(|\xi|_\infty^\beta + ||\xi||^\alpha)} \mathcal{F}_{\xi \rightarrow \xi} u_0 \right)
\]

for \( t \geq 0 \).

Proof. We first verify that \( u(\cdot, t) \in D(D^{\alpha, \beta}) \) for \( t \geq 0 \). Without loss of generality we may assume that \( u_0(x) = u_\infty(x) u_f(x_f) \) with \( u_\infty \in S(\mathbb{R}) \) and \( u_f \in S(\mathbb{A}) \). Since

\[
\mathcal{F}_{\xi \rightarrow \xi} u(x, t) = e^{-t(|\xi|_\infty^\beta + ||\xi||^\alpha)} u_\infty(\xi) \tilde{u}_f(\xi),
\]

we have

\[
\left\| (|\xi|_\infty^\beta + ||\xi||^\alpha) \mathcal{F}_{\xi \rightarrow \xi} u \right\|_{L^2(\mathbb{A})} \\
\leq \left\| \xi_f^\alpha e^{-t(|\xi|_\infty^\beta + ||\xi||^\alpha)} \tilde{u}_f(\xi) \right\|_{L^2(\mathbb{A})} \cdot \left\| e^{-t(|\xi|_\infty^\beta + ||\xi||^\alpha)} u_\infty(\xi) \right\|_{L^2(\mathbb{R})} \\
+ \left\| e^{-t||\xi||^\alpha} \tilde{u}_f(\xi) \right\|_{L^2(\mathbb{A})} \cdot \left\| \xi_f^\beta e^{-t(|\xi|_\infty^\beta + ||\xi||^\alpha)} u_\infty(\xi) \right\|_{L^2(\mathbb{R})} \\
\leq \left\| \xi_f^\alpha \tilde{u}_f(\xi) \right\|_{L^2(\mathbb{A})} \cdot \left\| u_\infty \right\|_{L^2(\mathbb{R})} + \left\| \tilde{u}_f \right\|_{L^2(\mathbb{A})} \cdot \left\| \xi_f^\beta u_\infty(\xi) \right\|_{L^2(\mathbb{R})} \\
= \left\| D^\alpha u_f \right\|_{L^2(\mathbb{A})} \left\| u_\infty \right\|_{L^2(\mathbb{R})} + \left\| u_f \right\|_{L^2(\mathbb{A})} \left\| D^\beta u_\infty \right\|_{L^2(\mathbb{R})},
\]

where we used the Parseval-Steklov equality and the equality $d\xi = d\xi_\beta d\xi_\infty$. Therefore $u(x, t) \in D(D^{\alpha, \beta})$ for $t \geq 0$ and formula (10.3) holds.

To verify the continuity, assume again that $u_0(x) = u_\infty(x_\infty)u_f(x_f)$ with $u_\infty \in S(\mathbb{R})$, $u_f \in S(A_f)$. With the use of the Parseval-Steklov equality and the Mean Value Theorem we obtain

$$
\lim_{t \to t_0} \|u(x, t) - u(x, t')\|_{L^2(\mathcal{A})} = \lim_{t \to t_0} \|e^{-t(|\xi_\beta|_\infty + |\xi_\infty|^\alpha)} - e^{-t'(|\xi_\beta|_\infty + |\xi_\infty|^\alpha)}\tilde{u}_\infty(\xi_\infty)\tilde{u}_f(\xi_f)\|_{L^2(\mathcal{A})}
$$

$$
= \lim_{t \to t_0} \|(t - t')\left(|\xi_\infty|^\beta + |\xi_f|^\alpha\right)e^{-t(|\xi_\beta|_\infty + |\xi_\infty|^\alpha)}\tilde{u}_\infty(\xi_\infty)\tilde{u}_f(\xi_f)\|_{L^2(\mathcal{A})}
$$

$$
\leq \lim_{t \to t_0} |t - t'| \cdot \left(|\xi_\infty|^\beta + |\xi_f|^\alpha\right)\tilde{u}_\infty(\xi_\infty)\tilde{u}_f(\xi_f)\|_{L^2(\mathcal{A})}
$$

where $\tilde{t} = \tilde{t}(|\xi_\beta|_\infty + |\xi_\infty|^\alpha)$ is a point between $t$ and $t'$.

**Lemma 10.2.** Let $u_0 \in S(\mathcal{A})$ and $u(x, t), t \geq 0$ is defined by (10.2). Then $u(x, t)$ is continuously differentiable in time for $t \geq 0$ and the derivative is given by

$$
\frac{\partial u}{\partial t}(x, t) = -F_{x \to \xi}(\left(|\xi_\infty|^\beta_2 + |\xi_f|^\alpha\right)e^{-t(|\xi_\beta|_\infty + |\xi_\infty|^\alpha)}F_{x \to \xi}u_0).
$$

**Proof.** Assume that $u_0(x) = u_\infty(x_\infty)u_f(x_f)$ with $u_\infty \in S(\mathbb{R})$, $u_f \in S(A_f)$. By reasoning as in the proofs of Lemmas 10.1 and 10.1, we have

$$
\lim_{t \to t_0} \left\|\frac{\tilde{u}(\xi, t) - \tilde{u}(\xi, t_0)}{t - t_0} + \left(|\xi_\infty|^\beta_2 + |\xi_f|^\alpha\right)e^{-t(|\xi_\beta|_\infty + |\xi_\infty|^\alpha)}F_{x \to \xi}u_0\right\|_{L^2(\mathcal{A})}
$$

$$
\leq \lim_{t \to t_0} |t - t_0| \cdot \left(|\xi_\infty|^\beta_2 + |\xi_f|^\alpha\right)^2F_{x \to \xi}u_0\|_{L^2(\mathcal{A})}
$$

where we have used the fact that $S(\mathbb{R}) \subset D(D^{\beta})$ for any $\beta > 0$ and $S(A_f) \subset D(D^{\alpha})$ for any $\alpha > 0$.

To verify the continuity of $\frac{\partial u}{\partial \xi}(x, t)$, we proceed similarly:

$$
\lim_{t \to t_0} \left\|\frac{\partial u}{\partial t}(x, t) - \partial u\right\|_{L^2(\mathcal{A})}
$$

$$
= \lim_{t \to t_0} |t - t_0| \cdot \left(|\xi_\infty|^\beta_2 + |\xi_f|^\alpha\right)^2e^{-t(|\xi_\beta|_\infty + |\xi_\infty|^\alpha)}\tilde{u}_\infty(\xi_\infty)\tilde{u}_f(\xi_f)\|_{L^2(\mathcal{A})}
$$

$$
\leq \lim_{t \to t_0} |t - t_0| \cdot \left(|\xi_\infty|^\beta_2 + |\xi_f|^\alpha\right)^2\tilde{u}_\infty(\xi_\infty)\tilde{u}_f(\xi_f)\|_{L^2(\mathcal{A})} = 0.
$$

As an immediate consequence from Lemmas 10.1 and 10.2 we obtain

**Proposition 10.3.** Let the function $u_0 \in S(\mathcal{A})$. Then the function $u(x, t)$ defined by (10.2) is a solution of Cauchy problem (10.7).

Consider the operator $T(t; \alpha, \beta), t \geq 0$ of convolution with the adelic heat kernel

$$
T(t; \alpha, \beta)u = Z_t * u.
$$
As in Section 7 the convolution $Z_t * u$ is a continuous function of $x$ for $t > 0$ and any $u \in L^2(\mathbb{A})$ and the operator $T(t; \alpha, \beta) : L^2(\mathbb{A}) \to L^2(\mathbb{A})$ is bounded.

By reasoning as in the proof of Theorem 7.5 we obtain

**Theorem 10.4.** Let $\alpha > 1$ and $\beta \in (0, 2]$. Then the following assertions hold.

(i) The operator $-D^{\alpha, \beta}$ generates a $C_0$ semigroup $(T(t; \alpha, \beta))_{t \geq 0}$. The operator $T(t; \alpha, \beta)$ coincides for each $t \geq 0$ with the operator $T(t; \alpha, \beta)$ given by (10.3).

(ii) Cauchy problem (10.1) is well-posed and its solution is given by $u(x, t) = Z_t * u_0$, $t \geq 0$.

10.2. **Non homogeneous equations.** Consider the following Cauchy problem

\begin{equation}
\begin{cases}
\frac{\partial u(x, t)}{\partial t} + D^{\alpha, \beta} u(x, t) = f(x, t), & x \in \mathbb{A}, \ t \in [0, T], \ T > 0, \\
u(x, 0) = u_0(x), & u_0(x) \in \mathcal{D}(D^{\alpha, \beta}).
\end{cases}
\end{equation}

We say that a function $u(x, t)$ is a solution of (10.6), if $u \in C([0, T], \mathcal{D}(D^{\alpha, \beta})) \cap C^1([0, T], L^2(\mathbb{A}))$ and if $u$ satisfies equation (10.6) for $t \in [0, T]$.

**Theorem 10.5.** Let $\alpha > 1$, $\beta \in (0, 2]$ and let $f \in C([0, T], L^2(\mathbb{A}))$. Assume that at least one of the following conditions is satisfied:

(i) $f \in L^1((0, T), \mathcal{D}(D^{\alpha, \beta}))$;

(ii) $f \in W^{1,1}((0, T), L^2(\mathbb{A}))$.

Then Cauchy problem (10.6) has a unique solution given by

$$u(x, t) = \int_\mathbb{A} Z(x - y, t; \alpha, \beta) u_0(y) \, dy + \int_0^t \left\{ \int_\mathbb{A} Z(x - y, t - \tau; \alpha, \beta) f(y, \tau) \, dy \right\} \, d\tau.$$

**Proof.** With the use of Theorem 10.4 the proof follows from well-known results of the semigroup theory, see e.g. [3 Proposition 3.1.16], [13 Proposition 4.1.6]. □

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