Proof of the simplicity conjecture

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Abstract

We show that the group of compactly supported area-preserving homeomorphisms of the two-disc is not simple; in fact, we prove the apriori stronger statement that this group is not perfect. This settles what is known as the “simplicity conjecture” in the affirmative. As a corollary, we find that for closed surfaces, the kernel of the “mass-flow” homomorphism is not simple. This answers a question of Fathi, and implies that the connected component of the identity in the group of area-preserving homeomorphisms of the two-sphere is not simple; the two-sphere is the only closed manifold for which the question of simplicity of the component of the identity in the group of volume-preserving homeomorphisms remained open — for other closed manifolds, this was settled by Fathi in 1980.

An important step in our proof involves verifying in a special case a conjecture of Hutchings concerning recovering the Calabi invariant from the asymptotics of spectral invariants defined using periodic Floer homology. Another key step involves proving that these spectral invariants extend continuously to area-preserving homeomorphisms of the disc. These two properties of PFH spectral invariants are potentially of independent interest. Our strategy is partially inspired by the approach of Oh towards the simplicity question.

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1 Introduction and main results

1.1 Background

Let \((S, \omega)\) be a surface with area form. An area-preserving homeomorphism is a homeomorphism which preserves the measure induced by \(\omega\).

Let \(\text{Homeo}_0(S^2, \omega)\) denote the connected component of the identity in the group of area-preserving homeomorphisms of the two-sphere, and let \(\text{Homeo}_c(\mathbb{D}, \omega)\) denote the group of area-preserving homeomorphisms of the two-disc which are the identity near the boundary. Recall that a group is simple if it does not possess a proper normal subgroup. The following long-open problem was raised by Fathi in 1980:

**Question 1.1** (Fathi, 80). Are the groups \(\text{Homeo}_0(S^2, \omega)\) and \(\text{Homeo}_c(\mathbb{D}, \omega)\) simple?

Indeed, the simplicity question for the connected component of the identity\(^1\) in the group of volume-preserving homeomorphisms is known for every

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\(^1\)The simplicity question is interesting only for the identity component of the group, because the identity component is a normal subgroup of the larger group. The group \(\text{Homeo}_c(\mathbb{D}, \omega)\) coincides with its identity component.
closed manifold other than $S^2$, as we will review in Section 1.2 below. And, the analogue of Question 1.1 for higher dimensional balls is also known \cite{Fat80}.

The main goal of the current paper is to show that neither of these groups are simple.

1.2 Historical remarks

The question of simplicity of groups of homeomorphisms and diffeomorphisms has a long history. In the 1930s, in the “Scottish Book”, Ulam asked if the identity component of the group of homeomorphisms of the $n$–dimensional sphere is simple. According to \cite{Bou08}, the first result obtained in this direction was an unpublished work of Ulam and Von Neumann who settled the question from the Scottish Book in the case $n = 2$. In the 50s and 60s, the works of Anderson \cite{And58}, Fisher \cite{Fis60}, Chernovski, Edwards and Kirby \cite{EK71} led to the proof of simplicity of the connected component of the identity in the group of compactly supported homeomorphisms of any manifold. In the 70s, the simplicity question was taken up in the smooth setting by Epstein \cite{Eps70}, Herman \cite{Her73}, Mather \cite{Mat74a, Mat74b, Mat75}, and Thurston \cite{Thu74} who showed that the connected component of identity in the group of compactly supported diffeomorphisms of any manifold is simple.

The connected component of identity in volume-preserving, and symplectic, diffeomorphisms admits a homomorphism, called flux, to a certain abelian group. Hence, these groups are not simple when this homomorphism is non-trivial. The kernel of flux, however, is always simple on a closed manifolds. This was proven by Thurston (See Chapter 5 of \cite{Ban97}) in the volume-preserving case and by Banyaga \cite{Ban78} in the symplectic case. We should mention that in the symplectic setting this kernel coincides with the group of Hamiltonian diffeomorphisms.

The simplicity question for volume-preserving homeomorphisms is well-understood in dimensions greater than two, and for closed surfaces of genus at least 1, thanks to the article \cite{Fat80}, in which Fathi constructs a “mass-flow” homomorphism that we introduce below. However, the case of area-preserving homeomorphisms of surfaces has been quite mysterious: in addition to the $S^2$ case, the simplicity question for the kernel of the mass-flow homomorphism has remained open.

\footnote{More precisely, Epstein, Herman and Thurston settled the question in the case of smooth diffeomorphisms, while Mather resolved the case of $C^r$ diffeomorphisms for $r < \infty$ and $r \neq \dim(M) + 1$. The case of $r = \dim(M) + 1$ remains open.}
1.2.1 The case of the disc

Question 1.1, especially the case of $\text{Homeo}_c(D,\omega)$, has received much attention; see for example [Ban78, Fat80, OM07, Ghy07, Oh10, Bou08, LR10a, LR10b, EPP12]. Indeed, the question appears on McDuff and Salamon’s list of open problems; see Section 14.7 in [MS17]. It is generally believed that the group $\text{Homeo}_c(D,\omega)$ is not simple — McDuff and Salamon refer to this as the simplicity conjecture.

Note that the group $\text{Diff}_c(D,\omega)$ of compactly supported area-preserving diffeomorphisms of the disc is not simple. Indeed, it admits a non-trivial homomorphism $\text{Cal} : \text{Diff}_c(D,\omega) \to \mathbb{R}$, called the Calabi homomorphism, which we will return to below.

There have been various attempts to settle the simplicity conjecture by constructing normal subgroups. Example of normal subgroups have been constructed by Oh-Müller [OM07, Oh10], Ghys [Ghy07], and Le Roux [LR10a]. The difficulty always lies in showing that these normal subgroups are proper. The article of Le Roux reduces the problem to a question about fragmentation properties of area-preserving diffeomorphisms. Yet another approach was suggested by Entov, Polterovich and Py [EPP12] who reduced the problem to a question about the existence of certain homogeneous quasi-morphisms on $\text{Homeo}_c(D,\omega)$; see Section 5 of [EPP12]. Here is our main result.

**Theorem 1.2.** The group $\text{Homeo}_c(D,\omega)$ is not simple.

Recall that a group $G$ is called perfect if its commutator subgroup $[G,G]$ satisfies $[G,G] = G$. The commutator subgroup $[G,G]$ is a normal subgroup of $G$. Thus, every simple group is perfect. However, in the case of certain transformation groups, such as $\text{Homeo}_c(D,\omega)$, a general argument due to Epstein and Higman [Eps70, Hig54], implies that perfectness and simplicity are equivalent; see Proposition 2.2. Hence, we obtain the following corollary.

**Corollary 1.3.** The group $\text{Homeo}_c(D,\omega)$ is not perfect.

1.2.2 The case of the sphere

There exists a circle of ideas and techniques, generally referred to as “Thurston tricks,” which can be used to prove that simplicity of $\text{Homeo}_0(S^2,\omega)$ is equivalent to perfectness of $\text{Homeo}_c(D,\omega)$; see Proposition 4.1.7 in [Bou08]. The next result then follows immediately.

**Corollary 1.4.** The group $\text{Homeo}_0(S^2,\omega)$ is not simple.

This stands in contrast to the case of higher-dimensional spheres where, by Fathi’s results involving the mass-flow homomorphism, explained below, the analogous groups are simple. Furthermore, the above theorem also contrasts the smooth setting where, as a consequence of Banyaga’s theorem
the group of area-preserving diffeomorphisms of the sphere is simple.

1.2.3 The case of surfaces of higher genus

Let $M$ denote a closed manifold equipped with a volume form $\omega$ and denote by $\text{Homeo}_0(M, \omega)$ the identity component in the group of volume-preserving homeomorphisms of $M$. In [Fat80], Fathi constructs the **mass-flow homomorphism**

$$\mathcal{F} : \text{Homeo}_0(M, \omega) \to H^1(M)/\Gamma,$$

mentioned above, where $H^1(M)$ denotes the first cohomology group of $M$ with coefficient in $\mathbb{R}$ and $\Gamma$ is a discrete subgroup of $H^1(M)$ whose definition we will not need here. Clearly, $\text{Homeo}_0(M, \omega)$ is not simple when the mass-flow homomorphism is non-trivial. This is indeed the case when $M$ is a closed surface other than the sphere. Fathi proved that $\ker(\mathcal{F})$ is simple if the dimension of $M$ is at least three. In his article, Fathi asks whether $\ker(\mathcal{F})$ is simple in the case of surfaces; see Appendix A.6 in [Fat80].

Note that in the case of the sphere, $\ker(\mathcal{F}) = \text{Homeo}_0(S^2, \omega)$, which by Corollary 1.4 is not simple. In fact, as in the case of the sphere, the Thurston tricks imply that, for any closed surface $M$, the kernel $\ker(\mathcal{F})$ is simple if and only if $\text{Homeo}_c(D, \omega)$ is perfect; see Proposition 4.1.7 in [Bou08]. Thus, we have:

**Corollary 1.5.** Let $M$ be any closed surface. The group of homeomorphisms with vanishing mass-flow, $\ker(\mathcal{F})$, is not simple.

**Remark 1.6.** In the case of surfaces, $\ker(\mathcal{F})$ is sometimes referred to as the group of Hamiltonian homeomorphisms; see for example [LC06]. It can be shown that $\ker(\mathcal{F})$ coincides with the $C^0$ closure of Hamiltonian diffeomorphisms.

1.3 Idea of the proof

Our goal here is to give an outline of the proof of Theorem 1.2.

1.3.1 A proper normal subgroup of $\text{Homeo}_c(D, \omega)$

To prove Theorem 1.2, we will define below a normal subgroup of $\text{Homeo}_c(D, \omega)$ which is a variation on the construction of Oh-Müller [OM07]. We will show that this normal subgroup is proper, and in fact, contains the commutator subgroup of $\text{Homeo}_c(D, \omega)$.

Denote by $C_c^\infty(S^1 \times D)$ the set of time-dependent Hamiltonians of the disc whose support is contained in the interior of $D$. As will be recalled in Section 2.3, one can associate to every $H \in C_c^\infty(S^1 \times D)$ a Hamiltonian
flow $\varphi^t_H$. Furthermore, every area-preserving diffeomorphism of the disc $\varphi \in \text{Diff}_c(\mathbb{D}, \omega)$ is a Hamiltonian diffeomorphism in the sense that there exists $H \in C^\infty_c(S^1 \times \mathbb{D})$ such that $\varphi = \varphi^1_H$.

The energy, or the Hofer norm, of a Hamiltonian $H \in C^\infty(S^1 \times S^2)$ is defined by the quantity

$$\|H\|_{(1, \infty)} = \int_0^1 \left( \max_{x \in S^2} H(t, \cdot) - \min_{x \in S^2} H(t, \cdot) \right) dt.$$ 

**Definition 1.7.** An element $\varphi \in \text{Homeo}_c(\mathbb{D}, \omega)$ is a finite-energy homeomorphism if there exists a sequence of smooth Hamiltonians $H_i \in C^\infty_c(S^1 \times \mathbb{D})$ such that the sequence $\|H_i\|_{(1, \infty)}$ is bounded, i.e. $\exists C \in \mathbb{R}$ such that $\|H_i\|_{(1, \infty)} \leq C$, and the Hamiltonian diffeomorphisms $\varphi^1_{H_i}$ converge uniformly to $\varphi$. We will denote the set of all finite-energy homeomorphisms by $\text{FHomeo}_c(\mathbb{D}, \omega)$.

We clarify the topology for convergence in the definition, namely the $C^0$ topology, in Section 2.2.

We will show in Proposition 2.1 and Corollary 2.3 that $\text{FHomeo}_c(\mathbb{D}, \omega)$ is a normal subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$ that contains the commutator subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$. Thus, Theorem 1.2 will follow from the following result that we show:

**Theorem 1.8.** $\text{FHomeo}_c(\mathbb{D}, \omega)$ is a proper normal subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$.

**Remark 1.9.** In defining $\text{FHomeo}_c(\mathbb{D}, \omega)$ as above, we were inspired by the article of Oh and Müller [OM07], who defined a normal subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$, denoted by $\text{Hameo}_c(\mathbb{D}, \omega)$, which is usually referred as the group of hameomorphisms. It has been conjectured that $\text{Hameo}_c(\mathbb{D}, \omega)$ is a proper normal subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$; see for example Question 4.3 in [OM07].

We will see in Section 2 that $\text{Hameo}_c(\mathbb{D}, \omega) \subset \text{FHomeo}_c(\mathbb{D}, \omega)$. Hence, it follows from the above theorem that $\text{Hameo}_c(\mathbb{D}, \omega)$ is a proper normal subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$.

**Remark 1.10.** Theorem 1.8 gives an affirmative answer to Question 1 in [LR10b] where Le Roux asks if there exist area-preserving homeomorphisms of the disc which are “infinitely far in Hofer’s distance” from area-preserving diffeomorphisms.

### 1.3.2 The Calabi invariant and the infinite twist

The hard part of Theorem 1.8 is to show properness. Here we describe the key example of an area-preserving homeomorphism that we later show is not in $\text{FHomeo}_c(\mathbb{D}, \omega)$. 


We first summarize some background that will motivate what follows. As mentioned above, for smooth, area-preserving compactly supported two-disc diffeomorphisms, the simplicity conjecture is known, via the Calabi invariant. More precisely, the Calabi invariant of \( \theta \in \text{Diff}_c(D, \omega) \) is defined as follows. Pick any Hamiltonian \( H \in C_c^\infty(S^1 \times \mathbb{D}) \) such that \( \theta = \varphi_H^1 \). Then,

\[
\text{Cal}(\theta) := \int_{S^1} \int_D H \omega \, dt.
\]

The above integral does not depend on the choice of \( H \) and so \( \text{Cal}(\theta) \) is well-defined. It is well-known that \( \text{Cal} : \text{Diff}_c(D, \omega) \to \mathbb{R} \) is a non-trivial group homomorphism, \( i.e. \text{Cal}(\theta_1 \theta_2) = \text{Cal}(\theta_1) + \text{Cal}(\theta_2) \). For further details on the Calabi homomorphism see [Cal70, MS17].

We will need to know the value of the Calabi invariant for the following class of area-preserving diffeomorphisms. Let \( f : [0, 1] \to \mathbb{R} \) be a smooth function vanishing near 1 and define \( \phi_f \in \text{Diff}_c(\mathbb{D}, \omega) \) by \( \phi_f(0) := 0 \) and \( \phi_f(r, \theta) := (r, \theta + 2\pi f(r)) \). If the function \( f \) is taken to be (positive/negative) monotone, then the map \( \phi_f \) is referred to as a (positive/negative) monotone twist.

Now suppose that \( \omega = \frac{1}{2\pi} r dr \wedge d\theta \). A simple computation (see our conventions in Section 2) shows \( \phi_f \) is the time–1 map of the flow of the Hamiltonian defined by

\[
F(r, \theta) = \int_r^1 sf(s)ds.
\]  

From this we compute:

\[
\text{Cal}(\phi_f) = \int_0^1 \int_r^1 sf(s)ds \, r \, dr.
\]

We can now introduce the element that will not be in \( \text{FHomeo}_c(D, \omega) \).

Let \( f : [0, 1] \to \mathbb{R} \) be a smooth function which vanishes near 1, is decreasing, and satisfies \( \lim_{r \to 0} f(r) = \infty \). Define \( \phi_f \in \text{Homeo}_c(\mathbb{D}, \omega) \) by \( \phi_f(0) := 0 \) and \( \phi_f(r, \theta) := (r, \theta + 2\pi f(r)) \). If the function \( f \) is taken to be (positive/negative) monotone, then the map \( \phi_f \) is referred to as a (positive/negative) monotone twist.

We will show that if

\[
\int_0^1 \int_r^1 sf(s)ds \, r \, dr = \infty,
\]

then

\[
\phi_f \notin \text{FHomeo}_c(D, \omega).
\]
Remark 1.11. The idea outlined in the above paragraph is not entirely new. Indeed, Fathi has conjectured that the Calabi homomorphism extends to $\text{Hameo}_c(D, \omega)$ which is the normal subgroup constructed by Müller and Oh described above; see Conjecture 6.8 in Oh’s article [Oh10]. Theorem 7.2 of the same article shows that if the Calabi invariant extended to $\text{Hameo}_c(D, \omega)$, then the homeomorphism $\phi_f$ would not be in $\text{Hameo}_c(D, \omega)$.

Hence, it seemed reasonable to conjecture that $\phi_f \notin \text{Hameo}_c(D, \omega)$. Indeed, McDuff & Salamon refer to this as the Infinite Twist Conjecture and it is Problem 46 on their list of open problems; see Section 14.7 of [MS17]. Clearly, since $\text{Hameo}_c(D, \omega) \subseteq \text{FHomeo}_c(D, \omega)$ implies the Infinite Twist Conjecture as well.

1.3.3 Spectral invariants from periodic Floer homology

To show that the infinite twist is not in $\text{FHomeo}_c(D, \omega)$, we use the theory of periodic Floer homology (PFH), reviewed in Section 3. PFH is a version of Floer homology for area-preserving diffeomorphisms which was introduced by Hutchings [Hut02, HS05]. As with ordinary Floer homology, PFH can be used to define a sequence of functions $c_d : \text{Diff}_c(D, \omega) \to \mathbb{R}$, where $d \in \mathbb{N}$, called spectral invariants, which satisfy various useful properties. We give the definition of $c_d$ in Section 3.4, see in particular Remark 3.8.

The definition of PFH spectral invariants is due to Michael Hutchings [Hut], but very few properties have been established about these. We will prove in Theorem 3.6 that the PFH spectral invariants satisfy the following properties:

1. $c_d(\text{Id}) = 0$,

2. Monotonicity: Suppose that $H \leq G$ where $H, G \in C^\infty_c(S^1 \times D)$. Then, $c_d(\varphi_H) \leq c_d(\varphi_G)$ for all $d \in \mathbb{N}$,

3. Hofer Continuity: $|c_d(\varphi_H) - c_d(\varphi_G)| \leq d\|H - G\|_{(1, \infty)}$,

4. Spectrality: $c_d(\varphi_H) \in \text{Spec}_d(H)$ for any $H \in \mathcal{H}$, where $\text{Spec}_d(H)$ is the order $d$ spectrum of $H$ and is defined in Section 2.4.

A key property, which allows us to use the PFH spectral invariants for studying homeomorphisms (as opposed to diffeomorphisms) is the following theorem, which we prove in Section 4 via the methods of $C^0$ symplectic topology.

Theorem 1.12. The spectral invariant $c_d : \text{Diff}_c(D, \omega) \to \mathbb{R}$ is continuous with respect to the $C^0$ topology on $\text{Diff}_c(D, \omega)$. Furthermore, it extends continuously to $\text{Homeo}_c(D, \omega)$.

Another key property is the following which was originally conjectured in greater generality by Hutchings [Hut]:

Remark 1.11. The idea outlined in the above paragraph is not entirely new. Indeed, Fathi has conjectured that the Calabi homomorphism extends to $\text{Hameo}_c(D, \omega)$ which is the normal subgroup constructed by Müller and Oh described above; see Conjecture 6.8 in Oh’s article [Oh10]. Theorem 7.2 of the same article shows that if the Calabi invariant extended to $\text{Hameo}_c(D, \omega)$, then the homeomorphism $\phi_f$ would not be in $\text{Hameo}_c(D, \omega)$. Hence, it seemed reasonable to conjecture that $\phi_f \notin \text{Hameo}_c(D, \omega)$. Indeed, McDuff & Salamon refer to this as the Infinite Twist Conjecture and it is Problem 46 on their list of open problems; see Section 14.7 of [MS17]. Clearly, since $\text{Hameo}_c(D, \omega) \subseteq \text{FHomeo}_c(D, \omega)$ implies the Infinite Twist Conjecture as well.
Theorem 1.13. The PFH spectral invariants $c_d : \text{Diff}_c(D,\omega) \to \mathbb{R}$ satisfy the Calabi property

$$\lim_{d \to \infty} \frac{c_d(\varphi)}{d} = \frac{1}{2} \text{Cal}(\varphi)$$

if $\varphi$ is a monotone twist map of the disc.

The property (6) can be thought of as a kind of analogue of the “Volume Property” for ECH spectral invariants proved in [CGHR15]. ECH has many similarities to PFH, which is part of the motivation for conjecturing that something like (6) might be possible.

Remark 1.14. In fact, Hutchings [Hut] has conjectured that the Calabi property in Theorem 1.13 holds more generally for all $\varphi \in \text{Diff}_c(D,\omega)$. The point is that we verify this conjecture for monotone twists; and, this is sufficient for our purposes.

As mentioned in Remark 1.11, it has been an open question whether the Calabi invariant extends to $\text{Homeo}_c(D,\omega)$. Hutchings’ conjecture, combined with Theorem 1.12, would imply that the Calabi invariant admits an extension to $\text{Homeo}_c(D,\omega)$.

Remark 1.15. A reader with experience in this topic might wonder if it is possible to prove Theorem 1.8 using other spectral invariants such as those arising from Hamiltonian Floer theory or embedded contact homology. This might be possible, however, to the best of our knowledge PFH spectral invariants are the only such invariants which simultaneously satisfy Theorems 1.12 & 1.13.

1.3.4 Basic idea of the proof

We can now summarize the idea of the proof.

As was already explained above, the challenge with our approach is to show that the infinite twist is not in $F\text{Homeo}_c(D,\omega)$. Here is how we do this. Theorem 1.12 allows to define the PFH spectral invariants for any $\psi \in \text{Homeo}_c(D,\omega)$. We will show, by using the Hofer Continuity property, that if $\psi$ is a finite-energy homeomorphism then the sequence of PFH spectral invariants $\{c_d(\psi)\}_{d \in \mathbb{N}}$ grows at most linearly. On the other hand, in the case of the monotone twist $\phi_f$, satisfying the condition in Equation (4), the sequence $\{c_d(\phi_f)\}_{d \in \mathbb{N}}$ has super-linear growth, as a consequence of the Calabi property (6). From this we can conclude that $\phi_f \notin F\text{Homeo}_c(D,\omega)$, as desired.

1.3.5 Proof of Theorem 1.8

Here, we will give a proof of Theorem 1.8 relying on the properties of PFH spectral invariants mentioned in Section 1.3.3.
We begin our proof with the following lemma which tells us that for a finite-energy homeomorphism $\psi$ the sequence of PFH spectral invariants $\{c_d(\psi)\}_{d \in \mathbb{N}}$ grows at most linearly.

**Lemma 1.16.** Let $\psi \in \text{FHomeo}_c(\mathbb{D}, \omega)$ be a finite-energy homeomorphism. Then, there exists a constant $C$, depending on $\psi$, such that

$$\frac{c_d(\psi)}{d} \leq C, \forall d \in \mathbb{N}.$$ 

**Proof.** By definition, $\psi$ being a finite-energy homeomorphism implies that there exist smooth Hamiltonians $H_i \in C^\infty_c(S^1 \times \mathbb{D})$ such that the sequence $\|H_i\|_{(1, \infty)}$ is bounded, i.e. $\exists C \in \mathbb{R}$ such that $\|H_i\|_{(1, \infty)} \leq C$, and the Hamiltonian diffeomorphisms $\varphi^1_{H_i}$ converge uniformly to $\phi$.

The Hofer continuity property and the fact that $c_d(Id) = 0$ imply that $c_d(\varphi^1_{H_i}) \leq d \|H_i\|_{(1, \infty)} \leq dC$, for each $d \in \mathbb{N}$.

On the other hand, by Theorem 1.12 $c_d(\varphi^1_{F}) = \lim_{i \to \infty} c_d(\varphi^1_{H_i})$. We conclude from the above inequality that $c_d(\psi) \leq dC$ for each $d \in \mathbb{N}$. \hfill \Box

We now turn our attention to showing that if the PFH spectral invariants of an infinite twist $\phi_f$, which satisfies Equations (3) and (4), violate the inequality from the above lemma. We will need the following.

**Lemma 1.17.** There exists a sequence of smooth monotone twists $\phi_{f_i} \in \text{Diff}_c(\mathbb{D}, \omega)$ satisfying the following properties:

1. The sequence $\phi_{f_i}$ converges in the $C^0$ topology to $\phi_f$,

2. There exist Hamiltonians $F_i$, compactly supported in the interior of the disc $\mathbb{D}$, such that $\varphi^1_{F_i} = \phi_{f_i}$ and $F_i \leq F_{i+1}$,

3. $\lim_{i \to \infty} \text{Cal}(\phi_{f_i}) = \infty$.

**Proof.** Recall that $f$ is a decreasing function of $r$ which vanishes near 1 and satisfies $\lim_{r \to 0} f(r) = \infty$. It is not difficult to see that we can pick smooth functions $f_i : [0, 1] \to \mathbb{R}$ satisfying the following properties:

1. $f_i = f$ on $[\frac{1}{i}, 1]$,

2. $f_i \leq f_{i+1}$.

Let us check that the monotone twists $\phi_{f_i}$ satisfy the requirements of the lemma. To see that they converge to $\phi_f$, observe that $\phi_f$ and $\phi_{f_i}$ coincide outside the disc of radius $\frac{1}{i}$. Hence, $\phi^{-1}_{f} \phi_{f_i}$ converges uniformly to $\text{Id}$ because it is supported in the disc of radius $\frac{1}{i}$. Next, note that by Formula (1), $\phi_{f_i}$ is
the time–1 map of the Hamiltonian flow of $F_i(r, \theta) = 2 \int_0^1 s f_i(s) ds$. Clearly, $F_i \leq F_{i+1}$ because $f_i \leq f_{i+1}$. Finally, by Formula (2) we have

$$\text{Cal}(\phi_{f_i}) = \int_0^1 \int_r^1 s f_i(s) ds \, rdr \geq \int_0^1 \int_r^1 s f_i(s) ds \, rdr = \int_0^1 \int_r^1 s f(s) ds \, rdr.$$ 

Recall that $f$ has been picked such that $\int_0^1 \int_r^1 s f(s) ds \, rdr = \infty$; see Equation (4). We conclude that $\lim_{i \to \infty} \text{Cal}(\phi_{f_i}) = \infty$. $\square$

We will now use Lemma 1.17 to complete the proof of Theorem 1.8. We showed in Proposition 2.1 that $\text{FHomeo}_c$ is a normal subgroup; it is certainly non-trivial, so it remains to show it is proper.

By the Monotonicity property, we have $c_d(\phi_{f_i}) \leq c_d(\phi_{f_{i+1}})$ for each $d \in \mathbb{N}$. Since $\phi_{f_i}$ converges in $C^0$ topology to $\phi_f$, we conclude from Theorem 1.12 that $c_d(\phi_f) = \lim_{i \to \infty} c_d(\phi_{f_i})$. Combining the previous two lines we obtain the following inequality:

$$c_d(\phi_{f_i}) \leq c_d(\phi_f), \forall d, i \in \mathbb{N}.$$ 

Now the Calabi property of Theorem 1.13 tells us that $\lim_{d \to \infty} \frac{c_d(\phi_{f_i})}{d} = \text{Cal}(\phi_{f_i})$. Combining this with the previous inequality we get $\text{Cal}(\phi_i) \leq \lim_{d \to \infty} \frac{c_d(\phi)}{d}$ for all $i$. Hence, by the third item in Lemma 1.17

$$\lim_{d \to \infty} \frac{c_d(\phi)}{d} = \infty,$$

and so by Lemma 1.16 $\phi$ is not in $\text{FHomeo}_c(D, \omega)$.

**Remark 1.18.** The proof outlined above does not use the full force of Theorem 1.13; it only uses the fact that $\lim_{d \to \infty} \frac{c_d(\phi)}{d} \geq \frac{1}{2} \text{Cal}(\phi)$. $\blacksquare$

**Organization of the paper**

In Section 2, we review some of the necessary background from symplectic geometry, especially the case of surfaces. In Section 3 of the paper we review the construction of periodic Floer homology and the associated spectral invariants. Some of the properties of PFH spectral invariants, such as Hofer continuity and spectrality, are proven in Section 3.4. We prove Theorem 1.12 on $C^0$ continuity of PFH spectral invariants, in Section 4. Section 5.2 of the article begins with a more precise version of Theorem 1.13. The rest of Section 5.2 is dedicated to the development of a combinatorial model of the periodic Floer homology of monotone twists. In Section 6.1 we first use the aforementioned combinatorial model of PFH to compute the PFH spectral invariants for monotone twists. We then use this computation to prove that the PFH spectral invariants for monotone twists satisfy the Calabi property of Theorem 1.13; this will be carried out in Section 6.2.
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2 Preliminaries about the symplectic geometry of surfaces

Here we collect some basic facts, and fix notation, concerning two-dimensional symplectic geometry and diffeomorphism groups.

2.1 Symplectic form on the disc and sphere

Let \( S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3 \) and \( D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \). We equip the sphere \( S^2 \) with the symplectic form \( \omega := \frac{1}{4\pi} d\theta \wedge dz \), where \((\theta, z)\) are cylindrical coordinates on \( \mathbb{R}^3 \). Note that
with this form, $S^2$ has area 1. Let

$$S^+ = \{(x, y, z) \in S^2 : z \geq 0\},$$

be the Northern hemisphere in $S^2$. In certain sections of the paper, we will need to identify the disc $D$ with $S^+$. To do this, we will take the embedding $\iota : D \to S^2$ given by the formula

$$\iota(r, \theta) = (\theta, 1 - r^2),$$  \hspace{1cm} (7)

where $(r, \theta)$ denotes the standard polar coordinates on $\mathbb{R}^2$. We will equip the disc with the area form given by the pullback of $\omega$ under $\iota$; explicitly, this is given by the formula $\frac{1}{2\pi} r dr \wedge d\theta$. We will denote this form by $\omega$ as well. Note that this gives the disc a total area of $\frac{1}{2}$.

Any area form on $S^2$ or $D$ is equivalent to the above differential forms, up to multiplication by a constant.

### 2.2 The $C^0$ topology

Here we fix our conventions and notation concerning the $C^0$ topology.

Denote by $\text{Homeo}(S^2)$ the group of homeomorphisms of the sphere and by $\text{Homeo}_c(D)$ the group of homeomorphisms of the disc whose support is contained in the interior of $D$.

Let $d$ be a Riemannian distance on $S^2$. Given two maps $\phi, \psi : S^2 \to S^2$, we denote

$$d_{C^0}(\phi, \psi) = \max_{x \in S^2} d(\phi(x), \psi(x)).$$

We will say that a sequence of maps $\phi_i : S^2 \to S^2$ converges uniformly, or $C^0$–converges, to $\phi$, if $d_{C^0}(\phi_i, \phi) \to 0$ as $i \to \infty$. As is well known, the notion of $C^0$–convergence does not depend on the choice of the Riemannian metric. The topology induced by $d_{C^0}$ on $\text{Homeo}(S^2)$ is referred to as the $C^0$ topology. The $C^0$ topology on $\text{Homeo}_c(D)$ is defined analogously.

### 2.3 Hamiltonian diffeomorphisms

Let

$$\text{Diff}(S^2, \omega) := \{\theta \in \text{Diff}(S^2) : \theta^* \omega = \omega\}$$

denote the group of area-preserving, in other words symplectic, diffeomorphisms of the sphere.

Let $C^\infty(S^1 \times S^2)$ denote the set of smooth time-dependent Hamiltonians on $S^2$. As alluded to in the introduction, a smooth Hamiltonian $H \in C^\infty(S^1 \times S^2)$ gives rise to a time-dependent vector field $X_H$, called the Hamiltonian vector field, defined via the equation

$$\omega(X_H(t), \cdot) = dH_t.$$
The Hamiltonian flow of $H$, denoted by $\varphi^t_H$, is by definition the flow of $X_H$. A Hamiltonian diffeomorphism is a diffeomorphism which arises as the time-one map of a Hamiltonian flow. It is easy to verify that every Hamiltonian diffeomorphism of $S^2$ is area-preserving. And, as is well-known, every area-preserving diffeomorphism of the sphere is in fact a Hamiltonian diffeomorphism.

As for the disc, as mentioned in the introduction, every $\theta \in \text{Diff}_c(D, \omega)$ is Hamiltonian, in the sense that one can find $H \in C^\infty_c(S^1 \times D)$ such that $\theta = \varphi_1^H$, where the notation is as in the sphere case. Here, $C^\infty_c(S^1 \times D)$ denotes the set of Hamiltonians of $D$ whose support is compactly contained in $D$.

Note that $\text{Diff}(S^2, \omega) \subset \text{Homeo}_0(S^2, \omega)$ and $\text{Diff}_c(D, \omega) \subset \text{Homeo}_c(D, \omega)$. It is well-known that $\text{Diff}(S^2, \omega)$ and $\text{Diff}_c(D, \omega)$ are dense, with respect to the $C^0$ topology, in $\text{Homeo}_0(S^2, \omega)$ and $\text{Homeo}_c(D, \omega)$, respectively.

### 2.4 The action functional and its spectrum

Spectral invariants take values in the “action spectrum”. We now explain what this spectrum is.

Denote by $\Omega := \{ z : S^1 \to S^2 \}$ the space of all loops in $S^2$. By a capping of a loop $z : S^1 \to S^2$, we mean a map

$$u : D^2 \to S^2,$$

such that $u|_{\partial D^2} = z$. We say two cappings $u, u'$ for a loop $z$ are equivalent if $u, u'$ are homotopic rel $z$. Henceforth, we will only consider cappings up to this equivalence relation. Note that given a capping $u$ of a loop $z$, all other cappings of $z$ are of the form $u \# A$ where $A \in \pi_2(S^2)$ and $\#$ denotes the operation of connected sum.

A capped loop is a pair $(z, u)$ where $z$ is a loop and $u$ is a capping for $z$. We will denote by $\tilde{\Omega}$ the space of all capped loops in the sphere.

Let $H \in C^\infty(S^1 \times S^2)$ denote a smooth Hamiltonian in $S^2$. Recall that $A_H : \tilde{\Omega} \to \mathbb{R}$, the action functional associated to the Hamiltonian $H$ is defined by

$$A_H(z, u) = \int_0^1 H(t, z(t))dt + \int_{D^2} u^* \omega.$$  \hspace{1cm} (8)

Note that $A_H(z, u \# A) = A_H(z, u) + \omega(A)$, for every $A \in \pi_2(S^2)$.

The set of critical points of $A_H$, denoted by $\text{Crit}(A_H)$, consists of capped loops $(z, u) \in \tilde{\Omega}$ such that $z$ is a 1–periodic orbit of the Hamiltonian flow $\varphi^1_H$. We will often refer to such $(z, u)$ as a capped 1–periodic orbit of $\varphi^1_H$.

The action spectrum of $H$, denoted by $\text{Spec}(H)$, is the set of critical values of $A_H$; it has Lebesgue measure zero. It turns out that the action spectrum $\text{Spec}(H)$ is independent of $H$ in the following sense: If $H'$ is another Hamiltonian such that $\varphi^1_H = \varphi^1_{H'}$, then there exists a constant...
Lemma 3.3 of \[Sch00\] proves this in the case where \(\omega\) which we call the higher order action spectrum and the proof generalizes readily to general symplectic manifolds.

The PFH spectral invariants will take values in a more general set, which we call the higher order action spectrum. To define it, let \(H, G\) be two Hamiltonians. The composition of \(H\) and \(G\) is the Hamiltonian \(H \# G(t, x) := H(t, x) + G(t, (\phi_H^t)^{-1}(x))\). It is known that \(\phi_{H \# G} = \phi_H^t \circ \phi_G^t\); see \[Pol01\], for example. Denote by \(H^k\) the \(k\)-times composition of \(H\) with itself. For any \(d > 0\), we now define the order \(d\) spectrum of \(H\) by

\[
\text{Spec}_{d}(H) := \cup_{k_1 + \ldots + k_d = d} \text{Spec}(H^{k_1}) + \ldots + \text{Spec}(H^{k_d}).
\]

Note that \(\text{Spec}_{d}(H)\) may equivalently be described as follows: For every value \(a \in \text{Spec}_{d}(H)\) there exist capped periodic orbits \((z_1, u_1), \ldots, (z_k, u_k)\) of \(H\) the sum of whose periods is \(d\) and such that

\[
a = \sum A_{H^{c_i}}(u_i, z_i).
\]

We can use the above to define the action spectrum for compactly supported disc maps. Recall from Section 2.3 our convention to identify the northern hemisphere of \(S^2\) with the disc; we will use this to define the action spectrum in the case of the disc.

More precisely, if \(H, H'\) are supported in the northern hemisphere \(S^+\subset S^2\), and generate the same time-1 map \(\phi\), then we in fact have \(\text{Spec}_{d}(H) = \text{Spec}_{d}(H')\) for all \(d > 0\). To see this, let \(H, H' \in C^\infty(S^1 \times S^+)\) be such that \(\varphi_H^1 = \varphi_H^1\). Let \((z, u)\) and \((z', u')\) be capped orbits of, respectively, \(\varphi_H^1\) and \(\varphi_{H'}^1\), which correspond to the same fixed point \(x\) of \(\varphi_H^1\), i.e., \(z(t) = \varphi_H^1(x), z'(t) = \varphi_{H'}^1(x)\). As we explain below, there exists \(A \in \pi_2(S^3)\) such that

\[
A_H(z, u) - A_{H'}(z', u') = \omega(A).
\]

It follows that if \(\phi \in \text{Diff}_{S^+}(S^2, \omega)\), then we can define the action spectra of \(\phi\) without any ambiguity by setting

\[
\text{Spec}_{d}(\phi) = \text{Spec}_{d}(H),
\]

where \(H\) is any Hamiltonian in \(C^\infty(S^1 \times S^+)\) such that \(\phi = \varphi_H^1\).

Let us now explain why Equation (9) is true. Observe that it is sufficient to show that \(A_H(z, u) - A_{H'}(z', u') = 0\) when \(u, u'\) are cappings in \(S^+\) because any other capping for \(z, z'\) may be obtained from such cappings by attachment of spheres. Since the group of compactly supported Hamiltonian diffeomorphisms of the disc is contractible, we can find Hamiltonians \(H_\lambda\), where \(\lambda \in [0, 1]\), which are compactly supported in \(S^+\) and such that \(H_0 = H, H_1 = H'\) and \(\varphi_{H_\lambda}^1 = \varphi_{H}^1 = \varphi_{H'}^1\) for all \(\lambda \in [0, 1]\). Let \(z_\lambda(t) = \varphi_{H_\lambda}^1(x)\);
this is a 1–parameter family of 1–periodic orbits such that $z_0(t) = z(t)$ and $z_1(t) = z'(t)$. It is easy to see that starting from the capping $u$ of $z$, using the family of orbits $z_\lambda$, we can construct a 1–parameter family of cappings $u_\lambda$ of $z_\lambda$. Note that we obtain a capping $u_1$ of $z_1 = z'$ which is equivalent to $u'$ because both of these cappings are contained in $S^+$ which is contractible. Hence, to prove $A_H(z, u) - A_H(z', u') = 0$ it is enough to show that

$$\frac{d}{d\lambda} A_{H_\lambda}(z_\lambda, u_\lambda) = 0.$$  

This is proven in Lemma 9.1.9 of [MS17].

### 2.5 Finite energy homeomorphisms

Recall that we defined finite-energy homeomorphisms in Definition 1.7. It is not hard to show that $\text{FHomeo}_c(\mathbb{D}, \omega)$ is a normal subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$ and $\text{FHomeo}_c(\mathbb{D}, \omega)$ contains the commutator subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$. In this section, we show that $\text{FHomeo}_c(\mathbb{D}, \omega)$ is a normal subgroup.

**Proposition 2.1.** $\text{FHomeo}_c(\mathbb{D}, \omega)$ is a normal subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$.

**Proof.** Consider smooth Hamiltonians $H, G \in C^\infty_c(S^1 \times \mathbb{D})$. As was partly mentioned in Section 2.4, it is well-known (see [Pol01], for example) that the Hamiltonians

$$H \# G(t, x) := H(t, x) + G(t, (\varphi_H^t)^{-1}(x)), \quad \tilde{H}(t, x) := -H(t, \varphi_H^t(x)), \quad (11)$$

generate the Hamiltonian flows

$$\varphi_H^t \circ \varphi_G^t, \quad (\varphi_H^t)^{-1},$$

respectively. Furthermore, given $\psi \in \text{Diff}_c(\mathbb{D}, \omega)$, the Hamiltonian

$$H \circ \psi(t, x) := H(t, \psi(x))$$

generates the flow $\psi^{-1} \varphi_H^t \psi$.

We now show that $\text{FHomeo}_c$ is closed under conjugation. Take $\phi \in \text{FHomeo}_c(\mathbb{D}, \omega)$ and let $H_i$ and $C$ be as in the definition of finite-energy homeomorphisms. Let $\psi \in \text{Homeo}_c(\mathbb{D}, \omega)$ and take a sequence $\psi_i \in \text{Diff}_c(\mathbb{D}, \omega)$ which converges uniformly to $\psi$. Consider the Hamiltonians $H_i \circ \psi_i$. The corresponding Hamiltonian diffeomorphisms are the conjugations $\psi_i^{-1} \varphi_H^t \psi_i$ which converge uniformly to $\psi^{-1} \phi \psi$. Furthermore,

$$\|H_i \circ \psi_i\|_{(1, \infty)} = \|H_i\|_{(1, \infty)} \leq C,$$

where the inequality follows from the definition of $\text{FHomeo}_c(\mathbb{D}, \omega)$.

We will next check that $\text{FHomeo}_c$ is a group. Take $\phi, \psi \in \text{FHomeo}_c$ and let $H_i, G_i \in C^\infty_c(S^1 \times \mathbb{D})$ be two sequences of Hamiltonians such that $\varphi_H^t, \varphi_G^t$,
converge uniformly to $\phi, \psi$, respectively, and $\|H_i\|_{(1, \infty)}, \|G_i\|_{(1, \infty)} \leq C$ for some constant $C$. Then, the sequence $\varphi_{H_i}^{-1} \circ \varphi_{G_i}^{1}$ converges uniformly to $\phi^{-1} \circ \psi$. Moreover, by the above formulas, we have $\varphi_{H_i}^{-1} \circ \varphi_{G_i}^{1} = \delta_i \# G_i$. Since $\|H_i\# G_i\|_{(1, \infty)} \leq \|H_i\|_{(1, \infty)} + \|G_i\|_{(1, \infty)} \leq 2C$, this proves that $\phi^{-1} \circ \psi \in \text{FHomeo}_c$ which completes the proof that $\text{FHomeo}_c$ is a group.

2.6 Equivalence of perfectness and simplicity

The goal of this section is to show that in the case of $\text{Homeo}_c(\mathbb{D}, \omega)$ perfectness and simplicity are equivalent.

**Proposition 2.2.** Any non-trivial normal subgroup $H$ of $\text{Homeo}_c(\mathbb{D}, \omega)$ contains the commutator subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$. Hence, $\text{Homeo}_c(\mathbb{D}, \omega)$ is perfect if and only if it is simple.

**Corollary 2.3.** The commutator subgroup of $\text{Homeo}_c(\mathbb{D}, \omega)$ is contained in $\text{FHomeo}_c(\mathbb{D}, \omega)$.

**Remark 2.4.** As a consequence, proving that the infinite twist $\phi_f$, introduced in Section 1.3.2, is not in $\text{FHomeo}_c(\mathbb{D}, \omega)$ proves also that $\phi_f$ cannot be written as a product of commutators.

The proof of Proposition 2.2 relies on a general argument, due to Epstein [Eps70] and Higman [Hig54], which essentially shows that perfectness implies simplicity for transformation groups satisfying certain assumptions. We will present a version of this argument, which we learned of in [Fat80], in our context.

**Proof.** Pick $h \in H$ such that $h \neq \text{Id}$. We can find a closed topological disc, that is a set which is homeomorphic to a standard Euclidean disc $\mathbb{D}' \subset \mathbb{D}$ such that $h(\mathbb{D}') \cap \mathbb{D}' = \emptyset$. Denote by $\text{Homeo}_c(\mathbb{D}', \omega)$ the subset of $\text{Homeo}_c(\mathbb{D}, \omega)$ consisting of area-preserving homeomorphisms whose supports are contained in the interior of $\mathbb{D}'$. We will first prove the following lemma.

**Lemma 2.5.** The commutator subgroup of $\text{Homeo}_c(\mathbb{D}', \omega)$ is contained in $H$.

**Proof.** We must show that for any $f, g \in \text{Homeo}_c(\mathbb{D}', \omega)$, the commutator $[f, g] := fgf^{-1}g^{-1}$ is an element of $H$.

First, observe that for any $f \in \text{Homeo}_c(\mathbb{D}', \omega)$ we have

$$[f, r] \in H \quad (12)$$

for any $r \in H$. Indeed, by normality, $frf^{-1} \in H$ and hence $frf^{-1}r^{-1} \in H$. Next, we claim that for any $f, g \in \text{Homeo}_c(\mathbb{D}', \omega)$

$$[f, g][g, hfh^{-1}] = f[g, [f^{-1}, h]]f^{-1}. \quad (13)$$
Postponing the proof of this identity for the moment, we will first show that it implies the lemma. Note that \( g \) and \( hf^{-1} \) are, respectively, supported in \( D' \) and \( h(D') \) which are disjoint. Thus, \( [g, hf^{-1}] = \text{Id} \). Hence, Identity (13) yields \( [f, g] = f[g, [f^{-1}, h]]f^{-1} \). Now, (12) implies that \( [g, [f^{-1}, h]] \in H \) which, by normality of \( H \), implies that \( f[g, [f^{-1}, h]]f^{-1} \in H \). This gives us the conclusion of the proposition. We complete the proof by proving Identity (13):

\[
[f, g][g, hf^{-1}] = (fgf^{-1}g^{-1})(ghfh^{-1}g^{-1}hf^{-1}h^{-1}) = fg(f^{-1}hf^{-1})g^{-1}hf^{-1}h^{-1} = fg[f^{-1}, h]g^{-1}hf^{-1}h^{-1} = fg[f^{-1}, h]g^{-1}[h, f^{-1}]f^{-1} = fg[g, [f^{-1}, h]]f^{-1}.
\]

We continue with the proof of Proposition 2.2. Fix a small \( \varepsilon > 0 \) and let \( S \) be the set consisting of all \( g \in \text{Homeo}_c(\mathbb{D}, \omega) \) whose supports are contained in some topological disc of area \( \varepsilon \). It is a well-known fact that the set \( S \) generates the group \( \text{Homeo}_c(\mathbb{D}, \omega) \). This is usually referred to as the fragmentation property and it was proven by Fathi; see Theorems 6.6, A.6.2, and A.6.5 of [Fat80].

We claim that \( [f, g] \in H \) for any \( f, g \in S \). Indeed, assuming \( \varepsilon \) is small enough, we can find a topological disc \( U \) which contains the supports of \( f \) and \( g \) and whose area is less than the area of \( D' \). There exists \( r \in \text{Homeo}_c(\mathbb{D}, \omega) \) such that \( r(U) \subset D' \). As a consequence, \( rfr^{-1}, rgr^{-1} \) are both supported in \( D' \) and hence, by Lemma 2.2, \( [rfr^{-1}, rgr^{-1}] \in H \). Since \( H \) is a normal subgroup of \( \text{Homeo}_c(\mathbb{D}, \omega) \), and \( [rfr^{-1}, rgr^{-1}] = r[f, g]r^{-1} \), we conclude that \( [f, g] \in H \).

Now, the set \( S \) generates \( \text{Homeo}_c(\mathbb{D}, \omega) \) and \([f, g] \in H \) for any \( f, g \in S \). Hence, the quotient group \( \text{Homeo}_c(\mathbb{D}, \omega)/H \) is abelian. Thus, \( H \) contains the commutator subgroup of \( \text{Homeo}_c(\mathbb{D}, \omega) \). □

3 Periodic Floer Homology and basic properties of the PFH spectral invariants

In this section, we recall the definition of Periodic Floer Homology (PFH), due to Hutchings [Hut02, HS05], and the construction of the spectral invariants which arise from this theory, also due to Hutchings [Hut]. We will then prove that PFH spectral invariants satisfy the Monotonicity, Hofer Continuity, and Spectrality properties which we mentioned in the introduction. The spectral invariants appearing in Section 1.3.3 are defined by identifying area-preserving maps of the disc, \( \text{Diff}_c(\mathbb{D}, \omega) \), with area-preserving maps of the sphere, which are supported in the northern hemisphere \( S^+ \), and using
the PFH of $S^2$. Thus, the three aforementioned properties will follow from related properties about PFH spectral invariants on $S^2$; see Theorem 3.6 below.

3.1 Preliminaries on $J$–holomorphic curves and stable Hamiltonian structures

A stable Hamiltonian structure (SHS) on a closed three-manifold $Y$ is a pair $(\alpha, \Omega)$, consisting of a 1–form $\alpha$ and a closed two-form $\Omega$, such that

1. $\alpha \wedge \Omega$ is a volume form on $Y$,
2. $\ker(\Omega) \subset \ker(d\alpha)$.

Observe that the first condition implies that $\Omega$ is non-vanishing, and as a consequence, the second condition is equivalent to $d\alpha = g\Omega$, where $g : Y \to \mathbb{R}$ is a smooth function.

A stable Hamiltonian structure determines a plane field $\xi := \ker(\alpha)$ and a Reeb vector field $R$ on $Y$ given by

$$R \in \ker(\Omega), \quad \alpha(R) = 1.$$ 

Stable Hamiltonian structures were introduced in [BEH+03, CM05] as a setting in which one can obtain general Gromov-type compactness results, such as the SFT compactness theorem, for pseudo-holomorphic curves in $\mathbb{R} \times Y$. Here are two examples of stable Hamiltonian structures which are relevant to our story.

**Example 3.1.** A contact form on $Y$ is a 1–form $\lambda$ such that $\lambda \wedge d\lambda$ is a volume form. The pair $(\alpha, \Omega) := (\lambda, d\lambda)$ gives a stable Hamiltonian structure with $g \equiv 1$. The plane field $\xi$ is the associated contact structure and the Reeb vector field as defined above gives the usual Reeb vector field of a contact form.

The contact symplectization of $Y$ is

$$X := \mathbb{R} \times Y_\varphi,$$

which has a standard symplectic form, defined by

$$\Gamma = d(e^s\lambda),$$

where $s$ denotes the coordinate on $\mathbb{R}$.

**Example 3.2.** Let $(S, \omega_S)$ be a closed surface and denote by $\varphi$ a smooth area-preserving diffeomorphism of $S$. Define the mapping torus

$$Y_\varphi := \frac{S \times [0, 1]}{(x, 1) \sim (\varphi(x), 0)}.$$
Let $r$ be the coordinate on $[0, 1]$. Now, $Y_\varphi$ carries a stable Hamiltonian structure $(\alpha, \Omega) := (dr, \omega_\varphi)$, where $\omega_\varphi$ is the canonical closed two form on $Y_\varphi$ induced by $\omega_S$. Note that the plane field $\xi$ is given by the vertical tangent space of the fibration $\pi : Y_\varphi \to S^1$ and the Reeb vector field is given by $R = \partial_r$. Here, $g \equiv 0$. Observe that the closed orbits of $R$ are in correspondence with the periodic points of $\varphi$.

We define the symplectization of $Y_\varphi$

$$X := \mathbb{R} \times Y_\varphi,$$

which has a standard symplectic form, defined by

$$\Gamma = ds \wedge dr + \omega_\varphi,$$

(15)

where $s$ denotes the coordinate on $\mathbb{R}$.

We say an almost complex structure $J$ on $X = \mathbb{R} \times Y$ is admissible, for a given SHS $(\alpha, \Omega)$, if the following conditions are satisfied:

1. $J$ is invariant under translation in the $\mathbb{R}$-direction of $\mathbb{R} \times Y$,
2. $J\partial_s = R$, where $s$ denotes the coordinate on the $\mathbb{R}$-factor of $\mathbb{R} \times Y$,
3. $J\xi = \xi$, where $\xi := \text{Ker}(\alpha)$, and $\Omega(v, Jv) > 0$ for all nonzero $v \in \xi$.

We will denote by $\mathcal{J}(\alpha, \Omega)$ the set of almost complex structures which are admissible for $(\alpha, \Omega)$. The space $\mathcal{J}(\alpha, \Omega)$, equipped with the $C^\infty$ topology is path connected, and even contractible.

Define a $J$-holomorphic map to be a smooth map

$$u : (\Sigma, j) \to (X, J),$$

satisfying the equation

$$du \circ j = J \circ du,$$

(16)

where $(\Sigma, j)$ is a closed Riemann surface (possibly disconnected), minus a finite number of punctures. As is common in the literature on ECH, we will sometimes have to consider $J$-holomorphic maps up to equivalence of currents, and we call such an equivalence class a $J$-holomorphic current; see [Hut14] for the precise definition of this equivalence relation. An equivalence class of $J$-holomorphic maps under the relation of biholomorphisms of the domain will be called a $J$-holomorphic curve; this relation might be more familiar to the reader, but is not sufficient for our needs.

For future reference, we will call a $J$-holomorphic curve or current irreducible when its domain is connected. A $J$-holomorphic map $u : (\Sigma, j) \to (X, J)$ is called somewhere injective if there exists a point $z \in \Sigma$ such that $u^{-1}(u(z)) = \{z\}$ and $du : T_z\Sigma \to T_{u(z)}X$ is injective.

In the lemma below we state a standard property of $J$-holomorphic curves which plays a key role in our arguments. For a proof see [Wen], Lemma 9.9, for example.
Lemma 3.3. Suppose $J \in J(\alpha, \Omega)$ where $(\alpha, \Omega)$ is a stable Hamiltonian structure on $Y$. If $C$ is a $J$-holomorphic curve in $\mathbb{R} \times Y$, then $\Omega$ is pointwise nonnegative on $C$. Furthermore, $\Omega$ vanishes at a point on $C$ only if the tangent space to $C$ at the point is in the span of $\partial_s$ and $R$.

3.1.1 Weakly admissible almost complex structures on mapping cylinders

In this article, we will be almost exclusively considering $J$–holomorphic curves and currents in the symplectization $X = \mathbb{R} \times Y$, introduced above in Example 3.2. Now, the usual SHS on $\mathbb{R} \times Y$ is $(dr, \omega \phi)$, defined above, and hence, we will be mostly considering almost complex structures $J$ which are admissible for this SHS, i.e. $J \in J(dr, \omega \phi)$. However, in Section 5 we will need the added flexibility of working with almost complex structures on $\mathbb{R} \times Y$ which are admissible for any SHS of the form $(\alpha, \Omega) = (\alpha, \omega \phi)$; we will refer to such almost complex structures as weakly admissible. Clearly, an admissible almost complex structure is weakly admissible.

We will now list several observations about weakly admissible almost complex structures which will be helpful in Section 5. Let $(\alpha_0, \omega \phi)$ and $(\alpha_1, \omega \phi)$ be SHSs inducing the same orientation and denote their Reeb vector fields by $R_0, R_1$.

1. There exists a positive function $\eta : Y \to \mathbb{R}$ such that $R_0 = \eta R_1$. This is because $R_0, R_1 \in \text{Ker}(\omega \phi)$ and $(\alpha_0, \omega \phi)$ and $(\alpha_1, \omega \phi)$ induce the same orientation. In particular, $R_0, R_1$ have the same closed Reeb orbits.

2. Define $\alpha_t = (1 - t)\alpha_0 + t\alpha_1$. Then, $(\alpha_t, \omega \phi)$ is a SHS for all $t \in [0, 1]$. In other words, the space of all SHS of the above form is convex, hence contractible.

3. As a consequence of the previous, we see the space of weakly admissible almost complex structures is path connected. Indeed, it is even contractible as it forms a fibration, over the space of SHS with $\Omega = \omega \phi$, whose base and fibres are contractible.

4. Lastly, Lemma 3.3 holds for weakly admissible $J$.

3.2 Definition of periodic Floer homology

Periodic Floer homology (PFH) is a version of Floer homology, defined by Hutchings [Hut02, HS05], for area-preserving maps of surfaces. The construction of PFH is closely related to the better-known embedded contact homology (ECH) and, in fact, predates the construction of ECH. We now review the definition of PFH; for further details on the subject we refer the reader to [Hut02, HS05].
Let \((S, \omega_S)\) be a closed\(^\text{\ref{footnote:nondeg}}\) surface with an area form, and \(\varphi\) a \textbf{nondegenerate} smooth area-preserving diffeomorphism. Non-degeneracy is defined as follows: A periodic point \(p\) of \(\varphi\), with period \(k\), is said to be non-degenerate if the derivative of \(\varphi^k\) at the point \(p\) does not have 1 as an eigenvalue. We say \(\varphi\) is \(d\)-nondegenerate if all of its periodic points of period at most \(d\) are non-degenerate; if \(\varphi\) is \(d\)-nondegenerate for all \(d\), then we say it is non-degenerate. A \(C^\infty\)-generic area-preserving diffeomorphism is nondegenerate.

Recall the definition of \(Y_\varphi\) from Example 3.2 and take \(0 \neq h \in H_1(Y_\varphi)\). If \(\varphi\) is nondegenerate and satisfies a certain “monotonicity” assumption, which we do not need to discuss here as it automatically holds when \(S = S^2\), the periodic Floer homology \(PFH(\varphi, h)\) is defined as the homology of a chain complex \(PFC(\varphi, h, J)\) which we define below.

**Remark 3.4.** If we carry out the construction outlined below, nearly verbatim, for a contact SHS structure \((\lambda, d\lambda)\), rather than the SHS \((d\tau, \omega_\varphi)\), then we would obtain the \textbf{embedded contact homology} \(ECH\); see [Hut14] for further details. ▽

### 3.2.1 PFH generators

The chain complex \(PFC\) is freely generated over \(Z_2\), by certain finite orbit sets \(\alpha = \{(\alpha_i, m_i)\}\) called \textbf{PFH generators}. Specifically, we require that each \(\alpha_i\) is a closed embedded orbit of the vector field \(R\), the \(\alpha_i\) are distinct, the \(m_i\) are positive integers, \(m_i = 1\) whenever \(\alpha_i\) is \textit{hyperbolic}\(^\text{\ref{footnote:hyperbolic}}\) and \(\sum m_i[\alpha_i] = h\).

### 3.2.2 The ECH index

The \(Z_2\) vector space \(PFC(\varphi, h, J)\) has a relative \(Z\) grading which we now explain. Let \(\alpha = \{(\alpha_i, m_i)\}, \beta = \{(\beta_j, n_j)\}\) be two \textbf{PFH generators} in \(PFC(\varphi, h)\). Define \(H_2(Y_\varphi, \alpha, \beta)\) to be the set of equivalence classes of 2–chains \(Z\) in \(Y_\varphi\) satisfying \(\partial Z = \sum m_i\alpha_i - \sum n_i\beta_i\). Note that \(H_2(Y_\varphi, \alpha, \beta)\) is an affine space over \(H_2(Y_\varphi)\).

We define the \textbf{ECH index}

\[
I(\alpha, \beta, Z) = c_\tau(Z) + Q_\tau(Z) + \sum_{i} \sum_{k=1}^{m_i} CZ_\tau(\alpha^k_i) - \sum_{j} \sum_{k=1}^{n_j} CZ_\tau(\beta^k_j), \quad (17)
\]

where \(\tau\) is a trivialization of the plane field \(\xi\) over all closed orbits of \(R\), \(c_\tau(Z)\) denotes the relative first Chern class of \(\xi\) over \(Z\), \(Q_\tau(Z)\) denotes the relative

---

\(^{\text{\ref{footnote:nondeg}}}\)PFH can still be defined if \(S\) is not closed, but we will not need this here.

\(^{\text{\ref{footnote:hyperbolic}}}\)Being hyperbolic means that the eigenvalues at the corresponding periodic point of \(\varphi\) are real. If the eigenvalues are non-real, the orbit is called \textit{elliptic}.
intersection pairing, and \( CZ(\gamma^k) \) denotes the Conley-Zehnder index of the \( k^{th} \) iterate of \( \gamma \); all of these quantities are computed using the trivialization \( \tau \). We will review the definitions of \( c_\tau \), \( CZ_\tau \), and \( Q_\tau \) in Section 5.3.

It is proven in [Hut02] that although the individual terms in the above definition do depend on the choice of \( \tau \), the ECH index itself does not depend on \( \tau \). According to Proposition 1.6 of [Hut02], the change in index caused by changing the relative homology class \( Z \) to another \( Z' \in H_2(Y_\varphi, \alpha, \beta) \) is given by the formula

\[
I(\alpha, \beta, Z) - I(\alpha, \beta, Z') = \langle c_1(\xi) + 2PD(h), Z - Z' \rangle.
\]

(18)

### 3.2.3 The differential

Let \( J \in J(\sigma, \omega_\varphi) \) be an admissible almost complex structure for the SHS \( (\sigma, \omega_\varphi) \) and define \( \mathcal{M}^{I=1}_J(\alpha, \beta) \) to be the space of \( J \)-holomorphic currents \( C \), modulo translation in the \( \mathbb{R} \) direction, with ECH index \( I(\alpha, \beta, [C]) = 1 \), which are asymptotic to \( \alpha \) as \( s \to +\infty \) and \( \beta \) as \( s \to -\infty \); we refer the reader to [HS05], page 307, for the precise definition of asymptotic in this context.

Assume now and below for simplicity that \( S = S^2 \). (For other surfaces, a similar story holds, but we will not need this.) Then, for generic \( J \), \( \mathcal{M}^{I=1}_J(\alpha, \beta) \) is a compact 0-dimensional manifold and we can define the PFH differential by the rule

\[
\langle \partial_{\alpha, \beta} \rangle = \# \mathcal{M}^{I=1}_J(\alpha, \beta),
\]

(19)

where \( \# \) denotes mod 2 cardinality. It is shown in [HT09] that \( \partial^2 = 0 \), hence the homology \( PFH \) is defined.

Lee and Taubes [LT12] proved that the homology does not depend on the choice of \( J \); in fact, Corollary 1.1 of [LT12] states that for any surface, the homology depends only on the Hamiltonian isotopy class of \( \varphi \) and the choice of \( h \in H_1(Y_\varphi) \). In our case, where \( S = S^2 \), all area-preserving diffeomorphisms are Hamiltonian isotopic, and so we obtain a well-defined invariant which we denote by \( PFH(Y_\varphi, h) \). For future motivation, we note that the Lee-Taubes invariance results discussed here come from an isomorphism of PFH and a version of the Seiberg-Witten Floer theory from [KM07].

Importantly, for the applications to this paper, we can relax the assumption that \( \varphi \) is nondegenerate to requiring only that \( \varphi \) is \( d \)-nondegenerate, where \( d \), called the degree, is the positive integer determined by the intersection of \( h \) with the fiber class of the map \( \pi : Y_\varphi \to S^1 \); note that any orbit

\footnote{More precisely, [HT09] proves that the differential in embedded contact homology squares to zero. As pointed out in [LT12] this proof carries over, nearly verbatim, to our setting.}
set $\alpha$ with $[\alpha] = h$ must correspond to periodic points with period no more than $d$.

### 3.2.4 The structure of PFH curves

We now explain part of the motivation for PFH, and for the ECH index $I$; we will use some of the results below later in the paper as well. For additional details on the account here, we refer the reader to [Hut14], for example.

Let $C$ be a $J$–holomorphic curve which is asymptotic to orbit sets $\alpha, \beta$ and denote $I([C]) := I(\alpha, \beta, [C])$. A key fact which powers the definition of PFH is the index inequality

$$\text{ind}(C) \leq I([C]) - 2\delta(C),$$

valid for any somewhere injective curve $C$, where $\delta(C) \geq 0$ is a count of singularities of $C$; see [Hut14] for the precise definition of $\delta(C)$; here $\text{ind}(C)$ refers to the Fredholm index

$$\text{ind}(C) = -\chi(C) + 2c_\tau(C) + CZ^{\text{ind}}(C),$$

where $\chi(C)$ denotes the Euler characteristic of $C$, and $CZ^{\text{ind}}$ is another combination of Conley-Zehnder terms that we will review in Section 5.3. The Fredholm index is the formal dimension of the moduli space of curves near $C$, and (20) therefore says that the ECH index bounds this dimension from above. In the case relevant here, when $I([C]) = 1$, we therefore get an important structure theorem for ECH index 1 curves. To simplify the exposition, we assume here and below that $S = S^2$.

**Proposition 3.5** (Lem. 9.5, [Hut02], Cor. 2.2 [HS05]). Assume that $J$ is admissible and generic, and let $C$ be a $J$-holomorphic curve with degree $d > 0$. Assume that $\phi$ is $d$-nondegenerate. Then $I([C]) \geq 0$, and if $I([C]) = 1$, then $C$ has exactly one embedded component $C'$ with $I(C') = 1$, and all other components, if they exist, are covers of $\mathbb{R}$-invariant cylinders that do not intersect $C'$.

We end this section by mentioning that for any SHS $(\alpha, \Omega)$ on $Y$, the formula (21) still gives the formal dimension of the moduli space of curves near $C$ when $J \in \mathcal{J}(\alpha, \Omega)$; see Sections 7 and 8 of [Wen]; later, we will make use of this.

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8In [Hut02], one also wants to assume a “local linearity” condition around periodic points; however, this condition is not necessary — as was stated in [Hut02], this condition was added to simplify the analysis, and it can be dropped, as in [LT12], using work of Siefring [Sie08, Sie11].
3.3 Twisted PFH and the action filtration

One can define a twisted version of PFH, where we keep track of the relative homology classes of $J$–holomorphic curves, that we will need to define spectral invariants. It has the same invariance properties of ordinary PFH, e.g. by Cor. 6.7, [LT12], where it is shown to agree with an appropriate version of Seiberg-Witten Floer cohomology. For the benefit of the reader, we provide a brief explanation of how the twisted version works in relationship with Seiberg-Witten theory in Remark 3.13 below. We refer the reader to [HS06] 11.2.1, [Tau10] 1.a.2.

The reason we want to use twisted PFH is because, for example, while PFH does not have a natural action filtration, the twisted PFH does. As above, we are continuing to assume $S = S^2$.

First, note that in this case, $Y_{\varphi}$ is diffeomorphic to $S^2 \times S^1$ and so $H_1(Y_{\varphi}) = \mathbb{Z}$. A class $h \in H_1(Y_{\varphi})$ is then determined by its intersection with the homology class of a fiber of the map $\pi : Y_{\varphi} \to S^1$, which we defined above to be the degree and denote by the integer $d$; from now on we will write the integer $d$ in place of $h$, since these two quantities determine each other.

Choose a reference cycle $\gamma_0$ in $Y_{\varphi}$ such that $\pi|_{\gamma_0} : \gamma_0 \to S^1$ is an orientation preserving diffeomorphism and fix a trivialization $\tau_0$ of $\xi$ over $\gamma_0$. We can now define the $\widetilde{PFH}$ chain complex $\widetilde{PFC}(\varphi, d)$. A generator of $\widetilde{PFC}(\varphi, d)$ is a pair $(\alpha, Z)$, where $\alpha$ is a PFH generator of degree $d$, and $Z$ is a relative homology class in $H_2(Y_{\varphi}, \alpha, d\gamma_0)$. The $\mathbb{Z}_2$ vector space $\widetilde{PFC}(\varphi, d)$ has a canonical $\mathbb{Z}$-grading $I$ given by

$$I(\alpha, Z) = c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k). \quad (22)$$

The terms in the above equation are defined as in the definition of the ECH index given by Equation (17). Note that the above index depends on the choice of the reference cycle $\gamma_0$ and the trivialization $\tau_0$ of $\xi$ over $\gamma_0$.

The index defined here is closely related to the ECH index of Equation (17): Let $(\alpha, Z)$ and $(\beta, Z')$ be two generators of $\widetilde{PFC}(\varphi, d)$. Note that $Z - Z'$ is a relative homology class in $H_2(Y_{\varphi}, \alpha, \beta)$. Then, it follows from Proposition 1.6 of [Hut02] that

$$I(\alpha, \beta, Z - Z') = I(\alpha, Z) - I(\beta, Z').$$

As a consequence, we see that the index difference $I(\alpha, Z) - I(\beta, Z')$ does not depend on the choices involved in the definition of the index.

We now define the differential of $\widetilde{PFC}(\varphi, d)$. We say that $C$ is a $J$–holomorphic curve, or current, from $(\alpha, Z)$ to $(\beta, Z')$ if it is asymptotic to $\alpha$ as $s \to +\infty$ and $\beta$ as $s \to -\infty$, and moreover satisfies
\[ Z' + [C] = Z, \]  

as elements of \( H_2(Y, \alpha, d\gamma_0) \).

Suppose that \( I(\alpha, Z) - I(\beta, Z') = 1 \) and let \( J \in J(dr, \omega_\varphi) \). We define \( \mathcal{M}_J((\alpha, Z), (\beta, Z')) \) to be the moduli space of \( J \)-holomorphic currents in \( X = \mathbb{R} \times Y_\varphi \), modulo translation in the \( \mathbb{R} \) direction, from \( (\alpha, Z) \) to \( (\beta, Z') \). As before, for generic \( J \in J(dr, \omega_\varphi) \), the above moduli space is a compact 0–dimensional manifold and we define the differential by the rule

\[ \langle \partial(\alpha, Z), (\beta, Z') \rangle = \# \mathcal{M}_J((\alpha, Z), (\beta, Z')), \]

where \( \# \) denotes mod 2 cardinality. As before, \( \partial^2 = 0 \) by [HT09], and so the homology \( \overline{PFH} \) is well-defined; by [LT12] it depends only on the degree \( d \). We will write it as \( \overline{PFH}(Y_\varphi, d) \).

By a direct computation in the case where \( \varphi \) is an irrational rotation of the sphere, i.e. \( \varphi(z, \theta) = (z, \theta + \alpha) \) with \( \alpha \) being irrational, we obtain

\[ \overline{PFH}_*(Y_\varphi, d) = \begin{cases} \mathbb{Z}_2, & \text{if } * = d \mod 2, \\ 0, & \text{otherwise}. \end{cases} \]  

(24)

The vector space \( \overline{PFC}(\varphi, d) \) carries a filtration, called the action filtration\(^9\), defined by

\[ A(\alpha, Z) = \int_Z \omega_\varphi. \]

We define \( \overline{PFC}(\varphi, d) \) to be the \( \mathbb{Z}_2 \) vector space spanned by generators \( (\alpha, Z) \) with \( A(\alpha, Z) \leq L \).

By Lemma 3.3, \( \omega_\varphi \) is pointwise nonnegative along any \( J \)-holomorphic curve \( C \), and so \( \int_C \omega_\varphi \geq 0 \). This implies that the differential decreases the action filtration, i.e.

\[ \partial(\overline{PFC}_L(\varphi, d)) \subset \overline{PFC}_L(\varphi, d). \]

Hence, it makes sense to define \( \overline{PFH}_L(\varphi, d) \) to be the homology of the subcomplex \( \overline{PFC}_L(\varphi, d) \).

We are now in position to define the PFH spectral invariants. There is an inclusion induced map

\[ \overline{PFH}_L(\varphi, d) \to \overline{PFH}(Y_\varphi, d). \]  

(25)

\(^9\)The relation between the quantity \( A(\alpha, Z) \) and the Hamiltonian action functional discussed in Section 2.4 will be clarified in Lemma 5.10.
If \( 0 \neq \sigma \in \widetilde{PFH}(\varphi, d) \) is any nonzero class, then we define the PFH spectral invariant \( c_\sigma(\varphi) \) to be the infimum, over \( L \), such that \( \sigma \) is in the image of the inclusion induced map \( (25) \) above. We remark that \( c_\sigma(\varphi) \) is given by the action of some \((\alpha, Z)\). Indeed, this is a consequence of the following observation:

If \( L < L' \) are such that there exists no \((\alpha, Z)\) with \( L \leq A(\alpha, Z) \leq L' \), then the two vector spaces \( \widetilde{PFC}^L(\varphi, d) \) and \( \widetilde{PFC}^{L'}(\varphi, d) \) coincide and so \( \widetilde{PFH}^L(\varphi, d) \rightarrow \widetilde{PFH}(Y_\varphi, d) \) and \( \widetilde{PFH}^{L'}(\varphi, d) \rightarrow \widetilde{PFH}(Y_\varphi, d) \) have the same image.

In Remark 3.11 below we show that this does not depend on the choice of the admissible almost complex structure \( J \). Note, however, that \( c_\sigma(\varphi) \) does depend on the choice of the reference cycle \( \gamma_0 \).

### 3.4 Initial properties of PFH spectral invariants

Let \( p_\varphi = (0, 0, -1) \in S^2 \). We denote

\[
S := \{ \varphi \in \text{Diff}(S^2, \omega) : \varphi(p_\varphi) = p_\varphi \}.
\]

Recall from the previous section that the spectral invariant \( c_\sigma \) depends on the choice of a reference cycle \( \gamma_0 \) in \( Y_\varphi \). For \( \varphi \in S \), we set the reference cycle \( \gamma_0 \) to be the preimage of \( p_- \) under \( \pi : Y_\varphi \rightarrow S^1 \); this is the Reeb orbit in \( Y_\varphi \) corresponding to \( p_- \). The grading on \( \widetilde{PFH} \) depends on the choice of \( \tau_0 \); fix a trivialization \( \tau_0 \).

Having set the above convention, we now define PFH spectral invariants without ambiguity. Suppose that \( \varphi \in S \) is non-degenerate. According to Equation (24), for every pair \((d, k)\) with \( k = d \mod 2 \), we have a distinguished nonzero class \( \sigma_{d,k} \) with degree \( d \) and grading \( k \), and so we can define

\[
c_{d,k}(\varphi) := c_{\sigma_{d,k}}(\varphi).
\]

Lastly, we also define

\[
c_d(\varphi) = \begin{cases} 
c_{d,0}(\varphi), & \text{if } d = 0 \mod 2, \\
c_{d,1}(\varphi), & \text{if } d = 1 \mod 2.
\end{cases}
\]

Denote

\[
\mathcal{H} := \{ H \in C^\infty(S^1 \times S^2) : \varphi^*_H(p_-) = p_-, H(t, p_-) = 0 \ \forall t \in S^1 \},
\]

and observe that \( S = \{ \varphi^*_H : H \in \mathcal{H} \} \). The theorem below, which is the main result of this section, establishes some of the key properties of the PFH spectral invariant and furthermore allows us to extend the definition of these invariants to all, possibly degenerate, \( \varphi \in S \).
Theorem 3.6. The PFH spectral invariants $c_{d,k}(\varphi)$ can be extended to all $\varphi \in S$ in a manner such that the following properties are satisfied:

1. Monotonicity: Suppose that $H \leq G$, where $H, G \in \mathcal{H}$. Then,
   $$c_{d,k}(\varphi_H^1) \leq c_{d,k}(\varphi_G^1).$$

2. Hofer Continuity: For any $H, G \in \mathcal{H}$, we have
   $$|c_{d,k}(\varphi_H^1) - c_{d,k}(\varphi_G^1)| \leq d\|H - G\|_{(1,\infty)}.$$

3. Spectrality: $c_{d,k}(\varphi_H^1) \in \text{Spec}_d(H)$ for any $H \in \mathcal{H}$.

Remark 3.7. Given a degenerate $\varphi \in S$, take a sequence of non-degenerate $\varphi_i \in S$ converging to $\varphi$ in the $C^\infty$ topology. As we will see in the course of the proof, $c_{d,k}(\varphi)$ is given by
   $$c_{d,k}(\varphi) = \lim_i c_{d,k}(\varphi_i).$$

In the case where $\varphi = \text{Id}$, one can take $\varphi_i$ to be irrational rotations of the sphere. By direct computation, $c_{d,k}(\varphi_i) \to 0$ and so
   $$c_{d,k}(\text{Id}) = 0.$$

Alternatively, one could deduce that $c_{d,k}(\text{Id}) = 0$ as a direct consequence of Theorem 6.1 below.

Remark 3.8. To define the PFH spectral invariant $c_{d,k}$ for $\varphi \in \text{Diff}_c(D,\omega)$, we use Equation (7) to identify $\text{Diff}_c(D,\omega)$ with area-preserving diffeomorphisms of the sphere which are supported in the interior of the northern hemisphere $S^+_1$, and we take as $\tau_0$ the trivialization of $\xi$ given by pulling back a a fixed frame of $T_{S^1}S^2$. We similarly define $c_d : \text{Diff}_c(D,\omega) \to \mathbb{R}$ which was introduced in Section 1.3.3. It is clear that $c_d : \text{Diff}_c(D,\omega) \to \mathbb{R}$ satisfies the properties listed in Section 1.3.3.

The rest of this section is dedicated to the proof of the above theorem. The proof requires certain preliminaries. First, it will be convenient to identify $Y_\varphi$ with $S^1 \times S^2$. Pick $H \in \mathcal{H}$ such that $\varphi = \varphi_H^1$. We define
   $$S^1 \times S^2 \to Y_\varphi$$
   $$(t,x) \mapsto ((\varphi_H^1)^{-1}(x),t),$$
   where $t$ denotes the variable on $S^1$. This identifies the Reeb vector field on $Y_\varphi$ with the vector field
   $$\partial_t + X_H$$
   (27)
on $S^1 \times S^2$. The 2-form $\omega_\varphi$ pulls back under this map to the form

$$\omega + dH \wedge dt$$

where $\omega$ is the area form on $S^2$. This allows us to identify $\mathbb{R} \times S^1 \times S^2$ with the symplectization $X$ via

$$\mathbb{R} \times S^1 \times S^2 \to X$$

$$(s, t, x) \mapsto (s, (\varphi_H^t)^{-1}(x), t).$$

The symplectic form $\Gamma$ on $X$ then pulls back to

$$\omega_H = ds \wedge dt + \omega + dH \wedge dt.$$  \hspace{1cm} (28)

Next, let $H, K$ be two Hamiltonians in $\mathcal{S}$. As mentioned earlier, $\widehat{PFH}(\varphi_H^t, d)$ is isomorphic to $\widehat{PFH}(\varphi_K^t, d)$. The proof of this uses Seiberg-Witten theory, and is carried out in Cor. 6.1 of [LT12], see also Remark 3.13 below. With the choice of a reference cycle in $H_2(S^1 \times S^2)$, which we have taken to be the cycle corresponding to $\gamma_0 = \{p_-\} \times S^1$, one obtains a canonical isomorphism

$$\widehat{PFH}(\varphi_H^t, d) \to \widehat{PFH}(\varphi_K^t, d),$$  \hspace{1cm} (29)

which preserves the $\mathbb{Z}$-grading on $\widehat{PFH}$.

As is generally the case with related invariants, one might expect this isomorphism to be induced by a chain map counting certain $J$-holomorphic curves. In fact, it is not currently known how to define the map $\text{(29)}$ this way; the construction uses Seiberg-Witten theory. Nevertheless, the map in $\text{(29)}$ does satisfy a “holomorphic curve” axiom which was proven by Chen [Che18] using variants of Taubes’ “Seiberg-Witten to Gromov” arguments in [Tau96]. A similar “holomorphic curve” axiom was proven in the context of embedded contact homology by Hutchings-Taubes; we compare the Chen proof to the Hutchings-Taubes one in Remark 3.12 below.

To state this holomorphic curve axiom in our context, take Hamiltonians $H, K \in \mathcal{S}$, and define for $s \in \mathbb{R}$

$$G_s = K + \beta(s) \cdot (H - K)$$

where $\beta : \mathbb{R} \to [0, 1]$ is some non-decreasing function that is 0 for $s$ sufficiently negative and 1 for $s$ sufficiently positive. Now consider the form

$$\omega_X = ds \wedge dt + \omega + dG \wedge dt,$$

where, as throughout this article, $dG$ denotes the derivatives in the $S^2$ directions. This is a symplectic form on $\mathbb{R} \times S^1 \times S^2$. Observe that, for $s >> 0$, the form $\omega_X$ agrees with the symplectization form $\omega_H$, and for $s << 0$, it
agrees with the symplectization form $\omega_K$. Let $J_X$ be any $\omega_X$-compatible\textsuperscript{10} almost complex structure that agrees with a generic $(dt, \omega_H)$ admissible almost complex structure $J_+$ for $s >> 0$ and with a generic $(dt, \omega_K)$ admissible almost complex structure $J_-$ for $s << 0$.

Then, the holomorphic curve axiom says that (29) is induced by a chain map

$$\Psi_{H,K} : \tilde{PFC}(\varphi^1_H, d, J_+) \to \tilde{PFC}(\varphi^1_K, d, J_-),$$

with the property that if $\langle \Psi_{H,K}(\alpha, Z), (\beta, Z') \rangle \neq 0$, then there is a $J_X$-holomorphic building $C$ from $\alpha$ to $\beta$ such that

$$Z' + [C] = Z,$$

as elements of $H_2(S^1 \times S^2, \alpha, d\gamma_0)$; we say more about this in Remark 3.13 below. Here, by a $J_X$-holomorphic building from $\alpha$ to $\beta$, we mean a sequence of $J_i$-holomorphic curves

$$(C_0, \ldots, C_i, \ldots, C_k),$$

such that the negative asymptotics of $C_i$ agree with the positive asymptotics of $C_{i+1}$, the curve $C_0$ is asymptotic to $\alpha$ at $+\infty$, and the curve $C_k$ is asymptotic to $\beta$ at $-\infty$; we refer the reader to [Hut14], Page 68 for more details.

We remark for future reference that the $C_i$ are called \textbf{levels}, and each $J_i$ is either $J_X$, $J_+$ or $J_-$.\textsuperscript{11}

We will want to assume that $J_X$ is \textbf{compatible with the fibration} $\mathbb{R} \times S^1 \times S^2 \to \mathbb{R} \times S^1$ in the following sense: Let $\mathbb{V}$ be the vertical tangent bundle of this fibration and denote by $\mathbb{H}$ the $\omega_X$-orthogonal complement of $\mathbb{V}$; observe that $\mathbb{H}$ is spanned by the vector fields $\partial_s$ and $\partial_t + X_G$. Then, we will want $J_X$ to preserve $\mathbb{V}$ and $\mathbb{H}$. Given any admissible $J_{\pm}$ on the ends, we can achieve this as follows. On the horizontal tangent bundle $\mathbb{H}$, we always demand that $J_X$ sends $\partial_s$ to $\partial_t + X_G$. On the vertical tangent bundle, we observe that $\omega_X|\mathbb{V} = \omega$, and in particular $\omega_X|\mathbb{V}$ is independent of $s$; we can then connect $J_+|\mathbb{V}$ to $J_-|\mathbb{V}$ through a path of $\omega$-tamed almost complex structures on $\mathbb{V}$.

We can now prove Theorem 3.6.

\textbf{Proof of Theorem 3.6}. We begin by first supposing that the three listed properties hold when $\varphi^1_K, \varphi^1_H$ are nondegenerate and explain how this implies the theorem in the degenerate case. To that end, let $H \in \mathcal{H}$, not necessarily nondegenerate, and take a sequence $H_i \in \mathcal{H}$ which $C^2$ converges to $H$ and such that $\varphi^1_{H_i}$ is nondegenerate. Then, we define

$$c_{d,k}(\varphi^1_H) = \lim_{i \to \infty} c_{d,k}(\varphi^1_{H_i}).$$

\textsuperscript{10}Recall that an almost complex structure $J$ is \textbf{compatible} with a symplectic form $\omega$ if $g(u, v) := \omega(u, Jv)$ defines a Riemannian metric.

\textsuperscript{11}More can be said, but we will not need this additional information.
This limit exists thanks to the inequality $|c_{d,k}(\varphi^1_H) - c_{d,k}(\varphi^1_{H_j})| \leq d\|H_i - H_j\|_{(1,\infty)}$. Moreover, the same inequality implies that the limit value does not depend on the choice of the sequence $H_i$ and so $c_{d,k}(\varphi^1_H)$ is well-defined for all $H \in \mathcal{H}$. Thus, we obtain a well-defined mapping

$$c_{d,k} : S \rightarrow \mathbb{R}.$$ 

It can be seen that $c_{d,k}$ continues to satisfy the monotonicity and Hofer continuity properties for degenerate $\varphi^1_K, \varphi^1_H$. The spectrality property is also satisfied; this is a consequence of the Arzela-Ascoli theorem under the assumption that spectrality is satisfied in the nondegenerate case; we will not provide the details here.

For the rest of the proof we will suppose that $\varphi^1_H, \varphi^1_K$ are nondegenerate. We will now prove the monotonicity and Hofer continuity properties. Let $(\alpha_1, Z_1) + \ldots + (\alpha_k, Z_m)$ be a cycle in $\tilde{PFC}(\varphi^1_H, d)$ representing $\sigma_{d,k}$, with

$$c_{\sigma_{d,k}}(\varphi^1_H) = A(\alpha_1, Z_1) \geq \ldots \geq A(\alpha_m, Z_m).$$

Let $(\beta, Z')$ be a generator in $\tilde{PFC}(\varphi^1_K, d)$ which has maximal action among generators which appear with a non-zero coefficient in

$$\Psi_{H,K} ((\alpha_1, Z_1) + \ldots + (\alpha_k, Z_m)).$$

Then, by the holomorphic curve axiom there is a $J_X$-holomorphic building from some $(\alpha_i, Z_i)$ to $(\beta, Z')$. For the rest of the proof we will write $(\alpha_i, Z_i) = (\alpha, Z)$ and will denote the $J_X$-holomorphic building by $C$.

For the arguments below, which only involve energy estimates, we can assume that $C$ consists of a single level – in other words, is an actual curve, rather than a building – so to simplify the notation, we assume this.

For the remainder of the proof we will need the following Lemma.

**Lemma 3.9.** *The following identity holds:*

$$A(\alpha, Z) - A(\beta, Z') = \int_C \omega + dG \wedge dt + G' ds \wedge dt,$$

where $G'$ denotes $\frac{\partial G}{\partial s}$. *Furthermore, we have*

$$\int_C \omega + dG \wedge dt \geq 0.$$

*Proof of Lemma* We will begin by proving that

$$A(\alpha, Z) - A(\beta, Z') = \int_C \omega + d(G dt), \quad (32)$$

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which establishes the first item because \( \omega + d(G dt) = \omega + dG \wedge dt + G' ds \wedge dt \). Note that we can write

\[
\mathcal{A}(\alpha, Z) = \int_Z \omega + d(H dt), \quad \mathcal{A}(\beta, Z') = \int_{Z'} \omega + d(K dt).
\]

Hence, Equation (32) would follow if we show that

\[
\int_C \omega = \int_Z \omega - \int_{Z'} \omega, \quad \text{and} \quad \int_C d(G dt) = \int_Z d(H dt) - \int_{Z'} d(K dt).
\]

The first identity holds because all of these integrals are determined by the homology classes, and we have \([C] = Z - Z'\). The second identity follows from the following chain of identities:

\[
\int_C d(G dt) = \int_\alpha G dt - \int_\beta G dt = \int_\alpha H dt - \int_\beta K dt = \int_Z d(H dt) - \int_{Z'} d(K dt),
\]

where the first equality holds by Stokes’ theorem, the second follows from the definition of \(G\), and the third is a consequence of Stokes’ theorem combined with the fact that \(H, K\) both belong to \(\mathcal{H}\) and so vanish on \(\gamma_0\). This completes the proof of the first item in the lemma.

Now, we will show that \(\int_C (\omega + dG \wedge dt) \geq 0\) by showing that the form \(\omega + dG \wedge dt\) is pointwise non-negative along \(C\). Indeed, at any point \(p \in X\), we can write any vector as \(v + h\), where \(v \in \mathcal{V}\) and \(h \in \mathcal{H}\) are vertical and horizontal tangent vectors as described in the paragraph before the proof of Theorem 3.6. Since \(C\) is \(J_X\)-holomorphic, it is sufficient to show that \((\omega + dG \wedge dt)(v + h, J_X v + J_X h) \geq 0\). We will show that

\[
(\omega + dG \wedge dt)(v + h, J_X v + J_X h) = \omega_X(v, J_X v),
\]

which proves the inequality because \(J_X\) is \(\omega_X\)-tame. Now, to simplify our notation we will denote \(\Omega = \omega + dG \wedge dt\) for the rest of the proof. Expanding the left hand side of the above equation we get

\[
\Omega(v + h, J_X v + J_X h) = \Omega(v, J_X v) + \Omega(h, J_X v) + \Omega(v, J_X h) + \Omega(h, J_X h).
\]

We will prove below that \(\Omega(v, J_X v) = \omega_X(v, J_X v)\) and \(\Omega(h, J_X h) = \Omega(v, J_X h) = \Omega(h, J_X v) = 0\) which clearly implies Equation (33). Note that \(v\) and \(J_X v\) are in the kernel of \(ds \wedge dt\), and this in turn implies that \(\Omega(v, J_X v) = \omega_X(v, J_X v)\), \(\Omega(h, J_X h) = \Omega(h, J_X v) = 0\). It remains to show that \(\Omega(h, J_X h) = 0\), that is \(\Omega|_{\mathcal{H}} = 0\). This follows from the fact that \(\mathcal{H}\) is spanned by \(\{\partial_s, \partial_t + X_G\}\) and \(\partial_s\) is in the kernel of \(\Omega\). Indeed, a 2–form on a 2–dimensional vector space with non-trivial kernel is identically zero. \(\Box\)
Note that \( c_{d,k}(\varphi^1_H) \geq \mathcal{A}(\alpha, Z) \) and \( c_{d,k}(\varphi^1_K) \leq \mathcal{A}(\beta, Z') \). Hence,

\[
c_{d,k}(\varphi_H) - c_{d,k}(\varphi_K) \geq \mathcal{A}(\alpha, Z) - \mathcal{A}(\beta, Z'). \tag{34}
\]

As a consequence of this inequality, Monotonicity would follow from proving that if \( H \geq K \), then \( \mathcal{A}(\alpha, Z) - \mathcal{A}(\beta, Z') \geq 0 \). By the above lemma we have

\[
\mathcal{A}(\alpha, Z) - \mathcal{A}(\beta, Z') \geq \int_C G' \, ds \wedge dt. \tag{35}
\]

If \( H \geq K \), then \( G' \geq 0 \), and so \( \int_C G' \, ds \wedge dt \geq 0 \), which proves Monotonicity.

As for Hofer Continuity, it is sufficient to show that

\[
\left| \int_C G' \, ds \wedge dt \right| \leq d\|H - K\|_{(1,\infty)}. \tag{36}
\]

Indeed, this inequality combined with Inequalities (34) & (35) implies that

\[
c_{d,k}(\varphi^1_H) - c_{d,k}(\varphi^1_K) \leq d\|H - K\|_{(1,\infty)}.
\]

Similarly, one gets \( c_{d,k}(\varphi^1_H) - c_{d,k}(\varphi^1_K) \leq d\|K - H\|_{(1,\infty)} \) which then implies Hofer Continuity. It remains to prove Inequality (36). We know that

\[
\left| \int_C G' \, ds \wedge dt \right| = \left| \int_C \beta'(s)(H - K) \, ds \wedge dt \right| \leq \int_C \beta'(s)|H - K| \, ds \wedge dt.
\]

Note that because \( H, K \) both vanish at the point \( p_- \), for all \( t, x \) we have

\[
|H(t, x) - K(t, x)| \leq \max_{x \in S^2} (H(t, \cdot) - K(t, \cdot)) - \min_{x \in S^2} (H(t, \cdot) - K(t, \cdot)).
\]

Hence, we get

\[
\left| \int_C G' \, ds \wedge dt \right| \leq \int_C \beta'(s) \left( \max_{x \in S^2} (H - K) - \min_{x \in S^2} (H - K) \right) \, ds \wedge dt.
\]

We can evaluate the second integral by projecting \( C \) to the \((s,t)\) plane. The projection of \( C \) to the \( t \) factor has degree \( d \), and so the second integral evaluates to

\[
d\|H - K\|_{(1,\infty)}.
\]

This completes the proof of Hofer Continuity.

It remains to prove Spectrality. As stated in Section 3.3, the spectral invariant \( c_{d,k}(\varphi^1_H) \) is the action of a PFH generator \((\alpha, Z)\) of degree \( d \). Spectrality, hence Theorem 3.6, is then a consequence of the following lemma.

**Lemma 3.10.** Let \((\alpha, Z)\) be a PFH generator of degree \( d \) for the mapping torus \( Y_{\varphi} \) of \( \varphi = \varphi^1_H \) with \( H \in \mathcal{H} \). Then, \( \mathcal{A}(\alpha, Z) \) belongs to \( \text{Spec}_d(H) \), as defined in Section 2.2.
Proof. For every orbit set $\alpha$ we will construct a specific relative class $Z_\alpha \in H_2(Y_\varphi, \alpha, \gamma_0)$ and will show that $\mathcal{A}(\alpha, Z_\alpha) \in \text{Spec}_d(H)$. Any other $Z \in H_2(Y_\varphi, \alpha, \gamma_0)$ is of the form $Z_\alpha + k[S^2]$ where $k \in \mathbb{Z}$. Hence, $\mathcal{A}(\alpha, Z) = \mathcal{A}(\alpha, Z_\alpha) + k$ and so we get that $\mathcal{A}(\alpha, Z) \in \text{Spec}_d(H)$ for all $Z$.

First, suppose that $d = 1$. Let $q \in \text{Fix}(\varphi)$ and suppose that $\alpha$ is the Reeb orbit in the mapping cylinder corresponding to $q$. The relative cycle $Z_\alpha$ will be of the form $Z_\alpha = Z_0 + Z_1 + Z_2$. We begin by choosing a path $\eta$ in $S^2 \times \{0\} \subset Y_{\varphi_H}$ such that $\partial \eta = (q,0) - (p_-,0)$. We parametrize this curve with a variable $x \in [0,1]$. We define $Z_0$ to be the chain induced by the map

$$[0,1]^2 \to Y_{\varphi_H}^1, \quad (x,t) \mapsto (\eta(x),t).$$

Its boundary is given by $\partial Z_0 = \alpha - \gamma_0 + (\eta,0) - (\varphi(\eta),0)$. Note that $\int_{Z_0} \omega_\varphi = 0$.

We define $Z_1$ to be the chain induced by the map

$$[0,1]^2 \to Y_{\varphi_H}^1, \quad (t,x) \mapsto (\varphi_H^t(\eta(x)),0).$$

Then, $\partial Z_1 = (\varphi(\eta),0) - (\eta,0) - (\varphi_H^t(q),0)$. We can compute $\int_{Z_1} \omega_\varphi$ as follows:

$$\int_{Z_1} \omega_\varphi = \int \int_{[0,1]^2} \omega(\partial_t \varphi_H^t(\eta(x)), \partial_x \varphi_H^t(\eta(x)))$$

$$= \int \int_{[0,1]^2} \omega(X_{H_1}(\varphi_H^t(\eta(x))), \partial_x \varphi_H^t(\eta(x)))$$

$$= \int \int_{[0,1]^2} dH_1(\partial_x \varphi_H^t(\eta(x))) = \int \int_{[0,1]^2} \partial_x H_1(\varphi_H^t(\eta(x)))$$

$$= \int_0^1 H_1(\varphi_H^t(q)) - H_1(\varphi_H^t(p_-))dt = \int_0^1 H_1(\varphi_H^t(q))dt.$$

Next, we define $Z_2$ to be the chain induced by the map $(u_{\alpha},0)$ where $u_{\alpha} : D^2 \to S^2$ is such that $u_{\alpha}|_{\partial D^2}$ is the Hamiltonian orbit $t \mapsto \varphi_H^t(q)$. Then, $\partial Z_2 = (\varphi_H^t(q),0)$ and $\int_{Z_2} \omega_\varphi = \int_{D^2} u_{\alpha}^* \omega$. We will furthermore require $u_\alpha$ to satisfy the following additional properties which will be used in Section 5.2

(i) If $\alpha = \gamma_0$, the Reeb orbit corresponding to $p_-$, then we take $u_{\alpha}$ to be the constant disc with image $p_-$.

(ii) If $\alpha \neq \gamma_0$, then we take $u_{\alpha}$ such that its image does not contain $p_-$.

Finally, we set $Z_\alpha = Z_0 + Z_1 + Z_2$. Adding up the above quantities we obtain $\partial Z_\alpha = \partial Z_0 + \partial Z_1 + \partial Z_2 = \alpha - \gamma_0$ and

$$\mathcal{A}(\alpha, Z_\alpha) = \int_{Z_0} \omega_\varphi + \int_{Z_1} \omega_\varphi + \int_{Z_2} \omega_\varphi = \int_{D^2} u_{\alpha}^* \omega + \int_0^1 H_1(\varphi_H^t(q))dt. \quad (37)$$
Clearly, $A(\alpha, Z_\alpha) \in \text{Spec}(H)$.

Next, let $q$ be a periodic point of period $m \in \mathbb{N}$ and suppose that $\alpha$ is the Reeb orbit in the mapping cylinder corresponding to $q$. Then, $q$ is a fixed point of $\varphi_H^m$. Consider the mapping torus $Y_{\varphi_H^m}$. There is a map $c : Y_{\varphi_H^m} \to Y_{\varphi_1^m}$, pulling back $\omega_{\varphi_1^m}$ to $\omega_{\varphi_H^m}$, given by mapping each interval $S^2 \times \left[\frac{k}{m}, \frac{k+1}{m}\right]$ onto $Y_\varphi$ via the map

$$(x, t) \mapsto (\varphi_H^k(x), m \cdot t - k).$$

Then, we can repeat the construction in the previous paragraph to associate a relative cycle $Z'$ in $Y_{\varphi^m}$. The map $\varphi_H^m$ is generated by the Hamiltonian $H^m$, so by the above discussion we have $\int_{Z'} \omega_{\varphi_H^m} \in \text{Spec}(H^m)$. Now, we define $Z_\alpha$ to be the pushforward of $Z'$ under the map $c$. We leave to the reader to check that $Z_\alpha$ is a relative cycle in $Y_\varphi$, from $\alpha$ to $m\gamma_0$ with $\int_{Z_\alpha} \omega_{\varphi_H^m} = \int_{Z'} \omega_{\varphi_H^m}$.

We note here that a calculation analogous to what was done above yields

$$A(\alpha, Z_\alpha) = \int_{D^2} u_\alpha^* \omega + \int_0^m H_t(\varphi_H^t(q)) dt,$$

where $u_\alpha$ is a disc whose boundary is the Hamiltonian orbit $t \mapsto \varphi_H^t(q), t \in [0, m]$, which satisfies the analogues of properties (i) and (ii) above.

Now let $\alpha = \{ (\alpha_i, m_i) \}$, where the $\alpha_i$ are simple closed orbits of $\partial_t$. So, each $(\alpha_i, m_i)$ corresponds to a (not necessarily simple) orbit of a periodic point $q_i$ of $\varphi_H^1$. By using the construction in the previous paragraph, we can associate a relative cycle to each $(\alpha_i, m_i)$; the sum, over $i$, of all of these cycles gives a relative cycle from $\alpha$ to $d\gamma_0$, where $d$ is the sum of the periods of the periodic points $q_i$. The arguments in the previous paragraphs show that the action of this cycle $A(\alpha, Z)$ is in $\text{Spec}_d(H)$.

**Remark 3.11.** In the special case where $H = K$, but the two $J_i$ are different, the Monotonicity argument above, applied first to $H \geq K$ and next to $K \geq H$, gives that the spectral invariant does not depend on $J$. □

**Remark 3.12.** It is perhaps instructive to the reader to compare the Chen proof of the holomorphic curve axiom to the Hutchings-Taubes proof of the holomorphic curve axiom, and also to briefly summarize how these proofs go. For brevity, we assume in this remark that the reader is familiar with some of the relevant background, referring the curious reader to [HTT13] for relevant terminology.

In the Hutchings-Taubes context, one starts with an exact symplectic cobordism between contact manifolds, and attaches symplectization-like ends, to get a non-compact symplectic manifold $(\overline{X}, \omega)$. Then, Hutchings-Taubes prove a "holomorphic curve axiom" for any $\omega$-compatible almost complex structure $J$ that agrees with symplectization-admissible almost
complex structures $J_{\pm}$ on the symplectization ends. Their proof goes via adapting a “Seiberg-Witten to Gromov” type degeneration that was needed by Taubes to show the equivalence of $ECH$ and $HM$, to this context. Essentially, just as Taubes’ proof that $ECH = HM$, they write down a family of perturbations to the four-dimensional Seiberg-Witten equations, so that when there is a solution counted by the Seiberg-Witten cobordism map, it degenerates after perturbation to a holomorphic building. It has been observed by Hutchings (see e.g. [Hut12]) that the analogous deformation should still work for a “strong” symplectic cobordism, in particular the exactness of the symplectic form is not essential.

In the Chen context, one starts with a fibered symplectic cobordism between (fiberwise symplectic) mapping tori, and again attaches symplectization-like ends, analogously to the Hutchings-Taubes context, to get a non-compact symplectic manifold $(X, \omega)$; the main difference here is that the symplectic form on the ends in the Hutchings-Taubes context is what one would get from a contact form, whereas in the Chen context, the symplectic form is what one gets from the mapping torus. Just like in Hutchings-Taubes, Chen proves a “holomorphic curve axiom” for any $\omega$-compatible almost complex structure $J$ that agrees with symplectization-admissible almost complex structures $J_{\pm}$ on the ends; this also goes via a “Seiberg-Witten to Gromov” type degeneration.

\[\blacksquare\]

**Remark 3.13.** On the Seiberg-Witten side, the twisted theory corresponds to a version of the Floer homology, where, instead of taking the quotient of solutions by the full gauge group $G = C^\infty(M, S^1)$, one only takes the quotient by the subgroup $G^0 \subset G$ of gauge transformations in the connected component of the identity. This has an $H^1(Y)$ action, induced by the action via gauge transformations, which corresponds to the $H_2(Y)$ action on twisted PFH given by adding a homology class.

As remarked by Taubes [Tau10], Sec. 1, the twisted invariant on the PFH/ECH side is canonical only up to a choice of element of $H_2(Y, \rho, \rho')$, where $\rho, \rho'$ are two reference cycles. With respect to cobordism maps, this corresponds to the following fact — to modify the holomorphic curve axiom for the twisted theory, given a cobordism $X$ from $Y_+$ to $Y_-$, and reference cycles $\rho_{\pm}$, one needs to choose a reference cycle $R \in H_2(X, \rho_+, \rho_-)$, so that one can look for buildings $C$ satisfying

\[Z + R = [C] + Z'.\]  

(39)

This is the best way to think about (31): this corresponds to the very special case of the cobordism map, for $X = Y \times [0, 1]$, where we are choosing $\rho_{\pm} = \rho$, and our reference cycle an $\mathbb{R}$-invariant cylinder from $\rho$ to itself.

It is perhaps instructive to consider the effect of changing (31), if we choose a different cycle $R$ in (39), for our particular $X = Y \times [0, 1]$ with
$Y = \mathbb{S}^1 \times \mathbb{S}^2$, as far as the proof of Theorem 3.6 is concerned. Such a cycle $R$ differs from the trivial cycle by $k[\mathbb{S}^2]$ for some $k$. Recall that (32) in Lemma 3.9 requires that

$$\int_C \omega = \int_{Z-Z'} \omega.$$  

If we write $[C] = Z - Z' + k[\mathbb{S}^2]$, then the addition of $k[\mathbb{S}^2]$ means that the right hand side of this integral must go up by $k$, so that since the rest of the proof of (32) is otherwise unchanged, (32) must be modified by subtracting $k$ from the right hand side in order for Lemma 3.9 to hold. All of this is consistent, for as $I([C]) = 0$, the grading difference between $Z$ and $Z'$ must be $-k(2d+2)$, and so to have the same grading we would have to add an additional $k$ spheres to $Z$. In particular, this argument shows that for our purposes, we do not actually need (31); it would suffice to prove Theorem 3.6 to know that we have some $I = 0$ holomorphic building from $\alpha$ to $\beta$.

For more about the connection between the twisted theory and the relevant Seiberg-Witten Floer homology, we refer the reader to [Tau10] Sections 1 - 2. In [Tau10] Sections 1 - 2, Taubes is writing about twisted ECH; we have adapted what is written there to the PFH context, as suggested by [LT12], Cor. 6.1.

\section{$C^0$ continuity}

Here we prove Theorem 1.12 using Theorem 3.6 from Section 3.

The central objects of Theorem 1.12 are the maps $c_d: \text{Diff}_c(D^2, \omega) \rightarrow \mathbb{R}$. Remember from Section 3.4 and Remark 3.8 that these maps are defined from the spectral invariants $c_d: S \rightarrow \mathbb{R}$, by identifying $\text{Diff}_c(D^2, \omega)$ with the group $\text{Diff}_{S^+}(\mathbb{S}^2, \omega)$ consisting of symplectic diffeomorphisms of $\mathbb{S}^2$, which are supported in the interior of the northern hemisphere $S^+$. In the present section, we directly work in the group $\text{Diff}_{S^+}(\mathbb{S}^2, \omega)$.

Our proof is inspired by the proof of the $C^0$-continuity of barcodes arising from Hamiltonian Floer theory (hence, of Hamiltonian spectral invariants) presented in [LRSV]. Other existing proofs of $C^0$-continuity of spectral invariants make use of the product structure of Hamiltonian Floer homology.\footnote{It might be possible to define a “quantum product” on PFH, see [HS05], but we give a much more direct proof here.}

Let $d$ be a positive integer. As in [LRSV], we treat separately the $C^0$-continuity of $c_d$ at the identity and elsewhere. Theorem 1.12 will be a consequence of the following two propositions.

\footnote{The product is usually used to deduce continuity everywhere from continuity at Id. Without a product, we need another argument to prove continuity in the complement of the identity.}
Proposition 4.1. The map $c_d : \text{Diff}_+(S^2, \omega) \to \mathbb{R}$ is continuous at $\text{Id}$ with respect to the $C^0$-topology on $\text{Diff}_+(S^2, \omega)$.

In the next proposition, we denote by $\text{Homeo}_{S^+}(S^2, \omega)$ the group of area-preserving homeomorphisms of $S^2$ which are supported in the interior of the northern hemisphere.

Proposition 4.2. Every area-preserving homeomorphism $\eta \in \text{Homeo}_{S^+}(S^2, \omega)$ with $\eta \neq \text{Id}$ admits a $C^0$-neighborhood $\mathcal{V}$ such that the restriction of $c_d$ to $\mathcal{V} \cap \text{Diff}_+(S^2, \omega)$ is uniformly continuous with respect to the $C^0$-distance.

This last proposition can be rephrased as follows. For any homeomorphism $\eta \in \text{Homeo}_{S^+}(S^2, \omega)$, $\eta \neq \text{Id}$, there exists $\delta > 0$ such that for all $\varepsilon > 0$ there exists $\delta' > 0$ satisfying:

$$\forall \phi, \psi \in \text{Diff}_+(S^2, \omega) \text{ s.t. } d_{C^0}(\phi, \eta) < \delta \text{ and } d_{C^0}(\psi, \eta) < \delta,$$

if $d_{C^0}(\phi, \psi) < \delta'$, then $|c_d(\phi) - c_d(\psi)| < \varepsilon$.

4.1 Continuity at the identity

We first prove Proposition 4.1 by adapting the $\varepsilon$-shift technique from [Sey13].

Proof. Let $\varepsilon > 0$. We will use an auxiliary Hamiltonian diffeomorphism $f \in \text{Diff}(S^2, \omega)$, constructed as follows. We first let $\chi$ be a uniformly continuous symplectic embedding from an open neighborhood of the northern hemisphere $S^+$ into $\mathbb{R}^2$; let $I \times J$ be a product of intervals which contains the image of $S^+$. We let $F$ be a $C^2$-small function on $S^2$, which vanishes near the south pole $p_-$, and such that the pushforward $\chi_* X_F$ of its Hamiltonian vector field by $\chi$ is a constant horizontal vector field $(\nu, 0)$, for some small $\nu > 0$, on some intermediate rectangle $I' \times J'$ such that $\chi(S^+) \subset I' \times J' \subset I \times J$. We define our diffeomorphism $f$ to be the time-1 map $f = \varphi_F^1$ for $F$ (hence $\nu$) sufficiently small.

More precisely, $F$ and $\nu$ should be picked so small that:

(i) the periodic points of $f$ of period $\leq d$ are nothing but the critical points of $F$; these critical points are fixed all along the isotopy $\varphi_F^t$;

(ii) for any $t \in [0, d]$, $\varphi_F^t(S^+) \subset I' \times J'$, and there exists a compact interval $[a, b]$ such that $\chi(\varphi_F^t(S^+)) \subset [a, b] \times J'$, for all $t \in [0, 1]$, and such that we have $[a, b + 2d\nu] \subset I'$;

(iii) the norm of $F$ satisfies $\|F\|_{(1, \infty)} \leq \frac{\varepsilon}{2\eta}$.

Denote by $q : (x_1, x_2) \mapsto x_1$ the first coordinate on $\mathbb{R}^2$. By uniform continuity of $\chi$, there exists $\delta > 0$ so small that for all $x, y \in S^+$, such that $d(x, y) < \delta$, we have $|q(\chi(x)) - q(\chi(y))| < \nu$. We fix such $\delta$. 38
Claim 4.3. Let $H$ be a Hamiltonian supported in $S^+$, such that for all $s \in [0,1]$, we have $d_{C^0}(\varphi^s_H, \text{Id}) < \delta$. Let $f$ be defined as above. Then, $\text{Spec}_d(\varphi^s_H \circ f) = \text{Spec}_d(f)$, for all $s \in [0,1]$, and $c_d(\varphi^s_H \circ f) = c_d(f)$.

Proof. We first prove that $\text{Spec}_d(\varphi^s_H \circ f) = \text{Spec}_d(f)$, after which the rest follows quickly.

Let $s \in [0,1]$. We first verify that for any $k \in \{1, \ldots, d\}$, $\varphi^s_H \circ f$ admits no $k$-periodic point in $S^+$. To see this, first note that for all $x \in \mathbb{R}^2$ such that $x$ is in the domain of $\chi$, and $\chi(x) \subset [a, b + 2(d-1)\nu] \times J'$, the conjugation $\chi \circ f \circ \chi^{-1}$ shifts the point $\chi(x)$ horizontally by $\nu$, whereas the horizontal shift due to $\chi \circ \varphi^s_H \circ \chi^{-1}$ is strictly less than $\nu$ in absolute value. Thus,

$$q(\chi(x)) < q(\chi \circ (\varphi^s_H \circ f) \circ \chi^{-1}(\chi(x))) < q(\chi(x)) + 2\nu.$$

For $x \in S^+$, we have $q(\chi(x)) \in [a, b]$, and we obtain by induction that

$$q(\chi(x)) < q(\chi \circ (\varphi^s_H \circ f)^k \circ \chi^{-1}(\chi(x))),$$

for all $k \in \{1, \ldots, d\}$. This shows that $(\varphi^s_H \circ f)^k(x) \neq x$ for all $x \in S^+$ and $k \in \{1, \ldots, d\}$.

For any $k \in \{1, \ldots, d\}$, since $\varphi^s_H \circ f$ has no $k$-periodic points in $S^+$ and since $H$ is supported in $S^+$, we conclude that the $k$-periodic points of $\varphi^s_H \circ f$ are the same as those of $f$. There remains to verify that the actions of these periodic points are the same for $\varphi^s_H \circ f$ and for $f$.

Consider the isotopy from $\text{Id}$ to $(\varphi^s_H \circ f)^k$ obtained by concatenating $2k$-isotopies as follows. We first follow $\varphi^f_H$ from $\text{Id}$ to $f$, then follow $\varphi^s_H \circ f$ from $f$ to $\varphi^s_H \circ f$ (this second part is supported in $S^+$ at all $t$), then follow $\varphi^f_H \circ \varphi^s_H \circ f$ from $\varphi^s_H \circ f$ to $f \circ \varphi^s_H \circ f$, then follow $\varphi^f_H \circ f \circ \varphi^s_H \circ f$ from $f \circ \varphi^s_H \circ f$ to $\varphi^s_H \circ f \circ \varphi^s_H \circ f$ (this fourth part is supported in $S^+$), etc, until we reach $(\varphi^s_H \circ f)^k$. More precisely, such an isotopy can be generated by a Hamiltonian $K$ constructed as follows\footnote{Our isotopy is obtained as a concatenation and not as a composition of several isotopies, therefore the formula for $K$ does not involve the operator $\sharp$ as in \cite{11}.}. Let $\rho : [0,1] \to [0,1]$ be a non-decreasing smooth function which is equal to 0 near 0 and equal to 1 near 1. Then for all $\ell \in \{0, \ldots, 2k - 1\}$ and all $t \in [\frac{\ell}{2k}, \frac{\ell + 1}{2k}]$, we set

$$K_t(x) = \begin{cases} \rho'(2kt - \ell) F^\rho_{2k t - \ell}(x), & \text{if } \ell \text{ is even}, \\ \rho'(2kt - \ell) H_{\rho^\prime(2k t - \ell)}(x), & \text{if } \ell \text{ is odd}. \end{cases}$$

In the above formula, the role of the time-reparametrization $\rho$ is simply to make $K$ smooth.

Given a fixed point $x$ of $f^k$ (hence of $(\varphi^s_H \circ f)^k$), a capped orbit $(z,u)$, where $z(t) = \varphi^s_K(x)$, has the following Hamiltonian action (defined in \cite{8}):
\[ \mathcal{A}_K(z, u) = \int_{\mathbb{R}^2} u^* \omega + \sum_{\ell=0}^{2k-1} \int_0^{\frac{\ell \pi}{2k}} K_t(\varphi_K^t(x)) \, dt \]
\[ = \int_{\mathbb{R}^2} u^* \omega + \sum_{j=0}^{k-1} \left( \int_0^1 F_t(\varphi_F^t \circ (\varphi_H^s \circ f)^j(x)) \, dt \right) \]
\[ + \int_0^1 sH_{st}(\varphi_H^s \circ f \circ (\varphi_H^s \circ f)^j(x)) \, dt. \]

Now note that \((\varphi_H^s \circ f)^j(x)\) does not belong to \(S^+\), since we showed above that \(\varphi_H^s \circ f\) has no periodic points in \(S^+\) of period less than \(d\), and so \(f \circ (\varphi_H^s \circ f)^j(x)\) does not belong to \(S^+\) either for any \(j = 0, \ldots, k - 1\), so the terms \(\int_0^1 sH_{st}(\varphi_H^s \circ f \circ (\varphi_H^s \circ f)^j(x)) \, dt\) all vanish. Therefore, the above formula reduces to:

\[ \mathcal{A}_K(z, u) = \int_{\mathbb{R}^2} u^* \omega + \sum_{j=0}^{k-1} \int_0^1 F_t(\varphi_F^t \circ f^j(x)) \, dt, \]

which does not depend on \(s\). We deduce that \(\text{Spec}((\varphi_H^s \circ f)^k) = \text{Spec}(f^k)\) for all \(k = 1, \ldots, d\), hence \(\text{Spec}_d(\varphi_H^s \circ f) = \text{Spec}_d(f)\).

By Theorem 3.6 the function \(s \mapsto c_d(\varphi_H^s \circ f)\) is continuous and takes its values in the measure 0 subset \(\text{Spec}_d(f)\). As a consequence, it is constant. This concludes the proof of Claim 4.3.

We now turn our attention to Proposition 4.2. Our proof requires three lemmas, the first of which is as follows.

4.2 Continuity away from the identity

We now turn our attention to Proposition 4.2. Our proof requires three lemmas, the first of which is as follows.

We can now finish the proof of Proposition 4.1. Let \(\phi \in \text{Diff} S^+(S^2, \omega)\) be such that \(d_{C^0}(\phi, \text{Id}) < \delta\). Then, using Alexander isotopy, one can construct a Hamiltonian isotopy \((\varphi_H^t)_{t \in [0,1]}\) in \(\text{Diff} S^+(S^2, \omega)\), such that \(d_{C^0}(\varphi_H^t, \text{Id}) < \delta\) for all \(t \in [0,1]\) and \(\varphi_H^1 = \phi\) (See Lemma 3.2 in [Sey13] for details). We may apply Claim 4.3 to this isotopy, and deduce

\[ c_d(\phi \circ f) = c_d(f). \]

Therefore, using the Hofer continuity of the spectral invariant \(c_d\) from Theorem 3.6 together with the first formula in [11] and the fact that \(c_d(\text{Id}) = 0\),

\[ |c_d(\phi)| \leq |c_d(\phi \circ f)| + d\|F\|_{(1,\infty)} = |c_d(f)| + d\|F\|_{(1,\infty)} \leq 2d\|F\| < \varepsilon. \]

We have proved that for any \(\varepsilon > 0\), there exists \(\delta > 0\), such that whenever \(d_{C^0}(\phi, \text{Id}) < \delta\), then \(|c_d(\phi)| < \varepsilon\). Thus, we have proved the \(C^0\)-continuity at \(\text{Id}\).
Lemma 4.4. Let \( \eta \in \text{Homeo}_c(\mathbb{D}, \omega) \) with \( \eta \neq \text{Id} \). Then, there exists a point \( x \in \mathbb{D} \) such that \( x, \eta(x), \eta^2(x), \ldots, \eta^d(x) \) are pairwise distinct points.

Proof. Let \( \eta \in \text{Homeo}_c(\mathbb{D}, \omega) \) with \( \eta \neq \text{Id} \) and let \( d \) be a positive integer. It is known (See [CK94]) that for any positive integer \( N \), \( \eta^N \neq \text{Id} \). Thus, there exists a point \( x \in \mathbb{D} \) such that \( \eta^d(x) \neq x \). For such a point, \( x, \eta(x), \eta^2(x), \ldots, \eta^d(x) \) are pairwise distinct. Indeed, otherwise, there would be integers \( 0 \leq k < \ell \leq d \) such that \( \eta^k(x) = \eta^\ell(x) \), and we would get \( \eta^{\ell-k}(x) = x \), in contradiction with \( \eta^d(x) \neq x \) since \( \ell - k |d| \).

Our second lemma is a fragmentation property for diffeomorphisms which are sufficiently close to identity. We denote by \( \omega_0 \) the standard symplectic form on \( \mathbb{R}^2 \).

Lemma 4.5 (Lemma 47 in (Le Roux-Seyfaddini-Viterbo)). Let \( m \) be a positive integer and \( \rho \) a positive real number. For \( i = 0, \ldots, m \), denote by \( U_i \) the open rectangle \((0,1) \times (\frac{i}{m}, \frac{i+1}{m})\). Then, there exists \( \delta > 0 \), such that for every \( g \in \text{Diff}_c((0,1) \times (0,1), \omega_0) \) with \( d_{C^0}(g, \text{Id}) < \delta \), there exist \( g_1 \in \text{Diff}_c(U_1, \omega_0) \), \ldots, \( g_m \in \text{Diff}_c(U_m, \omega_0) \) and \( \theta \in \text{Diff}_c((0,1) \times (0,1), \omega_0) \) supported in a disjoint union of topological disks whose total area is less than \( \rho \), such that \( g = g_1 \circ \cdots \circ g_m \circ \theta \).

Finally, our third Lemma is inspired by Lemma 3.2 in [Ush10] which treats the case \( d = 1 \).

Lemma 4.6. Let \( \phi \in \text{Diff}_+(S^2, \omega) \) and \( B \) be a small topological disk in \( S^2 \) with the property that the disks \( B, \phi(B), \ldots, \phi^d(B) \) are pairwise disjoint. Then, for all \( g \in \text{Diff}_+(S^2, \omega) \) compactly supported in \( B \), we have

\[
c_d(g \circ \phi) = c_d(\phi).
\]

Proof. Let \( H \) be a Hamiltonian supported in \( S^+ \) with \( \varphi_H^1 = \phi \) and let \( G \) be a Hamiltonian supported in \( B \) with \( \varphi_G^1 = g \). We will prove that for all \( s \in [0,1] \), \( \text{Spec}_d(\varphi_G^s \circ \phi) = \text{Spec}_d(\phi) \). This implies, as in Claim 4.3, that the map \( s \mapsto c_d(\varphi_G^s \circ \phi) \) is constant, hence the lemma.

Let \( s \in [0,1] \). We will first verify that the diffeomorphism \( \varphi_G^s \circ \phi \) admits the same \( k \)-periodic points as \( \phi \), for all \( k \in \{1, \ldots, d\} \). For all \( \ell \in \{0, \ldots, d\} \), we have \( B \cap \phi^\ell(B) = \emptyset \) and \( B \cap \phi^{-\ell}(B) = \emptyset \). It follows that \( \varphi_G^s(\phi^\ell(B)) = \phi^\ell(B) \), for all \( \ell \in \{-d, \ldots, d\} \), hence

\[
(\varphi_G^s \circ \phi)^k(\phi^{-\ell}(B)) = \phi^{k-\ell}(B), \quad \forall k \in \{1, \ldots, d\}, \forall \ell \in \{0, \ldots, d\}.
\]

Since \( \phi^{-\ell}(B) \cap \phi^{k-\ell}(B) = \emptyset \) for such \( k, \ell \), this implies that \( \varphi_G^s \circ \phi \) has no \( k \)-periodic points with \( 1 \leq k \leq d \) in \( \bigcup_{\ell=0}^{d} \phi^{-\ell}(B) \).

We now fix \( k \in \{1, \ldots, d\} \). If \( x \notin \bigcup_{\ell=0}^{d} \phi^{-\ell}(B) \), then \( (\varphi_G^s \circ \phi)^k(x) = \phi^k(x) \). As a consequence, the \( k \)-periodic points of \( \varphi_G^s \circ \phi \) are exactly those of \( \phi \).
We will now prove that the corresponding action values coincide as well. Similarly to the proof of Claim 4.3 above, \((\varphi_G^s \circ \phi)^k\) is the time-1 map of the Hamiltonian \(K\) given by the formula

\[
K_t(x) = \begin{cases} 
\rho'(2kt - \ell)H_{\rho(2kt-\ell)}(x), & \text{if } \ell \text{ is even,} \\
sp'(2kt - \ell)G_{\rho(2kt-\ell)}(x), & \text{if } \ell \text{ is odd.}
\end{cases}
\]

for \(\ell \in \{0, \ldots, 2k-1\}\) and \(t \in \left[\frac{\ell}{2k}, \frac{\ell+1}{2k}\right]\) and where \(\rho : [0, 1] \to [0, 1]\) is a non-decreasing smooth function which is equal to 0 near 0 and equal to 1 near 1. We will compute the spectrum of \(\varphi_s \circ G \circ \phi\) with the help of this particular Hamiltonian. Again, as in the proof of Claim 4.3, the action of a capped 1-periodic orbit \((z, u)\) of \(K\), with \(z = \varphi_t K(x)\), is given by

\[
A_K(z, u) = \int_{\mathbb{R}^2} u^* \omega + \sum_{j=0}^{k-1} \left( \int_0^1 H_t(\varphi_H^t \circ (\varphi_G^s \circ \phi)^j)(x) \right) dt
\]

And, just as in the proof of Claim 4.3, since we showed above that \(\varphi_G^s \circ \phi\) has no \(k\)-periodic points in \(\phi^{-1}(B)\), we know that \(\phi \circ (\varphi_G^s \circ \phi)^j(x)\) does not belong to \(B\), hence to the support of \(G\), and so the integrand for the third integral above has to vanish and the integrand for the second integral above can be simplified, so that we get

\[
A_K(z, u) = \int_{\mathbb{R}^2} u^* \omega + \sum_{j=0}^{k-1} \int_0^1 H_t(\varphi_H^t \circ \phi^j)(x) dt,
\]

We see that this action does not depend on \(s\). As a consequence, we get \(\text{Spec}_d(\phi \circ \varphi_G^s) = \text{Spec}_d(\phi)\) for all \(s \in [0, 1]\). \(\square\)

We can now give the proof of Proposition 4.2. Our proof of Proposition 4.2 will use the Hofer norm of a Hamiltonian diffeomorphism \(\phi\) which is defined as

\[
\|\phi\| = \inf\{\|H\|_{(1, \infty)}\},
\]

where the infimum runs over all Hamiltonians whose time-1 map is \(\phi\).

It satisfies a triangle inequality

\[
\|\phi \circ \psi\| \leq \|\phi\| + \|\psi\|,
\]

for all Hamiltonian diffeomorphisms \(\phi, \psi\), it is conjugation invariant and moreover, we have \(\|\phi^{-1}\| = \|\phi\|\) for all Hamiltonian diffeomorphism \(\phi\).

The displacement energy of a closed subset \(A\) of the ambient symplectic manifold is by definition the quantity

\[
e(A) := \inf\{\|\phi\| : \phi(A) \cap A = \emptyset\}\]
On a surface, it is known that for a disjoint union of closed discs, with each disc having area \(a\), and whose union covers less than half the area of the surface, the displacement energy is \(a\). We refer the reader to [Pol01] and the references therein for a general introduction on the Hofer norm and displacement energy.

Note that the second item of Theorem 3.6 can be reformulated as

\[ |c_d(\psi) - c_d(\phi)| \leq d \cdot \|\psi^{-1} \circ \phi\|, \tag{40} \]

for all Hamiltonian diffeomorphisms \(\phi, \psi \in S\). To see this, let \(K\) be a Hamiltonian such that \(\psi = \varphi_K^1\). Then for any Hamiltonian \(H\) such that \(\psi^{-1} \circ \phi = \varphi_H^1\), then \(\phi = \varphi_K^1 \# H\) by \(1\), thus by Theorem 3.6 we get

\[ |c_d(\psi) - c_d(\phi)| = |c_d(\varphi_K^1) - c_d(\varphi_K^1 \# H)| \leq d \cdot \|K - K^1_H\|_{(1, \infty)} = d \cdot \|H\|_{(1, \infty)}. \]

Inequality \(40\) follows.

**Proof of Proposition 4.2.** Let \(\eta \in \text{Homeo}_S(S^2, \omega), \eta \neq \text{Id}\). As a consequence of Lemma 4.3 there exists a disk \(B\) in \(S^+\), whose iterates \(B, \eta(B), \ldots, \eta^d(B)\) have pairwise disjoint closures. Let \(V\) be the \(C^0\)-neighborhood of \(\eta\) given by the set of all \(\phi \in \text{Homeo}_S(S^2, \omega)\) such that \(B, \phi(B), \ldots, \phi^d(B)\) are pairwise disjoint. We will prove that \(c_d\) is uniformly continuous with respect to the \(C^0\)-distance on \(V \cap \text{Diff}_S(S^2, \omega)\).

We fix such \(\delta > 0\) and let \(\phi, \psi \in \text{Diff}_S(S^2, \omega)\) be such that \(d_{C^0}(\phi, \psi) < \delta\). We will prove that \(|c_d(\phi) - c_d(\psi)| < \varepsilon\).

Let \(g = \psi \circ \phi^{-1}\), so that \(\psi = g \circ \phi\). Then, as

\[
d_{C^0}(g, \text{Id}) = \max_{x \in S^2} d(g \circ \phi^{-1}(x), x) = \max_{x \in S^2} d(\psi \circ \phi^{-1}(\phi(x)), \phi(x)) = d_{C^0}(\phi, \psi),
\]

we can decompose \(g\) as above in the form \(g = g_1 \circ \cdots \circ g_m \circ \theta\).

For \(j = 1, \ldots, N\), denote by \(f_j\) the composition \(f_j := \prod_{i=0}^{j-1} g_{j+iN}\). Also write \(f_{N+1} := \theta\), so that the we have the following formula

\[ g = \prod_{j=1}^{N+1} f_j. \]
Each $f_j$ for $j = 1, \ldots, N$ is supported in $V_j = \bigcup_{j=0}^{m-1} U_{j+iN}$ whose area is $\frac{1}{N} < \text{area}(B)$. Note that $V_j$ is a disjoint union of disks of area $\frac{1}{2m}$. By assumption, the support of $f_{N+1} = \emptyset$, which we denote by $V_{N+1}$, is also included in a disjoint union of disks of area smaller that $\frac{1}{2m}$. This will allow us to prove the following claim.

**Claim 4.7.** For all $j = 1, \ldots, N+1$, there exists a Hamiltonian diffeomorphism $h_j \in \text{Diff}_+ (S^2, \omega)$, such that $h_j(V_j) \subset B$ and $\|h_j\| \leq \frac{1}{m}$.

**Proof.** Let $\phi$ be any Hamiltonian diffeomorphism which maps $V_j$ into $B$, such that the complement of the support of $\phi$ contains a disk of area bigger than the area of $V_j$. As explained before the proof of Proposition 4.2, the displacement energy of each $V_j$, $j = 1, \ldots, N + 1$ is less than $\frac{1}{2m}$. Thus, there exists a Hamiltonian diffeomorphism $\ell$ such that $\ell(V_j) \cap V_j = \emptyset$ and $\|\ell\| \leq \frac{1}{2m}$. By conjugating $\ell$ with a Hamiltonian diffeomorphism which fixes $V_j$ and maps $\ell(V_j)$ in the complement of the support of $\phi$, we may further assume that $\ell(V_j) \cap \text{Supp}(\phi) = \emptyset$.

Then $\ell \circ \phi^{-1} \circ \ell^{-1}$ and $\phi$ have disjoint support, hence denoting $h_j := \phi \circ \ell \circ \phi^{-1} \circ \ell^{-1}$, we have $h_j(V_j) = \phi(V_j) \subset B$. On the other hand, by triangle inequality and conjugation invariance, $\|h_j\| \leq 2\|\ell\| \leq \frac{1}{m}$. \hfill $\Box$

We can now conclude the proof of Proposition 4.2. Consider the diffeomorphism

$$g' = \prod_{j=1}^{N+1} h_j \circ f_j \circ h_j^{-1}.$$ 

By construction, $g'$ is supported in $B$, thus $c_d(g' \circ \phi) = c_d(\phi)$, by Lemma 4.2.

Therefore,

$$|c_d(\phi) - c_d(\psi)| = |c_d(\phi) - c_d(g \circ \phi)|$$

$$\leq |c_d(\phi) - c_d(g' \circ \phi)| + |c_d(g' \circ \phi) - c_d(g \circ \phi)|$$

$$\leq d \cdot \|g' - g\|,$$

where the last inequality follows from (10).

In order to estimate the quantity $\|g' - g\|$, note that for any Hamiltonian diffeomorphisms $\rho, \rho', f, h$, if we set $\chi := \rho \circ f$ and $\chi' := \rho' \circ h \circ f \circ h^{-1}$, then

$$\|\chi' - \chi\| \leq 2\|h\| + \|\rho' - \rho\|. \quad (41)$$

Indeed, using the triangle inequality and the conjugation invariance of Hofer norm,

$$\|\chi' - \chi\| = \|hf^{-1}h^{-1} \rho' - \rho f\| = \|hf^{-1}h^{-1}ff^{-1} \rho' - \rho f\|$$

$$\leq \|h\| + \|f^{-1}h^{-1}f\| + \|f^{-1} \rho' - \rho f\|$$

$$= 2\|h\| + \|\rho' - \rho\|.$$
Now, Inequality (41) yields by induction:

$$\|g^{-1} \circ g\| \leq \sum_{j=1}^{N+1} 2\|h_j\| \leq 2 \frac{N + 1}{m}.$$ 

With the previous estimate, we obtain

$$|c_d(\phi) - c_d(\psi)| \leq 2d \frac{N + 1}{m} < \varepsilon.$$ 

This finishes the proof of Proposition 4.2. \(\square\)

5 The periodic Floer homology of positive monotone twists

The goal of this section and the next is to prove Theorem 1.13 which establishes the Calabi property for monotone twist maps of the disc. Recall from Remark 3.8 that we define PFH spectral invariants for maps of the disc by identifying \(\text{Diff}_c(D, \omega)\) with maps of the sphere supported in the northern hemisphere \(S^+\). In particular, we will view any monotone twist as an element of the set \(S\) appearing in Theorem 3.6.

Theorem 1.13 will follow from the following theorem for the invariants \(c_{d,k}\), which, as alluded to in the introduction, was originally conjectured in greater generality by Hutchings [Hut].

Theorem 5.1. Let \((d, k)\) be any sequence, with \(k = d \mod 2\). Then, for any positive monotone twist \(\phi\) we have:

$$\text{Cal}(\phi) = \lim_{d \to \infty} \left( \frac{2c_{d,k}(\phi)}{d} - \frac{k}{d^2 + d} \right).$$

(42)

A first observation, concerning Equation (42), which was part of Hutchings’ motivation for the conjecture, is that it suffices to establish (42) for a single such sequence \((d, k)\). Indeed, for \(d\)-nondegenerate diffeomorphisms \(\phi\), there is an automorphism of the twisted PFH chain complex given by

\((\alpha, Z) \mapsto (\alpha, Z + [S^2])\),

where \([S^2]\) denotes the class of the sphere. This increases the grading by \(2d + 2\), by Formula (18). It also increases the action by 1. So, we have

$$c_{d,k+2d+2} = c_{d,k} + 1$$

for all \(\phi\). Now, the right hand side of Equation (42) is invariant under increasing the numerator of the first fraction by one, and increasing the
numerator of the second fraction by $2d + 2$. Moreover, as a corollary of Theorem 6.1 we will obtain

$$c_{d,k} \leq c_{d,k'},$$

(43)

when $k' \geq k$, with $k = k' = d \mod 2$ and $\varphi$ is a positive monotone twist; see Remark 6.2.

5.1 Perturbations of rotation invariant Hamiltonian flows

The remainder of Section 5 is dedicated to describing a combinatorial model of $\widetilde{PFH}(Y_\varphi, d)$ where $\varphi$ belongs to a certain class of Hamiltonian diffeomorphisms which has the property that its closure in the $C^\infty$–topology contains all positive monotone twists. The main result of Section 5 is Proposition 5.2 which describes this combinatorial model. Our model is inspired by similar combinatorial models that have been developed for computing the PFH of a Dehn twist [HS05], the ECH of $T^3$ [HS06], and the ECH of toric domains [Cho16], and our methods in this section are inspired by the techniques used to establish these combinatorial models.

We will use this combinatorial model in Section 6 to prove Theorem 5.1. Here, and through the end of Section 5, we consider Hamiltonian flows on $(S^2, \omega = \frac{1}{4\pi} d\theta \wedge dz)$, for an autonomous Hamiltonian

$$H = \frac{1}{2} h(z),$$

where $h$ is some function of $z$. We have

$$X_H = 2\pi h'(z) \partial_z.$$

(44)

Hence,

$$\varphi^1_H(\theta, z) = (\theta + 2\pi h'(z), z)$$

We further restrict $h$ to satisfy

$$h' > 0, h'' \geq 0, h(-1) = 0.$$

Furthermore, we demand that $h'(-1), h'(1)$ be irrational numbers satisfying $h'(-1) \leq \frac{\varepsilon_0}{d}$ and $[h'(1)] - h'(1) \leq \frac{\varepsilon_0}{d}$, where $\varepsilon_0$ is a small positive number and $[\cdot]$ denotes the ceiling function. Let $D$ denote the set of such $H$ that satisfy all of these conditions and observe that $D \subset H$ where $H$ was defined in Section 3.4 As a consequence of Theorem 3.6 we have well-defined PFH spectral invariants $c_{d,k}(\varphi^1_H)$ for all $H \in D$.

The periodic orbits of $\varphi^1_H$ are then as follows:

1. There are elliptic orbits $p_+$ and $p_-$, corresponding to the north and south poles, respectively.
2. For each \( p/q \) in lowest terms such that \( h' = p/q \) is rational, there is a circle of periodic orbits, all of which have period \( q \).

These circles of periodic orbits are familiar from Morse-Bott theory, and are sometimes referred to as “Morse-Bott circles”. There is also a standard \( \varphi_H^1 \) admissible almost complex structure \( J_{std} \) respecting this symmetry; its action on \( \xi = T(S^2 \times \{ pt \}) = TS^2 \) is given by the standard almost complex structure on \( S^2 \).

As is familiar in this context (see Section 3.1 of [HS05]), we can perform a \( C^2 \)-small perturbation of \( H \), to split such a circle corresponding to the locus where \( h' = p/q \) into one elliptic and one hyperbolic periodic points, such that the elliptic one \( e_{p,q} \) has slightly negative monodromy angle, and the eigenvalues for the hyperbolic one \( h_{p,q} \) are positive. Furthermore, the \( C^2 \) small perturbation can be taken to be supported in an arbitrarily small neighborhood of the circle where \( h' = p/q \). More precisely, given a \( \varphi_H^1 \) such as above, for any positive \( d \) and arbitrarily small \( \epsilon > 0 \), we can find another area-preserving diffeomorphism \( \varphi_0^1 \) of \( S^2 \), which we call a nice perturbation of \( \varphi_H^1 \), such that:

1. The only periodic points of \( \varphi_0 \) which are of degree at most \( d \) are \( p_+, p_- \), and the orbits \( e_{p,q} \) and \( h_{p,q} \) from above such that \( q \leq d \). Furthermore, all of these orbits are non-degenerate.

2. The eigenvalues of the linearized return map for \( e_{p,q} \) are within \( \epsilon \) of 1.

3. \( \varphi_0(\theta, z) = \varphi_H^1(\theta, z) \) as long as \( z \) is not within \( \epsilon \) of a value such that \( h' = p/q \) where \( q \leq d \).

Observe that we can pick a time-dependent Hamiltonian \( \tilde{H} \) such that \( \varphi_0 = \varphi_H^1 \) and \( \tilde{H} - H = 0 \) as long as \( z \) is not within \( \epsilon \) of a value such that \( h' = p/q \) where \( q \leq d \).

It is also familiar from the work of Hutchings-Sullivan, see [HS05], Lemma A.1, that we can choose our perturbation of \( \varphi_H^1 \) such that we can assume the following:

4. \( \varphi_0 \) is chosen so that “Double Rounding”, defined in 5.2 below, can not occur for generic \( J \) close to \( J_{std} \).

Later, it will be clear why it simplifies the analysis to rule out Double Rounding.

5.2 The combinatorial model

We now aim to describe the promised combinatorial model of \( \widetilde{PFH} \) for the Hamiltonians described in the previous section. Fix \( d \in \mathbb{N} \) and \( \varphi_H^1 \), where \( H \in \mathcal{D} \), for the remainder of Section 5.
To begin, define a concave lattice path to be a piecewise linear, continuous path $P$, in the $xy$-plane, such that $P$ starts and ends on integer lattice points, its starting point is on the $y$–axis, the nonsmooth points of $P$ are also at integer lattice points, and $P$ is concave, in the sense that it always lies above any of the tangent lines at its smooth points. Lastly, every edge of $P$ is labelled by either $e$ or $h$. Below, we will associate one such lattice path to every $\tilde{PFH}$ generator $(\alpha, Z)$.

Let $\alpha = \{(\alpha_i, m_i)\}$ be an orbit set of degree $d$, for a nice perturbation $\varphi_0$ of $\varphi_1$. First of all, recall that the simple Reeb orbits for $Y_{\varphi_0}$, with degree no more than $d$, are as follows:

1. The Reeb orbits $\gamma_{\pm}$ corresponding to $p_{\pm}$.
2. For each $z$ such that $h'(1) = p/q$ in lowest terms, there are Reeb orbits of degree $q$ corresponding to the periodic points $e_{p,q}$ and $h_{p,q}$, that we will also denote by $e_{p,q}$ and $h_{p,q}$.

We will now associate to the orbit set $\alpha = \{(\alpha_i, m_i)\}$ a concave lattice path $P_{\alpha}$ whose starting point we require to be $(0,0)$. If $(\gamma_{-}, m_{-}) \in \alpha$, we set $v_{-} = m_{-}(1,0)$ and label it by $e$. If $(\gamma_{+}, m_{+}) \in \alpha$ we set $v_{+} = m_{+}(1, l(1))$ and label it by $e$. Next, consider the orbits in $\alpha$ corresponding to $z = z_{p,q}$ such that $h'(z) = p/q$; note that there are at most two such entries in $\alpha$: one corresponding to $e_{p/q}$ and another corresponding to $h_{p/q}$. To these entries, we associate the labeled vectors $v_{p,q} = m_{p,q}(q,p)$, where $m_{p,q}$ is the sum of multiplicities of $e_{p/q}$ and $h_{p/q}$; the vector is labeled $h$ if $(h_{p/q}, 1) \in \alpha$, and $e$ otherwise. (For motivation, note that by the conditions on the PFH chain complex, an $m_i$ corresponding to a hyperbolic orbit must equal 1.) To build the concave lattice path $P_{\alpha}$ from all of the data in $\alpha$, we simply concatenate the vectors $v_{-}, v_{p,q}, v_{+}$ into a concave lattice path. Note that there is a unique way to do this: The path must begin with $v_{-}$, it must end with $v_{+}$, and the vectors $v_{p,q}$ must be concatenated in the increasing order with respect to the ratios $p/q$.

Now, given a chain complex generator $(\alpha, Z)$ for $\tilde{PFH}$, we define an assignation

$$(\alpha, Z) \mapsto P_{\alpha, Z}$$

which associates a concave lattice path $P_{\alpha, Z}$ to the generator $(\alpha, Z)$. More specifically, when $Z = Z_{\alpha}$, we define $P_{\alpha, Z_{\alpha}}$ to be the concave lattice path $P_{\alpha}$. Since $H_2(Y_{\varphi_0}) = Z$, generated by the class of $S^2 \times \{pt\}$, for any other $(\alpha, Z)$, we have $Z = Z_{\alpha} + y[S^2]$. We then define $P_{\alpha, Z}$ to be $P_{\alpha}$ shifted by the vector $(0, y)$.

We now state some of the key properties of the above assignation

$$(\alpha, Z) \mapsto P_{\alpha, Z}.$$
Degree: Note that since we are fixing the degree of \( \alpha \) to be \( d \), the horizontal displacement of \( P_{\alpha,Z} \) must be \( d \); we therefore call the horizontal displacement of a concave lattice path its degree. Clearly, the degree of \( P_{\alpha,Z} \) agrees with the degree of the \( \text{PFH} \) generator \((\alpha, Z)\).

Action: Define the action \( A(P_{\alpha,Z}) \) as follows. We first define the actions of the edges of \( P_{\alpha,Z} \) by the formulae:

\[
A(v_-) = 0, \quad A(v_+) = m_+ \frac{h(1)}{2} \\
A(v_{p,q}) = \frac{m_{p,q}}{2} \left( p(1 - z_{p,q}) + qh(z_{p,q}) \right)
\]

where \( v_- = m_-(1,0), v_+ = m_+(1,\lceil h'(1) \rceil), v_{p,q} = m_{p,q}(q,p) \). We then define the action of \( P_{\alpha,z} \) to be

\[
A(P_{\alpha,Z}) = y + m_+ \frac{h(1)}{2} + \sum_{v_{p,q}} A(v_{p,q}),
\]

where \( y \) is such that \( P(\alpha, Z) \) begins at \((0,y)\).

We claim that by picking the nice perturbation \( \varphi_0 \) to be sufficiently close to \( \varphi_1^H \) we can arrange for \( A(\alpha, Z) \) to be as close to \( A(P_{\alpha,Z}) \) as we wish. To show this it is sufficient to prove it when \( \alpha \) is a simple Reeb orbit and \( Z = Z_\alpha \), where \( Z_\alpha \) is the relative class constructed in the proof of Lemma 3.10. We have to consider the following three cases:

- If \( \alpha = \gamma_- \), then \( A(\alpha, Z_\alpha) = 0 \), by Equation (37), which coincides with \( A(1,0) \).
- If \( \alpha = \gamma_+ \), then \( A(\alpha, Z_\alpha) = h(1) \), by Equation (37), which coincides with \( A(1,\lceil h'(1) \rceil) \). Note that, in Equation (37), the term \( \int_{D^2} u_\alpha^* \alpha \omega \) is zero.
- The remaining case is when \( \alpha = e_{p,q} \) or \( h_{p,q} \); here, it is sufficient to show that the action of the Reeb orbits at \( z_{p,q} \), for the unperturbed diffeomorphism \( \varphi_1^H \) is exactly the quantity \( \frac{1}{2}(p(1 - z_{p,q}) + qh(z_{p,q})) \). This follows from Equation (38): the term \( \int_{D^2} u_\alpha^* \omega \) is exactly \( \frac{1}{2} p(1 - z_{p,q}) \) and the term \( \int_0^q H_t(\varphi_t^H(q))dt \) is exactly \( \frac{1}{2} qh(z_{p,q}). \)

Index: Next, we associate an index to a concave lattice path \( P \) which begins at a point \((0,y)\), on the \( y \)-axis, and has degree \( d \).

First, we form (possibly empty) regions \( R_{\pm} \), where \( R_- \) is the closed region bounded by the \( x \)-axis, the \( y \)-axis, and the part of \( P \) below the \( x \)-axis, while \( R_+ \) is the closed region bounded by the \( x \)-axis, the line \( x = d \), and the part of \( P \) above the \( x \)-axis. Let \( j_+ \) denote the number of lattice points in the region \( R_+ \), not including lattice points on \( P \), and let \( j_- \) denote the
number of lattice points in the region $R_-$, not including the lattice points on the $x$-axis. We now define

$$j(P) := j_+(P) - j_-(P).$$  \hfill (46)

This definition of $j$ is such that if one shifts $P$ vertically by 1, then $j(P)$ increases by $d + 1$.

Given a path $P_{\alpha,Z}$, associated to a PFH generator $(\alpha, Z)$, we define its index by

$$I(P_{\alpha,Z}) := 2j(P_{\alpha,Z}) - d + h,$$  \hfill (47)

where $h$ denotes the number of edges in $P_{\alpha,Z}$ labelled by $h$. We will show in Section 5.3 that $I(P_{\alpha,Z})$ coincides with the PFH index of $I(\alpha, Z)$ as defined in Equation (22).

**Corner rounding and the differential:** Lastly, we define a combinatorial process which corresponds to the PFH differential. Let $P_\beta$ be a concave lattice path of degree $d$ which begins on the $y$-axis. Then, if we attach vertical rays to the beginning and end of $P_\beta$, in the positive $y$ direction, we obtain a closed convex subset $R_\beta$ of the plane. For any given corner of $P_\beta$, where we include the initial and final endpoints of $P_\beta$ as corners, we can define a **corner rounding** operation by removing this corner, taking the convex hull of the remaining integer lattice points in $R_\beta$, and taking the lower boundary of this region, namely the part of the boundary that does not consist of vertical lines. Note that the newly obtained path is of degree $d$.

We now say that another concave lattice path $P_\alpha$ is obtained from $P_\beta$ by **rounding a corner and locally losing one $h$**, if $P_\alpha$ is obtained from $P_\beta$ by a corner rounding such that the following conditions are satisfied:

(i) Let $k$ denote the number of edges in $P_\beta$, with an endpoint at the rounded corner, which are labelled $h$. We require that $k \geq 1$.

(ii) Of the new edges in $P_\alpha$, created by the corner rounding operation, exactly $k - 1$ are labelled $h$.

The notions introduced above and the proposition below give a complete combinatorial interpretation of the PFH chain complex:

**Proposition 5.2.** Fix $d > 0$ and let $\varphi_0$ be a nice perturbation of $\varphi_H^1$, where $H \in \mathcal{D}$. Then, for generic $\varphi_0$-admissible almost complex structure $J$ close to $J_{\text{std}}$, there is an assignation

$$(\alpha, Z) \mapsto P_{\alpha,Z}$$

with the following properties:
1. $A(\alpha, Z) \sim A(P_\alpha, Z)$,
2. $I(\alpha, Z) = I(P_\alpha, Z)$,
3. $(\partial(\alpha, Z), (\beta, Z')) \neq 0$ if and only if $P_{\alpha, Z}$ is obtained from $P_{\beta, Z'}$ by rounding a corner and locally losing one $h$.

Here, by $A(\alpha, Z) \sim A(P_\alpha, Z)$, we mean that by choosing our nice perturbation $\varphi_0$ sufficiently close to $\varphi_H$, we can arrange for $A(\alpha, Z)$ to be as close to $A(P_\alpha, Z)$ as we wish.

We have already proven the first of the three listed properties in the above proposition. The second will be proven below in Section 5.3. The proof of the third takes up the remainder of Section 5.3.

Finally, we end this section by defining the Double Rounding operation which will appear in the following sections and which has already been introduced in Section 5.1. Namely, if $P_{\beta, Z'}$ has three consecutive edges, all labeled by $h$, we say that $P_{\alpha, Z}$ is obtained from $P_{\beta, Z'}$ by double rounding if we remove both interior lattice points for these three edges, take the convex hull of the remaining lattice points (in the region formed by adding vertical lines, as above), and label all new edges by $e$.

5.3 Computation of the index

In this section, we prove the second item in Proposition 5.2. Before giving the proof, we first summarize the definitions of the various terms of the PFH grading as defined in Equation (22), which we recall here:

$$I(\alpha, Z) = c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k).$$

These definitions can be found explained in detail, for example, in Sec. 2 [Hut02].

To define the relative Chern class, $c_\tau(Z)$, we first take a surface $S$, representing $Z$ in $[-1, 1] \times S^2 \times S^1$, assumed transverse to the boundary $\{-1, 1\} \times S^2 \times S^1$ and embedded in $(−1, 1) \times S^2 \times S^1$. We then define $c_\tau(Z)$ to be a signed count of zeros of a generic section $\Psi$ of $V|S$, where $V$ is the vertical bundle of the fibration $[-1, 1] \times S^2 \times S^1 \rightarrow [-1, 1] \times S^1$, such that the restriction of $\Psi$ to $\partial S$ is non-winding with respect to the trivialization $\tau$. We similarly define the relative intersection number $Q_\tau(Z)$ by the formula

$$Q_\tau(Z) := c_1(N, \tau) - w_\tau(S),$$

where $c_1(N, \tau)$, the relative Chern number of the normal bundle, is a signed count of zeros of a generic section of $N|S$, such that the restriction
of this section to \( \partial S \) is non-winding with respect to \( \tau \); note that the normal bundle \( N \) can be canonically identified with \( V \) along \( \partial S \). Meanwhile, the term \( w_{\tau}(S) \), the asymptotic writhe, is defined by using the trivialization \( \tau \) to identify a neighborhood of each Reeb orbit with \( S^1 \times D^2 \subset \mathbb{R}^3 \), and then computing the writhe\(^{14}\) of a constant \( s \) slice of \( S \) near the boundary using this identification.

Finally, to define the Conley-Zehnder index, we first clarify the definitions of elliptic and hyperbolic Reeb orbits, and define the rotation number \( \theta \) for a simple orbit, relative to the trivialization \( \tau \). Specifically, the elliptic case is characterized by the property that the linearized return map has eigenvalues on the unit circle; in this case, one can homotope the trivialization so that the linearized flow at time \( t \) with respect to the trivialization is always a rotation by \( 2\pi \theta_t \), for a continuous function \( \theta_t \), and then the rotation number is the change in \( \theta_t \) as one goes around the orbit once.

In the hyperbolic case, the linearized return map has real eigenvalues, and the linearized return map rotates by angle \( 2\pi k \) for some half-integer \( k \in \frac{1}{2} \mathbb{Z} \) as one goes around the orbit; the integer \( k \) is the rotation number in this case. In either case, denoting by \( \theta \) the rotation number, for any cover of \( \gamma \), we have

\[
CZ_{\tau}(\gamma^n) := \lfloor n\theta \rfloor + \lceil n\theta \rceil, \tag{49}
\]

Proof of the second item in Proposition 5.2. By the index ambiguity formula, Equation (18), we have

\[
I(\alpha, Z + a[\mathbb{S}^2]) = I(\alpha, Z) + a(2d + 2). \tag{50}
\]

Therefore, we only have to compute the index \( I(\alpha, Z'_{\alpha}) \) for a given relative class \( Z'_{\alpha} \in \mathcal{H}^2(\mathbb{S}^2 \times \mathbb{S}^1, \alpha, d\gamma) \).

Let us now define the relative class \( Z'_{\alpha} \) we will be using. Write \( \alpha = \{(\gamma_-, m_-)\} \cup \{(\alpha_i, m_i)\} \cup \{(\gamma_+, m_+)\} \), where each \( (\alpha_i, m_i) \) is either an \( (h_{p_i/q_i}, 1) \) or an \( (e_{p_i/q_i}, m_{p_i/q_i}) \). We define \( Z'_{\alpha} = m_- Z'_- + m_+ Z'_+ + \sum_i m_i Z'_{\alpha_i} \), where

- \( Z'_- \in \mathcal{H}^2(\mathbb{S}^2 \times \mathbb{S}^1, \gamma_-, \gamma_-) \) is the trivial class,
- \( Z'_+ \in \mathcal{H}^2(\mathbb{S}^2 \times \mathbb{S}^1, \gamma_+, \gamma_-) \) is represented by the map

\[
S_+: [0, 1] \times [0, q] \to \mathbb{S}^2 \times \mathbb{S}^1, \quad (s, t) \mapsto (R_{\ell(h(1))}(\eta(s)), t),
\]

where \( \eta \) is a meridian from the South pole \( p_- \) to the North pole \( p_+ \), and \( R_{\ell k} \) denotes the rotation on \( \mathbb{S}^2 \) by the angle \( 2\pi \ell k \).

\(^{14}\)This is defined by identifying \( \mathbb{S}^1 \times D^2 \) with the product of an annulus and interval, projecting to the annulus, and counting crossings with signs.
• for $\alpha_i = e_{p,q}$ or $h_{p,q}$, the relative class $Z'_{\alpha_i} \in H^2(S^2 \times S^1, \alpha_i, \gamma_-)$ is represented by the map

$$S_{\alpha_i} : [0, 1] \times [0, q] \to S^2 \times S^1, \quad (s, t) \mapsto (R_{t \frac{q}{q}}(\eta(s)), t),$$

where $\eta$ is a portion of the great circle which begins at $p_-$ and ends at $z_{\frac{q}{q}}$.

The class $Z_\alpha$ is related to the class $Z'_{\alpha}$ as follows. We have $Z_- = Z'_- = Z'_0 + \lceil h'(1) \rceil[S^2]$ and for $\alpha_i = e_{p,q}$ or $h_{p,q}$, then $Z_{\alpha_i} = Z'_{\alpha_i} + p[S^2]$. If we denote by $(0, y_\alpha)$ and $(d, w_\alpha)$ the endpoints of $P_{\alpha,Z}$, we thus obtain $Z_\alpha = Z'_\alpha + (w_\alpha - y_\alpha)[S^2]$. Using (50), this yields

$$I(\alpha, Z_\alpha) = I(\alpha, Z'_\alpha) + (w_\alpha - y_\alpha)(2d + 2). \quad (51)$$

We will now compute the index $I(\alpha, Z'_\alpha)$. For that purpose, we first need to make choices of trivializations along periodic orbits.

Along the orbit $\gamma_-$, the trivialization is given by any frame of $T_{p_+}S^2$ independent of $t$; note that for this specific orbit the trivialization was fixed in Remark 3.8.

Along $\gamma_+$, we take a frame which rotates positively with rotation number $\lceil h'(1) \rceil$. Along other orbits, we use as trivializing frame $(\partial_y, \partial_z) \in T_{S^2}$. We denote by $\tau$ these choices of trivialization.

Recall that we are also assuming for simplicity that $h'(-1)$ is arbitrarily close to 0 and $h'(1)$ is arbitrary close (but not equal) to its ceiling $\lceil h'(1) \rceil$.

In order to compute the grading, we now have to compute the Conley-Zehnder index, the relative Chern class, and the relative self-intersection; we then have to put this all together to give the stated interpretation in terms of a count of lattice points.

Step 1: The Conley-Zehnder index.

We begin by computing the Conley-Zehnder index of each orbit, relative to the trivialization above.

1. The North pole orbit $\gamma_+$ is elliptic with rotation number $h'(1) - \lceil h'(1) \rceil$.
   Picking $h'(1)$ to be sufficiently close to its ceiling, we then find by (49) that
   $$CZ(\gamma_+^k) = [k(h'(1) - \lceil h'(1) \rceil)] + [k(h'(1) - \lceil h'(1) \rceil)] = -1,$$
   for any $k = 1, \ldots, d$.

2. The South pole orbit $\gamma_-$ is elliptic with rotation number $-h'(-1)$ with respect to the considered trivialization. Since $h'(-1)$ is positive but arbitrary small, we then find by (49) that:
   $$CZ(\gamma_-^k) = [-k h'(-1)] + [-k h'(-1)] = -1.$$
3. For other orbits, we are in the same settings as [HS05]. Namely, for hyperbolic orbits, the rotation number is 0, so from (49) we have
\[ CZ(h_{p/q}) = 0, \]
and for the elliptic orbits \( e_{p/q} \), the rotation number is slightly negative, so that from (49) we have
\[ CZ(e_{k_{p/q}}) = -1, \]
for any \( k = 1, \ldots, d \).

It follows from the above that the contribution of the Conley-Zehnder part to the index in (22) is given by
\[ CZ(\tau(\alpha)) = \sum_i m_i \sum_{k=1}^{m_i} CZ(\alpha_i^k) + \sum_{k=1}^{m_-} CZ(\gamma^-_k) + \sum_{k=1}^{m_+} CZ(\gamma^+_k) = -M + h, \] (52)
where \( M \) denotes the total multiplicity of all orbits, and \( h \) denotes the total number of hyperbolic orbits.

**Step 2: The relative Chern class.**

The relative Chern class \( c_r(Z'_\alpha) \) is obviously 0. For \( \alpha_i = e_{p,q} \) or \( h_{p,q} \), we consider the representative \( S_{\alpha_i} \) of \( Z'_\alpha \) given above. We choose the section \( \partial_\theta \) as non-winding section of \( V|_{S_i} \) along \( \alpha_i \), and any constant non-zero vector along \( q\gamma_- \). Then, the section \( \partial_\theta \) over the orbit \( \alpha_{p/q} \) has index \(-p\) while the section over \( q\gamma_- \) has index 0. It follows that any extension of these sections over \( S_{\alpha_i} \) must have \(-p\) zeros. Hence,
\[ c_r(Z'_\alpha) = -p. \]

For \( Z_+ \), an argument analogous to that of the previous paragraph gives
\[ c_r(Z'^+_\alpha) = -[h'(1)]. \]

The Chern class is additive, so we conclude from the above that the \( c_r \) term of the index is
\[ c_r(Z'_\alpha) = \sum m_i c_r(Z'_{\alpha_i}) + m_- c_r(Z'_-\gamma) + m_+ c_r(Z'_+\gamma) \]
\[ = -\sum m_i p_i - m_+[h'(1)] = -w_\alpha + y_\alpha. \] (53)

**Step 3: The relative self-intersection**

Inspired by an analogous construction performed in [HS05] Lemma 3.7, we construct a representing surface \( S \subset [0,1] \times S^2 \times S^1 \) of \( Z'_\alpha \) as a movie of curves as follows. Denote by \( \sigma \) the variable in \([0,1]\), and \( S_\sigma = \{ \sigma \} \cap S^2 \times S^1 \). We will describe \( S_\sigma \) as \( \sigma \) decreases from 1 to 0.

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• For $\sigma = 1$, $S_1$ is the union of the orbits appearing in $\alpha$ with non-zero multiplicity.

• For values of $\sigma$ close to 1, $S_\sigma$ consists of

(a) $m_i$ circles, parallel to the orbit $\alpha_i$, in the torus $\{z = z_{p_i}/q_i\} \times S^1 \subset S^2 \times S^1$ (these circles have slope $\frac{q_i}{p_i}$ if we see this torus as $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$),

(b) $m_+$ parallel circles with slope $\frac{1}{[h'(1)]}$ in the torus $\{z = \sigma\} \times S^1$,

(c) $m_-$ parallel “vertical” circles $\{pt\} \times S^1$ in the torus $\{z = -\sigma\} \times S^1$.

• As $\sigma$ decreases to $1/2$, we move all these circles to the same $\{z = \text{constant}\} \times S^1$ torus. As in [HS05], we perform negative surgeries, around $\sigma = 1/2$ to obtain an embedded union of circles in a single $\{z = \text{constant}\} \times S^1$ torus; this union will consist of $k$ (straight) parallel embedded circles directed by a primitive integral vector $(a, b)$. The vector $(a, b)$ and the number $k$ of circles are determined by our data: for homological reasons we must have

$$kb = d, \quad ka = \sum_i m_ip_i + m_+[h'(1)] = w_\alpha - y_\alpha.$$  

• As $\sigma$ decreases from $1/2$ to 0, we simply modify the torus in which the curves are located, without changing the curves themselves, so that at $\sigma = 0$, and make this torus shrink towards $\gamma_-$.

We will compute $Q_\tau(Z'_\alpha)$ using the above surface $S$ and the formula (48).

To compute $c_1(N, \tau)$, we take $\psi \in \Gamma(N)$ as follows: we take $\psi = \pi_N \partial_\theta$ everywhere on $S$ except in a small neighborhood of the boundary components $\gamma_+, \gamma_-$ where the vector $\partial_\theta$ is not well-defined. The surface is constructed such that $\psi$ may be extended to a $\tau$-trivial section of $N$ over $\gamma_+$ without introducing any zeroes. However, extending $\psi$ to a $\tau$–trivial section of $N$ over $\gamma_-$ necessarily creates $-a$ zeroes for each of the $k$ embedded circles which gives a total of $-ka$ zeroes.

As in [HS05], the other zeroes of $\psi$ appear at the surgery points with negative signs and their number is given by

$$-\sum \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix},$$

where the sum runs over all pairs of distinct edges $v_{p,q}, v_{p',q'}$ in $P_{\alpha,Z}$, with $\frac{p'}{q'} < \frac{p}{q}$. Geometrically, this sum can be interpreted as $-2\text{Area}(R'_\alpha)$, where $R'_\alpha$ is the region between $P_{\alpha,Z}$ and the straight line connecting $(0, y_\alpha)$ to $(d, w_\alpha)$. Thus, we obtain:

$$c_1(N, \tau) = -(w_\alpha - y_\alpha) - 2\text{Area}(R'_\alpha).$$
We must now compute the writhe $w_\tau(S)$. By construction, there is no writhe near $\sigma = 1$. Near $\sigma = 0$ the writhe is given by the writhe of the braid $k(a,b)$ on the torus which is $(w_\alpha - y_\alpha)(1 - d)$, so we get

$$w_\tau(S) = (w_\alpha - y_\alpha)(d - 1).$$

Summing the above, we therefore get

$$Q_\tau(Z'_\alpha) = -(w_\alpha - y_\alpha) - 2\text{Area}(R'_\alpha) - (w_\alpha - y_\alpha)(d - 1). \quad (54)$$

**Step 4: The combinatorial interpretation**

We now put all of this together to prove the first item in Proposition 5.2. By combining (52), (53) and (54) and the definition of the grading (22), we have

$$I(\alpha, Z'_\alpha) = -M + h - (w_\alpha - y_\alpha)(d + 1) - 2\text{Area}(R'_\alpha).$$

Using Equation (51), we obtain

$$I(\alpha, Z_\alpha) = -M + h + 2\text{Area}(R_\alpha) + (w_\alpha - y_\alpha),$$

where $R_\alpha$ denotes the region between $P_{\alpha,Z}$ and the $x$-axis.

By Pick’s theorem,

$$2\text{Area}(R_\alpha) = 2T - (M + d + (w_\alpha - y_\alpha)) - 2,$$

where $T$ denotes the total number of lattice points in the closed region $R_\alpha$. So, by combining the previous two equations, we get

$$I(\alpha, Z_\alpha) = 2(T - M - 1) - d + h.$$ 

Now $(T - M - 1)$ is exactly the number of lattice points in the closed region $R_\alpha$, not including the lattice points on the path, and so, $T - M - 1 = j$, hence the second item of Proposition 5.2 is proved for $Z = Z_\alpha$.

Now remember that if we shift our path upwards by $(0,1)$, then $j$ increases by $d + 1$. Thus, using (50), we deduce that the second item of Proposition 5.2 is satisfied for all relative class $Z$.

**5.3.1 Fredholm index in the combinatorial model**

The goal of this section is to give a simple formula for the Fredholm index which relates it to our combinatorial model.

Let $C$ be a $J$–holomorphic curve in $\mathcal{M}_J((\alpha,Z), (\beta,Z'))$, where $J$ is weakly admissible. Recall from Equation (21) that the Fredholm index of $C$ is given by the formula

$$\text{ind}(C) = -\chi(C) + 2\text{c}_\tau(C) + CZ_\tau^{\text{ind}}(C),$$

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where $\chi(C)$ denotes the Euler characteristic of the curve, $c_\tau(C)$ is the relative first Chern class which we discussed above, and $CZ^\text{ind}_\tau$ is a term involving the Conley-Zehnder index defined as follows: Write $\alpha = \{(\alpha_i, m_i)\}$ and $\beta = \{(\beta_j, n_j)\}$. Suppose that $C$ has ends at $\alpha_i$ with multiplicities $q_{i,k}$ and ends at $\beta_j$ with multiplicities $q'_{j,k}$; note that we must have $\sum_k q_{i,k} = m_i$ and $\sum_k q'_{j,k} = n_j$. Then,

$$
CZ^\text{ind}_\tau(C) := \sum_i \sum_k CZ_\tau(\alpha_{i,k}^{q_i}) - \sum_j \sum_k CZ_\tau(\beta_{j,k}^{q'_{j,k}}).
$$

The next lemma explains how to compute $\text{ind}(C)$ from the combinatorial model. In this lemma, we denote the starting points of $P_{\alpha,Z}$ and $P_{\beta,Z'}$ by $(0, y_\alpha)$ and $(0, y_\beta)$, and their endpoints by $(d, w_\alpha)$ and $(d, w_\beta)$, respectively.

**Lemma 5.3.** Let $\varphi_0$ be a nice perturbation of $\varphi^1_H$, where $H \in \mathcal{D}$, let $J$ be any weakly admissible almost complex structure, and let $C$ be any irreducible $J$-holomorphic curve from $(\alpha, Z)$ to $(\beta, Z')$. Then,

$$
\text{ind}(C) = -2 + 2g + 2e_- + h + 2v,
$$

where $g$ is the genus of $C$, $e_-$ is the number of negative ends of $C$ which are at elliptic orbits, $h$ is the number of ends of $C$ at hyperbolic orbits, and $v = (y_\alpha - y_\beta) + (w_\alpha - w_\beta)$.

**Proof.** We will prove the lemma by describing each of the three terms $\chi(C), c_\tau(C)$, and $CZ^\text{ind}_\tau(C)$, which appear in $\text{ind}(C)$, in terms of our combinatorial model.

The number of ends of the curve $C$ is given by the sum $e_- + e_+ + h$, where $e_+$ denotes the number of positive ends of $C$ which are at elliptic orbits. The Euler characteristic of $C$ is given by the formula

$$
\chi(C) = 2 - 2g - e_- - e_+ - h.
$$

As for the Chern class, because $[C] = Z - Z'$, we can write $c_\tau(C) = c_\tau(Z) - c_\tau(Z')$. Now, by the index computations of the previous section, $c_\tau(Z) = w_\alpha + y_\alpha$. Similarly, $c_\tau(Z') = w_\beta + y_\beta$. It follows that

$$
c_\tau(C) = v.
$$

To compute $CZ^\text{ind}_\tau(C)$, note that by our computations from the previous section, we have $CZ_\tau(\alpha_{i,k}^{q_i}) = \text{ if } \alpha_i \text{ is elliptic and } 0, \text{ otherwise; a similar formula holds for } CZ_\tau(\beta_{j,k}^{q'_{j,k}})$. This implies that

$$
CZ^\text{ind}_\tau(C) = -e_+ + e_-.
$$

Combining the above, we get

$$
\text{ind}(C) = -2 + 2g + 2e_- + h + 2v.
$$

\[\square\]
Remark 5.4. As already mentioned in Section 3.1.1, in some situations we want the added flexibility of being able to work with weakly admissible almost complex structures. Lemma 5.3 above is also stated for weakly admissible almost complex structures, as are the forthcoming Lemma 5.5 and Lemma 5.9. Ultimately, the reason we want to be able to work with weakly admissible almost complex structures is because of the very useful Lemma 5.14 below, which will allow us to connect an admissible \( J \) to a weakly admissible one in order to facilitate computations; the point is that the proof of Lemma 5.14 requires Lemma 5.3, Lemma 5.5 and Lemma 5.9 in the weakly admissible case.

5.4 Positivity

We now begin the proof of the third item of Proposition 5.2; this will take several subsections and will require a close examination of those \( J \)-holomorphic curves in \( \mathbb{R} \times S^1 \times S^2 \) which appear in the definition of the PFH differential. (Recall from Section 3.3 that \( X \) is identified with \( \mathbb{R} \times S^1 \times S^2 \).)

In this section, we prove a very useful lemma which puts major restrictions on what kind of \( J \)-holomorphic curves can appear.

To prepare for the lemma of this section, we need to introduce some new terminology. For \( -1 < z_0 < 1 \), define the slice

\[
S_{z_0} := \{(s, t, \theta, z) \in \mathbb{R} \times S^1 \times S^2 : z = z_0\}.
\]

This is homotopy equivalent to a two-torus, and in particular we have \( H_1(S_{z_0}) = H_1(S^1_t \times S^1_\theta) \); we identify \( H_1(S^1_t \times S^1_\theta) \) with \( \mathbb{Z}^2 \) so that the positively oriented circle factors \( S^1_t \) and \( S^1_\theta \) correspond to the vectors \((1, 0)\) and \((0, 1)\), respectively.

Let \( C \) be a \( J \)-holomorphic curve whose domain is a Riemann surface \( \Sigma \). If \( C \) is transverse to \( S_{z_0} \) (which happens for generic \( z_0 \)) and \( C \) has no ends near \( z_0 \), then \( \Sigma_{z_0} = C^{-1}(S_{z_0}) \) is a (possibly empty) compact 1-dimensional submanifold of \( C \). When non-empty, it is the boundary of the subdomain of \( \Sigma \) given by \( C^{-1}\{z \leq z_0\} \). Thus, the orientation of \( \Sigma \) induces an orientation on \( \Sigma_{z_0} \), with the convention “outer normal first”\(^\text{15}\). Therefore, we get a well-defined class \([C_{z_0}] = C_*[\Sigma_{z_0}] \in H_1(S_{z_0}) = \mathbb{Z}^2\), which we call the slice class.

Lemma 5.5. Let \( \varphi_0 \) be a nice perturbation of \( \varphi_1^H \), where \( H \in \mathcal{D} \). Let \( C \) be a \( J \)-holomorphic curve, where \( J \) is weakly admissible, and let \( z_0 \in (-1, 1) \) be such that \( C \) intersects \( S_{z_0} \) transversally, and the nice perturbation vanishes in an open neighborhood of \( z_0 \). (In particular, \( C \) has no ends at Reeb orbits near \( z_0 \).) Then,

\[
(1, h'(z_0)) \times [C_{z_0}] \geq 0, \tag{56}
\]

with equality only if \( C \) does not intersect \( S_{z_0} \).

\(^{15}\) Equivalently, a vector \( v \) on \( \Sigma_{z_0} \) is positive if for an inner normal vector \( w \), the frame \((v, w)\) is positive.
Here, \((a, b) \times (c, d)\), where \((a, b) \in \mathbb{R}^2\), is defined to be the quantity \(ad - bc\).

**Remark 5.6.** In the context of the above lemma, suppose that \(C\) is irreducible and let \(z_{\text{min}} := \inf\{z_0 : [Cz_0] \neq 0\}\) and \(z_{\text{max}} := \sup\{z_0 : [Cz_0] \neq 0\}\); here, we consider the \(z_0\) such that the above lemma is applicable. The curve \(C\) is connected, because it is irreducible, and thus its projection to \(S^2\) is also connected. Therefore, \(C\) is contained in \(\{(s, t, \theta, z) \in \mathbb{R} \times S^1 \times S^2 : z_{\text{min}} \leq z \leq z_{\text{max}}\}\).

**Remark 5.7.** In the context of the above lemma, suppose that \(C\) is a \(J\)-holomorphic curve such that \(Cz_0 = 0\) for all \(z_0\) satisfying the conditions of the lemma. Then, as a consequence of the above lemma, \(C\) must be a local curve in the following sense: there exists \(z_{p,q}\) such that all ends of \(C\) are either at \(e_{p,q}\) or \(h_{p,q}\).

**Proof.** We use the fact that, by Lemma 3.3, the canonical 2-form \(\omega_\varphi\) is pointwise nonnegative on \(C\), with equality only if the tangent space is the span of \(\partial_s\) and \(R\). Namely, as a function of \(z > z_0\), close to \(z_0\), we have that the mapping

\[
\rho : z \mapsto \int_{C^{-1}(\mathbb{R} \times S^1 \times S^1 \times \mathbb{R})} C^* \omega_\varphi
\]

is non-decreasing, as \(z\) increases. Hence, its derivative with respect to \(z\) is nonnegative. We will prove Equation (56) by showing that

\[
\rho'(z) = \frac{1}{2} (1, h'(z)) \times [Cz_0],
\]

for \(z\) close to \(z_0\) and \(z \geq z_0\).

Recall from Section 3.3 that \(\omega_\varphi\) is identified with \(\omega + d\tilde{H} \wedge dt\). Since \(\tilde{H}\) coincides with \(H\) near \(z_0\) we can write \(\omega_\varphi = \omega + dH \wedge dt\). Now, we have

\[
\omega_\varphi = \omega + dH \wedge dt = \frac{1}{4\pi} d(-z d\theta) + d(H dt) = d(\frac{-1}{4\pi} z d\theta + H(z) dt).
\]

Let \(\alpha = H(z) dt - \frac{1}{4\pi} z d\theta\) and note that \(\alpha\) restricts to a closed 1-form on any slice.

Our choice of orientation (see Footnote 15) gives:

\[
\partial(C^{-1}(\mathbb{R} \times S^1 \times S^1 \times \mathbb{R}) \times [z_0, z]) = \Sigma_{z_0} - \Sigma_z.
\]

Thus it follows from the above:

\[
\rho(z) = \int_{C^{-1}(\mathbb{R} \times S^1 \times S^1 \times \mathbb{R})} C^* d\alpha = \int_{\Sigma_{z_0}} C^* \alpha - \int_{\Sigma_z} C^* \alpha
\]

\[
= \langle \alpha_{z_0} - \alpha_z, [Cz_0] \rangle.
\]
The rate of change of the expression above with respect to \( z \) is given by
\[
\rho'(z) = -\left(\frac{1}{2}h'(z), -\frac{1}{2}\right) \cdot [C_{z_0}] = \frac{1}{2} (1, h'(z)) \times [C_{z_0}],
\]
which extends by continuity to \( z_0 \). This proves (56).

To prove that equality in Equation (56) forces the slice to be empty, assume that \( \rho'(z_0) = 0 \). We have shown that \( \rho' \geq 0 \), and so \( \rho' \) must have a local minimum at \( z_0 \). Hence, \( \rho'' \) and \( \rho' \) vanish at \( z_0 \). We claim that this implies \([C_{z_0}] = 0\). To prove this, write \([C_{z_0}] = (a, b)\) and note that, by Equation (57), we have
\[
\rho'(z) = \frac{1}{2}(b - a h'(z)) \quad \text{and} \quad \rho''(z) = -\frac{1}{2} a h''(z).
\]
Since we are assuming \( h''(z_0) \neq 0 \), the vanishing of both of the above quantities can only take place if \( a = b = 0 \). We can therefore conclude that \([C_{z_0}] = 0\).

By the assumption that \( C \) intersects \( S_{z_0} \) transversely, we conclude that \([C_z] = 0\) for \( z \) sufficiently close to \( z_0 \). Hence, \( \rho(z) = 0 \) for \( z \) sufficiently close to \( z_0 \). But, by Lemma 5.3 this can only occur if the tangent space to \( C \cap ( \mathbb{R} \times S^1_x \times S^1_y \times \{z_0, z\} ) \) is always in the span of the Reeb vector field \( R \) and \( \partial_s \), which are tangent to \( S_z \). But, since \( C \) is transverse to \( S_{z_0} \), this cannot happen for \( z \) sufficiently close to \( z_0 \), unless the intersection \( C_{z_0} = C \cap S_{z_0} \) is empty.

The next lemma allows us to compute the slice class \([C_{z_0}]\) from our combinatorial model. We suppose here that \((\alpha, Z), (\beta, Z')\) are two PFH generators for \( \varphi_0 \), the nice perturbation of \( \varphi^1_H \), as described in Section 5.1 and that \( C \) is a \( J \)-holomorphic curve from \((\alpha, Z)\) to \((\beta, Z')\); recall that this means that \( C \) is a \( J \)-holomorphic curve from \( \alpha \) to \( \beta \) such that \([C] = Z - Z'\).

Let \( P_{\alpha, Z} \) and \( P_{\beta, Z'} \) be the concave lattice paths associated to \((\alpha, Z)\) and \((\beta, Z')\), respectively, as described in Section 5.2. For any \( z_0 \), let \( P^0_{\alpha, z_0} \) be the vector obtained by summing all of the vectors in the underlying path \( P_{\alpha} \) which correspond to Reeb orbits that arise from \( z < z_0 \). Denote
\[
P^0_{\alpha, Z} = (0, y_\alpha) + P^0_{\alpha},
\]
where \((0, y_\alpha)\) denotes the starting point of \( P_{\alpha, Z} \) on the \( y \)-axis. We define \( P^0_{\beta, z_0} \) and \( P^0_{\beta, Z'} \), analogously.

**Lemma 5.8.** Let \( C \) be a \( J \)-holomorphic curve from \((\alpha, Z)\) to \((\beta, Z')\) and let \( z_0 \in (-1, 1) \) be such that \( \alpha, \beta \) have no Reeb orbits near \( z_0 \) and suppose that \( C \) intersects \( S_{z_0} \) transversally. Then,
\[
[C_{z_0}] = P^0_{\alpha, Z} - P^0_{\beta, Z'}.
\]
Proof. First, note that the first coordinate of $P_{\alpha}^{z_0}$ (hence that of $P_{\alpha,Z}^{z_0}$, too) corresponds to the class in $H_1(S^1 \times S^2) \simeq \mathbb{Z}$ obtained by summing the contributions of the orbits in $\alpha$ that belong to the domain $\{z < z_0\}$. The second coordinate of $P_{\alpha}^{z_0}$ is given by the $\theta$ component of the class in $H_1(S^1 \times S^2)$ obtained by summing the contributions of the orbits in $\alpha$, other than $\gamma_-$, which are included in $\{z < z_0\}$. Analogous statements hold for $P_{\beta}^{z_0}$.

Pick some $z_- < z_+$ and denote the part of $C$ with $z_- \leq z \leq z_+$ by $C_{[z_-,z_+]}$. This is asymptotic to some orbit set $\alpha'$ at $+\infty$ and some orbit set $\beta'$ at $-\infty$.

The boundary of $C_{[z_-,z_+]}$ has components corresponding to $\alpha', \beta', C_{z_+}$ and $C_{z_-}$. The positive ends $\alpha'$ have the orientation coming from the Reeb vector field and the negative ends $\beta'$ have the opposite orientation. We therefore have

$$[C_{z_+}] = [\alpha'] - [\beta'] + [C_{z_-}]$$

(60)

in $H_1(S^1 \times S^2)$. Note that if $z_- > -1$, Equation (60) also holds in $H_1(S^1 \times S^2) \simeq \mathbb{Z}^2$.

If $z_- = -1$, and $z_+ = z_0$, this implies that the first component of $[C_{z_0}]$ is that of $[\alpha'] - [\beta']$, which is exactly the first coordinate of $P_{\alpha,Z}^{z_0} - P_{\beta,Z}^{z_0}$.

We now turn our attention to the second coordinate of $[C_{z_0}]$. Note that $[C_{z_0}]$ is determined by $[C] = Z - Z'$ in the following way. Consider the Mayer-Vietoris sequence associated to the cover by the two open subsets $\mathbb{R} \times S^1 \times \{z \in S^2 : z < z_0 + \delta\}$ and $\mathbb{R} \times S^1 \times \{z \in S^2 : z > z_0 - \delta\}$. The slice class $[C_{z_0}]$ is the image of $[C]$ under the connecting map

$$H_2(\mathbb{R} \times S^1 \times S^2, \alpha, \beta) \to H_1(\mathbb{R} \times S^1 \times \{z \in S^2 : z_0 - \delta < z < z_0 + \delta\}) \simeq H_1(S^2_{z_0}),$$

where $\delta > 0$ is small enough. In particular, adding $y[S^2]$ with $y \in \mathbb{Z}$ to the class $[C]$, adds $y$ to the second component of $[C_{z_0}]$.

To compute the second component of $[C_{z_0}]$, we apply Equation (60) in the case where $z_+ = z_0$ and $z_-$ is such that $C_{[z_0]}$ has no ends other than possibly $\gamma_-$. We obtain in this case that the second component of $[C_{z_0}]$ is the second coordinate of $P_{\alpha}^{z_0} - P_{\beta}^{z_0}$.

There only remains to establish that the second component of $[C_{z_-}]$ is $y_\alpha - y_\beta$. We will use here the fact that $[C_{z_-}]$ is determined by $Z$ and $Z'$. In the case, where $Z = Z_\alpha$ and $Z' = Z_\beta$ (as defined in Section 4.4), the second coordinate of $[C_{z_-}]$ vanishes. In general, $[C] = Z_\alpha + y_\alpha[S^2] - (Z_\beta + y_\beta[S^2])$. Thus, the second component of $[C_{z_-}]$ is $y_\alpha - y_\beta$, which concludes the proof.

\[\Box\]

5.5 Paths can not cross

As a consequence of the results in Section 4.4 we can prove the following useful fact which will play an important role in our proof of Proposition 5.2.
Lemma 5.9. Let $\varphi_0$ be a nice perturbation of $\varphi^1_H$, where $H \in \mathcal{D}$, and let $J$ be any weakly admissible almost complex structure. Then, if there exists a $J$-holomorphic curve $C$ from $(\alpha, \gamma)$ to $(\beta, \gamma')$, then $P^0_{\beta,Z'}$ is never above $P_{\alpha,Z}$.

Proof. Let $(0, y_\alpha)$ and $(0, y_{\beta})$ denote the starting points of $P_{\alpha,Z}$ and $P_{\beta,Z'}$, respectively. We will first show that $y_\alpha \geq y_{\beta}$. Denote by $m_\alpha, m_{\beta}$ the multiplicities of $\gamma_-$ in $\alpha$ and $\beta$, respectively. Let $z_0 = -1 + \varepsilon$ for some small $\varepsilon > 0$. We have $P^0_{\alpha,Z} = (m_\alpha, y_\alpha)$ and $P^0_{\beta,Z'} = (m_{\beta}, y_{\beta})$. Hence, by Lemma 5.8, $[C_{z_0}] = (m_\alpha - m_{\beta}, y_\alpha - y_{\beta})$. Applying Lemma 5.8, we obtain

$$(1, h'(z_0)) \times [C_{z_0}] = (y_\alpha - y_{\beta}) - h'(z_0)(m_\alpha - m_{\beta}) \geq 0.$$ 

By our conventions from Section 5.1, $h'(z_0) \approx 0$ and thus $(1, h'(z_0)) \times [C_{z_0}] \approx y_\alpha - y_{\beta}$. Since $y_\alpha - y_{\beta}$ is integer valued, the above inequality yields $y_\alpha \geq y_{\beta}$.

Next, let $(d, w_\alpha)$ and $(d, w_{\beta})$ denote the endpoints of $P_{\alpha,Z}$ and $P_{\beta,Z'}$, respectively. We will now show that $w_\alpha \geq w_{\beta}$. Denote by $n_\alpha, n_{\beta}$ the multiplicities of $\gamma_+$ in $\alpha$ and $\beta$, respectively. Let $z_0 = 1 - \varepsilon$ for some small $\varepsilon > 0$. We have $P^0_{\alpha,Z} = (d, w_\alpha) - n_\alpha (1, [h'(1)])$ and $P^0_{\beta,Z'} = (d, w_{\beta}) - n_{\beta} (1, [h'(1)])$.

Hence, by Lemma 5.8,

$$[C_{z_0}] = (n_\beta - n_\alpha, w_\alpha - w_{\beta} + (n_\beta - n_\alpha)[h'(1)]).$$

Now, applying Lemma 5.8, we obtain

$$(1, h'(z_0)) \times [C_{z_0}] = w_\alpha - w_{\beta} + (n_\beta - n_\alpha) ([h'(1)] - h'(z_0)) \geq 0.$$ 

By our conventions from Section 5.1, $[h'(1)] - h'(z_0) \approx 0$ and thus $(1, h'(z_0)) \times [C_{z_0}] \approx w_\alpha - w_{\beta}$. Since $w_\alpha - w_{\beta}$ is integer valued, the above inequality yields $w_\alpha \geq w_{\beta}$.

To complete the proof, suppose that the conclusion of the lemma does not hold. We have shown that $P_{\beta,Z'}$ cannot begin or end above $P_{\alpha,Z}$. Hence, we can find two intersection points $(a, b)$ and $(c, d)$, with $a < c$, between the two paths, such that the path $P_{\beta,Z'}$ is strictly above $P_{\alpha,Z}$ in the strip $\{(x, y) \in \mathbb{R}^2 : a < x < c\}$. Let $U, L$ denote the parts of $P_{\beta,Z'}, P_{\alpha,Z}$, respectively, which are contained in $\{(x, y) \in \mathbb{R}^2 : a \leq x \leq c\}$. Consider the line connecting $(a, b)$ and $(c, d)$. We can find a point $z_0$, with $a < z_0 < c$, such that $h'(z_0) = \frac{d - b}{c - a}$. We will compute the slice class $[C_{z_0 + \varepsilon}]$, for sufficiently small $\varepsilon > 0$ and will show that

$$(1, h'(z_0 + \varepsilon)) \times [C_{z_0 + \varepsilon}] < 0,$$

which contradicts Lemma 5.9. The reason for considering $[C_{z_0 + \varepsilon}]$ instead of $[C_{z_0}]$ itself is that there might be Reeb orbits in $\alpha, \beta$ corresponding to $z_0$ in which case we cannot apply Lemmas 5.3 & 5.8.

To compute the slice class $[C_{z_0 + \varepsilon}]$, we will compute $P^0_{\alpha,Z} + \varepsilon, P^0_{\beta,Z'} + \varepsilon$, and use Lemma 5.8. We begin with $P^0_{\alpha,Z}$. Let $(p, q)$ be the corner of $P_{\alpha,Z}$,
on L, with the following property: the edge in \( P_{\alpha,Z} \) to the left of \((p,q)\) has slope at most \( \frac{d-b}{c-a} \), and the edge to the right of \((p,q)\) has slope strictly larger than \( \frac{d-b}{c-a} \). The corner \((p,q)\) exists because \( L \) is strictly below the line passing through \((a,b)\) and \((c,d)\). Then, \( P_{\alpha,Z} = (p,q) \). Now, denote \( P_{\alpha,Z} = (p,q) \); this vector may be computed as follows: If the line passing through \((a,b)\) and \((c,d)\) is strictly above \( U \), then \((p',q')\) is computed exactly as above. If not, the line passing through \((a,b)\) and \((c,d)\) must coincide with \( U \); then \((p',q')\) is the endpoint of the edge in \( P_{\alpha,Z} \) containing \( U \).

We obtain
\[ [C_{z_0+\epsilon}] = (p - p', q - q') \]
with \((p,q)\) and \((p',q')\) as described in the previous paragraph. Now, we have
\[ (1, h'(z_0 + \epsilon)) \times [C_{z_0+\epsilon}] = \left(1, \frac{d-b}{c-a}\right) \times (p - p', q - q') .\]
This quantity is negative because \( L \) is strictly below \( U \). Indeed, one can see this by applying a rotation, which does not change the determinant, so that \((1, \frac{d-b}{c-a})\) is rotated to a positive multiple of \((1,0)\), and \((p - p', q - q')\) is rotated to a vector with a negative second component. Hence, \((1, h'(z_0 + \epsilon)) \times [C_{z_0+\epsilon}] < 0\) which contradicts Lemma 5.5.

5.6 Curves correspond to corner rounding

Using the results we have obtained thus far, we can now describe the configurations of concave paths which could give rise to a non-trivial term in the PFH differential. More precisely, we can now prove the “only if” part of Proposition 5.2, which we state as a lemma below:

Lemma 5.10. Let \( \varphi_0 \) be a nice perturbation of \( \varphi_1^H \), where \( H \in \mathcal{D} \). Assume that \( I(P_{\alpha,Z}) - I(P_{\beta,Z'}) = 1 \). Then, for generic admissible \( J \) close to \( J_{\text{std}} \),
\[ \langle \partial(\alpha,Z), (\beta,Z') \rangle \neq 0 \]
only if \( P_{\alpha,Z} \) is obtained from \( P_{\beta,Z'} \) by rounding a corner and locally losing one \( h \).

Proof. Assume that
\[ \langle \partial(\alpha,Z), (\beta,Z') \rangle \neq 0 .\]
for some generically chosen \( J \) and some generators \((\alpha,Z)\) and \((\beta,Z')\). We first choose \( J \) generically to rule out double rounding, which we can do by the argument in [HS05], Lemma A.16. In this argument, other than notational changes, we need to make one minor modification: the 2-form \( dt \wedge dy - ds \wedge dx \) in the proof of Lemma A.2 must be replaced with the 2-form \( dt \wedge d\theta - f(z) ds \wedge dz \), for a function \( f \) determined by \( h \). (We could give an explicit formula for \( f \), but it is not necessary for what we write here.) The reason we need
By Lemma 5.9 we know that \( P_{\beta,Z'} \) is never above \( P_{\alpha,Z} \). Consider the region between \( P_{\alpha,Z} \) and \( P_{\beta,Z'} \). We can take this region and decompose it into two kinds of subregions: Non-trivial subregions where \( P_{\alpha,Z} \) is above \( P_{\beta,Z'} \) — meaning that the parts of \( P_{\alpha,Z} \) and \( P_{\beta,Z'} \) intersect at most at two points in these regions; and, trivial subregions where the concave paths (without the labels) coincide.

We will first show that there is at least one non-trivial region. Arguing by contradiction, assume there is no non-trivial region, hence that \( P_{\alpha,Z} \) and \( P_{\beta,Z'} \) coincide as unlabeled concave paths. Let \( C \) be the unique embedded component of a given \( J \)-holomorphic curve from \((\alpha,Z)\) to \((\beta,Z')\); see Proposition 3.5. We claim that \( C \) must be local in the following sense: It is a \( J \)-holomorphic cylinder from the hyperbolic orbit, near some \( z = z_{p,q} \), that arises after the good perturbation, to the elliptic orbit near \( z = z_{p,q} \); furthermore, it does not leave the neighborhood of \( z_{p,q} \) where our good perturbation is non-trivial. Indeed, if \( C \) were not local, then we could find \( z_0 \in (-1,1) \) such that both of Lemmas 5.5 & 5.8 would be applicable at \( z_0 \). Now, Lemma 5.3 would imply \([C,z_0] \neq 0\), while Lemma 5.3 would imply \([C,z_0] = 0\) because the two (unlabeled) concave paths coincide. Hence, \( C \) must be local. As explained in the proof of Lemma 3.14 in [HS05] local curves appear in pairs and so their mod 2 count vanishes.

We will prove that if there exists \( C \in \mathcal{M}_J((\alpha,Z),(\beta,Z')) \), then \( P_{\alpha,Z} \) is obtained from \( P_{\beta,Z'} \) by rounding a corner and locally losing one \( h \). First, observe that it suffices to prove this under the assumption that \( C \) is irreducible. Indeed, if \( C \) is not irreducible, consider its embedded component \( C' \). Then, as a consequence of Proposition 3.3 there exists PHF generators \((\alpha_1,Z_1)\) and \((\beta_1,Z_1')\) such that \( C' \in \mathcal{M}_J((\alpha_1,Z_1),(\beta_1,Z_1')) \). Furthermore, \( P_{\alpha_1,Z_1} \) is obtained from \( P_{\beta_1,Z_1'} \) by rounding a corner and locally losing one \( h \) if and only if \( P_{\alpha,Z} \) is obtained from \( P_{\beta,Z'} \) via the same operation. We will suppose for the rest of the proof that \( C \) is irreducible.

**Claim 5.11.** Under the assumption that \( C \) is irreducible, the region between \( P_{\alpha,Z} \) and \( P_{\beta,Z'} \) contains no trivial regions and one non-trivial region.

**Proof of Claim.** Lemma 5.3 implies that if the number of non-trivial regions between \( P_{\alpha,Z} \) and \( P_{\beta,Z'} \) is at least 2, then the Fredholm index of \( C \) is also at least 2, because a non-trivial region makes a contribution of size at least 2 to the sum \( 2e_1 + h + 2v \) from Equation (55). Since \( \text{ind}(C) = 1 \), we conclude that the number of non-trivial regions must be one.

Next, we will prove that the number of trivial regions must be zero. First, we will show that an edge in \( P_{\beta,Z'} \) cannot have lattice points in its interior. Indeed, if such an edge existed it would make a contribution of to the function \( f \) because the almost complex structure \( J_{std} \) does not map \( \partial_z \) to \( \partial_\theta \), in contrast to the almost complex structure \( J_0 \) from [HS05], Lemma A.1; however, the rest of the argument there can be repeated essentially verbatim, since \( \int_C f(z) dsdz = 0 \), by Stokes’ theorem.
size at least 3 to the term $2e_- + h$ in Equation (55). This would then force $2e_- + h + 2v$ to be at least four which cannot happen because $\text{ind}(C) = 1$. Now, suppose that there exists a trivial region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$. We will treat the case where there is a trivial region to the left of the non-trivial region, leaving the remaining case, which is very similar, to the reader. There can be at most one trivial region because each trivial region makes a contribution of size at least 1 to $2e_- + h$, and so if there were two or more such regions $2e_- + h + 2v$ would be at least four. Let $v_{p,q}$ the vector/edge of the trivial region. Then, the edge in $P_{\beta,Z'}$ immediately to the right of $v_{p,q}$ corresponds to a vector $v_{p_1,q_1}$ with $\frac{p}{q} < \frac{p_1}{q_1}$; this inequality is strict because otherwise $P_{\beta,Z'}$ would have an edge with an interior lattice point. It follows that we can find $z_0$ such that $z_{p,q} < z_0 < z_{p_1,q_1}$ and Lemma 5.5 is applicable at $z_0$; moreover, $P_{\alpha,Z} = P_{z_0} = v_{p,q}$. Hence, $[C_{z_0}] = 0$. This contradicts Remark 5.4 because $C$ is asymptotic to orbits in both of $\{z \in S^2 : z \geq z_0\}$ and $\{z \in S^2 : z \leq z_0\}$. We therefore conclude that the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$ consists of a single non-trivial region.

Continuing with the proof of Lemma 5.10 observe that by the second item of Proposition 5.2, we have

$$I(\alpha, Z) - I(\beta, Z') = 2j + h_\alpha - h_\beta,$$

where $j$ is the number of lattice points in the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$, not including lattice points on $P_{\alpha,Z}$, and $h_\alpha, h_\beta$ denote the number of edges labeled $h$ in $P_{\alpha,Z}, P_{\beta,Z'}$, respectively. The number of edges in $P_{\beta,Z'}$, which we denote by $r_\beta$, satisfies the following inequality: $h_\beta \leq r_\beta \leq j + 1$. Hence, we have

$$I(\alpha, Z) - I(\beta, Z') \geq 2(r_\beta - 1) - r_\beta = r_\beta - 2,$$

with equality if and only if the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$ contains no interior lattice points, every edge of $P_{\beta,Z'}$ is labelled $h$, and no edge of $P_{\alpha,Z}$ is labelled $h$. Since $I(\alpha, Z) - I(\beta, Z') = 1$, we can rewrite the above inequality as $r_\beta \leq 3$.

If $r_\beta = 1$, then the equality $1 = 2j + h_\alpha - h_\beta$ can hold if and only if $j = 1, h_\alpha = 0, h_\beta = 1$. This implies that there are no interior lattice points in the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$; moreover, the two paths either begin at the same lattice point or end at the same lattice point. We see that in both cases $P_{\alpha,Z}$ is obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one $h$. Note that the corner rounding takes place at the extremity of $P_{\beta,Z'}$ which is not on $P_{\alpha,Z}$.

If $r_\beta = 2$, and if the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$ has at least one interior lattice point, then $j \geq 2$, so as $h_\beta \leq 2$, the index difference must be at least 2, which can not happen. Thus, the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$ must have no interior lattice points, and $h_\alpha = h_\beta - 1$. Thus, in this case
$P_{\alpha,Z}$ must be obtained from $P_{\beta,Z}$ by rounding a corner and locally losing one $h$.

If $r_\beta = 3$, then equality holds, so every edge of $P_{\beta,Z'}$ must be labelled $h$, no edge of $P_{\alpha,Z}$ can be, and there are no interior lattice points between $P_{\alpha,Z}$ and $P_{\beta,Z'}$; thus $P_{\alpha,Z}$ must be obtained from $P_{\beta,Z'}$ by double rounding. As explained in appendix A of [HS05], for a generic choice of the almost complex structure $J$ and a nice perturbation of Section 5.1, there exist no $J$–holomorphic curves corresponding to the double rounding operation. □

5.7 Corner rounding corresponds to curves

In the previous section we proved that if $\langle \partial(\alpha, Z), (\beta, Z') \rangle \neq 0$, then $P_{\alpha,Z}$ is obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one $h$. To complete the proof of Proposition 5.2, we must show the converse:

Lemma 5.12. Let $\varphi_0$ be a nice perturbation of $\varphi_1^H$, where $H \in D$. If $P_{\alpha,Z}$ is obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one $h$, then $\langle \partial(\alpha, Z), (\beta, Z') \rangle \neq 0$. In other words, counting mod 2 we have

$$\# M_J((\alpha, Z), (\beta, Z')) = 1,$$

for generic admissible $J \in \mathcal{J}(dr, \omega_\varphi)$.

The proof of the above lemma takes up the rest of this section. As we will now explain, it is sufficient to prove the lemma under the assumption that every $C \in M_J((\alpha_1, Z_1), (\beta_1, Z'_1))$ is irreducible: We can write $P_{\alpha,Z}$ and $P_{\beta,Z'}$ as concatenations

$$P_{\alpha,Z} = P_{\text{in}} P_{\alpha_1,Z_1} P_{\text{fin}},$$
$$P_{\beta,Z'} = P_{\text{in}} P_{\beta_1,Z'_1} P_{\text{fin}},$$

where $P_{\text{in}}$ and $P_{\text{fin}}$ correspond to the (possibly empty) trivial subregions between $P_{\alpha,Z}$ and $P_{\beta,Z'}$, and $P_{\alpha_1,Z_1}, P_{\beta_1,Z'_1}$ correspond to the non-trivial subregion; here we are using the terminology of Section 5.6. The concave path $P_{\alpha_1,Z_1}$ is obtained from $P_{\beta_1,Z'_1}$ by rounding a corner and locally losing one $h$, and the region between $P_{\alpha_1,Z_1}$ and $P_{\beta_1,Z'_1}$ consists of a single non-trivial subregion.

Claim 5.13. Every curve $C_1 \in M_J((\alpha_1, Z_1), (\beta_1, Z'_1))$ is irreducible.

Proof. By the structure of the corner rounding operation, $C_1$ has at most two negative ends. Thus, by degree considerations it must have at most two irreducible components; and, if it has exactly two irreducible components, then it must have exactly two negative ends, with one component corresponding to each end; assume this for the sake of contradiction, and write the components as $D_0$ and $D_1$. 66
By Lemma 5.5, each of the $D_i$ must correspond to a region between concave paths that do not go above $P_{\alpha_1,Z_1}$, and do not go below $P_{\beta_1,Z_1}'$. In fact, there are no such concave lattice paths, other than $P_{\alpha_1,Z_1}$ and $P_{\beta_1,Z_1}'$, because $P_{\alpha_1,Z_1}$ is obtained from $P_{\beta_1,Z_1}'$ by rounding a corner. Therefore, the region corresponding to each $D_i$ has a lower edge corresponding to one of the two edges of $P_{\beta_1,Z_1}'$ and the upper edges of each region must be on the edges of $P_{\alpha_1,Z_1}$. It follows from the concavity of the paths that each region must have $v \geq 1$, where $v$ is defined as in Lemma 5.3. Hence, by the Fredholm index formula (55), each $D_i$ must have index at least one and so $C_1$ must have index 2 which is not possible for generic $J$, by (20).

Next, note that as a consequence of Proposition 3.5, every curve $C \in \mathcal{M}_J((\alpha,Z),(\beta,Z'))$ can be written as a disjoint union

$$C = C_{\text{in}} \sqcup C_1 \sqcup C_{\text{fin}},$$

where $C_1 \in \mathcal{M}_J((\alpha_1,Z_1),(\beta_1,Z_1'))$ is the irreducible component of $C$ and $C_{\text{in}}, C_{\text{fin}}$ are unions of covers of trivial cylinders. It follows from Lemma 5.8 and the equality case of Lemma 5.3 that $C_{\text{in}}, C_{\text{fin}}$ correspond to the orbits in $P_{\text{in}}, P_{\text{fin}}$.

Combining Claim 5.13 and Equation (61), we obtain a canonical bijection

$$\mathcal{M}_J((\alpha,Z),(\beta,Z')) \leftrightarrow \mathcal{M}_J((\alpha_1,Z_1),(\beta_1,Z_1'))$$

given by removal of covers of trivial cylinders.

We conclude from the above discussion that it is indeed sufficient to prove Lemma 5.12 under the assumption that, for generic admissible $J$, every $C \in \mathcal{M}_J((\alpha,Z),(\beta,Z'))$ is irreducible. In terms of our combinatorial model, this assumption is equivalent to requiring that the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$ consists of one non-trivial subregion and no trivial subregions. This will be our standing assumption for the rest of this section.

5.7.1 Deformation of $J$

The next ingredient, which we will need to prove the Lemma 5.12, allows us to deform $J$ within the class of weakly admissible almost complex structures, while keeping the count of curves the same.

Recall our standing assumption that the region between $P_{\alpha,Z}$ and $P_{\beta,Z'}$ consists of one non-trivial subregion and no trivial subregions.

Lemma 5.14. Let $\varphi_0$ be a nice perturbation of $\varphi^1_H$, where $H \in D$. Let $J_0$ be an admissible and $J_1$ a weakly admissible almost complex structure. Assume that $P_{\alpha,Z}$ is obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one $h$. Then if $J_0$ and $J_1$ are generic,

$$\#\mathcal{M}_{J_0}((\alpha,Z),(\beta,Z')) = \#\mathcal{M}_{J_1}((\alpha,Z),(\beta,Z')).$$
Remark 5.15. We will apply the above lemma in a setting where $J_0, J_1$ are only defined on $\mathbb{R} \times X_1$, where $X_1$ is a subset of $Y_\varphi$ with the following property: There exists a compact subset $K \subset X_1$ such that for any weakly admissible almost complex structure $J$ on $\mathbb{R} \times Y_\varphi$, every $C \in \mathcal{M}_J((\alpha, Z), (\beta, Z'))$ is contained in $\mathbb{R} \times K$.

Because all $J$-holomorphic curves are contained in $\mathbb{R} \times K$, with $K \subset X_1$ compact, the proof we give below for Lemma 5.14 works verbatim in the setting of the previous paragraph as well. However, for clarity of exposition we give the proof in the setting were $J_0, J_1$ are globally defined.

To prove the above lemma, we will need the following claim which will be used in the proof below and the next section.

**Claim 5.16.** Suppose that $P_{\alpha, Z}$ and $P_{\beta, Z'}$ are as in Lemma 5.14 and recall the definitions of $P^\alpha_{\alpha, Z}, P^\beta_{\beta, Z'}$ from Lemma 5.8. Let $z_{\min}$ and $z_{\max}$ be the minimum and maximum values of $z_0 \in [-1, 1]$ such that $P^\alpha_{\alpha, Z} - P^\beta_{\beta, Z'} \neq 0$. Then, either $-1 < z_{\min}$ or $z_{\max} < 1$.

**Proof of Claim.** Denote by $(0, y_\alpha)$ and $(0, y_\beta)$ the starting points of $P_{\alpha, Z}$ and $P_{\beta, Z'}$, respectively, and by $(0, w_\alpha)$ and $(0, w_\beta)$ their endpoints. We begin by supposing $z_{\min} = -1$, and we will show this entails $z_{\max} < 1$. By Lemma 5.8 if $z_{\min} = -1$ then at least one of the following two scenarios must hold:

First, the path $P_{\beta, Z'}$ begins with a horizontal edge $(1, 0)$. Second, $y_\alpha > y_\beta$.

In the first scenario, $P_{\beta, Z'}$ must have a second edge $(q, p)$ labelled $h$; this edge corresponds to some $z_{p/q} < 1$. Note that we must also have $w_\beta = w_\alpha$. It then follows that $P^\alpha_{\alpha, Z} - P^\beta_{\beta, Z'} = 0$ for $z_{p/q} < z_0$ and so $z_{\max} < 1$.

In the second scenario, $P_{\beta, Z'}$ has only one $(q, p)$ and this edge is labelled $h$. Since it is labelled by $h$, this edge must correspond to some $z_{p/q} < 1$. As in the first scenario, we must also have $w_\beta = w_\alpha$. It then follows that $P^\alpha_{\alpha, Z} - P^\beta_{\beta, Z'} = 0$ for $z_{p/q} < z_0$ and so $z_{\max} < 1$.

The case where $z_{\max} = 1$ is similar to above and hence, we will not provide a proof.

**Proof of Lemma 5.14** Any curve $C$ in $\mathcal{M}_{J_0}$ or $\mathcal{M}_{J_1}$ has Fredholm index 1, and so it follows from Equation (5.5) that the genus of $C$ must be zero; this is because $2e_\pm + h + 2v \geq 2$ in our setting.

We next argue as in the proof of Lemma 3.17 of [HS05]. As we explained in Section 3.1.1 we can connect $J_0$ and $J_1$ with a smooth family of weakly admissible almost complex structures $J_s, s \in [0, 1]$. Consider the moduli space $\mathcal{M} := \cup_s \mathcal{M}_{J_s}((\alpha, Z), (\beta, Z'))$, for a generic choice of $J_s, s \in [0, 1]$. There exists a global bound on the energy of curves in $\mathcal{M}$: Indeed, any two $C, C'$ are homologous, as elements of $H_2(Y_\varphi, \alpha, \beta)$, and so have the same energy $\int_C \omega_\varphi = \int_{C'} \omega_\varphi$. Moreover, these curves all have genus zero as

\[ \int_C \omega_\varphi = \int_{C'} \omega_\varphi. \]

More precisely, this argument shows that, in the language of [BEH+03], $C, C'$ have the same $\omega$-energy. It then follows from Proposition 5.13 of [BEH+03] that there exists a global bound on the $\lambda$-energy of the curves in $\mathcal{M}$, as well.
we explained in the previous paragraph. Hence, we can appeal to the SFT compactness theorem\(^{18}\) from [BEH+03] to conclude that if a degeneration of the moduli space \(\mathcal{M}\) occurs at some \(s\), then there is convergence to a broken \(J\)-holomorphic building \((C_0, \ldots, C_k)\); here, \(C_i\) is the \(i^{th}\) level of the building and it is a \(J\)-holomorphic curve between PFH generators \((\alpha_i, Z_i)\) and \((\alpha_{i+1}, Z_{i+1})\) with \((\alpha_0, Z_0) = (\alpha, Z)\) and \((\alpha_{k+1}, Z_{k+1}) = (\beta, Z')\). This building is in the homology class \(Z - Z'\), has genus 0, and the top and bottom levels must have at most one end at any Reeb orbit, by the partition conditions provided by Theorem 1.7 and Definition 4.7 in [Hut02] (here we are using the fact that the monodromy angles of our elliptic periodic orbits are close to zero and negative, and that since the curve is supposed irreducible, the orbits constituting \(\beta\) have multiplicity 1).

By Lemma 5.5 any \(C_i\) must correspond to a region between paths that do not go above \(P_{\alpha, Z}\), and do not go below \(P_{\beta, Z'}\). In fact, there are no such concave lattice paths, other than \(P_{\alpha, Z}\) and \(P_{\beta, Z'}\) because \(P_{\alpha, Z}\) is obtained from \(P_{\beta, Z'}\) by rounding a corner. It follows that there exists \(0 \leq j < k\) such that the paths \(P_{\alpha_0, Z_0}, \ldots, P_{\alpha_j, Z_j}\) coincide with \(P_{\alpha, Z}\) and the remaining paths \(P_{\alpha_{j+1}, Z_{j+1}}, \ldots, P_{\alpha_k, Z_k}\) coincide with \(P_{\beta, Z'}\), as unlabeled lattice paths. In other words, the curve \(C_j\) is the unique curve in our building whose asymptotics correspond to two distinct unlabeled lattice paths.

We will now explain that bubbling does not occur in our setting. Let \(z_{\min}\) and \(z_{\max}\) be as in Claim 5.16. According to Lemma 5.8 and Remark 5.6 the curves \(C_i\) are all contained in

\[\mathcal{Y} = \{(s, t, \theta, z) \in \mathbb{R} \times S^1 \times S^2 : z_{\min} \leq z \leq z_{\max}\}\]

If a bubble were formed it would be contained in the above set \(\mathcal{Y}\) and therefore, it would be null homologous because, by Claim 5.16 either \(-1 < z_{\min}\) or \(z_{\max} < 1\). However, closed non-trivial pseudo-holomorphic maps are never null homologous and we conclude that no bubbles are formed.

We next prove that every irreducible curve in our building must have Fredholm index 0 or 1. To see this, first note that Equation (55) applies to any such curve \(C\). Since any such \(C\) has both positive and negative ends, \(C\) must have index at least \(-1\), and the only way for \(C\) to have index \(-1\) is for \(C\) to have exactly one negative end at a hyperbolic orbit, all positive ends at elliptic orbits, and \(v = \beta = 0\). If \(C\) is such a curve, then by Equation (55) the index of any somewhere injective curve that it covers must be \(-1\): such somewhere injective curves do not exist, even in (generic) one parameter families of \(J\), as observed in Lemma 3.15 of [HS05], because this is the index before modding out by translation. In particular, since the index is additive, and the index of the building is 1, one of the irreducible curves must have index 1, and all other irreducible curves must have index 0.

\(^{18}\) The SFT compactness theorem holds in our setting where we are allowing the stable Hamiltonian structures to varying smoothly; see, for example, page 170 of [Wen].
The conclusion of the previous paragraph implies that the curve $C_j$, whose asymptotics correspond to two distinct unlabelled lattice paths, must be irreducible. Indeed, we can repeat the argument given in the proof of Claim 5.13 to conclude that if $C_j$ were not irreducible it would then have index 2 which is not possible by the previous paragraph.

Next, we claim that every irreducible curve in the building must be a cylinder, other than $C_j$. To see this, let $P_{\alpha,Z}$ have $k$ edges and $P_{\beta,Z'}$ have $k'$ edges. Remember that since the curve is irreducible, $\beta$ admits either only 1 edge or 2 distinct edges. Then, by the partition condition considerations mentioned above, the Euler characteristic of the building must be $-k - k' + 2$. Thus, the sum of the Euler characteristic of all curves in the building must also be $-k - k' + 2$. Each curve must have at least one positive end and one negative end, so the Euler characteristic of each curve must be non-positive. And, since $C_j$ is irreducible, with ends at $k + k'$ distinct orbits, the Euler characteristic of $C_j$ must be at most $k + k' - 2$. It follows that every component other than $C_j$ must have Euler characteristic 0, hence must be a cylinder; and, $C_j$ has exactly one end at each of its Reeb orbits.

For, $i \neq j$, the slice class is always zero, and therefore, $C_i$ consists of cylinders which are either trivial or local; see Remark 5.7. We proved above that each irreducible component of our building, and in particular these cylinders, can have Fredholm index 0 or 1. We will now show that the only non-trivial cylinders which could exist are local cylinders of Fredholm index 1 with a positive end at a hyperbolic orbit and a negative end at an elliptic orbit. First, observe that any local cylinder with both ends at the same orbit must in fact be trivial; see for example Proposition 9.1 of [Hut02]. Thus, a non-trivial cylinder must have one elliptic end and one hyperbolic end, and, by Formula (55), for the Fredholm index to be non-negative, the positive and negative ends must be, respectively, hyperbolic and elliptic. The index will then be 1.

Since the index of the building is 1, we therefore learn that if the building is non-trivial, then it must consist of a single non-trivial index 1 local cylinder, with a positive end at a hyperbolic orbit, and a negative end at an elliptic orbit, and the curve $C_j$ must have index 0, and so it must have two negative ends, both at hyperbolic orbits, and all positive ends at elliptic orbits. Thus the building must have only two levels, and it is possible that in the level structure, $C_j$ could be either on top or on bottom. In either case, as observed in Lemma 3.15 of [HS05], there are two such local cylinders, that cancel, so standard gluing arguments show that the mod 2 count is unaffected by this degeneration. □
5.7.2 Existence of $J$–holomorphic curves

In this Section, we complete the proof of Lemma 5.12 by showing that $\#M_{J_1}((\alpha, Z), (\beta, Z')) = 1$ for generic weakly admissible $J_1$.

Below, we will introduce an open subset $X_1$ of $Y_\varphi$ satisfying the conditions of Remark 5.15. More specifically, it will satisfy the following property: There exists a compact subset $K \subset X_1$ such that for any weakly admissible almost complex structure $J$ on $\mathbb{R} \times Y_\varphi$, every $C \in M_J((\alpha, Z), (\beta, Z'))$ is contained in $\mathbb{R} \times K$. We will then prove the following lemma.

**Lemma 5.17.** For generic weakly admissible almost complex structure $J_1$ on $\mathbb{R} \times X_1$,

$$\#M_{J_1}((\alpha, Z), (\beta, Z')) = 1.$$ 

We now explain why the above lemma completes the proof of Proposition 5.2. Let $J_0$ be a generic and admissible almost complex structure on $Y_\varphi$, and connect its restriction to $K$ to $J_1$ via a generic smooth path $J_s$ of weakly admissible almost complex structures. Then, appeal to Lemma 5.14, Lemma 5.17 and Remark 5.15 to conclude that, counting mod 2, we have

$$\#M_{J_0}((\alpha, Z), (\beta, Z')) = \#M_{J_1}((\alpha, Z), (\beta, Z')) = 1,$$

which implies that $\langle \partial(\alpha, Z), (\beta, Z') \rangle = 1$.

It remain to prove Lemma 5.17. This will occupy the remainder of this section. We begin by recalling certain preliminaries which will be used in the course of the proof. Let $\Omega \subset \mathbb{R}^2$ be a subset of the first quadrant, and consider the set

$$X_\Omega := \{ (z_1, z_2) | \pi(|z_1|^2, |z_2|^2) \in \Omega \} \subset \mathbb{C}^2 = \mathbb{R}^4.$$ 

When $\Omega$ is the region bounded by the axes and the graph of a function $f$ with $f'' \leq 0$, then we call $X_\Omega$ a **convex toric domain**. When $\Omega$ is the region bounded by the axes and the graph of a function $f$ with $f'' \geq 0$, then we call $X_\Omega$ a **concave toric domain**. Much work has been done about these domains, see [CCGF + 14, Hut16, CG19].

The boundary $\partial X_\Omega$ of a toric domain is a contact manifold with the contact form $\lambda$ being the restriction to $\partial \Omega$ of the standard one-form on $\mathbb{R}^4$

$$\frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2).$$

Recall from Section 5.1 that $\mathbb{R} \times \partial X_\Omega$ is referred to as the **contact symplectization** and its symplectic form is given by $\omega = d(e^s \lambda)$, where $s$ denotes the coordinate on $\mathbb{R}$. In the argument below, we will be considering $J$–holomorphic curves in $\mathbb{R} \times \partial X_\Omega$ where $J$ is admissible for the SHS determined by the contact structure. Here, we will refer to such $J$ as **contact admissible**.

We now begin the proof of Lemma 5.17.
Proof. Recall the definitions of \( P_{\alpha,Z}^{\pm} \), \( P_{\beta,Z}^{\pm} \) from Lemma \( 5.8 \). Let \( z_{\text{min}} \) and \( z_{\text{max}} \) be the minimum and maximum value of \( z_0 \in [-1,1] \) such that \( P_{\alpha,Z}^{\pm} - P_{\beta,Z}^{\pm} \neq 0 \). In view of Claim 5.16 either \(-1 < z_{\text{min}} \leq 1 \). We will treat the two cases separately.

**Case 1:** \( z_{\text{min}} > -1 \). Define \( X_1 := \{(t,\theta,z) \in S^1 \times S^2 : -1 < z \} \). Let \( J \) be a weakly admissible almost complex structure on \( \mathbb{R} \times Y_{\phi} \) and denote by \( M_J(\alpha,\beta; X_1) \) the set of \( J \) holomorphic curves from \( \alpha \) to \( \beta \) with image contained in \( \mathbb{R} \times X_1 \). We claim that

\[
M_J((\alpha,Z),(\beta,Z')) = M_J(\alpha,\beta; X_1). \tag{62}
\]

To prove the above, first note that any \( J \)-holomorphic curve from \( (\alpha,Z) \) to \( (\beta,Z') \) is contained in \( \mathbb{R} \times X_1 \) because we have \( [C_z] \neq 0 \) only if \( z \geq z_{\text{min}} > -1 \); see Remark 6.6. This proves \( M_J((\alpha,Z), (\beta,Z')) \subseteq M_J(\alpha,\beta; X_1) \).

As for the other inclusion, observe that, because \( P_{\alpha,Z} \) is obtained from \( P_{\beta,Z'} \) by corner rounding, \( Z - Z' \) is an element of \( H_2(Y_{\phi},\alpha,\beta) \) which can be represented by a chain in \( X_1 \). Moreover, it is the only such element of \( H_2(Y_{\phi},\alpha,\beta) \) as other elements are of the form \( [Z - Z'] + k[S^2] \), where \( k \in \mathbb{Z} \) is nonzero. This proves \( M_J(\alpha,\beta; X_1) \subseteq M_J((\alpha,Z), (\beta,Z')) \).

Observe that the previous paragraph implies, in particular, that \( X_1 \) has the property stated in Remark 5.14 with \( K = \{(t,\theta,z) : z \geq z_{\text{min}} \} \).

We will next identify \( X_1 \) with a subset of the concave toric domain \( X_\Omega \) defined below. Consider the concave toric domain \( X_\Omega \), where \( \Omega \) is the region bounded by the axes and the graph of \( u \mapsto \frac{f(u)}{2} \), where \( f(x) := h(1-x) \) for \( 0 \leq x \leq 2 \), so that \( 0 \leq u \leq 1 \). (The factor of 2 here is merely a convenience to simplify the calculations that will follow; we could if we had preferred work with the graph of \( f \).

The boundary \( \partial X_\Omega \) is a contact manifold as described above. Consider the subset of \( \partial X_\Omega \) given by

\[
X_2 := \{(z_1,z_2)|\pi(|z_1|^2,|z_2|^2) \in \partial \Omega \setminus \{(1,0)\} \}.
\]

Note that this is \( \partial X_\Omega \) with a Reeb orbit removed. Define the mappings

\[
\psi : X_1 \rightarrow X_2, \quad (t,\theta,z) \mapsto \left( \frac{1}{2}(1-z), \theta, \frac{1}{2}h(z), 2\pi t \right),
\]

\[
\Psi : \mathbb{R} \times X_1 \rightarrow \mathbb{R} \times X_2, \quad (s,t,\theta,z) \mapsto \left( s, \frac{1}{2}(1-z), \theta, \frac{1}{2}h(z), 2\pi t \right). \tag{63}
\]

Here, we are regarding \( \partial X_\Omega \subseteq \mathbb{C}^2 \), and we are equipping \( \mathbb{C}^2 \) with coordinates \((\rho_1 := \pi|z_1|^2, \theta_1, \rho_2 := \pi|z_2|^2, \theta_2)\). Note that the standard-one form \( \lambda \), defined above, is given in these coordinates by

\[
\lambda = \frac{1}{2\pi}(\rho_1 d\theta_1 + \rho_2 d\theta_2). \tag{64}
\]

The above diffeomorphisms have the following properties:

(i) The Reeb vector field \( R \) on \( X_1 \) pushes forward under \( \psi \) to a positive multiple of the contact Reeb vector field \( \hat{R} \) on \( X_2 \).
(ii) The two-form $d\lambda$ on $X_2$ pulls back under $\psi$ to $\omega_\varphi$ on $X_1$. Thus, the SHS $(\lambda, d\lambda)$ on $X_2$ pulls back under $\psi$ to the SHS $(\psi^* \lambda, \omega_\varphi)$ on $X_1$. Item (i) above holds because at a point $(x, \theta_1, f(x), \theta_2)$, $R'$ is a positive multiple of

$$-f'(x)\partial_{\theta_1} + \partial_{\theta_2},$$

see for example Eq. 4.14 in [Hut14], while

$$R = \partial_t + 2\pi h'(z)\partial_\theta,$$

by combining (27) and (44), and $f'(x) = -h'(z)$. Item (ii) holds because $d\lambda$ is the restriction of

$$\frac{1}{2\pi}(d\rho_1 \wedge d\theta_1 + d\rho_2 \wedge d\theta_2),$$

which pulls back to

$$\frac{1}{4\pi}(d\theta \wedge dz + 2\pi h'(z)dz \wedge dt),$$

which is exactly $\omega_\varphi$.

Having established the above properties of the diffeomorphisms $\psi, \Psi$, we now proceed with the proof of Lemma 5.17. By property (i), $\psi$ induces a bijection between the Reeb orbit sets of $\hat{R}$ in $X_2$ and the Reeb orbit sets of $R$ in $X_1$. We’ll denote the induced bijection by

$$\alpha \mapsto \hat{\alpha}.$$

Now, suppose $P_{\alpha, Z}$ is obtained from $P_{\beta, Z'}$ via rounding a corner and locally losing one $h$. Let $\hat{J}$ be a contact admissible almost complex structure on the symplectization $\mathbb{R} \times \partial X_1$ and consider $\mathcal{M}^I_{\hat{J}}(\hat{\alpha}, \hat{\beta})$ the space of $\hat{J}$-holomorphic currents $C$, modulo translation in the $\mathbb{R}$ direction, with ECH index $I(\alpha, \beta, [C]) = 1$, which are asymptotic to $\alpha$ as $s \to +\infty$ and $\beta$ as $s \to -\infty$. For a generic choice of contact admissible $\hat{J}$, this moduli space is finite and, as alluded to in Remark 3.4, its mod 2 cardinality determines the ECH differential in the sense that

$$\langle \partial_{ECH} \hat{\alpha}, \hat{\beta} \rangle = \# \mathcal{M}^I_{\hat{J}}(\hat{\alpha}, \hat{\beta}).$$

As we will explain below, it follows from results proven in [Cho16] that the following hold:

A1. The image of every curve in $\mathcal{M}^I_{\hat{J}}(\hat{\alpha}, \hat{\beta})$ is contained in $\mathbb{R} \times X_2$.

B1. The mod 2 count of curves in $\mathcal{M}^I_{\hat{J}}(\hat{\alpha}, \hat{\beta})$ is 1.

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We will now explain why $A1 & B1$ imply Lemma 5.17. Indeed, define $J_1$ to be the almost complex structure on $\mathbb{R} \times X_1$ given by the pull back under $\Psi$ of a generic $\hat{J}$ as above. Then, $J_1$ is an almost complex structure on $\mathbb{R} \times X_1$ which is admissible for the SHS given by $(\psi^*\lambda, \psi^*d\lambda) = (\psi^*\lambda, \omega_\phi)$, which means that $J_1$ is weakly admissible; see Section 3.1.1. Next, by property A1, $\Psi$ induces a bijection between $\mathcal{M}_{J_1}(\hat{\alpha}, \hat{\beta})$ and $\mathcal{M}_{J_1}(\alpha, \beta, X_1)$. Property A2 then implies that $\mathcal{M}_{J_1}(\alpha, \beta, X_1) = 1$. Lastly, by Equation (62), we have

$$\mathcal{M}_{J_1}(((\alpha, Z), (\beta, Z'))) = 1,$$

which proves Lemma 5.17 in the case $z_{\min} > -1$.

Before elaborating on $A1 & B1$, we will treat Case 2.

**Case 2:** $z_{\max} < 1$.

The proof of Case 2 is similar, and in a sense dual, to that of Case 1.

Define $X_1 := \{(t, \theta, z) \in S^1 \times S^2 : z < 1\}$. Let $J$ be a weakly admissible almost complex structure on $\mathbb{R} \times Y_\phi$ and denote by $\mathcal{M}_J(\alpha, \beta; X_1)$ the set of $J$ holomorphic curves from $\alpha$ to $\beta$ with image contained in $\mathbb{R} \times X_1$. As in Case 1, we have

$$\mathcal{M}_J((\alpha, Z), (\beta, Z')) = \mathcal{M}_J(\alpha, \beta; X_1). \quad (65)$$

As before, $X_1$ has the property stated in Remark 5.15 with $K = \{(t, \theta, z) : z \leq z_{\max}\}$. We will next identify $X_1$ with a subset of the convex toric domain $X_\Omega$ defined below.

Consider the convex toric domain $X_\Omega$, where $\Omega$ is the region bounded by the axes and the graph of $u \mapsto g(2u^2)$, where

$$g(x) := h(1) - h(x - 1),$$

for $0 \leq x \leq 2$, so that $0 \leq u \leq 1$. The boundary $\partial X_\Omega$ is a contact manifold as described above. Consider the subset of $\partial X_\Omega$ given by

$$X_2 := \{(z_1, z_2) | |z_1|^2, |z_2|^2) \in \partial \Omega - \{(1, 0)\}.$$

Define the mappings

$$\psi : X_1 \to X_2, \quad (t, \theta, z) \mapsto \left(\frac{1}{2}(z + 1), \theta, \frac{1}{2}(h(1) - h(z)), 2\pi t\right),$$

$$\Psi : \mathbb{R} \times X_1 \to \mathbb{R} \times X_2, \quad (s, t, \theta, z) \mapsto (-s, \frac{1}{2}(z + 1), \theta, \frac{1}{2}(h(1) - h(z)), 2\pi t). \quad (66)$$

Here, we are regarding $\partial X_\Omega \subset \mathbb{C}^2$, and we are equipping $\mathbb{C}^2$ with coordinates $(\rho_1 := \pi |z_1|^2, \theta_1, \rho_2 := \pi |z_2|^2, \theta_2)$. Similarly (but dual) to Case 1, these diffeomorphisms have the following properties:

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(i) The Reeb vector field $R$ on $X_1$ pushes forward under $\psi$ to a positive multiple of the contact Reeb vector field $R$ on $X_2$.

(ii) The two-form $d\lambda$ on $X_2$ pulls back under $\psi$ to $-\omega_\phi$ on $X_1$. Thus, the SHS $(\lambda, -d\lambda)$ on $X_2$ pulls back under $\psi$ to the SHS $(\psi^* \lambda, \omega_\phi)$ on $X_1$.

By property (i), $\psi$ induces a bijection

$$\alpha \mapsto \hat{\alpha},$$

between the Reeb orbit sets of $R$ in $X_1$ and the Reeb orbit sets of $\hat{R}$ in $X_2$.

Now, suppose $P_{\alpha, Z}$ is obtained from $P_{\beta, Z'}$ via rounding a corner and locally losing one $h$. As in Case 1, let $\hat{J}$ be a contact admissible almost complex structure on $R \times \partial X_2$ and consider the moduli space $\mathcal{M}_{\hat{J}}^f(\hat{\beta}, \hat{\alpha})$ which for a generic choice of contact admissible $\hat{J}$ is finite and its mod 2 cardinality determines the ECH differential in the sense that

$$\langle \partial_{\text{ECH}} \hat{\beta}, \hat{\alpha} \rangle = \# \mathcal{M}_{\hat{J}}^f(\hat{\beta}, \hat{\alpha}).$$

As we will explain below, it follows from results proven in [Cho16] that the following hold:

A2. The image of every curve in $\mathcal{M}_{\hat{J}}^f(\hat{\beta}, \hat{\alpha})$ is contained in $R \times X_2$.

B2. The mod 2 count of curves in $\mathcal{M}_{\hat{J}}^f(\hat{\beta}, \hat{\alpha})$ is 1.

We will now explain why A2 and B2 imply 5.17. Define $J_1$ to be the almost complex structure on $R \times X_1$ given by the pull back under $\Psi$ of $-\hat{J}$ for a generic $\hat{J}$ as above. Then, $J_1$ is an almost complex structure on $R \times X_1$ which is admissible for the SHS given by $(\psi^* \lambda, \psi^*(-d\lambda)) = (\psi^* \lambda, \omega_\phi)$, which means that $J_1$ is weakly admissible. Next, by property A2, $\Psi$ induces a bijection between $\mathcal{M}_{\hat{J}}^f(\hat{\beta}, \hat{\alpha})$ and $\mathcal{M}_{J_1}(\alpha, \beta, X_1)$; and, there is a canonical bijection between $\mathcal{M}_{\hat{J}}^f(\hat{\beta}, \hat{\alpha})$ and $\mathcal{M}_{J_1}^f(\beta, \alpha)$ given by associating a $\hat{J}$-holomorphic curve

$$u : (\Sigma, j) \to (X_2, \hat{J})$$

to the $-\hat{J}$ holomorphic curve

$$u : (\Sigma, -j) \to (X_2, -\hat{J}).$$

We can now repeat the rest of the argument given in Case 1 to see that A2 & B2 imply Lemma 5.17.

As promised, we will next elaborate on properties A1, B1, from Case 1, and A2, B2, from Case 2.
5.7.3 Elaborations on properties A1, B1 and A2, B2

For the benefit of the reader, we now elaborate on A1, B1, A2, and B2 hold. We first explain items A1 and A2. We showed above, while proving Equations (62) and (65), that as a consequence of Lemma 5.5, any J–holomorphic curve from $(\alpha, Z)$ to $(\beta, Z')$ is contained in $\mathbb{R} \times X_1$. A reasoning similar to what we have given above, proves that items A1, A2 are consequences of Lemma 3.5 in [Cho16] which is analogous to our Lemma 5.5.

We now explain items B1 and B2, beginning with some context. The purpose of [Cho16] is to give a combinatorial realization of the ECH chain complex for any toric contact three-manifold. Here, we use Choi’s results in the special case of concave and convex toric domains.

It turns out that in these cases, the ECH chain complex has a particularly nice form, as was originally conjectured by Hutchings [Hut16], Conj. A.3.

We first explain the ECH combinatorial model in the case where $X_\Omega$ is a convex toric domain, which appears in Case 2 above. Let $X_\Omega$ be such a domain, described by a smooth function $f : [0, a] \to [0, b]$. In this case, we can represent an ECH generator $\hat{\alpha}$ by an ECH convex lattice path $P_{\hat{\alpha}}$: this is a piecewise linear lattice path that starts on the y-axis, ends on the x-axis, stays in the first quadrant, and is the graph of a concave function; we usually orient this convex lattice path “to the right”. Note that the region bounded by such a path and the coordinate axes is a convex region.

Hutchings conjectured ([Hut16], Conj. A.3), and Choi proved ([Cho16], Cor. 4.15) that the ECH chain complex differential in this case has a particularly nice form: it is given by rounding a corner and locally losing one $h$.

More precisely, we can augment any convex lattice path to a closed lattice path by adding edges along the axes, and then there is a nonzero differential coefficient $\langle \partial_{ECH} \hat{\beta}, \hat{\alpha} \rangle$ if and only if this augmentation of $P_{\hat{\alpha}}$ is obtained from the augmentation of $P_{\hat{\beta}}$ by rounding a corner and locally losing one $h$.

The ECH combinatorial model in the case where $X_\Omega$ is concave, which appears in Case 1 above, is dual to this. We represent an ECH generator by an ECH concave lattice path, and we augment by adding infinite rays along the axes. Then, by [Cho16], Prop. 4.14, Thm.4.8 there is a nonzero differential coefficient $\langle \partial_{ECH} \hat{\alpha}, \hat{\beta} \rangle$ if and only if this augmentation of $P_{\hat{\beta}}$ is obtained from the augmentation of $P_{\hat{\alpha}}$ by rounding a corner and locally losing one $h$.

We can now explain the proofs of B1 and B2. We begin with B1. In the assignation between an ECH orbit set $\hat{\alpha}$ and a concave lattice path $P_{\hat{\alpha}}$ in the combinatorial model of [Cho16], an edge $(p, q)$ corresponds to a point $r$ on the graph of $f$ with slope $q/p$, or to the two points $(0, a)$ and $(b, 0)$ where the graph of $f$ meets the axes. We also handle the elliptic orbits corresponding to $(0, a)$ and $(b, 0)$ as in the PFH case: in the case of an m-fold cover of the elliptic orbit corresponding to $(0, a)$ (which is the only case relevant to the proof), we concatenate the line segment with slope $f'(0)$ and horizontal
displacement $m$ to the beginning of the rest of the lattice path. Now, this
does not necessarily yield a lattice path, because $f'(0)$ could be irrational,
and so we take the lowest lattice path above this concatenated path and
then translate so that the path starts on the $y$-axis and ends on the $x$-axis.

We now describe how the identification of the previous section between
PFH and ECH orbit sets $\alpha \mapsto \hat{\alpha}$, translates to an identification of the cor-
responding lattice paths $P_{\alpha,Z} \mapsto P_{\hat{\alpha}}$: We take $P_{\alpha,Z}$ and translate it so that
it starts on the $x$-axis and ends on the $y$-axis, and reflect across the $y$-axis,
to get $P_{\hat{\alpha}}$, the ECH lattice path corresponding to $\hat{\alpha}$. Thus, since $P_{\alpha,Z}$ is
obtained from $P_{\beta,Z'}$ by rounding a corner and locally losing one $h$, the ECH
lattice path $P_{\beta}$ is obtained from $P_{\hat{\beta}}$ by rounding a corner and locally losing
one $h$, and so the above results apply to prove B1.

The proof of B2 is similar. The identification $\alpha \mapsto \hat{\alpha}$ translates to an
identification of the corresponding lattice paths $P_{\alpha,Z} \mapsto P_{\hat{\alpha}}$: given a PFH
lattice path $P_{\alpha,Z}$, we translate it so that it begins on the $y$-axis and ends
on the $x$-axis, we then reflect it across the $x$-axis to get $P_{\hat{\alpha}}$. As with B1,
this now implies B2.

Remarks 5.18.

To make our paper as self-contained as possible (while maintaining some
amount of brevity), we sketch those parts of Choi’s argument that are relevant
for what we need.

The first point to make is a historical one. Many of our arguments in
are inspired by arguments in [Cho16], which are themselves inspired by
arguments in Hutchings-Sullivan: in particular, most of the ideas needed
for Prop. 4.14 & Cor. 4.15 of [Cho16] have already been presented here,
although we presented them in the PFH case rather than the ECH case. In
particular, Choi proves analogues of all of the results in 5.2 through 5.7.1
and indeed his argument for reducing to considering regions given by locally
rounding a corner and losing one $h$ is quite similar to the argument we give.
So, the only differences really worthy of comment is on involve how Choi shows
that such a region actually carries a nonzero count of curves — here there
are two main differences between our argument and Choi’s that we should
highlight.

1. Choi finds his curves by referencing a paper by Taubes [Tau02], which
works out various moduli spaces of curves for a particular contact
form on $S^1 \times S^2$. Taubes’ contact form is Morse-Bott, so Choi does
a perturbation as in [HS05, Lem. 3.17], [HS06, Thm. 11.11, Step 4]
to break the Morse-Bott symmetry so as to obtain a nondegenerate
contact form: Choi then does a deformation argument that is very
analogous to what we do. We remark that this strategy was pioneered
by Hutchings and Sullivan in the series of papers [HS05] [HS06].
In contrast, we find our curves by referencing Choi’s paper rather than Taubes’. This means that we do not need any Morse-Bott argument.

2. Choi uses an inductive argument to reduce to considering moduli spaces of twice and thrice punctured spheres. The reason he does this is to be able to use the paper by Taubes, which does not directly address all the curves needed to analyze corner rounding operations, which could lead for example to curves with an arbitrary number of ends. This induction works by using the fact that the differential is already known to square to 0, by [HT07, HT09] — once one shows the result for the curves one gets from Taubes, given an arbitrary region that one wishes to show corresponds to a nonzero count of curves, one proves that if it did not have a nonzero count, the differential could not possibly square to zero, essentially by concatenating with a region that can be analyzed through the Taubes curves. As with the above item, we again remark that this strategy was previously pioneered by Hutchings and Sullivan in [HS05, HS06].

In contrast, we have no need to use this inductive argument, since we can reference Choi’s work for our needed curves.

As should be clear from the above, the main reason we have chosen to take a slightly different tactic from Choi concerning these two points is for brevity — we could have also used a strategy like the above, if for some reason we had wanted to. Another point from the above is that the ideas in the part of Choi’s paper relevant here already appear in the Hutchings-Sullivan papers [HS05, HS06]: what is especially new and impressive in Choi’s paper is the analysis of very general toric contact forms, (for example, the contact form on a toric domain that is neither concave nor convex) for which there could be curves corresponding to regions more general than those that come from rounding a corner and locally losing one $h$ — however, we do not need to consider these particular kinds of contact forms in this paper.

6 The Calabi property for positive monotone twists

In this section we provide a proof of Theorem 5.1. We will first use the combinatorial model of PFH, from the previous section, to compute the PFH spectral invariants for monotone twists; this is the content of Theorem 6.1. We will then use this computation to prove Theorem 5.1 this will be carried out in Section 6.2.
6.1 Computation of the spectral invariants

We will need to introduce some notations and conventions before stating, and proving, the main result of this section. Throughout this section, we fix \( \varphi \) to be a (smooth) positive monotone twist map of the disc. Recall from Remark 3.8 that we define PFH spectral invariants for maps of the disc by identifying \( \text{Diff}_c(\mathbb{D}, \omega) \) with maps of the sphere supported in the northern hemisphere \( S^+ \subset S^2 \), where the sphere \( S^2 \) is equipped with the symplectic form \( \omega = \frac{1}{4\pi}d\theta \wedge dz \).

Now, \( \varphi \) can be written as the time–1 map of the flow of an autonomous Hamiltonian

\[
H = \frac{1}{2}h(z),
\]

where \( h : S^2 \to \mathbb{R} \) is a function of \( z \) satisfying

\[
h' \geq 0, \quad h'' \geq 0, \quad h(-1) = 0.
\]

Note that the only difference between the Hamiltonians \( H \) as above and those considered in Section 5.1 is that here we allow \( h' \) to vanish.

Although \( \varphi \) is degenerate, we can still define the notion of a concave lattice path for \( \varphi \), by copying the definition at the beginning of Section 5.2, but not having any labelling of the edges. Although \( \varphi \) is degenerate, we can still define the notion of a concave lattice path for \( \varphi \) as any lattice path obtained from a starting point \((0, y), y \in \mathbb{Z}\), and a finite sequence of consecutive edges \( v_{p_i, q_i}, i = 0, \ldots, \ell \), such that:

- \( v_{p_i, q_i} = m_{p_i, q_i}(q_i, p_i) \) with \( q_i, p_i \) coprime,
- the slopes \( p_i/q_i \) are in increasing order,
- we have \( 0 \leq p_0/q_0 \) and \( p_\ell/q_\ell \) is either \( \lceil h'(1) \rceil \) or less than \( h'(1) \).

If \( p_0 = 0 \), we will denote \( v_- = m_-(1, 0) = v_{p_0, q_0} \). If \( p_\ell/q_\ell = \lceil h'(1) \rceil \), we will denote \( v_+ = m_+(1, \lceil h'(1) \rceil) = v_{p_\ell, q_\ell} \). We also let \( z_{p, q} \) be such that \( h'(z_{p, q}) = p/q \).

We can also define the action of such a path just as in Section 5.2. We first define

\[
A(v_-) = 0, \quad A(v_+) = m_+ \frac{h(1)}{2}, \quad A(v_{p, q}) = \frac{m_{p,q}}{2} (p(1 - z_{p,q}) + qh(z_{p,q})).
\]  

We then define the action of a concave lattice path \( P \) to be

\[
A(P) = y + m_+ \frac{h(1)}{2} + \sum_{v_{p,q}} A(v_{p,q}).
\]

The definition of \( j(P) \) from Section 5.2 (see Equation (46)) is still valid here. With this in mind, we have the following:
Theorem 6.1. Let $\varphi$ be a positive monotone twist. Then, for all integers $d > 0$ and $k = d \mod 2$,
\[
c_{d,k}(\varphi) = \max\{A(P) : 2j(P) - d = k\}, \tag{69}
\]
where the max is over all concave lattice paths $P$ for $\varphi$ of horizontal displacement $d$.

Proof. We can take a $C^\infty$ small perturbation of $\varphi$ to a $d$–nondegenerate Hamiltonian diffeomorphism $\varphi_0$ which itself is a nice perturbation of some $\varphi^1_H$, where $H \in \mathcal{D}$ as in Section 5.1.

Since $c_{d,k}(\varphi)$ is the limit of $c_{d,k}(\varphi_0)$, as we take smaller and smaller perturbations, it suffices to show that the analogous formula to (69) holds for $\varphi_0$. In other words, we wish to show
\[
c_{d,k}(\varphi_0) = \max\{A(P_{\alpha, Z}) : I(P_{\alpha, Z}) = k\}, \tag{70}
\]
where the max is over all concave lattice paths of horizontal displacement $d$.

To prove (70), given $(d, k)$, consider the element $\sigma$ of the PFH chain complex for $\varphi_0$ given by
\[
\sigma := \sum (\alpha, Z)
\]
where the sum is over all PFH generators $(\alpha, Z)$ where $\alpha$ consists of only elliptic orbits, is of degree $d$ and $I(\alpha, Z) = k$. Equivalently, the corresponding concave lattice path $P_{\alpha, Z}$ has edges which are all labelled $e$, it has degree $d$ and index $k$.

We first claim that $\sigma$ is in the kernel of the PFH differential. Indeed, by Proposition 5.2, the differential is the mod 2 sum over every $(\beta, Z')$ such that $P_{\alpha, Z}$ can be obtained from $P_{\beta, Z'}$ by rounding a corner and locally losing one $h$. Fix one such $P_{\beta, Z'}$. It has exactly one edge labelled $h$ and so there are exactly two concave paths, say $P_{\alpha, Z}$ and $P_{\beta, Z'}$, which are obtained from $P_{\beta, Z'}$ by rounding a corner and locally losing one $h$. The two paths $P_{\alpha, Z}$ and $P_{\beta, Z'}$ are different, because for example when you round the two corners for an edge, one rounding contains one of the corners, and the other contains the other corner. Now, $(\alpha, Z)$ and $(\beta, Z')$ both contribute to $\sigma$ and thus, $(\beta, Z')$ appears exactly twice in the differential of $\sigma$; hence, its mod 2 contribution to the differential is zero. Consequently, we see that $\sigma$ is in the kernel of the PFH differential.

Now, by Proposition 5.2, no concave path with all edges labeled $e$ is ever in the image of the differential, because the concave path corresponding to the negative end of a holomorphic curve counted by the differential has more edges labeled $h$ than the concave path corresponding to the positive end, and in particular has at least one edge labeled $h$. So, $[\sigma] \neq 0$ in homology. In fact, $\sigma$ must carry the spectral invariant for similar reasons. Specifically, if there is some other chain complex element $\sigma'$ homologous to $\sigma$, then $\sigma + \sigma'$ must
be in the image of the differential. Nothing in the image of the differential has a path with all edges labeled by \( e \), so \( \sigma' \) must contain all possible paths of degree \( d \) and index \( k \) with all edges labeled by \( e \), and so its action must be at least as much as \( \sigma \).

In view of the above paragraph, to prove Equation (70), we must show that the supremum in the equation is attained by a path whose edges are labelled \( e \). To see this, consider a path of degree \( d \) and index \( k \) with some edges labelled \( h \). As a consequence of the combinatorial index formula in Proposition 5.2, the number of edges labelled \( h \) must be even; denote this number by \( 2r \). If we round \( r \) corners, all of them being endpoints of edges with label \( h \), and remove all \( h \) labels, we obtain a path of the same grading all of whose edges are labelled \( e \). Now, the newly obtained path has larger action because the corner rounding operation increases action. This completes the proof. □

Remark 6.2. As a corollary to Theorem 6.1, we find that

\[ c_{d,k} \leq c_{d,k+2}, \]

as promised in Equation (43). Indeed, if we take an action maximizing path of index \( k \), with all edges labeled \( e \) as above, and round a corner, then the grading increases by 2, and the action increases as well. □

6.2 Proof of Theorem 5.1

We now give a proof of Theorem 5.1. Our proof relies on a version of the isoperimetric inequality for non-standard norms which we now recall.

Let \( \Omega \subset \mathbb{R}^2 \) be a convex subset. Using the standard Euclidean inner product, the dual norm associated to \( \Omega \), denoted \( || \cdot ||_{\Omega}^* \), is defined for any \( v \in \mathbb{R}^2 \) by

\[ ||v||_{\Omega}^* = \max \{ v \cdot w : w \in \partial \Omega \}. \] (71)

Let \( \Lambda \subset \mathbb{R}^2 \) be an oriented, piecewise smooth curve and denote by \( \ell_{\Omega}(\Lambda) \) its length measured with respect to \( || \cdot ||_{\Omega}^* \). It is remarkable that when \( \Lambda \) is closed, its length remains unchanged under translation of \( \Omega \).

For our proof, we will suppose that \( \Omega \) is the region bounded by the graph of \( h \), the horizontal line through \((1, h(1))\), and the vertical line through \((-1, 0)\). Denote by \( \hat{\Omega} \) the region obtained by rotating \( \Omega \) clockwise by ninety degrees. We orient the boundary \( \partial \hat{\Omega} \) counterclockwise with respect to any point in its interior.

Proof of Theorem 5.1. Let \( P \) be a concave lattice path of horizontal displacement \( d \) for \( \varphi \). Complete the path \( P \) to a closed, convex polygon by adding a vertical edge at the beginning of \( P \) and a horizontal edge at the
end; orient this polygon counterclockwise, relative to any point in its interior; and, rotate it clockwise by ninety degrees. Call the resulting path \( \Lambda \).

We will need the following lemma.

**Lemma 6.3.** The following identities hold:

1. \( \ell_\Omega(\partial \Omega) = 2(2h(1) - I) \), where \( I := \int_{-1}^{1} h(z)dz \).

2. \( \ell_\Omega(\Lambda) = dh(1) + 2V - 2A(P) \), where \( V \) denotes the vertical displacement of the path \( P \).

**Proof of Lemma 6.3.** According to the Isoperimetric Theorem [BM94], for a simple closed curve \( \Gamma \), we have

\[
\ell^2(\Gamma) \geq 4A(\Omega)A(\Gamma),
\]

where \( A(\Omega) \) and \( A(\Gamma) \) denote the Euclidean areas of \( \Omega \) and the region bounded by \( \Gamma \), respectively. Moreover, equality holds when \( \Gamma \) is a scaling of a ninety degree clockwise rotation of \( \partial \Omega \); see Example 8.3 in [Hut11].

The first item follows immediately from the equality case of the theorem applied to \( \Gamma = \partial \hat{\Omega} \) because \( A(\hat{\Gamma}) = A(\Gamma) = A(\Omega) = 2h(1) - I \). Alternatively, Item 1 could be obtained via direct computation.

We now prove the second item. The length of the polygon \( \Lambda \) is given by the sum \( \sum_{e \in \Lambda} ||e||_\Omega^* \), where the sum is taken over the edge vectors \( e \) of \( \Lambda \).

It follows from the method of Lagrange multipliers that

\[
||e||_\Omega^* = e \cdot p_e,
\]

for some point \( p_e \in \partial \Omega \) where \( e \) points in the direction of the outward normal cone at \( p_e \). Hence, we can write

\[
\ell_\Omega(\Lambda) = \sum_{e \in \Lambda} e \cdot p_e = \sum_{e \in \Lambda} e \cdot (p_e - m), \tag{72}
\]

where the second equality holds, for any \( m \in \mathbb{R}^2 \), because \( \Lambda \) is closed. We will calculate \( \ell_\Omega(\Lambda) \) using the choice \( m = (1, 0) \).

Let \( e \) denote one of the edges of \( \Lambda \) corresponding to a vector \( v_{p,q} = m_{p,q}(q, p) \) in \( P \). Now, we have \( e = m_{p,q}(p, -q) \), since we are taking a ninety degree clockwise rotation; moreover, \( p_e - m = (z_{p,q} - 1, h(z_{p,q})) \). Thus,

\[
e \cdot (p_e - m) = m_{p,q}(p, -q) \cdot (z_{p,q} - 1, h(z_{p,q}))
\]

\[
= m_{p,q}(p(z_{p,q} - 1) - qh(z_{p,q}))
\]

\[
= -2A(v_{p,q}),
\]

where the final equation follows from (67).

If \( e \) is an edge of \( \Lambda \) corresponding to either of the vectors \( v = v_- = m_-(1, 0) \) or \( v = v_+ = m_+(1, [h'(1)]) \), then a similar computation to the
above yields $e \cdot (p_e - m) = -2A(v)$. Summing over all of the edges $e$ of $\Lambda$, corresponding to vectors in $P$, we obtain the quantity

$$-2A(P).$$

(73)

The remaining two edges of $\Lambda$ are the vectors $e_1 = (0, d)$ and $e_2 = (-V, 0)$ for which we have

$$e_1 \cdot (p_{e_1} - m) = (0, d) \cdot (-1, h(1)) = dh(1),$$
$$e_2 \cdot (p_{e_2} - m) = (-V, 0) \cdot (-2, 0) = 2V.$$

(74)

We obtain from Equations (72), (73), and (74) that $\ell_\Omega(\Lambda) = dh(1) + 2V - 2A(P)$.

Step 1: Calabi gives the lower bound. Here, we will prove the lower bound needed for establishing Equation (72). In other words, we will show that for any sequence $(d, k)$, such that $d \to \infty$, we have

$$\text{Cal}(\varphi) \leq \liminf_{d \to \infty} \left( \frac{2c_{d,k}(\varphi)}{d} - \frac{k}{d^2 + d} \right),$$

(75)

To prove the above, fix $\epsilon > 0$. We will show that there exists a sequence of concave lattice paths $\{P_{\epsilon,d}\}$, indexed by $d$ the sufficiently large degree of $P_d$, such that $d \to \infty$, we have

$$\left| \text{Cal}(\varphi) - \left( \frac{2A(P_{\epsilon,d})}{d} - \frac{k_d}{d^2 + d} \right) \right| \leq 4\epsilon,$$

(76)

for sufficiently large $d$, where $k_d = 2j(P_{\epsilon,d}) - d$ denotes the combinatorial index of $P_{\epsilon,d}$. By Theorem 6.1, we have $A(P_{\epsilon,d}) \leq c_{d,k_d}(\varphi)$, and, as we explained in Section 5, see the discussion after Theorem 5.1, proving (76) for one sequence $\{((d, k))\}$, with $d \to \infty$, proves it for all such sequences, and so we conclude (76) from the above, since $\epsilon$ was arbitrary.

We now turn to the description of the concave paths $P_{\epsilon,d}$. Let $P$ be a concave path approximating the graph of $h$ such that it begins at $(-1, 0)$, is piecewise linear, and its vertices are rationals with numerator an even integer and denominator $d$. Let $\Lambda$ be the convex polygon obtained as follows: Add a vertical edge at the beginning of $P$ and a horizontal edge at the end of it; orient this polygon counterclockwise, relative to any point in its interior; and, rotate it clockwise by ninety degrees. The convex polygon $\Lambda$ approximates $\partial\hat{\Omega}$. More precisely, given $\epsilon$, we pick $d$ sufficiently large and then $P$ such that

(A) $P$ is within $\epsilon$ of the graph of $h$,

(B) $|\ell_\Omega(\Lambda) - \ell_\Omega(\partial\hat{\Omega})| \leq \epsilon$ which by Lemma 5.3 is equivalent to

$$|\ell_\Omega(\Lambda) - 2(2h(1) - I)| \leq \epsilon,$$
The area of the region under the path $P$, and above the $x$–axis, is within $\varepsilon$ of $I$.

Let $P_{\varepsilon,d}, \Lambda_{\varepsilon,d}$ be the images of $P, \Lambda$, respectively, under the mapping

$$(x, y) \mapsto \frac{d}{2}(x + 1, y).$$

For $d$ sufficiently large, the path $P_{\varepsilon,d}$ is a concave lattice path of degree $d$. Recall that $\text{Cal}(\varphi) = \frac{1}{2}I$. We will prove the two inequalities below, which will imply Equation (76):

$$\left| \frac{2A(P_{\varepsilon,d})}{d} - I \right| \leq 2\varepsilon,$$

$$\left| \frac{k_d}{d^2 + d} - \frac{I}{2} \right| \leq \frac{\varepsilon}{2}. \quad (77)$$

We first examine the term $\frac{2A(P_{\varepsilon,d})}{d}$. By Lemma 6.3 and using the fact that $\ell_{\Omega}(\Lambda_{\varepsilon,d}) = \frac{d}{2}\ell_{\Omega}(\Lambda)$, we obtain

$$\frac{2A(P_{\varepsilon,d})}{d} = \frac{dh(1) + 2V - \ell_{\Omega}(\Lambda_{\varepsilon,d})}{d} = \frac{h(1) + 2V}{d} - \frac{\ell_{\Omega}(\Lambda)}{d}.$$

By item (A) above, the term $\frac{2V}{d}$ is within $\varepsilon$ of $h(1)$. And, by item (B) above, the term $\ell_{\Omega}(\Lambda)$ is within $\varepsilon$ of $2(2h(1) - I)$, hence the first inequality in (77).

As for the second inequality, we know from the index computations of Section 5.3 that, up to an error of $O(d)$, the index $k_d$ is twice the area between the $x$–axis and the path $P_{\varepsilon,d}$. Because $P_{\varepsilon,d}$ is a scaling of $P$ by a factor of $d^2$, item (C) above implies

$$-\frac{d^2}{2} \varepsilon + O(d) \leq k_d - \frac{d^2}{2} I \leq \frac{d^2}{2} \varepsilon + O(d),$$

which for sufficiently large $d$ yields the second inequality in (77).

**Step 2: The upper bound.**

We begin by recalling that this step of the proof is in fact not necessary for the proof of our main result; see Remark 1.18.

To complete the proof, we need to show that

$$\text{Cal}(\varphi) \geq \lim sup_{d \to \infty} \left( \frac{2c_{d,k}(\varphi)}{d} - \frac{k}{d^2 + d} \right). \quad (78)$$

We will in fact show that

$$\text{Cal}(\varphi) \geq \lim sup_{d \to \infty} \left( \frac{2A(P)}{d} - \frac{k}{d^2 + d} \right). \quad (79)$$
for all degree $d$ concave lattice paths $P$ of combinatorial index $k$.

To do this, we recall the concave toric domains introduced in Section 5.7.2. To the region under the graph of $h$, we associate a concave toric domain $\Omega$ as follows: we first reflect the graph across the $y$ axis, and then translate by the vector $(1, 0)$. In other words, $\Omega$ is the concave toric domain associated to the region under the graph of $g(x) := h(-x+1)$, for $0 \leq x \leq 2$.

As explained in [CCGF+14], a concave region $\Omega$ defines a length function $\ell_{\Omega}$. We now recall this definition in the case at hand. Given a vector $(q, p)$, we find a point $x$ with $g'(x) = p/q$, and similarly to the definitions we gave above, we define

$$\ell_{\Omega}(q, p) := (-p, q) \cdot (x, g'(x)).$$

(80)

We extend this linearly to lattice paths as in the previous step; see (72).

Recall (see Section 5.7.3) that, in the combinatorial model for ECH, an ECH concave lattice path is a piecewise linear, convex lattice path that starts on the positive $y$ axis and ends on the positive $x$ axis. We call the region bounded by this path and the axes an ECH concave lattice region. The motivation for these definitions is that there are a series of numbers, called ECH capacities $c_k(\Omega)$, associated to $\Omega$; they are defined as the maximum length of a concave lattice path, measured with respect to $\ell_{\Omega}$, such that the associated concave lattice region contains $k$ lattice points, not including lattice points on the boundary.

It is suffices to prove (79) for concave lattice paths $P$ that start at $(0, 0)$; indeed, this follows from the discussion after Theorem 5.1. Let $\Lambda$ be a region bounded by the path $P$, part of the $x$-axis, and a vertical line. We can associate a new region $\tilde{\Lambda}$ by reflecting across the vertical line, and translating so that the vertical and horizontal lines for $\tilde{\Lambda}$ agree with the $x$ and $y$ axes.

This operation preserves lattice points. The region $\tilde{\Lambda}$ is an ECH concave lattice region.

**Lemma 6.4.** We have

$$\ell_{\Omega}(\tilde{\Lambda}) = 2A(P).$$

(81)

**Proof.** Note first that $g'(x) = -h'(-x + 1)$. Hence, if $z_{p,q}$ is a point with $h'(z_{p,q}) = p/q$, then $\hat{z}_{p,q} := -z_{p,q} + 1$ satisfies $g'(\hat{z}_{p,q}) = -p/q$ and $g(\hat{z}_{p,q}) = h(z_{p,q})$.

Hence, if $(q, p)$ is any vector in $P$, then its contribution to $2A(P)$, which is given by twice the right hand side of (67), agrees with the contribution of $(q, -p)$ to $\ell_{\Omega}(\tilde{\Lambda})$, since this is given by (80) with $x = \hat{z}_{p,q} = 1 - z_{p,q}$.

We now consider any sequence of concave lattice paths $P_d$ as above, with $d$ tending to infinity. Recall that $k_d = 2j(P_d) - d$; we should think of this as the combinatorial index of $P_d$; to simplify the notation, we write $j_d$ for $j(P_d)$.
We first consider the case where $k_d$ is uniformly bounded. Then, there must be a uniformly bounded number of edges in $P_d$ with $p \neq 0$. The edges with $p = 0$ do not contribute to the action, by (67), since $z_{p,q} = -1$ for these edges. Thus, in this case, $A$ stays uniformly bounded in $d$ and so the right hand side of (79) is 0. But the left hand side of (79) is nonnegative, so that (79) holds.

It remains to consider the case where $k_d$ also tends to infinity with $d$. Given $P_d$, construct concave ECH lattice regions $\tilde{\Lambda}_d$ as above. These regions have $j_d$ lattice points. We now need to recall a few more properties about concave ECH lattice regions:

(i) Each $c_k(X_\Omega)$ is the maximum of $\ell_\Omega(R)$, over all concave ECH lattice regions containing $k$ lattice points, not including lattice points on the upper boundary.

(ii) We have
\[
\lim_{k \to \infty} \frac{c_k(X_\Omega)}{k} = \frac{4A(\Omega)}{d},
\]
where $A(\Omega)$ denotes the area of $\Omega$.

The first property was proved in [CCGF+14]. The second is a special case of the volume property for ECH capacities from [CGHR15] that was already mentioned in the introduction.

We return now to the case of paths $P_d$ with $k_d$ tending to infinity, and their associated concave ECH lattice regions $\tilde{\Lambda}_d$.

First note that we must have
\[
\ell_\Omega(\tilde{\Lambda}_d) \leq 2\sqrt{A(\Omega)}\sqrt{j_d} + o(\sqrt{j_d}).
\]
Otherwise, by Property (i) above, $\frac{c_{k_d}(X_\Omega)}{\sqrt{j_d}}$ would be at least as big as
\[
2\sqrt{A(\Omega)} + e(j_d),
\]
for some $e(j_d)$ with a positive lim sup as $j_d$ tends to infinity, contradicting (82).

We now combine (83) with (81) to conclude that
\[
\frac{2A(P_d)}{d} - \frac{k_d}{d^2 + d} \leq \frac{2\sqrt{A(\Omega)}\sqrt{j_d} + o(\sqrt{j_d})}{d} - \frac{2j_d - d}{d^2 + d} = \frac{2\sqrt{A(\Omega)}\sqrt{j_d}(d+1)}{d^2 + d} - \frac{2j_d - d}{d^2 + d} + \frac{d}{d^2 + d}.
\]

We know that $j_d \leq 4d^2h'(1)$. Hence, the term $\frac{o(\sqrt{j_d})}{d} + \frac{d}{d^2 + d}$ goes to 0 as $d \to \infty$. Thus, to prove (79) it remains to show that
\[
\frac{2\sqrt{A(\Omega)}\sqrt{j_d}(d+1)}{d^2 + d} - \frac{2j_d - d}{d^2 + d} \leq \frac{1}{2}A(\Omega) + \epsilon,
\]
for some $\epsilon$. This follows from the inequality above.
for sufficiently large $d$, for any $\epsilon > 0$. Indeed, $\frac{1}{2}A(\Omega)$ is exactly the Calabi invariant.

To prove (84), we use an elementary argument. As a function of $j_d \geq 0$, the numerator of the above fraction is maximized when its derivative is equal to 0, which occurs when

$$\sqrt{\frac{A(\Omega)(d + 1)}{j_d}} = 2,$$

in other words when

$$\sqrt{j_d} = \frac{1}{2} \sqrt{A(\Omega)(d + 1)}.$$

Plugging this equation in for $j_d$ gives that the left hand side of (84) is bounded from above by

$$\frac{A(\Omega)(d + 1)^2 - (1/2)A(\Omega)(d + 1)^2}{d^2 + d}$$

which converges to $\frac{1}{2}A(\Omega)$ as $d \to \infty$, hence (84).

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