Disorder effects on thermal transport on the surface of topological superconductors by the self-consistent Born approximation

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We have studied the longitudinal thermal conductivity of the surface of a three-dimensional time-reversal symmetric topological superconductor with random disorder. Majorana fermions on the surface of topological superconductors have a response to the gravitational field, which is realized as a thermal response to the temperature gradient inside of the material. Because of the presence of both time-reversal symmetry and particle-hole symmetry, disorder on the surface emerges in the Hamiltonian only as spatial deformations of the pair potential. In terms of the gravitational field, the disorder results in spatial fluctuations of the metric. We consider disorder effects on the thermal conductivity perturbatively within the self-consistent Born approximation. The density of states is calculated with the Green’s function technique and the thermal conductivity of the surface modes is derived through the electronic conductivity using Wiedemann-Franz law for the Majorana fermions.

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I. INTRODUCTION

The concept of the topological phase provides a new class of quantum states of matter where the conventional understanding of phases due to the spontaneous symmetry breaking can not be applied. A topological phase and a trivial gapped phase are separated by a quantum critical point, and they can not be continuously connected without closing the energy gap. As for non-interacting fermions, the topological insulator is a topological phase of the electron system and the topological superconductor is that of the system of Bogoliubov quasi-particles in the BCS mean field theory of superconductors. Both topological phases are distinguished from usual insulator and superconductor phases by the topology of the occupied Bloch bands.

The topological phases of non-interacting fermionic systems are classified by discrete symmetries such as time-reversal symmetry, particle-hole symmetry, chiral symmetry, reflection symmetry, point group symmetries, and so on. The former three symmetries were first brought into the classification of topological insulators and topological superconductors\textsuperscript{a} according to the Altland-Zirnbauer symmetry classes.\textsuperscript{\textsuperscript{2}} The topological phases appear in three spatial dimensions in five out of the ten of Altland-Zirnbauer symmetry classes. In symmetry class AII, the three-dimensional $\mathbb{Z}_2$ topological insulator has been theoretically predicted\textsuperscript{3} and experimentally examined.\textsuperscript{4} In symmetry class DIII, the $^3$He-B phase is considered to be a realization of a topological superfluid phase, and Cu-doped Bi$_2$Se$_3$ has been proposed to be a candidate material of topological superconductors.\textsuperscript{5,6}

One of the intriguing properties of the topological insulators and the topological superconductors is emergence of gapless surface modes, while the bulk is fully gapped as in usual insulators and superconductors. There is one-to-one correspondence between nontrivial bulk topological numbers and the appearance of surface modes (bulk-boundary correspondence), and symmetries of the bulk topological phases protect the surface modes from opening a gap and localization.

The surface of the topological superconductor hosts the Majorana fermion whose anti-particle is itself. The topological superconductor and superfluid in symmetry class DIII have surface Majorana fermions in a time-reversal pair. Robustness of the surface Majorana fermion modes in symmetry class DIII is partially understood as follows. Under both time-reversal symmetry and particle-hole symmetry, only the variations of the Fermi velocity of the Dirac equation are allowed as perturbations, and thus the perturbations can not open an energy gap. These types of perturbations are described in terms of the gravitational field, which couples to the Majorana fermions. While charge neutrality of the Majorana particles prevents direct detection by an external electromagnetic field, transport phenomena of the surface Majorana fermions could be observed by applying a gravitational field, realizations of which are temperature gradients and rotations inside of the material.

The electronic conductivity of the Dirac (Majorana) fermion systems at the exact zero Fermi energy is known to be of the order of a universal constant $e^2/h$, which is referred to as the minimal conductivity.\textsuperscript{7,8} The surface of topological insulators and topological superconductors naturally realize this situation. The relation between the electronic conductivity and the thermal conductivity is given by the Wiedemann-Franz law.\textsuperscript{9} The Wiedemann-Franz law for Majorana particles\textsuperscript{10} reads as

\begin{equation}
    \kappa = \frac{\pi^2 k_B^2 T}{6e^2} \sigma.
\end{equation}

Note that the electronic conductivity of charge neutral Majorana fermions means just the response function derived from the Kubo formula, the calculation of which...
is not restricted to charged particles. The coefficient of the right-hand side of the above law is half of that of the complex fermion case, since a Majorana (real) fermion is equal to a half of a complex fermion.

In this paper, static and dynamical properties of the surface Majorana fermions of the topological superconductor in symmetry class DIII with disorder are studied through the Green’s function technique. With respect to small deformations of the pair potential, a part of the perturbation series that is significant in the conductive regime is picked up by the self-consistent Born approximation (SCBA). This paper is organized as follows. In section 2, the model Hamiltonian of the three-dimensional topological superconductor in symmetry class DIII is introduced with its defining symmetries. Possible terms that are allowed to be added to the non-perturbed Hamiltonian under two discrete symmetries and the perturbation terms that we consider in this paper are shown. In section 3, the disorder averaged Green’s function is derived by solving equations of the SCBA. The density of states is obtained through the averaged Green’s function. In section 4, the electronic conductivity is derived by summing up infinite series of perturbation terms of the SCBA, and the thermal conductivity is obtained via the Wiedemann-Franz law.

II. SURFACE STATES OF TOPOLOGICAL SUPERCONDUCTORS IN CLASS DIII

In this section, the low-energy effective Hamiltonian of the three-dimensional topological superconductor in symmetry class DIII is reviewed with its symmetry classification. The surface of topological superconductors hosts two-dimensional Majorana surface modes. We will show that characterizing symmetries of the symmetry class DIII puts constraints on possible perturbations on the surface modes. Disorder on the surface is then regarded as spatial deformations of the pair potential.

A. Model Hamiltonian of topological superconductors in class DIII

Symmetry classes of the random matrix theory are brought to classification of the topological insulators and the topological superconductors, which are defined with three types of discrete symmetries: time-reversal symmetry, particle-hole symmetry, and a combination of them called chiral symmetry. Symmetry class DIII corresponds to time-reversal-symmetric superconductors with the triplet Cooper pairing. We consider a bulk 4 × 4 matrix Hamiltonian in the momentum space \( H_{\text{bulk}}(k) \). Four fermion flavors in the Hamiltonian are the product of two flavors of spin up and down, and two flavors of the particle and the hole. Time-reversal symmetry for spin 1/2 is defined by

\[
(is^y)H_{\text{bulk}}^T(-k)(-is^y) = H_{\text{bulk}}(k),
\]

where \( s^l(i = x, y, z) \) is a set of the Pauli matrix for the spin degrees of freedom and \( T \) denotes the transpose of matrices. The square of the time-reversal operation is \((is^yK)^2 = -1\), where \( K \) denotes the complex conjugation operator. Particle-hole symmetry for triplet superconductors is defined by

\[
t^\varepsilon H_{\text{bulk}}^T(\varepsilon)(-t^\varepsilon) = -H_{\text{bulk}}(\varepsilon),
\]

where \( t^\varepsilon(i = x, y, z) \) is a set of the Pauli matrix for the particle-hole degrees of freedom. The square of the particle-hole conjugation operator is \((t^\varepsilon K)^2 = +1\).

As a low-energy effective model of topological superconductors, we consider the Bogoliubov-de Gennes Hamiltonian of the form

\[
\mathcal{H}_{\text{bulk}} = \frac{1}{2} \sum_k \Psi_k^\dagger H_{\text{bulk}}(k) \Psi_k,
\]

where \( \Psi_k = (c_{ks}^\dagger, c_{ks}^-)^T \) \( (c_{ks}^\dagger, c_{ks}^-) \) \( c_{ks}^\dagger \) and \( c_{ks}^- \) are the creation and the annihilation operators of a complex fermion with the momentum \( k \) and the spin \( s \). From Hermiticity and particle-hole symmetry, the matrix elements of the Hamiltonian are written as

\[
H_{\text{bulk}}(k) = \begin{pmatrix} \Xi_k & \Delta_k \\ \Delta_k^\dagger & -\Xi_k^T \end{pmatrix},
\]

where \( \Xi_k \) and \( \Delta_k \) are 2 × 2 matrices satisfying \( \Delta_k^T = -\Delta_k \). Time reversal symmetry for \( \Xi_k \) and \( \Delta_k \) is given by \( is^y \Xi_k = \Xi_k \) and \( is^y \Delta_k^T = -\Delta_k \). We consider the forms of \( \Xi_k \) and \( \Delta_k \) given by

\[
\Xi_k = \Xi_k^T = \left( \frac{\hbar^2 k^2}{2m} - \mu \right) s^0,
\]

\[
\Delta_k = \Delta_0 \mathbf{s}(is^y) = \Delta_0 \frac{-k_x + ik_y}{k_z} \begin{pmatrix} k_x - ik_y \\ k_z \end{pmatrix},
\]

which satisfy all the symmetries required for the Hamiltonian in symmetry class DIII. Note that we only consider real and positive \( \Delta_0 \) without loss of generality since the phase factor of \( \Delta_0 \) can be removed by a U(1) gauge transformation of the fermion operators. The Hamiltonian \( \Xi_k \) represents, for example, the B phase of superfluid \(^3\)He.

B. Surface states of topological superconductor

Since the kinetic energy in the diagonal blocks of the Hamiltonian matrix \( \Xi_k \) can be ignored in the long wavelength limit, the diagonal blocks are regarded as a mass term with respect to the Dirac cone structure of the off-diagonal pair potential term \( \Xi_k \). The boundary of a material is, for convenience, defined by a position dependent mass term \( \mu(z) \), where the \( z \) direction is normal to the surface of the material. Putting a material in the region of \( z < 0 \), \( \mu \) smoothly changes its sign from negative to positive when moving from the material to the vacuum.
(outside of the material), and converges to a finite value away from the surface:

\[
\mu(z) \rightarrow \begin{cases} 
\mu & (z \rightarrow \infty) \\
-\mu & (z \rightarrow -\infty) 
\end{cases},
\]

where \( \mu > 0 \).

In the coordinate space description, the bulk Hamiltonian is written as

\[
H_{\text{bulk}} = \frac{1}{2} \int d^3r \bar{\Psi}^1(r) H_{\text{bulk}}(r) \Psi(r),
\]

where

\[
H_{\text{bulk}}(r) = \begin{pmatrix}
-\mu(z)s^0 & \Delta_0(-i \partial \cdot \mathbf{s}(i s^y)) \\
\Delta_0(-i s^y)(-i \partial) \cdot \mathbf{s} & \mu(z)s^0
\end{pmatrix}.
\]

Using the fermion operator in the coordinate space \( c_s(r) = \langle 1/E \rangle \sum_k \exp(i \mathbf{k} \cdot \mathbf{r}) c_k \), the spinor is \( \Psi(r) = (c_\uparrow(r), c_\downarrow(r), c_\uparrow^\dagger(r), c_\downarrow^\dagger(r)) \). The eigenfunctions of the Hamiltonian near the surface are the product of the plane waves in surface direction (\( x \)- and \( y \)-direction), and a bound function normal to the surface (\( z \)-direction):

\[
e^{i(k_x x + k_y y)} \psi^{1(2)}(z),
\]

with

\[
\psi^{1(2)}(z) = \exp \left[ - \int dz \frac{\mu(z')}{\Delta_0} \right] |1(2)\rangle.
\]

Four-component spinors \( |1\rangle \) and \( |2\rangle \) are the basis vectors that span the eigenspace of \( (s^x \otimes t^y) \) with the eigenvalue \(-1\) which are assigned to bound functions, while those with the eigenvalue \(+1\) are diverging functions which cannot be normalized.

The Hamiltonian for surface modes is reduced from the bulk Hamiltonian by projecting the Hilbert space onto the subspace spanned by the product of \( \psi^{1(2)}(z) \) and functions of \( x \) and \( y \). Through this process, the Hamiltonian becomes independent of \( z \), and the four-component spinor degrees of freedom are reduced to the two-component ones. We use the basis vectors \( |1(2)\rangle \) as

\[
|1\rangle = \begin{pmatrix} i \\ 1 \\ -i \\ 1 \end{pmatrix} / 2, \quad |2\rangle = \begin{pmatrix} 1 \\ i \\ 1 \\ -i \end{pmatrix} / 2.
\]

With these vectors, operators of the surface Majorana fermions at the position \((x, y)\) are given by \( \gamma^{1(2)}(x, y) \propto \int dz \psi^{1(2)}(z) \Psi(r) \), or explicitly,

\[
\gamma^1 \propto \int dz \exp \left[ - \int dz' \frac{\mu(z')}{\Delta_0} \right] (\gamma_{\uparrow\downarrow} + \gamma_{\downarrow\uparrow}),
\]

\[
\gamma^2 \propto \int dz \exp \left[ - \int dz' \frac{\mu(z')}{\Delta_0} \right] (\gamma_{\uparrow\uparrow} + \gamma_{\downarrow\downarrow}),
\]

where \( \gamma_{\alpha\beta} = c_\alpha(r) + c_\beta^\dagger(r) \) and \( \gamma_{\alpha\beta} = (c_\alpha(r) - c_\beta^\dagger(r))/i \) are Majorana fermion operators generated from complex fermion operators \( c_\alpha \) with spin \( s = \uparrow, \downarrow \). Obviously, the operators \( \gamma^1 \) and \( \gamma^2 \) satisfy the Majorana condition \( \gamma^\dagger = \gamma^\dagger \). The wavefunctions of the surface mode can be written as \( u(x, y) \psi^\dagger(z) + v(x, y) \psi^\dagger(z) \). Therefore the Hamiltonian for the surface modes is given by

\[
H_0(r) = -i \Delta_0 (\partial_x \sigma^z + \partial_y \sigma^x),
\]

where \( \sigma^i (i = x, y) \) is the set of the Pauli matrix for the two-component spinor \( (u(x, y), v(x, y))^T \). After projecting onto the surface modes, the time-reversal operator becomes \( T = i \sigma^y K \), and the particle-hole conjugation operator becomes \( C = K \). In the following, we use the notation \( (\tilde{\sigma}^x, \tilde{\sigma}^y) \) in place of the Pauli matrix \( (\sigma^x, \sigma^y) \) for convenience, and the Hamiltonian \( (15) \) is briefly rewritten as \( H_0(r) = -i \Delta_0 \tilde{\sigma} \cdot \tilde{\sigma} \). Implicitly \( \sigma^y \) is replaced by \( \sigma^z \) accordingly, while \( \sigma^0 \) is unchanged. This replacement does not affect the following calculations since the algebra that the Pauli matrix obeys is invariant under this replacement. Also the physical meaning of perturbation terms added to the surface Hamiltonian \( (15) \), like the chemical potential term and the mass term, is conserved under the replacement since their meaning is dependent on the explicit form of the unperturbed Dirac Hamiltonian.

C. Deformation of the pair potential

Under both of time-reversal symmetry and particle-hole symmetry, the possible perturbation terms that are allowed to be added to the surface Hamiltonian \( (15) \) are strictly limited. Prohibited terms are, for example, a chemical potential \( \mu \sigma^0 \), which breaks particle-hole symmetry, a mass term \( m \sigma^x \), which breaks time-reversal symmetry, and \( U(1) \) gauge potential terms \( A_\mu \sigma^x, A_\mu \sigma^y \), which break both of them. Note that symmetries of each perturbation term must be examined before replacing the Pauli matrix since the action of the time-reversal and particle-hole conjugation operators is dependent on the elements of the \( 2 \times 2 \) Pauli matrix. In the momentum-space representation, the available terms are ones listed in the following:

\[
\text{(odd function of } k_x, k_y \text{) } \times \tilde{\sigma}^x, \quad \text{(odd function of } k_x, k_y \text{) } \times \tilde{\sigma}^y.
\]

Among them, only the terms proportional to \( k_x \) or \( k_y \) have significant contributions, since higher order terms can be neglected in the long wave-length limit. The full Hamiltonian that we consider in this paper is as follows:

\[
H = H_0 + U = -\frac{i}{2} \left( \Delta(r), \partial_x \tilde{\sigma}^x + \partial_y \tilde{\sigma}^y \right).
\]

The Hamiltonian \( (17) \) means that the pair potential is spatially deformed by random disorders, which will be
explained in the following. A deformation of the pair potential represented by a small conformal factor \( \Lambda(r) \) \((\ll 1)\) as \( \Delta(\mathbf{r}) = \Delta_0 e^{\Lambda(\mathbf{r})} \) can be undertaken by the Pauli matrix with the vierbein field \( e^a(\mathbf{r}) = \delta^a_b e^{\Lambda(\mathbf{r})} \) as \( \tilde{\sigma}^a(\mathbf{r}) = e^a(\mathbf{r}) \delta^b \). Thus the Hamiltonian \(^{17}\) describes the surface Majorana fermions on the curved space with the metric

\[
g^{ab}(\mathbf{r}) = \{ \tilde{\sigma}^a(\mathbf{r}), \tilde{\sigma}^b(\mathbf{r}) \} = e^{2\Lambda(\mathbf{r})} \delta^{ab}, \tag{18}\]

where \( a, b \) are indices of spatial coordinates. The metric \(^{14}\) indicate that the Hamiltonian \(^{17}\) is continuously connected from the unperturbed Hamiltonian \(^{15}\) by a conformal transformation of the spatial coordinates. The perturbation term is given by

\[
U(\mathbf{r}) \simeq -\frac{i\Lambda_0}{2} \{ \Lambda(\mathbf{r}), \partial_x \tilde{\sigma}^x + \partial_y \tilde{\sigma}^y \}, \tag{19}\]

where all terms second order in \( \Lambda(\mathbf{r}) \) or higher are neglected. We should note that the spin connection does not contribute to the Hamiltonian for the case of a single Dirac cone Hamiltonian (Hamiltonian described by the \( 2 \times 2 \) Pauli matrices) \(^{13}\).

Consider situation where time-reversal symmetry preserved scatterers are contained inside the topological superconductor. Only scatterers near the surface have an influence on the surface modes. Point-like scatterers are randomly distributed on the surface, and a single scatterer at the position \( \mathbf{r}_i = (x_i, y_i) \) \((i = 1, 2, \ldots, N_i)\) affects the pair potential around \( \mathbf{r}_i \) as the form of Gaussian

\[
\Lambda(\mathbf{r}) = -c \sum_{i=1}^{N_i} e^{-|\mathbf{r} - \mathbf{r}_i|^2/2R^2}, \tag{20}\]

where \( c \) is a dimensionless parameter that represents the strength of disorder, \( R \) is an effective range of the influence of a scatterer, both of which are assumed to be common for all scatterers, and \( N_i \) is the number of the of scatterers. We consider the effective range \( R \) to be much larger than the lattice spacing so as to justify the continuous description of the conformal factor \(^{23}\). By the Fourier transformation, the conformal factor in the momentum space is

\[
\Lambda(\mathbf{k}) = \frac{1}{L} \int d^2r e^{ik\cdot r} \Lambda(\mathbf{r}) = -\frac{2\pi c R^2}{L} \sum_{i=1}^{N_i} e^{-R^2|\mathbf{k}|^2/2+i\mathbf{k}\cdot \mathbf{r}_i}. \tag{21}\]

On the surface of the topological superconductor, distribution of the positions of the scatterers is random in two-dimensional space. The average of physical quantities over the position of the scatterers \( \mathbf{r}_i \) is denoted by \( \langle A \rangle \), and its definition is

\[
\langle A \rangle = \left[ \prod_{i=1}^{N_i} \frac{1}{L^2} \int d^2 r_i \right] A. \tag{22}\]

First, the disorder average of a single conformal factor gives \( \langle \Lambda(\mathbf{k}) \rangle = 2\pi N_i c R^2 \omega_0/L \). Next the disorder average of the product of two conformal factors, that is, the correlator of conformal factors is

\[
\langle \Lambda(\mathbf{k}) \Lambda(\mathbf{k}') \rangle = \delta_{k+k',0} n_1 (2\pi c R^2)^2 e^{-R^2|\mathbf{k}|^2}, \tag{23}\]

where \( n_1 = N_i/L^2 \) is the number of scatterers per unit area. For further simplification, we assume the effective range of a single scatterer \( R \) to be infinity. In this long-ranged limit, the correlator of the conformal factor becomes

\[
\langle \Lambda(\mathbf{k}) \Lambda(\mathbf{k}') \rangle \rightarrow \delta_{k+k',0} n_1 (2\pi c R^2)^2 \delta_{k,0} \equiv \delta_{k+k',0} L^2 v_1 \delta_{k,0}. \tag{24}\]

where \( v_1 = n_1 (2\pi c R^2)^2 / L^2 \) is a parameter that represents the intensity of disorder.

### III. Averaged Green’s Function by the Self-Consistent Born Approximation

In this section, the disorder effects on the surface Majorana fermions are studied through the disorder-averaged Green’s function

\[
\tilde{G}(\mu) \equiv \langle G(\mu) \rangle = \left\langle \frac{1}{\mu - H} \right\rangle \tag{25}\]

within the SCBA \(^{22}\). Note that, in (25) and hereafter, \( \sigma^0 \) is not written explicitly. For the long-ranged limit of the deformations of the pair potential, resultant difference from the Green’s function in the clean limit is fully described by a single parameter \( A \). The density of state is obtained by the averaged Green’s function.

#### A. Self-consistent Born approximation

Consider the situation that the deformations of the pair potential is much smaller than the original superconducting energy gap \( \langle \Lambda(\mathbf{r}) \rangle \ll 1 \). The disorder term \(^{13}\) is treated perturbatively with respect to the bare Green’s function

\[
G_0(\mu) = \frac{1}{\mu - H_0}. \tag{26}\]

The self-energy \( \Sigma(\mu) \) is introduced by the Dyson’s equation

\[
\tilde{G}(\mu) = G_0(\mu) + G_0(\mu) \Sigma(\mu) \tilde{G}(\mu). \tag{27}\]

Here, we should note that although the nonzero Fermi energy term is prohibited from the symmetry argument in the previous section, we will relax this condition for a while in order to examine properties away from \( \mu = 0 \). The disorder average of a single \( U \) term is omitted since it only shifts the energy due to the fact.
\[ \langle \Lambda(\mathbf{k}) \rangle = 2\pi N_F R^2 \delta_{k,0}/L, \] and also the disorder average of the product of more than three of the \(U\) term can be neglected when the scatterers are not densely distributed. The SCBA for the self-energy is given by

\[ \Sigma(\mu) = \langle U \tilde{G}(\mu) U \rangle. \] (28)

The averaged Green’s function and the self-energy are derived self-consistently by combining (25) with the alternative representation of the definition of the self-energy (27) as

\[ \tilde{G}(\mu) = \frac{1}{\mu - H_0 - \Sigma(\mu)}. \] (29)

We will use the momentum-space representation of the Hamiltonian to apply the above method to the current situation, i.e.,

\[ \mathcal{H} = \sum_{k,k'} \Psi_k^\dagger (H_{0kk'} + U_{kk'}) \Psi_{k'}, \] (30)

where the \((k,k')\) component of each term is

\[ H_{0kk'} = \delta_{kk'} v_F \hat{\sigma} \cdot \mathbf{k}, \] (31)

\[ U_{kk'} = \frac{v_F h}{2L} \Lambda(\mathbf{k} - \mathbf{k}') \hat{\sigma} \cdot (\mathbf{k} + \mathbf{k}'). \] (32)

In the above equation, the pair potential \(\Delta_0/2\) is replaced by the Fermi velocity \(v_F\) in order to fit the notation of the Dirac equation. Note that (22) satisfies Hermiticity \((U_{kk'}^\dagger = U_{k'k})\) since \(\Lambda^*(\mathbf{k}) = \Lambda(\mathbf{k})\).

After averaging over the positions of the scatterers, the translational invariance is recovered, and thus the averaged Green’s function \(\tilde{G}\) and the self-energy \(\Sigma\) are diagonal with respect to the momentum. Introducing the averaged Fermi energy \(F_0\) and the averaged momenta \(\mathbf{F} = (F_x, F_y)\) by \(\mathcal{G}^{-1} = F_0 - \tilde{\sigma} \cdot \mathbf{F}\), the self-energy is given by

\[ \Sigma(k,\mu) = G_{0kk}^{-1}(\mu) - \tilde{G}_{k}^{-1}(\mu) = \mu - F_{0k} - \tilde{\sigma} \cdot (v_F h \mathbf{k} - \mathbf{F}_k), \] (33)

where \(F_{0k}\) and \(F_k\) are, respectively, the \(k\) component of \(F_0\) and that of \(\mathbf{F}\). Similarly, by substituting (24), (25) becomes

\[ \Sigma(k,\mu) = \sum_{k'} (U_{kk'} \tilde{G}_{k'}(\mu) U_{k'k}) \]

\[ = \frac{v_F (v_F h \hat{\sigma} \cdot k)(F_{0k} \pm \hat{\sigma} \cdot \mathbf{F}_k)(v_F h \hat{\sigma} \cdot k)}{F_{0k}^2 - |\mathbf{F}_k|^2}. \] (34)

Removing the self-energy by equating the right-hand side of (33) and that of (34), and decomposing them into equations proportional to \(\sigma^0, \hat{\sigma}^x\) and \(\hat{\sigma}^y\), we obtain a set of self-consistent equations as

\[ \begin{align*}
F_{0k} &= \frac{\mu}{v_F h \mathbf{k}} \\
F_k &= \frac{\mu}{v_F h \mathbf{k}}
\end{align*} \] (35)

where \(A = v_F (v_F h |\mathbf{k}|)^2/(F_{0k}^2 - |\mathbf{F}_k|^2)\) is a single parameter that represents the disorder effects calculated within the SCBA.

**B. Solutions of \(A\)**

The relation between the bare and the averaged Green’s function (25) indicates that a couple of self-consistent integral equations of the averaged Green’s function and the self-energy are reduced to algebraic equations of \(A\). The equation of \(A\) is given by the definition of \(A\) as

\[ A = \frac{\langle (v_F h |\mathbf{k}|)^2 \rangle}{v_F^2 (1 + A)^2 - (v_F h |\mathbf{k}|)^2/(1 - A)^2}. \] (36)

Note that the equation of \(A\) is determined by two parameters, the intensity of disorder \(v_1\) and the ratio of the momentum to the Fermi energy \(\kappa = v_F h |\mathbf{k}|/\mu\), except for a point \(\mu = 0\) (or equivalently we can consider a parameter \(|\mu|/v_F h |\mathbf{k}|\) except for \(|\mathbf{k}| = 0\). Then the problem is decomposed into two parts, one for \(\mu \neq 0\) and the other for \(\mu = 0\).

First, we consider the case \(\mu \neq 0\). The equation (36) has four branches of solutions. The explicit forms of the solutions are given by

\[ A_{\pm \pm} = - \left( \frac{l_1^{1/2} - l_2^{1/2}}{l_1^{1/2} + l_2^{1/2}} \right) \pm 1 \left( \frac{l_3^{1/2} - l_4^{1/2}}{l_3^{1/2} + l_4^{1/2}} \right) \pm 1, \] (37)

where the two signs in the subscript of \(\mathbf{A}\) in the left-hand side correspond, respectively, to the two signs in right-hand side, and they can be taken independently. In the following, the square root of a negative value indicates a square root that has a positive imaginary part. The four variables are

\[ \begin{align*}
l_1 &= (1 + 2v_1^{1/2})\kappa + 1 \\
l_2 &= (1 - 2v_1^{1/2})\kappa + 1 \\
l_3 &= (1 + 2v_1^{1/2})\kappa - 1 \\
l_4 &= (1 - 2v_1^{1/2})\kappa - 1
\end{align*} \] (38)

As can be readily seen from the expression (37), \(A\) is real when the signs of both \(l_1 \times l_2 \times l_3 \times l_4\) are positive. Otherwise, that is, when at least one of the signs of \(l_1 \times l_2 \times l_3 \times l_4\) is negative, \(A\) can be complex valued. The signs of the above four variables are shown in the \(\kappa\)-\(v_1\) space in Fig.4. The boundaries where the sign of one of the variables changes are \(l_3 = 0\), \(l_2 = 0\) and \(l_4 = 0\). Thus, the lower region (below \(l_4 = 0\)) and upper-left region (left side of \(l_3 = 0\)) in Fig.4 have real solutions of \(A\), and in the other regions, complex solutions of \(A\) are realized. There is an upper bound of the intensity of disorder at \(v_1 = 0.25\) (shown in Fig.4) by a dotted line) up to which the solutions of \(A\) are continuously connected from the clean limit. Throughout this paper, we consider only \(v_1 < 0.25\).

Among the four branches of the solutions of \(A\), only two ones that converge to zero in the limit of \(v_1 \to 0\) make physical sense, since \(A\) represents deviation from the clean limit. In the complex \(A\) region, however, we do not impose this condition, since the complex \(A\) region
in $\kappa$ space shrinks to a point $\kappa = 1$ in the clean limit, and the Green's functions have singular behavior there. When $l_1 \times l_2$ and $l_3 \times l_4$ are positive, we observe
\begin{align}
l_{1/2}^{1/2} + l_{2/2}^{1/2} & \simeq 2(\kappa + 1)^{1/2} \sim O(1), \\
l_{3/2}^{1/2} - l_{4/2}^{1/2} & \simeq \frac{2\kappa}{(\kappa + 1)^{1/2}} l_{1/2}^{1/2} \sim O(v_1^{1/2}),
\end{align}
and also
\begin{align}
l_{3/2}^{1/2} + l_{4/2}^{1/2} & \simeq 2(\kappa - 1)^{1/2} \sim O(1), \\
l_{3/2}^{1/2} - l_{4/2}^{1/2} & \simeq \frac{2\kappa}{(\kappa - 1)^{1/2}} l_{1/2}^{1/2} \sim O(v_1^{1/2}),
\end{align}
Note again that we took a positive imaginary part branch for the square root of a negative value. The appropriate choice is therefore $A_{++}$ for the real $A$ regions.

In the complex $A$ region, the solutions will be those that are continuously connected to $A_{++}$ at the two boundaries intervening the real and complex $A$ regions, that is, $l_3 = 0$ and $l_4 = 0$. From (38), an identity $A_{++} = A_{+-}$ holds when $l_3 = 0$ or $l_4 = 0$. Thus, two possibilities arise for the solutions in the complex $A$ region: $A_{++}$ and $A_{+-}$, which are related by the complex conjugation. In the following, we will show that these two solutions correspond to the retarded or the advanced averaged Green’s functions.

The averaged Green’s function is obtained with the solution of $A$ as
\begin{equation}
\tilde{G}_k^{-1}(\mu) = \frac{\mu}{1 + A} - \frac{v_F h \hat{\sigma} \cdot \mathbf{k}}{1 - A}.
\end{equation}
The two branches of the solutions of $A$ in the complex $A$ region, $A_{++}$ and $A_{+-}$, are assigned to the retarded or the advanced Green’s functions by comparing the signs of the imaginary part of the inverse of Green’s function between the one in the clean limit and the averaged one. Here, we consider one of the eigenvalues of the Green’s function in which the sign of the eigenvalue of $\tilde{\sigma} \cdot \mathbf{k}$ is equal to the sign of $\mu$ in place of the matrix valued Green’s function. This side of the eigenvalue has a nonzero imaginary part when the chemical potential is slightly shifted to the imaginary direction $\mu \rightarrow \mu \pm i\delta$. The inverse of the bare Green’s function has the sign of the imaginary part as follows:
\begin{equation}
\text{sgn}[\text{Im}[G_{\delta k}^{-1}(\mu \pm i\delta)]] = \mp 1,
\end{equation}
where $\delta$ is a positive infinitesimal parameter. The sign of the imaginary part of the averaged Green’s function with the complex $A_{++}$ is
\begin{align}
\text{sgn}[\text{Im}[\tilde{G}_k^{-1}(\mu)]] &= \text{sgn} \left( \text{Im} \left[ \frac{\mu}{1 + A_{++} - 1 - A_{++}} \right] \right) \\
&= \text{sgn} \left[ \text{Im} \left[ -\mu + \frac{1}{2} \kappa (l_3^{1/2} + l_4^{1/2})^2 \right] \right] \\
&= \mp \text{sgn}[\mu].
\end{align}
Therefore, the appropriate choices of the branches of $A$ for the retarded (denoted by $A^R$) and the advanced (denoted by $A^A$) averaged Green’s function turn out to be, $A^R = A^A = A_{++}$ for $\kappa < (1 + 2v_1^{1/2})^{-1}$ or $\kappa > (1 - 2v_1^{1/2})^{-1}$, and for the interval $(1 + 2v_1^{1/2})^{-1} < \kappa < (1 - 2v_1^{1/2})^{-1}$,
\begin{align}
A^R &= \begin{cases} 
A_{+-} & (\mu > 0) \\
A_{++} & (\mu < 0)
\end{cases}, \\
A^A &= \begin{cases} 
A_{++} & (\mu > 0) \\
A_{+-} & (\mu < 0)
\end{cases}.
\end{align}
For a positive Fermi energy, one of the eigenvalues of the averaged Green’s functions multiplied by the Fermi energy $\mu \tilde{G}_k$, where the sign of the eigenvalue of $\tilde{\sigma} \cdot \mathbf{k}$ is equal to $\text{sgn}[\mu]$ (in this case +1), is drawn in Fig. 2 [(b), (c), (d)] as a function of $\kappa = v_F h |\mathbf{k}|/|\mu|$. Each line in Fig. 2 [(b), (c), (d)] corresponds to the intensity of disorder $v_1 = 2^{-n}$ ($n = 1, 2, \ldots, 10$). Fig. 2 (a) shows the boundaries of the complex and the real $A$ regions. At the boundaries, the Green’s functions have singular behavior. For a negative Fermi energy, graphs are obtained by inverting the signs of the imaginary parts. It is easily seen that in the low $v_1$ limit, the graphs of the averaged Green’s functions converge to those in the clean limit, that is, the real part converges to $(1 - \kappa)^{-1}$, and the imaginary part of the retarded [advanced] Green’s function to $-\pi \delta(\kappa - 1)$ [\$\mp \pi \delta(\kappa - 1)$], since
\begin{align}
\mu G_{\delta k}(\mu \pm i\delta) &= \frac{1}{1 - \kappa \pm i\delta} \\
&= \frac{P}{1 - \kappa} \mp \pi \delta(\kappa - 1).
\end{align}
Next, we consider the case $\mu = 0$. The algebraic equation of $A$ can be reduced from (33) by taking $\mu = 0$. The equation is given by
\begin{equation}
A = -v_1(1 - A)^2,
\end{equation}
and the solutions of (51) have two branches as

\[ A_{\pm} = \frac{1}{2v_1} \left( 2v_1 - 1 \pm \sqrt{1 - 4v_1^2} \right). \]  

(50)

Since the solutions (51) are real when \( v_1 < 0.25 \), we always consider real solutions of (51). A branch of the solutions that converges to zero in the clean limit is \( A_+ \). Note that the solutions (51) are reduced from the solutions (57) by taking \( (1 \pm 2v_1^{1/2}) \kappa \pm 1 \rightarrow (1 \pm 2v_1^{1/2}) \kappa \).

C. Density of states

The density of states of the Majorana surface modes at the energy \( \epsilon \) is obtained with the averaged Green’s function \( \tilde{G}(\mu = \epsilon) \) by the formula

\[ D(\epsilon) = -\frac{1}{\pi L^2} \sum_k \text{Im} \text{Tr} \tilde{G}_k(\epsilon + i\delta). \]  

(51)

The sum over discrete \( k \) is replaced by the integration over continuous \( k \) by taking the limit of \( L \rightarrow \infty \:

\[ \frac{1}{L} \sum_k \rightarrow \frac{1}{2\pi} \int d\kappa. \]

Substituting the averaged Green’s function (53) with the branch for \( A^R \) given in (49) into the above formula and using the identity (50), the density of states is given by

\[ D(\epsilon) = -\frac{\epsilon}{\pi^2 v_F^2} \int_{0}^{\infty} \frac{dk}{v_1 \kappa} \text{Im} \frac{A^R}{1 + A^R} \]

\[ = \frac{|\epsilon|}{\pi^2 v_F^2} \int_{(1 - 2v_1^{1/2})^{-1}}^{(1 + 2v_1^{1/2})^{-1}} \frac{d\kappa}{4v_1 \kappa} (4v_1^2 \kappa^2 - (\kappa - 1)^2)^{1/2}. \]  

(52)

Here we have used the fact that, from the solutions of \( A \) in (57) with the signs shown in Fig. 1, the imaginary part of the Green’s function is nonzero only in the interval \( (1 + 2v_1^{1/2})^{-1} < \kappa < (1 - 2v_1^{1/2})^{-1} \). Since \( A^R \) for \( \mu > 0 \) and that for \( \mu < 0 \) are related by complex conjugation, the imaginary part of the averaged Green’s function for \( \mu > 0 \) is equal to that for \( \mu < 0 \) multiplied by \(-1\). The sign of the imaginary part thus cancels the sign of \( \epsilon \) in front of the integral in the first line of (52). Therefore the density of states is an even function of the energy \( \epsilon \), and it depends only on the absolute value of the energy \(|\epsilon|\). The expression of the density of states is written by

\[ D(\epsilon) = \frac{|\epsilon|}{2\pi v_F^2} \frac{1 - (1 - 4v_1^{1/2})^{1/2}}{2v_1(1 - 4v_1^{1/2})^{1/2}} \]

\[ = \frac{|\epsilon|}{2\pi v_F^2} (1 + 3v_1 + O(v_1^2)). \]  

(53)

In the clean limit, the density of states of the surface Majorana fermions converges to \(|\epsilon|/2\pi v_F^2\), which is half of the density of states of \(4 \times 4\) Dirac fermions systems, like graphene. However, since two Majorana fermions are equivalent to a single complex fermion, the density of states of the complex fermions composed by the surface Majorana fermions is quarter of that of the \(4 \times 4\) Dirac fermions.

IV. THERMAL CONDUCTIVITY

The electronic conductivity is obtained from the Green’s functions by the following formula,

\[ \sigma(\epsilon) = \frac{1}{2} \text{Re}[I(\epsilon + i\delta, \epsilon - i\delta) - I(\epsilon + i\delta, \epsilon + i\delta)], \]  

(54)

where

\[ I(\epsilon, \epsilon') = \frac{e^2 \hbar}{\pi L^2} \sum_k \text{Tr} \langle v_x G_k(\epsilon) v_x G_k(\epsilon') \rangle, \]  

(55)

and the velocity operator is defined by

\[ v_x = \frac{i}{\hbar} [H_0, x] = v_F \sigma \hat{x}. \]  

(56)

The quantity \( I(\epsilon, \epsilon') \) contains the disorder average of the product of two Green’s functions and \( v_x \) between them: \( K(\epsilon, \epsilon') = \langle G(\epsilon)v_xG(\epsilon') \rangle \). This is another quantity aside from the averaged Green’s function to be calculated perturbatively. Within the SCBA, \( K(\epsilon, \epsilon') \) is self-consistently determined with use of the averaged single Green’s function as

\[ K(\epsilon, \epsilon') = \tilde{G}(\epsilon)v_x \tilde{G}(\epsilon') + \tilde{G}(\epsilon)(UK(\epsilon, \epsilon')U') \tilde{G}(\epsilon'). \]  

(57)

Then, the \( k \) component of the above equation is as follows:

\[ K_k(\epsilon, \epsilon') = v_F \tilde{G}_k(\epsilon) \hat{\sigma} \tilde{G}_k(\epsilon'). \]
Iteratively substituting $K_k(\epsilon, \epsilon')$ in (58), the formal solution of $K_k(\epsilon, \epsilon')$ is described by the sum of infinite series as

$$K_k(\epsilon, \epsilon') = v_F \sum_{n=1}^{\infty} v_n^{n-1} \tilde{G}_k^{(n)}(\epsilon) \tilde{\sigma}^n \tilde{G}_k^{(n-1)}(\epsilon'),$$

(59)

where

$$\tilde{G}_k^{(1)}(\epsilon) = \tilde{G}_k(\epsilon),$$

(60)

$$\tilde{G}_k^{(n)}(\epsilon) = \tilde{G}_k(\epsilon)(v_F \hbar \sigma \cdot k) \tilde{G}_k^{(n-1)}(\epsilon).$$

(61)

Introducing new variables $E$ and $\varphi$ by $\epsilon/(1 + A) = E \sinh \varphi$ and $v_F \hbar |k|/(1 + A) = \cosh \varphi$, and a matrix valued variable $s_k = \tilde{\sigma} \cdot k/|k|$, the averaged Green’s function is written by

$$\tilde{G}_k(\epsilon) = -(\sinh \varphi + s_k \cosh \varphi)/E.$$

(62)

Then, $\tilde{G}_k^{(n)}(\epsilon)$ is recursively given by $E$ and $\varphi$ as

$$\tilde{G}_k^{(n)}(\epsilon) = \frac{(v_F \hbar |k|)^{n-1}}{(-E)^n} (\sinh [n \varphi] + s_k \cosh [n \varphi]).$$

(63)

The sum of the infinite series can be calculated with the identity of the sum of power series $\sum_{n=1}^{\infty} x^n = x/(1 - x)$, as

$$K_k(\epsilon, \epsilon') = \frac{v_F}{4} \left[\tilde{\sigma}^x + s_k \tilde{\sigma}^x + \tilde{\sigma}^x s_k + s_k \tilde{\sigma}^x s_k\right]$$

$$+ \frac{\tilde{\sigma}^x - s_k \tilde{\sigma}^x + \tilde{\sigma}^x s_k + s_k \tilde{\sigma}^x s_k}{EE' e^{-\varphi - \varphi'} - v_1(v_F \hbar |k|)^2}$$

$$+ \frac{\tilde{\sigma}^x + s_k \tilde{\sigma}^x - \tilde{\sigma}^x s_k + s_k \tilde{\sigma}^x s_k}{EE' e^{\varphi - \varphi'} - v_1(v_F \hbar |k|)^2}$$

$$+ \frac{\tilde{\sigma}^x - s_k \tilde{\sigma}^x - \tilde{\sigma}^x s_k + s_k \tilde{\sigma}^x s_k}{EE' e^{\varphi + \varphi'} - v_1(v_F \hbar |k|)^2}$$

$$+ \frac{\tilde{\sigma}^x + s_k \tilde{\sigma}^x + \tilde{\sigma}^x s_k + s_k \tilde{\sigma}^x s_k}{EE' e^{-\varphi + \varphi'} - v_1(v_F \hbar |k|)^2},$$

(64)

where $E, \varphi$ are for $\epsilon$, and $E', \varphi'$ are for $\epsilon'$. After the substitution in (54), only the imaginary part of the inverse of the averaged Green’s function $E e^{\pm \varphi} = \pm \epsilon/(1 + A) + v_F \hbar |k|/(1 - A)$ contributes. Thus we restrict our discussion to the case with complex $A$. The sum of infinite power series $\sum_{n=1}^{\infty} x^n$ converges when $|x| < 1$. For $v_1 < 0.25$ and when $A$ is complex valued, the sum appearing in (54) does not converge since, for $\epsilon, \epsilon' > 0$,

$$\frac{v_1(v_F \hbar |k|)^2}{EE' e^{-\varphi - \varphi'}} = 1$$

holds when $A$ for $\epsilon$ and $A$ for $\epsilon'$ are related by complex conjugation, the case of which appears in $I(\epsilon + i\delta, \epsilon - i\delta)$. The same result is true for $\epsilon, \epsilon' < 0$ by replacing $e^{-\varphi - \varphi'}$ by $e^{\varphi + \varphi'}$. Since no other terms that cancel the infinity appear, we conclude that the conductivity away from $\epsilon = 0$ is always infinity. This result indicates that the conductivity is unaffected by the disorder at $\epsilon \neq 0$.

Then, we proceed to the case $\epsilon = 0$, which is exactly the case of the surface of the topological superconductor. $\epsilon = 0$ can be realized by taking the limit of $\varphi \to 0$. For $\epsilon = 0 \pm i\delta$, we obtain

$$E e^{\varphi} \to \frac{v_F \hbar |k|}{1 - A} \pm i\delta, \quad E e^{-\varphi} \to \frac{v_F \hbar |k|}{1 - A} \mp i\delta.$$

(66)

Here, we should note that since the parameter $A$ for $\epsilon = 0$ is real and $\epsilon = 0$, we need an infinitesimal imaginary parameter $\pm i\delta$ to avoid the singularity. The conductivity is then given by

$$\sigma(0) = \frac{e^2 v_F^2 \hbar}{(2\pi)^2} \int_0^{\infty} kdk$$

$$\times \left[\frac{2}{[v_F \hbar k/(1 - A) + i\delta]^2 - v_1(v_F \hbar k)^2} - \frac{1}{(v_F \hbar k/(1 - A) + i\delta)^2 - v_1(v_F \hbar k)^2}\right].$$

(67)

Finally we obtain the electronic conductivity at $\epsilon = 0$ as

$$\sigma(0) = \frac{e^2}{\pi \hbar} \frac{1}{4v_1^{1/2}} \left[\frac{1}{1/(1 - A) - v_1^{1/2}} + \frac{1}{1/(1 - A) + v_1^{1/2}}\right]$$

$$\times \log \left[\frac{1/(1 - A) + v_1^{1/2}}{1/(1 - A) - v_1^{1/2}}\right]$$

$$= \frac{e^2}{\pi \hbar} \left[1 + (10/3)v_1 + O(v_1^2)\right].$$

(68)

In the clean limit, the electronic conductivity of the surface Majorana modes converges to $e^2/\pi \hbar$.

The electronic conductivity of Dirac fermions in the zero-energy limit is known to be a universal value of the order of $e^2/\hbar$, which is referred to as the minimal conductivity. The minimal conductivity calculated from the Kubo formula is sensitive to the order of taking limits of zero temperature, non-perturbative (clean limit), and zero frequency (dc limit) [35]. So far, two coefficients of the minimal conductivity have been reported.

$$\sigma_{\text{min}}^1 = \frac{1}{\pi} \frac{e^2}{\hbar},$$

(69)

$$\sigma_{\text{min}}^2 = \frac{1}{8} \frac{e^2}{\hbar}.$$ (70)

When the dc limit is taken before the zero temperature limit and finally the clean limit is taken, $\sigma_{\text{min}}^1$ is yielded. Conversely, taking the clean limit before the zero-temperature limit and then taking the dc limit results in $\sigma_{\text{min}}^2$. The minimal conductivity obtained in this paper is consistent with $\sigma_{\text{min}}^1$.

With the help of the Wiedemann-Franz law for the Majorana fermions, the thermal conductivity of the surface of time-reversal-symmetric topological superconductors is given by

$$\kappa = \frac{1}{\pi} \frac{\pi^2 k_B^2 T}{6\hbar} \left[1 + (10/3)v_1 + O(v_1^2)\right].$$

(71)
Note that the number of degrees of freedom that contribute to the thermal conductivity is a quarter of $4 \times 4$ Dirac fermions, since the Hamiltonian is $2 \times 2$ and the fermions are real (Majorana).

V. CONCLUSION

We have studied the disorder effects on the longitudinal thermal conductivity of the Majorana surface modes of the three-dimensional time-reversal symmetric topological superconductor within the SCBA. Due to the two defining symmetries of the topological superconductor in symmetry class DIII, disorder appears in the Hamiltonian only as spatial deformations of the pair potential. For the long-ranged limit of the Gaussian deformations around each scatterer, the self-consistent Born equations are reduced to an algebraic equation that can be exactly solved.

We have derived the density of states and the electric conductivity of the surface Majorana fermions by means of the Green’s function technique. The density of states is only modified by its coefficient, while its dependence on the energy is unchanged.

The thermal conductivity is calculated from the electronic conductivity via the Wiedemann-Franz law for Majorana fermions. The electronic conductivity away from $\mu = 0$ remains infinity, which means that it is unaffected by disorder that is written by the gravitational field. However, the electronic conductivity at $\mu = 0$ (minimal conductivity), which is realized in the surface of the topological superconductor takes the finite value of the order of $e^2/h$. In the clean limit, the minimal conductivity with a coefficient $1/\pi$ appears. The thermal conductivity in the clean limit is given by $(1/\pi) \cdot \pi^2 k_B^2 T/6h$.

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