Error-and-erasure Decoding of Product and Staircase Codes with Simplified Extrinsic Message Passing

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Abstract—The decoding performance of product codes and staircase codes based on iterative bounded-distance decoding (iBDD) can be improved with the aid of a moderate amount of soft information, maintaining a low decoding complexity. One promising approach is error-and-erasure (EaE) decoding, whose performance can be reliably estimated with density evolution (DE). However, the extrinsic message passing (EMP) decoder required by the DE analysis entails a much higher complexity than the simple intrinsic message passing (IMP) decoder. In this paper, we simplify the EMP decoding algorithm for the EaE channel for two commonly-used EaE decoders by deriving the EMP decoding results from the IMP decoder output and some additional logical operations based on the algebraic structure of the component codes and the EaE decoding rule. Simulation results show that the number of BDD steps is reduced to being comparable with IMP. Furthermore, we propose a heuristic modification of the EMP decoder that reduces the complexity further. In numerical simulations, the decoding performance of the modified decoder yields up to 0.2 dB improvement compared to standard EMP decoding.

I. INTRODUCTION

Product codes (PCs) [1] and staircase codes (SCCs) [2] are powerful code constructions often used in optical communications. Conventionally, hard-decision decoding (HDD) of PCs and SCCs is based on efficient iterative bounded distance decoding (BDD) with algebraic component decoders. Recently, to meet the requirements of ultra high-speed optical communications, several hybrid algorithms have been proposed aiming to improve the decoding performance of HDD with a certain amount of soft information without increasing the decoding complexity and internal decoder data flow significantly [3]–[7]. We focus on a promising approach using error-and-erasure (EaE) decoding of the component codes using a 3-level (ternary) channel output. It was shown that EaE decoding improves the coding gain of PCs [3] and SCCs [9] based on simulation and stall pattern analysis assuming miscorrection-free decoding. In [7], the performance of EaE decoding is improved with additional miscorrection control. In [10], we have analyzed the decoding behavior of EaE decoding for both PCs and SCCs using density evolution (DE) formulated for the corresponding generalized LDPC (GLDPC) and spatially-coupled GLDPC (SC-GLDPC) ensembles including miscorrections. DE has been extensively applied in the analysis of low-density parity-check (LDPC) codes and it is well-known that DE requires extrinsic message passing (EMP). In the context of PCs and SCCs, EMP also usually yields better decoding performance than simple intrinsic message passing (IMP) [10]. Moreover, the advantages of EMP compared to IMP for the binary symmetric channel (BSC) have been shown in [11]. The major obstacle in applying EMP is that the number of component code decoder executions is proportional to \( n^2 \) in every half-iteration with \( n \) being the block length of the component codes, while IMP requires only a linear number of decoding steps. To reduce the complexity, a simplification of EMP has been investigated for the BSC in [11] such that the result of EMP decoding can be obtained from the IMP decoding result together with some simple logical operations. In this paper, we generalize the simplification of EMP in [11] to the EaE channel to obtain low-complexity EMP decoding algorithms with linear complexity. Moreover, we also propose a modification of the algorithm which reduces the complexity even further while delivering improved decoding performance.

II. PRELIMINARIES

We consider PCs of rate \( r = k_2/n^2 \), which can be seen as 2-D arrays of size \( n \times n \) where each row/column vector \( x \) is a codeword of an \( (n, k_2, t) \) component code \( \mathcal{C} \). \( \mathcal{C} \) is either a \((2^l - 1, k_0, t)\) binary Bose–Chaudhuri–Hocquenghem (BCH) code or its \((2^l - 1, k_0 - 1, t)\) even-weight subcode, both able to correct \( t \) errors. The designed distance of \( \mathcal{C} \) is denoted as \( d_{\text{des}} = 2t + 1 \). An SCC of an \((n, k_2, t)\) component code \( \mathcal{C} \) and length \( L \) consists of a chain of \( L \) matrices \( \mathbf{B}_i \) of size \( n/2 \times n/2 \) where \( i \in \{1, 2, \ldots, L\} \). Every row of the matrix \( [\mathbf{B}_{i-1}^T, \mathbf{B}_i] \) is a valid codeword of the component code \( \mathcal{C} \). We consider \( \mathcal{C} \) being either shortened BCH codes or shortened even-weight BCH subcodes. To decode a PC or SCC codeword, the rows and columns of the blocks are alternately decoded with a component code decoder \( \mathcal{D}_C \).

A PC can be interpreted as a GLDPC code and an SCC can be viewed as an SC-GLDPC code [11]. Thus, the decoding performance of PCs and SCCs can be predicted via DE formulated on an adequate GLDPC or SC-GLDPC ensemble, respectively. A detailed description of constructing the random Tanner graphs of such ensembles can be found in [10]–[12].

The codewords are transmitted over a binary-input additive white Gaussian noise (BI-AWGN) channel which outputs \( \tilde{y}_i = (-1)^{x_i} + n_i \), where \( n_i \) is (real-valued) AWGN with noise variance \( \sigma^2 = \frac{1}{2}(E_s/N_0)^{-1} \). To obtain the discrete channel output \( y_i \in \{0, 1\} \), the values \( \tilde{y}_i \in [-T, T] \) are declared...
Algorithm 1: EaED (input: $y \in \{0, ?, 1\}^{n}$)

1. $E \leftarrow$ number of erasures in $y$
2. If $E \geq d_{des}$ then $w = y$ // failure
3. else
4. $p_1, p_2 \leftarrow$ two random complementary vectors in $\{0, 1\}^{E}$
5. $y_1, y_2 \in \{0, 1\}^{n} \leftarrow y$ with erasures replaced by $p_1, p_2$
6. for $i = 1, 2$ do
7. $w_i \leftarrow$ BDD($y_i$), $d_i \leftarrow \infty$
8. if BDD($y_i$) $\in \mathcal{C}$ then $d_i \leftarrow d_{E}(y_i, w_i)$
9. if $w_1 \notin \mathcal{C}, w_2 \notin \mathcal{C}$ then $w \leftarrow y$ // failure
10. else if $w_1 \in \mathcal{C}, w_2 \notin \mathcal{C}$ (i, j $\in \{1, 2\}, i \neq j$) then $w \leftarrow w_i$
11. else if $d_1 > d_2$ then $w = w_2$
12. else if $d_2 > d_1$ then $w = w_1$
13. else $w \leftarrow$ random choice from $\{w_1, w_2\}$

as erasures “?” , where $T$ is a configurable threshold to be optimized. Values outside this interval are mapped to 0 and 1 by the usual HDD rule. For a fixed $T$, the capacity of this EaE channel is

$$C\left(\frac{E}{N_0}, T\right) = c_{C}\log_2\left(\frac{2c_{e}}{1-c_{e}}\right) + \delta_{C}\log_2\left(\frac{2\delta_{C}}{1-\epsilon_{C}}\right),$$

where the probability for an error $\delta_{C}$ and the probability for an erasure $\epsilon_{C}$ are given by

$$\delta_{C} = Q\left(\sqrt{2E/N_0(T + 1)}\right),$$
$$\epsilon_{C} = 1 - Q\left(\sqrt{2E/N_0(T - 1)}\right) - Q\left(\sqrt{2E/N_0(T + 1)}\right),$$

and $c_{C} := 1 - \delta_{C} - \epsilon_{C}$. Numerical optimization of $C(E/N_0, T)$ with respect to $T$ results in a capacity gain compared to the BSC ($T = 0$) and an optimal threshold $T_{opt}$.

III. ERROR-AND-ERASURE DECODING

We consider two commonly used EaE decoders as component code decoder $D_C$. For both algorithms, let $y \in \{0, ?, 1\}^{n}$ be the received row/column vector and define the decoding result $w$ as

$$w := D_C(y) \in \mathcal{C} \cup \{y\},$$

where $y$ is returned unchanged upon decoding failure.

Similar to the Hamming sphere $S_t(c)$ in $\{0, 1\}^{n}$, we define

$$S_t^{\mathcal{C}_{I}}(c) := \{ y \in \{0, ?, 1\}^{n} : 2d_{E}(y, c) + |E(y)| < d_{des}(t) \}$$

as the Hamming sphere in $\{0, ?, 1\}^{n}$, where $|E(y)|$ is the number of erasures of $y$ and $d_{E}(y)$ is the Hamming distance at the non-erased coordinates of $y$.

The first EaE decoder (EaE1) is a modification of [13, Sec. 3.8.1] and is described in Algorithm [1].

The second one, referred to as EaE2+, is an algebraic EaE decoding algorithm that requires only one decoding step [14]. For EaE2+, the decoding result $w$ is obtained by

$$w := D_{EaE2+}(y) = \begin{cases} c & \text{if } \exists c \in \mathcal{C} \text{ such that } y \in S_t^{\mathcal{C}_{I}}(c) \\ y & \text{otherwise} \end{cases}.$$

Similar to EaE1+, EaE guarantees that $w = c$ when there exists a $c \in \mathcal{C}$ such that $y \in S_t^{\mathcal{C}_{I}}(c)$. The difference between both decoders is that EaED, with higher complexity, can sometimes still decode when EaED+ fails (i.e., beyond the designed distance). This subtle difference causes a notable iterative decoding performance gain for EaED compared to EaED+ [10].

IV. MESSAGE-PASSING DECODING FOR GLDPC CODES

Iterative decoding of PCs and SCCs can be formulated as a message passing decoding process of corresponding GLDPC and SC-GLDPC codes [11].

We denote by $\nu_{i,j}^{(t)}$ the message passed from variable node (VN) $i$ to constraint node (CN) $j$ and by $\nu_{i,j}^{(t)}$ the message passed from CN $j$ to VN $i$ in the $t$-th iteration. Define $i := \sigma(j)(k)$ where $k \in \{1, 2, \ldots, n\}$ such that $i$ is the index of the VN that is connected to the $k$-th socket of the CN $j$. Upon initialization, the outgoing message of a VN $i$ is set to the channel output $r_i \in \{0, ?, 1\}$.

In the $t$-th CN update, each CN $j$ receives $n$ incoming messages from all its neighboring VNs. For IMP decoding [2], the messages are combined into

$$y_{j,IMP}^{(t)} := \left(\nu_{\sigma(1),j}(1), \ldots, \nu_{\sigma(n),j}(n)\right),$$

and are decoded by the component decoder $D_C \in \{D_{EaE1}, D_{EaE2+}\}$. The CN $j$ then sends the message $\nu_{i,j}^{(t)} = [D_C(y_j^{(t),IMP})]_k$ back to VN $i = \sigma(j)(k)$, where $[y_k]$ denotes the $k$-th component of vector $y$.

For EMP decoding [11], one component code decoding is performed to calculate $\nu_{i,j}^{(t)}$ for each of the $n$ VNs. For computing $\nu_{i,j}^{(t)}$, the $k$-th position of $y_j^{(t),IMP}$ is replaced by the channel output $r_i$, yielding

$$y_j^{(k,t)} := \left(\nu_{\sigma(1),j}(1), \ldots, \nu_{\sigma(k-1),j}(k-1), r_i, \nu_{\sigma(k+1),j}(k+1), \ldots\right).$$

The CN $j$ then sends $\nu_{i,j}^{(t)} = [D_C(y_j^{(k,t)})]_k$ to VN $i$.

In the VN update, each VN $i$ receives two messages from its connected CNs $j, j'$ and forwards to each CN the message that it has received from the respective other CN: $\nu_{i,j}^{(t+1)} = \nu_{i,j}^{(t)}$, $\nu_{i,j'}^{(t+1)} = \nu_{i,j'}^{(t)}$.

At the end of message passing, each VN randomly chooses one of the incoming messages as its final value. If the message is erased, it is replaced by a random binary value.

EMP guarantees the independence of the messages, which enables the DE analysis [10]. Moreover, in EMP, the hard channel outputs are used twice: Once as the initial VN value and a second time as a replacement of the intermediate VN value, avoiding miscorrections to some extent.

V. LOW-COMPLEXITY EMP ALGORITHMS (LCEAs)

To fully use the benefits of EMP and of an accurate performance prediction via DE, we need a low-complexity version of the EMP decoder. As described in Sec. [IV] the number of component code decoding steps of EMP is proportional to $n^2$ instead of $n$ as for IMP. This is the major obstacle for EMP decoding in practical applications. For the BSC, the EMP decoding result can be calculated from the IMP decoding result and some simple logical operations, such that the complexity...
becomes comparable to IMP \[\text{[I]}\]. In this section, we show that similar simplifications can be performed for the EaE channel.

We consider one CN update and try to predict the EMP decoding result \(\hat{v}^{(\ell)}_{\sigma_j(k), j}\) destined for VN \(\sigma_j(k)\). To simplify the notation, we omit the index of CN node \(j\) and iteration \(\ell\) and use the upper-script \(k\) to differentiate between IMP- and EMP-associated intermediate variables and vectors during the decoding. For example, \(y\) stands for \(y^{(\ell)}_{\sigma_j(k), j}\) and \(y^k\) stands for \(y^{(\ell)}_{\sigma_j(k), j} \). Let \(r_k \in \{0, ?, 1\}\) denote the channel output at the \(\sigma_j(k)\)-th VN. The IMP decoding result is \(w_{\text{IMP}} := D_{\text{C}}(y)\) and the final EMP decoding result is denoted by \(w\). We denote by \(w_k\) the \(k\)-th position of \(w\) such that \(\hat{v}^{(\ell)}_{\sigma_j(k), j} = w_k\). Additionally, for EaED (Algorithm \[\text{I}\]), let \(y_1\) and \(y_2\) be the vector where the erasure positions in \(y\) are replaced by two random complementary vectors \(p_1\) and \(p_2\), and similarly, define \(y^1_k\) and \(y^2_k\) for \(y^k\). Let \(y^1_k\) be the \(k\)-th component of \(y^k\) for \(i \in \{1, 2\}\). Finally, \(w_{\text{IMP}, i} := \text{BDD}(y_i)\) and \(w^k_i := \text{BDD}(y^k_i)\).

A. Low-complexity EMP Algorithm for the BSC

We first revisit the low-complexity EMP algorithm for the BSC \[\text{[I]}\] from a slightly different perspective. The decoding result is \(w_{\text{IMP}} := \text{BDD}(y) \in \mathcal{C} \cup \{y\}\). We can obtain the distance \(d(w_{\text{IMP}}, y)\) from the IMP decoder and we set \(d(w_{\text{IMP}}, y) = \infty\) in case of a decoding failure. For every position \(k\), the distance \(d(w_{\text{IMP}}, y^k)\) can be obtained from \(d(w_{\text{IMP}}, y)\) by

\[
d(w_{\text{IMP}}, y^k) = d(w_{\text{IMP}}, y) + \begin{cases} 0 & y_k = r_k \\ -1 & y_k \neq r_k, w_{\text{IMP}} = r_k \\ +1 & y_k \neq r_k, w_{\text{IMP}} \neq r_k. \end{cases}
\]

If \(d(w_{\text{IMP}}, y^k) \leq t\), then \(\text{BDD}(y^k) = \text{BDD}(y) = w_{\text{IMP}}\). Thus, \(w_k = w_{\text{IMP}, k}\).

If \(d(w_{\text{IMP}}, y^k) > t\), there are only two possible results: 1) If \(y^k \in \mathcal{S}_i(c)\) for some \(c \in \mathcal{C}\), then it must hold that \(c_k = r_k\) and it follows that \(w_k = r_k\). 2) If \(y^k \not\in \mathcal{S}_i(c)\) for any \(c \in \mathcal{C}\), then \(\text{BDD}(y^k)\) will fail and \(w_k = r_k\).

In summary, we have

\[
w_k = \begin{cases} w_{\text{IMP}, k} & d(w_{\text{IMP}}, y^k) \leq t \\ r_k & d(w_{\text{IMP}}, y^k) > t. \end{cases}
\]  \(\text{(1)}\)

B. Low-complexity EMP Algorithm with EaED

Now we analyze the relation between \(w_{\text{IMP}}\) and \(w\) for the EaED (Algorithm \[\text{I}\]). The goal is to obtain \(w^k\) and the distance \(d_{\text{EMP}}(y^k_i, y^k_j)\) from the IMP decoding result and then predict \(w\).

We first consider the case when the number of erasures \(E\) in \(y\) is too large, i.e., \(E \geq d_{\text{des}}\). As the decoding process will not turn a non-erased bit into an erasure (EaED does not introduce new erasures), the number of erasures \(E^k\) in any \(y^k\) is at least \(E\). No decoding will happen and \(w_k = r_k\) for all \(k\). Thus, we output \(w = r\).

For \(E < d_{\text{des}}\), we observe the following facts for every \(k \in \{1, 2, \ldots, n\}\), which lead to the low-complexity EMP algorithm:

If \(y_k = r_k\), we have \(w_k = w_{\text{IMP}, k}\) since \(y^k = y\).

If \(E^k = E + 1\{y_k \neq r_k\} \geq d_{\text{des}}\), then \(w_k = r_k\).

Since the EaED is based on two BDD outcomes, we first obtain the distance \(d(w_{\text{IMP}, i}, y^k_i)\) from \(d(w_{\text{IMP}, i}, y_i)\) calculated in the BDD step to predict the result of BDD \(y^k_i\). We have

\[
d(w_{\text{IMP}, i}, y^k_i) = d(w_{\text{IMP}, i}, y_i) + \begin{cases} 0 & r_k = ?, y_i = p_i \\ -1 & r_k = ?, y_i \neq p_i, w_{\text{IMP}, i} = p_i \\ +1 & r_k = ?, y_i \neq p_i, w_{\text{IMP}, i} \neq p_i \\ 0 & r_k \neq ?, y_k = r_k \\ -1 & r_k \neq ?, y_k \neq r_k, w_{\text{IMP}, k} = r_k \\ +1 & r_k \neq ?, y_k \neq r_k, w_{\text{IMP}, k} \neq r_k. \end{cases}
\]  \(\text{(2)}\)

Finally, \(w_k \in \{1, 2\}\). We are left to determine which one of the \(w^k_i\) will be chosen for the following three cases.

Case 1: \(d(w_{\text{IMP}, i}, y^k_i) \leq t\) for both \(i \in \{1, 2\}\). In this case, \(w^k_i = w_{\text{IMP}, i}\). We just need to compare the distance between \(w^k_i\) and \(y^k_j\) at the unerased coordinates of \(y^k\). We observe that

\[
d_{\text{EMP}}(y^k_i, y^k_j) = \min \begin{cases} 0 & y_k = ? \\ 1 & y_k \neq ?, r_k = ?, w_{\text{IMP}, k} = y^k_i \\ 2 & y_k \neq ?, r_k = ? , w_{\text{IMP}, k} \neq y^k_i \end{cases}.
\]

Case 2: \(d(w_{\text{IMP}, i}, y^k_i) > t\) for both \(i \in \{1, 2\}\). Then \(w_k = r_k\) as in Sec. \(\text{[V-A]}\)

Case 3: \(d(w_{\text{IMP}, i}, y^k_i) \leq t\) and \(d(w_{\text{IMP}, j}, y^k_j) > t\) for all \(i, j \in \{1, 2\}, i \neq j\). This is the most complicated case and \(w_k\) is only solvable for some special cases:

Case 3.1: If \(r_k = w_{\text{IMP}, i, k}\), then \(w_k = r_k\), because we have \(w_{i, k} = r_k\) \(d(w_{\text{IMP}, i, y^k_i}) \leq d(w_{\text{IMP}, i}, y_i))\) and \(w_{j, k} = r_k\) (using \[\text{[I]}\]). Both \(w_{i, k}\) and \(w_{j, k}\) are consistent.
Algorithm 2: LCEA and h-LCEA with EaED

1. If $E \geq d_{des}$, then $w = r$, return.
2. For $i = 1, 2$, do $w_{IMP,i} \leftarrow \text{BDD}(y_i)$
3. $w_{IMP} \leftarrow D_{EAd}(y)$
4. For $k = 1, 2, \ldots, n$ do
5. If $y_k = r_k$ then $w_k \leftarrow w_{IMP,k}$. continue
6. $E_k \leftarrow E + 1$ if $y_k \neq r_k$.
7. If $E_k \geq d_{des}$ then $w_k \leftarrow r_k$. continue
8. For $i = 1, 2$, do $d_i \leftarrow d(w_{IMP,i}, y_i)$ using (2).
9. If $d_1 \leq t, d_2 \leq t$ then  // Case 1
10. calculate $w_k$ using (3) and (4).
11. else if $d_1 \leq t, d_2 > t (i,j \in \{1,2\}, i \neq j)$ then  // Case 3
12. if $\{(r_k = w_{IMP,i,k}) \lor (r_k = ? \land y_j = p_j)\}$ then
13. $w_k = w_{IMP,k}$. else
14. $LCEA: w_k \leftarrow D_{EAd}(y_k)$
15. h-LCEA: $w_k \leftarrow w_{IMP,k}$
16. else
17. $LCEA: w_k \leftarrow D_{EAd}(y_k)$
18. h-LCEA: $w_k \leftarrow r_k$  // Case 2
19. else
20. $LCEA: w_k \leftarrow D_{EAd}(y_k)$
21. End For

Case 3.2: If $(r_k = ?, y_j, k = r_k)$ or $(r_k = ?, y_j, k = p_j)$, then
$w_k = w_{IMP,i,k}$ as $\text{BDD}(y_k) = \text{BDD}(y_k) \notin C$ and
$\text{BDD}(y_k) = \text{BDD}(y_k) = w_{IMP,i} \in C$. Note that the first condition implies $y_j, k = y_i, k = y_k = r_k$, which has already been covered above.

General case: $w_k$ is not solvable and an extra decoding is required. As we have $w_i = \text{BDD}(y_k) = w_{IMP,i}$, $w_k \neq w_{IMP,i,k}$ if and only if $w_i = \text{BDD}(y_k) \in C$ and
$d_{EAd}(y_k) \leq d_{EAd}(y_k) < d_{EAd}(w_i)$. This is possible when $y_j, k \neq y_j, k$ which is true if none of the conditions above holds.
A heuristic approach is to set $w_k = w_{IMP,i,k}$ to avoid additional decoding as $\text{BDD}(y_k)$ is more prone to a miscorrection than $\text{BDD}(y_k)$.

We call the decoding process described above low-complexity EMP algorithm (LCEA) with EaED. Additionally, heuristic-LCEA (h-LCEA) is a simplified version of LCEA where the extra decoding step is avoided by setting $w_k$ with a heuristic value. Both algorithms are summarized in Algorithm 2. The difference lies on lines 14, 15, and lines 19, 20.

C. Low-complexity EMP Algorithm with EaED+

For EaED+, a similar analysis as for the BSC can be performed with
$$\tilde{d}(w_{IMP}, y) = 2d_{EAd}(w_{IMP}, y) + |E(y)|,$$
which involves both errors and erasures. We calculate
$$\tilde{d}(w_{IMP}, y_k) = \tilde{d}(w_{IMP}, y)$$
+ $2$ $y_k !?; y_k = w_{IMP,k}, r_k \neq w_{IMP,k}, r_k \neq ?$
+ $1$ $y_k !?; y_k = w_{IMP,k}, r_k = ?$
+ $0$ otherwhise.

If $\tilde{d}(w_{IMP}, y_k) < d_{des}$, then $w_k = w_{IMP,k}$.

If $\tilde{d}(w_{IMP}, y_k) \geq d_{des}$, two cases may occur.

Case 1: If $r_k \neq ?$, then $D_{EAd}(y_k)$ will either fail ($w_k = r_k$) or succeed with $w_k = r_k$ (Sec. V-A).

Case 2: $r_k = ?$: If $D_{EAd}(y_k)$ fails, then $w_k = r_k$. We need to determine if it is possible that $D_{EAd}(y_k)$ succeeds.

Case 2.1: We first assume that $D_{EAd}(y_k)$ succeeded. With $c_1 = w_{IMP}, d(c_1, y) = d_{des} - 1$. Let $c_2 \in C$ such that $\tilde{d}(c_2, y_k) \leq d_{des} - 1$. This is only possible when $\tilde{d}(c_2, y) \leq d_{des}$ and $y_k \neq c_2, k$. We can see that $\tilde{d}(c_1, c_2) \leq \tilde{d}(c_1, y) + \tilde{d}(c_2, y) = 2d_{des} - 1$ as
$$\tilde{d}(c_1, c_2) = 2\tilde{d}(c_1, c_2)$$
+ $2(\tilde{d}(e(y_k, c_1) + d_{EAd}(c_1, c_2))$
+ $\leq 2d_{eAd}(c_1, c_2) + 2|E(y)|$
+ $\leq 2(\tilde{d}(E_{Ad}(c_1, y) + d_{EAd}(c_2, y)) + 2|E(y)|)$
+ $= \tilde{d}(c_1, y) + \tilde{d}(c_2, y)$.

Since $C$ is a linear code, we know that $\tilde{d}(c_1, c_2) \leq 2d_{des}$, resulting in a contradiction. Hence, this is an impossible case.

Case 2.2: If $D_{EAd}(y_k)$ fails, $D_{EAd}(y_k)$ can succeed if $\exists c \in C, \tilde{d}(c, y) = d_{des}$ and $\tilde{d}(c, y_k) = d_{des} - 1$. This is possible when one error position in $y$ is replaced by an erasure. If this happens, then it must hold that $c_k = \bar{y}_k$. Hence, $w_k = y_k$.

In summary, for EaED+, we have
$$w_k = \begin{cases} w_{IMP,k} & \tilde{d}(w_{IMP}, y_k) < d_{des} \\ r_k & \tilde{d}(w_{IMP}, y_k) \geq d_{des} \land \bar{a} \\ r_k \lor y_k & \tilde{d}(w_{IMP}, y_k) \geq d_{des} \land a. \end{cases}$$
where $a := (r_k = ?) \land (y_k \neq r_k) \land (D_{EAd}(y) = \text{fail})$ is a condition. A re-decoding is required for the third case.

VI. SIMULATION RESULTS

We evaluate the performance of h-LCEA with EaED by simulation and compare it with conventional EMP decoding. We calculate the noise threshold defined as the minimal $E_v/N_0$ with which the target bit error rate (BER) of $10^{-4}$ after a fixed number of iterations (for PCs 20 iterations and for SCC's 3 iterations with window length 7) is achieved numerically by a Monte Carlo approach along with a binary search. The component codes are as described in Sec. II with parameters $n \in \{63, 127, 255, 511\}$ and $t \in \{2, 3, 4\}$. We further find the optimal erasure threshold $T_{opt}$ during the search. The noise threshold difference (gain) $\Delta(E_v/N_0)*$ compared to an iterative HDD EMP decoder is calculated and shown in Fig. I. Additionally, the noise threshold difference for the conventional EMP (implemented using the LCEA) and the respective $T_{opt}$ are shown. The results of LCEA with EaED+ are not shown for the sake of clarity as the EaED+ decoder usually yields smaller performance gains than the EaED decoder. The h-LCEA yields a larger noise threshold gain compared to conventional EMP for most of the PCs and some of the SCCs. For small $t$, the gain is relatively large. This is due to the fact that the heuristic value avoids certain miscorrections to some extent, which is particularly beneficial as miscorrections occur approximately with probability $1/t!$.
Fig. 1. Noise threshold gain and $T_{\text{opt}}$ for PCs and SCCs in the EaED decoding with h-LCEA and conventional EMP.

Our proposed h-LCEA decoder achieves roughly half of the capacity gain. As the increased number of component code decoding steps is the major cause of the EMP complexity overhead, we compare the number of BDD steps of the proposed LCEAs based on the EaE decoders and BDD in a 20 half-iterations decoding for a PC constructed with the (511, 3) even-weight BCH subcode as an example. The optimal erasure thresholds $T_{\text{opt}}$ are used. The results are plotted in Fig. 2 together with their respective noise thresholds for a target BER of $10^{-4}$. By setting $T = 0$, the result of LCEA with HDD [11], which is comparable to a normal iterative HDD with IMP, is obtained and used as a baseline. Due to algorithm termination upon decoding success, the curves converge at high $E_s/N_0$. The number of BDD steps required for LCEA with EaED is still several times higher than for HDD because of the required re-decoding for some bits in EMP and the two BDD steps for words with erasures. The complexity can be reduced further with h-LCEA. For EaED+, the increased complexity for re-decoding is relatively small.

VII. CONCLUSION

In this paper, we analyzed EMP decoding over the EaE channel. This essentially comes down to the question: how will the decoding result change if we change one bit in the vector to be decoded? While this question has a simple and deterministic answer for the BSC with BDD, it is unfortunately not the case for the EaE channel due to the uncertainty introduced by the erasures. However, we observe that EMP decoding achieves larger coding gains over the EaE channel than over the BSC channel. Furthermore, replacing the uncertain result with a value that is more likely to be (closer to) the correct value further improves the decoding performance and reduces the complexity.
REFERENCES

[1] P. Elias, “Coding for noisy channels,” in IRE Convention Record, Part IV, Mar. 1955, pp. 37–46.
[2] B. P. Smith, A. Farhood, A. Hunt, F. R. Kschischang, and J. Lodge, “Staircase codes: FEC for 100 Gb/s OTN,” J. Lightw. Technol., vol. 30, no. 1, pp. 110–117, 2012.
[3] Y. Lei, B. Chen, G. Liga, X. Deng, Z. Cao, J. Li, K. Xu, and A. Alvarado, “Improved decoding of staircase codes: The soft-aided bit-marking (SABM) algorithm,” IEEE Trans. Commun., vol. 67, no. 12, pp. 8220–8232, 2019.
[4] A. Sheikh, A. Graell i Amat, G. Liva, and A. Alvarado, “Refined reliability combining for binary message passing decoding of product codes,” J. Lightw. Technol., vol. 39, no. 15, pp. 4958-4973, 2021.
[5] A. Sheikh, A. Graell i Amat, G. Liva, C. Häger, and H. D. Pfister, “On low-complexity decoding of product codes for high-throughput fiber-optic systems,” in Proc. ISTC, 2018.
[6] A. Sheikh, A. Graell i Amat, and G. Liva, “Binary message passing decoding of product-like codes,” IEEE Trans. Commun., vol. 67, no. 12, pp. 8167–8178, 2019.
[7] A. Sheikh, A. Graell i Amat, and A. Alvarado, “Novel high-throughput decoding algorithms for product and staircase codes based on error-and-erasure decoding,” J. Lightw. Technol. vol. 39, no. 15, pp. 4909-4922, 2021.
[8] D. K. Soma, A. K. Pradhan, and K. Narayanan, “Errors and erasures decoding of product codes for optical transport networks,” IEEE Commun. Lett., vol. 25, no. 8, pp. 2482-2486, 2021.
[9] A. Y. Sukmadji, “Zipper codes: High-rate spatially-coupled codes with algebraic component codes,” Master’s thesis, University of Toronto, 2020.
[10] L. Rapp and L. Schmalen, “Error-and-erasure decoding of product and staircase codes,” IEEE Trans. Commun., vol. 70, no. 1, pp. 32-44, 2022.
[11] Y. Y. Jian, H. D. Pfister, and K. R. Narayanan, “Approaching capacity at high rates with iterative hard-decision decoding,” IEEE Trans. Inf. Theory, vol. 63, no. 9, pp. 5752–5773, 2017.
[12] L. M. Zhang, D. Truhachev, and F. R. Kschischang, “Spatially coupled split-component codes with iterative algebraic decoding,” IEEE Trans. Inf. Theory, vol. 64, no. 1, pp. 205–224, 2017.
[13] T. K. Moon, Error Correction Coding - Mathematical Methods and Algorithms. John Wiley & Sons, Inc., 2005.
[14] G. Forney, “On decoding BCH codes,” IEEE Trans. Inf. Theory, vol. 11, no. 4, pp. 549–557, 1965.
[15] R. McEliece and L. Swanson, “On the decoder error probability for Reed-Solomon codes (corresp.),” IEEE Trans. Inf. Theory, vol. 32, no. 5, pp. 701–703, 1986.
[16] J. Justesen, “Performance of product codes and related structures with iterated decoding,” IEEE Trans. Commun., vol. 59, no. 2, pp. 407–415, 2010.