Vertiport Selection in Hybrid Air–Ground Transportation Networks via Mathematical Programs With Equilibrium Constraints

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Abstract—Urban air mobility is a concept that promotes aerial modes of transport in urban areas. In these areas, the location and capacity of the vertiports—where the travelers embark and disembark the aircraft—not only affect flight delays, but can also aggravate the congestion of ground vehicles by creating extra ground travel demands. In this article, we introduce a mathematical model for selecting the location and capacity of the vertiports that minimizes traffic congestion in hybrid air–ground transportation networks. Our model is based on a mathematical program with bilinear equilibrium constraints. Furthermore, we show how to compute a global optimal solution to this mathematical program by solving a mixed-integer linear program. We demonstrate our results via the Anaheim transportation network model, which contains more than 400 nodes and 900 links.

Index Terms—Traffic equilibria, transportation network optimization, urban air mobility (UAM).

NOMENCLATURE

Parameters

\( n_t \) Number of different equilibria.
\( n_n \) Number of nodes.
\( n_v \) Number of candidate vertiports.
\( n_g \) Number of ground and air links.
\( n_d \) Number of destination nodes.
\( n_a \) Number of ground links.
\( n_a \) Number of air links.
\( n_c \) Number of options for vertiport capacity.
\( n_b \) Number of logical constraints.
\( \gamma \) Budget for selecting vertiport capacity.
\( \mu \) A large positive scalar.

Variables

\( g \) Vertiport capacity vector variable.
\( p \) Dual variable for link capacity constraint.
\( q \) Dual variable for vertiport capacity constraint.
\( B \) Binary selection matrix.
\( U \) Dual variable for nonnegative flow constraint.
\( V \) Dual variable for flow conservation constraint.
\( X \) Flow matrix variable.
\( Y \) Auxiliary variable for linearizing constraints.

Subscripts

\( [a]_j \) \( j \)th element of vector \( a \).
\( [A]_{ij} \) \( ij \)th element of matrix \( A \).

Sets

\( \mathbb{R}^n \) Set of \( n \)-dimensional real vectors.
\( \mathbb{R}^{m \times n} \) Set of \( m \times n \) real matrices.
\( \{0, 1\}^n \) Set of \( n \)-dimensional binary vectors.
\( \{0, 1\}^{m \times n} \) Set of \( m \times n \) binary matrices.

Operations

\( \odot \) Hadamard product.
\( \text{vec} \) Vectorization function.

I. INTRODUCTION

URBAN air mobility (UAM) is a concept that promotes short-range aerial travel in urban areas [1], [2]. By adding
alternative air modes of transportation—mainly supported by electric vertical takeoff and landing (eVTOL) aircraft—to the existing ground transportation networks, UAM has the potential to alleviate ground traffic congestion. The latter has become a growing concern for both travelers and transportation authorities [3].

An integral part of the operation of eVTOL aircraft is to build vertiports where passengers or cargo embark and disembark the aircraft. Many UAM industries, including Ferrovial, Urban-Air Port Ltd., and Skyports, are actively investigating the possibility of ultra-compact, rapidly deployable, multifunctional vertiports for both manned and unmanned aircraft around the world [4].

One challenge in UAM is to select the locations of the vertiports optimally among candidate options. The input of the selection includes a set of candidate vertiports (usually generated by clustering algorithms [5], [6], [7]) and the budget of the total number of vertiports. The output of the selection is a set of selected vertiports that optimizes certain performance metrics of the air transportation network supported by the selected vertiports, such as the savings in total travel time [8] and the packaging demand served by the aircraft [9].

Since the candidate locations for vertiports are often limited by safety, accessibility, and noise emission factors, most door-to-door travel in UAM will require transportation via both aircraft and ground vehicles [5], [6], [7], [10]. Consequently, the vertiport locations can also affect the traffic in the existing ground transportation network by creating additional travel demands on the way to and from the vertiports. Recent studies investigate the potential of UAM as a complement to the existing transportation systems [11]. However, how to select the vertiport locations by optimizing its impacts on the congestion in a network that allows both aerial and ground modes of transport is, to our best knowledge, still an open question.

There have been various studies on transportation network design based on mathematical programs with equilibrium constraints (MPEC), a nonconvex optimization problem with bilinear complementarity constraints (see [12], [13], [14], and references therein). However, the existing results have the following limitations when applied to vertiport selection. First, to our best knowledge, these results are all based on traffic equilibrium models that only include link parameters, such as the number of lanes and the speed limit in ground transportation networks [15], [16], [17], [18], [19], but not node parameters, such as the number of touch-down and lift-off pads and the total number of scheduled flights at a vertiport in an air transportation network. Consequently, despite the success of link parameter design, such as the addition or expansion of candidate road segments, the design of node parameters, such as the location and capacity of the vertiports, has not been investigated in depth. Second, the number of bilinear complementarity constraints in the MPEC for transportation network design depends on the number of nodes and links in the network, which grows rapidly as the network size increases [13].

We introduce a mathematical model for selecting the location and capacity of the vertiports in a hybrid air–ground transportation network. Our objective is to reduce the congestion in the network. Our contributions are threefold and are as follows:

1) First, we developed a linear program-based model for the static traffic equilibria in a hybrid air–ground transportation network. This model extends the Nesterov and de Palma model by adding node capacity constraints.

2) Second, we proposed a mathematical program with bilinear equilibrium constraints that optimizes the location and capacity of the vertiports subject to budget and logical constraints. In addition, we showed how to compute a global optimal solution of this mathematical program by solving a mixed-integer linear program (MILP). This MILP does not contain any bilinear complementarity constraints, and the number of integer variables only depends on the number of candidate vertiports, which is typically much smaller than the total number of nodes or links in the network.

3) Finally, we demonstrated our results using the Anaheim transportation network, which contains more than 400 nodes, 900 links, and nine candidate vertiports where each vertiport has two candidate capacity values. Our MILP contains only 18 binary integer variables.

Our work is the first step in adapting the mathematical tools for ground transportation network analysis and design in the age of UAM. In particular, we showed how to extend the ground traffic equilibria model to predict traffic equilibria in hybrid air–ground transportation networks. We also provide a numerical tool to design the vertiports as an extension of an existing ground transportation network.

The rest of this article is organized as follows. Section II briefly reviews some existing results in static traffic equilibria and transportation network design. Section III introduces an extended Nesterov and de Palma model for static traffic equilibria in hybrid air–ground transportation networks. Section IV introduces the mathematical program with equilibrium constraints for vertiports selection. We demonstrate this mathematical program using the Anaheim transportation network in Section V. Finally, Section VI concludes the article.

II. RELATED WORK

Transportation network design is the problem of determining the optimal modification of an existing ground transportation network [12], [13]. These modifications can be expanding the capacity of existing links or adding new links to the network. The quality of the modifications is evaluated via the congestion of the travelers in the modified network and the cost of the modifications. The input of the problem includes the following:

1) the existing transportation network topology;
2) the travel demand between each origin–destination pair for a specific time interval;
3) the characteristics of roads, such as flow capacity and free travel time;
4) the set of candidate options for modifications and their cost;
5) the total budget for modifications.

The outcome of the problem is a set of modifications that satisfies the budget constraint and minimizes the congestion of
III. STATIC TRAFFIC EQUILIBRIA IN HYBRID AIR–GROUND TRANSPORTATION NETWORKS

We first introduce the static traffic equilibria model of a hybrid air–ground transportation network. This model predicts the static regime of the traffic patterns, where the number of travelers entering and exiting a road segment (or a flight leg) per unit of time are the same. We will later use this model to evaluate the performance of a given transportation network. Our model is based on the following three assumptions.

1) For each origin–destination pair, only the routes with the minimum accumulated travel time are used.

2) The traffic flow on each road segment or flight leg never exceeds its capacity; the total incoming and outgoing air traffic flow at each vertiport never exceeds its capacity.

3) If the ground traffic flow on a road segment is below its capacity, its travel time equals a nominal value; if the capacity is reached, the travel time is higher than the nominal value. If the total incoming and outgoing air traffic flow at a vertiport is below its capacity, the delay (for embarkation and disembarkation) at this vertiport is zero; if the capacity is reached, the delay is nonnegative.

The above three assumptions have the following implications. The first assumption characterizes the selfish and competitive nature of the travelers’ behavior; it is also known as the Wardrop equilibrium principle [25] and has been the basis of static traffic equilibria models [15]. The second assumption states that the traffic flow on a road segment is upper bounded—typically due to the number of lanes and the green light time—and the total air traffic flow at a vertiport is upper bounded—typically due to the number of touch-down and lift-off pads. The third assumption is based on the empirical observation that the travel time on a road segment (or the delay at a vertiport) is at its minimum when there is no congestion and increases with the congestion level. Similar assumptions were first introduced in the Nesterov and de Palma model for ground traffic equilibria [18]. Here, we add two additional assumptions on the capacity and delay at vertiports.

**Remark 1:** Notice that the first assumption above is not reasonable at all when understood literally: Other than the time consumed during travel, operating cost, such as the fare of a trip, is also an important factor that affects the travelers’ decision. However, one can convert such an operating cost to an additional effective time using the travelers’ average value of time based on their annual income; a similar conversion was used in [26]. Therefore, without loss of generality, we refer to the term “travel time” as an effective travel time that accounts for both the operating costs and the actual time of travel.

In the following, we will introduce a mathematical model for static traffic equilibria that satisfies all the aforementioned assumptions. Our model is based on the Nesterov and de Palma model for ground traffic equilibria.

A. Hybrid Air–Ground Transportation Networks

We first introduce some basic network concepts: Nodes, links, incidence matrices, link and node capacities, and travel time.

1) **Nodes and Links:** We let $\mathcal{N} = \{1, 2, \ldots, n_n\}$ denote the set of nodes. We let $\mathcal{V} = \{v(1), v(2), \ldots, v(n_v)\}$ denote the set of nodes that contain a candidate vertiport location, where $v(i) \in \mathcal{N}$ for all $i = 1, 2, \ldots, n_v$.

We let $\mathcal{L} = \{1, 2, \ldots, n_l\}$ denote the set of links. Each link is an ordered pair of distinct nodes, where the first and second nodes are the “tail” and “head” of the link, respectively. In addition, we let $n_g \leq n_g$ denotes the number of ground links, and $n_a := n_l - n_g$ denote the number of air links. The presence of link $k = (i, j)$ with $1 \leq k \leq n_a$ means that any ground travelers can travel from node $i$ to node $j$, and the presence of link $k = (i, j)$ with $n_a + 1 \leq k \leq n_l$ means any aircraft can fly from node $i$ to node $j$.

2) **Incidence Matrices:** We represent the topology of the hybrid air-ground network using the node-edge incidence matrix $E \in \mathbb{R}^{n_n \times n_l}$. The entry $[E]_{ik}$ in matrix $E$ is associated with node $i$ and link $k$ as follows:

$$
[E]_{ik} = \begin{cases} 
1, & \text{if node } i \text{ is the tail of link } k, \\
-1, & \text{if node } i \text{ is the head of link } k, \\
0, & \text{otherwise}. 
\end{cases}
$$
Note that $[E]_{ik} \neq 0$ for some $n_g + 1 \leq k \leq n_t$ only if $i \in \mathcal{V}$.

We represent the topology of the air links and vertiports using the following unsigned incidence matrix $D \in \mathbb{R}^{n_s \times n_l}$ for air links. The entry $[D]_{ik}$ is associated with node $i$ and link $k$ as follows:

$$[D]_{ik} = \begin{cases} 1, & \text{if } g \geq n_g + 1 \text{ and } [E]_{i,v},k \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(2)

3) Demand Matrix: We distinguish different travelers in the network using their destinations, denoted by a subset of nodes $\{s(1), s(2), \ldots, s(n_d)\} \subset \mathcal{N}$. We denote the number of trips per unit time, also known as the traffic demand, between different origin and destination nodes using a demand matrix $S \in \mathbb{R}^{n_s \times n_d}$ defined as follows. For any $i \in \mathcal{N}$ with $i \neq s(j)$, we let the entry $[S]_{ij}$ in matrix $S$ denote the traffic demand from node $i$ to node $s(j)$, i.e., the amount of travelers leaving node $i$ heading towards node $s(j)$ per unit time. If $[S]_{ij} = 0$, then $(i, s(j))$ is also known as an origin–destination pair. Finally, we let $[S]_{s(j),j} = \sum_{i \neq s(j)} [S]_{ij}$ for all $j = 1, 2, \ldots, n_d$ such that the sum of each column in matrix $S$ equals zero. Such an assignment is convenient for defining the flow conservation constraints in matrix form, as we will show.

4) Flow Matrix: At a static traffic equilibrium, the amount of travelers entering and exiting the same link are the same. We represent the amount of travelers on different links per unit of time using the flow matrix $X \in \mathbb{R}^{n_s \times n_d}$. In particular, the entry $[X]_{kj}$ in matrix $X$ denotes the amount of travelers exiting link $k$ while heading towards destination node $s(j)$ per unit time.

By construction, the demand matrix $S$, flow matrix $X$, and incidence matrices $E$ together satisfy the following flow conservation constraint:

$$EX = S, \quad X \geq 0.$$  

(3)

We note that the above constraints implicitly imply that the sum of each column in matrix $S$ equals zero. This observation justifies our definition of the negative entries in matrix $S$.

Example 1: To illustrate the aforementioned network concepts, we consider the example network in Fig. 1. In this case, we have $\mathcal{N} = \{1, 2, 3, 4\}$, $\mathcal{V} = \{2, 3\}$, $\mathcal{L} = \{1, 2, 3, 4, 5, 6\}$, and matrices $E$ and $D$ are as follows:

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

Furthermore, a possible choice of demand matrix $S$ and flow matrix $X$ that satisfy the constraints in (3), are as follows:

$$S = \begin{bmatrix} 5 & -5 & 0 & 0 \\ 10 & 0 & -10 \end{bmatrix}^T, X = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 3 & 7 & 7 & 3 & 0 & 4 \end{bmatrix}^T.$$  

5) Capacity and Free Travel Time: The link capacity of a (ground or air) link is the maximum amount of travelers existing on this link per unit of time. For a ground link, this capacity depends on the number of lanes and cycle time of traffic signals; for an air link, this capacity depends on the available airspace and the maximum allowed aircraft density of each flight leg. We denote the link capacity of all the air and ground links using the link capacity vector $f \in \mathbb{R}^{n_l}$, whose entry $[f]_k$ denotes the capacity on link $k$.

The free travel time of a (ground or air) link is the time consumed by each traveler on this link when there is no traffic congestion. We denote the free travel time of all links using a vector $c \in \mathbb{R}^{n_l}$, whose $k$th entry denotes the free travel time on link $k$.

The Nesterov and de Palma model [17, 18] assumes that the link capacity, the free travel time, and the flow matrix are coupled as follows. First, the traffic flow on each link never exceeds its capacity, i.e.,

$$\sum_{j=1}^{n_d} [X]_{kj} \leq [f]_k$$  

(4)

for all $k \in \mathcal{L}$. Second, if the traffic flow on a link is below its capacity, then the traffic delay of this link equals the corresponding free travel time; if the traffic flow on a link equals its capacity, then the average travel time of this link is lower bounded by the corresponding free travel time. In other words, if vector $\bar{c} \in \mathbb{R}^{n_l}$ is such that $[\bar{c}]_k$ denotes the traffic time on link $k$, then the following conditions hold for all $k \in \mathcal{L}$:

$$\sum_{j=1}^{n_d} [X]_{kj} < [f]_k \Rightarrow [\bar{c}]_k = [c]_k,$$  

(5a)

$$\sum_{j=1}^{n_d} [X]_{kj} = [f]_k \Rightarrow [\bar{c}]_k \geq [c]_k.$$  

(5b)

In addition to the above link capacity, here we also consider the additional capacity of vertiports in the hybrid air–ground transportation network. Each vertiport can accommodate a maximum amount of takeoff and landing per unit of time, due to the limited number of touch-down and lift-off pads. Similar to those in Nesterov and de Palma model, we make the following assumptions. First, the total amount of air traffic entering and exiting a vertiport never exceeds its capacity, i.e.,

$$\sum_{k=1}^{n_l} \sum_{j=1}^{n_d} [D]_{ik} [X]_{kj} \leq [g]_i.$$  

(6)

For all $i = 1, 2, \ldots, n_v$. Second, if the traffic flow on a vertiport is below its capacity, then the delay at this vertiport equals zero; if the traffic flow on a vertiport reaches its capacity, then the
average flight delay at this vertiport is nonnegative. In other words, if vector \( \hat{e} \in \mathbb{R}^{n_v} \) is such that \( [\hat{e}]_i \) denote the average flight delay at vertiport \( i \), then the following condition holds for all \( i = 1, 2, \ldots, n_v \):

\[
\sum_{k=1}^{n_k} \sum_{j=1}^{n_j} [D]_{ik} [X]_{kj} < [g]_i \Rightarrow [\hat{e}]_i = 0 \quad (7a)
\]

\[
\sum_{k=1}^{n_k} \sum_{j=1}^{n_j} [D]_{ik} [X]_{kj} = [g]_i \Rightarrow [\hat{e}]_i \geq 0. \quad (7b)
\]

In practice, link and node capacities are defined by the number of (ground or air) vehicles rather than the number of travelers or passengers in the vehicles. Hence, the value of the link and node capacities above often depends on the average number of passengers per ground vehicle and air vehicle. The latter increases, e.g., with the capacity of the vehicle and the average level of ridesharing.

**Remark 2:** Several studies in the literature have considered the link capacity constraints (4) in ground transportation network models [27], including the Beckmann model [16] and the Nesterov and de Palma model [17], [18]. We refer interested readers to [19] for a detailed numerical comparison of the effects of these constraints in different transportation models.

**B. Traffic Equilibria With Node and Link Capacities**

We are now ready to introduce the concept of *static equilibrium matrix*.

**Definition 1:** Matrix \( X \in \mathbb{R}^{n_l \times n_v} \) is a static equilibrium matrix defined by the tuple \( \{S, E, D, c, f, g\} \) if it is the optimizer of the following linear program:

\[
\begin{align*}
\text{minimize} & \quad \mathbf{c}^\top \mathbf{X} \mathbf{1}_d \\
\text{subject to} & \quad EX = S, \quad X \geq 0, \quad X \mathbf{1}_n \leq f, \quad DX \mathbf{1}_d \leq g. \quad (8)
\end{align*}
\]

**Remark 3:** Optimization (8) augments the multicommodity min-cost flow problem [28, Ch. 4] with additional node capacity constraints. The main difference between optimization (8) and previous work on the Nesterov and de Palma model for ground traffic [17], [18] is that optimization (8) contains the vertiport capacity constraints in (6), which, unlike the link capacity well studied in the literature, are defined on the nodes of the network rather than the links.

The linear program in Definition 1 is our prediction model for the traffic patterns—including flow and travel cost—of a hybrid air–ground transportation network. The following proposition provides two equivalent characterizations of static equilibrium matrix based on the optimality condition of linear programs.

**Proposition 1:** Matrix \( X \in \mathbb{R}^{n_l \times n_v} \) is an static equilibrium matrix associated with the tuple \( \{S, E, D, c, f, g\} \), and only if there exists \( V \in \mathbb{R}^{n_l \times n_v}, U \in \mathbb{R}^{n_l \times n_v}, p \in \mathbb{R}^n, \) and \( q \in \mathbb{R}^n \) such that the two conditions hold simultaneously.

1) The following constraints are satisfied:

\[
EX = S, \quad X \mathbf{1}_d \leq f, \quad DX \mathbf{1}_d \leq g. \quad \text{(9a)}
\]

\[
(c + p + D^\top q) \mathbf{1}_d \mathbf{1}_n^\top = E^\top V + U, \quad X \geq 0, \quad U \geq 0, \quad p \geq 0, \quad q \geq 0. \quad \text{(9b)}
\]

2) One of the following two set of constraints are satisfied: Either

\[
\begin{align*}
\text{tr}(X^\top U) = 0, & \quad p^\top X \mathbf{1}_d = f^\top p, \\
q^\top DX \mathbf{1}_d = g^\top q.
\end{align*} \quad \text{(10)}
\]

or

\[
c^\top X \mathbf{1}_d + f^\top p + g^\top q = \text{tr}(V^\top S). \quad \text{(11)}
\]

**Proof:** See Appendix A.

The conditions in (10) and (11) are also known as the complementary slackness condition and the zero duality gap condition. For linear programs, these two conditions are equivalent [29, Th. 1.3.3]. Later we will use both conditions to define and simplify the mathematical program with equilibrium constraints for vertiport selection.

Let \( \hat{c} = c + p \) and \( \hat{e} = q \). One can verify that the conditions in (9a), (9c), and (10) together imply the constraints in (3), (4), (6), (5), and (7). Hence the equilibria model in Definition 1 satisfies the second and the third assumptions we introduced at the beginning of this section.

Furthermore, the conditions in Proposition 1 also imply that only routes with the minimum accumulated travel time are used, a property known as the Wardrop equilibrium principle [25]. To see this implication, we define the set of *route vectors* from node \( i \) to destination node \( s(j) \) as follows:

\[
P(i, s(j)) = \left\{ u \in \{0, 1\}^{n_l} \mid [Eu]_i = 1, [Eu]_{s(j)} = -1, \quad [Eu]_k = 0 \forall k \neq i, s(j) \right\}. \quad \text{(12)}
\]

Intuitively, each vector \( u \) in set \( P(i, s(j)) \) defines a sequence of links connecting node \( i \) and node \( s(j) \) in a head-to-tail fashion; link \( k \) is on the route defined by \( u \), if and only if \( [u]_k = 1 \). Note that the set \( P(i, s(j)) \) is not necessarily a singleton, since there can be multiple routes—routes composed of ground links, air links, or a combination of both—between each origin–destination pairs.

Based on the above definition, the following corollary shows that any tuple \( \{X, U, V, p, q\} \) satisfying the conditions in Proposition 1 implies that any used routes have the lowest accumulated travel time, where the travel time of link \( k \) is given by \( [c + p + Dq]_k \).

**Corollary 1:** Let \( \{X, U, V, p, q\} \) satisfy the conditions in (9) and (10), and \( \overline{\tau} := c + p + Dq \). Let \( i \in \{1, 2, \ldots, n_u\} \) and \( j \in \{1, 2, \ldots, n_d\} \) such that \( i \neq s(j) \) and \( |S|_{u,s(j)} > 0 \). If \( u^* \in P(i, s(j)) \) and \( |X|_{kj} > 0 \) for all \( k \) such that \( [u^*]_k = 1 \), then the following condition holds for all \( u \in P(i, s(j)) \):

\[
\overline{\tau} u^* \leq \overline{\tau} u. \quad \text{(13)}
\]

Corollary 1 shows that the equilibria model in Definition 1 also satisfies the first assumption we introduced at the beginning of this section: Any routes with positive traffic flow have the lowest accumulated time of travel.

Alternatively, one can predict the traffic equilibria using an extension of the Beckmann model rather than an extension of
the Nesterov and de Palma model [16]. However, the Beck- 
mann model results in a set of equilibrium conditions with 
more nonlinear equality constraints than those in Nesterov and 
de Palma model [12], [13]. The equilibrium conditions in the 
Beckmann model are the KKT conditions of a nonlinear convex 
optimization, which contain nonlinear constraints; in contrast, 
the equilibrium conditions in the Nesterov and de Palma model, 
as we showed in Proposition 1, only contain linear constraints. 
On the other hand, studies have shown that the Nesterov and de 
Palma model and the Beckmann model give similar prediction 
results [19]. Therefore, here we chose the Nesterov and de Palma 
model as the basis of our equilibria model.

IV. VERTIPORT SELECTION VIA MIXED-INTEGER PROGRAMS

We now introduce a mathematical model that selects the 
location and capacity of vertiports in a hybrid air–ground trans-
portation network as an effort to optimize the resulting traffic 
equilibria. In particular, we aim to change the optimal solution of 
the linear program (8) by choosing the entries in the vertiport-capacity vector $g$ among discrete values—including zero values, 
in which case the corresponding vertiport location is discarded.

Throughout, we make the following assumptions on linear 
program (8).

**Assumption 1:** Linear program (8) is feasible and has a 
bounded optimal value.

Assumption 1 implies that link capacity and vertiport capacity 
in the hybrid air-ground transportation network are large enough 
to accommodate the traffic demand, i.e., the flow conservation 
constraints in (3) and capacity constraints in (4) and (6) hold 
simultaneously. Such an assumption trivially holds in practice 
since the ground transportation network alone can accommodate 
the traffic demand, even without adding any vertiports and air 
transportation networks.

Based on the above assumption, we will first define the objective 
function for the vertiport selection problem in Section IV-A, 
then define an MPEC for vertiport selection. We further prove 
that this MPEC is equivalent to a MILP in Section IV-B.

A. Vertiport Selection Via MPEC

We now introduce the mathematical problem for vertiport 
location and capacity selection. To this end, we first introduce 
three components of the vertiport selection problem: the design 
variables, the objective function, and the constraints.

1) **Design Variables:** First, we introduce the design variable 
of the vertiport selection problem. To this end, we start with the 
following assumption on the vertiport capacity vector $g$.

**Assumption 2:** There exists $G \in \mathbb{R}^{n_c \times n_c}$ such that the 
vertiport capacity vector $g$ in Definition 1 satisfies the following 
constraints:

$$ [g]_i \in \{0, [G]_{i1}, [G]_{i2}, \ldots, [G]_{i,n_c} \} $$

where $[G]_{i1} < [G]_{i2} < \cdots < [G]_{i,n_c}$ for all $i = 1, 2, \ldots, n_v$.

Assumption 2 states that the capacity of the $i$th vertiport 
is selected from an increasing sequence $\{0, [G]_{i1}, [G]_{i2}, \ldots, [G]_{i,n_c} \}$. For example, if $n_c = 3$, then the capacity of the $i$th 
vertiport can be zero—in this case, this vertiport is discarded—or 
a small, medium, or large value, denoted by $[G]_{i1}$, $[G]_{i2}$, and 
$[G]_{i3}$, respectively.

Here, we assume the capacity of each vertiport can only be 
discrete values rather than continuous ones for the following 
reasons. First, the capacity of a candidate vertiport—which is the 
upper bound of the total incoming and outgoing air traffic per unit 
time—is zero if this candidate is not selected and strictly positive 
otherwise. A discrete value of the capacity can capture such 
discrete change when a candidate vertiport changes from being 
not selected to selected. Second, the value of vertiport capacity 
often can only change discontinuously in practice. For example, 
increasing the capacity of a vertiport requires increasing the 
number of touch-down and lift-off pads, which can only change 
discretely rather than continuously.

Based on Assumption 2, we define the selection matrix as 
follows. Let $B \in \mathbb{R}^{n_v \times n_c}$ be a binary matrix such that $[g]_i = 
[G]_{ij}$ if and only if $[B]_{ij} = 1$. Then Assumption 2 holds, if and only if

$$ g = (B \odot G)^{1,v}, \ B^{1,v} \leq 1 v, \ B \in \{0, 1\}^{n_v \times n_c}. \tag{14} $$

In other words, each choice of $B$ that satisfies the constraints 
in (14) corresponds to a value of capacity vector $g$ that satisfies 
Assumption 2. In the following, we will use binary matrix $B$ as 
our design variable in the vertiport selection problem.

2) **Objective Function:** Given a set of vertiport with corre-
sponding capacity, we will introduce a quantitative measure for 
the quality of the traffic equilibria. To this end, given a selected 
capacity vector $g$, let $\{X, U, V, p, q\}$ be a tuple that satisfies the 
equilibrium conditions in Proposition 1. We evaluate the quality 
of this tuple using the following network loading function:

$$ \ell(X, p, q) := (c + p + D^T q)^T X^{1d} $$

$$ = \sum_{k=1}^{n_l} \frac{c + p + D^T q)_k ^T [X^{1d}]_k}{\pi_k}. \tag{15} $$

Here, the value of $\pi_k$ is the travel time on link $k$ at the 
equilibrium: It is the sum of the free travel time $[c]_k$ and the 
extra time delay caused by the congestion on the link and nodes, 
given by $[p + D^T q)_k$. The value of $\pi_k$ is the total amount of 
travelers entering or exiting link $k$ per unit time.$^1$

Assumption 2 states that the location and capacity of the verti-
ports depend on a binary selection matrix $B$: if $\sum_{i=1}^{n_c} [B]_{ij} = 0$, 
then vertiport $i$ is not selected; if $[B]_{ij} = 1$, then vertiport $i$ is 
selected with capacity $[G]_{ij}$ at the cost of $[K]_{ij}$. In addition, 
the capacity selection for all the vertiports is subject to a budget 
constraint defined by parameter $\gamma$.

3) **Constraints:** The first set of constraints in our selection 
problem are given in (9), (10) [or (11)], and (14). Together these 
constraints define the coupling relation among the selection 
matrix $B$, the capacity vector $g$, and the static traffic equilibria 
that correspond to the tuple $\{X, U, V, p, q\}$.

In addition, we also consider the following budget and logical 
constraints on the selection matrix $B$. First, constructing and

$^1$At a static equilibrium, the number of travelers entering and exiting the same 
link are the same; see [18].
maintaining a vertiport comes at a cost—which typically increases with the vertiport capacity. To impose a budget constraint in the vertiport selection problem, we introduce a cost matrix \( K \in \mathbb{R}^{n_v \times n_v} \), where its entry \( [K]_{ij} \) is the cost of selecting capacity \( [G]_{ij} \) for the \( i \)th vertiport. We let \( \gamma \in \mathbb{R} \) denote the upper bound on the total cost of vertiport selection, then a budget constraint takes the following form:

\[
1_v^\top (K \odot B) 1_c \leq \gamma.
\] (16)

Second, the choice of vertiport location is often subject to additional logical constraints: e.g., two locations close to each other cannot be selected simultaneously due to noise management regulations, and some locations must be selected as an air traffic hub. To account for these logical constraints, we consider the following linear constraints on the selection matrix \( B \):

\[
A \text{vec}(B) \leq b
\] (17)

where \( \text{vec} : \mathbb{R}^{n_v \times n_v} \rightarrow \mathbb{R}^{n_v n_e} \) is a vectorization map such that \( \text{vec}(B) = (i-1)n_v + j = B_{ij} \) for all \( i = 1, 2, \ldots, n_v \) and \( j = 1, 2, \ldots, n_e \), \( A \in \mathbb{R}^{n_v \times (n_v n_e)} \), and \( b \in \mathbb{R}^{n_e} \) defines all the logical constraints on matrix \( B \).

**Example 2:** To illustrate the logical constraints on vertiport location, we consider the case with two candidate vertiport locations, and each vertiport has two candidate capacity value, i.e., \( n_v = n_e = 2 \). In this case, if we let

\[
A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\] (18)

then, the constraint in (17) implies that one and only one of the two candidate vertiport location can be selected, i.e.,

\[
[B]_{11} + [B]_{12} + [B]_{21} + [B]_{22} = 1.
\] (19)

**4) MPEC for Vertiport Selection:** We now introduce a mathematical program that selects the value of capacity vector \( g \). The idea is to optimally choose the value of vector \( g \) such that the resulting equilibrium minimizes a weighted sum of the network loading function in (15) and the selection cost defined in the left-hand side of (16). To this end, we consider the following optimization problem, where \( \omega \in \mathbb{R}_+ \) is a weighting parameter:

**Problem:**

\[
\begin{align*}
\text{minimize} & \quad (c + p + D^\top q)^\top X 1_d + \omega 1_v^\top (K \odot B) 1_c \\
\text{subject to} & \quad EX = S, \quad X 1_d \leq f, \quad DX 1_d \leq g \\
& \quad (c + p + D^\top q) 1_d = E^\top V + U \\
& \quad X \geq 0, \quad U \geq 0, \quad p \geq 0, \quad q \geq 0 \\
& \quad c^\top X 1_d + f^\top p + g^\top q = \text{tr}(V^\top S) \\
& \quad g = (G \odot B) 1_c, \quad B 1_c \leq 1_v \\
& \quad 1_v^\top (K \odot B) 1_c \leq \gamma, \quad A \text{vec}(B) \leq b \\
& \quad B \in \{0, 1\}^{n_v \times n_e}.
\end{align*}
\] (20)

Optimization (20) is an MPEC: It includes the equilibrium conditions in (9) and (11) as part of its constraints. Proposition 1 shows that these constraints—which jointly depend on the primal and dual variables for linear program (8)—together ensure that matrix \( X \) is a static equilibrium matrix in the sense of Definition 1; similar constraints are common in MPEC, see [30, Sec. 7.1]. According to Proposition 2, one can alternatively replace the duality gap constraint in optimization (20)—which was first introduced in (11)—with the complementarity constraints in (10). However, such replacement introduces even more bilinear functions of the unknowns. Hence, we choose to write optimization in its current form; a similar MPEC was also used in electrified ground network design [21].

A globally optimal solution of optimization (20) is challenging to compute since its objective function, and constraints of optimization (20) contains bilinear functions of unknowns, such as \( p^\top X 1_d \) and \( g^\top q \).

**B. Reformulation of MPEC as an Equivalent MILP**

We now show that the MPEC in (20), a bilinear mixed integer optimization problem, is equivalent to a MILP. As a result, one can compute a global optimal solution of optimization (20) using off-the-shelf optimization software, such as GUROBI [31].

As our first step, the following proposition shows how to replace the bilinear constraints in optimization (20) with a linear one.

**Proposition 2:** Let \( G \in \mathbb{R}^{n_v \times n_e} \). There exists a large enough \( \mu \in \mathbb{R}_{++} \) such that the following two sets of conditions are equivalent.

1) There exists \( \delta \in \mathbb{R}, g \in \mathbb{R}^{n_e}, B \in \{0,1\}^{n_v \times n_e} \) and \( g \in \mathbb{R}^{n_e} \) such that

\[
\delta = g^\top q, \quad g = (G \odot B) 1_c, \quad B 1_c \leq 1_v, \quad q \geq 0.
\] (21)

2) There exists \( \delta \in \mathbb{R}, g \in \mathbb{R}^{n_e}, B \in \{0,1\}^{n_v \times n_e} \), and \( Y \in \mathbb{R}^{n_v \times n_e} \), such that

\[
\delta = 1_v^\top Y 1_c, \quad 0 \leq Y \leq \mu B, \quad B 1_c \leq 1_v, \quad 0 \leq G \odot (q 1_c^\top) - Y \leq \mu(1_v^\top 1_c - B), \quad q \geq 0.
\] (22)

**Proof:** See Appendix C.

Proposition 2 allows us to replace the bilinear function \( g^\top q \), appearing in the constraints of optimization (20), with a linear function of an auxiliary matrix \( Y \).

Our next step is to show the bilinear objective function of optimization (20) is also equivalent to a linear one. To this end, by using Proposition 1 again we can show the following:

\[
\begin{align*}
p^\top X 1_d & = f^\top p, \quad q^\top DX 1_d = g^\top q.
\end{align*}
\] (23)

Next, using Proposition 2, we can further replace the inner product \( q^\top g \) with a linear function of the auxiliary matrix \( Y \). By combining these results together, we can replace the bilinear objective function in (20) with a linear one.

Equipped with these results, we can reformulate optimization (20) as the following equivalent MILP, where \( \mu \) is a large
enough positive scalar.

\[
\begin{aligned}
\text{minimize } & \quad c^\top Xd + f^\top p + 1_d^\top Y1_c + \omega 1_c^\top (K \odot B)1_c \\
\text{subject to } & \quad EX = S, \quad X1_d \leq f, \quad DX1_d \leq (G \odot B)1_c,
\end{aligned}
\]

\[
(c + p + D^\top q)1_d^\top = E^\top V + U
\]
\[
X \geq 0, \quad U \geq 0, \quad p \geq 0, \quad q \geq 0,
\]
\[
c^\top X1_d + f^\top p + 1_d^\top Y1_c = \text{tr}(V^\top S),
\]
\[
0 \leq G \odot (q1_c^\top) - Y \leq \mu (1_n, 1_m^\top - B),
\]
\[
B1_c \leq 1_v, \quad 1_c^\top (K \odot B)1_c \leq \gamma,
\]
\[
0 \leq Y \leq \mu B, \quad A \text{ vec}(B) \leq b,
\]
\[
B \in \{0, 1\}^{n_x \times n_c}.
\]

Optimization (23) is a MILP: Its objective function and constraints only depend on linear functions of the unknowns, and it contains a binary unknown matrix \(B\). One can solve such MILP and obtain a globally optimal solution using off-the-shelf optimization software.

One challenge in solving optimization (23) is to choose an appropriate value for the scalar parameter \(\mu\). For the equivalence in Proposition 2 to hold, one must choose \(\mu\) to be large enough such that \(\mu\) upper bounds each element in matrix \(G \odot (q1_c^\top)\) and matrix \(Y\). On the other hand, choosing \(\mu\) to be too large can cause slow convergence and memory error when solving optimization (23). This phenomenon is well known in the mixed integer programming literature. For the guidelines on diagnosing and preventing large values of parameter \(\mu\), we refer the interested readers to [32, Sec. 3.4].

In practice, one can choose an appropriate value for \(\mu\) by letting \(\mu = \bar{\tau} \max_i[G]_{i,n_x}\), where \(\bar{\tau} \in \mathbb{R}_+\) is an estimate of the maximum delay among all vertiports at equilibrium. One can empirically estimate the magnitude of \(\bar{\tau}\) using the magnitude of free travel time \(c\). In our numerical experiments, we find that letting \(\bar{\tau}\) be one order of magnitudes higher than the value of \(c^\top 1_n\) usually gives a valid estimate. On the other hand, if \(\mu\) is too small or too large, the MILP in (23) will become infeasible or ill-conditioned, and numerical MILP solvers will fail to provide a solution.

V. NUMERICAL EXPERIMENTS

We demonstrate our vertiport selection approach using the Anaheim ground transportation network model developed in [33], which contains more than 400 nodes and 900 links. Our goal is to numerically demonstrate the effects of adding different vertiports to an existing ground transportation network in terms of traffic loading in the network.

A. Anaheim Transportation Network With Additional Air Links

The Anaheim ground transportation network model consists of a well-defined arterial grid system integrated with an extensive freeway system. See Fig. 2 for an illustration.\(^2\) The model includes the data for the following:

1) the incidence matrix;
2) the demand matrix;
3) the free travel time;
4) the link capacity.

Based on these data, we construct the Nesterov and de Palma model for the ground transportation network, which is known to produce similar results as the Beckmann model [19].

In addition to the Anaheim ground transportation network, we construct an air transportation network as follows. Based on their location and travel demands, we choose nine different destination nodes in the Anaheim network as candidate locations for vertiports (see Fig. 2). The capacity of each vertiport can be either 600 or 1200 takeoffs and landings per hour; choosing these capacities will take 1 or 2 units of cost. We add an air link to each pair of vertiports if their physical distance is greater than the median of the pairwise distance of all the nodes in the Anaheim network. The free travel time of these air links is set to be proportional to the corresponding distance, and the flow capacity is fixed to be 80 flights per hour for all air links.

We also consider the following budget and logical constraints on the vertiport locations. First, the total selection budget \(\gamma\) is chosen such that \(\gamma \in [5, 11]\). Second, the locations marked in Fig. 2 are subject to the logical constraints listed in Table I.

\(^2\)Map images we used are generated by Mapbox https://www.mapbox.com.
Optimal air and ground traffic network loading when vertiport selection budget $\gamma \leq 1$.

By choosing the elements in vector $B \leq \text{vec}(G)$, we have clear interpretations in the context of selecting vertiports, where $A \leq g \leq \gamma$.

The link loading in the ground and air networks is shown by solving the following MILP in integer programs.

The shape of the marker indicates the capacity value, with costs one unit in the budget; the triangle marker denotes the capacity value of 1200, which costs 2 units in the budget.

(a) Air traffic network loading. (b) Ground traffic network loading.

### B. Selection Based on the Knapsack Problem

As a benchmark approach, we consider selecting vertiport locations using the variation of the Knapsack problem, a classical model in integer programs [34, Sec. 1.3]. To this end, we define a value vector $w \in \mathbb{R}^{n v}$, where $[w]_k$ denotes the value of the unit capacity at the $k$th candidate vertiport. Based on this vector, we compute the selection matrix $B$ in (14) by solving the following MILP:

$$
\begin{align*}
\text{maximize} & \quad w^\top g \\
\text{subject to} & \quad g = (B \odot G)1_m, \quad B1_m \leq 1_v, \\
& \quad 1_v^\top (K \odot B)1_m \leq \gamma, \quad A \text{vec}(B) \leq b, \\
& \quad B \in \{0, 1\}^{n_v \times n_m}.
\end{align*}
$$

(24)

Notice that optimization (24) contains the discrete capacity constraints in (14), the budget constraints in (16), the logical constraints in (17).

Compared with the one in (23), the MILP in (24) has several advantages. It contains fewer variables and fewer constraints. It does not require any network parameters, such as free travel time vector $c$ and link capacity vector $f$. It does not require the tuning of the scalar parameter $\mu$. Its solutions satisfy the budget and logical constraints in (23).

However, these advantages come at the price of the suboptimality of the vertiport selection. Particularly, a solution to the MILP in (23) is optimal in terms of minimizing the weighted sum of network loading and the selection cost in (23), as a result of Proposition 2. On the other hand, a solution to the MILP in (24) lacks such guarantees on optimality, regardless of how one chooses the value vector $w$ in (24). Furthermore, the network parameters used in (23) have clear interpretations in the context of transportation networks—such as free travel time and link flow capacity—and there are comprehensive means to estimate these parameters in the literature [33]. On the other hand, to the best of our knowledge, there is no systematic way to estimate the value vector $w$ in (24).

In our numerical experiments, we consider a heuristics estimate for the value vector $w$ by choosing the elements in vector $w$ to be the total traffic demand at the candidate vertiport. The idea behind this heuristics is that the value of the unit capacity at a candidate vertiport should increase with the travel demand, the higher the demand, the more beneficial to provide air travel as an alternative. In particular, we choose the candidate vertiport nodes $V = \{v(1), v(2), \ldots, v(n_v)\}$ from the set of destination nodes $s(1), s(2), \ldots, s(n_d)$ such that there exists $1 \leq d \leq n_d$, with $v(k) = s(d)$ for all $k = 1, 2, \ldots, n_v$. Furthermore, we let $[w]_k = S(s(d), d)$.

We note that other heuristics for choosing the value vector $w$ exist, and some may be better than the ones we used in this work. However, the MILP (23) eliminates the need for searching for better heuristics and guarantees the optimal selection with respect to network loading and selection cost.

### C. Numerical Comparison

With the above choices of parameters, we solve optimization (23). To demonstrate our results, we define the following notion of link loading for each link $k = 1, 2, \ldots, n_l$:

$$
\ell_k(X, p, q) = [c + p + D^\top q]_k[X1_d]_k.
$$

(25)

Intuitively, $\ell_k$ denotes the number of vehicles traveling on link $k$ at the equilibrium—which is also the summand in the total link loading defined in (15).

Fig. 3 shows the link loading in the ground and air networks when we let choose the budget to be $\gamma = 8$. In this case, a total of six vertiports are selected, and only two of them have the larger capacity value of 1200. The one near Westminster and the one near Villa Park are connecting some of the flight legs with the highest loading; hence, they necessarily need larger capacity.

We also show how the budget value $\gamma$ in vertiport selection affects the link loading in the ground traffic network. Intuitively, adding vertiports will reduce ground link loading by providing...
alternative means of transportation. Furthermore, as the budget increases, the selected vertiports can support an air transportation network with a larger volume of air traffic, and consequently, the ground link loading will decrease more. These intuitions are confirmed by Figs. 4 and 5, which shows the sum of the link loading reduction in the ground network increases with the budget value, and so does the number of ground links with decreased loading.

We also compare the performance of the results based on (23) and the results based on the Knapsack problem in (24); both of which contain 18 binary integer variables in this problem. Figs. 4 and 5 show that the MPEC approach is better than the Knapsack problem approach in terms of the total link loading reduction in the ground network as well as the number of ground links with decreased loading.

We similarly compare the performance of the results based on (23) and the results based on the Knapsack problem in (24); both of which contain 18 binary integer variables in this problem. Figs. 4 and 5 show that the MPEC approach is better than the Knapsack problem approach in terms of the total link loading reduction in the ground network as well as the number of ground links with decreased loading. These results confirm the advantage of the MPEC approach. We note that it may be possible for the Knapsack problem to produce similar results to those of the MPEC approach via a better estimate of the value vector—rather than directly using the total travel demand—in optimization (24).

However, to our best knowledge, there is no systematic method to compute these estimates. Hence, MPEC is more useful for vertiport selection.

VI. CONCLUSION

In this article, we introduce a mathematical model to select the optimal vertiport location and capacity for minimizing traffic congestion in a hybrid air-ground transportation network. Our model is equivalent to a MILP, and we demonstrate this model using the Anaheim transportation network.

Our work also opens some new research questions. For example, although the identification of the parameters for ground transportation networks—such as free travel time and link capacity—are well studied in the literature, similar results are still missing for air transportation networks. In order to use the mathematical models we developed, it is important to identify these parameters using realistic air traffic data. Another example is to consider the impacts of different weather conditions on the vertiport selection problem. Since weather conditions are more likely to affect aircraft than automobiles, it is critical to ensure the air transportation network is robust against temporary capacity decreases caused by extreme weather conditions. We aim to answer these open questions in our future work.

APPENDIX

A. Proof of Proposition 1

We start by deriving the dual of linear program (8). Let the Lagrangian be defined as follows:

\[
L(X,U,V,p,q) = c^\top X_1 - \text{tr}(V^\top E X) + \text{tr}(V^\top S) \\
- \text{tr}(U^\top X) + p^\top (X_1 - f) \\
+ q^\top (D X_1 - g).
\]

(26)

The dual of linear program is given by

\[
\begin{align*}
\text{maximize} & \quad \psi(U,V,p,q) \\
\text{subject to} & \quad U \geq 0, \quad p \geq 0, \quad q \geq 0
\end{align*}
\]

(27)

where \( \psi(U,V,p,q) = \min_X L(X,U,V,p,q) \). Since matrix trace is invariant under cyclic permutation, we have

\[
\begin{align*}
&c^\top X_1 = \text{tr}(1_d c^\top X), \\
p^\top X_1 = \text{tr}(1_d p^\top X), \\
&q^\top D X_1 = \text{tr}(1_d q^\top DX).
\end{align*}
\]

Substituting the above equalities into (26), we can show the following:

\[
\begin{align*}
\frac{\partial}{\partial X} L(X,U,V,p,q) &= \frac{\partial}{\partial X} \text{tr}((1_d (c^\top + p^\top + q^\top D) - V^\top E - U^\top) X) \\
&= (c + p + D^\top q) 1_d - E^\top V - U.
\end{align*}
\]

Since \( L(X,U,V,p,q) \) is a linear function of \( X \), we have \( \psi(U,V,p,q) = L(X,U,V,p,q) \) if and only if \( \frac{\partial}{\partial X} L(X,U,V,p,q) = 0 \). Therefore we can rewrite optimization (27) equivalently as
follows:

\[
\begin{align*}
\text{maximize} & \quad \text{tr}(V^\top S) - f^\top p - g^\top q \\
\text{subject to} & \quad (c + p + D^\top q)1_d = E^\top V + U \\
& \quad U \geq 0, \quad p \geq 0, \quad q \geq 0.
\end{align*}
\] (28)

Using [29, Th. 1.3.3], we conclude that \( X \) and \( U, V, p, q \) are optimal for linear program (8) and (28), respectively, if and only if the primal and dual feasibility condition in (9) and the complementary slackness condition (10) are satisfied. Furthermore, the complementary slackness conditions in (10) are equivalent to the zero duality gap condition in (11).

**B. Proof of Corollary 1**

Since \( u^*, u \in \mathcal{P}(i, s(j)) \), by premultiplying (9) with \( u^* \) and \( u \) and we can show the following:

\[
\begin{align*}
(u^*)^\top \tau &= V_{ij} - V_{s(j),j} + \sum_{k=1}^{n_1} [u^*]_j [U]_{kj}, \\
(u)^\top \tau &= V_{ij} - V_{s(j),j} + \sum_{k=1}^{n_1} [u]_j [U]_{kj}.
\end{align*}
\] (29a)

In addition, the constraints in (9c) and (10) together implies that \([U]_{kj} = 0 \) for all \( k \) such that \([X]_{kj} > 0 \). Combining this fact with the assumption that \([X]_{kj} > 0 \) for all \( k \) such that \([u^*]_k = 1 \), we conclude that \([U]_{kj} = 0 \) for all \( k \) such that \([u^*]_k = 1 \). Hence

\[
(u^*)^\top \tau = V_{ij} - V_{s(j),j} + \sum_{k=1}^{n_1} [u^*]_j [U]_{kj} = V_{ij} - V_{s(j),j}.
\] (29b)

By combining (29) and (30), we obtain the following:

\[
(u^*)^\top \tau = V_{ij} - V_{s(j),j} = u^\top \tau - \sum_{k=1}^{n_1} [u]_j [U]_{kj} \leq u^\top \tau
\]

where the last step is because \( u \) and \( U \) are both elementwise nonnegative.

**C. Proof of Proposition 2**

First, suppose \( \delta, q, B, \) and \( g \) satisfy the constraints in (21). Let \([Y]_{ij} = [g]_i [q]_j [B]_{ij} \) for all \( i = 1, 2, \ldots, n_v \) and \( j = 1, 2, \ldots, n_m \), and \( \mu = \max_{i,j} [g]_i [G]_{ij} \). Then, one can verify that \( \delta, q, B, \) and \( Y \) satisfy the constraints in (22).

Second, suppose \( \delta, q, B, \) and \( Y \) satisfy the constraints in (22) for some sufficiently large \( \mu \in \mathbb{R}_{++} \). The constraints \( B \in \{0, 1\}^{n_v \times n_m} \) and \( B1_{nm} \leq 1 \), implies that each row of matrix \( B \) can have at most one entry equals one. Hence, we can obtain an unique vector \( g \) by defining its \( i \)th entry as follows:

\[
[g]_i = \begin{cases} [G]_{ij}, & \text{if } [B]_{ij} = 1, \\
0, & \text{if } [B]_{ij} = 0 \end{cases} \quad \text{for all } j = 1, 2, \ldots, n_m.
\] (31)

Next, since \( \mu \in \mathbb{R}_{++} \) is sufficiently large, an upper bound of \( \mu \) can be treated as redundant. As a result, if \([B]_{ij} = 0 \), then the constraints in (22) implies that \([Y]_{ij} = 0 \) and \([G]_{ij} [g]_i \geq 0 \). Since \( q \geq 0 \) and \( G \geq 0 \), the latter constraint is redundant.

Furthermore, if \([B]_{ij} = 1 \), then the constraints in (22) implies that \( 0 \leq [Y]_{ij}, [G]_{ij} [q]_j \leq [Y]_{ij} \).

By combining the above two cases with the definition in (31), we conclude that \( \sum_{i=1}^{n_v} \sum_{j=1}^{n_m} [Y]_{ij} = \sum_{i=1}^{n_v} [g]_i \) for all \( i = 1, 2, \ldots, n_v \) and \( j = 1, 2, \ldots, n_m \). Therefore, \( \delta, q, B, \) and \( g \) satisfy the constraints in (21).

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