A SHORT PROOF OF A SHARP WEYL LAW FOR THE
SPECIAL ORTHOGONAL GROUP

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Abstract. We give a short proof of a strong form of Weyl’s law for SO($N$)
using well known facts of the theory of modular forms. The exponent of the
error term is sharp when the rank is at least 4. We also discuss the cases
with smaller rank improving previous results.

1. Introduction

If $M$ is a compact Riemannian manifold of dimension $d$ then the spectrum of
Laplace-Beltrami operator $-\Delta$ on $M$ is discrete and corresponding to nonnegative
eigenvalues. Let $N(\lambda)$ be the counting function, namely the cardinality of the
eigenvalues less or equal than $\lambda$ counted with multiplicity. Weyl’s law states the
asymptotic behavior

$$\lim_{\lambda \to \infty} \lambda^{-d/2} N(\lambda) = \frac{2 \Vol(M)}{d(4\pi)^d \Gamma(d/2)}$$

where $\Vol(M) = \int_M \omega$ with $\omega$ the Riemannian volume form. It comes from early
works by Weyl related to mathematical physics but it seems that in this generality
the first proof was not given until 1949 [20]. The study of sharper asymptotics
in this setting or allowing boundaries has revealed a deep connection between
analysis, geometry and mathematical physics [15]. In this interplay, arithmetic
has not stood aside [13] [5] [25] and it has a main role in this paper.

We use very basic properties of modular forms. Recall that they are holomorphic
functions on the upper half plane with a Fourier expansion $\sum_{k=0}^{\infty} a_k e^{2\pi i k z}$. They
have, in some sense, symmetries given by a finite index subgroup of $SL_2(\mathbb{Z})$. In
particular, if $f$ is a modular form of weight $r$ the function $|z|^r \overline{f(z)}$ is invariant
under the transformations of the group and if it is a so-called cusp form, it remains
bounded. An elementary argument [3, Lem. 3.2] [17, Th. 5.3] proves

$$\sum_{k \leq K} a_k = O\left(K^{r/2} \log K\right) \quad \text{and} \quad C^{-1} < K^{-r} \sum_{k \leq K} |a_k|^2 < C$$

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for $K$ large and certain $C > 0$ depending on $f$. The connection with modular forms we exploit in this work, is that when $P$ is a harmonic polynomial of degree $\nu > 0$ the function

$$\Theta_P(z) = \sum_{k=1}^{\infty} k^{\nu/2} r_n(k, P) e^{2\pi i k z}$$

where

$$r_n(k, P) = \sum_{\|\vec{m}\|^2 = k} P\left(\frac{\vec{m}}{\sqrt{k}}\right)$$

(here it is assumed $\vec{m} \in \mathbb{Z}^n$) is a cusp form of weight $\nu + n/2$. The proof is essentially an involved application of the Poisson summation formula [17, Th. 10.9] being the hardest part the computation of the multiplier that is irrelevant for (2).

Our goal is to provide a short proof of

**Theorem 1.1.** Let $M = SO(N)$, $n$ its rank and $C_d$ the right hand side of (1). Then

$$N(\lambda) - C_d\lambda^{d/2} = \begin{cases} O(\lambda^{d/2 - 1}) & \text{if } n > 4, \\ O(\lambda^{d/2 - 1} \log \lambda) & \text{if } n = 4. \end{cases}$$

Recall that the rank of $SO(N)$ is $n = N/2$ for even $N$ and $n = (N - 1)/2$ for odd $N$. As explained later, $\log \lambda$ can be replaced by $(\log \lambda)^{\alpha}$ for some $\alpha < 1$ but it would require non-elementary techniques that would contrast with the simple arguments employed in our proof in the next section. We think that our approach, with non-essential modifications, may also cover the cases $U(N)$ and the classical groups (in the tables [24, VIII.1, IX.8]).

For $n > 4$ in principle this result follows from [22, Th. 5.1] but we think that there is a gap in the proof provided there related with the equidistribution of lattice points on spheres. It can be quantified with homogeneous polynomials $P$ in $n$ variables through

$$E_n(k, P) = \frac{r_n(k, P)}{r_n(k)} - \int_{S^{n-1}} P \tilde{\omega}$$

where we have abbreviated, as usual, $r_n(k) = r_n(k, 1)$ and $\tilde{\omega}$ is the normalized volume form of $S^{n-1}$, the standard one divided by $\text{Vol}(S^{n-1})$. In [22] it is claimed $E_n(k, P) = O(k^{-n/4})$. Using that for $n > 4$ it is known [17, Cor. 11.3] that $r_n(k) < C_n k^{n/2 - 1}$, it would imply when $P$ is harmonic and non-constant that $\sum_{k \leq K} \left(\frac{k^{\nu/2} r_n(k, P)}{r_n(k)}\right)^2 = O(K^{\nu + (n-1)/2})$, and this contradicts the second formula of (3) and finer estimates based on the Rankin-Selberg convolution. As the matter of fact $E_3(k, P) = o(1)$, the so-called “Linnik problem”, remained as a conjecture during many years. Finally it was proved in 1988 by W. Duke [7] using a breakthrough due to H. Iwaniec [16] and still the best known result is $E_3(k, P) = O_\alpha(k^{-\alpha})$ for any $\alpha < 1/28$.

### 2. Proof of Theorem 1.1

After the fundamental work of Weyl on the theory of compact Lie groups, the irreducible representations are determined by their highest weights [23]. On the other hand, we know that the entries of the matrices of the irreducible representations are the eigenfunctions of the $-\Delta$ that is the differential form of the quadratic Casimir operator [19 VIII.3], [26 §32]. The outcome is that the eigenvalues are
given by \( \|\mu + \rho\|^2 - \|\rho\|^2 \) for \( \mu \) in certain sectors of a lattice and \( \rho \) a constant vector (see [5] for generalizations). Moreover, the multiplicities are given by Weyl’s dimension formula [23, §7.3.4]. In the case of \( SO(N) \) the eigenvalues are indexed by \( \vec{b} \in \mathbb{Z}^n \) where \( n \) is the rank, with \( b_1 \geq b_2 \geq \cdots \geq |b_n| \) for \( N = 2n \) and with \( b_1 \geq b_2 \geq \cdots \geq b_n \geq 0 \) for \( N = 2n + 1 \). In the even case, writing \( x_j = b_j + n - j \), for each choice of \( \vec{b} \) we have the eigenvalue \( \lambda_{\vec{b}} \) and its multiplicity \( m \)

\[
\lambda_{\vec{b}} = \sum_{j=1}^n x_j^2 - \frac{n(n-1)(2n-1)}{6} \quad \text{and} \quad m = \left( \prod_{i=0}^{n-1} \prod_{j=i+1}^{n-1} \frac{x_{n-j}^2 - x_{n-i}^2}{j^2 - i^2} \right)^2.
\]

Similarly, in the odd case, writing \( x_j = 2b_j + 2n - 2j + 1 \), the formulas are

\[
\lambda_{\vec{b}} = \frac{1}{4} \sum_{j=1}^n x_j^2 - \frac{n(4n^2 - 1)}{12} \quad \text{and} \quad m = \left( \prod_{i=0}^{n-1} \frac{x_{n-i}^2}{2i+1} \prod_{j=i+1}^{n-1} \frac{x_{n-j}^2 - x_{n-i}^2}{(2j+1)^2 - (2i+1)^2} \right)^2.
\]

The relation with lattice points problems is given through the following result [22]. We provide a proof not appealing to the properties of the underlying Lie algebra.

**Lemma 2.1.** Consider \( m \) in [4] and [6] as a polynomial in the \( x_1, \ldots, x_n \) and let \( \chi_R \) the characteristic function of the ball of radius \( R \) in \( \mathbb{R}^n \). Then for \( N \) even

\[
\mathcal{N}(\lambda) = \frac{1}{2^{n-1} n!} \sum_{\vec{x} \in \mathbb{Z}^n} m(\vec{x}) \chi_R(\vec{x}) \quad \text{where} \quad R^2 = \lambda + \frac{n(n-1)(2n-1)}{6},
\]

and for \( N \) odd,

\[
\mathcal{N}(\lambda) = \frac{1}{2^n n!} \sum_{\vec{x} \in \mathbb{Q}^n} m(\vec{x}) \chi_R(\vec{x}) \quad \text{where} \quad R^2 = 4\lambda + \frac{n(4n^2 - 1)}{3}
\]

where \( \mathbb{Q} \) denotes the odd integers.

**Proof.** Due to the relationship between \( b_j \)’s and \( x_j \)’s, the eigenvalues of \( SO(N) \) can also be indexed by the elements of the set \( \mathcal{E} = \{ \vec{x} \in \mathbb{Z}^n : x_1 > x_2 > \cdots > |x_n| \} \)

if \( N \) is even, and by those of \( \mathcal{C}_0 = \{ \vec{x} \in \mathbb{Q}^n : x_1 > x_2 > \cdots > x_n > 0 \} \) if \( N \) is odd. As \( m(\vec{x}) \) is invariant under ordering and sign changes in the \( x_j \)’s in both cases, we can consider for \( N \) even all \( n \)-tuples in \( \mathbb{Z}^n \) and then divide by the \( 2^{n-1} n! \) possibilities for those changes on \( \mathcal{E} \); and for \( N \) odd all \( n \)-tuples in \( \mathbb{Q}^n \) and then divide by the \( 2^n n! \) possibilities for those changes on \( \mathcal{C}_0 \). Thus, we obtain the expression for \( \mathcal{N}(\lambda) \), just by noting that \( \lambda_{\vec{b}} \leq \lambda \) implies \( \|\vec{x}\|^2 \leq R^2 \) in each case. \( \square \)

After these preliminary considerations, we can deduce Theorem [1.1] for even \( N \) in few lines from [2] and a classic and basic result about an arithmetic function.

Each homogeneous polynomial \( P \) of degree \( g \) can be written uniquely as [17]

\[
P(\vec{x}) = \sum_{1 \leq i \leq g/2} \|\vec{x}\|^{2i} P_{g-2i}(\vec{x}) \quad \text{with} \quad P_j \text{ a harmonic polynomial of degree } j.
\]

When we apply this to \( m(\vec{x}) \) that has even degree \( d - n \), with \( d = \dim (SO(2n)) \), separating the contribution of the constant harmonic polynomial, we deduce from
Lemma 2\textsuperscript{[2]} that there exists $C$ and some harmonic polynomial $P$ of degree $0 < \nu \leq d - n$ such that

$$(7) \quad \mathcal{N}(\lambda) = C \sum_{\bar{x} \in \mathbb{Z}^n} \|\bar{x}\|^{d-n} \chi_R(\bar{x}) + O\left( \sum_{\bar{x} \in \mathbb{Z}^n} \|\bar{x}\|^{d-n-\nu} P(\bar{x}) \chi_R(\bar{x}) \right).$$

Grouping together the terms with $\|\bar{x}\|^2 = k$, we have, with the notation as in 3\textsuperscript{[3]},

$$(8) \quad \mathcal{N}(\lambda) = C \sum_{k \leq R^2} k^{(d-n)/2} r_n(k) + O\left( \sum_{k \leq R^2} k^{(d-n)/2} r_n(k, P) \right).$$

By the first formula in 2\textsuperscript{[2]} the $O$-term is $O\left(R^{d-n/2} \log R\right)$. On the other hand, a classic result states that the average behavior of the arithmetic function $r_n(k)$ is

$$\sum_{k \leq R^2} r_4(k) = C_4 R^4 + O\left(R^2 \log R\right) \quad \text{and} \quad \sum_{k \leq R^2} r_n(k) = C_n R^n + O\left(R^{n-2}\right) \quad \text{for} \ n > 4,$$

where $C_n$ is the volume of the unit ball. The proof is a simple partial summation from Jacobi’s formula for $r_4(n)$ [13 p. 22] and the simplest exponential sum method to avoid the logarithm when passing from $n = 4$ to 5 (for the details, see for instance [9 §15, Satz 2]). In fact, there are also elementary proofs of the formula for $r_4(n)$ [27] and of the exponential sums estimation [10 Th. 2.2].

When we substitute in (8) the modular form estimate and the average result for $r_n(k)$, we conclude Theorem 1\textsuperscript{[1]} with an unidentified constant that has to be $C_d$ to match (1).

The odd case follows with some technical modifications. In this case one obtains 7\textsuperscript{[7]} with $\mathbb{Z}$ replaced by $\mathbb{Q}$. The first sums gives readily the first sum in 3\textsuperscript{[3]} with $r_n(k)$ replaced by $r^*_n(k)$, defined as the number of representations as a sum of $n$ odd squares. In connection with this, 9\textsuperscript{[9]} still holds (with different constants) when $r_n$ is replaced by $r^*_n$ because there is an analog for $r_4^*$ of Jacobi’s formula (that it is indeed simpler 2\textsuperscript{[2]}).

On the other hand, given $B \subset \{1, 2, \ldots, n\}$ define $T_B$ as the diagonal matrix $\text{diag}(a_1, \ldots, a_n)$ where $a_j = 2$ if $j \in B$ and $a_j = 1$ otherwise. By the inclusion-exclusion principle

$$\sum_{\bar{x} \in \mathbb{Q}^n} \|\bar{x}\|^{d-n-\nu} P(\bar{x}) \chi_R(\bar{x}) = \sum_{B \subseteq \{1, \ldots, n\}} (-1)^{|B|} \sum_{\bar{x} \in \mathbb{Z}^n} \|T_B \bar{x}\|^{d-n-\nu} P_B(\bar{x}) \chi_R(T_B \bar{x})$$

with $P_B = P \circ T_B$. Then the O-term in (8) holds replacing $r_n(k, P)$ by

$$r^*_n(k, P_B) = \sum_{Q(\bar{m}) = k} P_B\left(\frac{\bar{m}}{\sqrt{k}}\right) \quad \text{with} \ Q(\bar{x}) = \|T_B \bar{x}\|^2.$$

The polynomial $P_B$ is a spherical function with respect to the quadratic form $Q$ and the theory assures that $k^{\nu/2} r^*_n(k, P_B)$ are the Fourier coefficients of a cusp form of weight $\nu + n/2$ 17\textsuperscript{[17]} and hence the same bound as in the even case applies.
3. Some remarks about the true order of the error term

A natural question is if the error term in Theorem 1.1 is sharp. The general result of [6] suggests that the exponent of $\lambda$ cannot be lowered. We give in this section a closer view taking advantage of the arithmetic interpretation.

As we saw in the previous section, when we substitute the first formula of (2) in (8) we get for even $N$

$$N(\lambda) = C \sum_{k \leq R^2} k^{(d-n)/2} r_n(k) + O\left(R^{d-n/2} \log R\right)$$

and a similar formula for odd $N$ replacing $r_n$ by $r_n^*$.

On the other hand, it is known for $n > 4$ that $r_n(k) > C_n k^{n/2 - 1}$ and $r_n^*(k) > C_n k^{n/2 - 1}$, under $4 \mid k - n$, for certain constant $C_n > 0$ [17, Cor. 11.3] [2]. It implies that when $R^2$ reaches an integer, $N(\lambda)$ increases by an amount comparable to $R^{d-2}$. In particular, in Theorem 1.1 it is neither possible to change $O\left(\lambda^{d/2 - 1}\right)$ by $o\left(\lambda^{d/2 - 1}\right)$ nor to extract a smooth secondary main term (even if $\lambda$ is restricted to integral values $r_n(k)$ [17, §11.5]).

The case $n = 4$ is more involved. Without entering into details, adapting the folklore techniques reviewed in [11] it is possible to go beyond the first bound in (2) to get

$$N(\lambda) = C \sum_{k \leq R^2} k^{(d-4)/2} r_4(k) + O\left(R^{d-\alpha_0} \log R\right)$$

for certain $\alpha_0 > 2$. When $k$ is the product the the first odd primes, from the formula for $r_4(k)$ and Mertens formula, it follows that $r_4(k) \sim \frac{48}{\pi^2} k \log \log k$. As before, one concludes as before that $N(\lambda)$ is not $o\left(\lambda^{d/2 - 1} \log \log \lambda\right)$. On the other hand, in [11] $\log R$ can be lowered to $(\log R)^{2/3}$ with advanced methods due to N.M. Korobov and I.M. Vinogradov [28]. In Weyl’s law it translates into the error term $O\left(\lambda^{d/2 - 1} \log \lambda\right)^{2/3}$ for $n = 4$. The gap between $(\log R)^{2/3}$ and the barrier $\log \log R$ in the error term for the average of $r_4(n)$ has remained unchanged during the last 50 years (see [14] for more information). The same uncertainty applies to the average of $r_4^*(n)$.

4. The cases $n < 4$

We now discuss the low dimensional cases that are not covered by the sharp estimates in [22].

For $n = 1$, SO(2) is just $S^1$ and the eigenfunctions correspond to the standard orthonormal system $\{e^{ikx}\}_{k \in \mathbb{Z}}$ for Fourier series. Then we have plainly

$$N(\lambda) = \#\{k \in \mathbb{Z} : k^2 \leq \lambda\} = 2\lambda^{1/2} + O(1)$$

and the $O(1)$ is sharp because $N(\lambda)$ increases in one when $\lambda$ is a square. Note that the constant 2 matches the right hand side of (1) for $M = S^1$.

In the odd case, for $n = 1$ we have to consider SO(3) with eigenvalues of the form $l(l + 1)$, corresponding to the angular momentum operator in quantum physics,
with multiplicity \((2l + 1)^2\) and clearly
\[
N(\lambda) = \sum_{l(l+1) \leq \lambda} (2l + 1)^2 = \frac{4}{3} \lambda^{3/2} + O(\lambda),
\]
that is sharp again.

It is known that \(\sum_{k \leq R^2} r_3(k) = \frac{4\pi}{3} R^3 + O(R^{\alpha_3})\) for certain \(\alpha_3 < 3/2\) and the same applies to \(\sum_{k \leq R^2} r_3^2(k)\), with a different constant in the main term (the best known result in this direction allows to take any \(\alpha_3 > 21/16\) \([12, 4]\)). Then \([10]\) after partial summation gives Weyl's law for \(n = 3\) in the form
\[
N(\lambda) = C_d \lambda^{d/2} + O(\lambda^{d/2-3/4} \log \lambda).
\]

With the aforementioned techniques in \([11]\) it seems possible to reduce the exponent (one should proceed as indicated at the end of the paper for Maass forms using \([7]\), see also \([11]\)) beating Theorem 4.1 of \([22]\) for \(N\).

By partial integration we have
\[
\int_{1/2}^1 \alpha\,d\alpha = 2 \sum_{1/2 \leq \alpha \leq 1} \alpha\,d\alpha = \frac{4}{3} \lambda^{3/2} + O(\lambda),
\]
and using the second part of Lemma 2.1. The multiplicity in \([5]\) is given by the polynomial \(m(x, y) = (x^2 - y^2)^2\). By Lemma 2.1 and the symmetry
\[
N(\lambda) = 2 \sum_{0 \leq x \leq R/\sqrt{2}} I_x \sum_{y \leq I_x} m(x, y) \quad \text{where} \quad I_x = \left(x, \sqrt{R^2 - x^2}\right)
\]
and the prime in the first sum means that the contribution of \(x = 0\) is halved. Using the variant of Euler-Maclaurin summation \(\sum_{a \leq n \leq b} f(n) = \psi(a) f(a) - \psi(b) f(b) + \int_a^b (f + \psi f') dt\) with \(\psi(t) = t - [t] - 1/2\) (cf. \([11]\) Th. 4.2), it was stated in 1885 by N.Y. Sonin, we have
\[
N(\lambda) = T_1 + T_2 + T_3 \quad \text{with} \quad T_1 = 2 \sum_{0 \leq x \leq R/\sqrt{2}} \int_{I_x} m(x, y) \, dy
\]
and
\[
T_2 = 2 \sum_{0 \leq x \leq R/\sqrt{2}} (R^2 - 2x^2) \psi(\sqrt{R^2 - x^2}), \quad T_3 = 2 \sum_{0 \leq x \leq R/\sqrt{2}} \int_{I_x} 4y(y^2 - x^2) \psi(y) \, dy.
\]

By partial integration \(T_3 = O(R^4)\). A new application of Euler-Maclaurin summation in \(T_1\) gives
\[
T_1 = 2 \int_0^{R/\sqrt{2}} \left( \int_0^{\sqrt{R^2 - x^2}} m(x, y) \, dy + O(R^4) \right) = \frac{1}{4} \int_{\mathbb{R}^2} m \chi_R + O(R^4)
\]
which is \(C_d \lambda^{d/2} + O(\lambda^{d/2-1})\). For \(T_2\) we employ the existence of two finite Fourier series \(Q^\pm(x) = \sum_{|m| \leq M} a_m^\pm e^{2\pi imx}\) for each \(M \in \mathbb{Z}^+\) such that \(Q^-(x) \leq \psi(x) \leq Q^+(x)\) with \(a_m^\pm = O(M^{-1})\) and \(a_m^\pm = O(m^{-1})\) (see for instance \([21]\)). Substituting in \(T_2\) and applying a van der Corput exponent pair \([10]\) \((\alpha, \beta)\) to the sum on \(x\), we have
\[
T_2 = O(M^\alpha R^{\beta+4} + M^{-1} R^5) \quad \text{that is optimal taking} \quad M = R^{(1-\beta)/(1+\alpha)}.
\]
The valid choice \(\alpha = 11/30, \beta = 16/30\) gives \(O(R^{191/41})\) and then for \(n = 2\)
\[
N(\lambda) = C_d \lambda^{d/2} + O(\lambda^{d/2-55/82}).
\]
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with tiny improvements for other choices of the exponent pair.

This slightly improves the cases $N = 4$ and $N = 5$ of Theorem 4.1 in [22].

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