Radiation reaction on a Brownian scalar electron in high-intensity fields

Keita Seto
Extreme Light Infrastructure – Nuclear Physics (ELI-NP) / Horia Hulubei National Institute for R&D in Physics and Nuclear Engineering (IFIN-HH), 30 Reactorului St., Bucharest-Magurele, jud. Ilfov, P.O.B. MG-6, RO-077125, Romania
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Radiation reaction against a relativistic electron is of critical importance since the experiment to check this “quantumness” becomes possible soon with an extremely high-intensity laser beam. However, there is a fundamental mathematical quest to apply any laser profiles to laser focusing and superposition beyond the Furry picture of its usual method by a plane wave. To give the apparent meaning of \( q(\chi) \) the quantumness factor with respect to a radiation process is absent. Thus for resolving the above questions, we propose stochastic quantization of the classical radiation reaction model for any laser field profiles, via the construction of the relativistic Brownian kinematics with the dynamics of a scalar electron and the Maxwell equation with a current by a Brownian quanta. This is the first proposal of the coupling system between a relativistic Brownian quanta and fields in Nelson’s stochastic quantization. Therefore, we can derive the radiation field by its Maxwell equation, too. This provides us the fact that \( q(\chi) \) produced by QED is regarded as \( \mathcal{P}(\Omega_{\text{sc}}^\text{ev}) \) of an existence probability such that a scalar electron stay on its average trajectory.

The case \( q(\chi) = 1 \) for \( \chi \sim 0 \) is regarded as no quantum correction. On the other hand if one uses an extremely high-intensity laser such as the 10PW laser of ELI-NP [1, 4, 5], the quantum correction \( q(\chi) = 0.3 \) appears for the laser intensity of \( 10^{22}\text{W/cm}^2 \) and an electron energy of 600MeV [1, 12] (see Fig.2). Therefore, the investigation of \( q(\chi) \) w.r.t. RR links to the interest in high-intensity laser science. However, Eq.(1) is derived by the Furry picture to employ a laser profile of a plane wave [17, 18]. Hence, there are several proposals for the non-plane wave condition of laser focusing and superposition [16, 21].

Anyway, the effective regime of RR locates at quantized, relativistic and high-intensity field interactions marked by “the star” in Fig.3 This Furry picture is the way from relativistic quantum dynamics. The one from classical, relativistic and high-intensity regime may be another candidate. Let us consider this second candidate for any laser profiles.

For the second idea, we should refer classical RR model. RR has been treated by the Lorentz-Abraham-
Dirac (LAD) equation as its standard model \[22\]:

\[
m_0 \frac{d\mu^\nu}{d\tau} = -e(F_{\text{ex}}^\mu^\nu + F_{\text{LAD}}^\mu^\nu)v_\nu
\]

(4)

\[
F_{\text{LAD}}^\mu^\nu(x) = -\frac{m_0 \gamma_0}{e c^2} \left[ \frac{d^3 x^\mu}{d\tau^3} \cdot \frac{dx^\nu}{d\tau} - \frac{d^3 x^\nu}{d\tau^3} \cdot \frac{dx^\mu}{d\tau} \right]
\]

(5)

With the metric \(g = (+, -, -, -)\) and \(\gamma_0 := c^2/6\pi\varepsilon_0 m_0 e^3\). The force \(-eF_{\text{LAD}}^\mu^\nu v_\nu\) represents the interaction of RR acting on a scalar electron. When we can find the quantization of Eq.\(2\) with Eq.\(3\), it is not known how this affects Eq.\(1\), especially, \(q(\chi)\) by any laser field profiles.

We are then to study quantum dynamics of RR by adopting Nelson’s stochastic quantization \[23\] \[24\] as the second candidate in Fig.\(8\) which can draw a real trajectory of a quanta by a Brownian motion. In addition, that achievement regarded as one of the few application of Nelson’s stochastic quantization.

However, its well-defined relativistic version and the Maxwell equation have been absent. Thus, we resolve the following issues for RR in this article. For \(\{\dot{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}\) a Brownian trajectory, (A) the set of the relativistic Brownian kinematics and the dynamics for a scalar electron

\[
d_{\pm} \dot{x}^\mu(\tau, \omega) = \mathcal{V}^\mu_{\pm}(\dot{x}(\tau, \omega))d\tau + \lambda \times dW^\mu_{\pm}(\tau, \omega)
\]

(6)

\[
m_0 \mathcal{D}_\tau \mathcal{V}^\mu_{\pm}(\dot{x}(\tau, \omega)) = -eV_\nu(\dot{x}(\tau, \omega))F_{\mu\nu}(\dot{x}(\tau, \omega))
\]

(7)

is regarded as the Klein-Gordon (KG) equation with \(\mathcal{V}^\mu_{\pm}(x) := 1/m_0 \times [i\hbar \partial^\mu \ln \phi(x) + eA^\mu(x)]\) given by its wave function \(\phi\) and \(\{W_{\pm}(\tau, \omega)\}_{\tau \in \mathbb{R}}\) of Wiener processes [Sect.\(IV\)]. And also (B) the Maxwell equation with a current of a Brownian scalar electron [Sect.\(IV\)]

\[
\partial_\mu [\mathcal{F}^\mu_{\tau}(x) + \delta f^\mu_{\tau}(x)]
\]

(8)

is constructed. We propose (C) an action integral to give the above Eq.\(7\) and Eq.\(8\) in this model, too [Sect.\(V\)]. In the fact, the consistent system of Eqs.\(6\) \[\[7\] and Eq.\(8\) has been absent after Nelson’s first article \[23\]. Especially, the charge current in Eq.\(8\) adapting stochastic quantization has not been discovered for a long time. So, this is the first proposal for the coupling system between a relativistic Brownian quanta and fields. To describe its interaction is, therefore, a new result. Namely by solving Eq.\(3\) in Sect.\(IV\) we derive (D) RR in quantum dynamics

\[
m_0 \mathcal{D}_\tau \mathcal{V}^\mu(\dot{x}(\tau, \omega))) = -eF_{\text{ex}}^\mu\nu(\dot{x}(\tau, \omega))V_\nu(\dot{x}(\tau, \omega))
\]

\[
- e\mathcal{F}^\mu_{\tau}(\dot{x}(\tau, \omega))V_\nu(\dot{x}(\tau, \omega))
\]

(9)

\[
\mathcal{F}^\mu_{\tau}(\dot{x}(\tau, \omega)) = -\frac{m_0 \gamma_0}{e c^2} \int_{\Omega(\tau, \omega)} d\mathcal{P}(\omega') \left[ \hat{a}^\mu(\dot{x}(\tau, \omega')) \cdot \text{Re}\{V^\nu(\hat{x}(\tau, \omega'))\} - \hat{a}^\nu(\dot{x}(\tau, \omega')) \cdot \text{Re}\{V^\mu(\hat{x}(\tau, \omega'))\} \right]
\]

(10)

with \(\hat{a}^\mu(x) \approx \text{Re}\{\mathcal{D}_\tau \mathcal{V}^\mu_{\tau}(x)\}\), i.e., the quantization of the LAD equation \[15\]. The readers can find the similarity between Eqs.\(14\) \[15\] and Eqs.\(10\) \[11\] by the comparison. In the fact, \(\lim_{\lambda \to 0} \mathcal{F} = F_{\text{LAD}}\) is ensured. Thus, Eqs.\(10\) \[11\] becomes Eqs.\(14\) \[15\] in the classical limit. This easy comparison is the reason why we study stochastic quantization for RR. By its Ehrenfest’s theorem, (E) a radiation formula

\[
\frac{d W_{\text{stochastic}}}{dt} = -m_0 \gamma_0 \mathcal{P}(f_{\text{rad}}^2) \frac{d^2(\hat{p}^\mu_{\tau})}{d\tau^2} \cdot \frac{d^2(\hat{p}^\mu_{\tau})}{d\tau^2}
\]

(11)
similar to Eq. (1) is imposed. This shows the fact that $g(\chi)$ is $\mathcal{P}(\Omega^\text{prob})$ a probability which a scalar electron stays at its average position. Finally, a possibility of its higher-order corrections is discussed.

Let us note a naive idea of quantum dynamics. The present proposal does not deny the previous formulations of quantum dynamics. As K. Yasue suggests [25], quantum dynamics is symbolically illustrated by

$$(\text{Quantum dynamics}) = \text{(Matrix mechanics)} \cup \text{(Wave mechanics)} \cup \text{(Path integral)} \cup \text{(Stochastic quantization)} \cup \text{(Something else)}.$$  

So, each expressions of quantum dynamics are complementary via a wave function.

Before discussing the relativistic regime, let us summarize Nelson’s model in the non-relativistic regime by a 1D stochastic process (an (S3)-processes) in the following Sect.II.

II. STOCHASTIC KINEMATICS AND DYNAMICS IN NON-RELATIVISTIC REGIME BY A 1D STOCHASTIC PROCESS

A. Kinematics

For $\omega$ the label of sample paths, let $\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}$ a trajectory of a quanta with its existence probability of $\mathcal{O}_{\text{prob}}^{1\text{-dim}}(\omega)$ be a 1D Nelson’s (S3)-process [24].

$$\hat{x}(\tau_b, \omega) - \hat{x}(\tau_a, \omega) = \int_{\tau_a}^{\tau_b} v_\pm(\hat{x}(\tau, \omega))d\tau + \lambda_0 \times \int_{\tau_a}^{\tau_b} dw_\pm(\tau, \omega) \quad (12)$$

With $\lambda_0 := \sqrt{\hbar/m_0}$, Equation (12) is also written like

$$\hat{x}(\tau_b, \omega) - \hat{x}(\tau_a, \omega) = \int_{\tau_a}^{\tau_b} d_\pm \hat{x}(\tau, \omega) \quad (13)$$

by introducing its differential form

$$d_\pm \hat{x}(\tau, \omega) := \pm [\hat{x}(\tau \pm d\tau, \omega) - \hat{x}(\tau, \omega)] = v_\pm(\hat{x}(\tau, \omega))d\tau + \lambda_0 \times dw_\pm(\tau, \omega). \quad (14)$$

This is a combination of a drift and its randomness governed by $\{w_\pm(\tau, \omega)\}_{\tau \in \mathbb{R}}$ of 1D Wiener processes (WP; or Brownian motion) such that

$$\mathbb{E}[dw_+(\tau, \bullet)] = 0, \quad (15)$$

$$\mathbb{E}[\{dw_+(\tau, \bullet)\}^2] = d\tau, \quad (16)$$

$$\mathbb{E}[d_{+}(\tau, \bullet) \cdot dw_-(\tau, \bullet)] = 0, \quad (17)$$

for $d\tau > 0$. Where, $\mathbb{E}[f(\bullet)] := \int_{a}^{b} f(\omega)d\mathcal{O}_{\text{prob}}^{1\text{-dim}}(\omega)$ the expectation of $\{f(\omega)\}_{\omega \in \Omega}$. The two types of “±” are derived from the randomness of $\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}$, for the time reversibility [26], let “$\{P_\tau\}$-progressive (prog.)” denoted by “+” be a diffusion from $\tau_a$ to $\tau_b > \tau_a$. And “$\{F_\tau\}$-prog.” by “−” is an inverse process of “$\{P_\tau\}$-prog.” shown in Fig. 4. Thus, an (S3)-process of Eqs. (12) (14) is $\{P_\tau\}$-prog. and $\{F_\tau\}$-prog. “Progressive” means that a stochastic process $\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}$ is integrable for $\tau$ and $\omega$, mathematically. For its probability density $p(x, \tau) := d\mathcal{O}_{\text{prob}}^{1\text{-dim}}/dx$, the difference between a $\{P_\tau\}$-prog. and an $\{F_\tau\}$-prog. appears in the forward (+) and backward (−) Fokker-Planck (FP) equations w.r.t. Eq. (14) [23, 24, 27].

$$\partial_\tau p(x, \tau) + \partial_x [v_\pm(x)p(x, \tau)] = \pm \frac{\lambda_0^2}{2} \partial^2_x p(x, \tau). \quad (18)$$

Figure 4. $d_+ \hat{x}(\tau, \omega)$ and $d_\pm \hat{x}(\tau, \omega)$ of the fractions of a continuous stochastic process $\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}$. Let us draw a sample path $\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}$, then, consider how to generate this by stochastic differential equations. (A) Due to the non-differentiability of $\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}$, we have to define Eq. (14) as the two types of its evolution at $\hat{x}(\tau, \omega)$; $d_\pm \hat{x}(\tau, \omega)$ for a $\{P_\tau\}$-prog. of a normal diffusion process, and $d_+ \hat{x}(\tau, \omega)$ for an $\{F_\tau\}$-prog. of an inverse process of a $\{P_\tau\}$-prog. See also Fig. 5 for this. (B) A curve $\{\hat{x}(\tau, \omega)\}_{\tau \in [\tau_a, \tau_b]}$ is imposed by (A). For the initial value $\hat{x}(\tau_a, \omega)$, $\hat{x}(\tau_b, \omega)$ is found by a forward (normal) diffusion as a $\{P_\tau\}$-prog., i.e., $\hat{x}(\tau_b, \omega) = \hat{x}(\tau_a, \omega) + \int_{\tau_a}^{\tau_b} d_+ \hat{x}(\tau, \omega)$. On the other hand for an $\{F_\tau\}$-prog. is regarded as a backward diffusion, $\hat{x}(\tau_b, \omega) = \hat{x}(\tau_a, \omega) + \int_{\tau_a}^{\tau_b} d_- \hat{x}(\tau, \omega)$ with the terminal value $\hat{x}(\tau_a, \omega)$. This is the reason why we have the two expressions of “+” and “−”. 
**Figure 5** shows the difference between \( \{ P_\tau \} \) and \( \{ F_\tau \} \)-WPs by their trajectories (the blue lines) and probabilities (the red lines) when \( v_\pm = 0 \) in Eq. (18): (a) a \( \{ P_\tau \} \)-WP of a forward diffusion and (b) an \( \{ F_\tau \} \)-WP of a backward diffusion. Equation (18) is derived by the following Itô formula for \( \{ \hat{x}(\tau, \omega) \}_{\tau \in \mathbb{R}} \) of a \( \{ P_\tau \} \)-prog. and an \( \{ F_\tau \} \)-prog., respectively [23, 24]:

\[
d_\pm f(\hat{x}(\tau, \omega)) = f'(\hat{x}(\tau, \omega)) \cdot d_\pm \hat{x}(\tau, \omega) + \pm \frac{\lambda^2}{2} f''(\hat{x}(\tau, \omega))d\tau \quad \text{a.s.} \tag{19}
\]

**B. Dynamics**

What Nelson performed after the above construction of the kinematics was the derivation of the Schrödinger equation \([i\hbar \partial_t + c\phi(x, t)]\psi(x, t) = -\hbar^2/2m_0 \times \partial_x^2 \psi(x, t)\) by the following with the field \( E(x, t) := -\partial_x \phi(x, t) \):

\[
m_0 \left[ \partial_t v(x, t) + v(x, t) \cdot \partial_x v(x, t) - u(x, t) \cdot \partial_x u(x, t) - \frac{\hbar}{2m_0} \partial_x^2 u(x, t) \right] = -eE(x, t) \tag{20}
\]

\[V(x, t) := -i \frac{\hbar}{m_0} \partial_x \ln \psi(x, t) \tag{21}\]

Equation (24) is similar to \( m_0 d\mu/d\tau = -eE \) in classical dynamics. Then, \( v_+ \) and \( v_- \) are reproduced by \( V \) for Eqs. (12-14). Since this is coupled with Eq. (18), he succeeded to demonstrate the question why \( \psi^*(x, t)\psi(x, t) \) is regarded as \( p(x, t) \).

### III. STOCHASTIC KINEMATICS AND DYNAMICS OF A SCALAR ELECTRON BY A 4D STOCHASTIC PROCESS

We give the set of the stochastic kinematics and dynamics of a scalar electron in this section, i.e., the quantization of \( d\sigma^\mu = \nu^\mu d\tau \) and \( m_0 d\nu^\mu = -e\nu^\mu E^\nu \). It is recommended to read from Sect. III to the readers who want to check its scheme briefly at first.

**A. Kinematics**

The following is a natural idea for a scalar electron in the 4D spacetime: We assume an expansion of Eq. (14) to the relativistic kinematics such that

\[
d_\pm \hat{x}^\mu(\tau, \omega) = \mathcal{V}^\mu_\pm(\hat{x}(\tau, \omega))d\tau + \lambda \times DW^\mu(\tau, \omega) \tag{26}
\]

for \( \mu = 0, 1, 2, 3 \) with its existence probability \( \mathcal{P}(\omega) \). It is coupled with \( m_0 \mathcal{D}_\tau \mathcal{V}^\mu(\omega) = -e\mathcal{V}_\nu E^\nu \) equivalent to the KG equation (see it later). As a mimic of the 1D case, we want to require the relativistic FP equation

\[
\partial_\tau p(x, \tau) + \partial_\nu [\mathcal{V}^\nu_\pm(x)p(x, \tau)] = \mp \frac{\lambda^2}{2} \partial_\nu \partial^- p(x, \tau) \tag{27}
\]

w.r.t. \( p(x, \tau) := d\mathcal{P}/d^4x \) and the Itô formula,

\[
d_\pm f(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega)) \cdot d_\pm \hat{x}^\mu(\tau, \omega) + \frac{\lambda^2}{2} \partial^- d_\mu f(\hat{x}(\tau, \omega))d\tau \quad \text{a.s.} \tag{28}
\]

We will demonstrate the validity of the above adapting the KG equation well in the later discussion.

Anyway for \( \{ \hat{x}(\tau, \omega) \}_{\tau \in \mathbb{R}} \) of a 4D \( \{ P_\tau \} \)-prog., it imposes the expanded formula of Eq. (19): \( d_\pm f(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega)) \cdot d_\pm \hat{x}^\mu(\tau, \omega) + \lambda^2/2 \partial^- \partial_\mu f(\hat{x}(\tau, \omega))d\tau \) a.s. By comparing it with Eq. (28), \( \delta_\mu^\nu \partial_\mu \partial_\nu f(\hat{x}(\tau, \omega)) \) has to

\[
v(x, t) = \text{Re} \{ V(x, t) \} \tag{22}
\]

\[
u(x, t) = -\text{Im} \{ V(x, t) \} \tag{23}
\]

Where, \( v := (v_+ + v_-)/2 \) and \( u := (v_+ - v_-)/2 \) [23, 24]. By \( V := v - iv \), its compact form [31] appears:

\[
m_0 D_t V(x, t) = -eE(x, t) \tag{24}
\]

\[
D_t := \partial_t + V(x, t)\partial_x - i\frac{\lambda_0}{2} \partial_x^2 \tag{25}
\]
be \((-g^\mu\nu)\partial_\mu\partial_\nu f(\hat{x}(\tau,\omega))\). Therefore, let us introduce a \(\{\mathcal{P}_\tau\}\)-prog. for "\(\pm\)" in Eq. (26)
\[
\hat{x}(\tau, \omega) := \left\{ \begin{array}{ll}
\hat{x}^0(\tau, \omega), & \hat{x}^1(\tau, \omega), \hat{x}^2(\tau, \omega), \hat{x}^3(\tau, \omega)
\end{array} \right. \\
\{\mathcal{P}_\tau\}\text{-prog.} \\
\{\mathcal{F}_\tau\}\text{-prog.} \\
\{\mathcal{R}_\tau\}\text{-prog.}
\]
and an \(\{\mathcal{F}_\tau\}\)-prog. by "\(-\)"
\[
\hat{x}(\tau, \omega) := \left\{ \begin{array}{ll}
\hat{x}^0(\tau, \omega), \hat{x}^1(\tau, \omega), \hat{x}^2(\tau, \omega), \hat{x}^3(\tau, \omega)
\end{array} \right. \\
\{\mathcal{P}_\tau\}\text{-prog.} \\
\{\mathcal{F}_\tau\}\text{-prog.} \\
\{\mathcal{R}_\tau\}\text{-prog.}
\]
where, let \(\{W_+(\tau, \omega)\}_{\tau \in \mathbb{R}}\) and \(\{W_-(\tau, \omega)\}_{\tau \in \mathbb{R}}\) of WP's in Eq. (26) be \(\{\mathcal{P}_\tau\}\) and \(\{\mathcal{F}_\tau\}\)-prog., respectively. Hereby, we name \(\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}\) of a \(\{\mathcal{P}_\tau\}\) and \(\{\mathcal{F}_\tau\}\)-prog. “a D-prog.” More precise construction is given by Appendix A. For \(d\tau \geq 0\) and \(\mathbb{E}[f(\bullet)] := \int_{\Omega} f(\omega) d\mathbb{P}(\omega)\), the following rules are satisfied as the expansion of the 1D case:
\[
\mathbb{E}[dW^\mu_{\pm}(\tau, \bullet)] = 0 \quad (29)
\]
\[
\mathbb{E}[dW^\mu_{\pm}(\tau, \bullet) \cdot dW^\nu_{\pm}(\tau, \bullet)] = \delta^{\mu\nu} \times d\tau \quad (30)
\]
\[
\mathbb{E}[dW^\mu_{\pm}(\tau, \bullet) \cdot dW^\nu_{\mp}(\tau, \bullet)] = 0 \quad (31)
\]
The definition of \(d_- f(\hat{x}(\tau, \omega))\) by Eq. (25) can be checked by the expansion for an \(\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}\) of \(\{\mathcal{F}_\tau\}\)-prog.,
\[
d_- f(\hat{x}(\tau, \omega)) = f(\hat{x}^0(\tau + d\tau, \omega), \hat{x}^{i=1,2,3}(\tau, \omega)) \\
\{\mathcal{P}_\tau\}\text{-prog.} \\
\{\mathcal{F}_\tau\}\text{-prog.} \\
- f(\hat{x}^0(\tau, \omega), \hat{x}^{i=1,2,3}(\tau - d\tau, \omega)) \quad (32)
\]
with the help by Eq. (26) (a 4D version of Eq. (14))
\[
d_- \hat{x}(\tau, \omega) = \left\{ \begin{array}{ll}
(\hat{x}^0(\tau + d\tau, \omega), \hat{x}^{i=1,2,3}(\tau, \omega))
\end{array} \right. \\
\{\mathcal{P}_\tau\}\text{-prog.} \\
\{\mathcal{F}_\tau\}\text{-prog.} \\
- (\hat{x}^0(\tau, \omega), \hat{x}^{i=1,2,3}(\tau - d\tau, \omega)) \quad (33)
\]
to first order of \(d\tau\). \(d_+ f(\hat{x}(\tau, \omega))\) is imposed by \(\{\hat{x}(\tau, \omega)\}_{\tau \in \mathbb{R}}\) of \(\{\mathcal{P}_\tau\}\)-prog. Let Eq. (25) be the general definition of \(d_\pm\). The probability \(\mathbb{P}(\omega)\) is calculated by the FP equation (27) since \(p(x, \tau) := d\mathbb{P}/d^4x\). Nelson introduced the drift velocities via the so-called mean derivatives (23). In the present case, it is evaluated by
\[
\mathbb{V}_\pm^\mu(\hat{x}(\tau, \omega)) := \mathbb{E} \left[ \frac{d_{\pm} \hat{x}^\mu}{d\tau}(\tau, \bullet) \left| \hat{x}(\tau, \omega) \right. \right] (\omega) \quad (34)
\]
where, \(\mathbb{E}[f(\hat{x}(\tau, \bullet))|\mathcal{G}]() = \int f(\hat{x}(\tau, \omega)) d\mathbb{P}(\omega)\) the conditional expectation for \(\mathbb{P}_\mathcal{G}(X)\) a conditional probability of \(X\) given \(\mathcal{G}\).

Then, we define the complex differential \(d := (d_+ + d_-)/2 - i(d_+ - d_-)/2\) symbolically such that
\[
df(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega)) \cdot d\hat{x}^\mu(\tau, \omega) \\
- i\lambda^2/2 \partial_\mu \partial^\mu f(\hat{x}(\tau, \omega)) d\tau \quad \text{a.s.} \quad (35)
\]
and the complex velocity
\[
\mathbb{V}_\mu(\hat{x}(\tau, \omega)) := \mathbb{E} \left[ \frac{d\hat{x}^\mu}{d\tau}(\tau, \bullet) \left| \hat{x}(\tau, \omega) \right. \right] (\omega) \quad (36)
\]
\[
= \frac{\mathbb{V}_+^\mu(\hat{x}(\tau, \omega)) + \mathbb{V}_-^\mu(\hat{x}(\tau, \omega))}{2} \\
- i\frac{\mathbb{V}_+^\mu(\hat{x}(\tau, \omega)) - \mathbb{V}_-^\mu(\hat{x}(\tau, \omega))}{2} \quad (37)
\]
with its other assumption [31]:
\[
\mathbb{V}_+^\mu(x) := \frac{1}{m_0} \times [i\hbar \partial^\mu \ln \phi(x) + eA^\mu(x)] \quad (38)
\]

Let \(||A||^2_{(g, C^1)} := A^*_\mu A^\mu\) (\(A^*\) is the complex conjugate of a vector \(A\)) on the Minkowski spacetime,
\[
d\tau := \frac{1}{c} \times \sqrt{\mathbb{E} \left[ ||d\hat{x}(\tau, \bullet) - \lambda \times dW(\tau, \bullet)||^2_{(g, C^1)} \right]} \quad (39)
\]
the proper time links to the Lorentz invariant
\[
\mathbb{E} \left[ \mathbb{V}_+^\mu(\hat{x}(\tau, \bullet)) \mathbb{V}^\mu(\hat{x}(\tau, \bullet)) \right] = c^2 \quad (40)
\]

For satisfying Eq. (10), \(\phi\) in Eq. (9) has to be a wave function of the KG equation with \(i\hbar \partial_\tau + eA_\tau(x)\),
\[
(i\hbar \partial_\tau) \cdot (i\hbar \partial^\mu) \phi(x) - m_0^2 c^2 \phi(x) = 0 \quad (41)
\]
as Ref. [32] suggests. Namely by
\[
\mathbb{V}_+^\mu(x) \mathbb{V}^\mu(x) = \frac{1}{m_0^2} \times \frac{\text{Re}\{\phi^*(x)(i\hbar \partial_\tau) \cdot (i\hbar \partial^\mu) \phi(x)\}}{\phi^*(x) \phi(x)} \\
+ \frac{\hbar^2}{2m_0^2} \times \frac{\partial_\mu \partial^\mu [\phi(x) \cdot \phi^*(x)]}{\phi^*(x) \phi(x)} \quad (42)
\]
the first term in RHS is \(c^2\) and the second term becomes zero by its expectation after the substitution \(x = \hat{x}(\tau, \omega)\). Let us demonstrate this in the end of the next small section for the probability density.

### B. Equations of probability density

By using this complex velocity, the FP equation of Eq. (26) derives the equation of continuity for \(p\), w.r.t. \(x \in \{\hat{x}(\tau, \omega)\}_{\omega \in \Omega}\).
\[
\partial_\tau p(x, \tau) + \partial_\mu \left[ \text{Re} \{\mathbb{V}^\mu(x)\} p(x, \tau) \right] = 0 \quad (43)
\]
Then for a naturally boundary condition \(p(x, \pm\infty) = 0\),
\[
\partial_\tau \left[ \text{Re} \{\mathbb{V}^\mu(x)\} \int_{\mathbb{R}} d\tau p(x, \tau) \right] = 0 \quad (44)
\]
is found. Or by using
\[
p(x, \tau) := \mathbb{E}[d^4(x - \hat{x}(\tau, \bullet))] \quad (45)
\]
due to the definition of \( \mathbb{E}[f(\hat{x}(\tau, \bullet))] \), i.e.,
\[
\mathbb{E}[f(\hat{x}(\tau, \bullet))] := \int_{\Omega} f(\hat{x}(\tau, \omega')) d\mathcal{P}(\omega')
\]
\[
= \int_{\mathbb{R}^4} f(x') p(x', \tau) d^4x',
\]
the following is imposed:
\[
\partial_{\mu} \mathbb{E} \left[ -ec \int d\tau \text{Re}\{\mathcal{V}^\mu(x)\} \delta^4(x - \hat{x}(\tau, \bullet)) \right] = 0
\]
(48)

This is the conservation law of the current density
\[
\mathcal{J}_{\text{stochastic}}(x) := \mathbb{E} \left[ -ec \int d\tau \text{Re}\{\mathcal{V}^\mu(x)\} \delta^4(x - \hat{x}(\tau, \bullet)) \right]
\]
in the 4D spacetime (see this role in Sect. IV). The following relation is also found by Eq. (27).
\[
\text{Im}\{\mathcal{V}^\mu(x)\} = \begin{cases} \\
\frac{\lambda^2}{2} \times \partial^\mu \ln p(x, \tau), & x \in \hat{x}(\tau, \Omega) \\
\frac{\lambda^2}{2} \times \partial^\mu \ln \int d\tau p(x, \tau), & x \in \bigcup_{\tau} \hat{x}(\tau, \Omega),
\end{cases}
\]
(50)

Where, \( \bigcup_{\tau} \hat{x}(\tau, \Omega) = \text{supp}(\int d\tau p(\circ, \tau)) \). Equation (50) is a mimic of the osmotic pressure formula [23, 24].

We can easily clarify that Eq. (53) satisfies the \( U(1) \)-gauge symmetry. Let us study the relation between Eq. (40) and Eq. (53). This relation corresponds to the one between \( v_\mu v^\mu = c^2 \) and \( d/d\tau(v_\mu v^\mu) = 0 \) in classical dynamics. The readers may find another definition of \( \mathcal{D}_\tau \):
\[
\mathcal{D}_\tau^\pm := \mathbb{E} \left[ \frac{d}{d\tau} \hat{x}(\tau, \omega) \right]
\]
(57)

\[
\mathcal{D}_\tau := \mathbb{E} \left[ \frac{d}{d\tau} \hat{x}(\tau, \omega) \right] = \frac{1-i}{2} \mathcal{D}_\tau^+ + \frac{1+i}{2} \mathcal{D}_\tau^-
\]
(58)

Nelson introduced the partial integral formula for his (S3)-process. In our case for \( \{\hat{x}(\tau, \omega)\} \) of a D-prog., the differential form of that formula for \( \{\alpha^\mu\}_{\mu=0,1,2,3} \) and \( \{\beta^\mu\}_{\mu=0,1,2,3} \) of complex functions is
\[
\frac{d}{d\tau} \mathbb{E}[\alpha^\mu(\hat{x}(\tau, \bullet))\beta^\mu(\hat{x}(\tau, \bullet))]
\]
\[
= \mathbb{E} \left[ \mathcal{D}_\tau \alpha^\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) \right]
\]
\[
= \mathbb{E} \left[ \mathcal{D}_\tau^+ \alpha^\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) \right] + \mathbb{E} \left[ \mathcal{D}_\tau^- \alpha^\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) \right].
\]

(59)

By linear combining the above "+" and "−" formulas,
\[
\frac{d}{d\tau} \mathbb{E}[\alpha^\mu(\hat{x}(\tau, \bullet))\beta^\mu(\hat{x}(\tau, \bullet))]
\]
\[
= \mathbb{E} \left[ \mathcal{D}_\tau \alpha^\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) \right]
\]
\[
+ \mathbb{E} \left[ \mathcal{D}_\tau^+ \alpha^\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) \right] + \mathbb{E} \left[ \mathcal{D}_\tau^- \alpha^\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) \right].
\]

(60)

The demonstration of the above formulas is shown in Appendix [25]. When \( \alpha^\mu = \mathcal{V}^\mu_\mu \) and \( \beta^\mu = \mathcal{V}^\mu \) in Eq. (61),
\[
\frac{d}{d\tau} \mathbb{E} \left[ \mathcal{V}^\mu_\mu(\hat{x}(\tau, \bullet)) \mathcal{V}^\mu(\hat{x}(\tau, \bullet)) \right]
\]
\[
= \mathbb{E} \left[ \mathcal{V}^\mu_\mu(\hat{x}(\tau, \bullet)) \cdot \mathcal{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \bullet)) \right]
\]
\[
+ \mathbb{E} \left[ \mathcal{D}_\tau^+ \mathcal{V}^\mu_\mu(\hat{x}(\tau, \bullet)) \cdot \mathcal{V}^\mu(\hat{x}(\tau, \bullet)) \right] + \mathbb{E} \left[ \mathcal{D}_\tau^- \mathcal{V}^\mu_\mu(\hat{x}(\tau, \bullet)) \cdot \mathcal{V}^\mu(\hat{x}(\tau, \bullet)) \right]
\]
\[
= \frac{\lambda^2 e}{2m_0} \times \int_{\mathbb{R}^4} d^4x p(x, \tau) \cdot \partial_{\mu} \partial_{\nu} F^{\mu \nu}(x) = 0
\]

(62)

is found. Thus, Eq. (40) is supported by Eq. (53) the dynamics of a Brownian quanta, too.

\[\mathcal{D}_\tau := \hat{\mathcal{V}}^\mu(x) \cdot \partial_{\mu}\]
Consider the expectation of Eq. (53), Ehrenfest’s theorem is imposed naturally for $\langle \hat{x}(\tau) \rangle := E[\hat{x}(\tau, \bullet)]$ and $\delta \hat{x}(\tau, \omega) := \hat{x}(\tau, \omega) - \langle \hat{x}(\tau) \rangle$:

$$m_0 \frac{d^2 \langle \hat{x}(\tau) \rangle}{d\tau^2} = E[\mu F^{\mu\nu}(\hat{x}(\tau, \bullet)) Re \{V_{\nu}(\hat{x}(\tau, \bullet))\}]$$

$$= -e F^{\mu\nu}(\hat{x}(\tau, \bullet)) \frac{d\langle \hat{x}_\nu(\tau) \rangle}{d\tau} + O((\partial^2 \delta \hat{x}(\tau)) \tau)$$

$$d\langle \hat{x}_\nu(\tau) \rangle / d\tau - Re \{V(\langle \hat{x}(\tau) \rangle)\} = O((\partial^2 \delta \hat{x}(\tau)) \tau)$$

$$d\langle \hat{x}_\nu(\tau) \rangle / d\tau = E[Re \{V(\hat{x}(\tau, \bullet))\}]$$

let us conclude the above discussion schematically. At the kinematics of a relativistic quanta from classical dynamics, the expectation of Eq. (26), namely, $\delta f^{\mu\nu}(x) \equiv \delta^4(x - x(\tau))$ corresponding to

$$\partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}(x)]$$

$$= \mu_0 \times \left[-ec \int R d\tau \frac{dx^{\nu}}{d\tau}(\tau) \times \delta^4(x - x(\tau))\right]$$

in classical physics. Where, we will use $\delta f$ as a singular field attached to a quanta like the Coulomb field. $j_{\text{stochastic}}$ of the current density given by Eq. (69) (and in Eq. (65)) is equal to one of a KG particle, i.e., $j_{\text{stochastic}} = j_{\text{K-G}}$.

$$j_{\text{K-G}}(x) = -\frac{i e c \lambda^2}{2} [\phi^*(x) \Omega^{\mu}(\phi(x) - \phi(x)(\Omega^*)^\mu \phi^*(x)]$$

by the assumption

$$\phi^*(x) \phi(x) = \int R d\tau p(x, \tau).$$

Since $\partial_\mu j_{\text{stochastic}}^\mu(x) = 0$, $\partial_\mu j_{\text{K-G}}(x) = 0$ and

$$\partial_\mu j_{\text{stochastic}}^\mu(x) = \int R d\tau p(x, \tau) \phi^*(x) \phi(x) x_j^\mu(x_j^\mu(x)$$

therefore, $\partial_\mu [\int R d\tau p(x, \tau) / \phi^*(x) \phi(x)] = 0$ has to be satisfied. We emphasize that Eq. (72) is considered as the normalization of $\phi$ a wave function of the KG equation corresponding to one of the Schrödinger equation. The quantum effect of RR appearing in Eq. (1) is derived by the definition of $j_{\text{stochastic}}^\mu$ obviously. When we write it such as

$$j_{\text{stochastic}}(x) = -ec \int \int R d\tau \frac{dx^{\nu}}{d\tau}(\tau) \times Re \{V(\mu)(\phi^*(x) - \phi(x)(\Omega^*)^\mu \phi^*(x)]$$

its classical limit ($h \to 0$) converge to a path $\omega_0$ a smooth trajectory of $\{x(\tau)\} \tau \in \mathbb{R} := \{\hat{x}(\tau, \omega_0)\}$ for $\tau \in \mathbb{R}$. Then, Eq. (74) is deduced from Eq. (69), namely,

$$\lim_{h \to 0} j_{\text{stochastic}}^\mu(x) = -ec \int \int R d\tau \frac{dx^{\nu}}{d\tau}(\tau) \times \delta^4(x - \hat{x}(\tau, \omega_0))$$

with Re $\{V(\mu)(x)\}$ $\to dx^{\nu}(x)/d\tau$. See Sect. VII B for more detail of the classical limit.

V. ACTION INTEGRAL

Let us give the systematic way to define Eq. (53) and Eq. (69) by the following action integral (issue-(C)):

$$S_0[\hat{x}, A] = \int \int R d\tau dA \left[\frac{m^0}{2} V_{\nu}(\hat{x}(\tau, \bullet)) V_{\nu}(\hat{x}(\tau, \bullet))\right]\right]$$

$$+ \int \int R d\tau dA \left[-e A_\nu(\hat{x}(\tau, \bullet)) Re \{V_{\nu}(\hat{x}(\tau, \bullet))\}\right]\right]$$

$$+ \int \int R d^4 x \frac{1}{4 \mu_0 c} [F(x) + \delta f(x)]^2$$

IV. MAXWELL EQUATION

The Maxwell equation (the issue-(B)) is also given by

$$\partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}(x)]$$

$$= \mu_0 \times \left[-ec \int \int R d\tau Re \{V(\mu)(x)\} \times \delta^4(x - \hat{x}(\tau, \bullet))\right].$$

By considering that this kinematics is deduced from the expectation of Eq. (20), namely, $E[d_+ \hat{x}(\tau, \bullet)] = v(\tau) d\tau$, the above classical dynamics is considered as the one by Ehrenfest’s theorem. Hence, the classical kinematics and dynamics are replaced by

$$E[d_+ \hat{x}(\tau, \bullet)] = \frac{d\langle \hat{x}(\tau, \bullet) \rangle}{d\tau} d\tau,$$

$$m_0 \frac{d^2 \langle \hat{x}(\tau, \bullet) \rangle}{d\tau^2} = E[\mu F^{\mu\nu}(\hat{x}(\tau, \bullet)) Re \{V_{\nu}(\hat{x}(\tau, \bullet))\}]$$

Since $d^2 \langle \hat{x}(\tau, \bullet) \rangle / d\tau^2 = E[Re \{\Omega_{\nu}(\hat{x}(\tau, \bullet))\}]$, Eq. (53) the dynamics of a Brownian scalar electron is found with its complex conjugate. This is the schematic method of stochastic quantization.

D. Schematic method of stochastic quantization

For obtaining Eq. (53), the equations of quantum dynamics of a relativistic quanta from classical dynamics, let us conclude the above discussion schematically. At first, we regard classical dynamics as the combination of the kinematics

$$dx(\tau) = v(\tau) d\tau$$

and its dynamics

$$m_0 \frac{dv(\tau)}{d\tau} = -e \nu v(\tau) F^{\mu\nu}(x(\tau))$$

By considering that this kinematics is deduced from the expectation of Eq. (20), namely, $E[d_+ \hat{x}(\tau, \bullet)] = v(\tau) d\tau$, the above classical dynamics is considered as the one by Ehrenfest’s theorem. Hence, the classical kinematics and dynamics are replaced by

$$E[d_+ \hat{x}(\tau, \bullet)] = \frac{d\langle \hat{x}(\tau, \bullet) \rangle}{d\tau} d\tau,$$

$$m_0 \frac{d^2 \langle \hat{x}(\tau, \bullet) \rangle}{d\tau^2} = E[\mu F^{\mu\nu}(\hat{x}(\tau, \bullet)) Re \{V_{\nu}(\hat{x}(\tau, \bullet))\}]$$

Since $d^2 \langle \hat{x}(\tau, \bullet) \rangle / d\tau^2 = E[Re \{\Omega_{\nu}(\hat{x}(\tau, \bullet))\}]$, Eq. (53) the dynamics of a Brownian scalar electron is found with its complex conjugate. This is the schematic method of stochastic quantization.
This is an analogy of the classical action integral:

\[
S_{\text{classical}}[x, A] = \int_{\mathbb{R}} d\tau \frac{m_0}{2} v_\alpha(\tau) v^\alpha(\tau) - \int_{\mathbb{R}} d\tau a_\alpha(x(\tau)) v^\alpha(\tau) + \int_{\mathbb{R}} d^4 x \frac{1}{4\mu_0 c} [F(x) + \delta f(x)]^2
\]  (77)

However to implement the sub-equations for a scalar electron,

\[
\mathcal{S}[\hat{x}, A, \lambda] = \mathcal{S}_0[\hat{x}, A] + \int_{\mathbb{R}} d\tau \frac{d}{d\tau} \mathcal{E} \left[ W(\hat{x}(\tau, \bullet)) \right] + \int_{\mathbb{R}} d\tau (\lambda(\tau) + \mathcal{E} \left[ \hat{V}_\alpha(\hat{x}(\tau, \bullet))\hat{V}^\alpha(\hat{x}(\tau, \bullet)) - c^2 \right])
\]  (78)

is hereby proposed in the present model. Where, \( \mathcal{E}[W(\hat{x}(\tau, \bullet))]/d\tau \) with \( W(x) = \text{Re} \{ i\hbar \ln \phi(x) \} \) is an uncertainty of its Lagrangian, and the final term in RHS \( \mathcal{E} \) represents its holonomic constraint. The following Euler-Lagrange-Yasue equation is derived by the variation of Eq. (78) w.r.t. \( \hat{x} \) (see also Ref. 24):

\[
\frac{\partial L_{\text{particle}}}{\partial \hat{x}^\mu} - \mathcal{D}_\tau \mathcal{D}^\tau \frac{\partial L_{\text{particle}}}{\partial \hat{v}^\mu} - \mathcal{D}_\tau \mathcal{D}^\tau \frac{\partial L_{\text{particle}}}{\partial \hat{V}^\mu} = 0
\]  (79)

\[
\frac{\partial L_{\text{particle}}}{\partial \hat{v}^\mu} + \frac{\partial L_{\text{particle}}}{\partial \hat{V}^\mu} = 0
\]  (80)

\[
L_{\text{particle}}(\hat{x}, \hat{V}, \hat{V}^\star) = \frac{m_0}{2} \hat{V}_\alpha \hat{V}^\alpha - \text{Re} \{ \hat{a}_\alpha(\hat{x}) \hat{V}^\alpha \}
+ \text{Re} \{ \hat{V}^\alpha \} \cdot \partial_\alpha W(\hat{x})
+ \lambda(\tau) \times (\hat{V}_\alpha \hat{V}^\alpha - c^2)
\]  (81)

In addition, let us also consider the variation of Eq. (78) w.r.t \( \lambda \) for its holonomic constraint. Then, the following equations are the results w.r.t. a scalar electron:

\[
\left| m_0 + 2\lambda(\tau) \right| \times \text{Re} \{ \mathcal{D}_\tau \hat{V}^\alpha(\hat{x}(\tau, \omega)) \}
= \text{Re} \{ -\hat{V}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega)) \}
\]  (82)

\[
\left| m_0 + 2\lambda(\tau) \right| \times \text{Re} \{ \hat{V}^\alpha(\hat{x}(\tau, \omega)) \}
= \text{Re} \{ i\hbar \hat{m}^\mu \ln \phi(\hat{x}(\tau, \bullet)) \} + \text{Re} \{ \hat{a}^\mu(\hat{x}(\tau, \bullet)) \}
\]  (83)

\[
\mathcal{E} \{ \hat{V}^\alpha(\hat{x}(\tau, \bullet)) \hat{V}^\alpha(\hat{x}(\tau, \bullet)) \} - c^2 = 0
\]  (84)

For satisfying the above three, \( \lambda = 0 \) is required. Thus, we can find not only Eq. (83), but Eq. (85) the definition of \( \hat{V}^\mu \) and Eq. (80) from the action integral.

The Maxwell equation (80) is derived by the variation of Eq. (78) for \( A; \partial_\mu \left( \partial_\nu \mathcal{L}_{\text{field}}/\partial \hat{v}^\nu - \partial \mathcal{L}_{\text{field}}/\partial A_\nu \right) - \partial \mathcal{L}_{\text{field}}/\partial A_\nu = 0 \) with the Lagrangian density

\[
\mathcal{L}_{\text{field}} = \mathcal{E} \left[ -\int_{\mathbb{R}} d\tau e A_\alpha(x) \text{Re} \{ \hat{V}^\alpha(x) \} \delta^4(x - \hat{x}(\tau, \bullet)) \right] + \frac{1}{4\mu_0 c} [F(x) + \delta f(x)]^2
\]  (85)

Thus, a radiating Brownian quanta is illustrated by Eq. (78) with Eq. (26) the kinematics of a scalar electron for defining its probability \( \mathcal{P} \) for the action integral \( \mathcal{E} \left[ \int_{\mathbb{R}} d\tau L_{\text{particle}}(\hat{x}, \hat{V}, \hat{V}^\star) \right] \).

A. Remark: derivation of Eq. (79) and Eq. (80):

For \( \hat{D}_\tau^\pm \hat{x}^\mu(\tau, \omega) = \hat{V}_\alpha(\hat{x}(\tau, \omega)) \), \( L_{\text{particle}}(\hat{x}, \hat{V}, \hat{V}^\star) = L_0(\hat{x}, \hat{V}, \hat{V}^\star) \), and the following relations such that

\[
\frac{\partial L_0}{\partial \hat{x}^\mu} = \frac{1 + i}{2} \frac{\partial L_{\text{particle}}}{\partial \hat{v}^\mu} + \frac{1 - i}{2} \frac{\partial L_{\text{particle}}}{\partial \hat{V}^\mu},
\]  (86)

\[
\frac{\partial L_0}{\partial \hat{V}^\mu} = \frac{1 - i}{2} \frac{\partial L_{\text{particle}}}{\partial \hat{v}^\mu} + \frac{1 + i}{2} \frac{\partial L_{\text{particle}}}{\partial \hat{V}^\mu},
\]  (87)

Eqs. (79) (80) are derived via

\[
\delta \int_{\mathbb{R}} d\tau \mathcal{E} \left[ L_0(\hat{x}, \hat{V}, \hat{V}^\star) \right] = \int_{\mathbb{R}} d\tau \mathcal{E} \left[ \frac{\partial L_0}{\partial \hat{x}^\mu} \delta \hat{x}^\mu + \frac{\partial L_0}{\partial \hat{v}^\mu} \delta \hat{v}^\mu + \frac{\partial L_0}{\partial \hat{V}^\mu} \delta \hat{V}^\mu \right] = \int_{\mathbb{R}} d\tau \mathcal{E} \left[ \left( \delta \hat{x}^\mu \cdot \frac{\partial}{\partial \hat{x}^\mu} + \mathcal{D}_\tau \delta \hat{x}^\mu \cdot \frac{\partial}{\partial \hat{V}^\mu} + \mathcal{D}_\tau^\star \delta \hat{V}^\mu \cdot \frac{\partial}{\partial \hat{V}^\mu} \right) L_{\text{particle}} \right]
\]  (88)

with Eq. (61) of Nelson’s partial integral formula, namely,

\[
\delta \int_{\mathbb{R}} d\tau \mathcal{E} \left[ L_{\text{particle}}(\hat{x}, \hat{V}, \hat{V}^\star) \right] = \int_{\mathbb{R}} d\tau \mathcal{E} \left[ \delta \hat{x}^\mu \left( \frac{\partial}{\partial \hat{x}^\mu} - \mathcal{D}_\tau \frac{\partial}{\partial \hat{V}^\mu} - \mathcal{D}_\tau^\star \frac{\partial}{\partial \hat{V}^\mu} \right) L_{\text{particle}} \right] + \int_{\mathbb{R}} d\tau \frac{d}{d\tau} \mathcal{E} \left[ \delta \hat{x}^\mu \left( \frac{\partial}{\partial \hat{v}^\mu} + \frac{\partial}{\partial \hat{V}^\mu} \right) L_{\text{particle}} \right].
\]  (90)

With respect to any \( \{ \delta \hat{x}(\tau, \omega) \} \) \( \in \mathbb{R} \times \Omega \). Eq. (79) and Eq. (80) are fulfilled for \( \delta \int_{\mathbb{R}} d\tau \mathcal{E} \left[ L_{\text{particle}}(\hat{x}, \hat{V}, \hat{V}^\star) \right] = 0 \).

VI. RADIATION REACTION

Since the Maxwell equation is given by Eq. (80), let us describe RR on a Brownian scalar electron as the quantization of the LAD equation (44) (the issue-(D)). We also discuss its Ehrenfest’s theorem, its classical limit, Landau-Lifshitz’s approximation, and the radiation formula corresponding to the well-known classical model in this section.
A. Quantization of the LAD equation

By recalling Eqs. (44,45) in the classical regime, $F_{\text{LAD}}$ by Eq. (55) is a homogeneous solution of $\partial_{\mu}F^{\mu\nu} = -e\kappa_0 \int_{\mathcal{R}} d\omega \nu'(\omega) \times \delta^4(x-x(\omega))$. The readers can find the derivation of $F_{\text{LAD}}$ in Ref. [11, 22, 33]. We explore $\mathcal{G}$ the RR field given by Eq. (69) corresponding to $F_{\text{LAD}}$. Let us solve Eq. (69) as the mimic of the LAD model at $x = \hat{x}(\tau, \omega)$ under the Lorenz gauge. Where, $\Omega$ is a set of all sample paths in our physics. Instead of Eq. (69),

\[
\partial_{\mu}F^{\mu\nu}_{(\pm)}(x) = \mu_0 J_{\text{stochastic}}^{\nu}(x) \tag{91}
\]

with $\partial_{\mu}A^{\mu}_{(\pm)}(x) = 0$. Where, $F^{\mu\nu}_{(\pm)} := \partial^\mu A^{\nu}_{(\pm)} - \partial^\nu A^{\mu}_{(\pm)}$ are the retarded $(\pm)$ / advanced $(\mp)$ fields, namely,

\[
\partial_{\mu}\partial^\nu A^{\mu}_{(\pm)}(x) = \mu_0 J_{\text{stochastic}}^{\nu}(x). \tag{92}
\]

Thus, their potentials are given by

\[
A_{(\pm)}^{\nu}(x) = -e\kappa_0 \int_{\mathcal{R}} d\omega \nu' \left\{ \int_{\Omega} d\omega \nu \mu \right\} G_{(\pm)}(x, \hat{x}(\tau, \omega')) \tag{93}
\]

or by $\int_{\mathcal{R}} f(\hat{x}(\tau, \omega')) d\omega = \int_{\mathcal{R}} f(x') p(x', \tau') d\tau' d^4x'$.

\[
A_{(\pm)}^{\nu}(x) = -e\kappa_0 \int_{\mathcal{R}} d\omega \nu' \left\{ \int_{\Omega} d\omega \nu \mu \right\} G_{(\pm)}(x, x', \tau'). \tag{94}
\]

Where, $G_{(\pm)}$ are the retarded/advanced Green functions defined by $\partial_{\mu}\partial^\nu G_{(\pm)}(x, x') = \delta^4(x-x')$. Therefore,

\[
F^{\mu\nu}_{(\pm)}(x) = -e\kappa_0 \int_{\mathcal{R}} d\omega \nu' \left\{ \int_{\Omega} d\omega \nu \mu \right\} \left[ \text{Re} \{\nu' \hat{x}(\tau, \omega') \} \cdot \partial^\nu - (\mu \leftrightarrow \nu) \right] G_{(\pm)}(x, \hat{x}(\tau, \omega')). \tag{95}
\]

For the calculation of $F^{\mu\nu}_{(\pm)}(\hat{x}(\tau, \omega))$, consider

\[
V(\tau, \omega) := \text{supp}(\int_{\mathcal{R}} p(\omega', \tau) d\tau') \cap \{ x' | ||x' - \hat{x}(\tau, \omega)||^2 = 0 \} \tag{96}
\]

a neighborhood of a quanta on its light cone. For each $\omega'$, there is the largest $T(\omega') := [\tau - \tau_1^{(\omega')}, \tau + \tau_2^{(\omega')}]$ such that $\hat{x}(\tau - \tau_1^{(\omega')}, \omega')$ and $\hat{x}(\tau + \tau_2^{(\omega')}, \omega')$ stay in $V(\tau, \omega)$. Thus, $\Omega(\tau, \omega) := \{ \omega' | V(\tau, \omega) \cap \{ \hat{x}(\tau, \omega') \}_{\tau \in \Omega(\tau, \omega)} \neq \emptyset \}$ is the largest set of feasible paths for Eq. (95) when $x = \hat{x}(\tau, \omega)$,

\[
F^{\mu\nu}_{(\pm)}(\hat{x}(\tau, \omega)) = -e\kappa_0 \int_{\Omega(\tau, \omega)} d\omega \nu' \left\{ \int_{T(\omega')} d\tau' \right\} \left[ \text{Re} \{\nu' \hat{x}(\tau, \omega') \} \cdot \partial^\nu - (\mu \leftrightarrow \nu) \right] G_{(\pm)}(x, \hat{x}(\tau, \omega')). \tag{97}
\]

Where, Eq. (96) restricts the domain of the integral for $\tau'$ on $T(\omega')$. Let us assume the each duration in $(\tau'(\omega'))_{\omega' \in \Omega(\tau, \omega)}$ is finite and enough short since $G_{(\pm)}$ decays as $(\text{distance of two points})^{-1}$. Thus, the stochastic-Taylor expansion of a function $\text{Re} \{ f(\hat{x}(\tau', \omega')) \}$ at $\tau$ is employed:

\[
\text{Re} \left\{ f(\hat{x}(\tau', \omega')) \right\} = \sum_{m=0}^{\infty} \frac{(\tau' - \tau)^m}{m!} \text{Re} \left\{ \mathcal{D}_\tau^m f(\hat{x}(\tau, \omega')) \right\} + R(f) \tag{98}
\]

Where, $\mathcal{D}_\tau = \int \nu' \times \tau'\partial^\nu - (\mu \leftrightarrow \nu)$, $\mathcal{D}_\tau^0 = \text{identity}$, and $R(f)$ is its reminder,

\[
R(f) = \lambda \times \sum_{m=0}^{\infty} \int_{\tau}^{\tau'} d\tau_1 \int_{\tau}^{\tau_1} d\tau_2 \cdots \int_{\tau}^{\tau_{m-1}} d\tau_m \int_{\tau}^{\tau_m} d\tau_m^{-1} \left\{ \mathcal{D}_\tau^m f(\hat{x}(\tau_m, \omega')) \right\}. \tag{99}
\]

This is produced by the iteration of the formula $f(\hat{x}(\tau', \omega')) = f(\hat{x}(\tau, \omega')) + \int_{\tau}^{\tau'} df(\hat{x}(\sigma, \omega'))$, the integral of Eq. (98) see also Definition 37 in Appendix 3, i.e.,

\[
f(\hat{x}(\tau', \omega')) = \int_{\tau}^{\tau'} \mathcal{D}_\tau f(\hat{x}(\tau'', \omega')) d\tau'' \tag{100}
\]

and its derivative by

\[
\partial^\mu G_{(\pm)}(\hat{x}(\tau', \omega')) \bigg|_{x = \hat{x}(\tau, \omega') \text{ for } \omega' \in \Omega(\tau, \omega)} = -\frac{\text{Re} \{ \hat{x}(\tau', \omega') \} \cdot \Delta \hat{x}^\alpha(\tau', \omega')}{\Delta \hat{x}_\alpha(\tau', \omega')} \cdot \frac{d \Delta \hat{x}^\alpha(\tau', \omega')}{d\tau'} + O(R(x)) \tag{101}
\]

and its derivative by

\[
\partial^\mu G_{(\pm)}(\hat{x}(\tau', \omega')) \bigg|_{x = \hat{x}(\tau, \omega') \text{ for } \omega' \in \Omega(\tau, \omega)} = \frac{\Delta \hat{x}^\alpha(\tau', \omega') \cdot \frac{d \Delta \hat{x}_\alpha(\tau', \omega')}{d\tau'}}{\Delta \hat{x}^\alpha(\tau', \omega')} \times \frac{d \Delta \hat{x}_\alpha(\tau', \omega')}{d\tau'} G_{(\pm)}(\hat{x}(\tau', \omega')) + O(R(x)). \tag{102}
\]
The calculation of $\mathcal{F}(\hat{x}(\tau, \omega))$ requires us to formulate
\[ \partial^\mu G(\pm) (x, \hat{x}(\tau, \omega') \bigg| x = \hat{x}(\tau, \omega')) \bigg|_{x = \hat{x}(\tau, \omega')} \]
We evaluate it by
\begin{align*}
\partial^\mu G(\pm) (x, \hat{x}(\tau, \omega')) & = \partial^\mu G'(\pm) (x, \hat{x}(\tau, \omega')) |_{x = \hat{x}(\tau, \omega')} \\
& + O(\delta_x) \end{align*}
(103)
with
\begin{align*}
\Delta \hat{x}(\tau, \omega') - [\hat{x}(\tau, \omega') - \hat{x}(\tau, \omega')] & = O(R(x), \hat{x}(\tau, \omega) - \hat{x}(\tau, \omega')). 
\end{align*}
(104)
\[ O(R(x), \hat{x}(\tau, \omega) - \hat{x}(\tau, \omega')) \]
means the declaration that we only investigate the paths in $\Omega(\tau, \omega) \cap \{\omega' | \hat{x}(\tau, \omega') = \hat{x}(\tau, \omega)\}$ at $\tau$. The complex velocity $\text{Re} \{\nabla^\mu (\hat{x}(\tau, \omega'))\}$ is stochastic-Taylor expanded by Eq. (97), too. Now, the RR field $\mathfrak{g}$ is evaluated such that
\[ \mathfrak{g}(\hat{x}(\tau, \omega)) = \frac{\mathcal{F}(\hat{x}(\tau, \omega)) - \mathcal{F}(\hat{x}(\tau, \omega))}{2} \\
= O(R(x), R(V), \hat{x}(\tau, \omega) - \hat{x}(\tau, \omega')) \]
by following the similar calculation given by Ref. [11, 22, 33]. For $F_{ex}$ the external field(s) such that $\partial^\mu F_{ex} = 0$ as a laser field, quantum dynamics corresponding to Eqs. (4)
\[ \text{is hereby imposed w.r.t. } F = F_{ex} + \mathfrak{g} \text{ in Eqs. (33) by rounding } O(R(x), R(V), \hat{x}(\tau, \omega) - \hat{x}(\tau, \omega')) \]
into $\delta f$ the singularity of the radiation field:
\[ m_0 \mathcal{D} \nabla^\mu \hat{x}(\tau, \omega) = - \hat{x}(F_{ex}^\mu (\hat{x}(\tau, \omega))) V_{\nu}(\hat{x}(\tau, \omega)) - c \mathfrak{g}^\mu (\hat{x}(\tau, \omega)) V_{\nu}(\hat{x}(\tau, \omega)) \]
(105)
\[ \mathfrak{g}^\mu \hat{x}(\tau, \omega) = \frac{-m_0 \tau_0}{2 c^2} \int_{\Omega(\tau, \omega)} d \mathcal{P}(\omega') \]
\[ \left[ \hat{a}^\mu (\hat{x}(\tau, \omega')) \cdot \text{Re} \left\{ \nabla^\nu (\hat{x}(\tau, \omega')) \right\} \right] \\
- \hat{a}^\nu (\hat{x}(\tau, \omega')) \cdot \text{Re} \left\{ \nabla^\nu (\hat{x}(\tau, \omega')) \right\} \]
(106)
\[ \hat{a}^\mu (x) := \frac{c^4}{\text{Re} \left\{ \nabla_{\nu}(x) \cdot \text{Re} \left\{ \nabla_{\nu}(x) \right\} \right\}^2} \text{Re} \left\{ \mathcal{D}_{\nu}^2 \nabla_{\nu}(x) \right\} \]
\[ - \frac{27 c^6 \text{Re} \left\{ \nabla_{\nu}(x) \right\} \cdot \text{Re} \left\{ \mathcal{D}_{\nu} \nabla_{\nu}(x) \right\}^3 \text{Re} \left\{ \nabla_{\nu}(x) \right\} \}
\] (107)
Where, $\text{Re} \left\{ \nabla(\hat{x}(\tau, \omega)) \right\}$ doesn’t satisfies $v_{ex}^{\mu \nu} = c^2$ of a common rule in classical dynamics. The formulation of RR with $v_{ex}^{\mu \nu} \neq c^2$ is found in Ref. [34].
Consider that average trajectory. By restricting
\[ \mathcal{P}_{\tau}^\text{ave} := \mathcal{P}((\hat{x})_{\tau})|_{\hat{x}(\tau, \omega) = (\hat{x})_{\tau}}, \]
(108)
$\mathcal{P}_{\tau}^\text{ave}$ is the probability which a quanta stays at $\langle \hat{x} \rangle_{\tau} := \mathbb{E}[\hat{x}(\tau, \omega)]$. Then, the lowest order of Ehrenfest’s theorem (Eq. (11)) is below:
\begin{align*}
m_0 \frac{d^2 \langle \hat{x}(\tau) \rangle_{\tau}}{d \tau^2} & = - \left[ F_{ex}^{\mu \nu} \langle \hat{x}(\tau) \rangle_{\tau} + \mathfrak{g}^{\mu \nu} \langle \hat{x}(\tau) \rangle_{\tau} \right] \frac{d \langle \hat{x}(\tau) \rangle_{\tau}}{d \tau} \\
& + O(\delta^2 \langle \hat{x}(\tau) \rangle_{\tau}) \end{align*}
(109)
$\delta^\mu \langle \hat{x}(\tau) \rangle_{\tau} = - \frac{m_0 \tau_0}{c^2} \mathcal{P}(\Omega_{\tau}^\text{ave}) \\
\times \left[ \frac{d^3 \langle \hat{x}(\tau) \rangle_{\tau}}{d \tau^3} \right]_{\tau} \frac{d \langle \hat{x}(\tau) \rangle_{\tau}}{d \tau} - \frac{d^3 \langle \hat{x}(\tau) \rangle_{\tau}}{d \tau^3} \frac{d \langle \hat{x}(\tau) \rangle_{\tau}}{d \tau} \right] \end{align*}
(110)
A trajectory of $\langle \hat{x}(\tau) \rangle_{\tau}$ is drawn by Eqs. (109), (110). The non-relativistic limit of Eqs. (109), (110) is found in Ref. [35]. Where, the following simple relation is obtained:
\[ \mathfrak{g}(\hat{x}(\tau)) = \mathcal{P}(\Omega_{\tau}^\text{ave}) \times F_{\text{LAD}}(\langle \hat{x} \rangle_{\tau}) \]
(111)

**B. Classical limit**

What will happen with Eq. (26) and Eqs. (109), (110) in $h \to 0$? The randomness of Eq. (26) is neglected since
\[ \lim_{h \to 0} d_{\pm} \hat{x}(\tau, \omega) = \lim_{h \to 0} \mathcal{P}_{\tau}^\text{ave} \hat{\mathcal{P}}(\langle \hat{x}(\tau, \omega) \rangle_{\tau}) = 0, \]
then, its trajectory becomes a smooth and differentiable function. Hence, the all sample paths are identified as $\mathcal{P}(\langle \hat{x}(\tau) \rangle_{\tau})$.
Let us express the classical limit in this model by employing the following equivalence relation
\[ \omega = \omega' \Leftrightarrow \forall \tau, \lim_{h \to 0} \mathfrak{g}(\hat{x}(\tau, \omega) - \hat{x}(\tau, \omega')) = 0 \]
(112)
and its equivalency class
\[ [\omega] = \{ \omega' | \omega' \in \Omega(\tau, \omega) \} . \]
(113)
Then there is $\omega_0$ the representative of $[\omega], \{\hat{x}(\tau, \omega_0)\}_{\tau \in \mathbb{R}}$ is the smooth trajectory given in the classical limit. Namely, we understand it as all of $\omega' \in [\omega]$ converge to $\omega_0$ in $h \to 0$. Thus, the probability becomes the Dirac measure $\delta_{\omega_0}$:
\[ \lim_{h \to 0} \mathcal{P}(\omega) = \delta_{\omega_0}(\omega) \]
(114)
Where,$\int_{\Omega} f(\omega) d \delta_{\omega_0}(\omega) = f(\omega_0)$. Alternatively, we can express the same thing by Eq. (26) with $h \to 0$,
\[ \partial^\mu \mathcal{P}(x, \tau) + \partial_\tau \left[ \mathcal{P}(x, \tau) V_{\mu}^\nu (x) p(x, \tau) \right] = 0 \]
(115)
the equation of continuity with $V_{\mu}(x) = V_{\mu}(x)$. In this case, an initial profile $d^4(x - x_{\text{initial}})$ propagates by keeping its profile of the delta distribution. This is represented by $\delta_{\omega_0}(\omega)$ with labels of sample paths. Of course, $\langle \hat{x} \rangle_{\tau}$ should be included in the class of $[\omega]$. Therefore by Eq. (111),
\[ \lim_{h \to 0} \mathfrak{g}(\hat{x}(\tau)) = \lim_{h \to 0} \mathfrak{g}(\langle \hat{x} \rangle_{\tau}) = F_{\text{LAD}}(x(\tau)). \]
(116)
Thus, the LAD equation (113) is derived from Eqs. (109), (110) by its classical limit. Figure 6 shows the relation between each models which we discussed.
C. Landau-Lifshitz’s approximation

The instability (run-away) of Eqs. (105, 106, 107) is expected like one of the LAD equation [22] since it includes the high-order derivative of Re{\(\mathcal{D}_\tau V\alpha(x)\)} + Re{\(\mathcal{D}_\tau^2 V\alpha(x)\)} for avoiding this complexity. The Landau-Lifshitz (LL) approximation, the perturbation w.r.t. \(\tau_0\) is normally applied to Eq. (117) avoiding this complexity. The version for Eqs. (105, 106, 107) is realized by the following:

\[
\dot{\alpha}(x) = \frac{e^4 \times \text{Re}\{\hat{\alpha}(x)\}}{[\text{Re}\{\alpha(x)\} \cdot \text{Re}\{\gamma(x)\}]^2} + O(\tau_0) \quad (117)
\]

Where, Re{\(\mathcal{V}_\alpha(x)\)} \cdot Re{\(\mathcal{D}_\tau V\alpha(x)\)} = O(\tau_0) for \(\partial_\mu F^{\mu\nu}_{\text{ex}} = 0\). Ehrenfest’s theorem of Eqs. (105, 106) with Eqs. (117) becomes

\[
m_0 \frac{d^2 (\hat{\alpha}(x))}{d\tau^2} = -e F^{\mu\nu}_{\text{ex}}(\hat{\alpha}(x)) \frac{d(\hat{\alpha}(x))}{d\tau} + O((\hat{\alpha}^2 \hat{\alpha})_\tau) - e \mathcal{P}(\Omega^\alpha_{\text{ex}}) F^{\mu\nu}_{\text{LL}}(\hat{\alpha}(x)) \frac{d(\hat{\alpha}(x))}{d\tau} + O(\tau_0^2 \langle \hat{\alpha}^2 \hat{\alpha} \rangle_\tau) \quad (119)
\]

since Im{\(\mathcal{V}(\hat{\alpha}(x))\)} = O((\hat{\alpha}^2 \hat{\alpha})_\tau), and

\[
F^{\mu\nu}_{\text{LL}}(x) := \tau_0 \partial_\alpha F^{\mu\nu}_{\text{ex}}(x) \cdot \frac{dx_\alpha}{d\tau}(\tau) - \frac{e \tau_0}{m_0 \tau} \Omega^\alpha_{\text{ex}} (\hat{\alpha}(x)) F^{\beta\gamma}_{\text{ex}}(x) \frac{dx_\beta}{d\tau}(\tau) \frac{dx_\gamma}{d\tau}(\tau)
\]
D. Radiation formula

By Eqs.\(^{(109,110)}\) for \(\{\langle \hat{x} \rangle_\tau \}_{\tau \in \mathbb{R}}\), the radiation formula in the present model (the issue-(E)) is found:

\[
m_0 \frac{d^2 \langle \hat{x}_\mu \rangle_\tau}{d\tau^2} \approx -eF^\mu_{\tau \nu}(\langle \hat{x} \rangle_\tau) \frac{d\langle \hat{x}_\nu \rangle_\tau}{d\tau} - \frac{dW_{\text{stochastic}}}{dt} \frac{d\langle \hat{x}_\mu \rangle_\tau}{d\tau} \tag{121}
\]

\[
dW_{\text{stochastic}} := -m_0\tau_0 \mathcal{P}(\Omega^\text{ave}_\tau) \frac{d^2 \langle \hat{x}_\mu \rangle_\tau}{d\tau^2} \cdot \frac{d^2 \langle \hat{x}_\nu \rangle_\tau}{d\tau^2} \tag{122}
\]

It should be compared with Eq.\(^{(11)}\) and Larmor’s formula

\[
dW_{\text{classical}} = -m_0\tau_0 \frac{dv_\mu}{d\tau} \frac{dv_\nu}{d\tau}. \tag{123}
\]

When an external field is a plane wave, this has to converge to one by Ref.\(^{(11)}\),

\[
\mathcal{P}(\Omega^\text{ave}_\tau) = q_{\text{QED}}(\chi)
\]

\[
= \frac{9\sqrt{3}}{8\pi} \int_0^{\chi^{-1}} dr \int_{r\tau_\gamma}^\infty dr' K_{5/3}(r'). \tag{124}
\]

For \(\mathcal{P}(\Omega^\text{ave}_\tau) = q_{\text{QED}}(\chi) + \delta\mathcal{P}(\Omega^\text{ave}_\tau)\) in the case of non-plane wave fields, the radiation spectrum becomes

\[
d^2W_{\text{stochastic}} = \frac{d q_{\text{QED}}(\chi)}{d(\hbar\omega)} + \frac{d\delta\mathcal{P}(\Omega^\text{ave}_\tau)}{d(\hbar\omega)} \frac{dW_{\text{classical}}}{dt}, \tag{125}
\]

thus, the observation of \(d\delta\mathcal{P}(\Omega^\text{ave}_\tau)/d(\hbar\omega)\) provides us an unknown correction in non-linear QED. By solving Eq.\(^{(27)}\), \(\delta\mathcal{P}(\Omega^\text{ave}_\tau)\) is found even in the case of general field profiles like laser focusing and superposition.

VII. SUMMARY

The topics discussed in this paper are illustrated by Fig.\(^{7}\). We derived the set of Eqs.\(^{(109,110)}\), which is the quantized equation of the LAD equation \([44]\) via the construction of Issue-(A) Eq.\(^{(26)}\) of the relativistic kinematics with Eq.\(^{(53)}\) of the dynamics for a Brownian scalar electron proposed in Sect.\(^{\text{III}}\) and Issue-(B) the Maxwell equation \([69]\) in Sect.\(^{\text{IV}}\). Issue-(C) of the action integral was introduced by Eq.\(^{(78)}\) in Sect.\(^{\text{V}}\). The consistent system w.r.t. Issues-(A-C) was first proposed by the present article. Hence, the description of RR by solving Eq.\(^{(69)}\) is that first example of stochastic quantization. Again, Issue-(D) the stochastic quantization of the LAD equation was given by Eqs.\(^{(105,106)}\) in Sect.\(^{\text{VI}}\). We also confirmed its classical limit becomes the LAD equation. The LL approximation was introduced by Eqs.\(^{(105,106)}\). The readers can understand the fact that we did not employ any restriction of the external laser fields except the Lorenz gauge in this article. We obtained Issue-(E) the radiation formula of Eq.\(^{(122)}\) in Sect.\(^{\text{VI}}\) by Ehrenfest’s theorem of Eqs.\(^{(109,110)}\). By the comparison between Eq.\(^{(1)}\) and Eq.\(^{(122)}\), we found \(q(\chi)\) is included in the existence probability \(\mathcal{P}(\Omega^\text{ave}_\tau)\) which a Brownian quanta stays at its average position \(\langle \hat{x} \rangle_\tau\). Hence, the observation of quantumness \(q(\chi)\) indicates the existence probability of a radiating charged quanta. The calculation of Eq.\(^{(27)}\) is the requirement to derive \(\mathcal{P}(\Omega^\text{ave}_\tau)\). Now, a trajectory of a radiating scalar electron can be drawn by a stochastic process. The precise analysis of \(\delta f\) the field singularity as the Coulomb field in Eq.\(^{(69)}\) should be performed. The existence of \(d\delta\mathcal{P}(\Omega^\text{ave}_\tau)/d(\hbar\omega)\) suggests a new correction beyond the Furry picture which may be found in high-intensity laser experiments.

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Appendix A: Mathematical supports

Since we have not discussed the detail of stochastic processes, let us see the precise construction from the 1D WP to the 4D kinematics in this appendix. Thus, our purpose in this Appendix A is to give Eq. (22), Eq. (23) and Eq. (35) mathematically.

a. Mathematical spaces

At first, we define the Minkowski spacetime and a probability space as measure spaces. Where, $\mathcal{B}(I)$ denotes a Borel $\sigma$-algebra of a topological space $I$.

Definition 1 (Minkowski spacetime). Let $\mathbb{A}^4(\mathbb{V}_M^4, g)$ be a 4D metric affine space w.r.t. a 4D standard vector space $\mathbb{V}_M^4$ and its metric $g$ on $\mathbb{V}_M^4$. By defining a measure space

$$(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu),$$

we regard this as the Minkowski spacetime when $g = (+, -, -, -)$. The measure $\mu : \mathbb{A}^4(\mathbb{V}_M^4, g) \to \mathbb{R}$ is defined as $d\mu(x) = dx^4$.

A metric affine space is an abstract mathematical space without its origin and its coordinate. By the coordinate mapping $\varphi(x) = (x^0, x^1, x^2, x^3)$ for all $x \in \mathbb{V}_M^4$ and $x \in \mathbb{A}^4(\mathbb{V}_M^4, g)$ with its origin, the readers can consider that $\mathbb{A}^4(\mathbb{V}_M^4, g) \cong \mathbb{R}^4$ with the metric $g$.

Definition 2 (Probability space). For a certain abstract non-empty set $\Omega$, $\mathcal{F}$ a $\sigma$-algebra of $\Omega$, and $\mathcal{P}$ a probability measure on the measurable space $(\Omega, \mathcal{F})$ such that $\mathcal{P}(\Omega) = 1$,

$$(\Omega, \mathcal{F}, \mathcal{P})$$

is a probability space. Especially, we use this for 4D stochastic processes, $(\Omega^{\text{1D}}, \mathcal{F}^{\text{1D}}, \vartheta^{\text{1D}})$ for 1D stochastic processes.

Then, a 4D continuous stochastic process

$$\hat{x} (\cdot, \cdot) := \{ \hat{x}(\tau, \omega) \}_{(\tau, \omega) \in \mathbb{R} \times \Omega}$$

is regarded as a $\mathcal{B}(\mathbb{R}) \times \mathcal{F}/\mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g))$-measurable mapping. Where for two measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$, an $\mathcal{X}/\mathcal{Y}$-measurable mapping $f$ is a mapping $f : X \to Y$ such that $f^{-1}(A) := \{ x \in X | f(x) \in A \} \subset \mathcal{X}$ for all $A \in \mathcal{Y}$.

b. Increasing family

Let us consider an usual 1D Wiener process

$$w_+(\cdot, \cdot) := \{ w_+(t, \omega) \in \mathbb{R} | t \in [0, \infty), \omega \in \Omega^{\text{1D}} \}$$
on $(\Omega^{\text{1D}}, \mathcal{F}^{\text{1D}}, \vartheta^{\text{1D}})$:

Definition 3 (Wiener process $w_+(\cdot, \cdot)$). When a 1D stochastic process $w_+(\cdot, \cdot)$ satisfies below, it is a 1D Wiener process (WP) or a 1D Brownian motion.

1. $w_+(0, \omega) = 0$ a.s.,
2. $t \mapsto w_+(t, \omega)$ is continuous,
3. For all times $0 = t_0 < t_1 < \cdots < t_n (n \in \mathbb{Z})$, the increments $\{ w_+(t_i, \omega) - w_+(t_{i-1}, \omega) \}_{i=1}^n$ are independent and each of them follows the normal distribution $\mathcal{N}(0, t_i - t_{i-1})$.

Instead of the above rule (1), we can choose $w_+(0, \omega) = x$ a.s., too. For $\mathbb{E}[\int_{t \in \Omega^{\text{1D}}} f(w_+(t, \omega)) d\vartheta^{\text{1D}}(\omega)]$, the following basic result is found:

Lemma 4. For $(\Omega^{\text{1D}}, \mathcal{F}^{\text{1D}}, \vartheta^{\text{1D}})$, a 1D WP $w_+(\cdot, \cdot)$ satisfies below for all $t$:

$$\mathbb{E}[w_+(t + \delta t, \cdot) - w_+(t, \cdot)] = 0 \quad (A1)$$

$$\lim_{\delta t \to 0^+} \mathbb{E} \left[ \frac{|w_+(t + \delta t, \cdot) - w_+(t, \cdot)|^2}{\delta t} \right] = 1 \quad (A2)$$

For the discussion of stochastic processes, the definition of an increasing family (a filtration) is important.

Definition 5 (Increasing family $\{ \mathcal{P}_t \}$). For a probability space $(\Omega^{\text{1D}}, \mathcal{F}^{\text{1D}}, \vartheta^{\text{1D}})$, $\{ \mathcal{P}_t \}_{t \in \mathbb{R}}$ is an increasing family of sub-$\sigma$-algebras on $\Omega$ such that $-\infty < s \leq t \Rightarrow \mathcal{P}_s \subset \mathcal{P}_t \subset \mathcal{F}^{\text{1D}}$.

$\mathcal{P}_s \subset \mathcal{P}_{t \geq s}$ denotes an increment of branches of sample paths. This is the characteristics of a forward (normal) diffusion process.

Definition 6 ($\{ \mathcal{P}_t \}$-adapted). When a stochastic process is $\mathcal{P}_t/\mathcal{F}(X)$-measurable for each $t \in \mathbb{R}$ in the present case, we call it a $\{ \mathcal{P}_t \}$-adapted.

Definition 7 ($\{ \mathcal{P}_t \}$-WP). When $w_+(\cdot, \cdot)$ satisfies the following, it is a $\{ \mathcal{P}_t \}$-WP.

1. $w_+(\cdot, \cdot)$ is $\{ \mathcal{P}_t \}$-adapted.
2. $w_+(t, \omega) - w_+(s, \omega)$ and $\mathcal{P}_s$ are independent for $0 \leq s \leq t$.

For $(\Omega, \mathcal{F}, \mathcal{P})$ and $T \subset \mathbb{R}$, consider $\mathbb{L}^p(T)$ a family of $\mathcal{B}(T) \times \mathcal{F}/\mathcal{B}(X)$-measurable mappings $(X$ is an $N$-dimensional topological space) such that

$$\mathbb{L}^p_T(X) := \left\{ \hat{f}(\cdot, \cdot) : \Omega^{\text{1D}} \to X, \quad t \mapsto f(t, \omega) \text{ is continuous}, \quad \sum_{i=1}^N \int_T |\varphi^i \circ f(t, \omega)|^p dt < \infty \text{ a.s.} \right\}$$
with \( \{ \varphi^i \}_{i=1}^N \) the coordinate mapping \( \varphi^i : X \to \mathbb{R} \). Then, its “adapted” class is

\[
\mathcal{L}^p_{loc}(\{P_t\}; X) := \left\{ \hat{f}(\omega, \cdot) \left| \forall t_1 \leq t_2 \in \mathbb{R}, \hat{f}(\omega, \cdot) \in L^p_{[t_1, t_2]}(X) \right. \right\}
\]

Definition 8 (\( \{P_t\} \)-prog.). If a stochastic process \( \hat{f}(\omega, \cdot) \in \mathcal{B}(s, t) \times P_t / \mathcal{B}(X) \)-measurable for each \( t \in \mathbb{R} \) and \( s \leq t \), \( \hat{f}(\omega, \cdot) \) is called \( \{P_t\} \)-progressively measurable, \( \{P_t\} \)-progressive or \( \{P_t\} \)-prog.

Theorem 9. A stochastic process \( \hat{f}(\omega, \cdot) \) is \( \{P_t\} \)-prog. when \( \hat{f}(\omega, \cdot) \) is continuous and \( \{P_t\} \)-adapted. Its converse is also satisfied.

Proof. Consider the domain of \([s, t]\) for each \( s \leq t \). If \( \hat{f}(\omega, \cdot) : \mathbb{R} \times \Omega \to \mathbb{R} \) is continuous and \( \{P_t\} \)-adapted, it is described by a simple function such as

\[
\hat{f}(\tau, \omega) = \xi_0(\omega) \times 1_{\{t_0\}}(\tau) + \sum_{j=1}^{n} \xi_j(\omega) \times 1_{\{t_{j-1}, t_j\}}(\tau)
\]

for \( s = t_0 \leq t_1 \leq \cdots \leq t_n = t \) and each of \( \{P_{t_{m=0, \ldots, n}}\} \)-adapted \( \xi_m(\cdot) \),

\[
\{(\tau, \omega) \in [s, t] \times \Omega | f(\tau, \omega) \leq a\} = \{t_0\} \times \{\omega \in \Omega | \xi_0(\omega) \leq a\} \cup \bigcup_{j=1}^{n} [t_{j-1}, t_j] \times \{\omega \in \Omega | \xi_j(\omega) \leq a\}
\]

with \( n \to \infty \). Where,

\[
1_A(\tau) = \begin{cases} 1 & \tau \in A \\ 0 & \tau \notin A \end{cases}
\]

with \( \forall A \subset \mathbb{R} \). Thus, \( \hat{f}(\omega, \cdot) \in \mathcal{B}(s, t) \times P_t / \mathcal{B}(X) \)-measurable, namely, it is a \( \{P_t\} \)-prog.. When \( \hat{f}(\omega, \cdot) \) is \( \{P_t\} \)-prog., it is \( \{P_t\} \)-adapted since \( f(\tau, \omega) \) is \( \{P_t\} \)-adapted.

Definition 10 (\( \{P_t\} \)-martingale). A stochastic process \( m(\omega, \cdot) \) is a \( \{P_t\} \)-martingale when \( m(\omega, \cdot) \) satisfies below:

1. \( m(t, \cdot) \) is integrable, i.e., \( \mathbb{E}[m(t, \cdot)] < \infty \),
2. \( m(\cdot, \cdot) \) is \( \{P_t\} \)-adapted,
3. For \(- \infty < s \leq t \), \( \mathbb{E}[m(t, \cdot) | P_s] = \hat{m}(s, \cdot) \) a.s.

Therefore, a \( \{P_t\} \)-WP \( w_+(\omega, \cdot) \) is a \( \{P_t\} \)-martingale.

Then, \( \int_t^s f(t', \omega)dw_+(t', \omega) \) is defined by means of an Itô integral of \( f \in \mathcal{L}^2_{loc}(\{P_t\}; \mathbb{R}) \). Since a \( \{P_t\} \)-prog. is expressed by a simple function

\[
\hat{f}(t, \omega) = \xi_0(\omega) \times 1_{\{t_0\}}(t) + \sum_{j=1}^{n} \xi_j(\omega) \times 1_{\{t_{j-1}, t_j\}}(t)
\]

for \( s = t_0 \leq t_1 \leq \cdots \leq t_n = T \), let us define

\[
S^+(s, t, \omega) := \sum_{j=0}^{n-1} \xi_j(\omega) \times \left[ w_+(t \land t_{j+1}, \omega) - w_+(t \land t_j, \omega) \right]
\]

with \( s \land t := \max\{s, t\} \). If there is \( S^+(s, t, \omega) \) such that

\[
\lim_{n \to \infty} \mathbb{E} \left[ \left[ S^+(s, t, \omega) - S^+(s, t, \cdot) \right]^2 \right] = 0,
\]

this \( S(s, t, \omega) \) is regarded as \( \int_s^t f(t', \omega)dw_+(t', \omega) \). Therefore, \( \int_s^t f(t', \omega)dw_+(t', \omega) \) is a \( \{P_t\} \)-martingale. Then, the following is found easily.

Lemma 11. For \( f(\omega, \cdot), g(\omega, \cdot) \in \mathcal{L}^2_{loc}(\{P_t\}; \mathbb{R}) \), the following relation is imposed \([23]\):

\[
\mathbb{E} \left[ \int_0^T f(t', \omega)dw_+(t', \cdot) \right] = \mathbb{E} \left[ \int_0^T f(t', \cdot) \cdot g(t', \cdot)dt' \right]
\]

Finally, the following well-known theorem of the so-called Itô formula is found:

Theorem 12 (\( \{P_t\} \)-Itô formula). For \( a(\omega, \cdot) \in \mathcal{L}^1_{loc}(\{P_t\}; \mathbb{R}) \) and \( b(\omega, \cdot) \in \mathcal{L}^2_{loc}(\{P_t\}; \mathbb{R}) \), consider a 1D stochastic process \( \hat{X}(t, \omega) \) of a \( \{P_t\} \)-prog. given by

\[
\hat{X}(t, \omega) = \hat{X}(0, \omega) + \int_0^t a(t', \omega)dt' + \int_0^t b(t', \omega)dw_+(t', \omega)
\]

w.r.t. its initial condition \( \hat{X}(0, \omega) \). Then for a function \( f \in C^2(\mathbb{R}^N) \), the Itô formula

\[
f(\hat{X}(t, \omega)) = f(\hat{X}(0, \omega)) + \int_0^t f'(\hat{X}(t', \omega))a(t', \omega)dt' + \int_0^t f''(\hat{X}(t', \omega))b(t', \omega)dw_+(t', \omega)
\]

is almost surely satisfied \([23]\). By introducing

\[
\int_0^t d_+ \hat{X}(t', \omega) := \hat{X}(t, \omega) - \hat{X}(0, \omega),
\]

\[
\int_0^t d_+ f(\hat{X}(t', \omega)) := f(\hat{X}(t, \omega)) - f(\hat{X}(0, \omega)),
\]

Eq. (A7) is also expressed as

\[
d_+ f(\hat{X}(t, \omega)) = f'(\hat{X}(t', \omega))d_+ \hat{X}(t, \omega) + \frac{1}{2} f''(\hat{X}(t', \omega)) [d_+ \hat{X}(t, \omega)]^2 \text{ a.s. (A8)}
\]
c. Decreasing family

We also introduced a backward diffusion. This is called as a decreasing family.

**Definition 13** (Decreasing family $\{F_t\}$). For a probability space $(Ω^{1\text{-dim}}, F^{1\text{-dim}}, P^{1\text{-dim}})$, $\{F_t\}_{t \in \mathbb{R}}$ is a decreasing family of sub-σ-algebras on $Ω$ such that $t \leq s < \infty \implies F^1 \supset F_t \supset F_s$.

This is regarded as an inverse process of a $\{P_t\}$-prog. Let us suggest its WP.

**Definition 14** ($\{F_t\}$-WP). Let us define a monotonically decreasing function $f_\downarrow : \mathbb{R} \to \mathbb{R}$ such that $df_\downarrow/dt = -1$. $w_\downarrow(\cdot, \omega)$ is an $\{F_t\}$-WP when $\{w_\downarrow(f_\downarrow(t), \omega)\}_{t \in \mathbb{R}}$ is a $\{P_t\}$-WP.

**Definition 15** ($\{F_t\}$-martingale). A stochastic process $m(\cdot, \omega)$ is a $\{F_t\}$-martingale when $m(\cdot, \omega)$ satisfies below:

1. $m(t, \omega)$ is integrable, i.e., $\mathbb{E}[m(t, \omega)] < \infty$,
2. $m(\cdot, \omega)$ is $\{F_t\}$-adapted,
3. For $t \leq s < \infty$, $\mathbb{E}[m(t, \omega)|F_s] = \hat{m}(s, \omega)$ a.s.

In order to the above definition, an $\{F_t\}$-WP is an $\{F_t\}$-martingale. That is confirmed by the following lemma.

**Lemma 16.** $w_\downarrow(\cdot, \omega)$ of an $\{F_t\}$-WP satisfies below:

\[
\mathbb{E}[w_\downarrow(t, \omega) - w_\downarrow(t - \delta t, \omega)] = 0 \quad (A9)
\]

\[
\lim_{\delta t \to 0^+} \mathbb{E} \left[ \frac{(w_\downarrow(t, \omega) - w_\downarrow(t - \delta t, \omega))^2}{\delta t} \right] = 1 \quad (A10)
\]

Then, the following function family is introduced:

\[
L^p_{\text{loc}}(\{F_t\}; X) := \left\{ f(\cdot, \omega) : \forall t_1 \leq t_2 \in \mathbb{R}, f(\cdot, \omega) \in L^p_{[t_1, t_2]}(X) \right\}
\]

\[
f(\cdot, \omega) \text{ is } \{F_t\}\text{-adapted.}
\]

**Definition 17** ($\{F_t\}$-prog.). If a stochastic process $f(\omega, \cdot) \in \mathcal{B}([s, t]) \times \mathcal{F}_t/\mathcal{B}(X)$-measurable for all $s \leq t \in \mathbb{R}$, let us call that $f(\cdot, \omega)$ is $\{F_t\}$-prog.

For an $\{F_t\}$-prog, $\hat{f}(\cdot, \omega) = \mathcal{B}([s, t]) \times \mathcal{F}_t/\mathcal{B}(X)$-measurable all $s \leq t \in \mathbb{R}$, let us introduce the following summand below:

\[
S_n^-(s, t, \omega) := \sum_{j=0}^{n-1} \xi_{j+1}(\omega) \times \left[ w_\downarrow(t \land t_{j+1}, \omega) - w_\downarrow(t \land t_j, \omega) \right].
\]

When there is $S^-(s, t, \omega)$ such that

\[
\lim_{n \to \infty} \mathbb{E} \left[ [S^+_n(s, t, \omega) - S^-(s, t, \omega)]^2 \right] = 0,
\]

this $S^-(s, t, \omega)$ is expressed by $\int_t^s f(t', \omega)dw_\downarrow(t', \omega)$.

**Theorem 18** ($\{F_t\}$-Itô formula). For $A(\omega, \cdot) \in L^1_{\text{loc}}(\{F_t\}; \mathbb{R})$ and $B(\omega, \cdot) \in L^2_{\text{loc}}(\{F_t\}; \mathbb{R})$, let $\hat{X}(\cdot, \omega)$ of an $\{F_t\}$-prog. be given by

\[
d_\downarrow \hat{X}(t, \omega) = A(t, \omega)dt + B(t, \omega)dw_\downarrow(t, \omega)
\]

with

\[
\int_a^b d_\downarrow \hat{X}(t, \omega) := \hat{X}(b, \omega) - \hat{X}(a, \omega).
\]

Then by $\int_a^b d_\downarrow f(\hat{X}(t', \omega)) := f(\hat{X}(b, \omega)) - f(\hat{X}(a, \omega))$ for $f \in C^2(\mathbb{R}^N)$, its Itô formula becomes

\[
d_\downarrow f(\hat{X}(t, \omega)) = f'(\hat{X}(t, \omega))d_\downarrow \hat{X}(t, \omega)
\]

\[
- \frac{1}{2} [f''(\hat{X}(t, \omega))][d_\downarrow \hat{X}(t, \omega}]^2
\]

a.s.

Consider the derivation of (A15) by using (A8). The decreasing family $\{F_t\}_{t \in \mathbb{R}}$ relates to an increasing family $\{P_t\}_{t \in \mathbb{R}}$ by a monotonically decreasing function $f_\downarrow : \mathbb{R} \to \mathbb{R}$ such that $df_\downarrow/dt = -1$. $(\mathcal{P}_{f_\downarrow(t)}(\cdot))_{t \in \mathbb{R}}$ becomes an increasing family since $t \leq s \Rightarrow \mathcal{P}_{f_\downarrow(t)}(\cdot) \supset \mathcal{P}_{f_\downarrow(s)}(\cdot)$. Thus, there is $f_\downarrow$ such that $\{\mathcal{P}_{f_\downarrow(t')}\}_{t' \in \mathbb{R}} \equiv \{F_t\}_{t \in \mathbb{R}}$. For $\hat{X}(\cdot, \omega)$ of an $\{F_t\}$-prog., there is $\hat{X}'(\cdot, \omega)$ of a $\{P_t\}$-prog. satisfying $\hat{X}'(t', \omega') = \hat{X}(t, \omega)$ with $t' := f_\downarrow(t) = t$ at a fixed $t$. For $\forall \alpha \geq 0$, let us set $f_\downarrow(t - a) = t + a$. Since $\int_{f_\downarrow(t + a)} f_\downarrow(\hat{X}(t', \omega')) = \hat{X}'(f_\downarrow(t - a), \omega') = \hat{X}(f_\downarrow(t + a), \omega') = \hat{X}(t - a, \omega) - \hat{X}(t + a, \omega) = - \int_{f_\downarrow(t + a)} f_\downarrow(\hat{X}'(t', \omega'))$ is derived. $d_\downarrow f(\hat{X}'(t', \omega')) = - \int_{f_\downarrow(t + a)} d_\downarrow(\hat{X}'(t', \omega'))$ is also imposed for $\forall f \in C^2(\mathbb{R}^N)$. Let us apply those relations to (A8), namely,

\[
d_\downarrow f(\hat{X}'(t', \omega')) = f'(\hat{X}'(t', \omega'))d_\downarrow \hat{X}'(t', \omega')
\]

\[
+ \frac{1}{2} [f''(\hat{X}'(t', \omega'))][d_\downarrow \hat{X}'(t', \omega')]^2
\]

a.s.

Finally, Eq. (A15) is found by the replacement from $\hat{X}'(\cdot, \omega')$ to $X(\cdot, \omega)$. Let $\omega' = \omega$ when $\{X'(t', \omega')\}_{t' \in \mathbb{R}} \equiv \{X(t, \omega)\}_{t \in \mathbb{R}}$. In this case, $X(\cdot, \omega)$ is $\{P_t\}$ and $\{F_t\}$-prog.

d. Forward-backward composition

Let us introduce the simplest example of a composition of stochastic processes for the relativistic kinematics. Consider the set of 1D stochastic processes, $\hat{X}(\cdot, \omega)$ and $\hat{Y}(\cdot, \omega)$ of a $\{P_t\}$-prog. and an $\{F_t\}$-prog., respectively:
\begin{align}
  d\dot{X}(t,\omega) &= a_+(\dot{X}(t,\omega))dt + \theta \times dw_+(t,\omega) \\
  d\dot{Y}(t,\omega) &= a_-\dot{Y}(t,\omega'))dt + \theta \times dw_-(t,\omega') 
\end{align}

(A17)

By a 2D vector \( \hat{\gamma}(t,\omega) := (\hat{X}(t,\omega), \hat{Y}(t,\omega)) \), Eq. (A17) becomes
\begin{align}
  \hat{X}(b,\omega) - \hat{X}(a,\omega) &= \int_a^b \left[ a_+(\hat{X}(t,\omega), \dot{Y}(t,\omega)) \right] dt \\
  \hat{Y}(b,\omega) - \hat{Y}(a,\omega) &= \int_a^b \left[ a_-\dot{Y}(t,\omega), \dot{Y}(t,\omega) \right] dt + \theta \times \int_a^b dw_+(t,\omega) \\
  \hat{Y}(b,\omega) - \hat{Y}(a,\omega) &= \int_a^b \left[ a_-\dot{Y}(t,\omega), \dot{Y}(t,\omega) \right] dt \tag{A18}
\end{align}

with \( a_+, a_- : \mathbb{R}^2 \rightarrow \mathbb{R} \). This is forward-diffused in \( \hat{X} \)-direction and backward-diffused in \( \hat{Y} \)-direction. Consider \((\mathcal{P}_t, \mathcal{F}_t)\) the set of the sub-\(\sigma\)-algebras for each \( t \). This is regarded as a new sub-\(\sigma\)-algebra of a 2D stochastic process denoted by \( \mathcal{M}_t := \mathcal{P}_t \otimes \mathcal{F}_t \) and its family \( \{ \mathcal{M}_t \}_{t \in \mathbb{R}} \).

**Definition 19** (Forward-backward composition). Let \( \{ \mathcal{M}_t \}_{t \in \mathbb{R}} \) be a family of sub-\(\sigma\)-algebras for a 2D stochastic process \( \hat{\gamma}(t,\omega) := (\hat{X}(t,\omega), \hat{Y}(t,\omega)) \) on \((\Omega, \mathcal{F}, \mathcal{F})\). Where, an \( \mathcal{M}_t/\mathcal{B}(\mathbb{R}^2) \)-measurable \( \hat{\gamma}(t,\omega) \) is \( \mathcal{P}_t \otimes \mathcal{F}_t/\mathcal{B}(\mathbb{R}^2) \)-measurable when \( \hat{X}(t,\omega) \) is \( \mathcal{P}_t/\mathcal{B}(\mathbb{R}) \)-measurable and \( \hat{Y}(t,\omega) \) is \( \mathcal{F}_t/\mathcal{B}(\mathbb{R}) \)-measurable. Hence, \( \mathcal{M}_t := \mathcal{P}_t \otimes \mathcal{F}_t \). When \( \hat{\gamma}(t,\omega) \) is \( \mathcal{P}_t \otimes \mathcal{F}_t \)-adapted for all \( t \), \( \hat{\gamma}(t,\omega) \) is \( \mathcal{P}_t \otimes \mathcal{F}_t \)-adapted for all \( s \leq t \), it is \( \{ \mathcal{P}_t \otimes \mathcal{F}_t \} \)-prograding.

For \( \gamma(o,\cdot) := (a_+(\gamma(o,\cdot), a_-\gamma(o,\cdot))) \) of a \( \{ \mathcal{P}_t \otimes \mathcal{F}_t \} \)-prograding, let us confirm the Itô formula.

\[
  f(\hat{\gamma}(b,\omega)) - f(\hat{\gamma}(a,\omega)) = \int_a^b \frac{\partial f}{\partial x}(\gamma(t,\omega))d_+\hat{X}(\gamma(t,\omega)) \\
  + \int_a^b \frac{\partial f}{\partial y}(\gamma(t,\omega))d_-\hat{Y}(\gamma(t,\omega)) \\
  + \frac{\partial^2 f}{\partial x \partial y}(\gamma(t,\omega))dt \\
  - \frac{\partial f}{\partial x}(\gamma(t,\omega))d_+\hat{X}(\gamma(t,\omega))dt \ a.s. \tag{A19}
\]

The appearance of \( \int_a^b (\partial^2_+ - \partial^2_-)f(\gamma(t,\omega))dt \) is useful to make the relativistic kinematics. The readers can understand the above formula by the following:

\[
  f(\hat{X}(t+\delta t,\omega), \hat{Y}(t+\delta t,\omega)) - f(\hat{X}(t,\omega), \hat{Y}(t,\omega)) \\
  = \begin{bmatrix}
    f(\hat{X}(t+\delta t,\omega), \hat{Y}(t+\delta t,\omega)) \\
    -f(\hat{X}(t,\omega), \hat{Y}(t+\delta t,\omega)) \\
    f(\hat{X}(t,\omega), \hat{Y}(t+\delta t,\omega)) \\
    -f(\hat{X}(t,\omega), \hat{Y}(t,\omega)) \\
  \end{bmatrix} \\
  \hat{X}(t,\omega) \text{ is fixed}
\]

Consider the following types of 4D processes on the Minkowski spacetime \((\mathbb{A}^4(V^4_{M1}, g), \mathcal{B}(\mathbb{A}^4(V^4_{M1}, g)), \mu)\).

- \( \varphi \circ \hat{\gamma}(t,\omega) := \hat{\gamma}(t,\omega), \hat{\gamma}(t,\omega), \hat{\gamma}(t,\omega), \hat{\gamma}(t,\omega) \)
  \( \{ \mathcal{P}_t \} \)-prograding
  \( \{ \mathcal{F}_t \} \)-prograding
  \( \{ \mathcal{P}_t \} \)-prograding

**Definition 20** (\( \{ \mathcal{P}_t \} \)-WPs and \( \{ \mathcal{F}_t \} \)-WPs). Let \( W_+(\cdot, \cdot) \) and \( W_-(\cdot, \cdot) \) be \( \{ \mathcal{P}_t \} \) and \( \{ \mathcal{F}_t \} \)-WPs. Let \( W_+\circ \hat{\gamma}(t,\omega) \) and \( W_-\circ \hat{\gamma}(t,\omega) \) be the 4D stochastic processes:

- \( \varphi \circ W_+(\tau,\omega) := \left( W_+^1(\tau,\omega), W_+^2(\tau,\omega), W_+^3(\tau,\omega) \right) \)
  \( \{ \mathcal{F}_t \} \)-WP
  \( \{ \mathcal{P}_t \} \)-WP

**Definition 21** (\( \{ \mathcal{P}_t \} \)-WPs and \( \{ \mathcal{F}_t \} \)-WPs). Let \( W_+\circ \hat{\gamma}(t,\omega) \) and \( W_-\circ \hat{\gamma}(t,\omega) \) satisfy below with \( W_\pm^\mu(\tau,\omega) := \varphi \circ W_\pm^\mu(\tau,\omega) \) for each \( \mu, \nu = 0, 1, 2, 3 \):

\[
  \mathbb{E} \left[ \int_{\tau}^{r+\delta \tau} dW_\pm^\mu(\tau',\cdot) \right] = 0 \tag{A20}
\]

\[
  \mathbb{E} \left[ \int_{\tau}^{r+\delta \tau} dW_\mp^\mu(\tau',\cdot) \times \int_{\tau}^{r+\delta \tau} dW_\pm^\nu(\tau'',\cdot) \right] = \delta^{\mu\nu} \times \delta \tau \tag{A21}
\]

**Theorem 23**. The Itô formula of a \( C^2 \)-function \( f : \mathbb{A}^4(V^4_{M1}, g) \rightarrow C \) w.r.t. \( W_\pm(\cdot, \cdot) \) is

\[
  f(W_\pm(\tau,\omega)) = f(W_\pm(\tau,\omega)) \\
  + \int_{\tau}^{\tau_\nu} \partial_\mu f(W_\pm(\tau,\omega))dW_\pm^\mu(\tau,\omega) \\
  + \frac{\lambda^2}{2} \int_{\tau}^{\tau_\nu} \partial_\mu \partial_\nu f(W_\pm(\tau,\omega))d\tau \ a.s. \tag{A22}
\]
Where, we emphasize that \( \partial_{\mu} \partial^{\mu} f(W_{3}(\tau, \omega)) \) appears in the Itô formula.

\[ f. \quad \text{D-prog. } \dot{x}(\omega, \cdot) \]

Then the stochastic differential equation
\[
d_{\tau} \dot{x}(\tau, \omega) = \nu_{+}(\dot{x}(\tau, \omega)) \, d\tau + \lambda \times dW_{\pm}(\tau, \omega)
\]
is defined as a relativistic kinematics of a stochastic scalar electron. Let us follow the construction by Nelson [24], however, it has to be on \((\mathcal{A}^{4}(V_{M}, g), \mathcal{B}(\mathcal{A}^{4}(V_{M}, g)), \mu)\).

**Definition 24 ((R0)-process).** For \((\Omega, \mathcal{F}, \mathcal{P})\), a \(\mathcal{B}(\mathbb{R}) \times \mathcal{F}/\mathcal{B}(\mathcal{A}^{4}(V_{M}, g))\)-measurable mappings for a topological space \(E\), let us introduce \(L_{\mathcal{P}}^{p}(\mathcal{P}_{T}; E)\) and \(L_{\mathcal{P}}^{p}(\mathcal{F}_{T}; E)\):

\[
L_{\mathcal{P}}^{p}(\mathcal{P}_{T}; E) := \left\{ \dot{x}(\omega, \cdot) \mid 0 \leq \tau_{1} \leq \tau_{2} \leq \tau \leq \tau_{3} \leq \tau_{4}, \dot{x}(\omega, \cdot) \in L_{\mathcal{P}}^{p}(\tau_{1}, \tau_{2}; E), \dot{x}(\omega, \cdot) \in \mathcal{P}_{T}\text{-adapted.} \right\}
\]

\[
L_{\mathcal{P}}^{p}(\mathcal{F}_{T}; E) := \left\{ \dot{x}(\omega, \cdot) \mid 0 \leq \tau_{1} \leq \tau_{2} \leq \tau \leq \tau_{3} \leq \tau_{4}, \dot{x}(\omega, \cdot) \in L_{\mathcal{P}}^{p}(\tau_{1}, \tau_{2}; E), \dot{x}(\omega, \cdot) \in \mathcal{F}_{T}\text{-adapted.} \right\}
\]

**Definition 25 ({\mathcal{P}_{T}})-prog. and {\mathcal{F}_{T}}-prog.** For all \(\sigma \leq \tau \leq \tau_{3}\), let \(\mathcal{P}_{T}\)-prog. on \(E\) be a \(\mathcal{B}(\{\sigma, \tau\}) \times \mathcal{P}_{T}/\mathcal{B}(E)\)-measurable process and \(\mathcal{F}_{T}\)-prog. on \(E\) be a \(\mathcal{B}(\{\sigma, \tau\}) \times \mathcal{F}_{T}/\mathcal{B}(E)\)-measurable process.

For the later discussion, \(\dot{\epsilon}\) is defined as \(\dot{\epsilon} = 1\) when a component \(\dot{x}_{\mu}(\cdot, \cdot) = \nu_{A^{4}(V_{M}, g)}^{\mu} \circ \dot{x}(\cdot, \cdot)\) as a 1D stochastic process is \(\mathcal{P}_{T}\)-adapted and \(\dot{\epsilon} = -1\) if \(\dot{x}_{\mu}(\cdot, \cdot) \in \mathcal{F}_{T}\)-adapted. Namely for \(\{V_{\mu}(\cdot, \omega)\}_{\mu=0,1,2,3}\), let us regard the above as follows:

\[
\mathbb{E}[V((\tau, \omega)) | \mathcal{P}_{T}] := \left\{ \begin{array}{ll}
\mathbb{E}[V_{0}(\cdot, \omega) | \mathcal{F}_{T}] & (\omega) \\
\mathbb{E}[V_{1,2,3}(\cdot, \omega) | \mathcal{P}_{T}] & (\omega)
\end{array} \right.
\]

The mean derivatives (see Eq. 23) are mathematically introduced by those ideas:

**Definition 26 ((R1)-process).** If \(\dot{x}(\cdot, \cdot)\) is an (R0)-process and a following \(V_{+}(\dot{x}(\cdot, \cdot)) \in L_{\mathcal{P}}^{1,2,3}(\mathcal{P}_{T}; V_{M}^{4})\) exists, \(\dot{x}(\cdot, \cdot)\) is an (R1)-process.

\[
V_{+}(\dot{x}(\tau, \omega)) := \lim_{\delta \tau \to 0^{+}} \mathbb{E} \left[ \frac{\dot{x}(\tau + \delta \tau, \cdot) - \dot{x}(\tau, \cdot)}{\delta \tau} \right] (\mathcal{P}_{T}) \quad (\omega)
\]

The each components of \(V_{+}(\dot{x}(\tau, \omega))\) are interpreted as below:

\[
V_{+}^{0}(\dot{x}(\tau, \omega)) = \lim_{\delta \tau \to 0^{+}} \mathbb{E} \left[ \frac{\dot{x}(\tau + \delta \tau, \cdot) - \dot{x}(\tau, \cdot)}{\delta \tau} \right] (\mathcal{P}_{T}) \quad (\omega)
\]

**Definition 27 ((S1)-process).** If \(\dot{x}(\cdot, \cdot)\) is an (R1)-process and a following \(V_{-}(\dot{x}(\cdot, \cdot)) \in L_{\mathcal{F}}^{1,2,3}(\mathcal{F}_{T}; V_{M}^{4})\) exists, \(\dot{x}(\cdot, \cdot)\) is named an (S1)-process.

\[
V_{-}(\dot{x}(\tau, \omega)) := \lim_{\delta \tau \to 0^{+}} \mathbb{E} \left[ \frac{\dot{x}(\tau + \delta \tau, \cdot) - \dot{x}(\tau, \cdot)}{\delta \tau} \right] (\mathcal{F}_{T}) \quad (\omega)
\]

The definition of an (S1)-process declare that this is \(\mathcal{P}_{T}\)-prog. and \(\mathcal{F}_{T}\)-prog. Therefore, an (S1)-process provides us the form of the stochastic integral on \(\tau_{a} \leq \tau_{b}\):

\[
\dot{x}_{\mu}(\tau_{b}, \omega) - \dot{x}_{\mu}(\tau_{a}, \omega) = \int^{\tau_{b}}_{\tau_{a}} d\tau \nu_{A^{4}(V_{M}, g)}^{\mu} \dot{x}(\tau, \omega) + \int^{\tau_{b}}_{\tau_{a}} dy_{\mu}(\tau, \omega)
\]

\[
= \int^{\tau_{b}}_{\tau_{a}} d\tau \nu_{A^{4}(V_{M}, g)}^{\mu} \dot{x}(\tau, \omega) + \int^{\tau_{b}}_{\tau_{a}} dy_{\mu}(\tau, \omega)
\]

Where, \(y_{+}(\cdot, \cdot)\) and \(y_{-}(\cdot, \cdot)\) of martingales satisfy below:
Definition 28 ((R2)-process). When \( \dot{x}(0, \cdot) \) is an (R1)-process and let \( y_{+}(\tau, \cdot) \) be its \( \{ \mathcal{F}_{\tau} \} \)-martingale part such that \( y_{+}(\tau + \epsilon \times \delta \tau, \cdot) - y_{+}(\tau, \cdot) \in \mathcal{L}^{2}_{\text{loc}}(\{ \mathcal{F}_{\tau} \}; V_{M}^{4}) \), then, \( \dot{x}(0, \cdot) \) is named an (R2)-process if

\[
\mathbb{E} \left[ \left( y_{+}(\tau + \epsilon \times \delta \tau, \cdot) - y_{+}(\tau, \cdot) \right) \mid \mathcal{F}_{\tau} \right] (\omega) = 0
\]  
(A27)

and a following \( \sigma_{+}^{2}(\tau, \omega) \in \mathcal{L}^{1}_{\text{loc}}(\{ \mathcal{F}_{\tau} \}; V_{M}^{4} \otimes V_{M}^{4}) \) exists such that \( \tau \mapsto \sigma_{+}^{2}(\tau, \omega) \) is continuous:

\[
\sigma_{+}^{2}(\tau, \omega) := \lim_{\delta \tau \to 0+} \mathbb{E} \left[ \left( y_{+}(\tau + \epsilon \times \delta \tau, \cdot) - y_{+}(\tau, \cdot) \right) \otimes \left( y_{+}(\tau + \epsilon \times \delta \tau, \cdot) - y_{+}(\tau, \cdot) \right) \mid \mathcal{F}_{\tau} \right] (\omega)
\]  
(A28)

Definition 29 ((S2)-process). When \( \dot{x}(0, \cdot) \) is an (R2) and (S1)-process and let \( y_{-}(\tau, \cdot) \) be its \( \{ \mathcal{F}_{\tau} \} \)-martingale part \( y_{-}(\tau + \epsilon \times \delta \tau, \cdot) - y_{-}(\tau, \cdot) \in \mathcal{L}^{2}_{\text{loc}}(\{ \mathcal{F}_{\tau} \}; V_{M}^{4}) \), then, \( \dot{x}(0, \cdot) \) is called an (S2)-process if

\[
\mathbb{E} \left[ \left( y_{-}(\tau + \epsilon \times \delta \tau, \cdot) - y_{-}(\tau, \cdot) \right) \mid \mathcal{F}_{\tau} \right] (\omega) = 0 ,
\]  
(A29)

and a following \( \sigma_{-}^{2}(\tau, \omega) \in \mathcal{L}^{1}_{\text{loc}}(\{ \mathcal{F}_{\tau} \}; V_{M}^{4} \otimes V_{M}^{4}) \) exists such that \( \tau \mapsto \sigma_{-}^{2}(\tau, \omega) \) is continuous:

\[
\sigma_{-}^{2}(\tau, \omega) := \lim_{\delta \tau \to 0+} \mathbb{E} \left[ \left( y_{-}(\tau + \epsilon \times \delta \tau, \cdot) - y_{-}(\tau, \cdot) \right) \otimes \left( y_{-}(\tau + \epsilon \times \delta \tau, \cdot) - y_{-}(\tau, \cdot) \right) \mid \mathcal{F}_{\tau} \right] (\omega)
\]  
(A30)

Definition 30 ((R3)-process). If \( \dot{x}(0, \cdot) \) is an (R2)-process and \( \det \sigma_{+}^{2}(\tau, \omega) > 0 \) is almost surely satisfied for all \( \tau \in \mathbb{R} \), then, \( \dot{x}(0, \cdot) \) is named an (R3)-process.

Definition 31 ((S3)-process). If \( \dot{x}(0, \cdot) \) is an (R3) and (S2)-process, and \( \det \sigma_{-}^{2}(\tau, \omega) > 0 \) is almost surely satisfied for all \( \tau \in \mathbb{R} \), then, \( \dot{x}(0, \cdot) \) is called an(S3)-process.

Where, \( y_{\pm}(\tau, \omega) := \lambda \times W_{\pm}(\tau, \omega) \) for \( \lambda > 0 \) satisfies the above definition of the (S3) process, i.e.,

\[
\det \sigma_{\pm}(\tau, \omega) \big|_{y_{\pm}(\tau, \omega) = \lambda \times W_{\pm}(\tau, \omega)} = 4 \times \lambda^{2} > 0 .
\]  
(A31)

Definition 32 (D-prog. \( \dot{x}(0, \cdot) \)). An (S3)-process \( \dot{x}(0, \cdot) \) on \( (\Omega, \mathcal{F}, \mathcal{F}, \mathcal{F}) \) is named “D-prog.” if \( y_{\pm}(\tau, \cdot) := \lambda \times W_{\pm}(\tau, \cdot) \) w.r.t. \( \lambda > 0 \). \( \dot{x}(0, \cdot) \) of a D-prog. is given by the following stochastic differential equation:

\[
d\dot{x}(\tau, \omega) = \mathcal{V}_{\pm}(\dot{x}(\tau, \omega))d\tau + \lambda \times dW_{\pm}(\tau, \omega)
\]  
(A32)

For \( \tau_{\alpha} \leq \tau_{\beta} \), \( \dot{x}(\cdot, \cdot) \) of D-prog. by Eq. (A32) is regarded as the symbolic expression w.r.t. the following integral:

\[
\dot{x}(\tau_{\beta}, \omega) - \dot{x}(\tau_{\alpha}, \omega) = \int_{\tau_{\alpha}}^{\tau_{\beta}} d\tau' \mathcal{V}_{\pm}(\dot{x}(\tau', \omega)) + \int_{\tau_{\alpha}}^{\tau_{\beta}} dW_{\pm}(\tau', \omega)
\]  
(A33)

\[
= \int_{\tau_{\alpha}}^{\tau_{\beta}} d\tau' \mathcal{V}_{\pm}(\dot{x}(\tau', \omega)) + \int_{\tau_{\alpha}}^{\tau_{\beta}} dW_{\pm}(\tau', \omega)
\]  
(A34)

Let \( d_{\pm}\dot{x}(\tau, \omega) \) be defined by \( d_{\pm}\dot{x}(\tau, \omega) := \mathcal{V}_{\pm}(\dot{x}(\tau + \epsilon \times \delta \tau, \omega) - \dot{x}(\tau, \omega)) \) with \( \epsilon = \pm \) for \( d_{\pm} \). Since \( \dot{x}(\tau, \cdot) \) is \( \mathcal{F}_{\tau}/\mathcal{F}(\mathcal{A}^{4}(V_{M}^{4}, g)) \) and \( \mathcal{F}_{\tau}/\mathcal{F}(\mathcal{A}^{4}(V_{M}^{4}, g)) \)-measurable for all \( \tau \), the following is imposed:

Theorem 33. A D-progressive \( \dot{x}(0, \cdot) \) is continuous and \( \{ \mathcal{F}_{\tau} \cap \mathcal{F}_{\tau} \} \)-adapted.

Finally, the construction of our Brownian and relativistic kinematics is mathematically completed by defining \( \lambda \). The following conjecture is demonstrated by Eqs. of Issue-\( \lambda \) in the main body.

Conjecture 34. A D-prog. \( \dot{x}(0, \cdot) \) is a trajectory of a scalar electron satisfying a Klein-Gordon equation when \( \lambda = \sqrt{h/m} \).

\( \dot{x}(\cdot, \cdot) \) of a D-prog. imposes the following Itô formula.

Theorem 35 (Itô formula). Consider a C2-function \( f : \mathbb{A}^{4}(V_{M}^{4}, g) \to \mathbb{C} \), the following Itô formula w.r.t. a D-prog. \( \dot{x}(0, \cdot) \) is found:

\[
d_{\pm} f(\dot{x}(\tau, \omega)) = \partial_{\mu} f(\dot{x}(\tau, \omega))d_{\pm}\dot{x}^{\mu}(\tau, \omega) + \frac{\lambda^{2}}{2} \partial_{\mu} \partial^{\mu} f(\dot{x}(\tau, \omega))d\tau
\]  
(A35)

This is given by the following stochastic integral:

\[
f(\dot{x}(\tau_{\beta}, \omega)) - f(\dot{x}(\tau_{\alpha}, \omega)) = \int_{\tau_{\alpha}}^{\tau_{\beta}} d_{\pm} f(\dot{x}(\tau, \omega))
\]  
(A36)

\[
= \int_{\tau_{\alpha}}^{\tau_{\beta}} \partial_{\mu} f(\dot{x}(\tau, \omega))d_{\pm}\dot{x}^{\mu}(\tau, \omega) + \frac{\lambda^{2}}{2} \int_{\tau_{\alpha}}^{\tau_{\beta}} \partial_{\mu} \partial^{\mu} f(\dot{x}(\tau, \omega))d\tau
\]  
(A37)
**Definition 36** (Mean derivatives \( \mathcal{D}_x^\pm \)). \( \mathcal{D}_x^\pm \) and \( \mathcal{D}_x^- \) the operators of the mean derivatives are defined by the following:

\[
\mathcal{D}_x^\pm f(\hat{x}(\tau, \omega)) := \left[ \mathcal{V}_x^\pm(\hat{x}(\tau, \omega)) + \frac{\lambda^2}{2} \partial^\mu \right] \partial_\mu f(\hat{x}(\tau, \omega)) \quad (A38)
\]

By using this,

\[
\int_{\tau_a}^{\tau_b} d_+ f(\hat{x}(\tau, \omega)) = \int_{\tau_a}^{\tau_b} \mathcal{D}_x^+ f(\hat{x}(\tau, \omega)) d\tau + \lambda \times \int_{\tau_a}^{\tau_b} \partial_\mu f(\hat{x}(\tau, \omega)) dW^\mu(\tau, \omega). \quad (A39)
\]

**Definition 37** (Complex differential \( \hat{d} \)). For a D-prog, \( \hat{x}(\omega, \bullet) \) and a \( C^2 \)-function \( f: \mathbb{A}_4^4(V_M^4, g) \to \mathbb{R} \), let \( \hat{d} \) be a differential mapping as follows:

\[
\hat{d} f(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega)) \hat{d} \mathcal{V}_x^\mu(\tau, \omega) + \frac{i \lambda^2}{2} \partial^\mu \partial_\mu f(\hat{x}(\tau, \omega)) d\tau \quad (A41)
\]

When \( \int_{\tau_a}^{\tau_b} \hat{d} f(\hat{x}(\tau, \omega)) \) satisfies

\[
\text{Re} \left\{ \int_{\tau_a}^{\tau_b} \hat{d} f(\hat{x}(\tau, \omega)) \right\} + \text{Im} \left\{ \int_{\tau_a}^{\tau_b} \hat{d} f(\hat{x}(\tau, \omega)) \right\} = \int_{\tau_a}^{\tau_b} d_{\pm} f(\hat{x}(\tau, \omega)),
\]

\( \hat{d} \) indicates \( \hat{d} = (d_+ + d_-)/2 - i(d_+ - d_-)/2 \) symbolically. Therefore,

\[
f(\hat{x}(\tau_b, \omega)) = f(\hat{x}(\tau_a, \omega)) + \int_{\tau_a}^{\tau_b} \hat{d} f(\hat{x}(\tau, \omega)) \text{ a.s.} \quad (A42)
\]

**Definition 38** (Mean derivative \( \mathcal{D}_x \)). \( \mathcal{D}_x \) is defined by the following:

\[
\mathcal{D}_x f(\hat{x}(\tau, \omega)) := \left[ \mathcal{V}_x(\hat{x}(\tau, \omega)) + \frac{i \lambda^2}{2} \partial^\mu \right] \partial_\mu f(\hat{x}(\tau, \omega)) \quad (A43)
\]

By employing this operator

\[
\int_{\tau_a}^{\tau_b} \hat{d} f(\hat{x}(\tau, \omega)) = \int_{\tau_a}^{\tau_b} \mathcal{D}_x f(\hat{x}(\tau, \omega)) d\tau + \lambda \times \int_{\tau_a}^{\tau_b} d_{\pm} f(\hat{x}(\tau, \omega)) dW^\mu(\tau, \omega). \quad (A44)
\]

Hence, \( \mathcal{D}_x \hat{x}(\tau, \omega) = \mathcal{V}(\hat{x}(\tau, \omega)) \). For \( f(\hat{x}(\tau_b, \omega)) = f(\hat{x}(\tau_a, \omega)) + \int_{\tau_a}^{\tau_b} \hat{d} f(\hat{x}(\tau, \omega)) \), its iteration is

\[
f(\hat{x}(\tau_b, \omega)) = \sum_{n=0}^{\infty} \frac{\tau_b - \tau_a}{n!} \times \mathcal{D}_x^n f(\hat{x}(\tau_a, \omega)) + \lambda \times \sum_{n=0}^{\infty} \int_{\tau_a}^{\tau_b} \int_{\tau_a}^{\tau_b} \cdots \int_{\tau_a}^{\tau_b} d\tau_1 \int_{\tau_a}^{\tau_b} \cdots \int_{\tau_a}^{\tau_b} d\tau_{n-1} \int_{\tau_a}^{\tau_b} dW(\tau, \omega) \cdot \partial_\mu \mathcal{D}_x^n f(\hat{x}(\tau_a, \omega)). \quad (A45)
\]

We named it the stochastic-Taylor expansion, however strictly speaking, the second term in the RHS has to be expanded more at \( \tau_a \), too.

**Appendix B: Partial integral formula**

Let us demonstrate Nelson’s partial integral formula of Eqs. 60, 61

**Lemma 39** (Nelson’s partial integral). For the spacetime \( (\mathbb{A}_4^4(V_M^4, g), \mathbb{R}(\mathbb{A}_4^4(V_M^4, g)), \mu) \), let \( \alpha \) and \( \beta \) be the complex-valued and \( C^2 \)-local square integrable functions on \( \mathbb{A}_4^4(V_M^4, g) \). Then, the following partial integral formula is fulfilled:

\[
\int_{\tau_1}^{\tau_2} d\tau E \left[ \mathcal{D}_x^+ \alpha(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) \right] = E \left[ \alpha(\hat{x}(\tau_2, \bullet)) \beta^\mu(\hat{x}(\tau_2, \bullet)) \right] - E \left[ \alpha(\hat{x}(\tau_1, \bullet)) \beta^\mu(\hat{x}(\tau_1, \bullet)) \right]. \quad (B1)
\]
Its differential form is

\[
\frac{d}{dt} \mathbb{E} \left[ \alpha_\mu (\dot{x}(\tau, \bullet)) \beta^\mu (\dot{x}(\tau, \bullet)) \right] = \mathbb{E} \left[ D^\pm_\tau \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right].
\]

(B2)

**Proof.** Confirm the following relation at first.

\[
\begin{align*}
\mathbb{E} \left[ D^+_\tau \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right] &+ \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot D^-_\tau \beta^\mu (\dot{x}(\tau, \bullet)) \\
&= \mathbb{E} \left[ D^-_\tau \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right] + \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot D^+_\tau \beta^\mu (\dot{x}(\tau, \bullet))
\end{align*}
\]

(B3)

since

\[
\begin{align*}
\mathbb{E} \left[ D^+_\tau \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right] &+ \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot D^-_\tau \beta^\mu (\dot{x}(\tau, \bullet)) \\
&= -\lambda^4 \times \int_{\mathcal{A}(V_{\mu}^\tau, g)} d\mu(x) \int_{\mathbb{A}^\tau} \partial^\nu \left\{ p(x, \tau) \left[ \frac{\partial_\nu \alpha_\mu (x) \cdot \beta^\mu (x)}{\beta^\mu (x)} - \alpha_\mu (x) \cdot \partial_\nu \beta^\mu (x) \right] \right\}
\end{align*}
\]

(B4)

Then by using the Fokker-Planck equation \[27\],

\[
\begin{align*}
\frac{d}{dt} \mathbb{E} \left[ \alpha_\mu (\dot{x}(\tau, \bullet)) \beta^\mu (\dot{x}(\tau, \bullet)) \right] &\int_{\mathcal{A}(V_{\mu}^\tau, g)} d\mu(x) \alpha_\mu (x) \beta^\mu (x) \partial_\tau p(x, \tau) \\
&\frac{1}{2} \times \mathbb{E} \left[ D^+_\tau \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right] - \mathbb{E} \left[ D^-_\tau \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right] \left( \partial^\nu \left\{ p(x, \tau) \left[ \frac{\partial_\nu \alpha_\mu (x) \cdot \beta^\mu (x)}{\beta^\mu (x)} - \alpha_\mu (x) \cdot \partial_\nu \beta^\mu (x) \right] \right\} \right.
\end{align*}
\]

(B5)

Where, we employ \[\int_{\mathcal{A}(V_{\mu}^\tau, g)} d\mathcal{P}(\omega) = \int_{\mathcal{A}(V_{\mu}^\tau, g)} p(x, \tau) d\mu(x).\]

By combining it with Eq. \[133\], Eq. \[122\] is demonstrated. \[\Box\]

By using the superposition of the above “±”-formulas, it can be switched to Eq. \[170\] the formula for the complex derivatives of \(D_{\tau}\) and \(D^*_{\tau}\).

\[
\begin{align*}
\frac{d}{dt} \mathbb{E} \left[ \alpha_\mu (\dot{x}(\tau, \bullet)) \beta^\mu (\dot{x}(\tau, \bullet)) \right] &\int_{\mathcal{A}(V_{\mu}^\tau, g)} d\mu(x) \alpha_\mu (x) \beta^\mu (x) \partial_\tau p(x, \tau) \\
&\frac{1}{2} \times \mathbb{E} \left[ D^+_\tau \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right] - \mathbb{E} \left[ D^-_\tau \alpha_\mu (\dot{x}(\tau, \bullet)) \cdot \beta^\mu (\dot{x}(\tau, \bullet)) \right] \left( \partial^\nu \left\{ p(x, \tau) \left[ \frac{\partial_\nu \alpha_\mu (x) \cdot \beta^\mu (x)}{\beta^\mu (x)} - \alpha_\mu (x) \cdot \partial_\nu \beta^\mu (x) \right] \right\} \right.
\end{align*}
\]

(B6)

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