All tight correlation Bell inequalities have quantum violations

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Introduction.— In 1964, Bell [1] proved that some predictions of quantum theory regarding the correlations between distant events cannot be explained by any classical, i.e., local realistic theory. He derived a simple observable criterion that any classical theory must obey, and showed that particular measurements performed by two parties on a maximally entangled state could violate it. What we now call a Bell inequality was introduced in [2], as an upper bound on a single linear function of observable probabilities, i.e., an operational expectation value. This quantity has been experimentally measured [3,4] and shown to exceed the classical upper bound, and thereby elevated Bell’s theorem to one of the deepest results in science, with a momentous impact on the way we understand the physical world. Quantum entanglement is responsible for these observed correlations and it is also the key ingredient in most of the quantum informational advantage in computation, communication, and sensing applications. Non-locality on its own has also been identified as a valuable resource in applications such as secure key distribution [5], certified randomness [6], reduced communication complexity [7], self-testing [8,9], and computation [10].

In order to advance in the fundamental understanding of the perplexing features of non-local correlations and their technological spin-offs, in the last decades important efforts have been devoted to their characterisation and exploitation (see [11] for a recent review). Tsirelson [12] computed the maximal violation of the CHSH inequality [2] attainable by quantum mechanics; later, Popescu and Rohrlich [13] (see also [14]) showed that although quantum correlations belong to the set of no-signalling correlations —they do not allow for instantaneous communication—, they do not attain the full strength allowed in principle by the no-signalling condition. These results reveal astonishing features of the convex sets of classical (C), quantum (Q) and no-signalling (NS) correlations, in particular the strict inclusion $C \subset Q \subset NS$. However, a lot remains to be understood, both at the conceptual and mathematical level. For instance, the fact that $Q \subset NS$ spurred the search for underlying operational principles that would single out quantum correlations among general no-signalling ones [15,17].

An approach that may assist in identifying such operationally defined principles and that may unveil new applications of non-locality is based on cooperative games of incomplete information [18], where two (or more) remote parties cooperate to win a probabilistic game against a referee. Indeed, an increased winning probability when the two parties use quantum resources (quantum entanglement) instead of classical ones (which includes shared randomness), is equivalent to the violation of a Bell inequality. Gill [19,20] asked the fruitful question whether all tight Bell inequalities are violated by quantum mechanics. Here, tightness means that the inequality cannot be expressed as a positive linear combination of other Bell inequalities, or in geometric terms, that the Bell inequality defines a facet of the polytope of classical correlations (see below). Linden et al. [21] (motivated by [22]) found the first class of two-player games, called non-local computation (NLC), that have no quantum advantage; the tightness of their Bell inequalities was posed as an open question in [21]. Almeida et al. [23] presented another case, the multi-party guess your neighbour’s input (GYNI) game, shown to define a tight Bell inequality without quantum violation. Around the same time, it was understood that NLC games never define facets of the Bell polytope [24], though this result was never written up; a proof was eventually published by Ramanathan et al. [25].

In this Letter we prove that XOR games never define a facet of the Bell polytope (thus extending the result for NLC), answering Gill’s question in the affirmative for the correlation polytope: all nontrivial tight correlation Bell inequalities have quantum violations. The remainder of this Letter is structured as follows: i) we first introduce the general formalism to describe the set of no-signalling, local classical and quantum correlations; ii) we briefly present the XOR
and denoted $C$. A polytope $\mathcal{P}$ can equivalently be defined as the convex hull of a finite set of points, $\mathcal{P} = \text{conv}\{v_j : j = 1, \ldots, k\}$, or as a bounded intersection of finitely many closed half-spaces, $\mathcal{P} = \{v : \forall i = 1, \ldots, \ell \; \bar{u}_i : \bar{v} \leq w_i\}$ [27]. A linear inequality for $\mathcal{P}$ is a $\bar{u} : \bar{v} \leq w$ that holds for all $\bar{v} \in \mathcal{P}$; in geometry, $H = \{\bar{v} : \bar{u} : \bar{v} = w\}$ is also called a supporting hyperplane of $\mathcal{P}$. For a given supporting hyperplane, the set of points $\bar{v} \in \mathcal{P}$ achieving the equality is called a face of the polytope $\mathcal{P}$, $F = \mathcal{P} \cap H$. In the case of $C$, its corresponding inequalities are precisely the Bell inequalities. The faces of maximum dimension $D - 1$ are called facets; when dealing with $C$, the corresponding inequalities are called tight, or facet, Bell inequalities. Facet inequalities give the minimal characterisation of the polytope in terms of half-spaces in the sense that any other inequality that holds for the polytope can be written as a positive linear combination of the facet inequalities.

To state and prove our results, we use a convenient minimal parametrisation of the no-signalling polytopes. Any no-signalling behaviour $p(a,b|x,y)$ is fully characterised by the first moments $\alpha_x = \sum_{a,b} (-1)^a p(a,b|x,y)$ and $\beta_y = \sum_{a,b} (-1)^b p(a,b|x,y)$ (which, due to no-signalling property, are independent of $y$ and $x$, resp.); and the correlators $c_{xy} = \sum_{a,b} (-1)^{a+b} p(a,b|x,y)$. Indeed, from these $D = m_A m_B + m_A + m_B$ values we recover $4p(a,b|x,y) = 1 + (-1)^a \alpha_x + (-1)^b \beta_y + (-1)^{a+b} c_{xy}$. The polytope of $\mathcal{N}S$ distributions can hence be described by the tuple $([\alpha], [\beta], C) \in \mathbb{R}^D$, where $[\alpha] = \sum_x \alpha_x |x\rangle \langle x|^A$ and $[\beta] = \sum_y \beta_y |y\rangle \langle y|^B$ are the local moment vectors, and $C = \sum_{x,y} c_{xy} |x\rangle \langle y|$ is the correlation matrix. The local, or Bell, polytope arises when restricting the strategies to convex combinations of local deterministic ones [25]. We use the subscript $c$ to label such extremal classical strategies $[\alpha_c] = \{(-1)^a \alpha_x|a\rangle \langle x|^A\}$, $[\beta_c] = \{(-1)^b \beta_y|b\rangle \langle y|^B\}$, which is $C = [\alpha_c][\beta_c]$. The Bell polytope is then given by the convex hull

$$C = \text{conv}\{[\alpha_c], [\beta_c], [\alpha_c][\beta_c]\}.$$  

Finally, there are at least two definitions of sets of quantum behaviours that we have to consider: the most general setting is of a state $|\psi\rangle$ in a Hilbert space $\mathcal{H}$, together with Alice’s and Bob’s observables $\hat{a}_x$ and $\hat{b}_y$, respectively, with eigenvalues 0 and 1 (i.e. they are projectors), and such that for all $x, y$, $[\hat{a}_x, \hat{b}_y] = 0$. Then, $p(a,b|x,y) = \langle \psi | \hat{a}_x \hat{b}_y | \psi \rangle$, and hence in the above parametrisation for $\mathcal{N}S$, we have $\alpha_x = \langle \psi | (-1)^a \hat{a}_x | \psi \rangle$, $\beta_y = \langle \psi | (-1)^b \hat{b}_y | \psi \rangle$, and $c_{xy} = \langle \psi | (-1)^{a+b} \hat{a}_x \hat{b}_y | \psi \rangle$. The set of such behaviours is denoted $Q_{\text{comm}}$, the subscript standing for “commuting” strategies, and it is known to be the closed convex set contained in $\mathcal{N}S$. The other, traditionally considered setting is that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is a tensor product Hilbert space, and that $\hat{a}_x = \hat{a}_x^A \otimes \mathbb{1}_B$ and $\hat{b}_y = \mathbb{1}_A \otimes \hat{b}_y^B$, with observables $\hat{a}_x^A$ on $\mathcal{H}_A$ and $\hat{b}_y^B$ on $\mathcal{H}_B$. 

FIG. 1. Three-dimensional schematic of a local correlation polytope and the convex body of quantum correlations, illustrating different types of regions where local and quantum boundaries coincide (corresponding to XOR games without quantum advantage): the green-striped region is a facet (face of dimension 2) of the Bell polytope, while the red line is a face (not a facet). Theorem 1 excludes the former, and hence implies that tight Bell inequalities (all facets) have quantum violations. Theorem 2 implies that all facets of the set of quantum correlations are trivial.

games and general expressions for the winning probabilities under different locality scenarios; iii) we present our main theorem and the main ideas of its proof; iv) we extend our result to the quantum set of correlations, and conclude with a discussion and outlook. In the Supplementary Material we give the full proofs of our result, and present a simplified analysis in the particular case of NLC, reconstructing the argument alluded to in [24], and improving [25] by giving a bound on the dimension of the face.

No-signalling behaviours.— Consider a bipartite system where two parties, Alice and Bob, can perform measurements $x \in [m_A]$ and $y \in [m_B]$, respectively, obtaining the respective outcomes $a$ and $b$, which are binary. The event of obtaining $a$ and $b$ when the local measurements $x$ and $y$ are performed, is given according to a conditional probability $p(a,b|x,y)$. In order to avoid instantaneous communication between the distant parties, forbidden by special relativity, this probability $p$ must satisfy the no-signalling property, i.e. $\sum_b p(a,b|x,y) = \sum_a p(a,b|x,y') \; \forall a,x,y,y'$ and analogously summing Alice’s outcomes, which in physical terms excludes that any party signals to another party by their choice of input.

The set of all probabilities satisfying the above no-signalling property, called the no-signalling set ($\mathcal{N}S$), constitutes a polytope of dimension $D = m_A m_B + m_A + m_B$ [26]. In an attempt to explain phenomena locally, one may consider the existence of classical local (hidden) variables $\lambda \in \Lambda$, distributed according to a probability law $p(\lambda)$, such that the probability of an observed event can be written as $p(a,b|x,y) = \int_{\Lambda} d\lambda p(\lambda)p(a|x,\lambda)p(b|y,\lambda)$. The set of all probabilities of this form is called the local set, which is also a polytope, the so-called Bell or local polytope, of the same dimension as the no-signalling polytope [26],
\( \hat{p} \) on \( \mathcal{H}_B \). The corresponding set of behaviours is convex, but recently has been shown not to be closed \([29]\) for \( m_A, m_B \geq 5 \) \([30]\), which is why we define \( Q_{\ominus} \) to be its closure. By definition, \( Q_{\ominus} \subseteq Q_{\text{com}} \), and while it is open whether the two sets are equal, this would be equivalent to Connes’ long-standing Embedding Problem in the theory of von Neumann algebras \([31]\) \([32]\). The sets of quantum behaviours are convex sets, but unlike the classical and no-signalling sets, they are not polytopes: they have uncountably many extremal points, and part of their boundary is curved.

The study of non-local correlations is often carried out in a simplified scenario, the so-called correlation polytope, which is given by the set of correlators \( C \) (without including the local terms). The corresponding linear criteria that define the set of classical/local correlations \( C_0 \) are called correlation Bell inequalities \([28] \ [33] \ [34]\). The projection of quantum and no-signalling behaviours onto the correlator subspace are the quantum \( Q_0 \) and no-signalling \( N S_0 \) correlations respectively. Note that by Tsirelson’s results \([12]\), both \( Q_{\text{com}} \) and \( Q_{\ominus} \) give rise to the same quantum correlator set, realized in fact with local Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) of bounded dimension.

\textbf{XOR games.}— Non-local games provide an intuitive operational setting in which to cast Bell inequalities, and relate those to the well-established field of interactive proofs in computer science. Here we will focus on the particular class of two-player XOR games \([18]\), where the outcomes of each party are binary and the winning condition depends on the exclusive disjunction (XOR) of the outcomes. XOR games have a prominent role in non-locality: the paradigmatic CHSH inequality \([2]\), the GHZ paradox \([33]\) and NLC \([21]\) can all be phrased as XOR games; and most importantly, they provide a characterisation of the correlation Bell polytope as will become apparent below.

![FIG. 2. Representation of a XOR game. The goal is that Alice and Bob output a and b such that a ⊕ b = f(x, y).](image)

In an XOR game, see Fig. 2 the referee provides queries \( x \in [m_A] \) to one player (Alice) and \( y \in [m_B] \) to the second player (Bob), with the promise that queries are sampled from some prior probability distribution \( q(x, y) \) known to both players. In order to win the game, upon receiving their inputs \( x \) and \( y \), Alice and Bob must produce a binary output \( a, b \in \{0, 1\} \), respectively, such that \( a \oplus b = f(x, y) \), where \( f \) is a given Boolean function also known to both players. The performance of Alice and Bob’s strategy is quantified by the average winning probability

\[
\omega = \sum_{x, y} q(x, y) p(a \oplus b = f(x, y) | x, y) = \frac{1}{2} (1 + \xi), \tag{2}
\]

where \( \xi = \sum_{x, y} q(x, y) (1 - f(x, y) \langle c(x) | c(y) \rangle) \) is the gain (or bias). Note that XOR games can always be won with at least probability \( \frac{1}{2} \) if Alice (or Bob) produces a random output independently of the input. Since \( q(x, y) \) and \( f(x, y) \) are given, we can characterise the game by the so-called game matrix

\[
\Phi = \sum_{x, y} (-1)^{f(x, y)} q(x, y) |x\rangle\langle y|,
\]

so that the gain can be written in terms of the correlation matrix

\[
\xi = \text{tr} C \Phi^T. \tag{3}
\]

Every correlation Bell inequality, as it is based on a linear function of the correlators \( C \), can be written in the form \( \text{tr} C \Phi^T \leq \xi \), and by rescaling if necessary, \( \Phi \) can be chosen as the game matrix of a suitable XOR game. The optimal classical success probability can always be attained by extremal (i.e. deterministic) strategies \( |\alpha_c\rangle \) and \( |\beta_c\rangle \), with \( C = |\alpha_c\rangle \langle \beta_c| \). Hence the gain of the local classical average winning probability can be written as:

\[
\xi_c = \max_{|\alpha_c\rangle \langle \beta_c|} \text{tr} C \Phi^T. \tag{3}
\]

In the Supplementary Material (A) we present various useful ways to write the quantum gain, which are employed in the proofs of our main results.

It is easy to see that no-signalling behaviours allow to win XOR games with \( \omega_{NS} = 1 \) \([36]\), and therefore any XOR game with \( \omega_c < 1 \) will correspond to a non-trivial Bell inequality, i.e. one that can potentially be violated quantumly. See the Supplementary Material (B) for further discussion of this point.

Observe finally that without loss of generality, we may restrict ourselves to XOR games with game matrices \( \Phi \) that have no all-zero rows or columns. Indeed, because such a row or column of zeros imply that the marginal \( q(x) \) or \( q(y) \) are zero for some inputs, we can redefine the set of possible queries (decreasing \( m_A \) and \( m_B \) accordingly) to obtain an equivalent game without all-zero rows or columns in its game matrix. We refer to such games with \( q(x) > 0 \) and \( q(y) > 0 \) for all \( x \in [m_A] \) and \( y \in [m_B] \) as exhaustive games.

\textbf{Results.}— For a long time, it was implicitly assumed that if Alice and Bob use entangled strategies, they can attain a greater success probability than if they are limited to classical resources, for any nontrivial Bell inequality. As explained in the introduction, it took a while to find examples of nontrivial games that do not show any quantum advantage.

Here we show that XOR games (which characterise the correlation polytope) without quantum advantage never define a facet of the Bell polytope (full behaviours or correlations). This in turn implies that all (nontrivial) tight correlation Bell inequalities have quantum violations.

In \([36]\) Ramanathan et al. derived a necessary and sufficient condition for a two-player XOR game to have no quantum advantage, which will turn out to be fundamental for the proof of our first result.

\textbf{Theorem 1.} If an exhaustive XOR game has no quantum advantage, the corresponding Bell inequality does not define a facet of the Bell polytope, nor of the correlation Bell polytope.
The proof—see the Supplementary Material (C) for full details—proceeds by bounding the dimension of the face $F$ in the Bell polytope corresponding to the maximum classical bias $\xi_c$ of the given XOR game:

$$F = \{(|\alpha\rangle, |\beta\rangle, C) \in \mathbb{C} : \text{tr} \, C \Phi^T = \xi_c\} = \text{conv}\{(|\alpha_c\rangle, |\beta_c\rangle, |\alpha\rangle|\beta\rangle) : \langle\alpha_c| \Phi |\beta_c\rangle = \xi_c\}.$$

The first ingredient is the characterisation of the maximum quantum bias $\xi_Q$ by semidefinite programming (SDP) \cite{1801.02420}, which by SDP duality leads to strong constraints on any optimal strategy via complementary slackness. From the assumption that $\xi_Q = \xi_c$, this leads to the second, and key, insight of the proof, namely that in any pair $\langle|\alpha\rangle, |\beta\rangle\rangle$ of optimal classical strategies, Alice’s and Bob’s local answers uniquely determine each other linearly: as we show in the proof, $|\beta\rangle = F|\alpha\rangle$ for a certain matrix $F$. Assuming w.l.o.g. $m_A \leq m_B$, we thus have

$$F = \text{conv}\{(|\alpha\rangle, F|\alpha\rangle, |\alpha\rangle|\alpha\rangle|F^T) : |\alpha\rangle \text{ opt.}\},$$

and its dimension can be upper bounded by that of

$$\text{aff}\{(|\alpha\rangle, F|\alpha\rangle, |\alpha\rangle|\alpha\rangle|F^T) : |\alpha\rangle \in \{\pm 1\}^{m_A}\},$$

which is $m_A + \frac{1}{2}m_A(m_A - 1) < D - 1$. In the case of the correlation polytope, the dimension is similarly upper bounded by $\frac{1}{2}m_A(m_A - 1) < m_A m_B$.

**Theorem 2.** All nontrivial tight correlation Bell inequalities for bipartite systems with binary outcomes have a quantum violation.

**Proof.** Consider a (non-exhaustive) XOR game with $M_A$ (Alice) and $M_B$ (Bob) inputs. W.l.o.g. the first $m_A$ ($m_B$) inputs of Alice (Bob) have non-zero probability, the rest are never asked, so we can apply Theorem 1 to the reduced exhaustive game, which relates the optimal strategies for the indices $x \in [m_A]$ and $y \in [m_B]$, but leaves completely unconstrained the remaining ones. Hence, given a strategy by Alice $|\alpha\rangle |\alpha\rangle \odot |\beta\rangle |\beta\rangle$, Bob’s strategy must be $\langle F|\alpha\rangle \odot |\beta\rangle |\beta\rangle$, where $|\alpha\rangle \in \{\pm 1\}^{M_A - m_A}$ and $|\beta\rangle \in \{\pm 1\}^{M_B - m_B}$.

We thus arrive at a codimension $\Delta = D - \dim F$ of the face of $\Delta \geq m_B + M_A(m_B - m_A) + \frac{1}{2}m_A(m_A + 1) > 1$. That is, XOR games with quantum equal to classical value do not define a facet of the Bell polytope. Following the same argument for the correlation polytope leads to the codimension $\Delta_0 \geq M_A(m_B - m_A) + \frac{1}{2}m_A(m_A + 1)$, and this is greater than 1 unless $m_A = m_B = 1$, corresponding precisely to the trivial inequalities $|e_{xy}| \leq 1$.

In the Supplementary Material (D) we give a different proof of the above result for non-local computation games, showing also that the dimension bounds are asymptotically attained for nontrivial games.

From the proof of Theorem 1 we learn that the optimal extremal behaviours in an exhaustive XOR game with no quantum advantage are fully determined by the strategy of one of the parties. We will now show that this feature actually extends to all optimal quantum behaviours of arbitrary XOR games.

To understand the following theorem, we recall the definition of a face $F \subset Q$ of a general compact convex set $Q$, namely, that whenever $F \ni \bar{q} = t\bar{q} + (1 - t)\bar{r}$, $0 < t < 1$, then both $\bar{q}, \bar{r} \in F$. An exposed face is obtained as $F = Q \cap H$ with a supporting hyperplane $H$ of $Q$; all exposed faces are faces of $Q$, but not vice versa. However, for polytopes every face is an exposed face \cite{0418.52004,0418.52005}. Also, facets, and more generally maximal faces, are always exposed.

Note that this result explains the previous two theorems on the classical behaviours as being due to broader properties of the quantum sets.

**Theorem 3.** Nontrivial XOR games, or equivalently nontrivial correlation Bell inequalities, never define a facet of the quantum sets of behaviours $\mathcal{Q}_{\text{com}}$ and $\mathcal{Q}_{\otimes}$. As a consequence, the set $\mathcal{Q}_0$ of quantum correlations has no nontrivial facets.

See the Supplementary Material (C) for the complete proof. To give the broad outline, we start with an exhaustive XOR game. The complementary slackness condition in the proof of Theorem 1 for the optimal quantum strategy leads to $|\beta\rangle = \sum_{x'} F_{yy'} |\alpha_{x'}\rangle$, with the same matrix $F$ as before. In other words, once again Alice’s optimal quantum strategy uniquely determines Bob’s, and vice versa.

Thus, we get for an optimal quantum correlation matrix $C_{xy} = |\alpha_x\rangle |\beta_y\rangle = \sum_{x'} F_{yy'} |\alpha_{x'}\rangle |\alpha_{x'}\rangle$, and hence the dimension of their affine span is bounded by that of the Gram matrices $|\langle \alpha_x | \alpha_{x'} \rangle|_{xx'}$, with dimension $\frac{1}{2}m_A(m_A - 1)$, leading to the same dimension bound as in the proof of Theorem 1. Non-exhaustive XOR games are treated as in the proof of Theorem 2.

**Discussion and outlook.**— We have shown that a two-party correlation Bell inequality (XOR game) with no quantum violation (quantum advantage) cannot define a facet of the Bell polytope. The contrapositive of this statement has deep physical implications: all tight correlation Bell inequalities exhibit a quantum violation. In fact, we have proven lower bounds on the codimension of the defined face (increasing with the number of inputs). As a consequence, when the codimension is lower bounded by $\Delta > 1$, not only all tight correlation inequalities will have quantum violations, but also those corresponding to faces $F$ with $\dim F \geq D - \Delta$.

On the way, we have proved that this in fact is due to a broader property of the convex set of quantum correlations, namely that it does not have any nontrivial facets, only lower-dimensional faces. It remains to be seen what the physical meaning of this curious geometric observation is.

Our results appear very much tied to the world of two-player, binary outcome XOR games. This leaves open the questions whether XOR games for more than two players can define common facets of the quantum and classical sets (note that GYNI defines such a facet, but it is not an XOR game), and whether
for two players there are any tight Bell inequalities at all without quantum violations. It might be possible to extend our results at least two two-players MOD-\(q\) games where each player has a \(q\)-ary outcome.

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**SUPPLEMENTARY MATERIAL**

A. Quantum bias

If Alice and Bob use quantum resources, i.e. a (possibly) entangled state $|\psi\rangle \in \mathcal{H}$, with with observables $a_x$ and $b_y$ depending on their respective inputs $x$ and $y$, to produce their measurement outcomes $a$ and $b$, then, using the expression given in the main text for the quantum correlations $c_{x,y}$, the quantum gain can be written as

$$\xi_Q = \sum_{x,y} q(x, y) \langle \psi | (-1)^{a_x + b_y + f(x,y)} | \psi \rangle$$

$$= \sum_{x,y} q(x, y) (-1)^{f(x,y)} \langle \alpha_x | \beta_y \rangle,$$

with $|\alpha_x\rangle = (-1)^{a_x} |\psi\rangle$ and $|\beta_y\rangle = (-1)^{b_y} |\psi\rangle$ unit vectors in $\mathcal{H}$. Vice versa, given any set of unit vectors in any complex Hilbert space, there exists an equivalent set, i.e. with the same pairwise inner products $\langle \alpha_x | \beta_y \rangle$, of the above form, on a tensor product Hilbert space $\mathcal{H}$. It will prove convenient to define the following states on an extended vector space,

$$|\alpha\rangle_Q = \sum_x (-1)^{a_x} |\psi\rangle \otimes |x\rangle,$$

$$|\beta\rangle_Q = \sum_y (-1)^{b_y} |\psi\rangle \otimes |y\rangle.$$  

With this, the expression for the optimal quantum gain reads

$$\xi_Q = \max_{|\alpha\rangle_Q,|\beta\rangle_Q} q(\alpha|\Phi|\beta)q,$$

which resembles the expression (3) for its classical counterpart.

B. XOR games can always be won using no-signalling behaviours

It is easy to understand the previous observation of $\omega_{NS} = 1$ for every XOR game, by realising that an XOR game can always be won with certainty using a strategy that outputs uniformly random local bits $a$ and $b$, and so is evidently no-signalling:

$$p(a, b|x, y) = \begin{cases} \frac{1}{2} & \text{if } a \oplus b = f(x, y), \\ 0 & \text{otherwise} \end{cases}$$

This behaviour has $|\alpha\rangle = |\beta\rangle = 0$, but its correlator term is remarkably $c_{xy} = (-1)^{f(x,y)}$, i.e. an arbitrary $\pm 1$-matrix. In geometric terms, this says that $\mathcal{NS}_0$ is the $\ell_\infty$ unit ball (aka hypercube), in other words $\mathcal{NS}_0$ is entirely characterised by the inequalities $-1 \le c_{xy} \le 1$. These are indeed the trivial Bell inequalities, since they follow from the non-negativity of the probabilities $p(a, b|x, y)$.

To back up the definite article in the previous statement, it is in fact known that they not only define facets of $\mathcal{NS}_0$, but also of $\mathcal{C}_0$ and thus of $\mathcal{Q}_0$; namely, for every pair $(x_0, y_0)$, the behaviour $C_0 = |x_0\rangle|y_0\rangle$ is in $\mathcal{C}_0$, and so is an entire sufficiently small neighbourhood of $C_0$ of correlators $C$ in the supporting hyperplane $H = \{ C : C_{x_0y_0} = 1 \}$ (and analogously for $\mathcal{C}_0$). Again, this is easy to see: consider local classical strategies $|\alpha_C\rangle = \sum_x \alpha_x |x\rangle$ and $|\beta_C\rangle = \sum_y \beta_y |y\rangle$ with $\alpha_0 = \beta_0 = 1$, so that we can write $|\alpha_C\rangle = |x_0\rangle + |\alpha\rangle$ and $|\beta_C\rangle = |y_0\rangle + |\beta\rangle$. Then,

$$\langle \alpha_C | \beta_C \rangle = C_0 + |x_0\rangle|\beta\rangle + |\alpha\rangle|y_0\rangle + |\alpha\rangle|\beta\rangle.$$  

By taking convex combinations over $|\beta\rangle' = \epsilon |y_1\rangle + \sum_{y \neq y_0, y_1} \beta_y |y\rangle$, with uniformly random $\beta_y = \pm 1$ ($y \neq y_0, y_1$) and uniformly random $|\alpha\rangle'$, we annihilate all terms except one, showing that $C_0 + \epsilon |x_0\rangle|y_1\rangle \in \mathcal{C}_0$, for $\epsilon = \pm 1$. Similarly, we find $C_0 + \epsilon |x_1\rangle|y_0\rangle \in \mathcal{C}_0$.

Finally, by taking convex combinations over $|\alpha\rangle' = \sum_{x \neq x_0} \alpha_x |x\rangle$ and $|\beta\rangle' = \epsilon \alpha_{x_0} |y_1\rangle + \sum_{y \neq y_0, y_1} \beta_y |y\rangle$, with uniformly random $|\alpha\rangle'$ and uniformly random $\beta_y = \pm 1$ ($y \neq y_0, y_1$), we get that $C_0 + \epsilon |x_1\rangle|y_1\rangle \in \mathcal{C}_0$. The convex hull of all these points, a hyper-octahedron, contains a small neighbourhood of $C_0$ in the hyperplane $H$.

C. Proofs

Here, we give the complete proofs of Theorems 1 and 3 in the main text.

**Proof of Theorem 1**. Our aim is to bound the dimension of the face $F$ defined by the maximum classical bias $\xi$ of a given XOR game:

$$F = \{ (|\alpha\rangle, |\beta\rangle, \epsilon) \in \mathcal{C} : tr C^{\Phi} F = \xi \}$$

$$= \text{conv} \{ (|\alpha_C\rangle, |\beta_C\rangle, |\alpha_C\rangle |\beta_C\rangle) : |\alpha_C\rangle |\Phi | \beta_C\rangle = \xi \}.$$  

In particular, we want to show that its dimension $\dim F$ is strictly lower than the dimension of a facet,
$D - 1$, or equivalently that the codimension of the face in the classical polytope is $\Delta = D - \dim F > 1$.

We start by considering the consequences of assuming no quantum advantage, i.e. $\xi_Q = \xi_Q$. The optimal quantum gain can be written as $\xi_Q = \max \tr C^T \Phi^T$, where $C = \sum_{x,y} |x\rangle\langle y| \otimes |x\rangle\langle y|$ is the quantum correlator matrix, which is fully characterised [12] by the inner products of an arbitrary set of unit vectors in $\mathbb{R}^{m_A + m_B}$. In terms of the characterisation in [12], $|x\rangle = \langle x|Q = (-1)^{k_x}|\psi\rangle$, and analogously for Bob’s strategy. In order to write this optimisation problem as a semidefinite program [18 37] we define the following Gram matrix

$$\tilde{Q} = \begin{pmatrix} R & C \\ C & S \end{pmatrix},$$

with $R_{xx'} = \langle x|\alpha_x\rangle$ and $S_{yy'} = \langle \beta_y|\beta_{y'}\rangle$. An equivalent characterisation of a Gram matrix of unit vectors is $\tilde{Q} \succeq 0$ and $\tilde{Q}_{ii} = 1$ for all $i \in \{m_A + m_B\}$, and consequently we can write the maximum quantum gain as:

$$\xi_Q = \max \tr \tilde{Q} \tilde{\Phi},$$

s.t. $\text{diag}(\tilde{Q}) = (1, \ldots, 1), \tilde{Q} \succeq 0,$

where $\tilde{\Phi} = \frac{1}{2} \begin{pmatrix} 0 & \tilde{\Phi}^T \\ \Phi^T & 0 \end{pmatrix}$.

Consider now the Lagrangian,

$$L = \tr \tilde{Q} \tilde{\Phi} - \sum_i t_i \left( \tr \tilde{Q} i\tilde{Q} - 1 \right)$$

$$= \tr \left[ \tilde{Q} (\tilde{\Phi} - \sum_i t_i i\tilde{Q}) \right] + \sum_i t_i,$$

where $t_i \in \mathbb{R}$ are the Lagrange multipliers. Therefore, 

$$\max_{\tilde{Q} \succeq 0} L = \begin{cases} +\infty & \text{if } \sum_i t_i i\tilde{Q} - \tilde{\Phi} \not\succeq 0, \\ \sum_i t_i & \text{if } \sum_i t_i i\tilde{Q} - \tilde{\Phi} \succeq 0, \end{cases}$$

and thus the original SDP [11] can be written in its dual form,

$$\min \sum_i t_i \text{ s.t. } \sum_i t_i i\tilde{Q} \succeq \tilde{\Phi}.$$  

This holds because the primal and dual SDPs satisfy the condition for strong duality.

From the above construction it follows that $\tr \tilde{Q} \tilde{\Phi} \leq \xi_Q \leq \sum_i t_i$ for any pair of primal and dual feasible solutions. Furthermore, by strong duality, the maximum primal value $\xi_Q$ equals the solution of the dual [14]. That is, the optimal value is attained if and only if $\tr \tilde{Q} \tilde{\Phi} = \sum_i t_i$, or equivalently if the spurious term in the Lagrangian vanishes: $\tr \tilde{Q} (\sum_i t_i i\tilde{Q} - \tilde{\Phi}) = 0$ (complementary slackness). Note that by the form of the dual problem [14], the $t_i$ are non-negative numbers; indeed, by our assumption that $\Phi$ has no all-zero rows or columns, we even can conclude that all $t_i > 0$.

Now, our hypothesis of no quantum advantage implies that $\xi_Q$ coincides with its classical counterpart $\xi_c = \langle \beta_c|\Phi|\alpha_c\rangle = \tr |s\rangle\langle s| \tilde{\Phi}^T$, where we have defined $|s\rangle = |\alpha_c\rangle \oplus |\beta_c\rangle$. In other words, $\xi_Q$ can be reached with a classical correlation matrix $Q = |s\rangle\langle s|$. Letting $\Gamma = \sum_i t_i i\langle i| = \frac{1}{2} \Lambda \oplus \Lambda > 0$, where $\Sigma$ and $\Lambda$ are diagonal positive definite matrices, the slackness condition reads $\tr (\Gamma - \Phi) |s\rangle\langle s| = 0$, which in turn implies that $(\Gamma - \Phi) |s\rangle = 0$, since both $\Gamma - \Phi$ and $|s\rangle\langle s|$ are positive semidefinite. Therefore, $\Sigma(\alpha_c) = \Phi|\beta_c\rangle$ and $\Lambda|\beta_c\rangle = \Phi^T|\alpha_c\rangle$, or equivalently

$$|\alpha_c\rangle = \Sigma^{-1} \Phi|\beta_c\rangle, \ |\beta_c\rangle = \Lambda^{-1} \Phi^T|\alpha_c\rangle := F|\alpha_c\rangle.$$

This means that once Alice’s optimal strategy is fixed, Bob’s best strategy is uniquely determined by her choice and vice versa. Assuming w.l.o.g. $m_A \leq m_B$, the dimension of the face $F$ generated by such strategies is then

$$\dim F = \dim \text{aff} \{ (|\alpha_c\rangle, |\alpha_c\rangle, |\alpha_c\rangle, |\alpha_c\rangle, |F^T) : |\alpha_c\rangle \text{ opt.} \} \leq \dim \text{aff} \{ (|\alpha_c\rangle, |\alpha_c\rangle, |\alpha_c\rangle, |\alpha_c\rangle) : |\alpha_c\rangle \in \{\pm 1\}^{m_A} \}$$

$$= m_A + \frac{m_A(m_A - 1)}{2},$$

where aff denotes the affine span, and where the second term in the dimension comes from the fact that the affine span of the matrices $|\alpha_c\rangle$ consists precisely of the real symmetric matrices with 1’s along the diagonal of the codimension of the face in the Bell polytope of $\Delta \geq m_B + m_A m_B - \frac{1}{2} m_A (m_A - 1) > 1$.

Similarly, we can bound the dimension of this facet in the correlation polytope, which is solely generated by the correlators $|\alpha_c\rangle \langle \alpha_c| \Phi^T$, leading to $\Delta_0 \geq m_A m_B - \frac{1}{2} m_A (m_A - 1) > 1$, unless $m_A = m_B = 1$, which is a trivial case.

Therefore, we see that the defined face is not a facet in either setting.

Proof of Theorem [8]. Starting from the definition $|q\rangle = |\alpha\rangle Q \oplus |\beta\rangle Q$, we can write the quantum bias as $\xi_Q = q^T \Phi|\alpha\rangle Q = (q^T \Phi Q)|q\rangle$. It is straightforward to check, following the steps in the proof of Theorem [8] that the complementarity slackness condition translates into $(\mathbb{I} \otimes \Gamma - \mathbb{I} \otimes \Phi)|q\rangle = 0$, which in the case of exhaustive games leads to

$$|\beta\rangle Q = (\mathbb{I} \otimes F)|\alpha\rangle Q,$$

or equivalently $|\beta\rangle = \sum_{x'} F_{x'x}|x\rangle$, with the same matrix $F$ as in the proof of Theorem [8]. This shows that Bob’s optimal quantum strategy (as encoded in the vectors $\{|\beta\rangle\}$) is fully determined by Alice optimal strategy (the vectors $\{|\alpha\rangle\}$).

In order to bound the dimensionality of the behaviours that determine the corresponding face we note that $c_{xy} = \langle x|\alpha\rangle \langle y|\beta\rangle = \sum_{x'} F_{x'x} \langle x|\alpha\rangle \langle x|\alpha\rangle$. The dimension of the affine span of the optimal behaviours
C is clearly bounded by that of the Gram matrices 
\[ [(\alpha_x|\alpha_x)]_x, \] which we recall to be real symmetric
matrices with diagonal elements \((\alpha_x|\alpha_x) = 1,\) thus
the dimension is bounded by \(\frac{1}{\lambda} m_A (m_A - 1)\) and
therefore the codimension of the face \(Q_0\) is bounded by
the classical bound derived in Theorem [1]. To get the
same for \(Q,\) we need to control also the marginal parts
of the behaviours, which can be written as
\[ \alpha_x = (\psi|\alpha_x) \text{ and } \beta_y = (\psi|\beta_y) = \sum_x F_{uxy} \alpha_x. \]
In other words, \(|\beta\rangle = F|\alpha\rangle\) is fully determined by
\(|\alpha\rangle,\) just as we had seen for the deterministic classical
strategies, hence only \(m_A\) is added to the dimension
of the face.

For non-exhaustive XOR games we then proceed as
in Theorem [2]. In particular, XOR games do not define
facets of either \(Q,\) or \(Q_0.\) In the quantum correlation
set \(Q,\) the XOR games that can define facets have
only one input each for Alice and Bob, and that
leaves only \(|c_{xy}\rangle| \leq 1,\) which are indeed facet defining
inequalities, but they are trivial, as they correspond to
the non-negativity of probability.

To arrive at the conclusion for \(Q_0,\) having no facets
at all, note that any purported facet is exposed, so
it has to be defined by an XOR game, and for those
we just showed that only the trivial inequalities define
facets. \(\Box\)

D. Dimension bound for non-local computation
and (asymptotic) attainability

In [21], Linden et al. introduced the cooperative
games of non-local computation (NLC) and
showed that quantum strategies provide no advantage
over classical ones, although stronger forms of non-
signaling correlations allowed perfect success. In
the problem of non-local computation Alice and Bob need
to collaborate in order to compute to a boolean function
\(f(z)\) of a string of \(n\) bits, \(z = z_1z_2...z_n,\) without
communicating with each other during the computation,
and without individually learning anything about
the input string \(z.\) The inputs are promised to be
given with an arbitrary probability distribution
\(p(z)\) known to Alice and Bob and are split in two
correlated signals \(x = x_1x_2...x_n\) and \(y = y_1y_2...y_n\)
so that \(z = x \oplus y\) (meaning \(x_i \oplus y_i = z_i,\) for all
\(i = 1, ..., n\)) which are given to Alice and Bob
respectively. In order to enforce that Alice and Bob
do not learn anything about \(z,\) it is necessary that
\(p(x_i = 0) = p(x_i = 1) = \frac{1}{2}\) for all \(x_i,\) and idem
for \(y_i.\) It is hence clear that NLC is a particular
instance of an XOR game with \(q(x, y) = \frac{1}{2} q(x \oplus y)\)
and \(a \oplus b = f(z = x \oplus y).\) The corresponding game matrix
is given by \(F_{NLC} = \sum_{x,y} (-1)^{f(x \oplus y)} q(x \oplus y)|x \oplus y\rangle \langle y |.\)

In [21], it is proved that the game matrix is diagonal
in the Fourier (Hadamard) basis, \(|\alpha\rangle = \sum (-1)^{ux} |x\rangle,\)
where \(u \cdot x\) is the inner product modulo 2 of the
bit strings \(u\) and \(x.\) As a consequence, \(\xi_Q \leq 2^{n-1}||\Phi_{NLC}|| := \xi^*,\) where \(||.||\) denotes the
operator norm. Moreover, it was shown that the upper bound
is attained by a classical local strategy with \(\xi_c = \xi^*,\)
i.e. by a suitably chosen pair of vectors \(|\alpha_c\rangle, |\beta_c\rangle,\) and
one concludes that NLC games present no quantum
advantage. Since the inequality
\[ \xi \leq \xi^* \quad (18) \]
holds for any classical local and quantum average success
probability, this Bell inequality is neither violated by
classical physics nor quantum mechanics and thus
we have that the Bell polytope and the quantum set share
a region of their boundary.

The result stated below as Corollary [2] (because it
follows from Theorem [1] was first proved in [23]. It
says that an NLC Bell inequality is never tight for
any number of inputs. Here we present an alternative
proof, based on directly bounding the dimension of the
face defined by the Bell inequality. This is a
reconstruction of the earlier unpublished proof referenced
in [24].

Corollary 4. For any number \(n\) of input bits, an
NLC game never defines a facet of the Bell polytope,
or of the correlation polytope.

Proof. As \(m_A = m_B = 2^n,\) the dimension of the Bell
polytope is \(D = 4^n + 2^{n+1}.\) In order to see that
the NLC Bell inequality does not define a facet, we
show that the dimension of the affine space generated
by the optimal classical strategies is strictly smaller
than \(D - 1.\) The local classical success probability is
bounded by \(\omega^* = \frac{1}{2} (1 + \xi^*),\) which only depends on
\(\lambda = ||\Phi_{NLC}||.\) Let \(|u_0\rangle\) be a corresponding eigenvector
of \(\pm \lambda.\)

Now, \(\xi\) achieves the maximum value when \(|\alpha_c\rangle\)
and \(|\beta_c\rangle\) are both proportional to \(|u_0\rangle,\) i.e. \(|\alpha_c\rangle = \pm |\beta_c\rangle = \pm |u_0\rangle.\) Thus, we consider the eigenspaces
of the two eigenvalues \(\pm \lambda,\) which we denote by
\(\text{Eig}(\lambda)\) and \(\text{Eig}(-\lambda),\) respectively. There are two cases:
either \(|\alpha_c\rangle = |\beta_c\rangle \in \text{Eig}(\lambda)\) or \(|\alpha_c\rangle = -|\beta_c\rangle \in \text{Eig}(-\lambda).\)

These we can write in a single equation as \(|\beta_c\rangle = F|\alpha_c\rangle,\) with \(F = 2\Pi_1 - 1\) and \(\Pi_1\) the projector onto
the eigenspace \(\text{Eig}(\lambda).\)

Therefore, the face \(F_{NLC}\) of these optimal strategies
define, is contained in the following affine subspace:
\[ \text{aff} \left\{(|\alpha_c\rangle, F|\alpha_c\rangle, |\alpha_c\rangle, |\alpha_c\rangle) : |\alpha_c\rangle \in \{\pm 1\}^{2^n}\right\}, \quad (19) \]
whose dimension we have already bounded before, in
the proof of Theorem [1] Eq. (16), and so
\[ \dim F_{NLC} \leq 2^n + 2^{n-1}(2^n - 1), \quad (20) \]
and codimension \(\Delta \geq 2^n + 2^{n-1}(2^n + 1) > 1.\)

For the correlation polytope, we get similarly
that the dimension of the face is upper bounded by
\(2^{n-1}(2^n - 1),\) resulting in a bound of \(\Delta_C \geq 2^{n-1}(2^n + 1) > 1\) for the codimension of the face.

The bound Eq. (20) is achievable with equality for
\(F = 2^n - 1;\) this describes a game where \(x\) is uniformly
distributed, and \(y = x,\) and to win, Alice and Bob
have to output the same bit \(a = b.\) This is evidently
possible with probability 1, using any local strategy
Note that the linear span, in Eq. (22), denote it \( L_1 \) the origin. This is due to the fact that otherwise the affine span of the latter does not contain \( \omega_c = 1 \), equal to the no-signalling bound, so in some sense it is trivial, but it is worth noting that the Bell inequality is not a trivial one (\(|c_{xy}| \leq 1\)).

We can get a slightly better bound, sometimes much better depending on the game matrix \( \Phi \), by exploiting the fact that the latter is Hermitian and that the optimal local strategies must lie either in \( \text{Eig}(\lambda) \) or in \( \text{Eig}(\lambda) \); denote their dimensions by \( k \) and \( \ell \), respectively, so that \( k + \ell \leq 2^n \). In the following we can discard the extreme cases \( k = 2^n \) and \( \ell = 2^n \), since those correspond to \( \Phi \propto I \), which we have just discussed.

From the previous analysis, we have

\[
F_{NL\text{C}} = \text{conv}\left\{ (|\alpha_c\rangle, \epsilon|\alpha_c\rangle, \epsilon|\alpha_c\rangle \mid \epsilon = \pm 1, |\alpha_c\rangle \in \text{Eig}(\epsilon\lambda) \cap \{\pm 1\}^{2^n} \right\} \quad (21)
\]

\[
\subset \text{span}\left\{ (|\alpha\rangle, \epsilon|\alpha\rangle, \epsilon|\alpha\rangle \mid \epsilon = \pm 1, |\alpha\rangle \in \text{Eig}(\epsilon\lambda) \right\} \quad (22)
\]

\[
= \text{span}\left\{ (|\alpha\rangle, |\alpha\rangle, |\alpha\rangle \mid |\alpha\rangle \in \text{Eig}(\lambda) \right\} \quad (23)
\]

Note that the linear span, in Eq. (22), denote it \( L \), has a dimension at least 1 larger than the face \( F_{NL\text{C}} \), because the affine span of the latter does not contain the origin. This is due to the fact that otherwise the optimal classical winning probability were \( \frac{1}{2} \), corresponding to a bias 0, but it is easily seen that XOR game always have some positive bias [22].

Now, the spaces in Eq. (23) have dimension \( \ell + \frac{1}{2}(\ell + 1) \) and \( k + \frac{1}{2}k(k + 1) \), respectively, and so

\[
\dim F_{NL\text{C}} \leq \dim L - 1 = k + \ell + \frac{k(k + 1)}{2} + \frac{\ell(\ell + 1)}{2} - 1. \quad (24)
\]

Among the pairs with \( k + \ell \leq 2^n \), and as explained before – excluding \( \Phi \propto I \), i.e. imposing \( k, \ell < 2^n \), the r.h.s. is maximised at \( k = 2^n - 1, \ell = 1 \), for which values it reproduces the previously obtained bound \( 2^n + 2^{n-1}(2^n - 1) \). Note that this restricts the game severely, since the game matrix has only the two possible eigenvalues \( \pm 1 \), w.l.o.g. with multiplicities \( 2^n - 1 \) and 1, respectively: this means \( \Phi = \lambda F = \lambda(2I_\lambda - I) \) and \( I - I_\lambda \) has rank 1. In all other cases, Eq. (24) is strictly better than the bound (20). For the correlation polytope, we get similarly that the dimension of the face is upper bounded by \( \frac{k(k + 1)}{2} + \frac{(\ell + 1)}{2} - 1 \leq 2^{n-1}(2^n - 1) \).

These bounds can be attained, if not exactly, then asymptotically, as we will show on the example \( k = 2^n - 1, \ell = 1 \). By the analysis of [24], this essentially leaves only the game matrix \( \Phi = \lambda(1 - 2^{-n}J) - \lambda 2^{-n}J \), where \( J = |1\rangle\langle 1| \) is the all-one-matrix, and \( |\lambda| = \sum_{n=1}^n |x⟩ \). Since the sum of the absolute values of the entries of \( \Phi \) has to be 1, this fixes the value of \( \lambda = \frac{1}{\sqrt{2^n - 1}} \), corresponding to the winning probability (classical and quantum) \( \omega_c = \frac{1}{2} \left( 1 + \frac{\sqrt{2^n - 1}}{2^n - 1} \right) \approx \frac{2}{3} < 1 \) for \( n \geq 2 \), i.e. the game, and with it its Bell inequality, is nontrivial. The game can be described as follows: \( x \) and \( y \) are jointly distributed according to \( q(x,y) > 0 \), and to win, Alice and Bob have to output the same bit \( a = b \) if \( x = y \), and different bits \( a \neq b \) if \( x \neq y \). The optimal classical local strategies are on the one hand \( |\alpha_c\rangle = |1\rangle = -|\beta_c\rangle \) (corresponding to the single negative eigenvalue \( -\lambda \) of \( \Phi \), and \( |\alpha_c\rangle = |\beta_c\rangle \perp |I\rangle \) (corresponding to the \( 2^n - 1 \)-fold eigenvalue \( \lambda \). The latter means that \( |\alpha_c\rangle = |\beta_c\rangle \) has to have exactly \( 2^{n-1} \) entries +1 and \( 2^{n-1} \) entries −1, of which there are \( 2^{n-1} \). With this, we determine the dimension of the corresponding face of the correlation Bell polytope as

\[
\dim F = 1 + \text{dim aff} \{ |\alpha_c\rangle, |\alpha_c\rangle : |\alpha_c\rangle \perp (|0\rangle + |1\rangle)^\otimes n \},
\]

where it is understood that \( |\alpha_c\rangle \in \{\pm 1\}^{2^n} \). The affine span on the r.h.s. is precisely the space \( \mathcal{G}_0 \) of real symmetric matrices with 1’s along the diagonal and with the property that all row and column sums are 0. By parameter counting, it is straightforward to see that \( \dim \mathcal{G}_0 = 2^{n-1}(2^n - 3) \), and so we get \( \dim F = 1 + 2^{n-1}(2^n - 3) \sim 2^{n-1}(2^n - 1) \), matching the upper bound to leading order. \( \square \)