Quasilinear Wave Equations and Microlocal Analysis

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Abstract

In this text, we shall give an outline of some recent results (see \cite{3} \cite{4} and \cite{5}) of local wellposedness for two types of quasilinear wave equations for initial data less regular than what is required by the energy method. To go below the regularity prescribed by the classical theory of strictly hyperbolic equations, we have to use the particular properties of the wave equation. The result concerning the first kind of equations must be understood as a Strichartz estimate for wave operators whose coefficients are only Lipschitz while the result concerning the second type of equations is reduced to the proof of a bilinear estimate for the product of two solutions for wave operators whose coefficients are not very regular. The purpose of this talk is to emphasise the importance of ideas coming from microlocal analysis to prove such results.

The method known to prove Strichartz estimates uses a representation eventually approximate of the solution. In the case of the wave equation, the approximation used is the one coming from the Lax method, namely the one connected to the geometrical optics. But it seems impossible, in the framework of the quasilinear wave equations, to construct a suitable approximation of the solution on some interval $[0, T]$, since the associate Hamilton-Jacobi equation develop singularities (it is the caustic phenomenon) at a time connected with the frequency size. We have then to microlocalize, which means to localize in frequencies, and then to work on time interval whose size depend on the frequency considered. It is the alliance of geometric optics and harmonic analysis which allow to establish a quasilinear Strichartz estimate and to go below this minimal regularity in the case of the first kind of equations.

To study the second kind of equations, we are confronted to an additional problem: Contrary to the constant case, the support of the Fourier transform is not preserved by the flow of the variable coefficient wave equation. To overcome this difficulty, we show that the relevant information in the variable case is the concept of microlocalized function due to J.M.Bony \cite{11}. The proof that for solutions of variable coefficient operators, microlocalization properties propagate nicely along the Hamiltonian flows related to the wave operator is the key point in the proof of the result in the second case.

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1. Introduction

In this paper, our interest is to prove local solvability for quasilinear wave equations of the type

\[ (E) \left\{ \begin{array}{l}
\partial_t^2 u - \Delta u - g(u) \cdot \nabla^2 u = Q(\partial u, \partial u) \\
(u, \partial_t u)_{t=0} = (u_0, u_1)
\end{array} \right. \]

where \( g \) is a smooth function vanishing at 0 with value in \( K \) such that \( \text{Id} + K \) is a convex compact subset of the set of positive symmetric matrices and \( Q \) is a quadratic form on \( \mathbb{R}^{d+1} \). Our interest proceeds also for cubic quasilinear wave equations of the type

\[ (EC) \left\{ \begin{array}{l}
\partial_t^2 u - \Delta u - \sum_{1 \leq j,k \leq d} g^{j,k} \partial_j \partial_k u = \sum_{1 \leq j,k \leq d} \bar{Q}_{j,k}(\partial g^{j,k}, \partial u) \\
\Delta g^{j,k} = \bar{Q}_{j,k}(\partial u, \partial u) \\
(u, \partial_t u)_{t=0} = (u_0, u_1)
\end{array} \right. \]

where \( Q_{j,k} \) and \( \bar{Q}_{j,k} \) are quadratic forms on \( \mathbb{R}^{d+1} \) and where all the quadratic forms are supposed to be smooth functions of \( u \).

The basic tool to prove local solvability for such equations is the following energy estimate, also valid for the symmetric systems

\[
\|\partial u(t,\cdot)\|_{H^{s-1}} \leq \|\partial u(0,\cdot)\|_{H^{s-1}} e^{CT} \int_0^T \|\partial g(\tau,\cdot)\|_{L^\infty} d\tau.
\]

So thanks to classical arguments, local solvability derives easily from the control of the quantity

\[
\int_0^T \|\partial g(\tau,\cdot)\|_{L^\infty} d\tau.
\]

In the framework of the equation \((E)\), the control of this key quantity requires initial data \((u_0, u_1)\) in \( H^s \times H^{s-1} \) for \( s > \frac{d}{2} + 1 \) while in the framework of \((EC)\) (with small data, which makes sense in this case) it only requires initial data \((u_0, u_1)\) in \( H^{\frac{s+1}{2}} \times H^{\frac{d}{2} - \frac{1}{2}} \).

The goal of this paper is to go below this regularity for the initial data. Let us first have a look at the scaling properties of equations \((E)\) and \((EC)\). If \( u \) is a solution of \((E)\) or \((EC)\), then \( u_\lambda(t,x) \equiv u(\lambda t, \lambda x) \) is also a solution of the same equation. The space which is invariant under this scaling for the couple \((u_0, u_1)\) is \( H^{\frac{s}{2}} \times H^{\frac{d}{2} - \frac{1}{2}} \). So the results given by the classical energy estimate appear to require more regularity than the scaling in the two cases.
In fact, the energy methods despise the particular properties of the wave equation. It is on the impulse of the pioneer work of S. Klainerman (see [19]) that a vast series of works have been attached to improve the span life time of regular solutions of quasilinear wave equations using the Lorentz invariance. Let us notice the results of S. Alinhac (see [1] and [2]), of L. Hörmander (see [14]), of F. John (see [15]), of F. John and S. Klainerman (see [16]), of S. Klainerman (see [20]) and of J-M. Delort ([12]) concerning the Klein-Gordon equation.

In this talk, we shall limit our self to the question of minimal regularity. Concerning this subject, the only case studied is the semilinear case, which means the case of the equation (E) with \( g \equiv 0 \). As it has been shown by S. Klainerman and M. Machedon (see [21] and [22]) we can, when the quadratic form \( Q \) verifies a structure condition known by “null condition”, nearly reach the space invariant by scaling. For any quadratic form \( Q \), we have the following theorem, proved by G. Ponce and T. Sideris in [27].

**Theorem 1.1** Let us define \( \tilde{s}_d \) by

\[
\tilde{s}_d = \frac{d}{2} + \frac{1}{2} \quad \text{if} \quad d \geq 3 \quad \text{and} \quad \tilde{s}_2 = \frac{7}{4}.
\]

Let \((u_0, u_1)\) be a Cauchy data in \( H^s \times H^{s-1} \) with \( s > \tilde{s}_d \) then there exists a time \( T \) such that there exists a unique solution \( u \) of the equation (E) such that

\[
u \in L^\infty([0, T]; H^s) \cap L^p([0, T]; H^{s-1}) \quad \text{and} \quad \partial_t u \in L^2([0, T]; L^\infty).
\]

The proof of this result lies on specific properties of the wave equation, namely the following Strichartz estimate

\[
\| \partial_t u \|_{L^2_x(L^\infty_t)} \leq C \left( \| \partial_x u(0, \cdot) \|_{H^{s-1}} + \| \square u \|_{L^1_x(H^{s-1})} \right),
\]

for \( d \geq 3 \) and \( s > \frac{d}{2} + \frac{1}{2} \). (2)

Indeed, if we couple it with the standard energy estimate

\[
\| \partial_t u \|_{L^p_x(H^{s-1})} \leq C \left( \| \partial_x u(0, \cdot) \|_{H^{s-1}} + \| \square u \|_{L^1_x(H^{s-1})} \right)
\]

we obtain, owing to the tame estimates and the Cauchy-Schwarz inequality

\[
\| \partial_t u \|_{L^2_x(L^\infty_t)} + \| \partial_t u \|_{L^p_x(H^{s-1})} \leq C \left( \| \partial_x u(0, \cdot) \|_{H^{s-1}} + T^\frac{1}{2} \| \partial_x u(t, \cdot) \|_{L^2_x(L^\infty_t)} \| \partial_t u \|_{L^2_x(H^{s-1})} \right),
\]

which ensures by the theory of evolution equations the local solvability for \( T \leq \frac{1}{\| \partial_x u(0, \cdot) \|_{H^{s-1}}^2} \).

In other respects, in [26], H. Linblad shows that for \( d = 3 \) the above result is optimum, which means that the problem (E) with \( g \equiv 0 \) is not wellposed in \( H^2 \).

Let us also notice that the same kind of result is also true on the Heisenberg group (see [19]).

The authors (see [3] and [4]) adjust a method followed by D. Tataru (see [32]) based on microlocal analysis to improve the minimal regularity for the equation (E) in the quasilinear case. Let us recall this result...
Theorem 1.2 If $d \geq 3$, let $(u_0, u_1)$ be in $H^s \times H^{s-1}$ for $s > s_d$ with $s_d = \frac{d}{2} + \frac{1}{2} + \frac{1}{6}$. Then, a positive time $T$ exists such that a unique solution $u$ of the equation $(E)$ exists such that

$$\partial u \in C([0, T]; H^{s-1}) \cap L^2([0, T]; L^\infty).$$

Remarks

- This theorem has been proved with $1/4$ instead than $1/6$ in [3] and then improved a little bit in [4] and proved with $1/6$ by D. Tataru in [32]. Strichartz estimates for quasilinear equations are the key point of the proofs.
- Let us notice that the improvement with $1/6$ of D. Tataru in [32] is due to a different manner of counting the intervals where microlocal estimates are true.
- Recently, S. Klainerman and I. Rodnianski in [24] have obtained a better index in dimension 3. Their proof is based on very different methods.
- Let us notice that we have also improved the minimal regularity in dimension 2, but the gain is only of $\frac{1}{8}$ derivative, this is explained by the mean dispersif effect in this dimension already known for the constant case.

The analogous theorem in the case of equation $(EC)$ is the following

Theorem 1.3 If $d \geq 4$, let $(u_0, u_1)$ be in $H^s \times H^{s-1}$ with $s > \frac{d}{2} + \frac{1}{6}$ such that $\|\gamma\|_{H^{s-1}}$ is small enough. Then, a positive times $T$ exists such that a unique solution $u$ of $(EC)$ exists such that

$$\partial u \in C([0, T]; H^{s-1}) \cap L^2_T(\dot{B}^\frac{d}{4} - \frac{1}{2}_{2,2}), \quad \text{for} \quad d \geq 5,$$

and

$$\partial u \in C([0, T]; H^{s-1}) \cap L^2_T(\dot{B}^\frac{1}{6} - \frac{1}{2}_{6,2}) \quad \text{and} \quad \partial g \in L^1_T(L^\infty) \quad \text{for} \quad d = 4.$$

Remarks

- The case when $d \geq 5$ can be treated only with Strichartz estimates simply because if $\partial u$ belongs to $L^2_T(\dot{B}^\frac{d}{4} - \frac{1}{2}_{2,2})$ then $\partial g$ is in $L^1_T(L^\infty)$.
- The case when $d = 4$ requires bilinear estimates. This fact appears in the statement of Theorem 1.3 through the following phenomenon: The fact that $\partial u$ is in $L^2_T(\dot{B}^\frac{1}{6} - \frac{1}{2}_{6,2})$ does not imply that the time derivative of $g$ belongs to $L^1_T(L^\infty)$. Of course this condition is crucial in particular to get the basic energy estimate. But we have been unable to exhibit a Banach space $\mathcal{B}$ which contains the solution $u$ and such that if a function $a$ is in $\mathcal{B}$, then $\partial \Delta^{-1}(a^2)$ belongs to $L^1_T(L^\infty)$.
- For technical obstructions, this theorem is limited to the dimensions $d \geq 4$.

2. Quasilinear Strichartz estimates

Following the process of G. Ponce and T. Sideris in [27], we reduce the proof of the theorem 1.2 to the following a priori estimate
Theorem 2.1 If \(d \geq 3\), a constant \(C\) exists such that, for any regular solution \(u\) of the equation \((E)\), if

\[
T^{d+ (s-s_d)} \left( \| \gamma \|_{H^d} + T^{s} \| \gamma \|_{H^{s_d-1}} \right) \leq C, \quad \text{with} \quad s > s_d
\]

then we have

\[
\| \partial u \|_{L^2_t (L^\infty)} \leq C \left( \| \gamma \|_{H^{s_d-1}} + \| Q(\partial u, \partial u) \|_{L^1_t (H^{s_d-1})} \right).
\]

The estimate in hand must be understood as a Strichartz estimate for wave equations with variable coefficients and not very regular. The Strichartz estimates have a long history beginning with Segal’s work \([28]\) for the wave equation with constant coefficients. After the fundamental work of Strichartz \([30]\), it was developed by diverse authors, we refer to the synthesis article of Ginibre and Velo \([13]\) to which it is advisable to add the recent works of Keel-Tao \([18]\) consecrated to some limited cases and of Bahouri, Gérard and Xu \([8]\) for the wave equation on the Heisenberg group. For Strichartz estimates with \(C^\infty\) coefficients, we refer to the result of L. Kapitanski (see \([17]\)). The article of H. Smith (see \([29]\)) constitutes an important step in the study of Strichartz estimates for operators with coefficients not very regular since it proves Strichartz estimates with coefficients only \(C^{1,1}\).

We shall now explain how to establish this quasilinear Strichartz estimate, showing where are the difficulties and what are the essential ideas which allow to overcome them. The method known to prove these estimates uses a representation, eventually approximate, but always explicit of the solution. In the case of the wave equation, the approximation used is the one coming from the Lax method, namely the one connected to the geometrical optics. To make such a method work in the framework of quasilinear wave equations requires a “regularization” of the coefficients also in time. This leads to the following iterative scheme introduced in \([4]\). Let us define the sequence \((u^{(n)})_{n \in \mathbb{N}}\) by the first term \(u^{(0)}\) satisfying

\[
\begin{cases}
\partial_t^2 u^{(0)} - \Delta u^{(0)} = 0 \\
(u^{(0)}, \partial_t u^{(0)})_{t=0} = (S_0 u_0, S_0 u_1),
\end{cases}
\]

and by the following induction

\[
(E_n) \begin{cases}
\partial_t^2 u^{(n+1)} - \Delta u^{(n+1)} - g_{n,T} \cdot \nabla^2 u^{(n+1)} = 0 \\
(u^{(n+1)}, \partial_t u^{(n+1)})_{t=0} = (S_{n+1} u_0, S_{n+1} u_1)
\end{cases}
\]

where \(g_{n,T} \overset{\text{def}}{=} \theta (T^{-1}) g_n\) with \(g_n \overset{\text{def}}{=} g(u^n)\) and \(\theta\) a function of \(\mathcal{D}([-1,1])\) whose value is 1 near 0 and where \(S_n\) is a frequencies truncated operator which only conserves the frequencies lower than \(C 2^{-n}\). Let us introduce some notations which will be used all along this section. If \(s = s_d + \alpha\) where \(\alpha\) is a small positive number, let us define

\[
N^\alpha_T (\gamma) \overset{\text{def}}{=} \| \gamma \|_{H^d} + T^{s} \| \gamma \|_{H^{s_d-1}}.
\]

The assertion we have to prove by induction, for \(T^{d+ \alpha} N^\alpha_T (\gamma)\) small enough, is the following: If \(d \geq 3\),

\[
P_n \begin{cases}
\| \partial u^{(n)} \|_{L^2_t (L^\infty)} \leq C_n T^\alpha N^\alpha_T (\gamma) \\
\| \partial u^{(n)} \|_{L^2_t (H^{s_d-1})} \leq C \| \gamma \|_{H^{s_d-1}}.
\end{cases}
\]
For this, we shall transform the equation \((E_n)\) into a paradifferential equation, more precisely an equation of the type

\[
\partial_t^2 \varphi^{(n+1)} - \Delta \varphi^{(n+1)} - (S_q g_{n,T}) \nabla^2 \varphi^{(n+1)} = \tilde{R}_q(n)
\]

where the term \(\tilde{R}_q(n)\) is a remainder term estimated as agreed and where \(\varphi\) denotes the part of \(u\) which is relative to the frequencies of size \(2^q\).

This transformation of the equation, which is the classical paralinearization defined by J.-M. Bony in \([10]\) is here not sufficient, since it is well known that the paradifferential operators defined in \([10]\) belong to a bad class of pseudodifferential operators \((\text{class } S^{1,1}_{1,1}\) of Hörmander), class in particular devoid of any asymptotic calculus, which forbids of course to envisage any approximate method of type “Lax method”.

The idea is as in \([25]\) to truncate more in the frequencies of the metric \(g\) and to transform the equation \((E_q)\) into the following equation \((EPM_q)\)

\[
\partial_t^2 \varphi^{(n+1)} - \Delta \varphi^{(n+1)} - (S_{\delta q} g_{n,T}) \nabla^2 \varphi^{(n+1)} = R_q(n);\]

where 0 < \(\delta < 1\) and \(S_{\delta q}\) is a frequencies truncated operator which conserves only the frequencies smaller than \(CT^{-1-\delta}2^{\delta q-1}\). We can interpret it as a localization in the pseudodifferential calculus sense \((1, \delta)\) of Hörmander.

This localization allows us to construct an approximation of the solution but engenders a loss in the remainder \(R_q(n)\). This approximation is on the form

\[
\int e^{i\Phi_q(t,x,\xi)} \sigma_q(t,x,\xi) \hat{\gamma}_q(\xi) d\xi
\]

where \(\Phi_q\) is a solution of the Hamilton-Jacobi equation and \(\sigma_q\) is a symbol calculated by resolving a sequence of transport equations; it is about a classical method. But, on account of the the caustic phenomenon, this approximation is microlocal, which means valid only a time interval whose length depends on the size of the frequencies we work with.

Nevertheless, following the classical method, we prove microlocal Strichartz estimates

\[
\|\partial_t \varphi^{(n+1)}\|_{L^2_q(L^\infty)} \leq C_{\beta}(2^q T)^{\beta} 2^{\frac{d}{2}} \left( \|\gamma_q\|_{L^2} + \|R_q(n)\|_{L^1_q(L^2)} \right)
\]

for any positive \(\beta\), where \(\gamma_q \equiv (\nabla(u_0)_q, (u_1)_q)\) and \(I_q\) satisfies

\[
\|\nabla^2 G_{\delta}^{(n)}\|_{L^1_q(L^\infty)} \leq \epsilon
\]

where \(G_{\delta}^{(n)} \equiv S_{\delta q} g(u^{(n)})\) and

\[
|I_q| \leq T(2^q T)^{1-2\delta-\epsilon}.
\]

The condition \([1]\) is imposed by the Hamilton-Jacobi equation while the condition \([3]\) is required by the asymptotic calculus to turn out the “Lax method”.

\[
[1]\text{condition imposed by the Hamilton-Jacobi equation.}
\]

\[
[3]\text{required by the asymptotic calculus to turn out the “Lax method”.}
\]
Finally to prove the complete estimate, the method we used consists in a decomposition of the interval \([0, T]\) on subintervals \(I_q\) on which the above microlocalized estimates are true. The key point is a careful counting of the number of such intervals, for this we shall use here D. Tataru’s version of the method we introduced in \([\text{II}3]\).

The idea consists to seize at the opportunity of this decomposition to compensate the loss on the remainder. To do so, we impose on the interval \(I_q\) the supplementary condition

\[
\|R_q(n)\|_{L^1_q(L^2)} \leq \lambda \|R_q(n)\|_{L^1_q(L^2)}
\]

where the parameter \(\lambda\) is to be determined in the interval \([0, 1]\). This constraint joint to the conditions \([\text{II}1]\) and \([\text{II}5]\) leads by optimization to the best choice

\[
\lambda = (2^q T)^{-\frac{1}{2}}, \quad \delta = \frac{2}{3},
\]

and allows to conclude that the number \(N\) of such intervals is less than \(C(2^q T)^{\frac{1}{2} + \epsilon}\).

If we denote by \((I_q, \ell)\) the partition of the interval \([0, T]\) on such intervals, we can write thanks to \([\text{II}3]\),

\[
\|\partial u_q^{(n+1)}\|_{L^2_q(L^\infty)}^2 \leq C_\beta \sum_{\ell=1}^{N} (2^q T)^{2\beta} 2q(d-1) \left( \|\gamma_q\|_{L^2} + \|R_q(n)\|_{L^1_{q, \ell}(L^2)} \right)^2
\]

\[
\leq C_\beta \sum_{\ell=1}^{N} (2^q T)^{2\beta} 2q(d-1) \left( \|\gamma_q\|_{L^2} + (2^q T)^{-\frac{1}{2}} \|R_q(n)\|_{L^1_{q, \ell}(L^2)} \right)^2.
\]

As \(N\) is less than \(C(2^q T)^{\frac{1}{2} + \epsilon}\), we obtain

\[
\|\partial u_q^{(n+1)}\|_{L^2_q(L^\infty)} \leq C_\beta (2^q T)^{\frac{3}{2}} 2q(d-1) (2^q T)^{\frac{1}{2} + \epsilon} \cdot \left( \|\gamma_q\|_{L^2} + (2^q T)^{-\frac{1}{2}} \|R_q(n)\|_{L^1_{q, \ell}(L^2)} \right). \tag{7}
\]

Now as the loss in the remainder \(R_q(n)\) is of order \(2^q(1-\beta)\) and more precisely, for \(N_0^2(\gamma)\) small enough, we have

\[
\|R_q(n)\|_{L^1_q(L^2)} \leq c_q C 2^{-q(d-1)} (2^q T)^{(s-\frac{1}{2}-\frac{1}{2}+\delta)} T^{s-\frac{1}{2}-\frac{1}{2}} \|\gamma\|_{H^{s-1}}
\]

\[
\cdot \left( 1 + T^\frac{3}{2} \|\partial u^{n+1}\|_{L^2_q(L^\infty)} \right), \tag{8}
\]

where \((c_q) \in \ell^2\). We deduce, owed to the choice of \(\delta\) that

\[
\|\partial u_q^{(n+1)}\|_{L^2_q(L^\infty)} \leq c_q C (2^q T)^{-(\alpha-\frac{1}{2}-\beta)} T^{s-\frac{1}{2}-\frac{1}{2}} \|\gamma\|_{H^{s-1}} \left( 1 + T^\frac{3}{2} \|\partial u^{n+1}\|_{L^2_q(L^\infty)} \right)
\]

which implies the result by summation.
3. Quasilinear bilinear estimates

The method used here is not without any interaction with the one used to prove the theorem 1.3. As in the case of equation \((E)\), the basic fact is the control of
\[
\int_0^T \|\partial g(\tau, \cdot)\|_{L^\infty} d\tau,
\]
and the proof of the theorem 1.3 follows from the following a priori estimate

**Theorem 3.1** If \(d \geq 4\), a constant \(C\) exists such that, for any regular solution \(u\) of the equation \((EC)\), if its initial data \(\gamma\) is small enough and
\[
T^{\frac{s-d}{2} + \frac{1}{6}} \|\gamma\|_{H^{s-1}} \leq C,
\]
then we have
\[
\|\partial \Delta^{-1} Q(\partial u, \partial u)\|_{L^1_T(L^\infty)} \leq C \|\gamma\|_{H^{s-1}}^2.
\]

This is the quasilinear version of the following bilinear estimate owed to D. Tataru and S. Klainerman (see [23]).

**Proposition 3.1** Let \(u\) be a solution of \(\partial_t^2 u - \Delta u = 0\) and \((\partial u)_{t=0} = \gamma\). Then, if \(d \geq 4\),
\[
\|\partial \Delta^{-1} Q(\partial u, \partial u)\|_{L^1_T(L^\infty)} \leq C_{\epsilon, T} \|\gamma\|_{H^{s-1}}^2.
\]

**Remark** We find a gain of one derivative from the regularity of the initial data compared with the product laws and a gain of half a derivative about the regularity of the initial data compared with purely Strichartz methods.

To explain the basic ideas of bilinear estimates, let us first consider the case of constant coefficients. As
\[
\partial_t \Delta^{-1} \left( \partial_j u(t, \cdot) \partial_k u(t, \cdot) \right) = \Delta^{-1} \left( \partial_j \partial_t \partial_k u(t, \cdot) \right) + \Delta^{-1} \left( \partial_j \partial_t \partial_k u(t, \cdot) \right),
\]
we have to control expression of the type
\[
\int_0^T \|\Delta^{-1} \left( \partial_t \partial_j \partial_k u(t, \cdot) \right)\|_{L^\infty} dt.
\]

For this we introduce Bony’s decomposition which consists in writing
\[
ab = \sum_q S_{q-1}^{a} \Delta_q b + \sum_q S_{q-1}^{b} \Delta_q a + \sum_{-1 \leq j \leq 1} \Delta_q \Delta_{q-j} b.
\]

When \(d \geq 4\), we have \(\|\partial^k u_q\|_{L^2_T(L^\infty)} \leq C 2^q \left( \frac{d}{2} - 1 + k - 1 \right) \|\gamma_q\|_{L^2}\), then it is easy to prove that
\[
\left\|\Delta^{-1} \left( \sum_q S_{q-1}^{a} \partial^2 u \partial u_q \right)\right\|_{L^1_T(L^\infty)} \leq C \|\gamma\|_{H^{1-1}}^2.
\]

The symmetric term can be treated exactly along the same lines. The remainder term
\[
\Delta^{-1} \left( \sum_{-1 \leq j \leq 1} \partial^2 u_q \partial u_{q-j} \right) \tag{9}
\]
is much more difficult to treat in particular in dimension 4. The idea introduced by
D. Tataru and S. Klainerman (see [23]) consists to treat this term using precised
Strichartz estimates and interaction lemma.

The precised Strichartz estimates are described by the following proposition.

**Proposition 3.2** Let $C$ be a ring of $\mathbb{R}^d$. If $d \geq 3$, a constant $C$ exists such that
for any $T$ and any $h \leq 1$, if $\text{Supp} \, \hat{u}_j$ are included in a ball of radius $h$ and in the
ring $C$, we have

$$
\|u\|_{L^2_T(L^\infty)} \leq C (h^{d-2} \log(e + T))^{\frac{1}{2}} (\|u_0\|_{L^2} + \|u_1\|_{L^2}),
$$

where $u$ denotes the solution of $\partial_t^2 u - \Delta u = 0$ and $\partial_t^j u_{t=0} = u_j$.

As usual it is deduced with the $TT^\star$ argument from the following dispersive
inequality.

**Lemma 3.1** A constant $C$ exists such that if $u_0$ and $u_1$ are functions in $L^1(\mathbb{R}^d)$
such that

$$
\text{Supp} \, \hat{u}_j \subset C \quad \text{and} \quad \max \{ \delta(\text{Supp} \, \hat{u}_0), \delta(\text{Supp} \, \hat{u}_1) \} \leq h,
$$

then, for any $\tilde{\alpha}$ between 0 and $d - 1$, we have

$$
\|u(t, \cdot)\|_{L^\infty} \leq \frac{C h^d \alpha}{t^{\frac{d-\alpha}{2}}} (\|u_0\|_{L^1} + \|u_1\|_{L^1}),
$$

where $u$ denotes the solution of $\partial_t^2 u - \Delta u = 0$ and $\partial_t^j u_{t=0} = u_j$.

This inequality is proved in [23] in the case $\alpha = d - 1$. The general case is
obtained by interpolation with the classical Sobolev embedding.

Let us now show how to take account the interactions of the solutions to control
the accumulation of frequencies at the origin in the study of the remainder term.

**Lemma 3.2** “Interaction Lemma” There exists a constant $C$ such that if $v_1$ and $v_2$
are two solutions of $\partial_t^2 v_j - \Delta v_j = 0$ satisfying $(\partial v_j)_{t=0} = \gamma_j$ with $\text{Supp} \, \hat{\gamma}_j \subset C$
we have for $0 < h < 1$

$$
\|\chi(h^{-1}D) (\partial^2 v_1 \partial v_2)\|_{L^1_T(L^\infty)} \leq C h^{d-2} \log(e + T) \|\gamma_1\|_{L^2} \|\gamma_2\|_{L^2},
$$

where $\chi$ is a radial function in $\mathcal{D}$ which is equal to 1 near the origin.

Let us define $(\phi_\nu)_{1 \leq \nu \leq N_h}$ a partition of unity of the ring $C$ such that $\text{Supp} \, \phi_\nu \subset B(\xi_\nu, h)$. Then, using the fact that the support of the Fourier transform of the
product of two functions is included in the sum of the support of their Fourier
transform, a family of functions $(\tilde{\phi}_\nu)_{1 \leq \nu \leq N_h}$ exists such that $\text{Supp} \, \tilde{\phi}_\nu \subset B(-\xi_\nu, 2h)$ and

$$
\chi(h^{-1}D) (\partial^2 v_1 \partial v_2) = \sum_{\nu=1}^{N_h} \chi(h^{-1}D) (\partial^2 \tilde{\phi}_\nu(D) v_1 \partial \phi_\nu(D) v_2).
$$
Applying Proposition 3.2 gives
\[ \| \chi (h^{-1}D)(\partial^2 v_1 \partial v_2) \|_{L^1_t(L^{\infty})} \leq C h^{d-2} \log(e + T) \sum_{\nu=1}^{N_h} \| \tilde{\phi}_\nu(D) \gamma_1 \|_{L^2} \| \phi_\nu(D) \gamma_2 \|_{L^2}. \]

The Cauchy Schwarz inequality implies that
\[ \| \chi (h^{-1}D)(\partial^2 v_1 \partial v_2) \|_{L^1_t(L^{\infty})} \leq C h^{d-2} \log(e + T) \left( \sum_{\nu=1}^{N_h} \| \tilde{\phi}_\nu(D) \gamma_1 \|_{L^2} \right)^{1/2} \left( \sum_{\nu=1}^{N_h} \| \phi_\nu(D) \gamma_2 \|_{L^2} \right)^{1/2}. \]

The almost orthogonality of \((\tilde{\phi}_\nu(D) \gamma_1)_{1 \leq \nu \leq N_h}\) and \((\phi_\nu(D) \gamma_2)_{1 \leq \nu \leq N_h}\) implies (11) and leads then to the estimate of the remainder term (10) by rescaling.

To establish the theorem 3.1, we shall follow the steps of the proof of the theorem 2.1 which consists owing to the gluing method to reduce the problem to the proof of “microlocal” bilinear estimates. The generalization of the precise Strichartz estimates to the framework of the equation \((EC)\) doesn’t cost more than the generalization of the Strichartz estimates to the framework of the equation \((E)\), the supplementary difficulty to study the equation \((EC)\) lies in the generalization of the interaction lemma. The preservation of the support of the Fourier transform by the flow of the wave equation is the crucial point in the proof of this lemma. The defect of this property in the case of the variable coefficients constitutes the additional major problem in the proof of the theorem 3.1.

To palliate this difficulty, we have used a finer localization in phase space. This localization is given by the concept of microlocalized function near a point \(X = (x, \xi)\) of the cotangent space \(T^*\mathbb{R}^d\) (the cotangent space of \(\mathbb{R}^d\)). More precisely, if we consider the positive quadratic form \(g\) on \(T^*\mathbb{R}^d\) defined by
\[ g(dy^2, d\eta^2) \overset{\text{def}}{=} \frac{dy^2}{K^2} + \frac{d\eta^2}{h^2} \quad \text{with} \quad \lambda \overset{\text{def}}{=} Kh \geq 1 \]
a function \(u\) in \(L^2(\mathbb{R}^d)\) is said to be microlocalized in \(X_0 = (x_0, \xi_0)\) a point of \(T^*\mathbb{R}^d\) if
\[ M_{X_0, N}^{C_{0,r}}(u) \overset{\text{def}}{=} \sup_{g(X-X_0) \geq C_0} \lambda^{2N} g(X-X_0)^N \sup_{\| \varphi \|_{k_{N,g} \leq 1}} \| \varphi^D u \|_{L^2} \]
are finite, where \(B_g(X, r)\) denotes the g-ball of center X and radius r, the operator \(\varphi^D\) is defined by
\[ (\varphi^D u)(x) = (2\pi)^{-d} \int_{T^*\mathbb{R}^d} e^{i(x-y)\xi} \varphi(y, \xi) u(y) dy d\xi, \]
and
\[ \| \varphi \|_{j,g} \overset{\text{def}}{=} \sup_{k \leq j} \sup_{X \in T^*\mathbb{R}^d} \| D^k \varphi (X)(T_1, \cdots, T_k) \|, \]
This notion due to J.-M. Bony ([11]) means that the function $u$ is concentrated in space near the point $x_0$ and in frequency near the point $\xi_0$ and behaves well against the product, namely, we show that if

$$g(\tilde{Y}_1 - Y_2)^{\frac{1}{2}} \geq C_0 r,$$

where $\tilde{Y} \overset{\text{def}}{=} (y, -\eta)$ if $Y = (y, \eta)$ then for any $N$, we have

$$\| \chi(h^{-1}D)(\varphi_1^D u_1 \varphi_2^D u_2)\|_{L^1} \leq C_N \| \varphi_1 \|_{k_{N,g}} \| \varphi_2 \|_{k_{N,g}} (1 + \lambda^2 g(\tilde{Y}_1 - Y_2))^{-N} \| u_1 \|_{L^2} \| u_2 \|_{L^2},$$

where $\varphi_i \in \mathcal{D}(B_g(Y_i, r))$.

This study of the interaction between two typical examples of microlocalized functions allows as in ([11]) to concentrate the bilinear estimate on real interaction.

Anyway, the choice of the localization metric $g$ is essential and it is crucial to impose that the size of the $g$-balls is preserved by Hamiltonian flow which leads to the only choice $K = C|2^q I_q|h$ thanks to the properties of the solution of the associate Hamilton Jacobi equation.

The key point in the generalization of the bilinear estimate is the proof that for solutions of a variable coefficients wave equation, microlocalization properties propagate nicely along the Hamiltonian flows related to the wave operator; this point follows from the choice of the metric used to localize in the cotangent space of $\mathbb{R}^d$.

Finally to end the proof of the microlocal bilinear estimate, the strategy consists to decompose the Cauchy data using unity partition whose elements are supported in $g$-balls and then to apply the product and the propagation theorems to concentrate on real interaction (see the proof in the constant coefficient case). Because of the fact that interaction in the product and propagation of microlocalization are badly related, we need at this step to recourse to a second microlocalization, which means that we have to decompose again the interval on which we work on sub intervals where the Hamiltonian flow is nearly constant.

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