BTW model with probabilistically nonuniform distribution of particles from the unstable sites

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Abstract: The two dimensional BTW model of SOC, with probabilistically nonuniform distribution of particles among the nearest neighbouring sites, is studied by computer simulation. When the value of height variable of a particular site reaches the critical value ($z_c = 4$), the value of height variable of that site is reduced by four units by distributing four particles among the four nearest neighbouring sites. In this paper, it is considered that, two particles are distributed equally among the two nearest neighbouring sites along X-axis. However, the distribution of other two particles along Y-axis, is probabilistically nonuniform. The variation of spatial average of the height variable with time is studied. In the SOC state, the distribution of avalanche sizes and durations is obtained. The total number of topplings occurred during the stipulated time of evolution is also calculated.

Keywords: BTW model, Height variable, Avalanche size, SOC
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I. Introduction

There exists some extended driven systems in nature, like forest fire, earthquake, growth of sandpile, which can be explained by using self-organised criticality (SOC). This phenomena of SOC is characterised by spontaneous evolution into a steady state which shows long-range spatial and temporal correlations. The concept of SOC was introduced by Bak, Tang and Wiesenfeld in terms of a simple cellular automata model[1, 2]. The steady state dynamics of the model shows a power law behaviour in the probability distributions for the occurrence of the relaxation (avalanches) clusters of a certain size, area, lifetime, etc. Extensive work has been done so far to study the properties of the model in the steady SOC state. The BTW model has been solved exactly using the commutative property of the particle addition operator[3]. Several properties of this critical state, e.g., entropy, height correlation, height probabilities, cluster statistics, etc have been calculated analytically[4, 5, 6]. And extensive numerical efforts have also been performed to estimate various critical exponents[7, 8, 9, 10, 11, 12, 13, 16]. Recently, the avalanche exponents were estimated using the renormalization scheme[14, 15]. The BTW model in dilute lattice has also been studied recently[17, 18].

It is important to mention that in the ricepile experiment, Oslo observed that SOC occurs in systems of granular materials. And in the case of earthquakes, the power law describing the magnitude of earthquake is similar to the power law followed in the case of frequency distribution of the size of avalanches in the sandpile model[19].

So far in all the studies related to BTW model, the interaction with the nearest neighbouring sites are considered to be identical, i.e, the particles from the unstable lattice site, as it topples, distributed uniformly among its nearest neighbouring sites. In case of BTW model on two dimensional regular lattice, the particles are distributed uniformly among the four nearest neighbouring sites from the unstable site. However, in this study, we have considered that two particles are distributed uniformly along one direction (say along X-axis) i.e., one particle moves along positive X-axis and the other moves along negative X-axis from the unstable lattice site. But, along other (Y) direction, there is a probability($P_r$) that the remaining two particles are distributed nonuniformly, (i.e., both particles will go towards positive Y-axis with probability $P_r$). Consequently, two particles are distributed uniformy, with probability ($1-P_r$), restoring original BTW model.

In the original BTW model (with uniform distribution of particles after topplings) in two dimensions, may be visualized as the sandpile formed on a horizontal plane. Whereas, a probabilistically nonuniform distribution of particle (considered in the present study), is a manifestation of formation of sandpile on an inclined plane.

In this work, we have first studied the time evolution of the spatial average of the height variable, $\bar{z}$, for a particular probability $P_r$, of nonuniform
distribution of particles along Y-axis. The distribution of size and duration of the avalanches in the critical state has also been obtained. Here, we have also studied the effect of probabilistically nonuniform distribution of particles along Y-axis through the spatial variation of the total number of topplings, \( N_t(i, j) \), occurred at a particular lattice site \((i, j)\), during the stipulated time of evolution of the system.

This paper is organised as follows. In Section I, the model and simulation is discussed. In section II, the results are described. And the paper ends with the conclusion, in Section III.

II. The model and simulation

The BTW model is a lattice automata model of sandpile growth, which evolves spontaneously into a critical state. We consider a two dimensional square lattice of size \( L \times L \). Each site \((i, j)\) of the lattice is associated with a variable (so called height) \( z(i, j) \) which can take positive integer values varying from 0 to \( z_c \). In every time step, one particle is added to a randomly selected site, which increases the value of the height of that site, according to

\[
\begin{align*}
\Delta z(i, j) &= z(i, j) + 1 \\
\end{align*}
\]  

If, at any site the height variable exceeds a critical value \( z_c \) (i.e., if \( z(i, j) \geq z_c \)), then that site becomes unstable and it relaxes by a toppling. When an unstable site topples, the value of the height variable of that site is reduced by 4 units and that of each of the four of its neighbouring sites increased by unity (local conservation), i.e.,

\[
\begin{align*}
z(i, j) &= z(i, j) - 4 \\
z(i \pm 1, j) &= z(i \pm 1, j) + 1 \\
z(i, j \pm 1) &= z(i, j \pm 1) + 1
\end{align*}
\]

for \( z(i, j) \geq z_c \). Each boundary site is attached to an additional site which acts as a sink. We use here the open boundary conditions so that the system can dissipate through the boundary. In our simulation, we have taken \( z_c = 4 \).

In original BTW model, when the unstable site topples, four particles from the unstable site are distributed uniformly among its four nearest neighbours. In this paper, we consider the distribution of two particles from the unstable site equally among the two nearest neighbouring sites along the directions of X-axis. But in case of distribution of particles along Y-axis, the distribution is probabilistically nonuniform. Thus, there is a probability \( P_r \), that two particles from the unstable site will move to the nearest neighbouring site along the positive direction of Y-axis, i.e., from \((i, j)\) to \((i, j + 1)\). And with probability \( (1 - P_r) \), two particles will be distributed equally among the two nearest neighbouring sites along both the directions of Y-axis. Thus in this work, as the unstable site topples, the particles from the unstable site will move towards its nearest neighbouring sites as follows:
\[ z(i, j) = z(i, j) - 4 \quad (4) \]
\[ z(i \pm 1, j) = z(i \pm 1, j) + 1 \quad \text{and} \quad (5) \]

There is a probability \( (P_r) \) that two particles from the unstable site will move along the direction of Y-axis according to
\[ z(i, j + 1) = z(i, j + 1) + 2 \quad (6) \]
\[ z(i, j - 1) = z(i, j - 1) \quad (7) \]

and with probability \( (1 - P_r) \), two particles will move along the two direction of Y-axis according to
\[ z(i, j \pm 1) = z(i, j \pm 1) + 1 \quad (8) \]

In this work, the system is evolved according to the dynamics (following eqns 4-8) starting from an initial condition with all the sites having \( z = 0 \).

With the evolution of time, the value of height variables \( (z(i, j)) \) of different sites first increases due to random addition of particles. As soon as the value of height variable of any site reaches (or exceeds) the critical value \( (z_c = 4) \), that site topples. Here, we have studied the following observations in BTW model with probabilistically nonuniform distribution of particles,

1. The time evolution of the average (spatial) value of \( z \), i.e.,
\[ \bar{z} = (1/N) \sum_{i=1}^{N} z_i \quad (N = L^2) \]

2. The fraction of sites, \( f_z \), having the height variable \( z = 0, 1, 2, 3 \), in the critical state.

3. The distribution of the avalanche size \( (D(s)) \), and avalanche time \( \tau \) \( (D(\tau)) \).

4. The spatial variation of the total number of topplings occurred at a site \( (i, j) \) during total time \( (t) \), \( N_t(i, j) \).

### III. Results

In this paper, we have studied the two dimensional BTW model, when the two particles are distributed uniformly along X-axis. But in case of distribution of particles along Y-axis, there is a probability, \( P_r = 0.4 \), that two particles are distributed nonuniformly (i.e, two particles move along the positive direction of Y-axis). Here we have considered a square lattice of size \( (L = 200) \). We first studied the time evolution of the average (spatial) value of \( z \), (i.e., \( \bar{z} \)), which is plotted in Figure-1(a). Figure-1(a) shows that the value of \( \bar{z} \) in critical state is 1.98. It is to be noted that this value is less than the value observed, 2.12\( ^2 \) in case of BTW model.
We have also studied how the value of $\bar{z}$ changes as the tendency of two particles to move along a particular direction of Y-axis increases. We have calculated the value of $\bar{z}$ at critical state for different values of the probability of occurring nonuniform distribution of particles, $(P_r)$. In Figure-1(b), the variation of the value of $\bar{z}$ at critical state is plotted for different values of $P_r$. The Figure-1(b) shows that the value of $\bar{z}$ decreases as $P_r$ increases. And the value of $\bar{z}$ becomes 1.7, when both the particles move along a particular direction of Y-axis from the unstable site, as it topples. Various studies related to the structure of the lattice at the critical state, like the fractal dimension and the fraction of the sites having different height variable has been studied in case of original BTW model\[8\]. Here, for BTW model with probabilistically nonuniform distribution, we have also calculated the fraction of sites($f_z$) for different height variables, $z = 0, 1, 2, 3$. We have calculated $f_z$ for different probability, $P_r$ and plotted in Figure-2. The calculation of $f_z$ is done for a lattice of size, $L=200$ and in the critical state reached at $t = 4 \times L^2$.

We have also calculated the distributions of duration($\tau$) and size($s$) of the avalanches at the critical state. The distribution is obtained for 80000 number of avalanches. The distribution of avalanche size, $D(s)$ is plotted, on a doubly logarithmic scale, in Figure-3(a). Similarly the distribution of avalanche time, $D(\tau)$, is plotted, on a doubly logarithmic scale, in Figure-3(b). We have estimated the value of the exponents within limited accuracy and given by, $D(s) \propto s^{-1.22}$ and $D(\tau) \propto \tau^{-1.60}$. The power law variations of the distribution of avalanche size ($s$) and avalanche time ($\tau$) given by: $D(s) \propto s^{-1.22}$ and $D(\tau) \propto \tau^{-1.60}$, indicates that the steady state is a critical state, however the exponents are different from the BTW model.

Here we have calculated the total number of topplings, $N_t(i,j)$, occurred at a site $(i,j)$ during the total time of evolution of the system and observed its spatial variation on the square lattice. In Figure-4(a), the value of total number of topplings occurred at any site, $N_t(i,j)$, is plotted for different lattice sites when the four particles are distributed equally among the four nearest neighbouring sites from the unstable site as it topples, i.e, for BTW model. Similarly, the Figure-4(b) plots the spatial variation of $N_t(i,j)$ when there is a probability $(P_r = 0.4)$, that two particles will be nonuniformly distributed along the Y-axis. Here, we have plotted the spatial variation of the total number of topplings occurred at a site, $N_t(i,j)$, for a square lattice of size, $L=50$.

It is interestingly observed that the spatial variation of the number of topplings, $N_t(i,j)$ is symmetric for the uniform distribution of particles from the unstable site. And it becomes asymmetric as the distribution of particles become probabilistically nonuniform along one direction from the unstable site as it topples.
IV. Summary

We studied here, the BTW model with probabilistically nonuniform distribution of two particles, from the unstable sites as it topples, among its nearest neighbouring sites along a particular direction. It is observed that in the case of nonuniform distribution, the spatial average value of the height variable, \( \bar{z} \), reaches a steady value, which is less than the value that obtained in the BTW model with uniform distribution of particles. The exponents of the power law obeyed by the distribution of size(\( s \)) and duration(\( \tau \)) of the avalanches are calculated as \( D(s) \propto s^{-1.22} \) and \( D(\tau) \propto \tau^{-1.55} \). The fractions of the lattice sites, \( f_z \), having different height variables, \( z = 0, 1, 2, 3 \), at the critical state have been calculated. And the variations of \( f_z \) with the probability \( P_r \) have been studied. The total number of topplings, \( N_t(i,j) \), occurred at any site, \((i,j)\), is calculated and plotted for different lattice sites for both the cases, uniform and probabilistically nonuniform distribution of particles among the four nearest neighbouring sites from the unstable site. The total number of topplings occurred at the central site is maximum and it is symmetric with respect to X and Y axis in case of uniform distribution of particles. But in case of probabilistically nonuniform distribution of particles from the unstable site the symmetry observed in the spatial variation of the quantity, total number of topplings occurred at any site, is broken.

References

1. P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett., 59 (1987) 381
2. P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. A, 38 (1988) 364.
3. D. Dhar, Phys. Rev. Lett., 64 (1990) 1613;
4. S. N. Majumder and D. Dhar, J. Phys. A 24 (1991) L357.
5. E. V. Ivashkevich, J. Phys. A 27 (1994) 3643;
6. V. B. Priezzhev, J. Stat. Phys. 74 (1994) 955.
7. S. S. Manna, J. Stat. Phys. 59 (1990) 509
8. P. Grassberger and S. S. Manna, J. De. Physique. 51 (1990) 1077;
9. S. S. Manna, Physica A 179 (1991) 249
10. V. B. Priezzhev, D. V. Ktitarev and E. V. Ivashkevich, Phys. Rev. Lett 76 (1996) 2093.
11. A. Benhur and O. Biham, Phys. Rev. E 53 (1996) R1317.
12. S. Lubek and K. D. Usadel, Phys. Rev. E 55 (1997) 4095.
13. E. V. Ivashkevich, J. Phys. A 27 (1994) L585.

14. L. Pietronero, A. Vespignani and S. Zapperi, Phys. Rev. Lett., 72 (1994) 1690.

15. E. V. Ivashkevich, Phys. Rev. Lett, 76 (1996) 3368.

16. A. B. Acharyya, Acta Phys. Pol. B, 42, (2011) 19.

17. A. B. Acharyya, J. Mod. Phys, 5, (2014) 1958.

18. M. N. Najafi, J. Phys. A 49 (2016) 335003.

19. Claudius Gros, Complex and Adaptive Dynamical Systems, Springer, 2008.
Fig-1. (a) Plot of time variation of the spatial average of height variable, \( \bar{z} \) for probabilistically nonuniform distribution of particles along Y-axis from the unstable site \( (P_r = .4) \). (b) Plot of \( \bar{z} \) against \( P_r \). \( (L=200) \)
Fig-2. Plot of $f_z$ against $P_r$ for $z = 0(\ast)$, $z = 1(\blacksquare)$, $z = 2(\bullet)$ and $z = 3(\blacktriangle)$. (L=200)
Fig-3. (a) Log-log plot of distribution of avalanche size \( s \) for probabilistically nonuniform distribution of particles along Y-axis for a particular probability, \( P_r = 0.4 \). Solid line represents \( y \sim x^{-1.22} \). (b) Log-log plot of distribution of avalanche time \( \tau \) for nonuniform distribution of particles along Y-axis for a particular probability, \( P_r = 0.4 \). Solid line represents \( y \sim x^{-1.60} \).
Fig-4. Plots the variation of $N_t(i,j)$ for different sites of the lattice for (a) uniform distribution of the particles among the nearest neighbouring sites and (b) for probabilistically nonuniform distribution of particles with a particular probability, $P_r = .4$. 