Generalization of Hamiltonian Mechanics to a Three-Dimensional Phase Space

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Abstract
Classical Hamiltonian mechanics is realized by the action of a Poisson bracket on a Hamiltonian function. The Hamiltonian function is a constant of motion (the energy) of the system. The properties of the Poisson bracket are encapsulated in the symplectic 2-form, a closed second order differential form. Due to closure, the symplectic 2-form is preserved by the Hamiltonian flow, and it assigns an invariant (Liouville) measure on the phase space through the Lie-Darboux theorem. In this paper we propose a generalization of classical Hamiltonian mechanics to a three-dimensional phase space: the classical Poisson bracket is replaced with a generalized Poisson bracket acting on a pair of Hamiltonian functions, while the symplectic 2-form is replaced by a symplectic 3-form. We show that, using the closure of the symplectic 3-form, a result analogous to the classical Lie-Darboux theorem holds: locally, there exist smooth coordinates such that the components of the symplectic 3-form are constants, and the phase space is endowed with a preserved volume element. Furthermore, as in the classical theory, the Jacobi identity for the generalized Poisson bracket mathematically expresses the closure of the associated symplectic form. As a consequence, constant skew-symmetric third order contravariant tensors always define generalized Poisson brackets. This is in contrast with generalizations of Hamiltonian mechanics postulating the fundamental identity as replacement for the Jacobi identity. In particular, we find that the fundamental identity represents a stronger requirement than the closure of the symplectic 3-form.

1 Introduction

In the bracket formalism \[1, 2, 3\], a classical Hamiltonian system is defined as a pair consisting of a smooth manifold \(\mathcal{M}\) of dimension \(n\) and a Poisson bracket \(\{\cdot, \cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})\) on the ring of real valued smooth functions \(C^\infty(\mathcal{M})\) on \(\mathcal{M}\). The Poisson bracket satisfies the following axioms:

\begin{align}
\{a F + b G, H\} &= a \{F, H\} + b \{G, H\}, \quad \{H, a F + b G\} = a \{H, F\} + b \{H, G\}, \quad (1a) \\
\{F, F\} &= 0, \quad (1b) \\
\{F, G\} &= - \{G, F\}, \quad (1c) \\
\{F G, H\} &= F \{G, H\} + \{F, H\} G, \quad (1d) \\
\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} &= 0, \quad (1e)
\end{align}

for all \(a, b \in \mathbb{R}\) and \(F, G, H \in C^\infty(\mathcal{M})\). Equations \((1a)\), \((1b)\), \((1c)\), \((1d)\), and \((1e)\) express bilinearity, alternativity, antisymmetry, Leibniz rule, and Jacobi identity respectively. Bilinearity \((1a)\) guarantees that the Poisson bracket defines an algebra over \(\mathbb{R}\). Given an observable \(F \in C^\infty(\mathcal{M})\) and an Hamiltonian (energy) \(H \in C^\infty(\mathcal{M})\), the time evolution of \(F\) is expressed as

\[
\frac{d F}{dt} = \{F, H\} = F_i \mathcal{J}^{ij} H_j.
\]
In this notation \( J^{ij} \) is a skew-symmetric second order contravariant tensor (the Poisson operator associated with the Poisson bracket), summation on repeated indexes is used, and lower indexes applied to a function specify partial derivatives, e.g. \( F_i = \partial F / \partial x^i \) with \( x^i \) the \( i \)th coordinate on \( \mathcal{M} \), \( i = 1, \ldots, n \). Then, alternativity (1b) ensures that energy is conserved, since \( dH / dt = \{ H, H \} = 0 \). Antisymmetry (1c) follows from the axioms (1a) and (1b). The Leibniz or derivation rule (1d) implies that the Poisson bracket behaves as a derivation since \( d (FG) / dt = F (dG / dt) + (dF / dt) G \). Finally, the Jacobi identity (1e) is equivalent to the existence of a closed 2-form of even rank \( 2n \). Here, \( s \) is the dimension of the kernel of the Poisson bracket. Due to the Lie-Darboux theorem [4, 5, 6], the closure of the 2-form \( \omega \) further implies that the phase space is locally spanned by \( 2n \) canonically conjugated variables \((p^i, q^i)\), \( i = 1, \ldots, n \), and \( s = n - 2m \) Casimir invariants \( C^i \), \( i = 1, \ldots, s \), which fill the center (kernel) of the Poisson bracket:

\[
\frac{dC^i}{dt} = \{ C^i, H \} = 0 \quad \forall H \in C^\infty (\mathcal{M}), \quad i = 1, \ldots, s.
\]

(3)

The naming Casimir invariant used in this context originates from the Lie algebra associated with angular momentum [7]. The equations of motion therefore take the local canonical form

\[
\frac{dp^i}{dt} = -\partial H / \partial q^i, \quad \frac{dq^i}{dt} = \partial H / \partial p^i, \quad \frac{dC^j}{dt} = 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, s.
\]

(4)

In his 1973 paper [8], Nambu proposed what he calls a ‘possible generalization of classical Hamiltonian dynamics to a three-dimensional phase space’. Here, a three-dimensional phase space means that the classical canonical pair \((p, q)\) is replaced by a canonical triplet \((p, q, r)\), while the number of generating functions (Hamiltonians) is increased to two. Then, for the basic \( n = 3 \) case with Hamiltonians \( G \) and \( H \), in place of Hamilton’s canonical equations, Nambu’s canonical equations are introduced as follows:

\[
\frac{dp}{dt} = \frac{\partial G}{\partial q} \frac{\partial H}{\partial r} - \frac{\partial G}{\partial r} \frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial G}{\partial p} \frac{\partial H}{\partial r} - \frac{\partial G}{\partial r} \frac{\partial H}{\partial p}, \quad \frac{dr}{dt} = \frac{\partial G}{\partial p} \frac{\partial H}{\partial q} - \frac{\partial G}{\partial q} \frac{\partial H}{\partial p}.
\]

(5)

Setting \((x^1, x^2, x^3) = (p, q, r)\), system (5) can be written through a ternary operation (Nambu bracket) \( \{ \circ, \circ, \circ \} : C^\infty (\mathcal{M}) \times C^\infty (\mathcal{M}) \times C^\infty (\mathcal{M}) \rightarrow C^\infty (\mathcal{M}) \) as below:

\[
\frac{dx^i}{dt} = \{ x^i, G, H \} = \hat{\epsilon}^{ijk} G_j H_k, \quad i = 1, 2, 3.
\]

(6)

In this equation, \( \hat{\epsilon}^{ijk} \) denotes the three-dimensional Levi-Civita symbol. More generally, the evolution of an observable \( F \in C^\infty (\mathcal{M}) \) takes the form

\[
\frac{dF}{dt} = \{ F, G, H \} = \hat{\epsilon}^{ijk} F_i G_j H_k.
\]

(7)

Notice that due to the skew-symmetry of \( \hat{\epsilon}^{ijk} \) both \( G \) and \( H \) are constant in time.

The generalization of the Nambu bracket occurring in equation (7) to an algebraic framework analogous to the Poisson bracket of classical Hamiltonian mechanics has proven difficult, especially in the context of quantization, because the generalization of the Poisson bracket axioms (11) (in particular, the generalization of the Jacobi identity (13)) to the Nambu bracket is nontrivial [9]. Historically, Nambu dynamics was first placed in a Lie algebraic setting in [11]. Here, the authors considered the properties of the bracket \( \{ \circ, \circ \}_G = \{ \circ, G, \circ \} \) obtained by fixing one of the Hamiltonians, and showed that \( \{ \circ, \circ \}_G \) satisfies the Jacobi identity of Poisson brackets when the Nambu bracket is defined through structure constants of Lie-Poisson brackets and \( G \) is the corresponding Casimir invariant (note that this construction is not limited to the Lie algebra of the rotation group chosen by Nambu). This problem is intimately related to the generalization of the Jacobi identity to Nambu mechanics because it is desirable that a triple bracket with an equivalent Poisson structure satisfies the generalized identity. In [11], the authors also derived a field theoretic (infinite-dimensional) Nambu bracket though the Lie algebra associated with the Weyl-Wigner representation (these aspects are also reviewed in [7][11]). Several authors [9][12] have proposed the following set of axioms for the
Nambu bracket: trilinearity, skew-symmetry, Leibniz rule, and fundamental identity. Respectively, they are written as

\[
\{a F_1 + b F_2, F_3, F_4\} = a \{F_1, F_3, F_4\} + b \{F_2, F_3, F_4\}, \tag{8a}
\]

\[
\{F_1, F_2, F_3\} = \epsilon^{ijk} \{F_1, F_j, F_k\}, \quad i, j, k = 1, 2, 3 \quad \text{(not summed)}, \tag{8b}
\]

\[
\{F_1 F_2, F_3, F_4\} = F_1 \{F_2, F_3, F_4\} + F_2 \{F_1, F_3, F_4\}, \tag{8c}
\]

\[
\{\{F_1, F_2, F_3\}, F_4, F_5\} = \{\{F_1, F_4, F_3\}, F_2, F_5\} + \{\{F_1, F_2, F_5\}, F_3\} + \{F_1, F_2, \{F_3, F_4, F_5\}\}, \tag{8d}
\]

for all \(F_1, F_2, F_3, F_4, F_5 \in C^\infty(M)\) and \(a, b \in \mathbb{R}\). Observe that in (8a) the linearity condition on the second and third argument has been omitted. As in the Poisson bracket case, linearity in the other arguments follows from linearity in the first argument and skew-symmetry. Alternativity has been absorbed in the second argument and (8c) reduces to (1e) when \(F_1 = F_4 = G\). However, the fundamental identity (8d) also leads to the fact that constant skew-symmetric 3-tensors do not define a Nambu bracket in general, a situation pointing to the fact that the axiom (8d) is more stringent than the Jacobi identity (1e) required for a Poisson bracket (on this point, see [12]).

As discussed in [9, 12, 13], the axioms (8) can be further generalized to the \(N\)-ary case \((N \leq n)\) in which the evolution of an observable \(F \in C^\infty(M)\) in \(n\)-dimensional phase space is determined by \(N - 1\) Hamiltonians, \(H_1, ..., H_{N-1} \in C^\infty(M)\) as

\[
\frac{dF}{dt} = \{F, H_1, ..., H_{N-1}\}. \tag{10}
\]

The main case discussed by Nambu corresponds to \(n = N = 3\). More generally, Nambu dynamics with \(n = N\) is associated with integrable systems endowed with \(n - 1\) constants of motion.

In this paper, we wish to address the following question: ‘How can we generalize Hamiltonian mechanics to allow triplets (and possibly \(N\)-tuples, \(N \geq 2\)) of variables as the building blocks for the phase space?’ To achieve this goal, we shall not consider the problem of quantization (see [14, 15] on the difficulties encountered in the quantization of Nambu brackets defined by (8)), but instead focus on the minimal differential-geometric structure that should be possessed by a generalized phase space, a closed symplectic form of degree \(N\). We find that, for \(N > 2\), the mathematical identity (Jacobi identity) expressing the closure of the symplectic form ceases to coincide with the fundamental identity (the bracket identity (8d) proposed as generalization of the Jacobi identity (1e) for the classical Poisson bracket). Hence, the fundamental identity represents a stronger condition than what is required for a system to be Hamiltonian in the geometric sense above, and the fundamental identity (8d) is replaced with a weaker condition expressing the closure of the relevant symplectic \(N\)-form. Here, the bracket formalism is not essential to the definition of the generalized Hamiltonian framework, and all bracket axioms are deducible from the underlying symplectic structure of the phase space.

Although the construction of generalized Hamiltonian mechanics presented here does not come with an immediate pathway to quantization, it possesses several merits: any constant skew-symmetric 3-tensor always satisfies the Jacobi identity and therefore represents a generalized Poisson operator, and a theorem, analogous to the classical Lie-Darboux theorem for closed 2-forms, holds, implying that the phase space has a local set of coordinates that define an invariant (Liouville) measure. We also take the view that the applicability of any generalization of Hamiltonian mechanics should not be limited to integrable dynamical systems with \(n = N\). Instead, it should include systems with arbitrary dimensionality \(n \geq N\) and a desired number of
Hamiltonians $H_1, ..., H_{N-1}$, $N - 1 \geq 1$. The theory discussed here applies to $n$-dimensional systems ($n \geq 3$) with $N - 1 = 2$ Hamiltonians, and the formalism has a potential extension to the general case with $n \geq N$ dimensions and $N - 1 \geq 1$ Hamiltonians.

One may wonder whether such generalizations of Hamiltonian mechanics represent redundant formulations of the classical theory, or whether they bring physical insight into practical problems. The relationship between classical Hamiltonian mechanics and the Nambu bracket defined through the axioms has been studied by several authors $[16, 17, 18, 19]$, who have highlighted the range of interchangeability of the formalisms and suggested applications to statistical mechanics, where the classical Liouville measure is replaced by the invariant measure associated with the volume element of Nambu’s phase space (an invariant measure is needed for the definition of entropy and the applicability of the ergodic ansatz $[20, 21]$), quantum mechanics, and fluid mechanics $[22, 23]$. As it will be shown in the following sections, the geometric generalization of Hamiltonian mechanics presented here has the property that, exception made for the three-dimensional case $n = 3$, the bracket obtained by fixing one of the constants, say $G$, according to $\{\circ, G\} = \{\circ, G, \circ\}$ fails to define a Poisson bracket. This fact holds true even when the generalized Poisson operator $J^{ijk}$ generating the dynamics as $dF/dt = \{F, G, H\} = J^{ijk}F_iG_jH_k$ can be transformed to a tensor whose components are given by Levi-Civita symbols (one for each canonical triplet). This suggests that there may be systems that possess a generalized Hamiltonian structure, but that do not exhibit a classical phase space. This fact would make the theory indispensable if such systems were found in the natural world.

The present paper is organized as follows. In section 2 we review the geometric formulation of classical Hamiltonian mechanics. In section 3, based on the same geometric construction, we define a generalization of Hamiltonian mechanics to dynamical systems possessing two Hamiltonians. In section 4, we extend the classical Lie-Darboux theorem, and show that phase space is locally spanned by a set of coordinates defining an invariant measure. In these coordinates, the components of the symplectic 3-form are constants. The conditions for the local existence of canonical triplets are also discussed. Concluding remarks are given in section 6.

Finally, we remark again that, for clarity of exposition, we are concerned only with the generalization of Hamiltonian mechanics to systems with two Hamiltonians. Nevertheless, systems with a higher number of Hamiltonians $N - 1 > 2$ are expected to admit analogous constructions.

## 2 Classical Hamiltonian Mechanics

For simplicity, we restrict our attention to Euclidean space, $\mathcal{M} = \mathbb{R}^n$. Consider a smooth bounded domain $\Omega \subset \mathbb{R}^n$. Let $x$ denote the position vector in $\mathbb{R}^n$, $(x^1, ..., x^n)$ a coordinate system in $\Omega$, $(\partial_1, ..., \partial_n)$ the associated tangent basis, $T\Omega$ the tangent space to $\Omega$, and $T^*\Omega$ the cotangent space to $\Omega$.

A bivector field $\mathcal{J} \in \bigwedge^2 T\Omega$ with expression

$$\mathcal{J} = \sum_{i<j} \mathcal{J}^{ij} \partial_i \wedge \partial_j,$$

where $\mathcal{J}^{ij} \in C^\infty(\Omega)$, $i, j = 1, ..., n$, are the components of a skew-symmetric second order contravariant tensor, is called a Poisson operator provided that the Jacobi identity

$$\mathcal{J}^{im} \frac{\partial \mathcal{J}^{jk}}{\partial x^m} + \mathcal{J}^{jm} \frac{\partial \mathcal{J}^{ki}}{\partial x^m} + \mathcal{J}^{km} \frac{\partial \mathcal{J}^{ij}}{\partial x^m} = 0, \quad i, j, k = 1, ..., n,$$

is satisfied. Using the Poisson bracket $\{F, G\} = F_i \mathcal{J}^{ij} G_j$, one can verify that equation (12) is equivalent to (14).

Given a smooth vector field $X = X^i \partial_i \in T\Omega$, equations of motion are assigned as follows:

$$\frac{dx}{dt} = X.$$
The vector field $X$ defines a Hamiltonian system whenever there exist a smooth function (Hamiltonian) $H \in C^\infty (\Omega)$ and a Poisson operator $\mathcal{J}$ such that

$$X^i = \mathcal{J}^i_j H_j, \quad i = 1, \ldots, n.$$  \hfill (14)

The central result stemming from the Jacobi identity (12) is the so-called Lie-Darboux theorem [4, 5, 6], which can be stated as follows:

**Theorem 1.** Let $\mathcal{J} \in \wedge^2 T\Omega$ denote a smooth bivector field of rank $2m$ on a smooth manifold $\Omega$ of dimension $n = 2m + s$, $s \geq 0$. Assume that the Jacobi identity (12) holds. Then, for every point $x \in \Omega$ there exist a neighborhood $U \subset \Omega$ of $x$ and local coordinates $(p^1, \ldots, p^m, q^1, \ldots, q^m, C^1, \ldots, C^s) \in C^\infty (U)$ with associated tangent basis $(\partial_{p^1}, \ldots, \partial_{p^m}, \partial_{q^1}, \ldots, \partial_{q^m}, \partial_{C^1}, \ldots, \partial_{C^s})$ such that

$$\mathcal{J} = \sum_{i=1}^m \partial_{q^i} \wedge \partial_{p^i}, \quad \text{in } U. \hfill (15)$$

Equation (15) implies that, in $U$,

$$\frac{dp^i}{dt} = i_X dp^i = \mathcal{J} (dp^i, dH) = \sum_{k=1}^m \left( \frac{\partial p^i}{\partial p^k} \frac{\partial H}{\partial p^k} - \frac{\partial p^i}{\partial q^k} \frac{\partial H}{\partial q^k} \right) = \frac{\partial H}{\partial q^i}, \hfill (16a)$$

$$\frac{dq^i}{dt} = i_X dq^i = \mathcal{J} (dq^i, dH) = \sum_{k=1}^m \left( \frac{\partial q^i}{\partial p^k} \frac{\partial H}{\partial q^k} - \frac{\partial q^i}{\partial p^k} \frac{\partial H}{\partial p^k} \right) = \frac{\partial H}{\partial p^i}, \hfill (16b)$$

$$\frac{dC^j}{dt} = i_X dC^j = \mathcal{J} (dC^j, dH) = \sum_{k=1}^m \left( \frac{\partial C^j}{\partial p^k} \frac{\partial H}{\partial p^k} - \frac{\partial C^j}{\partial q^k} \frac{\partial H}{\partial q^k} \right) = 0, \hfill (16c)$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, s$. In this notation $i$ represents the contraction operator (interior product), and the bivector field $\mathcal{J}$ acts on arbitrary 1-forms $dF$ and $dG$ according to $\mathcal{J} (dF) = \mathcal{J}^i_j F_i \partial_t$ and $\mathcal{J} (dG, dF) = \mathcal{J}^{ij} G_i F_j$. Hence, the equations of motion (14) can be locally written in the canonical form (14). Notice that any differentiable function $f (C^1, \ldots, C^s)$ of the coordinates $C^j$, $j = 1, \ldots, s$, is preserved by the Hamiltonian system (14) regardless of the specific form of the Hamiltonian $H$. Indeed,

$$\mathcal{J} (dC^j) = \sum_{i=1}^m \left( \frac{\partial C^j}{\partial p^i} \partial_{q^i} - \frac{\partial C^j}{\partial q^i} \partial_{p^i} \right) = 0, \quad j = 1, \ldots, s. \hfill (17)$$

The coordinates $C^j$, $j = 1, \ldots, s$, are called Casimir invariants.

Consider the $2m$-dimensional submanifold $\Omega_C \subset \Omega$ defined by $\Omega_C = \{ x \in \Omega \mid C^j = c^j, j = 1, \ldots, s \}$ with $c^j \in \mathbb{R}, j = 1, \ldots, s$. Let $H_C = H (p^1, \ldots, p^m, q^1, \ldots, q^m, c^1, \ldots, c^s)$ be the value of the Hamiltonian $H$ on $\Omega_C$. Then, in $\Omega_C$, the equations of motion (14) can be cast in the form

$$i_X \omega = -dH_C, \hfill (18)$$

where $\omega \in \wedge^2 T^* \Omega$ is the closed 2-form with local expression

$$\omega = \sum_{i=1}^m dp^i \wedge dq^i. \hfill (19)$$

Equation (19) represents an alternative statement of the Lie-Darboux theorem. More precisely, one has:

**Theorem 2.** Given a smooth closed 2-form $\omega$ of rank $2m$ on a smooth manifold $\Omega$ of dimension $n = 2m + s$, $s \geq 0$, for every point $x \in \Omega$ there exist a neighborhood $U \subset \Omega$ of $x$ and local coordinates $(p^1, \ldots, p^m, q^1, \ldots, q^m, C^1, \ldots, C^s) \in C^\infty (U)$ such that

$$\omega = \sum_{i=1}^m dp^i \wedge dq^i \quad \text{in } U. \hfill (20)$$
The closure of the 2-form $\omega$, which is called symplectic 2-form, is expressed by the condition $d\omega = 0$. In the coordinate system $(x^1, ..., x^n)$, the 2-form $\omega$ has expression

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j,$$

(21)

where $\omega_{ij}$ is a skew-symmetric second order covariant tensor, $\omega_{ij} = -\omega_{ji}$, $i, j = 1, ..., n$. Then, the closure condition can be written as

$$d\omega = \sum_{i<j<k} \left( \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} \right) dx^i \wedge dx^j \wedge dx^k = 0.$$  

(22)

Recalling Cartan’s homotopy formula for the Lie derivative of a differential form,

$$\mathcal{L}_X \omega = (di_X + i_X d) \omega,$$

(23)

and using the closure of $\omega$, equation (18) implies that $\omega$ is invariant along the vector field $X$,

$$\mathcal{L}_X \omega = 0.$$  

(24)

In addition to the conservation of the symplectic 2-form $\omega$, Hamilton’s canonical equation (3) imply that the phase space (Liouville) measure

$$d\Pi = dp^1 \wedge ... \wedge dp^n \wedge dq^1 \wedge ... \wedge dq^m \wedge dC^1 \wedge ... \wedge dC^s,$$

(25)

is preserved by the flow $X$, i.e.

$$\mathcal{L}_X d\Pi = 0.$$  

(26)

The phase space measure $d\Pi$ is the cornerstone of classical statistical mechanics.

To summarize, there are two dual geometrical formulations of classical Hamiltonian mechanics, one based on the Poisson operator $\mathcal{J}$, the other centered on the symplectic 2-form $\omega$. The first formulation has the merit that the role of the kernel of $\mathcal{J}$, which is spanned by the Casimir invariants, is made explicit, while to define a Hamiltonian system by $\mathcal{J}$ one must require that $X \perp \ker(\omega)$ and that the Hamiltonian is independent of the Casimirs, $\partial H/\partial C^j = 0$, $j = 1, ..., s$. This is because a non-empty kernel in $\omega$ results in ambiguity in the definition of $X$ (equation $\mathcal{J}$ is symmetric with respect to transformations $X \rightarrow X + Y$, with $Y \in \ker(\omega)$). On the other hand, it should be noted that the proof of theorem 1 ultimately relies on the proof of theorem 2 to find local canonical pairs with $Y \in \ker(\omega)$.

This fact will be the basic principle for the generalization of Hamiltonian mechanics presented in the following sections.
3 Generalized Hamiltonian Mechanics

As anticipated in the introduction, the discussion will be limited to the generalization of Hamiltonian mechanics to a three-dimensional phase space, i.e. \( N = 3 \) and \( n \geq 3 \). Nevertheless, the same construction applies to \( N > 3 \). We conjecture that a generalized Hamiltonian system should have the following properties.

1. Given two Hamiltonians \( G, H \in C^\infty(\Omega) \), the equations of motion are given by
   \[
   X^i = \mathcal{J}^{ijk} G_j H_k = \sum_{j < k} \mathcal{J}^{ijk} (G_j H_k - G_k H_j), \quad i = 1, \ldots, n.
   \] (29)
   Here, \( \mathcal{J}^{ijk} \in C^\infty(\Omega) \) are the components of a skew-symmetric third order contravariant tensor. The corresponding trivector field \( \mathcal{J} \in \Lambda^3 T\Omega \) is given by
   \[
   \mathcal{J} = \sum_{i < j < k} \mathcal{J}^{ijk} \partial_i \wedge \partial_j \wedge \partial_k.
   \] (30)

2. The trivector field \( \mathcal{J} \) (generalized Poisson operator) satisfies a generalized Jacobi identity. This identity expresses the closure of a smooth 3-form \( w \) (symplectic 3-form).

3. The symplectic 3-form \( w \) is Lie-invariant,
   \[ \mathcal{L}_X w = 0. \] (31)

4. Given a generalized Poisson operator \( \mathcal{J} \) with associated symplectic 3-form \( w \), there exists a local coordinate system \( (y^1, \ldots, y^n) \) such that system (29) takes the generalized canonical form
   \[
   \frac{dy^i}{dt} = \sum_{j < k} B^{ijk} \left( \frac{\partial G}{\partial y^j} \frac{\partial H}{\partial y^k} - \frac{\partial G}{\partial y^k} \frac{\partial H}{\partial y^j} \right), \quad i = 1, \ldots, n.
   \] (32)
   Here, \( B^{ijk} \in \mathbb{R}, \ i, j, k = 1, \ldots, n, \) are constant components of a skew-symmetric third order contravariant tensor \( B \). Furthermore, the generalized Poisson operator and the symplectic 3-form \( w \) have local expressions
   \[
   \mathcal{J} = \sum_{i < j < k} B^{ijk} \partial_{y^i} \wedge \partial_{y^j} \wedge \partial_{y^k}, \quad w = \sum_{i < j < k} A_{ijk} dy^i \wedge dy^j \wedge dy^k,
   \] (33)
   where \( (\partial_{y^1}, \ldots, \partial_{y^n}) \) are tangent vectors and \( A \) is the constant inverse of \( B \) with skew-symmetric covariant components \( A_{ijk} \in \mathbb{R}, \ i, j, k = 1, \ldots, n \) (the definition of inverse and the conditions of invertibility will be given later). When \( n = 3 \), \( B^{ijk} = b \epsilon^{ijk} \) for some constant \( b \in \mathbb{R} \), and equation (32) gives Nambu’s canonical equations (5).

5. The local coordinate system \( (y^1, \ldots, y^n) \) defines a phase space (Liouville) measure
   \[ d\Xi = dy^1 \wedge \ldots \wedge dy^n, \] (34)
   which is Lie invariant, i.e.
   \[ \mathcal{L}_X d\Xi = 0. \] (35)

In essence, the geometric construction outlined above can be thought of as a noncanonical formulation of Nambu mechanics. For this construction to be consistent, we must show that the analogous of the Lie-Darboux theorem discussed in the previous section applies to three-dimensional phase spaces. The proof of this theorem, which is given in section 5, can be obtained once certain subtleties concerning the notion of inverse and rank for tensors like \( \mathcal{J}^{ijk} \) are settled. Then, the properties of the generalized formulation follow automatically.
In the remainder of this section, we describe basic properties of the theory. First, notice that the generalized Poisson bracket (Nambu bracket) is defined as

\[ \{F, G, H\} = \mathcal{J}^{ijk} F_i G_j H_k, \quad \forall F, G, H \in C^\infty(\Omega). \] (36)

Observe that, while trilinearity, skew-symmetry, and Leibniz rule are satisfied by construction, the same is not true for the fundamental identity [22]. This aspect will be discussed in detail in section 4.

A Casimir invariant \( C \) is characterized by the property that it is a constant of the motion for any choice of Hamiltonians \( G \) and \( H \), i.e.

\[ \frac{dC}{dt} = \{C, G, H\} = 0, \quad \forall G, H \in C^\infty(\Omega). \] (37)

In terms of the generalized Poisson operator, the equation above implies that the 1-form \( dC \) belongs to \( \ker(\mathcal{J}) \), \( \mathcal{J}(dC) = 0 \). In components,

\[ \mathcal{J}^{ijk} C_k = 0, \quad i, j = 1, \ldots, n. \] (38)

At this point a remark on null spaces is useful. Since \( \mathcal{J} \) is a trivector field, the following scenario may arise:

\[ \mathcal{J}(dC^1), dC^2 \neq 0, \quad \mathcal{J}(dC^1) \neq 0, \quad \mathcal{J}(dC^2) \neq 0. \] (39)

Neither \( C^1 \) or \( C^2 \) is a Casimir invariant, but they return 0 when combined. For example, consider the following trivector field on \( \mathbb{R}^6 \),

\[ \mathcal{J} = \partial_1 \land (\partial_2 \land \partial_3 + \partial_4 \land \partial_5). \] (40)

Clearly, \( \mathcal{J}(dx^6) = 0 \). However, we also have \( \mathcal{J}(dx^2 - dx^4, dx^3 + dx^5) = 0 \) with \( \mathcal{J}(dx^2 - dx^4) = \partial_1 \land (\partial_2 - \partial_4) \) and \( \mathcal{J}(dx^3 + dx^5) = \partial_1 \land (\partial_2 + \partial_4) \). \( x^1 \) behaves as a Casimir invariant, but \( x^2 - x^4 \) and \( x^3 + x^5 \) do not. We shall refer to quantities like \( x^2 - x^4 \) and \( x^3 + x^5 \) as semi-Casimir invariants. If one of the Hamiltonians, say \( G \), happens to be a semi-Casimir invariant, it follows that the other semi-Casimir invariant is a constant of the motion independent of \( H \).

Suppose that there exists a local coordinate system \( (p^1, \ldots, p^m, q^1, \ldots, q^m, r^1, \ldots, r^m, C^1, \ldots, C^n) \) with \( n = 3m + s \) and tangent vectors \( (\partial_1, \ldots, \partial_m, \partial_{q^1}, \ldots, \partial_{q^m}, \partial_{r^1}, \ldots, \partial_{r^m}, \partial_{C^1}, \ldots, \partial_{C^n}) \) such that

\[ \mathcal{J} = \sum_{i=1}^m \partial_{p^i} \land \partial_{q^i} \land \partial_{r^i}. \] (41)

In these coordinates, the equations of motion take Nambu’s canonical form

\[ \frac{dp^i}{dt} = \frac{\partial G}{\partial q^i} \frac{\partial H}{\partial p^j} - \frac{\partial G}{\partial p^i} \frac{\partial H}{\partial q^j}, \quad \frac{dq^i}{dt} = \frac{\partial G}{\partial r^i} \frac{\partial H}{\partial q^j} - \frac{\partial G}{\partial q^i} \frac{\partial H}{\partial r^j}, \quad \frac{dr^i}{dt} = \frac{\partial G}{\partial p^i} \frac{\partial H}{\partial r^j} - \frac{\partial G}{\partial r^i} \frac{\partial H}{\partial p^j}, \quad \frac{dC^i}{dt} = 0, \] (42)

with \( i = 1, \ldots, m \) and \( j = 1, \ldots, s \). Let \( \tilde{\mathcal{J}}^{abc} \) denote the \( (a, b, c) \) component of \( \mathcal{J} \) with respect to the coordinate system \( (p^1, \ldots, p^m, q^1, \ldots, q^m, r^1, \ldots, r^m, C^1, \ldots, C^n) \). From equation (41), it follows that

\[ \tilde{\mathcal{J}}^{abc} = E^{abc}, \quad a, b, c = 1, \ldots, n, \] (43)

where

\[ E^{abc} = \begin{cases} \epsilon^{abc} & \text{if } \sigma(a, b, c) = (i, m + i, 2m + i), \quad i = 1, \ldots, m \\ 0 & \text{otherwise} \end{cases} \] (44)

In this notation, \( \sigma \) is any rearrangement of the indexes \( a, b, c \). In the next section it will be shown that skew-symmetric third order contravariant tensors with constant entries satisfy the Jacobi identity. Therefore, (44) is a generalized Poisson operator. Observe that when a generalized Poisson operator can be transformed in...
the form \([\mathcal{J}^G]_{abc} = \epsilon^{abc}, \quad a, b, c = 1, 2, 3,\) (45)

which is the generalized Poisson operator originally introduced by Nambu. Let \(\{o, o, o\}\), denote the Nambu bracket defined by a generalized Poisson operator with components given by \(\[\mathcal{J}\]_{abc}\). We shall refer to such bracket as canonical Nambu bracket. Given a generalized Poisson operator \(\mathcal{J}\) with Nambu bracket \(\{o, o, o\}\), the local coordinate system \((p^1, ..., p^m, q^1, ..., q^m, r^1, ..., r^m, C^1, ..., C^n)\) transforms \(\{o, o, o\}\) into \(\{o, o, o\}\). Fixing one of the Hamiltonians, say \(G\), define the bracket

\[\{o, o\}_G = \{o, G, o\}_c,\] (46)

Let us show that, if \(n = 3\), \(\{o, o\}_G\) is a Poisson bracket, and that the same is not true, in general, for \(n > 3\).

The candidate Poisson operator is

\[\mathcal{J}_G = \sum_{a < c} J_{abc} G_a \partial_a \wedge \partial_c = \sum_{a < c} E_{abc} G_a \partial_a \wedge \partial_c.\] (47)

When \(n = 3\), we have two cases, \(m = 0\) and \(m = 1\). The case \(m = 0\) is trivial because the dimension of \(\ker(\mathcal{J})\) is \(s = n = 3\), implying \(\mathcal{J} = 0\). If \(m = 1\), the expression above implies \(\mathcal{J}_G^{bc} = \epsilon^{abc} G_b\). Then, the Jacobi identity \([12]\) reads as

\[\mathcal{J}_G^{am} \frac{\partial \mathcal{J}_G^{cd}}{\partial x^m} + \mathcal{J}_G^{cm} \frac{\partial \mathcal{J}_G^{da}}{\partial x^m} + \mathcal{J}_G^{dm} \frac{\partial \mathcal{J}_G^{ac}}{\partial x^m} = \epsilon^{abc} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{mn} G_{em} + \mathcal{J}_G^{cm} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{dm} G_{en} + \mathcal{J}_G^{dm} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{ac} G_{fn} \]

\[= \epsilon^{abc} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{mn} G_{em} + \mathcal{J}_G^{cm} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{dm} G_{en} + \mathcal{J}_G^{dm} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{ac} G_{fn} \]

\[= \partial_h \left[ \epsilon^{abc} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{mn} G_{em} + \mathcal{J}_G^{cm} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{dm} G_{en} + \mathcal{J}_G^{dm} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{ac} G_{fn} \right] = 0, \quad a, c, d = 1, 2, 3.\] (48)

Hence, for \(n = 3\), \(\{o, o\}_G\) is a Poisson bracket. Next, suppose that \(n > 3\). Following similar steps as above with \(\mathcal{J}_G^{am} = E_{abc} G_b\), one obtains

\[\mathcal{J}_G^{am} \frac{\partial \mathcal{J}_G^{cd}}{\partial x^m} + \mathcal{J}_G^{cm} \frac{\partial \mathcal{J}_G^{da}}{\partial x^m} + \mathcal{J}_G^{dm} \frac{\partial \mathcal{J}_G^{ac}}{\partial x^m} = \partial_h \left[ \epsilon^{abc} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{mn} G_{em} + \mathcal{J}_G^{cm} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{dm} G_{en} + \mathcal{J}_G^{dm} \epsilon^{def} \epsilon^{gmn} \mathcal{J}_G^{ac} G_{fn} \right], \quad a, c, d = 1, ..., n.\] (49)

Due to the skew-symmetry of \(\mathcal{J}_G\), the indexes \(a, c, d\) must be different from each other for the right-hand side of the equation to be different from zero. In addition, at least one pair of these three indexes must belong to the same canonical triplet due to the terms \(E^{cde}\), \(E^{dcf}\), and \(E^{ace}\). If all three indexes belong to the same triplet, the corresponding term of the Jacobi identity identically vanishes as in the previous case \(n = 3\). Hence, suppose that \(a\) and \(c, d\) belong to two different canonical triplets \(m_1\) and \(m_2\). Then, equation (49) reduces to

\[\mathcal{J}_G^{am} \frac{\partial \mathcal{J}_G^{cd}}{\partial x^m} + \mathcal{J}_G^{cm} \frac{\partial \mathcal{J}_G^{da}}{\partial x^m} + \mathcal{J}_G^{dm} \frac{\partial \mathcal{J}_G^{ac}}{\partial x^m} = E_{abc} E_{dce} G_b G_{em} = (G_{\beta G_{\theta}} - G_{\theta G_{\beta}}), \quad a, c, d = 1, ..., n,\] (50)

where \((a, \mu, \beta)\) and \((c, d, \epsilon)\) are the indexes of the canonical triplets \(m_1\) and \(m_2\). The quantity \([59]\) does not vanish in general. Notice that this result in agreement with the fact that the direct sum of canonical Nambu brackets with operator \(\mathcal{J} = \partial_1 \wedge \partial_2 \wedge \partial_3 + \partial_4 \wedge \partial_5 \wedge \partial_6 + ... \wedge \partial_L \wedge \partial_{L-1} \wedge \partial_L\) on \(\mathbb{R}^3L\) does not satisfy the fundamental identity \([81]\) when \(L > 1\) (see [12]). This is because, as mentioned in the introduction, the fundamental identity \([81]\) for the Nambu bracket \(\{o, o, o\}\) reduces to the Jacobi identity \([16]\) for the bracket
It is worth observing that, however, the right-hand side of (50) identically vanishes whenever the Hamiltonian $G$ is of the type

$$G = \sum_{i=1}^{m} G^i \left(p^i, q^i, r^i, C^1, ..., C^s\right),$$

where the $G^i$ are $m$ functions depending only on the $i$th canonical triplet and the Casimir invariants.

Finally, a remark on the quantization of the theory. The definition of Hamiltonian system provided by the present construction is weaker than that resulting from the axioms (8). In particular, the fundamental identity is replaced by a weaker condition, the closure of the symplectic 3-form (see sections 4 and 5 for details). Therefore, we expect additional freedom in the derivation of a quantized bracket.

### 4 Symplectic 3-Form and Jacobi identity

The purpose of the present section is to rigorously formulate the generalization of Hamiltonian mechanics introduced above in terms of a closed 3-form, the symplectic 3-form $w$, and to obtain the Jacobi identity for the generalized Poisson operator $\mathfrak{J}$ by using the closure of $w$. Consider again a smooth bounded domain $\Omega \subset \mathbb{R}^n$. The core of theory lies in the assumption that, given two Hamiltonians $G, H \in C^\infty(\Omega)$, a vector field $X \in T\Omega$ defines a generalized Hamiltonian system provided that there exist a smooth 3-form $w \in \bigwedge^3 T\Omega$ with the following properties,

$$i_X w = -dH \wedge dG,$$  

and

$$dw = 0.$$  

The 3-form $w$ can be expressed as

$$w = \sum_{i<j<k} w_{ijk} dx^i \wedge dx^j \wedge dx^k,$$

where $w_{ijk} \in C^\infty(\Omega)$, $i, j, k = 1, ..., n$, are the components of a skew-symmetric third order covariant tensor. Using (54), equations (52) becomes

$$\sum_{j<k} X^i w_{ijk} dx^j \wedge dx^k = \sum_{j<k} (H_k G_j - H_j G_k) dx^j \wedge dx^k,$$

or,

$$X^i w_{ijk} = H_k G_j - H_j G_k, \quad j, k = 1, ..., n.$$  

Similarly, equation (54) gives

$$dw = -\sum_{i<j<k<\ell} \left( \frac{\partial w_{ijk}}{\partial x^\ell} + \frac{\partial w_{ij\ell}}{\partial x^k} + \frac{\partial w_{ik\ell}}{\partial x^j} + \frac{\partial w_{jk\ell}}{\partial x^i} \right) dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell = 0,$$

or,

$$\frac{\partial w_{ijk}}{\partial x^\ell} + \frac{\partial w_{ij\ell}}{\partial x^k} + \frac{\partial w_{ik\ell}}{\partial x^j} + \frac{\partial w_{jk\ell}}{\partial x^i} = 0, \quad i, j, k, \ell = 1, ..., n.$$  

At this point, a notion of inverse for third order tensors of the type $w_{ijk}$ is needed. Suppose that there exists a skew-symmetric third order contravariant tensor $\mathfrak{J}$ with components $\mathfrak{J}^{\ell jk}$ such that

$$\sum_{j<k} w_{ijk} \mathfrak{J}^{\ell jk} = \delta^\ell_i, \quad i, \ell = 1, ..., n.$$  

Then, we say that $\mathfrak{J}$ is the inverse of $w$. More generally, we say that $\mathfrak{J} \in \bigwedge^3 T\Omega$ is a weak inverse of $w$ whenever the solution $X$ of the equation $i_X w = -dH \wedge dG$ can be cast in the form $X^i = \mathfrak{J}^{ijk} G_j H_k$ (the
In terms of the components $\mathcal{J}^{ijk}$, summing over repeated indexes, and using (59), one obtains
\[
X^\ell = \sum_{j<k} \mathcal{J}^{jk\ell} (H_k G_j - H_j G_k) = \mathcal{J}^{ijk} G_j H_k, \quad \ell = 1, \ldots, n.
\] (60)

This shows that system (58) leads to the set of equations (24) if the inverse $\mathcal{J}$ exists. Let us derive necessary conditions for the existence of the inverse $\mathcal{J}$. First, observe that tensors like $w_{ijk}$ can be thought of as matrices that have rows, columns, and ‘depth’. For example, the Levi-Civita symbol $\epsilon_{ijk}$ can be represented as
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}_1, \quad \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_2, \quad \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}_3.
\] (61)

Here, each matrix is numbered by the index $i$ (the depth of the matrix), while rows and columns by the indexes $j$ and $k$. To the tensor $\epsilon_{ijk}$ we can also assign in a unique manner a conventional matrix having $n$ rows and $n^2$ columns as follows:
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\] (62)

Here, the rows are numbered by the index $i$, while columns by the pair $j, k$. The components of this $n \times n^2$ matrix will be denoted as $\epsilon_{i(jk)}$. Notice that the same construction applies to arbitrary tensors $A_{ijk}$, the associated $n \times n^2$ matrix being $A_{i(jk)}$. Non-square matrices do not have a classical inverse. However, they may possess a left or right inverse [24]. In particular, given a $n \times n^2$ matrix $A$ with components $A_{i(jk)}$ of rank $n$, the matrix $A$ has a right inverse, i.e. a matrix $B$ with components $B^{(jk)\ell}$ such that $AB = 2I_n$, where $I_n$ is the $n$-dimensional identity matrix [24] (the factor 2 before $I_n$ is useful when handling skew-symmetric tensors). As an example, consider the matrix $\epsilon_{i(jk)}$ above. Evidently, its rank is $n = 3$ because three of the columns are linearly independent vectors in $\mathbb{R}^3$. Hence, $\epsilon_{i(jk)}$ has a right inverse. The components of the right inverse are given by $B^{(jk)\ell} = \epsilon^{(jk)\ell}$ because
\[
\epsilon_{ijk} \epsilon^{i\ell k} = 2 \sum_{j<k} \epsilon_{ijk} \epsilon^{i\ell k} = 2 \delta_{\ell 1}, \quad i, \ell = 1, 2, 3.
\] (63)

Therefore, the notion of invertibility [23] for the tensor $w_{ijk}$ is related to the existence of a right inverse for the $n \times n^2$ matrix $w_{i(jk)}$. The right inverse $\mathcal{J}^{(jk)\ell}$ exists when $w_{i(jk)}$ has rank $n$. In such case $w_{i(jk)}$ is the left inverse of $\mathcal{J}^{(jk)\ell}$.

Next, we move to the Jacobi identity. In the present construction, the Jacobi identity is nothing but the closure condition [58] for the symplectic 3-form $w$ expressed in terms of the inverse $\mathcal{J}$. If the Jacobi identity is satisfied, we say that $\mathcal{J}$ is a generalized Poisson operator. Although there is no simple way to write [58] in terms of the components $\mathcal{J}^{ijk}$ (except directly substituting the expressions of the components $w_{ijk}$ as functions of $\mathcal{J}^{ijk}$), a necessary condition for equation [58] to hold where the components $w_{ijk}$ are partially removed can be derived as follows. Multiplying each side of (58) by $-\mathcal{J}^{aij} \mathcal{J}^{bkl}$ and summing over repeated
indexes, we have:

\[
0 = -3^{ijk} \delta^{kl} \left( \frac{\partial w_{ijk}}{\partial x^j} + \frac{\partial w_{ikj}}{\partial x^k} + \frac{\partial w_{jki}}{\partial x^l} \right) - w_{ijk} \delta^{l} \delta_{jk} \delta_{il} + w_{ikj} \delta^{l} \delta_{il} \delta_{jk} - w_{jki} \delta^{l} \delta_{jk} \delta_{il} = 2 (\delta^{l} \delta_{jk} \delta_{il} - \delta^{i} \delta_{j} \delta_{lk} - \delta^{l} \delta_{ij} \delta_{lk} + \delta^{i} \delta_{jl} \delta_{ik} - \delta^{l} \delta_{ij} \delta_{kl} + \delta^{i} \delta_{jl} \delta_{ki})
\]

where \( \alpha, \beta = 1, \ldots, n \). Equation (64) is a necessary condition that must be satisfied for the symplectic 3-form \( w \) to be closed. Observe that any invertible skew-symmetric third order tensor \( \omega \) that satisfies (65) is closed and therefore defines a symplectic 3-form. We stress that, however, equation (68) as well. Hence, the generalization of Hamiltonian mechanics following from the present construction is weaker than that resulting from enforcing the axioms (8), since skew-symmetric third order tensors with constant entries do not satisfy the fundamental identity (8d) in general. Equation (64) becomes a sufficient condition when \( w \) and \( \omega \) satisfy the additional property

\[
w_{ijk} \omega^{kl} = f \left( \delta^{l} \delta_{jk} \delta_{il} - \delta^{i} \delta_{jl} \delta_{ik} + \delta^{l} \delta_{ij} \delta_{kl} - \delta^{i} \delta_{jk} \delta_{il} \right), \quad \alpha, \beta = 1, \ldots, n, \quad f \in C^\infty (\Omega).
\]

Indeed, using (68) one can recover the closure condition (65) by multiplying each side of (64) by \( w_{\sigma \tau \alpha} w_{\pi \rho \beta} \) and summing on repeated indexes. Notice that the property (65) is satisfied by the Levi-Civita symbol. Indeed,

\[
\epsilon_{ijkl} \omega^{kl} = \delta^{l} \delta_{ij} \delta_{kl} - \delta^{i} \delta_{jl} \delta_{kl}, \quad \alpha, \beta = 1, \ldots, n.
\]

When (65) holds, the right hand side of equation (64) reduces to

\[
2 f \left( \delta^{l} \delta_{ij} \delta_{kl} - \delta^{k} \delta_{ij} \delta_{il} \right) \left( \frac{\partial \alpha_{ij}}{\partial x^k} + \frac{\partial \alpha_{ij}}{\partial x^l} - \frac{\partial \alpha_{kl}}{\partial x^j} \right) = 4 f \left( \frac{\partial \alpha_{ij}}{\partial x^k} + \frac{\partial \alpha_{ij}}{\partial x^l} \right) = 0, \quad \alpha, \beta = 1, \ldots, n,
\]

This result implies that a smooth invertible 3-form \( w \) satisfying (65) is closed and therefore defines a symplectic 3-form. We stress that, however, equation (65) is only a sufficient condition for the closure of \( w \), and it is not needed for \( w \) to possess a right inverse.

A final remark concerns how to identify a generalized Hamiltonian system from a given set of equations \( X \in T\Omega \). While in classical Hamiltonian mechanics the Hamiltonian nature of a dynamical system can be verified by checking the Poisson operator axioms, and, in particular, the Jacobi identity (12), the procedure for the generalized case requires the determination of the symplectic 3-form \( w \). Once the symplectic 3-form \( w \) and the Hamiltonians \( G \) and \( H \) such that (72) holds have been identified, one must check that \( w \) is a closed differential form, i.e. that \( dw = 0 \). Notice that, in the presence of Casimir invariants, one may need to reduce the dynamics to a Casimir leaf in order to find the Hamiltonian structure.

5 Lie-Darboux and Liouville Theorems in Generalized Phase Space

This section is dedicated to the proof of Lie-Darboux type theorems (theorems 3 and 4) in the generalized Hamiltonian framework with a three-dimensional phase space \( N = 3, n \geq 3 \). A direct consequence of these theorems is the local existence of an invariant (Liouville) measure. In particular, we prove a Lie-Darboux theorem (theorem 4) for closed 3-forms of the type \( w = \omega \wedge dG \), with \( \omega \) a 2-form and \( dG \) an exact 1-form. A
further result (proposition 1) is also proven explaining the relevance of this class of 3-forms for generalized Hamiltonian mechanics, intended as the ideal dynamics of systems with 2 invariants.

Below, we consider a smooth manifold Ω of dimension n and assume smoothness of the involved quantities. We have the following:

**Theorem 3.** Let $w \in \Lambda^3 T^*\Omega$ be a closed 3-form. Let $w_{ijk}$, $i, j, k = 1, ..., n$ denote the components of $w$ with respect to a coordinate system $(x^1, ..., x^n)$ in $\Omega$,

$$w = \sum_{i<j<k} w_{ijk} dx^i \wedge dx^j \wedge dx^k. \quad (68)$$

Suppose that the $n \times n^2$ matrix $w_{ijk}$ has rank $n$. Take a sufficiently small neighborhood $U$ of any $x_0 \in \Omega$. Let $w_0 = w_{0ijk} dy^i \wedge dy^j \wedge dy^k$ denote the constant (flat) 3-form with components $w_{0ijk} = w_{ijk}(x_0)$ in a coordinate system $(y^1, ..., y^n)$. Further assume that Moser’s 2-form $\sigma_t$, $t \in [0, 1]$, such that $d\sigma_t = dw_t/dt$ in $U$, belongs to the image of the map $\hat{w}_t : T\Omega \to \Lambda^2 T^*\Omega$ defined by $\hat{w}_t(X_t) = -i_{X_t}w_t$, i.e. $\sigma_t \in \text{Im}(\hat{w}_t)$ for some $X_t \in T\Omega$. Then, $w_t$ has a right inverse $\tilde{J}_t$ in $U$. Furthermore, there exists a coordinate change $(x^1, ..., x^n) \to (y^1, ..., y^n)$ generated by the vector field $X_t = -\tilde{J}_t^{ikl}\sigma_{ijk} \partial_k$ such that

$$w = w_0 \quad \text{in} \quad U. \quad (69)$$

**Proof.** We follow the steps of the classical proof of the Lie-Darboux theorem based on Moser’s method [6, 25]. Let $w_0$ denote the constant form on $\mathbb{R}^n$,

$$w_0 = \sum_{i<j<k} A_{ijk} dy^i \wedge dy^j \wedge dy^k, \quad (70)$$

with $A_{ijk}$, $i, j, k = 1, ..., n$, real constants. Consider a family of vector fields $X_t \in T\Omega$, $0 \leq t \leq 1$, defined in a neighborhood $U$ of a point $x_0 \in \Omega$ that generates a one-parameter group of diffeomorphisms $g_t$ as follows,

$$\frac{d}{dt}g_t(x_0) = X_t(g_t(x_0)), \quad g_0(x_0) = x_0. \quad (71)$$

Next, define the family of 3-forms

$$w_t = w_0 + t(w - w_0). \quad (72)$$

We wish to obtain $X_t$, and thus $g_t$, so that the following property is satisfied

$$g_t^* w_t = w_0. \quad (73)$$

Here $g_t^* w_t$ denotes the pullback of $w_t$ by $g_t$. Equation (73) implies that

$$\frac{d}{dt} g_t^* w_t = g_t^* \left( \frac{dw_t}{dt} + dX_t w_t \right) = 0, \quad (74)$$

where we used the fact that $w_t$ is a closed differential form. By the Poincaré lemma, in a sufficiently small neighborhood $W$ of $x_0$, the closed differential form $dw_t/dt$ is exact, i.e. there exists a 2-form $\sigma_t = \sum_{j<k} \sigma_{ijk} dx^j \wedge dx^k$ such that

$$\frac{dw_t}{dt} = d\sigma_t \quad \text{in} \quad W. \quad (75)$$

Hence, equation (74) can be solved in $W$ by finding a vector field $X_t$ satisfying

$$\sigma_t = -i_{X_t}w_t. \quad (76)$$

In components, equation (76) is equivalent to

$$\sigma_{ijk} = -X_t^i w_{tijk}, \quad j, k = 1, ..., n. \quad (77)$$
Next, observe that by hypothesis the $n \times n^2$ matrix $w_{ijk}$ has rank $n$. Similarly, setting the components of $w_0$ in the variables $(y^1, ..., y^n)$ to be given by the constant tensor $A_{ijk} = w_{ijk}(x_0)$, the $n \times n^2$ matrix $A_{ijk}$ has rank $n$. Furthermore, at the point $x_0$ we may assume $w(x_0) = w_0(x_0)$ because the matrices $w_{ijk}$ and $A_{ijk}$ coincide there. Then, for $0 \leq t \leq 1$,

$$w_t(x_0) = w_0(x_0).$$

(78)

This implies that the $n \times n^2$ matrix $w_{ti(jk)}(x_0)$ has rank $n$ at $x_0$. By continuity of the tensor $w_{ti(jk)}$ it follows that there exists a neighborhood $V$ of $x_0$ where the rank of the $n \times n^2$ matrix $w_{ti(jk)}$ is $n$. Define $U = W \cap V$. Then, the matrix $w_{ti(jk)}$ has a right-inverse inverse $\hat{\sigma}^{i(jk)}$. Since by hypothesis $\sigma_1 \in \text{Im}(\hat{w}_t)$, in $U$ the solution $X_t$ of equation (77) can be written in terms of the right inverse as

$$X_t^i = -\hat{\sigma}^{i(jk)} \sigma_{jk}, \quad t = 1, ..., n.$$  

(79)

The vector field (79) gives the desired local change of coordinates.

**Theorem 4.** Let $\omega \in \wedge^2 T^*\Omega$ denote a (not necessarily closed) 2-form of constant rank $2m = n - s$ and $dG \in T^*\Omega$ an exact 1-form such that $(x^1, ..., x^n)$ defines a coordinate system in $\Omega$ with $x^n = G$. Define the 3-form $w \in \wedge^3 T^*\Omega$ as $w = \omega \wedge dG$ and suppose that $dw = 0$. Then, for every $x_0 \in \Omega$ there exist a neighborhood $U$ of $x_0$ and a coordinate system $(p^1, ..., p^\ell, q^1, ..., q^\ell, G^1, ..., G^s)$ with $n = 2\ell + \tau$ such that

$$w = \omega_0 \wedge dG, \quad \omega_0 = \sum_{i=1}^{\ell} dp^i \wedge dq^i \quad \text{in } U,$$

(80)

with $\ell = m$ if $\partial_n \in \ker(\omega)$ and $2\ell \leq n - 1$ if $\partial_n \notin \ker(\omega)$. Furthermore, given a 1-form $dH \in T^*\Omega$, linearly independent from $dG$, the phase space measure $d\Pi = dp^1 \wedge ... \wedge dp^\ell \wedge dq^1 \wedge ... \wedge dq^\ell \wedge dG^1 \wedge ... \wedge dG^s$ is an invariant measure in $U$ for the generalized Hamiltonian system $X \in TU$ such that

$$i_X w = -dH \wedge dG,$$

(81)

provided that such $X$ exists. In addition,

$$i_X \omega_0 = -\tilde{d}H \quad \text{in } \Sigma_G,$$

(82)

where $\Sigma_G = \{x \in U : G(x) = c \in \mathbb{R}\}$ and $\tilde{d}$ denotes the differential operator on $\Sigma_G$.

**Proof.** Since $dw = dw \wedge dG = 0$, it follows that $\tilde{d} \omega = 0$ in any level set $\Sigma_G$. On the other hand,

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j = \sum_{i=1}^{n-1} \omega_{i0} dx^i \wedge dG + \sum_{i<j} \omega_{ij} dx^i \wedge dx^j.$$  

(83)

Define $\tilde{\omega} = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$. Evidently $w = \tilde{\omega} \wedge dG$. Since $w$ is closed, this implies $\tilde{d} \tilde{\omega} = 0$. If $\partial_n \in \ker(\omega)$, from (83) it follows that $\omega = \tilde{\omega}$ and rank $(\tilde{\omega}) = 2m = n - s$. Conversely, if $\partial_n \notin \ker(\omega)$ the forms $\omega$ and $\tilde{\omega}$ are different, with rank $(\tilde{\omega}) = 2\ell = n - 1 - u \leq n - 1$. In either case, by the Lie-Darboux theorem for all $x_0 \in \Omega$ there exist a neighborhood $U$ of $x_0$ and $n - 1$ local coordinates $(p^1, ..., p^\ell, q^1, ..., q^\ell, G^1, ..., G^{s-1})$ or $(p^1, ..., p^\ell, q^1, ..., q^\ell, G^1, ..., G^s)$ such that

$$\tilde{\omega} = \omega_0 = \sum_{i=1}^{\ell} dp^i \wedge dq^i \quad \text{in } \Sigma_G.$$  

(84)

By smoothness, the coordinates $p^i, q^i : C^\infty(\Sigma_G) \to \mathbb{R}$ also define smooth functions $p^i, q^i : C^\infty(U) \to \mathbb{R}$. Then,

$$w = \sum_{i=1}^{\ell} dp^i \wedge dq^i \wedge dG = \sum_{i=1}^{\ell} dp^i \wedge dq^i \wedge dG = \omega_0 \wedge dG.$$

(85)
Now consider a solution $X \in TU \text{ of system } (81)$. Recalling that, by hypothesis, $dH$ and $dG$ are linearly independent and noting that $0 = i_X i_X w = -(i_X dH) dG + (i_X dG) dH$, it follows that $i_X dH = i_X dG = 0$. On the other hand, $i_X w = i_X \omega_0 \wedge dG = -\hat{d} H \wedge dG$, which implies

\[ i_X \omega_0 = -\hat{d} H \quad \text{in } \Sigma_G. \tag{86} \]

Since $\hat{d} \omega = 0$, equation (86) defines a Hamiltonian system with invariant measure $(\bigwedge_{i=1}^{\ell} dp^i \wedge dq^i) \wedge \hat{d} G^1 \wedge \ldots \wedge \hat{d} G^{s-1}$ if $\partial_n \in \ker(\omega)$ or $(\bigwedge_{i=1}^{\ell} dp^i \wedge dq^i) \wedge \hat{d} G^1 \wedge \ldots \wedge \hat{d} G^s$ if $\partial_n \notin \ker(\omega)$ on $\Sigma_G$. Set $\zeta = (G^1, \ldots, G^s) = (G^1, \ldots, G^s-1, G)$ if $\partial_n \in \ker(\omega)$ and $\zeta = (G^1, \ldots, G^s)$ if $\partial_n \notin \ker(\omega)$. It follows that

\[ d\Pi = \left( \bigwedge_{i=1}^{\ell} dp^i \wedge dq^i \right) \wedge dG^1 \wedge \ldots \wedge dG^s, \tag{87} \]

defines an invariant measure for $X$ in $U$.

**Proposition 1.** Let $w \in \bigwedge^3 T^*\Omega$ denote a closed 3-form and $dG \in T^*\Omega$ an exact 1-form such that $(x^1, \ldots, x^n)$ defines a coordinate system in $\Omega$ with $x^n = G$. Suppose that for any exact 1-form $dH \in T^*\Omega$ such that $dH$ and $dG$ are linearly independent there exists a vector field $X \in T\Omega$ solving

\[ i_X w = -dH \wedge dG. \tag{88} \]

Further assume that the 2-tensor $\omega_{ij} = w_{ijn}$ is invertible on the level sets $\Sigma_G = \{ x \in \Omega : G(x) = c \in \mathbb{R} \}$ with inverse $J \in \bigwedge^2 T\Sigma_G$ such that

\[ \sum_{j=1}^{n-1} \omega_{ij} J^{jk} = \delta_i^k, \quad i, k = 1, \ldots, n-1. \tag{89} \]

Then, on each level set $\Sigma_G$ there exists a closed 2-form $\hat{\omega} \in \bigwedge^2 T^*\Sigma_G$ such that

\[ i_X \hat{\omega} = -dH, \tag{90} \]

where $\hat{d}$ denotes the differential operator on $\Sigma_G$. Furthermore,

\[ w = \hat{\omega} \wedge dG, \tag{91} \]

and

\[ X = \zeta (dH, dG), \tag{92} \]

with $\zeta = J \wedge \partial_n$.

**Proof.** Equation (88) implies that

\[ X^i w_{ijk} = H_k G_j - H_j G_k. \tag{93} \]

Since $x^n = G$, we have $X^i w_{ijn} = -H_j$ for $j = 1, \ldots, n-1$. Hence,

\[ i_X \omega = -dH, \tag{94} \]

where $\omega \in \bigwedge^2 T^*\Omega$ is the 2-form $\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$ and $\hat{d}$ is the differential operator on the level sets $\Sigma_G$. Since $i_X dG = 0$, the equations of motion (81) and (94) give

\[ i_X (w - \omega \wedge dG) = 0. \tag{95} \]

Let $\xi \in \bigwedge^3 T^*\Omega$ denote a 3-form such that

\[ i_X \xi = \sum_{j<k} X^i \xi_{ijk} dx^j \wedge dx^k = \sum_{j<k} \zeta^i H_k \xi_{ijk} dx^j \wedge dx^k = 0. \tag{96} \]
It follows that
\[ w - \omega \wedge dG = \xi. \]  
(97)

On the other hand \( w \), and thus \( \xi \), cannot depend on \( H \) by construction. Therefore, we must have
\[ J^i_{\ell} \xi_{ijk} = 0 \quad \forall \ \ell = 1, \ldots, n - 1, \ j, k = 1, \ldots, n. \]  
(98)

However, the tensor \( J \) is invertible on \( \Sigma_G \) by hypothesis (equation (89)). Hence, \( \xi = 0 \) must be the zero 3-form. Then, equation (95) can be expressed in the form
\[ w = \omega \wedge dG. \]  
(99)

Using the closure of \( w \), we therefore arrive at the equation
\[ 0 = d\omega \wedge dG = \tilde{d}\omega \wedge dG. \]  
(100)

However, the 3-form \( \tilde{d}\omega \) can be expanded on the basis elements \( dx^i \wedge dx^j \wedge dx^k \) with \( i < j < k \) and \( i, j, k = 1, \ldots, n - 1, n \), which satisfy \( dx^i \wedge dx^j \wedge dx^k \wedge dG \neq 0 \). It follows that
\[ \tilde{d}\omega = 0, \]  
(101)
i.e. the 2-form \( \omega \in \bigwedge^2 T^* \Sigma_G \) is closed. The theorem is proven by noting that \( X = \mathfrak{J} (dH, dG) \) with \( \mathfrak{J} = J \wedge \partial_n \)
and by setting \( \tilde{\omega} = \omega \).

We remark that proposition 1 applies to the case in which \( n \) is odd, because the invertibility of \( \omega \) implies that \( n = 2m + 1 \) for some \( m \in \mathbb{N} \). The case in which \( n \) is even can be handled by a further integrability assumption on the kernel of \( \omega \).

We conclude this section with some observation concerning invertible 3-forms \( w \) that admit a constant (flat) expression \( w_0 = \sum_{i<j<k} A_{ijk} dy^i \wedge dy^j \wedge dy^k, A_{ijk} \in \mathbb{R} \), by a suitable change of coordinates. First, any such form induces an invariant (Liouville) measure given by the phase space volume
\[ d\Xi = dy^1 \wedge \ldots \wedge dy^n. \]  
(102)

To see this observe that the equations of motion (52) take the local form
\[ A_{ijk} \frac{dy^i}{dt} = \frac{\partial G}{\partial y^j} \frac{\partial H}{\partial y^k} - \frac{\partial G}{\partial y^k} \frac{\partial H}{\partial y^j}, \ j, k = 1, \ldots, n. \]  
(103)

The existence of the inverse \( B^{\ell j k} \) of \( A_{ijk} \) implies that
\[ \frac{dy^\ell}{dt} = B^{\ell j k} \frac{\partial G}{\partial y^j} \frac{\partial H}{\partial y^k}, \ \ell = 1, \ldots, n. \]  
(104)

It follows that,
\[ \mathcal{L}_X d\Xi = \frac{\partial}{\partial y^\ell} \left( \frac{dy^i}{dt} \right) d\Xi = B^{\ell j k} \left( \frac{\partial^2 G}{\partial y^j \partial y^k} \frac{\partial H}{\partial y^\ell} + \frac{\partial G}{\partial y^k} \frac{\partial^2 H}{\partial y^j \partial y^\ell} \right) d\Xi = 0. \]  
(105)

Notice that, in addition to \( d\Xi \), the closure condition implies that the symplectic 3-form \( w \) is Lie-invariant as well,
\[ \mathcal{L}_X w = di_X w = -d (dH \wedge dG) = 0. \]  
(106)

Therefore, in the generalized Hamiltonian framework developed here both the symplectic form \( w \) and the Liouville measure \( d\Xi \) are preserved as in the classical formulation. However, a difference exists with respect to the existence of canonical variables. Indeed, while in the classical proof of the Lie-Darboux theorem the skew symmetry of the tensor \( \omega_{ij} \) associated with the symplectic 2-form \( \omega \) is sufficient to ensure that there exists a linear change of basis transforming the skew-symmetric matrix \( \omega_{ij} (x_0) \) into block diagonal form at any \( x_0 \in \Omega \), an analogous result is not available for third order tensors like \( w_{ijk} (x_0) \). For this reason, one
cannot guarantee the local invertibility of the tensor $\tilde{w}_{ijk}$ associated with the 3-form $\tilde{w}_t = \tilde{w}_0 + t (w - \tilde{w}_0)$, with

$$\tilde{w}_0 = \sum_{i=1}^{m} dp^i \wedge dq^i \wedge dr^i. \quad (107)$$

Hence, the proof of theorem 3 breaks down, i.e. local canonical triplets $(p_i, q_i, r_i)$, $i = 1, ..., m$, are not available in general. Furthermore, even if $n = 3m$ with $m$ an integer, for canonical triplets of variables $(p^1, ..., p^m, q^1, ..., q^m, r^1, ..., r^m)$ to locally exist in the neighborhood of all points $x_0 \in \Omega$, it is not sufficient that $w_{ijk}(x_0)$ can be transformed by a linear change of basis into the generalized Levi-Civita symbol $E_{ijk}$ (the covariant version of the tensor $\tilde{w}_t$ introduced in section 3), because the applicability of theorem 3 also requires that the relevant Moser 2-form $\tilde{\sigma}_t$ belongs to the image of the map $\tilde{w}_t$.

6 Concluding Remarks

In this paper, we have formulated a generalization of classical Hamiltonian mechanics to a three-dimensional phase space. The generalized theory relies on a symplectic 3-form $w$ and a pair of Hamiltonian functions $G, H$. The closure of the symplectic 3-form $w$ ensures that there exist a local coordinate system $(y^1, ..., y^n)$ such that the components of $w$ are constants, and the volume form $d\Xi = dy^1 \wedge ... \wedge dy^n$ is Lie-invariant with respect to the generalized Hamiltonian flow. The invariant volume element can be used to define the phase space measure of statistical mechanics. When the components of $w$ define an $n \times n^2$ matrix of rank $n$, the form $w$ has a right inverse. If the right-inverse corresponds to a an antisymmetric 3-tensor, it defines a generalized Poisson operator $\mathcal{J}$. In analogy with the classical theory, the Jacobi identity is identified with the closure of the symplectic 3-form $w$ expressed in terms of $\mathcal{J}$. As a consequence, the Jacobi identity is a weaker condition than the fundamental identity, and any skew-symmetric third order contravariant tensor with constant entries defines a generalized Poisson operator. Skew-symmetric third order tensors also exhibit a richer kernel structure. In particular, semi-Casimir invariants produce conservation laws when they appear together in the generalized Poisson bracket.

There are aspects of the theory that deserve further investigation. On one hand it is desirable to identify the mathematical conditions for the existence of a linear change of basis transforming the constant skew-symmetric third order covariant tensor $w_{ijk}(x_0)$, $x_0 \in \Omega$, into the generalized Levi-Civita symbol $E_{ijk}$. Such conditions will provide necessary conditions for the local existence of canonical triplets $(p^i, q^i, r^i)$, $i = 1, ..., m$. On the other hand, it would be useful to identify systems that are Hamiltonian in the generalized sense, but that do not possess a classical Hamiltonian structure. Such systems would give the theory a physical foundation. Finally, the quantization of the generalized Poisson bracket resulting from the present construction has not been investigated.

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Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.
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