Numerical Evaluation of Exact Person-by-Person Optimal Nonlinear Control Strategies of the Witsenhausen Counterexample

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Abstract—Witsenhausen’s 1968 counterexample is a simple two-stage decentralized stochastic control problem that highlighted the difficulties of sequential decision problems with non-classical information structures. Despite extensive prior efforts, what is known currently, is the exact Person-by-Person (PbP) optimal nonlinear strategies, which satisfy two nonlinear integral equations, announced in 2014, and obtained using Girsanov’s change of measure transformations. In this paper, we provide numerical solutions to the two exact nonlinear PbP optimal control strategies, using the Gauss Hermite Quadrature to approximate the integrals and then solve a system of nonlinear equations to compute the signaling levels. Further, we analyse and compare our numerical results to existing results previously reported in the literature.

I. INTRODUCTION

The Witsenhausen’s counterexample [1] is a two-stage stochastic control problem, shown in Fig. 1 described by the following (state and output) equations, admissible strategies and pay-off.

State Equations:
\[ x_1 = x_0 + u_1, \quad x_0 \sim \mathcal{N}(\cdot) \]
\[ x_2 = x_1 - u_2 \]  \hspace{1cm} (1)

Output Equations:
\[ y_0 = x_0, \]
\[ y_1 = x_1 + v, \quad v \sim \mathcal{N}(0, \sigma^2) \] \hspace{1cm} (2)

Admissible Borel Measurable Strategies:
\[ u_1 = \gamma_1(y_0), \quad u_2 = \gamma_2(y_1) \] \hspace{1cm} (3)

Cost function:
\[ J(u_1, u_2) = J(\gamma_1, \gamma_2) \triangleq \mathbb{E}\left\{ k^2 u_1^2 + x_2^2 \right\}, \quad k \in \mathbb{R} \] \hspace{1cm} (4)

Fig. 1: Witsenhausen’s decentralized stochastic system

The exact form of the nonlinear strategies \((\gamma_1^*, \gamma_2^*)\) is currently unknown; the difficulty is attributed to the fact that, \(y_0\) is known to the control strategy \(\gamma_1\) but not to the control strategy \(\gamma_2\), i.e., the information structure is nonclassical [1].

A. Prior Literature

Hans Witsenhausen in [1], analyzed the counterexample extensively; he showed that optimal strategies exist, and for certain parameters \((k, \sigma^2)\), constructed a sub-optimal tuple of nonlinear strategies that outperform the tuple of optimal affine or linear strategies (these are recalled in Section III see (16) and (17)). We should emphasize that Theorem 2 of [1] does not claim that nonlinear strategies outperform all affine strategies for all values of parameters \((k, \sigma^2)\); rather, it is only for certain parameters that the sub-optimal nonlinear strategies outperform the optimal affine strategies [16]. One of the main results of Witsenhausen is: for a fixed \(\gamma_1\) the optimal strategy \(\gamma_2^*(y_1)\) is [1]:
\[ \gamma_2^*(y_1) = \mathbb{E}\{\gamma_1(x_0)|y_1\}, \quad \gamma_1(x_0) = x_0 + \gamma_1(x_0). \] \hspace{1cm} (6)

However, the optimal strategy \(\gamma_1^*(y_1)\) is currently unknown.

The Witsenhausen’s counterexample received much attention over the years by the control and information theory communities. [2] parameterized the tuple of strategies \((\gamma_1, \gamma_2)\) by partitioning the parameter space into two regions: one with an affine strategy and the other with a nonlinear strategy.

[3] applied finite element methods to develop numerical schemes to compute the optimal pay-off, when \(\gamma_2^*\) is given by \(\mathbb{E}\{\gamma_1(x_0)|y_1\}\). [4] developed a numerical scheme to compute the pay-off by employing one-hidden-layer neural network, making use of \(\gamma_2^*\) given by \(\mathbb{E}\{\gamma_1(x_0)|y_1\}\). [5] applied approaches an iterative source-channel coding method to quantize the strategies. [6] developed numerical methods based on nonconvex optimization. [7] applied making use of \(\gamma_2^*\) given by \(\mathbb{E}\{\gamma_1(x_0)|y_1\}\), and transformed the problem to an optimization problem over the space of quantile functions, and provided a numerical scheme that generates the approximate pay-off.

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Charalambous and Ahmed [8] in 2014 computed the exact nonlinear strategies \((\gamma^*_1, \gamma^*_2)\) using the notion of Person-by-Person (PbP) optimality. The approach in [8] is the formulation of an equivalent optimization problem, under a reference probability measure, such that the observation \(y_t\) is independent of the strategies \((\gamma_1, \gamma_2)\). This approach is fully described in [8], [9], [10] for decentralized problems described by stochastic differential equations.

The exact optimal PbP strategies are given by [8]

\[
\gamma^*_1(x_0) = -\frac{1}{2k^2} \int_{-\infty}^{\infty} \left\{ y_1 - \gamma^*_1(x_0) \right\} \left( \gamma^*_2(x_0) - \gamma^*_2(\gamma^*_1(x_0)) \right)^2
\]

\[
- \frac{1}{k^2} \int_{-\infty}^{\infty} \gamma^*_2(x_0) - \gamma^*_2(y_1) | x_0 \right) \tag{7}
\]

\[
\gamma^*_2(y_1) = \mathbb{E}\{ \gamma^*_1(y_1) | x_0 \}, \quad \gamma^*_1(y_1) = x_0 + \gamma^*_1(x_0). \tag{8}
\]

The above PbP optimal strategies are equivalently expressed in terms of \(\gamma^*_1 = x_0 + \gamma^*_1\) and \(\gamma^*_2\) by the two nonlinear integral equations:

\[
\gamma^*_1(x_0) = x_0 - \frac{1}{k^2} \int_{-\infty}^{\infty} \left\{ \frac{1}{2\sigma^2} \left( \xi - \gamma^*_1(x_0) \right) \left( \gamma^*_2(x_0) - \gamma^*_2(\xi) \right)^2
\]

\[
+ \left( \gamma^*_1(x_0) - \gamma^*_2(\xi) \right) \right\} \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^2 \exp \left( -\frac{1}{2\sigma^2} \right) d\xi \tag{9}
\]

\[
\gamma^*_2(y_1) = \frac{\int_{-\infty}^{\infty} \gamma^*_1(\xi) \exp \left( -\frac{1}{2\sigma^2} \left( y_1 - \gamma^*_1(\xi) \right)^2 \right) d\xi}{\int_{-\infty}^{\infty} \exp \left( -\frac{1}{2\sigma^2} \right) d\xi} \tag{10}
\]

It is important to mention that the PbP strategy \(\gamma^*_2\), i.e., \((\gamma^*_1, \gamma^*_2)\) is derived by applying PbP optimality, i.e., calculus of variations, and that \(\gamma^*_2\) has the form derived by Witsenhausen in [1], i.e., \(\gamma^*_2\). That is, the functional forms satisfy \(\gamma^*_2 = \gamma^*_2\). However, we do not yet know, if Witsenhausen’s global optimal strategy \(\gamma^*_1\) is identical to the PbP optimal strategy \(\gamma^*_1\).

### B. Contributions of the Paper

In this paper we undertake the study of calculating the optimal PbP strategies, by approximating the integrals \((9)\) and \((10)\), using the Gauss Hermite Quadrature numerical integration method. The resulting coupled approximations are then solved by posing them as a system of nonlinear equations; this method is detailed in Section II.

One of the main contributions is to evaluate the performance of the exact nonlinear PbP strategies with respect to the properties of the global optimal strategies \((\gamma^*_1, \gamma^*_2)\) derived by Witsenhausen in [1].

The findings are presented in Section III for different parameter values \((k, \sigma^2)\). The conclusions are found in Section IV.

### II. Numerical Integration of the Optimal Strategies

Consider the optimal strategies in their integral form \((9)\) and \((10)\). Recognizing that with the exponential function within the integral, the integral form can be reformulated to have a Gaussian exponential function, we employ the Gauss Hermite Quadrature (GHQ) method to implement the optimal strategies.

First, we briefly review the Gauss Hermite Quadrature method. The approximate numerical integration formula for a function \(f(x)\) on the infinite range \((-\infty, \infty)\) with the weight function \(e^{-x^2}\) is [11]:

\[
\int_{-\infty}^{\infty} f(x) e^{-x^2} dx \approx \sum_{i=1}^{n} f(x_i) \lambda_i, \tag{11}
\]

where the abscissas \(\{x_i, n\}\) are the roots of the \(n\)th order Hermite polynomial \(H_n(x)\) is defined by

\[
H_n(x) = -\sqrt{2^n} h_n(\sqrt{2}x) = 0
\]

with \(h_n(x) = e^{x^2} \frac{d^n}{d x^n} e^{-x^2}\) and the weights \(\{\lambda_{i, n}\}\) are given by

\[
\lambda_{i, n} = \frac{\sqrt{\pi} 2^{n+1} n!}{H_n(x_i)^2}
\]

where \(H'_n(x) = 2n H_{n-1}(x)\). For \(n \leq 10\), the zeros \(x_i, n\) of the Hermite polynomial \(H_n(x)\) and the weights \(\lambda_{i, n}\) are calculated in [11]. For higher orders, the zeros and weights are calculated in [12]. It is shown in [13] that the Gauss quadrature rule \((11)\) is exact for all continuous functions \(f\) that are polynomials of degree \(\leq 2n - 1\). The implications of quadrature rule to approximate a discontinuous function will be discussed in Section III.

It is in general a difficult problem to compute zeros and weights for any Hermite polynomial and any weight function. Therefore, since the zeros and weights for the aforementioned \(H_n(x)\) are calculated in the literature, we transform the optimal strategies \((9)\) and \((10)\) to have the standard Gaussian function \(e^{-z^2}\) as the weight function.

Consider the first law \((9)\) and the change of variables as \(z = \frac{\xi - \gamma^*_1(x_0)}{\sqrt{2\sigma^2}}\) and \(du = \frac{dx}{\sqrt{2\sigma^2}}\). Then,

\[
\gamma^*_1(x_0) = x_0 - \frac{1}{\sqrt{\pi}k^2} \int_{-\infty}^{\infty} \left\{ \frac{z}{\sqrt{2\sigma^2}} \right\} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{1}{2\sigma^2} \right) d\xi
\]

Using Gauss Hermite Quadrature approximation \((11)\),

\[
\gamma^*_1(x_0) \approx x_0 - \frac{1}{\sqrt{\pi}k^2} \sum_{i=1}^{n} \left\{ \frac{z_i}{\sqrt{2\sigma^2}} \right\} \lambda_i
\]

Similarly approximating the second law \((10)\) with the change of variable \(z = \frac{\xi - \gamma^*_1(x_0)}{\sqrt{2\sigma^2}}\) we get:

\[
\gamma^*_2(y_1) \approx \frac{\int_{-\infty}^{\infty} \gamma^*_1(\xi) \exp \left( -\frac{1}{2\sigma^2} \left( y_1 - \gamma^*_1(\xi) \right)^2 \right) d\xi}{\int_{-\infty}^{\infty} \exp \left( -\frac{1}{2\sigma^2} \right) d\xi}
\]
\[
\gamma_2^o(y_1) = \frac{\int_{-\infty}^{\infty} \gamma_0^o(\xi) \exp \left( -\frac{(y_1 - \gamma_0^o(\xi))^2}{2\sigma^2} \right) \exp \left( -\frac{\xi^2}{2\sigma^2} \right) d\xi}{\int_{-\infty}^{\infty} \exp \left( -\frac{(y_1 - \gamma_0^o(\xi))^2}{2\sigma^2} \right) \exp \left( -\frac{\xi^2}{2\sigma^2} \right) d\xi}
\]

(13)

Consider (12), since \( z_i \) and \( \lambda_i \) are the (known) nodes and weights, for certain \( x_0 \in \mathbb{R} \), the unknowns are \( \gamma_0^o(x_0) \) and \( \gamma_0^o(\sqrt{2\sigma^2 z_i + \gamma_0^o(x_0)}) \) (whose argument is in turn a function of \( \gamma_0^o(x_0) \)). In order to solve this equation, we employ the expression for \( \gamma_2^o(y_1) \) from (13) by having \( y_1 = \sqrt{2\sigma^2 z_i + \gamma_0^o(x_0)} \). Substituting \( \gamma_2^o(y_1) = \sqrt{2\sigma^2 z_i + \gamma_0^o(x_0)} \) from (13) in (12) to get:

\[
\gamma_0^o(x_0) \approx x_0 - \frac{1}{\sqrt{\pi k^2}} \sum_{i=1}^{n} \lambda_i \left\{ \frac{z_i}{2\sigma^2} \right\}
\]

\[
\left( \gamma_0^o(x_0) - \left( \sum_{j=1}^{n} (\gamma_1^o(\sqrt{2\sigma^2 z_j}) \left( \frac{\exp (-\frac{2\sigma^2 z_i + \gamma_0^o(x_0) - \gamma_0^o(\sqrt{2\sigma^2 z_j})^2}{2\sigma^2})}{\gamma_2^o(y_1)} \right)} \right) \right)^2
\]

\[
+ \left( \gamma_0^o(x_0) - \left( \sum_{j=1}^{n} (\gamma_1^o(\sqrt{2\sigma^2 z_j}) \left( \frac{\exp (-\frac{2\sigma^2 z_i + \gamma_0^o(x_0) - \gamma_0^o(\sqrt{2\sigma^2 z_j})^2}{2\sigma^2})}{\gamma_2^o(y_1)} \right)} \right) \right)^2
\]

(14)

While \( x_0 \in \mathbb{R} \) and \( \sqrt{2\sigma^2 z_i} \) are known, \( \gamma_1^o(\sqrt{2\sigma^2 z_i}) \) and \( \gamma_0^o(\sqrt{2\sigma^2 z_i}) \) are unknown. Let \( s_i = \gamma_1^o(\sqrt{2\sigma^2 z_i}) \), \( \forall i \). For each \( x_0 \), (14) hence contains \((n+1)\) number of unknowns, i.e., \( n \) \( s_i \)'s and one \( \gamma_1^o(x_0) \):

\[
\gamma_0^o(x_0) \approx x_0 - \frac{1}{\sqrt{\pi k^2}} \sum_{i=1}^{n} \lambda_i \left\{ \frac{z_i}{2\sigma^2} \right\}
\]

\[
\left( \gamma_1^o(x_0) - \left( \sum_{j=1}^{n} (s_j \exp \left( -\frac{2\sigma^2 z_i + \gamma_1^o(x_0) - s_j^2}{2\sigma^2} \right) \right) \right)^2
\]

\[
+ \left( \gamma_1^o(x_0) - \left( \sum_{j=1}^{n} (s_j \exp \left( -\frac{2\sigma^2 z_i + \gamma_1^o(x_0) - s_j^2}{2\sigma^2} \right) \right) \right)^2
\]

Substituting \( x_0 = x_{0l} = \sqrt{2\sigma^2 z_i} \) for each \( l \in \{1, 2, \ldots, n\} \), we obtain \( n \) nonlinear equations with \( n \) \( s_i \)'s that are unknown, given in (13). Each \( s_i \), which is the value of \( \gamma_1^o(x_0) \) at nodes selected according to Gauss-Hermite Quadrature, is the signaling level of the control action. Rearranging (13) to move all terms on one side, we denote the resulting system of nonlinear equations as \( f_{\text{sysnonlin}} : \mathbb{R}^n \rightarrow \mathbb{R}^n \).

\[
t_l \approx \sqrt{2\sigma^2 z_i} - \frac{1}{\sqrt{\pi k^2}} \sum_{i=1}^{n} \lambda_i \left\{ \frac{z_i}{2\sigma^2} \right\}
\]

\[
\left( t_l - \left( \sum_{j=1}^{n} (t_j \exp \left( -\frac{2\sigma^2 z_i + t_j - t_j^2}{2\sigma^2} \right) \right) \right)^2
\]

\[
+ \left( t_l - \left( \sum_{j=1}^{n} (t_j \exp \left( -\frac{2\sigma^2 z_i + t_j - t_j^2}{2\sigma^2} \right) \right) \right)^2
\]

(15)

The solution of the system of \( n \) nonlinear equations (15) results in \( n \) explicit points, i.e., \( n \) signaling levels \( s^*_l \), \( \forall l = 1, 2, \ldots, n \), such that \( ||f_{\text{sysnonlin}}(s^*_1, s^*_2, \ldots, s^*_n)|| \) is close to zero. Using these \( n \) signaling levels, we obtain the value of \( \gamma_1^o(x_0) \), \( \forall x_0 \), by substituting \( (s^*_1, s^*_2, \ldots, s^*_n) \) in (14) which results in one unknown \( \gamma_2^o(x_0) \) and solving the resulting nonlinear equation for \( \gamma_2^o(x_0) \) for each \( x_0 \). This is similar to the collocation method used to solve integral equations, [14]. Here, \( x_0 = x_{0l} = \sqrt{2\sigma^2 z_i} \) for each \( l \in \{1, 2, \ldots, n\} \) are the collocation points and signaling levels are the values of \( \gamma_1^o(x_0) \) at the collocation points.

To obtain the strategy of the second controller, we substitute the signaling levels \( s^*_1, s^*_2, \ldots, s^*_n \) in (13). This directly gives the expression for \( \gamma_2^o(y_1) \) which is evaluated at \( y_1 \). It is worth noting here that although \( y_1 \in \mathbb{R} \), but because \( y_1 = \gamma_1^o(x_0) + v \) from (2), the values taken by \( y_1 \) are dictated by the strategy of the first controller \( \gamma_1^o(x_0) \). Once both the strategies \( \gamma_1^o, \gamma_2^o \) are obtained, we calculate the total cost \( J \) from (4).

We now briefly summarize the methodology to numerically integrate the derived optimal strategies (9) and (10).

### III. RESULTS

We employ the software MATLAB to implement the solution strategies (9) and (10). The command f.solve is used.

#### Input parameters:
- \( k, \sigma, \sigma_z, n \)
- Input signals: \( x_0, v \)
  - Solve \( f_{\text{sysnonlin}} \) to obtain the signaling levels \( (s^*_1, s^*_2, \ldots, s^*_n) \)
  - For each \( x_0 \), compute \( \gamma_1^o(x_0) \)
  - For all \( y_1 = \gamma_1^o(x_0) + v \), compute \( \gamma_2^o(y_1) \)
to solve the system of nonlinear equations \( J_{sysnonlin} \) and \( lsqnonlin \) to solve for \( \bar{\gamma}_1(x_0) \).

The set of parameters in the Witsenhausen counterexample \([1-4]\) are \((k, \sigma, \sigma_x)\). For certain sets of values of these parameters, the optimal law is affine while for the rest of the region of parameters, the optimal law is non-linear. In Lemma 1 in [1], Witsenhausen derived the optimal affine laws as:

\[
\begin{align*}
\bar{\gamma}_1^{aff}(x_0) &= \nu x_0 \\
\bar{\gamma}_2(y_1)^{aff} &= \mu y_1
\end{align*}
\]

(16)

where \( \bar{\gamma}_1(x_0) = x_0 + \gamma_1(x_0) \),

\[
\mu = \frac{\sigma^2\nu^2}{1 + \sigma^2\nu^2}
\]

and \( t = \sigma_x \nu \) is a real root of the equation

\[
(t - \sigma_x)(1 + t^2)^2 + \frac{1}{k^2}t = 0
\]

We denote the cost obtained from the optimal affine laws as \( J^{aff} = J(\bar{\gamma}_1^{aff}, \bar{\gamma}_2^{aff}) \). In Theorem 2 of [1] he considers the sample non-linear laws:

\[
\begin{align*}
\bar{\gamma}_1^{wit}(x_0) &= \sigma_x \text{sgn}(x_0) \\
\bar{\gamma}_2^{wit}(y_1) &= \sigma_x \tanh(\sigma_x y_1)
\end{align*}
\]

(17)

and shows that \( J^{wit} < J^{aff} \) as \( k \to 0 \), where \( J^{wit} \) is the cost resulting from the nonlinear laws \([17]\).

We denote the cost we obtain from the derived optimal laws \([9]\) and \([10]\) and implemented using the Gauss Hermite Quadrature numerical integration method detailed in Section II as \( J^\circ \). We consider different parameter values of \((k, \sigma, \sigma_x)\) and compare the cost we obtain \( J^\circ \) with \( J^{aff}, J^{wit} \) and some other costs previously reported in the literature. For additional insight into the results, the total cost is broken into two stages: Stage 1 and Stage 2 costs are the first and the second term, respectively, in the total cost:

\[
J(\gamma_1, \gamma_2) = \mathbb{E}\left\{k^2(\gamma_1(x_0) - x_0)^2 + (\gamma_1(x_0) - \gamma_2(y_1))^2\right\}
\]

(18)

We have employed 600,000 samples for \( x_0 \) and \( v \) generated according to \( \mathcal{N}(0, \sigma) \) and \( \mathcal{N}(0, \sigma) \) respectively. The order of the Hermite polynomial in GHQ method is \( n=7 \). As stated in Lemma 1 of [1], the optimal cost is less than \( \min(1, k^2\sigma_x^2) \). Accordingly, we verify if the cost \( J^\circ \) is less than \( \min(1, k^2\sigma_x^2) \).

1) Parameters \( k = 0.001, \sigma_x = 1000, \sigma = 1 \): The total costs obtained are reported in Table I. Note that \( J^\circ < \min(1, k^2\sigma_x^2) \) and so are \( J^{wit} \) and \( J^{aff} \). The optimal PbP strategies \( \bar{\gamma}_1^{aff} \) and \( \bar{\gamma}_2^{aff} \) obtained are shown in Fig. 2. As pointed in [1], we observe that PbP \( \bar{\gamma}_1^{aff} \) is indeed symmetric around the origin. Moreover, we obtain four signaling levels, compared to one resulting from \( \bar{\gamma}_1^{wit} \). We also observe that the derived strategies result in a strategy such that \( \bar{\gamma}_1^{aff}(x_0) \approx \gamma_2^{aff}(y_1) \) leading to near zero Stage 2 cost. It is worth pointing out that since \( \gamma_2^{aff} \) admits \( y_1 = \bar{\gamma}_1^{aff}(x_0) + v \) as the input, the behaviour of \( \gamma_2^{aff} \) over the entire real line \( \mathbb{R} \) is not apparent. Moreover, despite the high value of \( \sigma_x = 1000 \), the presented methodology is not numerically unstable.

2) Parameters \( k = 1, \sigma_x = 1, \sigma = 1 \): As pointed in [15], this set of parameter values \((k \neq 0.56 \text{ and } \sigma_x \text{ is not large}) \) is in the region where affine laws are optimal. The optimal control laws \([9]\) and \([10]\) are compared with optimal affine laws in Fig. 3. It is seen that the resulting laws are almost the same as the optimal affine laws. We further compare the cost with \( J^{aff} \) and \( J^{wit} \) in Table II. The negligible difference in \( J^{aff} \) and \( J^\circ \) is attributed to numerical inaccuracy in the implementation of \([9]\) and \([10]\) through approximate numerical integration method.

3) Comparison with [16]: A class of nonlinear policies initially introduced in [1] and further analyzed and improved upon in [2] is given by:

\[
\begin{align*}
\gamma_1^{bb}(x_0) &= \epsilon^{bb}\text{sgn}(x_0) + \lambda^{bb}x_0 \\
\gamma_2^{bb}(y_1) &= \mathbb{E}[\epsilon^{bb}\text{sgn}(x_0) + \lambda^{bb}x_0|y_1]
\end{align*}
\]

(19)

where \( \epsilon^{bb} \) and \( \lambda^{bb} \) are parameters to be optimized over. For \( k = 0.01, \sigma_x = \sqrt{80} \text{ and } \sigma = 1 \), [16] picks \( \epsilon^{bb} = 5 \) and \( \lambda^{bb} = 0.01006 \) in the law \([19]\) and reports the cost to be

| Stage 1 | Stage 2 | Total Cost |
|---------|---------|-----------|
| \( J^{aff} \) | 0.9984 | 9.9843 x 10^-7 | 0.9984 |
| \( J^{wit} \) | 0.4041 | 0 | 0.4041 |
| \( J^\circ \) | 0.1137 | 1.1368 x 10^-7 | 0.1137 |

TABLE II: Total cost, \( k = 1, \sigma_x = 1 \)
find the signaling levels (value of \( \gamma_1 \) at the step) and the breakpoints (\( x_0 \) where the step change occurs). They also find that the cost objective is lower for slightly sloped steps than perfectly leveled steps. Through comparison of their costs for different number of steps, they find that 7-step solution yields the lowest cost. The cost obtained in [6] is denoted \( J^{\text{th}} \) here and the signaling levels therein are \( s^* = \{0, \pm 6.5, \pm 13.2, \pm 19.9\} \).

In our work, the solution of (15) yields the signaling levels \( s^{**} = \{0, \pm 6.15, \pm 12.8, \pm 19.8\} \) and \( ||f_{\text{synnonlin}}(s^{**})|| = 10^{-15} \) while \( ||f_{\text{synnonlin}}(s^*)|| = 0.7 \). Following up on the notes from Section II, the Gauss quadrature rule is not exact for the set of parameters \( k = 0.2, \sigma_x = 5, \sigma = 1 \) because this parameter set lies in the region where the optimal laws are non-linear. Moreover, the optimal non-linear laws are not continuous; they are only piecewise continuous. As a result, the inaccuracy in the approximation using Gauss quadrature rule is apparent. The cost we obtain for signaling levels \( s^* \) and \( s^{**} \) are \( J^0 = 0.16 \) and \( J^*_{**} = 0.1712 \) respectively.

The optimal PnP strategy, \( \tilde{\gamma}_1^*(x_0) \), we have obtained for the signaling levels \( s^* \) and \( s^{**} \), are shown in Fig 5. Although we do not externally impose symmetry, it can be observed that \( \tilde{\gamma}_1^* \) is symmetric around the origin and is non-decreasing. We zoom in on one of the 7 steps and observe in the left column of Fig 5 that the steps are slightly sloped. Further zooming in, we see in the right column of Fig 6 that each signaling level is further comprised of a number of closely spaced steps. Similar to this result, the authors in [6] added segments in each of the 7 steps to obtain the cost \( J^{\text{th}} = 0.167313205338 \). We compare both the costs with previously reported costs in the literature in Table IV. Further in agreement with the findings in [6], we obtain the lowest cost for 7 steps, \( J^*_{**} = 0.1712 \).

With the parameter set \( k = 0.2, \sigma_x = 5, \sigma = 1 \), the number of steps we obtain is the same as the value of the Gauss quadrature rule parameter \( n \). However, this is not necessarily the case for all parameter sets; for example see Section III.2. The parameter set \( k = 1, \sigma_x = 5, \sigma = 1 \) is known to lie in a region where the optimal law is affine, and even though we employ \( n = 7 \) order for GHQ, the resulting control laws are affine. Likewise, as seen in Fig 4 the parameter set lies in the region where the optimal law is non-linear and we obtain a three-step control strategy for \( \tilde{\gamma}_1^* \) for the GHQ order \( n = 7 \).

**IV. Conclusion**

Computed are the exact optimal PnP strategies of the Witsenhausen counterexample derived in [8], that satisfy the

| Parameter Set | Total Cost |
|---------------|------------|
| \( k = 0.01, \sigma = 1 \) | \( 0.007986277332674 \) |
| \( k = 0.001566775786064 \) | \( 0.003232551870223 \) |
| \( k = 0.3309 \) | \( 0.3309 \) |

**TABLE III:** Total cost obtained from different solutions

Fig. 3: Comparison of the optimal PnP strategies and the optimal affine strategies

Fig. 4: Optimal PnP strategies for the parameters in [16]
tuples of nonlinear integral equations (9) and (10), using the Gauss hermite quadrature scheme, to transform the integral equations to a system of nonlinear equations.

Comparison to various costs obtained in the literature show that strategies (9) and (10) outperform previously reported results for most parameters values. Moreover, PbP strategies (9) and (10) reduce to optimal affine laws, for certain parameters.

The computed optimal PbP strategies are approximations of the exact PbP optimal strategies, because our numerical scheme, based on the Gauss Hermite Quadrature numerical integration, is not exact, when the underlying functions are not continuous.

Since the tuple of PbP optimal strategies (9) and (10) predict the properties of global optimal strategies \((\gamma_1^*, \gamma_2^*)\) of the Witsenhausen problem defined by (5), it is natural to investigate, in future work, whether \((\gamma_1^*, \gamma_2^*) = (\gamma_1^0, \gamma_2^0)\).

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