Mach-Zehnder Interferometry at the Heisenberg Limit
with coherent and squeezed-vacuum light

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We show that the phase sensitivity $\Delta \theta$ of a Mach-Zehnder interferometer fed by a coherent state in one input port and squeezed-vacuum in the other one is i) independent from the true value of the phase shift and ii) can reach the Heisenberg limit $\Delta \theta \sim 1/N_r$, where $N_r$ is the average number of particles of the input states. We also show that the Cramer-Rao lower bound, $\Delta \theta \propto 1/\sqrt{|\alpha|^2 e^{2r} + \sinh^2 r}$, can be saturated for arbitrary values of the squeezing parameter $r$ and the amplitude of the coherent mode $|\alpha|$ by a Bayesian phase inference protocol.

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Introduction. The goal of quantum interferometry is to estimate phases beyond the shot-noise (“standard quantum”) limit. The quest requires proper non-classical states, as was first shown by Caves in 1981 [1], who considered a Mach-Zehnder (MZ) fed by coherent ⊗ squeezed-vacuum light. This benchmark generated a large body of theoretical [2, 3, 4, 5] and experimental [6] studies, including the demonstration of sub-shot-noise sensitivity [6] using parametric down-conversion in a cavity as a source of squeezed vacuum [8]. The quest requires proper non-classical vacuum light. This benchmark generated a large body of theoretical [2, 3, 4, 5] and experimental [6] studies, including the demonstration of sub-shot-noise sensitivity [6] using parametric down-conversion in a cavity as a source of squeezed vacuum [8].

The scheme proposed by Caves is sketched in Fig. 1. One of the inputs of the linear loss-less MZ is the coherent state $|\alpha\rangle_a \equiv \sum_{m=0}^{+\infty} C_m |m\rangle_a$, with $\alpha \equiv e^{i\theta_0}|\alpha|$ and $C_m \equiv \frac{e^{-|\alpha|^2/2}}{\sqrt{m!}}$. The second input state is the squeezed-vacuum $|\zeta\rangle_b \equiv \sum_{m=0}^{+\infty} S_m |m\rangle_b$, with $\zeta \equiv re^{i\theta_0}$ and $S_m \equiv \left( \frac{e^{\zeta} \tanh r}{2^{m/2} \sqrt{m!}} \right) H_m(x)$ being the Hermite polynomials. Following the current literature, based on the original works of the 80’s, the phase estimate of this system is retrieved from the measurement of the relative number of particles at the output ports $N_{out} = N_c - N_d$. Fluctuations on the results obtained in $p$ independent measurements propagate to the estimated value of the phase shift $\theta [10]$, which can be eventually determined with uncertainty [11]:

$$\Delta \theta = \frac{1}{\sqrt{p}} \sqrt{\frac{|\alpha|^2 e^{-2r} + \sinh^2 r}{(|\alpha|^2 - \sinh^2 r)^2} + \frac{|\alpha|^2 + 2 \sinh^2 r \cosh^2 r}{(|\alpha|^2 - \sinh^2 r)^2 \tan \theta}} (1)$$

According to Eq. (1), we can appreciate an increase of phase sensitivity with respect to the shot-noise only when the true value of the phase shift is sufficiently close to the $\theta = \pi/2$ [1, 8] (dark fringe), where $\langle M_{out} \rangle = \langle N_c - N_d \rangle = 0$. On the other hand, $\langle M_{out} \rangle$ depends weakly on the phase shift when $\theta \approx 0, \pi$ and the error propagation formula Eq. (1) predicts large phase fluctuations around these points. Asymptotically in the amplitude of the coherent state, $|\alpha|^2 \gg \sinh^2 r$, and for a fixed squeezing parameter $r$, Eq. (1) predicts a sub shot-noise sensitivity [1, 12]

$$\Delta \theta = \frac{1}{\sqrt{p}} \frac{e^{-r}}{\sqrt{n}}, \quad (\theta = \pi/2), \quad (2)$$

with the average number of photons injected in the MZ $\bar{n} \approx |\alpha|^2$.

In this Letter we show that the choice of the average relative number of photons as phase estimator is not optimal. Quantum fluctuations also contains information on the true value of the phase shift, which can be retrieved by taking in account the higher moments of the measured number of particles at the output ports. We will show that the ultimate phase sensitivity of a Mach-Zehnder fed by coherent ⊗ squeezed-vacuum light is

$$\Delta \theta = \frac{1}{\sqrt{p}} \frac{1}{\sqrt{|\alpha|^2 e^{2r} + \sinh^2 r}} \quad (0 \leq \theta \leq \pi). \quad (3)$$

The phase sensitivity Eq. (3) is i) independent from the true value of the phase shift over the whole interval $0 \leq \theta \leq \pi$ and ii) it reaches, at the optimal point $|\alpha|^2 = \sinh^2 r$, the Heisenberg limit:

$$\Delta \theta = \frac{1}{\sqrt{p}} \frac{1}{\sqrt{n}} \quad (0 \leq \theta \leq \pi), \quad (4)$$

asymptotically in the average number of photons $\bar{n} = |\alpha|^2 + \sinh^2 r$ and with a number of independent measurements $p \gtrsim 30$.

In the following, we will first analytically calculate the Cramer-Rao lower bound (CRLB), Eq. (3), and then demonstrate that it is saturated by a Bayesian phase inference approach. A proof of principle of Eq. (3) can be obtained within current technology, at least in the limit of small $\bar{n}$: high-efficiency number-resolving photodetectors have been recently applied to interferometry [13, 14] and high squeezing has been obtained with parametric down-conversion [15]. Out results can be relevant, for instance, to improve the efficiency of the large scale interferometers dedicated to the detection of gravitational waves [16], which would not require phase-stabilization techniques to lock at the optimal point $\theta = \pi/2$ [3] and which can significantly increase their sensitivity.

The Cramer-Rao lower bound. The output state of a loss-less Mach-Zehnder interferometer is given by $|\psi_{out}\rangle = e^{-i\theta_0} |\psi_{in}\rangle$ [17], where, in our case, $|\psi_{in}\rangle = |\alpha\rangle_a |\zeta\rangle_b$. The conditional probability to measure $N_c$ and $N_d$ particles at the output ports, given an unknown phase
values of the parameters \( \theta \), is

\[
P(N_c, N_d|\theta) = \left| \sum_{n=0}^{N} C_{N-n} S_n \frac{d^{N/2}}{d \theta^{N/2}} (\cos \theta) \right|^2, \tag{5}
\]

where \( \mu = (N_c - N_d)/2 \) and \( \frac{d^{N/2}}{d \theta^{N/2}} (\cos \theta) \) are rotation matrix elements. The Fisher information, \( F(\theta) = \sum_{N_c=0}^{\infty} \sum_{N_d=0}^{\infty} P(N_c, N_d|\theta) \left( \frac{\partial P(N_c, N_d|\theta)}{\partial \theta} \right)^2 \), turns out to be independent from the true value of the phase shift \( \theta \), see Fig. (2), and an analytical calculation gives \( F(\theta) = |\alpha|^2 e^{2r} + \sin^2 r \). According to Cramer and Rao, the phase sensitivity of an unbiased estimator is bounded by \( \Delta \theta = \frac{1}{\sqrt{F(\theta)}} \), which, after replacing the previous expression for the Fisher information, gives Eq. (3). There are interesting limit regimes recovered by this equation: i) When \( r = 0 \) or \( \alpha = 0 \) we get the (\( \theta \)-independent) shot-noise limit \( \Delta \theta = 1/\sqrt{\mu n} \). The phase independence of the case \( r = 0 \) has been studied and experimentally demonstrated in [13]. ii) When \( \sin^2 r \ll |\alpha|^2 \) we obtain the sub shot-noise limit discussed by Caves, \( \Delta \theta = e^{-r} / \sqrt{\mu n} \).

The most important regime predicted by Eq. (3) is obtained when \( |\alpha|^2 \sim \sin^2 r = \bar{n}/2 \) (i.e., with half of the input intensity provided by the coherent state and half by the squeezed light). This gives \( \Delta \theta = 1/\sqrt{\mu n} \) when \( \bar{n}, p \gg 1 \). It is interesting to notice that, for these optimal values of the parameters \( \alpha \) and \( r \), the error propagation formula Eq. (1) predicts a divergence. In Fig. (2a) we compare, as a function of \( r \) and for \( \theta = \pi/2 \), the quantity \( \sqrt{\mu n} \Delta \theta \) calculated with Eq. (1) (dotted line) compared with Eq. (3) (solid line).

Why does the error propagation formula Eq. (1) provides such a poor phase sensitivity with respect to the CRLB? The answer is that an estimation of the phase shift based only on the measurement of the average relative number of particles does not exploit all the available information contained in the detection of \( N_c \) and \( N_d \). It forgets about the information contained in the fluctu-
For larger values of $p$, we saturate the Fisher information and obtain $\Delta \theta = \sqrt{p}/N_T$. The prefactor $\sqrt{p}$ arises from the statistics of independent measurements. As shown in figure (3a), the optimal value is $p_{opt} \sim 30$. The crucial point to notice is that $p_{opt}$ does not depend on $N_T$. If it would, we could not claim the Heisenberg limit. The phase sensitivity calculated at $p_{opt}$ is plotted in figure (3b) as a function of $N_T$ (circles). The dashed line is the Heisenberg limit $\Delta \theta = 7.12/N_T$, while the solid line is $\Delta \theta \approx \sqrt{3}/\pi \sqrt{\sqrt{p}/N_T}$. For comparison, we include in the figure the shot-noise limit (dot-dashed line).

We emphasize that an enhancement of phase sensitivity can be obtained also when only one output port is monitored (reduced MZ configuration). A numerical calculation of the Fisher information for $|\alpha|^2 \sim \sinh^2 r$ shows a strong dependence on $\theta$, the optimal working point being close to 0 or $\pi$, depending on the port which is monitored. Even if we were not able to numerically investigate large values of $\bar{n}$, we have strong evidences that, asymptotically in $\bar{n}$, we obtain a phase sensitivity $\Delta \theta \sim 1/N_T$, with a prefactor larger than the one obtained with the Mach-Zehnder interferometer.

**Discussion.** What is the physics underlying the increase in phase sensitivity using squeezed vacuum light? In Caves associated sub-shot-noise to quadrature squeezing. Indeed, under the conditions $\theta = \pi/2$, and $|\alpha|^2 \gg \sinh^2 r$, Eq. (1) reduces to $\Delta \theta = \Delta \hat{x}_{opt} = \sqrt{\bar{n}}/p_{opt}$, with the quadrature $\hat{x}_{opt} = \langle \hat{b}^\dagger + \hat{b} \rangle/2$. With squeezed-vacuum light $\Delta \hat{x}_{opt} = e^{-r}$ and we recover Eq. (2). Conversely, we can understand the saturation at the Heisenberg limit by quantum interference effects created by the beam splitter. The key point is to notice that the input squeezed state has the components $S_m = 0$ when $m$ is odd. For the sake of simplicity, we discuss this problem by fixing (post-selecting) a total number of particles $N = \bar{n}$. The input $|\psi_N\rangle = \sum_{\mu=-N/2}^{N/2} A_{\mu}|N/2-\mu\rangle|N/2+\mu\rangle$ is characterized by a relative number of particles distribution $P(\mu) = |A_\mu|^2$, where $A_\mu = 0$ for odd values of $N/2 - \mu$, see Fig. (4a). This creates a relative number of particles distribution after the first beam splitter characterized by a mean-square fluctuation of the order of $N$. In particular, the distribution has the largest peaks centered at $\mu = \pm N/2$, see Fig. (4b), which indicates that the corresponding quantum state after the beam splitter contains a large “NOON” component $|\psi_{N}\rangle \sim |(N,0)+(0,N)\rangle$. Such a distribution is typical of states attaining the Heisenberg limit $\Delta \theta \sim 1/N$. Intuitively, the phase distribution $P(\phi)$ obtained by projecting a state with heavily weighted components at $\mu = \pm N/2$ over phase states $|\phi\rangle = \sum_{|\nu| = N/2} c_{\nu} |\nu\rangle$ is characterized by oscillations of frequency $2\pi/N$. This typical structure is illustrated in Fig. (4c) where we plot the phase distribution obtained by projecting $|\psi_{N}\rangle = e^{-iJ_F}|\psi_N\rangle$ over $|\phi\rangle$. Finally, it is interesting to notice that the highest “NOON” component is obtained when $\alpha^2 = \bar{n}/2$, which precisely corresponds to the optimal conditions discussed in Eq. (4). This is illustrated in Fig. (4d), where $P_{NOON} \equiv |(NOON)e^{-iJ_F}|\psi_N\rangle|^2$ is shown as a function of $|\alpha|^2/\bar{n}$.

**Conclusions.** The discovery that interferometric measurements can be dramatically improved by non-classical light has been crucial for the development of modern quantum optics. Several states and strategy have been proposed in the literature to beat the shot-noise limit. Here we have shown that the oldest of these proposals, a linear lossless Mach-Zehnder interferometer fed by a coherent-squeezed-vacuum light $|\psi_N\rangle$, can indeed reach the Heisenberg limit Eq. (1), but only if the whole information included in the measurement of the number of particles at the output ports is taken into account. This requires a feasible analysis of the interferometric

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**FIG. 3:** (Color online) Demonstration of the Heisenberg limit $\Delta \theta \sim 1/N_T$. A) Circles are the phase sensitivity obtained with the Bayesian analysis as a function of the number of measurements $p$ with fixed total number of particles $N_T = \bar{n}$. The optimal value, $p_{opt} = 30$, corresponds to the minimum of $\Delta \theta$ and does not depend on $\bar{n}$. Dashed lines are guides to the eye. B) Corresponding optimal sensitivity as a function of $N_T$. The dashed line is the asymptotic limit $\Delta \theta = 7.12/N_T$, the solid line is $\Delta \theta \sim \sqrt{p}/N_T$. Shot-noise has been included for comparison (dot-dashed line).

**FIG. 4:** (Color online) Relative number of particles distribution $P(\mu)$ for A) the input state $|\psi_N\rangle$ with post-selected $N = \bar{n}$ and optimal conditions $|\alpha|^2 = \sinh^2 r$ and B) state after the first beam splitter, $|\psi_{N}\rangle$. C) Phase distribution $P(\phi)$ obtained after a projection of $|\psi_{N}\rangle$ over phase states. D) Quantity $P_{NOON}$ as a function of $|\alpha|^2/\bar{n}$. The maximum is reached at $|\alpha|^2 \sim \bar{n}/2$. Here $\bar{n} = 20$. 

For larger values of $p$, we saturate the Fisher information and obtain $\Delta \theta = \sqrt{p}/N_T$. The prefactor $\sqrt{p}$ arises from the statistics of independent measurements. As shown in figure (3a), the optimal value is $p_{opt} \sim 30$. The crucial point to notice is that $p_{opt}$ does not depend on $N_T$. If it would, we could not claim the Heisenberg limit. The phase sensitivity calculated at $p_{opt}$ is plotted in figure (3b) as a function of $N_T$ (circles). The dashed line is the Heisenberg limit $\Delta \theta = 7.12/N_T$, while the solid line is $\Delta \theta \approx \sqrt{\sqrt{p}/N_T}$. For comparison, we include in the figure the shot-noise limit (dot-dashed line).

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data which is provided, for instance, by a Bayesian protocol. Moreover, we have also shown that the phase sensitivity is independent from the true value of the phase shift for arbitrary values of squeezing.

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