Finite particle creation in 3+1 de Sitter space

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Abstract

In this paper we calculate the particle creation as seen by a stationary observer in 3+1 de Sitter space. This particle creation is calculated using an observer dependent geometrically based definition of time which is used to quantize a field on two different spacelike surfaces. The Bogolubov transformation relating these two quantizations is then calculated and the resulting particle creation is shown to be finite.

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I. INTRODUCTION

In this paper we calculate the particle creation as seen by a stationary observer in 3+1 de Sitter space. This particle creation is calculated by looking at the Bogolubov transformation relating the observer’s different definitions of particle states on two different spacelike hypersurfaces. The definition of particle states used is that proposed by Capri and Roy [1] and is equivalent to the definition proposed by Massacand and Schmid [2]. This definition of particle states uses a coordinate independent definition of time which one uses to decompose the field into positive and negative frequency parts. This time is defined as being normal to the spacelike geodesic hypersurface which intersects the observer’s worldline orthogonally. In this way the spacetime is spanned by geodesics. If there is a geodesically complete coordinatization for the spacetime this is the coordinatization that will be picked out by this definition of time. In de Sitter space this implies that the radial coordinate is compact even though the coordinatization we start with would not suggest this. It is this compact coordinatization that allows us to eventually integrate by parts the expression for the total particle production and show that it is finite. Similar results were obtained in an earlier paper [3] for a 1 + 1 dimensional model which was compact in space.

The particle production is shown to be finite as the Bogolubov $\beta(N, N', l)$ coefficient drops off faster than any inverse power of $N$ or $N'$. If this drop off is actually an exponential then the particle production would be consistent with a thermal distribution which is what is expected for the large momenta limit. This finite particle creation agrees with the analysis presented in Fulling’s book for expanding isotropic universes [4].

This calculation is not a calculation of the Bogolubov transformation relating two different coordinatizations of, the same spacetime or, different portions of the same spacetime.

II. THE MODEL

We start with the following coordinatization of de Sitter space,
\[ ds^2 = dT^2 - e^{\lambda T} \left( (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \right) \] (2.1)

To calculate the coordinates which provide the foliation mentioned in the introduction we must first calculate the geodesic equations. The first integrals of the geodesic equations are,

\[ \frac{dX^i}{ds} = c^i e^{-\lambda T} \quad \text{and} \quad \frac{dT}{ds} = \sqrt{\epsilon + e^{-\lambda T} c^2} \] (2.2)

where \( i = 1 \) to \( 3 \) and \( \epsilon = \pm 1 \) depending on whether the geodesic is timelike or spacelike respectively. The preferred coordinates on the hypersurface of instantaneity are constructed using a 4-bein of orthonormal basis vectors based at \( P_0 \), the observer’s position. These vectors are chosen to be,

\[

e^0_\mu(P_0) = (1, 0, 0, 0) \quad e^1_\mu(P_0) = (0, e^{-\lambda T_0}, 0, 0) \\
e^2_\mu(P_0) = (0, 0, e^{-\lambda T_0}, 0) \quad e^3_\mu(P_0) = (0, 0, 0, e^{-\lambda T_0})
\] (2.3)

In this way \( e^0_\mu(P_0) \) is tangent to the worldline of an observer which is stationary with respect to the coordinates of (2.1). To construct a spacelike geodesic which is orthogonal to the observer’s worldline it is required that

\[ \left. \frac{dT}{ds} \right|_{P_0} = 0 \quad \text{which implies} \quad c^2 = e^{\lambda T_0}. \] (2.4)

The preferred coordinates on the spacelike hypersurface are chosen to be Riemann coordinates based on the observer’s position \( P_0 = (T_0, X^1_0, X^2_0, X^3_0) \). The coordinates are constructed using the point \( P_1 = (T_1, X^1_1, X^2_1, X^3_1) \) which is the point at which a timelike geodesic “dropped” from an arbitrary point \( P = (T, X^1, X^2, X^3) \) intersects the spacelike hypersurface orthogonally. The Riemann coordinates \( \eta^\alpha \) of the point \( P_1 \) are given by,

\[ s_s p^\mu = \eta^\alpha e^\mu_\alpha(P_0) \] (2.5)

where \( s_s \) is the distance along the geodesic \( P_0 - P_1 \) and \( p^\mu \) is the vector tangent to the geodesic connecting \( P_0 \) to \( P_1 \), at \( P_0 \). These equations can be solved for the coordinates \( \eta^\alpha \) using the orthogonality of \( p^\mu \) to \( e_0(P_0) \) and the identity \( e^\mu_\alpha e_\beta^\mu = \eta_{\alpha\beta} \) (Minkowski metric) to give,
\[ \eta^0 = s_s p^\mu e^0_\mu(P_0) \quad \eta^i = -s_s p^\mu e^i_\mu(P_0) \]  

(2.6)

The surface of instantaneity is then just the surface \( \eta^0 = 0 \) and the preferred spatial coordinates are given by,

\[ x^i = s_s c^i e^{\frac{\lambda T_0}{c}} \]  

(2.7)

The preferred time coordinate \( t \) of an arbitrary point \( P \) is then given by the geodesic distance along the timelike geodesic connecting \( P \) to \( P_1 \). This timelike geodesic is also determined by (2.2) with a different set of constants \( b^i \) and \( \epsilon = 1 \). The condition that this timelike geodesic is orthogonal to the spacelike hypersurface is,

\[ \sqrt{e^{\lambda(T_0-T_1)} - 1} \sqrt{c^2 e^{-\lambda T_1} + 1} = c \cdot b e^{-\lambda T_1} \]  

(2.8)

There is an arbitrary choice involved in how one solves these two equations for the constants \( b \) and \( c \). This freedom can be understood as the ability to rotate the hypersurface of instantaneity through a reparametrization of the surface. The choice which we make for reasons of calculational simplicity is that

\[ b^i = \sqrt{1 - e^{-\lambda(T_0-T_1)}} c^i. \]  

(2.9)

At this point it convenient to to introduce the variable \( r \),

\[ r^2 = x \cdot x = s^2_s c \cdot c = s^2_s. \]  

(2.10)

We can now calculate the metric in terms of the preferred coordinates \((t, x^i)\) by calculating \((T(t, x^i), X(t, x))\).

\[ X^i = X^i_0 + \int_{T_0}^{T_1} dT \frac{c e^{-\lambda T}}{\sqrt{e^{\lambda(T_0-T)} - 1}} + \int_{T_1}^{T} dT \frac{b e^{-\lambda T}}{\sqrt{1 + b^2 e^{-\lambda T}}} \]  

(2.11)

We also need to calculate \( t \) and \( s_s \),

\[ s_s = \int_{T_0}^{T_1} \frac{dT}{\sqrt{e^{\lambda(T_0-T)} - 1}} \]  

(2.12)

\[ t = \int_{T_1}^{T} \frac{dT}{\sqrt{1 + b^2 e^{-\lambda T}}} \]  

(2.13)
One can now obtain the coordinate transformations,

\[
\begin{align*}
\lambda T(T-T_0) = \cosh\left(\frac{\lambda t}{2}\right) \cos\left(\frac{\lambda r}{2}\right) + \sinh\left(\frac{\lambda t}{2}\right) \\
\frac{\lambda}{2}(X^i - X^i_0)e^{-\frac{r}{T}} = \frac{x^i}{r} \cosh\left(\frac{\lambda t}{2}\right) \sin\left(\frac{\lambda r}{2}\right).
\end{align*}
\] (2.14)

We can see here by looking at a particular \( t = \)constant surface that the range of \( r \) is now compact and the range \( 0 \leq \frac{\lambda r}{2} < \pi \) covers the entire manifold which was covered by the original coordinates \((T, X)\). It is now easy to put the preferred coordinates into polar form,

\[
\begin{align*}
x^1 &= r \sin(\theta) \sin(\phi) \\
x^2 &= r \sin(\theta) \cos(\phi) \\
x^3 &= r \cos(\theta)
\end{align*}
\] (2.15)

In terms of these preferred coordinates the metric is,

\[
ds^2 = dt^2 - \cosh^2\left(\frac{\lambda t}{2}\right) \left( dr^2 + \frac{4}{\lambda^2} \sin^2\left(\frac{\lambda r}{2}\right) \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right) \right)
\] (2.16)

This result is of course no surprise to anyone familiar with different coordinatizations of de Sitter space, given that the space was being coordinatized in terms of geodesics. The point here is not what the final form of the metric is as much as how these transformations will change as our observer moves to a different point and the entire construction is repeated.

### III. MODES AND INITIAL CONDITIONS

In the coordinates constructed above, the minimally coupled massless Klein Gordon equation is,

\[
\partial_t^2 \phi + \frac{1}{\sqrt{g}} \partial_t \left( \sqrt{g} \right) \partial_t \phi + \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \right) \partial_j \phi = 0
\] (3.1)

where \(|g|\) and the \(g^{ij}\) can be read off from (2.16). To quantize a scalar field on the \( t = 0 \) surface we now define positive the frequency modes as those which satisfy the initial conditions,
\[ \phi_{Nln}^+ = A_{Nln}(0, r, \theta, \phi) \quad \text{and} \quad \partial_t \phi_{Nln}^+|_{t=0} = -i \omega_N(0) A_{Nln}(0, r, \theta, \phi). \] (3.2)

Where \( A_{Nln}(0, r, \theta, \phi) \) are the instantaneous eigenmodes of the spatial part of the Laplace-Beltrami operator, and \( \omega_N(t)^2 \) are the corresponding eigenvalues,

\[ \frac{1}{\sqrt{g}} \partial_i \left( \sqrt{g} g^{ij} \right) \partial_j A_{Nln}(t, r, \theta, \phi) = \omega_N(t)^2 A_{Nln}(t, r, \theta, \phi). \] (3.3)

Henceforth we write \( \omega_N \) for \( \omega_N(0) \).

\[ \omega_N \equiv \omega_N(0) = \sqrt{\lambda^2 + N(N + 2)} \] (3.4)

The differential equations (3.1) and (3.3) must now be solved and the appropriate initial conditions imposed. The positive frequency solution to these differential equations which satisfies the correct initial conditions as just stated is,

\[ \phi_{Nln}^+(t, r, \theta, \phi) = F_{Nl} Y_{ln}(\theta, \phi) \sin^l \left( \frac{\lambda r}{2} \right) \text{sech} \left( \frac{\lambda t}{2} \right) \times \]

\[ \left( LP_{\frac{1}{2}+N}^\frac{3}{2} \left[ \tanh \left( \frac{\lambda t}{2} \right) \right] + MQ_{\frac{1}{2}+N}^\frac{3}{2} \left[ \tanh \left( \frac{\lambda t}{2} \right) \right] \right) \] (3.5)

where

\[ L = - \frac{(2 + N) \lambda Q_{\frac{3}{2}+N}^\frac{3}{2} - 2 i Q_{\frac{3}{2}+N}^\frac{3}{2}(0) \omega_N}{\lambda (2 + N) \left( -P_{\frac{3}{2}+N}^\frac{3}{2}(0) Q_{\frac{3}{2}+N}^\frac{3}{2}(0) + P_{\frac{3}{2}+N}^\frac{3}{2}(0) Q_{\frac{3}{2}+N}^\frac{3}{2}(0) \right)} \]

\[ M = \frac{(2 + N) \lambda P_{\frac{3}{2}+N}^\frac{3}{2} - 2 i P_{\frac{3}{2}+N}^\frac{3}{2}(0) \omega_N}{\lambda (2 + N) \left( -P_{\frac{3}{2}+N}^\frac{3}{2}(0) Q_{\frac{3}{2}+N}^\frac{3}{2}(0) + P_{\frac{3}{2}+N}^\frac{3}{2}(0) Q_{\frac{3}{2}+N}^\frac{3}{2}(0) \right)} \]

\[ F_{Nl} = \frac{2^{\frac{l}{2}+l} \sqrt{1+N} \Gamma(1+l) \sqrt{\Gamma(1-l+N)}}{\sqrt{\pi} \sqrt{\Gamma(2+l+N)}}, \] (3.6)

\( C_m^n[x] \) are Gegenbauer polynomials and \( P_n^m[x] \) and \( Q_n^m[x] \) are associated Legendre functions.

We can now write out the field which has been quantized on the \( t = 0 \) surface which corresponds to the geodesic surface passing through the point \((T_0, X_0)\).

\[ \Psi_1 = \sum_{N=0}^{\infty} \sum_{l=0}^{N} \sum_{n=-l}^{l} \left\{ a_{Nln} \phi_{Nln}^+(t, r, \theta, \phi) + a_{Nln}^+ \phi_{Nln}^+(t, r, \theta, \phi) \right\} \] (3.7)
IV. PARTICLE CREATION

To investigate the particle creation in this universe, as observed by an observer stationary with respect to the original coordinates \((T, X)\), we calculate the Bogolubov transformation relating the annihilation and creation operators from two different surfaces of quantization that the observer passes through. To calculate the coefficients of this transformation we equate the same field from two different quantizations on a common surface,

\[
\Psi_1(t, r, \theta, \phi) = \Psi_2(t'(t, r, \theta, \phi), r'(t, r, \theta, \phi), \theta'(t, r, \theta, \phi), \phi'(t, r, \theta, \phi)).
\] (4.1)

Here \(\Psi_1(t, r, \theta, \phi)\) is the field written out in (3.7) and \(\Psi_2(t', r', \theta', \phi')\) is the same field which has been quantized on a second surface \(t' = 0\). The “second” field is therefore quantized for the same observer as the first but at some later time \(T'_0\) with \(X_0 = X'_0\). All the physics of the observations made by this observer are determined by the functions \(t'(t, r, \theta, \phi), r'(t, r, \theta, \phi), \theta'(t, r, \theta, \phi), \phi'(t, r, \theta, \phi), \) and the derivatives of these functions with respect to \(t\).

In this way the geometry of the spacetime via the coordinate independent prescription we have used, determines the spectrum of created particles. This is the reason for the comment at the end of Section II about the form of the metric not being as important as the transformations that gave that form of the metric. These functions take on a fairly simple form for the stationary observer,

\[
t' = \frac{2}{\lambda} \sinh^{-1}\left[ \sinh\left(\frac{\lambda t}{2}\right) \cosh(\tau) - \cosh\left(\frac{\lambda r}{2}\right) \cos\left(\frac{\lambda r}{2}\right) \sinh(\tau) \right]
\]

\[
r' = \frac{2}{\lambda} \tan^{-1}\left[ \frac{\cosh\left(\frac{\lambda t}{2}\right) \sin\left(\frac{\lambda r}{2}\right)}{\cosh\left(\frac{\lambda t}{2}\right) \cos\left(\frac{\lambda r}{2}\right) \cosh(\tau) - \sinh\left(\frac{\lambda r}{2}\right) \sinh(\tau)} \right]
\]

\[
\theta' = \theta
\]

\[
\phi' = \phi \quad \text{where} \quad \tau = \frac{\lambda}{2}(T'_0 - T_0)
\] (4.2)

We calculate the Bogolubov transformation by “matching” the field and its first derivative with respect to \(t\) at \(t = 0\). This allows us to calculate the \(\beta\) coefficient of the Bogolubov transformation which gives rise to the particle creation. In calculating the Bogolubov \(\beta\)
coefficient we are able to perform the $\theta$ and $\phi$ integrals of the spherical harmonics because of the simplicity of the coordinate transformations \((4.2)\) leaving,

$$
\beta(N, N', l) = \frac{i}{2\omega_N} \int_0^\pi d\chi \sin^2(\chi) R_{NI}(\chi) \left( -i\omega_N f_N^{(+)}(t') R_{NI}(\chi') + \partial_t f_N^{(+)}(t') R_{NI}(\chi') \right) \Big|_{t=0}
$$

(4.3)

here $\chi = \frac{\lambda r}{2}$ and $\chi' = \frac{\lambda r'}{2}$. For notational convenience we have split up the radial and time functions as

$$
R_{NI}(\chi) = F_{NI} \sin^l(\chi) C_{N-l}^{l+1} (\cos(\chi))
$$

$$
f_N^{(+)}(t') = \text{sech} \left( \frac{\lambda t}{2} \right) \left( LP_{\frac{1}{2} + N}^{\frac{3}{2}} (\tanh(\frac{\lambda t}{2})) + MQ_{\frac{3}{2} + N}^{\frac{3}{2}} (\tanh(\frac{\lambda t}{2})) \right)
$$

(4.4)

In the next section we examine the structure of $\beta$ in detail.

V. TOTAL NUMBER OF PARTICLES CREATED

To show that the total number of particles created is finite we must show that the Bogolubov transformation is Hilbert-Schmidt namely,

$$
\sum_{NN'\ell} |\beta(N, N', \ell)|^2 < \infty.
$$

(5.1)

Since the sum on the left hand side of this inequality gives the number of particles created this inequality, if it holds, implies that the total number of particles created is finite and that the Bogolubov transformation is unitarily implementable. To show this one need only be concerned with the large $N, N'$ and $l$ behaviour. As the sum over $l$ is a finite sum and $\beta(N, N', l)$ decreases with $l$ when $l$ is large then one only need be concerned with the large $N$ and $N'$ behaviour of $\beta(N, N', l)$. By looking at this asymptotic behaviour one is left with simpler functions that may be integrated exactly. We now show that indeed when looking at the large $N$ and $N'$ behaviour the integrals defining $\beta$ may be bounded by terms implying that $|\beta(N, N', l)|^2$ drops off faster than any inverse power of $N$ and $N'$. This also implies that the finite sum over $l$ does not change this result as it only introduces a simple power.
of $N$. Using the following relations for the functions that the modes are constructed from, we are able to obtain an approximate form of $\beta(N, N', l)$ valid for large $N$ and $N'$,

$$
C_n^m = \frac{\Gamma(2m + n)\Gamma(m + 1)}{\pi^2 \Gamma(2m)\Gamma(n + 1)} \left[ \frac{1}{4} x^2 - 1 \right]^{\frac{1}{2}} \frac{x^{\frac{m}{2} - m}}{P_{m+n}^{\frac{1}{2}}(x)}
$$

$$
P_{\nu}^m[\cos(x)] \approx \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \left( \frac{1}{2\pi \sin(x)} \right)^{\frac{1}{2}} \cos \left( (\nu + \frac{1}{2})x - \frac{\pi}{4} + \frac{\mu\pi}{2} \right) \quad \text{for large } \nu
$$

$$
Q_n^m[\cos(x)] \approx \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \left( \frac{\pi}{2 \sin(x)} \right)^{\frac{1}{2}} \cos \left( (\nu + \frac{1}{2})x + \frac{\pi}{4} + \frac{\mu\pi}{2} \right) \quad \text{for large } \nu
$$

$$
\Gamma(ax + b) \approx \sqrt{2\pi e^{-ax}} (ax)^{a+b-\frac{1}{2}} \quad \text{for large } a \text{ and } x > 0. \quad (5.2)
$$

The expression for $\beta$ now involves many terms but is still simple enough to see what is required.

$$
\beta(N, N', l) = \int_0^{2\pi} d\chi K((A(A_1 \times L + A_2 \times M) + B(B_1 \times L + B_2 \times M)) M_1 \\
+ (A(C_1 \times L + C_2 \times M) + B(D_1 \times L + D_2 \times M)) N_1)
$$

(5.3)

where

$$
A = \cos(l \frac{\pi}{2}) - \cos^{-1}\left( \frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \sech(\tau)^2 \sin(\chi)^2}} \right) - N' \cos^{-1}\left( \frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \sech(\tau)^2 \sin(\chi)^2}} \right)
$$

$$
B = \sin(l \frac{\pi}{2}) - \cos^{-1}\left( \frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \sech(\tau)^2 \sin(\chi)^2}} \right) - N' \cos^{-1}\left( \frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \sech(\tau)^2 \sin(\chi)^2}} \right)
$$

$$
M_1 = \cos(N'\pi - N' \cos^{-1}\left( \frac{\cos(\chi) \sinh(\tau)}{\sqrt{1 + \cos(\chi)^2 \sinh(\tau)^2}} \right)) \sin(l \frac{\pi}{2} - \chi - N\chi)
$$

$$
N_1 = \sin(l \frac{\pi}{2} - \chi - N\chi) \sin(N'\pi - N' \cos^{-1}\left( \frac{\cos(\chi) \sinh(\tau)}{\sqrt{1 + \cos(\chi)^2 \sinh(\tau)^2}} \right))
$$

$$
A_1 = 16 (1 + l) \lambda N' \Gamma(2(1 + l)) \Gamma(\frac{5}{2} + l) \sin(\chi) \sinh(\tau)
$$

$$
A_2 = -8 (1 + l) \lambda N' \pi \cos(\chi) \Gamma(2(1 + l)) \Gamma(\frac{5}{2} + l) \sin(\chi) \sinh(\tau)^2
$$

$$
B_1 = -4 \cosh(\tau)^2 \Gamma(\frac{3}{2} + l) \Gamma(2(2 + l)) \left( -2i \omega_N \sech(\tau)^2 + 2\lambda \cos(\chi) \tanh(\tau) \\
+ l\lambda \cos(\chi) \tanh(\tau) + \lambda N' \cos(\chi) \tanh(\tau) - 2i \cos(\chi)^2 \omega_N \tanh(\tau)^2 \right)
$$

$$
B_2 = 2\pi \Gamma(\frac{3}{2} + l) \Gamma(2(2 + l)) \left( -2\lambda \cosh(\tau) - \lambda N' \cosh(\tau) - 2i \cos(\chi) \omega_N \sinh(\tau) \\
+ l\lambda \cos(\chi)^2 \cosh(\tau) \sinh(\tau)^2 - 2i \cos(\chi)^3 \omega_N \sinh(\tau)^3 \right)
$$
\[ C_1 = -16 (1 + l) \lambda N' \cos(\chi) \Gamma(2 (1 + l)) \Gamma(\frac{5}{2} + l) \sin(\chi) \sinh(\tau)^2 \]
\[ C_2 = -8 (1 + l) \lambda N' \pi \Gamma(2 (1 + l)) \Gamma(\frac{5}{2} + l) \sin(\chi) \sinh(\tau) \]
\[ D_1 = 4 \Gamma(\frac{3}{2} + l) \Gamma(2 (2 + l)) \left( -2 \lambda \cosh(\tau) - \lambda N' \cosh(\tau) - 2i \cos(\chi) \omega_N \sinh(\tau) \right) \]
\[ + l \lambda \cos(\chi)^2 \cosh(\tau) \sinh(\tau)^2 - 2i \cos(\chi)^3 \omega_N \sinh(\tau)^3 \]
\[ D_2 = \pi \Gamma(\frac{3}{2} + l) \Gamma(2 (2 + l)) \left( -3i \omega_N + i \cos(2\chi) \omega_N - i \cosh(2\tau) \omega_N - i \cos(2\chi) \cosh(2\tau) \omega_N \right) \]
\[ + 2 \lambda \cos(\chi) \sinh(2\tau) + l \lambda \cos(\chi) \sinh(2\tau) + \lambda N' \cos(\chi) \sinh(2\tau) \]
\[ K = \frac{2^l \sqrt{1 + N} \sqrt{N'} \sqrt{1 + N'} \sqrt{\frac{2}{\pi}} \Gamma(1 + l) \Gamma(\frac{3}{2} + l)}{\frac{1}{4} \sqrt{N} \Gamma(2 (1 + l)) \Gamma(2 (2 + l)) \left( 1 + \cos(\chi)^2 \sinh(\tau)^2 \right)^{\frac{3}{2}}} \quad (5.4) \]

The exact form of the above expressions are not important to understanding the large \( N \) and \( N' \) behaviour of \( |\beta(N, N', l)|^2 \). What is important is to notice that the expressions \( A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, K \) do not change as far as their \( N \) and \( N' \) behaviour is concerned when differentiated with respect to \( \chi \). This implies that one can integrate the expression by parts indefinitely to observe that the expression must drop off faster than any inverse power of \( N \) and \( N' \). A typical term after writing out the trigonometric functions in terms of exponentials reads,
\[
\int_0^{2\pi} d\chi e^{\pm iN\chi} e^{\pm iN' \cos^{-1}(p(\chi))} e^{\pm iN' \cos^{-1}(q(\chi))} F(N, N', \chi). \quad (5.5)
\]

Here
\[
p(\chi) = \frac{\cos(\chi)}{\sqrt{\cos(\chi)^2 + \text{sech}(\tau)^2}} \]
\[
q(\chi) = \frac{\cos(\chi) \sinh(\tau)}{\sqrt{1 + \cos(\chi)^2 \sinh(\tau)^2}}. \quad (5.6)
\]

In the above expression the exponentials represent the contributions from the combinations of \( A, B, M_1, N_1 \) and \( F(N, N', \chi) \) represents the contribution from the functions \( A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, K \). Equation (5.5) can be rewritten,
\[
\int_0^{2\pi} d(\pm e^{\pm iN\chi} e^{\pm iN' \cos^{-1}(p(\chi))} e^{\pm iN' \cos^{-1}(q(\chi))}) \sqrt{1-p^2} \sqrt{1-q^2} d\chi \quad (5.7)
\]
Thus an integration by parts produces a terms which drops off like,

$$\frac{d}{d\chi} \left( \pm iN \mp iN'\left( \frac{1}{\sqrt{1-p^2}} \frac{dp}{d\chi} \pm \frac{1}{\sqrt{1-q^2}} \frac{dq}{d\chi} \right) \right)$$

(5.8)

Because the behaviour of $F'(N, N', \chi)$ for large $N$ and $N'$ is no worse than $F(N, N', \chi)$ this procedure can be repeated indefinitely showing that $\beta(N, N', l)$ drops off faster than any inverse power of $N$ and $N'$ for large $N, N'$. We can then conclude that the particle creation is finite and that the Bogolubov transformation is unitarily implementable.

Concerning the $l$ dependence in $\beta(N, N', l)$ we only have a finite sum for the total particle creation. It is easy to show that if one uses the same approximations (5.2) for the gamma functions involving the $l$’s which are valid for large $l$, $\beta(N, N', l)$ drops off for large $l$ as $l$ increases. Thus, the probability of finding particles created with angular momentum $l$ decreases as $l$ increases. This means that when one does the finite sum over $l$ the result will not grow any quicker than $N$. Therefore because the particle density in $N$ and $N'$ drops off faster than any inverse power of $N$ and $N'$ the total particle creation remains finite.

VI. CONCLUSIONS

We have calculated explicitly the particle creation observed by an observer which is stationary in $3 + 1$ de Sitter space. We calculate this particle creation by calculating the Bogolubov transformation relating the annihilation and creation operators from two different quantizations. These different quantizations are constructed using the same procedure on two different spacelike surfaces. Physically this particle creation can be understood as the particle creation seen by an observer moving from one of these surfaces to the next. By looking at the large momenta behaviour for the Bogolubov transformations we are able to show that the transformation is unitarily implementable and therefore the particle creation is finite. Because $\beta(N, N', l)$ drops off faster than any inverse power of $N$ and $N'$ it may be that it drops as an exponential suggesting a thermal spectrum.
It should be emphasized what this calculation is not a calculation of. Many calculations have been done calculating the Bogolubov transformations relating the creation and annihilation operators due to two different coordinatizations of similar spacetimes. One coordinatization usually covering the entire spacetime and the other only covering a portion of the spacetime. These calculations seem to require the observers to have a split personality, so that at one time they think they are in a geodesically complete spacetime but at the same time are in only a portion of the spacetime. The procedure advocated in this paper requires that one use the geodesically complete coordinatization as the spacetime is spanned by geodesics in the preferred coordinates. In this particular example this means that that the preferred coordinatization is compact. It is this compactness that allows us to integrate by parts the expression for the total particle creation and show it is finite.

In spacetimes where there is a boundary present such as an horizon one may have to impose boundary conditions at the horizon [6]. In fact comparing coordinatizations where one coordinatization implies a boundary and therefore does not cover the entire manifold has been investigated in a clear paper by Salaev and Krustalev [7]. In this paper the authors conclude that either one has a boundary in the spacetime or one does not, there is no in between. This is the reason for the split personality analogy made above.

The alternative to the split personality scenario is that the observer somehow moves from one spacetime to the other, an issue that has been addressed earlier by Massacand and Schmid [8] and argued to be unreasonable.

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