Local Hawking temperature for dynamical black holes

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Abstract
A local Hawking temperature is derived for any future outer trapping horizon in spherical symmetry, using a Hamilton–Jacobi variant of the Parikh–Wilczek tunneling method. It is given by a dynamical surface gravity as defined geometrically. The operational meaning of the temperature is that Kodama observers just outside the horizon measure an invariantly redshifted temperature, diverging at the horizon itself. In static, asymptotically flat cases, the Hawking temperature as usually defined by the Killing vector agrees in standard cases, but generally differs by a relative redshift factor between the horizon and infinity, this being the temperature measured by static observers at infinity. Likewise, the geometrical surface gravity reduces to the Newtonian surface gravity in the Newtonian limit, while the Killing definition instead reflects measurements at infinity. This may resolve a long-standing puzzle concerning the Hawking temperature for the extremal limit of the charged stringy black hole, namely that it is the local temperature which vanishes. In general, this confirms the quasi-stationary picture of black-hole evaporation in early stages. However, the geometrical surface gravity is generally not the surface gravity of a static black hole with the same parameters.

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1. Introduction
Since the discovery of quantum black-hole radiance by Hawking [1], it has been widely seen as a key area to generate and test ideas concerning the interface of gravity, quantum theory and thermodynamics. In the usual picture, a radiating black hole loses energy and therefore shrinks, evaporating away to an unknown fate. However, the classical derivations
of Hawking temperature applied only to stationary black holes, while the above picture uses quasi-stationary arguments. In actuality, an evaporating black hole is non-stationary. So the question arises: is there in any sense a Hawking temperature for dynamical black holes?

In seeking to generalize from statics to dynamics, one immediately faces the fundamental conceptual issue of what constitutes a black hole. The traditional definition was by an event horizon [2]. However, this is an abstract definition with an essentially teleological nature, unlocatable by any mortal observer and devoid of any local, physical relevance. Certainly, it is generally not applicable in cosmology. Moreover, the concept leads naturally to the infamous information paradox.

A more practical theory has been developed by refining the concept of apparent horizon to trapping horizon [3, 4]. Trapping horizons are locally defined and have physical properties such as mass, angular momentum and surface gravity, satisfying conservation laws [5, 6]. They are a geometrically natural generalization of Killing horizons, which are stationary trapping horizons. In a dynamical regime, an evolving trapping horizon is not a null hypersurface, although it is still one of infinite redshift, in a sense which will be made precise below.

For stationary black holes, Parikh & Wilczek [7] pioneered a tunneling method to derive Hawking temperature, which made precise the intuitive picture of Hawking radiation in terms of virtual pair creation. A Hamilton–Jacobi variant has recently been applied to examples of non-stationary black holes [8–11]. The results all apply to trapping horizons of some sort. However, there have been derivations of several inequivalent temperatures or surface gravities [12]. Still different definitions were advocated for expanding cosmological black holes [13, 14].

In this communication, we apply the Hamilton–Jacobi method to general spherically symmetric spacetimes. We find that the method works if and only if there is a trapping horizon of the future outer type, as proposed some time ago as a local definition of black hole [3, 4]. The temperature so derived is given by the surface gravity as defined geometrically [4]. We discuss the operational meaning of the temperature and compare with other definitions, including the usual Killing temperature in the static case.

2. Geometry

In spherical symmetry, the area $A$ of the spheres of symmetry is a geometrical invariant. It is convenient to use the area radius $r = \sqrt{A/(4\pi)}$. A sphere is said to be untrapped, marginal or trapped if $g^{-1}(dr)$ is spatial, null or temporal, respectively. If the spacetime is time orientable and $g^{-1}(dr)$ is future (respectively past) causal, then the sphere is said to be future (respectively past) trapped or marginal. A hypersurface foliated by marginal spheres is called a trapping horizon [3, 4].

The active gravitational mass $m$ [16] is defined by

$$1 - \frac{2m}{r} = g^{-1}(dr, dr), \quad (1)$$

where spatial metrics are positive definite, and the Newtonian gravitational constant is unity. Various properties were derived in [4, 17, 18], to which we refer for more detailed motivation of the definitions here.

The Kodama vector [19] is

$$K = g^{-1}(*dr), \quad (2)$$

where $*$ is the Hodge operator in the space normal to the spheres of symmetry, i.e.

$$K \cdot dr = 0 \quad \text{and} \quad g(K, K) = -g^{-1}(dr, dr). \quad (3)$$
Then \( m \) is the Noether charge of the conserved energy–momentum density with respect to \( K \).
The Kodama vector gives a preferred flow of time, coinciding with the static Killing vector of standard black holes such as Schwarzschild and Reissner–Nordström. Note that \( K \) is temporal, null or spatial on untrapped, marginal or trapped spheres, respectively.

The geometrical surface gravity was defined as \[ \kappa = \frac{1}{2} (\ast d\ast dr), \tag{4} \]
where \( d \) is the exterior derivative in the normal space, or in terms of the metric \( \gamma \) normal to the spheres of symmetry,
\[ \kappa = \frac{1}{2} \square r. \tag{5} \]
Note also that \( \kappa \) satisfies
\[ K^a \nabla_b [\kappa_a] \cong \pm \kappa K_b, \tag{6} \]
where \( \cong \) denotes evaluation on a trapping horizon \( r \cong 2m \), similarly to the usual Killing identity.

Then a trapping horizon is said to be outer, degenerate or inner if \( \kappa > 0, \kappa = 0 \) or \( \kappa < 0 \), respectively. Examples are provided by Reissner–Nordström solutions. In vacuo, \( \kappa = m/r^2 \), therefore reducing to the Newtonian surface gravity in the Newtonian limit, since \( m \) reduces to the Newtonian mass [4]. Thus it provides a relativistic definition of the surface gravity of planets and stars as well as black holes.

Any spherically symmetric metric can locally be written in dual-null coordinates \( x^\pm \) as
\[ ds^2 = r^2 d\Omega^2 - 2 e^{2\Psi} dx^+ dx^- \tag{7} \]
where \( d\Omega^2 \) refers to the unit sphere and \( (r, \varphi) \) are functions of \( (x^+, x^-) \). There is still the freedom to rescale functionally \( x^\pm \rightarrow \tilde{x}^\pm(x^\pm) \). We wish to use the generalized advanced Eddington–Finkelstein form
\[ ds^2 = r^2 d\Omega^2 + 2 e^{2\Psi} dv dr - e^{2\Psi} C dv^2 \tag{8} \]
with \( (C, \Psi) \) being functions of \( (r, v) \). Transforming from dual-null coordinates with \( v = x^+ \):
\[ dx^+ = dv \quad \text{and} \quad dx^- = \partial_v x^- dv + \partial_r x^- dr, \tag{9} \]
so one identifies
\[ e^\Psi = - e^{2\Psi} \partial_v x^- \quad \text{and} \quad e^{2\Psi} C = 2 e^{2\Psi} \partial_r x^-, \tag{10} \]
which is possible if and only if \( \partial_r x^- < 0 \). Since we are assuming that \( v \) is an advanced time, this means that any trapped \( (C < 0) \) or marginal \( (C = 0) \) surface will be future trapped or marginal, as appropriate for black holes rather than white holes.

Note that
\[ C = 1 - \frac{2m}{r} \tag{11} \]
is an invariant, but \( \Psi \) is not, due to the freedom \( v \rightarrow \tilde{v}(v) \). We also have \( K = e^{-\Psi} \partial_v \) and
\[ \kappa = \frac{1}{2} e^{-\Psi} \partial_r (e^{\Psi} C) = \frac{\partial_r C + C \partial_r \Psi}{2}, \tag{12} \]
so
\[ \kappa \cong \frac{\partial_r C}{2} \cong \frac{1 - 2\partial_r m}{2r}. \tag{13} \]
3. Hamilton–Jacobi tunneling method

The tunneling approach uses the fact that the WKB approximation of the tunneling probability along the classically forbidden trajectory from inside to outside the horizon has the form

$$\Gamma \propto \exp \left( -\frac{\Im I}{\hbar} \right),$$

(14)

where \(\Im I\) is the imaginary part of the action \(I\) on the classical trajectory, to leading order in \(\hbar\), henceforth set to unity. If \(\Im I\) is proportional to an energy parameter \(\omega\), it takes a thermal form

$$\Gamma \propto \exp \left( -\frac{\omega}{T} \right),$$

(15)

which defines the temperature \(T\).

Consider a massless scalar field \(\phi = \phi_0 \exp(iI)\) in the eikonal (or geometric optics) approximation, so that the amplitude \(\phi_0\) is slowly varying and the action,

$$I = \int \omega e^{\Psi} dv - \int k dr,$$

(16)

is rapidly varying. Here \(e^{\Psi}\) is included to make \(\omega\) and \(I\) invariant, recalling the freedom \(v \rightarrow \tilde{v}(v)\). Equivalently, \(\omega = K \cdot dI = e^{-\Psi} \partial_\nu I, k = -\partial_r I\). Then the wave equation \(\nabla^2 \phi = 0\) yields the Hamilton–Jacobi equation

$$g^{-1}(\nabla I, \nabla I) = 0.$$  

(17)

In fact, the action (16) and this equation are all that we need to assume here. It becomes simply

$$2\omega k - Ck^2 = 0.$$  

(18)

Then \(k = 0\) yields the ingoing modes, while \(k = 2\omega/C\) yields the outgoing modes. Since \(C \cong 0\) at a trapping horizon \(r \cong r_0\), \(I\) has a pole, which can be evaluated by \(C \approx (r - r_0)\partial_r C\). Thus \(k \approx \omega/\kappa (r - r_0)\) if \(\kappa \neq 0\). Deforming the contour into the lower \(\omega\) half-plane corresponds to deforming the contour into the upper \(r\) half-plane, yielding an imaginary contribution,

$$\Im I \approx \frac{\pi \omega}{\kappa}.$$  

(19)

Then the particle production rate takes the thermal form (15) if the temperature is

$$T \cong \frac{\kappa}{2\pi}.$$  

(20)

For this to be positive, \(\kappa > 0\), so the trapping horizon is of the outer type. Thus the method has derived a positive temperature if and only if there is a future outer trapping horizon. This remarkably confirms the local definition of black hole which was proposed previously on purely geometrical grounds [3, 4].

4. Operational meaning: redshift

Having derived the temperature \(T\) formally, one may ask what it means operationally, i.e. in terms of what observers measure. First note that there is a preferred class of observers even in a non-static spacetime, whose worldlines are the integral curves of \(K\), who become static observers in the static case. These Kodama observers lie outside the horizon and have the velocity vector \(\hat{K} = K/\sqrt{C}\). Since \(-I\) is the phase, the frequency measured by such observers is

$$\dot{\omega} = \hat{K} \cdot dI = \frac{\omega}{\sqrt{C}}.$$  

(21)
Following the above method, such observers measure a thermal spectrum with temperature,

$$\hat{T} \approx \frac{T}{\sqrt{C}}, \quad (22)$$

to leading order near the horizon. The invariant redshift factor $\sqrt{C}$ (11) is familiar from the Schwarzschild case [20], where it reflects the acceleration required to keep an observer static. So this is the operational meaning of $T$: not that someone is measuring $T$ directly, but that the preferred observers just outside the horizon measure $T/\sqrt{C}$, which diverges at the horizon. Then $T$ itself can be interpreted as a redshift-renormalized temperature, which is finite at the horizon.

We note that $\kappa$ (13) is inequivalent to the Nielsen–Visser surface gravity [9], which in these coordinates takes the form

$$\tilde{\kappa} \approx \frac{1}{2r}(1 - 2\partial_r m - e^{-\Psi} \partial_r m), \quad (23)$$

though they coincide in the static case. Also, both are inequivalent to the Visser surface gravity $e^{\Psi} \tilde{\kappa}$ [8], which was derived as a temperature by essentially the same method as above, but in Painlevé–Gullstrand coordinates. The relative factor $e^{\Psi}$ is explained in the following section. The remaining difference can be traced to choice of time, since the action (16) defines a frequency with respect to the time coordinate, and consequently different temperatures can be obtained. This reflects the generally unresolved issue of choice of time in quantum field theory on non-stationary spacetimes. One physical argument for choosing advanced time $v$ for a massless scalar field is that, in the classical limit, a particle follows null geodesics. A more thorough discussion will be given in the longer article [15]; here we merely stress the operational meaning of $T$.

5. Static, asymptotically flat spacetimes

The geometrical surface gravity $\kappa$ coincides with the usual definition of the Killing surface gravity $\kappa_{\infty}$ for standard examples of static black holes such as Schwarzschild and Reissner–Nordström. However, it does not coincide if $\Psi \neq 0$, requiring further explanation.

Note first that the unit normalization of the Killing vector $K_{\infty} = \partial_t$ at infinity is crucial to the definition of $\kappa_{\infty}$, since otherwise $K_{\infty}$ can be rescaled by a positive constant and $\kappa_{\infty}$ scales likewise. This scaling, however, cannot be known locally. This problem is illustrated by two Schwarzschild regions matched across a static shell outside the horizon, such that $\Psi \neq 0$: if surface gravity is to be a local quantity, one would expect to define it with the normalization appropriate to the interior region containing the horizon, rather than the normalization at infinity.

Consider static metrics in the form

$$\text{d} s^2 = r^2 \text{d} \Omega^2 + C^{-1} \text{d} r^2 - C e^{2\Psi} \text{d} t^2, \quad (24)$$

where $(C, \Psi)$ are, henceforth, functions of $r$ alone, the notation being consistent with the metric (8). Then $\kappa_{\infty}$ is defined by

$$K^a_{\infty} \nabla_b K_{\infty a} \equiv \kappa_{\infty} K_{\infty b} \quad (25)$$

and yields

$$\kappa_{\infty} \equiv e^{\Psi} \kappa. \quad (26)$$

This relative factor stems from our use of the Kodama vector $K$ instead of the static Killing vector $K_{\infty} = e^{\Psi} K$, since the latter does not exist in dynamic cases. Thus, we can deal in a
unified way with such situations as an accreting black hole which settles down to a static state, or a static black hole which starts to evaporate. In our opinion, the relative factor $e^\Psi/\Psi_1$ can be explained as follows, first noting that a textbook method derives the gravitational redshift of light along a given ray [21]:

$$\sqrt{-g(\partial_t, \partial_t)} \hat{\omega} = e^\Psi/\Psi_1 \sqrt{C} \hat{\omega}$$

is constant along the ray.

If the spacetime is asymptotically flat, with $(t, r)$ being Minkowski coordinates as $r \to \infty$, then $C \to 1$, $\Psi \to 0$ and $\partial_t \to K$. Note that it is precisely here where the generally non-invariant $\Psi$ acquires a specific meaning. Then the frequency measured by static observers at infinity is

$$\omega_\infty = e^\Psi \sqrt{C} \hat{\omega} \quad (27)$$

and the corresponding temperature measured by such observers is

$$T_\infty = e^\Psi \sqrt{C} T, \quad (28)$$

which is the famous Tolman relation [22]. Thus

$$T_\infty \simeq e^\Psi T, \quad (29)$$

which indeed corresponds to $\kappa_\infty \equiv 2\pi T_\infty$. Such considerations suggest to interpret $e^\Psi$, which in general measures the discrepancy between the Killing and the Kodama temperatures, as an interpolating factor between $T_\infty$ and $T$, appearing as a relative redshift between the horizon and infinity, or as a gravitational dressing effect, since $e^\Psi$ enhances the redshift over the spatial curvature.

These results seem to suggest that the appropriate local temperature at the horizon is $T$ and generally not $T_\infty$ even in the static case. Likewise, the local surface gravity is $\kappa$ and generally not the textbook definition $\kappa_\infty$. Recall that the physical interpretation of $\kappa_\infty$ is the force at infinity per unit mass required to suspend an object from a massless rope just outside the horizon [21]. This ‘surface gravity at infinity’ would seem to be an oxymoron. Certainly this is not how Newtonian surface gravity is defined, as the local gravitational acceleration. Recall as above that $\kappa$ reduces to the latter in vacuo.

6. Extremal limit

As an example, consider the charged stringy black hole, which represents a non-vacuum solution of Einstein–Maxwell dilaton gravity in the string frame [23, 24]:

$$ds^2 = r^2 d\Omega^2 + \frac{dr^2}{(1-a/r)(1-b/r)} - \left( \frac{1-a/r}{1-b/r} \right) dt^2 \quad (30)$$

where $a > b > 0$. The horizon radius is $r \approx a$.

For this example, the extremal limit as defined by a global structure is $b \to a$. However, the Killing surface gravity,

$$\kappa_\infty \simeq \frac{1}{2a}, \quad (31)$$

does not vanish in this limit. Garfinkle et al [24] noted this as puzzling, since extremal black holes are expected to be zero-temperature objects.

Remarkably, the geometrical surface gravity (4)

$$\kappa \simeq \frac{a-b}{2a^2} \quad (32)$$

vanishes in the extremal limit. Thus the gravitational dressing effect lowers the temperature to its theoretically expected value.
We conjecture that this is true in general. Indeed, past experience with extremal black holes showed that the horizon of these objects is not only a zero but also a minimum of the expansion \( \theta_+ = \partial_+ A/A \) of the radially outgoing null geodesics, \( \theta_+ \) becoming positive again on crossing the horizon. Thus \( \partial_- \theta_+ \geq 0 \) should be the appropriate definition of an extremal black hole. Since \( \kappa = -e^{-2\varphi} \partial_+ \partial_+ r \), this is equivalent to \( \kappa \equiv 0 \).

7. Concluding remarks

We conclude that dynamical black holes do indeed possess a local temperature \( T \), with the operational meaning that it determines the redshifted temperature \( T/\sqrt{1 - 2m/r} \) measured by Kodama observers just outside a trapping horizon. Moreover, the method works precisely for future outer trapping horizons, as proposed previously to define black holes on purely geometrical grounds, and \( T = \kappa/2\pi \) in terms of the geometrically defined surface gravity \( \kappa \). This confirms the quasi-stationary picture of black-hole evaporation in the early stages.

The derivation holds formally even in regimes where one normally expects a semi-classical approximation to break down. With this qualification, it strongly suggests that evaporation proceeds until \( \kappa \to 0 \). While this is reminiscent of quasi-stationary arguments, it has a different meaning, since \( \kappa \) is generally not the surface gravity of a static black hole with the same mass, charge or whatever other parameters in a given model. It also encodes information about the dynamic spacetime geometry, such as the rate of evaporation. This may be of relevance to the information puzzle.

A common idea is that evaporation results in an extremal remnant [25, 26]. For instance, an outer (\( \kappa > 0 \)) and an inner (\( \kappa < 0 \)) trapping horizon might asymptote to the same null hypersurface, effectively forming a degenerate (\( \kappa = 0 \)) trapping horizon. Another idea is that the outer and inner trapping horizons merge smoothly at a single moment of extremality where \( \kappa \) vanishes [27]. The results here are consistent with either picture.

Finally, we note that a minor modification derives a positive temperature for past inner trapping horizons, namely to use a retarded time \( \nu \) instead of \( v \), as will be discussed in more detail elsewhere. For future inner or past outer trapping horizons, there is formally a negative temperature, but the physical meaning is debatable.

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Note added in proof. More precisely, equation (22) means that \( \hat{T} \sqrt{C} \to T \) as \( r \to 2m \).

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