Renormalization of coupling constants in the minimal SUSY models

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Abstract

The considerable part of the parameter space in the MSSM corresponding to the infrared quasi fixed point scenario is excluded by LEP II bounds on the lightest Higgs boson mass. In the NMSSM the mass of the lightest Higgs boson reaches its maximum value in the strong Yukawa coupling limit when Yukawa couplings are essentially larger than gauge ones at the Grand Unification scale. In this case the renormalization group flow of Yukawa couplings and soft SUSY breaking terms is investigated. The quasi–fixed and invariant lines and surfaces are briefly discussed. The coordinates of the quasi–fixed points, where all solutions are concentrated, are given.
1 Introduction

The search for the Higgs boson remains one of the top priorities for existing accelerators as well as for those still at the design stage. This is because this boson plays a key role in the Standard Model which describes all currently available experimental data with a high degree of accuracy. As a result of the spontaneous symmetry breaking $SU(2) \otimes U(1)$ the Higgs scalar acquires a nonzero vacuum expectation value without destroying the Lorentz invariance, and generates the masses of all fermions and vector bosons. An analysis of the experimental data using the Standard Model has shown that there is a 95% probability that its mass will not exceed 210 GeV \cite{1}. At the same time, assuming that there are no new fields and interactions and also no Landau pole in the solution of the renormalisation group equations for the self-action constant of Higgs fields up to the scale $M_{Pl} \approx 2.4 \cdot 10^{18}$ GeV, we can show that $m_h < 180$ GeV \cite{2},\cite{3}. In this case, physical vacuum is only stable provided that the mass of the Higgs boson is greater than 135 GeV \cite{2}-\cite{6}. However, it should be noted that this simplified model does not lead to unification of the gauge constants \cite{7} and a solution of the hierarchy problem \cite{8}. As a result, the construction of a realistic theory which combines all the fields and interactions is extremely difficult in this case.

Unification of the gauge constants occurs naturally on the scale $M_X \approx 3 \cdot 10^{16}$ GeV within the supersymmetric generalisation of the Standard Model, i.e., the Minimal Supersymmetric Standard Model (MSSM) \cite{7}. In order that all the fundamental fermions acquire mass in the MSSM, not one but two Higgs doublets $H_1$ and $H_2$ must be introduced in the theory, each acquiring the nonzero vacuum expectation value $v_1$ and $v_2$ where $v^2 = v_1^2 + v_2^2 = (246 \text{ GeV})^2$. The spectrum of the Higgs sector of the MSSM contains four massive states: two CP-even, one CP-odd, and one charged. An important distinguishing feature of the supersymmetric model is the existence of a light Higgs boson in the CP-even sector. The upper bound on its mass is determined to a considerable extent by the value $\tan \beta = v_2/v_1$. In the tree-level approximation the mass of the lightest Higgs boson in the MSSM does not exceed the mass of the Z-boson ($M_Z \approx 91.2$ GeV): $m_h \leq M_Z |\cos 2\beta| \cite{9}$. Allowance for the contribution of loop corrections to the effective interaction potential of the Higgs fields from a $t$–quark and its superpartners significantly raises the upper bound on its mass:

$$m_h \leq \sqrt{M_Z^2 \cos^2 2\beta + \Delta}. \quad (1)$$

Here $\Delta$ are the loop corrections \cite{10},\cite{11}. The values of these corrections are proportional to $m_t^4$, where $m_t$ is the running mass of $t$–quark which depends logarithmically on the supersymmetry breaking scale $M_S$ and is almost independent of the choice of $\tan \beta$. In \cite{3},\cite{5},\cite{6} bounds on the mass of the Higgs boson were compared in the Minimal Standard and Supersymmetric models. The upper bound on the mass of the light CP–even Higgs boson in the MSSM increases with increasing $\tan \beta$ and for $\tan \beta \gg 1$ in realistic supersymmetric models with $M_S \leq 1000$ GeV reaches $125 – 128$ GeV.

However, a considerable fraction of the solutions of the system of MSSM renormalisation group equations is focused near the infrared quasi-fixed point at $\tan \beta \sim 1$. In the region of parameter space of interest to us ($\tan \beta \ll 50$) the Yukawa constants of a $b$–quark ($h_b$) and a $\tau$–lepton ($h_\tau$) are negligible so that an exact analytic solution can be obtained for the one–loop renormalisation group equations \cite{12}. For the Yukawa constants of a
$t$–quark $h_t(t)$ and the gauge constants $g_i(t)$ its solution has the following form:

$$Y_i(t) = \frac{E(t)}{6F(t)} \left(1 + \frac{1}{6Y_2(0)F(t)}\right), \quad \tilde{\alpha}_i(t) = \frac{\tilde{\alpha}_i(0)}{1 + b_i\tilde{\alpha}_i(0)t},$$

$$E(t) = \left[\frac{\tilde{\alpha}_3(t)}{\tilde{\alpha}_3(0)}\right]^{16/9} \left[\frac{\tilde{\alpha}_2(t)}{\tilde{\alpha}_2(0)}\right]^{-3} \left[\frac{\tilde{\alpha}_1(t)}{\tilde{\alpha}_1(0)}\right]^{-13/99}, \quad F(t) = \int_0^t E(t')dt',$$

where the index $i$ has values between 1 and 3,

$$b_1 = 33/5, \quad b_2 = 1, \quad b_3 = -3$$

$$\tilde{\alpha}_i(t) = \left(\frac{g_i(t)}{4\pi}\right)^2, \quad Y_i(t) = \left(\frac{h_t(t)}{4\pi}\right)^2.$$  

The variable $t$ is determined by a standard method $t = \ln(M_X^2/q^2)$. The boundary conditions for the renormalisation group equations are usually set at the grand unification scale $M_X (t = 0)$ where the values of all three Yukawa constants are the same: $\tilde{\alpha}_1(0) = \tilde{\alpha}_2(0) = \tilde{\alpha}_3(0) = \tilde{\alpha}_0$. On the electroweak scale where $h_t^2(0) \gg 1$ the second term in the denominator of the expression describing the evolution of $Y_i(t)$ is much smaller than unity and all the solutions are concentrated in a narrow interval near the quasi–fixed point $t = 0$ of the system of MSSM renormalisation group equations. In other words in the low-energy range the dependence of $Y_i(t)$ on the initial conditions on the scale $M_X$ disappears. In addition to the Yukawa constant of the $t$–quark, the corresponding trilinear interaction constant of the scalar fields $A_i$ and the combination of the scalar masses $\mathcal{M}_i^2 = m_Q^2 + m_U^2 + m_D^2$ also cease to depend on $A_i(0)$ and $\mathcal{M}_i^2(0)$ as $Y_i(0)$ increases. Then on the electroweak scale near the infrared quasi–fixed point $A_i(t)$ and $\mathcal{M}_i^2(t)$ are only expressed in terms of the gaugino mass on the Grand Unification scale. Formally this type of solution can be obtained if $Y_i(0)$ is made to go to infinity. Deviations from this solution are determined by ratio $1/6F(t)Y_i(0)$ which is of the order of $1/10h_t^2(0)$ on the electroweak scale.

The properties of the solutions of the system of MSSM renormalisation group equations and also the particle spectrum near the infrared quasi–fixed point for $\tan\beta \sim 1$ have been studied by many authors [14,15]. Recent investigations [15–17] have shown that for solutions $Y_i(t)$ corresponding to the quasi–fixed point regime the value of $\tan\beta$ is between 1.3 and 1.8. These comparatively low values of $\tan\beta$ yield significantly more stringent bounds on the mass of the lightest Higgs boson. The weak dependence of the soft supersymmetry breaking parameters $A_i(t)$ and $\mathcal{M}_i^2(t)$ on the boundary conditions near the quasi–fixed point means that the upper bound on its mass can be calculated fairly accurately. A theoretical analysis made in [15,16] showed that $m_h$ does not exceed $94 \pm 5$ GeV. This bound is $25 - 30$ GeV below the absolute upper bound in the Minimal Supersymmetric Model. Since the lower bound on the mass of the Higgs boson from LEP II data is 113 GeV [1], which for the spectrum of heavy supersymmetric particles is the same as the corresponding bound on the mass of the Higgs boson in the Standard Model, a considerable fraction of the solutions which come out to a quasi–fixed point in the MSSM, are almost eliminated by existing experimental data. This provides the stimulus for theoretical analyses of the Higgs sector in more complex supersymmetric models.
The simplest expansion of the MSSM which can conserve the unification of the gauge constants and raise the upper bound on the mass of the lightest Higgs boson is the Next–to–Minimal Supersymmetric Standard Model (NMSSM) [18]-[20]. By definition the superpotential of the NMSSM is invariant with respect to the discrete transformations $y'_\alpha = e^{2\pi i/3} y_\alpha$ of the $Z_3$ group [19] which means that we can avoid the problem of the $\mu$-term in supergravity models. The thing is that the fundamental parameter $\mu$ should be of the order of $M_{Pl}$ since this scale is the only dimensional parameter characterising the hidden (gravity) sector of the theory. In this case, however, the Higgs bosons $H_1$ and $H_2$ acquire an enormous mass $m_{H_1,H_2}^2 \sim \mu^2 \sim M_{Pl}^2$ and no breaking of $SU(2) \otimes U(1)$ symmetry occurs. In the NMSSM the term $\mu(\hat{H}_1 \hat{H}_2)$ in the superpotential is not invariant with respect to discrete transformations of the $Z_3$ group and for this reason should be eliminated from the analysis ($\mu = 0$). As a result of the multiplicative nature of the renormalisation of this parameter, the term $\mu(q)$ remains zero on any scale $q \leq M_X \div M_{Pl}$. However, the absence of mixing of the Higgs doublets on electroweak scale has the result that $H_1$ acquires no vacuum expectation value as a result of the spontaneous symmetry breaking and $d$–type quarks and charged leptons remain massless. In order to ensure that all quarks and charged leptons acquire nonzero masses, an additional singlet superfield $\hat{Y}$ with respect to gauge $SU(2) \otimes U(1)$ transformations is introduced in the NMSSM. The superpotential of the Higgs sector of the Nonminimal Supersymmetric Model [18]-[20] has the following form:

$$W_h = \lambda \hat{Y} (\hat{H}_1 \hat{H}_2) + \frac{\kappa}{3} \hat{Y}^3.$$  \hspace{1cm} (3)

As a result of the spontaneous breaking of $SU(2) \otimes U(1)$ symmetry, the field $Y$ acquires a vacuum expectation value ($\langle Y \rangle = y/\sqrt{2}$) and the effective $\mu$-term ($\mu = \lambda y/\sqrt{2}$) is generated.

In addition to the Yukawa constants $\lambda$ and $\kappa$, and also the Standard Model constants, the Nonminimal Supersymmetric Model contains a large number of unknown parameters. These are the so-called soft supersymmetry breaking parameters which are required to obtain an acceptable spectrum of superpartners of observable particles form the phenomenological point of view. The hypothesis on the universal nature of these constants on the Grand Unification scale allows us to reduce their number in the NMSSM to three: the mass of all the scalar particles $m_0$, the gaugino mass $M_{1/2}$, and the trilinear interaction constant of the scalar fields $A$. In order to avoid strong CP–violation and also spontaneous breaking of gauge symmetry at high energies ($M_{Pl} \gg E \gg m_t$) as a result of which the scalar superpartners of leptons and quarks would require nonzero vacuum expectation values, the complex phases of the soft supersymmetry breaking parameters are assumed to be zero and only positive values of $m_0^2$ are considered. Naturally universal supersymmetry breaking parameters appear in the minimal supergravity model [23] and also in various string models [22], [24]. In the low-energy region the hypothesis of universal fundamental parameters allows to avoid the appearance of neutral currents with flavour changes and can simplify the analysis of the particle spectrum as far as possible. The fundamental parameters thus determined on the Grand Unification scale should be considered as boundary conditions for the system of renormalisation group equations which describes the evolution of these constants as far as the electroweak scale or the supersymmetry breaking scale. The complete system of the renormalisation group equations of the Nonminimal Supersymmetric Model can be found in [25], [26]. These experimental data impose various constraints on the NMSSM parameter space which were analysed in [27], [28].
The introduction of the neutral field $Y$ in the NMSSM potential leads to the appearance of a corresponding $F$–term in the interaction potential of the Higgs fields. As a consequence, the upper bound on the mass of the lightest Higgs boson is increased:

$$m_h \leq \sqrt{\frac{\lambda^2}{2} v^2 \sin^2 2\beta + M_Z^2 \cos^2 2\beta + \Delta_{11}^{(1)} + \Delta_{11}^{(2)}}. \quad (4)$$

The relationship (4) was obtained in the tree-level approximation ($\Delta_{11} = 0$) in [20]. However, loop corrections to the effective interaction potential of the Higgs fields from the $t$–quark and its superpartners play a very significant role. In terms of absolute value their contribution to the upper bound on the mass of the Higgs boson remains approximately the same as in the Minimal Supersymmetric Model. When calculating the corrections $\Delta_{11}^{(1)}$ and $\Delta_{11}^{(2)}$ we need to replace the parameter $\mu$ by $\lambda y/\sqrt{2}$. Studies of the Higgs sector in the Nonminimal Supersymmetric model and the one–loop corrections to it were reported in [21],[25],[28]–[31]. In [6] the upper bound on the mass of the lightest Higgs boson in the NMSSM was compared with the corresponding bounds on $m_h$ in the Minimal Standard and Supersymmetric Models. The possibility of a spontaneous CP–violation in the Higgs sector of the NMSSM was studied in [31],[32].

It follows from condition (4) that the upper bound on $m_h$ increases as $\lambda$ increases. Moreover, it only differs substantially from the corresponding bound in the MSSM in the range of small tan $\beta$. For high values (tan $\beta \gg 1$) the value of $\sin 2\beta$ tends to zero and the upper bounds on the mass of the lightest Higgs boson in the MSSM and NMSSM are almost the same. The case of small tan $\beta$ is only achieved for fairly high values of the Yukawa constant of a $t$–quark $h_t$ on the electroweak scale ($h_t(t_0) \geq 1$ where $t_0 = \ln(M_X^2/m_t^2)$), and tan $\beta$ decreases with increasing $h_t(t_0)$. However, an analysis of the renormalisation group equations in the NMSSM shows that an increase of the Yukawa constants on the electroweak scale is accompanied by an increase of $h_t(0)$ and $\lambda(0)$ on the Grand Unification scale. It thus becomes obvious that the upper bound on the mass of the lightest Higgs boson in the Nonminimal Supersymmetric model reaches its maximum on the strong Yukawa coupling limit, i.e., when $h_t(0) \gg g_t(0)$ and $\lambda(0) \gg g_t(0)$.

2 Renormalization of the Yukawa couplings

From the point of view of a renormalisation group analysis, investigation of the NMSSM presents a much more complicated problem than investigation of the minimal SUSY model. The full set of renormalization group equations within the NMSSM can be found in [25],[26]. Even in the one–loop approximation, this set of equations is nonlinear and its analytic solution does not exist. All equations forming this set can be partitioned into two groups. The first one contains equations that describe the evolution of gauge and Yukawa coupling constants, while the second one includes equations for the parameters of a soft breakdown of SUSY, which are necessary for obtaining a phenomenologically acceptable spectrum of superpartners of observable particles. Since boundary conditions for three Yukawa coupling constants are unknown, it is very difficult to perform a numerical analysis of the equations belonging to the first group and of the full set of the equations given above. In the regime of strong Yukawa coupling, however, solutions to the renormalisation group equations are concentrated in a narrow region of the parameter
space near the electroweak scale, and this considerably simplifies the analysis of the set of equations being considered.

In analysing the nonlinear differential equations entering into the first group, it is convenient to use the quantities \( \rho_t, \rho_\lambda, \rho_\kappa, \rho_1, \) and \( \rho_2, \) defined as follows:

\[
\rho_t(t) = \frac{Y_t(t)}{\alpha_3(t)}, \quad \rho_\lambda(t) = \frac{Y_\lambda(t)}{\alpha_3(t)}, \quad \rho_\kappa(t) = \frac{Y_\kappa(t)}{\alpha_3(t)}, \quad \rho_1(t) = \frac{\tilde{\alpha}_1(t)}{\alpha_3(t)}, \quad \rho_2(t) = \frac{\tilde{\alpha}_2(t)}{\alpha_3(t)},
\]

where \( \tilde{\alpha}_i(t) = g_i^2(t)/(4\pi)^2, \) \( Y_t(t) = h_t^2(t)/(4\pi)^2, \) \( Y_\lambda(t) = \lambda^2(t)/(4\pi)^2, \) and \( Y_\kappa(t) = \kappa^2(t)/(4\pi)^2. \)

Let us first consider the simplest case of \( \kappa = 0. \) The growth of the Yukawa coupling constant \( \lambda(t_0) \) at a fixed value of \( h_t(t_0) \) results in that the Landau pole in solutions to the renormalization group equations approaches the Grand Unification scale from above. At a specific value \( \lambda(t_0) = \lambda_{\text{max}}, \) perturbation theory at \( q \sim M_X \) cease to be applicable. With increasing (decreasing) Yukawa coupling constant for the \( b-\)quark, \( \lambda_{\text{max}} \) decreases (increases). In the \((\rho_t, \rho_\lambda)\) plane, the dependence \( \lambda_{\text{max}}^2(h_t^2) \) is represented by a curve bounding the region of admissible values of the parameters \( \rho_t(t_0) \) and \( \rho_\lambda(t_0). \) At \( \rho_\lambda = 0, \) this curve intersects the abscissa at the point \( \rho_t = \rho_t^{QFP}(t_0). \) This is the way in which there arises, in the \((\rho_t, \rho_\lambda)\) plane, the quasi–fixed (or Hill) line near which solutions to the renormalization group equations are grouped (see Fig. 1). With increasing \( \lambda^2(0) \) and \( h_t^2(0), \) the region where the solutions in questions are concentrated sharply shrinks, and for rather large initial values of the Yukawa coupling constants they are grouped in a narrow stripe near the straight line

\[
\rho_t(t_0) + 0.506\rho_\lambda(t_0) = 0.91,
\]

which can be obtained by fitting the results of numerical calculations (these results are presented in Fig. 2). Moreover, the combination \( h_t^2(t_0) + 0.506\lambda^2(t_0) \) of the Yukawa coupling constants depends much more weakly on \( \lambda^2(0) \) and \( h_t^2(0) \) than \( \lambda^2(t_0) \) and \( h_t^2(t_0) \) individually. In other words, a decrease in \( \lambda^2(t_0) \) compensates for an increase in \( h_t^2(t_0), \) and vice versa. The results in Fig. 3, which illustrate the evolution of the above combinations of the Yukawa coupling constants, also confirm that this combination is virtually independent of the initial conditions.

In analysing the results of numerical calculations, our attention is engaged by a pronounced nonuniformity in the distribution of solutions to the renormalization group equations along the infrared quasi–fixed line. The main reason for this is that, in the regime of strong Yukawa coupling, the solutions in question are attracted not only to the quasi–fixed but also to the infrared fixed (or invariant) line. The latter connects two fixed points. Of these, one is an infrared fixed point of the set of renormalization group equations within the NMSSM \((\rho_t = 7/18, \rho_\lambda = 0, \rho_1 = 0, \rho_2 = 0)\) \(\text{[33]},\) while the other fixed point \((\rho_\lambda/\rho_t = 1)\) corresponds to values of the Yukawa coupling constants in the region \( Y_t, Y_\lambda \gg \tilde{\alpha}_i, \) in which case the gauge coupling constants on the right–hand sides of the renormalization group equations can be disregarded. For the asymptotic behaviour of the infrared fixed line at \( \rho_t, \rho_\lambda \gg 1 \) we have

\[
\rho_\lambda = \rho_t - \frac{8}{15} - \frac{2}{75}\rho_1,
\]

while in the vicinity of the point \( \rho_t = 7/18, \rho_\lambda = 0 \) we have

\[
\rho_\lambda \sim (\rho_t - 7/18)^{25/14}.
\]
The infrared fixed line is invariant under renormalization group transformations – that is, it is independent of the scale at which the boundary values \( Y_t(0) \) and \( Y_\lambda(0) \) are specified and of the boundary values themselves. If the boundary conditions are such that \( Y_t(0) \) and \( Y_\lambda(0) \) belong to the fixed line, the evolution of the Yukawa coupling constants proceeds further along this line toward the infrared fixed point of the set of renormalization group equations within the NMSSM. With increasing \( t \), all other solutions to the renormalization group equations are attracted to the infrared fixed line and, for \( t/(4\pi) \gg 1 \), approach the stable infrared fixed point. From the data in Figs. 1 and 2, it follows that, with increasing \( Y_t(0) \) and \( Y_\lambda(0) \), all solutions to the renormalization group equations are concentrated in the vicinity of the point of intersection of the infrared fixed and the quasi–fixed line:

\[
\rho_t^{\text{QFP}}(t_0) = 0.803, \quad \rho_\lambda^{\text{QFP}}(t_0) = 0.224.
\]

Hence, this point can be considered as the quasi–fixed point of the set of renormalization group equations within the NMSSM at \( \kappa = 0 \).

In a more complicated case where all three Yukawa coupling constants in the NMSSM are nonzero, analysis of the set of renormalization group equations presents a much more difficult problem. In particular, invariant (infrared fixed) and Hill surfaces come to the fore instead of the infrared fixed and quasi–fixed lines. For each fixed set of values of the coupling constants \( Y_t(t_0) \) and \( Y_\kappa(t_0) \), an upper limit on \( Y_\lambda(t_0) \) can be obtained from the requirement that perturbation theory be applicable up to the Grand Unification scale \( M_X \). A change in the values of the Yukawa coupling constants \( h_t \) and \( \kappa \) at the electroweak scale leads to a growth or a reduction of the upper limit on \( Y_\lambda(t_0) \). The resulting surface in the \((\rho_t, \rho_\kappa, \rho_\lambda)\) space is shown in Fig. 4. In the regime of strong Yukawa coupling, solutions to the renormalization group equations are concentrated near this surface. In just the same way as in the case of \( Y_\kappa = 0 \), a specific linear combination of \( Y_t, Y_\lambda, \) and \( Y_\kappa \) is virtually independent of the initial conditions for \( Y_i(0) \to \infty \):

\[
\rho_t(t_0) + 0.72\rho_\lambda(t_0) + 0.33\rho_\kappa(t_0) = 0.98. \quad (6)
\]

The evolution of this combination of Yukawa couplings at various initial values of the Yukawa coupling constants is illustrated in Fig. 5.

On the Hill surface, the region that is depicted in Fig. 4 and near which the solutions in question are grouped shrinks in one direction with increasing initial values of the Yukawa coupling constants, with the result that, at \( Y_t(0), Y_\kappa(0), \) and \( Y_\lambda(0) \sim 1 \), all solutions are grouped around the line that appears as the result of intersection of the quasi–fixed surface and the infrared fixed surface, which includes the invariant lines lying in the \( \rho_\kappa = 0 \) and \( \rho_\lambda = 0 \) planes and connecting the stable infrared point with, respectively, the fixed point \( \rho_\lambda/\rho_t = 1 \) and the fixed point \( \rho_\kappa/\rho_t = 1 \) in the regime of strong Yukawa coupling. In the limit \( \rho_t, \rho_\lambda, \rho_\kappa \gg 1 \), in which case the gauge coupling constants can be disregarded, the fixed points \( \rho_\lambda/\rho_t = 1, \rho_\kappa/\rho_t = 0 \) and \( \rho_\kappa \rho_t = 1, \rho_\lambda/\rho_t = 0 \) cease to be stable. Instead of them, the stable fixed point \( R_\lambda = 3/4, R_\kappa = 3/8 \) appears in the \((R_\lambda, R_\kappa)\) plane, where \( R_\lambda = \rho_\lambda/\rho_t \) and \( R_\kappa = \rho_\kappa/\rho_t \). In order to investigate the behaviour of the solutions to the renormalization group equations within the NMSSM, it is necessary to linearise the set of these equations in its vicinity and set \( \alpha_i = 0 \). As a
result, we obtain

\[
R_\lambda(t) = \frac{3}{4} + \left( \frac{1}{2} R_{\lambda 0} + \frac{1}{\sqrt{5}} R_{\sigma 0} - \frac{3(\sqrt{5} + 1)}{8\sqrt{5}} \right) \left( \frac{\rho_t(t)}{\rho_{t0}} \right)^{\lambda_1} + \left( \frac{1}{2} R_{\lambda 0} - \frac{1}{\sqrt{5}} R_{\sigma 0} - \frac{3(\sqrt{5} - 1)}{8\sqrt{5}} \right) \left( \frac{\rho_t(t)}{\rho_{t0}} \right)^{\lambda_2},
\]

\[
R_\kappa(t) = \frac{3}{8} + \frac{\sqrt{5}}{2} \left( \frac{1}{2} R_{\lambda 0} + \frac{1}{\sqrt{5}} R_{\sigma 0} - \frac{3(\sqrt{5} + 1)}{8\sqrt{5}} \right) \left( \frac{\rho_t(t)}{\rho_{t0}} \right)^{\lambda_1} - \frac{\sqrt{5}}{2} \left( \frac{1}{2} R_{\lambda 0} - \frac{1}{\sqrt{5}} R_{\sigma 0} - \frac{3(\sqrt{5} - 1)}{8\sqrt{5}} \right) \left( \frac{\rho_t(t)}{\rho_{t0}} \right)^{\lambda_2},
\]

where \( R_{\lambda 0} = R_\lambda(0) \), \( R_{\sigma 0} = R_\kappa(0) \), \( \rho_{t0} = \rho_t(0) \), \( \lambda_1 = \frac{3 + \sqrt{5}}{9} \), \( \lambda_2 = \frac{3 - \sqrt{5}}{9} \), and \( \rho_t(t) = \frac{\rho_{t0}}{1 + 7\rho_{t0}t} \). From (7), it follows that the fixed point \( R_\lambda = 3/4 \), \( R_\kappa = 3/8 \) arises as the result of intersection of two fixed lines in the \((R_\lambda, R_\kappa)\) plane. The solutions are attracted most strongly to the line \( \frac{1}{2} R_\lambda + \frac{1}{\sqrt{5}} R_\kappa = \frac{3}{8} \left( 1 + \frac{1}{\sqrt{5}} \right) \), since \( \lambda_1 \gg \lambda_2 \).

This line passes through three fixed points in the \((R_\lambda, R_\kappa)\) plane: \((1, 0)\), \((3/4, 3/8)\), and \((0, 1)\). In the regime of strong Yukawa coupling, the fixed line that corresponds, in the \((\rho_t, \rho_\kappa, \rho_\lambda)\) space, to the line mentioned immediately above is that which lies on the invariant surface containing a stable infrared fixed point. The line of intersection of the Hill and the invariant surface can be obtained by mapping this fixed line into the quasi–fixed surface with the aid of the set of renormalization group equations. For the boundary conditions, one must then use the values \( \lambda^2(0) \), \( \kappa^2(0) \), and \( \lambda^4_1(0) \gg 1 \) belonging to the aforementioned fixed line.

In just the same way as infrared fixed lines, the infrared fixed surface is invariant under renormalization group transformations. In the evolution process, solutions to the set of renormalization group equations within the NMSSM are attracted to this surface. If boundary conditions are specified on the fixed surface, the ensuing evolution of the coupling constants proceeds within this surface. To add further details, we note that, near the surface being studied and on it, the solutions are attracted to the invariant line connecting the stable fixed point \((\rho_\lambda/\rho_t = 3/4, \rho_\kappa/\rho_t = 3/8)\) in the regime of strong Yukawa coupling with the stable infrared fixed point within the NMSSM. In the limit \( \rho_t, \rho_\kappa, \rho_\lambda \gg 1 \), the equation for this line has the form

\[
\rho_\lambda = \frac{3}{4} \rho_t - \frac{176}{417} + \frac{3}{139} \rho_2 - \frac{7}{417} \rho_1,
\]

\[
\rho_\kappa = \frac{3}{8} \rho_t - \frac{56}{18} - \frac{18}{139} \rho_2 - \frac{68}{2085} \rho_1.
\]

As one approaches the infrared fixed point, the quantities \( \rho_\lambda \) and \( \rho_\kappa \) tend to zero: \( \rho_\lambda \sim (\rho_t - 7/18)^{25/14} \) and \( \rho_\kappa \sim (\rho_t - 7/18)^{9/7} \). This line intersects the quasi–fixed surface at the point

\[
\rho_{t0}^{QFP} = 0.82, \quad \rho_{\kappa0}^{QFP} = 0.087, \quad \rho_{\lambda0}^{QFP} = 0.178.
\]

Since all solutions are concentrated in the vicinity of this point for \( Y_t(0), Y_\lambda(0), Y_\kappa(0) \to \infty \), it should be considered as a quasi–fixed point for the set
of renormalization group equations within the NMSSM. We note, however, that the solutions are attracted to the invariant line \( \mathbf{S} \) and to the quasi–fixed line on the Hill surface. This conclusion can be drawn from the analysis of the behaviour of the solutions near the fixed point \( (R_\lambda = 3/4, \ R_\kappa = 3/8) \) (see \( \mathbf{4} \)). Once the solutions have approached the invariant line \( \frac{1}{2} R_\lambda + \frac{1}{\sqrt{3}} R_\kappa = \frac{3}{8} \left(1 + \frac{1}{\sqrt{3}}\right) \), their evolution is governed by the expression \( (\epsilon(t))^{0.085} \), where \( \epsilon(t) = \rho_i(t)/\rho_{i0} \). This means that the solutions begin to be attracted to the quasi–fixed point and to the invariant line \( \mathbf{S} \) with a sizable strength only when \( Y_i(0) \) reaches a value of 10^2, at which perturbation theory is obviously inapplicable. Thus, it is not the infrared quasi–fixed point but the quasi–fixed line on the Hill surface (see Fig. 4) that, within the NMSSM, plays a key role in analysing the behaviour of the solutions to the renormalization group equations in the regime of strong Yukawa coupling, where all \( Y_i(0) \) are much greater than \( \tilde{\alpha}_0 \).

3 Renormalization of the soft SUSY breaking parameters

If the evolution of gauge and Yukawa coupling constants is known, the remaining subset of renormalization group equations within the MNSSM can be treated as a set of linear differential equations for the parameters of a soft breakdown of supersymmetry. For universal boundary conditions, a general solution for the trilinear coupling constants \( A_i(t) \) and for the masses of scalar fields \( m_i^2(t) \) has the form

\[
A_i(t) = e_i(t) A + f_i(t) M_{1/2},
\]

\[
m_i^2(t) = a_i(t)m_0^2 + b_i(t)M_{1/2}^2 + c_i(t)AM_{1/2} + d_i(t)A^2.
\]

The functions \( e_i(t), f_i(t), a_i(t), b_i(t), c_i(t), \) and \( d_i(t) \), which determine the evolution of \( A_i(t) \) and \( m_i^2(t) \), remain unknown, since an analytic solution to the full set of renormalization group equations within the NMSSM is unavailable. These functions greatly depend on the choice of values for the Yukawa coupling constants at the Grand Unification scale \( M_X \). At the electroweak scale \( t = t_0 \), relations \( \mathbf{9} \) and \( \mathbf{10} \) specify the parameters \( A_i^2(t_0) \) and \( m_i^2(t_0) \) of a soft breaking of supersymmetry as functions of their initial values at the Grand Unification scale.

The results of our numerical analysis indicate that, with increasing \( Y_i(0) \), where \( Y_i(t) = \frac{h_i^2(t)}{(4\pi)^2}, \ Y_\lambda(t) = \frac{\lambda^2(t)}{(4\pi)^2}, \) and \( Y_\kappa(t) = \frac{\kappa^2(t)}{(4\pi)^2} \), the functions \( e_i(t_0), c_i(t_0), \) and \( d_i(t_0) \) decrease and tend to zero in the limit \( Y_i(0) \to \infty \), relations \( \mathbf{9} \) and \( \mathbf{10} \) becoming much simpler in this limit. Instead of the squares of the scalar particle masses, it is convenient to consider their linear combinations

\[
\mathcal{M}_i^2(t) = m_i^2(t) + m_0^2(t) + m_i^2(t),
\]

\[
\mathcal{M}_\lambda^2(t) = m_1^2(t) + m_2^2(t) + m_y^2(t),
\]

\[
\mathcal{M}_\kappa^2(t) = 3m_y^2(t)
\]

in analysing the set of renormalization group equations. In the case of universal boundary conditions, the solutions to the differential equations for \( \mathcal{M}_i^2(t) \) can be represented in the
same form as the solutions for $m_i^2(t)$ (see (10)); that is

$$\mathcal{M}_i^2(t) = 3\tilde{a}_i(t)m_0^2 + \tilde{b}_i(t)M_{1/2}^2 + \tilde{c}_i(t)AM_{1/2} + \tilde{d}_i(t)A^2.$$  \hspace{1cm} (12)

Since the homogeneous equations for $A_i(t)$ and $\mathcal{M}_i^2(t)$ have the same form, the functions $\tilde{a}_i(t)$ and $e_i(t)$ coincide; in the limit of strong Yukawa coupling, the $m_0^2$ dependence disappears in the combinations (11) of the scalar particle masses as the solutions to the renormalization group equations for the Yukawa coupling constants approach quasi–fixed points. This behaviour of the solutions implies that $A_i(t)$ and $\mathcal{M}_i^2(t)$ corresponding to $Y_0(0) \gg \tilde{a}_i(0)$ also approach quasi–fixed points. As we see in the previous section, two quasi–fixed points of the renormalization group equations within the NMSSM are of greatest interest from the physical point of view. Of these, one corresponds to the boundary conditions $Y_0(0) = Y_\lambda(0) \gg \tilde{a}_i(0)$ and $Y_\kappa(0) = 0$ for the Yukawa coupling constants. The fixed points calculated for the parameters of a soft breaking of supersymmetry by using these values of the Yukawa coupling constants are

$$
\rho_{A_\kappa}^{\text{QFP}}(t_0) \approx 1.77, \quad \rho_{2\kappa}^{\text{QFP}}(t_0) \approx 6.09, \\
\rho_{A_\lambda}^{\text{QFP}}(t_0) \approx -0.42, \quad \rho_{\kappa}^{\text{QFP}}(t_0) \approx -2.28,
$$  \hspace{1cm} (13)

where $\rho_{A_\kappa}(t) = A_\kappa(t)/M_{1/2}$ and $\rho_{2\kappa}(t) = \mathcal{M}_2^2/M_{1/2}^2$. Since the coupling constant $\kappa$ for the self–interaction of neutral scalar fields is small in the case being considered, $A_\kappa(t)$ and $\mathcal{M}_2^2(t)$ do not approach the quasi–fixed point. Nonetheless, the spectrum of SUSY particles is virtually independent of the trilinear coupling constant $A_\kappa$ since $\kappa \to 0$.

In just the same way, one can determine the position of the other quasi–fixed point for $A_i(t)$ and $\mathcal{M}_i^2(t)$, that which corresponds to $R_{\lambda 0} = 3/4$, $R_{\kappa 0} = 3/8$. The results are

$$
\rho_{A_\kappa}^{\text{QFP}}(t_0) \approx 1.73, \quad \rho_{A_\lambda}^{\text{QFP}}(t_0) \approx -0.43, \quad \rho_{A_\kappa}^{\text{QFP}}(t_0) \approx 0.033, \\
\rho_{2\kappa}^{\text{QFP}}(t_0) \approx 6.02, \quad \rho_{\kappa}^{\text{QFP}}(t_0) \approx -2.34, \quad \rho_{\kappa}^{\text{QFP}}(t_0) \approx 0.29,
$$  \hspace{1cm} (14)

where $R_{\lambda 0} = Y_\lambda(0)/Y_0(0)$ and $R_{\kappa 0} = Y_\kappa(0)/Y_0(0)$. It should be noted that, in the vicinities of quasi–fixed points, we have $\rho_{2\kappa}^{\text{QFP}}(t_0) < 0$. Negative values of $\mathcal{M}_2^2(t_0)$ lead to a negative value of the parameter $m_2^2(t_0)$ in the potential of interaction of Higgs fields. In other words, an elegant mechanism that is responsible for a radiative violation of $SU(2) \otimes U(1)$ symmetry and which does not require introducing tachyons in the spectrum of the theory from the outset survives in the regime of strong Yukawa coupling within the NMSSM. This mechanism of gauge symmetry breaking was first discussed in [35] by considering the example of the minimal SUSY model.

By using the fact that $\mathcal{M}_i^2(t)$ as determined for the case of universal boundary conditions is virtually independent of $m_0^2$, we can predict $a_i(t_0)$ values near the quasi–fixed points (see [37]). The results are

1) $R_{\lambda 0} = 1$, $R_{\kappa 0} = 0$, 

$$a_0(t_0) = a_0(t_0) = \frac{1}{7}, \quad a_1(t_0) = a_4(t_0) = \frac{4}{7}, \quad a_2(t_0) = -\frac{5}{7};$$

2) $R_{\lambda 0} = 3/4$, $R_{\kappa 0} = 3/8$, 

$$a_0(t_0) = 0, \quad a_1(t_0) = -a_2(t_0) = \frac{2}{3}, \quad a_4(t_0) = \frac{5}{9}, \quad a_0(t_0) = \frac{1}{9}.$$  \hspace{1cm} (15)
To do this, it was necessary to consider specific combinations of the scalar particle masses, such as \( m_{t^2} - 2m_Q^2, m_Q^2 + m_{\tilde{t}}^2 - m_0^2, \) and \( m_0^2 - 2m_1^2 \) (at \( \varkappa = 0 \)), that are not renormalized by Yukawa interactions. As a result, the dependence of the above combinations of the scalar particle masses on \( m_0^2 \) at the electroweak scale is identical to that at the Grand Unification scale. The predictions in [15] agree fairly well with the results of numerical calculations.

Let us now consider the case of nonuniversal boundary conditions for the soft SUSY breaking parameters. The results of our numerical analysis, which are illustrated in Figs. 6 and 7, indicate that, in the vicinity of the infrared fixed point at \( Y_\varkappa = 0 \), solutions to the renormalization group equations at the electroweak scale are concentrated near some straight lines for the case where the simulation was performed by using boundary conditions uniformly distributed in the \( (A_t, A_\lambda, A_\varkappa) \) space, and \( (\mathcal{M}_t^2, \mathcal{M}_\lambda^2, \mathcal{M}_\varkappa) \) space (see Figs. 8-10):

\[
\begin{align*}
A_t + 0.147A_\lambda &= 1.70M_{1/2}, \\
\mathcal{M}_t^2 + 0.147\mathcal{M}_\lambda^2 &= 5.76M_{1/2}^2.
\end{align*}
\]

For \( Y_\varkappa(0) \gg \delta_0 \) solutions to the renormalization group equations are grouped near planes in the space of the parameters of a soft breaking of supersymmetry \( (A_t, A_\lambda, A_\varkappa) \) and \( (\mathcal{M}_t^2, \mathcal{M}_\lambda^2, \mathcal{M}_\varkappa) \) (see Figs. 8-10):

\[
\begin{align*}
A_t + 0.128A_\lambda + 0.022A_\varkappa &= 1.68M_{1/2}, \\
\mathcal{M}_t^2 + 0.128\mathcal{M}_\lambda^2 + 0.022\mathcal{M}_\varkappa &= 5.77M_{1/2}^2.
\end{align*}
\]

It can be seen from Figs. 8 and 9 that, as the values of the Yukawa coupling constants at the Grand Unification scale are increased, the areas of the surfaces near which the solutions are attracted to them grows with increasing \( Y_t(0) \). The equations for the lines being considered can be obtained by fitting the numerical results displayed in Figs. 6 and 7. This yields

\[
\begin{align*}
A_t + 0.147A_\lambda &= 1.70M_{1/2}, \\
\mathcal{M}_t^2 + 0.147\mathcal{M}_\lambda^2 &= 5.76M_{1/2}^2.
\end{align*}
\]

The numerical calculations also showed that, with increasing \( Y_\varkappa(0) \), only in the regime of infrared quasi–fixed points (that is, at \( R_{\lambda_0} = 1, R_{\varkappa_0} = 0 \) or at \( R_{\lambda_0} = 3/4, R_{\varkappa_0} = 3/8 \)) \( \epsilon_i(t_0) \) and \( \tilde{\epsilon}_i(t_0) \) decrease quite fast, in proportion to \( 1/Y_\varkappa(0) \). Otherwise, the dependence on \( A \) and \( m_0^2 \) disappears much more slowly with increasing values of the Yukawa coupling constants at the Grand Unification scale – specifically, in proportion to \( (Y_\varkappa(0))^{-\delta} \), where \( \delta < 1 \) (for example, \( \delta = 0.35 - 0.40 \) at \( \varkappa = 0 \)). In the case of nonuniversal boundary conditions, only when solutions to the renormalization group equations approach quasi–fixed points are these solutions attracted to the infrared quasi–fixed points (that is, at \( \varkappa = 0 \)). The parameters \( A_i(t) \) and \( \mathcal{M}_i^2(t) \) cease to be dependent on the boundary conditions.

For the solutions of the renormalization group equations for the soft SUSY breaking parameters near the electroweak scale in the strong Yukawa coupling regime one can construct an expansion in powers of the small parameter \( \epsilon_i(t) = Y_t(t)/Y_{\varkappa}(0) \):

\[
\begin{align*}
\begin{pmatrix} A_i(t) \\ \mathcal{M}_i^2(t) \end{pmatrix} = \sum_k u_{ik}v_{ik}(t) \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \left( \epsilon_i(t) \right)^{\lambda_k} + \ldots,
\end{align*}
\]

(18)
where $\alpha_i$ and $\beta_i$ are constants of integration that can be expressed in terms of $A_i(0)$ and $M_i^2(0)$. The functions $v_{ij}(t)$ are weakly dependent on the Yukawa coupling constants at the scale $M_X$, and $v_{ij}(0) = 1$. They appear upon renormalizing the parameters of a soft breaking of supersymmetry from $q \sim 10^{12} - 10^{13}$ GeV to $q \sim m_t$. In equations \ref{18}, we have omitted terms proportional to $M_{1/2}$, $M_{1/2}^2$, $A_i(0)M_{1/2}$, and $A_i(0)A_j(0)$.

At $\varpi = 0$, we have two eigenvalues and two corresponding eigenvectors:

$$\lambda = \left( \begin{array}{c} \frac{1}{3/7} \\ \frac{1}{9} \frac{3 + \sqrt{5}}{2} \\ \frac{1}{9} \frac{3 - \sqrt{5}}{2} \end{array} \right), \quad u = \left( \begin{array}{ccc} 1 & 1 & \sqrt{5} - 1 \\ \frac{\sqrt{5}}{6} & 1 & \frac{\sqrt{5}}{6} \\ \frac{1}{6} & 1 & 1 \end{array} \right),$$

whose components specify $(A_t, A_\lambda)$ and $(M_t^2, M_\lambda^2)$. With increasing $Y_i(0) \simeq Y_\lambda(0)$, the dependence on $\alpha_0$ and $\beta_0$ becomes weaker and the solutions at $t = t_0$ are concentrated near the straight lines $(A_t(\alpha_1, A_\lambda(\alpha_1)))$ and $(M_t^2(\beta_1), M_\lambda^2(\beta_1))$. In order to obtain the equations for these straight lines, it is necessary to set $A_\lambda(0) = -3A_t(0)$ and $M_\lambda^2(0) = -3M_t^2(0)$ at the Grand Unification scale. At the electroweak scale, there then arise a relation between $A_t(t_0)$ and $A_\lambda(t_0)$ and a relation between $M_t^2(t_0)$ and $M_\lambda^2(t_0)$:

$$A_t + 0.137A_\lambda = 1.70M_{1/2},$$
$$M_t^2 + 0.137M_\lambda^2 = 5.76M_{1/2}^2.$$ \tag{19}

These relations agree well with the equations deduced for the straight lines at $Y_i(0) \sim 1$ by fitting the results of the numerical calculations \ref{18}. When the Yukawa coupling constant $\varpi$ is nonzero, we have three eigenvalues and three corresponding eigenvectors:

$$\lambda = \left( \begin{array}{c} 1 \\ 3/7 \\ 1/9 \end{array} \right), \quad u = \left( \begin{array}{ccc} 1 & -1 + \frac{\sqrt{5}}{2} & \frac{\sqrt{5} - 1}{2} \\ 1 & \frac{\sqrt{5}}{6} & \frac{\sqrt{5}}{6} \\ 1 & 1 & 1 \end{array} \right),$$

whose components specify $(A_t, A_\lambda, A_\varpi)$ and $(M_t^2, M_\lambda^2, M_\varpi^2)$. An increase in $Y_\lambda(0) \simeq 2Y_\varpi(0) \simeq \frac{3}{4}Y_t(0)$ leads to the following: first, the dependence of $A_t(t)$ and $M_t^2(t)$ on $\alpha_0$ and $\beta_0$ disappears, which leads to the emergence of planes in the space spanned by the parameters of a soft breaking of supersymmetry:

$$A_t + 0.103A_\lambda + 0.0124A_\varpi = 1.69M_{1/2},$$
$$M_t^2 + 0.103M_\lambda^2 + 0.0124M_\varpi^2 = 5.78M_{1/2}^2.$$ \tag{20}

After that, the dependence on $\alpha_1$ and $\beta_1$ becomes weaker at $Y_t(0) \sim 1$. This means that, with increasing initial values of the Yukawa coupling constants, solutions to the renormalization group equations are grouped near some straight lines and we can indeed see precisely this pattern in Figs. 8-10. All equations presented here for the straight lines and planes in the $M_t^2$ space were obtained at $A_i(0) = 0$.

From relations \ref{18} and \ref{20}, it follows that $A_t(t_0)$ and $M_t^2(t_0)$ are virtually independent of the initial conditions; that is, the straight lines and planes are orthogonal to the $A_t$ and $M_t^2$ axes. On the other hand, the $A_\varpi(t_0)$ and $M_\varpi^2(t_0)$ values that correspond to the Yukawa self–interaction constant $Y_\varpi$ for the neutral fields are fully determined by the boundary conditions for the parameters of a soft breaking of supersymmetry. For this reason, the planes in the $(A_t, A_\lambda, A_\varpi)$ and $(M_t^2, M_\lambda^2, M_\varpi^2)$ spaces are virtually parallel to the $A_\varpi$ and $M_\varpi^2$ axes.
4 Conclusions

In the strong Yukawa coupling regime in the NMSSM, solutions to the renormalisation group equations for $Y_i(t)$ are attracted to quasi–fixed lines and surfaces in the space of Yukawa coupling constants and specific combinations of $\rho_i(t)$ are virtually independent of their initial values at the Grand Unification scale. For $Y_i(0) \to \infty$, all solutions to the renormalisation group equations are concentrated near quasi–fixed points. These points emerge as the result of intersection of Hill lines or surfaces with the invariant line that connects the stable fixed point for $Y_i \gg \bar{\alpha}_i$ with the stable infrared fixed point. For the renormalisation group equations within the NMSSM, we have listed all the most important invariant lines and surfaces and studied their asymptotic behaviour for $Y_i \gg \bar{\alpha}_i$ and in the vicinity of the infrared fixed point.

With increasing $Y_i(0)$, the solutions in question approach quasi–fixed points quite slowly; that is, the deviation is proportional to $(\epsilon_i(t))^\delta$, where $\epsilon_i(t) = Y_i(t)/Y_i(0)$ and $\delta$ is calculated by analysing the set of the renormalisation group equations in the regime of strong Yukawa coupling. As a rule, $\delta$ is positive and much less than unity. By way of example, we indicate that, in the case where all three Yukawa coupling constants differ from zero, $\delta \approx 0.085$. Of greatest importance in analysing the behaviour of solutions to the renormalisation group equations within the NMSSM at $Y_i(0), Y_{\lambda}(0), Y_{\kappa}(0) \sim 1$ is therefore not the infrared quasi–fixed point but the line lying on the Hill surface and emerging as the intersection of the Hill and invariant surface. This line can be obtained by mapping the fixed points $(1,0)$, $(3/4,3/8)$, and $(0,1)$ in the $(R_{\lambda}, R_{\kappa})$ plane for $Y_i \gg \bar{\alpha}_i$ into the quasi–fixed surface by means of renormalisation group equations.

While $Y_i(t)$ approach quasi–fixed points, the corresponding solutions for the trilinear coupling constants $A_i(t)$ characterising scalar fields and for the combinations $M_i^2(t)$ of the scalar particle masses (see (12)) cease to be dependent on their initial values at the scale $M_X$ and, in the limit $Y_i(0) \to \infty$, also approach the fixed points in the space spanned by the parameters of a soft breaking of supersymmetry. Since the set of differential equations for $A_i(t)$ and $m_i^2(t)$ is linear, the $A$, $M_{1/2}$, and $m_0^2$ dependence of the parameters of a soft breaking of supersymmetry at the electroweak scale can be explicitly obtained for universal boundary conditions. It turns out that, near the quasi–fixed points, all $A_i(t)$ and all $M_i^2(t)$ are proportional to $M_{1/2}$ and $M_{1/2}^2$, respectively. Thus, we have shown that, in the parameter space region considered here, the solutions to the renormalization group equations for the trilinear coupling constants and for some combinations of the scalar particle masses are focused in a narrow interval within the infrared region. Since the neutral scalar field $Y$ is not renormalized by gauge interactions, $A_{\kappa}(t)$ and $M_{\kappa}^2(t)$ are concentrated near zero; therefore they are still dependent on the initial conditions. The parameters $A_i(t_0)$ and $M_i^2(t_0)$ show the weakest dependence on $A$ and $m_0^2$ because these parameters are renormalized by strong interactions. By considering that the quantities $M_i^2(t_0)$ are virtually independent of the boundary conditions, we have calculated, near the quasi–fixed points, the values of the scalar particle masses at the electroweak scale.

In the general case of nonuniversal boundary conditions, the solutions to the renormalization group equations within the NMSSM for $A_i(t)$ and $M_i^2(t)$ are grouped near some straight lines and planes in the space spanned by the parameters of a soft breaking of supersymmetry. Moving along these lines and surfaces as $Y_i(0)$ is increased, the trilinear coupling constants and the above combinations of the scalar particle masses approach
quasi–fixed points. However, the dependence of these couplings on $A_i(0)$ and $M^2_i(0)$ dies out quite slowly, in proportion to $(\epsilon_i(t))^\lambda$, where $\lambda$ is a small positive number; as a rule, $\lambda \ll 1$. For example, $\lambda = 3/7$ at $Y_\lambda(0)$ and $Y_\kappa(0) \approx 0$. The above is invalid only for the solutions $A_i(t)$ and $M^2_i(t)$ that correspond to universal boundary conditions for the parameters of a soft breaking of supersymmetry and to the initial values of $R_{X_0} = 1$, $R_{X_0} = 0$ and $R_{X_0} = 3/4$, $R_{X_0} = 3/8$ for the Yukawa coupling constants at the Grand Unification scale. They correspond to quasi–fixed points of the renormalization group equations for $Y_i(t)$. As the Yukawa coupling constants are increased, such solutions are attracted to infrared quasi–fixed points in proportion to $\epsilon_i(t)$.

Straight lines in the $(A_t, A_\lambda, A_\kappa)$ and $(M^2_t, M^2_\lambda, M^2_\kappa)$ spaces play a key role in the analysis of the behaviour of solutions for $A_i(t)$ and $M^2_i(t)$ in the case where $Y_i(0), Y_\lambda(0), Y_\kappa(0) \sim 1$. In the space spanned by the parameters of a soft breaking of supersymmetry, these straight lines lie in the planes near which $A_i(t)$ and $M^2_i(t)$ are grouped in the regime of strong Yukawa coupling at the electroweak scale. The straight lines and planes that were obtained by fitting the results of numerical calculations are nearly orthogonal to the $A_t$ and $M^2_t$ axes. This is because the constants $A_t(t_0)$ and $M^2_t(t_0)$ are virtually independent of the initial conditions at the scale $M_X$. On the other hand, the parameters $A_\kappa(t_0)$ and $M^2_\kappa(t_0)$ are determined, to a considerable extent, by the boundary conditions at the scale $M_X$. At $R_{X_0} = 3/4$ and $R_{X_0} = 3/8$, the planes in the $(A_t, A_\lambda, A_\kappa)$ and $(M^2_t, M^2_\lambda, M^2_\kappa)$ spaces are therefore parallel to the $A_\kappa$ and $M^2_\kappa$ axes.

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Figure captions

Fig. 1. The values of the Yukawa couplings at the electroweak scale corresponding to the initial values at the GUT scale uniformly distributed in a square $2 \leq h_t^2(0), \lambda^2(0) \leq 10$. The thick and thin curves represent, respectively, the invariant and the Hill line. The dashed line is a fit of the values $(\rho_t(t_0), \rho_\lambda(t_0))$ for $20 \leq h_t^2(0), \lambda^2(0) \leq 100$.

Fig. 2. The values of the Yukawa couplings at the electroweak scale corresponding to the initial values at the GUT scale uniformly distributed in a square $20 \leq h_t^2(0), \lambda^2(0) \leq 100$. The dashed line is a fit of the values $(\rho_t(t_0), \rho_\lambda(t_0))$ for $20 \leq h_t^2(0), \lambda^2(0) \leq 100$.

Fig. 3. Evolution of the combination $\rho_t(t) + 0.506\rho_\lambda(t)$ of the Yukawa couplings from the GUT scale ($t = 0$) to the electroweak scale ($t = t_0$) for $\varepsilon^2 = 0$ and for various initial values $h_t^2(0)$ – Fig. 3a, $\lambda^2(0)$ – Fig. 3b.

Fig. 4. Quasi-fixed surface in the $(\rho_t, \rho_\kappa, \rho_\lambda)$ space. The shaded part of the surface represents the region near which the solutions corresponding to the initial values $2 \leq h_t^2(0), \varepsilon^2(0), \lambda^2(0) \leq 10$ – Fig. 4a, $20 \leq h_t^2(0), \varepsilon^2(0), \lambda^2(0) \leq 100$ – Fig. 4b are concentrated.

Fig. 5. Evolution of the combination $\rho_t + 0.720\rho_\lambda + 0.3330\rho_\kappa$ of the Yukawa couplings from the GUT scale ($t = 0$) to the electroweak scale ($t = t_0$) for various initial values $h_t^2(0)$ – Fig. 5a, $\lambda^2(0)$ – Fig. 5b, $\varepsilon^2(0)$ – Fig. 5c.

Fig. 6. The values of the trilinear couplings $A_t$ and $A_\lambda$ at the electroweak scale corresponding to the initial values uniformly distributed in the $(A_t, A_\lambda)$ plane, calculated at $\varepsilon^2 = 0$ and $h_t^2(0) = \lambda^2(0) = 20$. The straight line is a fit of the values $(A_t(t_0), A_\lambda(t_0))$.

Fig. 7. The values of the combinations of masses $M_t^2$ and $M_\lambda^2$ at the electroweak scale corresponding to the initial values uniformly distributed in the $(M_t^2/M_{1/2}, M_\lambda^2/M_{1/2})$ plane, calculated at $\varepsilon^2 = 0$, $h_t^2(0) = \lambda^2(0) = 20$, and $A_t(0) = A_\lambda(0) = 0$. The straight line is a fit of the values $(M_t^2(t_0), M_\lambda^2(t_0))$.

Fig. 8. Planes in the parameter spaces $(A_t/M_{1/2}, A_\lambda/M_{1/2}, A_\kappa/M_{1/2})$ – Fig. 8a, and $(M_t^2/M_{1/2}, M_\lambda^2/M_{1/2}, M_\kappa^2/M_{1/2})$ – Fig. 8b. The shaded parts of the planes correspond to the regions near which the solutions at $h_t^2(0) = 16$, $\lambda^2(0) = 12$, and $\varepsilon^2(0) = 6$ are concentrated. The initial values $A_t(0)$ and $M_t^2(0)$ vary in the ranges $-M_{1/2} \leq A \leq M_{1/2}$ and $0 \leq M_t^2(0) \leq 3M_{1/2}^2$, respectively.

Fig. 9. Planes in the parameter spaces $(A_t/M_{1/2}, A_\lambda/M_{1/2}, A_\kappa/M_{1/2})$ – Fig. 9a, and $(M_t^2/M_{1/2}, M_\lambda^2/M_{1/2}, M_\kappa^2/M_{1/2})$ – Fig. 9b. The shaded parts of the planes correspond to the regions near which the solutions at $h_t^2(0) = 32$, $\lambda^2(0) = 24$, and $\varepsilon^2(0) = 12$ are concentrated. The initial values $A_t(0)$ and $M_t^2(0)$ vary in the ranges $-M_{1/2} \leq A \leq M_{1/2}$
and $0 \leq M_i^2(0) \leq 3M_{1/2}^2$, respectively.

**Fig. 10.** Set of points in planes $(0.0223(A_\kappa/M_{1/2}) + 0.1278(A_\lambda/M_{1/2}), A_t/M_{1/2})$ – Fig. 10a, and $(0.0223(M_\kappa^2/M_{1/2}^2) + 0.1278(M_\lambda^2/M_{1/2}^2), M_t^2/M_{1/2}^2)$ – Fig. 10b, corresponding to the values of parameters of soft SUSY breaking for $h_t^2(0) = 32$, $\lambda^2(0) = 24$, $\kappa^2(0) = 12$, and for a uniform distribution of the boundary conditions in the parameter spaces $(A_t, A_\lambda, A_\kappa)$ and $(M_\kappa^2, M_\lambda^2, M_t^2)$. The initial values $A_i(0)$ and $M_i^2(0)$ vary in the ranges $-M_{1/2} \leq A \leq M_{1/2}$ and $0 \leq M_i^2(0) \leq 3M_{1/2}^2$, respectively. The straight lines in Figs. 10a and 10b correspond to the planes in Figs. 9a and 9b, respectively.
Fig. 4a.

Fig. 4b.
Fig. 5a.

Fig. 5b.

Fig. 5c.
Fig. 6.

Fig. 7.
Fig. 8a.

Fig. 8b.
Fig. 9a.

Fig. 9b.
\[ 0.1278 \left( \frac{A_\lambda}{M_{1/2}} \right) + 0.0223 \left( \frac{A_\times}{M_{1/2}} \right) \]

Fig. 10a.

\[ 0.1278 \left( \frac{m_\lambda^2}{M_{1/2}^2} \right) + 0.0223 \left( \frac{m_\times^2}{M_{1/2}^2} \right) \]

Fig. 10b.