Generalized mechanics of flexure hinges for unified simulations

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Abstract. Flexure hinges are widely introduced into precision mechanisms to perform the smooth motion without backlash and friction. In this paper, a generalized beam element based on the weak form quadrature element method is proposed to assess the compliances of the flexure hinges with different profiles. The flexure hinge is regarded as the Timoshenko beam taking the shearing effect into account. The elemental stiffness matrix is obtained with solution of algebraic equations and the explicit mesh structure is not required for compliance modelling of the flexure hinge with an arbitrary profile. The compliances of two typical types of flexure hinges are computed based on the proposed element. The results are compared with ones yielded by analytical equations. Then the high accuracy and effectiveness of the proposed element are validated.

1. Introduction

Flexure hinges are critical components in compliant mechanisms since they achieve smooth precision motion, virtually free of backlash, hysteresis and friction. In contrast to traditional joints assembled with different parts, flexure hinges act as elastic joints, directly connect two adjacent rigid members, and provide mobility by their elastic deformation. Besides, flexure-based monolithic mechanisms are lightweight, compact, tractable and low-cost [1]. Therefore, compliant mechanisms are extensively utilized in a variety of applications, such as atomic force microscope (AFM) [1] and stereo lithography [2].

Compliance, known as the displacement-load ratio, is the most important mechanical parameter for flexure hinges. Hinge’s compliances are defined by their geometric configuration. Flexure hinges are divided generally into two categories: leaf-type hinges and notch hinges. The leaf-type hinges possess constant cross sections, so their compliances are computed with ease. The notch hinges are obtained through removing two symmetric cutouts in a rectangular block. Diverse hinge designs have been presented including the right circular profile [3], the parabolic and hyperbolic configurations [4, 5], the elliptical-arc-fillet shape family [6-8], the V-shaped geometry [9], the Bézier-profile hinge [10], and the power-function profile variant [11]. Those flexure hinges give designers much flexibility and convenience to choose suitable flexure hinges for desired compliant mechanisms. However, it increases difficulties in optimizing mechanisms with those hinge design equations.

Many methods have been introduced to model hinge compliances, including analytical, numerical and empirical ones. The analytic methods are commonly used to obtain various compliance equations, including the classical beam theory based full integration [3, 6-8], the Castigliano’s displacement theorem [4, 5, 9] and the unit-load method [11]. In numerical methods, Shen [12] presented a force-
based beam element modeling circular flexure hinges. However, these approaches (based on elliptic integrals) are laborious in practice, especially for the flexure hinges with complicated geometrical profiles. The dimensionless empirical equations are expressed in succinct forms of the high order polynomial or exponent [13-16]. Due to the limit of the theoretical basis, the method fails to combine with the optimization of the flexure hinges.

Recently, researchers have attempted to construct the generalized compliance formulations, which is a challenging work. Lobontiu [5] defined a family of symmetric conic-section hinges. Chen [7] introduced conic hinge equations in polar coordinates. In addition, Chen [8] proposed a set of elliptical-arc-fillet hinges. Lobontiu [17] developed a generalized three-segment notch flexure hinge exhibiting transverse symmetry. Then they obtained analytically general equations. Vallance [10] used Bézier curves to represent a family of conic-section hinges and derived numerical design compliances by a mixing algorithm. However, it would be difficult or even impossible to derive the unified mathematical formulations for flexure hinges with different geometry features. Finite element methods [18, 19] are also an alternative to formulating general models of flexure hinges. Friedrich [18] and Yang [19] applied displacement-based beam elements to modeling various flexure hinges. However, their accuracy is resulted from the resolution of element discretization.

This work supplements those generalized models and extend the weak form quadrature element method (QEM) [20] to construct a generalized beam element, where the explicit mesh and laborious integral/differential operations are not required to efficiently calculate the compliances of the flexure hinges with arbitrary profiles. The advantage of the weak form QEM is that only a series of algebraic operations are conducted by introducing the differential quadrature method (DQM) and Gauss-Lobatto quadrature rule into the theorem of minimum potential energy.

2. Mathematical Model and Formulation

Fig.1 shows a flexure hinge of variable rectangular cross section, regarded as a Timoshenko beam ignoring the torsion effect. The flexure hinge is divided by N sampling points. The coordinate system (x, y, z) is fixed on the element. In this study, the torsion is not considered. The dimensionless coordinate system \((\xi, y, z)\) is defined as \(\xi=2x/L-1\), where \(L\) denotes the hinge length. The nodal displacement vector of the element is then indicated as

\[
\mathbf{d}^T = \begin{bmatrix}
u_1 & u_1 & \alpha_1 & u_2 & u_2 & \alpha_2 & u_3 & u_3 & \alpha_3 & \ldots & u_N & u_N & \alpha_N \\
\end{bmatrix}
\]

Where the superscript of \(u\) represents sampling point number, the subscript of \(u\) represents translations along the coordinate, and the subscript of \(\alpha\) represents the nodal rotation about the coordinate.

![Figure 1. Discretization of a variable cross-section flexure hinge](image-url)
For the variable cross-section hinge, \( t(x) \) and \( w(x) \) denote the thickness and width, which are depicted as the analytical formulas \( t(x) = f_2(x) - f_1(x) \) and \( w(x) = g_2(x) - g_1(x) \), respectively. The cross-sectional area and moments of inertia are:

\[
A = w(x)t(x), \quad I_z = \frac{w(x)t(x)^3}{12}, \quad I_y = \frac{w(x)^3t(x)}{12}
\]  

Considering the small deformation and linear analysis, the element strain energy can be expressed as integral over the dimensionless interval \([-1, 1]\):

\[
U' = \int_{-1}^{1} \mathbf{I}' d\xi, \quad \mathbf{I}' = \frac{1}{2} \mathbf{e}'^T \mathbf{J} \mathbf{e}' \frac{L}{2}
\]  

With

\[
\mathbf{e}' = \begin{bmatrix}
\frac{2}{L} \frac{\partial u}{\partial \xi} - \alpha_x \\
\frac{2}{L} \frac{\partial u}{\partial \xi} - \alpha_y
\end{bmatrix}, \quad \mathbf{J} = \text{diag}(EA, \beta GA, EI, \beta GA, EI_y)
\]  

Where \( E \) represents the Young’s modulus, \( G \) represents the shear modulus, \( \beta \) represents the coefficient of the shearing effect. When the beam cross-section is rectangular, \( \beta = 10(1+\mu)/(12+11\mu) \) is defined [21], where \( \mu \) is the Poisson’s ratio.

An important step in the proposed element is to use an efficient numerical scheme to assess the integrals in Eq. (3). For simplicity and precision, the Gauss-Lobatto quadrature rule is applied. Accordingly, Eq. (3) is rewritten as follows

\[
U' = \sum_{i=1}^{N} \mathbf{I}'_i W_i = \frac{\sum_{i=1}^{N} \mathbf{e}'_i \mathbf{J} \mathbf{e}'_i \frac{L}{2} W_i}{2} \quad (i = 1, 2, \ldots, N)
\]  

Where \( W_i \) denotes the weighting coefficients of the numerical approximation. The sampling point coordinates along the dimensionless axis \( \xi \) are denoted as

\[
\xi_i = -1, K, \xi_j, K, \xi_N = 1, \quad i = 2, K, N - 1
\]  

Derivations of \( \xi_i \) and \( W_i \) can be found in [22]. Approximation of the derivatives in Eq. (4) is efficiently conducted through the differential quadrature method [23]. For the axial strain component, the derivative at a sampling point \( \xi_i \) can be approximated by a weighted linear sum of axial displacements at all sampling points.

\[
\left. \frac{d\mathbf{u}_x}{d\xi} \right|_{\xi=\xi_i} = \sum_{j=1}^{N} \mathbf{G}_{ij}^{(l)} \mathbf{u}_x^l, \quad i = 1, 2, K, N.
\]  

To obtain the first-order derivative weighting coefficients \( \mathbf{G}_{ij}^{(l)} \) in Eq. (8), Lagrange interpolation basis functions are selected. Then the coefficients can be explicitly expressed [23]. Approximation of the first-order derivative of the translations and rotations in Eq. (4) is similar to that of the axial strain.
component. Application of the differential quadrature rule into Eq. (4) results in the discrete strain vector at a sampling point

\[ e_i = \left[ \frac{2}{L} \sum_{j=1}^{N} G_{ij}^{(1)} u_j^{(1)} \right] - \frac{2}{L} \sum_{j=1}^{N} G_{ij}^{(2)} \phi_j^{(2)} - \frac{2}{L} \sum_{j=1}^{N} G_{ij}^{(3)} \phi_j^{(3)} - \frac{2}{L} \sum_{j=1}^{N} G_{ij}^{(4)} \phi_j^{(4)} \] = \mathbf{B} \, d^i, \quad (i = 1, K, N) \tag{9} 

Substituting Eq. (9) into Eq. (3) yields

\[ L^e = \frac{1}{2} (d^e)^T \mathbf{B}^T \mathbf{J} \mathbf{B} d^e \frac{L}{2} = \frac{1}{2} (d^e)^T \mathbf{H} d^e, \quad (i = 1, K, N); \tag{10} \]
\[ U^e = \int_{-\xi}^{\xi} L^e \, d\xi = \frac{1}{2} (d^e)^T \mathbf{K} d^e = \frac{1}{2} (d^e)^T \left( \sum_{j=1}^{N} \mathbf{H} W_j \right) d^e. \]

Where \( K_e \) is the stiffness matrix.

Considering the applied forces, the same numerical integration rule is applied to deriving the external force energy \( V_e \) and element force vector \( F_e \) as follows

\[ V^e = -(d^e)^T F^e; \]
\[ (F^e)^T = \begin{bmatrix} 0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 & L & F_x & F_y & M_x & F_z & M_y \end{bmatrix} \tag{11} \]

For the static analysis, we can accordingly transform the total potential energy into \( \Pi = U_e + V_e = (d e)TK e d e / 2 - (d e)T F e \). Substituting the stationary condition \( \delta \Pi = 0 \) into it yields \( K e d e = F e \). The nodal generalized displacement is obtained through enforcement of the different boundary conditions. Then the compliance terms are the corresponding displacement at the sampling point \( N \) divided by the applied force. As for the sampling point number \( N \), one can see that larger \( N \) yields more accurate solutions. In this study, \( N \) is specified as 25 for accuracy and generality.

According to \( dN = [C] FN \), the compliance matrix \( C \) establishes a mapping between the displacement vector \( dN \) and the force vector \( FN \) at the free end. Therefore, \( C \) is derived.

3. **Benchmark examples**

The compliances of two typical types of flexure hinges including elliptical-arc-fillet flexure hinges [6,8], and two-axis flexure hinges with inconstant thickness and width [24-26] are evaluated by the proposed element. ANSYS results are considered as a benchmark for comparison. We make a comparison between analytical results and ones derived by the proposed element. All calculations are conducted on a desktop PC (CPU frequency: 2.20 GHz, RAM: 8 GB) with 64-bit Windows 7 operating system.

3.1. Example one: elliptical-arc-fillet flexure hinges

The elliptical-arc-fillet flexure hinge with the constant width, as shown in Fig. 2, is investigated first. It can be seen that the specified geometric parameters of \( a, b, t, l \) and \( \theta m \) lead to various hinge profiles including the circular (C) shape, circular fillet (CF) shape, elliptical (E) shape, elliptical arc (EA) shape, elliptical arc fillet (ECF) shape.

In the computational analysis, the left end of the hinge is rigidly fixed and the force is applied at the point \( A \) of the free end, as shown in Fig. 2. The Young’s modulus \( E \) and the shear modulus \( G \) are specified as 2.07×10^11 and 8.1×10^10 N/m^2, respectively. The width is set as 10mm. The geometric parameters from Chen [6,8] are tabulated in Table 1.
Compliance results calculated by ANSYS (AN), the closed-form formulation (CF) and proposed element (QEM) are listed in Table 2. It is shown that the compliance results derived from QEM agree well with those of AN. The maximum relative error is 0.06% compared with CF \cite{6, 8}. The ANSYS simulation contains approximately 18000 hexahedron elements for meshing flexure hinges and takes more than 20 seconds to complete the process. However, the proposed approach only needs 1 element and the elapsed time is about 0.4 seconds. It implies that the proposed approach is both accurate and effective.

### Table 2. Compliances calculated by AN, CF and QEM

|     | $\Delta_x/F_x$   | $\Delta_y/F_y$   | $\Delta_z/F_z$   | $\alpha_x/M_x$ | $\alpha_y/M_y$ | $\alpha_z/M_z$ |
|-----|------------------|------------------|------------------|----------------|----------------|----------------|
| 1   | AN: $2.774 \times 10^{-9}$ | $5.152 \times 10^{-7}$ | $1.918 \times 10^{-6}$ | $3.378 \times 10^{-4}$ | $9.366 \times 10^{-5}$ | $1.853 \times 10^{-2}$ |
|     | Error: 2.42%     | 1.71%            | 1.77%            | 0.92%          | 1.67%          | 2.75%          |
|     | AN: $2.841 \times 10^{-9}$ | $5.240 \times 10^{-7}$ | $1.705 \times 10^{-6}$ | $3.409 \times 10^{-4}$ | $9.522 \times 10^{-5}$ | $1.904 \times 10^{-2}$ |
|     | Error: 2.42%     | 1.70%            | 1.77%            | 0.92%          | 1.67%          | 2.75%          |
| 2   | AN: $2.519 \times 10^{-9}$ | $4.446 \times 10^{-7}$ | $1.619 \times 10^{-6}$ | $3.141 \times 10^{-4}$ | $8.230 \times 10^{-5}$ | $1.628 \times 10^{-2}$ |
|     | Error: 1.83%     | 0.56%            | 4.43%            | 0.01%          | 0.39%          | 1.47%          |
|     | AN: $2.565 \times 10^{-9}$ | $4.471 \times 10^{-7}$ | $1.539 \times 10^{-6}$ | $3.078 \times 10^{-4}$ | $8.262 \times 10^{-5}$ | $1.652 \times 10^{-2}$ |
|     | Error: 1.83%     | 0.56%            | 4.43%            | 0.01%          | 0.39%          | 1.47%          |
| 3   | AN: $1.109 \times 10^{-8}$ | $4.578 \times 10^{-5}$ | $5.976 \times 10^{-6}$ | $1.328 \times 10^{-3}$ | $0.974 \times 10^{-2}$ | 2.182          |
|     | Error: 0.18%     | 6.42%            | 0.71%            | 0.08%          | 6.47%          | 6.46%          |
|     | AN: $1.107 \times 10^{-8}$ | $4.872 \times 10^{-5}$ | $5.931 \times 10^{-6}$ | $1.329 \times 10^{-3}$ | $1.037 \times 10^{-2}$ | 2.323          |
|     | Error: 0.18%     | 6.42%            | 0.71%            | 0.08%          | 6.47%          | 6.46%          |
| 4   | AN: $4.459 \times 10^{-9}$ | $4.355 \times 10^{-6}$ | $5.682 \times 10^{-4}$ | $1.704 \times 10^{-3}$ | $0.625$         | 0.656          |
|     | Error: 4.98%     | 4.68%            | 1.94%            | 1.14%          | 4.93%          | 4.96%          |
|     | AN: $4.681 \times 10^{-9}$ | $4.559 \times 10^{-6}$ | $5.167 \times 10^{-4}$ | $1.704 \times 10^{-3}$ | $0.625$         | 0.656          |
|     | Error: 4.98%     | 4.68%            | 1.94%            | 1.14%          | 4.93%          | 4.96%          |
| 5   | AN: $9.393 \times 10^{-9}$ | $2.723 \times 10^{-8}$ | $4.163 \times 10^{-5}$ | $4.143 \times 10^{-3}$ | $7.118 \times 10^{-3}$ | 1.978          |
|     | Error: 1.27%     | 6.35%            | 1.07%            | 0.17%          | 6.41%          | 6.42%          |
|     | AN: $9.512 \times 10^{-9}$ | $2.896 \times 10^{-8}$ | $4.107 \times 10^{-5}$ | $4.143 \times 10^{-3}$ | $7.574 \times 10^{-3}$ | 2.105          |
|     | Error: 1.27%     | 6.35%            | 1.07%            | 0.17%          | 6.41%          | 6.42%          |
3.2. Example two: two-axis flexure hinges

The two-axis flexure hinges with axially-collocated notches can rotate about either the primary sensitive axis or the secondary sensitive axis, as sketched in Fig. 3, where \( l \) represents the length, \( w \) represents the minimum width, and \( t \) represents the minimum thickness. The collocated notches are expressed by analytical curves, such as right circle formula [24], elliptical formula [25], and parabolic formula [26]. Generally, the cross-sections of the two-axis flexure hinges are rectangle, where the thickness and width all vary along the longitudinal axis, so it is more difficult to predict their elastic behavior. We choose three kinds of the two-axis flexure hinges [24-26] for comparison. Their material and geometrical parameters are illustrated in Table 3, where TE, TRC, and TP represent elliptical profiles, right circle profiles, and parabolic profiles, respectively.

Figure 3. Diagram of a two-axis flexure hinge

Table 4 shows the compliance results obtained from the proposed element (QEM), ANSYS (AN) and closed-form expressions (CF). These results indicate that all the compliances based on QEM match with those of AN. For the TRC and TE cases, the relative errors of QEM solutions are less than or equal to those of closed-form expressions. For the TP curve case, their relative errors are the same. Therefore, it can be seen that the proposed approach is simple and yields satisfactory results.

Table 3. Geometric parameters of the two-axis flexure hinges

|   | \( E (\text{N/m}^2) \) | \( \mu \) | \( t (\text{mm}) \) | \( w (\text{mm}) \) | \( c (\text{mm}) \) | \( c' (\text{mm}) \) | \( l (\text{mm}) \) | Profile |
|---|------------------|---|---|---|---|---|---|---|
| 1 | 2.2 \( \times \) 10\(^{11} \) | 0.28 | 3.2 | 4.1 | 9 | 9 | 18 | TRC |
| 2 | 2.2 \( \times \) 10\(^{11} \) | 0.28 | 1 | 1.5 | 4 | 6 | 30 | TE |
| 3 | 2.0 \( \times \) 10\(^{11} \) | 0.3 | 0.4 | 0.6 | 1 | 1.5 | 2.5 | TP |

Table 4. Compliances calculated by AN, CF and QEM

|   | \( \Delta_{x}/F_x \) | \( \Delta_{y}/F_y \) | \( \Delta_{z}/F_z \) | \( \Delta_{M_y} \) | \( \alpha_x/M_x \) | \( \Delta_{M_z} \) | \( \alpha_y/M_y \) |
|---|------------------|---|---|---|---|---|---|
| 1 | AN | 2.810 \( \times \) 10\(^{-9} \) | 2.064 \( \times \) 10\(^{-7} \) | 1.385 \( \times \) 10\(^{-5} \) | 1.349 \( \times \) 10\(^{-5} \) | 0.0015 | 2.064 \( \times \) 10\(^{-3} \) | 2.27 \( \times \) 10\(^{-3} \) |
|   | Chen [24] | 2.753 \( \times \) 10\(^{-9} \) | 1.839 \( \times \) 10\(^{-7} \) | 1.194 \( \times \) 10\(^{-5} \) | 1.242 \( \times \) 10\(^{-5} \) | 0.0014 | 1.924 \( \times \) 10\(^{-3} \) | 2.10 \( \times \) 10\(^{-3} \) |
|   | Error | 2.03% | 10.90% | 13.79% | 7.93% | 6.67% | 6.78% | 7.49% |
|   | QEM | 2.753 \( \times \) 10\(^{-9} \) | 1.922 \( \times \) 10\(^{-7} \) | 1.277 \( \times \) 10\(^{-5} \) | 1.242 \( \times \) 10\(^{-5} \) | 0.0014 | 1.924 \( \times \) 10\(^{-3} \) | 2.14 \( \times \) 10\(^{-3} \) |
|   | Error | 2.03% | 6.88% | 7.80% | 7.93% | 6.67% | 6.78% | 7.49% |
| 2 | AN | 3.333 \( \times \) 10\(^{-8} \) | 6.212 \( \times \) 10\(^{-5} \) | 2.778 \( \times \) 10\(^{-5} \) | 0.0018 | 0.1178 | 0.0040 | 0.2639 |
|   | Cao [25] | 3.300 \( \times \) 10\(^{-8} \) | 6.167 \( \times \) 10\(^{-5} \) | 2.741 \( \times \) 10\(^{-5} \) | 0.0018 | 0.1167 | 0.0039 | 0.2626 |
|   | Error | 0.99% | 0.72% | 1.33% | 0 | 0.93% | 2.50% | 0.49% |
|   | QEM | 3.301 \( \times \) 10\(^{-8} \) | 6.178 \( \times \) 10\(^{-5} \) | 2.751 \( \times \) 10\(^{-5} \) | 0.0018 | 0.1167 | 0.0039 | 0.2626 |
|   | Error | 0.96% | 0.55% | 0.97% | 0 | 0.93% | 2.50% | 0.49% |
| 3 | AN | 3.28 \( \times \) 10\(^{-8} \) | 3.23 \( \times \) 10\(^{-6} \) | 1.51 \( \times \) 10\(^{-4} \) | 10.12 \( \times \) 10\(^{-4} \) | 8.23 \( \times \) 10\(^{-3} \) | 2.3 \( \times \) 10\(^{-3} \) | 1.80 |
|   | Lobontiu [26] | 3.04 \( \times \) 10\(^{-8} \) | 3.11 \( \times \) 10\(^{-6} \) | 1.42 \( \times \) 10\(^{-4} \) | 9.66 \( \times \) 10\(^{-4} \) | 7.73 \( \times \) 10\(^{-3} \) | 2.17 \( \times \) 10\(^{-3} \) | 1.74 |
|   | Error | 7.32% | 4.60% | 5.96% | 4.55% | 6.07% | 5.62% | 3.33% |
|   | QEM | 3.04 \( \times \) 10\(^{-8} \) | 3.11 \( \times \) 10\(^{-6} \) | 1.42 \( \times \) 10\(^{-4} \) | 9.66 \( \times \) 10\(^{-4} \) | 7.73 \( \times \) 10\(^{-3} \) | 2.17 \( \times \) 10\(^{-3} \) | 1.74 |
|   | Error | 7.32% | 4.60% | 5.96% | 4.55% | 6.07% | 5.65% | 3.33% |
4. Conclusion  
In the present study, a generalized beam element based on the weak form quadrature element method is proposed to numerically derive compliances of the flexure hinges with arbitrary profiles. Compared with the conventional finite element method, the explicit mesh is not required to model the flexure hinges. Moreover, the proposed approach significantly decreases computational effort to obtain the element stiffness matrix with only algebraic operations. Based on the proposed element, the compliances of two typical types of flexure hinges are calculated. All the numerical compliances match well with the ANSYS results. For the elliptical-arc-fillet flexure hinges, the compliances from the proposed approach almost equate with those yielded by closed-form equations. For two-axis flexure hinges, although the compliances based on the proposed approach deviate slightly from ones provided by analytical formulas, the former is closer to the ANSYS results. All the comparisons verify highly accuracy and effectiveness of the proposed element.

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