Towards a Theory of the
Integer Quantum Hall Transition:
From the Nonlinear Sigma Model
to Superspin Chains

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ABSTRACT

A careful study of the supersymmetric version of Pruisken’s nonlinear \( \sigma \) model for the integer quantum Hall effect is presented. The lattice regularized model is cast in Hamiltonian form by taking the anisotropic limit and interpreting the topological density as an alternating sum of Wess-Zumino terms. It is argued that the relevant large-scale physics of the model is preserved by projection of the quantum Hamiltonian on its sector of degenerate strong-coupling ground states. For values of the Hall conductivity close to \( e^2/2h \) (mod \( e^2/h \)), where a delocalization transition occurs, this yields the Hamiltonian of a quantum superspin chain which is closely related to an anisotropic version of the Chalker-Coddington model. The relation implies that the ratio of magnetic length over potential correlation length is an irrelevant parameter at the transition. The superspin chain resembles a 1\( d \) isotropic antiferromagnet with spin 1/2. It has an alternating structure which however permits an invariance under translation by one site. The conductance coefficients of a quantum Hall system with \( N \) small contacts translate into \( N \)-superspin correlation functions which are governed by conformal invariance. The superspin formalism provides a framework for studying the crossover from classical to quantum percolation. It does not however encompass the frequency-dependent correlations of wave amplitudes at criticality.

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1 Introduction

When a two-dimensional (2d) electron gas is placed in a strong magnetic field $B$, the Hall conductance $\sigma_H$ is observed to be a rational multiple of $e^2/h$ near certain magic values of the density \[1\]. While Coulomb interactions are essential for the occurrence of fractional quantization, it is generally agreed that the observation of integer values $\sigma_H = n e^2/h \ (n \in \mathbb{Z})$ in low-mobility samples can be understood from a model of noninteracting electrons with potential disorder. It was first proposed in \[2\] that such a model undergoes a sequence of localization-delocalization transitions as the Fermi-energy $E_F \sim 1/B$ is varied. Given the localization of all states in the Landau band tails, quantization of the Hall conductance has been explained by various theories, of which we mention the topological argument \[3\] and the scattering-theoretic approach \[4\]. What is less well understood is the critical phenomenon which occurs at energies $E_c$ close to a Landau level center where some electron states become delocalized. The picture that has emerged from experiments \[5\] and numerical simulations for noninteracting systems \[6,7\] is that of some kind of second-order phase transition with a divergent correlation length $\xi \sim |E - E_c|^{-\nu}$, $\nu = 2.3 \pm 0.1$.

We begin with some general statements concerning conformal invariance at the integer quantum Hall delocalization transition and other metal-insulator transitions in disordered electron systems, in Sect. 2. Two ways of regularizing the electron Green’s functions are distinguished. The system may either be open with absorbing (i.e. conducting) boundary conditions at the leads or, alternatively, it can be taken to be closed but perturbed by an external field with finite frequency. Both an external field and absorption at a boundary cut off the long-time correlations and in this sense act as regulators. The introduction of a frequency $\omega$ sets a length scale, $L_\omega$, which breaks conformal invariance. Moreover, the amplitude of critical Green’s functions diverges for $L_\omega \to \infty$ (or $\omega \to 0$). An interpretation of these Green’s functions in the framework of conformal field theory \[8\] must involve primary fields with scaling dimensions that are negative. The situation is somewhat different when the first option of regularizing by absorption at a boundary is utilized. We suggest the novel procedure of keeping the size and relative distance of the absorbing contacts finite while sending the system size to infinity. In this limit the d.c. conductance, or current-current correlation function, decays algebraically with increasing distance between the contacts, and we expect it to expand in conformal fields that have positive scaling dimensions only.

To promote the reasoning of Sect. 2 to a quantitative theory with the possibility of getting exact analytical solutions, we must turn to field-theoretic techniques. An effective field theory for the integer quantum Hall effect was first formulated by Pruisken and collaborators \[9,10\]. The field theory is of the nonlinear $\sigma$ model type and has the action

$$S[Q] = \frac{1}{8} \int_M dxdy \left( \sigma_{xx}^0 \tr(\partial Q)^2 - \sigma_{xy}^0 \tr Q[\partial_x Q, \partial_y Q] \right).$$

(1)

The matrix-valued fields $Q$ map the configuration space $\mathcal{M}$ of the 2d electron gas into a coset space $U(2n)/U(n) \times U(n)$ with $n = 0$ due to the use of the replica trick. By the derivation of the model, the bare coupling constants $\sigma_{xx}^0$ and $\sigma_{xy}^0$ are identified as the longitudinal and Hall conductivities, respectively, calculated in the self-consistent Born approximation and measured in units of $e^2/h$. The field theory with action \(\square\) attributes
to $\sigma_{xy}^0$ the meaning of a topological angle $\theta = 2\pi\sigma_{xy}^0$ multiplying the topological density $\text{tr}Q[\partial_x Q, \partial_y Q]/8$. When $\mathcal{M}$ has no boundary, the topological density integrates to $2\pi i$ times an integer and the physics of the model \([\Pi]\) is periodic in $\sigma_{xy}^0$ with period 1.

Pruisken’s work on the model \([\Pi]\) focussed mainly on the semiclassical instanton approximation \([11,12]\), from which he proposed a two-parameter renormalization group flow with one (or possibly two) unstable fixed point(s) at $\sigma_{xy}^0 = 1/2 \pmod 1$ and some $\sigma_{xx}^0 = \sigma_{xx}^*$. This (or one of these) fixed point(s) is conjectured to be the integer quantum Hall delocalization fixed point seen in numerical and real experiments. While Pruisken’s conjecture has great intuitive appeal and is consistent with experiments, it has not led to quantitative results, beyond predicting the critical point to be isolated – with no continuous symmetry spontaneously broken and Goldstone bosons absent. In particular, the observed universality $\sigma_{xx}^* = 0.5 \pm 0.1$ \([13]\) was not foreseen, and a computation of critical indices is still lacking. This is not surprising since the critical region is beyond the reach of the instanton approximation, which is valid for $\sigma_{xx}^0 \gg 1$ while the integer quantum Hall (IQHE) transition occurs at a critical value $\sigma_{xx}^*$ of order unity. Another, more serious difficulty arises from the constraints imposed by conformal invariance. The field theory \([\Pi]\) at $n = 0$ and $\sigma_{xy}^0 = 1/2 \pmod 1$ must be critical with a divergent correlation length – or else it would not describe the IQHE transition. We therefore expect it to flow under renormalization to a conformally invariant field theory \([8]\). Now, as was pointed out by Affleck \([14]\), conformal invariance of a 2d field theory with continuous symmetry implies that there exist two sets of conserved currents, one for the right-moving and another one for the left-moving charges. Unfortunately, the nonlinear $\sigma$-model \([\Pi]\) has just one manifestly conserved current, which is its Noether current. (Presumably, in order for the required separation into left and right movers to become visible, one has to restrict the $\sigma$ model to its low-energy limit.) What happens is well understood for the case of one replica \((n = 1)\). Affleck and Haldane \([15]\) have argued convincingly that the $\text{U}(2)/\text{U}(1) \times \text{U}(1)$ nonlinear $\sigma$ model (also known as the O(3) model) at $\theta = \pi$ is not scale-invariant but renormalizes to the $\text{SU}(2) k = 1$ WZW model – a conformal field theory with central charge $c = 1$ and two independent conserved currents $J(z)$ and $\bar{J}(\bar{z})$. This conclusion was confirmed by Shankar and Read \([16]\). By analogy, we expect Pruisken’s model at $\theta = \pi$ and $n = 0$ to flow to some conformal field theory of the Wess-Zumino type. We are thus led to reject Pruisken’s assertions about the fixed-point structure of the model \([\Pi]\) at $n = 0$. This model has neither one nor two nontrivial fixed points at $\theta = \pi$, but has no such fixed points at all! (Note that we are not calling into question the two-parameter scaling theory \([17]\), which is arguably correct when proper meaning is given to the parameters $\sigma_{xx}$ and $\sigma_{xy}$. What we are saying is that these parameters have no interpretation as running coupling constants of the model \([\Pi]\).)

The lesson learned is that, if our goal is to approach the IQHE fixed point by analytic means, we had better abandon the critical (at $\theta = \pi$) but scale-dependent field theory \([\Pi]\). This logical step was taken by Affleck \([18]\) who exploited the relation of Pruisken’s $\text{U}(2n)/\text{U}(n) \times \text{U}(n)$ replica field theory to certain $\text{SU}(2n)$-invariant quantum “spin” chains. Such a relation first appeared in the work of Haldane \([19]\) who suggested that antiferromagnetic spin chains with $\text{SU}(2)$-invariance and large spin $S$ map on the $\text{U}(2)/\text{U}(1) \times \text{U}(1)$ nonlinear $\sigma$ model with topological angle $\theta = 2\pi S$. Applying an inverse of this mapping \([16]\) to Pruisken’s model for $n = 1$, one gets an interpretation of the Hall conductivity
as the spin $S = 1/2$ of a 1d isotropic antiferromagnet. Even more intriguingly, the hypothesis of a sequence of localization-delocalization transitions occurring at $\sigma_{xy}^0 = \pm 1/2, \pm 3/2, \pm 5/2$ etc. translates into the accepted statement that a 1d isotropic $S = 1/2$ antiferromagnet, while being in a massive dimerized phase for a staggered site-site interaction, becomes critical as the staggering is turned off.

Affleck’s approach, though well conceived, ultimately failed. An early prediction ($\nu = 1/2$, gotten from a further mapping of SU($2n$) spin chains onto SU($2n$) WZW models) was later withdrawn on the grounds that the phase transition at $\theta = \pi$ is likely to be of first order for $n \geq 2$, making it difficult if not impossible to extrapolate to the replica limit $n = 0$. This did not come as a surprise. Problems with the replica $\sigma$ models of localization theory occur not only for the integer quantum Hall transition, but are notorious and show up whenever one tries to make calculations in a nonperturbative regime where observables are sensitive to the global topology of the field space. (The first example of a discrepancy with exact results was reported in [20].) Here then is a summary of how I view the situation: i) The nonlinear $\sigma$ model (1) does not have the right symmetry structure to accommodate conformal invariance and is therefore a bad starting point for extracting the critical properties. ii) Replica field theory has failed. The replica limit is not smooth and one does not know how to extrapolate to $n = 0$.

Clearly, a new analytical approach to the IQHE transition is very much called for. Such an approach was recently proposed by Ludwig et al. [21] who introduced a class of model systems which undergo a phase transition with a jump in the Hall conductivity even in the absence of disorder. This transition is in the Ising universality class and has $\nu = 1$. Adding various types of disorder to the pure system, Ludwig et al. found a rich array of fixed points and lines, but they were unable to control the perturbations that drive the system to the generic IQHE fixed point of interest. It remains to be seen whether any quantitative results for the proper IQHE transition will be obtained from this approach. In the present paper we will follow a more conservative line of attack. It is well known that, in addition to the replica trick, there exists another (and actually superior) method for calculating disorder averages of products of Green’s functions for noninteracting electron systems. This is the supersymmetry technique pioneered by Efetov [22], and it was used in [23] for the derivation of a supersymmetric analog of Pruisken’s model (1). This latter model, which enjoys a mathematically sound foundation, is the object we shall study in the present paper.

The supersymmetric model was rejected by Affleck on the grounds that its fermions do not satisfy the spin-statistics theorem. While this is indeed so, I fail to see why it should disqualify the model. To be sure, the supersymmetric model was not constructed so as to satisfy the narrow requirements of canonical quantum field theory! Rather, it arose from a careful study of noninteracting disordered electron systems, mapping the problem of calculating ensemble averages of products of single-electron Green’s functions on the statistical mechanics problem of computing the Green’s functions of a supersymmetric field theory. When viewed as a statistical mechanics model, the supersymmetric model makes perfect sense. In fact, it possesses a transfer matrix that passes for a well-defined object even by the most rigorous of mathematical standards. It is therefore natural to go back to Affleck’s perceptive ideas [18] and try and get them straightened out by using supersymmetry. This is what we shall attempt to do in this paper. Of course, things will
be rough going at times because we’ll have to make do without the luxuries of canonical quantum field theory (Hilbert space, normalizable states, canonical quantization, unitary representations etc.) Nevertheless, as we shall see, the first half of Affleck’s programme – mapping of the nonlinear $\sigma$ model on a quantum spin chain – can be executed. The second step – mapping of the spin chain on a field theory of the Wess-Zumino type and computation of the critical indices – is currently under investigation.

In Sect. 3 we carefully pass from the $2d$ Euclidean supersymmetric field theory to its $1d$ quantum Hamiltonian. Our treatment was inspired by a pedagogical paper of Shankar and Read [14] on the O(3) nonlinear $\sigma$ model. Setting up the transfer matrix on a mesh of lattice points and taking the anisotropic limit, we obtain a $1d$ lattice Hamiltonian which is a sum of two terms: a “kinetic energy” governing the field dynamics on individual lattice sites and a “potential energy” which couples fields on neighboring sites and enforces local order. The kinetic energy operator for a site located on one of the boundaries of the quantum Hall system is a supersymmetric generalization of the Hamiltonian of a charged particle moving in the field of a (fictitious) magnetic monopole, with the monopole charge being the integer $m$ closest to $2\sigma_{xy}^0$. We call this operator the “monopole Hamiltonian” for short. The value of the monopole charge is quantized by the condition that the Dirac string singularities of a corresponding magnetic vector potential be unobservable. For half-integral values of $\sigma_{xy}^0$, the single-site dynamics in the bulk of the system is governed by the monopole Hamiltonian with charge $m = \pm 1$, and the potential energy, which is staggered in general, becomes translationally invariant.

The ground state of the monopole Hamiltonian is constructed in Sect. 4. The outcome of this calculation can be understood from the simple analogy to the Landau problem of a single 2$d$ electron in a magnetic field. In a suitable gauge, the lowest Landau level is spanned by holomorphic functions of the complex coordinate $z = x + iy$, multiplied by a Gaussian. Reversal of the magnetic field turns holomorphic functions into antiholomorphic ones. Similarly, the lowest level of the monopole Hamiltonian of the $\sigma$ model is a space $V (V^*)$ of holomorphic (antiholomorphic) functions for monopole charge $m > 0 (m < 0)$. The Gaussian gets replaced by something else since the field space is not flat but curved.

The significance of the integer $m$ becomes particularly clear in Sect. 4.6. In previous work on Pruisken’s nonlinear $\sigma$ model [11], only the local (longitudinal and Hall) conductivities were calculated. However, it is clear that these are insufficient for a complete characterization of transport in a quantum Hall system – as in any mesoscopic conductor – at $T = 0$. What is needed is the entire set of conductance coefficients [24]. Computing these for a circular geometry with $N$ probes, we obtain a result which agrees with what is expected from phenomenological theory [3]: the nonvanishing conductance coefficients for even $m$ are given by $|m|$ – the number of “edge states”.

The sign of the monopole charge $m = \pm 1$ alternates on sites in the bulk, resulting in an alternating sequence of single-site ground state spaces $V, V^*, V, V^*$ etc. Our calculation of Sect. 4.6 suggests that a state in $V \otimes V^* \otimes V \otimes V^* \otimes ...$ can be thought of as representing a snapshot of the evolution in “time” (i.e. along a quasi-1$d$ strip) of the distribution of squared transfer matrix elements in an anisotropic version of the Chalker-Coddington model [6]. Guided by this interpretation, we argue that the relevant large-scale physics of the $\sigma$ model is captured by simply projecting the site-site interaction on the space of strong-coupling ground states $V \otimes V^* \otimes V \otimes V^* \otimes ...$. The result of doing such a
projection is a quantum (super-)spin chain Hamiltonian specified in detail in Sect. 5.1. The possibilities for obtaining exact solutions of this or related Hamiltonians are under current investigation.

We conclude this introduction with some comments on the style of presentation. This paper is intended to be fairly self-contained and complete, which explains its length. In Sects. 3-4, familiarity with supersymmetry is needed only for some calculational details, many of which have been relegated to several appendices. The methods used in this paper are robust and apply to any symmetric space \( G/K \), whether it is super or not. Indeed, everything we do is easily transcribed to the replica \( \sigma \) model by simply replacing supertrace by ordinary trace, superdeterminant by ordinary determinant etc. and omitting the sign factors from superparity. We take advantage of this fact by illustrating many of the concepts at the familiar example of a two-sphere \( S^2 \approx U(2)/U(1) \times U(1) \).

The material of Sect. 2 is largely independent of the rest of the paper and will not be used until Sect. 5.1.

### 2 Conformal Invariance of Critical Disordered Electron Systems

Conformal invariance, which occurs in models of field theory and statistical mechanics at a critical point \([8]\), is an especially powerful symmetry in two space dimensions, the group of conformal transformations being in this case infinite-dimensional. Conformal invariance connects correlation functions in different geometries with each other; for example, the conformal mapping \( z \mapsto \ln z \) relates the correlation functions in the plane to those on a cylinder. Relations of this kind are being widely used for the computation, numerical and otherwise, of critical indices.

This second section of the paper is concerned with the question whether conformal field theory has applications to critical wave propagation in \( 2d \) disordered media. It is by now well known \([25]\) that the interference of waves multiply scattered by strong disorder leads to localization, if the coherence volume is large enough. The points in parameter space where the transition from diffusive motion to localization occurs, are characterized by a diverging correlation length. It is therefore reasonable to ask, and the question has been asked before \([26]\), whether conformal field theory makes statements about the critical correlations at such a point. To be specific, let us consider in the sequel the motion of a single electron in a disordered solid, although the reasoning used may have generalizations to other systems. Then there are (at least) two universality classes where a critical point occurs in two dimensions: (i) systems in a strong magnetic field (quantum Hall systems), and (ii) time-reversal invariant systems with spin-orbit scattering.

For a system of class (i) or (ii), we take \( \langle \cdot \rangle \) to mean the average over some statistical ensemble of Hamiltonians \( \hat{H} \) which is invariant under the group of Euclidean motions of the infinite plane. Let \( \delta(E) = \pi^{-1} \varepsilon/(\varepsilon^2 + E^2) \) be a regularization of the \( \delta \)-distribution, and let \( S_E(x, y) \) denote the matrix element of \( \delta(E - \hat{H}) \) between the eigenstates of the position operator at \( x \) and \( y \). With these definitions we consider the averages

\[
\langle S_E(x, x)^p S_{E+h\omega}(y, y)^q \rangle \quad (p, q \in \mathbb{N}) \quad \text{and} \quad \langle S_E(x, y) S_{E+h\omega}(y, x) \rangle.
\]

The first of these measures the correlations of the local density of states, while the Fourier transform of the second one yields the dynamic structure factor, which can be parametrized...
in terms of a wave-vector and frequency-dependent diffusion constant \[27\]. By the spectral expansion \( S_E(x, y) = \sum \psi_i(x) \delta(E-E_i) \psi_i(y) \), the correlation functions \( \mathcal{E} \) can expressed in terms of the eigenfunctions of \( H \) with energies \( E \) and \( E + \hbar \omega \). Introducing the symbol \( \langle \langle \cdot \rangle \rangle \) to denote ensemble averaging in combination with energy averaging by the regularized \( \delta \)-distribution, we can write the correlators \( \mathcal{E} \) in the short-hand form

\[
K_{AB}(x - y; E, \omega) = \langle \langle A(x) B(y) \rangle \rangle
\]  

where \( A(x) \ (B(y)) \) is some polynomial in the eigenfunctions and their complex conjugates evaluated at the point \( x \ (y) \). \( K_{AB} \) is translationally and rotationally invariant, by the Euclidean invariance of the probability measure defining \( \langle \cdot \rangle \).

At a critical point, \( K_{AB}(x - y) \) is expected to show algebraic decay \( \sim |x - y|^{-\eta} \) with some positive exponent \( \eta \). When the infinite plane is replaced by a cylinder with circumference \( L \), finite-size scaling theory states that \( K_{AB} \) should decay exponentially \( \sim \exp(-|x - y|/\lambda L) \). One might now think that, by the usual hypothesis of conformal invariance at a critical point, \( \eta \) and \( \lambda \) are related by

\[
\eta = 1/\lambda \pi.
\]  

This however is false and we will review the reason why in Sect. 2.1, by giving a simplified presentation of the work of Wegner \[28\]. The lesson learnt from this will prompt us to point out in Sect. 2.2 a different class of correlation functions where the relation \( \eta \) does hold.

### 2.1 Finite frequency breaks conformal invariance

The message of this subsection is easy to state. In addition to the correlation length \( \xi \) there exists a second length scale, \( L_\omega \), which is set by the frequency \( \omega \). The correlation functions \( K_{AB} \), Eq. \( \mathcal{E} \), retain their dependence on \( L_\omega \) at the critical point, even in the limit \( L_\omega \to \infty \). The precise form of these correlation functions has been calculated by Wegner \[28\] in some cases. We will now repeat Wegner’s basic argument.

Let \( L_\omega \) be defined as the linear size of a system with level spacing \( \hbar \omega \): \( L_\omega = (\hbar \omega \rho)^{-1/2} \), where \( \rho \) is the density of states per unit of energy and area. At the critical energy \( E_c \) where the correlation length \( \xi \sim |E - E_c|^{-\nu} \) diverges, \( K_{AB} \) takes the form

\[
K_{AB}(z, E_c, \omega) = k_{AB}(|z|, L_\omega) \quad (z = x - y).
\]  

A necessary condition in order for arguments from conformal invariance to apply is that \( k_{AB} \) should be ignorant of the length scale \( L_\omega \). For \( L_\omega \) large and fixed, let us therefore specialize to the limit \( |z| \ll L_\omega \) where we expect power law behavior

\[
k_{AB}(|z|, L_\omega) \sim c_0 |z|^{-\eta_{AB}}
\]  

with some exponent \( \eta_{AB} \).

We can obtain more information about \( \eta_{AB} \) and the dependence of \( c_0 \) on \( L_\omega \) by using the real-space renormalization group (RG) transformation for disordered single-particle
systems introduced by Wegner [29]. To keep the technicalities to a minimum, we assume that $A$ and $B$ are scaling fields (i.e. eigenfields of the RG transformation) with scaling dimensions $\Delta_A$ and $\Delta_B$, respectively. A change of cutoff scale $a \mapsto b_1 a$ then results in

$$A(x) \mapsto b_1^{-\Delta_A} A(x/b_1), \quad B(y) \mapsto b_1^{-\Delta_B} B(y/b_1).$$

Choosing $b_1 = |x-y|/a = |z|/a$, we get

$$k_{AB}(|z|, L_\omega) = (|z|/a)^{-(\Delta_A+\Delta_B)} k_{AB}(a, L_\omega a/|z|).$$

Note the dependence of the right-hand side on $|z|$ through both the algebraic prefactor and the second argument of $k_{AB}$, which is $L_\omega a/|z|$. To extract the full $|z|$-dependence of the correlation function (4), we must therefore perform a second RG transformation $b_1 a \mapsto b_2 b_1 a$, taking $L_\omega a/|z|$ to some fixed value $L_\omega a/(|z|b_2) = L_0$. In calculating the effect of this transformation, we may no longer renormalize $A$ and $B$ separately, since the RG transformation $a \mapsto b_1 a = |z|$ has already brought the points at which $A$ and $B$ are evaluated within a distance of one unit of the cutoff scale. Instead, we must renormalize the local object $A(x/b_1)B(y/b_1) \simeq (AB)(x/b_1)$. The product $AB$ will not in general be a scaling field, but it can be expanded in terms of such fields. If $\Delta_{AB}$ denotes the scaling dimension of the most relevant field occurring in this expansion, $(AB)(x)$ is renormalized under $a \mapsto b_2 a$ by

$$(AB)(x) \mapsto b_2^{\Delta_{AB}} (AB)(x/b_2) + \text{subdominant terms}. $$

Taking $b_2 = L_\omega a/(L_0 |z|) = \text{const} \times L_\omega /|z|$, we finally obtain:

$$k_{AB}(|z|, L_\omega) \sim |z|^{-\Delta_A-\Delta_B+\Delta_{AB}} L_\omega^{-\Delta_{AB}},$$

which expresses the dependence of $k_{AB}$ on both $|z|$ and $L_\omega$ by three scaling dimensions $\Delta_A$, $\Delta_B$, and $\Delta_{AB}$. On physical grounds, the correlation function $k_{AB}$ must neither increase with distance $|z|$ nor decrease with $L_\omega$, $\omega \sim L_\omega^{-2}$ being a distance in energy space. Therefore, $\Delta_{AB} \leq 0$ and $\Delta_{AB} \leq \Delta_A + \Delta_B$. For the case of potential scattering (orthogonal universality class), Wegner has shown the scaling dimensions $\Delta_A$ and $\Delta_B$ for the first choice in (2) to be zero or negative in $2 + \epsilon$ dimensions. (Moreover, the scaling dimensions decrease without bound as the powers $p$ and $q$ increase.) We expect this to be true in general. Thus, the scaling dimensions $\Delta_O$ ($O = A, B, \text{or } AB$) are subject to the inequalities

$$\Delta_O \leq 0, \quad -\Delta_{AB} \geq -\Delta_A - \Delta_B.$$

The second of these can be understood independently of (3) by observing that the limit $L_\omega \to \infty$ restricts the contributions to $k_{AB}$ to those from a single eigenfunction $\psi$ with energy close to $E_c$. The inequality is a manifestation of the existence of large amplitude fluctuations of such an eigenfunction, indicating its multifractal nature [30].

We now return to the question concerning the relation of $\eta_{AB} = \Delta_A + \Delta_B - \Delta_{AB}$ to the Lyapunov exponent for exponential decay of $k_{AB}$ on the cylinder. The derivation [3] of relation (4) assumes that $A$ and $B$ are (spinless) primary fields which are dual to each other
and belong to a unitary representation of the Virasoro algebra, so that $\Delta_A = \Delta_B = \Delta > 0$ and $\langle A(x)B(y) \rangle \sim |x - y|^{-2\Delta}$. Clearly, these assumptions are violated in the present context. We are therefore ill-advised [26] to expect a relation of the form (4) to hold. (Of course, this does not yet rule out the possibility of some generalization of (4) still being valid.)

Could there exist some sort of sensible, nonunitary conformal field theory that has an infinite number of primary fields with negative scaling dimensions to accommodate the correlation functions (6)? Whatever the answer to this question may turn out to be, it will be constrained by the observation [28] that $\langle \langle A(x) \rangle \rangle$ for $A(x) = S_E(x, x)^p$ and $p \geq 2$ diverges as the regularizing imaginary frequency $i\varepsilon \to 0$. This divergence is suggestive of a field theory with unbroken noncompact symmetry – just as $\langle A \rangle = 0$ for an order parameter field $A$ would signal unbroken compact symmetry. In fact, ever since the seminal work of Schäfer and Wegner [31] and of Efetov [22] it has been understood that the problem of calculating disorder averages of powers of wave amplitudes maps on field theories of the nonlinear $\sigma$ model type with noncompact symmetry. We will study such a field theory in Sects. 3-4. The nonpositivity of the dimensions of a subset of its scaling fields turns out to be intimately related to the fact that these fields are improper, i.e. their application to the vacuum produces states that are non-normalizable in the absence of external frequency as a regulator for non-compactness.

### 2.2 Conformal invariance and d.c. conductance

We consider again a 2$d$ noninteracting disordered electron gas, either in a strong magnetic field or with spin-orbit scattering. The configuration space $\mathcal{M}$ for one electron may be finite or infinite. An important difference to earlier is that we now envisage the $N$-terminal geometry shown in Figure 1. There are $N$ contact areas where electrons are exchanged between $\mathcal{M}$ and the leads. Applying a small voltage (or, more precisely, a small shift of chemical potential) to lead $q$ we get a small current flowing in lead $p$. The constant of proportionality is called the d.c. conductance, or conductance coefficient, $g_{pq}$. From linear response theory one can derive a formula for $g_{pq}$ as a current-current correlation function [24]: $g_{pq} \sim \text{Tr} (v_p G^+ v_q G^-)$ in symbolic notation where $v_p$ is the component of the velocity operator normal to the contact area $C_p$, and the Green’s functions $G^\pm$ are those of the total system (including the leads) evaluated at the Fermi energy $E_F$. It turns out that an alternative formulation on reduced configuration space $\mathcal{M}$ (without the leads) is possible. If $\Gamma$ is a certain nonlocal kernel determined by the lead geometry, it has been shown [32] that

$$g_{pq} = \int_{C_p} dw \int_{C_p} dx \int_{C_q} dy \int_{C_q} dz \; \Gamma(w, x) G^+(x, y) \Gamma(y, z) G^-(z, w). \tag{7}$$

$G^\pm$ are now the Green’s functions of the isolated system subject to certain boundary conditions on the contact areas. Qualitatively, $G^\pm = (E_F - H |_{\mathcal{M}} \pm i\varepsilon)^{-1}$. The exact formulation has been given in [32]. (Note that Eq. (8) of [32] is misprinted: $G^-_c$ should read $G^+_c$ instead.) Formula (7) is exact for noninteracting electrons at zero temperature and for a magnetic field of arbitrary strength. We are measuring conductance in its natural units $e^2/h$.

The energy argument of the reduced Green’s functions $G^\pm$ in (7) is a real number $E = E_F$. Why are we permitted to set to zero the infinitesimal imaginary part $\pm i\varepsilon$ which
is usually added to energy to make Green’s functions well-defined? The answer is that the boundary conditions satisfied by the reduced Green’s functions spell out the “openness” of the system, i.e. the possibility for electrons to leave $\mathcal{M}$ at a contact area $C_r$ and enter the lead $r$. The damping caused by the escape of probability from $\mathcal{M}$ shifts the spectrum of $H|_{\mathcal{M}}$ off the real energy axis and makes the limit $\varepsilon \to 0$ exist. This is the crucial point: conductance – as opposed to the Green’s functions for an isolated system – does not require any regularizing real or imaginary frequency in order to be well-defined.

Let us now take the disorder average $\langle g_{pq} \rangle$ and set energy right to its critical value $E_c$ where the correlation length diverges. When the spatial extension of the contact areas is comparable to the system size, as is the case in most experimental geometries, the average conductance will become some scale-invariant number $\langle g_{pq} \rangle = g_{pq}^*$. We find it more interesting however to consider the opposite case of small contacts, see Figure 1. A procedure that makes perfect sense is to take the thermodynamic limit of infinite system size while keeping the relative distance and size of the contacts fixed. We expect the conductance coefficients to be well-behaved and finite in this limit. With system size having gone to infinity, there do not remain any scales over which to measure correlations. (Of course, one can still assign some length scale to the “strength” or “decay width” of each contact. However, since the contacts are no more than a local perturbation on the system, they do not set any scale in the sense of scaling theory. Note that, in spite of this, they remain effective as regulators for conductance even in the thermodynamic limit.) With all scales absent, the average conductance $\langle g_{pq} \rangle$, and all moments of conductance, must decay algebraically with increasing distance between the contacts. (This will remain true even for large contacts, provided that their size is kept fixed as the distance between them is sent to infinity.)

Within the framework of the effective field theory described in Sect. 3, we can rephrase our assertion as follows. If the linear extension of the contacts is less than or comparable to the microscopic cutoff scale of the field theory, which is set either by the magnetic length or by the elastic mean free path, the contacts can be regarded as point-like. The conductance coefficients $\langle g_{pq} \rangle$ for $N$ terminals then translate into $N$-point functions

$$
\langle O_p(x_p)O_q(x_q) \prod_r O_r(x_r) \rangle,
$$

with local operators $O_i(x)$, of the field theory. On physical grounds, these $N$-point functions must decay algebraically on all but the shortest distances when $E = E_c$. There aren’t any length scales left and, moreover, the disease of divergent amplitudes is now absent. It is therefore reasonable to expect that the $N$-point functions (8) can be expanded in terms of scaling fields with positive dimensions of an underlying conformal field theory. (As a matter of fact, we shall see in Sect. 5.1 that the $N$-point functions (8) restrict to dynamical $N$-spin correlation functions of a superspin chain Hamiltonian for quantum Hall systems.) More precisely, because of the noncompact symmetry of the field theory, we expect a continuum of primary fields – with an associated continuum of scaling dimensions – to occur in the expansion of $O_i(x)$.

In summary, what we are saying is the following. Critical disordered electron systems possess a number of interesting correlation functions whose definition involves two (or
more) energies $E_+$ and $E_-$. The effect of frequency $\omega = (E_+ - E_-)/\hbar$ is similar to the effect of an external magnetic field on a critical spin system: it sets a length scale and breaks conformal invariance. While conformal invariance is restored for the spin system when the external field is turned off, the same is not guaranteed to happen for the disordered electron system. Many of the correlation functions of interest diverge as $\omega \to 0$. For example, the dynamic structure factor does. (This is a consequence of critical long-time correlations at the metal-insulator transition.) Any consistent conformal field theory interpretation of these correlation functions will have to confront the task of defining operator product expansion for an infinite set of primary fields whose spectrum of negative dimensions is unbounded from below. The situation is much better for correlation functions that can be defined at a single energy $E = E_c$. An example is the d.c. conductance – or current-current correlation function – of an open system. While conformal invariance will govern the geometric variation of conductance for any contact size, things become particularly simple for point contacts: in this case the conductance coefficients for an $N$-terminal geometry map on $N$-point functions for local fields of an effective field theory. These $N$-point functions are expected to have a conformal field theory representation involving primary fields with positive scaling dimensions only.

I have included the material of Sects. 2.1-2 in this paper to motivate why the quest for the conformal limit of the 2$d$ nonlinear $\sigma$ model holds more promise than just an explanation of the critical indices $\nu = 2.3 \pm 0.1$ [6,7], $y_{\text{irr}} = -0.40 \pm 0.04$ [20,21] for the quantum Hall effect and $\nu = 2.75 \pm 0.15$ for systems with spin-orbit scattering [36]. Conformal field theory will also predict interesting algebraic decay of the conductance and its moments for an $N$-terminal geometry with small contacts.

3 Hamiltonian limit of the nonlinear $\sigma$ model

In the remainder of this paper we will make a fresh field-theoretic attack on the integer quantum Hall effect, starting from a supersymmetric version [23] of Pruisken’s nonlinear $\sigma$ model [9]. Most of what we do in Sects. 3-5 is general and can be transcribed to systems with spin-orbit scattering without difficulty. I hope to return to such systems in a future publication.

We start out by introducing the nonlinear $\sigma$ model from a gauge theory point of view in Sects. 3.1-2. We are not aiming at a complete solution of this model here. All we care about is its low-energy limit, describing the long wave length physics of integer quantum Hall systems. This is important to keep in mind, since restriction to the low-energy domain gives us much freedom to manipulate the field theory. We use this freedom in Sects. 3.3-5, where a careful derivation of the quantum Hamiltonian of the $\sigma$ model is presented. (The well-known connection between Feynman’s path integral and Schrödinger wave mechanics allows us to relate the 2$d$ classical field theory to a 1$d$ quantum field theory.) The possibility of such a Hamiltonian formulation was first discussed briefly in [10]. Our approach follows the work of Shankar and Read [14] on the O(3) nonlinear $\sigma$ model with topological angle $\theta = \pi$ to some extent. The essential idea is to rewrite the topological density as an alternating sum of Wess-Zumino terms. Gauge ambiguity of the Wess-Zumino term quantizes the values of the topological coupling constant for which such a rewriting can be done. The O(3) model is somewhat special in that its field space is a two-sphere and the topological density has an intuitive meaning as (the pullback of) the
solid-angle two-form. The simplifications resulting from this allow Shankar and Read to take some short cuts that are not available to us here.

### 3.1 Review of Pruisken’s model (supersymmetric version)

An effective field theory for noninteracting electrons moving in two dimensions under the influence of a strong magnetic field and a Gaussian white noise potential, was first derived by Pruisken [9]. He pointed out that a strong magnetic field has the effect of adding a so-called topological term to the Lagrangian of the nonlinear $\sigma$ model for noninteracting disordered electron systems developed by Wegner, Efetov, and others [37,31,22]. A closely related model with internal supersymmetry, avoiding the replica trick used in [9,10], was subsequently derived by Weidenmüller [23]. In the present paper we shall study the latter model, extended by the addition of an extra pair of indices to the $\sigma$ model matrix field. Such an extension is necessary for the calculation of disorder averages of products of $n$ retarded and $n$ advanced Green’s functions ($n \geq 1$), which is what is needed when one wants to go beyond the average of the conductance and calculate its variance and higher cumulants as well [38]. (Another benefit from making the extension is that it forces us to use the generic structures of the theory.) We will not give yet another derivation of the $\sigma$ model here but refer to the original papers for this [31,22,9,23]. The nonlinear $\sigma$ model is usually presented in terms of matrix fields $Q$ satisfying the nonlinear constraint $Q^2 = 1$. This presentation, while adequate for some purposes, is rather useless for most of what we intend to do in this paper. So, rather than wasting time by first writing down the $Q$-field formulation and then changing to something better, we shall present the “good” formulation and the proper structures of the model right away.

We start from the associative algebra of $4n \times 4n$ supermatrices acting on the tensor product of three linear spaces: (i) superspace or Boson-Fermion (BF) space, (ii) Advanced-Retarded (AR) space and (iii) replica or Extra (E) space. The first two have dimension 2 and the last one dimension $n$. Invertible $4n \times 4n$ supermatrices $g$ satisfying the condition

$$g^{-1} = \eta g^\dagger \eta \quad \text{with} \quad \eta = ((E_{11})_{BF} \otimes (\sigma_3)_{AR} + (E_{22})_{BF} \otimes 1_{AR}) \otimes 1_E$$

form a noncompact Lie supergroup $G = U(n,n|2n)$. (Here as always $\{E_{ij}\}_{i,j=1,\ldots,N}$ is the canonical basis of $\text{End}(\mathbb{C}^N)$ and $\sigma_1 = E_{12} + E_{21}, \sigma_2 = -iE_{12} + iE_{21}, \sigma_3 = E_{11} - E_{22}$ for $N = 2$ are the Pauli matrices.) The subgroup of elements in $G$ that commute with

$$\Lambda = 1_{BF} \otimes (\sigma_3)_{AR} \otimes 1_E$$

is denoted by $K = U(n|n) \times U(n|n)$. $G$ is projected onto $G/K$, the space of left cosets $gK$ ($g \in G$), by $g \mapsto \pi(g) := gK$. It is this coset space $G/K$ which is the field space of the nonlinear $\sigma$ model. The base manifold (or “body” in the language of deWitt [39]) of $G/K$ is the direct product of $U(n,n)/U(n) \times U(n)$ with $U(2n)/U(n) \times U(n)$. Both of these are Riemannian symmetric spaces [40], the former of the noncompact and the latter of the compact type. We sometimes refer to them as BB (Boson-Boson) and FF (Fermion-Fermion) space, respectively.

To write down the $\sigma$ model action – and at the same time prepare later analysis – we identify various structures on $G = \text{Lie}(G)$, the Lie algebra of $G$. Defining orthogonality by the natural symmetric form

$$B(X,Y) = \text{STr} \ XY/2$$

13
where STTr is the supertrace, we denote by \( \mathcal{P} \) the orthogonal complement of \( \mathcal{K} = \text{Lie}(\mathbf{K}) \) in \( \mathcal{G} \). The orthogonal decomposition of an element \( X \in \mathcal{G} \) by \( \mathcal{G} = \mathcal{K} + \mathcal{P} \) is written \( X = X_\mathcal{K} + X_\mathcal{P} \). (The explicit formulas are \( X_\mathcal{K} = (X + \Lambda X \Lambda)/2 \) and \( X_\mathcal{P} = (X - \Lambda X \Lambda)/2 \). With these definitions the principal part of the \( \sigma \) model Lagrangian can be written in the form

\[
L_0(g, \partial g) = B \left( (g^{-1} \partial_\mu g) \mathcal{P}, (g^{-1} \partial_\nu g) \mathcal{P} \right).
\]

(Repeated Greek indices are implicitly summed over the values \( \mu = 0, 1 \).) Pruisken’s Lagrangian \([9]\) contains in addition to \( L_0 \) another term, the so-called topological density \( L_{\text{top}} \), which is formulated as follows. Let a linear operator \( J : \mathcal{P} \to \mathcal{P} \) be defined by \( JX = i[\Lambda, X]/2 \). \( J \) squares to minus the identity and preserves \( B \). With its help we define \( \Omega(X, Y) \) for \( X, Y \in \mathcal{P} \) by \( \Omega(X, Y) = B(JX, JY) \). This two-linear form is skew symmetric:

\[
\Omega(Y, X) = B(X, JY) = B(JX, J^2Y) = -\Omega(X, Y),
\]

owing to the symmetry of \( B \) and the properties of \( J \). It induces the topological density by

\[
L_{\text{top}}(g, \partial g) = \epsilon^{\mu\nu} \Omega \left( (g^{-1} \partial_\mu g) \mathcal{P}, (g^{-1} \partial_\nu g) \mathcal{P} \right)
\]

where \( \epsilon^{\mu\nu} \) is the antisymmetric tensor. Both \( L_0 \) and \( L_{\text{top}} \) are invariant under a local change of representative \( g(x) \mapsto g(x)k(x) \) \((k(x) \in \mathbf{K})\) and are therefore well-defined as Lagrangians on the coset field space \( \mathbf{G}/\mathbf{K} \). In addition, they are invariant under global transformations \( g(x) \mapsto g_0g(x) \) \((g_0 \in \mathbf{G})\). Eqs. (11) and (12) follow from the \( Q \)-field expressions \([9]\) \( L_0 = -\text{STr}(\partial Q)^2/8 \) and \( L_{\text{top}} = \epsilon^{\mu\nu} \text{STr} Q \partial_\mu Q \partial_\nu Q/8i \) on making the substitution \( Q = g\Lambda g^{-1} \).

After these preparations, we introduce the nonlinear \( \sigma \) model for an integer quantum Hall system with longitudinal conductivity \( \sigma_{xx} \) and Hall conductivity \( \sigma_{xy} \) by the functional integral

\[
Z = \int \mathcal{D}[g] \exp -\int_{\mathcal{M}} d^2x \left( \sigma_{xx} L_0 + i \sigma_{xy} L_{\text{top}} \right).
\]

\( \sigma_{xx} \) and \( \sigma_{xy} \) are measured in their natural units \( e^2/h \). \( \mathcal{M} \) is the configuration space of the two-dimensional electron gas, and the functional integration measure \( \mathcal{D}[g] \) equals \( \prod_x dg_K(x) \) where \( dg_K \) is the uniform super-integration measure for \( \mathbf{G}/\mathbf{K} \). (Note that the derivation of the nonlinear \( \sigma \) model from a Gaussian white noise potential yields for \( \sigma_{xx} \) and \( \sigma_{xy} \) their bare (or SCBA) values, denoted by \( \sigma_{xx}^0 \) and \( \sigma_{xy}^0 \) in \([9,10]\).) We here imagine that these coupling constants have already been renormalized and simplify the notation by dropping the superscript \( \text{o} \).

The functional integral (13) calls for some additional explanation. First of all, the invariant symmetric form \( B \), Eq. (11), has the crucial property of being positive definite on the ordinary part (or body) of \( \mathcal{P} \). This means that \( \mathbf{G}/\mathbf{K} \) is Riemann in its natural geometry induced by \( B \). Second, an experimental quantum Hall system has a boundary \( \partial \mathcal{M} \), which is composed of insulating and conducting parts, giving rise to corresponding boundary conditions for the \( \sigma \) model field in (13). What is particularly important is that the conducting boundary conditions cut off the spatially uniform fluctuations of the noncompact \( \sigma \) model field – which would otherwise cause a divergence – by breaking the global \( \mathbf{G} \)-symmetry of (13) down to a global \( \mathbf{K} \)-symmetry. (The microscopic origin of the
damping is the absorptive term $i\Gamma$ in the reduced Green’s functions $G^\pm = (E_F - H)_{\mathcal{M}} \pm i\Gamma)^{-1}$. This, in conjunction with the Riemannian nature of $G/K$, makes the functional integral (13) well-defined and convergent after UV regularization. Third, for the sequel we adopt the Corbino geometry $\mathcal{M} = [R_1, R_2] \times S^1$, with two or more contacts placed along the boundary $\partial \mathcal{M}$. Alternatively, for a particularly neat theoretical model, we might imagine $\mathcal{M}$ to be a two-torus $S^1 \times S^1$, with several point contacts attached to it to cut off the divergent $\sigma$ model fluctuations. What is important for our derivation of the quantum Hamiltonian in Sect. 3.5 is that one of the two directions on $\mathcal{M}$ be periodic. This direction $\sim S^1$ will be assigned the role of imaginary time. Finally, we remark that the functional integral $Z$, Eq. (13), equals unity by supersymmetry [12] which serves to cancel all vacuum graphs. (This is true even in the presence of a boundary since the boundary conditions preserve $K$-supersymmetry.) To obtain the Green’s functions of the nonlinear $\sigma$ model one adds sources to the Lagrangian and takes derivatives as usual.

For $\sigma_{xy} = 0$, (13) is the nonlinear $\sigma$ model describing the long wave length excitations of noninteracting disordered 2d electron systems with time-reversal invariance broken by a weak magnetic field (unitary universality class). Perturbative renormalization group analysis [37,22] shows its coupling constant $1/\sigma_{xx}$ to be driven to strong coupling at large scales. This has led to the widely accepted conjecture that all stationary single-particle states of such systems are localized in $d = 2$. At the same time, the plateau-to-plateau transition in integer quantum Hall systems cannot be understood if all bulk electronic states are localized. There is ample numerical studies for the lowest Landau level, we know the correlation length exponent of this delocalization phase transition to be $\nu = 7/3$ (or close to that value). Hence, if (13) is to describe the underlying physics, the addition of the topological density $L_{\text{top}}$ must cause critical behavior, i.e. vanishing masses for some of the $\sigma$ model fields, at certain values of $\sigma_{xy}$. At first sight, though, $L_{\text{top}}$ appears to be an unlikely candidate for the job it is supposed to do. $L_{\text{top}}$ does not change the classical equations of motion, for it can be written locally as a total derivative. This is closely related to the fact that, for suitable boundary conditions, $\int d^2x L_{\text{top}}$ is topologically quantized in integer multiples of $2\pi$ [10,12], making it invisible in weak-coupling perturbation theory. Nevertheless, Pruiisken and collaborators [10] have argued emphatically that the presence of the topological density does lead to decisive changes in the large-scale physics of the replica analog of the nonlinear $\sigma$ model (13). One of their arguments uses twisted boundary conditions and ideas stimulated by ’t Hooft duality to show that some kind of phase transition must indeed occur somewhere in the range from $\sigma_{xy} = m$ to $\sigma_{xy} = m + 1$. However, the precise nature of the phase transition is not specified by the duality argument. It could, in principle, be of first order with a finite correlation length. (This is in fact the conclusion reached by Affleck [18]: the $U(2n)/U(n) \times U(n)$ nonlinear $\sigma$ model for $n \geq 2$ has a first-order transition at $\theta = \pi$.) As was mentioned in the introduction, the current algebra of the model (13) fails to meet the strict requirements set by conformal invariance in $d = 2$. By implication, the model (13) cannot possess any nontrivial RG fixed points, contrary to what has been asserted by Pruiisken et al. [10,11]. Rather, what must happen is that the model at $\sigma_{xy} = 1/2 \pmod{1}$ is driven under renormalization to the fixed point of some other field theory, which is not
known at present but is likely to be of the Wess-Zumino type. To summarize the situation, a sceptic will say that fifteen years after the discovery of the integer quantum Hall effect, one of its crucial features, the occurrence of a localization-delocalization transition in the center of a Landau band, is still lacking an analytic description. This very unsatisfactory state of affairs has been the motivation for the efforts reported in the present paper.

### 3.2 Gauge fixing

The nonlinear $\sigma$ model for noninteracting disordered electron systems is usually presented as a theory of fields $Q$ taking values in some nonlinear space isomorphic to a coset space $G/K$. In Sect. 3.1 we gave its alternative presentation [10] as a theory of $G$-valued fields $g$ with an invariance under local gauge transformations $g(x) \mapsto g(x)k(x)$ for $k(x) \in K$. In other words, we are here viewing the nonlinear $\sigma$ model as a kind of gauge theory with the gauge group $K$ acting on the field space $G$ by right multiplication. (However, as the gauge field has no dynamics, the nonlinear $\sigma$ model is not a gauge theory in the strict sense of the word.) The practical implementation of the gauge theory point of view requires gauge fixing, i.e. some prescription for separating the unphysical (or gauge) degrees of freedom from the physical ones. Gauge fixing in the present context is best described as follows. We choose some (locally) smooth map $s : G/K \rightarrow G$. The only requirement on $s$ is that, with $\pi$ being the canonical projection $G \rightarrow G/K$, $s$ has to satisfy $\pi \circ s = \text{id}$. Equivalently, for every $g \in G$ the element $k(g)$ defined by the equation

$$g = s(\pi(g))k(g)$$

must be an element of $K$. Technically speaking, an $s$ with this property is called a section of the bundle $\pi : G \rightarrow G/K$. For the purpose of illustration, consider the example $G = SU(2)$ and $K = U(1)$ (generated by $i\sigma_3$). The coset space $SU(2)/U(1)$ is isomorphic to the two-sphere $S^2$ by the map $gU(1) \mapsto g\sigma_3g^{-1} = \sum_{i=1}^{3} n_i \sigma_i$, assigning to every coset $gU(1)$ a unit vector $n = (n_1, n_2, n_3)$. If we parametrize $S^2$ by a polar angle $\theta$ and an azimuthal angle $\phi$ in the usual way, two examples of a section $s : S^2 \rightarrow SU(2)$ are

$$s = e^{i\phi\sigma_3/2}e^{i\theta\sigma_2/2} \quad \text{and} \quad s = e^{i\phi\sigma_3/2}e^{i\theta\sigma_2/2}e^{-i\phi\sigma_3/2}.$$  

The property $\pi \circ s = \text{id}$ is easy to check using the isomorphism $SU(2)/U(1) \simeq S^2$.

After this example we continue our general considerations. By definition, a section $s$ assigns to a coset $\pi(g) = gK$ one representative $s(\pi(g)) \in G$. Since this assignment is (locally) one-to-one, $s$ distinguishes some submanifold of $G$ which is (locally) isomorphic to $G/K$. Thus, making a choice of $s$ fixes the gauge: $s$ identifies the subspace of physical degrees of freedom on the field space $G$, which are integrated over in the functional integral, while $k : G \rightarrow K$ defined by Eq. (14) accounts for the unphysical gauge degrees of freedom (not integrated over). There is of course much arbitrariness in the choice of $s$. However, given any two sections $s$ and $s_1$ we can always find a map $k_1 : G/K \rightarrow K$ such that $s$ and $s_1$ are related by a gauge transformation $s = s_1k_1$. Thus, the arbitrariness of $s$ precisely reflects the freedom we have in choosing the gauge.

We were careful to insert the qualification “locally” in various places of the above discussion. As a matter of fact, $s$ cannot be defined globally when $G/K$ is topologically
nontrivial. To see this by way of example, let us focus again on \( \text{SU}(2)/\text{U}(1) \simeq S^2 \) and consider the first of the two sections given in (13). Since the azimuthal angle \( \phi \) is ill-defined on the north and south pole, \( s \) is singular at these points. (It is perhaps helpful to mention that a choice of \( s \) specifies, by formula (27) below, a choice of vector potential \( A \) for a magnetic monopole placed at the center of \( S^2 \). The singularities of \( s \) then translate into Dirac string singularities of \( A \).) We can move the singularities of \( s \) around at will by a gauge transformation, but a basic result of fibre bundle theory \([14]\) tells us that we can never make them disappear altogether. The same kind of topological obstruction occurs for our quantum Hall model space \( G/K \), due to the presence of the topologically nontrivial space \( \text{U}(2n)/\text{U}(n) \times \text{U}(n) \) in its base manifold. (In this case however the singularities are not points but are submanifolds of codimension two on \( \text{U}(2n)/\text{U}(n) \times \text{U}(n) \).)

We need to make a small addendum concerning the most general choice of section \( s : G/K \to G \) that will be permitted in this paper. It so turns out that one crucial step – going from Eq. (25) to Eq. (28) below – requires the values taken by \( s \) to be restricted to \( G_0 = \text{SU}(n,n|2n) \), the subgroup of elements of \( G \) with superdeterminant equal to unity. There are good reasons for not making the restriction to \( G_0 \) throughout our development. (Working with \( G_0 \) instead of \( G \) would cause a certain amount of inconvenience in Sect. 5.) Thus, we will base all our considerations on the pair \( G \) and \( K \), except that a section \( s \) of the bundle \( G \to G/K \) will always have to be a map \( s : G/K \to G_0 \).

### 3.3 Quantum Hamiltonian for \( \sigma_{xy} = 0 \)

To make a continuum field theory such as (13) well-defined, one needs to specify some prescription for regularization of the functional integral in the ultraviolet. Elementary particle physics has come up with quite a number of different regularization schemes for us to use. The approach we adopt here is to reformulate the two-dimensional Euclidean field theory as a one-dimensional quantum theory defined on a lattice in space. This particular regularization scheme has been used successfully by Shankar and Read \([16]\) in their frontal assault on the \( \text{O}(3) \) nonlinear \( \sigma \) model with topological angle \( \theta = \pi \). It is the optimal choice in the present context, too, as it will allow us to clearly identify the relevant low-energy degrees of freedom.

The quantum Hamiltonian of a canonical 2d Euclidean field theory can be constructed in the following standard way. First, one re-interprets the Euclidean theory as a Lorentz-invariant field theory in 1+1 dimensional Minkowski space-time. Then, the Lagrangian of the latter is converted into a Hamiltonian function by carrying out a Legendre transform. And finally, the classical Hamiltonian is turned into a quantum operator by the process of canonical quantization. What is attractive about this procedure is that it avoids detailed manipulations of the functional integral. It turns out, however, that the supersymmetric field theory (13) does not have a Hilbert space for its space of quantum states, so it is not a quantum field theory of the canonical type. Canonical quantization, though applicable as a purely formal procedure, is therefore not really appropriate here. (The noncanonicity is probably one of the reasons why the model (13) has not yet been embraced by the community at large but has been studied by a rather small group of people.) I emphasize that there is nothing ill-defined or suspect about our noncanonical field theory. After lattice regularization, we are faced with the bona fide statistical mechanics problem of computing a finite number of well-defined and convergent superintegrals.
The foregoing discussion motivates the no-nonsense approach we will adopt. Having UV-regularized the field theory by placing its fields on a rectangular lattice, we will take the anisotropic (or Hamiltonian) limit and use no more – and no less – than straightforward manipulations of the integrals over field variables to derive the quantum Hamiltonian. More precisely, what we will do is to carry out the familiar steps leading from Feynman’s path integral (in imaginary time) to the Schrödinger picture of quantum mechanics. Aside from getting the proper structures into play, this approach has the added advantage of leading directly and without further ado to the expression for the quantum Hamiltonian optimally suited for later analysis. There exists one delicate aspect however: the topological term needs to be handled with care. To avoid dealing with too many issues at once, we here set \( \sigma_{xy} = 0 \) and postpone the inclusion of the topological term to Sect. 3.5.

As is usual in statistical mechanics, we define the quantum Hamiltonian \( H \) as (minus) the logarithm of the transfer matrix \( T \) connecting two neighboring rows of lattice sites, both at constant time. (The first of the two directions on \( \mathcal{M} = I \times S^1 \) is decreed to be “space” \( I = [R_1, R_2] \), the other “time” or, rather, imaginary time.) In order for \( H = -\ln T \) to come out to be a local and tractable operator, we need to make the system anisotropic by lowering the coupling, or “temperature”, in time direction while strengthening it in space direction. Such an ad hoc modification of the system is permissible if we are concerned (as we are) with the study of critical or low-energy properties only, since spatial anisotropy is an irrelevant perturbation at a critical point. Imagining the anisotropy to be caused by our use of two different lattice constants – a small one, \( a_0 \), in time direction and a larger one, \( a_1 \), in space direction – we make the replacement

\[
\int_{I \times S^1} dx d\tau \ L_0(g, \partial g) \longrightarrow \sum_n \left( \frac{a_1}{a_0} \Delta(g_n, g_{n+e_0}) + \frac{a_0}{a_1} \Delta(g_n, g_{n+e_1}) \right)
\]

with the “distance” \( \Delta(g, h) \) between \( g \) and \( h \) defined by \( \Delta(g, h) = \text{STr} g^{-1} h \Lambda h^{-1} g \Lambda / 4 \). The sum runs over all lattice points \( n \), and \( e_0 \) and \( e_1 \) are the lattice vectors in time and space direction. Periodic boundary conditions \( g_{n+N_1 e_0} = g_n \) are assumed. Our discretization scheme has the right naive continuum limit, as is easily seen from \( \Delta(g, ge^{aX}) = a^2 \mathcal{B}(X_P, X_P) \mod a^3 \). Note also that, since \( \Delta(hg_1, hg_2) = \Delta(g_1, g_2) = \Delta(g_1 k_1, g_2 k_2) \) for \( h \in \mathbf{G} \) and \( k_1, k_2 \in \mathbf{K} \), the replacement (16) preserves the global and local symmetries of the continuum theory.

By its origin from \( \mathcal{B} ((g^{-1} \partial g)_P, (g^{-1} \partial g)_P) \) with \( \partial \) differentiating in time (space) direction, the summand with \( a_0 \) in the denominator (numerator) is naturally identified as a “kinetic energy” (“potential energy”). We now wish to extract the logarithm of the transfer matrix in the Hamiltonian limit \( a_0/a_1 \rightarrow 0 \). To do so, we must first perform a standard transformation, turning kinetic energy into a second-order differential operator while inverting the coefficient \( a_1/a_0 \). A crucial simplification gained from making the system very anisotropic is that we can ignore the potential energy terms while doing this transformation. (Both potential energy and the differential operator that will eventually represent kinetic energy carry a factor of \( a_0 \), so their commutator is of negligible order \( a_0^2 \).)

Thus, concentrating on a single site \( n \), setting \( g_n = g \) and \( g_{n+e_0} = h \), and defining \( \epsilon \) by \( a_0/a_1 = \sigma_{xx} \epsilon \), what we need to do is to work out the integral

\[
(\tau f)(\pi(h)) := \int_{\mathbf{G}/\mathbf{K}} dg_K \ e^{\frac{1}{\epsilon} \Delta(h, g)} f(\pi(g)),
\]

18
which transfers some function \( f(\pi(g_n)) \) to the neighboring site \( n + e_0 \) in time direction. This is prepared by the following little calculation:

\[
(\tau f)(\pi(h)) = \int_{G/K} dg_K \ e^{-\Delta(1,g)/\epsilon} f(\pi(hg)) = \int_{\mathcal{P}} dX \ e^{-B(X,X)/\epsilon} f(\pi(he^X)).
\]  

(18)

The first equality sign follows from the invariance of \( dg_K \) under the substitution \( g \mapsto hg \), and from the symmetry property \( \Delta(h,hg) = \Delta(1,g) \) with 1 being the unit element in \( G \). For the second equality sign we set \( g = e^X \), parametrizing \( G/K \) by its tangent space \( \mathcal{P} \) at the origin \( \pi(1) \). In the small-\( \epsilon \) limit under consideration, the integral receives contributions only from the immediate vicinity of \( \pi(1) \). We may therefore replace \( \Delta(1,e^X) = \text{STr} e^X \Lambda e^{-X} \Lambda/4 = \text{STr} e^{2X}/4 = \text{STr} \cosh(2X)/4 \) by \( B(X,X) = \text{STr} X^2/2 \) and \( dg_K \) by \( dX \), the Euclidean integration measure on \( \mathcal{P} \).

It remains to convert the last integral in (18) into a differential operator acting on \( f \). To that end, we express \( X \) in some basis \( \{e_i\}_{i=1,\ldots,\text{dim} \mathcal{P}} \) as

\[
X = \sum x^i e_i,
\]

with \( g_{ij} = B(e_i,e_j) \). By Taylor expanding \( f \) and doing the resulting Gaussian integral, we easily get

\[
(\tau f)(\pi(h)) = (f + 4\epsilon L f)(\pi(h)) \mod \epsilon^2
\]

(20)

where \( L \) is the second-order differential operator defined by

\[
(Lf)(\pi(g)) = \lambda(\partial/\partial x)f(\pi(g \exp \sum x^i e_i)) \bigg|_{x=0} \quad \text{with} \quad \lambda(\partial/\partial x) = \sum_{ij} g_{ij} \frac{\partial^2}{\partial x^j \partial x^i}.
\]  

(21)

(The indices of the metric tensor are raised by \( \sum_j g^{ij} g_{jl} = \delta^i_j \) as usual.) Note that, in view of its derivation from a manifestly gauge-invariant expression, \( L \) is independent of the choice of gauge and thus makes sense as an operator acting on functions on \( G/K \). It is not difficult to see directly from (21) how this gauge invariance comes about. A change of representative \( g \mapsto gk (k \in K) \) amounts to the same as a coordinate transformation \( \sum x^i e_i \mapsto \sum y^i e_i = k(\sum x^i e_i)k^{-1} \). Such a transformation leaves \( B(X,X) \), Eq. (19), invariant and the process of raising the indices of the metric tensor confers this invariance upon \( \lambda \).

We can now read off the quantum Hamiltonian \( H \) from Eqs. (13), (16), (17) and (20):

\[
a_0 H := -\ln T = \frac{a_0}{a_1} \sum_{l=1}^{N_1} \left( -\frac{4}{\sigma_{xx}} \mathcal{L}_l + \sigma_{xx} \Delta(g_l,g_{l+1}) \right) \mod (a_0/a_1)^2.
\]  

(22)

The first term is site-diagonal, and for a single site is defined by the formula (21). The second term couples neighboring sites and acts on “wave functions” \( f(\pi(g_1),\ldots,\pi(g_{N_1})) \) by ordinary multiplication.

We should like to point out that the operator \( \mathcal{L} \) is nothing but the Laplace operator \( \nabla^2 = \text{sdiv grad} \) (gradient followed by superdivergence) for \( G/K \). To see this we note that,
since a left translation \( L_g : \pi(h) \mapsto \pi(gh) \) acts on the left while \( L \) acts on the right, \( L_g \) and \( L \) commute. Now, by a basic result of the theory of homogeneous spaces \( G/K \), any second-order differential operator with the property of commuting with left translations is a multiple of the Laplace operator. Hence \( L = c \nabla^2 \) \((c \in \mathbb{R})\). The constant of proportionality \( c \) equals unity since \( \lambda \) is the correctly normalized differential operator associated with the quadratic form defining the metric. Choosing some system of super-coordinates \( \xi^i \) in which the metric \( B \left( (g^{-1}dg)_P, (g^{-1}dg)_P \right) \) is expressed by \( \sum d\xi^i g_{ij}^\xi(\xi)d\xi^j \), we have for \( \nabla^2 \) the familiar coordinate expression

\[
\nabla^2 = J(\xi)^{-1/2} \sum_{ij} (-)^{|i|} \frac{\partial}{\partial \xi^i} g^{ij}(\xi) J(\xi)^{1/2} \frac{\partial}{\partial \xi^j}
\]

(23)

where \( J(\xi) = \text{SDet}(g_{ij}(\xi)) \) and \( \text{SDet} \) is the superdeterminant. (The sign factor involving the superparity \( |i| \) of \( \xi^i \) is due to supersymmetry.) The coordinate expression (23) illustrates the meaning of the less familiar formula (21). For practical calculations, it is much less useful than (21) and we will actually never use it.

What can we say about the Hamiltonian \( H \), Eq. (22)? The normalization of the functional integral \( Z = 1 \) translates into trace \( \exp(-\beta H) = 1 \) for all \( \beta, \sigma_{xx} \) and \( N_1 \). We therefore expect \( H \) to have a unique singlet ground state with zero energy and excited states organized into supermultiplets consisting of an equal number of bosonic and fermionic states, whose contributions to the partition sum exactly cancel each other. Apart from these special features due to supersymmetry, we expect the physics of \( H \) to be similar, at least qualitatively, to that of the quantum Hamiltonian of a conventional (i.e. non-super) nonlinear \( \sigma \) model such as the \( \text{O}(3) \) one. The nature of the ground state will be determined by the battle between kinetic energy and interaction energy (formerly called potential energy). The first of the two makes the field diffuse on \( G/K \), thereby causing strong fluctuations in the ground state when \( \sigma_{xx} \) is small. The interaction energy, in contrast, has the opposite tendency of aligning neighboring field variables. For a space dimension larger than the lower critical dimension of 2, either one of the two tendencies may win. If \( \sigma_{xx} \) is small enough, the quantum fluctuations caused by \( \sum L_l \) disorder the ground state. As \( \sigma_{xx} \) increases beyond some critical value \( \sigma_{xx}^* \), the interaction energy takes over, driving a phase transition to an ordered ground state with spontaneously broken symmetry. This phase transition is the Anderson (metal-insulator) transition in noninteracting disordered electron systems with time reversal invariance broken by magnetic impurities or a weak magnetic field. In \( d = 1 + 1 \), the case at hand, renormalization group analysis of the continuum theory (13) leads us to expect that quantum fluctuations always disorder the ground state on sufficiently large length scales, no matter how big is the value of \( \sigma_{xx} \). In other words, the Hamiltonian (22) is expected to have a nondegenerate and disordered ground state with massive excitations for any \( \sigma_{xx} \). This scenario must be changed drastically by the inclusion of the topological term.

3.4 Topological density and Berry phase

In Sect. 3.3, we put the Euclidean field theory (13) for \( \sigma_{xy} = 0 \) on a rectangular lattice with lattice constants \( a_0 \) and \( a_1 \), and then took the extreme anisotropic limit \( a_0/a_1 \to 0 \) to derive the expression (22) for the quantum Hamiltonian \( H \). Our present goal is to extend
this procedure to the case of half-integral $\sigma_{xy}$. We can imagine several ways of discretizing the topological density. Recalling the $Q$-field expression $L_{\text{top}} = \epsilon^{\mu\nu} \text{STr} \partial_{\mu} Q \partial_{\nu} Q / 8i$ we might be led to the scheme

$$\int_M d^2 x \; \epsilon^{\mu\nu} \text{STr} \partial_{\mu} Q \partial_{\nu} Q \rightarrow \sum_n \text{STr} \, Q_n \left( Q_n + e_0 - Q_n \right) \left( Q_n + e_1 - Q_n \right).$$  \hspace{1cm} (24)$$

Note the conspicuous absence of the lattice constants $a_0$ and $a_1$, indicating the topological (i.e. metric free) nature of the term. Although the naive choice (24) pays no attention to the geometric meaning of the topological density, it should work just fine when field configurations are slowly varying. For our purposes, this would actually be good enough since we will be satisfied with a quantum Hamiltonian that gives the correct low-energy physics near criticality. However, since triples of field variables $Q$ separated by one unit in both space and time direction are coupled together, the discretized topological density (24) is hard to make any analytic progress with. Fortunately, a discretization scheme much better than (24) exists. To formulate this scheme and understand its true meaning, we have to work through some differential geometry. The crucial step will be to take a curve $S^1 \rightarrow G/K, \, t \mapsto \gamma(t)$ and lift it to a curve $t \mapsto g(t)$ on $G$. In this way we will be able to relate the topological density to the geometric phase of Berry [42]. Discretization and the Hamiltonian limit will then be almost immediate.

Before delving into our differential geometric project, let me inject a few words of explanation concerning the use of terminology. What we are dealing with is a supermanifold (more precisely, a supermanifold of the Berezin-Kostant type [43]). Such an object is not made from points – anticommuting c-numbers, unlike real numbers, can’t be considered as points on a line – so the reader may wonder just what is the meaning of the word “curve” in the present context?! The answer is this. When constructing an ordinary (i.e. nonsuper) differential manifold one has two options: one may describe the manifold either by its points (i.e. as a topological space with a differentiable structure) or by its functions. Since a function $f$ can be applied to a point $x$ to produce the number $f(x)$, functions are dual to the elements of the space they are functions on. Because of this relationship, an ordinary manifold can be reconstructed from the knowledge of its functions. In the super case the point option is nonexistent, which is of course why Berezin [43] defines a supermanifold by its (super-)functions. This is not as horrible as it may sound, as the vast space of functions – on a supermanifold just as on an ordinary manifold – can be generated from a basic set of coordinate functions. The message following from this is that, on a pedantic level, we ought to be using coordinate language throughout. For example, when we talk about a “curve” on a supermanifold, what we should really do is to describe the object “curve” in terms of some set of superfunctions $\xi^i(t)$ depending on a parameter $t$ and taking values in the underlying parameter Grassmann algebra. (In the ordinary case, the $\xi^i(t)$ would result from evaluation of a set of coordinate functions $x^i : G/K \rightarrow \mathbb{R}$ along the curve $t \mapsto \gamma(t).$) Clearly, our notation will be more transparent, and our general train of thought much easier to follow, if we say “curve” rather than “a set of functions depending on a parameter and taking values in ...” This is why we shall stick to point manifold language for the most part, keeping however in mind that everything we do should be expressible (and ultimately must be expressed) in terms of supercoordinates.
After this preamble, we consider for some smooth field configuration the integral of the
topological density \( L_{\text{top}} \) over some connected and simply connected region \( \Gamma \) of space-time \( \mathcal{M} \):

\[
\varphi_\Gamma = \int_\Gamma d^2 x \, L_{\text{top}}.
\]  

(25)

We wish to elucidate the meaning of \( \varphi_\Gamma \) as the geometric phase acquired by a "G/K-
superparticle" when transported once around the curve \( \partial \Gamma \rightarrow \pi_0(g(\partial \Gamma)) \). We begin by
writing \( L_{\text{top}} \), Eq. (12), in the form

\[
L_{\text{top}}(g, \partial g) = \frac{i}{2} \epsilon^\mu\nu \text{STr} \Lambda(g^{-1}\partial_\mu g)_\nu(g^{-1}\partial_\nu g)_\mu.
\]

Our first step will be to convert the integral over \( \Gamma \) into a line integral along the boundary
\( \partial \Gamma \). Since \( L_{\text{top}} \) is induced by the two-form

\[
\omega = \frac{i}{2} \text{STr} \Lambda(g^{-1}dg)_\nu \wedge (g^{-1}dg)_\mu,
\]

(26)

we will be able to apply Stokes' theorem if we can find a one-form \( A \) with exterior derivative
d\( A = \omega \). We claim that if \( \pi_+ = (1 + \Lambda)/2 \) and \( s \) is any section \( G/K \rightarrow G_0 \), the choice

\[
A = -i \text{STr} \pi_+ s^{-1}ds
\]

(27)
does the job. The argument is as follows. Because \( s^{-1}ds \) takes values in \( \text{Lie}(G_0) = \text{su}(n,n|2n) \), the supertrace \( \text{STr} \, s^{-1}ds \) vanishes and we may replace the expression for \( A \) by

\[
A = \text{STr} \Lambda s^{-1}ds/2i.
\]

Differentiating the relation \( g = s(\pi(g))k(g) \) we get \( s^{-1}ds = k(g^{-1}dg - k^{-1}dk)k^{-1} \), which we insert into
d\( A = i \text{STr} \Lambda(s^{-1}ds) \wedge (s^{-1}ds)/2 \). Evaluation of the two-
form \( dA \) on a pair of tangent vectors \( sX \) and \( sY \) gives \( i \text{STr} \Lambda[X,Y]/2 = i \text{STr} \nu[A,X]/2 \). This vanishes for \( X \in \mathcal{K} \) and \( Y \in \mathcal{K} + \mathcal{P} \), so only the \( \mathcal{P} \)-component of \( s^{-1}ds \) need be
kept. Hence, \( dA = i \text{STr} \Lambda(s^{-1}ds)_\nu \wedge (s^{-1}ds)_\mu/2 \), and since \( (s^{-1}ds)_\mu = k(g^{-1}dg)_\mu k^{-1} \)
and \( k^{-1}\Lambda k = \Lambda \), comparison with (26) shows that \( dA = \omega \), as claimed.

Naive application of Stokes’ theorem would now yield \( \varphi_\Gamma = -i \int_0^1 dt \text{STr} \pi_+ s^{-1}s \),
where \( s \) is the total derivative of \( s \) along the curve \( \partial \Gamma \), which we parametrize by \( t \in [0,1] \).
This, however, is not quite correct. There exists a difficulty of topological origin. For a fixed choice of gauge \( s \), the gauge potential \( -i \text{STr} \pi_+ s^{-1}s \) induced from \( A \) by the field
configuration along the loop \( \partial \Gamma \), may not have a smooth extension to the interior of the loop.
This is because in the presence of topological excitations the field may wind once –
or several times – around a Dirac-string type singularity of \( s \) as we move along \( \partial \Gamma \). Correct
application of Stokes’ theorem would then require subtraction of contributions from the
topological singularity. Fortunately, there exists a way to get around this problem, which
is to use a gauge transformation to move the string singularities somewhere else on the
manifold, far from where the fields are. This works locally, i.e. for a small loop \( \Gamma \). In
general we are forced to patch together several coordinate charts in each of which \( s \) is well-
defined. The transition functions for going from one chart to another are given by gauge
transformations. Thus, we are led to consider what happens if we make a \( t \)-dependent
gauge transformation \( s(\gamma(t)) \rightarrow s(\gamma(t))u(t) \). Such a transformation changes the expression
for \( \varphi_\Gamma \) by the addition of the integral \(-i \int_0^1 dt \ln \det A(t)\). This integral is quantized in integer multiples of \(2\pi\), by the following argument. In a matrix notation where \( \Lambda = 1_{BF} \otimes (\sigma_3)_{AR} \otimes 1_E \) is represented by \(
abla_{AB} = (1_{2n} \otimes 0) \) and \( u(t) \) by \( A(t) \), we have \( \ln \det A(t) = \ln \det \). The curve \( t \mapsto \gamma(t) \) traced out on \( G/K \) by the \( \sigma \) model field along \( \partial \Gamma \) is closed, so \( u(1) = u(0) \) and \( A(1) = A(0) \). Integrating the total derivative we get

\[
-i \int_0^1 dt \frac{d}{dt} \ln \det A(t) = i \ln \det A(0) - i \ln \det A(1),
\]

which proves the statement by the single-valuedness of the function \( U(n|n) \to U(1), A \to \det A \). We conclude that the correct formula for \( \varphi_\Gamma \) is

\[
\varphi_\Gamma = -i \int_0^1 dt \ln \det A(t) s^{-1} \dot{s} + 2\pi q, \quad (28)
\]

with \( q \) some integer. The gauge-ambiguous integral \( \oint dt \ln \det A(t) \) is a one-dimensional version of what is called a Wess-Zumino term in field theory.

In the next step we perform one further manipulation, which will eventually allow us to completely integrate the expression for \( \varphi_\Gamma \). (This is not yet possible in the present form since \( s \) and \( \Lambda \) do not commute.) Given the curve \( t \mapsto \gamma(t) \) on the base space \( G/K \), we define the lifted curve \( t \mapsto g(t) \) on the total space \( G \) by the equations

\[
\begin{align*}
(A) \quad & \pi(g(t)) = \gamma(t), \\
(B) \quad & (g(t)^{-1} \dot{g}(t))_{\mathcal{K}} = 0.
\end{align*}
\]

Eq. (B) has the following geometric interpretation. The time derivative \( \dot{g}(t) \) is tangent to \( G \) at \( g(t) \). The left translated vector \( g(t)^{-1} \dot{g}(t) \) lies in the tangent space of \( G \) at the group unit. It can therefore be regarded as an element of \( G = \text{Lie}(G) \). Now recall the orthogonal decomposition \( G = \mathcal{K} + \mathcal{P} \) where \( \mathcal{K} = \text{Lie}(K) \). \( \dot{g}(t) \) is called vertical if \( g(t)^{-1} \dot{g}(t) \) lies in \( \mathcal{K} \). (\( \dot{g}(t) \) is then tangent to a fibre of the \( K \)-bundle \( G \to G/K \).) In contrast, Eq. (B) requires \( g(t)^{-1} \dot{g}(t) \) to be orthogonal to \( \mathcal{K} \) for all \( t \). This is expressed by saying that a solution of Eq. (B) is a curve on \( G \) which is everywhere horizontal (Figure 2).

To solve Eqs. (A) and (B), we use the ansatz \( g(t) = s(\gamma(t)) u(t) \) with unknown \( u(t) = k(g(t)) \in K \). (A) is then satisfied automatically while (B) turns into

\[
(B') \quad \dot{u}(t)u(t)^{-1} = -(s^{-1} \dot{s})(\gamma(t))_{\mathcal{K}},
\]

which is a first-order differential equation determining \( u(t) \) uniquely once \( u(0) \) has been specified. Using \( (B') \) in Eq. (28), we can do the integral for \( \varphi_\Gamma \):

\[
\varphi_\Gamma = i \int_0^1 dt \ln \det A(t) s^{-1} \dot{s} + 2\pi q = i \ln \det A(0) - i \ln \det A(1) + 2\pi q \quad (29)
\]
since $\pi_+$ and $u(t) \in K$ commute.

We conclude this subsection with a more “global” view of what we have done. Given a smooth field configuration and some value for $\sigma_{xy}$, the topological density assigns to a region $\Gamma \subset M$ the topological phase $p_{\Gamma} = \exp(-i \sigma_{xy} \int_{\Gamma} d^2x L_{\text{top}})$. The field configuration maps the boundary of $\Gamma$ onto a closed curve on $G/K$ which we parametrize by $t \in [0, 1] \mapsto \gamma(t)$. This curve is lifted to a curve $\partial \Gamma \to G$ (not closed in general) by solving $(g^{-1}\dot{g})_K = 0$. Making some choice of gauge $k(g) = s(\pi(g))^{-1}g$ (more precisely, patching together local charts) we get $p_{\Gamma} = \exp(\sigma_{xy} \text{Str} \pi_+ \ln k(g(0))^{-1}k(g(1)))$. This way of writing $p_\Gamma$ identifies it as a Berry phase. The expression for $p_\Gamma$ is gauge-invariant if $\sigma_{xy} = m \in \mathbb{Z}$, in which case we can write

$$p_{\Gamma} = \mu(k(g(0))^{-1}k(g(1)))$$

(30)

where

$$\mu(k) = \exp(m \text{Str} \pi_+ \ln k)$$

(31)

is globally well-defined as a representation $\mu : K \to U(1)$. By the analogy with the Dirac monopole problem that results on setting $G/K = SU(2)/U(1)$, we call the integer $m$ the monopole charge. The calculation we have done in this subsection is an application of the beautiful theory [44] of connections on the hermitean line bundles that are associated with a principal fibre bundle. We refrain from elaborating on this wider context here.

### 3.5 Quantum Hamiltonian for $\sigma_{xy} \neq 0$

The functional integral (13) carries the topological phase factor $p_M = \exp(-i \sigma_{xy} \int_M d^2x L_{\text{top}})$. As was explained in the preceding subsection, we can use Stokes’ theorem to convert $\int_M d^2x L_{\text{top}}$ into a gauge-ambiguous line integral along the boundary of $M$. The gauge ambiguity is unobservable if and only if $\sigma_{xy}$ is an integer. We will derive the quantum Hamiltonian of the $\sigma$ model for this special case first. The generalization to the more interesting case of half-integral $\sigma_{xy}$ will be presented afterwards.

Let $\sigma_{xy} = m \in \mathbb{Z}$. For a configuration space $M$ without boundary, Eq. (28) leads to $p_M = \exp(-im\varphi(M)) = 1$, which is just another way of stating the quantization of the topological term. Hence, the quantum Hamiltonian $H$ of the $\sigma$ model is given by Eq. (22), just as for the case of vanishing $\sigma_{xy}$. As was said earlier, $H$ is expected to have a disordered ground state with an excitation gap, corresponding to localization of all states. A less dull situation arises when $M$ does have a boundary $\partial M \neq 0$. For the Corbino disk geometry, $\partial M$ consists of two pieces, an inner and an outer edge, both of which are circles $\sim S^1$ with radii $R_1$ and $R_2$ respectively. The orientations they inherit from $M$ are opposite to each other. Using Eq. (28) we get a sum of two Wess-Zumino terms, one for each of the two edges:

$$\int_M d^2x L_{\text{top}} = -i \int d\tau \text{Str} \pi_+ (s_N^{-1} \partial_\tau s_N - s_1^{-1} \partial_\tau s_1) \mod 2\pi \mathbb{Z}$$

(32)

where $s_1 (s_N)$ is $s$ evaluated at $x = R_1$ ($x = R_2$). The appearance of these terms changes the $\sigma$ model dynamics at the edges only. More precisely, the time evolution of the edge degrees of freedom is governed no longer by the free-particle Hamiltonian $-\mathcal{L}$ but by another operator which we are now going to construct. As before, we discretize the field theory (13) on a rectangular lattice with lattice vectors $e_0, e_1$ and lattice constants $a_0, a_1$ in time and
space direction, respectively. With all the preparations that were made in Sect. 3.3, we can focus attention on the kinetic edge and take its orientation to follow the positive direction of time. Visiting a set of lattice sites \( n \rightarrow n + e_0 \rightarrow n + 2e_0 \rightarrow \ldots \) we encounter a sequence of field variables which we parametrize by \( \pi(g_n) \rightarrow \pi(g_ne^{X_1}) \rightarrow \pi(g_ne^{X_1}e^{X_2}) \rightarrow \ldots \) \((X_i \in \mathcal{P})\). In the time-continuum limit \( a_0 \rightarrow 0 \) this discrete sequence condenses into a continuous curve \( S^1 \rightarrow G/K \) which is closed by periodic boundary conditions for time. To get a discrete approximation to the line integral (29), we need to lift the discrete sequence of field variables to the total space \( G \). Since \( t \rightarrow g(t) = g(0)e^{tX} \) satisfies \( (g^{-1})_K = 0 \) for \( X \in \mathcal{P} \), the lifted sequence is \( g_n \rightarrow g_ne^{X_1} \rightarrow g_ne^{X_1}e^{X_2} \) etc. By formula (30), the total topological phase factor accumulated along the outer edge is \( \mu(k(g_n)^{-1}k(g_ne^{X_1})) \) (time-ordered product). Using the representation property \( \mu(k_0^{-1}k_2) = \mu(k_0^{-1}k_1)\mu(k_1^{-1}k_2) \) we see that the kinetic plus topological part of the transfer matrix \( \tau \) for a single site and a single step in negative time direction is given by

\[
(\tau f)(\pi(g)) = \int \mathcal{P} dX \ e^{-B(X,X)/\epsilon} \mu(k(ge^X)) f(\pi(ge^X)),
\]

compare Eq. (13). Reversal of the time direction is equivalent to the substitution \( \mu \rightarrow \mu^{-1} \). Taking the time-continuum limit \( \epsilon \sim a_0 \rightarrow 0 \) we obtain for the \( \sigma \) model degrees of freedom at the outer edge the single-site Hamiltonian \(-4L^m/a_1\sigma_{xx} \) where \( L^m \) is the second-order differential operator defined by

\[
(L^m f)(\pi(g)) = \lambda(\partial/\partial x)\mu(k(g)k(\exp \sum x^ie_i)^{-1}) f(\pi(\exp \sum x^ie_i)) \bigg|_{x=0}.
\]

By setting \( \mu = \exp m \text{STr} \pi_+ \ln 1 = 1 \) we retrieve Eq. (21), defining the Laplace operator \( L \). In view of the relation between \( \mu \) and the gauge connection \( A \), Eq. (27), it should not be surprising that \( L^m \) can be shown to be identical to the operator \((\nabla - imA)^2 \) (symbolic notation). (More precisely, if \( A = \sum A_j(\xi)d\xi^j \), \( L^m \) is the operator that results from (23) by making the replacement \( \partial/\partial \xi^j \rightarrow \partial/\partial \xi^j - imA_j(\xi) \). We will never use this equivalence.) Eq. (33) expresses \( L^m \) through derivatives acting on the right, but for some purposes it is better to switch to derivatives acting on the left. This is done in Appendix A, where we derive the alternative formula

\[
(L^m f)(\pi(g)) = \lambda_G(\partial/\partial x)\mu(k(g)k(e^{\Sigma x^ie_i)}g)^{-1}) f(\pi(e^{\Sigma x^ie_i}g)) \bigg|_{x=0}.
\]

Here \( \lambda_G(\partial/\partial x) = \sum_{i,j=1}^{\dim G} g^{ij}\partial^2/\partial x^j\partial x^i \) is defined by inverting the quadratic form \( B \) on the Lie algebra \( G \) with basis \( \{e_i\} \). We will see in Sect. 4.6 that the single-site Hamiltonian \(-4L^m/a_1\sigma_{xx} \) describes uni-directional transport through a number \(|m| \) of edge channels.

To get the site-diagonal part of the Hamiltonian for the degrees of freedom at the inner edge of the Corbino disk, we simply reverse the sign of the monopole charge \( m \). (Recall that there is a relative minus sign in Eq. (32) from opposite orientation of the two edges.) We will construct the ground states of \(-L^m \) for all \( m \) in Sect. 4.

Let us consider half-integral values of \( \sigma_{xy} \) next. In this case, the gauge ambiguity of the Wess-Zumino integral (28) prevents us from wiping the interior of \( \mathcal{M} \) clean of topological...
phase. Decomposing the topological density $\sigma_{xy} L_{\text{top}}$ into its integer and fractional parts, we can convert the former part into a boundary term in the same fashion as before, but the latter part cannot be removed. We must treat it differently. This we do by modifying the continuum field theory (13) as follows. We organize the space-time manifold $\mathcal{M}$ into a set of equidistant strips. The strips have a width of one lattice spacing $a_1$ and are parallel to the time direction. Every other strip is shaded, see Figure 3. We set the fractional part (= 1/2) of the topological coupling $\sigma_{xy}$ to zero in the empty regions while raising it to twice its original value in the shaded ones. We are allowed to make such a drastic modification since, once again, all we are concerned with are the universal low-energy properties. These are determined by slowly varying fields, for which the staggered value of the modified topological coupling balances out. Put in formulas, we replace $0.5 \int_{\mathcal{M}} d^2 x \ L_{\text{top}}$ in (13) by $\int_{\mathcal{M}_1} d^2 x \ L_{\text{top}}$ where $\mathcal{M}_1$ is the union of the shaded regions. By using Stokes’ theorem for $\mathcal{M}_1$ we get a sum of line integrals:

$$\frac{1}{2} \int_{\mathcal{M}} d^2 x \ L_{\text{top}} \rightarrow -i \sum_l (-)^l \oint d\tau \text{STr} \pi_+ s_{l}^{-1} \partial_\tau s_l . \quad (35)$$

The sign of the line integrals alternates because the orientation of the loops forming the boundary of $\mathcal{M}_1$ does.

The sceptical reader might have the impression that by modifying the topological coupling we are making a novel and unnecessary approximation. This impression is wrong. A close look at the literature (see [45] for a very readable account) reveals that the same approximation is always made in this context and, as a matter of fact, there is no way of avoiding it. Our presentation is simply inverted compared to most published ones, which start from the sum of line integrals (35) (derived by using coherent states on an antiferromagnetic quantum spin chain) and end up with the topological term of the $\sigma$ model. The argument usually given is the following. One takes the alternating sum (35) and breaks translational invariance by organizing the line integrals into groups of two:

$$i \sum_l a_1 \oint d\tau \text{STr} \Lambda[s^{-1}_l \partial_\tau s, s^{-1}_l \partial_x s]_{2l} .$$

By Taylor expanding up to linear order in the lattice constant $a_1$, one gets $i \sum_l a_1 \oint d\tau \text{STr} \pi_+(s^{-1}_{2l+1} \partial_\tau s + s^{-1}_{2l} \partial_\tau s_{2l+1} - s^{-1}_{2l} \partial_x s_{2l})$. Because periodic boundary conditions in time direction are used, this expression is not changed by subtraction of a term with $\partial_x \leftrightarrow \partial_\tau$, whence it becomes

$$i \sum_l a_1 \oint d\tau \text{STr} \Lambda[s^{-1}_l \partial_\tau s, s^{-1}_l \partial_x s]_{2l} .$$

The expression $i \text{STr} \Lambda[s^{-1}_l \partial_\tau s, s^{-1}_l \partial_x s]$ is recognized as twice the topological density $L_{\text{top}}$. Note in particular that the sum runs over even lattice sites only (i.e. the topological coupling is staggered). By replacing $\sum_{\text{even } l} \rightarrow \sum_{\text{all } l/2}$ and taking the naive continuum limit, one recovers the topological term $\int_{\mathcal{M}} dx \partial_\tau \ L_{\text{top}}/2$.

We are finally ready to write down the quantum Hamiltonian $H$ for $\sigma_{xy} \in \mathbb{Z} + 1/2$. Putting everything together, we get the formula

$$a_1 H = -\frac{4}{\sigma_{xx}} \left( \sum_{\text{even } l} \mathcal{L}_l^+ + \sum_{\text{odd } l} \mathcal{L}_l^- \right) + \sigma_{xx} \sum_l \Delta(g_l, g_{l+1}) . \quad (36)$$
where $\mathcal{L}^\pm := \mathcal{L}^{\pm 1}$ is defined by Eq. (33). This expression for $H$ replaces the formula (22) and is correct modulo terms of higher order in the small parameter $a_0/a_1$. The sign of the monopole charge $m = \pm 1$ alternates on even and odd sites because the sign multiplying the line integrals (35) does. Eq. (36) applies to case of a two-torus $\mathcal{M} = S^1 \times S^1$. For the disk geometry it must be modified at the edges to account for the integer part of $\sigma_{xy}$ in the manner described above.

So far, we have explained how to pass to the Hamiltonian formulation of the $\sigma$ model when $\sigma_{xy} \in \mathbb{Z} + 1/2$. This is not yet entirely satisfactory since, in order to compute critical indices such as the correlation length exponent $\nu$, we may wish to study the $\sigma$ model slightly off these points. We can do so as follows [18]. Suppose $\sigma_{xy} = (1 - \epsilon)/2$ with $|\epsilon| \ll 1$. For definiteness, let $\epsilon > 0$. If we choose $\mathcal{M}_1$, the union of the shaded regions in Figure 3, to satisfy the condition area($\mathcal{M}_1$) = area($\mathcal{M}$)/2 as before, the alternating sum of line integrals (35) will produce an excess topological phase. We can eliminate the excess by making area($\mathcal{M}_1$) smaller by a factor of $1 - \epsilon$. This modification amounts to choosing an alternating sequence of lattice constants $(1 - \epsilon)a_1$ (width of the shaded regions) and $(1 + \epsilon)a_1$ (width of the empty regions). It is clear that the alternating spatial distance between sites causes a staggering of the site-site interaction. Thus, to perturb the $\sigma$ model off the point $\sigma_{xy}^* = 1/2$, we may simply add to the quantum Hamiltonian (36) an alternating term $\tau \sum_l (-1)^l \Delta(g_l, g_{l+1})$ where $\tau \sim \sigma_{xy} - \sigma_{xy}^*$.

## 4 Strong-Coupling Ground States

Low-energy properties of quantum chromodynamics in $3 + 1$ dimensions, such as confinement of quarks and gluons, are beyond the reach of weak-coupling perturbation theory. In the late seventies, it was hoped that the incorporation of topological excitations such as instantons would lead to progress. These hopes, however, were frustrated by severe infrared divergences which make it difficult if not impossible to formulate a sensible expansion around the dilute instanton gas limit. On the other hand, confinement occurs very naturally in lattice QCD, by the area law for Wilson loops in the strong-coupling limit. The situation is similar for the field theory of the integer quantum Hall effect.

Although weak-coupling perturbation theory augmented by instanton calculus [11,12] has yielded some qualitative insights, it has not led to any conclusive or quantitative statements about low-energy (or large-scale) phenomena such as quantization of the Hall conductance and the delocalization transition between Hall plateaus. In this case, too, an approach starting from the opposite limit of strong coupling looks more promising. Diagrammatic perturbation theory predicts that the renormalization group flow takes $\sigma_{xx}$ to small values. Therefore, an expansion around the small-$\sigma_{xx}$ (or strong-coupling) limit is more likely to capture the essential long wave length physics.

The quantum Hamiltonian $H$, Eq. (36), is a sum of two terms: $a_1 H = 4 H_0/\sigma_{xx} + \sigma_{xx} H_1$. For $\sigma_{xx} \rightarrow 0$, the term $H_0 = -\sum_l (\mathcal{L}_{2l} + \mathcal{L}_{2l+1}^-)$ dominates and $H_1 = \sum_l \Delta(g_l, g_{l+1})$ can be treated as a perturbation. The first step towards making a strong-coupling expansion is to diagonalize $H_0$. Since $H_0$ does not couple fields on neighboring lattice sites, diagonalization of $H_0$ reduces to a one-site problem. We shall denote the one-site Hamiltonian by $H_s = -\mathcal{L}^m = -(\nabla - A)^2$, concealing the dependence on the monopole charge $m = \pm 1$ but making the dependence on the choice of gauge $s$ by $A = m \text{STr} \pi_\perp s^{-1} ds$.
explicit. The purpose of the current section is to construct the ground state of $\mathcal{H}_s$ for all $m \in \mathbb{Z}$. (Doing the calculation for a general value of $m$ will enable us to discuss edge state transport in the Hall plateau regions in Sect. 4.6.) We will see that this ground state is infinitely degenerate unless $m = 0$ in which case $\mu = \exp m \text{Str} \pi_+ \ln$ is trivial.

States in the degenerate ground state (as in any eigenspace) of $\mathcal{H}_s$ transform according to some representation of the symmetry group $\mathbf{G}$. It turns out that for $m > 0$ ($m < 0$), this representation possesses a lowest (highest) weight. Our strategy will make use of this fact: we will first identify the lowest (highest) weight and then generate the entire space of ground states by symmetry transformations. Part of what we do takes place at the edge of current mathematical understanding. This is why we need to begin with some preparation in Sect. 4.1.

### 4.1 Principal remarks

We begin by recalling some of the obvious structures at hand. The basis for our considerations is the super coset space $\mathbf{G}/\mathbf{K} = \text{U}(n,n|2n)/\text{U}(n|n) \times \text{U}(n|n)$. $\mathbf{G}$ acts on $\mathbf{G}/\mathbf{K}$ by left translation $x = \pi(h) \mapsto g \cdot x = \pi(gh)$. This induces an action of $\mathbf{G}$ on functions on $\mathbf{G}/\mathbf{K}$ by

$$ (L_g f)(x) = f(g^{-1} \cdot x). $$

(37)

(Since the construction of $L_g f$ from $f$ involves Grassmann-analytic continuation of $f$ w.r.t. the Grassmann parameters of $g$, $f$ must have sufficiently good analyticity properties in order for $L_g f$ to be well-defined.) The action (37) has the representation property $L_{gh} = L_g L_h$.

Integration of functions on $\mathbf{G}/\mathbf{K}$ is defined in the sense of Berezin. (Again, since the definition of the Berezin integral involves differentiation w.r.t. the fermionic coordinates, functions must be analytic.) $\mathbf{G}/\mathbf{K}$ carries with it a Berezin integration measure $dx$ which is invariant under left translations. $dx$ is unique within multiplication by a constant and is alternatively denoted by $dg_K$. By the invariance of $dx$,

$$ \int_{\mathbf{G}/\mathbf{K}} f(x) dx = \int_{\mathbf{G}/\mathbf{K}} (L_g f)(x) dx \quad (g \in \mathbf{G}). $$

Given two analytic functions $f_1$ and $f_2$, we define their scalar product $(f_1, f_2)$ by

$$ (f_1, f_2) = \int_{\mathbf{G}/\mathbf{K}} \overline{f_1(x)} f_2(x) dx $$

(38)

whenever the right-hand side exists. The bar denotes complex conjugation. The scalar product (38) is preserved by left translation,

$$ (L_g f_1, L_g f_2) = (f_1, f_2) \quad (g \in \mathbf{G}). $$

We will see later that the operator $\mathcal{H}_s = - (\nabla - A)^2$ is hermitean with respect to it. In this respect, the scalar product (38) is similar to the scalar product on the space of square-integrable functions in standard quantum mechanics. There is, however, one important difference: the scalar product (38) is not positive definite and does not define a Hilbert
space. (See Eq. (69) below.) From a rigorous mathematical point of view, this raises a number of interesting questions: convergence of sequences of functions, completeness of the function space etc. These are the standard questions of functional analysis that physicists never need to worry about because they always deal with Hilbert spaces where these questions have been answered by mathematicians long ago. It seems to me that the proper definition of norm and topology for function spaces on supermanifolds is not well understood at present. (In any case, I don’t understand it very well.)

The loss of positivity is worrysome not just on a rigorous level but may sometimes cause severe problems for the practical user. As an example we mention the difficulties that one encounters when trying to establish a workable theory of Fourier analysis on compact supergroups [46]. (By Fourier analysis we mean expansion in eigenfunctions of the Laplace operators.) In this case the lack of a positive definite invariant hermitean form causes Laplace-Casimir operators to be generically non-diagonalizable! Fortunately, the situation for Riemannian supersymmetric spaces $G/K$ – as opposed to supergroups $G$ – appears to be much better, in spite of the lack of positivity. The first proof of a Fourier expansion theorem, for the simple case of a rank-one supersymmetric space, was published in [47]. The method of [47] was subsequently extended [48] to all of Efetov’s spaces [22], i.e. the supersymmetric spaces $G/K$ that appear in the theory of noninteracting disordered electron systems. It was found that the problem of non-diagonalizability of Laplace operators is absent for these spaces. It turns out that the method invented in [47] can be further generalized to prove a spectral expansion theorem for the operator $\mathcal{H}_s$. (Note that the hermiticity of $\mathcal{H}_s$ implies orthogonality of its eigenspaces w.r.t. the scalar product (38).) In the present paper we are concerned with the strong-coupling limit of the Hamiltonian $H$, Eq. (36). For that, all we need are the ground states of $\mathcal{H}_s$ and its conjugate. We will therefore concentrate on these ground states and relegate the full spectral expansion of $\mathcal{H}_s$ to a future complete account of my work on supersymmetric Fourier analysis.

With superanalytic theory being incompletely developed at present, even the modest goal of constructing the ground state of $\mathcal{H}_s$ is not entirely straightforward. Suppose we have found some eigenfunction of $\mathcal{H}_s$. How do we know it occurs in the spectral expansion? (Note that invariant differential operators on noncompact spaces are bounded neither from above nor from below. They have a large number of eigenfunctions that do not occur in their spectral expansion. Examples of such eigenfunctions are the functions that correspond to the scaling fields of Sect. 2.1.) On a Hilbert space the answer is easy: we simply look to see whether the function is (properly or improperly) normalizable. If it is not, we reject it. For reasons that are not entirely understood, the same principle works in the present case, in spite of the lacking Hilbert space property. Although the proof of the Fourier expansion theorem for Efetov’s spaces makes no reference to the hermitean form (38), it turns out that the eigenfunctions occurring in the Fourier expansion are precisely those with “proper” behavior with respect to (38). (By this I mean that eigenfunctions belonging to a discrete series have finite norm in a certain sense, while those belonging to a continuous series are normalized to a $\delta$-function.) This then is the heuristic principle we adopt: given some eigenfunction of $\mathcal{H}_s$ we shall declare it to be a “physical state” (occurring in the spectral expansion) if it is normalizable w.r.t. (38).
4.2 Induced Representation

We again begin with some motivation. Eq. (37) defines an action of the symmetry group $G$ on functions on $G/K$. This action is not yet the “good” one for our purposes, since it does not commute with the Hamiltonian $H_s = -(\nabla - A)^2$. If this is not obvious, consider the simple example of the motion of a charged particle in the field of a magnetic monopole. Such a system is invariant under rotations that fix the magnetic monopole. However, rotational invariance is not manifest in the Hamiltonian formulation since writing $(\nabla - A)^2$ requires making some choice of $A$, which inevitably breaks spherical symmetry. Consequently, the symmetry transformations of $H_s$ are not just rotations but are rotations followed by a suitable gauge transformation. The same happens in the present case. Thus, the good action of $G$ will combine left translations with gauge transformations.

To construct the good action we proceed as follows. For a fixed value of the monopole charge $m$, let us denote the linear space of functions which $H_s$ acts on by $W_s$. The subscript $s$ does not mean inequivalence of $W_s$ and $W_{s'}$, but is simply a reminder that the elements of $W_s$ are tied to some choice of gauge $A = m \text{STr} \pi_s s^{-1} ds$. When $s$ is changed, the functions change. Thus, $W_s$ is a space of gauge-dependent functions on $G/K$. Since a change of section $s \mapsto sk$ takes $A$ into $A + d(m \text{STr} \pi_s \ln k)$, we expect the behavior under gauge transformations to be governed by the one-dimensional representation $\mu$ of the gauge group $K$ which is defined by $\mu(k) = \exp(m \text{STr} \pi_s \ln k)$. To confirm this expectation we use an idea that has been described to physicists in Refs. [41,49]. It is to get rid of the gauge dependence by “working with all gauges at once” and regarding an element $f \in W_s$ as a function $F$ on the total space $G$. In detail this works as follows. Recall that a choice of section $s$ gives rise to a map $k : G \to K$ by the equation $g = s(\pi(g))k(g)$. Given $k(g)$, we associate with $f \in W_s$ a function $F$ on $G$ by

$$F(g) = \mu(k(g))^{-1} f(\pi(g)).$$

Since $k(gk_1) = k(g)k_1$ for $k_1 \in K$ and since $\mu$ is a representation, $F$ satisfies

$$F(gk_1) = \mu(k_1)^{-1} F(g).$$

In this way, $W_s$ corresponds to the linear space, $W$, of functions $F$ on $G$ subject to the symmetry condition (40). The correspondence is one-to-one, with the inverse of relation (39) being

$$f(x) = \mu(k(s(x))) F(s(x)) \quad (x \in G/K).$$

In other words, Eqs. (41) and (39) define a bijection $\psi_s : W \to W_s$. (An element $f \in W_s$ is technically called a section of an associated bundle [4].)

Let us illustrate the above correspondence between $f$ and $F$ at the simple example of SU(2)/U(1). With the choices

$$g = e^{i\phi}e^{i\theta}e^{-i\psi}, \quad s = e^{i\phi}e^{i\theta}e^{-i\psi}, \quad \mu = e^{iK\alpha},$$

the eigenfunctions of the Dirac monopole Hamiltonian are $\mathcal{D}_{KM}^J(s^{-1}) = \tilde{\mathcal{D}}_{KM}^J(0,-\theta,-\phi) = f$ where $\mathcal{D}$ are Wigner’s D-functions expressed in Euler angles $\phi, \theta, \psi$. They extend to

30
functions $\mathcal{D}^J_{KM}(g^{-1}) = \tilde{\mathcal{D}}^J_{KM}(-\psi, -\theta, -\phi) = F$, which are functions on SU(2) obeying the symmetry condition

$$\mathcal{D}^J_{KM}((ge^{i\alpha\sigma_3/2})^{-1}) = e^{-iK\alpha}D^J_{KM}(g^{-1}).$$

How does the monopole Hamiltonian operate on functions $F \in W$? Making the similarity transformation $H := \psi_s^{-1}H_s\psi_s$ we get from Eq. (33) for $H_s = -L^m$:

$$(\mathcal{H}F)(g) = -\lambda(\partial/\partial x)F(g \exp \sum x^i \epsilon_i) \bigg|_{x=0}.$$

(42)

Note that $\mathcal{H}$ preserves the symmetry condition (40). In fact, assuming $F$ to satisfy (40) and using the invariance of $\lambda(\partial/\partial x)$ under $\sum x^i \epsilon_i \mapsto k(\sum x^i \epsilon_i)k^{-1}$, we get

$$\mathcal{H}(F)(gk) = -\mu(k)^{-1}\lambda(\partial/\partial x)F(g \exp(k\sum x^i \epsilon_i k^{-1}) \bigg|_{x=0} = \mu(k)^{-1}(\mathcal{H}F)(g).$$

Therefore, $\mathcal{H}$ is well-defined as an operator $W \rightarrow W$.

Consider now the natural representation of the symmetry group $G$ on $W$ by left translation:

$$(L_gF)(h) = F(g^{-1}h).$$

(43)

(Right translations are unnatural because they interfere with the constraint (40).) The representation (43) commutes with the operator $\mathcal{H}$, Eq. (12). This is obvious since left translations act on the left while $\mathcal{H}$ acts on the right. At this point we recognize the advantage of making the similarity transformation from $H_s$ to $\mathcal{H}$: the symmetry transformations of $\mathcal{H}$ are simply the left translations! The good action, $T_g$, of the symmetry group $G$ on gauge-dependent functions $f \in W_s$ now follows by transforming from $W$ back to $W_s$. We have $T_g = \psi_s L_g \psi_s^{-1}$ or, in explicit terms,

$$T_g f(\pi(h)) = \mu(k(h)k(g^{-1}h)^{-1}) f(\pi(g^{-1}h)).$$

(44)

These are the symmetry transformations of $H_s$. They form a representation of $G$, by virtue of their construction from (13) via similarity transformation. The action (44) is called an induced representation. It depends on $\mu$ and on the choice of gauge $s$. This fact is not made explicit in our notation.

We claim that the action (44) preserves the hermitean form (38):

$$(T_g f_1, T_g f_2) = (f_1, f_2) \quad (g \in G).$$

(45)

(If $(\cdot, \cdot)$ were positive definite this would amount to saying that the representation is unitary.) In view of Eq. (44) and the invariance of $dx$ under left translations $x = \pi(h) \mapsto g^{-1} \cdot x = \pi(g^{-1}h)$, our statement is obvious if we can show that

$$\mu(k(g)) = \mu(k(g))^{-1}$$

(45)
holds for \( g \in G \). The relation (15) is established as follows. From Eq. (34) and \( \text{STr} \, A = \text{STr} \, A^T \) (superscript \( T \) denoting supertranspose) and \( A^T = A^\dagger \), we get \( \mu(k(g)) = \mu(k(g)) = \mu(k(g)^\dagger) \). Now, by Eq. (9) \( k(g) \in K \subset G \) satisfies \( k(g)^\dagger = \eta k(g)^{-1} \eta \) and, since \( \mu \) is a representation and \( \mu(\eta)^2 = 1 \) (note \( \eta \in K \)), \( \mu(k(g)^\dagger) = \mu(k(g))^{-1} \), so (15) is proved.

It follows from (45) that the monopole Hamiltonian \( \mathcal{H}_s \) is hermitian w.r.t. the scalar product (38). The proof is simply a matter of writing down

\[
(f_2, \mathcal{H}_s f_1) = \int_{G/K} dg_K \frac{1}{f_2(\pi(g)) \mathcal{H}_s f_1(\pi(g))},
\]

using the formula (34) for \( \mathcal{H}_s = -\mathcal{L}^m \), making the change of integration variable \( g \mapsto e^{-\Sigma x^i e_i} g \) (which leaves \( dg_K \) invariant), and then using Eq. (15).

### 4.3 Choice of gauge, and coordinate presentation

To proceed further, we pick suitable coordinates for \( G/K \), and we fix the gauge by choosing some section \( s : G/K \to G \). Our choice of coordinates is motivated by the requirement that the projection \( \pi : G \to G/K \) be expressed in a particularly simple form.

An element \( g \in G \) is written

\[
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]  

Here, and in all that follows, the displayed matrix structure refers to advanced-retarded space. Each of the blocks \( A, B, C, D \) is a \( 2n \times 2n \) supermatrix acting in the tensor product of Boson-Fermion and Extra space. They are subject to a certain set of constraints that follow from the defining equation for \( G \), Eq. (9). In particular, these constraints imply that both \( A \) and \( D \) have an inverse in some neighborhood of the unit element of \( G \). Since \( \Lambda = \text{diag}(1_{BF \otimes E}, -1_{BF \otimes E}) \) in the presentation adopted, an element \( k \in K \) is of the block-diagonal form \( k = \begin{pmatrix} k_+ & 0 \\ 0 & k_- \end{pmatrix} \). Invariants for the action \( g \mapsto gk \) (\( k \in K \)) are the combinations \( CA^{-1} \) and \( BD^{-1} \). Counting shows that the number of independent variables in \( CA^{-1} \) and \( BD^{-1} \) equals the number of degrees of freedom on \( G/K \). We put \( Z := CA^{-1}, \bar{Z} := BD^{-1} \) and take \( Z, \bar{Z} \) as local coordinates for the coset space \( G/K \). Thus, \( f(\pi(g)) \) will from now on be written \( f(Z, \bar{Z}) \). (For clean mathematics we should work with several coordinate charts. We will however get away with using just one.) Recalling the defining equation for \( G \) one can show with a little calculation that \( Z \) and \( \bar{Z} \) are related by \( \bar{Z} = Z^\dagger \tau_3 \) where \( \tau_3 = (\sigma_3)_{BF} \otimes 1_E \).

The meaning of the coordinates \( Z, \bar{Z} \) is best understood by looking at some examples. Let us specialize to \( n = 1 \), in which case \( G/K \) has a two-sphere \( S^2 \simeq SU(2)/U(1) \simeq U(2)/U(1) \times U(1) \) for its FF-space. By projecting the pair of supermatrices \( (Z, \bar{Z}) \) on the FF-sector, we get a pair of complex numbers \( (-z, \bar{z}) \). (The minus sign is dictated by the unitarity condition \( \bar{Z} = Z^\dagger \tau_3 \).) The special element \( \Lambda \) is replaced by the Pauli matrix \( \sigma_3 \). Parametrizing \( SU(2) \) by Euler angles

\[
g = \exp(i\phi \sigma_3/2) \exp(i\theta \sigma_2/2) \exp(i\psi \sigma_3/2) = \begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix},
\]

(47)
we get $z = \beta/\alpha = \tan(\theta/2) \exp(-i\phi)$. It follows that, if $n = (n_1, n_2, n_3) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is a unit vector parametrizing the two-sphere $S^2$, $z$ has the expression $z = (n_1 - i n_2)/(1 + n_3)$. In other words, $z$ is the usual complex coordinate associated with the stereographic mapping of $S^2 \simeq SU(2)/U(1)$ onto the complex plane $\mathbb{C}$. Consider next the noncompact analog space $U(1, 1)/U(1) \times U(1) \simeq SU(1, 1)/U(1)$, which is the BB-space of $G/K$ for $n = 1$ and is isomorphic to the two-hyperboloid (or hyperbolic plane) $H^2$. For this we replace $(Z, \tilde{Z}) \rightarrow (z, \tilde{z})$ (without the minus sign). Changing to Euler angles for $SU(1, 1)$ by making in the exponent of the middle factor in Eq. (47) the substitution $G/K \rightarrow G/0$, we get $z = \tanh(\theta/2) \exp(-i\phi)$. $H^2$ can be parametrized by a hyperbolic unit vector $n = (n_1, n_2, n_3) = (\sinh \theta \cos \phi, \sinh \theta \sin \phi, \cosh \theta)$. As before, $z = (n_1 - i n_2)/(1 + n_3)$. What this means is that $z$ is the complex coordinate of the Poincaré disk model $|z|^2 < 1$ of $H^2$. Finally, we mention that $Z$ and $\tilde{Z}$ coincide with the matrices $w_{21}$ and $-w_{12}$ used for the parametrization of $Q$ in Ref. [12] for $n = 1$. To verify this, one expresses $Q = gA \cdot g^{-1}$ in terms of $Z, \tilde{Z}$ and compares the result with Eq. (3.2) of that reference.

Let us now fix $s : G/K \rightarrow G_0 = SU(n, n|2n)$. For various reasons that will become clear as we go along, the optimal choice is $s(\pi(g)) = (gA \cdot g^{-1})^{1/2}$ with coordinate expression

$$s(Z, \tilde{Z}) = \begin{pmatrix} 1 & \tilde{Z} \\ Z & 1 \end{pmatrix} \begin{pmatrix} (1 - ZZ)^{-1/2} \\ 0 \end{pmatrix}. \tag{48}$$

It is easy to verify that (i) $s(Z, Z^\dagger \tau_3) \in G_0$ and (ii) $\pi \circ s = id$. (These properties are sufficient in order for $s$ to be a section of the bundle $G \rightarrow G/K$.) Specializing to the FF-sector for $n = 1$ and using the Euler angle parametrization (47), we see that the choice $s(\pi(g)) = (g\sigma_3 g^{-1} \sigma_3)^{1/2}$ corresponds to the second of the two sections in (47).

We now wish to re-express the central formulas (33) and (44) in terms of the complex coordinates $Z$ and $\tilde{Z}$, starting with the latter. For that we need to figure out how left translations $\pi(h) \rightarrow \pi(gh)$ act on $Z$ and $\tilde{Z}$. We put $h = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ and characterize the coset $\pi(h)$ by $Z = C' A''^{-1}$ and $\tilde{Z} = B' D''^{-1}$. For the product $gh$ with $g$ given by Eq. (40) we write $gh = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}$. The complex coordinates of the left translated coset $\pi(gh)$ are denoted by $g \cdot Z = C'' A''^{-1}$ and $g \cdot \tilde{Z} = B'' D''^{-1}$. Carrying out the matrix multiplication we find

$$g \cdot Z = (DZ + C)(A + BZ)^{-1},$$

$$g \cdot \tilde{Z} = (A\tilde{Z} + B)(D + C\tilde{Z})^{-1}. \tag{49}$$

We mention in passing that these relations generalize the rational action of $SL(2, \mathbb{C})$ on $\mathbb{C}$ by $z \mapsto \frac{az + b}{cz + d}$. By solving Eqs. (49) for $Z, \tilde{Z}$ we get the inverse transformations

$$g^{-1} \cdot Z = (D - ZB)^{-1}(ZA - C),$$

$$g^{-1} \cdot \tilde{Z} = (A - \tilde{Z}C)^{-1}(\tilde{Z}D - B). \tag{50}$$

The other ingredient on the right-hand side of Eq. (44) is the multiplier built from $\mu$ and $k(\cdot)$. Unfortunately, the derivation of its explicit expression in terms of $Z, \tilde{Z}$ requires a certain amount of algebra. To avoid cluttering the main text with a lot of formulas, I have
relegated these details to Appendix B. Using the information given there, we arrive at the following coordinate expression for the representation (44):

\[(T_g f)(Z, \tilde{Z}) = (\text{SDet } g)^{m/2} \text{SDet } \left( \frac{A - \tilde{Z} C}{D - Z B} \right)^{m/2} f(g^{-1} \cdot Z, g^{-1} \cdot \tilde{Z}). \]  

(Eq. (46) is understood.) Next, we write down the coordinate expression of the second-order differential operator \(H_s = -\mathcal{L}^m\) defined in (33). Again, the calculations are somewhat involved which is why they are done in Appendix B. The result however is quite transparent and can be stated as follows. Let \(X, \tilde{X}\) be a pair of supermatrices with the same dimensions and symmetries as \(Z, \tilde{Z}\). If we define a differential operator \(\partial_{X, \tilde{X}}\) analogous to \(\lambda(\partial/\partial x)\) in (33) by

\[\partial_{X, \tilde{X}} = \sum_{ij} (-)^{|i|} \frac{\partial^2}{\partial X_{ij} \partial \tilde{X}_{ji}},\]

the operator \(H_s\) has the coordinate expression

\[(H_s f)(Z, \tilde{Z}) = -\partial_{X, \tilde{X}} \text{SDet } \left( \frac{1 + Z \tilde{X}}{1 + \tilde{Z} X} \right)^{m/2} f((Z + X)(1 + \tilde{Z} X)^{-1}, \ldots) \big|_{X = \tilde{X} = 0}. \]

The second argument of \(f\) on the right-hand side is obtained from the first one by \(Z \leftrightarrow \tilde{Z}\) and \(X \leftrightarrow \tilde{X}\).

Since the formula (53) for the monopole Hamiltonian \(H_s\) may look unfamiliar we turn, once again, to the prototypical example \(\text{SU}(2)/U(1)\) for illustration. In this case it is well-known [16] that, for \(m = 1\), \(H_s\) has a spin 1/2 ground state spanned by the functions \(D_{1/2, -1/2}(s^{-1}) = \cos(\theta/2)\) and \(D_{1/2, 1/2}(s^{-1}) = \sin(\theta/2) \exp(-i\phi)\). To verify this from the present formalism, we use

\[(H_s f)(z, \bar{z}) = -\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} \left( \frac{1 - \bar{z} \zeta}{1 - z \bar{\zeta}} \right)^{m/2} f\left( \frac{z + \zeta}{1 - z \bar{\zeta}}, \text{c.c.} \right) \big|_{\zeta = \bar{\zeta} = 0}, \]

as results from (53) by specialization to the FF-sector. (The multiplier got inverted because \(\text{SDet}\) puts the determinant of the FF-block in the denominator.) The expressions for the functions \(\cos(\theta/2)\) and \(\sin(\theta/2) \exp(-i\phi)\) in terms of the complex coordinates \(z, \bar{z}\) are \((1 + \bar{z} z)^{-1/2}\) and \(z(1 + \bar{z} z)^{-1/2}\). From Eq. (54) these are easily seen to be eigenfunctions, for \(m = 1\), of \(H_s\) with eigenvalue 1/2. More generally, for any positive monopole charge \(m\) the functions \(z^k(1 + \bar{z} z)^{-m/2}\) \((0 \leq k \leq m)\) are seen to be eigenfunctions with eigenvalue \(m/2\). They form an irreducible spin \(m/2\) multiplet of \(\text{SU}(2)\). Indeed, by specializing Eq. (51) and making the similarity transformation \(f(z, \bar{z}) = (1 + \bar{z} z)^{-m/2} \varphi(z)\) we get the transformation law

\[\left(T'_{\alpha \beta} \varphi \right)(z) = (\alpha + \bar{\beta} z)^m \varphi\left( \frac{\alpha z - \beta}{\alpha + \bar{\beta} z} \right). \]
From this we read off the representation of the generators \( \sigma_+ = (\sigma_1 + i\sigma_2)/2 \), \( \sigma_- = (\sigma_1 - i\sigma_2)/2 \), and \( \sigma_3 \) by the differential operators \(-z^2\partial_z + mz, \partial_z, \text{ and } 2z\partial_z - m\), respectively. It can be shown that the \((m+1)\)-dimensional space of functions \( f(z, \bar{z}) = (1 + \bar{z}z)^{-m/2}\varphi(z) \) with holomorphic \( \varphi(z) = \sum_{k=0}^m c_k z^k \) is the ground state of \( \mathcal{H}_s \). (Note that in the zero curvature limit \( m \to \infty \) we recover the lowest Landau level wave functions \( f(z, \bar{z}) = \exp(-\bar{z}z)\varphi(z) \) of the Hamiltonian \(-(\bar{\nabla} - A)^2\) on the complex plane in the symmetric gauge \( A = (zd\bar{z} - zd\bar{z})/2 \).) The representation of SU(2) on holomorphic functions has been used \cite{[51]} in numerical studies of the fractional quantum Hall effect. (\( z = v/u \) in the notation of Ref. \cite{[51]}.) Eq. (53) applies to the case of positive \( m \). When \( m \) is negative, we get a ground state spanned by antiholomorphic functions.

All of the above can be transcribed without difficulty to the noncompact analog space \( SU(1,1)/U(1) \). To cut things short, we just mention that the ground states \( z^k(1+\bar{z}z)^{-m/2} \) \((0 \leq k \leq m)\) translate into the ground states \( z^k(1-\bar{z}z)^{m/2} \) \((0 \leq k < \infty)\). The latter transform according to a unitary infinite-dimensional lowest-weight representation of the noncompact group SU(1,1).

4.4 Holomorphic sections

At the end of the preceding subsection we recalled the fact that the degenerate ground state of the monopole Hamiltonian \( \mathcal{H}_s \) for \( S^2 \) and monopole charge \( m > 0 \) is spanned by the functions \((1 + \bar{z}z)^{-m/2}\varphi(z) \) with holomorphic \( \varphi(z) = \sum_{k=0}^m c_k z^k \). This observation holds the clue how to make progress with \( \mathcal{H}_s \) in the general case, Eq. (53), and nonnegative \( m \). The functions \((1 + \bar{z}z)^{-m/2}\) for \( S^2 \) and \((1 - \bar{z}z)^{m/2}\) for \( H^2 \), generalize to SDet\((1 - \bar{Z}Z)^{m/2}\) for our quantum Hall model space \( G/K \). Let us therefore set \( v_m(Z, \bar{Z}) = \text{SDet}(1 - \bar{Z}Z)^{m/2} \) and make a similarity transformation from \( \mathcal{H}_s \) to \( \mathcal{H}_+ = v_m^{-1}\mathcal{H}_sv_m \). The transformation is carried out with the help of the identity

\[
1 - (Z + X)(1 + \bar{Z}X)^{-1}(\bar{Z} + \bar{X})(1 + Z\bar{X})^{-1} = (1 - Z\bar{Z})(1 + X\bar{Z})^{-1}(1 - X\bar{X})(1 + Z\bar{X})^{-1}.
\]

Substituting \( f = v_mw \) into Eq. (55) we obtain

\[
(\mathcal{H}_+w)(Z, \bar{Z}) = -\partial_{X,\bar{X}} \text{SDet} \left( \frac{(1 - X\bar{X})^{1/2}}{1 + \bar{Z}X} \right)^m w((Z + X)(1 + \bar{Z}X)^{-1}, \ldots) \bigg|_{X = \bar{X} = 0}. \tag{56}
\]

What happens for holomorphic \( w(Z, \bar{Z}) = \varphi(Z) \)? The crucial feature of Eq. (56) is the absence of any dependence on \( \bar{X} \) in the denominator of the multiplier. With this observation, and recalling the definition of \( \partial_{X,\bar{X}} \) in (52), we immediately find \( \varphi(Z) \) to be an eigenfunction of \( \mathcal{H}_+ \) with eigenvalue

\[
-\partial_{X,\bar{X}} \text{SDet}(1 - X\bar{X})^{m/2} \bigg|_{X = \bar{X} = 0}. \tag{57}
\]

By the property of perfect grading, i.e. the equal number of bosonic and fermionic degrees of freedom, the eigenvalue (57) vanishes. We thus arrive at the important conclusion that functions \( f \) of the form

\[
f(Z, \bar{Z}) = \text{SDet}(1 - Z\bar{Z})^{m/2}\varphi(Z)
\]

35
with holomorphic $\varphi$ are zero modes of the Hamiltonian $\mathcal{H}_s$ for monopole charge $m > 0$. For the general reason of stability of the functional integral (13), we expect only eigenfunctions with nonnegative eigenvalues to contribute to the spectral expansion of $\mathcal{H}_s$. (The operator $\mathcal{H}_s$ does have a vast number of eigenfunctions with negative eigenvalues. These however are not normalizable and do not occur in its spectral expansion.) We thus perceive the central role played by holomorphic functions: they span the degenerate ground state of $\mathcal{H}_s$ not only for $\text{SU}(2)/\text{U}(1)$ but for our quantum Hall model space $G/K$ (and other spaces $G/K$) as well. The story does not end here, since the requirement of decent behavior under transition to another coordinate chart imposes a constraint on $\varphi(Z)$. (Recall that for $\text{SU}(2)/\text{U}(1)$ the degree of the polynomial $\varphi(z)$ must not exceed $m$.) We can avoid getting into the boring technicalities of switching charts by first identifying some special zero mode and then generating the entire space of zero modes by the action (51) of the symmetry group $G$. This is what we will do in Sect. 4.5. To prepare the second step, we now specialize the representation (51) to the case of holomorphic sections.

As is shown in Appendix B, by making the similarity transformation $T_g^+ = v_m^{-1} T_g v_m$ and keeping only the dependence on the holomorphic coordinate $Z$, we obtain from (51):

$$
(T_g^+ \varphi)(Z) = (\text{SDet} g)^m \text{SDet}(D - ZB)^{-m} \varphi(g^{-1} \cdot Z).
$$
(58)

(Again, Eq. (46) is understood.) Eq. (58) gives rise to a representation of $\mathcal{G} = \text{Lie}(G)$ by the usual procedure of linearization around the group unit, and this representation extends to the complexified Lie algebra $\mathcal{G}_C = \mathfrak{gl}(2n, 2n)$ by linearity. The motivation for going to the complexification is that we desire the convenience of working with the usual raising and lowering operators. (These do not exist inside $\mathcal{G}$.) Making the extension and then returning to group level, we get a representation of the complexified group $G_C = \text{GL}(2n, 2n)$. It is given by the formula (58) without any change. (In fact, the condition $g^{-1} = \eta g^\dagger \eta$ defining the pseudo-unitary group $G$ is never used in the derivation of (58). All we need is the existence of the inverse $g^{-1}$.) Having made the extension to $G_C$ we can now write down the action of raising operators $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ and lowering operators $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$ at the group level. These are seen from (58) and (51) to be represented by

$$
(T_g^+ B \varphi)(Z) = \text{SDet}(1 - ZB)^{-m} \varphi((1 - ZB)^{-1} Z),
$$
(59a)

$$
(T_g^+ C \varphi)(Z) = \varphi(Z - C).
$$
(59b)

Another special case of importance is the action of elements of $K_C$:

$$
(T_g^+ A \varphi)(Z) = \text{SDet} A^m \varphi(D^{-1} Z A).
$$
(60)

All of the above applies to the case of positive monopole charge $m > 0$ but is easily transcribed to $m < 0$. In the latter case the zero modes of $\mathcal{H}_s$ are functions $f = v_{-m} \varphi$ with antiholomorphic $\varphi(\tilde{Z})$. Carrying out the similarity transformation $T_{g^-} = v_{-m}^{-1} T_g v_{-m}$, we obtain

$$
(T_{g^-} \varphi)(\tilde{Z}) = \text{SDet}(A - \tilde{Z} C)^m \varphi(g^{-1} \cdot \tilde{Z}).
$$
(61)
4.5 Zero modes and coherent states

Our results of Sect. 4.4 reveal the ground state of \( \mathcal{H}_s \) for \( m \neq 0 \) to be vastly degenerate: any function \( f \) of the form \( f = v_m \varphi \) (\( f = v_{-m} \varphi \)) with holomorphic (antiholomorphic) \( \varphi \) is a zero mode of \( \mathcal{H}_s \) for \( m > 0 \) (\( m < 0 \)). However, not all of these are physical states and contribute to the spectral expansion of \( \mathcal{H}_s \). It is the purpose of the current subsection to describe which of them do.

We put \( m \geq 1 \) for definiteness and consider the special vector \( v_m(Z, \bar{Z}) = \text{SDet}(1 - Z \bar{Z})^{m/2} \) obtained by setting \( \varphi = 1 \). Our first task will be to show that \( v_m \) has finite length. To do this with a minimum amount of effort, we put \( v_m \) in coordinate-free form. Using \( g = s(\pi(g)) \) and the expression (48) for \( s \), we get

\[
v_m(\pi(g)) = \text{SDet}(2 + g\Lambda g^{-1} \Lambda + \Lambda g \Lambda g^{-1})^{-m/2}.
\]  

(62)

Note that the coordinate-free formula (62) guarantees good behavior of \( v_m \) under transition to another coordinate chart. Aside from decaying sufficiently fast at infinity – so that \((v_m, v_m)\) in fact exists – \( v_m \) has the special property \( v_m(k \cdot x) = v_m(x) \) for \( k \in K \). Functions with this property are called \( K \)-radial. For \( f \) any such function (with the additional property of being integrable), the so-called Parisi-Sourlas-Efetov-Wegner supersymmetric integral formula states that

\[
\int_{G/K} f(x) \, dx = f(o)
\]

(63)

where \( o = \pi(1) \) is the unique element of \( G/K \) which is left invariant by \( K \). (Incidentally, Eq. (63) fixes a natural normalization of \( dx \).) A pedestrian proof of (63) for \( n = 1 \) can be found in [52]. The general proof has been worked out in [53]. Application of theorem (63) to \( f = \bar{v}_m v_m \) yields \((v_m, v_m) = 1\) since \( v(o) = 1 \) from Eq. (62). This says that \( v_m \) is a physical state.

Given \( v_m \), we can apply a symmetry transformation \( T_g \) to form the vector \( v_g = T_g v_m \). Since \( T_g \) preserves the scalar product (33) for \( g \in G \) we have \((v_g, v_g) = (v_m, v_m) = 1\) and, since \( T_g \) commutes with \( \mathcal{H}_s \), \( v_g \) is an eigenstate of \( \mathcal{H}_s \) with zero energy. Under the assumption of irreducibility of the representations entering the spectral expansion of \( \mathcal{H}_s \), we can generate the entire zero-energy sector in this way.

For the explicit construction of all zero-energy states it is convenient to go to the complexification \( G_C \) and work with raising and lowering operators, which are norm-changing. A unique feature of \( v_m \) is that it satisfies the equations

\[
T \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} v_m = v_m \quad \text{and} \quad T_k v_m = \mu(k) v_m,
\]

(64)

which are easily verified using (59b), (61) and (31). Thus, \( v_m \) is a lowest-weight vector carrying the one-dimensional representation \( \mu \) of \( K \). To see how the raising operators act on \( v_m \) we set \( \varphi = 1 \) in Eq. (59a) and find

\[
(T \begin{pmatrix} 1 & \bar{B} \\ 0 & 1 \end{pmatrix} v_m)(Z, \bar{Z}) = \text{SDet}(1 - BZ)^{-m} v_m(Z, \bar{Z}).
\]

(65)
The space of states that can be reached by repeated application of a raising operator to $v_m$ will be denoted by $V_m$. It is called a lowest-weight module of $G$.

A vector of the form (65), obtained by applying a raising operator $\exp(0 \cdot X)$ parametrized by $X$ to a lowest-weight vector, is commonly called a coherent state. To enhance the readability of our notation, we temporarily promote the subscript of $T$ to argument and write

$$v_X := T \left( \begin{array}{c} 1 \\ X \end{array} \right) v_m.$$  

We will now calculate the coherent state overlap function $N(X, Y) = (v_X, v_Y)$. Knowing it we will be able to deduce the norms and overlaps of vectors by Taylor expansion. Since

$$\left( \begin{array}{c} 0 \\ \tau_3 X^\dagger \\ 0 \end{array} \right)$$

is an element of $\text{Lie}(G)$ and $T_g$ for $g \in G$ preserves the scalar product,

$$N(X, Y) = (v_m, T \left( \begin{array}{c} 1 \\ -\tau_3 X^\dagger \\ Y \\
1 - \tau_3 X^\dagger Y \end{array} \right) v_m).$$

Using the representation property and carrying out the matrix multiplication, we get

$$N(X, Y) = (v_m, T \left( \begin{array}{c} 1 \\ -\tau_3 X^\dagger \\ Y \\
1 - \tau_3 X^\dagger Y \end{array} \right) v_m).$$

To calculate this, we use the following trick. There exist matrices $B$, $C$, $k_+$ and $k_-$ such that

$$\left( \begin{array}{c} 1 \\ -\tau_3 X^\dagger \\ Y \\
1 - \tau_3 X^\dagger Y \end{array} \right) = \left( \begin{array}{c} 1 \\ B \\ 0 \\ C \end{array} \right) \left( \begin{array}{c} k_+ \\ 0 \\ k_- \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 1 \end{array} \right).$$  (66)

From (64) we have

$$(v_m, T \left( \begin{array}{c} 1 \\ 0 \\ B \\ 1 \end{array} \right) \left( \begin{array}{c} k_+ \\ 0 \\ k_- \\ 1 \end{array} \right) v_m) = \mu(\left( \begin{array}{c} k_+ \\ 0 \\ k_- \end{array} \right))(v_m, v_m).$$

Hence, $N = \exp(m \text{Str} \ln k_+) = (\text{SDet} k_+)^m$, and solving Eq. (66) for $k_+$ we obtain

$$N(X, Y) = \text{SDet}(1 - \tau_3 X^\dagger Y)^{-m}.\quad (67)$$

From this result in combination with Eqs. (59) and (60), all desired information about the state vector content and norms and overlaps of vectors in $V_m$ can be deduced by Taylor expansion.

Let us now take a close-up look at $V_m$ for the minimal model $n = 1$. The base manifold of the super coset space $G/K$ for $n = 1$ is the direct product of a two-sphere $S^2 \simeq \text{SU}(2)/\text{U}(1)$ and a two-hyperboloid $H^2 \simeq \text{SU}(1,1)/\text{U}(1)$. The Cartan algebra of $\text{Lie}(G)$ is four-dimensional in this case and can be taken to be generated by

$$H_i = (E_{ii})_{BF} \otimes (\sigma_3)_{AR} \quad (i = 1, 2), \quad H_3 = (\sigma_3)_{BF} \otimes 1_{AR}, \quad H_4 = 1_{BF} \otimes 1_{AR}.$$  

The eigenvalues (also called weights) of the generators $H_1$ and $H_2$ will be used to label vectors in $V_m$. $H_1$ ($H_2$) is the generator of the $\text{U}(1)$ subgroup of rotations fixing the north
pole on $H^2 (S^2)$. Introducing $B_{ij} = (E_{ij})_{BF} \otimes (E_{12})_{AR}$ ($i, j = 1, 2$), we define differential operators $B_{ij} : V_m \rightarrow V_m$ by

$$B_{ij} f = dT(B_{ij}) f = \frac{d}{dt} T_{\exp t B_{ij}} f \bigg|_{t=0}$$

where $t$ is a commuting or anticommuting element of the parameter Grassmann algebra according to whether $B_{ij}$ is even or odd, respectively. Their coordinate expressions are easily found from Eq. (59a). Acting repeatedly with elements of the set $\{B_{ij}\}$ on $v_m$, we get $v_m$ times a polynomial in the complex coordinates $Z$. By applying $B_{ij}$ once, we raise the degree of the polynomial by one unit. (We say that $v_m$ is the vacuum and one action of an operator $B_{ij}$ creates one quantum of excitation.) It is quite straightforward to construct all polynomials that are generated in this way. The degree in the anticommuting variables $Z_{ij}$ ($i \neq j$) cannot exceed one, and the degree in the variable $Z_{22}$ (coordinatizing the compact FF-space) turns out to be $\leq m$. Omitting the details of this straightforward calculation, we now give a summary of the results found. For every pair of integers $p, q$ in the range $p \geq m, -m \leq q \leq m$ ($a_{pq}$), $p \geq m + 1, -m + 1 \leq q \leq m - 1$ ($\beta_{pq}$ and $\gamma_{pq}$), and $p \geq m + 2, -m + 2 \leq q \leq m - 2$ ($d_{pq}$), we define the state vectors

$$a_{pq} = B_{11}^{(p-m)/2} B_{22}^{(q+m)/2} v_m,$$
$$\beta_{pq} = B_{11}^{(p-m-1)/2} B_{22}^{(q+m-1)/2} B_{12} v_m,$$
$$\gamma_{pq} = B_{11}^{(p-m-1)/2} B_{22}^{(q+m-1)/2} B_{21} v_m,$$
$$d_{pq} = B_{11}^{(p-m-2)/2} B_{22}^{(q+m-2)/2} (B_{12} B_{21} - m^{-1} B_{11} B_{22}) v_m.$$

These vectors form an orthogonal basis of $V_m$. (The allowed values of $p$ and $q$ differ by two units in each case since exponents must be integer-valued.) All of them are eigenvectors of $dT(H_1)$ and $dT(H_2)$ with eigenvalue $p$ and $q$, respectively. Furthermore, $dT(H_3) a_{pq} = dT(H_3) d_{pq} = 0$ and $dT(H_3) \beta_{pq} = +2 \beta_{pq}$, $dT(H_3) \gamma_{pq} = -2 \gamma_{pq}$. All vectors are annihilated by $dT(H_4)$. The vectors $a_{pq}$, $\beta_{pq}$, $\gamma_{pq}$, $d_{pq}$ for fixed $p$ and variable $q$ form an SU(2)-multiplet with spin $S = m/2, (m - 1)/2, (m - 1)/2, (m - 2)/2$, respectively. When $q$ is held fixed and $p$ is varied, they form a unitary lowest-weight representation of SU(1, 1), again with spin $S = m/2, (m - 1)/2, (m - 1)/2, (m - 2)/2$, respectively. The norms $|f|^2 = (f, f)$ are

$$|a_{pq}|^2 = N \left( \frac{p-m}{2}, \frac{q+m}{2} \right),$$
$$|\beta_{pq}|^2 = -|\gamma_{pq}|^2 = m \ N \left( \frac{p-m-1}{2}, \frac{q+m-1}{2} \right),$$
$$|d_{pq}|^2 = (m^2 - 1) \ N \left( \frac{p-m-2}{2}, \frac{q+m-2}{2} \right),$$

where $N$ is defined by

$$N(p, q) = p!^2 \binom{m+p-1}{p} q!^2 \binom{m}{q}.$$
The \( d_{pq} \) are seen to be null vectors for \( m = 1 \). The weight diagram for \( m = 2 \) is shown in Figure 4.

We thus have a complete and detailed description of the lowest-weight module \( V_m \) for \( m > 0 \) and \( n = 1 \). The transcription to \( m < 0 \) can be made very simply by complex conjugation, taking the lowest-weight vector \( v_m \) into the highest-weight vector \( v_{-m} \) and holomorphic functions into antiholomorphic ones. There remain two important questions that are left open by our analysis. i) Does the module \( V_m \) really exhaust the space of zero-energy states? (It is conceivable in principle that \( V_m \) might be an invariant subspace of a bigger indecomposable representation. If this were the case, there would exist other zero-energy states that cannot be reached by applying raising operators to the lowest-weight vector \( v_m \).) ii) Are the excited states of \( \mathcal{H}_s \) separated from the ground state by a gap? Unfortunately, giving a convincing answer to these questions is beyond the scope of this paper. Without proof I claim that the answer to both questions is yes. As was explained in Sect. 4.1, I hope to publish the proof in a future complete account of my work on supersymmetric Fourier analysis.

### 4.6 Edge dynamics and Hall conductance

Our derivation of the quantum Hamiltonian of the nonlinear \( \sigma \) model goes through even when the configuration space \( \mathcal{M} \) of the 2d electron gas has a boundary. The only condition on \( \mathcal{M} \) is that one of its two directions be periodic. We will now use this flexibility for a semi-realistic calculation of the quantized Hall conductance in the plateau regions. The experimental setup we imagine is shown in Figure 5. We take for \( \mathcal{M} \) a disk \([R_1, R_2] \times S^1\) and attach various voltage and current probes to its outer edge.

Let us begin by summarizing what phenomenological theory \([4]\) tells us about this kind of system. When the magnetic field is located in a Hall plateau region, all stationary electron states in the interior of the disk are localized. Transport proceeds via edge states and is effectively one-dimensional. The number \( M \) of edge channels equals the number of Landau levels below the Fermi energy. Edge channels are directed, i.e. electrons can propagate in one direction only. The direction of propagation is determined by the sign of the magnetic field (\( E \times B \) drift). For ideal probes, the conductance coefficients \( g_{pq} \) are given by

\[
\begin{align*}
    g_{21} = g_{32} = \ldots = g_{1N} &= M, \\
    g_{12} = g_{23} = \ldots = g_{N1} &= 0, \\
    g_{11} = g_{22} = \ldots = g_{NN} &= -M,
\end{align*}
\]

(70)

if the ordering of the contacts follows the orientation induced by the magnetic field. All other conductance coefficients vanish. We will now show how the relations (70) follow from the effective field theory \([13]\).

The disorder averages of the \( g_{pq} \) are given by certain correlation functions of the field theory whose form is specified below. To take the Hamiltonian limit of the field theory, we assign to the radial and angular directions of \( \mathcal{M} \) the roles of space and time (or, rather, imaginary time), respectively. By proceeding as in Sects. 3.3-5, we get a quantum Hamiltonian \( \mathcal{H} \) governing the evolution in time (i.e. around the disk) of a discrete set of degrees of freedom. Each of these takes values in \( \mathbf{G}/\mathbf{K} \) and is located on one site of a
lattice in radial direction. The degrees of freedom will be referred to as $\mathbf{G}/\mathbf{K}$-superparticles (SPs). We are going to consider values of the magnetic field at the center of a Hall plateau region, where the Hall conductivity $\sigma_{xy}$ is an integer $m \in \mathbb{Z}$. The dynamics of the SPs in the interior of the disk is then governed by the Hamiltonian specified in Eq. (22). In contrast, the two SPs at the inner and the outer edge sense the additional field of a fictitious “magnetic monopole” – the remnant of the topological density $L_{\text{top}}$ – with charge $\pm m$. The sign of the monopole charge is determined by the sign of $\sigma_{xy}$ and the orientation of the integration contour for the corresponding Wess-Zumino term $\oint dt \text{Str} \pi_+ s^{-1} \dot{s}$, see Figure 5 and Sect. 3.4. For definiteness, we take $m < 0$ and $m > 0$ for the inner and outer edge, respectively.

Because of the influence of the probes, the SP at the outer edge gets perturbed in some way every time it passes a contact. (Qualitatively, passage past the contact collapses the wave function and puts the SP close to the origin of $\mathbf{G}/\mathbf{K}$; see Eq. (71) below.) When $\sigma_{xx}$ is large (high Landau level), the edge SP passes the perturbation on to its neighbors by the action of the coupling term of the Hamiltonian (22), thereby producing a complicated many-superparticle problem which we are unable to solve. Nevertheless, we can deal with such a situation by the following argument. We expect that there exists a mass gap, that is to say, a finite distance over which correlations decay. (This expectation comes from weak-coupling perturbation theory for the field theory (13), as was mentioned earlier. Ultimately, we would like to establish the existence of a mass gap by direct calculation for the quantum Hamiltonian $H$. This however remains as a project for the future. Here we will simply assume the existence of a gap and see what we can deduce from it.) When the length over which correlations decay is much larger than the lattice spacing, we are facing the many-SP problem we are unable to solve. However, we can use the renormalization group to improve the situation. Let us imagine changing the short-distance cutoff of the field theory (13). (We may think of this cutoff scale as being the lattice constant $a_1$ for $H$.) As we increase the cutoff, the correlation length (being a length) becomes shorter when measured in units of the cutoff. Continuing the process, we eventually get a correlation length of order unity or less. In this limit, the SPs propagate independently of each other, and the perturbation caused by the probes is not transferred to the interior of the disk but affects only the SP at the outer edge. With the assistance of the renormalization group, we thus get a one-SP problem. This problem we can solve. (Of course, our use of the renormalization group is naive and overly simplistic. No argument from the renormalization group should be trusted without further work when the correlation length is of the order of the lattice spacing or less. Nevertheless, the independent edge superparticle approximation is not without justification, for this approximation corresponds to transport occurring only via edge states localized near the sample boundary, see below.)

In the independent edge-SP approximation, the average conductance coefficients reduce to the dynamic one-SP correlation functions $\text{Tr} \prod_i O_i \exp \left( - (t_i - t_{i-1}) \mathcal{H}_s \right)$ with each operator insertion $O_i$ representing one contact. $\mathcal{H}_s = -\mathcal{L}^m$ is the one-SP Hamiltonian, Eq. (33). The imaginary time intervals are given by $t_i - t_{i-1} = 4L_{i,i-1}/a_1 \sigma_{xx}$ where $\sigma_{xx}$ is the renormalized longitudinal conductivity, and $L_{i,i-1}/a_1$ is the distance between the $(i-1)^{\text{th}}$ and the $i^{\text{th}}$ contact, measured in units of the lattice constant $a_1$.

We now need the expressions for the $\sigma$ model operators $O_i$ when the conductance coefficient is $g_{pq}$. It will be sufficient to consider the case $p \neq q$ since the diagonal coefficients
\( g_{qq} \) are determined through the off-diagonal ones by current conservation \( \sum_p g_{pq} = 0 \). To avoid getting into lengthy calculations that would detract from our main purpose, we will give a heuristic derivation. Let us first consider a diffusive quantum dot in a weak magnetic field (unitary universality class). In this case the formulas for the operators \( O_i \) can be extracted from Appendix B of Ref. [54]. Expressing them in terms of the complex coordinates \( Z, \tilde{Z} \) we find

\[
O_p = M_p \tilde{Z}_{11} \text{SDet}(1 - \tilde{Z}Z)^{M_p} \quad \text{(current)},
\]
\[
O_q = M_q Z_{11} \text{SDet}(1 - \tilde{Z}Z)^{M_q} \quad \text{(voltage)},
\]
\[
O_r = \text{SDet}(1 - \tilde{Z}Z)^{M_r} \quad \text{(none)},
\]

where \( M_i \) is the number of scattering channels in lead \( i \). (We are assuming the ideal limit of unit transmission into every open channel.) From these equations we can guess the analogous expressions for quantum Hall systems. The main novelty that occurs for a strong magnetic field as compared to a weak field is spatial separation of incoming and outgoing channels in the leads. Figure 6 depicts the situation when the incoming and outgoing channels are located on the left and right edges of the lead, respectively. For an ideal contact there are no direct transitions between the incoming and outgoing channels of the same lead. As a result, every contact operator must separate into two factors, one each for the left (L) and the right (R) side of the contact. For example, the expression for a voltage contact factors as

\[
O_q = M_q (Z_L)_{11} \text{SDet}(1 - \tilde{Z}_L Z_L)^{M_q/2} \times \text{SDet}(1 - \tilde{Z}_R Z_R)^{M_q/2}.
\]

The number of edge channels in the leads equals the number of occupied Landau levels \( m = \sigma_{xy} \). Making the identification \( M_p = M_q = \ldots = m \) and changing to coordinate-free notation we obtain

\[
O_p = v_m (B_{11} v_m, \cdot) \quad \text{(current)},
\]
\[
O_q = B_{11} v_m (v_m, \cdot) \quad \text{(voltage)},
\]
\[
O_r = v_m (v_m, \cdot) \quad \text{(none)},
\]

(71)

where \( v_m (Z, \tilde{Z}) = \text{SDet}(1 - \tilde{Z}Z)^{m/2}, (B_{11} v_m) (Z, \tilde{Z}) = m Z_{11} \text{SDet}(1 - \tilde{Z}Z)^{m/2}, \) and \( (\cdot, \cdot) \) is the inner product (38).

After these preparations, we are ready to proceed to the essential point of this subsection. We are going to calculate the correlation function \( \text{Tr} \prod_i O_i \exp(- (t_i - t_{i-1}) \mathcal{H}_s) \), using the expressions (71) for the operators \( O_i \). The following two observations enable us to do the calculation with ease. First, since \( m > 0 \), the ground state of \( \mathcal{H}_s \) is spanned by functions \( f = v_m \varphi \) with holomorphic (as opposed to antiholomorphic) \( \varphi \). The space of such functions is denoted by \( V_m \) as before, and we set \( v = v_m \) for brevity. The state \( B_{11} v \) lies in \( V_m \). Second, when inserting a complete set of intermediate states in the expression for the correlation function, we may restrict the sum over states to \( V_m \), since any contribution from the excited states of \( \mathcal{H}_s \) will be exponentially small of order \( \exp(- \text{const} \times \Delta L / \sigma_{xx}) \), where \( \Delta L \) is the distance between neighboring contacts. This allows us to replace \( \exp(- (t_i - t_{i-1}) \mathcal{H}_s) \) by the projection operator onto \( V_m \), denoted by \( P \).
We must now distinguish cases. Let us first consider \( g_{pq} \) for \( p = q + 1 \). As the time evolution outside the interval from \( t_q \) to \( t_{q+1} \) does no more than project onto \( v \), we obtain

\[
(g_{q+1,q}) = (B_{11}v, e^{-(t_{q+1}-t_q)\mathcal{H}_s}B_{11}v).
\]

Making an exponentially small error by replacing \( \exp(-(t_{q+1}-t_q)\mathcal{H}_s) \) by the projector \( P \), and using \( PB_{11}v = B_{11}v \), we get \( (g_{q+1,q}) = (B_{11}v, B_{11}v) \). Since \( (B_{11}v, B_{11}v) = |a_m|^2 = m \) by the first of Eqs. (59), we arrive at \( (g_{q+1,q}) = m \), which is precisely the result expected from the phenomenological picture of transmission through edge states. What happens when we interchange \( p \leftrightarrow q \)? As before, the time evolution outside the interval \([t_{q-1}, t_{q+2}]\) projects on \( v \), but now \( (g_{q,q+1}) = (v, PB_{11}v)(v, Pv)(B_{11}v, Pv) \) vanishes by the orthogonality relation \( (v, B_{11}v) = 0 \). Similarly, all \( (g_{pq}) \) with \( p \) and \( q \) separated by more than one unit vanish by orthogonality.

In the same way, we can calculate the conductance fluctuations. To get the second moment, what we do [48] is to replace \( Z_{11}(\bar{Z}_{11}) \) in the expression for a voltage (current) contact by \( Z_{11}Z_{22}(\bar{Z}_{11}(\bar{Z}_{22})) \). This amounts to substituting \( B_{11}B_{22}v \) for \( B_{11}v \) in (71). The factorization property \( (B_{11}B_{22}v, B_{11}B_{22}v) = (B_{11}v, B_{11}v)(B_{22}v, B_{22}v) \) leads to \( \text{var}(g_{pq}) = 0 \), so conductance coefficients do not fluctuate. All of the above results hold modulo exponentially small corrections and for ideal contacts and leads.

As it stands, our argument applies only to integral values of \( \sigma_{xy} \). It extends to other values by Khmel’nit’skii’s flow diagram [53] or, more precisely, by assuming the nonlinear \( \sigma \) model at \( (\sigma_{xx}, \sigma_{xy}) = (0, m \in \mathbb{Z}) \) to be an attractive fixed point of the renormalization group – with the basin of attraction being the strip \( m - 1/2 < \sigma_{xy} < m + 1/2 \). (In the introduction to this paper we reviewed the argument why the critical nonlinear \( \sigma \) model at \( \sigma_{xy} = 1/2 \) (mod 1) cannot be renormalizable but must flow under renormalization to another, conformal invariant field theory. There is no doubt, however, that the noncritical system will be attracted to the \( \sigma \) model with \( \sigma_{xx} \rightarrow 0 \) and \( \sigma_{xy} \) an integer.)

In summary, for \( \sigma_{xy} \) close to \( m \in \mathbb{N} \) we have argued that the conductance coefficients

\[
g_{21} = g_{32} = \ldots = m
\]

are integers, while all others (except for the diagonal ones, which are not easily accessible by direct calculation but are determined by current conservation) vanish. By the reasoning of Ref. [4], this explains the experimental observation of the integer quantum Hall effect for systems with ideal probes.

5 Quantum Spin Chain

Apart from demonstrating how quantization of the Hall conductance arises in the \( \sigma \) model framework, the calculation of Sect. 4.6 serves another useful purpose by throwing much light on the physical meaning of the holomorphic zero-energy module \( V_m \) \( (m > 0) \) and its antiholomorphic dual \( V_{-m} \). The lesson we have learnt is this. The voltage contact \( q \) injects probability flux into the \( m \) edge states emanating from the lead \( q \). In our field-theoretic formalism the process of injection corresponds to preparing the edge superparticle (SP) in the quantum state \( B_{11}v_m \). This state, being a zero-energy state of the one-SP Hamiltonian \( -\mathcal{L}^m \), evolves in imaginary time without attenuation. A rather different
behavior obtains when we replace $L^m$ by its dual $L^{-m}$, which corresponds to reversing the sign of the magnetic field. In this case $B_{11} v_m$ lies outside the zero-energy space of $-L^{-m}$, and therefore $\exp(t L^{-m}) B_{11} v_m$ decays exponentially as $t$ increases, with the decay time being determined by the smallest excitation energy of $-L^{-m}$. (Of course, here as always “time” is one of the two directions of the 2d quantum Hall system.) We thus see that evolution of edge SP states over long times (i.e., distances) is directed by the magnetic field. This directedness is the field-theoretic equivalent of $E \times B$ drift along a boundary.

Having understood this, we now turn to the critical value $\sigma_{xy} = 1/2$ (mod 1) corresponding to a point of transition between Hall plateaus. The quantum Hamiltonian, $H$, of the nonlinear $\sigma$ model for such a point has been given in Eq. (36). $H$ differs from the Hamiltonian for a plateau region by the alternating sequence of monopole Laplacians $L^\pm$ replacing the plain Laplacian $L$. This difference is crucial: while $-L$ has a nondegenerate ground state with a gap, the operators $H := -L^+$ and $H^* := -L^-$ possess infinitely many zero-energy states, see Sect. 4.5. These are given by holomorphic and antiholomorphic functions for $H$ and $H^*$, respectively. In the absence of the coupling $H_1 = \sum_i \Delta(g_i, g_{i+1})$, a superparticle prepared in such a zero-energy state evolves in time without attenuation. Note that the unattenuated propagation of superparticles in holomorphic/antiholomorphic states in the bulk is strongly reminiscent of the semiclassical picture of electron guiding center drift along percolating equipotential lines. Our interest is in the critical long-range correlations. These will not be affected by the high-energy modes of $H$ and $H^*$ – they do no more than give rise to complicated and irrelevant short-distance physics – but are determined by the zero-energy modes coupled together by the interaction Hamiltonian $H_1$. Degenerate perturbation theory tells us how to construct the critical low-energy theory: we are to project the full space of quantum states of the $\sigma$ model on the alternating product of zero-energy modules $V_{+1} \otimes V_{-1} \otimes V_{+1} \otimes V_{-1} \otimes ...$. Calculating the matrix elements of $H_1$ between the states of this multi-"spin" space, we will get the Hamiltonian of a quantum spin chain.

To provide further motivation and background to Sect. 5.1, let us anticipate how the spin chain Hamiltonian is related to established theory. Starting from a quantum Hall system in the semiclassical high-field limit, where electrons drift slowly along the equipotential lines of a smooth random potential while performing rapid cyclotron motion, Chalker and Coddington [6] formulated a network model that mimics the relevant effects of quantum tunnelling near saddle points. The links of the network are directed, see Figure 7a. Disorder is modelled by random 2x2 scattering matrices $S$ connecting the incoming and outgoing channels at each node of the network. To bring the Chalker-Coddington model closer to the model that will emerge in Sect. 5.1, we take its Hamiltonian (or anisotropic) limit. This is done by setting $S = \exp(-i \epsilon h)$, with $h$ some random hermitean 2x2 matrix, and sending $\epsilon \to 0$. The resulting Hamiltonian dynamics is illustrated graphically in Figure 7b. Electrons run up or down on straight lines of alternating direction, and with a small probability amplitude ($\sim \epsilon$) they tunnel from one line to a neighboring one, thereby reversing their direction of motion and acquiring a random change of phase. One can carry out the disorder average for this Hamiltonian version of the Chalker-Coddington model by using the supersymmetry method. Doing so, one arrives rather directly at the spin chain Hamiltonian of Sect. 5.1 [56], with electrons running up and down straight lines translating into degrees of freedom whose quantum states are specified by holomorphic
and antiholomorphic functions, respectively.  

The nonlinear $\sigma$ model is derived from the quantum limit of a random potential with correlation length $l_c$ short compared to the magnetic length $l_B$. We are discovering that its low-energy limit converges to a model associated with the opposite limit $l_c \gg l_B$. This coincidence is a beautiful vindication of the hypothesis of low-energy universality at a critical point.

5.1 Mapping on a Spin Chain

We start with some basic definitions underlying the notion of what we are going to call a generalized “spin” chain. (We really ought to say “superspin” instead of just “spin”, but we prefer to use the shorter and less pretentious word when there is no risk of confusion.) As always, let $E_{ij}$ $(i,j = 1,...,p+q)$ be a set of canonical generators of $\text{End}(\mathbb{C}^{p+q})$ obeying the multiplication law $E_{ij} E_{kl} = \delta_{jk} E_{il}$. $\text{End}(\mathbb{C}^{p+q})$ is $\mathbb{Z}_2$-graded by $|E_{ij}| = |i| + |j|$ mod 2, where $|i| = 0$ for $1 \leq i \leq p$ and $|i| = 1$ for $p+1 \leq i \leq p+q$. The introduction of a bracket operation (“supercommutator”)

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-)^{|i|+|j|(|k|+|l|)} \delta_{il} E_{kj}$$

(72)

turns $\text{End}(\mathbb{C}^{p+q})$ into the Lie superalgebra $\text{gl}(p, q; \mathbb{C})$, with the quadratic Laplace-Casimir element being $\sum_{ij} (-)^{|j|} E_{ij} E_{ji}$. $\text{gl}(2n, 2n; \mathbb{C})$ can be identified with the complex Lie superalgebra of the symmetry group $G$ of the quantum Hall model space $G/K$ by the canonical isomorphism

$$\text{gl}(2n, 2n; \mathbb{C}) \simeq \text{gl}(1, 1; \mathbb{C}) \otimes \text{gl}(2) \otimes \text{gl}(n).$$

(73)

(Recall that the three factors in the tensor product relate to superspace, advanced-retarded space and extra space, in this order.)

With $V := V_{+1}$ and the isomorphism (73) being understood, we introduce $\mathcal{E}_{ij} : V \rightarrow V$ by $\mathcal{E}_{ij} = dT(E_{ij})$, where $dT$ is the differential taken at the unit element of the representation $g \mapsto T_g$ defined by Eq. (71). The $\mathcal{E}_{ij}$ are linear differential operators on the space of functions $f \in V$. We shall refer to them as “spin operators”. They obey the supercommutation relations

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = (-)^{|i|+|j|(|k|+|l|)} \delta_{jk} \mathcal{E}_{il} - \delta_{il} \mathcal{E}_{kj}.$$  

(74)

It is seen that these do not properly represent the superalgebra (72), because of the appearance of overall extra minus signs. (Their origin is explained in Appendix C.) Nevertheless, the bracket (74) does define a Lie superalgebra and, as a matter of fact, it is a perfectly sensible object to work with. Operators $\mathcal{E}_{ij}^*$ on the dual space $V^* := V_{-1}$ are defined in identical fashion. They, too, obey the superalgebra (74). By Eq. (71), the set $\{\mathcal{E}_{ij}\}_{i,j=1,...,4n}$ naturally decomposes into four sets of operators $\mathcal{A}_{kl}$, $\mathcal{B}_{kl}$, $\mathcal{C}_{kl}$, and $\mathcal{D}_{kl}$ $(k, l = 1,...,2n)$. The same decomposition is made for the spin operators $\mathcal{E}_{ij}^*$. For future reference we note the action of the spin operators on the special vectors $v = v_1 \in V$ and $v^* = v_1 \in V^*$:

$$\mathcal{C}_{ij} v = \mathcal{B}_{ij}^* v^* = 0,$$

$$\mathcal{A}_{ij} v = \mathcal{D}_{ij} v = \mathcal{A}_{ij}^* v^* = \mathcal{D}_{ij}^* v^* = 0 \quad (i \neq j),$$

$$\mathcal{A}_{ii} v = (-)^{|i|} v/2, \quad \mathcal{D}_{ii} v = 0,$$

$$\mathcal{A}_{ii}^* v^* = (-)^{|i|+1} v^*/2, \quad \mathcal{D}_{ii}^* v^* = 0.$$  

(75)
These express the fact that \( v \) and \( v^* \) are lowest- and highest-weight vectors carrying the one-dimensional \( K \)-representations \( \mu = \exp \text{STr} \pi_+ \ln \) and \( \mu^{-1} = \exp -\text{STr} \pi_+ \ln \), respectively.

We turn to the formulation of the two-spin Hamiltonian. For two sites, the \( \sigma \) model Hamiltonian \( H_1 \), Eq. (36), has \( V \otimes V^* \) for its degenerate space of strong-coupling ground states. The degeneracy is split by the two-site interaction \( H_1 = \sigma_{xx} \Delta(g,h)/a_1 \) orthogonally projected onto \( V \otimes V^* \). Orthogonality is defined w.r.t. to the invariant scalar product (38), extended to tensor product space by

\[
(a \otimes b^*, c \otimes d^*) = (-)|b^*||c| (a, c)(b^*, d^*).
\]  

(76)

We wish to express the projected Hamiltonian \( H_{\text{eff}} := PH_1 P \) in terms of spin operators. Since \( \Delta(g,h) \) is \( G \)-invariant and invariance is preserved by orthogonal projection, \( H_{\text{eff}} \) must be some Laplace-Casimir element of \( G \) formed from the spin operators on \( V \) and \( V^* \). The question is now, which Laplace-Casimir? Introducing \( Q(g) = g \Lambda g^{-1} \) we can write \( \Delta(g,h) = \text{STr} Q(g) Q(h)/4 \). A symmetry transformation \( g \mapsto xg \) \((x \in G)\) takes \( Q(g) \) into \( xQ(g)x^{-1} \). The unique set of operators \( V \rightarrow V \) \((V^* \rightarrow V^*)\) with this transformation behavior, which is identical to that of the adjoint representation, are the spin operators \( E_{ij} \) \((E_{ij}^*)\). It follows that \( H_{\text{eff}} \) must be proportional to the quadratic Laplace-Casimir element:

\[
H_{\text{eff}} = J \sum_{ij} (-)|i|^{\pm 1} E_{ij} \otimes E_{ij}^*.
\]

This operator commutes with the two-spin generators \( E_{kl} \otimes 1 + 1 \otimes E_{kl}^* \), as is easily verified by using the supercommutation relations (74). By construction, \( H_{\text{eff}} \) is hermitean w.r.t. to the invariant scalar product (76). (The interaction Hamiltonian \( \Delta(g,h) \) obviously is, and hermiticity is preserved by orthogonal projection.) The constant of proportionality \( J = 4\sigma_{xx}/a_1 > 0 \) is finite and is found by calculating some special nonzero matrix element of \( H_1 \). This is done in Appendix D. From now on we will often omit the tensor-product symbol in multi-spin operators, keeping it for clarity however when writing multi-spin states.

It is clear how to generalize all this to the case of \( 2N \) spins. The degenerate space of strong-coupling ground states becomes \( V \otimes V^* \otimes V \otimes V^* \otimes \ldots \) \((2N \text{ factors})\), and the scalar product (76) is extended in the natural way. (We just keep track of the minus signs that arise from changing the ordering of the fermions.) Orthogonal projection yields the multi-spin Hamiltonian

\[
H_S = \sum_{\text{even}} \sum_{l \text{ even}} (-)|i|^{\pm 1} \left( J_+ E_{l,ij} E_{l+1,ji}^* + J_- E_{l,ij} E_{l-1,ji}^* \right), \quad J_+ = J_- = J.
\]

(77)

We have thus arrived at a spin chain reformulation of the nonlinear \( \sigma \) model with topological coupling \( \sigma_{xy}^\pi = 1/2 \) \((\text{mod } 1)\). The spin Hamiltonian (77) is completely specified by the commutation relations (74) for the spin operators \( E_{ij} \) and \( E_{ij}^* \) and their action (74) on the lowest- and highest-weight vectors \( v \) and \( v^* \). The spatial boundary conditions are chosen to be periodic. (In the case of open boundary conditions, we need to modify the modules \( V, V^* \) for the edge spins in general; see Sect. 4.6.) According
to what was said at the end of Sect. 3.5, we can incorporate deviations of $\sigma_{xy}$ from $\sigma^*_{xy}$ by staggering the spin-spin interaction, i.e. by putting $J_\pm = (1 \pm \epsilon)J$ with $\epsilon \sim \sigma_{xy} - \sigma^*_{xy}$.

Because the spin chain is built from alternating modules $V$ and $V^*$, translation by one site cannot ever be a symmetry of the Hamiltonian $H_S$. However, for $J_+ = J_-$ and periodic boundary conditions, $H_S$ possesses another symmetry which is just as good. There exists a canonical isomorphism $T : V \rightarrow V^*$, $f \mapsto \bar{f}$ (complex conjugation). $T$ squares to superparity by our using an adjoint of the second kind, $\bar{f} = (-)^{|f|} f$. It extends to a map $T : V \otimes V^* \otimes ... \rightarrow V^* \otimes V \otimes ...$ in the natural way. From Eqs. (44) and (45) one can show that the two-spin Hamiltonian behaves under complex conjugation as follows:

$$T^{-1} \left( \sum_{ij} (-)^{|i|} \epsilon_{ij} \otimes \epsilon^*_{ji} \right) T = \sum_{ij} (-)^{|j|} \epsilon^*_{ji} \otimes \epsilon_{ij}.$$ 

This equation implies that the operator composed of translation by one site followed by complex conjugation, *commutes* with the Hamiltonian $H_S$ of a periodic chain with $J_+ = J_-$. The Hamiltonian $H_S$ has been defined over the spin spaces $V = V_{+1}$ and $V^* = V_{-1}$. Actually, we can formulate a whole family of Hamiltonians $H^{(m)}_S$ by taking $V = V_m$ and $V^* = V_{-m}$ with $m$ any positive integer. Since we know $m$ to be the number of “edge states”, it is clear which role to assign to $H^{(m)}_S$: it applies to a situation where $m$ Landau levels are strongly mixed by the disorder. This is of experimental relevance for $m = 2$ since, for a weak magnetic field, the broadening caused by the disorder may exceed the Zeeman splitting and render Landau levels spin-degenerate. A plausible global phase diagram for this case has been suggested by D.K.K. Lee and Chalker [57]. (See also a recent preprint by D.H. Lee [58] in this context.) Here we simply mention that, by using the coherent states of Sect. 4.5 to derive a functional integral representation of the $m = 2$ chain, we retrieve the nonlinear $\sigma$ model ([13]) with $\sigma_{xy} = 0$ (mod 1). This implies a finite localization length in the center of the Landau band, in accordance with the conclusions reached in Ref. [57].

We have argued that projection of the $\sigma$ model on the superspin chain should retain the essential long wave length quantum Hall physics. Therefore, we expect the chain for $m = 1$ and $J_+ = J_- $ where a delocalization transition is known to occur, to be massless. Clearly, a direct proof of masslessness would be very welcome. A useful pool of ideas we can draw from is provided by the Lieb-Schultz-Mattis theorem [59] for isotropic antiferromagnetic (ordinary) spin $S = 1/2$ chains, which has been generalized to arbitrary half-integer $S$ by Affleck and Lieb [60]. What these authors do is to construct a variational state (orthogonal to the ground state) with excitation energy of order $1/L$ for a chain of length $L$. By the variational principle, the existence of such a state implies that the Hamiltonian either has zero gap, or else a doubly degenerate ground state with a finite gap, in the thermodynamic limit $L \rightarrow \infty$. Because of the resemblance between the $S = 1/2$ spin chain and the superspin chain with Hamiltonian ([77]), it seems that it should be possible to transcribe the theorem and its proof. Unfortunately, the transcription has not been done successfully up to now, for but one reason which however is vital: in order for the variational principle to apply, we need the Hamiltonian to be hermitean with respect to a quadratic form.
which is positive definite. As we have seen, positivity is not a property enjoyed by the invariant scalar product \( \langle \rangle \). Nevertheless, as we shall see, the two-superspin Hamiltonian is completely diagonalizable and has positive excitation energies. We are thus encouraged to think that the difficulty is not of a fundamental kind but is a technical complication that can be overcome.

To compute from the (super-)spin chain such physical observables as the conductance coefficients \( \langle g_{pq} \rangle \), we need the spin chain expressions for the “contact operators” \( O_i \) of Sect. 2.2. For an ideal contact connected to a single spin, we know these to be given by \( O_p = v(B_{11} v, \cdot) \) for current, \( O_q = B_{11} v(v, \cdot) \) for voltage, and \( P_v = v(v, \cdot) \) or \( P^*_v = v^*(v^*, \cdot) \) for a contact which is neither voltage nor current; see Sect. 4.6. In that subsection, we calculated the Hall conductance for a disk geometry using the independent edge spin approximation. Having constructed the Hamiltonian \( H_S \) governing the evolution of the interacting spin chain, we can now study the conductance coefficients in principle in the following, more interesting setting. We assume periodic boundary conditions in both time and space direction with lengths \( L_0 \) and \( L_1 \), respectively. (Periodicity in space eliminates the boundary current.) \( L_0 \) is chosen to be very large. We select two times \( t_i < t_f \) where all spins are connected to “particle reservoirs”. Recall that the effect of the reservoirs is to hit every degree of freedom in \( V \) or \( V^* \) with a projector \( P_v \) or \( P^*_v \), at the corresponding time. For \( L_0 \rightarrow \infty \), the time evolution outside the fixed interval \([t_i, t_f]\) projects on the ground state of \( H_S \). In Sect. 5.2 we will show that this ground state has zero energy and contains the state \( \psi_0 = v \otimes v^* \otimes v \otimes v^* \otimes \ldots \) with unit amplitude. To measure a conductance \( \langle g \rangle \), we connect the \( k^{\text{th}} \) \( V \)-spin, say, to a voltage probe at \( t_i \) and the \( l^{\text{th}} \) \( V \)-spin to a current probe at \( t_f \). From the above expressions for \( O_p \) and \( O_q \), we then get

\[
\langle g \rangle = \langle \psi_f, \exp (-|t_f - t_i| H_S) \psi_i \rangle \tag{78}
\]

where \( \psi_i \) (\( \psi_f \)) is the multi-spin state obtained from \( \psi_0 \) by creating one quantum of excitation \( v \rightarrow B_{11} v \) in the \( k^{\text{th}} \) (\( l^{\text{th}} \)) \( V \)-spin. When \( L_1 \) is held fixed, \( \langle g \rangle \) must decay exponentially with distance \( |t_f - t_i| \) over some length \( \xi \), the localization length. For the critical chain \( \xi \) will be proportional to \( L_1 \) for \( L_1 \) large. \( \langle g \rangle \) is expected to be governed by the laws of conformal invariance. However, since the definition of \( \langle g \rangle \) involves two strings of operator insertions at the times \( t_i \) and \( t_f \), its behavior under conformal transformations is not elementary. (The two strings impose boundary conditions.) Things simplify if we close off all the contacts which are neither voltage nor current. Then \( \langle g \rangle \) turns into a spin-spin correlation function which measures conductance in a two-terminal (small contact) geometry of the kind discussed in Sect. 2.2. In this case, the hypothesis of conformal invariance relates the constant of proportionality between \( L_1 \) and the localization length \( \xi \) to the critical index governing the algebraic decay of \( \langle g \rangle \) in the infinite system \( \mathbb{R} \).

To conclude this subsection, we mention that the \( \sigma \) model expressions for the scaling fields of Sect. 2.1 are polynomials in the elements of the matrix-valued function \( Q_{1j}(g) = (g \Lambda g^{-1})_{ij} \); see e.g. Ref. [28]. Calculating the matrix element of such a polynomial \( Q_{1i,j} Q_{1j,2i} \ldots Q_{1i,j} \), between two states in \( V \) or \( V^* \), we get an integral which is divergent for \( r \geq 2 \) because of the noncompactness of \( \mathbb{G}/\mathbb{K} \). (The matrix elements between excited states of the monopole Hamiltonian are divergent for all \( r \geq 1 \).) In other words, these \( \sigma \) model operators do not have a sensible restriction to the spin chain. (They are well-defined only when the system is perturbed by a finite frequency \( \omega + i \varepsilon \).) This is a technical
manifestation of the distinction between the different observables discussed in more direct physical terms in Sects. 2.1-2.

5.2 Solution of the two-spin problem

In Sect. 5.1 we constructed a spin Hamiltonian $H_S$ by projection of the quantum Hamiltonian of the nonlinear $\sigma$ model. Solving for the ground state and the low-energy excitations of $H_S$ we will get an analytic description of the critical properties of integer quantum Hall systems. However, such a solution is not easy to find and has not been found yet. Here we will have to be satisfied with doing some preparatory work toward that ambitious goal.

The two-spin Hamiltonian $H := \sum_{ij} (-)^{|i|+1} E_{ij} E_{ji}^*$ is an invariant second-order differential operator acting on wave functions

\[ (f \otimes f^*)(Z, \tilde{Z}, Z', \tilde{Z}') = v(Z, \tilde{Z})v(Z', \tilde{Z}')\varphi(Z)\varphi^*(\tilde{Z}) \]

with holomorphic $\varphi$ and antiholomorphic $\varphi^*$. The following trick allows us to reduce the problem of diagonalizing $H$ to a problem that has been solved before. Because $f$ and $f^*$ are completely specified by the dependence of $\varphi$ and $\varphi^*$ on $Z$ and $\tilde{Z}'$, respectively, we may identify $Z \equiv Z'$ and $\tilde{Z} \equiv \tilde{Z}'$ without incurring any loss of information. On making this identification, $H$ turns into an invariant second-order differential operator acting on functions

\[ F(Z, \tilde{Z}) = v(Z, \tilde{Z})^2 \varphi(Z)\varphi^*(\tilde{Z}). \] (79)

There exists only one such invariant differential operator. This is the Laplacian $\mathcal{L}$ for $G/K$ or, to be precise, $\mathcal{L}$ restricted to the truncated space of functions on $G/K$ of the form (79). Therefore, $H = c\mathcal{L}$ with some constant $c$ and in the restricted sense just specified. In Appendix E we show $c = -1/2$.

The Laplacian $\mathcal{L}$ for Riemannian supersymmetric spaces $G/K$ has been the object of some amount of study. A general procedure for the construction of its eigenfunctions, using such concepts as the Iwasawa decomposition of a semisimple complex Lie group and Harish Chandra’s formula, has been described in [48]. We are not going to review this procedure here but will specialize to $n = 1$ for brevity. In this case, an accidental simplification occurs which enables us to short cut the general theory.

The key to making progress – for $n = 1$ as in general – is to consider the action of $\mathcal{L}$ on $K$-radial functions $F(\pi(g)) = F(\pi(kg)) \ (k \in K)$. By the Cartan decomposition $G = KA^+K$ [10] such functions can be regarded as functions on the positive part $A^+$ of a maximal abelian subgroup $A \subset G$ with Lie algebra contained in $\mathcal{P}$, the tangent space of $G/K$ at $\pi(1)$. In the present case, $A \simeq \mathbb{R} \times S^1$, and $A^+$ is conveniently parametrized by two coordinates $x, y$ in the range $1 \leq x < \infty$ and $-1 \leq y \leq 1$. (These are the polar coordinates of Efetov, denoted by $\lambda_1, \lambda$ in [22].) The origin $\pi(1)$ has the coordinates $x = y = 1$. The invariant Berezin measure $dg_K$ on $G/K$ induces a scalar product on the space of $K$-radial functions defined by

\[ (F_1, F_2)\# = \int_1^\infty dx \int_{-1}^1 dy \ (x - y)^{-2} \frac{F_1(x, y)}{F_1(x, y)} F_2(x, y). \] (80)

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With $S$ being the function $S(x, y) = x - y$, the $K$-radial part $L^\#$ of $L$ is the operator
\[ L^\# = S \left( L^\#_{H^2} + L^\#_{S^2} \right) S^{-1} \]
where $L^\#_{H^2} = 4\partial_x(x^2 - 1)\partial_x$ and $L^\#_{S^2} = 4\partial_y(1 - y^2)\partial_y$ are the radial parts of the Laplacians on the two-hyperboloid and the two-sphere, respectively. Note that $L^\#$ is hermitean w.r.t. the scalar product $(80)$ on the space of $K$-radial functions that vanish as $x - y$ at the point $x = y = 1$.

The separable form of $L^\#$ makes it possible to write down its eigenfunctions $\phi_{\lambda, l}$ immediately:
\[ \phi_{\lambda, l}(x, y) = (x - y)P_{-(\lambda + 1)/2}(x)P_{(l-1)/2}(y). \]
Here, $P_\nu(x)$ is a Legendre function and $P_n(y)$ a Legendre polynomial $[61]$. The eigenvalue is $-(\lambda^2 + l^2)$, and the quantum numbers have the range $\lambda \in \mathbb{R}^+$ and $l \in 2\mathbb{N} - 1$. Because of the special form of the functions $(79)$, not all of these quantum numbers occur in the spectral expansion of the two-spin Hamiltonian $H$. Since $V = V_{+1}$ and $V^* = V_{-1}$ both carry angular momentum (or SU(2)-spin) $\leq 1/2$, the tensor product $V \otimes V^*$ carries at most angular momentum 1. The largest angular momentum present in $\phi_{\lambda, l}$ is seen to be $(l + 1)/2$. Therefore, the only $l$-quantum number that occurs for $m = 1$ is $l = 1$. (There is no restriction on $\lambda \in \mathbb{R}^+$.)

The functions $\phi_{\lambda, l}$ all vanish at the point $x = y = 1$ corresponding to $Z = \bar{Z} = 0$. The two-spin quantum space $V \otimes V^*$ contains the state $v \otimes v^*$ whose wave function does not vanish at $Z = \bar{Z} = 0$ so, clearly, something is still missing. What is missing and must be added for completeness is the constant function $\phi_0 = 1$, which trivially is an eigenfunction of
\[ L^\# = 4(x - y)^2 \left( \frac{\partial}{\partial x} \frac{x^2 - 1}{(x - y)^2} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{1 - y^2}{(x - y)^2} \frac{\partial}{\partial y} \right) \]
with eigenvalue zero. Using $\phi_0 = 1$ together with the functions $\phi_{\lambda, l}$ we can expand any $K$-radial function on $G/K$ with “good” behavior at infinity. (See [47] for more insight into the precise role played by $\phi_0$ in a prototypical example.)

Given a $K$-invariant stationary two-spin state $\psi$ we can generate $K$-noninvariant stationary states by applying symmetry transformations to the equation $H\psi = E\psi$. Doing this in a systematic fashion by using the methods of supersymmetric Fourier analysis $[17,15]$, we obtain a continuous set of $G$-multiplets labelled by $\lambda, l$ (with $l = 1$ being the only quantum number occurring for $m = 1$) that are eigenspaces of $H$ and completely exhaust $V \otimes V^*$.

To put this in perspective, recall the indefiniteness of the invariant scalar product $(83)$, which jeopardizes the diagonalizability of $H : V \otimes V^* \to V \otimes V^*$. (In fact, it is precisely because of this indefiniteness that Laplace-Casimir elements of Lie superalgebras generically cannot be brought to diagonal form.) What saves the day is the existence of an auxiliary scalar product $(\cdot, \cdot)^\#$ on a restricted space $(V \otimes V^*)^\#$ of $K$-invariant states. This auxiliary scalar product has all the attributes of a (Hilbert space) scalar product, and $H : (V \otimes V^*)^\# \to (V \otimes V^*)^\#$ is hermitean with respect to it. This in conjunction with $G$-symmetry guarantees the diagonalizability of $H$.  

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Note the pronounced “antiferromagnetic” character of $H$: it has a $(G^{-})$-singlet ground state separated by a gap from a continuum of non-singlet excited states. The existence of a gap implies exponential decay of correlations in the two-spin system, which is related to exponential localization of all states in a Chalker-Coddington network model with only two links in the short, spatial direction. For the spin chain, exponential decay comes about because the repeated action of $H$ on the state $\psi_i = (B_{11}v) \otimes v^*$ creates on average an ever increasing number of quanta of excitation, eventually producing a time-evolved state which has exponentially small overlap with $\psi_f = \psi_i$. In the network model, exponential localization sets in when the elements of the transfer matrix become exponentially large. Thus, growth in the number of $B$-(and $C^*$-) quanta is the spin chain equivalent of growth in the transfer matrix amplitudes of the network model. Using the Fourier decomposition of the Dirac delta distribution $\delta_o$ for $G/K$ [43],

$$\delta_o = \phi_0 + 2^2 \sum_{l \in 2 \mathbb{N} - 1} \int_0^\infty d\lambda \frac{\lambda \tanh(\pi \lambda/2)}{(\lambda^2 + l^2)^2} \phi_{\lambda,l},$$

and doing the same calculations as in [48], one can derive an exact expression for the (two-spin) conductance $\langle g \rangle$ of Eq. (78). $\langle g \rangle$ decays as $|t_f - t_i|^{-3/2} \exp(-J|t_f - t_i|/2)$ for large distance $|t_f - t_i|$.

Let us now present an independent argument showing why the ground state of $H$ has zero energy. This argument has the merit of permitting easy generalization to many spins. We write the two-spin Hamiltonian in the form

$$H = \sum_{ij} (-)^{|i|+1} (A_{ij}A^*_{ji} + B_{ij}C^*_{ji} + C_{ij}B^*_{ji} + D_{ij}D^*_{ji})$$

and consider the state $v \otimes v^*$. It is a $K$-singlet since $T_kv = \mu(k)v$ and $T_kv^* = \mu(k)^{-1}v^*$. Acting with $H$ on it we get

$$H(v \otimes v^*) = \sum_{ij} (-)^{|i|+1} (B_{ij}v) \otimes (C^*_{ij}v^*),$$

as follows from Eqs. (73). Let $L_M$ ($M \geq 0$) denote the $(M+1)$-dimensional space of states generated by at most $M$ actions of $H$ on $v \otimes v^*$. Since $H$ commutes with the generators of $K$, all elements in $L_M$ are $K$-invariant. Taking $M \to \infty$ we obtain a space, $L_\infty$, which is closed under the action of $H$. (We here pay no attention to mathematical subtleties that might arise from the infinite-dimensionality of $L_\infty$.) This space is decomposed by $L_\infty = L_{M=0} \oplus L_c$ (direct and orthogonal sum). We claim that $HL_\infty \subset L_c$. To prove this statement it suffices to prove $H\psi \in L_c$ for $\psi \in L_c$, since $H(v \otimes v^*) \in L_c$ from Eq. (82). Let $N = \sum_i (A_{ii} - D_{ii})/2$ be the operator counting the number, $N$, of quanta of excitation in the first factor of $V \otimes V^*$. We have $N(v \otimes v^*) = 0$. The commutation relations $[N, B_{ij}] = B_{ij}, [N, C_{ij}] = -C_{ij}, [N, A_{ij}] = [N, D_{ij}] = 0$ imply that $H$ changes $N$ by at most one unit. $L_c$ is a direct sum of spaces with $N \geq 1$. Let $\psi$ now be a state with $N = 1$. There exists only one such state which has the additional property of being $K$-invariant. This is the state on the right-hand side of Eq. (82). Using the relations (74) and (75), we
easily see that \( \sum_{ij} (-)^{|i|+1} C_{ij} B_{ji}^* \) annihilates \( \psi = \sum_{ij} (-)^{|i|+1} (B_{ij} v) \otimes (C_{ji}^* v^*) \), so \( H \psi \) has zero intersection with \( v \otimes v^* \). But the image \( H \psi \) of any state \( \psi \in L_c \) with more than one quantum of excitation in \( V \) carries at least one such quantum, which proves \( HL_c \subset L_c \).

We conclude that the cokernel of \( H : L_\infty \to L_\infty \) is (at least) one-dimensional. Therefore, if \( H \) can be brought to diagonal form – and we know it can be from the previous discussion – it must have an eigenstate with zero energy. On physical grounds, this eigenstate is expected to be the ground state. (Because the partition sum equals unity, the existence of a negative-energy state would give rise to unphysical spin correlations that increase exponentially with distance.)

The result \( HL_\infty \subset L_c \) has the following consequence which is worth mentioning. \( L_0 \) and \( L_c \) are orthogonal spaces w.r.t. the scalar product \( \langle \psi_1, \psi_2 \rangle \). Hence, by the hermitecity of \( H \),

\[
(H^N(v \otimes v^*), H^N(v \otimes v^*)) = (v \otimes v^*, H^{2N}(v \otimes v^*)) = 0,
\]

so the space \( L_c \) is null, i.e. all of its elements have vanishing norm.

The above argument readily generalizes to an arbitrary even number of spins. Let \( L_0 \) denote the linear space spanned by \( \psi_0 = v \otimes v^* \otimes ... \otimes v \otimes v^* \). If \( \psi \) is some state that differs from \( \psi_0 \) on more than two neighboring sites, then clearly \( HS \psi \cap L_0 = \{0\} \). If \( \psi \in H_{SM}^{M-1} L_0 \) \( (M \geq 1) \) differs from \( \psi_0 \) on no more than two neighboring sites, we are back to the two-spin problem analyzed before and again we have \( HS \psi \cap L_0 = \{0\} \). This shows \( H_{SM}^M L_0 \) to have zero intersection with \( L_0 \). Hence, the rank of \( H_S \) is reduced by one and we expect a zero-energy (ground) state as before.

A neat application of the results of this subsection is the following. When the process of quantum tunnelling of electrons across saddle points is switched off, the Chalker-Coddington model (or, rather, a slight generalization thereof \[62\]) reduces to a network model which undergoes a classical percolation transition. This phase transition is in a different universality class and has correlation length exponent \( \nu = 4/3 \) \[63\]. With a complete solution of the two-spin problem in hand, we can now write down a spin chain Hamiltonian for the classical percolation transition, too. To do so, it is easiest to make a checkerboard decomposition of the Hamiltonian and pass to a vertex model \[77\] by exponentiation. The vertex model is specified by its \( R \)-matrix \( R : V \otimes V^* \to V^* \otimes V \) (Figure 8). Because of the connection of the supersymmetric vertex model with an underlying disordered network model, it is clear that the \( R \)-matrix at the critical point of the percolation transition must be

\[
R_{ab^*,c^*d} = \frac{1}{2} (\delta_{ad} \delta_{bc^*} + \delta_{ab} \delta_{c^*d}).
\]

(In the network model, electrons turn either right or left at a node, with equal probabilities. Correspondingly, the spin degrees of freedom of the vertex model must be transferred by the \( R \)-matrix to the right or left, with equal weights.) On general grounds, the \( R \)-matrix \[83\] must meet the requirement of \( G \)-invariance, and it does. With \( P : a \otimes b^* \to (-)^{|b^*||a|} b^* \otimes a \) the graded permutation operator, \( R \) can be written in the form \( R = P(1 + \Pi_0)/2 \) where \( \Pi_0 \) projects on the unique \( G \)-singlet state contained in the tensor product \( V \otimes V^* \). (Of course, this singlet state is nothing but the ground state of the two-spin Hamiltonian discussed above.) Restricting the resolution of unity \( \xi_\lambda \) to \( V \otimes V^* \), we obtain

\[
R = P \tilde{R}, \quad \tilde{R} = 1 - \frac{1}{2} \int_{\mathbb{R}^+} d\mu(\lambda) \Pi_\lambda.
\]
Here $d\mu(\lambda) = 2^2 \lambda \tanh(\pi \lambda/2)(\lambda^2 + 1)^{-2}d\lambda$, and $\Pi_\lambda$ is the spectral projector onto total $SU(1,1)$-spin with quantum number $\lambda$, normalized by the condition

$$
\left( \int_{\mathbb{R}^+} d\mu(\lambda) h(\lambda) \Pi_\lambda \right)^2 = \int_{\mathbb{R}^+} d\mu(\lambda) h(\lambda)^2 \Pi_\lambda.
$$

To return to a Hamiltonian formulation, we have to compute the logarithm of the transfer matrix. We put $\check{R}^\epsilon = \exp(-\epsilon H_{cp})$ and get the two-spin Hamiltonian

$$
H_{cp} = \int_{\mathbb{R}^+} d\mu(\lambda) h(\lambda) \Pi_\lambda, \quad h(\lambda) = 1/2, \quad (84)
$$

in the limit $\epsilon \to 0$. The Hamiltonian of the percolating chain is a translationally invariant sum of two-spin Hamiltonians. To move the system off the critical point, we stagger the interaction strength.

The constancy of the excitation energy $h(\lambda)$ for $H_{cp}$ reflects the additional symmetries that distinguish classical percolation from the generic case of quantum percolation, where $h(\lambda)$ is nonconstant. We note that $H_{qp} = \sum_{ij} (-)^{|i|+1} E_{ij} E_{ji}^*$ can be written in the form (84) with $h(\lambda) = \lambda^2 + 1$.

6 Summary and Outlook

In this paper the study of the supersymmetric version of Pruisken’s nonlinear $\sigma$ model for the integer quantum Hall effect has been resumed. By generalizing the work of Shankar and Read, we derived a 1$d$ Hamiltonian lattice formulation of the $\sigma$ model which has the right (naive) continuum limit. The lattice Hamiltonian consists of a site-diagonal part and a site-site interaction, the former being a supersymmetric generalization of the Hamiltonian for a charged particle moving in the field of a magnetic monopole. The monopole charge $m$ is determined by the value of the Hall conductivity $\sigma_{xy}$. In the center of a Hall plateau region, where $\sigma_{xy}$ is an integer, $m$ vanishes for all sites in the bulk of the system and equals $\pm \sigma_{xy}$ at the edges. In this special case we used a renormalization group assisted argument to calculate the conductance coefficient $g$ pertaining to neighboring probes in a Corbino disk geometry. As is expected from the microscopic picture of directed transmission through channels localized near the sample boundary, we obtain either $g = 0$ or $g = |\sigma_{xy}|$ depending on the sign of $\sigma_{xy}$. This result suggests an interpretation of the monopole charge as the number of “edge states”.

For half-integral values of the Hall conductivity $\sigma_{xy}$ corresponding to the points of transition between neighboring plateaus, the monopole charge in the bulk of the system alternates on sites ($m = \pm 1$). Projection onto the degenerate strong-coupling ground states of the lattice Hamiltonian yields a superspin chain Hamiltonian effectively describing the critical point of the $\sigma$ model. The space of quantum states for even and odd sites of the chain is a space of holomorphic and antiholomorphic functions $V$ and $V^*$, respectively. This alternating structure corresponds to the alternating orientation of links in the Chalker-Coddington network model, reflecting the directedness of guiding center motion in a strong magnetic field and a smooth random potential. A direct derivation of the superspin chain from the network model is possible [56]. While Pruisken’s nonlinear $\sigma$ model is derived...
from the “quantum” limit of a random potential with correlation length $l_c$ short compared to the magnetic length $l_B$, the network model applies to the opposite semiclassical limit $l_c \gg l_B$. Since both models map on the very same superspin chain, we conclude that the ratio $l_c/l_B$ is an irrelevant parameter for the IQHE transition. By the hypothesis of universality at a critical point, such irrelevance is not unexpected.

We recalled the fact that the dynamic structure factor and other frequency-dependent correlation functions as well, have critical amplitudes that diverge at zero frequency. Within the field-theoretic framework this divergence is related to the fact that such correlators are not expressible as superspin correlation functions: the matrix elements of the corresponding $\sigma$ model operators between states in the superspin quantum spaces $V$ and $V^*$ just do not exist. A quantity that does restrict to the superspin chain in a natural manner is d.c. conductance.

While the nature of the subject is such as to force extensive use of mathematical argument, we tried to keep the mathematics to the minimum needed for clarity and completeness. For example, we did not mention the role of the topological density as pullback of the Kähler form of the Kähler supermanifold $G/K$. The theory of connexions on hermitian line bundles associated with a principal fibre bundle (central to Sects. 3.4-5) and of holomorphic sections of such bundles (central to Sects. 4.2-5), and the theory of Fourier analysis on supersymmetric spaces $G/K$ (central to Sects. 4 and 5.2) were only mentioned parenthetically. We made these omissions in an effort to keep the paper within reasonable size and accessible to readers without a background in mathematical physics.

Although the main body of this paper was concerned with the $\sigma$ model for 2$d$ electrons in a strong magnetic field, most of what we did can be transcribed with minor modifications to the symplectic $\sigma$ model describing time-reversal invariant systems with spin-orbit scattering. The role played by the state vectors $v$ and $v^*$ is taken by the zero mode of the Laplacian discovered in Ref. [48]. This zero mode is one state in an infinite-dimensional space of strong-coupling ground states of the quantum Hamiltonian of the symplectic $\sigma$ model. The strong-coupling ground states transform according to a representation of the symmetry group which is neither lowest-weight nor highest-weight nor irreducible. It is not clear at present whether the resulting superspin chain can serve as the universal critical theory of the Anderson metal-insulator transition for symplectic symmetry in $d = 2$. (The transition caused by spin-orbit scattering differs from that for quantum Hall systems in that a continuous symmetry is spontaneously broken.)

An issue we briefly addressed is how the classical limit $l_B \to 0$ fits into the superspin chain picture. This limit is not immediately accessible from our approach since the derivation of Pruisken’s model, as it stands, applies only to the limit $l_B \gg l_c$. Nevertheless, by using the relation to the Chalker-Coddington model we were able to write down a superspin chain Hamiltonian for the classical percolation transition, too. This Hamiltonian has a higher degree of symmetry than the generic critical Hamiltonian describing quantum percolation. In this way, the superspin chain provides a unified framework in which to discuss both quantum and classical percolation and the crossover from the latter to the former.

Since the mapping on a superspin chain retains the essential long wave length quantum Hall physics, the chain for $\sigma_{x,y}^* = 1/2 \pmod 1$, where a delocalization transition occurs, should be critical. More precisely, we expect a unique $G$-singlet ground state with local
“antiferromagnetic” order and massless (non-Goldstone) excitations. The staggering of the spin-spin interaction for $\sigma_{xy} \neq \sigma_{xy}^*$ favors singlets on strong bonds and opens a mass gap. Gaplessness of the critical chain should be guaranteed by the same mechanism that underlies the Lieb-Schultz-Mattis theorem for isotropic antiferromagnetic spin chains with half-integer spin. In spite of the tantalizing analogy, I have not yet succeeded in transcribing the theorem and its proof. The problem is not lack of translational invariance but the indefiniteness of the invariant scalar product (38), which prevents the application of the variational principle in its usual form.

The ground-state energy of the superspin Hamiltonian vanishes identically for a chain of any length and for arbitrary boundary conditions, as is guaranteed by unbroken $K$-supersymmetry. The Virasoro algebra underlying the massless chain for $m = 1$ must therefore have central charge $c = 0$. Note that this does not imply triviality of a theory which is nonunitary. (The Kac-Moody algebra of conserved currents will have a nontrivial central extension.)

In addition to the Hamiltonian (77) for $m = 1$, we can formulate quantum superspin chain Hamiltonians for all values of the monopole charge $m > 0$. By the interpretation of $m$ as the number of transmitting channels, we know that such a generalized Hamiltonian describes a situation where $m$ Landau levels are strongly mixed by the disorder. The chain for monopole charge $m$ resembles a 1$d$ isotropic antiferromagnet with spin $S = m/2$. Large values of $m$ offer the possibility of making a $1/m$ expansion. A natural starting point for such an expansion is the Néel state $v \otimes v^* \otimes v \otimes v^* \otimes \ldots$. Approximating the spin operators by their large-$m$ limits one gets a quadratic Hamiltonian which can be diagonalized by a Boboliubov transformation in the usual manner. The low-energy excitations of this Hamiltonian, the superspin waves, are found to have a linear dispersion relation, as one would expect from the close analogy to antiferromagnetic spin chains. These low-energy modes are the Goldstone modes restoring the continuous $G$-symmetry broken by the Néel state. Unfortunately, this type of analysis is not too useful for understanding the critical properties, for the $1/m$ expansion is ignorant of the expected profound difference between massive superspin chains for even $m$ and massless ones for odd $m$.

What are the prospects for obtaining some sort of exact solution? Although not too much work in this direction has been done as yet, a few statements can be made. The superspin chain Hamiltonian (77) is unlikely to be integrable. (The corresponding ordinary spin chains for spin $S > 1/2$ are not integrable.) It is therefore very fortunate that the universality we expect for $m = 1$ gives us much freedom to modify the superspin chain: any Hamiltonian which has the “right” symmetries and is not too remote in parameter space, should give the same low-energy physics — the physics of the unique RG-fixed point that describes the IQHE transition. Making use of this freedom, one may be able to construct a one-parameter family of commuting transfer matrices (with the superspin Hamiltonian being the logarithmic derivative at a special point) by solving the quantum Yang-Baxter equation (QYBE). This, unfortunately, is not immediately possible but requires an extension of existing theory, for the following reason. We are looking for a $G$-invariant solution of QYBE. An efficient method for constructing such solutions has been described by Drinfel’d [65], generalizing earlier work of Kulish et al. [66]. The basis of the method is an expansion of the $R$-matrix $R(u) : W_1 \otimes W_2 \rightarrow W_1 \otimes W_2$ around $u = \infty$ in the form $R(u) = 1 + u^{-1} r + \ldots$ where $u^{-1} r$ is a solution of the classical Yang-Baxter equation. The
crucial difference for our superspin chains is that no such classical limit exists. Because of
the alternating structure of the chain, the \( R \)-matrix is a map \( R(u) : V \otimes V^* \rightarrow V^* \otimes V \)
(Figure 8), which implies \( R(u) \neq 1 \) for all \( u \). (One might think that the isomorphism
\( T : V \rightarrow V^* \), \( f \mapsto f \) could be used instead of the identity. However, \( R(\infty) = T \) fails to
meet the requirement of \( G \)-invariance.)

In my opinion, the most promising line of further attack is to study superspin chains
with \( 1/r^2 \) exchange of the Haldane-Shastry type \[67\] and/or explore the possible relation
to a gauged WZW model. (Gauging is necessary to combat the indefinite metric on the
noncompact symmetry group \( G \).

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**Appendix A: Proof of Formula (34)**

Eq. (33) expresses \( \mathcal{L}^m \) by left-invariant vector fields. To switch to right-invariant vector
fields we proceed as follows. In Eq. (21) we introduced a second-order differential operator
\( \lambda_P(\partial/\partial x) := \lambda(\partial/\partial x) = \sum g^{ij} \partial^2/\partial x^i \partial x^j \) by inverting the invariant quadratic form \( B \)
on \( P \). Analogous differential operators \( \lambda_G \) and \( \lambda_K \) are defined by inverting \( B \) on \( G = \text{Lie}(G) \) and \( K = \text{Lie}(K) \), respectively. These operators are related by the identity \( \lambda_P = \lambda_G - \lambda_K \), which we insert into Eq. (33). By definition of the projection \( \pi : G \rightarrow G/K \),
we have \( f(\pi(g \exp \sum x^i e_i)) = f(\pi(g)) \) for \( \sum x^i e_i \in K \). Moreover, \( \mu(k(g \exp \sum x^i e_i)) = \\
\mu(k(g)) \mu(\exp \sum x^i e_i) \) for \( \sum x^i e_i \in K \), and

\[
\lambda_K(\partial/\partial x) \mu(\exp \sum x^i e_i))^{-1} \bigg|_{x=0} = \lambda_K(\partial/\partial x) \exp \left( -m \text{Str} \pi_+ \sum x^i e_i \right) \bigg|_{x=0} = 0
\]

by a simple calculation. Therefore, we may drop \( \lambda_K \). Doing so we arrive at a formula for
\( \mathcal{L}^m \) identical to (33) except that \( \lambda = \lambda_P \) is now replaced by \( \lambda_G \). After this preparation
we can move \( \exp \sum x^i e_i \) to the left of \( g \) by using the invariance of \( \lambda_G \) under \( \exp \sum x^i e_i \rightarrow \\
g^{-1}(\exp \sum x^i e_i) g \). This proves Eq. (34).

**Appendix B: Derivation of Eqs. (51), (53) and (58)**

We shall derive the coordinate expression (31) of the representation \( g \mapsto T_g \), Eq. (44).
Recall the definition of \( s : G/K \rightarrow G_0 = SU(n, n|2n) \) by

\[
s(\pi((A \ B) \ C \ D)) = \begin{pmatrix}
1 & BD^{-1} \\
CA^{-1} & 1
\end{pmatrix} \begin{pmatrix}
(1 - BD^{-1}CA^{-1})^{-1/2} & 0 \\
0 & (1 - CA^{-1}BD^{-1})^{-1/2}
\end{pmatrix}.
\]

Solving the equation \( g = s(\pi(g))k(g) \) for \( k \), we get

\[
k((A \ B) \ C \ D) = \begin{pmatrix}
(1 - BD^{-1}CA^{-1})^{1/2}A & 0 \\
0 & (1 - CA^{-1}BD^{-1})^{1/2}D
\end{pmatrix}.
\]

(B.1)
From $\pi_+ = (1 + \Lambda)/2$ we have $\mu(k) = \exp(m \text{STr } \pi_+ \ln k) = (\text{SDet } k)^{m/2} \exp \left( \frac{m}{2} \text{STr } \Lambda \ln k \right)$. For the second factor we obtain from (B.1) the simple formula

$$\exp \left( \frac{m}{2} \text{STr } \Lambda \ln k \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \right) = \text{SDet}(A/D)^{m/2}.$$  

To express the factor SDet $k(g)$, we take the superdeterminant of both sides of the equation $g = s(\pi(g))k(g)$, which gives SDet $k(g) = \text{SDet } g$, since $s(\pi(g)) \in \mathbb{G}_q$. Hence,

$$\mu(k(\left( \begin{array}{cc} A & B \\ C & D \end{array} \right))) = \text{SDet} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)^{m/2} \text{SDet}(A/D)^{m/2} \quad \text{(B.2)}$$

From this we infer $\mu(k(s(Z, \tilde{Z}))) = 1$. Moreover, Eq. (B.2) in combination with

$$\left( \begin{array}{cc} A & B \\ C & D \end{array} \right)^{-1} = \left( \begin{array}{cc} 1 & -A^{-1}B \\ -D^{-1}C & 1 \end{array} \right) \left( \begin{array}{cc} (A - BD^{-1}C)^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{array} \right) \quad \text{(B.3)}$$

yields $\mu(k(x)^{-1}) = \mu(k(x^{-1}))$ for any $x \in \mathbb{G}$. Applying this identity to $x = g^{-1}h$ with $h = s(Z, \tilde{Z})$, and using Eqs. (10), (13) and (B.2) to calculate $\mu(k(s(Z, \tilde{Z})^{-1}g))$, we arrive at

$$\mu(k(h)k(g^{-1}h)^{-1}) = (\text{SDet } g)^{m/2} \text{SDet} \left( \frac{A - ZC}{D - ZB} \right)^{m/2},$$

which proves Eq. (51).

The second calculation we do in this appendix is to work out the similarity transformation $T_g^+ = v_m^{-1}T_g v_m$ for $v_m(Z, \tilde{Z}) = \text{SDet}(1 - \tilde{Z}Z)^{m/2}$ ($m > 0$). The effect of this transformation is to make the multiplier in Eq. (51) pick up an extra factor $v_m(Z, \tilde{Z})^{-1}v_m(g^{-1} \cdot Z, g^{-1} \cdot \tilde{Z})$. To calculate it, we use the matrix function $Q = g\Lambda g^{-1} = s\Lambda s^{-1}$. From Eq. (D.1) it is seen that $1 - \tilde{Z}Z = ((1 + Q)_{BB}/2)^{-1}$ where the indices $BB$ mean the upper left (or Boson-Boson) block. Writing $g = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$, $g^{-1} = \left( \begin{array}{cc} A' & B' \\ C' & D' \end{array} \right)$ and doing matrix multiplications, we find

$$1 - (g^{-1} \cdot \tilde{Z})(g^{-1} \cdot Z) = \left( \frac{1}{2}(1 + g^{-1}Qg)_{BB} \right)^{-1} = (A - \tilde{Z}C)^{-1}(1 - \tilde{Z}Z)(A' + B'Z)^{-1}.$$  

Now, from Eq. (B.3) we read off $A' = (A - BD^{-1}C)^{-1}$ and $B' = -A'BD^{-1}$, so taking the superdeterminant we obtain

$$v_m(Z, \tilde{Z})^{-1}v_m(g^{-1} \cdot Z, g^{-1} \cdot \tilde{Z}) = \text{SDet} \left( \frac{AD(1 - CA^{-1}BD^{-1})}{(A - ZC)(D - ZB)} \right)^{m/2}.$$  

Multiplying this with the multiplier in Eq. (51) and using $\text{SDet } g = \text{SDet } (AD) \text{SDet } (1 - CA^{-1}BD^{-1})$ we get the multiplier in Eq. (58).

Finally, we derive the coordinate expression (53) of the monopole Hamiltonian $H_s = -\mathcal{L}^m$, Eq. (53). The pair of matrices $X, \tilde{X}$ appearing in (53) is defined by the equation
\[ \exp(\sum x^i e_i) = s(D^{-1}XA, A^{-1}XD). \] Within the accuracy required by setting \( X = \tilde{X} = 0 \) after differentiation, and because of the invariance of the differential operator \( \partial_{X,\tilde{X}} \), Eq. (52), under \( X \mapsto D^{-1}XA, \tilde{X} \mapsto A^{-1}XD \), we may replace \( \lambda(\partial/\partial x) \) in Eq. (33) by \( \partial_{X,\tilde{X}} \). Again, let \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). Using Eqs. (49) we get

\[
\begin{align*}
 g \cdot (D^{-1}XA) &= (Z + X)(1 + \tilde{Z}X)^{-1}, \\
 g \cdot (A^{-1}XD) &= (\tilde{Z} + \tilde{X})(1 + Z\tilde{X})^{-1},
\end{align*}
\]

where \( Z = CA^{-1} \) and \( \tilde{Z} = BD^{-1} \) are the coordinates of the coset \( \pi(g) \). This explains the argument of the function \( f \) on the right-hand side of Eq. (53). We turn to the multiplier \( \mu(k(g)k(g \exp \sum x^i e_i)^{-1}) \) in Eq. (33). The first factor, \( \mu(k(g)) \), has been given in Eq. (B.2). For the second factor, we use

\[
\mu(k(gs(X, \tilde{X}))) = (\text{SDet } g)^{m/2} \text{SDet } \left( \frac{A + BX}{D + CX} \right)^{m/2},
\]

as follows easily from (B.2). Making the substitutions \( X \mapsto D^{-1}XA, \tilde{X} \mapsto A^{-1}XD \) and putting the factors together, we get the multiplier in Eq. (53).

**Appendix C: Representation of Lie superalgebras on superfunctions**

We denote the superparity on any \( \mathbb{Z}_2 \)-graded vector space \( V \) by \( | \cdot | \), i.e. \( |X| = 0 \) if \( X \in V \) is even and \( |X| = 1 \) if \( X \) is odd. A complex Lie superalgebra \( G \) is a \( \mathbb{Z}_2 \)-graded vector space over \( \mathbb{C} \) with a bracket operation satisfying the axioms

\[
\begin{align*}
|[[X,Y]]| &= |X| + |Y| \mod 2, \\
[X,Y] &= (-)^{|X|^{|Y|}|+1}|Y,X|, \\
[X,[Y,Z]] &= [[X,Y],Z] + (-)^{|X|^{|Y|}|}[Y,[X,Z]].
\end{align*}
\]

(C.1)

An example is \( \mathfrak{gl}(p,q; \mathbb{C}) \) defined by the bracket (72). Let now some Grassmann algebra \( \Lambda \) be given. Following Berezin [43], we define the Grassmann envelope \( G(\Lambda) \) of \( G \) as the even part of the tensor product \( \Lambda \otimes G \), that is to say, as the linear space of objects \( aX \) with \( a \in \Lambda \), \( X \in G \) and \( |a| = |X| \). When \( G = \mathfrak{gl}(p,q; \mathbb{C}) \), an element of \( G(\Lambda) \) is what is commonly called a supermatrix of dimension \((p+q) \times (p+q)\). The Grassmann envelope \( G(\Lambda) \) can be turned [43] into a Lie algebra with Grassmann structure in either one of two ways:

\[
\begin{align*}
[aX, bY] &:= ab[X,Y], \\
[aX, bY] &:= ab(-)^{|X|^{|b|}}[X,Y].
\end{align*}
\]

(1st kind)

(2nd kind)

For the definition of the second kind we imagine \( X \) and \( b \) to anticommute when both \( X \) and \( b \) are odd. The usual multiplication rules for supermatrices used in the physics literature
imply \( aXbY = abXY \) for \( a, b \in \Lambda \) and \( X, Y \in \mathfrak{gl}(p, q; \mathbb{C}) \). Therefore, the natural convention is to take the Grassmann envelope \( \mathfrak{gl}(p, q|\Lambda) \) to be a Lie algebra with Grassmann structure of the first kind. This is the convention we adopt.

Now let \( G_C \) be a Lie group which has \( \mathfrak{gl}(p, q|\Lambda) \) for its Lie algebra, and let \( g \mapsto T_g \) be a representation of \( G_C \) on some space of superfunctions. (For example, for \( p = q = 2n \), \( G_C \) might be the complexified symmetry group of the quantum Hall model space \( G/K \), and \( T \) might be the representation defined by Eq. (68).) Taking the differential at the group unit \((t \in \mathbb{R})\)

\[
dT(aX)f := \left. \frac{d}{dt} T_{\exp taX} f \right|_{t=0},
\]

we get a representation of the Lie algebra:

\[
[dT(aX),dT(bY)] = dT([aX,bY]).
\]  

(\text{C.2})

\( \text{d}T(X) \) for \( X \in \mathfrak{gl}(p, q; \mathbb{C}) \) is defined by linearity, \( \text{d}T(aX) = \text{ad}T(X) \). (This is the precise meaning of Eq. (68).) Thus, \( \text{d}T \) assigns to \( X \) a differential operator \( \text{d}T(X) \) acting on superfunctions \( f \). Such an operator differentiates with respect to the commuting and anticommuting coordinates. If \( X \) is odd, so is \( \text{d}T(X) \). Consistency now requires us to observe the graded commutation law \( \text{d}T(X)b = (-)^{|X||b|}b \text{d}T(X) \) where \( b \in \Lambda \) is some parameter. Hence, from (\text{C.2}) we deduce

\[
[dT(X),dT(Y)] = (-)^{|X||Y|}dT([X,Y]).
\]

It is easy to see that this bracket obeys the axioms (\text{C.1}), i.e. it defines a Lie superalgebra. Setting \( \mathcal{E}_{ij} = \text{d}T(E_{ij}) \) with \( E_{ij} \) the canonical generators of \( \mathfrak{gl}(p, q; \mathbb{C}) \), we obtain the supercommutation relations \( [\mathcal{E}_{ij},\mathcal{E}_{kl}] = (-)^{(|i|+|j|)(|k|+|l|)}\delta_{jk}\mathcal{E}_{il} - \delta_{il}\mathcal{E}_{kj} \), which are different from those for the \( E_{ij} \). We thus need to distinguish between two Lie superalgebras. The first one is generated by the \( E_{ij} \), the second one by the \( \mathcal{E}_{ij} \). Our discussion clarifies how the two are connected: constructing from the former and the latter a Lie algebra with Grassmann structure of the first and second kind, respectively, we get Lie algebras that match, i.e. the latter represents the former.

The sceptical reader might be reluctant to accept the violation of the representation property at the Lie superalgebra level. He is invited to convince himself that the only consistent way to fix the disease is to modify the basic law of multiplication of supermatrices. Clearly, such a modification would be very undesirable.

**Appendix D: Calculation of a Reduced Matrix Element**

We calculate the value of the exchange coupling constant \( J \) of the spin chain for \( m > 0 \). (Recall \( m = 1 \) for a spin-split and \( m = 2 \) for a spin-degenerate Landau level.) The matrix-valued function \( Q(g) = g\Lambda g^{-1} \) is expressed by

\[
Q = \begin{pmatrix}
1 + \tilde{Z}Z & -2\tilde{Z} \\
2Z & -1 - ZZ
\end{pmatrix}
\begin{pmatrix}
(1 - \tilde{Z}Z)^{-1} & 0 \\
0 & (1 - ZZ)^{-1}
\end{pmatrix}
\]

(\text{D.1})

in the coordinates \( Z, \tilde{Z} \). We denote the matrix element \( (2Z(1 - \tilde{Z}Z)^{-1})_{11} \) by \( Q^{0}_{11} \). From Sect. 5.1 we have the relation \( J = c_0^2\sigma_{xx}/a_1 \) where \( c_0 \) is the constant of proportionality in
\( P Q_{11}^{21} v = c_0 B_{11} v. \) (\( P \) projects onto \( V \), and \( v = v_m \).) Making use of the invariant quadratic form \((\cdot, \cdot)\) we get \( c_0 = (B_{11} v, Q_{11}^{21} v)/(B_{11} v, B_{11} v). \) From Eq. (59a) for \( T_g^+ = v^{-1} T_g v \) we have \((B_{11} v)(Z, \tilde{Z}) = m Z_{11} v(Z, \tilde{Z}). \) The invariant Berezin integral in the coordinates \( Z, \tilde{Z} \) by the flat measure \( dZ d\tilde{Z} \) [12], so

\[
(B_{11} v, Q_{11}^{21} v) = \int dZ d\tilde{Z} \ m \tilde{Z}_{11} (2Z(1 - \tilde{Z} Z)^{-1})_{11} v(Z, \tilde{Z})^2. \tag{D.2}
\]

To calculate this integral, we use the identity

\[
m(2Z Z^{-1})_{11} v(Z, \tilde{Z})^2 = \frac{d}{dt} v(Z, \tilde{Z} - tE_{11}) \bigg|_{t=0}
\]

which follows from \( v(Z, \tilde{Z})^2 = \text{SDet}(1 - \tilde{Z} Z)^m \). By partial integration we get

\[
(B_{11} v, Q_{11}^{21} v) = 2 \int dZ d\tilde{Z} v(Z, \tilde{Z})^2 = 2(v, v) = 2. \tag{D.3}
\]

On the other hand, Eq. (63) for \( a_{m+2, -m} = B_{11} v \) gives \((B_{11} v, B_{11} v) = m. \) Hence, \( c_0 = 2/m \) and \( J = 4 \sigma_{xx}/m^2 a_1. \)

The above calculation does not stand up to close scrutiny, for it turns out that the invariant Berezin integral in the coordinates \( Z, \tilde{Z} \) is plagued by boundary ambiguities [68] in general. This means that the integrals in (D.2) and (D.3) must be corrected by the addition of extra terms. (I have analyzed the situation for \( n = 1 \) in some detail. In this case I find the correction terms for (D.2) to be nonvanishing only for \( m = 1, 2 \). For \( m = 2 \), the correction term is precisely cancelled by the boundary term resulting from partial integration, so the first equality sign in (D.3) is correct as it stands in this case.) One can avoid dealing with this issue by switching to Efetov’s polar coordinates [22] where the problem of boundary ambiguities is well understood [52, 53]. Doing so, I find the final result to be correct for all \( m \geq 1 \) (and all \( n \)).

**Appendix E: Relation between the two-spin Hamiltonian and the Laplacian for \( G/K \)**

From Eq. (34), the monopole Hamiltonian \( \mathcal{H}_s = -\mathcal{L}^m \) is seen to have the expression \( \mathcal{H}_s = \sum_{ij} (-)^{|i|+1} E_{ij} E_{ji} \), with the spin operators \( E_{ij} = dT(E_{ij}) \) being defined by the differential of \( T_g \), Eq. (14). Since \( \mathcal{H}_s \) is zero on its ground-state module \( V = V_m \), we may write the two-spin Hamiltonian \( H : V \otimes V^* \rightarrow V \otimes V^* \) in the form

\[
H = \frac{1}{2} \sum_{ij} (-)^{|i|+1} (E_{ij} + E_{ij}^*)(E_{ij} + E_{ij}^*). \tag{E.1}
\]

Returning to the notation used in Eq. (34) we get

\[
(H(f \otimes f^*)) (\pi(g), \pi(h)) = -\frac{1}{2} \lambda g \left( \partial \phi/\partial x \right) \mu(k(g)k(e^{\Sigma x^i e_i} g)^{-1})
\times \mu(k(h)^{-1} k(e^{\Sigma x^i e_i} h)) (f \otimes f^*) (\pi(e^{\Sigma x^i e_i} g), \pi(e^{\Sigma x^i e_i} h)) \bigg|_{x=0}. \tag{E.2}
\]
Now, following Sect. 5.2, we use the trick of identifying \( g \equiv h \). The multipliers then cancel each other, and switching from right- to left-invariant vector fields we arrive at the formula

\[
(H(f \otimes f^*)) (\pi(g)) = -\frac{1}{2} \lambda_G (\partial/\partial x) (f \otimes f^*) (\pi(e^{\Sigma x^i e_i} g)) \bigg|_{x=0} = -\frac{1}{2} \lambda_P (\partial/\partial x) (f \otimes f^*) (\pi(g \exp \sum x^i e_i)) \bigg|_{x=0}.
\]

Comparison with formula (21) proves that \( H \) coincides with minus one half times the Laplacian \( L \) on \( G/K \), after restriction to the subspace of functions which have the quantum numbers that are generated by the tensor product \( V \otimes V^* \).

**Figure Captions**

Fig. 1: Three-terminal device (schematic) with small contacts. The configuration space \( \mathcal{M} \) of the 2d electron gas is finite or infinite. The contacts \( C_i \) connect \( \mathcal{M} \) with particle reservoirs.

Fig. 2: Lifting of a curve \( t \mapsto \gamma(t) \) on a principal fibre bundle \( G \to G/K \) with gauge connection \((g^{-1}dg)_{\mathcal{K}}\).

Fig. 3: Modification of the topological density. \( \mathcal{M}_1 \) is the union of the shaded regions. The topological coupling constant is doubled on \( \mathcal{M}_1 \) and set to zero on the empty regions.

Fig. 4: Weight diagram of the lowest-weight module \( V_2 \). Weights are marked by full black dots. Weights enclosed by an additional circle are doubly degenerate. The lowest weight is \((2, -2)\).

Fig. 5: Corbino disk geometry with four terminals. Fat black lines symbolize \( G/K \) superparticles propagating in imaginary time. The particles at the outer and inner edges and in the bulk see monopole charges \(+m, -m\), and 0, respectively. The sign of the monopole charge at the edges is indicated by arrows.

Fig. 6: Ideal lead connected to a disordered quantum Hall sample. There are no direct transitions between incoming and outgoing channels.

Fig. 7: (a) The network model of Chalker and Coddington. (b) Anisotropic limit of the network model. The dotted lines indicate tunnelling occurring with a small probability amplitude.

Fig. 8: \( R \)-matrix of the supersymmetric vertex model at the classical percolation transition.
References

[1] K. v. Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. 45 (1980) 494; D.C. Tsui, H.L. Stoermer, and A.C. Goddard, Phys. Rev. Lett. 48 (1982) 1559.
[2] T. Ando and H. Aoki, Sol. State Commun. 38 (1981) 1079.
[3] Q. Niu, D.J. Thouless, and Y.S. Wu, Phys. Rev. B31 (1985) 3372; Y. Avron and R. Seiler, Phys. Rev. Lett. 54 (1985) 259.
[4] M. Böttiker, Phys. Rev. B38 (1988) 9375.
[5] H.P. Wei, D.C. Tsui, M. Palaanen, and A.A.M. Pruisken, Phys. Rev. Lett. 61 (1988) 1294; S. Koch, R. Haug, K. v. Klitzing, and K. Ploog, Phys. Rev. Lett. 67 (1991) 883.
[6] J.T. Chalker and P.D. Coddington, J. Phys. C21 (1988) 2665.
[7] B. Huckestein and B. Kramer, Phys. Rev. Lett. 64 (1990) 1437; B. Mieck, Europhys. Lett. 13 (1990) 453.
[8] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333; J. Cardy, in: Phase Transitions and Critical Phenomena vol. 11, eds. C. Domb and J.L. Lebowitz (1987).
[9] A.A.M. Pruisken, Nucl. Phys. B235 (1984) 277.
[10] H. Levine, S.B. Libby, and A.A.M. Pruisken, Phys. Rev. Lett. 51 (1983) 1915; Nucl. Phys. B240 (1984) 30; 49; 71.
[11] A.A.M. Pruisken, Nucl. Phys. B285 (1987) 719; B290 (1987) 61.
[12] H.A. Weidenmüller and M.R. Zirnbauer, Nucl. Phys. B305 (1988) 339.
[13] Y. Huo, R.E. Hetzel, and R.N. Bhatt, Phys. Rev. Lett. 70 (1993) 481.
[14] I. Affleck, Phys. Rev. Lett. 56 (1986) 746.
[15] I. Affleck and F.D.M. Haldane, Phys. Rev. B36 (1987) 5291.
[16] R. Shankar and N. Read, Nucl. Phys. B336 (1990) 457.
[17] A.A.M. Pruisken, Phys. Rev. Lett. 61 (1988) 1297.
[18] I. Affleck, Nucl. Phys. B257 (1985) 397; B265 (1986) 409; B305 (1988) 582.
[19] F.D.M. Haldane, Phys. Lett. 93A (1983) 464; Phys. Rev. Lett. 50 (1983) 1153; J. Appl. Phys. (USA) 57 (1985) 3359.
[20] J.J.M. Verbaarschot and M.R. Zirnbauer, J. Phys. A17 (1985) 1093.
[21] A.A.W. Ludwig, M.P.A. Fisher, R. Shankar and G. Grinstein, Princeton University Preprint 9/1993.
[22] K.B. Efetov, Adv. in Phys. 32 (1983) 53.
[23] H.A. Weidenmüller, Nucl. Phys. B290 (1987) 87.
[24] H. Baranger and A.D. Stone, Phys. Rev. B40 (1989) 8169.
[25] P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. 57 (1985) 287.
[26] J.T. Chalker, J. Phys. C21 (1988) L119.
[27] J.T. Chalker, Physica A167 (1990) 253.
[28] F. Wegner, in: *Localisation and Metal Insulator Transitions*, eds. H. Fritzsche and D. Adler, Institute for Amorphous Studies Series (Plenum, New York, 1985).

[29] F. Wegner, Z. Phys. B25 (1976) 327.

[30] U. Fastenrath, M. Janssen, and W. Pook, Physica A191 (1992) 401; B. Huckestein and L. Schweitzer, Physica A191 (1992) 406; Phys. Rev. Lett. 72 (1994) 713.

[31] L. Schäfer and F. Wegner, Z. Phys. B38 (1980) 113.

[32] M.R. Zirnbauer, Nucl. Phys. A560 (1993) 95.

[33] H.A. Weidenmüller, Physica A167 (1990) 28.

[34] M.R. Zirnbauer, Physica A167 (1990) 132.

[35] B. Huckestein, Phys. Rev. Lett. 72 (1994) 1080.

[36] U. Fastenrath, Physica A189 (1992) 27.

[37] F. Wegner, Z. Phys. B35 (1979) 207.

[38] B.L. Al’tshuler, V.I. Kravtsov, and I. Lerner, in: *Mesoscopic Phenomena in Solids*, eds. B.L. Al’tshuler, R.A. Lee, and R.A. Webb, North Holland, Amsterdam 1991.

[39] B. DeWitt, *Supermanifolds*, Cambridge University Press, Cambridge 1984.

[40] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York 1978; *Groups and geometric analysis*, Academic Press, Orlando 1984.

[41] T. Eguchi, P. Gilkey, and A.J. Hanson, Phys. Rep. 66, (1980) 213.

[42] M.V. Berry, Proc. Roy. Soc. (London) A392 (1984) 45; B. Simon, Phys. Rev. Lett. 51 (1983) 2167.

[43] F.A. Berezin, *Introduction to Superanalysis*, D. Reidel Publishing Co., Doordrecht, Holland 1987.

[44] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. I, John Wiley & Sons, New York 1963.

[45] E. Fradkin, *Field theories of condensed matter systems*, Addison-Wesley, Redwood City 1991.

[46] A. Hüffmann, Ph.D. Thesis, University of Cologne (1992); J. Phys. A (in press).

[47] M.R. Zirnbauer, Commun. Math. Phys. 141 (1991) 503.

[48] M.R. Zirnbauer, Phys. Rev. Lett. 69 (1992) 1584; A.D. Mirlin, A. Müller-Groeling, and M.R. Zirnbauer, Ann. Phys. (in press).

[49] M. Stone, Nucl. Phys. B314 (1989) 557.

[50] N.L. Balazs and A. Voros, Phys. Rep. 143 (1986) 109.

[51] F.D.M. Haldane, Phys. Rev. Lett. 51 (1983) 605; F.D.M. Haldane and E. Rezayi, Phys. Rev. Lett. 54 (1985) 237.

[52] M.R. Zirnbauer, Nucl. Phys. B265 (1986) 375.

[53] R. Bundschuh, Diploma thesis, University of Cologne (1993).

[54] A. Altland, Z. Phys. B86 (1991) 105.

[55] D.E. Khmel’nitskii, JETP Lett. 38 (1983) 454.
The symmetries of the superspin chain were first communicated to me by N. Read (Aspen 1993).

D.K.K. Lee and J.T. Chalker, Phys. Rev. Lett. 72 (1994) 1510.

D.H. Lee, *Network models of quantum percolation and their field-theory representation*, cond-mat/9404011.

E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. 16 (1961) 407.

I. Affleck and E.H. Lieb, Lett. Math. Phys. 12 (1986) 57.

M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington 1964.

D.H. Lee, Z. Wang, and S.A. Kivelson, Phys. Rev. Lett. 70 (1993) 4130.

S.A. Trugman, Phys. Rev. B27 (1983) 7539.

R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London 1982.

V.G. Drinfel’d, Soviet Math. Dokl. 32 (1985) 254.

P.P. Kulish, N. Yu. Reshetikhin, and E.K. Sklyanin, Lett. Math. Phys. 5 (1981) 393.

F.D.M. Haldane, Phys. Rev. Lett. 60 (1988) 635; B.S. Shastry, Phys. Rev. Lett. 60 (1988) 639.

M.J. Rothstein, Trans. Am. Math. Soc. 299 (1987) 387.
Figure 1
Figure 2

\[ g(t) = s(\gamma(t))u(t) \]
Figure 3
Figure 4
Figure 5
Figure 7
Figure 8

\[ \begin{align*}
V & \quad V^* \\
\quad V & \quad 1/2 \\
V^* & \quad 1/2
\end{align*} \]