PHASE DIAGRAM OF AN SU(2) x SU(2) SCALAR-FERMION MODEL WITH MASSLESS DECOUPLED DOUBLERS

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Abstract. We present the phase structure of the chiral SU(2) × SU(2) scalar-fermion model on the lattice using the Zaragoza proposal for chiral fermions. The numerical result agrees with an analytic study based on the use of weak and strong yukawa coupling expansions combined with the mean-field approach. The phase diagram consists of four phases: paramagnetic (PM), ferromagnetic (FM), antiferromagnetic (AFM) and ferrimagnetic (FI). The transition lines separating these four phases intersect at one quadruple point.

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1. Introduction

The formulation of a Chiral Gauge Theory (CGT) on the lattice suffers from the well-known doubling problem [1]. Several ways of dealing with this problem have been reviewed in reference [2]. Among them, the Zaragoza proposal [3,4] is a viable candidate for describing chiral fermions on the lattice. This proposal belongs to a category of models wherein, as in the Rome approach [5], gauge symmetry is explicitly broken by the regularization and therefore additional counterterms must be included in order to restore the gauge invariance in the continuum limit. The distinguishing feature of this approach is that the global chiral symmetry is preserved, and the doubler fermions are free and massless. These properties simplify the calculation of some important parameters of the Standard Model (the $S$, $U$ and $\Delta \rho$ parameters)[6].

In the case of Chiral Yukawa models (CYM), this regularization method preserves the invariance under discrete rotations and translations, but also all the important symmetries present in the continuum: hermiticity and global chiral symmetry. Hence it is particularly well suited to study those models [4].

In this paper, we investigate the phase structure of a $SU(2)_L \times SU(2)_R$ CYM which is essentially the fermion-scalar sector of the electroweak theory. We decided to freeze the radial mode of the scalar field, which corresponds to the choice of an infinite bare quartic coupling. Knowledge from the pure $\Phi^4$ theory suggests that such a model belongs to the same universality class as the models with finite quartic coupling. For a comparison of the phase structure of different models see references [2,7-10].

2. Model, symmetries and limiting case

Our action $S(\Psi, \Phi)$ is given by

$$S(\Psi, \Phi) = S_B(\Phi) + S_F(\Psi) + S_Y(\Psi, \Phi),$$

(1)
where
\[ S_B(\Phi) = -\frac{k}{2} \sum_{x,\mu} \text{Tr} \left( \Phi^+_x \Phi_x + \Phi^+_x \Phi_x + \hat{\Phi}_x \right) \] (2)
is the kinetic term for the scalar fields, \( \Phi \), which are \( 2 \times 2 \) \( SU(2) \) matrices.

\[ S_F(\Psi) = \frac{1}{2} \sum_{\Psi,x,\mu} \left( \overline{\Psi}_x \gamma_\mu \Psi_{x+\mu} - \overline{\Psi}_{x+\mu} \gamma_\mu \Psi_x \right) \] (3)
is the fermionic kinetic term where \( \sum_{\Psi} \) stands for the summation over \( n_f \) doublets of Dirac fermions, and finally,

\[ S_Y(\Psi, \Phi) = y \sum_{\Psi,x} \left( \overline{\Psi}^{(1)}_L \Phi_x + \overline{\Psi}^{(1)}_R \Phi_x + \overline{\Psi}^{(1)}_L \Phi_x \right) \] (4)
is the Yukawa interaction. The \( L \) and \( R \) indices refer to the left and right components of the fermion fields. In this interaction term, the way to implement the decoupling of the doubler fermions is based on the use of the quasi-local field component \( \Psi^{(1)} \) given by

\[ \Psi^{(1)}_x = \frac{1}{2^d} \sum_{x' \in hc(x)} \Psi_{x'} \] (5)

Here \( hc(x) \) is the elementary hypercube that starts from the site \( x \) in the positive direction. To understand why we have chosen \( \Psi^{(1)} \) to describe the fermion interactions in Eq. (4) let us go to momentum space. There \( \Psi^{(1)} \) is given by

\[ \Psi^{(1)}(q) = F(q) \Psi(q), \quad F(q) = \prod_\mu f(q_\mu), \quad f(q_\mu) = \cos\left(\frac{q_\mu}{2}\right), \quad q \in (-\pi, \pi]^d. \] (6)

Note that \( \Psi^{(1)} \) is the original field \( \Psi \) modulated by a form factor \( F \) which is responsible for the decoupling of the doublers at the tree level in the continuum limit. This is achieved by forcing the form factor to vanish for momenta corresponding to the doublers (note that \( f(0) = 1 \) and \( f(\pi) = 0 \)). Other form factors are possible but eq. (6) corresponds to the most local possible choice for the smearing in position space, given by eq. (5). We have proved perturbatively \([3,4]\) that in the continuum limit only one fermion (for each flavour) is coupled to the scalar. The doublers stay free and massless.

The action (1) has a global chiral symmetry and is also invariant under a shift transformation of the fermion fields. This shift-symmetry can be used to argue, if a
continuum limit exists, the non-perturbative decoupling of the doublers (see Ref. 4 for more details).

A few words are needed about the symmetries of the phase diagram because, in this model, they are not exactly the same as in other regularizations and have relevant consequences.

Some of the usual symmetries are present. The action is invariant under the transformation $\Phi_x \rightarrow -\Phi_x$, $y \rightarrow -y$, so we can restrict ourselves to $y \geq 0$ with no loss of generality. For $y = 0$ the action is invariant under $k \rightarrow -k$, $\Phi_x \rightarrow \epsilon_x \Phi_x$, $\epsilon_x = (-1)^{x_1+x_2+x_3+x_4}$; the scalar and fermionic fields are decoupled and the model reduces to a $\Phi^4$ model with the radial mode frozen: two critical points are present at $k = \pm k_c$.

But the symmetry $k \rightarrow -k$, $y \rightarrow -iy$, usual in other models, is absent here [4]. As a consequence the phase transition line PM–AFM is not determined by the phase transition line PM–FM, as often happens [7]. The simple mean field computation that we shall present in sect. 3 is able to detect the crossing of these lines and the presence of a ferrimagnetic phase.

Concerning the limiting case $y \rightarrow \infty$, we have a peculiarity similar to that in the Chiral Yukawa model with hypercubic coupling [9]. A rescaling of the fermionic field $\Psi \rightarrow \frac{1}{\sqrt{y}} \Psi$, suppresses, as always, the fermion kinetic term but, in our case, the fermionic fields on different sites are still coupled by the Yukawa interaction $S_Y$ (because of the smearing in $\Psi^{(1)}$). The fermion can propagate and does not decouple from the scalar field when $y$ goes to $\infty$. In this limit the model is not equivalent to a pure $\Phi^4$ model.

3. Yukawa coupling expansions and the mean field technique

To evaluate the free energy, we have used conventional mean-field methods for the scalar field [11] together with weak and strong coupling expansions for the Yukawa inter-
action. At small $y$, the Yukawa term:

$$\exp \left[ -y \sum_{\Psi, x} \left( \Psi_{Lx}^{(1)} \Phi_{x}^{(1)} + \Psi_{Rx}^{(1)} \Phi_{x}^{+} \Psi_{Lx}^{(1)} \right) \right],$$  

(7)

is straightforwardly expanded in powers of $y$ up to the four fermion interaction (i.e. up to $y^2$).

On the other hand, for large values of $y$, we first introduce auxiliary fermionic fields $\eta_x$ and $\eta_x$ using the following identity:

$$\exp \{ -y \left( \Psi_{x}^{(1)} \Phi_{x} P_{R} \Psi_{x}^{(1)} + \Psi_{x}^{(1)} \Phi_{x}^{+} P_{L} \Psi_{x}^{(1)} \right) \} = \int d\eta_x d\eta_x \exp \left\{ \frac{1}{2} \sum_{x, \mu} \text{Tr} \left[ \Psi_{x}^{(1)} \Phi_{x} \eta_x \Phi_{x}^{+} \eta_x + \Phi_{x}^{+} \Psi_{x}^{(1)} \right] \right\},$$  

(8)

and proceed with an expansion of the exponential term in powers of $1/y^2$. Here again, we have kept all the terms up to the four fermion interaction (i.e. up to $1/y^2$).

The next step is to make the scalar field dependence of $S_B(\Phi)$ linear by using the auxiliary fields $V_x, V_x^{\dagger}, A_x, A_x^{\dagger}$ $(N$ is the number of sites):

$$\exp \left\{ \frac{k}{2} \sum_{x, \mu} \text{Tr} \left[ \Phi_{x+\mu} \Phi_{x}^{+} + \Phi_{x+\mu}^{+} \Phi_{x} \right] \right\} =$$

$$\left( \frac{1}{2\pi} \right)^{8N} \int [dV dV^{\dagger} dAdA^{\dagger}] \exp \left\{ \frac{k}{2} \sum_{x, \mu} \text{Tr} \left[ V_{x}^{\dagger} V_{x+\mu} + V_{x+\mu}^{\dagger} V_{x} \right] \right\}$$

$$+ \frac{i}{4} \sum_{x} \text{Tr} \left[ A_{x}^{\dagger} (\Phi_{x} - V_{x}) + A_{x} (\Phi_{x}^{+} - V_{x}^{\dagger}) \right].$$  

(9)

In both cases (for small or large $y$), the functional integration over the scalar field $\phi$ can be performed and the fermion terms exponentiated. Before doing the integration over the fermion fields (in order to convert the Yukawa model into a purely bosonic model), we need to decouple the composite fields of the type $\Psi_{x}^{(1)} \Psi_{x}$ (which appear in the four fermion interaction) by using identities like [12]:

$$\exp \left\{ \frac{1}{2} \left( \sum_{x} \Psi_{x}^{(1)} \Psi_{x}^{(1)} \right)^2 M_x^{-1} \right\} = \int \frac{d\lambda_x}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \lambda_x^2 + M_x^{-1/2} \sum_{x} \Psi_{x}^{(1)} \Psi_{x}^{(1)} \lambda_x \right\}$$  

(10)
As a result of the bilinear structure of the action in the fermionic fields, it is now possible to carry out the fermionic integrations in the partition function. The remaining integrals are approximated by the mean field saddle point method: we look for a translationally invariant saddle point with a constant and a staggered piece, we then substitute the auxiliary fields \( V_x \) and \( V_x^\dagger \) with \( v + v_{st} \epsilon_x, \) \( A_x \) and \( A_x^\dagger \) with \( -i(\alpha + \alpha_{st} \epsilon_x) \) and \( \lambda_x \) with \( \lambda + \lambda_{st} \epsilon_x. \) This approach yields the free energy per unit volume,

\[
\mathcal{F} = -\frac{1}{N} \log Z = \frac{1}{2}(\lambda^2 + \lambda_{st}^2) - 2k(v^2 - v_{st}^2) + \alpha v + \alpha_{st} v_{st} - \frac{1}{2} [u(\alpha + \alpha_{st}) + u(\alpha - \alpha_{st})] - \frac{n_f}{2} 2^{d/2} I,
\]

with \( k = kd, \) \( u(\beta) = \log \frac{2}{\beta} I_1(\beta), \) where \( I_1 \) is the modified Bessel function of order 1. The function \( I \) comes from the fermionic determinant and has different expressions for strong and weak yukawa coupling, \( I = I_S \) and \( I = I_W \) respectively with:

\[
I_W = \int \frac{d^d p}{(2\pi)^d} \log \left\{ [s^2(p) + y^2 z \bar{z} F^2_F(p) F^2_F(p)]^2 + y^2 \left( \frac{z + \bar{z}}{2} \right)^2 s^2(p)(F^2_F(p) - F^2_F(p))^2 \right\},
\]

\[
I_S = \int \frac{d^d p}{(2\pi)^d} \log \left\{ \left[ \bar{z} s^2(p) + y^2 F^2(p) F^2_F(p) \right]^2 + y^2 \left( \frac{z + \bar{z}}{2} \right)^2 s^2(p)(F^2(p) - F^2_F(p))^2 \right\}.
\]

where \( F_F(p) = F(p_1 + \pi, \ldots, p_d + \pi) \), and

\[
s^2(p) = \sum_\lambda \sin^2 p_\lambda,
\]

\[
z = \dot{u}(\alpha + \alpha_{st}) - (\lambda + \lambda_{st}) \sqrt{\ddot{u}(\alpha + \alpha_{st})},
\]

\[
\bar{z} = \dot{u}(\alpha - \alpha_{st}) - (\lambda - \lambda_{st}) \sqrt{\ddot{u}(\alpha - \alpha_{st})}.
\]

The saddle point equations are obtained by requiring that the free energy is extremal for \( v, \alpha, \lambda, v_{st}, \alpha_{st}, \) and \( \lambda_{st}. \) The free energy in the presence of source terms can be evaluated following the same steps, and the mean field predictions for the order parameters are
obtained by taking derivatives with respect to these sources. For instance, for a lattice with $N$ sites, with the definitions

$$\langle \Phi \rangle = \left\langle \frac{1}{N} \sum_x \Phi_x \right\rangle,$$
$$\langle \Phi_{st} \rangle = \left\langle \frac{1}{N} \sum_x \epsilon_x \Phi_x \right\rangle,$$  
(15)

we find $\langle \Phi \rangle = v$, $\langle \Phi_{st} \rangle = v_{st}$ if we align the symmetry breaking direction along the unity matrix. Looking numerically for the saddle point values and the free energy for several choices for the parameters $\kappa$ and $y$, we have found four phases, (see fig. 1):

- **Paramagnetic (PM):** Here, $\langle \Phi \rangle = \langle \Phi_{st} \rangle = 0$.
- **Ferromagnetic (FM):** In this phase $\langle \Phi \rangle \neq 0$ and $\langle \Phi_{st} \rangle = 0$.
- **Antiferromagnetic (AFM):** This phase is characterized by $\langle \Phi \rangle = 0$ and $\langle \Phi_{st} \rangle \neq 0$.
- **Ferrimagnetic (FI):** Both parameters different from zero.

The expansion of the term $e^{-S_y}$ is essentially an expansion in $y \Psi^{(1)}_x \Psi^{(1)}_x$ or in $1/y \eta_x \eta_x$, and as a consequence we expect that our results are not too bad in regions where $\left| y \left\langle \Psi^{(1)}_x \Psi^{(1)}_x \right\rangle \right| < 1$ at small $y$ or where $\left| (1/y) \left\langle \eta^{(1)}_x \eta^{(1)}_x \right\rangle \right| < 1$ at large $y$. In the figure we have only shown the regions of $y$ where these conditions are fulfilled.

One last comment about the mass of the physical fermions in the FM phase, $m_F$ (in lattice units). For small $y$, it can be shown, from eqs. (11,12), that $m^2_F = y^2 z^2$. For large $y$, eqs. (11,13) gives $m^2_F = y^2/z^2$. Therefore, as usual [7], we can distinguish a weak and a strong FM regions.

### 4. Phase transition lines

A sampling of the values of the order parameters for several values of $y$ suggests that the change in the order parameters occurs continuously when crossing the phase transitions. Close to the phase transitions, this allows a linearization of the saddle point equations in the mean field variables which vanish at the phase transition (those are $v$, $\alpha$, $\lambda$ for the FM–PM and AFM–FI phase transitions and $v_{st}$, $\alpha_{st}$, $\lambda_{st}$ for the AFM–PM and FM–FI phase transitions). In the phase diagram plotted in the figure, the
FM–FI and AFM–FI transition lines have been obtained with a numerical solution of the linearized equations. For the FM–PM and the PM–AFM transitions an analytical solution is possible; the transitions are of second order and the critical lines are:

\[ \overline{k_c} = \pm 1 - y^2 2^{d/2} n f I \pm, \] (16)

where the upper signs stand for the FM–PM transition and the lower signs for the PM-AFM. For \( d = 4 \),

\[ I_+ = \int \frac{d^4 \theta}{(2\pi)^4} \frac{F^4(\theta)}{s^2(\theta)} \approx 1.61 \times 10^{-2} \] (17)

\[ I_- = \int \frac{d^4 \theta}{(2\pi)^4} \frac{F^2(\theta) F_\pi^2(\theta)}{s^2(\theta)} \approx 8.4 \times 10^{-5} \] (18)

Because of the very small value of \( I_- \) the PM–AFM transition line is almost straight. One would expect this behaviour in any theory in which the doubler fermions are decoupled. In fact, in the mean field approximation, without any form factor, in the AFM phase, the Yukawa term couples the scalar mean field, \( v_{st} \), to the fermion sector through a coupling of the type \( \overline{\Psi}_{k'} v_{st} \Psi_k \delta(k - k' - \pi) \). Because of the factor \( \delta(k - k' - \pi) \), the only coupling between the scalar and the fermion is through a vertex involving simultaneously the physical and the doubler fermions. Therefore, if somehow the doubler fermions are effectively decoupled from the scalar sector, simultaneously the scalar sector is decoupled from the physical fermion, and the PM–AFM phase transition line will be straight.

In our model it is easy to understand how this phenomenon occurs. In \( I_+ \), \( F^4(\theta) \) admits the contribution of the physical fermions and kills the contribution of the doublers, whereas in \( I_- \), \( F^2(\theta) F_\pi^2(\theta) \) kills the contribution of the doublers but also that of the physical fermions. The PM–AFM transition is nearly independent of \( y \). We can compare with the case of the naive fermions where \( F^4(\theta) = F^2(\theta) F_\pi^2(\theta) = 1 \), so \( I_+ = I_- \) and we recover the \( k \to -k, y \to -iy \) symmetry; the transition lines PM-FM and PM-AFM are now parallel, the FI phase and the quadruple point are not seen in the mean field approximation.

Note also that the average slope of the PM–FM transition line in our case is smaller than when the doublers are coupled. Consequently the value of the bare Yukawa coupling
at the quadruple point is large. Moreover, perturbative renormalisation group arguments are in favor of an increase of the renormalized coupling constant when the number of coupled fermions decreases. This could be relevant for the upper bounds of the physical masses.

5. Monte-Carlo results

We have performed an exploratory numerical simulation of this model on lattices of size $4^4$ and $8^4$. We have used the Hybrid Monte Carlo algorithm [13] with two flavour doublets (an even number is required for the algorithm). The fermion matrices was inverted with the Conjuguate Gradient algorithm. The length of the molecular dynamics trajectories was selected with a random distribution with a mean value of 10 steps. For chosen values of $y$, we scan over $k$ in a region where a phase transition is predicted by the mean field results given in the previous section. For each simulation in the $(y, k)$ plane, the number of configurations varies between 3000 and 14000 after having discarded about 1000 trajectories for thermalisation. We have measured the order parameters and the associated susceptibilities. In the large $y$ region, we have only checked that we have a ferrimagnetic phase when $k$ is sufficiently negative without determining precisely the transition point between the FM and the FI phases. In the small $y$ region, the measured transition points are plotted on the figure and compare well with the results from the mean field predictions given by the dashed lines.

6. Conclusions

We have determined the phase structure of an $SU(2) \times SU(2)$ fermion-scalar model with the globally chiral invariant model proposed in [3,4] to decouple the doublers. The numerical and mean field results are in good agreement. We have found four phases (PM,
FM, AFM, FI) with a quadruple point. The mean field calculation is able to predict the FI phase and the quadruple point. The PM–AFM transition line is nearly independent of \( y \), this property is intimately related with the decoupling of the doublers.

The global chiral symmetry is preserved by this model and the decoupling mechanism allows the PM–FM transition line to reach rather large bare yukawa couplings. Thus, the questions of a bound on the mass of a heavy fermion induced by a strong Yukawa interaction and its influence on the Higgs mass should be fruitfully investigated along this way. These issues are currently under investigation.

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**Figure Caption**

Fig.1 Phase diagram for the $SU(2)_L \times SU(2)_R$ Chiral Yukawa model, for $n_f = 2$ doublets of Zaragoza fermions. The dashed lines are the mean field results for the transition lines. The open symbols are the transition points determined by the Monte Carlo simulation. The square correspond to the FM–PM or AFM–FI transitions, the circles to the AF–FM or FI–FM ones. Because of CPU limitation, we have not succeeded in a precise determination of the position of the quadruple point.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9402001v2