Minimum Decision Cost for Operators

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State discrimination with the aim to minimize the error probability is a well studied problem. Instead, here the binary decision problem for operators with a given prior is investigated. A black box containing the unknown operator is probed by selected wave functions. The output is analyzed with conventional methods developed for state discrimination. An error probability bound for all binary operator choices is provided, and it is shown how probe entanglement enhances the result.

I. INTRODUCTION

Quantum metrology encompasses quantum decision problems and has gained prominence with the rise of quantum engineering. One common task is to determine an optimal observational strategy. This has been carried out in various frameworks. Here we focus on the Bayes procedure, which is natural if prior information is available. This leads one to seek a strategy minimizing the expected cost, or in other words to minimize the error probability. The cost function is chosen to be 0 - 1, where 0 is associated with the correct and 1 to the incorrect choice. A detailed exposition of the general quantum decision strategy for wave functions can be found in the book of Helstrom [1], papers by Holevo [2] and Yuen et al. [3], where independently much of the groundwork of the field was laid.

In the present paper instead of distinguishing states we optimally distinguish between operators. We study the Bayesian approach to a binary decision problem for a probe made up of an ensemble of particles being sent through a black box containing an operator selected from a set of two with known prior. The insight employed is that the binary operator decision problem can be mapped into a binary state decision problem. This mapping is allowed, because the unknown operator in the black box transforms the input state into one of two possible output states. The two possible output states have a well-defined transition probability, which is all one needs to apply the standard state decision machinery. The challenge is to find the optimal state in the set of all allowed inputs to obtain the smallest transition probability in the output states. To solve this we use techniques previously applied to the quantum brachistochrone.

The operator decision problem is interesting in its own right, since the space of operators has a Finsler metric and is the natural space for quantum algorithms, e.g. in Grover’s algorithm different positions of the marked state are associated with different operators acting on an initial state, and one can rewrite the algorithm as a multiple operator decision problem[4].

A relevant application of the Bayesian approach to quantum hypothesis testing for multiple polarised spin 1/2 particles was given in Brody et al. [5]. Two strategies were sketched out and compared. In one the experimenter is provided the input as a sequence of identical particles with an adjustable measurement for each particle and in the other one joint measurement is carried out on all particles. It was shown that the optimal Bayes cost for separate sequential measurements of the individual particles is the same as that of a combined measurement. Intermediate strategies were also shown to be unable to reduce the Bayes cost.

In the next section the Bayes cost calculation will be carried out for two of the simplest possible operators acting on an input, i.e. either the 2-dimensional identity operator or an operator which leaves one of the two eigenstates unchanged and adds a phase shift to the other eigenstate. It is shown how entanglement reduces the Bayes cost. In the penultimate section the binary decision problem is extended to a larger class of operators. In the last section the results will be summarized and some additional points briefly discussed.

II. EXAMPLE CALCULATION IN THE BINARY OPERATOR DECISION CASE

In this section one particularly simple binary operator decision problem, generalized in the next section, is described to clarify the procedure. The calculation is divided into three subsections. One each dealing with the entangled and unentangled case, and in the last subsection the results are compared.

As part of the set up we choose a diagonal 2-dimensional operator of the form

\[ U_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha_1} \end{pmatrix} , \]

with \( \alpha_1 = 0 \) and \( \alpha_2 = 2\pi \). The prior probability is given by \( \xi \) for the first and hence \( 1 - \xi \) for the other case. The experimenter has to find out by sending probes through the black box, if it contains \( U_1 \) or \( U_2 \). The advantage of this set up is that a closely related binary state decision problem has been analyzed in Brody et al. [5], and the remaining challenge is to understand how entanglement effects the transition probability.

The decision strategy has various parts. One has to decide on the probe state, on the multiple interactions...
between parts of the probe and the black box, and on
the multiple measurements of the output to optimally
determine the content of the black box. The whole range
of positive operator valued measures are allowed in the
decision strategy as was the case in the earlier paper.

Next more details are provided about the components.
The experimenter can choose as a probe a wave func-
tion freely from the set of \( N \) particle ensembles. These
\( N \) particles can be entangled or sent as a product state.
Naturally there are some constraints on the allowed input
states, e.g. the dimension of the Hilbert space of the
input particles and the black box Hamiltonian have to
match. In the current example each particle is repre-
sented by a 2-dimensional wave function. In addition
to the probe the experimenter is provided with one
black box that can be used multiple times, but at most
\( N \)-times, to modify the probe. As an additional con-
straint each particle, entangled or not, can at most be
sent through the black box once \([\mathbb{I}]\). The final step is to
choose how the probe after passing through the black box
is measured and how the result is analyzed, i.e. in terms
of preposterior analysis.

One should keep in mind that in principle part of
probe preparation as well as probe and black box in-
teraction can be influenced by earlier measurements.
This would complicate the situation, since partial mea-
surements with binary outputs followed by passing the probe
again through the black box increases the number of pos-
sible outputs. Here instead we assume that probe and
black box interaction are completed before any measure-
ment takes place.

A. Probe: One or more unentangled particle

We consider first the simplest case of sending just one
particle through the black box. Any 2-dimensional wave
function of the form \((a|0⟩ + b|1⟩)\) with \(|a|^2 + |b|^2 = 1\)
is allowed as an input. The two possible outputs are
\((a|0⟩ + b|1⟩)\). The transition probability between the
possible outputs is

\[
(|a|^2 + |b|^2)e^{2i\delta}(|a|^2 + |b|^2)e^{-i2\delta} = 1 - 4|a|^2|b|^2\sin^2(\delta),
\]

which is minimal for \(|a|^2 = |b|^2 = 1/2\) leading to a transi-
tion probability of \(\cos^2(\delta)\).

After the one particle probe has been successful sent
through black box, and either changed or left unchanged,
the next step is to carry out a measurement. An opera-
tor decision problem has then turned into a problem of
distinguishing between two different wave functions.

The optimal procedure for states was developed by
Helstrom and others, and the optimal cost function, also
called the Helstrom bound, has for a transition proba-
bility of \(\cos^2(\delta)\) and prior \(\xi\) the form

\[
C_{unEnt}(\xi, 1) = \frac{1}{2} \left(1 - \sqrt{1 - 4\xi(1 - \xi)\cos^2(\delta)}\right).
\]

The strategy minimizing the error probability between
the output states also minimizes the error in the operator
decision problem.

Next we consider the multi-particle case. Two types
of probes are considered. First, a sequence of \( N \)
independent 2-dimensional particles each of the form
1/√2(0| + 1|); second we take the direct product state
\((1/\sqrt{2}0| + 1/\sqrt{2}1|)^\otimes N\). As a result the total transi-
tion probability is in each case \(\cos^{2N}(\delta)\), i.e. either one
has \(N\)-times a transition probability of \(\cos^2(\delta)\) or in the
product state case one transition probability of \(\cos^{2N}(\delta)\).

Following very closely the argument in a paper by
Brody et al. [5] one can carry out either a sequence or a
combined measurement to optimally determine between
the 2 cases. This leads in both cases to an optimal binary
decision cost of

\[
C_{unEnt}(\xi, N) = \frac{1}{2} \left(1 - \sqrt{1 - 4\xi(1 - \xi)\cos^{2N}(\delta)}\right).
\]

This finishes the analysis of the unentangled case. In the
next subsection we deal with entangled probes.

B. Probe: Entangled particles

The question, if entanglement, a term introduced by
Schrödinger as ‘Verschränkung’ in the thirties, can be
utilized to reduce the minimal decision cost, is of inter-
est due the ongoing fascination with the concept, which
now lies at the core of the burgeoning field of quantum
information theory. Therefore, we next consider the cost
of the fully entangled \( N \) particle state.

Following very closely the argument for product states
one can show that the optimal entangled \( N \) particle probe
state to maximize the transition probability between the
possible outcomes is 1/√2(0|^N + 1|^N). The resulting
output states are therefore

\[
U_i^\otimes N 1/2^\left(0|^N + 1|^N\right) = 1/2\left(0|^N + e^{iN\alpha}|1|^N\right)
\]

with \( i \) taking the value one or two. As a result the transi-
tion probability between the possible two outputs is
\(\cos^2(N\delta)\) and the associated decision cost is

\[
C_{Ent}(\xi, N) = \frac{1}{2} \left(1 - \sqrt{1 - 4\xi(1 - \xi)\cos^{2N}(\delta)}\right).
\]

If probe modification is completed before any measure-
ment takes places, then the bound applies without re-
striction. A comparison of the two cost functions follows
next.

C. Comparison of the unentangled and entangled
case

If one compares the two cost functions \(C_{unEnt}(\xi, N)\)
and \(C_{Ent}(\xi, N)\), one notices that they only differ in
the size of the transition probability, which is either \( \cos^{2N}(\delta) \) or \( \cos^{2}(N\delta) \). In the case of \( N\delta \ll \pi/2 \) we have \( \cos^{2N}(\delta) \geq \cos^{2}(N\delta) \) and as a consequence \( C_{\text{Ent}}(\xi, N) \geq C_{\text{Ent}}(\xi, N) \). Therefore the optimal entangled cost is always lower or equal than what is possible for any product state with the same number of particles, if the candidate operators are ‘close’ together, i.e. the operators are difficult to distinguish and the black box output transition probability for any input is small.

After having dealt with the two extreme cases of either a fully entangled or unentangled states (either in sequence or as a product state), one can study if any intermediate case can give a better outcome. This cannot be the case for unentangled states as shown in Brody et al. [5]. In the entangled case for the same reason this is also impossible. This follows from the following two facts. The transition probability of product states is the product of the transition probability, and second that for appropriately small product of \( \delta \) and \( N \), i.e. for \( N\delta \ll \pi/2 \), the following always holds

\[
\prod_{j=1}^{K} \cos^{2n_j}(m_j\delta) \geq \cos^{2}(N\delta)
\]

for \( N = \sum_{j=1}^{K} n_j m_j \). This immediately means that standard partial measurement strategies, even optimally implemented, always increase the cost, since

\[
C_{\text{Ent}}(\xi, N) \leq \frac{1}{2} \left( 1 - \sqrt{1 - 4\xi(1 - \xi) \prod_{j=1}^{K} \cos^{2n_j}(m_j\delta)} \right).
\]

This completes the calculation for this simple example. Subsequently, we build on this result.

III. ANALYSIS OF THE GENERAL BINARY OPERATOR DECISION PROBLEM

In this section we extend the earlier results to the general binary operator decision problem, where the decision is between any two operators of the same dimension. The two operators are again called \( U_1 \) and \( U_2 \) and the prior is still \( \xi \) and \( 1 - \xi \). The aim is to find a state \( \phi \) for which the transition probability between the two possible output states

\[
|\langle \phi | U_1 U_2 | \phi \rangle|^2
\]

is minimal. The reason, why this is sufficient to solve the decision problem is the same as in the special example discussed above. Instead of tackling the problem directly, let us rewrite the operator \( U_i \) as \( e^{iH_1 t/\hbar} \), where \( H_1 \) is the related Hamiltonian, i.e. a Hermitian operator, and \( t \) is a real parameter. Here we choose \( t \) to reflect the time a one particle probe spends in the black box. For the rest of the paper \( \hbar \) is set to one.

Next we exploit the similarity between our problem and the brachistochrome. Here we really study the dual of the brachistochrome, where instead of the optimal Hamiltonian within the constraints, one has to find the optimal input state within the constraints. In the brachistochrome problem eigenvalues play an important role. In particular it is true for all Hamiltonians that an equal weighted superposition of the eigenstates with largest and smallest eigenvalue evolves faster away from the input state than any other state, i.e. the resulting output state has the smallest transition probability to the input state for small time intervals. For a concise geometric derivation of the speed limit \( \sqrt{2} \) in the brachistochrome problem and the connection to the Anandan-Aharonov relation see Brody [6]. In our setting this means that an equal superposition of any maximal eigenvalue and minimal eigenvalue eigenstate is the optimal input state for minimizing the output transition probability.

In general, one can for any unitary \( U_i \) write \( U_1 U_2 = U_{\text{new}} = e^{iH_{\text{new}} t} \) with \( H_{\text{new}} \) Hermitian and \( t \) the time spent inside the black box. The largest and smallest eigenvalue of \( H_{\text{new}} \) are called \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) respectively. Therefore, the optimal state in the one particle case is an equal superposition of the eigenstate with largest and eigenstate with the smallest eigenvalue. The optimal state can be highly non-unique, if there is degeneracy in the eigenstates. Independent of the size of the Hilbert space the problem reduces, once the minimal and maximal eigenvalue has been chosen, to two dimensions. The transition probability between the two possible output states is \( \cos^2((\lambda_{\text{max}} t - \lambda_{\text{min}} t)/2) \). To clarify the result, one can just look at the special example analyzed initially, where the black box either does or does not produce a phase-shift. In that particular case the matrix \( H_{\text{new}} \) was of rank one, \( t \) was set to 1, \( \lambda_{\text{max}} \) was \( 2\delta \), and \( \lambda_{\text{min}} \) was zero.

Next, we look at the special case where the parameter \( t \) is small, then

\[
|\langle \phi | U_1 U_2 | \phi \rangle|^2
\]

simplifies to

\[
|\langle \phi | e^{-iH_1 t + iH_2 t + \frac{i}{2}[H_2, H_1]}t^2 + O(t^3) | \phi \rangle|^2
\]

according to the Baker-Campbell-Haussdorff formula. For the minimal transition probability one has to find the lowest and highest eigenvalues of the matrix \( H_{\text{new}} = H_1 t - H_2 t + O(t^3) \), which can again be viewed as a Hamiltonian in its own right. If the expansion of \( H_{\text{new}} \) up to linear order in \( t \) is not at least rank 1, then one needs to further expand the matrix by including higher order terms like \( \frac{1}{2} [H_2, H_1]^2 \), and more if necessary, until one obtains at least a rank 1 matrix.

Once the optimal minimal transition probability is known in the one particle case, one can, in an analog way to what was done in the first example, extend the result to multi-particle product and entangled states. The benefits associated with entanglement again become evident. The general minimal cost function is for unentan-
gled states

\[ C_{UnEnt}(\xi, N) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\xi(1 - \xi)\cos^2(N(\lambda_{max} - \lambda_{min})t/2)}, \]

and if entangled states are allowed it becomes

\[ C_{Ent}(\xi, N) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\xi(1 - \xi)\cos^2(N(\lambda_{max} - \lambda_{min})t/2)}. \]

As before the entangled cost is smaller than the entangled cost as long as \( N(\lambda_{max} - \lambda_{min})t \ll \pi \), i.e. \( C_{Ent}(\xi, N) \leq C_{UnEnt}(\xi, N) \). This concludes the analysis of the general case, even if one can with the help of the well-developed machinery of matrix theory describe interesting properties of the minimal and maximal eigenvalues of \( H_{new} \), and analyze special cases, where \( H_{new} \) is of non-maximal rank. This will be done in a different setting, where the focus will be on applications.

IV. CONCLUSION

The paper solves the binary operator decision problem by translating it into the well understood problem of distinguishing states. Once this is recognized, the rest of the analysis is mainly cranking the handle of a well-oiled machine.

The general problem of distinguishing arbitrary numbers of operators of possibly unknown dimension is much more challenging, as is the related problem of distinguishing arbitrary number of states. Any progress made on the general state decision problem has an immediate application on the general operator decision problem. As far as the author is aware, no comprehensive and fast algorithm for choosing optimal measurement directions and calculating the Bayes cost has or can maybe be found in the discrete general state decision case with arbitrary prior, since the combinatorics of the measurement possibilities and the related posterior calculations increase markedly as the number of states rises and the probe size increases. Only if there is some conducive structure, is will it be possible to find a self-contained and efficient solution.

Throughout the present paper we have only considered the cost associated with making decisions. In any practical situation one must take into consideration other costs, e.g., the observational cost, state preparation cost, etc., which as constraints might tilt the result in favour of one or another of the strategies. This is maybe an avenue for more applied research.

As a point of departure from the specific problem considered here, the time spent inside the black box could be made to vary. This would make the situation even more similar to the brachistochrone problem.

Let us recap the logic behind the paper, and explain why it was possible to avoid elaborate calculations. This relies on three insights. First, the operator decision problem can be mapped into a state decision problem. Second, the result of Brody et al. [5] is applicable that a range of measurement strategies have the same minimal Bayes cost. Third, the close connection with the brachistochrone, which reduces the problem of finding the optimal input state to finding eigenstates with extremal eigenvalues and combining them in a well-defined way.

In this paper we always assumed that probe and black box interaction is completed before any measurement takes place. What happens to the cost function, if probe modifications are interspersed with measurements will be studied in a separate paper.

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