DUALS OF HARDY-AMALGAM SPACES $H^{(q,p)}_{\text{loc}}$ AND PSEUDO-DIFFERENTIAL OPERATORS

ZOBO VINCENT DE PAUL ABLÉ AND JUSTIN FEUTO

Abstract. In this paper, we carry on with the study of the Hardy-Amalgam spaces $H^{(q,p)}_{\text{loc}}$ spaces introduced in [1]. We investigate their dual spaces and establish some results of boundedness of pseudo-differential operators in these spaces.

1. Introduction

The celebrated paper of C. Fefferman and E. Stein [11] has been crucial in recent developments of the real variable theory of Hardy spaces. Given a function $\varphi \in C^\infty(R^d)$ with support on $B(0,1)$ such that $\int_{R^d} \varphi(x)dx = 1$, where $B(0,1)$ is the unit open ball centered at 0 and $C^\infty(R^d)$ denotes the space of infinitely differentiable complex valued functions on $R^d$, and $t > 0$, we denote by $\varphi_t$ the dilated function $\varphi_t(x) = t^{-d/2} \varphi(x/t)$, $x \in R^d$. The Hardy space $H^q := H^q(R^d)$ is defined as the space of tempered distributions $f$ such that the maximal function

\[ M_\varphi(f) := \sup_{t > 0} |f \ast \varphi_t| \]

is in $L^q(R^d)$, while its local version $H^q_{\text{loc}} := H^q_{\text{loc}}(R^d)$ introduced by D. Goldberg in [14] is that for which $M_{\text{loc}} \varphi(f) \in L^q(R^d)$, where $M_{\text{loc}} \varphi(f)$ is defined as in (1.1) with $0 < t \leq 1$. These spaces have been the subject of several studies and generalizations (see [3], [5], [6], [9], [13], [19], [20], [21], [23], [27], [31], [37] and [38]). Also, following the maximal function approach and using Amalgams, we introduced in [1] the spaces $H^{(q,p)}$ and $H^{(q,p)}_{\text{loc}}$ which generalize $H^q$ and $H^q_{\text{loc}}$ spaces.

For recall, Amalgams arise naturally in harmonic analysis and were introduced by N. Wiener [35] in 1926. For a systematic study of these spaces and their role in Fourier analysis, we refer to [1], [12] and [18]. For $0 < q, p \leq +\infty$, the amalgam of $L^q := L^q(R^d)$ and $L^p := L^p(R^d)$ is the space $(L^q, \ell^p) := (L^q, \ell^p)(R^d)$ of measurable functions $f : R^d \to C$ which are locally in $L^q$ and such that the sequence $\{ \| f \chi_{Q_k} \|_q \}_{k \in Z^d}$ belongs to $\ell^p := \ell^p(Z^d)$, where $Q_k := k + [0,1)^d = \prod_{i=1}^d [k_i, k_i + 1)$, $\chi_{Q_k}$ denotes the characteristic function of $Q_k$ and $\| f \|_q := (\int_{R^d} |f(x)|^q dx)^{1/q}$.

1991 Mathematics Subject Classification. 42B30, 46E30, 42B35, 47G30.

Key words and phrases. Amalgam spaces, Hardy-Amalgam spaces, Atomic decomposition, Molecular decomposition, Duality, Pseudo-differential operators.
with the usual modification when \( q = +\infty \). In [1], we defined the Hardy-amalgam spaces \( H^{(q,p)} \) and \( H^{(q,p)}_{\text{loc}} \), for \( 0 < q, p < +\infty \), by taking the Wiener-amalgam ”norm” in the definition of Hardy space \( H^q \) and local Hardy space \( H^{q}_{\text{loc}} \) instead of the Lebesgue ”norm”. Next, in [2], we characterized the dual space of \( H^{(q,p)} \), whenever \( 0 < q \leq p \leq 1 \), and obtained some results of boundedness of some classical linear operators such as Calderón-Zygmund, convolution and Riesz potential operators, when \( 0 < q \leq 1 \) and \( q \leq p < +\infty \).

The aim of this paper is to extend some well known results for \( H^{q}_{\text{loc}} \) spaces to Hardy-amalgam spaces \( H^{(q,p)}_{\text{loc}} \), when \( 0 < q \leq 1 \) and \( q \leq p < +\infty \), as we did in [2] for \( H^{(q,p)} \). This article is organized as follows.

In Section 2, we recall some properties of Wiener amalgam spaces \( (L^q, L^p) \) and Hardy-amalgam spaces \( H^{(q,p)} \) and \( H^{(q,p)}_{\text{loc}} \) obtained in [1] we need. In Section 3, we establish the atomic and molecular decompositions of \( H^{(q,p)}_{\text{loc}} \) spaces, when \( 0 < q \leq 1 \) and \( q \leq p < +\infty \), which correspond to the local versions of those obtained in [1] for \( H^{(q,p)} \).

Next, in Section 4, with some results of Sections 2 and 3, we characterize the dual space of \( H^{(q,p)}_{\text{loc}} \), whenever \( 0 < q \leq p \leq 1 \). Lastly, in Section 5, as applications of the results of Sections 3 and 4, we study the boundedness of pseudo-differential operators in \( H^{(q,p)}_{\text{loc}} \) spaces, when \( 0 < q \leq 1 \) and \( q \leq p < +\infty \).

Throughout the paper, we let \( \mathbb{N} = \{1, 2, \ldots \} \) and \( \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \). We use \( S := S(\mathbb{R}^d) \) will denote the Schwartz class of rapidly decreasing smooth functions endowed with the topology defined by the family of norms \( \{\mathcal{N}_m\}_{m \in \mathbb{Z}_+} \), where for all \( m \in \mathbb{Z}_+ \) and \( \psi \in S \),

\[
\mathcal{N}_m(\psi) := \sup_{x \in \mathbb{R}^d} (1 + |x|)^m \sum_{|\beta| \leq m} |\partial^\beta \psi(x)|
\]

with \( |\beta| = \beta_1 + \ldots + \beta_d \), \( \partial^\beta = (\partial/\partial x_1)^{\beta_1} \ldots (\partial/\partial x_d)^{\beta_d} \) for all \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d \) and \( |x| := (x_1^2 + \ldots + x_d^2)^{1/2} \). The dual space of \( S \) is the space of tempered distributions denoted by \( S' := S'(\mathbb{R}^d) \) equipped with the weak-* topology. If \( f \in S' \) and \( \theta \in S \), we denote the evaluation of \( f \) on \( \theta \) by \( \langle f, \theta \rangle \). The letter \( C \) will be used for non-negative constants independent of the relevant variables that may change from one occurrence to another. When a constant depends on some important parameters \( \alpha, \gamma, \ldots \), we denote it by \( C(\alpha, \gamma, \ldots) \). Constants with subscript, such as \( C_{\alpha, \gamma, \ldots} \), do not change in different occurrences but depend on the parameters mentioned in it. We propose the following abbreviation \( A \lesssim B \) for the inequalities \( A \leq CB \), where \( C \) is a positive constant independent of the main parameters. If \( A \lesssim B \) and \( B \lesssim A \), then we write \( A \approx B \). For any given (quasi-) normed spaces \( A \) and \( B \) with the corresponding (quasi-) norms \( \|\cdot\|_A \) and \( \|\cdot\|_B \), the symbol \( A \hookrightarrow B \) means that for all \( f \in A \), then \( f \in B \) and \( \|f\|_B \lesssim \|f\|_A \).
For $\lambda > 0$ and a cube $Q \subset \mathbb{R}^d$ (by a cube we mean a cube whose edges are parallel to the coordinate axes), we write $\lambda Q$ for the cube with same center as $Q$ and side-length $\lambda$ times side-length of $Q$, while $|\lambda|$ stands for the greatest integer less or equal to $\lambda$. Also, for $x \in \mathbb{R}^d$ and $\ell > 0$, $Q(x, \ell)$ will denote the cube centered at $x$ and side-length $\ell$.

We use the same notations for balls. We denote by $E(c)$ the set $\mathbb{R}^d \setminus E$ and $\text{diam}(E) := \sup_{x, y \in E} |x - y|$. For a measurable set $E \subset \mathbb{R}^d$, we denote by $\chi_E$ the characteristic function of $E$ and $|E|$ for the Lebesgue measure. Also, for any set $A$, we denote by $\#A := \text{card}(A)$ its cardinality.

Throughout this paper, without loss of generality and unless otherwise specified, we assume that cubes are closed and denote by $Q$ the set of all cubes.

2. Recalls on Wiener amalgam spaces $(L^q, \ell^p)$ and Hardy-amalgam spaces $\mathcal{H}^{(q,p)}$ and $\mathcal{H}^{(q,p)}_{\text{loc}}$

2.1. Wiener amalgam spaces $(L^q, \ell^p)$. For $f \in (L^q, \ell^p)$, we set

$$\|f\|_{q,p} := \left\{ \left\| f(x) \chi_{Q_k} \right\|_q \right\}_{k \in \mathbb{Z}^d} \ell^p.$$ 

Endowed with the (quasi)-norm $\|\cdot\|_{q,p}$, the amalgam space $(L^q, \ell^p)$ is a complete space. It is well known that it is a Banach space if $1 \leq q, p \leq +\infty$ and its dual is $(L^q, \ell^p)^\prime$ for $1 \leq q, p < +\infty$, where $\frac{1}{q} + \frac{1}{p} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$ with the usual conventions (see [4], [7], [12], [18] and [30]). The following result is also well known (see [4], [7], [12], [18] and [30]).

**Proposition 2.1.** Let $0 < q, q_1, p, p_1 \leq +\infty$. We have

1. $(L^q, \ell^p) = L^q$ with $\|\cdot\|_{q,q} = \|\cdot\|_q$.
2. $\|f\|_{q,p_1} \leq \|f\|_{q,p}$, if $p \leq p_1$ and $f \in (L^q, \ell^p)$.
3. $\|f\|_{q,p} \leq \|f\|_{q_1,p}$, if $q \leq q_1$ and $f \in (L^q, \ell^p)$.

Also, we have the following reverse Minkowski’s inequality for $(L^q, \ell^p)$, with $0 < q < 1$ and $0 < p \leq 1$.

**Proposition 2.2** ([2], Proposition 2.2). Let $0 < q < 1$ and $0 < p \leq 1$. For all finite sequence $\{f_n\}_{n=0}^m$ of elements of $(L^q, \ell^p)$, we have

$$\sum_{n=0}^m \|f_n\|_{q,p} \leq \left( \sum_{n=0}^m |f_n| \right)_{q,p}. \quad (2.1)$$

**Proposition 2.3** ([1], (2.4) and (2.5), p. 1902). We have $S \hookrightarrow (L^q, \ell^p)$, for $0 < q, p \leq +\infty$. Also, $(L^q, \ell^p) \hookrightarrow S'$, for $1 \leq q, p \leq +\infty$.

**Remark 2.4.** For $0 < q, p \leq +\infty$, the convergence in $S$ implies the convergence in $(L^q, \ell^p)$.

Likewise, for $1 \leq q, p \leq +\infty$, the convergence in $(L^q, \ell^p)$ implies the convergence in $S'$. Consequently, for all $1 \leq q \leq +\infty$ and $0 < p \leq +\infty$.
+∞, the convergence in \((L^q, \ell^p)\) implies the convergence in \(S'\), more precisely \((L^q, \ell^p) \hookrightarrow S'\).

Another important property of amalgam spaces is the boundedness of the Hardy-Littlewood maximal operator \(\mathcal{M}\). Let \(f\) be a locally integrable function. The centralized Hardy-Littlewood maximal function \(\mathcal{M}(f)\) is defined by

\[
\mathcal{M}(f)(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^d.
\]

Carton-Lebrun and al. proved in [8], Theorems 4.2 and 4.5, the following result.

**Proposition 2.5.** Let \(1 < q, p < +\infty\). Then, for any locally integrable function \(f\),

\[
\|\mathcal{M}(f)\|_{q,p} \lesssim \|f\|_{q,p}.
\]

A very useful generalization of this result is the following

**Proposition 2.6** ([22], Proposition 11.12). Let \(1 < p, u \leq +\infty\) and \(1 < q < +\infty\). Then, for all sequence of measurable functions \(\{f_n\}_{n \geq 0}\),

\[
\left(\sum_{n=0}^{+\infty} \|\mathcal{M}(f_n)\|^{q_u}\right)^{\frac{1}{q_u}} \lesssim \left(\sum_{n=0}^{+\infty} \|f_n\|^{q_u}\right)^{\frac{1}{q_u}},
\]

with the implicit equivalent positive constants independent of \(\{f_n\}_{n \geq 0}\).

2.2. **Hardy-amalgam spaces** \(\mathcal{H}^{(q,p)}\) and \(\mathcal{H}^{(q,p)}_{\text{loc}}\). Let \(0 < q, p < +\infty\) and \(\varphi \in S\) with \(\text{supp}(\varphi) \subset B(0,1)\) and \(\int_{\mathbb{R}^d} \varphi(x)dx = 1\). In [1], we defined the space \(\mathcal{H}^{(q,p)}\) as the space of tempered distributions \(f\) having their maximal function \(\mathcal{M}_\varphi(f)\) in \((L^q, \ell^p)\). For \(f \in \mathcal{H}^{(q,p)}\), we set

\[
\|f\|_{\mathcal{H}^{(q,p)}} := \|\mathcal{M}_\varphi(f)\|_{q,p}.
\]

Similarly, we defined the space \(\mathcal{H}^{(q,p)}_{\text{loc}}\) by replacing the maximal operator \(\mathcal{M}_\varphi\) by its local version \(\mathcal{M}_{\varphi,\text{loc}}\). Also, \(\|\cdot\|_{\mathcal{H}^{(q,p)}}\) and \(\|\cdot\|_{\mathcal{H}^{(q,p)}_{\text{loc}}}\) respectively define norms on \(\mathcal{H}^{(q,p)}\) and \(\mathcal{H}^{(q,p)}_{\text{loc}}\), whenever \(1 \leq q, p < +\infty\), and quasi-norms otherwise. Moreover,

\[
\|f\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \leq \|f\|_{\mathcal{H}^{(q,p)}}\)

for all \(f \in \mathcal{H}^{(q,p)}\). We obtained the following result in [1].

**Theorem 2.7** ([1], Theorem 3.2). Let \(1 \leq q, p < +\infty\).

1. If \(1 < q, p < +\infty\), then \(\mathcal{H}^{(q,p)} = \mathcal{H}^{(q,p)}_{\text{loc}} = (L^q, \ell^p)\), with norms equivalences.
(2) If \( q = 1 \), then \( \mathcal{H}^{(1,p)} \subset \mathcal{H}_{\text{loc}}^{(1,p)} \subset (L^1, \ell^p) \). Furthermore,

\[
\|f\|_{1,p} \leq \|f\|_{\mathcal{H}_{\text{loc}}^{(1,p)}} \leq \|f\|_{\mathcal{H}^{(1,p)}},
\]

for all \( f \in \mathcal{H}^{(1,p)} \).

**Corollary 2.8** ([1], Remark 3.3). Let \( 1 \leq q < +\infty \) and \( 0 < p < +\infty \). Then, \( \mathcal{H}^{(q,p)} \subset \mathcal{H}_{\text{loc}}^{(q,p)} \subset L^1_{\text{loc}} \).

Also, in [1], we showed that \( \mathcal{H}^{(q,p)} \) and \( \mathcal{H}^{(q,p)}_{\text{loc}} \) can be characterized with other maximal functions and do not depend on the choice of the function \( \varphi \). Endeed, given \( a, b > 0 \), \( \Phi \in \mathcal{S} \) and \( f \in \mathcal{S}' \), we define the non-tangential maximal function \( M_{\Phi,a}^*(f) \) of \( f \) with respect to \( \Phi \) as

\[
M_{\Phi,a}^*(f)(x) := \sup_{t>0} \left\{ \sup_{|x-y| \leq at} |(f * \Phi_t)(y)| \right\}, \quad x \in \mathbb{R}^d
\]

and the auxiliary maximal function \( M_{\Phi,b}^{**}(f) \) of \( f \) with respect to \( \Phi \) as

\[
M_{\Phi,b}^{**}(f)(x) := \sup_{t>0} \left\{ \sup_{y \in \mathbb{R}^d} \frac{|(f * \Phi_t)(x-y)|}{(1+t^{-1}|y|)^b} \right\}, \quad x \in \mathbb{R}^d.
\]

Moreover, given a positive integer \( N \), we define

\[
\mathcal{N}_N(\phi) := \int_{\mathbb{R}^d} (1 + |x|)^N \left( \sum_{|\alpha| \leq N+1} |\partial^\alpha \phi(x)| \right) dx, \quad \phi \in \mathcal{S}
\]

and

\[
\mathcal{F}_N := \{ \phi \in \mathcal{S} : \mathcal{N}_N(\phi) \leq 1 \}.
\]

The non-tangential grand maximal function of \( f \) is defined as

\[
M_{\mathcal{F}_N}^*(f)(x) := \sup_{\psi \in \mathcal{F}_N} M_{\psi,1}^*(f)(x), \quad x \in \mathbb{R}^d
\]

and the radial grand maximal function of \( f \) as

\[
M_{\mathcal{F}_N}^0(f)(x) := \sup_{\psi \in \mathcal{F}_N} M_{\psi}(f)(x), \quad x \in \mathbb{R}^d.
\]

Similarly, we define the local versions of these maximal functions by taking \( t \) in \((0; 1]\). We have the following theorem (see [1], Theorem 3.7, p. 1910).

**Theorem 2.9.** Let \( 0 < q, p < +\infty \) and \( \Phi \in \mathcal{S} \) with \( \int_{\mathbb{R}^d} \Phi(x)dx \neq 0 \). Then, for any \( f \in \mathcal{S}' \), the following assertions are equivalent:

1. \( M_{\Phi}(f) \in (L^q, \ell^p) \).
2. For all \( a > 0 \), \( M_{\Phi,a}^*(f) \in (L^q, \ell^p) \).
3. For all \( b > \max \left\{ \frac{d}{q}, \frac{d}{p} \right\} \), \( M_{\Phi,b}^{**}(f) \in (L^q, \ell^p) \).
4. There exists an integer \( N > \max \left\{ \frac{d}{q}, \frac{d}{p} \right\} \) such that \( M_{\mathcal{F}_N}(f) \in (L^q, \ell^p) \).
Moreover, for any $N \geq \max \left\{ \left\lfloor \frac{d}{q} \right\rfloor, \left\lfloor \frac{d}{p} \right\rfloor \right\} + 1$ and max $\left\{ \frac{d}{q}, \frac{d}{p} \right\} < b < N$ with $N = [b] + 1$,
\[
\| \mathcal{M}^0_{\mathcal{F}_N}(f) \|_{q,p} \approx \| \mathcal{M}_{\mathcal{F}_N}(f) \|_{q,p} \approx \| \mathcal{M}_\phi(f) \|_{q,p} \approx \| \mathcal{M}_{\psi,b}(f) \|_{q,p} \approx \| \mathcal{M}_{\psi,a}(f) \|_{q,p}.
\]

**Remark 2.11.** Theorem 2.9 holds when the maximal functions are replaced by their local versions. Thus, we obtain a variety of maximal characterizations for $\mathcal{H}^{(q,p)}$ and $\mathcal{H}^{(q,p)}_{\text{loc}}$ and their independence of the choice of the function $\varphi$.

For the sequel, we retain three equivalences in Theorem 2.9 and their local versions; more precisely, for any $N \geq \max \left\{ \left\lfloor \frac{d}{q} \right\rfloor, \left\lfloor \frac{d}{p} \right\rfloor \right\} + 1$,
\[
\| \mathcal{M}^0_{\mathcal{F}_N}(f) \|_{q,p} \approx \| \mathcal{M}_{\mathcal{F}_N}(f) \|_{q,p} \approx \| \mathcal{M}_\phi(f) \|_{q,p}.
\]

We have the following remark (see [1], (3.12) and (3.13), pp. 1911-1912).

**Remark 2.11.** Let $f \in \mathcal{H}^{(q,p)}$ and $\phi \in \mathcal{S}$. We have
\[
|f \ast \Phi(x)| \leq C(\Phi, d, q, p) \| \mathcal{M}_{\mathcal{F}_N}(f) \|_{q,p},
\]
for all $x \in \mathbb{R}^d$, with $C(\Phi, d, q, p) := C(d, q, p) \mathfrak{M}_N(\Phi)$ a nonnegative constant independent of $x$ and $f$. Thus, $f$ is a bounded distribution,
\[
|\langle f, \Phi \rangle| \leq C(d, q, p) \mathfrak{M}_N(\Phi) \| \mathcal{M}_{\mathcal{F}_N}(f) \|_{q,p}
\]
and $\mathcal{H}^{(q,p)} \hookrightarrow \mathcal{S}'$. Also,
\[
\|f \ast \Phi\|_{q,p} \leq \mathfrak{M}_N(\Phi) \| \mathcal{M}_{\mathcal{F}_N}(f) \|_{q,p}.
\]

In Remark 2.11 we can replace $\mathcal{H}^{(q,p)}$ by $\mathcal{H}^{(q,p)}_{\text{loc}}$ and $\mathcal{M}_{\mathcal{F}_N}$ by its local version $\mathcal{M}_{\text{loc,}\mathcal{F}_N}$.

**Corollary 2.12** ([1], Proposition 3.8). The convergence in $\mathcal{H}^{(q,p)}$ and $\mathcal{H}^{(q,p)}_{\text{loc}}$ implies the convergence in $\mathcal{S}'$. Furthermore, $\mathcal{H}^{(q,p)}$ and $\mathcal{H}^{(q,p)}_{\text{loc}}$ are Banach spaces whenever $1 \leq q, p < +\infty$ and quasi-Banach spaces otherwise.

We give a very useful lemma in passing from $\mathcal{H}^{(q,p)}$ to $\mathcal{H}^{(q,p)}_{\text{loc}}$.

**Lemma 2.13.** Let $0 < q, p < +\infty$ and $f \in \mathcal{H}^{(q,p)}_{\text{loc}}$. Then, there exist $u \in \mathcal{H}^{(q,p)}$ and $v \in \mathcal{H}^{(q,p)}_{\text{loc}} \cap C^\infty(\mathbb{R}^d)$ such that $f = u + v$ and
\[
\|f\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \approx \|u\|_{\mathcal{H}^{(q,p)}} + \|v\|_{\mathcal{H}^{(q,p)}_{\text{loc}}},
\]
with the implicit equivalent positive constants independent of $f$.

**Proof.** The idea of our proof is the one of the classical case, namely in passing from $\mathcal{H}^d$ to $\mathcal{H}^{\text{loc}}_d$ (see [14], Lemma 2.2.5, p. 55). One can also see [14], Lemma 4. Fix an integer $N \geq \max \left\{ \left\lfloor \frac{d}{q} \right\rfloor, \left\lfloor \frac{d}{p} \right\rfloor \right\} + 1$ with $N > d$. Let $\psi \in C^\infty(\mathbb{R}^d)$ with supp$(\psi) \subset B(0, 2)$, $0 \leq \psi \leq 1$ and $\psi \equiv 1$.
on $B(0, 1)$. Set $\theta(x) := (\mathcal{F}^{-1}(\psi))(x)$, $v(x) := (f * \theta)(x)$, for all $x \in \mathbb{R}^d$, and $u := f - f * \theta$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform $\mathcal{F}$ defined by $\mathcal{F}^{-1}(\psi)(x) = (\mathcal{F}(\psi))(−x) =: \hat{\psi}(−x)$ and

$$(\mathcal{F}(\psi))(x) =: \hat{\psi}(x) = \int_{\mathbb{R}^d} e^{-2\pi i x \xi} \psi(\xi) d\xi, \quad x \in \mathbb{R}^d,$$

with $i^2 = −1$ (see [15], Definitions 2.2.8 and 2.2.13). Clearly, we have $v \in C^\infty(\mathbb{R}^d) \cap S'$, since $\theta \in S$ by its definition and $f \in S'$. Fix $\phi \in S$ with $\text{supp}(\hat{\phi}) \subset B(0, 1)$, $0 \leq \hat{\phi} \leq 1$ and $\hat{\phi} \equiv 1$ on $B(0, 1/2)$. Then, there exists a constant $C > 0$ such that

$$C(\theta * \phi_t) \in \mathcal{F}_N,$$

for all $0 < t \leq 1$. Indeed, since $\int_{\mathbb{R}^d} \phi(x) dx = \hat{\phi}(0) = 1$, we have $\theta * \phi_t \to \theta$ in $S$ as $t \to 0$. Therefore, for all multi-indexes $\beta$ and all $0 < t \leq 1$, we have

$$|\partial^\beta (\theta * \phi_t)(x)| \leq C_{\beta,N}(1 + |x|)^{-2N},$$

for all $x \in \mathbb{R}^d$, where $C_{\beta,N} > 0$ is a constant independent of $t$ and $x$. Thus,

$$\mathcal{N}_N(\theta * \phi_t) = \int_{\mathbb{R}^d} (1 + |x|)^N \left( \sum_{|\beta| \leq N+1} |\partial^\beta (\theta * \phi_t)(x)| \right) dx$$

$$\leq C_N \int_{\mathbb{R}^d} (1 + |x|)^{-N} dx = C(N) < +\infty,$$

since $N > d$. Hence $C(N)^{-1}(\theta * \phi_t) \in \mathcal{F}_N$. This establishes (2.7).

From (2.7), it follows that

$$|C((f * \theta) * \phi_t)(x)| = ||((C(\theta * \phi_t))_1 \ast f)(x)|| \leq M^{0}_{\mathcal{F}_N}(f)(x),$$

for all $x \in \mathbb{R}^d$ and all $0 < t \leq 1$. Hence

$$M_{\mathcal{F}_N}(v)(x) = \sup_{0 < t \leq 1} |((f * \theta) * \phi_t)(x)| \leq C M^{0}_{\mathcal{F}_N}(f)(x),$$

for all $x \in \mathbb{R}^d$. Thus,

$$\|M_{\mathcal{F}_N}(v)\|_{q,p} \leq C \|M^{0}_{\mathcal{F}_N}(f)\|_{q,p} \approx \|f\|_{\mathcal{H}^{(q,p)}},$$

by the local version of (2.3). Since $\int_{\mathbb{R}^d} \phi(x) dx = \hat{\phi}(0) = 1 \neq 0$, it follows from (2.8) that $v \in \mathcal{H}^{(q,p)}_{\mathcal{F}_N}$ and

$$\|v\|_{\mathcal{H}^{(q,p)}_{\mathcal{F}_N}} \approx \|M_{\mathcal{F}_N}(v)\|_{q,p} \leq C \|f\|_{\mathcal{H}^{(q,p)}},$$

for all $x \in \mathbb{R}^d$. Thus,
by the local version of (2.3). Moreover,

\[(u * \phi_t)(x) = F^{-1}(u * \phi_t)(x) = F^{-1}(\hat{u}(\hat{\phi}_t))(x) = F^{-1}(\hat{u}(\hat{\phi}(t)))(x)\]

\[= F^{-1}((\hat{f} - f \ast \hat{\theta})(\hat{\phi}(t)))(x) = F^{-1}((\hat{f} - f \ast \hat{\theta})(\hat{\phi}(t)))(x)\]

\[= F^{-1}(\hat{f}(1 - \hat{\theta})(\hat{\phi}(t)))(x) = F^{-1}(\hat{f}(1 - \hat{F}^{-1}(\psi))(\hat{\phi}(t)))(x)\]

\[= F^{-1}(\hat{f}(1 - \psi))(\hat{\phi}(t))(x)\]

and \((1 - \psi)[\hat{\phi}(t)] \equiv 0\) if \(t > 1\), since \(\psi \equiv 1\) on \(B(0, 1)\) and \(\text{supp}(\hat{\phi}(t)) \subset B(0, 1/t) \subset B(0, 1)\). Hence \((u * \phi_t)(x) = 0\), for all \(t > 1\). Thus,

\[\mathcal{M}_\phi(u)(x) = \sup_{t > 0} |(u * \phi_t)(x)| = \sup_{0 < t \leq 1} |(u * \phi_t)(x)| = \mathcal{M}_{\text{loc}}(u)(x)\]

and

\[\|\mathcal{M}_\phi(u)\|_{q,p} = \|\mathcal{M}_{\text{loc}}(u)\|_{q,p} = \|\mathcal{M}_{\text{loc}}(f - v)\|_{q,p} \approx \|f - v\|_{\mathcal{H}^{(q,p)}},\]

since \(f, v \in \mathcal{H}^{(q,p)}\) and \(\int_{\mathbb{R}^d} \phi(x)dx \neq 0\). It follows that \(u \in \mathcal{H}^{(q,p)}\),

\[(2.10) \quad \|u\|_{\mathcal{H}^{(q,p)}} \approx \|\mathcal{M}_\phi(u)\|_{q,p} = \|\mathcal{M}_{\text{loc}}(u)\|_{q,p} \approx \|u\|_{\mathcal{H}^{(q,p)}}\]

and

\[\|u\|_{\mathcal{H}^{(q,p)}} \approx \|u\|_{\mathcal{H}^{(q,p)}} = \|f - v\|_{\mathcal{H}^{(q,p)}} \leq C(q, p) \left(\|f\|_{\mathcal{H}^{(q,p)}} + \|v\|_{\mathcal{H}^{(q,p)}}\right),\]

by (2.3) and its local version. Hence

\[\|u\|_{\mathcal{H}^{(q,p)}} + \|v\|_{\mathcal{H}^{(q,p)}} \leq C \|f\|_{\mathcal{H}^{(q,p)}} + C \|v\|_{\mathcal{H}^{(q,p)}} \leq C \|f\|_{\mathcal{H}^{(q,p)}},\]

by (2.7). Conversely, since \(v, u \in \mathcal{H}^{(q,p)}\), we have

\[\|f\|_{\mathcal{H}^{(q,p)}} = \|u + v\|_{\mathcal{H}^{(q,p)}} \leq C \left(\|u\|_{\mathcal{H}^{(q,p)}} + \|v\|_{\mathcal{H}^{(q,p)}}\right),\]

by (2.10). This completes the proof of Lemma 2.13.

From now on, for simplicity, we denote \(\mathcal{M}_{\mathcal{F}_q}^0\) by \(\mathcal{M}^0\) and assume that \(0 < q \leq 1\) and \(q \leq p < +\infty\). We recall the following definition which is also the one of an atom for \(\mathcal{H}^q\).

**Definition 2.14.** Let \(1 < r \leq +\infty\) and \(s \geq \left\lfloor d \left(\frac{1}{q} - 1\right)\right\rfloor\) be an integer. A function \(a\) is a \((q, r, s)\)-atom on \(\mathbb{R}^d\) for \(\mathcal{H}^{(q,p)}\) if there exist a cube \(Q\) such that

1. \(\text{supp}(a) \subset Q\),
2. \(\|a\|_r \leq |Q|^\frac{1}{r} \cdot \frac{1}{\delta}\),
3. \(\int_{\mathbb{R}^d} x^\beta a(x)dx = 0\), for all multi-indexes \(\beta\) with \(|\beta| \leq s\).

Condition (3) is called the vanishing condition or the vanishing moment. We denote by \(A(q, r, s)\) the set of all \((a, Q)\) such that \(a\) is a \((q, r, s)\)-atom and \(Q\) is the associated cube (with respect to Definition 2.14). We obtained the following result in [1].
**Theorem 2.15** ([1], Theorem 4.4). Let \( \delta \geq \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor \) be an integer.

For all \( f \in \mathcal{H}^{(q,p)} \), there exist a sequence \( \{ (a_n, Q^n) \}_{n \geq 0} \) in \( \mathcal{A}(q, \infty, \delta) \) and a sequence of scalars \( \{ \lambda_n \}_{n \geq 0} \) such that

\[
    f = \sum_{n \geq 0} \lambda_n a_n \quad \text{in } S' \text{ and } \mathcal{H}^{(q,p)},
\]

and, for any real \( \eta > 0 \),

\[
    \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\| \chi_{Q^n} \|_q} \right)^{\frac{\eta}{d}} \chi_{Q^n} \right\|_{\frac{d}{d-1}, \frac{\eta}{d}} \lesssim \|f\|_{\mathcal{H}^{(q,p)}}.
\]

Notice that in Theorem 2.15 the sequence of elements of \( \mathcal{A}(q, \infty, \delta) \) can be replaced by a sequence of elements of \( \mathcal{A}(q, r, \delta) \), with \( 1 < r < +\infty \), since \( \mathcal{A}(q, \infty, \delta) \subset \mathcal{A}(q, r, \delta) \).

Also, we have the following result (see [1], Proposition 4.5, p. 1920) which will be very useful for the next sections.

**Proposition 2.16.** Let \( 1 < u, v < s < +\infty \). Then, for every sequence of scalars \( \{ \delta_n \}_{n \geq 0} \) and every sequence \( \{ b_n \}_{n \geq 0} \) in \( L^s \) with the support of each \( b_n \) embedded in a cube \( Q^n \) and \( \| b_n \|_s \leq |Q^n|^{\frac{1}{s} - \frac{1}{u}} \), we have

\[
    \left\| \sum_{n \geq 0} |\delta_n b_n| \right\|_{u,v} \lesssim \left\| \sum_{n \geq 0} \frac{|\delta_n|}{\| \chi_{Q^n} \|_u} \chi_{Q^n} \right\|_{u,v},
\]

with implicit constant independent of \( \{ b_n \}_{n \geq 0} \) and \( \{ \delta_n \}_{n \geq 0} \).

### 3. Atomic and molecular Decompositions of \( \mathcal{H}^{(q,p)}_{\text{loc}} \)

In this section, our aim is to give atomic and molecular decompositions of \( \mathcal{H}^{(q,p)}_{\text{loc}} \) spaces, namely reconstruction and decomposition theorems, when \( 0 < q \leq 1 \) and \( q \leq p < +\infty \). In fact, in our earlier paper [1], p. 1929, the idea of such decompositions for \( \mathcal{H}^{(q,p)}_{\text{loc}} \) spaces has been mentioned without more details. In the sequel, unless otherwise specified, we assume that \( 0 < q \leq 1 \) and \( q \leq p < +\infty \). Also, for simplicity, we adopt the following notations:

\[
    \mathcal{M}_{\text{loc}} := \mathcal{M}_{\text{loc}, \mathcal{F}_N}, \quad \mathcal{M}^0_{\text{loc}} := \mathcal{M}^0_{\text{loc}, \mathcal{F}_N} \quad \text{and} \quad \mathcal{M}_{\text{loc}_0} := \mathcal{M}_{\text{loc}_0}. \]

#### 3.1. Atomic Decomposition

Our definition of atom on \( \mathbb{R}^d \) for \( \mathcal{H}^{(q,p)}_{\text{loc}} \) is the one of \( \mathcal{H}^{(q,p)} \) except that, when \( |Q| \geq 1 \), the vanishing condition (3) in Definition 2.14 is not required. Likewise, we denote by \( \mathcal{A}_{\text{loc}}(q, r, s) \) the set of all \( (a, Q) \) such that \( a \) is a \( (q, r, s) \)-atom on \( \mathbb{R}^d \) for \( \mathcal{H}^{(q,p)}_{\text{loc}} \) and \( Q \) is the associated cube. It follows from this definition the following remark.

**Remark 3.1.** Let \( 1 < r \leq v \leq +\infty \). Then,
(1) \((a, Q) \in A_{\text{loc}}(q, v, s) \Rightarrow (a, Q) \in A_{\text{loc}}(q, r, s)\).

(2) \(A(q, r, s) \subset A_{\text{loc}}(q, r, s)\).

Also, it is easy to see that, for all \((a, Q) \in A_{\text{loc}}(q, r, s)\),

\[
\|a\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \leq C,
\]

where \(C > 0\) is a constant independent of the \((q,r,s)\)-atom \(a\).

We have the following reconstruction theorems.

**Theorem 3.2.** Let \(0 < \eta \leq 1\) and \(\delta \geq \left\lfloor \frac{d}{q} - 1 \right\rfloor\) be an integer.

Then, for all sequences \(\{(a_n, Q^n)\}_{n \geq 0}\) in \(A_{\text{loc}}(q, \infty, \delta)\) and all sequences of scalars \(\{\lambda_n\}_{n \geq 0}\) such that

\[
\sum_{n \geq 0} \left( \frac{|\lambda_n|}{\|\chi_{Q^n}\|_q} \right)^\eta \chi_{Q^n} < +\infty,
\]

the series \(f := \sum_{n \geq 0} \lambda_n a_n\) converges in \(S'\) and \(\mathcal{H}^{(q,p)}_{\text{loc}}\), with

\[
\|f\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \simeq \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\|\chi_{Q^n}\|_q} \right)^\eta \chi_{Q^n} \right\|_{\frac{d}{q} + \frac{\eta}{q}}.
\]

**Proof.** For any \((a_n, Q^n) \in A_{\text{loc}}(q, \infty, \delta)\), we have

\[
\mathcal{M}_{\text{loc}}(a_n)(x) \lesssim \left[ \mathfrak{M}(\chi_{Q^n})(x) \right]^{\frac{d + \frac{\delta + 1}{\eta}}{d}} \left\| \chi_{Q^n} \right\|_q^{-1},
\]

for all \(x \in \mathbb{R}^d\). For the proof of (3.3), we distinguish two cases: \(|Q^n| < 1\) and \(|Q^n| \geq 1\). Denote by \(x_n\) and \(\ell_n\) respectively the center and the side-length of \(Q^n\). When \(|Q^n| < 1\), we know that the atom \(a_n\) satisfies the vanishing condition. Thus, setting \(\widetilde{Q^n} := 2^\ell Q^n\), we obtain, as in the proof of [1], Theorem 4.3, pp. 1914-1915,

\[
\mathcal{M}_{\text{loc}}(a_n)(x) \lesssim \left[ \mathfrak{M}(\chi_{Q^n})(x) \right]^{\frac{d + \frac{\delta + 1}{\eta}}{d}} \left\| \chi_{Q^n} \right\|_q^{-1},
\]

for all \(x \in \mathbb{R}^d\). When \(|Q^n| \geq 1\), the vanishing condition of the atom \(a_n\) not being required, we consider \(\widetilde{Q^n} := 4^\ell Q^n\). Then, it is straightforward to see that

\[
\mathcal{M}_{\text{loc}}(a_n)(x) = 0,
\]
for all \( x \notin \tilde{Q}^n \). Thus,
\[
M_{\text{loc}}(a_n)(x) = M_{\text{loc}}(a_n)(x)\chi_{\tilde{Q}^n}(x)
\]
\[
\lesssim \mathcal{M}(a_n)(x)\chi_{\tilde{Q}^n}(x)
\]
\[
\lesssim \|a_n\|_\infty \chi_{\tilde{Q}^n}(x)
\]
\[
\lesssim \frac{\ell_n^{d+\delta+1}}{\ell_n^{d+\delta+1} + |x - x_n|^{d+\delta+1}} \|\chi_{Q^n}\|^{-1}_q
\]
\[
\lesssim \left[\mathcal{M}\chi_{Q^n}(x)\right]^{\frac{d+\delta+1}{d}} \|\chi_{Q^n}\|^{-1}_q,
\]
for all \( x \in \mathbb{R}^d \). Combining these two cases, we obtain (3.3).

With (3.3), we end as in the proof of [1], Theorem 4.3. \( \square \)

**Theorem 3.3.** Let \( \max\{p, 1\} < r < +\infty \), \( 0 < \eta < q \) and \( \delta \geq \left\lfloor d \left(\frac{1}{\eta} - 1\right)\right\rfloor \) be an integer. Then, for all sequences \( \{(a_n, Q^n)\}_{n \geq 0} \) in \( A_{\text{loc}}(q, r, \delta) \) and all sequences of scalars \( \{\lambda_n\}_{n \geq 0} \) such that
\[
(3.4) \quad \left\| \sum_{n \geq 0} \left( \frac{\|\lambda_n\|}{\|\chi_{Q^n}\|_q} \right)^\eta \chi_{Q^n} \right\|_{\frac{\mu}{\eta}} < +\infty,
\]
the series \( f := \sum_{n \geq 0} \lambda_n a_n \) converges in \( S' \) and \( \mathcal{H}^{(q, p)}_{\text{loc}} \), with
\[
\|f\|_{\mathcal{H}^{(q, p)}_{\text{loc}}} \lesssim \left\| \sum_{n \geq 0} \left( \frac{\|\lambda_n\|}{\|\chi_{Q^n}\|_q} \right)^\eta \chi_{Q^n} \right\|_{\frac{\mu}{\eta}}.
\]

*Proof.* For any \( (a_n, Q^n) \in A_{\text{loc}}(q, r, \delta) \), we have
\[
(3.5) \quad M_{\text{loc}}(a_n)(x) \lesssim \mathcal{M}(a_n)(x)\chi_{\tilde{Q}^n}(x) + \left[\mathcal{M}\chi_{Q^n}(x)\right]^{\mu} \|\chi_{Q^n}\|^{-1}_q,
\]
for all \( x \in \mathbb{R}^d \), where \( \tilde{Q}^n := 4\sqrt{d}Q^n \) and \( \mu = \frac{d+\delta+1}{d} \). To prove (3.5), denote by \( x_n \) and \( \ell_n \) respectively the center and the side-lenght of \( Q^n \). It’s clear that
\[
(3.6) \quad M_{\text{loc}}(a_n)(x) \lesssim \mathcal{M}(a_n)(x),
\]
for all \( x \in \tilde{Q}^n \). Also,
\[
(3.7) \quad M_{\text{loc}}(a_n)(x) \lesssim \left[\mathcal{M}\chi_{Q^n}(x)\right]^{\frac{d+\delta+1}{d}} \|\chi_{Q^n}\|^{-1}_q,
\]
for all \( x \notin \tilde{Q}^n \). Endeed, when \( |Q^n| < 1 \), by proceeding as in the proof of [1], Theorem 4.3, pp. 1914-1915, we obtain (3.7). And when \( |Q^n| \geq 1 \), (3.7) is clearly satisfied since
\[
M_{\text{loc}}(a_n)(x) = 0,
\]
for all \( x \notin \tilde{Q}^n \). Thus, combining (3.6) and (3.7), we obtain (3.5).

With (3.5), we end as in the proof of [1], Theorem 4.6, p. 1921. \( \square \)
For our decomposition theorem, we need some lemmas. Also, we borrow some ideas from [31] and [37]. But before, we have the useful following remark.

**Remark 3.4.** In our earlier paper [1], Lemma 4.1 has been very useful to establish the decomposition theorem of \( H^{(q,p)} \) spaces ([1], Theorem 4.4, p. 1916). Also, in the same paper (see p. 1929), we noticed that, in Lemma 4.1, the maximal operator \( M^0 \) can be replaced by its local version \( M^0_{\text{loc}} \), more precisely with \( b_k = (f - c_k)\eta_k \), if \( |Q_k^0| < 1 \), and \( b_k = f\eta_k \), if \( |Q_k^0| \geq 1 \), for each integer \( k \geq 0 \). That allowed us to show that \( L^1_{\text{loc}} \cap H^{(q,p)} \) is dense in \( H^{(q,p)}_{\text{loc}} \) in the quasi-norm \( \| \cdot \|_{H^{(q,p)}_{\text{loc}}} \) (see [1], Proposition 4.11, p. 1929). However, we discovered a mistake in the proof of [1], Proposition 4.11, p. 1929. Here, we correct this error.

**Proposition 3.5.** Suppose that \( 0 < q \leq 1 \) and \( 0 < p < +\infty \). Let \( p \leq s \leq +\infty \) and \( 0 < r \leq +\infty \). Then, \( L^1_{\text{loc}} \cap H^{(q,p)} \), \( (L^{q,s}) \cap H^{(q,p)}_{\text{loc}} \), and \( (L^r,\ell^s) \cap H^{(q,p)}_{\text{loc}} \) are dense subspaces of \( H^{(q,p)}_{\text{loc}} \).

**Proof.** Let \( f \in H^{(q,p)}_{\text{loc}} \) and \( \delta \geq 1 \). Let \( j \in \mathbb{Z}_+ \), we consider the level set

\[
Q_j^0 := \{ x \in \mathbb{R}^d : M^0_{\text{loc}}(f)(x) > 2^j \}.
\]

By Remark 3.4, \( f \) can be decomposed as follows: \( f = g_j + b_j \) with \( b_j = \sum_{n \geq 0} b_{j,n} \) and

\[
b_{j,n} = \begin{cases} (f - c_{j,n})\eta_{j,n}, & \text{if } |Q^0_{j,n}| < 1 \\ f\eta_{j,n}, & \text{if } |Q^0_{j,n}| \geq 1, \end{cases}
\]

where each \( \eta_{j,n} \) is supported in \( Q^0_{j,n} \), with

\[
(3.8) \quad \bigcup_{n \geq 0} Q^0_{j,n} = Q^0_j, \quad \sum_{n \geq 0} \chi_{Q^0_{j,n}} \lesssim 1; \quad \sum_{n \geq 0} \eta_{j,n} = \chi_{Q^0_j}; \quad 0 \leq \eta_{j,n} \leq 1;
\]

\[
(3.9) \quad M^0_{\text{loc}}(b_{j,n})(x) \lesssim M^0_{\text{loc}}(f)(x)\chi_{Q^0_{j,n}}(x) + \frac{2^j\ell_{j,n}^{d+\delta+1}2^{-j}2^{j\ell_{j,n}}}{|x - x_{j,n}|^{d+\delta+1}\chi_{\mathbb{R}^d \setminus Q^0_{j,n}}}(x)
\]

and

\[
(3.10) \quad M^0_{\text{loc}}(g_j)(x) \lesssim M^0_{\text{loc}}(f)(x)\chi_{\mathbb{R}^d \setminus Q^0_j}(x) + 2^j \sum_{n \geq 0} \frac{\ell_{j,n}^{d+\delta+1}2^{-j}2^{j\ell_{j,n}}}{|x - x_{j,n}|^{d+\delta+1}}
\]

for all \( x \in \mathbb{R}^d \), where \( x_{j,n} \) and \( \ell_{j,n} \) denote respectively the center and the side-length of \( Q^0_{j,n} \) and \( \mathbb{R}^d \setminus Q^0_j = \{ x \in \mathbb{R}^d : M^0_{\text{loc}}(f)(x) \leq 2^j \} \).

Set \( u = \frac{d+\delta+1}{d} \). We have \( 1 < u \) and \( 1 < \min \{ qu, pu \} \), since \( \delta \geq 1 \). Thus, with (3.9) and (3.8), by arguing as in the proof of [1], Theorem 4.4, p. 1916, we obtain

\[
\| b_j \|_{H^{(q,p)}_{\text{loc}}} \lesssim \| \chi_{Q^0_j} M^0_{\text{loc}}(f) \|_{q,p},
\]

\[
\| b_j \|_{H^{(q,p)}_{\text{loc}}} \lesssim \| \chi_{Q^0_j} M^0_{\text{loc}}(f) \|_{q,p},
\]
Hence \( \| \mathbf{M}^0_{\text{loc}}(g_j) \|_{1, \frac{p}{q}} \leq \| \mathbf{M}^0_{\text{loc}}(f) \|_{q, p} \to 0 \) as \( j \to +\infty \). Hence the sequence \( \{g_j\}_{j \geq 0} \subset \mathcal{H}^{(q, p)}_{\text{loc}} \) converges to \( f \) in \( \mathcal{H}^{(q, p)}_{\text{loc}} \). Moreover,

\[
\| \mathbf{M}^0_{\text{loc}}(g_j) \|_{1, \frac{p}{q}} \leq \| \mathbf{M}^0_{\text{loc}}(f) \chi_{\mathbb{R}^n \setminus \mathcal{O}} + 2^j \sum_{n \geq 0} \left[ \mathbf{M}(\chi_{Q_j^n}) \right]^{\frac{d+4+1}{d}} \|_{1, \frac{p}{q}} + \sum_{n \geq 0} \left[ \mathbf{M}(\chi_{Q_j^n}) \right]^{\frac{d+4+1}{d}} \|_{1, \frac{p}{q}},
\]

for all \( j \in \mathbb{Z}_+ \), by \( (3.11) \). But, since \( 0 < q \leq 1 \), we have

\[
\| \mathbf{M}^0_{\text{loc}}(f) \chi_{\mathbb{R}^n \setminus \mathcal{O}} \|_{1, \frac{p}{q}} \leq 2^{j(1-q)} \| \mathbf{M}^0_{\text{loc}}(f) \|_{q, p} \leq 2^{j(1-q)} \| f \|_{\mathcal{H}^{(q, p)}_{\text{loc}}}.
\]

Also, since \( 0 < q \leq 1 < pu \), we have \( \frac{p}{q} u > 1 \). Thus, by Proposition \( 2.6 \) and \( (3.8) \),

\[
\left\| \sum_{n \geq 0} \left[ \mathbf{M}(\chi_{Q_j^n}) \right]^{\frac{d+4+1}{d}} \right\|_{1, \frac{p}{q}} = \left\| \left( \sum_{n \geq 0} \left[ \mathbf{M}(\chi_{Q_j^n}) \right] u \right)^{\frac{1}{p}} \right\|_{u, \frac{p}{q} u} \leq \left\| \sum_{n \geq 0} \chi_{Q_j^n} \right\|_{1, \frac{p}{q}} \leq \| \chi_{\mathcal{O}} \|_{1, \frac{p}{q}}
\]

and

\[
\| \chi_{\mathcal{O}} \|_{1, \frac{p}{q}} = \left[ \sum_{k \in \mathbb{Z}^d} \left( \int_{Q_k} \chi_{\mathcal{O}}(x) \right) dx \right]^{\frac{q}{p}} \leq \left[ 2^{-j pu} \| \mathbf{M}^0_{\text{loc}}(f) \|_{q, p} \right]^{\frac{p}{q}} \approx 2^{-jq} \| f \|_{\mathcal{H}^{(q, p)}_{\text{loc}}}.
\]

Hence

\[
\left\| \sum_{n \geq 0} \left[ \mathbf{M}(\chi_{Q_j^n}) \right]^{\frac{d+4+1}{d}} \right\|_{1, \frac{p}{q}} \approx 2^{-jq} \| f \|_{\mathcal{H}^{(q, p)}_{\text{loc}}}.
\]

Thus,

\[
\| \mathbf{M}^0_{\text{loc}}(g_j) \|_{1, \frac{p}{q}} \leq 2^{j(1-q)} \| f \|_{\mathcal{H}^{(q, p)}_{\text{loc}}} < +\infty,
\]

for all \( j \in \mathbb{Z}_+ \). Therefore, for all \( j \in \mathbb{Z}_+ \),

\[
g_j \in \mathcal{H}^{(1, \frac{p}{q})}_{\text{loc}} \subset L^1_{\text{loc}},
\]

for all \( j \in \mathbb{Z}_+ \), so that

\[
(3.12) \quad g_j = f - b_j \in \mathcal{H}^{(q, p)}_{\text{loc}},
\]
by Corollary 2.8. Combining (3.11) and (3.12), we obtain
\[(3.13)\]
g_j \in L^1_{\text{loc}} \cap \mathcal{H}_{\text{loc}}^{(q,p)},
for all \( j \in \mathbb{Z}_+ \). Thus, \( L^1_{\text{loc}} \cap \mathcal{H}_{\text{loc}}^{(q,p)} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \).

Also, since \( g_j \in L^1_{\text{loc}} \) for all \( j \in \mathbb{Z}_+ \), we have \(|g_j(x)| \leq \mathcal{M}_{\text{loc}_0}(g_j)(x)\), for almost all \( x \in \mathbb{R}^d \) and all \( j \in \mathbb{Z}_+ \). It follows from this estimate that
\[
\|g_j\|_{q,s} \leq \|\mathcal{M}_{\text{loc}_0}(g_j)\|_{q,p} = \|g_j\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} < +\infty,
\]
all \( j \in \mathbb{Z}_+ \). Hence \( g_j \in (L^q, \ell^s) \cap \mathcal{H}_{\text{loc}}^{(q,p)} \), for all \( j \in \mathbb{Z}_+ \). Thus, \( (L^q, \ell^s) \cap \mathcal{H}_{\text{loc}}^{(q,p)} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \).

For the proof of the density of \( (L^r, \ell^\infty) \cap \mathcal{H}_{\text{loc}}^{(q,p)} \) in \( \mathcal{H}_{\text{loc}}^{(q,p)} \), our approach in the proof of [1], Proposition 4.11 is not correct, since the estimate \( \mathcal{M}_{\text{loc}_0}(g_j)(x) \leq 2^j \), for almost all \( x \in \mathbb{R}^d \), used in [1] is not valid when the tempered distribution \( f \notin L^1_{\text{loc}} \). Here, we close this gap by showing that for any \( \epsilon > 0 \), there exists \( f_\epsilon \in (L^r, \ell^\infty) \cap \mathcal{H}_{\text{loc}}^{(q,p)} \) such that
\[(3.14)\]
\[\|f - f_\epsilon\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \leq \epsilon,\]
which implies that \( (L^r, \ell^\infty) \cap \mathcal{H}_{\text{loc}}^{(q,p)} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \). For the proof of (3.14), we distinguish two cases.

Case 1: Assume that \( f \in L^1_{\text{loc}} \). Then,
\[(3.15)\]
\[|g_j(x)| \leq \mathcal{M}_{\text{loc}_0}(g_j)(x) \leq 2^j,\]
for almost all \( x \in \mathbb{R}^d \) and all \( j \in \mathbb{Z}_+ \). Hence \( g_j \in L^\infty \subset (L^r, \ell^\infty) \), for all \( j \in \mathbb{Z}_+ \). Thus, \( g_j \in (L^r, \ell^\infty) \cap \mathcal{H}_{\text{loc}}^{(q,p)} \), for all \( j \in \mathbb{Z}_+ \). Furthermore, by the above results, we know that \( \lim_{j \to +\infty} \|f - g_j\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} = 0 \). Hence for any \( \epsilon > 0 \), there exists an integer \( j_\epsilon \geq 0 \) such that \( g_{j_\epsilon} \in (L^r, \ell^\infty) \cap \mathcal{H}_{\text{loc}}^{(q,p)} \) and
\[\|f - g_{j_\epsilon}\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \leq \epsilon.\]

Case 2: Assume that \( f \notin L^1_{\text{loc}} \). Then, we have no longer \( g_j \in L^\infty \), since (3.15) does not hold. However, by the above results, we know that \( g_j \in L^1_{\text{loc}} \cap \mathcal{H}_{\text{loc}}^{(q,p)} \), for all \( j \in \mathbb{Z}_+ \), and
\[(3.16)\]
\[\lim_{j \to +\infty} \|f - g_j\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} = 0.\]
Since \( g_j \in L^1_{\text{loc}} \cap \mathcal{H}_{\text{loc}}^{(q,p)} \), we can apply the preceeding argument to each \( g_j \). Thus, for each \( g_j \), there exists a sequence \( \{g_{j_k}\}_{k \geq 0} \subset (L^r, \ell^\infty) \cap \mathcal{H}_{\text{loc}}^{(q,p)} \) with
\[(3.17)\]
\[\lim_{k \to +\infty} \|g_j - g_{j_k}\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} = 0.\]
Let \( \epsilon > 0 \) and \( C_{q,p} \geq 1 \) the modulus of concavity of the quasi-norm \( \|\cdot\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \). Then, by (3.16), there exists an integer \( j_{\epsilon} \geq 0 \) such that
\[\|f - g_{j_{\epsilon}}\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \leq \frac{1}{2C_{q,p}}.\]
Also, by (3.17), there exists an integer \( k_{j_{\epsilon}} \geq 0 \)
such that $g_{k_j}^{j_k} \in (L^r, \ell^\infty) \cap \mathcal{H}_{loc}^{(q,p)}$ and $\|g_{j_\star} - g_{k_j}^{j_k}\|_{\mathcal{H}_{loc}^{(q,p)}} \leq \frac{\varepsilon}{2^{k_j}}$. Thus, $g_{k_j}^{j_k} \in (L^r, \ell^\infty) \cap \mathcal{H}_{loc}^{(q,p)}$ and

$$\|f - g_{k_j}^{j_k}\|_{\mathcal{H}_{loc}^{(q,p)}} \leq C_{q,p} \left( \|f - g_{j_\star}\|_{\mathcal{H}_{loc}^{(q,p)}} + \|g_{j_\star} - g_{k_j}^{j_k}\|_{\mathcal{H}_{loc}^{(q,p)}} \right) \leq \varepsilon.$$

Combining the cases 1 and 2, we obtain (3.14). This completes the proof of Proposition 3.5. □

Now, consider $f \in \mathcal{H}_{loc}^{(q,p)}$ and an integer $\delta \geq 0$. For each $j \in \mathbb{Z}$, we set $O^j := \{x \in \mathbb{R}^d : M_{\text{loc}}^0(f)(x) > 2^j\}$. By Remark 3.3, $f$ can be decomposed as follows: $f = g_j + b_j$, with $b_j = \sum_{k \geq 0} b_{j,k}$, $b_{j,k} = (f - c_{j,k})\eta_{j,k}$, if $|Q_{j,k}^*_s| < 1$, and $b_{j,k} = f\eta_{j,k}$, if $|Q_{j,k}^*_s| \geq 1$, where each function $\eta_{j,k}$ is supported on $Q_{j,k}^*$. Also, $c_{j,k} \in \mathcal{P}_\delta := \mathcal{P}_\delta(\mathbb{R}^d)$ (the space of polynomial functions of degree at most $\delta$) satisfies

$$\int_{\mathbb{R}^d} \left( |f(x) - c_{j,k}(x)|\eta_{j,k}(x) \right) p(x) dx = 0, \quad (3.18)$$

for all $p \in \mathcal{P}_\delta$; namely $c_{j,k} = P_j^k(f)$, where $P_j^k$ is the projection operator from $\mathcal{S}'$ onto $\mathcal{P}_\delta$ with respect to the norm

$$\|p\| = \left( \int_{\mathbb{R}^d} |p(x)|^2 \eta_{j,k}(x) dx \right)^{1/2}, \quad p \in \mathcal{P}_\delta, \quad (3.19)$$

with $\eta_{j,k} = \frac{\eta_{j,k}}{\int_{Q_{j,k}^*} \eta_{j,k}(x) dx}$. (3.18) is understood as

$$\langle f, \eta_{j,k} p \rangle = \int_{\mathbb{R}^d} c_{j,k}(x) \eta_{j,k}(x) p(x) dx.$$

We denote by $c_k^j$ the unique polynomial of $\mathcal{P}_\delta$ which satisfies

$$\langle (f - c_{j+1,\ell}) \eta_{j,k}, p \eta_{j+1,\ell} \rangle = \int_{\mathbb{R}^d} c_k^j(x) p(x) \eta_{j+1,\ell}(x) dx, \quad (3.20)$$

for all $p \in \mathcal{P}_\delta$; namely $c_k^j = P_{j+1,\ell}^j((f - c_{j+1,\ell})\eta_{j,k})$, where $P_{j+1,\ell}^j$ is the projection operator from $\mathcal{S}'$ onto $\mathcal{P}_\delta$ with respect to the norm

$$\|p\| = \left( \int_{\mathbb{R}^d} |p(x)|^2 \eta_{j+1,\ell}(x) dx \right)^{1/2}, \quad p \in \mathcal{P}_\delta. \quad (3.21)$$

We will often write (3.20) as follows:

$$\int_{\mathbb{R}^d} \left( f(x) - c_{j+1,\ell}(x) \right) \eta_{j,k}(x) p(x) \eta_{j+1,\ell}(x) dx = \int_{\mathbb{R}^d} c_k^j(x) p(x) \eta_{j+1,\ell}(x) dx.$$
We have
\begin{equation}
(3.22) \quad c_{j,k} = \sum_{n=1}^{m} \left( \int_{\mathbb{R}^d} f(x)e_n(x)\tilde{\eta}_{j,k}(x)\,dx \right)\overline{e_n},
\end{equation}
and
\begin{equation}
(3.23) \quad \sum_{n=1}^{m} \left( \int_{\mathbb{R}^d} (f(x) - c_{j+1,\ell}(x)) \eta_{j,k}(x)\pi_n(x)\tilde{\eta}_{j+1,\ell}(x)\,dx \right)\overline{\pi_n},
\end{equation}
where \{e_1, \ldots, e_m\} and \{\pi_1, \ldots, \pi_m\} are orthonormal bases of \( P_\delta \) respectively with respect to the norms \((3.19)\) and \((3.21)\) with \( m = \dim P_\delta \), and \( \overline{p} \) stands for the conjugate of the polynomial \( p \in P_\delta \). The integrals in \((3.22)\) and \((3.23)\) are understood respectively as \( \langle f, e_n\tilde{\eta}_{j,k} \rangle \) and \( \langle f - c_{j+1,\ell}, \eta_{j,k}, \pi_n\tilde{\eta}_{j+1,\ell} \rangle \). We have
\begin{equation}
(3.24) \quad Q_\ast^{j,k} \cap Q_\ast^{j+1,\ell} = \emptyset \Rightarrow c_\ell^k = 0,
\end{equation}
by \((3.23)\). We have the following lemma whose proof is identical to those of \([1]\), (4.9), (4.10) and (4.11), p. 1918.

**Lemma 3.6.**

1. If \( Q_\ast^{j,k} \cap Q_\ast^{j+1,\ell} \neq \emptyset \), then \( \text{diam}(Q_\ast^{j+1,\ell}) \leq C \text{diam}(Q_\ast^{j,k}) \) and \( Q_\ast^{j+1,\ell} \subset C_0 Q_\ast^{j,k} \) with \( 1 < C < C_0 \) constants independent of \( f \), \( j \), \( k \) and \( \ell \).
2. \( \# \{ k \in \mathbb{Z}_+ : Q_\ast^{j,k} \cap Q_\ast^{j+1,\ell} \neq \emptyset \} \leq 1 \), for each \( j \in \mathbb{Z} \) and \( \ell \in \mathbb{Z}_+ \).
3. Let \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}_+ \). For any real \( \lambda \geq 9d \), \( \lambda Q_\ast^{j,k} \cap (\mathbb{R}^d \setminus \mathcal{O}^\ell) \neq \emptyset \).

Set \( E_{\ast}^j := \{ k \in \mathbb{Z}_+ : |Q_\ast^{j,k}| < 1 \} \) and \( E_{\ast}^j := \{ k \in \mathbb{Z}_+ : |Q_\ast^{j,k}| \geq 1 \} \), for each \( j \in \mathbb{Z} \). Also, \( F_{\ast}^j := \{ k \in \mathbb{Z}_+ : |Q_\ast^{j,k}| < 1/C_0 \} \) and \( F_{\ast}^j := \{ k \in \mathbb{Z}_+ : |Q_\ast^{j,k}| \geq 1/C_0^d \} \), for each \( j \in \mathbb{Z} \), where \( C_0 > 1 \) is the constant \( C_0 \) in Lemma 3.6. We have the following lemmas whose proofs are similar to those of \([1]\), (4.12), (4.13), p. 1918 and \([23]\), Lemma 3.7. We omit details.

**Lemma 3.7.** Let \( j \in \mathbb{Z} \) and \( k, \ell \in \mathbb{Z}_+ \). If \( \ell^{j,k} := \ell(Q_\ast^{j,k}) < 1 \), then
\begin{equation}
(3.25) \quad \sup_{x \in \mathbb{R}^d} |c_{j,k}(x)\eta_{j,k}(x)| \lesssim \sup_{y \in \partial Q_\ast^{j,k} \cap (\mathbb{R}^d \setminus \mathcal{O}^\ell)} \mathcal{M}_{\text{loc}}(f)(y) \lesssim 2^j.
\end{equation}
If \( \ell^{j+1,\ell} := \ell(Q_\ast^{j+1,\ell}) < 1 \), then
\begin{equation}
(3.26) \quad \sup_{x \in \mathbb{R}^d} |c_\ell^k(x)\eta_{j+1,\ell}(x)| \lesssim \sup_{y \in \partial Q_\ast^{j+1,\ell} \cap (\mathbb{R}^d \setminus \mathcal{O}^\ell)} \mathcal{M}_{\text{loc}}(f)(y) \lesssim 2^j.
\end{equation}

**Lemma 3.8.** For every \( j \in \mathbb{Z} \),
\[ \sum_{k \geq 0} \sum_{\ell \in B_{\ell}^{j+1}} c_\ell^k \eta_{j+1,\ell} = 0, \]
where the series converges pointwise and in $\mathcal{S}'$.

Now, we can give our decomposition theorem.

**Theorem 3.9.** Let $\delta \geq \left\lfloor d \left( \frac{4}{q} - 1 \right) \right\rfloor$ be an integer. For every $f \in \mathcal{H}^{(q,p)}_{\text{loc}}$, there exist a sequence $\{(a_n, Q^n)\} \forall n\geq 0$ in $\mathcal{A}_{\text{loc}}(q, \infty, \delta)$ and a sequence $\{\lambda_n\}_{n \geq 0}$ of scalars such that

$$f = \sum_{n \geq 0} \lambda_n a_n \quad \text{in } \mathcal{S}' \text{ and } \mathcal{H}^{(q,p)}_{\text{loc}},$$

and, for all $\eta > 0$,

$$\left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\|\lambda_n\|_q} \right)^\eta \chi_{Q^n} \right\|_{L^{\frac{1}{\eta}}_q} \lesssim \|f\|_{\mathcal{H}^{(q,p)}_{\text{loc}}}.$$  

**Proof.** First, we assume that $f \in L^{(q,p)}_{\text{loc}} \cap \mathcal{H}^{(q,p)}_{\text{loc}}$. For each $j \in \mathbb{Z}$, we consider the level set $\mathcal{O}^j := \{ x \in \mathbb{R}^d : M^0_{\text{loc}}(f)(x) > 2^j \}$. By Remark 3.4 we have $f = g_j + b_j$, with $b_j = \sum_{k \geq 0} b_{j,k}$ and

$$b_{j,k} = \begin{cases} (f - c_{j,k}) \eta_{j,k}, & \text{if } |Q_{*}^{j,k}| < 1 \\ f \eta_{j,k}, & \text{if } |Q_{*}^{j,k}| \geq 1 \end{cases},$$

where each $\eta_{j,k}$ is supported in $Q_{*}^{j,k}$, with analogous estimates described in [1], Lemma 4.1, pp. 1913-1914. Thus, arguing as in the proof of Theorem 2.15 (see [1], Theorem 4.4, p. 1916), we obtain

$$(3.27) \quad f = \sum_{j = -\infty}^{+\infty} (g_{j+1} - g_j) \quad \text{almost everywhere and in } \mathcal{S}'.$$

By Lemma 3.8 and the fact that $\sum_{k \geq 0} \eta_{j,k} b_{j+1,\ell} = x_{\mathcal{O}^j} b_{j+1,\ell} = b_{j+1,\ell}$ for all integer $\ell \geq 0$ (since $\text{supp}(b_{j+1,\ell}) \subset \text{supp}(\eta_{j+1,\ell}) \subset Q_{*}^{j+1,\ell} \subset \mathcal{O}^{j+1}$), we have

$$g_{j+1} - g_j = (f - \sum_{\ell \geq 0} b_{j,\ell}) - (f - \sum_{\ell \geq 0} b_{j,\ell})$$

$$= \sum_{\ell \geq 0} b_{j,\ell} - \sum_{\ell \geq 0} b_{j+1,\ell} + \sum_{k \geq 0} \sum_{\ell \in E^j_{k+1}} c_{k} \eta_{j+1,\ell}$$

$$= \sum_{\ell \geq 0} b_{j,\ell} - \sum_{k \geq 0} \sum_{\ell \geq 0} \eta_{j,k} b_{j+1,\ell} + \sum_{k \geq 0} \sum_{\ell \in E^j_{k+1}} c_{k} \eta_{j+1,\ell}$$

$$= \sum_{k \geq 0} \left[ b_{j,k} - \sum_{\ell \in E^j_{k+1}} \eta_{j,k} b_{j+1,\ell} - \sum_{\ell \in E^j_{k+1}} (\eta_{j,k} b_{j+1,\ell} - c_{k} \eta_{j+1,\ell}) \right]$$

$$= \sum_{k \geq 0} A_{j,k},$$
where all series converge almost everywhere and in $S'$, and
\begin{equation}
(3.28) \quad A_{j,k} := b_{j,k} - \sum_{\ell \in E_2^{j+1}} \eta_{j,k} b_{j+1,\ell} - \sum_{\ell \in E_1^{j+1}} (\eta_{j,k} b_{j+1,\ell} - c_{j,k} \eta_{j+1,\ell}).
\end{equation}

By Lemma 3.6 (1) and (3.24), we see that
\begin{equation}
(3.29) \quad \text{supp}(A_{j,k}) \subset C_0 Q_{*}^{j,k} =: \tilde{Q}_{j,k}.
\end{equation}

Also, we claim that
\begin{equation}
(3.30) \quad |A_{j,k}| \leq C_1 2^j \text{ almost everywhere},
\end{equation}
where $C_1 > 0$ is a constant independent of $f$, $j$ and $k$. To prove (3.30), we distinguish two cases for $k$.

Case 1: $k \in E_1^j$. Then, $b_{j,k} = (f - c_{j,k}) \eta_{j,k}$. Thus, since $\sum_{\ell \geq 0} \eta_{j+1,\ell} = \chi_{\omega^{j+1}}$, by rewriting (3.28), we obtain
\begin{equation}
A_{j,k} = f \chi_{\mathbb{R}^d \setminus C_0^{j+1}, -j} \eta_{j,k} - c_{j,k} \eta_{j,k} + \eta_{j,k} \sum_{\ell \in E_1^{j+1}} c_{j+1,\ell} \eta_{j+1,\ell} + \sum_{\ell \in E_1^{j+1}} c_{j,k} \eta_{j+1,\ell}.
\end{equation}

Furthermore, $f \in L^1_{\text{loc}}$ implies that
\begin{equation}
|f(x)| \leq M_{\text{loc}}(f)(x) \lesssim M_{\text{loc}}^0(f)(x) \lesssim 2^{j+1},
\end{equation}
for almost all $x \in \mathbb{R}^d \setminus C_0^{j+1}$. This, together with Lemma 3.7 and $\sum_{\ell \geq 0} \chi_{Q_{*}^{j+1,\ell}} \lesssim 1$, implies that $|A_{j,k}(x)| \lesssim 2^j$, for almost all $x \in \mathbb{R}^d$.

Case 2: $k \in E_2^j$. Then, $b_{j,k} = f \eta_{j,k}$ and we obtain
\begin{equation}
A_{j,k} = f \chi_{\mathbb{R}^d \setminus C_0^{j+1}, -j} \eta_{j,k} + \eta_{j,k} \sum_{\ell \in E_1^{j+1}} c_{j+1,\ell} \eta_{j+1,\ell} + \sum_{\ell \in E_1^{j+1}} c_{j,k} \eta_{j+1,\ell}.
\end{equation}

Thus, as in Case 1, we have $|A_{j,k}(x)| \lesssim 2^j$, for almost all $x \in \mathbb{R}^d$.

Combining these two cases, we obtain (3.30).

Set
\begin{equation}
(3.31) \quad \lambda_{j,k} := C_1 2^j |\tilde{Q}_{j,k}|^\frac{1}{\tau} \quad \text{and} \quad a_{j,k} := \lambda_{j,k}^{-1} A_{j,k}.
\end{equation}

Then, $\text{supp}(a_{j,k}) \subset \tilde{Q}_{j,k}$ and $\|a_{j,k}\|_\infty \leq |\tilde{Q}_{j,k}|^{-\frac{1}{\tau}}$, by (3.29) and (3.30).

When $k \in F_2^j$, we have
\begin{equation}
|\tilde{Q}_{j,k}| = |C_0 Q_{*}^{j,k}| = C_0^d |Q_{*}^{j,k}| \geq C_0^d / C_0^d = 1.
\end{equation}

Hence $(a_{j,k}, \tilde{Q}_{j,k}) \in A_{\text{loc}}(q, \infty, \delta)$. When $k \in F_1^j$, we have
\begin{equation}
|\tilde{Q}_{j,k}| = |C_0 Q_{*}^{j,k}| = C_0^d |Q_{*}^{j,k}| < C_0^d / C_0^d = 1
\end{equation}
and
\begin{equation}
(3.32) \quad \int_{\mathbb{R}^d} A_{j,k}(x)p(x)dx = 0, \ \forall \ p \in P_\delta.
\end{equation}
For the proof of (3.32), notice that \( F_j^1 \subset E_j^j \), since \( 1/C_0^d < 1 \) (see Lemma 3.6 (1)). Hence \( \beta_{j,k} = (f - c_{j,k}) \eta_{j,k} \) and

\[
A_{j,k} = (f - c_{j,k}) \eta_{j,k} - \sum_{\ell \in E_{j+1}^j} f \eta_{j,k} \eta_{j+1,\ell} - \sum_{\ell \in E_{j+1}^j} \left( (f - c_{j+1,\ell}) \eta_{j+1,\ell} \eta_{j,k} - c_k^\ell \eta_{j+1,\ell} \right),
\]

by (3.28). Moreover, for every \( \ell \in E_{j+1}^j \), we have

\[
C^d |Q_{j,k}^*| < |Q_{j+1,\ell}^*|,
\]

since \(|Q_{j,k}^*| < 1/C_0^d < 1/C^d \) (see Lemma 3.6 (1)) and \( 1 \leq |Q_{j+1,\ell}^*| \). Therefore, for all \( \ell \in E_{j+1}^j \),

\[
diam(Q_{j+1,\ell}^*) > C \diam(Q_{j,k}^*),
\]

which implies that

\[
Q_{j,k}^* \cap Q_{j+1,\ell}^* = \emptyset,
\]

for all \( \ell \in E_{j+1}^j \), by Lemma 3.6 (1). Hence

\[
\sum_{\ell \in E_{j+1}^j} f \eta_{j,k} \eta_{j+1,\ell} = 0.
\]

Thus,

\[
A_{j,k} = (f - c_{j,k}) \eta_{j,k} - \sum_{\ell \in E_{j+1}^j} \left( (f - c_{j+1,\ell}) \eta_{j+1,\ell} \eta_{j,k} - c_k^\ell \eta_{j+1,\ell} \right).
\]

Then, (3.32) follows from (3.18) and (3.20). Thus, \((a_{j,k}, \tilde{Q}_{j,k}) \in A_{loc}(q, \infty, \delta)\). Therefore, by (3.27),

\[
f = \sum_{j=-\infty}^{+\infty} \sum_{k \geq 0} \lambda_{j,k} a_{j,k}
\]

almost everywhere and in \( S' \), where each \((a_{j,k}, \tilde{Q}_{j,k}) \in A_{loc}(q, \infty, \delta)\).

Let us fix a real \( \eta > 0 \). Arguing as in the proof of Theorem 2.15, we obtain

\[
\left\| \sum_{j=-\infty}^{+\infty} \sum_{k \geq 0} \left( \frac{\lambda_{j,k}}{\| \chi_{\tilde{Q}_{j,k}} \|_q} \right)^\frac{1}{q} \chi_{\tilde{Q}_{j,k}} \right\|_{H_{loc}^{(q,p)}} \lesssim \| f \|_{H_{loc}^{(q,p)}}
\]

and the convergence of (3.33) also holds in \( H_{loc}^{(q,p)} \). Indeed, for each \((a_{j,k}, \tilde{Q}_{j,k}) \in A_{loc}(q, \infty, \delta)\), we have

\[
\mathcal{M}_{loc} (a_{j,k})(x) \lesssim \left[ \mathcal{M} \left( \chi_{\tilde{Q}_{j,k}} \right)(x) \right]^u \| \chi_{\tilde{Q}_{j,k}} \|_q^{-1},
\]

for all \( x \in \mathbb{R}^d \), with \( u = \frac{d+\delta+1}{d} \), by (3.3) (see the proof of Theorem 3.2). With (3.34) and (3.35), the convergence in \( H_{loc}^{(q,p)} \) easily follows.
The general case \( f \in \mathcal{H}_{\text{loc}}^{(q,p)} \) follows from the density of \( L_{\text{loc}}^1 \cap \mathcal{H}_{\text{loc}}^{(q,p)} \) in \( \mathcal{H}_{\text{loc}}^{(q,p)} \) with respect to the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \) as in the classical case with appropriate modifications. This completes the proof of Theorem 3.9. \( \square \)

**Remark 3.10.** By the definition of \( a_{j,k} \), it is clear that \( \text{supp}(a_{j,k}) \subset \Omega^j \). Moreover, for every \( j \in \mathbb{Z} \), the family \( \{ \text{supp}(a_{j,k}) \}_{k \geq 0} \) has the bounded intersection property, namely,

\[
\sum_{k \geq 0} \chi_{\text{supp}(a_{j,k})} \lesssim 1.
\]

Also, in Theorem 3.9, we can replace \( \mathcal{A}_{\text{loc}}(q, \infty, \delta) \) by \( \mathcal{A}_{\text{loc}}(q, r, \delta) \), for any \( 1 < r < +\infty \), since \( \mathcal{A}_{\text{loc}}(q, \infty, \delta) \subset \mathcal{A}_{\text{loc}}(q, r, \delta) \).

We give another proof of Theorem 3.9. This approach essentially uses Lemma 2.13, Theorem 2.15 and Plancherel-Polya-Nikols’kij’s inequality that we recall.

**Lemma 3.11.** (Plancherel-Polya-Nikols’kij’s inequality, [33], Theorem, p. 16). Let \( \Omega \) be a compact subset of \( \mathbb{R}^d \) and \( 0 < s < +\infty \). Then, there exist two constants \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
\sup_{z \in \mathbb{R}^d} \frac{|\nabla \phi(x - z)|}{1 + |z|^2} \leq c_1 \sup_{z \in \mathbb{R}^d} \frac{|\phi(x - z)|}{1 + |z|^2} \leq c_2 [\mathcal{M}(|\phi|^s)(x)]^\frac{1}{s},
\]

for all \( \phi \in \mathcal{S}' \) and all \( x \in \mathbb{R}^d \), where \( \mathcal{S}' = \{ \phi \in \mathcal{S} : \text{supp}(\mathcal{F}(\phi)) \subset \Omega \} \).

One can also see [34], 1.5.3, (iii) (4)-(6), p. 31, for this inequality. Now, we can give our second approach of the proof of theorem 3.9.

**Proof.** Let \( f \in \mathcal{H}_{\text{loc}}^{(q,p)} \). Then, by Lemma 2.13, there exist \( u \in \mathcal{H}^{(q,p)} \) and \( v \in \mathcal{H}_{\text{loc}}^{(q,p)} \cap C^\infty(\mathbb{R}^d) \) such that \( f = u + v \) and

\[
\|u\|_{\mathcal{H}^{(q,p)}} + \|v\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \lesssim \|f\|_{\mathcal{H}_{\text{loc}}^{(q,p)}}.
\]

Moreover, by Theorem 2.15, there exist a sequence \( \{(a_n, Q^n)\}_{n \geq 0} \) in \( \mathcal{A}(q, \infty, \delta) \) and a sequence of scalars \( \{\lambda_n\}_{n \geq 0} \) such that \( u = \sum_{n \geq 0} \lambda_n a_n \in \mathcal{S}' \) and \( \mathcal{H}^{(q,p)} \), and for any real \( \eta > 0 \),

\[
\left( \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\|\chi_{Q^n}\|_q} \right) \frac{\eta}{\eta} \right)^{\frac{1}{\eta}} \lesssim \|u\|_{\mathcal{H}^{(q,p)}} \lesssim \|f\|_{\mathcal{H}_{\text{loc}}^{(q,p)}},
\]

by (3.37). Also \( u = \sum_{n \geq 0} \lambda_n a_n \) in \( \mathcal{H}_{\text{loc}}^{(q,p)} \), since \( \mathcal{H}^{(q,p)} \hookrightarrow \mathcal{H}_{\text{loc}}^{(q,p)} \).

To end, it suffices to show that there exist a sequence \( \{(b_n, R^n)\}_{n \geq 0} \) in \( \mathcal{A}_{\text{loc}}(q, \infty, \delta) \) and a sequence of scalars \( \{\sigma_n\}_{n \geq 0} \) such that \( v = \sum_{n \geq 0} \sigma_n b_n \).
To prove this, we borrow some ideas from [25], Lemma 7.9. Let us fix
\[ R_{\text{loc}} \] and for any real \( \eta > 0, \)
\begin{equation}
(3.39) \quad \left\| \sum_{n \geq 0} \left( \frac{\|\sigma_n\|}{\|R_n\|} \right)^\eta \chi_{R_n} \right\|_{\frac{\tilde{p}}{\eta}} \lesssim \|f\|_{H^{(q,p)}_{\text{loc}}}. \end{equation}

This establishes (3.40). Also, from the definition of \( v \) and \( \theta \), it follows
that \( \text{supp}(F(v)) \subseteq \text{supp}(\psi) \subseteq B(0,2) \). Moreover, for all \( x \in R_m \),
\[ \sup_{y \in R_m} \left| v(y) \right| = \sup_{y \in R_m} \left| v(x - (x - y)) \right| = \sup_{z \in Q(0,2)} \left| v(x - z) \right|, \]
since \( x, y \in \mathbb{R}^n \) imply that \( z = x - y \in Q(0, 2) \). Therefore, for any \( 0 < s < q \), we have

\[
\sum_{m \in \mathbb{Z}^d} \left( \frac{|\sigma_m|}{\|\chi_{R_m}\|_q} \right) \eta \chi_{R_m}(x) = \sum_{m \in \mathbb{Z}^d} \left( \sup_{y \in \mathbb{R}^n} |v(y)| \right) \eta \chi_{R_m}(x)
\]

\[
= \sum_{m \in \mathbb{Z}^d} \left( \sup_{z \in Q(0,2)} |v(x - z)| \right) \eta \chi_{R_m}(x)
\]

\[
\subset \sum_{m \in \mathbb{Z}^d} \left( \sup_{z \in Q(0,2)} \frac{|v(x - z)|}{1 + |z|^{\frac{d}{s}}} \right) \eta \chi_{R_m}(x)
\]

\[
\subset \sum_{m \in \mathbb{Z}^d} \left( \sup_{z \in \mathbb{R}^d} \frac{|v(x - z)|}{1 + |z|^{\frac{d}{s}}} \right) \eta \chi_{R_m}(x)
\]

\[
\subset (\mathcal{M}(|v|^s)(x))^{\frac{q}{p}},
\]

for all \( x \in \mathbb{R}^d \), by Lemma 3.11 and (3.41). Thus,

\[
\left\| \sum_{m \in \mathbb{Z}^d} \left( \frac{|\sigma_m|}{\|\chi_{R_m}\|_q} \right) \eta \chi_{R_m} \right\|_{\frac{q}{s}} \lesssim \left\| (\mathcal{M}(|v|^s))(x) \right\|_{\frac{q}{s}}
\]

\[
\lesssim \|v\|_{q,p} \lesssim \|f\|_{\mathcal{H}_{loc}^{(q,p)}},
\]

by Proposition 2.3 and the local version of Remark 2.11 (2.0). We rearrange the \( b^m \)'s, \( R_m \)'s and \( \sigma_m \)'s to obtain \( \{(b_n, R_n)\}_{n \geq 0} \subset A_{loc}(q, \infty, \delta) \) and a sequence of scalars \( \{\sigma_n\}_{n \geq 0} \) such that \( v = \sum_{n \geq 0} \sigma_n b_n \) in \( \mathcal{H}_{loc}^{(q,p)} \) and (3.39) holds. Thus,

\[
f = \sum_{n \geq 0} \lambda_n a_n + \sum_{n \geq 0} \sigma_n b_n \text{ in } S' \text{ and } \mathcal{H}_{loc}^{(q,p)},
\]

with

\[
\left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\|\chi_{Q^n}\|_q} \right) \eta \chi_{Q^n} \right\|_{\frac{q}{s}} \lesssim \|f\|_{\mathcal{H}_{loc}^{(q,p)}},
\]

by (3.38) and (3.39). This finishes the second proof of Theorem 3.9.

**Remark 3.12.** Let \( 0 < \eta \leq 1, 1 < r \leq +\infty \) and \( \delta \geq \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor \) be an integer. For simplicity, we denote by \( \mathcal{H}_{loc,f}^{(q,p)} \) the subspace of \( \mathcal{H}_{loc}^{(q,p)} \) consisting of finite linear combinations of \((q, r, \delta)\)-atoms (for \( \mathcal{H}_{loc}^{(q,p)} \)). By Theorems 3.2, 3.3 and 3.9, we have
(1) If \( r = +\infty \), then for all \( f \in \mathcal{H}_{\text{loc}}^{(q,p)} \),
\[
\|f\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \approx \inf \left\{ \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\|X_n\|_q} \right)^{\eta} \chi_{Q_n} \right\|_{\frac{1}{\eta}} : f = \sum_{n \geq 0} \lambda_n a_n \right\},
\]
where the infimum is taken over all decompositions of \( f \) using \((q,\infty,\delta)\)-atom \( a_n \) supported on the cube \( Q^n \).

(2) If \( \max \{ p, 1 \} < r < +\infty \) and \( 0 < \eta < q \), then for all \( f \in \mathcal{H}_{\text{loc}}^{(q,p)} \),
\[
\|f\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \approx \inf \left\{ \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\|X_n\|_q} \right)^{\eta} \chi_{Q_n} \right\|_{\frac{1}{\eta}} : f = \sum_{n \geq 0} \lambda_n a_n \right\},
\]
where the infimum is taken over all decompositions of \( f \) using \((q,r,\delta)\)-atom \( a_n \) supported on the cube \( Q^n \).

(3) \( \mathcal{H}_{\text{loc},\text{fin}}^{(q,p)} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \) in the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \).

(4) \( \mathcal{H}_{\text{loc}}^{q} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \) in the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \).

(5) \( \mathcal{H}_{\text{loc}}^{(q,p)} \cap L^r \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \) in the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \).

Corollary 3.13. The Schwartz space \( \mathcal{S} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \) with respect to the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \), for \( 0 < q < +\infty \) and \( q \leq p < +\infty \).

Proof. We distinguish two cases for \( q \).

Case 1: \( 0 < q \leq 1 \). It’s well known that \( \mathcal{S} \) is dense in \( \mathcal{H}_{\text{loc}}^{q} \) with respect to the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{q}} \), by [14], p. 35. It follows that \( \mathcal{S} \) is dense in \( \mathcal{H}_{\text{loc}}^{q} \) with respect to the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \), since \( \mathcal{H}_{\text{loc}}^{q} \hookrightarrow \mathcal{H}_{\text{loc}}^{(q,p)} \). Furthermore, \( \mathcal{H}_{\text{loc}}^{q} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \) with respect to the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \), by Remark 3.12. Therefore, \( \mathcal{S} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \) in the quasi-norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \).

Case 2: \( 1 < q < +\infty \). Then, \( 1 < q \leq p < +\infty \). Hence \( \mathcal{H}_{\text{loc}}^{(q,p)} = (L^q, \ell^p) \) with norms equivalence, by Theorem 2.7. Denote by \( \mathcal{C}_{\text{comp}}(\mathbb{R}^d) \) the space of continuous complex-valued functions on \( \mathbb{R}^d \) with compact support. It’s clear that \( \mathcal{C}_{\text{comp}}(\mathbb{R}^d) \subset L^q \subset (L^q, \ell^p) \). Moreover, \( \mathcal{C}_{\text{comp}}(\mathbb{R}^d) \) is dense in \( (L^q, \ell^p) \) with respect to the norm \( \| \cdot \|_{q,p} \), by [1], Section 7, p. 77. Hence \( L^q \) is dense in \( (L^q, \ell^p) \) in the norm \( \| \cdot \|_{q,p} \). Also, \( \mathcal{S} \) is dense in \( L^q \) in the norm \( \| \cdot \|_{q,p} \), since \( \mathcal{S} \) is dense in \( L^q \) in the norm \( \| \cdot \|_q \) and \( L^q \hookrightarrow (L^q, \ell^p) \). Therefore, \( \mathcal{S} \) is dense in \( (L^q, \ell^p) \) in the norm \( \| \cdot \|_{q,p} \).

It finally follows that \( \mathcal{S} \) is dense in \( \mathcal{H}_{\text{loc}}^{(q,p)} \) with respect to the norm \( \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \), since \( \mathcal{H}_{\text{loc}}^{(q,p)} = (L^q, \ell^p) \) and \( \| \cdot \|_{q,p} \approx \| \cdot \|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \). This finishes the proof in Case 2 and hence, the proof of Corollary 3.13.

We end this subsection by giving the local version of [1], Theorem 4.9, p. 1924. For this, we define a quasi-norm on \( \mathcal{H}_{\text{loc},\text{fin}}^{(q,p)} \) the subspace...
Theorem 3.16. Let \( r < H \) of \( \mathcal{H}(q,p) \) consisting of finite linear combinations of \((q,r,\delta)\)-atoms, with 
\( 1 < r \leq +\infty \) and \( \delta \geq \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor \) an integer fixed.

(1) When \( r = +\infty \), we fix \( 0 < \eta \leq 1 \). For every \( f \in \mathcal{H}(q,p) \), we set
\[
\|f\|_{\mathcal{H}(q,p)} := \inf \left\{ \left\| \sum_{n=0}^{\infty} \left( \frac{1}{\|x_n\|_q} \right)^\eta x_n \right\|^{\frac{1}{\eta}} : f = \sum_{n=0}^{\infty} \lambda_n a_n, \ m \in \mathbb{Z}_+ \right\},
\]
where the infimum is taken over all finite decompositions of \( f \) using \((q,\infty,\delta)\)-atom \( a_n \) supported on the cube \( Q^n \).

(2) When \( \max \{p,1\} < r < +\infty \), we fix \( 0 < \eta < q \). we set
\[
\|f\|_{\mathcal{H}(q,p)} := \inf \left\{ \left\| \sum_{n=0}^{\infty} \left( \frac{1}{\|x_n\|_q} \right)^\eta x_n \right\|^{\frac{1}{\eta}} : f = \sum_{n=0}^{\infty} \lambda_n a_n, \ m \in \mathbb{Z}_+ \right\},
\]
where the infimum is taken over all finite decompositions of \( f \) using \((q,r,\delta)\)-atom \( a_n \) supported on the cube \( Q^n \).

It’s straightforward to see that, in the two cases, \( \|\cdot\|_{\mathcal{H}(q,p)} \) defines a quasi-norm on \( \mathcal{H}(q,p) \).

Remark 3.14. With the above assumptions, for any \( (a,Q) \in \mathcal{A}(q,r,\delta) \),
\[
\|a\|_{\mathcal{H}(q,p)} \leq \left( \frac{1}{\|x_Q\|_q} \right)^\eta \|x_Q\|_{q,p}^{\frac{1}{\eta}} = \frac{1}{\|x_Q\|_q} \|x_Q\|_{q,p},
\]
since \( a = 1a \) is a finite decomposition of \((q,r,\delta)\)-atoms.

Also, we will need the following lemma whose proof is similar to the one of \[\[\], Lemma 4.8, p. 1923. We omit details.

Lemma 3.15. Let \( f \in \mathcal{H}(q,p) \) such that \( \|f\|_{\mathcal{H}(q,p)} = 1 \) and \( \text{supp}(f) \subset B(0,R) \) with \( R > 1 \). Then, for all \( x \notin B(0,4R) \),
\[
\mathcal{M}(f)(x) \leq C(\varphi, d, q, p)R^{-\frac{d}{r}},
\]
where \( C(\varphi, d, q, p) > 0 \) is a constant independent of \( f \) and \( x \).

Now, we can give our result which is the following.

Theorem 3.16. Let \( \delta \geq \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor \) be an integer and \( \max \{p,1\} < r \leq +\infty \).

(1) If \( r < +\infty \), we fix \( 0 < \eta < q \). Then \( \|\cdot\|_{\mathcal{H}(q,p)} \) and \( \|\cdot\|_{\mathcal{H}(q,p)} \) are equivalent on \( \mathcal{H}(q,p) \).

(2) If \( r = +\infty \), we fix \( 0 < \eta \leq 1 \). Then \( \|\cdot\|_{\mathcal{H}(q,p)} \) and \( \|\cdot\|_{\mathcal{H}(q,p)} \) are equivalent on \( \mathcal{H}(q,p) \cap C(\mathbb{R}^d) \), where \( C(\mathbb{R}^d) \) denotes the space of continuous complex values functions on \( \mathbb{R}^d \).
Proof. We first prove the part (1). We consider an integer \( \delta \geq \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor \), \( \max \{ 1, p \} < r < +\infty \) and \( 0 < \eta < q \). Let \( f \in \mathcal{H}_{loc}^{(q,p)} \). From Theorem 3.3 we deduce that

\[
\| f \|_{\mathcal{H}_{loc}^{(q,p)}} \lesssim \| f \|_{\mathcal{H}_{loc,f}^{(q,p)}}.
\]

It remains to show that

\[
\| f \|_{\mathcal{H}_{loc}^{(q,p)}} \lesssim \| f \|_{\mathcal{H}_{loc,f}^{(q,p)}}.
\]

By homogeneity of the quasi-norm \( \| \cdot \|_{\mathcal{H}_{loc}^{(q,p)}} \), we may assume that \( \| f \|_{\mathcal{H}_{loc}^{(q,p)}} = 1 \). Since \( f \) is a finite linear combination of \( (q,r,\delta) \)-atoms, there exists a real \( R > 1 \) such that \( \text{supp}(f) \subset B(0, R) \). Hence

\[
\mathcal{M}_{loc}^{0}(f)(x) \leq C_{\varphi,d,q,p}R^{-\frac{d}{r}},
\]

for all \( x \notin B(0, 4R) \), by Lemma 3.15. For each \( j \in \mathbb{Z} \), we set

\[
\mathcal{O}^{j} := \{ x \in \mathbb{R}^d : \mathcal{M}_{loc}^{0}(f)(x) > 2^j \}.
\]

Denote by \( j' \) the largest integer \( j \) such that \( 2^j < C_{\varphi,d,q,p}R^{-\frac{d}{r}} \). Then,

\[
\mathcal{O}^{j} \subset B(0, 4R),
\]

for all \( j > j' \), by (3.43). Furthermore, since \( f \in L^r \), by the proof (first approach) of Theorem 3.9 there exist a sequence \( \{(a_{j,k}, Q_{j,k})\}_{(j,k)\in \mathbb{Z} \times \mathbb{Z}^d} \) in \( \mathcal{A}_{loc}(q,r,\delta) \) and a sequence of scalars \( \{\lambda_{j,k}\}_{(j,k)\in \mathbb{Z} \times \mathbb{Z}^d} \) such that \( f = \sum_{j=-\infty}^{+\infty} \sum_{k \geq 0} \lambda_{j,k} a_{j,k} \) almost everywhere and in \( S' \), with

\[
\text{supp}(a_{j,k}) \subset \mathcal{O}^{j}, \quad |\lambda_{j,k} a_{j,k}| \lesssim 2^j \text{ a.e and } Q_{j,k} = C_d Q_{j,k}^* ,
\]

for each \( j \in \mathbb{Z} \), where the family of cubes \( \{Q_{j,k}^*\}_{k \geq 0} \) satisfies

\[
\bigcup_{k \geq 0} Q_{j,k}^* = \mathcal{O}^{j} \text{ and } \sum_{k \geq 0} \chi_{Q_{j,k}^*} \lesssim 1.
\]

Also,

\[
\left\| \sum_{j=-\infty}^{+\infty} \sum_{k \geq 0} \left( \frac{|\lambda_{j,k}|}{\| \chi_{Q_{j,k}} \|_q} \right) \chi_{Q_{j,k}} \right\|_{\frac{1}{\eta}}^{\frac{1}{\eta}} \lesssim \| f \|_{\mathcal{H}_{loc}^{(q,p)}}.
\]

Set \( h := \sum_{j \leq j'} \sum_{k \geq 0} \lambda_{j,k} a_{j,k} \) and \( \ell := \sum_{j > j'} \sum_{k \geq 0} \lambda_{j,k} a_{j,k} \), where the series converge almost everywhere and in \( S' \). We have \( f = h + \ell \) and \( \text{supp}(\ell) \subset \bigcup_{j > j'} \mathcal{O}^{j} \subset B(0, 4R) \), by (3.45) and (3.44). It follows that \( \text{supp}(h) \subset B(0, 4R) \), since \( f = \ell = 0 \) sur \( \mathbb{R}^d \setminus B(0, 4R) \). Also, with (3.45) and (3.46), arguing as in the proof of [1], Theorem 4.9, p. 1924, we have

\[
\| f \|_r \lesssim \| \mathcal{M}_{loc}^{0}(f) \|_r , \approx \| f \|_r < +\infty.
\]
since $f \in L^r$, $r > 1$ and $N \geq \left\lfloor \frac{d}{2} \right\rfloor + 1 > \left\lfloor \frac{d}{r} \right\rfloor + 1$. Hence $\ell \in L^r$ and the series $\sum_{j>j'} \sum_{k \geq 0} \lambda_{j,k} a_{j,k}$ converges to $\ell$ in $L^r$, by the Lebesgue dominated convergence theorem in $L^r$. Thus, $h = f - \ell \in L^r$. Moreover, since $\sum_{k \geq 0} \chi_{\text{supp}(a_{j,k})} \lesssim 1$ (see Remark 3.10 (3.36)), arguing as in the proof of [11], Theorem 4.9, p. 1924, we obtain $|h(x)| \leq C_{\varphi,d,q,p,\delta} R^{-\frac{d}{p}}$, for almost all $x \in \mathbb{R}^d$, by the choice of $j'$. Setting

$$C_2 := \left( C_{\varphi,d,q,p,\delta} R^{-\frac{d}{p}} \right)^{-1} \left\| \chi_{Q(0,R)} \right\|_q^{-1},$$

we have $|C_2 h(x)| \leq \left\| \chi_{Q(0,R)} \right\|_q^{-1}$, for almost all $x \in \mathbb{R}^d$. Therefore, $(C_2 h, Q(0,8R)) \in A_{\text{loc}}(q, \infty, \delta) \subset A_{\text{loc}}(q, r, \delta)$, since $\text{supp}(h) \subset B(0,4R) \subset Q(0,8R)$.

Now, we rewrite $\ell$ as a finite linear combination of $(q, r, \delta)$-atoms. For every positive integer $i$, we set

$$F_i := \{(j, k) \in \mathbb{Z} \times \mathbb{Z}_+ : j > j', |j| + k \leq i\}$$

and define the finite sum $\ell_i$ by $\ell_i = \sum_{(j,k) \in F_i} \lambda_{j,k} a_{j,k}$. The convergence of the series $\sum_{j>j'} \sum_{k \geq 0} \lambda_{j,k} a_{j,k}$ to $\ell$ in $L^r$ implies that there exists an integer $i_0 > 0$ such that $\left\| \ell - \ell_{i_0} \right\|_r \leq \left| Q(0,8R) \right|^{\frac{1}{r} - \frac{1}{q}}$. Hence $(\ell - \ell_{i_0}, Q(0,8R)) \in A_{\text{loc}}(q, r, \delta)$, since $\text{supp}(\ell - \ell_{i_0}) \subset B(0,4R) \subset Q(0,8R)$. Thus, $\ell = (\ell - \ell_{i_0}) + \ell_{i_0}$ is a finite linear combination of $(q, r, \delta)$-atoms, and

$$f = h + \ell = C_2^{-1}(C_2 h) + (\ell - \ell_{i_0}) + \ell_{i_0}$$

is a finite decomposition of $f$ as $(q, r, \delta)$-atoms. Therefore, arguing as in the proof of [11], Theorem 4.9, p. 1924, we obtain

$$\left\| f \right\|_{H_{\text{loc},\text{fin}}}^{(q,p)} \lesssim I_1 + I_2 + I_3 \lesssim 1,$$

where

$$I_1 = \left\| \left( \frac{C_2^{-1}}{\left\| \chi_{Q(0,R)} \right\|_q} \right)^{\eta} \chi_{Q(0,R)} \right\|_{q, \frac{\eta}{\bar{p}, \eta}}, \ I_2 = \left\| \left( \frac{1}{\left\| \chi_{Q(0,R)} \right\|_q} \right)^{\eta} \chi_{Q(0,R)} \right\|_{q, \frac{\eta}{\bar{p}, \eta}},$$

and

$$I_3 = \left\| \sum_{(j,k) \in F_{i_0}} \left( \frac{|\lambda_{j,k}|}{\left\| \chi_{Q(0,R)} \right\|_q} \right)^{\eta} \chi_{Q_{j,k}} \right\|_{\frac{\eta}{\bar{p}, \eta}}.$$

This implies (3.42) which completes the proof of (11).

For the part (2), we fix $0 < \eta \leq 1$. Let $f \in H_{\text{loc},\text{fin}}^{(q,p)} \cap C(\mathbb{R}^d)$. As in (11), we have $\left\| f \right\|_{H_{\text{loc},\text{fin}}}^{(q,p)} \lesssim \left\| f \right\|_{H_{\text{loc},\text{fin}}}^{(q,p)}$, by Theorem 3.2. For the
reverse inequality, we also may assume that \(\|f\|_{H^1_{\text{loc}}} = 1\), by homogeneity. We have \(f \in L^1_{\text{loc}}\). Thus, as in the proof of (1), there exist a sequence \(\{(a_{j,k}, Q_{j,k})\}_{(j,k) \in \mathbb{Z} \times \mathbb{Z}_+}\) in \(A_{\text{loc}}(q, \infty, \delta)\) and a sequence of scalars \(\{\lambda_{j,k}\}_{(j,k) \in \mathbb{Z} \times \mathbb{Z}_+}\) such that \(f = \sum_{j=-\infty}^{+\infty} \sum_{k \geq 0} \lambda_{j,k} a_{j,k}\) almost everywhere and in \(S'\), and (3.45), (3.46) hold. Notice that each \((q, \infty, \delta)\)-atom \(a_{j,k}\) is continuous by examining its definition (see (3.28) and (3.31)), since \(f\) is continuous. Also, since \(\|\Phi\|_1 \leq \mathcal{H}_N(\Phi) \leq 1\) for any \(\Phi \in \mathcal{F}_N\), it’s easy to see that

\[
\mathcal{M}_{\text{loc}}^0(f)(x) \leq \|f\|_{\infty}.
\]

for all \(x \in \mathbb{R}^d\). Then, it follows from (3.45) that \(\mathcal{O}^j = \emptyset\), for all \(j \in \mathbb{Z}\) such that \(\|f\|_{\infty} \leq 2^j\). Moreover, as in the part (1), there exists a real \(R > 1\) such that \(\text{supp}(f) \subset B(0, R)\) so that (3.43) holds. Let \(h\) and \(\ell\) be the functions as defined in the part (1). Arguing as in (1), we obtain that \(\text{supp}(\ell) \subset B(0, 4R)\) and \((C_0, q, 0, 8R) \in A_{\text{loc}}(q, \infty, \delta)\) with

\[
C_2 := \left( C_{\varepsilon, d, q, R} R^{-\frac{d}{p}} \right)^{-1} \left\| \chi_{Q(0, 8R)} \right\|_{q}^{-1}.
\]

Denote by \(j''\) the largest integer \(j\) such that \(2^j < \|f\|_{\infty}\). Then

\[
\ell := \sum_{j > j''} \sum_{k \geq 0} \lambda_{j,k} a_{j,k} = \sum_{j' < j''} \sum_{k \geq 0} \lambda_{j,k} a_{j,k}.
\]

Let \(\epsilon > 0\). Since \(f\) is uniformly continuous, there exists a real \(\gamma > 0\) such that, for all \(x, y \in \mathbb{R}^d\), \(|x - y| < \gamma\) implies that \(|f(x) - f(y)| < \epsilon\). Without loss of generality, we may assume that \(\gamma < 1\). Write \(\ell = \ell_1 + \ell_2\), with

\[
\ell_1 := \sum_{(j,k) \in G_1} \lambda_{j,k} a_{j,k} \quad \text{and} \quad \ell_2 := \sum_{(j,k) \in G_2} \lambda_{j,k} a_{j,k},
\]

where

\[
G_1 := \{(j, k) \in \mathbb{Z} \times \mathbb{Z}_+ : \text{diam}(Q_{j,k}) \geq \gamma, \ j' < j \leq j''\}
\]

and

\[
G_2 := \{(j, k) \in \mathbb{Z} \times \mathbb{Z}_+ : \text{diam}(Q_{j,k}) < \gamma, \ j' < j \leq j''\}.
\]

Using (3.45), the construction of cubes \(Q^*_{j,k}\) \(Q^*_{j,k} = cQ_{j,k}\), for some \(1 < c < 5/4\), where the family of cubes \(\{Q_{j,k}\}_{k \geq 0}\) corresponds to the Whitney decomposition of \(\mathcal{O}^j\) and (3.44), we see that \(\text{card}(G_1) < +\infty\). Therefore, \(\ell_1\) is continuous.

For any \((j, k) \in G_2\) and \(x \in Q_{j,k}\), we clearly have \(|f(x) - f(x_{j,k})| < \epsilon\), where \(x_{j,k}\) stands for the center of \(Q_{j,k}\). Set

\[
\hat{f}(x) := [f(x) - f(x_{j,k})] \chi_{Q_{j,k}}(x) \quad \text{and} \quad \hat{c}_{j,k}(x) := c_{j,k}(x) - f(x_{j,k}),
\]

for all \(x \in \mathbb{R}^d\), where \(c_{j,k}\) is the polynomial defined by (3.18). We have

\[
|\hat{f}(x) - \hat{c}_{j,k}(x)| \eta_{j,k}(x) = |f(x) - c_{j,k}(x)| \eta_{j,k}(x), \quad \text{for all} \ x \in \mathbb{R}^d,
\]

where \(\eta_{j,k}\)
is the function in (3.18), since supp(ηj,k) ⊂ Q∗ j,k ⊂ Q j,k. Hence
\[ \int_{\mathbb{R}^d} [\hat{f}(x) - \tilde{c}_{j,k}(x)]\eta_{j,k}(x)\mathbf{p}(x)dx = 0, \]
for all \( \mathbf{p} \in \mathcal{P}_\delta \), by (3.18). Since \( |\hat{f}(x)| < \epsilon \) for all \( x \in \mathbb{R}^d \) implies that \( \mathcal{M}^0_{\text{loc}}(\hat{f})(x) \leq \epsilon \) for all \( x \in \mathbb{R}^d \), and \( \ell^{j,k} := \ell(Q_{j,k}) < \frac{2}{C_0\sqrt{d}} < 1 \), then by Lemma 3.7 (3.25), we have
\[ \sup_{x \in \mathbb{R}^d} |\tilde{c}_{j,k}(x)| \eta_{j,k}(x) | \lesssim \sup_{y \in \mathbb{R}^d} \mathcal{M}^0_{\text{loc}}(\hat{f})(y) \lesssim \epsilon. \]

Let \( i \geq 0 \) be an integer. Denote by \( \tilde{c}^i_k \) the polynomial of \( \mathcal{P}_\delta \) such that
\[ \int_{\mathbb{R}^d} [\hat{f}(x) - \tilde{c}^{i+1}(x)]\eta_{j,k}(x)\mathbf{p}(x)\eta_{j+1,i}(x)dx = \int_{\mathbb{R}^d} \tilde{c}^i_k(x)\mathbf{p}(x)\eta_{j+1,i}(x)dx, \]
for all \( \mathbf{p} \in \mathcal{P}_\delta \), where \( \tilde{c}^{j+1,i}(x) = c^{j+1,i}(x) - f(x_{j,k}) \). The above expression means that \( \tilde{c}^i_k \) is the orthogonal projection of \( \hat{f} - \tilde{c}^{j+1,i} \eta_{j,k} \) on \( \mathcal{P}_\delta \) with respect to the norm \( \mathcal{L}_{\text{loc}}(\hat{f})(x) \leq \epsilon \). Since supp(ηj,k) ⊂ Q∗ j,k ⊂ Q j,k, we have \( \hat{f}(x) - \tilde{c}^{j+1,i}(x)\eta_{j,k}(x) = [f(x) - c^{j+1,i}(x)]\eta_{j,k}(x) \), for all \( x \in \mathbb{R}^d \). Hence \( \tilde{c}^i_k = c^i_k \), by (3.20). Then, we infer from Lemma 3.7 (3.26) that
\[ \sup_{y \in \mathbb{R}^d} |\tilde{c}^i_k(y)| \eta_{j+1,i}(y) | \lesssim \sup_{y \in \mathbb{R}^d} \mathcal{M}^0_{\text{loc}}(\hat{f})(y) \lesssim \epsilon. \]
Endeed, since \( \tilde{c}^i_k = c^i_k \), we have \( \tilde{c}^i_k = 0 \), if \( Q_{j,k}^* \cap Q_{j+1,i}^* = \emptyset \), by (3.24). In this case, (3.50) is clearly verified. If \( Q_{j,k}^* \cap Q_{j+1,i}^* \neq \emptyset \), then \( \text{diam}(Q_{j+1,i}^*) \leq C\text{diam}(Q_{j,k}^*) \), by Lemma 3.6 (1). Thus,
\[ \ell^{j+1,i} := \ell(Q_{j+1,i}^*) \leq C(\frac{d}{\text{diam}(Q_{j,k}^*)}) < \frac{C}{C_0\sqrt{d}} < 1, \]
since \( 1 < C < C_0 \) (see Lemma 3.6 (1)). Moreover,
\[ \int_{\mathbb{R}^d} [\hat{f}(x) - \tilde{c}^{j+1,i}(x)]\eta_{j+1,i}(x)\mathbf{p}(x)dx = 0, \]
for all \( \mathbf{p} \in \mathcal{P}_\delta \). Endeed, \( Q_{j,k}^* \cap Q_{j+1,i}^* \neq \emptyset \) implies that supp(\( \eta_{j+1,i} \)) ⊂ \( Q_{j+1,i}^* \subset C_0Q_{j,k}^* =: Q_{j,k} \), by lemma 3.6 (1). Hence \( [\hat{f}(x) - \tilde{c}^{j+1,i}(x)]\eta_{j+1,i}(x) = [f(x) - c^{j+1,i}(x)]\eta_{j+1,i}(x), \) for all \( x \in \mathbb{R}^d \), which implies (3.51), by (3.18). Therefore, we can apply Lemma 3.7 (3.26) to obtain (3.50).

Also, when \( Q_{j,k}^* \cap Q_{j+1,i}^* \neq \emptyset \), we have
\[ \sup_{y \in \mathbb{R}^d} |\tilde{c}^{j+1,i}(y)\eta_{j+1,i}(y)| \lesssim \sup_{y \in \mathbb{R}^d} \mathcal{M}^0_{\text{loc}}(\hat{f})(y) \lesssim \epsilon, \]
by Lemma 3.7 (3.25), since (3.51) holds and \( \ell^{j+1,i} < 1 \). Moreover, since \( \ell^{j,k} < 1 \), it follows from the definition of \( \lambda_{j,k}a_{j,k} \) (see (3.28) and (3.31))
that

\[ \lambda_{j,k} a_{j,k} = (f - c_{j,k}) \eta_{j,k} - \sum_{i \in E_1^{j+1}} (f - c_{j+1,i}) \eta_{j+1,i} \eta_{j,k} + \sum_{i \in E_1^{j+1}} c_i \eta_{j+1,i} = \hat{f} \eta_{j,k} (1 - \sum_{i \in E_1^{j+1}} \eta_{j+1,i}) - \hat{c}_{j,k} \eta_{j,k} + \hat{c}_{j+1,i} \eta_{j+1,i} + \sum_{i \in E_1^{j+1}} c_i \eta_{j+1,i}. \]

Also, by \( 0 \leq \sum_{i \in E_1^{j+1}} \eta_{j+1,i}(x) \leq \sum_{i \geq 0} \eta_{j+1,i}(x) = \chi_{c_{j+1}}(x) \leq 1 \), we have

\[ \left| \hat{f}(x) \eta_{j,k}(x) \left( 1 - \sum_{i \in E_1^{j+1}} \eta_{j+1,i}(x) \right) \right| \leq |\hat{f}(x)| \leq \epsilon, \]

for all \( x \in Q_{j,k} \) and \( (j,k) \in G_2 \). From this together with (3.49), (3.50); (3.52) and the fact that \( \sum_{i \geq 0} \chi_{Q_{j+1}}(x) \lesssim 1 \), it follows that

\[ |\lambda_{j,k} a_{j,k}(x)| \lesssim \epsilon, \]

for all \( x \in Q_{j,k} \) and \( (j,k) \in G_2 \). Thus, using the fact that \( \sum_{k \geq 0} \chi_{\text{supp}(a_{j,k})} \lesssim 1 \) (see Remark 3.10 (3.36)), we obtain

\[ (3.53) \]

\[ |\ell_{2}(x)| \leq C_3 \sum_{j' < j \leq j''} \epsilon = C_3 (j'' - j') \epsilon, \]

for all \( x \in \mathbb{R}^d \), where \( C_3 > 0 \) is a constant independent of \( x, f; k \) and \( \epsilon \). We deduce from this estimate that \( \ell \) is continuous. Therefore, \( h = f - \ell \) is continuous and \( C_3 h \) is a continuous \((q,\infty,\delta)\)-atom.

Now, we find a finite decomposition of \( \ell \) as continuous \((q,\infty,\delta)\)-atoms by using again the splitting \( \ell = \ell_1' + \ell_2' \). Indeed, it is clear that, for all \( \epsilon > 0 \), \( \ell_1' \) is a finite linear combination of continuous \((q,\infty,\delta)\)-atoms. Also, \( \ell_2' = \ell - \ell_1' \) is continuous and supp \((\ell_2') \subset B(0,4R) \subset Q(0,8R) \). Hence taking \( \epsilon = |C_3 (j'' - j')|^{-1} |Q(0,8R)|^{-1/q} \), we have \( |\ell_2'(x)| \leq |Q(0,8R)|^{-1/q} \), by (3.53), so that \((\ell_2',Q(0,8R)) \in A_{\text{loc}}(q,\infty,\delta) \). This establishes the finite decomposition of \( \ell \) as continuous \((q,\infty,\delta)\)-atoms.

Thus, \( f = h + \ell = C_2'^{-1}(C_2' h) + \ell_1' + \ell_2' \) is a finite decomposition of \( f \) as continuous \((q,\infty,\delta)\)-atoms and as in the part (1), we obtain

\[ \|f\|_{H^{(q,p)}_{\text{loc,fin}}} \lesssim 1. \]

This finishes the proof of the part (2) and hence of Theorem 3.10. \( \square \)

We give a useful result of density.

**Lemma 3.17.** Let \( H^{(q,p)}_{\text{loc,fin}} \) be the subspace of \( H^{(q,p)}_{\text{loc}} \) consisting of finite linear combinations of \((q,\infty,\delta)\)-atoms. Then, \( H^{(q,p)}_{\text{loc,fin}} \cap C^\infty(\mathbb{R}^d) \) is a dense subspace of \( H^{(q,p)}_{\text{loc}} \) in the quasi-norm \( \| \cdot \|_{H^{(q,p)}_{\text{loc}}}. \)
Proof. We use some arguments of the proof of [37], Theorem 6.4. Let $f \in \mathcal{H}_{loc,fin}^{(q,p)}$. Then, $f$ is a finite linear combination of $(q, \infty, \delta)$-atoms. Therefore, there exists a real $R > 0$ such that $\text{supp}(f) \subset B(0,R)$. Furthermore, for all $0 < t \leq 1$, we have

$$f \ast \varphi_t \in C^\infty(\mathbb{R}^d) \quad \text{and} \quad \text{supp}(f \ast \varphi_t) \subset B(0,R + 1).$$

Assume that $f = \sum_{n=0}^j \lambda_n a_n$ with $\{(a_n, Q^n)\}_{n=0}^j \subset A_{loc}(q, \infty, \delta)$ and $\{\lambda_n\}_{n=0}^j \subset \mathbb{C}$. Then, for all $0 < t \leq 1$,

$$f \ast \varphi_t = \sum_{n=0}^j \lambda_n (a_n \ast \varphi_t).$$

Now, for any $0 < t \leq 1$ and $n \in \{0, 1, \ldots, j\}$, we prove that there exists a constant $c_{t,n} > 0$ such that $c_{t,n}(a_n \ast \varphi_t)$ is a $(q, \infty, \delta)$-atom lying in $C^\infty(\mathbb{R}^d)$, which implies that for any $0 < t \leq 1$,

$$f \ast \varphi_t \in \mathcal{H}_{loc,fin}^{(q,p)} \cap C^\infty(\mathbb{R}^d).$$

For $n \in \{0, 1, \ldots, j\}$, we have $\text{supp}(a_n) \subset Q^n := Q(x^n, \ell_n)$. Then $\text{supp}(a_n \ast \varphi_t) \subset \tilde{Q}^n := Q(x^n, \ell_n + 2t)$. Moreover, $a_n \ast \varphi_t \in C^\infty$ and

$$\|a_n \ast \varphi_t\|_\infty \leq \|\varphi\|_1 \|a_n\|_\infty \leq \left(\|\varphi\|_1 \frac{|Q^n|^{-\frac{1}{q}}}{|Q^n|^{-\frac{1}{q}}}\right)|\tilde{Q}^n|^{-\frac{1}{q}}.$$  

Also, for all $\beta \in \mathbb{Z}^d_+$ with $|\beta| \leq \delta$, $\int_{\mathbb{R}^d} x^\beta a_n(x)dx = 0$ implies that

$$\int_{\mathbb{R}^d} x^\beta (a_n \ast \varphi_t)(x)dx = 0.$$ 

Thus, with $c_{t,n} := \left(\|\varphi\|_1 \frac{|Q^n|^{-\frac{1}{q}}}{|Q^n|^{-\frac{1}{q}}}\right)^{-1} = \|\varphi\|_1^{-1} \left(\frac{|Q^n|}{|Q^n|}\right)^{\frac{1}{q}}, c_{t,n}(a_n \ast \varphi_t)$ is a $(q, \infty, \delta)$-atom lying in $C^\infty(\mathbb{R}^d)$.

Likewise, $\text{supp}(f - f \ast \varphi_t) \subset B(0,R + 1)$, by (3.54), and $f - f \ast \varphi_t$ has the same vanishing moments as $f$. Let $1 < s < \infty$. We have

$$\lim_{t \to 0} \|f - f \ast \varphi_t\|_s \to 0,$$

since $f \in L^s$ and $\int_{\mathbb{R}^d} \varphi(x)dx = 1$. Without loss of generality, we may assume that $\|f - f \ast \varphi_t\|_s > 0$, when $t$ is small enough. Set

$$c_t := \|f - f \ast \varphi_t\|_s \|Q(0,2(R+1))\|^{1/q-1/s}$$

and $a_t := (c_t)^{-1}(f - f \ast \varphi_t)$. Then, $a_t$ is a $(q, s, \delta)$-atom, $f - f \ast \varphi_t = c_t a_t$ and $c_t \to 0$ as $t \to 0$, by (3.55). Thus, $\|f - f \ast \varphi_t\|_{L^q_{loc}} \leq c_t \to 0$ as $t \to 0$. The proof of Lemma 3.17 is complete. \hfill \Box

If $\mathcal{H}_{loc,fin}^{(q,p)}$ is the subspace of $\mathcal{H}_{loc}^{(q,p)}$ consisting of finite linear combinations of $(q, \infty, \delta)$-atoms, then it follows from Lemma 3.17 that
\( \mathcal{H}^{(q,p)}_{\text{loc}, \text{fin}} \cap C(\mathbb{R}^d) \) is dense in \( \mathcal{H}^{(q,p)}_{\text{loc}} \) with respect to the quasi-norm \( \| \cdot \|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \), since \( \mathcal{H}^{(q,p)}_{\text{loc}, \text{fin}} \) is dense in \( \mathcal{H}^{(q,p)}_{\text{loc}} \) in the quasi-norm \( \| \cdot \|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \).

3.2. Molecular Decomposition. Our definition of molecule for \( \mathcal{H}^{(q,p)}_{\text{loc}} \) spaces is the one of \( \mathcal{H}^{(q,p)} \) spaces (see [1], Definition 5.1, p. 1931) except that, when \( |Q| \geq 1 \), the condition (3) in [1], Definition 5.1, is not required. Thus, a \((q, r, \delta)\)-atom for \( \mathcal{H}^{(q,p)}_{\text{loc}} \) is a \((q, r, \delta)\)-molecule for \( \mathcal{H}^{(q,p)}_{\text{loc}} \). We denote by \( \mathcal{M}_{\text{loc}}^{(q,p)}(q, r, \delta) \) the set of all \((m, Q)\) such that \( m \) is a \((q, r, \delta)\)-molecule centered at \( Q \) (for \( \mathcal{H}^{(q,p)}_{\text{loc}} \)).

We have the following decomposition theorem as an application of Theorem 3.18.

**Theorem 3.18.** Let \( 1 < r \leq +\infty \) and \( \delta \geq \left\lfloor \frac{d}{r} - 1 \right\rfloor \) be an integer. Then, for every \( f \in \mathcal{H}^{(q,p)}_{\text{loc}} \), there exist a sequence \( \{(m_n, Q^n)\}_{n \geq 0} \) in \( \mathcal{M}^{(q,p)}_{\text{loc}}(q, r, \delta) \) and a sequence of scalars \( \{\lambda_n\}_{n \geq 0} \) such that

\[
 f = \sum_{n \geq 0} \lambda_n m_n \quad \text{in } S' \text{ and } \mathcal{H}^{(q,p)}_{\text{loc}},
\]

and, for all \( \eta > 0 \),

\[
 \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\| \chi_{Q^n} \|_q} \right)^{\eta} \chi_{Q^n} \right\|_{\frac{1}{q}, \frac{1}{r} } \lesssim \| f \|_{\mathcal{H}^{(q,p)}_{\text{loc}}}. 
\]

Also, we have the two following reconstruction theorems which are the local analogues of [1], Theorems 5.2 and 5.3, pp. 1931-1932.

**Theorem 3.19.** Let \( 0 < \eta \leq 1 \) and \( \delta \geq \left\lfloor \frac{d}{q} - 1 \right\rfloor \) be an integer. Then, for all sequences \( \{(m_n, Q^n)\}_{n \geq 0} \) in \( \mathcal{M}^{(q,p)}_{\text{loc}}(q, \infty, \delta) \) and all sequences of scalars \( \{\lambda_n\}_{n \geq 0} \) satisfying

\[
(3.56) \quad \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\| \chi_{Q^n} \|_q} \right)^{\eta} \chi_{Q^n} \right\|_{\frac{1}{q}, \frac{1}{r} } < +\infty,
\]

the series \( f := \sum_{n \geq 0} \lambda_n m_n \) converges in \( S' \) and \( \mathcal{H}^{(q,p)}_{\text{loc}} \), with

\[
\| f \|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \lesssim \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\| \chi_{Q^n} \|_q} \right)^{\eta} \chi_{Q^n} \right\|_{\frac{1}{q}, \frac{1}{r} }^{\frac{1}{\eta}}.
\]

**Proof.** Let \( \{(m_n, Q^n)\}_{n \geq 0} \) be a sequence of elements of \( \mathcal{M}^{(q,p)}_{\text{loc}}(q, \infty, \delta) \) and \( \{\lambda_n\}_{n \geq 0} \) be a sequence of scalars satisfying (3.56). Consider an integer \( j \geq 0 \). Set \( J_1 = \{n \in \{0, 1, \ldots, j\} : |Q^n| < 1\} \) and \( J_2 = \)
\{ n \in \{0, 1, \ldots, j \} : |Q^n| \geq 1 \}. \text{ We have}

\[ M_{\text{loc}_0}\left( \sum_{n \in J_1} \lambda_n m_n \right)(x) \leq \sum_{n \in J_1} |\lambda_n| \left( M(m_n)(x) \chi_{\tilde{Q}^n}(x) + \frac{[M(\chi_{Q^n})(x)]^u}{\| \chi_{Q^n} \|_q} \right), \]

for all \( x \in \mathbb{R}^d \), where \( \tilde{Q}^n = 2 \sqrt{d} Q^n \) and \( u = \frac{d+\delta+1}{d} \), by \cite{24}, (5.2), p. 3698, and

\[ M_{\text{loc}_0}\left( \sum_{n \in J_2} \lambda_n m_n \right)(x) \leq \sum_{n \in J_2} |\lambda_n| \left( M(m_n)(x) \chi_{\tilde{Q}^n}(x) + \frac{[M(\chi_{Q^n})(x)]^u}{\| \chi_{Q^n} \|_q} \right), \]

for all \( x \in \mathbb{R}^d \), where \( \tilde{Q}^n = 4 \sqrt{d} Q^n \) and \( u = \frac{d+\delta+1}{d} \), by \cite{24}, (7.6), p. 955. Hence

\[ M_{\text{loc}_0}\left( \sum_{n=0}^j \lambda_n m_n \right)(x) \leq M_{\text{loc}_0}\left( \sum_{n \in J_1} \lambda_n m_n \right)(x) + M_{\text{loc}_0}\left( \sum_{n \in J_2} \lambda_n m_n \right)(x) \]

\[ \leq \sum_{n=0}^j |\lambda_n| \left( M(m_n)(x) \chi_{\tilde{Q}^n}(x) + \frac{[M(\chi_{Q^n})(x)]^u}{\| \chi_{Q^n} \|_q} \right), \]

for all \( x \in \mathbb{R}^d \), by the proof of \cite{1}, Theorem 5.2, p. 1931. And we end as in the proof of \cite{1}, Theorem 4.3, pp. 1914-1915. \qed

**Theorem 3.20.** Let \( \max \{p, 1\} < r < +\infty \), \( 0 < \eta < q \) and \( \delta \geq \left\lfloor \frac{d}{q} - 1 \right\rfloor \) be an integer. Then, for all sequences \( \{ (m_n, Q^n) \}_{n \geq 0} \) in \( M\ell_{\text{loc}}(q, r, \delta) \) and all sequences of scalars \( \{ \lambda_n \}_{n \geq 0} \) such that

\[ (3.57) \quad \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\| \chi_{Q^n} \|_q} \right)^{\eta/\delta} \chi_{Q^n} \right\|_{\frac{\eta}{\delta}} < +\infty, \]

the series \( f := \sum_{n \geq 0} \lambda_n m_n \) converges in \( S' \) and \( H^{(q,p)}_{\text{loc}} \), with

\[ \| f \|_{H^{(q,p)}_{\text{loc}}} \leq \left\| \sum_{n \geq 0} \left( \frac{|\lambda_n|}{\| \chi_{Q^n} \|_q} \right)^{\eta/\delta} \chi_{Q^n} \right\|_{\frac{\eta}{\delta}}. \]

**Proof.** Let \( \{ (m_n, Q^n) \}_{n \geq 0} \) be a sequence of elements of \( M\ell_{\text{loc}}(q, r, \delta) \) and \( \{ \lambda_n \}_{n \geq 0} \) be a sequence of scalars satisfying \( (3.57) \). Consider an integer \( j \geq 0 \). Proceeding as in the proof of Theorem 3.19, we have

\[ M_{\text{loc}_0}\left( \sum_{n=0}^j \lambda_n m_n \right)(x) \leq \sum_{n=0}^j |\lambda_n| \left( M(m_n)(x) \chi_{\tilde{Q}^n}(x) + \frac{[M(\chi_{Q^n})(x)]^u}{\| \chi_{Q^n} \|_q} \right), \]
for all $x \in \mathbb{R}^d$, where $\widetilde{Q}^n = 4\sqrt{d}Q^n$ and $u = \frac{d+\delta+1}{d}$. Thus, arguing as in the proof of [1], Theorem 5.3, p. 1932, we obtain
\[
\left\| \mathcal{M}_{loc_0} \left( \sum_{n=0}^{j} \lambda_n m_n \right) \right\|_{q,p} < \left\| \sum_{n=0}^{j} \left( \frac{|\lambda_n|}{\|\chi_{Q^n}\|_q} \right)^{\frac{1}{q}} \chi_{Q^n} \right\|_{q,p}.
\]
And we end as in the proof of [1], Theorem 4.3, pp. 1914-1915.

4. Characterization of the dual space of $\mathcal{H}_{loc}^{(q,p)}$

In this section, we consider an integer $\delta \geq \left\lfloor \frac{d(1/q - 1)}{q} \right\rfloor$. Let $1 < r \leq +\infty$. We denote by $L_{comp}^r(\mathbb{R}^d)$ the subspace of $L^r$-functions with compact support, and, for all cube $Q$, $L^r(Q)$ stands for the subspace of $L^r$-functions supported in $Q$. If $f \in L^1_{loc}$ and $Q$ is a cube, then, by Riesz’s representation theorem, there exists an unique polynomial of $P_{\delta}(\mathbb{R}^d)$ (we recall that $P_{\delta}(\mathbb{R}^d)$ is the space of polynomial functions of degree at most $\delta$) that we denote by $P_{\delta}^Q(f)$ such that, for all $q \in P_{\delta}$,
\[
(4.1) \quad \int_{Q} [f(x) - P_{\delta}^Q(f)(x)] q(x)dx = 0.
\]

By following [37], Definition 7.1, p. 51, we introduce the following definition.

**Definition 4.1.** ($L_{r,\delta,\phi}^{loc}(\mathbb{R}^d)$). Let $1 \leq r \leq +\infty$ and $\phi : Q \rightarrow (0, +\infty)$ be a function. The space $L_{r,\delta,\phi}^{loc}(\mathbb{R}^d)$ is the set of all $f \in L^r_{loc}$ such that $\|f\|_{L_{r,\delta,\phi}^{loc}} < +\infty$, where
\[
\|f\|_{L_{r,\delta,\phi}^{loc}} := \sup_{Q \subset \mathbb{R}^d} \frac{1}{\phi(Q)} \left( \frac{1}{|Q|} \int_Q |f(x)|^r dx \right)^{\frac{1}{r}} + \sup_{Q \subset \mathbb{R}^d} \frac{1}{\phi(Q)} \left( \frac{1}{|Q|} \int_Q \left| f(x) - P_{\delta}^Q(f)(x) \right|^r dx \right)^{\frac{1}{r}},
\]
when $r < +\infty$, and
\[
\|f\|_{L_{r,\delta,\phi}^{loc}} := \sup_{Q \subset \mathbb{R}^d} \frac{1}{\phi(Q)} \|f\|_{L^\infty(Q)} + \sup_{Q \subset \mathbb{R}^d} \frac{1}{\phi(Q)} \|f - P_{\delta}^Q(f)\|_{L^\infty(Q)},
\]
when $r = +\infty$.

For simplicity, we just denote $L_{r,\delta,\phi}^{loc}(\mathbb{R}^d)$ by $L_{r,\delta,\phi}^{loc}$. We consider the function $\phi_1 : Q \rightarrow (0, +\infty)$ defined by
\[
(4.2) \quad \phi_1(Q) = \frac{\|\chi_Q\|_{q,p}}{|Q|},
\]
for all \( Q \in \mathcal{Q} \). Also, we set, for every \( T \in (\mathcal{H}^{(q,p)})^* \),
\[
\|T\| := \|T\|_{(\mathcal{H}^{(q,p)})^*} = \sup_{f \in \mathcal{H}^{(q,p)}} |T(f)|
\]
and \( \mathcal{L}_{r,\phi_1,\delta} := \mathcal{L}_{r,\phi_1,\delta}(\mathbb{R}^d) \) the set of all \( f \in L^1_{\text{loc}} \) such that \( \|f\|_{\mathcal{L}_{r,\phi_1,\delta}} < +\infty \) (see [2], Definition 3.2). In [2], we obtained the following duality result for \( \mathcal{H}^{(q,p)} \), with \( 0 < q \leq p \leq 1 \).

**Theorem 4.2** ([2], Theorem 3.3). Suppose that \( 0 < q \leq p \leq 1 \). Let \( 1 < r \leq +\infty \). Then, \( (\mathcal{H}^{(q,p)})^* \) is isomorphic to \( \mathcal{L}_{r,\phi_1,\delta} \) with equivalent norms, where \( \frac{1}{r} + \frac{1}{p} = 1 \). More precisely, we have the following assertions:

1. Let \( g \in \mathcal{L}_{r,\phi_1,\delta} \). Then, the mapping
   \[
   T_g : f \in \mathcal{H}^{(q,p)}_{\text{fin}} \mapsto \int_{\mathbb{R}^d} g(x)f(x)dx,
   \]
where \( \mathcal{H}^{(q,p)}_{\text{fin}} \) is the subspace of \( \mathcal{H}^{(q,p)} \) consisting of finite linear combinations of \( (q,r,\delta) \)-atoms, extends to a unique continuous linear functional on \( \mathcal{H}^{(q,p)} \) such that
   \[
   \|T_g\| \leq C \|g\|_{\mathcal{L}_{r,\phi_1,\delta}},
   \]
where \( C > 0 \) is a constant independent of \( g \).

2. Conversely, for any \( T \in (\mathcal{H}^{(q,p)})^* \), there exists \( g \in \mathcal{L}_{r,\phi_1,\delta} \) such that
   \[
   T(f) = \int_{\mathbb{R}^d} g(x)f(x)dx, \quad \text{for all } f \in \mathcal{H}^{(q,p)}_{\text{fin}},
   \]
   and
   \[
   \|g\|_{\mathcal{L}_{r,\phi_1,\delta}} \leq C \|T\|,
   \]
where \( C > 0 \) is a constant independent of \( T \).

Now, for all \( T \in (\mathcal{H}^{(q,p)}_{\text{loc}})^* \), for simplicity, we set
\[
\|T\| := \|T\|_{(\mathcal{H}^{(q,p)}_{\text{loc}})^*} = \sup_{f \in \mathcal{H}^{(q,p)}_{\text{loc}}} |T(f)|.
\]

With Definition 4.1 and Theorem 4.2, we have our following characterization of the dual of \( \mathcal{H}^{(q,p)}_{\text{loc}} \), when \( 0 < q \leq p \leq 1 \).

**Theorem 4.3.** Suppose that \( 0 < q \leq p \leq 1 \). Let \( 1 < r \leq +\infty \). Then, \( (\mathcal{H}^{(q,p)}_{\text{loc}})^* \) is isomorphic to \( \mathcal{L}^{\text{loc}}_{r,\phi_1,\delta} \) with equivalent norms, where \( \frac{1}{r} + \frac{1}{p} = 1 \). More precisely, we have the following assertions:
(1) Let \( g \in \mathcal{L}_{r', \phi_1, \delta}^{\text{loc}} \). Then, the mapping

\[
T_g : f \in \mathcal{H}_{\text{loc}, \text{fin}}^{(q,p)} \mapsto \int_{\mathbb{R}^d} g(x) f(x) dx,
\]

where \( \mathcal{H}_{\text{loc}, \text{fin}}^{(q,p)} \) is the subspace of \( \mathcal{H}_{\text{loc}}^{(q,p)} \) consisting of finite linear combinations of \((q,r,\delta)\)-atoms, extends to a unique continuous linear functional on \( \mathcal{H}_{\text{loc}}^{(q,p)} \) such that

\[
\|T_g\| \leq C \|g\|_{\mathcal{L}_{r', \phi_1, \delta}^{\text{loc}}},
\]

where \( C > 0 \) is a constant independent of \( g \).

(2) Conversely, for any \( T \in \left( \mathcal{H}_{\text{loc}}^{(q,p)} \right)^* \), there exists \( g \in \mathcal{L}_{r', \phi_1, \delta}^{\text{loc}} \) such that

\[
T(f) = \int_{\mathbb{R}^d} g(x) f(x) dx, \quad \text{for all } f \in \mathcal{H}_{\text{loc}, \text{fin}}^{(q,p)},
\]

and

\[
\|g\|_{\mathcal{L}_{r', \phi_1, \delta}^{\text{loc}}} \leq C \|T\|,
\]

where \( C > 0 \) is a constant independent of \( T \).

**Proof.** We borrow some ideas from [37], Theorem 7.5. We distinguish two cases: \( 1 < r < +\infty \) and \( r = +\infty \).

Suppose that \( 1 < r < +\infty \). First, we prove (1). Fix \( 0 < \eta < q \). Let \( g \in \mathcal{L}_{r', \phi_1, \delta}^{\text{loc}} \), where \( \frac{1}{r} + \frac{1}{r'} = 1 \). Consider the mapping \( T_g \) defined on \( \mathcal{H}_{\text{loc}, \text{fin}}^{(q,p)} \) by

\[
T_g(f) = \int_{\mathbb{R}^d} g(x) f(x) dx, \quad \forall f \in \mathcal{H}_{\text{loc}, \text{fin}}^{(q,p)}.
\]

It is easy to verify that \( T_g \) is well defined and linear. Let \( f \in \mathcal{H}_{\text{loc}, \text{fin}}^{(q,p)} \).

Then, there exist a finite sequence \( \{(a_n, Q^n)\}_{n=0}^m \) of elements of \( \mathcal{A}_{\text{loc}}(q, r, \delta) \) and a finite sequence of scalars \( \{\lambda_n\}_{n=0}^m \) such that \( f = \sum_{n=0}^m \lambda_n a_n \). Set \( J_1 = \{n \in \{0, 1, \ldots, m\} : |Q^n| < 1\} \) and \( J_2 = \{n \in \{0, 1, \ldots, m\} : |Q^n| \geq 1\} \).

We have

\[
|T_g(f)| \leq I_1 + I_2,
\]

with

\[
I_1 = \left| \int_{\mathbb{R}^d} \left( \sum_{n \in J_1} \lambda_n a_n(x) \right) g(x) dx \right| \quad \text{and} \quad I_2 = \left| \int_{\mathbb{R}^d} \left( \sum_{n \in J_2} \lambda_n a_n(x) \right) g(x) dx \right|.
\]
With Proposition 2.2 by arguing as in the proof of Theorem 4.2 we obtain

\[ I_1 \leq \left\| \sum_{n \in J_1} \left( \frac{|\lambda_n|}{\|X_{Q^n}\|_q} \right)^{\eta} \chi_{X_{Q^n}} \right\|^{\frac{1}{\eta}}_{\frac{q}{\eta}; \frac{q}{\eta}} \left[ \sup_{Q \in \mathcal{Q}} \frac{1}{\phi_1(Q)} \left( \frac{1}{|Q|} \int g(x) - P_{Q}^\delta(g)(x) \right| r' \, dx \right]^{\frac{1}{\eta}} \]

\[ \leq \left\| \sum_{n=0}^{m} \left( \frac{|\lambda_n|}{\|X_{Q^n}\|_q} \right)^{\eta} \chi_{X_{Q^n}} \right\|^{\frac{1}{\eta}}_{\frac{q}{\eta}; \frac{q}{\eta}} \left[ \sup_{Q \in \mathcal{Q}} \frac{1}{\phi_1(Q)} \left( \frac{1}{|Q|} \int g(x) \right| r' \, dx \right]^{\frac{1}{\eta}}. \]

Also,

\[ I_2 \leq \left\| \sum_{n \in J_2} \left( \frac{|\lambda_n|}{\|X_{Q^n}\|_q} \right)^{\eta} \chi_{X_{Q^n}} \right\|^{\frac{1}{\eta}}_{\frac{q}{\eta}; \frac{q}{\eta}} \left[ \sup_{Q \in \mathcal{Q}} \frac{1}{\phi_1(Q)} \left( \frac{1}{|Q|} \int g(x) \right| r' \, dx \right]^{\frac{1}{\eta}} \]

\[ \leq \left\| \sum_{n=0}^{m} \left( \frac{|\lambda_n|}{\|X_{Q^n}\|_q} \right)^{\eta} \chi_{X_{Q^n}} \right\|^{\frac{1}{\eta}}_{\frac{q}{\eta}; \frac{q}{\eta}} \left[ \sup_{Q \in \mathcal{Q}} \frac{1}{\phi_1(Q)} \left( \frac{1}{|Q|} \int g(x) \right| r' \, dx \right]^{\frac{1}{\eta}}. \]

Hence

\[ |T_g(f)| \leq \|f\|_{\mathcal{H}^{(q,p)}_{\text{loc, fin}}} \|g\|_{\mathcal{L}^\text{loc}_{r', \phi_1, \delta}} \lesssim \|f\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \|g\|_{\mathcal{L}^\text{loc}_{r', \phi_1, \delta}}, \]

by Theorem 3.16. This shows that \( g \in \left( \mathcal{H}^{(q,p)}_{\text{loc}} \right)^* \) and

\[ \|g\| := \|T_g\| \lesssim \|g\|_{\mathcal{L}^\text{loc}_{r', \phi_1, \delta}}, \]

since \( \mathcal{H}^{(q,p)}_{\text{loc, fin}} \) is dense in \( \mathcal{H}^{(q,p)}_{\text{loc}} \) with respect to the quasi-norm \( \|\cdot\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \).

Now, prove (2). Let \( T \in \left( \mathcal{H}^{(q,p)}_{\text{loc}} \right)^* \). Fix \( 0 < \eta < q \). Let \( Q \) be a cube such that \( \ell(Q) \geq 1 \). We first prove that

\[ \left( \mathcal{H}^{(q,p)}_{\text{loc}} \right)^* \subset (L^r(Q))^*. \]

For any \( f \in L^r(Q) \setminus \{0\} \), we set

\[ a(x) := \|f\|_{L^r(Q)}^{-1} |Q|^\frac{1}{\eta - \frac{1}{r}} f(x), \]

for all \( x \in \mathbb{R}^d \). Clearly, \( (a, Q) \in \mathcal{A}_{\text{loc}}(q, r, \delta) \). Thus, \( f \in \mathcal{H}^{(q,p)}_{\text{loc, fin}} \subset \mathcal{H}^{(q,p)}_{\text{loc}} \) and

\[ \|f\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} = \|f\|_{L^r(Q)} |Q|^\frac{1}{\eta - \frac{1}{r}} \|a\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \lesssim |Q|^\frac{1}{\eta - \frac{1}{r}} \|f\|_{L^r(Q)}. \]

Hence

\[ |T(f)| \leq \|T\| \|f\|_{\mathcal{H}^{(q,p)}_{\text{loc}}} \lesssim |Q|^\frac{1}{\eta - \frac{1}{r}} \|T\| \|f\|_{L^r(Q)}, \]
for all $f \in L^r(Q)$. Thus, $T$ is a continuous linear functional on $L^r(Q)$, with

$$\|T\|_{(L^r(Q))^*} := \sup_{f \in L^r(Q)} \frac{|T(f)|}{\|f\|_{L^r(Q)} \leq 1} \lesssim |Q|^{\frac{1}{r} - \frac{1}{s}} \|T\|.$$ 

This proves (4.3).

Therefore, since $(L^r(Q))^*$ is isomorphic to $L^{r'}(Q)$, there exists $g^Q \in L^{r'}(Q)$ such that

$$T(f) = \int_Q f(x) g^Q(x) dx, \text{ for all } f \in L^r(Q),$$

(4.5)

Now, for every cube $Q$ such that $\ell(Q) \geq 1$, we consider the fonction $g^Q$ as defined above. Let $\{Q_n\}_{n \geq 1}$ be an increasing sequence of cubes which converges to $\mathbb{R}^d$ and $\ell(Q_1) \geq 1$. For each cube $Q_n$, we have

$$T(f) = \int_{Q_n} f(x) g^{Q_n}(x) dx,$$

(4.6)

for all $f \in L^r(Q_n)$, by (4.5).

Now, we construct a fonction $g \in L^{r'}_{\text{loc}}(\mathbb{R}^d)$ such that

$$T(f) = \int_{Q_1} f(x) g(x) dx,$$

(4.7)

for all $f \in L^r(Q_1)$ and all $n \geq 1$. First, assume that $f \in L^r(Q_1)$. Then,

$$T(f) = \int_{Q_1} f(x) g^{Q_1}(x) dx,$$

by (4.6). Since $L^r(Q_1) \subset L^r(Q_2)$, also we have

$$T(f) = \int_{Q_2} f(x) g^{Q_2}(x) dx = \int_{Q_1} f(x) g^{Q_2}(x) dx,$$

by (4.6). Thus, for all $f \in L^r(Q_1)$,

$$\int_{Q_1} f(x) \left[ g^{Q_1}(x) - g^{Q_2}(x) \right] dx = 0.$$

This implies that $g^{Q_1}(x) = g^{Q_2}(x)$, for almost all $x \in Q_1$. Thus, after changing values of $g^{Q_1}$ (or $g^{Q_2}$) on a set of measure zero, we have $g^{Q_1}(x) = g^{Q_2}(x)$, for all $x \in Q_1$. Arguing as above, we obtain

$$g^{Q_n}(x) = g^{Q_{n+1}}(x),$$

(4.8)

for all $x \in Q_n$ and all $n \geq 1$. Set

$$g_1(x) := g^{Q_1}(x), \text{ if } x \in Q_1$$

(4.9)
and
\[ g_{n+1}(x) := \begin{cases} 
  g_{Q_n}(x), & \text{if } x \in Q_n, \\
  g_{Q_{n+1}}(x), & \text{if } x \in Q_{n+1} \setminus Q_n,
\end{cases} \]
for all \( n \geq 1 \). We have
\[ (4.10) \quad g_{n+1}(x) = g_{Q_{n+1}}(x), \]
for all \( x \in Q_{n+1} \) and all \( n \geq 1 \), by (4.8). With (4.9) and (4.10), we define the function \( g \) on \( \mathbb{R}^d \) by
\[ g(x) := g_n(x) = g_{Q_n}(x), \quad \text{if } x \in Q_n, \]
for all \( n \geq 1 \). Then, \( g \in L^r_{\text{loc}} \), since \( g_n \in L^r_{\text{loc}} \), for all \( n \geq 1 \), and
\[ \int_{Q_n} f(x) g(x) dx = \int_{Q_n} f(x) g_n(x) dx = T(f), \]
for all \( f \in L^r(Q_n) \), for all \( n \geq 1 \), by (4.6). Thus, the function \( g \) satisfies (4.7).

To end, we show that \( g \in L^r_{\text{loc}}' \), and
\[ (4.11) \quad T(f) = \int_{\mathbb{R}^d} f(x) g(x) dx, \]
for all \( f \in \mathcal{H}_{\text{loc}, fin}^{(q,p)} \). Let \( f \in \mathcal{H}_{\text{loc}, ffin}^{(q,p)} \). We have \( f \in L^r_{\text{comp}}(\mathbb{R}^d) \), since \( \mathcal{H}_{\text{loc}, ffin}^{(q,p)} \subset L^r_{\text{comp}}(\mathbb{R}^d) \). Thus, there exists an integer \( n \geq 1 \) such that \( f \in L^r(Q_n) \). Hence (4.11) holds, by (4.7).

Now, we prove that \( g \in L^r_{\text{loc}}' \). Let \( Q \) be a cube with \( \ell(Q) \geq 1 \), and \( f \in L^r(Q) \) such that \( \|f\|_{L^r(Q)} \leq 1 \). Set
\[ a(x) := \frac{1}{|Q|} \int_Q f(x), \]
for all \( x \in \mathbb{R}^d \). We have \( (a, Q) \in A_{\text{loc}}(q, r, \delta) \). Hence
\[ T(a) = \int_{\mathbb{R}^d} a(x) g(x) dx = \int_{Q} a(x) g(x) dx, \]
by (4.11). Since \( T \in (\mathcal{H}_{\text{loc}}^{(q,p)})^* \), we have
\[
\int_{Q} a(x) g(x) dx = |T(a)| \leq \|T\| \|a\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} \\
\lesssim \|a\|_{\mathcal{H}_{\text{loc}, ffin}^{(q,p)}} \|T\| \lesssim \frac{1}{\|\chi_Q\|_q} \|\chi_Q\|_{q,p} \|T\|,
\]
by Theorem 3.16 and Remark 3.14. Thus, for all \( f \in L^r(Q) \) with \( \|f\|_{L^r(Q)} \leq 1 \), we have
\[
\left| \int_Q f(x)g(x)\,dx \right| \lesssim |Q|^{\frac{1}{r} - \frac{1}{q}} \left\| \chi_Q \right\|_{q,p} \|T\|
= C|Q|^{\frac{1}{r} - \frac{1}{q}} \left\| \chi_Q \right\|_{q,p} \|T\|.
\]
This implies that
\[
\frac{1}{\phi_1(Q)} \left( \frac{1}{|Q|} \int_Q |g(x)|^{r'} \,dx \right)^{\frac{1}{r'}} \lesssim \|T\|.
\]
Therefore,
\[
(4.12) \quad \sup_{|Q| \geq 1} \frac{1}{\phi_1(Q)} \left( \frac{1}{|Q|} \int_Q |g(x)|^{r'} \,dx \right)^{\frac{1}{r'}} \lesssim \|T\|.
\]
Moreover, since \( H^{(q,p)} \subset H^{(q,p)}_{\text{loc}} \) and \( \|f\|_{H^{(q,p)}_{\text{loc}}} \leq \|f\|_{H^{(q,p)}} \) for all \( f \in H^{(q,p)} \), we have \( \left( H^{(q,p)}_{\text{loc}} \right)^* \subset \left( H^{(q,p)} \right)^* \) and \( T|_{H^{(q,p)}} \in \left( H^{(q,p)} \right)^* \). Since (4.11) holds for all \( f \in H^{(q,p)}_{\text{fin}} \), from Theorem 4.2 (2), we deduce that \( g \in L_{r',\phi_1,\delta} \) and
\[
\|g\|_{L_{r',\phi_1,\delta}} \lesssim \|T\|_{H^{(q,p)}} \left( H^{(q,p)} \right)^* \lesssim \|T\|.
\]
Hence
\[
(4.13) \quad \sup_{|Q| \leq 1} \frac{1}{\phi_1(Q)} \left( \frac{1}{|Q|} \int_Q \left| g(x) - P_Q^\delta(g)(x) \right|^{r'} \,dx \right)^{\frac{1}{r'}} \lesssim \|T\|.
\]
Combining (4.12) and (4.13), we obtain \( \|g\|_{L_{r',\phi_1,\delta}} \lesssim \|T\| \) and \( g \in L_{r',\phi_1,\delta} \). This finishes the proof of (2) and hence, the proof in Case \( 1 < r < +\infty \).

The proof in Case \( r = +\infty \) is similar to the one of Theorem 4.2, we omit details. This completes the proof of Theorem 4.3. \( \square \)

**Remark 4.4.**

1. The space \( L_{r',\phi_1,\delta} \) is independent of \( 1 \leq r < +\infty \) and the integer \( \delta \geq \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor \), by Theorem 4.3.

2. When \( q = p = 1 \), we have \( H^{(q,p)}_{\text{loc}} = H^{1}_{\text{loc}} \) and \( \phi_1 \equiv 1 \). Thus, taking \( r = +\infty \) and \( \delta = 0 \) in Theorem 4.3, we obtain \( L_{r',\phi_1,0} = \text{bmo}(\mathbb{R}^d) \) the dual of \( H^{1}_{\text{loc}} \) introduced by Goldberg in [14] and
which is the set of all $f \in L^1_{\text{loc}}$ such that

$$\|f\|_{bmo} := \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q |f(x)| \, dx + \sup_{Q \in \mathcal{Q}, |Q| < 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < +\infty,$$

where $f_Q = \frac{1}{|Q|} \int_Q f(x) \, dx$.

(3) When $q = p < 1$, we have $H^1_{\text{loc}} = H^\rho_{\text{loc}}$ and $\phi_1(Q) = |Q|^\frac{1}{\rho} - 1$, for all $Q \in \mathcal{Q}$. Thus, taking $r = +\infty$ and $\delta = \left[ d \left( \frac{1}{q} - 1 \right) \right]$ in Theorem 4.3, we obtain $L^1_{\text{loc}} = N_d(\frac{1}{q} - 1)$ the dual of $H^\rho_{\text{loc}}$ defined by Goldberg in [14], Theorem 5, p. 40.

(4) Also, when $q = p < 1$, by taking $1 < r < +\infty$, we obtain $L^r_{\rho,\delta} = \text{bmo}_{\rho,\omega}^r(\mathbb{R}^d)$ the dual of $h^0_{\omega}(\mathbb{R}^d) = H^\rho_{\text{loc}}$ defined in [37], in the case where $\Phi(t) = t^q$ and $\omega \equiv 1$, with $\rho(t) = t^{-1} \Phi^{-1}(t^{-1})$, $0 < t < +\infty$.

(5) As in [2], Remark 3.4, we do not know to characterize the dual space of $H^1_{\text{loc}}$, whenever $0 < q \leq 1 < p < +\infty$.

5. Boundedness of Pseudo-differential Operators

In this section, unless otherwise specified, we assume that $0 < q \leq 1$ and $q \leq p < +\infty$. In [14], Theorem 4, D. Goldberg showed that pseudo-differential operators of a certain class $(\mathcal{S}^\rho_{\rho,\omega})$ are bounded on $H^\rho_{\text{loc}}$. This result has been extended to other generalizations of $H^\rho_{\text{loc}}$ spaces (see [37, 38]). We prove that this result extends to $H^1_{\text{loc}}$ spaces. To see this, we recall the definition of a pseudo-differential operator and some results we will need. For more informations about pseudo-differential operators, the reader can refer to [10, 26, 29, 32, 34] and [36].

**Definition 5.1** $(\mathcal{S}^\mu_{\rho,\sigma})$. Let $\mu \in \mathbb{R}$, $0 \leq \rho \leq 1$ and $0 \leq \sigma \leq 1$. $\mathcal{S}^\mu_{\rho,\sigma}$ is the collection of all complex-values $C^\infty$ functions $\psi(x, \xi)$ in $\mathbb{R}^d \times \mathbb{R}^d$, such that, for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ there exists a positive number $C_{\alpha,\beta}$ with

$$|\partial_\xi^\alpha \partial_x^\beta \psi(x, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{\mu + \sigma |\alpha| - |\rho|\beta}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$

Every element of $\mathcal{S}^\mu_{\rho,\sigma}$ is called symbol. Let $\psi(x, \xi) \in \mathcal{S}^\mu_{\rho,\sigma}$. The operator $T$ defined by

$$T(f)(x) = \int_{\mathbb{R}^d} \psi(x, \xi) e^{2\pi i x \xi} \hat{f}(\xi) \, d\xi,$$

for all $f \in \mathcal{S}$ and all $x \in \mathbb{R}^d$, is called a $\mathcal{S}^\mu_{\rho,\sigma}$ pseudo-differential operator. For every $\mathcal{S}^\mu_{\rho,\sigma}$ pseudo-differential operator $T$, (5.1) also writes

$$T(f)(x) = \int_{\mathbb{R}^d} K(x, x-y) f(y) \, dy,$$
where

\[
K(x, z) = \int_{\mathbb{R}^d} \psi(x, \xi) e^{2\pi i x \cdot \xi} d\xi.
\]

For (5.2) and (5.3), see [10], Chap. 5, pp. 113-114 or [29], Chap.VI, (8), p. 235; (43), p. 250. When \( \mu = \sigma = 0 \) and \( \rho = 1 \), we just denote by \( S^0 \) the class \( S^0_{1,0} \).

In the rest of this section, we are interested in \( S^0 \) pseudo-differential operators. It’s well known that, when \( \psi \in S^0 \), for every \( f \in L^2 \), (5.2) is valid for almost all \( x \notin \text{supp}(f) \), and \( T \) is a bounded operator on \( L^r \), for any \( 1 < r < +\infty \) (see [10], Chap. 5, pp. 113-114 and [29], Chap.VI, Proposition 4, p. 250). Also, we have the following

**Lemma 5.2** ([14], Lemma 6). Let \( T \) be a \( S^0 \) pseudo-differential operator. If \( \phi \in S \), then \( T_0 (f) = \phi \ast T(f) \) has a symbol \( \psi_1 \) which satisfies

\[
|\partial_\xi^a \partial_\xi^b \psi_1(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\beta|}, \quad x \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d,
\]

and a kernel \( K_1(x, z) = FT_\xi \psi_1(x, \xi) \) which satisfies

\[
|\partial_\xi^a \partial_\xi^b K_1(x, z)| \leq C_{\alpha, \beta} |z|^{-d-|\beta|}, \quad x \in \mathbb{R}^d, \quad z \in \mathbb{R}^d \setminus \{0\},
\]

where \( C_{\alpha, \beta} > 0 \) is a constant independent of \( t \) if \( 0 < t \leq 1 \).

Our main result in this section is the following theorem.

**Theorem 5.3.** Let \( T \) be a \( S^0 \) pseudo-differential operator. Then, \( T \) extends to a bounded operator from \( \mathcal{H}^{(q,p)}_{\text{loc}} \) to \( \mathcal{H}^{(q,p)}_{\text{loc}} \), for all \( 0 < q \leq 1 \).

**Proof.** Let \( \delta \geq \left| d \left( \frac{1}{q} - 1 \right) \right| \) be an integer and \( r > \max \{2; p\} \) be a real.

Consider the set \( A_{\text{loc}}(q, r, \delta) \). Let \( f \in \mathcal{H}^{(q,p)}_{\text{loc,fm}} \). Then, there exist a finite sequence \( \{ (a_n, Q_n^r) \}_{n=0}^j \) of elements of \( A_{\text{loc}}(q, r, \delta) \) and a finite sequence of scalars \( \{ \lambda_n^{j} \}_{n=0}^j \) such that \( f = \sum_{n=0}^j \lambda_n^{j} a_n \). Set \( Q^n : = 4 \sqrt{d} Q^n \), \( n \in \{0, 1, \ldots, j\} \), and denote by \( x_n \) and \( \ell_n \) respectively the center and side-length of \( Q^n \). We have

\[
\mathcal{M}_{\text{loc}}(T(f))(x) \leq \sum_{n=0}^j |\lambda_n| \left[ \mathcal{M}_{\text{loc}}(T(a_n))(x) \chi_{Q^n}(x) + \mathcal{M}_{\text{loc}}(T(a_n))(x) \chi_{\mathbb{R}^d \setminus Q^n}(x) \right],
\]

for all \( x \in \mathbb{R}^d \). Hence

\[
\left\| \mathcal{M}_{\text{loc}}(T(f)) \right\|_{q,p} \lesssim I + J,
\]

with

\[
I = \left\| \sum_{n=0}^j |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n)) \chi_{Q^n} \right\|_{q,p} \quad \text{and} \quad J = \left\| \sum_{n=0}^j |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n)) \chi_{\mathbb{R}^d \setminus Q^n} \right\|_{q,p}.
\]
Fix $0 < \eta < q$. Arguing as in the proof of [2], Theorem 4.20, we obtain
\[
I \leq \left( \sum_{n=0}^{j} |\lambda_n|^{\eta} \left( \mathcal{M}_{\text{loc}}(T(a_n))\chi_{Q^n} \right) \right)^{\frac{1}{\eta}} \lesssim \left( \sum_{n=0}^{j} \left( \frac{|\lambda_n|}{\|\chi_{Q^n}\|_q} \right)^{\eta} \chi_{Q^n} \right)^{\frac{1}{\eta}}.
\]

To estimate $J$, it suffices to prove that
\[
\mathcal{M}_{\text{loc}}(T(a_n))(x) \lesssim \frac{[\mathcal{M}(\chi_{Q^n})(x)]^\vartheta}{\|\chi_{Q^n}\|_q},
\]
for all $x \notin \hat{Q}^n$, where $\vartheta = \frac{d+\delta+1}{d}$. To show (5.6), we distinguish two cases: Case $\ell_n < 1$ and Case $\ell_n \geq 1$.

Case: $\ell_n < 1$. Then, the atom $a_n$ satisfies the vanishing condition. Consider $x \notin \hat{Q}^n$. Let $0 < t \leq 1$. We have
\[
|T(a_n) \ast \varphi_t(x)| = \int_{Q^n} K_t(x, x - z)a_n(z)dz \leq \int_{Q^n} K_t(x, x - z)a_n(z)dz \lesssim \frac{[\mathcal{M}(\chi_{Q^n})(x)]^\vartheta}{\|\chi_{Q^n}\|_q},
\]
with $\vartheta = \frac{d+\delta+1}{d}$. Hence
\[
\mathcal{M}_{\text{loc}}(T(a_n))(x) \lesssim \frac{[\mathcal{M}(\chi_{Q^n})(x)]^\vartheta}{\|\chi_{Q^n}\|_q},
\]
with $\vartheta = \frac{d+\delta+1}{d}$, when $\ell_n < 1$.

Case: $\ell_n \geq 1$. Consider $x \notin \hat{Q}^n$. Let $0 < t \leq 1$. We have
\[
|T(a_n) \ast \varphi_t(x)| \leq \int_{Q^n} K_t(x, x - z)a_n(z)dz.
\]
Also, for all $z \in Q^n$, we have
\[
1 < |x - z| \approx |x - x_n|.
\]
Combining (5.7) with Lemma 5.2 (5.4), we have, by [23], (9), p. 235 (also see [32], (0.5.5), p. 16), for all integer $M > 0$, the existence of a constant $C_M > 0$ independent of $t$, such that
\[
|K_t(x, x - z)| \leq C_M|x - z|^{-M} \leq C(M)|x - x_n|^{-M}.
\]
Taking $M = d + \delta + 1$, it follows from (5.8) that
\[ |(T(a_n) \ast \varphi_t)(x)| \leq \left( C(M)|x - x_n|^{-M} \right) \int_{\Omega^n} |a_n(z)| dz \]
\[ \leq \frac{1}{\|\chi_{\Omega^n}\|_q} \frac{\ell_n^{d+\delta+1}}{|x - x_n|^{d+\delta+1}} \]
\[ \leq \frac{1}{\|\chi_{\Omega^n}\|_q} \frac{\ell_n^{d+\delta+1}}{|x - x_n|^{d+\delta+1}} \leq \frac{\|M(\chi_{\Omega^n})(x)\|^\vartheta}{\|\chi_{\Omega^n}\|_q}, \]
with $\vartheta = \frac{d+\delta+1}{d}$, since $\ell_n \geq 1$ implies that $\ell_n^d \leq \ell_n^{d+\delta+1}$. Hence
\[ M_{\text{loc}}(T(a_n))(x) \leq \frac{\|M(\chi_{\Omega^n})(x)\|^\vartheta}{\|\chi_{\Omega^n}\|_q}, \]
with $\vartheta = \frac{d+\delta+1}{d}$, when $\ell_n \geq 1$.

Combining these two cases, we obtain (5.6).

With (5.6), we obtain
\[ J \leq \left( \sum_{n=0}^{j} \left( \frac{|\lambda_n|}{\|\chi_{\Omega^n}\|_q} \right)^\eta \chi_{\Omega^n} \right)^{\frac{1}{\eta}}. \]

Finally,
\[ \|M_{\text{loc}}(T(f))\|_{q,p} \leq \left( \sum_{n=0}^{j} \left( \frac{|\lambda_n|}{\|\chi_{\Omega^n}\|_q} \right)^\eta \chi_{\Omega^n} \right)^{\frac{1}{\eta}}. \]

Thus,
\[ \|T(f)\|_{\mathcal{H}_{\text{loc}}^{(q,p)}} = \|M_{\text{loc}}(T(f))\|_{q,p} \leq \|f\|_{\mathcal{H}_{\text{loc,f}}^{(q,p)}} \leq \|f\|_{\mathcal{H}_{\text{loc}}^{(q,p)}}, \]
by Theorem 3.16. Therefore, $T$ is bounded from $\mathcal{H}_{\text{loc,f}}^{(q,p)}$ to $\mathcal{H}_{\text{loc}}^{(q,p)}$ and the density of $\mathcal{H}_{\text{loc,f}}^{(q,p)}$ in $\mathcal{H}_{\text{loc}}^{(q,p)}$ with respect to the quasi-norm $\|\cdot\|_{\mathcal{H}_{\text{loc}}^{(q,p)}}$ yields the result.

**Corollary 5.4.** Let $T$ be a $\mathcal{S}^0$ pseudo-differential operator. Then, $T$ extends to a bounded operator from $\mathcal{H}_{\text{loc,f}}^{(q,p)}$ to $\mathcal{H}_{\text{loc}}^{(q,p)}$, if $q_1 \leq q$ and $p \leq p_1$, for all $0 < q \leq 1$.

**Proof.** Corollary 5.4 follows from Proposition 2.1 and Theorem 5.3. □

As consequence of the proof of Theorem 5.3, every convolution operator $T$ as in [2], Theorem 4.20, which satisfies in addition a certain condition, can be extended to a bounded operator from $\mathcal{H}_{\text{loc}}^{(q,p)}$ to $\mathcal{H}_{\text{loc}}^{(q,p)}$. More precisely, we have the following result.

**Proposition 5.5.** Let $T$ be a convolution operator, $T(f) = K \ast f$, $f \in \mathcal{S}$, where the distribution $K$ satisfies the following conditions:
(1) \(|\hat{K}(x)| \leq A\),
(2) \(|\partial^\beta \hat{K}(x)| \leq B|x|^{-d-|\beta|}, \text{ for all } |\beta| \leq \left\lfloor d \left(\frac{1}{q} - 1\right)\right\rfloor + 1;
(3) \(|K(x)| \leq C_K|x|^{-\frac{d}{q} - \gamma}, \text{ with } \gamma \geq 1,\]
where \(A > 0, B > 0\) and \(C_K > 0\) are constants independent of \(x\) and \(\beta\). Then, \(T\) extends to a bounded operator from \(\mathcal{H}^{(q,p)}_{\text{loc}}\) to \(\mathcal{H}^{(q,p)}_{\text{loc}}\).

**Proof.** Conditions (1) and (2) imply that \(T\) is bounded on \(L^r\), for any \(1 < r < +\infty\), by [2], Remark 4.14 and [10], Theorem 5.1. Set \(\delta = \left\lfloor d \left(\frac{1}{q} - 1\right)\right\rfloor\). Let \(r > \max\{2; p\}\) be a real. Consider the set \(A_{\text{loc}}(q, r, \delta)\). Let \(f \in \mathcal{H}^{(q,p)}_{\text{loc, fin}}\). Then, there exist a finite sequence \(\{(a_n, Q^n)\}_{n=0}^{j}\) in \(A_{\text{loc}}(q, r, \delta)\) and a finite sequence of scalars \(\{\lambda_n\}_{n=0}^{j}\) such that \(f = \sum_{n=0}^{j} \lambda_n a_n\). Set \(\tilde{Q}^n := 4\sqrt{d}Q^n, n \in \{0, 1, \ldots, j\}\), and denote by \(x_n\) and \(\ell_n\) respectively the center and side-length of \(Q^n\). We have
\[
\|\mathcal{M}_{\text{loc}}(T(f))\|_{q,p} \lesssim I + J,
\]
with
\[
I = \left\| \sum_{n=0}^{j} |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n))\chi_{\tilde{Q}^n} \right\|_{q,p} \quad \text{and} \quad J = \left\| \sum_{n=0}^{j} |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n))\chi_{\tilde{Q}^n} \right\|_{q,p}.
\]
Fix \(0 < \eta < q\). As in the proof of Theorem 5.3, we have
\[
I \lesssim \left\| \sum_{n=0}^{j} \left( \frac{|\lambda_n|}{\|\chi_{Q^n}\|_q} \right)^{\eta} \chi_{Q^n} \right\|_{q,p}^{\frac{1}{\eta}}.
\]
To estimate \(J\), it suffices to show that
\[
\mathcal{M}_{\text{loc}}(T(a_n))(x) \lesssim \left( \frac{\mathcal{M}(\chi_{Q^n})(x)}{\|\chi_{Q^n}\|_q} \right)^{\vartheta},
\]
for all \(x \notin \tilde{Q}^n\), where \(\vartheta = \frac{d+\delta+1}{d}\). To prove (5.9), we distinguish two cases, as in the proof of Theorem 5.3.

Case: \(\ell_n < 1\). Consider \(x \notin \tilde{Q}^n\). For all \(0 < t \leq 1\), we put
\(K^{(t)} := K * \varphi_t\).

We have
\[
\sup_{0 < t \leq 1} |\hat{K}^{(t)}(z)| \leq \|\hat{\varphi}\|_\infty A,
\]
for all \(z \in \mathbb{R}^d\), and
\[
\sup_{0 < t \leq 1} |\partial_\xi^\beta K^{(t)}(z)| \leq C_{\varphi, A, B, \delta, d} \frac{|z|^{d+|\beta|}}{|z|^{d+|\beta|}},
\]
for all \( z \neq 0 \) and all multi-indices \( \beta \) with \( |\beta| \leq \delta + 1 \). The proof of (5.10) and (5.11) is similar to the one of (4.41) and (4.42) of the proof of [2], Theorem 4.20, we omit details. With (5.11), we obtain

\[
\mathcal{M}_{\text{loc}}(T(a_n))(x) \lesssim \left( \frac{\|\mathcal{M}(\chi_{Q^n})(x)\|}{\|\varphi\| q} \right)^{\vartheta},
\]

where \( \vartheta = \frac{d+\delta+1}{d} \).

Case: \( \ell_n \geq 1 \). Consider \( x \not\in \tilde{Q}^n \). For all \( 0 < t \leq 1 \), we have

\[
|(T(a_n) \ast \varphi_t)(x)| \leq \int_{Q^n} |K^{(t)}(x-z)||a_n(z)|dz.
\]

Furthermore, for all \( z \in Q^n \), we have

\[
(5.12) \quad \frac{3}{2} < |x-z| \approx |x-x_n|
\]

and

\[
(5.13) \quad |K^{(t)}(x-z)| \leq C(\varphi, d, q, \gamma, C_K)|x-z|^{-\frac{d}{\delta} - \gamma},
\]

where \( C(\varphi, d, q, \gamma, C_K) > 0 \) is a constant independent of \( t, x \) and \( z \). We prove a general version of (5.13), namely,

\[
(5.14) \quad |K^{(t)}(z)| \leq C(\varphi, d, q, \gamma, C_K)|z|^{-\frac{d}{\delta} - \gamma},
\]

for all \( |z| \geq \frac{3}{2} \). For the proof of (5.14), consider \( z \in \mathbb{R}^d \) such that \( |z| \geq \frac{3}{2} \). Then,

\[
K^{(t)}(z) = \int_{\mathbb{R}^d} K(z-y)\varphi_t(y)dy = \int_{|y|<t} K(z-y)\varphi_t(y)dy,
\]

since \( \text{supp}(\varphi) \subset B(0,1) \) and \( |z-y| \geq |z| - |y| > \frac{3}{2} - t > 0 \), for all \( |y| < t \). Hence

\[
|K^{(t)}(z)| \leq \int_{|y|<t} \frac{C_K}{|z-y|^{\frac{d}{\delta} + \gamma}} |\varphi_t(y)|dy,
\]

by (3). But, for all \( |y| < t \), we have

\[
|z-y| \geq |z| - |y| > |z| - \frac{2}{3}|z| = \frac{1}{3}|z|,
\]

since \( |z| \geq \frac{3}{2} \geq \frac{3}{2} t > \frac{3}{2} |y| \). Therefore,

\[
|K^{(t)}(z)| \leq 3^{\frac{d}{\delta} + \gamma} \frac{C_K}{|z|^{\frac{d}{\delta} + \gamma}} \int_{|y|<t} |\varphi_t(y)|dy
\]

\[
\leq C(\varphi, d, q, \gamma, C_K)|z|^{-\frac{d}{\delta} - \gamma},
\]

with \( C(\varphi, d, q, \gamma, C_K) := 3^{\frac{d}{\delta} + \gamma} \|\varphi\|_1 C_K > 0 \). This establishes (5.14). Then, (5.13) immediately follows from (5.14), by (5.12).
With (5.12) and (5.13), by proceeding as in the proof of Theorem 5.3, we obtain

\[ |(T(a_n) \ast \varphi_t)(x)| \lesssim \frac{\ell_n^d}{\|\chi_Q^n\|_q} \frac{|x - x_n|^{\frac{d}{q} + \gamma}}{|x - x_n|^{\frac{d}{q} + \gamma}} \lesssim \frac{\ell_n^{d+\delta+1}}{\|\chi_Q^n\|_q} \frac{|x - x_n|^{d+\delta+1}}{|x - x_n|^{d+\delta+1}}, \]

with \( \vartheta = \frac{d+\delta+1}{d} \), since \( d + \delta + 1 \leq \frac{d}{q} + \gamma \) and \( |x - x_n| > 1 \). Hence

\[ \mathcal{M}_{\text{loc}}(T(a_n))(x) \lesssim \frac{[\mathcal{M}(\chi_Q^n)(x)]^\vartheta}{\|\chi_Q^n\|_q}, \]

with \( \vartheta = \frac{d+\delta+1}{d} \).

Combining these two cases, we obtain (5.9).

With (5.9), we end as in the proof of Theorem 5.3.

In Proposition 5.5, the hypothesis on \( \gamma \) can be weakened by taking, in (3),

\[ \gamma > d \left( \frac{1}{q} - 1 \right) - \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor. \]

(5.15)

In fact, when \( \gamma \geq 1 \), we clearly have (5.15), since

\[ 0 \leq d \left( \frac{1}{q} - 1 \right) - \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor < 1. \]

We are going to prove Proposition 5.5 under Condition (5.15). The proof is similar to the previous, we only present points which change.

**Proof.** With the same elements of the previous proof, only the estimation of \( J \) presents some modifications. Endeed, in Case : \( \ell_n \geq 1, \)

\[ \mathcal{M}_{\text{loc}}(T(a_n))(x) \leq C(\varphi, d, q, \gamma, C_K) \frac{[\mathcal{M}(\chi_Q^n)(x)]^\vartheta}{\|\chi_Q^n\|_q}, \]

with \( \vartheta = \frac{d+\delta+\gamma}{d} \). Thus, the expression of \( \vartheta \) is no longer the same in the two cases (Case: \( \ell_n < 1 \) and Case: \( \ell_n \geq 1 \)). And hence, we have no longer (5.9), for all \( x \not\in \widetilde{Q}^n, n \in \{0, 1, \ldots, j\} \). To overcome the problem, we write

\[ \sum_{n=0}^{j} |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n)) \chi_{x^n \setminus \widetilde{Q}^n} = \sum_{n: \ell_n < 1} |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n)) \chi_{x^n \setminus \widetilde{Q}^n} + \sum_{n: \ell_n \geq 1} |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n)) \chi_{x^n \setminus \widetilde{Q}^n}. \]
Thus, $J \lesssim J_1 + J_2$, with $J_1 = \left\| \sum_{n: \epsilon_n < 1} |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n)) \chi_{\mathbb{R}^d \setminus \mathcal{Q}_n} \right\|_{q,p}$ and $J_2 = \left\| \sum_{n: \epsilon_n \geq 1} |\lambda_n| \mathcal{M}_{\text{loc}}(T(a_n)) \chi_{\mathbb{R}^d \setminus \mathcal{Q}_n} \right\|_{q,p}$. We have

$$J_1 \lesssim \left\| \sum_{n: \epsilon_n < 1} \left| \lambda_n \right| \frac{\mathcal{M}(\chi_{\mathcal{Q}_n})}{\| \chi_{\mathcal{Q}_n} \|_q} \right\|_{q,p} \lesssim \left\| \sum_{n: \epsilon_n < 1} \left( \frac{|\lambda_n|}{\| \chi_{\mathcal{Q}_n} \|_q} \right)^{\vartheta} \chi_{\mathcal{Q}_n} \right\|_{q,p} \leq \left\| \sum_{n=0}^j \left( \frac{|\lambda_n|}{\| \chi_{\mathcal{Q}_n} \|_q} \right)^{\vartheta} \chi_{\mathcal{Q}_n} \right\|_{q,p},$$

where $\vartheta = \frac{d+\delta+1}{d}$. Likewise,

$$J_2 \lesssim \left\| \sum_{n: \epsilon_n \geq 1} \left| \lambda_n \right| \frac{\mathcal{M}(\chi_{\mathcal{Q}_n})}{\| \chi_{\mathcal{Q}_n} \|_q} \right\|_{q,p} \lesssim \left\| \sum_{n=0}^j \left( \frac{|\lambda_n|}{\| \chi_{\mathcal{Q}_n} \|_q} \right)^{\vartheta} \chi_{\mathcal{Q}_n} \right\|_{q,p},$$

where $\vartheta = \frac{d+\delta+\gamma}{d}$. Hence

$$J \lesssim \left\| \sum_{n=0}^j \left( \frac{|\lambda_n|}{\| \chi_{\mathcal{Q}_n} \|_q} \right)^{\vartheta} \chi_{\mathcal{Q}_n} \right\|_{q,p}.$$  

And we end as in the previous proof. □

Another consequence of Theorem 5.3 is that $\mathcal{H}_{\text{loc}}^{(q,p)}$ ($0 < q \leq 1, q \leq p$) is stable under multiplication by the Schwartz class $\mathcal{S}$. Endeed, let $\phi \in \mathcal{S}$. The operator $T$ defined by

$$T(f)(x) = \phi(x) f(x), \ x \in \mathbb{R}^d,$$

for all $f \in \mathcal{S}$, is clearly a $\mathcal{S}^0$ pseudo-differential operator. Therefore, by Theorem 5.3, $T$ extends to a bounded operator on $\mathcal{H}_{\text{loc}}^{(q,p)}$. Thus, we can define by extension the product of an element $f \in \mathcal{H}_{\text{loc}}^{(q,p)}$ and a function $\phi \in \mathcal{S}$ such that the product also denoted by $\phi f \in \mathcal{H}_{\text{loc}}^{(q,p)}$.

This result extends the one of [14] to $\mathcal{H}_{\text{loc}}^{(q,p)}$ spaces, for $0 < q \leq 1$ and $q \leq p$.

Let $T_\psi$ be the $\mathcal{S}^0$ pseudo-differential operator associated with $\psi \in \mathcal{S}^0$. By [29], (4), p. 233,

$$(5.16) \quad \langle T_\psi f, g \rangle = \langle f, (T_\psi)^* g \rangle,$$

for all $f, g \in \mathcal{S}$, where $(f, g)$ denotes $\int_{\mathbb{R}^d} f(x) g(x) dx$ and $(T_\psi)^*$ is the adjoint of $T_\psi$. Also, there exists $\psi^* \in \mathcal{S}^0$ such that $(T_\psi)^* = T_{\psi^*}$, by [29], Proposition, p. 259. Combining these facts with Theorems 5.3 and 13 we have the following result.
Corollary 5.6. Suppose that $0 < q \leq p \leq 1$. Let $\delta \geq \left\lfloor d \left( \frac{1}{q} - 1 \right) \right\rfloor$ be an integer, $1 < r \leq +\infty$ and $T$ be a $S^0$ pseudo-differential operator. Then, there exists a positive constant $C$ such that, for all $f \in L^r_{r', \phi_1, \delta}^\text{loc}$,

$$\|T(f)\|_{L^r_{r', \phi_1, \delta}^\text{loc}} \leq C \|f\|_{L^r_{r', \phi_1, \delta}^\text{loc}}.$$ 

REFERENCES

[1] Z. V. d. P. Able, J. Feuto, Atomic decomposition of Hardy-amalgam spaces, J. Math. Anal. Appl. 455 (2017), 1899-1936.
[2] Z. V. d. P. Able, J. Feuto, Duals of Hardy-amalgam spaces $H^{(q,p)}$ and boundedness of operators, preprint.
[3] W. Abu-Shammala and A. Torchinsky, The Hardy-Lorentz spaces $H^{p,q}$, Studia Math. 182 3 (2007), 283-294.
[4] J. P. Bertrandias, C. Datry and C. Dupuis, Unions et intersections desespaces $L^p$ invariants par translation ou convolution, Ann. Inst. Fourier (Grenoble), 28 (1978), 53-84.
[5] M. Bownik, Anisotropic Hardy spaces and waveletes, Mem. Amer. Math. Soc., 164 (2003).
[6] M. Bownik, Baode Li, Dachun Yang, and Yuan Zhou, Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators, Indiana Univ. Math. J., 57 (2008) 3065-3100.
[7] R. C. Busby and H. A. Smith, Product-convolution operators and mixed-norm spaces, Trans. Amer. Math. Soc., 263 (1981), 309-341.
[8] C. Carton-Lebrun, H. P. Heinig and S. C. Hofmann, Integral operators on weighted amalgams, Studia Math., 109 (1994), 133-157.
[9] D. Cruz-Uribe, Sfo and L. D. Wang, Variable Hardy spaces, Indiana Univ. Math. J., 63 (2014), 447-493.
[10] J. Duoandikoetxea, Fourier Analysis, Grad. Stud. Math., vol. 29, American Mathematical Society, Providence, RI, (2001), translated and revised from the 1995 Spanish original by D. Cruz-Uribe.
[11] C. Fefferman, E. M. Stein, $H^p$ spaces of several variables, Acta. Math., 129 (1972), 137-193.
[12] J. J. F. Fournier and J. Stewart, Amalgams of $L^p$ and $l^p$, Bull. Amer. Math. Soc., 13 (1) (1985), 1-21.
[13] J. García-Cuerva, Weighted $H^p$ spaces. Dissertationes Math., 162 (1979), 1-63.
[14] D. Goldberg, A local version of real Hardy spaces, Duke Math. Journal., 46 (1979), 27-42.
[15] L. Grafakos, Classical Fourier analysis, Second edition. Graduate Texts in Mathematics, 250. Springer, New York, 2009.
[16] L. Grafakos, Modern Fourier analysis, Second edition. Graduate Texts in Mathematics, 250. Springer, New York, 2009.
[17] G. Hoepfner, Hardy spaces, its variants and applications, IV Workshop on Geometric Analysis of PDE and Several Complex Variables, Serra Negra-SP, Brazil, (2007).
[18] F. Holland, Harmonic analysis on amalgams of $L^p$ and $l^q$, J. London Math. Soc., 2 (10) (1975), 295-305.
[19] J. Houyu, W. Henggeng, Decomposition of Hardy-Morrey spaces, J. Math. Anal. Appl. 354 (2009), 99-110.
DUAL OF $H_{loc}^{(q,p)}$ AND PSEUDO-DIFFERENTIAL OPERATORS

[20] L. D. Ky, New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators, Integral Equations Operator Theory 78 (2014), no. 1, 115-150.

[21] R. H. Latter, A characterization of $H^p(\mathbb{R}^d)$ in terms of atoms, studia Math., 62 (1978), 93-101.

[22] Y. Liang, Y. Sawano, T. Ulrich, D. Yang and W. Yuan, A new framework for generalized Besov-type and Triebel-Lizorkin-type spaces, Dissertationes Math. (Rozprawy Mat.). 489 (2013), 1-114.

[23] S. Z. Lu, Four Lectures on Real $H^p$ Spaces, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[24] E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, J. Funct. Anal., 262, (2012), 3665-3748.

[25] E. Nakai and Y. Sawano, Orlicz-Hardy spaces and their dual, Sci. China Math., 62 (2014), 903-962, doi: 10.1007/s11425-014-4798-y.

[26] M. Ruzhansky, V. Turunen, Pseudo-differential operators and symmetries: Background Analysis and Advanced Topics, Vol. 2 of Pseudo-Differential Operators Theory and Applications, Birkhäuser Verlag AG, Basel, 2010.

[27] Y. Sawano Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators, Integral Equations Operator Theory, 77 (2013), 123-148.

[28] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, N. J., 1970.

[29] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integral, Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[30] J. Stewart, Fourier transforms of unbounded measures, Canad. J. Math., 31 (1979), 1281-1292.

[31] L. Tang, Weighted local Hardy spaces and their applications, Illinois J. Math., 56 (2012), 453-495 (or arXiv: 1004.5294).

[32] M. E. Taylor, Pseudodifferential operators and nonlinear PDE, Progress in Mathematics, 100, Birkhäuser, Boston, 1991.

[33] H. Triebel, Theory of Function Spaces. Basel: Birkhäuser Verlag, 1983.

[34] H. Triebel, Theory of Function Spaces II. Basel: Birkhäuser Verlag, 1992.

[35] N. Wiener, On the representation of functions by trigonometric integrals, Math. Z., 24 (1926), 575-616.

[36] M. W. Wong, An Introduction to Pseudo-Differential Operators, Second Edition, World Scientific, 1999.

[37] D. Yang, S. Yang, Weighted local Orlicz-Hardy spaces with applications to pseudo-differential operators, Dissertationes Math. (Rozprawy Mat.), 478 (2011), 1-78.

[38] D. Yang, S. Yang, Local Hardy spaces of Musielak-Orlicz type and their applications. Sci China Math, 2012, 55, doi: 10.1007/s11425-000-0000-0.

Laboratoire de Mathématiques Fondamentales, UFR Mathématiques et Informatique, Université Félix Houphouët-Boigny Abidjan-Cocody, 22 B.P 582 Abidjan 22. Côte d’Ivoire
E-mail address: vincentdepaulzobo@yahoo.fr

Laboratoire de Mathématiques Fondamentales, UFR Mathématiques et Informatique, Université Félix Houphouët-Boigny Abidjan-Cocody, 22 B.P 1194 Abidjan 22. Côte d’Ivoire
E-mail address: justfeuto@yahoo.fr