Level crossing of fermions coupled with the $\mathbb{C}P^N$ Skyrme-Faddeev model

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Abstract. The extended Skyrme-Faddeev model possesses vortex solutions with target space $\mathbb{C}P^N$. In terms of the coupling with fermions, the spectral flow and the level crossing occurs which corresponds to fermionic quantum number nonconservation.

1. Introduction

The Skyrme-Faddeev model is an example of a field theory that supports the finite-energy knotted solitons. The classical soliton solutions of the Skyrme-Faddeev model can play a role of adequate normal models useful in description of the strong coupling sector of the Yang-Mills theory. It has been shown that also in the case of the complex projective target space $\mathbb{C}P^N$ the Skyrme-Faddeev type model possesses an infinite number of exact soliton solutions in the integrable sector if coupling constants satisfy a special relation [1]. The existence of vortex solutions of the model outside the integrable sector has been confirmed numerically for appropriate choice of potentials [2].

It is widely known that quantum aspects of the soliton solutions exhibit a special property (‘fractional’ spin-statistics) when the Hopf term (theta term) is included in the action of the model [3]. Since $\Pi_3(\mathbb{C}P^1) = \mathbb{Z}$, then such a term became the Hopf invariant and therefore it can be represented as a total derivative which has no influence on classical equations of motion. On the other side, since $\Pi_4(\mathbb{C}P^1)$ is trivial, the coupling constant (prefactor) $\Theta$ is not quantized. As shown in [3], when the Hopf lagrangian is included in the model, the solitons with unit topological charge possess spin $\frac{\Theta}{2\pi}$, which can be fractional. For a fermionic model coupled with a $\mathbb{C}P^N$ valued field, $\Theta$ can be determined at least perturbatively [4, 5].

For $N > 1$, $\Pi_3(\mathbb{C}P^N)$ is already trivial and then the Hopf term is perturbative, i.e., it is not a $y$ invariant, what in general means that the contribution from the term may always be fractional even though one choose an integer $n$ in the anyon angle $\Theta = n\pi$. It was pointed out in [6] that an analogue of the Wess-Zumino-Witten term appears in $\mathbb{C}P^N$ field which plays a similar role as the Hopf term for $N = 1$ [7]. As a result, the soliton can be quantized as an anyon with statistics angle $\Theta$ and also such Hopf-like term.

In this paper we solve the fermionic model coupled with the $\mathbb{C}P^N$ Skyrme-Faddeev model. Basic property of localizing mode of fermions on a topological soliton is understood in terms of basis of Atiyah-Singer Index theorem [8]. The index for the Dirac operator $D$ may be defined as $\dim \ker D - \dim \ker D^\dagger$, which is related with the Casimir energy of the fermions. The spectral flow analysis in the chiral invariant model (the Skyrme model) is a simple realization of the
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In terms of this parametrization, the lagrangian (1) reads

\[ L_{\text{SF}} = \frac{M^2}{2} \text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) + \frac{1}{e^2} \text{Tr}([\partial_\mu \Phi, \partial_\nu \Phi]^2) \]

where the symbols $M$ is a coupling constant with dimension of mass whereas the coupling constants $e^{-2}$, $\beta$, $\gamma$ are dimensionless. $V$ denotes a potential term which contains no derivative term and does not break local symmetries of the model. For the field $\Phi$, we employ the following parametrization with $N$ complex fields $u := \{u_i\}$, $i = 1, \ldots, N$

\[ \Phi = I_{N+1 \times N+1} + \frac{2}{\sqrt{2}} \left( \begin{array}{cc} -u \otimes u^\dagger & iu \\ -iu^\dagger & -1 \end{array} \right), \quad \eta := \sqrt{1 + u^\dagger \cdot u} \]

In terms of this parametrization, the lagrangian (1) reads

\[ L_{\text{SF}} = -\frac{1}{2} \left[ M^2 \eta_{\mu \nu} + C_{\mu \nu} \right] \tau^{\mu \nu} - \mu^2 V \]

where the symbols $C_{\mu \nu}$ and $\tau_{\mu \nu}$ are defined as follows

\[ C_{\mu \nu} := M^2 \eta_{\mu \nu} - \frac{4}{e^2} \left[ (\beta e^2 - 1) \tau^{\mu \nu} \eta_{\mu \nu} + (\gamma e^2 - 1) \tau_{\mu \nu} + (\gamma e^2 + 2) \tau_{\mu \nu} \right], \quad (3) \]

\[ \tau_{\mu \nu} := -\frac{4}{\sqrt{2}} \left[ \partial^2 \partial_\nu u^\dagger \cdot \partial_\mu u - (\partial_\nu u^\dagger \cdot u)(u^\dagger \cdot \partial_\mu u) \right]. \quad (4) \]

A variation with respect to $u_i^*$ leads to the equations which can be cast in the form

\[ (1 + u^\dagger \cdot u) \partial^\mu (C_{\mu \nu} \partial^\nu u_i) - C_{\mu \nu} \left[ (u^\dagger \cdot \partial^\mu u) \partial^\nu u_i + (u^\dagger \cdot \partial^\nu u)(\partial^\mu u_i) \right] \]

\[ + \frac{\mu^2}{4} (1 + u^\dagger \cdot u)^2 \sum_{k=1}^N \left( \delta_{ik} + u_i u_k^* \right) \frac{\delta V}{\delta u_k} = 0 \]

where we have already multiplied the resulting equations by inverse of $\Delta_{ij}^2$, i.e., $\Delta_{ij}^{-2} = \frac{1}{1 + u^\dagger \cdot u}(\delta_{ij} + u_i u_j^*)$. We shall discuss some examples of the potential in the further part of the paper. In the simplest case when the potential is a function of absolute values of the fields $V(|u_1|^2, \ldots, |u_N|^2)$ the contribution from the potential becomes

\[ \sum_{k=1}^N \left( \delta_{ik} + u_i u_k^* \right) \frac{\delta V}{\delta u_k} = u_i \left[ \frac{\delta V}{\delta |u_i|^2} + \sum_{k=1}^N |u_k|^2 \frac{\delta V}{\delta |u_k|^2} \right]. \]

2. $\mathbb{C}P^N$ Skyrme-Faddeev type model

We briefly sketch the Skyrme-Faddeev model on the target space $\mathbb{C}P^N$. We introduce the lagrangian of the form

\[ L_{\text{SF}} = \frac{M^2}{2} \text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) + \frac{1}{e^2} \text{Tr}([\partial_\mu \Phi, \partial_\nu \Phi]^2) \]

but generalization to higher $N$ is straightforward.
We introduce the dimensionless coordinates \((t, \rho, \phi)\) defined as
\[
x^0 = r_0 t, \quad x^1 = r_0 \rho \cos \phi, \quad x^2 = r_0 \rho \sin \phi
\]
where the length scale \(r_0\) is defined in terms of coupling constants \(M^2\) and \(e^2\), i.e.,
\[
r_0^2 = -\frac{4}{M^2 e^2}
\]
and the light speed is \(c = 1\) in the natural units. The linear element \(ds^2\) reads
\[
ds^2 = r_0^2 (dt^2 - d\rho^2 - \rho^2 d\phi^2).
\]

We shall consider the following ansatz
\[
u_j = f_j(\rho) e^{i n_j \phi}
\]
The constants \(n_i\) form the set of integer numbers. We define the diagonal matrix
\[
\text{diag}(n_1, \ldots, n_N)
\]
in order to simplify the form of some formulas below. The expressions \(\tau_{\mu\nu}\) have the following form
\[
\tau_{\rho\rho} \equiv \theta(\rho) = -\frac{4}{\sqrt{1 + \frac{1}{\rho^2}}} \left[ \vartheta^2 (f^T \cdot f') - (f^T \cdot f)(f^T \cdot f') \right]
\]
\[
\tau_{\phi\phi} \equiv \omega(\rho) = -\frac{4}{\sqrt{1 + \frac{1}{\rho^2}}} \left[ \vartheta^2 (f^T \cdot \lambda^2 \cdot f) - (f^T \cdot \lambda \cdot f)^2 \right]
\]
\[
\tau_{\phi\rho} = -\tau_{\rho\phi} \equiv i \zeta(\rho) = -i \frac{4}{\sqrt{1 + \frac{1}{\rho^2}}} \left[ \vartheta^2 f^T \cdot \lambda \cdot f - (f^T \cdot \lambda \cdot f)(f^T \cdot f) \right]
\]
where derivative with respect to \(\rho\) is denoted by \(\frac{d}{d\rho} = '\) and \(T\) stands for matrix transposition.

The equations of motion written in dimensionless coordinates take the form
\[
(1 + f^T \cdot f) \left[ \frac{1}{\rho} \left( \rho \tilde{C}_{\rho\rho} f'_k \right) ' + \frac{i}{\rho} \left( \tilde{C}_{\phi\rho} \right) ' \left( \lambda \cdot f \right) k - \frac{1}{\rho^2} \tilde{C}_{\phi\phi} \left( \lambda^2 \cdot f \right) k \right]
\]
\[
-2 \left[ \tilde{C}_{\rho\rho} (f^T \cdot f') f'_k - \frac{1}{\rho^2} \tilde{C}_{\phi\phi} (f^T \cdot \lambda \cdot f)(\lambda \cdot f) k \right] + \tilde{\mu}^2 f_k \left( 1 + f^T \cdot f \right) 2 \left[ \frac{\delta V}{\delta f_k} + \sum_{i=1}^{N} f_i^2 \frac{\delta V}{\delta f_i^2} \right] = 0
\]
for each \(k = 1, \ldots, N\), where we have introduced the symbols \(\tilde{C}_{\mu\nu} := \frac{1}{\tilde{\mu}^2} C_{\mu\nu}\), and also \(\tilde{\mu}^2 := \frac{\sqrt{2}}{4\pi} \mu^2\). The components \(\tilde{C}_{\mu\nu}\) which appear in the equations of motion read
\[
\tilde{C}_{\rho\rho} = -1 + (\beta e^2 - 1) \left( \theta + \frac{\omega}{\rho^2} \right) + (2\gamma e^2 + 1)\theta
\]
\[
\tilde{C}_{\phi\phi} = -\rho^2 + \rho^2 (\beta e^2 - 1) \left( \theta + \frac{\omega}{\rho^2} \right) + (2\gamma e^2 + 1)\omega
\]
\[
\tilde{C}_{\phi\rho} = -\tilde{C}_{\rho\phi} = -3i\zeta .
\]

In the numerical computation it is useful to introduce the scaled coordinate and the variables
\[
\rho = \sqrt{\frac{1-y}{y}}, \quad f_j = \frac{1}{\sqrt{N}} \left( \frac{1-g_i}{g_j} \right)
\]
where $y \in (0, 1]$ and $g_j \in (0, 1]$. According to the discussions in [10] and also in [1], we can define the topological charge in the present model. The field $u_i$ provide a mapping from $x^1 x^2$ plane into $\mathbb{C} P^N$. However, for the finiteness of the energy, the field should go to a constant at space infinity. Then the plane is topologically compactified into $S^2$ and the finite energy field configuration define the mapping $S^2 \to \mathbb{C} P^N$ which is classified into the homotopy classes of $\pi_2(\mathbb{C} P^N)$. There exists a theorem describing in [10], $\pi_2(G/H) = \pi_1(H)_G$ where $\pi_1(H)_G$ is the subset of $\pi_1(H)$ formed by closed paths in $H$ which can be contracted to a point in $G$. Thus, in the present case, the homotopy group is given by

$$\pi_2(\mathbb{C} P^N) = \pi_1(SU(N) \otimes U(1))_{SU(N+1)} = \mathbb{Z}. \quad (12)$$

The topological charge, an element of the homotopy group, is given by the integral of the topological current defined in terms of the field $\Phi$ as

$$j^\mu(\Phi) = \frac{i}{16\pi} e^{\mu\lambda} \text{Tr}(\Phi \partial_\lambda \Phi \partial_\mu \Phi). \quad (13)$$

As discussed in [11, 1], in fact, the topological charge is equal to the number of poles of $u_i$, including those at infinity. Since the solutions behave as a holomorphic function near boundaries, i.e., $u_i \rho^n e^{im}$ near the origin and at space infinity where $n_i \in \mathbb{Z}$, the topological charge is given by

$$Q_{\text{top}} = n_{\text{max}} \pm |n_{\text{min}}| \quad (14)$$

where the highest positive integer in the set $n_i, i = 1, 2, \ldots, N$ and $n_{\text{min}}$ is the lowest negative integer in the same set.

Now we introduce the explicit form of the potential term. Generally speaking, potential terms are a function of fields, which should vanish at space infinity and preserve the local symmetries of the model. In this model, the simplest choice is the old baby type potential $\text{Tr}(1 - \Phi^{-1} \Phi)$ where $\Phi_\infty$ is value of the field at space infinity, i.e., $\Phi_\infty := \lim_{\rho \to \infty} \Phi(\rho)$. By assuming that the solution and its holomorphic counterpart have the same asymptotic behaviour at the spatial infinity, one gets that inverse of the principal variable $\Phi$ goes to $\Phi^{-1} := \text{diag}(1, 1, 1)$, or $\Phi^{-1} := \text{diag}(1, 1, -1)$, then the expression $\text{Tr}(1 - \Phi^{-1} \Phi)$ can be included as the “new-baby” potential which has two vacua [12]. Finally, the following expression can be considered as a general form of the potential

$$V = \frac{[\text{Tr}(1 - \Phi_0^{-1} \Phi)]^a [\text{Tr}(1 - \Phi^{-1} \Phi)]^b}{(1 + |u_1|^2 + |u_2|^2)^{a+b}} = \frac{(g_1 + g_2 - 2g_1g_2)^a g_1^b (1 + g_2)^b}{(g_1 + g_2)^{a+b}}$$

where the integers $a, b$ satisfy $a \geq 0, b > 0$.

Assuming that for $n_2 < 0$ the field $u_2$ behaves at zero as its holomorphic counterpart, i.e., $\sim \rho^{n_2}$ one gets that it tends to diverge as $\rho \to 0$. Then inverse of the principal variable $\Phi$ goes to $\Phi^{-1} := \text{diag}(1, -1, 1)$ as $\rho \to 0$. The general form of the potential takes the form

$$V = \frac{(1 + |u_1|^2)^a (1 + |u_2|^b)}{(1 + |u_1|^2 + |u_2|^2)^{a+b}} = \frac{g_1^b g_1^a (1 + g_2)^b}{(g_1 + g_2)^{a+b}}$$

where the integers satisfy $a \geq 0, b > 0$.

We show results for the solutions of the topological charges $Q_{\text{top}} = 3, 4$ in Fig.1. We employ the potential of $a = 0, b = 2$ of (15), or (16).
Figure 1. The plot of the solutions $g_1, g_2$ with the potential $(a, b) = (0, 2)$. The other parameters are $(\beta e^2, \gamma e^2, \mu^2) = (2.0, 2.0, 1.0)$.

3. Normalizable modes of fermions

The fermion-vortex system was first studied by Jackiw and Rossi in [13]. For the target space $\mathbb{C}P^N$, fermions with chiral symmetry coupled with the soliton was initially discussed in [10]. It was confirmed that the normalizable zero mode of the fermion appears thanks to the Index theorem.

We consider a gauged model corresponding to (1)

$$L_{\text{fermi}} = \bar{\psi} \gamma^\alpha (\partial_\alpha - iA_\alpha) \psi - m \bar{\psi} \Phi \psi \equiv \bar{\psi} iD_A \psi$$

where $\alpha = 1, 2, 3$. The gamma matrices are the standard prescription such that $\gamma^1 = -i\sigma_1, \gamma^2 = -i\sigma_2, \gamma^3 = \sigma_3$ where $\sigma_\alpha$ are standard Pauli matrices. Under an appropriate rescaling of the lagrangian, i.e., $\psi \rightarrow r_0^{-1} \psi, A_\alpha \rightarrow r_0^{-1} A_\alpha$, $m \rightarrow r_0^{-1} m$, the system moves to the dimensionless one.

The Euclidean partition function is

$$Z = e^{\omega(\Phi)} = \int D\psi D\bar{\psi} \exp(\bar{\psi} iD_A \psi) = \det iD_A.$$

We separate the effective action $\omega(\Phi)$ into real and imaginary part such that

$$\omega(\Phi) = \text{Tr} \ln(iD_A) = \omega(\Phi)_{\text{Re}} + \omega(\Phi)_{\text{Im}}$$

$$\omega(\Phi)_{\text{Re}} = \frac{1}{2} \text{Tr} \ln D_A^\dagger D_A,$$

$$\omega(\Phi)_{\text{Im}} = \frac{1}{2i} \text{Tr} \ln \frac{iD_A}{(iD_A)^\dagger}.$$

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Figure 2. The spectral flow corresponding to the solutions of Fig. 1. For $\lambda = 0$, the background field is the vacuum $\Phi_\infty$ and for $\lambda = 1$, it becomes the solitonic ones $\Phi_i$. We plot the first 34 levels (which means 17 positive and 17 negative energy levels at the vacuum). The coupling constant is chosen as $m = 2.0$. The 2 levels of the zero crossing in the case of $(n_1, n_2) = (3, 1)$ and the 3 levels in the $(2, -2)$ take very close values but are not degenerate ones.

It is well known that for a fermionic effective model coupled to the baby-skyrmion with a constant gap $mU$ the integrating out the Dirac field leads to an effective lagrangian containing
a baby-Skyrme like model and some topological terms including the Hopf term \([4, 5]\). A number of articles extensively describe the derivative expansion of the effective action \(H\) that appears in \(Z = \exp(\omega(\Phi))\). It contains both the action of the model (in the real part) and the topological terms (in the imaginary part). After a bit lengthy calculation one gets

\[
\omega(\Phi)_{\text{Re}}|_{A=0} = n_c \frac{|m|}{8\pi} \int d^3 x \text{Tr}(\partial_\mu \Phi \partial^\mu \Phi) + O(\partial^4 \Phi),
\]

\[
\omega(\Phi)_{\text{Im}} = -n_c \int d^3 x \left( j^\mu(\Phi) A_\mu + \pi \text{sgn}(m) L_{\text{Hopf}}(\Phi) \right).
\]

where \(n_c\) is a degeneracy of the fermions. The explicit form of the current \(j^\mu\) coincides with (13). Consequently, as pointed out in \([7, 4, 5]\), the anyon angle \(\Theta\) is determinable in this fermionic context. One can fix the anyon angle \(\Theta\) as \(\Theta \equiv \pi \text{sgn}(m)\) provided that the vortices are coupled with fermionic field. However, since \(I_3(\mathbb{C}P^n)\) is trivial, the Hopf term itself \(L_{\text{Hopf}}\) is perturbative and the value of the integral depends on the background classical solutions. Consequently, one can not expect that this value becomes an integer number. As a result, the solitons are always anyons even when \(\Theta = n\pi, n \in \mathbb{Z}\) \([14]\).

The hamiltonian is defined in terms of \(iD^{A=0} := \gamma_3(i\partial_3 - H_{\text{fermi}})\) as

\[
\mathcal{H}_{\text{fermi}} = -i\gamma_3 \gamma_k \partial_k + \gamma_3 m \Phi
\]

\[
= \begin{pmatrix}
m \Phi & -e^{-i\varphi} \left( \partial_\rho - \frac{i\partial_\rho}{\rho} \right) \\
e^{i\varphi} \left( \partial_\rho + \frac{i\partial_\rho}{\rho} \right) & -m \Phi
\end{pmatrix}
\]

One can confirm that the \(\mathcal{H}_{\text{fermi}}\) commutes with the following angular momentum which we call a grand spin

\[
\mathcal{K} = \ell_3 + \frac{\sigma_3}{2} = \frac{\ell_1}{2} \left( \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right) + \frac{\ell_2}{2} \left( \lambda_3 - \frac{1}{\sqrt{3}} \lambda_8 \right)
\]

where \(\ell_3\) is a third component of an orbital angular momentum, i.e., \(\ell_3 = (r \times p)_3 = -i \frac{\partial}{\partial \sigma_3}\) and \(\lambda_i\) are the components of Gell-Mann matrices.

We briefly explain the numerical method for the spectrum of this fermions. According to the Rayleigh-Ritz variational principle, the upper bound of the spectrum can be obtained from the secular equation for each \(\mathcal{K}\):

\[
\det (\mathcal{A} - \epsilon \mathcal{B}) = 0
\]

where

\[
\mathcal{A}_{k^{(p)}k^{(q)}} = \int d^3 x \phi^{(p)}_k(k^{(p)}, x) \mathcal{H}_{\text{fermi}} \phi^{(q)}_k(k^{(q)}, x)
\]

\[
\mathcal{B}_{k^{(p)}k^{(q)}} = \int d^3 x \phi^{(p)}_k(k^{(p)}, x) \phi^{(q)}_k(k^{(q)}, x)
\]

The plain wave basis are defined as

\[
\phi^{(u)}_{\mathcal{K}}(k^{(u)}, x) = N^{(u)}_i \begin{pmatrix}
\omega_{i,e}^{(u)} - J_{\mathcal{K}, \frac{1}{2} - \frac{n_1 - n_2}{2}}(k^{(u)}_i) e^{i(k^{(u)}_i - \frac{1}{2} + \frac{n_1 - n_2}{2}) \varphi} \\
\omega_{i,e}^{(u)} + J_{\mathcal{K} + \frac{1}{2} + \frac{n_1 - n_2}{2}}(k^{(u)}_i) e^{i(k^{(u)}_i + \frac{1}{2} + \frac{n_1 - n_2}{2}) \varphi}
\end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
\phi^{(d)}_{\mathcal{K}}(k^{(d)}, x) = N^{(d)}_i \begin{pmatrix}
\omega_{i,e}^{(d)} + J_{\mathcal{K}, -\frac{1}{2} - \frac{n_1 + n_2}{2}}(k^{(d)}_i) e^{i(k^{(d)}_i - \frac{1}{2} - \frac{n_1 + n_2}{2}) \varphi} \\
\omega_{i,e}^{(d)} - J_{\mathcal{K} + \frac{1}{2} + \frac{n_1 + n_2}{2}}(k^{(d)}_i) e^{i(k^{(d)}_i + \frac{1}{2} + \frac{n_1 + n_2}{2}) \varphi}
\end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
\phi^{(s)}_{\mathcal{K}}(k^{(s)}, x) = N^{(s)}_i \begin{pmatrix}
\omega_{i,e}^{(s)} + J_{\mathcal{K}, -\frac{1}{2} - \frac{n_1 + n_2}{2}}(k^{(s)}_i) e^{i(k^{(s)}_i - \frac{1}{2} - \frac{n_1 + n_2}{2}) \varphi} \\
\omega_{i,e}^{(s)} - J_{\mathcal{K} + \frac{1}{2} + \frac{n_1 + n_2}{2}}(k^{(s)}_i) e^{i(k^{(s)}_i + \frac{1}{2} + \frac{n_1 + n_2}{2}) \varphi}
\end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
Figure 3. The plot of the fermion density defined by (34). $m = 0.5$. We adopt same color notation with Fig.2.

where

$$\omega_{i,\epsilon > 0}^{(p)+} = \omega_{i,\epsilon < 0}^{(p)-} = 1, \quad \omega_{i,\epsilon > 0}^{(p)-} = \omega_{i,\epsilon < 0}^{(p)+} = \frac{-\text{sgn}(\epsilon)k_i^{(p)}}{|\epsilon| + m}, \quad p = u, d, s$$  (29)

The plain wave basis can be constructed via a large circular box with radius $\rho = D$. The wave number $k^{(u)}, k^{(d)}, k^{(s)}$ are discretized by the boundary conditions

$$J_{K_{-\frac{1}{2} - \frac{n_1 + n_2}{3}} (k_i^{(u)} D)} = 0, \quad J_{K_{-\frac{1}{2} - \frac{n_1 - 2n_2}{3}} (k_i^{(d)} D)} = 0, \quad J_{K_{-\frac{1}{2} - \frac{n_1 + n_2}{3}} (k_i^{(s)} D)} = 0. \quad (30)$$

The orthogonal conditions are then

$$\int_0^D dp \rho J_{\nu}^{(k_i^{(p)} \rho)} J_{\nu}^{(k_j^{(p)} \rho)} = \int_0^D dp \rho J_{\nu+1}^{(k_i^{(p)} \rho)} J_{\nu+1}^{(k_j^{(p)} \rho)} = \delta_{ij} \frac{D^2}{2} [J_{\nu+1}^{(k_i^{(p)} D)}]^2. \quad (31)$$
Eq. (25) can be solved numerically. For the infinite number of the wave number (which means the infinite size of the matrices), the spectrum $\epsilon$ becomes exact. The normalization constants of the basis are thus

$$N_{i}^{(u)} = \frac{[2\pi D^2\epsilon]\left(J_{\frac{k_{i}+\frac{1}{3}+2n_{1}-n_{2}}{3}}(k_{i}^{(u)}D)\right)^{2}}{[\epsilon + m]}^{1/2}$$
$$N_{i}^{(d)} = \frac{[2\pi D^2\epsilon]\left(J_{\frac{k_{i}+\frac{1}{3}-n_{1}-2n_{2}}{3}}(k_{i}^{(d)}D)\right)^{2}}{[\epsilon + m]}^{1/2}$$
$$N_{i}^{(s)} = \frac{[2\pi D^2\epsilon]\left(J_{\frac{k_{i}+\frac{1}{3}-n_{1}+n_{2}}{3}}(k_{i}^{(s)}D)\right)^{2}}{[\epsilon + m]}^{1/2}$$

(32)

In order to implement the spectral flow, we employ following linearly interpolated field

$$\Phi(x, \lambda)_{\text{intp}} = (1 - \lambda)\Phi_{\infty} + \lambda\Phi(x), \quad 0 \leq \lambda \leq 1$$

(33)

where $\Phi(x)$ is the field with nontrivial topology and $\Phi_{\infty}(x)$ is the field at the large asymptotics. For changing $\lambda$, we smoothly connect the vacuum and the solitonic states. We plug $\Phi_{\text{intp}}$ into the hamiltonian (23) and solve the eigenproblem (25) by the standard matrix diagonalization algorithm of LAPACK. In Fig.2, we present some typical results of the spectral flow. Some special energy levels are diving from the negative to the positive continuum. Number of fermionic levels of the zero crossing always are equal to the topological charges of the field $\Phi(x)$. As we will see that those levels are normalizable modes and then the behaviour is a realization of the Index theorem.

In terms of the eigenfunction of the eigenequation (25), we directly obtain the eigenfunction $\psi(x)$ of the hamiltonian (23). Therefore we are able to define normalizable density of the one fermion mode

$$d(\rho) = \frac{1}{2\pi n_{c}} \int d\varphi \langle \psi(x)\gamma_{3}\psi(x) \rangle.$$  (34)

The plot is given in Fig.3. One easily see that except for the small fluctuations, only just the modes of spectral flowing are the localizing normalizable modes. Such fluctuations of the non-localizing modes are due to the size effects and the cut off in the numerical analysis. For our analysis we introduce $D = 20.0$ for the radius of circular box and $n_{\text{mode}} = 192$ for the number of discretized momenta for saving computer time. As we growing these values, the fluctuations always tend to reduce.

4. Summary

In this paper, we studied the spectrum of the fermions coupled with the vortices of a $\mathbb{C}P^{N}$ Skyrme-Faddeev type model. We computed the spectral flow of the fermionic one-particle state which realized the level crossing picture. The $\mathbb{C}P^{N}$ solitons are anyon because their Hopf term is perturbative. On the other hand, the number of $Q$ states are always localized on the soliton which emerges a discrepancy between the statistical nature of the soliton and of the constituents (fermions). There are many possible resolutions or interpretations but we will report them in subsequent papers.

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