Tests for Group-Specific Heterogeneity in High-Dimensional Factor Models∗

Antoine A. Djogbenou
Department of Economics, York University
and
Razvan Sufana
Department of Economics, York University

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Abstract

Standard high-dimensional factor models assume that the comovements in a large set of variables could be modeled using a small number of latent factors that affect all variables. In many relevant applications in economics and finance, heterogenous comovements specific to some known groups of variables naturally arise, and reflect distinct cyclical movements within those groups. This paper develops two new statistical tests that can be used to investigate whether there is evidence supporting group-specific heterogeneity in the data. The first test statistic is designed for the alternative hypothesis of group-specific heterogeneity appearing in at least one pair of groups; the second is for the alternative of group-specific heterogeneity appearing in all pairs of groups. We show that the second moment of factor loadings changes across groups when heterogeneity is present, and use this feature to establish the theoretical validity of the tests. We also propose and prove the validity of a permutation approach for approximating the asymptotic distributions of the two test statistics. The simulations and the empirical financial application indicate that the proposed tests are useful for detecting group-specific heterogeneity.

Keywords: Factor models, group-specific heterogeneity, statistical test, permutation.

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1 Introduction

Large datasets of financial and macroeconomic variables are currently easily accessible. In many policy-relevant applications, empirical researchers have found it useful to summarize the information in a high-dimensional set of economic variables by extracting common underlying latent factors. As a consequence, the modeling and extraction of underlying latent factors is important for econometricians and practitioners alike. Factor models have been widely used since the seminal paper of Stock and Watson (2002) to model the dynamics of the common factors, and the principal components method (PCM) has remained a popular factor extraction procedure. The PCM generally assumes a one-level structure, where comovements in a large panel of variables are summarized by a few latent factors affecting all variables. In particular, each variable in a large panel can be decomposed into an idiosyncratic error component and a common component (for details, see Stock and Watson (2002); Bai and Ng (2002); Bai (2003)). The popularity of the PCM as a factor extraction procedure relies on its computational simplicity and its properties, established in Bai (2003) and Bai and Ng (2006).

In many applications, such as asset pricing and studies of real international economic activity, heterogeneous group structures may naturally arise in the specification of the factor models. In such cases, cyclical movements specific to some groups of variables can be considered. In a study of real international economic activity, Kose, Otrok, and Prasad (2012) used these models to investigate the decoupling of developed and emerging economy activity factors. Although globalization has shaped the world economy in recent decades, emerging economies have experienced impressive growth relative to developed economies, suggesting the existence of separate business cycles. Using a large set of observable economic activity variables, Djogbenou (2020) proposed a statistical test to investigate whether the latter assertion is supported by observed data, and found evidence that suggests the existence of specific comovement within the group of emerging economy variables and the group of developed economy variables. However, this test is only applicable to the case of two groups of variables.

This paper makes two contributions. The first is to extend the analysis in Djogbenou (2020) to situations where there may be heterogeneity specific to multiple groups of
variables. We assume that the number of groups of variables and the group membership of each variable are known, which is the case in certain real-world situations where these features are already predetermined, such as the one considered in the application section. These assumptions distinguish the framework of this paper from that used in Ando and Bai (2017), where the number of groups is determined using a model selection criterion, and the group membership of each variable is estimated from historical data.

We define the situations of group-specific heterogeneity that can occur, some of which can only appear in settings with three or more groups. We develop tests of the null hypothesis of no group-specific heterogeneity against the alternative that some form of group-specific heterogeneity is present. Two types of alternative hypotheses are considered. Under the first alternative, group-specific heterogeneity emerges in at least one pair of groups; under the second, group-specific heterogeneity emerges in all pairs of groups. We propose two Lagrange Multiplier (LM) type tests, derive their asymptotic distributions under the null hypothesis, and show that each test statistic diverges under the corresponding type of alternative hypothesis. Compared to Djogbenou (2020), where two groups are considered, our generalization to the multiple groups setting leads to some non-standard asymptotic distributions. These asymptotic distributions are based on Chi-square random variables, some of which are dependent, and can be simulated taking their dependence structure into account in order to generate critical values for the tests.

The second contribution of the paper is to consider and show the validity of a permutation approach for approximating the asymptotic distributions of the two test statistics. This approach has the advantage of avoiding the simulation of the asymptotic distributions. Permutation tests are nonparametric tests that provide a superior control of the test size, and are based on a standard exchangeability assumption on the distribution of the factor loadings under the null hypothesis. For details regarding permutation inference, see, for example, Lehmann and Romano (2005), Canay, Romano, and Shaikh (2017), and Hemerik and Goeman (2018).

Simulation experiments are used to study the finite sample performance of all tests. We also consider an application in which we test for the existence of heterogeneous industry groups in two classifications of industries in the U.S. financial sector. We use the Standard
Industrial Classification (SIC) to determine the groupings. The first consists of six major groups within the Finance, Insurance, and Real Estate division of the SIC, and the second includes nine industry groups within the same division. The tests reveal that there is group heterogeneity in all pairs of major groups. For the second grouping, they show that there is group heterogeneity in at least one pair of industry groups.

The rest of the paper is organized as follows. In Section 2, we present the model. In Section 3, we describe the test statistics and the asymptotic validity results. Section 4 presents the permutation versions of the tests. We investigate the finite sample properties of the proposed tests in Section 5. An empirical illustration is included in Section 6. Section 7 concludes the paper. Proofs are relegated to the Appendix. Throughout the paper, the scalar $C$ denotes a generic constant, $I(\cdot)$ is the indicator function and $\|\cdot\|$ denotes the Euclidean norm. We also use the notation $\delta_{NT} = \min \left[ \sqrt{N}, \sqrt{T} \right]$.

### 2 The Model

The goal of this paper is to develop tests for group-specific heterogeneity in high-dimensional factor models. We assume the underlying structure without group-specific heterogeneity of a variable $i$ can be modeled using a large panel factor model given by

$$X_{it} = \lambda_{c,i} f_{c,t} + e_{it}, i = 1, \ldots, N, t = 1, \ldots, T,$$

where $N$ is the number of variables, $T$ is the number of time periods, and $t$ denotes the time period index. The vector $f_{c,t}$ of dimensions $r_c \times 1$ contains common latent factors affecting all variables. The factor loadings $\lambda_{c,i} : r_c \times 1$ capture the exposure of the variable $i$ to the common factors, and $e_{it}$ denotes an idiosyncratic error.

We assume there are $S$ groups and let the index $g_i \in \{j, j = 1, \ldots, S\}$ denote the group that includes the variable $i$. We let $N_j$ denote the number of variables in group $j$ such that $N = \sum_{j=1}^{S} N_j$, and let $M_j = \sum_{l=1}^{j} N_l, j = 1, \ldots, S$, with $M_0 = 0$. We also assume that for any $j, \frac{N_j}{N}$ converges to a fixed number $\pi_j \in [\alpha_1, \alpha_2] \subset (0, 1)$.

The factor model can be written in matrix form as

$$X = \Lambda F_c' + e = [\Lambda_1' \cdots \Lambda_S'] F_c' + e,$$
where \( \mathbf{X} = (X_{it}) : N \times T \) is the matrix of observed variables, \( \mathbf{F}'_c = [f_{c,1} \cdots f_{c,T}] : r_c \times T \) contains the common factors, \( \mathbf{e} = (e_{it}) : N \times T \) collects the idiosyncratic errors,

\[
\Lambda = \begin{bmatrix} \lambda_{c,1} \\ \vdots \\ \lambda_{c,N} \end{bmatrix} : N \times r_c
\]

is the matrix of factor loadings, and

\[
\Lambda_j = \begin{bmatrix} \lambda'_{c,(M_{j-1}+1)} \\ \vdots \\ \lambda'_{c,M_j} \end{bmatrix} : N_j \times r_c, j = 1, \ldots, S.
\]

captures the exposure of variables in group \( j \) to the factors in \( \mathbf{F}_c \).

We impose the following assumptions, where we let \( \varpi_{ji} \equiv \mathbb{I}(M_j - 1 \leq i \leq M_j) \), with \( j = 1, \ldots, S \), and \( i = 1, \ldots, N \).

**Assumption 1.** (Factor model and idiosyncratic errors)

(a) \( \mathbb{E} \| \mathbf{f}_{c,t} \|^4 \leq C \) and \( \frac{1}{T} \mathbf{F}'_c \mathbf{F}_c = \frac{1}{T} \sum_{t=1}^{T} \mathbf{f}_{c,t} \mathbf{f}_{c,t}' \xrightarrow{P} \Sigma_F > 0 \), where \( \Sigma_F \) is non-random.

(b) The factor loadings \( \{\lambda_{c,i}\}_{i=1,\ldots,N} \) are independent across \( i \), and \( \mathbb{E} (\lambda_{c,i} \lambda'_{c,i}) = \Sigma_\Lambda > 0 \), where \( \Sigma_\Lambda \) is non-random, such that \( \mathbb{E} (\|\lambda_{c,i}\|^8) \leq C \).

(c) The eigenvalues of the \( r \times r \) matrix \( (\Sigma_F \times \Sigma_\Lambda) \) are distinct.

(d) \( \mathbb{E} (e_{it}) = 0, \mathbb{E} |e_{it}|^8 \leq C \).

(e) \( \mathbb{E} (e_{it}e_{js}) = \sigma_{ij,ts}, |\sigma_{ij,ts}| \leq \bar{\sigma}_{ij} \) for all \( (t,s) \) and \( |\sigma_{ij,ts}| \leq \tau_{st} \) for all \( (i,j) \), with \( \frac{1}{N} \sum_{i,j=1}^{N} \bar{\sigma}_{ij} \leq C, \frac{1}{T} \sum_{t,s=1}^{T} \tau_{st} \leq C \) and \( \frac{1}{NT} \sum_{i,j,t,s=1} \sigma_{ij,ts} \leq C \).

(f) \( \mathbb{E} \left( \frac{1}{\sqrt{N}} \sum_{t=1}^{N} (e_{it}e_{is} - \mathbb{E} (e_{it}e_{is})) \right)^4 \leq C \) for all \( (t,s) \).

(g) The limit \( S_0 \) of \( \text{Var} \left( \frac{\sqrt{N}}{N} \mathbf{A}_0 \left( j, \Lambda H_0' - 1 \right) \right) \) is bounded and positive definite, where

\[
\mathbf{A}_0 \left( j, \Lambda H_0' - 1 \right) = \text{Vech} \left( \frac{\sqrt{N}}{N_j} \sum_{i=M_{j-1}+1}^{M_j} H_0^{-1} \left( \lambda_{c,i} \lambda'_{c,i} - \Sigma_\Lambda \right) H_0^{-1} \right),
\]

and \( H_0 \) is defined in Section 2.2.
(h) \( \mathbb{E} \left( \text{Vech} \left( H^{-1}_0 \lambda_{c,i} \lambda'_{c,i} H^{-1}_0 \right) \right) \text{Vech} \left( H^{-1}_0 \lambda_{c,i} \lambda'_{c,i} H^{-1}_0 \right)' = \Sigma_{\Lambda\Lambda} \).

**Assumption 2.** (Moment conditions and weak dependence among \( \{f_{c,t}\}, \{\lambda_{c,i}\} \) and \( \{e_{it}\} \))

(a) \( \mathbb{E} \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} f_{c,t} e_{it} e_{it'} \right\|^2 \right) \leq C, \ m = 1, \ldots, S, \) where \( \mathbb{E} (f_{c,t} e_{it}) = 0 \) for every \((i, t)\).

(b) For each \( t \), \( \mathbb{E} \left\| \frac{1}{\sqrt{T N}} \sum_{s=1}^{T} \sum_{i=1}^{N} f_{c,s} (e_{it} e_{is} - \mathbb{E} (e_{it} e_{is})) \right\|^2 \leq C \).

(c) \( \mathbb{E} \left\| \frac{1}{\sqrt{T N}} \sum_{t=1}^{T} \sum_{i=1}^{N} f_{c,t} \lambda_{c,i} e_{it} e_{it'} \right\|^2 \leq C, \ m = 1, \ldots, S, \) where \( \mathbb{E} (f_{c,t} \lambda_{c,i} e_{it}) = 0 \) for all \((i, t)\).

(d) \( \mathbb{E} \left( \frac{1}{T} \sum_{t=1}^{T} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{c,i} e_{it} e_{it'} \right\|^2 \right) \leq C, \ m = 1, \ldots, S, \) where \( \mathbb{E} (\lambda_{c,i} e_{it}) = 0 \) for all \((i, t)\).

Assumptions 1 and 2 allow for weak dependence and heteroskedasticity in the idiosyncratic errors and are similar to the assumptions A–D of Bai and Ng (2002), 1–3 of Djogbenou, Gonçalves, and Perron (2015) and 1–2 of Djogbenou (2021). Assumption 1 (b), (g) and (h) are useful for deriving the asymptotic distribution of our proposed test statistics under the null hypothesis and impose the existence of slightly more than the fourth moment of \( \lambda_{c,i} \). Using a law of large numbers, Assumption 1 (b) implies a standard assumption for factor models, stating that \( \frac{1}{N} \Lambda' \Lambda = \frac{1}{N} \sum_{i=1}^{N} \lambda_{c,i} \lambda'_{c,i} \xrightarrow{P} \Sigma_{\Lambda} > 0 \), where \( \Sigma_{\Lambda} \) is non-random. Moreover, Assumption 2 (a), (c) and (d), which restricts the dependence between \( f_{c,t}, \lambda_{c,i} \) and \( e_{it} \) among specific groups of variables, is slightly stronger.

In the model presented thus far, the only factors driving the variables are common factors. In this case, the exposure of the variables to the factors is homogenous across groups, and all variables share common cyclical movements. However, in the presence of group-specific heterogeneity, we expect some factors to appear and affect the groups differently, implying distinct cyclical movements across the groups of variables.

As shown in the next section, the second moment of factor loadings changes in presence of group-specific heterogeneity. Nevertheless, under the null hypothesis \( H_0 \) of no group-specific heterogeneity, the scaled product of the factor loadings for groups \( j \) is

\[
\frac{1}{N_j} \Lambda_j' \Lambda_j = \frac{1}{N_j} \sum_{i=M_{j-1}+1}^{M_j} \lambda_{c,i} \lambda'_{c,i} = \Sigma_{\Lambda} + o_P(1),
\]
under Assumption 1(b), and therefore the difference between the scaled products for groups $j$ and $k$ ($j < k$) is

$$\frac{1}{N_j} \Lambda_j^t \Lambda_j - \frac{1}{N_k} \Lambda_k^t \Lambda_k = o_P(1).$$

We distinguish two types of alternative hypotheses. The first, denoted $H_1$, corresponds to the case where group-specific heterogeneity emerges in at least one pair of groups. The second, denoted $H_2$, corresponds to the case where group-specific heterogeneity emerges in all pairs of groups. Given the model, the hypothesis $H_2$ also refers to the case where there is no pair of groups without group-specific heterogeneity. For convenient reference, we list these hypotheses together:

- $H_1$: Group-specific heterogeneity emerges in at least one pair of groups.
- $H_2$: Group-specific heterogeneity emerges in all pairs of groups.

Section 2.1 formally presents the model when different types of group heterogeneity are present.

### 2.1 Group-specific heterogeneity

Group heterogeneity in factor models may arise in the form of group-specific factors. In this case, the specification of the variable $i$ is

$$X_{it} = \lambda_{c,i} f_{c,t} + \lambda_{g,i}^t f_{g,i,t} + e_{it}, i = 1, \ldots, N, t = 1, \ldots, T. \tag{3}$$

The vector $f_{g,i,t}$ of dimensions $r_{g,i} \times 1$ includes latent specific factors affecting only the variables in group $g$. The factor loadings $\lambda_{g,i} : r_{g,i} \times 1$ capture the exposure of the variable $i$ to the specific factors in group $g$.

The matrix equivalent of Equation (3) is

$$X = \Phi G' + e = [\Phi_1 \cdots \Phi_S]' [F_c F_1 \cdots F_S]' + e, \tag{4}$$

where $F_j' = [f_{j,1} \cdots f_{j,T}] : r_j \times T$ is the matrix of factors specific to group $j$, $j = 1, \ldots, S$, and the matrices

$$\Phi_j = [\phi_{M,j-1+1}^t \cdots \phi_{M_j}^t]' = \begin{bmatrix} \lambda_{c,M_j-1+1}^t & \lambda_{j,M_j-1+1}^t & 0 \\ \vdots & \vdots & \vdots \\ \lambda_{c,M_j}^t & 0 & \lambda_{j,M_j}^t \\ 0 & 0 & 0 \end{bmatrix} : N_j \times r, j = 1, \ldots, S.
with \( r = r_c + r_1 + \ldots + r_S \), reflect the exposure to the latent factors in the factor model representation in Equation (4). The first \( r_c \) columns contain the factor loadings that capture the exposure of the variables in group \( j \) to the common factors. The following \( \sum_{l=1}^{j-1} r_l \) columns and the last \( \sum_{j=1}^{S} r_l \) columns have only zero elements because the variables in group \( j \) have zero exposure to the specific factors \( F_1, \ldots, F_{j-1}, F_{j+1}, \ldots, F_S \). The columns of factors loadings between the zero columns reflect how the variables in group \( j \) react to specific factors in \( F_j \).

**Assumption 3.** (Additional conditions for group heterogeneous factor models)

(a) \( \frac{1}{T} G'G = \frac{1}{T} \sum_{t=1}^{T} \mathbf{g}_t \mathbf{g}_t' \xrightarrow{P} \Sigma_G > 0 \), where \( \Sigma_G \) is non-random and \( \mathbf{g}_t \) denotes the \( t \)th row of the matrix \( G \).

(b) The factor loadings \( \{ \phi_i \}_{i=1, \ldots, N} \) are independent across \( i \), \( \text{E} \left( \| \phi_i \|^4 \right) \leq C \), and for \( i \) in group \( j \), \( \text{E} (\phi_i \phi_j') \) is the limit defined in Equation (5) or Equation (8), such that \( \frac{1}{N} \Phi' \Phi = \frac{1}{N} \sum_{i=1}^{N} \phi_i \phi_i' \xrightarrow{P} \Sigma_\phi > 0 \), where \( \Sigma_\phi \) is non-random.

(c) The eigenvalues of the matrix \( \Sigma_G \times \Sigma_\phi \) are distinct.

(d) \( \text{E} \left( \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \mathbf{g}_t e_{it} t_{ji} \right\|^2 \right) \leq C, j = 1, \ldots, S \), where \( \text{E} (\mathbf{g}_t e_{it}) = 0 \) and \( \text{E} (\mathbf{g}_t \phi_j' e_{it}) = 0 \) for every \( (i, t) \).

(e) For each \( t \), \( \text{E} \left( \frac{1}{\sqrt{TN}} \sum_{s=1}^{N} \sum_{i=1}^{N} \mathbf{g}_s (e_{it} e_{is} - \text{E} (e_{it} e_{is})) \right) \leq C \).

(f) \( \text{E} \left( \frac{1}{\sqrt{TN}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathbf{g}_t \phi_j' e_{it} t_{ji} \right) \leq C, j = 1, \ldots, S \).

(g) \( \text{E} \left( \frac{1}{\sqrt{TN}} \sum_{t=1}^{T} \sum_{i=1}^{N} \mathbf{g}_t \phi_j' e_{it} t_{ji} \right) \leq C, j = 1, \ldots, S \), where \( \text{E} (\mathbf{g}_t \phi_j' e_{it}) = 0 \) for all \( (i, t) \).

(h) For any \( j \neq k \), the limit in probability \( S_0(j, k) \) of \( S(j, k, \Phi \Xi^{-1}) \) is positive definite, where \( \Xi_0 \) is defined in Section 2.2 and \( S_0 \) is defined in Section 3.

Assumption 3 (a)–(g) complements the previous assumptions for the factor model representation in Equation (4) in the presence of group-specific heterogeneity. Assumption 3 (h) imposes positive definiteness on the limit of the variance estimator under the alternative.
Let us now consider a pair of groups \((j, k)\) \((j < k)\), such that group-specific factors emerge in group \(j\) or group \(k\). According to Assumption 3 (b), the scaled product of the factor loadings for group \(j\), \(\frac{1}{N_j} \Phi_j' \Phi_j\), is

\[
\begin{bmatrix}
\frac{1}{N_j} \sum_{i=M_j-1+1}^{M_j} \lambda_{c,i} \lambda'_{c,i} & 0 & \cdots & 0 \\
0 & \frac{1}{N_j} \sum_{i=M_j-1+1}^{M_j} \lambda_{c,i} \lambda'_{j,i} & 0 & \cdots \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{N_j} \sum_{i=M_j-1+1}^{M_j} \lambda_{j,i} \lambda'_{j,i}
\end{bmatrix}
= \begin{bmatrix}
\Sigma_{\Lambda} & \Sigma_{c\jmath} & 0 \\
0 & 0 & 0 \\
\Sigma'_{cj} & 0 & \Sigma_{jj}
\end{bmatrix} + o_P(1),
\]

where \(\Sigma_{\Lambda} \equiv \mathbb{E}(\lambda_{c,i} \lambda'_{c,i})\), \(\Sigma_{jj} \equiv \mathbb{E}(\lambda_{j,i} \lambda'_{j,i})\), and \(\Sigma_{cj} \equiv \mathbb{E}(\lambda_{c,i} \lambda'_{j,i})\). Then \(\Sigma_{jj} \neq 0\) or \(\Sigma_{kk} \neq 0\) since the factor loadings associated with specific factors within groups \(j\) and \(k\) would be \(0\) otherwise. Therefore, the difference between the scaled products for groups \(j\) and \(k\)

\[
\frac{1}{N_j} \Phi_j' \Phi_j - \frac{1}{N_k} \Phi_k' \Phi_k,
\]

converges in probability to a nonzero matrix. When the difference in (6) is scaled by \(\sqrt{N}\), it diverges. As described in Section 3, we construct a test statistic which diverges when at least one of the groups \(j\) and \(k\) is heterogeneous, that is, contains group-specific factors.

Group heterogeneity in factor models may also arise in the form of factors affecting the variables within groups differently and which can be identified through distinct second moments of loadings or distinct cross moments between loadings and global loadings. In order to formally express this situation, let us assume that a factor \(F_s: T \times r_s\) affects both groups in some pair of groups \((j, k)\) \((j < k)\). For simplicity of exposition, suppose the first type of heterogeneity discussed above is not also present. The matrix notation in Equation (4) becomes \(G = [F_c F_s]: T \times (r_c + r_s)\) and the factor loadings of the variables in group \(j\) are

\[
\Phi_j = [\phi_{M_j-1+1} \cdots \phi_{M_j}]':
\begin{bmatrix}
\lambda'_{c,M_j-1+1} & \lambda'_{j,M_j-1+1} \\
\vdots & \vdots \\
\lambda'_{c,M_j} & \lambda'_{j,M_j}
\end{bmatrix}: N_j \times (r_c + r_s),
\]

which leads to

\[
\frac{1}{N_j} \Phi_j' \Phi_j = \begin{bmatrix}
\Sigma_{\Lambda} & \Sigma_{c\jmath} \\
\Sigma'_{cj} & \Sigma_{jj}
\end{bmatrix} + o_P(1).
\]
\( \Phi_k \) and \( \frac{1}{N_k} \Phi'_k \Phi_k \) can be expressed in a similar way. The difference between the scaled products for the groups in the pair \((j,k)\) is

\[
\frac{1}{N_j} \Phi'_j \Phi_j - \frac{1}{N_k} \Phi'_k \Phi_k = \begin{bmatrix} 0 & \Sigma_{cj} - \Sigma_{ck} \\ \Sigma'_{cj} - \Sigma'_{ck} & \Sigma_{jj} - \Sigma_{kk} \end{bmatrix} + o_P(1).
\] (9)

Heterogeneity in the pair of groups \((j,k)\) arises if \( \Sigma_{jj} \neq \Sigma_{kk} \) or \( \Sigma_{cj} \neq \Sigma_{ck} \). When this occurs, the difference in (9) converges in probability to a nonzero matrix, which diverges when scaled by \( \sqrt{N} \).

To summarize, the first type of group-specific heterogeneity is induced by some factors affecting only the variables in each individual group, while the second type of group-specific heterogeneity is induced by some factors affecting the groups of variables in different ways. Note that a combination of both types of heterogeneity can be present at the same time. The proposed statistical tests developed in the next section can be used to detect group-specific heterogeneity of either type discussed above.

### 2.2 Model estimation

In practice, the factors and factor loadings are latent and need to be estimated. A popular approach consists of using the PCM to obtain \( \hat{\Lambda} = [\hat{\lambda}_1 \cdots \hat{\lambda}_N]' \) and \( \hat{F} = [\hat{f}_1 \cdots \hat{f}_r]' \), which represent the estimates of \( \Lambda \) and \( F_c \) when there is no group heterogeneity (\( \Phi \) and \( G \) when group heterogeneity is present). These estimates can be obtained by minimizing the sum of squared idiosyncratic residuals under the restriction \( \hat{\Lambda}' \hat{\Lambda} / N = I \). The estimated factor loadings are given by \( \sqrt{N} \) times the \( r \) eigenvectors associated with the \( r \) largest eigenvalues of \( XX' \), while the estimated factors are given by \( X' \hat{\Lambda} / N \) (see Bai and Ng (2008)). We use the information criteria suggested by Bai and Ng (2002) to select the number of latent factors and let this number be \( r \geq r_c \) in general. As shown in Bai and Ng (2002) and Bai (2003), at any time period \( t \), \( \hat{f}_t \) consistently estimates a rotation of the true factor space, given by \( \hat{H}' f_{c,t} \) under no group heterogeneity and \( \hat{\Xi}' g_t \) under group heterogeneity, where \( g'_t \) is a typical row of \( G \), and \( \hat{H} : r_c \times r_c \) and \( \hat{\Xi} : r \times r \) are the rotation matrices. In the former case, we have

\[
X = \Lambda F'_c + e = \left( \Lambda \hat{H}'^{-1} \right) \left( F_c \hat{H} \right)' + e, \tag{10}
\]
and in the latter case

\[ X = \Phi G' + e = (\Phi \hat{\Xi}'^{-1}) (G \hat{\Xi})' + e. \]  

(11)

The factor loading estimates \( \hat{\Lambda} \) and \( \hat{\Phi} \) in Equation (10) and Equation (11) converge to their rotated versions \( \Lambda \hat{H}'^{-1} \) and \( \Phi \hat{\Xi}'^{-1} \), respectively. The invertible limits of \( \hat{H} \) and \( \hat{\Xi} \) will be denoted \( H_0 \) and \( \Xi_0 \), respectively.

We consider two LM-type test statistics for each alternative hypothesis. The proposed testing procedures have the advantage that the estimation of the group-specific factors, which may be computationally intensive, is not required.

### 3 Test Statistics and Asymptotic Validity Results

In this section, we describe the test statistics and establish their formal validity. We consider two types of LM statistics that correspond to the alternative hypotheses \( H_1 \) and \( H_2 \), derive their limiting distributions under the null hypothesis, and show that they tend to infinity under the corresponding alternative hypothesis. To construct the statistics, we consider the pair of groups \((j,k)\) and replace the latent factor loadings in Equation (6) by their estimated version to have the scaled difference

\[ A(j, k, \hat{\Lambda}) = \text{Vech} \left( \sqrt{N} \left( \frac{1}{N_j} \sum_{i=M_{j-1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}'_i - \frac{1}{N_k} \sum_{i=M_{k-1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}'_i \right) \right). \]

(12)

If group-specific heterogeneity emerges in the pair of groups \((j,k)\), Appendix A shows that the difference

\[ \frac{1}{N_j} \sum_{i=M_{j-1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}'_i - \frac{1}{N_k} \sum_{i=M_{k-1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}'_i \xrightarrow{P} \Xi_0^{-1} \text{plim} \left( \frac{1}{N_j} \Phi'_j \Phi_j - \frac{1}{N_k} \Phi'_k \Phi_k \right) \Xi_0'^{-1} = R_0(j, k), \]

(13)

where \( \Xi_0 \) is positive definite and \( \text{plim} \left( \frac{1}{N_j} \Phi'_j \Phi_j - \frac{1}{N_k} \Phi'_k \Phi_k \right) \) is nonzero as explained in Section 2. This implies that \( \frac{1}{N_j} \sum_{i=M_{j-1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}'_i - \frac{1}{N_k} \sum_{i=M_{k-1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}'_i \) has a nonzero limit.

Thus, Equation (12) diverges as the sample sizes increase.

Under the null hypothesis, group-specific heterogeneity does not emerge in any pair \((j,k)\), and using a decomposition similar to the one in Equation (A.13), we have

\[ \frac{1}{N_j} \sum_{i=M_{j-1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}'_i - \frac{1}{N_k} \sum_{i=M_{k-1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}'_i \xrightarrow{P} H_0^{-1} \text{plim} \left( \frac{1}{N_j} \Lambda'_j \Lambda_j - \frac{1}{N_k} \Lambda'_k \Lambda_k \right) H_0'^{-1} = 0. \]

(14)
In this case, we show that the expression in Equation (12) no longer diverges.

For two groups $j$ and $k$, we normalize $A(j,k,\hat{\Lambda})$ using its variance estimator

$$S(j,k,\hat{\Lambda}) = \left(\frac{N}{N_j} + \frac{N}{N_k}\right) \frac{1}{N} \sum_{i=1}^{N} \text{Vech} (\hat{\Lambda}_i \hat{\Lambda}_i' - I) \text{Vech} (\hat{\Lambda}_i' \hat{\Lambda}_i - I)'$$

and consider the LM-type statistic given by

$$LM_N(j,k,\hat{\Lambda}) = A(j,k,\hat{\Lambda})' (S(j,k,\hat{\Lambda}))^{-1} A(j,k,\hat{\Lambda}). \quad (15)$$

We establish under Assumptions 1 and 2 that $LM_N(j,k,\hat{\Lambda})$ follows asymptotically a Chi-square distribution with $\frac{r(r+1)}{2}$ degrees of freedom. To ensure that the suggested test statistic will diverge under a group heterogeneity alternative, we propose two different test statistics given by

$$LM_{1N}(\hat{\Lambda}) = \max_{1 \leq j < k \leq S} LM_N(j,k,\hat{\Lambda}), \quad (16)$$

when testing for $H_1$, and

$$LM_{2N}(\hat{\Lambda}) = \min_{1 \leq j < k \leq S} LM_N(j,k,\hat{\Lambda}), \quad (17)$$

when testing for $H_2$. The following theorem derives the limiting distributions of these two test statistics under the null hypothesis.

**Theorem 1.** Suppose that Assumptions 1 and 2 are satisfied. As $N,T \to \infty$, if $\sqrt{N}/T \to 0$, then under the null, it holds that

$$LM_{1N}(\hat{\Lambda}) \xrightarrow{d} \max_{1 \leq j < k \leq S} Q_{j,k}, \quad (18)$$

and

$$LM_{2N}(\hat{\Lambda}) \xrightarrow{d} \min_{1 \leq j < k \leq S} Q_{j,k}, \quad (19)$$

where

$$Q_{j,k} = \left(\pi_j^{-1/2} Z_j - \pi_k^{-1/2} Z_k\right)' \left(\pi_j^{-1} + \pi_k^{-1}\right)^{-1} \left(\pi_j^{-1/2} Z_j - \pi_k^{-1/2} Z_k\right), \quad (20)$$

for $1 \leq j < k \leq S$, with $Z_j \sim N\left(0_{r_c(r_c+1)/2}, I_{r_c(r_c+1)/2}\right)$, $j = 1, \ldots, S$, are $\frac{S(S-1)}{2}$ random variables with a Chi-square distribution with $\frac{r_c(r_c+1)}{2}$ degrees of freedom.

The proof of Theorem 1 is in Appendix A. Theorem 1 suggests that we could test the null hypothesis using critical values from the derived asymptotic distributions under the
stated assumptions. These distributions are based on \( \frac{S(S-1)}{2} \) Chi-square random variables, some of which are dependent as shown in Equation (20). The Chi-square variables can be simulated taking their dependence into account by generating the normal vectors \( Z_j, j = 1, \ldots, S \), and using the estimators \( \frac{N_j}{N} \) of \( \pi_j \), for \( j = 1, \ldots, S \).

Theorem 2 below shows that, under the appropriate alternative hypothesis, \( LM_{1N} (\hat{\Lambda}) \) and \( LM_{2N} (\hat{\Lambda}) \) diverge as sample sizes increase. Note that the condition \( \sqrt{N}/T \to 0 \) is mild and is useful to ensure the consistency of the estimated factor loadings (Bai, 2003).

**Theorem 2.** Suppose that Assumptions 1–3 are satisfied. As \( N, T \to \infty \), if \( \sqrt{N}/T \to 0 \), it holds that

\[
LM_{1N} (\hat{\Lambda}) = N\delta_1 + o_P(N),
\]

under the group heterogeneity alternative \( H_1 \), with \( \delta_1 \) a positive scalar, and

\[
LM_{2N} (\hat{\Lambda}) = N\delta_2 + o_P(N),
\]

under the group heterogeneity alternative \( H_2 \), with \( \delta_2 \) a positive scalar.

The proof of Theorem 2 is in Appendix A. In particular, we show that

\[
\delta_1 = \max_{1 \leq j < k \leq S} \left( (\text{Vech} (\mathbf{R}_0(j,k)))'(\mathbf{S}_0(j,k))^{-1} \text{Vech} (\mathbf{R}_0(j,k)) \right)
\]

and

\[
\delta_2 = \min_{1 \leq j < k \leq S} \left( (\text{Vech} (\mathbf{R}_0(j,k)))'(\mathbf{S}_0(j,k))^{-1} \text{Vech} (\mathbf{R}_0(j,k)) \right).
\]

Hence, under \( H_1 \), there is at least one group pair \((j,k)\) such that \( \mathbf{R}_0(j,k) \neq \mathbf{0} \), while under \( H_2 \), all group pairs \((j,k)\) are such that \( \mathbf{R}_0(j,k) \neq \mathbf{0} \). Consequently, \( \delta_1 > 0 \) and \( \delta_2 > 0 \), under \( H_1 \) and under \( H_2 \), respectively. Thus, \( N\delta_1 \) and \( N\delta_2 \) diverge as \( N \) goes to infinity, and ensure that the test statistics \( LM_{1N} (\hat{\Lambda}) \) and \( LM_{2N} (\hat{\Lambda}) \), respectively, have power against the alternatives. The decision to reject or not may be made using simulated critical values from \( \max_{1 \leq j < k \leq S} Q_{j,k} \) or \( \min_{1 \leq j < k \leq S} Q_{j,k} \). The following algorithm summarizes the proposed testing procedure.

**Algorithm for Implementing the LM Test Procedure.**
1. Compute the estimated factor loadings (\( \hat{\Lambda} \)) : \( \sqrt{N} \) times the eigenvectors corresponding to the \( r \) largest eigenvalues of \( XX' \), in decreasing order, and using the normalization \( \hat{\Lambda}'\hat{\Lambda}/N = I \).\(^1\)

2. Find for \( 1 \leq j < k \leq S \),

\[
LM_N(j, k, \hat{\Lambda}) = A(j, k, \hat{\Lambda})' \left( \hat{S}(j, k, \hat{\Lambda}) \right)^{-1} A(j, k, \hat{\Lambda}),
\]

where

\[
A(j, k, \hat{\Lambda}) = \sqrt{N} \text{Vech} \left( \frac{1}{N_j} \sum_{i=M_{j-1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}_i' - \frac{1}{N_k} \sum_{i=M_{k-1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}_i' \right),
\]

and

\[
S(j, k, \hat{\Lambda}) = \left( \frac{N}{N_j} + \frac{N}{N_k} \right) \frac{1}{N} \sum_{i=1}^{N} \text{Vech} \left( \hat{\lambda}_i \hat{\lambda}_i' - I \right) \text{Vech} \left( \hat{\lambda}_i \hat{\lambda}_i' - I \right)'.
\]

3. Obtain the test statistic

\[
LM_{1N}(\hat{\Lambda}) = \max_{1 \leq j < k \leq S} LM_N(j, k, \hat{\Lambda})
\]

or

\[
LM_{2N}(\hat{\Lambda}) = \min_{1 \leq j < k \leq S} LM_N(j, k, \hat{\Lambda}).
\]

4. Reject \( H_0 \) or not using critical values from \( \max_{1 \leq j < k \leq S} Q_{j,k} \) or \( \min_{1 \leq j < k \leq S} Q_{j,k} \).

Note that the number of factors is first selected using existing selection criteria for factor models. To facilitate testings based on the proposed test statistics, we also propose a permutation approach in the next section.

4 Permutation Inference

The testing procedure described in Section 3 relies on critical values obtained by simulating the asymptotic distributions derived in Theorem 1. In this section we consider an alternative inference procedure that avoids the simulation of the asymptotic distributions, and is based on a permutation approach that approximates the cumulative distribution

\(^1\)Note that the results remain valid if the normalization \( \hat{F}'\hat{F}/T = I \) is used. However, in this case, the identity matrix in the variance formula should be replaced with \( \hat{\Lambda}'\hat{\Lambda}/N \).
functions $F_1(x)$ and $F_2(x)$ of $\max_{1 \leq j < k \leq S} Q_{j,k}$ and $\min_{1 \leq j < k \leq S} Q_{j,k}$, respectively. Permutation inference has been considered in other contexts to test parametric or distributional hypotheses related to observed variables. See, for example, Fisher (1936), Lehmann and Romano (2005) and Hemerik and Goeman (2018). An important feature of a permutation test is its ability to have level $\alpha$ whenever the $S$ populations of factor loadings have distributions that are (approximately) the same under the null hypothesis.

Let $G_N$ be the set of all possible permutations of the set $\{1, \ldots, N\}$ and $g(i)$ denote the value assigned by the permutation $g$ to $i = 1, \ldots, N$. We also let $g\hat{\Lambda} = [\hat{\lambda}_{g(1)} \cdots \hat{\lambda}_{g(N)}]'$ denote the matrix of factor loadings based on the permutation $g$.

The permutation approach relies on Assumption 4 (a) (Permutation Hypothesis) stating that the factor loadings $\{\lambda_{c,i}\}_{i=1}^{N}$ are identically distributed according to some distribution $Q$.

Assumption 4. (Permutation Hypothesis)

(a) The factor loadings $\{\lambda_{c,i}\}_{i=1}^{N}$ are identically distributed according to some distribution $Q$.

This assumption, when combined with Assumption 1 (b), ensures the exchangeability hypothesis widely used in permutation contexts. Under this assumption, the joint distribution of $(\lambda_{c,1}, \ldots, \lambda_{c,N})'$ is invariant under the permutations in $G_N$. Therefore, for a random permutation $g \in G_N$, $(\lambda_{c,g(1)}, \ldots, \lambda_{c,g(N)})'$ has the same distribution as $(\lambda_{c,1}, \ldots, \lambda_{c,N})'$.

The algorithm below summarizes the testing procedure based on the permutation approach.

**Algorithm for Implementing the Permutation Procedure.**

All steps are identical to those in the algorithm in Section 3, except for step 4, which is replaced by the following:

4. A permutation approach is used to compute $P-$values by resampling from $\{\hat{\lambda}_1, \ldots, \hat{\lambda}_N\}$ without replacement to obtain $B$ random samples and computing the $P-$values:

$$P\text{-value}_1 = \frac{1}{B+1} \left( 1 + \sum_{b=1}^{B} \mathbb{I} \left( LM_{1N} (\hat{\Lambda}) \leq LM_{1N} (g_b\hat{\Lambda}) \right) \right)$$

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and
\[ P\text{-value}_2 = \frac{1}{B+1} \left( 1 + \sum_{b=1}^{B} \mathbb{I} \left( LM_{2N} \left( \hat{A} \right) \leq LM_{2N} \left( g_b \hat{A} \right) \right) \right), \]

with \( g_b \hat{A} = [\hat{\lambda}_{g_b(1)} \ldots \hat{\lambda}_{g_b(N)}]' \), \( g_b \in \mathcal{G}_N \).

We now show that the constructed procedure is asymptotically valid. For \( m = 1, 2 \), consider the permutation distribution \( R_{mN}(x) \) of \( LM_{mN} \left( \hat{A} \right) \) defined by
\[ R_{mN}(x) = \frac{1}{N!} \sum_{g \in \mathcal{G}_N} \mathbb{I} \left( LM_{mN} \left( g \hat{A} \right) \leq x \right). \tag{21} \]
We prove that, under the null, the permutation distributions \( R_{1N}(x) \) and \( R_{2N}(x) \) asymptotically approximate the cumulative distribution functions \( F_1(x) \) and \( F_2(x) \) of \( \max_{1 \leq j < k \leq S} Q_{j,k} \) and \( \min_{1 \leq j < k \leq S} Q_{j,k} \) respectively.

**Theorem 3.** Suppose that Assumptions 1, 2 and 4 are satisfied. As \( N,T \to \infty \), if \( \sqrt{N}/T \to 0 \), then under the null, the permutation distribution of \( LM_{mN} \left( \hat{A} \right) \), given in Equation (21), satisfies
\[ R_{mN}(x) \xrightarrow{P} F_m(x), \tag{22} \]
for \( m = 1, 2 \), and any real \( x \).

The proof of Theorem 3 is in Appendix B. Theorem 3 allows us to test for group-specific heterogeneity using all permutations in \( \mathcal{G}_N \). If all distinct permutations of \( \hat{\lambda}_1, \ldots, \hat{\lambda}_N \) into \( S \) groups of sizes \( N_1, \ldots, N_S \) were drawn and the factor loadings were observed, then the resulting test would be exact. However, this may be computationally intensive since we are in a high-dimensional setting where \( N \) goes to infinity and the number of possible permutations may be very large. For this reason, it is common practice to use a stochastic approximation of \( R_{mN}(x) \) defined by
\[ R_{mN,B}(x) = \frac{1}{B+1} \left( 1 + \sum_{b=1}^{B} \mathbb{I} \left( LM_{mN} \left( g_b \hat{A} \right) \leq x \right) \right), m = 1, 2, \]
where the permutations \( g_1, \ldots g_B \) are IID and uniform over \( \mathcal{G}_N \). See, for example, Canay, Romano, and Shaikh (2017, Remark 2.2). Similar approaches were used in different situations in Romano (1990) and Székely and Rizzo (2004).

In the next section, we investigate the size and the power of the permutation tests by comparing their performance to the results from the asymptotic framework developed in Section 3.
5 Monte Carlo Illustrations

We use Monte Carlo simulations of six data-generating processes (DGP) to evaluate the small-sample properties of the proposed testing procedures described in Section 3 and Section 4. The DGPs are denoted DGP 1-a, DGP 2-a, DGP 1-b, DGP 2-b, DGP 1-c, and DGP 2-c. DGP 1-a and DGP 2-a are used to investigate the size of the test while the other DGPs help evaluate the power of the test. In specifying the DGPs below, NID means normally and identically distributed, and U is the uniform distribution.

DGP 1-a assumes

\[ X_{it} = \lambda_{c,i} f_{c,t} + \kappa e_{it}, \quad i = 1, \ldots, N \text{ and } t = 1, \ldots, T, \]  

(23)

with

\[ e_{it} \sim \text{NID}(0,1), \quad f_{c,t} \sim \text{NID}(0,1), \quad \lambda_{c,i} \sim \text{NID}(b,1). \]

We choose \( \kappa = \sqrt{1 + b^2} \), where \( b \) is the mean of the factor loadings, such that \( R^2 = 1 - \frac{\text{trace}(E(\varepsilon \varepsilon'))}{\text{trace}(E(XX'))} = 0.50 \). DGP 2-a allows cross-sectional dependence in idiosyncratic errors

\[ e_{it} = \sigma_i \left( u_{it} + \sum_{1 \leq |j| \leq P} \theta u_{(i-j)t} \right), \quad u_{it} \sim \text{NID}(0,1), \]  

(24)

\[ \sigma_i \sim U(0.5, 1.5) \text{ and } \kappa = \sqrt{\frac{12(1 + b^2)}{13(1 + 2P\theta^2)}}, \]  

(25)

where \( \theta = 0.1 \) and \( P = 4 \).

In DGP 1-b and DGP 2-b, each group has a distinct specific factor. We assume group-specific heterogeneity involving four groups and one specific factor in each group, such that \( g_i = 1 \) if \( i \in \{1, \ldots, N/4\} \), \( g_i = 2 \) if \( i \in \{N/4 + 1, \ldots, N/2\} \), \( g_i = 3 \) if \( i \in \{N/2 + 1, \ldots, 3N/4\} \) and \( g_i = 4 \) if \( i \in \{3N/4 + 1, \ldots, N\} \), and

\[ X_{it} = \lambda_{c,i} f_{c,t} + \lambda_{g_i,i} f_{g_i,t} + \kappa e_{it}, \quad i = 1, \ldots, N \text{ and } t = 1, \ldots, T, \]  

(26)

with

\[ f_{g_i,t} \sim \text{NID}(0,1), \quad \lambda_{g_i,i} \sim \text{NID}(b,1). \]

We introduce a parameter \( \rho \) representing the correlation between any two specific factors and set \( \rho = 0.3 \). Similarly to DGP 2-a, DGP 2-b adds the cross-sectional dependence in idiosyncratic errors.
In all settings, we simulate the data $M = 10,000$ times, set $b = 1$ and use sample sizes $(N, T)$ that belong to $\{80, 120, 160, 200\} \times \{50, 100\}$. We simulate the limit distribution for the asymptotic results. The considered level is 5%. The critical values are computed using 500,000 simulated data.

When there is no group-specific heterogeneity, the $LM_{1N}$ and $LM_{2N}$ tests tend to provide rejection frequencies close to the level of the test (see Tables 1 and 2). When there is group-specific heterogeneity, Tables 3 and 4 show that the power of the tests generally increases with the number of variables and the sample size. Also, the $LM_{1N}$ test is shown to have a superior power compared to the $LM_{2N}$ test in all cases. However, when the number of variables increases, the results show that the power of the $LM_{2N}$ test increases rapidly and virtually catches up with the power of the $LM_{1N}$ test. The findings are consistent with the fact that the second test has a stronger alternative hypothesis than the first one. These results also suggest that the testing procedures exhibit good control of size and power.

When the asymptotic distribution is approximated using the permutation procedure, the results in Tables 1–4 show that the size is often around the 5% level. Also, the power increases toward 100% as the sample sizes increase, suggesting a good approximation of the asymptotic distributions derived in Section 3.

To illustrate a situation where the $LM_{1N}$ and $LM_{2N}$ tests may provide different outcomes, we consider the designs DGP 1-c and DGP 2-c. They are constructed by setting $f_{1,t} = f_{2,t} = f_{3,t}$, in DGP 1-b and DGP 2-b, respectively, so that the first three groups of variables are affected by the same specific factor. Similarly to DGP 2-a, DGP 2-c adds the cross-sectional dependence in idiosyncratic errors. As expected, the $LM_{2N}$ test has very low rejection frequencies, while the $LM_{1N}$ test shows very high rejection frequencies (see Tables 5 and 6). Indeed, by construction, the $LM_{1N}$ should be used if the purpose is to test whether group-specific heterogeneity appears in at least one pair of groups of variables and the $LM_{2N}$ is useful for testing if group-specific heterogeneity appears in all pairs of groups.
Table 1: Test rejection frequencies (%) for DGP 1-a

|                  | $LM_{1N}$ Asymptotic Test | $N = 80$ | $N = 120$ | $N = 160$ | $N = 200$ |
|------------------|---------------------------|---------|---------|---------|---------|
| $T = 50$         |                           | 4.35    | 4.47    | 4.54    | 4.53    |
| $T = 100$        |                           | 4.13    | 4.32    | 4.41    | 4.71    |

|                  | $LM_{2N}$ Asymptotic Test | $N = 80$ | $N = 120$ | $N = 160$ | $N = 200$ |
|------------------|---------------------------|---------|---------|---------|---------|
| $T = 50$         |                           | 4.87    | 4.66    | 5.14    | 4.72    |
| $T = 100$        |                           | 4.93    | 4.77    | 4.61    | 5.04    |

|                  | $LM_{1N}$ Permutation Test | $N = 80$ | $N = 120$ | $N = 160$ | $N = 200$ |
|------------------|---------------------------|---------|---------|---------|---------|
| $T = 50$         |                           | 4.86    | 4.50    | 4.73    | 4.56    |
| $T = 100$        |                           | 4.60    | 4.54    | 4.47    | 4.65    |

|                  | $LM_{2N}$ Permutation Test | $N = 80$ | $N = 120$ | $N = 160$ | $N = 200$ |
|------------------|---------------------------|---------|---------|---------|---------|
| $T = 50$         |                           | 4.86    | 4.60    | 4.99    | 4.54    |
| $T = 100$        |                           | 5.01    | 4.80    | 4.54    | 4.91    |

Note: This table presents the rejection frequencies over 10,000 simulated datasets when there is no group specific factor and the level of the test is 5%.
Table 2: Test rejection frequencies (%) for DGP 2-a

|                  | $N = 80$ | $N = 120$ | $N = 160$ | $N = 200$ |
|------------------|----------|-----------|-----------|-----------|
| $LM_{1N}$ Asymptotic Test |          |           |           |           |
| $T = 50$         | 5.49     | 4.97      | 5.89      | 5.64      |
| $T = 100$        | 5.27     | 5.16      | 5.48      | 5.42      |
| $LM_{2N}$ Asymptotic Test |          |           |           |           |
| $T = 50$         | 5.03     | 5.55      | 5.43      | 5.33      |
| $T = 100$        | 5.20     | 5.33      | 5.48      | 5.40      |
| $LM_{1N}$ Permutation Test |        |           |           |           |
| $T = 50$         | 5.88     | 5.15      | 5.99      | 5.82      |
| $T = 100$        | 5.74     | 5.42      | 5.54      | 5.50      |
| $LM_{2N}$ Permutation Test |        |           |           |           |
| $T = 50$         | 5.15     | 5.46      | 5.33      | 5.07      |
| $T = 100$        | 5.26     | 5.22      | 5.28      | 5.14      |

Note: See note for Table 1.
Table 3: Test rejection frequencies (%) for DGP 1-b

|                  | LM1N Asymptotic Test | N = 80 | N = 120 | N = 160 | N = 200 |
|------------------|----------------------|--------|---------|---------|---------|
|                  | T = 50               | 82.23  | 95.03   | 98.32   | 99.40   |
|                  | T = 100              | 94.94  | 99.48   | 99.99   | 100.00  |

|                  | LM2N Asymptotic Test | N = 80 | N = 120 | N = 160 | N = 200 |
|------------------|----------------------|--------|---------|---------|---------|
|                  | T = 50               | 50.50  | 75.09   | 86.68   | 92.32   |
|                  | T = 100              | 76.95  | 94.00   | 99.10   | 99.85   |

|                  | LM1N Permutation Test| N = 80 | N = 120 | N = 160 | N = 200 |
|------------------|----------------------|--------|---------|---------|---------|
|                  | T = 50               | 83.99  | 95.43   | 98.37   | 99.46   |
|                  | T = 100              | 95.48  | 99.48   | 99.99   | 100.00  |

|                  | LM2N Permutation Test| N = 80 | N = 120 | N = 160 | N = 200 |
|------------------|----------------------|--------|---------|---------|---------|
|                  | T = 50               | 50.24  | 74.80   | 86.47   | 92.14   |
|                  | T = 100              | 76.72  | 93.91   | 99.08   | 99.86   |

Note: This table presents the rejection frequencies over 10,000 simulated datasets when group specific factors arise and the level of the test is 5%.
Table 4: Test rejection frequencies (%) for DGP 2-b

|                  | $N = 80$ | $N = 120$ | $N = 160$ | $N = 200$ |
|------------------|----------|----------|----------|----------|
| $LM_{1N}$ Asymptotic Test |          |          |          |          |
| $T = 50$         | 94.50    | 98.65    | 99.46    | 99.80    |
| $T = 100$        | 99.55    | 99.93    | 100.00   | 100.00   |
| $LM_{2N}$ Asymptotic Test |          |          |          |          |
| $T = 50$         | 76.77    | 89.29    | 93.99    | 96.88    |
| $T = 100$        | 94.43    | 98.74    | 99.81    | 99.97    |
| $LM_{1N}$ Permutation Test |          |          |          |          |
| $T = 50$         | 95.07    | 98.48    | 99.48    | 99.80    |
| $T = 100$        | 99.63    | 99.93    | 100.00   | 100.00   |
| $LM_{2N}$ Permutation Test |          |          |          |          |
| $T = 50$         | 76.51    | 89.06    | 93.87    | 96.82    |
| $T = 100$        | 94.34    | 98.76    | 99.80    | 99.96    |

Note: See note for Table 3.
Table 5: Test rejection frequencies (%) for DGP 1-c

|                | \( N = 80 \) | \( N = 120 \) | \( N = 160 \) | \( N = 200 \) |
|----------------|--------------|--------------|--------------|--------------|
| **\( LM_{1N} \) Asymptotic Test** |              |              |              |              |
| \( T = 50 \)  | 81.27        | 94.96        | 98.58        | 99.43        |
| \( T = 100 \) | 93.42        | 99.55        | 99.55        | 100.00       |
| **\( LM_{2N} \) Asymptotic Test** |              |              |              |              |
| \( T = 50 \)  | 14.54        | 11.50        | 11.31        | 9.58         |
| \( T = 100 \) | 10.84        | 8.75         | 7.40         | 7.25         |
| **\( LM_{1N} \) Permutation Test** |              |              |              |              |
| \( T = 50 \)  | 83.96        | 95.52        | 98.76        | 99.44        |
| \( T = 100 \) | 94.70        | 99.63        | 99.98        | 100.00       |
| **\( LM_{2N} \) Permutation Test** |              |              |              |              |
| \( T = 50 \)  | 14.01        | 11.04        | 10.91        | 9.33         |
| \( T = 100 \) | 10.29        | 8.30         | 7.06         | 6.95         |

Note: See note for Table 3.
Table 6: Test rejection frequencies (%) for DGP 2-c

|                  | $N = 80$ | $N = 120$ | $N = 160$ | $N = 200$ |
|------------------|----------|-----------|-----------|-----------|
| $LM_{1N}$ Asymptotic Test |         |           |           |           |
| $T = 50$        | 91.06    | 97.69     | 99.15     | 99.60     |
| $T = 100$       | 97.64    | 99.84     | 99.99     | 100.00    |
| $LM_{2N}$ Asymptotic Test |         |           |           |           |
| $T = 50$        | 23.08    | 20.47     | 19.85     | 18.32     |
| $T = 100$       | 20.55    | 17.41     | 16.29     | 16.06     |
| $LM_{1N}$ Permutation Test |       |           |           |           |
| $T = 50$        | 92.44    | 97.92     | 99.30     | 99.59     |
| $T = 100$       | 98.16    | 99.86     | 99.99     | 100.00    |
| $LM_{2N}$ Permutation Test |       |           |           |           |
| $T = 50$        | 22.04    | 19.94     | 19.41     | 17.67     |
| $T = 100$       | 19.53    | 16.51     | 15.53     | 15.0      |

Note: See note for Table 3.
6 Empirical Illustration

This section illustrates the proposed statistical tests with an empirical application. The purpose of this application is to study the existence of heterogeneous groups of industries within the U.S. financial sector. It contributes to the understanding of industry groups and is related to other financial applications that analyze the performance of stocks relative to their membership in industry groups, or the propagation of systematic risks to a whole industry and the overall economy. See, for example, Kahle and Walkling (1996), Negro and Brooks (2005), and Hrazdil and Scott (2013), for the role played by industry groups in earnings management, event studies, and risk management.

Originally created in 1937 in the United States, the Standard Industrial Classification (SIC) was designed to identify homogenous groups of companies based on the nature of the production process and product characteristics. SIC codes for U.S. stocks are available from the Center for Research in Security Prices (CRSP). To conduct our empirical analysis, we use data on daily stock prices from the CRSP U.S. stock database, for the companies in the Finance, Insurance, and Real Estate division of the SIC. The sample consists of daily log returns for the period from January 3, 2018, to December 31, 2019, and includes 502 observations for each stock. Stocks with insufficient or missing observations are discarded.

The SIC system classifies companies into industries, industries into industry groups, industry groups into major groups, and major groups into divisions. We consider the classifications according to major groups and industry groups and, in both cases, keep only the groups for which at least 30 stocks are available in the sample. The definitions of the resulting groups and the corresponding number of stocks are shown in Table 7 and Table 8. The number of stocks in the groups ranges from 30 to 2331.

Using these data, we answer the following question. Can we find statistical evidence of heterogeneous SIC major groups and SIC industry groups? To answer this question, we apply the two statistical tests to check for evidence of group-specific heterogeneity. The results for the asymptotic tests are summarized in Table 9, which also includes the simulated critical values. For the current application, we have $S = 6$ and $r = 10$ for the major groups classification, and $S = 9$ and $r = 12$ for the industry groups classification.

Table 9 shows that, for both group classifications, the $LM_{1N}$ test clearly rejects the null
Table 7: The SIC major groups in the application

| SIC Name                                                      | Number of stocks in the sample |
|--------------------------------------------------------------|--------------------------------|
| Depository institutions                                    | (307)                          |
| Nondepository credit institutions                           | (36)                           |
| Security and commodity brokers, dealers, exchanges, and services | (101)                         |
| Insurance carriers                                         | (94)                           |
| Real estate                                                 | (45)                           |
| Holding and other investment offices                       | (2331)                         |

Table 8: The SIC industry groups in the application

| SIC Name                                                      | Number of stocks in the sample |
|--------------------------------------------------------------|--------------------------------|
| Commercial banks                                             | (237)                          |
| Savings institutions                                         | (67)                           |
| Security brokers, dealers, and flotation companies           | (30)                           |
| Services allied with the exchange of securities or commodities | (51)                           |
| Fire, marine, and casualty insurance                        | (39)                           |
| Holding offices                                             | (36)                           |
| Investment offices                                           | (2045)                         |
| Trusts                                                       | (48)                           |
| Miscellaneous investing                                     | (201)                          |
hypothesis of no group-specific heterogeneity. The $LM_{2N}$ test rejects the null hypothesis for the major groups, but not for the industry groups. These results suggest the following: there is at least one pair of heterogeneous industry groups, and all pairs of major industry groups are heterogeneous. When the permutation approach is used, we obtain the same results. In particular, for the major groups classification, the $P$ values for the $LM_{1N}$ and $LM_{2N}$ tests are both 0.001, while for the industry groups classification, the $P$ values for the two tests are 0.001 and 0.991, respectively.

Table 9: The results of the statistical tests

| Test          | SIC major groups | SIC industry groups |
|---------------|------------------|---------------------|
| $LM_{1N}$ Test| 1492.64          | 1380.83             |
| Critical value| 87.76            | 119.11              |
| $LM_{2N}$ Test| 87.94            | 38.67               |
| Critical value| 47.41            | 65.44               |

7 Conclusion

This article contributes to the literature on factor models by developing two tests for group-specific heterogeneity. Both tests have the same null hypothesis under which there is no group-specific heterogeneity. They differ in the type of alternative hypothesis considered. For the first test, it states that group-specific heterogeneity emerges in at least one pair of groups; for the second test, it states that group-specific heterogeneity emerges in all pairs of groups. We derive the asymptotic distributions of the test statistics under the null hypothesis, and prove that each test statistics diverges under the corresponding alternative hypothesis.

In addition, we propose and show the validity of permutation tests that can be used as an alternative to the asymptotic tests, if an exchangeability condition on the factor loadings
is satisfied. Interestingly, given the normalization of the proposed test statistics, arguments similar to those in Chung and Romano (2013) can be used to allow more heterogeneous factor loadings at the cost of more cumbersome derivations. However, this point is beyond the scope of this paper and is left for future research.

The proposed tests have the advantage that the estimation of the factors and factor loadings under the alternatives is not required. Results based on several simulation experiments are presented to illustrate the performance of the tests. An empirical investigation that detects the existence of heterogeneous groups of industries within the U.S. financial sector shows the empirical relevance of the proposed tests.

Appendix: Proofs of Results in Sections 3 and 4

Appendix A: Proofs of Results in Section 3

This appendix proves the validity of the suggested test statistics under the null and under the different alternatives. To prove Theorem 1 and Theorem 2, we rely on the following auxiliary results.

**Lemma A.1.** Suppose that Assumptions 1 and 2 are satisfied. If as \( N, T \to \infty, \sqrt{N}/T \to 0 \), then under the null, it holds that

\[
\frac{1}{N_j} \sum_{i=M_{j-1}+1}^{M_j} (\hat{\lambda}_i - \hat{H}^{-1}\Lambda_c,i) \lambda'_i = O_P \left( \frac{1}{\delta^2_{NT}} \right),
\]

\[
\frac{1}{N_j} \sum_{i=M_{j-1}+1}^{M_j} \|\hat{\lambda}_i - \hat{H}^{-1}\Lambda_c,i\|^2 = O_P \left( \frac{1}{\delta^2_{NT}} \right),
\]

for any \( j = 1, \ldots, S \).

**Lemma A.2.** Suppose that Assumption 1 (b), (g) and (h) are satisfied. As \( N \to \infty \), under the null, it holds that

\[
\lim_{N \to \infty} S \left( j, k, \Delta H_0^{-1} \right) = \lim_{N \to \infty} \text{Var} \left( A \left( j, k, \Delta H_0^{-1} \right) \right),
\]

for any \( j \) and \( k \) such that \( 1 \leq j < k \leq S \), and

\[
A_0 \left( j, \Delta H_0^{-1} \right) \xrightarrow{d} N \left( 0, \lim_{N \to \infty} \text{Var} \left( A_0 \left( j, \Delta H_0^{-1} \right) \right) \right),
\]

for any \( j \) such that \( 1 \leq j \leq S \).
Lemma A.3. Suppose that Assumptions 1 and 2 are satisfied. As \( N, T \to \infty, \) if \( \sqrt{N}/T \to 0, \) then for any \( j \) and \( k \) such that \( 1 \leq j < k \leq S, \) under the null, it holds that

\[
\| A(j, k, \hat{\Lambda}) - A(j, k, \Lambda H_0'^{-1}) \| = o_P(1) \tag{A.5}
\]

and

\[
\| S(j, k, \hat{\Lambda}) - S(j, k, \Lambda H_0'^{-1}) \| = o_P(1). \tag{A.6}
\]

Lemma A.4. Suppose that Assumptions 1–3 are satisfied. If as \( N, T \to \infty, \) \( \sqrt{N}/T \to 0, \) then for any positive integer \( j \) such that \( 1 \leq j \leq S, \) under the alternatives, it holds that

\[
\left| \frac{1}{N_j} \sum_{i=Mj-1+1}^{Mj} \left( \hat{\lambda}_i - \hat{\Xi}^{-1} \phi_i \right) \phi_i' \right| = O_P \left( \delta_{NT}^{-2} \right), \tag{A.7}
\]

\[
\left| \frac{1}{N_j} \sum_{i=Mj-1+1}^{Mj} \left\| \hat{\lambda}_i - \hat{\Xi}^{-1} \phi_i \right\|^2 \right| = O_P \left( \delta_{NT}^{-2} \right), \tag{A.8}
\]

and

\[
\left| S(j, k, \hat{\Lambda}) - S(j, k, \Phi \Xi_0'^{-1}) \right| = o_P(1). \tag{A.9}
\]

Since Lemma A.1, Lemma A.2 Equation (A.3), Lemma A.3 and Lemma A.4 can be proved following steps nearly identical to those in Djogbenou (2020, Lemmas A1, A2, 2.1 and A3), they are omitted. Moreover, the proof for Lemma A.2 Equation (A.4), which is similar to the proof that (A.21) in Djogbenou (2020) is asymptotically normal, is also omitted. We next present the proofs of Theorem 1 and Theorem 2.

Proof of Theorem 1

The proof consists of three parts. In the first and the second parts, we show that

\[
\max_{1 \leq j < k \leq S} \left| LM_N(j, k, \hat{\Lambda}) - LM_N(j, k, \Lambda H_0'^{-1}) \right| = o_P(1)
\]

and

\[
\max_{1 \leq j < k \leq S} \left| LM_N(j, k, \Lambda H_0'^{-1}) - LM_0N(j, k, \Lambda H_0'^{-1}) \right| = o_P(1),
\]

respectively, where

\[
LM_0N(j, k, \Lambda H_0'^{-1}) = A(j, k, \Lambda H_0'^{-1}) \left( \left( \frac{1}{\pi_j} + \frac{1}{\pi_k} \right) S_0 \right)^{-1} A(j, k, \Lambda H_0'^{-1}).
\]

In the third part, we derive the asymptotic distribution in (18) and (19) after using the previous result to establish that \( LM_1N(\hat{\Lambda}) \) and \( LM_2N(\hat{\Lambda}) \) have the same asymptotic
distributions as \( \max_{1 \leq j < k \leq S} LM_{0N}(j, k, \mathbf{H}_0^{-1}) \) and \( \min_{1 \leq j < k \leq S} LM_{0N}(j, k, \mathbf{H}_0^{-1}) \), respectively.

**Part 1:** Using the triangle inequality, we have

\[
\max_{1 \leq j < k \leq S} \left| LM_N(j, k, \hat{\mathbf{A}}) - LM_N(j, k, \mathbf{H}_0^{-1}) \right| \leq M_1 + M_2 + M_3,
\]

where

\[
M_1 = \max_{1 \leq j < k \leq S} \left( \left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) \right\| \left\| \mathbf{S}(j, k, \hat{\mathbf{A}})^{-1} - \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1} \right\| \left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) \right\| \right),
\]

\[
M_2 = \max_{1 \leq j < k \leq S} \left( \left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) - \mathbf{A}(j, k, \mathbf{H}_0^{-1}) \right\| \left\| \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1} \right\| \left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) \right\| \right)
\]

and

\[
M_3 = \max_{1 \leq j < k \leq S} \left( \left\| \mathbf{A}(j, k, \mathbf{H}_0^{-1}) \right\| \left\| \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1} \right\| \left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) - \mathbf{A}(j, k, \mathbf{H}_0^{-1}) \right\| \right).
\]

The proof uses the following auxiliary results that hold for any \( j \) and \( k \) such that \( 1 \leq j < k \leq S \): (a) \( \left\| \mathbf{A}(j, k, \mathbf{H}_0^{-1}) \right\| = O_P(1) \), (b) \( \left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) - \mathbf{A}(j, k, \mathbf{H}_0^{-1}) \right\| = o_P(1) \), and \( \left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) \right\| = O_P(1) \), and finally (c) \( \left\| \mathbf{S}(j, k, \hat{\mathbf{A}})^{-1} - \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1} \right\| = o_P(1) \) and \( \left\| \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1} \right\| = O_P(1) \). Result (a) follows from Lemma A.2 (A.4) and \( \mathbf{A}(j, k, \mathbf{H}_0^{-1}) = \mathbf{A}_0(j, \mathbf{H}_0^{-1}) - \mathbf{A}_0(k, \mathbf{H}_0^{-1}) \). Result (b) follows from Lemma A.3 (A.5) and

\[
\left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) \right\| \leq \left\| \mathbf{A}(j, k, \hat{\mathbf{A}}) - \mathbf{A}(j, k, \mathbf{H}_0^{-1}) \right\| + \left\| \mathbf{A}(j, k, \mathbf{H}_0^{-1}) \right\| = O_P(1) + O_P(1) = O_P(1),
\]

for any \( j \) and \( k \). Result (c) holds since for any \( j \) and \( k \), \( \left\| \mathbf{S}(j, k, \hat{\mathbf{A}})^{-1} - \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1} \right\| \)

is bounded by

\[
\left\| \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1}(\mathbf{S}(j, k, \mathbf{H}_0^{-1}) - \mathbf{S}(j, k, \hat{\mathbf{A}})) \mathbf{S}(j, k, \hat{\mathbf{A}})^{-1} \right\|
\]

\[
\leq \left\| \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1} \right\| \left\| \mathbf{S}(j, k, \hat{\mathbf{A}}) - \mathbf{S}(j, k, \mathbf{H}_0^{-1}) \right\| \left\| \mathbf{S}(j, k, \hat{\mathbf{A}})^{-1} \right\|,
\]

\[
\left\| \mathbf{S}(j, k, \hat{\mathbf{A}}) - \mathbf{S}(j, k, \mathbf{H}_0^{-1}) \right\| \xrightarrow{P} 0 \text{ by Lemma A.3 (A.6), and } \mathbf{S}(j, k, \mathbf{H}_0^{-1})^{-1} \text{ and } \mathbf{S}(j, k, \hat{\mathbf{A}})^{-1} \text{ are bounded in probability by Lemma A.2 (A.3) and Lemma A.3 (A.6), and Assumption 1 (g). From (a), (b) and (c), } \max_{1 \leq j < k \leq S} \left| LM_N(j, k, \hat{\mathbf{A}}) - LM_N(j, k, \mathbf{H}_0^{-1}) \right| = o_P(1).
\]

**Part 2:** Noting that for any \( j \) and \( k \) such that \( 1 \leq j < k \leq S \),

\[
\left| LM_N(j, k, \mathbf{H}_0^{-1}) - LM_{0N}(j, k, \mathbf{H}_0^{-1}) \right| \leq J_1 + J_2,
\]

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where \( J_1 \) equals
\[
\max_{1 \leq j < k \leq S} \left( \left\| A \left( j, k, \Lambda H_0^{-1} \right) \right\| \left( \text{Var} \left( A \left( j, k, \Lambda H_0^{-1} \right) \right) \right)^{-1} - \left( \frac{1}{\pi_j} + \frac{1}{\pi_k} \right) S_0 \right)^{-1} \left( \left\| A \left( j, k, \Lambda H_0^{-1} \right) \right\| \right)
\]
and \( J_2 \) equals
\[
\max_{1 \leq j < k \leq S} \left( \left\| A \left( j, k, \Lambda H_0^{-1} \right) \right\| \left( S \left( j, k, \Lambda H_0^{-1} \right) \right)^{-1} - \left( \text{Var} \left( A \left( j, k, \Lambda H_0^{-1} \right) \right) \right)^{-1} \left( \left\| A \left( j, k, \Lambda H_0^{-1} \right) \right\| \right) \right).
\]
Because \( \max_{1 \leq j < k \leq S} \| A \left( j, k, \Lambda H_0^{-1} \right) \| \) is \( O_P \left( 1 \right) \) from Part 1, we only need to show that
\[
\max_{1 \leq j < k \leq S} \left( \left( \text{Var} \left( A \left( j, k, \Lambda H_0^{-1} \right) \right) \right)^{-1} - \left( \frac{1}{\pi_j} + \frac{1}{\pi_k} \right) S_0 \right)^{-1} = o_P \left( 1 \right) \quad (A.10)
\]
and
\[
\max_{1 \leq j < k \leq S} \left( \left( S \left( j, k, \Lambda H_0^{-1} \right) \right)^{-1} - \left( \text{Var} \left( A \left( j, k, \Lambda H_0^{-1} \right) \right) \right)^{-1} \right) = o_P \left( 1 \right). \quad (A.11)
\]
Using Assumption 1 \( (g) \), we find that
\[
\lim_{N \to \infty} \text{Var} \left( A_0 \left( j, \Lambda H_0^{-1} \right) \right) = \frac{1}{\pi_j} S_0,
\]
and we deduce that
\[
\text{Var} \left( A \left( j, k, \Lambda H_0^{-1} \right) \right) = \text{Var} \left( A_0 \left( j, \Lambda H_0^{-1} \right) \right) + \text{Var} \left( A_0 \left( k, \Lambda H_0^{-1} \right) \right) \to \left( \frac{1}{\pi_j} + \frac{1}{\pi_k} \right) S_0,
\]
where the limit is positive definite. Therefore, Equation \( (A.10) \) holds. Furthermore, Lemma A.2 \( (A.3) \) implies \( S \left( j, k, \Lambda H_0^{-1} \right) \) converges in probability to the limit of \( \text{Var} \left( A \left( j, k, \Lambda H_0^{-1} \right) \right) \), which is positive definite. Thus, Equation \( (A.11) \) also holds.

**Part 3:** To show Equation \( (18) \), we first prove that \( \text{LM}_{1N} \left( \Lambda \right) - \max_{1 \leq j < k \leq S} \text{LM}_{0N} \left( j, k, \Lambda H_0^{-1} \right) = o_P \left( 1 \right) \), and then derive the asymptotic distribution of \( \max_{1 \leq j < k \leq S} \text{LM}_{0N} \left( j, k, \Lambda H_0^{-1} \right) \).

Similarly, to show Equation \( (19) \), we prove that \( \text{LM}_{2N} \left( \Lambda \right) - \min_{1 \leq j < k \leq S} \text{LM}_{0N} \left( j, k, \Lambda H_0^{-1} \right) = o_P \left( 1 \right) \), and derive the asymptotic distribution of \( \min_{1 \leq j < k \leq S} \text{LM}_{0N} \left( j, k, \Lambda H_0^{-1} \right) \). We use the fact that \( \left| \text{LM}_{1N} \left( \Lambda \right) - \max_{1 \leq j < k \leq S} \text{LM}_{0N} \left( j, k, \Lambda H_0^{-1} \right) \right| \) is bounded by
\[
\left| \text{LM}_{1N} \left( \Lambda \right) - \max_{1 \leq j < k \leq S} \text{LM}_{N} \left( j, k, \Lambda H_0^{-1} \right) \right| + \left( \max_{1 \leq j < k \leq S} \text{LM}_{N} \left( j, k, \Lambda H_0^{-1} \right) - \max_{1 \leq j < k \leq S} \text{LM}_{0N} \left( j, k, \Lambda H_0^{-1} \right) \right) \leq \left| \text{LM}_{N} \left( j, k, \Lambda \right) - \text{LM}_{N} \left( j, k, \Lambda H_0^{-1} \right) \right| + \left( \text{LM}_{N} \left( j, k, \Lambda H_0^{-1} \right) - \text{LM}_{0N} \left( j, k, \Lambda H_0^{-1} \right) \right) = o_P \left( 1 \right)
\]
and \( |LM_{2N} (\hat{\Lambda}) - \min_{1 \leq j < k \leq S} LM_{0N} (j, k, \Delta H_0^{-1})| \) is bounded by

\[
|LM_{2N} (\hat{\Lambda}) - \min_{1 \leq j < k \leq S} LM_{N} (j, k, \Delta H_0^{-1})| \\
+ |\min_{1 \leq j < k \leq S} LM_{N} (j, k, \Delta H_0^{-1}) - \min_{1 \leq j < k \leq S} LM_{0N} (j, k, \Delta H_0^{-1})| \\
\leq \max_{1 \leq j < k \leq S} |LM_{N} (j, k, \hat{\Lambda}) - LM_{N} (j, k, \Delta H_0^{-1})| \\
+ \max_{1 \leq j < k \leq S} |LM_{N} (j, k, \Delta H_0^{-1}) - LM_{0N} (j, k, \Delta H_0^{-1})| \\
= o_p (1),
\]

where the orders in probability come from Part 1 and Part 2. Therefore, by the asymptotic equivalence lemma, the asymptotic distributions of \( LM_{1N} (\hat{\Lambda}) \) and \( LM_{2N} (\hat{\Lambda}) \) are given by those of \( \max_{1 \leq j < k \leq S} LM_{0N} (j, k, \Delta H_0^{-1}) \) and \( \min_{1 \leq j < k \leq S} LM_{0N} (j, k, \Delta H_0^{-1}) \), respectively.

Furthermore, we observe that

\[
LM_{0N} (j, k, \Delta H_0^{-1}) = \left( \pi_j^{-1/2} V_j - \pi_k^{-1/2} V_k \right) \left( \pi_j^{-1} + \pi_k^{-1} \right)^{-1} \left( \pi_j^{-1/2} V_j - \pi_k^{-1/2} V_k \right),
\]

where \( V_j = \pi_j^{1/2} S_0^{-1/2} A_0 (j, \Delta H_0^{-1}) \). Moreover,

\[
\left( \text{Var} \left( A_0 (j, \Delta H_0^{-1}) \right) \right)^{-1/2} A_0 (j, \Delta H_0^{-1}) \xrightarrow{d} N \left( 0_{r_c(r_c+1)/2}, I_{r_c(r_c+1)/2} \right),
\]

given Lemma A.2 (A.4). Therefore,

\[
V_j \xrightarrow{d} Z_j \sim N \left( 0_{r_c(r_c+1)/2}, I_{r_c(r_c+1)/2} \right). \tag{A.12}
\]

Since, \( \max_{1 \leq j < k \leq S} LM_{0N} (j, k, \Delta H_0^{-1}) \) and \( \min_{1 \leq j < k \leq S} LM_{0N} (j, k, \Delta H_0^{-1}) \) are both continuous functions of \( V_j; j = 1, \ldots, S \), which are independent based on Assumption 1 (b), we obtain using Equation (A.12) that \( \max_{1 \leq j < k \leq S} LM_{0N} (j, k, \Delta H_0^{-1}) \) converges in distribution to

\[
\max_{1 \leq j < k \leq S} \left( \left( \pi_j^{-1/2} Z_j - \pi_k^{-1/2} Z_k \right) \left( \pi_j^{-1} + \pi_k^{-1} \right)^{-1} \left( \pi_j^{-1/2} Z_j - \pi_k^{-1/2} Z_k \right) \right)
\]

and \( \min_{1 \leq j < k \leq S} LM_{0N} (j, k, \Delta H_0^{-1}) \) converges in distribution to

\[
\min_{1 \leq j < k \leq S} \left( \left( \pi_j^{-1/2} Z_j - \pi_k^{-1/2} Z_k \right) \left( \pi_j^{-1} + \pi_k^{-1} \right)^{-1} \left( \pi_j^{-1/2} Z_j - \pi_k^{-1/2} Z_k \right) \right).
\]

Furthermore, \( \pi_j^{-1/2} Z_j - \pi_k^{-1/2} Z_k \sim N \left( 0_{r_c(r_c+1)/2}, \left( \pi_j^{-1} + \pi_k^{-1} \right) I_{r_c(r_c+1)/2} \right) \). Therefore, we obtain the results in Theorem 1. \qed
**Proof of Theorem 2**

Consider a pair of groups \((j, k)\). We use the decomposition

\[
\frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}_i' - \frac{1}{N_k} \sum_{i=M_{k_1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}_i' = R_1(j, k) + R_2(j) - R_2(k),
\]

where

\[
R_1(j, k) = \frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} \hat{\Xi}^{-1} \phi_i \phi_i' \hat{\Xi}'^{-1} - \frac{1}{N_k} \sum_{i=M_{k_1}+1}^{M_k} \hat{\Xi}^{-1} \phi_i \phi_i' \hat{\Xi}'^{-1}
\]

\[
R_2(j) = \frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}_i' - \frac{1}{N_k} \sum_{i=M_{j_1}+1}^{M_j} \hat{\Xi}^{-1} \phi_i \phi_i' \hat{\Xi}'^{-1}
\]

\[
R_2(k) = \frac{1}{N_k} \sum_{i=M_{k_1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}_i' - \frac{1}{N_j} \sum_{i=M_{k_1}+1}^{M_k} \hat{\Xi}^{-1} \phi_i \phi_i' \hat{\Xi}'^{-1},
\]

and analyze the limiting behavior of \(R_1(j, k)\), \(R_2(j)\) and \(R_2(k)\). Starting with \(R_2(j)\), we note that

\[
R_2(j) = \frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} (\hat{\lambda}_i - \hat{\Xi}^{-1} \phi_i) (\hat{\lambda}_i - \hat{\Xi}^{-1} \phi_i)' + \frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} \hat{\Xi}^{-1} \phi_i (\hat{\lambda}_i - \hat{\Xi}^{-1} \phi_i)'
\]

\[+ \frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} (\hat{\lambda}_i - \hat{\Xi}^{-1} \phi_i) \phi_i \hat{\Xi}'^{-1}.
\]

Hence, applying Lemma A.4 (A.7) and (A.8) and noting that \(\hat{\Xi}^{-1} = \Xi_0^{-1} + o_P(1)\), we obtain \(R_2(j) = O_P\left(\delta_{NT}^{-2}\right)\). Similarly, \(R_2(k) = O_P\left(\delta_{NT}^{-2}\right)\). In consequence,

\[
\frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}_i' - \frac{1}{N_k} \sum_{i=M_{k_1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}_i' = R_1(j) + O_P\left(\delta_{NT}^{-2}\right).
\]

Using the moment conditions in Assumption 3 (b), we have \(R_1(j, k) = R_0(j, k) + o_P(1)\), where

\[
R_0(j, k) = \Xi_0^{-1} \lim_{N \to \infty} \left( \frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} \phi_i \phi_i' - \frac{1}{N_k} \sum_{i=M_{k_1}+1}^{M_k} \phi_i \phi_i' \right) \Xi_0'^{-1}.
\]

Hence, for any pair of groups \(j\) and \(k\),

\[
\frac{1}{N_j} \sum_{i=M_{j_1}+1}^{M_j} \hat{\lambda}_i \hat{\lambda}_i' - \frac{1}{N_k} \sum_{i=M_{k_1}+1}^{M_k} \hat{\lambda}_i \hat{\lambda}_i' = R_0(j, k) + o_P(1),
\]

with \(R_0(j, k) \neq 0\) when group-specific heterogeneity emerges in the pair \((j, k)\), while \(R_0(j, k) = 0\) when group-specific heterogeneity does not emerge in the pair \((j, k)\). Indeed, for any pair \((j, k)\) with group-specific heterogeneity, \(R_0(j, k)\) is different from 0 since
the rows of $\Xi_0^{-1}$ are linearly independent and $\frac{1}{N} \Phi_j^T \Phi_j - \frac{1}{N} \Phi_k^T \Phi_k$ has a nonzero limit in probability. Noting from Lemma A.4 (A.9) that
\[
\| S (j,k, \hat{\Lambda}) - S (j,k, \Phi \Xi_0^{-1}) \| = o_P (1),
\]
and that $S (j,k, \Phi \Xi_0^{-1})$ has a positive definite limit in probability according to Assumption 3 (h), we obtain that $S (j,k, \hat{\Lambda})$ has a positive definite limit in probability, equal to $S_0 (j,k)$. From Equations (A.15) and (A.16), there are positive constants $\delta_m, m = 1, 2$, such that
\[
LM_{mN} (\hat{\Lambda}) = N \delta_m + o_P (N), m = 1, 2,
\]
with
\[
\delta_1 = \max_{1 \leq j < k \leq S} \left( (\text{Vech} (R_0(j,k)))' (S_0(j,k))^{-1} \text{Vech} (R_0(j,k)) \right)
\]
and
\[
\delta_2 = \min_{1 \leq j < k \leq S} \left( (\text{Vech} (R_0(j,k)))' (S_0(j,k))^{-1} \text{Vech} (R_0(j,k)) \right).
\]
Therefore, the results in Theorem 2 follow. \qed

Appendix B: Proofs of Results in Section 4

This appendix proves the asymptotic validity of the permutation approach.

Proof for Theorem 3

Suppose $G_N$ is a permutation uniformly distributed over $G_N$. Let
\[
LM_{0N} (j,k, G_N \Lambda H_0^{-1}) = A (j,k, G_N \Lambda H_0^{-1})' \left( \frac{1}{\pi_j} + \frac{1}{\pi_k} \right) S_0^{-1} A (j,k, G_N \Lambda H_0^{-1})
\]
Using Assumptions 1, 2 and 4, and proceeding as in the proof of Theorem 1, we get
\[
\left| LM_{1N} (G_N \hat{\Lambda}) - \max_{1 \leq j < k \leq S} LM_{0N} (j,k, G_N \Lambda H_0^{-1}) \right| = o_P (1)
\]
and
\[
\left| LM_{2N} (G_N \hat{\Lambda}) - \min_{1 \leq j < k \leq S} LM_{0N} (j,k, G_N \Lambda H_0^{-1}) \right| = o_P (1).
\]
Thus, $LM_{1N} (G_N \hat{\Lambda})$ and $LM_{2N} (G_N \hat{\Lambda})$ have the same asymptotic limits as $\max_{1 \leq j < k \leq S} LM_{0N} (j,k, G_N \Lambda H_0^{-1})$ and $\min_{1 \leq j < k \leq S} LM_{0N} (j,k, G_N \Lambda H_0^{-1})$, respectively.

In order to prove Theorem 3, we verify the following conditions of Chung and Romano (2013, Theorem 5.1), written for two permutations $G_N$ and $G'_N$, that are independent,
uniformly distributed over $G_N$, and independent of $A$:

$$(T_{1N}, U_{1N}) \equiv \left( \max_{1 \leq j < k \leq S} LM_{0N} \left( j, k, G_N \Lambda H^{-1}_0 \right), \max_{1 \leq j < k \leq S} LM_{0N} \left( j, G_N \Lambda H^{-1}_0 \right) \right) \xrightarrow{d} (T_1, U_1),$$

where $T_1 = \max_{1 \leq l \leq \frac{s(s-1)}{2}} Q_{r_n(r_n+1)} (l)$ and $U_1 = \max_{1 \leq l \leq \frac{s(s-1)}{2}} Q'_{r_n(r_n+1)} (l)$ are independent with the same CDF $F_1(\cdot)$ and

$$(T_{2N}, U_{2N}) \equiv \left( \min_{1 \leq j < k \leq S} LM_{0N} \left( j, k, G_N \Lambda H^{-1}_0 \right), \min_{1 \leq j < k \leq S} LM_{0N} \left( j, G_N \Lambda H^{-1}_0 \right) \right) \xrightarrow{d} (T_2, U_2),$$

where $T_2 = \min_{1 \leq l \leq \frac{s(s-1)}{2}} Q_{r_n(r_n+1)} (l)$ and $U_2 = \min_{1 \leq l \leq \frac{s(s-1)}{2}} Q'_{r_n(r_n+1)} (l)$ are independent with the same CDF $F_2(\cdot)$.

We recall that

$$(T_{1N}, U_{1N}) = (f_1(V_{N1}, \ldots, V_{NS}), f_1(W_{N1}, \ldots, W_{NS})), $$

and

$$(T_{2N}, U_{2N}) = (f_2(V_{N1}, \ldots, V_{NS}), f_2(W_{N1}, \ldots, W_{NS})), $$

with

$$f_1(x_1, \ldots, x_S) = \max_{1 \leq j < k \leq S} \left( \left( \pi_j^{-1/2} x_j - \pi_k^{-1/2} x_k \right)' \left( \pi_j^{-1} + \pi_k^{-1} \right)^{-1} \left( \pi_j^{-1/2} x_j - \pi_k^{-1/2} x_k \right) \right),$$

and

$$f_2(x_1, \ldots, x_S) = \min_{1 \leq j < k \leq S} \left( \left( \pi_j^{-1/2} x_j - \pi_k^{-1/2} x_k \right)' \left( \pi_j^{-1} + \pi_k^{-1} \right)^{-1} \left( \pi_j^{-1/2} x_j - \pi_k^{-1/2} x_k \right) \right),$$

$$V_{Nj} = \pi_j^{1/2} S_0^{-1/2} \sqrt{N} \sum_{i=1}^N l_{jG_N(i)} \text{Vech} \left( H_0^{-1} \left( \lambda_c, \lambda_c, \Sigma_A \right) H_0^{-1} \right),$$

and

$$W_{Nj} = \pi_j^{1/2} S_0^{-1/2} \sqrt{N} \sum_{i=1}^N l_{jG_N(i)} \text{Vech} \left( H_0^{-1} \left( \lambda_c, \lambda_c, \Sigma_A \right) H_0^{-1} \right).$$

In order to show that equations Equation (B.1) and Equation (B.2) hold, we show that $(V'_{N1}, \ldots, V'_{NS}, W'_{N1}, \ldots, W'_{NS})'$ is asymptotically normal. To this end, we use the Cramér-Wold device and let $a$ and $b$ be two $\frac{(r_n+1)r_n}{2} S$-dimensional vectors of real values and prove that

$$\Xi^{-1/2} \sum_{j=1}^S \left( V'_{Nj} a_j + W'_{Nj} b_j \right) \xrightarrow{d} N(0, 1),$$

(B.5)
Hence, it is enough to prove that use Davidson (1994, Theorem 24.2 (iii)) by showing that normal with mean 0 limiting distribution. In the second part, we establish that finally show that Equation (B.1) and Equation (B.2) hold.

To see that, we note that

\[
\Xi = \sum_{j=1}^{S} \frac{\sqrt{N}}{N_j} \pi_j^{1/2} l_j G_N(t) b_j S_0^{-1/2} \operatorname{Vech} \left( H_0^{-1} \left( \lambda_{c,l} \lambda'_{c,l} - \Sigma_{\Lambda} \right) H_0^{-1} \right),
\]

\[
W_{NI} = \sum_{j=1}^{S} \frac{\sqrt{N}}{N_j} \pi_j^{1/2} l_j G_N'(t) b_j' S_0^{-1/2} \operatorname{Vech} \left( H_0^{-1} \left( \lambda_{c,l} \lambda'_{c,l} - \Sigma_{\Lambda} \right) H_0^{-1} \right),
\]

and

\[
\Xi^{-1/2} \sum_{j=1}^{N} \left( V'_{Nj} a_j + W'_{Nj} b_j \right) = \sum_{l=1}^{N} Z_{NIl}.
\]

Given these notations, we complete the proof of Theorem 3 in three parts. In the first part, we prove that \( \sum_{l=1}^{N} Z_{NI} \) and \( \sum_{l=1}^{N} Z_{Nl} \), where \( Z_{NI} = Z_{NI} \left( \sum_{m=1}^{l-1} Z_{Nml}^{2} \leq 2 \right) \), have the same limiting distribution. In the second part, we establish that \( \sum_{l=1}^{N} Z_{Nl} \), is asymptotically normal with mean 0 and variance 1. In the third part, we deduce Equation (B.5) and finally show that Equation (B.1) and Equation (B.2) hold.

**Part 1:** To prove that \( \sum_{l=1}^{N} Z_{NI} \) and \( \sum_{l=1}^{N} Z_{Nl} \) have the same limiting distribution, we use Davidson (1994, Theorem 24.2 (iii)) by showing that \( \sum_{l=1}^{N} Z_{NI} \xrightarrow{P} 1 \) when \( N \to \infty \).

To see that, we note that

\[
\sum_{l=1}^{N} Z_{NI}^{2} = \Xi^{-1} \sum_{l=1}^{N} \left( V_{NI}^{2} + W_{NI}^{2} + 2 V_{NI} W_{NI} \right).
\]

Hence, it is enough to prove that

\[
\sum_{l=1}^{N} V_{NI}^{2} \xrightarrow{P} \sum_{j=1}^{S} a_j' a_j, \quad (B.6)
\]

\[
\sum_{l=1}^{N} W_{NI}^{2} \xrightarrow{P} \sum_{j=1}^{S} b_j' b_j, \quad (B.7)
\]

and

\[
\sum_{l=1}^{N} V_{NI} W_{NI} \xrightarrow{P} \sum_{j=1}^{S} \sum_{k=1}^{S} \pi_j^{1/2} \pi_k^{1/2} a_j' b_k. \quad (B.8)
\]

Note that

\[
\sum_{l=1}^{N} V_{NI}^{2} = \sum_{j=1}^{S} \sum_{k=1}^{S} \pi_j^{1/2} \pi_k^{1/2} N_j N_k \frac{a_j' b_k}{\sqrt{N_j N_k}} \left( \frac{1}{N} \sum_{l=1}^{N} l_j G_N(t) b_l b_l' H_0^{-1} \left( \lambda_{c,l} \lambda'_{c,l} - \Sigma_{\Lambda} \right) H_0^{-1} \right) S_0^{-1/2}.
\]

where

\[
B_l = \operatorname{Vech} \left( H_0^{-1} \left( \lambda_{c,l} \lambda'_{c,l} - \Sigma_{\Lambda} \right) H_0^{-1} \right).
\]

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Since $\pi_{j,k}^{1/2} N^{-1/2}$, we can show that $\sum_{j=1}^{\infty} V_{N}^{2} \rightarrow \sum_{j=1}^{\infty} a_{j}^{*} a_{j} = \sum_{j=1}^{\infty} a_{j}^{*} a_{j} = 0$ if $j = k$.

If $a_{j}^{*} S_{0}^{-1/2} \left( \frac{1}{N} \sum_{i=1}^{N} t_{jG_{N}(l)kG_{N}(l)} B_{i} B_{i}^{*} \right) S_{0}^{-1/2} a_{k} \rightarrow a_{j}^{*} a_{k} \pi_{j,k}^{1/2} \pi_{k,j}^{1/2} (j = k)$. To get the results, we write that $a_{j}^{*} S_{0}^{-1/2} \left( \frac{1}{N} \sum_{l=1}^{N} t_{jG_{N}(l)kG_{N}(l)} B_{i} B_{i}^{*} \right) S_{0}^{-1/2} a_{k} = a_{j}^{*} a_{k} \pi_{j,k}^{1/2} \pi_{k,j}^{1/2} (j = k) + \frac{1}{N} \sum_{l=1}^{N} t_{jG_{N}(l)kG_{N}(l)} a_{j}^{*} \left( S_{0}^{-1/2} B_{i} B_{i}^{*} S_{0}^{-1/2} - I \right) a_{l} + a_{j}^{*} a_{k} \frac{1}{N} \sum_{l=1}^{N} \left( t_{jG_{N}(l)kG_{N}(l)} - \pi_{j}^{1/2} \pi_{k}^{1/2} (j = k) \right) .

So it will be sufficient to prove that

$$\frac{1}{N} \sum_{l=1}^{N} t_{jG_{N}(l)kG_{N}(l)} a_{j}^{*} \left( S_{0}^{-1/2} B_{i} B_{i}^{*} S_{0}^{-1/2} - I \right) a_{l} \rightarrow 0.$$  \hspace{1cm} (B.9)

and

$$a_{j}^{*} a_{k} \frac{1}{N} \sum_{l=1}^{N} \left( t_{jG_{N}(l)kG_{N}(l)} - \pi_{j}^{1/2} \pi_{k}^{1/2} (j = k) \right) \rightarrow 0.$$

To see Equation (B.9), we use $E \left( \frac{1}{N} \sum_{l=1}^{N} t_{jG_{N}(l)kG_{N}(l)} a_{j}^{*} \left( S_{0}^{-1/2} B_{i} B_{i}^{*} S_{0}^{-1/2} - I \right) a_{k} \right)^{2}$ equals

$$\frac{1}{N^{2}} \sum_{l=1}^{N} E \left( t_{jG_{N}(l)kG_{N}(l)} \right) E \left( \left( a_{j}^{*} S_{0}^{-1/2} B_{i} B_{i}^{*} S_{0}^{-1/2} - I \right) a_{k} \right)^{2},$$

where we employ the law of iterated expectation, the independence across $l$ of $\lambda_{c,l}$ conditional on the permutation $G_{N}$, and the independence of $G_{N}$ and $\lambda_{c,l}$. Because $E \left( t_{jG_{N}(l)kG_{N}(l)}^{2} \right) = \frac{N_{j}}{N}$ if $j = k$, and $E \left( t_{jG_{N}(l)kG_{N}(l)}^{2} \right) = 0$ if $j \neq k$, and $E \left( \left( a_{j}^{*} \left( S_{0}^{-1/2} B_{i} B_{i}^{*} S_{0}^{-1/2} - I \right) a_{j} \right)^{2} \right) \leq C$ given Assumption 1 (b), we have that

$$E \left( \frac{1}{N} \sum_{l=1}^{N} t_{jG_{N}(l)kG_{N}(l)} a_{j}^{*} \left( S_{0}^{-1/2} B_{i} B_{i}^{*} S_{0}^{-1/2} - I \right) a_{k} \right)^{2} \leq C \frac{N_{j}}{N} \frac{1}{N} = o(1) .$$

Thus, Equation (B.9) holds. For Equation (B.10), we also observe that

$$E \left( \frac{1}{N} \sum_{l=1}^{N} \left( t_{jG_{N}(l)kG_{N}(l)} - \pi_{j}^{1/2} \pi_{k}^{1/2} (j = k) \right) \right)^{2} \rightarrow 0$$

$$\quad \frac{1}{N^{2}} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left( t_{jG_{N}(l)G_{N}(m)lG_{N}(m)} \right) - 2 \pi_{j}^{1/2} \pi_{k}^{1/2} (j = k) \frac{1}{N} \sum_{l=1}^{N} E \left( t_{jG_{N}(l)kG_{N}(l)} \right) + \frac{1}{N} \sum_{l=1}^{N} E \left( t_{jG_{N}(l)kG_{N}(l)} \right) \rightarrow \pi_{j}^{2} \pi_{k} \pi_{j} \pi_{k}. $$

Since $\frac{1}{N} \sum_{l=1}^{N} E \left( t_{jG_{N}(l)kG_{N}(l)} \right) = \frac{N_{j}}{N} \pi_{j} (j = k) \rightarrow \pi_{j} \pi_{j}$, and

$$\frac{1}{N^{2}} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left( t_{jG_{N}(l)G_{N}(m)lG_{N}(m)} \right) \rightarrow \frac{1}{N^{2}} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left( t_{jG_{N}(l)G_{N}(m)} \right) \rightarrow \pi_{j}^{2} \pi_{j} (j = k),$$

the latter using $E \left( t_{jG_{N}(l)G_{N}(m)} \right) = \frac{N_{j} N_{j-1}}{N^{2}}$ for $l \neq m$, we have

$$E \left( \frac{1}{N} \sum_{l=1}^{N} \left( t_{jG_{N}(l)kG_{N}(l)} - \pi_{j}^{1/2} \pi_{k}^{1/2} (j = k) \right) \right)^{2} \leq 2 \pi_{j}^{2} \pi_{j} (j = k) - 2 \pi_{j}^{2} \pi_{j} (j = k) + o(1) = o(1) .$$
Given Equation (B.9) and Equation (B.10), Equation (B.6) follows. Equation (B.7) also follows with identical steps.

Let us now look at Equation (B.8). Since \( \frac{\pi_j^{1/2} \pi_k^{1/2} N^2}{N_j N_k} \to \frac{1}{\pi_j^{1/2} \pi_k^{1/2}} \) and

\[
\sum_{l=1}^{N} V_{Nl} W_{Nl} = \sum_{j=1}^{S} \sum_{k=1}^{S} \frac{\pi_j^{1/2} \pi_k^{1/2} N^2}{N_j N_k} a_j^t S_0^{-1/2} \left( \frac{1}{N} \sum_{l=1}^{N} t_{jG_N(l)} t_{kG'_N(l)} B_l B_l' \right) S_0^{-1/2} b_k,
\]

we can obtain that \( \sum_{l=1}^{N} V_{Nl} W_{Nl} \to \sum_{j=1}^{S} \sum_{k=1}^{S} \frac{1}{\pi_j^{1/2} \pi_k^{1/2}} a_j^t b_k \) if

\[
a_j^t S_0^{-1/2} \left( \frac{1}{N} \sum_{l=1}^{N} t_{jG_N(l)} t_{kG'_N(l)} B_l B_l' \right) S_0^{-1/2} b_k \to a_j^t b_k \pi_j \pi_k.
\]

To do so, we decompose \( a_j^t S_0^{-1/2} \left( \frac{1}{N} \sum_{l=1}^{N} t_{jG_N(l)} t_{kG'_N(l)} B_l B_l' \right) S_0^{-1/2} b_k \) as

\[
a_j^t b_k \pi_j \pi_k + \frac{1}{N} \sum_{l=1}^{N} t_{jG_N(l)} t_{kG'_N(l)} a_j^t \left( S_0^{-1/2} B_l B_l' S_0^{-1/2} - I \right) b_k + a_j^t b_k \frac{1}{N} \sum_{l=1}^{N} \left( t_{jG_N(l)} t_{kG'_N(l)} - \pi_j \pi_k \right),
\]

and show that

\[
\frac{1}{N} \sum_{l=1}^{N} t_{jG_N(l)} t_{kG'_N(l)} a_j^t \left( S_0^{-1/2} B_l B_l' S_0^{-1/2} - I \right) b_k \to 0 \tag{B.11}
\]

and

\[
\frac{1}{N} \sum_{l=1}^{N} \left( t_{jG_N(l)} t_{kG'_N(l)} - \pi_j \pi_k \right) \to 0. \tag{B.13}
\]

To get Equation (B.12), we use the equality

\[
E \left( \frac{1}{N} \sum_{l=1}^{N} t_{jG_N(l)} t_{kG'_N(l)} a_j^t \left( S_0^{-1/2} B_l B_l' S_0^{-1/2} - I \right) b_k \right)^2
= \frac{1}{N^2} \sum_{l=1}^{N} E \left( t_{jG_N(l)} t_{kG'_N(l)} \right) E \left( a_j^t \left( S_0^{-1/2} B_l B_l' S_0^{-1/2} - I \right) b_k \right)^2,
\]

where we apply again the law of iterated expectation, the independence across \( l \) of \( \lambda_{c,l} \) conditional on the permutations \( G_N \) and \( G'_N \), and the independence of \( G_N, G'_N, \) and \( \lambda_{c,l} \). In addition, \( E \left( a_j^t \left( S_0^{-1/2} B_l B_l' S_0^{-1/2} - I \right) b_k \right)^2 \leq C \) given Assumption 1 (b), and

\[
E \left( t_{jG_N(l)} t_{kG'_N(l)} \right) = E \left( t_{jG_N(l)} \right) E \left( t_{kG'_N(l)} \right) = \frac{N_j N_k}{N}
\]

due to the independence of \( G_N \) and \( G'_N \). As a result, we get

\[
E \left( \frac{1}{N} \sum_{l=1}^{N} t_{jG_N(l)} t_{kG'_N(l)} a_j^t \left( S_0^{-1/2} B_l B_l' S_0^{-1/2} - I \right) b_k \right)^2
\leq C \frac{1}{N} \frac{N_j N_k}{N} = o(1).
\]
For Equation (B.13), we noted that \( E \left( \frac{1}{N} \sum_{l=1}^{N} \left( t_j G_N(l) t_k G_N^*(l) - \pi_j \pi_k \right) \right)^2 \) equals

\[
\frac{1}{N^2} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left( t_j G_N(l) t_j G_N(m) t_k G_N^*(l) t_k G_N^*(m) \right) - 2 \pi_j \pi_k \frac{1}{N} \sum_{l=1}^{N} E \left( t_j G_N(l) t_k G_N^*(l) \right) + \pi_j^2 \pi_k^2.
\]

Since \( \frac{1}{N} \sum_{l=1}^{N} E \left( t_j G_N(l) t_k G_N^*(l) \right) = \frac{1}{N} \sum_{l=1}^{N} E \left( t_k G_N^*(l) \right) = \frac{N_j N_k}{N} \rightarrow \pi_j \pi_k \), and

\[
\frac{1}{N^2} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left( t_j G_N(l) t_j G_N(m) \right) \bigg( \left( t_k G_N^*(l) t_k G_N^*(m) \right)
= \frac{1}{N^2} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left( t_j G_N(l) t_j G_N(m) \right) + \frac{1}{N^2} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left( t_j G_N(l) t_j G_N(m) \right) E \left( t_k G_N^*(l) t_k G_N^*(m) \right)
= \frac{N_j N_k}{N} + \frac{N(N-1)N_j N_k - 1}{N^2} \frac{N_k}{N-1} \frac{N_k}{N-1} + o(1)
= \frac{N_j N_k - 1}{N} \frac{N_k}{N-1} + o(1) = (\pi_j \pi_k)^2 + o(1).
\]

Thus, we have \( E \left( \frac{1}{N} \sum_{l=1}^{N} \left( t_j G_N(l) t_k G_N^*(l) - \pi_j \pi_k \right) \right)^2 = o(1) \), which implies Equation (B.13). Hence, we have Equation (B.8). Since Equation (B.6), Equation (B.6) and Equation (B.8) hold, we conclude that \( \sum_{l=1}^{N} \tilde{Z}_{Nl} \rightarrow 1 \). Therefore, from Davidson (1994, Theorem 24.2 (iii)), we conclude that \( \sum_{l=1}^{N} Z_{Nl} \) and \( \sum_{l=1}^{N} \tilde{Z}_{Nl} \) have the same limiting distribution.

**Part 2:** Second, we verify that \( \tilde{Z}_{Nl}, 1 \leq l \leq N \), satisfy the conditions of McLeish (1974, Theorem 2.1), implying that \( \sum_{l=1}^{N} \tilde{Z}_{Nl} \) is asymptotically \( N(0, 1) \). In particular, we have to prove that (a) \( \sum_{l=1}^{N} \tilde{Z}_{Nl}^2 \rightarrow 1 \) as \( N \rightarrow \infty \), (b) \( \max_{1 \leq l \leq N} \left| \tilde{Z}_{Nl} \right| \rightarrow 0 \) as \( N \rightarrow \infty \), (c) for any real \( v \), \( E(T_N) \rightarrow 1 \) as \( N \rightarrow \infty \) and \( \{T_N\} \) is uniformly integrable, where \( T_N = \prod_{l=1}^{N} (1 + iv \tilde{Z}_{Nl}) \), and \( i \) denotes the complex number \( \sqrt{-1} \) in Part 2 of the proof.

For (a), we use the fact that \( \sum_{l=1}^{N} \tilde{Z}_{Nl}^2 \rightarrow 1 \) and Davidson (1994, Theorem 24.2 (ii)), to conclude that \( \sum_{l=1}^{N} \tilde{Z}_{Nl}^2 \rightarrow 1 \). To prove (c), we define \( \tilde{T}_{l-1} = \prod_{m=1}^{l-1} (1 + iv \tilde{Z}_{Nm}) \) and use the identity

\[
\tilde{T}_N = \prod_{l=1}^{N} (1 + iv \tilde{Z}_{Nl}) = 1 + iv \sum_{l=1}^{N} \tilde{T}_{l-1} \tilde{Z}_{Nl} = 1 + iv \sum_{l=1}^{N} \tilde{T}_{l-1} Z_{Nl} \mathbb{1} \left( \sum_{m=1}^{l-1} Z_{Nm}^2 \leq 2 \right).
\]  

(B.14)
Furthermore, we have

\[ E(Z_{Nl} | \lambda_{c,1}, \ldots, \lambda_{c,l-1}) = \Xi^{-1/2} E(V_{Nl} | \lambda_{c,1}, \ldots, \lambda_{c,l-1}) \]

\[ = \Xi^{-1/2} \sum_{j=1}^{S} a_j S^{-1/2} \pi_j^{1/2} \sqrt{N} E_{lG_N(t)} \left( \text{Vech} \left( H_0^{-1} \left( \lambda_{c,l} \lambda_{c,l}^T - \Sigma_A \right) H_0^{-1} \right) | \lambda_{c,1}, \ldots, \lambda_{c,l-1} \right) \]

\[ + \Xi^{-1/2} \sum_{j=1}^{S} b_j S^{-1/2} \pi_j^{1/2} \sqrt{N} E_{lG_N(t)} \left( \text{Vech} \left( H_0^{-1} \left( \lambda_{c,l} \lambda_{c,l}^T - \Sigma_A \right) H_0^{-1} \right) | \lambda_{c,1}, \ldots, \lambda_{c,l-1} \right) \]

\[ = 0. \]

Combining this result with Equation (B.14), we get from the law of iterated expectations that \( E(T_N) = E(\tilde{T}_N) = 1 \). In addition, \( \text{sup}_N E(\max_{1 \leq l \leq N} Z_{Nl}^2) \leq \text{sup}_N E(\sum_{l=1}^{N} Z_{Nl}^2) \leq C < \infty \). Hence, Davidson (1994, Theorem 24.2 (i)) implies that \( T_N \) is uniformly integrable, completing the proof of (c).

For (b), we note that \( \max_{1 \leq l \leq N} |\tilde{Z}_{Nl}| P \rightarrow 0 \) if we can show that \( \max_{1 \leq l \leq N} |Z_{Nl}| P \rightarrow 0 \). Consequently, we observe that for any \( \varepsilon > 0 \), and for some scalar \( \eta > 0 \),

\[ 0 \leq P \left( \max_{1 \leq l \leq N} |Z_{Nl}| > \varepsilon \right) = P \left( \bigcup_{1 \leq l \leq N} \{|Z_{Nl}| > \varepsilon\} \right) \leq \sum_{l=1}^{N} P \left( |Z_{Nl}| > \varepsilon \right) \leq \varepsilon^{-2+\eta} \sum_{l=1}^{N} E \left( |Z_{Nl}|^{2+\eta} \right), \]

where the last inequality follows from Markov’s inequality. Therefore, (b) is satisfied if we prove that as \( N \rightarrow \infty \), we have

\[ \sum_{l=1}^{N} E \left( |Z_{Nl}|^{2+\eta} \right) \rightarrow 0, \]

for any \( \varepsilon > 0 \) and for some scalar \( \eta > 0 \). Using the c-r inequality, we have

\[ E \left( |Z_{Nl}|^{2+\eta} \right) = E \left( \Xi^{-1/2} V_{Nl} + \Xi^{-1/2} W_{Nl} \right)^{2+\eta} \leq 2^{1+\eta} E \left( |\Xi^{-1/2} V_{Nl}|^{2+\eta} \right) + 2^{1+\eta} E \left( |\Xi^{-1/2} W_{Nl}|^{2+\eta} \right). \]

We note that \( E \left( |\Xi^{-1/2} V_{Nl}|^{2+\eta} \right) \) can be expressed as

\[ \Xi^{-(2+\eta)/2} S^{2+\eta} E \left( \frac{1}{S} \sum_{j=1}^{S} N^{1/2} \pi_j^{1/2} a_j S^{-1/2} \sqrt{N} E_{lG_N(t)} \left( \text{Vech} \left( H_0^{-1} \left( \lambda_{c,l} \lambda_{c,l}^T - \Sigma_A \right) H_0^{-1} \right) \right) \right) \]

Using the Jensen inequality, the Cauchy-Schwarz inequality, and the independence of \( G_N \).
and the factor loadings, we find that $E \left( \left| \Xi^{-1/2} V_{NI} \right|^{2+\eta} \right)$ is bounded from above by

$$
\Xi^{-(2+\eta)/2} S^{1+\eta} \sum_{j=1}^{S} E \left( \left\| N^{1/2} \frac{\pi_j}{N_j} a_j S_0^{-1/2} \right\|^{2+\eta} \right) \leq \Xi^{-(2+\eta)/2} S^{1+\eta} \sum_{j=1}^{S} E \left( \left\| \frac{N^{1/2} \pi_j}{N_j} a_j S_0^{-1/2} \right\|^{2+\eta} \right) E \left( \left\| \text{Vech} \left( \begin{pmatrix} \lambda_{c,t} x_{c,t} - \Sigma_\lambda \end{pmatrix} H_0^{-1} \right) \right\|^{2+\eta} \right).
$$

In addition,

$$
E \left( \left\| \frac{N^{1/2} \pi_j}{N_j} a_j' S_0^{-1/2} \right\|^{2+\eta} \right) = \left\| \frac{N^{1/2} \pi_j}{N_j} a_j' S_0^{-1/2} \right\|^{2+\eta} \left( \frac{N^{1/2}}{N_j} \right)^{2+\eta} E \left( \left\| \text{Vech} \left( \begin{pmatrix} \lambda_{c,t} x_{c,t} - \Sigma_\lambda \end{pmatrix} H_0^{-1} \right) \right\|^{2+\eta} \right) \leq C \left( \frac{N^{1/2}}{N_j^{1+\eta}} \right)
$$
and

$$
E \left( \left| \Xi^{-1/2} V_{NI} \right|^{2+\eta} \right) \leq \Xi^{-(2+\eta)/2} S^{1+\eta} C^2 \sum_{j=1}^{S} \left( \frac{N^{1/2}}{N_j^{1+\eta}} \right) \text{A similar inequality can be obtained for } E \left( \left| \Xi^{-1/2} W_{NI} \right|^{2+\eta} \right).
$$

We finally have

$$
\sum_{l=1}^{N} E \left( \left| Z_{NI} \right|^{2+\eta} \right) \leq 2^{2+\eta} \Xi^{-(2+\eta)/2} S^{1+\eta} C^2 \sum_{j=1}^{S} \left( \frac{N^{1+\eta/2}}{N_j^{1+\eta}} \right) = O \left( \frac{1}{N^{\eta/2}} \right) = o(1).
$$

**Part 3:** We showed in Part 1 that $\sum_{l=1}^{N} Z_{NI}$ and $\sum_{l=1}^{N} \tilde{Z}_{NI}$ have the same limiting distribution, and proved in Part 2 that $\sum_{l=1}^{N} \tilde{Z}_{NI}$ is asymptotically $N(0, 1)$. It follows that

$$
\sum_{j=1}^{S} \left( V_{Nj}' a_j + W_{Nj}' b_j \right) \xrightarrow{d} N(0, \Xi).
$$

Consequently, we deduce that the limit in distribution of $(V_{N1}', \ldots, V_{NS}', W_{N1}', \ldots, W_{NS}')'$, denoted $(V_1', \ldots, V_S', W_1', \ldots, W_S')'$, is multivariate normal with mean $0$ and covariance matrix such that

$$
\text{Cov} (V_j, V_k) = 0 \text{ if } j \neq k, \quad (B.15)
$$

$$
\text{Cov} (W_j, W_k) = 0 \text{ if } j \neq k, \quad (B.16)
$$

$$
\text{Cov} (V_j, W_k) = \pi_j^{1/2} \pi_k^{1/2} I, \quad (B.17)
$$

$$
\text{Var} (V_j) = I, \quad (B.18)
$$

$$
\text{Var} (W_j) = I. \quad (B.19)
$$
As a result, we have \((T_{1N}, U_{1N}) \xrightarrow{d} (T_1, U_1)\), and \((T_{2N}, U_{2N}) \xrightarrow{d} (T_2, U_2)\).

We still need to show the independence of \(T_1\) and \(U_1\) and of \(T_2\) and \(U_2\). For this purpose, we write

\[
(T_1, U_1) = (g_1(y_{1,2}, \ldots, y_{S-1,S}), g_1(z_{1,2}, \ldots, z_{S-1,S}))
\]

where

\[
y_{j,k} = \pi_{j}^{-1/2}v_j - \pi_{k}^{-1/2}v_k, \quad j = 1, \ldots, S-1, k = j+1, \ldots, S,
\]

\[
z_{j,k} = \pi_{j}^{-1/2}w_j - \pi_{k}^{-1/2}w_k, \quad j = 1, \ldots, S-1, k = j+1, \ldots, S,
\]

and

\[
g_1(x_{1,2}, \ldots, x_{S-1,S}) = \max_{1 \leq j < k \leq S} \left( (x_{j,k})' \left( \pi_j^{-1} + \pi_k^{-1} \right)^{-1} x_{j,k} \right).
\]

From the normality of \((V'_1, \ldots, V'_S, W'_1, \ldots, W'_S)\)', it follows through a linear transformation that \((y'_{1,2}, \ldots, y'_{S-1,S}, z'_{1,2}, \ldots, z'_{S-1,S})\)' is jointly normal. Therefore, to show that \(T_1\) and \(U_1\) are independent, it is sufficient to show that \(\text{Cov}(y_{j_1,k_1}, z_{j_2,k_2}) = 0\), for any \(j_1, k_1, j_2\) and \(k_2\). We can express \(\text{Cov}(y_{j_1,k_1}, z_{j_2,k_2})\) as

\[
\pi_{j_1}^{-\frac{1}{2}} \pi_{j_2}^{-\frac{1}{2}} \text{Cov}(V_{j_1}, W_{j_2}) - \pi_{j_1}^{-\frac{1}{2}} \pi_{k_2}^{-\frac{1}{2}} \text{Cov}(V_{j_1}, W_{k_2}) - \pi_{k_1}^{-\frac{1}{2}} \pi_{j_2}^{-\frac{1}{2}} \text{Cov}(V_{k_1}, W_{j_2}) + \pi_{k_1}^{-\frac{1}{2}} \pi_{k_2}^{-\frac{1}{2}} \text{Cov}(V_{k_1}, W_{k_2}),
\]

which equals 0 due to the fact that \(\text{Cov}(V_j, W_k) = \pi_j^{-\frac{1}{2}} \pi_k^{-\frac{1}{2}}\), for any \(j\) and \(k\). The independence of \(T_2\) and \(U_2\) follows using similar steps. Therefore, equations Equation (B.1) and Equation (B.2) hold.

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