Tunable zero and first sounds in ultracold Fermi gases with Rabi coupling

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Abstract. We consider a weakly-interacting fermionic gas of alkali-metal atoms characterized by two hyperfine states which are Rabi coupled. By using a Born approximation for the repulsive interaction we determine the zero-temperature equation of state of this Fermi gas in $D$ spatial dimensions ($D = 1, 2, 3$). Then, adopting the Landau–Vlasov equation and hydrodynamic equations, we investigate the speed of first sound and zero sound. We show that the two sounds, which occur respectively in collisional and collisionless regimes, crucially depend on the interplay between interaction strength and Rabi coupling. Finally, we discuss for some experimentally relevant cases the effect of a trapping harmonic potential on the density profiles of the fermionic system.

Keywords: cold atoms, quantum gases, quantum quenches
1. Introduction

In the last few years the study of ultracold alkali-metal atoms was made even more stimulating by the experimental advent of synthetic gauge potentials, applied on multi-component gases in different hyperfine levels [1-4]. A related ingredient, experimentally quite simple to implement because it does not involves space-dependent tunnelling processes, is the Rabi coupling. This technique is nowadays a common tool for experimental and theoretical investigations involving multi-component gases. Some examples are the control of the population of the hyperfine levels [5], the formation of localized structures [6], and the mixing–demixing dynamics of Bose–Einstein condensates [7, 8].

Due to the wide applicability of the Rabi coupling, it is particularly interesting to investigate how the presence of a Rabi term affects equilibrium and collective dynamical properties of interacting atomic gases. In the present paper we analyze the effect of a Rabi coupling on a two-hyperfine-component Fermi gas. In particular we focus on zero sound and first sound [11], which can be experimentally obtained with a local perturbation of the gas density. In general, the study of sound propagation in Fermi liquids, from Helium to electrons to cold gases, is of extreme importance to understand the physical properties of the system [11, 12].

We determine at first the equation of state for the gas, in the presence of a weak repulsive interaction within a Born approximation. We find that the Rabi coupling

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term divides the spectrum in two branches with different energies. Later on, we use this equation of state to derive the behavior of the zero and first sounds. In this analysis we adopt hydrodynamic equations in the collisional regime and the Landau–Vlasov equation in the collisionless regime [11, 12]. Finally, in the Appendix, density profiles in presence of an harmonic trap, mostly used in current experiments, are derived within the local density approximation, as well as the densities and the chemical potentials at the center of the trap, where the measurements for the sounds are usually performed.

2. The model

We consider a two-spin-component Fermi gas of alkali-metal atoms with mass \( m \), described by the Hamiltonian

\[
\hat{H} = \int \text{d} \mathbf{r} \left\{ \sum_{\sigma=\uparrow, \downarrow} \hat{\psi}^\dagger_\sigma(\mathbf{r}) \left( -\frac{\hbar^2}{2m} \nabla^2 - \mu \right) \hat{\psi}_\sigma(\mathbf{r}) + \hbar \Omega \left( \hat{\psi}^\dagger_\uparrow(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) + \hat{\psi}^\dagger_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}) \right) + g \hat{n}_\uparrow(\mathbf{r}) \hat{n}_\downarrow(\mathbf{r}) \right\},
\]

where \( \hat{\psi}_\sigma(\mathbf{r}) \) is the fermionic field operator for atoms with spin \( \sigma \), \( \hat{n}_\sigma(\mathbf{r}) = \hat{\psi}^\dagger_\sigma(\mathbf{r}) \hat{\psi}_\sigma(\mathbf{r}) \) is the local number density operator, and \( \mu \) is the chemical potential [4]. The term proportional to \( \hbar \Omega \) corresponds to the Rabi coupling inducing a spin flip between the components, while the term proportional to \( g \) models the density-density inter-component repulsion (\( g > 0 \)).

We assume that system is in a \( D \)-dimensional space. Moreover, we work in the weak-coupling regime \( gn \ll 1 \), with \( n = \langle \hat{n}_\uparrow(\mathbf{r}) \rangle + \langle \hat{n}_\downarrow(\mathbf{r}) \rangle \) the average total number density, to avoid Stoner instability [9] and itinerant ferromagnetism [10]. In this way, at equilibrium, we can safely set \( \langle \hat{n}_\uparrow(\mathbf{r}) \rangle = \langle \hat{n}_\downarrow(\mathbf{r}) \rangle \).

3. Equation of state

3.1. Non-interacting case

Let us start with the very simple case of non-interacting fermions. If \( g = 0 \) the Hamiltonian in equation (1) is quadratic, then it can be diagonalized in momentum space by a global unitary transformation, obtaining

\[
\hat{H} = \sum_{\sigma=\pm} \sum_k \lambda_\sigma(\mathbf{k}) \hat{\eta}^\dagger_\sigma(\mathbf{k}) \hat{\eta}_\sigma(\mathbf{k})
\]

with

\[
\lambda_\pm(\mathbf{k}) = \frac{\hbar^2 |\mathbf{k}|^2}{2m} - (\mu \pm \hbar \Omega).
\]

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We have in particular that $\hat{\eta}_{\pm}(k) = (\hat{\psi}_{\uparrow}(k) \mp \hat{\psi}_{\downarrow}(k))/\sqrt{2}$ and the application of the Rabi coupling leads to an effective unbalance equal to $2\Omega$ in the rotated components $\eta_{\pm}$, a trivial result for people working on quantum optics. More in detail, in the present case we see from equation (3) that the spectrum divides in two branches with different energies. However when $\mu < \hbar\Omega$ only the eigenstates with energies $\lambda_{\pm}(k)$ are occupied.

Equivalently, we obtain two free fermionic gases with the same energies $\frac{\hbar^2 k^2}{2m}$ and different chemical potentials $\mu_{\pm} = (\mu \pm \hbar\Omega)$.

As for a free one component Fermi gas, in this case the effective chemical potentials $\mu_{\pm}$ are linked to the zero-temperature average densities $n_{\pm}$ by the equations

$$\mu_{\pm} = (\mu \pm \hbar\Omega) = \left(\frac{n_{\pm}}{A_D}\right)^D,$$

where the constant $A_D$ depends on the space dimension $D$. In particular, one has $A_1 = \frac{\sqrt{2}}{\pi\hbar}$, $A_2 = \frac{m}{2\pi\hbar^2}$, and $A_3 = \frac{\sqrt{2}}{3\pi\hbar^2} m^2$.

These equations fix the chemical potential $\mu$, via the related equations

$$\frac{n}{A_D} = (\mu + \hbar\Omega)^D + (\mu - \hbar\Omega)^D.$$

(5)

When $\mu < \hbar\Omega$, only the eigenstates with energies $\lambda_{\pm}(k)$ are occupied, then equation (5) reduce to

$$\frac{n}{A_D} = (\mu + \hbar\Omega)^D.$$

(6)

It is useful to rescale the energies $\mu$ and $\hbar\Omega$ by an energy scale $\Lambda_D$, introducing in equations (5) and (6) the adimensional quantities $\tilde{\mu} \equiv \frac{\mu}{\Lambda_D}$, $\tilde{n} \equiv \frac{n}{A_D\Lambda_D^D}$, $\tilde{\Omega} \equiv \frac{\hbar\Omega}{\Lambda_D}$. The energy scale $\Lambda_D$ is defined finally as $\Lambda_D = \frac{\hbar^2}{2m} n\tilde{\Omega}$.

Equation (5), rescaled by $\Lambda_D$, read:

$$\tilde{n} = (\tilde{\mu} + \tilde{\Omega})^D + (\tilde{\mu} - \tilde{\Omega})^D \equiv \tilde{\mu}^D + \tilde{\Omega}^D.$$

(7)

In the particular case $D = 2$ we obtain immediately $n = 2\tilde{\mu}$, not depending on $\Omega$.

3.2. Interacting case

If $g \neq 0$ the global unitary transformation diagonalizing the quadratic part of equation (1) maps the interaction term as follows:

$$g \int d\mathbf{r} \hat{n}_{\uparrow}(\mathbf{r}) \hat{n}_{\downarrow}(\mathbf{r}) \rightarrow g \int d\mathbf{r} \hat{n}_{\uparrow}(\mathbf{r}) \hat{n}_{\downarrow}(\mathbf{r}).$$

(8)

We see that the interaction term transforms covariantly, as a consequence of the fermionic nature of field operators and of the global nature of the unitary transformation leading to equation (2). The latter features occurs only because the Rabi term proportional to $\Omega$ in equation (1) is the same in every point of the space. This is not the case for instance in the presence of a spin–orbit coupling.
We simplify the repulsive interaction term keeping only the Hartree terms in the first Born approximation: 

\[ \hat{g} \hat{n}_+(\mathbf{r}) \hat{n}_-(\mathbf{r}) \simeq \hat{g} \hat{n}_+(\mathbf{r}) n_- + \hat{g} \hat{n}_-(\mathbf{r}) n_+ , \]

where \( n_\pm = \langle \hat{n}_\pm(\mathbf{r}) \rangle \). 

In this way the chemical potentials \( \mu_\pm \) of the interacting system become

\[ \mu_\pm \rightarrow \mu_\pm - g n_\mp = \mu_\pm - g (n - n_\pm) \equiv \mu_\pm^{(f)}. \]

The label (I), denoting the chemical potentials shifted by the effect of the interaction, will be also used in the following for the potentials \( \mu_\pm^{(f)} \). The (dependent) quantities \( \mu, \mu_\pm \) and \( \mu_\pm^{(f)} \), as well as the densities \( n_\pm \), can be obtained solving two equations, similar to equation (4):

\[ n_\pm = A_D (\mu \pm h \Omega - g (n - n_\pm))^\frac{D}{2}, \]

or, in adimensional form (\( \bar{n}_- = \bar{n} - \bar{n}_+ \)):

\[ \bar{n}_\pm = (\bar{\mu} \pm \bar{\Omega} - \bar{g} (\bar{n} - \bar{n}_\pm))^\frac{D}{2}, \]

where \( \bar{g} \equiv g (A_D A_D^{-1/2}) \).

If \( D = 2 \) equation (10) can be solved also analytically, yielding the results

\[ n_\pm = \frac{n}{2} \pm \frac{h \Omega A_2}{1 - g A_2} \]  

and

\[ \mu = \frac{n}{2 A_2} (1 + g A_2). \]

No dependence on \( \Omega \) is found for \( \mu \) in this case.

If instead \( D = 1, 3 \), the same equations can be solved numerically. Clearly, in this case it is much better to work with the rescaled equations (11). For \( D = 3 \), the dependence of \( \bar{\mu} \) on \( \bar{n} \) at fixed \( \bar{\Omega} = 0.2 \) and \( \bar{g} = 0.1 \) is reported in figure 1, while the dependence on \( \bar{\Omega} \) at fixed \( \bar{n} = 2 \) and \( \bar{g} = 0.1 \) is shown in figure 2.

Numerical solutions for \( \bar{n}_\pm \) are shown instead in figure 3. We see that as \( \Omega \) increases, the same trend occurs for \( \bar{n}_+ \), while \( \bar{n}_- \) decreases (so that the total rescaled density \( \bar{n} \) stays constant).

Finally, beyond a critical value for \( \bar{\Omega} \) only the band \( \lambda_+(\mathbf{k}) \) is populated (\( \bar{n}_- = 0 \)). Notice also that the regime with only a band is achieved at a lower critical \( \bar{\Omega} \) when the dimension of the system increases.

4. Collective dynamics

The equation of state for the Rabi-coupled Fermi gas determined in the last section allows to turn on the investigation of some non-equilibrium properties. Along this line, on the next sections we will study the first sound and the zero sound. We focus mainly in the regime where both the branches \( s = \pm \) are macroscopically populated, so that speaking about collective spatial oscillations in their densities makes sense.
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The density perturbation giving rise to the sounds can be created by a suitable blue-detuned laser applied on the gas, producing an effective local repulsive potential and a gaussian density hole in the atomic cloud (see [13, 14] and citing articles)

Given an interacting system with collisional time $\tau$, a sound mode of frequency $\omega$ is in the so-called collisionless regime if $\tau \omega \gg 1$. In this case the mode is called zero sound [11] and the collisionless dynamics is accurately described by the Landau–Vlasov equation [11]. Notice that in the three-dimensional case the collisional time $\tau$ at $T = 0^4$ should scale as

3 Another net effect of the initial perturbation is to create locally superpositions $\alpha |+\rangle + \beta |-\rangle$, $\beta = \sqrt{1 - \alpha^2}$, with $\alpha$ randomly distributed in the complex domain of radius 1. This fact does not spoil the description of the (zero and first) sound propagation by the interactions between $|\pm\rangle$ packets. Indeed it is straightforward to check that, thanks to the Fermi statistic, any state (in particular imagined before the interaction) composed by the tensor product of two superpositions as above can be rewritten (in second quantization) as $\epsilon |+\pm\rangle$.

4 Notice that the temperature plays the most relevant role for the collisional time in a Fermi liquid [11, 24].

Figure 1. Chemical potential $\tilde{\mu}$ of the three-dimensional ($D = 3$) fermionic gas as a function of the average total number density $\tilde{n}$. Interaction strength $g = 0.1$ and Rabi frequency $\tilde{\Omega} = 0.2$.

Figure 2. Chemical potential $\tilde{\mu}$ of the three-dimensional ($D = 3$) fermionic gas as a function of the Rabi frequency $\tilde{\Omega}$. Interaction strength $g = 0.1$, average total number density $\tilde{n} = 2$. 

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where $a$ is the scattering length (proportional to the interaction strength $g$), and $v_F^{\pm} = \sqrt{\frac{2\mu^{(3)}_I}{m}}$ is the Fermi velocity. Instead, under the condition $\tau \omega \ll 1$, the sound mode of frequency $\omega$ is in the collisional regime and the mode is called first sound. The sound depends here on the collective wave motion of the Fermi gas, describable by ordinary Navier–Stokes equations (with suitable quantum corrections to be included).

We stress at the end that, for both the sounds described above, the simultaneous appearance of the two chemical potentials $\mu^{(3)}_I$ in the equilibrium equations of state (as in equation (11)) allows to include a priori density oscillations of the two species $\pm$ with arbitrary relative phases.

5. First sound

5.1. Derivation of the first sound velocities

We derive in this section the expressions for the first sound velocity in the two-components Fermi gas. The knowledge of the equilibrium thermodynamical quantities is again supposed, in the light of the studies performed in the section 3.

In principle, since two fermionic components are involved, two sounds could be expected, corresponding to local fluctuations of the total number density $n = n_+ + n_-$ and of the difference $\delta n = n_+ - n_-$. However, the single particle energies $\lambda^{(3)}(k)$ in equation (3) differ by an amount $2 \Delta$ and the sound related with the fluctuations of $\delta n$ is gapped. Consequently, an initial very small perturbation does not excite the gapped mode. For this reason, we concentrate below on the fluctuations of $n$.

Figure 3. Average number densities $\tilde{n}_\pm$ of the three-dimensional ($D = 3$) fermionic gas as a function of the Rabi frequency $\tilde{\Omega}$. We set in the regime with two bands and we also assumed $\tilde{g} = 0.1$, and $\tilde{n} = 2$. 

$$\tau \sim \frac{1}{a^2 \sqrt{v_F^{\pm} v_F^-} \sqrt{n_+ n_-}} \sim \frac{1}{a^2 (n_+ n_-)^{\frac{3}{2}}}$$

(14)
As claimed in the previous section, an hydrodynamic approach, based on the Navier–Stokes equations, is appropriate in the first sound regime. Here we use the equations of rotational hydrodynamics [15] for the local total particle densities \( n(\mathbf{r}, t) \) and local velocity \( \mathbf{v}(\mathbf{r}, t) \). These equations read

\[
\frac{\partial}{\partial t} n(\mathbf{r}, t) + \nabla \cdot (n(\mathbf{r}, t) \mathbf{v}(\mathbf{r}, t)) = 0
\]

and

\[
m \frac{\partial}{\partial t} \mathbf{v}(\mathbf{r}, t) + \nabla \left( \frac{1}{2} m |\mathbf{v}(\mathbf{r}, t)|^2 + \mu(\mathbf{r}, t) \right)
= m \mathbf{v}(\mathbf{r}, t) \wedge (\nabla \wedge \mathbf{v}(\mathbf{r}, t)),
\]

where \( \mu(\mathbf{r}, t) \) is the chemical potential of the bulk system, defined in the Hamiltonian in equation (1).

Notice that on the right side of the equality in equation (16) a rotational term appears. This term makes the difference between normal collisional hydrodynamics and superfluid (irrotational) hydrodynamics [15], since in the latter case \( \nabla \wedge \mathbf{v}(\mathbf{r}, t) = 0 \). However, even for rotational fluids, the rotational term, quadratic in the velocity field, can be neglected in the framework of linear-response theory, valid for small perturbations of the equilibrium configuration, as the ones considered here.

In the following we will work under this assumption, neglecting the rotational term and performing the expansions:

\[
\begin{cases}
n(\mathbf{r}, t) = n + \delta n(\mathbf{r}, t) \\
v(\mathbf{r}, t) = 0 + \delta v(\mathbf{r}, t),
\end{cases}
\]

where the symbols \( n \) denote the constant equilibrium density defined in section 2, moreover \( \delta n(\mathbf{r}, t) \) and \( \delta v(\mathbf{r}, t) \) represent small variations with respect to the equilibrium configuration. Similarly, the chemical potential can be expanded around the equilibrium value \( \mu = \mu(n) \) as:

\[
\mu(n(\mathbf{r}, t)) \simeq \mu(n) + \frac{\partial \mu}{\partial n}(n) \delta n(\mathbf{r}, t).
\]

We insert now in equation (25) the linearizations in equations (17) and (18), then we derive in \( t \) the first obtained equation and the second one in \( \mathbf{r} \). Finally we sum each other the two so-obtained expressions, arriving to the equation:

\[
\frac{\partial^2 \delta n(\mathbf{r}, t)}{\partial t^2} + \frac{n}{m} \frac{\partial \mu(\mathbf{r}, t)}{\partial n} \nabla^2 \delta n(\mathbf{r}, t) = 0.
\]

The same equation can be solved imposing

\[
\delta n(\mathbf{r}, t) = C e^{i(k \cdot \mathbf{r} - \omega t)},
\]

and solving the consequent algebraic equation for \( \omega \). The final result for the first sound velocity is (see e.g. [16]):

\[
v_1 = \frac{\omega}{k} = \sqrt{\frac{n}{m} \frac{\partial \mu(n)}{\partial n}}.
\]
In the $D = 2$ case, equation (13) yields:

$$v_1^{(2D)} = \sqrt{\frac{n}{2m A_2}} (1 + g A_2).$$

(22)

Notice that the latter result is not valid limit $g \to 0$, where the zero sound equations hold instead.

For the case $D = 3$, the explicit dependence of the first sound (rescaled in unity of $\sqrt{\frac{\Lambda_2}{m}}$) on $\tilde{\Omega}$ at fixed $\tilde{n} = 2$ and $\tilde{g} = 0.1$ is shown in figure 4. We find an increase of $\tilde{v}_1$ with $\tilde{\Omega}$. By increasing $\tilde{\Omega}$ one has more and more atoms in only the state $+$. At some point the ground state is formed only by atoms in this $+$ state and, as a consequence, there is no interaction between fermions and the first sound does not exist anymore. Thus, for large $\tilde{\Omega}$ the sound can be collisional, and described by figure 4, only for very small momenta.

We also show in figure 5 the dependence of $\tilde{v}_1$ on $\tilde{g}$ at fixed $\tilde{n} = 2$ and $\tilde{\Omega} = 0.5$, finding the opposite behaviour of $\tilde{v}_1$. Notice that in both of the plots, regimes where $n_{\pm} = 0$ are considered.

We comment finally that in equation (16) we neglected viscosity terms $\eta \nabla^2 \mathbf{v}$, the same will be done in the following. Strictly speaking, this is possible with good accuracy close to unitarity [17–19]. On the contrary, far from unitary this term leads generally to a damping of the oscillations proportional to the factor $e^{-\eta |\mathbf{r}|}$, nevertheless not affecting the first sound velocity.

6. Zero sound

We study in this section the zero sound in the cases $D = 2$ and $D = 3$. If instead $D = 1$ the Fermi liquid is sustained mostly if $g = 0$, when the zero sound coincides with the Fermi velocity $v_\text{F}^{(\pm)} = \frac{2\mu_{\pm}}{m}$. 
The zero sound can be derived exploiting the Boltzmann equation in the collisionless regime, named Landau–Vlasov equation:\cite{11}:

\[
\left( \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla - \mathbf{v} U_\pm(\mathbf{r}) \cdot \nabla_p \right) f_\pm(\mathbf{r}, \mathbf{p}, t) = 0. \tag{23}
\]

In this equation $f_\pm(\mathbf{r}, \mathbf{p}, t)$ is the phase-space distribution of the fermionic quasiparticles $\pm$ (assumed both occupied) with position $\mathbf{r}$ and momentum $\mathbf{p}$, having the property

\[
\int d\mathbf{p} f_\pm(\mathbf{r}, \mathbf{p}, t) = n_\pm(\mathbf{r}, t) \tag{24}
\]

and similarly for the same integral in $\mathbf{r}$.

The quantities $U_\pm(\mathbf{r})$ in equation (23) are generally the potentials acting on the components $\pm$; when the specific interaction in equation (1) is assumed, then $U_\pm(\mathbf{r}) = g n_\pm(\mathbf{r})$ are the Born potentials introduced in section 3.2.

Expanding $f_\pm(\mathbf{r}, \mathbf{p}, t)$ and $n_\pm(\mathbf{r})$ around their equilibrium values for general $U_\pm(\mathbf{r})$

\[
\begin{cases}
  n_s(\mathbf{r}, t) = n_s + \delta n_s(\mathbf{r}, t) \\
  f_s(\mathbf{r}, \mathbf{p}, t) = f_{s,eq}(\mathbf{p}) + \delta f_s(\mathbf{r}, \mathbf{p}, t),
\end{cases} \tag{25}
\]

with $s = \pm$, $f_{s,eq}(\mathbf{p}) = \delta(\mu^{(I)}_s - \epsilon_{s,eq}(\mathbf{p}))$ and $\epsilon_{s,eq}(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + g n_\pm$. We obtain (after relabelling for sake of brevity $\delta f_s(\mathbf{r}, \mathbf{p}, t) \equiv f_\pm(\mathbf{r}, \mathbf{p}, t)$):

\[
\left( \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla \right) f_\pm(\mathbf{r}, \mathbf{p}, t) + \delta(\mu^{(I)}_\pm - \epsilon_{\pm,eq}(\mathbf{p})) \frac{\mathbf{p}}{m} \times \nabla \int d\mathbf{p'} F(\mathbf{p}, \mathbf{p'}) f_\pm(\mathbf{r}, \mathbf{p'}, t) = 0. \tag{26}
\]

The quantity $F(\mathbf{p}, \mathbf{p'})$ is the so-called interaction function between the components. This function measures the change of the quasiparticle energies for small deviations

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**Figure 5.** First sound velocities $\tilde{v}_1$ as a function of the repulsive interaction strength $\tilde{g}$ for the Fermi gas with $D = 3$. We assumed the Rabi frequency $\tilde{\Omega} = 0.5$ and the average number density $\tilde{n} = 2$.**
of the distribution functions $n_{\pm}(p) = \int \text{d}r f_{\pm}(r, p, t)$ from the step function $\theta(|p| - p_{\pm}^{(F)})$ ($p_{\pm}^{(F)} = \sqrt{2 m \mu_{\pm}}$ being the Fermi momenta) typical of Fermi gases in equilibrium:

$$\delta \epsilon_{\pm, \text{eq}}(p, t) = \int \text{d}p' F(p, p') f_{\pm}(r, p', t)$$

(27)

and it vanishes in absence of interaction [11]. For most of the practical cases, one can fix $|p| = |p'| = p_{\pm}^{(F)}$, so that the interaction function can be supposed depending only on $\hat{p} \cdot \hat{p}'$. In these pretty general hypothesis, equations (23) can be simplified by the linearizing ansatz:

$$f_{\pm}(r, p, t) = \delta(\mu_{\pm}^{(I)} - \epsilon_{\pm, \text{eq}}(p)) \Phi_{\pm}(\hat{p}) e^{i(k \cdot r - \omega t)},$$

(28)

valid under the further hypothesis that the initial perturbation giving rise to the zero-sound has a typical energy scale $U$ such that $U \ll \mu_{\pm}^{(I)}$ and $U \ll \mu_{\mp}^{(I)}$. Assuming the equation (28), we obtain:

$$\left(-\omega + \frac{p_{\pm}^{(F)} \cdot k}{m}\right) \Phi_{\pm}(\hat{p}_{\pm}^{(F)}) + \frac{1}{(2\pi \hbar)^3} p_{\pm}^{(F)} p_{\mp}^{(F)} (\hat{p}_{\pm}^{(F)} \cdot k)$$

$$\times \int \Phi_{\pm}(\hat{p}') F(\hat{p}_{\pm}^{(F)} \cdot \hat{p}') \text{d}p' = 0.$$

(29)

In the absence of interaction, $F(\hat{p}_{\pm}^{(F)} \cdot \hat{p}') = 0$, equations (29) yield correctly two Fermi velocities:

$$\frac{\omega}{|k|} = v = v_{\pm}^{(F)} = \sqrt{\frac{2 \mu_{\pm}^{(I)}}{m}} = \frac{p_{\pm}^{(F)}}{m}.$$  

(30)

Assuming now the point-like repulsion in equation (1), so that in equations (23) it results $U_{\pm}(r) = g n_{\pm}(r)$, the interaction function turns out to be constant and equal to $2g$, and the latter equation reduces to

$$(-r_{\pm} + \cos \theta) \Phi_{\pm}(\hat{p}_{\pm}^{(F)}) + \cos \theta F_{0}^{(\pm)}$$

$$\times \int \frac{\text{d}p'}{4\pi} \Phi_{\mp}(\hat{p}') = 0,$$

(31)

with the adimensional quantities

$$F_{0}^{(\pm)} = g \frac{p_{\pm}^{(F)} m}{\pi \hbar^3} = \left(\frac{6}{\pi}\right)^{\frac{1}{3}} \frac{g m}{\hbar^2} n_{\pm}^{\frac{1}{3}},$$

(32)

and

$$r_{\pm} = \frac{\omega}{|k| v_{\pm}^{(F)}} = \frac{v}{v_{\pm}^{(F)}},$$

(33)

Equation (31) can be also derived by a random phase approximation (RPA) approach for unbalanced Fermi gases, used e.g. in [20]. Moreover they can be solved directly by the substitution [11]
\[ \Phi_{\pm}(\beta) = C_{\pm} \frac{\cos \beta}{r_{\pm} - \cos \beta}, \] (34)

leading finally to the system\textsuperscript{5}

\[
\begin{align*}
\Gamma &= F_{0}^{(-)} \left( -2 + r_{-} \ln \frac{r_{-} + 1}{r_{-} - 1} \right), \\
\frac{1}{\Gamma} &= F_{0}^{(+)}, \left( -2 + r_{+} \ln \frac{r_{+} + 1}{r_{+} - 1} \right),
\end{align*}
\] (35)

where \( \Gamma = \frac{C_{-}}{C_{+}} \). We also define:

\[
\mu_{\pm}^{\pm} = \frac{\mu_{+}^{(F)}}{\mu_{-}^{(F)}} = \sqrt{\frac{\rho_{+}^{(F)}}{\rho_{-}^{(F)}}}. \] (36)

In this way, the condition

\[ v = v_{+}^{(F)} r_{+} = v_{-}^{(F)} r_{-} \] (37)

(we look for a unique sound, corresponding with the situation in the collisional regime) results into the other one:

\[ r_{-} = a r_{+}. \] (38)

We insert now the last in relation in equation (35) and we solve the so-obtained system in \( r_{+} \). In this way, exploiting equation (37), we arrive finally to the zero sound velocity (rescaled in unity of \( \sqrt{\frac{2 \lambda_{0}}{m}} \))

\[ \bar{v} = r_{+} \sqrt{\rho_{+}^{(F)}}. \] (39)

The results are reported in figure 6 for \( \Omega \) varying and fixed \( \bar{n} = 2 \) and \( \bar{g} = 0.2 \) and in figure 7 for \( \bar{g} \) varying and fixed \( \bar{n} = 2 \) and \( \Omega = 0.5 \). Again in both of the plots, regimes where \( n_{\pm} \neq 0 \) are considered.

\textbf{6.2.} \( D = 2 \) case

In this case equation (31) becomes:

\[
(-r_{\pm} + \cos \theta) \Phi_{\pm}(\hat{\rho}_{\pm}^{(F)}) + \cos \theta F_{0}^{(\pm)}
\]
\[
\times \int \frac{dp^{\prime}}{2\pi} \Phi_{\pm}(p^{\prime}) = 0, \] (40)

\textsuperscript{5} The right-hand side of equations (35) differs from the corresponding equations in [11] for a \( \frac{1}{2} \) factor. This difference depends on the notations adopted here. To verify the correctness of our expressions, we checked that from them we obtain \( a \to 1 \) and \( \Gamma \to 1 \) when \( \Omega \to 0 \) (see definitions below equations (35)), as expected from physical considerations. Vice-versa, the mentioned \( \frac{1}{2} \) factors do not allow to recover the same limit conditions.
and the substitution in equation (34) leads to the system:

\[
\begin{align*}
\Gamma &= F_0^{(-)} \left( -1 + \frac{r_-}{\sqrt{r_-^2 - 1}} \right) \\
\frac{1}{\Gamma} &= F_0^{(+)} \left( -1 + \frac{r_+}{\sqrt{r_+^2 - 1}} \right).
\end{align*}
\]  

Figure 6. Zero sound velocities \( \tilde{v} = r_+ \sqrt{\mu_+^{(j)}} \) as a function of the Rabi frequency \( \tilde{\Omega} \) for the Fermi gas with \( D = 3 \). We assumed \( \tilde{g} = 0.2 \), \( \tilde{n} = 2 \), and \( F_0^{(-)} = 1 \) (\( m \) chosen accordingly).

Figure 7. Zero sound velocities \( \tilde{v} = r_+ \sqrt{\mu_+^{(j)}} \) versus \( \tilde{g} \). We also assumed \( \tilde{\Omega} = 0.5 \), \( \tilde{n} = 2 \), \( D = 3 \), and \( F_0^{(-)} = 1 \).

Notice that the differences between the systems in equation (41) and in equation (35) are due to the different angular integrations in equation (31) and in equation (40). The system in equation (41) can be solved numerically in \( r_+ \), as for the \( D = 3 \) case. The behaviours for the final rescaled velocity \( \tilde{v} = r_+ \sqrt{\mu_+^{(j)}} \) are very qualitatively similar to
the $D = 3$ case, reported in figures 6 and 7. We notice finally that for the present case, $D = 2$, a study of the zero sound has been performed by a RPA approach in [20] and [21, 22] also in the presence of a Rashba coupling.

7. Conclusions

In this paper we have analyzed the equation of state of a two-component repulsive Fermi gas under the application of a Rabi coupling. As main application we have investigated the behavior of first sound and zero sound after a local perturbation of the uniform density. Notably the application of the Rabi coupling appears as an effective experimental strategy to tune the sounds by hand, varying the Rabi frequency $\Omega$. Finally, in the appendix some density profile in presence of an harmonic external trap, mostly used in current experiments, are derived in local density approximation, as well as the densities and the chemical potentials at the center of the trap, where the measurements for the sounds are usually performed. Our findings can be relevant for current experiments in ultracold gases, where the application of Rabi couplings is widely used for various purposes, as mentioned in the Introduction.

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Appendix. Trap effects

In all the examples considered so far we analyzed uniform gases in a continuous $D$-dimensional space. At variance, in many other real experiments the space-density is made non uniform by the presence of a confining harmonic potential. In these conditions, the sound velocity is generally measured in the centre of the trap, so that the results of the previous sections hold using the atomic densities and chemical potentials in this point. Although these quantities can be as well measured directly in the centre of the trap, it is interesting to derive analytically some density profile in the presence of an harmonic trap; this is the aim of this section.

The trapping harmonic potential has the general form:

$$U(r) = \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2).$$  \hspace{1cm} (A.1)

The effect on the trap can be taken into account in local-density approximation, defining two space dependent chemical potentials as [23]:

$$\mu^{(I)}(r) = \mu^{(I)} - U(r),$$  \hspace{1cm} (A.2)
being $\mu^{(I)}_{\pm}$ the equilibrium chemical potentials (defined as in equation (9)) at the center of the trap for the components $\eta_{\pm}$, and two space-dependent densities $n_{\pm}(r)$ related to $\mu^{(I)}_{\pm}(r)$ as in the previous section: $\mu^{(I)}_{\pm}(r) = \left(\frac{n_{\pm}(r)}{A_0}\right)^2$. The quantities $\mu_{\pm}$ can be found by solving the equation:

$$\int_{V_+} dr \, n_+(r) + \int_{V_-} dr \, n_-(r) = N,$$

(A.3)

being $N$ the total number of loaded atoms, assumed known, and $V_{\pm}$ are the volumes of the trap, whose extensions, characterized by the radii $R_F^{(\pm)} = \sqrt{\frac{2\mu^{(I)}_{\pm}}{m_2\omega^2}}$, are limited by the condition $\mu^{(I)}_{\pm}(r) > 0$.

The total density $n(r)$ is then $n(r) = n_+(r) + n_-(r)$, with $n_{\pm}(r)$ related to $\mu^{(I)}_{\pm}(r)$ as above.

We assume first the non interacting case $g = 0$ (where $\mu^{(I)}_{\pm} = \mu_{\pm} = \mu + h\Omega$). If $D = 1$ ($\omega_y, \omega_z \gg \omega_x \equiv \omega$ and $h\omega_y, h\omega_z \gg \mu$) the integration of $n_{\pm}(r)$ up to $R_F^{(\pm)}$ yields:

$$\mu = \frac{3}{8\sqrt{2}} \frac{N}{A_1} \frac{m_2\omega}{\omega^2}.$$

(A.4)

If $D = 2$ ($\omega_z \gg \omega_x = \omega_y \equiv \omega$ and $h\omega_z \gg \mu$), similar calculations lead to:

$$\mu = \frac{N}{2\pi A_2} \frac{m_2\omega^2 - (h\Omega)^2}{\omega^2}.$$

(A.5)

Notice that in this case the positiveness of the argument in the square root in equation (A.5) is always fulfilled when the two branches $\pm$ in equation (3) are both populated.

In the case $D = 3$, $\mu$ can be obtained as the unique real solution of the third order algebraic equation

$$\mu^3 + 3(h\Omega)^2 \mu = \frac{15}{32\pi \sqrt{2}} \frac{N}{A_3} \frac{3 m_2^2 \omega^3}{\omega},$$

(A.6)

where $\omega_x = \omega_y = \omega_z \equiv \omega$. Although the analytic expression of this solution is not particularly enlightening, its qualitative analysis indicates a growth for $\mu$ as $\omega$ increases, and the opposite trend when $\Omega$ is varied. Both these behaviours are the ones expected from intuition and are also found for the cases $D = 1, 2$, equations (A.4) and (A.5). Remarkably, if $D = 1$ no dependence on $\Omega$ is found: this effect has the same origin of the homogeneous 2D case and it is simply due to a constant density of states. Notice that for every $\omega > 0$ and $\Omega > 0$, it always exists a solution $\mu > 0$.

In the interacting case $g \neq 0$, $\mu$ and $\mu^{(I)}_{\pm}$ can be found from the equations:

$$\begin{cases}
    n_+(r) = A_D \left( \mu + h\Omega - g \ n_-(r) - \frac{1}{2} m_2 |r|^2 \right)^\frac{D}{2}, \\
    n_-(r) = A_D \left( \mu - h\Omega - g \ n_+(r) - \frac{1}{2} m_2 |r|^2 \right)^\frac{D}{2}.
\end{cases}$$

(A.7)
A semi-analytic solution of the system in equation (A.7) is available in general, we give here details about the case $D = 2$. Under this condition we obtain ($g A_2 \ll 1$):

$$n_{\pm}(r) = A_2 \left( \frac{\mu}{1 + g A_2} \pm \frac{\hbar \Omega}{1 - g A_2} - \frac{1}{2(1 + g A_2)} m \omega^2 |r|^2 \right).$$

(A.8)

In this way, the condition $\mu^{(f)}(r) = 0$ (equivalently $n_{\pm}(r) = 0$) gives:

$$K_{F}^{(\pm)}(\mu) = \sqrt{\frac{2(1 + g A_2)}{m \omega^2}} \left( \frac{\mu}{1 + g A_2} \pm \frac{\hbar \Omega}{1 - g A_2} \right).$$

(A.9)

As functions of $K_{F}^{(\pm)}(\mu)$, exploiting equations (A.3) and (A.8) we obtain finally a second order algebraic equation for the chemical potential $\mu \neq 0$ at the center of the trap:

$$\frac{4}{m \omega^2} \mu^2 - \frac{2}{m \omega^2} \frac{\hbar \Omega}{1 - g A_2} \mu + \frac{4}{m \omega^2} (\hbar \Omega)^2 \frac{1 + g A_2}{(1 - g A_2)^2} \frac{N}{\pi A_2} = 0.$$

(A.10)

When $g A_2 \ll 1$ and $N \gg 1$, the last equation has a positive solution only. Exploiting it, the density profile $n(r)$ can be found by summing equations (A.8).

References

[1] Lin Y-J, Jimenez-Garcia K and Spielman I B 2011 Nature 471 83–6
[2] Galitski V and Spielman I B 2013 Nature 494 49–54
[3] Dalibard J, Gerbier F, Juzeliūnas G and Öhberg P 2011 Rev. Mod. Phys. 83 1523
[4] Lewenstein M, Sanpera A and Ahufinger V 2012 Ultracold Atoms in Optical Lattices: Simulating Quantum Many-Body Systems (Oxford: Oxford University Press)
[5] Steck D A 2017 Quantum and Atom Optics (available online at http://atomoptics-nas.uoregon.edu/~dsteck/teaching/quantum-optics/)
[6] Horstmann B, Durr S and Roscilde T 2010 Phys. Rev. Lett. 105 160402
[7] Nicklas E, Strobel H, Zibold T, Gross C, Malomed B A, Kevrekidis P G and Oberthaler M K 2011 Phys. Rev. Lett. 107 193001
[8] Abad M and Recati A 2013 Eur. Phys. J. D 67 148
[9] Stoner E C 1947 Rep. Prog. Phys. 11 43
[10] Jo G-B, Lee Y-R, Choi J-H, Christensen C A, Kim T H, Thywissen J-H, Pritchard D E and Ketterle W 2009 Science 325 1521152
[11] Lifschitz E M and Pitaevskii L P 1980 Statistical Physics part 2 (Oxford: Pergamon)
[12] Leggett A J 2006 Quantum Liquids (Oxford: Oxford University Press)
[13] Joseph J, Clancy B, Luo L, Kinast J, Turlapov A and Thomas J E 2007 Phys. Rev. Lett. 98 170401
[14] Andrews M R, Kurn D M, Miesner H J, Durfee D S, Townsend C G, Inouye S and Ketterle W 1997 Phys. Rev. Lett. 79 553
[15] Cozzini M and Stringari S 2003 Phys. Rev. Lett. 91 070401
[16] Lipparrini E 2008 Modern Many-Particle Physics 2nd edn (Singapore: World Scientific)
[17] Turlapov A, Kinast J, Clancy B, Luo L, Joseph J and Thomas J E 2008 J. Low Temp. Phys. 150 567
[18] Cao C, Elliott E, Joseph J, Wu H, Petricka J, Schaefer T and Thomas J E 2011 Science 331 58
[19] Cao C, Elliott E, Wu H and Thomas J E 2011 New J. Phys. 13 075007
[20] Stringari S 2009 Phys. Rev. Lett. 102 110406
[21] Salasnich L 2013 Phys. Rev. A 88 055601
[22] Gigli L and Toigo F 2015 J. Phys. B: At. Mol. Opt. Phys. 48 245302
[23] Capuzzi P, Vignolo P, Federici F and Tosi M P 2006 Phys. Rev. A 73 021603
[24] Baym G and Pethick C 1991 Landau Fermi-Liquid Theory: Concepts and Applications (New York: Wiley)