THE COMPLEX BUSEMANN-PETTY PROBLEM FOR ARBITRARY MEASURES.

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Abstract. The complex Busemann-Petty problem asks whether origin symmetric convex bodies in \( \mathbb{C}^n \) with smaller central hyperplane sections necessarily have smaller volume. The answer is affirmative if \( n \leq 3 \) and negative if \( n \geq 4 \). In this article we show that the answer remains the same if the volume is replaced by an "almost" arbitrary measure. This result is the complex analogue of Zvavitch’s generalization to arbitrary measures of the original real Busemann-Petty problem.

1. Introduction

In 1956 the Busemann-Petty problem was posed (see [BP]), asking the following question: suppose that \( K \) and \( L \) are two origin symmetric convex bodies in \( \mathbb{R}^n \) such that for every \( \xi \in S^{n-1} \),

\[
\text{Vol}_{n-1}(K \cap \xi^\perp) \leq \text{Vol}_{n-1}(L \cap \xi^\perp).
\]

Does it follow that

\[
\text{Vol}_n(K) \leq \text{Vol}_n(L) ?
\]

The answer is affirmative if \( n \leq 4 \) and negative if \( n \geq 5 \). The problem was solved in the late 90’s as a result of a series of papers ([LR], [Ba], [Gi], [Bu], [Lu], [Pa], [Ga], [Zh1], [K1], [K2], [Zh2], [GKS]; see [K5, p.3] for the history of the solution).

A few years later Zvavitch [Zv] showed that one can replace the volume by essentially any measure on \( \mathbb{R}^n \). Namely, if we consider any even continuous positive function \( f \) on \( \mathbb{R}^n \) and denote by \( \mu \) the measure with density \( f \), we can define

\[
\mu(D) = \int_D f(x)dx \quad \text{and} \quad \mu(D \cap \xi^\perp) = \int_{D \cap \xi^\perp} f(x)dx,
\]

for every closed bounded invariant with respect to all \( R_\theta \) set \( D \) in \( \mathbb{R}^n \) and every \( \xi \in S^{n-1} \). Then the Busemann-Petty problem for general measures is stated as follows:

Suppose that \( K \) and \( L \) are two origin symmetric convex bodies in \( \mathbb{R}^n \) such that, for every \( \xi \in S^{n-1} \),

\[
\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp).
\]

Does it follow that

\[
\mu(K) \leq \mu(L) ?
\]
Surprisingly, the answer remains the same as in the original problem. It is affirmative for \( n \leq 4 \) and negative for \( n \geq 5 \).

Zvavitch’s ideas for general measures were applied and further developed in [R], [Y1] and [Y2], for hyperbolic and spherical spaces and for sections of lower dimensions.

In this article we study the complex version of the Busemann-Petty problem for arbitrary measures.

Let \( \xi \in \mathbb{C}^n \) with \( |\xi| = 1 \). We denote by

\[
H_\xi = \{ z \in \mathbb{C}^n : (z, \xi) = 0 \}
\]

the complex hyperplane perpendicular to \( \xi \).

Origin symmetric convex bodies in \( \mathbb{C}^n \) are the unit balls of norms on \( \mathbb{C}^n \). We denote by \( \| \cdot \|_K \) the norm corresponding to the body \( K \)

\[
K = \{ z \in \mathbb{C}^n : \| z \|_K \leq 1 \}.
\]

We identify \( \mathbb{C}^n \) with \( \mathbb{R}^{2n} \) using the mapping

\[
\xi = (\xi_1, \ldots, \xi_n) = (\xi_{11} + i\xi_{12}, \ldots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2})
\]

and observe that under this mapping the complex hyperplane \( H_\xi \) turns into a \((2n - 2)\)-dimensional subspace of \( \mathbb{R}^{2n} \) orthogonal to the vectors

\[
\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^\perp = (-\xi_{12}, \xi_{11}, \ldots, -\xi_{n2}, \xi_{n1}).
\]

Since norms on \( \mathbb{C}^n \) satisfy the equality

\[
\| \lambda z \| = |\lambda| \| z \|, \quad \forall z \in \mathbb{C}^n, \; \forall \lambda \in \mathbb{C}^n,
\]

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies \( K \) in \( \mathbb{R}^{2n} \) that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each \( \theta \in [0, 2\pi] \) and each \( x = (x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \in \mathbb{R}^{2n} \)

\[
\| x \|_K = \| R_\theta(x_{11}, x_{12}), \ldots, R_\theta(x_{n1}, x_{n2}) \|_K,
\]

where \( R_\theta \) stands for the counterclockwise rotation of \( \mathbb{R}^2 \) by the angle \( \theta \) with respect to the origin. If a convex body satisfies (1) we will say that it is invariant with respect to all \( R_\theta \).

The complex Busemann-Petty problem ([KKZ]) can now be formulated as follows: Suppose \( K \) and \( L \) are origin symmetric invariant with respect to all \( R_\theta \) convex bodies in \( \mathbb{R}^{2n} \) such that

\[
\text{Vol}_{2n-2}(K \cap H_\xi) \leq \text{Vol}_{2n-2}(L \cap H_\xi)
\]

for each \( \xi \) from the unit sphere \( S^{2n-1} \) of \( \mathbb{R}^{2n} \). Does it follow that

\[
\text{Vol}_{2n}(K) \leq \text{Vol}_{2n}(L) ?
\]

As it is proved in [KKZ] the answer is affirmative if \( n \leq 3 \) and negative if \( n \geq 4 \).

Let \( f \) be an even positive and continuous function on \( \mathbb{R}^{2n} \). We define a measure \( \mu \) on \( \mathbb{R}^{2n} \) with density \( f \), so that
\[ \mu(D) = \int_D f(x)dx \quad \text{and} \quad \mu(D \cap H) = \int_{D \cap H} f(x)dx \]

for every closed bounded invariant with respect to all \( R_\theta \) set \( D \) in \( \mathbb{R}^{2n} \) and every \((2n - 2)\)-dimensional subspace \( H \) of \( \mathbb{R}^{2n} \). As it is proved in Section 3 (Lemma 2), one may assume, without loss of generality, that the density \( f \) is also invariant with respect to all rotations \( R_\theta \). We will call such a function \( R_\theta \)-invariant. Then, the complex Busemann-Petty problem for arbitrary measures is stated as follows:

Suppose \( K \) and \( L \) are origin symmetric invariant with respect to all \( R_\theta \) convex bodies in \( \mathbb{R}^{2n} \) so that for every \( \xi \in S^{2n-1} \)

\[ \mu(K \cap H_\xi) \leq \mu(L \cap H_\xi), \]

does it follow that

\[ \mu(K) \leq \mu(L) ? \]

In this article we prove that, analogously to the real case, the solution remains the same for arbitrary measures with a positive continuous density.

Note that, the positivity assumption on \( f \) is necessary, because otherwise one may assume that the density is identically zero where the affirmative answer to the problem holds trivially in all dimensions.

2. The Fourier analytic connection to the problem

Throughout this paper we use the Fourier transform of distributions. The Schwartz class of rapidly decreasing infinitely differentiable functions (test functions) in \( \mathbb{R}^n \) is denoted by \( S(\mathbb{R}^n) \), and the space of distributions over \( S(\mathbb{R}^n) \) by \( S'(\mathbb{R}^n) \). The Fourier transform \( \hat{f} \) of a distribution \( f \in S'(\mathbb{R}^n) \) is defined by \( \langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle \) for every test function \( \phi \). A distribution is called even homogeneous of degree \( p \in \mathbb{R} \) if \( \langle f(x), \phi(x/\alpha) \rangle = |\alpha|^{n+p} \langle f, \phi \rangle \) for every test function \( \phi \) and every \( \alpha \in \mathbb{R}, \alpha \neq 0 \). The Fourier transform of an even homogeneous distribution of degree \( p \) is an even homogeneous distribution of degree \( -n - p \). A distribution \( f \) is called positive definite if, for every test function \( \phi \), \( \langle f, \phi \ast \overline{\phi(-x)} \rangle \geq 0 \). By Schwartz’s generalization of Bochner’s theorem, this is equivalent to \( \hat{f} \) being a positive distribution in the sense that \( \langle \hat{f}, \phi \rangle \geq 0 \) for every non-negative test function \( \phi \), (see [K5, section 2.5] for more details).

A compact set \( K \subset \mathbb{R}^n \) is called a star body, if every straight line that passes through the origin crosses the boundary of the set at exactly two points and the boundary of \( K \) is continuous in the sense that the Minkowski functional of \( K \), defined by

\[ \|x\|_K = \min\{\alpha \geq 0 : x \in \alpha K\} \]

is a continuous function on \( \mathbb{R}^n \).

A star body \( K \) in \( \mathbb{R}^n \) is called \( k \)-smooth (infinitely smooth) if the restriction of \( \|x\|_K \) to the sphere \( S^{n-1} \) belongs to the class of \( C^k(S^{n-1}) \) (\( C^\infty(S^{n-1}) \)). It is well-known that one can approximate any convex body in \( \mathbb{R}^n \) in the radial metric, \( d(K, L) = \sup\{|\rho_K(\xi) - \rho_L(\xi)|, \xi \in S^{n-1}\} \), by a sequence of
infinitely smooth convex bodies. The proof is based on a simple convolution argument (see for example [Sch, Theorem 3.3.1]). It is also easy to see that any convex body in $\mathbb{R}^{2n}$ invariant with respect to all $R_\theta$ rotations can be approximated in the radial metric by a sequence of infinitely smooth convex bodies invariant with respect to all $R_\theta$. This follows from the same convolution argument, because invariance with respect to $R_\theta$ is preserved under convolutions.

If $D$ is an infinitely smooth origin symmetric star body in $\mathbb{R}^n$ and $0 < k < n$, then the Fourier transform of the distribution $\|x\|_D^{-k}$ is a homogeneous function of degree $-n + k$ on $\mathbb{R}^n$, whose restriction to the sphere is infinitely smooth (see [K5, Lemma 3.16]).

The following Proposition is a spherical version of Parseval’s formula established in [K3], (see also [K5, Lemma 3.22]):

**Proposition 1.** Let $D$ be an infinitely smooth origin symmetric star body in $\mathbb{R}^n$ and $g \in C^{k-1}(\mathbb{R}^n)$ even homogeneous of degree $-n + k$ function. Then

$$\int_{S^{n-1}} g(\theta)\|\theta\|_D^{-k}d\theta = (2\pi)^n \int_{S^{n-1}} \hat{g}(\xi)(\|\theta\|_D^{-k})^\wedge(\xi)d\xi.$$  

The concept of an intersection body was introduced by Lutwak [Lu]. This concept was generalized in [K3], as follows: Let $1 \leq k < n$, and let $D$ and $L$ be two origin symmetric star bodies in $\mathbb{R}^n$. We say that $D$ is the $k$-intersection body of $L$ if for every $(n-k)$-dimensional subspace $H$ of $\mathbb{R}^n$

$$\text{Vol}_k(D \cap H^\perp) = \text{Vol}_{n-k}(L \cap H).$$

We introduce the class of $k$-intersection bodies, as those star bodies that can be obtained as the limit, in the radial metric, of a sequence of $k$-intersection bodies of star bodies. A Fourier analytic characterization of $k$-intersection bodies was proved in [K4].

**Proposition 2.** An origin symmetric star body $D$ in $\mathbb{R}^n$ is a $k$-intersection body, $1 \leq k \leq n-1$, if and only if $\|\cdot\|_D^{-k}$ is a positive definite distribution.

Let $1 \leq k < 2n$ and let $H$ be an $(2n-k)$-dimensional subspace of $\mathbb{R}^{2n}$. We denote by $\chi(\cdot)$ the indicator function on $[-1,1]$ and by $|\cdot|_2$ the Euclidean norm in the proper space. We fix an orthonormal basis $e_1, \ldots, e_k$ in the orthogonal subspace $H^\perp$. For any convex body $D$ in $\mathbb{R}^{2n}$ and any even positive continuous function $f$ on $\mathbb{R}^{2n}$ we define the $(2n-k)$-dimensional parallel section function $A_{f,D,H}$ as a function on $\mathbb{R}^k$ such that

$$A_{f,D,H}(u) = \int_{\{x \in \mathbb{R}^{2n} : (x,e_1)=u_1,\ldots,(x,e_k)=u_k\}} \chi(\|x\|_D) f(x)dx,$$  

$u \in \mathbb{R}^k$. (2)

The original lower dimensional parallel section function that corresponds to the $(n-k)$-dimensional volume of the section of $D$ with a subspace $H$ (put $n$ instead of $2n$ and $f = 1$), was defined in [K4]. Note that at 0 the
function $A_{f,D,H}$ measures the central section of the body $D$ by the subspace $H$. Passing to polar coordinates on $H$ we have that

$$A_{f,D,H}(0) = \mu(D \cap H) = \int_H \chi(||x||D)f(x)dx$$

$$= \int_{S^{2n-1} \cap H} \left( \int_{||\theta||_D^{-1}} r^{2n-3} f(r\theta)dr \right) d\theta.$$  \hspace{1cm} (3)

If $D$ is infinitely smooth and $f \in C^\infty(\mathbb{R}^{2n})$, the function $A_{f,D,H}$ is infinitely differentiable at the origin (see [K5, Lemma 2.4]). So we can consider the action of the distribution $|u|^{-q-k}/\Gamma(-q/2)$ on $A_{f,D,H}$ and apply a standard regularization argument (see for example [K5, p.36] and [GS, p.10]).

Then the function

$$q \mapsto \left\langle \frac{|u|^{-q-k}}{\Gamma(-\frac{q}{2})}, A_{f,D,H}(u) \right\rangle$$  \hspace{1cm} (4)

is an entire function of $q \in \mathbb{C}$. If $q = 2m$, $m \in \mathbb{N} \cup \{0\}$, then

$$\left\langle \frac{|u|^{-q-k}}{\Gamma(-\frac{q}{2})} \big|_{q=2m}, A_{f,D,H}(u) \right\rangle = \frac{(-1)^m |S^{k-1}|}{2^{m+1}k(k+2) \cdots (k+2m-2)} \Delta^m A_{f,D,H}(0),$$

where $|S^{k-1}| = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of the unit sphere in $\mathbb{R}^k$, and $\Delta = \sum_{i=1}^k \partial^2/\partial u_i^2$ is the $k$-dimensional Laplace operator (see [GS, p.71-74]). Note that the function (4) is equal, up to a constant, to the fractional power of $\Delta^{q/2}A_{f,D,H}$ (see [KKZ] or [K4] for complete definition).

**Remark.** If a body $D$ is $m$-smooth (or infinitely smooth) and $f \in C^m(\mathbb{R}^{2n})$ (or $C^\infty(\mathbb{R}^{2n})$) it is easy for one to see that the function

$$x \mapsto |x|^{-m} \int_0^{1/|x|^k} r^{2n-3} f \left( \frac{x}{r|x|} \right) dr$$

is also $m$-times (infinitely) continuously differentiable on $\mathbb{R}^{2n} \setminus \{0\}$.

The proof of following proposition is similar to that of Proposition 4 in [KKZ]. So we omit it here.

**Proposition 3.** Let $D$ be an infinitely smooth origin symmetric convex body in $\mathbb{R}^{2n}$, $f \in C^\infty(\mathbb{R}^{2n})$, and $1 \leq k < 2n$. Then for every $(2n-k)$-dimensional subspace $H$ of $\mathbb{R}^{2n}$ and any $q \in \mathbb{R}$, $-k < q < 2n - k$,

$$\left\langle \frac{|u|^{-q-k}}{\Gamma(-\frac{q}{2})}, A_{f,D,H}(u) \right\rangle$$
\[ \frac{2^{-q-k \pi - \frac{k}{2}}}{\Gamma \left( \frac{2+k}{2} \right)} \int_{S^{2n-1} \cap H^\perp} \left( |x|_2^{-2n+k+q} \int_0^{\|x\|_D} r^{2n-k-1-q} f \left( \frac{x}{|x|_2} \right) dr \right)^{\wedge} (\theta) d\theta. \]

(5)

Now, if \( m \in \mathbb{N} \cup \{ 0 \} \),

\[ \Delta^m A_{f,D,H}(0) = \frac{(-1)^m}{(2\pi)^k} \int_{S^{2n-1} \cap H^\perp} \left( |x|_2^{-2n+k+2m} \int_0^{\|x\|_D} r^{2n-k-2m} f \left( \frac{x}{|x|_2} \right) dr \right)^{\wedge} (\theta) d\theta. \]

(6)

The following (elementary) inequality is similar to Lemma 1 in [Zv].

**Lemma 1.** Let \( a, b > 0 \) and let \( \alpha \) be a non-negative function on \((0, \max\{a, b\})\) so that the integrals below converge. Then

\[ \int_0^a t^{2n-1} \alpha(t) dt - a^2 \int_0^a t^{2n-3} \alpha(t) dt \leq \int_0^b t^{2n-1} \alpha(t) dt - a^2 \int_0^b t^{2n-3} \alpha(t) dt. \]

(7)

### 3. Connection with \( k \)-intersection bodies

As mentioned in the Introduction, we can assume that the density function is \( R_\theta \)-invariant. This simple observation plays an important role to the solution of the problem.

**Lemma 2.** Suppose \( f \) is an even non-negative continuous function on \( \mathbb{R}^{2n} \) and \( \mu \) is a measure with density \( f \). Then there exists an even non-negative continuous function \( \tilde{f} \) that is invariant with respect to all rotations \( R_\theta \) such that

\[ \mu(D) = \int_D \tilde{f}(x) dx \quad \text{and} \quad \mu(D \cap H_\xi) = \int_{D \cap H_\xi} \tilde{f}(x) dx, \]

for every closed bounded invariant with respect to all \( R_\theta \) set \( D \) in \( \mathbb{R}^{2n} \) and \( \xi \in S^{2n-1} \).

**Proof.** We define its average over the unit circle, \( \tilde{f}(x) = \frac{1}{2\pi} \int_0^{2\pi} f(R_\theta x) d\theta \), for every \( x \in \mathbb{R}^{2n} \). Then for every compact invariant with respect to all \( R_\theta \) set \( D \) in \( \mathbb{R}^{2n} \),

\[ \int_D \tilde{f}(x) dx = \frac{1}{2\pi} \int_D \int_0^{2\pi} f(R_\theta x) d\theta dx = \frac{1}{2\pi} \int_0^{2\pi} \int_{R_\theta^{-1}D} f(y) dy d\theta = \mu(D) \]

since \( R_\theta^{-1}D = D \), for all \( \theta \in [0, 2\pi] \).
Moreover, since central sections of complex convex bodies by complex hyperplanes correspond to convex bodies in $\mathbb{R}^{2n-2}$ that are also invariant with respect to the $R_\theta$ rotations, we similarly get that for every $\xi \in S^{2n-1}$,

$$\mu(D \cap H_\xi) = \int_{D \cap H_\xi} \tilde{f}(x)dx.$$  

Now, we are ready to express the measure of the central sections in terms of the Fourier transform.

**Theorem 1.** Suppose $K$ is an infinitely smooth origin symmetric invariant with respect to all $R_\theta$ convex body in $\mathbb{R}^{2n}$, $n \geq 2$, and $f$ is an infinitely differentiable even positive and $R_\theta$-invariant function on $\mathbb{R}^{2n}$. Then for every $\xi \in S^{2n-1}$

$$\mu(K \cap H_\xi) = \frac{1}{2\pi} \left(|x|_\xi^{-2n+2} \int_0^{\||x||_K} r^{2n-3} f\left(r \frac{x}{|x|_\xi}\right)dr\right)^\wedge (\xi)$$  

In order to prove Theorem 1 we need the following:

**Lemma 3.** Let $K$ and $f$ as in Theorem 1. Then for every $\xi \in S^{2n-1}$ the Fourier transform of the distribution

$$|x|^{-2n+2} \int_0^{||x||_K} r^{2n-3} f \left(r \frac{x}{|x|_\xi}\right)dr$$

is a constant function on $S^{2n-1} \cap H^\perp_\xi$.

**Proof.** The function $\|x\|^{-1}_K$ is invariant with respect to all $R_\theta$ (see Introduction), so, since $f$ is $R_\theta$-invariant it is easy to see that the distribution in (9) is a continuous function which is also invariant with respect to all rotations $R_\theta$. By the connection between the Fourier transform of distributions and linear transformations, its Fourier transform is also invariant with respect to all $R_\theta$. As mentioned in the Introduction, the space $H^\perp_\xi$ is spanned by the vectors $\xi$ and $\xi^\perp$. So every vector in $S^{2n-1} \cap H^\perp_\xi$ is a rotation $R_\theta$, for some $\theta \in [0, 2\pi]$, of $\xi$ and hence the Fourier transform of

$$|x|^{-2n+2} \int_0^{||x||_K} r^{2n-3} f \left(r \frac{x}{|x|_\xi}\right)dr$$

is a constant function on $S^{2n-1} \cap H^\perp_\xi$.  

**Proof of Theorem 1.** Let $\xi \in S^{2n-1}$. In formula (8) we put $H_\xi = H$, $k = 2$ and $m = 0$. Then, by the definition of the lower dimensional section function $A_{f,D,H}(0)$, equation (3), we have that

$$\mu(K \cap H_\xi) = \frac{1}{(2\pi)^2} \int_{S^{2n-1} \cap H^\perp_\xi} \left(|x|^{-2n+2} \int_0^{||x||_K} r^{2n-3} f \left(r \frac{x}{|x|_\xi}\right)dr\right)^\wedge (\eta)d\eta.$$
By Lemma 3, the function under the integral is constant on the circle $S^{2n-1} \cap H_{\xi}$. Since $\xi \in H_{\xi}$ we have that

\[ \mu(K \cap H_{\xi}) = \frac{1}{(2\pi)^2} 2\pi \left( |x|^{-2n-2} \int_0^{\frac{|x|}{2}} r^{2n-3} f(r \frac{x}{|x|}) \, dr \right)^\wedge (\xi) \]

which proves the theorem. \qed

As in the case of the complex Busemann-Petty problem the property of a body to be a 2-intersection body is closely related to the solution of the complex Busemann-Petty problem for arbitrary measures.

**Theorem 2.** The solution of the complex Busemann-Petty problem for arbitrary measures in $\mathbb{C}^n$ has an affirmative answer if and only if every origin symmetric invariant with respect to all $R_\theta$ convex body in $\mathbb{R}^{2n}$ is a 2-intersection body.

The proof of Theorem 2 will follow from the Remarks and the next lemmas.

**Remark 1.** To prove the affirmative part of the problem it is enough to consider infinitely smooth origin symmetric invariant with respect to all $R_\theta$ bodies. This is true because one can approximate, in the radial metric, from inside the body $K$ and from outside the body $L$ by infinitely smooth convex invariant with respect to all $R_\theta$ bodies. Then if the affirmative answer holds for infinitely smooth bodies it also holds in the general case.

**Remark 2.** Let $D$ be an origin symmetric convex body which is not a $k$-intersection body. Then, there exists a sequence of infinitely smooth convex bodies with strictly positive curvature which are not $k$-intersection bodies that converges in the radial metric to $D$, (see [K5, Lemma 4.10]). If, in addition, $D$ is invariant with respect to all $R_\theta$, one can choose a sequence of bodies with the same property.

**Remark 3.** A simple approximation argument allows us to prove Theorem 2 only for measures whose density is an infinitely differentiable even positive and $R_\theta$-invariant function on $\mathbb{R}^{2n}$. Let $f$ be the even positive continuous $R_\theta$-invariant density function of a measure $\mu$, as it is defined in the Introduction. Then there exists an increasing sequence $g_n$ of even positive functions in $C^\infty(\mathbb{R}^{2n})$ such that $g_n(x)\chi(\|x\|_D) \to f(x)\chi(\|x\|_D)$, a.e., for every compact set $D$. Then by the Monotone Convergence Theorem we have that

\[ \int_{\mathbb{R}^{2n}} g_n(x)\chi(\|x\|_D) \, dx \to \mu(D) \quad \text{and} \quad \int_{H} g_n(x)\chi(\|x\|_D) \, dx \to \mu(H \cap D), \]

as $n \to \infty$, for every subspace $H$ of $\mathbb{R}^{2n}$. In addition, by Lemma 2 we may assume that every $g_n$ is also $R_\theta$-invariant.

Now we are ready to prove the affirmative part of the complex Busemann-Petty problem for arbitrary measures.
Lemma 4. Suppose \( K \) and \( L \) are infinitely smooth origin symmetric invariant with respect to all \( R_\theta \) convex bodies in \( \mathbb{R}^{2n} \) so that \( K \) is a 2-intersection body and let \( f \) be an infinitely differentiable even positive \( R_\theta \)-invariant function on \( \mathbb{R}^{2n} \). Then, if for every \( \xi \in S^{2n-1} \)

\[
\mu(K \cap H_\xi) \leq \mu(L \cap H_\xi)
\]

then

\[
\mu(K) \leq \mu(L).
\]

Proof. By the remark before Proposition 3 and [K5, Lemma 3.16], the Fourier transform of the distributions

\[
|x|^{-2n+2} \int_0^{\|x\|_K} r^{2n-3} f(r \frac{x}{|x|_2}) dr,
\]

and

\[
|x|^{-2n+2} \int_0^{\|x\|_L} r^{2n-3} f(r \frac{x}{|x|_2}) dr
\]

are homogeneous of degree \(-2\) and continuous functions on \( \mathbb{R}^{2n} \setminus \{0\} \). So, by Theorem 1 the inequality (10) becomes

\[
\left( |x|^{-2n+2} \int_0^{\|x\|_K} r^{2n-3} f(r \frac{x}{|x|_2}) dr \right)^\wedge (\xi) \leq \left( |x|^{-2n+2} \int_0^{\|x\|_L} r^{2n-3} f(r \frac{x}{|x|_2}) dr \right)^\wedge (\xi).
\]

Since \( K \) is an infinitely smooth 2-intersection body, by Proposition 2 and [K5, Theorem 3.16] the Fourier transform of the distribution \( \|x\|_K^{-2} \) is a non-negative continuous, outside the origin, function on \( \mathbb{R}^{2n} \). Multiplying both sides of the latter inequality by \( (\|x\|_K^{-2})^\wedge \) and applying the spherical version of Parseval, Proposition 1 we have that

\[
\int_{S^{2n-1}} \left( \|x\|_K^{-2} \right)^\wedge (\xi) \left( |x|^{-2n+2} \int_0^{\|x\|_K} r^{2n-3} f(r \frac{x}{|x|_2}) dr \right)^\wedge (\xi) d\xi \leq \int_{S^{2n-1}} \left( \|x\|_K^{-2} \right)^\wedge (\xi) \left( |x|^{-2n+2} \int_0^{\|x\|_L} r^{2n-3} f(r \frac{x}{|x|_2}) dr \right)^\wedge (\xi) d\xi,
\]

which gives

\[
\int_{S^{2n-1}} \|x\|_K^{-2} \int_0^{\|x\|_K} r^{2n-3} f(r x) dr dx \leq \int_{S^{2n-1}} \|x\|_K^{-2} \int_0^{\|x\|_L} r^{2n-3} f(r x) dr dx.
\]

We use the elementary inequality, equation (7), with \( a = \|x\|_K^{-1} \), \( b = \|x\|_L^{-1} \) and \( \alpha(r) = f(r x) \) and integrate over \( S^{2n-1} \). Then

\[
\int_{S^{2n-1}} \int_0^{\|x\|_K^{-1}} r^{2n-3} f(r x) dr dx - \int_{S^{2n-1}} \|x\|_K^{-2} \int_0^{\|x\|_K^{-1}} r^{2n-3} f(r x) dr dx
\]
\[
\leq \int_{S^{2n-1}} \left( \int_0^{\|x\|^2_{L^1}} r^{2n-1} f(rx)dr \right)dx - \int_{S^{2n-1}} \|x\|^2_{K}^{-1} \left( \int_0^{\|x\|^2_{L^1}} r^{2n-3} f(rx)dr \right)dx
\]

(12)

We add the equations (11) and (12) and have that
\[
\int_{S^{2n-1}} \left( \int_0^{\|x\|^2_{K}} r^{2n-1} f(rx)dr \right)dx \leq \int_{S^{2n-1}} \left( \int_0^{\|x\|^2_{K}} r^{2n-1} f(rx)dr \right)dx
\]
which immediately implies that
\[
\mu(K) \leq \mu(L).
\]

For the negative part we need a perturbation argument to construct a body that will give a counter-example to the problem. The following lemma (without the assumption of invariance with respect to \(R_\theta\) rotations) was proved in [Zv, Proposition 2] (see also [K5, Lemma 5.16]). The new body immediately inherits the additional property of invariance with respect to all \(R_\theta\) of the original convex body.

**Lemma 5.** Let \(L\) be an infinitely smooth origin symmetric convex body with positive curvature and let \(f, g \in C^2(\mathbb{R}^{2n})\), such that \(f\) is strictly positive on \(\mathbb{R}^{2n}\). For \(\varepsilon > 0\) we define a star body \(K\) so that
\[
\int_0^{\|x\|^2_{K}} r^{2n-3} f(tx)dt = \int_0^{\|x\|^2_{L}} r^{2n-3} f(tx)dt - \varepsilon g(x), \forall x \in S^{2n-1}.
\]
Then, if \(\varepsilon\) is small enough the body \(K\) is convex. Moreover, if \(L\) is invariant with respect to all \(R_\theta\), and \(f, g\) are \(R_\theta\)-invariant then \(K\) is also invariant with respect to all \(R_\theta\).

**Lemma 6.** Let \(f \in C^\infty(\mathbb{R}^{2n})\) is an even positive \(R_\theta\)-invariant function. Suppose \(L\) is an infinitely smooth origin symmetric invariant with respect to all \(R_\theta\) convex body in \(\mathbb{R}^{2n}\) with positive curvature which is not a 2-intersection body. Then there exists an origin symmetric invariant with respect to all \(R_\theta\) convex body \(K\) in \(\mathbb{R}^{2n}\) so that for every \(\xi \in S^{2n-1}\)
\[
\mu(K \cap H_\xi) \leq \mu(L \cap H_\xi)
\]
but
\[
\mu(K) > \mu(L).
\]

**Proof.** The body \(L\) is infinitely smooth, so, by [K5, Lemma 3.16], the Fourier transform of \(\|x\|^2_{L}\) is a continuous function on \(\mathbb{R}^{2n}\). Since \(L\) is not a 2-intersection body, by Proposition 2 there exists an open set \(\Omega \subset S^{2n-1}\) where the Fourier transform of \(\|x\|^2_{L}\) is negative. We can assume that \(\Omega\) is invariant with respect to rotations \(R_\theta\) since \(L\) is.

Using a standard perturbation procedure for convex bodies, see for example [KKZ, Lemma 5] and [K5, p.96], we define an even non-negative invariant with respect to all \(R_\theta\) function \(h \in C^\infty(S^{2n-1})\) whose support is
in \( \Omega \). We extend \( h \) to an even homogeneous function \( h(\frac{x}{|x|_2})|x|^{-2}_2 \) of degree \(-2\) on \( \mathbb{R}^{2n} \). Then, by [K5, Lemma 3.16] the Fourier transform of \( h(\frac{x}{|x|_2})|x|^{-2}_2 \) is an even homogeneous function \( g(\frac{x}{|x|_2})|x|^{-2n+2}_2 \) of degree \(-2n+2\) on \( \mathbb{R}^{2n} \), with \( g \in C^\infty(S^{2n-1}) \). Moreover, \( g \) is also invariant with respect to rotations \( R_\theta \).

The assumptions for the body \( L \) allow us to apply Lemma 5 and take \( \varepsilon > 0 \) small enough to define a convex body \( \varepsilon h \). On the other hand, the function \( \mu \) is positive only where \( (\| \cdot \|^{-2}_L)^\wedge \) is negative.

So, for every \( \xi \in S^{2n-1} \),

\[
(\| \cdot \|^{-2}_L)^\wedge(\xi)\left(|x|^{-2n+2}_2 \int_0^{\|\cdot\|_L} r^{2n-3} f\left(r\frac{x}{|x|_2}\right) dr\right)^\wedge(\xi)
= (\| \cdot \|^{-2}_L)^\wedge(\xi)\left(|x|^{-2n+2}_2 \int_0^{\|\cdot\|_L} r^{2n-3} f\left(r\frac{x}{|x|_2}\right) dr\right)^\wedge(\xi)
- (2\pi)^{2n} (\| \cdot \|^{-2}_L)^\wedge(\xi) \varepsilon h(\xi)
> (\| \cdot \|^{-2}_L)^\wedge(\xi)\left(|x|^{-2n+2}_2 \int_0^{\|\cdot\|_L} r^{2n-3} f\left(r\frac{x}{|x|_2}\right) dr\right)^\wedge(\xi),
\]

Now, we integrate the latter inequality over \( S^{2n-1} \) and apply the spherical version of Parseval’s identity. Then similarly to Lemma 3 we apply the elementary inequality for integrals, Lemma 1 and conclude that

\[ \mu(K) > \mu(L). \]
4. The solution of the problem

To prove the main result of this paper we need to determine the dimensions in which an origin symmetric invariant with respect to all $R_\theta$ convex body in $\mathbb{R}^{2n}$ is a 2-intersection body.

**Main Theorem.** The solution to the complex Busemann-Petty problem for arbitrary measures is affirmative if $n \leq 3$ and negative if $n \geq 4$.

**Proof.** It is known that an origin symmetric invariant with respect to $R_\theta$, convex body in $\mathbb{R}^{2n}$, $n \geq 2$, is a $k$-intersection body if $k \geq 2n - 4$ (see [KKZ]). Hence, we obtain an affirmative answer to the complex Busemann-Petty problem for arbitrary measures if $n \leq 3$.

Now, suppose that $n \geq 4$. The unit ball $B_q^n$ of the complex space $l_q^n$, $q > 2$, considered as a subset of $\mathbb{R}^{2n}$:

$$B_q^n = \{ x \in \mathbb{R}^{2n} : \|x\|_q = ((x_{11}^2 + x_{12}^2)^{q/2} + \cdots + (x_{n1}^2 + x_{n2}^2)^{q/2})^{1/q} \leq 1 \}$$

provides a counter-example for the Lebesgue measure ($f = 1$), of a body that is not a $k$-intersection body for $k < 2n - 4$ (see [KKZ, Theorem 4]). By Proposition [2] this implies that for $n \geq 4$ the distribution $\|x\|^{-2}$ is not positive definite. Then the result follows by Theorem [2].

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