PRIMITIVE ROOT BIAS FOR TWIN PRIMES II: SCHINZEL-TYPE THEOREMS FOR TOTIENT QUOTIENTS AND THE SUM-OF-DIVISORS FUNCTION

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Abstract. Garcia, Kahoro, and Luca showed that the Bateman–Horn conjecture implies \( \varphi(p-1) \geq \varphi(p+1) \) for a majority of twin-primes pairs \( p, p+2 \) and that the reverse inequality holds for a small positive proportion of the twin primes. That is, \( p \) tends to have more primitive roots than does \( p+2 \). We prove that Dickson’s conjecture, which is much weaker than Bateman–Horn, implies that the quotients \( \frac{\varphi(p+1)}{\varphi(p-1)} \), as \( p, p+2 \) range over the twin primes, are dense in the positive reals. We also establish several Schinzel-type theorems, some of them unconditional, about the behavior of \( \frac{\varphi(p+1)}{\varphi(p)} \) and \( \frac{\sigma(p+1)}{\sigma(p)} \), in which \( \sigma \) denotes the sum-of-divisors function.

1. Introduction

The number of primitive roots modulo a prime \( p \) is \( \varphi(p-1) \), in which

\[
\varphi(n) = \left| \{ i \in \{1, 2, \ldots, n\} : (i, n) = 1 \} \right| = n \prod_{q \mid n} \left( 1 - \frac{1}{q} \right)
\]

is the Euler totient function. In other words, \( \varphi(p-1) \) is the number of generators of the multiplicative group \( (\mathbb{Z}/p\mathbb{Z})^\times \). We reserve \( p, q \) for prime numbers and use \( (a, b) \) to denote the greatest common divisor of \( a \) and \( b \).

For twin primes \( p, p+2 \), it is natural to ask about the relationship between \( \varphi(p-1) \) and \( \varphi(p+1) \). Assuming the Bateman–Horn conjecture, Garcia, Kahoro, and Luca proved that

\[
\varphi(p-1) \geq \varphi(p+1)
\]

for a majority of the twin primes \([10]\). Such proportions are computed relative to the conjectured twin-prime counting function

\[
\pi_2(x) \sim 2C_2 \int_2^x \frac{dt}{(\log t)^2},
\]

in which

\[
C_2 = \prod_{p \geq 3} \frac{p(p-2)}{(p-1)^2} \approx 0.660161815
\]

is the twin primes constant \([1, 13]\). Here \( \sim \) stands for asymptotic equivalence: \( f \sim g \) means \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1 \). The proportion of twin primes that satisfy (1.2) is at least 65% (assuming the Bateman–Horn conjecture), although computations
suggest something around 98%. Moreover, at least 0.46% of the twin primes satisfy the reverse inequality $\varphi(p - 1) \leq \varphi(p + 1)$ [10]. Analogous results for prime pairs $p, p + k$ were obtained by Garcia, Luca, and Schaaff [12]. Garcia and Luca showed unconditionally that the split is 50/50 if only $p$ is assumed to be prime [11].

A glance at the numerical evidence suggests that $\varphi(p + 1) / \varphi(p - 1)$ is bounded as $p, p + 2$ range over the twin primes; see Table 1. Our first theorem, whose proof is in Section 2, demonstrates that this is far from the truth.

Theorem 1. Dickson’s conjecture implies that

$$\left\{ \frac{\varphi(p + 1)}{\varphi(p - 1)} : p, p + 2 \text{ are prime} \right\} \text{ is dense in } [0, \infty).$$

Before proceeding, we require a few words about Dickson’s conjecture. The assertion that there are infinitely many twin primes is the twin prime conjecture, which remains unresolved despite significant recent work [16,18,19,27]. Thus, some unproved conjecture must be assumed to say anything nontrivial about the large-scale behavior of the twin primes. Dickson’s conjecture is among the weakest general assertions that implies the twin prime conjecture [1,6,20].
Dickson’s Conjecture. If \( f_1, f_2, \ldots, f_k \in \mathbb{Z}[t] \) are linear polynomials with positive leading coefficients and \( f = f_1 f_2 \cdots f_k \) does not vanish identically modulo any prime, then \( f_1(t), f_2(t), \ldots, f_k(t) \) are simultaneously prime infinitely often.

The twin prime conjecture is the special case \( f_1(t) = t \) and \( f_2(t) = t + 2 \). Dickson’s conjecture is weaker than the Bateman–Horn conjecture, which concerns polynomials of arbitrary degree and makes asymptotic predictions [1–3]. More extensive computations suggest the truth of Theorem 1; see Table 2.

Totient quotients have a long and storied history [21, Ch. 1]. Schinzel established a curious result in 1954 [22], when he showed that

\[
\left\{ \frac{\varphi(n + 1)}{\varphi(n)} : n = 1, 2, \ldots \right\} \text{ is dense in } [0, \infty).
\]

(1.3)

This inspired later research by Schinzel, Sierpińska, Erdős, and others [7, 8, 23–26].

The prime analogue of (1.3) is false since \( \limsup_{p \to \infty} \frac{\varphi(p+1)}{\varphi(p)} \leq \frac{1}{2} \) because \( p + 1 \) is even when \( p \) is odd. Taking this into account, we prove in Section 3 that the following modified analogue of Schinzel’s theorem holds unconditionally. The main ingredient is a generalization of Chen’s theorem [9, Thm. 25.11].

Theorem 2. (Unconditional)

\[
\left\{ \frac{\varphi(p+1)}{\varphi(p)} : p \text{ prime} \right\} \text{ is dense in } [0, \frac{1}{2}].
\]

The corresponding twin-prime analogue of Schinzel’s theorem (1.3) is the following result, whose proof is in Section 4.

Theorem 3. Dickson’s conjecture implies that

\[
\left\{ \frac{\varphi(p+1)}{\varphi(p)} : p, p+2 \text{ prime} \right\} \text{ is dense in } [0, \frac{1}{3}].
\]

Our proofs are transparent enough to permit the construction of striking numerical examples that cannot be obtained easily through brute force alone. For example, the twin primes

\[
p = 7642856398602124688629749934198565871312540429046303895770916192
95190673487065915056193496684466602708496281503155825518749784559242
263591099165956612523533130299687015551343007872907253311773591116917
\]
and $p + 2$ yield a ratio $\frac{\varphi(p+1)}{\varphi(p)} = 3.11615 \ldots$, which is far larger than those displayed in Table 2. As another example, consider $\pi_{10} = 0.31415 \ldots \in [0,1]$. The method of proof of Theorem 3 (with slight modifications) and a computer search yields the twin prime pair $p = 77262748210044748520865661013827857576370163133157$ and $p + 2$, which satisfies (the underlined digits agree with those of $\pi_{10}$)

$$\frac{\varphi(p+1)}{\varphi(p)} = 0.31415926535897921341 \ldots .$$

Theorem 1, Theorem 2, and Theorem 3 each have analogues for the sum-of-divisors function $\sigma(n) = \sum_{d \mid n} d$. We collect these results in the following theorem, whose proof is in Section 5.

**Theorem 4.**

(a) Dickson's conjecture implies that

$$\left\{ \frac{\sigma(p+1)}{\varphi(p)} : p, p+2 \text{ prime} \right\} \text{ is dense in } [0, \infty).$$

(b) (Unconditional)

$$\left\{ \frac{\sigma(p+1)}{\sigma(p)} : p \text{ prime} \right\} \text{ is dense in } \left[ \frac{3}{2}, \infty \right).$$

(c) Dickson's conjecture implies that

$$\left\{ \frac{\sigma(p+1)}{\sigma(p)} : p, p+2 \text{ prime} \right\} \text{ is dense in } [2, \infty).$$

### 2. Proof of Theorem 1

**A folk lemma.** Mertens' third theorem asserts that

$$\prod_{q \leq x} \left(1 - \frac{1}{q}\right) \sim \frac{e^{-\gamma}}{\log x},$$

in which $\gamma$ is the Euler–Mascheroni constant [14,17]. A more elementary proof of the following lemma can be based on [5, Prop. 8.8] instead.

**Lemma 5.** Let $\mathcal{P}$ denote a finite set of primes. Then

$$\left\{ \frac{\varphi(n)}{n} : n \text{ squarefree, } p \nmid n \text{ for all } p \in \mathcal{P} \right\} \text{ is dense in } [0,1].$$
Proof. Let \( \xi \in (0, 1] \) and \( n_t = \prod_{e^t \leq q < e^t} q \), in which \( t > \frac{1}{\xi} \log \max P \). Then \( n_t \) is squarefree, \( p \nmid n_t \) for all \( p \in P \), and

\[
\frac{\varphi(n_t)}{n_t} = \prod_{e^t \leq q < e^t} \left( 1 - \frac{1}{q} \right) \sim \frac{e^{-\gamma}/\log(e^t)}{e^{-\gamma}/\log(e^{e^t})} \sim \frac{\log(e^{e^t})}{\log(e^t)} = \xi
\]

as \( t \to \infty \). \( \square \)

Initial setup. It suffices to show that Dickson’s conjecture implies that for each fixed \( \xi \in (0, \infty) \) and \( 0 < \delta < 1 \), there is a twin-prime pair \( p, p + 2 \) such that

\[
\frac{\varphi(p+1)}{\varphi(p)} \in (\xi(1-\delta), \xi(1+\delta)).
\]

Let \( 0 < x \leq \min \left\{ \frac{\xi}{\delta}, \xi \right\} \). Lemma 5 provides a squarefree \( b \) such that

\[
(b, 6) = 1 \quad \text{and} \quad \frac{\varphi(b)}{b} \in \left( \frac{x}{\xi} \left( \frac{1}{1+\delta} \right), \frac{x}{\xi} \right).
\]

A second appeal to Lemma 5 yields a squarefree \( a' \) such that

\[
(a', 6b) = 1 \quad \text{and} \quad \frac{\varphi(a')}{a'} \in \left( \frac{3x}{2} \left( 1-\delta \right), \frac{3x}{2} \right).
\]

Our choice of \( x \) ensures that the intervals specified are contained in \((0, 1)\). Let \( a = 3a' \) and observe that

\[
\frac{\varphi(a)}{a} = \frac{2 \varphi(a')}{3 a'} \in (x(1-\delta), x).
\]

Consequently,

\[
\frac{\varphi(a) b}{a \varphi(b)} \in (\xi(1-\delta), \xi(1+\delta)). \quad (2.2)
\]

The polynomials. Our strategy is to produce linear polynomials \( f_1, f_2, \ldots, f_4 \) to which Dickson’s conjecture can be applied, using \( f_1, f_2 \) to produce twin primes \( p, p + 2 \), and using \( f_3, f_4 \) to ensure that \( \frac{\varphi(p+1)}{\varphi(p)} \) falls in the desired interval.

Since \( 8, a^2, b^2 \) are pairwise relatively prime, the Chinese remainder theorem provides \( c \) such that

\[
c \equiv 5 \pmod{8}, \quad (2.3)
\]

\[
c \equiv a - 1 \pmod{a^2}, \quad \text{and} \quad (2.4)
\]

\[
c \equiv b + 1 \pmod{b^2}. \quad (2.5)
\]

Since \( 3 \mid a \), it follows from \((2.4)\) that

\[
c \equiv 2 \pmod{3}. \quad (2.6)
\]

Define

\[
h(t) = 24a^2b^2 t + c
\]

and let

\[
f_1(t) = h(t),
\]

\[
f_2(t) = h(t) + 2,
\]

\[
f_3(t) = \frac{h(t) - 1}{4b} = \frac{24a^2b^2 t + (c - 1)}{4b} = 6a^2bt + \frac{c - 1}{4b}, \quad \text{and}
\]

\[
f_4(t) = \frac{h(t) + 1}{2a} = \frac{24a^2b^2 t + (c + 1)}{2a} = 12ab^2 t + \frac{c + 1}{2a}.
\]
Clearly $f_1, f_2 \in \mathbb{Z}[t]$. Observe that (2.3) and (2.5) ensure that $f_3$ has integral coefficients. Similarly, (2.3) and (2.4) ensure that $f_4$ has integral coefficients. Thus, all four polynomials are in $\mathbb{Z}[t]$ and have positive leading coefficients.

**Nonvanishing product.** We claim that $f = f_1 f_2 f_3 f_4$ does not vanish identically modulo any prime. Since

$$f_1(t) \equiv f_2(t) \equiv c \equiv 1 \pmod{2}, \quad \text{by (2.3)},$$

$$f_3(t) \equiv \frac{c - 1}{4b} \equiv 1 \pmod{2}, \quad \text{by (2, 2) = 1 and (2.3)},$$

$$f_4(t) \equiv \frac{c + 1}{2a} \equiv 1 \pmod{2}, \quad \text{by (a, 2) = 1 and (2.3)},$$

it follows that $f$ does not vanish modulo 2. Similarly,

$$f_1(t) \equiv c \equiv 2 \pmod{3}, \quad \text{by (2.6)},$$

$$f_2(t) \equiv c + 2 \equiv 1 \pmod{3}, \quad \text{by (2.6)},$$

$$f_3(t) \equiv \frac{c - 1}{4b} \equiv b \not\equiv 0 \pmod{3}, \quad \text{by (b, 3) = 1 and (2.6)},$$

$$f_4(t) \equiv \frac{c + 1}{2a} \equiv 2 \pmod{3}, \quad \text{by (2.4)},$$

so $f$ does not vanish modulo 3. The final statement perhaps deserves a bit of explanation. From (2.4) we have $c + 1 \equiv a \pmod{a^2}$ and hence $\frac{c + 1}{a} \equiv 1 \pmod{a}$. Since $3 \mid a$, it follows that $\frac{c + 1}{a} \equiv 1 \pmod{3}$ from which the desired statement follows.

For any prime $q \geq 5$ such that $q \nmid ab$, the polynomial $f$ has degree four and hence cannot vanish identically modulo $q$. Now suppose that $q \geq 5$ is prime and $q \mid ab$. Then $h(t) \equiv c \pmod{q}$. Since (2.4) and (2.5) ensure that

$$c \equiv \begin{cases} -1 \pmod{q} & \text{if } q \mid a, \\ 1 \pmod{q} & \text{if } q \mid b, \end{cases} \tag{2.7}$$

it follows that $f_1$ and $f_2$ do not vanish modulo $q$. Similarly,

$$f_3(t) \equiv \frac{c - 1}{4b} = \begin{cases} -2^{-1}b^{-1} \pmod{q} & \text{if } q \mid a \text{ (by (2.7))}, \\ 4^{-1} \pmod{q} & \text{if } q \mid b \text{ (by (2.5))}, \end{cases}$$

$$f_4(t) \equiv \frac{c + 1}{2a} = \begin{cases} 2^{-1} \pmod{q} & \text{if } q \mid a \text{ (by (2.4))}, \\ a^{-1} \pmod{q} & \text{if } q \mid b \text{ (by (2.7))}. \end{cases}$$

Thus, $f$ does not vanish identically modulo any prime.

**Conclusion.** Dickson’s conjecture provides infinitely many $T$ such that $f_1(T)$, $f_2(T)$, $f_3(T)$, and $f_4(T)$ are prime. For such $T$, the primes

$$p = f_1(T) \quad \text{and} \quad p + 2 = f_2(T)$$

satisfy

$$p + 1 = 2a f_4(T) \quad \text{and} \quad p - 1 = 4b f_3(T).$$

Consequently, (2.2) ensures that

$$\frac{\varphi(p + 1)}{\varphi(p - 1)} = \frac{\varphi(p + 1)}{p + 1} \frac{p - 1}{\varphi(p - 1)} \frac{p + 1}{p - 1}$$
with coprime integers with \(s\) in which \(\text{Thm. 25.11}\). We require a generalization of Chen’s theorem to linear forms.

Theorem 6 (Chen, Friedlander–Iwaniec). Let \(a, c \geq 1\) and \(b \neq 0\) be pairwise coprime integers with 2 \(|\) abc. For \(t\) sufficiently large (in terms of abc),

\[
|\{p \leq t : ap + b = cs\}| \geq \frac{W(abc) Bt}{31e} \frac{1}{(\log t)^2},
\]

in which \(s\) has at most two prime factors, each one larger than \(t^{3/11}\).

\[
W(d) = \prod_{p \mid d, p \geq 2} \left(1 - \frac{1}{p - 1}\right)^{-1} \quad \text{and} \quad B = 2 \prod_{p \geq 3} \left(1 - \frac{1}{(p - 1)^2}\right).
\]

Let \(\xi \in [0, \frac{1}{2}]\) and \(\delta > 0\). For \(x \geq \frac{\log 2}{\xi}\), the integer \(Q = Q(x)\) defined by

\[
Q = 2 \prod_{e^{2\xi x} < q \leq e^x} q
\]

is divisible by 2, but not 4. As \(x \to \infty\), (1.1) and (2.1) imply

\[
\frac{\varphi(Q)}{Q} = \frac{1}{2} \prod_{e^{2\xi x} < q \leq e^x} \left(1 - \frac{1}{q}\right) \sim \frac{1}{2} \frac{e^{-\gamma}}{\log(e^x)} \frac{\log(e^{2\xi x})}{e^{-\gamma}} = \xi.
\]

For each \(x\), apply Theorem 6 with \(a = b = 1\) and \(c = Q\) to obtain an \(S = S(x)\) with at most two prime factors, both of which are greater than \(\max\{Q, x\}\), such that \(p = p(x) = QS - 1\) is prime. Then

\[
\lim_{x \to \infty} \frac{\varphi(S)}{S} = \lim_{x \to \infty} \prod_{q \mid S} \left(1 - \frac{1}{q}\right) = 1
\]

and hence

\[
\frac{\varphi(p + 1)}{\varphi(p)} = \frac{\varphi(QS)}{\varphi(QS - 1)} = \frac{\varphi(Q) \varphi(S)}{QS - 2} = \frac{\varphi(Q) \varphi(S)}{S} \frac{S}{S - 2/Q} \to \xi
\]
as \( x \to \infty \). This concludes the proof. \( \square \)

4. Proof of Theorem 3

Fix \( \xi \in (0, \frac{1}{3}) \) and let \( 0 < \delta < \frac{1-3\xi}{3\xi} \). Lemma 5 yields a squarefree \( Q' \) such that

\[
(Q', 6) = 1 \quad \text{and} \quad \frac{\varphi(Q')}{Q'} \in (3\xi(1-\delta), 3\xi(1+\delta)).
\]

Let \( Q = 6Q' \), and observe that

\[
\frac{\varphi(Q)}{Q} = \frac{1}{3} \frac{\varphi(Q')}{Q'} \in (\xi(1-\delta), \xi(1+\delta)).
\]

Define the polynomials

\[
f_1(t) = t, \quad f_2(t) = Qt - 1, \quad \text{and} \quad f_3(t) = Qt + 1.
\]

If \( q \geq 5 \) and \( q \nmid Q \), then \( f_1f_2f_3 \) has degree three and cannot vanish identically modulo \( q \). If \( q \mid Q \), then \( f_1(1) = Q^2 - 1 \equiv -1 \pmod{q} \) and hence \( f \) does not vanish identically modulo \( q \). In particular, \( f \) does not vanish identically modulo 2 or 3. Thus, \( f \) does not vanish identically modulo any prime.

Dickson’s conjecture provides infinitely many \( T \) such that \( f_1(T), f_2(T), \) and \( f_3(T) \) are prime. In particular, we may assume that the prime \( f_1(T) = T \) is greater than \( Q \) so that \( (Q, T) = 1 \). Then \( p = QT - 1 \) and \( p + 2 = QT + 1 \) are twin primes and \( p + 1 = QT \). Then

\[
\frac{\varphi(p+1)}{\varphi(p)} = \frac{\varphi(QT)}{\varphi(QT-1)} = \frac{\varphi(Q)\varphi(T)}{\varphi(QT-1)} = \frac{\varphi(Q)(T-1)}{QT-2} = \frac{\varphi(Q)}{Q} \frac{Q}{T-2/Q} = \frac{\varphi(Q)}{Q} (1 + o(1))
\]

is in \((\xi(1-\delta), \xi(1+\delta))\) for sufficiently large \( T \). \( \square \)

5. Proof of Theorem 4

Proof of Theorem 4a. The proof of Theorem 4a is similar to the proof of Theorem 1. We first require the following version of Lemma 5 for the sum-of-divisors function.

Lemma 7. Let \( \mathcal{P} \) denote a finite set of primes. Then

\[
\left\{ \frac{\sigma(n)}{n} : n \text{ squarefree, } p \nmid n \text{ for all } p \in \mathcal{P} \right\}
\]

is dense in \([1, \infty)\).

Proof. Let \( Q = Q(t) = \prod_{q \leq t} q \). Then Mertens’ third theorem (2.1), the Euler product formula, and the evaluation \( \zeta(2) = \frac{T^2}{6} \) yield

\[
\frac{\sigma(Q)}{Q} = \prod_{q \leq t} \left(1 + \frac{q}{Q} \right) = \prod_{q \leq t} \left(1 + \frac{1}{q} \right) = \frac{\prod_{q \leq t} (1 - 1/q^2)}{\prod_{q \leq t} (1 - 1/q)} \sim \frac{6/\pi^2}{e^{-\gamma}/\log t} \sim \frac{6e^\gamma}{\pi^2 \log t}
\]
as \( t \to \infty \). Let \( \xi \geq 1 \) and define \( n_t = \prod_{t \leq q < e^t} q \), in which \( \log t > \max P \). Then \( n_t \) is squarefree, \( p \nmid n_t \) for all \( p \in P \), and

\[
\frac{\sigma(n_t)}{n_t} \sim \frac{\log(e^t)}{\log(e^t)} = \xi
\]
as \( t \to \infty \). \( \square \)

Fix \( \xi \in (0, \infty) \) and \( 0 < \delta < 1 \). Let \( x \geq \max \left\{ \frac{4}{3}, \frac{2 \xi}{6} \right\} \). Then Lemma 7 provides a squarefree \( b \) such that

\[
(b, 6) = 1 \quad \text{and} \quad \frac{\sigma(b)}{b} \in \left( \frac{6x}{7\xi}, \frac{6x}{7\xi} \left( \frac{1}{1 - \delta} \right) \right).
\]

A second appeal to Lemma 7 yields a squarefree \( a' \) such that

\[
(a', 6b) = 1 \quad \text{and} \quad \frac{\sigma(a')}{a'} \in \left( \frac{3x}{4}, \frac{3x}{4} (1 + \delta) \right).
\]

Our choice of \( x \) ensures that the intervals specified are contained in \((1, \infty)\). Let \( a = 3a' \) and observe that

\[
\frac{\sigma(a)}{a} = \frac{4}{3} \frac{\sigma(a')}{a'} \in (x, x(1 + \delta)).
\]

Consequently,

\[
\frac{\sigma(a)}{a} \cdot \frac{b}{\sigma(b)} \in \left( \frac{7}{6} \xi (1 - \delta), \frac{7}{6} \xi (1 + \delta) \right). \tag{5.1}
\]

Define the polynomials \( f_1, f_2, f_3, f_4 \) as in the proof of Theorem 1, in which we showed that the application of Dickson’s conjecture to this family is permissible. Dickson’s conjecture provides infinitely many \( T \) such that \( f_1(T), f_2(T), f_3(T), \) and \( f_4(T) \) are prime. For such \( T \), the primes

\[
p = f_1(T) \quad \text{and} \quad p + 2 = f_2(T)
\]
satisfy \( p + 1 = 2a f_4(T) \) and \( p - 1 = 4b f_3(T) \). Consequently, (5.1) ensures that

\[
\frac{\sigma(p + 1)}{\sigma(p - 1)} = \frac{\sigma(p + 1)}{\sigma(p - 1)} \cdot \frac{p - 1}{p + 1} \cdot \frac{p + 1}{p - 1} = \frac{\sigma(p + 1)}{\sigma(p - 1)} \cdot \frac{p - 1}{p + 1} \cdot \frac{1 + o(1)}{1 + o(1)} = \frac{\sigma(2a f_4(T))}{\sigma(4b f_3(T))} \cdot \frac{4b f_3(T)}{2a f_4(T)} \cdot \frac{1 + o(1)}{1 + o(1)} = \frac{3\sigma(a) f_4(T) + 1}{2a f_4(T)} \cdot \frac{4b f_3(T)}{\sigma(b) f_3(T) + 1} \cdot \frac{1 + o(1)}{1 + o(1)} = \frac{6 \sigma(a) b}{7 - a} \cdot \frac{b}{\sigma(b)} \cdot \frac{1 + o(1)}{1 + o(1)}
\]

belongs to \((\xi(1 - \delta), \xi(1 + \delta))\) for large \( T \). \( \square \)
Theorem 4b. Since the proof of Theorem 4b is similar to the proof of Theorem 2, we only sketch the details. First, a simple modification of Lemma 7 shows that for any finite set $\mathcal{P}$ of primes that does not contain 2, the set
\[
\left\{ \frac{\sigma(n)}{n} : n \text{ squarefree and even, } p \nmid n \text{ for all } p \in \mathcal{P} \right\}
\]
is dense in $[\frac{3}{2}, \infty)$. Let $\xi \in [\frac{3}{2}, \infty)$ and mimic the proof of Theorem 2 to find an even squarefree $Q = Q(x)$ such that $\frac{\sigma(Q)}{Q} \to \xi$ as $x \to \infty$. Apply Theorem 6 and obtain an $S = S(x)$ with at most two prime factors, both of which are greater than $\text{max}\{Q, x\}$, such that $p = p(x) = QS - 1$ is prime. Then $\frac{\sigma(S)}{S} \to 1$ as $x \to \infty$ and hence
\[
\frac{\sigma(p + 1)}{\sigma(p)} = \frac{\sigma(QS)}{\sigma(QS - 1)} = \frac{\sigma(Q)\sigma(S)}{QS} = \frac{\sigma(Q)}{Q} \cdot \frac{\sigma(S)}{S} \to \xi. \ \Box
\]

Proof of Theorem 4c. Since the proof of Theorem 4c is similar to the proof of Theorem 3, we only sketch the details. Let $\xi \in [2, \infty)$ and mimic the proof of Theorem 3 to find a squarefree $Q = Q(x)$ which is divisible by 6 such that $\frac{\sigma(Q)}{Q} \to \xi$ as $x \to \infty$.

Define the polynomials $f_1, f_2, f_3$ as in the proof of Section 4 in which we showed that the application of Dickson’s conjecture to this family is permissible. Thus, we can find arbitrarily large $T$ such that $f_1(T) = T$, $p = f_2(T) = QT - 1$, and $p + 2 = f_3(T) = QT + 1$ are simultaneously prime and hence
\[
\frac{\sigma(p + 1)}{\sigma(p)} = \frac{\sigma(QT)}{\sigma(QT - 1)} = \frac{\sigma(Q)\sigma(T)}{QT} = \frac{\sigma(Q)}{Q} \cdot \frac{T + 1}{T} \to \xi. \ \Box
\]

6. Numerical examples

Our methods of proof are transparent enough that they permit us to construct numerical examples whose totient and divisor-sum quotients approximate various mathematical constants surprisingly well (much better than can be obtained by brute force alone). Tables 3, 4, 5, 6, 7, and 8 showcase various examples for each of the theorems proven above.

Computational differences. For the sake of optimization, our computation of numerical examples involves slightly different methods than those provided in the proofs. In particular, our provided proofs of Lemmas 5 and 7 construct a product of consecutive primes between $e^{3t}$ and $e^t$. Our computation takes a more naïve but more efficient process: begin with 1, and repeatedly multiply by the next smallest $q \notin \mathcal{P}$ so that $\frac{\varphi(n)}{n} \geq \xi$ (resp., for Lemma 7, $\frac{\sigma(n)}{n} \leq \xi$); convergence of this process is guaranteed by the fact that $\prod_q (1 - 1/q)$ diverges to 0 (resp., $\prod_q (1 + 1/q)$ diverges to $\infty$), so the sequence we construct is monotonically decreasing (resp., increasing) and is bounded tightly below (resp., above) by $\xi$.

For Theorem 1, Theorem 3, Theorem 4a, and Theorem 4c, the method of construction is otherwise the same, relying on the same polynomial-based approach together with Dickson’s conjecture. For Theorem 2 and Theorem 4b, instead of the unconditional method of proof based on Theorem 6 provided in the paper, we instead took a polynomial/Dickson approach similar to that of Theorem 3 and Theorem 4c based on Lemma 5 and Lemma 7, since we found no straightforward numerical implementation of Theorem 6.
\[ \gamma = 0.577215664901532960 \ldots \]
\[ \frac{\pi}{2} = 0.31415926535897932 \ldots \]
\[ \frac{\pi}{2} = 1.570796326794 \ldots \]
\[ \frac{1}{\pi} = 0.36787944117144232 \ldots \]
\[ \sqrt{2} = 1.414213562373095048 \ldots \]
\[ \sqrt{3} = 1.732050807568877 \ldots \]
\[ \frac{\sqrt{\pi^2} + 1}{2} = 1.618033988749894 \ldots \]
\[ \frac{\sqrt{\pi^2} - 1}{2} = 0.61803398874989484 \ldots \]
\[ \log 2 = 0.693147180 \ldots \]
\[ \log 3 = 1.0986122886681096 \ldots \]

| $\xi$ | $p$ | $\varphi(p+1)$ | $\varphi(p-1)$ |
|-------|-----|----------------|----------------|
| 0.577215664901532960 \ldots | 42372380883463421517 | 32927172834786166829 | 166699917024666639 |
| 0.31415926535897932 \ldots | 39578784498983486285 | 34075187932170983182 | 0.271828182845904501 |
| 1.570796326794 \ldots | 166699917024666639 | 99425902809932261 | 1.570796326782 |
| 0.36787944117144232 \ldots | 06668541034063696514 | 34075187932170983182 | 0.3678794411714429 |
| 0.36787944117144232 \ldots | 9936342696417150404 | 9296419440286092113 | 0.36787944117144229 |
| 1.414213562373095048 \ldots | 68410491409666004609 | 2671592712932741 | 1.414213562373095034 |
| 1.732050807568877 \ldots | 30598653842274957605 | 135626552504989629676 | 1.732050807568862 |
| 1.618033988749894 \ldots | 63690469408652430795 | 81891430565948113897 | 1.6180339887498932 |
| 0.61803398874989484 \ldots | 41813794511453008263 | 7029847514969990486 | 0.6180339887498949 |
| 0.693147180 \ldots | 76327508082936140771 | 3803734229981 | 0.693147172 |
| 1.0986122886681096 \ldots | 86435730621522915217 | 5362742119712964679 | 1.09861228866810905 |

Table 3. Numerical examples for Theorem 1.
\[ \xi \cdot p = \frac{10}{3} \]

| \( \xi \)            | \( p \)       | \( \varphi(p+1) \)  |
|----------------------|--------------|---------------------|
| 0.314159265…        | 1902037158772097 | 0.314159233…        |
| 0.1570796326704…    | 6523051094813143551387418989 | 0.1570796169722…    |
| 0.2718281828…       | 9240530296299581  | 0.2718281556…       |
| 0.367879441…        | 5309646891817189  | 0.367879404…        |
| 0.141421356…        | 4002770936541226705231153047269 | 0.141421342…        |
| \( e - 1 = 0 \)      | 233570456771714761 | 0.414213520…        |
| \( e - 3 = 0 \)      | 72122196308966185995482309 | 0.173205063…        |
| \( e - 5 = 0 \)      | 1061017350953476949129 | 0.223606775…        |
| \( e - 7 = 0 \)      | 184295506315169  | 0.264575104…        |
| \( e - 10 = 0 \)     | 9472109613048558305686109 | 0.161803382…        |

Table 4. Numerical examples for Theorem 2

| \( \xi \) | \( p \)       | \( \varphi(p+1) \)  |
|---|--------------|---------------------|
| 0.314159265… | 1902037158772097 | 0.314159233…        |
| 0.141592653… | 378509219999916257860282849163969 | 0.141592639…        |
| 0.157079632… | 2762774807373943331969 | 0.157079474…        |
| 0.271828182… | 9240621837106421  | 0.2718281556…       |
| 0.141421356… | 400359963884787589865194818589  | 0.1414213429…       |
| 0.173205080… | 72123162724845651734440289  | 0.173205063…        |
| 0.223606797… | 1061064248215841845709  | 0.223606775…        |
| 0.264575131… | 184327276293689  | 0.264575104…        |
| 0.161803398… | 9472109613048558305686109 | 0.161803382…        |

Table 5. Numerical examples for Theorem 3
\[
\begin{array}{|c|c|c|}
\hline
\xi & p & \frac{\sigma(p+1)}{\sigma(p-1)} \\
\hline
\gamma = 0.577215664901532\ldots & 2856597151653495728962024 & 0.577215664901527\ldots \\
& 3848158380563455580813926 & 941 \\
& 37502571622323523026578053 & \\
\pi = 3.1415\ldots & 264692122256841983860921 & 3.1407 \\
& 2640874086000298158238701 & \\
& 0967174823412688455798495 & \\
& 6967613160956665455736936 & \\
& 7349279138922992270763544 & 429 \\
\zeta = 1.57079632679489661\ldots & 271445221589612969174543 & 1.57079632679489675\ldots \\
& 9828024978610834124808317 & 986556098778989 \\
& 2014265861114963900599828 & \\
& 5905237872952857257936121 & 229 \\
& 6852839150744818500432120 & \\
\sqrt{2} = 1.4142135623709733173\ldots & 1300421137603424629894146 & 1.414213562370899\ldots \\
& 7255162665877976721099405 & 92798687340114429 \\
\sqrt{3} = 1.73205080756887729\ldots & 5001304679832232446811636 & 1.73205080756877744\ldots \\
& 91329222196818061901730529 & 8767132709 \\
& 1391240861141698770219064 & \\
\sqrt{5} = 2.2360679774997897\ldots & 2013817027153422288458176 & 2.236067774980\ldots \\
& 8517669084996402903328 & \\
& 707829576024653829857989 & \\
\sqrt{5}+1 = 3.2360679774997897\ldots & 2768409745128994233528433 & 1.61803398876489888\ldots \\
& 3387767866165212510206810 & 96425364422069 \\
& 10854285409242522657480 & \\
\sqrt{5}-1 = 1.2360679774997897\ldots & 809616152917531651234309 & 0.6180339887498954\ldots \\
& 7606604285214472315419401 & 5414 \\
\log 2 = 0.6931471805599\ldots & 1583151329945483227597515 & 0.6931471805601\ldots \\
& 3677620193994400861623411 & 4504225821 \\
\log 3 = 1.09861228886810699\ldots & 2122045296052350978265208 & 1.0986122886810989\ldots \\
& 866926588247982830213319 & 5312459828943557 \\
\hline
\end{array}
\]

Table 6. Numerical examples for Theorem 4a.
ξ | p | $\sigma(p+1)$
--- | --- | ---

10γ = 5.77215... | 82873459080807955117965 | 5.77221...
| 6289238279189797435695514 | 9748729208974570049161227 | 89562389

π = 3.1415926... | 203351964077675489 | 3.1415957...
| 6551931232808483387020 | 66088021133128673476600309 | 2904430789475344820749138

2π = 6.283185... | 69570660616809198888379 | 6.283191...
| 69570660616809198888379 | 487214797161 | 1.6180341...

$\frac{\pi}{2}$ = 1.57079632... | 22955076440560177 | 1.57079648...
| 17165749754384369 | 2.7182821...

$e$ = 2.7182818... | 156513047792653 | 1.73205098...
| 156513047792653 | 4852141797161 | 2.230607...

$\sqrt{5}$ = 2.236067... | 18160683326748793 | 1.6180341...
| 18160683326748793 | 1.6180341...

Table 7. Numerical examples for Theorem 4b.

ξ | p | $\sigma(p+1)$
--- | --- | ---

10γ = 5.7721566... | 259568486506848043313701 | 5.7721572...
| 555218369928655989702069 | 444615323144556788787804 | 2009056424590269

π = 3.1415926... | 2007224256303311429 | 3.14159296...
| 6561189247647575857894512 | 417012719766272845928619 | 487116702096826558631970

2π = 6.2831853... | 9280421920002799868898085 | 6.28318593...
| 9337896045501356911046437 | 15461617425356689 | 15461617425356689

$e$ = 2.7182818... | 1717018302510268229 | 2.7182821...
| 28500982420045757101 | 2.23606682...

Table 8. Numerical examples for Theorem 4c.

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