FLOW BY MEAN CURVATURE INSIDE A MOVING AMBIENT SPACE

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ABSTRACT. We show some computations related in particular to the motion by mean curvature flow of a submanifold inside an ambient Riemannian manifold evolving by Ricci or backward Ricci flow. Special emphasis is given to the analogous of Huisken’s monotonicity formula and its connection with the validity of some Li–Yau–Hamilton Harnack-type inequalities in a moving manifold.

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1. Static Ambient Space

First we show the extension of Huisken’s monotonicity formula by Hamilton [3]. Let \( u \) be a positive solution of the backward heat equation on a Riemannian manifold \((M, g)\),

\[ u_t = -\Delta_M u. \]

Let us assume we have a smooth, compact, immersed submanifold \( N \) with \( \dim N = n \) evolving by the mean curvature flow in the ambient space \( M \) with \( \dim M = m \), the metric on \( N \) is the induced metric and we let \( \mu \) be the associated measure. We denote the normal indices with \( \alpha, \beta, \gamma, \ldots \) and the tangent ones with \( i, j, k, \ldots \), then,

\[ \Delta^M u = \Delta^N u + g^{\alpha\beta} \nabla_\alpha \nabla_\beta u - H^\alpha \nabla_\alpha u. \] (1.1)

Now we compute

\[ \frac{d}{dt} \int_N u \, d\mu = \int_N u_t + H^\alpha \nabla_\alpha u - H^2 u \, d\mu = \int_N -\Delta^M u + H^\alpha \nabla_\alpha u - H^2 u \, d\mu. \]

Using (1.1) and integrating by parts we obtain

\[ \frac{d}{dt} \int_N u \, d\mu = \int_N -g^{\alpha\beta} \nabla_\alpha \nabla_\beta u + 2H^\alpha \nabla_\alpha u - H^2 u \, d\mu. \]

Adding and subtracting the quantity \( \frac{\nabla_\alpha u \nabla_\alpha u}{u} \) we get

\[ \frac{d}{dt} \int_N u \, d\mu = \int_N - \left( H^2 u - 2H^\alpha \nabla_\alpha u + \frac{\nabla_\alpha u \nabla_\alpha u}{u} \right) - \int_N \nabla_\alpha \nabla^\alpha u - \frac{\nabla_\alpha u \nabla^\alpha u}{u} \, d\mu. \]
This becomes

\[
\frac{d}{dt} \int_{N} u \, d\mu = - \int_{N} \left( H - \frac{\nabla^2 u}{u}\right)^2 u \, d\mu - \int_{N} \nabla_\alpha u \nabla_\alpha u - \frac{\nabla_\alpha u \nabla_\alpha u}{u} \, d\mu.
\]

Finally, setting \( \tau = T - t \) for some constant \( T \in \mathbb{R} \) one obtains, for every \( t < T \),

\[
\frac{d}{dt} \left( \tau \frac{m-n}{2} \int_{N} u \, d\mu \right) = - \frac{m-n}{2} \int_{N} H - \frac{\nabla^2 u}{u} \, u \, d\mu - \frac{\nabla_\alpha u \nabla_\alpha u}{u} + \frac{u}{2\tau} (m-n) \, d\mu
\]

\[
= - \frac{m-n}{2} \int_{N} H - \frac{\nabla^2 u}{u} \, u \, d\mu - \frac{\nabla_\alpha u \nabla_\alpha u}{u^2} \, d\mu + \left( \nabla_\alpha \nabla_\beta u - \frac{g_{\alpha\beta}}{2\tau} \right) g^{\alpha\beta} \, d\mu
\]

where in the last passage we substituted \( u = e^{-f} \), as \( u > 0 \). Notice that \( f_t = -\Delta f + |\nabla f|^2 \).

This is Hamilton’s result in [3].

2. Moving Ambient Space

Let us assume now that the metric of the ambient space evolves by the rule \( g_t = -2Q \) (if \( Q = \text{Ric} \) we have the Ricci flow) and the backward heat equation is modified to

\[
u_t = -\Delta^M u + Ku
\]

for some function \( K \).

If now we repeat the previous computation we have two extra terms, the first is coming from the modification to the equation for \( u \), the second from the time derivative of the measure on \( N \). Indeed, the associated metric on \( N \) is affected not only by the motion of the submanifold but also by the evolution of the ambient metric on \( M \). After some computation we have

\[
\frac{d}{dt} \mu = (-H^2 - g^{ij} Q_{ij}) \mu = (-H^2 - \text{tr} Q + g^{\alpha\beta} Q_{\alpha\beta}) \mu.
\]

Therefore we get

\[
\frac{d}{dt} \left( \tau \frac{m-n}{2} \int_{N} u \, d\mu \right) = - \tau \frac{m-n}{2} \int_{N} \left( H - \frac{\nabla^2 u}{u}\right)^2 u \, d\mu
\]

\[
= - \tau \frac{m-n}{2} \int_{N} \left( \nabla^2_\alpha u - \frac{\nabla_\alpha u \nabla_\alpha u}{u^2} + g_{\alpha\beta} \frac{g^{\alpha\beta}}{2\tau} \right) g^{\alpha\beta} \, d\mu
\]

\[
+ \tau \frac{m-n}{2} \int_{N} \left( K - \text{tr} Q + g^{\alpha\beta} Q_{\alpha\beta} \right) u \, d\mu
\]

\[
= - \tau \frac{m-n}{2} \int_{N} \left( H + \nabla^2 f \right)^2 e^{-f} \, d\mu
\]

\[
+ \tau \frac{m-n}{2} \int_{N} \left( \nabla^2_\alpha f + Q_{\alpha\beta} - g_{\alpha\beta} \frac{g^{\alpha\beta}}{2\tau} \right) g^{\alpha\beta} e^{-f} \, d\mu
\]

(2.1)
where we substituted \( u = e^{-f} \), hence, \( f_t = -\Delta_M f + |\nabla f|^2 - K \).

This computation suggests that a good choice is \( K = \text{tr} \, Q \) as the last term vanishes and we get
\[
\frac{d}{dt} \left( \tau \frac{m-n}{2} \int_N u \, d\mu \right) = -\tau \frac{m-n}{2} \int_N |H + \nabla^\perp f|^2 \, e^{-f} \, d\mu + \tau \frac{m-n}{2} \int_N \left( \nabla^2 \alpha_\beta f + Q_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau} \right) g^{\alpha\beta} e^{-f} \, d\mu. \tag{2.2}
\]

Moreover, notice that with the choice \( K = \text{tr} \, Q \), we have
\[
\frac{d}{dt} \int_M u = \int_M u_t - \text{tr} \, Qu = \int_M -\Delta M u = 0,
\]
at least when the ambient manifold \( M \) is compact, hence the integral \( \int_M u = \int_M e^{-f} \) is constant during the flow.

3. RICCI AND BACK–RICCI FLOW

3.1. Ricci Flow Case. We choose now \( Q = \text{Ric} \), that is, the metric \( g \) on \( M \) evolves by the Ricci flow in some time interval \((a, b) \subseteq \mathbb{R} \), and we set \( K = R \) to be the scalar curvature. By the previous computation we get
\[
\frac{d}{dt} \left( \tau \frac{m-n}{2} \int_N u \, d\mu \right) = -\tau \frac{m-n}{2} \int_N |H + \nabla^\perp f|^2 \, e^{-f} \, d\mu + \tau \frac{m-n}{2} \int_N \left( \nabla^2 \alpha_\beta f + R_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau} \right) g^{\alpha\beta} e^{-f} \, d\mu, \tag{3.1}
\]
for a positive solution of the conjugate backward heat equation
\[
u_t = -\Delta u + Ru \tag{3.2}
\]
and \( f = -\log u \). Hence,
\[
f_t = -\Delta f + |\nabla f|^2 - R. \tag{3.3}
\]

Monotonicity of \( \tau \frac{m-n}{2} \int_N u \, d\mu \) is so related to the nonpositivity of the Li–Yau–Hamilton type expression \( \left( \nabla^2 \alpha_\beta f + R_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau} \right) g^{\alpha\beta} \). Notice that the same conclusion holds also if \( u_t \leq -\Delta u + Ru \).

If \((M, g(t))\) is a gradient soliton of Ricci flow and \( f \) its “potential” function, it is well known that \( u = e^{-f} \) satisfies the conjugate heat equation \((3.2)\) and we have
- Expanding Solitons: flow defined on \((T_{\text{min}}, +\infty)\) and \( \nabla^2 f + \text{Ric} = g/2(T_{\text{min}} - t) \)
- Steady Solitons: eternal flow and \( \nabla^2 f + \text{Ric} = 0 \)
- Shrinking Solitons: flow defined on \((-\infty, T_{\text{max}})\) and \( \nabla^2 f + \text{Ric} = g/2(T_{\text{max}} - t) \)

Substituting, in the three cases, the above expression becomes
- Expanding Soliton: \( m-n \left( \frac{1}{T_{\text{min}} - t} - \frac{1}{T} \right) \) which is always negative as \( t \in (T_{\text{min}}, T) \).
- Steady Soliton: \( m-n \left( - \frac{1}{T} \right) \) which is always negative as \( t \in (-\infty, T) \).
- Shrinking Soliton: \( m-n \left( \frac{1}{T_{\text{max}} - t} - \frac{1}{T} \right) \) which is nonpositive if \( T \leq T_{\text{max}} \) as \( t \in (-\infty, \min\{T, T_{\text{max}}\}) \).

**Proposition 3.1.** If \((M, g(t))\) is a steady or expanding gradient soliton and \( f \) is its potential function, then monotonicity holds for every \( T \geq T_{\text{min}} \). If \((M, g(t))\) is a shrinking gradient soliton on \((-\infty, T_{\text{max}})\) and \( f \) is its potential function, then monotonicity holds for every \( T \leq T_{\text{max}} \).
3.2. Back–Ricci Flow Case. If we choose $Q = -\text{Ric}$, that is, the metric $g$ evolves by back–Ricci flow in some time interval $(a, b) \subset \mathbb{R}$, and we set $K = R$ to be the scalar curvature. By the previous computation we get

$$
\frac{d}{dt} \left( \tau \int_N u \, d\mu \right) = -\tau \int_N \left[ H + \nabla^2 f \right]^2 e^{-f} \, d\mu + \tau \int_N \left( \nabla^2_{\alpha\beta} f - R_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau} \right) g^{\alpha\beta} e^{-f} \, d\mu ,
$$

for a positive solution of the conjugate backward heat equation

$$
u_t = -\Delta u - Ru
$$

and $f = -\log u$. Hence,

$$
f_t = -\Delta f + |\nabla f|^2 + R
$$

Monotonicity of $\tau \int_N u \, d\mu$ is so related to the nonpositivity of the Li–Yau–Hamilton type expression

$$
\left( \nabla^2_{\alpha\beta} f - R_{\alpha\beta} - \frac{g_{\alpha\beta}}{2\tau} \right) g^{\alpha\beta}.
$$

Notice that the same conclusion holds also if $u_t \leq -\Delta u - Ru$.

4. Li–Yau–Hamilton Harnack Inequalities and Ricci Flow

- We denote with $f_{ij} = \nabla_i^2 f$ the second covariant derivative of $f$, then

$$
\nabla_i^2 f = \frac{\partial^2 f}{\partial x_i \partial x_j} - \Gamma_{kl}^i \frac{\partial f}{\partial x_k}.
$$

- Let $\omega_i$ a 1–form, then we have the following formula for interchanging of covariant derivatives

$$
\nabla_{pq} \omega_i - \nabla_{qp} \omega_i = R_{pqij}^s \omega_s.
$$

Let $\omega_{ij}$ a 2–form, then

$$
\nabla_{pq} \omega_{ij} - \nabla_{qp} \omega_{ij} = R_{pqij}^s \omega_s + R_{pqjs}^i \omega_s.
$$

- II Bianchi Identity:

$$
\nabla_s R_{ijkl} + \nabla_l R_{ijks} + \nabla_k R_{ijls} = 0
$$

contracted,

$$
g^{js} \nabla_s R_{ijkl} = -\nabla_l \text{Ric}_{ik} + \nabla_k \text{Ric}_{il} = 0
$$

that is,

$$
\text{div Riem}_{kl} = \nabla_k \text{Ric}_{il} - \nabla_l \text{Ric}_{ik}
$$

contracted again (Schur Lemma),

$$
\text{div Ric}_k = \nabla_k R - \text{div Ric}_k
$$

that is,

$$
\text{div Ric} = \nabla R/2.
$$

- Evolution equations for Ricci tensor and scalar curvature under Ricci flow:

$$
\partial_t \text{Ric}_{ij} = \Delta \text{Ric}_{ij} + 2 \text{Ric}^{pq} \text{R}_{ijpq} - 2 g^{pq} \text{Ric}_{ip} \text{Ric}_{aj}
$$

$$
\partial_t R = \Delta R + 2 |\text{Ric}|^2.
$$

- Evolution equations for Christoffel symbols under Ricci flow:

$$
\partial_t \Gamma^k_{ij} = -g^{kl}(\nabla_i \text{Ric}_{jl} + \nabla_j \text{Ric}_{il} - \nabla_l \text{Ric}_{ij}).
$$
• Interchange of Laplacian and second derivatives:

\[
\nabla^2_{ij} \Delta f = \nabla_i \nabla_j \nabla_k \nabla_l f \\
= \nabla_i (R_{jklp} \nabla_p f) + \nabla_j \nabla_k \nabla_l f \\
= - \nabla_i \left( \text{Ric}_{jp} \nabla_p f \right) + \nabla_j \nabla_k \nabla_l f \\
= - \nabla_i \text{Ric}_{jp} \nabla_p f - \text{Ric}_{jp} f_{ip} + \nabla_i \nabla_k \nabla_l f \\
= - \nabla_i \text{Ric}_{jp} \nabla_p f - \text{Ric}_{jp} f_{ip} + \nabla_i \text{Ric}_{jp} \nabla_p f - \text{Ric}_{ip} f_{jp} + \nabla_i \nabla_k \nabla_l f \\
= - \nabla_i \text{Ric}_{jp} \nabla_p f - \text{Ric}_{jp} f_{ip} - \text{Ric}_{ip} f_{jp} - \text{Ric}_{ip} f_{jp} + \nabla_i \nabla_k \nabla_l f \\
= - (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}) \nabla_k f - \text{Ric}_{jp} f_{ip} - \text{Ric}_{ip} f_{jp} - 2\text{Ric}_{kp} f_{kp} + \Delta \nabla_i \nabla_j f
\]

where in the last passage we used the II Bianchi identity. Hence,

\[
\nabla^2_{ij} \Delta f - \Delta \nabla_i \nabla_j f = - (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}) \nabla_k f - \text{Ric}_{jp} f_{ip} - \text{Ric}_{ip} f_{jp} - 2\text{Ric}_{kp} f_{kp}.
\]

4.1. Computation I: Ricci Flow. Suppose that \(u_t = -\Delta u + Ru\) and \(u > 0\), we want to show the nonpositivity of the term

\[
\nabla^2_{ij} f = \nabla_i \nabla_j f + \text{Ric}_{ij} - \frac{g_{ij}}{2\tau}
\]

for \(f = -\log u\) which satisfies

\[
f_t = -\Delta f + |\nabla f|^2 - R.
\]

Equivalently, if we had chosen \(f = \log u\), we can show the positivity of

\[
\nabla^2_{ij} f - \text{Ric}_{ij} + \frac{g_{ij}}{2\tau}
\]

for \(f = \log u\) which satisfies

\[
f_t = -\Delta f - |\nabla f|^2 + R.
\]

We set \(\tau = T - t\), \(L_{ij} = f_{ij} - \text{Ric}_{ij}, \ H_{ij} = \tau L_{ij} + g_{ij}/2 = \tau [f_{ij} - \text{Ric}_{ij}] + g_{ij}/2\).
\((\partial_t + \Delta)H_{ij} = - L_{ij} - \text{Ric}_{ij} \)
\begin{align*}
+ \tau [\Delta f_{ij} + \nabla^2 f_h + (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}) \nabla^k f]
- \tau [\partial_t \text{Ric}_{ij} + \Delta \text{Ric}_{ij}]
\end{align*}
\begin{align*}
= - L_{ij} - \text{Ric}_{ij} \end{align*}
\begin{align*}
+ \tau [\Delta f_{ij} - \nabla^2 \Delta f - \nabla^2 f |\nabla f|^2]
+ (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}) \nabla^k f]
- \tau [2 \Delta \text{Ric}_{ij} + 2 \text{Ric}_{pq} R_{ipjq} - 2 \text{Ric}_{ip} \text{Ric}_{pj} - \nabla^2 \text{R}]
\end{align*}
\begin{align*}
= - L_{ij} - \text{Ric}_{ij} \end{align*}
\begin{align*}
+ \tau [(\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}) \nabla_k f]
+ \text{Ric}_{jp} f_{jp} + \text{Ric}_{ip} f_{pj} + 2 \text{R} \leq f_{kp}
- \nabla^2 f |\nabla f|^2 + (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}) \nabla^k f]
- \tau [2 \Delta \text{Ric}_{ij} + 2 \text{Ric}_{pq} R_{ipjq} - 2 \text{Ric}_{ip} \text{Ric}_{pj} - \nabla^2 \text{R}]
\end{align*}
\begin{align*}
= - L_{ij} - \text{Ric}_{ij} \end{align*}
\begin{align*}
+ \tau [\text{Ric}_{jp} f_{jp} + \text{Ric}_{ip} f_{pj} + 2 \text{R} \leq f_{kp}
- 2 f_{ip} f_{jp} - 2 \nabla^2 f |\nabla f|^2 + (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}) \nabla^k f]
- \tau [2 \Delta \text{Ric}_{ij} + 2 \text{Ric}_{pq} R_{ipjq} - 2 \text{Ric}_{ip} \text{Ric}_{pj} - \nabla^2 \text{R}]
\end{align*}
\begin{align*}
= - L_{ij} - \text{Ric}_{ij} \end{align*}
\begin{align*}
+ \tau [\text{Ric}_{jp} f_{ip} + \text{Ric}_{ip} f_{pj} + 2 \text{R} \leq f_{kp}
- 2 f_{ip} f_{jp} - 2 \nabla^2 f |\nabla f|^2 + 2 \text{R} \leq f_{kp} \nabla f |\nabla f|^2 + (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}) \nabla^k f]
- \tau [2 \Delta \text{Ric}_{ij} + 2 \text{Ric}_{pq} R_{ipjq} - 2 \text{Ric}_{ip} \text{Ric}_{pj} - \nabla^2 \text{R}]
\end{align*}
\begin{align*}
= - L_{ij} - \text{Ric}_{ij} \end{align*}
\begin{align*}
+ \tau [\text{Ric}_{jp} f_{ip} + \text{Ric}_{ip} f_{pj} - 2 f_{ip} f_{jp} - 2 \nabla^2 f |\nabla f|^2]
- \tau [2 \Delta \text{Ric}_{ij} + 2 \text{Ric}_{pq} R_{ipjq} - 2 \text{Ric}_{ip} \text{Ric}_{pj} - \nabla^2 \text{R}]
+ \tau [2 \nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij}] \nabla^k f
- 2 \text{R} \leq f_{kp} \nabla f |\nabla f|^2
\end{align*}

substituting, \(L_{ij} = [H_{ij} - g_{ij}/2]/\tau\) and \(f_{ij} = [H_{ij} - g_{ij}/2]/\tau + \text{Ric}_{ij}\), we get
\[(\partial_t + \Delta)H_{ij} = -H_{ij}/\tau + g_{ij}/2\tau - \text{Ric}_{ij} + \tau[\text{Ric}_{ip}\text{Ric}_{jp} + \text{Ric}_{kp}\text{Ric}_{pj} - 2\nabla_k\text{Ric}_{ij}\nabla^k f] - 2\tau[H_{ij}^2/\tau^2 - H_{ij}/\tau^2 + g_{ij}/4\tau^2 + \text{Ric}_{ik}\text{Ric}_{kj} + \text{Ric}_{ik}\text{H}_{jk}/\tau + \text{Ric}_{jk}\text{H}_{ik}/\tau - \text{Ric}_{ij}/\tau] + [\text{Ric}_{ip}\text{H}_{ip} + \text{Ric}_{kp}\text{H}_{pj} - 2\nabla_k\text{H}_{ij}\nabla^k f] - \text{Ric}_{ij} - \tau[2\Delta\text{Ric}_{ij} + 2\text{Ric}_{pq}\text{R}_{ipjq} - 2\text{Ric}_{ip}\text{Ric}_{pj} - \nabla^2_{ij} R] + \tau[2(\nabla_i\text{Ric}_{jk} + \nabla_j\text{Ric}_{ik} - \nabla_k\text{Ric}_{ij})\nabla^k f] - 2\tau\text{R}_{ikjp}\nabla_p f\nabla_k f = [H_{ij} - 2H_{ij}^2]/\tau - 2\nabla_k\text{H}_{ij}\nabla^k f - \tau[\text{Ric}_{ik}\text{H}_{jk} + \text{Ric}_{jk}\text{H}_{ik} + 2\text{R}_{ikjp}\text{H}_{pk}] - \tau[2\Delta\text{Ric}_{ij} - 2\text{Ric}_{pq}\text{R}_{ipjq} - 4\text{Ric}_{ip}\text{Ric}_{pj} - \nabla^2_{ij} R - \text{Ric}_{ij}/\tau] + \tau[2(\nabla_i\text{Ric}_{jk} + \nabla_j\text{Ric}_{ik} - 2\nabla_k\text{Ric}_{ij})\nabla^k f] - 2\tau\text{R}_{ikjp}\nabla_p f\nabla_k f \]

so finally, we get

\[(\partial_t + \Delta)H_{ij} = [H_{ij} - 2H_{ij}^2]/\tau - 2\nabla_k\text{H}_{ij}\nabla^k f - \text{Ric}_{ij}\text{H}_{kj} - \text{Ric}_{ij}\text{H}_{ik} - 2\text{R}_{ipjq}\text{H}^{pq} - \tau[2\Delta\text{Ric}_{ij} - 2g^{pq}\text{Ric}_{ip}\text{Ric}_{jq} + 4\text{Ric}^{pq}\text{R}_{ipjq} - \nabla^2_{ij} R - \text{Ric}_{ij}/\tau] + 2\tau(\nabla_i\text{Ric}_{jk} + \nabla_j\text{Ric}_{ik} - 2\nabla_k\text{Ric}_{ij})\nabla^k f - 2\tau\text{R}_{ipjq}\nabla^p f\nabla^q f . \]

Notice that the second and third lines gives the Hamilton’s Harnack quadratic with a wrong term \(-\text{Ric}_{ij}/\tau\).

4.2. Computation II: Back–Ricci Flow. Suppose that \(u_t = -\Delta u - R u \) and \(u > 0\), we want to show the nonpositivity of the term

\[\nabla^2_{ij} f - \text{R}_{ij} - g_{ij}/2\tau\]

for \(f = -\log u\) which satisfies

\[f_t = -\Delta f + |\nabla f|^2 + R . \]

Equivalently, if we had chosen \(f = \log u\) we can show the positivity of

\[\nabla^2_{ij} f + \text{R}_{ij} + g_{ij}/2\tau\]

for \(f = \log u\) which satisfies

\[f_t = -\Delta f - |\nabla f|^2 - R . \]

• Evolution equations for Ricci tensor and scalar curvature under back–Ricci flow:

\[\partial_t\text{Ric} = -(\Delta\text{Ric} + 2\text{Ric}^{pq}\text{R}_{ipjq} - 2g^{pq}\text{Ric}_{ip}\text{Ric}_{jq})\]

\[\partial_t R = -(\Delta R + 2|\text{Ric}|^2) . \]
Evolution equations for Christoffel symbols under back-Ricci flow:

\[ \partial_t \Gamma^k_{ij} = g^{kl}(\nabla_i \text{Ric}_{jl} + \nabla_j \text{Ric}_{il} - \nabla_l \text{Ric}_{ij}) . \]

We set \( f_i = \nabla_i f, f_{ij} = \nabla^2_{ij} f \) and \( L_{ij} = f_{ij} + \text{Ric}_{ij}, H_{ij} = \tau L_{ij} + g_{ij}/2 = \tau [f_{ij} + \text{Ric}_{ij}] + g_{ij}/2, \)

\[ (\partial_t + \Delta)H_{ij} = -L_{ij} + \text{Ric}_{ij} \]

\[ + \tau [(\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij})\nabla^k f] \]

\[ + \tau [\partial_t \text{Ric}_{ij} + \Delta \text{Ric}_{ij}] \]

\[ = -f_{ij} + \tau [\Delta f_{ij} - \nabla^2_{ij} \Delta f - \nabla^2_{ij} |\nabla f|^2 - (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij})\nabla^k f] \]

\[ - \tau [2 \text{Ric}_{pq} \text{Ric}_{ipjq} - 2g^{pq} \text{Ric}_{ipjq} + \nabla^2_{ij} R] \]

\[ = -f_{ij} + \tau [(\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij})\nabla^k f + g^{pq} \text{Ric}_{ipjq} f_{ij} + g^{pq} \text{Ric}_{ipjq} f_{ij} - 2 \text{Ric}_{ipjq} f_{pq}] \]

\[ + \tau [\nabla^2_{ij} |\nabla f|^2 - (\nabla_i \text{Ric}_{jk} + \nabla_j \text{Ric}_{ik} - \nabla_k \text{Ric}_{ij})\nabla^k f] \]

\[ - \tau [2 \text{Ric}_{pq} \text{Ric}_{ipjq} - 2g^{pq} \text{Ric}_{ipjq} + \nabla^2_{ij} R] \]

\[ = -f_{ij} + \tau [g^{pq} \text{Ric}_{ipjq} f_{ij} + g^{pq} \text{Ric}_{ipjq} f_{ij} - 2 \text{Ric}_{ipjq} f_{pq}] \]

\[ - \tau [2 \text{Ric}_{pq} \text{Ric}_{ipjq} - 2g^{pq} \text{Ric}_{ipjq} + \nabla^2_{ij} R] \]

\[ = -f_{ij} + \tau [g^{pq} \text{Ric}_{ipjq} f_{ij} + g^{pq} \text{Ric}_{ipjq} f_{ij} - 2 \text{Ric}_{ipjq} f_{pq}] \]

\[ - \tau [2 \text{Ric}_{pq} \text{Ric}_{ipjq} - 2g^{pq} \text{Ric}_{ipjq} + \nabla^2_{ij} R] \]

\[ = (-f_{ij} + \tau [g^{pq} \text{Ric}_{ipjq} f_{ij} + g^{pq} \text{Ric}_{ipjq} f_{ij} - 2 \text{Ric}_{ipjq} f_{pq}]) V^i V^j \]

Suppose now that at time \( t > 0 \), the tensor \( H_{ij} \) (which goes \( +\infty \) as \( t \to T^- \)) get its “last” zero eigenvalue at some point \( (p, t) \) in space and time, with \( V^i \) unit zero eigenvector. We extend \( V^i \) in space such that \( \nabla V(p) = \nabla^2 V(p) = 0 \) and constant in time. Then if \( Z = H_{ij} V^i V^j \) we have that \( Z \) has a global minimum on \( M \times [t, T] \) at \( (p, t) \). At such point we have \( Z = 0, \nabla Z = 0 \) and \( \Delta Z \geq 0 \), hence, \( f_{ij} V^i V^j = -\text{Ric}_{ij} V^i V^j - 1/2\tau, \) and as \( \nabla V = 0, \nabla_k f_{ij} V^i V^j = -\nabla_k \text{Ric}_{ij} V^i V^j \). Then

\[ 0 \leq \partial_t Z + \Delta Z = (\partial_t H_{ij} + \Delta H_{ij}) V^i V^j \]

\[ = (-f_{ij} + \tau [g^{pq} \text{Ric}_{ipjq} f_{ij} + g^{pq} \text{Ric}_{ipjq} f_{ij} - 2 \text{Ric}_{ipjq} f_{pq}]) V^i V^j \]

By this computation, it follows that we would get a contradiction by maximum principle, if the following Hamilton–Harnack type inequality is true.

\[ \nabla^2_{ij} R + 2 \text{Ric}_{ij}^2 + \text{Ric}_{ij}/\tau - 2 \nabla_k \text{Ric}_{ij} U_k + 2 \text{Ric}_{ipjq} U^p U^q \geq 0. \]

See [4] and also [2].

4.3. Dimension 2. In the special two–dimensional case of a surface with bounded and positive scalar curvature this inequality holds, see [1] Chapter 15, Section 3.

If a positive function \( u \) satisfies

\[ u_t = -\Delta u - Ru \]
for a closed curve moving by its curvature $k$ inside a surface evolving by $g_t = 2\text{Ric} = Rg$, we have
\[
\frac{d}{dt} \left( \sqrt{\tau} \int_\gamma u \, ds \right) \leq -\sqrt{\tau} \int_\gamma \left| k - \nabla^\perp \log u \right|^2 u \, ds,
\]
where $\nu$ is the unit normal to the curve $\gamma$.

4.4. A Very Special Case. In dimension 2, for a surface with positive and bounded scalar curvature, we consider the scalar curvature function $u = R > 0$. It satisfies
\[
\frac{\partial}{\partial t} R = -\Delta R - Ru
\]
as, under the back–Ricci flow, we have
\[
\partial_t R = -\Delta R - R^2.
\]
In this case we can get directly the monotonicity formula
\[
\frac{d}{dt} \left( \sqrt{\tau} \int_\gamma R \, ds \right) \leq -\sqrt{\tau} \int_\gamma \left| k - \nabla^\perp \log R \right|^2 R \, ds,
\]
as the Li–Yau quadratic in this case, that is,
\[
\nabla^\nu \log R + R / 2 + \frac{1}{2\tau},
\]
is nonnegative being exactly the “special” form of Hamilton–Harnack inequality for surfaces with positive scalar curvature (see [11]).

This inequality becomes an equality (for every curve) iff $M$ is a gradient expanding Ricci soliton with $R > 0$ and $k = \nabla^\perp \log R$.

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