A CRITERION FOR ASYMPTOTIC PRESERVING SCHEMES OF KINETIC EQUATIONS TO BE UNIFORMLY STATIONARY PRESERVING

CASIMIR EMAKO
Sorbonne University, UPMC Univ Paris 06, UMR 7598
Laboratoire Jacques-Louis Lions, F-75005, Paris, France

FARAH KANBAR* AND CHRISTIAN KLINGENBERG
University of Würzburg, Germany

MIN TANG
Department of Mathematics and Institute of Natural Sciences
Jiao Tong Univ., Shanghai, China

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ABSTRACT. In this work we are interested in the stationary preserving property of asymptotic preserving (AP) schemes for kinetic models. We introduce a criterion for AP schemes for kinetic equations to be uniformly stationary preserving (SP). Our key observation is that as long as the Maxwellian of the distribution function can be updated explicitly, such AP schemes are also SP. To illustrate our observation, three different AP schemes for three different kinetic models are considered. Their SP property is proved analytically and tested numerically, which confirms our observations.

1. Introduction. Kinetic models describe the time evolution of probability density distribution of particles that travel freely for a certain distance and then change their directions due to collision or scattering. They usually include a transport term that takes into account the movement of the particles and integral terms that take into account the scattering, tumbling or colliding. When the average distance between two successive velocity change is small, i.e. the mean free path is small, one has to use resolved space and time steps that are less than the mean free path. Moreover, the probability density function in kinetic models depends not only on space and time but also on velocity. The high dimensionality and the small mean free path lead to an extremely high computational cost and AP schemes that allow mean free path independent meshes become popular in last decades.

AP schemes were first proposed in [23, 24] for the neutron transport equation and have been successfully extended to a lot of applications, we refer to the review paper

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* Corresponding author: Farah Kanbar.
for more discussions. Different AP schemes have been developed for various kinetic models, including the neutron transport equation \([1, 22, 23, 25]\), the velocity jump model for E.coli chemotaxis \([5, 8]\) and the Boltzmann equation \([10, 30, 6, 18]\). The Knudsen number \(\varepsilon\) is the ratio of the mean free path and the domain typical length scale \([24]\). To prove that a scheme is AP, one has to show that when the Knudsen number goes to zero in the discretized scheme, it converges to a good discretization of the corresponding limit model. The main advantage of AP schemes is that their stability and convergence are independent of the Knudsen number. On the other hand, there are situations when in applications the solution after some time reaches a quasi-stationary state, meaning that numerically the difference between the global equilibrium and the solution after finite time is smaller than machine precision. Thus it is of interest to have a numerical scheme that maintains stationary solutions up to machine precision as known as stationary preserving (SP) schemes.

In the literature, the main motivation of designing SP schemes is that, due to the stiff source term in a hyperbolic equation or system, the low regularity of equilibrium solution requires refinement of the time steps or meshes and special treatment of source terms \([4]\). Similar problems arise for kinetic models when \(\varepsilon << 1\), the right hand side of the steady state equation is stiff and one has to be careful about the space discretizations. For kinetic equations the large time asymptotics are of interest. An AP discretization is then given by a particular discretization in time.

For kinetic equations, stationary solutions are space and velocity dependent solutions. The SP property of a scheme is given by a certain space and velocity discretization. We call a scheme SP if the following two requirements are satisfied:

- The discrete stationary solution provides a good approximation for the steady state solution.
- Starting from a discrete stationary solution, the solution of the time evolutionary problem does not change.

Combining a SP (in space and velocity) discretizations with an AP (in time) discretization gives a scheme we seek in this paper.

Our key observation is that as far as the Maxwellian of the distribution function of an AP scheme can be updated explicitly, the second requirement of the SP property is satisfied immediately. Our proof of the SP property is independent of \(\varepsilon\) and applicable whenever the discretization linearly depends on the Maxwellian of the collision operator. Numerically, one can check that the time evolutionary problem converges to a discrete stationary solution after finite time, and their difference is smaller than machine precision. In the subsequent part, we will consider three different classes of AP schemes for which one can prove their SP properties as examples on which our criterion applies. Once we are able to show that: for an AP scheme, 1) the space and velocity discretization of the stationary equation provides a good approximation to the steady state solution for all \(\varepsilon\); 2) the Maxwellian of the distribution function is updated explicitly, the SP property follows immediately. To show the universality of our observation, we test different kinetic models for different AP schemes, as listed in Table 1.

In recent literature, researchers have looked at AP schemes that are also SP\([14, 15, 16, 3, 9]\). But most are for some particular schemes or systems. Gosse and Toscani have proposed different AP schemes for the one-dimensional Goldstein–Taylor system \([13]\), the quasi-monotone discrete kinetic models \([14]\) and the radiative transfer model \([15]\). In all three papers, the authors present an SP scheme in the rarefied
regime ($\varepsilon$ is $\mathcal{O}(1)$) and then modify it in the diffusive regime ($\varepsilon << 1$) to give an AP scheme. The models considered in [14, 15] are one-dimensional two velocity kinetic equations, we can check that their modified AP schemes satisfy our criterion. On the other hand, one main characteristic of the AP schemes satisfying our criterion is that they are SP uniformly in $\varepsilon$. Therefore, we claim that the modified AP schemes in [14, 15] are uniformly SP. In [16] the authors present an AP scheme for the M1 model for the electronic transport. The scheme is SP in the sense that it preserve particular stationary states. In contrast, in this paper we consider schemes preserve all stationary solutions. Higher order time discretizations that are AP and SP are proposed in [3, 9]. Boscarino and Pareschi introduce an IMEX AP scheme for hyperbolic balance laws that is uniformly SP[3]. Dimarco and Pareschi pointed out the SP property in their paper [9] for an AP IMEX multistep scheme for the BGK and the Boltzmann equation. The main contribution of this paper is the introduction of the criterion which can be applied to different models and schemes. Schemes that have both AP and SP properties are more than previously observed.

The structure of the paper from section 2-4 is presented in Table 1 and we present some numerical results to show the AP and SP properties of each numerical scheme in section 5. All three different strategies of developing AP schemes (Parity-equations based scheme, Unified gas kinetic scheme(UGKS), IMEX penalization method) have been extended to various kinetic models and thus the extension of our observation is natural. Since we are using discrete ordinate method for the velocity discretization, as far as the quadrature sets are AP, they do not effect the SP property. Thus we do not consider velocity discretizations in the subsequent part.

| Section | Kinetic Model         | Scheme                   | Reference     |
|---------|-----------------------|--------------------------|---------------|
| 2       | Neutron transport equation | Parity-equations based  | [20]          |
| 3       | Chemotaxis kinetic model | UGKS             | [25, 30, 31]   |
| 4       | Boltzmann equation    | IMEX Penalization method | [10]          |

Table 1. A list of kinetic models together with their corresponding schemes.

2. Parity equations-based scheme for the Neutron transport equation.
In this section we check the Parity equations-based AP scheme for the neutron transport equation in [20, 21]. This scheme is then proved to be SP as well.

2.1. The neutron transport equation. Consider the one-dimensional neutron transport equation:

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f = \frac{\sigma_T}{\varepsilon^2} \left( \frac{1}{2} \int_{-1}^{1} f dv' - f \right) - \sigma_a \left( \frac{1}{2} \int_{-1}^{1} f dv' \right) + q$$

(1)

with $x \in [x_L, x_R]$ and $v \in [-1,1]$. We present the scheme for a simplified neutron transport equation with $\sigma_T = 1$, $\sigma_a = 0$, $q = 0$. The extension to more general cases does not add any difficulties.

2.2. Discretization of the model. When $\sigma_T = 1$, $\sigma_a = 0$, $q = 0$ in (1), the Parity equations-based scheme in [21] can be summarized by the following steps:
• Rewrite (1) into two equations. For \( v \geq 0 \),
\[
\varepsilon \partial_t f(v) + v \partial_x f(v) = \frac{1}{\varepsilon} \left( \frac{1}{2} \int_{-1}^{0} f dv - f(v) \right),
\]
\[
\varepsilon \partial_t f(-v) - v \partial_x f(-v) = \frac{1}{\varepsilon} \left( \frac{1}{2} \int_{0}^{1} f dv - f(-v) \right). \tag{2}
\]

• Introduce the even and odd parities that are
\[
\begin{align*}
  r(t, x, v) &= \frac{1}{2} [f(t, x, v) + f(t, x, -v)], \\
  j(t, x, v) &= \frac{1}{2} [f(t, x, v) - f(t, x, -v)].
\end{align*}
\]

• Add and subtract the equations in (2) and rewrite them into the following diffusive relaxation system,
\[
\begin{align*}
  \partial_t r + v \partial_x j &= -\frac{1}{\varepsilon^2} (r - \rho_r), \\
  \partial_t j + \eta v \partial_x r &= -\frac{1}{\varepsilon^2} [j + (1 - \varepsilon^2 \eta) v \partial_x r],
\end{align*} \tag{3}
\]
where \( \rho_r = \int_0^1 r dv' \) and \( \eta(\varepsilon) = \min(1, \frac{1}{\varepsilon^2}) \).

• Split the equations (3) into two steps:
  - Relaxation step:
    \[
    \begin{cases}
      \partial_t r = -\frac{1}{\varepsilon^2} (r - \rho_r), \\
      \partial_t j = -\frac{1}{\varepsilon^2} [j + (1 - \varepsilon^2 \eta) v \partial_x r],
    \end{cases}
    \]
  - Transport step:
    \[
    \begin{cases}
      \partial_t r + v \partial_x j = 0, \\
      \partial_t j + \eta v \partial_x r = 0.
    \end{cases}
    \]

• Discretize the two steps as follows:
  - For the transport step, we use an explicit first order upwind scheme on its diagonal from such that
\[
\begin{align*}
  r_i^{n+\frac{1}{2}} &= r_i^n - v \frac{\Delta t}{\Delta x} D^n j_i^n, \\
  j_i^{n+\frac{1}{2}} &= j_i^n - \eta v \frac{\Delta t}{\Delta x} D^n r_i^n. \tag{4}
\end{align*}
\]
where \( D^n f_i^n = f_{i+1}^n - f_i^n \) and \( D^n f_i^n = \frac{f_{i+1}^n - f_i^n}{\Delta x} \) are respectively the upwind and central spatial differences.

  - For the relaxation step, we use an implicit backward Euler method that writes
\[
\begin{align*}
  \frac{r_i^{n+1} - r_i^n}{\Delta t} &= -\frac{1}{\varepsilon^2} (r_i^{n+1} - \rho_{r_i}^{n+1}), \\
  \frac{j_i^{n+1} - j_i^n}{\Delta t} &= -\frac{1}{\varepsilon^2} (j_i^{n+1} + (1 - \varepsilon^2 \eta) v \frac{D^n}{\Delta x} r_i^{n+1}).
\end{align*}
\]
By integrating the above first equation over \( V \) we find, \( \rho_{r_i}^{n+1} = \rho_{r_i}^{n+\frac{1}{2}} \).
Then,
\[
\begin{align*}
  r_i^{n+1} &= A r_i^{n+\frac{1}{2}} + B \rho_{r_i}^{n+1}, \\
  j_i^{n+1} &= A j_i^{n+\frac{1}{2}} - B (1 - \varepsilon^2 \eta) v \frac{D^n}{\Delta x} r_i^{n+1}, \tag{5}
\end{align*}
\]
with \( A \) and \( B \) being defined as:
\[
A = \frac{\varepsilon^2}{\varepsilon^2 + \Delta t} \text{ and } B = \frac{\Delta t}{\varepsilon^2 + \Delta t}.
\]
The fully space-time discretized parity equations-based AP scheme is given by the transport step (4) and the relaxation step (5). The boundary conditions for \( r \) and \( j \) are the same as in [21] and are obtained using the following relations:

\[
\begin{align*}
\frac{r_{i+1}^{n+1} - r_i^n}{\Delta t} + v \frac{D^u}{\Delta x} j_i^n &= -\frac{1}{\varepsilon^2} \left(r_i^{n+1} - \rho_{r_i}^n\right), \\
\frac{j_i^{n+1} - j_i^n}{\Delta t} + \eta v \frac{D^u}{\Delta x} r_i^n &= -\frac{1}{\varepsilon^2} \left(j_i^{n+1} + (1 - \varepsilon^2 \eta) v \frac{D^c}{\Delta x} r_i^n\right).
\end{align*}
\]

(10a) and (10b)

A discrete stationary solution to (10) are \( r_i^n \) and \( j_i^n \) that satisfies:

\[
\begin{align*}
\frac{D^u}{\Delta x} j_i^n &= -\frac{1}{\varepsilon^2} \left(r_i^n - \rho_{r_i}^n\right),& (11a) \\
\eta v \frac{D^u}{\Delta x} r_i^n &= -\frac{1}{\varepsilon^2} \left[j_i^n + (1 - \varepsilon^2 \eta) v \frac{D^c}{\Delta x} r_i^n\right].& (11b)
\end{align*}
\]

Lemma 2.1. When \( r_i^n \) and \( j_i^n \) are solutions of the steady state equation discretization (11), then \( r_i^{n+1} = r_i^n \) and \( j_i^{n+1} = j_i^n \). Hence the parity equations-based scheme satisfies the second requirement of the SP property.

Proof. • For \( r \): Since \( \rho_{r_i}^n = \int_0^1 r_i^n \), inserting (11a) in (10a) and integrating over \([0, 1]\) yields \( \rho_{r_i}^{n+1} = \rho_{r_i}^n \).

Using (11a) and \( \rho_{r_i}^{n+1} = \rho_{r_i}^n \), (10a) gives

\[
\frac{r_{i+1}^{n+1} - r_i^n}{\Delta t} - \frac{1}{\varepsilon^2} (r_i^n - \rho_{r_i}^n) = -\frac{1}{\varepsilon^2} (r_i^{n+1} - \rho_{r_i}^n).
\]

Hence,

\[
\frac{1}{\Delta t} + \frac{1}{\varepsilon^2} (r_{i+1}^{n+1} - r_i^n) = 0.
\]

and then \( r_{i+1}^{n+1} = r_i^n \).
For \( j \): Using \( \Delta t + 1 \), (10b) becomes
\[
\frac{j_i^{n+1} - j_i^n}{\Delta t} + \frac{\eta v D^u}{\Delta x} r_i^n = -\frac{1}{\epsilon^2} [j_i^{n+1} + (1 - \epsilon^2 \eta) v D^e r_i^n].
\] (12)

From (11b), (12) writes,
\[
\frac{j_i^{n+1} - j_i^n}{\Delta t} - \frac{1}{\epsilon^2} [j_i^n + (1 - \epsilon^2 \eta) v D^e r_i^n] = -\frac{1}{\epsilon^2} [j_i^{n+1} + (1 - \epsilon^2 \eta) v D^e r_i^n].
\]

Then,
\[
\left(\frac{1}{\Delta t} + \frac{1}{\epsilon^2}\right)(j_i^{n+1} - j_i^n) = 0
\]
and thus \( j_i^{n+1} = j_i^n \).

Using this Lemma, the scheme satisfies the both requirements of the SP property as mentioned in the bullet points in the introduction. This is because starting from a discrete stationary solution, our discretization of the time evolutionary problem does not change this discrete stationary solution. This way we have shown that our parity equations-based scheme (which is AP) has both the AP and SP properties.

3. UGKS scheme for the chemotaxis kinetic model. In this section we first extend the UGKS in [25, 30, 31] to the time evolutionary chemotaxis model, then show its AP and SP properties. The following AP scheme is a new scheme developed in this paper in contrast to the schemes in section 2 and section 4 that were taken from the literature.

3.1. The chemotaxis kinetic model. The chemotaxis kinetic model models bacteria that undergo run and tumble process as mentioned in [17, 28, 29]. During the run phase, bacteria move along a straight line and change their directions during the tumble phase. This is called the velocity jump process and can be modeled by the Othmer-Dunbar-Alt model that writes [2, 26]:
\[
\begin{aligned}
\partial_t f + \frac{1}{\epsilon} v \cdot \nabla_x f &= \frac{1}{|V|} \int_V (1 + \epsilon \phi(v') \partial_x S) f(v') dv' - (1 + \epsilon \phi(v \cdot \partial_x S)) f(v) [1], \\
\partial_t S - D \Delta S + \alpha S &= \beta \rho, \\
\rho(x, t) := \frac{1}{|V|} \int_V f(v) dv.
\end{aligned}
\] (13)

Here \( f(x, v, t) \) is the probability density function at time \( t \), position \( x \) and moving with velocity \( v \); \( \phi \) is an odd decreasing function such that \( \phi(-u) = -\phi(u) \); \( S(x, t) \) is the concentration of a chemical substance where the parameters \( D, \alpha, \beta \) are positive constants; \( \epsilon \) is the Knudsen number. When \( \phi = 0 \), the chemotaxis kinetic model reduces to the neutron transport equation.

As \( \epsilon \to 0 \), \( f(x, v, t) \) converges to \( \rho_0(x, t) \) where \( \rho_0(x, t) \) solves the following Keller-Segel equation [7, 19, 27]:
\[
\begin{aligned}
\partial_t \rho_0 &= \frac{1}{3} \Delta \rho_0 + \nabla((\frac{1}{|V|} \int_V v \phi(v \cdot \partial_x S) dv) \rho_0), \\
\partial_t S - D \Delta S + \alpha S &= \beta \rho_0.
\end{aligned}
\] (14)
3.2. Discretization of the model. Before discussing about the more complex equation for \( f \), we first discretize the equation for the chemical concentration \( S \). Let \( S^n \approx S(x, t^n) \), the following centered finite difference method is used to update \( S \):

\[
\frac{S^{n+1}_i - S^n_i}{\Delta t} = D \frac{S^{n+1}_{i+1} - 2S^n_{i+1} + S^{n+1}_{i-1}}{\Delta x^2} - \alpha S^{n+1}_i + \beta \rho^n_i. \tag{15}
\]

After \( S^{n+1} \) are obtained, we approximate \( \partial_x S^{n+1} \) by a piecewise constant function such that

\[
\partial_x S(x, t^{n+1}) \approx \partial_x S(x_i, t^{n+1}) \approx \frac{S^{n+1}_{i+1} - S^{n+1}_i}{\Delta x} := \sigma_{i+\frac{1}{2}}, \quad \text{for } x \in [x_i, x_{i+1}).
\]

The UGKS is a finite volume approach for discretizing the kinetic equation \( f \). By integrating the chemotaxis kinetic model (13) over \([x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}] \times V \) and letting \( f^n_i = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} f(x, v, t^n) \, dx \), \( \rho^n_i = \frac{1}{|V|} \int_V f^n_i \, dv \), the total density \( \rho^n_i \) and density fluxes \( f^n_i \) are updated as follows

\[
\frac{\rho^{n+1}_i - \rho^n_i}{\Delta t} + \frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x} = 0,
\]

\[
\frac{f^{n+1}_i - f^n_i}{\Delta t} + \frac{\Phi^n_{i+\frac{1}{2}} - \Phi^n_{i-\frac{1}{2}}}{\Delta x} = \frac{1}{\varepsilon^2} \left( \rho^{n+1}_i - f^{n+1}_i \right)
\]

\[
+ \frac{1}{\varepsilon} \left( \frac{1}{|V|} \int_V \phi(v' \sigma_{i+\frac{1}{2}}) f^n_i(v') \, dv' - \phi(v \sigma_{i+\frac{1}{2}}) f^n_i \right). \tag{18}
\]

Here the numerical fluxes are given by

\[
\phi^n_{i+\frac{1}{2}} = \frac{1}{\varepsilon \Delta t} \int_{t^n}^{t^{n+1}} v f(x_{i+\frac{1}{2}}, v, t) \, dt,
\]

\[
\Phi^n_{i+\frac{1}{2}} = \frac{1}{|V|} \int_V \left( \frac{1}{\varepsilon \Delta t} \int_{t^n}^{t^{n+1}} v f(x_{i+\frac{1}{2}}, v, t) \, dt \right) \, dv. \tag{19}
\]

It is important to note that \( \sigma_{i+\frac{1}{2}} \) approximates \( \partial_x S \) in the interval \([x_i, x_{i+1})\) while \( f^n_i \) is the average density over the cell \([x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\). This choice is important to get the correct advection term in the limit Keller-Segel model when \( \varepsilon \) becomes small.

We use discrete ordinates method for the velocity discretization, but for convenience of explanation, we write the scheme in continuous velocity. The most crucial step for UGKS is to determine \( \Phi^n_{i+\frac{1}{2}} \) and \( F^n_{i+\frac{1}{2}} \). The details are listed below:

- **Find the approximation of** \( f(x_{i+\frac{1}{2}}, v, t) \). The 1d chemotaxis model (13) can be rewritten as:

\[
\partial_t f + \frac{1 + \varepsilon \phi(v \sigma_i S)}{\varepsilon^2} f + \frac{v}{\varepsilon} \partial_x f = \frac{1}{\varepsilon^2} T^1 f, \tag{20}
\]

where \((T^1 f)(x, t) := \frac{1}{|V|} \int_V (1 + \varepsilon \phi(v' \partial_x S)) f(x, v', t) \, dv'\). Consider the interval \([x_i, x_{i+1})\), multiplying both sides of (20) by \( \exp \left( \frac{(1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}})) t}{\varepsilon^2} \right) \) yields

\[
\frac{d}{dt} \left[ f(x + \frac{v}{\varepsilon} t, v, t) \exp \left( \frac{(1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}})) t}{\varepsilon^2} \right) \right] = T^1 f(x, t) \exp \left( \frac{(1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}})) t}{\varepsilon^2} \right). \]
Integrating the above equation over \((t^n, t)\) yields to,

\[
f(x_{i+\frac{1}{2}}, v, t) = f(x_{i+\frac{1}{2}} - \frac{v}{\varepsilon}(t - t^n), v, t^n) \exp \left( -\frac{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right)
+ \frac{1}{\varepsilon^2} \int_{t^n}^{t} T^1 f(x_{i+\frac{1}{2}} - \frac{v}{\varepsilon}(t - s), s) \exp \left( -\frac{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - s) \right) ds.
\] (21)

This is an exact expression for \(f(x_{i+\frac{1}{2}}, v, t)\) that will be used to determine \(\Phi^n_{i+\frac{1}{2}}, F^n_{i+\frac{1}{2}}\) in (19). At this stage, we need to approximate \(f(x, v, t^n)\) and \((T^1 f)(x, t)\) on the right hand side of (21). \(f\) is approximated by a piecewise constant function and \(T^1 f\) by a piecewise linear function as follows:

\[
f(x, v, t^n) = \begin{cases} f_i^n, & x < x_{i+\frac{1}{2}}, \\ f_{i+1}^n, & x > x_{i+\frac{1}{2}}, \end{cases}
\]

\[
T^1 f(x, t) = \begin{cases} T^1 f_i^n + \delta^L T^1 f_i^n (x - x_{i+\frac{1}{2}}), & x < x_{i+\frac{1}{2}}, \\ T^1 f_{i+1}^n + \delta^R T^1 f_{i+1}^n (x - x_{i+\frac{1}{2}}), & x > x_{i+\frac{1}{2}}. \end{cases}
\]

Here \(T^1 f_i^n, \delta^L T^1 f_i^n, \delta^R T^1 f_i^n\) are defined by:

\[
\begin{align*}
T^1 f_i^n & := \frac{1}{|V_i|} \int_{V_i} (1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}})) f_{i+1}^n + \frac{1}{|V_i|} \int_{V_i} (1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}})) f_i^n, \\
\delta^L T^1 f_i^n & := \frac{T^1 f_i^n - T^1 f^n}{\Delta x/2}, \\
\delta^R T^1 f_i^n & := \frac{T^1 f_{i+1}^n - T^1 f_i^n}{\Delta x/2},
\end{align*}
\]

with \(V^+ = V \cap \mathbb{R}^+\) and \(V^- = V \cap \mathbb{R}^-.\)

Substituting the above approximations into (21) yields an expression for \(f(x_{i+\frac{1}{2}}, v, t)\) such that:

For \(v > 0\),

\[
f(x_{i+\frac{1}{2}}, v, t) = f_i^n \exp \left( -\frac{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right) + \frac{T^1 f_i^n}{1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}})}
\]
\[
\times \left[ 1 - \exp \left( -\frac{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right) \right] + \varepsilon \frac{\delta^L T^1 f_i^n}{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))^2}
\]
\[
\times \left[ \left( 1 + \frac{1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2} (t - t^n) \right) \exp \left( -\frac{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right) - 1 \right],
\] (22)

and for \(v < 0\),

\[
f(x_{i+\frac{1}{2}}, v, t) = f_{i+1}^n \exp \left( -\frac{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right) + \frac{T^1 f_{i+1}^n}{1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}})}
\]
\[
\times \left[ 1 - \exp \left( -\frac{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right) \right] - \varepsilon \frac{\delta^R T^1 f_{i+1}^n}{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))^2}
\]
\[
\times \left[ \left( 1 + \frac{1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}})}{\varepsilon^2} (t - t^n) \right) \exp \left( -\frac{(1 + \varepsilon \phi(v\sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right) - 1 \right].
C. For the second requirement we assume that we start from a steady state solution property. The AP discretization of the steady state equation is given in Appendix 3.3. Integrating (27) over yields

\[
\times \left(1 - \exp \left( - \frac{(1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right) \right) + v \varepsilon \frac{\delta^R T^1 f^n_{i+\frac{1}{2}}}{(1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}}))^2} \\
\times \left[1 + \frac{1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}})}{\varepsilon^2} (t - t^n) \exp \left( - \frac{(1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}}))}{\varepsilon^2} (t - t^n) \right) - 1 \right].
\]

(23)

- **Determine** \(\Phi^n_{i+\frac{1}{2}}, F^n_{i+\frac{1}{2}}\). The flux \(\Phi^n_{i+\frac{1}{2}}(v)\) in (19) can be approximated by

\[
\Phi_{i+\frac{1}{2}}(v) = A v f^n_{i+1} + B v T^1 f^n_{i+\frac{1}{2}} + C v^2 \delta^R T^1 f^n_{i+\frac{1}{2}}, \quad \text{for } v < 0,
\]

\[
\Phi_{i+\frac{1}{2}}(v) = A v f^n_{i} + B v T^1 f^n_{i+\frac{1}{2}} + C v^2 \delta^R T^1 f^n_{i+\frac{1}{2}}, \quad \text{for } v > 0,
\]

(24)

where the coefficients \(A(v, \varepsilon, \Delta t), B(v, \varepsilon, \Delta t), C(v, \varepsilon, \Delta t)\) can be determined explicitly such that

\[
A(v, \varepsilon, \Delta t) := \frac{\varepsilon}{\Delta t (1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}}))} \left(1 - \exp \left( - \frac{1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}})}{\varepsilon^2} \Delta t \right) \right),
\]

\[
B(v, \varepsilon, \Delta t) := \frac{1}{\varepsilon (1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}}))} \\
- \frac{\varepsilon}{\Delta t (1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}}))^2} \left(1 - \exp \left( - \frac{1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}})}{\varepsilon^2} \Delta t \right) \right),
\]

(25)

\[
C(v, \varepsilon, \Delta t) := \frac{2\varepsilon^2}{\Delta t (1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}}))^2} \left(1 - \exp \left( - \frac{1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}})}{\varepsilon^2} \Delta t \right) \right) \\
- \frac{1}{(1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}}))^2} \left(1 + \exp \left( - \frac{1 + \varepsilon \phi(v \sigma_{i+\frac{1}{2}})}{\varepsilon^2} \Delta t \right) \right).
\]

Furthermore, \(F^n_{i+\frac{1}{2}}\) in (19) is given by

\[
F^n_{i+\frac{1}{2}} = \frac{1}{|V|} \int_{V^-} A v f^n_{i+1} dv + \frac{1}{|V|} \int_{V^+} A v f^n_{i} dv + \frac{1}{|V|} T^1 f^n_{i+\frac{1}{2}} \int_{V} v B dv \\
+ \frac{1}{|V|} \delta^R T^1 f^n_{i+\frac{1}{2}} \int_{V^-} C v^2 dv + \frac{1}{|V|} \delta^L T^1 f^n_{i+\frac{1}{2}} \int_{V^+} C v^2 dv.
\]

(26)

This concludes the construction of the scheme. For the proof of its AP property, one can refer to Appendix A.

3.3. **SP property.** The UGKS scheme has to meet the two requirements of the SP property. The AP discretization of the steady state equation is given in Appendix C. For the second requirement we assume that we start from a steady state solution that at the discrete level satisfies,

\[
\frac{\Phi^n_{i+\frac{1}{2}} - \Phi^n_{i-\frac{1}{2}}}{\Delta x} = \frac{1}{\varepsilon^2} (\rho^n_{i+\frac{1}{2}} - f^n_{i+\frac{1}{2}}) + \frac{1}{\varepsilon} \left(\frac{1}{|V|} \int_{V} \phi(v' \sigma_{i+\frac{1}{2}}) f^n_{i} (v') dv' - \phi(v \sigma_{i+\frac{1}{2}}) f^n_{i} \right).
\]

(27)

Integrating (27) over \(v\) yields

\[
\frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x} = 0.
\]
From (17) one can deduce that,
\[ \rho_i^{n+1} = \rho_i^n, \]
which indicates that the macroscopic density is preserved. Using (27), the equation of updating \( f_i^{n+1} \) in (18) can be written as,
\[ \frac{f_i^{n+1} - f_i^n}{\Delta t} = \frac{1}{\varepsilon^2} \left( (\rho_i^{n+1} - \rho_i^n) - (f_i^{n+1} - f_i^n) \right). \]
Then from (28),
\[ (1 + \frac{\Delta t}{\varepsilon^2})(f_i^{n+1} - f_i^n) = 0, \]
which gives \( f_i^{n+1} = f_i^n \). This concludes the SP property of the UGKS.

4. IMEX scheme with the Penalization method for the Boltzmann equation.

In this section, we consider the penalization method developed in [10] for the Boltzmann equation. This method together with an IMEX discretization of the equation give an AP scheme for the Boltzmann equation. One can find the AP proof in [10]. Here we show that the penalization method is not only AP but also SP. In [9], the authors propose a multistep high order IMEX AP scheme for the BGK model and the Boltzmann equation. The scheme is originally developed for the BGK model and then extended by the penalization method to the Boltzmann equation. One can think of the scheme proposed in [9] as the high order version of the scheme in [10]. The authors prove that the IMEX AP scheme, without penalization, is SP uniformly in \( \varepsilon \). Our criterion can be applied successfully to the high order IMEX AP scheme in [9] after penalization. Our proof, in contrast to theirs, requires the linear dependence of the Maxwellian of the collision operator.

4.1. The Boltzmann equation. The Boltzmann equation describes the time evolution of the density distribution of gas particles. It is given by
\[ \partial_t f + v \cdot \nabla_x f = \frac{Q(f)}{\varepsilon}. \]
Here \( f(x,v,t) \) is the probability density distribution of particles at time \( t \), position \( x \) and with velocity \( v \). \( Q \) is the Boltzmann collision operator where only binary interactions are considered. Let \((v,v_*)\) and \((v',v'_*)\) be respectively the velocities of the two colliding particles before and after the collision related by
\[ \begin{aligned}
v' &= \frac{1}{2}((v - v_*) - |v - v_*|\sigma), \\
v'_* &= \frac{1}{2}((v - v_*) + |v - v_*|\sigma).
\end{aligned} \]
with \( \sigma \in S^{d_v-1} \). \( Q \) is given by
\[ Q(f)(v) = \int_{B(0,|v-v_*|,\cos \theta)} \int_{S^{d_v-1}} B(|v-v_*|, \cos \theta)(f(v'_*)f(v') - f(v_*)f(v))d\sigma d\sigma'. \]
The collision kernel \( B \) is a non-negative function given by \( B(|u|, \cos \theta) = C_\lambda |u|^{\lambda} \) where \( u = \frac{(v-v_*)}{|v-v_*|} \) and \( \cos \theta = u \cdot \sigma \) for some \( \lambda \in [0,1] \) and a constant \( C_\lambda > 0 \). For more details, one can look at the Boltzmann equation description in [10]. \( \varepsilon \) is the dimensionless Knudsen number and \( \int v \omega(v)Q(f)dv = 0 \) for \( \omega(v) = (1,v,|v|^2) \). The equilibrium distribution of \( Q \) is the Maxwellian distribution \( M_{\rho,u,T} \), i.e. \( Q(M_{\rho,u,T}) = 0 \). As \( \varepsilon \to 0 \), the zeroth, first and second moments of the distribution function solve the Euler equations.
4.2. IMEX scheme with the penalization method. The penalization method was originally developed in [10, 20]. The idea is to split the collision term of the Boltzmann equation into a stiff part and less stiff part. More precisely, the Boltzmann equation is written in the following form:

\[
\partial_t f + v \cdot \nabla_x f = \frac{Q(f) - P(f)}{\varepsilon} + \frac{P(f)}{\varepsilon}.
\]

where \(Q(f)\) is the Boltzmann collision operator and \(P(f)\) is a relaxation operator, namely \(P(f) = \beta [M_{\rho,u,T}(v) - f(v)]\) where \(\beta\) is a strictly positive parameter. \(P(f)\) has the same equilibrium as \(Q(f)\). It satisfies \(\int_v P(f) \omega(v) dv = 0\) for \(\omega(v) = (1, v, |v|^2)\) and \(P(M_{\rho,u,T}) = 0\). As in [10], \(\beta^n\) is chosen to be \(2\pi \rho^n\) such that both operators \(P(f)\) and the full Boltzmann operator \(Q(f)\) have the same loss term corresponding to the dissipative part.

The following IMEX discretization of the Boltzmann equation is proposed in [10]:

\[
\frac{f^{n+1} - f^n}{\Delta t} + v \cdot \nabla_x f^n = \frac{Q(f^n) - P(f^n)}{\varepsilon} + \frac{P(f^{n+1})}{\varepsilon}. \tag{30}
\]

For the discretization of the Boltzmann operator one can use a fast spectral Fourier-Galerkin method [11], and for the transport part, a first or second order finite volume scheme can be employed. This gives an AP discretization for the Boltzmann equation as proved in [10].

4.3. SP property. As in section 2 and 3, the IMEX scheme has to satisfy the two requirements as well. Because we computed our numerical results in a space homogeneous set up, there is no point in proving that the discretization of the steady state equation is AP knowing that the full scheme is AP [10]. We only need to prove that starting from a discrete stationary solution, the solution of the time evolutionary problem does not change. Suppose that the solution satisfies the stationary equation at time \(t^n\), i.e.

\[
v \cdot \nabla_x f^n = \frac{Q(f^n) - P(f^n)}{\varepsilon} + \frac{P(f^n)}{\varepsilon}. \tag{31}
\]

It follows from the properties of the collision operator \(Q\) and the relaxation operator \(P\) that:

\[
\int_v \omega(v) v \cdot \nabla_x f^n = 0, \tag{32}
\]

with \(\omega(v) = (1, v, |v|^2)\).

Now multiply (30) by \(\omega(v)\) and integrate over the velocity space. Using the conservation properties of \(Q\), \(P\) and (32), one gets that the Maxwellian of the distribution function is preserved. Then \(M^{n+1} = M^n\) and \(\beta^{n+1} = \beta^n\). By substituting (31) in (30),(30) can be written as:

\[
\frac{f^{n+1} - f^n}{\Delta t} = -\frac{P(f^n)}{\varepsilon} + \frac{P(f^{n+1})}{\varepsilon},
\]

\[
\frac{f^{n+1} - f^n}{\Delta t} = -\beta^n [M^n - f^n] + \beta^{n+1} [M^{n+1} - f^{n+1}] \varepsilon.
\]

Thus \(f^{n+1} = f^n\) and the steady state is preserved.

5. Experimental results. Three test cases are considered in this section, each validates the AP and SP properties of one scheme presented in section 2, 3 or 4. For the velocity space, the standard Gaussian quadrature set is used. The numerical stationary solution is the solution of the time independent discrete equation.
5.1. Neutron transport equation - Parity equations-based scheme. To validate the AP and SP properties of the parity equations-based scheme, we use the same initial and boundary conditions as problem 1 in section 6 in [21]. The initial condition is given by the distribution is
\[ f(x,v,t=0) = 0 \]
and the computational domain is \( x \in [0,1] \). The boundary conditions are as in (8) and (9) with
\[ F_L(v) = 1 \quad \text{and} \quad F_R(v) = 0. \]
This data is consistent as can be seen by (8) and (9). The mesh and time step sizes are respectively \( \Delta x = 0.025 \) and \( \Delta t = 0.0002 \) with the \( S_{16} \) Gaussian quadrature points for the velocity. In Figure 1, we plot the density at time \( t = 0.05 \) for \( \varepsilon = 10^{-2}, \varepsilon = 10^{-3}, \varepsilon = 10^{-6} \) and compare it to its diffusion limit. The curves get close to each other when \( \varepsilon \) gets very small. The curve corresponding to \( \varepsilon = 10^{-6} \) is exactly on top of the curve of the diffusion limit equation. This verifies the AP property. Furthermore, we plot in Figure 2 the time evolution of the distance between the numerical stationary solution \( \rho^*_r \) and the numerical solution \( \rho_r \) of the time evolutionary equation given by the \( L^\infty \) norm
\[ ||\rho_r - \rho^*_r||_\infty = \max_j \{\rho_{rj} - \rho^*_{rj}\} \]
One can see that this distance does not change after we reach the steady state. After that we give the norm at discrete times in Table 2 where we also show that the SP property is valid for all \( \varepsilon << 1 \). Figure 2 and Table 2 indicate that the SP property is well satisfied.

**Figure 1.** Neutron Transport. Left: The density \( \rho_r \) at time \( t = 0.05 \) for \( \varepsilon = 10^{-2}, \varepsilon = 10^{-3}, \varepsilon = 10^{-6} \) and the solution of the diffusion limit equation; right: a zoomed part of the left plot.

**Table 2.** Neutron Transport: \( L^\infty \)-norm of the difference between the solution and the stationary solution in the time interval \([0,8]\) for \( \varepsilon = 10^{-2} \) (top) and \( \varepsilon = 10^{-8} \) (bottom).
5.2. Chemotaxis kinetic model - UGKS scheme. Parameters in (13) are chosen as in [12] such that,
\[ \chi_S = 1, D = 15, \beta = 60, \alpha = 3.\]
and \( \phi \) is of the form \( \phi(u) = -\chi_S \tanh u. \)
The computational domain is set to be \( x \in [-1,1] \). We impose specular boundary conditions for \( f \) and Dirichlet conditions for \( S \). The initial density distribution is composed of two bumps located at \( x = \pm 0.65 \) given by:
\[ f(x, v, 0) = 5(\exp(-10(x-0.65)^2 - 20(v+0.45)^2) + \exp(-10(x+0.65)^2 - 20(v-0.45)^2)). \]
We use \( \Delta x = 2/500 \) for the space discretization and \( v \in [-1,1] \) with the \( S_{32} \) Gaussian quadrature points for the velocity. The limiting scheme of the UGKS is an explicit solver for the diffusion equation. Therefore, to ensure the stability of the numerical scheme, the time step \( \Delta t \) is chosen as below
\[ \Delta t = \begin{cases} 
0.5\Delta x^2, & \text{for } \varepsilon < \Delta x, \\
0.5\varepsilon \Delta x, & \text{else.}
\end{cases} \]
In order to verify the AP property of our scheme, the total densities \( \rho \) at time \( t = 1 \) are displayed in Figure 3 for different values of \( \varepsilon \) ranging from \( 10^{-2} \) to \( 10^{-6} \).
In order to check the SP property, we give the time evolution of the \( L^\infty \)-norm of the difference between the solution and the stationary solution in the time interval \([0,100]\) in Table 3 for \( \varepsilon = 1 \) and \( \varepsilon = 10^{-3} \). These results ensure that the SP property is independent of \( \varepsilon \).

5.3. The Boltzmann equation - IMEX scheme with the penalization method. In this section we consider the 2D Bose gas experiment 3.3 in [18] to test the AP and the SP property of the penalization method presented in [10]. We solve the space homogeneous quantum Boltzmann equation in 2D velocity space...
Figure 3. Verification of the AP property of the UGKS for the chemotaxis kinetic model. Left: The density $\rho$ at time $t = 1$ for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$; right: a zoomed part of the left plot.

Table 3. Chemotaxis: $L^\infty$-norm of the difference between the solution and the stationary solution in the time interval $[0,100]$ for $\varepsilon = 1$ (top) and $\varepsilon = 10^{-3}$ (bottom).

| $T$ (s) | $L^\infty$ |
|---------|------------|
| 0       | $0.9064 \times 10^{-7}$ |
| 30      | $8.260 \times 10^{-11}$ |
| 60      | $7.474 \times 10^{-12}$ |
| 65      | $1.662 \times 10^{-12}$ |

which is a special case of the classical Boltzmann equation for a particular collision operator $Q_q$.

$$\partial_t f = \frac{Q_q(f) - P(f)}{\varepsilon} + \frac{P(f)}{\varepsilon}.$$

As defined in [18], the quantum collision operator is another version of the collision operator (29) and given by

$$Q_q(f)(v) = \int_{\mathbb{R}^{d_v}} \int_{S^{d_{v-1}}} B(|v - v_*|, \cos \theta) \left(f(v) f'(1 \pm \theta_0 f)(1 \pm \theta_0 f_*) - f_* f(1 \pm \theta_0 f')(1 \pm \theta_0 f'_*)\right) d\sigma dv_*$$

where $\theta_0 = h\delta_v$ and $h$ is the rescaled Planck constant. The upper sign correspond to the Bose gas while the lower sign to the Fermi gas. In this experiment we consider the Bose gas case. The idea can be extended to more general collision operators. Hence, scheme (30) is simplified to

$$f^{n+1} = \frac{\varepsilon}{\varepsilon + \beta^{n+1}\Delta t} f^n + \Delta t \frac{Q_q(f^n) - P(f^n)}{\varepsilon + \beta^{n+1}\Delta t} + \frac{\beta^{n+1}\Delta t}{\varepsilon + \beta^{n+1}\Delta t} M^{n+1}.$$

The initial distribution function is given as in [18],

$$f_0(v) = \frac{\rho_0}{4\pi T_0} \left( \exp \left( -\frac{|v - u_0|^2}{2T_0} \right) + \exp \left( -\frac{|v + u_0|^2}{2T_0} \right) \right).$$
where \( \rho_0 = 1 \), \( T_0 = 3/8 \), and \( u_0 = (1, 1/2) \). The computational domain is \([-8, 8]^2\) with 64 grid points. The quantum Maxwellian [18] is given as,

\[
\mathcal{M}_q(v) = \frac{1}{\theta_0} \frac{1}{z^{-1}} \exp \left( \frac{|v-u|^2}{2T} \right) - 1.
\]

where \( \theta_0 = 0.1^2 \), \( z = 0.001590 \), \( T = 1 \) is the temperature and \( u = 0 \) is the macroscopic velocity. In Figure 4 we test the AP property of the penalization method. A cross section of the distribution function for different values of \( \varepsilon \) is plotted on the left and a zoomed part of the plot on the right. The curves are getting closer to each other as \( \varepsilon \) converges to 0 which implies the AP property. Moreover, we investigate the SP property. Figure 5 shows contours of the 2D distribution function and the contour lines of the difference between the distribution function \( f \) and its equilibrium at \( t = 200 \). We computed the \( L^\infty \)-norm of the difference between \( f \) and its equilibrium in the time interval \([0, 200]\) in Figure 6 as an evidence that \( f \) converges exponentially to the equilibrium. Table 4 presents the \( L^\infty \) norm of the distances between the time evolutionary simulation and the equilibrium at some discrete times, where one can find exactly when the initial distribution function reaches its equilibrium.

| \( T \)  | 0    | 20   | 50   | 100  | 150  | 200  |
|--------|------|------|------|------|------|------|
| \( L^\infty \) | 0.5453 | \( 1.2 \times 10^{-4} \) | \( 6.581 \times 10^{-7} \) | \( 3.495 \times 10^{-12} \) | \( 7.619 \times 10^{-14} \) | \( 5.623 \times 10^{-13} \) |

Table 4. Boltzmann: \( L^\infty \)-norm of the difference between \( f \) and its equilibrium starting from \( t=0 \) till the final time \( t=200 \) for \( \varepsilon = 1 \).

6. Conclusion. In this work, we introduce a criterion for AP schemes of kinetic equations to be uniformly SP. Once one is able to show that the Maxwellian of the distribution function can be updated explicitly and the discretization of the stationary equation is a good approximation independent of \( \varepsilon \), then we are able to prove that scheme is SP. This is illustrated by three examples in the three sections 2 - 4. The criterion we propose in this paper gives a general technique to check if an AP scheme is SP. Extensions to nonlinear partial differential equations will be studied in future work.
Appendix A. AP property of the UGKS. In this part, we give a formal derivation of the AP property for the UGKS proposed in (17)–(18).

When \( \varepsilon \) goes to zero, asymptotic expansions of \( A, B, C \) given in (25) read \( A = O(\varepsilon), B = \frac{1}{\varepsilon} - \phi(v\sigma_{i+\frac{1}{2}}) + O(\varepsilon), C = -1 + O(\varepsilon) \). The leading order term of (18) yields \( f^n_{i+1} = \rho^n_{i+1} + O(\varepsilon) \) and we only need to show that (17) satisfies the equation for \( \rho \) in (14), at the discrete level. Suppose that \( f^n_i = \rho^n_i + O(\varepsilon) \), then

\[
\begin{align*}
\mathcal{T} f^n_{i+\frac{1}{2}} &= \frac{1}{2} (\rho^n_i + \rho^n_{i+1}) + O(\varepsilon), \\
\delta^L \mathcal{T} f^n_{i+\frac{1}{2}} &= \frac{\rho^n_{i+1} - \rho^n_i}{\Delta x} + O(\varepsilon), \\
\delta^R \mathcal{T} f^n_{i+\frac{1}{2}} &= \frac{\rho^n_{i+1} - \rho^n_i}{\Delta x} + O(\varepsilon).
\end{align*}
\]
We deduce that the expansion of $F_{i+\frac{1}{2}}^n$ reads:

$$F_{i+\frac{1}{2}}^n = \frac{\rho_i^n + \rho_{i+1}^n}{2|V|} \left( \int_V \phi(v \sigma_{i+\frac{1}{2}}') dv \right) - \frac{\rho_i^n - \rho_i^{n+1}}{3\Delta x} + O(\varepsilon).$$

Therefore,

$$\frac{F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n}{\Delta x} = -\frac{\rho_i^{n+1} - 2\rho_i^n + \rho_{i-1}^n}{3(\Delta x)^2} + \left( \frac{1}{|V|} \int_V \phi(v \sigma_{i+\frac{1}{2}}') dv \right) \frac{\rho_i^n + \rho_{i+1}^n}{2} + \frac{1}{|V|} \left( \int_V \phi(v \sigma_{i-\frac{1}{2}}') dv \right) \frac{\rho_i^n + \rho_{i-1}^n}{2} + O(\varepsilon).$$

In the limit of $\varepsilon \to 0$, the discretization (17) becomes

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{\rho_i^{n+1} - 2\rho_i^n + \rho_{i-1}^n}{3(\Delta x)^2} + \left( \frac{1}{|V|} \left( \int_V \phi(v \sigma_{i+\frac{1}{2}}') dv \right) \frac{\rho_i^n + \rho_{i+1}^n}{2} \right. \left. - \frac{1}{|V|} \left( \int_V \phi(v \sigma_{i-\frac{1}{2}}') dv \right) \frac{\rho_i^n + \rho_{i-1}^n}{2} \right).$$

which is a consistent discretization of the equation for $\rho$ in (14). Therefore, the proposed scheme is AP after coupling with the discretization for $S(x,t)$ in (15).

**Appendix B. AP property of the stationary discretization of the parity equations-based scheme.** Consider the behavior of the scheme as $\varepsilon \to 0$ for a stationary discretization of the fully space-time discretized parity equations based scheme. Equations (4) and (5) are then,

- **Transport step:**
  $$\left\{ \begin{array}{l}
  v \frac{D^n}{\Delta x} j_i^n = 0 \\
  \eta v \frac{D^n}{\Delta x} r_i^n = 0
  \end{array} \right.$$ 

- **Relaxations step:**
  $$\left\{ \begin{array}{l}
  -\frac{1}{\epsilon^2} (r_i^n - \rho_i^n) = 0 \\
  -\frac{1}{\epsilon^2} (j_i^n + (1 - \varepsilon^2 \eta) v \frac{D^n}{\Delta x} r_i^n) = 0
  \end{array} \right.$$ 

Consider the relaxation step as $\varepsilon \to 0$,

$$r_i^n = \rho_{r_i}^n \quad j_i^n = -v \frac{D^n}{\Delta x} \rho_{r_i}^n$$

So,

$$v \frac{D^n}{\Delta x} j_i^n = 0$$

$$\int_0^1 v \frac{D^n}{\Delta x} \left( -v \frac{D^n}{\Delta x} \rho_{r_i}^n \right) = 0$$

$$\frac{D^n D^n c}{\Delta x^2} \rho_{r_i}^n \int_0^1 v^2 dv = 0$$

$$\frac{1}{3} \frac{D^n D^n c}{\Delta x^2} \rho_{r_i}^n = 0$$

which is a consistent discretization of the stationary equation of the diffusion limit. Hence, the discretization of the stationary equation is AP.
Appendix C. AP property of the stationary discretization of UGKS. In this part, we give a formal derivation of the AP property of the stationary discretization of the UGKS which results from setting $\rho^{n+1} = \rho^n$ and $f^{n+1} = f^n$ in (17)–(18),

$$\frac{F_{n+\frac{1}{2}} - F_{n-\frac{1}{2}}}{\Delta x} = 0,$$  

(33)

$$\frac{\Phi_{n+\frac{1}{2}} - \Phi_{n-\frac{1}{2}}}{\Delta x} = \frac{1}{\varepsilon^2} (\rho^n_i - f^n_i) + \frac{1}{\varepsilon} \left( \frac{1}{|V|} \int_V \phi(v' \sigma_{i+\frac{1}{2}}) f^n_i (v') dv' - \phi(v \sigma_{i+\frac{1}{2}}) f^n_i \right).$$  

(34)

Formulas $A$, $B$ and $C$ given in (25) are time dependent but when $\varepsilon$ goes to zero, asymptotic expansions of $A, B, C$ read $A = O(\varepsilon), B = \frac{1}{\varepsilon} - \phi(v \sigma_{i+\frac{1}{2}}) + O(\varepsilon), C = -1 + O(\varepsilon)$ which is time independent. Hence, choosing $A, B, C$ as in (25) for the stationary discretization will not affect the AP proof. We only need to show that (33) satisfies the stationary equation of the Keller-Segel equation (14) for $\rho$, at the discrete level. Suppose that $f^n_i = \rho^n_i + O(\varepsilon)$, then

$$\begin{cases}
T^1 f^n_{i+\frac{1}{2}} = \frac{1}{2} (\rho^n_i + \rho^n_{i+1}) + O(\varepsilon), \\
\delta L T^1 f^n_{i+\frac{1}{2}} = \frac{\rho^n_{i+1} - \rho^n_i}{\Delta x} + O(\varepsilon), \\
\delta R T^1 f^n_{i+\frac{1}{2}} = \frac{\rho^n_{i+1} - \rho^n_i}{\Delta x} + O(\varepsilon).
\end{cases}$$

We deduce that the expansion of $F^n_{i+\frac{1}{2}}$ reads:

$$F^n_{i+\frac{1}{2}} = -\frac{\rho^n_i + \rho^n_{i+1}}{2|V|} \left( \int_V v\phi(v \sigma_{i+\frac{1}{2}}) dv \right) - \frac{\rho^n_{i+1} - \rho^n_i}{3\Delta x} + O(\varepsilon).$$

Therefore,

$$\frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x} = -\frac{\rho^n_{i+1} - 2\rho^n_i + \rho^n_{i-1}}{3(\Delta x)^2} + \left( -\frac{1}{|V|} \int_V v\phi(v \sigma_{i+\frac{1}{2}}) dv \right) \frac{\rho^n_i + \rho^n_{i+1}}{2} \left( \int_V v\phi(v \sigma_{i-\frac{1}{2}}) dv \right) \frac{\rho^n_i + \rho^n_{i-1}}{2} + O(\varepsilon).$$

In the limit of $\varepsilon \to 0$, the discretization (33) becomes

$$\frac{\rho^n_{i+1} - 2\rho^n_i + \rho^n_{i-1}}{3(\Delta x)^2} + \left( -\frac{1}{|V|} \int_V v\phi(v \sigma_{i+\frac{1}{2}}) dv \right) \frac{\rho^n_i + \rho^n_{i+1}}{2} \left( \int_V v\phi(v \sigma_{i-\frac{1}{2}}) dv \right) \frac{\rho^n_i + \rho^n_{i-1}}{2} = 0.$$
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E-mail address: kanbar.farah@gmail.com
E-mail address: klingen@mathematik.uni-wuerzburg.de
E-mail address: tangmin@sjtu.edu.cn