Two examples of discrete-time quantum walks taking continuous steps

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Abstract

This note introduces some examples of quantum random walks in $\mathbb{R}^d$ and proves the weak convergence of their rescaled $n$-step densities. One of the examples is called the Plancherel quantum walk because the “quantum coin flip” is the Fourier Integral (or Plancherel) Transform. The other examples are the Birkhoff quantum walks, so named because the coin flips are effected by means of measure preserving transformations to which the Birkhoff’s Ergodic Theorem is applied.

Quantum walks of the type we consider in this note were introduced in [1], which defined and analyzed the Hadamard quantum walk on $\mathbb{Z}$, and a “new type of convergence theorem” for such quantum walks on $\mathbb{Z}$ was discovered by Konno [4, 5]. A much simpler proof of Konno’s theorem has recently appeared in [3], allowing the theorem to be generalized to quantum walks in $\mathbb{Z}^d$. Inspired by the technique of [3], I have proven that Konno’s theorem also holds for an analog of the quantum walk that takes steps in $\mathbb{R}^d$ instead of $\mathbb{Z}^d$.

In this note, I describe a couple of quantum walks that take steps in $\mathbb{R}^d$: the Birkhoff quantum walk and the Plancherel quantum walk. These are analogs of the Hadamard quantum walk of [1], which is reviewed next.

The Hadamard random walker steps along the lattice $\mathbb{Z}$, carrying with her a “quantum coin.” Formally, the Walker&Coin state is specified by a unit vector in $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$; the standard basis vectors of the auxiliary “coin space” $\mathbb{C}^2$ will be denoted $|H\rangle$ for “Heads” and $|T\rangle$ for “Tails.” A complete measurement of the walker’s position would find her at $j \in \mathbb{Z}$ with probability

$$P(j; \psi) = \left| \langle (j \otimes H) | \psi \rangle \right|^2 + \left| \langle (j \otimes T) | \psi \rangle \right|^2$$

if the state of the Walker&Coin is $\psi \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$. But the walker walks unobserved, and her position will become entangled with the state of her coin. To take a step, the quantum walker flips her coin by a Hadamard transform

$$|H\rangle \mapsto \frac{1}{\sqrt{2}} (|H\rangle + |T\rangle)$$
$$|T\rangle \mapsto \frac{1}{\sqrt{2}} (|H\rangle - |T\rangle)$$

and takes one step to the left or right depending on the outcome. This conditional step is implemented by the unitary operator $S$ on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$ defined by

$$S(|j\rangle \otimes |H\rangle) = |j + 1\rangle \otimes |H\rangle$$
$$S(|j\rangle \otimes |T\rangle) = |j - 1\rangle \otimes |T\rangle,$$
so that a single step of the quantum random walk changes the Walker&Coin state from \( \psi \) to \( S(I \otimes F)\psi \), where \( F \) is the Hadamard operator of \( [2] \) and \( I \) denotes the identity operator on \( L^2(\mathbb{Z}) \). Taking \( n \) unobserved steps of the Hadamard random walk changes the initial state \( \psi_0 \) into \( U^n\psi_0 \), where \( U \) denotes \( S(I \otimes F) \). Konno’s theorem states that the probability measures

\[
\sum_{j \in \mathbb{Z}} P(j; U^n\psi_0) \delta(j/n)
\]

converge weakly to a probability measure depending on \( \psi_0 \), but supported in any case on the interval \( \left[ \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right] \). In \( [3] \), the probabilities \( P(\cdot; U^n\psi_0) \) are as defined in \( [1] \) and \( \delta(x) \) denotes a point-mass at \( x \).

Weak convergence \( Q_n \rightarrow Q \) of probability measures means that \( \int f dQ_n \rightarrow \int f dQ \) for all bounded and continuous functions \( f \) on \( \mathbb{R} \).

Now let us introduce a couple of analogs of Hadamard random walk that take steps in \( \mathbb{R}^d \). Instead of a quantum coin with the alternatives \( H \) and \( T \), the walker will use another copy of \( \mathbb{R}^d \) to choose her next step; instead of the Hadamard transform on \( \mathbb{C}^2 \), she will use a unitary operator on an infinite dimensional Hilbert space. In Plancherel quantum walk, the “coin space” is \( L^2(\mathbb{R}^d) \) and the “coin flip” operator is the Fourier transform on that space. (The fact that the Fourier transform is a unitary operator on \( L^2(\mathbb{R}^d) \) is known as Plancherel’s Theorem \([7]\).) In a Birkhoff quantum walk, the coin space is \( L^2(\Omega, \mathcal{B}, \mathbb{P}) \) where \( (\Omega, \mathcal{B}, \mathbb{P}) \) is a probability space, and the coin flip operator is the unitary map

\[
(F_T f)(\omega) = f(T(\omega))
\]

generated by a measure-preserving transformation \( T \) (i.e., a measurable map from \( \Omega \) to itself, whose inverse exists and is also measurable, and such that \( \mathbb{P}(T(E)) = \mathbb{P}(E) \) for all measurable \( E \subset \Omega \) \([6]\)).

In a Plancherel quantum walk, the Hilbert space for the Walker&Coin is \( \mathbb{H} = L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d) \). This space is isomorphic to \( L^2(\mathbb{R}^{2d}) \) and its members may be represented by wavefunctions \( \psi(x,y) \) with \( x, y \in \mathbb{R}^d \). The unitary operator \( \psi \mapsto U\psi \) on \( \mathbb{H} \) with

\[
(U\psi)(x,y) = (2\pi)^{-d/2} \int \psi(x-y,t) e^{-i ty} dt
\]

determines the single step of the Plancherel quantum random walk. This is the composition of the “conditional step” operator

\[
(S\psi)(x,y) = \psi(x-y,y)
\]

with the “coin flip” operator \( I \otimes \mathcal{F} \), where \( \mathcal{F} \) denotes the Fourier transform on \( L^2(\mathbb{R}^d) \).

In a Birkhoff quantum walk, the Hilbert space for the Walker&Coin is \( \mathbb{H} = L^2(\mathbb{R}^d) \otimes L^2(\Omega, \mathcal{B}, \mathbb{P}) \). This space is isomorphic to \( L^2(\mathbb{R}^d \times \Omega) \) and its members may be represented by wavefunctions \( \psi(x,\omega) \) with \( x \in \mathbb{R}^d, \omega \in \Omega \). Let \( h \) be an integrable function on \( \Omega \) with values in \( \mathbb{R}^d \). The unitary operator \( \psi \mapsto U\psi \) on \( \mathbb{H} \) with

\[
(U\psi)(x,\omega) = \psi(x-h(\omega), T(\omega))
\]

determines the single step of the Birkhoff quantum random walk. This is the composition of the conditional step operator

\[
(S\psi)(x,\omega) = \psi(x-h(\omega), \omega)
\]
with the coin flip operator $I \otimes F_T$, where $F_T$ is defined in [4].

The following are the analogs of Konno’s theorem for the Birkhoff and Plancherel quantum walks.

**Proposition 1** Let $U$ be as in [3]. For an arbitrary but fixed initial state $\psi_0 \in H$, define the probability densities

$$P_n(x) = \int |U^n \psi_0(x, \omega)|^2 \mathbb{P}(d\omega)$$

(7)
on $\mathbb{R}^d$. Then the rescaled probability measures $n^d P_n(nx)dx$ converge weakly as $n \to \infty$. Their weak limit is the image under

$$\overline{h}(\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} h(T^{-j}(\omega)).$$

(8)
of the probability measure on $(\Omega, \mathcal{B})$ that has density $\int |\psi_0(x, \omega)|^2 dx$ relative to $\mathbb{P}$.

**Proposition 2** Let $U$ be as in [3]. For an arbitrary but fixed initial state $\psi_0 \in H$, define the probability densities

$$P_n(x) = \int |U^n \psi_0(x, y)|^2 dy$$

(9)
on $\mathbb{R}^d$. Then the rescaled probability measures $Q_n(x)dx = n^d P_n(nx)dx$ converge weakly to the probability measure with density

$$Q(x) = \frac{2}{(2\pi)^d} \int \int \psi_0(t, y)e^{2itx}dt\ dy$$

as $n \to \infty$. Note that the limiting density $Q(x)$ is independent of $\chi_0$ if $\psi_0(x, y) = \phi_0(x)\chi_0(y)$.

**Proof of Proposition 1** From (3), $(U^n \psi_0)(x, \omega) = \psi_0(x - \sum_{j=0}^{n-1} h(T^j(\omega)), T^n(\omega))$, and the rescaled probability density $n^d P_n(nx)$ is

$$n^d \int \left| \psi_0(nx - \sum_{j=0}^{n-1} h(T^j(\omega)), T^n(\omega)) \right|^2 \mathbb{P}(d\omega).$$

For any test function $\phi(x) \in C_b(\mathbb{R}^d)$,

$$\langle n^d P_n(nx)dx, \phi(x) \rangle = n^d \int \phi(x) \int \left| \psi_0(nx - \sum_{j=0}^{n-1} h(T^j(\omega)), T^n(\omega)) \right|^2 \mathbb{P}(d\omega)dx$$

$$= \int \int \phi \left( \frac{1}{n} y + \frac{1}{n} \sum_{j=0}^{n-1} h(T^j(\omega)) \right) \left| \psi_0(y, T^n(\omega)) \right|^2 \mathbb{P}(d\omega)dy$$

$$= \int \int \phi \left( \frac{1}{n} y + \frac{1}{n} \sum_{j=0}^{n} h(T^{-j}(\omega')) \right) \left| \psi_0(y, \omega') \right|^2 \mathbb{P}(d\omega')dy$$

(10)

making the changes of variables $y = nx - \sum_{j=0}^{n-1} h(T^j(\omega))$ and $\omega' = T^n(\omega)$. By Birkhoff’s Ergodic Theorem, the limit [3] exist almost everywhere and defines an integrable function. Applying the Dominated Convergence Theorem to (10) yields

$$\lim_{n \to \infty} \langle n^d P_n(nx)dx, \phi(x) \rangle = \int \int \phi(\overline{h}(\omega')) \left| \psi_0(y, \omega') \right|^2 dy \mathbb{P}(d\omega').$$
which shows that \( n^d P_n(nx) \) converges weakly to the probability measure described in the theorem.

\[ \square \]

**Proof of Proposition 2**

The plan of the proof is to show that the Fourier transforms of the probability densities \( Q_n \) converge pointwise to the Fourier transform of the probability density \( Q \), for this would imply that \( Q_n(x) \) converges weakly to \( Q(x) \). We denote the Fourier transform on \( L^2(\mathbb{R}^d) \) by \( \mathcal{F} \), and we also define two unitary operators \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) on \( L^2(\mathbb{R}^{2d}) \) by

\[
(\mathcal{F}_1 f)(\zeta, y) = (2\pi)^{-d/2} \int f(x, y) e^{-ix \cdot \zeta} dx
\]

\[
(\mathcal{F}_2 f)(x, \zeta) = (2\pi)^{-d/2} \int f(x, y) e^{-iy \cdot \zeta} dy.
\]

**Step 1:** For the first step we will assume that \( (\mathcal{F}_1 \psi_0)(\zeta, y) \) is bounded and continuous, and we will prove that \( \mathcal{F} Q_{4m} \) converges to \( \mathcal{F} Q \) as \( m \to \infty \).

Since the integrand in (9) is the square of the modulus of \( U_n \psi_0 \), the Fourier transform of \( P_n(x) \) is

\[
(2\pi)^{-d/2} \int \int (\mathcal{F}_1 U^n \psi_0)(\eta, y)(\mathcal{F}_1 U^n \psi_0)(\eta + \zeta, y) d\eta dy \tag{11}
\]

Let \( \widetilde{U} = \mathcal{F}_1 U \mathcal{F}_2^* \). It may be verified that

\[
(\widetilde{U}^4 \phi)(\zeta, y) = e^{i\zeta^2} \phi(\zeta, y)
\]

(to this end it may be helpful to note that \( \widetilde{U} = M \mathcal{F}_2 \) where \( M \) denotes the multiplication operator \( M \phi(\zeta, y) = e^{-i\zeta y} \phi \)). It follows that

\[
(\mathcal{F}_1 U^{4m} \psi_0)(\zeta, y) = (\mathcal{F}_1 U \psi_0)(\zeta, y) = e^{im\zeta^2} (\mathcal{F}_1 \psi_0)(\zeta, y).
\]

Substituting this into (11) shows that

\[
(\mathcal{F} P_{4m})(\zeta) = (2\pi)^{-d/2} \int \int e^{-im\eta^2(\mathcal{F}_1 \psi_0)(\eta, y)} e^{im(\eta + \zeta)^2} (\mathcal{F}_1 \psi_0)(\eta + \zeta, y) d\eta dy
\]

\[
= (2\pi)^{-d/2} \int \int e^{im(2\zeta^2)}(\mathcal{F}_1 \psi_0)(\eta, y)(\mathcal{F}_1 \psi_0)(\eta + \zeta, y) d\eta dy
\]

and therefore the Fourier transform of the rescaled density \( n^d P_n(nx) \) is

\[
(2\pi)^{-d/2} \int \int e^{im(2\zeta^2/n + (\zeta/n)^2)}(\mathcal{F}_1 \psi_0)(\eta, y)(\mathcal{F}_1 \psi_0)(\eta + \zeta/n, y) d\eta dy \tag{12}
\]

when \( n = 4m \). The integrand in (12) tends pointwise to \( e^{i\zeta^2/2}|(\mathcal{F}_1 \psi_0)(\eta, y)|^2 \) as \( m \to \infty \). If \( \mathcal{F}_1 \psi_0 \) is both bounded and integrable for a.e. \( y \), then the Dominated Convergence Theorem implies that

\[
(\mathcal{F} Q_n)(\zeta) \to (2\pi)^{-d/2} \int \int e^{i\zeta^2/2}|(\mathcal{F}_1 \psi_0)(\eta, y)|^2 dy d\eta
\]
by the Dominated Convergence Theorem. But the latter is the Fourier transform of

\[ Q(x) = 2 \int \left| (F_1 \psi_0)(-2x, y) \right|^2 dy. \]  \(\text{(13)}\)

This proves that the probability measures

\[ Q_n(x) dx = n^d P_n(nx) dx \]  \(\text{(14)}\)

converge weakly to \(Q(x) dx\) along the subsequence \(n = 4m\) as \(m \to \infty\).

**Step 2:** Next, a density argument removes the restriction on \(\psi_0\) in Step 1:

Let \(\psi_0^{(j)}\) be a sequence of normalized wavefunctions that converges to an arbitrary \(\psi_0 \in L^2(\mathbb{R}^d)\). The \(\psi_0^{(j)}\) may be chosen from the Schwartz class, which is dense in \(L^2(\mathbb{R}^d)\). By Step 1, the probability measures \(Q_{4m}^{(j)}(x) dx\) converge weakly to \(Q^{(j)}(x) dx\) as \(m \to \infty\), where \(Q_{4m}^{(j)}\) and \(Q^{(j)}\) are defined as in (13) and (14) with \(\psi_0^{(j)}\) in place of \(\psi_0\). On the other hand, the Cauchy-Schwartz inequality implies that \(\|Q^{(j)} - Q\|_1 = 2\|\psi_0^{(j)} - \psi_0\|_2\) and indeed

\[ \|Q_{4m}^{(j)} - Q_{4m}\|_1 \leq 2\|\psi_0^{(j)} - \psi_0\|_2 \]

for all \(m\). The weak convergence \(Q_{4m}^{(j)} dx \to Q^{(j)} dx\) and the preceding uniform bound on \(\|Q_{4m}^{(j)} - Q_{4m}\|_1\) imply that \(Q_{4m} dx\) convergences weakly to \(Q dx\) weakly.

**Step 3:** Finally, we will prove that \(Q_n\) tends to \(Q\) along all subsequences, having already shown that \(Q_{4m}(x) dx \to Q(x) dx\) for any initial state \(\psi_0\). It will help to have the notation for \(P_n\) and \(Q_n\) display the dependence on the initial state; from now on we will write \(P_n(x ; \psi)\) and \(Q_n(x ; \psi)\) to indicate this dependence. From (13) and (14), one has that

\[ \|Q_{n+p}(x ; \psi_0) - Q_n(x ; U^p \psi_0)\|_1 \]

\[ = \int \left| (n+p)^d P_{n+p}((n+p)x ; \psi_0) - n^d P_n(nx ; U^p \psi_0) \right| dx \]

\[ = \int \left| (1 + \frac{p}{n})^d P_n((1 + \frac{p}{n})u ; U^p \psi_0) - P_n(u ; U^p \psi_0) \right| du \]

for any positive integer \(p\), and therefore \(\|Q_{n+p}(x ; \psi_0) - Q_n(x ; U^p \psi_0)\|_1\) tends to 0 as \(n \to \infty\) for fixed \(p\) since translation acts continuously on \(L^1\). Steps 1 and 2 of this proof and the estimate (15) imply that

\[ Q(x ; U^p \psi_0) dx = \lim_{m \to \infty} Q_{4m}(x ; U^p \psi_0) dx = \lim_{m \to \infty} Q_{4m+p}(x ; \psi_0) dx. \]

On the other hand, one may show by induction that \(Q(x ; U^p \psi_0) = Q(x ; \psi_0)\) for all \(p\). It follows that \(Q_n(x ; \psi_0) dx \to Q(x ; \psi_0) dx\) weakly along every subsequence. \(\square\)

**References**

[1] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous. One-dimensional quantum walks. *Proceedings of STOC’01*, 37-49 (2001)
[2] R. Dudley. Real Analysis and Probability. Cambridge University Press, Cambridge, UK (2002)

[3] G. Grimmett, S. Janson, P. Scudo. Weak limits for quantum random walks. Preprint: quant-ph/0309135 (2003)

[4] N. Konno. Quantum random walks in one dimension. Quantum Information Processing 1: 345-354 (2002)

[5] N. Konno. A new type of limit theorems for the one-dimensional quantum random walk. Preprint: quant-ph/0206103 (2002)

[6] K. Petersen. Ergodic Theory. Cambridge University Press (1983)

[7] N. Wiener. The Fourier Integral and Certain of Its Applications. Dover Publications, New York (1933)