R. M. TRIGUB

SOME TOPICS IN FOURIER ANALYSIS
AND APPROXIMATION THEORY

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References
Preface.

This manuscript presents shortly the results obtained by participants of the scientific seminar which is held more than twenty years under leadership of the author at Donetsk University. In the list of references main publications are given. These results are published in serious scientific journals and reported at various conferences, including international ones at Moscow, ICM66; Kaluga, 1975; Kiev, 1983; Haifa, 1994; Zürich, ICM94; Moscow, 1995. As for the area of investigation this is the Fourier analysis and the theory of approximation of functions. Used are methods of classical analysis including special functions, Banach spaces, etc., of harmonic analysis in finitedimensional Euclidean space, of Diophantine analysis, of random choice, etc.

The results due to the author and active participants of the seminar, namely E. S. Belinskii, O. I. Kuznetsova, E. R. Liflyand, Yu. L. Nosenko, V. A. Glukhov, V. P. Zastavny, Val. V. Volchkov, V. O. Leontyev, and others, are given. Besides the participants of the seminar and other mathematicians from Donetsk, many mathematicians from other places were speakers at the seminar, in particular, A. A. Privalov, Z. A. Chanturia, Yu. A. Brudnyi, N. Ya. Krugljak, V. N. Temlyakov, B. D. Kotlyar, A. N. Podkorytov, M. A. Skopina, A. A. Ligun, A. S. Romanyuk, V. A. Martirosyan.

Besides the papers of the participants of the seminar, only monographs and survey papers are given in the list of references. Relative results of other mathematicians are described in the text without details, but the year of publication is given.

The outline of the work is seen from the contents. Some unsolved problems are given as well.

Main notation is given in the very beginning of §1. Bibliographical remarks are given in the end of every section.

The most part of this preprint will appear soon in extended form in the author’s book ”Fourier Analysis and Approximation Theory”, World Federation Publishers, Inc., P.O.Box 48654, Atlanta, GA 30362-0654, USA.

The author thanks E. R. Liflyand (Bar-Ilan University, Israel) for the translation from Russian to English.
1. The Fourier transform. Absolute convergence.

We denote absolute positive constants by the letter $C$. Let $\gamma(\alpha, \beta)$ be a positive constant depending only on $\alpha$ and $\beta$. Let $\mathbb{R}^m$ be the $m$-dimensional real Euclidean space, $\mathbb{T}^m = [-\pi, \pi]^m$ be the torus, and $\mathbb{Z}^m$ be the lattice of points in $\mathbb{R}^m$ with integer coordinates. For $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ denote by $(x,y) = \sum_{j=1}^m x_j y_j$ the inner product of $x$ and $y$, by $|x| = (x,x)^{1/2}$ the Euclidean norm of $x$.

We write $x \leq y$ when $x_j \leq y_j$ for all $j \in [1,m]$. Denote also $\mathbb{R}_+^m = \{ x \in \mathbb{R}^m : x \geq 0 \}$, and $\| \cdot \|_p$ means the $L_p$-norm. We omit superscript $m$ for $m = 1$. The abbreviation a.e. means almost everywhere with respect to the Lebesgue measure.

For every function $f \in L_p(\mathbb{R}^m)$, with $p \in [1,2]$, the Fourier transform

$$\hat{f}(x) = (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} f(u) e^{-i(x,u)} \, du$$

is defined as usual (see e.g., [M22]). Let

$$B(\mathbb{R}^m) = \{ f : f(x) = \int_{\mathbb{R}^m} e^{-i(x,u)} \, d\mu(u), \quad \|f\|_B = \text{var} \mu < \infty \}$$

be the space of the Fourier transforms of all the finite complex valued Borel measures on $\mathbb{R}^m$. If also a measure $\mu$ is absolutely continuous with respect to Lebesgue measure, that is $d\mu(u) = g(u) \, du$, where $g \in L_1(\mathbb{R}^m)$, then we shall write $f \in A(\mathbb{R}^m)$ and $\|f\|_A = \|g\|_1$.

For $f : \mathbb{T}^m \to \mathbb{C}$ set for $p > 0$

$$A_p(\mathbb{T}^m) = \{ f : f(x) = \sum_{k \in \mathbb{Z}^m} c_k e^{i(k,x)}, \quad \|f\|_{A_p} = \left( \sum_{k} |c_k|^p \right)^{\frac{1}{p}} < \infty \}.$$

Notice that $\| \cdot \|_{A_p}$ is the quasinorm for $p \in (0,1)$. Set also for $p \in (0,1)$

$$A_p(\mathbb{R}^m) = \{ f : f = \hat{g}, \quad \|f\|_{A_p} = \|g\|_p < \infty \}$$

in assumption, e.g., $g \in L_p \cap L_1(\mathbb{R}^m)$. In particular, for a function with compact support $f \in A_p$ is equivalent to $\hat{f} \in L_p(\mathbb{R}^m)$. It is well known that $B(\mathbb{R}^m), A(\mathbb{R}^m)$ and $A(\mathbb{T}^m)$ are Banach algebras, and their local structure is the same (see e.g. [M10]). A difference in behavior of functions from $B(\mathbb{R}^m)$ and $A(\mathbb{R}^m)$ occurs only near infinity.

1.1. If $f \in B(\mathbb{R}^m)$ and is of bounded Vitali variation outside some neighborhood of the origin, and $\lim_{|x| \to \infty} f(x) = 0$, then $f \in A(\mathbb{R}^m)$ and $\|f\|_B = (2\pi)^{-m} \|\hat{f}\|_1$.

The following two statements are connected with discretization and we shall give them only for $m = 1$ (for $p = 1$ the case 1.2b) is Wiener's result).
1.2. a) If \( f \in L_1(\mathbb{R}) \) and supported in \([-\pi, \pi]\), then for every \( p > 0 \)
\[
\hat{f} \in L_p(\mathbb{R}) \iff f \quad \text{and} \quad f_1 \in A_p(\mathbb{T})
\]
where \( f_1(x) = e^{ix/2}f(x) \).

b) If either \( p > 1 \), or \( p \in (0, 1] \) and diameter of the support of \( f \) is less than \( 2\pi \), then for \( p \in (0, 1] \)
\[
\hat{f} \in L^p(\mathbb{R}) \iff f \in A_p(\mathbb{T}).
\]

1.3. Let \( n \in \mathbb{Z} \) and for some \( r \in \mathbb{Z}_+ \) the functions \( f \) and \( f^{(r)} \) be of bounded variation on \([n, +\infty)\), and \( \lim f^{(\nu)}(x) = 0 \) as \( x \to +\infty \) and \( \nu \in [0, r] \). Then for \( 0 < |x| \leq \pi \)
\[
\sum_{k=n}^{\infty} f(k)e^{ikx} = \int_n^{\infty} f(u)e^{ixu} \, du + \frac{1}{2}f(n)e^{inx} + e^{inx} \sum_{p=0}^{r-1} \frac{(-i)^{p+1}}{p!} h^{(p)}(x)f^{(p)}(n) + \frac{\theta}{\pi^r} V_n^{\infty} f^{(r)},
\]
where \( V_n^{\infty} f^{(r)} \) is the total variation of \( f^{(r)} \) on \([n, \infty)\), the function \( h(x) = \frac{1}{x} - \frac{1}{2} \cot \frac{x}{2} \), and \( |\theta| \leq 3 \).

Tending to the limit as \( x \to 0 \) we get the classical Euler-Maclaurin formula as a corollary.

Let us go on to necessary and sufficient conditions of belonging to the algebras defined above.

1.4. Let \( E \) be a closed set in \( \mathbb{R}^m \) and \( A(E) \) be a linear set of functions \( E \to \mathbb{C} \), which is containing restrictions of all boundedly supported functions from \( C^\infty(\mathbb{R}^m) \) as well as those multiplied by elements of \( A(E) \). Then \( A(E) \) possesses a local property, namely if for each \( x \in E \), including the infinity point when \( E \) is non-compact, there exist a neighborhood \( V_x \) of \( x \) and a function \( g_x \in A(E) \) such that \( f = g_x \) on \( V_x \), then \( f \in A(E) \).

1.5. Let \( f \in A(\mathbb{R}^m) \) and \( f_0 \) be its radial part, that is \( f_0(t) \) is an integral average of \( f \) over the sphere \(|x| = t\). Then \( f_0 \in C^{m_1}(0, \infty) \), where \( m_1 = \left\lceil \frac{m-1}{2} \right\rceil \), when \( t \to \infty \)
we have \( \lim_{t \to \infty} t^p f_0^{(p)}(t) = 0 \) for \( 0 \leq p \leq m_1 \). Besides that, for all \( t > 0 \) the following integral converges:
\[
\int_0^t u^{m_1 - \frac{m+1}{2}} \left[ f_0^{(m_1)}(t + u) - f_0^{(m_1)}(t - u) \right] \, du.
\]

The following (approximate) criterion is similar for \( p = 1 \) to that for the absolute convergence of orthogonal series due to S. B. Stechkin (1955; see e.g., [M10]).

Taking \( f \in L_2(\mathbb{R}^m) \), let us assume that for each \( \sigma > 0 \)
\[
a_\sigma(f) = \inf \{ ||f - g||_2 \quad \text{for} \quad g \quad \text{such that} \quad \text{mes supp } \hat{g} \leq \sigma \},
\]
where \( \text{mes} \) denotes the Lebesgue measure, be best approximation to \( f \) in \( L_2(\mathbb{R}^m) \) by functions with the spectrum in a set of measure less or equal to \( \sigma \).
1.6. If \( f \in L_p(\mathbb{R}^m) \), then for \( p \in (0, 2) \) we have \( \hat{f} \in L_p(\mathbb{R}^m) \) if and only if

\[
\int_0^\infty \sigma^{-p/2} (a_\sigma(f)_2)^p \, d\sigma < \infty.
\]

1.7. Let \( f \in C(\mathbb{R}^m) \), \( \lim f(x) = 0 \) as \( |x| \to \infty \), and for \( r \in \mathbb{R} \)

\[
\Omega_r(h_1, \ldots, h_m) = \Omega_r(f; h) = ||\hat{\Delta}_r f(\cdot)||_2,
\]

where

\[
\hat{\Delta}_r f(x) = \left( \prod_{j=1}^m \hat{\Delta}_r h_j \right) f(x), \quad \hat{\Delta}_r h_j f(x) = f(x + h_j e_j^0) - f(x - h_j e_j^0)
\]

is the symmetric mixed difference, \( e_j^0 \) is the orth of the axis \( Ox_j \).

a) If for at least one \( r \)

\[
\sum_{s_1=-\infty}^{\infty} \ldots \sum_{s_m=-\infty}^{\infty} 2^{1/2} \sum_{s_j=0}^{\infty} \Omega_r(f; \pi/2^{s_1}, \ldots, \pi/2^{s_m}) < \infty
\]

and for some \( p \in (0, 2) \) and some \( r \)

\[
\sum_{s_1=-\infty}^{\infty} \ldots \sum_{s_m=-\infty}^{\infty} 2^{(1-\xi)} \sum_{s_j=0}^{\infty} \Omega_r^p(f; \pi/2^{s_1}, \ldots, \pi/2^{s_m}) < \infty,
\]

then \( f = \hat{g} \), where \( g \in L_1 \cap L_p(\mathbb{R}^m) \).

b) If \( f = \hat{g} \), where \( g \in L_\infty \cap L_p(\mathbb{R}^m) \) for some \( p \in (0, 2) \), and in addition \( |u_j| \geq |v_j| \), with sign \( u_j = \text{sign } v_j \) and \( 1 \leq j \leq m \), implies \( |g(u)| \leq |g(v)| \), then the first series in a) converges for each \( r \) while the second one for \( r > (1/p - 1/2)m \).

The result 1.7 immediately yields a generalization to the multiple case of one theorem of A. Beurling, 1949 (see the beginning of §13).

Bernstein’s precise result for a function to be of smoothness greater than \( m/2 \) in \( C \) is sufficient for belonging to \( A(\mathbb{R}^m) \) in a small neighborhood of the endpoint, see [M22].

1.8. Let \( p \in (0, 1] \) and \( q = \left[ \frac{m}{p} - \frac{m+1}{2} \right] \) (the integral part). If a boundedly supported function \( f \in C^q(\mathbb{R}^m) \), and its partial derivative \( D_{q,j} f = \frac{\partial f}{\partial x_j} \) considered as a function of \( x_j \) has a finite number of points of inflection with respect to the other variables (more precisely, points separating intervals of convexity) and \( \omega(D_{q,j}; t)_\infty \) is the modulus of continuity in the space \( C(\mathbb{R}^m) \), for \( 1 \leq j \leq m \), then the condition

\[
\max_j \int_0^1 t^{q(p+(m+1)(p/2-1))} \omega^p(D_{q,j}; t) \, dt < \infty
\]

implies \( f \in A_p(\mathbb{R}^m) \) (or \( \hat{f} \in L_p(\mathbb{R}^m) \)).

In the neighborhood of \( \infty \), and for even \( f \) also near the origin, these conditions can be weakened.
1.9. If for every \( x \in \mathbb{R}^m \) and for all \( 1 \leq j \leq m \)
\[
f(x) = \int_{|x_j| \leq |u_j| < \infty} g(u) \, du \quad \text{and} \quad \int_{\mathbb{R}^m} \text{ess sup} \, |g(u)| \, dv < \infty,
\]
then \( f \in A(\mathbb{R}^m) \).

Given conditions of convexity type, sharper results are obtained.

1.10. Let \( f \) be a locally absolutely continuous function on \([0, +\infty)\), and \( \lim f(x) = 0 \) as \( x \to +\infty \) and
\[
||f||_{V^*} = \int_0^\infty \text{ess sup} \, |f'(u)| \, dx < \infty.
\]

a) Let for every \( x < 0 \) we have \( f(x) = 0 \). Then for every \( y \in \mathbb{R} \setminus \{0\} \)
\[
\hat{f}(y) = -\frac{i}{y\sqrt{2\pi}} f\left(\frac{\pi}{2|y|}\right) + F(y), \quad \text{where} \quad ||F||_1 \leq C||f||_{V^*}.
\]

b) Let \( f(x) = f_0(|x|) \), with \( f_0 \in C^{m+1}[0, \pi] \) for \( m = \left[\frac{m-1}{2}\right] \) and the derivative \( f_0^{(m+1)}(0) < \infty \). In order that \( \hat{f} \in L(\mathbb{R}^m) \) \( (or \ f \in A(\mathbb{R}^m)) \) it is necessary and sufficient that
\[
\left|\int_0^\pi u^{-m+1} f_0(\pi - u) \, du\right| < \infty.
\]

Notice that it was G. E. Shilov, 1942, who first investigated the asymptotics of Fourier coefficients of a convex function.

In the theory of partial differential equations, one has to study the absolute convergence of the spectral resolution of an elliptic operator. In the case of the Laplace operator on \( \mathbb{R}^m \) this means the absolute convergence of the Fourier integral with respect to spheres. V. P. Maslov (Dokl. AN SSSR, 1970) considered such a convergence for functions of two variables with a weak discontinuity on a given curve and clarified a role of the evolute of this curve for the divergence problem. The first result on convergence (see the case \( k = 0 \) in the next theorem) has been obtained by E. M. Nikishin and G. I. Osmanov (Sib. Math. J., 1976).

1.11. Let \( l \) be \( k + 2 \) times continuously differentiable curve in \( \mathbb{R}^2 \). Let \( E \) be the evolute of \( l \) and \( \kappa \) be the curvature of \( l \). Denote by \( E_k \) the set of points of \( E \) corresponding to points of \( l \) so that \( \kappa^{(p)}(s) = 0 \) for \( p = 1, \ldots, k-1 \), and \( \kappa^{(k)}(s) \neq 0 \), where \( s \) is natural parametrization of \( l \).

Let \( f \) be boundedly supported continuous function in \( \mathbb{R}^2 \), continuously differentiable outside \( l \). Let also for \( j = 1, 2 \) and some \( \varepsilon > 0 \) and \( \sigma < \frac{k+4}{2(k+2)} \) the following inequalities hold:
\[
|\Delta_h \frac{\partial f}{\partial x_j}(x)| \leq \frac{|h|^\varepsilon}{\sigma(\varepsilon)} \left( 1 + |(l - \sigma h)\varepsilon - l| \right).
\]
where $x, h \in \mathbb{R}^2$ and $r(x, l) = \inf_{y \in l} |x - y|$. Then

$$J_0(x_0) = (2\pi)^{-1/2} \int_0^\infty |\int \hat{f}(t)e^{itx_0} dt| dz < \infty$$

for every $x_0 \in E^k$.

For $\mathbb{R}^m$, $m > 2$, this theorem is generalized for functions weakly discontinuous on arbitrary compact l-dimensional surface $L_l$, $1 \leq l \leq n - 1$. Sets $E_k$ are defined with respect to different curves on $L_l$ (for $l > 1$). The theorem itself can be formulated in the same manner, interesting is the inequality for the order of weak discontinuity of $[m/2]$-th derivative:

$$\sigma < \frac{m - l}{m - 1} - \frac{k}{2(m - 1)(k + 2)}.$$

One gets the two-dimensional case taking $m = 2$, $l = 1$.

Let us consider also a question as to the structure of the set of real zeros of the Fourier transform of the indicator function of a convex planar body $K$.

1.12. Let $h(\varphi)$ be the support function of $K$. Let $d(\varphi) = h(\varphi) + h(\varphi + \pi)$ be the width of $K$ in the given direction and $\delta(\varphi) = \frac{1}{2}(h(\varphi + \pi) - h(\varphi))$. If $K$ is not a polygon, then the set $N(K)$ of real roots of the equation $\int e^{i(u, x)} du = 0$ is contained in $\bigcup_{p=1}^\infty M_p$, where $M_p$ is a closed continuous curve symmetric with respect to the origin and defined in the polar coordinates by the equation $r = r_p(\varphi)$. For each $p \in \mathbb{N}$ we have $2p\pi < d(\varphi)r_p(\varphi) < 2(p + 1)\pi$. If $\delta \in C^k(\mathbb{R})$ (analytic), then also $r_p \in C^k(\mathbb{R})$ (analytic) for every $p \in \mathbb{N}$. If the set $N(K) \cap M_p$ is infinite, then $M_p \subseteq N(K)$ and $r_p$ is an analytic function. If besides $K$ possesses a center of symmetry, then $N(K) = \bigcup_{p=1}^\infty M_p$. The curve $M_1$ may be not convex.

In connection with one Beurling’s theorem (see the beginning of §13) the following function space arised:

$$A^*_p(\mathbb{T}) = \{ f : f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx}, \quad ||f||_{A^*_p} = \sum_{n=0}^\infty \sup_{|k| \geq n} |c_k|^p < \infty \}.$$

The space $A^*_p(\mathbb{R})$ is defined similarly.

For $p \in (0, 1]$, when passing from the inequality

$$||fg||_{A^*_p(\mathbb{T})} \leq ||f||_{A^*_p(\mathbb{T})} ||g||_{A^*_p(\mathbb{T})}$$

to that for $A^*_p(\mathbb{T})$ a factor appears depending only on $p$.

Notice that an analog of the Wiener-Ditkin theorem holds for $A^*_p(\mathbb{T})$ as well as analogs of closed results, and the sharp sufficient condition of belonging to $A^*_p(\mathbb{T})$ is the smoothness in $L$ (or $C$) is greater than $1/p$.

The integral operator of order $\alpha$

$$f(x) \rightarrow \int_0^1 (1 - t)^{\alpha-1} f(xt) dt$$

takes $A(\mathbb{T})$ continuously into $A(\mathbb{T})$ for all $\alpha > 0$ and into $A^*(\mathbb{T})$ for all $\alpha \geq 1$.

The description of the dual space of $A^*(\mathbb{T})$ is contained in the following proposition.
1.13. For each two sequences

\[
\sup_{\sum_{k=0}^{\infty}} \sum_{n=0}^{\infty} \alpha_k \beta_k = \sup_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \beta_k,
\]

\[
\sup_{\sum_{k=0}^{\infty}} \sum_{n=0}^{\infty} \alpha_k \beta_k = \sum_{n=0}^{\infty} \sup_{k=n}^{\infty} \alpha_k.
\]

Notice in conclusion that there are several definitions of \( A^* \) in the multidimensional case and some of them are applied to the study of summability of the multiple Fourier series at Lebesgue points (see 3.4 and 3.5 later on).

Bibliographical remarks.
For 1.1, 1.7 – 1.9, see [T17]; for 1.2 see [T8] (for a generalization to the multiple case, see [L1]); for 1.3 see [T25] (the case \( r = 0 \) is contained, in essence, in [Be2]); for 1.5 – 1.6, see [T13]; for 1.10a, see [T10] and [T24] (much more general result can be found in [L9]); for 1.10b, see [T13] (for a certain refinement, see [L2]); for 1.11, see [L3] in the two-dimensional case and [L5] for \( \mathbb{R}^m \) when \( m \geq 2 \); for 1.12, see [Z1-4]. Properties of \( A^*(\mathbb{T}) \) are given in [T9]; for a generalization to the multiple case, see [L10]); for 1.13, see [BT1]. See also [T18].

2. Multipliers and comparison of various methods of summability of Fourier series as a whole.

Let \( \{e_k\} \) be an orthogonal system of functions and \( \{c_k\} = \{c_k(f)\} \) be the Fourier coefficients of \( f \) with respect to this system.

A number sequence \( \{\lambda_k\} \) is called a multiplier in \( L_p \), written \( \{\lambda_k\} \in M_p \), if the operator

\[
\sum c_k e_k \to \sum \lambda_k c_k e_k
\]

takes \( L_p \) continuously into \( L_p \). We write \( \{\lambda_k\} \in M \) if the multiplier operator takes \( C \) into \( C \). Multipliers form a special class of integral operators. Observe, that the norm of such integral operator can be expressed explicitly via the kernel only in the cases \( p = 1 \) and \( p = \infty \).

Let us start with the trigonometric system \( e_k = e^{i(k,x)} \), where \( k \in \mathbb{Z}^m \) and \( x \in \mathbb{T}^m \). In this case the multiplier operator taking \( L_p(\mathbb{T}^m) \) into \( L_p(\mathbb{T}^m) \) commutes with shifts and is expressed as the convolution (see e.g., [M22]). J. Marcinkiewicz indicated convenient sufficient conditions for multipliers in \( L_p(\mathbb{T}^m) \) with \( p \in (1, \infty) \) and gave certain applications of multipliers (see [M21]). For general properties of multipliers of trigonometric Fourier series, see [M7], Ch.16.

In particular, in order a sequence \( \{\lambda_k\}_{k \in \mathbb{Z}^m} \) to be a multiplier in \( C \) (\( L_1 \) or \( L_\infty \)), it is necessary and sufficient that there exists on \( \mathbb{T}^m \) a finite complex valued Borel measure \( \mu \) such that for each \( k \in \mathbb{Z}^m \) we have \( \lambda_k = \int_{\mathbb{T}^m} e_{-k} d\mu \). In this case the operator is the convolution of a function with the measure \( \mu \) and

\[
||\{\lambda_k\}||_M = ||\{\lambda_k\}||_{M_1} = ||\{\lambda_k\}||_{M_\infty} = \var\mu
\]

(see also 2.4 below).

Observe that there exist plenty many of such multiplier operators.
2.1. For every linear bounded operator $A$ taking $L_p(\mathbb{T}^m)$ into $L_p(\mathbb{T}^m)$ for some $p \in [1, +\infty]$, or taking $C(\mathbb{T}^m)$ into $C(\mathbb{T}^m)$, there exists a multiplier operator $A_0$ such that
\[ ||A_0|| \leq ||A||, \quad ||f - A_0f||_p \leq \sup_{\theta} ||f^\theta - A(f^\theta)||_p, \]
where $f^\theta(x) = f(x + \theta)$. And if also $A \geq 0$, then $A_0 \geq 0$.

As it is well known, every bounded operator is weakly compact in the space $L_\infty(\mathbb{T}^m) = (L_1(\mathbb{T}^m))^*$. The picture is different for $C(\mathbb{T}^m)$.

2.2. If the multiplier $\{\lambda_k\}_{k \in \mathbb{Z}^m}$ is weakly compact in $C(\mathbb{T}^m)$, then it is expressed as a convolution with some kernel from $L_1(\mathbb{T}^m)$ and hence is compact.

The set of those $k \in \mathbb{Z}^m$ for which the Fourier coefficients $c_k = c_k(f)$ do not vanish is called the spectrum of a function $f \in L_1(\mathbb{T}^m)$.

Let $S$ be a non-empty set of $\mathbb{Z}^m$ and $L_p(\mathbb{T}^m, S)$, where $p \in [1, +\infty]$, be a subspace of functions in $L_p(\mathbb{T}^m)$ with a spectrum in $S$. Similarly $C(\mathbb{T}^m, S)$ is defined. A number sequence $\{\lambda_k\}_{k \in S}$ is called a multiplier in $L_p(\mathbb{T}^m, S)$, written $\{\lambda_k\}_{k \in S} \in M_p(S)$, if for every $f \in L_p(\mathbb{T}^m, S)$ the series $\sum \lambda_k c_k(f) e_k$ is the Fourier series of some function $\Lambda f \in L_p(\mathbb{T}^m, S)$ and $||\{\lambda_k\}||_{M_p(S)} = ||\Lambda||_{L_p \to L_p}$. For the space $C(\mathbb{T}^m, S)$, we shall write $M(S)$ in contrast to $M_\infty(S)$ when $p = \infty$.

2.3. For every $S \subset \mathbb{Z}^m$ and every $p \in (1, +\infty)$, and $q$ such that $1/p + 1/q = 1$ we have
\[ ||\{\lambda_k\}||_{M_p(S)} = ||\{\lambda_k\}||_{M_q(S)}. \]

2.4. Every multiplier in $C(\mathbb{T}^m, S)$ can be extended to a norm-preserving multiplier in $C(\mathbb{T}^m)$ and
\[ ||\{\lambda_k\}||_{M(S)} = \min\{\text{var } \mu : \text{ for } \mu \text{ such that for all } k \in S \text{ we have } \int_{\mathbb{T}^m} e^{-k} d\mu = \lambda_k\} = \min_{\{\lambda_k\}_{k \in S} \subset \mathbb{C}} \sup_{n} (2\pi)^{-m} ||\sigma_n(\Lambda)||_1 = \min_{\{\lambda_k\}_{k \in S} \subset \mathbb{C}} \lim_{n \to \infty} (2\pi)^{-m} ||\sigma_n(\Lambda)||_1, \]
where
\[ \sigma_n(\Lambda) = \sum_{|k_j| \leq n} \lambda_k \prod_{j=1}^m \left(1 - \frac{|k_j|}{n + 1}\right) e_k. \]

Notice two corollaries; the first one answers a question posed in [M7], Vol.2.

2.5. a) Given a sequence $\{\lambda_k\}_{k \in S}$, in order that for each function $f \in C(\mathbb{T}^m)$ there exists at least one function $\Lambda f \in C(\mathbb{T}^m)$ with the Fourier coefficients $c_k(\Lambda f) = \lambda_k c_k(f)$ for all $k \in S$, it is necessary and sufficient that for some finite Borel measure $\mu$ we have $\lambda_k = \int_{\mathbb{T}^m} e^{-k} d\mu$ for all $k \in S$.

b) For every $S \subset \mathbb{Z}^m$ and every $p \in [1, +\infty)$ we have
\[ ||\{\lambda_k\}||_{M_p(S)} \leq ||\{\lambda_k\}||_{M(S)} = ||\{\lambda_k\}||_{M_\infty(S)}. \]

Will a multiplier be compact after extension from the spectrum if it was such? Let us give an example in case $m = 1$. 

2.6. Let $S$ be a finite subset of $\mathbb{Z}$. In order that a multiplier $\{\lambda_k\}_{k \in S}$ can be extended to a compact multiplier on $C(\mathbb{T})$ with the same norm, it is necessary and sufficient that there exist $p \in S$, $\alpha \in \mathbb{R}$, and a function $f \in L_1(\mathbb{T})$ such that $e^{i\alpha} e^{-ipx} f(x) \geq 0$ a.e. By this, $||\{\lambda_k\}||_{M(S)} = |\lambda_p|$. 

2.7. Let $p$ be even and $p \neq 2$.

a) There exist $S \subset \mathbb{Z}$ such that there exist an isometric multiplier in $L_p(\mathbb{T}, S)$ which is not extendable to a norm-preserving linear operator in $L_p(\mathbb{T})$.

b) Let $\{\lambda_k\}_{k \in \mathbb{Z}} \subset M_p$ and let exist $k_0 \in \mathbb{Z}$ such that $|\lambda_{k_0}| = |\lambda_{k_0+1}| = ||\{\lambda_k\}||_{M_p}$. Then the multiplier $\{\lambda_k\}$ is a shift, up to a factor.

The question on extention of a multiplier in $C(\mathbb{T}, S)$ with preservation of norm and compactness is equivalent to the following one: whether $L_1(\mathbb{T}, \mathbb{Z} \setminus S)$ is an existence space, written e. s., in $L_1(\mathbb{T})$? For the definition of e.s., see e.g., [M13].

2.8. a) If a set $S \subset \mathbb{Z}$ is such that every measure with spectrum in $S$ is absolutely continuous with respect to Lebesgue measure, or $S$ is the Sidon set (for definition, see [M10]), then $L(\mathbb{T}, S)$ is e.s. in $L(\mathbb{T})$.

b) If $S = -S$ and $\mathbb{Z} \setminus S$ is lacunary in the Hadamard sense, then $L(\mathbb{T}, S)$ is not e.s. in $L(\mathbb{T})$.

Obviously, one can always assume $\lambda_k = \varphi(k)$ for $\varphi$ sufficiently smooth. Consider more general case $\lambda_k = \varphi(\varepsilon k)$ where $\varepsilon$ is a positive parameter.

2.9. a) If $\varphi \in B(\mathbb{R}^m)$, that is the Fourier transform of a finite Borel measure on $\mathbb{R}^m$, then for each $\varepsilon > 0$ we have $||\{\varphi(\varepsilon k)\}||_M \leq ||\varphi||_B$.

b) If $\varphi$ is continuous a.e. on $\mathbb{R}^m$ and for some sequence $\varepsilon_n \to 0$

$$H = \sup_n ||\{\varphi(\varepsilon_n k)\}||_M < \infty,$$

then the function $\varphi$ can be corrected on the set of its points of discontinuity so that it will be in $B(\mathbb{R}^m)$ and $||\varphi||_B \leq H$.

Hence we established connection between the algebras $M(\mathbb{R}^m)$ and $B(\mathbb{R}^m)$: if $\varphi \in C(\mathbb{R}^m)$, then

$$||\varphi||_M = \sup_{\varepsilon > 0} ||\{\varphi(\varepsilon k)\}||_M = ||\varphi||_B.$$

2.10. Let $\mathbb{Z}_0 = \mathbb{Z}^m \setminus \{0\}$.

a) If a function $\varphi$ is continuous a.e. on $\mathbb{R}^m \setminus \{0\}$, and

$$\lim_{\varepsilon \to 0^+} ||\{\varphi(\varepsilon k)\}||_{M(\mathbb{Z}_0)} = H < \infty,$$

the it can be corrected at the points of discontinuity and defined at the origin by continuity so that $||\varphi||_M = H$.

b) If $\varphi \in C(\mathbb{R}^m)$, then

$$\lim_{\varepsilon \to 0^+} ||\{\varphi(\varepsilon k)\}||_{M(\mathbb{Z}_0)} = \sup_{\varepsilon > 0} ||\{\varphi(\varepsilon k)\}||_{M(\mathbb{Z}_0)} = ||\varphi||_M.$$

Let us give one sufficient condition of boundedness of the norms of of a sequence of multipliers.
2.11. Let $|\lambda_0| + \ln(n+1)(|\lambda_n| + |\lambda_{-n}|) \leq 1$. If also for some $\delta > 0$ one of the following two conditions, either a) or b), hold

a) $|\lambda_k - \lambda_{k+s}| \leq \sqrt{s/n} \ln(s/3n)^{-1-\delta}, \quad 1 \leq s \leq n, \quad -n \leq k \leq n - s$,

b) $\sum_{k=-n}^{n} |\lambda_k - \lambda_{k+1}| \leq 1, \quad |\lambda_k - \lambda_{k+s}| \leq \ln(s/3n)^{-2-\delta}, \quad 1 \leq s \leq n, \quad -n \leq k \leq n - s$,

then $\|\{\lambda_k\}_{n}^{-1,n}\|_M$ are bounded by a constant depending only on $\delta$. For $\delta = 0$ no one of these statements valids.

Let us go on to comparison of the two multiplier operators

$$\Lambda f \sim \sum \lambda_k c_k e_k \quad \text{and} \quad \tilde{\Lambda} f \sim \sum \tilde{\lambda}_k c_k e_k$$

and formulate a comparison principle.

2.12. Let $S_0 = \{k \in \mathbb{Z}^m : \lambda_k = 0\}$.

a) If $\tilde{\lambda}_k = 0$ for every $k \in S_0$, which is also necessary, and $K = \inf_{0/0} \|\{\tilde{\lambda}_k/\lambda_k\}\|_M < \infty$, where the least lower bound is taken with respect to choice of values of fractions of type $0/0$, then for each function $f$ such that $\Lambda f \in C(\mathbb{T}^m)$ we have $\|\Lambda f\|_\infty \leq K \|\Lambda f\|_\infty$. And vice versa, if this inequality holds for each function $f$ such that $\Lambda f \in C(\mathbb{T}^m)$ and $\{1/\lambda_k\} \in M(\mathbb{Z}^m \setminus S_0)$, then

$$\inf_{0/0} \|\{\tilde{\lambda}_k/\lambda_k\}\|_M = \min_{0/0} \|\{\tilde{\lambda}_k/\lambda_k\}\|_M \leq K.$$

b) If $\tilde{\lambda}_k = 0$ for every $k \in S_0$ and $\{1/\lambda_k\} \in M(\mathbb{Z}^m \setminus S_0)$, while $\{\tilde{\lambda}_k/\lambda_k\} \in M(\mathbb{Z}^m \setminus S_0)$ and compact, then

$$\sup_{f \in C: \|\Lambda f\|_\infty \leq 1} \|\tilde{\Lambda} f\|_C = \sup_{f \in L_1: \|\Lambda f\|_\infty \leq 1} \|\tilde{\Lambda} f\|_C \leq \min_{0/0} \|\{\tilde{\lambda}_k/\lambda_k\}\|_M.$$

c) If $\Lambda$ is a compact operator in $C(\mathbb{T}^m)$ and the equality $\lambda_k = 1$ implies $\tilde{\lambda}_k = 1$, then we have $\|f - \tilde{\Lambda} f\|_C \leq K \|f - \Lambda f\|_C$ for each $f \in C(\mathbb{T}^m)$ if and only if

$$\inf_{0/0} \|\{(1 - \tilde{\lambda}_k)/(1 - \lambda_k)\}\|_M \leq K.$$

Example. Continuity of

$$D_{2r}(f) = \sum_{j=1}^{m} \frac{\partial^{2r} f}{\partial x_j^{2r}}$$

for $r \geq 2$ and $m \geq 2$ does not imply continuity of $\Delta^r f$, where $\Delta$ is the Laplace operator.

To prove this, we consider the functions to be periodic and compare the Fourier series of $D_{2r}$ and $\Delta^r$. Then 2.12a), 2.10a) and 2.9b) yield that the function $\varphi(x) = (\sum x_j^{2r})(\sum x_j^{2r})^{-r}$ must have a limit as $x \to 0$. But this function is homogenious of zero order and hence constant.

Let us generalize now the comparison principle.
Let $E$ be a complex Banach space and $L(E)$ the Banach algebra of linear continuous operators acting in $E$. Let further $\{P_k\}_0^\infty$ be a complete sequence of mutually orthogonal projectors in $E$. This means that

$\alpha)$ $P_k \in L(E)$ for every $k \in \mathbb{Z}_+$;

$\beta)$ $P_k f = 0$ for each $k \in \mathbb{Z}_+$ implies $f = 0$;

$\gamma)$ $P_k P_s = \delta_{ks} P_s$ for every $k, s \in \mathbb{Z}_+$, where $\delta_{ks}$ is the Cronecker delta;

$\delta)$ $\|P_k\| \neq 0$ for every $k \in \mathbb{Z}_+$.

Let us associate with $f \in E$ the series $\sum_{k=0}^\infty P_k f$. By $\beta)$ the correspondence between the elements and the series indicated is one-to-one correspondence. Such series may be called ”the Fourier series”. A number sequence $\{\lambda_k\}$ is called the multiplier in $E$, with respect to a given sequence of orthoprojectors, if for every $f \in E$ there exists $g \in E$ satisfying the condition $P_k g = \lambda_k P_k f$ for every $k \in \mathbb{Z}_+$, that is $g = \Lambda f \sim \sum \lambda_k P_k f$ and $\Lambda \in L(E)$, and $\|\{\lambda_k\}\|_M = \|\Lambda\|$.

2.13. a) If $\{\lambda_k\}_0^\infty$ is a multiplier and $\lambda_k = 1$ implies $\tilde{\lambda}_k = 1$, the latter is also necessary, and

$$K = \inf_{0/0} \|\{(1 - \tilde{\lambda}_k)/(1 - \lambda_k)\}\|_M < \infty,$$

then for every $f \in E$ we have $\|f - \tilde{\Lambda} f\| \leq K \|f - \Lambda f\|$.

b) If the inequality indicated holds for each $f \in E$ and $\Lambda$ is a compact operator, then

$$\inf_{0/0} \|\{(1 - \tilde{\lambda}_k)/(1 - \lambda_k)\}\|_M \leq K + \sum_{\nu : \lambda_{\nu} = 1} \|P_{\nu}\|,$$

where in case $\lambda_k \neq 1$ for each $k \in \mathbb{Z}_+$ the sum is absent on the right-hand side as well as the least lower bound on the left-hand side.

With the availability of multiplier theorems it will be possible to apply the comparison principle to basis expansions, to orthogonal and biorthogonal expansions, etc.

Let us compare now classical summability methods as regards to the rate of convergence to elements of Banach space.

2.14. If the $(C, 1)$ method is regular, that is

$$\sigma_n(f) = \sum_{k=0}^n (1 - \frac{k}{n + 1}) P_k f \to f \quad \text{as} \quad n \to \infty$$

in the norm of a given space $E$ for every $f \in E$, then it is equivalent to the Abel-Poisson method, that is for every $r \in (0, 1)$ and for each $f \in E$ we have

$$\|f - \sigma_n(f)\| \asymp \|f - \sum_{k=0}^\infty r^k P_k f\|,$$

where $n = \lfloor 1/(1 - r) \rfloor$ is the integral part of $1/(1 - r)$ and in the two-sided inequality replaced by $\asymp$ positive constants do not depend on $f$ and $r$.

To replace $\sigma_n$ by $\sigma_n^\alpha$, that is $(C, \alpha)$ method, for small $\alpha$ more restrictive assumptions on $\sum P_k f$ are needed.
2.15. a) Let for some $\alpha > 0$ the following condition $(S, \alpha)$ hold: if for $\lambda_{n+1} = 0$

$$\sum_{k=0}^{n} |\lambda_k - \lambda_{k+1}| \leq 1 \quad \text{and} \quad |\lambda_k - \lambda_{k+s}| \leq (s/n)^\alpha$$

for $1 \leq s \leq n$ and $0 \leq k \leq n+1-s$, then

$$\sup_n \sup_{||f|| \leq 1} \left| \sum_{k=0}^{n} \lambda_k \mathcal{P}_k f \right| < \infty.$$ 

Then

$$||f - \sigma_n^\alpha(f)|| \asymp ||f - \sigma_n(f)||.$$ 

b) Let the condition $(S, \alpha)$ hold and let $\varphi \in \text{Lip} \alpha$ and be a function of bounded variation on $[0, 1]$. Suppose further that $0 \leq \varphi(1) \leq \varphi(x) < \varphi(0)$ on $(0, 1)$. Then either for each $\delta > 1$, or for each $\delta > 0$ provided $\varphi$ is piecewise monotone on $[0, 1]$, for which the condition $(S, \delta \alpha)$ holds, we have

$$||f - \sum_{k=0}^{n} \varphi^\delta(k/n) \mathcal{P}_k f|| \asymp ||f - \sum_{k=0}^{n} \varphi(k/n) \mathcal{P}_k f||.$$ 

For instance, the condition $(S, \alpha)$, for each $\alpha > 0$, is satisfied by the trigonometric system (see 2.11b)) and by the Walsh system (see 8.5).

For $\varphi(x) = (1 - x^\alpha)^\delta$, we get the Riesz means. Hardy established that the Riesz methods, as applied to number series, become stronger for larger $\delta$ and fixed $\alpha$ and are equivalent one to another, that is the summability by one of them yields the summability by another, for different $\delta$ and fixed $\alpha$. The comparison of summability methods defined as above is of other type. For the same Riesz methods as applied to the Fourier series, another picture occurred to be true: the growth of $\delta$ does not change anything while the growth of $\alpha$ improves the convergence (more precisely, a speed of convergence). See also §5.

To obtain more general sufficient conditions than $(S, \alpha)$, the strong summability can be applied. This notion was introduced by Hardy and Littlewood for trigonometric series, the so-called condition $(H, p), p > 0$:

$$\sup_n \sup_{||f|| \leq 1} \left| \frac{1}{n+1} \sum_{s=0}^{n} \left| \sum_{k=0}^{s} \mathcal{P}_k f \right|^p \right| < \infty.$$ 

The case $E = C$ may be understood, for example. The case $p = 1$ can be found in essence in 1.13.

2.16. Let $x = \{x_k\}^\infty_1$ and $y = \{y_k\}^\infty_1$. Define

$$||x||_{h_p} = \sup_n \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p}$$

and

$$||y||_{b_p} = \sum_{k=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{\infty} |y_k|^p \right)^{1/p}.$$
For $p \in (1, +\infty)$ and $1/p + 1/q = 1$ the following inequalities hold.

\[ |\sum_{k=1}^{\infty} x_k y_k| \leq \gamma_1(p) \|x\|_b \|y\|_{h_q}, \]

\[ \sup_{\|x\|_b \leq 1} |\sum_{k=1}^{\infty} x_k y_k| \geq \gamma_2(p) \|y\|_{h_q}, \]

\[ \sup_{\|y\|_{h_q} \leq 1} |\sum_{k=1}^{\infty} x_k y_k| \geq \gamma_3(p) \|x\|_b. \]

Let us go on to orthogonal series in the space $L_{2,h}(a,b)$, where $(a,b)$ is the finite or infinite interval in $\mathbb{R}$. Let \{\varphi_k\}^\infty_{0} be orthonormal system of complex valued functions on $(a,b)$ with weight $h$. This means that

\[ \int_a^b \varphi_k(t)\overline{\varphi_s(t)}h(t)\ dt = \delta_{ks}. \]

Let us assume that $\varphi_k \in L_{\infty}(a,b)$ for all $k \in \mathbb{Z}_+$ while $h \in L(a,b)$ and $h(t) > 0$ a.e. on $(a,b)$.

To each measurable function $f$ satisfying \( \int_a^b |f(t)|h(t)\ dt < \infty \) its Fourier series with respect to the system \{\varphi_k\}^\infty_{0} corresponds:

\[ f \sim \sum_{k=0}^{\infty} c_k \varphi_k, \quad c_k = c_k(f) = \int_a^b f(t)\overline{\varphi_k(t)}h(t)\ dt. \]

In the following statement, where also a question on multipliers with regard to the position of a point $x \in (a,b)$ is considered, the $(C,1)$ method is used which can be replaced by any Toeplitz regular method.

**2.17. a) If**

\[ \sup_n \int_a^b \left| \sum_{k=0}^{n} \lambda_k (1 - \frac{k}{n+1}) \varphi_k(t)\overline{\varphi_k(x)} \right| h(t)\ dt = K(x) \]

is in $L_{\infty}(a,b)$, then for every $f \in L_{\infty}(a,b)$ the series \( \sum \lambda_k c_k(f) \varphi_k \) is the Fourier series of some function $\Lambda f$ to which it converges in average, that is in the space $L_2$ with the weight $h$, and a.e. on $(a,b)$

\[ |(\Lambda f)(x)| \leq K(x)\|f\|_{\infty}. \]

**b) If in addition to that $\varphi_0 \equiv C$ and for all $n \in \mathbb{Z}_+$, for all $x \in (a,b)$, and for almost all $t \in (a,b)$ we have**

\[ \sum_{k=0}^{n} (1 - \frac{k}{n+1}) \varphi_k(t)\overline{\varphi_k(x)} \geq 0, \]
then
\[ \|\{\lambda_k\}\|_{M_\infty} = \text{ess sup}_{x \in (a,b)} \sup_{n} \frac{1}{b-a} \int_{a}^{b} \left| \sum_{k=0}^{n} \lambda_k \left(1 - \frac{k}{n+1}\right) \varphi_k(t) \varphi(x) \right| h(t) \, dt. \]

\[ \lambda \]

\[ \text{c) If in addition } \varphi_k \in C(a,b) \text{ for all } k \in \mathbb{Z}_+ \text{ and for every } f \in C(a,b) \]
\[ \lim_{n \to \infty} \|f - \sum_{k=0}^{n} (1 - \frac{k}{n+1}) c_k \varphi_k\|_C = 0, \]
\[ \text{then } \{\lambda_k\}_{0}^{\infty} \text{ is a multiplier taking } C \text{ into } C \text{ and} \]
\[ \|\{\lambda_k\}\|_M = \|\{\lambda_k\}\|_{M_\infty}. \]

It is known that when \([a,b] = [-1,1]\) for the system of Jacobi polynomials, that is when \(h(t) = (1+t)^{\alpha}(1-t)^{\beta}\) (see [M20]), nonnegative are \((C,\alpha + \beta + 2)\) means (see G. Gasper, SIAM J.Math.An.,1977). For the system of Hermite polynomials, that is for \((a,b) = (-\infty, +\infty)\) and \(h(t) = e^{-t^2}\), nonnegative is the Poisson kernel.

Let us return to the trigonometric system for which a greater progress can be achieved.

\[ \text{2.18. a) If } \{1/\lambda_k\}_{k \in \mathbb{Z}^m} |_{S_0} \in M(\mathbb{Z}^m \setminus S_0) \text{ (see 2.12), then the inequality } ||\tilde{\Lambda} f||_\infty \leq K ||\Lambda f||_\infty \text{ being true with the finite right-hand side for each } f \in C(\mathbb{T}^m) \text{ implies that for every } p \in [1, +\infty] \text{ the inequality } ||\tilde{\Lambda} f||_p \leq K ||\Lambda f||_p \text{ holds with the finite right-hand side for each } f \in L_p(\mathbb{T}^m). \]

\[ \lambda \]

\[ \text{b) If } \varphi \in B(\mathbb{R}^m), \psi \in A(\mathbb{R}^m), \text{ and } \psi(x) \neq 1 \text{ for all } x \neq 0, \text{ then the least constant in the inequality} \]
\[ \|f - \sum_{k} \varphi(k/n) c_k e_k\|_\infty \leq K \|f - \sum_{k} \psi(k/n) c_k e_k\|_\infty \]
\[ \text{being true for all } f \in C(\mathbb{T}^m) \text{ for all } n \text{ is} \]
\[ K = \|(1 - \varphi)/(1 - \psi)\|_B. \]

\[ \text{c) Let the inequality from b) hold with some constant independent of } f \text{ and } n. \text{ If in addition } \varphi \in C(\mathbb{R}^m), \psi \in B(\mathbb{R}^m), \text{ and } \psi(x) \neq 1 \text{ for all } x \neq 0 \text{ and there exist limits of } \psi \text{ and } \varphi \text{ as } |x| \to \infty \text{ and the first of them equals to zero, and besides, outside of some neighborhood of the origin } \varphi \text{ and } \psi \text{ are of bounded variation in the sense of Vitali and } |1 - \psi(x)| \leq C|1 - \varphi(x)| \text{ for all } x \in \mathbb{R}^m, \text{ then the inverse inequality to that indicated in b) holds with some other positive constant independent of } f \text{ and } n. \]

Let us give an additional result in case of L metric and \(m = 1\).

\[ \text{2.19. a) If } \lim \lambda_k = 0 \text{ as } |k| \to \infty \text{ and } \sum_{-\infty}^{\infty} |\lambda_k - \lambda_{k+1}| < \infty, \text{ then} \]
\[ \|\{\lambda_k\}\|_{M_1(Z_0)} = \inf_{\lambda_0} \frac{1}{2\pi} \int_{0}^{\pi} \left| \sum_{k=-\infty}^{\infty} \lambda_k e^{ikt} \right| \, dt - \theta \sum_{k=-\infty}^{\infty} |\lambda_k - \lambda_{k+1}|, \]
where \( \theta \in [0, C] \).

b) Let for \( f \in L[a, b] \) we have \( ||f - \lambda_0||_1 = \min_{\lambda \in \mathbb{C}} \int_a^b |f(x) - \lambda| \, dx \) and there exist a subset \( e \subset [a, b] \) on which \( f \) is bounded such that \( \gamma \operatorname{mes} e = b - a \) for some \( \gamma \in [1, 2) \). Then

\[
\sup_{x \in e} |f(x) - \lambda_0| \leq \frac{3}{2\sqrt{2} - \gamma} \omega^e(f)
\]

where \( \omega^e(f) \) is the oscillation of \( f \) on \( e \).

**Bibliographic remarks.**

The comparison principle for various trigonometric Fourier series and first results like 2.14 and 2.15 appeared in [T3]. The comparison principle in other form was published also by H. Shapiro (Bull.AMS, 1968). It turned out that the question on comparison of series with respect to their approximation properties was posed by J. Favard as early as in 1963. A generalization to series in Banach spaces was formulated by P. Butzer, R. Nessel, and W. Trebels (see [M29]). Statement 2.13 from [T12] is a refinement of this result.

For 2.2–2.6, 2.10, 2.12, see [T29]; for 2.7, see [Le2]; for 2.8, see [T22]; for 2.9, 2.18, see [T17]; for 2.11, see [T29]; for 2.14, 2.15, 2.17, see [T18,27]; 2.16 is due to E. Belinskii (see [BT1]).

It is known that in order that a sequence \( \{\lambda\} \) is a multiplier from \( L^\infty \) into \( C \) with respect to the trigonometric system, it is necessary and sufficient that the series \( \sum \lambda_k e_k \) is the Fourier series (see, e.g., [M7]). Thus various sufficient conditions for multipliers from \( L^\infty \) into \( C \) may be found in papers of S. A. Telyakovskii, G. A. Fomin, Ya. S. Bugrov, Yu. L. Nosenko ([N7,8]), O. I. Kuznetsova ([Ku4,6]), E. R. Liflyand ([L9]), P. V. Zaderey. See also [S14,4].

### 3. Regularity of summability methods of Fourier series.

Let \( f \in L(T^m) \) and \( \sum c_k e_k \) be its trigonometric Fourier series. Let us consider for \( n \in \mathbb{R}^m_+ \) a matrix \( \{\lambda_{n,k}\}_{k \in \mathbb{Z}^m} \) and introduce the following polynomial means

\[
\Lambda^0_n(f) = \sum_{-n \leq k \leq n} \lambda_{n,k} c_k e_k.
\]

For \( \lambda_{n,k} = 1 \) or 0 we obtain various partial sums (rectangular, circular, etc.) for \( m = 2 \).

Let in addition \( E \subset L(T^m) \). Summability method is called regular (in one sense or another) if for every \( f \in E \) we have \( \Lambda^0_n(f) \to f \) as \( n \to \infty \) at all the points of a subset of \( T^m \) or in norm if one is defined in \( E \). Such a regularity is not connected with the Toeplitz regularity and thus is called \( F \)-regularity.

An investigation of summability methods of simple series, \( m = 1 \), has a long history (see, e.g., [S14]).

In 1936, S. Bochner introduced the following spherical means of Riesz type

\[
S^\delta_n(f) = \sum_{|k| \leq n} (1 - \frac{|k|^2}{n^2})^\delta c_k e_k,
\]

with \( n \in \mathbb{N} \), and clarified a role of the critical order \( \delta = \frac{m-1}{2} \) in the multiple case (see, e.g., [M29]). In 1972, Cb. Fefferman proved that \( S^0 \), that is spherical
partial sums, converge on \( L_p(\mathbb{T}^m) \) for \( m \geq 2 \) when \( p = 2 \) only (see, e.g., [S6]). Up to the present the regularity in \( L - p(\mathbb{T}^m) \) of the Bochner-Riesz means \( S_n^\delta \) is not investigated in full yet for all \( \delta > 0 \) and \( m \geq 3 \) (see, e.g., [M5]).

If \( E \) is Banach space, the question is reduced to the boundedness of the sequence of operator norms \( ||\Lambda_n^0||\), the so-called Lebesgue constants. If the trigonometric system is closed in \( E \), then the condition \( \lambda_{n,k} \to 1 \), as \( n \to \infty \), for all \( k \in \mathbb{Z}^m \) is the one that should be added to the boundedness of ||\( \Lambda_n^0 || \|. \) The Lebesgue constants ||\( \Lambda_n^0 || \) are the norms of multipliers \( \{ \lambda_{n,k} \} \) which in the case of spaces \( C \) or \( L \) and \( \lambda_{n,k} = \varphi(k/n) \) are closely connected with belonging of \( \varphi \) to \( B \) or \( A \) (see 2.9 and 1.2).

3.1. a) If \( \varphi \in C(\mathbb{T}) \) and \( \varphi(\pi) = \varphi(0) = 0 \), then

\[
||\varphi||_M = \sup_{\varepsilon > 0} ||\varphi(\varepsilon k)||_M \leq C \sum_{k=-\infty}^{\infty} |c_k(\varphi)| \ln(|k| + 1).
\]

And if in addition \( \varphi \) is continuous real even functions with alternating, with respect to sign, Fourier coefficients in cosines starting with the first one, then the opposite inequality valids too.

b) If \( \varphi \in C(\mathbb{T}) \) and \( \varphi(\pi) = \varphi(-\pi) = \varphi(0) = 0 \), then denoting \( \varphi_0(x) = \varphi(x) \text{sign} x \) we have

\[
||\varphi||_M \leq C \sum_{k=-\infty}^{\infty} |c_k(\varphi_0)| \ln(|k| + 1),
\]

under assumption that the series on the right-hand side converges.

c) If for \( m_1 = \left[ \frac{m-1}{2} \right] \) we have \( \varphi_0 \in C^{m_1}[0, +\infty) \) and \( \varphi_0 \) being supported in \( [0, \pi] \) is such that for every \( t \in [0, \pi] \)

\[
\varphi_0(t) = \sum_{\nu=0}^{\infty} \alpha_\nu \cos \nu t \quad \text{and} \quad \sum_{\nu=0}^{\infty} \nu^{\frac{m_1-1}{2}} |\alpha_\nu| \ln(\nu + 1) < \infty,
\]

then \( \varphi(x) = \varphi_0(|x|) \in M(\mathbb{R}^m) \).

Several authors, R. P. Boas, J.-P. Kahane, S. I. Izumi and T. Tsuchikura, I. Wik, independently and almost simultaneously considered the following question: known is the behavior of absolute values of the Fourier coefficients of \( \varphi \), how is it possible to derive that \( \varphi_0(x) = \varphi(x) \text{sign} x \in A(\mathbb{T}) \). The answer obtained (see [M10, Ch.6, §2, Cor.4,5]) immediately follows from the results given above.

3.2. Let \( \varphi \) be continuous a.e. on \( \mathbb{R}^m \) and \( \{ \varphi(\varepsilon k) \} \in M_1 \) for every \( \varepsilon > 0 \). In order that \( \sum \varphi(\varepsilon k) c_k(f) e_k \) converges to \( f \), as \( \varepsilon \to +0 \), on \( L(\mathbb{T}^m) \) or \( C(\mathbb{T}^m) \) it is necessary and sufficient that after correction at the points of discontinuity \( \varphi \in B(\mathbb{R}^m) \) and \( \varphi(0) = 1 \).

Example. \( \varphi(x) = (1 - \sum_{j=1}^{m} |x_j|^{\alpha})^{\delta} \) for \( \alpha = 1 \) and \( \delta > 0 \) or for positive \( \alpha \neq 1 \) and \( \delta > \frac{m-1}{2} \).

Let us go on to summability a.e. Sufficient conditions can be derived from Marcinkiewicz’s theorem on the strong summability a.e. and 2.16 (see [BT1]). Let us give a necessary condition.
3.3. Let $p \in [1, 2)$ and $\varphi$ be continuous a.e. on $\mathbb{R}^m$ and boundedly supported. If on some subset of $\mathbb{T}^m$ of positive measure $\sum \varphi(\varepsilon_k)c_k(f)e_k$ converges to $f$, as $\varepsilon \to +0$, for all $f \in L_p(\mathbb{T}^m)$, then for each $q > p$

$$\int_{\mathbb{R}^m} |\hat{\varphi}(u)|^q \, du < \infty.$$ 

This yields immediately K. I. Babenko’s result, 1971: for every $p \in [1, \frac{2m}{m+1})$ for $m \geq 2$ there exists a function $f \in L_p(\mathbb{T}^m)$ such that its spherical Bochner-Riesz means $S^d_n(f)$ do not converge to $f$ on a set of positive measure if $d \in [0, \frac{m}{p} - \frac{m+1}{2})$.

As for the summability at Lebesgue points, criteria do exist. Let us start with the compact case.

3.4. In order that $\sum \varphi(k/n)c_k(f)e^{i(k,x)}$ converges to $f$ as $n \to \infty$ for each $f \in L(\mathbb{T}^m)$ at all its Lebesgue points, that is those for which

$$\lim_{r \to +0} r^{-m} \int_{|x-u| \leq r} |f(x) - f(u)| \, du = 0,$$

it is necessary and sufficient that $\varphi \in A^*(\mathbb{R}^m)$ (see after 1.12) and $\varphi(0) = 1$.

There exists a similar criterion for ”strong” Lebesgue points, that is those for which the ball in the definition is replaced by arbitrary parallelepiped with faces parallel to coordinate planes (see [Be1]), and 3.4 in terms of belonging to $A^*(\mathbb{T}^m)$ (see [T9] and [L1]).

The general case is investigated in the following theorem.

3.5. Let $\varphi \in B(\mathbb{R}^m)$. In order that for each $f \in L(\mathbb{T}^m)$ its Fourier series is summable at all Lebesgue points by a method generated by the function $\varphi$ it is necessary and sufficient that $\varphi \in A^*(\mathbb{R}^m)$ and $\varphi(0) = 1$.

Observe that if $\varphi$ satisfies convexity type conditions, then $\varphi \in A(\mathbb{T})$ and $\varphi \in A^*(\mathbb{T})$ simultaneously. The point is that in the asymptotics 1.10a) for this case, but not for the general one, we have in addition that also $\sup_{|y| \geq x} |F(y)| \in L[0, +\infty)$.

Let $\{\nu_k\}_0^\infty$ be strictly increasing integer-valued sequence. When does the following statement

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^n |f(x) - S_{\nu_k}(f; x)| = 0$$

valids? R. Salem, 1955, proved that in order that this equality holds for every $f \in C(\mathbb{T})$ it suffices that the sequence $\{\nu_k\}$ is of power growth.

3.6. a) If the sequence $\{\nu_k\}$ is convex, that is $\nu_{k+2} - 2\nu_{k+1} + \nu_k \geq 0$ for all $k \geq 0$, then in order that the afore-mentioned limiting equality holds for every $f \in C(\mathbb{T})$ everywhere or uniformly on $\mathbb{T}$, it is necessary and sufficient that $\ln \nu_n = O(\sqrt{n})$.

b) For the multiple case, when cubic partial sums of the Fourier series of $f$ are taken as $S_{\nu_{k}}$, the answer is the following under the same conditions: $\ln \nu_n = O(n^{1/2m})$. 


c) For \( m = 1 \) and \( \nu_k = [2^k\alpha] \), it is impossible to take \( \alpha > 1/2 \) even in the case when the sign of absolute value is removed outside the sign of sum, that is approximation by arithmetic means with gaps.

Let us investigate as an example the question of regularity of the following Bernstein-Rogosinski type means in rather general form on \( \mathbb{T}^2 \):

\[
R_n(f; x) = R_n(f; W; \gamma, \mu, x) = \int_{\mathbb{R}^2} S_n(f; W; x - \gamma u/n) \, d\mu(u),
\]

where for \( n \in \mathbb{N} \)

\[
S_n(f; W; x) = \sum_{k \in nW} c_k(f)e^{i(k,x)}
\]

is the partial sum of the Fourier series of a function \( f \in L(\mathbb{T}^2) \) generated by a bounded set \( W \subset \mathbb{R}^2 \), \( \gamma > 0 \), and \( \mu \) is a finite complexvalued Borel measure on \( \mathbb{R}^2 \) and \( \int_{\mathbb{R}^2} d\mu = 1 \).

Let us start with the case of discrete measure, that is a measure concentrated at finite number of points.

**3.7.** a) Let \( W \) be a parallelogram symmetric with respect to the origin. If a measure \( \mu \) is concentrated at four points, which is the least number, then the regularity in \( C \) valids if and only if these points are of equal measure and are the vertices of a parallelogram \( W_1 \) with the sides perpendicular to those of \( W \) and ratio of the product of lengths of the two perpendicular sides of \( W \) and \( W_1 \) equals to the ratio of odd numbers.

b) Let \( W \) be a polygon with the origin inside such that no one of the sides does not lie on the line passing through the origin. If \( m \) is a maximal number of its sides no one of which are not parallel one to another, then it is possible to choose \( 2^m \) points with the same measure so that the means \( R_n \) are regular in \( C(\mathbb{T}^2) \).

c) If \( W \) is a circle, then for every discrete measure and for every \( \gamma \) the regularity in \( C(\mathbb{T}^2) \) is impossible.

**3.8.** a) Let measure \( \mu \) be with compact support and \( W \) be a bounded connected set containing a neighborhood of the origin and satisfying another two conditions: the boundary \( \partial W \) is of zero two-dimensional Lebesgue measure and in every neighborhood of each boundary point there are both inner points from \( W \) and \( \mathbb{R}^2 \setminus W \). Then for regularity of \( R_n \) in \( C(\mathbb{T}^2) \) it is necessary and sufficient that

\[
\int_{\mathbb{R}^2} e^{-i\gamma(x,u)} \, d\mu(u) = 0
\]

for all \( x \in \partial W \).

b) If \( K \) is a subgraph of a nonnegative function continuous on an interval and non-constant, and measure \( \mu \) is uniformly distributed on the area of \( K \), the the method \( R_n \) is not regular in \( C(\mathbb{T}^2) \) for any choice of \( \gamma > 0 \) and \( W \).

c) Let \( K \) be strictly convex bounded body in \( \mathbb{R}^2 \) symmetric with respect to the origin and the boundary \( \partial K \in C^\infty \). Let measure \( \mu \) be uniformly distributed on its area and \( \gamma \partial W = M_\gamma \) where \( M_\gamma \) is the curve from 1.12. If in addition \( \gamma \) is big
enough, then for every $f \in L(T^2)$ at each of its Lebesgue points $\lim_{n \to \infty} R_n(f;W;x) = f(x)$.

Note that in order to get a regular method in $C(T^m)$ for $m > 2$ measure $\mu$ averaging of $S_n$ should be done several times.

**Bibliographical remarks.**
For results on multiple Fourier series obtained during the last 20-25 years, see [S6, M22, M5].

For 3.1a) and b); see [T3], for 3.1c), see [T18]; for 3.2, see [T17]; for 3.3, see [Be6]; for 3.4, see [Be1,4]; for 3.5, see [BT2]; for 3.6a), see [ZgT], and independently L. Carleson (conference in Budapest, 1979); for 3.6b), see [Ku3], and also [Ku6,8,10] where some other problems of strong summability of multiple Fourier series are investigated; for 3.6c), see [Be9].

The means $R_n$ in the partial case of $W$ being a circle and measure $\mu$ is uniformly distributed on $\partial W$ were investigated by S. Minakshisundaram and K. Chandrasekharan in 1947.

For the general form of $R_n$, examples and necessary condition from 3.8a), see [T17]; for 3.7a),b), see [N4] (see also [N6,9]); for 3.7c) and for 3.8a)-c), see [Za1,2,4].

### 4. Lebesgue constants and approximation of function classes.

A. Lebesgue found the asymptotics of operator norms of partial sums $S_n$ as follows:

$$
\sup_{\|f\|_C \leq 1} \|S_n(f)\|_C = (2\pi)^{-1} \int_T \left| \sum_{k=-n}^n e^{ikt} \right| dt = 4\pi^{-2} \ln n + O(1).
$$

It is easy now to write any number of members of the series $4\pi^{-2} \ln n + C_0 + C_1 n^{-1} + \ldots$.

E. Landau found the asymptotics $\|S_n\|$ for power series, or for trigonometric series with spectrum in $\mathbb{Z}_+$ : the factor $4\pi^{-2}$ is replaced by $\pi^{-1}$.

If the Lebesgue constants $\|\Lambda_n\|$ increase infinitely as $n$ increases, then there is no regularity of the means $\Lambda_n(f)$. Then for the convergence of $\Lambda_n(f)$ to $f$, some smoothness should be added to $f$ so that it is connected with the rate of growth of $\Lambda_n$.

A. N. Kolmogorov, 1935, has found the asymptotics of approximation of the class $W^r$, with $r \in \mathbb{N}$, by partial sums $S_n$. Here

$$
W^r = \{ f : f^{(r-1)} \text{ is absolutely continuous and } \|f^{(r)}\|_\infty \leq 1 \}
$$

and

$$
\sup_{f \in W^r(T)} \|f - S_n(f)\|_C = 4\pi^{-2} n^{-r} \ln n + O(n^{-r}).
$$

This Kolmogorov’s problem has a long history (for refinement of this formula, generalizations, etc., see, e.g., [S15]).

**4.1.** For every $r > 0$, it is not obligatory for $r$ to be integer, four numbers $A$, $B$, $C$, and $D$ can be pointed out so that

$$
\sup_{f \in W^r(T)} \|f - S_n(f)\|_C = 4\pi^{-2} n^{-r} \ln n + An^{-r} + Bn^{-r-1} \ln n + Cn^{-r-1} + Dn^{-r-2} \ln n + O(n^{-r-2}).
$$
Let us go on to the case of the space $C(T^m)$, or $L(T^m)$, for $m \geq 2$. The growth of the Lebesgue constants of the spherical Bochner-Riesz means (see the beginning of §3) was obtained by V. A. Ilyin, 1968, for $\delta = 0$, and by K. I. Babenko, 1971, for $\delta \in [0, \frac{m-1}{2})$, as follows:

$$\sup_{||f||_C \leq 1} ||S_n^\delta(f)||_C \asymp n^{\frac{m-1}{2} - \delta}$$

with $\delta \in [0, \frac{m-1}{2})$. For $\delta = \frac{m-1}{2}$ the asymptotics is known: $\gamma(m) \ln n + O(1)$, E. Stein, 1961.

Let for $n \in \mathbb{N}$

$$S_n(f; W) = \sum_{k \in nW} c_k(f)e_k.$$

The growth of the Lebesgue constants is minimal when $W$ is polygon.

4.2. If $W$ is the $m$-dimensional polygon in $\mathbb{R}^m$, then

$$\sup_{||f||_C \leq 1} ||S_n(f; W)||_C = \sup_{||f||_1 \leq 1} ||S_n(f; W)||_1 \asymp \ln^m n.$$

4.3. If for $m = 2$ for the rombic partial sums

$$S_n^\diamond(f) = \sum_{|k_1/n_1|+|k_2/n_2| \leq 1} c_k(f)e_k$$

we have $n_2/n_1 \in \mathbb{N}$, then

$$||S_n^\diamond||_{C \to C} = 32\pi^{-4} \ln n_1 \ln n_2 - 16\pi^{-4} \ln^2 n_1 + O(\ln n_2).$$

Note that for $m \geq 3$ a similar result like in Theorem 4.3 apparently is not obtained, unlike the following Theorem 4.4.

4.4. Let for $m = 2$, $n \in \mathbb{N}$, and $\alpha \geq 1$

$$H_n^\alpha(f) \sim \sum_{|k_1|^\alpha|k_2| \leq n} c_k(f)e_k.$$

a) We have $||H_n^\alpha||_{C \to C} \asymp n^{1/(2+2\alpha)}$ with positive constants depending only on $\alpha$.

b) Let the spectrum of $f$ is outside coordinate axes and

$$D_{r,\alpha}(f) \sim \sum_k (ik_1)^r(ik_2)^r c_k(f)e_k.$$

Then

$$\sum_{||D_{r,\alpha}(f)||_\infty \leq 1} ||f - H_n^\alpha(f)||_C \asymp n^{-r+1/(2+2\alpha)}.$$

c) If the matrix $A$ with the entries $\{a_{ij}\}_{i,j=1,2}$ cannot become a matrix with integer entries after multiplication its lines by some numbers, then the operator

$$H_n^A(f) \sim \sum_{|(a_{11}k_1+a_{21}k_2)(a_{12}k_1+a_{22}k_2)| \leq n} c_k(f)e_k$$

is not bounded in $C(T^2)$ for $n$ sufficiently large.

Let us consider now step-wise hyperbolic sums which give the approximation of the best order for the class of functions with bounded mixed derivative in $L_p$ with $p \in (1, +\infty)$. 


4.5. Let for \( f \in L(\mathbb{T}) \)

\[
H_n(f) = \sum_{s_j \geq 0, \sum s_j \leq n} \sum_{|k_j| < 2^{s_j+1}} c_k(f)e_k.
\]

Then

\[
||H_n||_{L_1 \to L_1} \lesssim n^{(3m-1)/2}.
\]

Let us go on to general estimates of the Lebesgue constants. Let us start with the upper bound for the \( L_p \) norm of a polynomial for \( p \in (0,2) \).

4.6. Let \( \{e_j^0\}_{j=1}^n \) be the standard basis in \( \mathbb{R}^m \), \( M_0 = (1, \ldots, m) \), and \( q = \sum q_je_j^0 \) where the \( q_j \) are natural numbers \( (j \in M_0) \); analogously \( h = \sum h_je_j^0 \) where the \( h_j \) are also natural numbers. Set

\[
\Delta_{h_j} \lambda_k = \lambda_k - \lambda_{k+h_je_j}
\]

(the difference operator with stepsize \( h_j \) in the direction \( e_j \)) and

\[
\Delta^{q_j}_{h_j} \lambda_k = \left( \prod_{j \in M_0} \Delta^{q_j}_{h_j} \right) \lambda_k
\]

("mixed" difference in the direction of all axes). For every \( p \in (0,2) \) and \( q \in \mathbb{N}^m \)

\[
\int_{\mathbb{T}^m} \left| \sum_{-n_j \leq k_j \leq n_j} \lambda_k e^{i(k,x)} \right|^p \, dx
\]

\[
\leq \gamma(p,q,m) \prod_j (n_j + 1)^{(p-2)/2} \sum_{0 \leq s_j \leq \lfloor \log_2(n_j + 1) \rfloor} 2^{(1-p/2)\sum s_j} \left( \sum_k |\Delta^{q_j}_{h_j} \lambda_k|^2 \right)^{p/2},
\]

where \( \lambda_k \) is taken to equal 0 for \( k_j \neq [-n_j, n_j] \), at least for one \( j \), in the sum \( \sum_k \), while \( h = h(s, n) \) is defined by the following conditions

\[
\frac{n_j + 1}{3 \cdot 2^{s_j}} \leq h_j \leq \frac{5(n_j + 1)}{6 \cdot 2^{s_j}}, \quad \frac{n_j + 1}{3 \cdot 2^{s_j}} \leq h_j \leq \frac{n_j + 1}{2^{s_j}}
\]

according as \( s_j < \lfloor \log_2(n_j + 1) \rfloor \) or \( s_j = \lfloor \log_2(n_j + 1) \rfloor \).

In the case \( \lambda_k = \varphi(k/n) \) there appears the Fourier transform \( \hat{\varphi} \). E. Belinskii was apparently the first began a systematic study of connections between summability and integrability of the Fourier transform of a function generating a method of summability, in the multi-dimensional case.

4.7. Let \( \varphi \) be a bounded measurable function with a compact support. Then for the norms of a sequence of linear operators

\[
L_n^\varphi : f \to L_n^\varphi(f, \cdot) = \sum \varphi(k/n)\hat{f}(k)e^{i(k, \cdot)}
\]
we have

\[
\| L_n^p \|_{L^p(\mathbb{T}^m) \to L^p(\mathbb{T}^m)} \leq (2\pi)^{-m} \int_{\mathbb{T}^m} \prod_{j=1}^{m} \frac{x_j}{2n \sin(x_j/2n)} |\hat{\varphi}(x)| \, dx
\]

\[
+ \sum_{j=1}^{r-1} (\pi/2)^{(j+1)m} \int_{\mathbb{T}^m} |\hat{\varphi}(x)||x/N|^j \, dx
\]

\[
+ \pi^{mr+m/2} - mr + m/2 \int_{\mathbb{T}^m/2\pi} \cdots \int_{\mathbb{T}^m/2\pi} \left( \sum_k \left| \Delta_{k/n}^r (\varphi; u_1/n, \ldots, u_r/n) \right|^2 \right)^{1/2} \, du_1 \cdots du_r,
\]

and

\[
\gamma(p) \| L_n^p \|_{L^p(\mathbb{T}^m) \to L^p(\mathbb{T}^m)} \geq \left\{ (2\pi)^{-m} \int_{\varepsilon \mathbb{T}^m} \prod_{j=1}^{m} \frac{x_j}{2n \sin(x_j/2n)} |\hat{\varphi}(x)| \, dx \right\}^{1/p}
\]

\[
- \sum_{j=1}^{r-1} (\pi/2)^{(j+1)m} \left\{ \int_{\varepsilon \mathbb{T}^m} |\hat{\varphi}(x)||x/n|^{jp} \, dx \right\}^{1/p}
\]

\[
- \frac{\pi^{mr+m/2p} \varepsilon^{mr+m/2p}}{n^{m-1/p}} \int_{\mathbb{T}^m/2\pi} \cdots \int_{\mathbb{T}^m/2\pi} \left( \sum_k \left| \Delta_{k/n}^r (\varphi; u_1/n, \ldots, u_r/n) \right|^2 \right)^{1/2} \, du_1 \cdots du_r.
\]

Here \( \varepsilon \in (0, 1) \) is an arbitrary real number, \( r \) is integer, and \( 1 \leq p \leq 2 \). The \( r \)-th difference \( \Delta_z^r(\varphi; h_1, \ldots, h_r) \) is defined recursively by the formulas

\[
\Delta_z^1(\varphi; h_1) = \varphi(z + h_1) - \varphi(z);
\]

\[
\Delta_z^r(\varphi; h_1, \ldots, h_r) = \Delta_{z+h_r}^{r-1}(\varphi; h_1, \ldots, h_{r-1}) - \Delta_z^{r-1}(\varphi; h_1, \ldots, h_{r-1}),
\]

with \( h_j, z \in \mathbb{R}^r \). When \( p > 2 \), in view of duality the lower bound still valids with \( p' = p/(p-1) \) instead of \( p \).

**4.8.** Let the boundary of the region \( B \) contain a simple (non-intersecting) piece of a surface of smoothness \([m+2]/2\) in which there is at least one point with non-vanishing principal curvatures. Then there exists a positive constant \( C \) depending only on \( B \) such that

\[
\int_{\mathbb{T}^m} \left| \sum_{k \in nB \cap \mathbb{Z}^m} e^{i(k,x)} \right| \, dx \geq Cn^{(m-1)/2}
\]

for large \( n \).
4.9. For each $r > 0$ and for every $\alpha > 2r$

$$\sup_{||\Delta^r f||_{\infty} \leq 1} \left| f - \sum_{|k| \leq n} (1 - |k|^\alpha/n^\alpha) \frac{m-1}{2} c_k e_k \right|_{C} = \gamma(m, r, \alpha) \frac{\ln n}{n^{2r}} + O(n^{-2r}).$$

Let us return to the one-dimensional case, that is $m = 1$. Let

$$\Lambda_0^n(f) = \sum_{k=-n}^{n} \lambda_{n,k} c_k(f) e_k$$

and

$$||\Lambda_0^n|| = \sup_{||f||_C \leq 1} ||\Lambda_0^n(f)||_C = \frac{1}{2\pi} \sup_{k=-n}^{n} \lambda_{n,k} e_k ||_{1}.$$

If to replace the Fourier coefficient $c_k$ (integrals) by $c_k^{(n)}(f)$ by rectangle formula for the uniform partition $x_p = \frac{2p\pi}{2n+1}$, with $|p| \leq n$, then we get

$$\tilde{\Lambda}_n^0(f; x) = \sum_{k=-n}^{n} \lambda_{n,k} c_k^{(n)}(f) e^{ikx}$$

where

$$c_k^{(n)}(f) = \frac{1}{2n+1} \sum_{p=-n}^{n} f(x_p) e^{-ikx_p}.$$

Coefficients $c_k^{(n)}(f)$ are called the Fourier-Lagrange coefficients. When $\lambda_{n,k} = 1$ for all $k \in [-n, n]$ we have that $\tilde{\Lambda}_n^0(f)$ is the interpolation polynomial defined by the values $f(x_p)$ for $p \in [-n, n]$.

The convergence of $\Lambda_n^0(f)$ to $f$ at a point $x$ for every $f \in C(\mathbb{T})$ is reduced to the boundedness in $n$ of the norms of functionals, the Lebesgue functions.

4.10. We have

$$\sup_{||f||_C \leq 1} |\tilde{\Lambda}_n^0(f; x)| = (\pi/2) |\sin(n + 1/2)x| \cdot ||\Lambda_n^0|| + \theta \sup_{||f||_C \leq 1} |\tilde{\Lambda}_n^0(f; 0)|$$

where $|\theta| \leq C$.

In the following theorem a trigonometric polynomial $T_n$ of order not greater than $n$ is replaced by piece-wise sinusoidal function (see a)), which allows to calculate the asymptotics if integral norms of $T_n$ (see b)). For this, let us introduce a sequence of functions $\varphi_n$ corresponding to $\{\lambda_{n,k}\}_{k=-n}^{n}$ and satisfying the only condition $\varphi_n(x_k) = \lambda_{n,k}$ for $k \in [-n, n]$. For instance, $\varphi_n$ may be a polynomial or piece-wise linear function.

4.11. a) For every $x \in [-\pi, \pi]$ and $p = [\frac{1}{2} + \frac{2n+1}{2\pi} x]$ the following inequality holds

$$|T_n(x) - \frac{(-1)^p}{2n+1} T_n'(x_p) \sin(n + 1/2)x|$$

$$\leq C \sum_{k=0}^{n} |T_n(x_k)| \frac{(2n+1)^2}{(2|p-k| + 1)^2(4n+1 - 2|n-k|)^2}. $$
b) The following asymptotic equality holds
\[ \int_{\mathbb{T}} \left| \sum_{k=-n}^{n} \lambda_{n,k} e^{ikx} \right| dx = \frac{4}{\pi} \sum_{k=-n}^{n} |c_{n,k}(x\varphi_{n})| + \theta \sum_{k=-n}^{n} |c_{n,k}(\varphi_{n})| \]
with $|\theta| \leq C$.

As an application, one can derive an asymptotics of the Lebesgue constants for the general sums of Bernstein-Rogosinski type $\sum_{k=-n}^{n} \mu_{k} S_{n}(f; x + x_{k})$ with the remainder term $\theta \sum |\mu_{k}|$.

Let us return to the problem of approximation of the class of functions with bounded derivative.

Let $\{\psi(k)\}_{1}^{\infty}$ be a convex downwards and decrease to zero sequence and $\beta \in \mathbb{R}$. Let us denote by $W_{\psi,\beta}$ the class of continuous functions on $\mathbb{T}$ for which the trigonometric series
\[ \sum_{k \neq 0} c_{k} \beta_{\psi} \frac{1}{\psi(|k|)} c_{|k|} e_{k} \sim f_{\psi,\beta} \]
is the Fourier series of the function $f_{\psi,\beta}$, derivative in some sense, and $||f_{\psi,\beta}||_{\infty} \leq 1$. If $\psi(k) = k^{-r}$ and $\beta = r$ or $\psi(k) = k^{-r}$ and $\beta = r + 1$ with $r > 0$, then this is the class of functions with the $r$-th derivative in the Weyl sense $f^{(r)}$ or conjugate to it $f^{(r)}$ bounded by 1 in both cases. The class $W_{\psi,\beta}$ was introduced and investigated by A. I. Stepanets, 1986, mainly with respect to approximation by partial sums (see [M24, Ch.3]).

A final general result (see a)) looks as follows. Let $E_{n}(f)$ be best approximation of $f$ by polynomials of order not greater than $n$ in $C(\mathbb{T})$.

4.12. a) Let $n \in \mathbb{Z}_{+}$, and $\{\psi(k)\}_{n+1}^{\infty}$ and $\beta$ be as above. Assume also in the case $\sin \frac{\beta \pi}{2} \neq 0$ that
\[ \sum_{k=1}^{\infty} \frac{1}{k} \psi(k) < \infty, \]
which is necessary. Then treating $0/0 = 0$ we have
\[ \sup_{f_{\psi,\beta} \in C} \frac{||f - S_{n}(f)||_{C}}{E_{n}(f_{\psi,\beta})} = \max_{f \in W_{\psi,\beta}} \frac{||f - S_{n}(f)||_{C}}{E_{n}(f_{\psi,\beta})_{\infty}} = \max_{f \in W_{\psi,\beta}} ||f - S_{n}(f)||_{C} = \frac{4}{\pi^{2}} \sum_{k=1}^{n+1} \psi(k + n) + \frac{2}{\pi} \sin \frac{\beta \pi}{2} \sum_{k=n+2}^{\infty} \frac{\psi(k + n)}{k} + \theta \psi(n + 1) \]
with $|\theta| \leq C$.

b) Under the same assumptions on $\psi$ and $\beta$ the following two-sided estimate holds with absolute constants:
\[ \sup_{f_{\psi,\beta} \in C} \frac{E_{n}(f)}{E_{n}(f_{\psi,\beta})} = \sup_{f_{\psi,\beta} \in C} E_{n}(f) \asymp \psi(n + 1) + |\sin \frac{\beta \pi}{2}| \sum_{k} \frac{\psi(k + n)}{k}. \]
Let make additional three remarks.
In the results given in 4.11 norms in $C$ or $L_\infty$ can be replaced by norm in $L$. Then in b) $E_n(f^\psi\beta)$ in the denominator can be replaced by $\omega_p(f^\psi\beta;\frac{1}{n+1})$, the modulus of smoothness of order $p$. But in this case constants may be dependent on $p \in \mathbb{N}$. At last, the convexity condition for $\{\psi(k)\}$ can be substituted by the weaker condition from 1.10a).

Bibliographical remarks.
For problems on approximation of classes of functions, see, e.g., [S15] and [M23].
Only recently steps to make a survey on Lebesgue constants in the multiple case were taken by E. Liflyand (see [L11]). Besides the results mentioned above, those due to V. A. Yudin, A. N. Podkorytov, M. A. Skopina, and others, are considered in that survey.
For 4.1, see [Le1,4]; for 4.2, see [Be5]; for 4.3, see [Ku1,2] (for $n_1 = n_2$ by I. K. Daugavet, 1970); for 4.4, see [Be7,8] (see also [BL3]); for 4.5, see [Be14]; for 4.6, see [T16]; for 4.7, see [Be5]; for 4.8, see [L7]; for 4.9, see [BL2] in the case of essentially more general radial methods; for 4.10, see [T31] (in the case $m = 2$ a similar result for rhombic sums from 4.3 was obtained in [Ku12]); for 4.11, see [T10]; for 4.12, see [T24,28] (the last inequality in 4.12a) was independently proved by S. A. Telyakovskii, 1987.

5. Two-sided estimates of approximation.
Moduli of smoothness and $K$-functionals.

Let us start with the following result.

5.1. For every $r \in \mathbb{N}$ there exists a continuous function $\varphi$ with compact support such that for each $p \in [1, +\infty]$ and for each $f \in L_p(T)$ we have the following two-sided estimate with positive constants depending only on $r$.

$$||f - \sum \varphi(k/n)c_k e_k||_p \approx \omega_r(f; 1/n)_p$$

$$= \sup_{0 < \delta \leq 1/n} ||\sum_{\nu=0}^r \binom{r}{\nu} (-1)^\nu f(\cdot + \nu \delta)||_p.$$

The upper bound is the well-known Jackson type theorem ($r = 1$—D. Jackson; $r = 2$—N. I. Akhiezer, 1947; $r \geq 3$—S. B. Stechkin, 1951; see [M26] or [M6]). Here the same lower bound is added. Such results allow to obtain:

1) constructive (approximate) characteristics of basic classes of functions, which are not described by the rate of decrease of their best approximations;

2) saturation classes of sequences of linear operators of convolution type; and, as it turned out,

3) formulas of new type for $K$-functionals of a couple of spaces (see below).

The first method of proving such inequalities is based on extremal properties of polynomials, like Bernstein’s inequalities, and use of the Jackson theorem itself (see, e.g., [M6], p.241). On this way necessary and sufficient (at once) conditions are found for validity of upper bounds, lower bounds, two-sided estimates, etc. Found is also the sharp order of approximation expressed by moduli of smoothness for each of the classical methods of summability of Fourier series, namely those of Riemann, Weierstrass, Lebesgue, Jackson, de la Vallée Poussin, Riesz, Rogosinski, Bernstein,
Favard, \((C, \alpha)\), and Abel-Poisson. The latter two methods are equivalent for \(\alpha > 0\) (see 2.14 and 2.15a)). For these, see [T2,18]. Exact order of approximation by the \((C,1)\)-means is found by V. V. Zhuk, 1967. See also [M31] where this method of proof is applied to nonlinear methods as well.

The second method is based on the comparison principle from \(\S 2\). On this way both the Jackson theorem and the needed properties of polynomials are obtained. Simplicity and exactness is its benefit. It was B. S. Mityagin, 1962, who first applied multipliers to problems of approximation theory in \(L_p\) for \(p \in (1, +\infty)\). Observe, that the inequality in the \(C\)-metric yields the inequality in \(L_p\) for all \( p \in [1, +\infty]\) with the same constant (see 2.18a)) but not vice versa. For instance, for \(p = 1\) and \(p = \infty\) and odd \(r\) the result 5.2b) is false.

5.2. a) Let \(r \in \mathbb{N}\) and \(\varphi \in C(\mathbb{R})\). In order that \(f \in W^r\) (for the definition of \(W^r\), see the beginning of \(\S 4\)) and for every \(n \in \mathbb{N}\)

\[
\|f - \sum \varphi(k/n)c_ke_k\|_\infty \leq Kn^{-r}\|f^{(r)}\|_\infty,
\]

it is necessary and sufficient that \(g(x) = (1 - \varphi(x))/x^r \in B(\mathbb{R})\). The minimal value of \(K\) is \(\|g\|_B\).

b) For every \(r \in \mathbb{N}\), for every \(f \in L_p(\mathbb{T})\) with \(p \in (1, +\infty)\), and for every \(n \in \mathbb{N}\) we have the following two-sided inequality with constants depending on \(r\) and \(p\)

\[
\|f - \sum_k (1 - |k|^r/n^r) + c_k e_k\|_p \asymp \omega_r(f; \pi/n)_p.
\]

Let us linearize the modulus of smoothness in \(C(\mathbb{T})\) as well as in \(L_p(\mathbb{T})\) as follows

\[
\tilde{\omega}(f; h) = \|\frac{1}{h} \int_0^h \sum_{\nu=0}^r (-1)^\nu \binom{r}{\nu} f(\cdot + \nu \delta)_d \delta\|,
\]

that is taking of the least upper bound in \(\delta \in (0, h]\) is replaced by the integral averaging.

5.3. a) For every \(r \in \mathbb{N}\) and for every \(f \in C(\mathbb{T})\)

\[
\gamma(r)\omega_{r}(f; h) \leq \tilde{\omega}_{r}(f; h) \leq \omega_{r}(f; h).
\]

b) The inequality \(\omega_{r}(f; h) \leq h^r\) holds if and only if the inequality \(\tilde{\omega}_{r}(f; h) \leq \frac{1}{\tau+1} h^r\) holds.

All this was the one-dimensional case. Let us go on to the multiple case. For the Jackson type theorem, see [M16]. Let \(r \in \mathbb{N}\), \(E \subset \mathbb{R}^m\), and \(h > 0\). Define

\[
\omega_{r}(f; E; h) = \sup_{u \in E} \|\sum_{\nu=0}^r \binom{r}{\nu} (-1)^\nu f(\cdot + \nu hu)\|,
\]

where the norm is taken in \(C(\mathbb{T}^m)\) or \(L_p(\mathbb{T}^m)\). For the monotonicity in \(h\), it should be assumed that \(E\) is star-like with respect to the origin. Let it also be compact. Then the biggest, in some sense, modulus is in the case of \(E\) is the ball, the full modulus \(\omega_r\); while the smallest one when \(E\) is the closed interval starting at the origin, the modulus of smoothness in the given direction. Denote by \(\omega_r^+\) the modulus of smoothness defined by the set \(E\) consisting of \(m\) unit intervals along the axes of the standard basis.
5.4. For every \( r \in \mathbb{N} \), for every \( p \in (1, +\infty) \), for every \( f \in L_p(\mathbb{T}^m) \), and for every \( n \in \mathbb{N} \) the two-sided estimate
\[
||f - \sum_k \prod_{j=1}^m (1 - |k_j|^r/n^r) c_k e_k||_p \simeq \omega^+_r(f; \pi/n)_p
\]
holds with the constants depending on \( r \) and \( p \).

5.5. For no one \( r \geq 1 \) and \( m \geq 2 \) the following inequalities
\[
\gamma_1(r, m, \varphi) \omega^+_r(f; \pi/n) \leq ||f - \sum \varphi(k/n) c_k e_k|| \leq \gamma(r, m, \varphi) \omega_0^+(f; \pi/n)
\]
cannot valid in \( C(\mathbb{T}^m) \), at least if \( \varphi \) is continuous at its support, the unit cube.

Thus the usual moduli in \( C(\mathbb{T}^m) \) and \( L_p(\mathbb{T}^m) \) are not suitable for \( m \geq 2 \). Let us introduce linearized moduli.

Restricting to the case of even \( r \), let us set for \( q \in \mathbb{N} \)
\[
\tilde{\omega}_{2q}(f; \mu; h) = || \int_{\mathbb{R}^m} \sum_{\nu=0}^{2q} \left( \begin{array}{c} 2r \\ \nu \end{array} \right) (-1)^\nu f(\cdot + (\nu - r)hu) \, d\mu(u)||,
\]
that is taking of the least upper bound in \( h \) is substituted by the integral averaging with respect to the finite Borel measure. If \( d\mu = \chi_E \, du \), where \( \chi_E \) is the indicator of \( E \), then we write \( \tilde{\omega}_{2q}(f; E; h) \). If \( E \) is the unit ball or the sum of \( m \) closed unit intervals starting at the origin in the directions of coordinate axes, then we write \( \tilde{\omega}_{2q}^0(f; h) \) or \( \tilde{\omega}_{2q}^+(f; h) \) respectively.

5.6. a) For every \( q \in \mathbb{N} \), for every \( \delta > \frac{m-1}{2} \), and for every \( f \in C(\mathbb{T}^m) \)
\[
||f - \sum_k (1 - |k|^{2q}/n^{2q})^\delta c_k e_k|| \simeq \tilde{\omega}_{2q}^0(f; \pi/n).
\]

b) For every \( q \in \mathbb{N} \), for every \( \delta > \frac{m-1}{2} \), and for every \( f \in C(\mathbb{T}^m) \)
\[
||f - \sum_k (1 - n^{-2q} \sum_{j=1}^m k_j^{2q} \delta c_k e_k|| \simeq \tilde{\omega}_{2q}^0(f; \pi/n),
\]
where these two-sided inequalities hold with constants depending on \( r, \delta, \) and \( m \).

Observe that the moduli \( \tilde{\omega}_{2q}^0 \) and \( \tilde{\omega}_{2q}^+ \) are almost monotone in \( h \).

5.7. For every \( q \in \mathbb{N} \), for every \( p \in (1, +\infty) \), for every \( f \in L_p(\mathbb{T}^m) \), and for every \( h > 0 \) the following two-sided estimate
\[
\tilde{\omega}_{2q}^0(f; h)_p \simeq \omega_{2q}^0(f; h)_p
\]
holds with constants depending on \( r, m, \) and \( p \).

When \( m \geq 2 \) the cases \( q = 1 \) and \( q \geq 2 \) differ for \( \tilde{\omega}_{2q}^0(f; E; h) \).
5.8. a) If $E$ is symmetric in the following sense: for any rearrangement of each two coordinates or change of the sign of any of the coordinates the point is still in $E$, then for every $p \in [1, +\infty]$

$$\tilde{\omega}_2(f; E; h)_p \simeq \tilde{\omega}_2^0(f; h)_p.$$ 

b) For $m \geq 2$ and $q \geq 2$ the moduli $\tilde{\omega}_2^0$ and $\tilde{\omega}_2^q$, where the latter modulus corresponds to the unit cube, are uncomparable on $C(\mathbb{T}^m)$ as $h \to 0$.

For odd $r$ and $p = \infty$ or $p = 1$, unlike the case $p \in (1, +\infty)$, in 5.2b) the two-sided estimates from 5.1 are impossible if $\varphi$ is continuous and even function with compact support (see [T6]).

Let $\tau_n$ be the Marcinkiewicz means, that is the $n$-th arithmetic means of the cubic partial sums.

5.9. For every $f \in C(\mathbb{T}^2)$

$$\|f - \tau_n(f)\| \lesssim \int_1^\infty \left| (\Delta^2_{(t/n)}(e_1^0 + e_2^0) + \Delta^2_{(t/n)}(e_1^0 - e_2^0)) f(\cdot) \frac{dt}{t^2} \right|,$$

where $\{e_1^0, e_2^0\}$ is the standard basis in $\mathbb{R}^2$ and

$$\Delta^2_h f(x) = f(x + 2h) - 2f(x) + f(x - 2h).$$

In order to find interpolation spaces by the real interpolation method J. Petre, 1963, has introduced $K$-functionals (see, e.g., [M4]). Z. Ciesielski, 1983, put the problem of finding of the $K$-functional via linear means of the Fourier series.

5.10. Under the above notation for every $\varepsilon > 0$

a) For $p \in [1, +\infty]$ we have

$$K(\varepsilon^{2r}, f, L_p, \Delta^r) \simeq \tilde{\omega}_{2r}^0(f; \varepsilon)_p.$$ 

b) For $p \in [1, +\infty]$ we have

$$K(\varepsilon^{2r}, f, L_p, D_{2r}) \simeq \tilde{\omega}_{2r}^+(f; \varepsilon)_p,$$

where $\Delta$ is the Laplace operator and $D_{2r} = \sum_{j=1}^m \frac{\partial^{2r}}{\partial x_j^{2r}}$.

For $p = \infty$ and $p = 1$ these formulas for the $K$-functionals of the indicated couples of spaces were not known earlier.

Observe, that for $p = \infty$ as well as for $p = 1$ it is impossible in 5.10b) to put $\omega_{2r}^+$ in view of 5.5. The modification needed for this was introduced by Z. Ditzian, TAMS 1991, for $r = 1$ in the space $C$. Let us generalize this modulus for any $r$ in the two cases indicated. Let us set

$$\Delta_{r,\delta}^+ f(x) = \sum_{j=1}^m \sum_{\nu=0}^{2r} \binom{2r}{\nu} (-1)^\nu f(x + (\nu - r) \delta e_j^0).$$
5.11. For every \( r \in \mathbb{N} \) and for every \( p \in [1, +\infty) \)

a) \( \tilde{\omega}_r^+(f; h) \geq \sup_{0 < \delta \leq h} ||\Delta_{r, \delta}^+ f(\cdot)||_p \)

b) \( \tilde{\omega}_r^0(f; h) \geq \sup_{0 < \delta \leq h} ||(\Delta_{1, \delta}^+)^r f(\cdot)||_p \).

Problems dealing with hyperbolic differential operators are usually more difficult than those for elliptic operators; see [S16].

A formula for the \( K \)-functional is found till now only as the approximation by stepwise-hyperbolic Riesz means and only for \( p \in (1, +\infty) \).

Let \( D_r = \frac{\partial^{1+r_m}}{\partial x_1^{r_1} \cdots \partial x_m^{r_m}} \) be the mixed derivative with \( 0 < r_1 = \ldots = r_\nu < r_{\nu + 1} \leq \ldots \leq r_m \). Let us assume that \( \int f(x) \, dx = 0 \) for each \( j \in [1, m] \) and set for \( s \in \mathbb{Z}_+ \)

\[
\delta_s(f) = \sum_{k \in \rho(s)} c_k(f)e_k,
\]

where \( \rho(s) = \{ k \in \mathbb{Z}^m : 2^{s_j - 1} \leq |k_j| < 2^{s_j} ; 1 \leq j \leq m \} \). Denote by \( I \) the identity operator.

5.12. For \( [1/\varepsilon] = 2^n \) and \( p \in (1, +\infty) \)

\[
K(\varepsilon^r; f; L_p, D_r) \geq ||f - \sum_{(s, r) \leq r_1 n} \left( I - \frac{1}{2r_1 n} D_r \right) \delta_j(f)||_p.
\]

the same methods are applicable also in the case of differential operators of fractional order. Let us restrict ourselves to the case \( m = 1 \).

5.13. Let us set for fractional \( r > 0 \) and for \( f \in C(\mathbb{T}) \)

\[
\tilde{\omega}_r(f; h) = \left| \int_1^\infty (\Delta_{2p, u}^2 f(\cdot) + \gamma \Delta_{2p+1, u} f(\cdot)) \frac{du}{u^{1+r}} \right|,
\]

where \( p \in \mathbb{N} \) satisfies \( p > r/2 \), and \( \Delta_{\delta} f(x) = f(x + \delta) - f(x - \delta) \) and

\[
\gamma = \frac{1}{2} \tan(r\pi/2) \int_0^\infty \frac{\sin^{2p} t}{t^{1+r}} \, dt \left( \int_0^\infty \frac{\sin^{2p+1} t}{t^{1+r}} \, dt \right)^{-1}.
\]

a) Setting \( \varphi_r(x) = (1 - |x|^r)_+ - i \tan(r\pi/2) |x|^r (1 - |x|)_+ \) sign \( x \), we have for every \( f \in C(\mathbb{T}) \) and for every \( n \)

\[
||f - \sum_k \varphi_r (k/n) e_k|| \geq \tilde{\omega}_r(f; 1/n).
\]

b) \( \tilde{\omega}_r(f; h) = O(h^r) \) as \( h \to 0 \) if and only if \( f^{(r)} \in L_\infty(\mathbb{T}) \).

c) \( K(\varepsilon^r; f; C, W^r) \geq \tilde{\omega}_r(f; \varepsilon) \).

Let us give one more comparatively general result on saturation class;
5.14. If a positive measure \( \mu \) satisfies \( \int |u|^{2r} \, d\mu < \infty \) and the support of this measure does not lie in any hyperplane passing through the origin, then for every \( \delta > \frac{m-1}{2} \)

\[
||f - \sum_k \left( 1 - \int \frac{(k,u)^{2r}}{n^{2r}} \, d\mu(u) \right)^{\delta} c_k \epsilon_k || = O(n^{-2r})
\]

if and only if

\[
\int \left( \sum_{j=1}^{m} u_j \frac{\partial}{\partial x_j} \right)^{2r} f(x) \, d\mu(u) \in L_\infty(T^m).
\]

Let us consider now the problem on approximation by the Bernstein method of the class of functions with given modulus of continuity in \( C(T) \) in the following form

\[
A \omega(f; \pi/n) \leq ||f(\cdot) - S_n(\cdot + \pi/n)/2|| \leq B \omega(f; \pi/n).
\]

The upper bound was known even to S. N. Bernstein; for the lower bound, see [T2] or [M6, p.241). The smallest constant \( B \) in this inequality under sufficiently general conditions on the modulus of continuity is found by V. T. Gavrilyuk and A. I. Stepanets, 1973 (see also [M23]). In the following statement the exact biggest constant \( A \) in approximation from below is found for the first time.

5.15. For every modulus of continuity the exact constant in the above given inequality is

\[
A = \left( 2 + \frac{4}{\pi} \int_0^\pi \frac{\sin t}{t} \, dt \right)^{-1}.
\]

A similar result in the lower bound is obtained also for the Rogosinski sums \( \frac{1}{2}(S_n(\cdot + \pi/2n) + S_n(\cdot - \pi/2n)) \) when \( \omega(f; \pi/n) \) is replaced by \( \omega_2(f; \pi/2n) \) in the above inequality.

Bibliographical remarks.

Two-sided estimates of approximation 5.1 (by the Bernstein sums, by the Rogosinski sums and the like) were firstly obtained by the author; see [T2] or [M6].

For 5.3, see [T3,15]; for 5.4, see [No2]; for 5.5, 5.6, 5.8, and 5.14, see [T17] (the case \( r = 1 \) in 5.6a) was obtained earlier in [Be3]); for 5.7 and 5.10, see [T23]; the result 5.9 was obtained by O. I. Kuznetsova (see [KT]). For \( p \in (1, +\infty) \) the rate of approximation by the J. Marcinkiewicz means was found by M. F. Timan and V. G. Ponomarenko in 1975.

For 5.12, see [Be19]; for 5.13, see [T32], see also [KT]; for 5.15, see [Kl].

General theorems of Voronovskaya type and converse to them are also obtained by the multiplier method, see [T2], [Ku5,7]. See also [T18] or [T27].

6. Hardy spaces \( H_p \).

Let us consider the same problems as in §§3 and 5 for the Hardy spaces \( H_p \) immediately in the multidimensional case and the first with \( p \in (0, 1] \). Our investigation is based on the multiplier method.
Let $D^m = \{ z = (z_1, \ldots, z_m) : |z_j| < 1, 1 \leq j \leq m \}$ be the unit polydisc in $\mathbb{C}^m$. Every function in $H_p(D^m)$, where $p > 0$, is expanded in $D^m$ in the absolutely convergent power series

$$f(z) = \sum_{k \in \mathbb{Z}^m_+} c_k z^k,$$

where $z^k = z_1^{k_1} \cdots z_m^{k_m}$ and

$$||f||_{H_p} = ||f(\cdot)||_{H_p} = \sup_{0 < r < 1} \left( \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |f(r_1 e^{iu_1}, \ldots, r_m e^{iu_m})|^p \, du_m \right)^{1/p} < \infty.$$

For $p \geq 1$ this is a subspace of $L_p(\mathbb{T}^m)$, namely every function from $H_p(D^m)$ is characterized in this case by the fact that its limit function on $\mathbb{T}^m$, the skeleton of the boundary of $D^m$, belongs to $L_p$ and has the Fourier series of the power type, that is with the spectrum in $\mathbb{R}_+^m$ (see, e.g., [M17]). Thus many results from §§2-3 and 5 are still valid in the same form as particular cases (for instance, 5.1, 5.3, 5.6, 5.9, 5.10, and the like). For $p \in (1, +\infty)$ also the Riesz projector theorem as well as more general Marcinkiewicz’ multiplier theorem (see [M21]) help. But in $C$ as well as in $L_1$ results may differ. Let us give an example (one-dimensional).

For every $f \in H_\infty(D) \cap C(\bar{D})$

$$\max_{z \in \bar{D}} |f(z) - \sum_{k=0}^{n} (1 - \frac{k}{n+1}) c_k z^k| \asymp \omega(f; 1/n)_\infty,$$

and for $f \in C(\mathbb{T})$, in assumption that also its trigonometrically conjugate $\tilde{f} \in C(\mathbb{T})$, we have only

$$||f - \sigma_n(f)||_\infty + ||\tilde{f} - \sigma_n(\tilde{f})||_\infty \asymp \omega(f; 1/n)_\infty + \omega(\tilde{f}; 1/n)_\infty.$$

For the second relation, see [T4], while the first one immediately follows from the second. Another situation is for $p \in (0, 1)$. A function from the quasinormed space $L_p$ is not expanded into the Fourier series if it is not from $L_1$. Moreover, as it is well-known, there are no in $L_p$ for $p \in (0, 1)$ nonzero linear continuous functionals. But in $H_p$ there are Tailor series for all $p > 0$ and thus there are also multipliers. It is possible to introduce multipliers in $L_p$, though: first on the dense set of polynomials and then to extend continuously.

A number sequence $\{ \lambda_k \}_{k \in \mathbb{Z}^m_+}$ is called the multiplier in $H_p(D^m)$, written $\{ \lambda_k \} \in M_p$, if for every $f \in H_p(D^m)$ with the Tailor coefficients at zero $\{ c_k \}$ we have

$$\Lambda f)(z) = \sum_{k \in \mathbb{Z}^m_+} \lambda_k c_k z^k \in H_p(D^m)$$

and there exists a positive constant $\gamma$ such that for every $f \in H_p(D^m)$

$$||\Lambda f||_{H_p} \leq \gamma ||f||_{H_p},$$

and

$$||\{ \lambda_k \}||_{M_p} = \inf \gamma.$$
6.1. Let $0 < p < q \leq 1 < r \leq \infty$. Then $M_p \subset M_q \subset M_r$ and this embedding is continuous.

For $p \in (1, +\infty)$, as it is follows from the Riesz projector theorem, a sequence $\{\lambda_k\}_{k \in \mathbb{Z}}$ is the multiplier in $H_p(D^m)$ if and only if its extension by zero on $\mathbb{Z}^m$ is the multiplier of Fourier series in $L_p(\mathbb{T}^m)$. For $p = \infty$ and $p = 1$ such extensions do exist (see 2.4 and [M7, II, 16.7.5], respectively) but it is difficult to describe them.

For $p \in (0, 1]$, let us start with the finite sequence.

6.2. For every $p \in (0, 1]$ and every $N \in \mathbb{N}^m$

$$||\{\lambda_k\}_{0 \leq k \leq N}\|_{M_p} \leq \gamma(m, p) \left( \int_{\mathbb{Z}^m} \sum_{-N \leq k \leq N} |\lambda_k|^p \, du \right)^{1/p},$$

where $\lambda_k$ may be arbitrary for $k \in [-N, N] \setminus [0, N]$ since $\lambda_k = 0$ for $k \notin [0, N]$.

Let $\varphi : \mathbb{R}^m_+ \to \mathbb{C}$. We will write $\varphi \in M_p$ if

$$||\varphi||_{M_p} = \sup_{\varepsilon > 0} ||\{\varphi(\varepsilon_k)\}\|_{M_p} < \infty.$$  

6.3. a) Let $\varphi \in C(\mathbb{R}^m)$ and $\text{supp } \varphi \subset [-N, N]$. If for some $p \in (0, 1]$ we have $\hat{\varphi} \in L^p(\mathbb{R}^m)$, then

$$||\varphi||_{M_p} \leq \gamma(m, p) \left( \prod_{j=1}^m N_j \right)^{1/p-1} \left( \int_{\mathbb{R}^m} |\hat{\varphi}(x)|^p \, dx \right)^{1/p}. $$

b) If $\varphi \in M_p$, then for every inner point from $\mathbb{R}^m_+$ there exists a function $\psi$ with compact support, which coincides with $\varphi$ in some neighborhood of this point, and $\hat{\psi} \in L^p(\mathbb{R}^m)$.

As it follows from 1.8 now, each function with compact support $\varphi \in C^\alpha(\mathbb{R}^m_+)$ for $\alpha > m/p - m/2$ belongs to $M_p$. P. Oswald, 1982, has proved by another method in essence that if $\varphi \in C^\infty(\mathbb{R}^m_+)$ equals to 1 on $[0, 1]$ and $\varphi$ vanishes on $[2, +\infty)$, then $\varphi \in M_p$ for every $p \in (0, 1)$ (see [S8]).

Example. Let $\varphi(x) = (1 - |x|^{2r})^\delta_+$, where $r \in \mathbb{N}$. It belongs to $M_p$ with $p \in (0, 1]$ if and only if $\delta > m/p - (m + 1)/2$.

6.4. The spherical Bochner-Riesz means

$$\sum_{k \in \mathbb{Z}^m_+} (1 - |k|^{2r}/n^{2r})^\delta_+ c_k e_k,$$

where $r \in \mathbb{N}$ and $\delta \geq 0$, are regular in $H_p(D^m)$ with $p \in (0, 1]$ if and only if $\delta > m/p - (m + 1)/2$.

Let us give now sufficient conditions on $\varphi$ near infinity.
6.5. Let \( p \in (0, 1] \) and \( \varphi \in C^r(\mathbb{R}_+^m) \) for some \( r > m(1/p - 1/2) \). If also

\[
|\varphi(x)| \leq A(1 + |x|^\alpha)^{-1}, \quad \sum_{j=1}^m |\frac{\partial^r \varphi}{\partial x_j^r}(x)| \leq B(1 + |x|^\beta)^{-1},
\]

where \( \alpha > 0 \) and \( \beta = r + \alpha \) or \( \alpha = \beta > m(1/p - 1/2) \), then

\[
||\varphi||_{M_p} \leq \gamma(m, p, r, \alpha, \beta)(A + B).
\]

Let us go on to two-sided estimates of approximation by the Bochner-Riesz means of power series and to moduli of smoothness.

Note that for \( m = 1 \) two-sided estimates of approximation by the Abel-Poisson means in \( H_p(D) \) with \( p \in (0, 1] \) were obtained earlier by E. A. Storozhenko, 1982 (see [S8]), and those by the Riesz means are due to P. Oswald, 1984 (upper bound) and to E. Belinskii, 1993 (lower bound). Another methods were used for these. When \( m = 1 \) no special moduli of smoothness are needed (see also 6.9 below).

Let us set

\[
(\Delta f)(z) = \sum_{j=1}^m (z_j \frac{\partial}{\partial z_j})^2 f(z),
\]

which is the Laplace operator on \( \mathbb{T}^m \), and introduce a linearized modulus of smoothness

\[
\tilde{\omega}^0_{2r}(f; h)_p = \| \int dx \int dy... \int |x| \leq 1 |y| \leq 1 |w| \leq 1 (2r)^{(2r)}(-1)^\nu f((\cdot)e^{ih(\nu-r)(x+y+...+w)}dw||_{H_p}.
\]

Here \( r \in \mathbb{N}, h > 0 \), and the integral averaging is taken over the Carthesian product of \( q \) unit balls in \( \mathbb{R}^m \) with \( q > 2m(1/p - 1/2)/(m + 1) \).

6.6. For every \( r \in \mathbb{N}, \) for every \( p \in (0, 1], \) for every \( f \in H_p(D^m) \), and for every \( \varepsilon > 0 \)

a) \( ||f(\cdot) - \sum_k (1 - \varepsilon^{2r})k|2r\delta \sum_k c_k(\cdot)^k||_{H_p} \approx \tilde{\omega}^0_{2r}(f; \varepsilon)_p \) for \( \delta > m/p - (m + 1)/2 \),

b) \( K(\varepsilon^{2r}; f; H_p(D^m), \Delta^r) \approx \tilde{\omega}^0_{2r}(f; \varepsilon)_p \),

where the two-sided estimates are with constants independent of \( f \) and \( \varepsilon \).

Let us set now

\[
D_{2r}(f; z) = \sum_{j=1}^m (z_j \frac{\partial}{\partial z_j})^{2r} f(z)
\]

and for \( h > 0 \)

\[
\tilde{\omega}^+_{2r}(f; h)_p = \| \int dx \int dy... \int |x| \leq 1 |y| \leq 1 |w| \leq 1 (2r)^{(2r)}(-1)^\nu f((\cdot)e^{ih(\nu-r)(x+y+...+w)}dw||_{H_p}.
\]

Here the integral averaging is taken over Carthesian product of \( q \) crosses in \( \mathbb{R}^m \) with \( q > 2m(1/p - 1/2)/(m + 1) \), where the cross is the sum of all unit intervals along the coordinate axes with center at the origin.
6.7. For every \( r \in \mathbb{N} \), for every \( p \in (0, 1] \), for every \( f \in H_p(D^m) \), and for every \( \varepsilon > 0 \)

\[
\begin{align*}
&\text{a)} \quad ||f(\cdot) - \sum_{k} (1 - \varepsilon^{2r} \sum_{j=1}^{m} k_j^{2r} \delta_{k} c_k(\cdot)^k)||_{H_p} \asymp \tilde{\omega}_{2r}^+(f; \varepsilon)_p \text{ for } \delta > m/p - (m + 1)/2, \\
&\text{b)} \quad K(\varepsilon^{2r}; f; H_p(D^m), D_{2r}) \asymp \tilde{\omega}_{2r}^+(f; \varepsilon)_p,
\end{align*}
\]

where the two-sided estimates are with constants independent of \( f \) and \( \varepsilon \).

Let us give in the multi-dimensional case the Hardy-Littlewood type inequalities; for \( m = 1 \) a similar inequality for every \( r \in \mathbb{N} \) is due to E. A. Storozhenko, 1982 (see [S8]).

6.8. For every \( r \in \mathbb{N} \), for every \( f \in H_p(D^m) \), and for every \( \delta \in [1/2, 1) \)

\[
\begin{align*}
&\text{a)} \quad ||\Delta^r f(\delta(\cdot))||_{H_p} \leq \gamma(r, m, p)(1 - \delta)^{-2r} \tilde{\omega}_{2r}^2(f; 1 - \delta)_p, \\
&\text{b)} \quad ||D_{2r} f(\delta(\cdot))||_{H_p} \leq \gamma(r, m, p)(1 - \delta)^{-2r} \tilde{\omega}_{2r}^+(f; 1 - \delta)_p.
\end{align*}
\]

The Hardy space is also studied on the ball \( \{z : \sum_{j=1}^{m} |z_j|^2 < 1 \} \) in \( \mathbb{C}^m \) (see [M18]). Recently Vit. Volchkov [ViV] proved 6.2 for this case, that is for Reinhardt domains, and consequently all what follows from this. Let us return to the one-dimensional case.

6.9. If \( E_n(f) \) is best approximation by polynomials of degree not higher than \( n \), then for every \( N \in \mathbb{N} \)

\[
\ln^{-1} N \sum_{n=1}^{N} n^{-1}||f(\cdot) - \sum_{k=0}^{n} (1 - k^2/n^2)^{1/p-1} c_k(\cdot)^k||_{H_p} \asymp \ln^{-1} N \sum_{n=1}^{N} n^{-1} E_n(f)_{H_p},
\]

where the constants in the two-sided inequality depend only on \( p \in (0, 1] \).

Let us consider different moduli of smoothness in the disc \( D \). For moduli of smoothness in \( C \) for analytic functions, see e.g., [M26]. We will study \( H_p(D) \) for all \( p \in (0, +\infty) \). The contour (boundary) modulus for \( f \in H_p(D) \) is defined as follows:

\[
\omega_r(f; h)_p = \sup_{0 < \delta < h} \left| \sum_{0 < \nu < h} \left( \frac{r}{\nu} \right) (-1)^\nu f((\cdot)e^{i\nu \delta}) \right|_{H_p}.
\]

One can replace \( f \) by the limit function \( f(e^{it}) \) and integrate in \( L_p \) over the circle \( \Gamma = \partial D \).

Let us introduce also a linearized boundary modulus (averaged contour modulus) for \( q \in \mathbb{N} \)

\[
\tilde{\omega}_r(f; h)_p = \left| \int_{[0,1]} \left| \sum_{0 < \nu < h} \left( \frac{r}{\nu} \right) (-1)^\nu f((\cdot)e^{i\nu h u}) \right| \, du \right|_{H_p}
\]

as well as radial one for \( h \in (0, 2/r) \)

\[
\omega_r(f; \text{rad}; h)_p = \left( \int_{\pi}^{\pi} \left| \sum_{0 < \nu < h} \left( \frac{r}{\nu} \right) (-1)^\nu f(e^{i(t)(1 - h\nu)}) \right|^p \, dt \right)^{1/p}.
\]

This modulus annulate all polynomials of degree not higher than \( r - 1 \).
6.10. For every \( r \in \mathbb{N} \), for every \( p \in (0, +\infty] \), for every \( f \in H^p(D) \), and for every \( h \in (0, 1/(r + 1)] \) we have

a) when \( q = 1 \) and for every \( p \geq 1 \) or for \( q = 1 + [1/p - 1/2] \) and \( p \in (0, 1) \)

\[
\omega_r(f; h)_p \preceq \tilde{\omega}_r(f; h)_p;
\]

b) when \( S_{r-1}(z) = \sum_{k=0}^{r-1} f^{(k)}(0) z^k / k! \) for \( r \geq 2 \) and \( S_0 = 0 \)

\[
\omega_r(f; \text{rad}; h)_p \preceq \omega_r(f - S_{r-1}; h)_p;
\]

where the constants in the two-sided inequalities depend only on \( r \) and \( p \).

Let us give also the Bernstein type inequality in \( L^p \) for \( p \in (0, 1) \) when the case of fractional \( r \) differs from that for \( p \geq 1 \).

6.11. For every \( r > 0 \) and for every \( p \in (0, 1) \)

\[
\sup_{||T_n||_p \leq 1} ||T_n^{(r)}||_p \asymp n^r; \quad n^{1/p-1}; \quad n^{1/p-1} \ln^{1/p} n
\]

for \( r \in \mathbb{Z}_+ \) or for \( r > 1/p-1 \), and \( r \not\in \mathbb{Z}_+ \) so that \( r < 1/p-1 \), and \( r = 1/p-1 \not\in \mathbb{Z}_+ \), respectively.

Bibliographical remarks.

For 6.1 - 6.6, 6.10, see [T35]; 6.7 - 6.8 is due to E. M. Klebanov (submitted for publication); for 6.9, see [Be19]; for 6.11, see [BL4].

7. Positive definite functions and splines.

Let \( E \) be a linear (vectorial) space over the field \( \mathbb{R} \). A function \( f : E \to \mathbb{C} \) is called positive definite on \( E \), written \( f \in \Phi(E) \), if for any set of elements \( \{x_k\}_1^n \) and for any set of numbers \( \{\zeta_k\}_1^n \) we have

\[
\sum_{k,s=1}^n f(x_k - x_s)\zeta_k\bar{\zeta_s} \geq 0.
\]

If \( E \) is the Hilbert space, then one can take, for example, \( f(x) = e^{i(x,y)} \) with \( y \in E \) (see also 7.14 below).

7.1. For every \( f \in \Phi(E) \) and for every \( x, y \in E \)

\[
|f(x + y) - 2f(x) + f(x - y)| \leq 2\Re(f(0) - f(y)).
\]

This simple necessary condition immediately yields the fact that \( e^{-|x|^\alpha} \in \Phi(\mathbb{R}) \) only for \( \alpha \in [0, 2] \).

For functions in \( \Phi(\mathbb{R}^m) \), which are the characteristic functions if \( f(0) = 1 \), see [M15] for \( m = 1 \) and [M1] for any \( m \).
7.2. In order that \( f \in \Phi(\mathbb{R}) \) it is necessary and sufficient that the following three conditions are satisfied:

a) \( f \in C(\mathbb{R}) \) and bounded,

b) the following improper integral converges

\[
\lim_{T \to +\infty} \left( 2T \right)^{-1} \int_{-T}^{T} f(t) \, dt \geq 0
\]

and there exists \( k_0 \in \mathbb{N} \) such that for every \( k \geq k_0 \) and for every \( x \neq 0 \)

\[
(\text{sign } x)^{k+1} \int_{-\infty}^{\infty} (x + it)^{-k-1} f(t) \, dt \geq 0.
\]

This new criterion proved to be fruitful during the proof of 7.5b) (see below).

Let us go on to the multiple case and radial functions. We will write \( f \in \Phi_m \) if \( f : [0, +\infty) \to \mathbb{R} \) and \( f(|x|) \in \Phi(\mathbb{R}^m) \). It is of certain interest that \( \Phi(\mathbb{R}^m) \subset \Phi(\mathbb{R}^{m+1}) \) but \( \Phi_{m+1} \subset \Phi_m \) (see [M1, Ch.5, §4]).

7.3. a) Let \( 1 \leq n < m \). In order that \( f \in \Phi_m \) it is necessary and sufficient that there exists a function \( g \in \Phi_n \) uniquely defined and such that

\[
f(t) = \int_{0}^{1} g(ut) w^{n-1} (1 - u^2)^{(m-n)/2-1} \, du.
\]

b) Let \( m \geq 3 \) and odd. In order that \( f \in \Phi_m \) it is necessary and sufficient that

\[
d^k \frac{d}{dt} \left\{ (m-2)/2 f_0(\sqrt{t}) \right\}_{t=0} = 0
\]

for \( 0 \leq k \leq (m-3)/2 \) and the function \( f_1 \), defined for \( t \geq 0 \) by the formula

\[
f_1(\sqrt{t}) = \sqrt{t} d^{(m-1)/2} \frac{d}{dt} \left\{ (m-3)/2 f_0(\sqrt{t}) \right\},
\]

belongs to \( \Phi_1 \).

For necessary conditions for membership in \( \Phi_m \), see [T21]. Let us give a simple sufficient condition which is Polya’s theorem when \( m = 1 \).

7.4. Let \( n = [(m + 2)/2] \). If \( f \in C[0, +\infty), \lim_{t \to +\infty} f(t) \geq 0, \) \( f \in C^{m-1}(0, +\infty), (-1)^{n-1} f^{(n-1)} \) is convex downwards on \((0, +\infty)\) and

\[
\lim_{t \to +0} t^n f^{(n)}(t) = \lim_{t \to +\infty} t^n f^{(n)}(t) = 0
\]

where, for example, the right derivative may be taken, then \( f \in \Phi_m \).

Shoenberg’s \( B \)-splines are very well-known (see [M30]), namely \( B_0 \) is the indicator function of the interval \((-1/2, 1/2)\), and for \( n \geq 1 \) we have \( B_n = B_{n-1} * B_0 \) (convolution). Evidently, \( B_n \) is a spline of compact support and of degree \( n \) which belongs to \( C^{n-1}(\mathbb{R}) \) and sewn by \( n + 1 \)-th algebraic polynomial. It is obvious also that for odd \( n \) the Fourier transform \( \hat{B}_n(x) \geq 0 \) for each \( x \in \mathbb{R} \) and consequently \( B_n \in \Phi(\mathbb{R}) \).

Let us consider compact splines in \( \Phi(\mathbb{R}) \) sewn only by two polynomials, the simplest in some sense. More precise they are of kind \( p_n(|x|) \) on \([-1, 1]\), where \( p_n \) are real algebraic polynomials of degree \( n \). An example for \( n = 1 \) is given by \( (1 - |x|) \).
7.5. Let us give two examples of sequences of such splines with \( n = 1, 2, \ldots \).

a) \( \tilde{v}_{2n+1} = (-1)^nP_n \ast P_n \), where \( P_n \) is Legendre’s polynomial for the interval \([-1/2, 1/2]\) extended to \( \mathbb{R} \) by zero.

b) \( e_n(\sqrt{t}) = \sqrt{t} \frac{d^n}{dt^n} \{t^{n-3/2}(1 - \sqrt{t})^n\} \).

A problem of splines of highest smoothness arises.

7.6. For each \( n \geq 2 \) there exists a unique spline of indicated type \( A_{3n-2} \) of degree \( 3n-2 \) in \( C^{2n-2}(\mathbb{R}) \) satisfying \( A_{3n-2}(0) = 1 \). For every \( x \in \mathbb{R} \) therewith \( A_{3n-2}(x) > 0 \). The graph of \( A \)-spline is, for any \( n \), of the bell form, that is increases on \([0,1]\) with one point of inflection.

There exist explicit formulas for \( A \)-splines. For example, \( A_4(x) = (1-|x|)^3_+(1+3|x|) \).

By Wiener’s theorem linear combination of shifts of \( A \)-spline are dense in \( L(\mathbb{R}) \). Studied is the question of degree of convergence of \( B \)-splines, see [M30]. The Fourier transform \( \hat{B} \) has zeros and shifts are supplemented by contractions.

7.7. Let \( n \geq 2 \) and \( A = A_{3n-2} \). If a function \( f \) is such that, for some \( r \in \mathbb{N} \) and for some \( \lambda > 1 \), both \( f(x) \) and \( x^r f(x) \in C(\mathbb{R}) \cap H_1^\lambda(\mathbb{R}) \), then for each \( h \in (0,1) \) there exist \( N \approx (1/h)^{1+2n/r} \) and \( \{c_k\}_N^\infty \) satisfying the inequality

\[
\sup_{x \in \mathbb{R}} |f(x) - \sum_{k=-N}^N c_k A(x + kh)| \leq Kh^\lambda - 1, \quad Kh^{2n} \ln(1/h), \quad \text{and} \quad Kh^{2n},
\]

for the cases \( \lambda < 2n+1, \lambda = 2n+1, \) and \( \lambda > 2n+1, \) respectively. A constant \( K \) depends only on \( \lambda, r, n, \) and the norms of two functions in the Nikolskii space \( H_1^\lambda(\mathbb{R}) \).

Note, that the change of degree of smoothness in the space \( L \), that is \( \lambda \) is replaced by \( \lambda - 1 \), is based on the embedding theorem from \( L \) into \( C \).

To apply splines to the finite element method it should be possible to represent any polynomial of corresponding degree as a linear combination of shifts of the spline.

7.8. Let \( n \geq 2 \) and \( A = A_{3n-2} \).

a) For every \( \varepsilon \in (0,1) \), for different points \( \{h_k\}_0^{3n-2} \) from \([0, \varepsilon]\), and for any polynomial \( p \) of degree \( 3n-2 \) there exists a unique representation of the form

\[
p(x) = \sum_{k=0}^{3n-2} c_k A(x + h_k)
\]

with \( x \in [0, 1 - \varepsilon] \).

b) If \( n \geq 3 \) and \( b - a > 1 \), then the fact \( \sum c_k A(x + h_k) \in C^{2n+1}[a,b] \) yields that this sum vanishes on \([a,b]\).

c) In the case \( n = 2 \) for any \( \varepsilon \in (0,1) \) and for any \( p \) of degree \( n \leq 4 \) there exists a representation of the form

\[
p(x) = \sum_{k=0}^N c_k A_4(x + kh)
\]
where \( x \in [0, 2 - \varepsilon] \) and \( h \) and \( N \) depend only on \( \varepsilon \). This is not so for \( \varepsilon = 0 \).

In the convex set of splines of the type in question of fixed degree and equal 1 at the origin, there exist extreme points. For example, \( \lambda e_n \) and \( A_{3n-2} \) are the extreme points in the set of splines of the same degree. But \( \lambda e_n \) is not an extreme point in the set of splines of degree not higher than \( n+1 \), while \( \mu e_{2n+1} \) is the extreme point even for all the (characteristic) functions in \( \Phi \) of support \([-1,1]\).

**7.9.** For every odd \( m \geq 3 \) and any \( n > (m+1)/2 \) there exists non-trivial spline of type \( p_n(|x|) \) for \( |x| \leq 1 \) and vanishing for \( |x| \geq 1 \) which belongs to \( \Phi(\mathbb{R}^m) \cap C^\alpha(\mathbb{R}^m) \), where \( r = 2[(2n - m - 1)/6] \), and this degree of smoothness is maximal.

In connection with one problem in approximation theory (see 7.11 below) arose a question of possible extension of a function defined, outside the interval \((-a,a)\), to a function in \( \Phi(\mathbb{R}) \). If such a continuation does exist, then it is desirable that its value at the origin is minimal.

**7.10.** a) If \( f \in \Phi(\mathbb{R}) \) and \( |\Re| + \Re f \in L(\mathbb{R}) \), then also \( \Re f \in L(\mathbb{R}) \).

b) If \( f \in L[a, +\infty) \), locally absolutely continuous, and there exists \( \varepsilon > 0 \) such that for \( h \to +0 \)

\[
\int_a^\infty |f'(x) - f'(x + h)| \, dx = O(\ln^{-1-\varepsilon}(1/h)),
\]

then \( f \) is extendable on \( \mathbb{R} \) so that it will belong to \( \Phi(\mathbb{R}) \). For \( \varepsilon = 0 \) a similar statement is false.

c) If on \([a, +\infty)\) we have \( f(x) = cx^{-r} \), where \( c \in \mathbb{C} \) and \( r > 0 \), then an extension into \( \Phi(\mathbb{R}) \) is possible only in the following two cases: 1) \( r > 1 \) and \( c \in \mathbb{C} \) is arbitrary, 2) \( r \in (0,1] \) and \( |\arg c| \leq r\pi/2 \).

**7.11.** Let \( r \) be an arbitrary positive number.

a) There exists a function \( \varphi_r \), supported on \([0,1]\) and depending only on \( r \) and \( m \), such that

\[
\|(-\Delta)^{r/2} f\|_\infty \leq M \quad \text{if and only if} \quad \|f - \sum_k \varphi_r(|k|/n)c_k e_k\|_\infty \leq Mn^{-r(r+1)\ldots(r+p)/p!}
\]

for every \( n \in \mathbb{N} \), where \( \Delta \) is the Laplace operator and \( p \) is the least integer satisfying the inequalities \( p \geq (m-1)/2 \) and \( p \geq (r+1)/2 \).

b) There exists a function \( \varphi_r \in C[-1,1] \) and a constant \( \gamma(r) \) such that

\[
\|f^{(r)}\|_\infty \leq M \quad \text{if and only if} \quad \|f - \sum_{k=-n}^n \varphi_r(k/n)c_k e_k\|_\infty \leq \gamma(r)Mn^{-r}
\]

for every \( n \in \mathbb{N} \).

Let \( E \) be a linear normed space. When \( f(||x||) \in \Phi(E) \)? This question is connected with the problem of isometric embedding of \( E \) into some \( L_p \) space (see [S9]).
7.12. If there exist three linear independent elements $a_1$, $a_2$, and $a_3$ from $E$ such that
\[ ||a_1 + a_2 y_1 + a_3 y_2||^{-1} \left\{ \partial_t ||ta_1 + a_2 y_1 + a_3 y_2|| \right\}_t=1 \in L_1(\mathbb{R}^2), \]
then $f(||x||) \in \Phi(E)$ only in the case $f \equiv \text{const}$.

7.13. The following spaces satisfy the condition 7.12:
1) $L_p(\Sigma)$, where $\Sigma$ is a space with measure either finite or infinite, when $\dim L_p(\Sigma) \geq 3$ and $2 < p \leq \infty$. In particular, $l_p^m$ with $m \geq 3$ and $2 < p \leq \infty$.
2) The space $C$ of continuous functions on metric space consisting of not less than three points.

Investigated are in the same way the spaces $E$ with $\dim = 2$.

In the next statement answered is one question due to Shoenberg, 1938.

7.14. For $E = l_p^m$ with $m \geq 2$ and $p \in (2, +\infty]$
\[ e^{-||x||_p^a} \in \Phi(\mathbb{R}^m) \text{ if and only if } \begin{cases} m = 2, \quad \alpha \in [0, 1] \\ m \geq 3, \quad \alpha = 0. \end{cases} \]

Bibliographical remarks.
For 7.1-7.3, 7.5b), 7.11a), see [T21]; for 7.2, 7.4, 7.5b), see [T26]; for 7.6, 7.7, 7.9, see the abstract by R. M. Trigub at the International Conference in Constructive Theory of Functions, Sophia, 1987, and [ZT1]; for 7.8, see [T33].
7.10, 7.11b) are due to V. P. Zastavny (see [ZT1]); for 7.12-7.14, see [Z6-8].
Shoenberg’s problem (see 7.14) was independently solved in other way by Koldobskii (Algebra i Analiz (St. Petersburg Math. J.), 1991).

8. Multiplicative Walsh system.

This system was introduced by J. Walsh in 1923 as an orthonormal system of functions complete in $L_2[0, 1]$ and such that each of the functions in this system takes only the meanings +1 and −1 (excluding some dyadic rational points at which it vanishes). One of the applications is the coding theory (see [M9]).

Here we consider the same questions we have studied earlier for the trigonometric system. Besides the natural similarity, one can find some special peculiarities in this.

To each number $x \in \mathbb{R}_+$ a dyadic expansion corresponds
\[ x = \sum_{k=0}^{\infty} \theta_{-k}(x)2^k + \sum_{k=1}^{\infty} \theta_k2^{-k}, \]
where $\theta_k(x) = 0$ or 1; the first sum is finite and for dyadic rational numbers the second sum is chosen to be finite as well.

Consider the Walsh system $\{\psi_n\}_0^\infty$ in Paley’s enumeration:
\[ \psi_0 \equiv 1, \quad \psi_{2^k}(x) = (-1)^{\theta_{k+1}(x)}, \quad \psi_n(x) = \prod_{k=1}^{\infty} (\psi_{2^k}(x))^{\theta_k(n)} = \prod_{k=1}^{\infty} (-1)^{\theta_{k+1}(x)}\theta_{-k}(n). \]
Let us introduce the following group operation on the interval \([0, 1]\)

\[
x \dot{+} y = \sum_{k=1}^{\infty} 2^{-k}(\theta_k(x) + \theta_k(y))(\text{mod } 2).
\]

Then \(\psi_n(x \dot{+} y) = \psi_n(x) \cdot \psi_n(y)\).

The Fourier-Walsh series of a function \(f : [0, 1] \rightarrow \mathbb{R}\) and \(f \in L_1[0, 1]\) (the Lebesgue measure is invariant with respect to the operation \(\dot{+}\)) may be written as follows:

\[
f \sim \sum_{k=0}^{\infty} c_k(f)\psi_k, \quad c_k(f) = \int_0^1 f(t)\psi_k(t) \, dt.
\]

The following theorem is an analog of Baire’s classical theorem on superposition (for the modern approach to this question, see [S11]).

8.1. For every \(f \in C[0, 1]\) there exists a function \(\varphi \in C[0, 1]\), which is strictly increasing from 0 to 1, so that for \(F = f \circ \varphi\) we have:

a) \(\sum_{k=0}^{n} k c_k(F) = o(n)\) as \(n \rightarrow \infty\);

b) The Fourier-Walsh series of \(F\) converges to it uniformly.

8.2. In order the Bernstein-Rogozinski method

\[
B_n(f; x, \alpha, \beta, \nu) = \alpha S_n(f; x) + \beta S_n(f; x \dot{+} \nu),
\]

where \(S_n\) is the partial sum of the Fourier-Walsh series of \(f\), and \(\alpha, \beta \in \mathbb{R}, \nu \in \mathbb{R}_+\), \(n \in \mathbb{N}\), to be regular in \(C[0, 1]\), it is necessary and sufficient that \(\alpha = \beta = 1/2\) and \(\nu = 1\).

A similar result is obtained also in the multidimensional case \(C[0, 1]^2\).

Let \(Q_k = [0, k-1]^m\), where \(k \in \mathbb{N}\) and \(Q_0 = \emptyset\), is the cube in \(\mathbb{R}^m\).

8.3. If \(\lim \lambda_k = 0\) as \(k \rightarrow \infty\) and \(\sum_{k=0}^{\infty} \max_{s \geq k} |\lambda_s - \lambda_{s+1}| < \infty\), then the series

\[
\sum_{k=0}^{\infty} \lambda_k \sum_{n \in Q_{k+1} \setminus Q_k} \psi_n(x), \quad \psi_n(x) = \psi_{n_1}(x_1) \ldots \psi_{n_m}(x_m),
\]

converges a.e. on \([0, 1]^m\) to an integrable function, is the Fourier-Walsh series of its sum and

\[
\int_{[0, 1]^m} \left| \sum_{k=0}^{\infty} \lambda_k \sum_{n \in Q_{k+1} \setminus Q_k} \psi_n(x) \right| \, dx \leq \sum_{k=0}^{\infty} \max_{s \geq k} |\lambda_s - \lambda_{s+1}|.
\]

This is an analog of the Sidon-Telyakovskii result for the one-dimensional trigonometric series.

Obtained is also an analog of the estimate from above for the Lebesgue constants from 4.6 when \(p = 1\).
8.4. If an integer valued sequence \( \{\nu_k\}_1^\infty \) is increasing and convex and \( \ln \nu_n = O(n^{1/2m}) \), then for every \( f \in C[0,1]^m \) we have
\[
\lim_{p \to \infty} \frac{1}{p} \sum_{k=1}^p |f(x) - S_{\nu_k}(f;x)| = 0,
\]
where \( S_{\nu} \) are cubic partial sums of the Fourier-Walsh series.

Let us return to one-dimensional series.

8.5. For every \( \alpha > 0 \) and every \( f \in L_q \), where \( q \in [1, +\infty] \),
\[
\gamma_1(\alpha) ||f - \sigma_n(f)||_q \leq ||f - \sigma_n^{\alpha}(f)||_q \leq \gamma_2(\alpha) ||f - \sigma_n(f)||_q.
\]

8.6. In \( C[0,1] \) we have for every \( n \in \mathbb{Z}_+ \), \( N \in (2^n, 2^{n+1}] \), and \( \alpha > 0 \)
\[
\gamma_1(\alpha)[\Omega_n(f) + \omega_{n+1}(f)] \leq ||f - \sigma_n^{\alpha}(f)|| \leq \gamma_2(\alpha)[\Omega_n(f) + \omega_n(f)],
\]
where
\[
\Omega_n(f) = \sup_{k \geq n} \left\| \frac{1}{2^{k+1}} \sum_{\nu=0}^{k} 2^{\nu-1}[f(\cdot) - f(\cdot + \frac{1}{2^{n+1}})] \right\|
\]
and
\[
\omega_n(f) = \sup_{0 < t \leq \frac{1}{2^n}} ||f(\cdot + t) - f(\cdot)||.
\]

For \( \alpha = 1 \) somewhat weaker result was obtained earlier by M. F. Timan and K. Tukhliev in 1982.

Now let us speak about the strong summability.

8.7. In order that for every \( f \in L_\infty[0,1] \) the following estimate
\[
\sup_N \frac{1}{N} \sum_{k=1}^N |S_{\nu_k}(f;0)| \leq \gamma ||f||_\infty
\]
holds, it is necessary, and in the case when \( \{\nu_k\} \) is convex, also sufficient that the following condition holds:
\[
\sup_N \frac{1}{2^{n+1}} \left( \frac{1}{N^2} \sum_{k=1}^N [(1 - \mu)\nu + \mu(2^s - \nu)]^2 \right)^{1/2} < \infty,
\]
where \( 2^n \leq \nu_N < 2^{n+1} \) and \( \nu_k = l2^{s+1} + \mu 2^s + \nu \), with \( l, \mu, \) and \( \nu \) integer defined by inequalities \( \mu = \mu(k,s) \in [1,2^s] \), \( \nu = \nu(k,s) \in [0,1] \).

Unlike the trigonometric system (see 3.6a)), also the arithmetic nature of \( \{\nu_k\}_1^\infty \) is important here. For instance, the strong summability takes place when \( 2^k \leq \nu_k \leq 2^k + \sqrt{k} \).

A similar result is obtained also for sums like
\[
\frac{1}{N} \sum_{k=1}^N |S_{\nu_k}(f;0)|^\theta
\]
with \( \theta > 1 \).

Really speaking, Theorems 8.3 and 8.7 are obtained for general orthonormal multiplicative systems. For the definition of the Vilenkin multiplicative systems, see [M9], §1.5.

Bibliographical remarks.
For 8.1, see [G6]; for 8.2-8.7, see [G1-5].
9. Pointwise approximation of functions by polynomials.
Approximation by polynomials with integral coefficients.

Let us consider approximation of functions by algebraic polynomials on an interval of real axis with regard to position of a point. Known are here the results due to S. M. Nikolskii, 1946, A. F. Timan, 1951 (direct theorems), and V. K. Dzyadyk, 1958 (inverse theorems); see e.g., [M26,6].

In connection with results of type 5.14 arose a question in the periodic case on characterization of the Lipschitz class of integral order as regard to degree of approximation. G. Alexits and G. Sunouchi, 1970, have made an attempt to answer this question. They succeeded only partially, namely they had to recede from the ends of the interval.

Let us denote by $W^r_0$, where $r \in \mathbb{N}$, the set of functions on $[-1, 1]$ which satisfy $f^{(r-1)} \in \text{Lip} 1$ and the following boundary conditions

$$
\sum_{p=1}^{r-1} a_{p,s} f^{(p)}(1)/p! = 0 \quad \text{and} \quad \sum_{p=1}^{r-1} (-1)^p a_{p,s} f^{(p)}(-1)/p! = 0.
$$

Here

$$
a_{p,s} = \sum_{q=1}^{p} \sum_{k=p}^{r-1} (-1)^{p+q+k+\nu} \binom{p}{k} \frac{2k}{k-\nu} 4^{-k\nu} 2^s.
$$

No conditions are for $r = 1$ and for $r \geq 2$ we have $s = [(r+1)/2], ..., r-1$.

9.1. We have $f \in W^r_0$ if and only if

$$
|f(x) - R_n(f; x)| = O(\sqrt{1 - x^2}/n + o(1/n))^r
$$

as $n \to \infty$ uniformly with respect to $x \in [-1, 1]$, where

$$
R_n(f; x) = a_0/2 + \sum_{k=1}^{n} (1 - k^2[(r+1)/2]/n^2[(r+1)/2]) a_k C_k(x)
$$

$$
+ \frac{1+(-1)^{r+1}}{2} \sqrt{1 - x^2} \sum_{k=1}^{n} (1 - k/n) k^r/n^r a_k S_{k-1}(x),
$$

where $C_k(x) = \cos k \arccos x$ is the Chebyshev polynomial, $a_k$ are the Fourier-Chebyshev coefficients, $S_{k-1} = C_k'/k$ is the Chebyshev polynomial of second order.

For another approach to this problem, see [M6], p.268.

Let us go on to approximation by polynomials with integral coefficients. Necessary and sufficient conditions of uniform approximation by such polynomials are found by Fekete, 1954, Hewitt-Zukerman, 1959, and those in complex domain by S. Ya. Alper, 1964 (see the survey [T5]). A peculiar kind of intertwining of analytic questions and arithmetic ones appears here. These conditions are such that for functions defined on sets of the complex plane we have the transfinite diameter to be less than 1 and the function, in addition, coincides with some polynomial of integral coefficients. The problems of degree of approximation of functions in various classes were studied as well. For instance, theorems on the rate of convergence to functions of finite smoothness as well as analytic on the interval $[0, 1]$ are obtained by A. O. Gelfond, 1955 (see [S5]).
9.2. a) Let \( f \) be analytic inside and continuous in the closed disk \( K_\rho = \{ z \in \mathbb{C} : |z| \leq \rho < 1 \} \). If, in addition, the numbers \( f^{(\nu)}(0)/\nu! \) are integers for all \( \nu = 0, 1, 2, \ldots \) (this is also necessary), then for every \( r \in \mathbb{N} \) there exists a constant \( \gamma \) independent of \( n \in \mathbb{N} \) such that

\[
E_n^e(f; K_\rho) = \inf_{Q_n} \| f - Q_n \|_{C(K_\rho)} \leq \gamma \omega_r(f; 1/n).
\]

Here \( Q_n(z) = \sum_{k=0}^{n} c_k z^k \) with \( \Re c_k, \Im c_k \in \mathbb{Z} \) for \( 0 \leq k \leq n \), and \( \omega_r \) is the boundary modulus of smoothness (see before 6.10).

b) Let \( X(z) \) be a polynomial with integral coefficients and the leading coefficient 1, and \( K_\rho = \{ z \in \mathbb{C} : |X(z)| \leq \rho < 1 \} \). Suppose, further, that \( f \) is analytic inside the lemniscate \( |X(z)| = 1 \). In order that

\[
\lim_{n \to \infty} (E_n^e(f; K_\rho))^{1/n} < 1
\]

it is necessary and sufficient that the polynomial \( q_r \) defined by conditions \( q_r^{(s)}(z_\nu) = f^{(s)}(z_\nu) \) for \( 0 \leq s \leq r \), where \( \{z_\nu\}_1^k \) are all the different zeros of \( X(z) \), has integral coefficients for any \( r \).

The following theorem is a general theorem on approximation of functions of finite smoothness on arbitrary interval of \( \mathbb{R} \).

9.3. Let both \( \{x_\nu\}_1^k \), the set of all integral algebraic numbers, and the set of their algebraically conjugate numbers situate on \( [a,b] \) with \( b - a < 4 \). If \( f \in C^r[a,b] \) for \( r \in \mathbb{Z}_+ \) and Hermitian polynomial \( q \), defined by conditions \( q^{(s)}(x_\nu) = f^{(s)}(x_\nu) \) for \( 1 \leq s \leq k \) and \( 0 \leq s \leq r \), has integral coefficients (this is also necessary), then a sequence of polynomials \( Q_n \) of degree not higher than \( n \) with integral coefficients can be picked out so that they satisfy on \( [a,b] \) the following inequalities for \( 0 \leq \nu \leq r \):

\[
|f^{(\nu)}(x) - Q_n^{(\nu)}(x)| = O\left[\sqrt{(x - a)(b - x)} + 1/n^2\right]^{r-\nu} \omega(f^{(r)}; \sqrt{(x - a)(b - x)} + 1/n^2).
\]

Besides that,

\[
|Q_n^{(r+1)}(x)| = O\left[\sqrt{(x - a)(b - x)} + 1/n^2\right]^{-1} \omega(f^{(r)}; \sqrt{(x - a)(b - x)} + 1/n^2).
\]

Although it is impossible to put \( \omega_k(f) \) with \( k \geq 2 \) in this theorem, the same theorem is true for functions satisfying Zygmund’s condition \( \omega_2(f^{(r)}; h) = O(h) \).

A similar theorem holds with \( r = 0 \) for functions with given majorants of partial moduli of smoothness on the parallelepiped with the edges parallel to the coordinate axes of length less than 4, see [T5].

9.4. Let \( f \) be analytic inside the square \( [0, 1] \times [0, 1] \) and \( f \in C^r \) on \( [0, 1] \times [0, 1] \).

Suppose, further, that Hermitian interpolation polynomial \( q(z) \), defined by conditions \( q^{(s)}(z_k) = f^{(s)}(z_k) \) for \( 0 \leq s \leq r \) and \( 1 \leq k \leq 4 \), \( z_1 = 0, z_2 = 1, z_3 = i, z_4 = 1 + i \), has integral coefficients. Then for each \( n \) there exists \( Q_n \) such that for any \( z \in \Gamma \), where \( \Gamma \) is the boundary of the square, we have

\[
|f^{(\nu)}(z) - Q_n^{(\nu)}(z)| \leq C \rho^{r-\nu} \omega(f^{(r)}; \rho)\exp(-\nu \log \rho).
\]
where $\rho_{1+\varepsilon}(z)$ is the distance from $z \in \Gamma$ to the level curve $\Gamma_{1+\varepsilon}$.

This is the first result of Dzyadyk’s type (see [M6], Ch.9) on approximation by polynomials with integral coefficients on a set in $\mathbb{C}$ with nonsmooth boundary.

In his lecture on the 1st All-Union Congress of Mathematicians, S. N. Bernstein posed, among other questions, a question on best approximation of arbitrary numbers $\lambda$ by polynomials $Q_n$ on an interval situated in $(0,1)$. A certain upper bound for the interval $[\delta, 1-\delta]$, where $\delta \in (0,1/2)$, was given not long after by R. O. Kuzmin and L. V. Kantorovich (see [S5]). Let us give one precise result.

**9.5.** For any $\lambda \in (0,1)$ we have

$$E_n^c(\lambda; \delta, 1-\delta) = \inf_{Q_n} \max_{[\delta, 1-\delta]} |\lambda - Q_n(x)| \leq 2n\rho^n,$$

where $\rho = \max\{1/2, (1-2\delta)/(1+2\sqrt{\delta(1-\delta)})\}$. Here $\rho = 1/2$ for $\delta \in [1/10, 1/2)$, and we have change-over for $\delta = 1/10$, and the factor $2n$ can be replaced by a bounded one if $\delta \neq 1/10$; it is impossible to make $\rho$ smaller if $\lambda$ is not dyadic.

The case of dyadic $\lambda$ is also investigated.

**9.6.** a) If inside $[a, b]$ with $b-a < 4$ there exists at least one integer point, then for every $\lambda \in (0, 1)$ and for every $p \in [1, +\infty)$ we have

$$E_n^c(\lambda; a, b)_p = \inf_{Q_n} \left( \int_a^b |\lambda - Q_n(x)|^p \, dx \right)^{1/p} \approx n^{-1/p}.$$

b) For any $b \in (0, 1]$ and for any $\lambda \in (0, 1)$ we have

$$E_n^c(\lambda; 0, b)_p \approx n^{-2/p}.$$

Somewhat weaker result was obtained earlier by E. Aparisio ($p = 2$) and A. O. Gelfond (for any $p \geq 1$) (see [S5]). A result similar to 9.6 is proved also for the ball in $\mathbb{R}^m$.

Strengthened is also one result due to A. O. Gelfond, 1966, on approximation of functions on $[\alpha, \beta] \subset (0, 1)$ by polynomials with coefficients from a given set (with restrictions to the growth of absolute values; see [T11]).

**9.7.** Let $E$ be a compactum in $\mathbb{C}$ with connected complement, the origin be a point of $\partial E$ and not being a limit point for inner points of $E$ if are those. If $\{w(k)\}_0^\infty$ is a positive sequence satisfying $\lim(w(k))^{1/k} = \infty$ as $k \to \infty$, then for every $f \in C(E)$ and analytic inside such that $f(0) = 0$ and for every $\varepsilon > 0$ there exists $\{c_k\}_0^n$ we have

$$\max_{z \in E} |f(z) - \sum_{k=0}^n c_k z^k| < \varepsilon \quad \text{and} \quad |c_k| \leq w(k) \quad \text{for all} \quad k \geq 0.$$

If a sequence $\{(w(k))^{1/k}\}$ is bounded, then this statement is false.

For sets without inner points this result was proved earlier by S. Ya. Khavinson, 1969, and more general result than 9.7 was obtained later by V. A. Martirosyan.

The problem of best approximation in $L_q(T)$ of periodic functions of class $W_p^r(T)$ by trigonometric polynomials of degree not higher than $n$ was solved for $q = \infty$ and $p = \infty$, in the space $C(T)$, and for $q = 1$ and $p \in [1, +\infty]$ (J. Favard, S. M. Nikolskii, L. V. Taikov; see [M13]). In the case of approximation by polynomials on interval with regard to position of a point no such results are known.
9.8. For every $r \in \mathbb{N}$, for every $f \in W^r_{\infty}[-1, 1]$, and for every $n \geq r - 1$ there exists a polynomial $p_n$ of degree not higher than $n$ satisfying the inequality

$$|f(x) - p_n(x)| \leq K_r(\sqrt{1 - x^2}/(n + 1))^r + \gamma(r)(\sqrt{1 - x^2})^{-r}/(n + 1)^{r+1},$$

where $K_r = (4/\pi) \sum_{k=0}^{\infty} (-1)^{k(r+1)}(2k + 1)^{-r-1}$ and necessarily $\gamma(r) \geq Ce^r$.

This result was known earlier only for $r = 1$ due to V. N. Temlyakov, 1981.

The problems of approximation by polynomials in integral metric with power weight were investigated by many mathematicians (see [S11]).

9.9. a) For every $p \in [1, +\infty)$, for every $r \in \mathbb{N}$, for every $f \in W^r_p[-1, 1]$, and for every $n \geq r - 1$ there exists a polynomial $p_n$ of degree not higher than $n$ satisfying the inequality

$$\int_{-1}^{1} |f(x) - p_n(x)|(1 - x^2)^{r/2} dx \leq ||\varphi_r||_{r'}(n + 1)^{-r} + o((n + 1)^{-r}),$$

where $1/p + 1/p' = 1$ and $\varphi_r(t)$ is the $r$-th $2\pi$-periodic integral of sign $\sin t$.

b) For every $f \in W^r_{\infty}[-1, 1]$, and for every $n \geq r - 1$ there exists a polynomial $p_n$ of degree not higher than $n$ satisfying the inequality

$$\int_{-1}^{1} |f(x) - p_n(x)|(1 - x^2)^{r/2} dx \leq 4K_{r+1}(n + 1)^{-r} + \gamma(r)(n + 1)^{-r-1}$$

and this result is asymptotically sharp on the class.

The proof of 9.8 differs from that of 9.9 mainly because the operator $p_n = p_n(f)$ is nonlinear, while in 9.9 it is linear.

Direct theorems on approximation by polynomials are applied, for example, to the investigation of the question of convergence of Fourier-Jacobi series (see [M20]).

9.10. a) If there exist $\gamma \in (0, 1]$, $\delta < 2\gamma$, and $\lambda < 2\gamma$ such that for any $x_1, x_2 \in [-1, 1]$

$$|f(x_1) - f(x_2)| \leq |x_1 - x_2|^\gamma(\sqrt{1 - x_1} + \sqrt{1 - x_2})^{-\delta}(\sqrt{1 + x_1} + \sqrt{1 + x_2})^{-\lambda},$$

then there exists a sequence of polynomials $p_n$ such that for each $x \in [-1, 1]$

$$|f(x) - p_n(x)| = O(n^{-\gamma}(\sqrt{1 - x} + 1/n)^{\gamma-\delta}(\sqrt{1 + x} + 1/n)^{\gamma-\lambda}).$$

b) Let $f(\pm 1) = 0$ and $f(x)(1 - x)^{\delta/2}(1 + x)^{\lambda/2} \in \text{Lip} \gamma$ for some $\gamma \in (0, 1]$, $\delta < 2\gamma$, and $\lambda < 2\gamma$, then its Fourier-Jacobi series orthonormal in $L_2$ with the weight $(1 - x)^{\alpha}(1 + x)^{\beta}$ converges uniformly on $[0, 1]$ for $\alpha - \beta < 1 + \lambda - 2\gamma$.

c) If the Fourier-Jacobi series of $f$ converges at the endpoints of $[-1, 1]$ and the Dini-Lipschitz condition $\omega(f; h)\ln(1/h) \rightarrow 0$ as $h \rightarrow 0$ holds, then this series converges uniformly on $[-1, 1]$.

Bibliographical remarks.
For 9.1, see [T7]; for 9.2, see [T15]; for 9.2, 9.5, 9.6, see [T1]; see also [T5]. 9.4 is due to Vit. Volchkov (the paper is accepted in Mat. Zametki); for 9.7, see [T11]; for 9.8, 9.9 $(p = 1)$, see [T34]; for 9.10, see [B1.2].

See also [S3], and for integral coefficients, see [M8], where some of the results mentioned are given with proof.
10. Approximation by trigonometric polynomials with a given number of harmonics.

Let $E$ be a Banach space, $W$ be a compactum in $E$ symmetric with respect to the origin, $L$ be a subspace of $E$. The (Kolmogorov) width of $W$ of order $n$ is defined as follows:

$$d_n(W)_E = \inf_{L: \dim L = n} \sup_{f \in W} \inf_{g \in L} \|f - g\|.$$  

The space of dimension $n$ on which the least lower bound is achieved, if one exists, is called extremal (the best for $W$). In the case $E = C(T)$ and $W = W^r$, that is the cylinder with compact base, extremal is the subspace of trigonometric polynomials with spectrum on the interval and the width is calculated (V. M. Tikhomirov, 1959).

K. I. Babenko, 1960, posed and investigated the problem of the choice of approximate polynomials for approximation in $L^p$ of function classes on $T^m$ which are defined by a given differential operator.

R. S. Ismagilov (see [S7]) has found out that for usual classes of periodic functions even for $m = 1$ polynomials with fixed spectrum do not form even asymptotically best subspace and introduced the following trigonometric widths:

$$d^T_n(W)_q = \inf_{\theta_n} \sup_{f \in W} \inf_{T(\theta_n)} \|f(\cdot) - T(\theta_n; (\cdot))\|_q,$$

where

$$T(\theta_n; x) = \sum_{k=1}^n c_k e^{i(p_k; x)},$$

and $\theta_n = \{p_1, \ldots, p_n\}$, that is minimization first with respect to $\{c_k\}^n_1$ and then with respect to the spectrum $\theta_n$ of $n$ harmonics.

The notion of trigonometric width makes a sense for separate functions as well and is as follows:

$$e_n(f)_q = \inf_{\theta_n} \inf_{T(\theta_n)} \|f - T(\theta_n)\|_q.$$  

For example, for $f_0(x) = |x|$ with $x \in T$ we have $e_n(f_0)_\infty \asymp n^{-3/2}$ (V. E. Majorov, 1986). This is essentially nonlinear problem of adaptive approximation.

Let us start with the problem on the Lebesgue constants.

In 1981, S. V. Konyagin and O.C.McGehee-L.Pigno-B.Smith independently proved Littlewood’s conjecture: for any collection $\theta_n = \{p_k\}_1^n$ of integers we have

$$\sum_{k=1}^n e^{ip_k(x)} \|_1 = \int_{-\pi}^{\pi} \sum_{k=1}^n e^{ip_k x} \, dx \geq \gamma \ln n.$$  

For this, see [M12]. Using S. A. Vinogradov’s version of the proof due to the three authors (oral communication, 1983) we obtain the following general fact.

**10.1.** For any $\{p_k\}_1^\infty$ such that $p_k \in \mathbb{N}$ and $p_{k+1} > p_k$ and for any $\{c_k\}_1^\infty$

$$\int_{-\pi}^{\pi} \left| \sum_{k=1}^n e^{ip_k x} \right| \, dx \geq \gamma \sum_{\nu=1}^\infty \left( \sum_{\theta \in \mathbb{N}} \nu^{-1} |\theta|^{2} \right)^{1/2}.$$
V. M. Tikhomirov suggested to find an order, as \( n \to \infty \), of the value

\[
L_n(r, q) = \inf_{\{p_k\}_n} ||(\sum_{k=1}^{n} e^{i p_k(\cdot)}(\cdot))||_q,
\]

where the \( r \)-th derivative in the Weyl sense for \( r \geq 0 \) is written inside the norm sign. The afore-mentioned results immediately give \( L_n(0, 1) \approx \ln n \).

**10.2.** The following relations are true with \( 1/q + 1/q' = 1 \)

\[
L_n(r, q) \asymp \begin{cases} 
    n^r \ln n, & q = 1, r \geq 0 \\
    n^{r+1/q'}, & 1 < q \leq 2, r \geq 0 \\
    n^{(r q + 1)/2}, & 2 < q < \infty, 0 \leq r < 1/q \\
    n(\ln n)^{-1/q'}, & 2 < q < \infty, r = 1/q.
\end{cases}
\]

The first two relations for any \( r > 0 \) and the third one for even \( q \) are established by V. E. Majorov. The other cases are due to E. Belinskii, 1988. The third relation in the general case was independently proved by S. V. Konyagin, 1988. There exist also a generalization to the multiple case due to E. Belinskii and E. M. Galeev, 1991.

Inequalities of different metrics for polynomials with floating spectrum look as follows.

**10.3.** For all \( 1 \leq p < q \leq \infty \) we have

\[
\sup_{\theta_n} ||T(\theta_n)||_q 
\approx \begin{cases} 
    n^{(1/p - 1/q)p/2}, & 2 < p < \infty \\
    n^{1/p - 1/q}, & 1 \leq p \leq 2.
\end{cases}
\]

The upper bounds are due partially to R. J. Nessel and G. Wilmes, 1978, and the rest to V. A. Rodin, 1983.

Let us go on to problems of approximation of basic classes.

We say that \( f \in W_{p}^{r, \beta} \) if

\[
\sum_{k \in \mathbb{Z}} |k|^r e^{i \beta \pi \text{sign} k/2} c_k(f) c_k \sim f^{r, \beta} \in L_p(\mathbb{T})
\]

and \( ||f^{r, \beta}||_p \leq 1 \). For definition of Nikolskii’s classes \( H_{p}^{r} \), see [M16].

**10.4.** For \( 1 \leq p \leq 2 \) and \( 2 < q < \infty \) the following relations hold:

\[
e_n(H_{p}^{r})_q = \sup_{f \in H_{p}^{r}} e_n(f)_q \asymp \begin{cases} 
    n^{-(r - 1/p + 1/2)}, & r > 1/p \\
    n^{-(r - 1/p + 1/q)/2}, & 1/p > r > 1/p - 1/q \\
    n^{-1/2} \ln n, & r = 1/p.
\end{cases}
\]
10.5. The following relations hold:

\[ e_n(W_{p}^{r,\beta})_q = \sup_{f \in W_{p}^{r,\beta}} e_n(f)_q \asymp \begin{cases} 
  n^{-(r-1/p+1/2)}, & r > 1/p \\
  n^{-(r-1/p+1/q)/2}, & 1/p > r > 1/p - 1/q \\
  n^{-1/2 \ln^{1-1/p} n}, & r = 1/p, p > 1 \\
  n^{-1/2 \ln n}, & r = p = 1.
\end{cases} \]

For the critical value \( r = 1/p \) the difference has come to light in approximate properties of \( H_{p}^{r} \) and \( W_{p}^{r,\beta} \) (cf. 10.4 and 10.5). This is for the first time in the one-dimensional case.

The same problem is solved also for the scale of Besov’s spaces (for definitions, see [M16]).

10.6. The following relations hold:

\[ e_n(B_{ps}^{r})_q = \sup_{f \in B_{ps}^{r}} e_n(f)_q \asymp \begin{cases} 
  n^{-(r-1/p+1/2)}, & r > 1/p \\
  n^{-(r-1/p+1/q)/2}, & 1/p > r > 1/p - 1/q \\
  n^{-1/2 \ln^{1-1/s} n}, & r = 1/p, 1 \leq s < \infty \\
  n^{-1/2}, & r = 1/p, 0 < s < 1.
\end{cases} \]

Hence for \( r \neq 1/p \) the degree of approximation is the same for each of the three classes and coincides with B. S. Kashin’s results, 1977-1981, on Kolmogorov’s widths for \( W_{p}^{r} \).

The proofs of the given theorems as well as the following ones are based on the next statement.

10.7. a) For any polynomial \( T(\theta_N) \) with spectrum \( \theta_N \), for every \( n \leq N \), and for every \( q \in (2, +\infty) \) there exist \( \theta_n \subset \theta_N \) and \( T(\theta_n) \) such that

\[ ||T(\theta_N) - T(\theta_n)||_q \leq \gamma(q)(N/n)^{1/2}||T(\theta_N)||_2. \]

b) For any polynomial \( T_N \) with spectrum in \( [0, N] \), for every \( n \in (1, N) \), and for every \( p \in [2, +\infty) \) there exist \( \theta_n \subset [0, 2N] \) and \( T(\theta_n) \) such that

\[ ||T_N - T(\theta_n)||_\infty \leq \gamma(p)((N/n) \ln(N/n))^{1/p}||T_N||_p. \]

It is obvious that \( e_n(f)_2 = (\sum_{k=n+1}^{\infty} (c_k^*)^2 2\pi)^{1/2} \), where \( \{c_k^*\}_\infty \) is decreasing rearrangement of \( \{|c_k(f)|\} \), the absolute values of Fourier coefficient of \( f \).

10.8. a) Let \( q \in (2, +\infty) \) and \( p \in (0, 1] \). Then

\[ \{c_k(f)\}_\infty \in l_p \text{ if and only if } \sum_{k=n+1}^{\infty} (2^{k(1/p-1/2)})^p e_{2^k}(f)_q < \infty. \]
b) If for some $\varepsilon > 0$ we have $\sum (|c_k(f)|(|\ln k|^{1+\varepsilon})^p < \infty$, then

$$\sum_{k=1}^{\infty} (2^{k(1/p-1/2)}e_{2^k}(f)_\infty)^p < \infty \quad \text{for } p \in (0,2/3),$$

$$\sum_{k=1}^{\infty} (2^{2k(1/p-1)}e_{2^k}(f)_\infty)^p < \infty \quad \text{for } p \in (2/3,1),$$

$$\sum_{k=1}^{\infty} (2^k e_{2^k}(f)_\infty)^{2/3k^{-1}} < \infty \quad \text{for } p = 2/3.$$

Let us go on to the multiple case. Let $r = (r_1,\ldots,r_m) \in \mathbb{R}^m_+$ and $0 < r_1 = r_2 = \ldots = r_\nu < r_{\nu+1} \leq \ldots \leq r_m$. We denote by $H_p^r$ the class of functions satisfying the conditions $\int_{-\pi}^{\pi} f(x) \, dx_j = 0$ for $1 \leq j \leq m$ and

$$||\Delta_h^l f||_p \leq \prod_{j=1}^{m} |h_j|^{r_j},$$

where $\Delta_h^l$ is the mixed difference with step $h_j$ in direction $Ox_j$, $1 \leq j \leq m$, and $\Delta_h^l = (\Delta_h^l)^l$ with $l \in \mathbb{N}$ and $l > r_m$. The study of problems of approximation of this class was started by K. I. Babenko and continued by S. A. Telyakovskii, Ya. S. Bugrov and others; see [M25].

10.9. Let $r_1 > 1/2$ and $p \in [2, +\infty]$. Then

$$\gamma_1(n^{-1} \ln^{\nu-1} n)^{r_1} \ln^{(\nu-1)/2} n \leq e_n(H_p^r)_\infty \leq \gamma_2(n^{-1} \ln^{\nu-1} n)^{r_1} \ln^{\nu/2} n.$$

A similar result is obtained also for any dimension for $e_n(W_{r,\beta}^p)_\infty$.

And at last let us give results in the final form for trigonometric widths in the multiple case. These results partially rehabilitate the trigonometric system as the best in the sense considered.

10.10. The following relations valid:

$$d_n^T(H_p^r)_q \leq \begin{cases} (n^{-1} \ln^{\nu-1} n)^{r_1-1/p+1/2} \ln^{(\nu-1)/2} n, & r_1 > 1, \ 1 \leq p < 2 < q < p' \\
(n^{-q/2} \ln^{(\nu-1)(q-1)/2} n)^{r_1-1/q+1/q} \ln^{(\nu-1)/q} n, & p = 1, \ 1 - 1/q < r_1 < 1, \ 2 < q < \infty \\
n^{-1/2} \ln^{\nu} n, & p = r_1 = 1, \ 2 < q < \infty. \end{cases}$$

The case $1 < p \leq q \leq 2$ was investigated earlier by V. N. Temlyakov, 1982.

10.11. The following relations valid:

$$d_n^T(W_{r,\beta}^p)_q \leq \begin{cases} (n^{-1} \ln^{\nu-1} n)^{r_1-1/p+1/2}, & r_1 > 1, \ 1 < p \leq 2 \leq q < p' \\
(n^{-1} \ln^{\nu-1} n)^{r_1-1/2} \ln^{(\nu-1)/2} n, & p = 1 < r_1, \ 2 < q < \infty \\
(n^{-q/2} \ln^{(\nu-1)(q-1)/2} n)^{r_1-1/q+1/q} \ln^{(\nu-1)/q} n, & p = 1, \ 1 - 1/q < r_1 < 1 \\
1/2, & p = 1, \ 1 < q < 2 \leq r_1. \end{cases}$$
Some problems are still open. But let us note that this field is ahead, in some respects, of results in Kolmogorov’s widths (for example, for \( r_1 = 1 \) having voluminous literature (see [M13] and [M27]).

**Bibliographical remarks.**

For 10.2-10.3, see [Be16]; for 10.4-10.6, 10.7a), see [Be12,13]; for 10.7b), see [Be10,20]; for 10.8, see [Be18]; for 10.9, see [Be15]; for 10.10, 10.11, see [Be11]. Pay attention also to [Be20].

For interesting applications to the problem of approximation by degenerate functions, see [M25].

Starting in 1984, methods of random choice are used here (E. Belinskii, Yu. Makovoz); see also [M11,12].

**11. The Pompeiu problem.**

Let \( A \) be an open bounded set in \( \mathbb{R}^m \) for \( m \geq 2 \), and \( M(m) \) be the group of Euclidean motions of \( \mathbb{R}^m \). The question arises (Pompeiu, 1929) if there exists a non-trivial locally integrable function \( f : \mathbb{R}^m \to \mathbb{C} \) such that

\[
(*) \quad \int_{\sigma A} f(x) \, dx = 0 \quad \text{for every } \sigma \in M(m).
\]

Are of certain interest connections of this problem and various of its versions with mean-value theorems for differential equations (see [S10]), with the classical Morera theorem in complex analysis (see 11.5 below), with uniqueness theorems (see 11.7), etc.

The existence of non-trivial function with (*) condition immediately yields an estimate of density of packing of an arbitrary compactum \( K \subset \mathbb{R}^m \) by sets \( \sigma A \in K \) (B. D. Kotlyar, 1984).

Taken in integral (*) a ball as \( A \) and \( f(x) = e^{i(x,y)} \) we get the Bessel (radial) function \( J_{m/2} \) (see [M22, Ch.IV, §4]), and hence condition (*) is satisfied on the countable set of balls (and their shifts) radia of which are defined by zeros of \( J_{m/2} \). There already exist a number of examples of sets \( A \) for which (*) implies that \( f \) is trivial, that is vanishes a.e. In particular, so is when the boundary of \( A \) is non-analytic (Williams, 1976; see also [S16]). Various criteria are also obtained.

**11.1.** a) If condition (*) is satisfied for some \( A \) and some non-trivial function of power growth, then the function \( f(x) = e^{i(x,y)} \) satisfies condition (*) for some \( y \in \mathbb{R}^m \).

b) Let for \( m = 2 \) a set \( A \) be star-like with respect to the origin, \( \rho = \rho(\varphi) \), with \( 0 \leq \varphi \leq 2\pi \), be the polar equation of \( \partial A \), where \( \rho \) is piece-wise continuous. In order that (*) implies that \( f \) is trivial, it is necessary and sufficient that there exists \( r > 0 \) such that

\[
\frac{2\pi}{0} e^{-ik\varphi} \, d\varphi \int_{0}^{\rho(\varphi)} tJ_k(tr) \, dt = 0
\]

for each \( k \in \mathbb{Z} \), where \( J_k \) is the Bessel function of order \( k \).

A similar criterion is obtained also for every \( m \geq 3 \).
11.2. Let $A$ be a set with connected complement. In order that there exists a non-
trivial function satisfying (*), it is necessary and sufficient the indicator of $A$ to be a limit in $L_1(\mathbb{R}^m)$ of a sequence of linear combinations of indicators of balls with radia proportional to positive zeros of $J_{m/2}$. The coefficient of proportionality depends only on $A$.

The problem becomes much more difficult if a function is defined not on the
whole space $\mathbb{R}^m$, where the Fourier transform methods are applicable, but on a
bounded domain.

11.3. Let $B_r$ be the open ball in $\mathbb{R}^m$ of radius $r$ with center at the origin and $f$ be
locally integrable in $B_r$.

a) If $A$ is the unit cube in $\mathbb{R}^m$ and (*) holds for every $\sigma \in M(m)$ such that
$\sigma A \subset B_r$, then for $r \geq \sqrt{m+3}/2$ we have $f = 0$ and for $r < \sqrt{m+3}/2$ there exist
nontrivial functions satisfying the given condition.

b) If $A$ is the unit semiball in $\mathbb{R}^m$ and (*) holds for every $\sigma \in M(m)$ such that
$\sigma A \subset B_r$, then for $r \geq \sqrt{m}/2$ we have $f = 0$ and for $r < \sqrt{m}/2$ there exist
nontrivial functions satisfying the given condition.

c) If $A$ is an ellipsoid in $\mathbb{R}^m$ and $\lambda$ and $\mu$ are its largest and its smallest halfaxis,
respectively; $R(x,y) = 2x$ for $y \leq \sqrt{2}x$ and $R(x,y) = y^2(y^2 - x^2)^{-1/2}$ for $y > \sqrt{2}$; and (*) holds for every $\sigma \in M(m)$ such that $\sigma A \subset B_r$, then for $r \geq R(\lambda, \mu)$ we have $f = 0$ and for $r < R(\lambda, \mu)$ there exist nontrivial functions satisfying the given condition.

Found is complete description of a class of functions with non-zero integrals over
all the balls of a given radius (in terms of coefficients of the Fourier series of the
function $f(\rho\eta)$ for $\rho > 0$ with respect to the spherical harmonic expansion of $\eta$ on
the unit sphere).

11.4. Let the integrals of $f$ over all the balls in $B_r$ of radia $r_1$ and $r_2$ vanish. If
$r_1 + r_2 < r$ and $r_1/r_2$ does not equal to ratio of two different zeros of $J_{m/2}$, then
$f = 0$. If $r_1 + r_2 > r$ or $r_1/r_2$ is equal to ratio of two different zeros of $J_{m/2}$, then
there exists a nontrivial function satisfying the condition indicated above.

The case $r_1 + r_2 = r$ is also completely investigated.

The Morera theorem can be strengthened as follows.

11.5. Let $D$ be the unit disc in $\mathbb{C}$ and $f \in C(D)$.

a) If the integrals over all the circles tangent to $\partial D$ vanish, then $f$ is analytic in $D$.

b) If the integral of $f$ over the boundary of every square in $D$ with a fixed side $d$
vanish, the $f$ is analytic for $d \leq 2/\sqrt{5}$ and not necessarily for $d > 2/\sqrt{5}$.

c) If the integral of $f$ over the boundary of every equisided triangular in $D$ with a
fixed side $d$ vanish, the $f$ is analytic for $d \leq 2/\sqrt{3}$ and not necessarily for $d > 2/\sqrt{3}$.

Similar results are obtained also for the multidimensional case in $\mathbb{C}^m$. Such
conditions are given for harmonic functions as well.

The following theorem strengthens a criterion of analyticity of functions due to
V. K. Dzyadyk, 1960.

11.6. Let real functions $u$ and $v$ be in $C^2(D)$, in the sense of real analysis, and
$r_1 + r_2 < 1$ and $r_1/r_2$ is not equal to ratio of two zeros of $J_1$. In order the function
$f = u + iv$ to be analytic in $D$, it is necessary and sufficient that areas of surfaces of
the form $x^2 + y^2 + r_1^2 = p(r_1/r_2, r_2)$ be connected for $p_0 < 1$.

Similar results are obtained also for the multidimensional case.
the graphs of functions \( u, v, \) and \( \sqrt{u^2 + v^2} \) placed over each disc from \( D \) of radius \( r_1 \) or \( r_2 \) to be equal.

Let us give an example of the uniqueness theorem for lacunary trigonometric system.

**11.7.** Let \( D \) be an arbitrary domain in \( \mathbb{R}^m \) and \( \lambda_k \in \mathbb{R}^m \) for all \( k \in \mathbb{N} \) be such that \( |\lambda_{k+1}| - |\lambda_k| \to \infty \) as \( k \to \infty \). Assume further that a sequence of polynomials \( \sum_{k=1}^{n} c_{k,n} e^{i(\lambda_k \cdot x)} \) converges weakly in \( L_1(D) \) as \( n \to \infty \) to a function \( f \) which is vanishing in some ball. Then \( f = 0 \) a.e. in \( D \).

**11.8.** Let \( f \) be locally integrable in \( \mathbb{R}^m \) and \( \{\nu_k\}_{1}^{\infty} \) be the sequence of all the positive zeros of the function \( J_{m/2} \). In order the integrals of \( f \) over all the balls from \( \mathbb{R}^m \) with radia \( \nu_1, \nu_2, ... \) to be zero, it is necessary and sufficient that \( f \) coincides a.e. with a solution of the Helmholtz equation \( \Delta f + f = 0 \).

**11.9.** Let \( G \) be a bounded Jordan domain in \( \mathbb{C} \) with rectifiable boundary \( \partial G \) and \( u = u(z) \) be a function harmonic in \( \mathbb{C} \setminus \overline{G} \) and continuous in \( \mathbb{C} \setminus G \). Let further \( u \) satisfy the following conditions: \( u = 0 \) on \( \partial G \), \( \frac{\partial u}{\partial n} = 1 \) a.e. on \( \partial G \), and \( u(z) = o(|z|^2) \) as \( z \to \infty \). Then \( G \) is the disk of center \( z_0 \) and radius \( R \) and \( u(z) = (1/R) \ln |(z - z_0)/R| \). The condition at infinity is sharp.

**Bibliographical remarks.**

For **11.1a),** see [Za2]; for **11.1b),** see [ZT2]; for **11.2, 11.8,** see [VaV8]; for **11.3,** see [VaV1,7]; for **11.4,** see [VaV9,10]; for **11.5,** see [VaV5,11]; for **11.6,** see [VaV5]; for **11.7,** see [VaV4]. See also [VaV3,6,12-14].

**12. Entire functions of exponential type and the Fourier integral.**

The Fourier transform of a function with compact support is the entire function of exponential type (EFET), see e.g., [M14].

Well-known are Wiener's and Cartright's theorems on sufficient conditions for membership to \( L_1 \) and \( L_\infty \), respectively, of restriction of EFET on \( \mathbb{R} \) according to the behavior of the function at equidistant points of \( \mathbb{R} \) (see the same references). Generalizations of these theorems are obtained. Let us give here only the asymptotic formula.

**12.1.** Let \( f \) be an entire function of exponential type not higher \( p\pi \) with \( p \in \mathbb{N} \),

\[
\lim_{y \to \pm \infty} |f(iy)| e^{-p|y|\pi} = 0 \quad \text{and} \quad \sum_{k \in \mathbb{Z}} |f^{(\nu)}(k)| < \infty
\]

for \( 0 \leq \nu \leq p - 1 \). Then for every \( n \in \mathbb{Z} \) and for every \( x \in [n, n + 1] \)

\[
f(x) = (-1)^n \pi^{-p} f^{(p)}(n) \sin^p(x\pi)/p! + \theta \sum_{k \in \mathbb{Z}} (k - n + 1/2)^{-2} \sum_{\nu=0}^{p-1} |f^{(\nu)}(k)|
\]

with \( |\theta| \leq \gamma(p) \).

If \( f \) is of type not higher than \( \sigma \), \( \lim x^{-1} f(x) = 0 \) as \( |x| \to \infty \), and \( \sup |f(k \pi/\sigma)| \to \infty \), then \( f \) is not obliged to be bounded on \( \mathbb{R} \) while the sum \( f(x) + f(x + \pi/\sigma) \) is always bounded.
bounded. This phenomenon was found by S. N. Bernstein, 1949, and he called it interference. There are general theorems on interference in $L_\infty$ (see [M2,13,23]). Let us consider the interference in $L$.

Let $L^1_{\pi}$ be a set of entire functions of exponential type not higher than $\pi$ such that $\lim f(x) = 0$ as $|x| \rightarrow \infty$ and $\sum_{k} |f(k)| < \infty$. Then

$$f(z) = (2\pi)^{-1} \int_{-\pi}^{\pi} \hat{f}(u)e^{-izu} du \quad \text{with} \quad \hat{f} \in A(\mathbb{T}).$$

Let us call a linear operator $\Lambda$ taking $L^1_{\pi}$ into itself to be interference if

$$\int_{-\infty}^{\infty} |(\Lambda f)(x)| \, dx \leq C \sum_{k=-\infty}^{\infty} |f(k)|.$$

12.2. In order that a linear operator $\Lambda$ continuous with respect to the uniform convergence on any compactum is interference and commuting with the unit shift operator, it is necessary and sufficient that it is representable in the form

$$(\Lambda f)(z) = \int_{-\pi}^{\pi} \lambda(u)\hat{f}(u)e^{-izu} du \quad \text{with} \quad \hat{\lambda} \in L(\mathbb{R}).$$

Besides that,

$$\int_{-\infty}^{\infty} |(\Lambda f)(x)| \, dx \leq \int_{-\infty}^{\infty} |\hat{\lambda}(u)| \, du \sum_{k=-\infty}^{\infty} |f(k)|$$

and the constant $||\hat{\lambda}||_1$ is exact on the class $L^1_{\pi}$.

The following theorem is the analog of B. Ya. Levin’s theorem on the $L_\infty$ space (see [M14]).

12.3. In order that there exists an entire function of exponential type not higher than $\pi$ in $L(\mathbb{R})$ and taking the values $c_k$ at the points $k \in \mathbb{Z}$ so that $\sum |c_k| < \infty$, it is necessary and sufficient that

$$\sum_{n \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} (-1)^{k+n}c_{k+n}(k^2 + 1) \right| < \infty.$$

Functions defined on $\mathbb{R}$ are approximated, following S. N. Bernstein, by EFET as their type $\sigma \rightarrow \infty$. In particular, if a function is periodical, then the approximating EFET in $L_\infty$ can be considered periodical, that is trigonometric polynomial.

The following theorem is an analog of 5.1.

12.4. Let $p \in [1, +\infty]$ and $f \in L^p(\mathbb{R})$. Then for any $r \in \mathbb{N}$ and for any $\sigma > 0$ there exists an entire function $g_\sigma$ of exponential type not higher than $\sigma$ such that

$$||f - g_\sigma||_p \asymp \omega_r(f; 1/\sigma)_p$$

with constants in the two-sided inequality depending only on $r$.

The following statements are very well known for $p \geq 1$ (see [M26,16]). The first one is on equivalence of integral and difference norms, while the second one is of Bernstein type.
12.5. a) For any entire function of exponential type not higher than \( \sigma \) and for any \( p \in (0, 1) \)

\[
\gamma_1(p) \|f\|_p \leq \left( \frac{\pi}{2\sigma} \right) \sum_k |f(k\pi/(2\sigma))|^{1/p} \leq \gamma_2(p) \|f\|_p.
\]

b) For any entire function of exponential type not higher than \( \sigma \) and for any \( p \in (0, 1) \)

\[
\|f'\|_p \leq \gamma(p) \sigma \|f\|_p.
\]

Statement a) for trigonometric polynomials was obtained by V. V. Peller (see [S13]), and b) for trigonometric polynomials was proved in 1975 independently by several authors, among them that with the exact constant \( \gamma(p) = 1 \) is due to V. V. Arestov, 1981. Also 12.5b) is apparently true with \( \gamma(p) = 1 \).

Methods of summability of Fourier integrals are often considered on place of the summability of the Fourier series. The question is when in some sense

\[
\int_{-\infty}^{\infty} \varphi(\varepsilon t) \hat{f}(t) e^{itx} dt \to f(x)
\]

as \( \varepsilon \to +0 \), where \( \varphi \) is a fixed function. If a (multiplier) function \( \varphi \) is of compact support, then the integral is EFET.

For all \( p > 0 \) the multiplier can be defined first on \( L_p \cap L_1(\mathbb{R}) \) which is dense in \( L_p(\mathbb{R}) \) and then to extend to \( L_p(\mathbb{R}) \) by continuity. The comparison principle may be formulated as well. Note only that for \( p \geq 1 \) there exists connection in both directions between summability methods (multipliers) of the Fourier series and the Fourier integrals generated by the same function \( \varphi \) (K. de Leeuw, 1965; see [M22]).

Let us consider now multipliers in the Hardy spaces \( H_p \) on the halfplane for \( p \in (0, 1] \).

Let \( f \) be analytic in the upper halfplane \( \{z : \Im z > 0\} \). We say that \( f \in H_p \) if

\[
\|f\|_{H_p}^p = \sup_{y > 0} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty.
\]

There exists a limit function, as \( y \to +0 \), which is in \( L_p(\mathbb{R}) \). A bounded measurable function \( \varphi : [0, +\infty) \to \mathbb{C} \) belongs to \( M_p \) if it defines a continuous multiplier taking \( H_p \) into \( H_p \).

12.6. If \( \varphi \in C(\mathbb{R}) \), \( \varphi(x) = 0 \) for \( |x| \geq \sigma \), and \( \hat{\varphi} \in L_p(\mathbb{R}) \), then \( \varphi \in M_p \) and for \( p \in (0, 1] \)

\[
\|\varphi\|_{M_p} \leq \gamma(p) \sigma^{1/p-1} \|\hat{\varphi}\|_p.
\]

Sufficient conditions similar to given in 6.5 are obtained too, and also in the multiple case. Weaker conditions of exponential decrease of \( \varphi \in C^\infty(\mathbb{R}) \) and its derivatives were obtained earlier by A. A. Solyanik, 1986.

Also an analog of the theorem 6.4 on approximation of the Fourier integrals of functions in \( H_p \) on the halfplane by the Bochner-Riesz means takes place. The following statement is similar to 9.8.
12.7. For every \( r \in \mathbb{N} \), for every \( f \in W^r_\infty[0, +\infty) \), and for every \( \sigma > 0 \) there exists an entire function \( g_\sigma \) of halfdegree not higher than \( \sigma \) satisfying for every \( x \in [0, +\infty) \) the inequality
\[
|f(x) - g_\sigma(x)| \leq 2^r K_r(\sqrt{x}/\sigma)^r + \gamma(r)(\sqrt{x}/\sigma)^{r-1}/\sigma^{r+1}.
\]

A general direct theorem, without exact constants in the main term, on approximation on the halfaxis by entire functions of given halfdegree as well as the inverse to it was obtained by Yu. A. Brudnyi, 1959 (see [M26]).

**Bibliographical remarks.**

For 12.1, see [T15,18]; 12.2-12.3 are due to G. Z. Ber, see [Ber] and 12.4 is due to V. V. Zhuk, Dokl. AN SSSR, 1967, in the case of nonlinear operators and due to G. Z. Ber [Ber] in the case of linear operators. 12.6 and further, 12.7 are due to A. V. Tovstolis, see [To], [TT].

**Concluding remarks.**

In conclusion, let us note that in these twelve sections some results are even not mentioned, such as those on approximation by Riesz means of the Fourier series with respect to eigenfunctions of the Sturm-Liouville operator (see [T15,18]), on the Hankel transform (see [T15,12]), on \( \varepsilon \)-entropy (see [Be15]), on best approximation by polynomials in \( L^2(\mathbb{R}) \) and Riemann’s hypothesis on zeta function (Val. Volchkov, Ukr. Math. J., 1995), on equivalence of differential operators in the complex and \( p \)-adic analysis (Vit. Volchkov), etc.

**13. Unsolved problems.**

1. A. Beurling, 1949, proved that if \( f \in C(\mathbb{R}) \), \( \lim_{|x| \to \infty} f(x) = 0 \), \( g \in A^*(\mathbb{R}) \) (for definition, see before 1.13), and for any \( x, h \in \mathbb{R} \) we have
\[
|f(x) - f(x + h)| \leq |g(x) - g(x + h)|,
\]
then \( f \in A(\mathbb{R}) \) (a new proof is given in [T17] for the multiple case at once). This yields that if \( \varphi \in \text{Lip} 1 \), \( \varphi(0) = 0 \), \( g \in A^*(\mathbb{R}) \), then \( \varphi \circ f \in A(\mathbb{R}) \).

   a) Is the converse statement true: if \( \varphi(g) \in A(\mathbb{R}) \) for any \( g \in A^*(\mathbb{R}) \), then \( \varphi \in \text{Lip} 1 \)?

   b) Does the following analog of Beurling’s theorem hold for summability of Fourier series (see 3.4 and 3.2): if \( \lim \lambda_{n,k} = 1 \) as \( n \to \infty \) for every \( k \), then we have for every \( f \in L_1(\mathbb{T}) \) at each of its Lebesgue points
\[
\sum \mu_{n,k} c_k(f)e^{ikx} \to f(x)
\]
as \( n \to \infty \) and \( |\lambda_{n,k} - \lambda_{n,k+s}| \leq |\mu_{n,k} - \mu_{n,k+s}| \), then for every \( f \in C(\mathbb{T}) \) and for every \( x \in \mathbb{T} \) we have
\[
\sum \lambda_{n,k} c_k(f)e^{ikx} \to f(x)
\]
as \( n \to \infty \)?

2. S. M. Nikolskii, 1946, has found the following formula for the norm of a multiplier in \( M_1(S) \) for \( S = \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\} \) in the compact case of \( A f = f * g \sim \sum \lambda_k c_k(f)e_k : \)
\[
||\{\lambda_k\}||_{M_1(\mathbb{Z}_0)} = (1/2) \max_x \int_0^\pi |g(x+t) - g(t)| \, dt,
\]
and by choice of the kernel \( g \) shown that the strict inequality \( \|\{\lambda_k\}\|_{M_1(Z_0)} < \|\{\lambda_k\}\|_{M(Z_0)} \). This yields immediately that in \( L_1 \) unlike in \( C \) not every multiplier is norm preserve extendible even in the case of the spectrum \( Z_0 \), the "closest" to \( Z \). A norm extension coefficient is connected in this case with the exact Jackson inequality in \( L_1 \) (see 2.4 and 2.5b)). In the same time for any \( S \subset Z \) every multiplier in \( M_1(S) \) is extendible to a multiplier in \( M_1 \) (see [M7]). For \( S = [-n, n] \) apparently there exist multipliers norms of which after extension enlarge to \( C \ln n \) times. For which \( p \neq 2 \) any multiplier in \( M_p(S) \) is extendible with preservation of norm or at least with the \( \gamma(p) \) growth of norm?

3. a) For which sequences \( \{\nu_k\}_0^\infty \) (see 3.6 we have

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^n S_{\nu_k}(f; x) = f(x)
\]

for every \( f \in L_1(\mathbb{T}) \) at all its Lebesgue points? This problem posed by Z. Zalcwasser in 1936 is still open. A similar question may be asked for \( \gamma \)-points introduced by O. D. Gabisonia (Math. Notes, 1973).

b) E. Landau, 1929, proved that for every \( f \in H_\infty = L_\infty(\mathbb{T}, \mathbb{Z}_+) \)

\[
\frac{1}{n+1} \sum_{k=0}^n |S_k(f; x)| \leq \|f\|_\infty.
\]

Is is true that for every \( f \in L_\infty(\mathbb{T}) \) the same inequality holds? I know this question from V. V. Zhuk. My conjecture is that the answer is negative.

c) For which \( \delta \geq 0 \) and \( p > 0 \) we have for every \( f \in L_\infty(\mathbb{T}^2) \)

\[
\frac{1}{n+1} \sum_{k=0}^n |S^\delta_k(f; x)|^p \leq \gamma(p, \delta)\|f\|_\infty^p.
\]

spherical Bochner-Riesz means are involved? For \( p = 1 \) and \( \delta = 0 \) (spherical partial sums) this inequality is apparently not true; cf. [Ku13].

d) All these questions have sense for functions in \( H_p \).

4. a) To construct a modulus of continuity \( \omega^* \) such that for every \( f \in C(\mathbb{T}) \)

\[
\|\frac{1}{n+1} \sum_{k=0}^n |f(\cdot) - S_k(f; \cdot(\cdot))|| \sim \omega^*(f; \pi/n).
\]

b) Which modulus of smoothness is defined by mixed derivative; see 5.6, 5.9, and 5.12?

c) Using the multiplier comparison principle, to prove that for Fourier-Chebyshev series on \([-1, 1]\)

\[
|f(x) - \sum_{k=0}^n (1 - k^2/n^2)c_k \cos k \arccos x| \leq \gamma \omega(f; \sqrt{1-x^2}/n + n^{-2}).
\]
To build in $C[-1,1]$ a sequence $\{p_n(f)\}$ of linear polynomial operators such that

$$|f(x) - p_n(f;x)| \asymp \omega_r(f; \sqrt{1 - x^2/n + n^{-2}}).$$

d) To formulate a comparison principle for Faber series, in accordance with position of a point.

5. a) To consider the problem of extension of multipliers with spectrum, in the Hardy space $H_p$ for $p > 0$.

b) For which $\alpha$ and $\beta$ the function $\varphi(x) = x^\alpha \sin x^{-\beta}$ belongs to $M_p$ in $H_p(D)$ near the origin and near the infinity? What are sufficient conditions of membership $\varphi \in M_p$, in the Hardy space for $p \in (0,1]$ near the origin and near the infinity (individually, on the class)?

c) The function $\varphi(x) = x_1(\sum_{j=1}^{m} x_j)^{-1} \in M_1$ in the Hardy space on the ball in $\mathbb{C}^m$ (see [M18], 6.6.3). To point out sufficient conditions of membership to $M_p$ which are satisfied by this function for $p = 1$.

6. Let $f$ be a function which is even, increasing, and convex upwards on $[0,\pi]$. To find an exact order of increase of $e_n(f)_\infty$.

7. a) Let $f^{(n-1)}(x) > 0$ on $[-1,1]$. What is the value

$$E_n(f)_1 = \inf_{p_n} \int_{-1}^{1} |f(x) - p_n(x)| \, dx$$

for $n \geq 2$? For $n = 0$ and $n = 1$ we have $E_0(f)_1 \leq (1/2) \int_0^{1} \omega(f; t)_\infty dt$ and $E_1(f)_1 \leq (1/4) \int_0^{1} \omega_2(f; t)_\infty dt$, respectively.

b) To find an analog of Müntz’ criterion, the completeness of the system of powers in $C[0,1]$, in $C[0,1]^m$ for $m \geq 2$, in $L_p$ with $p \in (0,1)$.

c) To prove general direct theorems on approximation by polynomials $Q_n$ with integral coefficients on sets in $\mathbb{C}$ (see §9). What is the rate of approximation of the constant by polynomials $Q_n$ on sets in $\mathbb{C}$? What is the value

$$q(a,b) = \lim_{n \to \infty} \inf_{Q_n \not\equiv 0} \max_{[a,b]} |Q_n(x)|^{1/n}$$

where $b - a < 4$? For discussion of these questions and their connection with distribution of prime numbers, see [T5].
References

Books:

[M1] N. I. Akhiezer, *Classical moment problem*, Fizmatgiz, Moscow, 1961. (Russian)

[M2] ———, *Lectures in Approximation Theory*, 2nd ed., Nauka, Moscow, 1965 (Russian); German transl. *Vorlesungen über Approximationstheorie*, Akademie Verlag, Berlin, 1967.

[M3] N. K. Bary, *Trigonometricheskie ryady*, Fizmatgiz, Moscow, 1961; English transl. *A treatise on trigonometric series*, Vol. I, II, Pergamon Press, New York, 1964.

[M4] J. Bergh, J. Lőfström, *Interpolation Spaces. An introduction.*, Springer-Verlag, Berlin, 1976.

[M5] K. M. Davis, Y.-C. Chang, *Lectures on Bochner-Riesz means*, London Math. Soc. Lecture Note Series 114, Cambridge Univ. Press, Cambridge, 1987.

[M6] V. K. Dzjadyk, *An Introduction to Theory of Uniform Approximation of Functions by Polynomials*, Nauka, Moscow, 1977. (Russian)

[M7] R. E. Edwards, *Fourier series*, 2nd ed. Vol.1-1979; Vol.2-1982, Springer-Verlag.

[M8] Le Baron O. Ferguson, *Approximation by Polynomials with Integral Coefficients*, Math. Surveys, 17, Amer. Math. Soc., Providence, 1980.

[M9] B. I. Golubov, A. V. Efimov, V. A. Skvortsov, *Walsh series and transforms. Theory and applications*, Nauka, Moscow, 1987. (Russian)

[M10] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Springer - Verlag, 1970.

[M11] ———, *Some random series of functions*, Heath, Lexington, Mass., 1968.

[M12] B. S. Kashin, A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow, 1984 (Russian); English transl. Transl. Math. Monographs, 75; AMS, Providence, 1989.

[M13] N. P. Korneichuk, *Exact Constants in Approximation Theory*, Nauka, Moscow, 1987 (Russian); English transl. Encycl. Math. 38; Cambridge Univ. Press, Cambridge, 1991.

[M14] B. Y. Levin, *Entire Functions (course of lectures)*, Moscow Univ. Press, Moscow, 1971. (Russian)

[M15] E. Lucacs, *Characteristic functions*, Griffin, London, 1970.

[M16] S. M. Nikolskii, *Approximation of Functions of Several Variables and Imbedding Theorems*, 2nd ed., Nauka, Moscow, 1977 (Russian); English translation of the 1st ed. Springer - Verlag, Berlin-Heidelberg-New York, 1975.

[M17] W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969.

[M18] ———, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Grund. der Math. Wiss. 241; Springer-Verlag, New York, 1980.

[M19] I. A. Shevchuk, *Approximation by Polynomials and Traces of Functions Continuous on Interval*, Naukova Dumka, Kiev, 1992. (Russian)

[M20] P. K. Suetin, *Classical orthogonal polynomials*, Nauka, Moscow, 1979. (Russian)

[M21] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J., 1970.

[M22] E. M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Presss, Princeton, N. J., 1971.

[M23] A. I. Stepanets, *Uniform Approximations by Trigonometric Polynomials*, Naukova Dumka, Kiev, 1981. (Russian)

[M24] ———, *Classification and Approximation of Periodic Functions*, Naukova Dumka, Kiev, 1987. (Russian)

[M25] V. N. Temlyakov, *Approximation of Functions with Bounded Mixed Derivative*, Vol. 178, Trudy Mat. Inst. im. V. A. Steklova, Moscow, 1986 (Russian); English translation in *Proc. of the Steklov Inst. of Math.*, 1989.

[M26] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Fizmatgiz, Moscow, 1960 (Russian); English transl. Oxford, 1963.

[M27] V. M. Tikhomirov, *Certain Questions of Approximation Theory*, Moscow Univ. Press, Moscow, 1976. (Russian)

[M28] N. Wiener, *The Fourier Integral and Certain of its Applications*, Camb. Univ. Press, Cambridge, 1933.

[M29] W. Trebels, *Multipliers for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation Theory*, Lecture Notes Math. 329; Springer, Berlin, 1973.

[M30] R. Varga, *Functional Analysis and Approximation Theory in Numerical Analysis*, SIAM, Philadelphia, 1971.
Surveys:

[S1] S. A. Alimov, R. R. Ashurov, A. K. Pulatov, *Multiple Fourier series and integrals*, Itogy Nauki i Techniki, vol. 42, VINITI, Moscow, 1989, pp. 7 – 104 (Russian); English translation in V. P. Khavin, N. K. Nikolskii (Eds.), *Commutative Harmonic Analysis IV*, Encycl. Math. Sciences, Vol.42; Springer-Verlag, New York, 1992, pp. 1–95.

[S2] K. I. Babenko, *Some Questions of Approximation Theory and Numerical Analysis*, Uspekhi Mat. Nauk 40:1 (1985), 3–27 (Russian); English translation in Russian Math. Surveys 40:1 (1985).

[S3] Y. A. Brudnyi, A. F. Timan’s results in polynomial approximation of functions, Materials of All-Union Conference in the Theory of Approximation of Functions. 26-29 of June, 1990, Dnepropetrovsk, 1991, pp. 13–17. (Russian)

[S4] M. I. Dyachenko, *Some problems in the theory of multiple trigonometric series*, Uspekhi Mat. Nauk 47:5 (1992), 97–162 (Russian); English translation in Russian Math. Surveys 47:5 (1992), 103–171.

[S5] A. O. Gelfond, *On uniform approximation by polynomials with integer coefficients* (1955), Selected papers, Nauka, Moscow, 1973, pp. 287–309. (Russian)

[S6] B. I. Golubov, *Multiple Fourier series and integrals*, Itogy Nauki i Techniki, Matematicheskii Analiz., vol. 19, VINITI, Moscow, 1982, pp. 3 – 54 (Russian); English translation in J. of Soviet Math. 24 (1984), no. 6, 639 – 673.

[S7] R. S. Ismagilov, *Widths of sets in linear normed spaces and approximation of functions by trigonometric polynomials*, Uspekhi Mat. Nauk 29:3 (1974), 161–178 (Russian); English translation in Russian Math. Surveys 29:3 (1974), 103–171.

[S8] V. G. Krotov, E. A. Storozenko, *Approximate and differential-difference properties of functions from Hardy spaces*, Theory of functions and approximations, Part 1, Proc. of Saratov winter school, Jan.25–Febr.5,1982, Saratov Univ. Press, 1983, pp. 65–80. (Russian)

[S9] J. Misiewicz, C. Scheffer, *Pseudo isotropic measures*, Nieuw Archief voor Wiskunde 8/2 (1990), 111–152.

[S10] I. Netuka, J. Vesely, *Mean value property and harmonic functions*, Preprint (1994).

[S11] A. M. Olevsky, *Modifications of functions and Fourier series*, Uspekhi Mat. Nauk 40:3 (1985), 157–193 (Russian); English translation in Russian Math. Surveys 40:3 (1985), 103–171.

[S12] M. K. Potapov, *Approximation by polynomials on a finite interval of the real axis*, Proc. of the Intern. Conference in Constructive Theory of Functions. 1981, Sofia, 1983, pp. 134–138. (Russian)

[S13] V. V. Peller, *Description of Hankel operators of the class $G_p$ for $p > 0$, investigation of speed of rational approximation and other applications*, Mat. Sbornik 122 (1983), no. 4, 485–510 (Russian); English translation in Soviet Math. Sbornik 122 (1983), no. 4.

[S14] S. A. Telyakovskii, *An estimate, of the norm of a function via the Fourier coefficients of the function, convenient in problems of Approximation Theory*, Trudy Mat. Inst. im. V. A. Steklova, vol. 109, Nauka, Moscow, 1971, pp. 645–97 (Russian); English translation in *Proc. of the Steklov Inst. of Math.*, vol. 109.

[S15] , *On approximation of functions of high smoothness by Fourier sums*, Ukrain. Mat. Z. 41 (1989), no. 4, 510–517 (Russian); English translation in Ukrainian Math. J. 41 (1989), no. 4.

[S16] L. Zalcman, *A bibliographic survey of the Pompeiu problem*, Approximation by Solutions of Partial Differential Equations, Kluwer Acad. Publ., 1992, pp. 185–194.

Papers

[T1] R. M. Trigub, *Approximation of function by polynomials with integral coefficients*, Izv. Akad. Nauk SSSR, ser. matem. 26 (1962), no. 2, 261–280. (Russian)

[T2] , *Constructive characteristics of some function classes*, Izv. Akad. Nauk SSSR, ser. matem. 29 (1965), no. 3, 645–679. (Russian)
Linear summation methods and absolute convergence of Fourier series, Izv. Akad. Nauk SSSR, ser. matem. 32 (1968), no. 1, 24–49. (Russian)

Summability and absolute convergence of Fourier series in whole, Metric Questions in the Theory of Functions and Mappings, Nauk. dumka, Kiev, 1971, pp. 173–266. (Russian)

Approximation of functions with Diophantine conditions by polynomials with integral coefficients, Metric Questions of the Theory of Functions and Mappings, Nauk. dumka, Kiev, 1971, pp. 267–333. (Russian)

On linear methods of summability of Fourier series and moduli of continuity of different orders, Sib. Mat. Zh. 12 (1971), no. 6, 1416–1421. (Russian)

Characteristics of Lipschitz classes of integer order on the interval by the rate of polynomial approximation, Theory of functions, functional analysis and their applications, vol. 18, Kharkov, 1973, pp. 63–70. (Russian)

Integrability of the Fourier transform of a boundedly supported function, Theory of functions, functional analysis and their applications, vol. 23, Kharkov, 1975, pp. 124–131. (Russian)

Summability of Fourier series at Lebesgue points and one Banach algebra, Metric Questions in the Theory of Functions and Mappings, vol. 6, Nauk. dumka, Kiev, 1975, pp. 125–135. (Russian)

On integral norms for polynomials, Mat. Sbornik 101(143) (1976), no. 3, 315–333 (Russian); Engl. transl. in, Math. USSR Sbornik 30 (1976), no. 3, 279–295.

On approximation of functions by polynomials with special coefficients, Izv. VUZ, matem. (1977), no. 1, 93–99. (Russian)

Linear methods for the summation of simple and multiple Fourier series and their approximative properties, Theory of Approximation of Functions (Proc. Internat. Conf., Kaluga, 1975), Nauka, Moscow, 1977, pp. 383–390. (Russian)

Integrability and asymptotic behavior of Fourier transform of radial function, Metric Questions of the Theory of Functions and Mappings, Nauk. dumka, Kiev, 1977, pp. 142–163. (Russian)

Summability of multiple Fourier series, Investigations in the theory of functions of many real variables, vol. 2, Yaroslavl, 1978, pp. 196–214. (Russian)

Comparison principle and some questions of the theory of approximation of functions, Theory of Functions and Mappings, Nauk. dumka, Kiev, 1979, pp. 149–173. (Russian)

Summability of multiple Fourier series. Growth of Lebesgue constants, Analysis Math. 6 (1980), no. 3, 255 – 267.

Absolute convergence of Fourier integrals, summability of Fourier series and polynomial approximation of functions on the torus, Izv.Akad. Nauk SSSR, Ser. Mat. 44 (1980), no. 6, 1378–1409 (Russian); Engl.transl. in, Math. USSR Izv. 17 (1981), no. 3, 567–593.

Summability of Fourier series and certain questions of approximation theory, All-Union Institute of Scientific and Technical Information, No. 5145–80, 1980. (Russian)

Absolute convergence of Fourier integrals and approximation of functions by the linear means of their Fourier series, Constructive Function Theory’81, Sofia, 1983, pp. 178–180. (Russian)

On the comparison principle for Fourier expansions, Proc. of the Seminar of the I. N. Vekua Inst. of Applied Math., vol. 1, 1985, pp. 439–443. (Russian)

Some properties of the Fourier transform of measure and their applications, Proc. Int. Conf. on the Theory of Approx. of Functions, Nauka, Moscow, 1987, pp. 439–443. (Russian)

On the comparison principle of the Fourier expansions and existence subspaces in integral metric, Trudy Mat. Inst. im. V. A. Steklova, vol. 180, 1987, pp. 151–152 (Russian); Engl.transl. in Proc. Steklov Math. Inst. 180 (1989), 176–177.

A formula for the K-functional of a couple of spaces of functions of several variables, Investigations in the theory of functions of many variables, Yaroslavl, 1988, pp. 122–127. (Russian)

Multipliers of Fourier series and approximation of functions by polynomials in spaces $C$ and $L^p$, Akad. Nauk SSSR 296 (1980), 293–296 (Russian); Engl. transl. in Proc. Steklov Math. Inst. 180 (1989), 176–177.
in, Soviet Math. Dokl. 39 (1989), no. 3, 494–498.

[T25] ______, Asymptotics of a sequence of the norms of multipliers of the Fourier series in the spaces $C$ and $L$, Abstracts of the All-Union School in Lutsk "Theory of Approximation of Functions", Math. Inst., Kiev, 1989. (Russian)

[T26] ______, Criterion of characteristic function and the Polya type test for radial functions of several variables, Probability theory and its applications 34 (1989), no. 4, 805–810. (Russian)

[T27] ______, Summability of the Fourier series and some questions in approximation theory, DS thesis, Kiev, 1984. (Russian)

[T28] ______, Approximation of continuous periodic functions with bounded derivative by polynomials, Theory of Mappings and Approximation of Functions, Nauk. dumka, Kiev, 1989, pp. 185–195. (Russian)

[T29] ______, Multipliers of the Fourier series, Abstracts of the Intern. Conf. in Constr. Theory of Functions, Varna, 1991. (Russian)

[T30] ______, Multipliers of the Fourier series, Ukr. Mat. Zh. (1991), no. 12, 1686–1693. (Russian)

[T31] ______, Investigations of A. F. Timan on the Lebesgue constants and linear methods of summability of the Fourier series, Materials of the All-Union Conference on the Theory of Approximation of Functions, Dneprpetrovsk, 1991, pp. 8–12. (Russian)

[T32] ______, Two-sided estimates of approximation by polynomials and interpolation of spaces of smooth functions on the torus, Materials of the All-Union Conference on the Theory of Approximation of Functions, Dneprpetrovsk, 1991, pp. 70–71. (Russian)

[T33] ______, Positive definite functions and splines, Theory of functions and approximations. Proc. of the 5th Saratov winter school Jan.25-Feb.4, 1990, Part 1, Saratov, 1992, pp. 68–75. (Russian)

[T34] ______, Direct theorems on approximation by algebraic polynomials of smooth functions on interval, Matem. Zametki 54 (1993), no. 6, 113–121 (Russian); Engl. transl. in, Math. Notes 54 (1993), no. 5-6.

[T35] ______, Multipliers in the Hardy spaces $H_p(D^m)$ for $p \in (0, 1]$ and approximate properties of summability methods of power series, Dokl. AN Rossii 335 (1994), no. 6, 697–699. (Russian)

[T36] ______, The Multipliers of Power Series in Hardy Spaces and Approximation Problems in the Polydisk, Special Semester in Approx. Theory, Tecnion, Haifa, 1994.

See also below [BT1,2], [BLT], [KT], [ZT1,2], [ZgT], [TT1].

[Be1] Belinskii E.S., Summability of multiple Fourier series at Lebesgue points, Theory of functions, functional analysis and their applications, vol. 23, Kharkov, 1975. (Russian)

[Be2] ______, On asymptotic behavior of integral norms of trigonometric polynomials, Metric Questions in the Theory of Functions and Mappings, Naukova dumka, Kiev, 1975. (Russian)

[Be3] ______, Approximation by Bochner-Riesz means and spherical modulus of continuity, Dolk. AN Ukraine, Ser. A (1975), no. 7. (Russian)

[Be4] ______, An application of the Fourier transform to summability of Fourier series, Sib. Mat. Zh. (1977), no. 3 (Russian); Engl. transl. in, Siberian Math. J. 18 (1977), no. 3, 353-363.

[Be5] ______, Behavior of Lebesgue constants of some methods of summation of multiple Fourier series, Metric Questions in the Theory of Functions and Mappings, Naukova Dumka, Kiev, 1977. (Russian)

[Be6] ______, An application of the Fourier transform to summability of Fourier series, PhD thesis, Donetsk, 1977. (Russian)

[Be7] ______, On some properties of hyperbolic partial sums, Theory of Functions and Mappings, Naukova dumka, Kiev, 1979. (Russian)

[Be8] ______, On the growth of Lebesgue constants of partial sums, generated by some unbounded sets, Theory of Mappings and Approximation of Functions, Naukova dumka, Kiev, 1983. (Russian)

[Be9] ______, On the summability of Fourier series with the method of lacunary arithmetic means, Analysis Math. 18 (1992) no. 4, 275–289.
Be10  Approximation of periodic functions by the "floating" system of exponents and trigonometric widths, Investigations in the Theory of functions of several real variables, Yaroslavl, 1984. (Russian)

Be11  Approximation of periodic functions of several variables by a "floating" system of exponentials and trigonometric widths, Dokl. AN SSSR 284 (1985), no. 6 (Russian); Engl. transl. in Soviet Math. Dokl. 32 (1985), no. 2, 571-574.

Be12  Approximation by the "floating" system of exponents on the classes of smooth periodic functions, Mat. sbornik 132 (1987), no. 1 (Russian); Engl. transl. in Math. USSR Sbornik 60 (1988), no. 1.

Be13  Approximation by "floating" system of exponents on the classes of periodic functions with bounded mixed derivative, Investigations in the Theory of Functions of several real variables, Yaroslavl, 1988. (Russian)

Be14  Lebesgue constants of "step-hyperbolic" partial sums, Theory of Functions and Mappings, Naukova dumka, Kiev, 1989. (Russian)

Be15  Approximation of functions of several variables by trigonometric polynomials with given number of harmonics, Analysis Math. 15 (1989), 67-74.

Be16  Two extremal problems for trigonometric polynomials with given number of harmonics, Mat. Zametki 49 (1991), no. 1, 12-19 (Russian); Engl. transl. in Math. Notes Acad. Sci. USSR 49 (1991).

Be17  Asymptotic characteristic of classes of functions with conditions on mixed derivatives (mixed difference), Investigations in the theory of functions of several real variables, Yaroslavl, 1990, pp. 22-37. (Russian)

Be18  Classes $A^p$ and their constructive characteristics, Materials of the All Union Conference on the Theory of Approximation of Functions, Dnepropetrovsk, 1991. (Russian)

Be19  On strong summability of periodic functions and embedding theorems, Dokl. Ross. Akad. Nauk 332 (1993), no. 2 (Russian); Engl. transl. in Russian Acad. Sci. Dokl. Math. 48 (1994), no. 2, 255-258.

Be20  Decomposition theorems and approximation by "floating" system of exponentials, Trans. Amer. Math. Soc. (accepted).

BL1  E. S. Belinskii and E. R. Liflyand, Lebesgue constants and integrability of Fourier transform of radial functions, Dokl. Acad. Sci. of Ukraine, Ser. A (1980), no. 6, 5–10. (Russian)

BL2  On asymptotic behavior of Lebesgue constants of radial summability methods, Constructive Theory of Functions and Theory of Mappings, Nauk. dumka, Kiev, 1981, pp. 49–70. (Russian)

BL3  Behavior of the Lebesgue Constants of Hyperbolic Partial Sums, Mat. Zametki 43 (1988), no. 2, 192-196 (Russian); Engl. transl. in Math. Notes Acad. Sci. USSR 43 (1988), 107–109.

BL4  Approximation properties in $L_p$, $0 < p < 1$, Funct. et Appr. XXII (1994).

BLT  E. S. Belinskii, E. R. Liflyand, R. M. Trigub, The Banach algebra $A^*$ and its properties, accepted in the J. Fourier Anal. Appl.

BT1  E. S. Belinskii and R. M. Trigub, Some numerical inequalities and their applications to the theory of summability of Fourier series, Constructive Theory of Functions and Theory of Mappings, Nauk. dumka, Kiev, 1981, pp. 70–81. (Russian)

BT2  Summability on Lebesgue set and one Banach algebra, Theory of functions and approximations, Part 2, Saratov, 1983, pp. 29–34. (Russian)

Ku1  O. I. Kuznetsova, paper On some properties of polynomial operators of triangular form in a space of continuous periodic functions of two variables, Dokl. Akad. Nauk SSSR 223 (1975), no. 6, 1304–1306 (Russian); Engl. transl. in Soviet Math. Dokl. 16 (1975), no. 4, 1080–1083.

Ku2  The Asymptotic Behavior of the Lebesgue Constants for a Sequence of Triangular Partial Sums of Double Fourier Series, Sib. Mat. Zh. XVIII (1977), no. 3, 629–636 (Russian); Engl. transl. in Siberian Math. J. 18 (1977), 440–454.
[Ku3] Lebesgue constants and approximate properties of linear means of multiple Fourier series, PhD thesis, Donetsk, 1985. (Russian)

[Ku4] On one condition of integrability of multiple trigonometric series, Proc. of the Seminar of the I. N. Vekua Inst. of Applied Math., vol. 1, 1985, pp. 87–90. (Russian)

[Ku5] On asymptotics of approximation of differentiable functions, Proc. Int. Conf. on the Theory of Approx. of Functions, Nauka, Moscow, 1987, pp. 243–245.

[Ku6] Integrability and strong summability of multiple trigonometric series, Trudy Mat. Inst. im. V. A. Steklova, vol. 180, 1987, pp. 143–144 (Russian); English translation in Proc. Steklov Math. Inst. 180 (1989), 168–169.

[Ku7] Asymptotic approximation of smooth functions, Theory of Mappings and Approximation of Functions, Nauk. dumka, Kiev, 1989, pp. 75–81. (Russian)

[Ku8] On the strong Karleman means of multiple trigonometric Fourier series, Ukr. Mat. Zh. 44 (1992), no. 2, 275–279. (Russian)

[Ku9] Strong summability with gaps of multiple Fourier series, Proc. of the Seminar of the I. N. Vekua Inst. of Applied Math., vol. 7, 1992. (Russian)

[Ku10] On partial sums with respect to polyhedra of the Fourier series of bounded functions, Anal. Math. 19 (1993), no. 4, 267–272.

[Ku11] On the growth of partial sums by polyhedra of Fourier series of bounded functions (1994), Abstracts of ICM94, Zürich.

[Ku12] The asymptotic behavior of the Lebesgue function of two variables, Ukr. Mat. Zh. 47 (1995), no. 2, 220–226. (Russian)

[Ku13] To the question of the strong summability by circles, accepted in Ukr. Mat. Zh.

[KT] O. I. Kuznetsova, R. M. Trigub, Two-sided estimates of approximation of functions by Riesz and Marzinkiewicz means, Dokl. AN SSSR 251 (1980), no. 1, 34–36 (Russian); Engl. transl. in, Soviet Math. Dokl.

[L1] E. R. Liflyand, On integrability of the Fourier transform of a function of compact support and summability of Fourier series of functions of two variables, Metric Questions in the Theory of Functions and Mappings, vol. 6, Nauk. dumka, Kiev, 1975, pp. 69–81. (Russian)

[L2] Some questions of absolute convergence of multiple Fourier integrals, Theory of Functions and Mappings, Nauk. dumka, Kiev, 1979, pp. 110–132. (Russian)

[L3] On certain conditions of integrability of the Fourier transform, Ukrainian Math. J. 32 (1980), no. 1, 110–118. (Russian)

[L4] Absolute convergence and summability of multiple Fourier series and Fourier integrals, PhD thesis, Donetsk, 1982. (Russian)

[L5] Absolute convergence and summability with respect to spheres of multiple Fourier integrals of weakly discontinuous functions, Theory of Mappings and Approximation of Functions, Nauk. dumka, Kiev, 1983, pp. 77–88. (Russian)

[L6] Exact order of the Lebesgue constants of hyperbolic partial sums of multiple Fourier series, Mat. Zametki 39 (1986), no. 5, 674–683 (Russian); Engl. transl. in Math. Notes Acad. Sci. USSR 39 (1986), no. 5-6, 369 – 374.

[L7] Sharp estimates of the Lebesgue constants of partial sums of multiple Fourier series, Trudy Mat. Inst. im. V. A. Steklova, vol. 180, 1987, pp. 151–152 (Russian); Engl. transl. in Proc. Steklov Math. Inst. 180 (1989), 176–177.

[L8] Order of growth of Lebesgue constants of hyperbolic means of multiple Fourier series, Materials of the All-Union Conference on the Theory of Approximation of Functions, Dneprpropetrovsk, 1991, pp. 59–61. (Russian)

[L9] On asymptotics of Fourier transforms for functions of certain classes, Anal. Math. 19 (1993), no. 2, 151–168.

[L10] On the operator of division by a power function in the multi-dimensional case, Mat. Zametki 56 (1994), no. 2, 82–90 (Russian); Engl. transl. in Math. Notes Acad. Sci. Russia 56 (1994), no. 1-2.

[L11] Estimates of Lebesgue constants via Fourier transforms. Many dimensions, Preprint, Bar-Ilan Univ. (1995).

See also [BL1-4], [BLT].
Yu. L. Nosenko, *On comparison of linear methods of summability of double Fourier series*, Metric Questions in the Theory of Functions and Mappings, vol. 6, Nauk. dumka, Kiev, 1975, pp. 89–101. (Russian)

Nosenko, *Approximate properties of the Riesz means of double Fourier series*, Ukr. Mat. Zh. (1979), no. 2, 157–165. (Russian)

Nosenko, *Concerning the Sidon-type conditions for integrability of double trigonometric series*, Theory of Functions and Mappings, Nauk. dumka, Kiev, 1979, pp. 132–149. (Russian)

Nosenko, *The exact order of deviation of continuous functions of two variables from their rectangular Riesz means*, Constructive Theory of Functions and Theory of Mappings, Nauk. dumka, Kiev, 1981, pp. 129–134. (Russian)

Nosenko, *Summability of double Fourier series by Bernstein-Rogozinski type methods*, Theory of functions and approximations. Proc. of the 1st Saratov winter school Jan.24-Feb.5, 1982, Part 2, Saratov, 1983, pp. 115–118. (Russian)

Nosenko, *Regularity and approximate properties of linear methods of summability of double Fourier series*, PhD thesis, Donetsk, 1983. (Russian)

Nosenko, *Regularity and approximate properties of Bernstein-Rogozinski type methods of double Fourier series*, Ukr. Mat. Zh. 37 (1985), no. 5, 599–604. (Russian)

Nosenko, *Sufficient conditions of integrability of double trigonometric series*, Theory of functions and approximations. Proc. of the 2nd Saratov winter school, 1984, Part 3, Saratov, 1986, pp. 55–59. (Russian)

Nosenko, *Sufficient conditions of integrability of double cosine trigonometric series*, Trudy Mat. Inst. im. V. A. Steklova, vol. 180, 1987, pp. 166–168 (Russian); Engl. transl. in Proc. Steklov Math. Inst. 180 (1989).

Nosenko, *Regularity of Bernstein-Rogozinski type means of double Fourier series of continuous functions*, Theory of Mappings and Approximation of Functions, Nauk. dumka, Kiev, 1989, pp. 133–142. (Russian)

Nosenko, *Approximation of functions by Riesz means of their Fourier series*, Harmonic Analysis and Progress in Approximate Methods, Math. Inst. AN Ukraine, Kiev, 1989, pp. 83–84. (Russian)

Nosenko, *Deviations of continuous functions from some means of their Fourier series*, Materials of the All-Union Conference on the Theory of Approximation of Functions, Dnepropetrovsk, 1991, pp. 61–62. (Russian)

Nosenko, *On approximation of functions by \((C, \alpha)\) means of their Fourier series*, Abstracts of the 6th annual IWAA, Maine, Orono, USA, 1992.

Nosenko, *Approximation of functions by Riesz, \((C, \alpha)\) and typical means of Fourier series of these functions*, Buletinul Stiintific al Univ. din Baia mare, ser B, Matem. Inform X, Romania, 1994, pp. 89–92.

See also [ZN]

V. A. Glukhov, *Comparison of \((C, \alpha)\) means and Nikolskii type summability test for Fourier series with respect to multiplicative systems*, Theory of functions and approximations. Proc. of the 2nd Saratov winter school, 1984, Part 3, Saratov, 1986, pp. 78–84. (Russian)

Glukhov, *On summability of multiple Fourier series with respect to multiplicative systems*, Mat. Zametki 39 (1986), no. 5, 665–673 (Russian); Engl. transl. in Math. Notes Acad. Sci. USSR 39 (1986), no. 5-6, 361–368.

Glukhov, *On summation of Fourier-Walsh series*, Ukr. Mat. Zh. (1986), no. 3, 303–309.

Glukhov, *Some questions of summability of simple and multiple Fourier series with respect to multiplicative systems*, PhD thesis, Donetsk, 1987. (Russian)

Glukhov, *Strong summability of Fourier series with respect to periodic multiplicative systems*, Theory of Mappings and Approximation of Functions, Nauk. dumka, Kiev, 1989, pp. 40–49. (Russian)

Glukhov, *Uniform convergence of Fourier-Walsh series of composition of functions*, Materials of the All-Union Conference on the Theory of Approximation of Functions, Dnepropetrovsk, 1991, pp. 36–37. (Russian)
Approximation of functions on bounded domains in $\mathbb{R}^n$ by linear combination of shifts, Dokl. AN Russia 334 (1994), no. 5, 560–561.

V. O. Leontyev, The second term in the Kolmogorov asymptotic formula for approximation by partial sums of the Fourier series, Dokl. AN Ukraine, ser. A (1990), no. 5, 17–21. (Russian)

Extension of a multiplier from the spectrum, Materials of the All-Union Conference on the Theory of Approximation of Functions, Dnipropetrovsk, 1991, pp. 58–59. (Russian)

Asymptotic approximations of differentiable functions by the Fourier series, PhD thesis, Donetsk, 1991. (Russian)

Asymptotic approximations of differentiable functions by the Fourier series, Dokl. AN Russia 326 (1992), no. 1. (Russian)

G. Z. Ber, On the interference phenomenon in integral metric and approximation by entire functions of exponential type, Theory of functions, functional analysis and their applications, vol. 34, Kharkov, 1980, pp. 11–24. (Russian)

A. M. Belenkii, On expansion of functions in the Fourier-Jacobi series, Constructive Theory of Functions and Theory of Mappings, Nauk. dumka, Kiev, 1981, pp. 35–48. (Russian)

On uniform convergence of Fourier-Jacobi series on the interval of orthogonality, Mat. Zametki 46 (1989), no. 6, 18–26 (Russian); Engl. transl. in Math. Notes Acad. Sci. USSR 46 (1989), no. 5-6.

N. A. Zagorodnii, R. M. Trigub, On one Salem’s question, Theory of Functions and Mappings, Nauk. dumka, Kiev, 1979, pp. 97–101. (Russian)

E. M. Klebanov, Estimates of approximation by linear means of Fourier series exact on class, Deposited in GNTB Ukraine No. 175-Uk, 1994. (Russian)

Vit. V. Volchkov, Multipliers of power series in Hardy spaces in the unit ball in $\mathbb{C}^n$, Abstracts of scientific conference of Donetsk Univ., April 1995, 186–187. (Russian)

A. V. Tovstolis, Multipliers in the Hardy spaces $H_p$ in the upper halfspace for $p \in (0,1]$ and approximation by the Bochner-Riesz means of Fourier integrals, Abstracts of scientific conference of Donetsk Univ., April 1995, 189–190. (Russian)

A. V. Tovstolis, R. M. Trigub, Pointwise approximation by polynomials on an interval and by entire functions on the exterior of interval and halfaxis, Theory of approximation and problems of numerical mathematics, Abstracts of Int. Conf., May 26-28, 1993, Dnipropetrovsk, 1993.