Theory of ground state cooling of a mechanical oscillator using dynamical back-action

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A quantum theory of cooling of a mechanical oscillator by radiation pressure-induced dynamical back-action is developed, which is analogous to sideband cooling of trapped ions. We find that final occupancies well below unity can be attained when the mechanical oscillation frequency is larger than the cavity linewidth. It is shown that the final average occupancy can be retrieved directly from the optical output spectrum.

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Mesoscopic mechanical oscillators are currently attracting interest due to their potential to enhance the sensitivity of displacement measurements [1] and to probe the quantum to classical transition of a macroscopic degree of freedom [2, 3]. A prerequisite for these applications is the capability of initializing an oscillator with a long phonon lifetime in its quantum ground state. So far this has not been demonstrated because the combination of sufficiently high mechanical frequencies (ω_m/2π) and quality factors in the relevant regime ω_m ≫ k_BT has not been reached [3]. In contrast, in atomic physics laser cooling has enabled the preparation of motional ground states [4, 5]. This has prompted researchers to study means of cooling a single mechanical resonator mode directly using laser radiation. Early work demonstrated cooling of a mechanical degree of freedom of a Fabry-Pérot mirror using a radiation pressure force controlled by an electronic feedback scheme [6, 7], in analogy to stochastic cooling. In contrast, the radiation pressure induced coupling of an optical cavity mode to a mechanical oscillator [cf. Fig. 1(a)] can give rise to self-cooling via dynamical back-action [8]. In essence, the cavity delay induces correlations between the radiation pressure force and the thermal Brownian motion that lead to cooling or amplification, depending on the laser detuning. In a series of recent experiments, these effects have been used to cool a single mechanical mode [8, 10, 11]. While classical and semiclassical analysis of dynamical back-action have been developed [12, 13], the question as to whether ground state cooling is possible has not been addressed.

Here a quantum theory of cooling via dynamical back-action is presented. We find that final occupancies below unity can indeed be attained when the optical cavity’s lifetime is comparable to or exceeds the mechanical oscillation period. Along these lines, an analogy between this mechanism and the sideband cooling of trapped ions in the Lamb-Dicke regime is elucidated [5]. In our setting the optical cavity mode plays the role of the ion’s pseudospin mediating the frequency up-conversion underlying the cooling cycle. Finally, we discuss how the average phonon occupancy can be retrieved from the spectrum of the optical cavity output. We note that these results can be applied to a wide range of experimental realizations of cavity self-cooling [8, 11, 12].

We treat the laser driven optical cavity mode coupled to the mechanical resonator mode as an open quantum system and adopt a rotating frame at the laser frequency ω_L. The system Hamiltonian is given by [16, 17]

\[ H' = -\hbar \Delta' L a_p^\dagger a_p + \hbar \eta \omega_m a_p^\dagger a_p (a_m + a_m^\dagger) \]

\[ + \hbar \frac{\Delta'}{2} (a_p + a_p^\dagger) + \hbar \omega_m a_m^\dagger a_m. \]  (1)

Here \( a_p \) (\( a_m \)) is the annihilation operator for the optical (mechanical) oscillator, \( \omega_p \) (\( \omega_m \)) its angular frequency and \( \Delta'_L \) the laser detuning from the optical resonance. We have also introduced the driving amplitude \( \Omega = 2 \sqrt{\gamma / \hbar \omega_L \tau_{ex}} \) where \( \gamma \) is the input laser power and \( 1/\tau_{ex} \) the photon decay rate into the associated outgoing modes (e.g. optical fiber modes [15]). The optomechanical coupling via radiation pressure can be characterized by the dimensionless parameter \( \eta \equiv (\omega_p/\omega_m)(l_m/L) \);
with \( l_m = \sqrt{\hbar/2m\omega_m} \) the zero point motion of the mechanical resonator mode, \( m \) its effective mass, and \( L \) an effective length that depends on how the radiation pressure force affects the optical cavity. For typical materials and dimensions, one obtains \( \eta \sim 10^{-4} \).

The optical cavity losses and the intrinsic dissipation of the mechanical resonator are characterized, respectively, by the cavity lifetime \( \tau \) and the mechanical quality factor \( Q_m \). These give rise to a dissipative contribution to the Liouvillian \( \mathcal{L}'_D \) (i.e., \( \mathcal{L}' = -\frac{i}{\hbar}(H', \ldots) + \mathcal{L}_p' \)) that is of Lindblad form \([25]\) with collapse operators \([17,18]\):

\[
\sqrt{1/\tau}a_p, \sqrt{\gamma_m n(\omega_m)} a_m \quad \text{and} \quad \sqrt{\gamma_m n(\omega_m) + 1} a_m.
\]

Here \( \gamma_m = \omega_m/Q_m \) is the mechanical oscillator’s natural linewidth and \( n(\omega_m) \) its Bose number at the environmental temperature. We will focus on the regime \( n(\omega_m) + 1 \gamma_m \ll \omega_m, \gamma_m \ll 1/\tau \) and \( \eta, |\alpha| \ll 1, 1/\omega_m \tau \). The first condition is necessary for ground state cooling (see below), the second one is satisfied in all recent experiments \([9,10,11]\), and the last one, given the smallness of \( \eta \), will hinge on having a sufficiently low input power.

To study the dynamics generated by the Liouvillian \( \mathcal{L}' \) it proves useful to apply a shift to the modes’ normal coordinates: \( a_p \rightarrow a_p + \alpha, a_m \rightarrow a_m + \beta \) with the \( c \)-numbers \( \alpha \) and \( \beta \) chosen to cancel out all the linear terms in the transformed Liouvillian \( \mathcal{L}' \rightarrow \mathcal{L} \). To lowest order in the small parameters \( \eta \) and \( 1/Q_m \) we have \([29]\):

\[
\alpha \approx \Omega \tau/(2\tau \Delta'_L + i), \quad \beta \approx -\eta|\alpha|^2.
\]

We include the radiation pressure induced optical resonance shift into the effective detuning \( \Delta'_L + 2\eta^2|\alpha|^2 \omega_m \rightarrow \Delta_L \) and perform the additional canonical transformation \( a_p \rightarrow (\alpha/|\alpha|)a_p \).

While the dissipative part of the Liouvillian \( \mathcal{L}' \) remains invariant, the Hamiltonians transform into

\[
H = -\hbar \Delta_L a_p^\dagger a_p + \hbar \omega_m a_m^\dagger a_m + \hbar\eta \omega_m [a_p^\dagger a_p + |\alpha|^2(a_p + a_p^\dagger)] (a_m + a_m^\dagger).
\]

Henceforth we will refer to the primed representation \([1]\) as the “physical” one and to the unprimed representation \([2]\) as the “shifted” one. The smallness of \( \eta^2 \) and \( n(\omega_m) + 1/Q_m \) imply a wide separation between the timescales for cooling and heating the mechanical oscillator and those characterizing the dynamics of the optical cavity mode and the mechanical oscillation period. Thus, the electromagnetic environment (including the optical cavity) can be regarded as a structured reservoir with which the mechanical mode interacts perturbatively [cf. Fig. 1(b,c)].

This prompts us to derive a “generalized quantum optical” master equation for the reduced density matrix \([18]\) of the latter: \( \mu = \text{Tr}_p[\rho] \). In our context, such a derivation can also be viewed as an adiabatic elimination \([18]\) of the optical cavity in the presence of fast rotating terms (\( \propto e^{\pm i\omega_m t} \)) in the optomechanical interaction. We note that while in the physical representation the steady state average occupancy of the optical cavity is given by \( |\alpha|^2 \), in the shifted one its steady state is simply the vacuum \( |0_p\rangle \). Thus, we obtain

\[
\mu = -i \left[ \omega_m a_m^\dagger a_m, \mu \right] + \frac{1}{2} \left[ \gamma_m n(\omega_m) + 1 \right] + A_+ \left( 2\eta \omega_m a_m^\dagger a_m - a_m a_m^\dagger \right) + \frac{1}{2} \left[ \gamma_m n(\omega_m) + 1 \right] a_m a_m^\dagger.
\]

In the first term we have redefined \( \omega_m \) to include the light-induced shift of the mechanical frequency. The second and third terms correspond, respectively, to cooling and heating induced by the coupling to the thermal bath (contributions \( \propto \gamma_m \)) and by inelastic light scattering processes [cf. Eq. (1b)] with rates

\[
A_\pm = \eta^2 \frac{4\Omega^2}{4\tau^2 \Delta'_L + 1} \left[ \frac{\omega_m^2 \tau^3}{4\tau^2 (\Delta_L \pm \omega_m)^2 + 1} \right] \pm \eta^2 \frac{4\Omega^2}{4\tau^2 \Delta'_L + 1} \frac{\omega_m^2 \tau^3}{4\tau^2 (\Delta_L \pm \omega_m)^2 + 1}.
\]

In the shifted representation it is simple to understand these cooling and heating rates in terms of perturbation theory in the small parameters \( \eta |\alpha| \) and \( \eta \) [cf. Eq. (2)].

To lowest order in \( \eta \) only the states \( |0_p\rangle \) and \( |1_p\rangle \) participate yielding the same results as for an equivalent dissipative two level system. Denoting by \( |n\rangle \) the number states of the mechanical oscillator we have anti-Stokes (Stokes) processes in which the transition \( |0_p\rangle |n\rangle \rightarrow |1_p\rangle |n-1\rangle \) (\( |0_p\rangle |n\rangle \rightarrow |1_p\rangle |n+1\rangle \)) followed by the decay of the cavity photon leads to cooling (amplification). This scenario is thus similar to the laser cooling of a trapped ion in the Lamb-Dicke regime \([1,2,13]\), or of a nanomechanical resonator coupled to an “artificial atom” \([20]\) or an ion \([21]\).

An important caveat in this analogy is that there is no external driving for \( \eta = 0 \). Furthermore, though the parameter \( \eta^2 \) will play a role reminiscent of the Lamb-Dicke parameter — determining for example the relative weights of the sidebands — the efficiency of the cooling process will depend solely on \( \eta^2 |\alpha|^2 \) and Eqs. (4) will remain valid for arbitrary \( \Omega \) provided \( \eta^2 \) is sufficiently small. This absence of a “direct” driving amplitude also implies that the cubic term in Hamiltonian (2) does not contribute to the master equation (3) as it only generates terms that are higher order in \( \eta^2 \). Thus to lowest order there is no “diffusive channel” and the theory is equivalent to a quadratic Liouvillian \([27]\).

Henceforth we focus on the regime \( \Delta_L < 0 \) for which there is a net laser cooling rate \( \Gamma = A_+ - A_- > 0 \). In this regime Eq. (3) has a well defined steady state that transformed back to the physical representation yields a shifted thermal state. The corresponding steady state average occupancy, to which the system converges on the timescale \( 1/(\Gamma + \gamma_m) \), is given by \( \langle a_m^\dagger a_m \rangle_{SS} = n_f + |\beta|^2 \) with \( n_f = [\gamma_m n(\omega_m) + A_+]/(\gamma_m + \Gamma) \) and \( |\beta|^2 = \eta^2 \Omega^4 \tau^4/(4\tau^2 \Delta'_L + 1)^2 \). However the final temperature should be defined in terms of \( n_f \) as the other contribution arises from a coherent shift. In fact, under the conditions underpinning our approximations, the latter could be undone by a “slow” switch-off of the laser.

As we start from thermal equilibrium, initially the number of phonons is given by \( n_i = n(\omega_m) \). Thus, from
the expression for \( n_f \) it is clear that for appreciable cooling (i.e. \( n_f \ll n_i \)) we need \( \Gamma \gg \gamma_m \). In this regime

\[
n_f = \left[ \frac{\gamma_m}{\Gamma} n_i + \tilde{n}_f \right] \left[ 1 + O \left( \frac{\gamma_m}{\Gamma} \right) \right],
\]

with

\[
\tilde{n}_f \equiv \frac{A_+}{\Gamma} = \frac{4r^2 (\Delta_L + \omega_m)^2 + 1}{16r^2 \omega_m (-\Delta_L)}.
\]

Note that from Eq. (4) it follows that \( \Gamma \) is exactly proportional to the input power \( P \). Since there are two terms in Eq. (5), there are two regimes depending on which contribution dominates the behavior of \( n_f \). In the first regime heating is dominated by the intrinsic dissipation of the mechanical resonance and the final average occupancy is proportional to the initial one. This behavior has already been demonstrated experimentally [10, 11, 12]. On the other hand for sufficiently low \( \gamma_m \) and input laser power, the heating is dominated by the scattering of laser light. In this regime the final occupancy is given by \( n_f \) (cf. Fig. 2). Thus the optimal value of \( n_f \) solely depends on the product \( \omega_m \tau \) and is found by minimizing with respect to the normalized detuning \( \delta \equiv \tau \Delta_L \) (i.e. it is power independent). This yields \( \delta_{\text{opt}} = -\sqrt{1 + 4\omega_m^2 \tau^2}/2 \), implying the fundamental temperature limit:

\[
\tilde{n}_{\text{TL}} = \min \{ \tilde{n}_f (\delta) \} = \frac{1}{2} \left( \sqrt{1 + 1/4\omega_m^2 \tau^2} - 1 \right).
\]

The regime \( \omega_m \tau \ll 1 \) is in essence the adiabatic limit [10], since the cavity dynamics is much faster than the mechanical oscillator's period. Several recent experiments fall into this regime [9, 12]. Expanding Eq. (7) we obtain \( \tilde{n}_{\text{TL}} \approx 1/4\omega_m \tau \gg 1 \) precluding ground state cooling in this parameter regime. The corresponding final temperature is of order \( h/\epsilon \tau \) in complete analogy with the Doppler limit of the laser cooling of harmonically bound atoms, where the sidebands are not resolved [3].

We turn now to the regime where retardation effects become significant (i.e. \( \omega_m \gtrsim 1/\tau \)). This has indeed been observed in recent experimental work pertaining to both amplification [12] and cooling [10, 11] of a mechanical oscillator mode. In this regime the optical cavity field cannot respond instantaneously to the mechanical motion and an entirely viscous radiation pressure force may arise [11]. In the frequency domain, this can be interpreted as having mechanical frequencies which exceed or are equal to the cavity linewidth [cf. Fig. 1(c)]. Accordingly, the asymmetry in the Stokes and anti-Stokes scattering rates becomes more pronounced leading into the analog of the “resolved sideband” limit of the laser cooling of harmonically bound atoms. More precisely for \( 4\omega_m^2 \tau^2 \gg 1 \), Eq. (7) yields \( \tilde{n}_{\text{TL}} \approx 1/16\omega_m^2 \tau^2 \ll 1 \) implying that in this limit one can reach arbitrarily low temperatures (ground state cooling). In a concrete realization, a sound benchmark to evaluate the cooling performance is whether occupancies below unity can be attained. Eq. (7) leads to the following criterion [28]: \( \tilde{n}_{\text{TL}} < 1 \iff \omega_m \tau > 1/\sqrt{32} \); and Eq. (6) implies that within this regime occupancies below unity are only possible for a certain “detuning window” \( -\sqrt{8\omega_m^2 - 1/4\tau^2} \leq \Delta_L \leq 3\omega_m \leq \sqrt{8\omega_m^2 - 1/4\tau^2} \).

Finally, we consider the impact of the intrinsic dissipation on the optimal value of \( n_f \) in the regime \( \omega_m \tau > 1/\sqrt{32} \) (analogous considerations can be done for the opposite regime). The situation is reminiscent of the “atomic” laser cooling of nanoresonators [20, 21] where the finite \( Q_m \) also plays a crucial role. However, in the present context, the analysis is simpler given that the optimal laser detuning to maximize the cooling rate \( \Gamma \) (and thus minimize the first term in Eq. (5)) is of the same order as \( \delta_{\text{opt}} \). Hence, the only relevant issue is the upper bound on \( P \) required by the wide timescale separation underpinning our adiabatic treatment of the cooling and heating processes. Given the quadratic nature of our approximate theory, already discussed, the Bose enhancement of the resonator mode plays no role and the adiabatic requirement reads \( A_m \ll 1/\tau \). Aside from the limitations of our approximate treatment, heuristic considerations imply that the timescale over which \( \langle a_m^\dagger a_m \rangle(t) \) reaches its steady state value can never be shorter than \( \tau \). Thus, our treatment provides an upper bound for the ultimate final temperature when the finite \( Q_m \) is considered. As an illustration we consider the parameters of Ref. [11] (i.e. \( \Delta_L \tau > 0.5, \omega_m/2\pi = 60 \text{ MHz}, \tau = 3 \text{ ns} \)). For a reservoir temperature of 4 K we have \( n_i \approx 1300 \). If we consider the improvements in the mechanical \( Q \)-values of toroid microcavities due to vacuum operation (\( Q_m = 30,000 \)) a cooling rate of 2.8 MHz is then required.

![FIG. 2: (color online) Final (steady state) average phonon number \( \tilde{n}_f \) as a function of normalized laser detuning \( (\Delta_L \tau) \) and normalized mechanical angular frequency \( (\omega_m \tau) \). The contour-lines indicate the values \( \tilde{n}_f = 1 \) and 10, respectively. Ground state cooling is only possible in a finite detuning window and for \( \omega_m \tau > 1/\sqrt{32} \).](image)
to reach \( n_i \gamma_m / \Gamma < 1 \) [cf. first term in Eq. (3)] 29.

The cooling process gives rise to photons which have frequencies that differ from the pump laser (\( \omega_L \)). Thus it can be studied in an experiment by measuring the spectrum of the scattered light. As depicted in Fig. 1(a), we consider a one-sided cavity and the relevant observable is the output power. The input-output formalism implies that in the physical representation its spectrum \( S( \omega ) \) is given by the Fourier transform of \( \exp[i \omega t] \left[ \sqrt{1/\tau_{\text{ex}} a(t + \tau) + a_\text{in}(t)} \right] \). In the shifted representation \( a_p(t) \rightarrow a_p(t) + \alpha \) and the classical input just adds a c-number to the cavity steady state amplitude. Along the lines of our derivation of Eq. (3): \( a_p(t), a^\dagger_p(t) \) are treated as environment operators to be reduced to the system operators \( a_m(t), a^\dagger_m(t) \) by integrating out the corresponding Heisenberg equations of motion. A calculation based on perturbation theory and the theory of quantum Markov processes 18 then yields 22:

\[
S(\omega) \approx \frac{\tau}{\tau_{\text{ex}}} \left\{ \frac{P}{\hbar \omega_L} \left[ \frac{1 - \frac{\tau_{\text{ex}}}{\tau}}{\tau^2 \Delta^2 + \frac{1}{4}} \right] \delta(\omega - \omega_L) + \frac{A_f - n_f}{\pi} \frac{\gamma_m}{\sqrt{\Delta^2 + \frac{1}{4}}} + \frac{A_\text{+}(n_f + 1)}{\pi} \frac{\gamma_m}{\sqrt{\Delta^2 + \frac{1}{4}}} \right\},
\]

where the relative order of the corrections is given by \( \eta^2 \) for all frequencies, and we have normalized to number of photons per unit time and unit frequency. Here \( A_\pm \) and \( n_f \) are given respectively by Eqs. (4) and (5) and \( \gamma_{\text{eff}} \equiv \gamma_m + \Gamma \).

The final occupancies can be retrieved by comparing the above spectra [Eq. (8)] for different input powers. The quantity \( [N_+(P) + N_-(P)] P_0 / [N_+(P) + N_-(P)] P \) provides an upper bound for the ratio \( n_f/n_i \). Here we have introduced a “reference” low power \( P_0 \) for which \( \Gamma, A_s \ll \gamma_m \) implying \( n_f(P_0) \approx n_i \), and assumed that the input power \( P \) induces appreciable cooling [i.e. \( n_f(P) \ll n_i \)]. It is important to note that (given \( n_i \)) this upper bound provides an accurate direct measurement of the final temperature for \( n_f \gg 1/2 \). On the other hand the worst case scenario occurs for \( \omega_m \tau \gg 1 \) and \( n_f \approx \tilde{n}_f \) where it yields \( 2n_f \). However for \( n_f \leq 1 \) an accurate measurement is afforded by the quantity

\[
\frac{N_-(P) N_+(P)}{N_+(P) N_-(P)} n_i \approx n_f(P) \frac{n_f(P) + \tilde{C} n_i + P A_\text{+}}{\tilde{C} (n_i + 1) + P A_\text{+}}. \tag{9}
\]

with \( \tilde{C} \equiv \hbar \omega_L \tau_{\text{ex}} \gamma_m (4\tau^2 \Delta^2 + 1) / 4\tau^2 \eta^2 \). Thus, the high power limit of Eq. (9) provides a clear signature for ground state cooling when it can be achieved. This is in stark contrast to the case of a laser cooled trapped ion where a well defined thermal reservoir associated to the intrinsic dissipation is lacking and detailed balance yields \( N_- = N_+ \). In a realistic scenario the central peak cannot be regarded as a delta function and the small relative weight of the sidebands poses an experimental challenge.

This could be overcome by combining state of the art low noise lasers and high resolution spectroscopy with a suitable lock-in technique. An advantage, as compared with the trapped ion case, is the larger number of photons contained by the sidebands for a given \( n_f \) 22 30.

In summary, we have derived a quantum mechanical model of the temperature limit for cooling using radiation pressure induced dynamical back-action and shown that ground state cooling can be achieved as the optical cavity linewidth becomes smaller than the mechanical frequency, in analogy to atomic sideband cooling. We find that the threshold to attain occupancies below unity is given by \( \omega_m \tau > 1 / \sqrt{2} \). Furthermore, we have shown how the spectrum of the optical cavity output could be used to measure the final temperatures achieved. Our results could apply to other systems exhibiting dynamical back-action such as an LC circuit with its capacitance modulated by a mechanical oscillator 24.

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[25] This requires rotating-wave-approximations that are warranted in the parameter regime of interest.
[26] The corrections do not affect the lowest order contributions in $\eta^2, 1/Q_m$ to the cooling and heating rates.
[27] While Eq. (4) is valid for arbitrarily low laser intensities it coincides with its counterpart in a classical treatment of the canonical variables as used in Ref. [11].
[28] This lower threshold, as compared with atomic laser cooling, is related to the absence of a diffusive channel.
[29] We note that $\Gamma \ll 1/\tau$ is satisfied. For comparison, the maximum cooling rate of Ref. [11] was 0.7 MHz.
[30] We note that $A_{\pm} \sim 1$MHz.