Canonical $p$-dimensions of algebraic groups and degrees of basic polynomial invariants

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Abstract

In the present notes we provide a new uniform way to compute a canonical $p$-dimension of a split algebraic group $G$ for a torsion prime $p$ using degrees of basic polynomial invariants described by V. Kac. As an application, we compute the canonical $p$-dimensions for all exceptional simple algebraic groups.

The notion of a canonical dimension of an algebraic structure was introduced by Berhuy and Reichstein [1]. For a split algebraic group $G$ and its torsion prime $p$ the canonical $p$-dimension of $G$ was studied by Karpenko and Merkurjev in [5]. In particular, this invariant was shown to be related with the size of the image of the characteristic map

$$\phi_G : S^*(\hat{T}) \to \text{CH}^*(X),$$

(1)

where $\hat{T}$ is the character group of a maximal split torus $T$, $X$ is the variety of complete flags and $S^*$ stands for the symmetric algebra. Namely, one has the following formula for the canonical $p$-dimension of a group $G$

$$\text{cd}_p(G) = \min\{i \mid \overline{\text{Ch}}_i(X) \neq 0\},$$

(2)

where $\overline{\text{Ch}}_i(X)$ stands for the image of $\phi_G$ in the modulo $p$ Chow group $\text{Ch}(X) = \text{CH}(X)/p \cdot \text{CH}(X)$ (see [5, Theorem 6.9]). Using this remarkable fact together with the explicit description of the image $R_p = \overline{\text{Ch}}(X)$ Karpenko and Merkurjev computed the canonical $p$-dimensions for all classical algebraic groups $G$ (see [5, Section 8]).

The goal of the present notes is to relate the work by V. Kac [4] devoted to the study of the $p$-torsion part of the cohomology ring of an algebraic
group $G$ with the canonical $p$-dimensions of $G$, hence, providing a different and uniform approach of computing those invariants. As a consequence, we compute canonical $p$-dimensions for all exceptional algebraic groups, hence, completing the computations started in [5].

**Theorem 1.** Let $G$ be a split simple algebraic group of rank $n$ and $p$ be an odd torsion prime. Then

$$cd_p(G) = N + n - (d_{1,p} + d_{2,p} + \ldots + d_{n,p}),$$

where $N$ stands for the number of positive roots of $G$ and integers $d_{1,p}, \ldots, d_{n,p}$ are the degrees of basic polynomial invariants modulo $p$.

**Proof.** Consider the characteristic map (1) modulo $p$

$$(\phi_G)_p: S^*(\hat{T}) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \to \text{Ch}^*(X)$$

According to [7, Cor. 3.9.(ii)] the kernel of this map $I_p$ is generated by a regular sequence of $n$ homogeneous polynomials of degrees $d_{1,p}, \ldots, d_{n,p}$.

Recall that a Poincare polynomial $P(A, t)$ for a graded module $M$ over a field $k$ is defined to be $\sum_i \dim_k M^i \cdot t^i$ (see [8]). Hence, the Poincare polynomial for the $\mathbb{Z}/p\mathbb{Z}$-module $R_p$ is equal to

$$P(R_p, t) = \prod_{i=1}^{n} \frac{1 - t^{d_{i,p}}}{1 - t}.$$  

Indeed, we identify $R_p$ with the quotient of the polynomial ring in $n$ variables $\mathbb{Z}/p\mathbb{Z}[\omega_1, \ldots, \omega_n]$ modulo the ideal $I_p$. The formula (4) then follows immediately by [6, Cor. 3.3].

According to (2) the canonical $p$-dimension is equal to the difference

$$\dim(X) - \deg P(R_p, t) = N - \sum_{i=1}^{n}(d_{i,p} - 1).$$

**Remark 2.** The fact that the ideal $I_p$ is generated by a regular sequence of elements was extensively used in [4]. Unfortunately, the original proof of it (provided in [4]) contains a mistake. Another proof of this fact which works only for odd torsion primes can be found in [7].

The next theorem relates $cd_p(G)$ with the Chow group of $G$ modulo $p$.

**Theorem 3.** Let $G$ be a split simple group and $p$ be its torsion prime. Then

$$cd_p(G) = \max\{i \mid \text{Ch}^i(G) \neq 0\}$$
Proof. It is known that \( \text{Ch}(G) = \text{Ch}(X)/J_p \) (see [2]), where \( J_p \) is the ideal generated by the non-constant part of \( R_p \). Since \( \text{Ch}(X) \) is a free \( R_p \)-module (see [4, Appendix]), we have \( P(\text{Ch}(X)/J_p, t) \cdot P(R_p, t) = P(\text{Ch}(X), t) \) and, hence, for the degrees \( \deg(P(\text{Ch}(G), t)) + \deg(P(R_p, t)) = \dim(X) \). The proof is completed since \( \deg(P(R_p, t)) = \dim(X) - \text{cd}_p(G) \). \( \square \)

Corollary 4. Let \( d_1, d_2, \ldots, d_r \) be the set of \( p \)-exceptional degrees (introduced in [4, Thm. 3]) for a group \( G \) and its odd torsion prime \( p \). Then the canonical \( p \)-dimension of \( G \) is equal to the sum

\[
\text{cd}_p(G) = \sum_{i=1}^{r} d'_i \cdot (p^{k_i} - 1),
\]

where the integers \( d'_i \) and \( p^{k_i} \) are the factors of the decompositions \( d_i = d'_i \cdot p^{k_i}, \) \( p \nmid d'_i \).

Proof. Follows by the last isomorphism of [4, Theorem 3.(ii)]. \( \square \)

Corollary 5. We obtain the following values for the canonical \( p \)-dimensions of groups of types \( F_4, E_6, E_7 \) and \( E_8 \) (here \( G^{sc} \) and \( G^{ad} \) stand for the simply-connected and adjoint forms of a group \( G \))

\begin{align*}
\text{cd}_2 F_4 &= 3, & \text{cd}_3 F_4 &= 8 \\
\text{cd}_2 E_6 &= 3, & \text{cd}_3 E_6^{sc} &= 8, & \text{cd}_3 E_6^{ad} &= 16 \\
\text{cd}_2 E_7^{sc} &= 17, & \text{cd}_2 E_7^{ad} &= 18 & \text{cd}_3 E_7 &= 8 \\
\text{cd}_2 E_8 &= 60, & \text{cd}_3 E_8 &= 28 & \text{cd}_5 E_8 &= 24
\end{align*}

Proof. The case of odd torsion primes follows immediately by Theorem 1 or Corollary 4 and Table 2 of [4].

The case \( p = 2 \) can be computed using Theorem 3 as follows. Consider the canonical map \( \pi : G \to X \). Note that the Chow group \( \text{Ch}(G) \) can be identified with the image \( \text{Im} \pi^* \) of the induced pull-back (see [2]). According to [3, Thm. 1.1 and Lem. 1.3], the cohomology ring of \( G \) can be represented as the tensor product of two algebras

\[
H(G; \mathbb{Z}/p\mathbb{Z}) \cong \text{Im} \pi^* \otimes \Delta(a_1, \ldots, a_n),
\]

where each \( a_i \) is of odd degree, all elements of \( \text{Im} \pi^* \) are of even degree and \( \Delta(a_1, \ldots, a_n) \) denotes the submodule spanned by the simple monomials \( a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n} (\epsilon_i = 0 \text{ or } 1) \) which are linearly independent. Knowing this
representation and the cohomology ring of $G$, one immediately obtains

$$cd_p(G) = \frac{1}{2}(\deg P(H(G; \mathbb{Z}/p\mathbb{Z}), t) - \sum_{i=1}^{n} \deg(a_i)).$$

To finish the proof observe that the cohomology ring of an exceptional algebraic group modulo 2 and degrees of the elements $a_i$ can be found in the literature (see the references of paper [3]).

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