SCATTERING IN THREE DIMENSIONAL EXTREMAL BLACK HOLES

J. Gamboa* and F. Méndez†
Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile

I. INTRODUCTION

Twenty five years ago Hawking [1] discovered that black holes (BH) could disappear by quantum effects, suggesting that in a full quantum theory of gravity the singularities could be smoothed out in the same sense as, for instance, the self-energy of the electron is smoothed out by renormalization in quantum electrodynamics.

However before disappearance and in the case of very small BH, quantum effects produced by these objects could be important and its one presence in the universe should be enough to detect new physical effects such as quantum corrections to scattering of particles, new effects produced by strong fields and so on.

The scattering of particles around BH is, nevertheless, very interesting because for \( D = 4 \) the classical absorption cross section is proportional to the BH area [2] (for results in higher dimensions see [3]) and one does not know if this result could survive at quantum level [1].

One can solve this problem, considering a particle moving around a very small BH. If the Schwarzschild radius is much bigger than the Compton length, then the BH can be considered as a classical effective potential and the wave equation

\[
\left[ \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu}) - m^2 \right] \psi = 0, \tag{1}
\]

can be written as (assuming spherical or axial symmetry)

\[
\frac{d^2 R}{dr^2} + (k^2 - V_{\text{eff}}(r))R = 0, \tag{2}
\]

where \( V_{\text{eff}} \) is the effective potential produced by the BH background. Module technical refinements, this is the point of view followed in the literature [6].

In the last years, intense research in lower dimensional gravity has been performed and, particularly, in three dimensions several classical and quantal results have been found. From the quantum mechanics point of view the equivalence between Chern-Simons theories and 3D quantum gravity have been established [7] and classically a three dimensional BH solution (3D BH) has been found [8].

Although three dimensional gravity is not related to the physical world, these results could be considered as a theoretical laboratory where the Hawking’s conjecture concerning violation of quantum mechanics and other expected properties could be proven [9].

The purpose of this paper is to study the scattering of particles by a three dimensional BH in the extremal limit. This is interesting by several reasons. Firstly one could expect that an extremal BH could be considered as a point particle and the scattering of particles by this BH when \(-1 < M < 0\) (\( M \) the mass of the BH) could be seen as a similar problem to the Aharonov-Bohm effect with the BH playing the role of the solenoid [10].

Secondly, if one can understand scattering theory on an AdS spacetime one should be able also to understand the technical and conceptual differences between the scattering by 3D BH and the Aharonov-Bohm effect. This is a quite non-trivial problem.

However, in order to formulate this problem one should understand firstly how to define scattering processes in an anti-Sitter (AdS) space and, consequently, how to define asymptotic states. In previous works have been considered in this problem has been partially considered and an expression for the absorption cross section has been given avoiding the formal definition of scattering processes [12] (see also [13]). More recently, following the last developments in the AdS-CFT correspondence [14], new approaches to the scattering theory in AdS have been proposed [15].

The main results that we will describe below are the following:

a) The wave equation is exactly solved. Our solution shows that outside the horizon the optical theorem is satisfied and unitarity is not violated. However inside the horizon the hamiltonian is not self-adjoint and there is not unitarity.

b) The total cross section vanishes.

c) We find a set of appropriates variables where the scattering problem is well defined; in some sense this variables are a sort of a reciprocal space.

In the next section we will exactly solve the Klein-Gordon equation for extremal 3D BH and compute the absorption cross section by means of

\[ ^2 \text{For the non-extremal case including subtle points see [14].} \]
\[ \sigma_{abs} = \frac{J(r \to r_+)}{J(r \to \infty)} \frac{J_{abs}}{J_{\infty}}, \]  
(3)

where \( r_+ \) is the horizon of the 3D BH. We find that \( J_{abs} = 0 \) implies \( \sigma = 0 \), i.e. the extremal 3D black hole is a transparent object. In section III we reinterpret the above problem in a space where is possible to define \textit{in} and \textit{out} states and the calculation of \( \sigma_{abs} \) is performed as in the standard scattering theory.

**II. SOLUTION OF KLEIN-GORDON EQUATION AND FLUX OF PARTICLES**

In order to prove the results mentioned above let us consider the 3D BH solution found in (1)

\[ ds^2 = -(N^2 - r^2 N^\phi)dt^2 + N^{-2}dr^2 + r^2 d\phi^2 + 2 r^2 N^\phi dt d\phi, \]  
(4)

where the lapse function \( N \) and \( N^\phi \) are defined as

\[ N^2 = -M + \frac{J^2}{r^2} + \frac{J}{4r^2}, \]  
\[ N^\phi = -\frac{J}{2r^2}. \]  
(5)

The lapse function vanishes for

\[ r_\pm = \frac{1}{2} \left[ \frac{M}{2} \left( 1 \pm \sqrt{1 - \frac{J^2}{M^2 r^2}} \right) \right]^2. \]  
(6)

The horizon is identified with \( r_+ \) and it will exist only if \( M \) and \( J \) satisfy the relations

\[ M > 0, \quad |J| \leq ML. \]  
(7)

The extreme case corresponds to \( |J| = ML \) and in such a case both roots in (7) coincide.

Using these results one could write the Klein-Gordon equation and study the scattering problem as it is usually realized in quantum mechanics. The equation (3), in the above metric (1) can be solved if we consider the ansatz

\[ \psi = e^{î\omega t} e^{în\phi} f_{nw}(r), \]  
(8)

where \( f_{nw}(r) \) is an unknown function. Once (3) is replaced in (1) one finds

\[ \left( \frac{(r^2 - r_+^2)^4}{r^2 r_+^4} \right) f''_{nw}(r) + \left( \frac{(r^2 - r_+^2)^3}{r^3 r_+^4} \right) (3r^2 - r_+^2) f'_{nw}(r) + \left[ (\omega l + n)(\omega l - n)r^2 + 2m^2 r_+^2 \right] - \frac{m^2}{r^2} (r^2 - r_+^2)^2 f_{nw}(r) = 0. \]  
(9)

In order to solve this equation, let us make the change of variable

\[ \xi = \frac{r_+^2}{r^2 - r_+^2}. \]  
(10)

Equation (9) now becomes

\[ \mathcal{L} f_{nw}(\xi) = 0, \]  
(11)

where the elliptic linear operator \( \mathcal{L} \) is

\[ \mathcal{L} = \frac{d^2}{d\xi^2} + k_0^2 + \frac{k_1}{\xi} - \frac{k_2}{\xi^2}, \]  
(12)

with the constants defined as

\[ k_0^2 = \Omega_+^2, \]  
\[ k_1 = \Omega_+ \Omega_-, \]  
\[ k_2 = \frac{m^2}{4}. \]  
(13)

where \( \Omega_\pm = \frac{1}{2\sqrt{2}} (\omega \pm n) \) (we have put \( l = 1 \)).

Equation (10) can be solved by standard methods \[17\]. The two linearly independent solutions are

\[ f_{nw}^{(1)}(\xi) = e^{-\Omega_+ \xi} \xi s_+ F[s_+ + \frac{\Omega_+}{2}, 2s_+, 2\Omega_+ \xi], \]  
\[ f_{nw}^{(2)}(\xi) = e^{-\Omega_+ \xi} \xi s_- F[s_- + \frac{\Omega_+}{2}, 2s_-, 2\Omega_+ \xi]. \]  
(14)

where \( s_\pm = \frac{1}{2}(1 \pm \sqrt{1 + m^2}) \) and \( F[a, c, z] \) is the confluent hypergeometric function (Kummer's solution).

In equation (11) we can see that the region \( r \to \infty \) corresponds to \( \xi \to 0 \), and \( r \to r_+ \), to \( \xi \to \infty \). Thus, the regular solutions are

\[ f_{nw}^{(1)}(\xi), \quad r \to \infty \quad (\xi \to 0), \]  
\[ f_{nw}^{(2)}(\xi), \quad r \to r_+ \quad (\xi \to \infty). \]  
(15)

In order to compute the absorption cross section in \( \xi \) coordinates is

\[ j_r(\xi) = f_{nw}^{*}(\xi) \frac{d}{d\xi} f_{nw}(\xi) - f_{nw}(\xi) \frac{d}{d\xi} f_{nw}^{*}(\xi). \]  
(16)

The solution of (10) is

\[ f_{nw}(\xi) = Af_{nw}^{(1)}(\xi) + Bf_{nw}^{(2)}(\xi), \]  
(17)

but near the horizon \( f_{nw}^{(1)}(\xi) \) goes to infinity and, in order to have a regular solution, one can choose \( A = 0 \). Thus, the flux of particles in the horizon comes from the regular part of (18). The calculation is straightforward if we use the Kummer’s transformation for the confluent hypergeometric function \( F(a, c; \xi), i.e. \)

\[ F(a, c; \xi) = e^\xi F(c-a, c; -\xi). \]  
(18)
Using this fact, one finds - for regular solutions in the horizon - that \( j_r = 0 \).

On the other hand since we are sending particles from infinity, \( J_\infty \neq 0 \) and consequently the absorption cross section is
\[
s(\sigma) = \frac{j_r}{J_\infty} = 0. \tag{24}
\]

Therefore, one finds that the total cross section vanishes and this extremal BH is transparent for any energy of particles. A similar phenomenon occurs in nature for the scattering of electrons by inert atoms where, for some values of the energy, there is not scattering for s-waves, a phenomenon known as Ramsauer-Townsend effect.\(^3\)

Thus, our calculation shows that the quantum scattering of particles with 3D extremal BH is always governed by a kind of Ramsauer-Townsend mechanism.

**III. RECIPROCAL SPACE**

Following the arguments given in section I, in an AdS space is not possible to define in and out states and, as a consequence, the definition of the scattering theory is more subtle. In this section we will reinterpret the previous results in order to find an alternative definition of the asymptotic states.

Indeed, equation (12) can formally be seen as a Schrödinger equation in the potential
\[
V(\xi) = \frac{k_1}{\xi} - \frac{k_2}{\xi^2}, \tag{25}
\]
with \( k_2^2 \) playing the role of the energy and "\( \xi \)" a radial coordinate. Of course, the next question is: what is the angle in this space? As we are interested in the calculation of the total cross section, this angle can be chosen equal to the \( \phi \) angle that appears in (14), i.e. taking values on \( 0 < \phi < 2\pi \). We will call \( H = (\xi, \phi) \) the reciprocal space.

Following general properties of elliptic differential equations [24], the eigenfunctions \( f_{n\omega} \) should be continuous everywhere, but as (23) is singular in \( \xi = 0 \), one must add an additional condition on \( f_{n\omega} \), i.e.
\[
f_{n\omega}(0) = 0, \tag{26}
\]
one could note that (27) is self-adjoint condition for the operator \( L [24] \).

This last condition assures us continuity in \( \xi = 0 \) and the vanishing of \( V(\xi) \) when \( \xi \) goes to infinity would permit to define asymptotic states as in the usual scattering theory, i.e. the existence of a general asymptotic solution of (12) like \( A(\phi)e^{ik_0\xi} \), where \( A(\phi) \) and \( e^{ik_0\xi} \) are the scattering amplitude and the asymptotic states, respectively. This last statement is heavily dependent on \( k_0 \) being a positive real number. Looking at (14), this mean of course that the inequality
\[
[n + \omega]^2 > 0, \tag{27}
\]
is verified.

One should also point out that (24) implies a physical condition on the system. If \( \xi \) takes values on \( \mathbb{R}^+ \) - i.e. if \( \xi \) is interpreted as a radial coordinate - (26) can be seen as a condition of probability conservation in the origin. By the contrary, if \( \xi \) takes positive and negative values - i.e. just when the particles cross the horizon, probability is not conserved and unitarity is violated [25].

Using (17), the calculation of the scattering cross section is straightforward. Indeed, the asymptotic behavior of \( F \) for \( \xi \gg 1 \) contains a term like \( 1/\xi \Delta + \) plus distorted waves corrections as in the Coulomb scattering. Nevertheless, the angular corrections only contain \( e^{im\phi} \) and, as a consequence, the total scattering amplitude simply becomes
\[
A(\phi) = \sum_{n=-\infty}^{\infty} e^{in\phi}, \tag{28}
\]
which can be computed using the regularization prescription (see e.g. [10])
\[
A(\phi) = 1 + \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} (e^{in(\phi + i\epsilon)} + e^{-in(\phi - i\epsilon)}),
\]
\[
= 1 + 2\pi i \delta(\phi). \tag{29}
\]

The optical theorem gives
\[
\sigma \sim \text{Im} A(0) = 0. \tag{30}
\]

Therefore, in the (\( \xi, \phi \)) reciprocal space we recuperate the Ramsauer-Townsend effect found in section II. Thus, one can assert that both descriptions are equivalents.

In conclusion, a) we have shown by two different methods that the absorption cross section for spinless relativistic particles in a 3D extremal BH background vanishes and b) we have found an alternative description for the scattering of particles in AdS spaces. The next step is to show the equivalence between our approach and the results found in [16].

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\(^3\)Here the reader should note that this result is independent of the normalization constants that appears in the solutions of (14).
Appendix

In this section we will show that the radial flux of the Klein-Gordon equation is identically equals to zero in the 3D black hole background. The argument is as follows; noticing that the solution (18) written in terms of $r$ also satisfy the property

$$f^{(2)*}(r) = f^{(2)}(r),$$

for $r = r_+ + \epsilon$, then the flux

$$j_r = f^*(rN^2)\partial_r f - f(rN^2)\partial_r f^*,$$

vanishes identically, independently of $\epsilon$. For incoming particles the incident flux $j_{\infty}$ is different from zero by definition, otherwise there are no particles (this point is also discussed in [12]).

For the massless particle one find the following solutions

$$f_{\nuomega}^{(1)}(\xi) = e^{-i\Omega \xi}F[1 + i\frac{\Omega}{2}, 2, 2\Omega_+\xi],$$

$$f_{\nuomega}^{(2)}(\xi) = e^{-i\Omega \xi}F[i\frac{\Omega}{2}, 0, 2i\Omega_+\xi].$$

but for the Kummer’s relation $F(a, 0; \xi) = e^\xi F(-a, 0; \xi)$ and (31) again is satisfied and the previous results are obtained, i.e. $\sigma = 0$.

ACKNOWLEDGMENTS

We would like to thank M. Bañados for useful discussions. This work was partially supported by grants 1980788, 3000005 from FONDECYT (Chile) and DICYT (USACH).

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