Cloud K-SVD: A Collaborative Dictionary Learning Algorithm for Big, Distributed Data

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Abstract

This paper studies the problem of data-adaptive representations for big, distributed data. It is assumed that a number of geographically-distributed, interconnected sites have massive local data and they are interested in collaboratively learning a low-dimensional geometric structure underlying these data. In contrast to some of the previous works on subspace-based data representations, this paper focuses on the geometric structure of a union of subspaces (UoS). In this regard, it proposes a distributed algorithm—termed cloud K-SVD—for collaborative learning of a UoS structure underlying distributed data of interest. The goal of cloud K-SVD is to learn an overcomplete dictionary at each individual site such that every sample in the distributed data can be represented through a small number of atoms of any one of the learned dictionaries. Cloud K-SVD accomplishes this goal without requiring communication of individual data samples between different sites. This paper also provides a rigorous analysis of cloud K-SVD that gives insights into its properties as well as deviations of the dictionaries learned at individual sites from a centralized solution in terms of different measures of local/global data and topology of the interconnections. Finally, the paper also numerically illustrates the efficacy of cloud K-SVD on both real and synthetic distributed data.

I. INTRODUCTION

Modern information processing is based on the axiom that while real-world data may live in high-dimensional ambient spaces, relevant information within them almost always lies near low-dimensional geometric structures. Knowledge of these (low-dimensional) geometric structures underlying data of interest is central to the success of a multitude of information processing tasks. But this knowledge is unavailable to us in an overwhelmingly large number of applications and a great deal of work has been done in the past to learn geometric structure of data from the data themselves. However, much of that work, often studied under rubrics such as principal component analysis (PCA) \cite{2}, generalized PCA \cite{3}, hybrid linear modeling \cite{4}, and dictionary learning \cite{5}–\cite{7}, has been focused on centralized settings in which the entire data are assumed available at a single location. In recent years, there has been some effort to extend these works to distributed settings; see, e.g., \cite{8}–\cite{16}. The setup considered in some of these works is that each distributed entity is responsible for either some dimensions of the data \cite{8}–\cite{10} or some

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part of the learned geometric structure \cite{9}, \cite{10}, \cite{15}. Other works in this direction also focus on learning under the assumption of data lying near \textit{(linear) subspaces} \cite{8}–\cite{12}, require extensive communications among the distributed entities \cite{13}, and ignore some of the technical details associated with processing among distributed entities having interconnections described by graphs of arbitrary, unknown topologies \cite{11}–\cite{14}.

In this paper, we are interested in a setting in which a number of geographically-distributed sites have massive local data and these sites are interested in collaboratively learning a geometric structure underlying their data by communicating among themselves over public and/or private networks. The key constraints in this problem, which we term \textit{data-adaptive representations in the cloud}, that distinguish it from some of the prior related works are that: (i) sites cannot communicate “raw” data among themselves; (ii) interconnections among sites are not described by a complete graph; and (iii) sites do not have knowledge of the global network topology. All these constraints are reflective of the future of big, distributed data in the world. In particular, the first constraint is justified because of the size of local data compilations as well as privacy concerns in the modern age. Similarly, the latter two constraints are justified because linking geographically-distributed sites into a complete graph can be cost prohibitive and since enterprises tend to be protective of their internal network topologies.

\textbf{A. Our Contributions}

The first main contribution of this paper is formulation of a distributed method, which we term as \textit{cloud K-SVD}, that enables data-adaptive representations in the cloud. In contrast to works that assume a linear geometric structure for data \cite{8}–\cite{12}, cloud K-SVD is based on the premise that data lie near a \textit{union} of low-dimensional subspaces. The \textit{union-of-subspaces} (UoS) model is a nonlinear generalization of the subspace model \cite{17} and has received widespread acceptance in the community lately. The task of learning the UoS underlying data of interest from data themselves is often termed \textit{dictionary learning} \cite{5}–\cite{7}, which involves data-driven learning of an overcomplete dictionary such that every data sample can be approximated through a small number of atoms of the dictionary. Dictionary learning—when compared to linear data-adaptive representations such as the PCA and the linear discriminant analysis \cite{18}—has been shown to be highly effective for information processing tasks such as compression \cite{5}, denoising \cite{19}, object recognition \cite{20}, and inpainting \cite{21}. Cloud K-SVD, as the name implies, is a distributed variant of the popular dictionary learning algorithm K-SVD \cite{6} and leverages a classical iterative eigenvector estimation algorithm, termed the \textit{power method} \cite{22} Ch. 8, and consensus averaging \cite{23} for collaborative dictionary learning.

The second main contribution of this paper is a rigorous analysis of cloud K-SVD that gives insights into its properties as well as deviations of the dictionaries learned at individual sites from the centralized K-SVD solution in terms of different measures of local/global data and topology of the interconnections. We noted earlier that a distributed variant of the power method that relies on consensus averaging lies at the heart of cloud K-SVD. Using tools from linear algebra, convex optimization, matrix perturbation theory, etc., our analysis shows that—under identical initializations—the dictionaries learned by cloud K-SVD come arbitrarily close to the one learned by (centralized) K-SVD as long as appropriate number of power method and consensus iterations are performed in

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each iteration of cloud K-SVD. Finally, the third main contribution of this paper involves numerical experiments on synthetic and real data that demonstrate both the efficacy of cloud K-SVD and the usefulness of collaborative dictionary learning over local dictionary learning.

B. Relationship to Previous Work

Some of the earliest works in distributed processing date back nearly three decades, such as distributed Kalman filtering [24] and consensus [25]. Since then a number of distributed methods have been proposed for myriad information processing tasks. Some recent examples of this that do not involve a centralized fusion center include methods for distributed classification [26]–[28], distributed localization [29], [30], distributed linear regression [31], and distributed estimation [32]. However, relatively little attention has been paid to the problem of data-driven distributed learning of the geometric structure of data. Most notable exceptions to this include [10]–[16]. While our work as well as [10]–[14] rely on consensus averaging for computing the underlying geometric structure, we are explicit in our formulation that perfect consensus under arbitrary, unknown topologies cannot be achieved. In contrast, developments in [10]–[14] are carried out under the assumption of infinite-precision consensus averaging. Further, [10]–[12] assume a subspace model for data, while [13] advocates the use of consensus averaging for computing sample covariance—an approach that requires extensive communications among the distributed entities. Our work is most closely related to that in [15], [16], which also study dictionary learning in distributed settings. However, [15] focuses only on learning parts of the dictionary at each site as opposed to the setup of this paper in which we are interested in learning a complete dictionary at each site. While [16] and this paper share the same setup, our approach is fundamentally different from that in [16]. Each iteration of the method proposed in [16] involves learning local dictionaries at different sites and then diffusing these local dictionaries to obtain a global dictionary. In contrast, cloud K-SVD involves updating each atom of the local dictionaries in a collaborative fashion. Such an approach helps us rigorously analyze the performance of cloud K-SVD in relation to the centralized solution, whereas no such analysis is provided in [16]. Finally, note that cloud K-SVD was initially proposed by us in [1], which predates both [15] and [16]. While [1] only analyzed the distributed power method component of cloud K-SVD, we extend that work in this paper to provide an analysis of the complete cloud K-SVD algorithm.

We conclude by noting that the distributed power method component of cloud K-SVD has similarities with the work in [11], [33]. However, unlike [11], [33], we do not assume perfect consensus during iterations of the power method, which leaves open the question of convergence of the distributed variant of the power method. While analyzing cloud K-SVD, we in fact end up addressing this question also. That part of our analysis is reminiscent of the one carried out in [34] in the context of convergence behavior of distributed eigenanalysis of a network using a power method-like iterative algorithm. However, there are fundamental differences in the analysis of [34] and our work because of the exact place where consensus averaging is carried out in the two works, which is dictated by the distinct nature of the two applications.
Fig. 1. A schematic representing global data $Y$ distributed across $N$ sites. Here, $n$ denotes the dimension of each data sample, while $S_i$ denotes the total number of data samples available at the $i^{th}$ site.

C. Notation and Paper Organization

We use lower-case letters to represent scalars and vectors, while we use upper-case letters to represent matrices. The operator $\text{sgn} : \mathbb{R} \rightarrow \{+1, -1\}$ is defined as $\text{sgn}(x) = x/|x|$, while $\text{supp}(v)$ returns indices of the nonzero entries in vector $v$. Superscript $(\cdot)^T$ denotes the transpose operation, $\| \cdot \|_0$ counts the number of nonzero entries in a vector, $\|v\|_p$ denotes the usual $\ell_p$ norm of vector $v$, and $\langle u, v \rangle$ denotes the inner product between vectors $u$ and $v$. Given a set $\mathcal{I}$, $v|_{\mathcal{I}}$ and $A|_{\mathcal{I}}$ denote a subvector and a submatrix obtained by retaining entries of vector $v$ and columns of matrix $A$ corresponding to the indices in $\mathcal{I}$, respectively, while $\|A\|_2$, $\|A\|_F$, and $\|A\|_{\text{max}}$ denote the operator norm, Frobenius norm, and max norm (i.e., maximum absolute value) of matrix $A$, respectively. Finally, given a matrix $A$, $a_j$ and $a_{j,T}$ denote the $j^{th}$ column and the $j^{th}$ row of $A$, respectively.

The rest of this paper is organized as follows. In Section II we formulate the problem of collaborative dictionary learning from big, distributed data. In Section III we describe the cloud K-SVD algorithm. In Section IV we provide an analysis of cloud K-SVD algorithm. We provide some numerical results in Section V and concluding remarks in Section VI. Finally, proofs of main theorems stated in Section IV are given in appendices.

II. Problem Formulation

In this paper, we consider a collection of $N$ geographically-distributed sites that are interconnected to each other according to a fixed topology. Here, we use “site” in the broadest possible sense of the term, with a site corresponding to a single computational system (e.g., sensor, drone, smartphone, tablet, server, database), a collection of co-located computational systems (e.g., data center, computer cluster, robot swarm), etc. Mathematically, we represent this collection and their interconnections through an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N} = \{1, 2, \cdots, N\}$ denotes the sites and $\mathcal{E}$ denotes edges in the graph with $(i, i) \in \mathcal{E}$, while $(i, j) \in \mathcal{E}$ whenever there is a connection between site $i$ and site $j$. The only assumption we make about the topology of $\mathcal{G}$ is that it is a connected graph.

Next, we assume that each site $i$ has a massive collection of local data, expressed as a matrix $Y_i \in \mathbb{R}^{n \times S_i}$ with $S_i \gg n$ representing the number of data samples at the $i^{th}$ site. We can express all of this distributed
data into a single matrix \( Y = \begin{bmatrix} Y_1 & Y_2 & \ldots & Y_N \end{bmatrix} \in \mathbb{R}^{n \times S} \), where \( S = \sum_{i=1}^{N} S_i \) denotes the total number of data samples distributed across the \( N \) sites; see Fig. [1] for a schematic representation of this. In this setting, the fundamental objective is for each site to collaboratively learn a low-dimensional geometric structure that underlies the global (distributed) data \( Y \). The basic premises behind collaborative structure learning of global data, as opposed to local structure learning of local data, are manifold. First, since the number of global samples is much larger than the number of local samples, we expect that collaborative learning will outperform local learning for data representations. Second, local learning will be strictly suboptimal for some sites in cases where sampling density, noise level, fraction of outliers, etc., are not uniform across all sites. Collaborative learning, on the other hand, will even out such nonuniformities within local data.

Our fundamental assumption is that the low-dimensional geometric structure underlying the global data corresponds to a union of \( T_0 \)-dimensional subspaces in \( \mathbb{R}^n \), where \( T_0 \ll n \). One possible means of learning such a structure is studied under the moniker dictionary learning, which learns an overcomplete dictionary \( D \) such that each data sample is well approximated by no more than \( T_0 \) columns (i.e., atoms) of \( D \) [5]–[7]. Specifically, assuming that the global data \( Y \) is available at a centralized location, the problem of dictionary learning can be expressed as

\[
(D, X) = \arg \min_{D,X} \| Y - DX \|_F^2 \quad \text{such that} \quad \forall s, \| x_s \|_0 \leq T_0, \tag{1}
\]

where \( D \in \mathbb{R}^{n \times K} \) with \( K > n \) is an overcomplete dictionary having unit \( \ell_2 \)-norm columns, \( X \in \mathbb{R}^{K \times S} \) corresponds to representation coefficients of the data having no more than \( T_0 \) nonzero coefficients per sample, and \( x_s \) denotes the \( s \)th column in \( X \). Note that (1) is non-convex in \((D, X)\), although it is convex in \( D \) alone. One of the most popular approaches to solving (1) involves alternate minimization in which one alternates between solving (1) for \( D \) using a fixed \( X \) and then solving (1) for \( X \) using a fixed \( D \) [6], [35].

Unlike classical dictionary learning, however, we do not have the global data \( Y \) available at a centralized location. Data aggregation either at a centralized location or at any one of the individual sites is also impractical due to communications and storage costs of big data. Therefore, our goal is to have individual sites collaboratively learn dictionaries \( \{ \hat{D}_i \}_{i \in N} \) from global data \( Y \) such that these collaborative dictionaries are close to a dictionary \( D \) that could have been learned from \( Y \) in a centralized fashion. In addition, we are interested in accomplishing this goal without ever exchanging individual samples between sites because of privacy concerns. In the following section, we present a distributed variant of a popular dictionary learning algorithm that accomplishes this goal. This is followed by a rigorous analysis of the proposed algorithm in Section [IV] which establishes that the collaborative dictionaries learned using our proposed algorithm can indeed be made to come arbitrarily close to a centralized dictionary.

### III. CLOUD K-SVD

In this paper, we focus on the well-studied K-SVD dictionary learning algorithm [6] as the basis for collaborative dictionary learning. We have chosen to work with K-SVD because of its iterative nature and its reliance on the singular value decomposition (SVD), both of which enable its exploitation for distributed purposes. In the following, we first provide a brief overview of K-SVD, which is followed by presentation of our proposed algorithm—termed cloud K-SVD—for collaborative dictionary learning.
A. Dictionary Learning Using K-SVD

The K-SVD algorithm initializes with a (often randomized) dictionary $D^{(0)}$ and solves (1) by iterating between two stages: a sparse coding stage and a dictionary update stage [6]. Specifically, for a fixed estimate of the dictionary $D^{(t-1)}$ at the start of iteration $t \geq 1$, the sparse coding stage in K-SVD involves solving (1) for $X^{(t)}$ as follows:

$$\forall s, x^{(t)}_s = \arg \min_{x \in \mathbb{R}^K} \|y_s - D^{(t-1)}x\|_2^2 \text{ such that } \|x\|_0 \leq T_0,$$

(2)

where $y_s$ denotes the $s^{th}$ column of $Y$. Note that while (2) in its stated form has combinatorial complexity, it can be easily solved by either convexifying [2] [36] or using greedy pursuit algorithms [37].

After the sparse coding stage, K-SVD fixes $X^{(t)}$ and moves to the dictionary update stage. The main novelty in K-SVD lies in the manner in which it carries out dictionary update, which involves iterating through the $K$ atoms of $D^{(t-1)}$ and individually updating the $k^{th}$ atom, $k = 1, 2, \ldots, K$, as follows:

$$d^{(t)}_k = \arg \min_{d \in \mathbb{R}^n} \left\| Y - \sum_{j=1}^{k-1} d^{(t)}_{j,T} x^{(t)}_j - \sum_{j=k+1}^{K} d^{(t-1)}_{j,T} x^{(t)}_j - d x^{(t)}_k \right\|_F^2 = \arg \min_{d \in \mathbb{R}^n} \left\| E^{(t)}_k - d x^{(t)}_{k,T} \right\|_F^2.$$

(3)

Here, $E^{(t)}_k$ is the representation error for $Y$ using first $k-1$ atoms of $D^{(t)}$ and last $k+1, \ldots, K$ atoms of $D^{(t-1)}$. In order to simplify computations, K-SVD in [6] further defines an ordered set $\omega^{(t)}_k = \{ s : 1 \leq s \leq S, x^{(t)}_{k,T}(s) \neq 0 \}$, where $x^{(t)}_{k,T}(s)$ denotes the $s^{th}$ element of $x^{(t)}_{k,T}$, and an $S \times |\omega^{(t)}_k|$ binary matrix $\Omega^{(t)}_k$ that has ones in $(\omega^{(t)}_k(s), s)$ locations and zeros everywhere else. Then, defining $E^{(t)}_{k,R} = E^{(t)}_k \Omega^{(t)}_k$ and $x^{(t)}_{k,R} = x^{(t)}_{k,T} \Omega^{(t)}_k$, it is easy to see from [3] that

$$d^{(t)}_k = \arg \min_{d \in \mathbb{R}^n} \left\| E^{(t)}_{k,R} - d x^{(t)}_{k,R} \right\|_F^2.$$

(4)

Solving (4) is equivalent to finding the best rank-one approximation of $E^{(t)}_{k,R}$, which is given by the Eckart–Young theorem as $d^{(t)}_k x^{(t)}_{k,R} = \sigma_1 u_1 v_1^T$, where $u_1$ and $v_1$ denote the largest left- and right-singular vectors of $E^{(t)}_{k,R}$, respectively, while $\sigma_1$ denotes the largest singular value of $E^{(t)}_{k,R}$. The $k^{th}$ atom of $D^{(t)}$ can now simply be updated as $d^{(t)}_k = u_1$. It is further advocated in [6] that the $k^{th}$ row of the “reduced” coefficient matrix, $x^{(t)}_{k,R}$, should be simultaneously updated to $x^{(t)}_{k,R} = \sigma_1 v_1^T$. The dictionary update stage in K-SVD involves $K$ such applications of the Eckart–Young theorem to update the $K$ atoms of $D^{(t-1)}$ and the $K$ “reduced” rows of $X^{(t)}$. The algorithm then moves to the sparse coding stage and continues alternating between the two stages till a stopping criterion (e.g., a prescribed representation error) is reached. We conclude our discussion of K-SVD by noting that it can be guaranteed to converge to a local minimum [6].

B. Collaborative Dictionary Learning Using Cloud K-SVD

In this section, we present our collaborative dictionary learning algorithm that is based on K-SVD. The key to distributing K-SVD is understanding ways in which both the sparse coding and the dictionary update stages can be distributed across the $N$ sites. To this end, we assume our collaborative dictionary learning is in iteration $t \geq 1$ and each site $i$ in this iteration has a local estimate $\hat{D}^{(t-1)}_i$ of the desired dictionary from the previous iteration.
order for the sparse coding stage to proceed, we propose that each site computes the representation coefficients of its local data without collaborating with other sites by locally solving

$$\forall s, \hat{x}_{i,s}^{(t)} = \arg \min_{x \in \mathbb{R}^k} \| y_{i,s} - \hat{D}_{i}^{(t-1)} x \|_2^2 \quad \text{such that} \quad \| x \|_0 \leq T_0,$$

where $y_{i,s}$ and $\hat{x}_{i,s}^{(t)}$ denote the $s^{th}$ sample and its coefficient vector at site $i$, respectively. Note that this “local” sparse coding for collaborative dictionary learning greatly simplifies the sparse coding stage and is justified as long as the local dictionary estimates $\hat{D}_{i}^{(t-1)}$ remain close to each other (which will be established in Section IV).

The next challenge in collaborative dictionary learning based on K-SVD arises during the dictionary update stage. Recall that the dictionary update stage in K-SVD involves computing the largest left- and right-singular vectors of the “reduced” error matrix $E_{k,R}^{(t)} = E_{k}^{(t)} \Omega_{k}^{(t)}$, $k = 1, 2, \ldots, K$. However, unless the local dictionary estimates $\hat{D}_{i}^{(t-1)}$ happen to be identical, we end up with $N$ such (reduced) error matrices in a distributed setting due to $N$ different local dictionary estimates. In order to resolve this, we propose to use the following definition of the reduced error matrix in a distributed setting $E_{k,R}^{(t)} = \begin{bmatrix} \hat{E}_{1,k,R}^{(t)} & \hat{E}_{2,k,R}^{(t)} & \cdots & \hat{E}_{N,k,R}^{(t)} \end{bmatrix}$, where

$$\hat{E}_{i,k,R}^{(t)} = Y_{i} \tilde{D}_{i,k,R}^{(t)} - \left( \sum_{j=1}^{k-1} \tilde{d}_{i,j,T}^{(t)} \tilde{x}_{i,j,T}^{(t)} + \sum_{j=k+1}^{K} \tilde{d}_{i,j,T}^{(t-1)} \tilde{x}_{i,j,T}^{(t)} \right) \tilde{x}_{i,k,R}^{(t)}.$$

Here, $\tilde{x}_{i,j,T}^{(t)}$ denotes the $j^{th}$ row of coefficient matrix $\tilde{X}_{i}^{(t)}$ computed at site $i$ during the sparse coding step performed on $Y_{i}$ using $\hat{D}_{i}^{(t-1)}$ at the start of iteration $t$, while $\tilde{x}_{i,k,R}^{(t)}$ denotes the $j^{th}$ row of the updated coefficient matrix $\tilde{X}_{i}^{(t)}$ available at site $i$ due to the update in coefficient matrix performed during the dictionary update step. Furthermore, $\tilde{\Omega}_{i,k}^{(t)}$ is similar to $\Omega_{k}^{(t)}$ defined for K-SVD except that it is defined for only local coefficient matrix $\tilde{X}_{i}^{(t)}$ at site $i$.

Next, in keeping with the K-SVD derivation in [6], we propose that each of the $N$ sites updates the $k^{th}$ atom of its respective local dictionary and the $k^{th}$ row of its respective “reduced” coefficient matrix, $\tilde{x}_{i,k,R}^{(t)} = \tilde{x}_{i,k,R}^{(t)} \hat{\tilde{d}}_{i,k,R}^{(t)}$, by collaboratively computing the dominant left- and right-singular vectors of the distributed error matrix $\hat{E}_{k,R}^{(t)}$, denoted by $u_1$ and $v_1$, respectively. In fact, since $u_1^{T} \hat{E}_{k,R}^{(t)} = \sigma_1 v_1$ with $\sigma_1$ being the largest singular value of $\hat{E}_{k,R}^{(t)}$, it follows that if a site has access to the dominant left-singular vector, $u_1$, of $\hat{E}_{k,R}^{(t)}$ then it can simply update the $k^{th}$ row of its respective “reduced” coefficient matrix by setting $\hat{d}_{i,k,R}^{(t)} = u_1$ and setting $\tilde{x}_{i,k,R}^{(t)} = \hat{d}_{i,k,R}^{(t)} \hat{E}_{i,k,R}^{(t)}$. Therefore, we need only worry about collaborative computation of $u_1$ in this setting. To this end, we define $\hat{M}_{i}^{(t)} = \hat{E}_{i,k,R}^{(t)} \hat{E}_{i,k,R}^{(t)}$ and notice that each $\hat{M}_{i}^{(t)} = \hat{E}_{i,k,R}^{(t)} \hat{E}_{i,k,R}^{(t)}$ is a matrix that is readily computable at each local site. Our goal now is computing the dominant eigenvector of $\hat{M}_{i}^{(t)} = \sum_{i=1}^{N} \hat{M}_{i}^{(t)}$ in a collaborative manner at each site. In order for this, we will make use of distributed power method, which has been invoked previously in [11], [12], [34] and which corresponds to a distributed variant of the classical power method for eigenanalysis [22].

**Distributed Power Method:** Power method is an iterative procedure for computing eigenvectors of a matrix. It is simple to implement and, assuming that the largest eigenvalue $\lambda_1$ of a matrix is strictly greater than its second-largest eigenvalue $\lambda_2$, it converges to the subspace spanned by the dominant eigenvector at an exponential rate. In this paper, we are interested in a distributed variant of the power method to compute the dominant eigenvector of $\hat{M}_{i}^{(t)} = \sum_{i=1}^{N} \hat{M}_{i}^{(t)}$, where the $\hat{M}_{i}^{(t)}$’s are distributed across $N$ sites. To this end, we proceed as follows.
First, all sites initialize to the same (unit-norm) estimate of the eigenvector $\hat{q}_i^{(0)} = q_i^{\text{init}}$. Next, assuming that the sites are carrying out iteration $t_p$ of the distributed power method, each site computes $\hat{M}_i(t) \hat{q}_i^{(t_p-1)}$ locally, where $\hat{q}_i^{(t_p-1)}$ denotes an estimate of the dominant eigenvector of $\hat{M}(t)$ at the $i^{th}$ site after $t_p - 1$ iterations. In the next step, the sites collaboratively attempt to compute an approximation $\hat{v}_i^{(t_p)}$ of $\sum_i \hat{M}_i(t) \hat{q}_i^{(t_p-1)}$ at each site. In the final step of the $t_p^{th}$ iteration of the distributed power method, each site normalizes its estimate of the dominant eigenvector of $\hat{M}(t)$ locally: $\hat{q}_i^{(t_p)} = \hat{v}_i^{(t_p)}/\|\hat{v}_i^{(t_p)}\|_2$.

It is clear from the preceding discussion that the key in distributed power method is the ability of the sites to collaboratively compute an approximation of $\sum_i \hat{M}_i(t) \hat{q}_i^{(t_p-1)}$ in each iteration. In order for this, we make use of the popular consensus averaging method [38]. To perform consensus averaging, we first design a doubly-stochastic weight matrix $W$ that adheres to the topology of the underlying graph $G$. In particular, we have that $w_{i,j} = 0$ whenever $(i, j) \notin \mathcal{E}$. We refer the reader to [38]-[40] for designing appropriate weight matrices in a distributed manner without relying on knowledge of the global network topology. In order to compute $\sum_i \hat{M}_i(t) \hat{q}_i^{(t_p-1)}$ using consensus averaging, each site is initialized with $z_i^{(0)} = \hat{M}_i(t) \hat{q}_i^{(t_p-1)}$. Next, let $\mathcal{N}_i = \{ j : (i, j) \in \mathcal{E} \}$ be the neighborhood of site $i$, define $Z(t)^c = [z_1^{(0)} z_2^{(0)} \ldots z_N^{(0)}]$, and assume we are in $t_c^{th}$ iteration of consensus averaging. Then consensus works by having each site carry out the following updates in each consensus iteration through communications with its neighbors: $z_i^{(t_c)} = \sum_{j \in \mathcal{N}_i} w_{i,j} z_j^{(t_c-1)}$. The dynamics of the overall system in this case evolve as $Z^{(t_c)} = W^{t_c} Z^{(0)}$. It then follows that $Z_i^{(t_c)} = 1^T N Z^{(0)}/N$ [38], where $Z_i^{(t_c)}$ denotes the $i^{th}$ row of $Z^{(t_c)}$ and $1 \in \mathbb{R}^N$ denotes a (column) vector of all ones. This in particular implies that each site achieves perfect consensus averaging as $t_c \to \infty$ and obtains $Z_i^{(\infty)} = 1/N \sum_{j=1}^N z_j^{(0)} = 1/N \sum_{j=1}^N \hat{M}_j(t) \hat{q}_j^{(t_p-1)}$.

But one can not perform infinite consensus iterations in practice within each iteration of the distributed power method. Instead, we assume a finite number of consensus iterations, denoted by $T_c$, in each power method iteration and make use of the modification of standard consensus averaging proposed in [34] to obtain $\hat{v}_i^{(t_p)} = Z_i^{(T_c)}/[W_1^{T_c}]_i$, where $W_1^{T_c}$ is the first column of $W^{T_c}$ and $[\cdot]_i$ denotes the $i^{th}$ entry of a vector. Note that this leads to an error $\epsilon_{i,c}^{(t_p)}$ within $\hat{v}_i^{(t_p)}$ at each site for any finite $T_c$, i.e.,

$$\hat{v}_i^{(t_p)} = Z_i^{(T_c)}/[W_1^{T_c}]_i = \sum_{j=1}^N \hat{M}_j \hat{q}_j^{(t_p-1)} + \epsilon_{i,c}^{(t_p)}.$$ (7)

After finishing consensus iterations, each site $i$ in iteration $t_p$ of power method normalizes this vector $\hat{v}_i^{(t_p)}$ to get an estimate of the dominant eigenvector of $\hat{M}(t)$. Finally, we carry out enough iterations of the distributed power method at each site that the error between successive estimates of the eigenvector falls below a prescribed threshold.

We have now motivated and described all the key components of our proposed algorithm and are ready to state the full collaborative dictionary learning algorithm, termed \textit{cloud K-SVD}, which is detailed in Algorithm 1. Note that the initialization of cloud K-SVD differs from K-SVD in the sense that each site also generates a common (random) reference vector $d^{ref} \in \mathbb{R}^n$ and stores it locally. The purpose of $d^{ref}$ is to ensure that the eigenvectors

This can be accomplished, for example, through the use of (local) random number generators initialized with the same seed. Also, note that a key requirement in power method is that $\langle u_1, q^{\text{init}} \rangle \neq 0$, which is ensured with very high probability in the case of random initialization.
Algorithm 1: Cloud K-SVD for collaborative dictionary learning

Input: Local data \(Y_1, Y_2, \ldots, Y_N\), problem parameters \(K\) and \(T_0\), and doubly-stochastic matrix \(W\).

Initialize: Generate \(d^{\text{ref}} \in \mathbb{R}^n\) and \(D^{\text{init}} \in \mathbb{R}^{n \times K}\) randomly, set \(t \leftarrow 0\) and \(\tilde{D}_i^{(t)} \leftarrow D^{\text{init}}, i = 1, \ldots, N\).

1: while stopping rule do
2: \(t \leftarrow t + 1\)
3: (Sparse Coding) The \(i\)th site solves \(\forall s, \tilde{x}_{i,s}^{(t)} = \arg \min_{x \in \mathbb{R}^K} \|y_{i,s} - \tilde{D}_i^{(t)} x\|_2^2\) such that \(\|x\|_0 \leq T_0\)
4: for \(k = 1\) to \(K\) (Dictionary Update) do
5: \(\tilde{E}_{i,k,R} = Y_i \tilde{q}_{i,k}^{(t)} - \sum_{j=1}^{K} \tilde{q}_{i,j}^{(t)} \tilde{E}_{i,j,R}^{(t)} - \sum_{j=k+1}^{K} \tilde{q}_{i,j}^{(t)} \tilde{E}_{i,j,R}^{(t)}\)
6: \(\tilde{M}_i \leftarrow \tilde{E}_{i,k,R}^{(t)} \tilde{M}_{i,k,R}\)
7: (Initialize Consensus Averaging) Set \(t_c \leftarrow 0\) and \(\tilde{z}_i^{(t_c)} \leftarrow \tilde{M}_i \tilde{q}_i^{(t_c-1)}\)
8: while stopping rule do
9: \(t_p \leftarrow t_p + 1\)
10: (Initialize Consensus Averaging) Set \(t_c \leftarrow 0\) and \(\tilde{z}_i^{(t_c)} \leftarrow \tilde{M}_i \tilde{q}_i^{(t_c-1)}\)
11: while stopping rule do
12: \(t_c \leftarrow t_c + 1\)
13: \(\tilde{z}_i^{(t_c)} = \sum_{j \in N_i} w_{i,j} \tilde{z}_i^{(t_c-1)}\)
14: end while
15: \(\tilde{v}_i^{(t_p)} \leftarrow \tilde{z}_i^{(t_c)} / \|W_{1:t_c}\|_2\)
16: \(\tilde{q}_i^{(t_p)} \leftarrow \tilde{v}_i^{(t_p)} / \|\tilde{v}_i^{(t_p)}\|_2\)
17: (Initialize Consensus Averaging) Set \(t_c \leftarrow 0\) and \(\tilde{z}_i^{(t_c)} \leftarrow \tilde{M}_i \tilde{q}_i^{(t_c-1)}\)
18: \(\tilde{d}_{i,k}^{(t)} \leftarrow \text{sgn}\left(\langle d^{\text{ref}}, \tilde{q}_i^{(t_p)} \rangle\right) \tilde{q}_i^{(t_p)}\)
19: \(\tilde{z}_{i,k,R}^{(t)} \leftarrow \tilde{d}_{i,k}^{(t)} \tilde{E}_{i,k,R}^{(t)}\)
20: end for
21: end while

Return: \(\tilde{D}_i^{(t)}, i = 1, 2, \ldots, N\).

computed by different sites using the distributed power method all point in the same quadrant, rather than in antipodal quadrants (see Step 18 in Algorithm 1). While this plays a critical role in our analysis, it does not have an effect on the fundamental workings of cloud K-SVD. Notice also that we have not defined any specific stopping rules in Algorithm 1. One set of rules could be to run the algorithm for fixed dictionary learning iterations \(T_d\), power method iterations \(T_p\), and consensus iterations \(T_c\). Our analysis of cloud K-SVD in the next section also assumes these stopping rules. We now comment on the communication costs of cloud K-SVD. During each iteration of cloud K-SVD, every site exchanges only an \(n\)-dimensional vector with its neighbors and the total number of such exchanges for each site is given by \(K T_d T_p T_c\). The most important thing to note here is that the amount of data exchanges among the distributed sites is not an explicit function of the number of data samples available at
each site, making cloud K-SVD a particularly useful algorithm for big data problems (also, see discussion after Theorem 1 in Section IV).

Remark 1. A careful reading of Algorithm 1 reveals that normalization by \([W^t_i]_i\) in Step 15 is redundant due to the normalization in Step 16. We retain the current form of Step 15 however to facilitate the forthcoming analysis.

IV. ANALYSIS OF CLOUD K-SVD

Since power method and consensus averaging in Algorithm 1 cannot be performed for an infinite number of iterations in practice, it is important for us to understand whether the dictionaries \(\{\hat{D}_i\}\) returned by cloud K-SVD approach the dictionary that could have been obtained by centralized K-SVD \([6]\). In order to address this question, we need to understand the behavior of major components of cloud K-SVD, which include sparse coding, dictionary update, and distributed power method within dictionary update. In addition, one also expects that the closeness of \(\hat{D}_i\)'s to the centralized solution will be a function of certain properties of local/global data. We begin our analysis of cloud K-SVD by first stating some of these properties in terms of the centralized K-SVD solution.

A. Preliminaries

The first thing needed to quantify deviations of the cloud K-SVD dictionaries from the centralized K-SVD dictionary is algorithmic specification of the sparse coding steps in both algorithms. Specifically, while the sparse coding steps as stated in (2) and (5) have combinatorial complexity, various low-complexity computational approaches can be used to solve these steps in practice. Our analysis in the following will be focused on the case when sparse coding in both cloud K-SVD and centralized K-SVD is carried out using the lasso \([41]\). Specifically, we assume sparse coding is carried out by solving

\[
x_{i,s} = \arg \min_{x \in \mathbb{R}^K} \frac{1}{2} \|y_{i,s} - Dx\|_2^2 + \tau \|x\|_1
\]  

with the regularization parameter \(\tau > 0\) selected in a way that \(\|x_{i,s}\|_0 \leq T_0 \ll n\). This can be accomplished, for example, by making use of the least angle regression algorithm \([42]\). Note that the lasso also has a dual, constrained form, given by

\[
x_{i,s} = \arg \min_{x \in \mathbb{R}^K} \frac{1}{2} \|y_{i,s} - Dx\|_2^2 \quad \text{subject to} \quad \|x\|_1 \leq \eta,
\]  

and the solutions of (8) and (9) are identical for an appropriate \(\eta_\tau = \eta(\tau)\) \([43]\).

Our analysis in the following is also based on the assumption that cloud K-SVD and centralized K-SVD are identically initialized, i.e., \(\hat{D}_i^{(0)} = D^{(0)}, i = 1, \ldots, N\), where \(D^{(t)}, t \geq 0\), in the following denotes the centralized K-SVD dictionary estimate in the \(t^{th}\) iteration. While both cloud K-SVD and centralized K-SVD start from the same initial estimates, the cloud K-SVD dictionaries get perturbed in each iteration due to imperfect power method and consensus averaging. In order to ensure these perturbations do not cause the cloud K-SVD dictionaries to

2Extensions of our analysis for the case when sparse coding is performed using greedy methods such as orthogonal matching pursuit (OMP) \([37]\) will be pursued in future works.
Let \( K\text{-SVD} \) in each iteration to satisfy certain properties. Below, we present and motivate these properties.

**[P1]** Let \( x_{i,s}^{(t)} \) denote the solution of the lasso (i.e., \((8)\)) for \( D = D^{(t-1)} \) and \( \tau = \tau^{(t)}, t = 1, \ldots, T_d \). Then there exists some \( C_1 > 0 \) such that the following holds:

\[
\min_{t,i,s,j \notin \text{supp}(x_{i,s}^{(t)})} \tau^{(t)} - \|d_j^{(t)} - y_{i,s} - D^{(t-1)}x_{i,s}^{(t)}\| > C_1. \tag{10}
\]

In our analysis in the following, we will also make use of the smallest regularization parameter among the collection \( \{\tau^{(t)}\}_{t=1}^{T_d} \), defined as \( \tau_{\min} = \min_t \tau^{(t)} \), and the largest dual parameter among the (dual) collection \( \{\eta^{(t)}\}_{t=1}^{T_d} \), defined as \( \eta^{(t)} = \max_t \eta^{(t)} \).

**[P2]** Define \( \Sigma_{T_0} = \{I \subset \{1, \ldots, K\} : |I| = T_0\} \). Then there exists some \( C_2' > \frac{c_{14}^2}{\sqrt{59.36}} \) such that the following holds:

\[
\min_{t = 1, \ldots, T_d, I \in \Sigma_{T_0}} \sigma_{T_0} \left( D^{(t-1)}_{|I|} \right) \geq \sqrt{C_2'}, \tag{11}
\]

where \( \sigma_{T_0}(\cdot) \) denotes the \( T_0^{th} \) (ordered) singular value of a matrix. In our analysis, we will be using the parameter \( C_2 = \left( \sqrt{C_2'} - \frac{c_{14}^2 \tau_{\min}}{44} \right)^2 \).

**[P3]** Let \( \lambda_{1,k}^{(t)} > \lambda_{2,k}^{(t)} \geq \ldots \geq \lambda_{n,k}^{(t)} \geq 0 \) denote the eigenvalues of the centralized “reduced” matrix \( E_{k,R}^{(t)}E_{k,R}^{(t)^T}, k \in \{1, \ldots, K\} \), in the \( t^{th} \) iteration, \( t \in \{1, \ldots, T_d\} \). Then there exists some \( C_3' < 1 \) such that the following holds:

\[
\max_{t,k} \frac{\lambda_{2,k}^{(t)}}{\lambda_{1,k}^{(t)}} \leq C_3'. \tag{12}
\]

Now define \( C_3 = \max \left\{ 1, \frac{1}{\min_{t,k} \lambda_{1,k}^{(t)}(1-C_3')} \right\} \), which we will use in our forthcoming analysis.

We now comment on the rationale behind these three properties. Properties P1 and P2 correspond to sufficient conditions for \( x_{i,s}^{(t)} \) to be a unique solution of \((8)\) [44] and guarantee that the centralized K-SVD generates a unique collection of sparse codes in each dictionary learning iteration. Property P3, on the other hand, ensures that algorithms such as the power method can be used to compute the dominant eigenvector of \( E_{k,R}^{(t)}E_{k,R}^{(t)^T} \) in each dictionary learning iteration [22]. In particular, P3 is a statement about the worst-case spectral gap of \( E_{k,R}^{(t)}E_{k,R}^{(t)^T} \).

In addition to these properties, our final analytical result for cloud K-SVD will also be a function of a certain parameter of the centralized error matrices \( \{E_{k}^{(t)}\}_{k=1}^{K} \) generated by the centralized K-SVD in each iteration. We define this parameter in the following for later use. Let \( E_{i,k}^{(t)}, i = 1, \ldots, N, \) denote part of the centralized error matrix \( E_{k}^{(t)} \) associated with the data of the \( i^{th} \) site in the \( t^{th} \) iteration, i.e., \( E_k^{(t)} = \left[ E_{1,k}^{(t)} \; E_{2,k}^{(t)} \; \cdots \; E_{N,k}^{(t)} \right], k = 1, \ldots, K, t = 1, \ldots, T_d. \) Then

\[
C_4 = \max \left\{ 1, \max_{t,i,k} \|E_{i,k}^{(t)}\|_2 \right\}. \tag{13}
\]
B. Main Result

We are now ready to state the main result of this paper. This result is given in terms of the $\| \cdot \|_2$ norm mixing time, $T_{\text{mix}}$, of the Markov chain associated with the doubly-stochastic weight matrix $W$, defined as

$$T_{\text{mix}} = \max_{i=1,\ldots,N} \inf \{ t : \| e_i^T W^t - \frac{1}{N} \|_2 \leq \frac{1}{2} \}.$$  

(14)

Here, $e_i \in \mathbb{R}^N$ denotes the $i^{th}$ column of the identity matrix $I_N$. Note that the mixing time $T_{\text{mix}}$ can be upper bounded in terms of inverse of the absolute spectral gap of $W$, defined as $1 - |\lambda_2(W)|$ with $\lambda_2(W)$ denoting the second largest (in modulus) eigenvalue of $W$. As a general rule, better-connected networks can be made to have smaller mixing times compared to sparsely connected networks.

**Theorem 1** (Stability of Cloud K-SVD Dictionaries). Suppose cloud K-SVD (Algorithm 1) and (centralized) K-SVD are identically initialized and both of them carry out $T_d$ dictionary learning iterations. In addition, assume cloud K-SVD carries out $T_p$ power method iterations during the update of each atom and $T_c$ consensus iterations during each power method iteration. Finally, assume the K-SVD algorithm satisfies properties P1–P3. Next, define $\alpha = \max_{i,t,k} \sum_{i=1}^N \| \hat{E}^{(t)}_{i,k,R} \hat{E}^{(t)}_{i,k,R}^T \|_2$, $\beta = \max_{i,t,p,k} \frac{1}{\| E^{(t)}_{i,k,R} E^{(t)}_{i,k,R}^T \|_2}$, $\gamma = \max_{i,t,k} \sqrt{\sum_{i=1}^N \| \hat{E}^{(t)}_{i,k,R} \hat{E}^{(t)}_{i,k,R}^T \|_2}$, $\nu = \max_{i,k} \frac{\hat{\lambda}_{2,k}^{(t)}}{\hat{\lambda}_{1,k}^{(t)}}$, $\tilde{\theta}_k^{(t)} \in [0, \pi/2]$ as $\tilde{\theta}_k^{(t)} = \arccos \left( \frac{\langle \hat{u}^{(t)}_{1,k}, \hat{q}^{(t)}_{\text{init}} \rangle}{\| \hat{u}^{(t)}_{1,k} \|_2 \| \hat{q}^{(t)}_{\text{init}} \|_2} \right)$, $\mu = \max \{ 1, \max_{i,k,t} \tan(\tilde{\theta}_k^{(t)}) \}$, and $\zeta = K \sqrt{2 S_{\text{max}} (\frac{6 \sqrt{K}}{\tau_{\text{min}}} + 1) + \eta_{\tau_{\text{max}}}}$, where $S_{\text{max}} = \max_i S_i$, $u^{(t)}_{1,k}$ is the dominant eigenvector of $\hat{E}^{(t)}_{i,k,R} \hat{E}^{(t)}_{i,k,R}^T$, $\hat{\lambda}_{1,k}$ and $\hat{\lambda}_{2,k}$ are first and second largest eigenvalues of $\hat{E}^{(t)}_{i,k,R} \hat{E}^{(t)}_{i,k,R}^T$ respectively, and $q^{(t)}_{\text{init}}$ denotes the iterates of a centralized power method initialized with $q^{\text{init}}$ for estimation of the dominant eigenvector of $\hat{E}^{(t)}_{i,k,R} \hat{E}^{(t)}_{i,k,R}^T$. Then, assuming $\min_{i,k} | \langle \hat{u}^{(t)}_{1,k}, q^{\text{init}} \rangle | > 0$, and fixing any $\epsilon \in \left( 0, \min \left\{ (10 \alpha^2 \beta^2)^{-1/3} T_p, (\frac{1 - \nu}{4})^{1/3} \right\} \right)$ and $\delta_d \in \left( 0, \min \left\{ \frac{1}{\sqrt{2}}, \frac{C_t \tau_{\text{min}}}{4 \sqrt{2} K} \right\} \right)$, we have

$$\max_{k=1,\ldots,K} \left\| \tilde{a}^{(T_d)}_{i,k} \tilde{a}^{(T_d)}_{i,k}^T - d^{(T_d)}_k d^{(T_d)}_k^T \right\|_2 \leq \delta_d$$

as long as the number of power method iterations $T_p \geq \frac{2 (T_d K - 2) \log(8 C_2 C_4^2 N + 5) + (T_d - 1) \log(1 + \epsilon) + \log(8 C_2 C_4^{2/3} \sqrt{\tau_{\text{mix}}})}{\log([\nu + (4 \tau_{\text{mix}})^3]^{-1})}$ and the number of consensus iterations $T_c = \Omega(T_p T_{\text{mix}} \log(2 \alpha \beta \epsilon^{-1}) + T_{\text{mix}} \log(\alpha^{-1} \gamma \sqrt{N})$.

The proof of this theorem is given in Appendix [C]. We now comment on the major implications of Theorem 1. First, the theorem establishes that the distributed dictionaries $\{ \hat{E}^{(T_d)}_i \}$ can indeed remain arbitrarily close to the centralized dictionary $D^{(T_d)}$ after $T_d$ dictionary learning iterations (cf. [15]). Second, the theorem shows that this can happen as long as the number of distributed power method iterations $T_p$ scale in a certain manner. In particular, Theorem 1 calls for this scaling to be at least linear in $T_d K$ (modulo the log $N$ multiplication factor), which is the total number of SVDs that K-SVD needs to perform in $T_d$ dictionary learning iterations. On the other hand, $T_p$ need only scale logarithmically with $S_{\text{max}}$, which is significant in the context of big data problems. Other main problem parameters that affect the scaling of $T_p$ include $T_0$, $n$, and $\delta_d^{-1}$, all of which enter the scaling relation in a logarithmic fashion. Finally, Theorem 1 dictates that the number of consensus iterations $T_c$ should also scale at least linearly with $T_p T_{\text{mix}}$ (modulo some log factors) for the main result to hold. In summary, Theorem 1 guarantees
that the distributed dictionaries learned by cloud K-SVD can remain close to the centralized dictionary without requiring excessive numbers of power method and consensus averaging iterations.

We now provide a brief heuristic understanding of the roadmap needed to prove Theorem $\dagger$ In the first dictionary learning iteration ($t = 1$), we have $\{\hat{D}_i^{t-1}\} \equiv D^{t-1}$ due to identical initializations. While this means both K-SVD and cloud K-SVD result in identical sparse codes for $t = 1$, the distributed dictionaries begin to deviate from the centralized dictionary after this step. The perturbations in $\hat{d}_{i,k}$ happen due to the finite numbers of power method and consensus averaging iterations for $k = 1$, whereas they happen for $k > 1$ due to this reason as well as due to the earlier perturbations in $\hat{d}_{i,j}^{(1)}, \hat{d}_{j,k}^{(1)}$, $j < k$. In subsequent dictionary learning iterations ($t > 1$), therefore, cloud K-SVD starts with already perturbed distributed dictionaries $\{\hat{D}_i^{t-1}\}$. This in turn also results in deviations of the sparse codes computed by K-SVD and cloud K-SVD, which then adds another source of perturbations in $\{\hat{d}_{i,k}^{(t)}\}$ during the dictionary update steps. To summarize, imperfect power method and consensus averaging in cloud K-SVD introduce errors in the top eigenvector estimates of (centralized) $E_{1,R}^{(t)} E_{1,R}^{(t)\top}$ at individual sites, which then accumulate for $(k, t) \neq (1, 1)$ to also cause errors in estimate $\hat{E}_{k,R}^{(t)} \hat{E}_{k,R}^{(t)\top}$ of the matrix $E_{k,R}^{(t)} E_{k,R}^{(t)\top}$ available to cloud K-SVD. Collectively, these two sources of errors cause deviations of the distributed dictionaries from the centralized dictionary and the proof of Theorem $\dagger$ mainly relies on our ability to control these two sources of errors.

C. Roadmap to Theorem $\dagger$

The first main result needed for the proof of Theorem $\dagger$ looks at the errors in the estimates of the dominant eigenvector $u_1$ of an arbitrary symmetric matrix $M = \sum_{i=1}^{N} M_i$ obtained at individual sites using imperfect power method and consensus averaging when the $M_i$’s are distributed across the $N$ sites (cf. Sec. III-B). The following result effectively helps us control the errors in cloud K-SVD dictionaries due to Steps 7–17 in Algorithm $\dagger$

**Theorem 2** (Stability of Distributed Power Method). Consider any symmetric matrix $M = \sum_{i=1}^{N} M_i$ with dominant eigenvector $u_1$ and eigenvalues $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_N|$. Suppose each $M_i, i = 1, \ldots, N$, is only available at the $i^{\text{th}}$ site in our network and let $\hat{q}_i$ denote an estimate of $u_1$ obtained at site $i$ after $T_p$ iterations of the distributed power method (Steps 7–17 in Algorithm $\dagger$). Next, define $\alpha_p = \sum_{i=1}^{N} ||M_i||_2$, $\beta_p = \max_{t_p=1,\ldots,T_p} \frac{1}{||M_{2p+\ell}||_2}$, and $\gamma_p = \sqrt{\sum_{i=1}^{N} ||M_i||_2^2}$, where $q^{(t)}_c$ denotes the iterates of a centralized power method initialized with $q^{\text{init}}$. Then, fixing any $\epsilon \in (0,1/33\beta_p^{-1/3T_p})$, we have

$$
\max_{i=1,\ldots,N} ||u_1 u_1^\top - \hat{q}_i \hat{q}_i^\top||_2 \leq \tan(\theta) \frac{\lambda_2^{T_p}}{\lambda_1} + 4\epsilon^{3T_p},
$$

as long as $||u_1, q^{\text{init}}|| > 0$ and the number of consensus iterations within each iteration of the distributed power method (Steps 10–14 in Algorithm $\dagger$) satisfies $T_c = \Omega(T_p T_{mix} \log(2\alpha_p \beta_p \epsilon^{-1}) + T_{mix} \log(\alpha_p^{-1} \gamma_p \sqrt{N}))$. Here, $\theta$ denotes the angle between $u_1$ and $q^{\text{init}}$, defined as $\theta = \arccos(||u_1||_2 ||q^{\text{init}}||_2)$.

The proof of this theorem is given in Appendix A. Theorem 2 states that $\hat{q}_i \xrightarrow{T_p} \pm u_1$ at an exponential rate at each site as long as enough consensus iterations are performed in each iteration of the distributed power method. In the case of a finite number of distributed power method iterations, (16) in Theorem 2 tells us that the maximum
error in estimates of the dominant eigenvector is bounded by the sum of two terms, with the first term due to finite number of power method iterations and the second term due to finite number of consensus iterations.

The second main result needed to prove Theorem 1 looks at the errors between individual blocks of the reduced distributed error matrix \( \hat{E}_{k,R}^{(t)} = \left[ \hat{E}_{1,k,R}^{(t)}, \hat{E}_{2,k,R}^{(t)}, \ldots, \hat{E}_{N,k,R}^{(t)} \right] \) and the reduced centralized error matrix \( E_{k,R}^{(t)} = \left[ E_{1,k,R}^{(t)}, E_{2,k,R}^{(t)}, \ldots, E_{N,k,R}^{(t)} \right] \) for \( k \in \{1, 2, \ldots, K\} \) and \( t \in \{1, 2, \ldots, T_d\} \). This result helps us control the error in step 5 of Algorithm 1 and, together with Theorem 2, characterizes the major sources of errors in cloud K-SVD in relation to centralized K-SVD. The following theorem provides a bound on error in \( \hat{E}_{i,k,R}^{(t)} \).

**Theorem 3** (Perturbation in the matrix \( \hat{E}_{i,k,R}^{(t)} \)). Recall the definitions of \( \Omega_k^{(i)} \) and \( \Omega_k^{(i)} \) from Sec. III-A and Sec. III-B respectively. Next, express \( \Omega_k^{(i)} = \left[ \Omega_{1,k}^{(i)}, \Omega_{2,k}^{(i)}, \ldots, \Omega_{N,k}^{(i)} \right] \), where \( \Omega_{i,k}^{(i)} \) corresponds to the data samples associated with the \( i \)-th site, and define \( E_{i,k,R}^{(t)} = \hat{E}_{i,k,R}^{(t)} - E_{i,k,R}^{(t)} \). Finally, let \( \zeta, \mu, \nu, \epsilon, \) and \( \delta_d \) be as in Theorem 1 define \( \epsilon = \mu \nu T_p + 4 \epsilon T_p \), and assume \( \epsilon \leq \frac{8N\sqrt{nC_3(1+\zeta)^{d-1}C_2^2(8C_3NC_4^2 + 5)^{2(4K-3)}}}{\delta_d} \). Then, if we perform \( T_p \) power method iterations and \( T_c = \Omega(T_p T_m \log (2\alpha \beta^{-1}) + T_m \log (\alpha^{-1}\gamma \sqrt{N})) \) consensus iterations in cloud K-SVD and assume P1–P3 hold, we have for \( i \in \{1, \ldots, N\} \), \( t \in \{1, 2, \ldots, T_d\} \), and \( k \in \{1, 2, \ldots, K\} \)

\[
\|B_{i,k,R}^{(t)}\|_2 \leq \begin{cases} 0, & \text{for } t = 1, k = 1, \\ \epsilon(1 + \zeta)^{-1}C_4(8C_3NC_4^2 + 5)^{(t-1)K+k-2}, & \text{otherwise}. \end{cases}
\]

Proof of Theorem 3 along with the proofs of supporting lemmas is given in Appendix B. Theorem 3 tells us that the error in matrix \( E_{i,k,R}^{(t)} \) can be made arbitrarily small through a suitable choice of \( T_p \) and \( \epsilon \) as long as all of the assumptions of Theorem 1 are satisfied. In proof of Theorem 1 we will show that assumption on \( \epsilon \) is satisfied as long as we are performing power method iterations and consensus iterations as required by Theorem 1. The proof of Theorem 1 given in Appendix C relies on these two aforementioned theorems.

**V. Numerical Experiments**

We present numerical results in this section for demonstrating the usefulness of cloud K-SVD and also validating some of our theoretical results. In the first set of experiments, synthetic data is used to demonstrate efficacy of cloud K-SVD for data representation. Furthermore, behavior of distributed power method (Steps 7–17 in Algorithm 1) as a function of the number of consensus iterations and deviations in cloud K-SVD dictionaries from centralized dictionary as a function of number of power method iterations are also shown with the help of simulations. In the second set of experiments, MNIST dataset is used to motivate an application of cloud K-SVD that can benefit from collaborations between distributed sites.

**A. Experiments Using Synthetic Data**

These experiments correspond to a total of \( N = 100 \) sites, with each site having \( S_i = 500 \) local samples in \( \mathbb{R}^{20} \) (i.e., \( n = 20 \)). Interconnections between the sites are randomly generated using an Erdős–Rényi graph with parameter \( p = 0.5 \). In order to generate synthetic data at individual sites, we first generate a dictionary with \( K = 50 \) atoms, \( D \in \mathbb{R}^{20 	imes 50} \), with columns uniformly distributed on the unit sphere in \( \mathbb{R}^{20} \). Next, we randomly select a
45-column subdictionary of $D$ for each site and then generate samples for that site using a linear combination of $T_0 = 3$ randomly selected atoms of this subdictionary, followed by addition of white Gaussian noise with variance $\sigma^2 = 0.01$. All data samples in our experiments are also normalized to have unit $\ell_2$ norms. Finally, in order to carry out distributed consensus averaging, we generate a doubly-stochastic weight matrix $W$ according to the local-degree weights method described in [38, Sec. 4.2]. For estimating eigenvectors during dictionary update step using distributed power method we use $T_p = 15$ power method iterations and $T_c = 15$ consensus iterations.

Based on above setup our first set of experiments demonstrate the effectiveness of collaboratively learning a dictionary using cloud K-SVD, as opposed to each site learning a local dictionary from its local data using the canonical K-SVD algorithm (referred to as local K-SVD in the following). In Fig. 2, we plot average representation error, defined as $\frac{1}{nS} \sum_{i=1}^{N} \sum_{j=1}^{S_i} \|y_{i,j} - Dx_{i,j}\|_2$, as a function of the number of dictionary learning iterations for three dictionary learning methods, namely, centralized (canonical) K-SVD, cloud K-SVD, and local K-SVD. It can be seen from this figure, which corresponds to an average of 100 Monte-Carlo trials, that cloud K-SVD and centralized K-SVD have similar performance and both of them perform better than local K-SVD. In particular, the local K-SVD error is $\approx 0.06$ after 50 iterations, while it is $\approx 0.03$ for cloud K-SVD and centralized K-SVD.

The second set of experiments that we perform illustrate the convergence behavior of the distributed power method within cloud K-SVD (Steps 7–17 in Algorithm 1) as a function of the number of consensus iterations. The results of these experiments, which are reported in Fig. 3, correspond to five different values of the number of consensus iterations (3, 4, 5, 10, 15) within each iteration of the distributed power method. Specifically, let $q$ denote the principal eigenvector of the matrix $\hat{M}$ in Algorithm 1 computed using Matlab and $\hat{q}_{1}^{(t_p)}$ denote an estimate of $q$ obtained at site 1 in iteration $t_p$ using the distributed power method. Then Fig. 3 plots $\|qq^T - \hat{q}_{1}^{(t_p)}\hat{q}_{1}^{(t_p)T}\|_2$ averaged over all distributed power method iterations, dictionary learning iterations, and 100 Monte-Carlo trials as a function of the number of distributed power method iterations. It can be seen from this figure that the distributed
power method hits an \textit{error floor} with increasing number of distributed power method iterations, where the floor is fundamentally determined by the number of consensus iterations, as predicted by Theorem 2.

Our third experiment highlights the average error in dictionary atoms learned using cloud K-SVD as compared to centralized K-SVD. Experimental setup is still the same as in first experiment. For this experiment number of consensus iterations $T_c$ are fixed to 15, while the number of power method iterations $T_p$ correspond to 3, 5, 7 and 9 to show the effect of $T_p$ on error in collaborative dictionaries as compared to centralized K-SVD. Error in dictionary atoms is averaged over all the atoms and sites for 100 Monte-Carlo trials, and error in each trial can be mathematically written as $E_{\text{average}}(t) = \frac{1}{NK} \sum_{k=1}^{K} \sum_{i=1}^{N} \| \tilde{d}_{i,k}^{(t)} d_{i,k}^{(t)^T} T - \tilde{d}_{i,k} \tilde{d}_{i,k}^{(t)^T} \|_{2}$. Simulation results in Fig. 4 show that error in dictionary atoms increases sharply at the start and after some iterations error becomes stable. Important point to note here is that as we increase the number of power method iterations we get smaller average error in dictionary atoms as predicted by our analysis.
Experiments Using Real Data: Classification of MNIST Images

For evaluation of cloud K-SVD for real dataset, we perform classification of digits \{0, 3, 5, 8, 9\} from MNIST dataset [46]. For each digit 6000 samples are used, where 5000 samples are used for training purposes and remaining 1000 for testing. Furthermore, the data are five-times randomly split into training and test samples. For cloud K-SVD, Erdős–Rényi graph with parameter \(p = 0.5\) is used to generate a network with 10 sites and data is equally distributed among these 10 sites. Before performing dictionary learning, data is down sampled from \(\mathbb{R}^{784}\) to \(\mathbb{R}^{256}\). After downsampling, a separate dictionary is learned for each digit using centralized K-SVD, cloud K-SVD, and K-SVD using only local data. Each dictionary has dimensions \(\mathbb{R}^{256 \times 400}\), i.e., \(K = 400\), and sparsity level of 10 is used, i.e., \(T_0 = 10\). Minimum residue based rule [47, Sec.II-A] is used for classification, more details on which are given in the following paragraph.

Let \(\{D_c\}_{c=1}^{5}\) be the set of dictionaries for 5 classes and let \(D = \begin{bmatrix} D_1 & D_2 & D_3 & D_4 & D_5 \end{bmatrix}\) be the complete dictionary. For any test sample \(y_s\), we perform sparse coding using dictionary \(D\) with sparsity constraint of \(T_0 = 10\) to get coefficients \(x_s \in \mathbb{R}^{2000}\). Then we partition \(x_s\) into five segment \(\{x_{s,c}\}_{c=1}^{5}\), where \(x_{s,c}\) are the coefficients corresponding to dictionary \(D_c\) of class \(c\). Next we define residue for class \(c\) as \(r_c = \|y_s - D_c x_{s,c}\|_2\). Finally, the detected class is given by \(c^* = \arg\min_c r_c\). Performance of each method (centralized K-SVD, cloud K-SVD, and local K-SVD) is measured in terms of average detection rate on the test samples, which is defined as,

\[
R_c = \frac{\text{Number of samples in class } c \text{ detected correctly}}{\text{Total number of samples of class } c}.
\]

Results of this experiment are given in Figure 5. We can see that centralized K-SVD and cloud K-SVD have comparable performance. While in the case of local K-SVD where we only use the local data for learning representations, classification rate deteriorates considerably. The bars in local K-SVD show the highest and lowest detection rates achieved among the 10 sites, which highlights the variation in effectiveness of models learned across different sites when using only the local data.
VI. CONCLUSION

In this paper, we have proposed a new dictionary learning algorithm, termed cloud K-SVD, that facilitates collaborative learning of a dictionary that best approximates massive data distributed across geographical regions. Mathematical analysis of proposed method is also provided, which under certain assumptions shows that if we perform enough number of power method and consensus iterations then the proposed algorithm converges to the centralized K-SVD solution. Furthermore, the efficacy of the proposed algorithm is demonstrated through extensive simulations on synthetic and real data.

APPENDIX A

PROOF OF THEOREM 2

The proof of this theorem relies on a lemma that guarantees that if the estimates obtained at different sites using the distributed power method are close to the estimate obtained using the centralized power method at the start of a power method iteration then the distributed estimates remain close to the centralized estimate at the end of that iteration. In order to prove such a lemma, we first need a result from the literature that characterizes the convergence behavior of vector consensus averaging as a function of the number of consensus iterations.

**Proposition 1** (Theorem 5 in [34]). Consider the \( n \times 1 \) vector sum \( z = \sum_{i=1}^{N} z_i^{(0)} \) and suppose each vector \( z_i^{(0)}, i = 1, \ldots, N, \) is only available at the \( i^{th} \) site in our network. Let \( b = \sum_{i=1}^{N} |z_i^{(0)}| \) be a vector whose entries are the sum of absolute values of the initial vectors \( z_i^{(0)} \) and \( z_i^{(t_c)} \) be the \( n \times 1 \) vector obtained at the \( i^{th} \) site after \( t_c \) consensus iterations. Then, fixing any \( \delta > 0, \) we have that \( \| \sum_{i=1}^{N} \frac{z_i^{(t_c)}}{|z_i^{(t_c)}|} - z \|_2 \leq \delta \| b \|_2 \) \( \forall i \) as long as the number of consensus iterations satisfies \( t_c = \Omega(T_{mix} \log \delta^{-1}) \).

We now make use of Proposition 1 to state and prove the desired lemma.

**Lemma 1.** Suppose we are at the start of \( (t_p + 1) \leq T_p \) power method iteration. Let \( q_c \) and \( q_{i,d} \) denote the outputs of centralized power method and distributed power method at \( i^{th} \) site after \( t_p \) iterations, respectively. Similarly, let \( q_c' \) and \( q_{i,d}' \) denote the outputs of centralized power method and distributed power method at \( i^{th} \) site after \( t_p + 1 \) iterations, respectively. Next, fix an \( \epsilon \in (0,1) \), define \( \delta = \frac{\alpha_p}{\gamma_p \sqrt{N}(2\alpha_p \beta_p)^3 T_p} \), and assume that \( \forall i, \|q_c - q_{i,d}\|_2 + \frac{\delta \gamma_p \sqrt{N}}{\alpha_p} \leq \frac{1}{2\alpha_p \beta_p (2\alpha_p \beta_p + 3 \gamma_p \sqrt{N})} \). Then, assuming \( \Omega(T_{mix} \log \delta^{-1}) \) consensus iterations, we have that

\[
\forall i, \|q_c' - q_{i,d}'\|_2 \leq (2\alpha_p \beta_p)^3 \left( \max_{i=1,\ldots,N} \|q_c - q_{i,d}\|_2 + \frac{\delta \gamma_p \sqrt{N}}{\alpha_p} \right).
\]

**Proof:** Define \( v = M q_c, r = \|v\|_2, \) and \( \widehat{v} = \sum_{i=1}^{N} M_i q_{i,d} \). Next, fix any \( i \in \{1, \ldots, N\} \) and let \( \widehat{\nu}_i \) be the vector obtained at the \( i^{th} \) site in Step 15 of Algorithm 1 during the \( (t_p + 1) \) iteration of distributed power method. Notice that \( \widehat{\nu}_i \) can be expressed as \( \widehat{\nu}_i = \widehat{v} + \epsilon_i,c \), where \( \epsilon_i,c \) denotes the error introduced in \( \widehat{v} \) at the \( i^{th} \) site due to finite number of consensus iterations. Next, define \( \widehat{r}_i = \|\widehat{\nu}_i\|_2 \) and notice that \( q_c' - q_{i,d}' = v(r^{-1} - \widehat{r}^{-1}_i) + (v - \widehat{\nu}_i)\widehat{r}^{-1}_i \).

It therefore follows from the triangle inequality that

\[
\|q_c' - q_{i,d}'\|_2 \leq \|v\|_2 r^{-1} - \widehat{r}^{-1}_i| + \|v - \widehat{\nu}_i\|_2 \widehat{r}^{-1}_i.
\]  

(18)
Since

\[ |\tilde{v}|^2 - r^2| \leq (2\alpha_p + \delta_p \sqrt{N}) \left( \sum_{i=1}^{N} \| M_i \|_2 \| q_c - q_i, d \|_2 + \delta_p \sqrt{N} \right) \leq (2\alpha_p + \delta_p \sqrt{N}) \left( \max_i \| q_c - q_i, d \|_2 + \delta_p \sqrt{N} \right). \]  

We can now use this inequality to obtain

\[ |r - \tilde{r}_i| \leq \tilde{r}_i \beta_p (2\alpha_p + \delta_p \sqrt{N}) (\alpha_p \max_i \| q_c - q_i, d \|_2 + \delta_p \sqrt{N}). \]

The only remaining quantity we need to bound is \( \tilde{r}_i \). To this end, notice that \( |r - \tilde{r}_i| \geq (r^{-1} - (\tilde{r}_i^{-1})^{-1}. \)

Since \( |r - \tilde{r}_i| \leq r^{-1} |\tilde{r}_i^2 - r^2| \), we obtain from (21) that

\[ (r^{-1})^{-1} - (\tilde{r}_i^{-1})^{-1} \leq \alpha_p r^{-1} (2\alpha_p + \delta_p \sqrt{N}) \left( \max_i \| q_c - q_i, d \|_2 + \delta_p \sqrt{N} \right) \].

It then follows from the lemma’s assumptions along with some algebraic manipulations that \( \tilde{r}_i^{-1} \leq 2\beta_p. \) Finally, plugging the bounds on \( \tilde{r}_i^{-1}, |r - \tilde{r}_i^{-1}|, \| v \|_2, \) and \( \| v - \tilde{v}_i \|_2 \) in (18), we obtain

\[ \| q_c - q_i, d \|_2 \leq 2\alpha_p \beta_p \left( \alpha_p \max_i \| q_c - q_i, d \|_2 + \delta_p \sqrt{N} \right) \left( 2\alpha_p + \delta_p \sqrt{N} \right) + 2\beta_p \left( \alpha_p \max_i \| q_c - q_i, d \|_2 + \delta_p \sqrt{N} \right) \]

\[ = \left( 4\alpha_p \beta_p^3 + 2\alpha_p \beta_p^3 \frac{\delta_p \sqrt{N}}{\alpha_p} + 2\alpha_p \beta_p \right) \left( \max_i \| q_c - q_i, d \|_2 + \delta_p \sqrt{N} \right). \]

Finally, \( \frac{\delta_p \sqrt{N}}{\alpha_p} \leq \left( \frac{\epsilon}{2} \right)^{3T} < 1 \) since \( \delta = \frac{\alpha_p}{\gamma_p \sqrt{N}} \left( \frac{\epsilon}{2\alpha_p \beta_p} \right)^{3T} \), \( \epsilon < 1 \), and \( \alpha_p r^{-1} \geq 1 \), which implies \( \alpha_p \beta_p \geq 1 \). Plugging this into the above expression and noting that \( \alpha_p \beta_p \leq \alpha_p^2 \beta_p \), we obtain the claimed result.

Lemma 1 provides an understanding of the error accumulation in the distributed power method due to finite number of consensus iterations in each power method iteration. And while the factor of \( (2\alpha_p \beta_p)^3 \) in the lemma statement might seem discouraging, the fact that the distributed power method starts with a zero error helps keep the total error in control. We now formally argue this in the proof of Theorem 2 below.

**Proof of Theorem 2** We begin by defining \( q_c \) as the estimate of \( u_1 \) obtained using \( T_p \) iterations of the centralized power method that is initialized with the same \( q^{init} \) as the distributed power method. Next, fix an \( i \in \{1, \ldots, N\} \) and notice that

\[ \| u_1 u_1^T - \tilde{q}_i \|_2 \leq \| u_1 u_1^T - q_c \|_2 + \| q_c \|_2 - \tilde{q}_i \|_2. \]
The convergence rate of the centralized power method is well studied and can be expressed as
\[
\|u_1 u_1^T - q_c q_c^T\|_2 \leq \tan(\theta) \left| \frac{\lambda_2}{\lambda_1} \right|^{T_p},
\]  
(24)

In order to bound \(\|q_c q_c^T - \hat{q}_t \hat{q}_t^T\|_2\), we make use of Lemma 1. To invoke this lemma, we first need to show that the main assumption of the lemma holds for all iterations \(t_p \leq (T_p - 1)\). We start with \(t_p = 0\) for this purpose and note that \(q_c^{(0)} = \hat{q}_t^{(0)} = q_{\text{init}}\), which trivially implies
\[
\|q_c^{(0)} - \hat{q}_t^{(0)}\|_2 + \frac{\delta \gamma_p \sqrt{N}}{\alpha_p} \leq \left( \frac{\epsilon}{2} \right)^{3T_p},
\]  
(25)

where \(\delta\) is as defined in Lemma 1. Further, under the assumptions of the theorem, it can be shown through elementary algebra that \(\left( \frac{\epsilon}{2} \right)^{3T_p} \leq \frac{1}{2\alpha_p \beta_p^2 (2\alpha_p + \delta \gamma_p \sqrt{N})}\). We now invoke mathematical induction and claim that the main assumption of Lemma 1 holds for all \(t_p \leq m < T_p\). Then we obtain from a recursive application of the statement of the lemma that for \(t_p = (m + 1)\), we have
\[
\|q_c^{(m+1)} - \hat{q}_t^{(m+1)}\|_2 + \frac{\delta \gamma_p \sqrt{N}}{\alpha_p} \leq \frac{\delta \gamma_p \sqrt{N}}{\alpha_p} \sum_{i=0}^m (2\alpha_p \beta_p)^{3i} (a) \leq 2 \cdot \frac{\delta \gamma_p \sqrt{N}}{\alpha_p} (2\alpha_p \beta_p)^{3m} \]
\[
= 2 \cdot 2^{3T_p} (2\alpha_p \beta_p)^{3m} (b) \leq \frac{1}{2\alpha_p \beta_p^2 (2\alpha_p + \delta \gamma_p \sqrt{N})},
\]  
(26)

where \((a)\) follows from the geometric sum and the fact that \((2\alpha_p \beta_p)^3 > 2\), while \((b)\) follows from the theorem assumptions and the fact that \(m < T_p\). We have now proved that the main assumption of Lemma 1 holds for all \(t_p \leq (T_p - 1)\). In order to compute \(\|q_c q_c^T - \hat{q}_t \hat{q}_t^T\|_2\), therefore, we can recursively apply the result of this lemma up to the \(T_p\)th iteration to obtain
\[
\|q_c - \hat{q}_t\|_2 \leq \frac{\delta \gamma_p \sqrt{N}}{\alpha_p} \sum_{i=0}^{T_p} (2\alpha_p \beta_p)^{3i} (c) \leq 2e^{3T_p},
\]  
(27)

where \((c)\) follows from the same arguments as in \(26\). The proof of the theorem now follows by noting the fact that \(\|q_c q_c^T - \hat{q}_t \hat{q}_t^T\|_2 \leq (\|q_c\|_2 + \|\hat{q}_t\|_2) \|q_c - \hat{q}_t\|_2 \leq 4e^{3T_p}\).

**APPENDIX B**

**PROOF OF THEOREM 3**

Notice from Algorithm 1 that sparse coding is always performed before update of the first dictionary atom. However, we do not perform sparse coding before updating any other dictionary atom. Due to this distinction, we answer how error is accumulated in matrix \(E_{i,k,R}^{(t)}\) for first dictionary atom differently than for any other dictionary atom. In the following, we first provide an overview of how to bound \(\|B_{i,k+1,R}^{(t)}\|_2\) when we know a bound on \(\|B_{i,k,R}^{(t)}\|_2\). Then we will talk about bounding \(\|B_{i,k+1,R}^{(t+1)}\|_2\) when we know bounds on \(\left\{\|B_{i,k+1,R}^{(t)}\|_2\right\}_{k=1}^K\).

Recall from Step 5 in Algorithm 1 that \(\hat{E}_{i,k,R}^{(t)} = \hat{Y}_{i,k,R}^{(t)} - \sum_{j=1}^{k-1} d_{i,j} x_{i,j,T}^{(t)} \hat{Y}_{i,k}^{(t)} - \sum_{j=k+1}^K d_{i,j}^{(t-1)} x_{i,j,T}^{(t)} \hat{Y}_{i,k}^{(t)}\). Now, if one assumes that \(\hat{\Omega}_k^{(t)} = \Omega_k^{(t)}\), which we will argue is true, then the error in \(E_{i,k,R}^{(t)}\) is due to errors in \(\{x_{i,j,T,R}^{(t)}\}_{j=1}^K\) and \(\{d_{i,j}^{(t)}\}_{j=1}^K\). In fact, we will show that \(\|B_{i,k+1,R}^{(t)}\|_2\) can be bounded by knowing bounds on errors in \(d_{i,k}^{(t)}\) and \(x_{i,k,T,R}^{(t)}\) only. Next, recall from Step 19 in Algorithm 1 that \(x_{i,k,R}^{(t)} = d_{i,k}^{(t)} \hat{E}_{i,k,R}^{(t)}\), which means we
only need to know a bound on \(d_k^{(t)}\) to bound \(\|B_{i,k+1,R}^{(t)}\|_2\). Another challenge for us will be to bound error in \(d_k^{(t)}\) from a given bound on \(\|B_{i,k,R}^{(t)}\|_2\). We will accomplish this by noting that there are two sources of error in \(d_k^{(t)}\). The first source is the difference in eigenvectors of \(\hat{E}_{k,R}^{(t)T}E_{k,R}^{(t)}\) and \(E_{k,R}^{(t)}E_{k,R}^{(t)T}\). We will bound this difference using Proposition 3 in Appendix D. In order to use this proposition, we will need a bound on \(\|\hat{E}_{k,R}^{(t)T}E_{k,R}^{(t)} - E_{k,R}^{(t)}E_{k,R}^{(t)T}\|_F\), which we will also prove using a given bound on \(\|B_{i,k,R}^{(t)}\|_2\) (Lemma 2). The second source of error in \(d_k^{(t)}\) is the error in eigenvector computation, which in our case is due to the distributed power method. It follows from Theorem 2 and statement of Theorem 3 that this error is bounded by \(\varepsilon\). Combining these two sources of error, we will first bound the error in \(d_k^{(t)}\) (Lemma 3), and will then use this bound to finally bound \(\|B_{i,k+1,R}^{(t)}\|_2\) (Lemma 4).

In order to bound \(\|B_{i,k+1}^{(t)}\|_2\) when we know bounds on \(\{\|B_{i,j,R}^{(t)}\|_2\}_{j=1}^K\), the difference from previous case is that now we cannot write sparse code \(\{\hat{x}_{i,j,T}^{(t)}\}_{j=1}^K\) in terms of dictionary atoms \(\{\hat{d}_{i,j}^{(t)}\}_{j=1}^K\). Therefore, in addition to bounding errors in dictionary atoms \(\{\hat{d}_{i,j}^{(t)}\}_{j=1}^K\), we also need to bound errors in sparse codes due to perturbations in dictionaries after iteration \(t\). Since we know \(\{\|B_{i,j,R}^{(t)}\|_2\}_{j=1}^K\), we can use the bounds on \(\{\hat{d}_{i,j}^{(t)}\}_{j=1}^K\) derived earlier (Lemma 3). Next, using error bounds on \(\{\hat{d}_{i,j}^{(t)}\}_{j=1}^K\), we can use Proposition 2 in Appendix D to bound errors in \(\{x_{i,j,T}^{(t+1)}\}_{j=1}^K\). Finally, using these error bounds on \(\{\hat{d}_{i,j}^{(t)}\}_{j=1}^K\) and \(\{x_{i,j,T}^{(t+1)}\}_{j=1}^K\) we will bound \(\|B_{i,k+1,R}^{(t)}\|_2\) (Lemma 4).

This will be followed by the remaining proof of Theorem 3.

Our first result in support of Theorem 3 shows that the assumption of Proposition 3 in Appendix D are satisfied under certain conditions, which will make it possible for us to bound the difference in the principal eigenvector of \(E_{k,R}^{(t)}E_{k,R}^{(t)T}\) and \(\hat{E}_{k,R}^{(t)}\hat{E}_{k,R}^{(t)T}\).

**Lemma 2.** Let \(\Omega_{i,k}^{(t)}\), \(\eta_{i,k}^{(t)}\), \(\varepsilon\) and \(\zeta\) be as defined in Theorem 3. Fix \(\delta_d\) as in Theorem 1 and suppose (i) P1–P3 are satisfied, (ii) \(\Omega_{i,k}^{(t)} = \Omega_{i,k}^{(t)}\) and (iii) \(\varepsilon \leq 8NC_3(1+\zeta)^{t-1}C_4^2(8C_3NC_4^2+5)(t-K)^{2t-K-2}\). Then for all \(i \in \{1, \ldots, N\}\) and for any \(t \in \{1, \ldots, T_d\}\) and \(k \in \{1, \ldots, K\}\), if
\[
\|B_{i,k,R}^{(t)}\|_2 \leq \begin{cases} 0, & t = 1 \text{ or } k = 1, \\ \varepsilon(1+\zeta)^{t-1}C_4(8C_3NC_4^2+5)(t-K)^{2t-K-2}, & \text{otherwise}, \end{cases}
\]
then \(\Delta M_k^{(t)} = E_{k,R}^{(t)}E_{k,R}^{(t)T} - \hat{E}_{k,R}^{(t)}\hat{E}_{k,R}^{(t)T}\) is bounded as \(\|\Delta M_k^{(t)}\|_F \leq \frac{1}{3\varepsilon_3}\).

**Proof:** Since our starting dictionaries are same, therefore, for \((t,k) = (1,1)\) we have \(E_{1,R}^{(1)} = \hat{E}_{1,R}^{(1)}\), which means \(\Delta M_k = 0\). Hence, claim is true for \((t,k) = (1,1)\). In the following, proof is provided for the claim for case \((t,k) \neq 1\).

Substituting the definition of \(B_{i,k,R}^{(t)}\) in the definition of \(\Delta M_k^{(t)}\), we get
\[
\Delta M_k^{(t)} = \sum_{i=1}^N E_{i,k,R}^{(t)}E_{i,k,R}^{(t)T} + B_{i,k,R}^{(t)}E_{i,k,R}^{(t)T} + B_{i,k,R}^{(t)}E_{i,k,R}^{(t)T}.
\]
Simple algebraic manipulations, along with submultiplicativity of matrix 2-norm, result in
\[
\|\Delta M_k^{(t)}\|_2 \leq 2\sum_{i=1}^N \left(\|E_{i,k,R}^{(t)}\|_2\|B_{i,k,R}^{(t)}\|_2 + \|B_{i,k,R}^{(t)}\|_2\right) \leq 2N \max_i \left(C_4\|B_{i,k,R}^{(t)}\|_2 + \|B_{i,k,R}^{(t)}\|_2\right),
\]
where the last inequality is due to (13). Now, using the assumptions on bound of $\|B_{i,k,R}^{(t)}\|_2$ and $\varepsilon$, it can be shown that

$$
\|\Delta M_k^{(t)}\|_2 \leq 2N\varepsilon(1 + \zeta)^{t-1} \left(C_4 C_4 (8C_3 NC_4^2 + 5)^{(t-1)K+k-2} + \varepsilon(1 + \zeta)^{t-1}C_4^2 (8C_3 NC_4^2 + 5)^{2(t-1)K+2k-4}\right)
$$

$$
\leq 2N\varepsilon(1 + \zeta)^{t-1} \left(C_4^2 (8C_3 NC_4^2 + 5)^{(t-1)K+k-2} + \frac{1}{8N\sqrt{nC_3}} (1 + \zeta)^{t-1}C_4^2 (8C_3 NC_4^2 + 5)^{2(t-1)K+2k-4}\right)
$$

$$
\leq 2N\varepsilon(1 + \zeta)^{t-1} \left(C_4^2 (8C_3 NC_4^2 + 5)^{(t-1)K+k-2} + \frac{1}{8N\sqrt{nC_3}}\right)
$$

$$
\leq 2N\varepsilon(1 + \zeta)^{t-1} \left(C_4^2 (8C_3 NC_4^2 + 5)^{(t-1)K+k-2} \right) \leq 4N\varepsilon(1 + \zeta)^{t-1}C_4^2 (8C_3 NC_4^2 + 5)^{(t-1)K+k-2},
$$

where (a) is true because

$$
\frac{(1+\varepsilon)^{t-1}C_4^2 (8C_3 NC_4^2 + 5)^{2(t-1)K+2k-4}}{(1+\varepsilon)^{t-1}C_4^2 (8C_3 NC_4^2 + 5)^{2(t-1)K+2k-4}} \leq 1.
$$

Finally, using once again the assumption on $\varepsilon$, performing algebraic manipulations and using the fact that $\delta_d \leq 1$, we get

$$
\|\Delta M_k^{(t)}\|_2 \leq \frac{(8C_3 NC_4^2 + 5)^{(t-1)K+k-2}}{2\sqrt{nC_3}(8C_3 NC_4^2 + 5)^{2(t-1)K+2k-2}} \leq \frac{(8C_3 NC_4^2 + 5)^{(t-1)K+k-2}}{2\sqrt{nC_3}(8C_3 NC_4^2 + 5)^{2(t-1)K+2k-2}}.
$$

$$
\leq \frac{1}{\sqrt{n}(8C_3 NC_4^2 + 5)^{(t-1)K+k-2}} \leq \frac{1}{\sqrt{n}(8C_3 NC_4^2 + 5)^{2(t-1)K+2k-2}} \leq \frac{1}{\sqrt{n}(8C_3 NC_4^2 + 5)^{(t-1)K+k-2}}.
$$

where (b) is true since $(8C_3 NC_4^2 + 5) > 1$ and (c) is mainly due to the fact that $N \geq 1$ and $C_4 \geq 1$. Now using the fact that rank$(\Delta M_k^{(t)}) \leq n$, we get

$$
\|\Delta M_k^{(t)}\|_F \leq \sqrt{\text{rank}(\Delta M_k^{(t)})\|\Delta M_k^{(t)}\|_F} = \sqrt{n}\|\Delta M_k^{(t)}\|_F \leq \frac{1}{\sqrt{C_4}}.
$$

We are now ready to prove one of our claims that if we know a bound on $\|B_{i,k,R}^{(t)}\|_2$ then we can bound the error in dictionary atom $\tilde{a}_{i,k}^{(t)}$. This result is given in the following lemma.

**Lemma 3.** Let $\Omega_{i,k}^{(t)}$, $\bar{\Omega}_{i,k}^{(t)}$, $\varepsilon$ and $\zeta$ be as defined in Theorem 3, also perform $T_c$ consensus iterations as given in Theorem 3. Now fix $\delta_d$ as in Theorem 7 and suppose (i) P1–P3 are satisfied, (ii) $\Omega_{i,k}^{(t)} = \bar{\Omega}_{i,k}^{(t)}$, and (iii) $\varepsilon \leq \frac{\delta_d}{8N\sqrt{nC_3}(1+\varepsilon)^{t-1}C_4^2 (8C_3 NC_4^2 + 5)^{2(t-1)K+2k-2}}$. Then for all $i \in \{1, \ldots, N\}$ and for any $t \in \{1, 2, \ldots, T_d\}$ and $k \in \{1, 2, \ldots, K\}$ if we know

$$
\|B_{i,k,R}^{(t)}\|_2 \leq \begin{cases} 
0, & t = 1, k = 1, \\
\varepsilon(1 + \zeta)^{t-1}C_4 (8C_3 NC_4^2 + 5)^{(t-1)K+k-2}, & \text{otherwise},
\end{cases}
$$

then, $\|\tilde{a}_{i,k}^{(t)} \bar{a}_{i,k}^{(t)^T} - d_{k}^{(t)^T} d_{k}^{(t)} \|_2 \leq \varepsilon(1 + \zeta)^{t-1}C_4 (8C_3 NC_4^2 + 5)^{(t-1)K+k-2}$.

**Proof:** To prove this lemma we first need to decompose error in dictionary atom into two different components i.e., error in principal eigenvector due to perturbation in $E_{k,R}^{(t)} F_{k,R}^{(t)^T}$ and error due to distributed power method. Let $d_k^{(t)}$ be the updated $k^{th}$ atom of centralized dictionary at iteration $t$, which is the principal eigenvector of $E_{k,R}^{(t)} E_{k,R}^{(t)^T}$. In cloud $K$-SVD, $\tilde{a}_{i,k}^{(t)}$ corresponds to the principal eigenvector estimate of $\tilde{E}_{k,R}^{(t)} \tilde{F}_{k,R}^{(t)^T}$ obtained at the $t^{th}$ site. Let us denote the true principal eigenvector of $\tilde{E}_{k,R}^{(t)} \tilde{F}_{k,R}^{(t)^T}$ by $\tilde{a}_{i,k}^{(t)}$ and let $a_{i,k}^{(t)}$ be the eigenvector of $E_{k,R}^{(t)} E_{k,R}^{(t)^T}$ computed using distributed power method at the $t^{th}$ site. Using this notation, notice that

$$
\|d_k^{(t)^T} d_k^{(t)} - \tilde{a}_{i,k}^{(t)^T} \tilde{a}_{i,k}^{(t)} \|_2 = \|d_k^{(t)^T} d_k^{(t)} - a_{i,k}^{(t)^T} a_{i,k}^{(t)} + \tilde{a}_{i,k}^{(t)^T} \tilde{a}_{i,k}^{(t)} - d_k^{(t)^T} d_k^{(t)} \|_2 \\
\leq \|d_k^{(t)^T} d_k^{(t)} - a_{i,k}^{(t)^T} a_{i,k}^{(t)} \|_2 + \|a_{i,k}^{(t)^T} a_{i,k}^{(t)} - \tilde{a}_{i,k}^{(t)^T} \tilde{a}_{i,k}^{(t)} \|_2.
$$
where the first term is due to perturbation in $E_{k,R}^{(t)}$, and the second term is due to imperfect power method and consensus iterates. We can now use Theorem 2 to obtain

$$\|d_k^{(t)}d_k^{(T)} - \tilde{d}_k^{(t)}\tilde{d}_k^{(T)}\|_2 \leq \|d_k^{(t)}d_k^{(T)} - \tilde{d}_k^{(t)}\tilde{d}_k^{(T)}\|_2 + \tan(\frac{\gamma_{2,K}^{(t)}}{\gamma_{1,K}^{(t)}})T + 4\varepsilon T + 3\varepsilon \leq \|d_k^{(t)}d_k^{(T)} - \tilde{d}_k^{(t)}\tilde{d}_k^{(T)}\|_2 + \varepsilon, \quad (29)$$

where (a) is due to definition of parameters $\mu$ and $\nu$ in Theorem 1 and (b) follows from the definition of $\varepsilon$ in Theorem 3.

Next, for symmetric matrices $M_k^{(t)} = \sum_i E_{i,k,R}^{(t)}$ and $\tilde{M}_k^{(t)} = \sum_i \tilde{E}_{i,k,R}^{(t)}$, such that $\tilde{M}_k^{(t)} = M_k^{(t)} + \Delta M_k^{(t)}$, we can use Lemma 2 and Proposition 3 to find a bound on deviation in principal eigenvector of $M_k^{(t)}$ due to perturbation $\Delta M_k^{(t)}$. Since we have from Lemma 2 that $\|\Delta M_k^{(t)}\|_F \leq \frac{1}{\delta C_5}$, it follows from Proposition 3 that

$$\|d_k^{(t)}d_k^{(T)} - \tilde{d}_k^{(t)}\tilde{d}_k^{(T)}\|_2 \leq 4C_3\|\Delta M_k^{(t)}\|_2 + \varepsilon \leq 8C_3N \max_i \left(C_4 B_i^{(1)} \frac{\|B_i^{(1)}\|_2}{2} + \frac{\|B_i^{(1)}\|_2}{2} + \frac{\|B_i^{(1)}\|_2}{2} \right) + \varepsilon, \quad (30)$$

where the last inequality is due to (28). Now using the bound on $\|B_i^{(1)}\|_2$ in the lemma statement, it can be show using some algebraic manipulations that

$$\|d_k^{(t)}d_k^{(T)} - \tilde{d}_k^{(t)}\tilde{d}_k^{(T)}\|_2 \leq \varepsilon(1 + \zeta)^{t_1-K-k-2} \left(8C_3NC_2^2(8C_3NC_2^2 + 5)^{(t_1-K-k-2)} + \varepsilon \right).$$

The claim in the lemma now follows by replacing the bound on $\varepsilon$ in the parentheses of the above inequality, followed by some manipulations.

The next lemma shows that if we know bounds on errors in $\{\tilde{E}_{i,k,R}^{(t)}\}_{k=1}^K$ for any $t$ then we can bound the error in $\tilde{E}_{i,1,R}^{(t+1)}$.

**Lemma 4.** Let $\Omega_{i,k}^{(t)}$, $\tilde{\Omega}_{i,k}^{(t)}$, $\varepsilon$ and $\zeta$ be as defined in Theorem 3 also perform $T_c$ consensus iterations as given in Theorem 3. Now fix $\delta_d$ as in Theorem 1 and suppose (i) $P1$–$P3$ are satisfied, (ii) $\Omega_{i,k}^{(t+1)} = \tilde{\Omega}_{i,k}^{(t+1)}$, (iii) $\Omega_{i,k}^{(t)} = \tilde{\Omega}_{i,k}^{(t)}$, and (iv) $\varepsilon \leq \frac{\delta_d}{8N^2NC_3(1+\zeta)^{t-1}C_4(8C_3NC_2^2N + 5)^{(t-1)K-k-2}}$, then for any $t \in \{1, \ldots, T_d - 1\}$, and for all $k \in \{1, \ldots, K\}$ and $i \in \{1, \ldots, N\}$, if $\|B_{i,k}^{(1)}\|_2 \leq \varepsilon (1 + \zeta)^{t-1}C_4(8C_3NC_2^2N + 5)^{(t-1)K-k-2}$ then, $\|B_{i,1,R}^{(t+1)}\|_2 \leq \varepsilon (1 + \zeta)^{t-1}C_4(8C_3NC_2^2N + 5)^{(t-1)K-k-2}. $

**Proof:** The error in $\tilde{E}_{i,1,R}^{(t+1)}$ is due to error in dictionary in the previous iteration $t$ and sparse coding at the start of iteration $(t+1)$. Specifically, $B_{i,1}^{(t+1)} = E_{i,1}^{(t+1)} - \tilde{E}_{i,1}^{(t+1)} = Y_i - \sum_{j=2}^K d_j^{(t+1)}x_{i,j,T} - Y_i + \sum_{j=2}^K \tilde{d}_j^{(t+1)}x_{i,j,T}$. It then follow that

$$\|B_{i,1}^{(t+1)}\|_2 \leq \sum_{j=2}^K \|d_j^{(t)}x_{i,j,T} - \tilde{d}_j^{(t)}x_{i,j,T}\|_2 \leq \sum_{i=1}^{K-1} \|d_j^{(t)}x_{i,j,T} - \tilde{d}_j^{(t)}x_{i,j,T}\|_2.$$

In reality we are interested in finding a bound on $\|B_{i,1,R}^{(t+1)}\|_2$. But since $\Omega_{i,k}^{(t+1)} = \tilde{\Omega}_{i,k}^{(t+1)}$ we can define $B_{i,1,R}^{(t+1)}$ as $B_{i,1,R}^{(t+1)} = \left(\sum_{j=2}^K \left(Y_i - \tilde{d}_j^{(t+1)}x_{i,j,T}\right) - \sum_{j=2}^K \left(Y_i - \tilde{d}_j^{(t+1)}x_{i,j,T}\right)\right) \Omega_{i,1}^{(t+1)}$. It can be seen from this definition that
$B_{i,1,R}^{(t+1)}$ is a submatrix of $B_{i,1}^{(t+1)}$, which implies

$$
\|B_{i,1,R}^{(t+1)}\|_2 \leq \|B_{i,1}^{(t+1)}\|_2 \leq \sum_{j=1}^{K} \|d_{i,j}^{(t)}x_{i,j,T}^{(t+1)} - d_{i,j}^{(t)}x_{i,j,T}^{(t+1)}\|_2.
$$

(31)

Now, defining $\tilde{d}_{i,j}^{(t)} = d_{i,j}^{(t)} + e_{i,j}^{(t)}$, where $e_{i,j}^{(t)}$ denotes the error in dictionary atom $d_{i,j}^{(t)}$, and substituting this in (31) we get

$$
\|B_{i,1,R}^{(t+1)}\|_2 \leq K \max_j \left\| (d_{i,j}^{(t)} + e_{i,j}^{(t)})x_{i,j,T}^{(t+1)} - d_{i,j}^{(t)}x_{i,j,T}^{(t+1)} \right\|_2 \leq K \max_j \left( \|d_{i,j}^{(t)}x_{i,j,T}^{(t+1)} - d_{i,j}^{(t)}x_{i,j,T}^{(t+1)}\|_2 + \|e_{i,j}x_{i,j,T}^{(t+1)}\|_2 \right)

\leq K \max_j \left( \|x_{i,j,T}^{(t+1)} - x_{i,j,T}^{(t+1)}\|_2 + \|\tilde{d}_{i,j}^{(t)} - d_{i,j}^{(t)}\|_2 \right)

= K \max_j \left( \|\tilde{d}_{i,j}^{(t)} - d_{i,j}^{(t)}\|_2 \|x_{i,j,T}^{(t+1)} - x_{i,j,T}^{(t+1)}\|_2 \right)

\leq K \max_j \left( \|\tilde{d}_{i,j}^{(t)} - d_{i,j}^{(t)}\|_2 \|x_{i,j,T}^{(t+1)} - x_{i,j,T}^{(t+1)}\|_2 \right)

\leq K \max_j \left( \|\tilde{d}_{i,j}^{(t)} - d_{i,j}^{(t)}\|_2 \right)

\leq K \max_j \left( \|\tilde{d}_{i,j}^{(t)} - d_{i,j}^{(t)}\|_2 \right)

\| \hat{\eta}_{i,j}^{(t)} \|_2 \leq \sqrt{\hat{S}_{\text{max}}} \|X^{(t+1)}\|_{\text{max}} \leq \sqrt{\hat{S}_{\text{max}}} \|X^{(t+1)}\|_{1}.

We therefore obtain under P1 that

$$
\|x_{i,j,T}^{(t+1)}\|_2 \leq \sqrt{\hat{S}_{\text{max}}} \|X^{(t+1)}\|_{\text{max}} \leq \sqrt{\hat{S}_{\text{max}}} \|X^{(t+1)}\|_{1}.
$$

(33)

Next, using the bound on $\|B_{i,k,R}^{(t+1)}\|_2$ and applying Lemma 3 we get

$$
\|d_{i,k}^{(t)}\|_2 \leq \|d_{i,k}^{(t)}\|_2 \leq \varepsilon \left( 1 + \zeta \right)^{-1} C_4 (8C_3 NC_4^2 + 5)^{(t-1)K+k-1}.
$$

Now, under the assumption that both cloud $K$-SVD and centralized $K$-SVD use the same $d_{\text{ref}}$, we have $\hat{d}_{i,k}^{(t)} d_{i,k}^{(t)} \geq 0$ and therefore it follows from Lemma 7 in Appendix B that

$$
\|\hat{\eta}_{i,k}^{(t)} - d_{i,k}^{(t)}\|_2 \leq \varepsilon \sqrt{2} \left( 1 + \zeta \right)^{-1} C_4 (8C_3 NC_4^2 + 5)^{(t-1)K+k-1} \leq \sqrt{2} \delta_d \leq 1,
$$

(34)

where (a) follows from the assumption on $\varepsilon$ and (b) is true for any fixed $\delta_d$ as defined in Theorem 1. Using this bound we can write

$$
\|D^{(t)} - \hat{D}^{(t)}\|_2 \leq \|D^{(t)} - \hat{D}^{(t)}\|_F = \sqrt{\sum_{j=1}^{K} \|d_{i,j}^{(t)} - d_{i,j}^{(t)}\|_2^2} \leq \sqrt{K} \max_j \|d_{i,j}^{(t)} - d_{i,j}^{(t)}\|_2

\leq \sqrt{K} \max_j \sqrt{2} \varepsilon \left( 1 + \zeta \right)^{-1} C_4 (8C_3 NC_4^2 + 5)^{(t-1)K+k-1} \leq \sqrt{2} \hat{K} \left( 1 + \zeta \right)^{-1} \varepsilon C_4 (8C_3 NC_4^2 + 5)^{(K-1)}.
$$

(35)

Furthermore, using lemma assumption on $\varepsilon$ we get

$$
\|D^{(t)} - \hat{D}^{(t)}\|_2 \leq \sqrt{2} \hat{K} \delta_d = \min \left\{ \sqrt{K}, \frac{C_4^2 \tau_{\min}}{44} \right\}.
$$

(36)
We can now use (36) and Proposition 2 in Appendix D to bound $\| x_{i,j,T}^{(t+1)} - \hat{x}_{i,j,T}^{(t+1)} \|_2$ in (32). Notice that Proposition 2 assumes the error in dictionary to be smaller than $C_2 T_0^{3 \min_{s \in S_i}}$, which is satisfied by (36). Other assumptions of Proposition 2 are satisfied due to P1 and P2. Therefore, we get

$\forall i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, S_i\}$, $\| x_{i,j,T}^{(t+1)} - \hat{x}_{i,j,T}^{(t+1)} \|_2 \leq 3 \sqrt{T_0} \frac{\| D(t) \|_2}{r_{\min} C_2} \| D(t) - \hat{D}_i(t) \|_2$.

Now defining $X_i^{(t+1)}$ and $\hat{X}_i^{(t+1)}$ as before, we note that

$$\| x_{i,j,T}^{(t+1)} - \hat{x}_{i,j,T}^{(t+1)} \|_2 \leq \sqrt{S_{\max} \| X_i^{(t+1)} - \hat{X}_i^{(t+1)} \|_{\max}^{\max} \max_{j \in \{1, \ldots, S_i\}} \| x_{i,j,T}^{(t+1)} - \hat{x}_{i,j,T}^{(t+1)} \|_2}$$

(37)

$$\leq \frac{3 \sqrt{S_{\max} T_0}}{r_{\min} C_2} \| D(t) - \hat{D}_i(t) \|_2 \leq \frac{3 \sqrt{2K S_{\max} T_0}}{r_{\min} C_2} \varepsilon_1 \| 1 + \eta \|^{t-1} C_4 (8C_3 NC_4^2 + 5)^{tK-1},$$

(38)

where the last inequality follows from (35). Now using bounds on $\| x_{i,j,T}^{(t+1)} \|_2$ and (37) we get the following from (32):

$$\frac{B(t+1)}{\| B(t+1)\|_2} = 2 \frac{K \max_j \| \hat{x}_{i,j,T}^{(t+1)} - x_{i,j,T}^{(t+1)} \|_2 + \max_j \| d_{i,j}^{(t)} - d_{i,j}^{(t+1)} \|_2 \| x_{i,j,T}^{(t+1)} \|_2}{\| x_{i,j,T}^{(t+1)} \|_2}$$

(39)

Now fix $\varepsilon_1$ as in Theorem 3 and suppose (i) P1–P3 are satisfied, (ii) $\Omega(t)_{i,k} = \Omega(t)_{i,k}^{(t)}$ and (iii) $\varepsilon_1 \leq \delta_2 \sum_{k=1}^{K} \max_{j \in \{1, \ldots, S_i\}} \| x_{i,j,T}^{(t+1)} \|_2$.

**Lemma 5.** Let $\Omega_{i,k}$, $\Omega^{(t)}_{i,k}$, $\varepsilon$ and $\xi$ be as defined in Theorem 3 and perform $T_c$ consensus iterations as given in Theorem 3. Now fix $\delta_d$ as in Theorem 1 and suppose (i) P1–P3 are satisfied, (ii) $\Omega^{(t)}_{i,k} = \Omega^{(t)}_{i,k}$ and $\varepsilon \leq \delta_2 \sum_{k=1}^{K} \max_{j \in \{1, \ldots, S_i\}} \| x_{i,j,T}^{(t+1)} \|_2$. For any fixed $k \in \{1, \ldots, K\}$, $t \in \{1, \ldots, T_d\}$, and all $i \in \{1, \ldots, N\}$, if $\| B^{(t)}_{i,k,R} \|_2 \leq \varepsilon (1 + \eta)^{-1} C_4 (8C_3 NC_4^2 N + 5)^{(t-1)K-k-2}$ then $\| B^{(t)}_{i,k+1,R} \|_2 \leq \varepsilon (1 + \eta)^{-1} C_4 (8C_3 NC_4^2 N + 5)^{(t-1)K-k-1}$.

**Proof:** Recall once again that we can write

$$B^{(t)}_{i,k+1,R} = \sum_{j=k+1}^{K} d_{i,j}^{(t)} x_{i,j,R}^{(t)} - \sum_{j=1}^{K} d_{i,j}^{(t)} x_{i,j,R}^{(t)} - \sum_{j=1}^{K} \eta^{(t)}_{i,k} + \sum_{j=1}^{K} \sum_{j=1}^{K} d_{i,j}^{(t)} x_{i,j,R}^{(t)} + \sum_{j=1}^{K} \sum_{j=1}^{K} d_{i,j}^{(t)} x_{i,j,R}^{(t)}$$

(39)
where the last equality is obtained by substituting $\hat{x}_{i,k}^{(t)} = \hat{d}_{i,k}^{(t)} \hat{y}_{i,k}^{(t)}$. It then follows that

$$\|B_{i,k,R}^{(t)}\|_2 \leq \|B_{i,k,R}^{(t)}\|_2 + \|\hat{d}_{i,k}^{(t)} \hat{y}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)} \hat{y}_{i,k}^{(t)}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$$

$$\leq \|B_{i,k,R}^{(t)}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$$

$$\leq 2\|B_{i,k,R}^{(t)}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$$

$$\leq 2\|B_{i,k,R}^{(t)}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$$

$$\leq 2\|B_{i,k,R}^{(t)}\|_2 + 8C_N \max_i \left( \|B_{i,k,R}^{(t)}\|_2 + \|B_{i,k,R}^{(t)}\|_2 \right) \|E_{i,k,R}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$$

$$\leq (8C_N \|E_{i,k,R}\|_2 + 2)\|E_{i,k,R}\|_2 \|E_{i,k,R}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$$

$$\leq (8C_N \|E_{i,k,R}\|_2 + 3)\|E_{i,k,R}\|_2 \|E_{i,k,R}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$$

$$\leq (8C_N \|E_{i,k,R}\|_2 + 3)\|E_{i,k,R}\|_2 \|E_{i,k,R}\|_2 + \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$$

Here (a) is due to the fact that $E_{i,k,R}^{(t)}$ is a submatrix of $E_{i,k,R}^{(t)}$ and the definition of $C_4$ in (13), (b) follows from (30), and (c) is due to the assumption on $\|B_{i,k,R}^{(t)}\|_2$ and using the same procedure as in Lemma 4 after (31) to bound $\sum_k \|\hat{d}_{i,k}^{(t)} - \hat{d}_{i,k}^{(t)}\|_2 \|E_{i,k,R}\|_2$. The proof of the lemma now follows by using the assumption on $\varepsilon$, noting that $1 + \zeta \geq 1$ and some algebraic manipulations.

The proof of Theorem 3 now can be given by combining Lemmas 2, 5. Since these lemmas require the supports of both centralized and distributed problems to be the same, the main challenge in proving Theorem 3 lies in showing this fact.

**Proof of Theorem 3**

We will prove this theorem by mathematical induction over $t$. To be specific, we will prove the following two cases:

1) For base case, we will show that the claim holds for $\|B_{i,k,R}^{(1)}\|_2 \forall k \in \{1, 2, \cdots, K\}$.

2) For induction step we assume that for any $q \in \{1, 2, \cdots, T_d - 1\}$ the claim is true for $\|B_{i,k,R}^{(q+1)}\|_2 \forall k \in \{1, 2, \cdots, K\}$ and $\Omega_{i,k}^{(q+1)} = \Omega_{i,k}^{(q+1)}$. Then we need to show that $\Omega_{i,k}^{(q+1)} = \Omega_{i,k}^{(q+1)}$ and claim holds for $\|B_{i,k,R}^{(q+1)}\|_2 \forall k \in \{1, 2, \cdots, K\}$.

**Base case: $t = 1 \forall k \in \{1, 2, \cdots, K\}$**

To prove the base case, we will do mathematical induction over $k$ by fixing $t = 1$. Hence, the first thing we need to prove is that the bound is true for $\|B_{i,k,R}^{(1)}\|_2$. Since both cloud $K$-SVD and Centralized $K$-SVD start with the same initial dictionary, we have $d_j^{(0)} = \tilde{d}_j^{(0)}$, $\forall j \in \{1, 2, \cdots, K\}$. Therefore, we get $\Omega_{i,k}^{(1)} = \tilde{\Omega}_{i,k}^{(1)}$, $\forall j \in \{1, 2, \cdots, K\}$. It then follows that $B_{i,1,R}^{(1)} = E_{i,1,R}^{(1)} - \tilde{E}_{i,1,R}^{(1)} = \sum_{j=1}^{K-1} \left( d_j^{(0)} x_{i,j,T}^{(1)} \Omega_{i,j}^{(1)} - \tilde{d}_j^{(0)} x_{i,j,T}^{(1)} \tilde{\Omega}_{i,j}^{(1)} \right) = 0$, thereby proving the claim.

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Next, for induction argument we fix \( k = p + 1 \). Using the induction assumption we have the following bound on \( \|B_{1,i,p,R}^{(1)}\|_2 \leq \varepsilon C_4 (8C_3 NC_4^2 + 5)^{p - 2} \). Since \( \Omega_{i,j}^{(1)} = \gamma_{i,j}^{(1)} \), we have \( B_{1,i,p+1,R}^{(1)} = E_{1,i,p+1,R}^{(1)} - E_{1,i,p+1,R}^{(1)} \). This results in

\[
\|B_{1,i,p+1,R}^{(1)}\|_2 = \|Y_{i} \Omega_{i,p+1} - \sum_{j=p+2}^{K} d_{i,j}^{(0)} x_{i,j}^{(1)} - \sum_{j=1}^{p} d_{i,j}^{(1)} x_{i,j,R}^{(1)} - Y_{i} \Omega_{i,p+1} + \sum_{j=p+2}^{K} d_{i,j}^{(0)} x_{i,j}^{(1)} + \sum_{j=1}^{p} d_{i,j}^{(1)} x_{i,j,R}^{(1)}\|
\]

\[
= \| \sum_{j=p+2}^{K} \left( d_{i,j}^{(0)} x_{i,j}^{(1)} - d_{i,j}^{(0)} x_{i,j,R}^{(1)} \right) - \sum_{j=1}^{p} \left( d_{i,j}^{(1)} x_{i,j}^{(1)} - d_{i,j}^{(1)} x_{i,j,R}^{(1)} \right) \|_2 = \| \sum_{j=1}^{p-1} \left( d_{i,j}^{(1)} x_{i,j}^{(1)} - d_{i,j}^{(1)} x_{i,j,R}^{(1)} \right) \|_2 = \| d_{i,p}^{(1)} x_{i,p}^{(1)} - d_{i,p}^{(1)} x_{i,p,R}^{(1)} + B_{1,i,p}^{(1)} \|_2,
\]

where (a) is true because \( d_{i,j}^{(0)} x_{i,j}^{(1)} = d_{i,j}^{(0)} x_{i,j,R}^{(1)} = 0 \). Substituting \( x_{i,p}^{(1)} = \gamma_{i,p}^{(1)} F_{i,p,R}^{(1)} \), we get

\[
\|B_{1,i,p+1,R}^{(1)}\|_2 \leq \|B_{1,i,p,R}^{(1)}\|_2 + \|d_{i,p}^{(1)} \gamma_{i,p}^{(1)\top} (E_{1,i,p,R}^{(1)} + B_{1,i,p,R}^{(1)}) - d_{i,p}^{(1)} \gamma_{i,p}^{(1)\top} E_{1,i,p,R}^{(1)}\|_2 \leq 2\|B_{1,i,p,R}^{(1)}\|_2 + \|d_{i,p}^{(1)} \gamma_{i,p}^{(1)\top} - d_{i,p}^{(1)} \gamma_{i,p}^{(1)\top} \|_2 \|E_{1,i,p,R}^{(1)}\|_2
\]

\[
\leq 2\|B_{1,i,p,R}^{(1)}\|_2 + \|d_{i,p}^{(1)} \gamma_{i,p}^{(1)\top} - d_{i,p}^{(1)} \gamma_{i,p}^{(1)\top} \|_2 C_4
\]

\[
\leq 2\|B_{1,i,p,R}^{(1)}\|_2 + 8C_3 N \max_i \left( \|B_{1,i,p,R}^{(1)}\|_2, \|B_{1,i,p,R}^{(1)}\|_2^2 \right) C_4 + \varepsilon C_4
\]

\[
\leq \max_i \left( 2\|B_{1,i,p,R}^{(1)}\|_2 + 8C_3 N C_4^2 \|B_{1,i,p,R}^{(1)}\|_2 + 8C_3 N \|B_{1,i,p,R}^{(1)}\|_2^2 + \varepsilon C_4 \right)
\]

\[
\leq \varepsilon C_4 \left((8C_3 N C_4^2 + 5)^{p - 2}(8C_3 N C_4^2 + 2) + 1 + 1\right) \leq \varepsilon C_4 (8C_3 N C_4^2 + 4)^{p - 1} \leq \varepsilon C_4 (8C_3 N C_4^2 + 5)^{p - 1}.
\]

**Induction step: Bound on \( \|B_{k,i,k,R}^{(q+1)}\|_2 \) holds for \( q \in \{1, \ldots, T_d - 1\} \) and \( \forall k \in \{1, 2, \ldots, K\} \).**

We need to show the bound holds for \( \|B_{k,i,k,R}^{(q+1)}\|_2 \forall k \in \{1, \ldots, K\} \). To show this, we will be using induction argument over \( k \) by fixing \( t = q + 1 \). Base case for this corresponds to a bound on \( \|B_{k,i,1,R}^{(q+1)}\|_2 \). To bound \( \|B_{k,i,1,R}^{(q+1)}\|_2 \), we will be using Lemma 4 which assumes \( \gamma_{i,1}^{(q+1)} = \Omega_{i,1}^{(q+1)} \). Using the induction assumptions, we get the following bound on error in dictionary \( \tilde{D}_{i}^{(q)} \) using Lemma 8 and performing same steps as we carried out in Lemma 4 to get (34) and (35). \( \|D^{(q)} - \tilde{D}_{i}^{(q)}\|_2 \leq \varepsilon \sqrt{2K} C_4 (8C_3 N C_4^2 + 5)^{q-1} K^{q-k-2} \). Using the assumption on \( \varepsilon \), we then have \( \|D^{(q)} - \tilde{D}_{i}^{(q)}\|_2 \leq \delta \sqrt{2K} \). It then follows from arguments similar to the ones made in Lemma 4 that \( \gamma_{i,1}^{(q+1)} = \gamma_{i,1}^{(q+1)} \). We can now use Lemma 8 to bound \( \|B_{k,i,1,R}^{(q+1)}\|_2 \) as follows: \( \|B_{k,i,1,R}^{(q+1)}\|_2 \leq \varepsilon (1 + \zeta)^q \varepsilon (8C_3 N C_4^2 + 5)^{qK-1}. \) Having proved the base case, we now suppose that the claim is true for some \( k = p \in \{1, \ldots, K - 1\} \). We then need to show it holds for \( \|B_{k,i,p+1,R}^{(q+1)}\|_2 \). That claim, however, simply follows from Lemma 5. This concludes the proof of theorem.
APPENDIX C
PROOF OF THEOREM 1

To prove Theorem 1 we need an upper bound on error in matrices \( \hat{E}^{(t)}_{i,k,R} \), which is given by Theorem 3. Applying Theorem 3 to get a bound on the error in dictionary atom \( \hat{d}^{(t)}_{i,k} \) is a trivial task. But before using Theorem 3 we need to show that our assumption on \( \hat{d} \) is indeed satisfied. In the following, we will prove that the assumption on \( \varepsilon \) is satisfied if we perform \( T_p \) power method and \( T_c \) consensus iterations that are given according to the statement of Theorem 1.

Proof of Theorem 1 After \( T_d \) iterations of cloud \( K \)-SVD, error in any \( k^{th} \) dictionary atom \( \hat{d}^{(T_d)}_{i,k} \) at site \( i \) is a function of the error in \( \hat{E}^{(T_d)}_{i,k,R} \). Specifically, notice from (30) that we can write
\[
\|d^{(T_d)}_{k,i}d^{(T_d)}_{k,i}^T - \hat{d}^{(T_d)}_{k,i}d^{(T_d)}_{k,i}^T\|_2 \leq 8NC_3 \max_i (\|B^{(T_d)}_{k,i,R}\|_2 C_4 + |B^{(T_d)}_{k,i,R}\|_2^2) + \varepsilon.
\]

We can now upper bound \( \|B^{(T_d)}_{k,i,R}\|_2 \) using Theorem 3 but we first need to show that the statement of Theorem 1 implies the assumption on \( \varepsilon \) in Theorem 3 is satisfied. That is, we need to show \( \varepsilon \leq \frac{\delta_d}{8NC_3(1 + \varepsilon)^{T_d} - 1C_4(8C_3NC_4^2 + 5)^{T_d/2}K^{T_d/2}2^{2(T_d/2) - 2}} \).

Recall that by definition \( \varepsilon = \mu \|T_p\| + 4e^{3T_p} \). Substituting this, we must show that
\[
\mu \|T_p\| + 4e^{3T_p} \leq \frac{\delta_d}{8NC_3(1 + \varepsilon)^{T_d} - 1C_4(8C_3NC_4^2 + 5)^{T_d/2}K^{T_d/2}2^{2(T_d/2) - 2}}.
\]

Since \( \nu > 0 \) and \( e > 0 \), therefore, \( \mu \|T_p\| + 4e^{3T_p} < \mu(\nu + 4e^3)^{T_p} \). It is therefore sufficient to show that \( \mu(\nu + 4e^3)^{T_p} \leq \frac{\delta_d}{8NC_3(1 + \varepsilon)^{T_d} - 1C_4(8C_3NC_4^2 + 5)^{T_d/2}K^{T_d/2}2^{2(T_d/2) - 2}} \) for our selected values of \( T_p \) and \( T_c \). Showing that, however, is a simple exercise in algebra and is left out for brevity. It therefore follows from Theorem 3 that \( \|d^{(T_d)}_{k,i}d^{(T_d)}_{k,i}^T - \hat{d}^{(T_d)}_{k,i}d^{(T_d)}_{k,i}^T\|_2 \leq \varepsilon(1 + \varepsilon)^{T_d} - 1C_4(8C_3NC_4^2 + 5)^{(T_d/2) - 1}K^{K/2} - 1 \). Finally, substituting upper bound on \( \varepsilon \) one more time, we get \( \|d^{(T_d)}_{k,i}d^{(T_d)}_{k,i}^T - \hat{d}^{(T_d)}_{k,i}d^{(T_d)}_{k,i}^T\|_2 \leq \delta_d \).

APPENDIX D
OTHER RESULTS

In this appendix, we collect some supporting results that are used in the proofs of our main results.

Lemma 6 (Perturbation of singular values). Let \( D_2 \) be a perturbed version of dictionary \( D_1 \) such that \( \|D_1 - D_2\|_2 \leq \varepsilon_2 \) and let \( \Sigma_{T_0} \) be as defined in Section IV-A. Then assuming \( \min_{\mathcal{I} \in \Sigma_{T_0}} \sigma_{T_0}(D_{1, \mathcal{I}}) \geq \sqrt{C_2} > \varepsilon_2 \), we have \( \min_{\mathcal{I} \in \Sigma_{T_0}} \sigma_{T_0}(D_{2, \mathcal{I}}) \geq \sqrt{C_2} - \varepsilon_2 \).

Proof: Using [48] Theorem 1], perturbation in \( T_0^{th} \) singular value of \( D_{1, \mathcal{I}} \) can be bounded as \( |\sigma_{T_0}(D_{1, \mathcal{I}}) - \sigma_{T_0}(D_{2, \mathcal{I}})| \leq \|D_{1, \mathcal{I}} - D_{2, \mathcal{I}}\|_2 \leq \|D_1 - D_2\|_2 \leq \varepsilon_2 \). Using reverse triangular inequality, we therefor get \( \forall \mathcal{I} \in \Sigma_{T_0}, \varepsilon_2 \geq |\sigma_{T_0}(D_{1, \mathcal{I}}) - |\sigma_{T_0}(D_{2, \mathcal{I}})| \geq \sqrt{C_2} - |\sigma_{T_0}(D_{2, \mathcal{I}})| \), which leads to the claimed result.

Proposition 2 (Stability of sparse coding). [49] Theorem 1] Let \( D_2 \) be a perturbed version of dictionary \( D_1 \) such that \( \|D_1 - D_2\|_2 \leq \varepsilon_2 \). Given any sample \( y \in \mathbb{R}^n \), suppose sparse codes \( x \in \mathbb{R}^K \) and \( \hat{x} \in \mathbb{R}^K \) are computed by solving the lasso problem \( \hat{x} \) using \( D_1 \) and \( D_2 \), respectively. Next, let \( \min_{j \in \text{supp}(x)} \tau - |\langle d_{1,j}, y - D_1 x \rangle| > C_1 \).
where $d_{1,j}$ denotes the $j^{th}$ atom of $D_1$, and suppose $D_1$ satisfies P2. Then, as long as $\epsilon_2 \leq \frac{C_2 \tau}{4T}$, we have that $\text{supp}(x) = \text{supp}(\hat{x})$ and

$$
\|x - \hat{x}\|_2 \leq \frac{3\|D_1 - D_2\|_2 \sqrt{T_0}}{\tau C_2},
$$

(43)

where $T_0 = |\text{supp}(x)| = |\text{supp}(\hat{x})|$.

Note that [49, Theorem 1] also requires $D_2$ to satisfy P2. Proposition 2 in its current form, however, is a simple consequence of [49, Theorem 1] and Lemma 6.

**Proposition 3** (Perturbation of principal eigenvector). ([22, Chap. 8]) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and define $\hat{A} = A + E$ to be a perturbed, but symmetric version of $A$. Define $Q = \begin{bmatrix} q_1 & Q_2 \end{bmatrix}$ to be an orthogonal matrix comprising eigenvectors of $A$, where $q_1$ denotes the principal eigenvector of $A$. Next, define $Q^T A Q = \begin{bmatrix} \lambda & 0 \\ 0 & \Lambda_2 \end{bmatrix}$ and $Q^T E Q = \begin{bmatrix} \epsilon & \epsilon^T \\ \epsilon & E_{22} \end{bmatrix}$. Then, using $\text{eig}(\Lambda_2)$ to denote the $(n-1)$ smallest eigenvalues of $A$, it follows that if $g = \min_{\epsilon \in \text{eig}(\Lambda_2)} |\lambda - \epsilon| > 0$, and $\|E\|_F \leq \frac{g}{4}$ then there exists $p \in \mathbb{R}^{n-1}$ satisfying $\|p\|_2 \leq \frac{\sqrt{g}}{2}\|e\|_2$, such that $\hat{q}_1 = \frac{q_1 + Q_2 p}{\sqrt{1+p^T p}}$ is a unit 2-norm principal eigenvector for $\hat{A}$. Moreover, $\|q_1 q_1^T - \hat{q}_1 \hat{q}_1^T\|_2 \leq \frac{\sqrt{g}}{2}\|e\|_2$.

**Lemma 7** (Errors in vectors and their outerproducts). For two unit $\ell_2$-norm vectors $u$ and $v$ if $\|uu^T - vv^T\|_2 \leq \epsilon$ and $u^T v \geq 0$ then $\|u - v\|_2 \leq \sqrt{2}\epsilon$.

**Proof:** Let $\theta = \angle(u, v)$ and notice that $\|uu^T - vv^T\|_2 = \sin\theta$. This implies $1 - \cos^2\theta = \sin^2\theta = \|uu^T - vv^T\|_2 \leq \epsilon^2$. Now, since $u$ and $v$ are unit norm and $u^T v \geq 0$, we can write $\cos\theta = u^T v$. It therefore follows that $1 - u^T v \leq \frac{\epsilon^2}{1+u^T v} < \epsilon^2$. The claim now follows by noting that we can write $\|u - v\|_2 = \sqrt{2(1-u^T v)}$.

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