The Balakrishnan-Alpha-Beta-Skew-Normal Distribution: Properties and Applications

Sricharan Shah1, Partha Jyoti Hazarika1,*, Subrata Chakraborty1 and M. Masoom Ali2

* Corresponding Author

1. Department of Statistics, Dibrugarh University, Dibrugarh, 786004, Assam, India, charan.shah90@gmail.com, parthajhazarika@gmail.com, and subrata.stats@dibru.ac.in
2. Department of Mathematical Sciences, Ball State University, Muncie, IN 47306 USA, mali@bsu.edu

Abstract

In this paper, a new form of alpha-beta-skew distribution is proposed under Balakrishnan (2002) mechanism and investigated some of its related distributions. The most important feature of this new distribution is that it is versatile enough to support both unimodal and bimodal as well as multimodal behaviors of the distribution. The moments, distributional properties and some extensions of the proposed distribution have also been studied. Finally, the suitability of the proposed distribution has been tested by conducting data fitting experiment and comparing the values of Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) with the values of some other related distributions. Likelihood Ratio test is used for discriminating between normal and the proposed distributions.

Keywords: Skew-Normal Distribution, Alpha-Beta-Skew-Normal Distribution, Bimodal Distribution, Likelihood Ratio Test, AIC, BIC.

Mathematical Subject Classification: 60E05, 62E10

1. Introduction:

The application of skewed distributions arises in every area of the sciences, engineering and medicine because the data are likely to come from asymmetrical populations. One of the simple and common approaches for the construction of skewed distributions is to introduce the skewness into some known symmetrical distributions. Azzalini (1985) first introduced the skew normal distribution using a natural extension of symmetrical normal distribution and adding an additional parameter to introduce asymmetry. Its probability density function (pdf) is given by

$$f_Z(z;\lambda) = 2\phi(z)\Phi(\lambda z); \quad -\infty < z < \infty$$ (1)

where \(\phi(\cdot)\) and \(\Phi(\cdot)\) are, respectively, the pdf and cumulative distribution function (cdf) of standard normal variable \(Z\) and \(\lambda \in R\), the skewness parameter. The family of distributions given by Equation (1) and the skew-normal class have been studied and extended by many authors (for detail see Chakraborty and Hazarika, 2011). Balakrishnan (2002) proposed the generalization of skew normal distribution (as a discussant in Arnold and Beaver, 2002) and studied its properties and the pdf of the same is given by

$$f_Z(z;\lambda,n) = \phi(z)[\Phi(\lambda z)]^n/C_n(\lambda); \quad -\infty < z < \infty, \lambda \in R$$ (2)

Where \(n\) is a positive integer and \(C_n(\lambda) = E[\Phi^n(\lambda U)], U \sim N(0,1)\).

Huang and Chen (2007) proposed a general methodology for the construction of skew-symmetric distributions by implementing the concept of skew function \(G(\cdot)\) instead of cdf in Equation (1) where, \(G(\cdot)\) is a Lebesgue
measurable function satisfying, \(0 \leq G(z) \leq 1\) and \(G(z) + G(-z) = 1, \ z \in R\), almost everywhere. The pdf of the same is given by

\[ f(z) = 2\varphi(z)G(z); \ z \in R. \]  

Obviously, by selecting different skew functions in Equation (3), one can develop a wide number of skewed distributions. Elal-Olivero (2010) introduced, using the approach of the Equation (3), a new form of skew distribution which has both unimodal as well as bimodal behavior and is known as alpha-skew-normal distribution, denoted by \(ASN(\alpha)\) and its pdf is given by

\[ f(z;\alpha) = \left(\frac{(1-\alpha z)^2 + 1}{2 + \alpha^2}\right)\varphi(z); \ z \in R. \]  

Shafiei et al. (2016) further extended the above family of skew distributions to provide more flexibility than the Azzalini (1985) and the Elal-Olivero (2010) distributions and called it the alpha-beta-skew-normal distribution, denoted by \(ABSN(\alpha, \beta)\) with pdf given by

\[ f(z;\alpha, \beta) = \left(\frac{(1-\alpha z - \beta z^3)^2 + 1}{2 + \alpha^2 + 15\beta^2 + 6\alpha \beta}\right)\varphi(z); \ z, \alpha, \beta \in R. \]  

Hazarika et al. (2020) introduced a new form of skew normal distribution known as Balakrishnan-alpha-skew-normal \(BASN_2(\alpha)\) distribution and the pdf is given by

\[ f(z;\alpha) = \frac{[1-\alpha z]^2 + 1]}{C_2(\alpha)} \varphi(z), \ z \in R \]  

where \(C_2(\alpha) = 4 + 8\alpha^2 + 3\alpha^4\).

There are more skewed distributions related to Huang and Chen (2007) by considering different skew functions such as Harandi and Alamatsaz (2013), Hazarika and Chakraborty (2014), Chakraborty et al. (2012, 2014, and 2015), Shah et al. (2020a, 2020b, 2020c), Shah and Hazarika (2019) etc.

In the present article, the Balakrishnan (2002) and the Shafiei et al. (2016) methodology have been implemented to propose a new alpha-beta-skew-normal distribution, known as Balakrishnan-alpha-beta-skew-normal distribution which is versatile enough to support both unimodal and bimodal behavior and investigate some of its distributional properties. To exhibit the applicability of the proposed distribution, three real life datasets are considered which give better fit when compared to some other known distributions. The article is organized as follows. In Section 2, the Balakrishnan-alpha-beta-skew-normal distribution is defined and some of its important distributional properties are discussed. The random number generation of the proposed distribution is defined in Section 3. The location-scale extension and maximum likelihood estimation are given in Section 4. In Section 5, some numerical examples based on real life data and Likelihood ratio test are provided. Finally, the article ends with conclusions in Section 6.

2. A New Alpha-Beta-Skew-Normal Distribution

In this section, we define a new family of distribution known as Balakrishnan-alpha-beta-skew-normal (BABS\(N\)) distribution and discuss some of its distributional properties.

**Definition 1:** If a random variable \(Z\) has a pdf given by

\[ f_Z(z;\alpha, \beta) = \frac{[1-\alpha z - \beta z^3]^2 + 1]}{C_2(\alpha, \beta)} \varphi(z); \ z \in R \]  

where \(C_2(\alpha, \beta) = C_2(\alpha) + 60\alpha^2 \beta + 12\alpha \beta (4 + 315\beta^2) + 630\alpha^2 \beta^2 + 15\beta^2 (8 + 693\beta^2)\), \(C_2(\alpha)\) is defined before and \(\varphi(.)\) is the pdf of standard normal distribution, then it is said to be a Balakrishnan-alpha-beta-skew-normal distribution with skewness parameters \(\alpha \in R\) and \(\beta \in R\). In the rest of this article we shall refer the distribution in Equation (7) as \(BABS\_2(\alpha, \beta)\).

**Remark 1:** The pdf of the proposed distribution is constructed using the formulae in Equation (2) and Equation (3), by taking \(\Phi(.) = G(.) = \frac{(1-\alpha z - \beta z^3)^2 + 1}{2 + \alpha^2 + 15\beta^2 + 6\alpha \beta}\) and \(n = 2\).

If \(Z \sim BABS\_2(\alpha, \beta)\), the following properties are deduced immediately from the definition:

- If \(\beta = 0\), then we get the \(BASN_2(\alpha)\) distribution of Hazarika et al. (2020) and is given by
  \[ f(z;\alpha) = \frac{[1-\alpha z]^2 + 1]}{C_2(\alpha)} \varphi(z)/C_2(\alpha). \]
• If $\alpha = 0$, then we get $f(z; \beta) = [(1 - \beta z^2)^2 + 1]^2 \phi(z) / [4 + 15\beta^2 (8 + 693\beta^2)]$.
  This is a new distribution referred to as Balakrishnan-beta-skew-normal $BBSN_2(\beta)$ distribution.

• If $\alpha = \beta = 0$, then we get the standard normal $N(0,1)$ distribution and is given by $f(z) = \phi(z)$.

• If $\alpha \to \pm\infty$, then we get the bimodal-normal $BN(4)$ distribution (see Hazarika et al. 2020) given by $f(z) = (z^4 / 3)\phi(z)$.

• If $\beta \to \pm\infty$, then we get the bimodal-normal $BN(12)$ distribution (see Hazarika et al. 2020) given by $f(z) = (z^{12} / 10395)\phi(z)$.

• If $Z \sim BBSN_2(\alpha, \beta)$, then $-Z \sim BBSN_2(-\alpha, -\beta)$.

The pdf of $BBSN_2(\alpha, \beta)$ for different choices of the parameters $\alpha$ and $\beta$ are plotted in Figure 1. It is observed from Figure 1 that the proposed distribution is very flexible to support unimodal, bimodal and multimodal behaviors and has at most four modes and is also proved in Proposition 2.1. The parameters of the proposed distribution have significant effects on the skewness and the probable number of modes.

**Figure 1**: Plots of the pdf of $BBSN_2(\alpha, \beta)$.

**Proposition 2.1**: The pdf of $BBSN_2(\alpha, \beta)$ distribution has at most four modes.

**Proof 2.1**: See Appendix A.

**Proposition 2.2**: The cdf of $BBSN_2(\alpha, \beta)$ distribution can be written as

$$F_Z(z; \alpha, \beta) = \Phi(z) + \frac{\phi(z)}{2C_2(\alpha, \beta)} \left[ \begin{array}{c}
2\alpha(8 + \alpha(8\alpha - z(8 - 4\alpha z + (3 + z^2)\alpha^2))) - 8(-2 + z^2) + 4\alpha \beta z + 3\alpha^2 b_1 + 3\alpha^2 b_2 - 4(4\alpha z - 6\alpha z + 3\alpha^2 b_3) - 8(-384 + z(-z(192 + 48\alpha^2 + 8\alpha z + 4\alpha z + z^2)) - 2z^4 b_6)
\end{array} \right]$$

(8)
where \( b_1 = (8 + 4z^2 + z^4), b_2 = (15 + 5z^2 + z^4), b_3 = (48 + 24z^2 + 6z^4 + z^6), b_4 = (105 + 35z^2 + 7z^4 + z^8), b_5 = (945 + 315z^2 + 63z^4 + 9z^6 + z^8) \) and \( b_6 = (10395 + 3465z^2 + 693z^4 + 99z^6 + 11z^8 + z^{10}) \). \( \varphi(\cdot) \) and \( \Phi(\cdot) \) are as defined before.

**Proof 2.2:** See Appendix B.

The cdf is plotted in Figure 2 for studying variation in its shape with respect to the parameters \( \alpha \) and \( \beta \).

**Corollary 1:** In particular, by taking the limit \( \alpha \to \pm \infty \) of \( F_Z(z; \alpha, \beta) \) in Equation (8), we get the cdf of \( BN(4) \) distribution as \( F_Z(z) = \Phi(z) - \frac{z(3 + z^2)}{3} \varphi(z) \). Again, in particular, by taking the limit \( \beta \to \pm \infty \) of \( F_Z(z; \alpha, \beta) \) in Equation (8), we get the cdf of \( BN(12) \) distribution as \( F_Z(z) = \Phi(z) - \frac{z b_6}{10395} \varphi(z) \) where \( b_6 \) is defined above.

**Proposition 2.3:** The moment generating function (mgf) of \( BABS_N(\alpha, \beta) \) distribution can be written as

\[
M_Z(t) = \frac{M_X(t)}{C_2(\alpha, \beta)} \begin{pmatrix}
4 + \alpha(-8t + 8\alpha(1 + t^2) - 4t\alpha^2(3 + t^2) + \alpha^3c_1) + 4(-2t(3 + t^2) + 4\alpha c_1 - 3\alpha^2c_2 + \alpha^3c_3)\beta + 2(60 + 4t^2c_4 - 6\alpha t c_5 + 3\alpha^2c_6)\beta^2 + 4(-t c_7 + \alpha c_8)\beta^3 + c_8 \beta^4
\end{pmatrix}
\]

where \( M_X(t) \) is the mgf of \( X \sim N(0, 1) \), \( c_1 = (3 + 6t^2 + t^4), c_2 = (15 + 10t^2 + t^4), c_3 = (15 + 45t^2 + 15t^4 + t^6), c_4 = (45 + 15t^2 + t^4), c_5 = (105 + 105t^2 + 210t^4 + t^6), c_6 = (105 + 420t^2 + 210t^4 + 28t^6 + t^8), c_7 = (945 + 1260t^2 + 3780t^4 + 360t^6 + t^8), c_8 = (945 + 4725t^2 + 3150t^4 + 630t^6 + 45t^8 + t^{10}) \) and \( c_9 = (10395 + 62370t^2 + 51975t^4 + 13860t^6 + 1485t^8 + 66t^{10} + t^{12}) \).

**Proof 2.3:** See Appendix C.

**Corollary 2:** In particular, by taking the limit \( \alpha \to \pm \infty \) of \( M_Z(t) \) in Equation (9), we get the mgf of \( BN(4) \) distribution as \( M_Z(t) = \frac{c_1}{3} M_X(t) \) where \( c_1 \) is defined above. Again, in particular, by taking the limit \( \beta \to \pm \infty \) of \( M_Z(t) \) in Equation (9), we get the mgf of \( BN(12) \) distribution as \( M_Z(t) = \frac{c_9}{10395} M_X(t) \) where \( c_9 \) is defined above.
Proposition 2.4: The $n^{th}$ order moment of $BABS_N(\alpha,\beta)$ distribution can be written as

$$E(Z^{2n}) = \frac{4E_N(Z^{2n}) + 8\alpha^2 E_N(Z^{2n+2}) + \alpha(\alpha^3 + 16\beta)E_N(Z^{2n+4}) + 4\beta(\alpha^3 + 2\beta)}{C_2(\alpha,\beta)}$$

and

$$E(Z^{2n-1}) = \frac{-8\alpha E_N(Z^{2n}) - 4(\alpha^3 + 2\beta)E_N(Z^{2n+2}) - 12\alpha^2 \beta E_N(Z^{2n+4})}{C_2(\alpha,\beta)}$$

where $E_N(Z^{2n}) = \frac{2^{(-1+\frac{1}{2})} [1+(-1)^n] \Gamma(\frac{n+1}{2})}{\sqrt{\pi}}$, $n > 0$ is the moment of standard normal distribution.

Proof 2.4: See Appendix D.

From the above equation of $n^{th}$ order moment, we get

$$E(Z) = \frac{-4[3\alpha^3 + 6\beta + 45\alpha^2 \beta + 945\beta^3 + \alpha(2 + 31\beta^2)]}{C_2(\alpha,\beta)}$$

$$E(Z^2) = \frac{4 + 15\alpha^4 + 420\alpha^2 \beta + 840\beta^2 + 13515\beta^4 + 60\alpha\beta(4 + 693\beta^2) + 6\alpha^2(4 + 945\beta^2)}{C_2(\alpha,\beta)}$$

$$E(Z^3) = \frac{-12[5\alpha^3 + 105\alpha^2 \beta + 5\beta(2 + 693\beta^2) + \alpha(2 + 945\beta^2)]}{C_2(\alpha,\beta)}$$

$$E(Z^4) = \frac{3[4 + 40\alpha^2 + 35\alpha^4 + 140\alpha\beta(4 + 9\alpha^2) + 630\beta^2(4 + 33\alpha^2) + 180180\alpha^2 + 67567\beta^4]}{C_2(\alpha,\beta)}$$

$$Var(Z) = \frac{-16[3\alpha^3 + 6\beta + 45\alpha^2 \beta + 945\beta^3 + \alpha(2 + 31\beta^2)]^2 + [4 + 3\alpha^4 + 60\alpha^3 \beta + 12\alpha\beta(4 + 315\beta^2) + \alpha^2(8 + 630\beta^2) + 15\beta^2(8 + 693\beta^2)][4 + 15\alpha^4 + 420\alpha^2 \beta + 840\beta^2 + 13515\beta^4 + 60\alpha\beta(4 + 693\beta^2) + 6\alpha^2(4 + 945\beta^2)]}{[C_2(\alpha,\beta)]^2}$$

By numerically optimizing $E(Z)$ and $Var(Z)$ with respect to $\alpha$ and $\beta$, we get the following bounds for mean and variance as $-1.52269 \leq E(Z) \leq 1.52269$ and $0.605494 \leq Var(Z) \leq 16.1254$. The plots of the mean and the variance are given respectively, in Figure 3 and Figure 4 to study their behaviors. These plots also verify these bounds.

![Figure 3: Plots of mean](image1)

![Figure 4: Plots of variance](image2)

**Remark 2:** By taking the limit $\alpha \to \pm \infty$ in the moments of $BABS_N(\alpha,\beta)$ distribution, we can derive the moments of $BN(4)$ distribution as $E(Z) = 0$, $Var(Z) = 5$. Again, by taking the limit $\beta \to \pm \infty$ in the moments of $BABS_N(\alpha,\beta)$ distribution, we can derive the moments of $BN(12)$ distribution as $E(Z) = 0$, $Var(Z) = 13$.

**2.1. Skewness and Kurtosis**

The skewness ($\beta_1$) and kurtosis ($\beta_2$) of $BABS_N(\alpha,\beta)$ distribution are respectively, given by
\[
\beta_1 = \frac{16(32d_1^3 + 3C_2(\alpha, \beta)^2d_2 - 3C_2(\alpha, \beta)d_1d_2^2)}{(-16d_1^2 + C_2(\alpha, \beta)d_2)^3}
\]

where \(d_1 = 3\alpha^3 + 6\beta + 45\alpha^2\beta + 945\beta^3 + \alpha(2 + 315\beta^2), d_2 = 5\alpha^3 + 105\alpha^2\beta + 5\beta(2 + 693\beta^2) + \alpha(2 + 945\beta^2),\) and \(d_3 = 4 + 15\alpha^4 + 420\alpha^3\beta + 840\beta^2 + 1351\beta^4 + 60\alpha\beta(4 + 693\beta^2) + 6\alpha^2(4 + 945\beta^2).\)

In particular, when \(\alpha = \beta = 1\), the value of \(\beta_1 = 0.0470217\) and when \(\alpha = \beta = 0\), then \(\beta_1 = 0\) which is the skewness of standard normal distribution and the distribution has symmetric normal curve. Again,

\[
\beta_2 = 3(C_2(\alpha, \beta))^3d_4 - 256d_1^3 - 64C_2(\alpha, \beta)^2d_2 + 32C_2(\alpha, \beta)d_1^2d_2)
\]

where \(d_4 = 4 + 40\alpha^2 + 35\alpha^4 + 140\alpha\beta(4 + 9\alpha^2) + 630\beta^2(4 + 33\alpha^2) + 180180\alpha\beta^3 + 675675\beta^4.\)

In particular, when \(\alpha = \beta = 1\), the value of \(\beta_2 = 1.22609\) and when \(\alpha = \beta = 0\), then \(\beta_2 = 3\) which is the kurtosis of standard normal distribution and the distribution has mesokurtic curve. The numerical optimization of \(\beta_1\) and \(\beta_2\) with respect to \(\alpha\) and \(\beta\), gives the following bounds for skewness and kurtosis as \(0 \leq \beta_1 \leq 6.27667\) and \(1.10797 \leq \beta_2 \leq 16.237\). The plots of the skewness and kurtosis are respectively given in Figure 5 and Figure 6 to study their behaviors. These plots also verify these bounds.

**Figure 5:** Plots of skewness

**Figure 6:** Plots of kurtosis

**Remark 3:** By taking the limit \(\alpha \to \pm \infty\) in the results of BABS\(N_2(\alpha, \beta)\) distribution, we can derive the skewness and kurtosis of \(BN(4)\) distribution as \(\beta_1 = 0, \beta_2 = 1.4\). Again, by taking the limit \(\beta \to \pm \infty\) in the results of BABS\(N_2(\alpha, \beta)\) distribution, we can derive the skewness and kurtosis of \(BN(12)\) distribution as \(\beta_1 = 0, \beta_2 = 1.15385\).

### 3. Random Number Generation

In this section, first we represent BABS\(N_2(\alpha, \beta)\) distribution into symmetric and asymmetric form and then generate a random number from BABS\(N_2(\alpha, \beta)\) distribution.

**Lemma1:** The density function in Equation (7) of model BABS\(N_2(\alpha, \beta)\) can be represented as sum of two functions

\[
f_Z(z, \alpha, \beta) = \frac{4 + z^2(\alpha + \beta z^2)^2 + (8 + z^2(\alpha + \beta z^2)^2)}{C_2(\alpha, \beta)} \varphi(z) + \frac{-4z(\alpha + \beta z^2)(2 + z^2(\alpha + \beta z^2)^2)}{C_2(\alpha, \beta)} \varphi(z)
\]

In Equation (10), the first term is symmetric and the second term is asymmetric and the symmetric part, which is defined below, is symbolically denoted by SCBABS\(N_2(\alpha, \beta)\). For \(\alpha = \beta = 0\), \(Z \sim \mathcal{N}(0,1) = \varphi(z)\).

**Remark 4:** To generate data from BABS\(N_2(\alpha, \beta)\) distribution, we use the acceptance-rejection algorithm which was introduced by Von Neumann (1951) as follows:

Let \(f(z)\) be the density function of \(Z \sim \text{BABS}\(N_2(\alpha, \beta)\) and \(f_1(z)\) the density function of \(S \sim \text{SCBABS}\(N_2(\alpha, \beta)\), with

\[
M = \sup \left[ \frac{f(z)}{f_1(z)} \right] = \frac{1}{3}(3 + 2\sqrt{2}).
\]

To generate a random variable \(Z \sim \text{BABS}\(N_2(\alpha, \beta)\), we shall carry out the following steps:

a) Generate a random variable \(S \sim \text{SCBABS}\(N_2(\alpha, \beta)\).

b) Generate \(U \sim \text{Uniform}(0,1)\) independently from \(S\).
c) If \( U < \frac{1}{M} \frac{f(S)}{f_1(S)} = \frac{3[(1-\alpha S - \beta S^3)^2 + 1]^2}{(3+2\sqrt{2})[4+S^2(\alpha + S^2 \beta)^2(8 + S^2(\alpha + S^2 \beta)^2)]} \), set \( Z = S \) accept; otherwise, go back to step one (reject).

By the acceptance-rejection method, any choice of this random variable will be accepted with probability \( \frac{1}{M} \), i.e.,

\[
P\left(U < \frac{1}{M} \frac{f(S)}{f_1(S)}\right) = \frac{1}{M} = \frac{3}{(3+2\sqrt{2})}. \]

Thus, since the number of trials is geometric with \( p = \frac{1}{M} \), the expected value for this number is \( M = \frac{3+2\sqrt{2}}{3} \approx 1.9428 \).

4. Parameter Estimation of BABS\(N_2(\alpha, \beta) \) Distribution

Here, we present the problem of parameter estimation of a location and scale extension of BABS\(N_2(\alpha, \beta) \) distribution. If \( Z \sim \text{BABS}N_2(\alpha, \beta) \) then \( Y = \mu + \sigma Z \) is said to be the location (\( \mu \in \mathbb{R} \)) and scale (\( \sigma > 0 \)) extension of \( Z \) and has the pdf given by

\[
f_Y(y; \mu, \sigma, \alpha, \beta) = \frac{1}{C_2(\alpha, \beta)} \left[ \left( 1 - \alpha \left( \frac{y - \mu}{\sigma} \right) - \beta \left( \frac{y - \mu}{\sigma} \right)^3 \right)^2 + 1 \right] \phi \left( \frac{y - \mu}{\sigma} \right) \tag{11} \]

where \( y \in \mathbb{R}, \alpha \in \mathbb{R}, \beta \in \mathbb{R} \) and \( C_2(\alpha, \beta) \) is as defined before and the distribution of \( Y \) is denoted by \( Y \sim \text{BABS}N_2(\mu, \sigma, \alpha, \beta) \).

4.1. Maximum Likelihood Estimation

The log-likelihood function of the random sample \( y_1, y_2, ..., y_n \) from \( Y \sim \text{BABS}N_2(\mu, \sigma, \alpha, \beta) \) distribution for the parameters \( \theta = (\mu, \sigma, \alpha, \beta) \) is given by

\[
\ell(\theta) = \sum_{i=1}^{n} \log \left[ \left( 1 - \alpha \left( \frac{y_i - \mu}{\sigma} \right) - \beta \left( \frac{y_i - \mu}{\sigma} \right)^3 \right)^2 + 1 \right] - n \log C_2(\alpha, \beta) - n \log(\sigma) \tag{12} \]

By differentiating Equation (12) partially with respect to the parameters \( \theta = (\mu, \sigma, \alpha, \beta) \), we get the following likelihood equations as follows:

\[
\frac{\partial \ell(\theta)}{\partial \mu} = \sum_{i=1}^{n} \frac{y_i - \mu}{\sigma^2} + 2 \sum_{i=1}^{n} \frac{2b_i}{(1+b_i^2)} \left( \alpha + 3\beta(y_i - \mu)^2 \right)
\]

\[
\frac{\partial \ell(\theta)}{\partial \sigma} = - \sum_{i=1}^{n} \frac{(y_i - \mu)^2}{\sigma^3} + 2 \sum_{i=1}^{n} \frac{2b_i}{(1+b_i^2)} \left( \frac{\alpha(y_i - \mu)}{\sigma^2} + \frac{3\beta(y_i - \mu)^3}{\sigma^4} \right)
\]

\[
\frac{\partial \ell(\theta)}{\partial \alpha} = - \frac{n[4(4\alpha + 3\alpha^3) + 12\beta + 45\alpha^2 \beta^2 + 315\alpha \beta^2 + 945\beta^3]}{C_2(\alpha, \beta)} + 2 \sum_{i=1}^{n} \frac{2(y_i - \mu)b_i}{\sigma(1+b_i^2)}
\]

\[
\frac{\partial \ell(\theta)}{\partial \beta} = - \frac{n[12(5\alpha^3 + 105\alpha^2 \beta + 5\beta(4 + 693\beta^2) + \alpha(4 + 945\beta^2))]}{C_2(\alpha, \beta)} + 2 \sum_{i=1}^{n} \frac{2(y_i - \mu)^3b_i}{\sigma^3(1+b_i^2)}
\]

where \( b_i = \left( 1 - \frac{\alpha(y_i - \mu)}{\sigma} - \frac{\beta(y_i - \mu)^3}{\sigma^3} \right) \).

Now, the solutions of the above system of likelihood equations by numerical maximization of the Equation (12) with respect to the parameters \( \theta = (\mu, \sigma, \alpha, \beta) \) gives the maximum likelihood estimates of the parameters \( \theta = (\mu, \sigma, \alpha, \beta) \).
5. Real life applications: comparative data fitting

Here we have considered three datasets, first (Dataset 1) is related to N latitude degrees in 69 samples from world lakes, which appear in Column 5 of the Diversity data set in website: http://users.stat.umn.edu/sandy/courses/8061/datasets/lakes.lsp, second (Dataset 2) is the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003. The data obtained from the website http://www.globalfindata.com and third (Dataset 3) consists of the velocities of 82 distant galaxies, diverging from our own galaxy. The data set is available at http://www.stats.bris.ac.uk/~peter/mixdata. The summary statistics of the datasets are given in Table 1 below.

| Datasets | Min. | Median | Mean | Max. | SD | Skewness | Kurtosis |
|----------|------|--------|------|------|----|----------|----------|
| 1        | 28   | 43     | 45.165 | 74.7 | 9.619 | 1.662 | 5.598 |
| 2        | 1.158 | 4.753 | 4.117 | 11.091 | 1.384 | 0.026 | 5.282 |
| 3        | 9.172 | 20.834 | 20.831 | 34.279 | 4.568 | -0.431 | 5.259 |

We then compared the proposed BABS\(_n\)(\(\mu, \sigma, \alpha, \beta\)) distribution with the normal distribution \(N(\mu, \sigma^2)\), the logistic distribution \(LG(\mu, \sigma)\), the Laplace distribution \(La(\mu, \sigma)\), the skew-normal distribution \(SN(\mu, \sigma, \lambda)\) of Azzalini (1985), the skew-logistic distribution \(SLG(\mu, \sigma, \lambda)\) of Wahed and Ali (2001), the skew-Laplace distribution \(SLa(\mu, \sigma, \lambda)\) of Aryal and Nadarajah (2005), the alpha-skew-normal distribution \(ASN(\mu, \sigma, \alpha)\) of Elal-Olivero (2010), the alpha-skew-Laplace distribution \(ASLa(\mu, \sigma, \alpha)\) of Harandi and Alamatsaz (2013), the alpha-skew-logistic distribution \(ASLG(\mu, \sigma, \alpha)\) of Hazarika and Chakraborty (2014), the alpha-beta-skew-normal distribution \(ABSN(\mu, \sigma, \alpha, \beta)\) and the beta-skew-normal distribution \(BSN(\mu, \sigma, \beta)\) of Shafiee et al. (2016), and the Balakrishnan-beta-skew-normal distribution \(BBSN(\mu, \sigma, \mu)\).

Using R software package (See GenSA package version 1.0.3, Xiang et al. 2013), the MLE of the parameters are obtained by using numerical optimization routine. AIC and BIC are used for comparison of the models. Tables 2, 3 and 4 give the parameter’s log-likelihood, AIC and BIC of the above mentioned distributions. The graphical representations of the results taking only the top three competitors for the proposed model are given in Figures 7, 8 and 9.

**Table 1:** Summary Statistic for the Datasets.

| Datasets | Min. | Median | Mean | Max. | SD | Skewness | Kurtosis |
|----------|------|--------|------|------|----|----------|----------|
| 1        | 28   | 43     | 45.165 | 74.7 | 9.619 | 1.662 | 5.598 |
| 2        | 1.158 | 4.753 | 4.117 | 11.091 | 1.384 | 0.026 | 5.282 |
| 3        | 9.172 | 20.834 | 20.831 | 34.279 | 4.568 | -0.431 | 5.259 |

**Table 2:** MLE’s, log-likelihood, AIC and BIC for N latitude degrees in 69 samples from world lakes.
The Balakrishnan-alpha-beta-skew-normal distribution: Properties and Applications

From Table 2, it is seen that the proposed Balakrishnan-alpha-beta-skew-normal $BABSN_2(\mu, \sigma, \alpha, \beta)$ distribution provides better fit to the data set under consideration in terms of all criteria, namely the log-likelihood, the AIC as well as the BIC. The plots of observed (in histogram) and expected (lines) densities presented in Figure 7, also confirm our findings.

Table 3: MLE’s, log-likelihood, AIC and BIC for the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003.

| Parameters Distributions | $\mu$ | $\sigma$ | $\lambda$ | $\alpha$ | $\beta$ | log $L$ | AIC     | BIC     |
|--------------------------|------|--------|---------|-------|-------|--------|--------|--------|
| $SN(\mu, \sigma, \lambda)$ | 3.589 | 1.478  | 0.501   | --    | --    | -355.217 | 716.434 | 726.388 |
| $N(\mu, \sigma^2)$       | 4.117 | 1.381  | --      | --    | --    | -355.265 | 714.529 | 721.165 |
| $LG(\mu, \sigma)$        | 4.251 | 0.753  | --      | --    | --    | -351.192 | 706.385 | 713.021 |
| $SLG(\mu, \sigma, \lambda)$ | 5.360 | 1.018  | -2.371  | --    | --    | -341.391 | 688.782 | 698.736 |
| $La(\mu, \sigma)$        | 4.754 | 0.971  | --      | --    | --    | -339.315 | 682.630 | 689.265 |
| $BSN(\mu, \sigma, \beta)$ | 4.526 | 0.974  | --      | --    | 0.181 | -334.487 | 674.975 | 684.929 |
| $BBSN_2(\mu, \sigma, \beta)$ | 4.689 | 0.844  | --      | --    | 0.085 | -325.817 | 657.635 | 667.589 |
| $ASN(\mu, \sigma, \alpha)$ | 3.656 | 0.883  | --      | -3.504 | --    | -317.946 | 641.892 | 651.847 |
| $SLa(\mu, \sigma, \lambda)$ | 4.855 | 1.000  | 1.506   | --    | --    | -311.318 | 628.636 | 638.590 |
| $ABSN(\mu, \sigma, \alpha, \beta)$ | 3.497 | 1.075  | --      | -7.583 | 1.196 | -301.347 | 610.695 | 623.968 |
| $ASLG(\mu, \sigma, \alpha)$ | 3.764 | 0.403  | --      | -2.025 | --    | -301.963 | 609.927 | 619.881 |
| $ASLa(\mu, \sigma, \alpha)$ | 4.861 | 0.677  | --      | 0.539  | --    | -301.443 | 608.885 | 618.840 |
| $BABSN_2(\mu, \sigma, \alpha, \beta)$ | 4.352 | 0.639  | --      | -0.631 | 0.157 | -294.9735 | 597.947 | 611.219 |
Figure 8: Plots of observed and expected densities for exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003.

From Table 3, it is observe that the proposed Balakrishnan-alpha-beta-skew-normal $BABSN_2(\mu, \sigma, \alpha, \beta)$ distribution provides better fit to the data set under consideration in terms of all criteria, namely the log-likelihood, the AIC as well as the BIC. The plots of observed (in histogram) and expected (lines) densities presented in Figure 8, also confirm our findings.

Table 4: MLE’s, log-likelihood, AIC and BIC for the velocities of 82 distant galaxies, diverging from our own galaxy.

| Parameters | Distributions | $\mu$ | $\sigma$ | $\lambda$ | $\alpha$ | $\beta$ | log $L$ | AIC | BIC |
|------------|---------------|-------|---------|----------|---------|--------|--------|-----|-----|
| $N(\mu, \sigma^2)$ | 20.832 | 4.540 | -- | -- | -- | -240.417 | 484.833 | 489.646 |
| $SN(\mu, \sigma, \lambda)$ | 24.610 | 5.907 | -1.395 | -- | -- | -239.21 | 484.420 | 491.640 |
| $SLG(\mu, \sigma, \lambda)$ | 21.532 | 2.219 | -0.154 | -- | -- | -233.314 | 472.628 | 479.849 |
| $LG(\mu, \sigma)$ | 21.075 | 2.204 | -- | -- | -- | -233.649 | 471.299 | 476.113 |
| $BSN(\mu, \sigma, \beta)$ | 20.596 | 3.260 | -- | -- | -0.158 | -232.220 | 470.440 | 477.660 |
| $BBSN_2(\mu, \sigma, \beta)$ | 21.602 | 3.017 | -- | -- | 0.070 | -230.666 | 467.332 | 474.552 |
| $ASN(\mu, \sigma, \alpha)$ | 17.417 | 3.869 | -- | -1.656 | -- | -230.088 | 466.175 | 473.395 |
| $ASLG(\mu, \sigma, \alpha)$ | 20.846 | 2.997 | 1.002 | -- | -- | -228.829 | 463.658 | 470.878 |
| $ASLa(\mu, \sigma)$ | 20.838 | 2.997 | -- | -- | -- | -228.830 | 461.660 | 466.474 |
| $ABSN(\mu, \sigma, \alpha, \beta)$ | 18.482 | 1.646 | -- | -0.833 | -- | -224.877 | 455.754 | 462.974 |
| $ABSLa(\mu, \sigma, \alpha, \beta)$ | 19.448 | 3.462 | -- | -1.392 | 0.323 | -220.055 | 448.109 | 457.736 |
| $ASLa(\mu, \sigma, \alpha, \beta)$ | 19.473 | 1.805 | -- | -0.842 | -- | -220.793 | 447.586 | 454.806 |
| $BABSN_2(\mu, \sigma, \alpha, \beta)$ | 15.933 | 4.137 | -- | -3.220 | 0.342 | -216.228 | 440.456 | 450.083 |
From Table 4, it is seen that the proposed Balakrishnan-alpha-beta-skew-normal $BABSN_2(\mu, \sigma, \alpha, \beta)$ distribution provides better fit to the data set under consideration in terms of all criteria, namely the log-likelihood, the AIC as well as the BIC. The plots of observed (in histogram) and expected (lines) densities presented in Figure 9, also confirm our findings.

5.1. Likelihood Ratio Test

Since $N(\mu, \sigma^2)$ and $BABSN_2(\mu, \sigma, \alpha, \beta)$ are nested models, the likelihood ratio (LR) test is used to discriminate between them. The LR test is carried out to test the following hypothesis:

$$H_0: \alpha = \beta = 0 \quad \text{against} \quad H_1: \alpha \neq 0, \beta \neq 0,$$

that is the sample is drawn from $N(\mu, \sigma^2)$; against the alternative $H_1: \alpha \neq 0, \beta \neq 0$, that is the sample is drawn from $BABSN_2(\mu, \sigma, \alpha, \beta)$. The values of the LR test statistic for the above three datasets are given in Table 5.

| Datasets | Dataset 1 | Dataset 2 | Dataset 3 | Degrees of Freedom | Critical value |
|----------|-----------|-----------|-----------|--------------------|---------------|
| LR test statistic | 55.322 | 120.583 | 48.378 | 2 | 9.210 |

The values of LR test statistic for the Datasets 1, 2 and 3 are respectively, 55.322, 120.583 and 48.378 which exceed the critical value at 1% level of significance for two (2) degrees of freedom, i.e., 9.210. Thus there is evidence in favor of the alternative hypothesis that the sampled data comes from $BABSN_2(\mu, \sigma, \alpha, \beta)$, and not from $N(\mu, \sigma^2)$.

6. Conclusions and Future Scope

In this study a new alpha-beta-skew-normal distribution is constructed which includes unimodal, bimodal as well as multimodal shapes and some of its properties are studied. Our findings adequately supported the proposed $BABSN_2(\mu, \sigma, \alpha, \beta)$ distribution as the better fitted one to the datasets under consideration in terms of model selection criteria, namely AIC and BIC. The plots of observed and expected densities presented above also confirm our findings. Furthermore, there is scope of extending the present work by considering the Logistic and the Laplace distributions. Moreover, logarithmic forms and bivariate generalizations can also be considered as future work.

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## Appendix

### A: Derivation of Mode (Proof 2.1.)

Differenitating the Equation (7) with respect to $z$, we have

\[
f_z'(z; \alpha, \beta) = \frac{\partial f_z(z; \alpha, \beta)}{\partial z} = \frac{\partial}{\partial z} \left[ 1 - \alpha z - \beta z^3 + 1 \right] \phi(z) \frac{1}{C_2(\alpha, \beta)}
\]

\[
= \frac{1}{C_2(\alpha, \beta)} \frac{\partial}{\partial z} \left[ (1 - \alpha z - \beta z^3)^2 + 1 \right] \phi(z)
= \frac{1}{C_2(\alpha, \beta)} \frac{\partial}{\partial z} \left[ 4 - 8\alpha z + 8\alpha^2 z^2 - 4(\alpha^3 - 2\beta) z^3 + \alpha(\alpha^3 + 16\beta) z^4 - 12\alpha^2 \beta z^5 + 4\beta(\alpha^3 + 2\beta) z^6 - 6\alpha^2 \beta^2 z^7 - 4\beta^3 z^8 + 4\alpha^3 z^{10} + \beta^4 z^{12} \right] \phi(z)
\]

\[
= \frac{1}{C_2(\alpha, \beta)} \left[ \frac{4}{\partial z} \left( \phi(z) \right) - 8\alpha \phi(z) + \alpha^2 \frac{\partial}{\partial z} \left( \phi(z) \right) - 4(\alpha^3 - 2\beta) \frac{\partial}{\partial z} \left( \phi(z) \right) + \alpha(\alpha^3 + 16\beta) \frac{\partial}{\partial z} \left( \phi(z) \right) 
- 12\alpha^2 \beta \frac{\partial}{\partial z} \left( \phi(z) \right) + 4\beta(\alpha^3 + 2\beta) \frac{\partial}{\partial z} \left( \phi(z) \right) - 12\alpha^2 \beta \frac{\partial}{\partial z} \left( \phi(z) \right) + 6\alpha^2 \beta^2 
\right]
\]

\[
\frac{\partial}{\partial z} \left[ \phi^2(z) \right] - 4\beta^3 \frac{\partial}{\partial z} \left( \phi(z) \right) + 4\alpha^3 \frac{\partial}{\partial z} \left( \phi(z) \right) + \beta^4 \frac{\partial}{\partial z} \left( \phi(z) \right)
\]

\[
(A1)
\]

Now, we have
\[ \frac{\partial}{\partial z} [\phi(z)] = -z \phi(z), \quad \frac{\partial}{\partial z} [\phi(z)] = -(z^2 - 1) \phi(z), \quad \frac{\partial}{\partial z} [z \phi(z)] = -z(z^2 - 2) \phi(z), \quad \frac{\partial}{\partial z} [z^3 \phi(z)] = -z^2 (z^2 - 3) \phi(z), \]
\[ \frac{\partial}{\partial z} [z^4 \phi(z)] = -z^3 (z^2 - 4) \phi(z), \quad \frac{\partial}{\partial z} [z^5 \phi(z)] = -z^4 (z^2 - 5) \phi(z), \quad \frac{\partial}{\partial z} [z^6 \phi(z)] = -z^5 (z^2 - 6) \phi(z), \]
\[ \frac{\partial}{\partial z} [z^7 \phi(z)] = -z^6 (z^2 - 7) \phi(z), \quad \frac{\partial}{\partial z} [z^8 \phi(z)] = -z^7 (z^2 - 8) \phi(z), \quad \frac{\partial}{\partial z} [z^9 \phi(z)] = -z^8 (z^2 - 9) \phi(z), \]
\[ \frac{\partial}{\partial z} [z^{10} \phi(z)] = -z^9 (z^2 - 10) \phi(z), \quad \frac{\partial}{\partial z} [z^{11} \phi(z)] = -z^{11} (z^2 - 12) \phi(z). \]

Putting these values in the Equation (A1), we get
\[ f_Z'(z; \alpha, \beta) = \frac{-\phi(z)}{C_2(\alpha, \beta)} \{ (2 - 2\alpha z + \alpha^2 z^2 - 2\beta z^3 + 2\alpha \beta z^4 + \beta^2 z^5) \{ 4\alpha + (2 - 4\alpha^2) z \} \}
\]
\[ -2(\alpha - 6\beta) z^2 + \alpha(\alpha - 16\beta) z^3 - 2\beta z^4 + 2\beta(\alpha - 6\beta) z^5 + \beta^2 z^7 \}
\]
(A2)

Since the Equation (A2) has at most seven zeros, the function \( f_Z(z; \alpha, \beta) \) can have at most four modes.

B: Derivation of cdf (Proof 2.2.)

\[ F_Z(z) = P(Z \leq z) = \int_{-\infty}^{z} \left( \frac{1}{C_2(\alpha, \beta)} \{ |1 - \alpha z - \beta z^2| + 1 \}^2 \right) \phi(z) dz \]
\[ = \frac{1}{C_2(\alpha, \beta)} \left[ \int_{-\infty}^{z} \left( 4 - 8az + 8a^2 z^2 + 2(\alpha^3 - 2\beta) z^3 + \alpha(\alpha^3 + 16\beta) z^4 - 12\alpha^2 \beta z^5 + \right. \right. \]
\[ \left. \left. 4\beta \{ \alpha^3 + 2\beta \} z^6 - 12\alpha^2 \beta^2 z^7 + 6\alpha^2 \beta^2 z^8 - 4\beta^3 z^9 + 4\alpha \beta^3 z^{10} + \beta^4 z^{12} \right) \phi(z) dz \right] \]
\[ = \frac{1}{C_2(\alpha, \beta)} \left[ 4 \int_{-\infty}^{z} \phi(z) dz - 8\alpha \int_{-\infty}^{z} z \phi(z) dz + 8\alpha \int_{-\infty}^{z} z^2 \phi(z) dz - 4(\alpha^3 + 2\beta) \int_{-\infty}^{z} z^3 \phi(z) dz + \right. \]
\[ \left. \alpha(\alpha^3 + 16\beta) \int_{-\infty}^{z} z^4 \phi(z) dz - 12\alpha^2 \beta \int_{-\infty}^{z} z^5 \phi(z) dz + 4\beta(\alpha^3 + 2\beta) \int_{-\infty}^{z} z^6 \phi(z) dz - 12\alpha^2 \beta^2 \int_{-\infty}^{z} z^7 \phi(z) dz + \right. \]
\[ \left. 6\alpha^2 \beta^2 \int_{-\infty}^{z} z^8 \phi(z) dz - 4\beta^3 \int_{-\infty}^{z} z^9 \phi(z) dz + 4\alpha \beta^3 \int_{-\infty}^{z} z^{10} \phi(z) dz + \beta^4 \int_{-\infty}^{z} z^{12} \phi(z) dz \right] \]
(B1)

Now, we have
\[ \int_{-\infty}^{z} \phi(z) dz = \Phi(z), \quad \int_{-\infty}^{z} z \phi(z) dz = -\Phi(z), \quad \int_{-\infty}^{z} z^2 \phi(z) dz = -z \phi(z) + \Phi(z), \quad \int_{-\infty}^{z} z^3 \phi(z) dz = -(2 + z^2) \phi(z), \]
\[ \int_{-\infty}^{z} z^4 \phi(z) dz = -(3 + z^2) \phi(z) + 3 \Phi(z), \quad \int_{-\infty}^{z} z^5 \phi(z) dz = -(8 + 4z^2 + z^4) \phi(z), \quad \int_{-\infty}^{z} z^6 \phi(z) dz = -z(15 + 5z^2 + z^4) \phi(z) + 15 \Phi(z) \]
\[ \int_{-\infty}^{z} z^7 \phi(z) dz = -z(48 + 24z^2 + 6z^4 + z^6) \phi(z), \quad \int_{-\infty}^{z} z^8 \phi(z) dz = -z(105 + 35z^2 + 7z^4 + z^6) \phi(z) + 105 \Phi(z), \]
\[ \int_{-\infty}^{z} z^9 \phi(z) dz = -z(384 + 192z^2 + 48z^4 + 8z^6 + z^8) \phi(z), \quad \int_{-\infty}^{z} z^{10} \phi(z) dz = -z(945 + 315z^2 + 63z^4 + 9z^6 + z^8) \phi(z) + 945 \Phi(z), \]
\[ \int_{-\infty}^{z} z^{12} \phi(z) dz = -z(10395 + 3465z^2 + 693z^4 + 99z^6 + 11z^8 + z^{10}) \phi(z) + 10395 \Phi(z) \]

Putting these values in the Equation (B1), we get the desired result in the Equation (8).

C: Derivation of mgf (Proof 2.3.)

\[ M_Z(t) = E(e^{t Z}) = \int_{-\infty}^{\infty} e^{t z} \left( \frac{1}{C_2(\alpha, \beta)} \{ |1 - \alpha z - \beta z^2| + 1 \}^2 \right) \phi(z) dz \]
\[ = \frac{1}{C_2(\alpha, \beta)} \int_{-\infty}^{\infty} e^{t z} \left( 4 - 8az + 8a^2 z^2 + 2(\alpha^3 - 2\beta) z^3 + \alpha(\alpha^3 + 16\beta) z^4 - 12\alpha^2 \beta z^5 + \right. \]
\[ \left. 4\beta \{ \alpha^3 + 2\beta \} z^6 - 12\alpha^2 \beta^2 z^7 + 6\alpha^2 \beta^2 z^8 - 4\beta^3 z^9 + 4\alpha \beta^3 z^{10} + \beta^4 z^{12} \right) \phi(z) dz \]
$$\int_{-\infty}^{\infty} e^{iz\phi(z)}dz = \frac{1}{C_2(\alpha, \beta)} \int_{-\infty}^{\infty} 4z^{4\alpha} e^{iz\phi(z)}dz + 8\alpha^2 \int_{-\infty}^{\infty} z^{2\alpha} e^{iz\phi(z)}dz - 4(\alpha^3 + 2\beta) \int_{-\infty}^{\infty} z^{3\alpha} e^{iz\phi(z)}dz +$$

$$(\alpha^3 + 16\beta) \int_{-\infty}^{\infty} z^{4\alpha} e^{iz\phi(z)}dz - 12\alpha^2 \beta \int_{-\infty}^{\infty} z^{5\alpha} e^{iz\phi(z)}dz + 4\beta(\alpha^3 + 2\beta) \int_{-\infty}^{\infty} z^{6\alpha} e^{iz\phi(z)}dz - 12\alpha\beta^2$$

$$\int_{-\infty}^{\infty} z^{7\alpha} e^{iz\phi(z)}dz + 6\alpha^2 \beta^2 \int_{-\infty}^{\infty} z^{8\alpha} e^{iz\phi(z)}dz - 4\beta^3 \int_{-\infty}^{\infty} z^{9\alpha} e^{iz\phi(z)}dz + 4\alpha\beta^3 \int_{-\infty}^{\infty} z^{10\alpha} e^{iz\phi(z)}dz +$$

$$e^{-\alpha^2 t^2}$$

Now, we have

$$\int_{-\infty}^{\infty} e^{iz\phi(z)}dz = e^{-\alpha^2 t^2} = M_X(t)$$

$$\int_{-\infty}^{\infty} z^{4\alpha} e^{iz\phi(z)}dz = t e^{-\alpha^2 t^2}$$

$$\int_{-\infty}^{\infty} z^{2\alpha} e^{iz\phi(z)}dz = (1 + t^2) e^{-\alpha^2 t^2}$$

$$\int_{-\infty}^{\infty} z^{3\alpha} e^{iz\phi(z)}dz = -(2 + z^2) e^{-\alpha^2 t^2}$$

Putting these values in the Equation (C1), we get the desired result in the Equation (9).

D: Derivation of Nth Moments (Proof 2.4.)

$$E(Z^{2n}) = \int_{-\infty}^{\infty} z^{2n} \frac{[1 - \alpha z - \beta z^3 + 1]^2}{C_2(\alpha, \beta)} \phi(z)dz$$

$$= \frac{1}{C_2(\alpha, \beta)} \int_{-\infty}^{\infty} \left[ 4z^{2n} - 8\alpha z^{2n+1} + 8\alpha^2 z^{2n+2} - 4(\alpha^3 - 2\beta)z^{2n+3} + \alpha(\alpha^3 + 16\beta)z^{2n+4} \
+ 12\alpha^2 \beta z^{2n+5} + 4\beta(\alpha^3 + 2\beta)z^{2n+6} - 12\alpha\beta^2 z^{2n+7} + 6\alpha^2 \beta^2 z^{2n+8} - 4\beta^3 z^{2n+9} + 4\alpha\beta^3 z^{10} + \beta^4 z^{12} \right] \phi(z)dz$$

$$= \frac{1}{C_2(\alpha, \beta)} \left[ 4E_N(z^{2n}) - 8\alpha E_N(z^{2n+1}) + 8\alpha^2 E_N(z^{2n+2}) - 4(\alpha^3 - 2\beta)E_N(z^{2n+3}) + \alpha(\alpha^3 + 16\beta)E_N(z^{2n+4}) - 12\alpha^2 \beta E_N(z^{2n+5}) + 4\beta(\alpha^3 + 2\beta)E_N(z^{2n+6}) - 12\alpha\beta^2 E_N(z^{2n+7}) + 6\alpha^2 \beta^2 E_N(z^{2n+8}) - 4\beta^3 E_N(z^{2n+9}) + 4\alpha\beta^3 E_N(z^{2n+10}) + \beta^4 E_N(z^{2n+12}) \right]$$

where $E_N(.)$ is defined above and using this formula into the last equation we get the required result.

Similarly, $E(Z^{2n-1})$ can be proved in the same way which is omitted for the sake of brevity.