ON THE DUALS OF NORMED SPACES AND QUOTIENT SHAPES

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Abstract. Some properties of the (normed) dual Hom-functor $D$ and its iterations $D^n$ are exhibited. For instance: $D$ turns every canonical embedding (in the second dual space) into a retraction (of the third dual onto the first one); $D$ rises the countably infinite (algebraic) dimension only; $D$ does not change the finite quotient shape type. By means of that, the finite quotient shape classification of normed vectorial spaces is completely solved. As a consequence, two extension type theorems are derived.

1. Introduction

The quotient shape theory is a genuine kind of the general shape theory (which began as a generalization of the homotopy theory such that the locally bad spaces can be also considered and classified in a very suitable “homotopical” way; [1], [2], [3], [5] and, especially, [11]). Although, in general, founded purely categorically, it is mostly well known only as the (standard) shape theory of topological spaces with respect to spaces having the homotopy types of polyhedra. The generalizations founded in [8] and [18] are, primarily, also on that line.

The quotient shape theory was introduced a few years ago by the author, [13]. Though it is a kind of the general (abstract) shape theory, it can be straightforwardly applied to any concrete category $C$, whenever an infinite cardinal $\kappa \geq \aleph_0$ is chosen. Concerning a shape of objects, in general, one has to decide which ones are “nice” absolutely and/or relatively (with respect to the chosen ones). In this approach, an object is “nice” if it is isomorphic to a quotient object belonging to a special

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full subcategory and if it (its “basis”) has cardinality less than (less than or equal to) a given infinite cardinal. It leads to the basic idea: to approximate a $\mathcal{C}$-object $X$ by a suitable inverse system consisting of its quotient objects $X_\lambda$ (and the quotient morphisms) which have cardinalities, or dimensions - in the case of vectorial spaces, less than (less than or equal to) $\kappa$. Such an approximation exists in the form of any $\kappa^{-}$-expansion ($\kappa$-expansion) of $X$,

$$p_{\kappa^{-}} = (p_{\lambda}) : X \to X_{\kappa^{-}} = (X_{\lambda}, p_{\lambda}^\prime, \Lambda_{\kappa^{-}})$$

$$p_{\kappa} = (p_{\lambda}) : X \to X_{\kappa} = (X_{\lambda}, p_{\lambda}^\prime, \Lambda_{\kappa})$$

where $X_{\kappa^{-}}$ ($X_{\kappa}$) belongs to the subcategory $pro-\mathcal{D}_{\kappa^{-}}$ ($pro-\mathcal{D}_\kappa$) of $pro-\mathcal{D}$, and $\mathcal{D}_{\kappa^{-}}$ ($\mathcal{D}_{\kappa}$) is the subcategory of $\mathcal{D}$ determined by all the objects having cardinalities, or dimensions - for vectorial spaces, less than (less than or equal to) $\kappa$, while $\mathcal{D}$ is a full subcategory of $\mathcal{C}$. Clearly, if $X \in \text{Ob}(\mathcal{D})$ and the cardinality $|X| < \kappa$ ($|X| \leq \kappa$) (or of the “basis" of $X$), then the rudimentary pro-morphism $\left[1_X\right] : X \to [X]$ is a $\kappa^{-}$-expansion ($\kappa$-expansion) of $X$. The corresponding shape category $Sh_{\mathcal{D}_{\kappa^{-}}}(\mathcal{C})$ ($Sh_{\mathcal{D}_\kappa}(\mathcal{C})$) and shape functor $S_{\kappa^{-}} : \mathcal{C} \to Sh_{\mathcal{D}_{\kappa^{-}}}(\mathcal{C})$ ($S_{\kappa} : \mathcal{C} \to Sh_{\mathcal{D}_{\kappa}}(\mathcal{C})$) exist by the general (abstract) shape theory, and they have all the appropriate general properties. Moreover, there exist the relating functors $S_{\kappa^{-}} : Sh_{\mathcal{D}_\kappa}(\mathcal{C}) \to Sh_{\mathcal{D}_{\kappa^{-}}}(\mathcal{C})$ and $S_{\kappa\kappa'} : Sh_{\mathcal{D}_{\kappa'}}(\mathcal{C}) \to Sh_{\mathcal{D}_{\kappa}}(\mathcal{C})$, $\kappa \leq \kappa'$, such that $S_{\kappa^{-}}S_{\kappa} = S_{\kappa^{-}}$ and $S_{\kappa\kappa'}S_{\kappa'} = S_{\kappa}$. Even in simplest case of $\mathcal{D} = \mathcal{C}$, the quotient shape classifications are very often nontrivial and very interesting. In such a case we simplify the notation $Sh_{\mathcal{D}_{\kappa^{-}}}(\mathcal{C})$ ($Sh_{\mathcal{D}_{\kappa}}(\mathcal{C})$) to $Sh_{\kappa^{-}}(\mathcal{C})$ ($Sh_{\kappa}(\mathcal{C})$) or to $Sh_{\kappa^{-}}(Sh_{\kappa}(\mathcal{C})$ when $\mathcal{C}$ is fixed.

In [13], several well known concrete categories were considered and many examples are given which show that the quotient shape theory yields classifications strictly coarser than those by isomorphisms. In [14] and [15] were considered the quotient shapes of (purely algebraic, topological and normed - the category $\mathcal{N}_F$) vectorial spaces and topological spaces, respectively. In paper [16], we have continued the studying of quotient shapes of normed vectorial spaces of [14], Section 4.1, primarily and separately focused to the well known $l_p$ and $L_p$ spaces. The main general result of [16] is that the finite quotient shape type of a normed spaces (over the field $F \in \{\mathbb{R}, \mathbb{C}\}$) reduces to that of its completion (Banach) spaces, and consequently, that the quotient shape theory of $(\mathcal{N}_F, (\mathcal{N}_F)_0)$ reduces to that of the full subcategory pair $(\mathcal{B}_F, (\mathcal{B}_F)_0)$ of Banach spaces. In the very recent paper [17] we have proven that the finite quotient shape type of normed spaces is an invariant of the (algebraic) dimension, but not conversely. The counterexamples exist, at least, in the dimensional par $\{\aleph_0, 2^{\aleph_0}\}$. Further,
in the case of separable Banach spaces, the classifications by dimension and by the finite quotient shape (as well as by the countable quotient shape) coincide. An application of those results has yielded two extension type theorems into lower dimensional Banach spaces.

In this paper we firstly consider the (iterated) dual normed spaces. We have used the functorial approach, i.e., we treat $D^{2n-1} : \mathcal{N}_F \to \mathcal{B}_F$ ($D^{2n} : \mathcal{N}_F \to \mathcal{B}_F$), $n \in \mathbb{N}$, as a contravariant (covariant) $\text{Hom}_{\mathcal{F}}$-functor. Among interesting and useful results, let us quote that, for every $X$, $D^{2n-1}$ turns the canonical embedding $j : X \to D^2(X)$ into $D^{2n-1}(j) : D^{2n+1}(X) \to D^{2n-1}(X)$ which is a retraction, while $D^{2n}(j) : D^{2n}(X) \to D^{2n+2}(X)$ is a section, and thus, one may consider $D^n(X)$ to be a retract of $D^{n+2}(X)$ admitting a closed direct complement. Therefore, the problem of the strict relation between $\dim D(X)$ and $\dim X$ (generally, $\dim X \leq \dim D(X)$) occurs in this consideration as a very significant one. We have solved it by means of the quotient shape theory technique as follows:

$\dim D(X) > \dim X$ if and only if $\dim X = \aleph_0$.

By applying that fact, we made an important step towards the main goal which was the complete finite quotient shape classification of all normed vectorial spaces over $F \in \{\mathbb{R}, \mathbb{C}\}$. Briefly, for all but countably infinite-dimensional normed spaces, the finite quotient shape types are their (algebraic) dimension classes, while all countably infinite-dimensional normed spaces belong to the dimension class $2^{\aleph_0}$. Consequently, every finite quotient shape type of normed spaces contains a representative that is a Hilbert space, and in addition, the finite quotient shape type of all $\aleph_0$- and all $2^{\aleph_0}$-dimensional normed spaces admits an $\aleph_0$-dimensional unitary representative.

At the end, we have established a significant improvements of the mentioned (previously obtained, [17]) extension type theorems. For instance (Theorem 6):

Let $X$ be a normed vectorial space, let $Y$ be a Banach space (over the same field) and let $f_n : D^n(X) \to Y$, $n \in \mathbb{N}$, be a continuous linear function. Then, for every $k \in \{0\} \cup \mathbb{N}$, $f_n$ admits a continuous linear norm-preserving extension $f_{n,k} : D^{n+2k}(X) \to Y$.

2. Preliminaries

We shall frequently use and apply in the sequel several general or special well known facts without referring to any source. So we remind a reader that

- our general shape theory technique is that of [11];
- the needed set theoretic (especially, concerning cardinals) and topological facts can be found in [4];
- the facts concerning functional analysis are taken from [6], [9], [10] or [12];
- our category theory language follows that of [7].

For the sake of completeness, let us briefly repeat the construction of a quotient shape category and a quotient shape functor, [13]. Given a category pair \((\mathcal{C}, \mathcal{D})\), where \(\mathcal{D} \subseteq \mathcal{C}\) is full, and a cardinal \(\kappa\), let \(\mathcal{D}_\kappa\) (\(\mathcal{D}_\kappa\)) denote the full subcategory of \(\mathcal{D}\) determined by all the objects having cardinalities or, in some special cases, the cardinalities of “bases” less than (less or equal to) \(\kappa\). By following the main principle, let \((\mathcal{C}, \mathcal{D}_\kappa)\) be such a pair of concrete categories. If

(a) every \(\mathcal{C}\)-object \((X, \sigma)\) admits a directed set \(R(X, \sigma, \kappa^-) \equiv \Lambda_{\kappa^-}\) (\(R(X, \sigma, \kappa) \equiv \Lambda_{\kappa}\)) of equivalence relations \(\lambda\) on \(X\) such that each quotient object \((X/\lambda, \sigma_\lambda)\) has to belong to \(\mathcal{D}_\kappa\) (\(\mathcal{D}_\kappa\)), while each quotient morphism \(p_\lambda : (X, \sigma) \to (X/\lambda, \sigma_\lambda)\) has to belong to \(\mathcal{C}\);
(b) the induced morphisms between quotient objects belong to \(\mathcal{D}_\kappa\) (\(\mathcal{D}_\kappa\));
(c) every morphism \(f : (X, \sigma) \to (Y, \tau)\) of \(\mathcal{C}\), having the codomain in \(\mathcal{D}_\kappa\) (\(\mathcal{D}_\kappa\)), factorizes uniquely through a quotient morphism \(p_\lambda : (X, \sigma) \to (X/\lambda, \sigma_\lambda)\), \(f = gp_\lambda\), with \(g\) belonging to \(\mathcal{D}_\kappa\) (\(\mathcal{D}_\kappa\)),

then \(\mathcal{D}_\kappa\) (\(\mathcal{D}_\kappa\)) is a pro-reflective subcategory of \(\mathcal{C}\). Consequently, there exists a (non-trivial) (quotient) shape category \(Sh_{\mathcal{C}, \mathcal{D}_\kappa^-} \equiv Sh_{\mathcal{D}_\kappa^-}(\mathcal{C})\) (\(Sh_{\mathcal{C}, \mathcal{D}_\kappa} \equiv Sh_{\mathcal{D}_\kappa}(\mathcal{C})\)) obtained by the general construction.

Therefore, a \(\kappa^-\)-shape morphism \(F_{\kappa^-} : (X, \sigma) \to (Y, \tau)\) is represented by a diagram (in \(pro-\mathcal{C}\))

\[
\begin{array}{ccc}
(X, \sigma)_{\kappa^-} & \xrightarrow{p_{\kappa^-}} & (X, \sigma) \\
\downarrow f_{\kappa^-} & & \\
(Y, \tau)_{\kappa^-} & \xrightarrow{q_{\kappa^-}} & (Y, \tau)
\end{array}
\]

(with \(p_{\kappa^-}\) and \(q_{\kappa^-}\) - a pair of appropriate expansions), and similarly for a \(\kappa\)-shape morphism \(F_{\kappa} : (X, \sigma) \to (Y, \tau)\). Since all \(\mathcal{D}_\kappa\)-expansions (\(\mathcal{D}_\kappa\)-expansions) of a \(\mathcal{C}\)-object are mutually isomorphic objects of \(pro-\mathcal{D}_\kappa\) (\(pro-\mathcal{D}_\kappa\)), the composition and identities follow straightforwardly. Observe that every quotient morphism \(p_\lambda\) is an effective epimorphism. (If \(U\) is the forgetful functor, then \(U(p_\lambda)\) is a surjection), and thus condition (E2) for an expansion follows trivially.

The corresponding “quotient shape” functors \(S_{\kappa^-} : \mathcal{C} \to Sh_{\mathcal{D}_\kappa^-}(\mathcal{C})\) and \(S_{\kappa} : \mathcal{C} \to Sh_{\mathcal{D}_\kappa}(\mathcal{C})\) are defined in the same general manner. That means,

\(S_{\kappa^-}(X, \sigma) = S_{\kappa}(X, \sigma) = (X, \sigma);\)
if \( f: (X, \sigma) \to (Y, \tau) \) is a \( C \)-morphism, then, for every \( \mu \in M_{\kappa^-} \), the composite \( g_\mu f: (Y, \tau) \to (Y_\mu, \tau_\mu) \) factorizes (uniquely) through a \( p_\lambda(\mu): (X, \sigma) \to (X_\lambda, \sigma_\lambda(\mu)) \), and thus, the correspondence \( \mu \mapsto \lambda(\mu) \) yields a function \( \phi : M_{\kappa^-} \to \Lambda_{\kappa^-} \) and a family of \( D_{\kappa^-} \)-morphisms \( f_\mu : (X_\phi(\mu), \sigma_\phi(\mu)) \to (Y_\mu, \tau_\mu) \) such that \( q_\mu f = f_\mu p_\phi(\mu) \); one easily shows that \((\phi, f_\mu): (X, \sigma)_{\kappa^-} \to (Y, \tau)_{\kappa^-}\) is a morphism of \( \text{inv}-D_{\kappa^-} \), so the equivalence class \( f_{\kappa^-} = [(\phi, f_\mu)] : (X, \sigma)_{\kappa^-} \to (Y, \tau)_{\kappa^-}\) is a morphism of \( \text{pro-}D_{\kappa^-} \); then we put \( S_{\kappa^-}(f) = (f_{\kappa^-}) \equiv F_{\kappa^-}: (X, \sigma) \to (Y, \tau) \) in \( Sh_{D_{\kappa^-}}(C) \).

The identities and composition are obviously preserved. In the same way one defines the functor \( S_\kappa \).

Furthermore, since \((X, \sigma)_{\kappa^-}\) is a subsystem of \((X, \sigma)_{\kappa}\) (more precisely, \((X, \sigma)_{\kappa}\) is a subobject of \((X, \sigma)_{\kappa^-}\) in \( \text{pro-}D\)), one easily shows that there exists a functor \( S_{\kappa^-}: Sh_{D_{\kappa}}(C) \to Sh_{D_{\kappa^-}}(C) \) such that \( S_{\kappa^-}S_\kappa = S_{\kappa^-}, \) i.e., the diagram

\[
\begin{array}{ccc}
S_{\kappa^-} & \xrightarrow{\sim} & S_{\kappa^-} \\
\downarrow & & \downarrow \\
Sh_{D_{\kappa}}(C) & \xrightarrow{\sim} & Sh_{D_{\kappa^-}}(C)
\end{array}
\]

commutes. Moreover, an analogous functor \( S_{\kappa^\prime}: Sh_{D_{\kappa^\prime}}(C) \to Sh_{D_{\kappa}}(C) \), satisfying \( S_{\kappa^\prime}S_\kappa = S_\kappa \), exists for every pair of infinite cardinals \( \kappa^\prime \leq \kappa \).

Generally, in the case of \( \kappa = \aleph_0 \), the \( \kappa^- \)-shape is said to be the finite (quotient) shape, because all the objects in the expansions are of finite (bases) cardinalities, and the category is denoted by \( Sh_{D_{\aleph_0}}(C) \) or by \( Sh_0(C) \equiv Sh_{\aleph_0} \) only, whenever \( D = C \).

Let us finally notice that, though \( D \nsubseteq C_{\kappa^-} \) \(((D \nsubseteq C_{\kappa})\)), the quotient shape category \( Sh_{C_{\kappa^-}}(D) \) \((Sh_{C_{\kappa}}(D))\) exists as a full subcategory of \( Sh_{C_{\kappa^-}}(C) \) \((Sh_{C_{\kappa}}(C))\), and, if \( D \) is closed with respect to quotients, then \( Sh_{C_{\kappa^-}}(D) = Sh_{D_{\kappa^-}}(D) \) \((Sh_{C_{\kappa}}(D) = Sh_{D_{\kappa}}(D))\).

3. SOME PROPERTIES OF THE (NORMED) DUAL FUNCTORS

We recall hereby the well known dual space of a normed vectorial space over \( F \in \{\mathbb{R}, \mathbb{C}\} \). For our purpose it is much more convenient to use the categorical approach as follows. There exists a contravariant structure preserving hom-functor, i.e., the contravariant Hom-functor \( Hom_F \equiv D : N_F \to N_F \),

\[
D(X) = X^* \text{ - the (normed) dual space of } X,
\]

\[
D(f: X \to Y) \equiv D(f) \equiv f^*: Y^* \to X^*, \quad D(f)(y^1) = y^1 f,
\]

and \( D[N_F] \subseteq B_F \). Furthermore, for every ordered pair \( X, Y \in Ob(N_F) \), the function

\[
D_X: N_F(X, Y) \equiv L(X, Y) \to L(Y^*, X^*) \equiv B_F(Y^*, X^*)
\]
is a linear isometry ($\|D(f)\| = \|f\|$), and hence, $D^2_f$ belongs to $\text{Mor}(\mathcal{N}_F)$ and $D$ is a faithful functor.

Further, there exists a covariant Hom-functor

$$\text{Hom}^2_F \equiv D^2 : \mathcal{N}_F \to \mathcal{N}_F,$$

$$D^2(X) = D(D(X)) \equiv X^{**} - \text{the (normed) second dual space of } X,$n

and $D^2[\mathcal{N}_F] \subseteq \mathcal{B}_F$. (Caution: The notation “$D(D(f)(x^2))$” makes no sense!) Furthermore, for every ordered pair $X, Y \in \text{Ob}(\mathcal{N}_F)$, the function

$$(D^2)_X^Y : \mathcal{N}_F(X, Y) \equiv L(X, Y) \to L(X^{**}, Y^{**}) \equiv \mathcal{B}_F(X^{**}, Y^{**})$$

is a linear isometry ($\|D^2(f)\| = \|f\|$), and thus, $(D^2)_X^Y$ belongs to $\text{Mor}(\mathcal{N}_F)$ and $D^2$ is a faithful functor.

The most useful fact hereby is the existence of a certain natural transformation $j : 1_{\mathcal{N}_F} \leadsto D^2$ of the functors, where, for every $X$, $j_X : X \to D^2(X)$ is an isometric embedding (the canonical embedding defined by $(j_X(x))(x^1) = x^1(x)$), and $\mathcal{C}(j_X[X]) \subseteq D^2(X)$ is the well known (Banach) completion of $X$. Namely, given a pair $X, Y$ of normed spaces, then

$$(\forall f \in \mathcal{N}_F(X, Y), j_Y f = D^2(f)j_X$$

holds true. Indeed, for every $x \in X$, every $y^1 \in D^1(Y)$ and every $x^2 \in D^2(X),$

$$(j_Y f)(x)(y^1) = y^1(f(x)) = j_X(x)(y^1f) = j_X(x)(D(f)(y^1)) = (j_X(x)D(f))(y^1) = (D^2(f)(j_X(x))(y^1) = ((D^2(f)j_X)(x))(y^1).$$

Clearly, if $X$ is a Banach space, then the canonical embedding $j_X$ is closed. Continuing by induction, for every $n \in \mathbb{N}$, $n > 2$, there exists a $\text{Hom}_F$-functor $D^n$ of $\mathcal{N}_F$ to $\mathcal{N}_F$ such that $D^n[\mathcal{N}_F] \subseteq \mathcal{B}_F$, $D^n$ is contravariant (covariant) whenever $n$ is odd (even), and for every ordered pair $X, Y$ of normed spaces, the function $(D^n)_X^Y$ is an isometric linear morphism of the normed space $L(X, Y)$ to the Banach space $L(D^n(X), D^n(Y)) (n \text{ odd})$ or $L(D^n(X), D^n(Y)) (n \text{ even})$. Consequently, every $(D^n)_X^Y$ preserves null-morphisms, i.e., $D^n(c_0) = c^n_0$. However, it holds much more than that.

**Lemma 1.** (i) The functor $D$ turns
- (open) epimorphisms into (closed) monomorphisms;
- open or closed monomorphisms and embeddings into (open) epimorphisms;
- isometric isomorphisms into isometric isomorphisms.

The functor $D^2$ maps
- open epimorphisms into (open) epimorphisms;
- open or closed monomorphisms and embeddings into closed monomorphisms;
- isometries into closed isometries.
(ii) In addition, the restriction functor \( D|B_F \) turns
- epimorphisms into closed monomorphisms;
- (isometric) monomorphisms with closed ranges into (closed) epimorphisms.

The restriction functor \( D^2|B_F \) maps
- epimorphisms into epimorphisms;
- monomorphisms with closed ranges into closed monomorphisms.
(iii) For all \( X, Y \in \text{Ob}N_F \), the canonical embedding

\[ \tilde{f}_{L(X,Y)} : L(X,Y) \to D^2(L(X,Y)) \]

factorizes through the linear isometry
\[ (D^2)^Y_X : L(X,Y) \to L(D^2(X), D^2(Y)), \quad D^2(f)(x^2) = x^2 D(f). \]

If \( Y \) is a Banach space, then the linear isometry
\[ D^Y_X : L(X,Y) \to L(D(Y), D(X)), \quad D(f)(y^1) = y^1 f, \]
is closed.

Proof. (i). Assume that \( f \in N_F(X,Y) \) is an epimorphism. Let \( y^1, y^{1''} \in Y^* \) such that \( D(f)(y^1) = D(f)(y^{1''}) \). It means that \( y^1 f = y^{1''} f \), implying that \( y^1 = y^{1''} \) because \( f \) is an epimorphism. Hence, \( D(f) \) is a monomorphism of the underlying abelian groups, and consequently, it is a monomorphism of \( B_F \subseteq N_F \). Assume, in addition, that \( f \) is open. It suffices to prove that the range \( R(D(f)) \subseteq X^* \) is a closed subspace, i.e., that \( \text{Cl}(R(D(f))) \subseteq R(D(f)) \). Namely, if it is so, then \( D(f) \) is a monomorphism of a Banach space with the range that is a Banach space too. Then,

\[ D(f)' : Y^* \to R(D(f)), \quad D(f)'(y^1) = D(f)(y^1), \]
is a continuous bijection of Banach spaces, and thus, an isomorphism, implying that \( D(f) \) is closed monomorphism. Let \( x^1 \in \text{Cl}(R(D(f))) \). Consider a sequence \( (x_n^1) \) in \( R(D(f)) \) such that \( \lim(x_n^1) = x^1 \). Since \( D(f) \) is a monomorphism, there exists a unique sequence \( (y_n^1) \) in \( Y^* \) such that, for each \( n \in \mathbb{N} \),

\[ D(f)(y_n^1) = y_n^1 f = x_n^1. \]

Recall that, algebraically, \( X = N(f) + W \), where \( W \cong R(f) = Y \), and that each fiber \( f^{-1}\{y\}, \quad y \in Y \), is the equivalence class \( [x]_f = x + N(f) \), where \( f(x) = y \). Thus, for every \( n \), and every \( y \in Y \),

\[ y_n^1(y) = x_n^1(x) = x_n^1(w), \]

where \( f(x) = y \) and \( x = z + w \) is the unique presentation of \( x \in X = N(f) + W \). It implies that, for each \( y \in Y \) and all \( x = z + w \in X \), such that \( f(x) = y \),

\[ \lim(y_n^1(y)) = \lim(x_n^1(x)) = x^1(x) = x^1(w) \]
holds true. Consequently, by putting

\[ y^1 : Y \to F, \quad y^1(y) = \lim(y_n^1(y)), \]

a certain function is well defined. Moreover, \( y^1 \) is linear, because it is a “copy” of the restriction \( x^1|W \), and \( y^1f = x^1 \) obviously holds. It remains to prove that \( y^1 \) is continuous. Let \( O \) be an open neighborhood of \( 0 \in F \). Since \( x^1 \) is continuous, there exists an open neighborhood \( U \) of \( \theta_X \in X \) such that \( x^1[U] \subseteq O \). Then \( V = f[U] \) is an open neighborhood of \( \theta_Y \in Y \), because \( f \) is open, and

\[ y^1[V] = (y^1f)[U] = x^1[U] \subseteq O. \]

Thus, \( y^1 \) is continuous, and hence \( y^1 \in Y^* \). Since \( D(f)(y^1) = y^1f = x^1 \), the additional statement is proven.

Assume that \( f : X \to Y \) is an open or closed monomorphism or an embedding. Then \( f \) admits the factorization

\[ X \xrightarrow{f'} f[X] \xrightarrow{i} Y, \quad f'(x) = f(x), \]

where \( f' \) is an isomorphism onto the subspace \( f[X] \subseteq Y \), and \( i \) is the inclusion. Given an \( x^1 \in X^* \), put \( y^1_{x^1} = x^1f^{-1} \in f[X]^* \). By the Hahn-Banach theorem, there exists an extension \( y^1 \in Y^* \) of \( y^1_{x^1} \), i.e., \( y^1i = y^1_{x^1} \). Then

\[ D(f)(y^1) = D(iy^1) = (D(f')D(i))(y^1) = D(f')(D(i)(y^1)) = \]

\[ = D(f')(y^1i) = D(f')(y^1_{x^1}) = y^1_{x^1}f' = x^1f^{-1}f' = x^1, \]

implying that \( D(f) : Y^* \to X^* \) is an epimorphism of the underlying abelian groups, and consequently, it is an epimorphism of \( B_F \subseteq N_F \).

Now, by Open-mapping theorem, \( D(f) \) is open. Finally, if \( f \) is an isometric isomorphism, then \( D(f) \) is an isomorphism and, for every \( y^1 \in D(Y) \),

\[ ||D(f)(y^1)|| = ||y^1f|| = \sup\{|y^1(f(x))| \mid x \in X, ||x|| = 1\} = \]

\[ = \sup\{|y^1(y)| \mid y \in Y, ||y|| = ||f(x)|| = ||x|| = 1\} = ||y^1||. \]

Hence, \( D(f) \) is an isometry as well. The statements concerning \( D^2 \) follow by \( D^2(f) = D(D(f)) \) and \( D^2(f)j_X = j_Yf \), where \( j_X \) and \( j_Y \) are the (isometric) canonical embeddings.

(ii). Assume that \( f \in B_F(X,Y) \) is an epimorphism. By Open-mapping theorem, \( f \) is open as well. Then, by (i), \( D(f) \) is a closed monomorphism.

Assume that \( f \in B_F(X,Y) \) is a monomorphism having the range \( R(f) \) closed in \( Y \). Then, as previously,

\[ f' : X \to R(f), \quad f'(x) = f(x), \]

is a continuous bijection of Banach spaces, and thus, an isomorphism. It follows that \( f \) is a closed monomorphism. Then, by (i), \( D(f) \) is an (open) epimorphism. If, in addition, \( f \) is an isometry, then \( f \) preserves Cauchy sequences, and one readily verifies that \( D(f) \) maps the sets
closed in $D(Y)$ into sets closed in $D(X)$. The statements concerning
$D^2|\mathcal{B}_F$ follow by those concerning $D|\mathcal{B}_F$.

(iii). Consider the range

$$R((D^2)_Y) \equiv (D^2)_Y[L((X,Y))] \subseteq L(D^2(X), D^2(Y))$$

and the function

$$u : R((D^2)_Y) \rightarrow D^2(L(X,Y))$$

well defined by $u(D^2(f)) = f^2_2$ such that, for each $f^1 \in D(L(X,Y))$, $f^2_2(f^1) = f^1(f)$. One readily sees that $u$ is linear and continuous, and that $j_{L(Y)} = u(D^2)_Y$.

Finally, let $Y$ be a Banach space, and let $C \subseteq L(X,Y)$ be a closed set. Let $(g_n)$ be sequence in $D[C]$ that converges in $L(D(Y), D(X))$, i.e., there exists $\lim(g_n) \equiv g \in L(D(Y), D(X))$. Since $D^2_Y$ is a linear isometry, it is a monomorphism, and there exists a unique Cauchy sequence $(f_n)$ in $C$ such that, for each $n \in \mathbb{N}$, $D(f_n) = g_n$. Notice that $L(X,Y)$ is a Banach space because such is $Y$, and thus, there exists $\lim(f_n) \equiv f \in L(X,Y)$. Since $C \subseteq L(X,Y)$ is closed, it follows that $f \in C$. Then $D(f) \in D[C]$, and the continuity implies that $D(f) = g$, which completes the proof.  

It is well known that there are Banach spaces (for instance $l_1$ and $c_0$) that are not isometrically isomorphic to any of the dual normed spaces. We shall now prove that it holds in general, i.e., without “isometrically”.

**Lemma 2.** (i) If a normed space $X$ is isomorphic to a dual space of a normed space, then $X$ is a Banach space, the canonical embedding $j_X : X \rightarrow D^2(X)$ is a section (of $\mathcal{B}_F$) and $X \equiv R(j_X)$ admits a closed direct complement in $D^2(X)$.

(ii) The (codomain restriction) functor $D : \mathcal{N}_F \rightarrow \mathcal{B}_F$ is not surjective onto the object class of any skeleton of $\mathcal{B}_F$, i.e.,

$$(\exists X \in Ob\mathcal{B}_F)(\forall Y \in Ob\mathcal{N}_F) X \not\equiv D(Y).$$

**Proof.** (i). Let $X$ be a normed space such that $X \cong D^n(Y)$ for some $Y \in Ob\mathcal{N}_F$ and $n \in \mathbb{N}$. Since $D^n(Y) = D(D^{n-1}(Y))$, one may assume that $n = 1$. Let $f : X \rightarrow D(Y)$ be an isomorphism of $\mathcal{N}_F$. Then $D^2(f) : D^2(X) \rightarrow D^3(Y)$ is an isomorphism of $\mathcal{B}_F \subseteq \mathcal{N}_F$ and the diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & D(Y) \\
\downarrow j_X & & \downarrow j_{D(Y)} \\
D^2(X) & \xrightarrow{D^2(f)} & D^3(Y)
\end{array}$$

in $\mathcal{N}_F$ commutes. We are to prove that $j_{D(Y)} : D(Y) \rightarrow D^3(Y)$ is a section (of $\mathcal{B}_F$) having $D(j_Y) : D^3(Y) \rightarrow D(Y)$ for a corresponding
retraction. Recall that \( j_Y : Y \to D^2(Y) \) is defined by \( j_Y(y^0) = y^2_{y^0} \), \( y^0 \in Y \), such that, for every \( y^1 \in D(Y), y^2_{y^0}(y^1) = y^1(y^0) \). Further, 
\[
D(j_Y) : D(D^2(Y)) = D^3(Y) \to D(Y)
\]
is determined by \( D(j_Y)(y^3) = y^3 j_Y, y^3 \in D^3(Y) \). In the same way, the canonical embedding
\[
j_{D(Y)} : D(Y) \to D^2(D(Y)) = D^3(Y) = D(D^2(Y))
\]
is determined by \( j_{D(Y)}(y^1) = y^3_{y^1}, y^1 \in D(Y) \), such that, for every \( y^2 \in D^2(Y), y^3_{y^1}(y^2) = y^2(y^1) \). Then, for every \( y^1 \in D(Y), \)
\[
(D(j_Y)j_{D(Y)})(y^1) = D(j_Y)(j_{D(Y)}(y^1)) = D(j_Y)(y^3_{y^1}) = y^3_{y^1} j_Y.
\]
Since, in addition, for every \( y^0 \in Y, \)
\[
(y^0_{y^0}, j_Y(y^0)) = y^3_{y^0}(y^2_{y^0}) = y^2_{y^0}(y^1) = y^1(y^0),
\]
then \( D(j_Y)j_{D(Y)} = 1_{D(Y)} \). This proves the claim. (Notice that \( j_{D(Y)}D(j_Y) : D^3(Y) \to D^3(Y) \) is a projection of norm 1.) Put
\[
r_X : D^2(X) \to X, r_X = f^{-1}D(j_Y)D^2(f).
\]
Then one readily verifies that \( r_X \) is a retraction of \( N_F \) having \( j_X \) for a corresponding section, i.e., \( r_X j_X = 1_X \). Consequently, \( R(j_X) \subseteq D^2(X) \) is a retract of \( D^2(X) \), and thus, a closed subspace, implying that it is a Banach space. Since the canonical embedding \( j_X \) is an isometry, it follows that \( X \) is a Banach space as well. Then, clearly, \( j_X \) and \( r_X \) belong to \( B_F.F \) Further, notice that the morphism
\[
p_X \equiv j_X r_X : D^2(X) \to D^2(X)
\]
is a continuous linear projection \( (p_X^2 = p_X) \) onto \( R(j_X) \). Therefore, the Banach space \( X \), identified with \( j_X[X] = R(j_X) \), admits a closed direct complement in \( D^2(X) \). (Notice that \( \|p_X\| = \|r_X\| = 1 \) regardless to \( \|f\| \)).

(ii). Assume to the contrary, i.e., that every Banach space is isomorphic to the dual space of a normed space. Then, since the dual of a space equals to the dual of its Banach completion, every Banach space would be isomorphic to the dual of a Banach space. Further, by iteration, every Banach space would isomorphic to the second dual of a Banach space. Let \( X \) be a non-bidual-like Banach space, i.e., \( D^2(X) \not\equiv X \) (see [17], Lemma 4. (ii)). Consider any closed subspace \( Z \subseteq D^2(X) \) such that \( X \equiv R(j_X) \subseteq Z \subseteq D^2(X) \), and denote by \( i : X \to Z \) the inclusion. Then we may assume that \( D^2(X) \subseteq D^2(Z) \) as well. By (i), there exists a retraction \( r_Z \) corresponding to the canonical embedding \( j_Z \), i.e.,
\[
r_Z : D^2(Z) \to Z, r_Z j_Z = 1_Z.
\]
Then the domain restriction
\[
r \equiv r_Z|D^2(X) : D^2(X) \to Z, r_i = 1_Z,
\]
is a (continuous linear) retraction of $D^2(X)$ onto the subspace $Z$, implying that

$$p \equiv ir : D^2(X) \to D^2(X)$$

is a continuous linear projection ($p^2 = p$) along $N(p) = N(r)$ onto $R(p) = R(r) = Z$. This implies that every such $Z$ admits a closed direct complement in $D^2(X)$. Finally, in order to get a contradiction, an appropriate pair $X, Z$ of concrete Banach spaces is needed. Let $X = c_0$ (the subspace of $l_\infty$ consisting of all null-convergent sequences in $F$). Recall that $c_0$ is not bidual-like because of $D^2(c_0) \cong l_\infty \not\cong c_0$. Namely, there are (isometric) isomorphisms $D(c_0) \cong l_1$ and $D(l_1) \cong l_\infty$. However, it is well known that there is a closed subspace $Z \leq l_\infty$, $c_0 \not\leq Z \leq l_\infty \cong D^2(c_0)$, which does not admit any closed direct complement in $l_\infty$ - a contradiction. This completes the proof. \hfill $\square$

**Remark 1.** (i) Though, by Lemma 2 (ii), there are Banach spaces that are not isomorphic to any of the dual spaces, we do not know whether every Banach space is a retract of its second dual space (the converse of Lemma 2 (ii) (?)).

(ii) Since $D(F^n) \cong F^n$ and, for all $1 < p, q < \infty$ such that $p^{-1} + q^{-1} = 1$,

$$D(F^n, \|\cdot\|_p) = D(Cl_{l_p}(F^n, \|\cdot\|_p)) = D(l_p) \cong l_q,$$

$$D(l_1) \cong l_\infty \quad \text{and}$$

$$Cl_{l_\infty}(F^n, \|\cdot\|_\infty) = c_0, \quad \text{implying}$$

$$D(F^n, \|\cdot\|_\infty) = D(c_0) \cong l_1$$

(see Lemma 4.1 (i) of [16]), the following (fundamental) question occurs: Does the functor $D$ rise an uncountably infinite algebraic dimension? In the next section we answer the question in negative.

Lemma 2 motivates the following consideration. Given a normed space $X$ and a $k \in \{0\} \cup \mathbb{N}$, let us denote by $j_{k,X} \equiv j_{D^k(X)} : D^k(X) \to D^{k+2}(X)$ the canonical embedding ($D^0 = !_{N_F}$). Then the class $\{ j_{k,X} \mid X \in Ob(N_F) \}$ determine a natural transformation $j_k : D^k \rightsquigarrow D^{k+2}$ of the functors. When there is no ambiguity, i.e., when a normed space $X$ is fixed, we simplify the notation $j_{k,X}$ to $j_k$. Notice that, for a given $X \in Ob(N_F)$, the following morphisms of $\mathcal{B}_F \subseteq N_F$ occur:

$$D(X) \xrightarrow{j_3} D^3(X) \xrightarrow{D(j_0)} D(X),$$

$$D^2(X) \xrightarrow{j_3} D^4(X) \xrightarrow{D(j_1)} D^2(X),$$

$$D^3(X) \xrightarrow{j_3} D^5(X) \xrightarrow{D(j_2)} D^3(X),$$

and, generally, for every $k \in \{0\} \cup \mathbb{N}$ and each $l, 0 \leq l \leq k,$
Let, for each \( k \), \( S_{2k+1}(X) \) be the set of all \( D^{2k-2l}(j_{2l+1}) \in L(D^{2k+1}(X), D^{2k+3}(X)) \), and let \( R_{2k+1}(X) \) be the set of all \( D^{2k+1-2l}(j_{2l+1}) \in L(D^{2k+3}(X), D^{2k+1}(X)) \),

\[
D^{2k+1}(X) \xrightarrow{D^{2k-2l}(j_{2l+1})} D^{2k+3}(X) \xrightarrow{D^{2k+1-2l}(j_{2l+1})} D^{2k+1}(X),
\]

\[
D^{2k-2l}(j_{2l+1}) D^{2k+3}(X) \xrightarrow{D^{2k+1-2l}(j_{2l+1})} D^{2k+1}(X). \]

Let, for each \( k \), \( S_{2k+1}(X) \) be the set of all \( D^{2k-2l}(j_{2l+1}) \in L(D^{2k+1}(X), D^{2k+3}(X)) \), and let \( R_{2k+1}(X) \) be the set of all \( D^{2k+1-2l}(j_{2l+1}) \in L(D^{2k+3}(X), D^{2k+1}(X)) \),

\[
0 \leq l \leq k. \]

Similarly, let \( S_{2k+2}(X) \) be the set of all \( D^{2k-2l+2}(j_{2l+2}) \in L(D^{2k+1}(X), D^{2k+3}(X)) \), and let \( R_{2k+2}(X) \) be the set of all \( D^{2k+1-2l+2}(j_{2l+2}) \in L(D^{2k+3}(X), D^{2k+1}(X)) \),

\[
0 \leq l \leq k. \]

Hence, for each \( n \in \mathbb{N} \), the sets \( S_n(X) \) and \( R_n(X) \) are well defined. By Lemma 1, since all \( j_k \) are isometries, all the morphisms belonging to \( S_n \cup R_n \) have norm 1.

**Theorem 1.** Let \( X \) be a normed space and let \( n \in \mathbb{N} \). Then

(i) each \( s \in S_n(X) \) is a (category) section, and each \( r \in R_n(X) \) is a (category) retraction;

(ii) \((\forall s \in S_n(X))(\exists r_s \in R_n(X)) \ r_s s = 1_{D^n(X)};\)

(iii) \((\forall r \in R_n(X))(\exists s_r \in S_n(X)) \ rs_r = 1_{D^n(X)};\)

(iv) \((\forall n \geq 3)(\exists s \in S_n(X))(\exists r \in R_n(X)) \ rs \) is not an epimorphism (especially, \( rs \neq 1_{D^n(X)} \)).

**Proof.** Let \( X \) be a normed space. Since, for every \( k \in \{0\} \cup \mathbb{N} \), the canonical morphism \( j_k : D^k(X) \to D^{k+2}(X) \) \( (D^0 = 1_{\mathcal{N}_F}) \) is an isometric embedding, Lemma 1 implies that \( D(j_k) : D^{k+3}(X) \to D^{k+1}(X) \) is an (open) epimorphism. Statements (i) and (ii) can be proved by “parallel” induction on \( 2n - 1 \) and on \( 2n \). Nevertheless, we provide an explicit proof. Let \( n = 1 \in \mathbb{N} \). By Lemma 2, \( D(j_0)j_1 = 1_{D(X)} \), i.e., \( j_1[D(X)] \) is a retract of \( D^3(X) \) with the retraction \( D(j_0) \) and the corresponding section \( j_1 \). Let \( n \geq 2 \). Since \( D \) is a contravariant functor, it follows that

\[
D(j_1)D^2(j_0) = D(D(j_0)j_1) = D(1_{D(X)}) = 1_{D^2(X)}.
\]

Therefore, \( D^2(j_0)[D^2(X)] \) is a retract of \( D^4(X) \) with the retraction \( D(j_1) \) having \( D^2(j_0) \) for a corresponding section. In general, by considering \( D^n(X) \) as \( D(D^{n-1}(X)) \), i.e., the canonical embedding \( j_n : D^n(X) \to D^{n+2}(X) \) as “\( j_1 : D(D^{n-1}(X) \to D^3(D^{n+1}(X)) \)” and \( D(j_{n-1}) : D^{n+2}(X) \to D^n(X) \) as “\( D(j_0) : D^3(D^{n-1}(X)) \to D^3(D^n(X)) \)” one proves (by mimicking the appropriate part of the proof of Lemma 2) that

\[
D(j_{n-1})j_n = 1_{D^n(X)}.
\]

holds true. Thus, for every \( n \in \mathbb{N} \), \( j_n[D^n(X)] \) is a retract of \( D^{n+2}(X) \) with the retraction \( D(j_{n-1}) \) having \( j_n \) for a corresponding section. Further, since \( D^2 \) is a (covariant) functor, one readily verifies that, for every \( k \in \mathbb{N} \) and every \( l \in \{0, \ldots, k\} \),

\[
D^{2k+1-2l}(j_{2l}) D^{2k-2l}(j_{2l+1}) = D^{2k-2l}(D(j_{2l})j_{2l+1}) = D^{2k-2l}(1_{D^{2l+1}(X)}) = 1_{D^{2k+1}(X)}.
\]
This shows that all $D_{2k+2}(j_{2t+1})$ and $D_{2k+2}(j_{2t+2})$ are sections having $D_{2t+1}(j_{2l})$ and $D_{2t+2}(j_{2l+1})$ for the corresponding retractions, respectively, and vice versa. Therefore, statements (i), (ii) and (iii) hold true.

Concerning statement (iv), let firstly $n = 3$. We are to show that, for the section $s = j_3 : D^3(X) \to D^5(X)$ and the retraction $r = D^3(j_0) : D^5(X) \to D^3(X)$, the composite

$$rs = D^3(j_0)j_3 : D^3(X) \to D^3(X)$$

may be not an epimorphism. Consider the following diagram

$$
\begin{array}{ccc}
D^3(X) & \xrightarrow{D(j_0)} & D(X) \\
\downarrow j_3 & & \downarrow j_1 \\
D^5(X) & \xrightarrow{D^3(j_0)} & D^3(X)
\end{array}
$$

in $B_F \subseteq N_F$. Since $D^3 = D^2D$, $D^5 = D^2D^3$, $j_1 = j_{D(X)}$, $j_3 = j_{D^3(X)}$ and $j : 1_{N_F} \simeq D^2$ is a natural transformation of the functors, the diagram commutes, i.e., $D^3(j_0)j_3 = j_1D(j_0)$. Notice that, in general, the canonical embedding $j_1$ is not an epimorphism, and the conclusion follows. If $n = 4$, then one similarly proves that, for instance, $D^3(j_1)j_4$ is not an epimorphism. Generally, if an $rs : D^n(X) \to D^n(X)$ factorizes through a $j_{n-2k}$, $1 \leq k \leq n - 2$, then, generally, it is not an epimorphism. Thus, statement (iv) follows.

The next corollary is an immediate consequence of our Theorem 1 and the known general facts (see Section 6 of Chapter 6 of [6]).

**Corollary 1.** (i) For every normed space $X$ and each $n \in \mathbb{N}$, the range $R(j_n)$ and the annihilator $R(j_{n-1})^0$ (of $R(j_{n-1})$ with respect to $D^{n+2}(X)$) are closed complementary subspaces of $D^{n+2}(X)$, i.e., by identifying $D^{n-1}(X)$ with $R(j_{n-1})$ and $D^n(X)$ with $R(j_n)$, the closed direct-sum presentation

$$D^{n+2}(X) = D^n(X) + D^{n-1}(X)^0$$

holds true. Consequently, by assuming the mentioned identifications,

$$D^{2n+1}(X) \equiv D(X) + X^0 + D^2(X)^0 + \cdots + D^{2n-2}(X)^0$$

and

$$D^{2n+2}(X) \equiv D^2(X) + D(X)^0 + D^3(X)^0 + \cdots + D^{2n-1}(X)^0.$$

(ii) If $X$ is a normed space admitting a retraction $r_0 : D^2(X) \to C(I(j_0[X]) \equiv X$ (in $B_F$), then

$$D^2(X) = X + N(r_0)$$

is a closed direct-sum presentation of $D^2(X)$. If, in addition, $X$ is a Banach space, then $D^2(X) = X + N(r_0)$ is a closed direct-sum presentation.
Proof. (i). Notice that, for a given $X$ and each $n \in \mathbb{N}$,
$$p_{n+2} \equiv j_n D(j_{n-1}) : D^{n+2}(X) \to D^{n+2}(X)$$
is a continuous linear projection. Indeed,
$$p_{n+2}^2 = (j_n D(j_{n-1}))(j_n D(j_{n-1})) = j_n 1_{D^n(X)} D(j_{n-1}) = j_n D(j_{n-1}) = p_{n+2}.$$  
Since $D(j_{n-1})$ is an epimorphism, and $j_n$ is an isometric embedding, it follows that
$$R(p_{n+2}) = R(j_n) \cong D^n(X).$$
Further,
$$N(p_{n+2}) = N(D(j_{n-1})) = \{ x^{n+2} \in D^{n+2}(X) \mid x^{n+2} j_{n-1} = c_0^{n-1} \} = \langle j_{n-1}[D^{n-1}(X)] \rangle^0 \cong R(j_{n-1})^0.$$
Now the conclusion follows by induction and the well known general facts. (Observe that, for instance,
$$p_{n+2}^2 \equiv D^2(j_{n-2}) D(j_{n-1}) : D^{n+2}(X) \to D^{n+2}(X), n > 1,,$$
is also a continuous linear projection yielding another closed direct-sum presentation of $D^{n+2}(X)$.)
(ii). If $X$ admits a retraction $r_0 : D^2(X) \to Cl(j_0[X]) \equiv \bar{X}$, then
$$p_2 \equiv j_0 r_0 : D^2(X) \to D^2(X)$$
is a continuous linear projection. Since the both $R(p_2) = Cl(R(j_0)) \equiv \bar{X}$ and $N(p_2) = N(r_0)$ are closed in $D^2(X)$, the stated closed direct-sum presentation follows. If such an $X$ is a Banach space, one may identify $X \equiv \bar{X} \subseteq D^2(X)$, and the conclusion follows. \qed

Example 1. Recall that $D(c) \cong l_1 \cong D(c_0)$ and $D(l_1) \cong l_\infty$. Then, by Corollary 1, $D(l_\infty) \cong l_1 + c_0^1 \cong l_1 + c_0^0$. This also shows that the annihilator does not preserve separability of a subspace.

By the mentioned identifications, Corollary 1 shows that $D^{n+2}(X)/D^n(X)$ is (isometrically) isomorphic to $D^{n-1}(X)^0$. Further, it is well known that $D(X)/Z^0, Z \subseteq X,$ is isometrically isomorphic to $D(Z)$, and that, for Banach spaces, $D(X/Z)$ is isometrically isomorphic to $Z^0$. These facts and Theorem 1 aim our attention at the behavior of the iterated dual functors on a quotient space (see also [6], Chapter 6., Sections 5. and 6.).

Lemma 3. Let $Z$ be a closed subspace of a normed space $X$, and denote by $i : Z \to X$ the inclusion, and by $q : X \to X/Z$ the quotient morphism. Then, for every $n \in \mathbb{N}$, the short sequences
$$D^{2n-1}(Z) \stackrel{D^{2n-1}(i)}{\leftarrow} D^{2n-1}(X) \stackrel{D^{2n-1}(q)}{\leftarrow} D^{2n-1}(X/Z) \quad \text{and}$$
$$D^{2n}(Z) \stackrel{D^{2n}(i)}{\to} D^{2n}(X) \stackrel{D^{2n}(q)}{\to} D^{2n}(X/Z)$$
in $\mathcal{B}_F$ are exact.

Proof. Clearly, the short sequence
\[ Z \xrightarrow{i} X \xrightarrow{\varphi} X/Z \]
in \( \mathcal{N}_F \) is exact, i.e., \( R(i) = N(\varphi) \). Since \( D \) is a contravariant functor and the function \( D^2_{X/Z} : L(X, X/Z) \to L(D(X/Z), D(X)) \) is linear, it follows that
\[ D(i)D(q) = D(qi) = D(c_\varphi) = c_0, \]
i.e., \( R(D(q)) \subseteq N(D(i)) \). We are to prove that the converse \( N(D(i)) \subseteq R(D(q)) \) holds as well. Let \( x^1 \in N(D(i)) \subseteq X^* \), i.e., \( D(i)(x^1) = x^1i = c_0 \) which implies that \( x^1[Z] = \{0\} \), i.e., \( x^1 \in Z^0 \). By the universal property of the quotient morphism \( q \), there exists a continuous linear function
\[ w_{x^1} : X/Z \rightarrow F, \quad w_{x^1}([x]) = w_{x^1}q(x) = x^1(x). \]
Then, clearly, \( w_{x^1} \in (X/Z)^* \) and, moreover,
\[ D(q)(w_{x^1}) = w_{x^1}q = x^1, \]
implying that \( x^1 \in R(D(q)) \), which proves the converse. Hence, the short sequence
\[ D(Z) \xrightarrow{D(i)} D(X) \xrightarrow{D(q)} D(X/Z) \]
in \( \mathcal{B}_F \) is exact. Further, by Lemma 1, \( D(i) : D(X) \rightarrow D(Z) \) is an epimorphism, and thus, the range \( R(D(i)) = D(Z) \) is (trivially) closed in \( D(Z) \). Then, by Proposition 6.5.13. of \([6]\), the short sequence
\[ D^2(Z) \xrightarrow{D^2(i)} D^2(X) \xrightarrow{D^2(q)} D^2(X/Z) \]
in \( \mathcal{B}_F \) is exact. Now, in general, by Lemma 1, \( D^{2n}(q) \) and \( D^{2n+1}(i) \) are epimorphisms, i.e., \( R(D^{2n}(q)) = D^{2n}(X/Z) \) and \( R(D^{2n+1}(i)) = D^{2n+1}(Z) \). (It suffices that \( R(D^{2n}(q)) \) is closed in \( D^{2n}(X/Z) \) and that \( R(D^{2n+1}(i)) \) is closed in \( D^{2n+1}(Z) \), which follows by Proposition 6.5.12. of \([6]\).) Then the final conclusion follows by Proposition 6.5.13. of \([6]\). \[\square\]

We can now state the following general facts concerning the iterated dual functors and quotients.

**Theorem 2.** Let \( Z \) be a closed subspace of a normed space \( X \), and denote by \( i : Z \hookrightarrow X \) the inclusion, and by \( q : X \to X/Z \) the quotient morphism. Then, for each \( n \in \mathbb{N} \),

(i) the functor \( D^{2n-1} \) permits cancellation on the quotient objects, i.e.,
\[ D^{2n-1}(X)/D^{2n-1}(X/Z) \cong D^{2n-1}(Z), \]
where \( D^{2n-1}(X/Z) \) is identified with \( R(D^{2n-1}(q)) \) in \( D^{2n-1}(X) \);

(ii) the functor \( D^{2n} \) “preserves” the quotient of objects, i.e.,
\[ D^{2n}(X/Z) \cong D^{2n}(X)/D^{2n}(Z), \]
where \( D^{2n}(Z) \) is identified with \( R(D^{2n}(i)) \) in \( D^{2n}(X) \);

(iii) \( D(X/Z) \cong Z^0, \ D^{2n+1}(X/Z) \cong D^{2n}(Z)^0, \ D^{2n}(Z) \cong D^{2n-1}(X/Z)^0, \)
where the mentioned identifications are assumed. Furthermore, all the isomorphisms are isometric.
Proof. Consider the short sequence
\[ Z \xrightarrow{1} X \xrightarrow{q} X/Z, \quad R(i) \equiv i[Z] = N(q), \]
in \( N_F \), which is exact, \( R(i) = Z = N(q) \), and \( i \) is the closed inclusion, while \( q \) is an open epimorphism. By Lemma 3, for each \( n \in \mathbb{N} \), the sequences
\[ D^{2n-1}(Z) \xleftarrow{D^{2n-1}(i)} D^{2n-1}(X) \xleftarrow{D^{2n-1}(q)} D^{2n-1}(X/Z), \]
\[ D^{2n}(Z) \xrightarrow{D^{2n}(i)} D^{2n}(X) \xrightarrow{D^{2n}(q)} D^{2n}(X/Z), \]
in \( B_F \) are exact, i.e.,
\[ R(D^{2n-1}(q)) \equiv D^{2n-1}(q)[D^{2n-1}(X/Z)] = N(D^{2n-1}(i)), \]
\[ R(D^{2n}(i)) \equiv D^{2n}(i)[D^{2n}(Z)] = N(D^{2n}(q)). \]
Observe that, by Lemma 1, the morphisms \( D^{2n-1}(q) \) and \( D^{2n}(i) \) are closed monomorphisms, while \( D^{2n-1}(i) \) and \( D^{2n}(q) \) are (open) epimorphisms. By the universal property of a quotient in \( N_F \), there exists a unique continuous linear (canonical) factorization of \( D^{2n-1}(i) \) through the quotient morphism
\[ q_{2n-1} : D^{2n-1}(X) \rightarrow D^{2n-1}(X)/N(D^{2n-1}(i)), \]
\[ q_{2n-1}(x^{2n-1}) = [x^{2n-1}], \]
such that \( D^{2n-1}(i) = h_{2n-1}q_{2n-1} \in Mor(B_F) \), where
\[ h_{2n-1} : D^{2n-1}(X)/N(D^{2n-1}(i)) \rightarrow D^{2n-1}(Z), \]
\[ h_{2n-1}([x^{2n-1}]) = D^{2n-1}(i)(x^{2n-1}) = x^{2n-1}D^{2n-2}(i). \]
By the same reason, there exists the canonical factorization \( D^{2n}(q) = h_{2n}q_{2n} \), where
\[ q_{2n} : D^{2n}(X) \rightarrow D^{2n}(X)/N(D^{2n}(q)), \]
\[ q_{2n}(x^{2n}) = [x^{2n}], \quad \text{and} \]
\[ h_{2n} : D^{2n}(X)/N(D^{2n}(i)) \rightarrow D^{2n}(X/Z), \]
\[ h_{2n}([x^{2n}]) = D^{2n}(i)(x^{2n}) = x^{2n}D^{2n-1}(q). \]
Since \( D^{2n-1}(i) \) and \( D^{2n}(q) \) are open epimorphisms, so are \( h_{2n-1} \) and \( h_{2n} \) (Open-mapping theorem). Further, the above exactness, i.e.,
\[ R(D^{2n-1}(q)) = N(D^{2n-1}(i)), \quad R(D^{2n}(i)) = N(D^{2n}(q)) \]
implies, respectively, that
\[ D^{2n-1}(X)/R(D^{2n-1}(q)) = D^{2n-1}(X)/N(D^{2n-1}(i)), \]
\[ D^{2n}(X)/R(D^{2n}(i)) = D^{2n}(X)/N(D^{2n}(q)). \]
Therefore, \( h_{2n-1} \) and \( h_{2n} \) are bijections. Finally, by the Banach inverse-mapping theorem, \( h_{2n-1} \) and \( h_{2n} \) are isomorphisms of \( B_F \). Since \( D^{2n-1}(q) \) and \( D^{2n}(i) \) are closed monomorphisms, one may identify \( D^{2n-1}(X/Z) \) with \( D^{2n-1}(q)[D^{2n-1}(X/Z)] \) in \( D^{2n-1}(X) \) as well as \( D^{2n}(Z) \) with \( D^{2n}(i)[D^{2n}(Z)] \) in \( D^{2n}(X) \). Consequently,
\[ D^{2n-1}(X)/D^{2n-1}(X/Z) \cong D^{2n-1}(Z) \]
and
\[ D^{2n}(X)/D^{2n}(Z) \cong D^{2n}(X/Z). \]
In this way we have proven the isomorphism relations in statements (i) and (ii).
In order to prove statement (iii) (see also Remark 2 below), let us again consider the starting exact sequence in $\mathcal{N}_F$, $R(i) = Z = N(q)$. Notice that

$Z^0 = \{ x^1 \in X^* \mid R(x^1) = \{ 0 \} \} = \{ x^1 \mid x^1i = c_0 \} = N(D(i)) = R(D(q))$, implying that

$R(D(q)) \equiv D(q)((X/Z)^*) = Z^0$ in $X^*$.

Since, by Lemma 1, $D(q)$ is a closed monomorphism, one may identify $((X/Z)^*)^0$ with $Z^0$ in $X^*$, and then (the well known) $D(X/Z) \cong Z^0$ holds. This proves the case $n = 1$ of statement (iii). If $n = 2$, the exactness (Lemma 3) and Lemma 1 imply that $R(D^2(i)) = ((X/Z)^*)^0$ and $D^2(i)$ is a closed monomorphism. Then, by identifying $Z^{**}$ with $((X/Z)^*)^0$ in $X^{**}$, it follows $D^2(Z) \cong D(X/Z)^0$. By arguing in the same manner through all the $D$-iterating exact sequences (Lemma 3), and assuming the mentioned identifications (Lemma 1), one obtains the remaining isomorphisms in (iii). Finally, since the isomorphisms $X^*/(X^*/Z^*) \cong X^*/Z^* \cong Z^*$ and $(X/Z)^{**} \cong X^{**}/Z^{**}$ are isometric (a non-zero quotient morphism has the norm 1), it follows, by applying the functor $D^{2n}$ inductively, that all the obtained isomorphisms are isometric. $\square$

Remark 2. We are aware of the well known fact (closely related to the case $n = 1$ of Theorem 2, (i) and (iii)), that $X^*/Y^0 \cong Y^*$ (isometrically), for every subspace $Y$ of $X$ ([10], Section 8. 12, Propozicija 17, p. 444). We did not use it in the proof. However, one can show that statements (i) and (ii) of Theorem 2 with that fact imply (iii), and conversely, statement (iii) with that fact implies (i) and (ii) of Theorem 2.

4. An application through quotient shapes

We shall now combine the obtained facts with those of [14], [16] and [17] in order to get a better insight in the quotient shapes of normed spaces (especially those considered in [16] and [17]) related to their (iterated) dual spaces. The first step in that direction is based on the fact that $D^2$ is a faithful functor.

**Theorem 3.** (i) Let $X$ be a normed space and let $p_0 = (p_\lambda) : X \rightarrow X_0 = (X_\lambda, p_{\lambda X}, \Lambda_0)$ be an $(\mathcal{N}_F)_0$-expansion of $X$. Then

$\text{pro-D}^2(p_0) = (D^2(p_\lambda)) : D^2(X) \rightarrow \text{pro-D}^2(X_0) = (D^2(X_\lambda), D^2(p_{\lambda X}), \Lambda_0)$

is an $D^2(\mathcal{N}_F)_0$-expansion of $D^2(X)$. 

(ii) Let $X$ and $Y$ be normed spaces of the same finite quotient shape type, i.e., $Sh_0(X) = Sh_0(Y)$. Then, for every $n \in \mathbb{N}$, $Sh_0(D^{2n}(X)) = Sh_0(D^{2n}(Y))$, i.e., $D^{2n}$ preserves the finite quotient shape type.

Proof. (i). According to Lemma 1 (i), given an $(\mathcal{B}_F)_0$-expansion (actually, a $(\mathcal{B}_F)_0$-expansion) $p_0 = (p_\lambda) : X \to X_0 = (X_\lambda, p_{\lambda\lambda}, \Lambda_0)$ of $X$, one has to verify the factorization property (E1) of $D^2(X) \to (D^2(X_\lambda), D^2(p_{\lambda\lambda}), \Lambda_0)$ with respect to the image subcategory $D^2(\mathcal{N}_F)_0$ only. Let $D^2(Y)$ be a finite-dimensional normed space (actually, a Banach space $Z \cong F^n$, where $n \in \mathbb{N}$) and let a morphism $D^2(f) \in \mathcal{N}_F(D^2(X), D^2(Y))$ be given. Then $Y$ is finite-dimensional and $f \in \mathcal{N}_F(X, Y)$. Since $p_0 : X \to X_0$ is an $(\mathcal{N}_F)_0$-expansion of $X$, there exist $\lambda \in \Lambda_0$ and an $f_\lambda : X_\lambda \to Y$ such that $f_\lambda p_\lambda = f$. Then $D^2(f_\lambda)D^2(p_\lambda) = D^2(f_\lambda p_\lambda) = D^2(f)$, and the claim follows.

(ii). It suffices to prove the claim in the case $n = 2$. Let $p_0 = (p_\lambda) : X \to X_0 = (X_\lambda, p_{\lambda\lambda}, \Lambda_0)$, 
$q_0 = (q_\mu) : Y \to Y_0 = (Y_\mu, q_{\mu\mu'}, \Lambda_0)$ be any $(\mathcal{N}_F)_0$-expansions (actually, $(\mathcal{B}_F)_0$-expansions) of $X$ and $Y$ respectively. Since $Sh_0(X) = Sh_0(Y)$, the expansion systems $X_0$ and $Y_0$ are isomorphic objects of $pro-(\mathcal{N}_F)_0$. Then $pro-D^2(X_0)$ and $pro-D^2(Y_0)$ are isomorphic objects of $pro-D^2(\mathcal{N}_F)_0$, because $pro-D^2 : pro-(\mathcal{N}_F)_0 \to pro-D^2(\mathcal{N}_F)_0$ is a functor (the restriction of the “prolongation” of $D^2$ to the pro-categories). Since $pro-D^2(\mathcal{N}_F)_0$ is a subcategory of $pro-(\mathcal{N}_F)_0$, it follows that $D^2(X_0)$ and $D^2(Y_0)$ are isomorphic in $pro-(\mathcal{N}_F)_0$ as well. Then, by (i), $Sh_0(D^2(X)) = Sh_0(D^2(Y))$. \qed

Corollary 2. All the $l_p$ spaces and all their normed duals belong to the same finite quotient shape type, i.e.,

$$(\forall 1 \leq p \leq \infty)(\forall n \in \{0\} \cup \mathbb{N}) \; Sh_0(D^n(l_p)) = Sh_0(F_0^n, \|\cdot\|_2)$$

The same holds for all the $L_p(n)$ spaces.

Proof. Recall that, by Theorem 2 (i) of [17], $Sh_0(l_p) = Sh_0(l_{p'})$ holds for all $1 \leq p, p' \leq \infty$. Since the spaces $l_p$, $1 < p < \infty$, are reflexive, Theorem 3 implies that all the even normed duals of all $l_p$ spaces belong to the same finite quotient shape type. Especially, $Sh_0(D^{2n}(l_1)) = Sh_0(D^{2n}(l_\infty)) = Sh_0(l_2) = Sh_0(F_0^n, \|\cdot\|_2)$. Further, since $l_p \cong l_{p'}$, for $1 < p, p' < \infty$ and $p^{-1} + (p')^{-1} = 1$, and since $l_\infty \cong D(l_1)$, it follows that all the $l_p$ spaces and all their normed duals belong to the same finite quotient shape type, i.e.,

$$(\forall 1 \leq p \leq \infty)(\forall n \in \{0\} \cup \mathbb{N}) \; Sh_0(D^n(l_p)) = Sh_0(F_0^n, \|\cdot\|_2)$$
Then, by Corollary 1 of [17] and Theorem 3, the same holds for all the $L_p(n)$ spaces. □

Now, the question about the quotient shapes of any normed (Banach, separable) space, occurs as the problem of the algebraic dimension(s) of its (iterated) normed dual space(s). This obstacle has essentially limited the obtained results of [17]. Namely, the restriction to separable or to bidual-like normed spaces (Theorems 2 and 4 of [17]) was necessary because, in essence, we did not know how to calculate $\dim D(X)$ (except for spaces of the mentioned classes, see Lemma 4 (ii) of [17]). We have hereby resolved that problem completely. Firstly, an auxiliary notion.

**Definition 1.** A normed vectorial space $X$ over $F \in \{\mathbb{R}, \mathbb{C}\}$ is said to be $\dim^*$-**stable** (or $dD$-**stable**) if $\dim D(X) \equiv \dim X^* = \dim X$.

Clearly, the functor $D$ does not diminish the algebraic dimension, and thus $\dim X \leq \dim D(X)$ holds generally.

**Example 2.** (i) Since $D(F^n) \cong F^n$, $n \in \mathbb{N}$, every finite-dimensional normed space over $F \in \{\mathbb{R}, \mathbb{C}\}$ is $\dim^*$-stable. Similarly, all separable and all bidual-like normed spaces are $\dim^*$-stable.

(ii) Since $l_p^* \cong l_{p'}$, $1/p + 1/p' = 1$, all $l_p$ spaces, $1 < p < \infty$, are $\dim^*$-stable, while $l_1$ is $\dim^*$-stable because $l_1^* \cong l_\infty$. The same holds true for all $L_p(n)$ spaces, $n \in \mathbb{N}$.

(iii) No direct sum normed space $(F_0^N, \|\cdot\|)$ is $\dim^*$-stable. (Namely, $\dim(F_0^N, \|\cdot\|) = \aleph_0$, while $\dim(F_0^N, \|\cdot\|)^* \neq \aleph_0$ (every dual space is a Banach space, and there is no Banach space having countably infinite algebraic dimension).

Since the finite quotient shape is an invariant of the algebraic dimension (Theorem 2 (i) of [17]), the next corollary follows immediately.

**Corollary 3.** If $X \in Ob(\mathcal{N}_F)$ is $\dim^*$-stable, then $Sh_0(X^*) = Sh_0(X)$.

Recall that the algebraic dual rises every infinite algebraic dimension. Thus, there is no countably infinite-dimensional algebraic dual space. Since there is no countably infinite-dimensional Banach space, there is no countably infinite-dimensional (algebraically) dual normed space as well. However, besides Example 2, (i) and (ii), we shall show that the class of all $\dim^*$-stable normed spaces is rather large. The main fact in that direction is the next lemma.

**Lemma 4.** Let $X, Y \in Ob(\mathcal{N}_F)$ such that $Sh_0(X) = Sh_0(Y)$. If $Y$ is $\dim^*$-stable, i.e., $\dim Y^* = \dim Y$, then so is $X^* \equiv D(X)$ and $\dim X^{**} = \dim X^* = \dim Y$. 
Proof. Clearly, the statement is not trivial in the infinite-dimensional case only. Since $Y$ is dim$^*$-stable, i.e., dim $Y = \dim Y^*$, and $Y^*$ is a Banach space, it follows that dim $Y = |Y| \geq 2^{\aleph_0}$ (see Lemma 3. 2 (iv) of [14]). Assume, firstly, that dim $X \geq 2^{\aleph_0}$ as well. Then dim $X = |X|$ and dim $X^* = |X^*| \geq |X|$. Let

$p_0 = (p_\lambda) : X \to X_0 = (X_\lambda, p_{\lambda\lambda'}, \Lambda_0),$  
$q_0 = (q_\mu) : Y \to Y_0 = (Y_\mu, q_{\mu\mu'}, M_0)$

be the canonical $(\mathcal{N}_F)_0$-expansions (actually, $(\mathcal{B}_F)_0$-expansions) of $X$, $Y$ respectively. Since dim $X \geq 2^{\aleph_0}$, the canonical construction of $p_0$ (see Section 12 of [13] and Section 4.1 of [14]) implies that the index set $\Lambda_0$ is the disjoint union of $\Lambda_0^{(n)}$, $n \in \{0\} \cup \mathbb{N}$, where $\Lambda_0^{(0)}$ is the singleton containing the first (minimal) element, while, for each $n \in \mathbb{N}$,

$\Lambda_0^{(n)} = \{ \lambda \in \Lambda_0 | \dim X_\lambda = n \},$  
and

$|\Lambda_0^{(1)}| = \cdots = |\Lambda_0^{(n)}| = \cdots = |\Lambda_0| \geq |X| \geq 2^{\aleph_0}.$

Indeed, given an $n \in \mathbb{N}$, for every $\lambda \in \Lambda_0^{(n)}$, $X_\lambda = X/Z_\lambda$ where $Z_\lambda \subseteq X$ is closed, dim $Z_\lambda = \dim X$ and dim$(X/Z_\lambda) = n$. Thus, there is a closed direct complement $W_\lambda \subseteq X$ of $Z_\lambda$, $X = Z_\lambda + W_\lambda$, dim $W_\lambda = n$.

Then, for $n = 1$, each continuous linear epimorphism $x^1 : X \to F$, i.e., $x^1 \in X^* \setminus \{0\}$, yields a unique $\lambda \in \Lambda_0^{(1)}$. Conversely, for every $\lambda \in \Lambda_0^{(1)}$, it can exist at most $2^{\aleph_0} \cdot |X|$ linear epimorphisms $x^1$ having $N(x^1) = Z_\lambda$, where $|X|$ counts all the 1-dimensional direct complements $W_\lambda$ of $Z_\lambda$. Similarly, each continuous linear epimorphism $f : X \to F^n$ yields a unique closed $Z_f \equiv N(f) \subseteq X$ such that $Z_f + W_f = X$, $W_f \cong R(f) = F^n$, while, since there are $2^{\aleph_0}$ linear epimorphisms of $F^n$ to $F^n$, for every such $Z_\lambda$, it can exist at most $2^{\aleph_0} \cdot |X|$ linear epimorphisms $f$ having $N(f) = Z_\lambda$, where $|X|$ counts all the $n$-dimensional direct complements $W_\lambda$ of $Z_\lambda$. Now, one readily sees that all $\Lambda_0^{(n)}$, have the same uncountable cardinality. Since $|\mathbb{N}| = \aleph_0$ and dim $X = |X| \geq 2^{\aleph_0}$, the above partition of $\Lambda_0$ follows by the cardinal arithmetic. Further, especially observe that

$|\Lambda_0^{(1)}| \leq |X^*| \leq |\Lambda_0^{(1)}| \cdot |X| \cdot 2^{\aleph_0} = |\Lambda_0^{(1)}| = |\Lambda_0|.$

Therefore,

$|\Lambda_0| = |X^*| = \dim X^* \geq \dim X = |X| \geq 2^{\aleph_0}.$

Further, notice that there is no bonding morphism between any pair of terms having indices $\lambda \neq \lambda'$ in the same $\Lambda_0^{(n)}$, while every $p_{\lambda\lambda'}$, $\lambda < \lambda'$, is an epimorphism, but not an monomorphism (i.e., not an isomorphism). The quite analogous partition

$|M_0| = |Y^*| = \dim Y^* = \dim Y = |Y| \geq 2^{\aleph_0}$
of $M_0$ and the properties of $Y_0$ hold as well. We shall prove that $|\Lambda_0| = |M_0|$. Firstly, let us pass to the isomorphic cofinite inverse systems ([11], Theorem I.1.2)

$$X'_0 = (X'_0, \rho_{\lambda \lambda'}, \bar{\Lambda}_0) \cong X_0,$$
$$Y'_0 = (Y'_0, \rho'_{\mu \mu'}, \bar{M}_0) \cong Y_0$$

(made of the same “term-bond material”) with $|\bar{\Lambda}_0| \leq |\Lambda_0|$ and $|\bar{M}_0| \leq |M_0|$ (actually, the both “$\leq$” are “$=$”). By this passage (the construction called “Mardešić trick”), $\bar{\Lambda}_0$ is the disjoint union of $\bar{\Lambda}_0^{(n)}$, $n \in \{0\} \cup \mathbb{N}$, where $\bar{\Lambda}_0^{(0)}$ is the singleton containing the first (minimal) element, while, for each $n \in \mathbb{N}$,

$$\bar{\Lambda}_0^{(n)} = \{ \bar{\lambda} \in \bar{\Lambda}_0 \mid |\bar{\lambda}| = n \}$$

$(|\bar{\lambda}|$ denotes the cardinal of the set of all predecessors $\bar{\lambda}_j < \bar{\lambda}$) and

$$|\bar{\Lambda}_0^{(0)}| = \cdots = |\bar{\Lambda}_0^{(n)}| = \cdots = |\bar{\Lambda}_0|.$$

Further, there is no bonding morphism between any pair of terms having indices $\bar{\lambda} \neq \bar{\lambda}'$ in the same $\bar{\Lambda}_0^{(n)}$, while every $\rho_{\lambda \lambda'}$, $\bar{\lambda} < \bar{\lambda}'$ (being $p_{\mathrm{max} \lambda \lambda'}$, is an epimorphism., and an monomorphism (i.e., an isomorphism) if and only if it is the identity, that occurs when $\bar{\lambda} \subset \bar{\lambda}'$ and $\max \bar{\lambda} = \max \bar{\lambda}'$ only. The quite analogous partition and properties hold for $M_0$ and $Y_0$. Therefore, we have to prove that $|\bar{\Lambda}_0| = |M_0|$. Since $\text{Sh}_0(X) = \text{Sh}_0(Y)$, there exist isomorphisms

$$f : X'_0 \to Y'_0, \ g = f^{-1} : Y'_0 \to X'_0$$

of pro-$(\mathcal{B}_F)_0 \subseteq \text{pro-}(\mathcal{N}_F)_0$. By [11], Lemmata I.1.2 and Remark I.1.8, there exist special (the appropriate square diagrams commute) representatives $(\phi, f_\mu), (\psi, g_\mu)$ in $\text{inv-}(\mathcal{B}_F)_0$ of $f$, $f^{-1}$ respectively. Then the subsystem

$$X'_0 = (X'_0, \phi_{\psi(\mu)}, \psi_{\phi(\mu)}, \rho'_{\mu \mu'}, \bar{M}_0)$$

of $X'_0$ is isomorphic to $X'_0$ in pro-$(\mathcal{B}_F)_0$. It implies, by the mentioned properties of the terms and bonds of $X'_0$, that the index function $\psi : \bar{M}_0 \to \bar{\Lambda}_0$ must be cofinal, i.e.,

$$(\forall \bar{\lambda} \in \bar{\Lambda}_0)(\exists \bar{\mu} \in \bar{M}_0) \ \psi(\bar{\mu}) \geq \bar{\lambda}.$$ 

Since $\bar{\Lambda}_0$ is cofinite, this readily implies that $|\bar{\Lambda}_0| \leq \aleph_0 \cdot |\bar{M}_0| = |\bar{M}_0|$. One can establish, in the same way, that $|\bar{M}_0| \leq |\bar{\Lambda}_0|$ holds, and the conclusion follows. Consequently, $|\bar{\Lambda}_0| = |\bar{M}_0|$, and therefore,

$$\dim X^* = \dim Y^* = \dim Y.$$ 

Then, by Theorem 2 (i) of [17], $\text{Sh}_0(X^*) = \text{Sh}_0(Y)$ holds. We may now apply the same proof (to $X^*$ and $Y$) and conclude that $\dim X^{**} = \dim Y = \dim X^*$. Therefore, $X^*$ is $\dim^*$-stable, whenever $\dim X \geq 2^\aleph_0$. Assume now that $\dim X = \aleph_0$. Then $X \cong (F_0^\aleph_0, \|\cdot\|)$, while $X^* \cong (F_0^\aleph_0, \|\cdot\|''$), and $\dim X < \dim X^* = 2^\aleph_0$. Let $Y$ be a Hilbert space such that $\dim Y = 2^\aleph_0$. Since, by Theorem 2 (i) of [17], $\text{Sh}_0(X^*) = \text{Sh}_0(Y)$,
and since $Y$ is dim$^*$-stable, the first part of the proof assures that $X^*$ is dim$^*$-stable, which completes the proof. \hfill $\square$

**Theorem 4.** Every normed vectorial space $X$ having (algebraic) dim $X \neq \aleph_0$ is dim$^*$-stable. Especially, every Banach space is dim$^*$-stable.

**Proof.** Firstly observe that in the special case of a dim$^*$-stable $Y = X$, Corollary 3 and Lemma 4 imply that $Y^*$ is dim$^*$-stable, i.e., dim $Y^{**} = dim Y$. Then, by induction and combining Lemma 4 with Theorem 2 (i) of [17], it follows that every iterated normed dual of $Y$ is dim$^*$-stable, i.e., dim $D^n(Y) = dim Y$, $n \in \mathbb{N}$. Further, in the special case of a dim$^*$-stable $Y$ and dim $X = dim Y$, Lemma 4 and Theorem 2 (i) of [17] imply that dim $X^{**} = dim X^* = dim Y = dim X$. Hence, $X$ is dim$^*$-stable as well, and consequently, so are its all iterated normed duals $D^n(X)$. In this way we have proven that the dim$^*$-stability is an invariant of the algebraic dimension, i.e., if dim $X$ = dim $Y$ and $Y$ is dim$^*$-stable, then so is $X$, as well as, that the functor $D^n$ preserves dim$^*$-stability. Clearly, the statement is not trivial in the infinite-dimensional case only. Let dim $X = \infty \neq \aleph_0$, implying that dim $X \geq 2^{\aleph_0}$ (GCH accepted). Let us choose a Hilbert space $Y$ (over the same $F$) such that dim $X = dim Y$. Such a $Y$ exists, for instance, by means of the usual construction. More precisely, let $J$ be an index set of cardinality $|J| = dim X$, and let the set

$$F^J = \{ y \equiv (y_j) | \ y : J \to F \}$$

be endowed with the usual vectorial (algebraic) structure (over $F$). Consider its subspace

$$F^J_2 = \{ y \in F^J | \sum_{j \in J} |y_j|^2 < \infty \} \leq F^J.$$

Then $Y = (F^J_2, \| \cdot \|_2)$, where

$$\| y \|_2 = (\sum_{j \in J} |y_j|^2)^{1/2},$$

is a Hilbert space. Moreover, since $|J| \geq 2^{\aleph_0}$ and every $y \in F^J_2$ contains at most $\aleph_0$ non-zero coordinates, one readily verifies that

$$|F^J_2| < |F|^{|J|} = 2^{|J|} = |F^J|.$$

Therefore, by Lemma 2 (iii) of [14],

$$\dim X \leq \dim F^J_2 < \dim F^J = 2^{|J|} = 2^{\dim X}.$$

It follows, by GCH, dim $F^J_2 = \dim X$, and thus, dim $Y = \dim X$. Since every Hilbert space is dim$^*$-stable, so is $X$. \hfill $\square$

Observe that the assumption dim $X \neq \aleph_0$ is essential because of, for instance, $Sh_0((F_0^{\aleph_0}, \| \cdot \|_p)^*) = Sh_0(l_p)$, $p > 1$, while

$$\dim(F_0^{\aleph_0}, \| \cdot \|_p) = \aleph_0 < 2^{\aleph_0} = \dim l_{p'} = \dim (F_0^{\aleph_0}, \| \cdot \|_p)^*,$$

whenever $p^{-1} + (p')^{-1} = 1$. 


An immediate consequence of Theorem 4 (and its proof) is the following fact.

**Corollary 4.** For every $X \in \text{Ob}(\mathcal{N}_F)$, the following properties are equivalent:

(i) $X$ is dim*-stable;
(ii) $(\forall k \in \{0\} \cup \mathbb{N}) D^k(X)$ is dim*-stable, i.e., $\dim D^{k+1}(X) = \dim D^k(X)$.

We can now improve Corollaries 1 and 2 as well as Theorem 2 of [17], and completely solve the finite quotient shape classification of normed vectorial spaces (GCH assumed) as follows.

**Theorem 5.** For every $X \in \text{Ob}(\mathcal{N}_F)$,
$$\dim X^* = 2^{\dim X} > \dim X \iff \dim X = \aleph_0.$$ 

Equivalently,
$$\dim X^* = \dim X \iff \dim X \neq \aleph_0.$$ 

Therefore, for every $n \in \mathbb{N}$, $D^n(X)$ is dim*-stable, $\dim D^n(X) = \dim X^*$ and $\text{Sh}_0(D^n(X)) = \text{Sh}_0(X)$. Consequently, given a pair $X,Y \in \text{Ob}(\mathcal{N}_F)$, then

(i) $\text{Sh}_0(X) = \text{Sh}_0(Y) \iff \begin{cases} \dim X = \dim Y \notin \{\aleph_0, 2^{\aleph_0}\} \\ \dim X, \dim Y \in \{\aleph_0, 2^{\aleph_0}\} \end{cases}$.

(ii) If $X,Y \in \text{Ob}(\mathcal{B}_F)$, then
$$(\dim X = \dim Y) \iff (\text{Sh}_0(X) = \text{Sh}_0(Y)) \iff (\text{Sh}_{\aleph_0}(X) = \text{Sh}_{\aleph_0}(Y)).$$

(iii) If $X$ and $Y$ are dim*-stable and if there exists a closed embedding $e : X \to Y$ such that $\dim(Y/e[X]) < \dim Y = \kappa \geq \aleph_0$, then
$$(\dim X = \dim Y) \iff (\text{Sh}_{\kappa^+}(X) = \text{Sh}_{\kappa^+}(Y)).$$

**Proof.** Concerning the first part and statement (i), i.e., the finite quotient shape classification, we only need to verify that $\text{Sh}_0(X^*) = \text{Sh}_0(X)$ in the case $\dim X = \aleph_0$ as well. Indeed, in that case, $X \cong (F_0^{\aleph_0}, \|\cdot\|)$, and $\dim X^* = 2^{\aleph_0}$. By Corollary 4, $\dim D^n(X) = \dim X^*$, for every $n \in \mathbb{N}$. Further, by Lemma 2 (iii) of [17], the Banach completion $\text{Cl}(X) \subseteq X^{**}$ rises (algebraic) dimension, i.e., $\dim \text{Cl}(X) = 2^{\aleph_0}$. Thus, $\dim \text{Cl}(X) = \dim X^*$. Since, by Theorem 3 (i) of [16], $\text{Sh}_0(\text{Cl}(X)) = \text{Sh}_0(X)$ holds, and by Theorem 2 (i) of [17], $\text{Sh}_0(\text{Cl}(X)) = \text{Sh}_0(X^*)$ holds, the conclusion follows. Statement (ii) follows by (i) because there is no countably infinite-dimensional Banach space. Statement (iii) generalizes Theorem 2 (iii) of [17] (for the bidual-like normed spaces) to the dim*-stable normed spaces. In the proof of that theorem (especially, that of Theorem 2 (iii)), Lemma 4 (ii) of [17] was used. However, its role in that proof is the same as that of the dim*-stability of $X$ and $Y$. The conclusion follows. $\square$
Remark 3. Although the second dual space $D^2(X) \equiv X^{**}$ is large enough to contain $X \subseteq \text{Cl}(X) \subseteq X^{**}$ (as the isometrically embedded subspaces), this enlargement does not rise dimension neither cardinality because $$|X^{**}| = \dim X^{**} = \dim \text{Cl}(X) = \dim X = |X|,$$ whenever $\dim X > \aleph_0$. Therefore, in an infinite-dimensional $\kappa$-expansion of $X$, the codimension of $\text{Cl}(X)$, i.e., $\dim(X^{**}/\text{Cl}(X))$, plays the most important role.

Observe that the cardinal arithmetic yield the following interesting consequences.

Corollary 5. Let $X, Y \in \text{Ob}(\mathcal{N}_F)$ such that $\dim X \geq \aleph_0$, $0 < \dim Y < \dim X$ and $\dim Y \neq \aleph_0$, and let $L(X, Y)$ be the (normed) space of all continuous linear functions of $X$ to $Y$. Then,

(i) $|L(X, Y)| = \dim L(X, Y) = \dim D^n(X) = 2^{\aleph_0}$, $n \in \mathbb{N}$, whenever $\dim X = \aleph_0$;

(ii) $|L(X, Y)| = \dim L(X, Y) = \dim D_n(X) = \dim X = |X|$, $n \in \mathbb{N}$, whenever $\dim X > \aleph_0$.

Proof. For statement (i), notice that $Y \cong F^k$ for some $k \in \mathbb{N}$. Then, since $\dim X \geq \aleph_0$, $L(X, Y)$ is an infinite-dimensional Banach space, and thus, $\dim L(X, Y) \geq 2^{\aleph_0}$. It follows, by Lemma 3. 2. (iv) of [16], that $\dim L(X, Y) = |L(X, Y)|$. Since $|L(X, Y)| = |L(\text{Cl}(X^{**})(X), Y)|$, the proof of Lemma 4 (the partition of the index set of the canonical $(\mathcal{N}_F)_0$-expansion of “$X^{**}$ = $\text{Cl}(X^{**})(X)$) shows that $|L(\text{Cl}(X^{**})(X), Y)| = |(\text{Cl}(X^{**})(X))^{**}|$. By Lemma 3. 2. (iv) of [16] again, $|(\text{Cl}(X^{**})(X))^{**}| = \dim(\text{Cl}(X^{**})(X))^{**}$. Finally, by Theorem 5,

$\dim(\text{Cl}(X^{**})(X))^{**} = \dim \text{Cl}(X^{**})(X) = 2^{\aleph_0}$,

and the conclusion follows. For statement (ii), let $\dim X = \kappa \geq 2^{\aleph_0}$ and $\dim Y = \kappa' < \kappa$, $\kappa' \neq \aleph_0$. Then, by Theorem 5 and Lemma 3. 2. (iv) of [16],

$$\kappa \leq \dim L(X, Y) = \dim L(X, Y)^* = \dim L(Y^*, X^*) = |L(Y^*, X^*)| \leq |X^*|^{|Y^*|} = |X|^{|Y|} = \kappa^{|Y'|} = \kappa,$$

and the conclusion follows. \qed

Concerning the extensions of morphisms, the following extension type theorem is an immediate consequence of Theorem 1 (i.e., Corollary 1).

Theorem 6. Let $X, Y \in \text{Ob}(\mathcal{N}_F)$ and let $f_n : D^n(X) \to Y$, $n \in \mathbb{N}$, be a continuous linear function. Then, for every $k \in \{0\} \cup \mathbb{N}$, $f_n$ admits a continuous linear extension $f_{n,k} : D^{n+2k}(X) \to Y$. If, in addition $Y$ is a Banach space, then $\|f_{n,k}\| = \|f_{n,k}\|$. 

Proof. Let an \( X, Y \in OBN_F \), an \( n \in \mathbb{N} \), and an \( f_n \in N_F(D^n(X), Y) = N_F(D^n(X), Y) \) be given. If \( k = 0 \), there is nothing to prove. Let \( k > 0 \). By Theorem 1 (and its proof), the canonical embedding \( j_n \) is a section having \( D(j_{n-1}) \) for an appropriate retraction, \( D(j_{n-1})j_n = 1_{D^n(X)} \). Then

\[
f_{n,1} = f_n D(j_{n-1}): D^{n+2}(X) \rightarrow Y
\]
is a desired extension when \( k = 1 \). Assume that \( Y \) is a Banach space. By the proof of Corollary 1, for every \( n \in \mathbb{N} \), the morphism

\[
p_{n+2} \equiv D^{n-1}(j_1 D(j_0)): D^{n+2}(X) \rightarrow D^{n+2}(X)
\]
is a continuous linear projection onto the retract \( R(j_n) \equiv D^n(X) \) of \( D^{n+2}(X) \). One readily sees that \( \|D(j_0)\| = 1 \), and thus,

\[
\|p_{n+2}\| = \|D^{n-1}(j_1 D(j_0))\| = \|D^{n+2}(X)\| = 1.
\]

This implies the existence of a desired extension \( f_{n,1}: D^{n+2}(X) \rightarrow Y \) (see also Proposition 6.6.18. of [6]). Thus, in the case \( k = 1 \), the statements are proven. The rest follows by induction on \( k \). \( \square \)

The following extension type theorem is an improvement of Theorem 4 of [17] (see also Lemma 3 (ii) of [17])

**Theorem 7.** Let \( X \) be a normed space, let \( Z \leq X \) be a subspace such that \( \dim Cl(Z) = \dim X \) and \( \dim(X/Cl(Z)) < \dim X \), and let \( Y \) be a Banach space (over the same field) having \( \dim Y < \dim X \). Then every continuous linear function \( f: Z \rightarrow Y \) admits a continuous linear norm-preserving extension \( \bar{f}: X \rightarrow Y \).

**Proof.** Clearly, only the infinite-dimensional case asks for a proof. Let \( \dim X = \kappa \geq \aleph_0 \), and let \( Z \leq X \), \( Y \), and \( f: Z \rightarrow Y \) be given according to the assumptions. By Lemma 1 of [17], there exists a (unique) continuous linear extension \( f': Cl(Z) \rightarrow Y \) of \( f \), and \( \|f'\| = \|f\| \). Firstly, consider the case \( \dim X = \aleph_0 \). Then, \( X \cong (F_0^N, \|\cdot\|) \) and \( \dim Cl(Z) = \aleph_0 \), while \( X/Cl(Z) \) and \( Y \) are finite-dimensional by assumption. By Theorem 5,

\[
\dim X = \dim Cl(Z) < \dim X^{**} = \dim Cl(Z)^{**} = 2^{\aleph_0},
\]
and, by Theorem 2 (ii),

\[
X^{**}/Cl(Z)^{**} \cong (X/Cl(Z))^{**}
\]

(that is, in this case, even isomorphic to \( X/Cl(Z) \)). Denote by

\[
Cl_{X^{**}}(Cl(Z)) \subseteq Cl_{X^{**}}(X) \subseteq X^{**}
\]
the closures (the Banach completions) of \( Cl(Z) \subseteq X \) in \( X^{**} \), and by \( f'' : Cl_{X^{**}}(Cl(Z)) \rightarrow Y \) the continuous (unique, linear) extension of \( f' \). Since the canonical embedding into the second dual space is an isometry, \( \|f''\| = \|f'\| \) holds. One readily sees (see also Lemma 2 (iii) of [17]) that

\[
\dim(Cl_{X^{**}}(Cl(Z))) = \dim(Cl_{X^{**}}(X)).
\]
and, since \( \dim(X/\text{Cl}(Z)) < \aleph_0 \), that
\[
\dim(\text{Cl}_{X^{**}}(X)/\text{Cl}_{X^{**}}(\text{Cl}(Z))) < \aleph_0.
\]
By applying Theorem 5 (iii) to \( \text{Cl}_{X^{**}}(\text{Cl}(Z)) \hookrightarrow \text{Cl}_{X^{**}}(X) \) (they are Banach spaces, hence \( \dim^{**} \)-stable) or Theorem 5 (ii) only, it follows that
\[
\text{Sh}_0(\text{Cl}_{X^{**}}(\text{Cl}(Z))) = \text{Sh}_0(\text{Cl}_{X^{**}}(X)).
\]
By means of that fact, we shall prove that \( f'' \) admits a continuous linear norm-preserving extension to \( \text{Cl}_{X^{**}}(X) \). Then a desired extension \( \bar{f} : X \to Y \) of \( f \) may be the restriction to \( X \) of that extension. Instead of proving this separately (in the special countably dimensional case), let us consider the general uncountably dimensional case, i.e., \( \dim X = \kappa > \aleph_0 \), and thus, \( \dim \text{Cl}(Z) = \kappa \), \( \dim(X/\text{Cl}(Z)) < \kappa \) and \( \dim Y < \kappa \).

As in the countably dimensional case before, \( f : Z \to Y \) admits a continuous linear norm-preserving extension \( f'' : \text{Cl}_{X^{**}}(Z) \to Y \). By Theorem 5,
\[
\dim X^{**} = \dim X = \dim \text{Cl}(Z) = \dim Cl(Z)^{**} = \kappa,
\]
and, by Theorem 2 (ii) and Lemma 2 (iii) of [17],
\[
\dim(\text{Cl}_{X^{**}}(X)/\text{Cl}_{X^{**}}(\text{Cl}(Z))) < \kappa.
\]
By Theorem 5 (iii),
\[
\text{Sh}_{\kappa^{-}}(\text{Cl}_{X^{**}}(\text{Cl}(Z))) = \text{Sh}_{\kappa^{-}}(\text{Cl}_{X^{**}}(X)).
\]
So, concerning the quotient shapes result, the uncountably dimensional case covers the countably dimensional case as well. According to Theorem 3 of [17], it suffices to prove that, by the inclusion \( i : \text{Cl}_{X^{**}}(\text{Cl}(Z)) \hookrightarrow \text{Cl}_{X^{**}}(X) \), induced quotient shape morphism
\[
\text{Sh}_{\kappa^{-}}(i) : \text{Cl}_{X^{**}}(\text{Cl}(Z)) \to \text{Cl}_{X^{**}}(X)
\]
is an isomorphism of \( \text{Sh}_{\kappa^{-}}(\text{Cl}_{Z^{**}}) \). To prove this, we may use the appropriate part of the proof of [17], Theorem 4. Indeed, that proof is based on Theorems 2 and 3 of [17], and the proof of [17], Theorem 2, depends on Lemmata 3 and 4 of [17]. Especially, Lemma 4 (ii) of [17] dictated the restriction to the separable or bidual-like normed spaces in order to keep control over the index sets in the considered quotient expansions. However, that control is nothing else but the \( \dim^{**} \)-stability of the considered normed spaces. The conclusion follows.

**Corollary 6.** Let \( Z \) be a dense subspace of an \( X \in \text{Ob}(\text{N}_F) \) and let \( Y \in \text{Ob}(\text{B}_F) \) having \( \dim Y < \dim \tilde{X} \), where \( \tilde{X} \) denotes the (canonical) Banach completion \( \text{Cl}_{D^2(\tilde{X})}(j_0[X]) \) of \( X \). If \( \dim(D^2(X)/X) < \dim D^2(X) \), then, for every \( n \in \mathbb{N} \), every continuous linear function \( f : Z \to Y \) admits a continuous linear norm-preserving extension \( f_{n-1} : D^2(X) \to Y \).

**Proof.** Firstly, if \( \dim X < \aleph_0 \), then \( Z = X \cong D^n(X) \), and there is nothing to prove. So, let \( \dim X = \infty \). Then, since \( \tilde{X} \) is a Banach
space, \( \dim X \geq 2^{\aleph_0} \) holds. Notice that, by Theorems 3 and 4, \( \dim X = 2^{\aleph_0} > \dim X \) if and only if \( \dim X = \aleph_0 \). Further, since \( Y \) is a Banach space, either \( \dim Y < \aleph_0 \) or \( \dim Y \geq 2^{\aleph_0} \) holds. Therefore, in any case, \( \dim Y < \dim X \) implies \( \dim Y < \dim X \). Let \( n = 1 \), and consider the corresponding diagram (in \( \mathcal{N}_F \)), i.e.,

\[
\begin{array}{c}
Z \hookrightarrow \text{Cl}(Z) = X \cong j_0[X] \hookrightarrow \bar{X} \hookrightarrow D^2(X) \\
f \searrow \\
\downarrow f' \\
\nearrow \quad \nearrow f'' \\
Y
\end{array}
\]

in which suitable extensions \( f' \) and \( f'' \), \( f'|Z = f \) and \( f''|\bar{X} = f' \), exist by Lemma 1 of [17], and then, a desired extension \( f_0 \) exists by Theorem 7. Now, for \( n > 1 \), one applies Theorem 6. \( \square \)

**Remark 4.** Since the finite quotient shape classification of normed spaces reduces to the relations of their algebraic dimensions established by Theorem 5 (similarly to that of all vectorial spaces - [14] Theorem 3, (i) \( \iff \) (v)), in order to get a deeper insight in this matter, one has to consider the higher dimensional quotient shape classifications. It asks, however, for an insight into the “dark area” of the structure of the set of all closed subspaces of an infinite \((\kappa-)\) dimensional normed space all having the (algebraic) dimension of the space and infinite codimension up to a given infinite cardinal less than \( \kappa \). Notice that the essential fact (for the finite quotient shape classification) that all norms on a finite-dimensional space are equivalent does not hold any more in the infinite-dimensional case. Further, the specific properties of the bonding morphisms between the terms having indices in a \( \Lambda^{(n)}_{\kappa} \) (used in the proof of Lemma 3) do not hold in the case of \( \Lambda^{(\kappa')}_{\kappa} \) whenever \( \dim X = \kappa' > \kappa \geq \aleph_0 \).

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O DUALIMA NORMIRANIH PROSTORA I KVOCIENTNIM OBICIMA

Nekoliko svojstava normiranoga dualnog Hom-funktora \( D \) i njegovih iteracija \( D^n \) je ustanovljeno. Primjerice: \( D \) preokreće svako kanonsko smještenje (u drugi dualni prostor) u retrakciju (trećega dualnog prostora na onaj prvi); \( D \) povišuje (algebrsku) dimenziju samo prebrojivo bezkonačno-dimenzionalnim normiranim prostorima; \( D \) ne mienja konačni kvocientni oblikovni tip. Spomoću toga je podpuno riješena
razredba svih normiranih vektorskih prostora po konačnom kvocientnom tipu. Kao primjena, za posljedicu su izvedena dva poučka o proširivanju neprekidnih linearnih funkcija.

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