Weak Nonlocal Extension of Classical Material Models
From Nucleus & Condensed Matter to Earth & Universe Scales

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Abstract

The various mathematical models developed over the years to interpret the behavior of materials and corresponding processes they undergo, were based on observations and experiments made at that time. Classical laws (such as Newton’s for gravity, Hooke’s for elasticity, Navier-Stokes for fluidity, Fick’s/Fourier’s for diffusion/heat transfer, Coulomb’s for electricity, as well as Maxwell’s for electromagnetism and Einstein’s for relativity) formed the basis of current technology and shaping of our civilization. The discovery of new phenomena with the aid of newly developed experimental probes have led to various modifications of these laws across the scale spectrum: from subatomic and elementary particle physics to cosmology and from atomistic and nano/micro to macro/giga scales. The emergence of nanotechnology and the further advancement of space technology are ultimately connected with the design of novel tools for observation and measurements, as well as the development of new methods and approaches for quantification and understanding. The paper first reviews the author’s previously developed weakly nonlocal or gradient models for elasticity, diffusion and plasticity. It then proposes a similar extension for fluids and Maxwell’s equations of electromagnetism. Finally, it suggests a gradient modification of Newton’s law of gravity and examines its implications to some problems of elementary particle physics.

1 Introduction

In a recent chapter in Advances of Applied Mechanics [1], a detailed account is presented of the author’s internal length gradient (ILG) mechanics framework. It is based on the assignment of internal lengths/ILs (associated with the local geometry/topology of material nano/microstructures) as scalar multipliers of extra Laplacian terms that are introduced to account for heterogeneity effects and weak nonlocality. Related background work for this framework can be found in the references quoted therein, as well as in earlier published articles by the author and his coworkers [2-10].

The motivation for the development of the author’s initial continuum mechanics-based ILG framework was the need for describing deformation pattern-forming instabilities that emerge when an externally applied stress reaches a certain threshold. Beyond that threshold, the evolution equations governing the system’s homogeneous response were becoming ill-posed and further analysis was not possible. The method proposed earlier by the author to overcome the difficulty for macroscopic deformation and fracture instabilities, was to introduce higher-order gradients (in the form of Laplacians) in the constitutive equations and corresponding ILs accounting for the heterogeneity of the underlying micro/nano structures. The resulting differential equations eliminate ill-posedness, estimate the width/spacings of deformation bands, dispense with the mesh-size
dependence in finite element calculations, and remove stress/strain singularities at crack tips. A similar approach has been employed by the author for higher-order diffusion and heat conduction theories, as well as for phase transitions by revisiting Van der Waals theory of liquid-vapor interfaces and Cahn-Hilliard theory of spinodal decomposition [11] through the introduction of chemical ILs. In these works, mechanical and chemical ILs were treated separately as phenomenological parameters, depending on the material local configuration and scale of observation. Their calibration and/or estimation was left to numerical and/or laboratory experiments. Moreover, statistical features emerging at sub-macroscopic scales were not considered. A preliminary effort to address these issues has been outlined in [1], and further been elaborated upon herein. In particular, the powerful multiscale technique of Kevrekidis [12] – the equation-free method/EFM – can be utilized for the hierarchical calibration of mechanical ILs. Their experimental estimation, usually inferred from “indirect” measurements of spatio-temporal features (width/spacing/velocity of deformation bands) and related size effects, can be based on “direct” measurements through novel nanoindentation/NI tests by monitoring the local strain gradients at various indentation depths.

The enhancement of the above deterministic ILG considerations to include stochastic effects associated with internal stress fluctuations that manifest as stress drops/strain bursts in micro/nanopillar experiments and popins/popouts in nanoindentation tests, may be pursued along the lines also outlined by in [1]. Corresponding gradient-stochastic models can be derived to capture incipient plasticity and serrated stress-strain graphs, as well as to determine statistical features such as fractal dimensions/FDs and probability density functions/PDFs. This task can be carried out by employing Tsallis q-statistics [13] – based on “nonextensive entropy” (as opposed to Boltzmann-Gibbs/B-G “extensive thermodynamics”) – resulting to q-dependent multifractal spectra and q-dependent PDFs, as well as q-generalization of B-G universal power laws. Novel NI tests may be conducted on multiple specimen sites and at different penetration depths for the determination of q-distributions by recording the observed popins/popouts and comparing them with corresponding determinations from micro/nanopillar serrated stress-strain curves in other Labs. Such a statistical mechanics enhanced ILG framework may also be conveniently employed to consider the Portevin Le Chatelier/PLC and Persistent slip band/PSB plastic instabilities, along with related size effects, as outlined below.

At very small scales, mechanical and chemical effects are often equipresent, and an extended chemomechanical ILG framework is necessary in order to consider higher-order IL couplings, as suggested in [1]. In view of the fact that mechanical and chemical ILs are introduced as scalar multipliers of corresponding Laplacian terms, it turns out that such coupled chemomechanical formulation is appealing and robust. Since in mathematical biology models cells are represented by scalar concentration fields (i.e. in the same way as chemical species), the formulation could be easily adapted for the description of higher-order couplings between mechanical and biochemical ILs. Such an extended ILG mechanics framework, including synergistic effects between mechanical and chemical or biological ILs, can be employed to consider chemomechanical instabilities in LiB anodes and biomechanical instabilities in brain tumors, as also outlined below.

As mentioned above, we conclude this introductory section by summarizing main results of the ILG framework and its potential to be employed for considering a variety of problems of current or emerging interest as follows:

• **Plastic Instabilities and Size Effects**: Recent experiments at micro/nano scales [14] have revealed a strong dependence on specimen size. Ongoing work in several Labs has revealed, in particular, that PLC and PSB instabilities may be suppressed when the ratio of the specimen size over the internal
length is reduced below a certain threshold. The previous deterministic ILG models earlier advanced by the author and collaborators for these instabilities at macro/meso scales can be revisited and evaluated for “small-volumes” and strain localization phenomena observed in NC and UFG polycrystals. New combined gradient-stochastic ILG models for both PLC and PSB instabilities can be employed to capture spatio-temporal periodicity, fractality, and transition to chaos. FDs for the observed deformation bands and PDFs for the recorded serrations in stress-strain curves can be determined through Tsallis q-statistics.

- **Chemomechanical Instabilities & Lithiation Fronts in LiB Anodes:** A deterministic version of our coupled chemomechanical ILG framework can be employed to address chemostress damage instabilities in nanostructured LiB anodes leading to cracking and capacity fade during Li insertion/de-insertion under electrochemical cycling. This is due to the huge local volume expansions (up to 400%) and associated internal stress generation occurring in Si active particles during lithiation [15]. A related issue is to understand the size/stress dependence of lithiation reaction, as well as the propagation of stress-assisted lithiated fronts which controls battery efficiency. The interplay between higher-order mechanical and chemical ILs has not yet been sufficiently considered to address these chemomechanical instabilities in LiBs, despite of their wide use in microelectronics, laptops and electric car technologies.

- **Biomechanical Instabilities & Cancer Growth/Metastasis in Human Brain:** A striking analogy exists between the Walgraef-Aifantis/W-A model [16] of dislocation patterning in PSBs and the Go or Grow/GoG model for glioblastoma cancer cells [17]. Both processes are described by similar reaction-diffusion/R-D type equations for mobile-immobile dislocations in the W-A model (under the action of applied stress) and the motile-immotile cancer cells in the GoG model (under the action of internally generated stress). Such internal stress effects have not been explicitly accounted for in the GoG model, despite of the fact that Murray [18] – the father of modern mathematical biology – had already introduced cell-tractions and corresponding strain gradients (in the form of Laplacians, as in the author’s work; see, for example, the related discussions in [1]) to revisit Turing’s seminal R-D work of morphogenesis. The interplay between higher-order mechanical and biochemical ILs in the GoG model can be studied, and the role of internal stress can thus be evaluated. The results can provide new insight on brain cancer progression and potential therapeutic procedures.

### 2 State-of-the Art: Previous Literature & Current State of Affairs

An extensive bibliography on gradient theories has already been mentioned that can be found in [1]. Specific aspects pertaining to the present review and related work on continuum mechanics models at micro/meso/macro scales are discussed in this section. For the convenience of the reader we present this section in two parts. In the first part we provide background on relatively recent phenomenological strain gradient models that have been developed to capture mechanically-induced plastic instabilities and size effects under the action of applied loads. In the second part we provide a brief account of earlier, more fundamental work on phase transitions which was a direct motivation for the author’s initial ILG deterministic models, as well as for the development of coupled chemomechanical/ biomechanical ILG models to be used for addressing instabilities in LiBs and brain tumors.
2.1 Plastic Instabilities & Size Effects:

The term “material instabilities” and “dislocation patterning” were quoted by the author in the mid 1980s [2-5,19,20] to denote the self-organization of localized strain bands and dislocations in deforming solids. Various gradient dislocation dynamics and gradient plasticity models were generated to deal with dislocation pattern formation and shear band thickness/spacing evolution, as well as for interpreting size effects [21-22]. Soon afterwards, in the beginning of the 1990s [2-5,19,23], the author incorporated the Laplacian of the Hookean stress into the standard constitutive equation of linear elasticity to remove the singularities from dislocation lines and crack tips. Some of this work is reviewed in a number of book chapters [14] by leading authors in the field.

Subsequently, or in parallel to the above developments, other types of gradient models have been developed such as the Fleck-Hutchinson and the Gao-Nix-Huang strain gradient theories, as well as improved gradient theories taking into account surface effects (Gudmundson, K.E. Aifantis/Willis, Polizzotto, Voyiadjis/Bammann et al). As an alternative to the initial W-A phenomenological model for dislocation patterning, a substantial effort has been initiated based on discrete dislocation dynamics (DDD) modeling (Kubin/Ghoniem/Bulatov and coworkers). Due to computational limitations of DDD for obtaining dislocation patterns and motivated by the initial W-A model of [16], alternative dislocation density based methods or continuum dislocation dynamics (CDD) have been pursued (Groma/El Azab/Zaiser and coworkers). Related references connected to the above named authors can be found in the bibliography listed in [21] and [24]. In this connection, it is pointed out that our gradient elasticity model has recently been successfully utilized by N. Ghoniem's group in UCLA to dispense with near-core singularities causing code-malfunctioning in 3D discrete dislocation dynamics simulations [25]. Moreover, our related non-singular strain/stress crack tip solutions have been successfully used by P. Isaksson's group in Uppsala to interpret experimental measurements on crack-tip profiles in micro-heterogeneous materials such as solid foams and bone tissue [26].

With the exception of the author’s preliminary efforts described in [1], all the above works on gradient models for addressing plastic instabilities and size effects do not account for internal stress/structural defect fluctuations and synergistic gradient-stochastic effects. There are no attempts for a hierarchical IL calibration through EFM multiscale simulations and novel NI tests. The same holds for the use of Tsallis q-statistics to determine FPs and PDFs. All these are open interdependent issues that need to be addressed. It is also pointed out that none of the above gradient elasticity/plasticity models incorporate diffusion and chemical reaction effects. An exception can again be found in [1] where higher-order chemomechanical IL couplings are discussed. This issue needs further addressed to consider chemomechanical and biomechanical instabilities as described below.

2.2 Chemomechanical Instabilities in LiBs & Biomechanical Instabilities in Brain

The main reason that the author’s Laplacian-based ILG models can be easily extended to include chemical and biochemical ILs is due to the fact that their motivation stems from his earlier treatment with Serrin [11] on Van der Waals theory of fluid interfaces, which was also the predecessor of Ginzburg-Landau theory of phase transitions and Cahn-Hilliard theory of spinodal decomposition [11]. This is in contrast to the aforementioned Fleck-Hutchinson and related strain gradient models which were motivated by Cosserat-type generalized continuum mechanics theories that do not contain explicitly the Laplacian and, thus, they do not exhibit the corresponding mathematical and physical properties that this operator implies. On the other hand, chemical reactions and phase transformations have traditionally been treated with R-D equations involving the Laplace operator.
The fact that both mechanical and chemical or biochemical ILs can be treated on the same footing through the introduction of the corresponding Laplacians allows for a robust formulation of a chemomechanical and biomechanical IL framework that can be used to consider corresponding instability phenomena in LiB anodes and brain glioblastomas, respectively.

There is a large number of recent articles on LiB capacity fade due to colossal volume changes in anodes (up to 400% for Li-Si based anodes) during lithiation/delithiation [15]. While in some of these works diffusion and coupled deformation-diffusion effects have been accounted for, higher-order strain gradients and corresponding mechanical ILs have not been considered. An exception is the recent article by the author and coworkers in [19] employing strain gradients and mechanical ILs to model size effects in LiB anodes, as well as in [20] employing both mechanical and chemical ILs to model the propagation of lithation fronts. This work can be used as a guide to develop criteria for the most optimum nanocomposite configuration (size/spacing of active Si-nanoparticles) for LiB anodes to prevent cracking and/or accelerate lithiation/delithiation.

Similarly to the case of LiBs, there is an abundance of mathematical models for brain cancer. However, related ILG models accounting for internal stress gradient effects due to tumor growth and cancer cell migration/proliferation are missing. This is also true for the aforementioned GoG [10] phenotypic plasticity model of cancer cell migration and its impact on tumor progression. It was found that low-grade tumor micro-ecology potentially exhibits an emergent Allee effect, i.e. a critical tumor cell density implying tumor growth or control. The precise quantification of this critical tumor cell density could be a relevant prognostic criterion for the tumor fate through biopsy measurements. It was shown that the GoG mechanism explains the fast tumor recurrence time of high-grade brain tumors after resection. These findings can be re-evaluated by transferring the stability analysis results earlier derived for the W-A model (of mobile-immobile dislocations) to the GoG model (of motile-immotile cells), by also considering stochastic heterogeneity and internal stress gradient effects.

3 Mechanics ILG Formulation through Continuum & Statistical Mechanics

ILs in Elasticity/Plasticity & Diffusion/Gradient Dislocation Dynamics: For elastic deformations, the term $\ell_\varepsilon^2 \nabla^2 [\lambda \varepsilon_{mm} \delta_{ij} + 2G \varepsilon_{ij}]$ – where $\ell_\varepsilon$ denotes elastic IL, $\varepsilon_{ij}$ is the elastic strain, and $(\lambda, G)$ are the Lamé constants – is incorporated into classical Hooke’s law. Previous results of the author and his coworkers (see [1] and references quoted therein) show that the resulting ILG model can eliminate stress/strain singularities from dislocation/disclination lines and crack tips, as well as interpret elastic size effects. Similarly, the term $\ell_p^2 \nabla^2 \gamma_p$ – where $\ell_p$ denotes plastic IL and $\gamma_p = \sqrt{2\varepsilon_{ij}^p \varepsilon_{ij}^p}$ is the equivalent plastic strain with $\varepsilon_{ij}^p$ denoting the plastic strain tensor – is introduced in the classical von-Mises yield condition or the flow rule to derive differential equations that remain well-posed into the unstable flow regime. Previous results of the author and his coworkers (e.g. [1] and refs therein) show that the resulting ILG model can determine shear band widths and spacings, as well as interpret plasticity induced size effects in micro-torsion/bending and micro/nano indentation experiments. For elastic deformations at the atomic scale (near dislocation lines in crystals), $\ell_\varepsilon$ relates to the subatomic configuration and electronic state (through DFT calculations), while at the microscale $\ell_\varepsilon$ relates to particle size/spacing (through MD simulations). For plastic deformations at micro/meso scales (deformation bands, dislocation cells), $\ell_p$ relates to dislocation source distance/pileup length/grain size (through DDD simulations). This suggests that our earlier practice of treating the ILs as “fitting” constants needs to be revised and consider them as evolving parameters in the course of deformation. This point of view can be adopted for exploring the IL-dependence on the current state of...
deformation and underlying micro/nanostructural configuration, also in relation to the size of the volume considered.

For diffusion problems, the ILs enter through the additional term \( \ell_d^2 \nabla^2 j_i \), which generalizes the classical Fick’s law (\( \ell_d \) is a diffusional internal length and \( j_i \) denotes the diffusion flux) in a manner similar to the Cahn-Hilliard theory [11] for spinodal decomposition. For collective dislocation phenomena, the IL enters through the extra Laplacian term \( D \nabla^2 \rho \), where \( \rho \) denotes dislocation density and \( D \) is an “effective” diffusion-like transport coefficient. Unlike random diffusion processes however, \( D \) is a strain rate driven parameter. Since the strain rate depends (through Orowan’s equation) linearly to the average dislocation velocity, which, in turn relates to the local stress, the coefficient \( D \) is treated as a stress-dependent parameter that relates to individual dislocation interactions. It is noted, in this connection, that the pioneering W-A model for dislocation patterning which is based on such type of \( D \nabla^2 \rho \) terms for the mobile and immobile dislocation densities was initially criticized for the phenomenological origin of these Laplacian terms. However, recent work on continuum dislocation density based dynamics (CDD) – in contrast to discrete dislocation based dynamics (DDD) simulations which were unable to produce dislocation patterns – generate such type of Laplacian terms which are necessary for dislocation pattern formation interpretations.

\textit{Stochasticity and Tsallis q-Statistics:} The enhancement of the above discussed ILG deterministic models through the incorporation of stochastic terms is necessary in order to account for the heterogeneity and fluctuations of internal stresses, as well as deformation-induced random micro/nanostructures. The resulting combined gradient-stochastic models can capture the observed behavior at micro/nano scales, including size dependent serrated stress-strain graphs and intermittent plasticity phenomena. Some initial results along this direction have recently been reported by the author and his coworkers [30] by resorting to empirical Weibull distribution functions, as also reviewed in [1]. This approach can be adopted to describe existing experimental data on stress drops/strain jumps routinely observed in micro tension/compression and nanoindentation laboratory tests. An additional issue that can be explored here is to employ time-dependent probability distributions guided by our earlier [31] and most recent [32] work based on the formalism of stochastic differential equations.

A convenient way to consider the competition between deterministic gradient and random effects is to introduce (in analogy to Wiener processes in statistical mechanics) an additive stochastic term of the form \( h(\gamma)g(x); \left\langle g(x) g(x') \right\rangle = l_{\text{corr}} \delta(x-x') \) – with \( l_{\text{corr}} \) denoting a correlation length, and \( \delta \) being the usual Dirac delta function – into the gradient expression of the flow stress. This is not an arbitrary assumption but emerges generically if one aims at a description above the scale of the discrete substructure which defines the correlation length – i.e. within a continuum model. The delta function then simply emerges because the individual volume elements of the continuum theory are effectively uncorrelated. The function \( h(\gamma) \) also covers the limiting case where only the material parameters fluctuate while the evolution is deterministic (e.g. in the case of flow stress fluctuations due to fluctuating grain orientation or in the presence of a chemical environment).

Standard deterministic ILG models cannot provide any information on measured statistical aspects of plastic deformation, such as fractal dimensions for deformation patterns; power-law exponents for dislocation avalanches [33]; and strain bursts recorded during nanoindentation [34] or micro/nanopillar compression tests [14,30]. When differential equations cannot be invented to
interpret experimental data and simulations, system characterization is left to statistical analyses for establishing fractality and universal power-laws. In many cases, however, the usual power-laws based on Boltzmann-Gibbs statistics exclude the regime of low intensity-high probability events. Tsallis q-statistics [13] based on nonextensive entropy thermodynamics remove this difficulty and can be employed here to analyze intermittent plasticity and deformation patterned images obtained experimentally. This information can also allow the construction of appropriate PDFs to be used in the aforementioned combined gradient-stochastic models. Tsallis nonextensive (non-additive) q-entropy reads $S_q = k \left(1 - \sum_i p_i^q\right)/(q-1)$ and by letting $q \to 1$ recovers the familiar Boltzmann-Gibbs extensive entropy. Corresponding q-distribution functions (q-Gaussian, q-exponential, q-Weibull) functions are obtained, which for $q \to 1$ reduce to their standard counterparts.

4 ILG Applications: Mechanics, ChemoMechanics, and BioChemoMechanics

In this section we discuss applications of the ILG framework to describe deformation instabilities and intermittent plasticity phenomena, as well as chemomechanical instabilities in lithium-ion battery anodes and tumor glioblastomas.

4.1 Mechanical Deformation Instabilities & Intermittent Plasticity

In this subsection we briefly discuss earlier developed ILG deformation models that were used to capture two types of propagating and stationary instabilities in metallic specimens under monotonic or cyclic applied loads. As the specimen size decreases these instabilities may be suppressed or manifest in a more complex non-deterministic manner when stochastic effects appear on equal footing as deterministic ones. This is the case for micro/nano pillar deformation where intermittent plasticity prevails and combined gradient-stochastic models are needed for interpreting size-dependent serrated stress-strain curves.

In order to provide insight on the applicability of ILG framework to capture propagating plastic deformation bands routinely observed in Al-Mg alloy specimens under tension, we list below an initial strain gradient model equation used by the author and coworkers for that purpose. It reads

$$\sigma = h \varepsilon + f(\dot{\varepsilon}) + c \varepsilon_x, \quad (1)$$

where $\sigma$ denotes stress, $\varepsilon$ denotes strain, $h$ a hardening modulus, $f(\dot{\varepsilon})$ a non-monotone function with a branch of negative slope modeling strain rate softening, and the gradient coefficient $c$ (units $[m]^2 \times [sec]$) is a phenomenological constant. For constant stress rate tests ($\sigma = \sigma_0 = h \dot{\varepsilon}_s$) and travelling wave solutions $\dot{\varepsilon} = z(x-Vt)$ – where $x$ denotes the spatial coordinate, $t$ time, and $V$ the band propagation velocity – we obtain the following Lienard type nonlinear equation

$$Z_{\eta\eta} - \mu f'(Z)Z_{\eta} + (Z - Z_s) = 0, \quad (2)$$

where $\eta = \sqrt{h/c}(x-Vt)$, $\mu = V/\sqrt{ch}$. This equation exhibits periodic solutions for propagating strain rate bands traveling through the specimen with constant velocity. It also leads to the staircase stress-strain graphs [3]. This model, which may be considered as a predecessor of later developed more elaborate PLC models, can be revisited for a strain-dependent gradient coefficient $c$ to account for the observed increase of the strain jumps in the course of deformation. In addition, it can be used for applied constant strain-rate conditions to interpret serrated stress-strain curves exhibiting stress drops (instead of strain jumps). Internal fluctuations can be accounted for by introducing a stochastic
term in Eq. (1) for the constitutive expression of the gradient stress. The resulting combined gradient-stochastic model can be evaluated according to the method discussed below to interpret non-deterministic serrations and intermittent plasticity phenomena in micro/nanopillar tests. Statistical characteristics for the serrations and corresponding PDFs can be obtained through Tsallis nonextensive q-entropy procedures. Additional typical experimental results for PLC bands and serrations in NC and UFG polycrystals can be analyzed in a similar way, as in the recent work of the author and coworkers [35].

Stationary Persistent Slip Bands/PSBs: Next, we briefly discuss the model equations describing the periodic ladder structure of stationary PSBs. The initial W-A model for the densities of immobile \((\rho_i)\) and mobile \((\rho_m)\) dislocations reads

\[
\dot{\rho}_i = g(\rho_i) + D_i \nabla^2 \rho_i - h(\rho_i, \rho_m) ; \quad \dot{\rho}_m = D_m \nabla^2 \rho_m + h(\rho_i, \rho_m),
\]

where \((D_i, D_m)\) denote transport stress-dependent gradient coefficients for the two dislocation populations; \(h(\rho_i, \rho_m)\) is an exchange term modeling dislocation reactions of the form \(h(\rho_i, \rho_m) = \beta \rho_i - \gamma \rho_m \rho_i^2\); and \(g(\rho_i)\) is a generation term for immobile dislocations. The coefficients \((\beta, \gamma)\) depend on stress with \(\beta\) measuring the rate of production of mobile dislocations at the expense of immobile, and \(\gamma\) measuring the rate of immobilization of mobile dislocations by immobile dipoles. Since the stress remains constant during PSB formation, all these model coefficients may be assumed as constants. Then, linear stability analysis of Eqs (3) around an equilibrium homogeneous state \((\rho_i^0, \rho_m^0)\) results to a Turing instability for a critical value of the bifurcation parameter

\[
\beta = \beta_c = \left[\sqrt{\alpha + \rho_i^0 \sqrt{\gamma (D_i/D_m)}}\right]^2, \quad \alpha = -g'(\rho_i^0).
\]

The critical wave number \(q_c\) is given by the expression \(q_c = \left[\alpha \gamma \rho_i^0 / D_i D_m\right]^{1/4}\) and the corresponding critical wavelength \(\lambda_c = 2\pi/q_c\) turns out to be of the same order of magnitude as in the experiments.

The above linear stability results were obtained for infinite domains, i.e. for specimen sizes much larger than the internal length. For finite size specimens, corresponding linear and nonlinear stability results were obtained recently by the author and coworkers [16c, 29]. The periodic ladder structure of PSBs is revealed again but it now turns out that below a critical specimen size comparable to the internal length, the PSB instability is suppressed. This size-dependence is consistent with recently obtained experimental results [36]. Stochasticity can also be introduced in Eq. (3) and the implication of a corresponding gradient-stochastic W-A model can be readily investigated.

An additional issue that can be considered is the coupling of the W-A model with an equation for the local stress \(\tau\) related to the macroscopic stress \(\sigma\) through the gradient expression

\[
\tau - l_t^2 \nabla^2 \tau = \sigma,
\]

where now the model parameters in Eq. (3) depend on \(\tau\) rather than \(\sigma\). This could offer an alternative simpler way to arrive at the result of the plateau stress during PSB formation than the method followed earlier by the author and coworkers [31].

Intermittent Plasticity: We conclude this section with some preliminary results on intermittent plastic instabilities by elaborating on a one dimensional combined gradient-stochastic model and Tsallis q-
statistics, as an illustrative example. The combined gradient-stochastic expression for the flow stress $\sigma$ reads

$$
\sigma = \sigma^{\text{ys}} + h\varepsilon - \ell_p^2 \left( \partial^2 \varepsilon / \partial x^2 \right),
$$

(5)

where the yield stress $\sigma^{\text{ys}}$ contains both an average and a fluctuating part given by $\sigma^{\text{ys}} = (1 + \delta)\bar{\sigma}^{\text{ys}}$ - where $\bar{\sigma}^{\text{ys}}$ denotes mean value and $\delta$ follows a Weibull distribution fitted to experimental data. The rest of the quantities have their usual meanings; $\varepsilon$ is the strain; $h$ is a linear hardening modulus, and $\ell_p$ the deterministic internal length.

When this model is incorporated into a cellular automaton (CA) grid, it results to serrated stress-strain curves and power-law interpretations of the corresponding statistical events. Appropriate expressions for the stochastic component of the flow stress can be more fundamentally deduced by employing the formalism of random processes and stochastic differential equations [31,32]. In this connection, reference is made to a class of Tsallis q-distributions that are used in many non-equilibrium physics problems where the usual power-laws based on Boltzmann-Gibbs statistics fail to predict observed behavior. An expression used for interpreting deformation behavior of Mo micropillar compression is Tsallis q-exponential PDF of the form

$$
P(s) = A[1 + (q-1)B_s]^{1/(1-q)};
$$

(A,B) are constants and the q-index is a measure of the system fractality, whereas $s$ denotes the burst size. A power-law relationship between the internal length $\ell_p$ and the entropic index q seems to hold, but this needs to be examined further [1].

Further elaboration along the above lines on combined gradient-stochastic models for the interpretation of size dependent serrated stress-strain graphs by employing Tsallis q-statistics and relate such findings with corresponding image observations on deformation patterns [1], is an open question that needs to be addressed in the future. In this connection, it is pointed out that the needed experimental information on appropriate PDF forms for the stochastic component of the flow stress can be deduced from multiple nanoindentation/NI tests and associated measurements of strain burst events. The PDF of the strain bursts would be related to a corresponding PDF for the flow stress on the assumption that a strain burst of a certain magnitude is the outcome of a number of material points yielding simultaneously. NI measurements at different locations and penetration depths can be conducted to deduce the statistical properties (mean, variance) for the local hardness which, in turn, one can extract information on the form of the stochastic component of the flow stress. From these multiple NI measurements one can extract direct information for both the deterministic ILs and the form of the stochastic contribution to the gradient dependent flow stress.

### 4.2 Chemomechanical Instabilities in LiB Anodes

In this subsection we provide elements of the ILG formulation that can be used to address chemomechanical instabilities in LiB anodes. In particular, we briefly present the basics of the stress-assisted diffusion and coupled ILG chemoelasticity theory that can be employed to consider local volume expansion in lithiated anodes and propagation of lithiation fronts.

**Size Dependent Stress-Assisted Diffusion:** The standard equations that are usually employed to model coupled elasto-diffusion processes are of the form

$$
\sigma_{ij} = \lambda \varepsilon_{mm} \delta_{ij} + 2G \mu \varepsilon_{ij} - \alpha p \delta_{ij}, \quad j = -D \nabla \rho + M \rho \nabla \sigma_{ij}
$$

(6)
for the chemostress \( \sigma_{ij} \) and the diffusive flux \( j \) where the coefficients \((\alpha, M)\) denote chemomechanical coupling constants and \( D \) is the diffusivity. The fields \((\rho, \varepsilon_{ij})\) denote concentration of the diffusing chemical agent and mechanical strain, while \((\lambda, G)\) are the Lamé constants. Since these constitutive equations do not contain higher-order ILs, related chemomechanical size effects may not be captured.

Within our Laplacian-based ILG formulation, it turns out that the above constitutive equations are generalized by replacing \( \sigma_{ij} \) with \( \sigma_{ij} - \ell_{\sigma}^2 \nabla^2 \sigma_{ij} \); \( \varepsilon_{ij} \) with \( \varepsilon_{ij} - \ell_{\varepsilon}^2 \nabla^2 \varepsilon_{ij} \); and \( \rho \) with \( \rho - \ell_{\rho}^2 \nabla^2 \rho \), with \((\ell_{\sigma}, \ell_{\varepsilon}, \ell_{\rho})\) denoting stress/strain and diffusional ILs. Under suitable assumptions, it is possible to uncouple the deformation and chemical fields by first computing a “ground” hydrostatic stress component \( \sigma_h^0 = \sigma_u^0 \) from a conventional or gradient elasticity theory, and then derive the concentration \( \rho \) from a stress-assisted diffusion equation of the form

\[
\partial \rho / \partial t = \left( D + N \sigma_h^0 \right) \nabla^2 \left[ \rho - \ell_{\rho}^2 \nabla^2 \rho \right] - M \nabla \cdot \left( \rho - \ell_{\rho}^2 \nabla^2 \rho \right),
\]

where \( N \) is a new phenomenological constant accounting for the effect of hydrostatic stress on diffusivity. This model with \( \ell_{\rho} = 0 \) has been used extensively to model hydrogen embrittlement and stress corrosion cracking in metals [37]. It can be adapted here, to consider chemomechanical damage and failure in LiB anodes.

**Size Dependent Lithiation Fronts:** To consider the propagation of lithiation fronts one may start with an expression for the free energy density \( \psi \) of the form

\[
\psi (\varepsilon, \nabla e, \rho, \nabla \rho) = f (\rho) + \frac{1}{2} \kappa \nabla \rho \cdot \nabla \rho + \frac{1}{2} \varepsilon : \varepsilon + \frac{1}{2} \varepsilon \cdot \varepsilon + \frac{1}{2} \varepsilon \cdot \varepsilon,
\]

where \((\kappa, c)\) are respectively chemical and mechanical gradient coefficients, \( \varepsilon \) denotes elasticity tensor, \( e \) is the strain tensor and \( e \) its hydrostatic part, while \( \rho \) denotes concentration as before. In this case, both chemical ILs (through \( \kappa \)) and mechanical ILs (through \( c \)) enter into the formulation. Minimization of a corresponding energy functional yields field equations (and associated boundary conditions) for the local stress/strain and concentration of Li species, including the synergistic effect or interplay between higher-order mechanical and chemical ILs. The resulting coupled chemoelasticity governing equations for the stress and chemical species read

\[
\sigma = 2G\varepsilon + \lambda (\text{tr} \varepsilon) \mathbf{1} - \ell_{\sigma}^2 \nabla^2 [2G\varepsilon + \lambda (\text{tr} \varepsilon) \mathbf{1}] - (2G + 3\lambda) M_o \rho \mathbf{1},
\]

\[
\mu = \mu^0 + RT \left[ \ln \left( \frac{\rho}{1-\rho} \right) + \alpha (1-2\rho) \right] - \kappa \nabla^2 \rho - \Omega_{ij} \sigma_{ij},
\]

where \((\lambda, G)\) are the Lamé constants, \( R \) is the universal gas constant, \( T \) is the absolute temperature, \( \mu^0 \) a reference value of the chemical potential, \( \sigma_h = \text{tr} \sigma / 3 \) the hydrostatic stress, and \( \Omega_{ij} = 3M_o / e_{\text{max}} \) is the partial molar volume of the diffusing species.
4.3 Glioblastoma Instabilities in Brain

In this final third subsection we present some details on the GoG model, along with its mathematical similarities to the W-A model, and outline the potential new results to be expected from this comparison. Recent evidence in glioblastoma shows that one-size-fits-all vaso-modulatory interventions usually fail because control of glioma invasion characteristics, such as tumor front speed and infiltration width, vary widely and may require more personalized therapeutic interventions, in contrast to existing GoG models, which assume that all glioma cells have an identical GoG mechanism. In reality, each cell may have an idiosyncratic migration and proliferation regulation due to internal stress dependence and associated intrinsic heterogeneity. The relevant question is “how can we model and analyze the impact of such internal stress dependence and intrinsic heterogeneity of a tumor cell population, where migration and proliferation are regulated by the GoG mechanism”.

This question can be addressed by incorporating internal stress effects in the D terms of the GoG model in analogy to the W-A model discussed earlier for structural defects. The GoG model as formulated by Hatzikirou and coworkers \[17\] reads

\[
\begin{aligned}
\frac{\partial \rho_m}{\partial t} & = D_m \Delta \rho_m + E(\rho_m, \rho_i) + g(\rho_i), \\
\frac{\partial \rho_i}{\partial t} & = D_i \Delta \rho_i - E(\rho_m, \rho_i),
\end{aligned}
\]

(11)

where the \((\rho_m, \rho_i)\) denote respectively the motile and immotile glioma cell densities, with \((D_m, D_i)\) being the corresponding diffusion coefficients. The term \(E(\rho_m, \rho_i)\) signifies the switching between the two different phenotypes. Finally, the function \(g(\rho_i)\) denotes the cell proliferation of the immotile population. The phenomenological resemblance of the GoG model for motile-immotile cells to the W-A model for mobile-immobile dislocations is striking. The results obtained from the earlier study of the W-A model to consider heterogeneity, stochasticity and local stress dependence can be used to improve predictions of the GoG model. It is expected that these predictive results can enable to understand the impact of intratumoral heterogeneity in glioma progression: in particular, the persistence and size dependence of the Allee effect under different heterogeneity and internal stress distributions, as well as the role of the pertinent spatio-temporal instabilities on potential therapeutic failures.

5 ILG and Rheology: Newtonian and Complex Fluids

In this section we suggest possibilities for a gradient enhancement of constitutive equations used in fluid mechanics and rheology. In this connection, it is pointed out that following the author’s work on gradient theory, a number of such generalizations have been proposed in these communities. For Newtonian fluids, such generalizations have been proposed by Silber and coworkers \[38\], as well as in more rigor and detail by Eliot and Gurtin \[39\]. For complex fluids, such generalizations can be found in the pioneering articles by Olmsted and coworkers (e.g. \[40\] and references quoted therein), as well as in the enlightening review by Cates and Fielding \[41\]; see also an earlier one by Dhont and Briels.

In the spirit of the ILG formulation such type generalizations can be readily deduced by replacing the local fields for the fluid density \(\rho\), stretching tensor \(D = 1/2[\text{grad } \mathbf{v} + (\text{grad } \mathbf{v})^T]\), and vorticity tensor \(W = 1/2[\text{grad } \mathbf{v} - (\text{grad } \mathbf{v})^T]\) with their gradient-dependent counterparts \(\rho - \ell_\rho \nabla^2 \rho\), \(D - \ell_D \nabla^2 D\), \(W - \ell_W \nabla^2 W\).
Another possibility is to include the Laplacian of the viscoelastic stress $\Sigma$ as proposed in the diffusive Johnson-Segalman (DJS) model employed to study shear banding flows of wormlike micelles or polymer solutions.

In such wormlike micellar systems, it is assumed [40] that the total stress $T$ is separated into contributions from the Newtonian solvent and a viscoelastic stress $\Sigma$ from the micelles, so that for creeping flow we have

$$T = -pI + 2\mu D + \Sigma; \text{div}T = 0 \quad (12)$$

with $p$ denoting the pressure, $\mu$ being the solvent’s shear viscosity, and the second equation standing for quasi-static equilibrium. It is further assumed that the viscoelastic stress $\Sigma$ obeys the following evolution equation

$$\dot{\Sigma} + \frac{1}{\tau} \Sigma = \frac{2\mu'}{\tau} + D\nabla^2\Sigma, \quad (13)$$

where $\tau$ denotes relaxation time, $\mu'$ is the micelle polymer-like viscosity and $D$ is a diffusion-like coefficient. The corotational time derivative $\dot{\Sigma}$ may assumed to take various forms depending on the local micelle microstructural configuration. The above model and variants of it has been used extensively to model shear banding in complex fluids. The introduction of the Laplacian is needed to deal with ill-posedness in the negative slope regime of the shear stress – shear strain rate regime, i.e. the nonmonotonicity of the flow curve that also required the introduction of Laplacians in the author’s gradient plasticity theory used to address shear banding in the deformation softening regime [3].

On returning to the topic of an appropriate generalization of the Navier-Stokes (N-S) equations for incompressible fluids, i.e. of the constitutive equation $T = -pI + 2\mu D$, we can propose, in analogy to the gradient elasticity [7], the following gradient model

$$T - \ell_T^2 \nabla^2 T = -pI + 2\mu (D - \ell_D^2 \nabla^2 D), \quad (14)$$

where $\ell_T$ and $\ell_D$ denote internal lengths associated with stress and strain inhomogeneities. On assuming that $\ell_T$ can be neglected and introducing Eq. (14) in the equation of momentum balance $\rho\ddot{v} = \text{div}T$, we obtain the following gradient generalization of the N-S equations

$$\rho\ddot{v} = -\nabla p + \mu(\Delta v - \ell_D^2 \Delta^2 v) \quad , (15)$$

where $\Delta = \nabla^2$ and $\Delta^2 = \nabla^4$ denote the Laplacian and biharmonic operators respectively. It is noted that Eq. (15) is identical to the equation used by Fried and Gurtin [39] to discuss plane Poiseuille liquid flow at small-length scales. A slightly generalized model was also used by the same authors to consider turbulence. The governing differential equations for this model read

$$\rho\ddot{v} = -\nabla p + \mu(1 - \alpha^2 \Delta v)\Delta v + 2\rho \alpha^2 \text{div} D \quad \ddot{v} \quad (16)$$

where the $\alpha$ parameter denotes a statistical correlation length and $\dot{D} = \dot{D} + DW - WD$ denotes the usual corotational Jaumann rate.
Steady-state solutions of Eq. (16) with $\alpha=0$, may be determined from operator split method (or the use of Ru-Aifantis theorem [43]) utilized to eliminate singularities from dislocation lines and crack tips in the theory of gradient elasticity (see also [1]). This same procedure leads to the cancelation of singularities in typical fluid flow calculations involving immersed objects. It turns out, for example, that the resulting gradient Oseen tensor $\mathcal{O}_0^G$, which generalizes its classical counterpart $\mathcal{O}_0$

$$
\mathcal{O}_0 = \frac{1}{8\pi\mu r} \left( \delta_{ij} + \frac{rr_{ji}}{r^2} \right),
$$

(17)

where $r_i$ denotes the position vector and $r$ its magnitude, reads

$$
\mathcal{O}_0^G = \frac{1}{8\pi\mu r} \left[ 1 - 2e^{-\epsilon \ell t} - \frac{2\ell}{r} e^{-\epsilon \ell t} + \frac{2\ell^2}{r^2} (1-e^{-\epsilon \ell t}) \right] \delta_{ij} + \left[ 1 + 2e^{-\epsilon \ell t} + \frac{6\ell}{r} e^{-\epsilon \ell t} - \frac{6\ell^2}{r^2} (1-e^{-\epsilon \ell t}) \right] \frac{rr_{ij}}{r^2},
$$

(18)

Which resembles the exponential regularization of the Green’s tensor in gradient elasticity and the resulting nonsingular gradient expresisons for the stresses and strains in dislocation lines and crack tips. More details can be found in [44] where the authors seemed to be unaware pf analogous developments in gradient elasticity.

6 ILG in Other Disciplines & Scales

In this section we summarize the applicability of the ILG framework to other disciplines and scales ranging from earth scales to quantum scales.

• ILG in Geology: Some initial work on introducing internal lengths and Laplacians of strain has been published by the author and coworkers to model shear banding and related instability phenomena in geomaterials including granular materials, soils, rocks and snow/ice (see, for example, [45-56]). Various types of gradient-dependent constitutive equations for such classes of geomaterials have also been proposed and elaborated upon in detail by many other authors. This was mainly due to the fact that the Laplacian was regularizing unstable behavior in the geomaterial’s softening regime and allowed for the determination of shear band thickness and convergence of corresponding finite element calculations. The popularization of the approach in the geomechanics community is mainly due to the follow-up works by Vardoulakis and collaborators for soils, as well as de Borst and collaborators for concrete. These are too many to mention and can be found in the web.

In connection with the above, it is worth noting that the W-A model for dislocation patterning has recently been used by Ord and Hobbs [57] to interpret fracture patterns in frictional, cohesive, granular materials. Their article was one contribution of seventeen to a Theme Issue “Patterns in our planet: applications of multi-scale non-equilibrium thermodynamics to Earth-system science”.

• ILG in Electrodynamics: The inclusion of higher-order gradients in deforming materials under the action of electromagnetic fields has also become very popular in recent years due to emerging applications and design of piezoelectric (induction of electricity due to applied pressure) and flexoelectric (induction of electricity due to strain gradients) components. The number of published articles is prohibitive to mention them here and we only refer to a couple of references by the author and coworkers [58-60], as well as the bibliography listed there for recent related literature on size effects.
In relation to the issue of eliminating singularities and introducing screening effects (e.g. Debye screening) in the electric and magnetic fields, the following gradient modification of Coulomb’s law of electrostatics has been proposed (see, for example, [61] where a fractional generalization of Debye screening is also discussed)

\[ \Delta \Phi(\mathbf{r}) - \frac{1}{r_0^2} \Phi(\mathbf{r}) = -\frac{1}{\varepsilon_0} \rho(\mathbf{r}), \]  

where \( \Phi \) is the electrostatic potential \( [E(\mathbf{r}) = -\nabla \Phi(\mathbf{r}); E(\mathbf{r}) \) is the electric field], \( \rho(\mathbf{r}) \) denotes the charge density, \( \varepsilon_0 \) is the vacuum permittivity, \( r_0 \) is the Debye screening distance, and \( \mathbf{r} \) denotes the spatial radial coordinate. The classical Coulomb’s potential for a point charge of strength \( Q \) has the form \( \Phi(\mathbf{r}) = Q / 4\pi\varepsilon_0 r \), while its Debye screened counterpart obtained from Eq. (19), which is identical in form to the reduced Ru-Aifantis equation for gradient elasticity [43], reads

\[ \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r} e^{-r/r_0}. \]  

(20)

In concluding this discussion on gradient electrodynamics, reference is made to an author’s unpublished work where MacCullagh’s 1850 proposal for an interesting formal analogy between elasticity and electromagnetism [62] is extended to include rotational gradients of the elastic aether. On assuming that the aether behaves as an elastic medium with its stress \( T \) depending linearly on rotations \( \omega \) (instead of strains), we have

\[ T = 2k\omega, \quad \omega = 1/2[\nabla \mathbf{u} - (\nabla \mathbf{u})^T], \quad \text{div} T = \rho \mathbf{u}, \]  

(21)

where \( \mathbf{u} \) denotes displacement, \( \rho \) density and \( k \) an elastic constant. These lead to the equation \( k \text{curl curl} \mathbf{u} + \rho \mathbf{u} = 0 \) and by setting the terms \( k \text{curl} \mathbf{u} \) and \( \rho \mathbf{u} \) to be proportional to the electric (\( \mathbf{E} \)) and magnetic (\( \mathbf{B} \)) fields respectively, we arrive at Maxwell’s equations

\[ \frac{\partial \mathbf{B}}{\partial t} + \text{curl} \mathbf{E} = 0; \quad \text{div} \mathbf{B} = 0 \quad \text{and} \quad \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\mu_0\varepsilon_0} \text{curl} \mathbf{B} = 0; \quad \text{div} \mathbf{E} = 0, \]  

(22)

where the identity \( \text{div curl} \mathbf{u} \) was used and a suitable identification of parameters \( (\beta = k\varepsilon_0, \mu_0 = \rho/\beta) \) is made. By using the above procedure as above but replacing Eq. (21) for the aether’s elastic stress with its gradient counterpart

\[ T = 2k(\omega - \ell^2 \nabla^2 \omega), \]  

(23)

we arrive at the following generalization of Maxwell’s equations

\[ \frac{\partial \mathbf{B}}{\partial t} + \text{curl} (1 - \ell^2 \nabla^2 \mathbf{E}) = 0; \quad \text{div} \mathbf{B} = 0 \]  

\[ \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\mu_0\varepsilon_0} \text{curl} \mathbf{B} = 0; \quad \text{div} \mathbf{E} = 0 \]  

(24)

It is noted that for electrostatics under the assumption that the electric field \( \mathbf{E} \) is proportional to a potential gradient \( \nabla \Phi \), Podolsky’s non-quantum electromagnetics [Podolsky B. A generalized electrodynamics Part I—Non-quantum. Phys. Rev. (1942) 62:68-71; Podolsky B, Schwed P. Review of a generalized electrodynamics. Rev. Mod. Phys. (1948) 20:40-50] equation \( \Delta(1 - \ell^2 \Delta \Phi) = 0 \) is obtained.
ILG in Atomistics and Quantum Mechanics: We conclude this section on applicability of the ILG framework to various disciplines and scales by focusing on two specific topics: A possible gradient generalization of the microscopic or molecular dynamics (MD) stress, and an analogous generalization of the quantum mechanical (QM) stress. In this connection, it is noted that the following expressions were proposed for these stresses:

\[
\langle \sigma \rangle = \frac{1}{V} \left[ \frac{1}{2} \sum_i \sum_j f_{ij} \otimes (r_i - r_j) \right] - \left( \sum_i m_i \mathbf{v}_i \otimes \mathbf{v}_i \right) \]  

in [63], and

\[
\sigma_{\text{eff}} = -\frac{1}{V} \sum_i \frac{p_{\alpha i} \beta}{m_i} \sum_{ij} \left( \frac{r_i - r_j}{|r_i - r_j|} \right) U_{ij} (|r_i - r_j|) \]  

in [64], where the various symbols have their usual meaning. The striking formal similarity between these two expressions and their resemblance with the virial stress and other statistical stress measures is noted. However, the problem to connect such discrete “microscopic” stress measures with the continuum “macroscopic” measure of Cauchy stress in a “seamless” way is a challenging issue. A gradient generalization of the force fields \( f_{ij} \) in Eq. (25) and the interaction potential \( U_{ij} \) in Eq. (26) may be appropriate which, among other things, could naturally introduce screening distances and eliminate associated singularities.

The effect of strain \( \varepsilon \) on the electronic structure has been described [65] through the equations

\[
(E_c - \frac{\hbar^2}{2m} \nabla^2) \psi(r) + a_{\varepsilon} \text{tr}(\varepsilon) \psi(r) = E \psi(r) \quad & \varepsilon = \varepsilon^0; \quad \sigma = [C] \varepsilon; \quad \text{div} \sigma = 0, \]  

where \( \psi(r) \) denotes the wavefunction, \( C \) is the Hookean elasticity matrix, \( a_{\varepsilon} \) the so-called deformation potential constant, and the rest of the symbols have their usual quantum mechanical meaning [65]. This is an uncoupled framework where strain can affect the electronic state but not vice-versa. A generalization to account for inverse effect on strain due to changes in the quantum field through the wavefunction \( \psi(r) \), has already proposed as follows [66]:

\[
\left( E_c - \frac{\hbar^2}{2m} \nabla^2 \right) \psi(r) + a_{\varepsilon} \text{tr}(\varepsilon) \psi(r) = E \psi(r) \quad & \varepsilon = \varepsilon^0 - \frac{a_{\varepsilon}}{3K} |\psi(r)|^2 \mathbf{1}; \quad \sigma = [C] \varepsilon; \quad \text{div} \sigma = 0, \]  

where \( K \) is the isotropic bulk elastic modulus. A possible gradient modification is then to replace \( \varepsilon \) with its gradient counterpart \( \varepsilon - \ell_{\varepsilon}^2 \nabla^2 \varepsilon \), and this formal generalization may be of interest to further explore.

7 ILG Modification of Newton’s Gradient Gravitation

In this final section we venture a gradient generalization of Newton’s Law which allows for the corresponding gravitational force to attain values larger than the electromagnetic force and even reach the levels of the nuclear and strong force which keeps matter together. The proposed modification is analogous to that earlier adopted by the author for gradient elasticity through the introduction of a Laplacian and a corresponding internal length. But the question of exploring the
consequences of such generalization to gravitation emerged during initial discussions with my daughter Katerina Aifantis during my visit in February 2019 to Florida and follow-up discussions with my classmate and academician Constantinos Vayenas during his visit in June 2019 to Thessaloniki. The initial numerical calculations reported herein started with the help of Katerina’s students in Florida and completed with the multifaceted assistance of my student Kostas Parisis in Thessaloniki.

We begin with the following nonlocal generalization of the gravitational force $f$ in its component form ($f_i$):

$$f_i(r) = \int G_i(r-r') F_j(r') d^3 r',$$

(29)

where $G_i(r-r')$ is a nonlocal interaction kernel and $F_j$ is the classical Newton’s force. By Fourier transforming Eq. (29), Taylor series expanding up to the second order term, and inverting results to the differential equation

$$f = (1 - \ell^2 \nabla^2) F \ ; \ \ell^2 = \sgn(\alpha) |\alpha|, \ \alpha = \left| \frac{d^2 G_i(0)}{d k^2} \right|,$$

(30)

where $k = |k|$ denotes wave vector and $\ell$ is an internal length. Such a formal derivation can be also established by considering the two point masses $M_0$ and $M$ in the classical Newton’s Law, as being distributed and bounded by spheres of finite radii. By considering, for example, the mass $M_0$ ( $M_0 = \sum m_i$ ) being distributed within a sphere of radius $\alpha_0$, summing up the interactions of each point mass $m_i$ located at distance $r_i$ from the center of the sphere with the point mass $M$ and expanding in Taylor series the density $\rho(r_i)$ around $\rho_0 = \rho(0)$ keeping terms up to the second order we obtain the following relationship

$$f = \frac{GMV}{R^2}(\rho_0 + \ell^2 \nabla^2 \rho_0) e_R \ ; \ \ell^2 = \frac{\alpha_0^2}{10},$$

(31)

where $e_R = R / R$ denotes the unit vector along the line connecting the center of $M_0$ with the point mass $M$. On setting $\int \rho_0 dV = M_0$ we then have $f = (1 + \ell^2 \nabla^2) F$ which by inversion leads to $f = (1 - \ell^2 \nabla^2) F = F$. This simplified rather intuitive calculation is similar to that earlier adopted by the author and coworkers (see, for example, [47]) to produce a corresponding gradient-dependent plastic strain.

On assuming a radial dependence of $f$ and $F$ [$f = f(r)e_r \ ; \ F = Fe_r$, $F = A / r^2$ with $A$ denoting Newton’s gravitational constant], we can readily solve the scalar counterpart of $(1 - \ell^2 \nabla^2)f = F$, i.e.

$$f - \ell^2 \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} - \frac{2 f}{r^2} = \frac{A}{r^2}$$

(32)

by also requiring that $F \rightarrow 0$ as $r \rightarrow \infty$. The result is

$$f = \frac{A}{r^2}[1 + Be^{-\ell}(1 + \frac{r}{\ell})],$$

(33)
where \( B \) is a new parameter to evaluate in connection with experiments. It is noted that the above expression of Eq. (33) reduces to Newton’s classical force \( F_N = A / r^2 \) as \( r \to \infty \) and to the expression \( F_{AB} = AB / r^2 \) as \( r \to 0 \). By adjusting the value of the new parameter \( B \) (\( B \gg 1 \)) we can attain values of the nuclear and strong force.

The internal length parameter can be identified with the de Broglie relativistic length, the Compton length, the Planck length or the Schwarzschild distance, according to the configuration at hand. For neutrino and proton related problems we then have

- **De Broglie**: \( \ell = \hbar / \gamma m_0 c \),
- **Compton**: \( \ell = \hbar / m_p c \),
- **Planck**: \( \ell = \sqrt{\hbar G / c^3} \),
- **Schwarzschild**: \( \ell = 2G m_0 / c^2 \).

where \( \hbar \) denotes the Planck constant, \( c \) is the speed of light; and \((m_0, m_p, m_{BH})\) denote rest masses for neutrino, proton, and black hole, respectively; whereas \( G \) is the classical Newton’s gravitational constant (not to be confused with the same symbol earlier used for the shear modulus), and \( \gamma \) is the Lorentz factor \( (\gamma = 1 / \sqrt{1 - (v/c)^2}) \); with \( v \) denoting particle speed.

On adopting the Vayenas and coworkers Rotating Neutrino model (RNM) for the nucleus \([67,68]\) we now utilize the above expression for the gravitational force given by Eq. (33), in conjunction with the centrifugal force \( F_C = \gamma m_0 c^2 / r \), where \( r \) denotes the radius of the nucleus modeled by the three rotating neutrinos whose total relativistic mass is \( m_N = 3 \gamma m_0 \). An estimate of \( \gamma \) can be obtained by equating the proton energy \( m_p c^2 \) with the relativistic neutrino mass. This gives the value of \( \gamma = m_p / m_N \) which, according to experimental measurements for \( m_p \) and \( m_0 \) turns out to be equal to \( 7.818 \times 10^9 \).

Having such a value of \( \gamma \) available, we can make effective use of the aforementioned equality between gravitational and centrifugal forces in Vayenas’ RNM to deduce the relationship

\[
f = \frac{A}{\sqrt{3}r^2} \left[ 1 + B e^{-rt} (1 + \frac{r}{\ell}) \right] = F_C = \frac{\gamma m_0 c^2}{r}
\]

with the constant \( A \) being set equal to \( A = G m_0^2 / \gamma^2 \) (\( G \) is the classical gravitational constant) to account for relativistic effects during the interaction of each pair of neutrinos in the assumed RNM configuration. The above relationship (for \( \ell \) identified with de Broglie’s relativistic length \( \ell = \hbar / \gamma m_0 c = 6.32 \times 10^{-16} m \)) gives the following value for the coefficient \( B \)

\[
B = \frac{\sqrt{5} \hbar c}{2 G m_0^2 \gamma^2} = 3.58 \times 10^{39}
\]

and a corresponding value of \( f \)

\[
f = 7.059 \times 10^4 \text{ N}
\]
i.e. the value of the strong force obtained by Vayenas [67,68] for his RNM by using an entirely different discrete particle method to modify Newton’s Law; i.e. \( F_v = \frac{G m_i \gamma^6}{\sqrt{3}r^2} = \frac{hc}{r^2}, \) with \( \gamma \) being determined from the relationship \( \gamma = \frac{3^{1/2} m_{\gamma}^{1/3} m_i^{1/3}}{7.38 \times 10^9} \) \( (m_{\gamma} = \sqrt{hc/G}) \), which also gives \( F_v = 7.059 \times 10^4 \) N for \( r = h / \gamma m_i c = 6.69 \times 10^{-16} \) m.

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