A VERY GENERAL QUARTIC DOUBLE FOURFOLD OR FIVEFOLD IS NOT STABLY RATIONAL

ARNAUD BEAUVILLE

1. Introduction

A projective variety $X$ is stably rational if $X \times \mathbb{P}^m$ is rational for some integer $m$. A stably rational variety is unirational; that the converse does not hold was shown by Artin and Mumford [A-M]. Their example is a double covering $X$ of $\mathbb{P}^3_C$ branched along a quartic symmetroid – a surface defined by the vanishing of a symmetric 4-by-4 determinant of linear forms. They prove that the torsion subgroup of $H^3(X, \mathbb{Z})$ is nonzero, while it is trivial for stably rational varieties.

Unfortunately this method applies only to rather particular varieties, and not to natural families like Fano threefolds, complete intersections, etc. A more powerful approach has been discovered recently by C. Voisin [V]: the existence of torsion in $H^3(X, \mathbb{Z})$ implies the non-triviality of a certain Chow group, a property which behaves better under specialization. She obtained the following beautiful consequence:

Theorem (Voisin). A double cover of $\mathbb{P}^3_C$ branched along a very general quartic surface is not stably rational.

Here “very general” means that the surface lies outside the union of countably many strict subvarieties in the space of quartic surfaces in $\mathbb{P}^3_C$.

The aim of this paper is to extend this result in higher dimension:

Theorem 1. For $n = 4$ or 5, a double cover of $\mathbb{P}^n_C$ branched along a very general quartic hypersurface is not stably rational.

These varieties are easily seen to be unirational (Proposition below); to our knowledge the weaker fact that they are not rational was not previously known.

To prove Theorem 1 we apply Voisin’s method, as extended in [C-P]. Here is the statement that we will use ([C-P], Théorème 1.12, plus [V], Remark 1.3):

Proposition 1. Let $B$ be a smooth curve, $o$ a point of $B$, $f : \mathcal{X} \to B$ a flat, projective morphism, such that the generic fiber of $f$ is smooth, and that the fiber $X := \mathcal{X}_o$ is integral and admits a desingularization $\sigma : \tilde{X} \to X$ with the following properties:

a) The torsion subgroup of $H^3(\tilde{X}, \mathbb{Z})$ is non trivial;

b) The fiber of $\sigma$ over any point $x \in X$ is a smooth rational variety over the residual field $\kappa(x)$.

Then for a very general point $b \in B$, the fiber $\mathcal{X}_b$ is not stably rational.

We stress that condition b) must hold for all points of the scheme $X$, not only for closed points. Actually Proposition gives the (possibly) stronger result that $\mathcal{X}_b$ is not retract rational. Thus in Theorem 1 one can replace “stably rational” by “retract rational”.

Voisin’s theorem follows at once from the Proposition by taking for $X$ the Artin-Mumford example. To treat the higher-dimensional case, we simply take the obvious generalization of that example,
namely a double covering \( X \to \mathbb{P}^n \) branched along a quartic symmetroid. The variety \( X \) is singular, but admits for \( n = 4 \) or \( 5 \) a simple desingularization\(^1\) which satisfies condition \( b \) of Proposition 1 (Proposition 3). To check condition \( a \), we view \( \mathbb{P}^n \) as a linear system \( L \) of quadrics in \( \mathbb{P}^3 \); then the smooth part \( X_{sm} \) of \( X \) parametrizes the quadrics of \( L \) of rank \( \geq 3 \) together with the choice of a system of generatrices. The generatrices in each system are parametrized by \( \mathbb{P}^1 \), so we get a \( \mathbb{P}^1 \)-bundle over \( X_{sm} \); this provides a 2-torsion class in \( H^3(\mathbb{X}_{sm}, \mathbb{Z}) \). We will prove that this class comes from a nontrivial torsion class in \( H^3(X, \mathbb{Z}) \) (Proposition 3), hence the result.

2. LINEAR SYSTEM OF QUADRICS

2.1. Let \( Q \) be the linear system of quadrics in \( \mathbb{P}^3_C \). We denote by \( Q_i \subset Q \) the subvariety of quadrics of rank \( \leq i \). We recall some basic properties of these varieties (see for instance [Va]):

- We have \( Q \cong \mathbb{P}^9 \), \( Q_3 \) is a quartic hypersurface in \( Q \), \( \dim Q_2 = 6 \), \( \dim Q_1 = 3 \).
- The singular locus of \( Q_i \) is \( Q_{i-1} \).
- The tangent cone \( TC_q(Q_3) \) at a point \( q \) of \( Q_2 \setminus Q_1 \) is a rank 3 quadric in \( T_q(Q) \).

2.2. For \( n = 3, 4 \) or \( 5 \), let \( L \) be a \( n \)-dimensional projective subspace of \( Q \). We assume that \( L \) does not meet \( Q_1 \) and is transverse to \( Q_2 - \) this is the case if \( L \) is sufficiently general. We put

\[
\Delta := L \cap Q_3, \quad \Sigma := L \cap Q_2.
\]

Thus \( \Delta \) is a quartic hypersurface in \( L \), with singular locus \( \Sigma \); \( \Sigma \) is smooth, of dimension \( n - 3 \). The tangent cone \( TC_q(\Delta) \) at a point \( q \) of \( \Sigma \) is a rank 3 quadric in \( T_q(L) \).

2.3. Desingularization of \( \Delta \). Let \( b : \tilde{L} \to L \) be the blowing up of \( L \) along \( \Sigma \); let \( E \) be the exceptional divisor and \( \tilde{\Delta} \) the strict transform of \( \Delta \).

**Proposition 2.** \( \tilde{\Delta} \) is smooth, and intersects \( E \) transversally, so that \( C := \tilde{\Delta} \cap E \) is smooth. Locally over \( \Sigma \) for the Zariski topology, the embedding \( C \hookrightarrow E \) is isomorphic to the embedding \( C_0 \times \Sigma \hookrightarrow \mathbb{P}^2 \times \Sigma \), where \( C_0 \) is a smooth conic in \( \mathbb{P}^2 \).

**Proof:** The fibration \( E \to \Sigma \) is the projectivization of the normal bundle \( N(\Sigma/L) = TL_{|\Sigma}/T\Sigma \), while the fibration \( C \to \Sigma \) is the projectivization of the normal cone \( NC(\Sigma/\Delta) = TC(\Delta)_{|\Sigma}/T\Sigma \); observe that at each point \( q \) of \( \Sigma \) \( T_q \Sigma \) is the vertex of the tangent cone \( TC_q(\Delta) \). By 2.2 \( C \) is a \( \mathbb{P}^1 \)-bundle over \( \Sigma \), in particular it is smooth. Since \( C \) is a Cartier divisor in \( \tilde{\Delta} \), this implies that \( \tilde{\Delta} \) is smooth along \( C \), and therefore everywhere.

There is another natural \( \mathbb{P}^1 \)-bundle over \( \Sigma \) : let \( C' \subset \mathbb{P}^3 \times \Sigma \) be the variety of pairs \((x,q)\) with \( x \in \text{Sing}(q) \). The projection \( C' \to \Sigma \) is a \( \mathbb{P}^1 \)-bundle, with fiber \( \text{Sing}(q) \) above \( q \in \Sigma \). It is easy to see that it is locally trivial for the Zariski topology. In fact, writing \( \mathbb{P}^3 = \mathbb{P}(V) \), we have a “universal quadric” \( q_L \in H^0(L, \text{Sym}^2V \otimes O_L(1)) \) over \( L \), or equivalently a symmetric map \( q_L^*: V^* \otimes O_L \to V \otimes O_L(1) \); the cokernel of \( q_L^* \mid \Sigma \) is a rank 2 vector bundle \( K \) on \( \Sigma \), and we have \( C' = \mathbb{P}_\Sigma(K) \). We will now compare the \( \mathbb{P}^1 \)-bundles \( C \) and \( C' \).

The projective tangent cone \( \mathbb{P}TC_q(\Delta) \) to \( \Delta \) at a singular point \( q \) can be viewed as the variety of lines in \( L \) passing through \( q \) and intersecting \( \Delta \) with multiplicity \( \geq 3 \). Let \( r \in L \); we denote by \( \hat{q}, \hat{r} \) the corresponding quadratic forms. The line \( (q,r) \) belongs to \( \mathbb{P}TC_q(\Delta) \) iff \( \det(\hat{q} + t\hat{r}) \) is divisible by \( t^3 \). Since \( \hat{q} \) is zero on \( \text{Sing}(q) \), this is equivalent to \( \det(\hat{r} |_{\text{Sing}(q)}) = 0 \), that is, the quadric \( r \) is tangent to \( \text{Sing}(q) \).

\(^1\)The desingularization becomes more complicated for \( n \geq 6 \), see [5,1].
Similarly, the line \( \langle q, r \rangle \) belongs to \( \mathbb{P}T_q(\Sigma) \) iff all 3-by-3 minors of \( q + t\hat{e} \) are divisible by \( t^2 \); this is equivalent to say that \( r \) contains the line \( \text{Sing}(q) \). Thus we have a canonical identification of the projectivization of the normal cone \( TC_q(\Delta)/T_q(\Sigma) \) with \( \text{Sing}(q) \), mapping a line \( \langle q, r \rangle \) not tangent to \( \Sigma \) to the point of contact of \( r \) with \( \text{Sing}(q) \). This shows that the \( \mathbb{P}^1 \)-bundle \( C \) is isomorphic to \( C' \), hence locally trivial for the Zariski topology.

Finally, put \( N := N(\Sigma/L) \); let \( p \) be the projection \( C \to \Sigma \). The embedding \( i : C \hookrightarrow E = \mathbb{P}_\Sigma(N^*) \) is determined by the line bundle \( M := i^*O_E(1) \) and the surjective homomorphism \( p^*N^* \to M \). The latter gives by adjunction an isomorphism \( N^* \cong p_*M \), so \( i \) is isomorphic to the embedding \( C \hookrightarrow \mathbb{P}_\Sigma(p_*M) \).

Let \( q \in \Sigma \). Replacing \( \Sigma \) by a Zariski open subset containing \( q \) we can assume that \( p \) is the projection \( \mathbb{P}^1 \times \Sigma \to \Sigma \) and that \( M \) is the pull back of \( O_{\mathbb{P}^1}(2) \). Then \( p_*M \cong \mathcal{O}_\Sigma^3 \), and \( i \) is isomorphic to the embedding \( C_0 \times \Sigma \hookrightarrow \mathbb{P}^2 \times \Sigma \) in a Zariski neighborhood of \( q \).

### 3. The Double Covering

3.1. The class \( b^* (\Delta) - 2E \) of \( \Delta \) in \( \text{Pic}(\tilde{L}) \) is divisible by \( 2 \), hence we can form the double covering \( \tilde{X} \to \tilde{L} \) branched along \( \Delta \). It gives a resolution \( \sigma : \tilde{X} \to X \) of \( X \), which is an isomorphism outside \( \Sigma \); the variety \( Q := \sigma^{-1}(\Sigma) \) is a double covering of \( E \) branched along \( C \).

**Proposition 3.** \( \sigma \) induces a smooth quadric fibration \( Q \to \Sigma \), locally trivial for the Zariski topology. In particular, for any \( q \in \Sigma \) the fiber \( \sigma^{-1}(q) \) is a smooth quadric, rational over \( \kappa(q) \).

**Proof:** Let \( q \in \Sigma \). In view of Proposition 2 replacing \( \Sigma \) by a Zariski open subset containing \( q \) we may assume that \( Q \) is a double covering of \( \mathbb{P}^2 \times \Sigma \) branched along \( C_0 \times \Sigma \). Such a double covering is determined by the branch locus \( C_0 \times \Sigma \) and a line bundle \( M \) on \( \mathbb{P}^2 \times \Sigma \) such that \( M^{-2} \cong \text{pr}_1^* O_{\mathbb{P}^2}(2) \). Shrinking again \( \Sigma \) we can assume \( M \cong \text{pr}_1^* O_{\mathbb{P}^2}(1) \); then the covering \( Q \to \mathbb{P}^2 \times \Sigma \) is isomorphic to \( Q_0 \times \Sigma \to \mathbb{P}^2 \times \Sigma \), where \( Q_0 \to \mathbb{P}^2 \) is the double covering of \( \mathbb{P}^2 \) branched along \( C_0 \). Since \( Q_0 \) is a smooth quadric, this implies the Proposition.

This gives us condition b) of Proposition 1; we now check condition a).

**Proposition 4.** The 2-torsion subgroup of \( H^2(\tilde{X}, \mathbb{Z}) \) is nontrivial.

**Proof:** We put \( U := \tilde{X} \setminus Q \). Let \( \mathcal{G} := \mathcal{G}(2, 4) \) denote the Grassmannian of lines in \( \mathbb{P}^3 \). We consider the incidence variety \( I := \{ (\ell, q) \in \mathcal{G} \times (L \setminus \Sigma) \mid \ell \subset q \} \), and the projection \( p : I \to L \setminus \Sigma \).

The fiber \( p^{-1}(q) \) is a disjoint union of two rational curves for \( q \in L \setminus \Delta \), one rational curve for \( q \in \Delta \setminus \Sigma \). Therefore \( p \) factors as a \( \mathbb{P}^1 \)-fibration \( \varphi : I \to U \), followed by the double covering \( U \to L \setminus \Sigma \). We will show first that the Severi-Brauer fibration \( \varphi \) provides a nonzero class in \( H^3(U, \mathbb{Z}) \), then that this class comes from a 2-torsion class in \( H^3(\tilde{X}, \mathbb{Z}) \).

The \( \mathbb{P}^1 \)-bundle \( \varphi \) gives a class \( [\varphi] \) in the 2-torsion subgroup \( \text{Br}_2(U) \) of the Brauer group of \( U \). The exact sequence of étale sheaves \( 0 \to \{ \pm 1 \} \to \mathcal{G}_m \to \mathcal{G}_m \to 0 \) gives an exact sequence
\[
\text{Pic}(U) \to H^2(U, \mathbb{Z}/2) \to \text{Br}_2(U) \to 0
\]
where the first map is the composition of the first Chern class map \( c_1 : \text{Pic}(U) \to H^2(U, \mathbb{Z}) \) and the reduction (mod. 2) \( r : H^2(U, \mathbb{Z}) \to H^2(U, \mathbb{Z}/2) \).

On the other hand the exact sequence \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0 \) for the classical topology gives an exact sequence
\[
H^2(U, \mathbb{Z}) \overset{r}{\longrightarrow} H^2(U, \mathbb{Z}/2) \overset{0}{\longrightarrow} H^3(U, \mathbb{Z})
\]
so that \( \partial \) induces a homomorphism \( \partial : \text{Br}_2(U) \to H^3(U, \mathbb{Z}) \), which is injective if \( c_1 : \text{Pic}(U) \to H^2(U, \mathbb{Z}) \) is surjective. The class \( \partial([\varphi]) \) is the topological Brauer class of the \( \mathbb{P}^1 \)-bundle \( \varphi \), see [N], 1.1.

Suppose first \( n = 3 \). Then \( c_1 \) is surjective. Indeed in the commutative diagram

\[
\begin{array}{ccc}
\text{Pic}(\tilde{X}) & \xrightarrow{c_1} & H^2(\tilde{X}, \mathbb{Z}) \\
\downarrow & & \downarrow \\
\text{Pic}(U) & \xrightarrow{c_1} & H^2(U, \mathbb{Z})
\end{array}
\]

the top horizontal arrow is surjective because \( H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0 \); the restriction map \( H^2(\tilde{X}, \mathbb{Z}) \to H^2(U, \mathbb{Z}) \) is surjective because of the Gysin exact sequence \( H^2(\tilde{X}, \mathbb{Z}) \to H^2(U, \mathbb{Z}) \to H^1(Q, \mathbb{Z}) \), where the last group is zero since \( Q \) is a disjoint union of smooth quadrics. Thus \( \partial : \text{Br}_2(U) \to H^3(U, \mathbb{Z}) \) is injective.

Now if the class of \( \varphi \) in \( \text{Br}(U) \) were zero, the \( \mathbb{P}^1 \)-bundle \( \varphi : I \to U \) would be a projective bundle. But \( I \) is a rational variety, because the projection \( I \to \mathbb{G} \) is birational [B], and we know that \( \tilde{X} \) is not rational [A-M]. We conclude that \( \partial([\varphi]) \neq 0 \) in \( H^3(U, \mathbb{Z}) \).

For \( n = 4 \) or \( 5 \), we choose a general 3-dimensional projective subspace \( L' \) in \( L \), and construct the corresponding subvarieties \( U' \subset U \), and \( I' := \varphi^{-1}(U') \). The class \( \partial([\varphi]) \) in \( H^3(U, \mathbb{Z}) \) restricts to \( \partial([\varphi|_{I'}]) \) in \( H^3(U', \mathbb{Z}) \), which is nonzero by the above; thus it is nonzero, and \( H^3(U, \mathbb{Z}) \) contains nonzero 2-torsion elements for \( 3 \leq n \leq 5 \).

To conclude it suffices to prove the following lemma:

**Lemma.** The torsion subgroups of \( H^3(\tilde{X}, \mathbb{Z}) \) and \( H^3(U, \mathbb{Z}) \) are isomorphic.

**Proof:** We consider the quadric fibration \( f : Q \to \Sigma \). The two system of generatrices of each fiber form a double covering of \( \Sigma \) which is locally trivial for the Zariski topology (Proposition [3]), hence trivial. We choose one of the two systems. In each fiber the generatrices of this system are parametrized by \( \mathbb{P}^1 \), and form a \( \mathbb{P}^1 \)-fibration \( g : G \to \Sigma \). For each point \( x \) of \( Q \) there is a unique generatrix of our system passing through \( x \); this gives again a \( \mathbb{P}^1 \)-fibration \( h : Q \to G \) such that \( g \circ h = f \). By Proposition [5] both fibrations are locally trivial for the Zariski topology, hence are projective bundles.

We claim that we can blow down \( \tilde{X} \) along the fibers of \( h \); more precisely, that there exists a compact complex manifold \( \tilde{X} \) and an embedding \( G \hookrightarrow \tilde{X} \) such that the diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{c} & \tilde{X} \\
\downarrow & & \downarrow h \\
G & \xrightarrow{c} & \tilde{X}
\end{array}
\]

is obtained by blowing up \( G \) in \( \tilde{X} \). According to the Fujiki-Nakano criterion [F-N], it suffices to prove that the restriction of the line bundle \( \mathcal{O}_{\tilde{X}}(Q) \) to a fiber \( \ell := h^{-1}(q) \) of \( h \) has degree \(-1\).

We have \( K_{Q|_{\ell}} \cong K_{Q_{\Delta|_{\ell}}} \cong \mathcal{O}_{\ell}(-2) \). Recall that \( \tilde{X} \) was obtained by first taking the blowing up \( b : \tilde{L} \to L \) along \( \Sigma \), with exceptional divisor \( E \), then by taking the double covering \( d : \tilde{X} \to \tilde{L} \) branched along the surface \( \tilde{\Delta} \in |b^*\Delta - 2E| \). Then

\[
K_{\tilde{L}} \cong b^*\mathcal{O}_L(-n-1)(2E) , \quad K_{\tilde{X}} \cong d^*b^*\mathcal{O}_L(-n+1)(Q) ;
\]

since \( \ell \) is contracted by \( b \circ d \), we find \( K_{\tilde{X}|_{\ell}} \cong \mathcal{O}_{\tilde{X}}(Q)|_{\ell} \). Using the adjunction formula \( K_Q \cong K_{\tilde{X}}(Q)|_Q \), we get \( 2 \deg \mathcal{O}_{\tilde{X}}(Q)|_{\ell} = -2 \), hence our claim.
Now since $G$ has codimension 2 in $X$ we have the Gysin exact sequence

$$0 \to H^3(\tilde{X}, \mathbb{Z}) \to H^3(U, \mathbb{Z}) \to H^3(G, \mathbb{Z})$$,

which implies that the torsion subgroups of $H^3(\tilde{X}, \mathbb{Z})$ and $H^3(U, \mathbb{Z})$ are isomorphic. Since $\tilde{X}$ is obtained by blowing up $X$ along $G$ we have $H^3(\tilde{X}, \mathbb{Z}) \cong H^3(X, \mathbb{Z}) \oplus H^3(G, \mathbb{Z})$, hence the torsion subgroups of $H^3(\tilde{X}, \mathbb{Z})$ and $H^3(X, \mathbb{Z})$ are isomorphic. This achieves the proof of the lemma, and therefore of the Proposition.

Thus our desingularization $\tilde{X} \to X$ satisfies the conditions stated in Proposition 1. Theorem 1 follows by taking for $B$ the space of quartic hypersurfaces in $L = \mathbb{P}^n$, for $\alpha \in B$ the point corresponding to $\Delta$, and for $\mathcal{X}$ the family of double coverings of $\mathbb{P}^n$ branched along those hypersurfaces.

4. Unirationality

The following result is classical for $n = 3$, and the proof extends easily to the general case:

**Proposition 5.** A double covering of $\mathbb{P}^n$ branched along an integral quartic hypersurface is unirational.

**Proof:** Let $\pi : X \to \mathbb{P}^n$ be the double covering, and let $\mathbb{G}$ be the Grassmannian of lines in $\mathbb{P}^n$. Consider the variety $X^* \subset X \times \mathbb{G}$ of pairs $(x, \ell)$ with $\pi(x) \in \ell$. The projection $p : X^* \to X$ is a projective $\mathbb{P}^{n-1}$-bundle, with fiber at $x \in X$ the space of lines passing through $\pi(x)$.

For $(x, \ell)$ general in $X^*$, the curve $E := \pi^{-1}(\ell)$ is a smooth genus 1 curve in $X$ passing through $x$; there is a unique point $f(x, \ell) \in E$ such that the divisors $\pi^*q + f(x, \ell)$, for $q \in \ell$, are linearly equivalent to $3x$. This defines a rational map $f : X^* \dashrightarrow X$.

Let $(y, \ell)$ be a general point of $X^*$. On the genus 1 curve $\pi^{-1}(\ell)$ there are 9 points $x$ such that $3x$ is linearly equivalent to $\pi^*q + y$ for $q \in \ell$, that is, such that $f(x, \ell) = y$. Thus $f$ is dominant, and a general fiber $f^{-1}(y)$ has dimension $n - 1$; in particular we have $p(f^{-1}(y)) \subsetneq X$.

Let $P$ be a general 2-plane in $\mathbb{P}^n$, and let $\tilde{P} := \pi^{-1}(P) \subset X$. We consider the restriction $f_P$ of $f$ to $p^{-1}(\tilde{P})$. We have $p(f_P^{-1}(y)) = \tilde{P} \cap p(f^{-1}(y)) \subsetneq \tilde{P}$. The projection $p : f_P^{-1}(y) \to \tilde{P}$ is generically injective (if $f(x, \ell) = y$, then $\ell = (\pi(x), \pi(y))$). Thus $\dim f_P^{-1}(y) \leq 1$. It follows that $f_P : p^{-1}(\tilde{P}) \dashrightarrow X$ is dominant. But $p^{-1}(\tilde{P})$ is a projective bundle over the rational surface $\tilde{P}$, hence is rational, and $X$ is unirational.

5. Questions

5.1. It might be possible to extend our main result in dimension $n = 6, \ldots, 9$, by taking a general linear system $L \subset \mathbb{Q}$ of dimension $n$. However for $n \geq 6$ this linear system contains rank one quadrics, which produce triple points of $\Delta$. The desingularization becomes much more intricate; we do not know whether the conditions $a)$ and $b)$ of Proposition 1 still hold.

5.2. In [C-P] the authors show that a very general quartic threefold is not stably rational, by applying Proposition 1 to a singular quartic birational to the Artin-Mumford threefold. It would be quite interesting to extend their result in dimension 4 and 5: on one hand the non-rationality of general quartic fourfolds or fivefolds is not known; on the other hand a general quartic fivefold is unirational [C-M]. However trying to adapt their argument in a direct way to get condition $b)$ of Proposition 1 fails at one crucial step, where they use that a conic bundle over a line is rational – a fact which does not extend in higher dimension. It may be that a more elaborate approach leads to the result.

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L A B O R A T O I R E J.-A. D I E U D O N N É, U M R 7 3 5 1 D U C N R S, U N I V E R S I T É D E N I C E, P A R C V A L R O S E, F - 0 6 1 0 8 N I C E C E D E X 2, F R A N C E
E-mail address: arnaud.beauville@unice.fr