On Word and Gómez Graphs and Their Automorphism Groups in the Degree-Diameter Problem

J. Fraser

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Abstract

One of the prominent areas of research in graph theory is the degree-diameter problem, in which we seek to determine how many vertices a graph may have when constrained to a given degree and diameter. Different variants of this problem are obtained by considering further restrictions on the graph, such as whether it is directed, undirected, mixed, vertex-transitive etc. The currently best known extremal constructions are the Gómez graphs, both when considered as directed and undirected. As the Gómez graphs are similar to the Faber-Moore-Chen graphs, we give a natural generalisation of their definition which we call word graphs. We then prove that the Gómez graphs are as large as possible word graphs for given degree and diameter. Further, we provide a test to determine the full automorphism group of a word graph, and apply it to the extremal directed Gómez graphs to settle the previously open question of when the Gómez graphs are also Cayley graphs. Finally, we conclude with a brief list of interesting related open problems.

1 Introduction

This paper aims to address two principle questions in the degree diameter problem (a survey of which can be found in [6]). The first question, posed in [7], concerns when the Gómez graphs are Cayley. To this question we provide a solution in the extremal case, though note that the paper of Gómez [5] defines a broader family of graphs than we address here. The second question we address regards the similarity in the construction methods used in the Faber-Moore-Chen graphs [3,4] and the Gómez graphs, and shows that with reasonable assumptions the Gómez graphs provide an optimal construction.

This paper is divided into material serving four logical purposes. The first section on tau and sigma sequences provides a technical proof necessary in subsequent work whose inclusion at a later stage would interrupt the natural flow of argument. The second section on word graphs contains a discussion of a natural generalisation of Faber-Moore-Chen graphs and Gómez graphs. This section achieves two main goals, firstly an optimality result for the Gómez graphs, and secondly providing the motivation to the argument pursued in the remaining section. Sections 4 through 7 deal with the problem of classifying the full automorphism group of extremal directed Gómez graphs, and then showing the limitation of the technique used for dealing with all Gómez graphs. The final section concludes with a list of relevant questions which remain open.

2 Tau and Sigma Sequences

In this section we define two special sequences, and we aim to count how many times a given initial value may occur in these sequences. We define a \( \tau \)-sequence as an ordered sequence of \( n \) integers \( a_1, a_2, \ldots, a_n \) such that

\[
\begin{align*}
(i) & \quad a_i \geq 0, \\
(ii) & \quad \text{if } a_i > 1 \text{ then } a_{i-1} = a_i - 1, \\
(iii) & \quad \text{there are at most three } i \text{ such that } a_i = 0,
\end{align*}
\]
(iv) if \(a_1a_2\ldots a_n\) is a \(\tau\)-sequence then \(a_na_1a_2a_3\ldots a_{n-1}\) must also be a \(\tau\)-sequence.

We shall use \(\tau(n, \alpha, \beta)\) to indicate the number of \(\tau\)-sequences of length \(n\) such that \(a_1 = \alpha\) and \(a_n = \beta\), and \(\tau(n, \alpha)\) the number of \(\tau\)-sequences of length \(n\) such that \(a_1 = \alpha\).

Informally we may think of a \(\tau\)-sequence of length \(n\) as being a concatenation of either 1, 2 or 3 sequences of ascending integers starting at 0, or a rotation thereof. We now aim to evaluate \(\tau(n, i)\) for each \(0 \leq i < n\).

**Lemma 1.** For \(n > 1\), \(\tau(n, 0, 0) = n - 1\).

**Proof.** We begin with the observation that in any \(\tau\)-sequence, if \(a_k = 0\) and \(a_{k+j} \neq 0\) for some range of \(j\), then we may repeatedly apply (i) and (ii) to show that \(a_{k+j} = j\). In particular, each \(a_{k+j}\) is uniquely defined.

Now suppose that \(a_1a_2\ldots a_n\) is a \(\tau\)-sequence with \(a_1 = a_n = 0\). First suppose that there is no \(j\) such that \(1 < j < n\) and \(a_j = 0\). If this is the case then each \(a_j = j - 1\), and there is exactly one \(\tau\)-sequence with this property.

Now suppose that there is some \(k\) such that \(1 < k < n\) and \(a_k = 0\). By (iii) there is no \(j \notin \{1, k, n\}\) such that \(a_j = 0\). Hence for \(1 < j < k\) we must have \(a_j = j - 1\) and for \(k < j < n\) we must have \(a_{k+j} = j\). Hence there is exactly one \(\tau\)-sequence with \(a_k = 1\) for each possible value of \(k\). This gives rise to \(n - 2\) possible \(\tau\)-sequences.

Hence, in total, we have \(n - 1\) possible \(\tau\)-sequences of length \(n\) with \(a_1 = a_n = 0\), hence \(\tau(n, 0, 0) = n - 1\).

**Lemma 2.** For \(1 < i \leq n\), we have \(\tau(n, 0, n-i) = i - 1\).

**Proof.** Suppose that \(a_1a_2\ldots a_n\) is a \(\tau\)-sequence with \(a_1 = 0\). If \(a_n = \alpha > 0\), then we see that \(a_1a_2\ldots a_{n-1}\) is a \(\tau\)-sequence of length \(n - 1\) if, and only if, \(a_1a_2\ldots a_{n-1}\) is a \(\tau\)-sequence of length \(n - 1\). Hence \(\tau(n, 0, \alpha) = \tau(n - 1, 0, \alpha - 1)\) for all \(\alpha > 1\). We may repeatedly apply this observation to show that \(\tau(n, 0, \alpha) = \tau(n - \alpha, 0, 0)\). Hence we have \(\tau(n, 0, n-i) = \tau(n - (n-i), 0, 0) = \tau(i, 0, 0) = i - 1\).

**Proposition 1.** For \(1 \leq i \leq n\), \(\tau(n, n-i) = \binom{i}{2}(i^2 - i + 2)\).

**Proof.** We proceed by induction on \(i\). We start with the case \(i = 1\). To calculate \(\tau(n, n-1)\), let \(a_1a_2\ldots a_n\) be a \(\tau\)-sequence with \(a_1 = n - 1\). By (iv) we equivalently have \(a_2a_3\ldots a_na_1\) is a \(\tau\)-sequence. Now we may repeatedly apply (i) and (ii) to show \(a_i = i - 1\), and that there is a unique possible \(\tau\)-sequence. Hence \(\tau(n, n-1) = 1\).

Now take \(i = k\), with the hypothesis given for \(i = k - 1\). Let \(a_1a_2\ldots a_n\) be a \(\tau\)-sequence with \(a_1 = (n-k)\). By (ii) we have \(a_2 = 0\) or \(a_2 = (n - (k - 1))\). In the first case, we may rotate to get a \(\tau\) sequence beginning with 0 and ending with \(n-k\). In the second case, we may rotate to get a \(\tau\)-sequence beginning with \(n-(k-1)\). This gives us

\[
\tau(n, n-k) = \tau(n, 0, n-k) + \tau(n, n-(k-1))
= k - 1 + ((k - 1)^2 - (k - 1) + 2)/2
= (k^2 - k + 2)/2.
\]

We now define a \(\sigma\)-sequence as a sequence of \(n = 2k + 1\) integers \(a_1a_2\ldots a_n\) such that

(i) \(a_i \geq 0\),
(ii) if \(a_i = 0\) then \(a_{i+(k+1)} = 1\),
(iii) if \(a_i = 1\) then either \(a_{i-1} = 0\) or \(a_{i+k} = 0\),
(iv) if \(a_i > 1\) then \(a_{i-1} = a_i - 1\),
(v) there are at most three \(i\) such that \(a_i = 0\),
(vi) if \(a_1a_2\ldots a_n\) is a \(\sigma\)-sequence then \(a_na_1a_2a_3\ldots a_{n-1}\) must also be a \(\sigma\)-sequence.
As this definition is not as readily visualisable as that of \( \tau \)-sequences, we give as examples each sequence for \( n \in \{9, 11\} \) without rotations.

\[
\begin{array}{c|c}
 n = 9 & n = 11 \\
012341234 & 01234512345 \\
010121212 & 01012312123 \\
012011231 & 01201212312 \\
001231123 & 01230112341 \\
001011121 & 00123411234 \\
001121101 & 00101211212 \\
011011011 & 00120111231 \\
00121112 & 00112311012 \\
& 00121211201 \\
& 01010112121 \\
& 01101210112 \\
& 00012311123 \\
\end{array}
\]

In our table we have highlighted a pattern made by the 0s and 1s in these sequences which we aim to formalise and prove. The patterns of 0s and 1s are of the following forms

\[
\begin{array}{c|c|c}
01 \ldots 1 & 001 \ldots 11 \ldots & 0001 \ldots 111 \ldots \\
\hline
k-1 & k-1 & k-2 \\
\hline
\end{array}
\]

We shall call these patterns 01-groups. We aim to show that each 0 or 1 in a \( \sigma \)-sequence occurs in a unique 01-group. In the following, suppose that \( a_1a_2 \ldots a_n \) is a \( \sigma \)-sequence with \( a_1 = 0 \) and \( a_n \neq 0 \).

**Lemma 3.** There is some \( 1 \leq \alpha \leq 3 \) such that for \( 1 \leq i \leq \alpha \) we have \( a_i = 0, a_{i+(k+1)} = 1 \) and \( a_{\alpha+1} = 1 \).

**Proof.** We let \( \alpha \) be the largest number such that \( a_i = 0 \) for \( 1 \leq i \leq \alpha \). From (v) we see that \( \alpha \leq 3 \). Hence, combining (i), (iv) and our definition of \( \alpha \), we see that \( 0 \leq a_{\alpha+1} < 2 \) and \( a_{\alpha+1} \neq 0 \), hence we must have \( a_{\alpha+1} = 1 \). Finally, we may apply (ii) and the fact \( a_i = 0 \) to show \( a_{i+(k+1)} = 1 \). \( \square \)

**Corollary 1.** Every 0 in a \( \sigma \)-sequence is in a unique 01-group.

**Proof.** Consider a \( \sigma \)-sequence with \( a_i = 0 \) for some \( i \). Using (vi) we may consider a rotation of this sequence which moves this 0 from \( i \) to 1, and then possibly further rotate \( a_1 \) to \( a_2 \) until \( a_n \neq 0 \). Then we may apply the previous lemma. \( \square \)

**Lemma 4.** Every 1 in a \( \sigma \)-sequence is in a unique 01-group.

**Proof.** Applying (iii), we have two possibilities if \( a_i = 1 \). In the first possibility, \( a_{i-1} = 0 \). In this case, \( a_{i-1+(k+1)} = a_{i+k} = 1 \) by (ii). Hence, the second possibility that \( a_{i+k} = 0 \) is mutually exclusive with the first. In the first possibility, we may use Lemma 3 to find the 01-group from \( a_{i-1} \) which contains \( a_i \), and in the second possibility we may do the same but from \( a_{i+k} \) instead of \( a_{i-1} \). \( \square \)

Let \( \sigma(i, n) \) where \( n = 2k + 1 \) be the number of \( \sigma \)-sequences of length \( n \) with \( a_1 = i \).

**Lemma 5.** \( \sigma(k, n) = 2 \).

**Proof.** If \( a_1 = k \) in a \( \sigma \)-sequence, by (vi) we may consider a rotation such that \( a_k = k \). Repeatedly applying (iv) we may show that for \( i \geq 1 \) we have \( a_i = i \). For \( 2 \leq i \leq k \), we have \( a_i > 1 \), and hence we have a block of \( k-1 \) numbers in our sequence not in a 01-group. Hence, the only possible 01-group in the sequence is \( 01 \ldots 1 \ldots \). As \( a_1 = 1 \), and each occurrence of 1 is in a 01-group, we must have this 01-group in our sequence and no other 01-group may be in this sequence. We may now consider rotating our \( \sigma \)-sequence again so that \( a_1 = 0 \). Now we may apply (iv) to show \( a_i = i \) and \( a_{i+(k+1)} = i \) for all \( 2 \leq i \leq k \), and this is the only \( \sigma \)-sequence containing \( k \) up to rotation. Finally, we may rotate this sequence in two ways to make \( a_1 = k \), hence \( \sigma(k,n) = 2 \). \( \square \)
Lemma 6. \(\sigma(0, n) \geq 3\).

Proof. For \(k \geq 3\) we consider the \(\sigma\)-sequence with \(a_1 = a_2 = a_3 = 0, a_4 = a_{k+2} = a_{k+3} = a_{k+4} = 1\) and all other \(a_i\) filled in using (iv). This sequence may be rotated to give \(a_1 = 0\) in three different ways, hence in this case \(\sigma(0, n) \geq 3\).

For \(k = 2\), we consider the \(\sigma\)-sequences 00111, 01110 and 01212 to see \(\sigma(0, n) \geq 3\). \(\square\)

Lemma 7. \(\sigma(i, n) < \sigma(i - 1, n)\) for \(1 < i \leq k\).

Proof. Consider the map \(\phi\) which takes a \(\sigma\)-sequence \(a_1a_2\ldots a_n\) to \(a_n a_1a_2\ldots a_{n-1}\). If \(a_1 = i\), then \(a_n = i - 1\) by (iv), hence \(\phi\) is an injective map from \(\sigma\)-sequences starting with \(i\) to those starting with \(i - 1\). Hence, to show \(\sigma(i, n) < \sigma(i - 1, n)\) we need only find a \(\sigma\)-sequence with \(a_1 = i - 1\) and \(a_2 \neq i\).

For \(i \leq k - 1\), the sequence \(a_1 = 0, a_2 = a_{k+2} = 1, a_{i+1} = 0, a_{i+2} = a_{i+k+2} = 1\) and all other \(a_j\) satisfying \(a_j = a_{j-1} + 1\) is a \(\sigma\)-sequence with \(a_1 = i - 1\) and \(a_{i+1} = 0 \neq i\). Hence, we can take a rotation of this by (vi) to find a \(\sigma\)-sequence with \(a_1 = i - 1\) and \(a_2 \neq i\).

Now, for \(i = k\), we consider the sequence \(a_1 = 0, a_2 = a_{k+2} = 1, a_{k-1} = 0, a_k = 1\) and \(a_n = 1\), and all other \(a_j\) satisfying \(a_j = a_{j-1} + 1\). We see this is a \(\sigma\)-sequence with \(a_{n-1} = i - 1\) and \(a_n = 1 \neq i\). Hence, again we can take a rotation of this by (vi) to find a \(\sigma\)-sequence with \(a_1 = i - 1\) and \(a_2 \neq i\). \(\square\)

3 Word Graphs

To facilitate our discussion of the Gómez graphs we first introduce the notion of a word graph and word graph families, which form a natural generalisation of the construction of the Gómez and Faber-Moore-Chen graphs (found in [5] and [3] respectively).

To define a word graph, fix some number \(n\), the word length, some set \(\Pi_n \subseteq S_n\), the rules, and some \(m > n\), the alphabet size. We define the word graph \(G_m = (V, E)\) as follows. Fix some arbitrary set \(B\) such that \(|B| = m\), let \(V = \{x_1x_2\ldots x_n|x_i \in B, x_i = x_j \Leftrightarrow i = j\}\), that is the vertices of \(G_m\) are the words of length \(n\) on \(B\) all of whose letters are distinct, and we form the directed adjacencies of \(G_m\) by the following rules

\[
\begin{align*}
\alpha &\colon x_1x_2\ldots x_n \to \begin{cases} x_2x_3\ldots x_ny, & y \in B \setminus \{x_1, x_2, \ldots, x_n\}, \\ x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(n)}, & \pi \in \Pi_n. \end{cases}
\end{align*}
\]

We define the word graph family of \(\Pi_n\) to be \(\{G_{n+1}, G_{n+2}, \ldots\}\) and denote it by \(\text{WG}(\Pi_n)\). For the following, let \(\Pi_n\) be an arbitrary rule set and \(G_m \in \text{WG}(\Pi_n)\).

We will refer to the rules of the form \(x_1x_2\ldots x_n \to x_2x_3\ldots x_ny\) for \(y \not\in \{x_1\}\) as alphabet changing and rules of the form \(x_1x_2\ldots x_n \to x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(n)}\) as alphabet fixing. For a vertex \(v = x_1x_2\ldots x_n\), we shall define \(\alpha\) by \(\alpha(v) = \{x_1, x_2, \ldots, x_n\}\) and refer to \(\alpha(v)\) as the alphabet of \(v\).

Lemma 8. For all \(m \geq 2n\) we have \(\text{Diam}(G_m) \geq n\).

Proof. Letting \(B = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}\) we consider any path from \(u = x_1x_2\ldots x_n\) to \(v = y_1y_2\ldots y_n\). As each rule of \(G_m\) introduces at most one new letter which is not in \(\alpha(u)\), and each letter of \(\alpha(v)\) is not in \(\alpha(u)\), we must have at least \(|\alpha(v)| = n\) rules in a path from \(u\) to \(v\). \(\square\)

Lemma 9. For all \(m \geq 3n\) we have \(\text{Diam}(G_m) \leq 2n\).

Proof. Consider \(u = x_1x_2\ldots x_n, v = y_1y_2\ldots y_n \in V(G_m)\), and let \(\{z_1, z_2, \ldots, z_n\} \subseteq B \setminus (\alpha(u) \cup \alpha(v))\). Letting \(w = z_1z_2\ldots z_n\), we can create a path of length \(n\) from \(u\) to \(w\) by using the alphabet changing rule to append \(z_i\) at the \(i\)th step in the path. This is always possible as \(\alpha(u) \cap \alpha(w) = \emptyset\). We then may form another such path of length \(n\) from \(w\) to \(v\) by the same logic. Concatenating these two paths gives us a path of length \(2n\) from \(u\) to \(v\). \(\square\)

Lemma 10. For all \(m \geq 4n\) we have \(\text{Diam}(G_m) = \text{Diam}(G_{4n})\).

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Proof. Take arbitrary $u, v \in G_m$. Lemma 9 tells us that $d(u, v) \leq 2n$, hence if we consider a shortest path connecting $u$ and $v$ we know it is of length at most $2n$. On such a path, whenever we encounter alphabet changing rules, denote by $z_i$ the element introduced by the $i^{th}$ alphabet changing rule. Now let $B' = \alpha(u) \cup \alpha(v) \cup \{z_i\}$, note that we have $|B'| \leq 4n$. Hence, we see that the shortest path connecting $u$ and $v$ is in the subgraph $H$ induced by the vertices $\{x_1 x_2 \ldots x_n | x_i \in B'\} \subseteq V(G_m)$, which is trivially isomorphic to a subgraph of $G_{4n}$. As $G_{4n}$ is also a subgraph of $G_m$, the result immediately follows. \(\square\)

From Lemma 10, for a given $\Pi_n$ we call the $\text{Diam}(G_{4n})$ the eventual diameter of $WG(\Pi_n)$, and note from Lemma 8 that the eventual diameter is at least $n$.

**Proposition 2.** A family of word graphs, $WG(\Pi_n)$, is asymptotically close to the Moore bound if, and only if, its eventual diameter is $n$.

**Proof.** Let $G_m \in WG(\Pi_n)$ for $m \geq 4n$. Suppose the eventual diameter of $WG(\Pi_n)$ is $n + \varepsilon$. The degree of $G_m$ is given by $|\Pi_n| + (m - n)$, hence letting $\alpha = |\Pi_n| - n$ we have the $\text{Deg}(G_m) = m + \alpha$. Finally, we may count the size of $G_m$ as follows

$$|V(G_m)| = |\Pi_n|^m \frac{m!}{(m - n)!} = m^n + O(m^{n-1}).$$

Now we recall the Moore bound for a directed graph of degree $d$ and diameter $k$ is given by $M(d, k) = d^k + d^{k-1} + \ldots + 1 = d^k + O(d^{k-1})$. Hence, letting $d_m = \text{Deg}(G_m)$ and $k_m = \text{Diam}(G_m)$, we have

$$\lim_{m \to \infty} \left\{ \frac{|V(G_m)|}{M(d_m, k_m)} \right\} = \lim_{m \to \infty} \left\{ \frac{m^n + O(m^{n-1})}{M(m + \alpha, n + \varepsilon)} \right\} = \lim_{m \to \infty} \left\{ \frac{m^n + O(m^{n-1})}{(m + \alpha)^{n+\varepsilon} + O(m^{n+\varepsilon-1})} \right\} = \begin{cases} 1, & \text{if } \varepsilon = 0, \\ 0, & \text{otherwise.} \end{cases} \square$$

Hence, we now introduce the restriction that a rule set $\Pi_n$ is admissible if the eventual diameter of $WG(\Pi_n)$ is $n$. For the rest of this section, we will only consider admissible sets $\Pi_n$.

We also now introduce the further restriction that, for all $\pi \in \Pi_n$, $\pi(i) \leq i + 1$. Informally this means that the alphabet fixing rules cannot “shift” any letter to the left more than one space at a time. We call this shift restriction and note that the Gómez and Faber-Moore-Chen graphs are shift restricted word graphs. For the remainder of this section, we will only consider shift restricted word graphs. We now show that the Gómez graphs are largest possible shift restricted word graphs for given degree and diameter.

Let $\Pi_n$ be admissible and shift restricted, and let $G_m \in WG(\Pi_n)$, where $m > n$. For each vertex $v \in V(G_m)$ and letter $x \in B$ we introduce the function $p_x(v)$ which is the position of the letter $x$ in $v$. The function is defined by $p_x(x_1 x_2 \ldots x_n) = i$ and $p_y(x_1 x_2 \ldots x_n) = 0$ where $y \notin \{x_1, x_2, \ldots, x_n\}$.

**Lemma 11.** For $u, v \in V$ with $u \to v$ and $y \in B$, we have $p_y(v) \geq p_y(u) - 1$.

**Proof.** If $p_y(u) = 0$ then the result is immediate as $p_y(v') \geq 0$ for all $v' \in V$. Hence, suppose $u = x_1 x_2 \ldots x_n$ and $y = x_i$. If $v = x_2 x_3 \ldots x_n y$, then $p_x(v) = p_x(u) - 1$. If $v = x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(n)}$ and $\pi(j) = i$ then $p_x(v) = p_x(u) = j \geq \pi(j) - 1 = p_x(u) - 1$. \(\square\)

**Corollary 2.** For any $u, v \in V$ and $y \in B$, all paths connecting $u$ to $v$ have length at least $p_y(u) - p_y(v)$ (if $p_y(u) = 0$ and $p_y(v) > 0$ this becomes $n + 1 - p_y(u)$).

**Lemma 12.** $\text{Diam}(G_m) \geq n$.

**Proof.** Let $u = x_1 x_2 \ldots x_n$ and $v = y x_1 x_2 \ldots x_{n-1}$. We have $p_x(u) = n$ and $p_x(v) = 0$, hence any path connecting $u$ and $v$ is at least length $n$. \(\square\)
Proposition 3. Diam \((G_m) = n\).

Proof. By our assumption of the eventual diameter being \(n\), we need only show this for \(m < 4n\). Hence, consider \(u,v \in V(G_m)\) where \(m < 4n\). Let \(\phi : G_m \to G_{4n}\) be the inclusion from \(G_m\) to \(G_{4n}\), and consider a path from \(u' = \phi(u)\) to \(v' = \phi(v)\) in \(G_{4n}\). Letting \(B' = \alpha(u') \cup \alpha(v')\), we can see that any vertex \(w \in G_{4n}\) satisfying \(\alpha(w) \subseteq B'\) is invertible by \(\phi\). Suppose that on a path from \(u'\) to \(v'\) we introduce a letter \(y \notin B'\) via an alphabet changing rule, call the vertex after this rule \(w\). We have \(p_y(w) = n\) and \(p_y(v') = 0\), hence the remainder of the path is at least length \(n\), but this contradicts that the path is of length at most \(n\). Therefore all vertices \(w\) between \(u'\) and \(v'\) satisfy \(\alpha(w) \subseteq B'\). Hence, we may use \(\phi^{-1}\) to find a path connecting \(u\) and \(v\) of length at most \(n\) in \(G_m\). This shows \(\text{Diam}(G_m) \leq n\), applying Lemma 12 gives us the result. \(\square\)

Corollary 3. If \(\Pi_n\) and \(T_n\) are admissible and shift restricted, and \(|\Pi_n| < |T_n|\), then each \(G_m \in \text{WG}(\Pi_n)\) and \(H_m \in \text{WG}(T_n)\) have the same diameter and are the same size for all \(m\), but \(\text{Deg}(G_m) < \text{Deg}(H_m)\).

Hence we now may make the definition that an admissible shift restricted rule set \(\Pi_n\) is optimal if there exists no rule set \(T_n\) with \(|T_n| < |\Pi_n|\) which is also both admissible and shift restricted.

Lemma 13. For each \(1 \leq i \leq n\) there exists some \(\pi \in \Pi_n\) which contains an \(i\)-cycle.

Remark. The proof of this lemma obscures that this is a fairly natural observation. We illustrate below an example path used in the lemma, noting that the key idea is that the letter \(x_n\) has to “jump” the block of \(y_i\)s by exactly \(k + 1\) spaces. At the point of the jump, we must have a permutation \(\pi \in \Pi_n\), and the jump is a \((k + 1)\)-cycle in that permutation.

\[
\begin{align*}
x_1 & \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8 \ x_9 \\
x_2 & \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8 \ x_9 \ y_1 \\
x_3 & \ x_4 \ x_5 \ x_6 \ x_7 \ x_8 \ x_9 \ y_1 \ y_2 \\
x_4 & \ x_5 \ x_6 \ x_7 \ x_8 \ x_9 \ y_1 \ y_2 \ y_3 \\
x_5 & \ x_6 \ x_7 \ x_8 \ x_9 \ y_1 \ y_2 \ y_3 \ y_1 \\
x_6 & \ x_7 \ x_8 \ x_9 \ y_1 \ y_2 \ y_3 \ x_1 \ x_2 \\
x_7 & \ x_8 \ x_9 \ y_1 \ y_2 \ y_3 \ x_1 \ x_2 \ x_3 \\
x_8 & \ y_1 \ y_2 \ y_3 \ x_9 \ x_1 \ x_2 \ x_3 \\
x_9 & \ y_1 \ y_2 \ y_3 \ x_9 \ x_1 \ x_2 \ x_3 \\
y_1 & \ y_2 \ y_3 \ x_9 \ x_1 \ x_2 \ x_3 \ x_4 \\
y_2 & \ y_3 \ x_9 \ x_1 \ x_2 \ x_3 \ x_4 \\
y_3 & \ x_9 \ x_1 \ x_2 \ x_3 \ x_4 \\
y_4 & \ x_9 \ x_1 \ x_2 \ x_3 \ x_4 \\
y_5 & \ x_9 \ x_1 \ x_2 \ x_3 \ x_4 \\

\end{align*}
\]

Proof. For \(k \geq 1\) we show the existence of a permutation containing a \((k + 1)\)-cycle, noting that when \(k + 1 = n - 1\) the remaining fixed element of the permutation provides the missing “1-cycle”. Let \(u = x_1x_2... x_n\) and \(v = y_1y_2... y_kx_nx_1x_2... x_{n-(k+1)}\). As \(p_{y_i}(u) = 0\) and \(p_{y_i}(v) = 1\), the shortest path that connects \(u\) and \(v\) is of length \(n\). Hence, let \(u = u_0 \to u_1 \to ... \to u_n = v\) be such a path. A trivial induction and Lemma 11 shows for \(1 \leq j \leq k\) we have \(p_{y_j}(u_{i+j}) = n - i\). A similar induction starting from \(u_0\) shows that \(p_{x_n}(u_i) \geq n - i\) and an induction working backwards from \(u_n\) shows that \(p_{x_n}(u_i) \leq n - i + k + 1\). As eventually we have \(p_{x_n}(u_n) = k + 1 > 0\) we know there exists \(c\) such that \(p_{x_n}(u_{c-1}) = n - c + 1\) and \(p_{x_n}(u_c) > n - c\). Now consider the first such \(c\). For each \(1 \leq j \leq k\) we cannot have \(p_{x_n}(u_c) = n - c - j\), as we have that \(p_{y_j}(u_c) = n - c + j\), and we cannot have some \(j > k + 1\) such that \(p_{x_n}(u_c) = n - c + j\), as we have \(p_{x_n}(u_c) \leq n - c + k + 1\). The only possibility this leaves is that \(p_{x_n}(u_c) = n - c + k + 1\). This can only happen if the edge connecting \(u_{c-1}\) and \(u_c\) is alphabet fixing, corresponding to some rule \(\pi\).

We see that in \(\pi\), each \(y_j\) went from \(n - c + i + j + 1\) to \(n - c + i + j\), and \(x_n\) went from \(n - c + 1\) to \(n - c + k + 1\), hence, letting \(\alpha = n - c + 1\), we have the \((k + 1)\)-cycle \(((\alpha + k) (\alpha + k - 1) ... \alpha)\). \(\square\)

Corollary 4. The Gómez graphs are optimal.

(The reader unfamiliar with the definition of the Gómez graphs will find the definition at the beginning of Section 4).

Proof. For \(n = 2k + 1\), the set \(\Pi_n\) which defines the Gómez graphs contains exactly one cycle of each length \(1 \leq i \leq n\).

For \(n = 2k\), the set \(\Pi_n\) which defines the Gómez graphs contains exactly one cycle of each length \(1 \leq i \leq n\), \(i \neq k\), and two cycles of length \(k\). As each permutation is a permutation on \(n = 2k\) elements, it is not possible to remove a permutation from \(\Pi_n\) by eliminating only one \(k\)-cycle. \(\square\)
Altogether, this shows that if we want to try to create new word graphs which are larger than Gómez graphs for a given degree and diameter, then we will either have to consider non-admissible $\Pi_n$ to find small examples, which would be limited to $m < 2n$, or consider word graphs which are not shift restricted.

We will now proceed in this section by establishing other properties shared by word graphs. In particular, we are interested in when they are Cayley, and shall provide a table of which values of $n$ and $m$ correspond to Cayley graphs. We note that it is possible that other values of $n$ and $m$ can correspond to Cayley graphs also, but this can only happen if the graph $G_m \in \text{WG}(\Pi_n)$ contains an automorphism outside of $S_m$.

Lemma 14. There is some group $H \leq \text{Aut}(G_m)$ with $H \cong S_m$.

Proof. We construct $H$ by taking all $\phi \in S_m$ acting naturally on $B'$ and defining $\phi' : V(G_m) \to V(G_m)$ by $\phi'(x_1x_2\ldots x_n) = \phi(x_1)\phi(x_2)\ldots \phi(x_n)$.

\[ \require{cancel} \]

In light of this lemma, we quote a result available in [1, 2, 8] and used in [7] and use it to classify when word graphs are Cayley. The following is an exhaustive table of values of $n$ and $m$ such that there is a subgroup of $S_m$ acting regularly on the tuples of length $n$, and the subgroups which have this action.

| \( n \) | \( m \) | Group                   |
|---------|-------|------------------------|
| 1       | 1     | $S_1$                  |
| 2       | 2     | $S_2$, $S_{k+1}$       |
| 3       | 3     | $A_{k+2}$, $P^\Delta L(2,q)$, PSL(2,q) |
| 4       | 4     | $M_{11}$               |
| 5       | 5     | $M_{12}$               |

Corollary 5. For $n, m$ in this table, the graph $G_m \in \text{WG}(\Pi_n)$ is Cayley.

Corollary 6. If $\text{Aut}(G_m) \cong S_m$, then the graph $G_m \in \text{WG}(\Pi_n)$ is Cayley if, and only if, $n$ and $m$ are in the given table.

Hence, we now conclude this section by establishing a test to determine whether a given family of word graphs satisfies $\text{Aut}(G_m) \cong S_m$. First we shall need some definitions. Let $\Gamma_n$ be the Cayley graph $\Gamma(\Pi_n, S_n)$.

Lemma 15. Letting $H$ be a subgraph of $G_m$ induced by vertices $\{x_1, x_2, \ldots, x_n\} \subset B$, we have $H \cong \Gamma_n$.

Proof. This is simply a relabelling.\[ \]

We will now refer to the graph $\Gamma_n$ as the alphabet fixing subgraph of $G$, noting that it is unique to isomorphism regardless of the choice of $\{x_i\}$. Now we make two further definitions. We shall call a word graph $G$ alphabet stable if there exists no automorphism $\phi \in \text{Aut}(G)$ such that there exist some $u, v \in V(G)$ with $\alpha(u) = \alpha(v)$ but $\alpha(\phi(u)) \neq \alpha(\phi(v))$. In other words, a word graph is alphabet stable if, and only if, it preserves whether arcs are alphabet changing or alphabet fixing. Second, we shall call a family of word graphs subregular if the alphabet fixing subgraph $\Gamma_n$ of $G_m$ is regular, i.e. $\text{Aut}(\Gamma_n) \cong S_n$. In the following let $G_m$ be a word graph which is alphabet stable and subregular. We now aim to show that $\text{Aut}(G_m) \cong S_m$.

Lemma 16. If $\phi \in \text{Aut}(G_m)$ fixes a vertex $u$, then $\phi$ fixes all $v$ such that $\alpha(u) = \alpha(v)$.

Proof. Let $V = \{v \in V(G_m) | \alpha(v) = \alpha(u)\}$. Consider $\psi = \phi|_V$, the restriction of $\phi$ to the vertices of $V$. For any $v \in V$ we have $\alpha(\psi(v)) = \alpha(\phi(v)) = \alpha(u)$, hence we have $\psi(v) \in V$. As $\phi$ is an automorphism, $\psi$ is injective, and therefore bijective as its image is its domain. Hence $\psi$ is a bijection on the subgraph induced by vertices of $V$, which is the alphabet fixing graph $\Gamma_n$. As $G_m$ is subregular, any automorphism of $\Gamma_n$ which fixes a vertex must fix all of $\Gamma_n$. Therefore, as $\psi(u) = u$ we must have that $\psi$ is the identity on $V$.\[ \]

Lemma 17. If $\phi \in \text{Aut}(G_m)$ and $X, Y, Z \subset B$ with the following properties

- $X = \{x_1, x_2, z_1, z_2, \ldots, z_{n-2}\}$,
- $Y = \{y_1, y_2, z_1, z_2, \ldots, z_{n-2}\}$,
• \( Z = \{x_2, y_2, z_1, z_2, \ldots, z_n - 2\} \),

• \( \phi \) fixes all \( v \in V(G_m) \) with \( \alpha(v) = X \) or \( \alpha(v) = Y \),

then \( \phi \) fixes all \( v \in V(G_m) \) with \( \alpha(v) = Z \).

Proof. Let \( u = x_1 z_1 z_2 \ldots z_{n-2} x_2, v = y_1 z_1 z_2 \ldots z_{n-2} y_2 \) and suppose we have \( w, w' \in V(G_m) \) such that \( u \rightarrow w, v \rightarrow w' \) and \( \alpha(w) = \alpha(w') \). As \( |X \cap Y| = 2 \), we must have that both \( u \rightarrow w \) and \( v \rightarrow w' \) are alphabet changing rules. Therefore, \( x_1 \notin \alpha(w), x_2 \in \alpha(w), y_1 \notin \alpha(w') \) and \( y_2 \notin \alpha(w') \). Hence we have \( \alpha(w) \supseteq (X \cap Y) \cup \{x_2, y_2\} \), but \( |\alpha(w)| = |(X \cap Y) \cup \{x_2, y_2\}| \), and hence we have equality. We now see the rule \( u \rightarrow w \) must introduce the letter \( y_2 \), and \( w = z_1 z_2 \ldots z_{n-2} y_2 x_2 \). Similarly, \( w' = z_1 z_2 \ldots z_{n-2} y_2 x_2 \). By our assumptions on \( \phi \) we have \( \phi(u) = u, \phi(v) = v \), and \( \alpha(\phi(w)) = \alpha(\phi(w')) \) as \( \alpha(w) = \alpha(w') \) and \( G_m \) is alphabet stable. Hence we have \( u \rightarrow \phi(w) \) and \( v \rightarrow \phi(w') \) with \( \alpha(\phi(w)) = \alpha(\phi(w')) \), so \( \phi(w) = w \) and \( \phi(w') = w' \). Now applying Lemma 16 we get the desired result.

Lemma 18. The only \( \phi \in \text{Aut}(G_m) \) which fixes a vertex \( u \in V(G_m) \) and all \( v \in V(G_m) \) such that \( u \rightarrow v \) is the identity.

Proof. We may label \( u = x_1 x_2 \ldots x_n \) taking \( B \) to be \( \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{m-n}\} \). For a vertex \( v \), define \( f(v) = \{|x_1, x_2, \ldots, x_n| \cap \alpha(v)\} \). We show by induction for \( n \geq k \geq 0 \) that \( \phi \) fixes all \( v \) such that \( f(v) = k \).

For \( k = n - 1 \), let \( n = n - 1 \). First we consider \( \alpha(v) = \{x_2, x_3, \ldots, x_n, y\} \). In this case the vertex \( v' = x_2 x_3 \ldots x_n y \) satisfies \( u \rightarrow v' \) and so \( \phi(v') = v' \). Hence, as \( \alpha(v) = \alpha(v') \) we use Lemma 16 again to show that \( \phi(v) = v \). For other \( v \) with \( f(v) = n - 1 \), without loss of generality we may assume \( \alpha(v) = \{x_1, x_2, \ldots, x_{n-1}, y\} \). Applying Lemma 17 to the sets \( \{x_1, x_2, \ldots, x_n\} \) and \( \{x_2, x_3, \ldots, x_n, y\} \) we see that \( v \) is fixed.

For \( k = c \) the inductive hypothesis for \( k = c + 1 \), \( v \in V(G_m) \) such that \( \alpha(v) = \{x_1, x_2, \ldots, x_{n-c}, y\} \).

By applying the inductive hypothesis and Lemma 16 to the sets

\[ \{x_1, \ldots, x_{c+1}, y_1, \ldots, y_{n-c-1}\} \quad \text{and} \quad \{x_1, \ldots, x_{c+1}, y_2, \ldots, y_{n-c}\} \]

we get the desired result.

Proposition 4. \( \text{Aut}(G_m) \cong S_m \).

Proof. Let \( H \leq \text{Aut}(G_m) \) as defined in Lemma 14. Suppose \( \phi \in \text{Aut}(G_m) \). Consider some \( u \in V(G_m) \), and define \( \psi \in H \) such that \( \psi(\phi(u)) = u \) and for all \( v \in V(G_m) \) with \( u \rightarrow v \) via an alphabet changing rule we have \( \psi(\phi(v)) = v \). Note that \( \psi \) is guaranteed to exist as alphabet stability guarantees this process of defining \( \psi \) corresponds to defining a unique permutation. We now consider the automorphism \( \psi \circ \phi \), which by Lemma 18 must be the identity. Hence \( \psi = \phi^{-1} \) and \( \phi \in H \).

Now we see that alphabet stability and subregularity are sufficient conditions to guarantee \( \text{Aut}(G_m) \cong S_m \), we devote the remainder of this section to creating tests to determine when a family of word graphs is alphabet stable and subregular. Our tests will only concern the counting of certain paths in the alphabet fixing subgraph. In the following we consider the word graph \( G_m \in \text{WG}(\Gamma_n) \) with alphabet fixing subgraph \( \Gamma_n \).

Lemma 19. If \( u, v \in V(G_m) \) such that \( u \rightarrow v \) and \( \alpha(u) \neq \alpha(v) \), then there is a unique path of length \( n \) from \( v \) to \( u \).

Proof. Without loss of generality we may take \( u = x_1 x_2 \ldots x_n \) and \( v = x_2 x_3 \ldots x_n y \). Considering a path \( u = u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_k = v \) with \( k \leq n \), we may repeatedly apply Corollary 2 whilst considering \( p_{x_i}(u_{i-1}) \) and \( p_{x_i}(u_k) \) to deduce that \( u_{i-1} \rightarrow u_i \) by the alphabet changing rule which introduces \( x_i \).
Now suppose for all \( u, v \in V(\Gamma_n) \) with \( u \rightarrow v \) we have either more than one path from \( v \) to \( u \) of length \( n \), or we have a path of length less than \( n \) from \( v \) to \( u \).

**Lemma 20.** There is no \( \phi \in V(G_m) \) with \( \phi(u) = u \) and \( \alpha(\phi(v)) \neq \alpha(u) \).

**Proof.** If such a \( \phi \) exists then \( u \rightarrow \phi(v) \) by an alphabet changing rule. Hence there is a unique shortest path of length \( n \) connecting \( \phi(v) \) to \( u \).

**Proposition 5.** \( G_m \) is alphabet stable.

**Proof.** Suppose \( G_m \) is not alphabet stable. Let \( \phi \in \text{Aut}(G_m) \) and \( u, v \in V(G_m) \) such that \( \alpha(u) = \alpha(v) \) and \( \alpha(\phi(u)) \neq \alpha(\phi(v)) \). Consider a path from \( u \) to \( v \) of length at most \( n \), say \( u = u_0 \rightarrow \cdots \rightarrow u_k = v \). Lemma 19 shows that \( \alpha(u) = \alpha(u_i) \) for each \( i \). Now consider the path \( \phi(u_0) \rightarrow \cdots \rightarrow \phi(u_k) \). As \( \alpha(\phi(u_0)) = \alpha(\phi(u)) \neq \alpha(\phi(u_i)) \) there must be some \( c \) such that \( \alpha(\phi(u_c)) \neq \alpha(\phi(u_{c+1})) \). Hence, we have \( u_c \rightarrow u_{c+1} \) and \( \alpha(u_c) = \alpha(u_{c+1}) \), but \( \alpha(\phi(u_c)) \neq \alpha(\phi(u_{c+1})) \), contradicting Lemma 20.

**Lemma 21.** If \( \Gamma_n \) is not regular, there is an automorphism \( \phi \in \text{Aut}(\Gamma_n) \) such that for some \( u, v \in V(\Gamma_n) \) with \( u \rightarrow v \) we have \( \phi(u) = u \) but \( \phi(v) \neq v \).

**Proof.** Let \( H < \text{Aut}(\Gamma_n) \) be regular, and let \( \phi \in \text{Aut}(\Gamma_n) \setminus H \). Consider \( u \in V(\Gamma_n) \) and let \( \psi \in H \) be the automorphism such that \( \psi(\phi(u)) = u \). Let \( \phi' = \psi \circ \phi \), so \( \phi' \) fixes \( u \). As \( \phi \not\in H \), we must have \( \phi' \) is not the identity. Hence there is some \( v \in V(\Gamma_n) \) such that \( \phi'(v) \neq v \). Consider a path from \( u \) to \( v \), we must encounter a pair of vertices on the path such that \( u' \rightarrow v' \), \( \phi(u') = u' \) and \( \phi(v') \neq v' \).

**Corollary 7.** If for all \( u, v, w \in V(\Gamma_n) \) with \( u \rightarrow v \) and \( u \rightarrow w \) there exists some \( k \) such that the number of paths of length \( k \) from \( v \) to \( u \) is different to the number of paths of length \( k \) from \( w \) to \( u \), then \( \Gamma_n \) is regular.

We now combine these results and state our test. Let \( G_m \in \text{WG}(\Pi_n) \) be a word graph with alphabet fixing subgraph \( \Gamma_n \). Let \( u \in V(\Gamma_n) \) be an arbitrary fixed vertex of \( \Gamma_n \) and let \( \{v_i\} \) be the set of vertices such that \( u \rightarrow v_i \).

**Proposition 6.** If the following conditions are satisfied, then \( \text{Aut}(G_m) \cong S_m \).

- each \( v_i, v_j \) has some \( k \) such that the number of paths of length \( k \) from \( v_i \) to \( v_j \) is different to the number of paths of length \( k \) from \( v_j \) to \( v_i \).
- each \( v_i \) has either a path of length less than \( n \) to \( u \) or has more than one path of length \( n \) to \( u \).

## 4 Introduction to Gómez Graphs

In our account of Gómez graphs, we shall use a modified notation to that of the original paper more appropriate to our purposes. We note that this paper will only deal with the Gómez graphs corresponding to the graphs \( \text{DG} \left( k, k \right) \) and \( \text{DG} \left( k, k + 1 \right) \). The technique used herein does not work for all \( \text{DG} \left( k, k' \right) \) where \( k' \geq k \), which we shall provide explicit examples to show, and runs into difficulty when pursued for the case \( \text{DG} \left( k, k + 2 \right) \). Hence, we only deal with the cases which provide the extreme examples in degree-diameter as opposed to dealing with the entire family.

We begin by giving a definition of the alphabet fixing subgraphs \( \Gamma_n \) of the Gómez graphs. For any \( n \), define \( k \) so that either \( n = 2k + 1 \) or \( n = 2k \), and let \( B \) be any set such that \( |B| = n \). We define the graph \( \Gamma_n = (V, E) \) as follows. The set \( V \) of vertices is given by \( V = \{x_1, x_2, \ldots, x_n | x_i \in B, x_i = x_j \Leftrightarrow i = j\} \), that is \( V \) is the set of all words of length \( n \) on the alphabet \( B \) with distinct letters, and the set \( E \) is given by the directed adjacencies

\[
x_1x_2\ldots x_n \rightarrow \begin{cases} x_2x_3\ldots x_{k-1}x_kx_{k-i+3}\ldots x_nx_{k-i+1}, & \text{for } 0 \leq i < k, \\ x_2x_3\ldots x_nx_1. \end{cases}
\]

Informally, each of these rules splits the word into a left and right half and rotates each half by one. The size of the left half is not allowed to exceed that of the right, and we also allow an empty left half. Now we give the example adjacencies for the cases \( n = 6 \) and \( n = 7 \) with the left and right halves coloured for clarity.
Now we introduce terminology and a visual representation of these rules which we will make use of throughout our proof. First, we note that the graph $\Gamma_n$ has $k+1$ rules, we shall call these rules $\pi_i$ for $0 \leq i \leq k$, where rule $\pi_i(x_1x_2\ldots x_n) = x_2x_3\ldots x_{k-i}x_1x_{k-i+2}x_{k-i+3}\ldots x_{n}x_{k-i+1}$, and $\pi_k(x_1x_2\ldots x_n) = x_2x_3\ldots x_nx_1$. In this notation, we now show our visual representation of the rules in the case $n = 8$.

Within this visual representation, we label the following features...
Lemma 22. The number of right arrows in a path of length $m$ is $m$, and the number of left arrows in a path of length $m$ is less than or equal to $m$.

Proof. Each of the rules $\pi_i$ for $0 \leq i \leq k$ contains exactly one right arrow, and either one or zero left arrows.

We represent the composition of rules as in the following diagram and call it a path. The following diagram shows the path $\pi_3\pi_0\pi_2$ when $n = 8$.

In subsequent diagrams, we may drop the explicit labelling of letters to present the same path in a more succinct manner as in the example below.

We will refer to the trail from position $i$ in a path to mean the concatenation of consecutive arrows in our diagram starting from the arrow at position $i$. In the following example we have highlighted the trail starting at position 2.
We will call a trail *closed* if it begins and ends at the same position. Here we illustrate a closed trail in blue, and a non-closed trail in red.

We will call a path *closed* if the trails starting at each position in the path are closed.

With the terminology established, we now briefly describe our motivation. In order to prove the Gómez graphs are subregular and alphabet stable, we aim to count paths of lengths $n$ and $n-1$ from each neighbour of an arbitrary vertex $v \in \Gamma_n$ back to $v$. All of these paths in $\Gamma_n$ correspond to cycles of lengths $n$ and $n + 1$ in $\Gamma_n$, which correspond exactly to the closed paths we have just defined. Hence, we now aim to count closed paths of lengths $n$ and $n + 1$, considering what the first rule is on those paths.

For a path $p = p_0p_1 \ldots p_n$ we shall call $p^i = p_{i+1}p_{i+2} \ldots p_n p_1 \ldots p_i$ the $i^{th}$ rotation of $p$.

**Lemma 23.** There is a bijection between the closed trails of a path $p$ and the closed paths of each of its rotations $p^i$.

**Proof.** We demonstrate such a bijection between the closed trails of some path $p$ and its rotation $p^2$, noting that the result follows from a trivial induction. Let $p$ be a path, first we show that the trail at $i$ is closed in $p$ if, and only if, the trail at $p_1(i)$ is closed in $p^2$.

$$p(i) = i \iff (p_2p_3 \ldots p_n)(p_1(i)) = i$$

$$\iff (p_2p_3 \ldots p_n p_1)(p_1(i)) = p_1(i) \iff p^2(p_1(i)) = p_1(i).$$

Hence, as $p_1$ is a bijection, we have a bijection between the closed trails of $p$ and $p^2$. \qed

In light of Lemma 23, we shall identify a closed trail starting at $i$ in a path $p$ with the closed trail starting at $(p_1p_2 \ldots p_{j-1})(i)$ in $p^j$, referring to them as the same trail. Here we illustrate an example of a closed trail and its rotations.
Corollary 8. A path $p$ is closed if and only if each of its rotations $p^i$ is closed.

Lemma 24. Any closed trail in a path of length $n + 1$ must contain at least two forward arrows.

Proof. For any closed trail we see that the distance traversed by backwards arrows is equal to the distance traversed by forwards arrows. If there are no forwards arrows, this obviously cannot happen. If there is only one forwards arrow, then there must be $n$ backwards arrows, hence the forwards arrow must correspond to travelling forwards $n$ places. However, the furthest that can be travelled forwards occurs in rule $\pi_k$, which travels forwards by $n - 1$ spaces. Hence, this cannot occur, and so any closed trail must contain at least two
forward arrows.

**Lemma 25.** Any closed trail of length \( n + 1 \) whose only forward arrows are left arrows contains at least three left arrows.

*Proof.* The furthest that can be travelled forwards by a left arrow occurs in the rule \( \pi_0 \) in which we travel forwards \( k - 1 \) spaces. If we have a closed trail which contains two left arrows, then it travels a distance of \( n - 1 \) with backwards arrows. Hence, we must have \( n - 1 \leq 2(k - 1) = 2k - 2 \leq n - 2 \).

**Lemma 26.** Any closed trail of length \( n + 1 \) whose only forward arrows are right arrows contains exactly two right arrows.

*Proof.* Consider a closed trail whose only forward arrows are right arrows. As the trail is closed, the sum of the forward arrows equals the sum of the backward arrows. Suppose there are three or more right arrows in the trail, then the sum of the forward arrows is at least \( 3(k - 1) \) and the sum of the backwards arrows is at most \( n - 3 \leq 3(k - 1) \), hence there must be fewer than three right arrows, and so by Lemma 24 there are two.

**Lemma 27.** In a closed path of length \( n + 1 \) there are at most three trails containing two right arrows.

*Proof.* Suppose we have a closed path of length \( n + 1 \) which contains at least four trails containing two right arrows. By Lemma 22 we now have \( (n + 1) - 8 = n - 7 \) unaccounted for right arrows, and at most \( n + 1 \) unaccounted for left arrows. Lemma 24 tells us that we need to use at least two forward arrows per remaining trail in the path. If each left arrow is in a trail with a right arrow, then such a trail requires only two arrows, otherwise it requires three or more. Hence, in order to minimise the number of required arrows, we may assume as many left arrows as possible are paired with right arrows. In this manner, we assume all \( n - 7 \) remaining right arrows are paired with left arrows. This leaves at most \( (n + 1) - (n - 7) = 8 \) unaccounted for left arrows. We have now accounted for \( (n - 7) + 4 = n - 3 \) of \( n \) trails, leaving three unaccounted for. Lemma 25 tells us that each of the three remaining trails requires at least three left arrows, but we only have eight unaccounted for left arrows.

**Lemma 28.** In a path of length \( n + 1 \), if the trail starting with the right arrow of \( p_1 \) contains no further right arrows, it contains the left arrow of \( p_{n+1} \).

*Proof.* After \( p_1 \), the trail is at position \( n \). As the trail contains no further right arrows, each \( p_{i+1} \) maps the trail from \( n - i \) to \( n - i - 1 \), provided that \( n - i > 1 \). Hence, the trail reaches position 1 after \( p_n \), and so the trail contains the left arrow of \( p_{n+1} \).

If in a path \( p \) of length \( n \) there is some \( i \) and \( j \) such that \( p_i = \pi_j \) and \( p_{i+1} = \pi_{j+1} \), then we call the left arrow of \( p_i \) and the right arrow of \( p_{i+1} \) paired and refer to them together as a pair. As a special case, we allow \( i = n \) and use \( p_1 \) instead of \( p_{i+1} \).

**Lemma 29.** If a closed trail in a path of length \( n + 1 \) contains both right and left arrows, then it contains one pair and no other forward arrows.

*Proof.* Suppose \( p \) is a path with such a trail. Let \( q \) be a rotation of \( p \) which puts a right arrow of the trail in position \( q_1 \). As the trail contains a left arrow, at some point we have some \( i \) such that \( (q_2 q_3 \ldots q_i)(n) = 1 \). As the most we can travel backwards in each \( q_j \) is one space, the soonest this can happen is by \( i = n - 1 \), if and only if each \( q_j \) introduces a backwards arrow into the trail. If this does not happen, then no \( q_j \) including \( j \in \{1, n + 1\} \) can contain a left arrow in the trail. Hence, this is the only possibility. If the position the trail starts at is \( k - \alpha \) then we know \( q_1(k - \alpha) = n \), so \( q_1 = \pi_\alpha \), and \( q_{n+1}(1) = k - \alpha \), hence \( q_{n+1} = \pi_{\alpha - 1} \). Hence the trail contains one pair and all other arrows are backwards arrows.

**Lemma 30.** If all right arrows in a path \( p \) of length \( n + 1 \) are either in closed trails or paired, then all pairs in \( p \) are in distinct closed trails.

*Proof.* Consider an arbitrary right arrow in \( p \), and the rotation of \( p \) which brings this right arrow to \( p_1 \). If the trail from this right arrow enters a forwards arrow before \( p_{n+1} \), the forwards arrow must be an unpaired right arrow, and so the trail is closed. Otherwise, the trail enters a left arrow at \( p_{n+1} \), which must be the pair of the right arrow of \( p_1 \), and the trail is closed.
We now note from what we have shown that if a closed trail contains a right arrow then either it contains exactly two right arrows or it contains one pair. Hence, we are in a position to easily deal with closed trails containing at least one right arrow. In order to settle the case of closed trails comprised entirely of left arrows, we now require a further definition to continue our discussion. We shall say that within a permutation in a path, a trail is on the left side if it is in a cycle containing a left arrow, and the right side otherwise. We shall say that between two rules a trail changes sides from right to left or from left to right between two permutations in the obvious manner. Below is a diagram to clarify the terminology, with arrows on the left drawn in red and arrows on the right drawn in blue.

For the remainder of this section, we will consider paths with the property that, letting \( p \) be our path, \( p_i = \pi_j \) and \( p_{i+1} = \pi_k \) implies \( j \geq k - 1 \) (note that this restriction applies to the last entry of \( p \), considering \( p_1 \) instead of \( p_{i+1} \)).

**Lemma 31.** If \( p \) is closed, any trail which changes sides contains a pair.

**Proof.** For any closed trail, the number of times the trail changes from left to right must equal the number of times the trail changes from right to left. Hence, if a trail changes sides at all we know at some point the trail must change sides from left to right. It is only possible for a trail to change sides from left to right between a pair of consecutive rules of the form \( p_i = \pi_j \) and \( p_{i+1} = \pi_k \) where \( k > j \). Given our assumption on our path \( p \), we see this happens only if \( k = j + 1 \), in which case there is only one path which changes sides from left to right, corresponding to the left arrow of \( p_i \) connecting to the right arrow of \( p_{i+1} \). Hence any trail which changes sides must contain a pair.$\square$

**Corollary 9.** If \( p \) is closed, any trail which contains only left arrows is always on the left side.

Now we shall define the closure of a path \( p \) to be \( p \) concatenated with itself the smallest number of times necessary to form a closed path. As \( p \) is a permutation, we know that the closure exists as each permutation has finite order.

**Lemma 32.** If \( p \) is a path in which every trail with a right arrow is closed, all trails with only left arrows are always on the left side.

**Proof.** Letting \( q \) be the closure of \( p \) we may use Corollary 9 to see that all trails of \( q \) with only left arrows are always on the left side. The closed trails of \( q \) which contain only left arrows correspond to the trails of \( p \) which contain only left arrows. This is because any trail containing a right arrow in \( q \) corresponds to one of the closed trails of \( p \). Now, we note that the property of being on the left or right side in a path only depends on the position of the trail at each \( i^{th} \) rule in the path, which are the same in \( p \) and \( q \).

**Lemma 33.** If \( p \) is a path with all trails containing right arrows closed, and the trails starting at positions \( a_1, a_2, \ldots, a_k \) are all the trails containing only left arrows, and there are \( m \) left arrows in total in these trails, then \( p \) maps \( a_i \) to \( a_{i-m} \) (subscripts considered modular).

**Proof.** This is provable by a trivial induction. Firstly, as all trails other than those starting at each \( a_i \) are closed, we see that \( p \) maps each \( a_i \) to some \( a_j \). Now, as all the trails are always on the left side, only two things may happen at each rule. Either all trails are mapped backward by backward arrows, and thus their left right ordering is preserved, or the trail on the far left is mapped by a left arrow and becomes the trail on the far right. The latter case happens exactly \( m \) times. Hence, the left right ordering of the trails starting at \( a_1, a_2, \ldots, a_k \) is cycled \( m \) times. $\square$
Corollary 10. If a path $p$ has all trails with right arrows closed, and contains exactly two trails whose only forward arrows are left arrows, and those trails together contain an even number of left arrows, then the path $p$ is closed.

Proof. Letting the trails starting at positions $a_1$ and $a_2$ be those containing only left forward arrows, we may apply the previous lemma to show that $a_1$ is mapped to $a_1 - 6 = a_1$ and $a_2$ is mapped to $a_2 - 6 = a_2$, hence the trails at $a_1$ and $a_2$ are closed.

5 The Odd Case

We now begin to count the closed paths of length $n + 1$ in the case where $n = 2k + 1$. Throughout this section, we consider the path $p = p_1p_2 \ldots p_{n+1}$ which is assumed to be closed.

Lemma 34. If $p_1 = \pi_0$, then $p_{k+1} = \pi_0$, and the trail beginning at $k + 1$ contains two right arrows.

Proof. We consider the trail starting with the right arrow of $p_1$. As this trail is closed, by Lemma 26 and Lemma 29 we see that it contains one more forward arrow which is either a right or a left arrow. Hence, this trail maps backwards $n - 1$ spaces, the right arrow at $p_1$ maps forward $k$ spaces, and therefore the other forwards arrow maps forward $(n - 1) - k = k$ spaces. The most any left arrow maps forward is $k - 1$ spaces, hence the other arrow in the trail is a right arrow. The only rule with a right arrow which maps forward $k$ spaces is $\pi_0$. To see $p_{k+1} = \pi_0$, we simply follow the backwards arrows after $p_1$.

Corollary 11. There are at most six occurrences of the rule $\pi_0$ in the path $p$.

Proof. This is the result of the combination of Lemma 34 and Lemma 27.

Lemma 35. If $p_1 = \pi_i$ for some $i \geq 1$, then $p_{n+1} = \pi_{i-1}$.

Proof. Consider the trail starting with the right arrow of $p_1$. If this trail contains a left arrow we apply Lemma 29 and are done. Otherwise, Lemma 26 shows that there is exactly one other right arrow in the trail. The distance mapped backward in the trail is $n - 1$, and the distance mapped forward by the right arrow in $p_1$ is $k + i$, hence the distance mapped forward by the right other arrow is $(n - 1) - (k + i) = k - i$, but all right arrows map forward at least $k$.

Corollary 12. If $p_i = \pi_j$ for $j \neq 0$, then $p_{i-1} = \pi_{j-1}$.

Proposition 7. The path $p$ is closed if, and only if,

$$p_i = \begin{cases} \pi_{a_i} & \text{for } 1 \leq i \leq k + 1, \\ \pi_{a_j} & \text{for } i = j + (k + 1), 1 \leq j \leq k + 1, \end{cases}$$

where $a_1, a_2, \ldots, a_{k+1}$ is a $\tau$-sequence.

Proof. To show the implication, we first note the combination of Lemma 34 and Corollary 12 show that $p_i = p_{i+(k+1)}$. Hence, we may now define the sequence $a_1a_2 \ldots a_{k+1}$ such that $p_i = \pi_{a_i}$, and therefore $p_{i+(k+1)} = \pi_{a_i}$. By Corollary 11, we see that $0$ may occur at most three times in $\langle a_i \rangle$. Finally, again from applying Corollary 12 and considering rotations of $p$, we see that $\langle a_i \rangle$ is a $\tau$-sequence.

Now, to show the reverse implication, let $\langle a_i \rangle$ be an arbitrary $\tau$-sequence and define the corresponding path $p$. As shown in Lemma 34 if some $a_i = 0$, this corresponds to a closed trail with the two right arrows in $p_i$ and $p_{i+(k+1)}$. Otherwise, if $a_i \neq 0$, then $a_{i-1} = a_i - 1$, and the right arrow of $p_i$ is paired with the left arrow of $p_{i-1}$. Therefore, all right arrows of $p$ are either in closed trails with right arrows only or are paired, hence by Lemma 30 each pair corresponds to a distinct closed trail.

Again by Lemma 34, each pair of zeroes introduces exactly one closed trail. In a $\tau$-sequence, we will have either one, two or three zeroes, and hence one, two or three pairs of zeroes in our path. We now consider each case separately, and count the number of closed trails containing right arrows.
• If we have one pair of zeroes in our path, these account for one closed trail. The remaining \((n+1) - 2 = n - 1\) right arrows in our path each correspond to a pair in a distinct closed trail. Hence we have found a total of \(n\) closed trails, and so \(p\) is closed.

• If we have two pairs of zeroes in our path, these account for two closed trails. The remaining \((n+1) - 4 = n - 3\) right arrows in our path each correspond to a pair in a distinct closed trail. Hence, we have found \(n - 1\) closed trails in our path. However, this means the remaining trail has to start and end in the same position, and so our path must be closed.

• If we have three pairs of zeroes in our path, these account for three closed trails. The remaining \((n+1) - 6 = n - 5\) right arrows in our path correspond to distinct pairs in our trail. Hence, we have found \(n - 2\) closed trails in our path. The remaining trails in our path use only left arrows. As the \(\tau\)-sequence in question has more than one zero, all other numbers in the sequence are less than \(k\), hence each corresponding rule in our path contains both a right and a left rule, hence there are 6 left arrows in these two trails, so we may apply Corollary 10 to show all trails in the path are closed.

\[\square\]

**Theorem 1.** For \(n = 2k + 1\), the Gómez graphs \(G_m \in \text{WG} (\Pi_n)\) satisfy \(\text{Aut}(G_m) \cong S_m\).

*Proof.* We have shown that the paths of length \(n + 1\) which lead from a vertex back to itself correspond to \(\tau\)-sequences. Hence, for the vertex \(e \in \text{V}(\Gamma_n)\) the number of paths from each \(\pi_i \in \Pi_n\) to \(e\) of length \(n\) is distinct for each \(i\). This shows that the Gómez graphs are subregular. Finally, we notice there is one path of length \(n\) from \(\pi_k\) to \(e\), but as \(\pi_k\) is simply an \(n\)-cycle we have \(\pi_k^n = e\), hence there is also a path of length \(n - 1\) from \(\pi_k\) to \(e\). Hence, each \(\pi_i\) satisfies either \(d(\pi_i, e) < n\) or there is more than one path of length \(n\) from \(\pi_i\) to \(e\). Hence the Gómez graphs are alphabet stable.

\[\square\]

6 The Even Case

In this section, we deal with the case where \(n = 2k\) and \(k > 1\). We will consider an arbitrary closed path \(p = p_1p_2 \ldots p_{n+1}\).

**Lemma 36.** If \(p_1 = \pi_0\), then \(p_{k+2} = \pi_1\), and the closed trail starting at \(k + 1\) contains two right arrows.

*Proof.* The closed trail starting at \(k + 1\) is the trail starting with the right arrow of \(p_1\). Hence, if we assume there are no further right arrows in this trail, we may apply Lemma 28 to show that \(p_{n+1}\) contains a left arrow mapping 1 to \(k + 1\). However, no such right arrow exists. Therefore we may use Lemma 26 to show to show the trail contains two right arrows. The second right arrow must have size \((n - 1) - (k - 1) = k\) so the number of spaces mapped forwards equals those mapped backwards. Hence, the other right arrow is from rule \(\pi_1\), and this is in the trail after being mapped backwards \(k + 1\) positions after \(\pi_0\), hence occurs in position \(p_{k+2}\).

\[\square\]

**Corollary 13.** The rule \(\pi_0\) occurs at most three times in \(p\).

*Proof.* This is an immediate consequence of Lemma 36 and Lemma 25.

\[\square\]

**Lemma 37.** If \(p_1 = \pi_1\), then either \(p_{k+1} = \pi_0\) or \(p_{n+1} = \pi_0\).

*Proof.* Consider the trail starting with the right arrow of \(p_1\). If this trail contains another right arrow, then Lemma 26 shows it contains exactly two right arrows, and following previous logic the other right arrow is \(\pi_0\) at position \(p_{k+1}\). Otherwise we may apply Lemma 29 to get that \(p_{n+1} = \pi_0\).

\[\square\]

**Lemma 38.** If \(p_1 = \pi_i\) for some \(i \geq 2\), then \(p_{n+1} = \pi_{i-1}\).

*Proof.* Here we follow the same reasoning as Lemma 35.

\[\square\]

**Lemma 39.** If \(p\) is a closed path of length \(n\), no rule \(\pi_i\) where \(0 < i < k\) may occur in \(p\).
Proof. Suppose $p$ contains some $\pi_i$ with $0 < i < k$. Rotate $p$ as necessary so that $p_1 = \pi_i$. Consider the trail starting with the right arrow of $\pi_i$. The right arrow of $\pi_i$ maps $(k - 1) + i$ spaces forward. If this trail contains another right arrow, the total distance mapped forward in the trail is at least $2(k - 1) + i > n - 2$. The total distance mapped backward in the, however, trail is at most $n - 2$. Therefore, this trail must contain a left arrow. This means that there is some $j$ such that $(p_2p_3 \ldots p_j)(n) = 1$. The smallest $j$ this can occur for is $j = n$, but this leaves no further rule to contain a left arrow in the trail. Therefore, no such path can exist. 

Proposition 8. The path $p$ is closed if, and only if, $p_i = \pi_a$, for some $\sigma$-sequence $(a_i)$.

Proof. We follow the same approach as the proof of Proposition 7.

Theorem 2. For $n = 2k$, the Gómez graphs $G_m \in \text{WG}(\Pi_n)$ satisfy $\text{Aut}(G_m) \cong S_m$.

Proof. We have shown that the paths of length $n + 1$ from each vertex in $\Gamma_n$ back to itself corresponds to a $\sigma$-sequence. Hence, for each $e \in \Gamma_n$ the number of paths from each $\pi_i \in \Pi_n$ to $e$ of length $n$ is distinct for each $i$, with the possible exception of $\pi_0$. Further, we always have at least two paths of length $n$ from each $\pi_i$ to $e$, hence we have alphabet stability. Now to show subregularity, we consider paths of length $n - 1$ from each $\pi_i$ to $e$. Lemma 39 shows that only $\pi_0$ and $\pi_k$ may have a path of length $n - 1$ to $e$, and $\pi_0$ is two disjoint $k$-cycles and hence $\pi_0^2 = e$ gives us a path of length $n - 1$ from $\pi_0$ to $e$. Therefore, if $\Gamma_n$ is not regular there is an automorphism which exchanges $\pi_0$ and $\pi_k$. However, there are only two paths of length $n$ from $\pi_k$ to $e$, and at least three paths of length $n$ from $\pi_0$ to $e$. Hence we have established subregularity. 

7 Problems in Other Cases

The above argument may seem somewhat unsatisfying as it begins with the observation of some reasonably general facts of Gómez graphs but then only goes on to make arguments of path counting in the cases $\text{DG}(k, k)$ and $\text{DG}(k, k + 1)$. However, we will now give an example to show that, though perhaps the counting argument could be generalised to count all similar paths in Gómez graphs, this would not serve our purpose of classifying when Gómez graphs have full automorphism group $S_n$.

We include here a computed table showing numbers of closed paths of length $k + 1$ starting with each $\pi_i$, written in the order $\pi_0, \pi_2, \ldots$. Noting that the cases we have addressed are $k = k'$ and $k = k' + 1$.

| $k'$ | 2   | 3   | 4     | 5    | 6   |
|------|-----|-----|-------|------|-----|
| 1    | 2,2 | 4,5 | 5,11,11 | 16,23,37,37,23 | 32,47,83,100,83,47 |
| 3    | -   | 1,2 | 2,5,3  | 4,12,12,12   | 8,27,35,44,33   |
| 5    | -   | -   | -      | 1,2,4         | 2,5,13,8        |

From this table we see that the first difficulty we encounter with the case $k = k' + 2$ occurs for $k' = 3$, where there are twelve closed paths of length 6 starting from each of $\pi_1, \pi_2$ and $\pi_3$. This problem cannot be resolved with the methods used to address the cases $k' = k$ and $k' = k + 1$.

In addition, this table highlights the interesting case of $k' = 1$. In this case, we see the number of closed paths of length $k + 1$ starting with each $\pi_i$ is equal to those starting from $\pi_{k-i}$ (taking $\pi_k = \pi_0$ in the special case). We shall provide an informal proof of this to demonstrate that this difficulty cannot possibly be overcome to make this style of proof work for the case $k' = 1$.

We consider the case $k = 8$ and $k' = 1$. In this case, we have the rules $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6$ and $\pi_7$ as follows:
We consider a diagram of the closed path $\pi_2 \pi_3 \pi_7 \pi_0 \pi_1 \pi_2 \pi_3 \pi_2$ and note that if we rotate it by 180 degrees and reverse the arrows we get another closed path. 

Hence from the closed path $p \triangleq \pi_2 \pi_3 \pi_7 \pi_0 \pi_1 \pi_2 \pi_3 \pi_2$ we form the closed path $q \triangleq \pi_6 \pi_5 \pi_6 \pi_7 \pi_0 \pi_1 \pi_5 \pi_6$. 

As we have highlighted in the above example, closed paths beginning with $\pi_2$ are in bijective correspondence with closed paths ending in $\pi_6$. Finally, if we consider the rotation of $q$ such that places the highlighted $\pi_6$ at the beginning, we have found a bijective correspondence between closed paths of a given length beginning with $\pi_2$ and those beginning with $\pi_6$. Hence, we cannot consider any length of path to differentiate these two rules under automorphisms of $\Gamma$.

8 Conclusion

In the context of the degree diameter problem in the directed case, the possibility of finding larger graphs for given degree and diameter than the Gómez graphs remains open (and, indeed, it appears highly likely that larger examples do exist). Hence the optimality result for the Gómez graphs primarily serves to demonstrate the limitations of this particular method of construction, and that the construction of larger graphs will likely require altogether new ideas.

Further, whilst we have shown an optimality result for the Gómez graphs we have not shown that they are unique with this property. The fact the cycles used in the Gómez graphs had the smaller cycle on the left and larger cycle on the right is not necessarily required. Therefore, an interesting question would be to determine which of the potential optimal constructions work. Beyond this, if other constructions work, it is possible that they could have larger automorphism groups.

In order to raise further questions, we now define Gómez like graphs. Suppose a set of permutations $\Pi \subseteq S_n$ contains at least one permutation containing a cycle of each length up to $n$, and $\Pi$ is as small as possible with this property (i.e. $\Pi$ meets our optimality condition from previously). If the word graph
family WG (Π) is diameter \( n \), then we shall call these graphs Gómez like. An obvious first question regarding Gómez like graphs is what conditions such a set Π needs to fulfil to be admissible.

With regards to the classification of the automorphism groups of Gómez graphs, what classification has been achieved misses out a number of important cases. In rough order of importance, these are

(i) the automorphism groups of undirected Gómez graphs,
(ii) the automorphism groups of the graphs DG \((k, k')\) for \( k \geq k' + 2 \),
(iii) the automorphism groups of DG \((k, 1)\),
(iv) the automorphism groups of Gómez like graphs.

A particular question considered by the author was whether a set of permutations Π is admissible for shift restricted word graphs if, and only if, the corresponding alphabet fixing subgraph \( \Gamma \) is \( n \)-reachable. The reason for this question is that both the Faber-Moore-Cheng graphs and the Gómez graphs have this property, and further a simple argument shows that admissibility implies a weaker but similar property as we now show.

**Lemma 40.** If \( \Pi \subseteq S_n \) is an admissible set of permutations, letting \( k < n \) and \( m = n - k \), then for any \( \tau \in S_n \) with

\[
\tau(i) = \begin{cases} 
  i - m, & \text{if } m < i \leq n, \\
  j, & \text{otherwise},
\end{cases}
\]

there are some \( \pi_1, \pi_2, \ldots, \pi_m \in \Pi \) such that \( \pi_1 \pi_2 \ldots \pi_m = \tau \).

**Proof.** Similar to Lemma 13. We simply consider a path from \( x_1 x_2 \ldots x_n \) to \( y_1 y_2 \ldots y_k x_{\tau(k+1)} x_{\tau(k+2)} \ldots x_{\tau(n)} \).

Consider the case \( n = 9, k = 3, m = 6 \) here

\[
\begin{align*}
  x_1 & \ x_2 & \ x_3 & \ x_4 & \ x_5 & \ x_6 & \ x_7 & \ x_8 & \ x_9 \\
  x_2 & \ x_3 & \ x_4 & \ x_5 & \ x_6 & \ x_7 & \ x_8 & \ x_9 & \ y_1 \\
  x_3 & \ x_4 & \ x_5 & \ x_6 & \ x_7 & \ x_8 & \ x_9 & \ y_1 & \ y_2 \\
  \pi_1 & \ x_4 & \ x_5 & \ x_6 & \ x_7 & \ x_8 & \ x_9 & \ y_1 & \ y_2 & \ y_3 \\
  x & \ x & \ x & \ x & \ x & \ y_1 & \ y_2 & \ y_3 & \ x \\
  x & \ x & \ x & \ x & \ y_1 & \ y_2 & \ y_3 & \ x & \ x \\
  x & \ x & \ y_1 & \ y_2 & \ y_3 & \ x & \ x & \ x & \ x \\
  x & \ y_1 & \ y_2 & \ y_3 & \ x & \ x & \ x & \ x & \ x \\
  \pi_2 & \ y_1 & \ y_2 & \ y_3 & \ x & \ x & \ x & \ x & \ x & \ x
\end{align*}
\]

Considering the path from \( \pi_1 \) to \( \pi_2 \) we find each permutation in this path must be in \( \Pi \) and the permutation of \( x_4, x_5, \ldots, x_9 \) is arbitrary.

**Corollary 14.** If \( \tau \in S_n \) such that there exists some \( k < n \) with

\[
\tau(i) = \begin{cases} 
  i' < k & \text{if } i < k \\
  i' \geq k & \text{if } i \geq k,
\end{cases}
\]

then there exist \( \pi_1, \pi_2, \ldots, \pi_n \in \Pi \) such that \( \tau = \pi_1 \pi_2 \ldots \pi_n \).

**Proof.** This is simply the concatenation of two of the previous permutations we showed exist in the previous lemma.

Hence, if \( \Pi \) is admissible, we can easily see “a lot” of permutations must be \( n \)-reachable. This taken in conjunction with the fact the known optimal admissible \( \Pi \) are \( n \)-reachable suggests that this may be a necessary requirement.
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