Dicke model for quantum Hall systems

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Abstract

In GaAs quantum Hall (QH) systems, electrons are coupled with nuclear spins through the hyperfine interaction, which is normally not strong enough to change the dynamics of electrons and nuclear spins. The dynamics of the QH systems, however, may drastically change when the nuclear spins interact with low-energy collective excitation modes of the electron spins. We theoretically investigate the nuclear–electron spin interaction in the QH systems as hybrid quantum systems driven by the hyperfine interaction. In particular, we study the interaction between the nuclear spins and the Nambu–Goldstone (NG) mode with the linear dispersion relation associated with the U(1) spin rotational symmetry breaking. We show that such an interaction is described as nuclear spins collectively coupled to the NG mode, and can be effectively described by the Dicke model. Based on the model we suggest that various collective spin phenomena realized in quantum optical systems can also emerge in the QH systems.

1. Introduction

Quantum Hall (QH) systems exhibit fascinating macroscopic quantum phenomena [1, 2]. Driven by the exchange Coulomb interactions, quantum coherence develops spontaneously and the system is organized into a QH ferromagnet. Particularly, the bilayer QH system presents rich physics due to the intralayer and interlayer phase coherence, which can be tuned by controlling the interplay between the spin and the layer (pseudospin) degrees of freedom. (Let us call the two layers the front and back layers.) The two-layer system acts as if it were a monolayer QH system. Then, the total filling factor defined by the summation of the individual filling factors in the front and back layers plays the role of the filling factor in the monolayer system. For instance, the total filling factor \( \nu = 1 \) bilayer QH system, which has well been studied both theoretically and experimentally, has exhibited many exciting coherent properties owing to the pseudospin [1, 2].

Research interests for the QH physics are not necessarily to be limited to the electron physics. The GaAs semiconductor with the s-type conduction band has a large natural abundance of nuclear spins with the 3/2 nuclear spin angular momentum \( ^{69}\text{Ga}, ^{71}\text{Ga}, \text{and} ^{73}\text{As} \). However, the electron-nuclear spin dynamics has not attracted much attention so far. This is because the Fermi contact hyperfine interaction is usually so weak that electron transport properties are not affected by this interaction. Almost all previous studies of QH physics have been focused on the electron physics, whereas the nuclear spins are utilized merely as a tool to investigate the electron magnetic properties [3]. The above situation may change when the nuclear spins interact coherently with low-energy excitation modes of electron spins such as Nambu–Goldstone (NG) modes. For instance, the NG modes appear in the canted antiferromagnetic (CAF) phase [4–12] due to the spontaneous breaking of the rotational U(1) symmetry. This phase is the most interesting one among the three phases of the total filling factor \( \nu = 2 \) bilayer QH systems: the ferromagnetic phase, the spin–singlet phase and the CAF phase. In the CAF phase, the antiferromagnetic correlations between the electron spins in the front layer and those in the back layer are...
generated, and the associated linear dispersing NG mode emerges. When the linear dispersing NG mode has a long wavelength, it can coherently couple with the nuclear spins all over the sample, and the electron-nuclear spin dynamics may become important, where a new physics is expected to emerge. In the related context, the nuclear spin relaxation was experimentally estimated by using the resistivity-detected nuclear spin relaxation measurement \[10\], where the longitudinal resistance $R_{\text{xx}}$ is used as a measure of the nuclear-spin polarization. It has been shown that the nuclear-spin relaxation time in the CAF phase is the shortest compared with those in the other two phases (the ferromagnetic phase and the spin-singlet phase). More recent experiment [12], where the spatial nuclear-spin polarization distribution was recorded after the exposure to the CAF phase, showed a sudden change in the nuclear-spin polarization distribution from the initial one. These experimental results suggest a unique many-body interaction between electron and nuclear spin systems.

It is worthwhile to investigate the nuclear spin physics mediated by the hyperfine interaction in the QH systems also from the following three perspectives: first, nonequilibrium phenomena of nuclear spins in the QH systems are still less understood to date. Second, we expect a rich variety of many body effects as well as cooperative phenomena in terms of a nuclear-spin ensemble driven by the magnetic properties of the QH systems, similar to dynamics seen in ultracold atoms [13] and in quantum dots [14, 15]. Recently, the creation of the helical nuclear spin order in the QH regime has been studied [16]. Third, the electron-nuclear spin hybrid system is a candidate for a spin-base quantum information processing and computing with a coherent manipulation [3, 17].

In this paper, motivated by the experiments [10, 12], as the first step of studying the electron-nuclear spin dynamics in the QH system as a hybrid quantum system, we theoretically study the interaction between the nuclear spins and the linear dispersing NG mode associated with the U(1) spin rotational symmetry breaking, a schematic illustration of this model is shown in figure 1. To make a detailed analysis and to clearly understand its physics behind, we focus on the CAF phase in the $2n=\text{bilayer QH system}$. We show that the NG mode couples with collective nuclear spins through the hyperfine interaction in the long-wavelength limit. This interaction is effectively described by the Dicke model, which has been extensively studied in the quantum optics \[18–21\]. It is interesting that the interaction between nuclear spins and the linear dispersing NG mode mediated by the hyperfine interaction in the QH systems can be described by the same model as the multi-level atomic system coupled with photonic modes.

Our analysis is not only applicable to the nuclear spin-NG mode interaction in the CAF phase, but also to other QH systems provided a linear dispersing NG mode is present due to the U(1) spin rotational symmetry breaking. For example, the skyrmion crystal formation in the monolayer QH system in the vicinity of the total filling factor $\nu = 1$ [22] may belong to this category, where a linear dispersing spin wave emerges and is expected to enhance the nuclear spin relaxation rate.

This paper is organized as follows. In section 2, we analyze the hyperfine interaction in the QH systems. Based on this analysis, in section 3 we derive the Dicke model as the effective Hamiltonian describing the interaction between the nuclear spins and the linear dispersing NG mode through the hyperfine interaction. It is the main result of this paper.

2. Hyperfine interaction in the QH systems

We first analyze the hyperfine interaction in QH systems to derive the interaction Hamiltonian between the nuclear spins and the NG mode. We may assume that all electrons are in the s-type conduction band. A high magnetic field is applied perpendicular to the plane, $\mathbf{B} = (0, 0, -B_z)$ with $B_z > 0$. 

![Figure 1. A schematic illustration of 'hybrid' quantum Hall (QH) systems. They consist of electron spins in the QH state and a large ensemble of nuclear spins in host crystals. They are coupled through the hyperfine interaction $H_{\text{HF}}$, and therefore, the total system can be regarded as a hybrid QH systems.](image-url)
The interaction between nuclear and electron spins in the QH state is described by the contact hyperfine interaction [23–25],

\[ H_{\text{HF}} = \frac{2\mu_0 g_e\gamma_e\hbar^2}{3N_g} \sum_{i=1}^{N_{\text{tot}}} \sum_{j=1}^{N_{\text{tot}}} |u(x_i, z_j)|^2 S_i \cdot (x_i - x_j, Z_i - z_j), \]

where \( S_i \) is the electron spin at \((x_i, z_i)\), \( I_i \) the nuclear spin at \((x_i, Z_i)\) assuming spin 1/2, and \( S(X_i, Z_i) \) the three-dimensional electron spin density. The quantities \( g_e \), \( \hbar \), \( \mu_0 \), \( u(x_i, z_i) \), \( N_{\text{tot}} \), and \( N_g \) are the gyromagnetic ratio for electron (nucleon), the permeability of the vacuum magnetic constant, the Bloch amplitude, the total number of nuclear spins residing in the quantum well, and the total electron number, respectively. When free electrons are within a narrow quantum well, the lowest energy level is well separated from all other energy levels at sufficiently low temperature. Electrons are essentially confined into two-dimensional plane since the motion of electrons along the growth \( z \) direction is frozen, and it is a good approximation to simulate the quantum well by the model where electrons are located at the center of the well, that is, \( z_j = z_0 \) for all \( j \), where the probability amplitude of the electronic wavefunction for the \( z \) direction takes the maximum. Then we can simulate the system by the model where electrons interact with nuclear spins effectively present at \( Z_i = z_0 \) for all \( i \).

The QH systems are hybrid systems consisting of many electrons and nuclei in the two-dimensional \( xy \) plane. Then we may set \( S(X_i, Z_i) \approx S(X_i) \left( L_z \right)^{-1} \) for \(-L_z/2 < Z_i < L_z/2\) with \( L_z \) the width of quantum well and \( S(X_i) \) the two-dimensional electron spin density. Since the Bloch amplitude is a periodic function with respect to the nuclear spin separation, we can set \( |u(x_i, z_0)|^2 = \eta = \text{const.} \) The hyperfine interaction (1b) is rewritten as

\[ H_{\text{HF}} = A \sum_{i=1}^{N} I_i \cdot S(X_i), \quad \text{with} \quad A = \frac{2\mu_0 g_e\gamma_e\hbar^2\eta}{3L_z}, \]

where \( N \) is the total number of nuclear spins effectively in the \( z = z_0 \) plane. The values of \( \eta \) for Ga and As are in the same order, given by \( \eta_{\text{Ga}} = 2.7 \times 10^3 \) and \( \eta_{\text{As}} = 4.5 \times 10^3 \), respectively [26], and within the current approximation we assume \( \eta = 10^3 \).

The hyperfine coupling is weak compared as the Landau-level energy as well as the thermal energy. It is reasonable to assume that the electronic ground state is unchanged by the hyperfine interaction, and to replace the electron spin density in equation (2) by the classical spin density \( S^c(X_i) = \langle \phi_{\text{QH}} | S(X_i) | \phi_{\text{QH}} \rangle \), with \( | \phi_{\text{QH}} \rangle \) denoting the QH state. We express the hyperfine interaction in terms of normalized spin density defined by \( S^c(X_i) = \rho_b \rho_i S^0(X_i) \), where \( \rho_b = \rho_i / \nu = 1/2 \pi \bar{g}^2 \) is the density of states of the Landau sites with \( \rho_i \) the total electron density, \( \nu \) the total Landau filling factor, and \( \bar{g} \) the magnetic length. Equation (2) becomes

\[ H_{\text{HF}} = \bar{g} \sum_{i=1}^{N} I_i \cdot S^c(X_i), \]

where \( \bar{g} = A \rho_b \). The Hamiltonian (3) describes the hyperfine interaction between the nuclear spins and the electron spins in the QH state. The order of the coupling \( \bar{g} \) in (3) is estimated by setting \( \rho_i = 4 \pi \times 10^{-7} \text{ N A}^{-2} \), \( g_e = 1.761 \times 10^{11} \text{ rad s}^{-1} \text{ T}^{-1} \), \( \gamma_e = 10 \times 10^5 \text{ rad s}^{-1} \text{ T}^{-1} \), \( \rho_b = 1 \times 10^{13} \text{ m}^{-2} \), \( L_z = 10^{-8} \text{ m} \), \( \eta = 10^3 \), and \( \bar{g} = 1.0546 \times 10^{-24} \text{ J s rad}^{-1} \), and we have \( \bar{g} / \hbar \sim \text{100} \text{ rad kHz} \). It is much smaller compared with the Larmor frequency \( \omega_L \sim \text{10} \text{ rad MHz} \). Note that in the QH system magnetic field is applied in the order of tesla.

3. Dicke model in the QH systems

We next derive the interaction Hamiltonian between the nuclear spins and the NG mode. We show that collective nuclear spins couple with the NG mode in the long-wavelength limit, and furthermore that it is effectively described by the Dicke model.

3.1. Electron spin configuration and effective Hamiltonian for the NG mode

When the U(1) spin rotational symmetry is spontaneously broken around the \( z \)-axis, the in-plane component of the classical electron spin density is expressed as

\[ S^{c, x}(x) = S \cos(\delta \theta + \delta \theta(x)), \quad S^{c, y}(x) = S \sin(\delta \theta + \delta \theta(x)), \]

where \( S \) is a constant in the range \( 0 < |S| < 1 \). In the ground state the electron spins are in a spatially homogeneous configuration with a fixed orientation angle \( \delta \theta \) of the in-plane spin component. The fluctuation field \( \delta \theta(x) \) is the associated NG mode. For the \( z \)-component, it is assumed that the fluctuation of the electron spin density is negligible, because it is gapped.

We expand the above spin densities in terms of \( \delta \theta(x) \) up to the first order and substitute it to the hyperfine interaction (3). The zeroth order terms with respect to \( \delta \theta(x) \) are
is the ground-state expectation value of the z component of the normalized electron spin density satisfying $0 < |\hat{S}_z^0| < 1$. The first term describes the in-plane magnetic field induced by the hyperfine interaction, which nuclear spins experience, while the second term generates the Knight shift $K_w$. These two terms have the same order of magnitude as the coupling $\tilde{g}/h \sim 100$ rad kHz, which was mentioned in the previous section, i.e., $K_w \sim 100$ rad kHz. Here we note that this value is comparable with the experimental result reported in [10], i.e., $(K_w)^{exp} \sim 10$ rad kHz. We can drop these terms because they are negligible compared with the nuclear-spin Larmor frequency, which is $\omega_L \sim 10$ rad MHz.

To understand the situation more clearly, we present an example, the CAF phase in the $\nu = 2$ bilayer QH state. As presented in figure 2, electron spins have ferromagnetic correlations in each layer, whereas they have antiferromagnetic correlations between the two layers. The electron spin configuration in the front (back) layer $S_{a}^{(f)}$ ($a = x, y, z$) is described as

$$S_{x}^{f} = -S_{x}^{b} = S \cos \vartheta_{0}, \quad S_{y}^{f} = -S_{y}^{b} = S \sin \vartheta_{0}, \quad S_{z}^{f} = \hat{S}_{z}^{0} = \hat{S}_{z}^{caf},$$

(7)

where $S_{a}^{(f)}$ ($a = x, y$) and $S_{z}^{caf}$ are the in-plane and z components of the electron spin in the front (back) layer, respectively. Here $S$ and $S_{z}^{caf}$ satisfy the conditions $0 < |S| < 1$ and $0 < |S_{z}^{caf}| < 1$. As a result, spins are canted coherently. By focusing on the in-plane component, electron spins orient homogeneously characterized by an angle $\vartheta_{0}$. Although the ground state energy does not depend on $\vartheta_{0}$, the ground state itself does, reflecting that the CAF state is the $U(1)$ spin rotational symmetry broken state around z axis. The small fluctuation mode $\delta \vartheta$ is the corresponding NG mode. We present the explicit formula of (7) in the case of the CAF phase in (A5) in appendix A.

The intriguing feature of the NG mode is that it has a linear dispersion in the CAF phase. We denote the fourier transform of $\delta \vartheta (x)$ as $\delta \vartheta_{k}$,

$$\delta \vartheta_{k} = \int \frac{d^2x}{2\pi} e^{-ikx} \delta \vartheta (x),$$

(8)

and introduce the canonical conjugate variable $\delta \sigma_{k}$ satisfying $[\delta \vartheta_{k}, \delta \sigma_{k'}] = i \delta (k - k')$. The effective Hamiltonian for the NG mode has the following form (in the case of the CAF phase see equation (B1) in appendix B)

$$H_{\delta} = \int d^2k \left[ a + b |k|^2 \right] \delta \sigma_{k}^{\dagger} \delta \sigma_{k} + \left( c |k|^2 \right) \delta \vartheta_{k} \delta \vartheta_{k}.$$  

(9)

where $a, b$, and $c$ are positive constants. This Hamiltonian is diagonalized by introducing another set of canonical variables.
\[ r_k = \frac{1}{\sqrt{2}} \left( \sqrt{G_k} \delta_\sigma k + \frac{i}{\sqrt{G_k}} \delta_\theta k \right), \quad r_k^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{G_k} \delta_\sigma k^\dagger - \frac{i}{\sqrt{G_k}} \delta_\theta k^\dagger \right), \]

satisfying \([r_k, r_k^\dagger] = \delta(k - k')\), where

\[ G_k = \left( a + b k^2 \right)^{1/2}. \]

Now the effective Hamiltonian (9) is diagonalized as

\[ H_R = \int d^3 k E_k r_k^\dagger r_k, \quad \text{with} \quad E_k = 2\sqrt{ck^2(a + b k^2)} \approx 2\sqrt{\alpha c} |k|, \]

with \(E_k\) a linear dispersion relation for the NG mode. We present the explicit formulas of equations (9), (11), and (12) for the case of the CAF phase in Appendix B: see equations (B4), (B7), and (B10), respectively.

### 3.2. Dicke model

We now describe the hyperfine interaction Hamiltonian (6) in terms of the NG mode \(r_k, r_k^\dagger\) and the nuclear spins \(I_i^z\). Using equations (8) and (10), we obtain

\[ H_{\text{IF}} = \frac{g}{2} \sum_{i=1}^{N} \left( \hat{I}_i^+ + \hat{I}_i^- \right) (R_i + R_i^\dagger), \]

\[ R_i = \int \frac{d^3 k}{2\pi} \sqrt{G_k} e^{-i k x_i - \frac{1}{2} i k^2 r_k}, \quad R_i^\dagger = \int \frac{d^3 k}{2\pi} \sqrt{G_k} e^{i k x_i - \frac{1}{2} i k^2 r_k^\dagger}, \]

where \(\hat{I}_i^\pm = e^{\pm i(\theta_i + \pi/2)} \hat{I}_i^z\) is the rotated in-plane nuclear spin, with \(\hat{I}_i^\pm = I_i^z \pm iI_i^\phi\). To derive (13b), we used the relations \(G_k = G_{k\cdot k}, \delta \sigma k = \delta \sigma_{k\cdot k}\) and \(\delta \theta k = \delta \theta_{k\cdot k}\). They follow from the fact that \(G_k\) are even function of \(k\), and \(\delta \sigma(x) = \int (d^3 k/2\pi) e^{-i k x} \delta \sigma_{k\cdot k}\) and \(\delta \theta(x)\) are real fields. Here, the \(i\) dependence disappears from \(R_i\) and \(R_i^\dagger\) in the long-wavelength limit \(e^{ikx} \approx 1\). We thus obtain

\[ H_{\text{SR}} = \frac{g}{2} \sum_{i=1}^{N} \left( \hat{I}_i^+ + \hat{I}_i^- \right) (R_i + R_i^\dagger) = \frac{g}{2} \left( J^+ + J^- \right) (R + R^\dagger), \]

\[ R = \int \frac{d^3 k}{2\pi} \kappa_k r_k, \quad R^\dagger = \int \frac{d^3 k}{2\pi} \kappa_k^\dagger r_k^\dagger, \]

\[ \kappa_k = \sqrt{\frac{G_k}{2}} e^{-i k^2 r_k}, \quad \kappa_k^\dagger = \sqrt{\frac{G_k}{2}} e^{i k^2 r_k^\dagger}, \]

where we have set

\[ J^\pm = \sum_{i=1}^{N} I_i^\pm. \]

Indeed, as the NG mode has a long wavelength, the approximation \(e^{ikx} \approx 1\) is valid in this system. For instance, in the case of the CAF phase the wavelength of the NG mode becomes \(\lambda_\gamma \sim 10^6 \text{ Å}\) for \(E_k = \hbar \omega_{\text{L}}\) with \(\omega_{\text{L}} = 100 \text{ rad MHz}\), where \(\omega_{\text{L}} = \gamma |k| (\gamma = 2\sqrt{\alpha c} > 0)\) is the linear dispersion relation for the NG mode. The value of \(\lambda_\gamma\) is about the same as the sample size \(L \sim 100 \mu\text{m}\) (see equation (B11)). Hence it is a good approximation at this energy scale. If the dispersion for the NG mode were quadratic with the same coefficient as the linear one, the wavelength at \(E_k = \hbar \omega_{\text{L}}\) would be around \(\lambda_\gamma \sim 10^4 \text{ Å}\), which is smaller than the sample size. Such a case, the long-wavelength limit is not a good approximation so that the interaction Hamiltonian \(H_{\text{SR}}\) cannot be expressed in terms of the NG mode and the collective spin.

We have shown that the interaction between the nuclear spins and the linear dispersing NG mode is described as an interaction between the collective nuclear spin \(J\) and the NG mode \(R\) with the coupling constant \(g\) in the long-wavelength limit. The resultant Hamiltonian (14a) has been extensively studied in the quantum optics [18–21]. The NG mode has a continuous spectrum containing the nuclear-spin Larmor frequency \(\omega_i\). In the regime where \(\hbar \omega_i \approx E_k\), the counter-rotated terms \((J^+ R^\dagger, J^- R^\dagger)\) can be averaged out with respect to time, and the interaction between the collective nuclear spin and the NG mode can be effectively described as

\[ H_{\text{SR}} = \frac{g}{2} \left( J^+ R^\dagger + J^- R^\dagger \right). \]

The nuclear-spin Hamiltonian \(H_5\) describing the Larmor precision is

\[ H_5 = -\hbar \gamma_0 B_z \sum_{i=1}^{N} I_i^z = \hbar \omega_i \sum_{i=1}^{N} I_i^z = \hbar \omega_i J^z, \]

where \(\gamma_0\) is the Larmor precision and \(B_z\) the magnetic field. Here, we have used the approximation \(\hbar \omega_i \approx E_k\) for \(i = 1, \ldots, N\), where \(N\) is the sample size.
and hence, the total effective Hamiltonian \( H = H_S + H_R + H_{SR} \) is given by

\[
H = \hbar \omega J^z + \int d^2 \mathbf{k} E_{\mathbf{k}} n_{\mathbf{k}} \mathbf{r}_k + \frac{g}{2} (J^+ R + J^- R^+).
\] (18)

Consequently, the interaction between the nuclear spins and the linear dispersing NG mode mediated by the hyperfine interaction is described effectively by the Dicke model \([18-21]\) with a continuous mode, where the collective spin operator \( J^i (i = x, y, z) \) with its magnitude \( N/2 \) interacts with the NG mode with the coupling \( g \). For an explicit derivation of this Dicke model in the case of the CAF phase, see appendix C.

It is interesting that the nuclear spin–NG mode interaction mediated by the hyperfine interaction in the QH systems can be described by the analogue model as the two-level atomic systems surrounded by the electromagnetic field in the vacuum: nuclear-spins 1/2 correspond to the two-level atoms, whereas the NG mode corresponds to the electromagnetic field in the vacuum \([21]\).

4. Conclusion

In this paper, we have presented the theoretical studies on the interaction between the nuclear spins and the linear dispersing NG mode due to the spin U(1) rotational symmetry breaking in the QH systems mediated by the hyperfine interaction. Since the NG mode has a long wavelength, the nuclear spins couple collectively with the NG mode all over the sample. Consequently, the nuclear spin–NG mode interaction can be described effectively by the multi-level Dicke model with a continuous NG mode. This collective interaction can effectively enhance the coupling strength due to the effect of the large number of nuclear spins, which implies that collective dynamics which can be seen in quantum optical systems, such as superradiant decay, could be observed in the QH system. Although here we focused on the CAF phase in the \( \nu = 2 \) bilayer QH system, this Dicke model could also be applicable to other QH systems with the linear dispersing NG mode due to the spin U(1) rotational symmetry breaking in general, for instance, to the nuclear spin–NG mode interaction in the vicinity of the \( \nu = 1 \) monolayer QH system in the presence of the skyrmion crystal formation.

We would like to emphasize that the Dicke model derived in this paper can lead to new directions in the study of QH physics, which cannot be obtained by solely focusing on the electron spin physics. Regarding the QH systems as hybrid quantum systems composed of electron and nuclear spins, as we discussed in this paper, new collective effects could appear from the interaction. To apply this model to analyze experimental results, we need to consider noise in the system. In the QH systems, dephasing in the nuclear spin ensemble can be considered as an important factor, however given the dephasing rate of nuclear spins is rather small, it is possible that the collective phenomenon predicted in this paper can be experimentally observed. To theoretically evaluate the noisy dynamics, we plan to extend the theory further in a future work. The study of the spin-boson dynamics inherent to the QH system might be another interesting direction.

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Appendix A. Order parameters in the CAF phase

We present the derivation of the Dicke model by constructing the Hamiltonian \( H_R \) for the NG mode and the interaction Hamiltonian \( H_{SR} \) between the nuclear spin and the NG mode in the case of the CAF phase. To derive the Dicke model, we make a concise review of the effective Hamiltonian density for the ground state and the NG modes in bilayer QH systems \([2, 8, 11]\). We start with the discussion on the phase structure as well as the spin density configuration, and the associated NG modes at \( \nu = 2 \). Then we present the effective Hamiltonian density for the linear dispersive NG mode in the CAF phase.

In the bilayer QH systems electrons possess the four internal degrees of freedom, the spin and the layer (pseudospin). We denote the two layers as the ‘front’ and ‘back’ layers. The electron field operator in the bilayer QH systems is represented as \( \Psi (\mathbf{x}) = (\psi_{1f} (\mathbf{x}), \psi_{1b} (\mathbf{x}), \psi_{2b} (\mathbf{x}), \psi_{2f} (\mathbf{x})) \). The physical operators in this system are expressed in terms of the following sixteen operators, the density operator and SU(4) isospin operators. In
terms of the electron field operator $\Psi(x)$, they are given by

$$\rho(x) = \Psi^\dagger(x)\Psi(x), \quad S_a(x) = \frac{1}{2} \Psi^\dagger(x)\tau_{n}^a\Psi(x),$$

$$P_a(x) = \frac{1}{2} \Psi^\dagger(x)\tau_{n}^{\text{pin}}\Psi(x), \quad R_{ab}(x) = \frac{1}{2} \Psi^\dagger(x)\tau_{n}^{\text{pin}}\tau_{n}^{b}\Psi(x),$$  \hspace{1cm} (A1)

where $a, b = x, y, z$ and

$$\tau_{n}^{\text{spin}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_{n}^{\text{pin}} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

$$\tau_{n}^{\text{pin}} = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix}, \quad \tau_{n}^{\text{pin}} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$  \hspace{1cm} (A2)

with $\tau_n$ denoting the Pauli matrices. The operator $\rho(x)$ is the density operator, while the SU(4) isospin density operator $S_a(x)$, $P_a(x)$ and $R_{ab}(x)$ represent the spin density operator, the pseudospin density operator, and the R-spin density operator, respectively. On the other hand, the total Hamiltonian in this system is given by [27],

$$H = H_0 + H_C + H_Z + H_{DZ},$$

where $H_0$ is the kinetic term which generates the Landau level, $H_C$ the Coulomb interaction term, $H_Z$ the Zeeman interaction term, and $H_{DZ}$ the pseudo-Zeeman term composed of the tunneling interaction term and the bias term, which represents the creation of the density-imbalanced configuration between the two layers. The Coulomb interaction is decomposed into the form $H_C = H_{C}^+ + H_{C}^-$, where $H_{C}^+ (-)$ represents the SU(4) invariant (non-invariant) term.

We analyze the classical Hamiltonian density of $H = \int d^2x H(x)$ by performing the lowest Landau level projection. We denote the generic lowest Landau level state as $|\mathcal{S}\rangle$. We derive the form of $H^{cl}(x) = \langle \mathcal{S}|H(x)|\mathcal{S}\rangle$, which are represented by the classical density operators $\rho^{cl}(x) = \langle \mathcal{S}|\rho(x)|\mathcal{S}\rangle$, $S^{cl}_a(x) = \langle \mathcal{S}|S_a(x)|\mathcal{S}\rangle$, $P^{cl}_a(x) = \langle \mathcal{S}|P_a(x)|\mathcal{S}\rangle$, and $R^{cl}_{ab}(x) = \langle \mathcal{S}|R_{ab}(x)|\mathcal{S}\rangle$. Actually we express $H^{cl}$ in terms of the normalized SU(4) operators defined by $S^{cl}_a(x) = \rho_0 S^{cl}_a(x)$, $P^{cl}_a(x) = \rho_0 P^{cl}_a(x)$, and $R^{cl}_{ab}(x) = \rho_0 R^{cl}_{ab}(x)$, by setting $\rho^{cl}(x) = \rho_0$ due to the incompressibility of the QH state.

In the $\nu = 2$ bilayer QH system the SU(4) order parameters, which are the expectation values of the normalized SU(4) operators in the ground state, are given by [8]

$$S^0_s = -\frac{\Delta_z}{\Delta_0} (1 - \alpha^2) \sqrt{1 - \beta^2},$$

$$P^0_s = \frac{\Delta_{\text{SAS}}}{\Delta_0} \alpha^2 \sqrt{1 - \beta^2}, \quad P^0_s = \frac{\Delta_{\text{SAS}}}{\Delta_0} \alpha^2 \beta = \sigma_0,$$

$$R^0_{sx} + iR^0_{sx} = -\frac{\Delta_{\text{SAS}}}{\Delta_0} \alpha \sqrt{1 - \alpha^2} e^{-i\omega},$$

$$R^0_{sy} + iR^0_{sy} = \frac{\Delta_z}{\Delta_0} \alpha \sqrt{1 - \alpha^2} \sqrt{1 - \beta^2} e^{i\omega},$$

$$R^0_{sz} + iR^0_{sz} = \frac{\Delta_z}{\Delta_0} \alpha \sqrt{1 - \alpha^2} \sqrt{1 - \beta^2} e^{-i\omega},$$  \hspace{1cm} (A3)

with

$$\Delta_0 = \sqrt{\Delta_{\text{SAS}}^2 + \Delta_z^2 (1 - \alpha^2) (1 - \beta^2)},$$  \hspace{1cm} (A4)

where $\Delta_z$ is the Zeeman gap and $\Delta_{\text{SAS}}$ the tunneling gap; $\alpha, \beta (|\alpha|, |\beta| \leq 1)$ and $\omega$ are real parameters. The quantity $\sigma_0$ is the imbalanced parameter, representing the density difference between the front and back layers, and defined by $\sigma_0 = (\rho_0^+ - \rho_0^-)/(\rho_0^+ + \rho_0^-)$, with $\rho_0^+$ the electron density in the front (back) layer. Here the parameters $\alpha$ and $\beta$ are determined by minimizing the classical Hamiltonian given by (B1), where the normalized isospin densities are in spatially homogeneous configurations. They satisfy the condition $(S^0_s)^2 + (P^0_s)^2 + (R^0_{sx})^2 = 1$. We demonstrate later that the parameter $\omega$ is associated with the NG mode in the CAF phase; see (B3).

It was shown [8] that for $\alpha = 0$, the ground state is the ferromagnetic phase, where only the spin is polarized as $S^0_s = 1$. On the other hand, for $\alpha = 1$ the ground state is the spin-singlet phase, where only the pseudospin is polarized as $P^0_s = \sigma_0$ and $P^0_s = \sqrt{1 - \sigma_0^2}$. The CAF phase is realized for $0 < \alpha < 1$.

From $2S^0_s = S^0_a + R^0_{sz}$ and $2\sigma^0_a = S^0_a - R^0_{sz}$, the relations between the spin densities in the front and back layers are
\[ S^t_x = -S^b_x = \frac{1}{2} \frac{\Delta_{\text{SAS}}}{\Delta_0} \sqrt{1 - \alpha^2} \sqrt{1 - \beta^2} \cos \omega, \]
\[ S^t_y = -S^b_y = -\frac{1}{2} \frac{\Delta_{\text{SAS}}}{\Delta_0} \sqrt{1 - \alpha^2} \sqrt{1 - \beta^2} \sin \omega, \]
\[ S^t_z = S^b_z = \frac{1}{2} S^0_z. \]  

(A5)

From the above equation, we see that in the CAF phase the antiferromagnetic correlation is built up between the two layers. Here the angle \( \omega \) describes the orientation angle of the in-plane spin component. The order parameters (A3) are obtained from the ones with \( \omega = 0 \) by the spin rotation \( \exp \{ iT_{\text{cap}} \omega \} \). Furthermore, the Hamiltonian density (B1) is invariant under this rotation. Thus, the CAF phase is the \( U_{T_{\text{cap}}}(1) \) spin rotational symmetry broken state.

At \( \nu = 2 \), four complex NG modes emerge due to the symmetry breaking pattern \( \text{SU}(4) \rightarrow \text{U}(1) \times \text{SU}(2) \times \text{SU}(2) \). All the NG modes get gapped due to the presence of the Coulomb interactions, the Zeeman and pseudo-Zeeman terms. We analyze the system in the limit \( \Delta_{\text{SAS}} \rightarrow 0 \), where only one NG mode responsible to the interlayer phase coherence becomes gapless. In this limit the order parameters (A3) are reduced to
\[ S^0_z = |\sigma_0| - 1, \quad \tau^0_z = \sigma_0, \]
\[ R_{xy} = -\text{sgn}(\sigma_0) R_{0,xx} = \sqrt{|\sigma_0|(1 - |\sigma_0|)} \cos \omega, \quad R_{xy} = \text{sgn}(\sigma_0) R_{0,xy} = \sqrt{|\sigma_0|(1 - |\sigma_0|)} \sin \omega, \]  

(A6)

with the relations
\[ \frac{\Delta_0}{\Delta_{\text{SAS}}} \sqrt{1 - \beta^2} = 1, \quad \frac{\Delta_{\text{SAS}}}{\Delta_0} = 1, \quad \alpha^2 = |\sigma_0|, \quad \sqrt{1 - \beta^2} = 0, \quad \beta = \text{sgn} \sigma_0, \]  

(A7)

which are obtained in the limit \( \Delta_{\text{SAS}} \rightarrow 0 \). We consider the case \( \sigma_0 > 0 \) explicitly, while the case \( \sigma_0 < 0 \) is similarly discussed. The order parameters (A6) imply that the \( \text{SU}(4) \) isospins are given by
\[ S^t_x(x) = \sigma(x) - 1, \quad \tau^t_z(x) = \sigma(x), \]
\[ R_{xy}(x) = -R_{0,xy}(x) = \sqrt{\sigma(x)(1 - \sigma(x))} \cos \vartheta(x), \]
\[ R_{xy}(x) = R_{xy}(x) = -\sqrt{\sigma(x)(1 - \sigma(x))} \sin \vartheta(x), \]  

(A8)

with all the others being zero, where \( \sigma(x) \) and \( \vartheta(x) \) are the canonical set of the NG mode. The ground-state expectation values of these fields must be \( \langle \sigma(x) \rangle = \sigma_0 \) and \( \langle \vartheta(x) \rangle = \vartheta_0 = -\omega \). Since \( \rho_{xy}(\sigma) \) represents the density while \( \vartheta(y) \) the angle variable, the following canonical commutation relation holds
\[ \rho_{xy}[\sigma(x), \vartheta(y)] = i\hbar(x - y). \]  

(A9)

We refer the detailed derivation of the \( \text{SU}(4) \) isospins (A8) and the canonical commutation relation (A9) to [11].

Appendix B. Effective Hamiltonian for the NG mode in the CAF phase

We next derive the effective Hamiltonian density and the dispersion relation. Apart from irrelevant constants, the basic Hamiltonian density for the ground state and the associated NG modes is given by [2]
\[ \mathcal{H}_{\text{eff}} = J_x \left( \sum \left( \partial_\xi S_0^x \right)^2 + \left( \partial_\xi P_x \right)^2 + \left( \partial_\xi R_{ab} \right)^2 \right) + 2J_y \left( \sum \left( \partial_\xi S_0^y \right)^2 + \left( \partial_\xi P_y \right)^2 + \left( \partial_\xi R_{ab} \right)^2 \right) + \rho_{xy} \left( \epsilon_{\text{cap}} (P_x)^2 - 2 \epsilon_X (S_0^y)^2 + \epsilon_{\text{cap}} (R_{ab})^2 \right), \]  

(B1)

where \( k = x, y, \Delta_{\text{bias}} \) the bias parameter, and
\[ J_x = J_x^0 + J_x^1 = \frac{1}{16 \sqrt{2\pi}} E_{\text{cap}}^0, \quad J_y = J_y^0 - J_y^1 = J_y \left[ -\frac{2}{\sqrt{2 \pi} \ell_0} + \left( 1 + \frac{d^2}{2 \ell_0^2} \right) \epsilon_{\text{cap}} \left( d / \sqrt{2} \ell_0 \right) \epsilon_X \right], \]
\[ \epsilon_X = \frac{1}{2} \frac{\pi}{2} E_{\text{cap}}^0, \quad \epsilon_{\text{cap}} = \frac{1}{2} \left[ 1 \pm e^{2i/2\ell_0^2} \text{erfc}(d / \sqrt{2} \ell_0) \right] \epsilon_X, \quad \epsilon_D = 4 \ell_0^2 E_{\text{cap}}^0, \quad \epsilon_{\text{cap}} = 4 \epsilon_D - 2 \epsilon_X, \]  

(B2)

with \( E_{\text{cap}}^0 = e^{2} / 4 \pi \ell_0^2 \) and \( d \) the layer separation.

We introduce the fluctuation fields \( b \sigma(x) \) and \( b \vartheta(x) \) around the ground state by
\[ \sigma(x) = \sigma_0 + b \sigma(x), \quad \vartheta(x) = \vartheta_0 + b \vartheta(x). \]  

(B3)

Note that \( \vartheta_0 \) is nothing but the the orientation angle of the in-plane spin component \( \vartheta(x) \) introduced in the main text: see (4). Substituting equation (A8) together with (B3) into the Hamiltonian density (B1), we obtain
\[ H_{\text{eff}} = \frac{I_0}{2} (\nabla \delta \sigma)^2 + \frac{2I_0}{\rho_0}(\nabla \phi)^2 + \frac{2\epsilon_{\text{cap}} - 1}{\rho_0} \delta \sigma^2, \]  
\tag{B4}

where \( \epsilon_{\text{cap}} = \epsilon_{\text{cap}} - 2\epsilon_{\text{gas}}, \sigma(x) = \rho_0 \delta \sigma(x) \) and

\[ J_0 = 4I_k \left( \frac{2\sigma_0 - 1}{\sigma_0(1 - \sigma_0)} \right) f_k^4, \quad J_0 = 4I_k^4 \sigma_0(1 - \sigma_0). \]  
\tag{B5}

The Hamiltonian density (B4) is written in the second-quantized form when the canonical commutation relation (A9) is imposed. We introduce the annihilation and creation operators

\[ r_k = \frac{1}{\sqrt{2}} \left( \sqrt{G_k} \sigma_k + i \frac{1}{\sqrt{G_k}} \delta \eta_k \right), \quad r_k^† = \frac{1}{\sqrt{2}} \left( \sqrt{G_k} \sigma_k^† - i \frac{1}{\sqrt{G_k}} \delta \eta_k \right), \]  
\tag{B6}

where \( \delta \eta_k \) and \( \delta \eta_k \) denoting the Fourier transforms of the fields \( \sigma(x) \) and \( \delta \eta(x) \), respectively, and

\[ G_k = \left( \frac{\lambda_\sigma}{\lambda_0} \right)^{1/2}, \quad \lambda_\sigma = \frac{2I_0}{\rho_0} k^2 + \frac{2\epsilon_{\text{cap}} - 1}{\rho_0}, \quad \lambda_0 = \frac{I_0}{2} k^2. \]  
\tag{B7}

From (A9), we obtain the commutation relations

\[ [r_k, r_k^†] = \delta (k - k'), \quad [\sigma_k, \delta \eta_k^†] = i\delta (k - k'). \]  
\tag{B8}

By using (B6), the Hamiltonian density (B4) in the momentum space is diagonalized as

\[ H_k = \int d^2k E_k r_k^† r_k, \]  
\tag{B9}

where \( E_k \) is given by

\[ E_k = |k| \sqrt{\frac{2I_0}{\rho_0} \left( \frac{2I_0}{\rho_0} k^2 + \frac{2\epsilon_{\text{cap}} - 1}{\rho_0} \right)} \approx 2|k| \sqrt{\frac{I_0 \epsilon_{\text{cap}} - 1}{\rho_0}} = \gamma |k|. \]  
\tag{B10}

Hence, the NG mode has the linear dispersion relation.

The wave length of the NG mode at \( \omega_i = 100 \text{ rad MHz} \) is estimated as

\[ \lambda_s = \frac{2\pi}{k_i} = 2.90454 \times 10^6 \text{ Å}, \]  
\tag{B11}

where \( k_i = h \omega_i \gamma^{-1} \), and we have set \( \hbar = \hbar = d = 230.967 \text{ Å}, \) or equivalently, \( \rho_0 = 0.59669 \times 10^{-5} \text{ Å}^{-2}, \) while \( \sigma_0 = 0.3676, \) as typical values for QH samples, and \( k_0 = 1.3807 \times 10^{-23} \text{ J K}^{-1} \). The wavelength (B11) is about the same size as the sample size, \( L \sim 100 \mu \text{m}. \) Thus the long-wavelength approximation is valid in this case.

Since \( \sigma(x) \) and \( \delta \eta(x) \) are real fields and \( G_k \) are even function of \( k \), we obtain the relations \( G_k = G_{-k}, \sigma_k = \sigma_{-k}, \delta \eta_k = \delta \eta_{-k}. \) Thus, by using them the phase field \( \delta \eta(x) \) is described in terms of (B6) as

\[ \delta \eta(x) = \int \frac{d^2k}{2\pi} e^{i\tilde{\kappa}_{x,k} (r_k - r_k^†)} = \int \frac{d^2k}{2\pi} (\kappa_{x,k} r_k + \kappa_{x,k}^* r_k^†), \]  
\tag{B12}

In the long wave-length limit, we have \( e^{i\tilde{\kappa}_{x,k}} \rightarrow 1 \), and there is no \( x \) dependence in \( \kappa_{x,k}. \) In this limit we just write it as \( \kappa_{x,k}. \)

**Appendix C. Dicke model in the CAF phase**

We proceed to derive the interaction Hamiltonian between the nuclear spins and the NG mode in the CAF phase based on (3). We assume that only the nuclear spins in one of the layers are dynamically polarized. This is indeed the case in the experiment [12]. Thus we consider the interaction between nuclear spins in the front layer and the NG mode \( \delta \eta \). From now on, we omit the pseudospin index in the spin density. First from (A8), we see that \( S_z \) is dynamically frozen, because the imbalanced field \( \sigma \) has a gap much larger than the thermal energy, and therefore, its excitation is suppressed. Thus the interaction between the nuclear spins and the NG mode is solely expressed by the spin–spin interaction for the in-plane component. For simplicity, we start from equation (A5) where any limits are not taken. By setting \( -\omega \rightarrow \phi(X) = \phi_0 + \delta \eta(X) \), and expanding \( S_z(X) \) in terms of \( \delta \eta \) up to the linear order, we have
\[ S_{\alpha}(\mathbf{X}_i) = S(\cos \vartheta_0 - \sin \vartheta_0 \cdot \delta \vartheta(\mathbf{X}_i)) + O(\delta \vartheta^2), \quad S_{\beta}(\mathbf{X}_i) = S(\cos \vartheta_0 + \cos \vartheta_0 \cdot \delta \vartheta(\mathbf{X}_i)) + O(\delta \vartheta^2), \]  

(C1)

where \( S = \Delta_{SASO} \sqrt{1 - \alpha^2} \sqrt{1 - \beta^2} / 2 \Delta_0 \) with \( 0 \leq S < 1 \). Then, from (3) the hyperfine interaction for the in-plane component becomes

\[ H_{\text{HF}} = \frac{g}{2} \sum_{i=1}^{N} \left( \cos \vartheta_0 I_{z}^i + \sin \vartheta_0 I_{x}^i \right) + \sum_{i=1}^{N} g \vartheta(\mathbf{X}_i)(-\sin \vartheta_0 I_{z}^i + \cos \vartheta_0 I_{x}^i), \]  

(C2)

where \( g = \frac{g'}{S} \). The first term in (C2) represents the in-plane Knight shift term and is much smaller compared with the Larmor frequency \( \omega_0 \) (see the discussion in section 2). With the same reason, we can neglect the interaction term between the \( z \) component of nuclear spins and that of electron spins. Hence we only retain the second term, representing the interaction between the nuclear spins and the NG mode. By introducing \( \mathbf{I}^\pm = \mathbf{I}^x \pm i \mathbf{I}^y \) and using (B12), we have

\[ H_{\text{HF}} = \frac{g}{2} \sum_{i=1}^{N} \left( \mathbf{I}_z^i + \mathbf{I}_x^i \right) \int \frac{d^3k}{2\pi} \left( \kappa_{X_i,k} \mathbf{R}_i + \kappa_{X_i,k}^* \mathbf{R}_i^* \right), \]  

(C3)

where \( \mathbf{I}^\pm = e^{i(\vartheta_0 + \pi/2)} \mathbf{I}^x \) is the rotated in-plane nuclear spin. By using the rotating-wave approximation, we obtain

\[ H_{\text{SR}} = \frac{g}{2} \sum_{i=1}^{N} \left( \mathbf{I}_z^i \mathbf{R}_i + \mathbf{I}_x^i \mathbf{R}_i^* \right), \quad \mathbf{R}_i = \int \frac{d^3k}{2\pi} \kappa_{X_i,k} \mathbf{R}_i^* \mathbf{R}_i \]  

(C4)

In the long-wavelength limit the \( i \) dependence disappears from \( \mathbf{R}_i \) and \( \mathbf{R}_i^* \). Thus we obtain the interaction Hamiltonian for the nuclear spins and the NG mode in the CAF phase

\[ H_{\text{SR}} = \frac{g}{2} \sum_{i=1}^{N} \left( \mathbf{I}_z^i \mathbf{R} + \mathbf{I}_x^i \mathbf{R}^* \right) = \frac{g}{2} (\mathbf{I}_z \mathbf{R} + \mathbf{I}_x \mathbf{R}^*). \]  

(C5)

By combining the Larmor–precision Hamiltonian (17), the effective Hamiltonian for the NG mode (B9), and the interaction Hamiltonian (C5), we have the Dicke model in the CAF phase.

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