A small-mode model for spatial regimes on a freely falling liquid film

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Abstract. Spatial wave modes of a viscous liquid film freely flowing down a vertical wall are considered. In the case of small Reynolds numbers the problem is reduced to considering solutions of a nonlinear equation for the film thickness deviation from the undisturbed level. It was used to obtain a model system of equations, describing spatially periodic solutions with wave numbers lying in the vicinity of special resonance points and symmetric in the transverse coordinate. Several solutions for this system are provided.

1. Introduction and problem statement
The study of waves on a viscous liquid film is known to be an extremely fruitful problem for researchers both experimentally and theoretically. In particular, it is a rich source of models of varying degrees of complexity. For the case of small fluid flows, the consideration of spatial long-wave disturbances of small but finite amplitude for a freely flowing film is reduced to the study of one equation for the deviation of the film thickness from the undisturbed level. It is obtained in [1], where it serves to study the stability of plane wave regimes with respect to spatial perturbations. After an appropriate transformation of the variables, this equation takes the form [2]:

\[
\frac{\partial H}{\partial t} + 4H \frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2} + \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right]^2 H = 0. \tag{1.1}
\]

Here \( H \) is the "stretched" deviation of the film thickness from the undisturbed level, the x-axis is the direction of undisturbed film flow, the z-axis is perpendicular to the x-axis and parallel to the surface of the undisturbed film. Equation (1.1) is written in the frame of reference moving relative to the wall with the velocity of plane neutral perturbations [1].

Equation (1.1) refers to a class of evolutionary equations modeling the behavior of perturbations in active-dissipative media: a trivial solution \( H = 0 \) is unstable with respect to linear perturbations \( \sim \exp[i\alpha(x-ct) + i\beta z] \) if the components of their wave vector \( \alpha \) and \( \beta \) on the half-plane \( (\alpha, \beta \geq 0) \) lie within the region bounded by the curve:

\[
\left( \alpha^2 + \beta^2 \right)^2 = \alpha^2. \tag{1.2}
\]
Here, perturbations are neutral if \((\alpha, \beta)\) belong to curve (1.2) and damp if their wave numbers lie above this curve.

Nepomnyashchy in [3] obtained some spatially periodic solutions of equation (1.1). In [2], steady-state periodic solutions of this equation were studied analytically and numerically. Several families of such solutions were constructed, and it was demonstrated that there are complex mutual transitions between them. In particular, the family, which in the limit passes into the soliton of the characteristic horseshoe-shaped form, was determined. It is of interest that such horseshoe-shaped soliton solutions were first obtained for equation (1.1) in [4]. Later, similar solutions were also found in more complex models applicable in the case of moderate Reynolds numbers (see, for example, [5]). The value of these solutions lies in the fact that in developed non-stationary flow regimes, similar horseshoe-shaped forms are often observed as a structural element. But only relatively recently in [6] in special precise experiments it has become possible to practically realize the steady-state travelling solitary horseshoe-shaped modes on a flowing film. They well coincided not only qualitatively, but also quantitatively, including with the corresponding solutions of equation (1.1) of [4].

As it known (see, for example, [2], [3]), the spatial steady-state travelling periodic solutions of equation (1.1) branch off from the trivial solution along curve (1.2). In [2] it was shown that the vicinity of the point \((\alpha = 1/2, \beta = 1/2)\) requires special consideration. This is due to the fact that because of the quadratic nonlinearity in equation (1.1) in one of the terms of the second approximation, the wave number falls into the vicinity of the point \(\alpha = 1\), also belonging to curve (1.2). The aim of this work is to obtain a model system of equations, describing the evolution of perturbations with wave numbers lying in the vicinity of these resonance points. In this case, we limit ourselves to considering spatially periodic \(z\)-symmetric solutions. Let us seek for solutions in the form of a series:

\[
H = \varepsilon H_1 + \varepsilon^2 H_2 + \varepsilon^3 H_3 + \ldots
\]

(1.3)

Acting similar to [2], we introduce a set of fast and slow variables

\[
t_n = \varepsilon^n t, \quad x_n = \varepsilon^n x, \quad z_n = \varepsilon^n z, \quad n = 0, 1, 2, \ldots
\]

(1.4)

Substituting (1.3) into equation (1.1), using (1.4) and equating the coefficients at the same powers \(\varepsilon\), we come to an infinite system of linear equations. The equation of the first order has the form

\[
\frac{\partial H_1}{\partial t_0} + \frac{\partial^2 H_1}{\partial x_0^2} + \left[\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial z_0^2}\right] H_1 = 0.
\]

(1.5)

The solutions of interest have the following form:

\[
H_1 = A_1 \exp(ix_0/2)\left[a \exp(iz_0/2) + \bar{a} \exp(-iz_0/2)\right] + A_2 \exp(ix_0) + K.C.
\]

(1.6)

Here K. C. is a complex conjugate expression. Hereinafter the bar over the letter denotes complex conjugate value. We emphasize that the amplitudes of harmonics \(A_1, A_2\) and \(a\) do not depend on the fast time \(t_0\), but can be functions of slower times and coordinates. If \(a\) does not depend on the slower coordinates \(z_n\), then in subsequent approximations it should be assumed a real function and included in the harmonic \(A_1\). If there is such a dependence on the slower \(z_n\) coordinates, then in order to provide a complete symmetry of the solution over \(z\) (i.e., for all \(z_n\) coordinates considered in the model), \(a\) has to satisfy the following relations:

\[
a(z_n) = \bar{a}(-z_n).
\]

(1.7)
Consideration of the second order relative to $\varepsilon$ leads to the equation:

$$\frac{\partial H_1}{\partial t} + 2 \frac{\partial (H^2)}{\partial x_0} + 2 \frac{\partial^2 H_2}{\partial x_0^2} + 2 \frac{\partial^2 H_1}{\partial x_0 \partial x_1} + \left[ \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial z_0^2} \right] H_2 + 4 \left[ \frac{\partial^4 H_1}{\partial x_0^2 \partial z_0 \partial z_1} + \frac{\partial^4 H_1}{\partial z_0^2 \partial x_0 \partial x_1} + \frac{\partial^4 H_1}{\partial z_0^2 \partial x_1} \right] = 0. \tag{1.8}$$

Substituting (1.6) into (1.8) and requiring secular terms to be zero, we obtain a system:

$$\frac{\partial A_1 a}{\partial t} + 2iA_2 \bar{A}_1 a - iA_4 \frac{\partial a}{\partial z_1} = 0,$$

$$\frac{\partial A_1 \bar{a}}{\partial t} + 2iA_2 \bar{A}_1 \bar{a} + iA_4 \frac{\partial \bar{a}}{\partial z_1} = 0,$$

$$\frac{\partial A_2}{\partial t} + 4iA^2 |\phi|^2 - 2i \frac{\partial A_2}{\partial x_1} = 0. \tag{1.9}$$

2. Results of solution of the model equation

For solutions symmetric with respect to the $z$-coordinate (1.6), by virtue of (1.7) it is obvious that when searching for solutions to the system (1.9) $a$ has to be represented as:

$$a = a_0 \exp(i\varphi z_1) \tag{2.1}$$

In this paper, higher orders relative to $\varepsilon$ are not considered. So slow $z_0$ coordinates with indices $n \geq 2$ do not appear, and $a_0$ may be assumed a real constant. After substitution (2.1), the first two equations of the system (1.9) are identical. As a result, after the inclusion of $a_0$ in $A_1$, it takes the form:

$$\frac{\partial A_1}{\partial t} + 2iA_2 \bar{A}_1 + A_4 \varphi = 0,$$

$$\frac{\partial A_2}{\partial t} + 4iA^2 |\phi|^2 - 2i \frac{\partial A_2}{\partial x_1} = 0. \tag{2.2}$$

Since in this paper only spatially periodic solutions of equation (1.1) are considered, we present the solution of system (2.2) as:

$$A_1 = A_{10}(t) \exp(i\delta x_1), \quad A_2 = A_{20}(t) \exp(2i\delta x_1). \tag{2.3}$$

Then from (2.2), (2.3) it follows that:

$$\frac{dA_{10}}{dt} + 2iA_{20} \bar{A}_{10} + A_{10} \varphi = 0,$$

$$\frac{dA_{20}}{dt} + 4iA_{10}^2 + 4A_{20} \delta = 0. \tag{2.4}$$

As it is clear from (1.6), (2.1) and (2.3), the system (2.4) describes the evolution of the first two harmonics $A_1$ and $A_2$. The values $e \delta$ and $e \varphi$ are corrections to the components of the wave vector $\alpha, \beta$, respectively.
For stationary solutions of the system (2.4) the following relations are obviously fair:

\[ \phi \delta = -2|A_{10}|^2, \]
\[ iA_{10}^2 = -A_{20}\delta. \]  

(2.5)

From the first relationship (2.5) it follows that such stationary regimes can exist in those areas of the vicinity of \((\alpha, \beta)\), where they have different signs (In Fig.1 these are quadrants II and IV). Assuming, without loss of generality, that the amplitude \(A_{10}\) is real, from (2.5) we have:

\[ A_{10} = \sqrt{-\phi \delta \over 2}, \quad A_{20} = i\phi \over 2. \]  

(2.6)

Let us consider the stability of solutions (2.3), (2.6) with respect to linear perturbations of the same period. Substituting into the system (2.4) the perturbed solution:

\[ \tilde{A}_1 = (A_{10} + a_{10}(t_1))\exp(i\delta x_1), \quad \tilde{A}_2 = (A_{20} + a_{20}(t_1))\exp(2i\delta x_1), \]

and linearizing it as regard to amplitudes of the perturbations \(a_{i0}\), we will receive:

\[ \frac{da_{10}}{dt_1} + 2i\left( A_{20}a_{10} + a_{20}A_{10} \right) + a_{10}\phi = 0, \]
\[ \frac{da_{20}}{dt_1} + 8iA_{10}a_{10} + 4a_{20}\delta = 0. \]  

(2.7)

Since time \(t_1\) is not explicitly included in (2.7), we present its solution as:

\[ a_{10}(t_1) = \tilde{a}_{10}\exp(\lambda t_1), \quad a_{20}(t_1) = \tilde{a}_{20}\exp(\lambda t_1). \]  

(2.8)

Without loss of generality we assume \(\tilde{a}_{10}\) a real constant, and then omit the sign \(\sim\). Substituting (2.8) into (2.7) taking into account (2.6) we obtain for \(\lambda\):

\[ \lambda^2 + 4\delta\lambda - 8\phi\delta = 0. \]  

(2.9)

The analysis of the roots of equation (2.9) in the regions of existence of solutions (2.6) shows that these solutions in quadrant II are unstable and in quadrant IV they are stable to the considered types of perturbations. On line \(\delta = -2\phi\) (Fig.1, curve 1) the behavior of disturbances is changing. In the II quadrant below this line (region II, a) both roots \(\lambda_1, \lambda_2 > 0\), i.e. perturbations grow monotonously. In the region II, b the roots become complex \((\lambda_1 = \bar{\lambda}_2, \quad \text{Re}(\lambda_1) > 0)\) and the perturbations grow oscillating. In the IV quadrant we have monotonic damping of disturbances in sector IV, a \((\lambda_1, \lambda_2 < 0)\) and damping with oscillations in the region IV, b \((\lambda_1 = \bar{\lambda}_2, \quad \text{Re}(\lambda_1) < 0)\).
Consider the solutions of system (2.4) for the special case when:
\[ \varphi = -2\delta. \]  

This case is of interest, since as it is shown in [2], on line (2.10) (Fig.1, curve 2) the steady-state travelling solutions with velocities \( \pm c \) generate. Considering (2.10) the system (2.4) takes the following form:
\[
\begin{align*}
\frac{dA_1}{dt} + 2iA_2\bar{A}_1 + A_0\varphi &= 0, \\
\frac{dA_2}{dt} + 4iA_1^2 - 2A_2\varphi &= 0.
\end{align*}
\]

We are seeking for solution (2.11) in the following form:
\[
A_1 = A_0 \exp(i(\psi(t_1) + \gamma)), \quad A_2 = A_0 \exp(i(\eta + 2\psi(t_1))).
\]

Where \( A_0, \gamma, \eta \) are the real constants and \( \psi = \psi(t_1) \) is the real function. From (2.11) – (2.12) we have:
\[
\begin{align*}
\frac{i}{t_1} \frac{d\psi}{dt_1} + 2iA_0e^{i(\eta - 2\gamma)} + \varphi &= 0, \\
\frac{i}{t_1} \frac{d\psi}{dt_1} + 2iA_0e^{-i(\eta - 2\gamma)} - \varphi &= 0.
\end{align*}
\]

As a result, for the system (2.13) we obtain a solution:
\[
\psi(t_1) = -2A_0 \cos(2\gamma - \eta) t_1.
\]

The requirement for consistency of equations of the system (2.13) results in the relationship between \( \varphi, \) amplitude \( A_0, \) and “phase shift” \( 2\gamma - \eta \):
\[
\varphi = 2A_0 \sin(2\gamma - \eta).
\]
Substituting (2.12), (2.14) and (2.15) into (1.6), we obtain for the case of (2.10) the steady-state travelling solutions of equation (1.1) whose velocities depend on the parameters of the problem as follows:

\[ c_\pm = \pm \epsilon \sqrt{16A_0^2 - 4\phi^2} \equiv \pm 4\epsilon \sqrt{A_0^2 - \delta^2}. \]

For solutions spatially uniform over the slow coordinates \( x_1 \) and \( z_1 \) after the obvious conversion:

\[ A_1 \rightarrow \frac{A_0}{\sqrt{8}}, \quad A_2 \rightarrow -\frac{A_0}{2}, \]

the system (2.2) takes the form:

\[ \frac{dA_1}{dt} - iA_2\bar{A}_1 = 0, \]
\[ \frac{dA_2}{dt} - iA_1^2 = 0. \quad (2.16) \]

Substituting its solutions in the form:

\[ A_1 = a_1(t_1) \exp (i\psi_1(t_1)), \quad A_2 = a_2(t_1) \exp (i\psi_2(t_1)), \]

where all functions are real, and selecting the real and imaginary parts in equations of the system (2.16) after simple transformation we obtain:

\[ \frac{da_1}{dt_1} = a_1a_2 \cos (\psi), \quad \frac{da_2}{dt_1} = -a_1^2 \cos (\psi), \]
\[ \frac{d\psi}{dt_1} = \left( \frac{a_1^2}{a_2} - 2a_2 \right) \sin (\psi) \quad (2.18) \]

Here \( \psi = \frac{\pi}{2} + \psi_2 - 2\psi_1 \) is the difference of “phases”.

The system (2.18) is equivalent to the model describing the stationary modes of degenerate parametric amplification [7]. This is the case when the three-wave interaction becomes degenerate and the problem is reduced to the consideration of the behaviour of two harmonics. It is the system of type (2.18) that is used to describe the stationary regime for these harmonics; but instead of differentiation in time, it contains derivatives of spatial coordinate (see [7]). From the first two equations of the system (2.18) it follows that it has an obvious conservation law (integral of motion):

\[ a_1^2 + a_2^2 = I \equiv \text{const}. \]

Finding the second integral requires some greater effort; it has the form [7]:

\[ a_1^2 a_2 \sin (\psi) = G \equiv \text{const}. \]

Using this analogy, some solutions of the system (2.18) are obtained analytically and numerically in this paper. For example, in the case of initial data

\[ a_1(0) = a_0, \quad a_2(0) = 0, \quad \psi(0) = 0, \]

the solution of the system (2.18) is analytical and has the form [7]:

\[ a_1 = a_0 \sec h (a_0t_1), \quad a_2 = a_0 \tanh (a_0t_1). \]

This is the mode of the second harmonic generation, in which the amplitude of the first harmonic tends to zero. For film flow regimes this result appears to be a routine. However, this is one of the
most impressive nonlinear optical effects first described in [8], where the transformation of the red laser beam into the blue one was observed when passing through a quartz crystal.

**Conclusion**

For the case of small Reynolds numbers, a model system of equations describing the evolution of spatial z-symmetric perturbations on a flowing film of a viscous liquid with wave numbers lying in the vicinity of special resonance points has been obtained. Several types of solutions of this system have been considered.

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