We investigate the dynamics of a two-level atom flying through a photonic cavity when the light-matter interaction is in the ultrastrong coupling regime. We adopt a closed full quantum description that takes into account the quantization of the atom center-of-mass motion in addition to its internal degree of freedom and to the quantized photonic cavity field. We find that multiple qualitatively different dynamical regimes are achievable according to two key figures of merit: the ratio between the kinetic energy and the bare excitation energies, and the product of these bare energies with the time the atom takes to fly through the cavity. According to the values of those figures of merit, the atom can be reflected by the dressed vacuum, or can convert part of its kinetic energy into real excitations which might be emitted out of the cavity. In the first case, the atom experiences a quantum regenerative braking mechanism, based on temporary storage of energy into virtual excitations.

In pioneering experiments, flying atoms passing through a cavity were used to investigate the light-matter strong coupling regime [1–4]. These experiments allowed to test fundamental aspects of measurement theory [5, 6], and led to proof-of-concept demonstrations of basic steps in quantum information processing [7, 8].

Flying atoms in a cavity have also been proposed to explore vacuum emission phenomena. For example, a theoretical study [9] has shown that ground-state atoms, if accelerated through a high-Q microwave cavity, can generate radiation with an intensity which can exceed that of the Unruh acceleration radiation in free space by many orders of magnitude.

Several other theoretical works predicted the generation of particles from the vacuum upon dynamical modulation of the cavity system: varying in time the boundary condition of an electromagnetic field (dynamical Casimir effect) [10–12], moving a qubit inside a single-mode cavity [13], having electrical currents transit through a photonic resonator [14, 15], or an atom in systems with broken Lorentz invariance [16].

In the ultrastrong coupling (USC) regime, in which the light-matter interaction strength is comparable or larger than the bare frequencies of the matter and/or light resonances, vacuum radiation has also been predicted to be emitted when the atom-light interaction is non-adiabatically modulated [17, 18] or switched on or off [19–21].

Since its first experimental observation in doped quantum wells [22], experimental progress in cavity quantum electrodynamics led to the achievement of USC in many different systems and with ever-increasing coupling strengths [23–42]. This in turn stimulated an intense theoretical effort [41–47] and a rich phenomenology has been predicted to occur in such a non-perturbative coupling regime. For example, a single photon can excite multiple atoms simultaneously [48, 49], lasing emission is modified [50], mechanical resonators can interact through virtual photons [51], electronic ground-state configurations can be modified [52], and Higgs-like phenomena can be observed breaking the parity symmetry of the system [53, 54]. The USC regime could be also applied in quantum technologies [55].

In this article, we investigate the effects caused by an atomic two-level system flying through a cavity, where the light-matter interaction is in the USC regime. We consider the system initially prepared in its lowest internal energy state (zero atomic and light excitations) with the atom flying toward the cavity, and we calcu-
late the quantum dynamics of the coupled system taking into account the interaction between center-of-mass and internal degrees of freedom of the atom. Such an interaction is due to the light-matter interaction changing the ground state energy inside the cavity, which also determines an effective potential barrier which can affect the center-of-mass motion.

Simulating the time evolution of the coupled system as the atom passes through the cavity, we find qualitatively different behaviours depending on two figures of merit. The first is the initial kinetic energy of the atom when compared to the height of the barrier. If the initial kinetic energy is much lower than the barrier, the atom slows down and gets reflected, with at least part of the kinetic energy temporarily stored in virtual excitations of the atom-photon system. If the kinetic energy is higher than the barrier, the atom passes instead through the cavity. The second figure of merit is instead the product between the bare excitation frequencies of the system and the time the atom takes to traverse the cavity. If this quantity is small, the atom experiences a non-adiabatic action is due to the light-matter interaction changing the internal degrees of freedom of the atom. Such an interaction between center-of-mass and the cavity, which is derived applying the generalized minimal coupling replacement [60],

\[ \hat{\mathcal{H}}_{R}^{(c)} = \omega_{c} \hat{a} \hat{a} + \frac{1}{2} \omega_{a} \left\{ \hat{\sigma}_{z} \cos \left[ 2 \eta \left( \hat{x} \right) \left( \hat{a} + \hat{a}^\dagger \right) \right] + \hat{\sigma}_{y} \sin \left[ 2 \eta \left( \hat{x} \right) \left( \hat{a} + \hat{a}^\dagger \right) \right] \right\}. \]  

Here \( \omega_{c} \) and \( \omega_{a} \) are, respectively, the cavity and atom frequencies, \( \hat{a} \) (\( \hat{a}^\dagger \)) is the annihilation (creation) operator for the single-mode cavity field, and \( \hat{\sigma}_{i} \) (\( i = x, y, z \)) are Pauli matrices operating on the internal states of the two-level atom \( \{|g\rangle, |e\rangle\} \).

For the sake of definiteness, we assume that the normalized space-dependent light-matter coupling strength has a Gaussian shape centered in the cavity center with width \( \mu_{c} \) and maximal intensity \( \eta_{0} [63, 64] \)

\[ \eta(x) = \eta_{0} \exp \left[ -\frac{x^{2}}{2 \mu_{c}^{2}} \right]. \]  

This assumption is not essential and different field profiles could be considered.

In the USC regime, the eigenstates of the Hamiltonian in Eq. (2) contain virtual excitations if represented in the bare atom and photon basis, with the \( n \)th state for the atom localised at position \( x \) having the form

\[ |l\rangle_{x} = \sum_{n=0}^{\infty} c_{l}^{(2n)}(x) |g, 2n \rangle + c_{l}^{(2n+1)}(x) |e, 2n + 1 \rangle, \]  

where the states \( |g, n \rangle \) and \( |e, n \rangle \) form a basis of the quantum Rabi Hamiltonian, describing the atom in the ground or excited state with \( n \) photons in the cavity. The exact form of the coefficients \( c_{l}^{(i)}(x) \) depend on the coupling strength at position \( x \), \( \eta(x) \), and on the chosen gauge.

**Dynamics.** We numerically study the full quantum dynamics of the system applying the time evolution operator, \( \hat{T}(t) = \exp[-i \hat{\mathcal{H}}(t) t] \), to the initial state. In our simulations we do not consider dissipation, which has been shown to have only a limited impact on the population of virtual excitations [65]. At the initial time \( t_{0} = 0 \), the system’s state is assumed to be in the factorised form

\[ |\Psi(t_{0})\rangle = |\varphi(t_{0})\rangle |g, 0 \rangle. \]  

The center-of-mass initial wave-function, centered at \( x_{0} \) and of width \( \mu_{s} \) is chosen as

\[ \langle x | \varphi(t_{0}) \rangle = G \left( \frac{x - x_{0}}{\mu_{s}} \right) \exp(i k_{0} x), \]  

where \( G(x) \) is a normalised symmetric Gaussian function of unit variance. Initially, the atom is placed outside the cavity, \( \langle \varphi(t_{0}) \rangle x | \varphi(t_{0}) \rangle = x_{0} \), with \( |x_{0}| \gg \mu_{c} \), implying \( \eta(x_{0}) \approx 0 \), and the initial momentum is directed towards the cavity, \( \langle \varphi(t_{0}) \rangle \hat{p} | \varphi(t_{0}) \rangle = k_{0} \), corresponding to an initial velocity \( v_{0} = k_{0} / \mu_{s} \).

In the USC coupling, the local vacuum energy depends on the strength of the light-matter coupling. This can be better understood in the dipole gauge (see Methods),

\[ \langle x | \varphi(t_{0}) \rangle = \frac{1}{\sqrt{\pi \mu_{s}^{2}}} \exp \left[ -\frac{x^{2}}{2 \mu_{s}^{2}} \right] \]  

This is the probability density for the initial momentum, \( \langle \hat{p} | \varphi(t_{0}) \rangle = k_{0} \).
where the quantum Rabi Hamiltonian has a term equal to $\omega c \eta^2(\hat{x})$ [60, 66]. In the case where the atom is considered to be fixed inside the cavity, this contribution introduces only an energy shift, and can often be neglected. However, when considering a space-dependent coupling strength $\eta(x)$ [67], this term corresponds to a space-dependent effective potential which generates a force affecting the atom dynamics. Working in the USC regime (in which we expect the maximum value of the normalized coupling $\eta_0$ to be of the order of 0.1 or more), a meaningful figure of merit to study the mechanical motion of the atom is therefore the initial kinetic energy normalised over the photon frequency

$$E_K = \frac{mv_0^2}{2\omega_c}. \quad (7)$$

Moreover, we expect that the atom-cavity system remains in its ground state only when the evolution is adiabatic, which in the present system should mean that the time the atom takes to see an increase of $\eta(x)$ from 0 to $\eta_0$, is substantially longer than the optical period of the system. In the quasi-resonant case $\omega_a \approx \omega_c$, which we will consider in the rest of this paper, we can then introduce the adiabatic figure of merit

$$\Xi = \frac{v_0}{\omega_c \mu_c}, \quad (8)$$

and expect the evolution to be adiabatic when $\Xi \ll 1$.

We start our simulation choosing two different values of the normalised kinetic energies $E_K = 0.02$ and $E_K = 40$, which will allow us to explore two completely different dynamical regimes. We will measure the time in units of $\tau_0 = |2v_0/\omega_0|$, which is the time the atom would take to fly unimpeded through the cavity to a position symmetric to the initial one with respect to the cavity center. In all the simulations we will fix the following parameters: $\omega_a = \omega_c$, $\eta_0 = 0.3$, $x_0 = -(25/6)\mu_c$, and $k_0 = 10 \times 2\pi/\mu_c$. The mass $m$ is varied to change the initial kinetic energy.

In Figure 2, we show three snapshots of the wave-packet evolution for the initial normalised kinetic energy $E_K = 0.02$, corresponding to $\Xi \approx 6.6 \times 10^{-4}$. At $t = 0$ the wave packet is outside the cavity, and it moves from the left to the right with initial velocity $v_0$. Before the atom enters the cavity, it remains in the initial state $|g,0\rangle$, since $\langle \varphi(t) | \eta(\hat{x}) | \varphi(t) \rangle \approx 0$. When the atom enters the cavity, as $\Xi \ll 1$, it sees the ground-state energy change adiabatically, and it thus remains approximately in its local ground state $|0\rangle$. We observe that, approaching the cavity center, the populations of virtual excitations forming the ground state (i.e., $|e, 1\rangle$ and $|g, 2\rangle$) increase, meanwhile the group velocity of the wave packet decreases (not shown in Fig. 2). At $t \approx \tau_0/2$ the atom stops and reverses its motion. The atom then moves backwards from right to left leaving the cavity with momentum opposite of the initial one. At $t \approx \tau_0$ the atom has left the cavity with the kinetic energy and absolute value of

![FIG. 2. Snapshots of the Gaussian wave-packets associated to the states $|g,0\rangle$, $|e,1\rangle$ and $|g,2\rangle$, when the initial normalized kinetic energy is $E_K = 0.02$ and $E_K = 40$, which will allow us to explore two completely different dynamical regimes. We will measure the time in units of $\tau_0 = |2v_0/\omega_0|$, which is the time the atom would take to fly unimpeded through the cavity to a position symmetric to the initial one with respect to the cavity center. In all the simulations we will fix the following parameters: $\omega_a = \omega_c$, $\eta_0 = 0.3$, $x_0 = -(25/6)\mu_c$, and $k_0 = 10 \times 2\pi/\mu_c$. The mass $m$ is varied to change the initial kinetic energy.](image-url)
the linear momentum essentially coincident with the initial ones. The reflection happens because the gradient of the vacuum energy as a function of the atom position is positive when the atom enters the cavity, resulting in a force pushing back the atom when the kinetic energy is lower than the effective potential barrier. It can be shown that excitations in Fig. 2 for \( t = 0.5\tau_0 \) are virtual. A hint comes from the fact that essentially no excitation is present when the atom leaves the cavity (\( t = \tau_0 \)). A more direct evidence will be provided later (see Fig. 6a).

In Figure 3 we show the numerical results obtained for an initial kinetic energy \( E_K = 40 \), corresponding to \( \Xi \approx 1.3 \). Not only in this case the kinetic energy is much larger than the barrier height, allowing the atom to pass through the cavity, but now the atom feels a non-adiabatic change of the local coupling strength \( \eta(x) \) entering the cavity. Consequently, the system’s state does not adiabatically follow the local ground state and the atomic and photonic excited states become populated. In this case the interaction of the atom with the cavity converts a fraction of the atomic kinetic energy into real excitations: both atomic excitations and real photons can persist after the atom leaves (\( \tau = \tau_0 \)) and eventually leaks out of the system.

**Emitted photon rates.** The rate of photon emission out of a cavity is usually assumed proportional to the mean cavity-photon number \( \langle \hat{a}^\dagger \hat{a} \rangle \). As better detailed in the Methods section, in the USC regime this ceases to be true and the output has to be calculated using instead the quantity \( \langle \hat{X}^- \hat{X}^+ \rangle \) [62, 68], where the operator \( \hat{X}^-\hat{X}^+ \) is related to the positive (negative) frequencies of the electric field operator, where

\[
\hat{X}^\pm = \sum_{l,k>\pm} i \langle l| (\hat{a} - \hat{a}^\dagger) |k \rangle |l \rangle |k \rangle ,
\]

and \( \hat{X}^- = (\hat{X}^+)\dagger \), with |\( l \rangle \) the system eigenstates ordered for increasing values of the energy. However, when the atom is outside the cavity, the cavity-matter coupling is zero, and we can retain \( \langle \hat{a}^\dagger \hat{a} \rangle \). In Figure 4, the mean number of real photons \( \langle \hat{X}^-\hat{X}^+ \rangle \) is shown as a function of time for different values of the adiabatic parameter \( \Xi \). We can clearly see that photons are emitted only when \( \Xi > 1 \). As \( \Xi \) increases, the emission slowly tapers off because the time the atom spends into the cavity grows shorter. Eventually, it becomes shorter than the Rabi oscillation period and the atom leaves the cavity before any meaningful interaction with the photonic field has the time to happen.

**Entanglement.** At every instant, the state of the system can be expressed as

\[
|\Psi(t)\rangle = \sum_i c_i(x,t) |\varphi(t)\rangle |R_i\rangle ,
\]

where |\( R_i \rangle \) are states which span a basis of the quantum Rabi Hamiltonian (e.g., |g,0\rangle, |g,1\rangle, |e,0\rangle, ...). Although we chose an initially factorizable state in Eq. 5, when the dynamics is non-adiabatic the atomic and photonic excitations generated in the process are entangled.
with the atom position. In Figure 5b we plot the entanglement calculated using the von-Neumann entropy \( S = - \text{Tr}[\rho_R \log_2(\rho_R)] \), where \( \rho_R = \text{Tr}_\rho[\rho] \) is the reduced density matrix obtained tracing out the atom center-of-mass motion. The coefficient \( m \) is a cut-off to the dimension of the Hilbert space describing the atom and the cavity mode. In our simulation, we choose \( m = 4 \), since higher lying states have vanishing populations for the parameter range considered. In Fig. 5b we see that, after the atom leaves the cavity, the entanglement remains constant.

**Discussion**

We investigated the dynamics of a flying atom prepared initially in its lowest internal energy state (e.g., the electronic ground state) that passes through a cavity with zero photons and interacts with it in the USC regime. We were thus able to show how the generation of excitations out of the vacuum affects the center of mass dynamics. We also found that any temporary reduction of the atomic kinetic energy due to a gradient of light-matter interaction strength results into a temporary energy storing in virtual excitations.

When the atom enters the cavity, it experiences a force due to the gradient in the vacuum energy felt by the atom. This leads to a quantum regenerative braking effect in which the kinetic energy is temporarily stored in virtual atom-light excitations inside the cavity. Different cases can take place, according to whether the final momentum and energy of the atom are changed by the interaction. If the kinetic energy is much lower than the difference between the ground state energies inside and outside the cavity, the atom is reflected. If instead its kinetic energy is large enough, the atom is transmitted. The condition to create real excitations is connected to the adiabatic parameter \( \Xi \). If it is small enough, the atom sees and adiabatic change in the light-matter interaction and in its ground state. In this case, the interaction is elastic, and the atom’s final kinetic energy is the same as the initial one. In the opposite case, the interaction is non-adiabatic and inelastic, as part of the initial kinetic energy is converted into real excitations.

These predictions might motivate new experimental ways to convert virtual photons into real ones. Our theoretical proposal could be experimentally observed in plasmonic picocavities or using single molecules trapped in plasmonic nanogaps. Nevertheless, the achievement of the USC with single molecule is still a challenging topic. Moreover, the experimental verification of this effect could verify also the signatures of the counter rotating terms, which are only present in the USC regime.

**Methods**

**Derivation of the quantum Rabi Hamiltonian.**

The quantum Rabi Hamiltonian in Eq. 2 approximates the physics of an atom coupled to a photonic resonator as a two-level system coupled to a single photonic mode. We are thus performing two key approximations: neglecting higher-lying matter states and photonic modes.

The first approximation leads to major problems when the coupling between light and matter enters in the USC regime, as gauge invariance breaks in the reduced Hilbert space and the two usually employed gauges, the Coulomb one and the dipolar one, provide completely different results, with the latter being the most correct one [59]. Recently, a consistent description of the quantum Rabi model was introduced, which is able to provide gauge-invariant physical results in any interaction regime [60]. It has been shown that this description is closely connected to lattice gauge theories [69]. Moreover, it was shown in Ref. [61] that adding a time dependent coupling may also lead to gauge-dependent predictions. These ambiguities have been solved in Ref. [62]. To derive the dynamics of our system, we take into account all these recent results regarding gauge invariance.

The neglect of higher photonic modes had also been recognised as potentially problematic for large values of the coupling strength [70], as it can lead to unphysical superluminal signalling. Still, the main issue for the present work regards the calculation of the energy barrier between the vacua outside and inside the cavity. A multimode exact calculation would have to consider that the cavity is effective at confining the electromagnetic radiation up to a cutoff frequency \( \omega_M \), and the energy difference felt by the atom inside and outside the cavity should then be calculated as a difference between a sum over the discrete modes \( \omega_c < \omega_M \) in the cavity and an integral over the continuum extending from 0 to \( \omega_M \) outside. This sort of calculation is similar to the ones performed when dealing with the static Casimir force, and the result would similarly depend on the specific geometry considered [71]. While a multimode quantum simulation is numerically infeasible in our case, its results would mainly boil down to a renormalization of the potential barrier felt by the atom, which can be taken into account in our model as a phenomenological renormalization of the coupling constant \( \eta_0 \) without qualitatively modify our results.

**The system Hamiltonian in the dipole gauge.**

By applying a Power-Zienau-Woolley (PZW) transformation to Eq. (1) [72], we can derive the full Hamiltonian in the dipole gauge, \( \hat{H}^{(d)} = \hat{T}\hat{H}^{(c)}\hat{T}^\dagger \), where

\[
\hat{T} = \exp \left[ -i\eta(\hat{x})\hat{\sigma}_x (\hat{a} + \hat{a}^\dagger) \right].
\]  

The transformed Hamiltonian \( \hat{H}^{(d)} \) has two terms: the quantum Rabi Hamiltonian in dipole gauge \( \hat{H}^{(d)}_R = \hat{T}\hat{H}^{(c)}_R\hat{T}^\dagger \), and the transformed kinetic contribution \( \hat{T}\hat{p}^2\hat{T}^\dagger/(2m) \). The first term becomes

\[
\hat{H}^{(d)}_R = \omega_c \hat{a}^\dagger \hat{a} + \frac{1}{2} \omega_0 \sigma_z - i\omega_c \eta(\hat{x})\sigma_x (\hat{a} - \hat{a}^\dagger) + \omega_c \eta^2(\hat{x}),
\]  

where \( \omega_0 \) is the atomic frequency and \( \omega_c \) is the cavity frequency.
while the second one is
\[
\hat{T}\hat{p}\hat{T}^\dagger = \hat{p}^2 + \hat{\sigma}_x (\hat{a} + \hat{a}^\dagger) [\eta'(\hat{x})\hat{p} + \hat{p}\eta'(|\hat{x}|)]
+ \eta^2(\hat{x}) (\hat{a} + \hat{a}^\dagger)^2,
\]
(13)
this follows from the property \(\hat{T}\hat{p}\hat{T}^\dagger = \hat{p} + \eta'(\hat{x})\hat{\sigma}_x (\hat{a} + \hat{a}^\dagger)\), where \(\eta'(x) = (d\eta(x))/dx\).

Figure 6 shows the mean photon number generated when the atom passes through the cavity as a function of time for different initial kinetic energies. The figure displays both the mean value of the real photons \(\langle \hat{X}^- \hat{X}^+ \rangle\) and the mean values of the bare photon number in the Coulomb and dipole gauge. In Figure 6a, the atom has an initial kinetic energy comparable to the bare energies of the system, \(E_K = 1\). When it enters the cavity, the evolution is adiabatic and no real photons are emitted. The expectation value \(\langle \hat{X}^- \hat{X}^+ \rangle\) is exactly zero during the entire dynamics and does not depend on the chosen gauge. On the contrary, \(\langle \hat{a}\hat{a}^\dagger \rangle\) is different from zero inside the cavity, and depends on the chosen gauge. A careful analysis displayed in Ref. [62] shows that the Coulomb gauge mean value \(\langle \hat{a}\hat{a}^\dagger \rangle_c\) represents the correct mean value for bare photons. Indeed, by changing to the dipole gauge, also the photonic operators need to be transformed \(\hat{a} \rightarrow \hat{a}' = \hat{T}\hat{a}\hat{T}^\dagger\), obtaining in this case the same results using the Coulomb gauge \(\langle \hat{a}\hat{a}^\dagger \rangle_d = \langle \hat{a}\hat{a}^\dagger \rangle_c\).

In Figure 6b, the atom has an initial normalised kinetic energy much higher than the bare energies of the system, \(E_K = 30\). When it enters the cavity, the evolution of the system is non-adiabatic and real photons are generated. In fact, the expectation value \(\langle \hat{X}^- \hat{X}^+ \rangle\) progressively increases till the moment that the center of the atomic wave packet leaves the cavity. Instead, the mean value \(\langle \hat{a}\hat{a}^\dagger \rangle_c\), in the Coulomb gauge, reaches its maximum when the center of the atomic wave packet arrives at the center of the cavity where the coupling is the largest, and decreases afterwards.

**Mapping to time-dependent perturbation theory.** Using a time-dependent perturbation theory, we analytically calculated the output photon rates. When the initial kinetic energy is large enough, the atom velocity can be considered constant along the trajectory, because the cavity does not affect significantly the atom motion. With this assumption, we remove the motion degree of freedom and consider only the simple quantum Rabi model with an effective time-dependent coupling. In particular, the coupling follows again a Gaussian shape (but in the time domain) \(\eta(t) = \eta_0 \exp[-(t^2/2\mu_t^2)]\), with \(\mu_t = \mu_c m / k_0\). As a starting point, we expand the quantum Rabi Hamiltonian in the Coulomb gauge in Eq. (2) up to the third order with respect to the photon operators. With these approximations, the Hamiltonian becomes \(\hat{H}^{(c)}(t) \approx \hat{H}_0 + \hat{V}(t)\), with \(\hat{H}_0 = \omega_c\hat{a}\hat{a} + (\omega_g/2)\hat{\sigma}_z\), and

\[
\hat{V}(t) = \frac{1}{2} \omega_0 \left[ 2\eta(t)(\hat{a} + \hat{a}^\dagger)\hat{\sigma}_y - 2\eta^2(t)(\hat{a} + \hat{a}^\dagger)^2\hat{\sigma}_z \right.
+ \left. \frac{8}{6} \eta^3(t)(\hat{a} + \hat{a}^\dagger)^3\hat{\sigma}_y \right].
\]
(14)

Using first-order time-dependent perturbation theory, the transition probability from the ground state \(|g,0\rangle\) to the excited states \(|n\rangle\) is given by

\[
e_n^{(1)} = -i \int_{-\infty}^{+\infty} \langle n | \hat{V}(t) | g,0 \rangle e^{i(\omega_n - \omega_g) t} dt.
\]

For the states \(|e,1\rangle, |g,2\rangle\) and \(|e,3\rangle\), and in the case of
resonant condition \( \omega_a = \omega_c \), we obtain

\[
\begin{align*}
\langle \hat{a} \rangle^{(1)} e, 1 & = -\frac{\eta_0}{\Xi} \sqrt{2\pi} e^{-2\sqrt{2}\text{i} \frac{\eta_0}{\Xi} \omega_c} + \frac{\eta_0^3}{9} \sqrt{\frac{8\pi}{3}} e^{-\frac{9}{2} \sqrt{\frac{\eta_0}{\Xi} \omega_c}}, \\
\langle \hat{a} \rangle^{(1)} e, 2 & = -i \frac{\eta_0^2}{\Xi} \sqrt{2\pi} e^{-\frac{1}{4} \sqrt{\frac{\eta_0}{\Xi} \omega_c}}, \\
\langle \hat{a} \rangle^{(1)} e, 3 & = \frac{\eta_0^3}{18} \sqrt{16\pi} e^{-\frac{2}{9} \sqrt{\frac{\eta_0}{\Xi} \omega_c}}. 
\end{align*}
\]  

(15)

The total photon rates can be calculated using \( \langle \hat{a} \rangle \), because, at the end of the process, it predicts correct results, as shown in Fig. 6. We obtain,

\[
\langle \hat{a} \rangle_{\text{out}} \approx \langle \hat{a} \rangle^{(1)} e, 1 \rangle^2 + 2 \langle \hat{a} \rangle^{(1)} e, 2 \rangle^2 + 3 \langle \hat{a} \rangle^{(1)} e, 3 \rangle^2. 
\]  

(16)

Therefore, we can analytically calculate the total photon rates substituting Eqs. (15) in Eq. (16). Figure 7(a) shows the comparison between analytical and numerical results of the output photon emission rate \( \langle \hat{a} \rangle_{\text{out}} \) as a function of the kinetic energy. As expected from the approximations made to obtain the analytical formula in Eq. (16), both numerical and analytical results coincide in the limit of high kinetic energies. Moreover, they remain in good agreement also for lower kinetic energies. We observe that the analytical calculation overestimates the generated photon rate, which is a direct consequence of neglecting the back-reaction effect on the atom motion. Lowering the kinetic energy, as expected, the generation of photons reduces and then tends to zero. Increasing the kinetic energy increases the photon rate until a maximum value, after that the photon rate decreases because the atom passes through the cavity faster than the time required to generate excitations.

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