A characterization of tightly triangulated 3-manifolds

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Abstract

For a field \( F \), the notion of \( F \)-tightness of simplicial complexes was introduced by Kühnel. Kühnel and Lutz conjectured that any \( F \)-tight triangulation of a closed manifold is the most economic of all possible triangulations of the manifold. The boundary of a triangle is the only \( F \)-tight triangulation of a closed 1-manifold. A triangulation of a closed 2-manifold is \( F \)-tight if and only if it is \( F \)-orientable and neighbourly. In this paper we prove that a triangulation of a closed 3-manifold is \( F \)-tight if and only if it is \( F \)-orientable, neighbourly and stacked. In consequence, the Kühnel-Lutz conjecture is valid in dimension \( \leq 3 \).

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1 Introduction

All simplicial complexes considered in this paper are finite and abstract. The vertex set of a simplicial complex \( X \) will be denoted by \( V(X) \). For \( A \subseteq V(X) \), the induced subcomplex \( X[A] \) of \( X \) on the vertex set \( A \) is defined by \( X[A] := \{ \alpha \in X : \alpha \subseteq A \} \). For \( x \in V(X) \), the subcomplexes \( \{ \alpha \in X : x \notin \alpha \} = X[V(X) \setminus \{ x \}] \) and \( \{ \alpha \in X : x \notin \alpha, \alpha \cup \{ x \} \in X \} \) are called the antistar and the link of \( x \) in \( X \), respectively. A simplicial complex \( X \) is said to be a triangulated (closed) manifold if it triangulates a (closed) topological manifold, i.e., if the geometric carrier \( |X| \) of \( X \) is a (closed) topological manifold. A triangulated closed \( d \)-manifold \( X \) is said to be \( F \)-orientable if \( H_d(X; F) \neq 0 \). If two triangulated \( d \)-manifolds \( X \) and \( Y \) intersect precisely in a common \( d \)-face \( \alpha \) then \( X \# Y := (X \cup Y) \setminus \{ \alpha \} \) triangulates the connected sum \( |X| \# |Y| \) and is called the connected sum of \( X \) and \( Y \) along \( \alpha \).

For our purpose, a graph may be defined as a simplicial complex of dimension \( \leq 1 \). For \( n \geq 3 \), the \( n \)-cycle \( C_n \) is the unique \( n \)-vertex connected graph in which each vertex lies on exactly two edges. For \( n \geq 1 \), the complete graph \( K_n \) is the \( n \)-vertex graph in which any two vertices form an edge. For \( m, n \geq 1 \), the complete bipartite graph \( K_{m,n} \) is the graph with \( m+n \) vertices and \( mn \) edges in which each of the first \( m \) vertices forms an edge with each of the last \( n \) vertices. Two graphs are said to be homeomorphic if their geometric carriers are homeomorphic. A graph is said to be planar if it is a subcomplex of a triangulation of the 2-sphere \( S^2 \). In this paper, we shall have an occasion to use the easy half of Kuratowski’s
famous characterization of planar graphs [5]: A graph is planar if and only if it has no homeomorph of $K_5$ or $K_{3,3}$ as a subgraph.

If $\mathbb{F}$ is a field and $X$ is a simplicial complex then, following Kühnel [9], we say that $X$ is $\mathbb{F}$-tight if (a) $X$ is connected, and (b) the $\mathbb{F}$-linear map $H_*(Y; \mathbb{F}) \to H_*(X; \mathbb{F})$, induced by the inclusion map $Y \hookrightarrow X$, is injective for every induced subcomplex $Y$ of $X$.

If $X$ is a simplicial complex of dimension $d$, then its face vector $(f_0, \ldots, f_d)$ is defined by $f_i = f_i(X) := \# \{ \alpha \in X : \dim(\alpha) = i \}$, $0 \leq i \leq d$. A simplicial complex $X$ is said to be neighbourly if any two of its vertices form an edge, i.e., if $f_1(X) = \binom{f_0(X)}{2}$.

A simplicial complex $X$ is said to be strongly minimal if, for every triangulation $Y$ of the geometric carrier $|X|$ of $X$, we have $f_i(X) \leq f_i(Y)$ for all $i$, $0 \leq i \leq \dim(X)$. Our interest in the notion of $\mathbb{F}$-tightness mainly stems from the following famous conjecture [10].

**Conjecture 1.1** (Kühnel-Lutz). For any field $\mathbb{F}$, every $\mathbb{F}$-tight triangulated closed manifold is strongly minimal.

Following Walkup [16] and McMullen-Walkup [12], a triangulated ball $B$ is said to be stacked if all the faces of $B$ of codimension 2 are contained in the boundary $\partial B$ of $B$. A triangulated sphere $S$ is said to be stacked if there is a stacked ball $B$ such that $S = \partial B$. This notion was extended to triangulated manifolds by Murai and Nevo [14]. Thus, a triangulated manifold $\Delta$ with boundary is said to be stacked if there is a stacked ball $B$ in $\Delta$ such that $S = \partial B$. A triangulated closed manifold $M$ is said to be locally stacked if all its vertex links are stacked spheres or stacked balls. The main result of this paper is the following characterization of $\mathbb{F}$-tight triangulated closed 3-manifolds, for all fields $\mathbb{F}$.

**Theorem 1.2.** A triangulated closed 3-manifold $M$ is $\mathbb{F}$-tight if and only if $M$ is $\mathbb{F}$-orientable, neighbourly and stacked.

The special case of Theorem 1.2, where $\text{char}(\mathbb{F}) \neq 2$, was proved in our previous paper [4]. In this paper we conjectured [4, Conjecture 1.12] the validity of Theorem 1.2 in general.

As a consequence of Theorem 1.2 we show that the Kühnel-Lutz conjecture (Conjecture 1.1) is valid up to dimension 3. Thus,

**Corollary 1.3.** If $M$ is an $\mathbb{F}$-tight triangulated closed manifold of dimension $\leq 3$, then $M$ is strongly minimal.

As a second consequence of Theorem 1.2 we show:

**Corollary 1.4.** The only closed topological 3-manifolds which may possibly have $\mathbb{F}$-tight triangulations are $S^3$, $(S^2 \times S^1)^k$ and $(S^2 \times S^1)^{\#k}$, where $k$ is a positive integer such that $80k + 1$ is a perfect square.

Kühnel conjectured that any triangulated closed 3-manifold $M$ satisfies $(f_0(M) - 4) \times (f_0(M) - 5) \geq 20\beta_1(M; \mathbb{F})$. (This is a part of his Pascal-like triangle of conjectures reported in [11].) This bound was proved by Novic and Swartz in [15]. Burton et al proved in [6] that if the equality holds in this inequality then $M$ is neighbourly and locally stacked. (Actually, these authors stated this result for $\mathbb{F} = \mathbb{Z}_2$, but their argument goes through for all fields $\mathbb{F}$.) In [11], the first author proved that the equality holds in this inequality if and only if $M$ is neighbourly and stacked. In [13], Murai generalized this to all dimensions $\geq 3$. Another consequence of Theorem 1.2 is:
Corollary 1.5. A triangulated closed 3-manifold $M$ is $\mathbb{F}$-tight if and only if $M$ is $\mathbb{F}$-orientable and $(f_0(M) - 4)(f_0(M) - 5) = 20\beta_1(M; \mathbb{F})$.

In [7], $\mathbb{Z}_2$-tight triangulations of $(S^2 \times S^1)^\#k$ were constructed for $k = 1, 30, 99, 208, 357$ and 546. However, we do not know any $\mathbb{F}$-tight triangulations of $(S^2 \times S^1)^\#k$.

Question 1.6. Is there any positive integer $k$ for which $(S^2 \times S^1)^\#k$ has an $\mathbb{F}$-tight triangulation?

2 Proofs

The following result is Theorem 3.5 of [4].

Theorem 2.1. Let $C$ be an induced cycle in the link $S$ of a vertex $x$ in an $\mathbb{F}$-tight simplicial complex $X$. Then the induced subcomplex of $X$ on the vertex set of the cone $x \ast C$ is a neighbourly triangulated closed 2-manifold.

If, in Theorem 2.1, $C$ is an $n$-cycle then the triangulated 2-manifold guaranteed by this theorem has $n$ vertices, $n(n + 1)/2$ edges and hence $n(n + 1)/3$ triangles. Thus $3$ divides $n(n + 1)$, i.e., $n \equiv 1 \pmod{3}$. Therefore, Theorem 2.1 has the following immediate consequence.

Corollary 2.2. Let $X$ be an $\mathbb{F}$-tight simplicial complex. Let $S$ be the link of a vertex in $X$. Then $S$ has no induced $n$-cycle for $n \equiv 1 \pmod{3}$.

We recall that the Möbius band has a unique 5-vertex triangulation $\mathcal{M}$. The boundary of $\mathcal{M}$ is a 5-cycle $C_5$. The simplicial complex $\mathcal{M}$ may be uniquely recovered from $C_5$ as follows. The triangles of $\mathcal{M}$ are $\{x\} \cup e_x$, where, for each vertex $x$ of $C_5$, $e_x$ is the edge of $C_5$ opposite to $x$. We also note the following consequence of Theorem 2.1.

Corollary 2.3. Let $S$ be the link of a vertex $x$ in an $\mathbb{F}$-tight simplicial complex $X$. Let $C$ be an induced cycle in $S$.

(a) If $C$ is a 3-cycle, then it bounds a triangle of $X$.

(b) If $C$ is a 5-cycle then it bounds an induced subcomplex of $X$ isomorphic to the 5-vertex Möbius band.

Proof. If $C$ is a 3-cycle, then the induced subcomplex of $X$ on the vertex set of $x \ast C$ is a neighbourly, 4-vertex, triangulated closed 2-manifold, which must be the boundary complex $\mathcal{T}$ of the tetrahedron. But all four possible triangles occur in $\mathcal{T}$, and $C$ bounds one of them. If $C$ is a 5-cycle then the induced subcomplex $X[V(x \ast C)]$ of $X$ is a neighbourly, 6-vertex, triangulated closed 2-manifold, which must be the unique 6-vertex triangulation $\mathbb{RP}^2_6$ of the real projective plane. Therefore, the induced subcomplex $X[V(C)]$ of $X$ is the antistar of the vertex $x$ in $\mathbb{RP}^2_6$, which is the 5-vertex Möbius band. $\square$

Let $\mathcal{T}$ and $\mathcal{I}$ denote the boundary complexes of the tetrahedron and the icosahedron, respectively. Thus the faces of $\mathcal{T}$ are all the proper subsets of a set of four vertices. Up to isomorphism, the 20 triangles of $\mathcal{I}$ are as follows:

$$
012, 015, 023, 034, 045, 124', 153', 13'4', 235', 24'5', 341', \\
31'5', 452', 41'2', 52'3', 0'1'2', 0'1'5', 0'2'3', 0'3'4', 0'4'5'.
$$

(1)

The following is Corollary 5.5 of [4].
Theorem 2.4. Let $S$ be a triangulated 2-sphere which has no induced $n$-cycle for any $n \equiv 1 \pmod{3}$. Then $S$ is a connected sum of finitely many copies of $T$ and $I$ (in some order).

As an immediate consequence of Corollary 2.2 and Theorem 2.4 we have:

Corollary 2.5. Let $S$ be the link of a vertex in an $F$-tight triangulated closed 3-manifold $M$. Then $S$ is a connected sum of finitely many copies of $T$ and $I$ (in some order).

Proof of Theorem 1.2. Let $M$ be an $F$-orientable, neighbourly, stacked, triangulated closed 3-manifold. Then $M$ is $F$-tight by the case $k = 1$ of Theorem 2.24 in [2]. This proves the “if part”. Conversely, let $M$ be $F$-tight. Since any $F$-tight triangulated closed manifold is neighbourly and $F$-orientable (Lemmas 2.2 and 2.5 in [4]), it follows that $M$ is $F$-orientable and neighbourly. To complete the proof of the “only if” part, it suffices to show that any connected sum of copies of $T$ is stacked (as may be seen by an easy induction on the number of summands). Hence Corollary 2.3(b) gives us eight more triangles of $M$ through the vertex 0, namely, $023'$, $034'$, $015'$, $034'$, $045'$, $032'$, $012'$, $023'$. Thus, if $S'$ is the link of the vertex 0 in $M$, then we have the graph of Fig. 2 as a subcomplex of $S'$.

Figure 2: A homeomorph of $K_{3,3}$

So we have a homeomorph of $K_{3,3}$ as a subcomplex of the triangulated 2-sphere $S'$. This is a contradiction since $K_{3,3}$ is not a planar graph.

Proof of Corollary 1.3. Let $M$ be an $F$-tight triangulated closed $d$-manifold, $d \leq 3$. By Lemma 2.2 in [4], $M$ is neighbourly. But the boundary complex of the triangle is the only
neighbourly triangulated closed 1-manifold. This is trivially the strongly minimal triangulation of $S^1$. So, we have the result for $d = 1$. Next let $d = 2$. Let $N$ be another triangulation of $|M|$. Let $(f_0, f_1, f_2)$ be the face vector of $N$. Let $\chi$ be the Euler characteristic of $M$ (hence also of $N$). Then $f_0 - f_1 + f_2 = \chi$ and $2f_1 = 3f_2$. Therefore we get $f_1 = 3(f_0 - \chi)$ and $f_2 = 2(f_0 - \chi)$. Thus, $f_1$ and $f_2$ are strictly increasing functions of $f_0$. So, it is sufficient to show that $f_0 \geq f_0(M)$. Now, trivially, $f_1 \leq f_0^2$, with equality if and only if $N$ is neighbourly. Substituting $f_1 = 3(f_0 - \chi)$ in this inequality, we get that $f_0(f_0 - 7) \geq -6\chi = f_0(M)(f_0(M) - 7)$. This implies that $f_0 \geq f_0(M)$. Thus, $M$ is strongly minimal. So we have the result for $d = 2$. If $d = 3$ then, by Theorem 2.24 of [2], $M$ is stacked and hence is locally stacked. But any locally stacked, $\mathbb{F}$-tight triangulated closed manifold is strongly minimal by Corollary 3.13 in [3]. So, we are done when $d = 3$.

**Proof of Corollary 1.4** Let $M$ be a closed 3-manifold which has an $\mathbb{F}$-tight triangulation $X$. By Theorem 2.24 $X$ is stacked. But, by Corollary 3.13 (case $d = 3$) of [8], any stacked triangulation of a closed 3-manifold can be obtained from a stacked 3-sphere by a finite sequence of elementary handle additions. It is easy to see by an induction on the number $k$ of handles added that $X$ triangulates either $S^3$ ($k = 0$) or $(S^2 \times S^1)^#k$ or $(S^2 \times S^1)^#k$ ($k \geq 1$). Let $X$ be obtained from the stacked 3-sphere $S$ by $k$ elementary handle additions. It follows by induction on $k$ that $f_0(S) = f_0(X) + 4k$ and $f_1(S) = f_1(X) + 6k = (f_0(X))^2 + 6k$. Since $S$ is a stacked 3-sphere, $f_1(S) = 4f_0(S) - 10$. Thus, $(f_0(X))^2 + 6k = 4f_0(X) + 4k - 10$. This implies $(f_0(X) - 4)(f_0(X) - 5) = 20k$ and hence $f_0(X) = 5\left(9 + \sqrt{80k + 1}\right)$. So, $80k + 1$ must be a perfect square.

**Proof of Corollary 1.5** If $(f_0(M) - 4)(f_0(M) - 5) = 20\beta_1(M; \mathbb{F})$ then Theorem 1.3 of [6] says that $M$ must be neighbourly and locally stacked. Therefore, the ‘if part’ follows from Theorem 2.24 of [2]. The ‘if part’ also follows from Theorem 1.12 of [1] and Theorem 2.24 of [2]. The ‘only if’ part follows from the proof of Corollary 1.4 above.

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