DIRICHLET PROBLEM FOR $f$-MINIMAL GRAPHS

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Abstract. We study the asymptotic Dirichlet problem for $f$-minimal graphs in Cartan-Hadamard manifolds $M$. $f$-minimal hypersurfaces are natural generalizations of self-shrinkers which play a crucial role in the study of mean curvature flow. In the first part of this paper, we prove the existence of $f$-minimal graphs with prescribed boundary behavior on a bounded domain $\Omega \subset M$ under suitable assumptions on $f$ and the boundary of $\Omega$. In the second part, we consider the asymptotic Dirichlet problem. Provided that $f$ decays fast enough, we construct solutions to the problem. Our assumption on the decay of $f$ is linked with the sectional curvatures of $M$. In view of a result of Pigola, Rigoli and Setti, our results are almost sharp.

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1. Introduction

In this paper we study the Dirichlet problem for the so-called $f$-minimal graph equation on a complete non-compact $n$-dimensional Riemannian manifold $M$ with the Riemannian metric given by $ds^2 = \sigma_{ij}dx^idx^j$ in local coordinates. We equip $N = M \times \mathbb{R}$ with the product metric $ds^2 + dt^2$ and assume that $f: N \to \mathbb{R}$ is a smooth function. The Dirichlet problem for $f$-minimal graphs is to find a solution $u$ to the equation

$$\begin{cases}
\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \langle \bar{\nabla} f, \nu \rangle & \text{in } \Omega \\
u|_{\partial \Omega} = \varphi,
\end{cases} \tag{1.1}$$

where $\Omega \subset M$ is a bounded domain, $\bar{\nabla} f$ is the gradient of $f$ with respect to the product Riemannian metric, and $\nu$ denotes the downward unit normal to the graph.

\[
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\]
of $u$, i.e.
\[ \nu = \frac{(\nabla u, -1)}{\sqrt{1 + |\nabla u|^2}}. \] (1.2)

The regularity assumptions on $f$, $\partial \Omega$, and on $\varphi$ will be specified in due course.

The equation (1.1) can be written in non-divergence form as
\[ \frac{1}{W} \left( \sigma^{ij} - u^i u^j \right) u_{i;j} = (\nabla f, \nu), \] (1.3)
where $W = \sqrt{1 + |\nabla u|^2}$, $(\sigma^{ij})$ stands for the inverse matrix of $(\sigma_{ij})$, $u^i = \sigma^{ij} u_j$, with $u_j = \partial u / \partial x^j$, and $u_{i;j} = u_{ij} - \Gamma^k_{ij} u_k$ denotes the second order covariant derivative of $u$.

We recall that an immersed hypersurface $\Sigma$ of a Riemannian manifold $(N, g)$ is called an $f$-minimal hypersurface if its (scalar) mean curvature $H$ satisfies an equation
\[ H = (\nabla f, \nu) \]
at every point of $\Sigma$. Here, too, $\nu$ is a unit normal vector field along $\Sigma$, $f$ is a smooth function on $N$, and $\nabla f$ denotes its gradient with respect to the Riemannian metric $g$. Hence the graph of a solution $u$ of (1.1) is an $f$-minimal hypersurface in $M \times \mathbb{R}$. Note that we define the mean curvature as the trace of the second fundamental form. Other examples of $f$-minimal hypersurfaces are
(a) minimal hypersurfaces if $f$ is identically constant,
(b) self-shrinkers in $\mathbb{R}^{n+1}$ if $f(x) = |x|^2/4$,
(c) minimal hypersurfaces of weighted manifolds $M_f = (M, g, e^{-f} d\text{vol}_M)$, where $(M, g)$ is a complete Riemannian manifold with the Riemannian volume element $d\text{vol}_M$.

We refer to [7, 6, 3, 4, 5, 15], and references therein for recent studies on self-shrinkers and $f$-minimal hypersurfaces. Let us just point out a recent result relevant to our paper. Wang in [22] investigated graphical self-shrinkers in $\mathbb{R}^n$ by studying the equation (1.1) in the whole $\mathbb{R}^n$ when $f(x) = |x|^2/4$. He proved that any smooth solution to this equation has to be a hyperplane improving an earlier result of Ecker and Huisken [11], where they made the extra assumption that the solution has polynomial growth. We will show that the situation is quite different when $\mathbb{R}^n$ is replaced by a Cartan-Hadamard manifold with strictly negative sectional curvatures and for more general $f$ satisfying some suitable assumptions. In particular, we impose that $\sup_{\mathbb{R}^n} |\nabla f| < \infty$ which is not valid for $f(x) = |x|^2/4$.

In our existence results we always assume that $f \in C^2(\bar{\Omega} \times \mathbb{R})$ is of the form
\[ f(x, t) = m(x) + r(t), \] (1.4)
Our first result is the following:

**Theorem 1.1.** Let $\Omega \subset M$ be a bounded domain with $C^{2, \alpha}$ boundary $\partial \Omega$. Suppose that $f \in C^2(\bar{\Omega} \times \mathbb{R})$ satisfies (1.4), with
\[ F = \sup_{\Omega \times \mathbb{R}} |\nabla f| < \infty, \quad \text{Ric}_\Omega \geq -\frac{F^2}{n-1}, \quad \text{and} \quad H_{\partial \Omega} \geq F, \]
where $\text{Ric}_\Omega$ stands for the Ricci curvature of $\Omega$ and $H_{\partial \Omega}$ for the inward mean curvature of $\partial \Omega$. Then, for all $\varphi \in C^{2, \alpha}(\partial \Omega)$, there exists a solution $u \in C^{2, \alpha}(\bar{\Omega})$ to the equation (1.1) with boundary values $\varphi$.

The proof of Theorem 1.1 is based on the Leray-Schauder method (see [12, Theorem 13.8]), and hence requires a priori height and gradient (both interior and boundary) estimates for solutions. It is worth noting already at this point that we cannot ask for the uniqueness of a solution if the function $f: M \times \mathbb{R} \to \mathbb{R}$ depends on
the $t$-variable since comparison principles fail to hold. Indeed, an easy computation shows that for the open disk $B(0,2) \subset \mathbb{R}^2$ and $f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $f(x,t) = ||(x,t)||^2/4$, both the upper and lower hemispheres and the disk $B(0,2)$ itself are $f$-minimal hypersurfaces with zero boundary values on the circle $\partial B(0,2)$.

Thanks to the interior gradient estimate Lemma [2.3] we can weaken the regularity assumption on the boundary value function.

**Theorem 1.2.** Let $\Omega \subset M$ be a bounded domain with $C^{2,\alpha}$ boundary $\partial \Omega$. Suppose that $f \in C^2(\bar{\Omega} \times \mathbb{R})$ satisfies (1.4), with

$$F = \sup_{\Omega \times \mathbb{R}} |\nabla f| < \infty, \quad \text{Ric}_\Omega \geq -\frac{F^2}{n-1}, \quad \text{and} \quad H_{\partial \Omega} \geq F.$$ 

Then, for all $\varphi \in C(\partial \Omega)$, there exists a solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ to the equation (1.1) with boundary values $\varphi$.

Let us point out that the assumption $H_{\partial \Omega} \geq F$ is necessary. Indeed, Serrin [20] has proved that the constant mean curvature equation

$$\text{div} \frac{\nabla u}{W} = H_0$$

is solvable on a bounded domain $\Omega \subset \mathbb{R}^n$ if and only if $H_{\partial \Omega} \geq |H_0|$; see also [13] for a related result.

Finally in Section [3] we consider the Dirichlet problem at infinity. Here we suppose that $M$ is a Cartan-Hadamard manifold, i.e. a complete, simply connected Riemannian manifold with non-positive sectional curvature. We denote by $M$ the compactification of $M$ in the cone topology (see [10]) and by $\partial_\infty M$ the asymptotic boundary of $M$. The Dirichlet problem at infinity consists in finding solutions to (1.1) in the case where $\Omega = M$ and $\partial \Omega = \partial_\infty M$. In order to formulate the assumptions on sectional curvatures of $M$ and on the function $f: M \times \mathbb{R} \to \mathbb{R}$, we first denote by $\rho(\cdot) = d(o,\cdot)$ the (Riemannian) distance to a fixed point $o \in M$. Then we assume that sectional curvatures of $M$ satisfy

$$-(b \circ \rho)^2(x) \leq K(P_x) \leq -(a \circ \rho)^2(x)$$

for all $x \in M$ and all 2-dimensional subspaces $P_x \subset T_x M$, where $a$ and $b$ are smooth functions subject to conditions (A1)-(A7); see Section [4]. Given a smooth function $k: [0,\infty) \to [0,\infty)$, we denote by $f_k: [0,\infty) \to \mathbb{R}$ the smooth non-negative solution to the initial value problem

$$\begin{cases} f_k(0) = 0, \\ f_k'(0) = 1, \\ f_k'' = k^2 f_k. \end{cases}$$

To state the main result on the solvability of the asymptotic Dirichlet problem requires a number of definitions. First of all we assume that there exists an auxiliary smooth function $a_0: [0,\infty) \to (0,\infty)$ such that

$$\int_1^\infty \left( \int_r^\infty \frac{ds}{s^{a_0-1}(s)} \right) a_0(r) f_a^{a-1}(r) dr < \infty.$$ 

Then we define $g: [0,\infty) \to [0,\infty)$ by

$$g(r) = \frac{1}{f_a^{a-1}(r)} \int_0^r a_0(t) f_a^{a-1}(t) dt.$$ 

The function $g$ was introduced in [15] where they studied some elliptic and parabolic equations with asymptotic Dirichlet boundary conditions on Cartan-Hadamard.
manifolds. In addition to [14], we assume that the function \( f \in C^2(\Omega \times \mathbb{R}) \) satisfies

\[
\sup_{\partial B(o,r) \times \mathbb{R}} |\nabla f| \leq \min \left\{ a_0(r) + (n-1) \frac{f_o(r)}{f_a(r)} \frac{g^3(r)}{(1 + g^2(r))^{3/2}}, (n-1) \frac{f_o(r)}{f_a(r)} \right\},
\]

for every \( r > 0 \), and

\[
\sup_{\partial B(o,r) \times \mathbb{R}} |\nabla f| = o \left( \frac{f_o(r)}{f_a(r)} r^{-\epsilon-1} \right)
\]

for some \( \epsilon > 0 \) as \( r \to \infty \).

The general solvability result for the asymptotic Dirichlet problem is the following.

**Theorem 1.3.** Let \( M \) be a Cartan-Hadamard manifold of dimension \( n \geq 2 \). Assume that

\[-(b \circ \rho)^2(x) \leq K(P_x) \leq -a \circ \rho^2(x)\]

for all \( x \in M \) and all 2-dimensional subspaces \( P_x \subset T_x M \) where \( a \) and \( b \) satisfy assumptions [A1]–[A7] and that the function \( f \in C^2(M \times \mathbb{R}) \) on the right side of [1.1] satisfies [1.4], [1.8], and [1.9]. Then the asymptotic Dirichlet problem for the equation [1.1] is solvable for any boundary data \( \varphi \in C(\partial_{\infty} M) \).

As a special case of the above theorem, we have:

**Corollary 1.4.** Let \( M \) be a Cartan-Hadamard manifold of dimension \( n \geq 2 \). Suppose that there are constants \( \phi > 1 \), \( \epsilon > 0 \), and \( R_0 > 0 \) such that

\[-\rho(x)^{2(\phi-2)-\epsilon} \leq K(P_x) \leq -\phi(\phi-1) \rho(x)^{2-\epsilon} ,
\]

for all 2-dimensional subspaces \( P_x \subset T_x M \) and for all \( x \in M \), with \( \rho(x) \geq R_0 \). Assume, furthermore, that \( f \in C^2(M \times \mathbb{R}) \) satisfies [1.4], [1.8], and [1.9], with \( f_a(t) = t \) for small \( t \geq 0 \) and \( f_a(t) = c_1 t^{\phi} + c_2 t^{1-\phi} \) for \( t \geq R_0 \). Then the asymptotic Dirichlet problem for the equation [1.1] is solvable for any boundary data \( \varphi \in C(\partial_{\infty} M) \).

In another special case we assume that sectional curvatures are bounded from above by a negative constant \( -k^2 \).

**Corollary 1.5.** Let \( M \) be a Cartan-Hadamard manifold of dimension \( n \geq 2 \). Assume that

\[-\rho(x)^{-2-\epsilon e^{2k}\rho(x)} \leq K(P_x) \leq -k^2
\]

for some constants \( k > 0 \) and \( \epsilon > 0 \) and for all 2-dimensional subspaces \( P_x \subset T_x M \), with \( \rho(x) \geq R_0 \). Assume, furthermore, that \( f \in C^2(M \times \mathbb{R}) \) satisfies [1.4], [1.8], and [1.9], with \( f_a(t) = t \) for small \( t \geq 0 \) and \( f_a(t) = c_1 \sinh(kt) + c_2 \cosh(kt) \) for \( t \geq R_0 \). Then the asymptotic Dirichlet problem for the equation [1.1] is solvable for any boundary data \( \varphi \in C(\partial_{\infty} M) \).

We refer to [14] Ex. 2.1, Cor. 3.22] and to [14] Cor. 3.23] for the verification of the assumptions [A1]–[A7] for the curvature bounds [1.10] and [1.11], respectively. We point out that, thanks to Examples 4.5 and 4.6 the assumption [1.8] in the above corollaries is weaker than [1.9] when \( r \to \infty \).

Let us discuss where the assumptions [1.8] and [1.9] will be used in our paper. First of all, we prove Theorem 1.3 by extending the boundary value function \( \varphi \) to \( M \), exhausting \( M \) by geodesic balls and solving the Dirichlet problem [1.1] in each ball. In this step, the assumption

\[
\sup_{\partial B(o,r) \times \mathbb{R}} |\nabla f| \leq (n-1) \frac{f_o(r)}{f_a(r)}
\]
is used. Secondly, the other assumption in (1.8),
\[
\sup_{\partial B(o,r)\times\mathbb{R}} |\bar{\nabla} f| \leq \frac{a_0(r) + (n-1)f'(r)g_3(r)}{(1 + g^2(r))^{1/2}},
\]
is used to prove that the sequence of solutions above is uniformly bounded, thus
allowing us to extract a subsequence converging towards a global solution. Finally,
we apply (1.9) to prove that this global solution has proper boundary values at
infinity. Furthermore, concerning (1.9), let us mention a result of Pigola, Rigoli,
and Setti in [19]. There they considered the equation
\[
\text{div} \sqrt{1 + |\nabla u|^2} = h(x),
\]
for a function \( h \in C^\infty(M) \). They proved that if \( \max_M |u| < \infty \), \( h \) has a constant
sign, and \( M \) satisfies one of the following growth assumptions:
\[
\text{vol}(\partial B(o,r)) \leq Cr^{\alpha}, \text{ for some } \alpha \geq 0 \quad (1.12)
\]
or
\[
\text{vol}(\partial B(o,r)) \leq Ce^{\alpha r}, \text{ for some } \alpha \geq 0 \quad (1.13)
\]
then necessarily we have
\[
\liminf_{\rho(x)\to\infty} \frac{|h(x)|}{\rho^{-2}(x)(\log \rho(x))^{1-t}} = 0,
\]
and
\[
\liminf_{\rho(x)\to\infty} \frac{|h(x)|}{\rho^{-1}(x)(\log r(x))^{1-t}} = 0,
\]
respectively. We notice that condition (1.12) (resp. (1.13)) is implied by (1.10)
(resp. (1.11)). On the other hand, assuming (1.10) (resp. (1.11)), we notice (using
Examples 4.5 and 4.6) that (1.9) reduces to \( \sup_{\partial B(o,r)\times\mathbb{R}} \bar{\nabla} f = o(r^{2-t}) \) (resp.
\( \sup_{\partial B(o,r)\times\mathbb{R}} \bar{\nabla} f = o(r^{-1-t}) \)) when \( r \to \infty \). Therefore, in these cases, (1.9)
is almost sharp.

The paper is organised as follows: in Section 2 we prove a priori height and
gradient estimates that are needed in Section 3 where we apply the Leray-Schauder
method and prove Theorem 1.1 and 1.2. Section 4 is devoted to the asymptotic
Dirichlet problem and proofs of Theorem 1.3 and Corollaries 1.4 and 1.5.

2. Height and Gradient Estimates

In this section we adapt methods from [8], [9], [16], and [21] to obtain a priori
height and gradient estimates.

2.1. Height estimate. We begin by giving an a priori height estimate for solutions
of the equation (1.1) in a bounded open set \( \Omega \subset M \) with a \( C^2 \)-smooth boundary
assuming the estimate (2.3) on the function \( f \). First we construct an upper barrier
for a solution \( u \) of (1.1) of the form
\[
\psi(x) = \sup_{\partial \Omega} \varphi + h(d(x)),
\]
where \( d = \text{dist}(\cdot, \partial \Omega) \) is the distance from \( \partial \Omega \) and \( h \) is a real valued function that
will be determined later. Denote by \( \Omega_0 \) the open set of all points \( x \in \Omega \) that can
be joined to \( \partial \Omega \) by a unique minimizing geodesic. It was shown in [17] that in \( \Omega_0 \)
the distance function \( d \) has the same regularity as \( \partial \Omega \).

In particular, now \( d \in C^2(\Omega_0) \) and straightforward computations give
\[
\psi_i = h'd_i \quad \text{and} \quad \psi_{ij} = h''d_id_j + h'd_{i,j}.
\]
Moreover, $|\nabla d|^2 = d^i d_i = 1$ and hence $d^i d_{i,j} = 0$. We also have that
\[ \sigma_{ij} d_{i,j} = \Delta d = -H, \]
where $H = H(x)$ is the (inward) mean curvature of the level set $\{ y \in \Omega_0 : d(y) = d(x) \}$.

Given a solution $u \in C^2(\Omega)$ of (1.1),
\[ Q[u] = \frac{1}{W} \left( \sigma_{ij} - \frac{u^i u^j}{W^2} \right) u_{i,j} - \langle \overline{\nabla} f, \nu \rangle = 0, \]
we define $b : \Omega \to \mathbb{R}$ by
\[ b(x) = \langle \overline{\nabla} f(x, u(x)), \nu(x) \rangle, \tag{2.1} \]
where $\nu(x)$ is the downward pointing unit normal to the graph of $u$ at $(x, u(x))$.

Next we define an operator \( \tilde{Q}[v] = \frac{1}{W} \left( \sigma_{ij} - \frac{v^i v^j}{W^2} \right) v_{i,j} - b \),
where $W = \sqrt{1 + |\nabla v|^2}$ and $b$ does not depend on $v$. The reason to define such an operator is that it allows us to use the comparison principle whereas the operator $Q$ need not satisfy the required assumptions, see e.g. [12, Theorem 10.1]. Then for a point $x \in \Omega_0$ we obtain
\[ \tilde{Q}[\psi] + b = \frac{1}{W} \left( \sigma_{ij} - \frac{(h')^2 d_i d_j}{W^2} \right) (h'' d_i d_j + h' d_{i,j}) \]
\[ = \frac{1}{W} \left( h'' + h' \Delta d - \frac{(h')^2 h''}{W^2} \right) \]
\[ = \frac{1}{W} \left( h'' \frac{W^2}{W^2} - h' H(x) \right) \]
\[ = \frac{h''}{W^3} - \frac{h'}{W} H(x), \tag{2.2} \]
where we used that $W^2 = 1 + (h')^2$.

Next we impose an extra condition on the function $f : M \times \mathbb{R} \to \mathbb{R}$ by assuming that
\[ \sup_{s \in \mathbb{R}} |\overline{\nabla} f(x, s)| \leq H(x) \tag{2.3} \]
for all $x \in \Omega_0$. Hence $|b(x)| \leq H(x)$ for all $x \in \Omega_0$. By choosing
\[ h = \frac{e^{AC}}{C} \left( 1 - e^{-Cd} \right), \]
where $A = \text{diam}(\Omega)$ and
\[ C > \sup_{\Omega_0 \times \mathbb{R}} |\overline{\nabla} f| \]
is a constant, we obtain
\[ h' = e^{C(A-d)} \geq 1 \quad \text{and} \quad h'' = -Ch', \]
and so
\[ \tilde{Q}[\psi] = - \frac{Ch'}{W^3} - \frac{h'}{W} H - b \]
\[ < -|b| \left( \frac{h'}{W^3} + \frac{h'}{W} - 1 \right) \]
\[ \leq 0. \]
Therefore we have
\[ \begin{cases} \tilde{Q}[\psi] < 0 = \tilde{Q}[u] = Q[u] \quad \text{in } \Omega_0 \\ \psi|_{\partial \Omega} \geq u|_{\partial \Omega} = \varphi|_{\partial \Omega}. \end{cases} \]
Next we observe that \( \psi \geq u \) in \( \bar{\Omega} \). Assume on the contrary that the continuous function \( u - \psi \) attains its positive maximum at an interior point \( x_0 \in \Omega \). As in [21 p. 795] (see also [8 pp. 239-240]), we conclude that, in fact, \( x_0 \) is an interior point of \( \Omega_0 \) that leads to a contradiction with the comparison principle [12 Theorem 10.1] which states that \( u - \psi \) cannot attain its maximum in the open set \( \Omega_0 \).

Similarly we deduce that \( \psi^- \),

\[
\psi^-(x) = \inf_{\partial\Omega} \varphi - h(d(x)),
\]

is a lower barrier for \( u \), i.e. \( \psi^- \leq u \) in \( \bar{\Omega} \). These barriers imply the following height estimate for \( u \).

**Lemma 2.1.** Let \( \Omega \subset M \) be a bounded open set with a \( C^2 \)-smooth boundary and suppose that

\[
\sup_{s \in \mathbb{R}} \|\nabla f(x, s)\| \leq H(x)
\]

in \( \Omega_0 \). Let \( u \in C^2(\Omega) \cap C(\bar{\Omega}) \) be a solution of \( Q[u] = 0 \) with \( u|\partial\Omega = \varphi \). Then there exists a constant \( C = C(\Omega) \) such that

\[
\sup_{\Omega} |u| \leq C + \sup_{\partial\Omega} |\varphi|.
\]

**2.2. Boundary gradient estimate.** In this subsection we will obtain an a priori boundary gradient estimate for the Dirichlet problem (1.1). We assume that \( \Omega \subset M \) is a bounded open set with a \( C^2 \)-smooth boundary and that \( \Omega_\varepsilon \) is a sufficiently small tubular neighborhood of \( \partial\Omega \) so that the distance function \( d \) from \( \partial\Omega \) is \( C^2 \) in \( \Omega_\varepsilon \cap \bar{\Omega} \).

Furthermore, we assume that the (inward) mean curvature \( H = H(x) \) of the level set \( \{ y \in \Omega_\varepsilon : d(y) = d(x) \} \) satisfies

\[
H(x) \geq \sup_{s \in \mathbb{R}} \|\nabla f(x, s)\| := F(x)
\]

for all \( x \in \Omega_\varepsilon \cap \bar{\Omega} \). Next we extend the boundary function \( \varphi \), which is assumed to be \( C^2 \)-smooth, to \( \Omega_\varepsilon \) by setting \( \varphi(\exp_y t\nabla d(y)) = \varphi(y) \), for \( y \in \partial\Omega \), where \( \nabla d(y) \) is the unit inward normal to \( \partial\Omega \) at \( y \in \partial\Omega \). We will construct barriers of the form \( w + \varphi \), where \( w = \psi \circ d \) and \( \psi \) is a real function that will be determined later.

We denote

\[
a^{ij} = a^{ij}(x, \nabla v) = \frac{1}{W} \left( \sigma^{ij} - \sigma^i \sigma^j \right), \quad W = \sqrt{1 + \|\nabla v\|^2},
\]

and, given a solution \( u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) of (1.1), we define an operator

\[
\tilde{Q}[v] = a^{ij}(x, \nabla v)w_{ij} - b,
\]

with \( b \) as in (2.1).

The matrix \( a^{ij}(x, \nabla v) \) is positive definite with eigenvalues

\[
\lambda = \frac{1}{W^3} \quad \text{and} \quad \Lambda = \frac{1}{W}
\]

with multiplicities 1 and \( n - 1 \) corresponding respectively to the directions parallel and orthogonal to \( \nabla v \). Hence a simple estimate gives

\[
\tilde{Q}[w + \varphi] = a^{ij}(w_{ij} + \varphi_{ij}) - b \leq a^{ij}w_{ij} + \Lambda\|\varphi\|_{C^2} - b,
\]

where \( a^{ij} = a^{ij}(x, \nabla w + \nabla \varphi) \), \( \Lambda = (1 + \|\nabla w + \nabla \varphi\|^2)^{-1/2} \), and \( \|\varphi\|_{C^2} \) denotes the \( C^2(\Omega_\varepsilon) \)-norm of \( \varphi \). Since in \( \Omega_\varepsilon \cap \bar{\Omega} \) we have \( \|\nabla d\|^2 = d'd_4 = 1 \), \( d'd_{ij} = 0 \), and \( \langle \nabla d, \nabla \varphi \rangle = 0 \), straightforward computations give that

\[
\Delta w = \psi'' + \psi' \Delta d,
\]

\[
w^i w^j w_{ij} = (\psi')^2 \psi'',
\]

\[
w^i \varphi^j w_{ij} = \psi' \psi'' (\nabla d, \nabla \varphi) = 0.
\]
and also
\[ \psi' \varphi^j w_{i,j} = \psi''(\nabla \varphi, \nabla d)^2 + \psi' \varphi^j \varphi^l d_{i,j} = \psi' \varphi^j \varphi^l d_{i,j}. \]

With these, and noticing that now \( W^2 = 1 + (\psi')^2 + |\nabla \varphi|^2 \), we obtain
\[ d^j_w w_{i,j} = \psi' \Delta d W^{-3} + \frac{\psi''(1 + |\nabla \varphi|^2)}{W^3} - \frac{\psi' \varphi^j \varphi^l d_{i,j}}{W^3}. \tag{2.9} \]

Putting (2.8) and (2.9) together, we arrive at
\[ \tilde{Q}[w + \varphi] \leq \psi' \Delta d W^{-3} + \frac{\psi''(1 + |\nabla \varphi|^2)}{W^3} - \frac{\psi' \varphi^j \varphi^l d_{i,j}}{W^3} + \Lambda \|\varphi\|_{C^2} + F. \tag{2.10} \]

Next we define
\[ \psi(t) = \frac{C \log(1 + K t)}{\log(1 + K)}, \]
where the constants \( C \geq 2(\max_{\Omega} |u| + \max_{\Omega} |\varphi|) \), \( K \geq (1 - 2\varepsilon)\varepsilon^{-2} \), and \( \varepsilon \in (0, 1/2) \) will be chosen later. Then
\[ \psi(\varepsilon) = \frac{C \log(1 + K \varepsilon)}{\log(1 + K)} \geq C/2 \]
and we have
\[ (w + \varphi)|\Gamma_\varepsilon = \psi(\varepsilon) + \varphi|\Gamma_\varepsilon \geq u|\Gamma_\varepsilon \]
on the “inner boundary” \( \Gamma_\varepsilon = \{ x \in \Omega : d(x) = \varepsilon \} \) of \( \Omega_\varepsilon \). On the other hand,
\[ (w + \varphi)|\partial \Omega = u|\partial \Omega. \tag{2.12} \]

We claim that \( \tilde{Q}[w + \varphi] \leq 0 \) in \( \Omega_\varepsilon \cap \Omega \) if \( C, K \), and \( \varepsilon \) are properly chosen. All the computations below will be done in \( \Omega_\varepsilon \cap \Omega \) without further notice. We first observe that
\[ \psi'(t) = \frac{CK}{(1 + K t) \log(1 + K)} \quad \text{and} \quad \psi''(t) = -\frac{\log(1 + K) \psi'(d)^2}{C}, \]
and therefore we have
\[ W \tilde{Q}[w + \varphi] \leq (W - \psi') H - \frac{\log(1 + K)}{C} \left( \frac{\psi'}{W} \right)^2 (1 + |\nabla \varphi|^2) + \|\varphi\|_{C^2} + |\nabla \varphi|^2 H \]
\[ \quad + \Lambda \|\varphi\|_{C^2} + |\nabla \varphi|^2 H \tag{2.13} \]
by (2.5), (2.7), and (2.10). We estimate
\[ \psi' \geq \frac{CK}{(1 + K \varepsilon) \log(1 + K)} = \frac{C}{(\varepsilon + 1/K) \log(1 + K)} = 1 \]
and consequently,
\[ \frac{\psi'}{W} \geq c_1 = c_1 \left( \max_{\Omega} |\nabla \varphi| \right) > 0 \]
and
\[ W - \psi' \leq c_2 = c_2 \left( \max_{\Omega} |\nabla \varphi| \right) \]
by choosing \( C = (\varepsilon + 1/K) \log(1 + K) \). The claim \( \tilde{Q}[w + \varphi] \leq 0 \) now follows from (2.13) since
\[ \frac{\log(1 + K)}{C} = \frac{1}{\varepsilon + 1/K} \geq \frac{c_2 H + \|\varphi\|_{C^2} + |\nabla \varphi|^2 H}{c_1^2 (1 + |\nabla \varphi|^2)} \]
by choosing sufficiently small \( \varepsilon \) and large \( K \) depending only on \( \max_{\Omega} |u|, \|\varphi\|_{C^2}, \) and \( H_{\partial \Omega} \).

Hence
\[ \tilde{Q}[w + \varphi] \leq 0 = \tilde{Q}[u]. \]
and therefore \( w + \varphi \) is an upper barrier in \( \Omega \cap \Omega \). Similarly, \(-w + \varphi\) is a lower barrier. Together these barriers imply that
\[
|\nabla u| \leq |\nabla w| + |\nabla \varphi| = \psi'(0) + |\nabla \varphi| = \frac{CK}{\log(1 + K)} + |\nabla \varphi|
\]
on \( \partial\Omega \).

We have proven the following boundary gradient estimate.

**Lemma 2.2.** Let \( \Omega \subset M \) be a bounded open set with a \( C^2 \)-smooth boundary and suppose that
\[
\sup_{s \in \mathbb{R}} |\nabla f(x, s)| \leq H(x)
\]
in some tubular neighborhood of \( \partial \Omega \). Let \( u \in C^2(\Omega) \cap C^1(\Omega) \) be a solution to \( Q[u] = 0 \) with \( u|\partial\Omega = \varphi \in C^2(\partial\Omega) \). Then
\[
\max_{\partial\Omega} |\nabla u| \leq C,
\]
where \( C \) is a constant depending only on \( \sup\Omega |u| \), \( H_{|\Omega} \), and \( \|\varphi\|_{C^2(\partial\Omega)} \).

### 2.3. Interior gradient estimate.

In this subsection we will assume that \( u \) is a \( C^3 \) function. The elliptic regularity theory will guarantee that the estimate holds also for \( C^{2,\alpha} \) solutions. We also assume that \( f: M \times \mathbb{R} \to \mathbb{R} \) is of the form
\[
f(x, t) = m(x) + r(t).
\]

In particular, all “space” derivatives
\[
f_i = \frac{\partial f}{\partial x_i}, \quad i = 1, \ldots, \dim M,
\]
are independent of \( t \); \( f_t = f_{tt} = 0 \).

For an open set \( \Omega \subset M \), we denote \( i(\Omega) = \inf_{x \in \Omega} i(x) \), where \( i(x) \) is the injectivity radius at \( x \). Thus \( i(\Omega) > 0 \) if \( \Omega \in M \) is relatively compact. Furthermore, we denote by \( R_\Omega \) the Riemannian curvature tensor in \( \Omega \).

**Lemma 2.3.** Let \( u \in C^3(\Omega) \) be a solution of \( \{1.1\} \) with \( u < m_u \) for some constant \( m_u < \infty \).

(a) For every ball \( B(o, r) \subset \Omega \), there exists a constant
\[
L = L(u(o), m_u, r, R_\Omega, \|f\|_{C^2(\Omega \times (-\infty, m_u))})
\]
such that \( |\nabla u(o)| \leq L \).

(b) If, furthermore, \( u \in C^1(\Omega) \), we have a global gradient bound
\[
|\nabla u(o)| \leq L \quad \text{for every } o \in \Omega, \text{ with}
\]
\[
L = L(u(o), m_u, i(\Omega), \text{diam}(\Omega), R_\Omega, \|f\|_{C^2(\Omega \times (-\infty, m_u))}, \max_{\partial\Omega} |\nabla u|) < \infty.
\]

**Proof.** We apply the method due to Korevaar and Simon [16]; see also [9]. Let \( 0 < r \leq \min\{i(\Omega), \text{diam}(\Omega)\} \), \( o \in \Omega \), and let \( \eta \) be a continuous non-negative function on \( M \), vanishing outside \( B(o, r) \) and smooth whenever positive. The function \( \eta \) will be specified later. Define
\[
h = \eta W
\]
and assume first that \( h \) attains its maximum at an interior point \( p \in B(o, r) \cap \Omega \). The case \( p \in B(o, r) \cap \partial\Omega \) and \( u \in C^1(\Omega) \) will be commented at the end of the proof.

We will first prove an upper bound for \( |\nabla u(p)| \). Therefore we may assume that \( |\nabla u(p)| \neq 0 \). We choose normal coordinates at \( p \) so that \( \nabla_1 = \nabla u/|\nabla u| \) at \( p \). All
the computations below will be made at \( p \) without further notice. Thus we have \( \sigma_{ij} = \sigma^{ij} = \delta^{ij} \), \( u_1 = u^1 = |\nabla u| \), and \( u_j = u^j = 0 \) for \( j > 1 \). Furthermore,

\[
a^{ij} = \frac{1}{W} \left( \delta^{ij} - \frac{|\nabla u|^2 \delta^{ij} \delta^{ij}}{W^2} \right),
\]

and therefore \( a^{11} = W^{-3} \), \( a^{ii} = W^{-1} \) for \( i > 1 \), and \( a^{ij} = 0 \) if \( i \neq j \). At the maximum point \( p \), we have \( h_i = 0 \) and \( h_{i;i} \leq 0 \) for all \( i \). Hence

\[
\eta_i W = -\eta W_i
\]

and

\[
a^{ij} h_{i;j} = a^{ii} h_{i;i} = a^{ii} (W \eta_{k;i} + 2 \eta_i W_i + \eta W_{i;i}) \leq 0.
\]

With \((2.15)\) we can write this as

\[
Wa^{ii} \eta_{k;i} + \frac{\eta a^{ii}}{W} (W W_{i;i} - 2 (W_i)^2) \leq 0. \tag{2.16}
\]

We have

\[
W_i = \frac{u^k u_{k;i}}{W} = |\nabla u|_{1;i} \frac{u_{1;i}}{W}
\]

and from \((1.2)\) we see that the \( k^{th} \) component of the unit normal is

\[
\nu^k = \frac{u^k}{W} = |\nabla u| \delta^{k1}.
\]

To scrutinize the second order differential inequality \((2.16)\), we first compute

\[
a^{ii} W_{i;i} = a^{ii} (W^{-1} u^k u_{k;i})_{,i} = -\frac{a^{ii} |\nabla u| u_{1;i}}{W} + \frac{a^{ii} u^k u_{k;i} W_i}{W} + \frac{a^{ii} |\nabla u| u_{1;i} W_i}{W} = -\frac{a^{ii} |\nabla u|^2 (u_{1;i})^2}{W^3} + \frac{a^{ii} u^k u_{k;i}}{W} + \frac{a^{ii} |\nabla u| u_{1;i} W_i}{W} = \frac{a^{ii} (u_{1;i})^2}{W^3} + \frac{a^{ii} \sum_{k \neq 1} (u_{k;i})^2}{W} + \frac{a^{ii} |\nabla u| u_{1;i} W_i}{W}.
\]

Hence

\[
Wa^{ii} W_{i;i} = A + a^{ii} |\nabla u| u_{1;i;i}, \tag{2.17}
\]

where

\[
A = a^{ii} (u_{1;i})^2 W^{-2} + a^{ii} \sum_{k \neq 1} (u_{k;i})^2 \geq 0.
\]

Using the Ricci identities for the Hessian of \( u \) we get

\[
u_{k;i;j} = u_{i;k;j} = u_{i;j;k} + R^l_{kij} u^l,
\]

where \( R \) is the curvature tensor in \( M \). This yields

\[
|\nabla u| a^{ii} u_{1;i} = |\nabla u| a^{ii} u_{i;i} + |\nabla u|^2 a^{ii} R^l_{1;i;i}. \tag{2.18}
\]

To compute \( |\nabla u| a^{ii} u_{i;i} \), we first observe that

\[
wa^{ii} u_{i;i} = \langle \nabla f, (\nabla u, -1) \rangle = f_i u^i - f_1.
\]
Since
\[ \nu^1(Wa^{ij})_{,i}u_{;ij} = \nu^1(\sigma^{ij} - u^i u^j W^{-2})_{,i}u_{;ij} \]
\[ = -\frac{|\nabla u|}{W} \left( \frac{2u^i u^j}{W^2} - \frac{2u^i u^j W}{W^3} \right) u_{;ij} \]
\[ = -2|\nabla u|^2 \frac{u^j u_{;ij}}{W^3} + 2|\nabla u|^4 (u_{1;1})^2 \]
\[ = -2|\nabla u|^2 \left( \sum_i (u_{1;i})^2 - \frac{|\nabla u|^2}{W^2} (u_{1;1})^2 \right) \]
\[ = -2|\nabla u|^2 a^{ii}(u_{1;1})^2 \]
\[ = -2a^{ii}(W_1)^2, \]
we obtain
\[ |\nabla u| a^{ii} u_{;i;i} = |\nabla u| a^{ij} u_{;i;j} = \nu^1 W a^{ij} u_{;i;j} \]
\[ = \nu^1 (W a^{ij} u_{;i;j})_{,i} - \nu^1 (W a^{ij})_{,i} u_{;i;j} \]
\[ = \nu^1 (f_i u^i - f_1)_{,i} + 2a^{ii}(W_1)^2 \]
\[ = \nu^1 (f_i u^i + (f_1),_1 u^f - (f_1)_{,1}) + 2a^{ii}(W_1)^2 \]
\[ = \frac{|\nabla u|}{W} (f_i u^i + (f_1),_1 u^f - f_1 u^f) + 2a^{ii}(W_1)^2 \]
\[ = W_i f^i + f_{11} |\nabla u|^2 - f_1 |\nabla u|^2 + 2a^{ii}(W_1)^2, \]
where we have denoted \((f_j)_{,1} = (x \mapsto f_j(x, u(x)))_{,1} \) and used the assumption \(f_\ell = f_{i\ell} = 0\). Putting together (2.15), (2.17), (2.18), and (2.19) we can estimate the inequality (2.16) as
\[ 0 \geq Wa^{ii} \eta_i + \frac{\eta a^{ii}}{W} (W W_{i;i} - 2(W_i)^2) \]
\[ = Wa^{ii} \eta_i + \frac{\eta}{W} \left( A + |\nabla u| a^{ii} u_{;i;i} - |\nabla u| a^{ii} u_{;i;i} + W_i f^i + \frac{|\nabla u|^2 (f_{11} - f_{1\ell})}{W} \right) \]
\[ = Wa^{ii} \eta_i + \eta \left( A + \frac{|\nabla u|^2 a^{ii} R_{11}^i}{W} + \frac{|\nabla u|^2 (f_{11} - f_{1\ell})}{W^2} \right) - f^i \eta_i \]
\[ \geq Wa^{ii} \eta_i - f^i \eta_i - N \eta, \]
where \(N\) is a positive constant depending only on the curvature tensor in \(\Omega\) and the \(C^2\)-norm of \(f\) in the cylinder \(\Omega \times (-\infty, m_u)\). Note that \(A \geq 0, a^{11} = W^{-3}\), and \(a^{ii} = W^{-1}\) for \(i \neq 1\).

Now we are ready to choose the function \(\eta\) as
\[ \eta(x) = g(\phi(x)), \]
where
\[ g(t) = e^{C_1 t} - 1 \]
with a positive constant \(C_1\) to be specified later and
\[ \phi(x) = (1 - r^{-2}d^2(x) + C(u(x) - m_u))^+. \]
Here \(d(x) = d(x, o)\) is the geodesic distance to \(o\) and
\[ C = \frac{-1}{2(u(o) - m_u)} > 0. \]
It follows that \( \eta \) fulfils the requirements and, moreover, \( \eta(\alpha) = e^{C_1/2} - 1 > 0 \). We have
\[
\eta_i = (-r^{-2}(d^2)_i + Cu_i) g' \tag{2.21}
\]
and
\[
\eta_{i;j} = (-r^{-2}(d^2)_{i;j} + Cu_{i;j}) g' + (-r^{-2}(d^2)_i + Cu_i) (-r^{-2}(d^2)_j + Cu_j) g''. \tag{2.22}
\]
A straightforward computation gives the estimate
\[
W a^{ii} (-r^{-2}(d^2)_{i;i} + Cu_{i;i}) = W a^{ii} (-r^{-2}(d^2)_{i;i} + 2Cr^{-2}(d^2)_i, u_i + C^2(u_i)^2)
\]
\[
= r^{-4} |\nabla d^2|^2 - 2Cr^{-2} |\nabla d^2, \nabla u| + C^2 |\nabla u|^2
\]
\[
= \frac{C^2 |\nabla u|^2}{W^2} - \frac{2C |\nabla d^2, \nabla u|}{r^2 W^2} + \frac{1}{r^4} \left( |\nabla d^2|^2 - \frac{C^2 |\nabla u|^4}{W^2} \right)
\]
\[
\ge \frac{C^2 |\nabla u|^2}{W^2} - \frac{2C |\nabla d^2, \nabla u|}{r^2 W^2} + \frac{1}{r^4} \left( |\nabla d^2|^2 - \frac{C^2 |\nabla u|^4}{W^2} \right). \tag{2.23}
\]
Next we observe that
\[
Wa^{ii} (-r^{-2}(d^2)_{i;i} + Cu_{i;i}) = -r^{-2} Wa^{ii}(d^2)_{i;i} + CW a^{ii} u_{i;i}
\]
\[
= -r^{-2} \Delta d^2 + \frac{|\nabla u|^2}{r^2 W^2} (d^2)_{1,1} + CW \langle \nabla f, \nu \rangle
\]
\[
= -r^{-2} \Delta d^2 + \frac{|\nabla u|^2}{r^2 W^2} \text{Hess} d^2(\partial_1, \partial_1) + CW \langle \nabla f, \nu \rangle.
\]
Putting together (2.21), (2.22), (2.23), and (2.24) we obtain
\[
Wa^{ii} \eta_{i;i} \ge g' \left( -r^{-2} \Delta d^2 + \frac{|\nabla u|^2}{r^2 W^2} \text{Hess} d^2(\partial_1, \partial_1) + CW \langle \nabla f, \nu \rangle \right)
\]
\[
+ g'' \left( \frac{C^2 |\nabla u|^2}{W^2} - \frac{2C |\nabla d^2, \nabla u|}{r^2 W^2} + \frac{1}{r^4} \left( |\nabla d^2|^2 - \frac{C^2 |\nabla u|^4}{W^2} \right) \right).
\]
Hence, by (2.20), we have
\[
g'' \left( \frac{C^2 |\nabla u|^2}{W^2} - \frac{2C |\nabla d^2, \nabla u|}{r^2 W^2} + \frac{1}{r^4} \left( |\nabla d^2|^2 - \frac{C^2 |\nabla u|^4}{W^2} \right) \right) + g' P - N g \le 0,
\]
where
\[
P = \frac{|\nabla u|^2}{r^2 W^2} \text{Hess} d^2(\partial_1, \partial_1) - \frac{\Delta d^2}{r^2} + \frac{f_i(d^2)}{r^2} - C f_i.
\]
It is easy to see that
\[
|P| \le \frac{(\text{Hess} d^2(\partial_1, \partial_1)) + |\Delta d^2|}{r^2} + \frac{2d^2 f_i d_i}{r^2} + C |f_i| \le C_0,
\]
with a constant \( C_0 = C_0(u(\alpha) - m_u, r, R_0, \|f\|_{C^1}) \).
In order to obtain an upper bound for \( |\nabla u(p)| \), we suppose that
\[
|\nabla u(p)| \ge \frac{16(m_u - u(\alpha))}{r}
\]
and derive a contradiction. Since \( |\nabla d^2(p)| \le 2r \), we see that
\[
|\nabla u(p)| \ge \frac{4 |\nabla d^2(p)|}{C r^2}
\]
and hence we have
\[
|\nabla u|^2 - \frac{2}{C r^2} (\nabla u, \nabla d^2) \ge \frac{1}{2} |\nabla u|^2
\]
at $p$. Therefore there exists a constant $D$ depending only on $m_u - u(o)$ and $r$ such that
\[
\frac{C^2}{W^2} \left( |\nabla u|^2 - \frac{2}{C^2} (\nabla u, \nabla d^2) \right) \geq D > 0.
\]
But now, taking $C_1 = C_1(C_0, D, N)$ large enough, we obtain
\[
D g''(\phi(p)) - C_0 g'(\phi(p)) - N g(\phi(p)) = (DC_1^2 - C_1 C_0 - N)e^{C_1 \phi(p)} + N > 0
\]
which is a contradiction with (2.25). Hence we have
\[
|\nabla u(p)| < \frac{16 (m_u - u(o))}{r}
\]
which implies
\[
W(p) \leq C_2 = 1 + \frac{16 (m_u - u(o))}{r}.
\]
Since $p$ is a maximum point of $h = \eta W$, we have
\[
\left( e^{C_1/2} - 1 \right) W(o) = \eta(o) W'(o) \leq \eta(p) W(p) \leq C_2 \left( e^{C_1} - 1 \right).
\]
This proves the case (a).

For the case (b), we assume, in addition, that $u \in C^1(\bar{\Omega})$ and we fix $r = \min\{d(\bar{\Omega}), \text{diam}(\Omega)\} > 0$. Let $o \in \bar{\Omega}$ and $h = \eta W$ be as above with the same constant $C_1$. If a maximum point $p$ of $h$ is an interior point of $\Omega$, the proof for the case (a) applies and we have a desired upper bound for $|\nabla u(o)|$. On the other hand, if $p \in \partial \Omega$ we have an upper bound
\[
|\nabla u(p)| \leq \max_{\partial \Omega} |\nabla u|
\]
and again we are done. \[\square\]

3. Existence of $f$-minimal graphs

In this section we will prove Theorem 1.1 and 1.2. Throughout this section we assume that $\Omega \subset M$ is a bounded open set with $C^{2,\alpha}$ boundary $\partial \Omega$. As in Subsection 2.1, we denote by $\Omega_0$ the open set of all those points of $\Omega$ that can be joined to $\partial \Omega$ by a unique minimizing geodesic. We start with the following lemma from [21] Lemma 4.2; see also [8, Lemma 5]. Since our definition of the mean curvature differs by a multiple constant from the one used in [21] and [8], we sketch the proof.

**Lemma 3.1.** Let $F = \sup\{|\nabla f(x, s)|: (x, s) \in \bar{\Omega} \times \mathbb{R}\} < \infty$ and suppose that $\text{Ric}_\Omega \geq -F^2/(n-1)$ and $H_{\partial \Omega} \geq F$. Then for all $x_0 \in \Omega_0$ the inward mean curvature $H(x_0)$ of the level set $\{ x \in \Omega: d(x) = d(x_0) \}$ passing through $x_0$ has a lower bound $H(x_0) \geq F$.

**Proof.** Denote by $H(t)$ the inward mean curvature of the level set $\Gamma_t = \{ x \in \Omega: d(x) = t \}$ at the point which lies on the unit speed minimizing geodesic joining $\gamma(0) \in \partial \Omega$ to $x_0$. Denote by $N = \gamma_t$ the inward unit normal to $\Gamma_t$ and by $S_t$ the shape operator, $S_t(X) = -\nabla_X N$, of the level set $\Gamma_t$. As in [8], we obtain the Riccati equation
\[
S'_t = S'^2 + R_t,
\]
where $R_t = R(\cdot, \gamma_t) \gamma_t$. Trace and derivative commute, but because of the term $S^2$, we need to substitute $s = \text{tr} S_t/(n-1)$ in order to get similar differential equation for the traces. Hence we have
\[
s' = s^2 + r,
\]
In particular, with \( \tau \in \Omega \), where \( a_{\tau} \) is as in (2.6) and \( \nu \) is the downward unit normal to the graph of \( u_k^\pm \). Since

\[
\mathrm{tr} \frac{S_1}{n-1} \leq \left( \frac{\mathrm{tr} S_1}{n-1} \right)^2 + \frac{1}{n-1} \mathrm{Ric}(\gamma, \gamma).
\]

Since \( H(t) = \mathrm{tr} S_1 \), we obtain the estimate

\[
\frac{H'(t)}{n-1} \geq \left( \frac{H(t)}{n-1} \right)^2 + \frac{1}{n-1} \mathrm{Ric}(\gamma, \gamma) \geq \frac{H^2(t)}{(n-1)^2} - \frac{F^2}{(n-1)^2}.
\]

On the boundary we have \( H(0) = H_{\partial \Omega} \geq F \) which implies that \( H'(t) \geq 0 \) and hence the claim follows. 

\[\square\]

**Proof of Theorem 1.2.** In order to prove Theorem 1.1 we assume that the given boundary value function is extended to a function \( \varphi \in C^{2,\alpha}(\overline{\Omega}) \) and we consider a family of Dirichlet problems

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} - \tau(\nabla f, \nu) &= 0 & \text{in } \Omega, \\
u = \tau \varphi & \text{in } \partial \Omega, \ 0 \leq \tau \leq 1.
\end{array} \right.
\end{align*}
\]

By Lemma 3.1,

\[
H(x) \geq F \geq \sup_{\Omega \times \mathbb{R}} |\nabla (\tau f)|
\]

for all \( x \in \Omega_0 \) and for all \( \tau \in [0, 1] \). Hence if \( u \in C^{2,\alpha}(\overline{\Omega}) \) is a solution of \( 3.1 \) for some \( \tau \in [0, 1] \), it follows from Lemmata 2.1, 2.2, and 2.3 that

\[
\|u\|_{C^1(\overline{\Omega})} \leq C
\]

with a constant \( C \) that is independent of \( \tau \). The Leray-Schauder method [12, Theorem 13.8] then yields a solution to the Dirichlet problem (3.1) for all \( \tau \in [0, 1] \). In particular, with \( \tau = 1 \) we obtain a solution to the original Dirichlet problem. 

\[\square\]

**Proof of Theorem 1.2.** Let \( \varphi \in C(\partial \Omega) \) and let \( \varphi_k^\pm \in C^{2,\alpha}(\partial \Omega) \) be two monotonic sequence converging uniformly on \( \partial \Omega \) to \( \varphi \) from above and from below, respectively. Denote

\[
F^+ = \sup_{\Omega \times \mathbb{R}} |\nabla f| \quad \text{and} \quad F^- = -F^+.
\]

By Theorem 1.1 there are functions \( u_k^+, v_k^+ \in C^{2,\alpha}(\overline{\Omega}) \) such that \( u_k^+|\partial \Omega = v_k^+|\partial \Omega = \varphi_k^\pm \) and

\[
\begin{align*}
a^{ij}(x, \nabla u_k^+)(u_k^+)_{ij} - (\nabla f, \nu_k^+) &= 0 \\
a^{ij}(x, \nabla v_k^+)(v_k^+)_{ij} + F^+ &= 0
\end{align*}
\]

in \( \Omega \), where \( a^{ij} \) is as in (2.6) and \( v_k^+ \) is the downward unit normal to the graph of \( u_k^+ \). Since

\[
a^{ij}(x, \nabla v_k^+)(v_k^+)_{ij} + F^- \leq a^{ij}(x, \nabla v_k^+)(v_k^+)_{ij} + F^+ = 0
\]

and \( v_k^+|\partial \Omega \geq v_j^+|\partial \Omega \) for all \( k, \ell \), we obtain from the comparison principle [12, Theorem 10.1] applied to the operator \( a^{ij} + F^- \) that

\[
v_k^- \leq v_k^+ \quad \text{in } \Omega.
\]

On the other hand, since \( \varphi_{k+1}^+ \leq \varphi_k^+ \) and \( \varphi_k^- \leq \varphi_{k+1}^- \) on \( \partial \Omega \), we have again by the comparison principle that

\[
v_1^- \leq \cdots \leq v_\ell^- \leq v_{\ell+1}^- \cdots \leq v_{k+1}^- \leq v_k^+ \cdots \leq v_1^+.
\]

(3.2)
Similarly, since
\[ a^{ij}(x, \nabla v_k^\pm)(v_k^\pm)_{i,j} - (\nabla f, v_k^\pm) \leq a^{ij}(x, \nabla v_k^\pm)(v_k^\pm)_{i,j} - F^- = 0 \]

\[ = a^{ij}(x, \nabla u_k^\pm)(u_k^\pm)_{i,j} - (\nabla f, v_k^\pm) \]

and \( v_k^\pm |_{\partial \Omega} = u_k^\pm |_{\partial \Omega} \), we get
\[ u_k^\pm \leq v_k^\pm \text{ in } \bar{\Omega}. \]

Similar reasoning implies that \( v_k^\pm \leq u_k^\pm \), and therefore
\[ v_k^- \leq u_k^- \leq u_k^+ \leq v_k^+ \text{ in } \bar{\Omega}. \] (3.3)

Hence the sequences \( u_k^\pm, v_k^\pm \) have uniformly bounded \( C^0 \) norms and the local interior gradient estimate (Lemma 2.3) together with [12, Corollary 6.3] imply that the \( u_k^\pm, v_k^\pm \) have equicontinuous \( C^{2, \alpha} \) norms on compact subsets \( K \subset \Omega \).

Taking an exhaustion of \( \Omega \) by compact sets we obtain, with a diagonal argument, that \( u_k^\pm \) and \( v_k^\pm \) contain subsequences that converge uniformly in compact subsets to functions \( u, v^\pm \in C^2(\Omega) \) with respect to the \( C^2 \) norm. Moreover, we have
\[ a^{ij}(x, \nabla u)u_{i,j} - (\nabla f, u) = 0 \quad \text{and} \quad a^{ij}(x, \nabla u^\pm)u_{i,j}^\pm + F^\pm = 0. \]

Since \( v_k^\pm |_{\partial \Omega} = \varphi_k^\pm \) convergences to \( \varphi \), \( (3.2) \) implies that \( v^\pm \) extends continuously to the boundary \( \partial \Omega \) and \( v^\pm |_{\partial \Omega} = \varphi \). In turn, this and \( (3.3) \) give that \( u \) extends continuously to \( \partial \Omega \) with \( u|_{\partial \Omega} = \varphi \). Furthermore, because \( f \in C^2(M \times \mathbb{R}) \), it follows that \( u \in C^{2, \alpha}(\Omega) \cap C(\bar{\Omega}) \) ([12, Theorem 6.17]).

\[ \square \]

4. Dirichlet problem at infinity

In this section we assume that \( M \) is a Cartan-Hadamard manifold of dimension \( n \geq 2 \), \( \partial_\infty M \) is the asymptotic boundary of \( M \), and \( M = M \cup \partial_\infty M \) the compactification of \( M \) in the cone topology. Recall that the asymptotic boundary is defined as the set of all equivalence classes of unit speed geodesic rays in \( M \); two such rays \( \gamma_1 \) and \( \gamma_2 \) are equivalent if \( \sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty \). The equivalence class of \( \gamma \) is denoted by \( \gamma(\infty) \). For each \( x \in M \) and \( y \in M \setminus \{x\} \) there exists a unique unit speed geodesic \( \gamma^{x,y} : \mathbb{R} \to M \) such that \( \gamma^{x,y}_0 = x \) and \( \gamma^{x,y}_t = y \) for some \( t \in (0, \infty] \).

If \( v \in T_x M \setminus \{0\} \), \( \alpha > 0 \), and \( r > 0 \), we define a cone
\[ C(v, \alpha) = \{ y \in M \setminus \{x\} : \angle(v, \gamma^{x,y}_0) < \alpha \} \]

and a truncated cone
\[ T(v, \alpha, r) = C(v, \alpha) \setminus B(x, r) \]

where \( \angle(v, \gamma^{x,y}_0) \) is the angle between vectors \( v \) and \( \gamma^{x,y}_0 \) in \( T_x M \). All cones and open balls in \( M \) form a basis for the cone topology on \( M \).

Throughout this section, we assume that the sectional curvatures of \( M \) are bounded from below and above by
\[ -(b \circ \rho)^2(x) \leq K(P_x) \leq -(a \circ \rho)^2(x) \] (4.1)

for all \( x \in M \), where \( \rho(x) = d(o, x) \) is the distance to a fixed point \( o \in M \) and \( P_x \) is any 2-dimensional subspace of \( T_x M \). The functions \( a, b : [0, \infty) \to [0, \infty) \) are assumed to be smooth such that \( a(t) = 0 \) and \( b(t) \) is constant for \( t \in [0, T_0] \) for some \( T_0 > 0 \), and \( b \geq a \). Furthermore, we assume that \( b \) is monotonic and that there exist positive constants \( T_1, C_1, C_2, C_3, \) and \( Q \in (0,1) \) such that
\[ a(t) \begin{cases} = C_1 t^{-1} & \text{if } b \text{ is decreasing,} \\ \geq C_1 t^{-1} & \text{if } b \text{ is increasing} \end{cases} \] (A1)
for all $t \geq T_1$ and

\begin{align*}
a(t) &\leq C_2, \\
 b(t+1) &\leq C_2 b(t), \\
 b(t/2) &\leq C_2 b(t), \\
 \frac{b(t)}{t} &\geq C_3 (1+t)^{-Q}
\end{align*}

for all $t \geq 0$. In addition, we assume that

\[ \lim_{t \to \infty} \frac{b'(t)}{b(t)^2} = 0 \]

and that there exists a constant $C_4 > 0$ such that

\[ \lim_{t \to \infty} \frac{t^{1+C_4} b(t)}{f_k(t)} = 0. \]

It can be checked from \cite{14} or from \cite{1} that the curvature bounds in Corollary 1.4 and Corollary 1.5 satisfy the assumptions (A1)-(A7).

### 4.1. Construction of a barrier.

The curvature bounds \cite{4,14} are needed to control the first two derivatives of the “barrier” functions that we will construct in this subsection. Recall from the introduction that for a smooth function $k: [0, \infty) \to [0, \infty)$, we denote by $f_k: [0, \infty) \to \mathbb{R}$ the smooth non-negative solution to the initial value problem

\[
\begin{cases}
f_k(0) = 0, \\
f_k(0) = 1, \\
f_k'' - k^2 f_k = 0
\end{cases}
\]

Following \cite{14}, we construct a barrier function for each boundary point $x_0 \in \partial_\infty M$. Towards this end let $v_0 = \gamma_0^{o,x_0}$ be the initial (unit) vector of the geodesic ray $\gamma^{o,x_0}$ from a fixed point $o \in M$ and define a function $h: \partial_\infty M \to \mathbb{R}$,

\[ h(x) = \min \left( 1, L \cdot \cos(v_0, \gamma^{o,x}_0) \right), \]

where $\frac{L}{(8/\pi, \infty)}$ is a constant. Then we define a crude extension $\tilde{h} \in C(M)$, with $\tilde{h}|_{\partial_\infty M} = h$, by setting

\[ \tilde{h}(x) = \min \left( 1, \max \left( 2 - 2\rho(x), L \cdot \cos(v_0, \gamma^{o,x}_0) \right) \right). \]

Finally, we smooth out $\tilde{h}$ to get an extension $h \in C^\infty(M) \cap C(M)$ with controlled first and second order derivatives. For that purpose, we fix $\chi \in C^\infty(\mathbb{R})$ such that

\[ 0 \leq \chi \leq 1, \quad \operatorname{spt} \chi \subset [-2, 2], \quad \chi([-1, 1]) \equiv 1. \]

Then for any function $\varphi \in C(M)$ we define functions $F_{\varphi}: M \times M \to \mathbb{R}$, $\mathcal{R}(\varphi): M \to M$, and $\mathcal{P}(\varphi): M \to \mathbb{R}$ by

\[
F_{\varphi}(x, y) = \chi \left( b(\rho(y)) d(x, y) \right) \varphi(y), \\
\mathcal{R}(\varphi)(x) = \int_M F_{\varphi}(x, y) dm(y), \quad \text{and} \\
\mathcal{P}(\varphi) = \frac{\mathcal{R}(\varphi)}{\mathcal{R}(1)},
\]

where

\[ \mathcal{R}(1) = \int_M \chi \left( b(\rho(y)) d(x, y) \right) dm(y) > 0. \]

If $\varphi \in C(M)$, we extend $\mathcal{P}(\varphi): M \to \mathbb{R}$ to a function $\tilde{M} \to \mathbb{R}$ by setting $\mathcal{P}(\varphi)(x) = \varphi(x)$ whenever $x \in M(\infty)$. Then the extended function $\mathcal{P}(\varphi)$ is $C^\infty$-smooth in
$M$ and continuous in $\bar{M}$; see [14, Lemma 3.13]. In particular, applying $\mathcal{P}$ to the function $h$ yields an appropriate smooth extension

$$h := \mathcal{P}(\bar{h})$$

of the original function $h \in C(\partial_{\infty}M)$ that was defined in (4.2).

We denote

$$\Omega = C(v_0, 1/L) \cap M \quad \text{and} \quad \ell \Omega = C(v_0, \ell/L) \cap M$$

for $\ell > 0$ and collect various constants and functions together to a data

$$C = (a, b, T_1, C_1, C_2, C_3, C_4, Q, n, L).$$

Furthermore, we denote by $\|\text{Hess}_x u\|$ the norm of the Hessian of a smooth function $u$ at $x$, that is

$$\|\text{Hess}_x u\| = \sup_{X \in T_x M, |X| \leq 1} |\text{Hess} u(X, X)|.$$  

The following lemma gives the desired estimates for derivatives of $h$. We refer to [14] for the proofs of these estimates; see also [2].

**Lemma 4.1.** [14, Lemma 3.16] There exist constants $R_1 = R_1(C)$ and $c_1 = c_1(C)$ such that the extended function $h \in C^\infty(M) \cap C(\bar{M})$ in (4.4) satisfies

$$|\nabla h(x)| \leq c_1 \frac{1}{(f_a \circ \rho)(x)},$$

$$\|\text{Hess}_x h\| \leq c_1 \frac{(b \circ \rho)(x)}{(f_a \circ \rho)(x)},$$

for all $x \in 3\Omega \setminus B(o, R_1)$. In addition,

$$h(x) = 1$$

for every $x \in M \setminus (2\Omega \cup B(o, R_1))$.

We define a function $F: M \to [0, \infty)$ and an elliptic operator $\tilde{Q}$ by setting

$$F(x) = \sup_{t \in \mathbb{R}} |\nabla f(x, t)|$$

and

$$\tilde{Q}[\psi] = \text{div} \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} + F(x).$$

Let then $A > 0$ be a fixed constant. We aim to show that

$$\psi = A(R_3^\delta \rho^{-\delta} + h)$$

is a supersolution $\tilde{Q}[\psi] < 0$ in the set $3\Omega \setminus B(o, R_3)$, where $\delta > 0$ and $R_3 > 0$ are constants that will be specified later and $h$ is the extended function defined in (4.4).

We shall make use of the following estimates obtained in [14]:

**Lemma 4.2.** [14, Lemma 3.17] There exist constants $R_2 = R_2(C)$ and $c_2 = c_2(C)$ with the following property. If $\delta \in (0, 1)$, then

$$|\nabla h| \leq c_2/(f_a \circ \rho),$$

$$\|\text{Hess} h\| \leq c_2 \rho^{-C_1-1}/(f_a \circ \rho),$$

$$|\nabla (\nabla h, \nabla h)| \leq c_2 \rho^{-C_2-2}/(f_a \circ \rho),$$

$$|\nabla (\nabla h, \nabla (\rho^{-\delta}))| \leq c_2 \rho^{-C_1-2}/(f_a \circ \rho),$$

$$\nabla (\nabla (\rho^{-\delta}), \nabla (\rho^{-\delta})) = -2\delta^2(\delta + 1) \rho^{-2\delta - 3} \nabla \rho$$

in the set $3\Omega \setminus B(o, R_2)$.
As in [14] we denote
\[
\phi_1 = 1 + \sqrt{1 + 4C_1^2} > 1, \quad \text{and} \quad \delta_1 = \min \left\{ C_4, \frac{-1 + (n - 1)\phi_1}{1 + (n - 1)\phi_1} \right\} \in (0, 1),
\]
where \(C_1\) and \(C_4\) are constants defined in (A1) and (A7), respectively.

**Lemma 4.3.** Let \(A > 0\) be a fixed constant and \(h\) the function defined in (4.4). Assume that the function \(F\) defined in (4.6) satisfies
\[
\sup_{\rho(x) = \rho} F(x) = o \left( f_\rho'(t)^{t-\varepsilon-1} \right)
\]
for some \(\varepsilon > 0\) as \(t \to \infty\). Then there exist two positive constants \(\delta \in (0, \min(\delta_1, \varepsilon))\) and \(R_3\) depending on \(C\) and \(\varepsilon\) such that the function \(\psi = A(R_3^3\rho^{-\delta} + h)\) satisfies \(\tilde{Q}[\psi] < 0\) in the set \(3\Omega \setminus \tilde{B}(\rho, R_3)\).

**Proof.** In the proof \(c\) will denote a positive constant whose actual value may vary even within a line. Since
\[
\tilde{Q}[\psi] = \frac{\Delta \psi}{\sqrt{1 + |\nabla \psi|^2}} = \frac{1}{2} \frac{\langle \nabla |\nabla \psi|^2, \nabla \psi \rangle}{(1 + |\nabla \psi|^2)^{3/2}} + F(x)
\]
it is enough to show that there exist \(\delta > 0\) and \(R_3\) such that
\[
(1 + |\nabla \psi|^2)\Delta \psi + (1 + |\nabla \psi|^2)^{3/2}F(x) - \frac{1}{2} \langle \nabla |\nabla \psi|^2, \nabla \psi \rangle < 0
\]
in the set \(3\Omega \setminus \tilde{B}(\rho, R_3)\).

First we notice that \(\psi\) is \(C^\infty\)-smooth and
\[
\nabla \psi = A(-R_3^3\delta^\rho^{-\delta-1}\nabla \rho + \nabla h)
\]
in \(M \setminus \{o\}\). Lemma 4.2 and our curvature assumption imply that \(|\nabla h| \leq c/\rho\) for \(\rho\) large enough, and therefore
\[
|\nabla \psi|^2 = (AR_3^6\delta^2\rho^{-2\delta-2} + A^2|\nabla h|^2 - 2A^2R_3^4\delta^\rho^{-\delta-1}\langle \nabla \rho, \nabla h \rangle \leq c\rho^{-2}
\]
in \(3\Omega \setminus \tilde{B}(\rho, R_3)\) for sufficiently large \(R_3\). Then, to estimate the term with \(\Delta \psi\) in (4.10), we first note that
\[
\Delta \psi = AR_3^6(\delta(\delta + 1)\rho^{-\delta-2} - \delta\rho^{-\delta-1}\Delta \rho) + A\Delta h.
\]
Furthermore, for every \(\delta \in (0, \delta_1)\), there exists \(R_3 = R_3(C, \delta)\) such that
\[
\Delta \rho \geq (n - 1)\frac{f_\rho'(t)^{t-\varepsilon-1}}{f_\rho'(t)^{t-\varepsilon-1}} \geq \frac{(n - 1)(1 - \delta)\phi_1}{\rho} > 0
\]
whenever $\rho \geq R_3$; see [11 (3.25)]. Therefore, using Lemma 4.2 we obtain

\[(1 + |\nabla\psi|^2)\Delta \psi \leq (1 + |\nabla\psi|^2)AR^3_\delta \left( \delta + 1 - (n - 1)\frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-\delta - 2} + (1 + |\nabla\psi|^2)\text{Anc}_2 \left( \frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-C-2} \]

\[\leq AR^3_\delta \left( \delta + 1 - (n - 1)\frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-\delta - 2} + (1 + \rho \delta^2 - 2)\text{Anc}_2 \left( \frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-C-2} \]

\[= - \left( \frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-\delta - 2} (AR^3_\delta \delta(n - 1) - (1 + \rho \delta^2)\text{Anc}_2 \rho^{-C-2}) \]

\[+ AR^3_\delta \delta(n - 1)\rho^{-\delta - 2} \leq c \left( \frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-\delta - 2} \]

whenever $\delta \in (0, \delta_1)$ is small enough and $\rho \geq R_3(C, \delta)$. These estimates hold since

\[\delta + 1 - (n - 1)\frac{\rho f'_a \circ \rho}{f_a \circ \rho} \leq \delta + 1 - (n - 1)(1 - \delta)\phi_1 \leq 0 \]

for a sufficiently small $\delta \in (0, \delta_1)$. Now taking into account our assumption (4.9) we obtain

\[(1 + |\nabla\psi|^2)\Delta \psi + (1 + |\nabla\psi|^2)^{3/2}F \leq -c \left( \frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-\delta - 2} + (1 + \rho \delta^2 - 2)F \]

\[(4.11)\]

whenever $\delta \in (0, \min(\epsilon, \delta_1))$ is small enough and $\rho \geq R_3(C, \delta)$.

It remains to estimate $|\nabla|\nabla\psi|^2, \nabla\psi|$ from above. Since

\[\nabla\psi = AR^3_\delta \nabla(\rho^{-\delta}) + A\nabla h,\]

we have

\[\nabla|\nabla\psi|^2 = A^2 \nabla (R^3_\delta \nabla(\rho^{-\delta})) + \nabla h, R^3_\delta \nabla(\rho^{-\delta}) + \nabla h)\]

\[= (AR^3_\delta)^2 \nabla \nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})) + 2A^2 R^3_\delta \nabla \nabla(\rho^{-\delta}), \nabla h) + A^2 \nabla(\nabla h, \nabla h).\]

By Lemma 4.2 we then get

\[|\nabla|\nabla\psi|^2, \nabla\psi| \leq c \rho^{-1} \left( 2(\delta AR^3_\delta)^2(\delta + 1)\rho^{-2\delta - 3} + A^2 c_2(2R^3_\delta + 1) \left( \frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-C-2} \right) \]

\[\leq c \delta^2(\delta + 1)\rho^{-2\delta - 4} + c \left( \frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-C-4} \]

\[(4.12)\]

Putting together (4.11) and (4.12) we finally obtain

\[(1 + |\nabla\psi|^2)\Delta \psi + (1 + |\nabla\psi|^2)^{3/2}F(x) - \frac{1}{2} |\nabla|\nabla\psi|^2, \nabla\psi| \leq -c \left( \frac{\rho f'_a \circ \rho}{f_a \circ \rho} \right) \rho^{-\delta - 2} < 0 \]

in $3\Omega \setminus B(a, R_3)$ for a sufficiently small $\delta > 0$ and large $R_3$. \hfill \square

Similarly, we have

\[\text{div} \frac{\nabla(-\psi)}{|\nabla(-\psi)|} = F(x) > 0 \]

\[(4.13)\]
in $3\Omega \setminus B(o, R_3)$.

4.2. Uniform height estimate. We will solve the asymptotic Dirichlet problem by solving the problem first in a sequence of balls with increasing radii. In order to obtain a converging subsequence of solutions, we need to have a uniform height estimate. This subsection is devoted to the construction of a barrier function that will guarantee the height estimate.

Since $f''_a - a^2 f_a = 0$, where $a(t) = 0$ for $t \in [0, T_0]$ and

$$a(t) \geq \frac{\sqrt{\phi(\phi - 1)}}{t}$$

for $t \geq T_1$ and some $\phi > 1$, we have $f_a(t) \geq ct^\phi$ for $t \geq T_1$. Therefore

$$\int_1^\infty \frac{dr}{f_a^{n-1}(r)} < \infty. \quad (4.14)$$

Let $\varphi: M \to \mathbb{R}$ be a bounded function. We aim to show the existence of a barrier function $V$ such that $Q[V] \leq 0$ and $V(x) > ||\varphi||_\infty$ in $M$. In order to define such a function $V$, we need an auxiliary function $a_0 > 0$, so that

$$\int_1^\infty \left( \int_r^\infty \frac{ds}{f_a^{n-1}(s)} \right) a_0(r) f_a^{n-1}(r) dr < \infty. \quad (4.15)$$

We will discuss about the choice of $a_0$ in Examples 4.5 and 4.6. Now, following [18], we can define

$$V(x) = V(\rho(x)) = \left( \int_0^{\rho(x)} \frac{ds}{f_a^{n-1}(s)} \right) \left( \int_0^{\rho(x)} a_0(t) f_a^{n-1}(t) dt \right)$$

$$- \int_0^{\rho(x)} \left( \int_t^\infty \frac{ds}{f_a^{n-1}(s)} \right) a_0(t) f_a^{n-1}(t) dt = H + ||\varphi||_\infty, \quad (4.16)$$

where

$$H := \limsup_{r \to \infty} \left\{ \int_r^\infty \frac{ds}{f_a^{n-1}(s)} \int_0^r a_0(t) f_a^{n-1}(t) dt \right. \left. - \int_0^r \int_t^\infty \frac{ds}{f_a^{n-1}(s)} a_0(t) f_a^{n-1}(t) dt \right\} \leq 0; \quad (4.17)$$

see [18] (4.5). From (4.14) and (4.15) we see that $H$ is finite and hence $V$ is well defined.

As in the proof of Lemma 4.3, we write

$$\tilde{Q}[V] = \frac{(1 + |\nabla V|^2) \Delta V + (1 + |\nabla V|^2)^{3/2} F(x) - \frac{1}{2} (\nabla V, \nabla V)}{(1 + |\nabla V|^2)^{3/2}} \quad (4.18)$$

where $F(x)$ is as in (4.6), and estimate the terms of the numerator. To begin, we notice that

$$V'(r) = -\frac{1}{f_a^{n-1}(r)} \int_0^r a_0(t) f_a^{n-1}(t) dt < 0,$$

$$V''(r) = (n - 1) f_a'(r) f_a^{n-1}(r) \int_0^r a_0(t) f_a^{n-1}(t) dt - a_0(r),$$

and

$$|\nabla V(\rho(x))| = |V'(\rho(x)) \nabla \rho(x)| = |V'(\rho(x))|.$$

Note that $-V(r) = g(r)$, the function (1.7) in Introduction. The Laplace comparison theorem implies that

$$\Delta \rho \geq (n - 1) \frac{f_a' \circ \rho}{f_a \circ \rho}.$$
Hence we can estimate the Laplacian of $V$ as
\[
\Delta V = V''(\rho) + \Delta \rho V'(\rho)
\]
\[
\leq V''(\rho) + (n - 1) \frac{f'_a(\rho)}{f_a(\rho)} V'(\rho)
\]
\[
= (n - 1) \frac{f'_a(\rho)}{f_a(\rho)} \int_0^\rho a_0(t) f_a^{n-1}(t) dt - a_0(\rho) - (n - 1) \frac{f'_a(\rho)}{f_a(\rho)} \int_0^\rho a_0(t) f_a^{n-1}(t) dt
\]
\[
= -a_0(\rho),
\]
and thus the first term of (4.18) can be estimated as
\[
(1 + |\nabla V|^2) \Delta V \leq -(1 + |\nabla V|^2)a_0(\rho) \leq -(1 + V'(\rho)^2)a_0(\rho).
\]
Then, for the last term of (4.18) we have
\[
-\frac{1}{2} \langle \nabla |\Delta V|^2, \nabla V \rangle = -\frac{1}{2} \langle \nabla (V''(\rho))^2, V'(\rho) \nabla \rho \rangle = -\frac{1}{2} (2V'(\rho)V''(\rho) \nabla \rho, V'(\rho) \nabla \rho)
\]
\[
= -(V'(\rho))^2 V''(\rho)
\]
\[
= -\frac{1}{f_a^{2n-2}(\rho)} \left( \int_0^\rho a_0(t) f_a^{n-1}(t) dt \right)^2 \cdot \left( (n - 1) \frac{f'_a(\rho)}{f_a(\rho)} \int_0^\rho a_0(t) f_a^{n-1}(t) dt - a_0(\rho) \right)
\]
\[
= a_0(\rho) V'(\rho)^2 - (n - 1) \frac{f'_a(\rho)}{f_a(\rho)} (-V'(\rho))^3.
\]
Collecting everything together, we obtain that $\bar{Q}[V] \leq 0$ if
\[
\sup_{\partial B(\rho, r) \times \mathbb{R}} |\nabla f| \leq \frac{a_0(\rho) + (n - 1) \frac{f'_a(\rho)}{f_a(\rho)} (-V'(\rho))^3}{(1 + V'(\rho)^2)^{3/2}}.
\]
Finally it is easy to check that, since $H$ is finite and $V$ is decreasing, we have $V(x) > ||\varphi||_{\infty}$ for all $x \in M$ and $V(x) \to ||\varphi||_{\infty}$ as $\rho(x) \to \infty$. Altogether, we have obtained the following.

**Lemma 4.4.** Let $\varphi: M \to \mathbb{R}$ be a bounded function and assume that the function $V$ defined in (4.16) satisfies
\[
\sup_{\partial B(\rho, r) \times \mathbb{R}} |\nabla f| \leq \frac{a_0(\rho) + (n - 1) \frac{f'_a(\rho)}{f_a(\rho)} (-V'(\rho))^3}{(1 + V'(\rho)^2)^{3/2}}. \quad (4.19)
\]
Then the function $V$ is an upper barrier for the Dirichlet problem such that
\[
\bar{Q}[V] = \text{div} \frac{\nabla V}{\sqrt{1 + |\nabla V|^2}} + F(x) \leq 0 \quad \text{in } M, \quad (4.20)
\]
\[
V(x) > ||\varphi||_{\infty} \quad \text{for all } x \in M \quad (4.21)
\]
and
\[
\lim_{r(x) \to \infty} V(x) = ||\varphi||_{\infty}. \quad (4.22)
\]
Furthermore,
\[
\text{div} \frac{\nabla (-V)}{\sqrt{1 + |\nabla (-V)|^2}} - F(x) \geq 0 \quad \text{in } M. \quad (4.23)
\]
Next we show by examples that in the situation of Corollaries 1.4 and 1.5 the condition (4.19) is not a stronger restriction than the assumption (4.9) in Lemma 4.3. First note that $V'(r) \to 0$ as $r \to \infty$, and hence the upper bound (4.19) for $|\bar{\nabla} f|$ is asymptotically the function $a_0$.

Example 4.5. Assume that the sectional curvatures of $M$ satisfy

$$K(P_x) \leq -a(\rho(x))^2 = -\frac{\phi(\phi - 1)}{\rho(x)^2}, \quad \phi > 1,$$

for $\rho(x) \geq T_1$. We need to choose the function $a_0$ such that (4.15) holds, and since this is a question about its asymptotical behaviour, it is enough to consider the integral

$$\int_{T_1}^{\infty} \left( \int_r^{\infty} \frac{ds}{f_u^n(s)} \right) a_0(r) f_u^{n-1}(r) dr.$$

For $t \geq T_1$, $f_u(t) = c_1 t^\phi + c_2 t^{1-\phi}$, and hence, by a straightforward computation, we have (4.15) if

$$\int_{T_1}^{\infty} a_0(r) r dr < \infty.$$

So it is enough to choose for example

$$a_0(r) = O \left( \frac{1}{r^2 (\log r)^\alpha} \right)$$

as $r \to \infty$ for some $\alpha > 1$. On the other hand, with this curvature upper bound, the assumption (4.9) requires decreasing of order $o(r^{-2-\epsilon})$.

Example 4.6. Assume that the sectional curvatures of $M$ satisfy

$$K \leq -k^2,$$

for $\rho(x) \geq T_1$ and some constant $k > 0$. Then, for large $t$, $f_u(t) = c_1 \sinh kt + c_2 \cosh kt \approx e^{kt}$. Therefore it is straightforward to see that we have (4.15) if

$$\int_{T_1}^{\infty} a_0(r) dr < \infty,$$

which holds by choosing, for example,

$$a_0(r) = O \left( \frac{1}{r (\log r)^\alpha} \right), \quad \alpha > 1,$$

as $r \to \infty$. On the other hand, with this curvature upper bound, the assumption (4.9) requires decreasing of order $o(r^{-1-\epsilon})$.

4.3. Proof of Theorem 1.3. We start with solving the Dirichlet problem in geodesic balls $B(o, R)$.

Lemma 4.7. Suppose that $f \in C^2(M \times \mathbb{R})$ is of the form $f(x, t) = m(x) + r(t)$ and satisfies

$$\sup_{\partial B(o, R) \times \mathbb{R}} |\nabla f| \leq (n - 1) \frac{f_u'(r)}{f_u(r)}$$

for all $r > 0$. Then for every $R > 0$ and $\varphi \in C(\partial B(o, R))$ there exists a solution $u \in C^{2, \alpha}(B(o, R)) \cap C(\overline{B(o, R)})$ of the Dirichlet problem

$$\begin{cases}
\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = \langle \nabla f, \nu \rangle & \text{in } B(o, R) \\
|u|\partial B(o, R) = \varphi.
\end{cases}$$
Proof. Assuming first that \( \varphi \in C^{2,\alpha}(\partial B(o,R)) \) the claim follows by the Leray-Schauder method. Indeed, for each \( x \in \tilde{B}(o,R) \setminus \{o\} \) the inward mean curvature \( H(x) \) of the level set \( \{ y \in \tilde{B}(o,R) : d(y) = d(x) \} = \partial B(o,\rho(x)) \) satisfies

\[
H(x) = \Delta \rho(x) \geq \left( n - 1 \right) \frac{f_\rho'(\rho(x))}{f_\rho(\rho(x))} \geq \sup_{\partial B(o,\rho(x)) \times \mathbb{R}} |\nabla f|.
\]

In other words, (2.4) and (2.14) hold and therefore we can apply the Leray-Schauder method as in the proof of Theorem 1.1. The general case \( \varphi \in C(\partial B(o,R)) \) follows by approximation as in the proof of Theorem 1.2. \( \square \)

Proof of Theorem 1.3. We extend the boundary data function \( \varphi \in C(\partial_{\infty}M) \) to a function \( \varphi \in \tilde{C}(M) \). Let \( \Omega_k = B(o,k), k \in \mathbb{N} \), be an exhaustion of \( M \). By Lemma 4.7, there exist solutions \( u_k \in C^{2,\alpha}(\Omega_k) \cap \tilde{C}(\Omega_k) \) to

\[
\begin{align*}
Q[\nabla u_k] = & \ \frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}} - \langle \nabla f, \nu_k \rangle \quad \text{in } \Omega_k \\
\nu_k |_{\partial \Omega_k} = & \ \varphi,
\end{align*}
\]

where \( \nu_k \) is the downward pointing unit normal to the graph of \( u_k \). Applying the uniform height estimate, Lemma 4.4, we see that the sequence \( (u_k) \) is uniformly bounded and hence the interior gradient estimate (Lemma 2.3), together with the diagonal argument, implies that there exists a subsequence, still denoted by \( u_k \), that converges locally uniformly with respect to \( C^2 \)-norm to a solution \( u \). Therefore we are left to prove that \( u \) extends continuously to \( \partial_{\infty}M \) and satisfies \( u|_{\partial_{\infty}M} = \varphi \).

Towards that end let us fix \( x_0 \in \partial_{\infty}M \) and \( \varepsilon > 0 \). Since the boundary data function \( \varphi \) is continuous, we find \( L \in (8/\pi, \infty) \) such that

\[
|\varphi(y) - \varphi(x_0)| < \varepsilon/2
\]

for all \( y \in \tilde{C}(v_0,4/L) \cap \partial_{\infty}M \), where \( v_0 = \gamma_{\omega}^*x_0 \) is the initial vector of the geodesic ray representing \( x_0 \). Moreover, by (4.22) we can choose \( R_3 \) in Lemma 4.3 so large that \( V(r) \leq \max_{\tilde{M}} |\varphi| + \varepsilon/2 \) for \( r \geq R_3 \).

We claim that

\[
w^-(x) := -\psi(x) + \varphi(x_0) - \varepsilon \leq u(x) \leq w^+(x) := \psi(x) + \varphi(x_0) + \varepsilon \quad (4.24)
\]

in the set \( U := 3\Omega \setminus B(o,R_3) \), where \( \psi = A(R_3^2 \rho^{-\delta} + h) \) is the supersolution \( \tilde{Q}[\psi] < 0 \) in Lemma 4.3 and \( A = 2 \max_{\tilde{M}} |\tilde{\varphi}| \). Recall the notation \( \Omega = C(v_0,1/L) \cap M \) and \( \tilde{\Omega} = C(v_0,1/L) \cap \tilde{M} \), \( \ell > 0 \), from Subsection 4.1.

The function \( \varphi \) is continuous in \( \tilde{M} \) so there exists \( k_0 \) such that \( \partial \Omega_{k_0} \cap U \neq \emptyset \), and

\[
|\varphi(x) - \varphi(x_0)| < \varepsilon/2 \quad (4.25)
\]

for all \( x \in \partial \Omega_k \cap U \) when \( k \geq k_0 \). Denote \( V_k = \Omega_k \cap U \) for \( k \geq k_0 \). We will conclude that

\[
w^- \leq u_k \leq w^+ \quad (4.26)
\]

in \( V_k \) by using the comparison principle for the operator \( \tilde{Q}_k \),

\[
\tilde{Q}_k[v] = \text{div} \frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} - \langle \nabla f, \nu_k \rangle,
\]

where \( \nu_k \) is the downward pointing unit normal to the graph of the solution \( u_k \). Notice that

\[
\partial V_k = (\partial \Omega_k \cap \tilde{U}) \cup (\partial U \cap \tilde{\Omega}_k).
\]

Let \( x \in \partial \Omega_k \cap \tilde{U} \) and \( k \geq k_0 \). Then (4.25) and \( u_k |_{\partial \Omega_k} = \varphi |_{\partial \Omega_k} \) imply that

\[
w^-(x) \leq \varphi(x_0) - \varepsilon/2 \leq \varphi(x) = u_k(x) \leq \varphi(x_0) + \varepsilon/2 \leq w^+(x).
\]
Moreover, by Lemma 4.1, we have 
\[ h|M \setminus (2\Omega \cup B(o, R_3)) = 1 \]
and \( R_3^\delta \rho^{-\delta} = 1 \) on \( \partial B(o, R_3) \), so 
\[ \psi \geq A = 2 \max_M |\varphi| \]
on \( \partial U \cap \Omega_k \). By Lemma 4.4, \( V \) is a supersolution \( \tilde{Q}[V] \leq 0 \) and hence 
\[
\text{div} \frac{\nabla V}{\sqrt{1 + |\nabla V|^2}} - \langle \nabla f, \nu_k \rangle \leq \text{div} \frac{\nabla V}{\sqrt{1 + |\nabla V|^2}} + F(x) \\
= \tilde{Q}[V] \leq 0 \\
= \text{div} \frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}} - \langle \nabla f, \nu_k \rangle.
\]
Since \( V \geq \max_M |\varphi| \) on \( \partial \Omega_k \), the comparison principle yields \( u_k|\Omega_k \leq V|\Omega_k \), and by the choice of \( R_3 \), we have 
\[ u_k \leq \max_M |\varphi| + \varepsilon/2 \]
in \( \Omega_k \setminus B(o, R_3) \).

Altogether, it follows that 
\[ w^+ = \psi + \varphi(x_0) + \varepsilon \geq 2 \max_M |\varphi| + \varphi(x_0) + \varepsilon \geq \max_M |\varphi| + \varepsilon \geq u_k \]
on \( \partial U \cap \Omega_k \), and similarly \( u_k \geq w^- \) on \( \partial U \cap \Omega_k \). Consequently \( w^\leq u_k \leq w^+ \) on \( \partial V_k \). By Lemma 4.3, \( \tilde{Q}[^{\psi}] < 0 \), and therefore 
\[
\tilde{Q}_k[w^+] = \text{div} \frac{\nabla w^+}{\sqrt{1 + |\nabla w^+|^2}} - \langle \nabla f, \nu_k \rangle \\
= \text{div} \frac{\nabla ^{\psi}}{\sqrt{1 + |\nabla \psi|^2}} - \langle \nabla f, \nu_k \rangle \\
\leq \text{div} \frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} + F(x) \\
= \tilde{Q}[\psi] < 0 \\
= \text{div} \frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}} - \langle \nabla f, \nu_k \rangle
\]
in \( U \). By the comparison principle, \( u_k \leq w^+ \) in \( U \). Similarly, using (4.13) we conclude that 
\[
\text{div} \frac{\nabla w^-}{\sqrt{1 + |\nabla w^-|^2}} - \langle \nabla f, \nu_k \rangle > \text{div} \frac{\nabla u_k}{\sqrt{1 + |\nabla u_k|^2}} - \langle \nabla f, \nu_k \rangle
\]
in \( U \). Hence \( u_k \geq w^- \) in \( U \) and we obtain (4.26). This holds for every \( k \geq k_0 \) and hence (4.24) follows. Finally, 
\[
\limsup_{x \to x_0} |u(x) - \varphi(x_0)| \leq \varepsilon
\]
since \( \lim_{x \to x_0} \psi(x) = 0 \). Because \( x_0 \in \partial \infty M \) and \( \varepsilon > 0 \) were arbitrary, this shows that \( u \) extends continuously to \( C(M) \) and \( u|\partial \infty M = \varphi \). \[\square\]
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