Degree Associated Edge Reconstruction Parameters of Strong Double Brooms

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Abstract

An edge deleted unlabeled subgraph of a graph $G$ is an ecard. A da-ecard specifies the degree of the deleted edge along with the ecard. The degree associated edge reconstruction number of a graph $G$, $dern(G)$, is the size of the smallest collection of da-ecards of $G$ that uniquely determines $G$. The adversary degree associated edge reconstruction number of a graph $G$, $adern(G)$, is the minimum number $k$ such that every collection of $k$ da-ecards of $G$ uniquely determines $G$. A strong double broom is the graph on at least 5 vertices obtained from a union of (at least two) internally vertex disjoint paths with same ends $u$ and $v$ by appending leaves at $u$ and $v$. In particular, $B(n,n,mP_k)$ is the strong double broom with $n$ leaves at both the ends $u$ and $v$ and with $m$ internally vertex disjoint paths of order $k$ joining $u$ and $v$. We show that $dern$ of strong double brooms is 1 or 2. We also determine $adern(B(n,n,mP_k))$. It is 3 in most of the cases and 1 or 2 for all the remaining cases, except $adern(B(1,1,2P_k)) = 5$ for $k \geq 4$.

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1 Introduction

All graphs considered in this paper are finite, simple and undirected. We shall mostly follow the graph theoretic terminology of [8]. A vertex-deleted subgraph or card $G - v$ of a graph (digraph) $G$ is the unlabeled graph (digraph) obtained from $G$ by deleting the vertex $v$ and all edges (arcs) incident with $v$. The deck of a graph (digraph) $G$ is its collection of cards. Following the formulation...
in [2], a graph (digraph) $G$ is reconstructible if it can be uniquely determined from its deck. The well-known Reconstruction Conjecture (RC) due Kelly [11] and Ulam [22] asserts that every graph with at least three vertices is reconstructible. The conjecture has been proved for many special classes, and many properties of $G$ may be deduced from its deck. Nevertheless, the full conjecture remains open. Surveys of results on the RC and related problems include [7, 14]. Harary and Plantholt [10] defined the reconstruction number of a graph $G$, denoted by $rn(G)$, to be the minimum number of cards which can only belong to the deck of $G$ and not to the deck of any other graph $H$, $H \not\cong G$, these cards thus uniquely identifying $G$. Reconstruction numbers are known for only few classes of graphs [5].

An extension of the RC to digraphs is the Digraph Reconstruction Conjecture (DRC), proposed by Harary [9], which asserts that every digraph with at least seven vertices is reconstructible. The DRC was disproved by Stockmeyer [21] by exhibiting several infinite families of counter-examples and this made people doubt the RC itself. To overcome this, Ramachandran [18] introduced degree associated reconstruction for digraphs and proposed a new conjecture in 1981. It was proved [18] that the digraphs in all these counterexamples to the DRC obey the new conjecture, thereby protecting the RC from the threat posed by these digraph counterexamples.

The ordered triple $(a, b, c)$ where $a$, $b$ and $c$ are respectively the number of unpaired outarcs, unpaired inarcs and symmetric pair of arcs incident with $v$ in a digraph $D$ is called the degree triple of $v$. The degree associated card or dacard of a digraph (graph) is a pair $(d, C)$ consisting of a card $C$ and the degree triple (degree) $d$ of the deleted vertex. The dadeck of a digraph is the multiset of all its dacards. A digraph is said to be $N$-reconstructible if it can be uniquely determined from its dadeck. The new digraph reconstruction conjecture [18] (NDRC) asserts that all digraphs are $N$-reconstructible. Ramachandran [19, 20] then studied the degree associated reconstruction number of graphs and digraphs in 2000. The degree (degree triple) associated reconstruction number of a graph (digraph) $D$ is the size of the smallest collection of dacards of $D$ that uniquely determines $D$. Articles [1], [2], [3], [6] and [13] are recent papers on the degree associated reconstruction number.

The edge card, edge deck, edge reconstructible graphs and edge reconstruction number are defined similarly with edge deletions instead of vertex deletions. The edge reconstruction conjecture, proposed by Harary [9], states that all graphs with at least 4 edges are edge reconstructible. The ordered pair $(d(e), G - e)$ is called a degree associated edge card or da-ecard of the graph $G$, where $d(e)$ (called the degree of $e$) is the number of edges adjacent to $e$ in $G$. The edeck (da-edeck) of a graph $G$ is its collection of ecards (da-ecards). For an edge reconstructible graph $G$, Molina studied [15] the edge reconstruction number of $G$, which is defined to be the size of the smallest subcollection of the edeck of $G$ which is not contained in the edeck of any other graph $H$, $H \not\cong G$. For an edge reconstructible graph $G$ from its da-edeck, the degree associated edge reconstruction number of a graph $G$, denoted by $dern(G)$, is the size of the smallest subcollection of the da-edeck of $G$ which is not contained in the da-edeck of any other graph $H$, $H \not\cong G$. The adversary degree associated edge reconstruction number
of a graph $G$, \( \text{adern}(G) \), is the minimum number $k$ such that every collection of $k$ da-ecards of $G$ is not contained in the da-edeck of any other graph $H$, $H \not\cong G$. Degree associated edge reconstruction parameters might be a strong tool for providing evidence to support or reject the Edge Reconstruction Conjecture that remains open. For very few classes of graphs, these edge reconstruction parameters have been determined \cite{4, 12, 13, 16, 17}.

A vertex of degree $m$ is called an $m$-vertex, and a 1-vertex is called an end vertex. The neighbour of a 1-vertex is called a base, and a base of degree $m$ is called an $m$-base. A neighbour of $v$ with degree $k$ is called a $k$-neighbour of $v$. A double broom is a tree obtained from a path by appending leaves at both ends of the path. A strong double broom, denoted by $B$, is the graph on at least 5 vertices obtained from a union of (at least two) internally vertex disjoint $(u,v)$-paths by appending leaves at $u$ and $v$. More precisely, $B(n_1, n_2, m_1 P_{k_1}, m_2 P_{k_2}, \ldots, m_t P_{k_t})$ denotes the strong double broom with $n_1$ leaves at one end $u$ and $n_2$ leaves at the other end $v$ and there are $m_i$ internally vertex disjoint $(u,v)$-paths on $k_i$ vertices for $1 \leq i \leq t$, $m_i \geq 0$ and $k_1 < k_2 < \ldots < k_t$. The vertices $u$ and $v$ are called the hub vertices and the 2-vertices are called the middle vertices. It is clear that $m_1 = 1$ when $k_1 = 2$.

Recently Ma et al. \cite{13} have determined \( \text{adern} \) of double brooms. In this paper, we determine \( \text{dern} \) and \( \text{adern} \) of strong double brooms. We show that \( \text{dern}(B(n_1, n_2, m_1 P_{k_1}, m_2 P_{k_2}, \ldots, m_t P_{k_t})) \) is 1 or 2 and that \( \text{adern}(B(n, n, mP_{k})) \) is 3 in most of the cases. For all the exceptional cases, usually \( \text{adern}(B(n, n, mP_{k})) \) is 1 or 2, except \( \text{adern}(B(1, 1, 2P_{k})) = 5 \) for $k \geq 4$.

### 2 Dern of Strong Double Brooms

The da-ecards of $B$ are classified into three types: a leaf da-ecard $L$, a middle da-ecard $M$ and a hub da-ecard $K$ are obtained, respectively, by deleting an edge incident to a leaf vertex and a hub vertex, an edge of degree sum 2 and an edge incident to a hub vertex and a 2-vertex (Figure 1).

![Figure 1. Strong Double Broom](image)

An extension of a da-ecard $(d(e), G - e)$ of $G$ is a graph obtained from the da-ecard by adding a new edge joining two non adjacent vertices whose degree sum is $d(e)$ and it is denoted by $H(d(e), G - e)$.
(or simply $H$). Throughout this paper, $H$ and $e$ are used in the sense of this definition. In the proof of every theorem, $G$ denotes the strong double broom considered in that theorem.

**Theorem 1.** \(\text{dern}(B(n, n, mP_k)) = \begin{cases} 1 & \text{if } n + m \geq 6 \text{ or } 'n + m < 6, (n, m) \neq (1, 2), (1, 3) \text{ and } k = 3' \\ 2 & \text{otherwise} \end{cases} \)

**Proof.** For $k = 2$, $m$ would be 1 (as $G$ is simple) which is excluded in the definition itself. Therefore $k > 2$.

Case 1. $n + m \geq 6$.

In any leaf da-ecard $(n + m - 1, L)$, exactly two vertices have degree sum $n + m - 1$ and hence $G$ can be obtained uniquely from the da-ecard $(n + m - 1, L)$ by adding an edge joining the unique isolated vertex and the unique $(n + m - 1)$-vertex and thus \(\text{dern}(G) = 1\) in this case.

Case 2. $n + m < 6$, $(n, m) \neq (1, 2), (1, 3)$ and $k = 3$.

In any hub da-ecard $(n + m, K)$, the degrees of the vertices are 1, 2, $n + m - 1$ and $n + m$; in fact only one vertex, say $v$ has degree $n + m - 1$. Hence all simple graphs obtained by adding an edge joining $v$ and any end vertex are isomorphic and they are $G$. Therefore \(\text{dern}(G) = 1\).

Case 3. ‘$n + m < 6$ and $k \geq 4$’ or ‘$(n, m) = (1, 2), (1, 3)$ and $k = 3$’.

Now consider a leaf da-ecard $(n + m - 1, L)$. It clearly forces $G$ to have \(\binom{m}{2}\) cycles of length $2k - 2$.

For $k \geq 4$, consider the middle da-ecard $(2, M)$ in addition with $L$. Then all extensions obtained from $M$ by adding a new edge joining two 1-vertices at distance $2k - 3$ are isomorphic and they are $G$. For $k = 3$, consider the hub da-ecard $(n + m, K)$ in addition with $L$. Now all extensions obtained from $K$ by adding a new edge joining the unique $(n + m - 1)$-base and any one of the two 1-vertices at distance 3 are isomorphic and they are $G$. Therefore \(\text{dern}(G) = 2\).

**Theorem 2.** For $n_1 < n_2$,

\[
\text{dern}(B(n_1, n_2, mP_k)) = \begin{cases} 2 & \text{if } 'n_1 + m \leq 5, n_2 - n_1 = 2 \text{ or } 3 \text{ and } k > 3' \\ 1 & \text{otherwise} \end{cases} \]

\[
\text{or } 'n_1 + m \leq 5, n_2 - n_1 = 1, n_2 + m \neq 6 \text{ and } k > 3' \]

**Proof.** For $k = 3$, any hub da-ecard $(n_2 + m, K)$ uniquely determines $G$. So consider $k > 3$.

Case 1. $n_1 + m \leq 5$.

Case 1.1. ‘$n_2 - n_1 = 2$ or 3’ or ‘$n_2 - n_1 = 1$ and $n_2 + m \neq 6$’.

Any leaf da-ecard forces $G$ to have \(\binom{m}{2}\) cycles of length $2k - 2$. Hence in any middle da-ecard, the newly added edge must be incident to two 1-vertices at distance $2k - 3$ and the resulting graph thus obtained is isomorphic to $G$ and hence \(\text{dern}(G) = 2\).

Case 1.2. ‘$n_2 - n_1 \geq 4$’ or ‘$n_2 - n_1 = 1$ and $n_2 + m = 6$’.

Any leaf da-ecard $(n_2 + m - 1, L)$ uniquely determines $G$. 

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Case 2. \( n_1 + m \geq 6 \).

Here any leaf da-ecard \( (n_1 + m - 1, L) \) will determine \( G \) uniquely.

**Theorem 3.** \( \text{dern}(B(n,n,m_1P_{k_1},m_2P_{k_2},\ldots,m_tP_{k_t})) = \begin{cases} 1 & \text{if } n + m \geq 6 \text{ or } n + m = 5, \; k_1 = 3 \\ n + m = 4, \; k_1 = k_2 - 1 = 3 \text{ and } n = 2 \\ 2 & \text{otherwise} \end{cases} \)

**Proof.** In view of Theorem 1 we can assume that at least two \( m_i \)'s are nonzero. For \( n + m \geq 6 \), any leaf da-ecard uniquely determines \( G \) and hence \( \text{dern}(G) = 1 \). If either ' \( n + m = 5 \) and \( k_1 = 3 \)' or ' \( n + m = 4, \; k_1 = k_2 - 1 = 3 \) and \( n = 2 \)', then any hub da-ecard, obtained by removing a hub edge lying on a path of length 3, uniquely determines \( G \) and so \( \text{dern}(G) = 1 \).

For all the remaining cases, we use a leaf da-ecard and a middle da-ecard (obtained by removing an edge lying on a path of length \( k_i \)) to determine \( G \). The leaf da-ecard forces \( G \) to have \( \binom{m_1}{2} \) cycles of length \( 2(k_1 - 1) \) (if \( k_1 > 2 \) ), \( \binom{m_2}{2} \) cycles of length \( 2(k_2 - 1) \) (if \( k_2 > 2 \) ), \( \ldots, \binom{m_t}{2} \) cycles of length \( 2(k_t - 1) \) (if \( k_t > 2 \) ) and to have \( m_1m_2 \) cycles of length \( k_1 + k_2 - 2 \), \( m_1m_3 \) cycles of length \( k_1 + k_3 - 2 \), \( \ldots, m_1m_t \) cycles of length \( k_1 + k_t - 2 \), \( m_2m_3 \) cycles of length \( k_2 + k_3 - 2 \), \( \ldots, m_2m_t \) cycles of length \( k_2 + k_t - 2 \), \( \ldots, m_{t-1}m_t \) cycles of length \( k_{t-1} + k_t - 2 \). Hence in the middle da-ecard, the newly added edge must be incident to two 1-vertices at distance \( k_1 + k_t - 3 \) and hence \( \text{dern}(G) = 2 \).

**Theorem 4.**

\[
\text{dern}(B(n_1,n_2,P_2,m_2P_{k_2},\ldots,m_tP_{k_t})) = \begin{cases} 2 & \text{if } \sum_{i=2}^{t} m_i = 1, \; k_j > 3 \; (2 \leq j \leq t), \; n_1 = 1, \; n_2 \leq 4 \\ 1 & \text{otherwise} \end{cases}
\]

**Proof.** Assume \( \sum_{i=2}^{t} m_i \) is 1 as otherwise the da-ecard \( (n_1 + n_2 + 2 \sum_{i=2}^{t} m_i, K) \) uniquely determines \( G \).

**Case 1.** \( k_j > 3, \; n_1 = 1 \) and \( n_2 \leq 4 \).

Then \( G = B(n_1, n_2, P_2, P_{k_j}) \). Consider a leaf da-ecard \( (n_2 + \sum_{i=2}^{t} m_i, L) \) and a hub da-ecard \( (n_1 + n_2 + 2 \sum_{i=2}^{t} m_i, K) \). The leaf da-ecard forces \( G \) to have a cycle of length \( k_j \). Hence in the hub da-ecard \( (n_2 + 3, K) \), the newly added edge must be incident to the two bases and thus \( \text{dern}(G) = 2 \).

**Case 2.** \( k_j > 3, \; n_1 = 1 \) and \( n_2 \geq 5 \).

The da-ecard \( (n_2 + \sum_{i=2}^{t} m_i, L) \) uniquely determines \( G \).

**Case 3.** \( k_j > 3 \) and \( n_1 \geq 2 \).

Here the da-ecard \( (n_1 + n_2 + 2 \sum_{i=2}^{t} m_i, K) \) will determine \( G \) uniquely.

**Case 4.** \( k_j = 3 \).

Here the da-ecard \( (n_1 + n_2 + 2 \sum_{i=2}^{t} m_i, K) \) will do so.

**Theorem 5.** For \( k_i > 2 \; (1 \leq i \leq t) \), \( \text{dern}(B(n_1,n_2,m_1P_{k_1},m_2P_{k_2},\ldots,m_tP_{k_t})) = 1 \) or 2.
Proof. In view of Theorems 1, 2 and 3 we assume that \( n_1 \neq n_2 \) and at least two \( m_i \)'s are nonzero. Let \( m = m_1 + m_2 + \ldots + m_t \).

**Case 1.** \( n_1 + m \geq 6 \) or \( n_1 + m \leq 5 \) and \( n_2 - n_1 \geq 4 \) or \( n_1 + m = 5 \) and \( n_2 - n_1 = 1 \).

The da-ecard \((n_2 + m - 1, L)\) uniquely determines \( G \).

**Case 2.** \( n_1 + m = 3 \) and either \( n_2 = k_2 - 2 = 2 \) or \( n_2 = k_1 + 1 = 4 \) or \( n_1 + m = 4, n_2 - n_1 < 4, (n_1, n_2) \neq (2, 4) \) and \( k_1 = 3 \).

Here the da-ecard \((n_2 + m, K)\) will do so.

**Case 3.** \( n_1 + m = 4 \), \((n_1, n_2) = (2, 4) \) and \( k_1 = 3 \).

The da-ecard \((n_1 + m, K)\) will do the job here.

For all the remaining cases, a leaf and a middle da-ecards determine \( G \) uniquely (as in Theorem 3). □

**Theorem 6.** The degree associated edge reconstruction number of strong double brooms is 1 or 2.

**Proof.** Follows by Theorems 4 and 5. □

### 3 Adern of Strong Double Brooms

For \( 2 \leq i \leq \lceil \frac{k}{2} \rceil \), we denote the da-ecard obtained from \( B(n, n, mP_k) \), by deleting an edge on \( mP_k \) whose ends are at distance respectively \( i - 1 \) and \( i \) from a hub vertex, by \( M'_i \).

In all graphs shown in this section, the dashed edge in extensions denotes the edge whose removal results in a de-ecard in common with \( G \).

**Theorem 7.**

\[
\text{adern}(B(n, n, 2P_k)) = \begin{cases} 5 & \text{if } n = 1 \text{ and } k \geq 4 \\ 2 & \text{if } 'n = 2, 3 \text{ and } k = 3' \text{ or } 'n \geq 3 \text{ and } k = 4' \\ 1 & \text{if } n \geq 4 \text{ and } k = 3 \\ 3 & \text{otherwise} \end{cases}
\]

**Proof.** Let \( D \) denote any da-ecard of \( G \).

**Lower bound:** For \( n = 1 \) and \( k \geq 4 \), the graph \( H_1 \) (Figure 2), obtained from \( K \) by adding an edge \( e_1 \) joining the 1-vertex at distance \( k \) from the 3-base and the 2-vertex at distance \( 2k - 3 \) from that 1-vertex, shares two leaf da-ecards and two hub da-ecards with \( G \). Hence \( \text{adern}(G) \geq 5 \).

For \( k = 3 \), the graph obtained from \( L \) by adding an edge joining a 1-vertex and a 2-vertex (when \( n = 2 \)) or two 2-vertices (when \( n = 3 \)), shares a leaf da-ecard with \( G \). For \( n = 1 \) and \( k = 3 \), the graph, obtained from \( L \) by adding an edge joining an isolated vertex and a 2-neighbour of the 3-base, has two leaf da-ecards in common with \( G \), which gives the desired lower bound.

For \( n = 2 \) and \( k \geq 4 \), consider the graph \( H_2 \) (Figure 2) obtained from \( L \) by adding an edge \( e_2 \)
joining the 1-neighbour of the 3-base and a 2-vertex at distance two from that 3-base. Clearly $H_2$ shares two leaf da-ecards with $G$. Hence $adern(G) \geq 2$.

Figure 2. The extensions

*Upper bound:* We proceed by thirteen cases and prove that the collection of da-ecards considered under each case determines $G$ uniquely.

We first give a table of outcome proof for the sake of readability.

| $k$ | $n$ | Cases | $adern(G)$ |
|-----|-----|-------|-----------|
| 3   | 1   | 3     | 3         |
|     | 2,3 | 2, 3  | 2         |
|     | $\geq 4$ | 1, 2 | 1         |
| 4   | 1   | 4, 5, 1, 6, 7, 10, 12 | 5 |
|     | 2   | 4, 5, 1, 9, 10, 13 | 3 |
|     | $\geq 3$ | 1, 4, 5, 1, 8, 10, 13 | 2 |
| 5   | 1   | 4, 6, 7, 12 | 5 |
|     | 2   | 4, 5, 2, 1, 9, 13 | 3 |
|     | $\geq 3$ | 1, 4, 5, 2, 8, 13 | 3 |
| $\geq 6$ | 1 | 4, 5, 2, 6, 7, 11, 12 | 5 |
|     | 2   | 4, 5, 2, 9, 11, 13 | 3 |
|     | $\geq 3$ | 1, 4, 5, 2, 8, 13 | 3 |

Let $S$ consist of the specified number of da-ecards from $G$.

*Case 1.* Any $L$ for $n \geq 4$.

In $L$, exactly two vertices have degree sum $n + 1$ and so $G$ can be determined uniquely.

*Case 2.* Any $K$ for $n \geq 2$ and $k = 3$.

In $K$, the ends of the newly added edge must be a 1-vertex and an $(n + 1)$-vertex. Since all 1-neighbours of the $(n + 2)$-base are similar and there is a unique $(n + 1)$-vertex, all extensions are isomorphic and they are $G$.

*Case 3.* For $k = 3$, $adern(G) = 2$ when $n = 2, 3$ and $adern(G) = 3$ when $n = 1$.

For $n = 2, 3$, the da-ecard $K$ uniquely determines $G$ by Case 2. Further, the da-ecard $L$ forces every extension other than $G$ to have exactly one da-ecard isomorphic to $L$ and no more da-ecards of $G$. Hence $G$ can be uniquely determined by $L$ along with one more da-ecard and $adern(G) = 2$.

For $n = 1$, any $L$ and $K$ together determine $G$ as in Case 3 of Theorem 1. Now we shall
show that any three isomorphic da-ecards together determine $G$. In $K$, the newly added edge must be incident to a 1-vertex and a 2-vertex. The extension (other than $G$), obtained by adding an edge joining a 1-neighbour of the unique 3-base to a 2-vertex, has exactly one hub da-ecard and the extension, obtained by joining the 1-neighbour of the 2-base to a 2-vertex, has exactly two da-ecards isomorphic to $K$. Every extension non isomorphic to $G$ of $L$ has exactly two da-ecards isomorphic to $L$.

Case 4. For $k \geq 4$, $L$ and $M$ (or $K$) when $n = 2$, 3, and $L$ and $M$ when $n = 1$.

For $n = 2, 3$, the da-ecard $M$, or $K$ forces $G$ to be connected and hence in $L$, the only possibility to join $e$ is joining the unique isolated vertex and the unique $(n+1)$-vertex.

For $n = 1$, the da-ecard $L$ forces $G$ to have a cycle of length $2(k-1)$ and hence in $M$, the only possibility to join $e$ is joining the two 1-vertices at distance $2k-3$.

Case 5. $\{K$ and $M_i, (2 \leq i \leq \lfloor \frac{k}{2} \rfloor)\}$ when $k > 4$, $n \geq 2$ or $k = 4$ and $\{M_i$ and $M_j, (2 \leq i < j \leq \lfloor \frac{k}{2} \rfloor)\}$ when $k > 5$.

Case 5.1. $k = 4$.

The da-ecard $K$ forces $G$ to have every base with at most $n$ neighbours of degree 1 and hence in $M_i$, the new edge $e$ must be joined two 1-neighbours of the bases with $n+1$ neighbours of degree 1.

Case 5.2. $k > 4$.

Case 5.2.1. $K$ and $M_i$ when $n \geq 2$.

The da-ecard $M_i$ forces $G$ to have two $(n+2)$-vertices and hence in $K$, the new edge $e$ must be joined the unique $(n+1)$-vertex and some 1-vertex. The extension other than $G$, obtained from $K$ by joining $e$ to a 1-neighbour of the $(n+2)$-base, has exactly one da-ecard isomorphic to $K$ (since the removal of any edge other than $e$ results in a disconnected da-ecard or a da-ecard having two bases of degree at least 3 at distance 2) and has no middle da-ecards (since the removal of any edge results in a disconnected da-ecard or a da-ecard with a 1-vertex at distance $k-2$ from the nearest $(n+2)$-base).

Case 5.2.2. $M_i$ and $M_j$ ($i \neq j$).

The da-ecards $M_i$ and $M_j$ have 1-vertices at distance $l_1 > 1$ and $l_2 \geq 1$, respectively from the nearest $(n+2)$-base such that $l_1 > l_2$ (say). Every extension other than $G$, of $M_i$ has no more middle da-ecards, since the removal of any 2-edge results in a disconnected da-ecard or a 1-vertex at distance $l_1$ or $k-l_1-2$ from the nearest $(n+2)$-vertex.

Case 6. $\alpha L$, $\beta K$, $\alpha, \beta \geq 1$, $\alpha + \beta = 4$ and $D$ when $n = 1$ and $k \geq 4$.

The da-ecard $L$ forces $G$ to have a cycle of length $2(k-1)$ and hence the only extension of $K$ non isomorphic to $G$ is obtained by adding an edge joining the 1-vertex at distance $k$ from the unique 3-base and to a 2-vertex at distance $2k-3$. This extension has exactly two da-ecards isomorphic to $L$ (obtained by removing each pendant edge) and exactly two da-ecards isomorphic to $K$ (obtained
by removing the 3-edge incident to base) and has no more da-ecards of $G$.

Case 7. $K$, $M_i$, and $D$ when $n = 1$ and $k > 4$.

The da-ecard $M_i$ has a 1-vertex at distance $l > 1$ from the nearest $(n + m)$-base. The extension other than $G$ obtained from $M_i$ by joining $e$ with a 1-vertex at distance $l$ or $k - l - 2$ from the nearest 3-base and a 1-neighbour of the other 3-base, then the resulting graph has exactly one da-ecard $M_i$ (obtained by removing $e$) and exactly one da-ecard $K$ (obtained by removing the 3-edge (non adjacent to $e$) lying on the cycle which is incident to a 3-vertex whose neighbours are 2-vertices). The above extensions have no more hub da-ecards (since the removal of any 3-edge results in a disconnected da-ecard or a da-ecard with no 3-base or a da-ecard having no 1-vertex at distance $k - 2$ from the nearest 3-base), no da-ecards isomorphic to $L$ (since the removal of any 2-edge results in a da-ecard with no cycle of length $2(k - 1)$) and no more middle da-ecards (since the removal of any other edge results in a da-ecard with an isolated vertex). For all other extensions, the resulting graph has exactly one da-ecard isomorphic to $M_i$ and has no more da-ecards of $G$.

Case 8. $2L$ determine $G$ when $k \geq 4$ and $n = 3$.

Every extension (other than $G$) of $L$ has exactly one da-ecard isomorphic to $L$ (obtained by removing $e$) as the removal of any other 4-edge results in a da-ecard with a cycle of length less than $2(k - 1)$.

Case 9. $2L$ and $D$ when $k \geq 4$ and $n = 2$.

Every extension (other than $G$) of $L$ obtained by joining $e$ with the 1-neighbour of the unique 3-base and with a 2-vertex at distance two from the 3-base has exactly two da-ecards $L$ (obtained by removing the edges, lying on the cycle $C_4$, incident to the 3-neighbour of the base) as the removal of any other 3-edge results in a da-ecard with a cycle $C_4$ or two bases at distance less than $k - 1$ and no da-ecards $K$ and $M$ (since the removal of any other edge results in a da-ecard with an isolated vertex). For any other extension, the resulting da-ecard has no leaf da-ecards (since the removal results in a da-ecard having a cycle of length less than $2(k - 1)$) and has no da-ecards $K$ and $M_i$ (since the removal of any other edge results in a da-ecard with an isolated vertex).

Case 10. $2M_i$ when $k = 4$.

The extension non isomorphic to $G$ of $M_i$ has exactly one middle da-ecard (obtained by removing $e$), since the removal of any other 2-edge results in a disconnected da-ecard.

Case 11. $2M_i$ and $D$ when $k > 5$.

Let $M_i$ have a 1-vertex at distance $l > 1$ from the nearest $(n + 2)$-base. Every extension of $M_i$ other than $G$, obtained by joining $e$ with a 1-vertex at distance $l$ or $k - l - 2 (> 1)$ from the nearest $(n + 2)$-vertex and a 1-neighbour of the same $(n + 2)$-base, has exactly two middle da-ecards and the remaining extensions other than $G$ has exactly one middle da-ecard as the removal of any other edge
results in a da-ecard with no 1-vertex at distance \( k - l - 2 \) (> 1) or \( l \) from the nearest \((n + 2)\)-vertex or \((n+3)\)-vertex or two 1-vertices at distance \( k - l - 2 \) (> 1) or \( l \) from the same \((n+2)\)-vertex. None of the extensions of \( M_i \) other than \( G \) have a leaf da-ecard (since the removal results in a da-ecard having a cycle of length less than \( 2(k-1) \)) and hub da-ecard (since the removal of any \((n+2)\)-vertex results in a disconnected da-ecard or a da-ecard with a 1-vertex at distance \( k - l - 2 \) (> 1) or \( l \) from the nearest \((n+2)\)-vertex or \((n+3)\)-vertex.

Case 12. \( 3K \) when \( n = 1 \) and \( k \geq 4 \).

Every extension of \( K \) other than \( G \) must be obtained by joining \( e \) with a 1-vertex and a 2-vertex. Clearly every extension has at most two hub da-ecards as the resulting extension has exactly two 3-edges or the removal of any 3-edge other than \( e \) results in a disconnected da-ecard or a da-ecard having a cycle or a base with two neighbours of degree 1 or two 3-bases or no 3-base.

Case 13. \( 2K \) when \( n \neq 1 \) and \( k \geq 4 \).

Every extension of \( K \) other than \( G \) obtained by joining \( e \) with two 2-vertices (when \( n = 2 \)) or a 1-vertex and an \((n+1)\)-vertex. Clearly every extension has exactly one hub da-ecard, since the resulting extension has exactly one \((n+2)\)-edge or the removal of any \((n+2)\)-edge other than \( e \) results in a disconnected da-ecard or a da-ecard having a cycle or a base with two neighbours of degree 1 or two 3-bases or no 3-base, which completes the proof of the theorem.

Theorem 8. For \( m \geq 3 \),

\[
\text{adern}(B(n, n, mP_k)) = \begin{cases} 
3 & \text{if '} k \geq 5 \text{' or '} m = 3, k = 4 \text{ and } n = 1 \text{' } \\
2 & \text{if '} m = k = n + 1 = 3 \text{' or '} m = k + 1 = n + 3 = 4 \text{' or '} m = k = n + 2 = 3 \text{' } \\
& \text{'} m \geq k = 4 \text{' or '} m = k - 1 = 3 \text{ and } n \geq 2 \text{' } \\
1 & \text{otherwise} 
\end{cases}
\]

Proof. Lower bound: For \( k \geq 5 \), the graph \( H_1 \) (Figure 3), obtained from \( M_i \) by joining a new edge \( e_1 \) between two 1-vertices at distance 3, shares two middle da-ecards with \( G \) and hence \( \text{adern}(G) \geq 3 \).

For \( m = 3 \), \( k = 4 \) and \( n = 1 \), the graph \( H_2 \) (Figure 3), obtained from \( L \) by joining a new edge \( e_2 \) between the 1-vertex and a 2-vertex at distance 3, shares two leaf da-ecards with \( G \). Hence \( \text{adern}(G) \geq 3 \).

For \( ' m \geq k = 4 \text{' or '} m = k - 1 = 3 \text{ and } n \geq 2 \text{' } \), the graph, obtained from \( K \) by joining a new edge between two 1-vertices at distance 2, has a hub da-ecard in common with \( G \) and so \( \text{adern}(G) \geq 2 \).

The graph, obtained from \( L \) by joining a new edge between two 2-vertices for \( m = k + 1 = n + 3 = 4 \) or \( m = k = n + 1 = 3 \) and joining the 1-vertex and a 2-vertex for \( m = k = n + 2 = 3 \), has a leaf da-ecard in common with \( G \), which gives the desired lower bound.
Figure 3. The extensions $H_1$ and $H_2$

**Upper bound:** We proceed by ten cases and prove that the collection of da-ecards considered under each case determines $G$ uniquely. We first give a table of outcome proof for the sake of readability.

| $k$ | $m$ | $n$ | Cases | $\text{ederu}(G)$ |
|-----|-----|-----|-------|-------------------|
| 3   | 1   | 3, 8| 2     |                   |
|     | 2   | 2, 6| 2     |                   |
|     | $\geq 3$ | 1, 2 | 1   |                   |
| 4   | 1   | 2, 6| 2     |                   |
|     | $\geq 2$ | 1, 2 | 1   |                   |
|     | $\geq 5$ | $\geq 1$ | 1   |                   |
| 3   | 1   | 3, 4, 7, 8, 9 | 3 |                   |
|     | 2   | 3, 4, 6, 8, 9 | 2 |                   |
|     | $\geq 3$ | 1, 4, 8, 9 | 2 |                   |
| 4   | 1   | 3, 4, 6, 8, 9 | 2 |                   |
|     | $\geq 2$ | 1, 4, 8, 9 | 2 |                   |
|     | $\geq 5$ | $\geq 1$ | 1, 4, 8, 9 | 2 |
| $\geq 5$ | 1 | 3, 4, 5 (if $k > 5$), 7, 8, 10 | 3 |                   |
|     | 2   | 3, 4, 5 (if $k > 5$), 6, 8, 10 | 3 |                   |
|     | $\geq 3$ | 1, 4, 5 (if $k > 5$), 8, 10 | 3 |                   |
| 4   | 1   | 3, 4, 5 (if $k > 5$), 6, 8, 10 | 3 |                   |
|     | $\geq 2$ | 1, 4, 5 (if $k > 5$), 8, 10 | 3 |                   |
|     | $\geq 5$ | $\geq 1$ | 1, 4, 5 (if $k > 5$), 8, 10 | 3 |

**Case 1.** Any $L$ for $n + m \geq 6$.

In $L$, exactly two vertices have degree sum $n + m - 1$ and so $G$ can be determined uniquely.

**Case 2.** Any $K$ for $n + m \geq 5$ and $k = 3$.

In $K$, the neighbour of the newly added edge must be a 1-vertex and an $(n + m - 1)$-vertex. Since all 1-neighbours of the $(n + m)$-base are similar and there is a unique $(n + m - 1)$-vertex, every extension obtained is unique and is isomorphic to $G$.

**Case 3.** ‘$L$ and $K$ (or $M_i$) when $n + m < 6$ and $k > 3$’ and ‘$L$ and $K$ when $n + m < 5$ and $k = 3$’.

Proof is similar to Case 3 of Theorem 1.

**Case 4.** $M_i$ and $K$ for $k > 3$.

The da-ecard $M_i$ forces $G$ to have two $(n + m)$-vertices. Hence in $K$, $e$ must be joined to the
\[(n + m - 1)\text{-vertex and some 1-vertex. The only extension non isomorphic to } G, \text{ obtained by joining} \]
\[e \text{ to a 1-neighbour of the } (n + m)\text{-base, has exactly one da-ecard isomorphic to } K \text{ (obtained by} \]
\[\text{removing } e \text{), no more da-ecards isomorphic to } K \text{ (since the removal of any } (n + m)\text{-edge results in a} \]
\[\text{da-ecard with cycle of length less than } 2(k - 1) \text{ or has at most } \binom{m - 1}{2} - 1 \text{ cycles of length } 2(k - 1) \],
\[\text{no middle da-ecards (since the removal of any 2-edge results in a disconnected da-ecard or a da-ecard} \]
\[\text{with 1-vertex at distance } k - 2 \text{ from the nearest } (n + m)\text{-base) and no leaf da-ecards as the removal} \]
\[\text{of any edge results in a da-ecards with at most } \binom{m - 1}{2} \text{ cycles of length } 2(k - 1). \]

**Case 5.** \(M_i\) and \(M_j\) for \(k > 5\).

Proof is similar to Case 5.2.2 of Theorem 7.

**Case 6.** Two isomorphic \(L\)'s when \('m = 3\) and \(n = 2\)' or \('m = 4\) and \(n = 1\)'.

Proof is similar to Case 8 of Theorem 7.

**Case 7.** For \(m = 3, n = 1\) and \(k > 3\), three isomorphic \(L\)'s.

The extension (other than \(G\)) of \(L\), obtained by adding an edge joining the 1-neighbour of the \((n+m)\)-base and a 2-vertex at distance 2 from the \((n+m)\)-base, has exactly two da-ecards isomorphic to \(L\) (obtained by removing the edges incident to 2-neighbour of the \((n+m)\)-vertex). The above extension and any other extensions (non isomorphic to \(G\)) have no more leaf da-ecards, since the removal of any 3-edge results in a da-ecard with cycle of length less than \(2(k - 1)\).

**Case 8.** Two isomorphic \(K\)'s when \(k > 3\) or \(n + m < 5\) and \(k = 3\).

Every extension other than \(G\) of \(K\), has exactly one hub da-ecard, since the removal of any \((n+m)\)-edge other than \(e\) results in a da-ecard with cycle of length less than \(2(k - 1)\) or at most \(\binom{m - 1}{2}\) cycles of length \(2(k - 1)\).

**Case 9.** Two isomorphic \(M_i\)'s for \(k = 4\).

Every extension other than \(G\) of \(M_i\), has exactly one middle da-ecard, since the removal of any 2-edge other than \(e\) results in a da-ecard containing a cycle of length 3.

**Case 10.** For \(k \geq 5\), three isomorphic \(M_i\)'s.

Proof is similar to Case 11 of Theorem 7. This case completes the proof of the theorem.

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