OPERATOR MAPS OF JENSEN-TYPE

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Abstract. Let $\mathcal{B}_J(\mathcal{H})$ denote the set of self-adjoint operators acting on a Hilbert space $\mathcal{H}$ with spectra contained in an open interval $J$. A map $\Phi: \mathcal{B}_J(\mathcal{H}) \to \mathcal{B}(\mathcal{H})_{sa}$ is said to be of Jensen-type if
\[
\Phi(C^* AC + D^* BD) \leq C^* \Phi(A) C + D^* \Phi(B) D
\]
for all $A, B \in B_J(\mathcal{H})$ and bounded linear operators $C, D$ acting on $\mathcal{H}$ with $C^* C + D^* D = I$, where $I$ denotes the identity operator.

We show that a Jensen-type map on a infinite dimensional Hilbert space is of the form $\Phi(A) = f(A)$ for some operator convex function $f$ defined in $J$.

1. Introduction

We recall that a function $f: J \to \mathbb{R}$ defined in a real interval $J$ is said to be $n$-convex if the inequality
\[
f(\lambda A + (1 - \lambda) B) \leq \lambda f(A) + (1 - \lambda) f(B)
\]
holds for all $\lambda \in [0, 1]$ and operators $A, B \in \mathcal{B}_J(\mathcal{H})$, when $\dim \mathcal{H} = n$. More generally, we call $f$ operator convex if the inequality (1) holds for all natural numbers $n$. It is known that the inequality in this case also holds for operators on an infinite dimensional Hilbert space. Hansen and Pedersen \cite{HansenPedersen1, HansenPedersen2} obtained the following characterization of operator convexity.

Theorem 1.1. Let $f: J \to \mathbb{R}$ be a continuous function defined in an interval $J$, and let $\mathcal{H}$ be an infinite dimensional Hilbert space. The following conditions are equivalent:

(i) $f$ is operator convex.

(ii) For each natural number $k$ the inequality
\[
f \left( \sum_{i=1}^{k} C_i^* A_i C_i \right) \leq \sum_{i=1}^{k} C_i^* f(A_i) C_i
\]

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holds for all $A_1, \ldots, A_k \in B_J(\mathcal{H})$ and arbitrary operators $C_1, \ldots, C_k$ on $\mathcal{H}$ with $C_1^*C_1 + \cdots + C_k^*C_k = I$.

(iii) For each natural number $k$ the inequality
\[
    f \left( \sum_{i=1}^{k} P_i A_i P_i \right) \leq \sum_{i=1}^{k} P_i f(A_i) P_i
\]
holds for all $A_1, \ldots, A_k \in B_J(\mathcal{H})$ and projections $P_1, \ldots, P_k$ on $\mathcal{H}$ with sum $P_1 + \cdots + P_k = I$.

One should note that the above result only holds on a Hilbert space of infinite dimensions. If $\mathcal{H}$ is of finite dimension $n$, then $f$ should be $2n$-convex for statement (ii) to hold with $k = 2$.

There are other equivalent conditions of operator convexity, cf. [3, 7]. We want to study operator maps $\Phi$ given on the form $\Phi(A) = f(A)$ in a more abstract setting, where $f(A)$ is defined by the functional calculus, and determine which general properties of $\Phi$ that entail this particular form. A related problem is to place maps of the said form, $\Phi(A) = f(A)$, in the context of other more general types of operator maps. To this end we first introduce the notion of a Jensen-type map.

**Definition 1.1.** Let $J$ be an open real interval, and let $\mathcal{H}$ be a Hilbert space. A (not necessarily linear) map $\Phi : B_J(\mathcal{H}) \to B(\mathcal{H})_{sa}$ is said to be of Jensen-type if
\[
    \Phi(C^* A C + D^* B D) \leq C^* \Phi(A) C + D^* \Phi(B) D
\]
for all $A, B \in B_J(\mathcal{H})$ and operators $C, D$ on $\mathcal{H}$ with $C^*C + D^*D = I$.

Note that a Jensen-type map is convex. It is also unitarily invariant. Indeed, by choosing $C$ as a unitary $U$ and setting $D = 0$, we obtain the inequality
\[
    \Phi(U^* A U) \leq U^* \Phi(A) U = U^* \Phi(UU^* AUU^*) U \leq \Phi(U^* A U),
\]
implying that $\Phi(U^* A U) = \Phi(U)$. We later realize that there exist unitarily invariant convex operator maps that are not of Jensen-type.

Robertson and Smith [8] showed that if $E$ is an operator system (i.e. a closed $*$-subspace of a unital $C^*$-algebra containing the identity), $B$ is a $C^*$-algebra and a linear map $\Psi : E \otimes \mathbb{M}_n \to B \otimes \mathbb{M}_n$ satisfies $\Psi(U^* X U) = U^* \Psi(X) U$ for all $X \in E \otimes \mathbb{M}_n$ and all unitaries $U \in \mathbb{M}_n$, then there exist $\phi, \lambda : E \to B$ such that
\[
    \Psi(X) = (\phi \otimes \text{id}_n)(X) + \lambda(\text{Tr} X) \otimes I_n
\]
for all $X \in E \otimes \mathbb{M}_n$, where $I_n$ is the identity in the $C^*$-algebra $\mathbb{M}_n$ of all complex $n \times n$ matrices. In addition, Bhat [1] proved that any bounded
unitarily invariant linear map $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is of the form
$$\alpha(X) = cX + d \operatorname{Tr} X \cdot I$$
for some $c, d \in \mathbb{C}$ if $\mathcal{H}$ is finite dimensional, and of the form
$$\alpha(X) = cX$$
for some $c \in \mathbb{C}$ if $\mathcal{H}$ is infinite dimensional.

2. UNITARILY INVARIANT CONVEX OPERATOR MAPS

Let $\Phi : B_J(\mathcal{H}) \to B(\mathcal{H})$ be a unitarily invariant (not necessarily linear) map.

**Lemma 2.1.** If operators $X \in B_J(\mathcal{H})$ and $Y \in B(\mathcal{H})$ commute, then so do $\Phi(X)$ and $Y$.

**Proof.** For any unitary operator $U$ commuting with $X$ we have
$$\Phi(X) = \Phi(U^* XU) = U^* \Phi(X) U.$$ Therefore, $U\Phi(X) = \Phi(X)U$ and $\Phi(X)$ is thus contained in the abelian double commutant $\{X\}''$. Since $Y \in \{X\}'$ it follows that $\Phi(X)$ and $Y$ commute. \qed

In particular, if $X = t \cdot I$ is a multiple of the identity operator for some $t \in J$, then we deduce that $\Phi(t \cdot I)$ commutes with every operator in $\mathcal{B}(\mathcal{H})$. It is therefore of the form
$$\Phi(t \cdot I) = f(t) \cdot I, \quad t \in J$$
for some function $f : J \to \mathbb{R}$. We realize that $f$ is convex if $\Phi$ is convex.

**Lemma 2.2.** Let $P_1, \ldots, P_k$ be projections with $P_1 + \cdots + P_k = I$ and put
$$U = \theta P_1 + \theta^2 P_2 + \cdots + \theta^{k-1} P_{k-1} + P_k,$$
where $\theta = \exp(2\pi i/k)$ is a $k$th root of unity. Then $U$ is unitary and
$$\sum_{j=1}^k P_j XP_j = \frac{1}{k} \sum_{j=1}^k U^{-j} XU^j$$
for any $X \in \mathcal{B}(\mathcal{H})$.

**Proof.** Take $r, s = 1, \ldots, k$. By computation we obtain
$$P_r \left( \sum_{j=1}^k U^{-j} XU^j \right) P_s = \sum_{j=1}^k \theta^{-jr} P_r X \theta^{js} P_s = P_r XP_s \sum_{j=1}^k \theta^{(s-r)}.$$
The sum
\[ \sum_{j=1}^{k} \theta_j s^{j(r-s)} = \sum_{j=1}^{k} \exp \left( j(s-r) \frac{2\pi i}{k} \right) = k \]
for \( s = r \). For \( s \neq r \) we set \( \omega = \exp \left( (s-r)2\pi i/k \right) \) and obtain
\[ \sum_{j=1}^{k} \theta_j s^{j(r-s)} = \sum_{j=1}^{k} \omega j \omega = \frac{\omega^{k+1} - \omega}{\omega - 1} = 0, \]
since \( \omega \neq 1 \) and \( \omega^k = 1 \). The assertion now follows. \( \square \)

The following result is well-known for spectral functions, but it holds under the weaker conditions of only unitary invariance and convexity.

**Proposition 2.1.** Let \( \mathcal{H} \) be a Hilbert space and \( \Phi: B_J(\mathcal{H}) \to B(\mathcal{H})_{sa} \) a unitarily invariant convex map. Then
\[ \Phi \left( \sum_{j=1}^{k} P_j XP_j \right) \leq \sum_{j=1}^{k} P_j \Phi(X) P_j \]
for positive integers \( k \), operators \( X \in B_J(\mathcal{H}) \), and projections \( P_1, \ldots, P_k \) on \( \mathcal{H} \) with \( P_1 + \cdots + P_k = I \).

**Proof.** By repeated application of Lemma 2.2 we obtain
\[ \Phi \left( \sum_{j=1}^{k} P_j XP_j \right) = \Phi \left( \frac{1}{k} \sum_{j=1}^{k} U^{-j} X U^j \right) \leq \frac{1}{k} \sum_{j=1}^{k} \Phi(U^{-j} X U^j) \]
\[ = \frac{1}{k} \sum_{j=1}^{k} U^{-j} \Phi(X) U^j = \sum_{j=1}^{k} P_j \Phi(X) P_j, \]
where we used the unitary invariance and convexity of \( \Phi \). \( \square \)

**Proposition 2.2.** Let \( \mathcal{H} \) be a Hilbert space, and let \( \Phi: B_J(\mathcal{H}) \to B(\mathcal{H})_{sa} \) be a unitarily invariant map. Then
\[ \Phi \left( \sum_{j=1}^{k} P_j XP_j \right) = \sum_{j=1}^{k} P_j \Phi \left( \sum_{i=1}^{k} P_i XP_i \right) \]
for \( X \in B_J(\mathcal{H})_{sa} \) and projections \( P_1, \ldots, P_k \) with \( P_1 + \cdots + P_k = I \).

**Proof.** The projections \( P_1, \ldots, P_k \) are necessarily mutually orthogonal and the sum
\[ \tilde{X} = \sum_{i=1}^{k} P_i XP_i \]
commutes with \( P_i \) for \( i = 1, \ldots, k \). It then follows by Lemma 2.1 that also \( Y = \Phi(\tilde{X}) \) commutes with every \( P_i \) and the assertion follows. \( \square \)
3. The structure of Jensen-type maps

Take an open real interval $J$, and let $\Phi: B_J(\mathcal{H}) \to B(\mathcal{H})_{sa}$ be a Jensen-type map. Since $\Phi$ is unitarily invariant we learned in (3) that $\Phi(t \cdot I) = f(t) \cdot I$, $t \in J$, for a function $f: J \to \mathbb{R}$. The convexity of $\Phi$ implies that $f$ is convex and thus continuous since $J$ is open.

**Lemma 3.1.** Let $\Phi: B_J(\mathcal{H}) \to B(\mathcal{H})_{sa}$ be a Jensen-type map. Then the following statements are true.

(i) Let $P$ be a projection on $\mathcal{H}$. The equality

$$P\Phi(tP + (I - P)Y(I - P))P = f(t)P, \quad t \in J$$

holds for any $Y \in B_J(\mathcal{H})$.

(ii) If $\lambda$ is an eigenvalue of an operator $X \in B_J(\mathcal{H})$ with corresponding eigenprojection $P$, then

$$P\Phi(X)P = P\Phi(\lambda P + (I - P)X(I - P))P = f(\lambda)P.$$

**Proof.** Since $\Phi$ is of Jensen-type we obtain

$$\Phi(tP + (I - P)Y(I - P)) \leq P\Phi(tP) + (I - P)\Phi(Y)(I - P) = f(t)P + (I - P)\Phi(Y)(I - P).$$

Furthermore,

$$f(t) = \Phi(P(tP + (I - P)Y(I - P))P + (I - P)t(I - P))$$

$$\leq P\Phi(tP + (I - P)Y(I - P))P + (I - P)\Phi(t)(I - P)$$

$$\leq P(f(t)P + (I - P)\Phi(Y)(I - P))P + f(t)(I - P)$$

$$= f(t)P + f(t)(I - P) = f(t).$$

Therefore we have the equality

$$f(t) = P\Phi(tP + (I - P)Y(I - P))P + f(t)(I - P)$$

and thus

$$P\Phi(tP + (I - P)Y(I - P))P = f(t)P$$

independent of $Y$, which proves (i). Statement (ii) follows from the spectral theorem and (i). \qed

**Theorem 3.1.** Let $J$ be an open real interval, and let $\mathcal{H}$ be a Hilbert space of finite dimension $n$. If $\Phi: B_J(\mathcal{H}) \to B(\mathcal{H})_{sa}$ is of Jensen-type, then

$$\Phi(A) = f(A) \quad A \in B_J(\mathcal{H}),$$

where $f$ is the function defined in (3). Furthermore, $f$ is $n$-convex.
Proof. Let $P_1, \ldots, P_k$ be the spectral projections of $X$. By the spectral theorem and Lemma 3.1 (ii) we obtain

$$\Phi(X) = \sum_{i=1}^{k} P_i \Phi(X) = \sum_{i=1}^{k} P_i \Phi(X) P_i = \sum_{i=1}^{k} f(\lambda_i) P_i = f(X),$$

where the second equality follows from Lemma 2.1. Since $\Phi$ is convex, it follows that $f$ is an $n$-convex function. \hfill \Box

Note that to obtain Theorem 3.1 we only used that $\Phi$ is unitarily invariant together with the inequality in (2) for projections $C = P$ and $D = I - P$. However, to conclude that a map of the form $\Phi(X) = f(X)$ is of Jensen-type, we need that $f$ is $2n$-convex, where $n$ is the dimension of the underlying Hilbert space.

Note also that even if when the underlying Hilbert space is infinite dimensional the proof of the preceding theorem implies that $\Phi(A) = f(A)$ for any finite rank operator $A \in B_J(\mathcal{H})$.

Lemma 3.2. Let $A, Y$ be self-adjoint operators on a Hilbert space with

$$\alpha < A \leq Y$$

for some constant $\alpha$. Then there exist operators $C$ and $D$ such that

$$A = C^* Y C + \alpha D^* D$$

and $C^* C + D^* D = I$.

Proof. Since $A - \alpha > 0$ we may set $C = (Y - \alpha)^{-1/2}(A - \alpha)^{1/2}$ and obtain

$$A - \alpha = C^* (Y - \alpha) C = C^* Y C - \alpha C^* C.$$

Since $C^* C \leq I$ we may put $D = (I - C^* C)^{1/2}$ and obtain

$$A = C^* Y C + \alpha D^* D$$

and $C^* C + D^* D = I$. \hfill \Box

Theorem 3.2. Let $\mathcal{H}$ be an infinite dimensional Hilbert space, and let $\Phi : B_J(\mathcal{H}) \to B(\mathcal{H})_{sa}$ be a Jensen-type map. Then

$$\Phi(A) = f(A) \quad A \in B_J(\mathcal{H}),$$

where $f$ is the function defined in (3). In addition, $f$ is operator convex.

Proof. Take $A \in B_J(\mathcal{H})$ and a constant $\alpha \in J$ with $\alpha < A$. We may determine an upper sum operator $Y_n = f_n(A)$ with spectrum in $J$ by choosing $f_n$ as an increasing step function defined on the convex hull of
the spectrum of $A$ corresponding to a subdivision with fineness $\varepsilon > 0$ such that $f_n(t) = t$ in the right hand side of each subinterval. Then

$$\alpha < A \leq Y_n$$

and $\|Y_n - A\| \leq \varepsilon$ such that $Y_n$ converges to $A$ in the norm topology as the fineness of the subdivision tends to zero. Furthermore, by Lemma 3.2 we obtain

$$A = C_n^* Y_n C_n + \alpha D_n^* D_n$$

for operators $C_n$ and $D_n$ with $C_n^* C_n + D_n^* D_n = I$, and thus

$$\Phi(A) \leq C_n^* \Phi(Y_n) C_n + D_n^* \Phi(\alpha) D_n = C_n^* f(Y_n) C_n + f(\alpha) D_n^* D_n,$$

where we first used that $\Phi$ is of Jensen-type, and then that $Y_n$ is a finite rank operator such that $\Phi(Y_n) = f(Y_n)$. Notice that $C_n$ by the spectral theorem converges to the identity operator in the norm topology, when the fineness of the subdivision tends to zero. In the limit we thus obtain

$$\Phi(A) \leq f(A).$$

We next choose a constant $\beta \in J$ such that

$$\beta < Z_n \leq A,$$

where in this case $Z_n = g_n(A)$ is an under sum operator of $A$ with spectrum in $J$ corresponding to a subdivision of $J$. We now obtain

$$Z_n = C_n^* A C_n + \beta D_n^* D_n$$

for operators $C_n$ and $D_n$ such that $C_n^* C_n + D_n^* D_n = 1$, and $C_n$ converges to the identity operator in the norm topology for $n$ tending to infinity. By using that $Y_n$ is finite rank and that $\Phi$ is of Jensen-type we obtain the inequality

$$f(Z_n) = \Phi(Z_n) \leq C_n^* \Phi(A) C_n + \Phi(\beta) D_n^* D_n$$

and thus in the limit $f(A) \leq \Phi(A)$. \qed

Note that a Jensen-type map automatically is strongly continuous by the preceding theorem.

**Remark 3.1.** If $\mathcal{H}$ is infinite dimensional and $\Phi: B_J(\mathcal{H}) \to B(\mathcal{H})_{sa}$ is unitarily invariant, we learned that the inequality

$$\Phi(P A P + (I - P) B (I - P)) \leq P \Phi(A) P + (I - P) \Phi(B) (I - P)$$

for all $A, B \in B_J(\mathcal{H})$ and projections $P$ on $\mathcal{H}$ is sufficient to conclude that $\Phi$ is of the form $\Phi(A) = f(A)$ for some operator convex function $f$, and it is therefore, by Theorem 1.1, of Jensen-type.

If $\Phi$ is just unitarily invariant and convex, then the more restricted inequality

$$\Phi(P A P + (I - P) A (I - P)) \leq P \Phi(A) P + (I - P) \Phi(A) (I - P)$$
holds for $A \in B_1(\mathcal{H})$ and projections $P$ on $\mathcal{H}$, cf. Proposition 2.7.

The difference between these two inequalities (the latter being more restricted than the former) elucidates the difference between the general class of unitarily invariant convex maps and the more restricted subset of Jensen-type maps.

The map $\Phi(X) = \text{Tr} X \cdot I$ is unitarily invariant and convex, but it is not of Jensen-type. To realize this, take $A = P$ and $B = 0$ in Definition 1.1.

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