Entropy fluctuations for directed polymers in 2+1 dimensions

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We find numerically that the sample to sample fluctuation of the entropy, $\Delta S$, is a tool more sensitive in distinguishing how from high temperature behaviors, than the corresponding fluctuation in the free energy. In 1+1 dimensions we find a single phase for all temperatures since $(\Delta S)^2$ is always extensive. In 2+1 dimensions we find a behavior may look at first sight as a transition from a low temperature phase where $(\Delta S)^2$ is extensive to a high temperature phase where it is subextensive. This is observed in spite of the relatively large system we use. The observed behavior is explained not as a phase transition but as a strong crossover behavior. We use an analytical agreement to obtain $(\Delta S)^2$ for high temperature and find that while it is always extensive it is also extremely small and the leading extensive part decays very fast to zero as temperature is increased.

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The problem of directed polymers has attracted much interest in recent years. It is relevant to many fields ranging from surface growth phenomena and spin glasses to flux lines in high-$T_c$ superconductors [1-4]. It is well known that the problem of directed polymers in a random medium is equivalent to the KPZ equation that describes surface growth [5-7]. Much is known about the KPZ system, in particular in 1+1 dimensions. The KPZ equation provides the exact dynamical exponent for directed polymers in 1+1 dimensions [5-9]. The situation for higher dimensions is more complex. Traditional approximation schemes like dynamical renormalization-group methods fail to produce the exponents obtained by simulations [1,2,10-15]. A self-consistent expansion of the correlation function introduced later [16] yields results compatible with simulations for 2+1 dimensions. Above 2+1 dimensions, the behavior of directed polymers has a phase transition when temperature is raised (in the directed polymer problem or the level of noise in the KPZ system) and the system goes over from a strong coupling to a perturbative weak coupling behavior [17-20]. All the field theoretical treatments agree that at 2+1 dimensions, namely at the lower critical dimension itself, no transition should occur and the strong coupling behavior exists at zero and any finite temperature [16,19-21]. Some authors have claimed in the past on the basis of numerical simulations, to have obtained a phase transition in 2+1 dimensional systems [22-28]. The system studied were, however, relatively small and all those claims were not pursued eventually. In this paper, we present numerical results obtained for much longer systems in 1+1 and 2+1 dimensions. We study the free energy fluctuations that is usually studied in literature but in addition we obtain numerically also the sample to sample fluctuations of the entropy. This is a quantity, that is less common in the literature and was introduced first, to the best of our knowledge, by Fisher and Huse [29]. The fluctuations in the entropy are more pronounced than those of the free energy and in fact, we expect it to be extensive in the size of the polymer in 1+1 and 2+1 dimensions. In higher dimensions we expect a transition from a low temperature extensive phase to a high temperature subextensive one. Namely defining the quantity $\lambda(t) = (\Delta S)^2/t$, its infinite volume limits, behaves like an order parameter. It is zero at high temperatures and is of order 1 at low temperatures. The numerical results concerning $\lambda(t)$ in 1+1 dimensions are those being expected. We find that $\lambda$ is of order 1 over all the temperature region. In 2+1 dimensions, the numerical results seem to indicate an unexpected transition from a low temperature phase in which $\lambda$ is of order 1 into a high temperature regime where it is approaching zero. Although the transition temperature seems to be defined very sharply and although $\lambda(t)$ can be fitted at high temperatures by $\lambda(t) \propto lnt/t$, we claim that it is just a crossover phenomenon and the reasons for it occurring in such a spectacular way will be discussed later.

Consider a directed polymer on a hyperpyramid lattice structure with the random energy assigned on each bond. The partition function $G(R,t)$ for directed polymers starting from $(0,0)$ and ending at $(R,t)$ is defined by $G(R,t) = \sum_{C} e^{-E_{C}/T}$ where $E_{C}$ is the sum of the energy on the path $C$ and $T$ is the temperature. For simplicity, we demonstrate our calculations using the transfer-matrix method for the case of 1+1 dimensions. A similar formalism has been used for 2+1 dimensions. The iteration relation for the partition function $G(R,t)$ is

\begin{equation}
G(R,t+1) = G(R-1,t) e^{-\alpha/T} + G(R+1,t) e^{-\alpha/T},
\end{equation}

(1)
in which, $\epsilon_l$ and $\epsilon_r$ are the energy assigned to the left and right bonds of the point $(R,t)$. The free energy $F(t)$ is given by $F(t) = -T \ln G(t)$, where $G(t) = \sum_R^{} G(R,t)$ is the total partition function. The free energy fluctuation $\Delta F = (\langle F^2 \rangle - \langle F \rangle^2)^{1/2}$ has been commonly studied ($\langle A \rangle$ is the ensemble average of the quantity $A$). We can also define the internal energy, $(E) \equiv \sum_{C}^{R} EC e^{-EC/T} / \sum_{C}^{R} e^{-EC/T}$. The internal energy fluctuation $(\Delta E)_T = \left( \langle (E^2) \rangle - \langle E \rangle^2 \right)^{1/2}$ is also an interesting quantity. In order to obtain the iteration relation for the internal energy $(E)$, we define $\hat{E}(R,t) = \sum_{C(R,t)}^{C} E_C(R,t) e^{-E_C(R,t)/T}/G(t)$. It is clear that $\langle E \rangle = \sum_R \hat{E}(R,t)$. The iteration relation for $\hat{E}(R,t)$ is:

$\hat{E}(R,t+1) = \left[ e^{-\epsilon_l/T} \hat{E}(R-1,t)G(t) + e^{-\epsilon_r/T} \hat{E}(R+1,t) \right] G(t) + \epsilon_l e^{-\epsilon_l/T} G(R-1,t) + \epsilon_r e^{-\epsilon_r/T} G(R+1,t) / G(t+1) \tag{2}$

To understand the difference between the free energy and internal energy fluctuations, we use the entropy fluctuation $\Delta S = \left( \langle S^2 \rangle - \langle S \rangle^2 \right)^{1/2}$ based upon equations (1) and (2) with the initial conditions $G(R,0) = \delta_{R0}$ and $\hat{E}(R,0) = 0$. The random energy assigned on the bond is assumed to be uniformly distributed in the interval $(-0.5,0.5)$ and uncorrelated in space and time. We use length up to $t = 2000$ ($d = 1 + 1$) and $t = 1000$ ($d = 2 + 1$) [30]. Six thousand configurations for $1+1$ dimensions and four thousand configurations for $2+1$ dimensions were collected to take the ensemble average.

In $1+1$ dimensions, the numerical results clearly show that the entropy fluctuation has the behavior $(\Delta S)^2 \propto t$ for any temperature (see Fig. 1) [30]. It is expected that the entropy fluctuation will tend to zero at the limits of zero and infinite temperature. Indeed, the slope of $(\Delta S)^2/t$ (Fig. 3(a)) is about 1 for $T < T_p$, i.e. $\lambda(t) \sim T$ and about $-4$ for $T > T_p$, i.e. $\lambda(t) \sim T^{-4}$. As a result, the free energy fluctuation and internal energy fluctuation will be the same at the two limits. About $T = T_p = 0.2$, the entropy fluctuation reaches a maximum. There is no evidence for a phase transition in $1+1$ dimensions.

The picture of directed polymers in $2+1$ dimensions is more complicated than that of $1+1$ dimensions (see Fig. 2). Similar to $1+1$ dimensions, the entropy fluctuation tends to zero at the limits of zero and infinite temperature and there is a peak at $T_p = 0.11$. For low temperatures $T \leq T_p = 0.11$, we see that the entropy fluctuation $(\Delta S)^2$ tends to $t$ and $(\Delta S)^2 \sim T$ as in $1+1$ dimensions (see Fig. 2(a) and Fig. 3(b)). However, for $T > T_p$, we find that the increase of entropy fluctuation as a function of $t$ becomes slower and slower as temperature is increased (see Fig. 2(b)). At the very high temperatures, e.g. $T = 10.0$, the entropy fluctuation is proportional to $\ln t$ for large $t$. It seems that $(\Delta S)^2/t$ will tend to the nonzero values only in the region $0 < T \leq T_p$. The fluctuations of the free energy $\Delta F$ are correlated with the fluctuations of the entropy $\Delta S$.

The high temperature behavior of $\Delta F$ gives for large $t$ a logarithmic dependence on $t$ while for lower temperatures we see a crossover from logarithmic behaviour at small $t$ to $\Delta F \sim t^{0.2}$ for larger $t$’s. We have not seen however, a sharp obvious temperature where this happens. In Fig. 3b we see a sharp transition from a low temperature region $T < T_p$, where $\lambda(t)$ is almost $t$ independent to a high temperature regime where $\lambda(t)$ is a decreasing function of $t$. The transition temperature is very close to the point where $\lambda(t,T)$ is maximal as a function of $T$ for all $t$.

The observed behavior may be explained as follows. The infinite system is characterized by a correlation length $\xi$. As long as the size of the system is adequately described in terms of the linear theory of deposition proposed by Edwards and Wilkinson [31]. It is easy to show that within the Edwards-Wilkinson theory $(\Delta S)^2$ is proportional, for a finite system to $\ln t$, as obtained by us for high temperatures. For $t$ longer than $\xi$ the nonlinearities become important and $(\Delta S)^2$ should be extensive in $t$. How is this related to the temperature dependence we find? The correlation length $\xi$ has a strong dependence on temperature, $\xi(T)$ is proportional to $\exp[(T/T_0)^\theta]$ with $\theta = 3$ according to Fisher and Huse [29] and $\theta = 2$ according to Kim, Bray and Moore [18]. In any case the temperature dependence of $\xi_T$ is so strong that a relatively small increase in temperature may result in an increase of $\xi_T$ by orders of magnitude increasing it from values below the minimal $t$ we are using ($t = 50$) to well above the maximal value ($t = 1000$). Indeed, a more careful examination of the data is consistent with the above explanation. In the temperature region $T_P < T < 2T_P$, all the lines but the one correspondings to the smallest $t$ merge. This suggest that the asymptotic value of $\lambda(t)$ has already been reached, so that the asymptotic value is of order 1. Thus above $T_p$ the asymptotic value is still finite. This may explain the former numerical results that claimed a phase transition in $2+1$ dimensions [22-26]. The strong dependence of the correlation length on temperature suggest that increasing the size of the system does not really undergo a phase transition.

It is not difficult to obtain the asymptotic (infinite $t$) high temperature form of $\lambda(T)$. As expected from the Edwards-Wilkinson model $\lambda(t)$ decrease as $\ln t/t$. This is true as long as $t > \xi_T$ but as $t$ becomes of the order of $\xi_T$ $\lambda$ must tend to a constant independent on $t$. Therefore,

$$\lambda(T) \propto \frac{\ln \xi_T}{\xi_T} = \left( \frac{T}{T_0} \right)^\theta \exp[-(\frac{T}{T_0})^\theta]. \tag{3}$$
Thus we see that although $\lambda$ is not zero it decreases extremely fast with temperature.

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FIG. 1. Plot of entropy fluctuation as a function of time $t$ for different temperatures in $d = 1 + 1$, (a) for $T = \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \frac{1}{25}$ (from top to bottom). (b) for $T = 10, 5, 2, 1, \frac{1}{2}, \frac{1}{5}, \frac{1}{10}$ (from bottom to top).

FIG. 2. Plot of entropy fluctuation as a function of time $t$ for different temperatures in $d = 2 + 1$, (a) for $T = \frac{1}{9}, \frac{1}{15}, \frac{1}{20}, \frac{1}{25}$ (from top to bottom). (b) for $T = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{9}, \frac{1}{3}$ (from bottom to top).
FIG. 3. Plot of entropy fluctuation per unit length of the polymer as a function of temperature (a) for the different time $t = 50, 500, 1000, 2000$ in $1+1$ dimensions. (b) for the different time $t = 50, 100, 300, 600, 1000$ (from top to bottom) in $2+1$ dimensions.