AN UPPER BOUND ON THE CHEEGER CONSTANT OF
A DISTANCE-REGULAR GRAPH

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ABSTRACT. We present an upper bound on the Cheeger constant of a
distance-regular graph. Recently, the authors found an upper bound on
the Cheeger constant of distance-regular graph under a certain restriction
in their previous work. Our new bound in the current paper is much better
than the previous bound, and it is a general bound with no restriction. We
point out that our bound is explicitly computable by using the valencies
and the intersection matrix of a distance-regular graph. As a major tool,
we use the discrete Green’s function, which is defined as the inverse of
$\beta$-Laplacian for some positive real number $\beta$. We present some examples
of distance-regular graphs, where we compute our upper bound on their
Cheeger constants.

1. Introduction

A notion of the Cheeger constant of a graph has an important geometric
meaning in graph theory. The Cheeger constant of a graph is closely related
to the problem of separating a graph into two large components by making a
small edge-cut. In fact, the Cheeger constant of a connected graph is strictly
positive. If the Cheeger constant of a connected graph is “small”, then it means
that there are two large sets of vertices with “few” edges between them. On
the other hand, if a graph has “large” Cheeger constant, then it indicates that
there are two sets of vertices with “many” edges between these two subsets. In
general, computation of the Cheeger constant of a graph is a hard task. Only
limited research has been done for finding the Cheeger constant of a graph. We
are interested in finding bounds of Cheeger constants of graphs.

We begin with introducing some definitions in graph theory. Let $\Gamma = (V, E)$
be a simple and connected graph, where $V$ is the vertex set of $\Gamma$ and $E$ is the
edge set of $\Gamma$. Let $S$ be a nonempty subset of $V$. The edge boundary of $S$, denoted by $\partial S$, is defined as follows:

$$\partial S = \{ \{x, y\} \in E \mid x \in S \text{ and } y \in V - S \}.$$ 

The volume of $S$, denoted by $\text{vol}(S)$, is defined as follows:

$$\text{vol}(S) = \sum_{u \in S} k_u,$$

where $k_u$ is the valency of $u$ in $\Gamma$. The Cheeger ratio of $S$, denoted by $h_S$, is defined as

$$h_S = \frac{|\partial S|}{\min\{\text{vol}(S), \text{vol}(\Gamma) - \text{vol}(S)\}}.$$ 

The Cheeger constant of $\Gamma$, denoted by $h_\Gamma$, is defined as

$$h_\Gamma = \min\{h_S \mid S \subseteq V\}.$$ 

Recent developments in [3, 4, 14] regarding distance-regular graphs show that there is a close connection between the Cheeger constant and vertex (or edge) connectivity. From Propositions A, B, and C we see that, for a distance-regular graph, there are close connections between the Cheeger constant and vertex and edge connectivity.

**Proposition A ([3]).** Let $\Gamma$ be a distance-regular graph with more than one vertex. Then its edge-connectivity equals its valency $k$, and the only disconnecting sets of $k$ edges are the sets of edges incident with a single vertex.

**Proposition B ([4]).** Let $\Gamma$ be a non-complete distance-regular graph of valency $k > 2$. Then the vertex-connectivity $\kappa(\Gamma)$ equals $k$, and the only disconnecting sets of vertices of size not more than $k$ are the point neighbourhoods.

**Proposition C ([14]).** Let $\Gamma = (V, E)$ be a simple graph with the vertex-connectivity $\kappa(\Gamma)$ and the edge-connectivity $\lambda(\Gamma)$. Then

$$\frac{2\kappa(\Gamma)}{|V|} \leq \frac{2\lambda(\Gamma)}{|V|} \leq \inf |\partial S| \leq \kappa(\Gamma) \leq \lambda(\Gamma),$$

where $S$ is a subset of $V$ with $|S| \leq \frac{|V|}{2}$.

The Cheeger constants [9, 15] are related to the eigenvalues of the Laplacians of distance-regular graphs, and their eigenvalues are also involved with the intersection numbers of distance-regular graphs [13, 16]. However, in general, it is a hard task to compute the Cheeger constant of a distance-regular graph. Distance-regular graphs introduced by Biggs [2] are connected with coding theory and design theory; well-known examples of distance-regular graphs are the Hamming graphs and the Johnson graphs. In [11, 12], by using the relationship between a discrete Green’s function and the Cheeger constant, we obtain an upper bound on the Cheeger constant of a distance-regular graph under a certain condition.
We find a general upper bound on the Cheeger constant of a distance-regular graph with no additional condition. Furthermore, our bound is a much more improved one comparing with the bound in [12] under the same additional condition; in Example 7 and Example 8, we show that our bound is much more improved one comparing with the bound in [12] under the same additional condition: $\beta vr_1^{(\beta)} > \frac{1}{1 + \lambda_1}$.

We point out that our bound is explicitly computable by using the valencies and the intersection matrix of a distance-regular graph; first, our bound is expressed in terms of $q$-numbers, and in general, it is not easy to compute the $q$-numbers. For resolving this problem, we obtain an alternative expression of our bound using the valencies and the intersection matrix of a distance-regular graph. In Example 10 and Example 11, we compute the upper bound on the Cheeger constant using the alternative expression in Theorem 2 and Remark 6. As a major tool, we use the discrete Green’s function, which is defined as the inverse of the $\beta$-Laplacian $L_{\beta}$ for some positive real number $\beta$.

We discuss our main result in more detail for the rest of this section. In this paper, we study distance-regular graphs. Let $\Gamma = (V, E)$ be a distance-regular graph of order $v$, diameter $d$ and valency $k$. Let $A_1$ be the adjacency matrix of $\Gamma$ and $P$ be the transition probability matrix of $\Gamma$. Two adjacent vertices $x, y$ are denoted by $x \sim y$. For a function $f : V \rightarrow \mathbb{R}$, we define a Laplace operator $\Delta$ by $\Delta f(x) = \frac{1}{k} \sum_{y \sim x} (f(x) - f(y))$. Then $\Delta = I - \frac{1}{k} A_1$. Let $L_{\beta}$ be the $\beta$-normalized Laplacian $\beta I + \Delta$. For $\beta > 0$, let $G_{\beta}$ be a discrete Green’s function denoted by the symmetric matrix which satisfies $L_{\beta} G_{\beta} = I$; that is, $G_{\beta}$ is defined as the inverse of the $\beta$-Laplacian $L_{\beta}$ [5, 6, 7]. As in [11], for any positive real number $\beta$, let $r_i^{(\beta)} (i = 0, 1, \ldots, d)$ denote the components of a Green’s function $G_{\beta}$. We define $\alpha_i$ to be the limit of a sequence $\{\alpha_i^{(\beta)}\}$ as $\beta$ goes to $0^+$, where

\begin{equation}
\alpha_i^{(\beta)} = \frac{\beta^2 vr_i^{(\beta)}}{1 - \beta vr_i^{(\beta)}} (i = 0, 1, \ldots, d).
\end{equation}

In fact, we can express $\alpha_i$’s by the eigenvalues $\lambda_j$ of the Laplacian $L_{\beta}$ and the $q$-numbers $q_j(i)$ of the P-polynomial scheme [1, 8, 10, 11]:

\begin{equation}
\alpha_i = \frac{1}{-q_1(i) \frac{1}{\lambda_1} - \cdots - q_d(i) \frac{1}{\lambda_d}}.
\end{equation}

We also see that $0 < \alpha_d < \alpha_{d-1} < \cdots < \alpha_1 < \lambda_1$ for some $\epsilon$.

The authors obtain the following result [12] on an upper bound on the Cheeger constant of a distance-regular graph with a certain restricted condition as follows.

**Theorem A** ([12]). Let $\Gamma$ be a distance-regular graph with diameter $d$ and $\beta vr_1^{(\beta)} > \frac{1}{1 + \lambda_1}$ for $\beta \leq \alpha_d$. Let $\lambda_1$ be the smallest eigenvalue of the Laplacian.
Then we have
\[ \lambda_1 h_\Gamma < \alpha_d < \alpha_{d-1} < \ldots < \alpha_e < \lambda_1 \]
for \( e \in \mathbb{C}_d' \).

Main results of this paper are the following Theorem 1, Theorem 2 and Corollary 3. Theorem 1 presents an upper bound on the Cheeger constant of a distance-regular graph. In Theorem 2, we find an explicit expression for the bound given in Theorem 1 by using the valencies \( k_j \) and the basis of nullspace \( \mathcal{N}(L^{(\alpha_d)}_{\text{sub}}) \). This shows that our new bound is a computable bound using the valencies and the intersection matrix of a distance-regular graph. Corollary 3 shows that our generalized bound in Theorem 1 and Theorem 2 improves the bound given in Theorem A [12] under the same additional condition.

**Theorem 1.** Let \( \Gamma \) be a distance-regular graph of diameter \( d \). Then we have the following upper bound:
\[ h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}}, \]
where \( \alpha_d = \lim_{\beta \to 0^+} \frac{\beta^2 \nu d(\beta)}{1 - \beta \nu d(\alpha_d)} \) and \( \alpha_d^{(\alpha_d)} = \frac{\alpha_d^2 \nu d(\alpha_d)}{1 - \alpha_d \nu d(\alpha_d)} \).

**Theorem 2.** Let \( \Gamma \) be a distance-regular graph of order \( v \) and diameter \( d \). Let \( (u_0^{(\alpha_d)}, u_1^{(\alpha_d)}, \ldots, u_d^{(\alpha_d)}) \) be a basis of \( \mathcal{N}(L^{(\alpha_d)}_{\text{sub}}) \) with \( u_d^{(\alpha_d)} = 1 \) as in Lemma 4. Then we have
\[ h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} = \alpha_d \left( \frac{1}{v} \sum_{j=0}^{d} k_j u_j^{(\alpha_d)} - 1 \right), \]
where \( k_j \) are valencies as in Lemma 4 and \( \alpha_d \) is the same as in Lemma 5.

**Corollary 3.** Let \( \Gamma \) be a distance-regular graph, and \( h_\Gamma \) be a Cheeger constant of \( \Gamma \). If \( \beta \nu d(\beta) > \frac{\lambda_1}{1 + \lambda_1} \) for \( \beta \leq \alpha_d \), then we have
\[ h_\Gamma < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} < \frac{\alpha_d}{\lambda_1}, \]
where \( \lambda_1 \) is the smallest positive eigenvalue of the Laplacian and \( \alpha_d \) is the same as in (1).

In Section 2, we introduce some notations and facts about distance-regular graph and some properties of the Green’s function \( G \). In Section 3, we find a new upper bound on the Cheeger constant of a distance-regular graph. We also obtain an alternative expression of our upper bound by using the valencies \( k_j \) and the basis of nullspace \( \mathcal{N}(L^{(\alpha_d)}_{\text{sub}}) \). Finally, in Section 4, we present some examples about our upper bound on the Cheeger constant of some distance-regular graphs.
2. Preliminaries and Green’s function

We introduce definitions of the distance-regular graphs and the $P$-polynomial schemes. A connected graph $\Gamma$ with diameter $d$ is called a distance-regular graph if there are constants $c_i, a_i, b_i$ such that for all $i = 0, 1, \ldots, d$, and all vertices $x$ and $y$ at distance $i = d(x, y)$, among the neighbors of $y$, there are $c_i$ at distance $i - 1$ from $x$, $a_i$ at distance $i$, and $b_i$ at distance $i + 1$. It follows that $\Gamma$ is a regular graph with valency $k = b_0$, and that $c_i + a_i + b_i = k$ for all $i = 0, 1, \ldots, d$.

By these equations, the intersection numbers $a_i$ can be expressed in terms of the others, and it is a standard to put these others in the so-called intersection array $(b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d)$. We describe the relations by its adjacency matrices $A_i \ (i = 0, 1, \ldots, d)$ which are $v \times v$ matrices defined by

\[
(A_i)_{x,y} = \begin{cases} 
1 & \text{if } (x, y) \in R_i, \\
0 & \text{otherwise,}
\end{cases}
\]

[1, 8]. Let $X$ be a nonempty finite set and $R = \{R_0, R_1, \ldots, R_d\}$ be a family of relations defined on $X$. We say that the pair $(X, R)$ is a symmetric association scheme with $d$ classes if it satisfies the following conditions.

1. $A_0 = I$ (identity matrix).
2. $A_0 + A_1 + \cdots + A_d = J$ (all 1 matrix).
3. $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$, where $p_{ij}^k$ is the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$.
4. $A_i^t = A_i$.
5. $A_i A_j = A_j A_i$.

A symmetric association scheme $\mathfrak{X} = (X, R)$ is called a $P$-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$, if there exist some complex coefficient polynomials $v_i(x)$ of degree $i \ (i = 0, 1, \ldots, d)$ such that $A_i = v_i(A_1)$, where $A_1$ is the adjacency matrix with respect to $R_1$.

It is known [1, 8] that a distance-regular graph is equivalent to a $P$-polynomial scheme $\mathfrak{X}$ with respect to some relations $R_0, R_1, \ldots, R_d$ on a vertex set $V$ with $|V| = v$. Thus, we can define the Green’s function over a $P$-polynomial scheme, and then by using the Green’s function we will obtain an upper bound on the Cheeger constant of a distance-regular graph.

The first intersection matrix $B_1$ of a distance-regular graph is a tridiagonal matrix with non-zero off diagonal entries:

\[
B_1 = \begin{pmatrix}
0 & k & 0 & 0 & \cdots & 0 \\
1 & a_1 & b_1 & 0 & \cdots & 0 \\
0 & c_2 & a_2 & b_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & & \cdots & c_{d-1} & a_{d-1} & b_{d-1} \\
0 & \cdots & \cdots & 0 & c_d & a_d
\end{pmatrix} (b_i \neq 0, c_i \neq 0).
\]

Let $\mathcal{A}$ be the algebra spanned by the adjacency matrices $A_0, A_1, \ldots, A_d$. Then $\mathcal{A}$ is called the Bose-Mesner algebra of $\mathfrak{X}$, and $\mathcal{A}$ has two distinguished
bases \{A_i\} and \{E_i\}, where the latter consists of primitive idempotent matrices. For \(A_i\) and \(E_i\), we can express one in terms of the other as the following:

\[
A_j = \sum_{i=0}^{d} p_j(i)E_i, \quad E_j = \frac{1}{|X|} \sum_{i=0}^{d} q_j(i)A_i
\]

for \(j = 0, 1, \ldots, d\). The \((d+1) \times (d+1)\) matrix \(P = (p_j(i))\) (respectively, \(Q = (q_j(i))\)) is called the first eigenmatrix (respectively, the second eigenmatrix) of the \(P\)-polynomial scheme \(X\), where \(p_j(i)\) (respectively, \(q_j(i)\)) is a \(p\)-number (respectively, a \(q\)-number).

As defined in Section 1, the Green’s function \(G_{\beta}\) is the inverse of the \(\beta\)-Laplacian \(L_{\beta}\). For \(\beta > 0\), we thus have \(G_{\beta}(\beta I + I - P) = I\). Therefore, we get

\[
L_{\beta} = \sum_{j=0}^{d} \left(\beta + 1 - \frac{1}{k_1} p_1(j)\right) E_j, \quad G_{\beta} = \sum_{j=0}^{d} \left(\frac{k_1}{(\beta + 1)k_1 - p_1(j)}\right) E_j,
\]

Since \(E_j = (1/v) \sum q_j(i)A_i\) and \(\lambda_j = 1 - p_1(j)/k_1\), we get

\[
G_{\beta} = \sum_{j=0}^{d} \left(\frac{k_1}{(\beta + 1)k_1 - p_1(j)}\right) \sum_{i=0}^{d} q_j(i)
= \sum_{i=0}^{d} \sum_{j=0}^{d} \left(\frac{1}{(\beta + 1)k_1 - p_1(j)}\right) q_j(i)A_i
= \sum_{i=0}^{d} \sum_{j=0}^{d} \left(\frac{1}{\beta + 1 - \frac{p_1(j)}{k_1}}\right) q_j(i)A_i
= \sum_{i=0}^{d} \sum_{j=0}^{d} \left(\frac{1}{\beta + \lambda_j}\right) q_j(i)A_i.
\]

That is, \(G_{\beta}\) is a linear combination of adjacency matrices \(A_i\) as follows:

\[
G_{\beta} = r^{(\beta)}_0 A_0 + r^{(\beta)}_1 A_1 + \cdots + r^{(\beta)}_d A_d,
\]

where \(r^{(\beta)}_i = \frac{1}{v} \left(\frac{1}{\beta} + q_i(i)\frac{1}{\beta + A_1} + \cdots + q_d(i)\frac{1}{\beta + A_d}\right)\) (\(i = 0, 1, \ldots, d\)).

In [10, 11], a \(d \times (d+1)\) matrix \(L_{sub}^{(\beta)}\) is introduced as a matrix obtained by the removal of the first row of \(B_1 - k_1(\beta + 1)I\).

**Lemma 4** ([11]). For \(\beta > 0\), let \(G_{\beta} = r^{(\beta)}_0 A_0 + r^{(\beta)}_1 A_1 + \cdots + r^{(\beta)}_d A_d\) be the Green’s function of a distance-regular graph \(\Gamma\) of order \(v\). Then we have

(a) \(G_{\beta}\) can be expressed as \(G_{\beta} = tu^{(\beta)}_0 A_0 + tu^{(\beta)}_1 A_1 + \cdots + tu^{(\beta)}_d A_d\) for some nonzero \(t \in \mathbb{R}\), where \((u^{(\beta)}_0, u^{(\beta)}_1, \ldots, u^{(\beta)}_d)\) is the unique basis of the nullspace \(N(L_{sub}^{(\beta)})\) of \(L_{sub}^{(\beta)}\) with \(u^{(\beta)}_d = 1\).
(b) \( k_0 r_0(\beta) + k_1 r_1(\beta) + \cdots + k_d r_d(\beta) = \frac{\lambda}{\beta} \), where \( k_j \) is the valency of \( A_j \) for \( j = 0, 1, \ldots, d \).
(c) \( r_0(\beta) > r_1(\beta) > \cdots > r_d(\beta) > 0 \).
(d) \( \lim_{\beta \to 0^+} |r_i(\beta) - r_j(\beta)| = 0 \) for \( 0 \leq i, j \leq d \).

3. A new improved bound on the Cheeger constant

In this section we prove Theorem 1 and Theorem 2. We need the following lemma for the proof of Theorem 2 and Theorem 3. We consider a set \( C_\beta = \{ i \mid \frac{1}{\beta} - vr_i(\beta) > 0 \} \) as a subset of \( \{ 0, 1, 2, \ldots, d \} \); then \( C_\beta \) is a non-empty set by Lemma 4. When \( \beta \) is sufficiently close to \( 0^+ \), we consider a set \( C'_\beta = \{ i \mid \beta vr_i(\beta) + (\beta + \lambda_1) < \lambda_1 \} \); then \( C'_\beta \) is a subset of \( C_\beta \).

Lemma 5 ([11]). For \( \beta > 0 \), let \( \Gamma \) be a distance-regular graph of order \( v \), and let \( G_\beta = r_0(\beta) A_0 + r_1(\beta) A_1 + \cdots + r_d(\beta) A_d \) be a Green's function of \( \Gamma \). We recall that \( \alpha_i(\beta) := \frac{\beta vr_i(\beta)}{1 - \beta vr_i(\beta)} (i = 0, 1, \ldots, d) \) as given in Eq. (1). Then for \( i \in C'_\beta \), we have the following:

(a) \( \lim_{\beta \to 0^+} \beta vr_i(\beta) = 1^- \), \( \lim_{\beta \to 0^+} \beta^2 vr_i(\beta) = 0^+ \).
(b) \( \alpha_i(\beta) \) is decreasing in \( i \in C'_\beta \).
(c) There exists \( i \in C'_\beta \) such that \( \lim_{\beta \to 0^+} \alpha_i(\beta) = \alpha_i < \lambda_1 \).
(d) \( \alpha_i(\beta) \) is decreasing in \( \beta > 0 \).

Proof of Theorem 1. Let \( S \) be a subset of the vertex set \( V \) of \( \Gamma \) with \( \text{vol}(S) \leq \text{vol}(V)/2 \) and \( \frac{\partial S}{\text{vol}(S)} = h \Gamma \). Let \( S' \) be a subset of \( V \) with \( \frac{\partial S'}{\text{vol}(S')} = h \Gamma \). Then there exists some \( \beta \) with \( 0 < \beta < 1 \) such that

\[
\frac{\partial S}{\text{vol}(S)} = \frac{\partial S'}{\text{vol}(S')},
\]

We first note that for a positive integer \( n \),

\[
\beta < \frac{\alpha_d(\alpha_d^n)}{\alpha_d(\alpha_d^n)};
\]

this follows immediately from \( \frac{\alpha_d(n)}{\alpha_d} < \frac{\alpha_d(\alpha_d^n)}{\alpha_d} \), which is clear since \( \alpha_d(x) \) is decreasing in \( x \) by Lemma 5.

We claim that for any \( \epsilon > 0 \), there exists a positive integer \( N_\epsilon \) such that

\[
\alpha_d(\alpha_d^n) < \alpha_d^2 + \epsilon
\]

for any \( n \geq N_\epsilon \); we use Lemma 5 for the proof as follows. Let \( f_n = \alpha_d \beta^n vr_d(\alpha_d^n) \). Then \( \lim_{n \to \infty} f_n = 1^- \) by Lemma 5(a). Thus, for sufficiently large positive integer \( n \), we obtain the following approximations:

\[
f_n(\beta^n + \alpha_d) \approx \alpha_d
\]
\[ f_n \beta^n \approx \alpha_d (1 - f_n) \]
\[ \Rightarrow \alpha_d^{(\alpha_d \beta^n)} = \frac{f_n \beta^n \alpha_d}{1 - f_n} \approx \alpha_d^2; \]
so our claim in Eq. (5) follows.

From Eq. (4) and Eq. (5), we thus have that

\[ (6) \]
\[ \beta < \frac{\alpha_d^2 + \varepsilon}{\alpha_d^{(\alpha_d)}}. \]

Taking \( \varepsilon > 0 \) to be such that \( \varepsilon < \left( \frac{\text{vol}(S)}{|\partial S|} - 1 \right) \alpha_d^2 \) (noting that the right hand side of this inequality is positive), we obtain

\[ h_T < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}}; \]
this is because from Eq. (3) and Eq. (6), we get the following:

\[ h_T = \frac{|\partial S| \beta}{\text{vol}(S)} < \frac{|\partial S|}{\text{vol}(S)} \left( \frac{\alpha_d^2 + \varepsilon}{\alpha_d^{(\alpha_d)}} \right) < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}}. \]
Consequently, the result follows as desired.

\[ \square \]

**Proof of Theorem 2.** From Theorem 1, we have

\[ h_T < \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}}. \]

By Lemma 5, we have

\[ \alpha_d = \lim_{\beta \to 0^+} \frac{\beta^2 \text{er}_d(\beta)}{1 - \beta \text{er}_d(\beta)}, \quad \alpha_d^{(\alpha_d)} = \frac{\alpha_d^2 \text{er}_d^{(\alpha_d)}}{1 - \alpha_d \text{er}_d^{(\alpha_d)}}. \]

Thus,

\[ \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} = \frac{1 - \alpha_d \text{er}_d^{(\alpha_d)}}{\text{er}_d^{(\alpha_d)}} = \frac{1}{\text{er}_d^{(\alpha_d)}} - \alpha_d, \]
where \( \text{er}_d^{(\alpha_d)} = \frac{1}{\alpha_d} + q_1(d) \frac{1}{\alpha_d + k_1} + \cdots + q_d(d) \frac{1}{\alpha_d + k_d} \). From Lemma 4, we have

\[ \sum_{j=0}^{d} k_j \text{er}_d^{(\alpha_d)} = \frac{1}{\alpha_d}, \]
and this implies that

\[ \text{er}_d^{(\alpha_d)} \sum_{j=0}^{d} k_j u_j^{(\alpha_d)} = \frac{1}{\alpha_d}; \]
so we get

\[ \alpha_d \sum_{j=0}^{d} k_j u_j^{(\alpha_d)} = \frac{1}{\text{er}_d^{(\alpha_d)}}. \]
It thus follows that

\[
\frac{1}{e_{d}(\alpha)} - \alpha_d = \alpha_d \left( \frac{1}{v} \sum_{j=0}^{d} k_j u_j^{(\alpha_d)} - 1 \right).
\]

The result follows immediately by combining Eq. (7) with Eq. (8).

The following remark shows that \( u_j^{(\beta)} \) can be expressed by a determinant of a submatrix \( L_j^{(\beta)} \) of \( N(L_{\sub}) \), and \( \alpha_d \) can be expressed in terms of a basis \( (u_0^{(\beta)}, u_1^{(\beta)}, \ldots, u_d^{(\beta)}) \) of \( N(L_{\sub}) \) and the valencies \( k_j \)'s as in Lemma 4.

**Remark 6.** (1) [10, 11] For \( \beta > 0 \), let \( L_0^{(\beta)} \) be the \( d \times d \) matrix obtained by the removal of the first column of \( L_{\sub}^{(\beta)} \) as in Lemma 4. Let \( L_j^{(\beta)} \) be the \((d-j)\times(d-j)\) matrix obtained by the removal from the first row(respectively, column) to the \( j \)-th row(respectively, column) of \( L_0^{(\beta)} \), and let \( (u_0^{(\beta)}, u_1^{(\beta)}, \ldots, u_d^{(\beta)}) \) be a basis of \( N(L_{\sub}) \) with \( u_d^{(\beta)} = 1 \). Then we have

\[
u_j^{(\beta)} = (-1)^{d-j} \frac{\det(L_j^{(\beta)})}{c_{j+1}c_{j+2}\cdots c_d}, \quad j = 0, 1, \ldots, d - 1,
\]

where \( \det(L_d^{(\beta)}) = 1 \).

(2) [11] Let \( \Gamma \) be a distance-regular graph of order \( v \), and let \( G_{\beta} = r_0^{(\beta)} A_0 + r_1^{(\beta)} A_1 + \cdots + r_d^{(\beta)} A_d \) be a Green’s function of \( \Gamma \) for \( \beta > 0 \). Then we have

\[
\alpha_d = \lim_{\beta \to 0^+} \frac{\beta v}{\sum_{j=0}^{d} k_j u_j^{(\beta)}} - v
\]

and \( \alpha_d < \alpha_d^{(\beta)} + \beta \).

**Proof of Corollary 3.** Since \( \beta v_d^{(\beta)} > \frac{\lambda_1}{1 + \lambda_1} \) for \( \beta \leq \alpha_d \), we have

\[
\lambda_1 < \frac{\beta v_d^{(\beta)}}{1 - \beta v_d^{(\beta)}} = \frac{\alpha_d^{(\beta)}}{\beta}
\]

Letting \( \beta = \alpha_d \), we get \( \frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} < \frac{1}{\lambda_1} \). It thus follows

\[
\frac{\alpha_d^2}{\alpha_d^{(\alpha_d)}} < \frac{\alpha_d}{\lambda_1}
\]

**4. Examples**

In this section we present some examples regarding our upper bound on the Cheeger constants for some distance-regular graphs. In particular, Example 7 and Example 8 show that our bound is much more improved one comparing with the bound in [11, 12] under the same additional condition.

Theorem 1 shows an upper bound on the Cheeger constant \( h_\Gamma \) in terms of \( \alpha_d \) and \( \alpha_d^{(\beta)} \). From Equations (2) and (3), if we know the \( q \)-numbers of the given \( P \)-polynomial scheme, then we can find \( \alpha_d \) and \( \alpha_d^{(\beta)} \) immediately.
In the Hamming scheme $H(d, q)$ (respectively, Johnson scheme $J(m, d)$), the $p$-number $p(i)$ is defined by the Krawtchouk polynomial (respectively, the Eberlein polynomial) [1]. Since $PQ = eI$, we can obtain the $q$-numbers $q(i)$ of the Hamming scheme $H(d, q)$ and the Johnson scheme $J(m, d)$. We present the following two examples for showing this case.

**Example 7.** Let $\Gamma$ be the graph of the Hamming scheme $H(d, q)$ with respect to $A_1$. Then $\Gamma$ is a distance-regular graph with $q^d$ vertices, valency $d(q - 1)$ and $d$ diameter. We consider two cases: (a) $d = 5$, $q = 4$ and (b) $d = 7$, $q = 3$, and in each case, our upper bound on the Cheeger constant is as follows:

(a) $H(5, 4) : v = 1024$, $k_1 = 15$, $\lambda_1 = 4/15$, $\alpha_d = 16/137$.
   And, for $\beta \leq \alpha_d$, $\beta v_d^{(\beta)} \geq \alpha_d v_d^{(\alpha_d)} \approx 0.41256 > \frac{\lambda_1}{1 + \lambda_1} \approx 0.21053$.
   Thus, $h_\Gamma < \frac{\alpha_d^2}{\alpha_d^2 + \beta} \approx 0.166291 < \frac{\alpha_d^2}{\alpha_1} \approx 0.43795$.

(b) $H(7, 3) : v = 2187$, $k_1 = 14$, $\lambda_1 = \frac{3}{14}$, $\alpha_d = \frac{10}{121}$.
   And, for $\beta \leq \alpha_d$, $\beta v_d^{(\beta)} \geq \alpha_d v_d^{(\alpha_d)} \approx 0.404240 > \frac{\lambda_1}{1 + \lambda_1} \approx 0.176471$.
   Thus, $h_\Gamma < \frac{\alpha_d^2}{\alpha_d^2 + \beta} \approx 0.12180 < \frac{\alpha_d^2}{\alpha_1} \approx 0.385675$.

**Example 8.** Let $\Gamma$ be a graph of the Johnson scheme $J(m, d)$ with respect to $A_1$. Then $\Gamma$ is a distance-regular graph with $\binom{m}{d}$ vertices, valency $d(m - d)$ and $d$ diameter. We consider two cases: (a) $m = 6$, $d = 3$ and (b) $m = 11$, $d = 5$. In each case, our upper bound on the Cheeger constant is as follows:

(a) $J(8, 4) : v = 126$, $k_1 = 20$, $\lambda_1 = 9/20$, $\alpha_d = 252/1325$.
   And, for $\beta \leq \alpha_d$, $\beta v_d^{(\beta)} \geq \alpha_d v_d^{(\alpha_d)} \approx 0.415036 > \frac{\lambda_1}{1 + \lambda_1} \approx 0.310345$.
   Thus, $h_\Gamma < \frac{\alpha_d^2}{\alpha_d^2 + \beta} \approx 0.268058 < \frac{\alpha_d^2}{\alpha_1} \approx 0.422642$.

(b) $J(11, 5) : v = 462$, $k_1 = 30$, $\lambda_1 = 11/30$, $\alpha_d = 11088/79091$.
   And, for $\beta \leq \alpha_d$, $\beta v_d^{(\beta)} \geq \alpha_d v_d^{(\alpha_d)} \approx 0.40805 > \frac{\lambda_1}{1 + \lambda_1} \approx 0.268923$.
   Thus, $h_\Gamma < \frac{\alpha_d^2}{\alpha_d^2 + \beta} \approx 0.203375 < \frac{\alpha_d^2}{\alpha_1} \approx 0.382344$.

**Example 9.** Let $\Gamma$ be a Taylor graph with intersection array $(275, 112, 1; 1, 112, 275)$. Then $\Gamma$ is a distance-regular graph with vertices 552, valency 275 and 3 diameter. Also, $\lambda_1 = 4/5$, $\alpha_3 = 3864/12475$, $\alpha_3^{(\alpha_3)} = 0.2265$. Thus we have an upper bound on the Cheeger constant of $\Gamma$ as follows:

$$h_\Gamma < \frac{\alpha_3^2}{\alpha_3^{(\alpha_3)}} \approx 0.42357.$$ 

In Theorem 2, we find an alternative upper bound, which is explicitly computable, by using $\alpha_d$, the valencies $k_j$, and the basis of nullspace $N(L_{\text{null}}^{(\beta)}).$ In Example 10 and Example 11, we compute the upper bound on the Cheeger constant using the alternative expression in Theorem 2 and Remark 6.

**Example 10.** Let $\Gamma$ be a graph with respect to $A_1$ of a Johnson scheme $J(8, 4)$. Then $\Gamma$ is a distance-regular graph with 70 vertices and valency 16. Also, the
valencies of $J(8, 4)$ are 1, 16, 36, 16, 1 and

$$L_{sub}^{(\beta)} = \begin{pmatrix} 1 & 6 - 16(\beta + 1) & 9 & 0 & 0 \\ 0 & 4 & 8 - 16(\beta + 1) & 4 & 0 \\ 0 & 0 & 9 & 6 - 16(\beta + 1) & 1 \\ 0 & 0 & 0 & 16 & -16(\beta + 1) \end{pmatrix}.$$  

Since

$$\alpha_d = \frac{1}{q_1(d) - q_2(d) - \cdots - q_d(d)}.$$  

$(q_1(4), \ldots, q_d(4)) = (-7, 20, -28, 14)$ and $(\lambda_1, \ldots, \lambda_4) = (\frac{8}{10}, \frac{14}{10}, \frac{18}{10}, \frac{20}{10})$, we get $\alpha_d = 315/1522$. Let $\beta = 315/1522$. By Remark 6, a basis $(u_0^{(\alpha_d)}, \ldots, u_4^{(\alpha_d)})$ for $\mathcal{N}(L_{sub}^{(\alpha_d)})$ is

$$\left(\frac{10692972602391}{335381132641}, \frac{3108779427}{881422162}, \frac{969476}{579121}, \frac{1837}{1522}, 1\right).$$

Thus, by Theorem 2, we have

$$h_\Gamma < (315/1522)\left(\frac{1}{70} \sum_{j=0}^{4} k_j u_j^{(\alpha_d)} - 1 \right) \approx 0.292388.$$

**Example 11.** Let $X$ be a set of $d \times n$ matrices over $GF(p^t)$ ($d \leq n$). We define the $i$-th relation $R_i$ on $X$ by $(x, y) \in R_i$ if and only if $\text{rank}(x - y) = i$. Then $X = (X, \{R_i\})$ ($0 \leq i \leq d$) is a $P$-polynomial scheme with respect to the ordering $R_0, R_1, \ldots, R_d$. Let $p = 2, t = 1, d = 4, n = 5$. Then $L_{sub}^{(\beta)}$ is obtained as follows:

$$\begin{pmatrix} 1 & 44 - 465(\beta + 1) & 420 & 0 & 0 \\ 0 & 6 & 123 - 465(\beta + 1) & 336 & 0 \\ 0 & 0 & 28 & 245 - 465(\beta + 1) & 192 \\ 0 & 0 & 0 & 120 & 345 - 465(\beta + 1) \end{pmatrix}.$$  

We have $|X| = v = 1048576, k_0 = 1, k_1 = 465, k_2 = 32550, k_3 = 390600$ and $k_4 = 624960$ by using $k_i = \prod_{c_2 \geq c_3 \geq \cdots \geq c_i} (i = 2, 3, \ldots, d)$.

Let $\beta = \frac{1}{100}$. Then by Lemma 4 and Remark 6 (1), we obtain the unique basis of $\mathcal{N}(L_{sub}^{(\beta)})$ as follows:

$$(u_0^{(\beta)}, u_1^{(\beta)}, u_2^{(\beta)}, u_3^{(\beta)}, u_4^{(\beta)}) = \left(\frac{3921317781669}{358400000}, \frac{486743013}{17920000}, \frac{661683}{448000}, \frac{831}{800}, 1\right).$$

Thus, by Remark 6, we find

$$\frac{(1048576) \frac{1}{100}}{(1) u_0^{(\beta)} + (465) u_1^{(\beta)} + (32550) u_2^{(\beta)} + (390600) u_3^{(\beta)} + (624960) u_4^{(\beta)} - 1048576 \approx 0.195023.}$$

Thus, we have $\alpha_d < \tilde{\alpha}_d \approx 0.195023 + 0.01 = 0.205023.$
Let $\beta = 0.205023$. Then, we obtain the unique basis of $N(L_{\tilde{\alpha}d})$ as follows:

\[
\begin{align*}
  u_0(\tilde{\alpha}d) &= \frac{227848229494208860060049660207}{512000000000000000000000000000}, \\
  u_1(\tilde{\alpha}d) &= \frac{2234200208459744136613}{256000000000000000000000000000}, \\
  u_2(\tilde{\alpha}d) &= \frac{85453392459301}{6400000000000}, \\
  u_3(\tilde{\alpha}d) &= \frac{14355713}{8000000}, \\
  u_4(\tilde{\alpha}d) &= 1.
\end{align*}
\]

Thus, from Theorem 2, we obtain

\[
h_{\Gamma} < (\alpha_d) \left( \frac{1}{1048576} \sum_{j=0}^{4} k_j u_j^{(\tilde{\alpha}d)} - 1 \right) \approx 0.305557.
\]

**Remark 12.** In general, it is a hard task to compute the Cheeger constant of the graph, and there is not much known about the actual value of the Cheeger constant of a graph. As far as we know, the only known case is the Cheeger constant of the Hamming graph $H(d, q)$ with $q$ even, which is $\frac{q}{2n(q-1)}$. For instance, we consider two cases $H(5, 2)$ and $H(5, 4)$ for comparing our bound with the actual Cheeger constant:

(a) $H(5, 2) : v = 512, k_1 = 5, \lambda_1 = 2/5, \alpha_d = 24/137, \alpha_d(\alpha_d) \approx 0.123033.$

Thus, $h_{\Gamma} = \frac{1}{5} = 0.2 < \frac{\alpha_d^2}{\alpha_d(\alpha_d)} \approx 0.249437.$

(b) $H(5, 4) : v = 1024, k_1 = 15, \lambda_1 = 4/15, \alpha_d = 16/137, \alpha_d(\alpha_d) \approx 0.083724.$

Thus, $h_{\Gamma} = \frac{2}{15} = 0.133333 \cdots < \frac{\alpha_d^2}{\alpha_d(\alpha_d)} \approx 0.166291.$

As we can see from these examples, our bound is close to the Cheeger constant, but it is not sharp yet.

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