GAUSS-MANIN CONNECTIONS AND LIE-RINEHART COHOMOLOGY

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Abstract. In this note we prove existence of a generalized Gauss-Manin connection for a class of Ext-sheaves extending the construction in Katz paper [12]. We apply this construction to give a Gysin map for the K-groups of categories of coherent flat connections. We also give explicit examples of a generalized Gauss-Manin connection for a family of curves.

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Introduction

In previous papers (see [17] and [18]) a theory of characteristic classes with values in Lie-algebra cohomology was developed: For an arbitrary Lie-Rinehart algebra \( L \) and any \( A \)-module \( W \) with an \( L \)-connection there exists Chern-classes (see [17])

\[ c_i(W) \in \mathbb{H}^{2i}(L|U, \mathcal{O}_U) \]

where \( U \) is the open subset where \( W \) is locally free, generalizing the classical Chern-classes with values in de Rham-cohomology. Moreover there exists a Chern-character (see [18])

\[ ch : K(L) \to \mathbb{H}^*(L, A) \]

generalizing the classical Chern-character with values in de Rham-cohomology. The cohomology theory \( \mathbb{H}^*(L, A) \) is a cohomology theory simultaneously generalizing de Rham-cohomology of \( A \) and Lie-algebra cohomology of \( L \).

In this paper we consider the problem of constructing a Gauss-Manin connection for the cohomology group \( \mathbb{H}^*(L, W) \) generalizing the Gauss-Manin connection existing for de Rham-cohomology. Consider the category of flat \( L \)-connections where
$\mathcal{L}$ is a quasi-coherent sheaf of Lie-Rinehart algebras on an arbitrary scheme $X$. Let $f : X \rightarrow S \rightarrow T$ be arbitrary maps of schemes, and let

\[
\begin{array}{ccc}
\mathcal{L} & \rightarrow & \Theta_X \\
\downarrow f & & \downarrow df \\
{f^*}\mathcal{H} & \rightarrow & {f^*}\Theta_S
\end{array}
\]

be any surjective map of Lie-Rinehart algebras. Let $\mathcal{K} = ker(F)$. We construct a class of Ext-sheaves giving rise to functors

\[
\text{Ext}^i_{\mathcal{L}(\mathcal{K})}(-, -) : \mathcal{L} - \text{conn} \times \mathcal{L} - \text{conn} \rightarrow \mathcal{H} - \text{conn}.
\]

hence we get for any pair of flat $\mathcal{L}$-connections $\mathcal{V}, \mathcal{W}$ a canonical flat connection - the generalized Gauss-Manin connection:

\[
\nabla : \mathcal{H} \rightarrow \text{End}_S(\text{Ext}^*_\mathcal{L}(\mathcal{K})(\mathcal{V}, \mathcal{W})).
\]

As an application we get in Theorem 2.2 Gysin-maps

\[
F_i : K(\mathcal{L}) \rightarrow K(\mathcal{H})
\]

defined by

\[
F_i[W, \nabla] = \sum_{i \geq 0} (-1)^i [\text{Ext}^i_{\mathcal{L}(\mathcal{K})}(\mathcal{X}, \mathcal{W})]
\]

between K-groups of categories of coherent flat connections.

We also construct in Theorem 3.2 a class of explicit examples of generalized Gauss-Manin connections on a Lie-Rinehart algebra $H \subseteq T_C$

\[
\nabla_i : H \rightarrow \text{End}_R(H^i(L, W)).
\]

associated to a family of curves $X \rightarrow C$, where $X$ is a quotient surface singularity.

The aim of the construction is to use the Gysin-map to later on prove Riemann-Roch formulas for Chern-classes with values in Lie-Rinehart cohomology (see [15]).

1. **GAUSS-MANIN CONNECTIONS AND LIE-RINEHART COHOMOLOGY**

In this section we define Lie-Rinehart algebras, connections and generalized Lie-algebra cohomology and state some general properties of the constructions introduced. We also prove existence of a class of Ext-sheaves equipped with canonical flat connections - generalized Gauss-Manin connections.

First some definitions:

**Definition 1.1.** Let $A$ be an $S$-algebra where $S$ is a commutative ring with unit.

A *Lie-Rinehart algebra* on $A$ is a $S$-Lie-algebra and an $A$-module $L$ with a map $\alpha : L \rightarrow \text{Der}(A)$ satisfying the following properties:

\[
\begin{align*}
&\alpha(a\delta) = a\alpha(\delta) \\
&\alpha([\delta, \eta]) = [\alpha(\delta), \alpha(\eta)] \\
&[\delta, a\eta] = a[\delta, \eta] + \alpha(\delta)(a)\eta
\end{align*}
\]

for all $a \in A$ and $\delta, \eta \in L$. Let $W$ be an $A$-module. An *$L$-connection* $\nabla$ on $W$, is an $A$-linear map $\nabla : L \rightarrow \text{End}_S(W)$ which satisfies the *Leibniz-property*, i.e.

\[
\nabla(\delta)(aw) = a\nabla(\delta)(w) + \alpha(\delta)(a)w
\]
for all $a \in A$ and $w \in W$. We say that $(W, \nabla)$ is an $L$-module if $\nabla$ is a homomorphism of Lie-algebras. The curvature of the $L$-connection, $R_\nabla$ is defined as follows:

$$R_\nabla(\delta \wedge \eta) = [\nabla_\delta, \nabla_\eta] - \nabla_{[\delta, \eta]}.$$ 

A Lie-Rinehart algebra is also referred to as a Lie-Cartan pair, Lie-algebroid, Lie-pseudo algebra or an LR-algebra. The notion of a Lie-Rinehart algebra and connection globalize to sheaves, hence we get a definition for any scheme $X/S$: It is a pair $(\mathcal{L}, \alpha)$ where

$$\alpha : \mathcal{L} \to \Theta_X$$

is a map of quasi-coherent sheaves satisfying the properties from Definition 1.1 locally. An $\mathcal{L}$-connection is a map of quasi-coherent sheaves

$$\nabla : \mathcal{L} \to \text{End}_{\mathcal{O}_S}(W)$$

satisfying the criteria of Definition 1.1 locally.

Assume for the rest of the paper that only morphisms of schemes with well-defined map on tangent-spaces are considered.

We next define morphisms of LR-algebras (see [1] and [5]): Let $f : X \to S$ be a map of schemes. Let furthermore $\mathcal{L} \to \Theta_X$ and $\mathcal{H} \to \Theta_S$ be LR-algebras.

**Definition 1.2.** A morphism of LR-algebras, is a pair of maps $(F, f)$ giving a commutative diagram

$$
\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\alpha} & \Theta_X \\
\downarrow F & & \downarrow df \\
\text{f}^*\mathcal{H} & \xrightarrow{f^*\beta} & \text{f}^*\Theta_S
\end{array}
$$

Here $F$ is a map of sheaves satisfying

$$(1.2.1) \quad F([x, y]) = [F(x), F(y)] + x.F(y) - y.F(x)$$

where $F(x) = \sum a_i \otimes x_i$ and $F(y) = \sum b_j \otimes y_j$ are local sections and the following holds:

$$x.F(y) = \sum x(b_j) \otimes y_j,$$

$$[F(x), F(y)] = \sum a_i b_j \otimes [x_i, y_j].$$

Let $\mathcal{U}(\mathcal{L}/\mathcal{H}) = f^*\mathcal{U}(\mathcal{H})$ be the transfer-module of the morphism $F$. Then $\mathcal{U}(\mathcal{L}/\mathcal{H})$ has a canonical $\mathcal{L}$-connection.

**Lemma 1.3.** The canonical $\mathcal{L}$-connection on $\mathcal{U}(\mathcal{L}/\mathcal{H})$ is flat.

**Proof.** Straight forward. \qed

In fact given any map $F : \mathcal{L} \to f^*\mathcal{H}$ it is easy to see that it is a map of LR-algebras if and only if the canonical connection on $\mathcal{U}(\mathcal{L}/\mathcal{H})$ is flat: flatness is equivalent to the equation 1.2.1. We get thus a category of LR-algebras, denoted $\text{LR-alg}$. We define cohomology of LR-algebras as follows:
Definition 1.4. Let $L$ be a Lie-Rinehart algebra and $(W, \nabla)$ a flat $L$-connection. The **standard complex** $C^\bullet(L, W)$ is defined as follows:

\[ C^p(L, W) = \text{Hom}_A(L^\wedge p, W), \]

with differentials $d^p : C^p(L, W) \to C^{p+1}(L, W)$ defined by

\[
d^p \phi(g_1 \wedge \cdots \wedge g_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} g_i \phi(g_1 \wedge \cdots \wedge \overline{g}_i \wedge \cdots \wedge g_{p+1}) + \sum_{i<j} (-1)^{i+j} \phi([g_i, g_j] \wedge \cdots \wedge \overline{g}_i \wedge \cdots \wedge g_{p+1}).
\]

Here $g \phi(g_1 \wedge \cdots \wedge g_p) = \nabla(g) \phi(g_1 \wedge \cdots \wedge g_p)$ and overlined elements should be excluded. It follows that $C^\bullet(L, W)$ is a complex if and only if $\nabla$ is flat. The cohomology $H^i(C^\bullet(L, W))$ is denoted $H^i(L, W)$ - the Lie-Rinehart cohomology of $(W, \nabla)$.

In the literature the standard complex is also referred to as the Chevalley-Hochschild complex. There exists a generalized universal enveloping algebra $U(L)$ with the property that there is a one to one correspondence between flat $L$-connections and left $U(L)$-modules.

**Proposition 1.5.** Let $L$ be locally free. There is an isomorphism

\[ H^i(L, W) \cong \text{Ext}^i_{U(L)}(A, W) \]

for all $i \geq 0$.

**Proof.** See [8]. \qed

Let for the rest of this section $f : X \to S$ be an arbitrary map of schemes defined over a base scheme $T$. Let $L \to \Theta_X$ and $H \to \Theta_S$ be LR-algebras, and let $F : L \to f^*H$ be a surjective map of LR-algebras with associated exact sequence

\[(1.5.1) \quad 0 \to K \to L \to F_B \otimes_A H \to 0.\]

**Proposition 1.6.** There is a functor

\[ F_l : \mathcal{L} - \text{conn} \to \mathcal{H} - \text{conn} \]

defined by

\[ F_l(W, \nabla) = (f_*(W^\nabla), \overline{\nabla}) \]

where $W^\nabla$ is the sheaf of horizontal sections with respect to the induced connection on $K$ and $\overline{\nabla}$ is a flat $H$-connection.

**Proof.** We assume $X = \text{Spec}(B), S = \text{Spec}(A)$ and $T = \text{Spec}(R)$ The exact sequence in 1.5.1 becomes

\[ 0 \to K \to L \to F_B \otimes_A H \to 0. \]

Let $W$ be a module with a flat $L$-connection $\nabla$ and consider $W^\nabla \subseteq W$ the sub-$A$-module of sections with $\nabla(y)(w) = 0$ for all $y \in K$. We seek to define a canonical flat $H$-connection

\[ \overline{\nabla} : H \to \text{End}_R(W^\nabla). \]

Let $x \in H$ and pick an element $y \in L$ with $F(y) = 1 \otimes x$. Define $\overline{\nabla}(x)(w) = \nabla(y)(w)$. This is well defined since $w \in W^\nabla$. We want to show we get an induced map

\[ \overline{\nabla}(x) : W^\nabla \to W^\nabla. \]
Pick \( h \in K \). We get
\[
\nabla(h) \nabla(y)(w) = \nabla(y) \nabla(h)(w) + \nabla([h, y])(w) = \nabla([h, y])(w).
\]
Since
\[
F[h, y] = [Fh, Fy] + h.F(y) - y.F(h) = h(1 \otimes x) = h(1) \otimes x = 0
\]
it follows \( [h, y] \in K \) hence \( \nabla([h, y])(w) = 0 \) and we have shown we get an induced map
\[
\nabla(x) : W^\nabla \to W^\nabla
\]
hence we get an induced flat connection
\[
\nabla : H \to \text{End}_R(W^\nabla).
\]
This argument globalize and the claim is proved.

We get thus in the situation where \( X \) and \( S \) are arbitrary schemes and \( F : \mathcal{L} \to f^*\mathcal{H} \) surjective map of LR-algebras a left exact functor
\[
F_! : \underline{\text{conn}} \to \underline{\text{conn}},
\]
defined by
\[
F_!(W, \nabla) = (W^\nabla, \nabla).
\]
In the case when \( \mathcal{L} = \Theta_X, \mathcal{H} = \Theta_S \) the functor \( F_! \) equals the functor introduced by Katz in [12].

Consider the LR-algebra \( \mathbb{K} \to \Theta_X \) with the obvious anchor map. We may consider its generalized sheaf of enveloping algebras \( U(\mathbb{K}) \) which is a sheaf of \( \mathcal{O}_X \)-algebras, and a flat \( \mathcal{L} \)-connection \( (\mathcal{W}, \nabla) \) gives in a natural way a flat \( \mathbb{K} \)-connection. We get in a natural way a left \( U(\mathbb{K}) \)-modulestructure on \( \mathcal{W} \). We may consider the category of sheaves of left \( U(\mathbb{K}) \)-modules witch has enough injectives. We can thus apply the machinery of derived functors. Let \( \mathcal{U}, \mathcal{V} \) be two left \( U(\mathbb{L}) \)-modules and consider the sheaf \( \text{Hom}_X(\mathcal{U}, \mathcal{V}) \) which in a natural way is a left \( U(\mathbb{K}) \)-module. There is by definition an equality
\[
\text{Hom}_X(\mathcal{U}, \mathcal{V})^\nabla = \text{Hom}_{U(\mathbb{K})}(\mathcal{U}, \mathcal{V})
\]
hence by Proposition 1.6 it follows that \( \text{Hom}_{U(\mathbb{K})}(\mathcal{U}, \mathcal{V}) \) is a left \( \mathcal{U}(\mathcal{H}) \)-module, hence we get a left exact functor
\[
\text{Hom}_{U(\mathbb{K})}(\mathcal{U}, -) : \underline{\text{conn}} \to \underline{\text{conn}}.
\]

**Definition 1.7.** Let \( \mathcal{U}, \mathcal{V} \) be left \( U(\mathbb{L}) \)-modules. We define for all \( i \geq 0 \) the cohomology-sheaves
\[
\text{Ext}_{U(\mathbb{K})}^i(\mathcal{U}, \mathcal{V}) = R^i \text{Hom}_{U(\mathbb{K})}(\mathcal{U}, \mathcal{V}).
\]
We write
\[
\text{Ext}_{U(\mathbb{K})}^*(\mathcal{U}, \mathcal{V}) = \oplus_{i \geq 0} \text{Ext}_{U(\mathbb{K})}^i(\mathcal{U}, \mathcal{V}).
\]
By construction there is for all \( i \geq 0 \) a canonical flat connection
\[
\nabla_i : \mathcal{H} \to \text{End}_{\mathcal{O}_T}(\text{Ext}_{U(\mathbb{K})}^i(\mathcal{U}, \mathcal{V})).
\]
We get thus a flat connection
\[
\nabla : \mathcal{H} \to \text{End}_{\mathcal{O}_T}(\text{Ext}_{U(\mathbb{K})}^*(\mathcal{U}, \mathcal{V})),
\]
called the generalized Gauss-Manin connection. We write
\[ \text{Ext}^i_{U(K)}(X, V) = \text{Ext}^i_{U(K)}(O_X, V). \]

**Proposition 1.8.** Assume \( U = \text{Spec}(A), V = \text{Spec}(S) \) are open affine subsets of \( X \) and \( S \) with \( f(U) \subseteq V \) and \( K \) a locally free LR-algebra. There is an isomorphism
\[ \text{Ext}^i_{U(K)}(X, W)|_U \cong H^i(K, W) \]
where \( K|_U = K \) and \( W|_U = W \).

**Proof.** This is straightforward: By Proposition 1.5 we get isomorphisms
\[ \text{Ext}^i_{U(K)}(X, W)|_U = R^i \text{Hom}_{U(K)}(O_X, W)|_U = R^i \text{Hom}_{U(K)}(A, W) = \text{Ext}^i_{U(K)}(A, W) = H^i(K, W) \]
and the claim follows. \( \square \)

Hence the cohomology-sheaves \( \text{Ext}^i_{U(K)}(X, W) \) may be calculated using the standard complex when \( U \) is locally free. We get in a natural way flat \( H \)-connections \( \nabla_i = R^i F(\nabla) \) on the cohomology sheaves \( \text{Ext}^i_{U(K)}(X, W) \) - generalized Gauss-Manin connections - for all \( i \geq 0 \).

Notice that the quasi-coherent \( O_S \)-modules
\[ \text{Ext}^i_{U(K)}(X, W) \]
specialize to \( \text{Ext}^i_{U(L)}(X, W) \) when \( S = T \). The cohomology sheaves \( \text{Ext}^i_{U(K)}(X, W) \) generalize simultaneously several cohomology theories as mentioned in [17]. In particular if \( L = \Theta_X, H = \Theta_S \) there are isomorphisms
\[ \text{Ext}^i_{U(K)}(X, W) \cong H^i(X/S, W, \nabla) \]
where \( H^i(X/S, W, \nabla) \) is the de Rham-cohomology sheaf introduced in 3.1 [12]. The generalized Gauss-Manin connection introduced in this section specialize to the Gauss-Manin connection introduced in [12] and [14].

**Example 1.9.** Let \( f : X = \text{Spec}(B) \rightarrow S = \text{Spec}(R) \) be a map of schemes and let \( W \) be a \( B \)-module with a flat connection
\[ \nabla : T_X \rightarrow \text{End}_R(W). \]
Assume \( L \subseteq T_X \) is a locally free LR-algebra, and consider the sub-R-algebra
\[ B^L \subseteq B. \]
We get a map of schemes
\[ \pi : X \rightarrow X/L = \text{Spec}(B^L) \rightarrow S \]
and an associated exact sequence
\[ 0 \rightarrow \text{Der}_{B^L}(B) \rightarrow \text{Der}_R(B) \rightarrow B \otimes_{B^L} \text{Der}_R(B^L) \]
Assume \( H \subseteq \text{Der}_R(B^L) \) is a sub-Lie algebra with \( T_X \rightarrow \pi^* H \) surjective. We get thus a right exact sequence
\[ 0 \rightarrow L \rightarrow T_X \rightarrow \pi^* H \rightarrow 0 \]
hence we get a functor
\[ F_1 : T_X - \text{conn} \rightarrow H - \text{conn} \]
as defined above. We get thus for all locally free Lie-algebras $L$ canonical flat connections on the $A^L$-modules $H^i(L,W)$
\[ \nabla_i : H \to \text{End}_R(H^i(L,W)) \]
were $(W, \nabla)$ is a flat $T_X$-connection.

We end this section with an analogue of a proposition found in [12].

**Proposition 1.10.** Suppose $X$ is proper and smooth over $S$ and $k$ is a field of characteristic zero. Then for any flat $T_X$-connection $W$ the sheaf $\text{Ext}^i_{(T_f)}(X, W)$ is a locally free finite rank $O_S$-module.

**Proof.** We have an isomorphism
\[ \text{Ext}^i_{(T_f)}(X, W) \cong H^i(X/S, W, \nabla) \]
hence since by properness $\text{Ext}^i_{(T_f)}(X, W)$ is a coherent $O_S$-module the statement follows from Proposition 8.8 [12]. □

Note: many constructions on the Ext-functor introduced in [3] globalize to give similar results (Kunneth-formulas etc) for the Ext-functor introduced in this section. There is similarly defined a Tor-functor (up to non-canonical isomorphism) and constructions introduced in [3] globalize to give change of rings-spectral sequences.

**2. Application: A Gysin map for $K$-theory of flat connections**

In this section we prove existence of a Gysin-map map between $K$-groups of categories of flat connections.

Let in the following $\mathcal{C}, \mathcal{D}$ be two arbitrary abelian categories with $\mathcal{C}$ having enough injectives and let $F : \mathcal{C} \to \mathcal{D}$ be a left exact functor with the property that $R^i F(C) = 0$ for all objects $C$ with $i >> 0$. We may thus construct the right derived functors $R^i F$ of $F$ (see [7]) having the property that for any exact sequence
\[ 0 \to C' \to C \to C'' \to 0 \]
in $\mathcal{C}$, we get a long exact sequence of objects in $\mathcal{D}$:
\[ 0 \to F(C') \to F(C) \to F(C'') \to R^1 F(C') \to R^1 F(C) \to R^1 F(C'') \to \cdots \]
It follows we get an induced push-forward map at the level of $K$-groups as follows: Let $K(\mathcal{C})$ and $K(\mathcal{D})$ be the $K$-group of the categories $\mathcal{C}$ and $\mathcal{D}$, ie $K(\mathcal{C})$ is the free abelian group on the isomorphism classes in $\mathcal{C}$ modulo the subgroup generated by elements of the type $[C] - [C'] - [C'']$ arising from an exact sequence in $\mathcal{C}$ as usual.

**Definition 2.1.** Let $C$ be an object of $\mathcal{C}$. Make the following definition:
\[ F_1[C] = \sum_{i \geq 0} (-1)^i [R^i F(C)] . \]

**Proposition 2.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a left exact functor. The map $F_1$ induces a well-defined map
\[ F_1 : K(\mathcal{C}) \to K(\mathcal{D}) \]
at the level of $K$-groups.
Proof. This is straightforward. □

Let \( K(X) \) (resp. \( K(S) \)) be the K-group of the category of coherent sheaves on a scheme \( X \) (resp. \( S \)) and let furthermore \( f : X \to S \) be a proper map of schemes.

**Corollary 2.3.** If \( F = f_* \) is the pushforward map arising from a map of schemes \( f : X \to S \) the induced map \( F_! : K(X) \to K(S) \),
defined by

\[
F_! [\mathcal{E}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{E}].
\]

**Proof.** See [7]. □

Let \((F, f) : (\mathcal{L}, X) \to (\mathcal{H}, S)\) be a surjective morphism of LR-algebras, where \( f : X \to S \) is a map of schemes over a scheme \( T \). Let furthermore \( \text{Ext}^i_{H(K)}(X, -) : \mathcal{L} - \text{conn} \to \mathcal{H} - \text{conn} \)
be the Ext-sheaves from Definition 1.7.

We get thus for any exact sequence of flat connections in \( \mathcal{L} - \text{conn} \) a long exact sequence in \( \mathcal{H} - \text{conn} \), hence we get an induced map at level of K-theory: Let \( K(L) \) (resp. \( K(H) \)) be the K-groups of the categories of coherent sheaves of flat \( \mathcal{L} \)-connections (resp. \( \mathcal{H} \)-connections). We can prove existence of a Gysin-map between these K-groups. Assume for simplicity for the rest of this section that \( \text{Ext}^i_{H(K)}(X, W) = 0 \) for all \( W \) and all \( i >> 0 \) (this holds for instance in the situation considered in Proposition 1.10).

**Theorem 2.4.** There is a Gysin map

\[
F_! : K(L) \to K(H)
\]
defined as follows:

\[
F_! [W, \nabla] = \sum_{i \geq 0} (-1)^i [\text{Ext}^i_{H(K)}(X, W)].
\]

**Proof.** The claim of the theorem follows from general properties of the Ext-functor, Proposition 2.2 as follows: For an exact sequence

\[
0 \to (W', \nabla') \to (W, \nabla) \to (W'', \nabla'') \to 0
\]
of horizontal maps of flat \( \mathcal{L} \)-connections, we always get by the functorial properties of Ext a long exact sequence

\[
0 \to \text{Ext}^0_{H(K)}(X, W') \to \text{Ext}^0_{H(K)}(X, W) \to \text{Ext}^0_{H(K)}(X, W'') \to \\
\text{Ext}^1_{H(K)}(X, W') \to \text{Ext}^1_{H(K)}(X, W) \to \text{Ext}^1_{H(K)}(X, W'') \to \cdots
\]
of \( \mathcal{H} \)-connections hence the map is well-defined. □

Assume \( X = \text{Spec}(B), S = \text{Spec}(A) \) and \( K = K \) where \( K \) is locally free. We get in this case a Gysin map

\[
F_! : K(L) \to K(H)
\]
defined by

\[
F_! [W, \nabla] = \sum_{i \geq 0} (-1)^i [\text{H}^i(K, W)].
\]
hence the standard complex calculates the Gysin-map in the case when $K$ is locally free.

In the special case when $L = \Theta_X$ and $H = \Theta_S$ we get a corollary:

**Corollary 2.5.** Assume the base field is of characteristic zero and that $X$ is proper over $S$. There is a Gysin-map

$$F_! : K(\Theta_X) \to K(\Theta_S)$$

defined by

$$F_!(W, \nabla) = \sum_{i \geq 0} (-1)^i H^i(X/S, W, \nabla),$$

where $H^i(X/S, W, \nabla)$ is the group introduced in 3.1.[12].

**Proof.** This follows from Proposition 1.10. □

There is ongoing work proving a Riemann-Roch formula using the Gysin-map introduced in this note and a refined version of the Chern-character introduced in [18].

### 3. Explicit examples: A Gauss-Manin connection for a family of curves

We consider examples of Gauss-Manin connections on a Lie-Rinehart algebra $H \subseteq T_C$ on a family of curves $X \to C$, using connections constructed in previous papers on the subject (see [16] and [17] for details on the calculations).

Consider the polynomial $B_{mn2} = x^m + y^n + z^2$ and let $A = \mathbb{C}[x, y, z]/(x^m + y^n + z^2)$ where $\mathbb{C}$ is the complex numbers. By the results of [17] we get a family of maximal Cohen-Macaulay modules on $A$ given by the two matrices

$$\phi = \begin{pmatrix} x^{m-k} & y^{n-l} & 0 & z \\ y^l & -x^k & z & 0 \\ z & 0 & -y^{n-l} & -x^k \\ 0 & z & x^{m-k} & -y^l \end{pmatrix}$$

and

$$\psi = \begin{pmatrix} x^k & y^{n-l} & z & 0 \\ y^l & -x^{m-k} & 0 & z \\ 0 & z & -y^l & x^k \\ z & 0 & -x^{m-k} & -y^{n-l} \end{pmatrix},$$

where $1 \leq k \leq m$ and $1 \leq l \leq n$. The matrices $\phi$ and $\psi$ give rise to an exact sequence of $A$-modules

(3.0.1) \[ \cdots \to \psi P \to \phi P \to \psi P \to \phi P \to W(\phi, \psi) \to 0. \]

And by the results in [6] the module $W = W(\phi, \psi)$ is a maximal Cohen-Macaulay module on $A$. The pair $(\phi, \psi)$ is a matrix-factorization of the polynomial $f$. In [17] we compute explicitly algebraic $V_W$-connections on the modules $W = W(\phi, \psi)$ for all $1 \leq k \leq m$ and $1 \leq l \leq n$. The calculation is as follows: We compute generators and syzygies of the derivation-modules $\text{Der}_C(A)$ and find it is generated by the derivations

$$\delta_0 = 2nx\partial_x + 2my\partial_y + mnz\partial_z,$$

$$\delta_1 = mx^{m-1}\partial_y - ny^{n-1}\partial_z,$$

$$\delta_2 = -2z\partial_x + mx^{m-1}\partial_z.$$
\( \delta_3 = -2z\partial_y + ny^{n-1}\partial_z. \)

The syzygy-matrix of \( \operatorname{Der}_C(A) \) is the following matrix

\[
\rho = \begin{pmatrix}
 y^{n-1} & z & 0 & x^{m-1} \\
 2x & 0 & -2z & -2y \\
 0 & nx & ny^{n-1} & -nz \\
 -mx & my & -m x^{m-1} & 0
\end{pmatrix}.
\]

A calculation using the Kodaira-Spencer map \( g : \operatorname{Der}_C(A) \to \operatorname{Ext}_A^1(W,W) \) gives the following connection:

\[
\begin{align*}
(3.0.2) & \quad \nabla_{\delta_0} = \delta_0 + A_0 = \\
& = \delta_0 + \begin{pmatrix}
 (nk + ml - \frac{1}{2} mn) & 0 & 0 & 0 \\
 0 & \frac{3}{2} mn - ml - nk & 0 & 0 \\
 0 & 0 & \frac{1}{2} mn + ml - nk & 0 \\
 0 & 0 & 0 & \frac{1}{2} mn + nk - ml
\end{pmatrix},
\end{align*}
\]

\[
(3.0.3) & \quad \nabla_{\delta_1} = \delta_1 + \begin{pmatrix}
 0 & b_2 & 0 & 0 \\
 b_1 & 0 & 0 & 0 \\
 0 & 0 & b_4 & 0 \\
 0 & 0 & b_3 & 0
\end{pmatrix} = \delta_1 + A_1,
\]

with \( b_1 = \frac{1}{4}(mn - 2nk - 2ml)x^{k-1}y^{l-1}, b_2 = \frac{1}{4}(3mn - 2ml - 2nk)x^{m-k-1}y^{n-l-1}, b_3 = \frac{1}{4}(2nk - mn - 2ml)x^{m-k-1}y^{l-1} \) and \( b_4 = \frac{1}{4}(2nk - 2ml + mn)x^{k-1}y^{n-l-1}. \)

\[
(3.0.4) & \quad \nabla_{\delta_2} = \delta_2 + \begin{pmatrix}
 0 & 0 & c_3 & 0 \\
 0 & 0 & 0 & c_4 \\
 c_3 & 0 & 0 & 0 \\
 c_2 & 0 & 0 & 0
\end{pmatrix} = \delta_2 + A_2,
\]

with \( c_1 = \frac{1}{4}(mn - 2nk - 2ml)x^{k-1}, c_2 = \frac{1}{4}(mn - 2ml - nk)x^{m-k-1}, c_3 = \frac{1}{4}(mn + ml - nk)x^{m-k-1} \) and \( c_4 = \frac{1}{4}(mn - 2nk - mn)x^{k-1}. \)

\[
(3.0.5) & \quad \nabla_{\delta_3} = \delta_3 + \begin{pmatrix}
 0 & 0 & 0 & d_4 \\
 0 & 0 & d_3 & 0 \\
 0 & d_2 & 0 & 0 \\
 d_1 & 0 & 0 & 0
\end{pmatrix} = \delta_3 + A_3,
\]

where \( d_1 = \frac{1}{4}(\frac{1}{2} mn - ml - nk)y^{l-1}, d_2 = \frac{1}{4}(ml + nk - \frac{3}{2} mn)y^{n-l-1}, d_3 = \frac{1}{4}(\frac{1}{2} mn + ml - nk)y^{l-1} \) and \( d_4 = \frac{1}{4}(\frac{1}{2} mn + ml + nk)y^{n-l-1}. \)

**Theorem 3.1.** For all \( 1 \leq k \leq m \) and \( 1 \leq l \leq n \) the equations 3.0.2-3.0.5 define algebraic \( \nabla_W : \nabla_W^{\phi, \psi} : V_W \to \operatorname{End}_C(W) \) where \( W = W(\phi, \psi). \)

**Proof.** See Theorem 2.1 in [17] \( \square \)

Note that the vectorfield \( \delta_0 \) - the Euler derivation - generates a locally free Lie-Rinehart algebra \( L \subseteq T_X. \) Let \( X = \operatorname{Spec}(A) \) and \( C = \operatorname{Spec}(A^L) \) with associated map of schemes

\( \pi : X \to C. \)

From Example 1.9 we get a right exact sequence

\[ L \to T_X \to \pi^* H \to 0 \]
where $H \subseteq T_C$ is a sub-Lie algebra, hence we get a functor

$$T_X - \text{conn} \to H - \text{conn}.$$  

We get thus connections

$$\nabla_i : H \to \text{End}(H^i(L,W))$$

for all $i \geq 0$.

**Theorem 3.2.** There exists for all $1 \leq k \leq m$ and $1 \leq l \leq n$ a Gauss-Manin connection

$$\nabla_i : H \to \text{End}(H^i(L,W))$$

for all $i \geq 0$.

**Proof.** This is immediate from the discussion above and the results in section one of this paper. \[\square\]

We illustrate the theory with some basic examples.

**Example 3.3.** Consider the following situation: Let $F = \mathbb{C}$ be the field of complex numbers, $A = F[x,y]$ and $X = \text{Spec}(A)$. Let $L$ be the Lie-algebra generated by $\partial_x + \partial_y \in \text{Der}_F(A)$ and $S = \text{Spec}(A^L) = \text{Spec}(F[x-y]) = \text{Spec}(F[T])$. We have a map of schemes $\pi : X \to S$ and there is an exact sequence

$$0 \to L \to T_X \to \pi^*T_S \to 0$$

of sheaves. Let $E$ be the free $A$-module of rank two. Then $E$ has a flat $T_X$-connection

$$\nabla : T_X \to \text{End}_F(E).$$

A calculation shows that $H^0(L,E)$ is the $F[T]$-module

$$E^L = A^L \oplus A^L = F[T] \oplus F[T]$$

and $H^1(L,E) = 0$. The Gauss-Manin-connection on

$$H^0(L,E) = E^L = F[T] \oplus F[T]$$

is the connection

$$\nabla : T_S \to \text{End}_F(H^0(L,E))$$

given by

$$\nabla(\partial)(a,b) = (\partial(a), \partial(b))$$

where $\partial = \frac{\partial}{\partial T}$.

**Example 3.4.** Consider the map

$$\pi : S \to C$$

where $S = \text{Spec}(F[x,y])$ and $C = \text{Spec}(F[t])$ defined by

$$\pi(t) = x^m - y^n = f(x,y).$$

We get a relative tangent sequence

$$0 \to T(\pi) \to T_S \to \pi^*T_C$$

which is not right exact, since the map $\pi$ is not smooth. We calculate, and find $T(\pi)$ to be defined by the vector field

$$\partial = mx^{m-1}\partial_x + ny^{n-1}\partial_y.$$
The vector field $\partial$ and the Euler-vectorfield $E = nx\partial_x + my\partial_y$
defines a Lie-sub-algebra
$$L = \{\partial, E\} \subseteq \text{Der}_F(B)$$
where $B = F[x, y]$, and we check we get an exact sequence
$$0 \rightarrow T(\pi) \rightarrow L \rightarrow \pi^* H \rightarrow 0$$
where $H = mnt\partial_t \subseteq F[t]\partial_t = T_C$. We get thus for any flat $L$-connection $W$ a flat $H$-connection
$$\nabla : H \rightarrow \text{End}_F(H^i(T(\pi), W)),$$
a logarithmic Gauss-Manin connection. The singular fibre $\pi^{-1}(0) = \text{Spec}(F[x, y]/x^m - y^n)$ of the map $\pi$ is defined by the ideal $I = (f) = (x^m - y^n)$, and there is an obvious flat $T(\pi)$-connection
$$\nabla : T(\pi) \rightarrow \text{End}_F(I)$$
defined by
$$\nabla(x)(b) = x(b).$$
We thus get natural flat $H$-connections
$$\nabla_i : H \rightarrow \text{End}_F(H^i(T(\pi), I))$$
for $i = 0, 1$, and we aim to calculate $\nabla_i$. A calculation shows that
$$H^0(T(\pi), I) = I^{T(\pi)} = F[t].$$
The standard complex $C^p(T(\pi), I)$ looks as follows:
$$I \rightarrow \text{Hom}_B(T(\pi), I)$$
where $B = F[x, y]$. It follows that $T(\pi) = B\partial$ and $I = (f)$. Hence
$$\text{Hom}_B(T(\pi), I) = Bf.$$
We get thus the complex
$$\partial : Bf \rightarrow Bf$$
and
$$H^1(T(\pi), I) = Bf/\text{Im}(\partial).$$
A calculation shows that if $m \leq n$
$$\dim(H^1(T(\pi), I)) = \frac{(n-2)(m-1)}{2}.$$
There is ongoing work implementing computer algorithms for calculations of Gauss-Manin connections.

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GAUSS-MANIN CONNECTIONS AND LIE-RINEHART COHOMOLOGY

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