2. Adelic constructions for direct images of differentials and symbols

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2.0. Introduction

Let $X$ be a smooth algebraic surface over a perfect field $k$.

Consider pairs $x \in C$, $x$ is a closed point of $X$, $C$ is either an irreducible curve on $X$ which is smooth at $x$, or an irreducible analytic branch near $x$ of an irreducible curve on $X$. As in the previous section 1 for every such pair $x \in C$ we get a two-dimensional local field $K_{x,C}$.

If $X$ is a projective surface, then from the adelic description of Serre duality on $X$ there is a local decomposition for the trace map $H^2(X, \Omega^2_X) \to k$ by using a two-dimensional residue map $\text{res}_{K_{x,C}/k(x)}: \Omega^2_{K_{x,C}/k(x)} \to k(x)$ (see [P1]).

From the adelic interpretation of the divisors intersection index on $X$ there is a similar local decomposition for the global degree map from the group $CH^2(X)$ of algebraic cycles of codimension 2 on $X$ modulo the rational equivalence to $\mathbb{Z}$ by means of explicit maps from $K_2(K_{x,C})$ to $\mathbb{Z}$ (see [P3]).

Now we pass to the relative situation. Further assume that $X$ is any smooth surface, but there are a smooth curve $S$ over $k$ and a smooth projective morphism $f: X \to S$ with connected fibres. Using two-dimensional local fields and explicit maps we describe in this section a local decomposition for the maps

$$f_*: H^n(X, \Omega^2_X) \to H^{n-1}(S, \Omega^1_S), \quad f_*: H^n(X, \mathcal{K}_2(X)) \to H^{n-1}(S, \mathcal{K}_1(S))$$

where $\mathcal{K}$ is the Zariski sheaf associated to the presheaf $U \to K(U)$. The last two groups have the following geometric interpretation:

$$H^n(X, \mathcal{K}_2(X)) = CH^2(X, 2 - n), \quad H^{n-1}(S, \mathcal{K}_1(S)) = CH^1(S, 2 - n)$$

where $CH^2(X, 2 - n)$ and $CH^1(S, 1 - n)$ are higher Chow groups on $X$ and $S$ (see [B1]). Note also that $CH^2(X, 0) = CH^2(X)$, $CH^1(S, 0) = CH^1(S) = \text{Pic}(S)$, $CH^1(S, 1) = H^0(S, \mathcal{O}_S^\ast)$.
Let $s = f(x) \in S$. There is a canonical embedding $f^*: K_s \to K_{x,C}$ where $K_s$ is the quotient of the completion of the local ring of $S$ at $s$.

Consider two cases:

1. $C \neq f^{-1}(s)$. Then $K_{x,C}$ is non-canonically isomorphic to $k(C)_x((t_C))$ where $k(C)_x$ is the completion of $k(C)$ at $x$ and $t_C$ is a local equation of $C$ near $x$.

2. $C = f^{-1}(s)$. Then $K_{x,C}$ is non-canonically isomorphic to $k(x)((u))((t_s))$ where $\{u = 0\}$ is a transversal curve at $x$ to $f^{-1}(s)$ and $t_s \in K_s$ is a local parameter at $s$, i.e. $k(s)((t_s)) = K_s$.

### 2.1. Local constructions for differentials

**Definition.** For $K = k((u))((t))$ let $U = u^i k[[u,t]][d k[[u,t]] + t^i k((u))][[t]]$ be a basis of neighbourhoods of zero in $\Omega^1_{k((u))[[t]]/k}$ (compare with 1.4.1 of Part I). Let $\Omega^1_{K/K} \equiv \Omega^1_{K/k} / (K \cdot \cap U)$ and $\hat{\Omega}^n_{K} = \wedge^n \hat{\Omega}^1_{K}$. Similarly define $\hat{\Omega}^n_{K_s}$. Note that $\hat{\Omega}^2_{K_s, C}$ is a one-dimensional space over $K_{x,C}$; and $\hat{\Omega}^n_{K_s, C}$ does not depend on the choice of a system of local parameters of $\hat{\Omega}_x$, where $\hat{\Omega}_x$ is the completion of the local ring of $X$ at $x$.

**Definition.** For $K = k((u))((t))$ and $\omega = \sum_i \omega_i(u) \wedge t^i dt = \sum_i u^i du \wedge \omega'_i(t) \in \hat{\Omega}^2_{K}$ put

$$\text{res}_t(\omega) = \omega_{-1}(u) \in \hat{\Omega}^1_{k((u))},$$

$$\text{res}_u(\omega) = \omega'_{-1}(t) \in \hat{\Omega}^1_{k((t))}.$$  

Define a relative residue map

$$f^x_{*}: \hat{\Omega}^2_{K_s, C} \to \hat{\Omega}^1_{K_s}$$

as

$$f^x_{*}(\omega) = \begin{cases} 
\text{Tr}_{k(C)_x/K_s} \text{res}_{t_C}(\omega) & \text{if } C \neq f^{-1}(s) \\
\text{Tr}_{k(x)((t_s))/K_s} \text{res}_{u}(\omega) & \text{if } C = f^{-1}(s).
\end{cases}$$

The relative residue map doesn’t depend on the choice of local parameters.

**Theorem** (reciprocity laws for relative residues). Fix $x \in X$. Let $\omega \in \hat{\Omega}^2_{K_s}$ where $K_x$ is the minimal subring of $K_{x,C}$ which contains $k(X)$ and $\hat{\Omega}_x$. Then

$$\sum_{C \supset x} f^x_{*}(\omega) = 0.$$
Fix \( s \in S \). Let \( \omega \in \widetilde{\Omega}_K^2 \) where \( K_F \) is the completion of \( k(X) \) with respect to the discrete valuation associated with the curve \( F = f^{-1}(s) \). Then
\[
\sum_{x \in F} f_{x,F}^*(\omega) = 0.
\]

See [O].

### 2.2. The Gysin map for differentials

**Definition.** In the notations of subsection 1.2.1 in the previous section put

\[
\Omega^1_{k,S} = \{(f_s dt_s) \in \prod_{s \in S} \tilde{\Omega}_K^1, \quad v_s(f_s) \geq 0 \text{ for almost all } s \in S \}
\]

where \( t_s \) is a local parameter at \( s \), \( v_s \) is the discrete valuation associated to \( t_s \) and \( K_s \) is the quotient of the completion of the local ring of \( S \) at \( s \). For a divisor \( I \) on \( S \) define

\[
\Omega^1_{k,S}(I) = \{(f_s) \in \Omega^1_{k,S} : v_s(f_s) \geq -v_s(I) \text{ for all } s \in S \}.
\]

Recall that the \( n \)-th cohomology group of the following complex

\[
\Omega^1_{k(S)/k} \oplus \Omega^1_{k,S}(0) \rightarrow \Omega^1_{k,S} \rightarrow f_0 + f_1.
\]

is canonically isomorphic to \( H^n(S, \Omega^1_S) \) (see [S, Ch.II]).

The sheaf \( \Omega^2_X \) is invertible on \( X \). Therefore, Parshin’s theorem (see [P1]) shows that similarly to the previous definition and definition in 1.2.2 of the previous section for the complex \( \Omega^2(A_X) \)

\[
\Omega^2_{A_0} \oplus \Omega^2_{A_1} \oplus \Omega^2_{A_2} \rightarrow \Omega^2_{A_{01}} \oplus \Omega^2_{A_{02}} \oplus \Omega^2_{A_{12}} \rightarrow \Omega^2_{A_{012}}
\]

\[
(f_0, f_1, f_2) \rightarrow (f_0 + f_1, f_2 - f_0, -f_1 - f_2) \quad (g_1, g_2, g_3) \rightarrow g_1 + g_2 + g_3
\]

where

\[
\Omega^2_{A_i} \subset \Omega^2_{A_{ij}} \subset \Omega^2_{A_{012}} = \Omega^2_{A_X} = \prod_{x \in C} \tilde{\Omega}^2_{K_{x,c}} \subset \prod_{x \in C} \tilde{\Omega}^2_{K_{x,c}}
\]

there is a canonical isomorphism

\[
H^n(\Omega^2(A_X)) \simeq H^n(X, \Omega^2_X).
\]

Using the reciprocity laws above one can deduce:
Theorem. The map $f_\ast = \sum_{C \ni x, f(x) = s}^\ast f_\ast^{x,C}$ from $\Omega^2_{A_X}$ to $\Omega^1_{A_S}$ is well defined. It maps the complex $\Omega^2(A_X)$ to the complex

$$0 \to \Omega^1_{k(S)/k} \oplus \Omega^1_{k_S}(0) \to \Omega^1_{k_S}.$$ 

It induces the map $f_\ast : H^n(X, \Omega^2_X) \to H^{n-1}(S, \Omega^1_S)$ of 2.0.

See [O].

2.3. Local constructions for symbols

Assume that $k$ is of characteristic 0.

**Theorem.** There is an explicitly defined symbolic map

$f_\ast( , , x,C) : K^*_x,C \times K^*_x,C \to K^*_S$

(see remark below) which is uniquely determined by the following properties

$$N_{k(x)/k(s)} t_{K_x,C}(\alpha, \beta, f^*(\gamma)) = t_{K_s}(f_\ast(\alpha, \beta)_x,C, \gamma) \quad \text{for all } \alpha, \beta \in K^*_x,C, \gamma \in K^*_s$$

where $t_{K_x,C}$ is the tame symbol of the two-dimensional local field $K_x,C$ and $t_{K_s}$ is the tame symbol of the one-dimensional local field $K_s$ (see 6.4.2 of Part I);

$$\text{Tr}_{k(x)/k(s)}(\alpha, \beta, f^*(\gamma))|_{K_x,C} = (f_\ast(\alpha, \beta)_x,C, \gamma)|_{K_s} \quad \text{for all } \alpha, \beta \in K^*_x,C, \gamma \in K_s$$

where $(\alpha, \beta, \gamma)_{K_x,C} = \text{res}_{K_x,C/k(x)}(\gamma d\alpha/\alpha \wedge d\beta/\beta)$ and

$(\alpha, \beta)_{K_s} = \text{res}_{K_s/k(s)}(\alpha d\beta/\beta)$.

The map $f_\ast( , , x,C)$ induces the map

$f_\ast( , , x,C) : K_2(K_x,C) \to K_1(K_s)$.

**Corollary (reciprocity laws).** Fix a point $s \in S$. Let $F = f^{-1}(s)$.

Let $\alpha, \beta \in K^*_F$. Then

$$\prod_{x \in F} f_\ast(\alpha, \beta)_x,F = 1.$$ 

Fix a point $x \in F$. Let $\alpha, \beta \in K^*_x$. Then

$$\prod_{C \ni x} f_\ast(\alpha, \beta)_x,C = 1.$$ 

**Remark.** If $C \neq f^{-1}(s)$ then $f_\ast( , , , x,C) = N_{k(C)/k_x} t_{K_x,C}$ where $t_{K_x,C}$ is the tame symbol with respect to the discrete valuation of rank 1 on $K_x,C$.

If $C = f^{-1}(s)$ then $f_\ast( , , , x,C) = N_{k(x)/k(s)}( , , )_f$ where $( , )_f^{-1}$ coincides with Kato’s residue homomorphism [K, §1]. An explicit formula for $( , )_f$ is constructed in [O, Th.2].
2.4. The Gysin map for Chow groups

Assume that \( k \) is of arbitrary characteristic.

**Definition.** Let \( K'_2(A_X) \) be the subset of all \( (f_{x,C}) \in K_2(K_{x,C}), \ x \in C \) such that
(a) \( f_{x,C} \in K_2(\mathcal{O}_{x,C}) \) for almost all irreducible curves \( C \) where \( \mathcal{O}_{x,C} \) is the ring of integers of \( K_{x,C} \) with respect to the discrete valuation of rank 1 on it;
(b) for all irreducible curves \( C \subset X \), all integers \( r \geq 1 \) and almost all points \( x \in C \)
\[
f_{x,C} \in K_2(\mathcal{O}_{x,C}, M_C) + K_2(\widehat{\mathcal{O}}_x[t_C^{-1}]) \subset K_2(K_{x,C})
\]
where \( M_C \) is the maximal ideal of \( \mathcal{O}_{x,C} \) and \( K_2(A, J) = \ker(K_2(A) \to K_2(A/J)) \).

This definition is similar to the definition of [P2].

**Definition.** Using the diagonal map of \( K_2(K_C) \) to \( \prod_{x \in C} K_2(K_{x,C}) \) and of \( K_2(K_x) \) to \( \prod_{C \ni x} K_2(K_{x,C}) \) put
\[
K'_2(A_{01}) = K'_2(A_X) \cap \text{image of } \prod_{C \ni x} K_2(K_{C}),
\]
\[
K'_2(A_{02}) = K'_2(A_X) \cap \text{image of } \prod_{x \in X} K_2(K_x),
\]
\[
K'_2(A_{12}) = K'_2(A_X) \cap \text{image of } \prod_{x \in C} K_2(\mathcal{O}_{x,C}),
\]
\[
K'_2(A_0) = K_2(k(X)),
\]
\[
K'_2(A_1) = K'_2(A_X) \cap \text{image of } \prod_{C \ni x} K_2(\mathcal{O}_C),
\]
\[
K'_2(A_2) = K'_2(A_X) \cap \text{image of } \prod_{x \in X} K_2(\widehat{\mathcal{O}}_x)
\]
where \( \mathcal{O}_C \) is the ring of integers of \( K_C \).

Define the complex \( K_2(A_X) \):
\[
k_2(A_0) \oplus k_2(A_1) \oplus k_2(A_2) \to k_2(A_{01}) \oplus k_2(A_{02}) \oplus k_2(A_{12}) \to k_2(A_{012})
\]
\[
(f_0, f_1, f_2) \mapsto (f_0 + f_1, f_2 - f_0, -f_1 - f_2)
\]
\[
(g_1, g_2, g_3) \mapsto g_1 + g_2 + g_3
\]
where \( k_2(A_{012}) = k_2(A_X) \).

Using the Gersten resolution from \( K \)-theory (see [Q, §7]) one can deduce:
Theorem. There is a canonical isomorphism
\[ H^n(K_2(\mathbb{A}_X)) \simeq H^n(X, \mathcal{K}_2(X)). \]

Similarly one defines \( K'_1(\mathbb{A}_S) \). From \( H^1(S, \mathcal{K}_1(S)) = H^1(S, \mathcal{O}_S^* ) = \text{Pic}(S) \) (or from the approximation theorem) it is easy to see that the \( n \)-th cohomology group of the following complex
\[
K_1(k(S)) \oplus \sum_{s \in S} K_1(\widehat{\mathcal{O}}_s) \rightarrow K'_1(\mathbb{A}_S)
\]
is canonically isomorphic to \( H^n(S, \mathcal{K}_1(S)) \) (here \( \widehat{\mathcal{O}}_s \) is the completion of the local ring of \( C \) at \( s \)).

Assume that \( k \) is of characteristic 0.

Using the reciprocity law above and the previous theorem one can deduce:

Theorem. The map \( f_* = \sum_{C \ni x, f(x)=s} f_*(\cdot, x, C) \) from \( K_2(\mathbb{A}_X) \) to \( K'_1(\mathbb{A}_S) \) is well defined. It maps the complex \( K_2(\mathbb{A}_X) \) to the complex
\[
0 \rightarrow K_1(k(S)) \oplus \sum_{s \in S} K_1(\widehat{\mathcal{O}}_s) \rightarrow K'_1(\mathbb{A}_S).
\]

It induces the map \( f_*: H^n(X, \mathcal{K}_2(X)) \rightarrow H^{n-1}(S, \mathcal{K}_1(S)) \) of 2.0.
If \( n = 2 \), then the last map is the direct image morphism (Gysin map) from \( CH^2(X) \) to \( CH^1(S) \).

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