A classical result of Huxley [5] states that for sufficiently large $x$ and any $\theta > 7/12$, the interval $[x, x + x^\theta]$ contains a rational prime. In this paper, we investigate an analogous problem about Gaussian primes. To be precise, let $\varphi \in \mathbb{R}$, $0 < \delta \leq \pi/2$, $0 < \theta \leq 1$, and $x$ large. We are interested in the cardinality of the set

$$\left\{ a + bi \in \mathbb{Z}[i] : (a + bi) \text{ is prime, } \varphi < \arg(a + bi) \leq \varphi + \delta, \ x - x^\theta < a^2 + b^2 \leq x \right\}.$$  

Here $(a + bi)$ denotes the ideal generated by $a + bi$. As is common in such problems, it is more convenient to count these ideals with a suitable weight. Denote by $a$ the ideal in $\mathbb{Z}[i]$ generated by $a + bi$ and by $N(a) = a^2 + b^2$ its norm. If we define

$$\Lambda(a) = \begin{cases} \log N(a) & \text{if } a = p^m \text{ with } p \text{ prime and } m \geq 1, \\ 0 & \text{otherwise}, \end{cases}$$

then our problem translates to obtaining an asymptotic estimate for

$$\psi(x, y; \varphi, \delta) = \sum_{x - x^\theta < N(a) \leq x \atop \varphi < \arg a \leq \varphi + \delta} \Lambda(a).$$

Ricci [11] has shown that for all $\varepsilon > 0$ and $\delta \geq x^{-3/10 + \varepsilon}$, one has

$$\psi(x, x; \varphi, \delta) \sim \frac{2\delta x}{\pi}.$$

We generalize this and prove the following

**Theorem 1.1.** For any $\varepsilon > 0$, $\varphi \in \mathbb{R}$, $x$ sufficiently large, $\theta > 7/10$, and $\delta x^\theta \geq x^{7/10 + \varepsilon}$, we have

$$\psi(x, x^\theta; \varphi, \delta) \sim \frac{2\delta x^\theta}{\pi}.$$  

Geometrically, the parameters $x, \theta, \varphi, \delta$ describe a sector centered at the origin. The inner and outer radii of this sector are $\sqrt{x - x^\theta}$ and $\sqrt{x}$, and the sector is cut by rays emanating from the origin with angles $\varphi$ and $\varphi + \delta$. Ricci’s result gives the expected number of prime ideals in a sector so long as the inner radius $\sqrt{x - x^\theta}$ is essentially 0 and the angle $\delta$ between the rays is sufficiently wide. Theorem 1.1 claims the more general result that one obtains the expected number of prime ideals so long as the area of the sector is sufficiently large.

**A note on the literature.** It should be noted that Maknys [10] has claimed a result similar to Theorem 1.1 but with the exponent $11/16$ in place of $7/10$. However, Heath-Brown [4] has found an error in Maknys’ proof of this result. He states that Maknys’ proof, when corrected, yields the exponent $(221 + \sqrt{201})/320 = 0.7349...$. However, the result is potentially worse than 0.7349...
because Maknys’s proof depends on a zero density estimate (Theorem 2 of [9]), the proof of which also contains an error. In particular, there is an incorrect application of Theorem 1 of [8]. For a version of Theorem 1 of [8] which is applicable in the proof of Maknys’ zero density result, see Theorem 6.2 and the end of Section 7 of [1].

Outline of the Paper

To orient the reader, we provide an outline of the paper. In Section 3, we begin by smoothing the angular and norm regions for \( \psi(x,x^0,\varphi,\delta) \), and then express these regions via a sum of Hecke characters \( \lambda^m \) and an integral of \( N(a)^{it} \). The main term in Theorem 1.1 then arises from the contribution of the principal character. After applying an analogue of Heath-Brown’s identity in \( \mathbb{Z}[i] \) (see Lemma 2.6 below), we are left to bound a sum of \( O((\log x)^{2J+2}) \) expressions roughly of the form

\[
\sum_{M \leq m \leq 2M} c_m \int_{T}^{2T} \hat{V}(\frac{1}{2} + it) \sum_{\substack{a = a_1 \cdots a_{2J} \in \mathbb{N} \cap N > N_j \mathbb{N}}} a_1(a_1) \cdots a_{2J}(a_{2J}) \frac{\lambda^m(a)}{N(a)^{1/2 + it}} dt
\]

for some parameters \( N_i \). Here the \( c_m \) are Fourier coefficients and \( \hat{V} \) is a Mellin transform. Using estimates for \( c_m \) and \( \hat{V} \), this reduces to showing that

\[
\sum_{M \leq m \leq 2M} \int_{T}^{2T} |F(\frac{1}{2} + it)| dt \ll \frac{x^{1/2}}{(\log x)^A},
\]

where \( F \) is the Dirichlet series appearing in the penultimate display.

In Section 4, we reduce this to bounding the number \( R \) of pairs \( m, t \) for which a particular factor \( f \) of \( F \) attains a large value. Specifically, for such a pair \( m, t \), we have

\[
\sum_{N(a) > N} c(a)\lambda^m(a)N(a)^{-it} \gg W
\]

for some divisor-bounded coefficients \( c(a) \) and \( W > 0 \). In Section 5, we use mean- and large-value estimates to bound \( R \). Specifically, we use a hybrid large sieve estimate due to Coleman and an analogue of Huxley’s large value result, also due to Coleman. Writing \( G = \sum |c(a)|^2 \), these yield

\[
R \ll NGW^{-2} + (M^2 + T^2)GW^{-2},
\]

\[
R \ll NGW^{-2} + (M^2 + T^2)NG^3W^{-6},
\]

respectively. We also use the “trivial” estimate

\[
R \ll \min(M,T)NGW^{-2} + MTGW^{-2},
\]

as well as a subconvexity result for \( L(s, \lambda^m) \) due to Ricci. There are a variety of ranges for \( N, M, T \) to consider when deciding which estimate to use. This requires a case analysis which is done in Sections 6 – 8. Here we also indicate the “worst cases” of \( N, M, T \) for which our estimates are sharp.

We note that with an optimal large sieve, one would have the estimate

\[
R \ll NGW^{-2} + MTGW^{-2}.
\]

Although such a large sieve is not available in the literature, this would not improve our results (it would, however, simplify the case analysis). This is because one of the worst cases in our analysis remains a worst case when using this estimate. See Section 8 for this discussion.

Acknowledgments. I would like to thank my advisor, Xiannan Li, for suggesting this problem to me and for many helpful comments in the development of these results.
2. Notation and Preliminary Lemmas

We collect here some additional notation and lemmas we will need throughout the paper. The symbols \( o, O, \ll, \gg, \asymp \) have their usual meanings. The letter \( \varepsilon \) denotes a sufficiently small positive real number, while \( A, B, C \) stand for an absolute positive constants, all of which may be different at each occurrence. For example, we may write

\[
x^\varepsilon \log x \ll x^\varepsilon, \quad (\log x)^B (\log x)^B \ll (\log x)^B
\]

Any statement in which \( \varepsilon \) occurs holds for each positive \( \varepsilon \), and any implied constant in such a statement is allowed to depend on \( \varepsilon \). The implied constants in any statement involving the letters \( A, B, C \) are also allowed to depend on these variables.

Similar to \( \Lambda(a) \), we define

\[
\mu(a) = \begin{cases} 
(-1)^r & \text{if } a = p_1 \cdots p_r \text{ with } p_i \text{ distinct primes}, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \arg a \) be the argument of any one of the generators of \( a \) (which is unique mod \( \pi/2 \)). For \( m \in \mathbb{Z} \), we define the angular Hecke characters

\[
\lambda^m(a) = e^{4im \arg a} = \left( \frac{\alpha}{|\alpha|} \right)^{4im},
\]

which are primitive with conductor (1). Note that the character is well-defined since the particular generator \( \alpha \) chosen for the definition above is immaterial. From these we get the Hecke \( L \)-functions, defined for \( \text{Re } s > 1 \) by

\[
L(s, \lambda^m) = \sum_a \frac{\lambda^m(a)}{N(a)^s}.
\]

Here the sum is over all nonzero ideals of \( \mathbb{Z}[i] \). These \( L \)-functions are absolutely convergent for \( \text{Re } s > 1 \), and Hecke showed that, for \( m \neq 0 \), they have analytic continuation to all of \( \mathbb{C} \) and satisfy a functional equation. We also have

\[
\frac{1}{L(s, \lambda^m)} = \sum_a \frac{\lambda^m(a)\mu(a)}{N(a)^s}, \quad \frac{L'(s, \lambda^m)}{L(s, \lambda^m)} = \sum_a \frac{\lambda^m(a)\Lambda(a)}{N(a)^s},
\]

which are also absolutely convergent for \( \text{Re } s > 1 \). We summarize these facts in the following

**Lemma 2.1.** The function \( L(s, \lambda^m) \) satisfies the functional equation

\[
L(s, \lambda^m) = \gamma(s, \lambda^m)L(1 - s, \lambda^m), \tag{2.1}
\]

where

\[
\gamma(s, \lambda^m) = \pi^{2s - 1} \frac{\Gamma(1 - s + 2|m|)}{\Gamma(s + 2|m|)}.
\]

If \( m \neq 0 \), then \( L(s, \lambda^m) \) is entire, and otherwise it is meromorphic with a simple pole at \( s = 1 \) with residue \( \pi/4 \). We also have

\[
L(s, \lambda^m) = L(s, \lambda^{-m}). \tag{2.2}
\]

**Proof.** These results are standard. See [6], for instance. \( \square \)
We will need several results on the behavior of these functions in the critical strip. These are given in the following pair of lemmas.

**Lemma 2.2.** Let \( V = (4m^2 + t^2)^{1/2} \). Then there exist absolute constants \( C, \delta > 0 \) such that

\[
L(\sigma + it, \lambda^m) \ll V^{C(1-\sigma)^{3/2}} (\log V)^{2/3},
\]

uniformly for \( 1 - \delta < \sigma < 1 \). It follows that there exists an absolute constant \( C > 0 \) such that \( L(s, \lambda^m) \) has no zeros in the region

\[
\sigma \geq 1 - C(\log V)^{-2/3} (\log \log V)^{-1/3}. \tag{2.3}
\]

**Proof.** These are special cases of Theorems 1 and 2 of [2]. \qed

**Lemma 2.3.** For \( \sigma \) in the region (2.3) with \( C \) replaced by \( C/4 \), we have

\[
\frac{L'(\sigma + it, \lambda^m)}{L(\sigma + it, \lambda^m)} \ll \log V, \quad \frac{1}{L(\sigma + it, \lambda^m)} \ll \log V.
\]

**Proof.** We follow closely the method of Titchmarsh ([13], Theorem 3.11). For \( t \) sufficiently large and

\[
\sigma \geq 1 - \left( \frac{\log \log V}{\log V} \right)^{2/3},
\]

we have by Lemma 2.2 that

\[
L(\sigma + it, \lambda^m) \ll (\log V)^A,
\]

for some constant \( A \). Let

\[
s_0 = 1 + \frac{C}{2} (\log V_0)^{-2/3} (\log \log V_0)^{-1/3} + it_0 \quad \text{and} \quad r = \left( \frac{\log \log V_0}{\log V_0} \right)^{2/3},
\]

where \( V_0 = V(2t_0 + 3) \). For \( m = 0 \), we have

\[
\frac{1}{L(s_0, 1)} \ll \frac{1}{\sigma_0 - 1},
\]

and for \( m \neq 0 \) we may bound \( L(s_0, \lambda^m)^{-1} \) trivially and obtain the same bound. Thus in the circle \( |s - s_0| \leq r \), we have

\[
\frac{L(s, \lambda^m)}{L(s_0, \lambda^m)} \ll \frac{(\log V_0)^A}{\sigma_0 - 1} \ll (\log V_0)^{A+1}.
\]

We also have

\[
\frac{L'(s_0, \lambda^m)}{L(s_0, \lambda^m)} \ll \frac{L'(s_0, 1)}{L(s_0, 1)} \ll \frac{1}{\sigma_0 - 1} \ll \frac{\log \log V_0}{r}
\]

for all \( k \). Now \( L(s, \lambda^m) \) has no zeros in the region

\[
t \leq t_0 + 1, \quad \sigma \geq 1 - C(\log V_0)^{-2/3} (\log \log V_0)^{-1/3}.
\]

We apply Lemma \( \gamma \) of Titchmarsh with \( 2r' = (3C/2)(\log V_0)^{-2/3} (\log \log V_0)^{-1/3} \) and \( M = O(\log \log V_0) \), getting

\[
\frac{L(s, \lambda^m)}{L(s, \lambda^m)} \ll (\log V_0)^{2/3} (\log \log V_0)^{1/3} \ll \log V_0
\]
for $|s - s_0| \leq r'$. In particular, this holds for

$$t = t_0, \quad \sigma \geq 1 - \frac{C}{4} (\log V_0)^{-2/3} (\log \log V_0)^{-1/3}.$$ 

To obtain the analogous bound for $L(s, \lambda^m)^{-1}$, let $\eta = (\log V_0)^{-2/3} (\log \log V_0)^{-1/3}$. In the region

$$1 - \frac{C}{4} \eta \leq \sigma \leq 1 + \eta$$

we have

$$\log \left| \frac{1}{L(s, \lambda^m)} \right| = -\text{Re} \log L(s, \lambda^m)$$

$$= -\text{Re} \log L(1 + \eta + it, \lambda^m) + \text{Re} \int_{\sigma}^{1+\eta} \log L(u + it, \lambda^m) du$$

$$\ll \log (1 + \eta, 1) + \int_{\sigma}^{1+\eta} \frac{L'(u + it, \lambda^m)}{L(u + it, \lambda^m)} du$$

$$\ll \log \frac{1}{\eta} + O(1),$$

which gives the desired estimate.

Next, we need an estimate for the number of lattice points in a suitably regular sector.

**Lemma 2.4.** Let $\varphi \in \mathbb{R}$, $x$ and $y$ be sufficiently large with $x^{1/2} \leq y \leq x$, and $x^{-1/2} \leq \delta \leq \pi/2$. If

$$N(x, y, \varphi, \delta) = \# \{ a + bi \in \mathbb{Z}[i] : \varphi \leq \arg(a + bi) \leq \varphi + \delta, \ x - y \leq a^2 + b^2 \leq x \},$$

then

$$N(x, y, \varphi, \delta) \ll \delta y.$$ 

**Proof.** Let $\mathcal{R}$ denote the region defined by the constraints in the set above. The area and perimeter of this region are

$$\frac{\delta y}{2}$$

and

$$2(\sqrt{x} - \sqrt{x - y}) + \delta(\sqrt{x} + \sqrt{x - y}),$$

respectively. The number $\mathcal{N}$ of lattice points in $\mathcal{R}$ is trivially bounded by the sum of the area and perimeter, which is

$$\ll \delta y + \frac{y}{\sqrt{x}} + \delta \sqrt{x} \ll \delta y.$$ 

We will also need some results on sums of the divisor functions $d_j(a)$.

**Lemma 2.5.** Let $d_j(a)$ be the $j$-divisor function for $\mathbb{Z}[i]$. We have $d_j(a) \ll N(a)^{\varepsilon}$, and for $y > x^{1/2}$ we also have

$$\sum_{x - y < N(a) \leq x} d_j(a) \ll y(\log x)^{j-1}.$$ 

For $\varphi \in \mathbb{R}$ and $x^{-1/2} < \delta \leq \pi/2$, we also have

$$\sum_{x - y < N(a) \leq x} d_j(a) \ll \delta yx^{\varepsilon},$$

where $\varepsilon$ and $j$.

The implied constants above depend only on $\varepsilon$ and $j$. 

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Proof. The proof is mostly an application of Shiu’s work [12]. We write
\[
\sum_{x-y<N(a)\leq x} d_j(a) = \sum_{x-y<n\leq x} f_j(n),
\]
where
\[
f_j(n) = \sum_{N(a)=n} d_j(a).
\]
For \( j \geq 2 \), have the convolution identity
\[
f_j(n) = \sum_{n=ab} r(a)f_{j-1}(b),
\]
where \( f_1(n) = r(n) \) is the number of ideals in \( \mathbb{Z}[i] \) with norm \( n \). Since
\[
r(n) = \sum_{d|n} \chi(d),
\]
where \( \chi \) is the nontrivial character mod 4, we see that \( r(n) \leq d(n) \), and so \( f_j(n) \leq d_j(n) \), where \( d_j(n) \) is the usual divisor function. This implies the estimate \( d_j(a) \ll N(a)^{\varepsilon} \) for any \( \varepsilon > 0 \). In [12], the class of functions \( M \) considered are those for which
\[
f(p^l) \leq A_1^l \quad \text{and} \quad f(n) \leq A_2 n^\varepsilon
\]
for all primes \( p \), integers \( l, n \geq 1 \) with constants \( A_1 \) and \( A_2 = A_2(\varepsilon) \). In particular, \( r(n) \) satisfies these conditions, and one may check that any convolution of such functions also satisfies these conditions. It follows that \( f_j \in M \), and so Theorem 1 of [12] gives
\[
\sum_{x-y<N(a)\leq x} d_j(a) \ll \frac{y}{\log x} \exp \left( \sum_{p\leq x} \frac{f_j(p)}{p} \right).
\]
To compute the exponential, we have
\[
f_j(p) = \begin{cases} 
  j & \text{if } p = 2, \\
  2j & \text{if } p \equiv 1 \pmod{4}, \\
  0 & \text{if } p \equiv 3 \pmod{4},
\end{cases}
\]
so
\[
\exp \left( \sum_{p\leq x} \frac{f_j(p)}{p} \right) \ll \exp \left( 2j \sum_{p\leq x} \frac{1}{p} \right) \ll (\log x)^j.
\]
This gives the first statement.

To prove the second statement, we have
\[
\sum_{x-y<N(a)\leq x} d_j(a) \ll \sum_{x-y<N(a)\leq x} N(a)^{\varepsilon} \ll x^{\varepsilon} N(x, y; \varphi, \delta),
\]
By Lemma 2.4 and the hypotheses of the lemma, we have \( N(x, y; \varphi, \delta) \ll \delta y \), which completes the proof.
Lastly, our analysis will make use of an analogue of Heath-Brown’s identity in \( \mathbb{Z}[i] \) (see \[3\]). For technical reasons, it is more convenient to have a smoothed version of this identity. This is given in the following

**Lemma 2.6 (Heath-Brown’s Identity).** Let \( X > 1 \) and \( J \) be a positive integer. Then for any \( a \) with \( N(a) \leq X^J \), we have

\[
\Lambda(a) = \sum_{j=1}^{J} \binom{J}{j} (-1)^{j-1} \sum_{a_1 \cdots a_{j-1}} \log(a_1) \mu(a_{j+1}) \cdots \mu(a_J) \\
\times \sum_{n_1, \ldots, n_J \geq 0} W \left( \frac{N(a_1)}{X^{j/2n_1}} \right) \cdots W \left( \frac{N(a_J)}{X^{j/2n_J}} \right) W \left( \frac{N(a_{j+1})}{X^{j+1/2n_{j+1}}} \right) \\
\times W(N(a_{j+1})) \cdots W(N(a_{J-1})) W(N(a_{J+1})) \cdots W(N(a_J)).
\]

Note that the terms on the last line simply force the ideals \( a_{j+1}, \ldots \) to have norm 1. The point of the lemma is that for \( N(a) \leq X^J \), the function \( \Lambda(a) \) can be decomposed into a linear combination of \( O((\log X)^{2J}) \) smooth sums of the form

\[
\sum_{a_1 \cdots a_{J-1}} \log(a_1) \mu(a_{J+1}) \cdots \mu(a_{2J}) W \left( \frac{N(a_1)}{N_1} \right) \cdots W \left( \frac{N(a_{2J})}{N_{2J}} \right),
\]

where \( N_j = X^{j/2n} \) or \( X/2^n \) for some integer \( n \), depending as \( j \leq J \).

**Proof.** Let \( W \) be a smooth function supported on \([\frac{3}{4}, 2]\) such that

\[
\sum_{n \geq 0} W(2^n t) = 1 \quad \text{and} \quad W^j(t) \ll t^{-j}
\]

for all \( 0 < t \leq 1 \). Then for all \( 1 \leq k \leq X \) we have

\[
\sum_{n \geq 0} W \left( \frac{k}{X/2^n} \right) = \sum_{0 \leq n \leq \left\lfloor \frac{\log X}{\log 2} \right\rfloor + 1} W \left( \frac{k}{X/2^n} \right) = 1.
\]

If we put

\[
M_X(s) = \sum_{n \geq 0} \sum_a \mu(a) N(a)^{-s} W \left( \frac{N(a)}{X/2^n} \right),
\]

then for any \( J \geq 1 \) we have

\[
- \frac{\zeta_K^J(s)}{\zeta_K(s)} = \sum_{j=1}^{J} \binom{J}{j} (-1)^{j-1} M_X(s) j \zeta_K^{j-1}(s) \zeta_K^j(s) \zeta_K'(s) \left( 1 - M_X(s) \zeta_K(s) \right)^J.
\]  

(2.4)

where \( \zeta_K \) is the Dedekind zeta function of \( K = \mathbb{Z}[i] \). Note that the coefficient \( N(a)^{-s} \) in \( M_X \) is \( \mu(a) \) for \( N(a) \leq X \), and so

\[
1 - M_X(s) \zeta_K(s) = \sum_{N(a) > X} c(a) N(a)^{-s}
\]
for some coefficients \(c(a)\). Here we have used the identity

\[
\sum_{b|a} \mu(b) = \begin{cases} 1 & \text{if } a = (1), \\ 0 & \text{otherwise}, \end{cases}
\]

which is the analogue of the familiar identity in \(\mathbb{Z}\). It follows that the last term in (2.4) does not contribute to the coefficient of \(N(a)^{-s}\) with \(N(a) \leq X^J\). For such \(a\), we are free to introduce dyadic partitions of unity as well. Using \(X^J\) in place of \(X\) in the partition of unity, inserting these into the sum on the right of (2.4), and comparing the coefficient of \(N(a)^{-s}\) on both sides, we derive the lemma.

\[\square\]

3. Initial Decomposition

To estimate \(\psi(x, x^\theta; \varphi, \delta)\), we begin by smoothing the angular region for \(a\). For this, we need

**Lemma 3.1.** Let \(k \in \mathbb{Z}\) with \(k \geq 0\) and let \(\alpha, \beta, \Delta, L\) be real numbers satisfying

\[
L > 0, \quad 0 < \Delta < \frac{L}{2}, \quad \Delta \leq \beta - \alpha \leq L - \Delta.
\]

Then there exists an \(L\)-periodic function \(P(x)\) with

\[
P(x) = \frac{1}{L} (\beta - \alpha) + \sum_{m \neq 0} c_m e^{4imx}
\]

which satisfies

\[
P(x) = 1 \quad \text{for } \alpha \leq x \leq \beta,
\]

\[
P(x) = 0 \quad \text{for } \beta + \Delta \leq x \leq L + \alpha - \Delta,
\]

\[
0 \leq P(x) \leq 1 \quad \text{for all } x,
\]

and where the coefficients \(c_m\) satisfy

\[
|c_m| \leq \begin{cases} \frac{1}{L} (\beta - \alpha), \\ \frac{L}{|m|} \left(\frac{kL}{\Delta|m|}\right)^k & \text{if } m \neq 0, \end{cases} \tag{3.1}
\]

where the factor involving \(k\) is taken to equal 1 when \(k = 0\).

**Proof.** This result is classical. See, for example, Lemma A of Chapter 1, Section 2 of [2]. The special case \(L = 1\) is proved there, but the lemma generalizes easily to arbitrary periods.

\[\square\]

Let \(P\) be as in the lemma with \(L = \pi/2, \alpha = \varphi, \beta = \varphi + \delta,\) and \(\Delta = \delta x^{-\varepsilon}\). Then

\[
\psi(x, x^\theta; \varphi, \delta) = \sum_{x - x^\theta < N(a) \leq x} \Lambda(a) P(\arg a) + O\left(\sum_{x - x^\theta < N(a) \leq x} \Lambda(a)\right) + O\left(\sum_{x - x^\theta < N(a) \leq x} \Lambda(a)\right).
\]
To estimate the error terms we note that the hypotheses of Theorem 1.1 imply that \( x^\theta \gg x^{7/10 + \varepsilon} \) and \( \delta \gg x^{-3/10 + \varepsilon} \). In particular, we have

\[
x^\theta \geq x^{1/2} \quad \text{and} \quad \Delta \geq x^{-1/2}.
\]

Since \( \Lambda(a) \leq \log x \), we have by Lemma 2.4 that

\[
\sum_{x - x^\theta < N(a) \leq x} \Lambda(a) \ll (\log x) N(x, x^\theta, \varphi - \Delta, \varphi) \ll (\log x) x^\theta \Delta = o(\delta x^\theta),
\]

and similarly for the other error term. We expand \( P(\arg a) \) using its Fourier series and write

\[
\sum_{x - x^\theta < N(a) \leq x} \Lambda(a)P(\arg a) = \sum_{x - x^\theta < N(a) \leq x} \Lambda(a) \sum_m c_m \lambda^m(a).
\]

We have

\[
\sum_{x - x^\theta < p \leq x} \Lambda(a) = 2 \sum_{p \equiv 1 (\mod 4)} \log p + O(x^{1/2} \log x),
\]

(see, for instance, display 7.4 in Chapter 2 of [11] for this computation). Since \( x^\theta \gg x^{7/12 + \varepsilon} \), the Siegel-Walfisz theorem in short intervals gives

\[
2 \sum_{x - x^\theta < p \leq x \equiv 1 (\mod 4)} \log p = x^\theta(1 + o(1)),
\]

and since \( c_0 = 2\delta \pi^{-1} \), we obtain

\[
\psi(x, x^\theta, \varphi, \delta) = \frac{2\delta x^\theta}{\pi}(1 + o(1)) + \sum_{x - x^\theta < N(a) \leq x} \Lambda(a) \sum_{m \neq 0} c_m \lambda^m(a).
\]

Using (3.1), we truncate the Fourier series at \( M_1 \) to obtain

\[
\psi(x, x^\theta, \varphi, \delta) = \frac{2\delta x^\theta}{\pi}(1 + o(1)) + \sum_{x - x^\theta < N(a) \leq x} \Lambda(a) \sum_{1 \leq |m| \leq M_1} c_m \lambda^m(a) + O(x^\theta \log x \left( \frac{\pi k x^\varepsilon}{2\delta M_1} \right)^k)
\]

for any \( k \geq 1 \). Choosing \( M_1 = \delta^{-1} x^\varepsilon \), a sufficiently large choice of \( k \) depending only on \( \varepsilon \) makes the error term \( o(\delta x^\theta) \), and so

\[
\psi(x, x^\theta, \varphi, \delta) = \frac{2\delta x^\theta}{\pi}(1 + o(1)) + \sum_{1 \leq |m| \leq M_1} c_m \sum_{x - x^\theta < N(a) \leq x} \Lambda(a) \lambda^m(a).
\]

Next, we smooth the norm-region for \( a \). Let \( V \) be a smooth function satisfying

\[
V(t) = 1 \text{ if } x \in [x - x^\theta, x],
\]

\[
V(t) = 0 \text{ if } x \in \mathbb{R} \setminus [x - x^\theta - x^\theta - \varepsilon, x + x^\theta - \varepsilon],
\]

Then \( \tilde{V} \) satisfies

\[
\tilde{V}(s) \ll x^{\sigma + \varepsilon - 1} \quad \text{and} \quad \tilde{V}(s) \ll \frac{x^{\sigma + (A-1)(1-\theta + \varepsilon)}}{(1 + |t|)^A}.
\]
for any real $A \geq 1$, where the implied constant depends only on $A$ and $\sigma$. We obtain
\[
\psi(x, x^\theta, \varphi, \delta) = \frac{2\delta x^\theta}{\pi}(1 + o(1)) + \sum_{1 \leq |m| \leq M_1} c_m \sum_a A(a)V(a)\lambda^m(a) = \frac{2\delta x^\theta}{\pi}(1 + o(1)) + S,
\]
say, where the error in replacing the sharp cutoff with the smoothing function $V$ has been absorbed into the error term $o(\delta x^\theta)$.

We now employ Lemma 2.6 with $X = (2x)^{1/J}$ for some integer $J \geq 1$ to be chosen. Then $S$ is a linear combination of $O((\log x)^{2J})$ sums of the form
\[
S = \sum_{1 \leq |m| \leq M_1} c_m \sum_{a = a_1 \cdots a_{2J}} a_1(a_1) \cdots a_{2J}(a_{2J}) W_1(N(a_1)) \cdots W_{2J}(N(a_{2J})) \lambda^m(a)V(N(a)),
\]
where
\[
a_j(a) = \begin{cases} 
\log N(a) & \text{if } j = 1, \\
1 & \text{if } 2 \leq j \leq J, \\
\mu(a) & \text{if } J + 1 \leq j \leq 2J,
\end{cases}
\]
and also let
\[
F_m(s) = \frac{2J}{\prod_{j=1}^{2J} f_{j,m}(s)} = \sum_a a(a)\lambda^m(a) \frac{W_j(N(a))}{N(a)^s},
\]
where the coefficients satisfy
\[
|a(a)| \ll d_{2J}(a) \log x.
\]
Then Mellin inversion gives
\[
S = \frac{1}{2\pi i} \int_{(1/2)} \tilde{V}(s) \sum_{1 \leq |m| \leq M_1} c_m F_m(s) \, ds.
\]
For Re $(s) = 1/2$, we have
\[
F_m(s) \ll \log x \sum_{N(a) \leq 2x} \frac{d_{2J}(a)}{N(a)^{1/2}} \ll x^{1/2+\varepsilon}.
\]
Also $|c_m| \ll \delta$. Truncating the integral at height $T_1$ and using (3.2) then gives
\[
S = \frac{1}{2\pi i} \int_{1/2 - iT_1}^{1/2 + iT_1} \tilde{V}(s) \sum_{1 \leq |m| \leq M_1} c_m F_m(s) \, ds + O\left(x^{1/2+\varepsilon} \frac{x^{1/2+(A-1)(1-\theta+\varepsilon)}}{T_1^{A-1}}\right)
\]
for any $A \geq 1$. Choosing $T_1 = x^{1-\theta+\varepsilon}$ and taking $A$ sufficiently large in terms of $\varepsilon$ makes the error term negligible. We have $|c_m| \ll \delta$ and $|\tilde{V}(1/2 + it)| \ll x^{\theta-1/2}$, so
\[
S \ll \frac{\delta x^\theta}{x^{1/2}} \sum_{1 \leq |m| \leq M_1} \int_{-T_1}^{T_1} |F_m(\frac{1}{2} + it)| \, dt \ll \frac{\delta x^\theta}{x^{1/2}} \sum_{1 \leq m \leq M_1} \int_0^{T_1} |F_m(\frac{1}{2} + it)| \, dt,
\]
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the last inequality following from \ref{22}. We divide the ranges of \(m\) and \(t\) into dyadic intervals \([M, 2M]\) and \([T, 2T]\) for \(M, T \geq 1\) along with the additional interval \([0, 1]\) for \(t\). Theorem 1.1 now follows from

**Lemma 3.2.** We have

\[
\sum_{M \leq m \leq 2M} \int_T^{2T} |F_m \left( \frac{1}{2} + it \right)| \, dt \ll \frac{x^{1/2}}{(\log x)^{2J+3}},
\]

uniformly for \(1 \leq M \leq M_1\) and \(1 \leq T \leq T_1\). The expression with an integral over \([0, 1]\) also satisfies this bound.

### 4. Reduction to Large Values

In this section, we reduce the proof of Lemma 3.2 to the estimation of the number of large values of a certain Dirichlet polynomials. We begin by letting \(\Delta\) be a small parameter to be chosen and write \(F_m(s) = G_m(s)H_m(s)\), where \(H_m(s)\) is the product of those factors for which the lengths \(N_j\) satisfy \(N_j \leq x^{\Delta/J}\). Since

\[
|f_{1,m} \left( \frac{1}{2} + it \right)| \ll N_1^{1/2} \log x; \quad |f_{j,m} \left( \frac{1}{2} + it \right)| \ll N_j^{1/2}, \quad (j \geq 2),
\]

we have

\[
|H_m \left( \frac{1}{2} + it \right)| \ll Z^{1/2} \log x,
\]

where \(Z\) is the product of those \(N_j\) with \(N_j \leq x^{\Delta/J}\). Then

\[
\int_T^{2T} \sum_{M \leq m \leq 2M} |F_m \left( \frac{1}{2} + it \right)| \, dt \ll Z^{1/2} \log x \int_T^{2T} \sum_{M \leq m \leq 2M} |G_m \left( \frac{1}{2} + it \right)| \, dt. \quad (4.1)
\]

We now bound the integral on the right \(I\), say) by a set of \(O(T)\) well-spaced points \(t_n\). We have

\[
I \ll \sum_n \sum_{M \leq m \leq 2M} |G_m \left( \frac{1}{2} + it_n \right)|,
\]

where \(|t_l - t_n| \geq 1\) for \(l \neq n\). For each triple \(j, m, n\), let

\[
\left| f_{j,m} \left( \frac{1}{2} + it_n \right) \right| = N_j^{\sigma(j,m,n)-1/2}.
\]

We need to show that \(\sigma(j,m,n)\) cannot be too close to 1. We treat the case \(j > J\), for which

\[
f_{j,m}(s) = \sum_a \frac{\mu(a)\lambda^m(a)W_j(a)}{N(a)^s}.
\]

The case \(j \leq J\) would be very similar. By Mellin inversion

\[
f_{j,m} \left( \frac{1}{2} + it \right) = \int_{(c)} L \left( \frac{1}{2} + it + s, \lambda^m \right)^{-1} N_j^s \tilde{W}_j(s) \, ds,
\]

where \(c = 1/2 + (\log x)^{-1}\). We have trivially that

\[
\frac{1}{|L(1 + (\log x)^{-1} + it, \lambda^m)|} \leq \zeta_K(1 + (\log x)^{-1}) \ll \log x,
\]

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(here again $\zeta_K$ is the Dedekind zeta function for $\mathbb{Z}[i]$). Truncating the integral at height $x^\varepsilon$ and using the rapid decay of $\tilde{W}$ gives

$$f_{j,m} \left( \frac{1}{2} + it \right) = \int_{c-ix^\varepsilon}^{c+ix^\varepsilon} L \left( \frac{1}{2} + it + s, \lambda^m \right)^{-1} N_j^{1/2} \tilde{W}_j(s) ds$$

with negligible error. We now use Lemmas 2.2 and 2.3 to move the line of integration to the left of $\Re s = 1$. Then in the region

$$1 - \eta \leq \Re s \leq \frac{1}{2} + c, \quad |\Im s - t| \leq x,$$

where

$$\eta = C (\log x)^{-2/3} (\log \log x)^{-1/3},$$

we have

$$\frac{1}{L(s, \lambda^m)} \ll \log x$$

Moving the line of integration to $1/2 - \eta$, we thus have

$$|f_{j,m} \left( \frac{1}{2} + it \right)| \ll (\log x)^{1/2 - \eta}$$

Since $N_j \geq x^{\Delta/J}$, we have

$$N_j^{\eta/2} \gg (x^\eta)^{\Delta/2J} \gg (\log x)^2.$$

Thus for $x$ sufficiently large, we have

$$N_j^{\sigma(j,m,n) - 1/2} = |f_{j,m} \left( \frac{1}{2} + it \right)| \leq N_j^{1/2 - \eta/2},$$

and so

$$\sigma(j,m,n) \leq 1 - \eta/2.$$

We now split the available range for $\sigma(j,m,n)$ into $O(\log x)$ ranges $I_0 = (-\infty, 1/2]$ and

$$I_l = \left( \frac{1}{2} + \frac{l - 1}{L}, \frac{1}{2} + \frac{l}{L} \right), \quad (1 \leq l \leq 1 + L/2, \ L = \lfloor \log x \rfloor),$$

For each $j,l$, let

$$C(j,l) = \{(m,t_n) : \sigma(j,m,n) \in I_l \text{ and } \sigma(j,m,n) \geq \sigma(k,m,n) \text{ for } 1 \leq k \leq 2J\}.$$

Since there are $O(\log x)$ classes $C(j,l)$, there must exist some class $C$ for which

$$I \ll (\log x) \sum_{(m,t) \in C} |G_m \left( \frac{1}{2} + it \right)|.$$

For $(m,t) \in C$, we have

$$|G_m \left( \frac{1}{2} + it \right)| = \prod N_j^{\sigma(j,m,n) - 1/2} \leq \prod N_j^{1/L} = Y^{1/L},$$

where $Y$ is the product of the $N_j$ with $N_j > x^{\Delta/J}$. To simplify notation, let

$$\sigma = \frac{1}{2} + \frac{l - 1}{L}, \quad f_m(s) = f_{j,m}(s), \quad N = N_j, \quad R = \#C.$$
If \( l = 0 \), then \( I \ll MT \log x \), so \( (4.1) \) gives

\[
\int_{T}^{2T} \sum_{M \leq m \leq 2M} \left| F_m \left( \frac{1}{2} + it \right) \right| dt \ll Z^{1/2} M T_1 (\log x)^2 \ll \delta^{-1} x^{1-\theta + \Delta + \varepsilon} \ll \frac{x^{1/2}}{(\log x)^4}
\]
since we may assume \( \Delta < 1/5 \) and \( x^\theta \delta > x^{7/10} \). If \( l \geq 1 \), we have

\[
I \ll (Y^{\sigma - 1/2} R \log x),
\]
and so

\[
\int_{T}^{2T} \sum_{M \leq m \leq 2M} \left| F_m \left( \frac{1}{2} + it \right) \right| dt \ll Z^{1/2} Y^{\sigma - 1/2} R (\log x)^2.
\]

Now since

\[
Z^{1/2} Y^{\sigma - 1/2} \ll Z^{1/2} (x Z^{-1})^{\sigma - 1/2} \ll x^{1/2} (Z x^{-1})^{1-\sigma} \ll x^{1/2 + (2\Delta - 1)(1-\sigma)},
\]
we find that

\[
\int_{T}^{2T} \sum_{M \leq m \leq 2M} \left| F_m \left( \frac{1}{2} + it \right) \right| dt \ll x^{1/2} (\log x)^2 \left( \frac{R}{x^{(1-2\Delta)(1-\sigma)}} \right).
\]

It thus remains to estimate \( R \). For each \((t, m) \in \mathcal{C}\), we have

\[
\left| f_m \left( \frac{1}{2} + it \right) \right| \gg N^{\sigma - 1/2},
\]
Since \( \sigma \leq 1 - \eta/2 \) we see that Lemma 3.2 follows from the bound

\[
R \ll x^{(1-3\Delta)(1-\sigma)} (\log x)^B \quad (4.3)
\]
for any fixed \( B > 0 \), since then the expression on the right of \( (4.2) \) is bounded by taking \( \sigma = 1 - \eta/2 \), and the definition of \( \eta \) allows us to save arbitrary powers of \( \log x \). To derive the requisite bound on \( R \), it is sufficient to show that

\[
R \ll (MT)^{10(1-\sigma)/3} (\log x)^B \quad (4.4)
\]
uniformly in \( M, T, \sigma \), since \( MT \leq M T_1 = x^{1-\theta + \varepsilon} \delta^{-1} \leq x^{3/10 - \varepsilon} \).

### 5. Mean and Large Value Results

To estimate \( R \), we will need several mean-value results of the form

\[
\sum_{|m| \leq M} \int_{-T}^{T} \left| \sum_{N(a) \leq N} c(a) \lambda^m(a) N(a)^{-it} \right|^2 dt \ll \mathcal{D} \sum_{N(a) \leq N} |c(a)|^2 \quad (5.1)
\]
for some \( \mathcal{D} = \mathcal{D}(N, M, T) \), where \( c(a) \) are arbitrary complex coefficients defined on the ideals of \( \mathbb{Z}[i] \). First, we have Coleman’s hybrid large sieve (Theorem 6.2 of \cite{coleman}).

**Lemma 5.1 (Coleman).** The estimate \( (5.1) \) holds with

\[
\mathcal{D} = M^2 + T^2 + N. \quad (5.2)
\]

Additionally, we also have the following trivial estimate.
Lemma 5.2. The estimate (5.1) holds with
\[ D = MT + N \min(M, T). \] (5.3)

Proof. For the case \( T \leq M \), see [11], Theorem C. For the other case, the mean-value theorem for Dirichlet polynomials gives
\[
\int_{-T}^{T} \left| \sum_{N(a) \sim N} c(a) \lambda^{m}(a) N(a)^{-it} \right|^2 dt = (T + O(N)) \sum_{N(a) \sim N} |c(a)|^2.
\]
Summing over \( m \) gives the other estimate.

Note that in each of the estimates above, the integral over \( t \) can be replaced by a sum over well-spaced points at the cost of a logarithmic factor, which will not affect our results.

For the problem at hand, the natural quantity to work with is \( MT \), rather than the minimum or maximum of \( M \) and \( T \). To this end, let
\[ \mathcal{L} = \mathcal{L}(M, T) = \frac{|\log(M/T)|}{\log MT} \]
so that
\[ \max(M^2, T^2) = (MT)^{1+\mathcal{L}} \quad \text{and} \quad \min(M^2, T^2) = (MT)^{1-\mathcal{L}}. \]
We will regard \( \mathcal{L} \) as an arbitrary parameter assuming values in \([0, 1]\). The estimates (5.2) and (5.3) become, respectively
\[ (MT)^{1+\mathcal{L}} + N \quad \text{and} \quad MT + N (MT)^{(1-\mathcal{L})/2}. \]
We will apply these estimates to suitable powers of the polynomial \( f_m \left( \frac{1}{2} + it \right) \). For any integer \( g \geq 1 \), we have
\[
RN^{g(2\sigma-1)} \ll \sum_{(m,t) \in \mathcal{L}} \left| \sum_{a} \frac{a(a)W(N(a))\lambda^{m}(a)}{N(a)^{1/2+it}} \right|^{2g} \ll D(N^g, M, T) \sum_{N(a) \sim N} \frac{|b(a)|^2}{N(a)},
\]
say where \( |b(a)| \leq d_g(a)(\log x)^g \). Using Lemma 2.5 and partial summation, we find that the coefficient sum on the right is \( O((\log x)^B) \) for some \( B \) which depends on \( g \). Since \( g \) is bounded in terms of \( \Delta \), we find that \( B \) and the implied constant depend at most on our choice of \( \Delta \). Thus
\[
RN^{g(2\sigma-1)} \ll \left( MT + N^g (MT)^{(1-\mathcal{L})/2} \right) (\log x)^B,
\]
\[
RN^{g(2\sigma-1)} \ll \left( (MT)^{1+\mathcal{L}} + N^g \right) (\log x)^B.
\]
We will also make use of the following large values result of Coleman (Theorem 7.3 of [1] with \( \theta = 0 \)) which is proved using Huxley’s subdivision method:
\[
R \ll \left( N^{2g(1-\sigma)} + (M^2 + T^2)N^{g(4-6\sigma)} \right) (\log x)^B \ll \left( N^{2g(1-\sigma)} + (MT)^{1+\mathcal{L}} N^{g(4-6\sigma)} \right) (\log x)^B.
\]
For any integer \( g \geq 1 \), the estimates above give
\[
R \ll \left( (MT)^{1+\mathcal{L}} N^{g(1-2\sigma)} + N^{2g(1-\sigma)} \right) (\log x)^B, \quad (5.4)
\]
\[ R \ll \left( MTN^g(1-2\sigma) + (MT)^{(1-L)/2}N^{2g(1-\sigma)} \right) (\log x)^B, \quad (5.5) \]

\[ R \ll \left( (MT)^{1+L}N^g(1-6\sigma) + N^{2g(1-\sigma)} \right) (\log x)^B. \quad (5.6) \]

To determine which estimate to apply, we divide into several cases. Writing \( N = (MT)^{\beta} \), these cases will depend on specific ranges of \( \beta, L, \sigma \). In the ensuing analysis, we will also use the following subconvexity estimate due to Ricci.

**Theorem 5.3 (Ricci).** If \((4m^2 + t^2) \geq 4\), then

\[ L \left( \frac{1}{2} + it, \lambda^m \right) \ll (m^2 + t^2)^{1/6} \log^3 (m^2 + t^2). \]

**Proof.** See [11], Chapter 2, Theorem 4, for instance. \( \square \)

To apply this estimate, we will need to Mellin invert the smoothing function \( W \) which will subsequently yield the integral of a certain Dirichlet series. To ensure that this Dirichlet series corresponds to \( L(s, \lambda^m) \), we will need the coefficients \( a(a) \) to be smooth. This is ensured by \( N > X \) (recall \( X \approx x^{1/J} \)). We will deal with the case \( N \leq X \) in Section 6 and the remaining case in Sections 7 and 8.

### 6. Short Polynomials

For \( N \leq X \), we divide into two cases. If \( N \leq MT \leq X \) or \( MT \leq N \leq X \), choose \( g \) so that

\[ X^2 \leq N^g \leq X^3. \]

Then \((MT)^{1+L} \leq (MT)^2 \leq X^2\), so by (5.5), we have

\[ R \ll \left( X^2 + 2(1-2\sigma) + X^6(1-\sigma) \right) (\log x)^B \ll x^{6(1-\sigma)/J} (\log x)^B. \]

This gives (4.3) so long as \( J > 6 \) and \( \Delta \) is sufficiently small.

Suppose now that \( N \leq X < MT \) so that \( \beta < 1 \). If \( \beta \geq 1/3 \), then \((MT)^{1+L} \leq X^6 \). Similar to the first case, we choose \( g \) so that

\[ X^6 \leq N^g \leq X^7 \]

and apply (5.4) to obtain

\[ R \ll x^{14(1-\sigma)/J} (\log x)^B. \]

We obtain (4.3) so long as \( J > 14 \) and \( \Delta \) is sufficiently small. From this and the previous case, we see that it is sufficient to choose \( J = 15 \) and \( \Delta < 1/45 \). If \( \beta < 1/3 \), then the choices

\[ g = \left\lfloor \frac{1 + L}{\beta} \right\rfloor + 1, \left\lfloor \frac{1 + L}{2\beta} \right\rfloor + 1, \left\lfloor \frac{1 + L}{\beta(4\sigma - 2)} \right\rfloor + 1, \]

give \( R \ll (MT)^{\alpha}(\log x)^B \), where

\[ \alpha = \min \left\{ \frac{2\beta}{1-\sigma} \left\lfloor \frac{1 + L}{\beta} \right\rfloor (1-\sigma), \frac{1}{2} (1-\sigma) + 2\beta \left\lfloor \frac{1 + L}{2\beta} \right\rfloor + 1 \right\} \]

\[ \leq \min \left\{ \frac{2(1 + L + \beta)(1-\sigma)}{(1-\sigma) + (1 + L + 2\beta)(1-\sigma)}, \left( \frac{1 + L}{\beta(4\sigma - 1)} + 2\beta \right) (1-\sigma), \right\} \]

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Here we have applied (5.4) – (5.6) in their respective order, the last only in the case \( \sigma > 3/4 \). Let us denote by \( B_1, B_2, B_3 \), respectively, the three expressions on the right. If \( L \leq 1/3 \), then

\[
B_1 \leq \frac{10}{3} (1 - \sigma).
\]

If \( L > 1/3 \), note that \( B_2 \) is decreasing in \( L \) and increasing in \( \beta \). It follows that \( B_2 \) is bounded above by taking \( L = \beta = 1/3 \), which is \( \frac{1}{3} + 2(1 - \sigma) \). This is less than \( 10(1 - \sigma)/3 \) so long as \( \sigma \leq 3/4 \). If \( \sigma > 3/4 \), we have \( B_2 \leq 10(1 - \sigma)/3 \) so long as

\[
\sigma \leq 1 - \frac{1 - L}{2(10/3 - (1 + L) - 2\beta)},
\]

and \( B_3 \leq 10(1 - \sigma)/3 \) so long as

\[
\sigma \geq \frac{1}{2} + \frac{1 + L}{2(10/3 - 2\beta)}.
\]

One may check that in the region \( 1/3 < L \leq 1 \), \( 0 < \beta \leq 1/3 \), we have

\[
\frac{1}{2} + \frac{1 + L}{2(10/3 - 2\beta)} \leq 1 - \frac{1 - L}{2(10/3 - (1 + L) - 2\beta)},
\]

and so in this range also we obtain (4.4).

### 7. Long Polynomials: Subconvexity

Suppose now that \( N > X \). Then the coefficients of \( f_m(1/2 + it) \) are smooth and we may write (in the case \( j > 1 \))

\[
f_m \left( \frac{1}{2} + it \right) = \frac{1}{2\pi i} \int_{(0)} L \left( \frac{1}{2} + it + s, \lambda^m \right) \tilde{W}(s) N^s ds. \tag{7.1}
\]

We have \( m \geq 1 \) always, so Theorem 5.3 yields

\[
f_m \left( \frac{1}{2} + it \right) \ll \int_{-\infty}^{\infty} \frac{(m^2 + t^2 + y^2)^{1/6} \log^3(m^2 + t^2 + y^2)}{(1 + |y|)^A} \, dy
\]

\[
\ll (M^2 + T^2)^{1/6} \log^3(M^2 + T^2)
\]

\[
\ll (MT)^{(1+L)/6} \log^3(M^2 + T^2).
\]

If \( j = 1 \), we write

\[
f_m \left( \frac{1}{2} + it \right) = \log N \sum_{a} W \left( \frac{N(a)}{N} \right) \frac{\lambda^m(a)}{N(a)^{1/2+it}} + \sum_{a} W \left( \frac{N(a)}{N} \right) \log \left( \frac{N(a)}{N} \right) \frac{\lambda^m(a)}{N(a)^{1/2+it}}.
\]

The first sum is handled in the same way as before. If \( W(y) \) is replaced by \( W^*(y) = W(y) \log y \), then \( \tilde{W}^* \) decays rapidly on vertical lines just as \( \tilde{W} \), and so in this case we obtain

\[
f_m \left( \frac{1}{2} + it \right) \ll (MT)^{(1+L)/6} \log^3(M^2 + T^2) \log N.
\]

Since \( |f_m \left( \frac{1}{2} + it \right)| = (MT)^{\beta(\sigma-1/2)} \), we deduce that

\[
\sigma \leq \frac{1}{2} + \frac{1}{\beta} \left( \frac{1 + L}{6} + \frac{\log C + \log \beta}{\log MT} + \frac{4 \log (M^2 + T^2)}{\log MT} \right).
\]
for some absolute constant $C$. We may always assume that at least one of $M$ or $T$ is bounded below by any prescribed absolute constant, for otherwise (4.2) follows trivially from $R \ll MT \ll 1$. Thus we may suppose

$$\log \log (M^2 + T^2) > 1 + \log C,$$

and if we let

$$L_0 = \frac{5 \log \log (M^2 + T^2)}{\log MT},$$

then we have

$$\sigma \leq \frac{1}{2} + \frac{1 + L}{6\beta} + \frac{L_0}{\beta^{1/2}}. \quad (7.2)$$

Lastly, we may assume

$$\sigma > \frac{7}{10} + C L_0,$$

for otherwise

$$R \leq MT \leq (MT)^{10(1-\sigma)/3} (\log x)^B.$$

In particular, this happens if $\beta > 5(1 + L)/6$, so we may always assume $\beta < 5(1 + L)/6 \leq 5/3$.

The remainder of the proof of Lemma 3.2 is now a matter of checking various cases, which is done in the next section.

8. Long Polynomials: Case Checking

1. $\sigma \leq 3/4$.

(i) $L \leq 1/4$. Since we may assume $\beta < 5/3$, choose $g$ so that

$$(MT)^{5/6} \leq N^g \leq (MT)^{5/3}.$$  

Then (5.4) gives (4.4) since $\sigma \leq 3/4$.

(ii) $L > 1/4$.

(a) $\beta \leq L + 2/3$. Choose $g$ so that

$$(MT)^{L/2 + 1/3} \leq N^g \leq (MT)^{L + 2/3}.$$  

Then (5.5) gives

$$R \ll \left((MT)^{1+(L/2+1/3)(1-2\sigma)} + (MT)^{(1-L)/2+2(L+2/3)(1-\sigma)}\right) (\log x)^B.$$  

The first exponent is

$$\leq 1 + \frac{11}{24}(1 - 2\sigma) \leq \frac{10}{3}(1 - \sigma)$$

so long as $\sigma \leq 45/58$, which is implied by $\sigma \leq 3/4$. The second exponent increases with $L$ since $\sigma \leq 3/4$, so it is bounded above by taking $L = 1$, which makes the exponent $10(1 - \sigma)/3$.

(b) $\beta > L + 2/3$. Choose $g = 1$ and use (5.5) to get

$$R \ll \left((MT)^{1+\beta(1-2\sigma)} + (MT)^{(1-L)/2+2\beta(1-\sigma)}\right) (\log x)^B.$$  

The first exponent is

$$\leq 1 + \left(\frac{11}{12}\right)(1 - 2\sigma) \leq \frac{10}{3}(1 - \sigma)$$
so long as \( \sigma \leq \frac{17}{18} \), which is implied by \( \sigma \leq \frac{3}{4} \). The second exponent is less than \( 10(1 - \sigma)/3 + CL_0 \) so long as

\[
\sigma \leq 1 - \frac{1 - L}{2(10/3 - 2\beta)} + \frac{CL_0}{10/3 - 2\beta}.
\]

In the region \( 2/3 < \beta \leq 5(1 + L)/6 \), this is implied by the subconvexity restriction \((7.2)\). Here the subconvexity restriction is essential, for if \( L = 3/5 \) and \( \beta = 4/3 \), the above restriction on \( \sigma \) and the subconvexity restriction each become

\[
\sigma \leq \frac{7}{10} + CL_0.
\]

2. \( \sigma > 3/4 \).

Choosing \( g \) as in case 1i and using \((5.6)\) in place of \((5.4)\) furnishes the case \( L \leq 1/4 \) in this range, so we may assume \( L > 1/4 \). We may also assume \( \beta > 1/3 \), for otherwise choose \( g \) so that

\[
(MT)^{4/3} \leq N^9 \leq (MT)^{5/3}.
\]

Then \((5.6)\) gives

\[
R \ll \left( (MT)^{1+L+4(4-6\sigma)/3} + (MT)^{10(1-\sigma)/3} \right) (\log x)^{B}.
\]

The first exponent is \( \leq 10(1 - \sigma)/3 \) so long as

\[
\sigma \geq \frac{9 + 3L}{14}.
\]

This is implied by \( \sigma > 3/4 \) if \( L \leq 1/2 \), so we may assume \( L > 1/2 \). If the condition on \( \sigma \) is not satisfied, instead choose \( g \) so that

\[
(MT)^{5/6} \leq N^9 \leq (MT)^{7/6}.
\]

Then \((5.5)\) gives

\[
R \ll \left( (MT)^{1+5(1-2\sigma)/6} + (MT)^{(1-L)/2+7(1-\sigma)/3} \right) (\log x)^{B}.
\]

The first exponent is \( \leq 10(1 - \sigma)/3 \) so long as \( \sigma \leq 9/10 \), and the second so long as \( \sigma \leq (1 + L)/2 \). Since \( 1/2 < L \leq 1 \), both conditions are implied by

\[
\sigma < \frac{9 + 3L}{14}.
\]

This will be our manner of reasoning throughout the remainder of the case analysis. Figure shows graphically what is happening in this case. We have

\[
\frac{9 + 3L}{14} \leq \min \left( \frac{9}{10}, \frac{1 + L}{2} \right)
\]

and so the minimum of the two bounds is less than \( 10(1 - \sigma)/3 \) in the relevant ranges of \( L \) and \( \sigma \).
\( \sigma = \frac{9+3L}{14} \)

\( \sigma = \min \left( \frac{9}{10}, \frac{1+L}{2} \right) \)

Figure 1: Preliminary estimates for \( \sigma > 3/4 \). Note that the minimum of the blue and orange lines lies below the black line.

(i) \( (1 + \mathcal{L})/6 < \beta \leq (1 + \mathcal{L})/4 \). Choose \( g \) so that

\[
(MT)^{7(1+\mathcal{L})/16} \leq N^g \leq (MT)^{11(1+\mathcal{L})/16}. 
\]

Then \((5.5)\) gives

\[
R \ll (MT)^{1+7(1+\mathcal{L})(1-2\sigma)/16} + (MT)^{(1-\mathcal{L})/2+11(1+\mathcal{L})(1-\sigma)/8}. 
\]

The exponents are less than \( 10(1-\sigma)/3 \)

\[
\sigma \leq \frac{35 - 21\mathcal{L}}{47 - 33\mathcal{L}}, \quad \sigma \leq \frac{91 - 21\mathcal{L}}{118 - 42\mathcal{L}}, 
\]

respectively. If one of these conditions fail, instead choose \( g \) so that

\[
(MT)^{5/3-(1+\mathcal{L})/4} \leq N^g \leq (MT)^{5/3}. 
\]

Then \((5.6)\) gives

\[
R \ll \left( (MT)^{1+\mathcal{L}+(5/3-(1+\mathcal{L})/4)(4-6\sigma)} + (MT)^{10(1-\sigma)/3} \right) (\log x)^B. 
\]

The first exponent is \( \leq 10(1-\sigma)/3 \) so long as

\[
\sigma \geq \frac{20}{31 - 9\mathcal{L}}. 
\]

But in the range \( 1/4 < \mathcal{L} \leq 1 \), we have

\[
\frac{20}{31 - 9\mathcal{L}} \leq \min \left( \frac{35 - 21\mathcal{L}}{47 - 33\mathcal{L}}, \frac{91 - 21\mathcal{L}}{118 - 42\mathcal{L}} \right), 
\]

and so we obtain \((4.4)\).
(ii) \((1 + \mathcal{L})/4 < \beta \leq (1 + \mathcal{L})/2\). This range includes two worst cases for our analysis, namely
\[
\beta = \mathcal{L} = 5/9, \quad \sigma = 4/5, \quad \text{and} \quad \beta = 5/6, \quad \mathcal{L} = 1, \quad \sigma = 9/10.
\]

To see this, note that the estimate (5.5) is optimized by choosing
\[
g_1 = \left\lfloor \frac{1 + \mathcal{L}}{2\beta} \right\rfloor \quad \text{or} \quad g_2 = g_1 + 1,
\]
and the estimate (5.6) is optimized by choosing
\[
g_3 = \left\lfloor \frac{1 + \mathcal{L}}{\beta(4\sigma - 2)} \right\rfloor \quad \text{or} \quad g_4 = g_3 + 1.
\]

These choices then give \(R \ll (MT)^\alpha (\log x)^B\) with
\[
\alpha = \min \left\{ \frac{1}{2} + 2g_2\beta(1 - \sigma), \quad 1 + \mathcal{L} + g_3\beta(4 - 6\sigma), \quad 2g_4\beta(1 - \sigma). \right. 
\]

If \(\beta = \mathcal{L} = 5/9\) and \(\sigma = 4/5\) then \(\alpha = 2/3 = 10(1 - 4/5)/3\), and if \(\beta = 5/6, \mathcal{L} = 1, \text{and} \sigma = 9/10\), then \(\alpha = 1/3 = (10 - 9/10)/3\). Moreover, the subconvexity restriction does not eliminate these ranges values \(\sigma\). We remark on this for two reasons. Firstly, it shows that the exponent \(10(1 - \sigma)/3\) cannot be improved in all ranges without some additional input. Second, this case requires us to choose \(g\) exactly, rather than to make \(N^g\) lie in a certain range, and so we must be rather careful in our reasoning.

Here we can also indicate why an optimal large sieve does not improve our results. Such a large sieve implies the estimate
\[
R \ll (MT)^{1+g\beta(1-2\sigma)} + (MT)^{2g\beta(1-\sigma)}) (\log x)^B,
\]
which is optimized by choosing \(g_5 = [1/\beta]\) or \(g_6 = g_5 + 1\). These give
\[
R \ll (MT)^{1+g_5\beta(1-2\sigma)} \quad \text{and} \quad R \ll (MT)^{2g_6\beta(1-2\sigma)}.
\]
If \(\beta = 5/6, \mathcal{L} = 1, \text{and} \sigma = 9/10\), then we obtain
\[
\min (\alpha, 1 + g_5\beta(1 - 2\sigma), 2g_6\beta(1 - 2\sigma)) = \frac{1}{3},
\]
and so our situation has not changed.

To handle the various ranges of \(\beta, \mathcal{L}, \sigma\), we divide into three subcases depending on the value of \(g_3\). Note that \(g_1 = 1\) for all \(\beta\) under consideration. We define
\[
A_{0,1} = 1 + \beta(1 - 2\sigma) \quad \text{and} \quad A_{0,2} = \frac{1 - \mathcal{L}}{2} + 4\beta(1 - \sigma)
\]
(i) \(\beta \in \left(\frac{1 + \mathcal{L}}{4}, \frac{1 + \mathcal{L}}{2}\right) \cap \left(\frac{1 + \mathcal{L}}{16\sigma - 8}, \frac{1 + \mathcal{L}}{12\sigma - 6}\right)\). This range is empty if \(\sigma \geq 5/6\). In this range we have \(g_3 = 3\), so
\[
\alpha = \min \left\{ \frac{1}{2} + 2\beta \right\} + 4\beta(1 - \sigma), \quad 1 + \mathcal{L} + 3\beta(4 - 6\sigma), \quad 8\beta(1 - \sigma). \right. 
\]

\[
\alpha \in \min \left\{ \frac{1}{2} + 2\beta \right\} + 4\beta(1 - \sigma), \quad 1 + \mathcal{L} + 3\beta(4 - 6\sigma), \quad 8\beta(1 - \sigma). \right.
\]
Note that $\beta \leq 2/3$. Consider $A_{0,2}$ and $A_{1,1}$. We have $A_{0,2} \leq 10(1 - \sigma)/3$ so long as

$$\sigma \leq 1 - \frac{1 - \mathcal{L}}{2(10/3 - 4\beta)}$$

and $A_{1,1} \leq 10(1 - \sigma)/3$ so long as

$$\sigma \geq 1 - \frac{6\beta - (1 + \mathcal{L})}{18\beta - 10/3}.$$ 

In the range $3/4 < \sigma < 5/6$, $1/4 < \mathcal{L} \leq 1$, and $\beta$ in the interval above, we have

$$1 - \frac{6\beta - 1 - \mathcal{L}}{18\beta - 10/3} \leq 1 - \frac{1 - \mathcal{L}}{2(10/3 - 4\beta)}.$$ 

(ii) $\beta \in \left(\frac{1 + \mathcal{L}}{4}, \frac{1 + \mathcal{L}}{2}\right] \cap \left(\frac{1 + \mathcal{L}}{12\sigma - 6}, \frac{1 + \mathcal{L}}{8\sigma - 4}\right]$. In this range we have $g_3 = 2$, so

$$\alpha = \min \begin{cases} 
1 + \beta(1 - 2\sigma), \\
1 - \frac{1}{2} + 4\beta(1 - \sigma), \\
1 + \mathcal{L} + 2\beta(4 - 6\sigma), \\
6\beta(1 - \sigma), 
\end{cases} = \begin{cases} 
A_{0,1}, \\
A_{0,2}, \\
A_{2,1}, \\
A_{2,2}. 
\end{cases}$$

We have $A_{2,2} \leq 10(1 - \sigma)/3$ so long as $\beta \leq 5/9$. If $\beta > 5/9$, we use a combination of the other three estimates above. We have $A_{0,1} \leq 10(1 - \sigma)/3$ so long as

$$\sigma \leq 1 - \frac{1 - \beta}{10/3 - 2\beta},$$

$A_{0,2} \leq 10(1 - \sigma)/3$ so long as

$$\sigma \leq 1 - \frac{1 - \mathcal{L}}{2(10/3 - 4\beta)},$$

and $\beta < 5/6$, and $A_{2,1} \leq 10(1 - \sigma)/3$ so long as

$$\sigma \geq 1 - \frac{4\beta - (1 + \mathcal{L})}{12\beta - 10/3}.$$ 

For $1/4 < \mathcal{L} \leq 1$ and $5/9 < \beta < 5/6$, we have

$$1 - \frac{4\beta - 1 - \mathcal{L}}{12\beta - 10/3} \leq \max \left(1 - \frac{1 - \beta}{10/3 - 2\beta}, 1 - \frac{1 - \mathcal{L}}{2(10/3 - 4\beta)}\right),$$

and for $1/4 < \mathcal{L} \leq 1$, $5/6 \leq \beta < 5/3$ we also have

$$1 - \frac{4\beta - (1 + \mathcal{L})}{12\beta - 10/3} \leq 1 - \frac{1 - \beta}{10/3 - 2\beta}.$$ 

(iii) $\beta \in \left(\frac{1 + \mathcal{L}}{4}, \frac{1 + \mathcal{L}}{2}\right] \cap \left(\frac{1 + \mathcal{L}}{8\sigma - 4}, \frac{1 + \mathcal{L}}{4\sigma - 2}\right]$. In this range we have $g_3 = 1$, so

$$\alpha = \min \begin{cases} 
1 + \beta(1 - 2\sigma), \\
\frac{1 - \mathcal{L}}{2} + 4\beta(1 - \sigma), \\
1 + \mathcal{L} + 2\beta(4 - 6\sigma), \\
4\beta(1 - \sigma), 
\end{cases} = \begin{cases} 
A_{0,1}, \\
A_{0,2}, \\
A_{3,1}, \\
A_{3,2}. 
\end{cases}$$
If $\beta \leq 5/6$, then $A_{3,2} \leq 10(1 - \sigma)/3$. If $\mathcal{L} \leq 2/3$, then $\beta \leq 5/6$, and so we may take assume $\mathcal{L} > 2/3$ and $\beta > 5/6$. In this case $A_{0,1} \leq 10(1 - \sigma)/3 + C\mathcal{L}_0$ so long as

$$\sigma \leq 1 - \frac{1 - \beta}{10/3 - 2\beta} + \frac{C\mathcal{L}_0}{10/3 - 2\beta}.$$ 

But if $\mathcal{L} > 2/3$ and $\beta > 5/6$, then

$$\frac{1}{2} + \frac{1 + \mathcal{L}}{6\beta} \leq 1 - \frac{1 - \mathcal{L}}{10/3 - 2\beta},$$

and so (4.3) follows from the subconvexity restriction (7.2).

(iii) $\beta > (1 + \mathcal{L})/2$. We have $g_1 = 0$, and in the relevant range of $\sigma$ given by the subconvexity restriction (7.2) we also have $g_3 = 1$. Thus

$$\alpha = \min \left\{ \frac{1 - \mathcal{L}}{2} + 2\beta(1 - \sigma), \frac{1 + \mathcal{L} + \beta(4 - 6\sigma)}{4\beta(1 - \sigma)} \right\}.$$ 

If $\beta \leq 5/6$, then the desired bound follows as before. If $\beta > 5/6$, then we use the first bound, which is $\leq 10(1 - \sigma)/3$ so long as

$$\sigma \leq 1 - \frac{1 - \mathcal{L}}{2(10/3 - 2\beta)}.$$ 

But

$$\frac{1}{2} + \frac{1 + \mathcal{L}}{6\beta} \leq 1 - \frac{1 - \mathcal{L}}{2(10/3 - 2\beta)}$$

so long as $\beta \leq 5(1 + \mathcal{L})/6$, which we may assume without loss.

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