A generalized Dynkin game of switching type for defaultable claims in presence of contingent CSA

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Abstract

In the present work we study the solution existence for a generalized Dynkin game of switching type which is shown to be the natural representation for general defaultable OTC contract in which a contingent CSA has been set between the parties. This is a theoretical counterparty risk mitigation mechanism that allows the counterparty of a general OTC contract to switch from zero to full/perfect collateralization and switch back whenever she wants until contract maturity paying some switching costs and taking into account the running costs that emerge over time. In this paper we allow for the strategic interaction between the counterparties of the underlying contract, which makes the problem solution much more tough. We are motivated in this research by the importance to show the economic sense - in terms of optimal contract design - of a contingent counterparty risk mitigation mechanism like our one. In particular, we show that the existence of the solution and the game Nash equilibrium is connected with the solution of a system of non-linear reflected BSDE which remains an problem. We have also proved its existence under strong condition (in the so called symmetric case) highlighting in conclusion some interesting applications in finance and future researches.

1 Introduction

1.1 Aim of the work

In this paper we analyze a theoretical contract in which counterparties want to set a contingent CSA (credit support annex) in order to gain the flexibility and the possibility to manage optimally the counterparty risk. We refer specifically to a contingent risk mitigation mechanism that allows the counterparties to switch from zero to full/perfect collateralization (or even partial) and switch back whenever until maturity $T$ paying some switching costs and taking into account the running costs that emerge over time. We can summarize the characteristics and the basic idea underlying the problem - that we show to admit a natural formulation as a stochastic differential game of switching type- through the so defined contingent CSA scheme shown below (Fig. 1.1), in which - by considering also the funding issue in the picture - is present a third party, an external funder assumed default free ($\lambda = 0$) in order to reduce dimension and technical issues.

Here, motivated by the results obtained in the unilateral case we analyze the problem in a generalized setting allowing for the strategic interplay between the parties of the contingent CSA scheme.

This has lead us to study the solution’s existence and the equilibrium for the stochastic differential game of switching type that we are going to define in section two. We can anticipate that solution of our game - its existence and uniqueness - remains an open problem, as far as we know. Further analysis on game equilibrium are taken over in section three and some model applications in section four.

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1We refer to the stochastic optimal control formulation highlighted in chapter three of Mottola (2013)
1.2 Literature review

Let us briefly review the body of literature related to stochastic differential games which is very wide. The roots of the theory are founded in the pioneering works of von Neumann\textsuperscript{2} for (mainly cooperative) zero sum game and Nash\textsuperscript{3} for (non cooperative) non-zero sum game - and the work of Isaacs\textsuperscript{4} who studied for first the differential games in a deterministic setting. In the stochastic framework it is worth of mention the seminal work of Eugene Dynkin who firstly analyzed stochastic differential games where the agent control set is given by stopping times, so that these "games on stopping" are known as Dynkin games in his honor.

Certainly, it is impossible to mention here all the numerous important contribution in this field of research and to give a systematic account of the theory and of the literature. For a complete treatment of the different type of games we refer to the book of Isaacs on differential games. So we give in the following just a simplified classification restricting ourselves to the literature more related to our stochastic switching control problem.

This classification is based on the following main categories:

a) game and equilibrium type: it includes zero and non-zero-sum games whose solution can be searched mainly in terms of cooperative or non-cooperative (Nash) equilibrium. This depends also on the characteristics of the game which are mainly the system dynamic - that can be markovian/non markovian - and controls which can be state controls, stopping controls (as in Dynkin games) or both which are called mixed control and stopping. Also the number of player is relevant, here we focus on the case $p = 2$.

\textsuperscript{2}“Theory of games and economic behavior” (1944), Princeton Press
\textsuperscript{3} “Equilibrium Points in N-person Games” (1950), Proceedings of the National Academy of Sciences
\textsuperscript{4}“Differential games” (1999), Dover.
b) solution approaches: these are mainly the analytical approach that allows - typically under a markovian framework - to formulate the stochastic differential game (SDG) as a system of (second order) Hamilton-Jacobi-Bellman equations or variational inequalities to solve, proving existence and (possibly) uniqueness of the solution, namely of the equilibrium of the game. The main solution techniques are the ones related to PDE theory, the dynamic programming principle and viscosity solution.

Worth of mention, from the analytical point of view, are the important works of Bensoussan and Friedman (1977) that for first showed the existence of a Nash equilibrium for a non-zero-sum SDG with stopping \( \{\tau_1, \tau_2\} \) as controls, formulating the problem as a system of quasi-variational inequalities (solved through fixed point methods), assuming continuous and bounded running rewards and terminal rewards; instead Fleming and Souganidis (1989) for first showed the existence and uniqueness of the solution/equilibrium for zero-sum SDG through dynamic programming and viscosity solution approach. These techniques have become very popular and used in the recent literature given the deep connection with probabilistic tools (see for example El Karoui et al. (1997)).

In fact, the probabilistic approach is the other more general one that makes use of the martingale (also via duality methods) and Snell envelope theory and, in addition, of the deep results of the forward-backward SDE theory in order to derive existence and uniqueness of the optimal control/stopping strategy for the game.

The main works worth of mention - other that the already mentioned work of Cvitanic and Karatzas (1996) that for first highlights the connection between the solution of zero-sum Dynkin game and that of doubly reflected BSDE (other than its analytical solution) - are that of: Hamadene (1998) that shows how the solution of a SDG is related to that of a backward-forward SDE; Hamadene and Lepeltier (2000) that extend the analysis through reflected BSDE to “mixed game” problems; El Karoui and Hamadene (2003) that generalize the existence and uniqueness results for zero and non-zero-sum game with “risk sensitive” controls; Hamadene and Zhang (2008) that use Snell envelope technique to show the existence of a Nash equilibrium for non-zero-sum Dynkin game in a non-markovian framework and Hamadene and Zhang (2010) that tackle the solution of a general switching control problem via systems of interconnected (nonlinear) RBSDE (with oblique reflection).

Much of these literature and works have been inspired also by valuation problems in the financial industry. We refer mainly to the american game option problem as defined in Kifer (2000) (also known as israeli option). This has given impulse to the literature related mainly to convertible (and switchable) bond valuation whose solution can be related to that of a zero-sum Dynkin game.

1.3 Some examples of Dynkin games

Let us briefly remind that stochastic differential games are a family of dynamic, continuous time versions of differential games (as defined by Isaacs) incorporating randomness in both the states and the rewards. The random states are described typically by adapted diffusion processes whose dynamics are known (or assumed). To play a game, a player receives a running reward cumulated at some rate till the end of the game and a terminal reward granted at the end of the game. The rewards are related to both the state process and the controls at the choice of the players, as deterministic or random functions or functionals of them. A control represents a player action in attempt to influence his rewards. Assuming his rationality, a player acts in the most profitable way based on his knowledge represented by his information filtration. Before starting the formulation and the analysis of our generalized Dynkin game of switching type, let us
- Non-zero-sum Dynkin game: Given a standard probability space represented by the triple $(\Omega, \mathcal{F}, \mathbb{P})$ where we define $W = (W_t)_{0 \leq t \leq T}$ be a standard $d$-dimensional Brownian motion adapted to the space filtration. Assuming as true the usual conditions on the drift function $\mu(\cdot)$ and volatility function $\sigma(\cdot)$, such that the following SDE admits a unique solution

\[
dy(t) = \mu(y(t), t)dt + \sigma(y(t), t)dW(t), \quad t \in [0, T]
\]

\[
y(0) = y_0.
\]

Let (for $p = 1, 2$) $f_p(y, t)$ the running reward function and $\phi_p(y, t)$, $\psi_p(y, t)$ the reward function obtained by the players upon stopping the game be continuous and bounded in $\mathbb{R}^d \times [0, T]$, with $f_p \in L^2$ square integrable and $\psi_p \leq \phi_p$ for all $(y, t) \in \mathbb{R}^d \times [0, T]$. Then let $g_p(y(T))$ the terminal reward function also continuous and bounded.

In a game of this kind, the two players have to decide optimally when to stop the game finding the optimal control given by the stopping times $(\tau, \tau_2)$ that give the maximum expected reward. So let us set the payoff functional for the two players of this Dynkin game as follows

\[
J^p(y, \tau_1, \tau_2) = \mathbb{E} \left[ \int_{\tau_1 \wedge \tau_2 \wedge T} f_p(y(s), s)ds + \mathbb{1}_{(\tau_1 < \tau_2)} \phi_p(y(\tau_1), \tau_1) + \mathbb{1}_{(\tau_1 \geq \tau_2, \tau > \tau_1)} \psi_p(y(\tau_2), \tau_2) + \mathbb{1}_{(\tau_1 = \tau_2 = T)} g_p(y(T)) \right] \quad \text{for} \quad j \neq i(\in \{1, 2\})
\]

and for $(t \leq \tau_1 \leq T)$. Being in a non-zero-sum game with the players that aims to maximize their payoff $J^p(\cdot)$ without cooperation, the problem here is to find a Nash equilibrium point (NEP) for the game, that is to determine the couple of optimal stopping times $(\tau_1^*, \tau_2^*)$ such that

\[
J^1(y, \tau_1^*, \tau_2^*) \geq J^2(y, \tau_1, \tau_2^*), \quad \forall \tau_1, \tau_2 \in [t, T]
\]

\[
J^2(y, \tau_1^*, \tau_2^*) \geq J^1(y, \tau_1^*, \tau_2), \quad \forall \tau_1, \tau_2 \in [t, T]
\]

namely the supremum of the payoff functional over the stopping time set. In other words, the NEP implies that every player has no incentive to change his strategy given that the other one has already defined optimally his strategy.

This type of game, as shown in Bensoussan and Friedman (1977), has an analytical representation given by a system of variational inequalities but it admits also a stochastic counterpart through system of BSDE with reflecting barrier. We return to its formal definition in the next section in relation to our problem. The Nash equilibrium defined above can be fairly generalized in the case of mixed game of control and stopping. We show this below in relation to zero-sum games.

- Zero-sum mixed game: A zero-sum game is characterized by the antagonistic interaction of the players that in this case has the same payoff functional but their objective are different because for one player the payoff is a reward (let’s think typically at the buyer of a convertible bond) that he wants to maximize, while for the other one is a cost that he intends to minimize.

In the generalized case of mixed games of control and stopping, the set of control will...
be enriched by the $\mathcal{F}_t$-progressively measurable process $(\alpha_t)_{t \leq T}$ and $(\beta_t)_{t \leq T}$ that are the intervention function namely the state controls respectively for the player $p_1$ and $p_2$. In addition, the players have to decide optimally when to stop the game setting the stopping times $\tau$ (for $p_1$) and $\sigma$ (for $p_2$). Indeed, the system dynamic being controlled by the agents can be expressed as the following controlled diffusion (remaining in a markovian framework):

$$
dy(t)^{\alpha,\beta} = \mu(t,y_t^{\alpha,\beta},\alpha_t,\beta_t)dt + \eta(t,y_t^{\alpha,\beta},\alpha_t,\beta_t)dW(t), \quad t \in [0,T]
$$

The zero-sum game payoff being the same for both the players will be

$$
\Gamma(\alpha,\tau;\beta,\sigma) := E \left[ \int_{t}^{T \wedge \tau \wedge \sigma} f(s,y_s^{\alpha,\beta},\alpha_s,\beta_s)ds + \mathbb{1}_{\{\tau \leq \sigma < T\}} \phi(\tau,y_\tau^{\alpha,\beta}) + \mathbb{1}_{\{\tau < \sigma\}} \psi(\sigma,y_\sigma^{\alpha,\beta}) + \mathbb{1}_{\{\tau = \sigma = T\}} g(y_\tau^{\alpha,\beta})(T) \right] \quad (t \leq \tau \leq \sigma < T)
$$

where the running and reward functions are intended to be the same as in the non-zero-sum case but clearly now they are the same for both players.

The solution of this SDG is typically tackled by studying the upper and lower value function of the players, which are

$$
\mathcal{U}(t,y) := \sup_{\alpha} \inf_{\beta} \sup_{\tau} \inf_{\sigma} \Gamma(\alpha,\tau;\beta,\sigma) \quad (upper\ value\ function) \\
\mathcal{L}(t,y) := \inf_{\beta} \sup_{\alpha} \inf_{\sigma} \sup_{\tau} \Gamma(\alpha,\tau;\beta,\sigma) \quad (lower\ value\ function)
$$

Under some standard condition on the reward function and on controls, the problem has been tackled analytically representing the lower and upper value of the game as a system of nonlinear PDE with two obstacles/barriers, defined as follows

$$
\begin{align*}
\min \left\{ u(t,y) - \phi(t,y), \max \left\{ \frac{\partial u}{\partial t}(t,y) - H^-(t,y,u,Du,D^2u), u(t,y) - \psi(t,y) \right\} \right\} = 0 \\
u(T,y) = g(y),
\end{align*}
$$

$$
\begin{align*}
\min \left\{ v(t,y) - \phi(t,y), \max \left\{ \frac{\partial v}{\partial t}(t,y) - H^+(t,y,v,Dv,D^2v), v(t,y) - \psi(t,y) \right\} \right\} = 0 \\
v(T,y) = g(y),
\end{align*}
$$

where $H^+(\cdot)$ and $H^-(\cdot)$ are the hamiltonian operators (as defined in chapter four) associated to the upper and lower value function of the SDG. To solve the system, the unknown solution function $u$ and $v$ can be shown (under some technical assumptions) to be viscosity solutions of the above two PDE with obstacles and to coincide with the value functions $\mathcal{U}(t,y)$ and $\mathcal{L}(t,y)$ of the game.

In particular, when the Isaacs condition holds, namely for $(t,y,u,q,X) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times (\mathcal{S}^d)$

$$
H^-(t,y,u,q,X) = H^+(t,y,u,q,X)
$$

then the two solution coincide and the SDG has a value namely

$$
V := \sup_{\alpha} \inf_{\beta} \sup_{\tau} \inf_{\sigma} \Gamma(\alpha,\tau;\beta,\sigma) = \inf_{\beta} \sup_{\alpha} \inf_{\sigma} \sup_{\tau} \Gamma(\alpha,\tau;\beta,\sigma)
$$
which is called the saddle point equilibrium of the mixed zero-sum game.

We mention also that in this case, the SDG has a stochastic representation which is expressed in terms of a doubly reflected BSDE (2RBSDE\textsuperscript{7}). In particular, setting the terminal reward \( \xi \), the early exercise rewards \( \phi_t = U_t \) and \( \psi_t = L_t \) which represent the two barriers of the value process, therefore the generator function \( f \), the value process \( Y_t, Z_t \) which is the conditional expectation/volatility process that helps \( Y_t \) to be \( \mathcal{F}_t \)-measurable and \( K \) the compensator process, it can be shown that the following 2RBSDE solution\textsuperscript{8}

\[
\begin{align*}
Y_t &= \xi + \int_t^T f(s,Y_s,Z_s) + (K^+_T - K^+_s) - (K^-_T - K^-_s) - \int_t^T Z_s dW_s \quad \forall t \leq T \\
L_t \leq Y_t \leq U_t, \quad \forall t \leq T,
\end{align*}
\]

provides the value function \( V \) of the associated zero sum (Dynkin) game. We underline that if we restrict to the case of data not depending on controls \( \alpha \) and \( \beta \) and consider the players criterion as the expectation of their payoff under the risk neutral measure, the game become a zero-sum Dynkin game whose solution is involved when dealing with the pricing of American game options, which is a natural representation for convertible bond valuation/pricing.

2 Defaultable Dynkin game of switching type.

2.1 Framework and assumptions

To begin is convenient to describe the framework in which we work and to give some useful definitions of the processes and variables involved. The framework is the typical one of the reduced-form models literature and we refer to Mottola (2013) for details. Let us just recall that in our usual probability space described by the triple \((\Omega, \mathcal{G}_t, \mathbb{Q})\) lie two strictly positive random time \( \tau_i \) for \( i \in \{A,B\} \), which represent the default times of the counterparties of the contract considered in our model, so that the market filtration is the enlarged one - which includeds the default filtration \( \mathbb{H} \) - and it is denoted by \( \mathcal{G} = \mathcal{F} \lor \mathbb{H}^A \lor \mathbb{H}^B \) where \( \mathcal{F} \) is the (risk-free) market filtration usually generated by a Brownian motion \( W \) (or a vector \( \hat{W} \)) adapted to it, under the real measure \( \mathbb{Q} \). All the processes considered were càdlàg semimartingales \( \mathcal{G} \) adapted and \( \tau^i \) are \( \mathcal{G} \) stopping times but for technical reasons, we also assume that the immersion property so that every càdlàg \( \mathcal{G} \)-adapted (square-integrable) process is also \( \mathcal{F} \)-adapted.

As already mentioned, counterparties \( \{A,B\} \) are both defaultable and are assumed to behave rationally and have the objective to minimize the overall costs related to counterparty risk - quantified through the BCVA - and those related to collateral and funding. The information flow is assumed symmetric.

2.2 Main definitions: BCVA, contingent CSA and funding

Let us state the main definitions that are needed to model our claim with contringent CSA. Also here we just state the objects involved, for proofs and details we refer to the already mentioned work and the reference therein.

\textsuperscript{7}Its connection with the analytical representation and viscosity solution of PDE with obstacles, as already mentioned, has been established in the work of Cvitanic and Karatzas (1996).

\textsuperscript{8}This is established in the work of Cvitanic and Karatzas (1996).
1) **BCVA definition** The bilateral CVA process of a defaultable claim with bilateral counterparty risk \((X; A, Z; \tau)\) maturing in \(T\) satisfies the following relation:\footnote{The formulation is seen from the point of view of \(B\). Being symmetrical between the party, just the signs change.}

\[
BCVA_t = CV A_t - DVA_t \\
= B_tE_{Q^*}\left[\mathbb{1}_{\{t<\tau=\tau_B \leq T\}}B_{\tau}^{-1}(1 - R^B_t)(S^t_f)^{-} \big| \mathcal{G}_t \right] + \\
- B_tE_{Q^*}\left[\mathbb{1}_{\{t<\tau=\tau_A \leq T\}}B_{\tau}^{-1}(1 - R^A_t)(S^t_f)^+ \big| \mathcal{G}_t \right]
\]  

(1)

for every \(t \in [0, T]\), where \(B_t\) indicates the discount factor, \(R^i_t\) for \(i \in \{A, B\}\) is the counterparty recovery rate (process), where the clean price process \(S^t_f\) would be simply represented by the integral over time of the dividend process under the relative pricing measure, that is

\[
S^t_f = B_tE_{Q^*}\left(\int_{[t,T]} B_u^{-1}dD^t_f \big| \mathcal{F}_t \right) \ t \in [0, T].
\]  

(2)

and \(D^t_f\) is the clean dividend process of the default-free contract

\[
D^t_f = X\mathbb{1}_{[T, \infty)}(t) + \sum_{i \in \{A, B\}} \left(\int_{[t,T]} (1 - H^i_u)dA^i_u + \int_{[t,T]} Z_u dH^i_u \right) \ t \in [0, T].
\]  

(3)

where \(X\) is the \(\mathcal{F}\)-adapted process indicating the final payoff, \(A\) the contract cashflows and \(\tau = \tau^f = \infty\). Let us state also the definitions of (bilateral) \textit{risky dividend and price process} of a defaultable claim:

\[
D_t = X\mathbb{1}_{\{T<\tau\}}\mathbb{1}_{[T, \infty)}(t) + \sum_{i \in \{A, B\}} \left(\int_{[t,T]} (1 - H^i_u)dA^i_u + \int_{[t,T]} Z_u dH^i_u \right) \ t \in [0, T].
\]  

(4)

for \(i \in \{A, B\}\) and

\[
NPV_t = S_t = B_tE_{Q^*}\left(\int_{[t,T]} B_u^{-1}dD_u \big| \mathcal{G}_t \right) \ t \in [0, T].
\]  

(5)

2) **BCVA definition with contingent CSA** It can be shown, using the definition of \(D^C\) and \(S^C\) as the dividend and price process in presence of the contingent CSA (see Mottola (2013) for details)

\[
D^C_t = D^t_f \mathbb{1}_{\{z_j = 0\}}\mathbb{1}_{\{\tau_j \leq t < \tau_{j+1}\}} + D_t \mathbb{1}_{\{z_j = 1\}}\mathbb{1}_{\{\tau_j \leq t < \tau_{j+1}\}}
\]

\[
S^C_t = S^t_f \mathbb{1}_{\{z_j = 0\}}\mathbb{1}_{\{\tau_j \leq t < \tau_{j+1}\}} + S_t \mathbb{1}_{\{z_j = 1\}}\mathbb{1}_{\{\tau_j \leq t < \tau_{j+1}\}}
\]

that the bilateral CVA of a contract with contingent CSA of switching type, say \(BCVA^C_t\), is the \(\mathcal{G}_t\)-adapted process defined, for any time \(t \in [0, T]\), for every switching time \(\tau_j \in [0, T]\) and \(j = 1, \ldots, M\), switching indicator \(z_j\) and default time \(\tau\) (defined as above), as follows

\[
BCVA^C_t = BCVA_t \mathbb{1}_{\{z_j = 1\}}\mathbb{1}_{\{\tau_j \leq t < \tau_{j+1}\}} + 0 \mathbb{1}_{\{z_j = 0\}}\mathbb{1}_{\{\tau_j \leq t < \tau_{j+1}\}}, \text{ for } t \in [0, T \wedge \tau],
\]  

(6)

where the expression for \(BCVA\) is known from the former point.
3) Collateral definition with contingent CSA 
In order to generalize collateralization in presence of contingency CSA, we recall the definition collateral account/process $Coll_t : [0,T] \rightarrow \mathbb{R}$ is a stochastic $\mathcal{F}_t$-adapted process defined as

$$
Coll_t = \mathbb{I}_{\{S^f_t > \Gamma_B + MTA\}}(S^f_t - \Gamma_B) + \mathbb{I}_{\{S^f_t < \Gamma_A - MTA\}}(S^f_t - \Gamma_A),
$$

(7)
on the time set $\{t < \tau\}$, and

$$
Coll_t = \mathbb{I}_{\{S^f_{\tau^-} = \Gamma_B + MTA\}}(S^f_{\tau^-} - \Gamma_B) + \mathbb{I}_{\{S^f_{\tau^-} < \Gamma_A - MTA\}}(S^f_{\tau^-} - \Gamma_A),
$$

(8)
on the set $\{\tau \leq t < \tau + \Delta t\}$, with thresholds $\Gamma_i$ for $i = \{A, B\}$ and positive minimum transfer amount $MTA$.

The perfect collateralization case, say $Coll^{perf}_t$, can be shown to be be always equal to the mark to market, namely to the (default free) price process $S^f_t$ of the underlying claim, that is formally

$$
Coll^{perf}_t = \mathbb{I}_{\{S^f_t > 0\}}(S^f_t - 0) + \mathbb{I}_{\{S^f_t < 0\}}(S^f_t - 0) = S^f_t \forall t \in [0,T], \text{ on } \{t < \tau\}. \quad (9)
$$

and

$$
Coll^{perf}_t = S^f_{\tau^-} \forall t \in [0,T], \text{ on } \{\tau \leq t < \tau + \delta t\}
$$

(10)

Let us remind that in presence of perfect/full collateralization one can easily show that

$$
BCV\ A_i^{Coll^{perf}} = S^f_t - S_t = 0 \iff S^f_t = S^f_t \forall t \in [0,T].
$$

(11)

Generalizing, the contingent collateral $Coll^C_t$ can be defined as the $\mathcal{F}_t$-adapted process defined for any time $t \in [0,T]$ and for every switching time $\tau_j \in [0,T]$ and $j = 1, \ldots, M$, switching indicator $z_j$ and default time $\tau$ (defined above as $\min\{\tau_A, \tau_B\}$), which is formally

$$
Coll^C_t = S^f_t \mathbb{I}_{\{z_j = 0\}} \mathbb{I}_{\{\tau_j \leq t < \tau_{j+1}\}} + 0 \mathbb{I}_{\{z_j = 1\}} \mathbb{I}_{\{\tau_j \leq t < \tau_{j+1}\}} \text{ on } \{t < \tau\}
$$

(12)

(on the set $\{t < \tau\}$).

4) Funding definition 
As regards funding, in our setting we allow for difference in the funding rates between counterparties. In particular, we assume the existence of the following funding asset $B^{opp^r}_t, B^{borr^r}_t$ and $B^{rem^r}_t$. In particular, under the assumptions of segregation (no collateral rehypotecation), collateral made up by cash and BCVA not funded let us highlight that if counterparty $i \in \{A, B\}$ has to post collateral in the margin account, she sustains a funding cost, applied by the external funder, represented by the borrowing rate $r_i^{borr^r} = r_t + s_t$ that is the risk free rate plus a credit spread (s) (that is usually different from the other party). By the other side, the counterparty receives by the funder the remuneration on the collateral post, that is usually defined in the CSA as a risk free rate plus some basis points, so that we can approximate it at the risk free rate, that is $r_t + bp_t \equiv r_t$. Hence we assume the following dynamics for the funding assets (which can be different between counterparties)

$$
dB^{borr^r}_t = (r_t + s_t)B^{borr^r}_t dt, \quad i \in \{A, B\}
$$

(13)

$$
dB^{rem^r}_t = (r_t + bp_t)B^{rem^r}_t dt, \quad i \in \{A, B\}
$$

(14)

\footnote{This assumption is relevant in order to simplify the problem formulation and to deal with its recursive nature.}
Instead, considering the counterparty that call the collateral, as above the collateral is remunerated at the rate given (as the remuneration for the two parties can be different) by $B^{rem}$, but she cannot use or invest the collateral amount (that is segregated), so she sustains an opportunity cost, that can be represented by the rate $r_{\text{opp}} = r_t + \pi_t$, where $\pi$ is a premium over the risk free rate. Hence, we assume the existence of the following asset too

$$dB_{t}^{\text{opp}} = (r_t + \pi_t)B_t^{\text{opp}} \, dt \quad i \in A, B$$

(15)

Let us underline that given the symmetrical nature of processes (except for the funding ones) defined above, the following relation are valid

$$BCVA_t^A = -BCVA_t^B \quad \forall t \in [0, T]$$

$$BCVA_t^{CA} = -BCVA_t^{CB} \quad \forall t \in [0, T]$$

$$\text{Coll}_t^A = \text{Coll}_t^B \quad \forall t \in [0, T].$$

2.3 Model dynamics, controls and cost functionals

In our contingent CSA model of multiple switching type (with finite horizon), both counterparties $A, B$ are free to switch from zero to perfect collateralization every time in $[0, T]$. Hence their control sets are made up by sequences of switching times - say $\tau_j \in \mathcal{T}$ - and switching indicator $z_j \in \mathcal{Z}$, with $\mathcal{T} \subset [0, T]$, that we define formally as follows

$$C^A = \{\mathcal{T}^A, \mathcal{Z}^A\} = \{\tau_j^A, z_j^A\}_{j=1}^M, \quad \forall \tau_j^A \in [0, T], \, z_j^A \in \{0, 1\};$$

(16)

$$C^B = \{\mathcal{T}^B, \mathcal{Z}^B\} = \{\tau_j^B, z_j^B\}_{j=1}^M, \quad \forall \tau_j^B \in [0, T], \, z_j^B \in \{0, 1\}.$$

(17)

with the last switching $\{\tau_M^i \leq T\} \; (M < \infty)$. We recall that $\tau_j^i$ are, by definition of stopping times, $\mathcal{F}_t$-measurable random variable, while the $z_j^i$ are $\mathcal{F}_{\tau_j^i}$-measurable switching indicators taking values in our model $\forall j = 1, \ldots, M$

$$\begin{cases} z_j = 1 & \text{"zero collateral" (full CVA)} \\ z_j = 0 & \text{"full collateral" (null CVA)} \end{cases}$$

Clearly controls affect also our model dynamic. As regards this point, we assume general markovian diffusions for $(X, \lambda^i) \; i \in \{A, B\}$ namely the interest rates and the default intensities of counterparties. From the definitions of contingent CSA and (bilateral) CVA given in the last section, we highlight that the switching controls enter and affect the dynamic of these processes. In fact, as we know, switching to full collateralization implies $BCVA = 0$, that is $S_t = S_t^D = \text{Coll}_t^{perf}$. This means no counterparty risk and so the default intensities’ dynamic $d\lambda^i$ (for $i \in \{A, B\}$) won’t be relevant, just $dX$ will be considered while $z = 0$ (namely until the collateralization will be kept active). So, formally, we have:

if $\{z_j = 1\}$ and $\{\tau_j \leq t < \tau_{j+1}\}$ $\Rightarrow$

$$D_t^C = D_t \Rightarrow$$
$$S_t^C = S_t \Rightarrow$$

$$BCVA_t^C = BCVA_t \forall t \in [0, T \land \tau]$$
so that the relevant dynamic to model the BCVA process in this regime is
\[
\begin{align*}
    dX_t &= \mu(t, X_t)X(t) dt + \sigma(t, X_t)X(t) dW^x_t; \quad X(0) = x_0 \\
    d\lambda^A_t &= \gamma(t, \lambda^A_t)\lambda^A(t) dt + \nu(t, \lambda_t)\lambda^A(t) dW^\lambda^A_t; \quad \lambda^A(0) = \lambda^A_0 \\
    d\lambda^B_t &= \chi(t, \lambda^B_t)\lambda^B(t) dt + \eta(t, \lambda_t)\lambda^B(t) dW^\lambda^B_t; \quad \lambda^B(0) = \lambda^B_0 \\
    d\langle X, \lambda^A \rangle_t &= d\langle W^x, W^\lambda^A \rangle_t = \rho_{X,\lambda^A} dt. \\
    d\langle X, \lambda^B \rangle_t &= d\langle W^x, W^\lambda^B \rangle_t = \rho_{X,\lambda^B} dt.
\end{align*}
\]
if \( \{z_j = 0\} \) and \( \{\tau_j \leq t < \tau_j + 1\} \) \( \Rightarrow \)
\[
\begin{align*}
    D^C_t &= D^{rf}_t \Rightarrow \\
    S^C_t &= S^{rf}_t = \text{Coll}^{perf}_t \Rightarrow \\
    BCVA^C_t &= 0 \forall t \in [0, T \wedge \tau] \\
\end{align*}
\]
so that the relevant dynamic to model the process in this regime will be just
\[
\begin{align*}
    dX_t &= \mu(t, X_t)X(t) dt + \sigma(t, X_t)X(t) dW^x_t; \quad X(0) = x_0 
\end{align*}
\]
Let us end up this section by expressing the generalized system dynamic - in vectorial form - as follows
\[
\begin{align*}
    dY^C_{ad}(t) := \begin{bmatrix}
    dt \\
    dX_t \\
    d\lambda^A_t \\
    d\lambda^B_t \\
    dZ
\end{bmatrix}, \quad Y^C_{ad}(0) = \begin{bmatrix}
    t = 0 \\
    x_0 \\
    \lambda^A_0 \\
    \lambda^B_0 \\
    Z_0 = 1
\end{bmatrix} \tag{22}
\end{align*}
\]
As regards counterparties cost functionals’ formulation, we remind that both are assumed coherently counterpart risk averse, but in this case they can have different preference/cost functions in which they need to take in account also the optimal control strategy of the other party, namely its best response function \( b^{-i}(u^i) \) where \( u^i := C^i \) and \( i \in \{A, B\} \). We are going to discuss more about it in the next section on game formulation. Here, let us be more explicit about the formulation of counterparties costs for which we assume - for convenience - quadratic preferences for both \( \{A, B\} \) generalized to take in account the optimal control strategy of the other party over time \( t \in [0, T] \), that is formally:\(^{12}\)
\[
\begin{align*}
    a) \text{ Running costs:} & \quad F^A(Y^u, b^B(u^A), t) = \begin{cases}
    \sum_j \left[ (CVA^A(s) - DVA^A(s)) - \delta^B(s, u^* - A) \right]^2 & \text{if } \{z_j = 1\} \\
    \sum_j \left( \int_u^{T \wedge \tau^B_{j+1}} R^A(s) [NPV^A(v)] du - NPV^A(s) \right) - \delta^B(s, \delta^B(s, u^* - A)) \right]^2 & \text{if } \{z_j = 0\}
    \end{cases}
\end{align*}
\]
\(^{11}\)Otherwise, it would not have sense for both to sign a contract wich give flexibility to activate collateralization whenever is optimal.
\(^{12}\)For all the details we refer to Mottola (2013).
∀ \(s, \tau \in [t, T] \) and \(i \in \{A, B\}\), and for counterparty \(B\)

\[
F^B(Y^u, b^A(u^B), t) = \begin{cases} 
\sum_j \left[ (CV^A(s) - DV^A(s)) - \delta^A(s, u^A) \right]^2 & \text{if } \{z^j = 1\} \\
\sum_j \left( \int_{\tau^{j+1}}^{\tau^B} R^B(s) [NPV^B(v)] du - NPV^B(s) - \delta^A(s, u^A) \right)^2 & \text{if } \{z^j = 0\}
\end{cases}
\]

∀ \(s, \tau \in [t, T] \) and \(i \in \{A, B\}\).

b) Terminal costs:

\[
G^A(Y^A(T), b^B(u^A)) = \begin{cases} 
\left( -NPV^A(T) - \delta^B(T, u^A) \right)^2 & \text{if collateral is active} \\
\left( 0 - \delta^B(T, u^A) \right)^2 & \text{otherwise no collateral}
\end{cases}
\]

and for the other party

\[
G^B(Y^B(T)), b^A(u^B) = \begin{cases} 
\left( -NPV^B(T) - \delta^A(T, u^B) \right)^2 & \text{if collateral is active} \\
\left( 0 - \delta^A(T, u^B) \right)^2 & \text{otherwise no collateral}
\end{cases}
\]

c) Instantaneous switching costs

\[
K(Y^u(\tau^A), z^j) = \sum_{j \geq 1} M e^{-r_j t_j} c_j(t) \mathbb{1}_{\{\tau^j < T\}}, \quad \forall \ \tau^j, z^j \in \{T^i, Z^i\},
\]

for \(i \in \{A, B\}\).

2.4 Game formulation and pure strategies definition

In this section we pass to give a generalized formulation for our contingent CSA scheme in which allowing for the strategic interaction between the players - which are the counterparties of this theoretical contract - we are lead to formulate our problem as a stochastic differential game whose study of the equilibrium is central for the existence of a solution and the optimal design of our contingent scheme.

In order to formulate the game, firstly, we recall that in our model for the contingent CSA scheme we assume no fixed times or other rules for switching, that is the counterparty can switch optimally every time until contract maturity \(T\) in order to minimize its objective functional. But the functionals, as set formally in the former section, now are generalized and not symmetrical between the parties: as already mentioned, both players are assumed to remain risk averse to the variance of bilateral CVA, collateral and funding costs, but depending on the different parametrization of the functionals (that we show below) and instantaneous switching costs other than the difference in default intensities, the problem can be naturally represented by a generalized non-zero-sum Dynkin game. This is a non-zero-sum game given that the player payoff functionals are not symmetrical and generalized in the sense that player controls are not just simple stopping times but sequences of random times that define the optimal times to switch \(\tau_j\).
from a regime to another one (together with switching indicators $z_j$). 
Therefore, given that the “right to switch” is bilateral and we assume no other rules/constraint on controls set by contract, the other “player” optimal strategy - and hence the strategic interaction with the other party - becomes central to define the own optimal switching strategy.

Let us be more formal and, building on the definitions of section 2.3, we define our model’s SDG as a generalized Dynkin game of switching type as follows.

**Definition 2.4.1 (Dynkin game of switching type definition).** Let us consider two players/counterparties \( \{ A, B \} \) that have signed a general contract with a contingent CSA of switching type. Given the respective payoff functionals \( F_i(z, \zeta) \) (or running reward) where \( i \in \{ A, B \} \) and \( Z \in \{ z, \zeta \} \) (being different also by switching regimes), terminal reward \( G_i(z, \zeta) \) and instantaneous switching costs \( c_{ij}(z, \zeta) \), counterparties (namely the players) are rational and interact strategically in a non cooperative way and they aim to minimize the following objective functional

\[
J^i(y, u^i) = \inf_{u^i \in U^i} \mathbb{E} \left[ \sum_j \int_t^{\tau_j^{-1}(T)^+} B_s \left[ F_{ij}(y_s, u^i, b^{-i}(u^i)) \right] ds + \sum_{j \geq 1} B_t c_{ij} \mathbb{1}_{\{ \tau_j^{-1}(T) < T \}} + G_i(y(T), b^{-i}(u^i)) \right] \text{ for } i \in \{ A, B \}, \ j = 1 \ldots M
\]

where we denote the discount factors with \( B(.) \), the system dynamic as defined by \( dY^{C_{ad}} \), controls defined in (16)-(17) and we have set for notational convenience \( u^i := \{ T^i, Z_i \} \) for \( i \in \{ A, B \} \).

We underline in definition 2.4.1 that the cost functions \( F_{ij}(.) \) are here generalized in order to take in account the strategic interaction with the other player represented formally by a response function \( b^{-i}(u^i) \), here left unspecified. Therefore, let us highlight that the payoffs functions, being intended as formulated in section 2.3, can differ here (between \( A \) and \( B \)) for the following terms

\[
\delta^A(.) \prec \delta^B(.) \quad R^A_t \prec R^B_t \quad c^A_z \prec c^B_z
\]

where

- a) \( R^i(.) \) is the funding-collateral cost factor (see Mottola (2013));
- b) \( \delta^i(.) \) is the running cost function threshold which is generalized her taking as argument the optimal response and control strategies of the other player;
- c) \( c^i_z \) are the well-known instantaneous costs from switching.

**Remarks 2.4.2.** The game as formulated above in (23) is fairly general; in addition, one could also introduce the possibility for the players to stop the game adding a stopping time (and the related reward/cost function) to the set of controls made up of switching times and indicators. From the financial point of view, this can be justified by a early termination clause set in the contingent CSA defined by the parties. Anyway, given the problem recursion, this would add greater complications that we leave for further research.

Actually this game is already complicated by the fact that, differently from the (non-zero-sum) Dynkin game as formulated in section 1.3, here the players control strategies affect also each
other payoffs. In fact, given that in our general formulation the players can switch optimally whenever over the life of the underlying contract, it is clear that - without setting any other “rules” for the game - the decision of one player to switch to a certain regime impose a different cost function $F_Z(.)$ also for the other player. So if A switches but for B the decision is not optimal, he is able to immediately switch back, taking in account the instantaneous switching costs\textsuperscript{13}. In this sense, the relative difference between players’ payoff (in the different regimes) and the strategic interaction between them over time are critical in order to understand and analyze the problem solution/equilibrium.

We return on this points later, here is important to mention that in order to highlight this strategic dependence in the game - that is assumed to be played by rational and non-cooperative players - we have enriched the running cost function $F_Z(.)$ by a response function $b^{-i}(u^i)$, which can be intended mainly in two way:

a) as the “classical” best response function to the other player strategy, which implies the complete information assumption in the game, that is the players have the same information set about the system dynamic and they know each other payoff;

b) if the game information is not complete and there is a degree of “uncertainty” over the players payoff and their switching strategy, the function $b^{-i}(u^i)$ can be intended in generalized terms as a probability distribution assigned by a player to the optimal response of the other one.

We discuss further on the game information flows below. Now, the main issue to tackle is to understand the condition under which this generalized game (23) have sense and it will be played, which means that it will be signed by counterparties. This takes to the problem definition of an equilibrium for this game and to the condition under which its existence and uniqueness are ensured.

Before giving the formal definition of the game equilibrium, let us highlight the game pure strategies at a given time $\{\tau_{j-1}^i < t \leq \tau_j^i\}$, under the assumption of simultaneous moves by players.

**Definition 2.4.3 (Pure strategies of the game of switching type).** For any given initial condition $z_0^A, z_0^B$ and $\forall z_j^A \in u^A$ and $z_j^B \in u^B$ and $\{\tau_{j-1}^i < t \leq \tau_j^i\}$, the pure strategies of our Dynkin game of switching type are defined as follows

\[
\begin{align*}
\{z_j^A = 0, z_j^B = 0\} & \iff \text{"no switch"} \\
\{z_j^A = 0, z_j^B = 1\} & \iff \text{"switch to 1"} \\
\{z_j^A = 1, z_j^B = 0\} & \iff \text{"switch to 1"} \\
\{z_j^A = 1, z_j^B = 1\} & \iff \text{"switch to 1"}.
\end{align*}
\]

\textsuperscript{13}Note that from (23) the indicator $1_{(\tau_{j-1}^i < \tau_j^i < T)}$ the instantaneous switching costs enter in the functional whoever of the players decides to switch
while if \( \{ z_{j-1} = 1 \} \implies \)

\[
\begin{align*}
\{ z_j^A = 0, z_j^B = 0 \} & \implies "switch to 0" \\
\{ z_j^A = 0, z_j^B = 1 \} & \implies "switch to 0" \\
\{ z_j^A = 1, z_j^B = 0 \} & \implies "switch to 0" \\
\{ z_j^A = 1, z_j^B = 1 \} & \implies "no switch".
\end{align*}
\]

In the table below we represent the standard game form at a given decision time with the possible (pure) strategies (namely the switching indicators) and the related random payoff between parenthesis.

|          | Switch          | No Switch        |
|----------|-----------------|------------------|
| Switch   | \( 1, 1 \) \( J^A, J^B \) | \( 1, 0 \) \( J^A, J^B \) |
| No Switch| \( 0, 1 \) \( J^A, J^B \) | \( 0, 0 \) \( J^A, J^B \) |

where we note that the players' strategies can be cast in these categories:

a) on the main diagonal of the table we have accommodation/peace type switching strategies played over time;

b) on the opposite diagonal of the table we have fighting/war type switching strategies played over time.

### 2.5 Game equilibrium and stochastic representation through system of RBSDE

From a static point of view the NEP for the game of definition 2.4.1 can be easily found once the payoff \( J^i \) are known. But the problem is that game configurations like these has to be played over time taking in account as key factors:

a) the payoff value that derives from switching at a given time;

b) the expected value from waiting until the next switching time;

c) the optimal responses, namely the other party optimal switching strategy which implies also to know the same points a) and b) for the other party.

The resulting equilibrium is an optimal sequence of switching over time for both players that needs a (backward) dynamically recursive valuation. On an heuristic base, we expect that if the relative difference (over time) in players payoff functionals - mainly due to different function parametrization, default intensities \( \lambda^i \) or switching costs \( c_{Z_i}^j \) - remains low, it is more likely that the switching strategies on the main diagonal of the game \( \{ 1,1; 0,0 \} \) will be played (given that both players would have also similar best responses over time). This should ease the search for the equilibrium of the game but this makes more likely to incur in banal solutions. Otherwise, one should verify a more complicated strategic behavior that needs a careful study depending also on the type of equilibrium that now we try to define.

Indeed, given the characteristics of our game namely a non-zero-sum game in which the agents are assumed rational and act in a non-cooperative way in order to minimize their objective
function knowing that also the other part do the same, the equilibrium/solution of this type of games is the celebrated \textit{Nash equilibrium point} (NEP). Actually, in our case the equilibrium is characterized by a sequence of optimal switching through time and being game 2.4.1 a generalization of a \textit{Dynkin game}, by similarity, we can state the following definition of a \textit{Nash equilibrium point} for a \textit{Dynkin game} of switching type.

\textbf{Definition 2.5.1 (NEP for Dynkin game of switching type).} \textit{Let us define the switching control sets for the player \{A, B\} of the generalized Dynkin game (23) as follows}

\[
\begin{align*}
&u_A := \{\tau^A_j, z^A_j\}_{j=1}^M, \quad \forall \tau^A_j \in [0, T], \quad z^A_j \in \{0, 1\}; \\
&u_B := \{\tau^B_j, z^B_j\}_{j=1}^M, \quad \forall \tau^B_j \in [0, T], \quad z^B_j \in \{0, 1\}.
\end{align*}
\]

A Nash equilibrium point for this game is given by the pair of sequences of switching times and indicators \{u^*_A, u^*_B\} such that for any control sequences \{u_A, u_B\} the following condition are satisfied

\[
\begin{align*}
J^A(y; u^*_A, u^*_B) &\leq J^A(y; u_A, u_B) \quad (27) \\
J^B(y; u^*_A, u^*_B) &\leq J^B(y; u_A, u_B) \quad (28)
\end{align*}
\]

(the signs will be reversed in the maximization case).

A formal and rigorous proof of the existence (and uniqueness) of a NEP for our game such that it is non trivial or \textit{banal} - in the sense that it is never optimal for both the parties to switch or when the switching control set reduce to a single switching/stopping time - is the big issue here.

In order to approach the solution of the game, as we know from the literature, one has in general two way: analytic or probabilistic which are deeply interconnected (working in a markovian framework).

In particular, from the theory of BSDE with reflection\textsuperscript{14}, we can state the next definition for the stochastic representation of our Dynkin game of switching type as a \textit{system of interconnected \textit{(non-linear) reflected BSDE}}.

\textbf{Definition 2.5.2 (RBSDE representation for game of switching type).} \textit{Let us define the vector triple \(Y_i, Z_i, N_i, \bar{Z}_i, K_i, \bar{K}_i\) for \(i \in \{A, B\}\) and \(Z \in \{z, \zeta\}\), with the same technical condition of definition 5.4.1. Then, given the standard Brownian motion vector \(W^i_t\), the terminal reward \(\xi^i\), the obstacles \(c^i_{t, Z}\) and the generator functions \(F^i_{t, Y}(., Y^{-1})\) both interconnected between the players, we can represent the Dynkin game of switching type formulated in (23) through the

\textsuperscript{14} See for details section five of Mottola (2013)
following system of interconnected non-linear reflected BSDE

\[
\begin{aligned}
Y_t^{A,z}, Y_t^{A,ζ} &\in \mathbb{K}^2; N_t^{A,z}, N_t^{A,ζ} \in \mathbb{M}^2; K_t^{A,z}, K_t^{A,ζ} \in \mathbb{K}^2, K \text{ non decreasing and } K_0 = 0, \\
Y_t^{A,Z} &= \xi^A + \int_s^T \Gamma(Y_{s}^{A,Z}, N_s^{A,Z}, Y_s^{A,Z}) ds - \int_s^T N_s^{A,Z} dW_s + K_T^{A,Z} - K_s^{A,Z}, \ t \leq s \leq T, Z \in \{z, ζ\} \\
Y_t^{A,z} &\geq (Y_{t}^{A,ζ} - c_t^{A,z}); \ \int_0^T |Y_t^{A,z} - (Y_{t}^{A,ζ} - c_t^{A,z})| dK_t^{A,z} = 0; \\
Y_t^{A,ζ} &\geq (Y_{t}^{A,z} - c_t^{A,ζ}); \ \int_0^T |Y_t^{A,ζ} - (Y_{t}^{A,z} - c_t^{A,ζ})| dK_t^{A,ζ} = 0; \\
Y_t^{B,z}, Y_t^{B,ζ} &\in \mathbb{K}^2; N_t^{B,z}, N_t^{B,ζ} \in \mathbb{M}^2; K_t^{B,z}, K_t^{B,ζ} \in \mathbb{K}^2, K \text{ non decreasing and } K_0 = 0, \\
Y_t^{B,Z} &= \xi^B + \int_s^T \Gamma(Y_{s}^{B,Z}, N_s^{B,Z}, Y_s^{B,Z}) ds - \int_s^T N_s^{B,Z} dW_s + K_T^{B,Z} - K_s^{B,Z}, \ t \leq s \leq T, Z \in \{z, ζ\} \\
Y_t^{B,z} &\geq (Y_{t}^{B,ζ} - c_t^{B,z}); \ \int_0^T |Y_t^{B,z} - (Y_{t}^{B,ζ} - c_t^{B,z})| dK_t^{B,z} = 0; \\
Y_t^{B,ζ} &\geq (Y_{t}^{B,z} - c_t^{B,ζ}); \ \int_0^T |Y_t^{B,ζ} - (Y_{t}^{B,z} - c_t^{B,ζ})| dK_t^{B,ζ} = 0
\end{aligned}
\]

From definition 2.5.2, it is evident that the system of RBSDE is a non-standard one given the characteristics of the generator functions (which are the cost function in our game) that are interdependent given the presence of the other player value process \(Y_i\) inside \(F_i(.)\) for \(i \in \{A, B\}\). This makes hard to show the existence and uniqueness of the solution of the system for the reason that we highlight below. In particular, the solution of this system of RBSDE is made up of a two-dimensional vector triple \((Y^{*,Z}, N^{*,Z}, K^{*,Z})\) where its dimension is given by the two switching regimes while the optimal switching sequences is determined by the value process crossing of the barriers indicated in the last two line of both the system RBSDEs. Therefore, the other main issue is to show that the solution value processes \(Y_t^{*,Z}\) coincide with the value functions of the two players of the non-zero-sum game of switching type (23).

As far as we know, these issues have been tackled - in relation to switching problems - in the already mentioned work of Hamadene and Zhang (2010). They study general system of m-dimensional BSDE called with "oblique reflection", which are RBSDE with both generator and barrier interconnected as in our case, showing existence and uniqueness of the solution while the optimal strategy in general does not exist but an "approximating optimal strategy" is constructed (through some technical estimates).

Let us briefly recall the main technical assumptions that are imposed in order to derive these results are:

a) **square integrability for both the generator function \(F(.)\) and terminal reward \(ξ\) while the obstacles function are continuous and bounded;**

b) **Lipschitz continuity of the generator function respect to its terms;**

c) **both the generator and the obstacles are assumed to be increasing function of the other players utility/value process.**

As also mentioned in the paper, the condition c) implies from a game point of view that the players are "partners", namely the interaction and the impact of the other players value
processes has a unique positive sign. This is not the case of our non-zero-sum game in which the interaction allowed between the two players is antagonistic and more complex. Therefore also the Lipschitz condition b) is hard to verify for our non-linear and recursive cost functions. Hence, as far as we know, the existence and uniqueness of the optimal switching strategy for our game formulated in definition 2.5.2 is an open problem, whose solution needs further studies. Probably a solution exists but it won’t be unique, indeed the classifications of solutions behavior and the conditions for their existence is an interesting and hard program to tackle analytically and also numerically. We leave this for further studies. Therefore, even though one could assume to simplify the problem in order to work under the same assumptions a)-c) that would ensure the existence and uniqueness of the solution, one has also to show that the solution of the system of RBSDE is the Nash equilibrium point for the game (23), which is complicated by the fact that the optimal control strategy may not exist. Formally, one should also prove the following theorem which is also an open problem.

**Theorem 2.5.3 (NEP and RBSDE system solution).** Let us assume the existence and uniqueness of the solution for the system of definition 2.5.2, under the assumption a)-c). Then the system RBSDE value processes $Y^*_A, Y^*_B$ coincide with the player value functions of the game of switching type, that is

$$
Y^*_A = J^A(y, u^*_A, u^*_B)
$$

$$
Y^*_B = J^B(y, u^*_B, u^*_A)
$$

and are such that condition (27) and (28) are satisfied, which implies the existence and uniqueness of a Nash equilibrium point for the game (23).

**Remark 7.2.3.** As we already know, in the markovian framework – thanks to El-Karoui et al. results – the solution of the system of RBSDE is connected with the viscosity solution of a generalized system of non-linear PDE with generator and obstacles different and interconnected between the two players, which is even harder to study analytically. The main alternative is to try to approach numerically the problem, searching for the conditions under which one can find the equilibrium. A possibility is to apply the same technique – Snell envelope and iterative optimal stopping technique – that we use in section six of Mottola (2013) adapted to study the our stochastic game’s solution. In this case, the algorithm and implementation are more complex than the unilateral case - where strategic interaction is not considered - and a careful study has to be undertook, so we leave this issue for a future paper.

3 Game solution in a ”special case” and further analysis.

In this section we make some further reasoning on the game characteristics in order to possibly simplify our general formulation (23) and to search for a solution. In particular, we focus the analysis on the following three main points - already mentioned in the past section- that have impact on the equilibrium characterization and existence:

a) **information set between the players/counterparties**;

b) **rules of the game**;

c) **differences in the objective functionals of the players/counterparties**.

---

15See Hamadene and Zhang (2010) for details.
a) Firstly, a careful analysis of the game information set is fundamental to characterize and understand the game itself and its equilibrium. In our game formulation (23) we have assumed symmetry in the information available for the players which helps to simplify the analysis, but in general one needs to specify what is the information available to them at all the stage of the game. Given that we have been working under the market filtration $(\mathcal{F}_t)_{t \geq 0}$, under symmetry we get that both player knows $\forall t \in [0,T]$ the values of the market variables and processes that enter the valuation problem, namely

$$ \mathcal{F}_t^A = \mathcal{F}_t^B = \mathcal{F}_t \forall t \in [0,T] $$

So, both players are able to calculate the outcomes/payoff of the game through time. This implies that they know each other cost functions so that the game is said information complete and it is easier to solve for a NEP knowing the best response function.

It is important to underline that the game is played simultaneously at the decision times but it is dynamic and recursive because the optimal strategy played today will depend not only on the initial condition (that is usually common knowledge) but on future decisions taken by both the players. Clearly, this complicates game characteristics, imposing to run a backward induction procedure to search for an equilibrium point.

Of course, the assumption to know the counterparty cost function is quite strong for our problem in which the parties of the underlying contract can operate in completely different markets or industries, but it can be not uncommon to verify a cooperative behavior between them. In particular, in cooperative games the players aim to maximize or minimize the sum of the values of their payoff over times, namely

$$ J^{\text{coop}}(y, u^*) := \inf_{u^* \in \{C^A_{ad} \cup C^B_{ad}\}} \left[ J^A(y, u^A) + J^B(y, u^B) \right] $$

$$ := \inf_{u^* \in \{C^A_{ad} \cup C^B_{ad}\}} \mathbb{E} \left[ \sum_j \int_1^{\tau^A_j \wedge \tau^B_j \wedge T} B_s \left[ F^A_{Z}(y_s, u^A, b^A(u^A)) + F^B_{Z}(y_s, u^B, b^A(u^B)) \right] ds + \sum_{j \geq 1} B_s \left( c^A_{Z_j}(t) 1_{\{\tau_j^A \wedge \tau_j^B < T\}} c^B_{Z_j}(t) 1_{\{\tau_j^A \wedge \tau_j^B < T\}} \right) \right] $$

$$ + \left( G^A(y(T), b^A(u^A)) + G^B(y(T), b^A(u^B)) \right) \Big|_{\mathcal{F}_t} \text{ for } j = 1 \ldots M $$

This type of equilibrium, depending on the type of game considered, is much more difficult to study in the stochastic framework, given the necessity to study and determine the conditions under which players cooperate over time (and paths) and have no incentives “to cheat” playing a different (non cooperative) strategy. Cooperative games and equilibrium can be an interesting topic to study in relation to our type of game and it would be important to examine this issue in major depth.

b) Also the rules of the game are important in order to simplify the analysis. In our model both the counterparties are able to switch optimally every time over the contract life. Discretizing the time domain, we have been lead to think at the game as played simultaneously through time over the switching time set that can be predefined in the contract or model specific. In terms of game theory, this means that a given decisional node of the game the players make their optimal choice based on the information available (which is common knowledge).
knowledge) at that node and at the subsequent node they observe the outcome of the last interaction and update their strategy.

An other possibility that may help to simplify things, is to assume that - by contract specifications - the counterparties can switch only at predefined times and that the two sets have null intersection, namely

$$\{\tau^A_j \cap \tau^B_j\}_{j=1}^M = \emptyset.$$ 

This happen for example if the right to switch is set as sequential. So, also in terms of game theory, the strategic interaction and the game become sequential: under the assumption of incomplete information, this type of games are generally solved via backward induction procedure - as it is necessary in our case - and one can search for a weaker type of Nash equilibrium. Clearly, we remind that in our case this type of equilibrium need to be studied under a stochastic framework which remains a cumbersome and tough task both analytically much and also numerically.

A strategic sequential interaction like that, can be also obtained by contract setting in the CSA some time rules like the so called "grace periods" namely a delta time $\Delta t$ that has to pass after a switching time before the other party can make its optimal switching decision.

c) The last point really relevant in our game analysis concerns the relative differences between counterparties objective functionals. As already mentioned above in relation to our model, the driving factors that have impact in this sense are given by:

- differences in the default intensities processes $\lambda^A_t, \lambda^B_t$;
- differences in the cost function thresholds/reaction functions $\delta^A(\cdot), \delta^B(\cdot)$;
- differences in funding/opportunity costs $R^A(t), R^B(t)$;
- differences in the instantaneous switching costs $c^A_z, c^B_z$.

To ease things we focus on some specific cases.

1) Symmetric case. Let us simplify things by considering a special case of our game (23) in which symmetry between the party of the contract is assumed. In this case, thanks to results of Mottola (2013) we are able to show the existence of the solution for the game and we also highlight the impact on the equilibrium behavior of just a simple constant threshold $\delta$ in the running cost functions. Under symmetry it is easy to show that the game solution coincides with that of our control problem on which we have focussed the analysis in the third chapter of Mottola (2013). In fact, considering the problem from just one side is equivalent to a game played by symmetric players with objective functional having the same parameters.

In economic terms, the reason to consider a game between two “symmetric” players can be justified if one thinks to two institutions with similar business characteristics other than risk worthiness, that operate in the same country/region/market with the objective to optimally manage the counterparty risk and the collateral and funding costs by signing a contingent CSA in which are defined all the relevant parameters necessary to know each other objective functional.

So, let us consider a game played under these “special” conditions, it is not difficult to see that the game payoffs will be the same for both the player: in fact, setting $\delta^A = \delta^B = 0$, being the square of BCVA and collateral costs functions equal by symmetry as well as the instantaneous costs. This implies that also the best response functions will be equal for both the player, which means they play the same switching strategy, however the game is
played simultaneously or sequentially. So, on the basis of this chain of thoughts we can state the following result.

**Proposition 7.2.3 (NEP Existence and uniqueness in ”symmetry” case.)** Assume ”symmetric” conditions for our Dynkin game of switching type (23), taking (24-26) with equality and setting $\delta^A = \delta^B$ a null constant. In addition, assume the same technical condition imposed in section 5.2 of Mottola (2013) for the solution of the switching control problem. Then exists and is unique a Nash equilibrium for this game and it coincides with the value function of the stochastic control problem derived in the unilateral case (in absence of strategic interaction), that is

$$J^A(y,u^*_A;u^*_B) = J^B(y,u^*_A;u^*_B) = V^*(y,u^*).$$

(29)

Proof. The proof is easy by the above reasonings. In fact, under the ”symmetry” conditions and recalling the notation from the general game formulation (23) we have that the following relations hold:

$$F^A_Z(y_s,u^A,B^A(u^A)) = F^B_Z(y_s,u^B,B^B(u^B))$$

$$c^A_{Zj}(t) \mathbb{1}_{\{\tau^*_j \wedge \tau^*_p < T\}} = c^B_{Zj}(t) \mathbb{1}_{\{\tau^*_j \wedge \tau^*_p < T\}}$$

$$G^A(y_T) = G^B(y_T)$$

and setting $\tau^*_j := \tau^*_j = \tau^*_j$ which implies $\mathbb{1}_{\{\tau^*_j \wedge \tau^*_p < T\}} = \mathbb{1}_{\{\tau^*_j < T\}}$, being the optimal control sequence optimal for both players, namely $u^*_A = u^*_B$, which implies also the equality of the best response functions $b^A(u^B) = b^B(u^A)$. This means also that the strategic interaction becomes irrelevant and the game solution can be reduced to that of an optimal switching control problem equivalent for both the players, so that game (23) is reduced to the following problem

$$J(y,u) := J^A(y,u^A) = J^B(y,u^B) \implies$$

$$J(y,u) = \inf_{u \in \{C_{ad}\}} \mathbb{E}\left[ \sum_j \int_{\tau^*_j \wedge T} B_s[F_Z(y_s,u)] ds + \sum_{j \geq 1} B_t c_{Zj}(t) \mathbb{1}_{\{\tau^*_j < T\}} + G(y_T) \right]$$

which is equivalent to the control problem formulated in Mottola (2013). Its solution is given by the value function $V^*(y,u^*)$ (derived in theorem 5.3.1 of the same work) to which is associated the optimal sequence of switching times and indicators $u^* = \{T^*, Z^*\}$. By the optimality of this sequence, conditions (27) and (28) of NEP definition 2.5.1 are satisfied, being the optimal strategy for both players (by symmetry). So being the two problems representation actually the same, the NEP exists and is unique (from the existence and uniqueness of $V^*(y,u^*)$) and equation (26) is true, as we wanted to show $\diamondsuit$.

2) Case $\delta^A = \delta^B \neq 0$. In general with different function parametrization between players and incomplete or asymmetric information, the equilibrium is much harder to find and different strategies has to be checked. To give an idea of this, let us consider just a slight modification of the symmetric case conditions, setting for example the cost functions

17We refer to the program (3.9-3.11) of the already mentioned paper.
threshold $\delta^A = \delta^B > 0$, and keeping the information incomplete and the game play simultaneous. By the symmetry of BCVA and (running) collateral/funding costs, we know that a positive value of one term for $A$, is negative for $B$ and vice versa. So introducing the threshold create different payoffs for the player, as we can easily see below

$$(BCVA^A - \delta)^2 \geq (BCVA^B - \delta)^2$$

given that if $BCVA^A > 0$ then $BCVA^B < 0$ and viceversa. So, even though they knew each other objective functional, there would be some paths and periods in which the strategic behavior of the players is in contrast, say of war type and others of peace type, making the analysis more complicate.

3) Game “banal” solution case. Worth of mention is also the possibility that the game is not played, namely it reveals to be never optimal to switch for both the players. It’s relevant to study the conditions under this kind of behavior of the solution come up, given that the scheme would lose its economic sense. This “singular game solution” can come up if we formulate to our simultaneous game as a zero sum game. This can happen by considering - for example - linear objective functionals with threshold $\delta \approx 0$. In fact - by symmetry of the BCVA and of funding cost function - a positive outcome for one player is negative for the counterpart. Assuming instantaneous switching costs $c^2 > 0$ for both players and - to simplify - that both know each other cost functions, we get that this game will never be played. The reason is that by the game zero sum structure, the optimal strategy for one is not optimal for the other, so every switch will be followed by the opposite switch at every switching time, like the sequence

$$\{z_1 = 1, z_2 = 0, z_3 = 1, \ldots, z_M = 1\}$$

But by rationality and taking in account the positive cost of switching, one can conclude that the game will never be played by a rational agent.

So, let us summarize this last logic chain of thoughts in the following proposition.

**Proposition 7.2.4. (Game banal solution in the zero-sum case).** *Let us assume that game (23) be a zero-sum game with linear functionals set for both the counterparties. Assuming in addition the same funding costs for both players, $\delta \approx 0$ and positive switching costs $c^2 > 0$ (for both), then the optimal strategy is to never play this game, namely the game has a banal solution.*

**Remarks 7.3.1** Let us recall that in the special case of zero-sum games the game equilibrium, is expressed as follows

$$J^{A=B}(y, u^*_A, u_B) \leq J^*(y, u^*_A, u^*_B) \leq J^{A=B}(y, u_A, u_B^*)$$

for all control sequences of $(u_A, u_B)$. In particular if

$$\inf_{u_A} \sup_{u_B} J^{A=B}(y, u_A, u_B) = \sup_{u_A} \inf_{u_B} J^{A=B}(y, u_A, u_B).$$

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18 This is true also for the other switching costs related to collateral and funding. Of course one should consider also the weight of the expected payoff value from keeping the strategy for an other period.

19 Obviously, when the value of the cost functions compensate each other, no switch take place
then the zero-sum game equilibrium - called a saddle point of the game - is said to have a value. As we know this depends on the verification of the Isaacs condition namely the equality between the lower and upper value functions for the players of the game. This type of equilibrium is central from the arbitrage free pricing point of view. In fact, the solution of the zero-sum game under the martingale pricing measure, that is
\[ J^*(y, u_A^*, u_B^*) := \mathbb{E}^{Q^*}[\Pi(t, y)], \]
implies also the existence and uniqueness of the arbitrage free price of the underlying claim which admits this stochastic zero-sum game representation.

We end the section by remarking the importance of the points highlighted in the construction of some kind of equilibrium for a Dinkyn game of switching type as our one. Although mainly theoretical, the existence of the equilibrium and the definition of the conditions under which a non banal solution is available, are relevant economically and in the contract design phase. Hence, the main task to pursue in future research are a rigorous proof of the existence and of the equilibrium for this type of game, and the definition of an efficient algorithm to check the model solutions in the bilateral (namely, stochastic game) setting.

4 Applications and further researches

Switching type mechanisms like the one we have analyzed can find different applications into the wild world of finance. The basic underlying idea is to ensure flexibility to agents investments decisions over time which is a usual objective in real option theory. Our problem has been thought mainly in a risk management view but with the development of new techniques and algorithms, also the related pricing problem will be tackled efficiently and more financial contract would find useful and convenient - in a optimal risk management view - this type of contingent mechanisms. As regards just a possible further application in risk management, it would be important to deepen the analysis of a switching type collateralization from a portfolio perspective, taking for example the view of a central clearing. In particular, it would be relevant to show possibly analytically but mainly with numerical examples, the greater convenience of the switching/contingent solution respect to a non contingent/standard collateral agreement like the partial or full one, including clauses like, early termination, netting and others. This is an hard program, which needs a generalized model formulation in order to include all the CSA clauses and in order to deal with the high recursion that characterizes the problem.

In a pricing view, we remind the example - of the fixed income market - some particular bonds called flippable or switchable, that are characterized by options to switch the coupon from fix to floating rate. Clearly, in this case the valuation is easier given that this securities have a market and are not traded OTC so one does not need to include in the picture counterparty risk, funding and CSA cashflows. Anyway, it would be interesting to delve into the valuation of an OTC contract in which also the dividend flows can be subject to contingent switching mechanism. As regards similar case, a problem that can be very interesting and difficult to tackle is the valuation of a flexi swap in presence of a contingent collateralization like our one.

The main characteristics of a flexi-swap are:

\begin{itemize}
\item[a)] the notional of the flexible swap at period \( n \) must lie (inclusively) between predefined bounds \( L_n \) and \( U_n \);
\item[b)] the notional of the flexible swap at period \( n \) must be less than or equal to the notional at the previous period \( n - 1 \);
\end{itemize}
c) The party paying fixed has the option at the start of each period $n$ to choose the notional, subject to the two conditions above.

In other words, we deal with a swap with multiple embedded option that allows one party to change the notional under certain constraints defined in the contract. This kind of interest rate swaps are usually used as hedging instruments of other swaps having notional linked to loans, especially mortgages. The underlying idea is that the fixed-rate payer (the option holder) will amortize as much as allowed if interest rates are very low, and will amortize as little as allowed if interest rates are very high.

Given a payment term structure $\{T_n\}_{n=0}^N$ and a set of coupons $X_n$ (with unit notional) fixing in $T_n$ e paying in $T_{n+1}$ ($n = 0, 1, \ldots, N - 1$), the flexible swap is a fixed vs floating swap where the fixed payer has to pay a net coupon $X_n R_n$ in $T_{n+1}$, the notional $R_0$ is fixed upfront at inception and for every $T_n$, $R_n$ can be amortized if it respects some given constraints defined as follows:

1. deterministic constraints: $R_n \in [g_n^{low}, g_n^{high}]$;

2. local constraints function of the current notional: $R_n \in [l_n^{low}(R_{n-1})l_n^{high}(R_{n-1})]$;

3. market constraints (libor, swap denoted with $X_n$): $R_n \in [m_n^{low}(X_n), m_n^{high}(X_n)]$.

The valuation procedure of this type of swap involves a backward recursion keeping track of the notional in every payment date. But introducing also the switching collateralization, the valuation become an intricate puzzle given the recursive relation between the optimal switching strategy, the price process of the claim which in addition depends on the optimal notional choice over time. Surely, one needs simplifying assumptions to break the curse of recursion.

5 Conclusions

In this work, we have generalized the contingent CSA scheme defined in our preceding work to the bilateral case allowing the strategic interaction between the counterparties of a defaultable (OTC) contract. The problem in this case has a natural formulation as a stochastic differential game - a generalized Dynkin game - of switching type, for which - as far as we know - no analytical solution for a Nash equilibrium point is known.

We have shown, in particular, that the game solution is strictly related to that of a system of reflected BSDE with interconnected barriers and generator functions. Only by imposing at cost strong assumptions and simplifications we are able to prove the game solution, in the so called symmetric case.

Further research are needed and addressed in the field of stochastic games and RBSDE and some interesting finance applications are highlighted in order to show also the importance in practice of our mainly theoretical problem.

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20 In this sense is like there were a third reference represented by the pool of loans.
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