A Local Lifting Theorem for Jointly Subnormal Families of Unbounded Operators

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Abstract. A local lifting theorem for bounded operators that intertwine a pair of jointly subnormal families of unbounded operators is proved. Each family in question is assumed to be composed of operators defined on a common invariant domain consisting of “joint” analytic vectors. This result can be viewed as a generalization of the local lifting theorem proved by Sebestyén, Thomson and the present authors for pairs of bounded subnormal operators.

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1. Introduction

A local lifting theorem, originally formulated for pairs of bounded subnormal operators (cf. [6, Theorem 4.2]), states that an intertwining operator between two subnormal operators lifts to an intertwining operator between their minimal normal extensions if and only if (1) the restriction of the intertwining operator to each cyclic invariant subspace lifts, and (2) the supremum of the norms of the cyclic lifts is finite (recall that the first commutant lifting theorem for bounded subnormal operators was proved by Bram in [2]). Our aim in this paper is to generalize the local lifting theorem of [6] to the case of pairs of jointly subnormal families of unbounded operators (see [12–14,17] for basic results on unbounded subnormal operators). The first problem to be overcome here concerns the question of which kind of minimality for normal extensions of unbounded operators should be chosen. A careful analysis

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of the results on lifting strong commutants contained in [11] reveals that the notion of minimality of cyclic type is appropriate for this purpose. The other difficulty is to guarantee the existence of minimal normal extensions of cyclic type for cyclic parts of a jointly subnormal family of operators. This can be achieved by assuming that each family in question have a dense set of “joint” analytic vectors (recall that the notion of an analytic vector was introduced by Nelson in [8]). The latter enables us to use a generalization of the Maserick theorem [7, Theorem 3.2], which was proved in [12], to establish the main result of the paper (cf. Theorem 5.3). In Sect. 6 we dwell upon a special kind of local intertwining referred to as local commutativity.

2. Preliminaries

As usual, \( \mathbb{C} \), \( \mathbb{R} \) and \( \mathbb{N} \) stand for the sets of complex numbers, real numbers and nonnegative integers, respectively. All linear spaces in this paper are assumed to be complex and all operators under consideration are assumed to be linear. Given two Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), we denote by \( B(\mathcal{H}, \mathcal{K}) \) the set of all bounded operators from \( \mathcal{H} \) into \( \mathcal{K} \). To simplify the writing, we put \( B(\mathcal{H}) := B(\mathcal{H}, \mathcal{H}) \). The identity operator on \( \mathcal{H} \) is denoted by \( I_{\mathcal{H}} \). If \( X \) is a subset of \( \mathcal{H} \), then \( \text{lin} X \) stands for the linear span of \( X \). A family \( \{f_{\gamma}\}_{\gamma \in \Lambda} \) of vectors in a linear space \( X \) is said to be finite if \( f_{\gamma} = 0 \) for all but a finite number of \( \gamma \)'s.

Let \( A \) be an operator in \( \mathcal{H} \). Denote by \( D(A) \), \( A^* \) and \( \bar{A} \) the domain, the adjoint and the closure of \( A \) (in case they exist). We write \( A \subseteq B \) if the operator \( B \) is an extension of \( A \) (\( A \) and \( B \) may act in distinct Hilbert spaces). We say that a closed linear subspace \( \mathcal{M} \) of \( \mathcal{H} \) reduces \( A \) if \( PA \subseteq AP \), where \( P \) is the orthogonal projection of \( \mathcal{H} \) onto \( \mathcal{M} \); if this is the case, then \( A|_{\mathcal{M}} \) stands for the restriction of \( A \) to \( \mathcal{M} \), i.e., \( D(A|_{\mathcal{M}}) = D(A) \cap \mathcal{M} \) and \( A|_{\mathcal{M}} f = Af \) for \( f \in D(A|_{\mathcal{M}}) \). If \( A \) is closable and \( \mathcal{E} \) is a linear subspace of \( D(A) \) such that \( A \subseteq A|_{\mathcal{E}} \) (or equivalently \( \bar{A} = \bar{A}|_{\mathcal{E}} \)), then \( \mathcal{E} \) is called a core of \( A \) (here \( A|_{\mathcal{E}} \) is the usual restriction of the mapping \( A \) to the set \( \mathcal{E} \)).

For an operator \( A \) in \( \mathcal{H} \), we put \( \mathcal{D}^{\infty}(A) = \bigcap_{n=1}^{\infty} D(A^n) \). Elements of \( \mathcal{D}^{\infty}(A) \) are referred to as \( C^{\infty} \)-vectors. In the present paper we explore a special class of \( C^{\infty} \)-vectors called analytic ones. Following [8], we say that \( f \in \mathcal{D}^{\infty}(A) \) is an analytic vector for \( A \) if there exists a positive real number \( t \) such that

\[
\sum_{n=0}^{\infty} \|A^n f\| \frac{t^n}{n!} < \infty.
\]

The set of all analytic vectors for \( A \), denoted by \( \mathcal{A}(A) \), is a linear subspace of \( \mathcal{H} \) which is invariant for \( A \), i.e. \( A(\mathcal{A}(A)) \subseteq \mathcal{A}(A) \).

Now, we recall some definitions from [12, Sect. 4]. Suppose that \( (\Omega, +, \ast) \) is a commutative \( \ast \)-semigroup with the zero element 0 and \( \mathcal{E} \) is a linear space. By a form over \( (\Omega, \mathcal{E}) \) we mean a mapping \( \varphi : \Omega \times \mathcal{E} \times \mathcal{E} \to \mathbb{C} \) such that \( \varphi(v; \cdot, \cdot) : \mathcal{E} \times \mathcal{E} \to \mathbb{C} \) is a sesquilinear form for all \( v \in \Omega \). A form \( \varphi \) over
(Ω, E) is called positive definite if
\[ \sum_{m,n=0}^{k} \varphi(v_n^* + v_m; f_m, f_n) \geq 0, \quad \{v_m\}_{m=0}^{k} \subseteq \Omega, \{f_m\}_{m=0}^{k} \subseteq E, \quad k \in \mathbb{N}. \]

We say that a form ϕ over (Ω, E) is weakly positive definite if for every f ∈ E, ϕ( : f, f) is positive definite as a form over (Ω, C). Given a weakly positive definite form ϕ over (Ω, E) and v ∈ Ω, we set
\[ A_ϕ(v) = \left\{ f \in E \mid \exists t > 0 : \sum_{n=0}^{\infty} \sqrt{\varphi(n \cdot (v^* + v); f, f)} \frac{r^n}{n!} < \infty \right\}. \]

Denote by \( \Omega \) the set of all characters of Ω (by a character of Ω we mean a nonzero additive involution preserving complex mapping on Ω). Let G be a set of *-generators of a *-semigroup Ω. As in [12, p. 40], we write \( \mathcal{C}_G \) for the Cartesian product \( \prod_{v \in G} \mathcal{C}_v \), where \( \mathcal{C}_v := C \) for every \( v \in G \). In what follows, we consider the mapping \( j_G : \Omega \to \mathcal{C}_G \) given by the formula \( j_G(\chi) = \{\chi(v)\}_{v \in G} \) for \( \chi \in \Omega \). The following lemma is a version of [12, Corollary 3] (see also [12, Remarks 3 and 5]).

**Lemma 2.1.** Let Ω, E and \( j_G \) be as above. Assume that \( j_G(\Omega) = \mathcal{C}_G \). If ϕ is a weakly positive definite form over (Ω, E) such that \( A_ϕ(v) = E \) for all \( v \in G \), then the form ϕ is positive definite.

Let \( \Sigma \) be a nonempty set. Denote by \( N_\Sigma \) the set of all mappings \( \alpha : \Sigma \to \mathbb{N} \) such that \( \alpha(\sigma) = 0 \) for all but a finite number of elements \( \sigma \in \Sigma \). The set \( \Omega_\Sigma := N_\Sigma \times N_\Sigma \) becomes a commutative *-semigroup when equipped with the standard coordinate-wise addition and involution \( (\alpha, \beta)^* = (\beta, \alpha) \) for \( \alpha, \beta \in N_\Sigma \). In this particular case, \( G := \{(\delta_\sigma, 0) : \sigma \in \Sigma\} \) is a set of *-generators of \( \Omega_\Sigma \) (\( \delta_\sigma \) is the characteristic function of the singleton set \( \{\sigma\} \)). As observed in [12, p. 50],
\[ j_G(\Omega_\Sigma) = \mathcal{C}_G. \]  

(2.1)

**3. Normal Extensions**

Let E be an inner product space and let \( \mathcal{H} \) be its Hilbert space completion. Denote by \( L(E) \) the algebra of all operators \( A : E \to E \) (with composition as multiplication), and by \( L^\#(E) \) the subalgebra of \( L(E) \) consisting of all operators \( A \in L(E) \) for which there exists an operator \( A^# \in L(E) \) such that \( \langle Af, g \rangle = \langle f, A^#g \rangle \) for all \( f, g \in E \). Such an operator \( A^# \) is uniquely determined. It is plain that \( L^\#(E) \) is a *-algebra with the involution \( A \mapsto A^# \).

Note that if E is a linear subspace of a Hilbert space K and A is a densely defined operator in K such that \( E \subseteq \mathcal{D}(A) \), then \( A|_E \) belongs to \( L^\#(E) \) if and only if \( E \subseteq \mathcal{D}(A^*) \), \( A(E) \subseteq E \) and \( A^*(E) \subseteq E \); if this is the case, then \( A^# = A^*|_E \).

Given a family \( A = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq L(E) \) of commuting operators (i.e. \( A_\sigma A_\tau = A_\tau A_\sigma \) for all \( \sigma, \tau \in \Sigma \)) and \( \alpha \in N_\Sigma \), we set \( A^\alpha = \prod_{\sigma \in \Sigma} A_\sigma^{\alpha(\sigma)} \).

A family \( A = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq L(E) \) is said to be jointly subnormal if there exist
a Hilbert space $\mathcal{K}$ and a family $\mathbf{M} = \{M_\sigma\}_{\sigma \in \Sigma}$ of spectrally commuting normal operators\(^1\) acting in $\mathcal{K}$ such that $\mathcal{E} \subseteq \mathcal{K}$ and $A_\sigma \subseteq M_\sigma$ for all $\sigma \in \Sigma$; $\mathbf{M}$ is then called a normal extension of $\mathbf{A}$ (cf. [12, p. 49]).

We say that a family $\mathbf{A} = \{A_\sigma\}_{\sigma \in \Sigma}$ of operators acts in $\mathcal{H}$ if each operator $A_\sigma$ acts in $\mathcal{H}$. Given a family $\mathbf{A} = \{A_\sigma\}_{\sigma \in \Sigma}$ of operators in $\mathcal{H}$, we set

$$D^\infty(\mathbf{A}) = \bigcap \{ \mathcal{D}(A_{\sigma_0} \cdots A_{\sigma_n}) : \sigma_0, \ldots, \sigma_n \in \Sigma, n \in \mathbb{N} \}.$$ 

Clearly, $D^\infty(\mathbf{A})$ is the greatest linear subspace of $\bigcap_{\sigma \in \Sigma} \mathcal{D}(A_\sigma)$ that is invariant (in the usual set-theoretical sense) for every $A_\sigma$. If each operator $A_\sigma$ is densely defined, then we write $\mathbf{A}^* := \{A_\sigma^*\}_{\sigma \in \Sigma}$.

Now we move on to the concept of minimality of a normal extension. By [12, Remark 8] and the results contained in [18, p. 423], we may formulate the ensuing lemma.

**Lemma 3.1.** Let $\mathbf{M} = \{M_\sigma\}_{\sigma \in \Sigma}$ be a family of spectrally commuting normal operators in $\mathcal{K}$. Then

(i) $D^\infty(\mathbf{M}) = D^\infty(\mathbf{M}^*) = D^\infty(\mathbf{M} \cup \mathbf{M}^*)$, 
(ii) $M_\sigma M_\tau f = M_\tau M_\sigma f$, $M_\sigma M_\tau^* f = M_\tau^* M_\sigma f$ and $M_\sigma^* M_\tau^* f = M_\tau^* M_\sigma^* f$ for all $f \in D^\infty(\mathbf{M})$ and $\sigma, \tau \in \Sigma$,
(iii) $\{M_\sigma|_{D^\infty(\mathbf{M})}\}_{\sigma \in \Sigma} \subseteq L^\#(D^\infty(\mathbf{M}))$,
(iv) $(M_\sigma|_{D^\infty(\mathbf{M})})^\# = M_\sigma^*|_{D^\infty(\mathbf{M})}$ for all $\sigma \in \Sigma$,
(v) if $\mathcal{E}$ is a subset of $\bigcap_{\sigma \in \Sigma} \mathcal{D}(M_\sigma)$ such that $M_\sigma(\mathcal{E}) \subseteq \mathcal{E}$ for all $\sigma \in \Sigma$, then $\mathcal{E} \subseteq D^\infty(\mathbf{M} \cup \mathbf{M}^*)$.

**Sketch of the proof.** Note first that if $N_1, \ldots, N_n$ are spectrally commuting normal operators in $\mathcal{K}$ and $E$ is the joint spectral measure\(^2\) of $(N_1, \ldots, N_n)$, then

$$\mathcal{D}(N_n \cdots N_1) = \mathcal{D}(N_n^* \cdots N_1^*)$$

\[
= \left\{ f \in \mathcal{K} : \int_{\mathbb{C}_n} \left( \sum_{k=1}^n \prod_{j=1}^n |z_j|^2 \langle E(dz)f, f \rangle \right) < \infty \right\}, \tag{3.1}
\]

\[
N_i N_j, N_j^* N_i \subseteq \int_{\mathbb{C}_n} z_i z_j E(dz), \quad N_i N_j^*, N_j^* N_i \subseteq \int_{\mathbb{C}_n} z_i \bar{z}_j E(dz), \tag{3.2}
\]

\[
N_i^* N_j^*, N_j N_i^* \subseteq \int_{\mathbb{C}_n} \bar{z}_i \bar{z}_j E(dz),
\]

for all $i, j \in \{1, \ldots, n\}$. It follows from (3.1) that $D^\infty(\mathbf{M}) = D^\infty(\mathbf{M}^*)$, which immediately implies that (i) holds. This combined with (3.2) gives (ii). The conditions (iii)–(v) follow from (i) and (ii). □

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\(^1\) We say that normal operators spectrally commute if their spectral measures commute.

\(^2\) We refer the reader to [1] for more information on joint spectral measures.
It is now clear from Lemma 3.1 that \( M^\alpha f \), \( M^{*\alpha} f \), and \( M^{*\alpha} M^\beta f \) make sense and that \( M^{*\alpha} M^\beta f = \tilde{M}^{\#\alpha} \tilde{M}^{\beta} f \) for all \( f \in \mathcal{D}^{\infty}(M) \) and \( \alpha, \beta \in \mathbb{N}_\Sigma \), where \( \tilde{M} := \{ M_\sigma \}_{\sigma \in (\mathcal{D}^{\infty}(M))} \) and \( \tilde{M}^{\#} := \{ (M_\sigma)_{\sigma \in (\mathcal{D}^{\infty}(M))} \} \). In view of Lemma 3.1, any jointly subnormal family \( \{ A_\sigma \}_{\sigma \in \Sigma} \subseteq L(\tilde{E}) \) is always commutative.

We say that a normal extension \( M = \{ M_\sigma \}_{\sigma \in \Sigma} \) of a jointly subnormal family \( A = \{ A_\sigma \}_{\sigma \in \Sigma} \subseteq L(\tilde{E}) \) is minimal of cyclic type if

\[
\mathcal{F}_M(\tilde{E}) := \text{lin}\{ M^{*\alpha} f : f \in \mathcal{E}, \alpha \in \mathbb{N}_\Sigma \}
\]

is a core of \( M_\sigma \) for every \( \sigma \in \Sigma \) (according to Lemma 3.1, this definition is correct). We now prove that each jointly subnormal family possessing a rich set of analytic vectors has a minimal normal extension of cyclic type which can be built up from a normal extension.

**Theorem 3.2.** Let \( A = \{ A_\sigma \}_{\sigma \in \Sigma} \subseteq L(\tilde{E}) \) be a jointly subnormal family of operators such that \( \tilde{E} = \mathcal{S}(A_\sigma) \) for all \( \sigma \in \Sigma \) and let \( M = \{ M_\sigma \}_{\sigma \in \Sigma} \) be a normal extension of \( A \). Then for every \( \sigma \in \Sigma \), the closed linear space \( \overline{\mathcal{F}_M(\tilde{E})} \) reduces \( M_\sigma \) to \( N_\sigma := M_\sigma |_{\mathcal{F}_M(\tilde{E})} \), i.e., \( N_\sigma = M_\sigma |_{\mathcal{F}_M(\tilde{E})} \). Moreover, \( N = \{ N_\sigma \}_{\sigma \in \Sigma} \) is a minimal normal extension of \( A \) of cyclic type.

**Proof.** Owing to Lemma 3.1, \( \mathcal{F}_M(\tilde{E}) \subseteq \mathcal{D}^{\infty}(M \cup M^*) \) and \( \{ \tilde{M}_\sigma \}_{\sigma \in \Sigma} \subseteq L^*(\mathcal{F}_M(\tilde{E})) \), where \( \tilde{M}_\sigma := M_\sigma |_{\mathcal{F}_M(\tilde{E})} \). By our assumption, \( \mathcal{E} \subseteq \mathcal{S}(\tilde{M}_\sigma) \) for all \( \sigma \in \Sigma \). Fix \( \sigma \in \Sigma \). Since \( \mathcal{F}_M(\tilde{E}) \) is the linear span of \( \bigcup_{\alpha \in \mathbb{N}_\Sigma} \tilde{M}^{\#\alpha}(\tilde{E}) \), and the operators \( \tilde{M}^{\#\alpha} \) commute with \( \tilde{M}_\sigma \) and \( \tilde{M}_\sigma^{\#} \) (use again Lemma 3.1), we deduce from [12, Theorem 1] that \( N_\sigma \) is a normal operator. Hence, by [15, Corollary 1], the closed linear space \( \overline{\mathcal{F}_M(\tilde{E})} \) reduces \( M_\sigma \) to \( N_\sigma \). This implies that \( N_\sigma^* \subseteq M_\sigma^* \). As \( \sigma \) is arbitrary, we deduce that \( \mathcal{F}_M(\tilde{E}) = \mathcal{F}_N(\tilde{E}) \). As a consequence, we see that \( \mathcal{F}_N(\tilde{E}) \) is a core of \( N_\sigma \) for every \( \sigma \in \Sigma \). The normal operators \( N_\sigma, \sigma \in \Sigma \), spectrally commute as restrictions of spectrally commuting normal operators \( M_\sigma, \sigma \in \Sigma \). This completes the proof. \( \square \)

In our paper we frequently consider unbounded subnormal operators \( A \) in a Hilbert space \( \mathcal{H} \) such that \( \mathcal{D}(A) = \mathcal{S}(A) \). The reader should be aware of the fact that such operators can never be closed (cf. [16, Theorem 7]).

Let \( A = \{ A_\sigma \}_{\sigma \in \Sigma} \subseteq L(\tilde{E}) \) be a jointly subnormal family, and let \( M = \{ M_\sigma \}_{\sigma \in \Sigma} \) and \( N = \{ N_\sigma \}_{\sigma \in \Sigma} \) be its normal extensions acting in Hilbert spaces \( \mathcal{K} \) and \( \mathcal{L} \) respectively. We say that \( M \) and \( N \) are \( \mathcal{H} \)-unitarily equivalent if there exists a unitary operator \( U \in \mathcal{B}(\mathcal{K}, \mathcal{L}) \) such that \( U|_{\mathcal{H}} = I_{\mathcal{H}} \) and \( UM_\sigma = N_\sigma U \) for all \( \sigma \in \Sigma \). We show that minimal normal extensions of cyclic type of a jointly subnormal family are determined up to \( \mathcal{H} \)-unitary equivalence (this was suggested in [12, Remark 8]).

**Proposition 3.3.** Let \( A = \{ A_\sigma \}_{\sigma \in \Sigma} \subseteq L(\tilde{E}) \) be a jointly subnormal family of operators, and let \( M \) and \( N \) be minimal normal extensions of cyclic type of \( A \). Then \( M \) and \( N \) are \( \mathcal{H} \)-unitarily equivalent.
The class of all operators

Definition 4.1. Let \( A, B \) be families of operators acting in Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively. If \( T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) intertwines the families \( A \) and \( B \), i.e. \( TA_\sigma \subseteq B_\sigma T \) for all \( \sigma \in \Sigma \), then we write \( TA \subseteq BT \).

The class of all operators \( T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) intertwining \( A \) and \( B \) will be denoted by \( \mathcal{I}(A, B) \).

Consider the following general situation.

\( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are closed linear subspaces of Hilbert spaces \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \), respectively; \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are dense linear subspaces of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively; \( A = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq L(\mathcal{E}_1) \) and \( B = \{B_\sigma\}_{\sigma \in \Sigma} \subseteq L(\mathcal{E}_2) \) are jointly subnormal families of operators; \( M = \{M_\sigma\}_{\sigma \in \Sigma} \) and \( N = \{N_\sigma\}_{\sigma \in \Sigma} \) are their minimal normal extensions of cyclic type acting in \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \), respectively.

Definition 4.1. We say that an operator \( T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) lifts to \( \mathcal{I}(M, N) \) if there exists an operator \( R \in \mathcal{I}(M, N) \) such that \( T \subseteq R \); \( R \) is called a lift of \( T \) to \( \mathcal{I}(M, N) \).

We first state a Bram type criterion for the existence of a lift, which is a multioperator version of \([11, \text{Proposition 4.1}]\) for intertwining operators.

Proposition 4.2. Suppose that \((4.1)\) holds and \( T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \) is an operator such that \( TE_1 \subseteq E_2 \). Then the following conditions are equivalent:

(i) there exists a (unique) lift \( \hat{T} \) of \( T \) to \( \mathcal{I}(M, N) \),

(ii) there exists a real number \( c \geq 0 \) such that

\[
\sum_{\alpha, \beta} \langle B^\alpha T f_\beta, B^\beta T f_\alpha \rangle \leq c \sum_{\alpha, \beta} \langle A^\alpha f_\beta, A^\beta f_\alpha \rangle
\]

for all finite families \( \{f_\alpha\}_{\alpha \in \Sigma} \subseteq \mathcal{E}_1 \).
Moreover, if (i) holds, then

(iii) \( TA \subseteq BT \),
(iv) \( \|\hat{T}\|^2 = \min\{c \geq 0 : c \text{satisfies (ii)}\} \).

Proposition 4.2 can be proved in a similar manner as [11, Proposition 4.1]: first, we may formulate a multioperator version of [11, Proposition 3.1] for intertwining operators, and then, with its help, adapt the proof of [11, Proposition 4.1] to the present context (use Lemma 3.1 to make the reasoning applicable).

Throughout what follows, a unique lift of \( T \) is denoted by \( \hat{T} \).

Now we are ready to formulate the main result of this section. Its proof resembles to some extent the proof of [6, Theorem 3.2]. To justify the implication (iv \( M \)) \( \Rightarrow \) (iii \( M \)) below, the pivotal part of the proof, we have to adapt the original idea from [6] to the context of unbounded operators, which seems to be the main difficulty in the whole procedure.

**Theorem 4.3.** Suppose that (4.1) holds. Assume that \( \mathcal{E}_1 = \mathcal{A}(A_\sigma) \) and \( \mathcal{E}_2 = \mathcal{A}(B_\sigma) \) for all \( \sigma \in \Sigma \). Let \( M \) be a linear subspace of \( \mathcal{E}_1 \) such that

\[
\mathcal{E}_1 = \text{lin}\{A^\alpha f : f \in M, \alpha \in \mathbb{N}_\Sigma\},
\]  

and let \( T \in B(\mathcal{H}_1, \mathcal{H}_2) \) be an operator such that \( T\mathcal{E}_1 \subseteq \mathcal{E}_2 \). Then the following conditions are equivalent:

(i) \( T \) lifts to \( I(M, N) \),
(ii) there exists a real number \( c \geq 0 \) such that (4.2) holds for all finite families \( \{f_\alpha\}_{\alpha \in \mathbb{N}_\Sigma} \subseteq \mathcal{E}_1 \),
(iii) there exists a real number \( c \geq 0 \) such that

\[
\sum_{\alpha,\beta,\gamma,\delta} \langle B^{\beta+\gamma} Tf_\alpha, B^{\alpha+\delta} Tf_\gamma, \delta \rangle \leq c \sum_{\alpha,\beta,\gamma,\delta} \langle A^{\beta+\gamma} f_\alpha, A^{\alpha+\delta} f_\gamma, \delta \rangle
\]

for all finite families \( \{f_\alpha\}_{\alpha,\beta \in \mathbb{N}_\Sigma} \subseteq \mathcal{E}_1 \),

\( \text{(iii) } TA \subseteq BT \) and there exists a real number \( c \geq 0 \) such that (4.4) holds for all finite families \( \{f_\alpha,\beta\}_{\alpha,\beta \in \mathbb{N}_\Sigma} \subseteq M \),
(iv) there exists a real number \( c \geq 0 \) such that

\[
\sum_{\alpha,\beta,\gamma,\delta} \langle B^{\beta+\gamma} Tf, B^{\alpha+\delta} Tf \rangle \lambda_\alpha,\beta \lambda_\gamma,\delta \leq c \sum_{\alpha,\beta,\gamma,\delta} \langle A^{\beta+\gamma} f, A^{\alpha+\delta} f \rangle \lambda_\alpha,\beta \lambda_\gamma,\delta
\]

for every \( f \in \mathcal{E}_1 \) and for all finite families \( \{\lambda_\alpha,\beta\}_{\alpha,\beta \in \mathbb{N}_\Sigma} \subseteq \mathbb{C} \),

(iv) \( TA \subseteq BT \) and there exists a real number \( c \geq 0 \) such that (4.5) holds for every \( f \in M \) and for all finite families \( \{\lambda_\alpha,\beta\}_{\alpha,\beta \in \mathbb{N}_\Sigma} \subseteq \mathbb{C} \).

Moreover, if (i) holds, then

(v) \( TA \subseteq BT \),
(vi) the smallest real number \( c \geq 0 \) satisfying the condition (ii) (resp. (iii), (iii) \( M \), (iv), (iv) \( M \)), is equal to \( \|\hat{T}\|^2 \).

**Proof.** The equivalence of conditions (i) and (ii) as well as the implication \( (i) \Rightarrow (v) \) follow from Proposition 4.2.
(ii)⇒(iii) From the implications (ii)⇒(i) and (i)⇒(v), we infer that $TA \subseteq BT$. Let $c \geq 0$ be as in (ii). Take a finite family $\{f_{\alpha,\beta}\}_{\alpha,\beta \in N_\sigma} \subseteq E_1$. We define a finite family $\{g_\alpha\}_{\alpha \in N_\sigma} \subseteq E_1$ by

$$g_\alpha = \sum_\beta A^\beta f_{\alpha,\beta}, \quad \alpha \in N_\sigma.$$  \hspace{1cm} (4.6)

Then $TA \subseteq BT$ implies that

$$\sum_{\alpha,\beta} \langle B^\alpha Tg_\beta, B^\beta Tg_\alpha \rangle = \sum_{\alpha,\beta} \langle B^\alpha T \left( \sum_\gamma A^\gamma f_{\beta,\gamma} \right), B^\beta T \left( \sum_\delta A^\delta f_{\alpha,\delta} \right) \rangle = \sum_{\alpha,\beta,\gamma,\delta} \langle B^{\gamma+\alpha} Tf_{\beta,\gamma}, B^{\beta+\delta} Tf_{\alpha,\delta} \rangle.$$  \hspace{1cm} (4.7)

Clearly, we have

$$\sum_{\alpha,\beta} \langle A^\alpha g_\beta, A^\beta g_\alpha \rangle = \sum_{\alpha,\beta,\gamma,\delta} \langle A^{\gamma+\alpha} f_{\beta,\gamma}, A^{\beta+\delta} f_{\alpha,\delta} \rangle.$$  \hspace{1cm} (4.8)

That (4.4) is satisfied with the constant $c$ follows from (4.2), (4.7) and (4.8).

(iii)⇒(ii) Let $c \geq 0$ be as in (iii). Take a finite family $\{f_\alpha\}_{\alpha \in N_\sigma} \subseteq E_1$. For $\alpha, \beta \in N_\sigma$, set $g_{\alpha,\beta} = f_\alpha$ if $\beta = 0$ and $g_{\alpha,\beta} = 0$ otherwise. Then, we have

$$\sum_{\alpha,\beta,\gamma,\delta} \langle B^{\gamma+0} Tg_{\alpha,\beta}, B^{0+\delta} Tg_{\alpha,\gamma} \rangle = \sum_{\alpha,\gamma} \langle B^\gamma Tf_\alpha, B^\gamma Tf_\gamma \rangle, \quad \sum_{\alpha,\beta,\gamma,\delta} \langle A^{\gamma+0} g_{\alpha,\beta}, A^{0+\delta} g_{\alpha,\gamma} \rangle = \sum_{\alpha,\gamma} \langle A^\gamma f_\alpha, A^\gamma f_\gamma \rangle.$$

This and (4.4) imply (4.2) with the same constant $c$.

Consider the commutative *-semigroup $\Omega_\Sigma = N_\Sigma \times N_\Sigma$ (see Section 2). Given a real number $c \geq 0$, we define the form $\varphi_c$ over $(\Omega_\Sigma, M)$ by

$$\varphi_c((\alpha, \beta); f, g) = c \langle A^\alpha f, A^\beta g \rangle - \langle B^\alpha Tf, B^\beta Tg \rangle, \quad f, g \in M, \quad (\alpha, \beta) \in \Omega_\Sigma.$$  \hspace{1cm} (iv)

(iii)⇒(ii) Let $c \geq 0$ be as in (iii). It follows from (iv) that the form $\varphi_c$ is weakly positive definite. Consider the set $\mathcal{F} = \{(\delta_\sigma, 0) : \sigma \in \Sigma\}$ of *-generators of the *-semigroup $\Omega_\Sigma$. We show that

$$\mathcal{A}_{\varphi_c}(\delta_\sigma, 0) = M \quad \text{for every } \sigma \in \Sigma.$$  \hspace{1cm} (4.9)

Indeed, if $\sigma \in \Sigma$ and $f \in M \subseteq \mathcal{A}_\sigma(A_\sigma)$, then there exists a real number $t = t(f) > 0$ such that

$$\sum_{n=0}^\infty \|A_\sigma^n f\| \frac{t^n}{n!} < \infty.$$  \hspace{1cm} (4.10)

Since $\varphi_c$ is weakly positive definite, we get

$$0 \leq \varphi_c(n \cdot ((\delta_\sigma, 0)^* + (\delta_\sigma, 0)); f, f) = \varphi_c(n \cdot (\delta_\sigma, \delta_\sigma); f, f) = c\|A_\sigma^n f\|^2 - \|B_\sigma^n Tf\|^2 \leq c\|A_\sigma^n f\|^2, \quad n \in N, \sigma \in \Sigma.$$  \hspace{1cm} (4.10)

This together with (4.10) yields

$$\sum_{n=0}^\infty \sqrt{\varphi_c(n \cdot ((\delta_\sigma, 0)^* + (\delta_\sigma, 0)); f, f)} \frac{t^n}{n!} < \infty.$$
A relationship between families $A$, $B$, and $C$ is elucidated below.

5. A Local Lifting Theorem for Unbounded Operators

Let $\mathcal{E}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$. Consider a family of commuting operators $A = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq L(\mathcal{E})$. For $f \in \mathcal{E}$ and $\sigma \in \Sigma$, we define

$$
\mathcal{E}_{A,f} = \{A_\alpha^* f : \alpha \in \mathbb{N}_\Sigma\}, \quad Q_{A,f} = \overline{\mathcal{E}_{A,f}} \quad \text{and} \quad A_{\sigma,f} = A_\sigma|_{\mathcal{E}_{A,f}}.
$$

The family $\{A_{\sigma,f}\}_{\sigma \in \Sigma}$ will be denoted briefly by $A_f$. Clearly, $A_f \subseteq L(\mathcal{E}_{A,f})$. Since $A$ is a family of commuting operators, so also is $A_f$. If $A$ is jointly subnormal and $M = \{M_\sigma\}_{\sigma \in \Sigma}$ is a normal extension of $A$, then for every $f \in \mathcal{E}$, we have

$$
A_{\sigma,f} \subseteq A_\sigma \subseteq M_\sigma, \quad \sigma \in \Sigma,
$$

so the family $A_f$ is jointly subnormal.

For $g \in \mathcal{D}^\infty(M)$ and $\sigma \in \Sigma$, we put

$$
\mathcal{F}_{M,g} = \{M^{\alpha,\beta} g : \alpha, \beta \in \mathbb{N}_\Sigma\}, \quad Q_{M,g} = \overline{\mathcal{F}_{M,g}}, \quad M_\sigma^g = M_\sigma|_{\mathcal{F}_{M,g}}.
$$

The definition of $M_\sigma^g$ makes sense due to Lemma 3.1. Let us denote by $M_f^g$ the family $\{M_{\sigma}^g\}_{\sigma \in \Sigma}$. If $f \in \mathcal{E}$, then $\mathcal{E}_{A,f} \subseteq \mathcal{F}_{M,f}$ and clearly $Q_{A,f} \subseteq Q_{M,f}$.

Therefore,

$$
A_{\sigma,f} \subseteq M_\sigma^f \subseteq M_\sigma, \quad \sigma \in \Sigma.
$$

A relationship between families $A_f$ and $M_f^g$ is elucidated below.

**Theorem 5.1.** Let $M = \{M_\sigma\}_{\sigma \in \Sigma}$ be a normal extension of a jointly subnormal family $A = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq L(\mathcal{E})$ such that $\mathcal{E} = \mathcal{A}(A_\sigma)$ for all $\sigma \in \Sigma$. If $f \in \mathcal{E}$, then

(i) $Q_{M,f}^g$ reduces $M_\sigma$ to $M_\sigma^f$ for every $\sigma \in \Sigma$,

(ii) $M_f^g$ is a minimal normal extension of $A_f$ of cyclic type.
Proof. Since $\mathcal{E} = \mathcal{A}(A_\sigma)$ for every $\sigma \in \Sigma$, we see that $\mathcal{E}_{A,f} = \mathcal{A}(A_{\sigma,f})$ for all $\sigma \in \Sigma$. Noticing that $\mathcal{F}_M(\mathcal{E}_{A,f}) = \mathcal{F}M,f$ (see (3.3) for the notation), we may apply Theorem 3.2 to the jointly subnormal family $A_f$ and its normal extension $M$. This completes the proof. \(\square\)

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be dense linear subspaces of Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Consider families of operators $A = \{A_\sigma\}_{\sigma \in \Sigma} \subseteq L(\mathcal{E}_1)$ and $B = \{B_\sigma\}_{\sigma \in \Sigma} \subseteq L(\mathcal{E}_2)$. Set $S = (A, B)$. Take an operator $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $TA \subseteq BT$. Then $T\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $T(\mathcal{E}_{A,f}) \subseteq \mathcal{E}_{B,Tf}$ for all $f \in \mathcal{E}_1$. Owing to the boundedness of $T$, this yields $T(Q_{A,f}) \subseteq Q_{B,Tf}$ for all $f \in \mathcal{E}_1$. Define the operator $T_{S,f}$ via

$$T_{S,f} = T|_{Q_{A,f}} \in B(Q_{A,f}, Q_{B,Tf}).$$

It follows from $TA \subseteq BT$ that

$$T_{S,f} A_f \subseteq B_{Tf} T_{S,f}, \quad f \in \mathcal{E}_1. \quad (5.1)$$

Now we formulate a version of [6, Lemma 4.1] for families of operators.

**Lemma 5.2.** Suppose that (4.1) holds. Assume that $\mathcal{E}_1 = \mathcal{A}(A_\sigma)$ and $\mathcal{E}_2 = \mathcal{A}(B_\sigma)$ for all $\sigma \in \Sigma$. Take an operator $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $TA \subseteq BT$ and fix a vector $f \in \mathcal{E}_1$. Then the following conditions are equivalent (with $S := (A, B)$):

(i) $T_{S,f}$ lifts to $\mathcal{I}(M^f, N^{Tf})$,

(ii) there exists a real number $c \geq 0$ such that (4.5) holds for all finite families $\{\lambda_{\alpha, \beta}\}_{\alpha, \beta \in \mathbb{N}_\Sigma} \subseteq \mathbb{C}$.

Moreover, if (i) holds, then $\|\hat{T}_{S,f}\|^2 = \min\{c \geq 0 : c$ satisfies (ii)$\}$.

**Proof.** In view of (5.1) and Theorem 5.1, we can apply Theorem 4.3 to $M = \mathbb{C}f$, the intertwining operator $T_{S,f}$, jointly subnormal families $A_f$ and $B_{Tf}$, and their minimal normal extensions of cyclic type $M^f$ and $N^{Tf}$, respectively. \(\square\)

The following theorem, which is the main result of the paper, generalizes a local lifting theorem (cf. [6, Theorem 4.2]) to the case of pairs of jointly subnormal families of unbounded operators.

**Theorem 5.3.** Suppose that (4.1) holds. Assume that $\mathcal{E}_1 = \mathcal{A}(A_\sigma)$ and $\mathcal{E}_2 = \mathcal{A}(B_\sigma)$ for all $\sigma \in \Sigma$. Let $\mathcal{M}$ be a linear subspace of $\mathcal{E}_1$ such that

$$\mathcal{E}_1 = \text{lin}\{A^\alpha f : f \in \mathcal{M}, \alpha \in \mathbb{N}_\Sigma\}$$

and let $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ be an operator such that $TE_1 \subseteq \mathcal{E}_2$. Then the following conditions are equivalent (with $S := (A, B)$):

(i) $T$ lifts to $\mathcal{I}(M, N)$,

(ii) $TA \subseteq BT$, the operator $T_{S,f}$ lifts to $\mathcal{I}(M^f, N^{Tf})$ for every $f \in \mathcal{M}$, and

$$\sup_{f \in \mathcal{M}} \|\hat{T}_{S,f}\| < \infty.$$
Moreover, if (i) holds, then
\[ \|\hat{T}\| = \sup_{f \in \mathcal{M}} \|\hat{T}s_{f}\|. \] (5.2)

**Proof.** The proof of the implication (i)⇒(ii) requires more care than in the case of bounded operators. Let \( \hat{T} \) be the lift of \( T \) to \( \mathcal{I}(\mathcal{M}, \mathcal{N}) \). Fix \( f \in \mathcal{M} \). By the Putnam-Fuglede theorem (cf. [10]), \( \hat{T}\mathcal{M} \subseteq \mathcal{N}\hat{T} \) implies \( \hat{T}\mathcal{M}^* \subseteq \mathcal{N}^*\hat{T} \). These two inclusions and \( T \subseteq \hat{T} \) guarantee that \( \hat{T}|_{\mathcal{M}, f} \in \mathcal{B}(\mathcal{M}, f, \mathcal{N}, Tf) \) and \( T_{s_{f}} \subseteq \hat{T}|_{\mathcal{M}, f} \). Next, we show that
\[ \hat{T}h \in \mathcal{D}(N^T_{\sigma}) \text{ for all } h \in \mathcal{D}(M^f_{\sigma}) \text{ and } \sigma \in \Sigma. \] (5.3)

Indeed, since \( h \in \mathcal{D}(M^f_{\sigma}) \subseteq \mathcal{D}(M^f_{\sigma}) \) and \( \hat{T}\mathcal{M} \subseteq \mathcal{N}\hat{T} \), we deduce that \( \hat{T}h \in \mathcal{D}(N^f_{\sigma}) \). As \( h \in \mathcal{Q}M^f_{\sigma} \), we have \( \hat{T}h = \hat{T}|_{\mathcal{M}, f} h \in \mathcal{Q}N^T_{\sigma}f \). This, combined with the fact that \( \mathcal{Q}N^T_{\sigma}f \) reduces the operator \( N^f_{\sigma} \) to \( N^T_{\sigma}f \) (cf. Theorem 5.1), leads to
\[ \hat{T}h \in \mathcal{D}(N^f_{\sigma}) \cap \mathcal{Q}N^T_{\sigma}f = \mathcal{D}(N^T_{\sigma}f), \]
which proves (5.3). The condition (5.3) together with \( \hat{T}\mathcal{M} \subseteq \mathcal{N}\hat{T} \) implies that \( \hat{T}|_{\mathcal{M}, f} M^f_{\sigma} \subseteq N^T_{\sigma}f \hat{T}|_{\mathcal{M}, f} \). Hence, by the uniqueness of \( T_{s_{f}} \), we have \( \hat{T}s_{f} = \hat{T}|_{\mathcal{M}, f} \), which also shows that the right-hand side of (5.2) is less than or equal to its left-hand side.

The implication (ii)⇒(i) can be deduced from Lemma 5.2 and Theorem 4.3. On the way we also verify that the left-hand side of (5.2) is less than or equal to its right-hand side. This completes the proof. \( \square \)

6. Local Commutativity

Let us concentrate on a single subnormal operator \( A \in \mathcal{B}(\mathcal{H}) \). For simplicity, we write “minimal normal extension” in place of “minimal normal extension of cyclic type”. Suppose for a moment that \( T \in \mathcal{B}(\mathcal{H}) \) is an operator that commutes with \( A \) (this requirement is necessary for \( T \) to lift to the commutant of a minimal normal extension of \( A \)). Then \( T\mathcal{Q}A_{f} \subseteq \mathcal{Q}A_{Tf} \) and consequently \( T|_{\mathcal{Q}A_{f}} A|_{\mathcal{Q}A_{f}} = A|_{\mathcal{Q}A_{Tf}} T|_{\mathcal{Q}A_{f}} \) for all \( f \in \mathcal{H} \). The question arises under what circumstances \( T\mathcal{Q}A_{f} \subseteq \mathcal{Q}A_{f} \) for all \( f \in \mathcal{H} \). Note that if this is the case, then \( T|_{\mathcal{Q}A_{f}} A|_{\mathcal{Q}A_{f}} = A|_{\mathcal{Q}A_{f}} T|_{\mathcal{Q}A_{f}} \) for all \( f \in \mathcal{H} \), which is referred to as the local commutativity of \( A \) and \( T \). This question can be extended to the case in which \( A \) and \( T \) are not assumed to commute. In Proposition 6.1 we provide a complete answer to the extended question. It is worth pointing out that the operators \( A \) and \( T \) must commute if all cyclic invariant subspaces \( \mathcal{Q}A_{f} \) of \( A \) are invariant for \( T \).

Given an operator \( A \in \mathcal{B}(\mathcal{H}) \) and a set \( \mathcal{A} \) of closed linear subspaces of \( \mathcal{H} \), we write \( \text{Lat}(A) \) for the set of all closed linear subspaces of \( \mathcal{H} \) invariant for \( A \) and \( \text{Alg}(A) \) for the set of all operators \( T \in \mathcal{B}(\mathcal{H}) \) such that \( \mathcal{A} \subseteq \text{Lat}(T) \). Denote by \( \mathcal{W}(A) \) the closure in the weak operator topology (equivalently: in the strong operator topology) of the algebra generated by \( \{A, I_{\mathcal{H}}\} \). It is clear that \( \mathcal{W}(A) \subseteq \text{Alg}(\text{Lat}(A)) \). In general, the reverse implication is not true.
The Olin-Thomson theorem states that \( \mathcal{W}(A) = \text{Alg} (\text{Lat}(A)) \) for any sub-normal operator \( A \in B(\mathcal{H}) \) (cf. [9, Theorem 3]).

**Proposition 6.1.** Let \( A \in B(\mathcal{H}) \) be a subnormal operator. If \( T \in B(\mathcal{H}) \), then the following conditions are equivalent:

(i) \( T \mathcal{Q}_{A,f} \subseteq \mathcal{Q}_{A,f} \) for all \( f \in \mathcal{H} \), where \( \mathcal{Q}_{A,f} \) is the smallest closed linear subspace of \( \mathcal{H} \) containing \( f \) and invariant for \( A \),

(ii) \( T \in \mathcal{W}(A) \).

**Proof.** (i)\( \Rightarrow \) (ii) In view of the Olin-Thomson theorem, it is enough to show that \( T \in \text{Alg} (\text{Lat}(A)) \). Take \( M \in \text{Lat}(A) \). If \( f \in M \), then \( f \in \mathcal{Q}_{A,f} \subseteq M \). Hence, by (i), we have \( Tf \in \mathcal{Q}_{A,f} \subseteq M \). This means that \( M \in \text{Lat}(T) \).

(ii)\( \Rightarrow \) (i) Take \( f \in \mathcal{H} \). Since \( \mathcal{Q}_{A,f} \in \text{Lat}(A) \), we see that \( \mathcal{Q}_{A,f} \in \text{Lat}(p(A)) \) for every complex polynomial \( p \) in one indeterminate, which implies that \( \mathcal{Q}_{A,f} \in \text{Lat}(R) \) for every \( R \in \mathcal{W}(A) \). This, together with (ii), gives (i). \( \square \)

We conclude the paper with a simple example of an operator \( T \) which lifts to the commutant of a minimal normal extension of a subnormal operator \( A \), and which has the property that not all cyclic invariant subspaces of \( A \) are invariant for \( T \). This shows that a (global) lift of a member \( T \) of the commutant of a subnormal operator \( A \) may exist though the local commutativity of \( A \) and \( T \) does not hold. In fact, it may even happen that for some vector \( f \) the linear spaces \( \mathcal{Q}_{A,f} \) and \( T \mathcal{Q}_{A,f} \) are orthogonal.

**Example 6.2.** Given a bounded Borel function \( \psi \) on \( T := \{ z \in \mathbb{C} : |z| = 1 \} \), we denote by \( M_\psi \) the operator of multiplication by \( \psi \) on \( L^2(T) \). Let \( \xi \) be the identity function on \( T \). Set \( T = M_\xi |H^2\) and \( A = T^2 \), where \( H^2 \) stands for the Hardy space regarded as a closed linear subspace of \( L^2(T) \). The operator \( A \) is subnormal (as an isometric operator) and \( TA = AT \). Observe that if \( f = 1 \), then \( A^n f = \xi^{2n} \) for all \( n \in \mathbb{N} \), which implies that \( \mathcal{Q}_{A,f} = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \cdot \xi^{2n} \). Since \( Tf = \xi \), we deduce that \( T \mathcal{Q}_{A,f} = \bigoplus_{n \in \mathbb{N}} \mathbb{C} \cdot \xi^{2n+1} \). Therefore, the linear spaces \( \mathcal{Q}_{A,f} \) and \( T \mathcal{Q}_{A,f} \) are orthogonal. Note that the unitary operator \( M_\xi \) is a minimal normal extension of the isometry \( A \) (because \( \bigvee_{n \in \mathbb{N}} M_\xi^n (H^2) = L^2(T) \), which follows from the equalities \( M_\xi^n f = \tilde{\xi}^{2n} \) and \( M_\xi^n \xi = \tilde{\xi}^{2n-1} \), \( n \geq 1 \)). Clearly, \( M_\xi \) is the lift of \( T \) to the commutant of \( M_\xi \).

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