ON THE BLOW-UP OF THE CAUCHY PROBLEM OF HIGHER-ORDER NONLINEAR VISCOELASTIC WAVE EQUATION

Mohammad Kafini

Department of Mathematics and Statistics
IR Center of Construction and Building Materials
KFUPM, Dhahran 31261, Saudi Arabia

Abstract. In this paper we consider the Cauchy problem for a higher-order viscoelastic wave equation with finite memory and nonlinear logarithmic source term. Under certain conditions on the initial data with negative initial energy and with certain class of relaxation functions, we prove a finite-time blow-up result in the whole space. Moreover, the blow-up time is estimated explicitly. The upper bound and the lower bound for the blow up time are estimated.

1. Introduction. In this paper, we are concerned with a higher-order Cauchy nonlinear viscoelastic problem of the form

\[
\begin{cases}
  u_{tt} + (-\Delta)^m u - \int_0^t g(t-s) (-\Delta)^m u(x,s) ds + u_t = u |u|^{p-2} \ln |u|^k, & x \in \mathbb{R}^n, t > 0, \\
  u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & x \in \mathbb{R}^n,
\end{cases}
\]  

where \( m \geq 1 \) is a natural number, \( p > 2 \) is a real number, \( g \) is the relaxation satisfying some conditions to be specified later and \( k \) is a positive constant. This type of problems with logarithmic nonlinearity is encountered in many branches of physics such as Nuclear Physics, Optics and Geophysics.

When \( m = 1 \) and \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) \((n \geq 1)\) with a smooth boundary \( \partial \Omega \), the problem have been studied extensively. For instance, Messaoudi [16], considered

\[
\begin{cases}
  u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(\tau)d\tau + u_t|u_t|^{\gamma-1} = b|u|^{p-1} u, & \Omega \times (0, \infty) \\
  u(x,0) = u_0(x), & u_t(x,0) = u_1(x), & x \in \Omega,
\end{cases}
\]  

where \( p > 1 \), \( \gamma \geq 1 \) and \( g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a positive nonincreasing function. He showed, under suitable conditions on \( g \), that solutions with negative initial energy blow up in finite time if \( p > \gamma \) and globally exist if \( \gamma \geq p \). With positive initial energy, same results obtained and developed in [17] and [31].

In the absence of the viscoelastic term \( (g = 0) \), many results concerning global existence and nonexistence have been proved. The equation

\[
u_{tt} - \Delta u + au_t|u_t|^{\gamma-1} = b|u|^{p-1} u, \quad \text{in} \ \Omega \times (0, \infty),\]

2020 Mathematics Subject Classification. 35B05, 35L05, 35L15, 35L70.

Key words and phrases. Blow up, Cauchy problem, upper bound, lower bound, nonlinear higher-order wave equation, memory.

The first author is supported by KFUPM project \# SB201026.

* Corresponding author: Mohammad Kafini.
finite-time blow-up result. Actually they extended the result of Zhou Yong [34]. In suitable conditions on the initial data and the relaxation function, they proved a
\begin{equation}
\gamma, p \geq 1,
\end{equation}
causes finite time blow up of solutions with negative initial energy (see [3]). The competition between the damping and the source terms was considered by Levine [12], [13] in the linear damping case ($\gamma = 1$). Georgiev and Todorova [8] extended Levine’s result to the nonlinear damping case ($\gamma > 1$). For results of same nature, we refer to Messaoudi [18], Levine and Serrin [14], Vitillaro [28], and Messaoudi and Said-Houari [20].

Problem with logarithmic nonlinearity of the form
\begin{equation}
u_{tt} - \Delta u + u - u \log |u|^p + u_t + u |u|^p = 0, \quad x \in \Omega, \quad t \in (0, T),
\end{equation}
where $\Omega$ is a smooth and bounded domain in $\mathbb{R}^3$, studied by Han in [9]. He showed the global existence of weak solutions of (4) by using Galerkin method, logarithmic Sobolev inequality and compactness theorem. Logarithmic Schrödinger equation
\begin{equation}
iu_{tt} + \Delta u + u \log |u|^2 = 0, \quad x \in \mathbb{R}^n, \quad t \in (0, T),
\end{equation}
also considered as in [6]. Some properties of the energy functional associated to (5) were established and applied to the study of behavior at infinity of solutions. Thierry and Alain [7], established the existence and uniqueness of a solution for the following Cauchy problem in the whole space ($\mathbb{R}^3$):
\begin{equation}
\begin{cases}
u_{tt} - \Delta u + u_t - u \log |u|^2 = 0, \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x).
\end{cases}
\end{equation}
In [10], the following Cauchy problem
\begin{equation}
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x, s)ds + u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n, \quad t > 0,
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n,
\end{equation}
where $g, u_0, u_1$ are specific functions, was discussed by Kafini and Messaoudi. Under suitable conditions on the initial data and the relaxation function, they proved a finite-time blow-up result. Actually they extended the result of Zhou Yong [34]. In [11], the same authors proved also a blow-up to the system
\begin{equation}
\begin{cases}
u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x, s)ds = f_1(u, v), \quad \text{in} \ \mathbb{R}^n \times (0, \infty), \\
u_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x, s)ds = f_2(u, v), \quad \text{in} \ \mathbb{R}^n \times (0, \infty), \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R}^n \\
v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \mathbb{R}^n.
\end{cases}
\end{equation}
For more results of problem (3) in $\mathbb{R}^n$, we refer to the work of Levine Serrin and Park [15], Todorova [26, 27], and Messaoudi [19].

The following Petrovsky equation with memory term and nonlinear source term
\begin{equation}
u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(x, s)ds = |u|^{p-1}u, \quad (x, t) \in \Omega \times \mathbb{R}^+,
\end{equation}
was studied by Tahamatanri et al [25]. They showed the existence of weak solutions with initial–boundary value conditions and proved that there are solutions under some conditions on initial data which blow up in finite time with non-positive initial energy as well as positive initial energy and give the lifespan estimates of solutions. In the absence of nonlinear source term, Rivera et al [23] considered (7) in a bounded domain $\Omega \subset \mathbb{R}^n$ and showed that the energy of solution decays exponentially provided the relaxation function $g$ decays exponentially. For more related results about the existence, blow-up and asymptotic properties of solutions of (7), the readers are also referred to [2, 5, 21, 29].
Higher-order case, the case of \( m \geq 1 \), problem (3) becomes an initial–boundary value problem of the form

\[
u_{tt} + (-\Delta)^m u + au_t|u_t|^{\gamma-1} = b|u|^{m-2}u, \quad (x,t) \in \Omega \times \mathbb{R}^+.
\]  

(8)

Brenner et al [4] proved the existence and uniqueness of classical solutions to (8) in Hilbert space. Pecher [24] investigated the existence and uniqueness of Cauchy problem for the equation in (8) by use of the potential well method.

Wang [30] showed that the scattering operators map a band in \( H^s \) into \( H^s \) if the nonlinearities have critical or subcritical powers in \( H^s \). Miao [22] obtained the scattering theory at low energy using time–space estimates and nonlinear estimates. He also established the global existence and uniqueness of solutions under the condition of low energy. Yaojun Ye [32] established the existence of global solutions for this problem by constructing the stable sets and showed the asymptotic stability of the global solutions as time goes to infinity by applying the multiplier method.

The case of \( m \geq 1 \), and \( g \neq 0 \), Yaojun Ye [33], studied the following initial–boundary value problem

\[
u_{tt} + (-\Delta)^m u - \int_0^t g(t-s) (-\Delta)^m u(x,s)ds = |u|^{p-2}u, \quad (x,t) \in \Omega \times \mathbb{R}^+.
\]  

(9)

The existence of global weak solutions for this problem is established by using the Galerkin method. Under suitable conditions on relaxation function \( g \) and the positive initial energy as well as non-positive initial energy, it is proved that the solution blows up in the finite time.

Recently, Algarabli [1], investigated the stability of the solutions of a viscoelastic plate equation with a logarithmic nonlinearity of the form

\[
u_{tt} - \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(x,s)ds = ku \ln |u|, \quad (x,t) \in \Omega \times \mathbb{R}^+.
\]

\[u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \quad u = \partial_u = 0, \quad x \in \partial \Omega, \quad t \geq 0.
\]

He assumed that the relaxation function \( g \) satisfies the minimal condition

\[g(t) \leq -\xi(t)G(g(t)),\]

where \( \xi \) and \( G \) satisfy some properties. With this very general assumption on the behavior of \( g \), he established explicit and general energy decay results from which the exponential and polynomial rates can be recovered when \( G(s) = s^p \) and \( p \) covers the full admissible range \([1,2)\).

Our aim is to extend the result of [33], established in bounded domains, to the problem in unbounded domains. Namely, we consider the Cauchy problem in (1). This type of problems is not well considered in the literature because Poincaré’s inequality and some Lebesgue and Sobolev embedding inequalities are no longer valid. We aim to obtain a blow up results for solutions with negative initial energy in a specific domain. To achieve this goal some conditions have to be imposed on the relaxation function \( g \), the initial data \( u_0, u_1 \) and on the exponents \( k, p \) as well.

2. Preliminaries. In this section we present some material needed in the proof of our result. First, we state without a proof the local existence result. It can be established by adopting the arguments of [26, 27, 33]. For that purpose, let us assume that
(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that
\[ 1 - \int_0^\infty g(s)ds = l > 0, \quad g'(t) \leq 0, \quad t \geq 0. \quad (10) \]

(G2) The natural number $m \geq 1$ satisfying
\[ 2 < p \leq \frac{2(n-m)}{n-2m} \quad \text{if} \quad n > 2m, \quad \text{and} \quad p > 2 \quad \text{if} \quad n \leq 2m. \]

Proposition 2.1. Assume that (G1) and (G2) hold. Then for any initial data
\[ u_0 \in H^{2m}(\mathbb{R}^n), \quad u_1 \in H^m(\mathbb{R}^n), \]
problem (1) has a unique local solution
\[ u \in C([0,T); H^{2m}(\mathbb{R}^n)), \quad u_t \in C([0,T); H^m(\mathbb{R}^n)), \]
for $T$ small enough.

In the proof of our main result, we shall make use of the following proposition.

Proposition 2.2 ([34]). Suppose that $\Phi$ is a twice continuously differentiable function satisfying
\[ \Phi''(t) + \Phi'(t) \geq C_0(t + L)^{\beta} \Phi^{1+\alpha}(t), \quad t > 0, \]
\[ \Phi(0) > 0, \quad \Phi'(0) \geq 0, \]
where $C_0, L > 0, -1 < \beta \leq 0, \alpha > 0$ are constants. Then $\Phi$ blows up in finite time. Moreover, the blow-up time $T_0$ can be estimated explicitly as follows.
\[ T_0 = \left( \frac{2\Phi(0)}{\varepsilon \alpha} + L^{\beta+1} \right)^{1/(\beta+1)} - L, \quad (11) \]
where $\varepsilon > 0$ is sufficiently small such that
\[ \varepsilon^2 \left( 1 + \frac{\alpha}{2} \right) + \varepsilon \Phi(0)^{-\frac{2}{\alpha}} \leq C_0. \]

We adopt the usual notations and convention. Let $H^m(\mathbb{R}^n)$ denote the Sobolev space with the usual scalar product and norm. For simplicity of notations, hereafter we denote by $\| D^m \|$ the equivalent norm $\| \cdot \|_{H^m(\mathbb{R}^n)}$ and $D$ denotes the gradient operator, that is
\[ D = \nabla = (\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_n). \]

Moreover,
\[ D^m = \Delta^j \quad \text{if} \quad m = 2j \quad \text{and} \quad D^m = D \Delta^j \quad \text{if} \quad m = 2j + 1. \]

At the end of this section, we introduce the “modified” energy functional
\[ E(t) = \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \left( 1 - \int_0^t g(s)ds \right) \| D^m u \|^2 + \frac{1}{2} (g \circ D^m u) \]
\[ + \frac{k}{p} \| u \|^{p-1} - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p \ln |u|^k \, dx, \quad (12) \]
where
\[ (g \circ D^m v)(t) = \int_0^t g(t-s) \| D^m v(t) - D^m v(s) \|^2 \, ds, \quad \forall v \in H^m(\mathbb{R}^n). \]
3. **Blow up.** Our main result reads as follows.

**Theorem 3.1.** Assume that (G1) hold and \(2 < p \leq \frac{2n+2}{n}\). Assume further

\[
\int_0^t g(s)ds < \frac{2p^2 - 2p}{2p^2 - 2p + 1}, \quad \forall t \geq 0, \tag{13}
\]

and the nonzero initial data

\[ (u_0, u_1) \in H^{2m}(\mathbb{R}^n) \times L^2(\mathbb{R}^n), \]

with compact supports, satisfying

\[ E(0) \leq 0 \quad \text{and} \quad \int_{\mathbb{R}^n} u_0 u_1 dx > 0. \tag{14} \]

Then any solution \(u\) of (1) blows up in finite time \(T_0\) determined by (11).

**Proof.** By multiplying the equation in (1) by \(u_t\) and integrating over \(\mathbb{R}^n\), we get

\[
\begin{align*}
\int_{\mathbb{R}^n} u_t u_t dx + \int_{\mathbb{R}^n} u_t (-\Delta)^m u dx - \int_{\mathbb{R}^n} u_t \int_0^t g(t-s) (-\Delta)^m u(x, s) ds dx & \\
+ \int_{\mathbb{R}^n} u_t^2 dx = \int_{\mathbb{R}^n} u_t |u_t|^{p-2} \ln |u| k dx,
\end{align*}
\]

then integration by parts, yields

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \int_{\mathbb{R}^n} |u_t|^2 dx + \int_{\mathbb{R}^n} |D^m u|^2 dx \right) & \\
- \int_0^t \int_{\mathbb{R}^n} D^m u_t(t) D^m u(s) dx ds + \int_{\mathbb{R}^n} u_t^2 dx & \\
= \frac{d}{dt} \left( -\frac{k}{p^2} ||u||_p^p + \frac{1}{p} \int_{\mathbb{R}^n} |u|^p \ln |u| k dx \right).
\end{align*}
\]

The third term in the left side of (15) can be estimated as follows.

\[
\begin{align*}
\int_0^t g(t-s) \int_{\mathbb{R}^n} D^m u_t(t) \cdot D^m u(s) dx ds & \\
= \int_0^t g(t-s) \int_{\mathbb{R}^n} D^m u_t(t) \cdot (D^m u(s) - D^m u(t)) dx ds & \\
+ \int_0^t g(t-s) \int_{\mathbb{R}^n} D^m u_t(t) \cdot D^m u(t) dx ds & \\
= -\frac{1}{2} \int_0^t g(t-s) \frac{d}{dt} \int_{\mathbb{R}^n} |D^m u(s) - D^m u(t)|^2 dx ds & \\
+ \int_0^t g(t-s) \left( \frac{d}{dt} \int_{\mathbb{R}^n} |D^m u(t)|^2 dx \right) ds & \\
= -\frac{1}{2} \frac{d}{dt} \left[ \int_0^t g(t-s) \int_{\mathbb{R}^n} |D^m u(s) - D^m u(t)|^2 dx ds \right] & \\
+ \frac{1}{2} \int_0^t g(t-s) \int_{\mathbb{R}^n} |D^m u(s) - D^m u(t)|^2 dx ds & \\
+ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |D^m u(t)|^2 dx ds - \frac{1}{2} g(t) \int_{\mathbb{R}^n} |D^m u(t)|^2 dx.
\end{align*}
\]
Insert (16) in (15) to get

\[
\frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} |u_t|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} |D^m u|^2 \, dx + \frac{k}{p^2} \|u\|^p_p - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p \ln |u|^k \, dx \right\} \\
+ \frac{1}{2} \int_0^t g(t-s) \int_{\mathbb{R}^n} |D^m u(s) - D^m u(t)|^2 \, dx \, ds - \frac{1}{2} \int_0^t g(s) \int_{\mathbb{R}^n} |D^m u(t)|^2 \, dx \, ds \\
= \frac{1}{2} \int_0^t g'(t-s) \int_{\mathbb{R}^n} |D^m u(s) - D^m u(t)|^2 \, dx \, ds \\
- \frac{1}{2} g(t) \int_{\mathbb{R}^n} |D^m u(t)|^2 \, dx - \int_{\mathbb{R}^n} u_t^2 \, dx,
\]

hence, using (12), we obtain

\[
E'(t) = \frac{1}{2} (g' \circ D^m u) - \frac{1}{2} g(t) \|D^m u\|^2 - \|\mu_t\|_2^2. \tag{17}
\]

It is clear that \(E'(t) \leq 0\) which means that the system is already dissipative. Because \((u_0, u_1) \in H^{2m}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) and of compact supports satisfying (14), we deduce that \(E(0)\) is finite and

\[
E(t) \leq E(0) \leq 0. \tag{18}
\]

In order to apply Proposition 2.2, we define

\[
\Phi(t) = \frac{1}{2} \int_{\mathbb{R}^n} |u(x,t)|^2 \, dx.
\]

Therefore,

\[
\Phi'(t) = \int_{\mathbb{R}^n} uu_t \, dx, \quad \Phi''(t) = \int_{\mathbb{R}^n} (uu_{tt} + |u_t|^2) \, dx. \tag{19}
\]

Using (14) and (19), it is easy to see that

\[
\Phi(0) > 0 \quad \text{and} \quad \Phi'(0) \geq 0.
\]

Multiplying (1) by \(u(t)\) and integrating over \(\mathbb{R}^n\) gives

\[
\int_{\mathbb{R}^n} uu_t \, dx = - \int_{\mathbb{R}^n} |D^m u|^2 \, dx - \int_{\mathbb{R}^n} uu_t \, dx + \int_{\mathbb{R}^n} |u|^p \ln |u|^k \, dx \\
+ \int_0^t g(t-s) \int_{\mathbb{R}^n} D^m u(x,t) \cdot D^m u(x,s) \, dx \, ds.
\]

By using

\[
\int_0^t g(t-s) \int_{\mathbb{R}^n} D^m u(x,t) \cdot D^m u(x,s) \, dx \, ds \\
= \int_0^t g(t-s) \int_{\mathbb{R}^n} D^m u(t) \cdot (D^m u(s) - D^m u(t)) \, dx \, ds \\
+ \left( \int_0^t g(s) \, ds \right) \int_{\mathbb{R}^n} |D^m u(t)|^2 \, dx,
\]
equation (19) becomes
\[
\Phi''(t) = - \left( 1 - \int_0^t g(s) ds \right) \int_{\mathbb{R}^n} |D^m u|^2 dx
\]
\[- \int_0^t g(t-s) \int_{\mathbb{R}^n} D^m u(t) \cdot (D^m u(t) - D^m u(s)) dx ds
\]
\[+ \int_{\mathbb{R}^n} |u|^p \ln |u|^k dx + \int_{\mathbb{R}^n} |u_t|^2 dx - \int_{\mathbb{R}^n} u_t dx.\]

Using Young’s inequality, we estimate
\[- \int_0^t g(t-s) \int_{\mathbb{R}^n} D^m u(t) \cdot (D^m u(t) - D^m u(s)) dx ds
\[\geq -\delta \int_{\mathbb{R}^n} |D^m u(t)|^2 dx - \frac{1}{4\delta} \left( \int_0^t g(s) ds \right) (g \circ D^m u)(t), \quad \forall \delta > 0.
\]

By combining (20) and (21), we get
\[
\Phi''(t) \geq - \left( 1 + \delta - \int_0^t g(s) ds \right) \|D^m u\|^2 - \frac{1}{4\delta} \left( \int_0^t g(s) ds \right) (g \circ D^m u)
\[+ \int_{\mathbb{R}^n} |u|^p \ln |u|^k dx + \int_{\mathbb{R}^n} |u_t|^2 dx - \int_{\mathbb{R}^n} u_t dx.
\]

So, from (19) and (22), we obtain
\[
\Phi''(t) + \Phi'(t) \geq - \left( 1 + \right. - \int_0^t g(s) ds \left) \|D^m u\|^2 - \frac{1}{4\delta} \left( \int_0^t g(s) ds \right) (g \circ D^m u)
\[+ \int_{\mathbb{R}^n} |u|^p \ln |u|^k dx + \int_{\mathbb{R}^n} |u_t|^2 dx.
\]

Now, we exploit (12) to substitute for \(\int_{\mathbb{R}^n} |u|^p \ln |u|^k dx\),
\[
\int_{\mathbb{R}^n} |u|^p \ln |u|^k dx
\[\geq p \left[ \|u_t\|^2 + \left( 1 - \int_0^t g(s) ds \right) \|D^m u\|^2 + \frac{1}{2} (g \circ D^m u) + \frac{2k}{p} \|u\|^p \right].
\]

Therefore,
\[
\Phi''(t) + \Phi'(t) \geq (1 + p) \|u_t\|^2 + \left[ \frac{p}{2} - \frac{1}{4\delta} \left( \int_0^t g(s) ds \right) \right] (g \circ D^m u)
\[+ \left[ (p - 1) \left( 1 - \int_0^t g(s) ds \right) - \delta \right] \|D^m u\|^2 + \frac{2k}{p} \|u\|^p.
\]

At this point, we aim to have
\[
\frac{p}{2} - \frac{1}{4\delta} \left( \int_0^t g(s) ds \right) > 0 \quad \text{and} \quad (p - 1) \left( 1 - \int_0^t g(s) ds \right) - \delta > 0. \tag{25}
\]

So, from the first part of (25), we need
\[
\delta > \frac{1}{2p} \left( \int_0^t g(s) ds \right).
\]
and from the second part that
\[ \delta < (p - 1) \left(1 - \int_0^t g(s) ds\right) \]
or
\[ \frac{1}{2p} \left(\int_0^t g(s) ds\right) < \delta < (p - 1) \left(1 - \int_0^t g(s) ds\right). \]
This implies that
\[ \frac{1}{2p} \left(\int_0^t g(s) ds\right) < (p - 1) \left(1 - \int_0^t g(s) ds\right) \]
or
\[ \int_0^t g(s) ds < \frac{2p^2 - 2p}{2p^2 - 2p + 1}, \quad \forall t \geq 0, \]
where it is guaranteed by (13).

Therefore, using (18), estimations (24) becomes
\[ \Phi''(t) + \Phi'(t) \geq \gamma \int_{\mathbb{R}^n} |u|^p dx, \quad (26) \]
for some \( \gamma \geq \frac{2k}{p} \).

Now, we use Hölder’s inequality to estimate
\[ \int_{\mathbb{R}^n} |u|^2 dx \leq \left(\int_{\mathbb{R}^n} |u|^p dx\right)^{\frac{2}{p}} \left(\int_{B(t+L)} 1 dx\right)^{\frac{p-2}{p}}, \]
where \( L > 0 \) is such that
\[ \text{supp}\{u_0(x), u_1(x)\} \subset B(L), \]
and \( B(t + L) \) is the ball with radius \( t + L \) centered at the origin. If we call \( W_n \) the volume of the unit ball then
\[ \left(\int_{\mathbb{R}^n} |u|^p dx\right)^{\frac{2}{p}} \geq \left(\int_{\mathbb{R}^n} |u|^2 dx\right) (W_n(t + L)^n)^{\frac{2-p}{p}} \quad (27) \]
\[ \int_{\mathbb{R}^n} |u|^p dx \geq \left(\int_{\mathbb{R}^n} |u|^2 dx\right)^{\frac{p}{2}} (W_n(t + L)^n)^{\frac{2-p}{p}}. \]
From the definition of \( \Phi(t) \), we have
\[ (2\Phi(t))^\frac{p}{2} = \left(\int_{\mathbb{R}^n} |u(x,t)|^2 dx\right)^{\frac{p}{2}}. \quad (28) \]
Combining (26), (27) and (28) yield
\[ \Phi''(t) + \Phi'(t) \geq 2^{\frac{p}{2}} \gamma (\Phi(t))^\frac{p}{2} (W_n(t + L)^n)^{\frac{2-p}{p}} \]
\[ \geq 2^{\frac{p}{2}} \gamma (W_n)^{\frac{2-p}{p}} (\Phi(t))^\frac{p}{2} (t + L)^{n(2-p)\frac{2}{p}}. \]
It is easy to verify that the requirements of Proposition 2.2. are satisfied with
\[ C_0 = 2^{\frac{p}{2}}(W_n)^{\frac{2-p}{p}} \gamma > 0, \quad -1 < \beta = \frac{n(2 - p)}{2} < 0, \quad \alpha = \frac{p - 2}{2} > 0. \]
Therefore \( \Phi \) blows up in finite time \( T_0 \) defined in (11). This completes the proof. \( \Box \)
4. Lower bound for the blow-up time. In this section, we estimate a lower bound for the blow-up time through the following theorem.

**Theorem 4.1.** Assume that conditions of theorem 3.1 are hold and the initial data 
\((u_0, u_1) \in H^m(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\)
are with compact support. Assume further \(u\) is a solution of (1) that does not exist after a time \(T^*\). Then there exists a positive constant \(C\) such that 
\[
\int_{\psi(0)}^{\infty} C \frac{dy}{y^2 + y^{(p-1-\varepsilon)}} \leq T^*,
\]
where
\[
\psi(t) = \frac{1}{p} \int_{\mathbb{R}^n} |u|^p \ln |u|^k \, dx.
\]

Before we start the proof, recall the following lemmas.

Note that the constant \(C > 0\) is a generic constant which may differ from place to another.

**Lemma 4.2.** *(Sobolev, Gagliardo, Nirenberg).* Suppose that \(1 \leq P < n\). If 
\(u \in W^{1,P}(\mathbb{R}^n)\), then \(u \in L^{p^*}(\mathbb{R}^n)\)
with \(\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}\). Moreover there exists a constant \(C = C(n, P)\) such that 
\[
|u|_{p^*} \leq C||\nabla u||_p, \quad \forall u \in W^{1,P}(\mathbb{R}^n).
\]

**Corollary 1.** For \(n > mP\), if \(u \in W^{m,P}(\mathbb{R}^n)\) then \(u \in L^{p^*}(\mathbb{R}^n)\). Hence,
\[
|u|_{p^*} \leq C||D^m u||_p, \quad \forall u \in W^{m,P}(\mathbb{R}^n).
\]

**Lemma 4.3.** Assume the initial data \(u_0, u_1\) are with compact support. Then for any solution \(u\) of (1) and for any \(2 < k < P^*\), we have 
\[
|u(t)|_k \leq C(L + t)^{\frac{2}{n}(1 - \frac{1}{P^*})} ||D^m u(t)||_2,
\]
where \(L > 0\) defined in the proof.

**Proof.** In lemma 4.2, if we let \(P = 2\) then we have \(P^* = \frac{2n}{n-2}, \quad n \geq 3\) and 
\[
|u|_{P^*} \leq C||D^m u||_2.
\]

Now
\[
\int_{\mathbb{R}^n} |u|^k \, dx = \int_{B(L+t)} |u|^k \, dx,
\]
where \(L > 0\) is such that 
\[
\text{Supp}\{u_0(x), u_1(x)\} \subset B(L)
\]
and \(B(L + t)\) is the ball, with radius \(L + t\), centered at the origin. Using Hölder inequality, we get 
\[
\int_{\mathbb{R}^n} |u|^k \, dx \leq \left( \int_{B(L+t)} 1 \, dx \right)^{1 - \frac{k}{P^*}} \left( \int_{B(L+t)} (|u|^k)^{\frac{P^*}{k}} \, dx \right)^{\frac{k}{P^*}} \leq C(L + t)^{n(1 - \frac{1}{P^*})} ||u(t)||_{P^*}^k,
\]
or 
\[
||u(t)||_k \leq C(L + t)^{\frac{2}{n}(1 - \frac{1}{P^*})} ||u(t)||_{P^*} \leq C(L + t)^{\frac{2}{n}(1 - \frac{1}{P^*})} ||D^m u(t)||_2.
\]
Hence, the result follows.
Proof of Theorem 4.1. A direct differentiation of $\psi(t)$ yields
\[
\psi'(t) = k \int_{\mathbb{R}^n} u |u|^{p-2} u_t \ln |u|^k \, dx + \frac{k}{p} \int_{\mathbb{R}^n} u |u|^{p-2} u_t \, dx.
\] (31)
As in [1], if we let
\[
\varepsilon_0 \in (0, 1) \quad \text{and} \quad f(s) = s^{\varepsilon_0} (|\ln s| - s),
\]
then it is easy to show that $f$ is continuous on $(0, \infty)$ and
\[
\lim_{s \to 0} f(s) = 0 \quad \text{and} \quad \lim_{s \to \infty} f(s) = -\infty.
\]
Hence $f$ attains a maximum value say $d_{\varepsilon_0}$ on $[0, \infty)$. So, the following estimate is hold
\[
s^{p-1} |\ln s| \leq s^p + d_{\varepsilon_0} s^{p-1-\varepsilon_0}, \quad \forall s > 0.
\]
Consequently, from (31), we have
\[
\psi'(t) \leq k \int_{\mathbb{R}^n} |u|^p + d_{\varepsilon_0} |u|^{p-1-\varepsilon_0} \, |u_t| \, dx + \frac{k}{p} \int_{\mathbb{R}^n} u |u|^{p-2} u_t \, dx
\leq k \left( \int_{\mathbb{R}^n} |u|^p \, |u_t| \, dx + d_{\varepsilon_0} \int_{\mathbb{R}^n} |u|^{p-1-\varepsilon_0} \, |u_t| \, dx + \frac{1}{p} \int_{\mathbb{R}^n} |u|^{p-1} \, |u_t| \, dx \right).
\]
Using Young’s inequality, we obtain
\[
\psi'(t) \leq C \left( \int_{\mathbb{R}^n} |u|^{2p} \, dx + \int_{\mathbb{R}^n} |u|^{2(p-1-\varepsilon_0)} \, dx + \int_{\mathbb{R}^n} |u|^{2(p-1)} \, dx + \int_{\mathbb{R}^n} |u_t|^2 \, dx \right).
\] (32)
Exploiting (30) for $k = 2p < p^*$, we have
\[
||u(t)||_{2p}^2 \leq C(L + t)^{\frac{n}{2p}(1 - \frac{2}{p^*})} ||D^m u(t)||_2^2.
\]
As $T^*$ is a finite blow up time hence $t < T^*$ and recalling (23), we deduce that
\[
||u(t)||_{2p}^2 \leq C(L + T^*)^{\frac{n}{2p}(1 - \frac{2}{p^*})} ||D^m u(t)||_2^2 \leq C ||D^m u(t)||_2^2 \leq C_{\psi}(t).
\]
Therefore,
\[
\int_{\mathbb{R}^n} |u|^{2p} \, dx \leq (||u(t)||_{2p}^2)^p \leq (C_{\psi}(t))^2.
\] (33)
Similarly, for $k = 2(p - 1 - \varepsilon_0) < p^*$, we have
\[
||u(t)||_{2(p-1-\varepsilon_0)} \leq C(L + t)^{\frac{n}{2(p-1-\varepsilon_0)}(1 - \frac{2(p-1-\varepsilon_0)}{p^*})} ||D^m u(t)||_2.
\]
hence,
\[
\int_{\mathbb{R}^n} |u|^{2(p-1-\varepsilon_0)} \, dx \leq \left( C ||u(t)||_{2(p-1-\varepsilon_0)} \right)^{(p-1-\varepsilon_0)} \leq (C_{\psi}(t))^{(p-1-\varepsilon_0)}.
\] (34)
and for $k = 2(p - 1) < p^*$, we have
\[
||u(t)||_{2(p-1-\varepsilon_0)} \leq C(L + t)^{\frac{n}{2(p-1)}(1 - \frac{2(p-1)}{p^*})} ||D^m u(t)||_2.
\]
and
\[
\int_{\mathbb{R}^n} |u|^{2(p-1)} \, dx \leq \left( C ||u(t)||_{2(p-1)} \right)^{(p-1)} \leq (C_{\psi}(t))^{(p-1)}.
\] (35)
Recalling (23), we deduce that
\[
\int_{\mathbb{R}^n} |u_t|^2 \, dx \leq C_{\psi}(t).
\] (36)
Inserting $\text{(33)} - \text{(36)}$ in $\text{(32)}$ and recalling $\text{(23)}$, yield
\[
\psi'(t) \leq C \left[ (C\psi(t))^2 + (C\psi(t))^{(p-1-\varepsilon_0)} + (C\psi(t))^{(p-1)} + \psi(t) \right].
\] (37)
We remark here as a result of $\text{(18)}$ that $\psi(0)$ is finite. So, if we let $y = \psi(t)$ in $\text{(37)}$ and integrate over $(0, T^*)$ yield
\[
\int_0^\infty dy \frac{C dy}{y^2 + y^{(p-1-\varepsilon_0)} + y^{(p-1)}} \leq T^*.
\]
This completes the proof. \(\Box\)

Acknowledgments. The author would like to express his sincere thanks to King Fahd University of Petroleum and Minerals-IR Center of Construction and Building Materials for their support. This work has been funded by KFUPM under Project # SB201026.

REFERENCES

[1] M. Al-Gharabli, New general decay results for a viscoelastic plate equation with a logarithmic nonlinearity, Boundary Value Problems, (2019), Paper No. 194, 21 pp.
[2] D. Andrade, L. H. Fatori and J. E. M. Rivera, Nonlinear transmission problem with a dissipative boundary condition of memory type, Electron J. Differential Equations, (2006), No. 53, 16 pp.
[3] J. M. Ball, Remarks on blow up and nonexistence theorems for nonlinear evolutions equations, Quart. J. Math. Oxford Ser., 28 (1977), 473–486.
[4] P. Bernner and W. von Whal, Global classical solutions of nonlinear wave equations, Mathematische Zeitschrift, 176 (1981), 87–121.
[5] M. M. Cavalcanti, V. N. Domingos Cavalcanti, T. F. Ma and J. A. Soriano, Global existence and asymptotic stability for viscoelastic problem, Differential Integral Equations, 15 (2002), 731–748.
[6] T. Cazenave, Stable solutions of the logarithmic Schrödinger equation, Nonlinear Anal., 7 (1983), 1127–1140.
[7] T. Cazenave and A. Haraux, Équations d’évolution avec non-linéarité logarithmique, Ann. Fac. Sci. Toulouse Math., 2 (1980), 21–51.
[8] V. Georgiev and G. Todorova, Existence of solutions of the wave equation with nonlinear damping and source terms, J. Diff. Eqs., 109 (1994), 295–308.
[9] X. Han, Global existence of weak solution for a logarithmic wave equation arising from Q-ball dynamics, Bull. Korean Math. Soc., 50 (2013), 275–283.
[10] M. Kafini and S. A. Messaoudi, A blow-up result in a Cauchy viscoelastic problem, Applied Mathematics Letters, 21 (2008), 549–553.
[11] M. Kafini and S. A. Messaoudi, A blow-up result for a viscoelastic system in $\mathbb{R}^n$, Elect. J. Diff. Eqs., 113 (2007), 1–7.
[12] H. A. Levine, Instability and nonexistence of global solutions of nonlinear wave equation of the form $P_{tt}u = Au + F(u)$, Trans. Amer. Math. Soc., 192 (1974), 1–21.
[13] H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equation, SIAM J. Math. Anal., 5 (1974), 138–146.
[14] H. A. Levine and J. Serrin, Global nonexistence theorem for quasilinear evolution equation with dissipation, Arch. Rational Mech. Anal., 137 (1997), 341–361.
[15] H. A. Levine, S. R. Park and J. Serrin, Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation, J. Math. Anal. Appl., 228 (1998), 181–205.
[16] S. A. Messaoudi, Blow up and global existence in a nonlinear viscoelastic wave equation, Mathematische Nachrichten, 260 (2003), 58–66.
[17] S. A. Messaoudi, Blow up of solutions with positive initial energy in a nonlinear viscoelastic equation, J. Math. Anal. Appl., 320 (2006), 902–915.
[18] S. A. Messaoudi, Blow up in a nonlinearly damped wave equation, Mathematische Nachrichten, 231 (2001), 105–111.
[19] S. A. Messaoudi, Blow up in the Cauchy problem for a nonlinearly damped wave equation, *Comm. On Applied. Analysis*, 7 (2003), 379–386.
[20] S. A. Messaoudi and B. Said Houari, Blow up of solutions of a class of wave equations with nonlinear damping and source terms, *Math. Methods Appl. Sci.*, 27 (2004), 1687–1696.
[21] S. A. Messaoudi and N.-e. Tatar, Exponential and polynomial decay for a quasilinear viscoelastic equation, *Nonlinear Anal.*, 68 (2008), 785–793.
[22] C. X. Miao, The time space estimates and scattering at low energy for nonlinear higher order wave equations, *Acta Math. Sin. Ser. A.*, 38 (1995), 708–717.
[23] J. E. Munoz Rivera, E. C. Lapa and R. Baretto, Decay rates for viscoelastic plates with memory, *J. Elasticity*, 44 (1996), 61–87.
[24] H. Pecher, Die existenz regulär Lösungen für Cauchy-und anfangs-randwertproble-me nicht-linear wellengleichungen, *Mathematische Zeitschrift*, 140 (1974), 263–279.
[25] F. Tahamatani and M. Shahrouzi, General existence and blow up of solutions to a Petrovsky equation with memory and nonlinear source, *Bound. Value Probl.*, 2012 (2012), 50, 15 pp.
[26] G. Todorova, Cauchy problem for a nonlinear wave with nonlinear damping and source terms, *C. R. Acad. Sci. Paris Ser. I*, 326 (1998), 191–196.
[27] G. Todorova, Stable and unstable sets for the Cauchy problem for a nonlinear wave with nonlinear damping and source terms, *J. Math. Anal. Appl.*, 239 (1999), 213–226.
[28] E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation, *Arch. Rational Mech. Anal.*, 149 (1999), 155–182.
[29] Y. Wang, A global nonexistence theorem for viscoelastic equation with arbitrary positive initial energy, *Appl. Math. Lett.*, 22 (2009), 1394–1400.
[30] B. Wang, Nonlinear scattering theory for a class of wave equations in $H^s$, *J. Math. Anal. Appl.*, 296 (2004), 74–96.
[31] S.-T. Wu, Blow-up of solutions for an integro-differential equation with a nonlinear source, *Elect. J. Diff. Eqs.*, (2006), No. 45, 9 pp.
[32] Y. Ye, Existence and asymptotic behavior of global solutions for a class of nonlinear higher-order wave equation, *Journal of Inequalities and Applications*, 2010 (2010), Article number: 394859.
[33] Y. Ye, Global existence and blow-up of solutions for higher-order viscoelastic wave equation with a nonlinear source term, *Nonlinear Analysis*, 112 (2015), 129–146.
[34] Y. Zhou, A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in $R^n$, *Applied Math Letters*, 18 (2005), 281–286.

Received April 2021; revised June 2021, early access August 2021.

*E-mail address: mكافین@kfupm.edu.sa*