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Master Bellman equation in the Wasserstein space: Uniqueness of viscosity solutions

Andrea COSSO* Fausto GOZZI† Idris KHRARROUBI‡
Huyên PHAM§ Mauro ROSESTOLATO¶

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Abstract

We study the Bellman equation in the Wasserstein space arising in the study of mean field control problems, namely stochastic optimal control problems for McKean-Vlasov diffusion processes. Using the standard notion of viscosity solution à la Crandall-Lions extended to our Wasserstein setting, we prove a comparison result under general conditions on the drift and reward coefficients, which coupled with the dynamic programming principle, implies that the value function is the unique viscosity solution of the Master Bellman equation. This is the first uniqueness result in such a second-order context. The classical arguments used in the standard cases of equations in finite-dimensional spaces or in infinite-dimensional separable Hilbert spaces do not extend to the present framework, due to the awkward nature of the underlying Wasserstein space. The adopted strategy is based on finite-dimensional approximations of the value function obtained in terms of the related cooperative \( \pi \)-player game, and on the construction of a smooth gauge-type function, built starting from a regularization of a sharpe estimate of the Wasserstein metric; such a gauge-type function is used to generate maxima/minima through a suitable extension of the Borwein-Preiss generalization of Ekeland’s variational principle on the Wasserstein space.

Keywords: viscosity solutions, Bellman equation, Wasserstein space, comparison theorem, Ekeland’s variational principle.

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*University of Bologna, Italy; andrea.cosso@unibo.it
†Luiss University, Roma, Italy; fgozzi@luiss.it
‡LPSM, UMR CNRS 8001, Sorbonne University and Université de Paris; idris.kharroubi@upmc.fr
§LPSM, UMR CNRS 8001, Sorbonne University and Université de Paris; pham@lpsm.paris; The work of this author is supported by FiME (Finance for Energy Market Research Centre) and the “Finance et Développement Durable - Approches Quantitatives” EDF - CACIB Chair.
¶University of Lecce; mauro.rosestolato@gmail.com
1 Introduction

The main goal of this paper is to develop a viscosity theory for second-order partial differential equations on the Wasserstein space related to the so-called mean field (or McKean-Vlasov) control problems, namely stochastic optimal control problems for McKean-Vlasov diffusion processes. Such partial differential equations are also known as Master Bellman equations or Bellman equations in the Wasserstein space, see for instance [6, 14, 46]. The topic of mean field optimal control is a very recent area of research, on which there are however already many papers and the two monographs [5, 14], to which we refer for a thorough introduction. Mean field control problems are strictly related to mean field games, developed by Lasry and Lions in [33, 34, 35] (see also Lions’ lectures at Collège de France [38]) and by Huang, Caines, Malhamé [29]. Both mean field control problems and mean field games can be interpreted as searches for equilibria of stochastic differential games with a continuum of players, symmetrically interacting each other through the empirical distribution of the entire population. These two problems differ because of the notion of equilibrium adopted. Mean field games arise when the concept of Nash’s non-cooperative equilibrium is employed, while mean field control problems are related to Pareto optimality where players can be identified with a single “representative agent”, see for instance [14, Section 6.2, pages 514-515]. In the latter case the stochastic differential game can be thought as an optimization problem of a central planner, who is looking for a common strategy in order to optimize some collective objective functional.

The state space of mean field control problems is the set of probability measures, and usually the Wasserstein space \( \mathcal{P}_2(\mathbb{R}^d) \) of probability measures having finite second moment is adopted. Various notions of differentiability for maps defined on spaces of probability measures are available, and some of them are particularly relevant in the theory of optimal transportation, see [3, 45] for a detailed presentation of these geometric approaches. The Master Bellman equation (see equation (3.3) below) adopts instead the notion of differentiability introduced by Lions [38] (see also [12, 13, 14], and Section 3), whose nature is more functional analytic than geometric. Such a definition seems to be the natural choice in the study of second-order Bellman equations in the Wasserstein space and related stochastic optimal control problems. In fact, it gave rise to a stochastic calculus on the space of probability measures, and in particular to an Itô formula (chain rule) for maps defined on the Wasserstein space (we recall it in our Theorem 3.3), which allows to relate the value function of the control problem to the Bellman equation (we recall it in our Theorem 3.8). Regarding the relation between partial differential equations adopting the derivatives introduced by Lions (as in the present paper) and equations using notions of differentiability as those adopted in optimal transport theory, we mention results in this direction in the first-order case in [28] and in a second-order semi-linear case in [24] (see also Remark 3.6).

The theory of partial differential equations in the Wasserstein space is an emerging research topic, whose rigorous investigation is still at an early stage. There are already well-posedness results in the first-order case, see [2, 25, 1, 26, 27, 28], even for equations adopting different notions of derivative with respect to the measure. They however do not admit an extension to the second-order case, which is notoriously a different and more challenging
problem. Concerning second-order equations, papers \[41, 42, 4, 17, 16\] focus on the existence of viscosity solutions, proving that the value function solves in the viscosity sense the Master Bellman equation. All those articles adopt the notion of viscosity solution à la Crandall-Lions, properly adapted to the Wasserstein space, as we do in the present paper (see Definition 3.5). Notice that, even if these papers dealt with the uniqueness property, they established it only for the so-called lifted Bellman equation, which is formulated on the Hilbert space of corresponding random variables so that standard results apply. We also recall that the relation between such a lifted equation and the original Bellman equation in the Wasserstein space is not rigorously clarified, and in particular whether the lifted value function is a viscosity solution to the lifted equation. Actually, it is not yet clear under which conditions test functions in the lifting Hilbert space are related to test functions in the Wasserstein space, see discussion in Remark 3.6.

Uniqueness for second-order equations in the Wasserstein space is only addressed in the two papers \[46\] and \[11\]. In \[46\], a new notion of viscosity-type solution is adopted, which differs from the Crandall-Lions definition since the maximum/minimum condition is formulated on compact subsets of the Wasserstein space. This modification makes easier to prove uniqueness, which is completely established in some specific cases. On the other hand, \[11\] studies viscosity solutions à la Crandall-Lions for a class of integro-differential Bellman equations of particular type. More precisely, the coefficients of the McKean-Vlasov stochastic differential equations, as well as the coefficients of the reward functional, do not depend on the state process itself, but only on its probability distribution. This allows to consider only deterministic functions of time as control processes in the mean field control problem, so that the Master Bellman equation has a particular form. Moreover, in \[11\] the Master Bellman equation is formulated on the subset of the Wasserstein space of probability measures having finite exponential moments, equipped with the topology of weak convergence, which makes such a space \(\sigma\)-compact and allows establishing uniqueness in this context.

In the present paper we prove, under general conditions on the drift and reward coefficients, existence and uniqueness of viscosity solutions for Master Bellman equations arising in the study of mean field optimal control problems. This is the first uniqueness result for such class of equations in the present context. Classical arguments based on Ishii’s lemma used in the standard cases of equations in finite-dimensional spaces or in infinite-dimensional separable Hilbert spaces seem hard to extend to the present framework, due to the awkward nature of the underlying Wasserstein space. The adopted strategy is instead based on refinements of early ideas from the theory of viscosity solutions \[37\] and relies on the existence of a candidate solution to the equation, which in our case is the value function \(v\) of the mean field control problem. In particular, we prove (see Theorem 5.1) that any viscosity subsolution \(u_1\) (resp. supersolution \(u_2\)) is smaller (resp. greater) than the candidate solution \(v\). In \[37\], the arguments for proving \(u_1 \leq v\) (or, similarly, \(v \leq u_2\)) are as follows: one performs a smoothing \(v_n\) of \(v\) through its control representation, take a maximum of \(u_1 - v_n\) (relying on the local compactness of the finite-dimensional space), and exploit the viscosity subsolution property of \(u_1\) with \(v_n\) as test function. In \[39\] such a methodology is extended to the infinite-dimensional case, relying on Ekeland’s variational principle in order to generate
maxima/minima.

In the context of equations in the Wasserstein space, the above arguments require the following adjustments. Firstly, the smoothing of $v$ is based on a propagation of chaos result [32], namely on a finite-dimensional approximation of the value function through value functions of non-degenerate cooperative $n$-player games. Secondly, in order to generate maxima/minima the idea is to perturb $u_1 - v_n$ (or $u_2 - v_n$) relying on a suitable extension of the Borwein-Preiss generalization of Ekeland’s principle, see [9, Theorem 2.5.2]. According to the latter, $u_1 - v_n$ can be perturbed using a so-called gauge-type function (see Definition 4.1). For the proof of the comparison theorem, such a perturbation has to be smooth. In an infinite-dimensional Hilbert space setting, an example of smooth gauge-type function is the square of the norm. In the present context, the main issue is to construct a smooth gauge-type function. This is achieved in Section 4, starting from a sharp estimate of the square of the Wasserstein metric (see (4.4)) and performing a smoothing of such a quantity (see Lemma 4.4). Due to the complexity of the techniques employed, our results are formulated under boundedness assumptions on the coefficients. The extension to more general cases covering path dependent cases (like in [16, 15, 43]) and/or applications like the ones mentioned in [20, Introduction] seems possible and will be the object of future research. Similarly refinements of the results showing that viscosity solutions can have a certain degree of regularity (on the line of what is done e.g. in [44]) seems possible and will be studied in further research.

The rest of the paper is organized as follows. In Section 2 we formulate the mean field optimal control problem and state the assumptions that are used throughout the paper; in such a section we also prove some properties of the value function $v$ and state the dynamic programming principle. In Section 3 we recall the notion of differentiability introduced by Lions, we state the Itô formula, we introduce the Master Bellman equation, and we give the definition of viscosity solution. Section 4 is devoted to the construction of the smooth gauge-type function, from which we derive the smooth variational principle on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, namely Theorem 4.5. In Section 5 we prove the comparison theorem (Theorem 5.1), from which we deduce the uniqueness result (Corollary 5.2). Finally, in Appendix A we perform the smooth finite-dimensional approximation of the value function; in particular, in subsection A.1 we approximate the mean field control problem with non-degenerate control problems; then, in subsection A.2 we introduce the related cooperative $n$-player game and state the propagation of chaos result.

2 Mean field optimal control problem

Wasserstein spaces of probability measures. Given a Polish space $(S, d_S)$, we denote by $\mathcal{P}(S)$ the set of all probability measures on $(S, \mathcal{B}(S))$. We also define, for every $q \geq 1$,

$$\mathcal{P}_q(S) := \left\{ \mu \in \mathcal{P}(S) : \text{for some (and hence for all) } x_0 \in S, \int_S d_S(x_0, x)^q \mu(dx) < +\infty \right\}.$$
The set \( \mathcal{P}_q(S) \) is endowed with the \( q \)-Wasserstein distance defined as
\[
\mathcal{W}_q(\mu, \mu') := \inf \left\{ \int_{S \times S} d_S(x, y)^q \pi(dx, dy) : \pi \in \mathcal{P}(S \times S) \right\}^{\frac{1}{q}}, \quad q \geq 1,
\]
for every \( \mu, \mu' \in \mathcal{P}_q(S) \). The space \((\mathcal{P}_q(S), \mathcal{W}_q)\) is a Polish space, see for instance [45, Theorem 6.18].

**Probabilistic setting and control processes.** We fix a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) on which a \( m \)-dimensional Brownian motion \( B = (B_t)_{t \geq 0} \) is defined. We denote by \( \mathbb{P}^B = (\mathcal{F}^B_t)_{t \geq 0} \) the \( \mathbb{P} \)-completion of the filtration generated by \( B \), which is also right-continuous, so that it satisfies the usual conditions. We assume that there exists a sub-\( \sigma \)-algebra \( \mathcal{G} \) of \( \mathcal{F} \) satisfying the following properties.

i) \( \mathcal{G} \) and \( \mathcal{F}^B_\infty \) are independent.

ii) \( \mathcal{G} \) is “rich enough”, namely \( \mathcal{P}_2(\mathbb{R}^d) = \{ \mathbb{P}_\xi \text{ such that } \xi : \Omega \to \mathbb{R}^d, \text{ with } \xi \text{ being } \mathcal{G} \text{-measurable and } E|\xi|^2 < \infty \} \). Recall from \([16, \text{Lemma 2.1}]\) that such a requirement is equivalent to the existence of a \( \mathcal{G} \)-measurable random variable \( U : \Omega \to \mathbb{R} \) having uniform distribution on \([0, 1]\).

We denote by \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) the filtration defined as
\[
\mathcal{F}_t = \mathcal{G} \lor \mathcal{F}^B_t, \quad t \geq 0.
\]
We observe that \( \mathbb{F} \) satisfies the usual conditions of \( \mathbb{P} \)-completeness and right-continuity. Finally, we fix a finite time horizon \( T > 0 \) and a Polish space \( A \). We then denote by \( \mathcal{A} \) the set of control processes, namely the family of all \( \mathbb{F} \)-progressively measurable processes \( \alpha : [0, T] \times \Omega \to A \).

**Assumptions and state equation.** We consider the functions \( b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \sigma : [0, T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d \times m}, f : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}, g : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) on which we impose the following assumptions (notice that \( \sigma \) does not depend on \( \mu \)).

**Assumption (A).**

(i) The functions \( b, \sigma, f, g \) are continuous.

(ii) There exists a constant \( K \geq 0 \) such that
\[
|b(t, x, \mu, a) - b(t, x', \mu', a)| + |\sigma(t, x, a) - \sigma(t, x', a)| \leq K(|x - x'| + \mathcal{W}_2(\mu, \mu')),
\]
\[
|b(t, x, \mu, a)| + |\sigma(t, x, a)| \leq K,
\]
for all \((t, a) \in [0, T] \times A, (x, \mu), (x', \mu') \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d), |x - x'| \) denoting the Euclidean norm of \( x - x' \) in \( \mathbb{R}^d \), \( \langle \cdot, \cdot \rangle \) denoting the scalar product, \( |\sigma(t, x, a)| := (\text{tr}(\sigma^\top)(t, x, a))^2 \) denoting the Frobenius norm of the matrix \( \sigma(t, x, a) \).
(iii) There exists a constant $K \geq 0$ such that
\[
|f(t, x, \mu, a) - f(t, x', \mu', a)| + |g(x, \mu) - g(x', \mu')| \leq K(|x - x'| + \mathcal{W}_2(\mu, \mu')),
\]
\[
|f(t, x, \mu, a)| + |g(x, \mu)| \leq K,
\]
for all $(t, a) \in [0, T] \times A$, $(x, \mu), (x', \mu') \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

**Assumption (B).** There exist constants $K \geq 0$ and $\beta \in (0, 1]$ such that
\[
|b(t, x, \mu, a) - b(s, x, \mu, a)| + |\sigma(t, x, a) - \sigma(s, x, a)|
\]
\[
+ |f(t, x, \mu, a) - f(s, x, \mu, a)| \leq K|t - s|^\beta,
\]
for all $t, s \in [0, T], (x, \mu, a) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A$.

**Assumption (C).** The Polish space $A$ is a compact subset of a Euclidean space.

**Assumption (D).** For any $a \in A$, the function $\sigma(\cdot, \cdot, a)$ belongs to $C^{1,2}([0, T] \times \mathbb{R}^d)$. Moreover, there exists some constant $K \geq 0$ such that
\[
|\partial_t \sigma(t, x, a)| + |\partial_x \sigma(t, x, a)| + |\partial^2_{x,x} \sigma(t, x, a)| \leq K,
\]
for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times A$ and any $i, j = 1, \ldots, d$.

**Remark 2.1.** Assumptions (B) and (D) are required in the proof of Theorem A.7 in order to exploit regularity results for uniformly parabolic Bellman equations. In particular, Assumption (D) is taken from [31, Section 7 of Chapter 4] in order to get suitable bounds on the second derivatives (see [31, Theorem 4.7.4]). On the other hand, Assumptions (A) and (C) are required in the propagation of chaos result, that is Theorem A.6. All these assumptions are therefore required in Theorem 5.1 and Corollary 5.2. Finally, notice that the results of the present section are stated under Assumption (A), however they hold under weaker assumptions, see [16].

For every $t \in [0, T], \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d), \alpha \in A$, the state process evolves according to the following controlled McKean-Vlasov stochastic differential equation:
\[
X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}_{X_r}, \alpha_r) \, dr + \int_t^s \sigma(r, X_r, \alpha_r) \, dB_r, \quad s \in [t, T].
\tag{2.2}
\]

**Proposition 2.2.** Suppose that Assumption (A) holds. For every $t \in [0, T], \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, $\alpha \in A$, there exists a unique (up to $\mathbb{P}$-indistinguishability) continuous $\mathbb{F}$-progressively measurable process $X^{t, \xi, \alpha} = (X_s^{t, \xi, \alpha})$ solution to equation (2.2) satisfying
\[
\mathbb{E}\left[ \sup_{s \in [t, T]} |X_s^{t, \xi, \alpha}|^2 \right]^{1/2} \leq C_1 \left( 1 + \mathbb{E}\left[ |\xi|_2^2 \right]^{1/2} \right),
\]
for some constant $C_1$, independent of $t, \xi, \alpha$.

Moreover, for every $\xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ it holds that
\[
\mathbb{E}\left[ \sup_{s \in [t, T]} |X_s^{t, \xi, \alpha} - X_s^{t, \xi', \alpha}|^2 \right]^{1/2} \leq C_2 \mathbb{E}\left[ |\xi - \xi'|_2^2 \right]^{1/2},
\tag{2.3}
\]
for some constant $C_2$, independent of $t, \xi, \xi', \alpha$.

**Proof.** See [16, Proposition 2.8].
Reward functional and lifted value function. We consider the reward functional $J$, given by

$$J(t, \xi, \alpha) = E\left[\int_t^T f\left(s, X_s^{t,\xi,\alpha}, P_{X_s^{t,\xi,\alpha}}, \alpha_s\right) ds + g\left(X_T^{t,\xi,\alpha}, P_{X_T^{t,\xi,\alpha}}\right)\right], \quad (2.4)$$

and the function $V$, to which we refer as the lifted value function, defined as

$$V(t, \xi) = \sup_{\alpha \in A} J(t, \xi, \alpha), \quad \forall (t, \xi) \in [0, T] \times L^2(\Omega, F_t, P; \mathbb{R}^d).$$

Proposition 2.3. Suppose that Assumption (A) holds. The function $V$ satisfies the following properties.

1) $V$ is bounded.

2) $V$ is jointly continuous, namely: for every $\{(t_n, \xi_n)\}_{n}, (t, \xi)$, with $t_n, t \in [0, T]$ and $\xi_n \in L^2(\Omega, F_{t_n}, P; \mathbb{R}^d), \xi \in L^2(\Omega, F_t, P; \mathbb{R}^d)$, such that $|t_n - t| + E|\xi_n - \xi|^2 \to 0$, it holds that $V(t_n, \xi_n) \to V(t, \xi)$.

3) There exists a constant $L \geq 0$ (depending only on $T, K, C_2$ in (2.3)) such that

$$|V(t, \xi) - V(t, \xi')| \leq L E\left[|\xi - \xi'|^2\right]^{1/2}, \quad (2.5)$$

for all $t \in [0, T], \xi, \xi' \in L^2(\Omega, F_t, P; \mathbb{R}^d)$. 

Proof. Item 1) is a direct consequence of the boundedness of $f$ and $g$, while item 2) follows from [16, Proposition 3.3]. Concerning item 3), we begin noticing that

$$|V(t, \xi) - V(t, \xi')| \leq \sup_{\alpha \in A} |J(t, \xi, \alpha) - J(t, \xi', \alpha)|.$$

Then, the Lipschitz continuity of $V$ follows from the Lipschitz continuity of $J$. To this regard, we have

$$|J(t, \xi, \alpha) - J(t, \xi', \alpha)| \leq \int_t^T E\left[|f\left(s, X_s^{t,\xi,\alpha}, P_{X_s^{t,\xi,\alpha}}, \alpha_s\right) - f\left(s, X_s^{t,\xi',\alpha}, P_{X_s^{t,\xi',\alpha}}, \alpha_s\right)|\right] ds$$

$$+ E\left[|g\left(X_T^{t,\xi,\alpha}, P_{X_T^{t,\xi,\alpha}}\right) - g\left(X_T^{t,\xi',\alpha}, P_{X_T^{t,\xi',\alpha}}\right)|\right].$$

By the Lipschitz continuity of $f$ and $g$, together with inequality $\mathcal{W}_2(X_s^{t,\xi,\alpha}, X_s^{t,\xi',\alpha}) \leq E[|X_s^{t,\xi,\alpha} - X_s^{t,\xi',\alpha}|^2]^{1/2}$, we obtain estimate (2.5). 

Law invariance property and dynamic programming principle. We recall from [16] that $V$ satisfies the fundamental law invariance property.

Theorem 2.4. Suppose that Assumption (A) holds. Then, the map $V$ satisfies the law invariance property: for every $t \in [0, T]$ and $\xi, \xi' \in L^2(\Omega, F_t, P; \mathbb{R}^d)$, with $P_\xi = P_{\xi'}$, it holds that

$$V(t, \xi) = V(t, \xi').$$

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Proof. See [16, Theorem 3.5].

As a consequence of Theorem 2.4, if Assumption (A) holds, we can define the value function \( v: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) as

\[
v(t, \mu) = V(t, \xi), \quad \forall (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d),
\]

for any \( \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d) \). By Proposition 2.5 we immediately deduce the following result.

**Proposition 2.5.** Suppose that Assumption (A) holds. The function \( v \) satisfies the following properties.

1) \( v \) is bounded.

2) \( v \) is jointly continuous on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\).

3) For all \( t \in [0, T], \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d) \),

\[
|v(t, \mu) - v(t, \mu')| \leq L \mathcal{W}_2(\mu, \mu'),
\]

with \( L \) as in (2.5).

**Proof.** The claim follows directly from Proposition 2.5, we only report the proof of item 3). By (2.5) and (2.6), we have

\[
|v(t, \mu) - v(t, \mu')| \leq |V(t, \xi) - V(t, \xi')| \leq L \mathbb{E}[(\xi - \xi')^2]^{1/2},
\]

for any \( \xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d) \), with \( \mathbb{P}_\xi = \mu \) and \( \mathbb{P}_{\xi'} = \mu' \). Hence

\[
|v(t, \mu) - v(t, \mu')| \leq L \inf \left\{ \mathbb{E}[(\xi - \xi')^2]^{1/2} : \xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d), \text{ with } \mathbb{P}_\xi = \mu \text{ and } \mathbb{P}_{\xi'} = \mu' \right\}
\]

\[= L \mathcal{W}_2(\mu, \mu').\]

Finally, we state the dynamic programming principle for \( v \).

**Theorem 2.6.** Suppose that Assumption (A) holds. Then, \( v \) satisfies the dynamic programming principle: for all \( t, s \in [0, T], \text{ with } t \leq s, \mu \in \mathcal{P}_2(\mathbb{R}^d) \), it holds that

\[
v(t, \mu) = \sup_{\alpha \in \mathcal{A}} \left\{ \mathbb{E} \left[ \int_t^s f(r, X_{t+}^r, \xi, \mathbb{P}_{X_{t+}\xi, \alpha}, \alpha_r) \, dr \right] + v(s, \mathbb{P}_{X_{t+}\xi, \alpha}) \right\},
\]

for any \( \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d) \) with \( \mathbb{P}_\xi = \mu \).

**Proof.** See [16, Corollary 3.8].
3 Master Bellman equation

$L$-derivatives and Itô’s formula along a flow of probability measures. We refer to [14, Section 5.2] for the definitions of the $L$-derivatives of first and second-order of a map $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ with respect to $\mu$, which are given by $\partial_\mu u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ and $\partial_x \partial_\mu u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$. We recall that such definitions are based on the notion of lifting of a map $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, which is a map $U: [0, T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to \mathbb{R}$ satisfying

$$U(t, \xi) = u(t, \mathbb{P}_\xi), \quad (3.1)$$

for every $t \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ (here, to alleviate notation, we have defined the lifting on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the mean field control problem was defined; however, any other probability space supporting a random variable with uniform distribution on $[0, 1]$ can be used). We observe that derivatives in the present context can be defined in different ways: the so-called “flat” derivative or the intrinsic notion of differential in the Wasserstein space. We refer for instance to [14, Chapter 5] for a survey and some equivalence results.

**Definition 3.1.** $C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ is the set of continuous functions $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that:

1) the lifting $U$ of $u$ admits a continuous Fréchet derivative $D_\xi U: [0, T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, in which case there exists, for any $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, a measurable function $\partial_\mu u(t, \mu) : \mathbb{R}^d \to \mathbb{R}^d$, such that $D_\xi U(t, \xi) = \partial_\mu u(t, \mu)(\xi)$, for any $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ with law $\mu$.

2) The map $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_x \partial_\mu u(t, \mu)(x) \in \mathbb{R}^d$ is jointly continuous;

3) $\partial_t u$ and $\partial_x \partial_x \partial_\mu u$ exist and the maps $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_t u(t, \mu) \in \mathbb{R}$, $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_x \partial_x \partial_\mu u(t, \mu)(x) \in \mathbb{R}^{d \times d}$ are continuous.

**Definition 3.2.** $C^{2,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ is the subset of $C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ of functions $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ satisfying, for some constant $C \geq 0$,

$$|\partial_\mu u(t, \mu)(x)| + |\partial_x \partial_\mu u(t, \mu)(x)| \leq C(1 + |x|^2),$$

for all $(t, x, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$.

**Theorem 3.3 (Itô’s formula).** Let $u \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$, $t \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$. Let also $\beta : [0, T] \times \Omega \to \mathbb{R}^d$ and $\vartheta : [0, T] \times \Omega \to \mathbb{R}^{d \times m}$ be bounded and $\mathbb{F}$-progressively measurable processes. Consider the $d$-dimensional Itô process

$$X_s = \xi + \int_t^s \beta_r \, dr + \int_t^s \vartheta_r \, dB_r, \quad \forall s \in [t, T].$$

Then, it holds that

$$u(s, \mathbb{P}_{X_s}) = u(t, \mathbb{P}_\xi) + \int_t^s \partial_t u(r, \mathbb{P}_{X_r}) \, dr + \int_t^s \mathbb{E}
\left[\langle \beta_r, \partial_\mu u(r, \mathbb{P}_{X_r})(X_r) \rangle \right] \, dr$$
\[ + \frac{1}{2} \int_t^s \mathbb{E} \left[ \text{tr} \left( \partial_x \partial_x^t \partial_x \partial_t u(r, \mathbb{P}_{X_t})(X_r) \right) \right], \]

for all \( s \in [t, T] \).

**Proof.** The claim follows from [14, Proposition 5.102] (see also [16, Theorem 4.15 and Remark 4.16]). \( \square \)

**Viscosity solutions.** Now, consider the second-order partial differential equation on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\):

\[
\begin{aligned}
\partial_t u(t, \mu) &= F(t, \mu, u(t, \mu), \partial_x u(t, \mu)(\cdot), \partial_x \partial_x u(t, \mu)(\cdot)), \quad (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \\
u(T, \mu) &= \int_{\mathbb{R}^d} g(x, \mu) \mu(dx), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d),
\end{aligned}
\tag{3.2}
\]

with \( F: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \times L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d) \times L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^{d \times d}) \rightarrow \mathbb{R} \), where \( \mathcal{B}(\mathbb{R}^d) \) are the Borel subsets of \( \mathbb{R}^d \), and \( L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d) \) is the set of \( \mathcal{B}(\mathbb{R}^d) \)-measurable functions that are square-integrable with respect to \( \mu \).

**Definition 3.4.** A function \( u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \) is a classical solution to equation (3.2) if \( u \in C^{1,2}_2([0, T] \times \mathcal{P}_2(\mathbb{R}^d)) \) and satisfies (3.2).

**Definition 3.5.** A continuous function \( u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \) is a viscosity subsolution (resp. supersolution) to equation (3.2) if:

- \( u(T, \mu) \leq (\text{resp.} \geq) \int_{\mathbb{R}^d} g(x, \mu) \mu(dx) \), for every \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \);
- for every \( (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \) and any \( \varphi \in C^{1,2}_2([0, T] \times \mathcal{P}_2(\mathbb{R}^d)) \) such that \( u - \varphi \) has a maximum at \((t, \mu) \) (with value 0), then (3.2) is satisfied with the inequality \( \geq (\text{resp.} \leq) \) instead of the equality and with \( \varphi \) in place of \( u \).

Finally, \( u \) is a viscosity solution of (3.2) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 3.6.** The above definition of viscosity solution is exactly in the spirit of the definition of Crandall and Lions for second-order equations (see for instance [18]) in finite dimension. In [18] it is proved that this definition is equivalent to the one using second-order semidifferentials (jets), while here such equivalence is not obvious.

We say that our definition is an “intrinsic” definition to distinguish it from the definition, adopted first in [41], which exploits the lifted equation (in the sense that a function is a viscosity solution if its lifting along (3.1) satisfies equation (3.2) with \( F \) substituted by its lifting \( \tilde{F} \)) and which, for this reason, we call “lifted” definition.

The relationship between these two definitions is not obvious. Indeed, as shown in Example 2.1 in [10], the lifted function of a smooth function on the Wasserstein space may not be smooth on the lifted Hilbert space, and so a viscosity solution in the intrinsic sense may
not be a viscosity solution in the lifted sense. In the first-order case a kind of equivalence result between two related definitions is provided in [28, Theorem 4.4]. In the second-order semi-linear case some results in this direction are provided in [24, Section 5]. We are not aware of any results on the fully non-linear second-order case.

As we recalled in the introduction an intrinsic notion of viscosity solution is employed also in the papers [46] and [11]. The definition introduced in [46, Definition 4.4], differs from our definition since test functions must satisfy the maximum/minimum condition on suitable compact subsets of the Wasserstein space, denoted by \( P_L \). Using this modification the authors prove first a comparison result among regular sub/supersolutions (“partial comparison”) and then a general comparison result with the assumption that the supremum of classical subsolutions and the infimum of classical supersolutions coincide. On the other hand, [11] studies viscosity solutions à la Crandall-Lions for a class of integro-differential Bellman equations of particular type. More precisely, the coefficients of the McKean-Vlasov stochastic differential equations, as well as the coefficients of the reward functional, do not depend on the state process itself, but only on its probability distribution. This allows to consider only deterministic functions of time as control processes in the mean field control problem, so that the Master Bellman equation has a particular form. Moreover, in [11] the Master Bellman equation is formulated on the subset of the Wasserstein space of probability measures having finite exponential moments, equipped with the topology of weak convergence, which makes such a space \( \sigma \)-compact and allows establishing uniqueness in this context.

Now, we consider the Master Bellman equation, namely equation (3.2) with

\[
F(t, \mu, r, p(x), M) = -\int_{\mathbb{R}^d} \sup_{a \in A} \left\{ f(t, x, \mu, a) + \langle b(t, x, \mu, a), p(x) \rangle + \frac{1}{2} \text{tr}[(\sigma(t, x, a)M(x))] \right\} \mu(dx).
\]

Therefore, equation (3.2) becomes

\[
\begin{align*}
\partial_t u(t, \mu) + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ f(t, x, \mu, a) + \langle b(t, x, \mu, a), \partial_x \mu(t, \mu)(x) \rangle + \frac{1}{2} \text{tr}[(\sigma(t, x, a)\partial_x \partial_\mu u(t, \mu)(x))] \right\} \mu(dx) &= 0, \quad (t, \mu) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d), \quad (3.3) \\
u(T, \mu) &= \int_{\mathbb{R}^d} g(x, \mu) \mu(dx), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d).
\end{align*}
\]

**Remark 3.7.** As described in [16, Section 5.2], to which we refer for more details, equation (3.3) can be written in various alternative forms. In particular, (3.3) corresponds to [16, equation (5.17)], the only difference being the presence of \( \sup_{a \in A} \) which in [16, equation (5.17)] is replaced by \( \text{ess sup}_{a \in A} \). However, as described in [16, Remark 5.8], under assumption (A), \( \text{ess sup}_{a \in A} \) can be replaced by \( \sup_{a \in A} \).

Finally, we mention that an alternative form of equation (3.3) is the following (corresponding
to equation [16, equation (5.16)]:

\[
\begin{align*}
\partial_t u(t, \mu) + \sup_{a \in A} & \left\{ \int_{\mathbb{R}^d} f(t, x, \mu, a(x)) \mu(dx) + \int_{\mathbb{R}^d} \langle b(t, x, \mu, a(x)), \partial_x u(t, \mu)(x) \rangle \mu(dx) \\
+ \frac{1}{2} & \int_{\mathbb{R}^d} \text{tr} \left[ (\sigma \sigma^T)(t, x, a(x)) \partial_x \partial_x u(t, \mu)(x) \right] \mu(dx) \right\} = 0, \\
u(T, \mu) &= \int_{\mathbb{R}^d} g(x, \mu) \mu(dx),
\end{align*}
\]

where \(M\) is the set of Borel-measurable maps \(a: \mathbb{R}^d \to A\).

**Theorem 3.8.** Let Assumptions (A) and (B) hold. Then, the value function \(v\), given by (2.6), is a viscosity solution of equation (3.3).

**Proof.** See [16, Theorem 5.5]. Notice that in [16] a different definition of viscosity solution is adopted, with \(\varphi \in C^{1,2}_b([0, T] \times \mathcal{P}_2(\mathbb{R}^d))\), i.e. \(\varphi \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))\) and has bounded derivatives, rather than \(\varphi \in C^{1,2}_b([0, T] \times \mathcal{P}_2(\mathbb{R}^d))\), but the arguments remain the same. \(\square\)

**Remark 3.9.** From [16, Theorem 5.5] we have that Theorem 3.8 still holds if we replace (B) with the following weaker assumption: the functions \(b, \sigma, f\) are uniformly continuous in the time variable \(t\), uniformly with respect to \((x, \mu, a)\). Similarly, Assumption (A) can be weakened, see [16].

## 4 Smooth variational principle

As described in the introduction, the comparison theorem (Theorem 5.1) relies on a smooth variational principle on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), to which the present section is devoted. Such a result is obtained from an extension of the Borwein-Preiss variational principle, for which we refer to [8] and, in particular, for its general form, to [9, Theorem 2.5.2]. An essential tool of [9, Theorem 2.5.2] is the concept of gauge-type function, whose definition is given below.

**Definition 4.1.** Let \(d_2\) be a metric on \(\mathcal{P}_2(\mathbb{R}^d)\) such that \((\mathcal{P}_2(\mathbb{R}^d), d_2)\) is complete. Consider the set \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) endowed with the metric \(((t, \mu), (s, \nu)) \mapsto |t - s| + d_2(\mu, \nu)\). A map \(\rho: ([0, T] \times \mathcal{P}_2(\mathbb{R}^d))^2 \to [0, +\infty)\) is said to be a **gauge-type function** if the following holds.

a) \(\rho((t, \mu), (t, \mu)) = 0\), for every \((t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\).

b) \(\rho\) is continuous on \(([0, T] \times \mathcal{P}_2(\mathbb{R}^d))^2\).

c) For all \(\varepsilon > 0\), there exists \(\eta > 0\) such that, for all \((t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), the inequality \(\rho((t, \mu), (s, \nu)) \leq \eta\) implies \(|t - s| + d_2(\mu, \nu) \leq \varepsilon\).

In the sequel we construct a gauge-type function on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), taking a particular metric \(d_2\) on \(\mathcal{P}_2(\mathbb{R}^d)\), namely the so-called Gaussian-smoothed 2-Wasserstein distance, see [40]. To this regard, we denote by \(\mathcal{N}_\varrho := \mathcal{N}(0, \varrho^2 I_d)\), for every \(\varrho > 0\), the \(d\)-dimensional
multivariate normal distribution with zero mean and covariance matrix \( \sigma^2 I_d \), with \( I_d \) being the identity matrix of order \( d \). Then, for every \( \sigma > 0 \), the Gaussian-smoothed 2-Wasserstein distance is defined as

\[
W_2^{(\sigma)}(\mu, \nu) := W_2(\mu \ast \mathcal{N}_\sigma, \nu \ast \mathcal{N}_\sigma), \quad \forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d),
\]

where \( \ast \) denotes the convolution of probability measures.

**Lemma 4.2.** For every \( \sigma > 0 \), \( W_2^{(\sigma)} \) is a metric on \( \mathcal{P}_2(\mathbb{R}^d) \), inducing the same topology as \( W_2 \). Moreover, \( (\mathcal{P}_2(\mathbb{R}^d), W_2^{(\sigma)}) \) is a complete metric space.

**Proof.** The first part follows from [40, Proposition 1]. It remains to prove that the metric space \( (\mathcal{P}_2(\mathbb{R}^d), W_2^{(\sigma)}) \) is complete. Let \( \{\mu_n\}_n \subset \mathcal{P}_2(\mathbb{R}^d) \) be a Cauchy sequence with respect to \( W_2^{(\sigma)} \). Then, \( \{\mu_n \ast \mathcal{N}_\sigma\}_n \) is a Cauchy sequence with respect to \( W_2 \). Since \( (\mathcal{P}_2(\mathbb{R}^d), W_2) \) is complete, there exists some \( \tilde{\nu} \in \mathcal{P}_2(\mathbb{R}^d) \) such that \( W_2(\mu_n \ast \mathcal{N}_\sigma, \tilde{\nu}) \to 0 \) as \( n \to \infty \). It follows (see for instance [3, Proposition 7.1.5]) that \( \{\mu_n \ast \mathcal{N}_\sigma\}_n \) has uniformly integrable second moments, namely

\[
\lim_{k \to \infty} \sup_n \int_{|y| \geq k} |y|^2 (\mu_n \ast \mathcal{N}_\sigma)(dy) = 0. \tag{4.1}
\]

Now notice that, given \( a, b, c \in \mathbb{R}_+, h \in \mathbb{N} \), if \( a \leq b + c \) then \( a1_{\{a \leq h\}} \leq 2b1_{\{b \geq h/2\}} + 2c1_{\{c \geq h/2\}} \). Hence, by the elementary inequality \( |x|^2 \leq 2|x|^2 + 2|z|^2 \), valid for every \( x, z \in \mathbb{R}^d \), we get

\[
|x|^2 1_{\{|x| \geq \sqrt{n}\}} \leq 4|x|^2 1_{\{|x|, z| \geq \sqrt{h/2}\}} + 4|z|^2 1_{\{|z| \geq \sqrt{h/2}\}}, \quad \forall x, z \in \mathbb{R}^d, h \in \mathbb{N}. \tag{4.2}
\]

Integrating the above inequality on \( \mathbb{R}^d \times \mathbb{R}^d \) with respect to the product measure \( \mu_n(dx)\mathcal{N}_\sigma(dz) \), we obtain (setting \( k := \sqrt{n} \), to simplify notation)

\[
\int_{|x| \geq k} |x|^2 \mu_n(dx) \leq 4 \int_{|x+z| \geq k/\sqrt{2}} |x+z|^2 \mu_n(dx)\mathcal{N}_\sigma(dz) + 4 \int_{|z| \geq k/\sqrt{2}} |z|^2 \mathcal{N}_\sigma(dz)
\]

\[
= 4 \int_{|y| \geq k/\sqrt{2}} |y|^2 (\mu_n \ast \mathcal{N}_\sigma)(dy) + 4 \int_{|z| \geq k/\sqrt{2}} |z|^2 \mathcal{N}_\sigma(dz).
\]

Then, by (4.1), we deduce that \( \{\mu_n\}_n \) has uniformly integrable second moments. This implies that \( \{\mu_n\}_n \) is tight, so that we can apply [3, Proposition 7.1.5], from which we deduce the existence of a subsequence \( \{\mu_{n_k}\}_k \) converging to some \( \nu \in \mathcal{P}_2(\mathbb{R}^d) \) with respect to \( W_2 \). Notice that (we denote by \( \varphi_\pi \) the characteristic function of the probability measure \( \pi \in \mathcal{P}(\mathbb{R}^d) \))

\[
\varphi_{\mu_{n_k} \ast \mathcal{N}_\sigma}(u) = \varphi_{\mu_{n_k}}(u) e^{-\frac{1}{2} \sigma^2 |u|^2} \xrightarrow{k \to \infty} \varphi_\nu(u) e^{-\frac{1}{2} \sigma^2 |u|^2} = \varphi_{\nu \ast \mathcal{N}_\sigma}(u).
\]

Then, by Lévy’s continuity theorem it follows that \( W_2(\mu_{n_k} \ast \mathcal{N}_\sigma, \nu \ast \mathcal{N}_\sigma) \to 0 \) as \( k \to \infty \). This implies that \( \nu \ast \mathcal{N}_\sigma = \tilde{\nu} \). By a standard argument, the entire sequence \( \{\mu_n \ast \mathcal{N}_\sigma\}_n \) converges to \( \nu \ast \mathcal{N}_\sigma \) with respect to \( W_2 \). This shows that \( W_2^{(\sigma)}(\mu_n, \nu) \to 0 \) as \( n \to \infty \) and concludes the proof. \( \square \)
Our aim is to find a gauge-type function \( \rho = \rho((t, \mu), (t_0, \mu_0)) \) smooth with respect to \((t, \mu)\), for every fixed \((t_0, \mu_0)\), on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) endowed with the metric \(((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^\phi(\mu, \nu)\). The construction of our smooth gauge-type function (whose definition is given in Lemma 4.4 below) relies on a sharp upper bound of \(\mathcal{W}_2\) obtained in [19, 22] (see also [14, Section 5.1.2]), which is valid in any dimension \(d\) and is reported in Lemma 4.3. Notice however that, in the particular case \(d = 1\), ad hoc gauge-type functions may be constructed in easier ways, as for instance relying on the following inequality (see [7, Proposition 7.14]):

\[
\mathcal{W}_2(\mu, \nu)^2 \leq 4 \int_{-\infty}^{+\infty} |x| |F_\mu(x) - F_\nu(x)| \, dx, \quad \forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}), \quad (4.3)
\]

where \(F_\mu\) and \(F_\nu\) are the cumulative distribution functions of \(\mu\) and \(\nu\), respectively. When \(d \in \mathbb{N}\), the upper bound of Lemma 4.3 can be viewed as a \(d\)-dimensional analogue of (4.3).

**Lemma 4.3.** For every integer \(\ell \geq 0\), let \(\mathcal{P}_\ell\) denote the partition of \((-1, 1]^d\) into \(2^{d\ell}\) translations of \((-2^{-\ell}, 2^{-\ell}]^d\). Moreover, let \(B_0 := (-1, 1]^d\) and, for every integer \(n \geq 1\), \(B_n := (-2^n, 2^n] \cap (-2^{n-1}, 2^{n-1}]^d\). Then, for every \(\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)\), the following inequality holds:

\[
(\mathcal{W}_2(\mu, \nu))^2 \leq c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |\mu((2^n B) \cap B_n) - \nu((2^n B) \cap B_n)|, \quad (4.4)
\]

where \(2^n B := \{2^n x \in \mathbb{R}^d : x \in B\}\) and \(c_d > 0\) is a constant depending only on \(d\).

**Proof.** Inequality (4.4) follows from [22, Lemma 5 and Lemma 6] (or, equivalently, [14, Lemma 5.11 and Lemma 5.12]).

Next lemma provides the claimed smooth gauge-type function and it is the main result of the present section. Notice that such a gauge-type function is obtained performing a smoothing of the right-hand side of (4.4), proceeding as follows.

a) Firstly, the absolute value of the difference \(\mu((2^n B) \cap B_n) - \nu((2^n B) \cap B_n)\) appearing in (4.4) is replaced by \(\sqrt{|\mu((2^n B) \cap B_n) - \nu((2^n B) \cap B_n)|^2 + \delta_{n, \ell}^2} - \delta_{n, \ell}\), with \(\delta_{n, \ell} = 2^{-(4n+2d\ell)}\). In other words, we replace \(\cdot\) by the smooth function \(\sqrt{\cdot + \delta_{n, \ell}^2} - \delta_{n, \ell}\). The particular choice of \(\delta_{n, \ell}\) will be used to obtain the convergence of a certain series (see (4.15)).

b) Secondly, as already mentioned, our function will be of gauge-type on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), with \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) endowed with the metric \(((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^\phi(\mu, \nu)\). As a consequence, we consider (4.4) for \(\mathcal{W}_2^\phi(\mu, \nu) = \mathcal{W}_2(\mu * \mathcal{N}_\phi, \nu * \mathcal{N}_\phi)\). This implies that \(\mu((2^n B) \cap B_n)\) and \(\nu((2^n B) \cap B_n)\) are replaced respectively by \((\mu * \mathcal{N}_\phi)((2^n B) \cap B_n)\) and \((\nu * \mathcal{N}_\phi)((2^n B) \cap B_n)\).
Lemma 4.4. We adopt the same notations as in Lemma 4.3. Let \( \varepsilon > 0 \) and \( \rho_{2,\varepsilon} : ([0, T] \times \mathcal{P}_2(\mathbb{R}^d))^2 \to [0, +\infty) \) be defined as

\[
\rho_{2,\varepsilon}( (t, \mu), (s, \nu) ) = |t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{B}_\ell} \left( \sqrt{\left| (\mu * \mathcal{N}_\varepsilon)((2^n B) \cap B_n) - (\nu * \mathcal{N}_\varepsilon)((2^n B) \cap B_n) \right|^2 + \delta_{n,\ell}^2 - \delta_{n,\ell}} \right),
\]

with \( \delta_{n,\ell} := 2^{-(4n+2d)} \). Then, the following holds.

1) \( \rho_{2,\varepsilon} \) is a gauge-type function on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), with \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) endowed with the metric \(((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^{d}(\mu, \nu)\);

2) for every fixed \((t_0, \mu_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), the map \((t, \mu) \mapsto \rho_{2,\varepsilon}((t, \mu), (t_0, \mu_0))\) is in \( C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))\);

3) there exists a constant \( C_d \) (depending only on the dimension \(d\)) such that

\[
\left| \partial_t \rho_{2,\varepsilon}((t, \mu), (t_0, \mu_0)) \right| \leq 2T, \quad (4.5)
\]

\[
\left| \partial_t \rho_{2,\varepsilon}((t, \mu), (t_0, \mu_0)) \right| \leq C_d \frac{\varepsilon}{\varphi^2} \left( \int_{\mathbb{R}^d} |y|^3 \zeta_\varphi(y) dy + |x|^2 \int_{\mathbb{R}^d} |y| \zeta_\varphi(y) dy \right), \quad (4.6)
\]

\[
\left| \partial_x \partial_t \rho_{2,\varepsilon}((t, \mu), (t_0, \mu_0)) \right| \leq C_d \left( \int_{\mathbb{R}^d} |y|^2 \left( \sqrt{d} \varphi^{-2} + |y|^2 \varphi^{-4} \right) \zeta_\varphi(y) dy \right)
\]

\[
+ |x|^2 \int_{\mathbb{R}^d} \left( \sqrt{d} \varphi^{-2} + |y|^2 \varphi^{-4} \right) \zeta_\varphi(y) dy), \quad (4.7)
\]

for all \((t, \mu), (t_0, \mu_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), x \in \mathbb{R}^d, where

\[
\zeta_\varphi(y) = \frac{1}{(2\pi)^{d/2} \varphi^d} e^{-\frac{1}{2} \frac{|y|^2}{\varphi^2}}, \quad \forall y \in \mathbb{R}^d. \quad (4.8)
\]

\textbf{Proof.} We split the proof into four steps.

\textit{Step 1. Uniform convergence of the series in }\rho_{2,\varepsilon}.\textit{ We prove a preliminary result concerning the series in }\rho_{2,\varepsilon}.\textit{ Let }\mathcal{M}\textit{ be a subset of }\mathcal{P}_2(\mathbb{R}^d)\textit{ such that }\{ \mu * \mathcal{N}_\varepsilon \}_{\mu \in \mathcal{M}}\textit{ has uniformly integrable second moments. Our aim is to prove that the series appearing in the definition of }\rho_{2,\varepsilon}\textit{ converges uniformly with respect to }\mu, \nu \in \mathcal{M}\.\textit{ More precisely, we prove that for every }\varepsilon > 0 \textit{ there exists }N = N(\varepsilon) \in \mathbb{N} \textit{ such that}

\[
\sup_{\mu, \nu \in \mathcal{M}} \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{B}_\ell} \left| (\mu * \mathcal{N}_\varepsilon)((2^n B) \cap B_n) - (\nu * \mathcal{N}_\varepsilon)((2^n B) \cap B_n) \right| \leq \varepsilon. \quad (4.9)
\]

Then, the claim follows from the elementary inequality \(\sqrt{a^2 + \delta_{n,\ell}^2 - \delta_{n,\ell}} \leq |a|\), valid for every \(a \in \mathbb{R}\). Let us prove (4.9). First of all, notice that

\[
\sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{B}_\ell} \left| (\mu * \mathcal{N}_\varepsilon)((2^n B) \cap B_n) - (\nu * \mathcal{N}_\varepsilon)((2^n B) \cap B_n) \right|
\]
\[ \leq \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} (\mu * \mathcal{N}_\delta)((2^n B) \cap B_n) + \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} (\nu * \mathcal{N}_\delta)((2^n B) \cap B_n). \]

Observe also that \( \sum_{B \in \mathcal{P}_\ell} (\mu * \mathcal{N}_\delta)((2^n B) \cap B_n) = (\mu * \mathcal{N}_\delta)(B_n) \), therefore \( \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} (\mu * \mathcal{N}_\delta)((2^n B) \cap B_n) = (4/3)(\mu * \mathcal{N}_\delta)(B_n) \), since \( \sum_{\ell \geq 0} 2^{-2\ell} = 4/3 \). So, in particular, (4.9) follows if we prove that for every \( \varepsilon > 0 \) there exists \( N = N(\varepsilon) \in \mathbb{N} \) such that

\[ \sup_{\mu \in \mathcal{M}} \sum_{n \geq N} 2^{2n}(\mu * \mathcal{N}_\delta)(B_n) \leq \varepsilon. \]

Recalling that \( B_n = (-2^n, 2^n)^d \setminus (-2^{n-1}, 2^{n-1})^d \), we obtain \( 2^{2n} \leq 4|x|^2/d, \forall x \in B_n \). Hence, for every \( N \in \mathbb{N} \),

\[ \sum_{n \geq N} 2^{2n}(\mu * \mathcal{N}_\delta)(B_n) \leq \frac{4}{d} \int_{\mathbb{R}^d \setminus (-2^{n-1}, 2^{n-1})^d} |x|^2 \mu * \mathcal{N}_\delta(dx). \]

Since the family \( \{\mu * \mathcal{N}_\delta\}_{\mu \in \mathcal{M}} \) has uniformly integrable second moments, the claim follows.

**Step II.** \( \rho_{2,\varrho} \) is a gauge-type function on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) with respect to the metric \(((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2(\mu, \nu)\). It is clear that \( \rho_{2,\varrho} \) satisfies item a) of Definition 4.1. Concerning items b) and c), we split the rest of the proof of Step II into two substeps.

\( \rho_{2,\varrho} \) satisfies item b) of Definition 4.1. Our aim is to prove that, given \( \{(t_k, \mu_k)\}_k \), \( \{(s_k, \nu_k)\}_k \subset [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \) and \( (t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \), if \( |t_k - t| + \mathcal{W}_2(\mu_k, \mu) + |s_k - s| + \mathcal{W}_2(\nu_k, \nu) \to 0 \) then \( \rho_{2,\varrho}((t_k, \mu_k), (s_k, \nu_k)) \to \rho_{2,\varrho}((t, \mu), (s, \nu)) \). In particular, we have to prove that, if \( \mathcal{W}_2(\mu_k, \mu) + \mathcal{W}_2(\nu_k, \nu) \to 0 \), then

\[ \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left( \sqrt{\left| (\mu_k * \mathcal{N}_\delta)(2^n B) \right| + B_n \right|^2 + \delta_{n,\ell}^2 \right) \]

\[ \to \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left( \sqrt{\left| (\mu * \mathcal{N}_\delta)(2^n B) \right| + B_n \right|^2 + \delta_{n,\ell}^2 \right). \]

Since \( \mathcal{W}_2(\mu_k, \mu) = \mathcal{W}_2(\mu_k, \nu_k, \nu_k) \) and \( \mathcal{W}_2(\nu_k, \nu) = \mathcal{W}_2(\nu_k, \nu_k, \nu_k) \), we have that \( \mathcal{W}_2(\mu_k * \mathcal{N}_\delta, \mu_k * \mathcal{N}_\delta) + \mathcal{W}_2(\nu_k * \mathcal{N}_\delta, \nu_k * \mathcal{N}_\delta) \to 0 \). Now, recall from [3, Proposition 7.1.5] that this implies that \( \{\mu_k * \mathcal{N}_\delta\}_k \) (resp. \( \{\nu_k * \mathcal{N}_\delta\}_k \)) weakly converges to \( \mu * \mathcal{N}_\delta \) (resp. \( \nu * \mathcal{N}_\delta \)) and has uniformly integrable second moments. Since both \( \mu * \mathcal{N}_\delta \) and \( \nu * \mathcal{N}_\delta \) are absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^d \), by the weak convergence (and, in particular, by the portmanteau theorem) we deduce that

\[ \lim_{k \to \infty} (\mu_k * \mathcal{N}_\delta)((2^n B) \cap B_n) = (\mu * \mathcal{N}_\delta)((2^n B) \cap B_n). \]

Similarly \( (\nu_k * \mathcal{N}_\delta)((2^n B) \cap B_n) \to (\nu * \mathcal{N}_\delta)((2^n B) \cap B_n) \). In addition, since \( \{\mu_k * \mathcal{N}_\delta\}_k \) and \( \{\nu_k * \mathcal{N}_\delta\}_k \) have uniformly integrable second moments, from Step I we can interchange the limit with the series, so that the claim follows.
where we have used the elementary inequality
\[
|t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left| (\mu * N_\rho)((2^n B) \cap B_n) - (\nu * N_\rho)((2^n B) \cap B_n) \right| \leq \frac{\varepsilon^2}{2}
\] (4.10)

As a matter of fact, recalling that \( W_2^{(g)}(\mu, \nu) = W_2(\mu * N_\rho, \nu * N_\rho) \), from inequality (4.4) we conclude that
\[
|t - s| + W_2^{(g)}(\mu, \nu) \leq \sqrt{2 |t - s|^2 + 2 W_2^{(g)}(\mu, \nu)^2} \leq \varepsilon.
\]

Let us prove that (4.10) holds with \( \eta_\varepsilon \) given by
\[
\eta_\varepsilon := \left( \sqrt{8c_d + \varepsilon^2/2} - \sqrt{8c_d} \right)^2.
\] (4.11)

To this end, denote by \( d_{2,\rho}((t, \mu), (s, \nu)) \) the left-hand side of (4.10), namely
\[
d_{2,\rho}((t, \mu), (s, \nu)) := |t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left| ((\mu * N_\rho) - (\nu * N_\rho))((2^n B) \cap B_n) \right|.
\]

Moreover, for every \( n \geq 0, \ell \geq 0, B \in \mathcal{P}_\ell, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \), denote
\[
a_{n,\ell}(\mu, \nu, B) := \left( \sqrt{|((\mu * N_\rho) - (\nu * N_\rho))((2^n B) \cap B_n)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \right),
\]
\[
b_{n,\ell}(\mu, \nu, B) := \left| ((\mu * N_\rho) - (\nu * N_\rho))((2^n B) \cap B_n) \right|.
\]

Notice that
\[
\rho_{2,\rho}((t, \mu), (s, \nu)) = |t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} a_{n,\ell}(\mu, \nu, B),
\] (4.12)
\[
d_{2,\rho}((t, \mu), (s, \nu)) = |t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} b_{n,\ell}(\mu, \nu, B).
\] (4.13)

Now, consider \((t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) such that \( \rho_{2,\rho}((t, \mu), (s, \nu)) \leq \eta_\varepsilon \), with \( \eta_\varepsilon \) given by (4.11). Then
\[
c_d 2^{2(n-\ell)} a_{n,\ell}(\mu, \nu, B) \leq \rho_{2,\rho}((t, \mu), (s, \nu)) \leq \eta_\varepsilon.
\] (4.14)

Since \( a_{n,\ell}(\mu, \nu, B) = \sqrt{|b_{n,\ell}(\mu, \nu, B)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \), we obtain
\[
b_{n,\ell}(\mu, \nu, B) = \sqrt{|a_{n,\ell}(\mu, \nu, B)|^2 + 2 \delta_{n,\ell} a_{n,\ell}(\mu, \nu, B)} \leq a_{n,\ell}(\mu, \nu, B) + \sqrt{2 \delta_{n,\ell} a_{n,\ell}(\mu, \nu, B)},
\]
where we have used the elementary inequality \( \sqrt{x + y} \leq \sqrt{x} + \sqrt{y} \), valid for every \( x, y \geq 0 \). Therefore, by (4.14) we get
\[
b_{n,\ell}(\mu, \nu, B) \leq a_{n,\ell}(\mu, \nu, B) + \sqrt{\frac{2}{c_d} \eta_\varepsilon \delta_{n,\ell} 2^{-2(n-\ell)}} = a_{n,\ell}(\mu, \nu, B) + \sqrt{\frac{2}{c_d} \eta_\varepsilon 2^{-(3n+(d-1)\ell)}}
\]
where the last equality follows from the fact that $\delta_{n,\ell} = 2^{-(4n+2d\ell)}$. Hence, from (4.12) and (4.13) we obtain

$$d_{2,\rho}((t, \mu), (s, \nu)) \leq \rho_{2,\rho}((t, \mu), (s, \nu)) + c_d \sqrt{\frac{2}{\eta_\varepsilon} \sum_{n \geq 0} \sum_{\ell \geq 0} 2^{2n} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} 2^{-(3n+(d-1)\ell)}}$$

Recalling that $\rho_{2,\rho}((t, \mu), (s, \nu)) \leq \eta_\varepsilon$ and also that $\mathcal{P}_\ell$ contains $2^{dt}$ sets (see the statement of Lemma 4.3), we get

$$d_{2,\rho}((t, \mu), (s, \nu)) \leq \eta_\varepsilon + \sqrt{2 c_d \eta_\varepsilon} \sum_{n \geq 0} \sum_{\ell \geq 0} 2^{2n} 2^{-2\ell} 2^{dt} 2^{-(3n+(d-1)\ell)}$$

(4.15)

$$= \eta_\varepsilon + \sqrt{2 c_d \eta_\varepsilon} \sum_{n \geq 0} \sum_{\ell \geq 0} 2^{-n} 2^{-\ell} = \eta_\varepsilon + 4 \sqrt{2 c_d \eta_\varepsilon} = \frac{\varepsilon^2}{2},$$

where the last equality follows from the definition of $\eta_\varepsilon$.

**Step III.** The map $(t, \mu) \mapsto \rho_{2,\rho}((t, \mu), (t_0, \mu_0))$ is in $C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$. Recall from (4.8) that $\zeta_\rho: \mathbb{R}^d \rightarrow \mathbb{R}$ denotes the density function of the multivariate normal distribution $\mathcal{N}_\rho = \mathcal{N}(0, \sigma^2 I_d)$. Then, the map $\rho_{2,\rho}$ can be written as

$$\rho_{2,\rho}((t, \mu), (t_0, \mu_0)) = |t - t_0|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left( \left| \int_{\mathbb{R}^d} \phi_n^B(y) \mu(dy) - \int_{\mathbb{R}^d} \phi_n^B(y) \mu_0(dy) \right|^2 + \delta_{n,\ell}^2 \right),$$

where

$$\phi_n^B(x) := \int_{(2^n B) \cap B_n} \zeta_\rho(z - x) \, dz, \quad \forall x \in \mathbb{R}^d.$$

We split the rest of the proof of Step III into two substeps.

**First-order derivatives.** By direct calculation, we have $\partial_t \rho_{2,\rho}((t, \mu), (t_0, \mu_0)) = 2(t - t_0)$. Moreover, we claim that $\partial_{(t, \mu)} \rho_{2,\rho}((t, \mu), (t_0, \mu_0))(x)$ is given by

$$c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{\int_{\mathbb{R}^d} \phi_n^B(y) \mu(dy) - \int_{\mathbb{R}^d} \phi_n^B(y) \mu_0(dy)}{\left| \int_{\mathbb{R}^d} \phi_n^B(y) \mu(dy) - \int_{\mathbb{R}^d} \phi_n^B(y) \mu_0(dy) \right|^2 + \delta_{n,\ell}^2} \partial_x \phi_n^B(x),$$

(4.16)

where $\partial_x \phi_n^B$ denotes the gradient of $\phi_n^B$. In order to prove (4.16), we denote, for every $n, \ell \geq 0$, $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, $B \in \mathcal{P}_\ell$,

$$u_{n,\ell,\mu_0}^B(\mu) = \sqrt{\left| \int_{\mathbb{R}^d} \phi_n^B(y) \mu(dy) - \int_{\mathbb{R}^d} \phi_n^B(y) \mu_0(dy) \right|^2 + \delta_{n,\ell}^2}, \quad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d).$$
Let us determine $\partial_t u_{n, t}^{B, µ_0}$. To this end, let us consider the lifting $U_{n, t}^{B, µ_0} : L^2(\Omega; \mathbb{R}^d) \to \mathbb{R}$ of $u_{n, t}^{B, µ_0}$, given by $U_{n, t}^{B, µ_0}(\xi) = u_{n, t}^{B, µ_0}(µ)$, for every $\xi \in L^2(\Omega; \mathbb{R}^d)$ having distribution $µ$. Recall from the definition of $\partial_t u_{n, t}^{B, µ_0}$ that, for every $\{η_k\}_{k \in \mathbb{N}} \subset L^2(\Omega; \mathbb{R}^d)$ such that $|η_k|_{L^2(\Omega; \mathbb{R}^d)} \to 0$, it holds that

$$\lim_{k \to \infty} \left| \frac{U_{n, t}^{B, µ_0}(\xi + η_k) - U_{n, t}^{B, µ_0}(\xi) - \mathbb{E}\left[ \langle \partial_t u_{n, t}^{B, µ_0}(µ) , η_k \rangle \right]}{|η_k|_{L^2(\Omega; \mathbb{R}^d)}} \right| = 0, \quad (4.17)$$

where $\xi \in L^2(\Omega; \mathbb{R}^d)$ has distribution $µ$. Then, we have

$$\partial_t u_{n, t}^{B, µ_0}(µ)(x) = \frac{\int_{\mathbb{R}^d} φ_n^B(y) (µ(µ_0) - µ_0)(dy)}{\sqrt{\int_{\mathbb{R}^d} φ_n^B(y) (µ(µ_0) - µ_0)(dy)^2 + δ_{n, t}^2}} \partial_x φ_n^B(x), \quad (4.18)$$

for every $(µ, x) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$. Now, by $(4.17)$ we see that $(4.16)$ follows if we prove that the series

$$c_d \sum_{n \geq 0} 2^{n^2} \sum_{\ell \geq 0} 2^{-\ell} \sum_{B \in \mathcal{B}_t} \left| \frac{U_{n, t}^{B, µ_0}(\xi + η_k) - U_{n, t}^{B, µ_0}(\xi) - \mathbb{E}\left[ \langle \partial_t u_{n, t}^{B, µ_0}(µ) , η_k \rangle \right]}{|η_k|_{L^2(\Omega; \mathbb{R}^d)}} \right| \quad (4.19)$$

converges uniformly with respect to $k$. To this end, denote

$$h(λ) := U_{n, t}^{B, µ_0}(\xi + λη_k), \quad 0 \leq λ \leq 1.$$ 

Since $h(1) = h(0) + \int_0^1 h'(λ)dλ$, we get

$$U_{n, t}^{B, µ_0}(\xi + η_k) = U_{n, t}^{B, µ_0}(\xi) + \int_0^1 \mathbb{E}\left[ \langle \partial_t u_{n, t}^{B, µ_0}(µ_k, λ) , η_k \rangle / |η_k|_{L^2(\Omega; \mathbb{R}^d)} \right]dλ,$$

where $µ_{k, λ}$ is the distribution of $\xi + λη_k$. Then, $(4.19)$ is bounded from above by

$$c_d \sum_{n \geq 0} 2^{n^2} \sum_{\ell \geq 0} 2^{-\ell} \sum_{B \in \mathcal{B}_t} \int_0^1 \mathbb{E}\left[ \langle \partial_t u_{n, t}^{B, µ_0}(µ_k, λ) , η_k / |η_k|_{L^2(\Omega; \mathbb{R}^d)} \rangle \right]dλ \quad (4.20)$$

$$+ c_d \sum_{n \geq 0} 2^{n^2} \sum_{\ell \geq 0} 2^{-\ell} \sum_{B \in \mathcal{B}_t} \mathbb{E}\left[ \langle \partial_t u_{n, t}^{B, µ_0}(µ) , η_k / |η_k|_{L^2(\Omega; \mathbb{R}^d)} \rangle \right].$$

Notice that $\{η_k\}_{k \in \mathbb{N}}$ has uniformly integrable second moments (see for instance [30, Theorem 4.12]), so that $\{ξ + λη_k\}_{k \in \mathbb{N}, λ \in [0, 1]}$ also has uniformly integrable second moments. Therefore, the two series in $(4.20)$ converge uniformly if we prove that $(ν$ denotes the distribution of $η)$

$$c_d \sum_{n \geq 0} 2^{n^2} \sum_{\ell \geq 0} 2^{-\ell} \sum_{B \in \mathcal{B}_t} \mathbb{E}\left[ \langle \partial_t u_{n, t}^{B, µ_0}(ν) , η \rangle \right]$$

converges uniformly with respect to $ν$, whenever $ν$ belongs to a subset $\mathcal{M}$ of $\mathcal{P}_2(\mathbb{R}^d)$ with uniformly integrable second moments, namely

$$\lim_{M \to \infty} \sup_{ν \in \mathcal{M}} \int_{|x| \geq M} |x|^2 ν(dx) = 0. \quad (4.21)$$
Then, the claim follows if we prove that for every $\varepsilon > 0$, there exists $N = N(\varepsilon, \mathcal{M}) \in \mathbb{N}$ such that, for every $\nu \in \mathcal{M}$, it holds that

$$
c_d \sum_{n \geq 2^{2n}} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{B}_\ell} \mathbb{E}
\left[ |\partial_\nu u_{n,\ell}^{B,\mu_0}(\nu)| \right] \leq \varepsilon,
$$

with $\eta \in L^2(\Omega; \mathbb{R}^d)$ having distribution $\nu$. Firstly, from (4.18) notice that $|\partial_\nu \phi_n^B(x)| \leq |\partial_\nu \phi_n^B(x)|, \forall \ x \in \mathbb{R}^d$. Moreover $\partial_\nu \phi_n^B(x) = \frac{1}{\varrho^2} \int_{(2^n B) \cap B_n} (z-x) \zeta_\nu(z-x) dz$. Therefore, the series (4.22) is bounded from above by

$$
c_d \sum_{n \geq 2^{2n}} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{B}_\ell} \mathbb{E}
\left[ \int_{(2^n B) \cap B_n} |z-\eta| \zeta_\nu(z-\eta) dz \right] = \frac{c_d}{\varrho^2} \sum_{n \geq 2^{2n}} \sum_{\ell \geq 0} 2^{-2\ell} \mathbb{E}
\left[ \int_{B_n} |z-\eta| \zeta_\nu(z-\eta) dz \right] = \frac{4}{3} \frac{c_d}{\varrho^2} \sum_{n \geq 2^{2n}} \mathbb{E}
\left[ \int_{B_n} |z-\eta| \zeta_\nu(z-\eta) dz \right].
$$

Recalling that $B_n = (-2^n, 2^n)^d \setminus (-2^{n-1}, 2^{n-1})^d$, we obtain $2^{2n} \leq 4|z|^2/d, \forall \ z \in B_n$, so that

$$
\frac{4}{3} \frac{c_d}{\varrho^2} \sum_{n \geq 2^{2n}} \mathbb{E}
\left[ \int_{B_n} |z-\eta| \zeta_\nu(z-\eta) dz \right] \leq \frac{16}{3d} \frac{c_d}{\varrho^2} \mathbb{E}
\left[ \int_{(-2^{n-1}, 2^{n-1})^d} |z|^2 |z-\eta| \zeta_\nu(z-\eta) dz \right] \leq \frac{16}{3d} \frac{c_d}{\varrho^2} \mathbb{E}
\left[ \int_{(-2^{n-1}, 2^{n-1})^d} |z-\eta| \zeta_\nu(z-\eta) dz \right] \leq \frac{16}{3d} \frac{c_d}{\varrho^2} \int \int_{(-2^{n-1}, 2^{n-1})^d} |z-\eta| \zeta_\nu(z-\eta) dz \nu(dx),
$$

Applying the elementary inequality (4.2) (with $x, x + z, z, \sqrt{\varepsilon}$ replaced respectively by $y + x, y, x, 2^{N-1}$), we obtain

$$
\frac{16}{3d} \frac{c_d}{\varrho^2} \int \int_{(-2^{n-1}, 2^{n-1})^d} |y + x|^2 |y| \zeta_\nu(y) dy \nu(dx) \leq \frac{64}{3d} \frac{c_d}{\varrho^2} \int \int_{(-2^{n-1}, 2^{n-1})^d} \left( |y|^2 + 1(|y| \geq 2^{n-3/2}) |x|^2 \right) |y| \zeta_\nu(y) dy \nu(dx) \leq \frac{64}{3d} \frac{c_d}{\varrho^2} \int \int_{(-2^{n-1}, 2^{n-1})^d} \left( |y|^2 \zeta_\nu(y) dy \nu(dx) \right) \left( \int |x|^2 \zeta_\nu(y) dy \nu(dx) \right).
$$

Then, (4.22) follows from (4.21).

Second-order derivatives. We claim that $\partial_\nu \partial_{\mu \mu_0}(t, \mu, (t_0, \mu_0))(x)$ is equal to

$$
c_d \sum_{n \geq 0} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{B}_\ell} \frac{\int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy)}{\sqrt{\int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy)^2 + \varrho_n^2 \phi_n^B(x)}} \partial_{xx} \phi_n^B(x)
$$

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We also observe that precisely, we prove that for all \( d \) the \(|\epsilon|\) and \((\)notice that the Frobenius norms \(|\epsilon|\)

\[
\text{where} \quad \partial_{xx}^2 \phi_n^B \text{ denotes the Hessian matrix of } \phi_n^B. \quad \text{Proceeding as in the previous substep, we see that this follows if we prove that the series } (|\partial_{xx}^2 \phi_n^B(x)| \text{ stands for the Frobenius norm of the } d \times d \text{ matrix } \partial_{xx}^2 \phi_n^B(x))
\]

converges uniformly with respect to \( x \), whenever \( x \) belongs to a bounded subset of \( \mathbb{R}^d \). More precisely, we prove that for all \( \epsilon > 0 \) and \( M \in \mathbb{N} \), there exists \( N = N(\epsilon, M) \in \mathbb{N} \) such that, for every \( x \in \mathbb{R}^d \), with \(|x| \leq M \), it holds that

\[
c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{|\int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy)|}{\sqrt{\int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy)}^2 + \delta_{n,\ell}^2} |\partial_{xx}^2 \phi_n^B(x)| \leq \epsilon. \quad (4.24)
\]

We begin noting that the latter series is bounded from above by

\[
c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |\partial_{xx}^2 \phi_n^B(x)|. \quad (4.25)
\]

We also observe that

\[
\partial_{xx}^2 \phi_n^B(x) = \frac{1}{\theta^2} I_d \int_{(2^n B) \cap B_n} \zeta_0(z - x) \, dz - \frac{1}{\theta^4} \int_{(2^n B) \cap B_n} (z - x) \otimes (z - x) \, \zeta_0(z - x) \, dz, \quad (4.26)
\]

where \( I_d \) denotes the identity matrix of order \( d \), while \((z - x) \otimes (z - x)\) is the \( d \times d \) matrix with \((i, j)\)-component equal to \((z_i - x_i)(z_j - x_j)\). Then, (4.25) is bounded from above by (notice that the Frobenius norms \(|I_d|\) and \(|(z - x) \otimes (z - x)|\) are given respectively by \(\sqrt{d}\) and \(|z - x|^2\), where \(|z - x|\) denotes the Euclidean norm of \(z - x\))

\[
c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \int_{(2^n B) \cap B_n} \left( \sqrt{d} \, g^{-2} + |z - x|^2 \, g^{-4} \right) \zeta(z - x) \, dz
\]

\[
= c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \int_{B_n} \left( \sqrt{d} \, g^{-2} + |z - x|^2 \, g^{-4} \right) \zeta(z - x) \, dz
\]

\[
= \frac{4}{3} c_d \sum_{n \geq N} 2^{2n} \int_{B_n} \left( \sqrt{d} \, g^{-2} + |z - x|^2 \, g^{-4} \right) \zeta(z - x) \, dz.
\]

Recalling that \(2^{2n} \leq 4|z|^2/d, \forall z \in B_n\), we find

\[
\frac{4}{3} c_d \sum_{n \geq N} 2^{2n} \int_{B_n} \left( \sqrt{d} \, g^{-2} + |z - x|^2 \, g^{-4} \right) \zeta(z - x) \, dz
\]
\[ \leq \frac{16}{3d} c_d \int_{\mathbb{R}^d \setminus (-2^{-N+1}, 2^{-N+1})^d} |z|^2 \left( \sqrt{d} \vartheta^{-2} + |z - x|^2 \vartheta^{-4} \right) \zeta(z - x) \, dz. \]  

(4.27)

Since \( |x| \leq M \), from the right-hand side of (4.27) we see that (4.24) follows.

**Step IV. Bounds.** The bound (4.5) for the time derivative follows directly from the definition of \( \rho_{2, \varrho} \). Let us now investigate the derivatives with respect to the measure. Recalling that \( \partial_{\mu} \rho_{2, \varrho}((t, \mu), (t_0, \mu_0))(x) \) is given by (4.16), we obtain (notice that \( \rho_n \phi_n^2 = \frac{1}{\varrho} \int_{(2^n B) \cap B_n} (z - x) \zeta \zeta(z - x) \, dz \))

\[
|\partial_{\mu} \rho_{2, \varrho}((t, \mu), (t_0, \mu_0))(x)| \leq \frac{c_d}{\varrho^2} \sum_{n \geq 0} 2^{2n} \sum_{t \geq 0} 2^{2t} \sum_{B \in \mathcal{B}} \int_{(2^n B) \cap B_n} |z - x| \zeta \zeta(z - x) \, dz
\]

\[
= \frac{4 c_d}{3 d \varrho^2} \sum_{n \geq 0} 2^{2n} \int_{B_n} |z - x| \zeta \zeta(z - x) \, dz.
\]

Since \( 2^{2n} \leq 4 |z|^2/d, \forall z \in B_n \), we get

\[
|\partial_{\mu} \rho_{2, \varrho}((t, \mu), (t_0, \mu_0))(x)| \leq \frac{16 c_d}{3d \varrho^2} \int_{\mathbb{R}^d} |z|^2 |z - x| \zeta \zeta(z - x) \, dz = \frac{16 c_d}{3d \varrho^2} \int_{\mathbb{R}^d} |y + x|^2 |y| \zeta \zeta(y) \, dy
\]

\[
\leq \frac{32 c_d}{3d \varrho^2} \int_{\mathbb{R}^d} |y|^3 \zeta \zeta(y) \, dy + \frac{32 c_d}{3d \varrho^2} |x|^2 \int_{\mathbb{R}^d} |y| \zeta \zeta(y) \, dy,
\]

which gives (4.6).

From similar calculations, by (4.23) and (4.26), we deduce that \( \partial_{\varrho} \partial_{\mu} \rho_{2, \varrho}((t, \mu), (t_0, \mu_0))(x) \) is bounded by

\[
|\partial_{\varrho} \partial_{\mu} \rho_{2, \varrho}((t, \mu), (t_0, \mu_0))(x)| \leq \frac{16 c_d}{3d \varrho^2} \int_{\mathbb{R}^d} |z|^2 \left( \sqrt{d} \varrho^{-2} + |z - x|^2 \varrho^{-4} \right) \zeta \zeta(z - x) \, dz
\]

\[
= \frac{16 c_d}{3d \varrho^2} \int_{\mathbb{R}^d} |y + x|^2 \left( \sqrt{d} \varrho^{-2} + |y|^2 \varrho^{-4} \right) \zeta \zeta(y) \, dy
\]

\[
\leq \frac{32 c_d}{3d \varrho^2} \int_{\mathbb{R}^d} |y|^2 \left( \sqrt{d} \varrho^{-2} + |y|^2 \varrho^{-4} \right) \zeta \zeta(y) \, dy + \frac{32 c_d}{3d \varrho^2} |x|^2 \int_{\mathbb{R}^d} \left( \sqrt{d} \varrho^{-2} + |y|^2 \varrho^{-4} \right) \zeta \zeta(y) \, dy.
\]

We conclude that (4.7) holds. \( \square \)

We are in a position to state the smooth variational principle on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\).

**Theorem 4.5.** Fix \( \delta > 0 \) and let \( G: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) be upper semicontinuous and bounded from above. Given \( \lambda > 0 \), let \((t_0, \mu_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) be such that

\[
\sup G - \lambda \leq G(t_0, \mu_0).
\]

Then, there exist \((\bar{t}, \bar{\mu}) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) and a sequence \( \{(t_k, \mu_k)\} \subset [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \) such that:

(i) \( \rho_{2,1/\delta}((\bar{t}, \bar{\mu}), (t_k, \mu_k)) \leq \frac{\lambda}{32\pi^2}, \) for every \( k \geq 0; \)
(ii) \( G(t_0, \mu_0) \leq G(\tilde{t}, \tilde{\mu}) - \delta^2 \varphi_\delta(\tilde{t}, \tilde{\mu}), \) with \( \varphi_\delta: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to [0, +\infty) \) given by

\[
\varphi_\delta(t, \mu) = \sum_{k=0}^{+\infty} \frac{1}{2^k} \rho_{2,1/\delta}( (t, \mu), (t_k, \mu_k) ), \quad \forall (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d);
\]

(iii) \( G(t, \mu) - \delta^2 \varphi_\delta(t, \mu) < G(\tilde{t}, \tilde{\mu}) - \delta^2 \varphi_\delta(\tilde{t}, \tilde{\mu}), \) for every \( (t, \mu) \in ([0, T] \times \mathcal{P}_2(\mathbb{R}^d)) \setminus \{ (\tilde{t}, \tilde{\mu}) \}. \)

Furthermore, the function \( \varphi_\delta \) satisfies the following properties.

1) \( \varphi_\delta \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d)); \)

2) its time derivative is bounded by \( 4T; \)

3) its measure derivative is bounded by

\[
|\partial_\mu \varphi_\delta(t, \mu)(x)| = 2C_d \delta^2 \left( \int_{\mathbb{R}^d} |y|^3 \zeta_{1/\delta}(y) \, dy + |x|^2 \int_{\mathbb{R}^d} |y| \zeta_{1/\delta}(y) \, dy \right), \quad (4.28)
\]

with the same constant \( C_d \) as in (4.6) and \( \zeta_{1/\delta} \) given by (4.8) with \( \varrho = 1/\delta; \)

4) its second-order measure derivative is bounded by

\[
|\partial_x \partial_\mu \varphi_\delta(t, \mu)(x)| = 2C_d \delta^2 \left( \int_{\mathbb{R}^d} |y|^2 \left( \sqrt{d} + |y|^2 \delta^2 \right) \zeta_{1/\delta}(y) \, dy \right) \quad (4.29)
\]

+ \( |x|^2 \int_{\mathbb{R}^d} \left( \sqrt{d} + |y|^2 \delta^2 \right) \zeta_{1/\delta}(y) \, dy \),

with the same constant \( C_d \) as in (4.7) and \( \zeta_{1/\delta} \) given by (4.8) with \( \varrho = 1/\delta. \)

**Proof.** Items (i)-(ii)-(iii) follow directly from the Borwein-Preiss variational principle [9, Theorem 2.5.2] applied on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) with gauge-type function \( \rho_{2,1/\delta} \) (we only remark that, concerning the sequence \( \{ \delta_i \}_{i \geq 0} \) appearing in the statement of [9, Theorem 2.5.2], here we take \( \delta_i = \delta^2/2^i, \) \( i \geq 0 \)). Finally, items 2)-3)-4) follow respectively from (4.5)-(4.6)-(4.7). \( \square \)

## 5 Comparison theorem and uniqueness

**Theorem 5.1 (Comparison).** Let Assumptions (A), (B), (C), (D) hold. Consider bounded and continuous functions \( u_1: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) and \( u_2: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}, \) with \( u_1 \) (resp. \( u_2 \)) being a viscosity subsolution (resp. supersolution) to equation (3.3). Then, it holds that \( u_1 \leq u_2 \) on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\).
**Proof.** Let \( v_0 \) be the map defined by (A.1) with \( \varepsilon = 0 \) (see also Remark A.1). Our aim is to prove that \( u_1 \leq v_0 \) and \( v_0 \leq u_2 \) on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) from which the claim follows.

**Step I. Proof of** \( u_1 \leq v_0 \). By contradiction, we suppose that there exists \((t_0, \bar{\mu}_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\) such that

\[
(u_1 - v_0)(t_0, \bar{\mu}_0) > 0.
\]

Since both \( u_1 \) and \( v_0 \) are continuous, we can find \( q > 2 \) and \( \mu_0 \in \mathcal{P}_q(\mathbb{R}^d) \) such that

\[
(u_1 - v_0)(t_0, \mu_0) > 0. \tag{5.1}
\]

As a matter of fact, let \( \xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \) be such that \( \mathbb{P}_\xi = \bar{\mu}_0 \). For every \( k \in \mathbb{N} \), let \( \mu_k^0 \in \mathcal{P}_2(\mathbb{R}^d) \) be the distribution of \( \xi_k := \xi 1_{\{|\xi| \leq k\}} \). We see that \( \mu_k^0 \in \mathcal{P}_q(\mathbb{R}^d) \), for any \( q \geq 1 \). Moreover, it holds that

\[
\mathcal{W}_2(\mu_k^0, \bar{\mu}_0)^2 \leq \mathbb{E}[|\xi_k - \xi|^2] = \int_{|x| > k} |x|^2 \bar{\mu}_0(dx) \xrightarrow{k\to+\infty} 0,
\]

from which we deduce that (5.1) holds with \( \mu_0 := \mu_k^0 \) for some \( k \) large enough.

We split the rest of the proof of Step I into four substeps.

**Substep I-A.** For every \( \varepsilon > 0 \) and \( n, m \in \mathbb{N} \), let \( v_{\varepsilon,n,m} \) be the map given by (A.7). Now, we define \( \bar{v}_1(t, \mu) := e^{\varepsilon t-a_0}u_1(t, \mu) \), for every \((t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\), and similarly \( \bar{v}_{\varepsilon,n,m}, \hat{f}_{n,m}, \hat{f}, v_{\varepsilon,n,m}, f_{n,m}, f \) from \( v_{\varepsilon,n,m}, f_{n,m}, f \), respectively. We also define \( \hat{g}(x, \mu) := e^{T-a_0}g(x, \mu) \) and \( \hat{g}_{\varepsilon,n,m}(x, \mu) := e^{T-a_0}\hat{g}_{n,m}(x, \mu) \), for every \((x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\). We observe that \( \bar{u}_1 \) is a viscosity subsolution of the following equation:

\[
\begin{aligned}
\partial_t \bar{u}_1(t, \mu) + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \hat{f}(t, x, \mu, a) + \frac{1}{2} \text{tr} \left[ (\sigma \sigma^T)(t, x, a) \partial_x \partial_\mu \bar{u}_1(t, \mu)(x) \right] \right\} \\
+ \langle b(t, x, \mu, a), \partial_\mu \bar{u}_1(t, \mu)(x) \rangle \mu(dx) = \bar{u}_1(t, \mu), \quad (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d),
\end{aligned}
\]

Moreover, by Theorem A.7 we deduce that \( \bar{v}_{\varepsilon,n,m} \) solves the following equation:

\[
\begin{aligned}
\partial_t \bar{v}_{\varepsilon,n,m}(t, \mu) - \bar{v}_{\varepsilon,n,m}(t, \mu) + \int_{\mathbb{R}^d} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \lambda_i \left( b_{n,m}(t, x_1, \ldots, x_n, a_i), \partial x_i \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \right) \right\} \\
+ \frac{1}{2} \text{tr} \left[ ((\sigma \sigma^T)(t, x_i, a_i) + \varepsilon^2) \partial_{x_i x_i} \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \right] \\
+ \frac{1}{n} \hat{f}_{n,m}(t, x_1, \ldots, x_n, a_i) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n) = 0, \quad (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d),
\end{aligned}
\]

\[
\bar{v}_{\varepsilon,n,m}(T, \mu) = \frac{1}{n} \sum_{i=1}^n \int_{\mathbb{R}^d} \hat{g}_{n,m}(\bar{x}) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d),
\]

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where \( \bar{\psi}_{\varepsilon,n,m}(t, \bar{x}) := e^{t-t_0} \psi_{\varepsilon,n,m}(t, \bar{x}) \), for every \((t, \bar{x}) \in [0, T] \times \mathbb{R}^n\), \( \bar{x} = (x_1, \ldots, x_n) \), with \( \bar{\psi}_{\varepsilon,n,m} \) being the same function appearing in Theorem A.7.

Finally, notice that, by Assumption (A)-(iii), \( \psi_{\varepsilon,n,m} \) is bounded by a constant independent of \( \varepsilon, n, m \). Since also \( u_1 \) is bounded, there exists \( \lambda \geq 0 \), independent of \( \varepsilon, n, m \), satisfying

\[
\sup(\hat{u}_1 - \bar{\psi}_{\varepsilon,n,m}) \leq (\hat{u}_1 - \bar{\psi}_{\varepsilon,n,m})(t_0, \mu_0) + \lambda.
\]

\text{Substep I-b.} Since \( \hat{u}_1 - \bar{\psi}_{\varepsilon,n,m} \) is bounded and continuous, by (5.4) and Theorem 4.5 with \( G = \hat{u}_1 - \bar{\psi}_{\varepsilon,n,m} \), we obtain that for every \( \delta > 0 \) there exist \( \{(t_k, \mu_k)\}_{k \geq 1} \subset [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \), converging to some \((\bar{t}, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \), and \( \varphi_\delta \) such that items (i)-(ii)-(iii) and 1)-2)-3) of Theorem 4.5 hold.

Now, recall from the proof of Lemma 4.4, and in particular from (4.10)-(4.11), that for all \((t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \) satisfying \( \rho_{2,1/\delta}((t, \mu), (s, \nu)) \leq \eta_\epsilon \), with \( \eta_\epsilon \) as in (4.11), namely

\[
\eta_\epsilon := \left( \sqrt{8c_d + \epsilon^2/2} - \sqrt{8c_d} \right)^2,
\]

it holds that \( \mathcal{W}_{2}^{(1/\delta)}((t, \mu), (s, \nu)) \leq \epsilon \). Since by item (i) of Theorem 4.5 we have \( \rho_{2,1/\delta}((\bar{t}, \bar{\mu}), (t_0, \mu_0)) \leq \lambda/\delta^2 \), we get

\[
\mathcal{W}_{2}^{(1/\delta)}(\bar{\mu}, \mu_0) \leq \frac{1}{\delta} \sqrt{2\lambda + 8\delta \sqrt{2c_d \lambda}}.
\]

Finally, by [40, Lemma 1] we obtain

\[
\mathcal{W}_{2}(\bar{\mu}, \mu_0) \leq \mathcal{W}_{2}^{(1/\delta)}(\bar{\mu}, \mu_0) + 2\sqrt{d} + 2 \leq \frac{1}{\delta} \left( \sqrt{2\lambda + 8\delta \sqrt{2c_d \lambda}} + 2\sqrt{d} + 2 \right).
\]

\text{Substep I-c.} Let us prove that \( \bar{t} < T \). If \( \bar{t} = T \), from item (ii) of Theorem 4.5 we have

\[
(u_1 - \psi_{\varepsilon,n,m})(t_0, \mu_0) = (\hat{u}_1 - \bar{\psi}_{\varepsilon,n,m})(t_0, \mu_0) \leq (\hat{u}_1 - \bar{\psi}_{\varepsilon,n,m} - \delta^2 \varphi_\delta)(T, \bar{\mu}) \leq (\hat{u}_1 - \bar{\psi}_{\varepsilon,n,m})(T, \bar{\mu}),
\]

where the last inequality follows from \( \varphi_\delta \geq 0 \). Hence

\[
(u_1 - \psi_{\varepsilon,n,m})(t_0, \mu_0)
\leq e^{T-t_0} \int_{\mathbb{R}^d} g(x, \bar{\mu}) \bar{\mu}(dx) - \frac{e^{T-t_0}}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^d} g_{n,m}^i(x_1, \ldots, x_n) \bar{\mu}(dx_1) \otimes \cdots \otimes \bar{\mu}(dx_n)
\]

\[
= \frac{e^{T-t_0}}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^d} (g(x_i, \bar{\mu}) - g_{n,m}^i(x_1, \ldots, x_n)) \bar{\mu}(dx_1) \otimes \cdots \otimes \bar{\mu}(dx_n)
\]

\[
= \frac{e^{T-t_0}}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^d} (g(x_i, \bar{\mu}) - g(x_i, \bar{\mu}^n)) \bar{\mu}(dx_1) \otimes \cdots \otimes \bar{\mu}(dx_n)
\]

\[
+ \frac{e^{T-t_0}}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^d} (g(x_i, \bar{\mu}^n) - g_{n,m}^i(x_1, \ldots, x_n)) \bar{\mu}(dx_1) \otimes \cdots \otimes \bar{\mu}(dx_n),
\]

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where \( \hat{\mu}^{n,\bar{x}} \) is given by

\[
\hat{\mu}^{n,\bar{x}} := \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j}
\]

for every \( n \in \mathbb{N}, \ \bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^d \) and \( x_1, \ldots, x_n \in \mathbb{R}^d \). Then, from the Lipschitz property of \( g \) we obtain

\[
(u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq e^{T-t_0} \int_{\mathbb{R}^d} KW_2(\bar{\mu}, \hat{\mu}^{n,\bar{x}}) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n) + \frac{e^{T-t_0}}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^d} (g(x_i, \hat{\mu}^{n,\bar{x}}) - g^i_{n,m}(x_1, \ldots, x_n)) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n),
\]

(5.6)

From [22, Theorem 1] we have that

\[
\int_{\mathbb{R}^d} W_2(\bar{\mu}, \hat{\mu}^{n,\bar{x}}) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n)
\]

\[
\leq c_d \left( \int_{\mathbb{R}^d} |x|^q \mu(dx) \right)^{1/q} \begin{cases}
\frac{1}{\sqrt{n}} + \frac{1}{n^{(q-1)/q}}, & \text{if } d = 1 \text{ and } q \neq 2, \\
\frac{1}{n^{1/d}} \log(1 + n) + \frac{1}{n^{(q-1)/q}}, & \text{if } d = 2 \text{ and } q \neq 2,
\end{cases}
\]

\[
\frac{1}{n^{1/d}} + \frac{1}{n^{(q-1)/q}}, \quad \text{if } d > 2 \text{ and } q \neq \frac{d}{d-1},
\]

with \( q \in (1,2] \) and for some constant \( c_d \geq 0 \), depending only on \( d \). So, in particular, there exists some \( q \in (1,2) \) such that

\[
\int_{\mathbb{R}^d} W_2(\bar{\mu}, \hat{\mu}^{n,\bar{x}}) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n) \leq c_d \left( \int_{\mathbb{R}^d} |x|^q \mu(dx) \right)^{1/q} h_n
\]

(5.7)

for some sequence \( \{h_n\}_n \) satisfying \( \lim_{n \to +\infty} h_n = 0 \).

Hence, plugging (5.7) and (A.5) into (5.6), we get

\[
(u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq c_d K e^{T-t_0} \left( \int_{\mathbb{R}^d} |x|^q \mu(dx) \right)^{1/q} h_n + K e^{T-t_0} m \sum_{i=1}^{n} \left| y_i \right| \prod_{j=1}^{n} \Phi(m y_j) dy_j.
\]

Now, recalling that \( q \in (1,2) \), we get (denoting by \( \delta_0 \) the Dirac measure centered at zero)

\[
\left( \int_{\mathbb{R}^d} |x|^q \mu(dx) \right)^{1/q} \leq \left( \int_{\mathbb{R}^d} |x|^2 \mu(dx) \right)^{1/2} = W_2(\bar{\mu}, \delta_0)
\]

\[
\leq W_2(\bar{\mu}, \mu_0) + W_2(\mu_0, \delta_0) \leq \frac{1}{\delta} \left( \sqrt{2\lambda + 8\delta(2c_d\lambda + 2\sqrt{d} + 2)} + W_2(\mu_0, \delta_0),
\right.
\]

(5.8)

where the last inequality follows from (5.5). Hence

\[
(u_1 - v_{\varepsilon,n,m})(t_0, \mu_0)
\]

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\[ \leq c_d Ke^{T_{-\delta}} \left( \frac{1}{\delta} \left( \sqrt{2\lambda + 8\delta \sqrt{2cd\lambda + 2\sqrt{d + 2}}} \right) + W_2(\mu_0, \delta_0) \right) h_n \]

\[ + Ke^{T_{-\delta}} \mathcal{M} \int_{\mathbb{R}^d} \left( \frac{1}{n} \sum_{i=1}^{n} |y_i| \right) \prod_{j=1}^{n} \Phi(my_j) dy_j. \]

Sending \( m \to +\infty \), then \( n \to +\infty \), and finally \( \varepsilon \to 0^+ \), we end up, using Theorem A.6 and Lemma A.2, with

\[ (u_1 - v_0)(t_0, \mu_0) \leq 0, \]

which gives a contradiction to (5.1).

**Substep I-D.** From item (iii) of Theorem 4.5 and the fact that \( \bar{u}_1 \) is a viscosity subsolution of (5.2), we find

\[- \partial_t (\bar{v}_{\varepsilon,n,m} + \delta^2 \varphi_\delta(t, \bar{\mu})) - \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\bar{t}, x, \bar{\mu}, a), \partial_\mu (\bar{v}_{\varepsilon,n,m} + \delta^2 \varphi_\delta)(\bar{t}, \bar{\mu})(x) \rangle \right\} \bar{\mu}(dx) + \bar{u}_1(\bar{t}, \bar{\mu}) \leq 0.\]

Then

\[ \bar{u}_1(\bar{t}, \bar{\mu}) \leq \delta^2 \partial_t \varphi_\delta(\bar{t}, \bar{\mu}) + \delta^2 \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\bar{t}, x, \bar{\mu}, a), \partial_\mu \varphi_\delta(\bar{t}, \bar{\mu})(x) \rangle \right\} \bar{\mu}(dx) + \bar{f}(\bar{t}, x, \bar{\mu}, a) \bar{\mu}(dx) + \partial_t \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu}). \]

Using that the \( \bar{v}_{\varepsilon,n,m} \) satisfies equation (5.3), the above implies

\[ (\bar{u}_1 - \bar{v}_{\varepsilon,n,m})(\bar{t}, \bar{\mu}) \leq \delta^2 \partial_t \varphi_\delta(\bar{t}, \bar{\mu}) \]

\[ + \delta^2 \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\bar{t}, x, \bar{\mu}, a), \partial_\mu \varphi_\delta(\bar{t}, \bar{\mu})(x) \rangle \right\} \bar{\mu}(dx) + \frac{1}{2} \text{tr} \left[ (\sigma \sigma^\top)(\bar{t}, x, a) \partial_\mu \varphi_\delta(\bar{t}, \bar{\mu})(x) \right] \bar{\mu}(dx) \]

\[ + \frac{1}{2} \text{tr} \left[ (\sigma \sigma^\top)(\bar{t}, x, a) \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu})(x) \right] + \frac{1}{2} \text{tr} \left[ (\sigma \sigma^\top)(\bar{t}, x, a) \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu})(x) \right] \]

\[ + \bar{f}(\bar{t}, x, \bar{\mu}, a) \bar{\mu}(dx) - \int_{\mathbb{R}^d} \sum_{i=1}^{n} \sup_{a_i \in A} \left\{ \frac{1}{n} \bar{f}_i(\bar{t}, \bar{x}, a_i) + \langle b_i(n, \bar{t}, \bar{x}, a_i), \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{x}) \rangle \right\} \bar{\mu}(dx_1) \otimes \cdots \otimes \bar{\mu}(dx_n), \]

with \( \bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^d \). Now, recalling item (ii) of Theorem 4.5 and that \( \varphi_\delta \geq 0 \), we find, using (5.9),

\[ (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) = (\bar{u}_1 - \bar{v}_{\varepsilon,n,m})(t_0, \mu_0) \leq (\bar{u}_1 - \bar{v}_{\varepsilon,n,m})(\bar{t}, \bar{\mu}) - \delta^2 \varphi_\delta(\bar{t}, \bar{\mu}) \]
\[ \leq (\bar{u}_1 - \bar{v}_{\varepsilon,n,m})(\bar{t}, \bar{\mu}) \leq \delta^2 \partial_i \varphi_\delta(\bar{t}, \bar{\mu}) + \delta^2 \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \left\langle b(\bar{t}, x, \bar{\mu}, a), \partial_\mu \varphi_\delta(\bar{t}, \bar{\mu})(x) \right\rangle + \frac{1}{2} \text{tr} \left[ (\sigma^\top)(\bar{t}, x, a) \partial_x \partial_\mu \varphi_\delta(\bar{t}, \bar{\mu})(x) \right] \right\} \bar{\mu}(dx) \\
+ \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \left\langle b(\bar{t}, x, \bar{\mu}, a), \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu})(x) \right\rangle + \frac{1}{2} \text{tr} \left[ (\sigma^\top)(\bar{t}, x, a) \partial_x \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu})(x) \right] \right\} \bar{\mu}(dx) \\
+ \frac{1}{2} \text{tr} \left[ (\sigma^\top)(\bar{t}, x, a_i) \partial^2_{x_i \bar{\mu}} \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu}) \right] \bar{\mu}(dx) \right\} \bar{\mu}(dx_1) \otimes \cdots \otimes \bar{\mu}(dx_n). \]

Now, recalling that \( b \) and \( \sigma \) are bounded, by item 2) and estimates \((4.28)-(4.29)\) of Theorem 4.5, we deduce that

\[ \partial_i \varphi_\delta(\bar{t}, \bar{\mu}) + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \left\langle b(\bar{t}, x, \bar{\mu}, a), \partial_\mu \varphi_\delta(\bar{t}, \bar{\mu})(x) \right\rangle + \frac{1}{2} \text{tr} \left[ (\sigma^\top)(\bar{t}, x, a) \partial_x \partial_\mu \varphi_\delta(\bar{t}, \bar{\mu})(x) \right] \right\} \bar{\mu}(dx) \]

\[ \leq 4T + \Lambda \delta^2 \left( \int_{\mathbb{R}^d} \|y\|^2 + |y|^3 + |y|^4 \delta^2 \right) \cdot \mathcal{W}_2(\bar{\mu}, \delta_0) \]

\[ \left( \int_{\mathbb{R}^d} (1 + |y| + |y|^2 \delta^2) \right) \cdot \mathcal{W}_2(\bar{\mu}, \delta_0) \]

for some constant \( \Lambda \geq 0 \), independent of \( \varepsilon, n, m, \delta \), where \( \delta_0 \) is the Dirac measure centered at zero, so that \( \mathcal{W}_2(\bar{\mu}, \delta_0)^2 = \int_{\mathbb{R}^d} |x|^2 \bar{\mu}(dx) \). Hence

\[ (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq 4\delta^2 T \]

\[ + \frac{1}{n} \sum_{i=1}^{n} \sup_{a_i \in A} \left\{ \left\langle b(\bar{t}, x, \bar{\mu}, a_i), \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu})(x) \right\rangle + \frac{1}{2} \text{tr} \left[ (\sigma^\top)(\bar{t}, x, a_i) \partial_x \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu})(x) \right] \right\} \bar{\mu}(dx) \]

\[ + \frac{1}{2} \text{tr} \left[ (\sigma^\top)(\bar{t}, x, a_i) \partial^2_{x_i \bar{\mu}} \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu}) \right] \bar{\mu}(dx) \right\} \bar{\mu}(dx) \right\} \bar{\mu}(dx_1) \otimes \cdots \otimes \bar{\mu}(dx_n). \]

By formulae (A.21) and (A.22), we have

\[ \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \left\langle b(\bar{t}, x, \bar{\mu}, a), \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu})(x) \right\rangle + \frac{1}{2} \text{tr} \left[ (\sigma^\top)(\bar{t}, x, a) \partial_x \partial_\mu \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu})(x) \right] \right\} \bar{\mu}(dx) \]

\[ + \frac{1}{2} \text{tr} \left[ (\sigma^\top)(\bar{t}, x, a) \partial^2_{x_i \bar{\mu}} \bar{v}_{\varepsilon,n,m}(\bar{t}, \bar{\mu}) \right] \bar{\mu}(dx) \right\} \bar{\mu}(dx) \right\} \bar{\mu}(dx_1) \otimes \cdots \otimes \bar{\mu}(dx_n). \]

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\[\begin{align*}
&\left\{\frac{1}{n}\tilde{f}(\tilde{t}, x, \tilde{\mu}, a)\right\}\tilde{\mu}(dx_1) \cdots \tilde{\mu}(dx_{i-1}) \tilde{\mu}(dx_{i+1}) \cdots \tilde{\mu}(dx_n) \right\} \tilde{\mu}(dx) \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d(n-1)}} \sum_{i=1}^{n} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_x \tilde{\nu}_{\varepsilon,n,m}(t, x, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \rangle \right. \\
&+ \frac{1}{2} \text{tr}[(\sigma^\top)(\tilde{t}, x, a) \partial_{x,x_i} \tilde{\nu}_{\varepsilon,n,m}(t, x, \ldots, x_i, x, x_{i+1}, \ldots, x_n)] \\
&+ \frac{1}{n} \tilde{f}(\tilde{t}, x, \tilde{\mu}, a)\right\} \tilde{\mu}(dx_1) \cdots \tilde{\mu}(dx_{i-1}) \tilde{\mu}(dx_{i+1}) \cdots \tilde{\mu}(dx_n) \tilde{\mu}(dx) \\
&= \int_{\mathbb{R}^{dn}} \sum_{i=1}^{n} \sup_{a_i \in A} \left\{ \langle b(\tilde{t}, x_i, \tilde{\mu}, a_i), \partial_x \tilde{\nu}_{\varepsilon,n,m}(t, x_i) \rangle \right. \\
&+ \frac{1}{2} \text{tr}[(\sigma^\top)(\tilde{t}, x_i, a_i) \partial_{x,x_i} \tilde{\nu}_{\varepsilon,n,m}(t, x_i)] \\
&+ \frac{1}{n} \tilde{f}(\tilde{t}, x_i, \tilde{\mu}, a_i)\right\} \tilde{\mu}(dx_1) \cdots \tilde{\mu}(dx_{i-1}) \tilde{\mu}(dx_{i+1}) \cdots \tilde{\mu}(dx_n).
\end{align*}\]}

Plugging (5.11) into (5.10), we obtain

\[\begin{align*}
(u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) &\leq 4\delta^2 T \\
+ \Lambda \delta^4 \left( \int_{\mathbb{R}^d} (|y|^2 + |y|^3 + |y|^4 \delta^2) \zeta_{1/\delta}(y) dy + \mathcal{W}_2(\tilde{\mu}, \delta_0)^2 \int_{\mathbb{R}^d} (1 + |y| + |y|^2 \delta^2) \zeta_{1/\delta}(y) dy \right) \\
+ \int_{\mathbb{R}^{dn}} \sum_{i=1}^{n} \sup_{a_i \in A} \left\{ \langle b(\tilde{t}, x_i, \tilde{\mu}, a_i), \partial_x \tilde{\nu}_{\varepsilon,n,m}(t, x_i) \rangle \right. \\
&+ \frac{1}{2} \text{tr}[(\sigma^\top)(\tilde{t}, x_i, a_i) \partial_{x,x_i} \tilde{\nu}_{\varepsilon,n,m}(t, x_i)] \\
&+ \frac{1}{n} \tilde{f}(\tilde{t}, x_i, \tilde{\mu}, a_i)\right\} \tilde{\mu}(dx_1) \cdots \tilde{\mu}(dx_{i-1}) \tilde{\mu}(dx_{i+1}) \cdots \tilde{\mu}(dx_n) - \frac{1}{2} \epsilon^2 \int_{\mathbb{R}^{dn}} \sum_{i=1}^{n} \text{tr} \left[ \partial_{x,x_i}^2 \tilde{\nu}_{\varepsilon,n,m}(\tilde{t}, x_i) \right] \tilde{\mu}(dx_1) \cdots \tilde{\mu}(dx_n) \\
&\leq 4\delta^2 T \\
+ \Lambda \delta^4 \left( \int_{\mathbb{R}^d} (|y|^2 + |y|^3 + |y|^4 \delta^2) \zeta_{1/\delta}(y) dy + \mathcal{W}_2(\tilde{\mu}, \delta_0)^2 \int_{\mathbb{R}^d} (1 + |y| + |y|^2 \delta^2) \zeta_{1/\delta}(y) dy \right) \\
+ \int_{\mathbb{R}^{dn}} \sum_{i=1}^{n} \sup_{a_i \in A} \left\{ |b(\tilde{t}, x_i, \tilde{\mu}, a_i) - b_{n,m}(\tilde{t}, x_i, a_i)| \right. \\
&+ \frac{1}{n} \tilde{f}(\tilde{t}, x_i, \tilde{\mu}, a_i) - \tilde{f}_{n,m}(\tilde{t}, x_i, a_i)\right\} \tilde{\mu}(dx_1) \cdots \tilde{\mu}(dx_{i-1}) \tilde{\mu}(dx_{i+1}) \cdots \tilde{\mu}(dx_n) \\
&- \frac{1}{2} \epsilon^2 \int_{\mathbb{R}^{dn}} \sum_{i=1}^{n} \text{tr} \left[ \partial_{x,x_i}^2 \tilde{\nu}_{\varepsilon,n,m}(\tilde{t}, x_i) \right] \tilde{\mu}(dx_1) \cdots \tilde{\mu}(dx_n) \\
&\leq 4\delta^2 T \\
+ \Lambda \delta^4 \left( \int_{\mathbb{R}^d} (|y|^2 + |y|^3 + |y|^4 \delta^2) \zeta_{1/\delta}(y) dy + \mathcal{W}_2(\tilde{\mu}, \delta_0)^2 \int_{\mathbb{R}^d} (1 + |y| + |y|^2 \delta^2) \zeta_{1/\delta}(y) dy \right)
\end{align*}\]
\begin{align*}
&+ \frac{C_K}{n} e^{t_{-\delta}t} \int_{\mathbb{R}^d} \sum_{i=1}^{n} \sup_{a \in A} \left\{ |b(t_i, x_i, \tilde{\mu}, a) - b_{n,m}(\tilde{t}, \tilde{x}, a)| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n) \\
&+ \frac{1}{n} e^{t_{-\delta}t} \int_{\mathbb{R}^d} \sum_{i=1}^{n} \sup_{a \in A} \left\{ |f(t_i, x_i, \tilde{\mu}, a) - f_{n,m}(\tilde{t}, \tilde{x}, a)| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n) \\
&- \frac{1}{2} \varepsilon^2 \int_{\mathbb{R}^d} \sum_{i=1}^{n} \text{tr} \left[ \delta_{x_i x_i} \tilde{\varepsilon}_{\varepsilon, n,m}(\tilde{t}, \tilde{x}) \right] \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n),
\end{align*}

where the last inequality follows from estimate (A.11). Recalling the left estimate in (A.12), we obtain

\begin{align*}
(u_1 - v_{\varepsilon, n,m})(t_0, \mu_0) &\leq \frac{1}{2} \varepsilon^2 n d C_{n,m} e^{t_{-\delta}t} + 4\delta^2 T \\
&+ \Lambda \delta^4 \left( \int_{\mathbb{R}^d} (|y|^2 + |y|^3 + |y|^4 \delta^2) \zeta_1/\delta(y) dy + \mathcal{W}_2(\tilde{\mu}, \delta_0) \int_{\mathbb{R}^d} (1 + |y| + |y|^2 \delta^2) \zeta_1/\delta(y) dy \right) \\
&+ \frac{C_K}{n} e^{t_{-\delta}t} \int_{\mathbb{R}^d} \sum_{i=1}^{n} \sup_{a \in A} \left\{ |b(t_i, x_i, \tilde{\mu}, a) - b_{n,m}(\tilde{t}, \tilde{x}, a)| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n) \\
&+ \frac{1}{n} e^{t_{-\delta}t} \int_{\mathbb{R}^d} \sum_{i=1}^{n} \sup_{a \in A} \left\{ |f(t_i, x_i, \tilde{\mu}, a) - f_{n,m}(\tilde{t}, \tilde{x}, a)| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n) \\
&\leq \frac{1}{2} \varepsilon^2 n d C_{n,m} e^{t_{-\delta}t} + 4\delta^2 T
\end{align*}
\[ + \frac{C_K}{n} e^{i-t_0} \int_{\mathbb{R}^d} \sum_{i=1}^n \sup_{a \in A} \left\{ \left| b(\tilde{t}, x, \hat{\mu}_n, a) - b_{n,m}(\tilde{t}, \tilde{x}, a) \right| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n) \]

\[ + \frac{1}{n} e^{i-t_0} \int_{\mathbb{R}^d} \sum_{i=1}^n K \mathcal{W}_2(\tilde{\mu}, \tilde{\mu}_n) \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n) \]

\[ + \frac{1}{n} e^{i-t_0} \int_{\mathbb{R}^d} \sum_{i=1}^n \sup_{a \in A} \left\{ \left| f(\tilde{t}, x, \hat{\mu}_n, a) - f_{n,m}(\tilde{t}, \tilde{x}, a) \right| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n), \]

where the last inequality follows from the Lipschitz property of \( b \) and \( f \). Recalling (5.7) and (5.8) we find

\[ (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq \frac{1}{2} \varepsilon^2 n d C_{n,m} e^{i-t_0} + 4\delta^2 T \] (5.12)

\[ + \Lambda \delta^4 \left( \int_{\mathbb{R}^d} (|y|^2 + |y|^3 + |y|^4 \delta^2) \zeta_{1/\delta}(y) dy + \mathcal{W}_2(\tilde{\mu}, \delta_0)^2 \right) \int_{\mathbb{R}^d} (1 + |y| + |y|^2 \delta^2) \zeta_{1/\delta}(y) dy \]

\[ + (C_K + 1) K e^{i-t_0} c_d \left( \frac{1}{\delta} \sqrt{2 \lambda + 8 \delta \sqrt{2 c_d \lambda + 2 \sqrt{d + 2}}} + \mathcal{W}_2(\mu_0, \delta_0) \right) h_n \]

\[ + \frac{1}{n} c_d \int_{\mathbb{R}^{d+1}} \sum_{i=1}^n \sup_{a \in A} \left\{ \left| b(\tilde{t}, x, \hat{\mu}_n, a) - b_{n,m}(\tilde{t}, \tilde{x}, a) \right| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n) \]

Now, from the Lemma A.3 we get

\[ |b(\tilde{t}, x, \hat{\mu}_n, a) - b_{n,m}(\tilde{t}, \tilde{x}, a)| \]

\[ \leq K n^{d+1} \int_{\mathbb{R}^{d+1}} \left( |\tilde{t} - T \wedge (\tilde{t} - s)^+|^{\beta} + |y| + \frac{1}{n} \sum_{j=1}^n |y_j| \right) \zeta(ms) \prod_{j=1}^n \Phi(my_j) dy_j ds. \]

An analogous estimate holds for \( |f(\tilde{t}, x, \hat{\mu}_n, a) - f_{n,m}(\tilde{t}, \tilde{x}, a)| \). Then, plugging these estimates into (5.12) we obtain

\[ (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq \frac{1}{2} \varepsilon^2 n d C_{n,m} e^{i-t_0} + 4\delta^2 T \]

\[ + \Lambda \delta^4 \left( \int_{\mathbb{R}^d} (|y|^2 + |y|^3 + |y|^4 \delta^2) \zeta_{1/\delta}(y) dy + \mathcal{W}_2(\tilde{\mu}, \delta_0)^2 \right) \int_{\mathbb{R}^d} (1 + |y| + |y|^2 \delta^2) \zeta_{1/\delta}(y) dy \]

\[ + (C_K + 1) K e^{i-t_0} c_d \left( \frac{1}{\delta} \sqrt{2 \lambda + 8 \delta \sqrt{2 c_d \lambda + 2 \sqrt{d + 2}}} + \mathcal{W}_2(\mu_0, \delta_0) \right) h_n \]

\[ + \frac{1}{n} c_d \int_{\mathbb{R}^{d+1}} \sum_{i=1}^n \sup_{a \in A} \left\{ \left| f(\tilde{t}, x, \hat{\mu}_n, a) - f_{n,m}(\tilde{t}, \tilde{x}, a) \right| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n) \]

\[ + \frac{1}{n} \sum_{j=1}^n |y_j| \zeta(ms) \prod_{j=1}^n \Phi(my_j) dy_j ds \]
\[ \leq \frac{1}{2} \varepsilon^2 n d C_{n,m} e^{-t_0} + 4 \delta^2 T + \Lambda \delta^4 \left( \int_{\mathbb{R}^d} |y|^2 + |y|^3 + |y|^{4\delta^2} \zeta_{1/\delta}(y) dy \right) \\
+ \frac{1}{\delta^2} \left( \sqrt{2\lambda + 8\delta \sqrt{2c_d \lambda} + 2\sqrt{d + 2}} \right)^2 \int_{\mathbb{R}^d} (1 + |y| + |y|^{2\delta^2}) \zeta_{1/\delta}(y) dy \]
\[
+ (C_K + 1) K e^{-t_0 c_d} \left( \frac{1}{\delta} \sqrt{2\lambda + 8\delta \sqrt{2c_d \lambda} + 2\sqrt{d + 2}} \right) + \mathcal{W}_2(\mu_0, \delta_0) h_n \]
\[
+ (C_K + 1) K e^{t_0} m^{n+1} \int_{\mathbb{R}^{dn+1}} (|\tilde{t} - T \wedge (\tilde{t} - s) + \sum_{i=1}^n |y_i| \zeta(\delta) \prod_{j=1}^n \Phi(my_j) dy_j ds, \\
+ \frac{1}{n} \sum_{j=1}^n |y_j| \right) \zeta(\delta \Phi(\delta)) \]
\[
\text{where in the last inequality we have used again (5.8).} 
\]
\[
\text{Now, we send } \varepsilon \to 0^+ \text{ so the left hand side goes to } u_1 - v_{0,n,m} \text{ and the first term after the last inequality above goes to zero. Then we send } m \to +\infty \text{ (so the last term above goes to zero), and afterwards } n \to +\infty \text{ (the second to last term above goes to zero) and use Theorem A.6. Finally, we send } \delta \to 0^+ \text{, from which we obtain (notice that } \int_{\mathbb{R}^d} |y|^{\ell} \zeta_{1/\delta}(y) dy = \int_{\mathbb{R}^d} |z|^{\ell} \delta^{-\ell} \zeta_1(z) dz, \text{ for every } \ell \in \mathbb{N}, \text{ with } \zeta_1 \text{ given by (4.8) with } \varrho = 1.}
\]
\[
(u_1 - v_0)(t_0, \mu_0) \leq 0. 
\]

This gives a contradiction to (5.1).

**Step II. Proof of } v_0 \leq u_2. \text{ Our aim is to prove that}

\[
u_2(t, \mu) \geq \mathbb{E} \left[ \int_t^s f(r, X_r^{t,\xi,\alpha}, \mathbb{P}_{X_r^{t,\xi,\alpha}}, \mathbf{a}) \, dr \right] + u_2(s, \mathbb{P}_{X_s^{t,\xi,\alpha}}), \tag{5.13}
\]

for every \((t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), s \in [t, T], \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d), \text{ with } \mathbb{P}_{\xi} = \mu, \text{ and } \mathbf{a} \in \mathcal{M}_t, \text{ where } \mathcal{M}_t \text{ denotes the set of } \mathcal{F}_t \text{-measurable random variables } \mathbf{a} : \Omega \to A. \text{ } (X_r^{t,\xi,\alpha})_{r \in [t, T]} \text{ is equal to the process } (X_r^{t,\xi,\alpha(r)})_{r \in [t, T]} \text{ with } \alpha_r = \mathbf{a} \text{ for } r \in [t, T].
\]

To see why \(u_2 \geq v_0\) follows from (5.13), we need to introduce some notation. First of all, following [31, Definition 3.2.3], we define on \(\mathcal{A}\) the metric \(\rho_{\mathcal{K}r}\) given by

\[
\rho_{\mathcal{K}r}(\alpha, \beta) := \mathbb{E} \left[ \int_0^T |\alpha_t - \beta_t| \, dt \right], \quad \forall \alpha, \beta \in \mathcal{A}. 
\]

Recall that, by Assumption (C) \(\mathcal{A}\) is a compact subset of some Euclidean space, so that \(|\alpha_t - \beta_t|\) denotes the Euclidean distance between \(\alpha_t\) and \(\beta_t\), moreover \(|\alpha_t - \beta_t|\) is bounded by some constant which depends only on \(\mathcal{A}\). Following [31], we also define the class of step control processes for the problem starting at \(t \in [0, T]\):

\[
\mathcal{A}_{\text{step}}(t) := \left\{ \alpha \in \mathcal{A} : \text{there exist } n \in \mathbb{N} \text{ and } t = t_0 < t_1 < \cdots < t_{n-1} < t_n = T \text{ such that } \alpha_s = \alpha_{t_i}, \forall s \in [t_i, t_{i+1}), i = 0, \ldots, n-1 \right\}. 
\]

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By [31, Lemma 3.2.6], we know that $A_{\text{step}}^t$ is dense in $A$ with respect to the metric $\rho_{Kr}$. Moreover, by similar arguments as in [31, Lemma 3.2.7], it is easy to prove that the reward functional $J = J(s, \xi, \alpha)$ in (2.4) is continuous in $\alpha$ with respect to the metric $\rho_{Kr}$. Then, using the density of $A_{\text{step}}^t$ in $A$ and the continuity of the reward functional with respect to $\rho_{Kr}$, we deduce that $v_0(t, \mu)$ can be equivalently defined as the supremum of the reward functional over $A_{\text{step}}^t$ (rather than $A$). Now, let $t \in [0, T]$ and $\alpha \in A_{\text{step}}^t$, so that there exist $n \in \mathbb{N}$, $t = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, and $a_0, \ldots, a_{n-1}: \Omega \to A$, with $a_i \in \mathcal{F}_{t_i}$, such that

$$\alpha_s = a_i, \quad \forall s \in [t_i, t_{i+1}), \quad i = 0, \ldots, n - 1.$$ 

If (5.13) holds true, applying it recursively on the intervals $[t_i, t_{i+1})$, $i = 0, \ldots, n - 1$ with $a = a_0, \ldots, a_{n-1}$, we get

$$u_2(t, \mu) \geq \mathbb{E} \left[ \int_t^T f \left( r, X_r^t, \xi, \alpha_r, \mathbb{P}_{X_r^t}, \alpha_r \right) dr + g \left( X_T^t, \mathbb{P}_{X_T^t} \right) \right].$$

Since $\alpha$ was arbitrary, the above inequality holds for every $\alpha \in A_{\text{step}}^t$, proving that $u_2 \geq v_0$. It remains to prove (5.13). To this end, for every $t \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, $a \in \mathcal{M}_t$, we consider the system of uncontrolled stochastic differential equations:

$$
\begin{cases}
X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}_{X_r}, Y_r) dr + \int_t^s \sigma(r, X_r, Y_r) dB_r, & s \in [t, T], \\
Y_s = a, & s \in [t, T].
\end{cases}
$$

We denote by $(X_t^{t, \xi, a}, Y_t^{t, a})$ the unique solution to the above system of equations. Then, fixed $t \in [0, T]$, $s \in [t, T]$, we set

$$v^s(t, \nu) := \mathbb{E} \left[ \int_t^s f \left( r, X_r^{t, \xi, a}, \mathbb{P}_{X_r^{t, \xi, a}}, Y_r^{t, a} \right) dr \right] + u_2(s, \mathbb{P}_{X_s^{t, \xi, a}}),$$

for all $(t, \nu) \in [t, s] \times \mathcal{P}_d(\mathbb{R}^d \times A)$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ and $a \in \mathcal{M}_t$ such that $\mathbb{P}_{(t, \nu)} = \nu$. Then, our aim is to prove that $u_2(t, \mu) \geq v^s(t, \nu)$, for every $(t, \nu) \in [t, s] \times \mathcal{P}_d(\mathbb{R}^d \times A)$, with $\mu$ being the first marginal of $\nu$, from which we get (5.13) for $t = t$.

Suppose for a moment that $u_2(s, \cdot)$ is Lipschitz continuous. Then, reasoning as in the proof of Proposition 2.5, we obtain that $v^s$ is bounded and Lipschitz continuous. If $u_2(s, \cdot)$ is not Lipschitz continuous, following [3, formula (5.1.4)] we can pointwise approximate $u_2(s, \cdot)$ from below with an increasing sequence of bounded Lipschitz functions $u_k$

$$u_{2,k}(\mu) := \inf_{\nu \in \mathcal{P}_d(\mathbb{R}^d)} \left\{ u_2(s, \nu) + k\mathcal{W}_2(\mu, \nu) \right\},$$

with

$$\left\{ \begin{array}{l}
\inf_{\nu \in \mathcal{P}_d(\mathbb{R}^d)} u_2(s, \nu) \leq u_{2,k}(\mu) \leq u_2(s, \mu) \leq \sup_{\nu \in \mathcal{P}_d(\mathbb{R}^d)} u_2(s, \nu), \\
u_2(s, \mu) = \lim_{k \to \infty} u_{2,k}(\mu) = \sup_{k \in \mathbb{N}} u_{2,k}(\mu).
\end{array} \right.$$ 

Then, we define

$$v^s_k(t, \nu) := \mathbb{E} \left[ \int_t^s f \left( r, X_r^{t, \xi, a}, \mathbb{P}_{X_r^{t, \xi, a}}, Y_r^{t, a} \right) dr \right] + u_{2,k}(\mathbb{P}_{X_s^{t, \xi, a}}).$$
If we prove that \( u_2(t, \mu) \geq v^s_k(t, \nu) \), for every \( k \in \mathbb{N} \), then sending \( k \to \infty \) we conclude that \( u_2(t, \mu) \geq v^s(t, \nu) \). In what follow we suppose that \( u_2(s, \cdot) \) is Lipschitz continuous and therefore we consider the function \( v^s \) and we prove that \( u_2(t, \mu) \geq v^s(t, \nu) \). This is not a loss of generality. As a matter of fact, if \( u_2(s, \cdot) \) is not Lipschitz continuous we repeat the same arguments reported below to \( v^s_k \) instead of \( v^s \), therefore proving that \( u_2(t, \mu) \geq v^s_k(t, \nu) \), for every \( k \in \mathbb{N} \). As already noticed, from the arbitrariness of \( k \), we conclude that \( u_2(t, \mu) \geq v^s(t, \nu) \).

Now, let us prove \( u_2(t, \mu) \geq v^s(t, \nu) \), for every \( t \in [0, T) \), \( s \in (t, T) \), \((t, \nu) \in [t, s] \times \mathcal{P}_2(\mathbb{R}^d \times A) \), with \( \mu \) being the first marginal of \( \nu \). We proceed by contradiction and suppose that there exist \( \tilde{t}_0 \in [0, T), \tilde{s}_0 \in (\tilde{t}_0, T], (\tilde{t}_0, \mu_0, \nu_0) \in [\tilde{t}_0, \tilde{s}_0) \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d \times A) \), with \( \mu_0 \) being the first marginal of \( \nu_0 \), such that

\[
 v^s(t_0, \nu_0) > u_2(t_0, \mu_0).
\]

As in Step I, we can suppose that there exists some \( q > 2 \) such that \( \nu_0 \in \mathcal{P}_q(\mathbb{R}^d) \).

For every \( n, m \in \mathbb{N} \), let \( v_{n,m}^s \) be the map given by (A.24). Now, we define \( \tilde{u}_2(t, \mu) := e^{t-t_0}u_2(t, \mu) \), for every \( (t, \mu) \in [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d) \), and similarly \( \tilde{v}_{n,m}^s \) from \( v_{n,m}^s \), \( f \), respectively. We also define \( \tilde{g}(x, \mu) := e^{t-t_0}g(x, \mu) \) and \( \tilde{u}_{n,m}(s_0, \mu) := e^{s_0-t_0}u_{n,m}(s_0, \mu) \), for every \( (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \). We observe that, given \( a_0 \in \mathcal{M}_t \) with distribution being equal to the marginal of \( \nu_0 \) on \( A \), \( \tilde{u}_2 \) is a viscosity supersolution of the following equation (see e.g. [16, Section 7]):

\[
 \begin{aligned}
 & \partial_t \tilde{u}_2(t, \mu) + \mathbb{E} \left\{ f(t, \xi, \mu, a_0) + \frac{1}{2} \text{tr} \left[ (\sigma \sigma^t)(t, \xi, a_0) \partial_{\sigma \sigma^t} \tilde{u}_2(t, \mu)(\xi) \right] \right. \\
 & + \left. \left\langle b(t, \xi, \mu, a_0), \partial_{\mu} \tilde{u}_2(t, \mu)(\xi) \right\rangle \right\} = \tilde{u}_2(t, \mu), \quad (t, \mu) \in [t_0, s_0) \times \mathcal{P}_2(\mathbb{R}^d), \\
 & \tilde{u}_2(s_0, \mu) = \tilde{u}_2(s_0, \mu), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d),
 \end{aligned}
\]

for any \( \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d) \), with \( \mathbb{P}_\xi = \mu \). Moreover, by Theorem A.8 we deduce that \( \tilde{v}_{n,m}^s \) solves the following equation:

\[
 \begin{aligned}
 & \partial_t \tilde{v}_{n,m}^s(t, \nu) + \mathbb{E} \sum_{i=1}^{n} \left\{ \frac{1}{n} \tilde{f}_{n,m}(t, \xi_i, \ldots, \xi_n, a_0^i) + \left\langle \tilde{b}_{n,m}(t, \xi_i, \ldots, \xi_n, a_0^i), \partial_{\xi_i} \tilde{v}_{n,m}^s(t, \xi_i, a_0) \right\rangle \right\} \\
 & + \frac{1}{2} \text{tr} \left[ (\sigma \sigma^t)(t, \xi_i, a_0^i) \partial_{\xi_i \xi_i} \tilde{v}_{n,m}^s(t, \xi_i, \tilde{a}_0) \right] = \tilde{v}_{n,m}^s(t, \nu), \quad (t, \nu) \in [t_0, s_0) \times \mathcal{P}_2(\mathbb{R}^d \times A), \\
 & \tilde{v}_{n,m}^s(s_0, \nu) = \mathbb{E} \left[ \tilde{u}_{n,m}(s_0, \tilde{\xi}) \right], \quad \nu \in \mathcal{P}_2(\mathbb{R}^d \times A),
 \end{aligned}
\]

for any \( \tilde{\xi} = (\xi_1, \ldots, \xi_n) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{dn}) \) and \( a_0 = (a_0^1, \ldots, a_0^n) \), with \( a_0 \in \mathcal{M}_t \), such that 

\[
 \mathbb{P}_{(\tilde{\xi}, a_0)} = \nu \otimes \cdots \otimes \nu,
\]

where \( \tilde{v}_{n,m}(t, \tilde{x}, \tilde{a}) := e^{t-t_0}\tilde{v}_{n,m}^s(t, \tilde{x}, a_0) \), for every \( (t, \tilde{x}, a_0) \in [t_0, s_0) \times (\mathbb{R}^d \times A)^n \), with \( \tilde{v}_{n,m}^s \) being the same function appearing in Theorem A.8.

In the sequel it is useful to see at \( u_2 \) as a function on \([0, T] \times \mathcal{P}_2(\mathbb{R}^d \times A)\) rather than \([0, T] \times \mathcal{P}_2(\mathbb{R}^d)\). In other words, it is useful to consider the function

\[
 \tilde{u}_2(t, \nu) := u_2(t, \mu), \quad \forall (t, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d),
\]

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with \( \mu \) being the first marginal of \( \nu \). To avoid introducing additional notations, we denote \( \hat{u}_2 \) still by \( u_2 \).

Now, notice that \( v^{s_0}_{n,m} \) is bounded by a constant independent of \( n, m \). As a consequence, there exists \( \lambda \geq 0 \), independent of \( n, m \), satisfying

\[
\sup_{[0,T] \times \mathcal{P}_2(\mathbb{R}^d \times A)} (\hat{v}^{s_0}_{n,m} - \hat{u}_2) \leq (\hat{v}^{s_0}_{n,m} - \hat{u}_2)(t_0, \nu_0) + \lambda. \tag{5.14}
\]

Since \( \hat{v}^{s_0}_{n,m} - \hat{u}_2 \) is bounded and continuous, by (5.14) and Theorem 4.5 applied on \([t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)\) with \( G = \hat{v}^{s_0}_{n,m} - \hat{u}_2 \), we obtain that for every \( \delta > 0 \) there exist \( \{(t_k, \nu_k)\}_{k \geq 1} \subset [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A) \) converging to some \( (\hat{t}, \hat{\nu}) \in [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A) \) and \( \varphi_\delta \) such that items (i)-(ii)-(iii) and 1)-2)-3) of Theorem 4.5 hold.

Now, as in the proof of STEP I we distinguish two cases. If \( \hat{t} = s_0 \) and, in addition, \( s_0 = T \), then we proceed as in SUBSTEP I-C to get a contradiction. On the other hand, if \( s_0 < T \), then we proceed as in SUBSTEP I-D in order to find a contradiction and conclude the proof.

**Corollary 5.2 (Uniqueness).** Let Assumptions (A), (B), (C), (D) hold. Then, the value function \( v \), given by (2.6), is the unique bounded and continuous viscosity solution of equation (3.3).

**Proof.** From Proposition 2.5 and Theorem 3.8 we know that \( v \) is bounded, continuous, and it is a viscosity solution of equation (3.3). Now, let \( u \) be another bounded and continuous viscosity solution of equation (3.3). Then, by Theorem 5.1 we deduce that \( u \leq v \) and \( v \leq u \) (in fact, both \( v \) and \( u \) are viscosity sub/supersolution of equation (3.3)), from which we conclude that \( v \equiv u \).

**A Smooth finite-dimensional approximations of the value function**

**A.1 Mean field control problem on a different probabilistic setting and approximation by non-degenerate control problems**

In the present appendix we formulate the mean field control problem on a different probabilistic setting, supporting an independent \( d \)-dimensional Brownian motion \( \hat{W} \).

Let \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \) be a complete probability space on which a \( m \)-dimensional Brownian motion \( \hat{B} = (\hat{B}_t)_{t \geq 0} \) and a \( d \)-dimensional Brownian motion \( \hat{W} = (\hat{W}_t)_{t \geq 0} \) are defined, with \( \hat{B} \) and \( \hat{W} \) being independent. We denote by \( \hat{\mathcal{F}}_{\infty}^{B,W} = (\hat{\mathcal{F}}_{t}^{B,W})_{t \geq 0} \) the \( \hat{P} \)-completion of the filtration generated by \( \hat{B} \) and \( \hat{W} \). We also assume that there exists a sub-\( \sigma \)-algebra \( \hat{\mathcal{G}} \) of \( \hat{\mathcal{F}} \) satisfying the following properties.

i) \( \hat{\mathcal{G}} \) and \( \hat{\mathcal{F}}_{\infty}^{B,W} \) are independent.

ii) \( \mathcal{P}_2(\mathbb{R}^d) = \{P_\xi \text{ such that } \xi: \hat{\Omega} \to \mathbb{R}^d, \text{ with } \xi \text{ being } \hat{\mathcal{G}} \text{-measurable and } \hat{E}|\xi|^2 < \infty\} \).
We denote by $\hat{\mathbb{P}} = (\hat{\mathcal{F}}_t)_{t \geq 0}$ the $\hat{\mathbb{P}}$-completed filtration of $(\hat{\mathcal{G}} \vee \hat{\mathcal{F}}_t^{B,W})_{t \geq 0}$, for all $t \geq 0$. Finally, we denote by $\hat{\mathcal{A}}$ the set of control processes, namely the family of all $\hat{\mathbb{P}}$-progressively measurable processes $\hat{\alpha} : [0, T] \times \hat{\Omega} \to A$.

Now, for every $\varepsilon \geq 0$, $t \in [0, T]$, $\hat{\xi} \in L^2(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}; \mathbb{R}^d)$, $\hat{\alpha} \in \hat{\mathcal{A}}$, let $\hat{X}^{\varepsilon,t,\hat{\xi},\hat{\alpha}} = (\hat{X}^{\varepsilon,t,\hat{\xi},\hat{\alpha}}_s)_{s \in [t, T]}$ be the unique solution to the following controlled McKean-Vlasov stochastic differential equation:

$$\hat{X}_s = \hat{\xi} + \int_t^s b(r, \hat{X}_r, \mathbb{P}_{\hat{X}_r}, \hat{\alpha}_r) \, dr + \int_t^s \sigma(r, \hat{X}_r, \hat{\alpha}_r) \, d\hat{B}_r + \varepsilon (\hat{W}_s - \hat{W}_t), \quad \forall s \in [t, T].$$

Moreover, consider the lifted value function

$$V_\varepsilon(t, \hat{\xi}) = \sup_{\hat{\alpha} \in \hat{\mathcal{A}}} \hat{\mathbb{E}} \left[ \int_t^s f(r, \hat{X}_r^{\varepsilon,t,\hat{\xi},\hat{\alpha}}, \mathbb{P}_{\hat{X}_r^{\varepsilon,t,\hat{\xi},\hat{\alpha}}}, \hat{\alpha}_r) \, dr + g(\hat{X}_T^{\varepsilon,t,\hat{\xi},\hat{\alpha}}, \mathbb{P}_{\hat{X}_T^{\varepsilon,t,\hat{\xi},\hat{\alpha}}}) \right],$$

for every $t \in [0, T]$, $\hat{\xi} \in L^2(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}; \mathbb{R}^d)$. Under Assumption (A), from Theorem 2.4 applied in the present probabilistic setting, with $\sigma$ and $\hat{B}$ replaced respectively by $(\sigma, \varepsilon I_d)$ and $(\hat{B}, \hat{W})$, we know that $V_\varepsilon$ satisfies the law invariance property. Therefore we can define the value function $v_\varepsilon : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ as follows:

$$v_\varepsilon(t, \mu) = V_\varepsilon(t, \hat{\xi}), \quad (A.1)$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and any $\hat{\xi} \in L^2(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}; \mathbb{R}^d)$ such that $\mathbb{P}_{\hat{\xi}} = \mu$. Moreover, applying Proposition 2.5 in the present probabilistic setting, it follows that $v_\varepsilon$ is bounded, jointly continuous on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and Lipschitz continuous in the measure: there exists $L > 0$ such that

$$|v_\varepsilon(t, \mu) - v_\varepsilon(t', \mu')| \leq LW_2(\mu, \mu'),$$

for any $t \in [0, T]$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$.

**Remark A.1.** Notice that, under Assumption (A), it is not immediately clear if $v_0 \equiv v$. However, under Assumptions (A) and (B), applying Theorem 3.8 in the present probabilistic setting we deduce that $v_0$ is a viscosity solution of the Master Bellman equation (3.3). As a consequence, under Assumptions (A)-(B)-(C)-(D), by Corollary 5.2 we conclude that $v_0 \equiv v$.

**Lemma A.2.** Suppose that Assumption (A) holds. Then, there exists a constant $C_{K,T} \geq 0$, depending only on $K$ and $T$, such that, for every $\varepsilon \geq 0$,

$$|v_\varepsilon(t, \mu) - v_\varepsilon(t, \mu)| \leq C_{K,T} \varepsilon,$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$. 

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Proof. By usual calculations (as in [31, Theorem 2.5.9]), we obtain

$$
\hat{E}\left[ \sup_{t \leq s \leq T} |\hat{X}_{s \xi}^{\epsilon, t, \xi, \hat{a}} - \hat{X}_{s}^{0, t, \xi, \hat{a}}|^2 \right] \leq C_K T e^{C_K T} \varepsilon^2,
$$

(A.2)

for every $\varepsilon \geq 0$, $t \in [0, T]$, $\hat{\xi} \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}_t; \tilde{P}; \mathbb{R}^d)$, $\hat{a} \in \hat{A}$, for some constant $C_K \geq 0$, depending only on $K$. Then, we have

$$
|v_\varepsilon(t, \mu) - v_0(t, \mu)| \leq \sup_{\hat{a} \in \hat{A}} \hat{E} \left[ \int_t^T \left| f(s, \hat{X}_{s \xi}^{\epsilon, t, \xi, \hat{a}}, \mathbb{P}_{\hat{X}_{s \xi}^{\epsilon, t, \xi, \hat{a}}}, \hat{a}_s) - f(s, \hat{X}_{s}^{0, t, \xi, \hat{a}}, \mathbb{P}_{\hat{X}_{s}^{0, t, \xi, \hat{a}}}, \hat{a}_s) \right| ds 
+ \left| g\left(\hat{X}_{T \xi}^{\epsilon, t, \xi, \hat{a}}, \mathbb{P}_{\hat{X}_{T \xi}^{0, t, \xi, \hat{a}}}\right) - g\left(\hat{X}_{T}^{0, t, \xi, \hat{a}}, \mathbb{P}_{\hat{X}_{T}^{0, t, \xi, \hat{a}}}\right) \right| \right]

\leq K \sup_{\hat{a} \in \hat{A}} \left\{ \int_t^T \left\{ \hat{E}\left[|\hat{X}_{s \xi}^{\epsilon, t, \xi, \hat{a}} - \hat{X}_{s}^{0, t, \xi, \hat{a}}|^2\right]^{1/2} + \hat{E}\left[|\hat{X}_{s \xi}^{\epsilon, t, \xi, \hat{a}} - \hat{X}_{s}^{0, t, \xi, \hat{a}}|^2\right]^{1/2} \right\} ds 
+ \hat{E}\left[|\hat{X}_{s \xi}^{\epsilon, t, \xi, \hat{a}} - \hat{X}_{s}^{0, t, \xi, \hat{a}}|^2\right]^{1/2} + \hat{E}\left[|\hat{X}_{s \xi}^{\epsilon, t, \xi, \hat{a}} - \hat{X}_{s}^{0, t, \xi, \hat{a}}|^2\right]^{1/2} \right\} \right\}

\leq 2K(T + 1) \sqrt{C_K T e^{C_K T}} \varepsilon,
$$

where the last inequality follows from estimate (A.8). \qed

A.2 Cooperative $n$-player stochastic differential game and propagation of chaos result

Let $n \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, \hat{P})$ be a complete probability space, supporting independent Brownian motions $\mathcal{B}^1, \ldots, \mathcal{B}^n, \mathcal{W}^1, \ldots, \mathcal{W}^n$, with $\mathcal{B}^i$ (resp. $\mathcal{W}^i$) being $m$-dimensional (resp. $d$-dimensional). Let also $\mathcal{F}^{B,W}_{\infty} = (\mathcal{F}^{B,W}_{t})_{t \geq 0}$ denote the $\mathcal{P}$-completion of the filtration generated by $\mathcal{B}$ and $\mathcal{W}$, with $\mathcal{B} = (\mathcal{B}^1, \ldots, \mathcal{B}^n)$ and $\mathcal{W} = (\mathcal{W}^1, \ldots, \mathcal{W}^n)$. Moreover, let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$ satisfying the following properties.

i) $\mathcal{G}$ and $\mathcal{F}^{B,W}_{\infty}$ are independent.

ii) $\mathcal{P}_2(\mathbb{R}^d) = \{\mathbb{P}_\xi \text{ such that } \tilde{\xi} : \tilde{\Omega} \to \mathbb{R}^d, \text{ with } \tilde{\xi} \text{ being } \mathcal{G} \text{-measurable and } \hat{E}|\tilde{\xi}|^2 < \infty\}$.

Furthermore, let $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ be given by $\mathcal{F}_t := \mathcal{G} \vee \mathcal{F}^{B,W}_{t}$, for every $t \geq 0$. Finally, let $\mathcal{A}^n$ be the family of all $\mathcal{P}$-progressively measurable processes $\hat{a} = (\hat{a}^1, \ldots, \hat{a}^n) : [0, T] \times \tilde{\Omega} \to \Lambda^n$. Now, for every $\varepsilon > 0$, $t \in [0, T]$, $\hat{a} \in \mathcal{A}^n$, $\hat{\xi}^1, \ldots, \hat{\xi}^n \in L^2(\tilde{\Omega}, \mathcal{F}_t, \hat{P}; \mathbb{R}^d)$, with $\hat{\xi} := \ldots$.
(ξ₁, ..., ξₙ), let \( \bar{X}^{ε,t,ξ,α} = (\bar{X}^{1,ε,t,ξ,α}, ..., \bar{X}^{n,ε,t,ξ,α}) \) be the unique solution to the following system of controlled stochastic differential equations:

\[
\bar{X}_s^i = \bar{ξ}^i + \int_t^s b(r, \bar{X}_r^i, \bar{µ}_r, \bar{α}_r) \, dr + \int_t^s \sigma(r, \bar{X}_r^i, \bar{α}_r) \, dB_r^i + \varepsilon (\bar{W}_s^i - \bar{W}_t^i), \quad \forall s \in [t, T], \quad (A.3)
\]

for \( i = 1, ..., n \), with

\[
\bar{µ}_r^i = \frac{1}{n} \sum_{j=1}^n \delta_{\bar{X}_r^j}, \quad \forall r \in [t, T].
\]

We denote \( \tilde{µ}^{n,ε,t,ξ,α} = \frac{1}{n} \sum_{j=1}^n \delta_{\bar{X}_r^j, ε,t,ξ,α} \). We consider the cooperative \( n \)-players game where a planner maximizes, over \( \bar{α} \in \bar{A}^n \), the payoff

\[
\tilde{J}_{\varepsilon,n}(t, \bar{µ}; \bar{α}) := \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \int_t^T f(s, \bar{X}_s^{i,ε,t,ξ,α}, \bar{µ}_s^{n,ε,t,ξ,α}, \bar{α}_s^i) \, ds + g(\bar{X}_T, \bar{µ}_T, \bar{α}_T) \right], \quad (A.4)
\]

Then, the value function \( \tilde{v}_{\varepsilon,n}: [0, T] \times \mathcal{P}_2(\mathbb{R}^{dn}) \to \mathbb{R} \) of such cooperative \( n \)-player game is given by

\[
\tilde{v}_{\varepsilon,n}(t, \bar{µ}) = \sup_{\bar{α} \in \bar{A}^n} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \int_t^T f(s, \bar{X}_s^{i,ε,t,ξ,α}, \bar{µ}_s^{n,ε,t,ξ,α}, \bar{α}_s^i) \, ds + g(\bar{X}_T, \bar{µ}_T, \bar{α}_T) \right], \quad (A.4)
\]

for every \( t \in [0, T] \), \( \bar{µ} \in \mathcal{P}_2(\mathbb{R}^{dn}) \), with \( \bar{ξ} \in L^2(\bar{Ω}, \bar{ℱ}_T, \bar{ℙ}; \mathbb{R}^{dn}) \) such that \( \bar{ℙ}_\xi = \bar{µ} \). We also introduce the following approximation of the value function: we call

\[
\tilde{v}_{\varepsilon,n,m}: [0, T] \times \mathcal{P}_2(\mathbb{R}^{dn}) \to \mathbb{R}
\]

the map given by

\[
\tilde{v}_{\varepsilon,n,m}(t, \bar{µ}) = \sup_{\bar{α} \in \bar{A}^n} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[ \int_t^T f_{n,m}^i(s, \bar{X}_s^{1,ε,t,ξ,α}, ..., \bar{X}_s^{n,ε,t,ξ,α}, \bar{α}_s^i) \, ds \right.
\]

\[
\left. + g_{n,m}^i(\bar{X}_T^{1,ε,t,ξ,α}, ..., \bar{X}_T^{n,ε,t,ξ,α}) \right],
\]

where \( \bar{X}_s^{i,m,ε,t,ξ,α} \) solves equation (A.3) with \( b \) replaced by \( b_{n,m}^i \), where

\[
b_{n,m}^i: [0, T] \times \mathbb{R}^{dn} \times A \to \mathbb{R}^d, \quad f_{n,m}^i: [0, T] \times \mathbb{R}^{dn} \times A \to \mathbb{R}, \quad g_{n,m}^i: \mathbb{R}^{dn} \to \mathbb{R}
\]

are smooth approximations of \( b, f, g \) defined as follows:

\[
b_{n,m}^i(t, \bar{x}, a) = m^{nd+1} \int_{\mathbb{R}^{dn+1}} b \left( T \wedge (t-s)^+, x_i - y_i, \frac{1}{n} \sum_{j=1}^n \delta_{x_j-y_j}, a \right) \zeta(ms) \prod_{j=1}^n \Phi(my_j) dy_j ds,
\]

\[
f_{n,m}^i(t, \bar{x}, a) = m^{nd+1} \int_{\mathbb{R}^{dn+1}} f \left( T \wedge (t-s)^+, x_i - y_i, \frac{1}{n} \sum_{j=1}^n \delta_{x_j-y_j}, a \right) \zeta(ms) \prod_{j=1}^n \Phi(my_j) dy_j ds,
\]

38
\[ g_{n,m}^i(\bar{x}) = m^{nd} \int_{\mathbb{R}^{dn}} g\left(x_i - y_i, \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j - y_j}\right) \prod_{j=1}^{n} \Phi(my_j)dy_j, \]

for all \( n, m \in \mathbb{N}, i = 1, \ldots, n, \bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^{dn}, (t, a) \in [0, T] \times A, \) with \( \Phi: \mathbb{R}^{d} \to [0, +\infty) \) and \( \zeta: \mathbb{R} \to [0, +\infty) \) being \( C^\infty \) functions with compact support satisfying \( \int_{\mathbb{R}^{d}} \Phi(y)dy = 1 \) and \( \int_{-\infty}^{+\infty} \zeta(s)ds = 1. \)

**Lemma A.3.** Suppose that Assumptions (A) and (B) hold. Let \( \hat{\mu}^{n,\bar{x}} \) be given by

\[ \hat{\mu}^{n,\bar{x}} := \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j} \]

Let \( i = 1, \ldots, n. \) We have

\[ \lim_{m\to+\infty} b_{n,m}^i(t, \bar{x}, a) = b(t, x_i, \hat{\mu}^{n,\bar{x}}), \quad \lim_{m\to+\infty} f_{n,m}^i(t, \bar{x}, a) = f(t, x_i, \hat{\mu}^{n,\bar{x}}), \]

uniformly for \((t, \bar{x}, a)\) in \([0, T] \times \mathbb{R}^{dn} \times A.\) Moreover,

\[ \lim_{m\to+\infty} g_{n,m}^i(\bar{x}) = g(x_i, \hat{\mu}^{n,\bar{x}}), \]

uniformly for \( \bar{x} \) in \( \mathbb{R}^{dn}. \)

Furthermore we have the following estimates

\[ |b(t, x_i, \hat{\mu}^{n,\bar{x}}, a) - b_{n,m}^i(t, \bar{x}, a)| \leq Km \int_{\mathbb{R}} |t - (T \wedge (t - s)^{+})|^{\beta} \zeta(ms)ds \]

\[ + K m^{nd} \int_{\mathbb{R}^{dn}} \left(|y_1| + \frac{1}{n} \sum_{j=1}^{n} |y_j|\right) \prod_{j=1}^{n} \Phi(my_j)dy_j \]

\[ |f(t, x_i, \hat{\mu}^{n,\bar{x}}, a) - f_{n,m}^i(t, \bar{x}, a)| \leq Km \int_{\mathbb{R}} |t - (T \wedge (t - s)^{+})|^{\beta} \zeta(ms)ds \]

\[ + K m^{nd} \int_{\mathbb{R}^{dn}} \left(|y_1| + \frac{1}{n} \sum_{j=1}^{n} |y_j|\right) \prod_{j=1}^{n} \Phi(my_j)dy_j \]

\[ |g(x_i, \hat{\mu}^{n,\bar{x}}) - g_{n,m}(x_1, \ldots, x_n)| \leq K m^{nd} \int_{\mathbb{R}^{dn}} \left(|y_1| + \frac{1}{n} \sum_{j=1}^{n} |y_j|\right) \prod_{j=1}^{n} \Phi(my_j)dy_j \quad (A.5) \]

Finally

\[ |b_{n,m}^i(t, \bar{x}, a) - b_{n,m}^i(t, \bar{z}, a)| \vee |f_{n,m}^i(t, \bar{x}, a) - f_{n,m}^i(t, \bar{z}, a)| \vee |g_{n,m}^i(\bar{x}) - g_{n,m}^i(\bar{z})| \]

\[ \leq K \left[ |x_i - z_i| + \frac{1}{n} \sum_{j=1}^{n} |x_j - z_j| \right]. \quad (A.6) \]
Proof. We first prove the claims for $g$. From the definition of $g^{i}_{n,m}$ we get
\[
\left|g(x_{i}, \hat{\mu}_{n,x}^{i}) - g_{n,m}(x_{1}, \ldots, x_{n})\right|
\leq m^{nd} \int_{\mathbb{R}^{dn}} \left|g(x_{i}, \hat{\mu}_{n,x}^{i}) - g(x_{i} - y_{i}, \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j} - y_{j}})\right| \prod_{j=1}^{n} \Phi(my_{j})dy_{j}
\leq K m^{nd} \int_{\mathbb{R}^{dn}} \left(|y_{i}| + \frac{1}{n} \sum_{j=1}^{n} |y_{j}|\right) \prod_{j=1}^{n} \Phi(my_{j})dy_{j},
\]
where the last inequality follows from the Lipschitz continuity of $g$, and also from the following:
\[
\mathcal{W}_{2}\left(\hat{\mu}_{n,x}^{i}, \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j} - y_{j}}\right) = \mathcal{W}_{2}\left(\frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j}}, \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j} - y_{j}}\right) \leq \frac{1}{n} \sum_{j=1}^{n} |y_{j}|,
\]
where the last inequality follows from the definition of $\mathcal{W}_{2}$ (see (2.1)) taking the probability measure $\pi$ on $\mathbb{R}^{d} \times \mathbb{R}^{d}$ satisfying $\pi(\{(x_{j}, x_{j} - y_{j})\}) = \frac{1}{n}, \forall j = 1, \ldots, n$.

For the other claim on $g$ we use that
\[
\left|g_{n,m}^{i}(\bar{x}) - g_{n,m}^{i}(\bar{z})\right| \leq m^{nd} \int_{\mathbb{R}^{dn}} \left|g(x_{i} - y_{i}, \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j} - y_{j}}) - g(z_{i} - y_{i}, \frac{1}{n} \sum_{j=1}^{n} \delta_{y_{j} - y_{j}})\right| \prod_{j=1}^{n} \Phi(my_{j})dy_{j} \leq K \left[|x_{i} - z_{i}| + \frac{1}{n} \sum_{j=1}^{n} |x_{j} - z_{j}|\right],
\]
for every $i = 1, \ldots, n$, where the last inequality follows from the Lipschitz continuity of $g$, and also from the following:
\[
\mathcal{W}_{2}\left(\frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j} - y_{j}}, \frac{1}{n} \sum_{j=1}^{n} \delta_{y_{j} - y_{j}}\right) \leq \frac{1}{n} \sum_{j=1}^{n} |x_{j} - z_{j}|,
\]
which follows from the definition of $\mathcal{W}_{2}$ (see (2.1)) taking the probability measure $\pi$ on $\mathbb{R}^{d} \times \mathbb{R}^{d}$ such that $\pi(\{(x_{j} - y_{j}, x_{j} - y_{j})\}) = \frac{1}{n}, \forall j = 1, \ldots, n$.

Now we prove the claims for $b$ (the ones for $f$ are proved exactly in the same way)
\[
b(t, x_{i}, \hat{\mu}_{n,x}^{i}, a) - b^{i}_{n,m}(t, \bar{x}, a) \leq m^{nd+1} \int_{\mathbb{R}^{dn+1}} \left|b(t, x_{i}, \hat{\mu}_{n,x}^{i}, a) - b\left(T \land (t - s)^{+}, x_{i} - y_{i}, \frac{1}{n} \sum_{j=1}^{n} \delta_{x_{j} - y_{j}}, a\right)\right| \zeta(m) \prod_{j=1}^{n} \Phi(my_{j})dy_{j}ds \leq m \int_{\mathbb{R}} \left|b(t, x_{i}, \hat{\mu}_{n,x}^{i}, a) - b(T \land (t - s)^{+}, x_{i}, \hat{\mu}_{n,x}^{i}, a)\right| \zeta(ms) \prod_{j=1}^{n} \Phi(my_{j})dy_{j}ds + m^{nd} \int_{\mathbb{R}^{dn}} \left|b(T \land (t - s)^{+}, x_{i}, \hat{\mu}_{n,x}^{i}, a)\right| \zeta(ms) \prod_{j=1}^{n} \Phi(my_{j})dy_{j}ds.
\]
Thanks to the Lipschitz property of \( b \) (see Assumption (A)-(ii)) the second integral is estimated as in the case of \( g \) considered above. Moreover, the first integral, thanks to the Assumption (B) is estimated by \( K|t-(T \land (t-s)^+)|^\beta \). Finally, the Lipschitz estimates for \( b \) and \( f \) are proved exactly in the same way as for \( g \) using Assumption (A) (ii)-(iii).

\[ \square \]

**Remark A.4.** Notice that it is not a priori clear the fact that the right-hand side of (A.4) depends on \( \xi \) only through its law \( \bar{\mu} \). However, as the cooperative \( n \)-player game is an example of mean field control problem (indeed, it is a standard stochastic optimal control problem) we can apply the results of Section 2 to it. In particular, from Theorem 2.4 we deduce the law invariance property, which explains why we can consider the value function \( \bar{v}_{\varepsilon,n} \) (and, similarly, \( \bar{v}_{\varepsilon,n,m} \)), which depends only on \( \bar{\mu} \) rather than on \( \xi \).

We also consider the functions \( v_{\varepsilon,n,m}, v_{\varepsilon,n} : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) defined as

\[ v_{\varepsilon,n,m}(t,\mu) = \bar{v}_{\varepsilon,n,m}(t,\mu \otimes \cdots \otimes \mu) \quad \text{and} \quad v_{\varepsilon,n}(t,\mu) = \bar{v}_{\varepsilon,n}(t,\mu \otimes \cdots \otimes \mu), \quad (A.7) \]

for every \( (t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \).

We first show that the analogous of Lemma A.2 holds for \( v_{\varepsilon,n,m} \).

**Lemma A.5.** Suppose that Assumption (A) holds. Then, there exists a constant \( C_{K,T} \geq 0 \), depending only on \( K \) and \( T \), such that, for every \( \varepsilon \geq 0 \),

\[ |v_{\varepsilon,n,m}(t,\mu) - v_{0,n,m}(t,\mu)| \leq C_{K,T} \varepsilon, \]

for every \( (t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \).

**Proof.** The proof is similar to the one of Lemma A.2. We provide a sketch for the reader convenience. By usual calculations (as in [31, Theorem 2.5.9]), we obtain

\[ \mathbb{E} \left[ \sup_{t \leq s \leq T} \left| \bar{X}^{i,m,0,t,\bar{\xi},\bar{\alpha}}_s - \bar{X}^{i,m,0,t,\bar{\xi},\bar{\alpha}}_s \right|^2 \right] \leq C_{K,T} e^{C_{K,T} \varepsilon^2}, \quad (A.8) \]

for every \( i = 1, \ldots, n \), \( m \in \mathbb{N}, \varepsilon \geq 0, t \in [0,T], \bar{\xi} \in L^2(\bar{\Omega},\mathcal{F}_t,\mathbb{P};\mathbb{R}^d), \bar{\alpha} \in \bar{\mathcal{A}}^n \), for some constant \( C_{K} \geq 0 \), depending only on \( K \). Then, we have, writing \( \bar{X}^{m,\varepsilon,t,\bar{\xi},\bar{\alpha}}_s \) for \( \bar{X}^{1,m,\varepsilon,t,\bar{\xi},\bar{\alpha}}_s, \ldots, \bar{X}^{n,m,\varepsilon,t,\bar{\xi},\bar{\alpha}}_s \),

\[ |v_{\varepsilon,n,m}(t,\mu) - v_{0,n,m}(t,\mu)| \leq \frac{1}{n} \sum_{i=1}^n \sup_{\bar{\alpha} \in \bar{\mathcal{A}}^n} \mathbb{E} \left[ \int_t^T \left| f_{i,n,m}^i(s,\bar{X}^{m,\varepsilon,t,\bar{\xi},\bar{\alpha}}_s,\bar{\alpha}_s^i) - f_{i,n,m}^i(s,\bar{X}^{m,0,t,\bar{\xi},\bar{\alpha}}_s,\bar{\alpha}_s^i) \right| ds \right. \]

\[ + \left| g_{i,n,m}^i(\bar{X}^{m,\varepsilon,t,\bar{\xi},\bar{\alpha}}_T) - g_{i,n,m}(\bar{X}^{m,0,t,\bar{\xi},\bar{\alpha}}_T) \right| \leq \frac{K}{n} \sum_{i=1}^n \sup_{\bar{\alpha} \in \bar{\mathcal{A}}^n} \left\{ \int_t^T \mathbb{E} \left[ \left| \bar{X}^{m,\varepsilon,t,\bar{\xi},\bar{\alpha}}_s - \bar{X}^{m,0,t,\bar{\xi},\bar{\alpha}}_s \right|^2 \right] \right\} \frac{1}{2} \]
\[ + \mathbb{E} \left[ \left| X_{s}^{\varepsilon,t,X_{s}} - X_{s}^{0,t,X_{s}} \right|^2 \right]^{1/2} \right) ds + \mathbb{E} \left[ \left| X_{T}^{\varepsilon,t,X_{s}} - X_{T}^{0,t,X_{s}} \right|^2 \right]^{1/2} + \mathbb{E} \left[ \left| X_{T}^{\varepsilon,t,X_{s}} - X_{s}^{0,t,X_{s}} \right|^2 \right]^{1/2} \right) \leq 2K(T + 1) \sqrt{C_{R}T \varepsilon}, \]

where the last inequality follows from estimate (A.8).

Now, we can state the following propagation of chaos result for \( v_{\varepsilon,n,m} \) (a more general propagation of chaos result holds for \( \tilde{v}_{\varepsilon,n,m} \), see [32, Theorem 2.12]).

**Theorem A.6.** Suppose that Assumptions (A), (B), (C) hold. Let \( \varepsilon \geq 0 \) and \( (t, \mu) \in \mathcal{P}_{2}(\mathbb{R}^{d}) \). If there exists \( q > 2 \) such that \( \mu \in \mathcal{P}_{q}(\mathbb{R}^{d}) \), then

\[ \lim_{n \to +\infty} \lim_{m \to +\infty} v_{\varepsilon,n,m}(t, \mu) = v_{\varepsilon}(t, \mu). \]

**Proof.** From the definitions of \( b_{i,n,m}^{i}, f_{i,n,m}^{i}, g_{i,n,m}^{i} \) we get, through straightforward arguments, the convergence, a.s., when \( m \to +\infty \), of \( X_{i,n}^{\varepsilon,t,X_{s}} \) to \( X_{i}^{\varepsilon,t,X_{s}} \). This implies, using the definitions of \( v_{\varepsilon,n,m} \) and \( v_{\varepsilon,n} \),

\[ \lim_{m \to +\infty} v_{\varepsilon,n,m}(t, \mu) = v_{\varepsilon,n}(t, \mu). \]

Then, the convergence

\[ \lim_{n \to +\infty} v_{\varepsilon,n}(t, \mu) = v_{\varepsilon}(t, \mu) \]

is a consequence of [32, Theorem 2.12]. More precisely, for every \( n \in \mathbb{N} \), \( \varepsilon > 0 \), denote

\[ J_{\varepsilon,n}(t, \mu, \tilde{\alpha}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{t}^{T} f \left( s, X_{s}^{\varepsilon,t,X_{s}}, \tilde{\mu}_{s}^{n,\varepsilon,t,X_{s}}, \tilde{\alpha}_{s}^{i} \right) ds + g \left( \hat{X}_{T}^{\varepsilon,t,X_{s}}, \hat{\mu}_{T}^{n,\varepsilon,t,X_{s}} \right) \right] , \]

for every \( t \in [0, T] \), \( \tilde{\alpha} \in \tilde{\mathcal{A}}^{n} \), \( \mu \in \mathcal{P}_{2}(\mathbb{R}^{d}) \), with \( \tilde{\xi} \in L^{2}(\tilde{\Omega}, \tilde{\mathcal{F}}_{t}, \tilde{\mathbb{P}}; \mathbb{R}^{dn}) \) such that \( \mathbb{P}_{\tilde{\xi}} = \mu \otimes \cdots \otimes \mu \). Notice that

\[ v_{\varepsilon,n}(t, \mu) = \sup_{\tilde{\alpha} \in \tilde{\mathcal{A}}^{n}} J_{\varepsilon,n}(t, \mu, \tilde{\alpha}), \quad (t, \mu) \in [0, T] \times \mathcal{P}_{2}(\mathbb{R}^{d}). \]

Now, by [32, Theorem 2.12], for every \( n \in \mathbb{N} \) there exist \( \varepsilon_{n} \geq 0 \) and \( \tilde{\alpha}_{n} \in \tilde{\mathcal{A}}^{n} \) such that \( \lim_{n \to +\infty} \varepsilon_{n} = 0 \) and \( \tilde{\alpha}_{n} \) is an \( \varepsilon_{n} \)-optimal control for (A.4), namely it holds that

\[ J_{\varepsilon,n}(t, \mu, \tilde{\alpha}_{n}) \leq v_{\varepsilon,n}(t, \mu) \leq J_{\varepsilon,n}(t, \mu, \tilde{\alpha}_{n}) + \varepsilon_{n}, \quad \forall n \in \mathbb{N}. \]  \( \text{(A.9)} \)

In addition, by the beginning of Step 3 of [32, Theorem 2.12] we have that \( \{ \tilde{\alpha}_{n} \}_{n} \) is converging in a suitable way to some optimal relaxed control \( m^{*} \), and also we have the convergence of the reward functionals: \( J_{\varepsilon,n}(t, \mu, \tilde{\alpha}_{n}) \to v_{\varepsilon}(t, \mu) \), where here we used that \( v_{\varepsilon}(t, \mu) \) coincides with the reward functional evaluated at the optimal relaxed control \( m^{*} \), that is \( v_{\varepsilon}(t, \mu) \) is equal to the value obtained optimizing over relaxed controls, see [32, Theorem 2.4]. Then, using the convergence \( J_{\varepsilon,n}(t, \mu, \tilde{\alpha}_{n}) \to v_{\varepsilon}(t, \mu) \), we see that the claim follows letting \( n \to \infty \) in (A.9). \( \square \)
A.3 Smooth finite-dimensional approximations

We consider the same probabilistic setting as in Section A.2.

**Theorem A.7.** Suppose that Assumptions (A), (B), (D) hold. Then, for every \( \varepsilon > 0 \), \( n, m \in \mathbb{N} \), there exists \( \bar{v}_{\varepsilon,n,m} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) such that

\[
v_{\varepsilon,n,m}(t, \mu) = \int_{\mathbb{R}^d} \bar{v}_{\varepsilon,n,m}(t, x_1, \ldots, x_n) \mu(dx_1) \cdots \mu(dx_n), \tag{A.10}
\]

for every \( (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \), with \( v_{\varepsilon,n,m} \) given by (A.7), and the following holds.

1) \( \bar{v}_{\varepsilon,n,m} \in C^{1,2}([0, T] \times \mathbb{R}^d) \) and \( v_{\varepsilon,n,m} \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d)) \).

2) For all \( (t, \bar{x}) \in [0, T] \times \mathbb{R}^d \), with \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_\ell, \ldots, \bar{x}_{dn}) = (x_1, \ldots, x_n) \) and \( \bar{x}_\ell \in \mathbb{R} \), \( x_i \in \mathbb{R}^d \), it holds that

\[
\left| \partial_x \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \right| \leq \frac{C_K}{n}, \tag{A.11}
\]

\[
-C_{n,m} \leq \partial_{x_\ell x_h} \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \leq \frac{1}{\varepsilon^2} C_{n,m}. \tag{A.12}
\]

for every \( i = 1, \ldots, n, \ell, h = 1, \ldots, dn \), for some constants \( C_K \geq 0 \) and \( C_{n,m} \geq 0 \), with \( C_K \) (resp. \( C_{n,m} \)) possibly depending on \( K \) (resp. \( K_n, m \)), but independent of \( \varepsilon, n, m \) (resp. \( \varepsilon \)), where \( K \) is as in Assumption (A).

3) \( v_{\varepsilon,n,m} \) solves the following equation:

\[
\begin{align*}
&\left\{ \partial_t v_{\varepsilon,n,m}(t, \mu) + \int_{\mathbb{R}^d} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \langle b^i_{n,m}(t, x_1, \ldots, x_n, a_i), \partial_{x_i} \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \rangle \right. \\
&\quad + \frac{1}{2} \text{tr} \left[ \left( (\sigma \sigma^T)(t, x_1, a_i) + \varepsilon^2 \right) \partial^2_{x_\ell x_h} \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \right] \right. \\
&\quad + \frac{1}{n} f^i_{n,m}(t, x_1, \ldots, x_n, a_i) \right\} \mu(dx_1) \otimes \cdots \otimes \mu(dx_n) = 0, \\
&v_{\varepsilon,n,m}(T, \mu) = \frac{1}{n} \sum_{i=1}^n \int_{\mathbb{R}^d} g^i_{n,m}(x_1, \ldots, x_n) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d),
\end{align*}
\]

for every \( n, m \in \mathbb{N}, \bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^d \) and \( x_1, \ldots, x_n \in \mathbb{R}^d \).

**Proof.** We split the proof into four steps.

**Step I. Definition of \( \bar{v}_{\varepsilon,n,m} \) and its properties.** Fix \( \varepsilon > 0 \) and \( n, m \in \mathbb{N} \). For every \( t \in [0, T] \), \( \bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^d \), let \( \bar{v}_{\varepsilon,n,m} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) be given by

\[
\bar{v}_{\varepsilon,n,m}(t, x_1, \ldots, x_n) = \bar{v}_{\varepsilon,n,m}(t, \delta_{x_1} \otimes \cdots \otimes \delta_{x_n}), \tag{A.14}
\]
with $\bar{v}_{\epsilon,n,m}$ defined by (A.4). In other words, $\bar{v}_{\epsilon,n,m}$ corresponds to the value function of the cooperative $n$-player game (see Section A.2) with deterministic initial state $\bar{x}$ in place of the random vector $\xi$. Hence

$$\bar{v}_{\epsilon,n,m}(t, x_1, \ldots, x_n) = \sup_{\alpha \in A^n} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{t}^{T} f_{n,m}^i (s, \bar{X}_s^{1,m,\epsilon,t,\bar{x},\bar{a}}, \ldots, \bar{X}_s^{n,m,\epsilon,t,\bar{x},\bar{a}}, \bar{\alpha}_s^i) \, ds \right. \\
\left. + g_{n,m}^i (\bar{X}_T^{1,m,\epsilon,t,\bar{x},\bar{a}}, \ldots, \bar{X}_T^{n,m,\epsilon,t,\bar{x},\bar{a}}) \right].$$

This optimal control problem involve coefficients satisfying Assumption (A). Therefore $\bar{v}_{\epsilon,n,m}$ is bounded, jointly continuous, and Lipschitz with respect to $\bar{x}$. Moreover, $\bar{v}_{\epsilon,n,m}$ is a viscosity solution of the following Bellman equation:

$$\begin{align*}
\partial_t \bar{v}_{\epsilon,n,m}(t, \bar{x}) + \sup_{(a_1, \ldots, a_n) \in A^n} \left\{ \frac{1}{n} \sum_{i=1}^{n} f_{n,m}^i (t, \bar{x}, a_i) \\
+ \sum_{i=1}^{n} \frac{1}{2} \text{tr} \left[ ((\sigma^\epsilon)(t, x_i, a_i) + \epsilon^2) \partial_{x_i} \bar{v}_{\epsilon,n,m}(t, \bar{x}) \right] \\
+ \sum_{i=1}^{n} \langle b_{n,m}^i (t, \bar{x}, a_i), \partial_{x_i} \bar{v}_{\epsilon,n,m}(t, \bar{x}) \rangle \right\} = 0, \quad \forall (t, \bar{x}) \in [0, T] \times \mathbb{R}^d,
\end{align*}
$$

(A.15)

We notice that equation (A.15) is uniformly parabolic, with coefficients satisfying Assumptions (A) and (B). It follows that $\bar{v}_{\epsilon,n,m} \in C^{1,2}([0, T] \times \mathbb{R}^d)$ (see [36, Theorem 14.15] and the comments just below Theorem 14.15 regarding the case with linear operators "$L_\nu". In addition, from [31, Theorem 4.7.4] we deduce estimate (A.12). Concerning (A.11), we just notice that it follows if we prove that

$$|\bar{v}_{\epsilon,n,m}(t, \bar{x}) - \bar{v}_{\epsilon,n,m}(t, \bar{z})| \leq \frac{C_K}{n} |\bar{x} - \bar{z}|,$$

whenever the components of $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{z} = (z_1, z_2, \ldots, z_n)$ are equal, apart for one component $x_k \neq z_k$. Such a Lipschitz continuity easily follows from the Lipschitz continuity estimates for $b_{n,m}^i$, $f_{n,m}^i$ and $g_{n,m}^i$ proved in Lemma A.3, formula (A.6).

Finally, we observe that equation (A.15) can be equivalently written as

$$\begin{align*}
\partial_t \bar{v}_{\epsilon,n,m}(t, \bar{x}) + \sum_{i=1}^{n} \sup_{a_i \in A} \left\{ \frac{1}{n} f_{n,m}^i (t, \bar{x}, a_i) + \langle b_{n,m}^i (t, \bar{x}, a_i), \partial_{x_i} \bar{v}_{\epsilon,n,m}(t, \bar{x}) \rangle \\
+ \frac{1}{2} \text{tr} \left[ ((\sigma^\epsilon)(t, x_i, a_i) + \epsilon^2) \partial_{x_i}^2 \bar{v}_{\epsilon,n,m}(t, \bar{x}) \right] \right\} = 0, \quad \forall (t, \bar{x}) \in [0, T] \times \mathbb{R}^d.
\end{align*}
$$

(A.16)
Step II. Proof of equality (A.10). We prove the more general equality

\[ \tilde{v}_{\varepsilon, n, m}(t, \tilde{\mu}) = \int_{\mathbb{R}^{dn}} \tilde{v}_{\varepsilon, n, m}(t, x_1, \ldots, x_n) \tilde{\mu}(dx_1, \ldots, dx_n), \]  

(A.17)

for every \((t, \tilde{\mu}) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^{dn})\), from which (A.10) follows. Notice that equality (A.17) can be equivalently written as

\[ \tilde{v}_{\varepsilon, n, m}(t, \tilde{\mu}) = \mathbb{E} [\tilde{v}_{\varepsilon, n, m}(t, \tilde{\xi})], \]

for every \(t \in [0, T]\), \(\tilde{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})\), with \(\mathbb{P}_\xi = \tilde{\mu}\). We split the rest of the proof of Step II into two substeps.

Step II-a. General case: \(\tilde{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})\). Observe that we can apply Proposition 2.5 to the cooperative \(n\)-player game, from which we deduce that \(\tilde{v}_{\varepsilon, n, m}\) is bounded, jointly continuous, and Lipschitz with respect to \(\tilde{\mu}\). Moreover, recall from Step I above that \(\tilde{v}_{\varepsilon, n, m}\) is also bounded, jointly continuous, and Lipschitz with respect to \(\bar{x}\). As a consequence, the general case with \(\tilde{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})\) can be deduced, relying on an approximation argument, from the case where \(\tilde{\xi}\) takes only a finite number of values, namely from the next Step II-b.

Step II-b. \(\tilde{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})\) taking only a finite number of values. Firstly, we fix some notation. For every \(t \in [0, T]\), let \(\bar{\mathbb{P}}^{B, W, t} = (\bar{\mathbb{F}}_{s}^{B, W, t})_{s \geq 0}\) be the \(\bar{\mathbb{P}}\)-completion of the filtration generated by \((B_{s \wedge t} - B_t)_{s \geq 0}\) and \((\bar{W}_{s \wedge t} - \bar{W}_t)_{s \geq 0}\), where we recall that \(B = (B^1, \ldots, B^n)\) and \(\bar{W} = (\bar{W}^1, \ldots, \bar{W}^n)\). Let also \(\text{Prog}(\bar{\mathbb{P}}^{B, W, t})\) denote the \(\sigma\)-algebra of \([0, T] \times \bar{\Omega}\) of all \(\bar{\mathbb{P}}^{B, W, t}\)-progressive sets.

Proof of the inequality \(\tilde{v}_{\varepsilon, n, m}(t, \tilde{\mu}) \leq \mathbb{E} [\tilde{v}_{\varepsilon, n, m}(t, \tilde{\xi})]\). Suppose that \(\tilde{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})\) takes only a finite number of values. In such a case, by [16, Lemma B.3] there exists a \(\bar{\mathcal{F}}_t\)-measurable random variable \(\bar{U}: \bar{\Omega} \rightarrow \mathbb{R}\), having uniform distribution on \([0, 1]\) and being independent of \(\tilde{\xi}\). Then, by [16, Lemma B.2], for every \(\bar{\alpha} \in \bar{A}^n\) there exists a measurable function

\[ a: ([0, T] \times \bar{\Omega} \times \mathbb{R}^{dn} \times [0, 1], \text{Prog}(\bar{\mathbb{P}}^{B, W, t}) \otimes \mathcal{B}(\mathbb{R}^{dn}) \otimes \mathcal{B}([0, 1])) \rightarrow (\bar{A}^n, \mathcal{B}(\bar{A}^n)) \]

such that \(\tilde{\beta} := (a_s(\tilde{\xi}, \bar{U}))_{s \in [0, T]} \in \bar{A}^n\) and

\[ \left( \tilde{\xi}, (a_s(\tilde{\xi}, \bar{U}))_{s \in [t, T]}, (B_s - B_t)_{s \in [t, T]}, (\bar{W}_s - \bar{W}_t)_{s \in [t, T]} \right) \]

\[ \equiv \left( \tilde{\xi}, (\bar{\alpha}_s)_{s \in [t, T]}, (\bar{B}_s - \bar{B}_t)_{s \in [t, T]}, (\bar{W}_s - \bar{W}_t)_{s \in [t, T]} \right), \]

where \(\equiv\) stands for equality in law. As a consequence, proceeding along the same lines as in [21, Proposition 1.137], we deduce that

\[ (\bar{X}^{m, \varepsilon, t, \tilde{x}, \bar{\alpha}}_{s \in [t, T]}, (\bar{\alpha}_s)_{s \in [t, T]} \equiv (\bar{X}^{m, \varepsilon, t, \tilde{x}, \tilde{\beta}}_{s \in [t, T]}), \tilde{\beta}_{s \in [t, T]}). \]
Moreover, since \( \bar{\xi} \) takes only a finite number of values, it holds that

\[
\bar{\xi} = \sum_{k=1}^{K} \bar{x}_k 1_{E_k}, \tag{A.18}
\]

for some \( K \in \mathbb{N}, \bar{x}_k \in \mathbb{R}^d, \bar{E}_k \in \sigma(\bar{\xi}), \) with \( \{ \bar{E}_k \}_{k=1}^{K} \) being a partition of \( \bar{\Omega} \). Let also

\[
\bar{\beta}_{k,s} := a_s(\bar{x}_k, U), \quad \forall \ s \in [0, T], \ k = 1, \ldots, K.
\]

It easy to see that \( \bar{X}^{m, \varepsilon, t, \bar{\xi}, \bar{\beta}} \) and \( \bar{X}^{m, \varepsilon, t, \bar{x}_1, \bar{\beta}_1} 1_{E_1} + \cdots + \bar{X}^{m, \varepsilon, t, \bar{x}_K, \bar{\beta}_K} 1_{E_K} \) satisfy the same system of controlled stochastic differential equations, therefore, by pathwise uniqueness, they are \( \bar{\mathbb{P}} \)-indistinguishable. Hence

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_t^T f_{n,m}^i(s, \bar{X}_s^{1,m,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \ldots, \bar{X}_s^{n,m,\varepsilon,t,\bar{\xi},\bar{\alpha}}) ds + g_{n,m}^i(\bar{X}_T^{1,m,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \ldots, \bar{X}_T^{n,m,\varepsilon,t,\bar{\xi},\bar{\alpha}}) \right]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_t^T f_{n,m}^i(s, \bar{X}_s^{1,m,\varepsilon,t,\bar{\xi},\bar{\beta}}, \ldots, \bar{X}_s^{n,m,\varepsilon,t,\bar{\xi},\bar{\beta}}) ds + g_{n,m}^i(\bar{X}_T^{1,m,\varepsilon,t,\bar{\xi},\bar{\beta}}, \ldots, \bar{X}_T^{n,m,\varepsilon,t,\bar{\xi},\bar{\beta}}) \right]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \sum_{k=1}^{K} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{s, \bar{X}_s^{1,m,\varepsilon,t,\bar{\xi},\bar{\beta}}, \ldots, \bar{X}_s^{n,m,\varepsilon,t,\bar{\xi},\bar{\beta}}; \bar{\beta}_{k,s} \right] ds
\]
Proof of the inequality \( \mathbb{E}[\tilde{v}_{\varepsilon, n, m}(t, \bar{\xi})] \leq \tilde{v}_{\varepsilon, n, m}(t, \bar{\mu}) \). Let \( \tilde{A}^n_t \) be the subset of \( \tilde{A}^n \) of all \( \tilde{\mathbb{F}}^{B, W, \bar{t}} \)-progressively measurable processes \( \tilde{\alpha} = (\tilde{\alpha}^1, \ldots, \tilde{\alpha}^n) : [0, T] \times \tilde{\Omega} \to A^n \). Then, it is well-known that the value function \( \tilde{v}_{\varepsilon, n, m} \) in (A.14) is also given by

\[
\tilde{v}_{\varepsilon, n, m}(t, x_1, \ldots, x_n) = \sup_{\tilde{\alpha} \in \tilde{A}^n_t} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{t}^{T} f_{n,m}^i (s, X_s^{1,m,\varepsilon,t,x,\tilde{\alpha}}, \ldots, X_s^{n,m,\varepsilon,t,x,\tilde{\alpha}}, \alpha_s^i) \, ds \right] \quad \text{(A.19)}
\]

where the supremum is taken on \( \tilde{A}^n_t \) rather than on \( A^n_t \). Now, let \( \bar{\xi} \in L^2(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}}, \mathbb{R}^{dn}) \) be given by (A.18). By (A.19), for every \( \delta > 0 \) and \( k = 1, \ldots, K \), there exists \( \bar{\beta}_k \in \tilde{A}^n_t \) (possibly depending on \( \delta \)) such that

\[
\tilde{v}_{\varepsilon, n, m}(t, \bar{x}_k) \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{t}^{T} f_{n,m}^i (s, X_s^{1,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k^i}, \ldots, X_s^{n,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k^i}, \bar{\beta}_k^i) \, ds \right] + \delta.
\]

Then, define

\[
\bar{\beta} := \sum_{k=1}^{K} \bar{\beta}_k 1_{E_k}.
\]

Notice that \( \bar{\beta} \in \tilde{A}^n_t \). Moreover, it is easy to see that \( \tilde{X}^{m,\varepsilon,t,\bar{\xi},\bar{\beta}} \) and \( \tilde{X}^{m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k} 1_{E_k} + \cdots + \tilde{X}^{m,\varepsilon,t,\bar{x}_k,\bar{\beta}_K} 1_{E_K} \) satisfy the same system of controlled stochastic differential equations, therefore, by pathwise uniqueness, they are \( \tilde{\mathbb{P}} \)-indistinguishable. Hence (using the independence of both \( \{\tilde{X}^{m,\varepsilon,t,\bar{\xi},\bar{\beta}}\}_k \) and \( \{\bar{\beta}_k\}_k \) from \( \{E_k\}_k \))

\[
\mathbb{E}[\tilde{v}_{\varepsilon, n, m}(t, \bar{\xi})] = \sum_{k=1}^{K} \mathbb{E} \left[ \tilde{v}_{\varepsilon, n, m}(t, \bar{x}_k) 1_{E_k} \right]
\]

\[
\leq \sum_{k=1}^{K} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{t}^{T} f_{n,m}^i (s, X_s^{1,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k^i}, \ldots, X_s^{n,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k^i}, \bar{\beta}_k^i) \, ds \right] \right] 1_{E_k} + \delta
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \sum_{k=1}^{K} \mathbb{E} \left[ \int_{t}^{T} f_{n,m}^i (s, X_s^{1,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k^i}, \ldots, X_s^{n,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k^i}, \bar{\beta}_k^i) \, ds \right] 1_{E_k} \right] + \delta
\]

\[
\quad + g_{n,m}^i \left( \tilde{X}_T^{1,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \ldots, \tilde{X}_T^{n,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k} \right) \right] 1_{E_k} \right] + \delta
\]

\[
\quad + g_{n,m}^i \left( \tilde{X}_T^{1,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \ldots, \tilde{X}_T^{n,m,\varepsilon,t,\bar{x}_k,\bar{\beta}_k} \right) \right] 1_{E_k} \right] + \delta
\]
Step III. Proof of item 1. We begin noting that, by equality (A.10), we have

\[
\partial_t v_{\varepsilon,n,m}(t, \mu) = \int_{\mathbb{R}^d} \partial_t \tilde{v}_{\varepsilon,n,m}(t, x_1, \ldots, x_n) \mu(dx_1) \cdots \mu(dx_n),
\]

(A.20)

which proves that \( \partial_t v_{\varepsilon,n,m} \) exists and is continuous. Now, for every \( \varepsilon > 0 \) and \( n \geq 2 \), let \( \tilde{v}_{\varepsilon,n,m}: [0, T] \times (\mathcal{P}_2(\mathbb{R}^d))^n \to \mathbb{R} \) be given by

\[
\tilde{v}_{\varepsilon,n,m}(t, \mu_1, \ldots, \mu_n) := \tilde{v}_{\varepsilon,n,m}(t, \mu_1 \otimes \cdots \otimes \mu_n) = \int_{\mathbb{R}^d} \tilde{v}_{\varepsilon,n,m}(t, x_1, \ldots, x_n) \mu_1(dx_1) \cdots \mu_n(dx_n),
\]

for every \( (t, \mu_1, \ldots, \mu_n) \in [0, T] \times (\mathcal{P}_2(\mathbb{R}^d))^n \). Then, by direct calculation, we obtain

\[
\partial_\mu \tilde{v}_{\varepsilon,n,m}(t, \mu_1, \ldots, \mu_n)(x)
= \int_{\mathbb{R}^{d(n-1)}} \partial_{x_i} \tilde{v}_{\varepsilon,n,m}(t, x_1, \ldots, x_i-1, x, x_{i+1}, \ldots, x_n) \mu_1(dx_1) \cdots \mu_{i-1}(dx_{i-1}) \mu_{i+1}(dx_{i+1}) \cdots \mu_n(dx_n),
\]

(A.21)

for every \( (t, \mu_1, \ldots, \mu_n, x) \in [0, T] \times (\mathcal{P}_2(\mathbb{R}^d))^n \times \mathbb{R}^d, i = 1, \ldots, n \). Since \( v_{\varepsilon,n,m}(t, \mu) = \tilde{v}_{\varepsilon,n,m}(t, \mu, \ldots, \mu) \), we obtain

\[
\partial_\mu v_{\varepsilon,n,m}(t, \mu)(x)
= \sum_{i=1}^n \int_{\mathbb{R}^{d(n-1)}} \partial_{x_i} \tilde{v}_{\varepsilon,n,m}(t, x_1, \ldots, x_i-1, x, x_{i+1}, \ldots, x_n) \mu(dx_1) \cdots \mu(dx_{i-1}) \mu(dx_{i+1}) \cdots \mu(dx_n),
\]

(A.22)

for every \( (t, \mu, x) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \). Hence

\[
\partial_x \partial_\mu v_{\varepsilon,n,m}(t, \mu)(x)
= \sum_{i=1}^n \int_{\mathbb{R}^{d(n-1)}} \partial_{x_i} \partial_{x_i} \tilde{v}_{\varepsilon,n,m}(t, x_1, \ldots, x_i-1, x, x_{i+1}, \ldots, x_n) \mu(dx_1) \cdots \mu(dx_{i-1}) \mu(dx_{i+1}) \cdots \mu(dx_n),
\]

for every \( (t, \mu, x) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \). In conclusion, we see that \( v_{\varepsilon,n,m} \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d)) \).
Step IV. Proof of item 3). Recall that \( \tilde{v}_{\varepsilon,n,m} \) solves equation (A.15). Fix \((t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)\). When \( t = T \), integrating the terminal condition of (A.15) with respect to \( \mu \otimes \cdots \otimes \mu \) on \( \mathbb{R}^{dn} \), we get

\[
v_{\varepsilon,n,m}(T, \mu) = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^{dn}} g_{n,m}(x_1, \ldots, x_n) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n),
\]

which corresponds to the terminal condition of equation (A.13). On the other hand, when \( t < T \), integrating equation (A.16) with respect to \( \mu \otimes \cdots \otimes \mu \) on \( \mathbb{R}^{dn} \), and using (A.20), we find

\[
\partial_t v_{\varepsilon,n,m}(t, \mu) + \int_{\mathbb{R}^{dn}} \sum_{i=1}^{n} \sup_{a_i \in A} \left\{ \frac{1}{n} f_{n,m}^i(t, \tilde{x}, a_i) + \langle b_{n,m}^i(t, \tilde{x}, a_i), \partial_{x_i} v_{\varepsilon,n,m}(t, \tilde{x}) \rangle \right\} \mu(dx_1) \otimes \cdots \otimes \mu(dx_n) = 0,
\]

which corresponds to equation (A.13).

We end this section with the next result, which is used in the proof of the comparison theorem, in order to prove that \( v_0 \leq u_2 \). We first need to regularize the coefficients also in the control variable. For that, we fix \( p \in \mathbb{N} \) such that \( A \subset \mathbb{R}^p \) and a function \( \zeta_p : \mathbb{R}^p \to [0, +\infty) \) being of class \( C^\infty \) with compact support and satisfying \( \int_{\mathbb{R}^p} \zeta_p(a) \, da = 1 \). Moreover, we extend the continuous and bounded functions \( b \) and \( f \) defined on \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A\) to some continuous and bounded functions, still denoted by \( b \) and \( f \), defined on \([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^p\). Then, as in Section A.2 we define the coefficients \( \tilde{b}_{n,m}^i \) and \( \tilde{f}_{n,m}^i \) on the entire space \([0, T] \times \mathbb{R}^{dn} \times \mathbb{R}^p\) (rather than \([0, T] \times \mathbb{R}^{dn} \times A\)). Afterwards, we define the coefficients \( \tilde{b}_{n,m}^i \) and \( \tilde{f}_{n,m}^i \) by

\[
\tilde{b}_{n,m}^i(t, \tilde{x}, a) = m_p \int_{\mathbb{R}^p} \tilde{b}_{n,m}^i(t, \tilde{x}, a - a') \zeta_p(ma') \, da',
\]

\[
\tilde{f}_{n,m}^i(t, \tilde{x}, a) = m_p \int_{\mathbb{R}^p} \tilde{f}_{n,m}^i(t, \tilde{x}, a - a') \zeta_p(ma') \, da',
\]

for all \( n, m \in \mathbb{N}, i = 1, \ldots, n, \tilde{x} = (x_1, \ldots, x_n) \in \mathbb{R}^{dn}, (t, a) \in [0, T] \times \mathbb{R}^p \). We can now state our last result.

**Theorem A.8.** Let Assumptions (A), (B), (C), (D) hold. For every \( t \in [0, T] \), let \( \mathcal{M}_t \) denote the set of \( \mathcal{F}_t \)-measurable random variables \( a : \Omega \to A \). Let \( u_2 : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) be a continuous and bounded function. For every \( t_0, s_0 \in [0, T] \), let \( v^{s_0} : [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A) \to \mathbb{R} \) be given by

\[
v^{s_0}(t, \nu) = \mathbb{E} \left[ \int_t^{s_0} f(r, X^r_{t, \xi, a_0, 0}, \mathbb{P}_{X^r_{t, \xi, a_0, 0}, Y^r_{t, \nu, a_0}}) \, dr \right] + u_2(s_0, \mathbb{P}_{X^t_{t_0, \xi, a_0}}),
\]

\[
49
\]
for all $(t, \nu) \in [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ and $a_0 \in \mathcal{M}_t$ such that $\mathbb{P}(\xi, a_0) = \nu$, where $(X^{t, \xi, a_0}, Y^{t, a_0})$ is the unique solution to the following system of McKean-Vlasov stochastic differential equations:

\[
\begin{aligned}
X_s &= \xi + \int_t^s b(r, X_r, \mathbb{P}_{X_r}, Y_r) \, dr + \int_t^s \sigma(r, X_r, Y_r) \, dB_r, \quad s \in [t, T], \\
Y_s &= a_0, \quad s \in [t, T].
\end{aligned}
\]  

(A.23)

Moreover, for every $n, m \in \mathbb{N}$, let $v^{s_0}_{n,m} : [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A) \to \mathbb{R}$ be given by

\[
v^{s_0}_{n,m}(t, \nu) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_t^{s_0} \tilde{f}_{i_{n,m}}(r, \tilde{X}^{1,m,t,\xi, a_0}, \ldots, \tilde{X}^{n,m,t,\xi, a_0}, \tilde{Y}^{i,t,\xi, a_0}) \, dr \right. \\
&\quad + u_{n,m}(s_0, \tilde{X}^{1,m,t,\xi, a_0}, \ldots, \tilde{X}^{n,m,t,\xi, a_0}) \right],
\]

for every $(t, \nu) \in [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)$, $\tilde{\xi} = (\xi_1, \ldots, \xi_n) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ and $\tilde{a}_0 = (a_0^1, \ldots, a_0^n)$, with $a_0^i \in \mathcal{M}_t$, such that $\mathbb{P}(\xi, a_0) = \nu \otimes \cdots \otimes \nu$. Moreover, $\tilde{Y}^{i,t,\xi, a_0}$ solves equation (A.3) with $\varepsilon = 0$, $\tilde{\alpha}_i = \tilde{Y}^{i,t,\xi, a_0}$ for $r \in [t, T]$, and $\tilde{X}^{i,m,t,\xi, a_0}$ solves equation (A.3) with $\varepsilon = 0$, $\tilde{a}_0^i = \tilde{b}_n, m, t, \nu$ replaced by $\tilde{b}_m, n, t, \nu$. Similarly, $u_{n,m}(s_0, \cdot) : \mathbb{R}^{d^n} \to \mathbb{R}$ is given by

\[
u_{n,m}(s_0, \bar{x}) = m^{nd} \int_{\mathbb{R}^{d^n}} u_2 \left( s_0, \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j-y_j} \right) \prod_{j=1}^{n} \Phi(my_j) \, dy_j,
\]

for all $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^{d^n}$, with $\Phi$ as in Section A.2.

Then, for every $n, m \in \mathbb{N}$, there exists $\tilde{v}^{s_0}_{n,m} : [t_0, s_0] \times (\mathbb{R}^d \times A)^n \to \mathbb{R}$ such that

\[
v^{s_0}_{n,m}(t, \nu) = \int_{\mathbb{R}^{d^n}} \tilde{v}^{s_0}_{n,m}(t, x_1, \ldots, x_n, a_1, \ldots, a_n) \nu(dx_1, da_1) \cdots \nu(dx_n, da_n),
\]

for every $(t, \nu) \in [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A)$, and the following holds.

1) $\tilde{v}^{s_0}_{n,m} \in C^{1,2}([t_0, s_0] \times (\mathbb{R}^d \times A)^n)$ and $v^{s_0}_{n,m} \in C^{1,2}([t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A))$.

2) For all $(t, \bar{x}, \bar{a}) \in [t_0, s_0] \times (\mathbb{R}^d \times A)^n$, with $\bar{x} = (x_1, \ldots, x_n)$, $\bar{a} = (a_1, \ldots, a_n)$ and $x_1, \ldots, x_n \in \mathbb{R}^d, a_1, \ldots, a_n \in A$, it holds that

\[
|\partial_{x_i} \tilde{v}^{s_0}_{n,m}(t, \bar{x}, \bar{a})| \leq \frac{C_K}{n},
\]

for every $i = 1, \ldots, n$, for some constant $C_K \geq 0$, possibly depending on $K$, but independent of $n, m$, where $K$ is as in Assumption (A).

3) $v^{s_0}_{n,m}$ solves the following equation:

\[
\begin{aligned}
&\partial_t v^{s_0}_{n,m}(t, \nu) + \mathbb{E} \sum_{i=1}^{n} \left\{ \frac{1}{n} \tilde{f}_{i_{n,m}}(t, \xi_1, \ldots, \xi_n, a_0^i) + \langle \tilde{b}_{n,m}^i(t, \xi_1, \ldots, \xi_n, a_0^i), \partial_{x_i} v^{s_0}_{n,m}(t, \bar{\xi}, \bar{a}_0) \rangle \right. \\
&\left. \quad + \frac{1}{2} \sigma^T(t, \xi_i, a_0^i) \sigma(t, \xi_i, a_0^i) \partial_{x_i} v^{s_0}_{n,m}(t, \bar{\xi}, \bar{a}_0) \right\} = 0, \quad (t, \nu) \in [t_0, s_0] \times \mathcal{P}_2(\mathbb{R}^d \times A), \\
v^{s_0}_{n,m}(s_0, \nu) = \mathbb{E} \left[ u_{n,m}(s_0, \bar{\xi}) \right], \quad \nu \in \mathcal{P}_2(\mathbb{R}^d \times A),
\end{aligned}
\]
for any \( \bar{\xi} = (\xi_1, \ldots, \xi_n) \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{dn}) \) and \( \bar{a}_0 = (a^{i}_0, \ldots, a^{n}_0) \), such that \( \bar{P}_{(\bar{\xi}, \bar{a}_0)} = \nu \otimes \cdots \otimes \nu \).

4) If there exists \( q > 2 \) such that \( \nu \in \mathcal{P}_q(\mathbb{R}^d) \), then

\[
\lim_{n \to +\infty} \lim_{m \to +\infty} v_{s_0}^{n,m}(t, \nu) = v^{s_0}(t, \nu).
\]

**Proof.** Items 1)-2)-3) follow from the same arguments as in Theorem A.7, taking into account that here we are in a “linear” context, while Theorem A.7 deals with the “fully nonlinear” case. Since we are in the linear case, the regularity results hold even if \( \varepsilon = 0 \) (that’s why here we do not need this extra parameter), as it can be deduced for instance from [23, Theorem 6.1, Chapter 5]. Finally, item 4) follows from the propagation of chaos result [32, Theorem 2.12] proceeding as in the proof of Theorem A.6 and noting that, in the present context, Assumption (B) in [32] can be neglected (that is Lipschitz continuity of the coefficients \( b \) and \( \sigma \) with respect to the extra state variable \( a \)). As a matter of fact, Assumption (B) in [32] is imposed to have uniqueness of the underlying McKean-Vlasov stochastic differential equations, which in our case correspond to system (A.23) and uniqueness clearly holds under our assumptions, without imposing in addition that \( b \) and \( \sigma \) are Lipschitz continuous with respect to \( a \).

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