Dirac Operators for Matrix Algebras Converging to Coadjoint Orbits

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Abstract: In the high-energy physics literature one finds statements such as “matrix algebras converge to the sphere”. Earlier I provided a general precise setting for understanding such statements, in which the matrix algebras are viewed as quantum metric spaces, and convergence is with respect to a quantum Gromov–Hausdorff-type distance. But physicists want even more to treat structures on spheres (and other spaces), such as vector bundles, Yang–Mills functionals, Dirac operators, etc., and they want to approximate these by corresponding structures on matrix algebras. In the present paper we provide a somewhat unified construction of Dirac operators on coadjoint orbits and on the matrix algebras that converge to them. This enables us to prove our main theorem, whose content is that, for the quantum metric-space structures determined by the Dirac operators that we construct, the matrix algebras do indeed converge to the coadjoint orbits, for a quite strong version of quantum Gromov–Hausdorff distance.

1. Introduction

In the literature of theoretical high-energy physics one finds statements along the lines of “this sequence of matrix algebras converges to the sphere” and “here are the Dirac operators on the matrix algebras that correspond to the Dirac operator on the sphere”. But one also finds that at least three inequivalent definitions of Dirac operators have been proposed in this context. See, for example, [1–8] and the references they contain. In [9–11] I provided definitions and theorems that give a precise meaning to the idea of the convergence of matrix algebras to spheres. This involved equipping the matrix algebras in a natural way with the structure of a non-commutative (or “quantum”) compact metric space (motivated by the “spectral triples” of Connes [12–15]), and developing non-commutative (or “quantum”) versions of the usual Gromov–Hausdorff distance between compact metric spaces. These results were developed in the general context of coadjoint orbits of compact Lie groups, which is the appropriate context for this topic, as is clear from the physics literature. (In [16] it is shown that the matrix algebras form a strict
quantization of the coadjoint orbits. A much-simplified proof of much of that fact, in the more general context of compact quantum groups, is given in proposition 4.13 of [17].

The purpose of the present paper is to provide a somewhat unified construction of Dirac operators (always \( G \)-invariant) that handles both the matrix algebras and the corresponding coadjoint orbits. This enables us to prove the main theorem of this paper, which states that for the quantum metrics determined by the Dirac operators on the matrix algebras and on the coadjoint orbits, the matrix algebras do indeed converge to the coadjoint orbits, in fact for a stronger version of quantum Gromov–Hausdorff distance than I had used earlier.

Roughly speaking, physicists who try to develop quantum field theory on spaces like the sphere, have found attractive the idea of approximating the spaces by means of matrix algebras because if they try “lattice approximations” by a collections of points, they lose the action of the symmetry group, whereas the matrix algebras can be viewed as quantum finite sets, on which the symmetry group still acts. It is my hope that when results of the kind contained in this paper are extended to further types of structure of interest to quantum physicists, such as Yang–Mills functionals [18,19] and other action functionals, then the fact that we have quantified the idea of distance between matrix algebras and spaces will help in quantifying the size of the error made by approximating quantities in quantum field theory on the spaces by corresponding quantities obtained for quantum field theory on the approximating matrix algebras. (Another type of structure for which this kind of approximation theory has already been worked out consists of vector bundles [20,21].)

Here is a brief imprecise sketch of what our construction of Dirac operators looks like for the case of the matrix algebras that converge to the (2-dimensional) sphere, for which this kind of approximation theory has already been worked out consists of quantum field theory on the approximating matrix algebras. (Another type of structure for which this kind of approximation theory has already been worked out consists of quantum finite sets, on which the symmetry group still acts. It is my hope that when results of the kind contained in this paper are extended to further types of structure of interest to quantum physicists, such as Yang–Mills functionals [18,19] and other action functionals, then the fact that we have quantified the idea of distance between matrix algebras and spaces will help in quantifying the size of the error made by approximating quantities in quantum field theory on the spaces by corresponding quantities obtained for quantum field theory on the approximating matrix algebras. (Another type of structure for which this kind of approximation theory has already been worked out consists of vector bundles [20,21].)

Let \( \mathcal{B}^n = B(\mathcal{H}^n) \) be the algebra of all linear operators on \( \mathcal{H}^n \) – our full matrix algebra. Then let \( \alpha \) be the action of \( G \) on \( \mathcal{B}^n \) by conjugation by the representation \( U^n \). We then somewhat follow the steps that are used in constructing Dirac operators on compact Riemannian manifolds, especially Riemannian homogeneous spaces. We think of the Lie algebra, \( \mathfrak{g} = su(2) \), as the tangent vector space at some nonexistent point, so that the (complexified) tangent bundle is \( \mathcal{B}^n \otimes_{\mathbb{R}} \mathfrak{g} \). Let \( \mathfrak{g}' \) be the vector space dual of \( \mathfrak{g} \), so that the cotangent bundle is \( \mathcal{O}^n = \mathcal{B}^n \otimes_{\mathbb{R}} \mathfrak{g}' \), viewed as a right module over \( \mathcal{B}^n \). We let \( \alpha \) also denote the corresponding infinitesimal action of \( \mathfrak{g} \) on \( \mathcal{B}^n \). Then for each \( T \in \mathcal{B}^n \) its total differential, \( dT \), is the element of \( \mathcal{O}^n \) defined by \( dT(X) = \alpha_X(T) \) for all \( X \in \mathfrak{g} \). As inner product on \( \mathfrak{g} \) we use the negative of the Killing form on \( \mathfrak{g} \), and as inner product on \( \mathfrak{g}' \) we use the dual of the inner product on \( \mathfrak{g} \). Then the Riemannian metric on \( \mathcal{B}^n \), viewed as a \( \mathcal{B}^n \)-valued inner product on the cotangent bundle \( \mathcal{O}^n \), is defined on elementary tensors by

\[
\langle S \otimes v, T \otimes \mu \rangle = S^* T \langle v, \mu \rangle
\]

for \( S, T \in \mathcal{B}^n \) and \( v, \mu \in \mathfrak{g}' \). We then let \( \mathcal{C}\ell(\mathfrak{g}') \) be the complex Clifford algebra corresponding to the inner product on \( \mathfrak{g}' \). The Clifford bundle over \( \mathcal{B}^n \) is then \( \mathcal{B}^n \otimes \mathcal{C}\ell(\mathfrak{g}') \). As spinors we can choose the Hilbert space \( S \) of one of the two irreducible \( \ast \)-representations of \( \mathcal{C}\ell(\mathfrak{g}') \) (necessarily 2-dimensional). The spinor bundle is then \( \mathcal{B}^n \otimes S \). The Dirac operator, \( D \), is then an operator on the spinor bundle. It is defined as follows. Let \( \{ E_j \}_{j=1}^3 \) be an orthonormal basis for \( \mathfrak{g} \), and let \( \{ \epsilon_j \}_{j=1}^3 \) be the dual orthonormal basis for \( \mathfrak{g}' \). Let \( \kappa \) denote the representation of \( \mathcal{C}\ell(\mathfrak{g}') \) on \( S \). Then for any elementary tensor
$T \otimes \psi$ in the spinor bundle (where $T \in B^n$ and $\psi \in S$) we set

$$D(T \otimes \psi) = \sum \alpha_{E_j}(T) \otimes \kappa_{\epsilon_j}(\psi).$$  \hspace{1cm} (1.1)

This Dirac operator has many attractive properties, which we will describe later. In particular, we will see that it is closely related to the Casimir element for the chosen inner product on $\mathfrak{g}$, and that this gives an attractive way of calculating the spectrum of $D$. When the main theorem of this paper, Theorem 17.7, is applied to the case of the sphere, it tells us that when the sphere is equipped with its usual metric (which corresponds to its Dirac operator), and the matrix algebras $B^n$ are equipped with the quantum metrics (defined in Sect. 7) determined by the Dirac operators defined just above, then the matrix algebras $B^n$ converge to the sphere for a quite strong form of quantum Gromov–Hausdorff distance (namely the propinquity of Latrémolière [22], which we will describe later). The proof of our main theorem makes essential use of coherent states and Berezin symbols.

Our construction of Dirac operators on matrix algebras sketched above is closely related to some of the proposals in the physics literature. At the end of Sect. 10 we give a substantial discussion of the relationship between our construction and various proposals, for the case of the sphere. In Sect. 20 we further discuss how our construction relates to proposals in the physics literature for Dirac operators on other coadjoint orbits, such as projective spaces.

Our general construction has some deficiencies. We discuss these deficiencies in some detail in Sect. 19. But briefly, Dirac operators on Riemannian manifolds are usually constructed using the the Levi–Civita connection, which is the unique torsion-free connection compatible with the Riemannian metric. But our construction in essence uses the “canonical connection”, and as is well-known, and explained in section 6 of [23], for homogeneous spaces the Levi–Civita connection agrees with the canonical connection only for symmetric spaces. The consequence is that our construction is very satisfactory for the case in which the coadjoint orbit is a symmetric space, such as the sphere or a projective space, but somewhat less satisfactory otherwise, though we will see that the metric from the canonical connection agrees with the metric from the Levi–Civita connection, so our sequence of matrix algebras does converge to the coadjoint orbit for the Dirac operator for its Levi–Civita connection.

A related deficiency, discussed further in Sect. 19, comes from the fact that each highest weight determines a $G$-invariant Kähler structure on its coadjoint orbit, which includes not just a Riemannian structure, but also a closely related complex structure and symplectic structure. But these Riemannian metrics are not related to the Killing form unless the coadjoint orbit is a symmetric space. When that is not the case, our sequence of matrix algebras will not converge to the coadjoint orbit equipped with the Riemannian metric from the Kähler structure, because it converges to the coadjoint orbit equipped with the Riemannian metric related to the Killing form.

This paper concludes with a final section that discusses how the main results of this paper are related to the very interesting “spectral propinquity” of Latrémolière [24], which is a metric on equivalent classes of spectral triples. The conclusion that is reached is that the spectral propinquity is too strong, in the sense that the spectral triples that we construct for matrix algebras do not converge to the spectral triples of coadjoint orbits for the spectral propinquity. On the other hand, the present paper says nothing about convergence of these spectral triples. It only shows that for the $C^*$-metrics coming from the spectral triples, the matrix algebras do converge to the coadjoint orbits for Latrémolière’s Gromov–Hausdorff propinquity [22]. It would be very desirable to have
a weaker form of convergence for spectral triples than the spectral propinquity, for which our spectral triples for the matrix algebras do converge to the spectral triples for the coadjoint orbits. Perhaps there might be something roughly along the lines that Lott used for ordinary Dirac operators [25,26].

There is a substantial relationship between the Dirac operators on matrix algebras that we construct here and the more general “matrix models” that are being intensively explored in the literature of theoretical high-energy physics. See [27] (some of whose examples are “fuzzy” spaces of the kind studied in the present paper) and its many references. It would be very interesting to explore how some of the ideas in the present paper might be extended to various matrix models, such as those in [27–33].

2. Ergodic Actions and Dirac Operators

In this section we construct a rough approximation to the Dirac operators that we seek. We do this in the general setting of ergodic actions of compact Lie groups on unital C*-algebras. This rough approximation will be crucial for the proofs of some of our main results.

Let G be a connected compact Lie group. In this paper we are concerned with two types of actions of G on unital C*-algebras. One type consists of actions by translation on the C*-algebras C(G/K) of continuous complex-valued functions on homogenous spaces G/K, where K is a closed subgroup of G. The other type consists of actions on the C*-algebra B(H) consisting of all the operators on the Hilbert space H of an irrep of G, where the action of G on B(H) is by conjugation by the representation. In both cases the action is ergodic in the sense that the only elements of the algebra that are invariant under the action of G are scaler multiples of the identity element. Consequently it is natural to begin by considering general ergodic actions on unital C*-algebras.

Thus let α be an ergodic action of G on a unital C*-algebra A. Using the fact that G is a Lie group, we can define the subalgebra A∞ of smooth elements of A, consisting of those elements a ∈ A such that the function x → α,(a) from G to A is infinitely differentiable for the norm on A. This means that for any X ∈ g, where g is the Lie algebra of G, there is an element, αX(a) ∈ A, such that DT0(αexp(tX)(a)) = αX(a), (where D+0 means “derivative in t at t = 0”), and similarly for higher derivatives. It is a standard fact that X ↦→ αX is a Lie algebra homomorphism from g into the Lie algebra of ∗-derivations of A∞, and that, using the Gårding smoothing argument, A∞ is dense in A. (See Sections 3 and 4 of chapter III of [34], whose discussion for unitary representations on Hilbert spaces adapts very easily to isometric actions on any Banach space.) It is easily verified that A∞ is a unital ∗-subalgebra of A.

Let g′ denote the dual vector space of g, and let Ωo = A∞ ⊗ g′ (where ⊗ is necessarily over R). Here and throughout the paper such tensor products are algebraic tensor products with one factor finite dimensional. View Ωo as an A∞-bimodule by letting A∞ act on itself by left and right multiplication. For any a ∈ A∞ let da : g → A∞ be defined by da(X) = αX(a). Then da can be viewed as an element of Ωo, and it is easily verified that d is then a derivation from A∞ into Ωo. (Thus (Ωo, d) can be viewed as a “first-order differential calculus” for A∞ [35,36].) We can view Ωo as a cotangent bundle for A. (But later we will want to do better.)

We remark that here, and in later paragraphs of this section, we do not need the more general structure of “connections”, because the A∞-modules we consider in this section are finitely generated free A∞-modules. Connections will be needed in Sect. 12 and later.
We want $d : \mathcal{A}^\infty \to \Omega_o$ to be equivariant for an action of $G$. This will usually not be true if we take the action on $\Omega_o$ to be $\alpha \otimes I^{\Omega_o}$, where $I^{\Omega_o}$ denotes the identity operator on $\Omega_o$. Let Cad denote the coadjoint representation of $G$ on $g'$, dual to the adjoint action $\text{Ad}$ on $g$ (so $\langle X, \text{Cad}_x(\mu) \rangle = \langle \text{Ad}^{-1}_x(X), \mu \rangle$ for $X \in g$, $x \in G$, and $\mu \in g'$, where here $\langle \cdot, \cdot \rangle$ denotes the pairing between $g$ and $g'$).

Notation 2.1. We let $\gamma$ be the diagonal action of $G$ on $\Omega_o$ defined by
$$\gamma_x = \alpha_x \otimes \text{Cad}_x$$
for any $x \in G$.

Proposition 2.2. For any $a \in \mathcal{A}^\infty$ and $x \in G$ we have
$$\gamma_x(da) = d(\alpha_x(a)),$$
that is, $d$ is equivariant for the actions $\alpha$ and $\gamma$.

Proof. Let $X \in g$. Then
$$\gamma_x(da)(X) = ((I^\mathcal{A} \otimes \text{Cad}_x)(\alpha_x \otimes I^{g'} )da)(X)$$
$$= (((\alpha_x \otimes I^{g'})da)(\text{Ad}^{-1}_x(X)) = \alpha_x(\alpha_x^{-1}(X)(a))$$
$$= \alpha_x(\alpha_x(a)) = d(\alpha_x(a))(X).$$

In section 4 of [37] we briefly constructed some Dirac-type operators for $\mathcal{A}^\infty$, and proved that for the corresponding Leibniz semi-norms $\mathcal{A}$ becomes a C*-metric space. More recently, in [38] the authors obtained important basic properties of these Dirac operators, such as their essential self-adjointness and having compact resolvant. We now recall the construction, slightly reformulated in a way that will be more convenient later, and influenced by [38]. The usual Dirac operators on a manifold are defined in terms of a Riemannian metric on the tangent bundle of the manifold, or equivalently, on the cotangent bundle. In non-commutative geometry “cotangent bundles” are more commonly available than “tangent bundles”, where now the analogs of vector bundles are modules, usually finitely generated projective, corresponding to the modules of smooth cross-sections of ordinary vector bundles. To aid the reader’s intuition, in this paper it seems best to still refer to these modules as “bundles”. Note that since $g'$ is finite-dimensional, $\Omega_o$ is a finitely-generated free (right or left) module over $\mathcal{A}^\infty$.

To construct a Dirac operator in our general framework, we need a “Riemannian metric” on $\Omega_o$. For this purpose we choose an inner product on $g'$ that is Cad-invariant. (Because $G$ is compact, these always exist, as seen by averaging any inner product on $g'$ using the action Cad and the Haar measure of $G$.) We fix such an inner product, and denote it just by $\langle \cdot, \cdot \rangle$. We view $\Omega_o$ as a right $\mathcal{A}^\infty$-module, and our Riemannian metric on $\Omega_o$ is the $\mathcal{A}^\infty$-valued inner product on $\Omega_o$ that is given on elementary tensors (necessarily over $\mathbb{R}$) by
$$\langle a \otimes \mu, b \otimes v \rangle_{\mathcal{A}} = a^* b \langle \mu, v \rangle$$
(so we define our inner product to be linear in the second variable, as done in [36,39–41]). This “bundle metric” is easily checked to respect the action $\gamma$ in the sense that
$$\langle \gamma_x(\omega_1), \gamma_x(\omega_2) \rangle_{\mathcal{A}} = \alpha_x(\langle \omega_1, \omega_2 \rangle_{\mathcal{A}})$$
for all $\omega_1, \omega_2 \in \Omega_\mu$ and all $x \in G$.

To construct a Dirac-type operator we must first define the Clifford bundle. For our chosen inner product on $g'$ we form its complex Clifford algebra. Much as in [36,42], we denote it by $\mathbb{C}\ell(g')$. It is the complexification of the real Clifford algebra for $g'$ with our chosen inner product. We follow the convention that the defining relation is

$$\mu \nu + v\mu = -2\langle \mu, v \rangle 1$$

for $\mu, v \in g'$. We include the minus sign for consistency with [23,42]. Thus if one wants to apply the results of the first pages of chapter 5 of [36] one must let the $g$ there to be the negative of our inner product. After exercise 5.6 of [36] it is assumed that the inner product is positive, so small changes are needed when one uses the later results in [36] but with our different convention. The consequence of including the minus sign is that in the representations which we will construct the elements of $g'$ will act as skew-adjoint operators, just as they do for $g$ for orthogonal or unitary representations of $G$, rather than as self-adjoint operators as happens when the minus sign is omitted. (The involution on $\mathbb{C}\ell(g')$ takes $\mu$ to $-\mu$ for $\mu \in g$.) The corresponding Clifford bundle is then the $C^*$-algebra $\mathcal{A} \otimes \mathbb{C}\ell(g')$, or its smooth version $\mathcal{A}^\infty \otimes \mathbb{C}\ell(g')$.

We need a $*$-representation, $\kappa$, of $\mathbb{C}\ell(g')$ on a finite-dimensional Hilbert space $S$ (for “spinors”). For the moment we do not require that it be irreducible or faithful. We then form $\mathcal{A}^\infty \otimes S$. It is a right module over $\mathcal{A}^\infty$, and is the analog of the bundle of “spinor fields” for a Riemannian manifold.

We are now in position to define a Dirac-type operator, $D_\circ$, on $\mathcal{A}^\infty \otimes S$. It is simply the composition of the following three operators. The first is the operator $d \otimes I^S$ from $\mathcal{A}^\infty \otimes S$ into $\Omega_\omega \otimes S = \mathcal{A}^\infty \otimes g' \otimes S$, where $I^S$ denotes the identity operator on $S$. The second is the operator $I^\mathcal{A}^\infty \otimes i \otimes I^S$ from $\mathcal{A}^\infty \otimes g' \otimes S$ into $\mathcal{A}^\infty \otimes \mathbb{C}\ell(g') \otimes S$, where $i$ is the inclusion of $g'$ into $\mathbb{C}\ell(g')$. The third is the operator $I^\mathcal{A}^\infty \otimes \kappa$ from $\mathcal{A}^\infty \otimes \mathbb{C}\ell(g') \otimes S$ into $\mathcal{A}^\infty \otimes S$ coming from the “Clifford multiplication” $\kappa$. Briefly:

$$\mathcal{A}^\infty \otimes S \xrightarrow{d} \mathcal{A}^\infty \otimes g' \otimes S \xrightarrow{i} \mathcal{A}^\infty \otimes \mathbb{C}\ell(g') \otimes S \xrightarrow{\kappa} \mathcal{A}^\infty \otimes S.$$ 

This can be expressed in the somewhat cryptic form

$$D_\circ \Psi = \kappa(d \Psi)$$

for all $\Psi \in \mathcal{A}^\infty \otimes S$. We can obtain a more explicit form for $D_\circ$ by choosing a basis, $\{E_j\}$, for $g$ and letting $\{\epsilon_j\}$ be the dual basis for $g'$. Then for any $X \in g$ we have $X = \sum \langle X, \epsilon_j \rangle E_j$, and so for any $a \in \mathcal{A}^\infty$ we have $da(X) = \sum \langle X, \epsilon_j \rangle \alpha_{E_j}(a)$, so that $da = \sum \alpha_{E_j}(a) \otimes \epsilon_j$. Then for $\psi \in S$ we have

$$D_\circ(a \otimes \psi) = \sum \alpha_{E_j}(a) \otimes \kappa_{\epsilon_j}(\psi). \tag{2.1}$$

Notice that $D_\circ$ only depends on the action of $g$ on $\mathcal{A}^\infty$, not on the action of $G$ itself. Thus we can replace $G$ by its simply-connected covering group whenever convenient. This observation will be useful later. It is also important to notice that the expression for $D_\circ$ is independent of the choice of the basis $\{E_j\}$.

To view $D_\circ$ as an (unbounded) operator on a Hilbert space, we let $\tau$ be the (unique by [43]) $\alpha$-invariant tracial state on $\mathcal{A}$ [43], and we define an inner product on $\mathcal{A} \otimes S$ by

$$\langle a_1 \otimes \psi_1, a_2 \otimes \psi_2 \rangle = \tau(a_1^{\ast} a_2) \langle \psi_1, \psi_2 \rangle$$
on elementary tensors. (Much as before, we choose our inner product on $S$ to be linear in the second variable.) On completing, we obtain a Hilbert space, $L^2(\mathcal{A}, \tau) \otimes S$ where $L^2(\mathcal{A}, \tau)$ is the GNS Hilbert space for $\tau$, with $D_o$ defined on the dense subspace $\mathcal{A}^\infty \otimes S$.

**Definition 2.3.** We will call the operator $D_o$ ($= D^A_o$) constructed above the **general Dirac-type operator** for the ergodic action $\alpha$ of $G$ on $\mathcal{A}$ and the given inner product on $g'$. A simple calculation (given in the proof of proposition 2.12 of [38]) shows that $D_o$ is symmetric. Because $\tau$ is $\alpha$-invariant, the action $\alpha$ is unitary with respect to the inner product that $\tau$ determines on $\mathcal{A}^\infty$, and so it extends to a unitary representation on $L^2(\mathcal{A}, \tau)$. Consequently, since $S$ is finite-dimensional, $L^2(\mathcal{A}, \tau) \otimes S$ decomposes into a direct sum of orthogonal finite-dimensional $\alpha$-invariant subspaces of $\mathcal{A}^\infty$. From the definition of $D_o$ it is evident that $D_o$ carries each of these finite-dimensional subspaces into itself. From this and the fact that $D_o$ is symmetric, it is easy to obtain:

**Proposition 2.4.** As an operator on $L^2(\mathcal{A}, \tau) \otimes S$ with dense domain $\mathcal{A}^\infty \otimes S$, the operator $D_o$ is essentially self-adjoint, and there is an orthonormal basis for $L^2(\mathcal{A}, \tau) \otimes S$ consisting of elements of $\mathcal{A}^\infty \otimes S$ that are eigenvectors of $D_o$.

What is not so easy to see is that when $\mathcal{A}$ is infinite-dimensional the eigenvalues of $D_o$, counted with multiplicity, converge in absolute value to $\infty$ (i.e. $D_o$ has “compact resolvent”). A somewhat indirect proof of this fact is given in theorem 5.5 of [38].

For $a \in \mathcal{A}^\infty$ let $M_a$ denote the operator on $\mathcal{A}^\infty \otimes S$ corresponding to the left regular representation of $\mathcal{A}^\infty$ on itself. As seen in [37], a simple calculation shows that for any $b \in \mathcal{A}^\infty$ and $\psi \in S$ we have

$$[D_o, M_a](b \otimes \psi) = \sum (M_{\alpha E_j(a)} \otimes \kappa \epsilon_j)(b \otimes \psi).$$

To simplify notation we will usually write $\alpha E_j(a)$ instead of $M_{\alpha E_j(a)}$ from now on. Then we see that we obtain:

**Proposition 2.5.** With notation as above, for any $a \in \mathcal{A}^\infty$ we have

$$[D_o, M_a] = \sum \alpha E_j(a) \otimes \kappa \epsilon_j,$$

acting on $\mathcal{A}^\infty \otimes S$.

From this it is clear that $[D_o, M_a]$ is a bounded operator. Because of this, we can define a seminorm on $\mathcal{A}^\infty$ that will play a central role in this paper.

**Notation 2.6.** For notation as above, define $L^{D_o}$ on $\mathcal{A}^\infty$ by

$$L^{D_o}(a) = \|[D_o, M_a]\|.$$
by the range of \( d \). As we will see, for many examples the bimodule \( \Omega_o \) that we have used above does not have this property. But before turning to those examples, we will soon consider two classes of example for which \( \Omega_o \) essentially does have this property. For now we will make a general observation about \( \Omega \) for our situation. Notice that its elements will be finite sums of the form \( \sum_k a_k b_k \) for \( a_k, b_k \in A^\infty \). (One uses the derivation property to show that right multiplication of such sums by elements of \( A^\infty \) are again of this form.) Let \( K \) be a Hilbert space, and let \( B(K) \) be the algebra of bounded operators on \( K \). Let \( A \) be a \(*\)-subalgebra of \( B(K) \), and let \( D \) be a (possibly unbounded) self-adjoint operator on \( K \) having the property that \([D, a]\) extends to a bounded operator on \( K \) for any \( a \in A^\infty \) (so that if a few more axioms are satisfied, \((A, K, D)\) is a spectral triple as defined by Connes [12–15]). We can view \( B(K) \) as an \( A\)-\( A\)-bimodule in the evident way. Then the map \( \delta : a \to [D, a] \) is a derivation from \( A^\infty \) into \( B(K) \). Connes sets \( \Omega_D \) to be the \( A^\infty\)-\( A^\infty\)-subbimodule of \( B(K) \) generated by the range of \( \delta \) (see section 7.2 of [44]), so that its elements are finite sums of the form \( \sum_k a_k [D, b_k] \) for \( a_k, b_k \in A^\infty \). Connes views \( \Omega_D \) as a space of first-order differential forms on \( A \).

Let us see what \( \Omega_{D_o} \) is for \( D_o \) as defined earlier in this section. Thus \( K = A^\infty \otimes S \), completed. For \( M \) denoting the left regular representation of \( A \) on itself, we see that \( M \otimes \kappa \) is a representation of \( A \otimes \mathfrak{C}(\ell(g')) \) on \( K \). Its restriction to \( A^\infty \otimes \mathfrak{C}(\ell(g')) \) is an \( A^\infty\)-\( A^\infty\)-bimodule map of \( \Omega_o \) into \( B(K) \). For an element \( \sum_k a_k b_k \) of \( \Omega \) we have

\[
(M \otimes \kappa)(\sum_k a_k b_k) = \sum_k M_{a_k} \sum_j M_{E_j(b_k)} \otimes \kappa_{e_j} = \sum_k M_{a_k}[D_o, M_{b_k}].
\]

We thus obtain:

**Proposition 2.7.** For notation as above, \( \Omega_{D_o} \) is the image of \( \Omega_o \) under the map \( M \otimes \kappa \).

3. Invariance Under the Group Actions

Since the action \( \text{Cad} \) of \( G \) on \( g' \) is by orthogonal operators for our chosen inner product, it extends to \( \mathfrak{C}(\ell(g')) \) as an action by \(*\)-algebra automorphisms (Bogoliubov automorphisms), which we still denote by \( \text{Cad} \). Since \( G \) acts on both \( A \) and \( \mathfrak{C}(\ell(g')) \), we let \( \gamma \) denote the corresponding diagonal action \( \alpha \otimes \text{Cad} \) on \( A \otimes \mathfrak{C}(\ell(g')) \). It is just an extension of our earlier action \( \gamma \) of \( G \) on \( \Omega_o \) defined in Notation 2.1. It carries the smooth version into itself.

We want to have a representation of \( G \) by unitary operators on the Hilbert-space completion of \( A^\infty \otimes S \) that is compatible with the action \( \alpha \) on \( A \), and that commutes with \( D_o \). As discussed in [38], the action of \( G \) on \( S \) corresponding to the action of \( G \) on \( \mathfrak{C}(\ell(g')) \) can, in general, at best be implemented by a projective representation on \( S \). In [38] it is shown how to handle the case of projective representations, but it is also remarked there that if \( G \) is a simply connected semisimple compact Lie group then every projective representation of \( G \) is equivalent to an ordinary (unitary) representation of \( G \), so that for these groups there is a unitary representations on \( S \) corresponding to the action of \( G \) on \( \mathfrak{C}(\ell(g')) \).

So for the moment, we will simply assume that there is a unitary representation, \( \sigma \), of \( G \) on \( S \) that is compatible with the action \( \text{Cad} \) of \( G \) on \( \mathfrak{C}(\ell(g')) \) and the representation \( \kappa \) of \( \mathfrak{C}(\ell(g')) \) on \( S \) in the sense that

\[
\kappa_{\text{Cad}_x(q)} = \sigma_x q_x \sigma_x^* \tag{3.1}
\]

for all \( q \in \mathfrak{C}(\ell(g')) \) and \( x \in G \). We then define a representation, \( \tilde{\sigma} \), of \( G \) on \( A^\infty \otimes S \) by \( \tilde{\sigma} = \alpha \otimes \sigma \).
Proposition 3.1. With assumptions and notation as above, the operator $D_o$ on $A^\infty \otimes S$ commutes with the representation $\tilde{\sigma}$ of $G$ on $A^\infty \otimes S$.

Proof. Let $a \in A^\infty$ and $\psi \in S$ and $x \in G$. Then

$$D_o(\tilde{\sigma}_x(a \otimes \psi)) = D_o(\alpha_x(a) \otimes \sigma_x(\psi)) = \sum \alpha_{E_j}(\alpha_x(a)) \otimes \kappa_{E_j}(\sigma_x(\psi))$$

$$= \sum \alpha_x(\alpha_{Ad_x^{-1}(E_j)}(a)) \otimes \sigma_x(\kappa_{Ad_x^{-1}(E_j)}(\sigma_x(\psi))))$$

$$= (\alpha_x \otimes \sigma_x)((\alpha_{Ad_x^{-1}(E_j)}(a)) \otimes (\kappa_{Ad_x^{-1}(E_j)}(\psi)))$$

$$= \tilde{\sigma}_x(D_o(a \otimes \psi)),$$

(3.2)

where for the third equality we have used the compatibility condition, and for the fifth equality we have used the fact that $\{Ad_x^{-1}(E_j)\}$ is equally well a basis for $g$, with dual basis $\{Cad_x^{-1}(E_j)\}$.

The representation $\tilde{\sigma}$ is unitary, so it extends to a unitary representation on the Hilbert space completion of $A^\infty \otimes S$. We will discuss further the possible existence of the representation $\sigma$ in Sect. 8.

Proposition 3.2. With assumptions and notation as above, the seminorm $L^{D_o}$ defined in Notation 2.6 is invariant under the action $\alpha$ in the sense that

$$L^{D_o}(\alpha_x(a)) = L^{D_o}(a)$$

for all $a \in A^\infty$ and $x \in G$.

Proof. Note that for any $a, b \in A^\infty$, $\psi \in S$ and $x \in G$ we have

$$(\tilde{\sigma}_x M_a \tilde{\sigma}_x^{-1})(b \otimes \psi) = (\alpha_x \otimes \sigma_x)(M_a \otimes I^S)(\alpha_x^{-1} \otimes \sigma_x^{-1})(b \otimes \psi)$$

$$= (\alpha_x(a(\alpha_x^{-1}(b)) \otimes \psi = M_{\alpha_x(a)}(b \otimes \psi),$$

where we have used $M$ to denote the left action of $A$ on both $A$ and $A \otimes S$. Thus

$$(\tilde{\sigma}_x M_a \tilde{\sigma}_x^{-1}) = M_{\alpha_x(a)}.$$

Since $\tilde{\sigma}_x$ commutes with $D_o$, it follows that

$$[D_o, M_{\alpha_x(a)}] = D_o \tilde{\sigma}_x M_a \tilde{\sigma}_x^{-1} - \tilde{\sigma}_x M_a \tilde{\sigma}_x^{-1} D_o = \tilde{\sigma}_x [D_o, M_a] \tilde{\sigma}_x^{-1}.$$

Since $\tilde{\sigma}_x$ is a unitary operator for the inner product on $A \otimes S$, we obtain the desired equality. □

4. Charge Conjugation

In [38] it is shown that a Dirac operator constructed in the way described above possesses an important structure on its domain, namely, a “real structure”, related to a charge conjugation operator, as first used by Connes [14] for non-commutative geometry. We do not need to use this structure later, so we will give here only a brief description of it, with references to the literature for further details.

Since $\mathbb{C}l(g')$ is the complexification of the Clifford algebra $\mathbb{C}l(g')$ over $\mathbb{R}$, it has the standard complex conjugation operator, which is a conjugate-linear algebra automorphism. We denote it by $q \mapsto \bar{q}$ for $q \in \mathbb{C}l(g')$. Let $\chi$ denote the usual grading
automorphism on $C\ell(g')$ determined by $\chi(\mu) = -\mu$ for $\mu \in g'$. By definition, the charge conjugation at the level of the Clifford algebra is the conjugate-linear automorphism, $c$, obtained by composing complex conjugation with $\chi$. Note that on the even subalgebra $C\ell^e(g')$ it is just complex conjugation.

But the operator that we need is a conjugate-linear operator $C_S$ on $S$ that implements $c$ for the representation $\kappa$, that is, such that

$$\kappa_{c(q)} = C_S q C_S^{-1}$$

for all $q \in C\ell(g')$. Furthermore, the operator $C_S$ is required to respect the inner product on $S$ in the sense that $(C_S(\psi), C_S(\phi)) = (\phi, \psi)$ for all $\psi, \phi \in S$. Then $C_S$ is unique up to a scaler multiple of modulus 1. It is normalized to satisfy $C_S^2 = \pm I^S$, where the sign depends on $\dim(g')$.

Then the charge-conjugation operator, $C$, on $A \otimes S$ is defined on elementary tensors by

$$C(a \otimes \psi) = a^* \otimes C_S(\psi).$$

It is conjugate-linear, and respects the $A$-valued inner product on $A \otimes S$. Its most important property is that

$$[a, Cb^*C^{-1}] = 0$$

for all $a, b \in A$, so that $b \mapsto Cb^*C^{-1}$ gives a right action of $A$ on $A \otimes S$ such that $A \otimes S$ is an $A$-$A$-bimodule. When $\dim(g')$ is even we also have $C\gamma = \pm \gamma C$, where the sign depends on $\dim(g')$. All this is summarized by saying that the charge conjugation operator $C$ provides a “real structure” on $A \otimes S$.

For the Dirac operator $D_o$ constructed above we then have the very important condition that

$$[[D_o, a], Cb^*C^{-1}] = 0$$

for all $a, b \in A^\infty$. It is called the “first-order condition”, and reflects the fact that $D_o$ is like a differential operator of order one. Furthermore, we have $CD_o = \pm D_o C$, where again the sign depends on $\dim(g')$. When $H$ is the Hilbert-space completion of $A \otimes S$, the triple $(A, H, D_o)$ is an example of the notion of “spectral triple” introduced by Connes, and $(A, H, D_o, C)$ is an example of a “real spectral triples”. For details about all of this see [13–15,36,45–48].

**Remark 4.1.** The results obtained in these first sections suggest the following questions:

1. For a given compact connected Lie group $G$, how can one characterize which of its ergodic actions $(A, a)$ have the property that the sub-bimodule, $\Omega$, of $\Omega_a = A^\infty \otimes g'$ generated by the range of the derivation $d$ is (finitely generated) projective as a right $A^\infty$-module?

2. Among those actions for which $\Omega$ is a projective module, how does one characterize those such that $\Omega$ also admits a “real structure” of the kind sketched above?

I have not investigated these questions.

Note that much concerning the classification of ergodic actions of connected compact Lie groups is a mystery. Good answers are known only for $G$ commutative [49], or for $G = SU(2)$ [50] (and subsequent papers) as far as I know.
5. Casimir Operators

Dirac found his famous equation that predicted the existence of the positron because he was looking for a first-order differential operator that is a square root of the Klein-Gordan operator, which is the appropriate Laplace-type operator for flat space-time. So it is of interest to examine the square of the Dirac operator that we have defined above to see whether it is related to a Laplace operator. For a Lie group, \( G \), whose Lie algebra \( \mathfrak{g} \) has a chosen \( Ad \)-invariant (non-degenerate) inner product, the appropriate Laplace operators come from the degree-2 Casimir element in the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) of \( \mathfrak{g} \). The Casimir element depends on the choice of inner product. Let \( \{ E_j \} \) be a basis for \( \mathfrak{g} \) that is orthonormal for the chosen inner product. Then the Casimir element, \( C \), is defined by

\[
C = \sum_j (E_j)^2.
\]

(We do not include the minus sign which is often used, so the image of \( C \) by the infinitesimal version of a unitary representations of \( G \) will be a non-positive operator.). Then \( C \) is in the center of \( \mathcal{U}(\mathfrak{g}) \), and so for any irrep \( (\mathcal{U}, \mathcal{H}) \) of \( G \), with corresponding representation of \( \mathfrak{g} \) and \( \mathcal{U}(\mathfrak{g}) \), the operator \( UC \) will be a scalar multiple of the identity operator on \( \mathcal{H} \). We will find this useful in Sect. 10 for determining the spectrum of the Dirac operator when \( G = SU(2) \).

\[
\text{Proposition 5.1. Let } \{ \epsilon_j \} \text{ be the basis for } \mathfrak{g}' \text{ dual to the orthonormal basis } \{ E_j \} \text{ for } \mathfrak{g}. \text{ Then}
\]

\[
D_0^2 = -\alpha C \otimes I^S + \sum_{j<k} \alpha_{[E_j, E_k]} \otimes \kappa_{\epsilon_j \epsilon_k}
\]

\[
\text{Proof. Since in } \mathbb{C}l(\mathfrak{g}') \text{ we have } \epsilon_j \epsilon_k = -\epsilon_k \epsilon_j \text{ and } \epsilon_j^2 = -1, \text{ we have}
\]

\[
D_0^2 = \sum_{j,k} \alpha_{E_j} \alpha_{E_k} \otimes \kappa_{\epsilon_j \epsilon_k}
\]

\[
= \sum_j (\alpha_{E_j})^2 \otimes (\kappa_{\epsilon_j})^2 + \sum_{j<k} \alpha_{E_j} \alpha_{E_k} \otimes \kappa_{\epsilon_j \epsilon_k} + \sum_{j>k} \alpha_{E_j} \alpha_{E_k} \otimes \kappa_{\epsilon_j \epsilon_k}
\]

\[
= -\alpha C \otimes I^S + \sum_{j<k} (\alpha_{E_j} \alpha_{E_k} - \alpha_{E_k} \alpha_{E_j}) \otimes \kappa_{\epsilon_j \epsilon_k}
\]

\[
= -\alpha C \otimes I^S + \sum_{j<k} \alpha_{[E_j, E_k]} \otimes \kappa_{\epsilon_j \epsilon_k},
\]

as desired. \( \square \)

It is appropriate to view \( \alpha C \) as the Laplace operator on \( \mathcal{A}^\infty \), and so the term \( -\alpha C \otimes I^S \) is very analogous to what one obtains for the square of the Dirac operator on flat \( \mathbb{R}^d \). The second term can be viewed as some kind of curvature term analogous to the curvature term in the Lichnerowicz formula \([42,51,52]\), but I do not know how to define a general version of curvature that would make this precise.

For later use in Sect. 10 we need the following result about the image of \( C \) under the representation \( \tilde{\sigma} = \alpha \otimes \sigma \) defined just before Proposition 3.1. We let \( \sigma \) also denote the corresponding representation of \( \mathfrak{g} \). Then \( \tilde{\sigma} \), as a representation of \( \mathfrak{g} \), is given by \( \tilde{\sigma}_X = \alpha_X \otimes I^S + I^A \otimes \sigma_X \) for any \( X \in \mathfrak{g} \), where \( I^S \) and \( I^A \) denote the identity operators on \( S \) and \( A \) respectively.
Proposition 5.2. For notation as above, we have
\[ \tilde{\sigma}_C = \alpha_C \otimes I^S + 2 \sum_j (\alpha_{E_j} \otimes \sigma_{E_j}) + I^A \otimes \sigma_C, \]

Proof.
\[
\tilde{\sigma}_C = \sum_j (\alpha_{E_j} \otimes I^S + I^A \otimes \sigma_{E_j})^2 \\
= \sum_j (\alpha_{E_j})^2 \otimes I^S + 2 \sum_j (\alpha_{E_j} \otimes \sigma_{E_j}) + \sum_j I^A \otimes (\sigma_{E_j})^2 \\
= \alpha_C \otimes I^S + 2 \sum_j (\alpha_{E_j} \otimes \sigma_{E_j}) + I^A \otimes \sigma_C.
\]

The operator \( \sum_j (\alpha_{E_j} \otimes \sigma_{E_j}) \) from the middle term above, looks somewhat like our expression for the Dirac operator. We will see in Sect. 10 that for the case of \( G = SU(2) \) it does in fact essentially coincide with the Dirac operator. This gives us a strong tool for computing the spectrum of the Dirac operator.

The results above have some resonance with results in the neighborhood of equation 2.7, theorem 2.13 and theorem 2.21 of [53] relating Dirac operators and Casimir elements, but that paper is aimed only at homogeneous spaces \( G/K \) where \( G \) is a compact semisimple group and \( K \) is a connected subgroup of \( G \) whose rank is the same as that of \( G \). That setting is very pertinent to the coadjoint orbits that we will consider later.

6. First Facts About Spinors

Let \( m \) be a finite-dimensional Hilbert space over \( \mathbb{R} \), and let \( \mathbb{C} \ell(m) \) denote the complex Clifford algebra for \( m \), much as discussed above for \( g' \). For its standard involution \( \mathbb{C} \ell(m) \) is a C*-algebra.

There is a special element, \( \gamma \), of \( \mathbb{C} \ell(m) \), called the “chirality element” [36], that is a suitably normalized product of all the elements of a basis for \( m \subset \mathbb{C} \ell(m) \), and that has the properties that \( \gamma^2 = 1, \gamma \neq 1 \) and \( \gamma^* = \gamma \). Let \( n = \dim(m) \). If \( n \) is even then \( \mathbb{C} \ell(m) \) is isomorphic to a full matrix algebra. For this case \( \gamma \) splits an irreducible representation of \( \mathbb{C} \ell(m) \) into two subspaces of equal dimension that are carried into themselves by the subalgebra \( \mathbb{C} \ell^e(m) \) of even elements of \( \mathbb{C} \ell(m) \). There is a standard way of explicitly constructing an irreducible representation of \( \mathbb{C} \ell(m) \) on a Fock-space. We will need to use some of the main steps of that construction in Sect. 14.

When \( n \) is odd, \( \gamma \) is in the center of \( \mathbb{C} \ell(m) \) and it splits \( \mathbb{C} \ell(m) \) into the direct sum of 2 full matrix algebras. Thus up to equivalence \( \mathbb{C} \ell(m) \) has two inequivalent irreducible representations, neither of which is faithful. The subalgebra, \( \mathbb{C} \ell^e(m) \), of even element of \( \mathbb{C} \ell(m) \) is itself a Clifford algebra on a vector space of even dimension, and so is a full matrix algebra, which has a unique irreducible representation (up to isomorphism). We will view the two irreducible representations of \( \mathbb{C} \ell(m) \) as being the irreducible representation of \( \mathbb{C} \ell^e(m) \), extended to \( \mathbb{C} \ell(m) \) by sending \( \gamma \) to \( +I \) for one of them, and to \( -I \) for the other. (Notice that the restriction to \( \mathbb{C} \ell^e(m) \) of any (unital) representation of \( \mathbb{C} \ell(m) \) will be a faithful representation of \( \mathbb{C} \ell^e(m) \).)
Now let $\kappa$ be a representation of $\mathbb{C}\ell(m)$ on a finite-dimensional Hilbert space $\mathcal{S}$. To deal with the fact that $\kappa$ need not be a faithful representation of $\mathbb{C}\ell(m)$, we need the following technical result.

**Lemma 6.1.** Let notation be as above, and let $\epsilon_1, \ldots, \epsilon_p$ be elements of $m$. Let $\mathcal{D}$ be any unital $C^\ast$-algebra, and let $(\mathcal{K}, M)$ be a faithful representation of $\mathcal{D}$. Let $d_1, \ldots, d_p$ be elements of $\mathcal{D}$, and let $t = \sum d_j \otimes \epsilon_j$, viewed as an element of the $C^\ast$-algebra $\mathcal{D} \otimes \mathbb{C}\ell(m)$. Then

$$\|(M \otimes \kappa)(t)\| = \|t\|.$$  

**Proof.** Since $\mathbb{C}\ell(m)$ is finite dimensional, there is a unique $C^\ast$-algebra norm on $\mathcal{D} \otimes \mathbb{C}\ell(m)$. If the dimension $n$ of $m$ is even, then the representation $\kappa$ of $\mathbb{C}\ell(m)$ must be faithful, and so the homomorphism $M \otimes \kappa$ between $C^\ast$-algebras must be faithful and so isometric. This gives the desired result.

If $n$ is odd, then because $\gamma$ is an odd unitary element of $\mathbb{C}\ell(m)$, left multiplication by $\gamma$ is an isometry from the odd subspace of $\mathbb{C}\ell(m)$ onto $\mathbb{C}\ell^e(m)$. Thus multiplication by $1_D \otimes \gamma$ is an isometry that carries $t$ into $\mathcal{D} \otimes \mathbb{C}\ell^e(m)$. But the restriction of $\kappa$ to $\mathbb{C}\ell^e(m)$ is faithful, and so $M \otimes \kappa$ is faithful and so an isometry from $\mathcal{D} \otimes \mathbb{C}\ell^e(m)$ into $\mathcal{B}(\mathcal{K}) \otimes \mathcal{B}(\mathcal{S})$. Consequently

$$\|t\| = \|(1_D \otimes \gamma)t\| = \|(M \otimes \kappa)(1_D \otimes \gamma)t\| = \|(M \otimes \kappa)(t)\|,$$

since $\kappa \gamma = \pm I^{\mathcal{S}}$.

The following proposition, which will be of use later, is a typical way in which we will apply the above Lemma.

**Proposition 6.2.** Let $\mathcal{A}$ be a unital $C^\ast$-algebra, and let $\alpha$ be an ergodic action of the Lie group $G$ on $\mathcal{A}$. Fix a choice of a finite-dimensional spinor space $\mathcal{S}$, and use it to construct the Dirac operators $D^\alpha_o$ for $\mathcal{A}$ as for Definition 2.3. Then

$$L^{D^\alpha_o}(a) = \|[D_o, M_a]\| = \|\sum \alpha_{E_j}(a) \otimes \epsilon_j\|,$$

where the norm on the right is that on the $C^\ast$-algebra $\mathcal{A} \otimes \mathbb{C}\ell(\mathfrak{g}')$. Thus $L^{D^\alpha_o}$ is independent of the choice of the spinor space $\mathcal{S}$.

**Proof.** We use Lemma 6.1 for the last equality in

$$\|[D_o, M_a]\| = \|\sum M_{\alpha_{E_j}(a)} \otimes \kappa_{\epsilon_j}\|
= \|(M \otimes \kappa)(\sum \alpha_{E_j}(a) \otimes \epsilon_j)\| = \|\sum \alpha_{E_j}(a) \otimes \epsilon_j\|.$$

The following proposition will be important for the proof of our main theorem. In that proof the $\theta$ here will be a Berezin symbol map.

**Proposition 6.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be unital $C^\ast$-algebras, and let $\alpha$ and $\beta$ be ergodic actions of the Lie group $G$ on $\mathcal{A}$ and $\mathcal{B}$. Fix a choice of a finite-dimensional spinor space $\mathcal{S}$, and use it to construct Dirac operators $D^\alpha_o$ and $D^\beta_o$ for $\mathcal{A}$ and $\mathcal{B}$ as above. Let $L^{D^\alpha_o}$ and $L^{D^\beta_o}$ be the seminorms determined by $D^\alpha_o$ and $D^\beta_o$ as in Notation 2.6. Let $\theta$ be a unital completely positive operator from $\mathcal{A}$ to $\mathcal{B}$ that intertwines the actions $\alpha$ and $\beta$, so that it carries $\mathcal{A}^\infty$ into $\mathcal{B}^\infty$. Then for any $a \in \mathcal{A}^\infty$ we have

$$L^{D^\beta_o}(\theta(a)) \leq L^{D^\alpha_o}(a).$$
Proof. According to Proposition 6.2
\[ L_D^B(\theta(a)) = \| \sum \beta_{E_j}(\theta(a)) \otimes \epsilon_j \| = \| \sum (\alpha_{E_j}(a)) \otimes \epsilon_j \| = \| (\theta \otimes I^{C_\ell(g^o)}) \sum \alpha_{E_j}(a) \otimes \epsilon_j \| \]
because \( \| (\theta \otimes I^{C_\ell(g^o)}) \| = 1 \) since \( \theta \) is a unital completely positive operator. (See section II.6.9 of [54].) ⊓⊔

Proposition 6.4. With the assumptions of Proposition 6.3, let \( \hat{\theta} = \theta \otimes I^S \), viewed as a map from \( A^\infty \otimes S \) to \( B^\infty \otimes S \). Then
\[ \hat{\theta} D_o^A = D_o^B \hat{\theta} \]
as operators from \( A^\infty \otimes S \) to \( B^\infty \otimes S \). Consequently if \( \Psi \) is an eigenvector for \( D_o^A \) with eigenvalue \( \lambda \) then \( \hat{\theta} \Psi \) is an eigenvector for \( D_o^B \) with eigenvalue \( \lambda \) if \( \hat{\theta} \Psi \neq 0 \).
Proof. For \( a \in A^\infty \) and \( \psi \in S \) we have
\[ \hat{\theta}(D_o^A(a \otimes \psi)) = \hat{\theta}(\sum \alpha_{E_j}(a) \otimes \kappa_{E_j}(\psi)) = \sum \theta(\alpha_{E_j}(a)) \otimes \kappa_{E_j}(\psi) = \sum \beta_{E_j}(\theta(a)) \otimes \kappa_{E_j}(\psi) = D_o^B(\hat{\theta}(a \otimes \psi)), \]
as needed. ⊓⊔

7. The C*-metrics
In this section we examine the C*-metrics that are determined by the Dirac operators constructed in Sect. 2. In the literature there are small variations in the definition of a “C*-metric”. The following definition is appropriate for this paper. For this purpose a “C*-normed ∗-algebra” means a normed ∗-algebra whose norm satisfies the C*-algebra identity \( \| a^*a \| = \| a \|^2 \), so that its completion is a C*-algebra. For us the main class of examples is \( A^\infty \) as used earlier.

Definition 7.1. Let \( A \) be a unital C*-normed ∗-algebra. By a C*-metric on \( A \) we mean a seminorm \( L \) on \( A \) having the following properties. For any \( a, b \in A \):
1. \( L(a) = 0 \) if and only if \( a \in C1_A \).
2. \( L(a^*) = L(a) \).
3. \( L \) is lower semi-continuous with respect to the norm of \( A \).
4. \( L \) satisfies the Leibnitz inequality
\[ L(ab) \leq L(a)\|b\| + \|a\|L(b) \]
5. Let \( S(A) \) be the state space of \( A \). Define a metric, \( \rho^L \), on \( S(A) \) by
\[ \rho^L(\mu, \nu) = \sup\{ |\mu(a) - \nu(a)| : a^* = a \text{ and } L(a) \leq 1 \}. \]
(7.1)

Without further hypotheses this metric can take value \( +\infty \). We require that the topology on \( S(A) \) from this metric coincides with the weak-* topology on \( S(A) \). Then, in particular, \( \rho^L \) will never take value \( +\infty \). (The condition \( a^* = a \) in the definition of \( \rho^L(\mu, \nu) \) can be omitted without changing \( \rho^L(\mu, \nu) \), as explained just before definition 2.1 of [9].)
If $A$ is actually a $C^*$-algebra, and if $L$ is a seminorm on $A$ that is permitted to take the value $+\infty$, but is semi-finite in the sense that $A_f = \{a : L(a) < \infty\}$ is dense in $A$, and if the restriction of $L$ to $A_f$ satisfies the 5 properties above, then we will also call $L$ a $C^*$-metric (on $A$).

A pair $(A, L)$ consisting of a unital $C^*$-normed $*$-algebra $A$ and a $C^*$-metric $L$ on $A$ is called a compact $C^*$-metric space.

Here is the motivating example. Let $(X, \rho)$ be a compact metric space, and let $A$ be the $C^*$-algebra $C(X)$ of all continuous complex-valued functions on $X$. Let $L^\rho$ assign to each function its Lipschitz constant, that is,

$$L^\rho(f) = \sup\{|f(x) - f(y)|/\rho(x, y) : x, y \in X \text{ and } x \neq y\}.$$ 

Then $L^\rho$ is a $C^*$-metric. Furthermore, one can recover $\rho$ from $L^\rho$. To see this, notice that the state space $S(A)$ is just the set of probability measures on $X$. Let $\rho^{L^\rho}$ be the metric on $S(A)$ defined by Eq. (7.1) for $L^\rho$. Then $\rho(x, y) = \rho^{L^\rho}(\delta_x, \delta_y)$, where $\delta_z$ denotes the delta-measure at $z$ for any $z \in X$.

We remark that property 5 is often the most difficult to verify for examples, but having $S(A)$ compact for the $\rho^L$-topology (property 5) is crucial for the definitions of quantum Gromov–Hausdorff distance which we will discuss later.

We will not explicitly need all of the next few remarks, but they give important context to the definition of $C^*$-metrics. Suppose that $L$ is a $C^*$-metric on a unital $C^*$-normed $*$-algebra $A$, and let

$$L^1_A = \{a \in A : L(a) \leq 1\}. \tag{7.2}$$

Let $\bar{A}$ be the completion of $A$, let $\bar{L}^1_A$ be the closure of $L^1_A$ in the $C^*$-algebra $\bar{A}$, and let $\bar{L}$ denote the corresponding “Minkowski functional” on $\bar{A}$, defined by setting, for $c \in \bar{A}$,

$$\bar{L}(c) = \inf\{r \in \mathbb{R}^+ : c \in r \bar{L}^1_A\},$$

with value $+\infty$ if there is no such $r$. Then $\bar{L}$ is a seminorm on $\bar{A}$ (often taking value $+\infty$), and the proof of proposition 4.4 of [55] tells us that because $L$ is lower semicontinuous, $\bar{L}$ is an extension of $L$. We call $\bar{L}$ the closure of $L$. We see that the set $\{c \in \bar{A} : \bar{L}(c) \leq 1\}$ is closed in $\bar{A}$. We say that the original seminorm $L$ on $A$ is closed if $L^1_A$ is already closed in $\bar{A}$, or, equivalently, is complete for the norm on $A$. Clearly if $L$ is closed, then it is lower semicontinuous. If $\bar{L}$ is closed and is not defined on all of $\bar{A}$, then $\bar{L}$ is obtained simply by giving it value $+\infty$ on all the elements of $\bar{A}$ that are not in $A$. It is clear that if $L$ is semifinite then so is $\bar{L}$. We recall that a unital subalgebra $B$ of a unital algebra $A$ is said to be spectrally stable in $A$ if for any $b \in B$ the spectrum of $b$ as an element of $B$ is the same as its spectrum as an element of $A$, or equivalently, that any $b$ that is invertible in $A$ is invertible in $B$. From proposition 3.1 of [11] one easily obtains:

**Proposition 7.2.** Let $L$ be a $C^*$-metric on a unital $C^*$-normed $*$-algebra $A$. Then the closure of $L$ is a $C^*$-metric. If $L$ is a closed $C^*$-metric, then $A_f$ is a spectrally-stable subalgebra of $\bar{A}$ that is carried into itself under the holomorphic functional calculus of $\bar{A}$.

We now continue our discussion of Dirac operators. Our discussion is very close to that given in section 4 of [37], but part of it will be general enough to also apply in later sections.
We continue with the notation of Sect. 2. Thus, for \( a \in \mathcal{A}^\infty \),

\[
[D_0, M_a] = \sum \alpha_{E_j}(a) \otimes \kappa_{\epsilon_j},
\]
acting on \( \mathcal{A}^\infty \otimes S \), and we define a seminorm, \( L^{D_0} \), on \( \mathcal{A}^\infty \) by

\[
L^{D_0}(a) = \|[D_0, M_a]\|.
\]

It is shown in theorem 4.2 of [37] that \( L^{D_0} \) satisfies property 5 of Definition 7.1. This will also follow from the considerations below.

For the proof of our main theorem we need certain bounds on \( L^{D_0} \). For this purpose we define a different seminorm, \( L_d \), on \( \mathcal{A}^\infty \) by

\[
L_d(a) = \|d_a\|,
\]
where we view \( d_a \) as a linear transformation from \( g \) to \( \mathcal{A}^\infty \), each of which is a normed space, with the norm on \( g \) coming from its inner product dual to the inner product on \( g' \). Thus

\[
L_d(a) = \sup\{\|\alpha_X(a)\| : X \in g \text{ and } \|X\| \leq 1\}. \tag{7.3}
\]

It is shown in theorem 3.1 of [37] that \( L_d \) satisfies property 5 of Definition 7.1. It is easy to check the other conditions of Definition 7.1, and thus we conclude that \( L_d \) is in fact a C*-metric.

We will obtain bounds on \( L^{D_0} \) in terms of \( L_d \). For this purpose we can for convenience choose the basis vectors \( \epsilon_j \) for \( g' \) to be orthonormal. Then as elements of \( C_\ell(g') \) they will satisfy the relations \( \epsilon_j^* = -\epsilon_j \), \( \epsilon_j^2 = -1 \) and \( \epsilon_j \epsilon_k = -\epsilon_k \epsilon_j \) if \( j \neq k \). The following lemma is probably well known, but I have not seen it in the literature.

**Lemma 7.3.** Let \( A \) and \( C \) be unital C*-algebras, and let \( \varepsilon_1, \ldots, \varepsilon_m \) be elements of \( C \) that satisfy the relations \( \varepsilon_j^* = -\varepsilon_j \), \( \varepsilon_j^2 = -1 \) and \( \varepsilon_j \varepsilon_k = -\varepsilon_k \varepsilon_j \) if \( j \neq k \). Let \( a_1, \ldots, a_m \) be elements of \( A \), and let \( t = \sum a_j \otimes \varepsilon_j \), an element of \( A \otimes C \) (for any C*-norm). Then

\[
\sup\{\|a_j\| : 1 \leq j \leq m\} \leq \|t\| \leq \sum\{\|a_j\| : 1 \leq j \leq m\}.
\]

**Proof.** For each \( j \) let \( p_j \) be the spectral projection of \( \varepsilon_j \) for the eigenvalue \(+i\). Note that \( p_j \neq 0 \). From the third relation above one quickly sees that \( p_j \varepsilon_k p_j = 0 \) if \( j \neq k \). It follow that \((1 \otimes p_j)t(1 \otimes p_j) = i(a_j \otimes p_j)\), so that \( \|a_j\| \leq \|t\| \), and the first inequality holds. The second inequality holds immediately from the definition of \( t \). \( \square \)

**Proposition 7.4.** In terms of the notation used before the lemma, for any \( a \in \mathcal{A}^\infty \) we have

\[
L_d(a) \leq L^{D_0}(a) \leq nL_d(a),
\]

where \( n = \dim(g) \).

**Proof.** According to Proposition 6.2 we have \( L^{D_0}(a) = \| \sum \alpha_{E_j}(a) \otimes \varepsilon_j \| \), where the norm is that on \( A \otimes C_\ell(g') \). So we can apply the above Lemma with \( C = C_\ell(g') \). Any \( X \in g \) with \( \|X\| = 1 \) can serve as one element, say \( E_1 \), of an orthonormal basis for \( g \). Thus for any such \( X \) we conclude from the above Lemma that \( \|\alpha_X(a)\| \leq \| \sum \alpha_{E_j}(a) \otimes \varepsilon_j \| \leq \sum \|\alpha_{E_j}(a)\| \).

\( \square \)
From the inequalities in Proposition 7.4 it is easily seen that the metric on \( S(A) \) from \( L^{D_o} \) is equivalent (not necessarily equal) to the metric from \( L_d \). Consequently we conclude, as in theorem 4.2 of [37], that \( L^{D_o} \) itself satisfies property 5 of Definition 7.1. Furthermore, \( L^{D_o} \) is lower semi-continuous, as seen in example 2.4 of [11]. It is easily verified that \( L^{D_o} \) satisfies the Leibniz inequality, and even is strongly Leibniz in the sense defined in definition 1.1 of [11]. Thus we have obtained:

**Proposition 7.5.** With notation as above, \( L^{D_o} \) is a C*-metric on \( A \).

But there is an even more basic general C*-metric on \( A \), which plays the principal role in [10,37,56], and which will be crucial for the proof of our main theorem. From the chosen inner product on \( g' \), and its dual on \( g \), we obtain a Riemannian metric on \( G \), which is both right and left-invariant because the inner product is Cad-invariant. From that Riemannian metric we obtain a corresponding continuous length-function, \( \ell \), on \( G \) (coming from path lengths determined by the Riemannian metric), which is constant on conjugacy classes. For any ergodic action \( \alpha \) of \( G \) on a unital C*-algebra \( A \) we define a seminorm \( L_\ell \) on \( \mathcal{A}\infty \) by

\[
L_\ell(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \in G \text{ and } x \neq 0\}.
\]  

(7.4)

(where the value \(+\infty\) is permitted). Notice that this seminorm is well-defined for any continuous length-function on \( G \) (of which there are many), and that \( G \) need not be a Lie group. In fact, for any compact group \( G \), any continuous length-function on \( G \) (which for our definition is constant on conjugacy classes), and any ergodic action of \( G \) on any unital C*-algebra \( A \) we define a seminorm \( L_\ell \) on \( \mathcal{A}\infty \) by

\[
L_\ell(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \in G \text{ and } x \neq 0\}.
\]  

(7.4)

(Proof. Let \( a \in \mathcal{A}\infty \), and let \( c \) be a smooth path in \( G \) from \( e_G \) to \( x \in G \). Then \( \phi \), defined by \( \phi(t) = \alpha_{c(t)}(a) \), is smooth, and so we have

\[
\|\alpha_x(a) - a\| = \|\int \phi'(t)dt\| \leq \|\alpha_{c(t)}(\alpha_{c'(t)}a)\|dt
\]

\[
= \int \|\alpha_{c(t)}(da(c'(t)))\|dt \leq \|da\| \int \|c'(t)\|dt.
\]

But the last integral is just the length of \( c \). Thus from the definition of the ordinary metric on \( G \) as in infimum over all smooth paths, with its length function \( \ell \) using paths from \( e_G \) to \( x \in G \), we obtain \( \|\alpha_x(a) - a\| \leq \|da\|\ell(x) \). Thus for all \( x \in G \)

\[
\|\alpha_x(a) - a\|/\ell(x) \leq \|da\|
\]

from which the desired result follows immediately. \( \square \)

On combining Propositions 7.4 and 7.6, we obtain:
Corollary 7.7. With notation as above, for any \( a \in A^\infty \) we have

\[
L_\ell(a) \leq L^{D_\ell}(a).
\]

By means of this key corollary we will in Sect. 17 be be able to apply the important bounds on \( L_\ell \) obtained in \([10,56]\) to prove that matrix algebras converge to coadjoint orbits for the \( C^* \)-metrics corresponding to Dirac operators.

8. More About Spinors

It is best if as our spinor bundle we can use a representation of the Clifford algebra on \( S \) that is irreducible. For some compact Riemannian manifolds this can not be done. In this section we collect further algebraic facts and establish the conventions and notation that we need in order to understand when this can be done. Proofs of the assertions made below for which no proof is given here can be found in \([51,52]\). Also useful are \([36,42]\), but they use slightly different conventions.

Let \( m \) be a finite dimensional Hilbert space over \( \mathbb{R} \) of dimension at least 3. Let \( \mathbb{C}\ell(m) \) denote the complex Clifford algebra for \( m \), much as discussed in Sect. 2 for \( g \). Let \( \text{Spin}(m) \) be the subgroup of the group of invertible elements of \( \mathbb{C}\ell(m) \) generated by products of two elements of \( m \) of length 1. Conjugation of \( \mathbb{C}\ell(m) \) by elements of \( \text{Spin}(m) \) carries \( m \) into itself, and this gives a group homomorphism of \( \text{Spin}(m) \) onto \( \text{SO}(m) \) whose kernel is \( \{1, -1\} \). In this way \( \text{Spin}(m) \) is the simply-connected covering group of \( \text{SO}(m) \), and these two groups have naturally isomorphic Lie algebras. The \( \mathbb{R} \)-linear span of products of two orthogonal elements of \( m \) is a Lie \( \mathbb{R} \)-sub-algebra, \( \text{spin}(m) \), of \( \mathbb{C}\ell(m) \) with its additive commutator as Lie bracket. Exponentiation in \( \mathbb{C}\ell(m) \) carries \( \text{spin}(m) \) onto \( \text{Spin}(m) \), and one finds in this way that \( \text{spin}(m) \) is the Lie algebra of \( \text{Spin}(m) \). It follows, in particular, that \( \text{spin}(m) \cong \text{so}(m) \) naturally.

Suppose now that \( S \) is a finite-dimensional Hilbert space over \( \mathbb{C} \), and that \( \kappa \) is a \( * \)-representation of the \( C^* \)-algebra \( \mathbb{C}\ell(m) \) on \( S \). Then the restrictions of \( \kappa \) to \( \text{Spin}(m) \) and \( \text{spin}(m) \) give a unitary representation of that group, and a corresponding representation of its Lie algebra, on \( S \). Let \( \beta \) denote the action of \( \text{Spin}(m) \) on \( \mathbb{C}\ell(m) \) by conjugation. Then the representation \( \kappa \) of \( \text{Spin}(m) \) on \( S \) is manifestly compatible with the the action \( \beta \) and the action \( \kappa \) of \( \mathbb{C}\ell(m) \) on \( S \), in the sense, much as given in Eq. 3.1, that

\[
\kappa_{\beta_y}(q) = \kappa_y \kappa_q \kappa_y^* 
\]

for all \( q \in \mathbb{C}\ell(m) \) and \( y \in \text{Spin}(m) \).

Suppose now that \( G \) is a connected Lie group with Lie algebra \( g \), and that \( \pi \) is a representation of \( G \) on \( m \) by orthogonal transformations. That is, \( \pi \) is a homomorphism from \( G \) into \( \text{SO}(m) \). Let \( \pi \) also denote the corresponding homomorphism from \( g \) into \( \text{so}(m) \). From our natural identification of \( \text{so}(m) \) with \( \text{spin}(m) \) we can view \( \pi \) as a homomorphism from \( g \) to \( \text{spin}(m) \). (A formula for that homomorphism can be obtained by applying formula 5.12 of \([36]\) or formula 3.4 of \([51]\).) From a fundamental theorem for Lie groups (theorem 5.6 of \([39]\)) it follows that there is a homomorphism from the simply connected covering group, \( \hat{G} \), of \( G \) into \( \text{Spin}(m) \) whose corresponding Lie algebra homomorphism is \( \pi \). Let \( \sigma \) denote the composition of this homomorphism with the action \( \kappa \) of \( \text{Spin}(m) \) on \( S \), so that \( \sigma \) is a unitary representation of \( \hat{G} \) on \( S \). When we combine this with the earlier observations, we obtain:
Proposition 8.1. Let $G$ be a connected simply-connected compact Lie group, and let $\pi$ be a representation of $G$ by orthogonal transformations on a finite-dimensional Hilbert space $m$ over $\mathbb{R}$, and so by automorphisms of $C\ell(m)$. Let $\tilde{\pi}$ denote the corresponding homomorphism from $G$ into $\text{Spin}(m) \subset C\ell(m)$. Let $\kappa$ be a $*$-representation of $C\ell(m)$ on a finite-dimensional Hilbert space $S$, and let $\sigma$ be the composition of $\tilde{\pi}$ with $\kappa$, so that $\sigma$ is a unitary representation of $G$ on $S$. Notice that the action $\pi$ of $G$ on $C\ell(m)$ is obtained by composing $\tilde{\pi}$ with the action of $\text{Spin}(m)$ on $C\ell(m)$ by conjugation. Then $\sigma$ satisfies the compatibility condition

$$\kappa_{\pi}(q) = \sigma_{\pi} \kappa_{\sigma}^*$$

for $q \in C\ell(m)$ and $x \in G$.

Example 8.2. Let $G = \text{SO}(3)$ and let $\pi$ be its standard representation on $m = \mathbb{R}^3$ with its standard inner product. Then $C\ell(m)$ is of dimension 8, and is isomorphic to the direct sum of 2 copies of $M_2(\mathbb{C})$, so its irreducible representations are of dimension 2. But $G$ does not have any irreducible representation of dimension 2 so it can not act on the spinor spaces for $C\ell(m)$. However, the simply connected covering group of $G$ is $SU(2)$, and it has irreducible representations of dimension 2. It will act on the spinor spaces for $C\ell(m)$, compatibly with its action on $C\ell(m)$ via $G$.

9. Dirac Operators for Matrix Algebras

In this section we consider the action $\alpha$ of $G$ on $\mathcal{A} = \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is the Hilbert space of an irrep $U$ of $G$, and the action $\alpha$ on $\mathcal{B}(\mathcal{H})$ is by conjugation by this representation. Since $\mathcal{A}$ is finite dimensional, we have $\mathcal{A}^\infty = \mathcal{A}$. For any $X \in g$ and $T \in \mathcal{A}$ we have $\alpha_X(T) = [U_X, T]$.

Since the center of $G$ will act trivially on $\mathcal{A}$, we can factor by the connected component of the center. The resulting group will be semisimple. Thus for the remainder of this section we will assume that $G$ is a compact connected semisimple Lie group.

In general the representation $(\mathcal{H}, U)$ need not be faithful. Its kernel at the Lie-algebra level is an ideal of $\mathfrak{g}$. But $\mathfrak{g}$, as a semisimple Lie algebra, is the direct sum of its minimal ideals, each of which is a simple Lie algebra. Denote the Lie-algebra-kernel of $U$ by $\mathfrak{g}_0$. It must be the direct sum of some of these minimal ideals. Denote the direct sum of the remaining minimal ideals by $\mathfrak{g}_U$, so that $\mathfrak{g} = \mathfrak{g}_U \oplus \mathfrak{g}_0$. Clearly $U$ is faithful on $\mathfrak{g}_U$ and trivial on $\mathfrak{g}_0$. We identify $\mathfrak{g}_U'$ with the subspace of $\mathfrak{g}'$ consisting of linear functionals on $\mathfrak{g}$ that take value 0 on $\mathfrak{g}_0$, and similarly for $\mathfrak{g}_0'$. Clearly $\mathfrak{g}_U$ and $\mathfrak{g}_0$ are Ad-invariant, and so $\mathfrak{g}_U'$ and $\mathfrak{g}_0'$ are Cad-invariant.

For any $T \in \mathcal{A}$ it is clear that $dT(X) = \alpha_X(T) = 0$ for any $X \in \mathfrak{g}_0$, so that $dT \in \mathcal{A} \otimes \mathfrak{g}_U'$. Thus the range of $d$ is in $\mathcal{A} \otimes \mathfrak{g}_U'$. In [41] it is shown that for the quotient of $G$ by the kernel of $U$, the sub-bimodule of its $\Omega_o$ that is generated by the range of $d$ is exactly $\Omega_o$ itself. But $\mathfrak{g}_U$ is exactly the Lie algebra of the quotient of $G$ by the kernel of $U$. We see in this way that the cotangent bundle, $\Omega(\mathcal{A})$, of $\mathcal{A}$ is exactly

$$\Omega(\mathcal{A}) = \mathcal{A} \otimes \mathfrak{g}_U'.$$

Notice that for the original $\Omega_o$ we have $\Omega_o = \mathcal{A} \otimes \mathfrak{g}' = \Omega(\mathcal{A}) \oplus (\mathcal{A} \otimes \mathfrak{g}_0')$.

We assume that, as before, we have chosen a Cad-invariant inner product on $\mathfrak{g}'$. It restricts to a Cad-invariant inner product on $\mathfrak{g}_U'$. We then form the complex Clifford
algebra $\mathbb{C}\ell(g'_{U})$. We note that it is a Cad-invariant unital C*-subalgebra of $\mathbb{C}\ell(g')$. Since $\Omega(A) = A \otimes g'_{U}$, the Clifford bundle for $A$ is exactly

$$\mathbb{C}\ell(A) = A \otimes \mathbb{C}\ell(g'_{U}).$$

Let $(S, \kappa)$ be a choice of an irreducible *-representation of $\mathbb{C}\ell(g'_{U})$. We then form the spinor bundle $A \otimes S$. It is a Hilbert space with its inner product coming from using the (unique) tracial state on $A$. We then form the corresponding Dirac operator, $D$, on $A \otimes S$ much as in Sect. 2, so that

$$D(T) = (I A \otimes \kappa) \circ (d \otimes I S)(T) = \sum \alpha_{E_j}(T) \otimes \kappa_{E_j}$$

for $T \in A$, where now $\{E_j\}$ is a basis for $g_{U}$.

**Definition 9.1.** With notation as above, the operator $D$ defined just above is the Dirac operator for $A = B(H)$ for the given inner product on $g'$. (Thus if the representation $(H, U)$ is faithful on $g$ then $D$ coincides with $D_{o}$.)

Let $\hat{G}$ be the simply-connected covering group of $G$. Then Proposition 8.1 tells us that there is a representation $\sigma$ of $\hat{G}$ on $S$ that is compatible with the action of $\hat{G}$ (through $G$) on $\mathbb{C}\ell(g'_{U})$ in the sense that

$$\kappa_{\text{Cad},(q)} = \sigma_{x}^{*} \kappa_{q} \sigma_{x}$$

for all $q \in \mathbb{C}\ell(g'_{U})$ and $x \in \hat{G}$. We then define a representation, $\tilde{\sigma}$, of $\hat{G}$ on $A \otimes S$ by $\tilde{\sigma} = \alpha \otimes \sigma$. It follows from a very slight modification of Proposition 3.1 to account for our using $g'_{U}$ instead of $g'$, that $D$ commutes with this action. Then, a very slight modification of Proposition 3.2 tells us that the corresponding C*-metric $L^{D}$ is invariant under the action $\alpha$ of $\hat{G}$ in the sense that for any $a \in A$ we have

$$L^{D}(\alpha_{x}(a)) = L^{D}(a)$$

for all $x \in \hat{G}$.

According to Proposition 6.2, for any $T \in A$

$$L^{D_{o}}(T) = \|[D_{o}, M_{a}]\| = \|\sum \alpha_{E_j}(T) \otimes \epsilon_{j}\|$$

where the norm on the right is that on $A \otimes \mathbb{C}\ell(g')$. If we choose a basis, $\{E_j : j \in I_{U}\}$, for $g_{U}$ and adjoin to it a basis for $g_{0}$, then from the fact that $\alpha_{X}(a) = 0$ for any $X \in g_{0}$ we see that

$$L^{D_{o}}(T) = \|\sum_{j \in I_{U}} \alpha_{E_j}(T) \otimes \epsilon_{j}\|.$$ 

But $\sum_{j \in I_{U}} \alpha_{E_j}(a) \otimes \epsilon_{j}$ is in $A \otimes \mathbb{C}\ell(g'_{U}) = \mathbb{C}\ell(A)$. Then regardless of the choice of spinors for $\mathbb{C}\ell(g'_{U})$, it follows from Proposition 6.2 that for the corresponding Dirac operator $D$ we have

$$L^{D}(T) = \|\sum_{j \in I_{U}} \alpha_{E_j}(T) \otimes \epsilon_{j}\|.$$ 

Since the norm on $\mathbb{C}\ell(g'_{U})$ is just the restriction to $\mathbb{C}\ell(g'_{U})$ of the norm on $\mathbb{C}\ell(g')$, the norm on the right can be taken to be the norm of $A \otimes \mathbb{C}\ell(g')$. We thus obtain the following proposition, which will be important for the proof of our main theorem.
Proposition 9.2. For any compact connected Lie group $G$ and any irrep $(\mathcal{H}, U)$ and corresponding action $\alpha$ of $G$ on $\mathcal{A} = \mathcal{B}(\mathcal{H})$, and for notation as above, we have

$$L^D = L^{D_0}.$$ 

10. The Fuzzy Sphere

We now illustrate our general results obtained so far by working them out explicitly for the case that has received the most attention in the quantum-physics literature, namely the case of $G = SU(2)$ and its irreps. This will permit us to compare our Dirac operator with those proposed in the physics literature.

We now establish our conventions and notation. As a basis for $\mathfrak{g} = su(2)$ we take the product by $i$ of the Pauli matrices [57], so

$$E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$ 

Then $E_1 E_2 = E_3$, and cyclic permutations of the indices. Consequently $[E_1, E_2] = 2E_3$ and cyclic permutations of this, which are Lie algebra relations. Furthermore $(E_j)^2 = -I_2$ for $j = 1, 2, 3$. We see that trace $((E_j)^2) = -2$. This leads us to define the $Ad$-invariant inner product on $\mathfrak{g}$ by $\langle X, Y \rangle = -(1/2)\text{trace}(XY)$, so that the $E_j$'s form an orthonormal basis for $\mathfrak{g}$. As Cartan subalgebra for $\mathfrak{g}^C = sl(2, \mathbb{C})$ we choose the $\mathbb{C}$-span of $E_3$.

Let $(\mathcal{H}^n, U^n)$ be an irrep of $G$, where the label $n$ is the highest weight of the representation. Thus dim$(\mathcal{H}^n) = n + 1$. We let $U^n$ also denote the corresponding “infinitesimal” representation of $\mathfrak{g}^C$ on $\mathcal{H}^n$. As done in the introduction, we set $\mathcal{B}^n = \mathcal{B}(\mathcal{H}^n)$, and we let $\alpha$ be the action of $G$ on $\mathcal{B}^n$ by conjugation by $U^n$. Its infinitesimal version is given by $\alpha_X(T) = [U^n_X, T]$ for all $T \in \mathcal{B}^n$ and $X \in \mathfrak{g}^C$.

According to Theorem 4.1 of [41], the cotangent bundle for $\mathcal{B}^n$ is $\Omega_n = \mathcal{B}^n \otimes_{\mathbb{R}} \mathfrak{g}' = \mathcal{B}^n \otimes_{\mathbb{C}} \mathfrak{g}^{C'}$. We let $\{\epsilon_j\}_{j=1}^3$ be the basis for $\mathfrak{g}'$ that is dual to the basis $\{E_j\}$ for $\mathfrak{g}$ chosen above. It will be orthonormal for the inner product on $\mathfrak{g}'$ that is dual to our chosen inner product on $\mathfrak{g}$. Any other $Ad$-invariant inner product on $\mathfrak{g}'$ will be a scalar multiple of this one. If needed, that scalar can be pulled along through the calculations we do below.

The Clifford algebra $\mathbb{C}\ell(\mathfrak{g}')$ is generated by $\{\epsilon_j\}_{j=1}^3$ with the relations

$$\epsilon_j \epsilon_k + \epsilon_k \epsilon_j = -2\langle \epsilon_j, \epsilon_k \rangle = -2\delta_{jk}.$$ 

Since $\mathfrak{g}$ is 3-dimensional, $\mathbb{C}\ell(\mathfrak{g}')$ is isomorphic to $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, while $\mathbb{C}\ell^e(\mathfrak{g}')$ is isomorphic to $M_2(\mathbb{C})$. Let $\kappa$ be an irreducible representation of $\mathbb{C}\ell^e(\mathfrak{g}')$ on a Hilbert space $\mathcal{S}$ (necessarily of dimension 2, and unique up to unitary equivalence). The chirality element for $\mathbb{C}\ell(\mathfrak{g}')$ is $\gamma = \epsilon_1 \epsilon_2 \epsilon_3$. We extend $\kappa$ to $\mathbb{C}\ell(\mathfrak{g}')$ by setting either $\kappa\gamma = I^S$ or $\kappa\gamma = -I^S$ (as discussed in Sect. 6) to obtain the two inequivalent irreducible representations of $\mathbb{C}(\mathfrak{g}')$. The Clifford bundle for $\mathcal{B}^n$ is $\mathbb{C}\ell(\mathcal{B}^n) = \mathcal{B}^n \otimes \mathbb{C}\ell(\mathfrak{g}')$, and the spinor bundle is $\mathcal{S}(\mathcal{B}^n) = \mathcal{B}^n \otimes \mathcal{S}$. The Dirac operator on $\mathcal{S}(\mathcal{B}^n)$ is defined on elementary tensors by

$$D(T \otimes \psi) = \sum_j \alpha_{E_j}(T) \otimes \kappa_{\epsilon_j}(\psi) = \sum [U^n_{E_j}, T] \otimes \kappa_{\epsilon_j}(\psi)$$ 

for $T \in \mathcal{B}^n$ and $\psi \in \mathcal{S}$, that is,

$$D = \sum_j \alpha_{E_j} \otimes \kappa_{\epsilon_j}.$$
Since $SU(2)$ is simply connected, Proposition 8.1 tells us that corresponding to its action $\text{Cad}$ on $\mathfrak{g}'$ there is a homomorphism of it into the subgroup $\text{Spin}(\mathfrak{g}')$ of the group of invertible elements of the even subalgebra $\mathcal{C}\ell'(\mathfrak{g}')$ of $\mathcal{C}\ell(\mathfrak{g}')$. This homomorphism, which we will denote by $\text{Cad}$, plays the role of the $\pi$ of Proposition 8.1. It is easy to describe, and we will need a precise description of it below. The $E_j$’s happen to also be elements of $SU(2)$. To try to avoid confusion, we denote them by $x_j = E_j$ when we view them as elements of $SU(2)$. We note that $(x_j)^{-1} = -x_j$. It is easily checked that $\text{Ad}_{x_j}(E_k) = E_k$ if $k = j$ and $= -E_k$ if $k \neq j$. Consequently $\text{Cad}_{x_j}(\epsilon_k) = \epsilon_k$ if $k = j$ and $= -\epsilon_k$ if $k \neq j$. Set $\hat{\epsilon}_1 = \epsilon_2\epsilon_3$ and cyclic permutations of the indices. The $\hat{\epsilon}_j$’s are elements of $\text{Spin}(\mathfrak{g}')$. It is easily checked that conjugation by $\hat{\epsilon}_j$ takes $\epsilon_k$ to $\epsilon_k$ if $k = j$ and to $= -\epsilon_k$ if $k \neq j$. Consequently $\text{Cad}_{x_j} = \hat{\epsilon}_j$ for each $j$.

Furthermore, if we set $x_0 = I_2$, then every element of $SU(2)$ can be expressed (see proposition VII.5.5 of [58]) uniquely as $\sum_{j=0}^{3} r_j x_j$ where the vector $r = (r_0, r_1, r_2, r_3)$ in $\mathbb{R}^4$ satisfies $\|r\| = 1$. (So here we view $SU(2)$ as the unit sphere in the quaternions.) Set $\hat{\epsilon}_0 = 1_{\mathcal{C}\ell}$ where $1_{\mathcal{C}\ell}$ is the identity element of $\mathcal{C}\ell(\mathfrak{g}')$. Thus we obtain the first statement of:

**Proposition 10.1.** With notation as above, $\text{Cad}$ is the mapping that takes $\sum_{j=0}^{3} r_j x_j$ to $\sum_{j=0}^{3} r_j \hat{\epsilon}_j$ for $\|r\| = 1$. It is an isomorphism from $SU(2)$ onto $\text{Spin}(\mathfrak{g}')$. The corresponding isomorphism, $\text{cad}$, from $\text{su}(2)$ to $\text{spin}(\mathfrak{g}')$ is the linear map that sends $E_j$ to $\hat{\epsilon}_j$ for $j = 1, 2, 3$.

*Proof.* The proof of the statement concerning $\text{cad}$ is easily obtained by examining, for each $j$, the derivative at 0 of the curve $t \mapsto \sin(t)x_j + \cos(t)x_0$ in $SU(2)$ and of the image of this curve under $\text{Cad}$. □

Let $\sigma$ be the representation of $SU(2)$ on $S$ obtained by composing $\text{Cad}$ with $\kappa$, much as done in Proposition 8.1. Then $\sigma$ and $\kappa$ manifestly satisfy the compatibility condition 3.1. We let $\bar{\sigma}$ be the action of $G$ on the spinor bundle $A \otimes S$ defined by $\bar{\sigma} = \alpha \otimes \sigma$. It follows from Proposition 3.1 that $D$ commutes with this action.

But from Proposition 10.1 we see that, in turn, we can let $S = \mathbb{C}^2$ and let $\kappa$ be the inverse of the isomorphism $\text{cad}$, so sending $\hat{\epsilon}_j$ to $E_j$, and then extending this to $\mathcal{C}\ell'(\mathfrak{g}')$ by sending $1_{\mathcal{C}\ell}$ to $I_2$. This then also gives the isomorphism from $\text{Spin}(\mathfrak{g}')$ onto $SU(2)$. We can then extend this $\kappa$ to $\mathcal{C}\ell(\mathfrak{g}')$ by sending $\gamma$ to $\pm I_2$.

The charge conjugation operator, $C^S$, on $\mathbb{C}^2$ is then defined by $C^S(v) = \sigma_2 \bar{v}$, where $\bar{v}$ is the standard complex conjugation on $\mathbb{C}^2$ applied to $v \in \mathbb{C}^2$, and $\sigma_2$ is the standard Pauli spin matrix. See the proof of Proposition 3.5 of [36], or for the general setting see section 2.3 of [46]. The charge-conjugation operator on $A \otimes S$ is then defined exactly as in Sect. 4.

We now compute the spectrum of $D$, by relating it to the Casimir element, $C$, somewhat as done in [59]. The following relation appears to be related to equation 99 of [45].

**Theorem 10.2.** For notation as above,

$$D = \pm (1/2)(\bar{\sigma} - \alpha_C \otimes I^S - I^A \otimes \sigma_C),$$

where the sign depends on the choice of the spinor representation on $S = \mathbb{C}^2$. 
Proof. We examine, for our case of $G = SU(2)$, the term $\sum_j (\alpha_{E_j} \otimes \sigma_{E_j})$ that appears in the formula for $\tilde{\sigma}_C$ in Proposition 5.2. From our discussion just above, $\sigma_{E_j} = \kappa(\hat{\text{cadj}}_{E_j})$, where $\kappa$ also denotes here the corresponding homomorphism of Lie algebras. Then from Proposition 10.1 we see that $\sigma_{E_j} = \kappa(\hat{\epsilon}_j)$. But it is easily checked that $\gamma \epsilon_j = -\hat{\epsilon}_j$, and so if we choose the spinor representation for which $\kappa(\gamma) = -I_2$, we find that $\kappa(\epsilon_j) = \kappa(\hat{\epsilon}_j) = \sigma_{E_j}$. Consequently

$$\sum_j (\alpha_{E_j} \otimes \sigma_{E_j}) = \sum_j (\alpha_{E_j} \otimes \kappa(\epsilon_j)) = D.$$ 

On the other hand, if we choose the spinor representation for which $\kappa(\gamma) = +I_2$, then we will obtain

$$\sum_j (\alpha_{E_j} \otimes \sigma_{E_j}) = -D.$$ 

When this is combined with the formula for $\tilde{\sigma}_C$ in Proposition 5.2, we obtain the formula in the statement of the theorem. \hfill \Box

To use the formula of the above theorem in order to determine the spectrum of $D$, we need to recall the well-known facts about the action of the Casimir element $C$ on irreducible representations of $SU(2)$. We define the following elements of $g^C$.

$$H = -iE_3, \quad E = -(E_2 + iE_1)/2, \quad F = (E_2 - iE_1)/2.$$ 

They satisfy the familiar relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$ 

Then in $U(g^C)$ we find (as in equation 1.3.8 of [60])

$$C = \sum_{j=1}^3 E^2_j = -(H^2 + 2(EF + FE)).$$ 

For our purposes it is useful to change this using $EF - FE = H$ to obtain

$$C = -(H^2 + 2H + 4FE).$$ 

This permits us to easily determine, as follows, the scaler multiple of the identity operator to which $C$ is taken by any irrep of $SU(2)$. Let $(\mathcal{H}^n, U^n)$ be the irrep with highest weight $n$. This means that $\mathcal{H}^n$ contains an eigenvector, $\xi_n$, for $H$ of eigenvalue $n$ such that $E\xi_n = 0$. Consequently, $U^n_C \xi_n = -(H^2 + 2H)\xi_n = -n(n+2)\xi_n$. Thus:

**Proposition 10.3.** For the irrep $(\mathcal{H}^n, U^n)$ of $SU(2)$ of highest weight $n$ we have $U^n_C = -n(n+2)I^{\mathcal{H}^n}$.

We now determine the spectrum of $D$. We choose the spinor representation for which $\kappa(\gamma) = -I_2$. The description of the spectrum of $D$ is most attractive if we state it in terms of $D' = D + 2$.

**Theorem 10.4.** The spectrum of $D'$ consists of the even numbers $\pm 2k$ for $1 \leq k \leq n$ together with $2(n+1)$. For each $k$ with $1 \leq k \leq n$ the multiplicity of the eigenvalue $\pm 2k$ is $2k = \dim(\mathcal{H}^{2k-1})$, while the multiplicity of $2(n+1)$ is $2(n+1) = \dim(\mathcal{H}^{2n+1})$. The spectrum of $D$ is just the spectrum of $D'$ shifted by $-2$. 

Proof. It suffices to determine the spectrum of
\[ \tilde{\sigma}_C - \alpha_C \otimes I^S - I^A \otimes \sigma_C, \]
which is an operator on \( B^0 \otimes S \). This operator involves three different representations of \( SU(2) \). Now for any compact group \( G \) and unitary representation \( (\mathcal{H}, U) \) the representation \( \alpha \) on \( B(\mathcal{H}) \) by conjugation by \( U \) is unitarily equivalent to the representation \( U \otimes U^* \) on \( \mathcal{H} \otimes \mathcal{H}^* \), where \( (\mathcal{H}, U^*) \) is the representation dual to \( (\mathcal{H}, U) \). But for \( G = SU(2) \) all representations are self-dual \([58]\). Thus for an irrepp \( (\mathcal{H}^n, U^n) \) of \( SU(2) \) the corresponding representation \( \alpha \) on \( B^n \) is equivalent to the representation \( U^n \otimes U^n \) on \( \mathcal{H}^n \otimes \mathcal{H}^n \). But this representation is known (see section 8-3 of \([57]\)) to decompose in such a way that \((\alpha, B^n)\) is equivalent to \( \bigoplus_{k=0}^n \mathcal{H}^{2k} \otimes U^{2k} \). Thus \( B^n \otimes S \) is equivalent to \( \bigoplus_{k=0}^n \mathcal{H}^{2k} \otimes \mathcal{H}^1 \) with corresponding representations \( \alpha \otimes I^S \), \( I^A \otimes \sigma \), and \( \tilde{\sigma} = \alpha \otimes \sigma \). For each \( k \) set \( W_k = \mathcal{H}^{2k} \otimes \mathcal{H}^1 \).

Since \( U_1^1 = -3I^H \), we see that
\[ I^A \otimes \sigma_C \text{ on } W_k \text{ is } -3I^{W_k}. \]

We also see that
\[ \alpha_C \otimes I^S \text{ on } W_k \text{ is } -2k(2k + 2)I^{W_k} = -4(k^2 + k)I^{W_k}. \]

Furthermore, \( W_k = \mathcal{H}^{2k} \otimes \mathcal{H}^1 \) decomposes for \( \tilde{\sigma} \) as \( \mathcal{H}^{2k+1} \oplus \mathcal{H}^{2k-1} \) for \( k \geq 1 \), while
\[ W_0 = \mathcal{H}^1. \]

Thus we see that for \( k \geq 1 \) we have
\[ \alpha_C \text{ on } \mathcal{H}^{2k+1} \subset W_k \text{ is } -(2k + 1)(2k + 3)I^{\mathcal{H}^{2k+1}}, \]
\[ \alpha_C \text{ on } \mathcal{H}^{2k-1} \subset W_k \text{ is } -(2k - 1)(2k + 1)I^{\mathcal{H}^{2k-1}}, \]
\[ \alpha_C \text{ on } W_0 = \mathcal{H}^1 \text{ is } -3I^{\mathcal{H}^1}. \]

It follows that for \( k \geq 1 \) we have
\[ \pm 2D \text{ on } \mathcal{H}^{2k+1} \subset W_k \text{ is } -(2k + 1)(2k + 3) - 4(k^2 + k) - 3)I^{\mathcal{H}^{2k+1}} = -4kI^{\mathcal{H}^{2k+1}}, \]
\[ \pm 2D \text{ on } \mathcal{H}^{2k-1} \subset W_k \text{ is } -(2k - 1)(2k + 1) - 4(k^2 + k) - 3)I^{\mathcal{H}^{2k-1}} = +4k + 1I^{\mathcal{H}^{2k-1}}, \]
\[ \pm 2D \text{ on } W_0 = \mathcal{H}^1 \text{ is } ((-3) - (0) - (-3))I^{\mathcal{H}^1} = 0I^{\mathcal{H}^1}. \]

We can conveniently assemble all of this in the following way. Let us keep the minus sign, and divide by 2. We obtain for \( k \geq 1 \)
\[ D \text{ on } \mathcal{H}^{2k+1} \subset W_k \text{ is } +2kI^{\mathcal{H}^{2k+1}}, \]
while
\[ D \text{ on } \mathcal{H}^{2k-1} \subset W_k \text{ is } -2k + 1I^{\mathcal{H}^{2k-1}}, \]
while
\[ D \text{ on } W_0 = \mathcal{H}^1 \text{ is } 0I^{\mathcal{H}^1}. \]

Then let \( D' = D + 2 \). For each \( k \) with \( 1 \leq k \leq n \) we see that on the copy of \( \mathcal{H}^{2k-1} \) in \( W_{k-1} \), we have \( D' = 2kI^{\mathcal{H}^{2k-1}} \), while on the copy of \( \mathcal{H}^{2k-1} \) in \( W_k \) we have \( D' = -2kI^{\mathcal{H}^{2k-1}} \).

Finally, for \( \mathcal{H}^{2n+1} \) in \( W_n \), we have \( D' = 2(n + 1)I^{\mathcal{H}^{2n+1}} \). \(\square\)

The asymmetry of the spectrum of \( D \) and \( D' \) precludes their having a grading (chirality) operator, so they do not give “even” spectral triples.

Let \( D_{DLV} \) denote the Dirac operator obtained in proposition 3.5 of \([59]\). Then part (iii) of that proposition makes clear that our \( D' \) is equal to \( 2D_{DLV} \). Thus we see that our \( D \) corresponds to \( D_{DLV} \) once differences in our conventions are taken into account. In
[59] the formula used to define $D_{DLV}$ is presumably guided in part by the formulas used in [53], but, as discussed in [59], their formula is closely related to the formula obtained in [61].

In section VI of [45] Barrett discusses various versions of fuzzy spheres, beginning with a Dirac operator that is essentially our Dirac operator $D'$ above. He points out that not only is there no chirality operator, but this Dirac operator also does not have the desired KO-dimension (defined there). He shows that both of these problems are solved by forming the direct sum of $D$ with its negative, so that the space of spinors is $C^4$ instead of $C^2$. He also proposes a corresponding modified Dirac operator for the 2-sphere itself and makes comments about why this may be a reasonable thing to do. It would be interesting to explore whether these ideas can be usefully extended to all coadjoint orbits to which matrix algebras converge.

In [59] there is a substantial discussion of how their Dirac operator compares to others in the literature. That discussion applies equally well to the Dirac operator we have constructed. We will not repeat here most of that discussion, but we now summarize some of it.

The Dirac operator constructed in section IIIB of [62] by using the Ginsparg-Wilson relations, which also appears in section 8.3.2 of [3], and is the spin-1/2 case in [63,64], is essentially the same as the Dirac operator that we have constructed above. However, the “chirality operator” defined there is not a true grading.

For the Dirac operator constructed in [4] and [65] and used in [66] there is a true chirality operator, but the Dirac operator is of “second-order” and has spectrum very different for the spectrum we described above. In section 3.7.2 of [7] there is a detailed discussion of the relations between the Dirac operator in [65] and that in [61].

The Dirac operators on matrix algebras defined in [67] are not mentioned in [59]. They are constructed by a quite different method, which gives a quite different formula for the Dirac operators. A useful discussion of the role of supersymmetry in the construction is given in the introduction to [68]. A good discussion of the contrast between that method and the methods used in the papers mentioned in the paragraphs above (and so in the present paper) is given in the introduction to [69], which lays the foundation for applying that method to complex projective spaces in [70]. Section 2 of [6] is devoted to an exposition of the construction in preparation for extending it to the q-deformed sphere (continued in [71–74]). In [75] the same method is used to treat the case in which $G = SU(2) \times SU(2)$ acting on $S^2 \times S^2$, a 4-dimensional space, so of special interest to physics. In the papers using that method there is no mention of charge conjugation. Thus it would be interesting to know whether the finite-dimensional spectral triples constructed by that method can be equipped with a “real” structure. It would also be interesting to know whether the method can be generalized to apply to all coadjoint orbits of integral weights of all compact semisimple Lie groups, and whether it even has a primitive version for ergodic actions of the kind discussed in the early sections of the present paper. It is not clear to me how to make a precise comparison between the Dirac operators obtained by that method and the ones constructed in the present paper.

11. Homogeneous Spaces and the Dirac Operator for $G$

In this and the next sections we reformulate the (usual) construction of Dirac operators on coadjoint orbits in terms just of algebras and modules, without any mention of coordinate patches or charts, refining the approach in Sect. 2. This is essential for the proof of our
main theorem. We follow fairly closely parts of [23, 40], but here we emphasize cotangent bundles rather than tangent bundles.

We begin working in the following generality. We let \( G \) be any compact connected Lie group. We let \( K \) be any closed subgroup of \( G \), and we form the corresponding homogeneous space \( G/K \). Many of the results in this section are adaptations to our context of some of the results in [23]. We let \( A = C(G/K) \) and we let \( \alpha \) be the action of \( G \) on \( A \) by left translation. Then \( A^\infty \), formed using \( \alpha \), is also the smooth structure on \( G/K \) coming from that on \( G \).

As in Sect. 2, we set \( \Omega_1 = A^\infty \otimes g' \), we chose a spinor space \( S \) for \( C\ell(g') \) (which need not be irreducible), and we define \( D_\alpha \) on \( A^\infty \otimes S \). Since now \( A^\infty = C^\infty(G/K) \), we can express \( D_\alpha \) in terms of functions as follows. Let \( f, g \in A^\infty \) and \( \psi \in S \). Then

\[
[D_\alpha, M_f](g \otimes \psi)(y) = \sum \alpha_E_j(f)(y)g(y) \otimes \kappa \epsilon_j \psi.
\]

But \( df_y \) is the element \( \sum \alpha_E_j(f)(y)\epsilon_j \) of \( g' \subset C\ell(g') \), so we see that \( [D_\alpha, M_f] \) is given by the operator-valued function

\[
[D_\alpha, M_f]_y = 1_A \otimes \kappa df_y.
\]

(Compare this with proposition 9.11 of [36].) Consequently \( \|[D_\alpha, M_f]\| = \|\kappa df_y\| \).

But for any \( \mu \in g' \) we have \( \|\kappa \mu\|^2 = \|\langle \mu, \mu \rangle 1_A\| = \|\mu\|^2 \), and it is easy to check that this extends to \( \mu \in g' \subset C\ell(g') \). Thus we obtain the following result, which will be of importance for the proof of our main theorem.

**Proposition 11.1.** With notation as above we have

\[
L_{D_\alpha}(f) = \|[D_\alpha, M_f]\| = \sup\{\|df_y\| : y \in G/K\},
\]

where the norm comes from the inner product on \( g' \).

We can put this in a more familiar form in terms of the dual inner product on \( g \). For all \( y \) there will exist an element, \( \text{grad}_y f \), of \( g \) such that \( df_y(X) = \langle X, \text{grad}_y f \rangle \) for all \( X \in g \). Thus

\[
L_{D_\alpha}(f) = \sup\{\|\text{grad}_y f\| : y \in G/K\}.
\]

All of the above applies to the case in which \( K = \{e_G\} \), that is, to the algebra \( A = C(G) \). Because any Lie group is parallelizable, the cotangent bundle for \( G \) actually is \( A^\infty \otimes g' \). So this case is also one to which all of the results of Sect. 2 immediately apply. Once one applies Proposition 8.1 concerning the possibility of choosing the space \( S \) to be irreducible, one obtains:

**Proposition 11.2.** Let \( G \) be a connected simply-connected compact Lie group, and let a Cad-invariant inner product be chosen for \( g' \). Then an irreducible spinor space \( S \) with compatible action of \( G \) can be found, and the operator \( D_\alpha \) on \( A^\infty \otimes S \) constructed as in Sect. 2 is the correct Dirac operator for \( G \), invariant under the action of \( G \) on itself by left translation. We denote it by \( D_G \).
We return to the general situation with \( \mathcal{A} = C(G/K) \). Since for any \( f \in \mathcal{A}^\infty \) and \( X \in \mathfrak{g} \) the expression \( \alpha_X(f) \) gives the same function regardless of whether \( f \) is viewed as an element of \( \mathcal{A}^\infty \) or of \( C^\infty(G) \), it is clear from the formulas above that \( D_\alpha \) is simply the restriction of \( D^G \) to \( \mathcal{A}^\infty \otimes \mathcal{S} \).

A standard argument (see the text around proposition 9.12 of [36]) shows that \( L_{D^G}(f) \) is the Lipschitz constant for any \( f \in C^\infty(G) \) for the ordinary metric on \( G \) coming from the Riemannian metric determined by our chosen inner product on \( g' \).

12. Dirac Operators for Homogeneous Spaces

We continue to work here in the following generality. We let \( G \) be any compact connected Lie group. We let \( K \) be any closed subgroup of \( G \), and we form the corresponding homogeneous space \( G/K \). Again, many of the results in this section are adaptations to our context of results in [23,40].

We let \( \mathfrak{k} \) be the Lie algebra of \( K \). Let \( f \in \mathcal{A}^\infty = C^\infty(G/K) \), and let \( X \in \mathfrak{g} \). As before, we view \( f \) as a function on \( G \) when convenient. Then for any \( x \in G \) we have

\[
d f_x(X) = D_\alpha^f ((\exp(-t X))x) = D_\alpha^f (x \exp(-t \text{Ad}_x^{-1}(X))).
\]

Since \( f(xs) = f(x) \) for all \( x \in G \) and \( s \in K \), we see from this that \( d f_{xs}(X) = d f_x(X) \) for all \( x \in G \) and \( s \in K \), and that if \( \text{Ad}_x^{-1}(X) \in \mathfrak{k} \) for some \( x \in G \) then \( d f_x(X) = 0 \). Let

\[
m' = \{ \mu \in \mathfrak{g}' : \mu(x) = 0 \text{ for all } x \in \mathfrak{k} \},
\]

and let \( m'^C \) be its complexification. Notice that \( m'^C \) is carried into itself by the restriction of \( \text{Cad} \) to \( K \). If \( X \in \mathfrak{k} \) then \( \text{Ad}_x^{-1}(\text{Ad}_x(X)) \in \mathfrak{k} \), so \( d f_x(\text{Ad}_x(X)) = 0 \). This means that \( \text{Cad}_x^{-1}(d f_x) \in m'^C \) for all \( x \in G \). Let

\[
\Omega(G/K) = \{ \omega \in C^\infty(G/K, m'^C) : \omega(x) \in \text{Cad}_x(m'^C) \text{ for all } x \in G \}.
\]

The calculations above show that \( d f \in \Omega(G/K) \). For pointwise operations, \( \Omega(G/K) \) is an \( \mathcal{A}^\infty \)-module. It is easy to check that \( \Omega(G/K) \) is carried into itself by the action \( \gamma \) of \( G \) on \( \Omega_\alpha(G) \). By very minor adjustments to the discussion in section 6 of [41] one sees that \( \Omega(G/K) \) is indeed the complexified cotangent bundle for \( G/K \) (as is basically well-known [76]).

However, the fact that the range spaces of the \( \omega \)'s depend on \( x \) complicates calculations. But by taking advantage of the fact that we are working on a homogeneous space, we can make a small change that makes the situation considerably more transparent (much as done in connection with notation 4.2 of [23]). For each \( \omega \in \Omega(G/K) \) define \( \hat{\omega} \) on \( G \) by \( \hat{\omega}(x) = \text{Cad}_x^{-1}(\omega(x)) \). Then it is easily checked that \( \hat{\omega} \) is in the \( \mathcal{A}^\infty \)-module

\[
\hat{\Omega}(G/K) = \{ \omega \in C^\infty(G, m'^C) : \omega(xs) = \text{Cad}_s^{-1}(\omega(x)) \text{ for } x \in G, s \in K \}.
\]

This is exactly the complexification of the space given in notation 6.1 of [41], where a proof is indicated for the well-known fact that it too can be viewed as the space of smooth cross sections of the cotangent bundle of \( G/K \). (It is an “induced representation” [23]. There are many more induced representations in the next pages.)
We then define \( \hat{\mathcal{A}} \) by \( \hat{\mathcal{A}} f = \hat{f} \) for all \( f \in \mathcal{A}^\infty \). Then for \( x \in G \) and \( X \in \mathfrak{g} \) we have

\[
\langle X, \hat{\mathcal{A}} f_x \rangle = \langle X, \mathcal{A}^{-1}(\mathcal{A} f_x) \rangle = \langle \mathcal{A}^{-1}(X), f_x \rangle = D_f^*(f(\exp(-t\mathcal{A}(X)x))) = D_f^*(f(x \exp(-tX)),
\]

so that \( \hat{\mathcal{A}} \) is defined in terms of the right action of \( G \) on itself. Clearly \( \hat{\mathcal{A}} \) is again a derivation, and the first-order differential calculi \( (\hat{\mathcal{A}}(G/K), \hat{\mathcal{A}}) \) and \( (\mathcal{A}(G/K), \mathcal{A}) \) are easily seen to be isomorphic. Thus we can use the latter. It is easily checked that

\[
(12.1)
\]

We now assume, as usual, that a \( \mathcal{A} \)-invariant inner product on \( \mathcal{A}^\infty \) has been chosen. We can then give a simple description of the Clifford bundle for \( \mathcal{A} \), much as done in section 7 of [23]. We first let \( \mathcal{A}^\infty(G/\mathcal{A}) \) be the complex Clifford algebra for \( \mathcal{A}^\infty \) of the given inner product on \( \mathcal{A}^\infty \), much as in Sect. 2. Since the restriction to \( K \) of the action \( \mathcal{A} \) on \( \mathcal{A}^\infty \) is by isometries, it extends (as Bogoliubov automorphisms) to an action on \( \mathcal{A}^\infty(G/\mathcal{A}) \), which we denote again by \( \mathcal{A} \). We then set

\[
\mathcal{A}^\infty(G/K) = \{ W \in C^\infty(G, \mathcal{A}^\infty(G/\mathcal{A})) : W(xs) = \mathcal{A}^{-1}(W(x)) \text{ for } x \in G, s \in K \}.
\]

Then \( \mathcal{A}^\infty(G/K) \) with pointwise product is easily seen to be a \( C^\ast \)-algebra that effectively contains \( \mathcal{A}^\infty \) in its center. Since \( \mathcal{A}^\infty \) generates \( \mathcal{A}^\infty \) as a unital algebra, it is also easily seen that \( \mathcal{A}^\infty(G/K) \subset \mathcal{A}^\infty(G/K) \) and that \( \mathcal{A}^\infty(G/K) \) together with \( \mathcal{A}^\infty \) generate \( \mathcal{A}^\infty(G/K) \) as a unital algebra. For our situation \( \mathcal{A}^\infty(G/K) \) is the appropriate Clifford-algebra bundle.

Much as in Sect. 8, we choose a spinor space \( S \) for \( \mathcal{A}^\infty(G/\mathcal{A}) \) (which need not be irreducible or faithful). We need to assume that there is a unitary representation, \( \rho \), of \( K \) on \( S \) that is compatible with the restricted action \( \mathcal{A} \) of \( K \) on \( \mathcal{A}^\infty(G/\mathcal{A}) \) and the representation \( \kappa \) of \( \mathcal{A}^\infty(G/\mathcal{A}) \) on \( S \), in the sense that

\[
k_{\mathcal{A}}(q) = \rho s_k \rho s^{-1}
\]

for all \( q \in \mathcal{A}^\infty(G/\mathcal{A}) \) and \( s \in K \). Since \( K \) is never semisimple in the situation in which we are most interested, namely that of coadjoint orbits, we can not in general apply the arguments used for Proposition 8.1 to obtain a spinor space that is irreducible. This is a serious issue, reflecting the fact that not all homogeneous spaces are Spin
c. Much of the next section will be devoted to dealing with this issue for coadjoint orbits. But we can always choose \( S = \mathcal{A}^\infty(G/\mathcal{A}) \) (with inner product from its canonical trace) with \( \rho = \mathcal{A} \) restricted to \( K \). This leads to the Dirac-Hodge operator, as discussed in section 8 of [23].

Whatever way \( S \) and \( \rho \) are chosen, we then set

\[
S(G/K) = \{ \Psi \in C^\infty(G, S) : \Psi(xs) = \rho s^{-1}\Psi(x) \text{ for all } s \in K \}.
\]

Proposition 12.1. The evident pointwise product of elements of \( S(G/K) \) by elements of \( \mathcal{A}^\infty(G/K) \), using the representation \( \kappa \) of \( \mathcal{A}^\infty(G/\mathcal{A}) \) on \( S \), carries \( S(G/K) \) into itself.

Proof. This uses the compatibility relation (12.1). Let \( W \in \mathcal{A}^\infty(G/K) \) and \( \Psi \in S(G/K) \). Then for \( x \in G \) and \( s \in K \) we have

\[
(W\Psi)(xs) = \kappa_{W(xs)}\Psi(xs) = \kappa_{\mathcal{A}^{-1}(W(x))}\rho s^{-1}(\Psi(x)) = \rho s^{-1}(\kappa_{W(x)}\Psi(x)) = \rho s^{-1}(W\Psi)(x)
\]

as desired. \( \square \)
There is an evident pointwise $A^\infty$-valued inner product on $S(G/K)$, using the inner product on $S$ and the fact that $\rho$ is unitary. We denote it again by $\langle \cdot, \cdot \rangle_A$.

Our Dirac operator $D$ will be an operator on $S(G/K)$. Because of the way $S(G/K)$ is defined, we cannot simply define $D$ on elementary tensors as done in previous sections. We need to use connections, as done in [23,40]. We follow somewhat closely the development in section 5 of [40].

Let $m$ be the orthogonal complement of $\mathfrak{k}$ for the inner product on $\mathfrak{g}$ that is dual to our chosen inner product on $\mathfrak{g}^\prime$. Then $m'$ can be viewed as the dual vector space to $m$. The tangent bundle, $T(G/K)$, of $G/K$ is given by

$$ T(G/K) = \{ V \in C^\infty(G, m) : V(xs) = \text{Ad}_x^{-1}(V(x)) \text{ for } x \in G, s \in K \}. $$

It is an $A^\infty$-module for the pointwise product, and $G$ acts on it by translation. We denote this translation action again by $\alpha$. Each $V \in T(G/K)$ determines a derivation, $\delta_V$, of $A^\infty$ by

$$ (\delta_V f)(x) = D^t_0(f(x \exp(tV(x))). $$

We denote the complexification of $T(G/K)$ by $T^C(G/K)$. For $V \in T^C(G/K)$ we define $\delta_V$ in terms of the real and imaginary parts of $V$.

There is an evident $A^\infty$-linear pairing of $T^C(G/K)$ with $\hat{\Omega}(G/K)$, coming from the pairing of $m$ with $m'$. We denote it by $\langle V, \omega \rangle_A$. Both $T^C(G/K)$ and $\hat{\Omega}(G/K)$ are finitely-generated projective $A^\infty$-modules (in accordance with Swan’s theorem) because they are induced modules. See proposition 2.2 of [23]. It is then easily verified that $\hat{\Omega}(G/K)$ is the $A^\infty$-module dual of $T^C(G/K)$ via the pairing mentioned above.

We define a connection, $\nabla^C$, on $\hat{\Omega}(G/K)$ by

$$ (\nabla^C_V(\omega))(x) = D^t_0(\omega(x \exp(tV(x)))) \tag{12.2} $$

for $V \in T(G/K)$ and $\omega \in \hat{\Omega}(G/K)$, extended to $T^C(G/K)$ by linearity. It is often called “the canonical connection”. (One can not expect that it is always a Levi-Civita connection in the sense of definition 8.8 of [36]).

We then extend $\nabla^C$ to $\mathbb{C}\ell(G/K)$, and define a connection, $\nabla^S$, on $S(G/K)$, by the same formula, setting

$$ (\nabla^S_V(\mathcal{W}))(x) = D^t_0(\mathcal{W}(x \exp(tV(x)))) \tag{12.3} $$

and

$$ (\nabla^S_V(\Psi))(x) = D^t_0(\Psi(x \exp(tV(x)))) \tag{12.4} $$

for $V \in T(G/K)$, $\mathcal{W} \in \mathbb{C}\ell(G/K)$, and $\Psi \in S(G/K)$, extended to $T^C(G/K)$ by linearity. These are all evidently $A^\infty$-linear in $V$. We will need shortly the evident Leibniz rule

$$ \nabla^S_V(f\Psi) = \delta_V(f)\Psi + f\nabla^S_V(\Psi). \tag{12.5} $$

We can view $\kappa$ as a bilinear form on $\mathbb{C}\ell(G/K) \times S(G/K)$ with values in $S(G/K)$, and so we have the Leibniz rule

$$ \nabla^C_V(\kappa\mathcal{W}\Psi) = \kappa\nabla^C_V(\mathcal{W})\Psi + \kappa\mathcal{W}(\nabla^C_V(\Psi)) \tag{12.6} $$
for any $V \in T^C(G/K)$ and $\Psi, \Phi \in \mathcal{S}(G/K)$, one can check that the connection on $\mathcal{S}(G/K)$ is compatible with the $\mathcal{A}^\infty$-valued inner product in the sense of the Leibniz rule
\[
d_\Psi(\langle \Psi, \Phi \rangle_\mathcal{A}) = \langle \nabla^S_V \Psi, \Phi \rangle_\mathcal{A} + \langle \Psi, \nabla^S_V \Phi \rangle_\mathcal{A}
\]
for any $V \in T^C(G/K)$ and $\Psi, \Phi \in \mathcal{S}(G/K)$, but we do not need this fact later.

We can now define the Dirac operator, $D$, on $\mathcal{S}(G/K)$ as follows. Let $\Psi \in \mathcal{S}(G/K)$. Define $d\Psi$ on $T^C(G/K)$ by $(d\Psi)(V) = \nabla^S_V \Psi$. Then $d\Psi$ is an $\mathcal{A}^\infty$-linear map from $T^C(G/K)$ into $\mathcal{S}(G/K)$, and so $d\Psi$ can be viewed as an element of $\hat{\Omega}(G/K) \otimes \mathcal{S}(G/K)$, since $\hat{\Omega}(G/K)$ is the $\mathcal{A}^\infty$-dual of $T^C(G/K)$. By means of the inclusion of $\hat{\Omega}(G/K)$ into $\mathbb{C}\ell(G/K)$ we can view $d\Psi$ as an element of $\mathbb{C}\ell(G/K) \otimes \mathcal{S}(G/K)$. We can then apply $\kappa$ to obtain an element of $\mathcal{S}(G/K)$. That is, with the above understanding, we define $D$ on $\mathcal{S}(G/K)$ by
\[
D(\Psi) = \kappa(d\Psi)
\]
for $\Psi \in \mathcal{S}(G/K)$. We can express $D$ in a more familiar and concrete form by using a biframe as follows (much as done in proposition 2.9 of [40]). Let $\{e_j\}$ be a basis for $\mathfrak{g}'$, and let $\{E_j\}$ be the dual basis for $\mathfrak{g}$. Let $P$ be the orthogonal projection from $\mathfrak{g}$ onto $\mathfrak{m}$, and let $P'$ be the orthogonal projection from $\mathfrak{g}'$ onto $\mathfrak{m}'$, extended to the complexifications. Notice that $P$ commutes with the restriction to $K$ of $\text{Ad}$, and similarly for $P'$. For each $j$ define an element, $E_j$, of $T^C(G/K)$ by
\[
E_j(x) = P \text{Ad}_x^{-1}(E_j),
\]
and an element, $\eta_j$, of $\Omega(G/K)$ by
\[
\eta_j(x) = P' \text{Cad}_x^{-1}(\epsilon_j).
\]
Then the pair $([E_j], \{\eta_j\})$ is a biframe for $T(G/K)$ in the sense that it has the reproducing property
\[
V = \sum \langle V, \eta_j \rangle_\mathcal{A} E_j
\]
for all $V \in T^C(G/K)$. Then
\[
(d\Psi)(V) = \nabla^S_V \Psi = \nabla^S_{\sum \langle V, \eta_j \rangle_\mathcal{A} E_j} \Psi = \sum \langle V, \eta_j \rangle_\mathcal{A} \nabla^S_{E_j} \Psi,
\]
so that
\[
d\Psi = \sum \eta_j \otimes \nabla^S_{E_j} \Psi,
\]
where $\eta_j$ can be viewed as an element of $\mathbb{C}\ell(G/K)$ via the inclusion $\Omega(G/K) \subset \mathbb{C}\ell(G/K)$. Then
\[
D \Psi = \sum \kappa_{\eta_j}(\nabla^S_{E_j} \Psi).
\]
When we combine the $\mathcal{A}^\infty$-valued inner product on $\mathcal{S}(G/K)$ with integration by the left-invariant probability measure on $G/K$, we obtain an ordinary inner product on $\mathcal{S}(G/K)$. On completing $\mathcal{S}(G/K)$ for this inner product, we obtain a Hilbert space. In this way we can view $D$ as a densely defined operator on a Hilbert space.
**Definition 12.2.** The Dirac operator, $D$, for the given Cad-invariant inner product on $\mathfrak{g}'$, and the given spinor bundle, is the operator $D$ defined above.

As explained in theorem 1.7i of [77] and proposition 9.4 of [36] and later pages, spinor bundles for Clifford bundles are not in general unique. The tensor product of a spinor bundle by a line bundle will be another spinor bundle, and all the irreducible spinor bundles are related in this way. Within our setting of equivariant bundles, we need to tensor with $G$-equivariant line bundles. These correspond exactly to the characters, that is, one-dimensional representations, of $K$. This is all relevant to coadjoint orbits because in that case $K$ always has non-trivial characters. All of our discussion of this following equation 5.2 of [40] carries over to the present situation with only very minor changes. So we will not discuss this further in this paper, and refer the interested reader to [40].

Because essentially all of the operations defined above commute with the actions of $G$ by left translation, it is easily checked that:

**Proposition 12.3.** The Dirac operator $D$ commutes with the action of $G$ on $\mathcal{S}(G/K)$ by left translation.

For $f \in \mathcal{A}^\infty$ let $M_f$ denote the operator on $\mathcal{S}(G/K)$ coming from $\mathcal{S}(G/K)$ being an $\mathcal{A}^\infty$-module. Notice that $\hat{d} f \in \mathcal{C}\ell(G/K)$ via $\hat{\Omega}(G/K) \subset \mathcal{C}\ell(G/K)$ so that $\kappa_{\hat{d} f}$ is an operator on $\mathcal{S}(G/K)$. Then much as in proposition 8.3 of [23] we have

**Proposition 12.4.** For any $f \in \mathcal{A}^\infty$ we have

$$[D, M_f] = \kappa_{\hat{d} f}.$$  

**Proof.** For $f \in \mathcal{A}^\infty$ and $\Psi \in \mathcal{S}(G/K)$ the Leibniz rule

$$d(f\Psi) = (\hat{d} f) \otimes \Psi + f d\Psi$$

follows from Eq. 12.5. Then, since $\kappa$ is $\mathcal{A}^\infty$-linear, we have

$$[D, M_f] \Psi = D(f\Psi) - f D(\Psi) = \kappa(d(f\Psi)) - f \kappa(d\Psi)$$

$$= \kappa(\hat{d} f \otimes \Psi + f d\Psi) - \kappa(f d\Psi) = \kappa_{\hat{d} f} \Psi.$$  

$\square$

Recall that for any $f \in \mathcal{A}^\infty$ and $x \in G$ we have $\hat{d} f_x \in \mathfrak{m}'^C$, so that $\|\hat{d} f_x\|_{\mathfrak{m}^C}$ is defined in terms of the inner product on $\mathfrak{m}^C$.

**Theorem 12.5.** For any $f \in \mathcal{A}^\infty$ we have

$$\| [D, M_f] \| = \sup \{ \| \hat{d} f_x \|_{\mathfrak{m}^C} : x \in G \} = \sup \{ \| d f_x \|_{\mathfrak{m}^C} : x \in G \}.$$  

**Proof.** Since $\mathcal{C}\ell(G/K)$ is a $*$-subalgebra of the $C^*$-algebra $\mathcal{A} \otimes \mathcal{C}\ell(\mathfrak{m}') = C(G, \mathcal{C}\ell(\mathfrak{m}'))$, and we are viewing $\hat{d} f$ as an element of $\mathcal{C}\ell(G/K)$, we have

$$\| \hat{d} f \| = \sup \{ \| \hat{d} f_x \|_{\mathcal{C}\ell(\mathfrak{m}')} : x \in G \}.$$  

But, just as seen before Proposition 11.1, if $\mu \in \mathfrak{m}^C$ then $\| \mu \|_{\mathcal{C}\ell(\mathfrak{m}')} = \| \mu \|_{\mathfrak{m}^C}$. The first equality in the statement of the theorem follows directly from this.

Because Cad is isometric on $\mathfrak{g}^C$, we have $\| \hat{d} f_x \|_{\mathfrak{m}^C} = \| d f_x \|_{\mathfrak{m}^C}$ for all $x \in G$. This gives the second equality in the statement of the theorem. $\square$
When we recall the notation of Proposition 11.1 and earlier, we see that we obtain:

**Corollary 12.6.** For any \( f \in \mathcal{A}^\infty \) we have

\[
\| [D, M_f] \| = \| [D_0, M_f] \|
\]

Thus the corresponding C*-metrics on \( \mathcal{A} \) are equal. This will be important for the proof of our main theorem.

For the reasons given immediately after Proposition 11.1, we then obtain:

**Corollary 12.7.** For any \( f \in \mathcal{A}^\infty \) we have

\[
\| [D, M_f] \| = \sup \{ \| \text{grad}_x f \|_{m_c} : x \in G \}.
\]

Now a standard argument (e.g., see the text between definition 9.12 and exercise 9.7 of [36]) shows that if we denote by \( d \) the ordinary metric on a Riemannian manifold \( N \) coming from its Riemannian metric, then for any two points \( p \) and \( q \) of \( N \) we have

\[
d(p, q) = \sup \{ |f(p) - f(q)| : \| \text{grad}_f \|_\infty \leq 1 \}.
\]

On applying this to \( G/K \), we obtain, for \( d \) now the ordinary metric on \( G/K \) from our Riemannian metric,

\[
d(p, q) = \sup \{ |f(p) - f(q)| : \| [D, M_f] \| \leq 1 \}.
\]

This is the formula on which Connes focused for general Riemannian manifolds [12,14], as it shows that the Dirac operator contains all of the metric information (and much more) for the manifold. This is his motivation for advocating that metric data for “non-commutative spaces” be encoded by providing them with a “Dirac operator”.

We do not need to know the formal self-adjointness of \( D \) in this paper. We refer the interested reader to theorem 6.1 of [40], and to theorem 8.4 of the most recent arXiv version of [23]. (The published version of [23] has a serious error in the proof of theorem 8.4.)

### 13. Complex Structure on Coadjoint Orbits

We want to show that in the case of coadjoint orbits we can choose a spinor bundle whose fibers are irreducible. The path to showing this involves showing that coadjoint orbits have complex structures. In fact, coadjoint orbits can be equipped with Kähler structures, as described in section 1 of [40] (and see also the theorem at the end of section 3 of [78]), but we do not need the full force of that fact. We will discuss in Sect. 19 why our approach does not always fit perfectly with the Kähler structures. So here we construct the complex structures directly, along the lines discussed in [79]. This requires the basic facts concerning weights and roots of compact semisimple Lie groups. In the next few paragraphs we follow somewhat closely the notation and development in section 1 of [40].

We assume now that \( G \) is a connected compact semisimple Lie group, with Lie algebra \( g \). The coadjoint orbits are the orbits in \( g' \) for the action \( \text{Cad} \). Fix \( \mu_\circ \in g' \), with \( \mu_\circ \neq 0 \). Let \( K \) denote the Cad-stability subgroup of \( \mu_\circ \), so that \( x \mapsto \text{Cad}_x(\mu_\circ) \) gives a \( G \)-equivariant diffeomorphism from \( G/K \) onto the Cad-orbit of \( \mu_\circ \). We will usually work with \( G/K \) rather than the orbit itself.
We choose an Ad-invariant inner product on \( g \), for example, the negative of the Killing form of \( g \) (since \( G \) is compact). This actually is not much more general than choosing the negative of the Killing form itself, because if \( g \) is simple, then every Ad-invariant inner product on \( g \) is just a scalar multiple of the negative of the Killing form, whereas if \( g \) is just semisimple, then every Ad-invariant inner product on \( g \) just arises from taking various scalar multiples of the negatives of the Killing forms on its simple ideals.

The action \( \text{Ad} \) of \( G \) on \( g \) is by orthogonal operators, and so the action \( \text{ad} \) of \( g \) on \( g \) is by skew-adjoint operators. There is a unique \( Z_\circ \in g \) such that

\[
\mu_\circ(X) = \langle X, Z_\circ \rangle \text{ for all } X \in g. \tag{13.1}
\]

It is easily seen that the Ad-stability subgroup of \( Z_\circ \) is again \( K \).

Let \( T_\circ \) be the closure in \( G \) of the one-parameter group \( r \mapsto \exp(rZ_\circ) \), so that \( T_\circ \) is a torus subgroup of \( G \). Then it is easily seen that \( K \) consists exactly of all the elements of \( G \) that commute with all the elements of \( T_\circ \). Note that \( T_\circ \) is contained in the center of \( K \) (but need not coincide with the center). Since each element of \( K \) will lie in a torus subgroup of \( G \) that contains \( T_\circ \), it follows that \( K \) is the union of the tori that it contains, and so \( K \) is connected (corollary 4.22 of [34]). Thus for most purposes we can just work with the Lie algebra, \( \mathfrak{t} \), of \( K \) when convenient. In particular, \( \mathfrak{t} = \{ X \in g : [X, Z_\circ] = 0 \} \), and \( \mathfrak{t} \) contains the Lie algebra, \( \mathfrak{t}_\circ \), of \( T_\circ \).

Let \( m = \mathfrak{t} \perp \) with respect to chosen inner product. Since Ad preserves the inner product, we see that \( m \) is carried into itself by the restriction of Ad to \( K \). Thus \( \{ \mathfrak{t}, m \} \subseteq m \). As we have seen in Sect. 12, \( m \) can be conveniently identified with the tangent space to \( G/K \) at the coset \( K \) (which corresponds to the point \( \mu_\circ \) of the coadjoint orbit). To define a complex structure on \( G/K \) we need to define a complex structure on \( m \) that commutes with the action of \( K \) via Ad.

Fix a choice of a maximal torus, \( T \), of \( G \) that contains \( T_\circ \). Then \( T \) is contained in \( K \), and is a maximal torus in \( K \). (Thus we are in the setting to which the results of [53] apply, since \( K \) has the same rank as \( G \). Our \( t \) is the \( \mathfrak{t} \) there, while our \( m \) is the \( \mathfrak{p} \) there, up to complexification. But we will not use results from that paper.) We denote the Lie algebra of \( T \) by the traditional \( \mathfrak{h} \). Since \( K \) is compact, it is reductive, and so \( \mathfrak{t} \) splits as \( \mathfrak{t} = \mathfrak{h}_m \oplus \mathfrak{t}_s \) where \( \mathfrak{t}_s \) is the semisimple subalgebra \( \mathfrak{t}_s = [\mathfrak{t}, \mathfrak{t}] \) of \( \mathfrak{t} \) and \( \mathfrak{h}_m \) is the center of \( \mathfrak{t} \) (so \( \mathfrak{h}_m \subseteq \mathfrak{h} \)). Thus \( g = \mathfrak{m} \oplus \mathfrak{h}_m \oplus \mathfrak{t}_s \). Furthermore, \( \mathfrak{h} \) splits as \( \mathfrak{h} = \mathfrak{h}_m \oplus \mathfrak{h}_s \) where \( \mathfrak{h}_s \) is a Cartan subalgebra of \( \mathfrak{t}_s \).

For any finite-dimensional unitary representation \( (\mathcal{H}, \pi) \) of \( T \) we let \( \pi \) also denote the corresponding representation of \( h \). For each \( H \in \mathfrak{h} \) the operator \( \pi_H \) is skew-adjoint, and so its eigenvalues are purely imaginary. Since the \( \pi_H \)'s all commute with each other, they are simultaneously diagonalizable. Because we need to keep track of the structure over \( \mathbb{R} \), we will use a convention for the weights of a representation that is slightly different from the usual convention. We used it previously in [41]. If \( \xi \in \mathcal{H} \) is a common eigenvector for the \( \pi_H \)'s, there will be a linear functional \( \alpha \) on \( \mathfrak{h} \) (with values in \( \mathbb{R} \)) such that

\[
\pi_H(\xi) = i\alpha(H)\xi
\]

for all \( H \in \mathfrak{h} \). For each \( \alpha \in \mathfrak{h}' \) (where \( \mathfrak{h}' \) denotes the dual vector space to \( \mathfrak{h} \)) we set

\[
\mathcal{H}_\alpha = \{ \xi \in \mathcal{H} : \pi_H(\xi) = i\alpha(H)\xi \text{ for all } H \in \mathfrak{h} \}.
\]

If there are non-zero vectors in \( \mathcal{H}_\alpha \) then we say that \( \alpha \) is a weight of the representation \( (\mathcal{H}, \pi) \). We denote the set of all weights for this representation by \( \Delta_\pi \). Then

\[
\mathcal{H} = \bigoplus \{ \mathcal{H}_\alpha : \alpha \in \Delta_\pi \}.
\]
We let \( g^C \) denote the complexification of \( g \), with inner product coming from that on \( g \), and corresponding unitary representation \( \text{Ad} \) of \( G \) on \( g^C \). The non-zero weights for \( \text{Ad} \) and \( \text{ad} \) acting on \( g^C \) are called the “roots” of \( G \). We denote the set of roots simply by \( \Delta \). Because we are dealing with the complexification of a representation over \( \mathbb{R} \), if \( \alpha \in \Delta \) then \( -\alpha \in \Delta \). For each root \( \alpha \) we let \( g^C_\alpha \) denote the corresponding root space.

It is a standard fact that these root spaces are all of dimension 1, and that \( \{ g^C_\alpha, g^C_{-\alpha} \} \) is not of dimension 0 (so is of dimension 1). In the standard way \([34,58,80]\) we make a choice, \( \Delta^+ \), of positive roots. We want to choose usual elements \( \alpha, C \) such that \( \alpha(C) \) is \( \Delta^+ \)-invariant in the sense that if \( \alpha \in \Delta^+ \), then \( \alpha(C) \) is \( \Delta^+ \)-invariant.

Thus it is appropriate to set \( \Delta_s = \{ \alpha \in \Delta : \alpha(Z_0) = 0 \} = \{ \alpha \in \Delta : \alpha(Z) = 0 \text{ for all } Z \in h^C_m \} \).

Notice that if \( \pi \) is a unitary representation of \( G \) on \( H \) for which \( \pi(\text{Ad}) \) is irreducible, then \( \pi(Z) \) is also irreducible, and \( \pi(C) \) is \( \Delta^+ \)-invariant in the sense that if \( \alpha \in \Delta^+ \), then \( \alpha(C) \) is \( \Delta^+ \)-invariant.

Proposition 13.1. With notation as above, for each \( \alpha \in \Delta^+ \) we can choose \( H_\alpha \in i \mathfrak{h} \) and \( E_\alpha \in g^C_\alpha \) such that \( [E_\alpha, E_\alpha^*] = H_\alpha \) and \( [H_\alpha, E_\alpha] = 2E_\alpha \). Setting \( F_\alpha = E_\alpha^* \), we then obtain \([H_\alpha, F_\alpha] = -2F_\alpha \).

Thus it is appropriate to define an involution on \( g^C \) by \( (X + iY)^* = -X + iY \), so that \( \pi(X)^* = \pi(X^*) \) for all \( X \in g^C \) (as in \([39,58]\)). Notice that for all \( W, Z \in g^C \) we have \([W, Z]^* = [Z^*, W^*] \).

The following result is well-known. It occurs with proof as proposition 1.1 of \([41]\).
coming from Weyl-chamber-type considerations. (See [79].) But there is one such order canonically associated with \( Z_\alpha \), namely specified by

\[
\Delta^+_m = \{ \alpha \in \Delta_m : \alpha(Z_\alpha) > 0 \}.
\]

Notice that this makes sense because \( \alpha(Z_\alpha) \in \mathbb{R} \) since \( Z_\alpha \in \mathfrak{h} \) and \( \alpha \in \mathfrak{h}' \). It is clear that \( \Delta^+_m \) is \( \Delta_s \)-invariant.

It is then natural to set \( n^+ = \bigoplus \{ \mathfrak{g}^C_{\alpha} : \alpha \in \Delta^+_m \} \), and similarly for \( n^- \). Then \( m^C = n^+ \oplus n^- \).

**Notation 13.2.** Define an operator, \( J \), on \( m^C \) by

\[
J(X) = \begin{cases} 
  iX & \text{if } X \in n^+, \\
  -iX & \text{if } X \in n^-.
\end{cases}
\]

Notice that \( J \) is isometric.

We show now that \( J \) commutes with the ad-action of \( \mathfrak{k}^C \) on \( m^C \). This uses the \( \Delta_s \)-invariance of \( \Delta^+_m \). Let \( \beta \in \Delta_s \). Then for any \( \alpha \in \Delta^+_m \) we have

\[
J(\text{ad}_{E_\beta}(E_\alpha)) = J([E_\beta, E_\alpha]) = i[E_\beta, iE_\alpha] = \text{ad}_{E_\beta}(J(E_\alpha))
\]

since \( \alpha + \beta \in \Delta^+_m \) if \( \alpha + \beta \in \Delta \). If instead \( Z \in \mathfrak{h}^C \subseteq \mathfrak{k}^C \) then

\[
J(\text{ad}_Z(E_\alpha)) = J(i\alpha(Z)E_\alpha) = i\alpha(Z)J(E_\alpha) = \text{ad}_Z(J(E_\alpha)).
\]

Thus \( J \) commutes with the ad-action of \( \mathfrak{k}^C \) on \( n^+ \). A similar calculation shows that \( J \) commutes with the ad-action of \( \mathfrak{k}^C \) on \( n^- \), and so on \( m^C \), as desired.

It is clear from the definition of \( J \) that \( J^2 = -I^{m^C} \), for \( I^{m^C} \) the identity operator on \( m^C \).

Ordinary complex conjugation on \( \mathfrak{g}^C \) with respect to \( \mathfrak{g} \) carries \( \mathfrak{g}^C_{\alpha} \) onto \( \mathfrak{g}^C_{-\alpha} \) for each \( \alpha \in \Delta \). It follows that complex conjugation carries \( n^+ \) onto \( n^- \), and consequently, complex conjugation commutes with \( J \). It follows that \( J \) carries \( m \) onto itself, and so is a complex structure on \( m \). Of course \( J \) commutes with the actions of \( K \) and \( \mathfrak{k} \) on \( m \).

Let \( \alpha \in \Delta^+_m \), and consider \( E_\alpha \in \mathfrak{g}^C_{\alpha} \). Then \( \tilde{E}_\alpha \in \mathfrak{g}^C_{-\alpha} \), where the bar denotes the ordinary complex conjugation. Set

\[
X_\alpha = E_\alpha + \tilde{E}_\alpha \quad \text{and} \quad Y_\alpha = i(E_\alpha - \tilde{E}_\alpha).
\]

They are invariant under complex conjugation, and so they are in \( \mathfrak{g} \), and they span \( (\mathfrak{g}^C_{\alpha} \oplus \mathfrak{g}^C_{-\alpha}) \cap \mathfrak{g} \) over \( \mathbb{R} \), which we denote by \( m_\alpha \). (So care must be taken not to confuse \( m_\alpha \) with root spaces of \( \mathfrak{g}^C \).) It is easily checked that

\[
J(X_\alpha) = Y_\alpha \quad \text{and} \quad J(Y_\alpha) = -X_\alpha,
\]

and that for any \( Z \in \mathfrak{h}_m \) we have

\[
[Z, X_\alpha] = \alpha(Z)Y_\alpha = \alpha(Z)J(X_\alpha) \quad \text{and} \quad [Z, Y_\alpha] = -\alpha(Z)X_\alpha = \alpha(Z)J(Y_\alpha).
\]

Thus if we view \( J \) as “multiplication by \( i \)”, we see that \( m_\alpha \) is a weight space (of dimension 1) for the representation of \( \mathfrak{h}_m \) on \( m \). Note that \( J \) is isometric on \( m \), since earlier we saw that it is isometric on \( m^C \).
Definition 13.3. The operator $J$ on $m$ defined above is the complex structure on $m$ canonically associated to the element $\mu_\circ$. When $m$ is equipped with this complex structure, we will denote it by $m_J$.

We remark that many elements of $g'$ can determine the same complex structure on $m$ — all the ones in the same Weyl-type chamber.

Definition 13.4. We will denote the adjoint of $J$ on $m'$ again by $J$. It is the complex structure on $m'$ canonically associated to the element $\mu_\circ$. When $m'$ is equipped with this complex structure, we will often denote it by $m'_J$.

Then $J$ on $m'$ is isometric, and will commute with the representation Cad of $K$ on $m'_J$.

14. Yet More About Spinors

We are now exactly in position to use some of the main results of section 4 of [40]. We consider a general even-dimensional Hilbert space $m$ over $\mathbb{R}$ (that will later be our $m'$), that is equipped with an isometric complex structure $J$. We form the complex Clifford algebra $\mathbb{C}\ell(m)$ for the given inner product. Let $O(m)$ be the group of orthogonal transformations of $m$. By the universal property of Clifford algebras, each element of $O(m)$ determines an automorphism of $\mathbb{C}\ell(m)$ (a “Bogoliubov automorphism”). We denote the corresponding action of $O(m)$ on $\mathbb{C}\ell(m)$ by $\beta$.

We seek to construct an irreducible representation, $\kappa$, of $\mathbb{C}\ell(m)$ on a Hilbert space $S$ that is suitably compatible with the action $\beta$. To construct this, we use the complex structure $J$. When we view $m$ as a complex Hilbert space using $J$, we will denote it by $m_J$. Let $U(m_J)$ denote the group of unitary operators on $m_J$. It is the subgroup of those elements of $O(m)$ that commute with $J$. Notice that the complex dimension of $m_J$ is half of the real dimension of $m$.

Let $S$ be the exterior algebra over $m_J$. By the universal property of exterior algebras, each element of $U(m_J)$ determines an automorphism of $S$ (a “Bogoliubov automorphism”). We denote the corresponding action of $U(m_J)$ on $S$ by $\rho$. There is a standard way of defining an irreducible representation, $\kappa$, of $\mathbb{C}\ell(m)$ on $S$, called the Fock representation, in terms of annihilation and creation operators. This is described in section 4 of [40], strongly influenced by the thorough exposition in [36] beginning with definition 5.6. (which uses the opposite sign convention than we use for the definition of Clifford algebras). See also the the discussion after corollary 5.17 of [42]. We will not describe the construction here. But by examining the explicit construction, as done in [40], we are able to obtain the following crucial result, which is just a restatement of proposition 4.4 of [40] with very minor changes of notation:

Proposition 14.1. The representation $\rho$ of $U(m_J)$ on $S$ is compatible with the action $\beta$ of $U(m_J)$ on $\mathbb{C}\ell(m)$ in the sense that

$$\kappa_\beta R(q) = \rho R \kappa q \rho_R^{-1}$$

for all $R \in U(m_J)$ and $q \in \mathbb{C}\ell(m)$.

We refer the reader to [40] for the proof.
15. Dirac Operators for Coadjoint Orbits

In this section we combine the results of several previous sections to construct Dirac operators for coadjoint orbits of compact semisimple Lie groups. We use the notation of Sect. 13. We apply the results of Sect. 13, but in the role of the $m$ in that section we will use $m' \subset g'$, with its complex structure $J$ defined at the end of Sect. 13. We form the complex Clifford algebra $\mathbb{C}\ell(m')$, the exterior algebra $\mathbb{S}$ over $m'J$, and the irreducible representation $\kappa$ of $\mathbb{C}\ell(m')$ on $\mathbb{S}$. We saw that the representation $\text{Cad}$ restricted to $K$ on $m'$ extends to an action $\beta$ of $K$ on $\mathbb{C}\ell(m')$. We also saw that this representation commutes with $J$, and so is a unitary representation of $K$ on $m'J$, that is, a homomorphism from $K$ into $U(m_J)$. Thus it extends to a unitary representation $\rho$ of $K$ on $\mathbb{S}$. Of crucial importance, from Proposition 14.1 we obtain the compatibility relation

$$\kappa_{\text{Cad}}(q) = \rho_s(\kappa q)\rho_s^{-1}$$

for all $s \in K$ and $q \in \mathbb{C}\ell(m)$. For the present situation, this is exactly the compatibility relation (12.1) that was assumed in Sect. 12.

As in Sect. 12, we form the Clifford bundle $\mathbb{C}\ell(G/K)$. Using the exterior algebra $\mathbb{S}$, we construct the spinor bundle $\mathbb{S}(G/K)$ as in Sect. 12, with the representation $\rho$ of $K$ on it. We let $\kappa$ be the pointwise action of $\mathbb{C}\ell(G/K)$ on $\mathbb{S}(G/K)$. As seen in Sect. 12, the compatibility relation is needed in order to ensure that $\kappa$ carries $\mathbb{S}(G/K)$ into itself.

We now let $D$ be the operator on $\mathbb{S}(G/K)$ constructed exactly as done in Sect. 12 before Definition 12.2.

**Definition 15.1.** The operator, $D$, defined above is the Dirac operator on $G/K$ (i.e. on $A = C(G/K)$) for the given element $\mu_0 \in g'$ and the given Cad-invariant inner product on $g'$.

As we will discuss in Sect. 19, $D$ is not always the Dirac operator corresponding to the Kähler structure on $G/K$ determined by $\mu_0$ and constructed in [40].

From Proposition 12.3 we know that $D$ commutes with the action of $G$ on $\mathbb{S}(G/K)$ by left translation. From Proposition 12.4 we immediately obtain:

**Proposition 15.2.** For any $f \in A^\infty$ we have

$$[D, M_f] = \kappa_{\tilde{\mu}_f}.$$

From Theorem 12.5 we immediately obtain

**Theorem 15.3.** For any $f \in A^\infty = C^\infty(G/K)$ we have

$$\|[D, M_f]\| = \sup\{\|\tilde{d}f_x\|_{m^c} : x \in G\} = \sup\{\|d_f\|_{m^c} : x \in G\}.$$

Then from Corollary 12.6 we immediately obtain:

**Corollary 15.4.** For any $f \in A^\infty$ we have

$$\|[D, M_f]\| = \|[D_0, M_f]\|.$$

Thus the corresponding C*-metrics on $A$ are equal. This fact will be important in the next sections. The comments after Corollary 12.7 apply equally well here.
16. Bridges with Symbols

The definition of quantum Gromov–Hausdorff distance between compact C*-metric spaces has evolved over the years since the first definition was proposed in [9]. At present the definition with the best properties is Latrémolière’s dual Gromov–Hausdorff propinquity [81], and it forms the base for the spectral propinquity [24]. But its definition can be somewhat difficult to work with directly for some classes of examples, including the examples we are considering. Latrémolière had slightly earlier introduced a somewhat stronger definition that he called the Gromov–Hausdorff propinquity [22]. This definition works well for our examples. So in this section we will recall the definition of the Gromov–Hausdorff propinquity and explain how it relates to our situation. With this as preparation, in the next section we will prove that a suitable sequence of matrix algebras, equipped with C*-metrics coming from Dirac operators, converges for the Gromov–Hausdorff propinquity to the coadjoint orbit for a given highest-weight vector. (This will imply convergence also for the dual Gromov–Hausdorff propinquity).

For any two unital C*-algebras $A$ and $B$, a bridge from $A$ to $B$ in the sense of Latrémolière [22] is a quadruple $(D, \pi_A, \pi_B, \omega)$ for which $D$ is a unital C*-algebra, $\pi_A$ and $\pi_B$ are unital injective homomorphisms of $A$ and $B$ into $D$, and $\omega$ is a self-adjoint element of $D$ such that $1$ is an element of the spectrum of $\omega$ and $\|\omega\| = 1$. Actually, Latrémolière only requires a looser but more complicated condition on $\omega$, but the above condition will be appropriate for our examples. Following Latrémolière, we will call $\omega$ the “pivot” for the bridge. We will often omit mentioning the injections $\pi_A$ and $\pi_B$ when it is clear what they are from the context, and accordingly we will often write as though $A$ and $B$ are unital subalgebras of $D$.

For our applications, $A$ will be $C(G/K)$ for $G$ and $K$ as in previous sections, and $B = B(H)$ will be the matrix algebra corresponding to an irrep $(H, U)$ of $G$. Let $\alpha$ be the action of $G$ on $B$ by conjugation by $U$. For our bridge we take $D$ to be the C*-algebra $D = A \otimes B = C(G/K, B)$. We take $\pi_A$ to be the injection of $A$ into $D$ defined by

$$\pi_A(a) = a \otimes 1_B$$

for all $a \in A$, where $1_B$ is the identity element of $B$. The injection $\pi_B$ is defined similarly. We define the pivot $\omega$ to be the coherent state associated to the irrep, that is, $\omega$ is the function in $C(G/K, B)$ defined by

$$\omega(x) = \alpha_x(P)$$

for all $x \in G/K$. We notice that $\omega$ is actually a non-zero projection in $D$, and so it satisfies the requirements for being a pivot. We will denote the bridge $(D, \omega)$ by $\Pi$.

**Definition 16.1.** We will call the bridge $\Pi$ constructed just above the bridge from $G/K$ (or from $A = C(G/K)$) to the irrep $(H, U)$ (or to $B = B(H)$).

Any choice of C*-metrics $L^A$ and $L^B$ on unital C*-algebras $A$ and $B$ can be used to measure any given bridge $\Pi = (D, \omega)$ between $A$ and $B$. Latrémolière [22] defines the “length” of the bridge by first defining its “reach” and its “height”.
Definition 16.2. Let $A$ and $B$ be unital C*-algebras, and let $\Pi = (D, \omega)$ be a bridge from $A$ to $B$. Let $L^A$ and $L^B$ be C*-metrics on $A$ and $B$. Set

$$\mathcal{L}^1_A = \{ a \in A : a = a^* \text{ and } L^A(a) \leq 1 \},$$

and similarly for $\mathcal{L}^1_B$. (This is slightly different from Eq. (7.2).) We can view these as subsets of $D$. Then the reach of $\Pi$ is given by:

$$\text{reach}(\Pi) = \text{Haus}_D(\mathcal{L}^1_A \omega \mathcal{L}^1_B),$$

where $\text{Haus}_D$ denotes the Hausdorff distance with respect to the norm of $D$, and where the product defining $\mathcal{L}^1_A \omega$ and $\omega \mathcal{L}^1_B$ is that of $D$.

Latrémiolère shows just before definition 3.14 of [22] that, under conditions that include the case in which $(A, L^A)$ and $(B, L^B)$ are C*-metric spaces, the reach of $\Pi$ is finite.

To define the height of $\Pi$ we need to consider the state space, $S(A)$, of $A$, and similarly for $B$ and $D$. Even more, we set

$$S_1(\omega) = \{ \phi \in S(D) : \phi(\omega) = 1 \},$$

the “level-1 set of $\omega$”. The elements of $S_1(\omega)$ are “definite” on $\omega$ in the sense [82] that for any $\phi \in S_1(\omega)$ we have

$$\phi(d\omega) = \phi(d) = \phi(\omega d),$$

for all $d \in D$. Since $L^A$ is a C*-metric, it determines, by formula (7.1), an ordinary metric, $\rho_A$, on $S(A)$, which metrizes the weak-* topology, for which $S(A)$ is compact. Define $\rho_B$ on $S(B)$ similarly.

Notation 16.3. We denote by $S^A_1(\omega)$ the restriction of the elements of $S_1(\omega)$ to $A$. We define $S^B_1(\omega)$ similarly.

Definition 16.4. Let $A$ and $B$ be unital C*-algebras and let $\Pi = (D, \omega)$ be a bridge from $A$ to $B$. Let $L^A$ and $L^B$ be C*-metrics on $A$ and $B$. The height of the bridge $\Pi$ is given by

$$\text{height}(\Pi) = \max\{ \text{Haus}_{\rho_A}(S^A_1(\omega), S(A)) \text{, } \text{Haus}_{\rho_B}(S^B_1(\omega), S(B)) \},$$

where the Hausdorff distances are with respect to the indicated metrics. The length of $\Pi$ is then defined by

$$\text{length}(\Pi) = \max\{ \text{reach}(\Pi) \text{, } \text{height}(\Pi) \}. $$

Latrémiolère defines the length of a finite path of bridges (a “trek”) to be the sum of the lengths of the individual bridges. He then defines the Gromov–Hausdorff propinquity between two compact C*-metric spaces to be the infimum of the lengths of all finite paths between them. (This gives the triangle inequality.) He proves the remarkable fact that if the propinquity between two compact C*-metric spaces is 0, then they are isometric in the sense that there is an isomorphism between the C*-algebras that carries the C*-metric on one to the C*-metric on the other. Thus the propinquity is a metric on the set of isometry classes of compact C*-metric spaces. But we will not need to deal directly with finite paths of bridges because we will prove that, for the sequences of bridges that we will construct, already their lengths will converge to 0.

For the main context of this paper there is extra structure available to help with measuring the lengths of bridges. The next paragraph is strongly motivated by the discussion of bridges with conditional expectations in sections 4 and 5 of [56].
Definition 16.5. Let $A$ and $B$ be unital $C^*$-algebras, and let $\Pi = (D, \omega)$ be a bridge from $A$ to $B$. By a pair of symbols for $\Pi$ we mean a pair of unital completely positive maps $(\sigma^A, \sigma^B)$ such that $\sigma^A$ maps $D$ to $A$ and $\sigma^A(\omega) = 1_A$, while $\sigma^B$ maps $D$ to $B$ and $\sigma^B(\omega) = 1_B$.

(Thus $\sigma^A$ and $\sigma^B$ are “definite” on $\omega$. We do not require any relation between these two maps.)

For our applications we define $\sigma^A$ by

$$\sigma^A(F)(x) = \text{tr}_B(F(x)\alpha_x(P))$$

for any $F \in D = C(G/K, B)$, where $\text{tr}_B$ is the un-normalized trace on $B$. We use the term "symbol" in definition 16.5 because the restriction of this particular $\sigma^A$ to $B$ is exactly the Berezin contravariant symbol map, and the restriction to $A$ of the particular $\sigma^B$ that we will define just below is exactly the Berezin covariant symbol map, that play an essential role in [10].

For our applications we define $\sigma^B$ by

$$\sigma^B(F) = d_H \int_{G/K} F(x)\alpha_x(P) \, dx,$$

where $d_H$ is the dimension of the Hilbert space of the irrep, and $dx$ refers to the $G$-invariant probability measure on $G/K$. It is easily seen [56] that $\sigma^A$ and $\sigma^B$ are unital and completely positive, and furthermore that they intertwine the actions of $G$ on $A$ and $B$ with the diagonal action of $G$ on $D = A \otimes B$.

Definition 16.6. Let $A$ and $B$ be unital $C^*$-algebras and and let $\Pi = (D, \omega)$ be a bridge from $A$ to $B$. Let $L_A$ and $L_B$ be $C^*$-metrics on $A$ and $B$. Let $(\sigma^A, \sigma^B)$ be a pair of symbols for $\Pi$. We say that this pair of symbols is compatible with the $C^*$-metrics $L_A$ and $L_B$ if their restrictions to $B$ and $A$ satisfy

$$L_A(\sigma^A(b)) \leq L_B(b) \quad \text{and} \quad L_B(\sigma^B(a)) \leq L_A(a)$$

for all $a \in A$ and $b \in B$. (Here we are viewing $A$ and $B$ as subalgebras of $D$ in the evident way discussed in the second paragraph of this section.)

We now show how to use a compatible pair of symbols to obtain an upper bound for the reach of a bridge $\Pi$. Let $b \in L_B^1$ be given. As an approximation to $\omega b$ by an element of the form $a \omega$ for some $a \in L_A^1$ we take $a = \sigma^A(b)$. It is indeed in $L_A^1$ by the compatibility condition. This prompts us to set

$$\gamma^B = \sup\{\|\sigma^A(b)\omega - \omega b\|_D : b \in L_B^1\}.\tag{16.4}$$

Interchanging the roles of $A$ and $B$, we define $\gamma^A$ similarly. We then see that

$$\text{reach}(\Pi) \leq \max\{\gamma^A, \gamma^B\}.\tag{16.5}$$

We will see shortly why this upper bound is useful for our applications.

We now consider the height of $\Pi$. For this we need to consider $S_1(\omega)$ as defined above. Because $\sigma^A$ is positive and unital, its composition with any $\mu \in S(A)$ is in
By definition \( \sigma^A(\omega) = 1_A \). Thus for every \( \mu \in S(A) \) we obtain an element, \( \phi_\mu \), of \( S_1( \omega ) \), defined by
\[
\phi_\mu(d) = \mu(\sigma^A(d))
\]
for all \( d \in D \). This provides us with a substantial collection of elements of \( S_1( \omega ) \). Since to estimate the height of \( \Pi \) we need to estimate the distance from each \( \mu \in S(A) \) to \( S_1^A( \omega ) \), we can hope that \( \phi_\mu \) restricted to \( A \) is relatively close to \( \mu \). Accordingly, for any \( a \in A \) we compute
\[
|\mu(a) - \phi_\mu(a)| = |\mu(a) - \mu(\sigma^A(a))| \leq \|a - \sigma^A(a)\|.
\]
This prompts us to set
\[
\delta^A = \sup\{\|a - \sigma^A(a)\| : a \in L^1_A\}.
\] (16.6)

Then we see that
\[
\rho_{L^A}(\mu, \phi_\mu|_A) \leq \delta^A
\]
for all \( \mu \in S(A) \). We define \( \delta^B \) in the same way, and obtain the corresponding estimate for the distances from elements of \( S(B) \) to the restriction of \( S_1( \omega ) \) to \( B \). In this way we see that
\[
\text{height}(\Pi) \leq \max\{\delta^A, \delta^B\}.
\] (16.7)
(Notice that \( \delta^A \) involves what \( \sigma^A \) does on \( A \), whereas \( \gamma^A \) involves what \( \sigma^B \) does on \( A \).)

While this bound is natural within this context, it turns out not to be so useful for our main applications. The following steps might not initially seem to be natural, but in the next section we will see that for our main applications they are quite useful. Our notation is as above. For any \( \nu \in S(B) \) we easily see that \( \nu \circ \sigma^B \circ \sigma^A \in S(D) \). But from the relation between the symbols and \( \omega \) it is easily seen that \( \nu \circ \sigma^B \circ \sigma^A(\omega) = 1 \), so that \( \nu \circ \sigma^B \circ \sigma^A \) is in \( S_1( \omega ) \). Let us denote its restriction to \( B \) by \( \psi_\nu \), so that \( \psi_\nu \in S_1^B( \omega ) \).

Then \( \psi_\nu \) can be used as an approximation to \( \nu \) by an element of \( S_1^B( \omega ) \). Now for any \( b \in B \) we have
\[
|\nu(b) - \psi_\nu(b)| = |\nu(b) - (\nu \circ \sigma^B \circ \sigma^A)(b)| \leq \|b - \sigma^B(\sigma^A(b))\|.
\]

**Notation 16.7.** In terms of the above notation we set
\[
\hat{\delta}^B = \sup\{\|b - \sigma^B(\sigma^A(b))\| : b \in L^1_B\}.
\]

It follows that \( \rho_{L^B}(\nu, \psi_\nu) \leq \hat{\delta}^B \) for all \( \nu \in S(B) \), so that
\[
\text{Haus}_{\rho_B}(S_1^B(\omega), S(B)) \leq \hat{\delta}^B.
\] (16.8)

We define \( \hat{\delta}^A \) in the same way, and obtain the corresponding bound for the distances from elements of \( S(A) \) to \( S_1^A(\omega) \). In this way we obtain:

**Proposition 16.8.** For notation as above,
\[
\text{height}(\Pi) \leq \max\{\min\{\delta^A, \hat{\delta}^A\}, \min\{\delta^B, \hat{\delta}^B\}\}.
\]
17. The Proof That Matrix Algebras Converge to Coadjoint Orbits

We continue with the notation of the previous sections, so \( G \) is a connected compact semisimple Lie group. Not every \( \mu \in \mathfrak{g}' \) is associated with an irrep of \( G \). To be associated with an irrep, the restriction of \( \mu \) to any Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) must exponentiate to a one-dimensional representation of the corresponding maximal torus, and so in particular must be “integral”. But then it can correspond to a weight of many different irreps. To be viewed as corresponding to a unique (equivalence class of an) irrep, \( \mu \) must be “dominant” with respect to some choice of Cartan subalgebra and positive root system.

So let us fix an irrep \((\mathcal{H}, U)\) of \( G \). Fix a choice of a maximal torus in \( G \) with Cartan subalgebra \( \mathfrak{h} \), and a choice of a positive root system \( \Delta^+ \) for it. Then choose elements \( H_\alpha, E_\alpha, F_\alpha \in \mathfrak{g}^C \) as described in Proposition 13.1. Let \( \xi_\phi \) be a highest-weight vector of length 1 for the irrep. As a weight vector it is an eigenvector of \( U_\mathcal{H} \) for all \( H \in \mathfrak{h} \). The fact that it is a highest-weight vector means exactly that \( U_\mathcal{H} \xi_\phi = 0 \) for all \( \alpha \in \Delta^+ \).

Define \( \mu_\phi \) in \( \mathfrak{g}' \) by
\[
\mu_\phi(X) = -i \langle \xi_\phi, UX \xi_\phi \rangle.
\]
(We take the inner product on \( \mathcal{H} \) to be linear in the second variable.) Up to the sign, \( \mu_\phi \) is exactly the “equivariant momentum map” of equation 23 of [16]. Because \( UX \) is skew-symmetric for all \( X \in \mathfrak{g} \), we see that \( \mu_\phi \) is \( \mathbb{R} \)-valued on \( \mathfrak{g} \). Note that \( \mu_\phi \) does not depend on the phase of \( \xi_\phi \). We extend \( \mu_\phi \) to \( \mathfrak{g}^C \) and \( \mathfrak{h}^C \) in the usual way. Because \( \xi_\phi \) is a highest-weight vector, we clearly have \( \mu_\phi(E_\alpha) = 0 \) for all \( \alpha \in \Delta^+ \), and then \( \mu_\phi(F_\alpha) = 0 \) for all \( \alpha \in \Delta^+ \) because \( F_\alpha = E_\alpha^* \). Furthermore, because \( [E_\alpha, E_\alpha^*] = H_\alpha \) and \( [H_\alpha, E_\alpha] = 2E_\alpha \) and \( [H_\alpha, F_\alpha] = -2F_\alpha \), the triplet \((H_\alpha, E_\alpha, F_\alpha)\) generates via \( U \) a representation of \( \mathfrak{sl}(2, \mathbb{C}) \), for which the spectrum of \( U_{H_\alpha} \) must consist of integers. In particular, \( i \mu_\phi(H_\alpha) \) is an integer, necessarily non-negative, in fact equal to \( \|F_\alpha \xi_\phi\|^2 \). We see in this way that \( \mu_\phi \) is a quite special element of \( \mathfrak{g}' \).

Let \( \mu \) denote the weight of \( \xi_\phi \), so that \( U_\mathcal{H}(\xi_\phi) = i \mu(H) \xi_\phi \) for all \( H \in \mathfrak{h} \). Comparison with the definition of \( \mu_\phi \) shows that \( \mu \) is simply the restriction of \( \mu_\phi \) to \( \mathfrak{h} \). It is clear that \( \mu_\phi \) is determined by \( \mu \) in the sense that \( \mu_\phi \) has value 0 on the Kil-orthogonal complement of \( \mathfrak{h}^C \). Thus from now on we will let \( \mu_\phi \) also denote the weight of \( \xi_\phi \). (Thus the special properties of \( \mu_\phi \) mean that, as a weight, \( \mu_\phi \) is a “dominant integral weight”.)

We now construct the sequence of matrix algebras that we will show converges to the coadjoint orbit of \( \mu_\phi \). For each positive integer \( m \) let \((\mathcal{H}^m, U^m)\) be an irreducible representation of \( G \) with highest weight \( m \mu_\phi \). All the \( m \mu_\phi \)'s will have the same Cad-stability group, \( K \). Then let \( \mathcal{B}^m = \mathcal{L}(\mathcal{H}^m) \) with action \( \alpha \) of \( G \) using \( U^m \), and let \( P^m \) be the projection on the highest-weight vector in \( \mathcal{H}^m \) (which is just the tensor product of \( m \) copies of \( \xi_\phi \)). As before, we let \( \mathcal{A} = C(G/K) \). Then for each \( m \) we construct as for Definition 16.1 the bridge \( \Pi_m = (\mathcal{D}^m, \omega^m) \) from \( \mathcal{A} \) to \( \mathcal{B}^m \), using \( P^m \).

We assume that a Cad-invariant inner product has been chosen for \( \mathfrak{g}' \). Let \( D^A \) be the corresponding Dirac operator for \( \mathcal{A} \) constructed for Definition 15.1. For each integer \( m \) let \( D^m = D_B^m \) be the corresponding Dirac operator for \( B^m \) constructed as for Definition 9.1. Let \( L_{DA} \) be the C*-metric corresponding to \( DA \), and for each \( m \) let \( L_{D^m} \) be the C*-metric corresponding to \( D^m \).

For each \( m \) we want to measure the bridge \( \Pi_m \) using the C*-metrics \( L_{DA} \) and \( L_{D^m} \). For this purpose, we want to show that the pair of symbols \((\sigma^A, \sigma^{B^m})\), defined as in Eqs. (16.2) and (16.3) using \( P^m \) (and restricted to \( B^m \) and \( A \) respectively, as before), is compatible with these C*-metrics, in the sense of Definition 16.6. But this is awkward to do directly because \( D^A \) and \( D^m \) have been constructed in somewhat
different ways. However, we were careful to show in Corollary 15.4 that \( L^{D^A} = L^{D^A_o} \), and in Proposition 9.2 that \( L^{D^m} = L^{D^m_o} \), where \( D^A_o \) and \( D^m_o \) are defined as in Definition 2.3. Thus it suffices to show that the pair of symbols \( (\sigma^A_m, \sigma^B_m) \) is compatible with the C*-metrics \( L^{D^A} \) and \( L^{D^m} \). But these C*-metrics were both defined by the same construction, that of Sect. 2, so we can apply the results of that section. Because the restriction of \( \sigma^A_m \) to \( B^m \) is unital completely positive and intertwines the actions of \( G \) on \( A \) and \( B^m \), it follows from Corollary 6.3 that for any \( b \in B^m \) we have

\[
L^{D^A_o} (\sigma^A_m(b)) \leq L^{D^m_o}(b).
\]

In the same way but using \( \sigma^B_m \), we find that for any \( a \in A \) we have

\[
L^{D^m_o} (\sigma^B_m(a)) \leq L^{D^A_o}(a).
\]

Since \( L^{D^A} = L^{D^A_o} \), and \( L^{D^m} = L^{D^m_o} \) for each \( m \), we see that we have obtained:

**Proposition 17.1.** With notation as above, for each \( m \) the pair of symbols \( (\sigma^A_m, \sigma^B_m) \) is compatible with the C*-metrics \( L^{D^A} \) and \( L^{D^m} \).

Thus we can use the pair of symbols \( (\sigma^A_m, \sigma^B_m) \) in conjunction with \( L^{D^A} \) and \( L^{D^m} \) to measure the bridge \( \Pi^1_m \). So in terms of them, define the constants \( \gamma^A_m \) and \( \gamma^B_m \) by Eq. (16.4), and \( \delta^A_m \) and \( \delta^B_m \) by Notation 16.7. Then by Eq. 16.2 and Proposition 16.8 we have

\[
\text{reach}_D(\Pi_m) \leq \max \{ \gamma^A_m, \gamma^B_m \} \quad \text{and} \quad \text{height}_D(\Pi_m) \leq \max \{ \delta^A_m, \delta^B_m \}
\]

(We will not use \( \delta^A_m \) and \( \delta^B_m \). The subscript D here on “reach” and “height” is to indicate that here we use the Dirac operators, in contrast to another bridge-length that we are about to introduce. We thus obtain:

**Proposition 17.2.** For notation as above, for each \( m \) we have

\[
\text{length}_D(\Pi_m) \leq \max \{ \gamma^A_m, \gamma^B_m, \delta^A_m, \delta^B_m \}.
\]

Our objective now is to maneuver so as to be able to take advantage of the bounds obtained in [9, 56] for the case in which the C*-metrics are defined in terms of continuous length-functions \( \ell \) on \( G \). The proof of those bounds is fairly complicated. But appealing to them permits us to avoid needing to give a quite complicated proof here.

Recall the definition of the C*-metric \( L^\ell \) defined by Eq. (7.4), where now our chosen \( \ell \) is defined in terms of the Riemannian metric on \( G \) corresponding to the chosen inner product on \( g' \). When we apply it to \( A = C(G/K) \) and to \( B^m = B(\mathcal{H}_m) \), we will denote it by \( L^A_\ell \) and \( L^m_\ell \) respectively. For each \( m \) the pair of symbols \( (\sigma^A_m, \sigma^B_m) \) is also compatible with the C*-metrics \( L^A_\ell \) and \( L^m_\ell \). This is already proposition 1.1 of [9] together with an argument in the paragraph before notation 2.1 of [9], but it is also easily checked directly. Thus we can use \( (\sigma^A_m, \sigma^B_m) \) in conjunction with \( L^A_\ell \) and \( L^m_\ell \) to measure the bridge \( \Pi_m \).

Recall the general definition

\[
L^1_A = \{ a \in A : a = a^* \quad \text{and} \quad L^A(a) \leq 1 \}
\]
given in Definition 16.2. When this definition of \( \mathcal{L}^{1}_{A} \) is applied to the C*-metrics \( L^{D_{A}} \)
and \( L^{\ell} \) we will denote it by \( \mathcal{L}^{1}_{D_{A}} \) and \( \mathcal{L}^{1}_{\ell A} \) respectively, while when it is applied to the C*-metrics \( L^{D_{m}} \) and \( L^{\ell_{m}} \) we will denote it by \( \mathcal{L}^{1}_{D_{m}} \) and \( \mathcal{L}^{1}_{\ell_{m}} \) respectively. From the fact
that \( L^{D_{A}} = L^{D_{0}}_{0} \) and \( L^{D_{m}} = L^{D_{0}}_{m} \), it follows from Corollary 7.7 that
\[
L^{A}_{\ell}(a) \leq L^{D_{A}}(a) \quad \text{and} \quad L^{m}_{\ell}(b) \leq L^{D_{m}}(b) \quad (17.1)
\]
for all \( a \in A \) and \( b \in B^{m} \). From the inequalities (17.1) it follows immediately that
\[
\mathcal{L}^{1}_{D_{A}} \subseteq \mathcal{L}^{1}_{\ell A} \quad \text{and} \quad \mathcal{L}^{1}_{D_{m}} \subseteq \mathcal{L}^{1}_{\ell_{m}}. \quad (17.2)
\]
Recall from Eq. (16.4) that the general definition of \( \gamma^{B} \) is given by
\[
\gamma^{B} = \sup \{ \| \sigma^{A}(b) \omega - \omega b \|_{D} : b \in \mathcal{L}^{1}_{B} \}.
\]
When this definition is applied using \( \sigma^{m}_{A} \), \( \omega^{m} \), and \( \mathcal{L}^{1}_{D_{m}} \) or \( \mathcal{L}^{1}_{\ell_{m}} \), we will denote it by \( \gamma^{D_{m}} \) (as above) or \( \gamma^{B^{m}} \) respectively; while when this definition is applied using \( \sigma^{B^{m}} \), \( \omega^{m} \), and \( \mathcal{L}^{1}_{D_{A}} \) or \( \mathcal{L}^{1}_{\ell_{A}} \), we will denote it by \( \gamma^{D_{A}} \) (as above) or \( \gamma^{A} \) respectively. From the containments of display (17.2) it follows immediately that
\[
\gamma^{D_{m}} (\omega_{m}) \leq \gamma^{A} \quad \text{and} \quad \gamma^{D_{m}} (\omega_{m}) \leq \gamma^{m}. \quad (17.3)
\]
From this and inequality (16.5) following Definition 16.6, we immediately obtain:

**Proposition 17.3.** For all \( m \) we have \( \text{reach}_{D}(\Pi_{m}) \leq \max \{ \gamma^{A}_{\ell_{m}}, \gamma^{m}_{\ell} \} \).

But the paragraph just before proposition 6.3 of [56] (where the \( \gamma^{A} \) and \( \gamma^{B^{m}} \) there
are our \( \gamma^{A}_{\ell_{m}} \) and \( \gamma^{m}_{\ell} \)) explains how the results in sections 10 and 12 of [11] prove that as
\( m \) goes to \( \infty \) both sequences \( \{ \gamma^{A}_{\ell_{m}} \} \) and \( \{ \gamma^{m}_{\ell} \} \) converge to 0. We thus obtain:

**Proposition 17.4.** The sequence \( \{ \text{reach}_{D}(\Pi_{m}) \} \) converges to 0 as \( m \) goes to \( \infty \).

We now turn to considering the height of \( \Pi_{m} \). For any \( \mu \in S(A) \) set \( \phi_{\mu} = \mu \otimes \tau^{m} \)
where \( \tau^{m} \) is the tracial state on \( B^{m} \). Then \( \phi_{\mu} \in S(D^{m}) \), and simple computations show
that \( \phi_{\mu}(\omega^{m}) = 1 \) so that \( \phi_{\mu} \) restricted to \( A \) is \( \mu \). It is easy to see that \( \phi_{\mu} \) restricted to \( A \) is. In
this way we see that \( S^{A}(\omega^{m}) = S(A) \). Thus Haus\( \rho_{A}(S^{A}(\omega), S(A)) = 0 \), regardless of
what the metric \( \rho_{A} \) is. It follows that
\[
\text{height}_{D}(\Pi_{m}) = \text{Haus}_{D^{m}}(S^{B^{m}}(\omega), S(B^{m}))
\]
where \( \rho_{D^{m}} \) is the metric on \( S(B^{m}) \) determined by the Dirac operator \( D^{m} \).

Recall from Notation 16.7 that the general definition of \( \hat{\delta}^{B} \) is given by
\[
\hat{\delta}^{B} = \sup \{ \| b - \sigma^{B} (\sigma^{A}(b)) \| : b \in \mathcal{L}^{1}_{B} \}.
\]
When this definition is applied using \( \sigma^{B^{m}} \circ \sigma^{A}_{m} \), and \( \mathcal{L}^{1}_{D_{m}} \) or \( \mathcal{L}^{1}_{\ell_{m}} \), we will denote it by \( \hat{\delta}^{D_{m}} \) (as above) or \( \hat{\delta}^{m}_{\ell} \) respectively. From Eq. (16.8) we see that
\[
\text{Haus}_{D^{m}}(S^{B^{m}}(\omega), S(B^{m})) \leq \hat{\delta}^{D_{m}}.
\]
But from the containments of display (17.2) it follows immediately that
\[
\hat{\delta}^{D_{m}} \leq \hat{\delta}^{m}_{\ell}.
\]
From this and Proposition 16.8 we immediately obtain:
Proposition 17.5. For all \( m \) we have \( \text{height}_D(\Pi_m) \leq \hat{\delta}_m^\ell \).

But the paragraph just before proposition 6.7 of [56] (where the \( \hat{\delta}^B_m \) there is our \( \hat{\delta}^m \)) explains how theorem 11.5 of [11] gives a proof that as \( m \) goes to \( \infty \) the sequence \( \{\hat{\delta}_m^\ell\} \) converge to 0. We thus obtain:

**Proposition 17.6.** The sequence \( \{\text{height}_D(\Pi_m)\} \) converges to 0 as \( m \) goes to \( \infty \).

Combining this with Proposition 17.4, we obtain the main theorem of this paper:

**Theorem 17.7.** The sequence \( \{\text{length}_D(\Pi_m)\} \) converges to 0 as \( m \) goes to \( \infty \). Thus the sequence \( \{(B^m, L^{D^m})\} \) of compact C*-metric spaces converges to the compact C*-metric space \( ((A, L^D), \omega) \) for Latrésolíère’s propinquity.

18. The Linking Dirac Operator

In this section we offer some generalities that place bridges in a more Dirac-operator-like setting.

Let \( A \) and \( B \) be unital C*-algebras, and let \( \Pi = (D, \omega) \) be a bridge from \( A \) to \( B \) (so we view \( A \) and \( B \) as unital subalgebras of \( D \)). Let \( E \) be a unital C*-algebra that contains \( D \) as a unital C*-subalgebra. Then \( (E, \omega) \) is equally well a bridge from \( A \) to \( B \). The corresponding bridge-norm \( N_\Pi \) on \( A \oplus B \) defined by

\[
N_\Pi(a, b) = \|a\omega - \omega b\|
\]

is the same regardless of whether we use the bridge \( (D, \omega) \) or the bridge \( (E, \omega) \). And every state in \( S_1(\omega) \) for \( D \) has an extension (perhaps not unique) to a state on \( E \), which is then clearly in the \( S_1(\omega) \) for \( E \). From these observations, it is easily seen that for any given C*-metrics \( L^A \) and \( L^B \) on \( A \) and \( B \), the reach and the height of the bridges \( (D, \omega) \) and \( (E, \omega) \) are the same. Thus these two bridges have the same length.

Suppose now that \( (\mathcal{H}, \pi) \) is a faithful (non-degenerate) representation of \( D \). We can then view \( D \) as a unital subalgebra of \( B(\mathcal{H}) \). The above comments then apply, and \( (B(\mathcal{H}), \omega) \) is a bridge from \( A \) to \( B \) that has the same length as the bridge \( (D, \omega) \).

**Definition 18.1.** Let \( A \) and \( B \) be unital C*-algebras. By a *Hilbert bridge* from \( A \) to \( B \) we mean a quadruple \( (\mathcal{H}, \pi^A, \pi^B, \omega) \), where \( \mathcal{H} \) is a Hilbert space, \( \pi^A \) and \( \pi^B \) are faithful (non-degenerate) representations of \( A \) and \( B \) on \( \mathcal{H} \), and \( \omega \) is a self-adjoint operator on \( \mathcal{H} \) that has 1 in its spectrum. We will often view \( A \) and \( B \) as unital C*-subalgebras of \( B(\mathcal{H}) \), and omit \( \pi^A \) and \( \pi^B \) in our notation.

As mentioned after Definition 16.4, Latrésolíère defines his propinquity in terms of finite paths (which he calls “treks”) of bridges between C*-metric spaces (See [22].) He defines the length of a trek to be the sum of the lengths of the bridges in the trek, and he defines the propinquity from \( A \) to \( B \) to be the infimum of the lengths of all treks from \( A \) to \( B \). Actually, Latrésolíère does not require that the pivot \( \omega \) be self-adjoint. I am unaware of an example where this makes a difference, but since the propinquity is defined as an infimum, the propinquity requiring pivots to be self-adjoint will be greater than or equal to the propinquity not requiring that.

From the comments made just before Definition 18.1, it is clear that if one insists on defining the propinquity using only Hilbert bridges, it will never-the-less coincide with the usual propinquity (using self-adjoint pivots).
Suppose now that $\Pi = (\mathcal{H}, \omega)$ is a Hilbert bridge from $A$ to $B$. Let $\mathcal{H}_o = \mathcal{H} \oplus \mathcal{H}$. For $A$ and $B$ viewed as subalgebras of $\mathcal{B}(\mathcal{H})$, define a representation of $A \oplus B$ on $\mathcal{H}_o$ by
\[ (a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \]
and define a bounded operator, $D_\omega$, on $\mathcal{H}_o$ by
\[ D_\omega = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}. \]
Then
\[ [D_\omega, (a, b)] = \begin{pmatrix} 0 & \omega b - a \omega \\ \omega a - b \omega & 0 \end{pmatrix}, \]
so that
\[ \| [D_\omega, (a, b)] \| = \| a \omega - \omega b \| \lor \| a^* \omega - \omega b^* \|, \]
where $\lor$ means “maximum”. We have used here the fact that $\omega$ is self-adjoint. Notice that the bridge seminorm $N_\Pi$ (equation (18.1)) of $\Pi$ is in general not a $*$-seminorm. Much as in theorem 6.2 of [11], define a $*$-seminorm, $\hat{N}_\Pi$, on $A \oplus B$ by
\[ \hat{N}_\Pi(a, b) = N_\Pi(a, b) \lor N_\Pi(a^*, b^*). \]
Then we see that
\[ \|[D_\omega, (a, b)]\| = \hat{N}_\Pi(a, b). \]
Of course, $\hat{N}_\Pi$ agrees with $N_\Pi$ on self-adjoint elements.

Let $(\mathcal{H}^A, D^A)$ and $(\mathcal{H}^B, D^B)$ be Dirac operators for $A$ and $B$, so that there are nice dense subalgebras $A^\infty$ and $B^\infty$ whose elements have bounded commutators with the Dirac operators. Let $L^A$ and $L^B$ be the corresponding $C^*$-metrics. Then for any $r \in \mathbb{R}^+$ we construct a Dirac-like operator, $D_r$, for $A \oplus B$ on $\mathcal{H}^A \oplus \mathcal{H}_o \oplus \mathcal{H}^B$ given by
\[ D_r = D^A \oplus r^{-1} D_\omega \oplus D^B. \]
We let $A \oplus B$ act on $\mathcal{H}^A \oplus \mathcal{H}_o \oplus \mathcal{H}^B$ in the evident way. Then for $a \in A^\infty$ and $b \in B^\infty$ we have
\[ [D_r, (a, b)] = [D^A, a] \oplus r^{-1} [D_\omega, (a, b)] \oplus [D^B, b]. \]
Thus
\[ \|[D_r, (a, b)]\| = L^A(a) \lor r^{-1} \hat{N}_\Pi(a, b) \lor L^B(b). \]
It is not difficult to check that $D_r$ determines a $C^*$-metric, $L_r$, on $A \oplus B$, and that $L_r$ is compatible with $L^A$ and $L^B$ (in the sense that its quotients on $A^\infty$ and $B^\infty$ coincide with $L^A$ and $L^B$, as discussed around definition 4.7 of [20]) if and only if $r \geq r_\Pi$,
where \( r_\Pi \) is the reach of the bridge. (This is proposition 4.8 of [20].) Thus \( D_r \) can be used to bound the quantum Gromov–Hausdorff distance (as defined in [9]) between the \( C^* \)-metric spaces \((A, L^A)\) and \((B, L^B)\). In particular, \( r_\Pi \) can be defined as the smallest \( r \) such that \( D_r \) is compatible with \( L^A \) and \( L^B \). Thus we can determine the reach of the bridge \( \Pi \) by considering the \( D_r \)'s. But I do not see a way of determining the height of \( \Pi \) directly in terms of the \( D_r \)'s. For that purpose one seems to need to remember \( \omega \) and the subspace \( H_\omega \).

19. Deficiencies

We have given above a somewhat unified construction of Dirac operators for matrix algebras and homogeneous spaces, in such a way that when the homogeneous space is the coadjoint orbit of an integral weight vector, the corresponding sequence of matrix algebras converges to the coadjoint orbit for Latrémoliére’s propinquity. Heuristically, the matrix algebras, with metric shape given by their Dirac operators, converge for a suitable quantum Gromov–Hausdorff distance to the coadjoint orbit with metric shape given by its Dirac operator.

There are several deficiencies with this picture, which we discuss here. The first deficiency is that Dirac operators for Riemannian metrics are usually defined by means of the Levi–Civita connection, which is the unique torsion-free connection compatible with the Riemannian metric. But our construction does not always yield that Dirac operator. In essence, we have been using the dual of the “canonical connection” on the tangent bundle. As is well-known, and explained in section 6 of [23], for homogeneous spaces the Levi–Civita connection agrees with the canonical connection exactly for the symmetric spaces. Many coadjoint orbits are not symmetric spaces, and so for those our approach does not give the usual Dirac operator. But our approach does give the usual Dirac operator for those coadjoint orbits that are symmetric spaces, which include the 2-sphere and complex projective spaces. For non-commutative algebras such as our matrix algebras it is far from clear how one might define “torsion-free” for connections. (But see definition 8.8 of [36].)

However, we saw in Theorem 15.3 that for the Dirac operators we constructed we have

\[
\|[D, M_f]\| = \sup \{ \| df_x \|_{m^C} : x \in G \}
\]

for any \( f \in A^\infty = C^\infty(G/K) \). Whereas on the last page of [23] we saw that the same result was obtained when using the Levi–Civita connection (though phrased in terms of the tangent bundle instead of the cotangent bundle, so \( \| df_x \|_{m^C} = \| \text{grad}_f(x) \| \)). In [23] only the Hodge-Dirac operator was considered, that is, the spinor bundle was assumed to be \( \ell_C(m) \) itself. But by arguments elaborating on the proof of Proposition 6.2 one can see that \( \| [D, M_f] \| \) is independent of the choice of spinor bundle. The same formula for \( \|[D, M_f]\| \) is discussed in the comments following proposition 5.10 of [40] specifically for the case of homogeneous spaces that can have a \( G \)-invariant almost complex structure, which as we saw includes the case of coadjoint orbits. Now a standard argument (e.g., following definition 9.13 of [36]) shows that if we denote by \( \rho \) the ordinary metric on a Riemannian manifold \( N \) coming from its Riemannian metric, then for any two points \( p \) and \( q \) of \( N \) we have

\[
\rho(p, q) = \sup \{ |f(p) - f(q)| : \| \text{grad}_f \|_\infty \leq 1 \}.
\]
On applying this to either the Dirac operator using the canonical connection, or the Dirac operator using the Levi–Civita connection we obtain

$$\rho(p, q) = \sup \{|f(p) - f(q)| : \|[D, M_f]\| \leq 1\}$$

(19.1)

for $\rho$ now the ordinary metric on $G/K$ from our Riemannian metric. Thus as far as the metric aspects are concerned, these two Dirac operators are equivalent, and we can recover the ordinary metric from either one. Consequently, the matrix algebras equipped with their Dirac operators from our construction do still converge to the coadjoint orbit equipped with the Dirac operator for the Levi–Civita connection. Formula (19.1) is the formula on which Connes focused for general Riemannian manifolds [12,14,15], as it shows that the Dirac operator contains all of the metric information (and much more) for the manifold. This is his motivation for advocating that metric data for “non-commutative spaces” be encoded by providing them with a “Dirac operator”.

A second deficiency of our somewhat unified approach is that coadjoint orbits actually carry a $G$-invariant Kähler structure, which includes not just a complex structure $J$, but also closely related symplectic and Riemannian structures. In the paragraph of [78] that contains equations 3.53 through 3.59 a proof is given that the inner product on $m$ (or equivalently on $m'$) corresponding to the Riemannian metric of the Kähler structure can not be extended to a $G$-invariant inner product on $g$ unless the coadjoint orbit is a symmetric space. Since the inner products on $m'$ that we have been using in our approach have always been restrictions of $G$-invariant inner products on $g'$, the Dirac operators we have constructed can not be the ones for the Riemannian metric of the Kähler structure except if the coadjoint orbit is a symmetric space. (The results in [53] do not apply to the Kähler Riemannian metrics for the same reason, as seen from condition (b) following equation 1.3 of [53]. Dirac operators for all the Kähler Riemannian metrics are constructed in [40]). In particular, if the coadjoint orbit corresponding to a highest-weight vector is not a symmetric space, then the sequence of matrix algebras equipped with the Dirac operators that we have constructed will not converge to the coadjoint orbit when that orbit is equipped with the Riemannian metric of the Kähler structure. This follows from Latréomolière’s remarkable theorem [22] that if the propinquity between two C*-metric spaces is 0 then they are isometrically isomorphic.

Thus our general approach works quite well for coadjoint orbits that are symmetric spaces, and somewhat less well otherwise.

At present it is far from clear to me how one might modify our general approach in a way that would correct either of these two deficiencies.

20. Comparisons with Dirac Operators in the Literature

At the end of Sect. 10 we already discussed for the case of the sphere the relations between our construction of Dirac operators and various proposals in the literature. Here we briefly discuss for other coadjoint orbits the relations between our construction and several proposals in the literature.

In [70] Dirac operators are constructed on matrix algebras related to complex projective spaces, using a Schwinger-Fock construction. The methods are quite different from those of the present paper, and our discussion for the case of the sphere given at the end of Sect. 10 applies here also. (Since $\mathbb{C}P^n$ is not a spin-manifold when $n$ is even, one does not expect to have a charge-conjugation operator in those cases.) One of the authors of [70] says in [83] that the “formulation” “seems ill suited to couple the fermions with
gauge fields”. In the last section of [83] that author briefly introduces a Dirac operator on the relevant matrix algebras that seems to be more closely related to that in the present paper. But that Dirac operator is not explored there, though some clarification of the notation used there is given in [84] (though that paper does not mention matrix algebras).

In [85] the full flag-manifold for $G = SU(3)$ is considered, so the subgroup $K$ is the maximal torus of $G$. Then $G/K$ is not a symmetric space, but it is a spin manifold. For the corresponding matrix algebras, twisted Dirac operators for many projective modules are constructed. The dimension of $g'$ is 8, and so the irreducible representation of $C\ell(g')$ has dimension 16. That is the dimension of the spinor spaces used in constructing the Dirac operators, as it would be for our construction in the sections above. The Ginsparg-Wilson method is used, and already in the untwisted case the formula for the Dirac operator has extra terms (that could be interpreted as related to curvature) in comparison to the Dirac operators in our present paper. Since $G/K$ is not a symmetric space, it is very unlikely that the Dirac operator would relate well to the Kähler structure. There is no discussion of charge conjugation. It would be interesting to find a natural framework that would lead to Dirac operators of the kind given in this paper.

Constructing matrix-algebra approximations to spheres of dimension at least 3 is more complicated since they do not admit a symplectic structure. But one can use their close relationship to coadjoint orbits, as shown, for example, in [86].

21. Spectral Propinquity

As in [24], we say that a spectral triple $(A, \mathcal{H}, D)$ is metric if the seminorm $L^D$ on $A$ defined by $L^D(a) = \|(D, a)\|$ is a $C^*$-metric, the issue being whether the topology it determines on $S(A)$ coincides with the weak-$*$ topology. In definition 4.2 of [24] (and just before theorem 1.21 of [87]) Latrémolière defines a metric on isomorphism classes of spectral triples, which he calls the spectral propinquity. In this section we will examine the spectral propinquity for the spectral triples studied in the earlier sections of this paper. We will not give here a complete explanation of the general spectral propinquity. We will give some definitions and proofs, but we refer the reader to Latrémolière’s papers for more details.

Latrémolière’s spectral propinquity is based on his dual Gromov–Hausdorff propinquity [81], which is based on “tunnels” instead of “bridges”. We recall its definition here, but in a restricted simpler form that is sufficient for our needs. We first recall:

**Definition 21.1.** Let $(\mathcal{D}, L^\mathcal{D})$ and $(A, L^A)$ be compact $C^*$-metric spaces (with $L^D$ and $L^A$ defined on all of $\mathcal{D}$ and $A$ respectively). By a quantum isometry from $(\mathcal{D}, L^\mathcal{D})$ to $(A, L^A)$ we mean a unital $*$-homomorphism, $\pi$, from $\mathcal{D}$ onto $A$ such that $L^A = L^\mathcal{D} \circ \pi$, that is, that for all $a \in A$ we have

$$L^A(a) = \inf \{ L^\mathcal{D}(d) : \pi(d) = a \}.$$ 

If $\pi$ is actually a $*$-isomorphism such that $L^\mathcal{D} = L^A \circ \pi$, then we say that $\pi$ is a full isometry.

The reason for using the term “isometry” here is that if $\pi$ is a quantum isometry then the corresponding map $\pi^*$ from $S(A)$ into $S(\mathcal{D})$ is an isometry for the metrics $\rho^{L^A}$ and $\rho^{L^\mathcal{D}}$, as already seen in proposition 3.1 of [9]. If $\pi$ is a full isometry, then $\pi^*$ is an isometry onto $S(\mathcal{D})$ (and conversely).
Definition 21.2. Let \((A, L^A)\) and \((B, L^B)\) be compact C*-metric spaces. By a *tunnel* from \((A, L^A)\) to \((B, L^B)\) we mean a compact C*-metric space \((D, L^D)\) together with quantum isometries \(\pi^A\) and \(\pi^B\) from \(D\) onto \(A\) and \(B\) respectively.

This definition is quite reminiscent of the set-up for the definition of ordinary Gromov–Hausdorff distance. For our setting in which \(A = C(G/K)\) and \(B^m = B(H^m)\) the tunnel we use is \((D^m, L^{D^m})\) where

\[
D^m = A \oplus B^m
\]

with its evident projections onto \(A\) and \(B\), while \(L^{D^m}\) is defined by

\[
L^{D^m}(a, b) = L^A(a) \lor L^{B^m}(b) \lor r^{-1}N(a, b)
\]

where \(N\) is the bridge-norm defined in Eq. (18.1) using the pivot as defined in Eq. (16.1), and \(r\) is large enough to ensure that the evident projections onto \(A\) and \(B\) are quantum isometries (and \(\lor\) means “max”). It is easy to see that this means exactly that \(r \geq r\Pi\) where \(r\Pi\) is the reach of \(\Pi\) as defined in Definition 16.2. (We remark that the seminorm defined by Eq. (21.1) looks just like the seminorm in the statement of theorem 5.2 of [9] for the “bridges” as defined in that paper, and used in definition 6.1 and theorem 6.2 of [11]. So the terminology of “bridges” and “tunnels” has gotten a bit scrambled.)

Latrémiolère then defines for any tunnel its “extent”, which is a specific way of defining its length. Let \(T = (D, L^D, \pi^A, \pi^B)\) be a tunnel from \((A, L^A)\) to \((B, L^B)\) as in Definition 21.2. Let \(\pi_*^A\) denote composition of elements of \(S(A)\) with \(\pi^A\). Then, as indicated above, \(\pi_*^A\) is an isometry from \(S(A)\) into \(S(D)\). In the same way we define \(\pi_*^B\). The extent of \(T\), \(\text{ext}(T)\), is defined to be

\[
\text{ext}(T) = \max\{\text{dist}_H(\pi_*^A(S(A)), S(D)), \text{dist}_H(\pi_*^B(S(B)), S(D))\},
\]

where \(\text{dist}_H\) denotes Hausdorff distance with respect to the metric on \(S(D)\) determined by \(L^D\). Latrémiolère then defines the “dual propinquity” between \((A, L^A)\) and \((B, L^B)\) to be the infimum of the extents of all tunnels from \((A, L^A)\) to \((B, L^B)\) (with no need to take treks etc). Latrémiolère showed in [81] the remarkable fact that the dual propinquity is a metric on isometry classes of compact C*-metric spaces, and that this metric is complete on the space of isometry classes. He also shows that the propinquity, which we used in earlier parts of this paper, dominates the dual propinquity, so our earlier convergence results, such as Theorem 17.7, imply convergence for the dual propinquity too. But the propinquity is probably not complete, though there is no known counter-example as far as I know.

Latrémiolère defines the spectral propinquity between metric spectral triples of the form \((\mathcal{A}, \mathcal{H}, D)\) in two steps. In the first step, the Hilbert space \(\mathcal{H}\) is viewed as a left module over \(\mathcal{A}\) that is equipped with a suitable seminorm, using \(D\). Latrémiolère had earlier [21] defined a metric on isometry classes of such modules, which he called the *modular propinquity*. (It uses “modular tunnels”.) In the second step one considers a covariance condition with respect to the one-parameter group of unitary operators generated by \(D\). Here we will just examine the modular propinquity for the spectral triples studied in the earlier sections of this paper.

In [88] Latrémiolère defines the modular propinquity for correspondences that are equipped with a suitable seminorm. We recall [54] that, given C*-algebras \(\mathcal{A}\) and \(\mathcal{C}\), an \(\mathcal{A}\)-\(\mathcal{C}\)-correspondence is a right Hilbert-\(\mathcal{C}\)-module \(\mathcal{M}\) (i.e. a right \(\mathcal{C}\)-module \(\mathcal{N}\) with a
\(C\)-valued inner product) together with a \(*\)-homomorphism of \(A\) into the \(C^*\)-algebra of adjointable \(C\)-endomorphisms of \(\mathcal{M}\).

We will only need the special case in which \(C = \mathbb{C}\), for which \(\mathcal{M}\) is a Hilbert space, and the action of \(A\) is through a \(*\)-representation on \(\mathcal{M}\). In fact we will often use the letter \(\mathcal{H}\) for \(\mathcal{M}\). But to align with the terminology for general correspondences, we will refer to this special case as a “special correspondence”. (These should not be confused with the “metrized quantum vector bundles” of definition 1.10 of [87], which are right Hilbert-\(C\)-modules, but there is no algebra \(A\) acting on the left.) We will tend to use Latréomlière’s notation from [24].

Here is the “special” version of definition 2.2 of [24] and definition 1.10 of [87].

**Definition 21.3.** We say that \((A, L^A, \mathcal{M}, D)\) is a special metrical \(C^*\)-correspondence if \((A, L^A)\) is a compact \(C^*\)-metric space, while \(\mathcal{M}\) is a Hilbert space on which there is a \(*\)-representation of \(A\), and \(D\) is a norm defined on a dense subspace \(\text{dom}(D)\) of \(\mathcal{M}\), such that:

1. For all \(\xi \in \text{dom}(D)\) we have \(\|\xi\| \leq D(\xi)\).
2. The subset \(\{\xi \in \text{dom}(D) : D(\xi) \leq 1\}\) of \(\mathcal{M}\) is totally bounded.
3. For all \(a \in A\) and all \(\xi \in \text{dom}(D)\) we have

\[
D(a\xi) \leq (\|a\| + L^A(a))D(\xi).
\]

(In particular, if \(L^A(a) < \infty\), then \(a\xi \in \text{dom}(D)\).)

For our purposes, the main general example is given by theorem 2.7 of [24], which we now state, with our notation and terminology.

**Theorem 21.4.** Let \((A, \mathcal{H}, D)\) be a metrical spectral triple, with \(L^D(a) = \|[D, a]\|\) the corresponding \(C^*\)-metric on \(A\). Define the norm \(D\) on \(\text{dom}(D)\) to be the graph-norm of \(D\), that is,

\[
D(\xi) = \|\xi\| + \|D\xi\|.
\]

Then \((A, \mathcal{H}, L^D, D)\) is a special metrical \(C^*\)-correspondence.

Accordingly we let \(D^A\) and \(D^m\) be the norms on \(S^A\) and \(S^m\) defined in terms of \(D^A_o\) and \(D^m_o\) much as in Theorem 21.4. We want to show that the special metrical \(C^*\)-correspondences \((B^m, L^D_o, S^m, D^m)\) converge to the special metrical \(C^*\)-correspondence \((A, L^D_o, S^A, D^A)\) for the modular propinquity.

The definition of the modular propinquity involves a definition of what is meant by a tunnel between two metrical \(C^*\)-correspondences. (Latrémolière calls these “modular tunnels”). We will not give the general definition. Instead we now begin constructing the modular tunnels we need. We start with the core of the construction. To some extent we follow the steps that Latréomlière used to construct the corresponding modular tunnels for the case of non-commutative tori in section 3.3 of [87]. We use the notation of Sect. 2 as well as that following Definition 21.2.

Recall that \(S^A = A^\infty \otimes S\) and \(S^m = B^m \otimes S\), dense subspaces of the corresponding Hilbert spaces. We let \(\theta^A_m = \sigma^A_m \otimes I^S\) and \(\theta^B_m = \sigma^B_m \otimes I^S\), where \(\sigma^A_m\) and \(\sigma^B_m\) are as defined in the paragraph preceding Proposition 17.1. So \(\theta^A_m\) maps \(S^m\) to \(S^A\), while \(\theta^B_m\) maps \(S^A\) to \(S^m\). Since \(\sigma^A_m\) and \(\sigma^B_m\) intertwine the actions \(\alpha\) of \(G\) on \(A\) (and so on \(A^\infty\)) and \(B^m\), as mentioned just before Definition 16.6, \(\theta^A_m\) and \(\theta^B_m\) intertwine the actions
$\alpha \otimes I^S$ of $G$ on $A_\infty \otimes S$ and $B^m \otimes S$. From formula 2.1 it follows that $D^A_0\theta^A_m = \theta^A_m D^m_0$ and $D^m_0 \theta^B_m = \theta^B_m D^A_0$. It is easily seen that $\sigma^A_m$ has operator norm 1 (because we are essentially using normalized traces on $A$ and $B^m$). Since $\sigma^B_m$ is easily seen to be the Hilbert-space adjoint of $\sigma^A_m$ (as shown early in section 2 of [9]), $\theta^B_m$ is the Hilbert-space adjoint of $\theta^A_m$, and they both have operator norm 1 (i.e., are contractions of norm 1) for the Hilbert-space norms on $S^A$ and $S^m$.

The following proposition is the analog of much of theorem 3.25 of [87]. Notice that it does not explicitly involve the C*-algebras $A$ and $B^m$, just as theorem 3.25 does not involve the C*-algebras $A_\infty$ and $A_\infty$ of [87].

**Proposition 21.5.** With notation as above, let $S_\epsilon = S^A \oplus S^m$, with its evident pre-Hilbert-space structure. For any $\epsilon > 0$ define a norm $TN$ (for “tunnel norm”) on $S_\epsilon$ by

$$
TN(\xi, \eta) = D^A(\xi) \vee D^m(\eta) \vee (1/\epsilon)\|\xi - \theta^A_m(\eta)\| 
$$

for $(\xi, \eta) \in S_\epsilon$. For any $\epsilon > 0$ there is a natural number $N$ such that if $m \geq N$ then $D^A$ and $D^m$ coincide with the quotient norms of $TN$ on $S^A$ and $S^m$ for the evident projections from $S_\epsilon$ onto $S^A$ and $S^m$.

**Proof.** Let $\eta \in S^m$ be given. Choose $\xi \in S^A$ to be $\xi = \theta^A_m\eta$, so that the third term in the definition of $TN$ is 0. Thus to show that the quotient norm of $TN$ on $S^m$ is $D^m$ we only need to show that $D^A(\xi) \leq D^m(\eta)$. But this follows immediately from the facts that $\theta^A_m$ is of norm 1 and $D^A_0\theta^A_m = \theta^A_m D^m_0$ as seen above. This part of the proof is independent of any choice of $\epsilon$.

Let $\epsilon > 0$ be given, and let $\xi \in S^A$ be given. Choose $\eta \in S^m$ to be $\eta = \theta^B_m\xi$. Then $D^m(\eta) \leq D^A(\xi)$ because $\theta^B_m$ is of norm 1 and $D^B_m\theta^B_m = \theta^B_m D^A_0$ as seen above. Thus we need to show that there is an $N$ such that if $m \geq N$ then $(1/\epsilon)\|\xi - \theta^A_m(\eta)\| \leq D^A(\xi)$, that is,

$$
\|\xi - \theta^A_m(\theta^B_m\xi)\| \leq \epsilon(\|\xi\| + \|D^A_0\xi\|).
$$

(21.3)

Notice that since $\theta^A_m$ and $\theta^B_m$ have norm 1, the left-hand side of Eq. (21.3) is $\leq 2\|\xi\|$. Thus Eq. (21.3) will be satisfied if

$$
2\|\xi\| \leq \epsilon(\|\xi\| + \|D^A_0\xi\|).
$$

Suppose now that $\xi$ is an eigenvector for $D^A_0$ with eigenvalue $\lambda$, so that the right-hand side of equation (21.3) is $\epsilon(1 + |\lambda|)\|\xi\|$. It is then clear that Eq. (21.3) is satisfied if $|\lambda| \geq 2/\epsilon$.

In Proposition 2.4 we saw that there is a basis for $S^A$ consisting of eigenvectors of $D^A_0$. As mentioned there, the eigenvalues of $D^A_0$, counted with multiplicity, converge in absolute value to $\infty$. (A somewhat indirect proof of this fact is given in theorem 5.5 of [38], whose method is to compare $D^A_0$ to the usual Dirac operator on $G$ itself, which is handled by the usual methods for Dirac operators on compact manifolds, as presented for example in section 4.2 of [52].) Accordingly, the span of the eigenspaces of $D^A_0$ for eigenvalues $\lambda$ for which $|\lambda| \leq 2/\epsilon$, is finite-dimensional. We denote this span by $K$.

Then for $K^\perp$ we can choose an orthonormal basis, $\{\xi_j\}$, consisting of eigenvectors of $D^A_0$, and for each $j$ the eigenvalues $\lambda_j$ for $\xi_j$ will satisfy $|\lambda_j| > 2/\epsilon$. If $\xi \in K^\perp$ is in the domain of $D^A_0$ so that for its expansion $\xi = \sum z_j\xi_j$ (with $z_j \in \mathbb{C}$) the sequence $\{|z_j\lambda_j|\}$ is square-summable, we have

$$
\|D^A_0\xi\|^2 = \sum |z_j|^2|\lambda_j|^2 \geq (\sum |z_j|^2)(2/\epsilon)^2 = (2/\epsilon)^2\|\xi\|^2.
$$
Thus \( \|\xi\| \leq (\varepsilon/2)\|D_o^A\xi\| \), so that \( \|\xi - \theta_m^A(\theta^B m \xi)\| \leq \varepsilon(\|\xi\| + \|D_o^A\xi\|) \), as desired.

Finally, let \( \xi \) be an element of \( \mathcal{K} \). In proposition 4.13 of [17] Sain proves, in the more general context of compact quantum groups, that (for our notation above) \( \sigma^A_m(\sigma^B m (a)) \) converges in norm to \( a \) for every \( a \in \mathcal{A} \) as \( m \) goes to \( \infty \). (The proof is not difficult.) It follows that \( \sigma^A_m(\sigma^B m (a)) \) converges to \( a \) with respect to the Hilbert-space norm on \( L^2(\mathcal{A}, \tau) \), and from this it follows that \( \theta_m^A(\theta^B m \xi) \) converges to \( \xi \) for each \( \xi \in \mathcal{K} \) (since \( \mathcal{K} \) is contained in the algebraic tensor product \( \mathcal{A}^\infty \otimes \mathcal{S} \) as follows from Proposition 2.4).

Since \( \mathcal{K} \) is finite-dimensional, it follows that we can find a natural number \( N \) such that for every \( m \geq N \) and every \( \xi \in \mathcal{K} \) we have

\[
\|\xi - \theta_m^A(\theta^B m \xi)\| \leq \varepsilon \|\xi\| \leq \varepsilon(\|\xi\| + \|D_o^A\xi\|). \tag{21.4}
\]

Putting together the steps above, we find that if \( m \geq N \) then \( D^A \) coincides with the quotient norm of \( T\mathcal{N} \) on \( \mathcal{S}^A \), as needed.

We now put the above result into the framework that Latrémoliére uses in [87] to treat the spectral propinquity for Dirac operators on quantum tori. We follow closely the pattern around theorem 3.25 of [87].

To begin with, according to the proof of the triangle inequality for the dual modular propinquity given in theorem 3.1 of [88], and as done for non-commutative tori in theorem 3.25 of [87], we must view \( \mathcal{S}_t = \mathcal{S}^A \oplus \mathcal{S}^m \) as a Hilbert module over the 2-dimensional C*-algebra \( \mathcal{C} = \mathbb{C} \oplus \mathbb{C} \) (i.e. the algebra of functions on a 2-point space). This means that we view \( \mathcal{S}_t \) as a module over \( \mathcal{C} \) in the evident way, and that we define a \( \mathcal{C} \)-valued inner product on \( \mathcal{S}_t \) by

\[
(\langle \xi, \eta \rangle, \langle \xi', \eta' \rangle)_\mathcal{C} = (\langle \xi, \xi' \rangle)_{\mathcal{S}^A}, (\eta, \eta')_{\mathcal{S}^m}.
\]

For each \( \varepsilon > 0 \) as used in the above proposition, we define a C*-metric, \( Q_\varepsilon \), on \( \mathcal{C} \) by \( Q_\varepsilon(z, w) = (1/\varepsilon)|z - w| \) (so the distance between the two points is \( \varepsilon \)). This makes \( \mathcal{C} \) into a compact quantum metric space. Then we have the evident projections of \( \mathcal{C} \) onto the two copies of \( \mathbb{C} \) (one-point spaces with trivial metric), so that \( (\mathcal{C}, Q_\varepsilon) \) is a tunnel between these two one-point spaces.

Altogether the above structures, with the properties we have obtained, form a modular tunnel between the special metrical C*-correspondences \( (\mathcal{B}^m, \mathcal{L}^D o^m, \mathcal{S}^m, \mathcal{D}^m) \) and \( (\mathcal{A}, \mathcal{L}^D o^A, \mathcal{S}^A, \mathcal{D}^A) \) for large enough \( m \), except that there is one further property that must be verified, namely what Latrémoliére calls the “inner Leibniz property”. This requires that

\[
Q_\varepsilon((\xi, \eta), (\xi', \eta')) \leq 2T\mathcal{N}((\xi, \eta)D\mathcal{N}((\xi', \eta')).
\]

For this we may need to make the natural number \( N \) of Proposition 21.5 somewhat larger. Specifically, we replace \( \varepsilon \) with \( \varepsilon/2 \) and choose \( N \) large enough that in Eq. (21.2) we have \( 1/\varepsilon \) replaced by \( 2/\varepsilon \). In particular, we have \( \|\xi - \theta_m^A\eta\| \leq (\varepsilon/2)T\mathcal{N}((\xi, \eta)) \) for all \( (\xi, \eta) \in \mathcal{S}_t \), and similarly for \( (\xi', \eta') \).

Then

\[
\|\langle \xi, \xi' \rangle_{\mathcal{S}^A} - \langle \eta, \eta' \rangle_{\mathcal{S}^m}\| \leq \|\langle \xi, \xi' \rangle_{\mathcal{S}^A} - \langle \theta_m^A\eta, \theta_m^A\eta' \rangle_{\mathcal{S}^A}\| + \|\langle \theta_m^A\eta, \theta_m^A\eta' \rangle_{\mathcal{S}^A} - \langle \eta, \eta' \rangle_{\mathcal{S}^m}\|, \tag{21.5}
\]

For this we may need to make the natural number \( N \) of Proposition 21.5 somewhat larger. Specifically, we replace \( \varepsilon \) with \( \varepsilon/2 \) and choose \( N \) large enough that in Eq. (21.2) we have \( 1/\varepsilon \) replaced by \( 2/\varepsilon \). In particular, we have \( \|\xi - \theta_m^A\eta\| \leq (\varepsilon/2)T\mathcal{N}((\xi, \eta)) \) for all \( (\xi, \eta) \in \mathcal{S}_t \), and similarly for \( (\xi', \eta') \). Then
and, for the first term,
\[
|\langle \xi, \xi' \rangle_{SA} - \langle \theta_m^A \eta, \theta_m^A \eta' \rangle_{SA} | \\
\leq |\langle \xi - \theta_m^A \eta, \xi' \rangle_{SA} | + |\langle \theta_m^A \eta, \xi' - \theta_m^A \eta' \rangle_{SA} |
\]
\[
\leq \| \xi - \theta_m^A \eta \| \| \xi' \| + \| \theta_m^A \eta \| \| \xi' - \theta_m^A \eta' \|
\]
\[
\leq (\varepsilon/2) TN(\xi, \eta) D^A(\xi') + D^m(\eta)(\varepsilon/2) TN(\xi', \eta')
\]
\[
\leq \varepsilon TN(\xi, \eta) TN(\xi', \eta'),
\]
(21.6)
while for the second term, since \(\theta^B_m\) is the adjoint of \(\theta_m^A\),
\[
|\langle \theta_m^A \eta, \theta_m^A \eta' \rangle_{SA} - \langle \eta, \eta' \rangle_{SM} | = |\langle \theta^B_m \theta_m^A \eta, \eta' \rangle_{SA} - \langle \eta, \eta' \rangle_{SM} |
\]
\[
= |\langle \theta^B_m \theta_m^A \eta - \eta, \eta' \rangle_{SA} | \leq \| \theta^B_m \theta_m^A \eta - \eta \| \| \eta' \|
\]
\[
\leq \| \theta^B_m \theta_m^A \eta - \eta \| \| \eta' \| |TN(\xi', \eta')|.
\]
(21.7)
To handle the term \(\| \theta^B_m \theta_m^A \eta - \eta \|\) we argue much as we did to obtain Eq. (21.3). We seek to show that
\[
\| \theta^B_m \theta_m^A \eta - \eta \| \leq \varepsilon (\| \eta \| + \| D_m^\eta \|).
\]
(21.8)
(so that \(\| \theta^B_m \theta_m^A \eta - \eta \| \leq \varepsilon TN(\xi, \eta)\). The left-hand side of Eq. (21.8) is \(\leq 2\| \eta \|\). Thus Eq. (21.8) will be satisfied if
\[
2\| \eta \| \leq \varepsilon (\| \eta \| + \| D_m^\eta \|).
\]
Suppose now that \(\eta\) is an eigenvector for \(D_m^\eta\) with eigenvalue \(\lambda\), so that the right-hand side of equation (21.8) is \(\varepsilon (1 + |\lambda|) \| \eta \|\). It is then clear that Eq. (21.8) is satisfied if \(|\lambda| \geq 2/\varepsilon\). Let \(K_m\) be the span of the eigenspaces of \(D_m^\eta\) for eigenvalues \(\lambda\) for which \(|\lambda| \leq 2/\varepsilon\). Then we can argue exactly as in the fourth paragraph of the proof of Proposition 21.5 to conclude that if \(\eta \in K_m^\perp\) then \(\| \eta \| \leq (\varepsilon/2) \| D_m^\eta \|\), so that
\[
\| \eta - \theta^B_m (\theta_m^A(\eta)) \| \leq \varepsilon (\| \eta \| + \| D_m^\eta \|).
\]
To treat \(K_m\), we recall first that \(\sigma_m^A\) is always injective. (This follows immediately from theorem 3.1 of [9], for which it is crucial that the range of \(P^m\) is spanned by a highest-weight vector.) Thus \(\theta_m^A\) is injective. Furthermore, from Proposition 6.4 we see that \(\theta_m^A\) carries eigenvectors of \(D_m^\eta\) to eigenvectors of \(D_m^\eta\) with the same eigenvalue (but maybe of different norm), so that \(\theta_m^A\) carries \(K_m\) into \(K\). We may now argue as in section 4 of [17]. As long as \(\varepsilon < 1/2\), which we now assume (increasing \(N\) if necessary), Eq. (21.4) implies that \(\theta_m^A \theta^B_m\), as an operator on \(K\), satisfies \(\| I_K - \theta_m^A \theta^B_m \| < 1/2\), so that \(\theta_m^A \theta^B_m\) is invertible and \(\| (\theta_m^A \theta^B_m)^{-1} \| < 2\). In particular, \(\theta_m^A\) is onto \(K\), and so is invertible as an operator from \(K_m\) onto \(K\). Consequently, \(\theta^B_m\) is invertible as an operator from \(K\) to \(K_m\), and \((\theta^B_m)^{-1} = (\theta_m^A \theta^B_m)^{-1} \theta_m^A\), so that \(\| (\theta^B_m)^{-1} \| < 2\). Since
\[
I_{K_m} - \theta^B_m \theta_m^A = \theta^B_m (I_{K_m} - \theta_m^A \theta^B_m) (\theta^B_m)^{-1},
\]
we see, using Eq. (21.4) and our replacement of \(\varepsilon\) by \(\varepsilon/2\), that
\[
\| I_{K_m} - \theta^B_m \theta_m^A \| \leq 2 \| I_{K_m} - \theta_m^A \theta^B_m \| \leq \varepsilon
\]
for all \(m \geq N\). Thus
\[
\| \eta - \theta^B_m \theta_m^A \eta \| \leq \varepsilon \| \eta \| \leq \varepsilon D_m^\eta(\eta)
\]
for all \( \eta \in \mathcal{K}_m \). Putting this together with the result of the previous paragraph, we obtain inequality (21.8), so that \( \| \theta \mathcal{E}_m^A - \eta \| \leq \varepsilon \mathbf{T} \mathbf{N}(\xi, \eta) \). Putting this together with the several inequalities obtained before inequality (21.8), we obtain

\[
|\langle \xi, \xi' \rangle_{\mathcal{S}^A} - \langle \eta, \eta' \rangle_{\mathcal{S}^m} | \leq 2\varepsilon \mathbf{T} \mathbf{N}(\xi, \eta) \mathbf{T} \mathbf{N}(\xi', \eta'),
\]

so that

\[
\mathcal{Q}_\varepsilon((\xi, \eta), (\xi', \eta')) = (1/\varepsilon)|\langle \xi, \xi' \rangle_{\mathcal{S}^A} - \langle \eta, \eta' \rangle_{\mathcal{S}^m} | \leq 2\mathbf{T} \mathbf{N}(\xi, \eta)\mathbf{T} \mathbf{N}(\xi', \eta'),
\]

as needed.

When we apply the definition of the “extent” of a modular tunnel, as given in definition 4.2 of [88], to the modular tunnel sketched above, the extent of this modular tunnel is the extent of the tunnel \((\mathcal{C}, \mathcal{Q}_\varepsilon)\), which is \(\varepsilon\) (just as in the case for the non-commutative tori, for which see the last paragraph in the proof of theorem 3.25 of [87]).

For metrical C*-correspondences the modular propinquity between them is defined to be the infimum of the extents of all modular tunnels between them. For the special metrical C*-correspondences corresponding to the spectral triples \((\mathcal{A}, S^A, D^A_\circ)\) and \((\mathcal{B}^m, S^m, D^m_\circ)\), the tunnels constructed in Proposition 21.5 and the following discussion show that the sequence \(\{(\mathcal{B}^m, L^{D^m_\circ}, S^m, D^m)\}\) of special metrical C*-correspondences converges to the special metrical C*-correspondence \((\mathcal{A}, L^{D^A_\circ}, S^A, D^A)\) for the dual modular propinquity.

In theorem 4.8 of [88] Latrémolière proves the remarkable fact that if the dual modular propinquity between two metrical C*-correspondences is 0 (using a change of terminology he made in later papers), then these two metrical C*-correspondences are fully isometric, meaning unitary equivalent in an appropriate sense. The application of this to spectral triples is given in proposition 2.25 of [24]. For \(\mathcal{A} = C(G/K)\) as usual, and with \(\mathcal{D}\) being the Dirac operator defined in Definition 15.1, acting on \(S(G/K)\), it is easily seen that the special metrical C*-correspondence defined as in Theorem 21.4 for the spectral triple \((\mathcal{A}, S(G/K), \mathcal{D})\) is not unitarily equivalent to the special metrical C*-correspondence for the spectral triple \((\mathcal{A}, S^A, D^A_\circ)\). Consequently the sequence of spectral triples \(\{(\mathcal{B}^m, S^m, D^m_\circ)\}\) can not converge to the spectral triple \((\mathcal{A}, S(G/K), \mathcal{D})\) for the spectral propinquity. Recall from Definition 9.1 that when the unitary representation \((\mathcal{H}, U)\) is faithful on \(g\) the Dirac operators for the corresponding matrix algebras coincide with the \(D^m_\circ\)'s (and when the unitary representation is not faithful, the corresponding Dirac operators are close to that).

I expect that when the covariance condition in the definition of the spectral propinquity, as described in section 3 of [24], is taken into account, it will be seen that the spectral triples \(\{(\mathcal{B}^m, S^m, D^m_\circ)\}\) do converge to the spectral triple \((\mathcal{A}, S^A, D^A_\circ)\) for the spectral propinquity. But I have not checked that.

**Data Availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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