A free boundary problem for the $p$-Laplacian with nonlinear boundary conditions

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Abstract

We study a nonlinear generalization of a free boundary problem that arises in the context of thermal insulation. We consider two open sets $\Omega \subseteq A$, and we search for an optimal $A$ in order to minimize a non-linear energy functional, whose minimizers $u$ satisfy the following conditions: $\Delta_p u = 0$ inside $A \setminus \Omega$, $u = 1$ in $\Omega$, and a nonlinear Robin-like boundary $(p, q)$-condition on the free boundary $\partial A$. We study the variational formulation of the problem in $SBV$, and we prove that, under suitable conditions on the exponents $p$ and $q$, a minimizer exists and its jump set satisfies uniform density estimates.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary, and let $A$ be a set containing $\Omega$. Consider the functional

$$F(A, v) = \int_A |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1} + C_0 \mathcal{L}^n(A),$$

(1.1)

with $v \in H^1(A)$, $v = 1$ in $\Omega$ and $\beta, C_0 > 0$ fixed positive constants. The problem of minimizing this functional arises in the environment of thermal insulation: $F$ represents the energy of a heat configuration $v$ when the temperature is maintained constant inside the body $\Omega$ and there’s a bulk layer $A \setminus \Omega$ of insulating material whose cost is represented by $C_0$ and the heat transfer with the external environment is conveyed by convection. For simplicity’s sake in the following we will set $C_0 = 1$. The variational formulation in (1.1) leads to an Euler-Lagrange equation, which is the weak form of the following problem:

$$\begin{cases}
\Delta u = 0 & \text{in } A \setminus \Omega, \\
\frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial A, \\
u = 1 & \text{in } \Omega,
\end{cases}$$

(1.2)

The problems we are interested in concern the existence of a solution and its regularity. In this sense, one could be interested in studying a more general setting in which it is possible to consider possibly irregular sets $A$. Specifically, we could generalize the problem into the context of $SBV$ functions, aiming to minimize the functional

$$F(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{\partial \nu} (v^2 + \pi^2) \, d\mathcal{H}^{n-1} + \mathcal{L}^n(\{v > 0\} \setminus \Omega)$$

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with \( v \in \text{SBV}(\mathbb{R}^n) \) and \( v = 1 \) in \( \Omega \). This problem has been studied by L. A. Caffarelli and D. Kriventsov in [5], where the authors have proved the existence of a solution \( u \) for the problem and the regularity of its jump set. Another similar problem, in a non-linear context, has been deepened by D. Bucur and A. Giacomini in [4] with a boundedness constraint.

In this paper, our main aim is to generalize the problem and techniques employed in [5] to a nonlinear formulation. In detail, for \( p, q > 1 \) fixed, we consider the functional

\[
\mathcal{F}(v) = \int_{\mathbb{R}^n} |\nabla v|^p \, d\mathcal{L}^n + \beta \int_{\Omega} (v^q + \nabla v) \, d\mathcal{H}^{n-1} + \mathcal{L}^n(\{ v > 0 \} \setminus \Omega),
\]

and in the following we are going to study the problem

\[
\inf \left\{ \mathcal{F}(v) \mid v \in \text{SBV}(\mathbb{R}^n), \quad v(x) = 1 \text{ in } \Omega \right\}.
\]

Notice that if \( v \in \text{SBV}(\mathbb{R}^n) \) with \( v = 1 \) a.e. in \( \Omega \), letting \( v_0 = \max\{0, \min\{v, 1\}\} \) we have that \( v_0 \in \text{SBV}(\mathbb{R}^n) \) with \( v_0 = 1 \) a.e. in \( \Omega \) and \( \mathcal{F}(v_0) \leq \mathcal{F}(v) \) so it suffices to consider the problem

\[
\inf \left\{ \mathcal{F}(v) \mid v(x) \in [0, 1] \mathcal{L}^n\text{-a.e.}, \quad v(x) = 1 \text{ in } \Omega \right\}.
\]

In a more regular setting, problem (1.4) can be seen as a PDE. Let us fix \( \Omega, A \) sufficiently smooth open sets, \( u \in W^{1,p}(A) \) with \( u = 1 \) on \( \Omega \), and let us define the functional

\[
F(u, A) = \int_{\Omega} |\nabla u|^p \, d\mathcal{L}^n + \beta \int_{\partial \Omega} |u|^q \, d\mathcal{H}^{n-1} + \mathcal{L}^n(A \setminus \Omega).
\]

minimizers \( u \) to (1.5) solve the following boundary value problem

\[
\begin{align*}
\text{div} \left( |\nabla u|^{p-2} \nabla u \right) &= 0 \quad &\text{in } A \setminus \Omega, \\
\frac{\partial u}{\partial \nu} + \beta \frac{q}{p} |u|^{q-2} u &= 0 \quad &\text{on } \partial A, \\
u &= 1 \quad &\text{in } \Omega.
\end{align*}
\]

In Section 2 we give some preliminary tools and definitions, and then we will prove the existence of a minimizer \( u \) of (1.4), under a prescribed condition on \( p \) and \( q \). Finally, we will prove density estimates for the jump set \( J_u \).

We resume in the following theorems the main results of this paper.

**Theorem 1.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded open set, and let \( p, q > 1 \) be exponents satisfying one of the following conditions:

- \( 1 < p < n \), and \( 1 < q < \frac{p(n-1)}{n-p} := p_* \);
- \( n \leq p < \infty \), and \( 1 < q < \infty \).

Then there exists a solution \( u \) to problem (1.4) and there exists a constant \( \delta_0 = \delta_0(\Omega, \beta, p, q) > 0 \) such that

\[
u > \delta_0
\]

\( \mathcal{L}^n \)-almost everywhere in \( \{ u > 0 \} \), and there exists \( \rho(\delta_0) > 0 \) such that

\[
\text{supp } u \subseteq B_{\rho(\delta_0)}.
\]
Theorem 1.2. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $p,q > 1$ be exponents satisfying the assumptions of Theorem 1.1. Then there exist positive constants $C(\Omega, \beta, p, q)$, $c(\Omega, \beta, p, q)$, $C_1(\Omega, \beta, p, q)$ such that if $u$ is a minimizer to problem (1.4), then
\[ cr^{n-1} \leq \mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq Cr^{n-1}, \]
and
\[ \mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq C_1 r^n, \]
for every $x \in J_u$ with $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$.
In particular, this implies the essential closedness of the jump set $J_u$, namely
\[ \mathcal{H}^{n-1}(J_u \setminus J_u) = 0. \]

In Section 3 we prove that the a priori estimate (1.7) holds for inward minimizers (see Definition 3.1), such an estimate will be crucial in the proof of Theorem 1.1 in Section 4. Finally, in Section 5 we prove Theorem 1.2.

Remark 1.3. Notice that the condition on the exponents is undoubtedly verified when $p \geq q > 1$. Furthermore, if $\Omega$ is a set with Lipschitz boundary, the exponent $p_*$ is the optimal exponent such that
\[ W^{1,p}(\Omega) \subset \subset L^q(\partial \Omega) \quad \forall q \in [1, p_*). \]

2 Notation and Tools

In this section, we give the definition of the space $SBV$, and some useful notations and results that we will use in the following sections. We refer to [1], [6], [2] for a more intensive study of these topics.

Definition 2.1 (BV). Let $u \in L^1(\mathbb{R}^n)$. We say that $u$ is a function of bounded variation in $\mathbb{R}^n$ and we write $u \in BV(\mathbb{R}^n)$ if its distributional derivative is a Radon measure, namely
\[ \int_{\Omega} u \frac{\partial \varphi}{\partial x_i} = \int_{\Omega} \varphi \, dD_i u \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n), \]
with $Du$ a $\mathbb{R}^n$-valued measure in $\mathbb{R}^n$. We denote with $|Du|$ the total variation of the measure $Du$. The space $BV(\mathbb{R}^n)$ is a Banach space equipped with the norm
\[ \|u\|_{BV(\mathbb{R}^n)} = \|u\|_{L^1(\mathbb{R}^n)} + |Du|(\mathbb{R}^n). \]

Definition 2.2. Let $E \subseteq \mathbb{R}^n$ be a measurable set. We define the set of points of density 1 for $E$ as
\[ E^{(1)} = \left\{ x \in \mathbb{R}^n \left| \lim_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 1 \right. \right\}, \]
and the set of points of density 0 for $E$ as
\[ E^{(0)} = \left\{ x \in \mathbb{R}^n \left| \lim_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 0 \right. \right\}. \]
Moreover, we define the essential boundary of $E$ as
\[ \partial^* E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}). \]
**Definition 2.3** (Approximate upper and lower limits). Let \( u : \mathbb{R}^n \to \mathbb{R} \) be a measurable function. We define the **approximate upper and lower limits** of \( u \), respectively, as

\[
\overline{u}(x) = \inf \left\{ t \in \mathbb{R} \left| \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ u > t \})}{\mathcal{L}^n(B_r(x))} = 0 \right\},
\]

and

\[
\underline{u}(x) = \sup \left\{ t \in \mathbb{R} \left| \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ u < t \})}{\mathcal{L}^n(B_r(x))} = 0 \right\}.
\]

We define the **jump set** of \( u \) as

\[
J_u = \{ x \in \mathbb{R}^n \mid \overline{u}(x) < \underline{u}(x) \}.
\]

We denote by \( K_u \) the closure of \( J_u \).

If \( \overline{u}(x) = \underline{u}(x) = l \), we say that \( l \) is the approximate limit of \( u \) as \( y \) tends to \( x \), and we have that, for any \( \varepsilon > 0 \),

\[
\limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ |u - l| \geq \varepsilon \})}{\mathcal{L}^n(B_r(x))} = 0.
\]

If \( u \in \text{BV}(\mathbb{R}^n) \), the jump set \( J_u \) is a \((n - 1)\)-rectifiable set, i.e. \( J_u \subseteq \bigcup_{i \in \mathbb{N}} M_i \), up to a \( \mathcal{H}^{n-1} \)-negligible set, with \( M_i \) a \( C^1 \)-hypersurface in \( \mathbb{R}^n \) for every \( i \). We can then define \( \mathcal{H}^{n-1} \)-almost everywhere on \( J_u \) a normal \( \nu_u \) coinciding with the normal to the hypersurfaces \( M_i \). Furthermore, the direction of \( \nu_u(x) \) is chosen in such a way that the approximate upper and lower limits of \( u \) coincide with the approximate limit of \( u \) on the half-planes

\[
H^+_{\nu_u} = \{ y \in \mathbb{R}^n \mid \nu_u(x) \cdot (y - x) \geq 0 \}
\]

and

\[
H^-_{\nu_u} = \{ y \in \mathbb{R}^n \mid \nu_u(x) \cdot (y - x) \leq 0 \}
\]

respectively.

**Definition 2.4.** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set, and \( E \subseteq \mathbb{R}^n \) a measurable set. We define the **relative perimeter** of \( E \) inside \( \Omega \) as

\[
P(E; \Omega) = \sup \left\{ \int_E \nabla \varphi \cdot d\mathcal{L}^n \left| \varphi \in C^1_c(\Omega, \mathbb{R}^n), |\varphi| \leq 1 \right\}. \]

If \( P(E; \mathbb{R}^n) < +\infty \) we say that \( E \) is a **set of finite perimeter**.

**Theorem 2.5** (Decomposition of BV functions). Let \( u \in \text{BV}(\mathbb{R}^n) \). Then we have

\[
dDu = \nabla u \, d\mathcal{L}^n + |\nabla - w| \nu_u \, d\mathcal{H}^{n-1}_{\nu_u} \, d\mathcal{H}^n u + D^c u,
\]

where \( \nabla u \) is the density of \( Du \) with respect to the Lebesgue measure, \( \nu_u \) is the normal to the jump set \( J_u \) and \( D^c u \) is the Cantor part of the measure \( Du \). The measure \( D^c u \) is singular with respect to the Lebesgue measure and concentrated out of \( J_u \).

**Definition 2.6.** Let \( v \in \text{BV}(\mathbb{R}^n) \), let \( \Gamma \subseteq \mathbb{R}^n \) be a \( \mathcal{H}^{n-1} \)-rectifiable set, and let \( \nu(x) \) be the generalized normal to \( \Gamma \) defined for \( \mathcal{H}^{n-1} \text{-a.e.} \ x \in \Gamma \). For \( \mathcal{H}^{n-1} \text{-a.e.} \ x \in \Gamma \) we define the traces \( \gamma^\pm_\Gamma(v)(x) \) of \( v \) on \( \Gamma \) by the following Lebesgue-type limit quotient relation

\[
\lim_{r \to 0} \frac{1}{r^n} \int_{B^+_r(x)} |v(y) - \gamma^+_\Gamma(v)(x)| \, d\mathcal{L}^n(y) = 0,
\]

where

\[
B^+_r(x) = \{ y \in B_r(x) \mid \nu(x) \cdot (y - x) > 0 \},
\]

\[
B^-_r(x) = \{ y \in B_r(x) \mid \nu(x) \cdot (y - x) < 0 \}.
\]

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Remark 2.7. Notice that, by [1, Remark 3.79], for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, $(\gamma^+_v(x), \gamma^-_v(x))$ coincides with either $(\overline{v}(x), \underline{v}(x))$ or $(\underline{v}(x), \overline{v}(x))$, while, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus J_v$, we have that $\gamma^+_v(x) = \gamma^-_v(x)$ and they coincide with the approximate limit of $v$ in $x$. In particular, if $\Gamma = J_v$, we have
\[
\gamma^+_v(x) = \overline{v}(x) = \underline{v}(x) = \gamma^-_v(x)
\]
for $\mathcal{H}^{n-1}$-a.e. $x \in J_v$. 

We now focus our attention on the BV functions whose Cantor parts vanish.

Definition 2.8 (SBV). Let $u \in \text{BV}(\mathbb{R}^n)$. We say that $u$ is a special function of bounded variation and we write $u \in \text{SBV}(\mathbb{R}^n)$ if $D^c u = 0$.

For SBV functions we have the following.

Theorem 2.9 (Chain rule). Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then if $u \in \text{SBV}(\mathbb{R}^n)$, we have \[
\nabla g(u) = g'(u) \nabla u.
\]
Furthermore, if $g$ is increasing, \[
g\left(\underline{u}\right) = g(u), \quad g\left(\overline{u}\right) = g(u)
\]
while, if $g$ is decreasing, \[
g\left(\underline{u}\right) = g(u), \quad g\left(\overline{u}\right) = g(u).
\]

We now state a compactness theorem in SBV that will be useful in the following.

Theorem 2.10. [Compactness in SBV] Let $u_k$ be a sequence in $\text{SBV}(\mathbb{R}^n)$. Let $p,q > 1$, and let $C > 0$ such that for every $k \in \mathbb{N}$
\[
\int_{\mathbb{R}^n} |\nabla u_k|^p \, d\mathcal{L}^n + \|u_k\|_\infty + \mathcal{H}^{n-1}(J_{u_k}) < C.
\]
Then there exists $u \in \text{SBV}(\mathbb{R}^n)$ and a subsequence $u_{k_j}$ such that

- Compactness: \[
u_{k_j} \overset{L^1(\mathbb{R}^n)}{\rightharpoonup} u
\]
- Lower semicontinuity: for every open set $A$ we have \[
\int_A |\nabla u|^p \, d\mathcal{L}^n \leq \liminf_{j \to +\infty} \int_A |\nabla u_{k_j}|^p \, d\mathcal{L}^n
\]
and \[
\int_{J_{u} \cap A} \left(\overline{u}^q + \underline{u}^q\right) \, d\mathcal{H}^{n-1} \leq \liminf_{j \to +\infty} \int_{J_{u_{k_j}} \cap A} \left(\overline{u}_{k_j}^q + \underline{u}_{k_j}^q\right) \, d\mathcal{H}^{n-1}
\]
We refer to [1, Theorem 4.7, Theorem 4.8, Theorem 5.22] for the proof of this theorem. We now conclude this section with the following proposition whose proof can be found in [5, Lemma 3.1].

Proposition 2.11. Let $u \in \text{BV}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then \[
\int_0^1 P(\{u > s\}; \mathbb{R}^n \setminus J_u) \, ds = |Du|(\mathbb{R}^n \setminus J_u).
\]
3 Lower Bound

In the following, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and that $p$ and $q$ are two positive real numbers such that

$$\frac{q'}{p'} > 1 - \frac{1}{n}$$

(3.1)

where $p'$ and $q'$ are the Hölder conjugates of $p$ and $q$ respectively.

**Definition 3.1.** Let $v \in \text{SBV}(\mathbb{R}^n)$ be a function such that $v = 1$ a.e. in $\Omega$. We say that $v$ is an inward minimizer if

$$\mathcal{F}(v) \leq \mathcal{F}(v\chi_A),$$

for every set of finite perimeter $A$ containing $\Omega$, where $\chi_A$ is the characteristic function of set $A$.

Let $A \subset \mathbb{R}^n$ be a set of finite perimeter such that $\Omega \subset A$, and let $v \in \text{SBV}(\mathbb{R}^n)$. We will make use of the following expression

$$\mathcal{F}(v\chi_A) = \int_A |\nabla v|^p \, d\mathcal{L}^n + \beta \int_{J_u \cap A(t)} (v^q + \pi^q) \, d\mathcal{H}^{n-1} + \beta \int_{\partial^* A \setminus J_v} v^q \, d\mathcal{H}^{n-1}$$

(3.2)

$$+ \beta \int_{J_u \cap \partial^* A} \gamma_{\partial A}(v)^q \, d\mathcal{H}^{n-1} + \mathcal{L}^n (\{v > 0\} \cap A \setminus \Omega),$$

Let $B$ be a ball containing $\Omega$, then $\chi_B \in \text{SBV}(\mathbb{R}^n)$ and $\chi_B = 1$ in $\Omega$, we will denote $\mathcal{F}(\chi_B)$ by $\tilde{\mathcal{F}}$.

**Theorem 3.2.** There exists a positive constant $\delta = \delta(\Omega, \beta, p, q)$ such that if $u$ is an inward minimizer with $\mathcal{F}(u) \leq 2\tilde{\mathcal{F}}$, then

$$u > \delta$$

$\mathcal{L}^n$-almost everywhere in $\{u > 0\}$.

**Proof.** Let $0 < t < 1$ and

$$f(t) = \int_{\{u \leq t\} \setminus J_u} u^{q-1} |\nabla u| \, d\mathcal{L}^n = \int_0^t s^{q-1} P(\{u > s\}; \mathbb{R}^n \setminus J_u) \, ds.$$

For every such $t$, we have

$$f(t) \leq \left( \int_{\{u \leq t\}} u^{(q-1)p'} \, d\mathcal{L}^n \right)^{\frac{1}{p'}} \left( \int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \leq \mathcal{F}(u) \leq 2\tilde{\mathcal{F}}. \quad (3.3)$$

Let $u_t = u\chi_{(u>t)}$. Using (3.2) we have

$$0 \leq \mathcal{F}(u_t) - \mathcal{F}(u)$$

$$= \beta \int_{\partial^* \{u > t\} \setminus J_u} \overline{u}^q \, d\mathcal{H}^{n-1} - \int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n - \beta \int_{J_u \cap \partial^* \{u > t\}} \overline{u}^q \, d\mathcal{H}^{n-1} +$$

$$- \beta \int_{J_u \cap \{u > t\}^{(0)}} (\overline{\pi}^q + \overline{u}^q) \, d\mathcal{H}^{n-1} - \mathcal{L}^n (\{0 < u \leq t\}),$$

and rearranging the terms,

$$\int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n + \beta \int_{J_u \cap \partial^* \{u > t\}} \overline{u}^q \, d\mathcal{H}^{n-1} + \beta \int_{J_u \cap \{u > t\}^{(0)}} (\overline{\pi}^q + \overline{u}^q) \, d\mathcal{H}^{n-1} +$$

$$+ \mathcal{L}^n (\{0 < u \leq t\}) \leq \beta t^q P(\{u > t\}; \mathbb{R}^n \setminus J_u) = \beta t f'(t). \quad (3.4)$$
On the other hand,
\[
f(t) = \int_{\{u \leq t\} \setminus J_u} u^{q-1} |\nabla u| \, d\mathcal{L}^n
\leq \left( \int_{\{u \leq t\}} u^{(q-1)p'} \, d\mathcal{L}^n \right)^{\frac{1}{p'}} \left( \int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}}
\]
\[
\leq \left( \mathcal{L}^n(\{0 < u \leq t\}) \right)^{\frac{1}{p'}} \left( \int_{\{u \leq t\} \setminus J_u} u^{q1^*} \, d\mathcal{L}^n \right)^{\frac{1}{q1^*}} \left( \int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}}
\]
where we used
\[1^* = \frac{n}{n-1}, \quad \text{and} \quad \gamma = \frac{q1^*}{(q-1)p'},\]
and \( \gamma > 1 \) by (3.1). By classical BV embedding in \( L^{1^*} \) applied to the function \( (u\chi_{\{u \leq t\}})^q \) and the estimate (3.4), we have
\[
f(t) \leq C(n, \beta) \left( tf'(t) \right)^{1 - \frac{n}{n+1}} \left( \int_{\mathbb{R}^n} d|D(u\chi_{\{u \leq t\}})^q| \right)^{\frac{1}{p'}}.
\]
We can compute
\[
\int_{\mathbb{R}^n} d|D(u\chi_{\{u \leq t\}})^q| \leq q \left( \mathcal{L}^n(\{0 < u \leq t\}) \right)^{\frac{1}{p'}} \left( \int_{\{u \leq t\} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} +
\]
\[
+ \int_{J_u \cap \{u > t\}^0} (\pi^q + u^q) \, d\mathcal{H}^{n-1} + \int_{J_u \cap \partial^* \{u > t\}} u^q \, d\mathcal{H}^{n-1} +
\]
\[
+ t^q P(\{u > t\}; \mathbb{R}^n \setminus J_u) \leq (2 + q\beta) tf'(t).
\]
We therefore get
\[
f(t) \leq C(n, \beta, q) \left( tf'(t) \right)^{1 + \frac{1}{q1^*}}.
\]
Let \( 0 < t_0 < 1 \) such that \( f(t_0) > 0 \), then for every \( t_0 < t < 1 \), we have \( f(t) > 0 \) and
\[
\frac{f'(t)}{f(t)^{\frac{q-1}{n(q+1)-1}}} \geq \frac{C(n, \beta, q)}{t},
\]
integrating from \( t_0 \) to 1, we have
\[
f(1)^{\frac{q-1}{n(q+1)-1}} - f(t_0)^{\frac{q-1}{n(q+1)-1}} \geq C(n, \beta, q) \log \frac{1}{t_0},
\]
so that, using (3.3),
\[
f(t_0)^{\frac{q-1}{n(q+1)-1}} \leq (2\tilde{F})^{\frac{q-1}{n(q+1)-1}} + C(n, \beta, q) \log t_0.
\]
Let
\[
\delta = \exp \left( -\frac{(2\tilde{F})^{\frac{q-1}{n(q+1)-1}}}{C(n, \beta, q)} \right),
\]
for every \( t_0 < \delta \) we would have \( f(t_0) < 0 \), which is a contradiction. Therefore \( f(t) = 0 \) for every \( t < \delta \), from which \( u > \delta \mathcal{L}^n \)-almost everywhere on \( \{u > 0\} \).
\[\square\]
Remark 3.3. From Theorem 3.2, if \( u \) is an inward minimizer with \( \mathcal{F}(u) \leq 2\bar{F} \), we have that

\[ \partial^* \{ u > 0 \} \subseteq J_u \subseteq K_u. \]

Indeed, on \( \partial^* \{ u > 0 \} \) we have that, by definition, \( u = 0 \) and that, since \( u \geq \delta \mathcal{L}^n \)-almost everywhere in \( \{ u > 0 \} \), we have \( \mathcal{L}^n \geq \delta \).

Proposition 3.4. There exists a positive constant \( \delta_0 = \delta_0(\Omega, \beta, p, q) < \delta \) such that if \( u \) is an inward minimizer with \( \mathcal{F}(u) \leq 2\bar{F} \), then \( u \) is supported on \( B_{\rho(\delta_0)} \), where \( \rho(\delta_0) = \delta_0^{1-q} \) and \( B_{\rho(\delta_0)} \) is the ball centered at the origin with radius \( \rho(\delta_0) \). Moreover there exist positive constants \( C(\Omega, \beta, p, q), C_1(\Omega, \beta, p, q) \) such that, for any \( B_r(x) \subseteq \mathbb{R}^n \), we have

\[ \mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq C(\Omega, p, q)^{n-1}, \]

and if \( x \in K_u \), then

\[ \mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq C_1(\Omega, p, q)r^n. \]

Proof. By Theorem 3.2, if \( u \) is an inward minimizer, we have

\[ \int_{J_u \cap B_r(x)} (\pi^l + u^q) \, d\mathcal{H}^{n-1} \geq \delta^q \mathcal{H}^{n-1}(J_u \cap B_r(x)), \]

on the other hand, using \( u\chi_{\mathbb{R}^n \setminus B_r(x)} \) as a competitor for \( u \), we have

\[ \int_{J_u \cap B_r(x)} (\pi^l + u^q) \, d\mathcal{H}^{n-1} \leq \int_{\partial B_r(x) \cap \{ u > 0 \}^{(1)}} (\pi^l + u^q) \, d\mathcal{H}^{n-1} \leq C(n)r^{n-1}. \]

Let now \( x \in K_u \) and consider \( \mu(r) = \mathcal{L}^n \left( B_r(x) \cap \{ u > 0 \}^{(1)} \right) \). Using the isoperimetric inequality and inequality (3.5), we have that for almost every \( r \in (0, d(x, \Omega)) \)

\[ 0 < \mu(r) \leq K(n) P \left( B_r(x) \cap \{ u > 0 \}^{(1)} \right)^{\frac{n}{n-1}} \]

\[ \leq K(\Omega, \beta, p, q) P \left( B_r(x); \{ u > 0 \}^{(1)} \right)^{\frac{n}{n-1}}. \]

Notice that we used Remark 3.3 in the last inequality. We have

\[ \mu(r) \leq K \mu'(r)^{\frac{n}{n-1}}. \]

Integrating the differential inequality, we obtain

\[ \mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq C_1(\Omega, \beta, p, q)r^n. \]

Finally, let \( \delta_0 > 0 \) and \( x \in K_u \) such that \( d(x, \Omega) > \rho(\delta_0) = \delta_0^{1-q} \). By (3.6)

\[ C_1(\Omega, \beta, p, q)\rho(\delta_0)^n \leq \mathcal{L}^n(\{ u > 0 \} \cap \Omega) \leq 2\bar{F}, \]

which leads to a contradiction if \( \delta_0 \) is too small, hence there exists a positive value \( \delta_0(\Omega, \beta, p, q) \) such that \( \{ u > 0 \} \subset B_{\rho(\delta_0)} \). \( \square \)
4 Existence

In this section, we are going to prove the existence of a solution $u$ to the problem (1.4). Let us denote

$$H_a = \left\{ u \in \text{SBV}(\mathbb{R}^n) \mid \begin{array}{l} u(x) = 1 \text{ in } \Omega \\ u(x) \in \{ 0 \} \cup [a, 1] \text{ } \mathcal{L}^n\text{-a.e.} \\ \text{supp } u \subseteq B \frac{1}{a^{q-1}} \end{array} \right\}.$$ 

We also denote by $H_0$ the set

$$H_0 = \left\{ u \in \text{SBV}(\mathbb{R}^n) \mid \begin{array}{l} u(x) = 1 \text{ in } \Omega \\ u(x) \in [0, 1] \text{ } \mathcal{L}^n\text{-a.e.} \end{array} \right\}.$$ 

Notice that if $u \in H_0$ is an inward minimizer, by Theorem 3.2 and Proposition 3.4, then $u \in H_{\delta_0}$.

**Proposition 4.1.** Let $u \in H_0$. Then $u$ is a minimizer for the functional (1.3) on $H_0$ if and only if $u \in H_{\delta_0}$ and

$$\mathcal{F}(u) \leq \mathcal{F}(v) \quad \forall v \in H_{\delta_0}.$$

**Proof.** As we observed before, if $u$ is a minimizer over $H_0$ then $u$ is in $H_{\delta_0}$, hence it is a minimizer over $H_{\delta_0}$. Conversely, let us take $u \in H_{\delta_0}$ a minimizer over $H_{\delta_0}$, and let us consider in addition $v \in H_0$. Without loss of generality assume $\mathcal{F}(v) \leq 2\tilde{\mathcal{F}}$. We will prove that there exists a sequence $w_k$ of inward minimizers such that

$$\mathcal{F}(w_k) \leq \mathcal{F}(v) + \frac{C}{k^{q-1}}.$$ 

We first construct a family of functions $v_a \in H_a$ such that

$$\mathcal{F}(v_a) \leq \mathcal{F}(v) + r(a),$$

with $\lim_{a \to 0} r(a) = 0$. Let $0 < a < 1$, and let $v_a = v\chi_{\{ v \geq a \} \cap B_{\rho(a)}}$, where $\rho(a) = a^{1-q}$, we have

$$\mathcal{F}(v_a) - \mathcal{F}(v) \leq \beta \int_{\partial^r(\{ v \geq a \} \cap B_{\rho(a)}) \setminus J_v} v^q \, d\mathcal{H}^{n-1}$$

$$\leq \beta a^q \mathcal{P}(\{ v \geq a \}) + \beta \int_{\partial B_{\rho(a)} \cap \{ v \geq a \} \setminus J_v} v^q \, d\mathcal{H}^{n-1}$$

$$\leq \beta a^q \left( \mathcal{P}(\{ v \geq a \}) + \frac{1}{a^q} \int_{\partial B_{\rho(a)} \cap \{ v \geq a \} \setminus J_v} v \, d\mathcal{H}^{n-1} \right).$$

(4.1)

In order to estimate the right-hand side, fix $t \in (0, 1)$, and observe that by the coarea formula

$$\int_0^t \mathcal{P}(\{ v \geq a \}) \, da \leq |Dv|(\mathbb{R}^n),$$

(4.2)

while, with a change of variables,

$$\int_0^t \frac{1}{a^q} \int_{\partial B_{\rho(a)} \cap \{ v \geq a \} \setminus J_v} v \, d\mathcal{H}^{n-1} \, da \leq (q-1) \int_0^{+\infty} \int_{\partial B_r \setminus J_v} v \, d\mathcal{H}^{n-1} \, dr = (q-1) \| v \|_{L^1(\mathbb{R}^n)},$$

$$\int_0^t \left( \mathcal{P}(\{ v \geq a \}) + \frac{1}{a^q} \int_{\partial B_{\rho(a)} \cap \{ v \geq a \} \setminus J_v} v \, d\mathcal{H}^{n-1} \right) \, da \leq q \| v \|_{BV}.$$
By mean value theorem, for every $k \in \mathbb{N}$ we can find $a_k \leq 1/k$ such that

\[
P(\{ v \geq a_k \}) + \frac{1}{a_k^q} \int_{\partial B_{a_k}(v \geq a_k)} v \, dH^{n-1} \leq \frac{q\|v\|_{BV}}{a_k},
\]

and in (4.1) we get

\[
\mathcal{F}(v_{ak}) \leq \mathcal{F}(v) + q\beta a_k^{q-1}\|v\|_{BV} \leq \mathcal{F}(v) + \frac{q\|v\|_{BV}}{k^{q-1}}.
\]

We now construct the aforementioned sequence of inward minimizers: let us consider the functional

\[
\mathcal{G}_k(A) = \mathcal{F}(v_{ak} \chi_A),
\]

with $A$ containing $\Omega$ and contained in $\{ v_{ak} > 0 \}$. If we consider $A_j$ a minimizing sequence for $\mathcal{G}_k$, then they are certainly equibounded. Moreover,

\[
\mathcal{G}_k(A_j) \geq L^n(A_j \setminus \Omega) + \beta \int_{J_{A_j} v_{ak}} \left( \chi_{A_j} v_{ak}^q + \chi_{A_j} v_{ak}^q \right) \, dH^{n-1}
\]

\[
\geq L^n(A_j) + \beta a_k^q H^{n-1}(J_{A_j} v_{ak}) - L^n(\Omega).
\]

Notice in addition that since $v_{ak} \geq a_k$ on its support, then the jump set $J_{A_j} v_{ak}$ clearly contains $\partial^* A_j$. We now have that $\chi_{A_j}$ satisfies the conditions of Theorem 2.10, and eventually extracting a subsequence we can suppose that

\[
A_j \overset{L^1}{\to} A^{(k)},
\]

with a suitable $A^{(k)}$, and moreover, letting $w_k = \chi_{A^{(k)}} v_{ak}$, we have

\[
\mathcal{F}(w_k) \leq \inf_{\Omega \subseteq A \subseteq \{ v_{ak} > 0 \}} \mathcal{G}_k(A) \leq \mathcal{F}(v_{ak}) \leq \mathcal{F}(v) + q\beta \frac{\|v\|_{BV}}{k^{q-1}}.
\]

By construction $w_k$ is an inward minimizer, therefore we have $w_k \in H_{\delta_0}$, and consequently, we can compare it with $u$, obtaining

\[
\mathcal{F}(u) \leq \mathcal{F}(w_k) \leq \mathcal{F}(v) + q\beta \frac{\|v\|_{BV}}{k^{q-1}}.
\]

Letting $k$ go to infinity we get the thesis. \[\square\]

**Proposition 4.2.** There exists a minimizer for problem (1.4).

**Proof.** By Proposition 4.1 and Theorem 3.2 it is enough to find a minimizer in $H_{\delta_0}$. Let $u_k$ be a minimizing sequence in $H_{\delta_0}$, then, for $k$ large enough, we have

\[
\beta \delta_0^q H^{n-1}(J_{u_k}) + \int_{\mathbb{R}^n} |\nabla u_k|^p \, dL^n \leq \mathcal{F}(u_k) \leq 2 \bar{\mathcal{F}}.
\]

From Theorem 2.10 we have that there exists $u \in H_{\delta_0}$ such that, up to a subsequence, $u_k$ converges to $u$ in $L^1_{loc}$ and

\[
\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k),
\]

therefore $u$ is a solution. \[\square\]

**Proof of Theorem 1.1.** The result is obtained by joining Proposition 4.2 and Theorem 3.2. \[\square\]
5 Density estimates

In this section, we prove the density estimates in Theorem 1.2 by adapting techniques used in [5] analogous to classical ones used in [7] to prove density estimates for the jump set of almost-quasi minimizers of the Mumford-Shah functional.

**Definition 5.1.** Let $u \in \text{SBV}(\mathbb{R}^n)$ be a function such that $u = 1$ a.e. in $\Omega$. We say that $u$ is a **local minimizer** for $F$ on a set of finite perimeter $E \subset \mathbb{R}^n \setminus \Omega$, if

$$F(u) \leq F(v),$$

for every $v \in \text{SBV}(\mathbb{R}^n)$ such that $u - v$ has support in $E$.

Let $E$ be a set of finite perimeter. We introduce the notation

$$F(u; E) = \int_E |\nabla u|^p d\mathcal{L}^n + \beta \int_{J_u \cap E} (\pi^l + \underline{u}^q) \, d\mathcal{H}^{n-1} + L^n \{u > 0 \} \cap E.$$

To prove Theorem 1.2 we will use the following Poincaré-Wirtinger type inequality whose proof can be found in [7, Theorem 3.1 and Remark 3.3]. Let $\gamma_n$ be the isoperimetric constant relative to the balls of $\mathbb{R}^n$, i.e.

$$\min \left\{ \mathcal{L}^n(E \cap B_r) \frac{n-1}{n}, \mathcal{L}^n(E \setminus B_r) \frac{n-1}{n} \right\} \leq \gamma_n P(E; B_r),$$

for every Borel set $E$, then

**Proposition 5.2.** Let $p \geq 1$ and let $u \in \text{SBV}(B_r)$ such that

$$\left( 2\gamma_n \mathcal{H}^{n-1}(J_u \cap B_r) \right)^{\frac{n}{n-1}} \leq \frac{\mathcal{L}^n(B_r)}{2},$$

Then there exist numbers $-\infty < \tau^- \leq m \leq \tau^+ < +\infty$ such that the function

$$\tilde{u} = \max \{ \min \{ u, \tau^+ \}, \tau^- \},$$

satisfies

$$||\tilde{u} - m||_{L^p} \leq C ||\nabla u||_{L^p}$$

and

$$\mathcal{L}^n \{ u \neq \tilde{u} \} \leq C \left( \mathcal{H}^{n-1}(J_u \cap B_r) \right)^{\frac{n}{n-1}},$$

where the constants depend only on $n, p, r$.

**Lemma 5.3.** Let $u \in H_s$ be a local minimizer on $B_r(x)$ in the sense of definition Definition 5.1. For sufficiently small values of $\tau$, there exist values $r_0, \varepsilon_0$ depending only on $n, \tau, \beta, p, q$ and $s$ such that, if $r < r_0$,

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq \varepsilon_0 r^{n-1},$$

and

$$F(u; B_r(x)) \geq r^{n-\frac{1}{2}},$$

then

$$F(u; B_{\tau r}(x)) \leq \tau^{n-\frac{1}{2}} F(u; B_r(x)).$$
Proof. Without loss of generality, assume $x = 0$. Assume by contradiction that the conclusion fails, then for every $\tau > 0$ there exists a sequence $u_k \in H_s$ of local minimizers on $B_{r_k}$, with $\lim_k r_k = 0$, such that
\[
\frac{\mathcal{H}^{n-1}(J_{u_k} \cap B_{r_k})}{r_k^{n-1}} = \varepsilon_k,
\]
with $\lim_k \varepsilon_k = 0$,
\[
\mathcal{F}(u_k; B_{r_k}) \geq r_k^{n-\frac{1}{2}}, \tag{5.2}
\]
and yet
\[
\mathcal{F}(u_k; B_{\tau r_k}) > \tau^{n-\frac{1}{2}} \mathcal{F}(u_k; B_{r_k}). \tag{5.3}
\]
For every $t \in [0, 1]$, we define the sequence of monotone functions
\[
\alpha_k(t) = \frac{\mathcal{F}(u_k; B_{\tau r_k})}{\mathcal{F}(u_k, B_{r_k})} \leq 1.
\]
By compactness of $\text{BV}([0, 1])$ in $L^1([0, 1])$, we can assume that, up to a subsequence, $\alpha_k$ converges $L^1$-almost everywhere to a monotone function $\alpha$. Moreover, notice that, by (5.3), for every $k$
\[
\alpha_k(\tau) > \tau^{n-\frac{1}{2}}. \tag{5.4}
\]
Our final aim is to prove that there exists a $p$-harmonic function $v \in W^{1,p}(B_1)$ such that for every $t$
\[
\lim_{k \to +\infty} \alpha_k(t) = \alpha(t) = \int_{B_1} |\nabla v|^p \, d\mathcal{L}^n.
\]
Let
\[
E_k = r_k^{p-n} \mathcal{F}(u_k; B_{r_k}), \quad v_k(x) = \frac{u_k(r_k x)}{E_k^{1/p}}.
\]
Then $v_k \in \text{SBV}(B_1)$, and
\[
\int_{B_1} |\nabla v_k|^p \, d\mathcal{L}^n \leq 1, \quad \mathcal{H}^{n-1}(J_{v_k} \cap B_1) = \varepsilon_k.
\]
Thus, applying the Poincaré-Wirtinger type inequality in Proposition 5.2 to functions $v_k$ we obtain truncated functions $\tilde{v}_k$ and values $m_k$, such that
\[
\int_{B_1} |\tilde{v}_k - m_k|^p \, d\mathcal{L}^n \leq C
\]
and
\[
\mathcal{L}^n(\{ v_k \neq \tilde{v}_k \}) \leq C \left( \mathcal{H}^{n-1}(J_{v_k} \cap B_1) \right)^{\frac{n}{n-1}} \leq C \varepsilon_k^{\frac{n}{n-1}}. \tag{5.5}
\]

**Step 1:** We prove that there exists $v \in W^{1,p}(B_1)$ such that
\[
\tilde{v}_k - m_k \overset{L^p(B_1)}{\longrightarrow} v,
\]
\[
\int_{B_1} |\nabla v|^p \, d\mathcal{L}^n \leq \alpha(\rho), \quad \text{for } \mathcal{L}^1\text{-a.e. } \rho < 1, \tag{5.6}
\]
and
\[
\lim_k \frac{r_k^{p-1}}{E_k} \mathcal{H}^{n-1}(\{ v_k \neq \tilde{v}_k \} \cap \partial B_{\rho}) = 0, \quad \text{for } \mathcal{L}^1\text{-a.e. } \rho < 1. \tag{5.7}
\]
Notice that
\[
\int_{B_1} |\nabla(\tilde{v}_k - m_k)|^p \, d\mathcal{L}^n \leq \int_{B_1} |\nabla v_k|^p \, d\mathcal{L}^n \leq 1,
\]
since \(\tilde{v}_k\) is a truncation of \(v\). From compactness theorems in SBV (see for instance [7, Theorem 3.5]), we have that \(\tilde{v}_k - m_k\) converges in \(L^p(B_1)\) and \(\mathcal{L}^n\)-almost everywhere to a function \(v \in W^{1,p}(B_1)\), since \(\mathcal{H}^{n-1}(J_{\tilde{v}_k})\) goes to 0 as \(k \to +\infty\). Moreover, for every \(\rho < 1\),
\[
\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \liminf_k \int_{B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n,
\]
and
\[
\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \liminf_k \int_{B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n \leq \liminf_k \alpha_k(\rho) = \alpha(\rho),
\]
since by definition
\[
\int_{B_\rho} |\nabla v_k|^p \, d\mathcal{L}^n = \frac{r_k^{p-n}}{E_k} \int_{B_{\rho r_k}} |\nabla u_k|^p \, d\mathcal{L}^n \leq \frac{r_k^{p-n}}{E_k} F(u_k; B_{\rho r_k}) \leq \alpha_k(\rho).
\]
Finally, up to a subsequence,
\[
\lim_k \frac{r_k^{p-1}}{E_k} \mathcal{L}^n(\{ v_k \neq \tilde{v}_k \}) = 0.
\]
Indeed, by (5.5),
\[
\frac{r_k^{p-1}}{E_k} \mathcal{L}^n(\{ v_k \neq \tilde{v}_k \}) \leq C \frac{r_k^{p-1}}{E_k} \varepsilon_k^{\frac{n}{n-1}},
\]
which tends to zero as long as \(r_k^{p-1}/E_k\) is bounded. On the other hand, if \(r_k^{p-1}/E_k\) diverges, we could use the fact that \(\varepsilon_k \leq s^{-q} F(u_k; B_{\rho r_k}) r_k^{1-n}\) and get
\[
\frac{r_k^{p-1}}{E_k} \mathcal{L}^n(\{ v_k \neq \tilde{v}_k \}) \leq C \left( \frac{E_k}{r_k^{p-1}} \right)^{\frac{n}{n-1}}
\]
which goes to zero. Using Fubini’s theorem we have (5.7).

Let \(\tilde{u}_k(x) = E_k^{1/p} \tilde{v}_k(\frac{x}{r_k})\), and for every \(t \in [0,1]\) we define
\[
\tilde{\alpha}_k(t) = \frac{F(\tilde{u}_k; B_{\rho r_k})}{F(u_k, B_{\rho r_k})}.
\]
The \(\tilde{\alpha}_k\) functions are also monotone and bounded: the jump set of \(\tilde{u}_k\) is contained in \(J_{u_k}\), therefore we can write
\[
\tilde{\alpha}_k(t) \leq \alpha_k(t) + \frac{2\beta \mathcal{H}^{n-1}(J_{u_k} \cap B_{\rho r_k})}{F(u_k; B_{\rho r_k})} \leq \left( 1 + \frac{2}{s^q} \right) \alpha_k(t),
\]
using the fact that \(u_k \in H_s\). As done for \(\alpha_k\), we can assume that \(\tilde{\alpha}_k\) converges \(\mathcal{L}^1\)-almost everywhere to a function \(\tilde{\alpha}\).

**Step 2:** Let \(I \subset [0,1]\) be the set of values \(\rho\) for which (5.7) holds, \(\alpha_k\) and \(\tilde{\alpha}_k\) converge and \(\alpha\) and \(\tilde{\alpha}\) are continuous. Notice that \(\mathcal{L}^1(I) = 1\). Fix \(\rho, \rho' \in I\) with \(\rho < \rho' < 1\) and let
\[
\mathcal{I}_k(\xi) = \beta E_k^{q/p-1} r_k^{p-1} \int_{J_\rho \cap (B_{\rho'} \setminus B_{\rho})} (\xi^q + \xi^{q'}) d\mathcal{H}^{n-1},
\]
with \( \xi \in \text{SBV}(B_1) \). Let \( w \in W^{1,p}(B_1) \) and consider \( \eta \) a smooth cutoff function supported on \( B_{r'} \) and identically equal to 1 in \( B_\rho \). Let
\[
\varphi_k = ((w + m_k)\eta + \tilde{v}_k(1 - \eta))\chi_{B_{r'}} + v_k\chi_{B_1 \setminus B_{r'}}.
\]

We want to prove that
\[
\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) \geq \int_{B_{r'} \setminus B_\rho} |\nabla \tilde{v}_k|^p d\mathcal{L}^n + I_k(\tilde{v}_k), \tag{5.8}
\]
and
\[
\alpha_k(\rho') \leq R_k + \int_{B_{r'}} |\nabla \varphi_k|^p d\mathcal{L}^n + I_k(\varphi_k), \tag{5.9}
\]
where \( R_k \) goes to zero as \( k \) goes to infinity. We immediately compute
\[
\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) = \mathcal{F}(u_k; B_{r_k})^{-1} \left[ \int_{B_{r'} \setminus B_{r_k}} |\nabla \tilde{u}_k|^p d\mathcal{L}^n + \beta \int_{J_{\tilde{u}_k}(B_{r'} \setminus B_{r_k})} (\tilde{u}_k^q + \tilde{v}_k^q) d\mathcal{H}^{n-1} \right]
+ \mathcal{F}(u_k; B_{r_k})^{-1} \mathcal{L}^n(\{ \tilde{u}_k > 0 \} \cap (B_{r'} \setminus B_{r_k}))
\geq \int_{B_{r'} \setminus B_\rho} |\nabla \tilde{v}_k|^p d\mathcal{L}^n + r_k^{q/p-1} \beta \int_{J_{\tilde{u}_k}(B_{r'} \setminus B_\rho)} (\tilde{u}_k^q + \tilde{v}_k^q) d\mathcal{H}^{n-1},
\]
and then we have (5.8). Now let \( \psi_k = E_{r_k}^{1/p} \varphi_k(x/r_k) \) and observe that \( \psi_k \) coincides with \( u_k \) outside \( B_{r'/r_k} \). We get from the local minimality of \( u_k \) that
\[
\mathcal{F}(u_k; B_{r_k}) \leq \mathcal{F}(\psi_k; B_{r_k}) = \mathcal{F}(\psi_k; B_{r'/r_k}) + \beta \int_{\{ u_k \neq \tilde{u}_k \} \cap \partial B_{r'/r_k}} (\psi_k^q + \tilde{u}_k^q) d\mathcal{H}^{n-1}
+ \mathcal{F}(u_k; B_{r_k} \setminus B_{r'/r_k}) \tag{5.10}
\]
So, in particular, we have
\[
\mathcal{F}(u_k; B_{r'/r_k}) = \mathcal{F}(u_k; B_{r_k}) - \mathcal{F}(u_k; B_{r_k} \setminus B_{r'/r_k}) - \beta \int_{J_u_k \cap \partial B_{r'/r_k}} (\tilde{u}_k^q + \tilde{v}_k^q) d\mathcal{H}^{n-1}
\leq 2\beta r_k^{-n+1} \mathcal{H}^{n-1}(\{ v_k \neq \tilde{v}_k \} \cap \partial B_{r'}) + \mathcal{F}(\psi_k; B_{r'/r_k}).
\]
Dividing by \( \mathcal{F}(u_k; B_{r_k}) \) and using (5.7) we get
\[
\alpha_k(\rho') \leq R_k + r_k^{q-n} E_{k}^{-1} \mathcal{F}(\psi_k; B_{r'/r_k}).
\]
With appropriate rescalings we have
\[
r_k^{q-n} E_{k}^{-1} \mathcal{F}(\psi_k; B_{r'/r_k}) = \int_{B_{r'}} |\nabla \varphi_k|^p d\mathcal{L}^n + I_k(\varphi_k) + r_k^n E_{k}^{-1} \mathcal{L}^n(\{ \varphi_k > 0 \} \cap B_{r'}).
\]
From (5.2) and the definition of \( E_k \), we have
\[
r_k^n E_{k}^{-1} \mathcal{L}^n(\{ \varphi_k > 0 \} \cap B_{r'}) \leq \omega_n r_k^{1/2},
\]
with \( \omega_n \) a constant depending on \( n \).
and then we get (5.9).

**Step 3:** We want to prove that for every $\phi \in W^{1,p}(B_1)$ such that $v - \phi$ is supported on $B_\rho$, we have

$$\alpha(\rho') \leq \int_{B_\rho} |\nabla \phi|^p \, d\mathcal{L}^n + C \left[ \tilde{\alpha}(\rho') - \tilde{\alpha}(\rho) \right] + C \int_{B_{\rho'} \setminus B_\rho} |\nabla \phi|^p \, d\mathcal{L}^n, \quad (5.11)$$

where $C$ does not depend on either $\rho$ or $\rho'$. From the definition of $\phi_k$, we have that on $B_\rho$

$$\nabla \phi_k = \nabla w$$

and on $B_{\rho'} \setminus B_\rho$

$$\nabla \phi_k = \eta \nabla w + (w + m_k - \tilde{v}_k) \nabla \eta + \nabla \tilde{v}_k(1 - \eta),$$

so that

$$\int_{B_{\rho'}} |\nabla \phi_k|^p \, d\mathcal{L}^n \leq \int_{B_\rho} |\nabla w|^p \, d\mathcal{L}^n$$

$$+ C \left[ \int_{B_{\rho'} \setminus B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n + \int_{B_{\rho'} \setminus B_\rho} (|\nabla w|^p + |w + m_k - \tilde{v}_k|^p |\nabla \eta|^p) \, d\mathcal{L}^n \right]. \quad (5.12)$$

We split the proof into two cases: either

$$\limsup_k E_k > 0 \quad (5.13)$$

or

$$\lim_k E_k = 0. \quad (5.14)$$

Assume (5.13) occurs. Notice that $s \leq u_k \leq 1$ for every $k$, then by definition we have that, for every $k$, $s \leq E_k^{1/p} \tilde{v}_k \leq 1$ and, since $m_k$ is a median of $v_k$, $0 \leq E_k^{1/p} m_k \leq 1$. In particular we have that

$$|\tilde{v}_k - m_k| \leq \frac{2}{E_k^{1/p}}.$$

passing to the limit when $k$ goes to infinity we have that

$$\|v\|_\infty \leq \liminf_k \frac{2}{E_k^{1/p}} < +\infty \quad \mathcal{L}^n\text{-a.e.}$$

then $v$ belongs to $L^\infty(B_1)$ and there exists a positive constant $C$ independent of $k$, and a natural number $\bar{k}$ such that

$$|v + m_k - \tilde{v}_k| \leq \frac{C}{E_k^{1/p}} \leq \frac{C}{s} \tilde{v}_k \quad \mathcal{L}^n\text{-a.e.}$$

for all $k > \bar{k}$. Let $\varphi \in W^{1,p}(B_1)$ with $v - \varphi$ supported on $B_\rho$, and let $w = \varphi$ in the definition of $\phi_k$, then, for every $k > \bar{k}$, we have

$$|\varphi_k| = |\tilde{v}_k + (v + m_k - \tilde{v}_k)\eta| \leq C\tilde{v}_k \quad (5.15)$$

$\mathcal{L}^n\text{-a.e.}$ on $B_\rho' \setminus B_\rho$. From (5.15) we have that

$$\mathcal{I}_k(\varphi_k) \leq C\mathcal{I}_k(\tilde{v}_k). \quad (5.16)$$
Notice, in addition, that (5.12) reads as
\[
\int_{B_{\rho'}} |\nabla \varphi_k|^p \, d\mathcal{L}^n \leq \int_{B_{\rho}} |\nabla \varphi|^p \, d\mathcal{L}^n + C \int_{B_{\rho}' \setminus B_{\rho}} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n + C \int_{B_{\rho}' \setminus B_{\rho}} |\nabla \varphi|^p \, d\mathcal{L}^n + R_k.
\]
(5.17)

Finally joining (5.9), (5.17), (5.16), and (5.8), we have
\[
\alpha_k(\rho') \leq \int_{B_{\rho}} |\nabla \varphi|^p \, d\mathcal{L}^n + C \left[ \tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) \right] + C \int_{B_{\rho}' \setminus B_{\rho}} |\nabla \varphi|^p \, d\mathcal{L}^n + R_k.
\]

Letting \( k \) go to infinity we get (5.11).

Suppose now that (5.14) occurs. The functions \(|\tilde{v}_k - m_k|^p, |v|^p\) are uniformly integrable, namely for every \( \varepsilon > 0 \) there exists a \( \sigma = \sigma_\varepsilon < \varepsilon \) such that if \( A \) is a measurable set with \(|A| < \sigma\), then
\[
\int_A |\tilde{v}_k - m_k|^p \, d\mathcal{L}^n + \int_A |v|^p \, d\mathcal{L}^n < \varepsilon.
\]
(5.18)

Since \( v \in L^p(B_1) \), we can find \( M > 1/\varepsilon \) such that
\[
\{|v| > M\} < \sigma.
\]
(5.19)

Setting \( w = \varphi_M = \max\{-M, \min\{\varphi, M\}\} \), then (5.12) reads as
\[
\int_{B_{\rho'}} |\nabla \varphi_k|^p \, d\mathcal{L}^n \leq \int_{B_{\rho} \cap \{|\varphi| \leq M\}} |\nabla \varphi|^p \, d\mathcal{L}^n + C \int_{(B_{\rho}' \setminus B_{\rho}) \cap \{|\varphi| \leq M\}} |\nabla \varphi|^p \, d\mathcal{L}^n + \int_{B_{\rho}' \setminus B_{\rho}} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n + \int_{B_{\rho}' \setminus B_{\rho}} |\varphi_M + m_k - \tilde{v}_k|^p |\nabla \eta|^p \, d\mathcal{L}^n.
\]
(5.20)

We can estimate the last integral as follows
\[
\int_{B_{\rho}' \setminus B_{\rho}} |\varphi_M + m_k - \tilde{v}_k|^p |\nabla \eta|^p \, d\mathcal{L}^n \leq C\varepsilon + \int_{(B_{\rho}' \setminus B_{\rho}) \cap \{|\varphi| \leq M\}} |v + m_k - \tilde{v}_k|^p |\nabla \eta|^p \, d\mathcal{L}^n.
\]
\[
= C\varepsilon + R_k,
\]
(5.21)

where we used (5.19) and (5.18), and \( C \) only depends on \( \rho \) and \( \rho' \). Furthermore, we have
\[
I_k(\varphi_k) \leq R_k + C I_k(\tilde{v}_k).
\]
(5.22)

Indeed, as before, \(|\tilde{v}_k - m_k| \leq C\tilde{v}_k\), while
\[
E_k^{q/p-1} r_k^{p-1} \int_{J_{\tilde{v}_k} \cap (B_{\rho}' \setminus B_{\rho})} |\varphi_M|^q \, d\mathcal{H}^{n-1} \leq M^q E_k^{q/p-1} r_k^{p-1} |\mathcal{H}^{n-1}(J_{\tilde{v}_k} \cap (B_{\rho}' \setminus B_{\rho}))| \leq M^q E_k^{q/p-1} \varepsilon_k E_k \leq M^q E_k^{\frac{q}{s} \frac{r_k^{p-1}}{E_k}} \leq M^q E_k^{\frac{q}{s} E_k^p},
\]

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which goes to 0 as $k \to \infty$. Finally, joining (5.9), (5.20), (5.21), (5.22), and (5.8), we have
\[\alpha_k(\rho') \leq R_k + \int_{B_r \cap \{|\varphi| \leq M\}} |\nabla \varphi|^p + C \left[\hat{\alpha}(\rho') - \hat{\alpha}(\rho)\right] + C \int_{(B_r \setminus B_{\rho'}) \cap \{|\varphi| \leq M\}} |\nabla \varphi|^p \, d\mathcal{L}^n + C \varepsilon.\]
Taking the limit as $k$ tends to infinity, and then the limit as $\varepsilon$ tends to 0, we get (5.11).

We are now in a position to prove that $v$ is $p$-harmonic: taking the limit as $\rho'$ tends to $\rho$ in (5.11), we have that if $\varphi \in W^{1,p}(B_1)$, with $v - \varphi$ supported on $B_\rho$, \[\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \alpha(\rho) \leq \int_{B_\rho} |\nabla \varphi|^p \, d\mathcal{L}^n,\] for every $\rho \in I$, therefore $v$ is $p$-harmonic in $B_1$. Notice that this implies that $v$ is a locally Lipschitz function (see [1, Theorem 7.12]) . Moreover, if $\varphi = v$, we have
\[\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n = \alpha(\rho)\]
for every $\rho \in I$, so that $\alpha$ is continuous on the whole interval $[0, 1]$, $\alpha(1) = 1$ and $\alpha(\tau) = \lim_k \alpha_k(\tau) \geq \tau^{n-1/2}$. Nevertheless, if $\tau$ is sufficiently small this contradicts the fact that $v$ is locally Lipschitz, since
\[\tau^{n-\frac{1}{2}} \leq \int_{B_\tau} |\nabla \varphi|^p \, d\mathcal{L}^n \leq C \tau^n,\]
where $C$ is a positive constant depending only on $n$ and $p$. \qed

Proof of Theorem 1.2. Let $u$ be a minimizer for the problem (1.4). By Proposition 3.4 there exist two positive constants $C(\Omega, \beta, p, q), C_1(\Omega, \beta, p, q)$ such that if $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$, then
\[\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq C(\Omega, \beta, p, q)r^{n-1},\]
and if $x \in K_u$
\[\mathcal{L}^n(B_r(x) \cap \{u > 0\}) \geq C_1(\Omega, \beta, p, q)r^n.\]

We now prove that there exists a positive constant $c = c(\Omega, \beta, p, q)$ such that
\[\mathcal{H}^{n-1}(J_u \cap B_r(x)) \geq c(\Omega, \beta, p, q)r^{n-1}\] (5.23)
for every $x \in K_u$ and $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$. Assume by contradiction that there exists $x \in J_u$ such that, for $r > 0$ small enough,
\[\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq \varepsilon_0 r^{n-1},\]
where $\varepsilon_0$ is the one in Lemma 5.3. Iterating Lemma 5.3 it can be proven (see [5, Theorem 5.1]) that
\[\lim_{r \to 0^+} r^{1-n} \mathcal{F}(u; B_r) = 0,\]
which, in particular, implies
(5.24)
\[\lim_{r \to 0^+} r^{1-n} \int_{B_r(x)} |\nabla u|^p \, d\mathcal{L}^n + \mathcal{H}^{n-1}(J_u \cap B_r(x)) = 0.\]
By [7, Theorem 3.6], (5.24) implies that $x \notin J_u$, which is a contradiction. Finally, if $x \in K_u$ and
\[\mathcal{H}^{n-1}(J_u \cap B_{2r}(x)) \leq \varepsilon_0 r^{n-1},\]

there exists \( y \in J_u \cap B_r(x) \) such that
\[
\mathcal{H}^{n-1}(J_u \cap B_r(y)) \leq \varepsilon_0 r^{n-1}
\]
which, again, is a contradiction. Then the assertion is proved. The density estimate (5.23) implies in particular that
\[
K_u \subset \left\{ x \in \mathbb{R}^n \middle| \limsup_{r \to 0^+} r^{1-n} \left[ \int_{B_r(x)} |\nabla u|^p dL^n + \mathcal{H}^{n-1}(J_u \cap B_r(x)) \right] > 0 \right\},
\]
hence \( \mathcal{H}^{n-1}(K_u \setminus J_u) = 0 \) (see for instance [7, Lemma 2.6]).

**Remark 5.4.** Let \( u \) be a minimizer for problem (1.4), then from Theorem 3.2 we have that the function \( u^\ast = (\beta \delta^q)^{-1/p} u \) is an almost-quasi minimizer for the Mumford-Shah functional
\[
MS(v) = \int_{\mathbb{R}^n} |\nabla v|^p dL^n + \mathcal{H}^{n-1}(J_v)
\]
with the Dirichlet condition \( u^\ast = (\beta \delta^q)^{-1/p} \) on \( \Omega \). If \( \Omega \) is sufficiently smooth we can apply the results in [4] to have that the density estimate for the jump set of minimizers holds up to the boundary of \( \Omega \).
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