Influence of Modification of Gravity on the Complexity Factor of Static Spherical Structures

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Abstract

The aim of this paper is to generalize the definition of complexity for the static self-gravitating structure in $f(R, T, Q)$ gravitational theory, where $R$ is the Ricci scalar, $T$ is the trace part of energy momentum tensor and $Q = R_{\alpha\beta}T^{\alpha\beta}$. In this context, we have considered locally anisotropic spherical matter distribution and calculated field equations and conservation laws. After the orthogonal splitting of the Riemann curvature tensor, we found the corresponding complexity factor with the help of structure scalars. It is seen that the system may have zero complexity factor if the effects of energy density inhomogeneity and pressure anisotropy cancel the effects of each other. All of our results reduce to general relativity on assuming $f(R, T, Q) = R$ condition.

Keywords: Gravitation; Self-gravitating Systems; Anisotropic Fluids.

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1 Introduction

The general relativity (GR) was proposed by Albert Einstein in 1915 in which he related matter and space-time through Einstein field equations. General Relativity could be recognized as the basis for gravitational physics and cosmology. It can used to express the history and expansion of our universe, the black hole phenomena and light that comes from distant galaxies. To get some feasible results about our universe on different scales, the $f(R)$ theory of gravity was introduced by replacing the Ricci scalar $R$ with its generic function in an action function. Nojiri and Odintsov [2] considered $f(R)$ gravity and described different phases of our universe like cosmological structure and phantom era. Capozziello et al. [3] considered the Lane-Emden equation with $f(R)$ corrections in order to study the hydrostatic equilibrium of stellar objects. They also found a mathematical connection of pressure with density and compared their results with that obtained in GR. Bamba et al. [4] explored the properties of different cosmologies with dark energy. They also investigated the $\Lambda$CDM-like universe by considering different cosmological models. Various researchers [5–13] investigated the effects of curvature terms on the formation and evolution of self-gravitating structures. After obtaining some exact solutions of $f(R)$ field equations, they also highlighted some applications of their results. Harko et al. [14] introduced the $f(R,T)$ theory of gravity which can be considered as the extension of $f(R)$ theory, where $T$ denotes the trace part of energy-momentum tensor. They presented the modified field equations in $f(R,T)$ gravity and analyzed the motion of test particles through variational principle. Baffou et al. [15] proposed a model to study the dynamics and stability of this theory through de Sitter and power-law solutions. They obtained few observationally compatible cosmological solutions.

The $f(R,T,Q)$ theory is based on the non-minimal coupling (NMC) between matter and geometry. In order to clarify the role of dark matter and dark energy in any stellar object without restoring to exotic matter, the formation of Einstein-Hilbert action is modified. Initially, the geometric part in the action was modified by replacing the Ricci scalar with its generic function. Later, $f(R)$ gravity was found not to meet the standard solar system constraints [16,17]. Consequently, Harko et al. [14] introduced the $f(R,T)$ theory of gravity which is the extension of $f(R)$ theory, where $T$ denotes the trace part of energy-momentum tensor. It is important to mention that the $f(R,T)$ gravity could not be able to encompass the NMC effects in the gravitational equations for $T = 0$, while an additional term $T^{\alpha\beta}R_{\alpha\beta}$ could provide these effects in this context. Haghani et al. [18] studied the role of strong NMC between geometry and fluid distribution by studying $f(R,T,Q)$ theory (where $Q \equiv T^{\alpha\beta}R_{\alpha\beta}$), which could be regraded as the generalization of $f(R,T)$ gravity. The fulfilment of the solar system tests and stability criteria are the fundamental requirements of any gravitational theory. Usually, the energy-momentum tensor is not conserved in $f(R,T,Q)$ theory. Haghani
et al. [18] found the stability conditions for this theory by interpreting the Dolgov-Kawasaki instability. In this kind of theories, the conservation of the stress energy tensor is also possible and in this case, throughout the high density era of our cosmic evolution, the existence of a de Sitter phase was to be found. They also utilized the Lagrange multiplier method to find the gravitational equations with conserved energy-momentum tensor.

Due to the non-conserved nature of this gravity and matter geometry interaction, an additional force is always present, even in our considered case $L_m = -\mu$, where $\mu$ is the energy density of the fluid, and thus the motion of particles does not follow geodesic path. The existence of an additional force could be helpful to study the galactic properties. The fluid-geometry coupling in $f(R,T,Q)$ theory may help us to analyze the reason of the late time acceleration of our cosmos. The cosmological aspects and the accelerating solutions of this theory has also been investigated by Odintsov and Sáez-Gómez [19]. The results of Dolgov-Kawasaki instability calculated in [18] and [19] are found to be same. Elizalde and Vacaru [20] considered some cosmological models and constructed off-diagonal analytical solutions in $f(R,T,Q)$ gravity. They also studied the FLRW cosmological model as well as $\Lambda$CDM universe, and the nonholonomic cosmological solutions are also described.

Haghani et al. [21] considered $f(R,T,Q)$ gravity and found some stability conditions regarding local perturbations. They examined the cosmological consequences which provide an exponential solution, and concluded that in the gravitational dynamics, the matter itself may play a key role. They also shown that de Sitter type solutions can also be conceded by modified field equations. Gama et al. [22] dealt with the $f(R,Q)$ gravity in order to calculate Gödel-type solutions and compared there stability with the GR solutions. They considered a particular model in this gravity to obtain causal solutions and found some conditions of their existence in the light of matter sources. They noticed that the rotating universe can be described by the Gödel-type metric, but its expansion was not taken into account. This theory may help us to explore the new aspects of the earliest stages of our cosmic evolution in near future.

Odintsov and Sáez-Gómez [19] determined numerical as well as analytical solutions of $f(R,T,Q)$ theory and checked the correspondence of their solutions with $\Lambda$CDM model. They also explained solutions for de Sitter universe and problems containing fluid instability. Ayuso et al. [23] studied the consistency and the stability of $f(R,T,Q)$ theory with an appropriate scalar/vector field. They produced higher order equations of motion describing the matter fields through the conformal and non-minimal couplings. Baffou et al. [24] discussed the stability through the de Sitter and power-law solution to explain the early evolution of our universe. They found numerical solution of some special models of $f(R,T,Q)$ theory to discuss their stability. In the gravitational collapse of spherical structure, Bhatti et al. and his collaborators [25-28] analyzed the role of the physical variables on the dynamical evolution of self-gravitating systems and found the relation of structural variables with that
of Weyl tensor in $f(R, T, Q)$ theory.

A complexity is the combination of various components which could be a source to trigger complications in any stable and static self-gravitating system. The definition of complexity has been examined in different fields of science. There are many definitions of complexity and one of them is introduced by López-Ruiz et al. [29–31] that was based on the concepts of entropy and information. Entropy gives us the disorderness of a system and information can be familiarly about any system. There are many other components to define the complexity of a system. The more appropriate way to define complexity was suggested by López-Ruiz et al. [29] which was based on the idea of disequilibrium.

Another way to interpret the definition of complexity in physics starts by dealing with isolated ideal gas and perfect crystal, as these systems have zero complexity. The isolated ideal gas is a system made-up of random moving molecules, so it is completely scattered. It gives us maximum information because all molecules participate equally. On the other hand, a perfect crystal is one whose constituents are organized in a highly ordered form and it is enough to study the small portion to describe its nature and hence it gives less data information. These two models are extreme in order and information. Hence, there would be maximum disequilibrium in case of perfect crystal and zero in case of the isolated ideal gas. As, they have zero complexity, so there is no complication in the behavior of these systems. In astrophysics, the complexity factor is significant in order to study the structure of the self-gravitating systems. In this scenario, the components which have been investigated usually are pressure, equilibrium, energy density and luminosity. In the absence of pressure factor in the energy-momentum tensor, energy density is not enough to express complexity.

Lloyd and Pagels [32] studied the observable states of self-gravitating structures and found their complexity factor that could be applicable to all physical structures. They also found the complexity factor for computational systems as a special case and discussed applications of some mathematical and physical problems. Crutchfield and Young [33] introduced the complexity of nonlinear dynamical structures. They proposed a measure of complexity of a system through entropy and dimension by a method that recreate minimal equations of a structure. Herrera et al. [34] investigated the aspects of inhomogeneous energy density and local pressure anisotropy on the spherical collapse of matter distribution and explained the active gravitational mass.

Herrera and Barreto [35] formulated polytropic spherical structures with pressure anisotropy and discussed their applications. They calculated the Tolman mass to interpret some properties of relativistic structures. Herrera et al. [36,37] proposed the orthogonal splitting of the Riemann curvature tensor and found some structure scalars. These scalars are then associated with the basic properties of matter distribution. They declared all possible solution of Einstein field equations in terms of these scalars through some examples in static case [38]. Yousaf and his collaborators [39,43] as well as Sharif and Manzoor [44,45] extended these
results for various cosmic models in modified theories.

Thirukkanesh and Ragel [46] studied spherically symmetric static geometry and interpreted compact structures of fluid distribution having pressure anisotropy. They obtained some exact models by writing another form of field equations using polytropic equation of state. Di Prisco et al. [47] dealt with two classes of homogeneous matter distribution in which the fluctuations induced by local pressure anisotropy may cause the cracking due to disequilibrium in spherical compact objects.

The aim of this work is to present the complexity factor for the locally anisotropic spherical matter configurations in $f(R, T, Q)$ gravity. We shall study the role of $f(R, T, Q)$ corrections in the modeling of relativistic spherical structure through complexity factor and structure scalars. The paper is outlined as under. In the next section, we propose the physical variables and modified field equations. Then we find one of the structure scalars known as the complexity factor from the curvature tensor in Sec. 3. After this in Sec. 4, we introduce the vanishing complexity factor condition and provide two exact solutions of modified field equations. Finally, we conclude all of our in Sec. 5.

2 The Physical Variables and Other Equations

We now model our system to be static spherically symmetric which is coupled with anisotropic matter configurations. We study the structure of such systems after considering $f(R, T, Q)$ equations of motion. We shall also express our results with the help of Tolman and Misner-Sharp formalisms. We describe various physical variables involved in the description of a static self-gravitating fluids. Further, we will evaluate few matching conditions on the three dimensional boundary surface Σ.

2.1 Modified Field Equations

The action for $f(R, T, Q)$ theory is [19, 23, 24]

$$S = \frac{1}{2} \int \sqrt{-g} [f(R, T, Q) + L_m] d^4x,$$

where $L_m$ is the matter Lagrangian and defined as $L_m = -\mu$. Here, $\mu$ denotes the energy density of the matter distribution, $g$ describes the determinant of the metric tensor $g_{\alpha\beta}$

The field equations corresponding to above action can be written as follows

$$G_{\mu\nu} = 8\pi T^{(\text{eff})}_{\mu\nu},$$
where $G_{\mu\nu}$ stands for Einstein tensor and the term $T^{(\text{eff})}_{\mu\nu}$ could be regarded as the energy-momentum tensor for $f(R, T, Q)$ theory, whose value can be given as follows

$$T^{(\text{eff})}_{\mu\nu} = \frac{1}{f_R - L_m f_Q} \left[ \left( f_T + \frac{1}{2} R f_Q + 1 \right) T^{(m)}_{\mu\nu} + \left\{ \frac{R}{2} (\frac{f}{R} - f_R) - L_m f_T \right. \right.$$ \n
$$- \frac{1}{2} \nabla_\alpha \nabla_\beta (f_Q T^{\alpha\beta}) \right\} g_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu f_R$$ \n
$$- 2 f_Q R_\alpha (\mu T^{\alpha}_\nu) + \nabla_\alpha \nabla_\nu (\mu T^{\alpha}_\nu) f_Q + 2 (f_Q R^{\alpha\beta} + f_T g^{\alpha\beta}) \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}} \right], \tag{3}$$

where $\nabla_\nu$, describes the covariant derivation and $\Box \equiv g^{\lambda\sigma} \nabla_\lambda \nabla_\sigma$. Moreover, the subscripts $R$, $T$ and $Q$ stand for partial differentiation with respect to their arguments. The trace of stress-energy tensor in GR provides a peculiar relationship between $R$ and $T$. However, in this case, we found from Eq.(3) as follows

$$3 \Box f_R + \frac{1}{2} \Box (f_Q T) - T (f_T + 1) + \nabla_\pi \nabla_\rho (f_Q T^{\pi\rho}) + R (f_R - \frac{T}{2} f_Q)$$ \n
$$+ (R f_Q + 4 f_T) L_m - 2 f + 2 R_\pi^{\pi\rho} f_Q - 2 \frac{\partial^2 L_m}{\partial g^{\lambda\sigma} \partial g^{\pi\rho}} (f_T g^{\pi\rho} + f_Q R^{\pi\rho}).$$

On assuming $Q = 0$ in the above equation, one can observe relativistic effects of $f(R, T)$ theory in the analysis, while the consideration of vacuum case in this theory describes the dynamical features of leads of $f(R)$ theory. The detailed analysis of their derivation and physical implication in the study of our cosmic structures are described in [19, 23, 24]. In Eq.(3), $T^{(m)}_{\mu\nu}$ is the usual energy-momentum tensor which in our case can be written as

$$T^{(m)}_{\mu\nu} = \mu u_\mu u_\nu - P h_{\mu\nu} + \Pi_{\mu\nu}, \tag{4}$$

where

$$\Pi_{\mu\nu} = \Pi \left( s_\mu s_\nu + \frac{1}{3} h_{\mu\nu} \right); \quad P = \frac{P_r + 2 P_\perp}{3}, \tag{5}$$

$$\Pi = P_r - P_\perp; \quad h_{\mu\nu} = \delta_{\mu\nu} - u_\mu u_\nu, \tag{6}$$

and $s^\mu$ and $u^\mu$ are the four vectors, $\Pi$ is the pressure anisotropy having radial pressure $P_r$ and tangential pressure $P_\perp$ and $h_{\mu\nu}$ is the projection tensor. We suppose that our geometry is characterized with a boundary surface $\Sigma$ which has demarcated the interior and exterior regions of spherical spacetimes. The geometry interior to $\Sigma$ can be given as follows

$$ds^2 = -e^\lambda dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + e^\nu dt^2, \tag{7}$$
where $\nu = \nu(r)$ and $\lambda = \lambda(r)$.

The four vectors corresponding to above system can be defined as

$$u^\mu = (e^{-\nu}, 0, 0, 0), \quad s^\mu = (0, e^{-\lambda}, 0, 0),$$

from which one can write the four-acceleration as $a^\alpha = u_\beta u^\beta$. We found only one non-zero component of the four acceleration which can be given as under

$$a_1 = -\frac{\nu'}{2}.$$

The 4-vectors satisfy the relations $s^\mu u_\mu = 0, s^\mu s_\mu = -1$.

The field equations in $f(R,T,Q)$ theory for the spherical system (4) and (7) are

$$-\left[e^{-\lambda} \left(1 - \frac{\lambda'}{r} \right) \right] + \frac{1}{r^2} = \frac{8\pi}{(f_R - L_m f_Q)} \mu^{(eff)}$$

$$-\left[- e^{-\lambda} \left(1 - \frac{\nu'}{r} \right) \right] + \frac{1}{r^2} = \frac{8\pi}{(f_R - L_m f_Q)} P_r^{(eff)},$$

$$\frac{1}{32\pi} e^{-\lambda} \left[2\nu'' + \nu'^2 - \nu'\lambda' + \frac{2(\nu' - \lambda')}{r} \right] = \frac{1}{(f_R - L_m f_Q)} P_\perp^{(eff)},$$

where $\mu^{(eff)}, P_r^{(eff)}$ and $P_\perp^{(eff)}$ describe the contribution of $f(R,T,Q)$ corrections in the physical variables of relativistic fluids. Their values are given in Appendix A. Here, prime indicates the derivative with respect to radial coordinate.

It is worthy to stress that, unlike GR and $f(R)$ theory, the divergence of effective energy momentum tensor in $f(R,T,Q)$ gravity is non-zero, which gives rise to the breaking of all equivalence principles. Thus, the present theory encompasses non-geodesic motion of the particles due to emergence of extra force acting on the moving particles in this gravitational field. Its value can be casted as

$$\nabla^\lambda T_{\lambda\sigma} = \frac{2}{R f_Q + 2 f_T + 1} \left[ \nabla_\sigma (L_m f_T) + \nabla_\sigma (f_Q R^\sigma \lambda T_{\pi\sigma}) - \frac{1}{2} (f_T g_{\pi\rho} + f_Q R_{\pi\rho}) \right. \times \nabla_\rho T_{\pi\rho} - G_{\lambda\sigma} \nabla^\lambda (f_Q L_m) \right].$$

For our observed system, the hydrostatic equilibrium can be studied from the conservation equation as

$$\left(\frac{P_r^{(eff)}}{H}\right)' = -\frac{\nu'}{2H} (\mu^{(eff)} + P_r^{(eff)}) + \frac{2(P_\perp^{(eff)} - P_r^{(eff)})}{rH} + Z e^\lambda,$$
where \( H = f_R - L_m f_Q \) and \( Z \) indicates extra curvatures terms of this theory described in Appendix A. This equation could be called as the generalized Tolman-Oppenheimer-Volkoff (TOV) equation for anisotropic matter which may help to understand the subsequent changes in the structure of the static spherical system.

From Eq.(11), the value of \( \nu' \) can be found as

\[
\nu' = 2 \frac{m + 4\pi r^3 P^{(\text{eff})}/H}{r(r - 2m)}.
\]

The substitution of Eq.(15) in Eq.(14), we get

\[
\left( \frac{P_r^{(\text{eff})}}{H} \right)' = - \frac{(m + 4\pi r^3 P_r^{(\text{eff})}/H)(\mu^{(\text{eff})} + P_r^{(\text{eff})})}{H r(r - 2m)} + \frac{2(P^{(\text{eff})} - P_r^{(\text{eff})})}{H r} + Z e^\lambda,
\]

where \( m \) can be expressed through metric coefficient of the spherical system as

\[
R_{232}^3 = 1 - e^{-\lambda} = \frac{2m}{r},
\]

which can be expressed through field equation (10) as

\[
m = 4\pi \int_0^r \tilde{r}^2 \tilde{\mu}^{(\text{eff})} \frac{H}{d\tilde{r}}.
\]

We now describe the geometric structure outside \( \Sigma \) with the help of the spacetime given below

\[
ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \frac{dr^2}{\left( 1 - \frac{2M}{r} \right)} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( M \) is a gravitating mass of the corresponding object. For the smooth matching of outer and inner manifold across the boundary, we consider Darmois junction conditions and matching criterion provided by Yousaf et al. [26] (after following Senovilla [48]) for \( f(R, T, Q) \) gravity. At boundary surface \( r = r_\Sigma = \), we found that

\[
e^\nu_\Sigma = 1 - \frac{2M}{r_\Sigma}, \quad e^{-\lambda_\Sigma} = 1 - \frac{2M}{r_\Sigma}, \quad [P_r]_\Sigma = -D_0,
\]

where \( D_0 \) is given in Appendix A. However, the boundary conditions for \( f(R, T, Q) \) gravity are found to be

\[
f,RR[\partial_\rho R]^+ = 0, \quad f,RR K_{\lambda\sigma}^+ = 0, \quad f,QQ[\partial_\rho Q]^+ = 0, \quad K|^+ = 0.
\]
along with
\[ R^+_{\alpha \beta} = 0, \quad Q^+_{\alpha \beta} = 0, \quad \gamma_{\lambda \sigma}^+ = 0, \tag{21} \]
where \( K^*_{\alpha \beta} \) is the trace-free and \( K \) is the trace component of the extrinsic curvature. The detailed analysis can be found in [26]. The conditions (20) and (21) hold only if \( f_{RR} \neq 0 \) and \( f_{QQ} \neq 0 \). Thus, the boundary conditions (18) comes after applying the Darmois matching condition, while the conditions (20) and (21) arises due to \( f(R, T, Q) \) theory. The satisfaction of such constraints over \( \Sigma \) are necessary for the smooth joining of manifolds even for matter thin shells in \( f(R, T, Q) \) gravity.

### 2.2 Curvature Tensors

It could be useful to express one of the well-known curvature tensor, i.e., the Riemann tensor through the Ricci tensor \( R^\rho_{\alpha \beta \mu} \), tensor \( C^\mu_{\nu \kappa \lambda} \) and the Ricci scalar \( R^\rho_{\alpha \beta} \) as
\[
R^\rho_{\alpha \beta \mu} = C^\rho_{\alpha \beta \mu} + \frac{1}{2} R^\rho_{\beta \mu} g_{\alpha \nu} - \frac{1}{2} R_{\alpha \beta} \delta^\rho_\mu + \frac{1}{2} R_{\alpha \mu} \delta^\rho_\beta - \frac{1}{2} R^\rho_{\mu \nu} g_{\alpha \beta} - \frac{1}{6} R (\delta^\rho_\mu g_{\alpha \beta} - g_{\alpha \beta} \delta^\rho_\mu) . \tag{22} \]

The magnetic part of Conformal tensor vanishes for spherically symmetric distribution, while its electric part \((E^\alpha_{\beta \gamma} = C^\alpha_{\beta \gamma \delta} u^\gamma u^\delta)\) can be given as
\[
C^\mu_{\nu \kappa \lambda} = (g_{\mu \nu \alpha \beta} g_{\kappa \lambda \delta} - \eta_{\mu \nu \alpha \beta} \eta_{\kappa \lambda \delta}) u^\alpha u^\gamma E^\delta_{\beta} , \tag{23} \]
where \( g_{\mu \nu \alpha \beta} = g_{\mu \alpha} g_{\nu \beta} - g_{\mu \beta} g_{\nu \alpha} \), and \( \eta_{\mu \nu \alpha \beta} \) is the Levi-Civita tensor. We can rewrite \( E_{\alpha \beta} \) as
\[
E_{\alpha \beta} = E \left( s_{\alpha} s_{\beta} + \frac{1}{3} h_{\alpha \beta} \right) , \tag{24} \]
with
\[
E = -\frac{e^{-\lambda}}{4} \left[ \nu'' + \frac{\nu'^2 - \lambda' \nu'}{2} - \frac{\nu' - \lambda'}{r} + \frac{2(1 - e^\lambda)}{r^2} \right] , \tag{25} \]
and satisfy the following conditions
\[
E^\alpha_{\alpha} = 0, \quad E_{\alpha \gamma} = E_{(\alpha \gamma)} , \quad E_{\alpha \gamma} u^\gamma = 0 . \tag{26} \]
2.3 The Mass Function

Here, we shall adopt the formalism provided by Misner-Sharp and Tolman to calculate few interesting equations that would assist us to study the structural properties of a relativistic fluid. After this, we will find some connection between mass function and Conformal tensor.

Utilizing Eqs. (2), (16), (22) and (24), one can write

\[ m = \frac{4\pi}{3H} r^3 (\mu^{\text{eff}} + P_r^{\text{eff}} - P_\perp^{\text{eff}}) + \frac{r^3 E}{3}, \]  

(27)

which can be manipulated as

\[ E = -\frac{4\pi}{r^3} \int_0^r \bar{r}^3 \left( \frac{\mu^{\text{eff}}}{H} \right)' d\bar{r} + \frac{4\pi}{H} (P_r^{\text{eff}} - P_\perp^{\text{eff}}). \]  

(28)

The above equation gives us the relation among Conformal tensor and spherical structural properties, like effective energy density inhomogeneity and effective form of the anisotropic pressure. By making use of Eq. (28) in Eq. (27), we have

\[ m(r) = \frac{4\pi}{3H} r^3 \mu^{\text{eff}} - \frac{4\pi}{3} \int_0^r \bar{r}^3 \left( \frac{\mu^{\text{eff}}}{H} \right)' d\bar{r}, \]  

(29)

which provides a peculiar relationship of the mass function with homogeneous energy density. One can study the effects of effective modified terms on the subsequent changes brought by energy density inhomogeneity in the spherical anisotropic self-gravitating system with the help of the above formula.

Tolman [49] introduced another formula of energy for a static spherical distribution of matter given by

\[ m_T = 4\pi \int_0^\Sigma r^2 e^{(\nu+\lambda)/2} (T_0^{0\text{eff}} - T_1^{1\text{eff}} - 2T_2^{2\text{eff}}) dr. \]  

(30)

Bhatti et al. [50,51] calculated the expressions for Tolman mass function for the case spherically symmetric systems in \( f(R) \) gravity with and without electromagnetic field. This was introduced to estimate the total energy of the structure and within the spherical configuration of radius \( r \). It can be described as

\[ m_T = 4\pi \int_0^r r^2 e^{(\nu+\lambda)/2} (T_0^{0\text{eff}} - T_1^{1\text{eff}} - 2T_2^{2\text{eff}}) dr. \]  

(31)

Using Eqs. (10)-(12) in above equation, one may obtain

\[ m_T = e^{(\nu+\lambda)/2} [m(r) + 4\pi r^3 P_r^{\text{eff}}/H]. \]  

(32)
Using the definition of mass function and the field equations, one can write

\[ m_T = e^{(\nu - \lambda)/2} \frac{\nu'}{2}. \]  

This equation gives us the physical importance of \( m_T \) as the effective inertial mass. In a static gravitational field (instantly at rest), the gravitational acceleration \( a = -s^\nu a_\nu \) of a test particle is followed by

\[ a = \frac{e^{-\lambda/2} \nu'}{2} = \frac{e^{-\lambda/2} m_T}{r^2}. \]  

The more suitable representation of \( m_T \) is,

\[
m_T = (m_T)_{\sum} \left( \frac{r}{r_{\Sigma}} \right)^3 - r^3 \int r_{\Sigma} \frac{e^{(\nu + \lambda)/2}}{\tilde{r}} \times \left[ \frac{8\pi}{H} (P_{(\perp)}^{(\text{eff})} - P_{(\text{eff})}^{(\parallel)}) + \frac{1}{r^3} \int_0^r 4\pi r^3 \left( \frac{\mu^{(\text{eff})}}{H} \right)' \text{d} \tilde{r} \right] \text{d} \tilde{r}. \]  

Using the information from Eq.(28), one can write

\[
m_T = (m_T)_{\sum} \left( \frac{r}{r_{\Sigma}} \right)^3 - r^3 \int r_{\Sigma} \frac{e^{(\nu + \lambda)/2}}{\tilde{r}} \times \left[ \frac{4\pi}{H} (P_{(\perp)}^{(\text{eff})} - P_{(\text{eff})}^{(\parallel)}) - E \right] \text{d} \tilde{r}. \]  

This relation could be helpful to understand the role of Weyl scalar, modified correction terms, effective pressure anisotropy and irregularity in the energy density of the static spherically symmetric spacetime on the Tolman mass. Thus relates the phenomenon of the occurrence of inhomogeneous energy density and local anisotropy of pressure through Tolman mass in \( f(R, T, R_{\mu\nu} T^{\mu\nu}) \) gravity.

### 3 The Orthogonal Splitting of The Riemann Tensor

The orthogonal splitting of Riemann curvature tensor was proposed by Bel [52] and Herrera et al. [36]. One can find three tensors obtained from the orthogonal decomposition of the Riemann tensor, as

\[
Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta} u^\gamma u^\delta, \tag{37}
\]

\[
Z_{\alpha\beta} = * R_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2} \eta_{\alpha\gamma\epsilon\mu} R^{\epsilon\mu}_{\beta\delta} u^\gamma u^\delta, \tag{38}
\]

\[
X_{\alpha\beta} = * R^*_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2} \eta_{\alpha\gamma\epsilon} R^*_{\epsilon\mu\beta\delta} u^\gamma u^\delta, \tag{39}
\]
where \( \eta_{\alpha\beta} \) represents the Levi-Civita symbol while the steric indicates the dual operation on the subsequent tensor. One can rewrite the Riemann tensor in terms of above mentioned tensors (see [53]), therefore following this we have calculated another form of Eq.(22) after using field equation as

\[
R^{\alpha\gamma}_{\beta\delta} = C^{\alpha\gamma}_{\beta\delta} + 16\pi T^{(eff)}_{[\beta\delta]} + 8\pi T^{(eff)} \left( \frac{1}{3} \delta^\alpha_{[\beta} \delta^\gamma_{\delta]} - \delta^{[\alpha}_{[\beta} \delta^{\gamma]}_{\delta]} \right),
\]

(40)

which further can be written through Eq.(3) as

\[
R^{\alpha\gamma}_{\beta\delta} = R^{\alpha\gamma}_{(I)\beta\delta} + R^{\alpha\gamma}_{(II)\beta\delta},
\]

(41)

where

\[
R^{\alpha\gamma}_{(I)\beta\delta} = \frac{16\pi}{H} \left( f_T + \frac{1}{2} Rf_T + 1 \right) \left[ \mu u^\alpha u_{[\beta} \delta^\gamma_{\delta]} - Ph^\alpha_{[\beta} \delta^\gamma_{\delta]} + \Pi^\alpha_{[\beta} \delta^\gamma_{\delta]} \right] + 8\pi \left( f_T + \frac{1}{2} Rf_T + 1 \right)(\mu - 3P) + 4 \left( \frac{R}{2} \left( \frac{1}{f_T - f_R} \right) + \mu f_T \right) - \frac{1}{2} \nabla_\mu \nabla_\nu (f_Q T^{\mu\nu}) \right) - \frac{1}{2} \Box \{ f_Q (\mu - 3P) \} - 3 \Box R - 2 \nabla_\mu \nabla_\nu (f_Q T^{\mu\nu} + \nabla_\mu (f_Q R^{\mu\nu} + f_T g^{\mu\nu}) \frac{\partial^2 L_m}{\partial g^{\alpha\xi} \partial g^{\mu\nu}}) \times \left( \frac{1}{3} \delta^\alpha_{[\beta} \delta^\gamma_{\delta]} - \delta^{[\alpha}_{[\beta} \delta^{\gamma]}_{\delta]} \right),
\]

(42)

\[
R^{\alpha\gamma}_{(II)\beta\delta} = \frac{4\pi}{H} \left[ 2 \left( \frac{R}{2} \left( \frac{f_T}{f_T - f_R} \right) + \mu f_T - \frac{1}{2} \nabla_\mu \nabla_\nu (f_Q T^{\mu\nu}) \right) \{ f_Q (T^\alpha_\beta \delta^\alpha_{\delta} - T^\alpha_\beta \delta^\alpha_{\delta} - T^\alpha_\beta \delta^\alpha_{\delta} + T^\alpha_\beta \delta^\alpha_{\delta}) \} \right] - 2 \Box \nabla_\beta (\delta^\alpha_\beta \delta^\alpha_{\delta} - \delta^\alpha_\beta \delta^\alpha_{\delta}) + \nabla_\alpha \nabla_\beta - \delta^\alpha_\beta \delta^\alpha_{\delta} \nabla_\beta - \delta^\alpha_\beta \delta^\alpha_{\delta} \nabla_\beta + \nabla_\alpha \nabla_\beta \nabla_\delta (f_Q - f_Q (R^\alpha_\beta T^\mu_\delta \delta^\alpha_{\delta} - R^\alpha_\beta T^\mu_\delta \delta^\alpha_{\delta} - R^\alpha_\beta T^\mu_\delta \delta^\alpha_{\delta} + R^\alpha_\beta T^\mu_\delta \delta^\alpha_{\delta})) - f_Q (R_{\mu\delta} T^{\mu\alpha} \delta^\alpha_\beta - R_{\mu\delta} T^{\mu\alpha} \delta^\alpha_{\beta} - R_{\mu\delta} T^{\mu\alpha} \delta^\alpha_{\beta} + R_{\mu\delta} T^{\mu\alpha} \delta^\alpha_{\beta}) + \nabla_\alpha \{ f_Q (T^\mu_\delta \delta^\alpha_{\delta} - T^\mu_\delta \delta^\alpha_{\delta}) \} + \nabla_\beta \nabla_\alpha \{ f_Q (T^\mu_\delta \delta^\alpha_{\delta} - T^\mu_\delta \delta^\alpha_{\delta}) \} + \nabla_\alpha \nabla_\beta \nabla_\delta \{ f_Q (T^\mu_\delta \delta^\alpha_{\delta} - T^\mu_\delta \delta^\alpha_{\delta}) \} + 2 g^{\alpha\gamma} (f_Q R^{\mu\nu} + f_T g^{\mu\nu}) \left( \delta^\alpha_\beta - \delta^\alpha_\beta \right) \frac{\partial^2 L_m}{\partial g^{\gamma\beta} \partial g^{\mu\nu}} - \delta^\alpha_\beta \frac{\partial^2 L_m}{\partial g^{\gamma\beta} \partial g^{\mu\nu}}.
\]
\[ R^{\alpha\gamma}_{(\Pi)\beta\delta} = 4u^{[\alpha} u_{\beta]} E^\gamma_{\delta]} - \epsilon^\alpha_{\mu} \epsilon_{\beta\delta\nu} E^{\mu\nu}, \]

with

\[ \epsilon_{\alpha\gamma\beta} = u^\mu \eta_{\mu\alpha\gamma\beta}, \quad \epsilon_{\alpha\gamma\beta} u^\beta = 0, \]

where we have used that the magnetic part of Conformal tensor is identically zero in case of spherical symmetry.

In order to find three important tensors, i.e., \(X_{\alpha\beta}, Y_{\alpha\beta}\) and \(Z_{\alpha\beta}\), we use above mentioned equation and get

\[
Y_{\alpha\beta} = E_{\alpha\beta} + \frac{1}{H} \left\{ \frac{4\pi}{3} (\mu + 3P) h_{\alpha\beta} + 4\pi \Pi_{\alpha\beta} \right\} (f_T + \frac{1}{2} R_{fQ} + 1)
- \frac{8\pi}{3H} \left\{ \frac{R}{2} \left( \frac{f}{R} - f_R \right) + \frac{1}{2} \nabla_\mu \nabla_\nu (f_T^{\mu\nu}) \right\} h_{\alpha\beta}
+ \frac{4\pi}{H} \left\{ \nabla_\alpha \nabla_\beta (f_Q T^T_{\delta}) + g_{\alpha\beta} u_\gamma u_\delta \nabla_\gamma \nabla_\delta (f_Q T^T_{\delta}) \right\}
+ \left\{ \nabla_\alpha \nabla_\beta f_R - u_\gamma u_\delta \nabla_\gamma \nabla_\delta f_R - u_\gamma u_\delta \nabla_\gamma \nabla_\delta f_R + g_{\alpha\beta} u_\gamma u_\delta \nabla_\gamma \nabla_\delta f_R \right\} + 2h^\epsilon (f_Q R^{\mu\nu} + f_T g^{\mu\nu}) \frac{\partial^2 L_m}{\partial g^{\beta\gamma} \partial g^{\mu\nu}}
+ \frac{8\pi}{3H} \left[ \frac{1}{2} \square (f_Q (\mu - 3P)) \right]
+ 2f_Q R_{\mu
u}(\mu u^\mu u^\nu - Ph^{\mu\nu} + \Pi^{\mu\nu}) - \nabla_\mu \nabla_\nu (f_Q T^T_{\epsilon})
- 2g^{\epsilon\xi} (f_Q R^{\mu\nu} + f_T g^{\mu\nu}) \frac{\partial^2 L_m}{\partial g^{\epsilon\xi} \partial g^{\mu\nu}} \right\} h_{\alpha\beta};
\]

\[
Z_{\alpha\beta} = 4\pi \left[ \frac{1}{H} \left\{ \frac{1}{2} u^\delta \square (f_Q T^T_{\delta}) - u^\delta \nabla_\epsilon \nabla_\delta f_R + f_Q \mu R^\epsilon_{\delta} u^\delta - f_Q P R^\epsilon_{\delta} u^\delta + \frac{1}{3} f_Q \Pi R^\epsilon_{\delta} u^\delta \right\}
- \frac{1}{2} u^\delta \nabla_\mu \nabla_\epsilon (f_Q T^T_{\delta}) - \frac{1}{2} u^\delta \nabla_\mu \nabla_\delta (f_Q T^T_{\epsilon}) \right\} \epsilon_{\epsilon\alpha},
\]
and

\[
X_{\alpha\beta} = -E_{\alpha\beta} + \frac{1}{H} \left( \frac{8\pi}{3} \mu h_{\alpha\beta} + 4\pi \Pi_{\alpha\beta} \right) \left( f_T + \frac{1}{2} R f_Q + 1 \right) \\
+ \frac{4\pi}{H} \left\{ -\frac{1}{2} \Box (f_Q T^\rho_\epsilon) + \nabla^\rho \nabla_\epsilon f_R + \frac{1}{2} \nabla_\mu \nabla^\rho (f_Q T^\rho_\mu) \\
+ \frac{1}{2} \nabla_\mu \nabla_\epsilon (f_Q T^\mu_\rho) \right\} \epsilon^\rho_\alpha \epsilon^\mu_\beta + f_Q R^\rho_\mu \left( P - \frac{\Pi}{3} \right) \epsilon^\rho_\alpha \epsilon^\mu_\beta \\
+ f_Q R_{\mu\epsilon} \left( P - \frac{\Pi}{3} \right) \epsilon^\epsilon_\alpha \epsilon^\mu_\beta \right\} + \frac{8\pi}{3} H \left\{ \frac{R}{2} \left( \frac{f}{R} - f_R \right) + \mu f_T \\
- \frac{1}{2} \nabla_\mu \nabla_\nu (f_Q T^\mu_\nu) \right\} - \frac{1}{2} \Box \{ f_Q (\mu - 3P) \} + 2 R f_Q \left( P - \frac{\Pi}{3} \right) \\
+ \nabla_\mu \nabla_\rho (f_Q T^\mu_\rho) + 2 g^{\rho\epsilon} (f_Q R^{\mu\nu} + f_T g^{\mu\nu}) \frac{\partial^2 L_m}{\partial g^{\rho\epsilon} \partial g^{\mu\nu}} \right\} h_{\alpha\beta}. \tag{48}
\]

Finally, we can obtain four structure scalars $X_T, X_{TF}, Y_T$ and $Y_{TF}$ from $X_{\alpha\beta}$ and $Y_{\alpha\beta}$, as

\[
X_T = \frac{8\pi\mu}{H} \left( f_T + \frac{1}{2} R f_Q + 1 \right) + \chi^{(D)}_1, \tag{49}
\]

\[
X_{TF} = -E + \frac{4\pi\Pi}{H} \left( f_T + \frac{1}{2} R f_Q + 1 \right), \tag{50}
\]

\[
Y_T = \frac{4\pi}{H} (\mu + 3P - 2\Pi) (f_T + \frac{1}{2} R f_Q + 1) + \chi^{(D)}_2, \tag{51}
\]

\[
Y_{TF} = E + \frac{4\pi\Pi}{H} \left( f_T + \frac{1}{2} R f_Q + 1 \right) + \chi^{(D)}_3. \tag{52}
\]

On utilizing Eq.(28) in Eq.(50), we obtain

\[
X_{TF} = \frac{4\pi}{r^3} \int_0^r \tilde{r}^3 \left( \frac{\mu^{(\text{eff})}}{H} \right)’ d\tilde{r} - \frac{4\pi\Pi^{(\text{eff})}}{H} + \frac{4\pi\Pi}{H} \left( f_T + \frac{1}{2} R f_Q + 1 \right), \tag{53}
\]

where $\chi^{(D)}_3 = \frac{1}{s_{\alpha\beta} + s_{\alpha\beta}} \chi^{(D)}_{\alpha\beta}$, while the values of $\chi^{(D)}_1, \chi^{(D)}_2$ and $\chi^{(D)}_{\alpha\beta}$ are given in Appendix B. Again utilizing Eq.(28), one can write Eq.(52) as

\[
Y_{TF} = -\frac{4\pi}{r^3} \int_0^r \tilde{r}^3 \left( \frac{\mu^{(\text{eff})}}{H} \right)’ d\tilde{r} + \frac{4\pi\Pi^{(\text{eff})}}{H} + \frac{4\pi\Pi}{H} \left( f_T + \frac{1}{2} R f_Q + 1 \right) + \chi^{(D)}_3. \tag{54}
\]

The anisotropic pressure plus $f(R, T, Q)$ corrections in terms of trace-free parts of structure scalars can be expressed, as

\[
X_{TF} + Y_{TF} = \frac{8\pi\Pi}{H} \left( f_T + \frac{1}{2} R f_Q + 1 \right) + \chi^{(D)}_3. \tag{55}
\]
To demonstrate the realistic meaning of $Y_{TF}$, we utilize Eq. (54) in Eq. (35) to yield

$$m_T = (m_T)_{\Sigma} \left( \frac{r}{r_{\Sigma}} \right)^3 + r^3 \int_{r}^{r_{\Sigma}} \frac{e^{(\nu+\lambda)/2}}{\tilde{f}} \left[ Y_{TF} + \frac{4\pi \Pi^{(eff)}}{H} \right] \left[ \chi_3^{(D)} \right] d\tilde{r}. \quad (56)$$

The matching the above equation with Eq. (35), we conclude that $Y_{TF}$ provides the effects of the local pressure anisotropy and inhomogeneous energy density on the Tolman mass under the effect of extra curvature terms in $f(R, T, Q)$ theory. Moreover, the Tolman mass can be expressed in an alternative way as

$$m_T = \int_0^r \tilde{r}^2 e^{(\nu+\lambda)/2} \left[ Y_T - \frac{4\pi}{H} (\mu + 3P_r - 2\Pi) (f_T + \frac{1}{2} Rf_Q + 1) \right. \quad \text{(57)}$$

This equation has directly relate $Y_T$ to the effective matter variables and Tolman mass function. It is well-known from the working of Herrera et al. [54,55] and Yousaf et al. [56–58] that $Y_T$ has been involved in the evolutionary equation of the expansion scalar which is widely known as Raychaudhuri equation. Equation (57) suggests that $m_T$ could be used to define the Raychaudhuri equation even in $f(R, T, Q)$ gravity.

### 4 Matter Distribution With Zero Complexity Factor

It is well-known that there are many components which could be reason to produce complexity in any static/non-static systems. The causes of complexity in our structure are the inhomogeneous energy density and pressure anisotropy contained in the structure scalar $Y_{TF}$ in the modified gravity. The corresponding equations of motion contain five unknowns $(\nu, P_r, \lambda, P_\perp, \mu)$ with $f(R, T, Q)$ theory. Therefore, in order to proceed our work, we need two more conditions. One of them can be obtained from the vanishing of complexity factor condition. After putting $Y_{TF} = 0$ in Eq. (54), we get

$$\Pi = \frac{H}{2} \left[ \frac{1}{r^3} \int_0^r \tilde{r}^3 \left( \frac{\mu^{(eff)}}{H} \right) d\tilde{r} - \frac{\Pi^{(D)}}{H} - \frac{\Pi}{H} (f_T + \frac{1}{2} Rf_Q) - \frac{1}{4\pi} \chi_3^{(D)} \right]. \quad (58)$$

This equation can be noticed as a non-local equation of state as it describes the radial pressure as a function of the energy density at a specific point on the manifold (under the constraint $f(R, T, Q) = R$). Now, we consider two subcases as follows:
4.1 The Gokhroo and Mehra Ansatz

In order to continue the systematic analysis, we take the assumption introduced by Gokhroo and Mehra, as

\[ e^{-\lambda} = 1 - \alpha r^2 + \frac{3Kr^4}{5r_\Sigma^2}, \]

where \( K \in (0, 1) \), \( \alpha = \frac{8\pi\mu_o}{3} \) and \( \mu_o \) is a constant. Utilizing Eq.(59) in Eqs.(10) and (17), we get

\[ \frac{\mu^{(\text{eff})}}{H} = \mu_o \left( 1 - \frac{Kr^2}{r_\Sigma^2} \right), \]

and

\[ m(r) = \frac{4\pi\mu_o r^3}{3} \left( 1 - \frac{3Kr^2}{5r_\Sigma^2} \right). \]

Further, from Eqs.(11) and (12), we obtain

\[ 8\pi \left[ P_r^{(\text{eff})} - P_\perp^{(\text{eff})} \right] = e^{-\lambda} \left[ -\frac{\nu''}{2} - \left( \frac{\nu'}{2} \right)^2 + \frac{\nu'}{2r} + \frac{1}{r^2} \left( \frac{\nu'}{2} + \frac{1}{r} \right) \right] - \frac{1}{r^2}. \]

It could be useful to introduce new structural variables as

\[ e^\nu(r) = e^{\int (2z(r) - 2/r)dr}, \quad e^{-\lambda(r)} = y(r), \]

In this context, after substituting Eq.(63) in Eq.(62), we obtain

\[ y' + y \left[ \frac{2z'}{z} + 2z - \frac{6}{r} + \frac{4}{r^2} \right] = \frac{2}{z} \left( \frac{1}{r^2} + \frac{8\pi}{H} (P_r^{(\text{eff})} - P_\perp^{(\text{eff})}) \right), \]

which appears to be in form proposed by Ricatti. The integration of this expression produces the line element in terms of \( z \) and \( \Pi \) in the background of \( f(R, T, Q) \) theory as

\[ ds^2 = -e^{(2z(r)-2/r)dr} dt^2 + \frac{z^2(r)e^{\int \left( \frac{4}{r^2z(r)} + 2z(r) \right)dr}}{r^6} \left( -2 \int \frac{z(r)\left( 1 + 8\pi\Pi^{(\text{eff})}(r)r^2/H \right)e^{\int \left( \frac{4}{r^2z(r)} + 2z(r) \right)dr}dr + C \right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \]

where \( C \) is integration constant. Thus, one may write the effective forms of physical variables as

\[ \frac{4\pi P_r^{(\text{eff})}}{H} = \frac{z(r - 2m) + m/r - 1}{r^2}, \]
\[
\frac{4\pi \mu^{(\text{eff})}}{H} = \frac{m'}{r^2},
\]
\[
\frac{8\pi P^{(\text{eff})}}{H} = \left(1 - \frac{2m}{r}\right) \left(z' + z^2 - \frac{z}{r} + \frac{1}{r^2}\right) + z \left(\frac{m}{r^2} - \frac{m'}{r}\right).
\]

Such kind of solutions could be helpful to understand some hidden and very interesting features of the spherical systems. Di Prisco et al. [59] calculated such results in the background of GR. These results were modified by Sharif and Yousaf for the cases of cylindrically [60] and shearfree spherical [61] structures. Yousaf [62] and Bhatti [63] further extended these results for Einstein-Λ gravity. To obtain non-singular characteristics of the obtained solutions, the obtained solutions (66)-(68) should satisfy Eqs.(19)-(21) at the boundary.

### 4.2 The Polytropic Fluid With Zero Complexity Factor

In this subsection, we study polytropic relativistic fluid in the presence of \(f(R, T, Q)\) corrections. To deal with the system of equations, we require vanishing complexity factor condition to supplement with the polytropic equation of state. Here, we consider two cases of polytropes separately. The first one is

\[
P^{(\text{eff})} = K[\mu^{(\text{eff})}]^\gamma = K[\mu^{(\text{eff})}]^{(1+1/n)}; \quad Y_{TF} = 0,
\]

with the polytropic constant \(K\), polytropic exponent \(\gamma\) and polytropic index \(n\).

To make the dimensionless form of TOV equation and the mass function, we may define some new variables as

\[
\alpha = P^{(\text{eff})}_{\text{rc}}/\mu^{(\text{eff})}_c, \quad r = \xi/A, \quad A^2 = 4\pi \mu^{(\text{eff})}_{\text{rc}}/\alpha(n+1),
\]

\[
\psi^n = \mu^{(\text{eff})}/\mu^{(\text{eff})}_c, \quad \nu(\xi) = m(r)A^3/(4\pi \mu^{(\text{eff})}_c),
\]

where subscript \(c\) shows that the quantity is calculated at the center. At the boundary \(r = r_\Sigma(\xi = \Sigma)\), we have \(\psi(\xi_\Sigma) = 0\). Putting these dimensionless variables in TOV equation, we get

\[
\xi^2 \frac{d\psi}{d\xi} \left[1 - \frac{2(n+1)\alpha\nu/\xi}{1 + \alpha \psi}\right] + \frac{2\Pi^{(\text{eff})}_{p\text{rc}}}{P^{(\text{eff})}_{\text{rc}}(n+1)} \left[1 - \frac{2(n+1)\alpha\nu/\xi}{1 + \alpha \psi}\right] 
\]

\+
\[\nu + \alpha \xi^3 \psi^{n+1} / H = \] \(\frac{\xi^2 \psi^n}{AP^{(\text{eff})}_{\text{rc}}(n+1)} \left(1 - \frac{2(n+1)\alpha\nu/\xi}{1 + \alpha \psi}\right)\]

\(\times \left[Z e^\lambda H + \frac{A}{H} \frac{dH}{d\xi} \frac{K \psi^{n+1}(\mu^{(\text{eff})}_c)^{(1+1/n)}}{1 + \alpha \psi}\right].\)
Differentiating \( \nu(\xi) \) and substituting the value of \( m(r) \), we get

\[
\frac{d\nu}{d\xi} = \frac{\xi^2 \psi^n}{H}. \tag{73}
\]

In two ordinary differential equations (72) and (73), there are three unknown functions \( \nu, \psi \) and \( \Pi \). For the unique solution of these equations, we need an extra condition. To obtain this, we assume the vanishing complexity factor condition in form of dimensionless variables. This gives

\[
\frac{6 \Pi}{n \mu_c^{(\text{eff})}} + \frac{2 \xi}{n \mu_c^{(\text{eff})}} \frac{d\Pi}{d\xi} = \frac{\psi^{n-1}}{1+\alpha \psi_b} \frac{d\psi}{d\xi} + \frac{2 \Pi \xi}{H n \mu_c^{(\text{eff})}} \frac{dH}{d\xi} - \frac{1}{n \mu_c^{(\text{eff})}} \left[ \Pi^{(D)} + \Pi(f_T + \frac{1}{2} R f_Q) + \frac{1}{4 \pi H} \chi^{(D)}_3 \right]
\]

\[
= \frac{H \xi}{n \mu_c^{(\text{eff})}} \left[ \mu_c^{(\text{eff})} \psi^n \frac{d}{d\xi} \left( \frac{1}{H} \right) - \frac{1}{d\xi} \left( \frac{\Pi^{(D)}}{H} \right) \right]
\]

\[
- \frac{d}{d\xi} \left[ \frac{\Pi}{H}(f_T + \frac{1}{2} R f_Q) \right] - \frac{1}{4 \pi} \frac{d\chi^{(D)}_3}{d\xi}.
\tag{74}
\]

These differential equations have a unique solution for arbitrary values of two parameters \( n \) and \( \alpha \). Any solution of these equations provide the mass, density, radius and pressure of peculiar stellar object for the specified values of \( n \) and \( \alpha \).

Now, we consider the second case of polytropic equation of state \( P^{(\text{eff})}_r = K[\mu_b^{(\text{eff})}]^\gamma = K[\mu_b^{(\text{eff})}]^{(1+1/n)} \), where \( \mu_b \) is the baryonic mass density. Equations (72) and (74) can be written in this case as

\[
\frac{6 \Pi}{n \mu_{bc}^{(\text{eff})}} + \frac{2 \xi}{n \mu_{bc}^{(\text{eff})}} \frac{d\Pi}{d\xi} = \frac{\psi^{n-1}}{1+\alpha \psi_b} \frac{d\psi}{d\xi} + \frac{2 \Pi \xi}{H n \mu_{bc}^{(\text{eff})}} \frac{dH}{d\xi} - \frac{1}{n \mu_{bc}^{(\text{eff})}} \left[ \Pi^{(D)} + \Pi(f_T + \frac{1}{2} R f_Q) + \frac{1}{4 \pi H} \chi^{(D)}_3 \right]
\]

\[
+ \frac{\psi^{n-1}}{1+\alpha \psi_b} \frac{d\psi}{d\xi} \left( 1 + K(n + 1)(\mu_{bc}^{(\text{eff})})^{1/n} \psi_b \right)
\]

\[
+ \frac{2 \Pi \xi}{H n \mu_{bc}^{(\text{eff})}} \frac{dH}{d\xi} - \frac{3}{n \mu_{bc}^{(\text{eff})}} \left[ \Pi^{(D)} + \Pi(f_T + \frac{1}{2} R f_Q) + \frac{1}{4 \pi H} \chi^{(D)}_3 \right].
\tag{75}
\]
\[ + \frac{H \xi}{n \mu_{bc}^{(\text{eff})}} \left[ \mu_{bc}^{(\text{eff})} \psi^n_b \{ 1 + n K (\mu_{bc}^{(\text{eff})})^{1/n} \psi_b \} \frac{d}{d \xi} \left( \frac{1}{H} \right) - \frac{d}{d \xi} \left( \frac{\Pi^{(D)}}{H} \right) \right] \]

\[ - \frac{d}{d \xi} \left\{ \frac{\Pi}{H} \left( f_T + \frac{1}{2} R f_Q \right) \right\} - \frac{1}{4 \pi} \frac{d \chi^{(D)}}{d \xi} , \quad (76) \]

with \( \psi^n_b = \mu_b^{(\text{eff})}/\mu_{bc}^{(\text{eff})} \).

5 Conclusions

The aim of this work is to understand the effects of \( f(R, T, Q) \) gravity on the structure of the self-gravitating spherical object. For this purpose, we have assumed static form of the spherical metric and then assumed that it is coupled with anisotropic matter configurations. The corresponding field as well as hydrostatic equilibrium equations are derived in the realm of \( f(R, T, Q) \) theory. After using formalisms provided by Misner-Sharp and Tolman, particular relations of \( m \) and \( m_T \) are derived, respectively. Five set of \( f(R, T, Q) \) scalar variables are derived from the orthogonal decomposition of the Riemann tensor. We then studied the impact of these variables in the emergence and maintenance of homogeneous distribution of matter content over the static relativistic spheres. Herrera [1] presented the concept of the complexity factor for static anisotropic self-gravitating spherically symmetric structure. The fundamental supposition is that the system with homogeneous energy density and anisotropic pressure is less complex. Then, the one of the derived scalar factor \( Y_{TF} \) is explored in our case in order to determine the complexity factor of the system. Now, we are going to point out some important results as follows.

(i) In background of \( f(R, T, Q) \) theory, the factor \( Y_{TF} \) involves inhomogeneous energy density and locally anisotropic pressure under the influence of modified corrections.

(ii) The quantity \( Y_{TF} \) measures the Tolman mass in terms of inhomogeneous energy density and anisotropic pressure with the contribution of extra curvature terms of modified gravity.

(iii) This scalar could contain the dissipative fluxes with inhomogeneous energy density and pressure anisotropy in non-static dissipative matter distribution with extra curvature terms of modified gravity.

(iv) A new definition of complexity for spherically symmetric static self-gravitating fluids has been introduced [1], which is sharply different from the definition given in [64]. Indeed,
the new concept of complexity [1], stems from the basic assumption that one of the less complex systems corresponds to a homogeneous (in the energy density) fluid distribution with isotropic pressure. So a zero value of the complexity factor is assigned for such a distribution. Then, as an obvious candidate to measure the degree of complexity, emerges a quantity which appears in the orthogonal splitting of the Riemann tensor and that was denoted by \( Y_{TF} \) and called the complexity factor.

After establishing the modified field equations and matter function, we obtain the complexity factor \( Y_{TF} \) given in Eq. (54) from the four structure scalars. This scalar has been arrived from the orthogonal splitting of the Riemann tensor which encompasses pressure anisotropy and density inhomogeneity. Furthermore, we propose two applications of physical systems through the vanishing complexity factor condition (58) by taking \( Y_{TF} = 0 \). The first one is Gokhroo and Mehra ansatz, in which we have observed the effects of extra curvature terms in stellar objects. In second example, we have considered the polytropic equation of state and introduced new variables to write TOV equation, vanishing complexity factor condition and mass function in dimensionless form. To explain the system, the above differential equations yield the solution for some physical constraints with zero complexity factor under the effect of modified corrections. All of our results reduce to GR [1] under the constraint \( f(R, T, Q) = R \).

**Appendix A**

The effective matter variables appearing in Eqs. (10)-(12) are

\[
\mu^{(eff)} = \mu \left[ 1 + 2f_T + f_Q \left( \frac{1}{2R} R - \frac{3\nu^2}{8\epsilon\lambda} - \frac{3\nu'}{2r\epsilon\lambda} + \frac{5\lambda\nu'}{8\epsilon\lambda} - \frac{3\nu''}{8e\lambda} \right) \right.
\]

\[
+ \left. f'_Q \left( \frac{1}{r\epsilon\lambda} - \frac{\lambda'}{4e\lambda} \right) + \frac{f''_Q}{2e\lambda} \right] + \mu' \left[ f_Q \left( \frac{1}{2e\lambda} - \frac{\lambda'}{4e\lambda} \right) + f'_Q \frac{e\lambda}{x^2} \right] + \mu'' f_Q \frac{2e\lambda}{x^2}
\]

\[
+ P_r \left[ f_Q \left( \frac{\nu^2}{8e\lambda} - \frac{1}{r^2e\lambda} - \frac{\lambda'\nu'}{8e\lambda} + \frac{\lambda'}{2r\epsilon\lambda} + \frac{\nu''}{4e\lambda} \right) + f'_Q \left( \frac{\lambda'}{4e\lambda} - \frac{2}{re\lambda} \right) \right.
\]

\[
- \left. \frac{f''_Q}{2e\lambda} \right] + P'_r \left[ f_Q \left( \frac{\nu'}{2e\lambda} - \frac{1}{r^2e\lambda} - \frac{\lambda'}{2r\epsilon\lambda} \right) + \frac{f'_Q}{e\lambda} \right] - \frac{P''_r f_Q}{2e\lambda}
\]

\[
+ P_{\perp} \left[ f_Q \left( \frac{\nu'}{2e\lambda} + \frac{1}{r^2e\lambda} - \frac{\lambda'}{2r\epsilon\lambda} \right) + f'_Q \frac{r\epsilon\lambda}{e\lambda} \right] + \frac{P'_r f_Q}{r\epsilon\lambda} + \frac{R}{2} \left( \frac{f}{R} - f_R \right)
\]

\[
+ \left. f'_R \left( \frac{2}{r\epsilon\lambda} - \frac{\lambda'}{2e\lambda} \right) + f''_R \frac{e\lambda}{x^2} \right],
\]  

(77)
The quantity $Z$ arises due to non-conserved nature of $f(R, T, Q)$ theory in Eq. (14) is

$$Z = \frac{2}{(2 + R f_Q + 2 f_T)} \left[ f_Q e^{-\lambda} P_r \left( \frac{\nu' \lambda'}{r^2} - \frac{\nu' \lambda'}{r^2} + 1 \right) + f_Q e^{-\lambda} P_r \left( \frac{\nu''}{r^2} - \frac{\nu' \lambda'}{r^2} - \frac{\nu' \lambda'}{r^2} \right) + \frac{2 e^{-\lambda}}{r^3} + \frac{2 e^{-\lambda}}{r^3} \right] + \frac{f_Q e^{-\lambda}}{2} P_r \left( \frac{\nu' \lambda'}{4} - \frac{\nu' \lambda'}{4} + \frac{\nu'}{r} - \frac{f_T}{2} \right) - \mu' f_T P_r \left( \lambda' - \frac{\nu' \lambda'}{2} - \frac{1}{r} \right) - \frac{f_Q e^{-\lambda}}{r^2} \left( \frac{\nu'^2}{2} + \frac{\nu'^2}{2} \right) - f_Q e^{-\lambda} P_r \left( \frac{\nu''}{r^2} - \frac{\nu' \lambda'}{r^2} - \frac{\nu' \lambda'}{r^2} \right).$$
The quantity $D_0$ appearing in Eq. (19) can be given as follows

$$D_0 = \mu \left[ -\tilde{f}_T + \tilde{f}_Q \left( \frac{\nu^2}{8e^\lambda} + \frac{\nu'}{2re^\lambda} - \frac{\chi'\nu'}{8e^\lambda} + \frac{\nu''}{4e^\lambda} \right) \right] - \frac{\mu\nu'\tilde{f}_Q}{4} + P_r \left[ \tilde{f}_T + \tilde{f}_Q \left( \frac{1}{2}R - \frac{3\nu^2}{8e^\lambda} + \frac{\nu'}{r^2e^\lambda} + \frac{1}{3}h R - \frac{3\chi'}{8e^\lambda} + \frac{3\chi''}{2r^2e^\lambda} \right) \right] + P' \left[ \tilde{f}_Q \left( \frac{1}{r^2e^\lambda} + \frac{\nu'}{4e^\lambda} \right) \right] + P' \left[ \tilde{f}_Q \left( -\frac{\nu'}{2r^2e^\lambda} - \frac{1}{r^2e^\lambda} + \frac{\chi}{2r^2e^\lambda} \right) \right] \right. \right. $$

$$+ \frac{P'_r}{r} \left[ -R \left( \frac{\tilde{f}}{R} - \tilde{f}_R \right) \right]. $$

(81)

**Appendix B**

The extra curvature terms appearing in the expressions of structure scalars (49)-(52) are found as follows

$$\chi^{(D)}_1 = \frac{4\pi}{H} \left[ \left\{ h^a_{\rho} \Box (f_Q T^\rho) - 2h^a_{\rho} \nabla^\rho \nabla \phi(f_Q T^\rho) - h^a_{\rho} \nabla K \nabla^\rho (f_Q T^\rho) \right\} \right. $$

$$- h^a_{\rho} \nabla K \nabla \phi(f_Q T^\rho) - 2f_Q R_{\mu}^a \left( P - \frac{\Pi}{3} \right) h^\nu \right] - \frac{8\pi}{H} \left[ \left\{ \frac{R}{2} \left( \frac{f}{R} - f_R \right) + \mu f_T - \frac{1}{2} \nabla \phi(f_Q T^{\mu \nu}) \right\} \right. $$

$$- \frac{1}{2} \Box \{ f_Q (\mu - 3P) \} + 2R f_Q \left( P - \frac{\Pi}{3} \right) \nabla \phi(f_Q T^{\mu \nu}) $$

$$+ 2g^{\rho \nu} (f_Q R^{\mu \nu} + f_T g^{\mu \nu}) \frac{\partial^2 L_m}{\partial g^{\mu \nu} \partial g^{\rho \nu}} \right], $$

(82)

$$\chi^{(D)}_2 = \frac{8\pi}{H} \left[ \left\{ \frac{R}{2} \left( \frac{f}{R} - f_R \right) + \mu f_T - \frac{1}{2} \nabla \phi(f_Q T^{\mu \nu}) \right\} \right. $$

$$\times \left\{ \Box (f_Q T^\rho) - u^\alpha u^\delta \Box (f_Q T_{\alpha \delta}) - u^\beta u_\gamma \Box (f_Q T_{\beta \gamma}) + 4u_\gamma u^\delta \Box (f_Q T_{\delta}) \right\} $$

$$+ \left\{ \Box f_R - u^\alpha u^\delta \Box \nabla \phi(f_Q T^\rho - u^\beta u_\gamma \nabla \gamma \nabla \beta f_R + 4u_\gamma u^\delta \nabla \gamma \nabla \delta f_R \right\} $$

$$+ f_Q \left\{ R_{\mu}^a (Ph - \Pi^\mu) - 3R_{\beta}^\gamma u^\mu u_\gamma \right\} + f_Q \left\{ R_{\mu}^a (Ph^\mu - \Pi^\mu) \right\} $$

$$- 3R_{\mu \delta} u^\mu u^\delta + \frac{1}{2} \left( \nabla \phi(f_Q T_{\alpha \beta}) + \nabla \phi(f_Q T_{\beta \gamma}) \right) $$

$$+ 4u_\gamma u^\delta \nabla \phi(f_Q T_{\delta}) + 4u_\gamma u^\delta \nabla \phi(f_Q T_{\delta}) - u^\alpha u^\delta \nabla \phi(f_Q T_{\delta}) \right], $$

(82)
\[
\chi^{(D)}_{\alpha\beta} = \frac{-2\pi}{H} \left[ h^\alpha_\lambda h^\beta_\pi \Box (f_Q T_{\lambda\pi}) - \Box (f_Q T_{\alpha\beta}) - u_\alpha u_\beta u_\gamma u_\delta \Box (f_Q T^\gamma_\delta) \right] \\
+ \frac{4\pi}{H} \left[ (h^\alpha_\lambda h^\beta_\pi \nabla_\lambda f_R - \nabla_\lambda f_R - u_\alpha u_\beta u_\gamma u_\delta \nabla^\gamma \nabla_\delta f_R) + f_Q (h^\alpha_\lambda h^\mu_\beta R_{\lambda\mu} P - h^\mu_\beta R_{\alpha\mu} P - h^\alpha_\lambda R_{\lambda\mu} \Pi^\mu_\beta + R_{\alpha\mu} \Pi^\mu_\beta) + f_Q (h^\mu_\beta h^\alpha_\lambda R_{\mu\alpha} P - h^\mu_\beta R_{\mu\alpha} \Pi^\alpha_\beta + R_{\mu\beta} \Pi^\mu_\alpha) + \frac{1}{2} \{ h^\alpha_\lambda h^\mu_\beta \nabla_\lambda (f_Q T^\mu_\beta) + h^\alpha_\lambda h^\mu_\beta \nabla_\mu (f_Q T^\mu_\lambda) - \nabla_\mu \nabla_\beta (f_Q T^\mu_{\alpha\beta}) \} + 2 (f_Q R^{\mu\nu} + f_T R^{\mu\nu}) h^\epsilon_\beta \left( h^\pi_\alpha \frac{\partial^2 L_m}{\partial g^{\pi\alpha} \partial g^{\mu\nu}} - \frac{\partial^2 L_m}{\partial g^{\epsilon\beta} \partial g^{\mu\nu}} \right) \right].
\]

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