On the convergence of discrete-time linear systems: A linear time-varying Mann iteration converges iff the operator is strictly pseudocontractive

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Abstract—We adopt an operator-theoretic perspective to study convergence of linear fixed-point iterations and discrete-time linear systems. We mainly focus on the so-called Krasnoselskij–Mann iteration \( x(k + 1) = (1 - \alpha_k)x(k) + \alpha_k Ax(k) \), which is relevant for distributed computation in optimization and game theory, when \( A \) is not available in a centralized way. We show that convergence to a vector in the kernel of \((I - A)\) is equivalent to strict pseudocontractiveness of the linear operator \( x \mapsto Ax \). We also characterize some relevant operator-theoretic properties of linear operators via eigenvalue location and linear matrix inequalities. We apply the convergence conditions to multi-agent linear systems with vanishing step sizes, in particular, to linear consensus dynamics and equilibrium seeking in monotone linear-quadratic games.

I. INTRODUCTION

State convergence is the quintessential problem in multi-agent systems. In fact, multi-agent consensus and cooperation, distributed optimization and multi-player game theory revolve around the convergence of the state variables to an equilibrium, typically unknown a-priori. In distributed consensus problems, agents interact with their neighboring peers to collectively achieve global agreement on some value [1]. In distributed optimization, decision makers cooperate locally to agree on primal-dual variables that solve a global optimization problem [2]. Similarly, in multi-player games, selfish decision makers exchange local or semi-global information to achieve an equilibrium for their inter-dependent optimization problems [3]. Applications of multi-agent systems with guaranteed convergence are indeed vast, e.g. include power systems [4], [5], demand side management [6], network congestion control [7], [8], social networks [9], [10], robotic and sensor networks [11], [12].

From a general mathematical perspective, the convergence problem is a fixed-point problem [13], or equivalently, a zero-finding problem [14]. For example, consensus in multi-agent systems is equivalent to finding a collective state in the kernel of the Laplacian matrix, i.e., in operator-theoretic terms, to finding a zero of the Laplacian, seen as a linear operator.

Fixed-point theory and monotone operator theory are then key to study convergence to multi-agent equilibria. For instance, Krasnoselskij–Mann fixed-point iterations have been adopted in aggregative game theory [16], [17], monotone operator splitting methods in distributed convex optimization [18] and monotone game theory [19], [3], [20].

The main feature of the available results is that sufficient conditions for convergence to equilibrium for their inter-dependent optimization problems include power systems [4], [5], demand side management [6], network congestion control [7], [8], social networks [9], [10], robotic and sensor networks [11], [12].

Our main contribution is to show that the Krasnoselskij–Mann fixed-point iterations, possibly time-varying, applied on linear operators converge if and only if the associated matrix has certain spectral properties (Section III). To motivate and achieve our main result, we adopt an operator-theoretic perspective and characterize some regularity properties of linear mappings via eigenvalue location and properties, and linear matrix inequalities (Section IV). In Section VII, we conclude the paper and indicate one future research direction.

Notation: \( \mathbb{R} \), \( \mathbb{R}_{\geq 0} \) and \( \mathbb{C} \) denote the set of real, non-negative real and complex numbers, respectively. \( \mathbb{D}_{r} := \{ z \in \mathbb{C} \mid |z - (1 - r)| \leq r \} \) denotes the disk of radius \( r > 0 \) centered in \((1 - r, 0)\), see Fig. 1 for some graphical examples. \( \mathcal{H}(\|\cdot\|) \) denotes a finite-dimensional Hilbert space with norm \( \|\cdot\| \). \( \mathbb{S}_{n}^{p} \) is the set of positive definite symmetric matrices and, for \( P \in \mathbb{S}_{n}^{p+} \), \( \|x\|_{P} := \sqrt{x^{\top}Px} \). Id denotes the identity operator. \( R(\cdot) := \frac{\cos(\cdot) - \sin(\cdot)}{\sin(\cdot) + \cos(\cdot)} \) denotes the rotation operator. Given a mapping \( T : \mathbb{R}^{n} \to \mathbb{R}^{n} \), \( \text{fix}(T) := \{ x \in \mathbb{R}^{n} \mid x = T(x) \} \) denotes the set of fixed points, and \( \text{zer}(T) := \{ x \in \mathbb{R}^{n} \mid 0 = T(x) \} \) the set of zeros. Given a matrix \( A \in \mathbb{R}^{n \times n} \), \( \ker(A) := \{ x \in \mathbb{R}^{n} \mid 0 = Ax \} = \text{zer}(A \cdot) \) denotes its kernel; \( \Lambda(A) \) and \( \rho(A) \) denote the spectrum and the spectral radius of \( A \), respectively. \( 0_{N} \) and \( 1_{N} \) denote vectors with \( N \) elements all equal to 0 and 1, respectively.
II. MATHEMATICAL DEFINITIONS

A. Discrete-time linear systems

In this paper, we consider discrete-time linear time-invariant systems,

\[ x(k + 1) = Ax(k), \]

and linear time-varying systems with special structure, i.e.,

\[ x(k + 1) = (1 - \alpha_k)x(k) + \alpha_k A x(k), \]

for some positive sequence \((\alpha_k)_{k \in \mathbb{N}}\). Note that for \(\alpha_k = 1\) for all \(k \in \mathbb{N}\), the system in (2) reduces to that in (1).

B. System-theoretic definitions

We are interested in the following notion of global convergence, i.e., convergence of the state solution, independently on the initial condition, to some vector.

Definition 1 (Convergence): The system in (2) is convergent if, for all \(x(0) \in \mathbb{R}^n\), its solution \(x(k)\) converges to some \(\bar{x} \in \mathbb{R}^n\), i.e., \(\lim_{k \to \infty} \|x(k) - \bar{x}\| = 0\).

Note that in Definition 1, the vector \(\bar{x}\) can depend on the initial condition \(x(0)\). In the linear time-invariant case, \(\bar{x}\), it is known that semi-convergence holds if and only if the eigenvalues of the \(A\) matrix are strictly inside the unit disk and the eigenvalue in 1, if present, must be semi-simple, as formalized next.

Definition 2 ((Semi-) Simple eigenvalue): An eigenvalue is semi-simple if it has equal algebraic and geometric multiplicities both equal to 1.

Lemma 1: The following statements are equivalent:

i) The system in (1) is convergent;

ii) \(\rho(A) \leq 1\) and the only eigenvalue on the unit disk is 1, which is semi-simple.

C. Operator-theoretic definitions

With the aim to study convergence of the dynamics in (1), (2), in this subsection, we introduce some key notions from operator theory in Hilbert spaces.

Definition 3 (Lipschitz continuity): A mapping \(T : \mathbb{R}^n \to \mathbb{R}^n\) is \(\ell\)-Lipschitz continuous in \(H(\|\cdot\|)\), with \(\ell \geq 0\), if \(\forall x, y \in \mathbb{R}^n, \|T(x) - T(y)\| \leq \ell \|x - y\|\).

Definition 4: In \(H(\|\cdot\|)\), an \(\ell\)-Lipschitz continuous mapping \(T : \mathbb{R}^n \to \mathbb{R}^n\) is

- \(\ell\)-Contractive (\(\ell\)-CON) if \(\ell \in [0, 1)\);
- NonExpansive (NE) if \(\ell \in [0, 1]\);
- \(\eta\)-Averaged (\(\eta\)-AVG), with \(\eta \in (0, 1)\), if \(\forall x, y \in \mathbb{R}^n\)

\[ \|T(x) - T(y)\| \leq \|x - y\| - \frac{1-\eta}{\eta} \| (I - \eta T)(x) - (I - \eta T)(y) \|, \]

or, equivalently, if there exists a nonexpansive mapping \(B : \mathbb{R}^n \to \mathbb{R}^n\) and \(\eta \in (0, 1)\) such that

\[ T = (1 - \eta)I + \eta B. \]

- \(\kappa\)-strictly Pseudo-Contractive (\(\kappa\)-sPC), with \(\kappa \in (0, 1)\), if \(\forall x, y \in \mathbb{R}^n\)

\[ \|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \kappa \| (I - T)(x) - (I - T)(y) \|^2. \]

Definition 5: A mapping \(T : \mathbb{R}^n \to \mathbb{R}^n\) is:

- Contractive (CON) if there exist \(\ell \in [0, 1)\) and a norm \(\|\cdot\|\) such that it is an \(\ell\)-CON in \(H(\|\cdot\|)\);
- Averaged (AVG) if there exist \(\eta \in (0, 1)\) and a norm \(\|\cdot\|\) such that it is \(\eta\)-AVG in \(H(\|\cdot\|)\);
- strict Pseudo-Contractive (sPC) if there exists \(\kappa \in (0, 1)\) and a norm \(\|\cdot\|\) such that it is \(\kappa\)-sPC in \(H(\|\cdot\|)\).

III. MAIN RESULTS: FIXED-POINT ITERATIONS ON LINEAR MAPPINGS

In this section, we provide necessary and sufficient conditions for the convergence of some well-known fixed-point iterations applied on linear operators, i.e.,

\[ A : x \mapsto Ax, \quad \text{with } A \in \mathbb{R}^{n \times n}. \]

First, we consider the Banach–Picard iteration [14, (1.69)] on a generic mapping \(T : \mathbb{R}^n \to \mathbb{R}^n\), i.e., for all \(k \in \mathbb{N}\),

\[ x(k + 1) = T(x(k)), \]

whose convergence is guaranteed if \(T\) is averaged, see [14, Prop. 5.16]. The next statement shows that averagedness is also a necessary condition when the mapping \(T\) is linear.

Proposition 1 (Banach–Picard iteration): The following statements are equivalent:

i) \(A\) in (5) is averaged;

ii) the solution to the system

\[ x(k + 1) = Ax(k) \]

converges to some \(\bar{x} \in \text{fix}(A) = \ker(I - A).\)

If the mapping \(T\) is merely nonexpansive, then the sequence generated by the Banach–Picard iteration in (6) may fail to produce a fixed point of \(T\). For instance, this is the case for \(T = -I\). In these cases, a relaxed iteration can be used, e.g. the Krasnoselskij–Mann iteration [14, Eqn. (5.15)]. Specifically, let us distinguish the case with time-invariant step sizes, known as Krasnoselskij iteration [13, Chap. 3], and the case with time-varying, vanishing step sizes, known as Mann iteration [13, Chap. 4]. The former is defined by

\[ x(k + 1) = (1 - \alpha)x(k) + \alpha T(x(k)), \]

for all \(k \in \mathbb{N}\), where \(\alpha \in (0, 1)\) is a constant step size.

The convergence of the discrete-time system in (8) to a fixed point of the mapping \(T\) is guaranteed, for any arbitrary \(\alpha \in (0, 1)\), if \(T\) is nonexpansive [14, Th. 5.15], or if \(T\), defined from a compact, convex set to itself, is strictly pseudo-contractive and \(\alpha > 0\) is sufficiently small [13, Theorem 3.5]. In the next statement, we show that if the mapping \(T : \mathbb{R}^n \to \mathbb{R}^n\) is linear, and \(\alpha\) is chosen small.
enough, then strict pseudo-contractiveness is necessary and sufficient for convergence.

**Theorem 1 (Krasnoselski iteration):** Let $\kappa \in (0,1)$ and $\alpha \in (0,1-\kappa)$. The following statements are equivalent:

(i) $A$ in (5) is $\kappa$-strictly pseudo-contractive;

(ii) the solution to the system

$$x(k+1) = (1-\alpha) x(k) + \alpha A x(k)$$  \tag{9}

converges to some $x \in \text{fix}(A) = \ker(I-A)$. \hfill \Box

In Theorem I the admissible step sizes for the Krasnoselski iteration depend on the parameter $\kappa$ that quantifies the strict pseudo-contractiveness of the mapping $A = A^\top$. When the parameter $\kappa$ is unknown, or hard to quantify, one can adopt time-varying step sizes, e.g., the Mann iteration:

$$x(k+1) = (1-\alpha_k)x(k) + \alpha_k T(x(k))$$  \tag{10}

for all $k \in \mathbb{N}$, where the step sizes $(\alpha_k)_{k \in \mathbb{N}}$ shall be chosen as follows.

**Assumption 1 (Mann sequence):** The sequence $(\alpha_k)_{k \in \mathbb{N}}$ is such that $0 < \alpha_k \leq \alpha^\text{max} < \infty$ for all $k \in \mathbb{N}$, for some $\alpha^\text{max}$, $\lim_{k \to \infty} \alpha_k = 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$. \hfill \Box

The convergence of (10) to a fixed point of the mapping $T$ is guaranteed if $T$, defined from a compact, convex set to itself, is strictly pseudo-contractive [13, Theorem 3.5]. In the next statement, we show that if the mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear, then strict pseudo-contractiveness is necessary and sufficient for convergence.

**Theorem 2 (Mann iteration):** Let $(\alpha_k)_{k \in \mathbb{N}}$ be a Mann sequence as in Assumption I. The following statements are equivalent:

(i) $A$ in (5) is strictly pseudo-contractive;

(ii) the solution to

$$x(k+1) = (1-\alpha_k)x(k) + \alpha_k A x(k)$$  \tag{11}

converges to some $x \in \text{fix}(A) = \ker(I-A)$. \hfill \Box

IV. OPERATOR-THEORETIC CHARACTERIZATION OF LINEAR MAPPINGS

In this section, we characterize the operator-theoretic properties of linear mappings via necessary and sufficient linear matrix inequalities and conditions on the spectrum of the corresponding matrices. We exploit these technical results in Section VII to prove convergence of the fixed-point iterations presented in Section III.

**Lemma 2 (Lipschitz continuous linear mapping):** Let $\ell > 0$ and $P \in \mathbb{S}^n_{\geq 0}$. The following statements are equivalent:

(i) $A$ in (5) is $\ell$-Lipschitz continuous in $\mathcal{H}(|\cdot|_P)$;

(ii) $A^\top P A \ll \ell^2 P$. \hfill \Box

**Proof:** It directly follows from Definition 5.

**Lemma 3 (Linear contractive/nonexpansive mapping):** Let $\ell \in (0,1)$. The following statements are equivalent:

(i) $A$ in (5) is an $\ell$-contraction;

(ii) $\exists P \in \mathbb{S}^n_{\geq 0}$ such that $A^\top P A \ll \ell^2 P$;

(iii) the spectrum of $A$ is such that

$$\begin{cases} \Lambda(A) \subset \ell \mathbb{D}_1 \\
\forall \lambda \in \Lambda(A) \cap \text{bdr}(\ell \mathbb{D}_1), \ \lambda \text{ semi-simple} \end{cases}$$  \tag{12}

If $\ell = 1$, the previous equivalent statements hold if and only if $A$ in (5) is nonexpansive. \hfill \Box

**Proof:** The equivalence between (i) and (ii) follows directly by inequality (5) in Definition 4. By the Lyapunov theorem, (iii) holds if and only if the discrete-time linear system $x(k+1) = \frac{1}{\ell} A x(k)$ is (at least marginally) stable, i.e., $\Lambda(A) \subset \ell \mathbb{D}_1$ and the eigenvalues of $A$ on the boundary of the disk $\Lambda(A) \cap \text{bdr}(\ell \mathbb{D}_1)$, are semi-simple. The last statement follows by noticing that an 1-contraction mapping is nonexpansive.

**Lemma 4 (Linear averaged mapping):** Let $\eta \in (0,1)$. The following statements are equivalent:

(i) $A$ in (5) is $\eta$-averaged;

(ii) $\exists P \in \mathbb{S}^n_{\geq 0}$ such that

$$A^\top P A \ll (2\eta - 1) P + (1-\eta) (A^\top P + PA);$$

(iii) $A_\eta := A_{1-\eta} := (1-\frac{1}{\eta}) I + \frac{\eta}{\lambda} A$ is nonexpansive;

(iv) the spectrum of $A$ is such that

$$\begin{cases} \Lambda(A) \subset \mathbb{D}_\eta \\
\forall \lambda \in \Lambda(A) \cap \text{bdr}(\mathbb{D}_\eta), \ \lambda \text{ semi-simple}. \end{cases}$$  \tag{13}

**Proof:** The equivalence (i) $\iff$ (ii) follows directly by inequality (5) in Definition 4. By [14, Prop. 4.35], $A$ is $\eta$-AVG if and only if the linear mapping $A_\eta$ is NE, which proves (i) $\iff$ (iii). To conclude, we show that (iii) $\iff$ (iv). By Lemma 3 the linear mapping $A_\eta$ is NE if and only if

$$\begin{cases} \Lambda(A_\eta) \subset \mathbb{D}_1 \\
\forall \lambda \in \Lambda(A_\eta) \cap \text{bdr}(\mathbb{D}_1), \ \lambda \text{ semi-simple} \end{cases}$$  \tag{14}

$\iff$

$$\begin{cases} \Lambda(A) \subset (1-\eta)[1] + \eta \mathbb{D}_1 = \mathbb{D}_\eta \\
\forall \lambda \in \Lambda(A) \cap \text{bdr}(\mathbb{D}_\eta), \ \lambda \text{ semi-simple} \end{cases}$$  \tag{15}

where the equivalence (14) $\iff$ (15) holds because $\Lambda(A_\eta) = (1-\frac{1}{\eta})[1] + \eta \Lambda(A)$, and because the linear combination with the identity matrix does not alter the geometric multiplicity of the eigenvalues.

**Lemma 5 (Linear strict pseudocontractive mapping):** Let $\kappa, \eta \in (0,1)$. The following statements are equivalent:

(i) $A$ in (5) is $\kappa$-strictly pseudo-contractive;

(ii) $\exists P \in \mathbb{S}^n_{\geq 0}$ such that

$$(1-\kappa) A^\top P A \ll (1+\kappa) P - \kappa (A^\top P + PA);$$

(iii) $A_{\kappa}^\# := A_{1-\kappa}^\# := \kappa I + (1-\kappa) A$ is nonexpansive;

(iv) the spectrum of $A$ is such that

$$\begin{cases} \Lambda(A) \subset \mathbb{D}_{1-\kappa} \\
\forall \lambda \in \Lambda(A) \cap \text{bdr}(\mathbb{D}_{1-\kappa}), \ \lambda \text{ semi-simple} \end{cases}$$  \tag{16}

(v) $A_\alpha := A_{1-\alpha} := (1-\alpha) I + \alpha A$ is $\eta$-averaged, with $\alpha = \eta(1-\kappa) \in (0,1)$. \hfill \Box
Proof: The equivalence (i) ⇔ (ii) follows directly by inequality [4] in Definition [5]. To prove that (ii) ⇔ (iii), we note that the LMI in (10) can be recast as

\[(\kappa I + (1 - \kappa)A)^{\top} P (\kappa I + (1 - \kappa)A) < P,\]

which, by Lemma [3] holds true if and only if the mapping \(A^\kappa_\eta\) is NE.

(iii) ⇔ (iv): By Lemma [3], \(A^\kappa_\eta\) is NE if and only if

\[
\begin{cases}
\forall \lambda \in \Lambda(A^\kappa_\eta) \subset D_1 \\
\forall \lambda \in \Lambda(A) \cap \text{bdr}(D_1), \quad \lambda \text{ semi-simple} \\
\forall \lambda \in \Lambda(A) \cap \text{bdr}(D_\eta), \quad \lambda \text{ semi-simple}
\end{cases}
\]

where the equivalence (19) ⇔ (20) holds because \(\Lambda(A^\kappa_\eta) = A^\kappa_\eta := \kappa I + (1 - \kappa)A\), and because the linear combination with the identity matrix does not alter the geometric multiplicity of the eigenvalues. (iii) ⇔ (v): By Definition [4] and [14, Prop. 4.35], \(A^\kappa_\eta\) is NE if and only if \(A^\kappa_\eta := (1 - \eta)I + \eta A^\kappa_\eta\) is \(\eta\)-AVG, for all \(\eta \in (0, 1)\). Since \(\alpha = \eta(1 - \kappa)\), \(A^\kappa_\eta := (1 - \eta(1 - \kappa))Id + \eta(1 - \kappa)A\), which concludes the proof.

V. PROOFS OF THE MAIN RESULTS

Proof of Proposition [2] (Banach–Picard iteration)

We recall that, by Lemma [4], \(A\) is AVG if and only if there exists \(\eta \in (0, 1)\) such that \(\Lambda(A) \subset D_\eta\) and \(\forall \lambda \in \Lambda(A) \cap \text{bdr}(D_\eta), \lambda\) is semi-simple and we notice that \(D_\eta \cap D_1 = \{1\}\) for all \(\eta \in (0, 1)\). Hence \(A\) is averaged if and only if the eigenvalues of \(A\) are strictly contained in the unit circle except for the eigenvalue in \(\lambda = 1\) which, if present, is semi-simple. The latter is a necessary and sufficient condition for the convergence of \(x(k+1) = A \cdot x(k)\), by Lemma [1].

Proof of Theorem [7] (Krasnoselskij iteration)

(i) ⇔ (ii): By Lemma [5], \(A\) is \(k\)-sPC if and only if \((1 - \alpha)Id + \alpha A\) is \(\eta\)-AVG, with \(\alpha = \eta(1 - \kappa)\) and \(\eta \in (0, 1)\); therefore, if and only if \((1 - \alpha)Id + \alpha A\) is AVG with \(\alpha \in (0, 1 - \kappa)\). By Proposition [1], the latter is equivalent to the global convergence of the Banach–Picard iteration applied on \((1 - \alpha)Id + \alpha A\), which corresponds to the Krasnoselskij iteration on \(A\), with \(\alpha \in (0, 1 - \kappa)\).

Proof of Theorem [8] (Mann iteration)

Proof that (i) ⇒ (ii): For the sake of contradiction, suppose that \(A\) is not sPC, i.e., at least one of the following facts must hold: 1) \(A\) has an eigenvalue in \(1\) that is not semi-simple; 2) \(A\) has a real eigenvalue greater than \(1\); 3) \(A\) has a pair of complex eigenvalues \(\sigma \pm j\omega\), with \(\sigma \geq 1\) and \(\omega > 0\). We show next that each of these three facts implies non-convergence of \(10\). Without loss of generality (i.e., up to a linear transformation), we can assume that \(A\) is in Jordan normal form.

1) \(A\) has an eigenvalue in \(1\) that is not semi-simple. Due to (the bottom part of) the associated Jordan block, the vector dynamics in \(10\) contain the two-dimensional linear time-varying dynamics

\[y(k+1) = \left(1 - \alpha_k\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_k\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} y(k)\]

\[= \begin{pmatrix} 1 & \alpha_k \\ 0 & 1 \end{pmatrix} y(k).\]

For \(y_2(0) := c > 0\), we have that the solution \(y(k)\) is such that \(y_2(k) = y_2(0) > 0\) and \(y_1(k+1) = y_1(k) + c\), which implies that \(y_1(k) = y_1(0) + kc\). Thus, \(x(k)\) diverges and we have a contradiction.

2) Let \(A\) has a real eigenvalue equal to \(1 + \epsilon > 1\). Again due to (the bottom part of) the associated Jordan block, the vector dynamics in \(10\) must contain the scalar dynamics

\[s(k+1) = (1 - \alpha_k)s(k) + \alpha_k(1 + \epsilon)s(k) = (1 + \epsilon \alpha_k)s(k).\]

The solution then reads as \(s(k+1) = (\prod_{h=0}^{k} (1 + \epsilon \alpha_h)) \cdot s(0)\).

Now, since \(\epsilon \alpha_k > 0\), it holds that \(\prod_{h=0}^{k} (1 + \epsilon \alpha_h) \geq \epsilon \sum_{h=0}^{k} \alpha_h = \infty\), by Assumption [1]. Therefore, \(s(k)\) and hence \(x(k)\) diverge, and we reach a contradiction.
3) $A$ has a pair of complex eigenvalues $\sigma \pm j\omega$, with $\sigma = 1 + \epsilon \geq 1$ and $\omega > 0$. Due to the structure of the associated Jordan block, the vector dynamics in (10) contain the two-dimensional dynamics

$$z(k + 1) = \left( (1 - \alpha_k) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_k \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} \right) z(k)$$

$$= \begin{bmatrix} 1 + \epsilon \alpha_k \\ \omega \alpha_k \end{bmatrix} \begin{bmatrix} -\omega \alpha_k \\ 1 + \epsilon \alpha_k \end{bmatrix} \cdot z(k).$$

Now, we define $\rho_k := \sqrt{(1 + \epsilon \alpha_k)^2 + \omega^2 \alpha_k^2} > \sqrt{1 + \omega^2 \alpha_k^2} > 1$, and the angle $\theta_k > 0$ such that $\cos(\theta_k) = (1 + \epsilon \alpha_k)/\rho_k$ and $\sin(\theta_k) = (\omega \alpha_k)/\rho_k$, i.e., $\theta_k = \arctan(\omega \alpha_k/(1 + \epsilon \alpha_k))$. Then, we have that $z(k + 1) = \rho_k R(\theta_k)z(k)$, hence the solution $z(k)$ reads as

$$z(k + 1) = \left( \prod_{h=0}^{k} \rho_h \right) R \left( \sum_{h=0}^{k} \theta_h \right) z(0).$$

Since $\|R(\cdot)\| = 1$, if the product $\left( \prod_{h=0}^{k} \rho_h \right)$ diverges, then $z(k)$ and hence $x(k)$ diverge as well. Thus, we assume that the product $\left( \prod_{h=0}^{k} \rho_h \right)$ converges. By the limit comparison test, the series $\sum_{h=0}^{\infty} \theta_h = \sum_{h=0}^{\infty} \arctan(\omega h)_{1+\epsilon \alpha h}$ converges (diverges) if and only the series $\sum_{h=0}^{\infty} \prod_{h=0}^{k} \rho_h$ converges (diverges). The latter diverges since $\sum_{h=0}^{\infty} \frac{\alpha_h}{1+\epsilon \alpha h} \geq \omega \sum_{h=0}^{\infty} \frac{\alpha_h}{1+\epsilon \alpha h} = \frac{\omega}{1+\epsilon \alpha h} \sum_{h=0}^{\infty} \alpha_h = \infty$. It follows that $\sum_{h=0}^{\infty} \theta_h$ diverges, hence $z(k)$ keeps rotating indefinitely, which is a contradiction.

VI. APPLICATION TO MULTI-AGENT LINEAR SYSTEMS

A. Consensus via time-varying Laplacian dynamics

We consider a connected graph of $N$ nodes, associated with $N$ agents seeking consensus, with Laplacian matrix $L \in \mathbb{R}^{N \times N}$. To solve the consensus problem, we study the following discrete-time linear time-varying dynamics:

$$x(k + 1) = x(k) - \alpha_k L x(k)$$

$$= (1 - \alpha_k) x(k) + \alpha_k (I - L) x(k),$$

where $x(k) := [x_1(k), \ldots, x_N(k)]^\top \in \mathbb{R}^N$ and, for simplicity, the state of each agent is a scalar variable, $x_i \in \mathbb{R}$.

Since the dynamics in (21) have the structure of a Mann iteration, in view of Theorem 2 we have the following result.

Corollary 1: Let $(\alpha_k)_{k \in \mathbb{N}}$ be a Mann sequence. The system in (21) asymptotically reaches consensus, i.e., the solution $x(k)$ to (21) converges to $\mathbf{1}_N$, for some $\mathbf{x} \in \mathbb{R}$. □

Proof: Since the graph is connected, $L$ has one (simple) eigenvalue at 0, and $N - 1$ eigenvalues with strictly-positive real part. Therefore, the matrix $I - L$ in (21) has one simple eigenvalue in 1 and $N - 1$ with real part strictly less than 1. By Lemma 5 $(I - L)(\cdot)$ is sPC and by Theorem 2 $x(k)$ globally converges to some $\mathbf{x} \in \text{fix}(I - L) = \{0\}$, i.e., $L \mathbf{x} = 0_N$. Since $L$ is a Laplacian matrix, $L \mathbf{x} = 0_N$ implies consensus, i.e., $\mathbf{x} = \mathbf{1}_N$, for some $\mathbf{x} \in \mathbb{R}$. □

We emphasize that via (21), consensus is reached without assuming that the agents know the algebraic connectivity of the graph, i.e., the strictly-positive Fiedler eigenvalue of $L$. We have only assumed that the agents agree on a sequence of vanishing, bounded, step sizes, $\alpha_k$. However, we envision that agent-dependent step sizes can be used as well, e.g., via matricial Mann iterations, see [13, §4.1].

Let us simulate the time-varying consensus dynamics in (21) for a graph with $N = 3$ nodes, adjacency matrix $A = [a_{i,j}]$ with $a_{1,2} = a_{1,3} = 1/2$, $a_{2,3} = a_{3,1} = 1$, hence with Laplacian matrix

$L = D_{\text{out}} - A = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{bmatrix}.$

We note that $L$ has eigenvalues $\Lambda(L) = \{0, 3/2 \pm j\sqrt{3}/2\}$. Since we do not assume that the agents know about the connectivity of the graph, we simulate with step sizes that are initially larger than the maximum constant-step value for which convergence would hold. In Fig. 2 we compare the norm of the disagreement vectors, $\|L x(k)\|$, obtained with two different Mann sequences, $\alpha_k = 2/k$ and $\alpha_k = 2/\sqrt{k}$, respectively. We observe that convergence with small tolerances is faster in the latter case with larger step sizes.

B. Two-player zero-sum linear-quadratic games:
Non-convergence of projected pseudo-gradient dynamics

We consider two-player zero-sum games with linear-quadratic structure, i.e., we consider $N = 2$ agents, with cost functions $f_1(x_1, x_2) := x_1^\top C x_2$ and $f_2(x_1, x_2) := -x_2^\top C^\top x_1$, respectively, for some square matrix $C = C^\top \neq 0$. In particular, we study discrete-time dynamics for solving the Nash equilibrium problem, that is the problem to find a pair $(x_1^*, x_2^*)$ such that:

$$\begin{align*}
  x_1^* &\in \arg\min_{x_1 \in \mathbb{R}^n} f_1(x_1, x_2^*) \\
  x_2^* &\in \arg\min_{x_2 \in \mathbb{R}^n} f_2(x_1^*, x_2).
\end{align*}$$

A classic solution approach is the pseudo-gradient method, namely the discrete-time dynamics

$$x(k + 1) = x(k) - \alpha_k F x(k)$$

$$= (1 - \alpha_k) x(k) + \alpha_k (I - F) x(k),$$

Fig. 2. Disagreement vector norm versus discrete time. Since the mapping $\text{Id} - L$ is strictly pseudocontractive, consensus is asymptotically reached.
where \( F \) is the so-called pseudo-gradient mapping of the game, which in our case is defined as
\[
F(x) := \begin{bmatrix}
\nabla x_1 f_1(x_1, x_2) \\
\nabla x_2 f_2(x_1, x_2)
\end{bmatrix} = \begin{bmatrix}
  Cx_2 \\
  -Cx_1
\end{bmatrix} \otimes C x,
\]
and \((\alpha_k)_{k \in \mathbb{N}}\) is a sequence of vanishing step sizes, e.g., a Mann sequence. In our case, \((x_1^*, x_2^*)\) is a Nash equilibrium if and only if \([x_1^*; x_2^*] \in \text{fix}(\text{Id} - F) = \text{zer}(F)\) [19, Th. 1].

By Theorem 2, convergence of the system in (22) holds if and only if \(I - F\) is strictly pseudocontractive. This result implies that Laplacian-driven linear time-varying consensus dynamics with Mann step sizes do converge. It also implies that projected pseudo-gradient dynamics for Nash equilibrium seeking in monotone games do not necessarily converge.

Future research will focus on studying convergence of other, more general, linear fixed-point iterations and of discrete-time linear systems with uncertainty, e.g., polytopic.

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