Infinite-time admissibility under compact perturbations

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We investigate the behavior of infinite-time admissibility under compact perturbations. We show, by means of two completely different examples, that infinite-time admissibility is not preserved under compact perturbations $Q$ of the underlying semigroup generator $A$, even if $A$ and $A + Q$ both generate strongly stable semigroups.

Index terms: infinite-time admissibility, compact perturbations, stabilization of collocated linear systems

1 Introduction

In this note, we investigate the behavior of infinite-time admissibility under compact perturbations of the underlying semigroup generator. So, we consider semigroup generators $A : D(A) \subset X \to X$ (with $X$ a Hilbert space) and possibly unbounded control operators $B$ (defined on another Hilbert space $U$) and we ask how the property of infinite-time admissibility of $B$ behaves under compact perturbations of the generator $A$. Infinite-time admissibility of $B$ for $A$ means that for every control input $u \in L^2([0, \infty), U)$ the mild solution of the initial value problem

$$x' = Ax + Bu(t) \quad \text{and} \quad x(0) = 0 \quad (1.1)$$

is a bounded function from $[0, \infty)$ with values in $X$. (A priori, the mild solution has values only in the extrapolation space $X_{-1}$ of $A$ and, a fortiori, need not be bounded in the norm of $X$, of course.)

It is well-known that (finite-time) admissibility is preserved under very general perturbations $Q$ of the generator $A$, in particular, under bounded perturbations. It is also clear that infinite-time admissibility, by contrast, is not preserved under bounded perturbations. Just think of a generator $A$ of an exponentially stable semigroup and a bounded perturbation $Q$ (for example, a sufficiently large multiple of the identity) such that $A + Q$ has spectral points in the right half-plane.
In this note, we will show by way of two completely different kinds of examples that infinite-time is also not preserved under compact perturbations $Q$ which are such that both $A$ and $A + Q$ generate strongly stable (but not exponentially stable) semigroups. So, in other words, we show that there exist semigroup generators $A$ and $A + Q$ with $Q$ being compact and a control operator $B$ such that

- the semigroups $e^{At}$ and $e^{(A+Q)t}$ are strongly stable but not exponentially stable
- $B$ is infinite-time admissible for $A$ but not infinite-time admissible for $A + Q$.

In our first – more elementary – example, we will use an old and well-known result from the 1970s, namely a stabilization result for collocated linear systems. In that example, the compact perturbation $Q$ will be of rank 1 and the control operator $B$ will be bounded. In particular, none of the technicalities coming along with unbounded control operators will bother us there. In our second – less elementary – example, we will use a more advanced result from the 1990s, namely a characterization of infinite-time admissibility for diagonal semigroup generators. In that example, the control operator $B$ will be unbounded and the compact perturbation $Q$ will be of rank $\infty$.

In the entire note, we will use the following notation.

$\mathbb{R}_0^+ := [0, \infty)$, \hspace{1em} $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Re } z > 0\}$, \hspace{1em} $\mathbb{C}^- := \{z \in \mathbb{C} : \text{Re } z < 0\}$.

As usual, $L(X,Y)$ denotes the Banach space of bounded linear operators between two Banach spaces $X$ and $Y$ and $\|\cdot\|_{X,Y}$ stands for the operator norm on $L(X,Y)$. Also, $\|u\|_2$ denotes the norm of a square-integrable function $u \in L^2(\mathbb{R}_0^+, U)$ with values in the Banach space $U$. And finally, for a semigroup generator $A$ and bounded operators $B, C$ between appropriate spaces, the symbol $\mathcal{S}(A, B, C)$ will stand for the state-linear system

\[ x' = Ax + Bu(t) \quad \text{with} \quad y(t) = Cx(t). \]

2 Some basic facts about admissibility and infinite-time admissibility

In this section, we briefly recall the definition of and some basic facts about admissibility and infinite-time admissibility. If $A : D(A) \subset X \to X$ is a semigroup generator on the Hilbert space $X$ and $X_{-1}$ is the corresponding extrapolation space, then an operator $B \in L(U, X_{-1})$ (with $U$ another Hilbert space) is called control operator for $A$. Also, $B$ is called a bounded control operator iff $B \in L(U, X)$ and an unbounded control operator iff $B \in L(U, X_{-1}) \setminus L(U, X)$. See [8] (Section 2.10) or [4] (Section II.5) for basic facts about extrapolation spaces.

**Definition 2.1.** Suppose $A : D(A) \subset X \to X$ is a semigroup generator on $X$ and $B \in L(U, X_{-1})$, where $X, U$ are both Hilbert spaces. Then $B$ is called admissible for $A$
iff for every $u \in L^2(\mathbb{R}_0^+, U)$

$$(0, \infty) \ni t \mapsto \Phi_t(u) := \int_0^t e^{A\cdot t} Bu(s) \, ds$$

(2.1)

is a function with values in $X$, where $A_{-1}$ is the generator of the continuous extension of the semigroup $e^A$ to $X_{-1}$.

Clearly, for a given semigroup generator $A$ every bounded control operator $B \in L(U, X)$ is admissible (because $e^{A_{-1}\cdot s}|_X = e^{A\cdot s}$ for $s \in \mathbb{R}_0^+$). It should also be noted that if $B \in L(U, X_{-1})$ is admissible for $A$, then for every $t \in (0, \infty)$ the linear operator $L^2(\mathbb{R}_0^+, U) \ni u \mapsto \Phi_t(u) \in X$ defined in (2.1) is closed and thus continuous by the closed graph theorem. Consequently, $B \in L(U, X_{-1})$ is admissible for $A$ if and only if

$$
\Phi_t \in L(L^2(\mathbb{R}_0^+, U), X) \quad (t \in (0, \infty)).
$$

(2.2)

**Definition 2.2.** Suppose $A : D(A) \subset X \to X$ is a semigroup generator on $X$ and $B \in L(U, X_{-1})$, where $X, U$ are both Hilbert spaces. Then $B$ is called infinite-time admissible for $A$ iff for every $u \in L^2(\mathbb{R}_0^+, U)$

$$(0, \infty) \ni t \mapsto \Phi_t(u) := \int_0^t e^{A\cdot t} Bu(s) \, ds$$

(2.3)

is a function with values in $X$ that is bounded (in the norm of $X$), where $A_{-1}$ is the generator of the continuous extension of the semigroup $e^A$ to $X_{-1}$.

Clearly, if $B \in L(U, X_{-1})$ is infinite-time admissible for a given semigroup generator $A$, then it is also admissible for $A$. It should also be noted that, by the uniform boundedness principle, $B \in L(U, X_{-1})$ is infinite-time admissible for $A$ if and only if

$$
\Phi_t \in L(L^2(\mathbb{R}_0^+, U), X) \quad (t \in (0, \infty)) \quad \text{and} \quad \sup_{t \in (0, \infty)} \|\Phi_t\|_{L^2(\mathbb{R}_0^+, U), X} < \infty. \quad (2.4)
$$

Some authors [8], [2], [10] use the term input-stability for the system $\mathfrak{S}(A, B)$ instead of infinite-time admissibility.

It is well-known that admissibility is preserved under bounded perturbations.

**Proposition 2.3.** Suppose $A : D(A) \subset X \to X$ is a semigroup generator on $X$ and $B \in L(U, X_{-1})$, where $X, U$ are both Hilbert spaces. Also, let $Q \in L(X)$. Then $B$ is admissible for $A$ if and only if $B$ is admissible for $A + Q$.

In fact, the conclusion of this proposition remains true for much more general perturbations $Q$, namely for perturbations of the (feedback) form $Q = B_0C_0$, where $B_0 \in L(U_0, X_{-1})$ is an admissible control operator for $A$ and $C_0 \in L(X, U_0)$ with $U_0$ an arbitrary Hilbert space. See Corollary 5.5.1 from [9], for instance.

**Proposition 2.4.** Suppose $A : D(A) \subset X \to X$ is the generator of an exponentially stable semigroup on $X$ and $B \in L(U, X_{-1})$ is admissible for $A$. Then $B$ is even infinite-time admissible for $A$. 

3
See Proposition 4.4.5 in [9], for instance, and notice that for bounded control operators $B$ the above proposition is trivial. In view of that proposition, it is clear that infinite-time admissibility – unlike admissibility – is not preserved under bounded perturbations. Choose, for example, a bounded generator $A$ of an exponentially stable semigroup and let $Q := -A \in L(X)$ and $B := I \in L(X, X)$ (identity operator on $X$).

3 An example using a stabilization result for collocated linear systems

3.1 Stabilization of collocated linear systems

We will use the following well-known stabilization result for collocated systems, that is, systems of the form $\mathcal{S}(A, B, B^*)$ with a bounded control operator $B$. It essentially goes back to [1] (Corollary 3.1) and, in the form below, can be found in [8] (Lemma 2.2.6), for instance. (Actually, for the more general version with the countability assumption on $\sigma(A_0) \cap \mathbb{R}$ we have to refer to [10], but this more general version will not be used in the sequel.)

**Theorem 3.1.** Suppose $A_0$ is a contraction semigroup generator on a Hilbert space $X$ with compact resolvent (or, more generally, with $\sigma(A_0) \cap \mathbb{R}$ being countable). Suppose further that $B \in L(U, X)$ with another Hilbert space $U$ and that $\mathcal{S}(A_0, B, B^*)$ is approximately controllable in infinite time or approximately observable in infinite time. Then

(i) $B$ is infinite-time admissible for $A_0 - BB^*$, more precisely,

$$\left\| \int_0^t e^{(A_0 - BB^*)s} Bu(s) \right\|_{X}^2 \leq \frac{1}{2} \| u \|_2^2 \quad (u \in L^2(\mathbb{R}_0^+, U), t \in \mathbb{R}_0^+).$$

(ii) $e^{(A_0 - BB^*)t}$ is a strongly stable contraction semigroup on $X$.

A far-reaching generalization of this result to the case of unbounded control operators was obtained by Curtain and Weiss [10]. See Theorem 5.1 and 5.2 in conjunction with Proposition 1.5 from [10]. We also refer to [2] for a parallel result on exponential stabilization.

3.2 Infinite-time admissibility under compact perturbations

**Example 3.2.** Set $X := \ell^2(\mathbb{N}, \mathbb{C})$ and let $A_0 : D(A_0) \subset X \to X$ be defined by

$$A_0 x := (\lambda_{0k} x_k)_{k \in \mathbb{N}} \quad (x \in D(A_0)),$$

where $D(A_0) := \{(x_k) \in X : (\lambda_{0k} x_k) \in X\}$ and $\lambda_{0k} := -\alpha_k + \beta_k$ with

$$\text{Re} \lambda_{0k} = -\alpha_k := -1/k \quad (k \in \mathbb{N}) \quad \text{and} \quad \text{Im} \lambda_{0k} = \beta_k \to \infty \quad (k \to \infty).$$
Set $U := \mathbb{C}$ and let $B : U \to \mathbb{C}^\mathbb{N}$ be defined by

$$Bu := (ub_k)_{k \in \mathbb{N}} \quad (u \in U),$$

where

$$b_k := 1/k^{3/8} \quad (k \in I_1) \quad \text{and} \quad b_k := 1/k \quad (k \in I_2)$$

$I_1 := \{l^2 : l \in \mathbb{N}\}$ and $I_2 := \mathbb{N} \setminus I_1$.

Clearly, $(b_k) \in X$ and therefore $B \in L(U, X)$. We now define

$$A := A_0 - BB^* \quad \text{and} \quad A' := A_0$$

and show, in various steps, that $A$ and $A'$ are generators of strongly but not exponentially stable contraction semigroups on $X$, that $A' = A + Q$ for a compact perturbation $Q$ of rank one, and that $B$ is infinite-time admissible for $A$ but not infinite-time admissible for $A'$.

As a first step, we observe that $A' = A + Q$ with $Q := BB^*$ and that $Q$ has rank one (because the same is true for $B$), whence $Q$ is compact.

As a second step, we observe from

$$\lambda_{0k} \in \mathbb{C}^- \quad (k \in \mathbb{N}) \quad \text{and} \quad \sup \{\Re \lambda_{0k} : k \in \mathbb{N}\} = 0$$

that $A'$ is the generator of a strongly stable but not exponentially stable contraction semigroup on $X$.

As a third step, we show that $B$ is not infinite-time admissible for $A'$. In view of (2.4) we have to show that

$$\sup_{\|u\|_2 = 1} \sup_{t \in (0, \infty)} \left\| \int_0^t e^{A_0 s} Bu(s)_{X} \right\|_X = \infty.$$  (3.1)

We first observe by Fatou’s lemma that

$$\liminf_{t \to \infty} \left\| \int_0^t e^{A_0 s} Bu(s)_{X} \right\|_X^2 \geq \sum_{k \in \mathbb{N}} \left\| \int_0^\infty u(s) e^{\lambda_{0k} s} \right\|_X^2 |b_k|^2$$

$$\geq \left\| \int_0^\infty u(s) e^{\lambda_{0n} s} \right\|_X^2 |b_n|^2$$  (3.2)

for every $u \in L^2(\mathbb{R}_0^+, U)$ and $n \in \mathbb{N}$. Setting $u_n(s) := n^{-1/2} \chi_{[0,n]}(s) \cdot e^{-\beta n s}$ for $s \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$, we see that

$$\|u_n\|_2 = 1$$

$$\left\| \int_0^n u_n(s) e^{\lambda_{0n} s} \right\|_X^2 = \frac{1}{n} \left\| \int_0^n e^{-\alpha n s} \right\|_X^2 = \frac{1}{\alpha_n^2} n (1 - e^{-\alpha n})^2 = n(1 - e^{-1})^2$$  (3.4)
for every \( n \in \mathbb{N} \). Combining now (3.2), (3.3) and (3.4) we get

\[
\sup_{\|u\|_2=1} \left\| \int_0^t e^{A_0 s} Bu(s) \, ds \right\|_X^2 \geq \sup_{n \in \mathbb{N}} \left( \liminf_{t \to \infty} \left\| \int_0^t e^{A_0 s} Bu_n(s) \, ds \right\|_X \right)^2 \\
\geq (1 - e^{-1})^2 \sup_{n \in \mathbb{N}} (n|b_n|^2).
\]

Since \( \sup_{n \in \mathbb{N}} (n|b_n|^2) \geq \sup_{n \in I_1} (n|b_n|^2) = \infty \), the desired relation (3.1) follows.

As a fourth step, we show that \( B \) is infinite-time admissible for \( A \) and that \( A \) is the generator of a strongly stable contraction semigroup on \( X \). In order to do so, we apply the stabilization theorem above (Theorem 3.1). Since \( \Re \lambda_0 k \leq 0 \) \((k \in \mathbb{N})\) and \( |\lambda_0 k| \to \infty \) \((k \to \infty)\), we see that \( A_0 \) is a contraction semigroup generator on \( X \) with compact resolvent, and since the eigenvalues \( \lambda_0 k \) of \( A_0 \) are pairwise distinct and \( b_k \neq 0 \) for every \( k \in \mathbb{N} \), we see that the collocated linear system \( \mathcal{S}(A, B, B^*) \) is approximately controllable and approximately observable in infinite time (Theorem 4.2.3 of [3]).

As a fifth and last step, we convince ourselves that the semigroup generated by \( A \) is not exponentially stable. Assume the contrary. Then there exist \( M \geq 1 \) and \( \omega < 0 \) such that \( \{ z \in \mathbb{C} : \Re z > \omega \} \subset \rho(A) \) and

\[
\| (A - z)^{-1} \| \leq \frac{M}{\Re z - \omega} \quad (\Re z > \omega).
\]

So, since \( \Re \lambda_0 n \to 0 \) as \( n \to \infty \), we conclude that

\[
\limsup_{n \to \infty} \| (A - \lambda_0 n)^{-1} \| \leq \limsup_{n \to \infty} \frac{M}{\Re \lambda_0 n - \omega} = \frac{M}{|\omega|}.
\]

We now observe that

\[
(A - \lambda_0 n)e_n = -BB^* e_n = -b_n \cdot b \to 0 \quad (n \to \infty).
\]

Combining (3.5) and (3.6) we arrive at

\[
1 = \limsup_{n \to \infty} \|e_n\| = \limsup_{n \to \infty} \| (A - \lambda_0 n)^{-1} b_n \cdot b \| \leq \frac{M}{|\omega|} \limsup_{n \to \infty} \| b_n \cdot b \| = 0.
\]

Contradiction! \( \blacksquare \)

4 An example using an admissibility result for diagonal linear systems

4.1 Characterization of infinite-time admissibility

We will use the following well-known characterization of infinite-time admissibility for diagonal semigroup generators \( A_0 \). It essentially goes back to [6] (Proposition 2.2) and can also be found in [9] (Theorem 5.3.9 in conjunction with Remark 4.6.5), for instance.
Theorem 4.1. Suppose \( X = \ell^2(I, \mathbb{C}) \) with a countable infinite index set \( I \) and let \( A_0 : D(A_0) \subset X \to X \) be the diagonal operator given by

\[
A_0 x := (\lambda_0 x_k)_{k \in I} \quad (x \in D(A_0)),
\]

where \( D(A_0) := \{ (x_k) \in X : (\lambda_0 x_k) \in X \} \) and \( \lambda_0 \in \mathbb{C}^+ \) for every \( k \in I \). Suppose further that \( B \in L(U, X_{-1}) \) with \( U := \mathbb{C} \), that is,

\[
Bu = (ub_k)_{k \in I} \quad (u \in U)
\]

for a unique sequence \( (b_k) \in X_{-1} = \{ (c_k) \in \mathbb{C}^I : \sum_{k \in I} |c_k|^2 / (1 + |\lambda_k|^2) < \infty \} \). Then the following statements are equivalent:

(i) \( B \) is infinite-time admissible for \( A_0 \)

(ii) there exists a constant \( M \in \mathbb{R}_0^+ \) such that

\[
\sum_{k \in I} \frac{|b_k|^2}{|z - \lambda_0|^2} \leq \frac{M}{\text{Re } z} \quad (z \in \mathbb{C}^+) \quad (4.1)
\]

Clearly, in the situation of the above theorem the condition (ii) is equivalent to the existence of a constant \( M \in \mathbb{R}_0^+ \) such that

\[
\left\| (z - A)^{-1} B \right\|_{U, X} \leq \frac{M}{\sqrt{\text{Re } z}} \quad (z \in \mathbb{C}^+) \quad (4.1)
\]

A far-reaching generalization of the above theorem to the case of general contraction semigroup generators \( A_0 \) on a separable Hilbert space \( X \) was obtained by Jacob and Partington \[6\]. See Theorem 1.3 from \[6\]. It states that for a contraction semigroup generator \( A_0 \) on a separable Hilbert space \( X \) a control operator \( B \in L(U, X_{-1}) \) with \( U := \mathbb{C} \) is infinite-time admissible if and only if there is a constant \( M \in \mathbb{R}_0^+ \) such that the resolvent estimate (4.1) is satisfied. We also refer to \[7\] and \[9\] (Section 5.6) for an overview of many more admissibility results, for example, for infinite-dimensional input-value spaces \( U \).

4.2 Infinite-time admissibility under compact perturbations

Example 4.2. Set \( X := \ell^2(\mathbb{Z}, \mathbb{C}) \) and let \( A : D(A) \subset X \to X \) and \( A' : D(A') \subset X \to X \) be defined by

\[
Ax := (\lambda_k x_k)_{k \in \mathbb{Z}} \quad (x \in D(A)) \quad \text{and} \quad A'x := (\lambda'_k x_k)_{k \in \mathbb{Z}} \quad (x \in D(A'))
\]

where \( D(A) := \{ (x_k) \in X : (\lambda_k x_k) \in X \} \) and \( D(A') := \{ (x_k) \in X : (\lambda'_k x_k) \in X \} \) with

\[
\lambda_k := \begin{cases} 
-1/k^{1/2} + 2k, & k \in \mathbb{N} \\
-(|k| + 1)^{1/2}, & k \in -\mathbb{N}_0
\end{cases} \quad \text{and} \quad 
\lambda'_k := \begin{cases} 
-e^{-k} + 2k, & k \in \mathbb{N} \\
-(|k| + 1)^{1/2}, & k \in -\mathbb{N}_0
\end{cases}
\]
Set \( U := \mathbb{C} \) and let \( B : U \to \mathbb{C}^\mathbb{Z} \) be defined by

\[
Bu := (ub_k)_{k \in \mathbb{Z}} \quad (u \in U),
\]

where

\[
b_k := 1/k \quad (k \in \mathbb{N}), \quad b_0 := 0, \quad b_k := 1/|k|^{1/2} \quad (k \in \mathbb{N}).
\]

Clearly, \( \sum_{k \in \mathbb{Z}} |b_k|^2/(1 + |\lambda_k|^2) < \infty \) and \( \sum_{k \in \mathbb{Z}} |b_k|^2 = \infty \) whence \( (b_k) \in X_{-1} \setminus X \). And therefore

\[
B \in L(U, X_{-1}) \setminus L(U, X).
\]

We now show, in various steps, that \( A \) and \( A' \) are generators of strongly but not exponentially stable contraction semigroups on \( X \), that \( A' = A + Q \) for a compact perturbation \( Q \) of infinite rank, and that \( B \) is infinite-time admissible for \( A \) but not infinite-time admissible for \( A' \).

As a first step, we observe from

\[
\lambda_k, \lambda'_k \in \mathbb{C}^- \quad (k \in \mathbb{Z}) \quad \text{and} \quad \sup \{ \text{Re} \lambda_k : k \in \mathbb{Z} \}, \sup \{ \text{Re} \lambda'_k : k \in \mathbb{Z} \} = 0
\]

that \( A \) and \( A' \) are generators of strongly stable but not exponentially stable contraction semigroups on \( X \).

As a second step, we observe that \( A' = A + Q \) for a compact operator \( Q \) of infinite rank. Indeed, the operator \( Q : X \to X \) defined by

\[
Qx := ((\lambda'_k - \lambda_k)x_k)_{k \in \mathbb{Z}} \quad (x \in X)
\]

is a bounded operator on \( X \) because \( (\lambda'_k - \lambda_k)_{k \in \mathbb{Z}} \) is a bounded sequence. Also, \( Q \) is the limit in norm operator topology of the finite-rank operators \( Q_N : X \to X \) defined by

\[
Q_Nx := (\ldots, 0, 0, (\lambda'_1 - \lambda_1)x_1, \ldots, (\lambda'_N - \lambda_N)x_N, 0, 0, \ldots) \quad (x \in X)
\]

and therefore \( Q \) is compact, as desired.

As a third step, we show that \( B \) is infinite-time admissible for \( A \). We have that

\[
\sum_{k \in \mathbb{Z}} \frac{|b_k|^2}{|z - \lambda_k|^2} \leq \sum_{k \in \mathbb{Z}} \frac{|b_k|^2}{(\text{Re } z + |\text{Re } \lambda_k|)^2} \leq \frac{1}{2 \text{Re } z} \sum_{k \in \mathbb{Z}} \frac{|b_k|^2}{|\text{Re } \lambda_k|} \quad (4.2)
\]

for every \( z \in \mathbb{C}^+ \) and that

\[
M := \sum_{k \in \mathbb{Z}} \frac{|b_k|^2}{|\text{Re } \lambda_k|} < \infty. \quad (4.3)
\]

So, by the admissibility theorem above (Theorem 4.1), the claimed infinite-time admissibility of \( B \) for \( A \) follows from (4.2) and (4.3).
As a fourth and last step, we show that $B$ is not infinite-time admissible for $A'$. We have that

$$
\sum_{k \in \mathbb{Z}} \frac{|b_k|^2}{|z - \lambda_k|^2} \geq \frac{|b_n|^2}{|z - \lambda_n|^2} = \frac{1}{(\text{Re } z + e^{-n})^2 + (\text{Im } z - n)^2 n^2} \quad (4.4)
$$

for every $z \in \mathbb{C}^+$ and $n \in \mathbb{N}$. Choosing $z_n := e^{-n} + \beta n \in \mathbb{C}^+$ for $n \in \mathbb{N}$, we see from (4.4) that

$$
\sup_{z \in \mathbb{C}^+} \left( \text{Re } z \sum_{k \in \mathbb{Z}} \frac{|b_k|^2}{|z - \lambda_k|^2} \right) \geq \sup_{n \in \mathbb{N}} \left( \frac{\text{Re } z_n}{(\text{Re } z_n + e^{-n})^2 + (\text{Im } z_n - n)^2 n^2} \right) = \sup_{n \in \mathbb{N}} \frac{e^n}{4n^2} = \infty. \quad (4.5)
$$

So, by the admissibility theorem above (Theorem 4.1), $B$ is not infinite-time admissible for $A'$, as desired. ▮

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