Abstract. Structure of the center of the quantum algebra $U_q(sl_2)$ is analyzed in detail for both cases when deformation parameter $q$ is/is not a root of unity. It is shown that the result can be obtained in a pure algebraic way without considering representation theory.

1. Introduction

In the past several decades many types of quantum deformations and other similar types of algebras appeared in the literature and shown its importance for various physics fields. For being able to take advantage of such particular structure, one often first needs to develop representation theory of the structure. Detailed knowledge about the properties of the center is important for developing the representation theory. On the example of the simplest quantum group $U_q(sl_2)$ we show how one can obtain information about the center having no preliminary knowledge about the representation theory. Similar approach can be successful in the case of the other types of algebras (see e. g. [1]).

$U_q(sl_2)$ is complex associative algebra with generators $E$, $F$, $K$, $K^{-1}$ and relations [2]

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}. \quad (1)$$

Poincaré-Birkhoff-Witt theorem is valid for this algebra. That means that the elements $E^kF^lK^m$, where $k, l \in \{0, 1, \ldots\}$ and $m \in \mathbb{Z}$ constitute vector space basis of the algebra. It can be shown by induction that formulas (1) imply

$$[E,F]^m = [m]_q E^{m-1}[K;1-m],$$

$$[E^n,F] = [n]_q E^{n-1}[K;n-1],$$

$$E^n F^m = \sum_{r=0}^{n} \frac{[n]_q! [m]_q!}{[r]_q! [n-r]_q! [m-r]_q!} E^{m-r} E^{n-r} \times [K; n-m] [K; n-m-1] \ldots [K; n-m-r+1], \quad n \leq m. \quad (2)$$
where \([a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}, \) \([m]_q! = [1]_q[2]_q \cdots [m]_q\) and \([K; n] = \frac{Kq^n - K^{-1}q^{-n}}{q - q^{-1}}.\) Using commutation relations one can easily see that the Casimir element

\[
C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}
\]  

lies in the center of the algebra, i.e. it commutes with \(E, F, K\) and \(K^{-1}.
\]

2. Center of \(U_q(\mathfrak{sl}_2)\) when \(q\) is not a root of unity

When \(q\) is not a root of unity, the center of the algebra \(U_q(\mathfrak{sl}_2)\) is generated by the element \(C_q\) (see also theorem 45’ in sect. 6.3.4 of \([2]\)). We will prove now this fact in purely algebraic way.

First let us choose the different vector space basis of \(U_q(\mathfrak{sl}_2)\), namely the set

\[
\{C_q^α E^k K^m \mid α, k ≥ 0, m ∈ \mathbb{Z}\} ∪ \{C_q^α F^k K^m \mid α ≥ 0, k ≥ 1, m ∈ \mathbb{Z}\}.
\]  

Each element \(X\) from the algebra can be then written as a linear combination of the elements from (4), that is

\[
X = \sum_{α,k,m} c_{α,k,m} C_q^α E^k K^m + \sum_{α,k,m,k>0} d_{α,k,m} C_q^α F^k K^m,
\]

\(c_{α,k,m}\) and \(d_{α,k,m}\) being arbitrary complex coefficients. For any element \(X\) of the form (5) and for any \(s ∈ \{0, 1, \ldots\}\) let us define

\[
\deg_s X = \sum_{α,k,m,2α+k+|m|=s} c_{α,k,m} C_q^α E^k K^m + \sum_{α,k,m,2α+k+|m|=s,k>0} d_{α,k,m} C_q^α F^k K^m
\]

and also

\[
\maxdeg X = \max\{2α + k + |m| \mid c_{α,k,m} ≠ 0 \text{ or } d_{α,k,m} ≠ 0\}.
\]

We have, for example,

\[
\maxdeg 0 = -\infty,
\]

\[
\maxdeg 1 = 0,
\]

\[
\maxdeg E = \maxdeg F = \maxdeg K = \maxdeg K^{-1} = 1,
\]

\[
\maxdeg C_q = 2,
\]

\[
\maxdeg(E^2 + 2EK^{-3} - C_qF^2) = 4,
\]

etc. Now let \(X\) belong to the center of the algebra. We want to show that the commutation relations

\[
[X, E] = 0, \quad [X, F] = 0, \quad [X, K] = 0
\]

imply

\[
c_{α,k,m} = 0 \quad \text{for all } (k, m) ≠ (0, 0) \quad \text{and} \quad d_{α,k,m} = 0 \quad \text{for all } k > 0.
\]

Let us remove from \(X\) the terms for which \((k, m) = (0, 0)\), denote

\[
X' = \sum_{α,k,m, (k,m)≠(0,0)} c_{α,k,m} C_q^α E^k K^m + \sum_{α,k,m, k>0} d_{α,k,m} C_q^α F^k K^m.
\]
We want to show $X' = 0$. We have

$$[X', K] = \sum_{\alpha, k, m \atop (k, m) \neq (0, 0)} c_{\alpha, k, m} C_q^{\alpha} [E^k, K] K^m + \sum_{\alpha, k, m \atop k > 0} d_{\alpha, k, m} C_q^{\alpha} [F^k, K] K^m.$$  

Let us assume $s = \text{maxdeg} X' > 0$. It is sufficient to show that $\text{deg}_s X' = 0$ (contradiction). By commuting with generators $E$, $F$, $K^{\pm 1}$ the maximum degree of $X'$ can rise up to $s + 1$. By induction we have from (1) for any $m \in \mathbb{Z}$

$$K^m E = q^{2m} E K^m, \quad K^m F = q^{-2m} F K^m.$$  

Similarly for all $k \geq 0$ we have

$$KE^k = q^{2k} E^k K, \quadKF^k = q^{-2k} F^k K.$$  

Hence

$$[K^m, E] = (q^{2m} - 1) E K^m, \quad [K^m, F] = (q^{-2m} - 1) F K^m,$$

$$[E^k, K] = (1 - q^{2k}) E^k K, \quad [F^k, K] = (1 - q^{-2k}) F^k K.$$  

Now using (8) we have

$$\text{deg}_{s+1}[X', K] = \text{deg}_{s+1} \sum_{\alpha, k, m \atop (k, m) \neq (0, 0)} c_{\alpha, k, m} C_q^{\alpha} [E^k, K] K^m + \sum_{\alpha, k, m \atop k > 0} d_{\alpha, k, m} C_q^{\alpha} [F^k, K] K^m$$

$$= \sum_{\alpha, k, m \atop 2\alpha + k + |m| = s} c_{\alpha, k, m} (1 - q^{2k}) C_q^{\alpha} E^k K^{m+1} + \sum_{\alpha, k, m \atop 2\alpha + k + |m| = s} d_{\alpha, k, m} (1 - q^{-2k}) C_q^{\alpha} F^k K^{m+1}.$$  

Because the elements in the sum (9) are linear independent, from the condition $\text{deg}_{s+1}[X', K] = 0$ we get the system of linear equations

$$c_{\alpha, k, m} = d_{\alpha, k, m} = 0 \text{ for } 2\alpha + k + |m| = s, \quad k > 0.$$  

(10)

Applying the conditions (10) to (6) we get

$$\text{deg}_s X' = \sum_{\alpha, m \atop 2\alpha + |m| = s} c_{\alpha, 0, m} C_q^{\alpha} K^m.$$  

(11)

To eliminate the remaining coefficients in (11) we commute $X'$ with the generator $E$. Using (7) we obtain

$$\text{deg}_{s+1}[X', E] = \text{deg}_{s+1} \sum_{\alpha, m \atop 2\alpha + |m| = s} c_{\alpha, 0, m} C_q^{\alpha} [K^m, E] = \sum_{\alpha, m \atop 2\alpha + |m| = s} c_{\alpha, 0, m} (q^{2m} - 1) C_q^{\alpha} E K^m.$$  

This way we obtain several new equations:

$$c_{\alpha, 0, m} = 0 \text{ for } 2\alpha + |m| = s, \quad m > 0.$$  

(12)

Putting the conditions (12) into (11) we get $\text{deg}_s X' = 0$. as desired. Therefore maxdeg $X' < s$ which is contradiction to the assumption. Hence $X' = 0$. This leads to the following theorem.

**Theorem 1.** Let $q \in \mathbb{C}$, $q \neq 0$, $q$ is not root of unity. The center of the algebra $U_q(sl_2)$ defined by the relations (1) is a free polynomial ring in variable $C_q$ defined by (3).
3. Center of $U_q(sl_2)$ when $q$ is a root of unity

When $q$ is a primitive root of unity (see also [4]), $q^n = 1, n > 2, q^j \neq 1$ for all $j \in \{1, 2, ..., n - 1\}$, one can easily see from (2) and (8) that center of the algebra $U_q(sl_2)$ contains besides the element $C_q$ additional four Casimir elements, namely

$$C_E = E^m', \ C_F = F^m', \ C_K^{\pm 1} = K^{\pm n'},$$

(13)

where $n' = \frac{n}{2}$ when $n$ is even and $n' = n$ when $n$ is odd. The set

$$\{C_E^\alpha C_F^\beta C_K^\gamma E^{l} F^{m} K^{n} \mid \alpha, \beta \geq 0, \gamma \in \mathbb{Z}, \ 0 \leq k, l, m \leq n' - 1\}$$

clearly constitutes new vector space basis of $U_q(sl_2)$. Let us denote

$$Z = \mathbb{C}[C_E, C_F, C_K^{\pm 1}]$$

commutative polynomial ring in variables $C_E, C_F, C_K^{\pm 1}$. $U_q(sl_2)$ can then be seen as a finite-dimensional module over the ring $Z$ with the dimension equal to $n'^3$ and a basis

$$\{E^{k} F^{l} K^{m} \mid 0 \leq k, l, m \leq n' - 1\}.$$ 

Trivial corollary of this fact is that any irreducible representation of $U_q(sl_2)$ is finite dimensional (see also corollary 16, p. 67 in [2]).

This module is also $Z$-spanned by the following $n'^3$ elements:

$$C_E^\delta E^k K^m, \quad 0 \leq \delta \leq n' - 1, \quad 0 \leq k \leq n' - 1 - \delta, \quad 0 \leq m \leq n' - 1,$$

$$C_F^\delta F^k K^m, \quad 0 \leq \delta \leq n' - 1, \quad 1 \leq l \leq n' - 1 - \delta, \quad 0 \leq m \leq n' - 1.$$

(14)

That means that we have also new vector space basis of the whole algebra $U_q(sl_2)$ consisting of the following elements:

$$C_E^\alpha C_F^\beta C_K^\gamma E^{k} F^{l} K^{m}, \quad \alpha, \beta \geq 0, \gamma \in \mathbb{Z}, \quad 0 \leq \delta \leq n' - 1,$$

$$0 \leq k \leq n' - 1 - \delta, \quad 0 \leq m \leq n' - 1.$$  

(15)

Because the element $C_F^\alpha$ does not belong to the basis (14) and it is of course element from the center of the algebra $U_q(sl_2)$, this implies algebraic dependency between $C_q$ and the elements $C_E, C_F$ and $C_K^{\pm 1}$. For each $k = 1, 2, ...$, we have

$$E^{k} F^{k} = \prod_{m=0}^{k-1} \left( C_q - \frac{q^{-2m_1} K + q^{2m+1} K^{-1}}{(q - q^{-1})^2} \right),$$

(16)

which is easily shown by induction: with the help of (8) one has

$$E^{k+1} F^{k+1} = E^{k+1} E F F^{k} = E^{k} \left( C_q - \frac{q^{-1} K + qK^{-1}}{(q - q^{-1})^2} \right) F^{k} = E^{k} F^{k} \left( C_q - \frac{q^{-1-2k} K + q^{1+2k} K^{-1}}{(q - q^{-1})^2} \right).$$

When we put $k = n'$ in (16) we obtain

$$C_E C_F = \prod_{m=0}^{n'-1} \left( C_q - \frac{q^{-2m_1} K + q^{2m+1} K^{-1}}{(q - q^{-1})^2} \right).$$

(17)
When we expand righthand side of (17), we find it is free of any powers of $K^j$, $j = -n' + 1, \ldots, -1, 1, \ldots, n' - 1$. Really, for any complex $x$ we have

$$\prod_{m=0}^{n'-1} (q^{2m+1} - x) = 1 + (-1)^n x^{n'},$$  \hspace{1cm} (18)

because the polynomials on both sides of (18) have the same roots $q, q^3, \ldots, q^{n-1}$.

For any nonzero complex $a, b$ we have

$$a - \frac{q^{-2m-1}b + q^{2m+1}b^{-1}}{(q - q^{-1})^2} = -aq^{-2m-1}\left(q^{2(2m+1)}a^{-1} - q^{2m+1} + ab\right) = -aq^{-2m-1}ab^{-1}\left(q^{2(2m+1)} - a^{-1}bq^{2m+1} + b^2\right) = -aq^{-2m-1}ab^{-1}\left(q^{2m+1} - b\frac{1 - \sqrt{1 - 4a^2}}{2a}\right) \times \left(q^{2m+1} - b\frac{1 + \sqrt{1 - 4a^2}}{2a}\right)$$  \hspace{1cm} (19)

where

$$\tilde{a} = \frac{1}{a(q - q^{-1})^2}.$$

Multiplying (19) with the help of (18) and binomial theorem we obtain

$$\prod_{m=0}^{n'-1} \left(a - \frac{q^{-2m-1}b + q^{2m+1}b^{-1}}{(q - q^{-1})^2}\right) = (-1)^n a^{n'} \prod_{m=0}^{n'-1} \left(q^{-2m-1}\right) a^{n'} b^{-n'} \left(1 + (-1)^n \left(b\frac{1 - \sqrt{1 - 4a^2}}{2a}\right)^n\right) \times \left(1 + (-1)^n \left(b\frac{1 + \sqrt{1 - 4a^2}}{2a}\right)^n\right) = a^{n'} \left(b^{-n'} a^{n'} + \frac{(-1)^n}{2^{n'}} \left((1 - \sqrt{1 - 4a^2})^n + (1 + \sqrt{1 - 4a^2})^n\right) + b^{n'} a^{n'}\right) =$$

$$= \frac{1}{b^{n'}(q - q^{-1})^{2n'}} + \frac{(-1)^n}{2^{n'}-1} \sum_{m=0}^{[n/2]} \binom{n'}{2m} a^{n'-2m} \frac{q^2(q - q^{-1})^4 - 4)^m}{(q - q^{-1})^{4m}} + \frac{b^{n'}}{(q - q^{-1})^{2n'}}.$$

Therefore, the relation (17) one can rewrite as

$$C_E C_F = \frac{C_K + C_K^{-1}}{(q - q^{-1})^{2n'}} + \frac{(-1)^n}{2^{n'-1}} \sum_{m=0}^{[n'/2]} \binom{n'}{2m} C_q^{n'-2m} \frac{(C_q^2(q - q^{-1})^4 - 4)^m}{(q - q^{-1})^{4m}}$$  \hspace{1cm} (20)

Using the combinatorial identity

$$\sum_{m=l}^{[n/2]} \binom{n}{2m} \binom{m}{l} = \frac{m \cdot 2^{n-2l-1}}{n-l} \binom{n-l}{n-2l},$$
we can rewrite (20) to the form

$$C_E C_F = \frac{C_K + C_K^{-1}}{(q - q^{-1})^{2l}} + (-1)^n n' \sum_{l=0}^{[n'/2]} \frac{(-1)^l(q - q^{-1})^{-4l}}{n' - l} \left(\frac{n' - l}{n' - 2l}\right)^{n'-2l}. $$

As a result we obtain there exists a complex polynomial $P$ in five variables such that

$$P(C_E, C_F, C_K, C_K^{-1}, C_q) = 0,$$

which explicit form is

$$P(x, y, z, \tilde{z}, w) = -xy + \frac{z + \tilde{z}}{(q - q^{-1})^{2n'}} + (-1)^n n' \sum_{l=0}^{[n'/2]} \frac{(-1)^l(q - q^{-1})^{-4l}}{n' - l} \left(\frac{n' - l}{n' - 2l}\right)^{n'-2l}. $$

Any other polynomial relation $Q$ between elements $C_E, C_F, C_K^{-1}$ and $C_q$ can be written in the form

$$Q(x, y, z, \tilde{z}, w) = \alpha(x, y, z, \tilde{z}, w)P(x, y, z, \tilde{z}, w) + R(x, y, z, \tilde{z}, w),$$

where $\alpha$ and $R$ are polynomials and the degree of $R$ is less than $n'$ with respect to variable $w$. The condition

$$Q(C_E, C_F, C_K, C_K^{-1}, C_q) = 0,$$

implies

$$R(C_E, C_F, C_K, C_K^{-1}, C_q) = 0.$$

This together with linear independence of $C_q^j$ for $j = 0, 1, \ldots, n' - 1$ gives

$$R(x, y, z, \tilde{z}, w) \equiv 0.$$

Any other relation between Casimir elements is then equal to the $P$ multiplied by a polynomial.

Our final task is to show that the center of $U_q(sl_2)$ has a basis

$$\{C_E^\alpha C_F^\beta C_K^\gamma C_q^\delta \mid \alpha, \beta \geq 0, \gamma \in \mathbb{Z}, 0 \leq \delta \leq n' - 1\},$$

i.e. there is no other algebraically independent Casimir element. Let us write an element $X$ from $U_q(sl_2)$ that is not in the span of (22) in the basis (15)

$$X = \sum_{\delta, k, m} \alpha_{\delta, k, m} C_q^\delta E^k K^m + \sum_{\delta, l, m} \beta_{\delta, l, m} C_q^\delta F^l K^m,$$

where we sum through

$$0 \leq \delta \leq n' - 1, 0 \leq k \leq n' - \delta - 1, 1 \leq l \leq n' - \delta - 1, 0 \leq m \leq n' - 1,$$

and where $\alpha_{\delta, k, m}$ and $\beta_{\delta, l, m}$ are arbitrary polynomials in elements $C_E, C_F$ and $C_K^{-1}$ except $\alpha_{0,0,0}$ which are identically zero. Commuting $X$ with the element $K$ we obtain

$$[X, K] = \sum_{\delta, k, m} \alpha_{\delta, k, m} (1 - q^{2k}) C_q^\delta E^k K^{m+1} + \sum_{\delta, l, m} \beta_{\delta, l, m} (1 - q^{-2l}) C_q^\delta F^l K^{m+1}.$$

Considering the bounds (23) only those coefficients $1 - q^{2k}$ can be equal to zero for which $k = 0$. Thus all polynomials $\alpha_{\delta, k, m}$ and $\beta_{\delta, l, m}$ must be identically zero except $\alpha_{\delta, 0, m}$. $X$ has now the form

$$X = \sum_{\delta, m} \alpha_{\delta, 0, m} C_q^\delta K^m.$$
Commuting with $E$ we obtain

$$[X, E] = \sum_{\delta, m} \alpha_{\delta,0,m}(1 - q^{2m})C^\delta_q E K^m = 0,$$

which implies

$$\alpha_{\delta,0,m} = 0 \text{ for all } 0 \leq \delta \leq n' - 1, 1 \leq m \leq n' - 1.$$

This yields $X = 0$, which proves that (22) is the vector space basis of the center of the algebra $U_q(sl_2)$. Thus we have shown the following.

**Theorem 2.** Let $q \in \mathbb{C}$ be a primitive root of unity, i.e. $q^n = 1$, $n > 2$, $q^j \neq 1$ for all $j \in \{1, 2, ..., n - 1\}$. The center of the algebra $U_q(sl_2)$ defined by the relations (1) is generated by the elements $C^\delta_q$ defined by (3) and $C_E, C_F$ and $C_{\pm 1}^\delta K$ defined by (13) and satisfy the relation $P(C_E, C_F, C_K, C_{-1}^\delta K, C_q) = 0$ where $P$ is given by (21).

**Acknowledgments**

We acknowledge financial support provided by Czech Technical University grant SGS10/210/OHK4/2T/14.

**References**

[1] Havlicek M and Posta S 2010 *Proc. Int. Conf. XXIX Workshop on Geometric Methods in Physics* ed V Buchstaber, A Odzijewicz et al (New York: AIP) pp 125-130

[2] Klimyk A U and Schmüdgen K 1997 *Quantum groups and their representations* (Berlin: Springer)

[3] Bergman G M 1978 *Adv. Math.* 29 178

[4] Concini C and Kac V G 1990 *Operator algebras, unitary representations, enveloping algebras, and invariant theory* (Boston: Birkhäuser)