Suppression of the Berezinskii-Kosterlitz-Thouless and Quantum Phase Transitions in 2D Superconductors by Finite Size Effects

T. Schneider$^1$ and S. Weyeneth$^1$

$^1$Physik-Institut der Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

We perform a detailed finite-size scaling analysis of the sheet resistance in Bi-films and the LaAlO$_3$/SrTiO$_3$ interface in the presence and absence of a magnetic field applied perpendicular to the system. Our main aim is to explore the occurrence of Berezinskii-Kosterlitz-Thouless (BKT) and quantum phase transition behavior in the presence of limited size, stemming from the finite extent of the homogeneous domains or the magnetic field. Moreover we explore the implications thereof. Above an extrapolated BKT transition temperature, modulated by the thickness $d$, gate voltage $V_g$ or magnetic field $H$, we identify a temperature range where BKT behavior occurs. Its range is controlled by the relevant limiting lengths, which are set by the extent of the homogeneous domains or the magnetic field. The extrapolated BKT transition lines $T_c(d, V_g, H)$ uncover compatibility with the occurrence of a quantum phase transition where $T_c(d, V_g, H) = 0$. However, an essential implication of the respective limiting length is that the extrapolated phase transition lines do not occur. Nevertheless, BKT and quantum critical behavior is observable, controlled by the extent of the relevant limiting length. Additional results and implications include: the magnetic field induced finite size effect generates a flattening out of the sheet resistance in the $T \to 0$ limit, while in zero field it exhibits a characteristic temperature dependence and vanishes at $T = 0$ only. The former prediction is confirmed in both, the Bi-films and the LaAlO$_3$/SrTiO$_3$ interface, as well as in previous studies. The latter is consistent with the LaAlO$_3$/SrTiO$_3$ interface data, while the Bi-films exhibit a flattening out.

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I. INTRODUCTION

Over the last two decades, electrical transport measurements of thin films near the onset of superconductivity have been studied extensively.$^{3, 4}$ Crucial observations include: the sheet resistance in zero magnetic field remains nearly temperature independent at the lowest attained temperature$^{5, 6}$ and remains ohmic below the expected normal state to superconductor transition temperature $T_c$. A magnetic field applied perpendicular to the film generates a flattening out of the sheet resistance in the $T \to 0$ limit.$^{10, 11}$ The occurrence of a smeared Nelson-Kosterlitz jump$^{12}$ in the superfluid density in the absence$^{15, 16}$ and presence of a magnetic field$^{17}$ Interprets of the saturation of the sheet resistance in the $T \to 0$ limit include the formation of a metallic phase,$^{10, 12, 18}$ the occurrence of quantum tunneling of vortices$^{6, 11}$ and the failure to cool the electrons$^{19}$.

On the other hand, more than three decades ago, Beasley, Mooij, and Orlando$^{20}$ suggested that the Berezinskii-Kosterlitz-Thouless$^{21, 22}$ (BKT) transition may be observable in sufficiently large and thin superconducting systems. They showed whenever the effective magnetic penetration depth $\lambda_{2D} = \lambda^2/d$ exceeds the sample size $[W_s, L_s]$, where $\lambda$ is the magnetic penetration depth, $d$ the thickness, $W_s$ the width and $L_s$ the length of the system, the vortices interact logarithmically over the entire sample, a necessary condition for a BKT transition to occur. Indeed, as shown by Pearl$^{23}$ vortex pairs in thin superconducting systems (charged superfluid) have a logarithmic interaction energy out to the characteristic length $\lambda_{2D} = \lambda^2/d$, beyond which the interaction energy falls off as $1/r$. Accordingly, as $\lambda_{2D}$ increases the diamagnetism of the superconductor becomes less important and the vortices in a thin superconducting film become progressively like those in $^4$He films. Invoking the Nelson-Kosterlitz relation$^{12}$ in the form $\lambda_{2D} = \lambda^2(T_c)/d = \Phi_0/(32\pi^2 k_B T_c)$, it is readily seen that for sufficiently low $T_c$'s, the condition $\lambda_{2D} > [W_s, L_s]$ is in practice accomplishable. Indeed, $T_c = 1K$ yields $\lambda_{2D} \approx 0.98$ cm. Additional limiting lengths include the magnetic length $L_H \propto (\Phi_0/H)^{1/2}$ associated with fields applied perpendicular to the film and in the case of ac measurements $L_f \propto f^{-1/2}$ where $f$ denotes the frequency. Concentrating on dc measurements of the sheet resistance one expects that the dimension of the homogeneous domains $L_h$ sets in zero magnetic field the smallest size so that $L = L_h = \min [W_s, L_s, \lambda_{2D}, L_h]$. As the magnetic field increases this applies as long as $L < L_H$, while for $L > L_H$ the magnetic field sets the limiting length. It controls the density of free vortices $n_p$ which determines the sheet resistance ($R \propto n_p$) as well as the correlation length ($\xi \propto n_p^{-1/2}$) at and above $T_s$. Accordingly, the correlation length cannot grow beyond $L$. In this context it is important to recognize that the finite size scaling approach adopted here is compatible with the Harris criterion$^{24, 25}$ stating that short-range correlated and uncorrelated disorder is irrelevant at the BKT critical point, contrary to approaches where the smearing of BKT criticality is attributed to a Gaussian-like distribution of the bare superfluid-stiffness around a given mean value$^{26}$.
be recognized that irrelevance of this disorder applies to the universal properties, while the nonuniversal parameters, including $T_c$ and the vortex core radius, may change. The finite size effects stemming from the limited extent of the homogeneous domains or the applied magnetic field have a profound influence on the observation of the BKT behavior and have been studied intensely in recent years.\textsuperscript{9,24,25,29,30} On the other hand, over the years, consistency with BKT behavior has been reported in thin films\textsuperscript{17,28–35} and in systems exhibiting interfacial superconductivity.\textsuperscript{7,9,24}

Here we extend previous work\textsuperscript{9,24,29,30} and analyze the sheet resistance data of Bi-films\textsuperscript{8} and the LaAlO$_3$/SrTiO$_3$ interface\textsuperscript{8,9} using the finite size scaling formulas appropriate for the BKT transition, which include multiplicative corrections when present.\textsuperscript{24,25} These systems have been chosen because the data comprise the low temperature limit, namely $T \ll T_c$ where $T$ is the extrapolated BKT transition temperature attained in the limit of an infinite limiting length $L$.

The paper is organized as follows. In Sec. II we sketch the finite size scaling behavior of the sheet resistance adapted to the BKT critical point and present the correspondent analysis of the thickness tuned Bi-films and the gate voltage tuned LaAlO$_3$/SrTiO$_3$ interface, in the presence and absence of a magnetic field, applied perpendicular to the film or interface. We observe remarkable consistency with the finite size scaling predictions. In the presence and absence of a magnetic field we identify a temperature range above the extrapolated $T_c$ where BKT behavior occurs. This temperature range is controlled by the relevant limiting length. In zero magnetic field it is the extent of the homogeneous domains. It turns out to decrease with the thickness $d$ or gate voltage $V_g$ tuned reduction of $T_c (d, V_g)$. The survival of BKT behavior in applied magnetic fields implies a smeared sudden drop in the superfluid stiffness at $T_c (H)$, where it adopts the universal value given by the Nelson-Kosterlitz relation.\textsuperscript{14} Recently, this behavior has been observed in MoGe and InO$_2$ thin films by means of low frequency measurements of the ac conductivity.\textsuperscript{17} Analogously, provided there is a temperature range above $T_c (d, V_g)$ where BKT behavior is present, the smeared jump should also occur in zero field, as observed in various films.\textsuperscript{15,16} An essential implication of the respective limiting length is that the extrapolated phase transition lines $T_c (d, V_g, H)$ are unattainable. As a consequence the occurrence of BKT transitions is suppressed and with that the occurrence of quantum phase transitions in the limit $T_c (d, V_g, H) \to 0$ as well. Nevertheless, in agreement with previous studies\textsuperscript{9,29,30} the lines $T_c (d, V_g, H)$ exhibit the characteristic quantum critical properties. Additional implications of finite size scaling adapted to the BKT transition include: the magnetic field induced finite size effect generates a flattening out of the sheet resistance in the $T \to 0$ limit, while in zero field it exhibits a characteristic temperature dependence and vanishes at $T = 0$ only. The former prediction is confirmed in both, the Bi-films and the LaAlO$_3$/SrTiO$_3$ interface, as well as in previous studies.\textsuperscript{9,29,30} The latter is consistent with the LaAlO$_3$/SrTiO$_3$ interface data, while the Bi-films exhibit a flattening out. Finally we explore the limitations of the quantum scaling approach.\textsuperscript{36}

II. THEORETICAL BACKGROUND AND DATA ANALYSIS

Since only the motion of free vortices dissipate energy, the sheet resistance should be proportional to the free vortex density.\textsuperscript{37}

$$R (T) \propto n_F (T).$$

(1)

On the other hand, dynamic scaling predicts the relationship\textsuperscript{38}

$$R (T) \propto \xi_f (T)^{-z},$$

(2)

between the sheet resistance above $T_c$ and the corresponding correlation length\textsuperscript{39}

$$\xi_f (T) = \xi_0 \exp \left( \frac{2\pi}{b_0 t^{1/2}} \right), t = T/T_c - 1.$$

(3)

$z$ is the dynamic critical exponent, the amplitude $\xi_0$ is related to the vortex core radius and $b$ is a nonuniversal parameter related to the vortex core energy.\textsuperscript{9,40} However, approaching $T_c$ from above, the aforementioned limiting lengths imply that the correlation length $\xi_f (T)$ cannot grow beyond $L = L_h = \min [W_s, L_s, \lambda_{2D}, L_h]$. According to this a finite size effect becomes visible around $T^* > T_c$ where

$$\xi_f (T^*) \approx L.$$  

(4)

It leads to a characteristic size dependence of the sheet resistance.\textsuperscript{9,24,25,29,30} Indeed, Eqs. (2) and (4) imply that for $\nu = 2$ at $T^* > T_c$ the sheet resistance adopts the size dependence

$$\sigma (T^*) \sigma_0 = \frac{R_0}{R (T^*)} = \left( \frac{L}{\xi_0} \right)^2$$

(5)

To illustrate the experimental situation we consider next the sheet resistance data of Yen-Hsiang Lin et al.\textsuperscript{44} for Bi films of various thickness and the heat conductance data of Agnolet et al.\textsuperscript{43} for a 23.42 Å thick $^4$He film. Both, the sheet resistance in thin superconducting films and the heat resistance in $^4$He film are supposed to be proportional to the to the free vortex density $n_F$ so that according to Eq. (2) the respective conductance scales of a homogeneous film with infinite extent scales for $\nu = 2$ as

$$\sigma (T) \sigma_0 = \frac{R_0}{R (T)} = \exp \left( b R t^{-1/2} \right),$$

(6)
where

\[ b_R = 4\pi/b. \]  

(7)

Supposing that the BKT regime is attainable, \( b_R \) is nearly independent of film thickness, \( R_0 \) and \( T_c \) adopt the appropriate values, the data plotted as \( \sigma (T) / \sigma_0 \) vs \( t^{-1/2} \) should then fall on the single curve \( \exp (b_R t^{-1/2}) \).

In Fig. 1(a) we depicted this plot for the Bi-films. As \( t^{-1/2} \) increases and with that \( T_c \) is approached the data no longer collapse, but run away and flatten out at \( \sigma (T) / \sigma_0 \) values which increase with film thickness \( d \). This behavior points to a finite size effect where the correlation length \( \xi_\perp (T) \) cannot grow beyond the limiting length \( L \) so that Eq. (5) applies. As a result the flattening out is controlled by the ratio \( L / \xi_0 \), which increases with film thickness and \( T_c \). In Fig. 1(b) we plotted the thickness dependence of \( R_0 \) and of the extrapolated BKT transition line \( R_0 (d) \). Apparently the decrease of \( T_c \) with reduced film thickness points to a quantum phase transition at a critical thickness \( d_c \) where \( T_c (d_c) \) = 0. Because the extrapolated BKT transition temperatures are not attainable due to the limiting length \( L \), it follows that these transitions, as well as the possible quantum phase transition at \( T_c (d_c) \) = 0 are suppressed. Nevertheless, slightly above \( T_c \), where the data tend to collapse on the BKT line, BKT fluctuations are present. This collapse attests the consistency with the universal and characteristic form of the BKT correlation length (Eq. (6)), while the nonuniversal parameters \( T_c \) and \( R_0 \) depend on the film thickness \( d \) (see Fig. 1b). The reduction of \( T_c \) and \( R_0 \) is attributable to disorder and quantum fluctuations. In particular, the strength of disorder is expected to increase with reduced film thickness \( d \). To quantify this expectation we consider

\[ k_F l = (h/e^2) / R_n, \]  

(8)

where \( k_F \) denotes the Fermi wavenumber, \( l \) the electron mean free path, and \( R_n \) the normal state sheet resistance. As disorder increases the mean free path \( l \) diminishes, \( k_F l \) decreases and the strength of disorder increases. In the Bi-films considered here \( k_F l \) varies from 3.8 for \( d = 22.2 \) Å to 17.4 for \( d = 23.42 \) Å. Accordingly, the strength of the disorder increases substantially with reduced film thickness or \( T_c \). Nevertheless, it does not affect the universal BKT properties but renormalizes the nonuniversal parameters.

To classify the relevance of the finite size effect in the Bi-films we show in Fig. 2 the corresponding scaling plot of the thermal conductance of a \(^4\)He film. Although the data attain the transition temperature rather closely there is now sign of a flattening out up to \( t^{-1/2} \approx 13 \), while in the Bi-films it sets in around \( 0.4 \leq t^{-1/2} \lesssim 0.75 \) (Fig. 1b), depending on the film thickness. Taking this dramatic difference as a generic fact, a finite scaling analysis of the sheet resistance data appears to be inevitable to uncover BKT behavior.

So far we considered finite size effects occurring at and above the transition temperature \( T_c \). In Fig. 3 we depicted \( R (d, T) / R_0 \) vs \( T_c (d) / T \) for the Bi films derived from Yen-Hsiang Lin et al.\(^{25,32,33}\) The solid line is the BKT behavior \( \sigma (d, T) / \sigma_0 (d) = \exp (b_R t^{-1/2}) \) for a homogeneous and infinite system with \( b_R = 5 \). In this temperature range the linear relationship \( T_c (d) / T \) vs \( d \) is well described by the data points. However, below \( T_c \) the dynamic scaling relation \( \xi_\perp (T) \) is no longer applicable because the correlation length is infinite there owing to the divergence of the susceptibility.\(^{22}\)

The BKT theory predicts that below \( T_c \) all vortices are bound in pairs by the logarithmic vortex interaction, whereupon the linear sheet resistance is zero. Instead there is a nonlinear dependence of the voltage on current since the current can unbind weakly bound pairs.\(^{22}\) Contrariwise, in a finite sample there will be a population of free vortices at and below the vortex unbinding transition temperature \( T_u \), where the vortex density still applies, while Eq. (4), relating the sheet resistance to the correla-
FIG. 2: (color online) Thermal conduction $\sigma_{th}(T)/\sigma_{th0}$ of a 23.42 Å thick $^4$He film vs $t^{-1/2}$ with $T_c = 1.2794$ K taken from Agnolet et al.\cite{Agnolet} The solid line is the BKT behavior $\sigma_{th}/\sigma_{th0} = \exp(b_R t^{-1/2})$ with $b_R = 1.762$ and $\sigma_{th0} = \exp(-24.13954) = 3.283 \cdot 10^{-11}$ W/K.

FIG. 3: (color online) $R(d,T)/R_0$ vs $T_c$ ($d$)/$T$ for the Bi films derived from Yen-Hsiang Lin et al.\cite{Lin} The solid line is the BKT behavior $R(T)/R_0 = \exp(-b_R (T/T_c - 1)^{-1/2})$ with $b_R = 5$.

To provide a rough estimate of the free vortex density we note that at low temperatures the energy change resulting from adding a single vortex in a system of size $L$ is given by $\Delta E = (J(T)/2) \int_0^{2\pi} d\Theta \int_0^L R dR/R^2 = \pi J(T) ln(L/\xi_0)$\cite{Houghton} where $\xi_0$ is the vortex core radius and

$$J(T) = \hbar^2 \rho_s(T)/2m = d\Phi_0^2/(16\pi^3\lambda^2(T)), \quad (9)$$

denotes the superfluid stiffness at low temperatures ($T << T_c$). An estimate for the free vortex density follows then from the probability of finding a free vortex from the Boltzmann factor

$$P(T) \propto n_F(T) \propto \exp(-\Delta E/k_B T) = (\xi_0/L)^{\pi J(T)/k_B T}. \quad (10)$$

Using Eq. (10) we obtain,

$$R(T) \propto n_F(T) \propto (\xi_0/L)^{\pi J(T)/k_B T}; \quad T << T_c. \quad (11)$$

Invoking the universal Nelson-Kosterlitz relation\cite{Nelson}

$$k_B T_c = \frac{\pi}{2} J(T_c^-), \quad (12)$$

the temperature range of validity is then restricted to $T << T_c = \pi J(T_c^-)/2k_B$. As it should be, for an infinite system, $n_F$ is zero for $T \leq T_c$. But if the limiting length $L$ is finite, the free vortex density vanishes at zero temperature only. This implies an ohmic tail in the IV characteristic below the extrapolated $T_c$\cite{Houghton,Callen}, and impedes a normal state to superconductor transition at finite temperature in a strict sense. In this context it is important to recognize that the standard finite size scaling outlined above neglects the multiplicative logarithmic corrections associated with BKT critical behavior\cite{Giamarchi,Giamarchi2}. A recent renormalization group treatment yields for $z = 2$ and free boundary conditions\cite{Giamarchi2}

$$R(T) \propto \begin{cases} (\xi_0/L)^{\pi J(T)/k_B T}; & L \gtrsim \xi_-(T) \\ (\xi_0/L)^2 \ln((L_{lim}/\xi_0)/b_0); & L \lessgtr \xi_+(T), \end{cases} \quad (13)$$

where

$$\xi_-(T) = \xi_0 \exp\left(\frac{1}{b_L |t|^{1/2}}\right), \quad (14)$$

is a diverging length below $T_c$\cite{Giamarchi2}. With Eq. (14) it follows that this thermal length is much smaller than the correlation length $\xi_+(T)$ for small $|t|$, because

$$\xi_+(t)/\xi_0 = (\xi_-(|t|)/\xi_0)^{2\pi}. \quad (15)$$

The parameter $b_0$ is fixed by the initial conditions of the renormalization group equations\cite{Giamarchi2} while the derivation of Eq. (11) identifies $\xi_0$ as vortex core radius. Furthermore, there is the upper bound $b_0 < L/\xi_0$ because $R(T) > 0$. Taking the multiplicative logarithmic correction into account Eq. (13) transforms with Eq. (13) to

$$\frac{R(T_c)}{R_0} = \frac{\sigma_0}{\sigma_T c} = \left(\frac{\xi_0}{L}\right)^2 \frac{1}{\ln((L/\xi_0)/b_0)}, \quad (16)$$

valid at $T \approx T_c$.

Given $R(T_c)/R_0$ and $b_0$, estimates for $L_{lim}/\xi_0$ are then readily obtained. Fig. 3 depicts the $T_c$ and $d$ dependence of $R(T_c)/R_0$ derived from Fig. 3 and the resulting $T_c$ dependence of $L_{lim}/\xi_0$ is shown in Fig. 3 for $b_0 = 0.05, 0.1$ and $1$ in comparison with the neglect of the multiplicative logarithmic correction. These $b_0$ values
satisfy the lower bound $b_0 < L/\xi_0$ resulting from the requirement, $R(d, T_c)/R_0(d) > 0$. Furthermore, $b_0 = 0.05$ is comparable to $b_0 \approx 0.07$, derived from large-scale numerical simulations. Striking features include the substantial decline of the ratio between limiting length and vortex core radius, $L/\xi_0$, with decreasing $T_c$, and the comparably low $L/\xi_0 < 80$ values. Indeed, the run away is controlled by the magnitude of $L/\xi_0$. The $^4$He data shown in Fig. 4 do not exhibit a sign of flattening out up to $\sigma_{th}(T)/\sigma_{th0} = 10^{10}$, yielding with Eq. 13 the lower bound $L/\xi_0 \gtrsim 10^{5}$. According to this, the run away observed in Fig. 1a and Fig. 3 stems from a limiting length $L$ where the ratio $L/\xi_0$ decreases with film thickness. Nevertheless, there is a temperature range where consistency with BKT behavior is observed, but in a strict sense a normal state to superconductor BKT transition is suppressed. As a consequence, there is also no film thickness driven quantum phase transition where the phase transition line $T_c(d)$ ends at $T_c(d_c) = 0$ vanishes at a critical film thickness $d_c$, as could be anticipated from the thickness dependence of the extrapolated $T_c$ shown in Fig. 4.

![FIG. 4: (color online) (a) $R(d, T_c)/R_0(d)$ vs $T_c$ and $d$ derived from the data shown in Fig. 3. (b) Estimates for the ratio $L/\xi_0$ between correlation length and vortex core radius without the multiplicative logarithmic correction term (○) and with this correction for different $b_0$ values entering Eq. 16.](image)

An essential issue left is the elucidation of the limiting length $L_{\text{min}}$. In principle the magnetic field induced finite size effect offers a direct estimate. A magnetic field applied perpendicular to the film leads to the limiting length

$$L_H = \left( \frac{\Phi_0}{aH} \right)^{1/2},$$

where $a \approx 4.8$ fixes the mean distance between vortices. It prevents the divergence of the correlation length at the extrapolated $T_c$. In analogy to Eq. 5 the sheet resistance is then expected to scale as

$$R(H, T_c) = \frac{1}{\sigma(H, T_c)} = \frac{f}{L_H} = \frac{a f H}{\Phi_0},$$

for $z = 2$ and low fields applied perpendicular to the film. In contrast to the zero field scaling form $|\mathbf{13}|$, this law holds below $T_c$ as well and the additive correction to the leading power law dependence is weak. The magnetic field induced finite sets then the limiting length as long as $L_H \propto H^{-1/2} < L$ whereby $L_H$ increases with decreasing field and approaches $L$. Here a runaway from the scaling behavior $|\mathbf{18}|$ sets in at $H^*$ providing for $L$ the estimate

$$L = \left( \frac{\Phi_0}{aH^*} \right)^{1/2}.$$

In Fig. 5 we depicted the magnetic field dependence of the sheet conductivity of the 23.42 Å thick Bi film at $T = 0.1$ K and 0.2 K where the latter is close to the extrapolated $T_c$. Even though the data are rather sparse we observe in an intermediate magnetic field range consistency with the predicted linear and nearly temperature independent field dependence of the sheet resistance. No, we focus on the low field behavior of the conductivity shown in Fig. 4. The run away from the $1/H$ dependence of the sheet conductivity occurs around $H = 0.01 T \approx H^*$, yielding with Eq. 19 for the limiting length the estimate

$$L \approx 208\AA.$$ (20)

With $L_{\text{min}}/\xi_0 \approx 32$, taken from Fig. 3b, we obtain for the magnitude of the radius of the vortex core radius

$$\xi_0 \approx 6.5 \AA.$$ (21)

The deviations from the finite size scaling behavior at higher fields are not unexpected because with increasing magnetic field the BKT regime is gradually left and the isotherms cross around $H \approx H^* \approx 0.4 \ T$, signaling the occurrence of a magnetic field driven quantum phase transition. In addition Eq. 18 captures the leading field dependence only. In the field range where it applies the plot $\sigma$ vs $1/H$ shown in Fig. 4a also reveals a nearly temperature independent coefficient of proportionality $\hat{\sigma}$. It implies that the temperature dependence of the sheet resistance at fixed field flattens out, as observed in the
23.42 Å thick Bi film. Analogous behavior was also observed in MoGe films and Ta-films in a field range where the magnetic field induced finite size scaling approach is no longer applicable. Indeed, in the MoGe films the temperature independent sheet resistance obeys the empirical form

\[ \sigma (H) = \sigma_0 \exp \left( -\frac{a H}{T} \right) . \]  

(22)

In the present case it applies according to Fig. 4 at best above the critical field only. The unusual empirical form was attributed to dissipative quantum tunneling of vortices from one "insulating" patch to another.

As the estimates for \( L_{\text{min}} \) and \( \xi_0 \) stem from rather sparse data a reliability check is inevitable. For this purpose we consider the temperature dependence of the correlation length \( \xi_+ (T) \) of the 23.42 Å thick Bi film in terms of \( \xi_+ (T) \) vs \( t^{-1/2} \) with \( \xi_0 = 6.5 \) Å shown in Fig. 5. As \( \xi_+ \) growth with increasing \( t^{-1/2} \) it approaches the limiting length \( L = 208 \) Å at \( t^{-1/2} \simeq 1.38 \), the range where in this film the run away from BKT behavior occurs (see Fig. 1b). Accordingly, we established for the 23.42 Å thick Bi-film reasonable consistency between the estimates for the vortex core radius \( \xi_0 \) and the limiting length \( L \), derived from the magnetic field induced finite size effect, and the observed zero field value of the sheet resistance. Unfortunately, this estimation of \( \xi_0 \) and \( L \) is restricted to this film because the magnetic field dependence of the sheet resistance appears to be missing for the other films. In any case, the rather small limiting length \( L = 208 \) Å points to an inhomogeneous film, with homogeneous patches of dimension \( L = L_h \).

In this context it should be kept in mind that there is the Harris criterion stating that short-range correlated and uncorrelated disorder is irrelevant at the unper-

![FIG. 5: (color online) Sheet conductivity \( \sigma \) of the 23.42 Å thick Bi-film vs magnetic field \( H \) at \( T = 0.1 \) K and 0.2 K derived from Yen-Hsiang Lin et al. The solid line is Eq. 18 in the form \( \sigma = \tilde{\sigma}/H \) where \( \tilde{\sigma} = 1.15 \cdot 10^{-4} \, (\Omega^{-1} \text{A}) \). The dashed line is Eq. 21 with \( \tilde{\sigma}_0 = 4.12 \cdot 10^{-4} \, \Omega^{-1} \) and \( \tilde{\sigma} = 2.25 \, \text{T}^{-1} \). The arrow indicates that this data point marks the zero field value of the sheet conductivity.

![FIG. 6: (color online) Correlation length \( \xi_+ = \xi_0 \exp \left( 2\pi (bt)^{-1/2} \right) \) vs \( t^{-1/2} \) of the 23.42 Å thick Bi-film with \( \xi_0 = 6.5 \) Å and \( 2\pi/b = \hbar k/2 = 2.5 \). The dashed line marks \( L = 208 \) Å. The crossing point at \( t^{-1/2} \simeq 1.38 \) corresponds to \( T_c/T \simeq 0.66 \).](image-url)

turbed critical point, provided that \( \nu > 2/D \), where \( D \) is the dimensionality of the system and \( \nu \) the critical exponent of the finite-temperature correlation length. With \( D = 2 \) and \( \nu = \infty \), appropriate for the BKT transition, this disorder should be irrelevant. Given the irrelevance of disorder, the reduction of the ratio \( L/\xi_0 \) with reduced film thickness or transition temperature (see Fig. 6) is then attributable to: (i) increasing vortex core radius \( \xi_0 \) with reduced \( T_c \) combined with a thickness independence \( L \); (ii) a limiting length \( L \) which decreases with film thickness combined with a \( T_c \) independent \( \xi_0 \); (iii) a thickness dependence of both, \( L \) and \( \xi_0 \), such that the ratio \( L/\xi_0 \) decreases with reduced transition temperature. Because the vortex core radius is known to increase with reduced \( T_c \) as \( \xi_0 \propto T_c^{-1/2} \) with \( z = 2 \) we are left with option (i) and (iii). In order to discriminate between these options we estimate \( \xi_0 (T_c) \) from the respective data for the 23.42 Å thick Bi-film, namely \( \xi_0 = 6.5 \) Å and \( T_c = 0.41 \) K, yielding \( \xi_0 = gT_c^{-1/2} \) with \( g = 4.19 \, \text{ÅK}^{1/2} \). The rough estimates for the thickness and \( T_c \) dependence of \( L \) shown in Fig. 7 are then readily obtained from the \( L/\xi_0 \) values depicted in Fig. 6b. In spite of the small total thickness increment of 1.18 Å there is a strong thickness dependence of \( L \), ranging from 50 Å to 200 Å. Direct experimental evidence for superconducting patches with an extent of 100 Å embedded in an insulating background stems from scanning tunneling spectroscopy investigations on TiN films. However, it should be kept in mind that transport measurements are sensitive to the phase and tunneling experiments to the magnitude of the order parameter. Furthermore, scanning SQUID measurements at the interface LaAlO\(_3\)/SrTiO\(_3\) covered superconducting regions occupying only a small fraction of the areas measured. In addition there are magnetic regions with patches of ferromagnetic re-
regions coexisting with a higher density of much smaller scale domains of fluctuating local magnetic moments.\footnote{28}

To explore the finite size scenario further we turn to the interface between LaAlO$_3$ and SrTiO$_3$, two excellent band insulators. It was shown that the electric-field effect can be used to map the phase diagram of this interface system revealing, depending on the gate voltage, a smeared BKT transition and evidence for quantum critical behavior.\footnote{8,22} Here we revisit the analysis of the temperature and gate voltage dependence of the sheet resistance data by invoking the approach outlined above.

In Fig. 7 we depicted $R(V_g,T)/R_0$ vs $T_c(V_g)$ for a homogenous and infinite system with $b_R = 3.43$. The solid and dashed lines indicate the approach of $T_c$ and $R_0$ to the extrapolated quantum phase transition.

In analogy to the Bi-films, important features include the substantial decline in $L/\xi_0$ with decreasing $T_c$, and the comparably low values of $L/\xi_0$ for $V_g = +80$ V. Accordingly, disorder is present, its strength is comparable to that in the Bi-films but increases only slightly by approaching the extrapolated quantum phase transition. In any case, it does not affect the universal BKT properties but renormalizes the nonuniversal parameters.

To unravel the consistency of the rounded transitions with a finite size effect, we invoke Eq. (16) to estimate the ratio $L_{\text{min}}/\xi_0$. Fig. 8 shows the $T_c$ and $d$ dependence of $R(V_g,T_c)/R_0(V_g)$ derived from Fig. 7. The resulting $T_c$ dependence of $L/\xi_0$ is shown in Fig. 8 for $b_0 = 0.05$ and 0.1 in comparison with the absence of the multiplicative logarithmic correction. Note that $b_0 = 0.05$ is comparable to $b_0 \approx 0.07$, derived from large-scale numerical simulations.\footnote{22} In analogy to the Bi-films, the run away from BKT behavior as observed in Fig. 8 is attributable to a limiting length $L$ where the ratio $L/\xi_0$ decreases with reduced $T_c$. Nevertheless, there is a temperature range where consistency with BKT behavior is observed, but in a strict sense a normal state to superconductor BKT transition is suppressed.
An independent confirmation of the finite size scenario demands the magnitude of $L$, allowing to determine $\xi_0$ and with that the temperature dependence of the correlation length $\xi_+(T)$, as well as $\xi_+(T^*) = L$, where the run away from BKT behavior should occur. Given the previous estimate derived from the magnetic field induced finite size effect

\[ L \approx 490 \, \text{Å}, \]  

for a LaAlO$_3$/SrTiO$_3$ interface with $T_c \approx 0.21$ K we obtain with $L/\xi_0 \approx 100$, taken from Fig. 9b, for the vortex core radius the value

\[ \xi_0 \approx 4.9 \, \text{Å}. \]  

The resulting temperature dependence of the correlation length is shown in Fig. 10 in terms of $\xi_+(T)$ vs $t^{-1/2}$. As the correlation length cannot grow beyond $L$ the run away from BKT behavior should occur around the crossing point between $\xi_+(T)$ and $L$ at $t^{-1/2} \approx 2.69$ corresponding to $T_c/T \approx 0.88$. A glance at Fig. 9 reveals that around this value the data of the LaAlO$_3$/SrTiO$_3$ interface at gate voltage $V_g = 80$ V ($T_c \approx 0.2$ K) run away from the BKT behavior. This agreement reveals that magnetic field and zero field finite size scaling yield consistent results. On this ground is the smeared BKT transition in both, the Bi-films and the LaAlO$_3$/SrTiO$_3$ interface, attributable to a finite size effect stemming from a limiting length $L$. In the samples with highest $T_c$ its dimension is $L \approx 208$ Å in the Bi-films and $L \approx 490$ Å in the LaAlO$_3$/SrTiO$_3$ interface.

Next we turn to the finite size behavior below the extrapolated transition temperature. Here the limiting length $L$ prevents the thermal length $\xi_-(|t|)$ to diverge. But compared to $\xi_+(|t|)$ the thermal length is much smaller for the same $|t|$ (Eq. 13). For this reason $L \geq \xi_-(T)$ is expected to hold already slightly below $T_c$. In this regime the sheet resistance is controlled by the free vortex density where Eq. 13 rewritten in the form

\[ \ln (R(T)) = r - \frac{s(T)}{T}, \quad s(T) = \frac{\pi J(T)}{k_B} \ln \frac{L}{\xi_0} \]  

applies. Accordingly, the coefficient $s(T)$ controls deviations from the $1/T$ temperature dependence. At zero temperature the superfluid stiffness given by Eq. 14 is fixed by the magnetic penetration depth in terms of $J(T = 0) \propto d/\lambda^2 (T = 0)$, expected to vanish as $J(T = 0) \propto d/\lambda^2 (T = 0) \propto T_c$\textsuperscript{26}. On the other hand, approaching $T_c$ from below, the superfluid stiffness tends according to Eq. 12 to $J(T^-) = 2k_B T_c/\pi$. In addition in both, the Bi-films (Fig. 11) and the LaAlO$_3$/SrTiO$_3$ interface (Fig. 9b), $(\ln (L/\xi_0))$ decreases with reduced $T_c$. As a consequence the magnitude of $s(T)$ is expected to decrease with reduced $T_c$. In Fig. 11 showing $\ln(R)$ vs $1/T$ of the LaAlO$_3$/SrTiO$_3$ interface for various gate
voltages, we observe that this supposition is well confirmed. On the other hand, in the temperature regime of interest the data exhibit jitter marking the characteristic temperature dependence of the superfluid stiffness in $s(T)$. Indeed, the straight lines, corresponding to the nearly temperature independent $s(T) \approx 2T_c \ln (L/\xi_0)$, describes the data quite well. To evidence the smeared BKT transition we included in Fig. 14 the characteristic BKT temperature dependence (14) in terms of the dash-dot-dot line. Additional confirmation of this finite size scenario below $T_c$ stems from the observation of an ohmic regime at small currents, because it uncovers according to Eq. 1 the presence of free vortices. The important implication then is: although BKT behavior is observable in an intermediate temperature regime above the extrapolated $T_c$, in a strict sense a BKT transition does not occur. It is smeared out and the sheet resistance vanishes at zero temperature only because Eq. 25 reduces in the zero temperature limit to

$$R(T) = r \exp - \left( \frac{\pi J(T = 0)}{k_B T} \ln L_{\text{lim}} \right) \left( \frac{\xi_0}{L_{\text{min}}} \right) \frac{\pi J(T = 0)}{\eta T} .$$

(26)

Contrariwise, the sheet resistance of the Bi-films shown in Fig. 3 does not exhibit a significant temperature dependence below $T \approx T_c/2$ down to $T \approx T_c/10$. To disentangle the scaling regimes below $T_c$ more quantitatively, we note that the plot $R(T)/R_0$ vs $T_c/T$ should exhibit a crossover from a temperature dependent to a temperature independent regime at $T^*$ where the diverging length $\xi(T)$ equals the limiting length $L_{\text{min}}$. According to Eqs. 13 and 14 $T^*$ follows from

$$\frac{L}{\xi_0} = \frac{\xi(T^*)}{\xi_0} = \exp \left( \frac{1}{b (1 - T^*/T_c)^{1/2}} \right).$$

(27)

To estimate $T^*$ we show in Fig. 12 the temperature dependence of $\xi(T)$ in terms of $\xi(T)/\xi_0$ vs $T/T_c$ for the Bi-films and the LaAlO$_3$/SrTiO$_3$ interface. Noting that the minimum value of $L/\xi_0$ in the Bi-films is around 3.8 (Fig. 1b) and in the LaAlO$_3$/SrTiO$_3$ interface around 5 (Fig. 1c) it becomes clear that in both systems $T^*$ is close and slightly below $T_c$. As a result, the temperature regime where $\xi(T) > L_{\text{lim}}$ holds is restricted to temperatures very close to $T_c$ only, while the regime where $\xi(T) < L_{\text{lim}}$ applies in slightly below $T_c$. It is the regime where the sheet resistance adopts the characteristic temperature dependence given by Eq. 25. A glance at Fig. 11 showing $\ln(R)$ vs $1/T$ of the LaAlO$_3$/SrTiO$_3$ interface, uncovers agreement with this temperature dependence, while the sheet resistance of the Bi-films shown in Fig. 3 does not exhibit a significant temperature dependence below $T \approx T_c/2$. Taking the saturation of the sheet resistance in the Bi-films for granted it implies the breakdown of the BKT behavior below $T_c$, while it applies above $T_c$. The breakdown may then be a clue that below $T_c$ a process is present which destroys BKT behavior. On the other hand we have seen that the LaAlO$_3$/SrTiO$_3$ interface data is at and below $T_c$ remarkably consistent with the predicted finite size BKT behavior. However, the absence of BKT behavior below $T_c$ is inconsistent with measurements of the superfluid stiffness, uncovering a smeared Nelson-Kosterlitz jump near $T_c$ and the presence of superfluidity down to the lowest attained temperatures. Given the odd behavior of the Bi-films it should be kept in mind that a failure to cool the electrons in the low temperature limit also implies a flattening of the sheet resistance.

Finally, to explore the implications of a magnetic field induced finite size effect below $T_c$ we depicted in Fig. 13 the temperature dependence of the sheet resistance of a LaAlO$_3$/SrTiO$_3$ interface with $T_c \approx 0.19$ K at various magnetic fields. Although the data exhibit jitter in the low field limit the predicted saturation of the sheet resistance in the $T \to 0$ limit (Eq. 18) is well established. On the other hand, considering the isotherm shown in Fig. 13b, the consistency with the finite size behavior (18) is restricted to low temperatures and low fields. Above $H = 30$ mT a crossover to the empirical form (22) can be surmised as the crossing point of the isotherms around $H_c = 110$ mT is approached. This crossing point is the direct consequence of the fact that in the covered $T$ range $R$ decreases with decreasing $T$ for $H < H_c$, increases with decreasing $T$ for $H > H_c$, and is temperature independent at $H_c$. Noting that the scaling form (18) presumes that density fluctuations are small (22) which is true for large limiting lengths $L_H = (\Phi_0/aH)^{1/2}$, but not for small, it becomes clear that the applicability...
FIG. 12: (color online) $\xi_0(T)/\xi_0 = \exp\left(1/(b(1 - T/T_c)^{1/2})\right)$ vs $T/T_c$ for the Bi-films with $1/b = b_R/4\pi \approx 0.398$ and the LaAlO$_3$/SrTiO$_3$ interface with $1/b = b_R/4\pi \approx 0.273$. The dash dot and dotted lines mark the minimum value of $L/\xi_0$. $L/\xi_0 \approx 3.8$ for the Bi-films (Fig. 12b) and $L/\xi_0 \approx 5$ for the LaAlO$_3$/SrTiO$_3$ interface (Fig. 12a).

of this approach is limited to the low field limit. Another essential feature emerging from Fig. 13a is the shift of the sheet resistance curves to lower temperatures with increasing magnetic field. This behavior uncovers the pair breaking effect of the magnetic field leading in a mean-field treatment to a reduction of $T_c$ according to $T_c(H = 0) - T_c(H) \propto H$ adopting the finite size point of view this behavior relies on the fact that an applied magnetic field sets an additional limiting length $L_H = (\Phi_0/aH)^{1/2}$, giving rise to a smeared BKT transition at a fictitious BKT transition temperature $T_c(H)$ below $T_c(H = 0)$. Contrariwise, in the standard finite size effect one attains $T_c$ in the $L \to \infty$ limit only. To quantify this option we performed fits to the characteristic BKT form of the sheet resistance. A glance at Fig. 13a reveals, in analogy to the zero field case (Fig. 3b), agreement in an intermediate temperature range below $T_c(H)$.

Given the consistency with the BKT expression and Fig. 13a) estimates for the fictitious lines $T_c(H)$ and $R_0(H)$ are readily obtained and shown in Fig. 13c) $T_c(H)$ extrapolates to zero around $H_c = 110$ mT where the isotherms cross. This behavior suggests a magnetic field induced quantum phase transition where superconducting behavior is lost at zero temperature and the amplitude $R_0$ approaches the critical value $R_0(H_c) \approx 1$ k$\Omega$ which is close to the normal state sheet resistance at $T = 0.5$ K. We note that $T_c(H)$ has properties compatible with a quantum critical point, where $T_c = T_0(H_c - H)^{\nu}$ applies, $\nu$ is the dynamic and $\nu$ the critical exponent of the zero temperature correlation length. The power law fit included in Fig. 13c) yields $\nu = 1.92 \pm 0.1$. It is interesting to note that this value is comparable with transport studies including MoGe, Nb$_{0.15}$Si$_{0.85}$, InO$_x$, and LaAlO$_3$/SrTiO$_3$ interface samples, though these studies have limited their analysis to exclude resistance data showing flattening in the zero temperature limit. In any case, BKT behavior occurs in an intermediate temperature range only. The extrapolated BKT line $T_c(H)$ is not attainable because the magnetic field induced finite size effect (Eq. 18) generates, as observed in Fig. 13a), the flattening out of the sheet resistance in the $T \to 0$ limit. Nevertheless, the established survival of BKT behavior in a magnetic field applied perpendicular to the film also implies a smeared sudden drop in the superfluid stiffness at $T_c(H)$, where the superfluid stiffness adopts the universal value given by the Nelson-Kosterlitz relation. Recently, this behavior has been observed in MoGe and InO$_x$ thin films by means of low frequency measurements of the ac conductivity. Although the low frequency $f = 20$ kHz implies an additional limiting length, namely $L_f \propto f^{-1/2}$, the magnetic field dependence of the blurred Nelson-Kosterlitz jump has been clearly de-
tected and the power law fits to \( T_c(H) \) yielded for \( z^\nu \) the estimates 1.25 ± 0.25 for MoGe and 1.3 ± 0.4 for InO\(_x\).

The solid line in Fig. 13a is depicted in Fig. 15a. For comparison we included the BKT scaling form (31). Apparently, the data do not collapse on a single curve because the amplitude \( R_0 \) exhibits a pronounced field dependence (see Fig. 14) and the sheet resistance flattens out for large and small values of the scaling argument \( z \). For fixed \( H_c - H \) this reflects the observed flattening out of the sheet resistance in the low and high temperature limits (Fig. 13a). A glance at Fig. 14 reveals that an improved data collapse is achieved by taking the field dependence of the amplitude \( R_0 \) into account. Clearly, the flattening out for small and large \( z \) values remains. Noting that for fixed \( H_c - H \) small \( z \) values are attainable at rather high temperatures only, the respective saturation reflects the fact that in this temperature regime BKT fluctuations no longer dominate. On the other hand large scaling arguments require the incidence of the zero temperature limit where the magnetic field induced finite size effect leads to a flattening out in the temperature dependence and with that in the \( z \) dependence of the sheet resistance in the \( z \rightarrow \infty \) limit. Furthermore, the field dependence of the amplitude \( R_0 \) also implies that the quantum scaling form holds in an unattainable regime close to quantum criticality only.

**III. SUMMARY AND DISCUSSION**

We analyzed sheet resistance data of thin Bi-films and the LaAlO\(_3/\)SrTiO\(_3\) interface near the onset of superconductivity to explore the compatibility with BKT behavior. On the Bi-films the onset temperature has been tuned by the film thickness, while on the LaAlO\(_3/\)SrTiO\(_3\) interface the gate voltage and the magnetic field, applied perpendicular to the interface, acted as tuning parameter. Noting that BKT behavior involves the transition from a low-temperature state in which only paired vortices exist to a high-temperature state in which free vortices occur, we demonstrated that finite size induced free vortices below \( T_c \) prevent the occurrence of a BKT transition in a strict sense. This does not mean, however, that the BKT vortex-unbinding mechanism does not occur and is not observable. Indeed our finite size analysis revealed that BKT behavior is present in an intermediate temperature range above the extrapolated BKT transition temperature. This temperature range depends on the magnitude of the limiting length \( L \) while the extrapolated transition temperature corresponds to the limit \( L \rightarrow \infty \). Limiting lengths include he effective magnetic penetration depth \( \lambda_{2D} = \lambda^2/d \), the dimension \( L_h \) of the homogeneous domains in the sample, the magnetic length \( L_H \propto (\Phi_0/H)^{1/2} \), and in the case of ac measurements \( L_f \propto f^{-1/2} \). \( L = \min[\lambda_{2D}, L_h, L_H, L_f] \) controls the density of free vortices \( n_c \) which determines

**FIG. 14:** (color online) Estimates for \( T_c \) and \( R_0 \) resulting from the fits to the BKT form (4) of the sheet resistance included in Fig. 13. The solid line is \( T_c = T_0(H_c - H)^{1/z^\nu} \) with \( T_0 = 3 \times 10^{-5} \) (KmT\(^{-1}\)) and \( z^\nu = 1.92 \pm 0.1 \) and the dashed line is \( R_0 = R_{0c} + (H_c - H)^{2/z^\nu} \) with \( R_{0c} = 0.96 \) k\( \Omega \), \( R = 0.16 \) \( \Omega \) mT\(^{1/z^\nu} \), and \( z^\nu = 2.78 \). These lines indicate the approach to the extrapolated quantum critical point.

Lastly we consider the limitations of the quantum scaling form

\[
R(H, T) = R_0 G(x), x = \frac{H_c - H}{T^{1/z^\nu}},
\]

applicable close to the quantum critical point. \( G(x) \) is a scaling function of its argument and \( G(0) = 1 \). It is essentially a finite size scaling function. Indeed at finite temperatures is the divergence of the zero temperature correlation length \( \xi(T = 0) \propto (H_c - H)^{-\nu} \) cut-off by the thermal length \( L_T \propto T^{-1/2} \) so that \( x \propto (L_T/\xi(T = 0))^{1/z^\nu} \propto (H_c - H)^{1/z^\nu} \) and the data for \( R(H, T) \) plotted as \( x = (H_c - H)/T^{1/z^\nu} \) should then collapse on a single curve. On the other hand BKT behavior uncovered in Fig. 13 implies the scaling form (4) rewritten in the form

\[
R(H, T) = R_0(H) \exp\left(-bR/\left(T/T_c(H) - 1\right)^{1/2}\right),
\]

where \( T_c(H) = T_0(H_c - H)^{z^\nu} \) is the transition line shown in Fig. 14. Noting that

\[
\frac{T}{T_c(H)} = \frac{1}{T_0^{1/z^\nu}},
\]

BKT behavior leads with Eqs. (28) and (29) to the explicit scaling form

\[
R(H, T) = R_0(H) \exp\left(-bR/\left(T_0 x^{2/z^\nu}\right)^{-1} - 1\right)^{1/2},
\]

valid for any \( T/T_c(H) = (T_0 x^{2/z^\nu})^{-1} > 1 \) where the universal critical behavior is entirely classical. The scaling plot \( R(H, T) vs z = (H_c - H)/T^{1/z^\nu} \) obtained from the LaAlO\(_3/\)SrTiO\(_3\) interface sheet resistance data shown in Fig. 13 is depicted in Fig. 15. For comparison we included the BKT scaling form (31). Apparently, the data do not collapse on a single curve because the amplitude \( R_0 \) exhibits a pronounced field dependence (see Fig. 14) and the sheet resistance flattens out for large and small values of the scaling argument \( z \). For fixed \( H_c - H \) this reflects the observed flattening out of the sheet resistance in the low and high temperature limits (Fig. 13a). A glance at Fig. 14 reveals that an improved data collapse is achieved by taking the field dependence of the amplitude \( R_0 \) into account. Clearly, the flattening out for small and large \( z \) values remains. Noting that for fixed \( H_c - H \) small \( z \) values are attainable at rather high temperatures only, the respective saturation reflects the fact that in this temperature regime BKT fluctuations no longer dominate. On the other hand large scaling arguments require the incidence of the zero temperature limit where the magnetic field induced finite size effect leads to a flattening out in the temperature dependence and with that in the \( z \) dependence of the sheet resistance in the \( z \rightarrow \infty \) limit. Furthermore, the field dependence of the amplitude \( R_0 \) also implies that the quantum scaling form holds in an unattainable regime close to quantum criticality only.

**III. SUMMARY AND DISCUSSION**

We analyzed sheet resistance data of thin Bi-films and the LaAlO\(_3/\)SrTiO\(_3\) interface near the onset of superconductivity to explore the compatibility with BKT behavior. On the Bi-films the onset temperature has been tuned by the film thickness, while on the LaAlO\(_3/\)SrTiO\(_3\) interface the gate voltage and the magnetic field, applied perpendicular to the interface, acted as tuning parameter. Noting that BKT behavior involves the transition from a low-temperature state in which only paired vortices exist to a high-temperature state in which free vortices occur, we demonstrated that finite size induced free vortices below \( T_c \) prevent the occurrence of a BKT transition in a strict sense. This does not mean, however, that the BKT vortex-unbinding mechanism does not occur and is not observable. Indeed our finite size analysis revealed that BKT behavior is present in an intermediate temperature range above the extrapolated BKT transition temperature. This temperature range depends on the magnitude of the limiting length \( L \) while the extrapolated transition temperature corresponds to the limit \( L \rightarrow \infty \). Limiting lengths include he effective magnetic penetration depth \( \lambda_{2D} = \lambda^2/d \), the dimension \( L_h \) of the homogeneous domains in the sample, the magnetic length \( L_H \propto (\Phi_0/H)^{1/2} \), and in the case of ac measurements \( L_f \propto f^{-1/2} \). \( L = \min[\lambda_{2D}, L_h, L_H, L_f] \) controls the density of free vortices \( n_c \) which determines
the sheet resistance \( R \propto n_p \) as well as the correlation length \( \xi \propto n_p^{-1/2} \) at and above \( T_c \). In this temperature range the limiting lengths prevent the correlation length to diverge. Concentrating on the dc sheet resistance we analyzed the data using finite size scaling formulas appropriate for the BKT transition.

The main results for zero magnetic fields include: Above \( T_c \) we observed in an intermediate temperature range consistency with the characteristic BKT behavior and a thickness or gate voltage dependent BKT transition temperature \( T_c \) (Figs. 1h and 3). However, in analogy to finite systems, the measured sheet resistance does not vanish at \( T_c \). In this context it should be kept in mind that there is the Harris criterion\(^{20,22}\), stating that short-range correlated and uncorrelated disorder is irrelevant at the unperturbed critical point, provided that \( \nu > 2/D \), where \( D \) is the dimensionality of the system and \( \nu \) the critical exponent of the finite-temperature correlation length. With \( D = 2 \) and \( \nu = \infty \), appropriate for the BKT transition\(^{22}\), this disorder should be irrelevant. Accordingly, the nonvanishing sheet resistance at \( T_c \) points to a finite size induced smeared BKT transition. Invoking the finite size scaling formula for the sheet resistance at \( T_c \) we obtained estimates for the \( T_c \) dependence of the ratio between the limiting length and the vortex core radius, namely \( L/\xi_0 \) (Figs. 3a and 3b). Striking features included the substantial decline of \( L/\xi_0|_{\max} \approx 10^2 \) with decreasing \( T_c \) and in comparison with \( L/\xi_0 > 10^5 \) in \(^4\)He the low value of \( L/\xi_0|_{\max} \). This difference and the \( T_c \) dependence of \( L/\xi_0 \) imply enhanced smearing of the BKT transition with reduced \( T_c \), as observed (Figs. 1h, 3 and 5). To disentangle the \( T_c \) dependence of the limiting length \( L \) and the vortex core radius \( \xi_0 \) we invoked the magnetic field induced finite size effect allowing to estimate the limiting length directly from magnetic dependence of the sheet conductivity at fixed temperature below \( T_c \). Unfortunately, in both the Bi-films and the LaAlO\(_3/\)SrTiO\(_3\) interface, the necessary data is available for the samples with highest \( T_c \) only. For the 23.42 Å thick Bi film we obtained \( L \approx 208 \) Å, \( \xi_0 \approx 6.5 \) Å (Eqs. 20 and 21) and for the LaAlO\(_3/\)SrTiO\(_3\) interface with \( T_c \approx 0.21 \) K the estimates \( L \approx 490 \) Å, \( \xi_0 \approx 4.9 \) Å (Eqs. 23 and 24). These values for the extent of the homogeneous domains are comparable with the dimension of the superconducting patches emerging from scanning tunneling spectroscopy investigations on TiN\(^{46}\) and InO\(^{47}\) films, as well as with scanning SQUID measurements at the interface LaAlO\(_3/\)SrTiO\(_3\). To disentangle the \( T_c \) dependence of \( L \) and \( \xi_0 \) we used the empirical relationship \( \xi_0 \propto T_c^{1/2} \) with \( z = 2.44 \),\(^{44,45}\) revealing that the extent of the homogenous domains decreases substantially with reduced \( T_c \). The breakdown may then be a clue that below \( T_c \) a process is present which destroys BKT behavior. On the other hand we have seen that the LaAlO\(_3/\)SrTiO\(_3\) interface data is at and below \( T_c \) remarkably consistent with the predicted finite size BKT predictions. In addition, an absence of BKT-behavior below \( T_c \) is also incompatible with measurements of the superfluid stiffness\(^{45,48,49}\) uncovering a smeared Nelson-Kosterlitz\(^{50}\) jump near \( T_c \) and the presence of superfluidity down to the lowest attained temperatures.

Subsequently we explored the implications of the magnetic field induced finite size effect. Considering the temperature dependence of the sheet resistance at various magnetic fields, applied perpendicular to the interface of LaAlO\(_3/\)SrTiO\(_3\), we observed in an intermedi-

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**FIG. 15:** (color online) (a) Scaling plot \( R(H,T) \) vs \( z = (H_c - H)/T^{1/\nu} \) with \( H_c = 110 \) mT, \( \nu = 1.92 \), and \( b_R = 3.43 \). The solid lines mark the respective BKT scaling form \((31)\) with \( R_0(H) \) taken from Fig. 14 and \( T_0 = 2 \cdot 10^{-5} \) (KmT\()^{1/\nu} \). (b) Scaling plot \( R(H,T)/R_0(H) \) vs \( z = (H_c - H)/T^{1/\nu} \). The solid line is the BKT scaling form \((31)\).
ate temperature range remarkable consistency with the characteristic BKT form (Fig. 13). Fits yielded the fictitious transition line \( T_c(H) \) extrapolating to zero at \( H_c \approx 110 \text{ mT} \) where a quantum phase transition is expected to occur (Fig. 13). Indeed, \( T_c(H) \) revealed properties compatible with a quantum critical point, near which \( \Delta_c = T_0(H_c - H)^z \) applies\(^{35}\) where \( z \) is the dynamic and \( \nu \) the critical exponent of the zero temperature correlation length. A power law fit yielded \( \nu = 1.92 \pm 0.1 \). However, this extrapolated line is not attainable because the magnetic field induced finite size effect (Eq. \( (13) \)) generates the observed flattening out of the sheet resistance in the \( T \to 0 \) limit (Fig. 13). This feature has been observed in the 23.42 Å thick Bi-film as well.\(^4\) The survival of BKT behavior in applied magnetic fields also implies a smeared sudden drop in the superfluid stiffness at \( T_c(H) \), where it adopts the universal value given by the Nelson-Kosterlitz relation \((12)\). Recently, this behavior has been observed in MoGe and InO\(_x\) thin films by means of low frequency measurements of the ac conductivity.\(^17\)

A key question our analysis raises is whether the homogeneity of 2D superconductors can be improved to reach the quality of \(^4\)He films. Analyzing the sheet resistance data of Bi-films and the LaAlO\(_3/SrTiO\(_3\) we have shown that the data are consistent with a finite size effect attributable to the limited homogeneity of the samples. The limited length of the homogenous domains impedes the occurrence of a BKT and quantum phase transitions in the strict sense of a true continuous phase transition. However, this strict interpretation of the definition of a continuous phase transition does not imply that the BKT vortex-unbinding mechanism is not observable and the reduction of the extrapolated \( T_c \) does not reveal properties compatible with a quantum critical point. Indeed, notwithstanding the comparatively small dimension of the homogeneous domains, our finite size analysis revealed reasonable compatibility with BKT and quantum critical point behavior. However, the reduction of the limiting length with decreasing \( T_c \) is an essential drawback (Fig. 7). Furthermore, considering the expected magnetic field tuned quantum phase transition in the LaAlO\(_3/SrTiO\(_3\) interface, it was shown that the standard quantum scaling form \((28)\) of the sheet resistance applies very close to the unattainable quantum critical point only (Fig. 13). Indeed, combining the BKT expression for the sheet resistance with the quantum scaling form of the extrapolated transition line \( T_c(H) \), we derived the explicit scaling relation \((31)\) uncovering the limitations of the standard quantum scaling form. Its main drawback was traced back to the neglect of the magnetic field dependence of the critical amplitude \( R_0 \) which varies substantially by approaching the critical value \( R_{0c} \) (Fig. 14).

Finally it should be noted that the finite size scaling approach adopted here is compatible with the Harris criterion\(^{26,27}\), stating that short-range correlated and uncorrelated disorder is irrelevant at the BKT critical point, contrary to approaches where the smearing of the BKT transition is attributed to a Gaussian-like distribution of the bare superfluid-stiffness around a given mean value.\(^{28}\) The irrelevance of this disorder implies, that the universal BKT properties still apply, while the nonuniversal parameters, including \( T_c \), the vortex core radius \( \xi_0 \) and the amplitude \( R_0 \), may change. Contrariwise, the relevance of disorder at the extrapolated quantum phase transition, separating the superconducting and metallic phase, depends on the universality to which it belongs. The relevance of disorder is again controlled by the Harris criterion\(^{26,27}\) if the zero-temperature correlation length critical exponent fulfills the Harris inequality \( \nu > 2/D = 1 \) the disorder does not affect the quantum critical behavior. Conversely, if \( \nu < 2/D = 1 \) disorder is relevant and affects the nonuniversal parameters \( R_0 \) and \( T_c \) in the BKT form \((2)\) of the sheet resistance and in particular the reduction of \( T_c \). In the magnetic field tuned case is the field dependence of \( R_0 \) and \( T_c \) attributable to Cooper pair breaking. However, another important feature of the of LaAlO\(_3/SrTiO\(_3\) interface is the large Rashba spin orbit interaction which originates from the broken inversion symmetry. It has been shown that its magnitude increases with reduced \( T_c \),\(^{56}\) suggesting that pair breaking occurs in zero magnetic field as well. Indeed, torque magnetometry measurement revealed that the LaAlO\(_3/SrTiO\(_3\) interface has a magnetic moment, which points in the plane, and has an onset temperature that is at least as high as 40 K and persists below the BKT transition temperature.

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\footnotetext*{Electronic address: toni.schneider@swissonline.ch}

1. N. Markovic, C. Christiansen, A. Mack, and A. M. Goldman, Phys. Status Solidi b \textbf{218}, 221 (2000).
2. A. M. Goldman, Physica E \textbf{18},1 (2003).
3. V. F. Gantmakher and V. T. Dolgopolov, Phys.—Usp. \textbf{53}, 1 (2010).
4. A. M. Goldman, Int. J. Mod. Phys. B \textbf{24}, 408 (2010).
5. H. M. Jaeger, D.B. Haviland, B. G. Orr, and A. M. Goldman, Phys. Rev. B \textbf{40}, 182 (1989).
6. Yen-Hsiang Lin, J. J. Nelson, and A. M. Goldman, Phys. Rev. Lett. \textbf{109}, 017002 (2012).
7. N. Reyren, S. Thiel, A. D. Caviglia, L. Fitting Kourkoutis,
