IDENTITIES OF CYCLE INTEGRALS OF WEAK MAASS FORMS

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Abstract. We prove identities between cycle integrals of non-holomorphic modular forms arising from applications of various differential operators to weak Maass forms.

1. Introduction

While investigating the Shintani lift of weakly holomorphic modular forms, Bringmann, Guerzhoy and Kane [1, 2] found surprising identities between cycle integrals of weakly holomorphic modular forms and cusp forms of the same weight. More precisely, they showed that if \( f \) is a weakly holomorphic modular form of weight \( 2k \in 2\mathbb{N} \) for \( \Gamma = \text{SL}_2(\mathbb{Z}) \) which is orthogonal to cusp forms and whose constant term in the Fourier expansion vanishes, then there exists a cusp form \( g \) of weight \( 2k \) such that the identity

\[
\int_{C_Q} f(z)Q(z,1)^{k-1}dz = -\frac{(2k-2)!}{(4\pi)^{2k-1}} \int_{C_Q} g(z)Q(z,1)^{k-1}dz
\]

holds for any integral binary quadratic form \( Q \) of positive non-square discriminant, where \( C_Q \) is the geodesic associated to \( Q \), see Section 2.1. Although the above identity only involves (weakly) holomorphic modular forms, the proof crucially uses the theory of harmonic weak Maass forms, which was first developed systematically by Bruinier and Funke [3]. Namely, by results of Bruinier, Ono and Rhoades [4], for every weakly holomorphic modular form \( f \) of weight \( 2k \) as above there exists a harmonic weak Maass form \( F \) of weight \( 2 - 2k \) such that \( f = \mathcal{D}^{2k-1}F \) and such that \( g = \xi_{2-2k}F \) is a cusp form of weight \( 2k \), with the differential operators \( \xi_{2-2k} = 2iy^{2-2k} \frac{d}{dz} \) and \( \mathcal{D} = \frac{i}{2\pi} \frac{d}{dz} \), \( z = x + iy \in \mathbb{H} \). The authors of [2] actually showed that the identity

\[
\int_{C_Q} (\mathcal{D}^{2k-1}F)(z)Q(z,1)^{k-1}dz = -\frac{(2k-2)!}{(4\pi)^{2k-1}} \int_{C_Q} (\xi_{2-2k}F)(z)Q(z,1)^{k-1}dz
\]

holds for all binary quadratic forms \( Q \) as above. This implies the formula (1). For the proof of (2), the authors of [2] defined (regularized) periods of weakly holomorphic modular forms and generalized identities between cycle integrals and periods of cusp forms proved by Kohnen and Zagier [5] to weakly holomorphic modular forms. Jens Funke informed us that he has obtained another proof of the identity (2) by comparing the cohomology classes of \( \mathcal{D}^{2k-1}F \) and \( \xi_{2-2k}F \).

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In the present note, we generalize the above identities by replacing \( f \) and \( g \) by non-holomorphic modular forms arising from applications of various differential operators to weak Maass forms. Furthermore, we prove our identities by direct calculations using explicit parametrizations of cycle integrals and commutation relations of differential operators, and thereby obtain refinements and shorter proofs of the above identities of Bringmann, Guerzhoy and Kane.

To simplify the notation, we define the cycle integral along \( C_Q \) of a smooth modular form \( F \) of weight \( 2k \in \mathbb{Z} \) for \( \Gamma \) by

\[
C(F, Q) = D \frac{1}{D} \int_{C_Q} F(z)Q(z, 1)^{k-1}dz.
\]

We prove the following identities of cycle integrals.

**Theorem 1.1.** Let \( F : \mathbb{H} \to \mathbb{C} \) be a smooth function which transforms like a modular form of weight \( 2 - 2k \in \mathbb{Z} \) for \( \Gamma \). Then the identity

\[
C(L_{2-2k}F, Q) = C(R_{2-2k}F, Q) = C(\xi_{2-2k}F, Q)
\]

of cycle integrals holds, where \( L_{2-2k} = -2iy^2 \frac{\partial}{\partial z} \) and \( R_{2-2k} = 2i \frac{\partial}{\partial z} + (2 - 2k)y^{-1} \) are the Maass lowering and raising operators.

Moreover, if \( F \) is a weak Maass form of weight \( 2 - 2k \) with eigenvalue \( \lambda \), then we have

\[
C(R_{2-2k}^{k-\ell-1}F, Q) = ((k + \ell)(k - \ell - 1) - \lambda)C(R_{2-2k}^{k-\ell-2}F, Q), \quad \ell \leq k - 2,
\]

\[
C(L_{2-2k}^{k-\ell+2}F, Q) = ((k + \ell)(k - \ell - 1) - \lambda)C(L_{2-2k}^{k-\ell}F, Q), \quad \ell \leq -k,
\]

where \( L_{2-2k}^n \) and \( R_{2-2k}^n \) are iterated versions of the lowering and raising operators, see Section 2.2.

Using the first relation of Theorem 1.1 and then repeatedly applying the second and third one, we obtain the following formulas.

**Corollary 1.2.** Let \( F \) be a harmonic weak Maass form of weight \( 2 - 2k \) for \( \Gamma \). For \( k \geq 1 \) and all integers \( 1 \leq j \leq k \) we have the identity

\[
C(R_{2-2k}^{2j-1}F, Q) = \frac{(j-1)!(k-j)!(2k-2)!}{(k-1)!(2k-2)!}C(\xi_{2-2k}F, Q).
\]

Furthermore, for \( k \leq 0 \) and all integers \( 0 \leq j \leq |k| \) we have the identity

\[
C(L_{2-2k}^{2j+1}F, Q) = \frac{(2j)!|k|!}{j!(|k|-j)!}C(\xi_{2-2k}F, Q).
\]

If we apply the first identity in the corollary with \( j = k \) and use Bol’s identity (5), we recover the identity (2) of Bringmann, Guerzhoy and Kane.

We start with a section on preliminaries about cycle integrals and weak Maass forms. In Section 3 we prove Theorem 1.1.
2. Cycle Integrals and Weak Maass Forms

2.1. Cycle integrals. Let \( Q(X, Y) = aX^2 + bXY + cY^2 \) be an integral binary quadratic form of non-square discriminant \( D = b^2 - 4ac > 0 \), and let \( \Gamma_Q \) be the stabilizer of \( Q \) in \( \Gamma \). Associated to \( Q \) is the semi-circle \( S_Q \) given by all \( z = x + iy \in \mathbb{H} \) satisfying \( a|z|^2 + bx + c = 0 \), which we orient counterclockwise if \( a > 0 \). Let \( C_Q = \Gamma_Q \setminus S_Q \) be the associated geodesic in the modular curve \( \Gamma \setminus \mathbb{H} \).

We now give an explicit parametrization of the cycle integral in [3]. Since the cycle integral only depends on the class of \( Q \) mod \( \Gamma \) and every class contains a quadratic form with positive \( a \)-entry, we can assume that \( a > 0 \). Then the two real endpoints \( w < w' \) of \( S_Q \) are given by \( w = -\frac{b + \sqrt{D}}{2a} \) and \( w' = -\frac{b - \sqrt{D}}{2a} \). The matrix

\[
\sigma = \begin{pmatrix} \frac{a+}{D^+} & w' & w' \xi \\ \frac{a-}{D^+} & 1 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{R})
\]

maps 0 to \( w \) and \( \infty \) to \( w' \), and hence maps the positive imaginary axis (oriented from \( i\infty \) to 0) to \( S_Q \) (oriented counterclockwise). Furthermore, we have \( Q \circ \sigma = [0, -\sqrt{D}, 0] \). The stabilizer of \([0, -\sqrt{D}, 0]\) in \( \sigma^{-1} \Gamma \sigma \) is generated by the two matrices \( \pm \left( \begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix} \right) \) for a suitable \( \varepsilon > 1 \).

**Lemma 2.1.** Let \( F : \mathbb{H} \to \mathbb{C} \) be a smooth function which transforms like a modular form of weight \( 2k \in 2\mathbb{Z} \) for \( \Gamma \). Then the cycle integral of \( F \) along \( C_Q \) is given by

\[
\mathcal{C}(F, Q) = (-i)^k \int_{1}^{e^2} F_{\sigma}(iy)y^{k-1}dy,
\]

where \( F_{\sigma} = F|_{2k}\sigma \) with the usual weight \( 2k \) slash operator \( |_{2k} \) of \( \text{SL}_2(\mathbb{R}) \).

**Proof.** We can parametrize \( C_Q \) by \( y \mapsto \sigma iy \) with \( y \) running from \( e^2 \) to 1. Using that \( Q(Miy, 1) = (ciy + d)^{-2}(Q \circ M)(iy, 1) \) and \( \frac{d}{dy}Miy = i(ciy + d)^{-2} \) for every matrix \( M = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{R}) \), we easily obtain the stated formula. \( \square \)

2.2. Weak Maass Forms. A weak Maass form of weight \( 2\kappa \in 2\mathbb{Z} \) for \( \Gamma \) is a smooth function \( F : \mathbb{H} \to \mathbb{C} \) which is an eigenform of the weight \( 2\kappa \) Laplace operator

\[
\Delta_{2\kappa} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2\kappa iy \left( \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right),
\]

which transforms like a modular form of weight \( 2\kappa \) for \( \Gamma \) and which has at most linear exponential growth at the cusp. The Maass lowering and raising operators

\[
L_{2\kappa} = -2iy^2 \frac{\partial}{\partial z}, \quad R_{2\kappa} = 2i \frac{\partial}{\partial z} + 2\kappa y^{-1},
\]

map weak Maass forms of weight \( 2\kappa \) with eigenvalue \( \lambda \) to weak Maass forms of weight \( 2\kappa - 2 \) with eigenvalue \( \lambda - 2\kappa + 2 \) and weight \( 2\kappa + 2 \) with eigenvalue \( \lambda + 2\kappa \), respectively. We write

\[
L_{2\kappa}^n = L_{2\kappa-2n+2} \circ \ldots \circ L_{2\kappa}, \quad R_{2\kappa}^n = R_{2\kappa+2n-2} \circ \ldots \circ R_{2\kappa}
\]

for \( n \in \mathbb{N}_0 \) (with
Here we also used that the lowering and raising operators commute with the slash action by Bol’s identity

\[ \lambda \] harmonic weak Maass forms of weight 2 \( \kappa \) and write \( \xi \)

The operator \( D^{1-2\kappa} \) maps harmonic weak Maass forms of weight 2\( \kappa \) to weakly holomorphic modular forms of weight 2 \( -2\kappa \), see [3], Theorem 3.7. The above differential operators are related by

\[ (4) \quad -\Delta_{2\kappa} = \xi_{2-2\kappa} \xi_{2\kappa} = L_{2\kappa+2} R_{2\kappa} + 2\kappa = R_{2\kappa+2} L_{2\kappa}. \]

Using the relation (4), the following lemma is easy to prove by induction.

**Lemma 2.2.** For \( \kappa, \ell \in \mathbb{Z} \) we have

\[ \Delta_{2\ell} R_{2\kappa} - \kappa + \ell = R_{2\kappa} - \kappa + \ell (\Delta_{2\kappa} - (\kappa - \ell)(\kappa + \ell - 1)), \quad \ell \geq \kappa, \]

\[ \Delta_{2\ell} L_{2\kappa} - \kappa - \ell = L_{2\kappa} - (\kappa - \ell)(\kappa + \ell - 1)), \quad \ell \leq \kappa. \]

If \( \kappa \leq 0 \), then the raising operator and the differential operator \( D = \frac{1}{2\pi i} \frac{\partial}{\partial z} \) are related by Bol’s identity

\[ (5) \quad D^{1-2\kappa} = \frac{1}{(-4\pi)^{1-2\kappa}} R_{2\kappa}^{1-2\kappa}. \]

The operator \( D^{1-2\kappa} \) maps harmonic weak Maass forms of weight 2\( \kappa \) to weakly holomorphic modular forms of weight 2 \( -2\kappa \), see [3], Theorem 1.1.

### 3. Proof of Theorem 1.1

The equality \( C(L_{2-2k} F, Q) = C(\xi_{2-2k} F, Q) \) becomes obvious if we use the parametrization of the cycle integrals given in Lemma 2.1 and write \( \xi_{2-2k} = y^{-2k} L_{2-2k}. \)

Using Lemma 2.1 we see that \( C(L_{2-2k} F, Q) = C(R_{2-2k} F, Q) \) is equivalent to

\[ \int_{1}^{e^2} (L_{2-2k} F_{\sigma})(iy) y^{-k-1} dy = - \int_{1}^{e^2} (R_{2-2k} F_{\sigma})(iy) y^{-k-1} dy. \]

Here we also used that the lowering and raising operators commute with the slash action of \( SL_2(\mathbb{R}) \). A short calculation shows that

\[ (R_{2-2k} F_{\sigma})(iy) y^{-k} + (L_{2-2k} F_{\sigma})(iy) y^{-k-1} = 2 \frac{\partial}{\partial y} (F_{\sigma}(iy) y^{-k}). \]

The integral

\[ \int_{1}^{e^2} \frac{\partial}{\partial y} (F_{\sigma}(iy) y^{-k}) dy = F_{\sigma}(i \varepsilon^2) \varepsilon^{-2-2k} - F_{\sigma}(i) = \left( F_{\sigma}|_{2-2k} \left( \begin{array}{cc} \varepsilon & 0 \\ 0 & -\varepsilon \end{array} \right) \right)(i) - F_{\sigma}(i) \]

vanishes since \( F_{\sigma} \) transforms like a modular form of weight 2 \( -2k \) for \( \sigma^{-1} \Gamma \sigma \). This yields the first identity in Theorem 1.1.
The second and the third identity in Theorem 1.1 easily follow from the first one applied to $R_{2-2k}^{k-\ell-1}F$ and $L_{2-2k}^{-k-\ell+1}F$. For example, for the third identity we obtain

$$C\left(L_{2-2k}^{-k-\ell+2}F, Q\right) = C\left(L_{2\ell}L_{2-2k}^{-k-\ell+1}F, Q\right) = C\left(R_{2\ell}L_{2-2k}^{-k-\ell+1}F, Q\right)$$

from the first identity in Theorem 1.1. Now, using (4), we further compute

$$C\left(R_{2\ell}L_{2-2k}^{-k-\ell+1}F, Q\right) = C\left(R_{2\ell}L_{2\ell+2}L_{2-2k}^{-k-\ell}F, Q\right) = C\left(-\Delta_{2\ell+2}L_{2-2k}^{-k-\ell}F, Q\right),$$

and then, using Lemma 2.2,

$$C\left(-\Delta_{2\ell+2}L_{2-2k}^{-k-\ell}F, Q\right) = C\left(-L_{2-2k}^{-k-\ell}(\Delta_{2-2k} - (k + \ell)(k - \ell - 1))F, Q\right)$$
$$= ((k + \ell)(k - \ell - 1) - \lambda)C\left(L_{2-2k}^{-k-\ell}F, Q\right).$$

The computation for the second identity in the theorem is similar, so we leave it to the reader. This finishes the proof of Theorem 1.1.

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