Group Field Theory: An overview

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We give a brief overview of the properties of a higher dimensional generalization of matrix model which arises naturally in the context of a background independent approach to quantum gravity, the so-called group field theory. We show that this theory leads to a natural proposal for the physical scalar product of quantum gravity. We also show in which sense this theory provides a third quantization point of view on quantum gravity.

I. INTRODUCTION

“Pluralitas non est ponenda sine necessitate”, William of Ockham (1285-1349).

Spin foam models describe the dynamics of loop quantum gravity in terms of state sum models. The purpose of these models is to construct the physical scalar product which is one of the main object of interest in quantum gravity. Namely, given a 4-manifold $M$ with boundaries $\Sigma_0$, $\Sigma_1$ and given a diffeomorphism class of 3 metric $[g_0]$ on $\Sigma_0$ and $[g_1]$ on $\Sigma_1$ we want to compute

$$\langle [g_0]|P|[g_1]\rangle = \int_M D[g]e^{iS(g)},$$

the integral being over $M$: The space of all metrics on $M$ modulo 4-diffeomorphism which agree with $g_0, g_1$ on $\partial M$. The action is the Einstein Hilbert action and $P$ denotes the projector on the kernel of the hamiltonian constraint. This expression is of course highly formal, there is no good non perturbative\(^1\) definition of the measure on $M$ and no good handle on the space of kinematical states $\langle |g]\rangle$.

In loop quantum gravity there is a good understanding of the kinematical Hilbert space (see\(^2\) for a review). In this framework the states are given by spin networks $\Gamma$, where $\Gamma$ is a graph embedded in a three space $\Sigma$ and $j$ denotes a coloring of the edges of $\Gamma$ by representations of a group\(^3\) $G$ and a coloring of the vertex of $\Gamma$ by intertwiners (invariant tensor) of $G$. These states are eigenvectors of geometrical operators, the representations labelling edges of the spin network are interpreted as giving a quanta of area $\sqrt{\hbar j(j+1)}$ to a surface intersecting $\Gamma$.

In this context the spacetime is obtained as a spin network history: If one evolve in time a spin network it will span a foam like structure i.e a combinatorial 2-complex denoted $\mathcal{F}$. The edges of the spin network will evolve into faces of $\mathcal{F}$ the vertices of $\Gamma$ will evolve into edges of $\mathcal{F}$ and transition between topologically different spin networks will occur at vertices of $\mathcal{F}$. The spin network induced a coloring of $\mathcal{F}$: The faces of $\mathcal{F}$ are colored by representation $j_f$ of $G$ and edges of $\mathcal{F}$ are colored by intertwiners $i_e$ of $G$. Such a colored two complex $\mathcal{F}(j_f, i_e)$ is called a spin foam\(^4\).

By construction the boundary of a spin foam is an union of spin networks.

The definition is so far purely combinatorial, however if one restricts the 2-dimensional complex $\mathcal{F}$ to be such that $D$ faces meet at edges of $\mathcal{F}$ and $D+1$ edges meet at vertices of $\mathcal{F}$ we can reconstruct from $\mathcal{F}$ a $D$ dimensional piecewise-linear pseudo-manifold $M_\mathcal{F}$ with boundary $\partial M_\mathcal{F}$. Roughly speaking, each vertex of $\mathcal{F}$ can be viewed to be dual to a $D$ dimensional simplex and the structure of the 2-dimensional complex gives the prescription for gluing these simplices together and constructing $M_\mathcal{F}$. The spin network states are dual to the boundary triangulation of $M_\mathcal{F}$.

A local spin foam model is characterized by a choice of local amplitudes $A_f(j_f), A_e(j_{fe}, i_e), A_v(j_{ve}, i_{ve})$ assigned respectively to the faces, edges and vertices of $\mathcal{F}$: $A_f$ depends only on the representation coloring the face, $A_e$ on the representations of the faces meeting at $e$ and the intertwiner coloring the edge $e$, likewise $A_v$ depends only on the representations and intertwiners of the faces and edges meeting at $v$.

Given a two complex $\mathcal{F}$ with boundaries $\Gamma_0, \Gamma_1$ colored by $j_0, j_1$ the Transition amplitude is given by

$$A(\mathcal{F}) = \langle \Gamma_0 |\mathcal{F}| \Gamma_1 \rangle = \sum_{j_f, i_e} \prod_f A_f(j_f) \prod_e A_e(j_{fe}, i_e) \prod_v A_v(j_{ve}, i_{ve}),$$

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\(^3\) except in 2+1 dimension

\(^4\) In conventional loop quantum gravity the group is $SL(2, \mathbb{C})$, more generally $G$ is a Lorentz group
the sum being over the labelling of internal faces and edges not meeting the boundary. Note that a priori the amplitude depends explicitly on the choice of the two complex $\mathcal{F}$.

There are many examples of such models. Historically, the first example is due to Ponzano and Regge [6]: They showed that the quantum amplitude for euclidean $2 + 1$ gravity with zero cosmological constant can be expressed as a spin foam model where the group $G$ is $SU(2)$, the faces are labelled by $SU(2)$ spin $j_f$ and the local amplitudes are given by $A_f(j_f) = (2j_f + 1)$, $A_v(j_f, j_v) = 1$ and the vertex amplitude $A_v(j_f, j_v)$ which depends on 6 spins is the normalized $6j$ symbol or Racah-Wigner coefficient. The remarkable feature of this model is that it doesn’t depend on the choice of the two complex $\mathcal{F}$ but only on $\mathcal{M}_F$. The inclusion of a cosmological constant or the description of lorentzian gravity can be implemented easily by taking the group to be a quantum group [7] or to be a non compact Lorentz group [8]. Along the same line, it was shown that 4d topological field theory called $BF$ theory [9] can be quantized in terms of triangulation independent spin foam model [10].

It was first realized by M. Reisenberger that spin foam models give a natural arena to deal with 4d quantum gravity [11]. Two seminal works triggered more interest on spin foam models. In the first one, Barrett-Crane [12] proposed a spin foam model for 4d general relativity. This model is obtained from the spin foam model of pure $BF$ by restricting the Lorentz representations to be simple so that the spin coloring the faces are $SU(2)$ representations in the Euclidean context. In the second one, it was shown by Reisenberger and Rovelli [13] that the evolution operator in loop quantum gravity can be expressed as a spin foam model and they propose an interpretation of the vertex amplitude in terms of the matrix elements of the hamiltonian constraint of loop quantum gravity [14]. The spin labelling the faces are also $SU(2)$ representations and are interpreted as quanta of area. This construction has been exemplified in 2+1 gravity [15].

It was then realized that spin foam models can naturally incorporate causality [16], Lorentzian signature [17] and coupling to gauge field theory [18]. The Barrett-Crane prescription was understood to be linked to the Plebanski formulation of gravity where the Einstein action is written as a $BF$ theory subject to constraints [20]. This formulation and the corresponding spin foam models were extended to gravity in any dimensions [21]. The main lesson is that spin foams are a very general framework which allows to address in a background independent manner the dynamical issues of a large class of diffeomorphism invariant models including gravity in any dimensions coupled to gauge fields [21]. This formulation naturally incorporated the fact that the kinematical Hilbert space of the theory is labelled by spin networks [20].

A different line of development originated from the detail study of the vertex amplitude proposed by Barrett-Crane and the corresponding higher dimensional quantum gravity models [22,24]. These studies shows that these amplitudes can be written as some Feynman graph evaluation. For instance in the original Barrett-Crane model

$$A_v(j_1, \ldots, j_{10}) = \int_{S^3} \prod_{i=1}^5 dx_i \prod_{i \neq j} G_{j_i}(x_i, x_j),$$

where the ten spins are simple representations of $SO(4)$ labelling the 10 faces of the 4-simplex and $G_j(x,y)$ is the Hadamard propagator of $S^3$. $(\Delta_{S^3} + j(j + 1))G_j = 0, G(x, x) = 1$.

This structure was calling a field theory interpretation of spin foam models. It was eventually found in [25] that the Barrett-Crane spin foam model can remarkably be interpreted as a Feynman graph of a new type of theory baptized ‘group field theory’ ($GFT$ for short). The $GFT$ structure was first discovered by Boulatov [26] in the context of three dimensional gravity where a similar connection was made and further developed by Ooguri in the context of 4d $BF$ theory [10]. Ambjorn, Durhuus and Jonnson [27] also pointed out similar structure in the context of dynamical triangulation. It is clear in this context that group field theory can be understood in a precise sense as a higher dimensional generalization of matrix models which generate a summation over 2d gravity models [28].

Reisenberger and Rovelli [24] showed, in a key work, that the appearance of $GFT$ in the context of spin foam models is not an accident but a generic feature. They proved that any local spin foam model of the form [23] can be interpreted as a Feynman graph of a group field theory. We have argued that spin foam models generically appear in the context of background independent approach to quantum gravity [21], this result shows that $GFT$ is an important and unexpected universal structure behind the dynamics of such models. A deeper understanding of this theory is clearly needed. $GFT$ was originally designed to
address one of the main shortcomings of the spin foam approach: namely the fact that the spin foam amplitude depends explicitly on the discrete structure (the two complex or triangulation). As we will now see in more details it does much more than that and give a third quantization point of view on gravity where spacetime is emergent and dynamical.

II. GROUP FIELD THEORY

A. definition

In this section we introduce the general GFT action that can be specialized to define the various spin foam models described in the introduction.

We consider a Lie group $G$ which is the Lorentz group in dimension $D$ ($G = SO(D)$ for Euclidean gravity models and $G = SO(D - 1, 1)$ for Lorentzian ones $^7$. $D$ is the dimension of the spacetime and we will call the corresponding GFT a $D$-GFT. The field $\phi(x_1, \ldots, x_D)$, denoted $\phi(x_i)$ where $i = 1 \ldots D$, is a function on $G^D$. The dynamics is defined by an action of the general form

$$S_D[\phi] = \frac{1}{2} \int dx_i dy_i \phi(x_i) K(x_i y_i^{-1}) \phi(y_i) + \frac{\lambda}{D + 1} \int \prod_{i \neq j = 1}^{D+1} dx_{ij} V(x_{ij} x_{ji}^{-1}) \phi(x_{ij}) \cdots \phi(x_{D+1j}),$$

where $dx$ is an invariant measure on $G$, we use the notation $\phi(x_{ij}) = \phi(x_{12}, \ldots, x_{1D+1})$. $K(X_i)$ is the kinetic and $V(X_{ij}) (X_{ij} = x_{ij} x_{ji}^{-1})$ the interaction kernel, $\lambda$ a coupling constant, the interaction is chosen to be homogeneous of degree $D + 1$. $K, V$ satisfy the invariance properties

$$K(gX_i g') = K(X_i), \quad V(g_i X_{ij} g_j^{-1}) = V(X_{ij}) \quad \forall g, g', g_i \in G. \quad (4)$$

This implies that the action is invariant under the gauge transformations $\delta \phi(x_i) = \psi(x_i)$, where $\psi$ is any function satisfying

$$\int_G dg \psi(gx_1, \ldots, gx_D) = 0. \quad (5)$$

This symmetry is gauge fixed if one restricts the field $\phi$ to satisfy $\phi(gx_i) = \phi(x_i)$. The action is also invariant under the global symmetry

$$\phi(x_1, \ldots, x_D) \rightarrow \phi(x_1 g, \ldots, x_D g). \quad (6)$$

The main interest of these theories resides in the following crucial properties they satisfy. Most of them are well established, some are new (property 4) and some (property 6) still conjectural. Altogether they give a picture of the relevance of GFT for background independent approach to quantum gravity and lead to the conclusion (more precisely the conjecture) that GFT provides a third quantization of gravity.

GFT properties:

1. The Feynman graphs of a $D$-GFT are cellular complexes $\mathcal{F}$ dual to a $D$ dimensional triangulated topological spacetime $M_\mathcal{F}$.

2. The Feynman graph evaluation of a GFT is equal to the spin foam amplitudes of a local spin foam model. Conversely, any local spin foam model can be obtained from the evaluation of a GFT Feynman graph.

3. Spin networks label polynomial gauge invariant operators of the GFT.

4. The tree level two-point function of GFT gauge invariant operators gives a proposal for the physical scalar product. This proposal involves spacetime of trivial topology and is triangulation independent.

$^7$ It is also possible to generalize the definition to quantum groups or fuzzy group, we will restrict the discussion to compact group to avoid unnecessary technical subtleties.
5. The full two-point function of GFT gauge invariant operators gives a prescription for the quantum gravity amplitude including a sum over all topologies.

6. The possible loop divergences of GFT Feynman graphs are interpreted to be a consequence of a residual action of spacetime diffeomorphisms on spin foam. One expect a relation between the renormalization group of GFT and the group of spacetime diffeomorphisms.

B. GFT: Examples and Properties

In this section we give some examples and illustrate the properties listed above.

**Some examples** The simplest examples comes from the choice

\[
K(x_i, y_i) = \int_G dg \prod_i \delta(x_i y_i^{-1} g), \quad \mathcal{V}(X_{ij}) = \int \prod_i dg_i \prod_{i<j} \delta(g_i X_{ij} g_j^{-1}),(7)
\]

\(\delta(\cdot)\) is the delta function on \(G\) and the integrals insure the gauge invariance \((5)\).

If one further restricts to dimension \(D = 2\) the symmetry property implies that \(\phi(x_1, x_2) = \tilde{\phi}(x_1^{-1} x_2)\). \(\tilde{\phi}\), being a function on the group, can be expanded in Fourier modes. Lets consider \(G = SU(2)\), denote by \(V_j\) the spin \(j\) representation, \(d_j\) its dimension, \(D_j(x) \in \text{End}(V_j)\) the group matrix element and define

\[
\Phi_j = \int dx \tilde{\phi}(x) D_j(x^{-1}), \quad \tilde{\phi}(x) = \sum_j d_j \text{Tr}(\Phi_j D_j(x)). \quad (8)
\]

One can readily see \([30]\) that the GFT reduces to a sum of matrix models:

\[
S_2[\phi] = \sum_j d_j \left( \text{Tr}(\Phi_j^2) + \frac{\lambda}{3} \text{Tr}(\Phi_j^3) \right). \quad (9)
\]

It is well known that the Feynman graph expansion of a matrix model is expressed in terms of fat graphs \([28]\), each edge can be represented as a double line each one carrying a matrix index, the trivalent interaction implies that this graph is dual to a triangulation of a two dimensional closed surface. Moreover if one computes the Feynman evaluation of a genus \(g\) diagram \(\Gamma\) diagram we find

\[
I(\Gamma) = \sum_j d_j^{2g}, \quad (10)
\]

which is the evaluation of the partition function of topological \(BF\) theory in 2 dimension on a surface of genus \(g^8\). This result exemplifies the properties \([12]\). This property generalizes to any \(D\), the Feynman graph evaluation of the example \([4]\) gives the partition function of \(BF\) theory in dimension \(D\).

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8 If one choose the propagator \(K(x_i, y_i) = \int_G dg \prod_i \delta_i(x_i y_i^{-1} g)\) with \(\delta_i\) the heat kernel on the group, \((\partial_i + \Delta_G)\delta_i = 0, \delta_0(g) = \delta(g)\) we obtain the partition function of 2D Yang-mills theory as a Feynman graph evaluation \([33]\).
Property 1. To illustrate this property in higher dimension lets consider the case of dimension \( D = 3\), the field \( \phi \) possesses three arguments, so each edge of a Feynman graph possesses three strands running parallel to it, 4 edges meet at each vertex and the form of the interaction \( V \) forces the strands to recombine as in figure 1. Each strand of the graph forms a closed loop which can be interpreted as the boundary of a 2d disk. These data are enough to reconstruct a topological 2d complex \( F \), the vertices and edges of this complex correspond to vertices and edges or the Feynman graph, the boundary of the faces of \( F \) correspond to the strands of the Feynman graph. As we have already emphasized we can reconstruct a triangulated 3d pseudo-manifold from such data. It can be understood as follows: The three strands running along the edges can be understood to be dual to a triangle and the propagator gives a prescription for the gluing of two triangles. At the vertex 4 triangles meet and their gluing form a tetrahedra (see figure 2). With this interpretation the Feynman graph of a GFT is clearly dual to a three dimensional triangulation. This is true in any dimension \( \geq 2\).

This means that the perturbative expansion of the partition function can be expressed as a sum over two complexes

\[
Z = \int D\phi e^{-S_D[\phi]} = \sum_F \frac{\lambda^{|V|}}{\text{sym}(F)} I(F),
\]

where the sum is over two complexes, \(|V|\) is the number of vertices of \( F \), \( \text{sym}(F) \) the symmetry factor of \( F \) and \( I(F) \) the GFT Feynman graph evaluation of the complex \( F \).

Property 2. Since the field \( \phi \) is a function on \( D \) copies of the group it can be expanded in Fourier modes \( \mathbb{R}^D \), the ‘momentum’ of this field are spins which circulate along the strands of the Feynman graph or equivalently which label the faces of the complex \( F \). From the quantum gravity point of view this spin is interpreted as a quanta of area.

In order to make this correspondence more precise lets give a geometrical interpretation of the invariance property of the interaction kernel \( V \): Let \( \Gamma_D \) be the graph of a \( D \) dimensional simplex which consists of \( D + 1 \) vertices and \( D(D + 1)/2 \) edges. \( V \) is a function of group elements associated with the edges of \( \Gamma_D \) which is invariant under an action of the gauge group at the vertices of \( \Gamma_D \). We denote the space of such function by \( L^2(\Gamma_D) \). \( L^2(\Gamma_D) \) admits an orthonormal basis labelled by spin networks \( (\Gamma_D, j_{ij}, t_i) \), where \( j_{ij} \) are spins labelling the edges and \( t_i \) are intertwiners labelling the vertices of \( \Gamma_D \). Given \( (\Gamma_D, j_{ij}, t_i) \), we can uniquely construct a spin network functional \( \Theta_{j_{ij}, t_i}(X_{ij})^9 \).

The interaction kernel can be expanded in terms of this basis as follows

\[
A_v(j_{ij}, t_i) = \int \prod_{i<j} dX_{ij} V(X_{ij}) \Theta_{j_{ij}, t_i}(X_{ij}), \quad V(X_{ij}) = \sum_{j_{ij}, t_i} \prod_{i<j} d_{j_{ij}} A_v(j_{ij}, t_i) \Theta_{j_{ij}, t_i}(X_{ij}).
\]

Similarly, the quadratic kernel \( K(X_i) \) can be expanded in terms of the spin network functional \( \Phi_{j_i}(X_i) \) associated with the ‘theta’ graph which consists of two vertex joined by \( D \) edges:

\[
1/A_v(j_i) = \int \prod_i dX_i K(X_i) \Theta_{j_i}(X_i), \quad K(X_i) = \sum_{j_i} \prod_i d_{j_i} 1/A_v(j_i) \Theta_{j_i}(X_i).
\]

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9 Given a spin network \( (\Gamma, j_e, t_\nu) \) the spin network functional \( \Theta_{\Gamma, j_e, t_\nu}(x_e) \) is obtained by contracting the matrix elements of \( x_e \) in the representation \( J_e \) with the intertwiners \( t_\nu \) according to the topology of the graph \( \Gamma \). By construction this functional is invariant under the action of the group at each vertex of \( \Gamma \).
It is now a direct computation to show that \( I(\mathcal{F}) \) is expressed as a local spin foam model with the edges and vertex amplitude determined by \([12,13]\) and the face amplitude being the dimension of the representation labelling the face\(^{10}\). Conversely, given a spin foam model we can reconstruct a GFT via \([12,13]\). This establishes the equivalence or duality between spin foam models and GFT, which was first proven in \([24]\). One can now check that the example \([7]\) gives in dimension 3 the Ponzano-Regge model and in higher dimensions the discretization of topological BF. With this prescription we can also reconstruct from the Barrett-Crane amplitude \([3]\) the interaction kernel:

\[
\mathcal{V}_{BC}(X_{ij}) = \int_G \prod_i dg_i \prod_i dh_i \int_H \prod_{i \neq j} dw_{ij} \prod_{i < j} \delta(g_i u_{ij} h_i X_{ij} (g_j u_{ji} h_j)^{-1}),
\]

(14)

where \( G = \text{SO}(4) \) and \( H = \text{SO}(3) \subset \text{SO}(4) \).

**Property 3.** We have discussed so far only the partition function of the GFT, but one should also consider expectation values of GFT operators. The physical operators \( \mathcal{O}(\phi) \) should be gauge invariant under \([5,6]\). Such operators can be constructed with the help of spin network: Let’s consider a spin network \((\Gamma, j_e, t_u)\) such that all its vertices have valency \( D \), and let’s denote \( \Theta_{(\Gamma, j_e, t_u)}(x_e) \) the corresponding spin network functional (see footnote \([9]\)). We denote by \( V_T, E_T \) the set of vertices of \( \Gamma \) and define the observable of the \( D \)-GFT

\[
\mathcal{O}_{(\Gamma, j_e, t_u)}(\phi) = \int_G \prod_{(ij) \in E_T} dx_{ij} dx_{ji} \Theta_{(\Gamma, j_e, t_u)}(x_{ij}) (x_{ji})^{-1} \prod_{i \in V_T} \phi(x_{ij}).
\]

(15)

The element \( x_{ij} \) are associated to all the edges of \( \Gamma \) meeting at the vertex \( i \), by construction there is always \( D \) such elements. This observable is homogenous in \( \phi \), the degree of homogeneity being the number of vertices of \( \Gamma \). It is straightforward to check that this observable respect the symmetries of the GFT.

**Property 4.** We can now come back to the original problem, that is the construction of the physical scalar product. Since spin networks label operators of the GFT, we propose to define this scalar product as the evaluation of the GFT two point function in the tree level truncation. Namely, given two \( D \) valent spin network \( \Gamma_1, \Gamma_2 \) having \( N_1, N_2 \) vertices we define

\[
\langle \Gamma_1 | \Gamma_2 \rangle_0 \equiv \langle \mathcal{O}_{\Gamma_1} | \mathcal{O}_{\Gamma_2} \rangle_{\text{tree}} = \sum_{\mathcal{F} \in \mathcal{T}_{N_1, N_2}} \frac{I(\mathcal{F})}{\text{sym}(\mathcal{F})}.
\]

(16)

\( \mathcal{T}_{N_1, N_2} \) denote the space of GFT Feynman graphs supported on connected trees having \( N_1 \) initial univalent vertices and \( N_2 \) final univalent vertices, we sum over all of them.

This simple proposal for the scalar product does not depend on a particular triangulation, and therefore it addresses one of the main shortcomings of the spin foam approach. A related idea has been already been considered by Gambini and Pullin in a different context \([32]\). This product satisfies two crucial properties: First, it is well defined and finite, since it is a tree level evaluation no infinite summation are involved. Second, it is positive but not strictly positive, it possesses a kernel. This kernel should be expected, since the physical scalar product of quantum gravity \([11]\) computes the matrix elements of the projector on the kernel of the hamiltonian constraints. This means that any vector in the image of the hamiltonian constraint belongs to the kernel of \([11]\). In our case, one can show that the GFT scalar product \([11]\) has a kernel which is in the image of the GFT equation of motion. Namely, the following gauge invariant observable

\[
\delta \mathcal{O}_{(\Gamma, j_e, t_u)}(\phi) = \int_G \prod_i dx_i \left( K^{-1} \frac{\delta S[\phi]}{\delta \phi} \right)(x_i) \frac{\delta \mathcal{O}_{(\Gamma, j_e, t_u)}(\phi)}{\delta \phi(x_i)}.
\]

(17)

which is proportional to the GFT equation of motion, is in the kernel of \([11]\). In this formula \( K^{-1} \) is the propagator, it is convoluted with the equation of motion. We can expand this observable as a linear combination of spin network observables, the first term in the expansion of \( \delta \mathcal{O}_\Gamma \) is \( \mathcal{O}_\Gamma \) the other terms are spin network observables containing \( D \) more fields.

The physical Hilbert space can be constructed as an application of the Gelfand-Naimark-Segel theorem. It is obtained from the kinematical Hilbert space spanned by spin networks, by quotienting out the vectors in the kernel of \([11]\): \( \mathcal{H}_{\text{phys}} = \mathcal{H}/\text{Ker}(\gamma|) \). \([33]\). The induced scalar product on \( \mathcal{H}_{\text{phys}} \) is positive definite.

The product \([11]\) involves only tree Feynman graphs. Using the correspondence between GFT Feynman graphs and discrete manifolds one sees that all the manifolds involved in the sum are of the same topology and describe

\(^{10}\) It is possible to produce different face amplitude by modifying the symmetry properties of the action \([23]\).
a ball on the boundary of which the operators are inserted. Since this product \( \text{(16)} \) is independent of the choice of triangulation it can be thought as a ‘continuous’ scalar product. One might worry that this is realized without taking any sort of continuum or refinement limit and therefore that this prescription describes some sort of topological field theory. This is not the case, in this prescription the complexity of the spacetime triangulation involved in the summation grows with the complexity of the boundary spin network state. If one considers highly complicated spin network states that approach a continuum geometry, the corresponding spin foams (discrete spacetimes) involved in the summation are also highly complicated and good approximation of a continuous geometry. Also, we have seen that \( \mathcal{O}_T \) is a linear combination of \( \mathcal{O}_T \) and higher order spin network operators. If we think of \( \Gamma \) as being dual to a space triangulation, the higher order spin network operators are dual to a refined triangulation. We can therefore replace in the computation of the scalar product, the state \( \Gamma \) by a linear combination of spin network states of higher degree, and continue this replacement in each new terms ad infinitum, therefore ending with a expression for the scalar product in terms of a linear combination of arbitrarily fine triangulations which gives a ‘true’ continuum limit expression of the scalar product. This use of the kernel expresses the fact that a subset of the Hilbert space based on a fine triangulation can be effectively described in terms of states leaving on a coarser one. The theory is not topological if this subset is a proper subspace (this is the case for the Barrett-Crane model for instance).

This proposal for a physical scalar product in the context of loop quantum gravity closes in some sense a long quest starting from the construction of the hamiltonian constraints, the search for its solutions and the construction of the physical scalar product. It gives us a hint on the answer to the last question (which contains the others), it doesn’t end the quest but provides a new starting point. The problem is now to understand the dynamical content which is contained in such a proposal and to see whether at least one GFT (the Barrett-Crane one for instance) possesses the right dynamical content and can reproduce the physics of general relativity in the infrared. This question is not new, but with the help of the GFT it can be asked for the first time in terms of a proposed physical scalar product. The difficulty resides in the fact that the dynamics is encoded in terms of spin networks transition amplitudes, a language far remote from semiclassical physics and one needs to design criteria to select the right model or to test and eventually refute a proposed one.

**Property 5.** The previous scalar product can be naturally extended to include a sum over all Feynman graphs of the GFT, this was the original proposal \( \text{(22)} \), the gravity amplitude is in this case

\[
\langle \Gamma_1|\Gamma_2 \rangle_\lambda = \ln \left[ \frac{\lambda^{N_1+N_2}}{Z} \int \mathcal{D}\phi \mathcal{O}_T(\phi) e^{-S_D[\phi]} \right] = \sum_{\mathcal{F}, \partial \mathcal{F} = \Gamma_1+\Gamma_2} \frac{\lambda^{V-\frac{N_1+N_2}{D-1}}}{\text{sym}(\mathcal{F})} I(\mathcal{F}).
\]

where \( |V| \) is the number of vertices of \( \mathcal{F} \), this correspond to the number of \( D \) simplex of the dual triangulation, the sum is over connected graph matching the given spin network on the boundary, \( N_1, N_2 \) are the degree of homogeneity of the operators \( \mathcal{O}_{\Gamma_1}, \mathcal{O}_{\Gamma_2} \) and \( Z \) is the partition function \( \text{(11)} \). The coupling parameter \( \lambda \) weights, in the perturbation expansion, the size of the discrete spacetime. Indeed its power in the perturbative expansion is the number of \( D \)-dimensional simplices, which can be understood as a ‘spacetime volume’ in discrete units. It can be given another interpretation: Let’s define \( \alpha = \lambda^{1/(D-1)} \) and lets redefine the fields \( \bar{\phi} = \alpha \phi \), the action becomes \( S_\alpha[\phi] = 1/\alpha^2 \bar{S}[\bar{\phi}] \) where \( \bar{S} = S_{\lambda=1} \) is independent of the coupling constant. The amplitude can be expanded in \( \alpha \), \( \langle \Gamma_1|\Gamma_2 \rangle_\alpha = \alpha^2 \sum_i \alpha^{2i} \langle \Gamma_1|\Gamma_2 \rangle_i \) where \( \langle \Gamma_1|\Gamma_2 \rangle_i \) is a sum of GFT Feynman graphs containing \( i \) loops. From the space time point of view adding a loop to a GFT Feynman diagram amounts to adding a handle to the discrete manifold. Hence \( \alpha \) controls the strength of topology change. In the limit \( \alpha = 0 \), we recover the classical evaluation \( \text{(19)} \) where topology change is suppressed. This can be also understood by looking at the Schwinger-Dyson equation of motion. Let’s focus for simplicity on the nucleation amplitude where \( \Gamma_1 \) is empty so that \( \mathcal{O}_{\Gamma_1} = 1 \) which describes the creation of a spacetime from nothing. The Schwinger-Dyson equation reads (see figure \( \text{(3)} \))

\[
\langle \delta \mathcal{O} \rangle_\alpha = \alpha^2 \langle \delta^2 \mathcal{O} \rangle_\alpha,
\]

where \( \delta \mathcal{O}_T \) \( \text{(17)} \) is in the kernel of the physical scalar product \( \langle \cdot | \cdot \rangle_0 \) as can be easily seen from taking the limit \( \alpha \to 0 \) in \( \text{(19)} \), and

\[
\delta^2 \mathcal{O}_T = \int_{\mathcal{G}} \prod_i dx_i dy_j K^{-1}(x_i, y_j) \frac{\delta^2 \mathcal{O}_T(\phi)}{\delta \phi(x_i) \delta \phi(y_j)}.
\]

We have seen that the field \( \phi \) is dual to a \( D - 1 \) simplex, the operator \( \delta \mathcal{O}_T \) corresponds to a sum of spin network boundary states, one being a triangulation dual to \( \Gamma \) the others obtained by subdividing one of the \( D - 1 \) simplex of this triangulation. The operator \( \delta^2 \mathcal{O}_T \) deletes two \( D - 1 \) simplices and glues the resulting holes together thus creating a handle. This handle creation is weighted by \( \alpha^2 \).
Note that the scalar product $\langle \cdot | \cdot \rangle_\alpha$ is strictly positive and doesn’t possess any non trivial kernel since $\langle O|O \rangle_\alpha$ is the integral of a positive quantity\(^{11}\). The states $\delta \Omega$ which are in the kernel of $\langle \cdot | \cdot \rangle_0$ now generates topology change \(^{19}\).

Since we now include Feynman graphs with loops we have to worry about potential perturbative divergences due to the Feynman graph evaluation. A careful analysis shows that the potential divergences of the GFT are not associated with loops but with higher dimensional analogs: The so called ‘bubbles’ of the spin foam \(^{34}\). A bubble is a collection of faces of the 2 complex $F$ which forms a closed surface. Each time a bubble appears the sum over spins (GFT momenta) is unrestricted and potentially infinite. A remarkable finiteness result was proven in \(^{35}\) for the Barrett-Crane model. It was shown that if one takes the interaction kernel \(^{14}\) and the propagator \(^{77}\), there are no divergences arising in the computation of Feynman graph (associated with regular 2d complex), the corresponding GFT is super-renormalizable in this case.

There is also a possibility of potential non-perturbative divergences which arises from the sum over topology. It is well known that the number of triangulated manifold of arbitrary topology grows factorially with the number of building block and the sum \(^{18}\) is therefore not convergent for any non zero value of $\alpha$. This is not surprising from the GFT point of view since we know that a perturbative expansion should be interpreted as an asymptotic series, not a convergent series. In some cases, this series is uniquely Borel summable and the GFT provides a non perturbative definition for the sum over all topologies. This was shown to be true in the context of 3d gravity where Borel summability of a mild modification of the Boulatov model was proven \(^{36}\). It is not known whether this can be achieved in 4d gravity.

**Property 6** As we already mentioned, the key open problem is to gain an understanding of the low energy effective physics from the GFT. One proposal for addressing this question is to focus on the issue of the diffeomorphism symmetry. We know that any theory which depends on a metric reproduces usual gravity plus higher derivative corrections in the infrared if it is invariant under spacetime diffeomorphism. In loop quantum gravity, spin networks label the gravitational degree of freedom, this suggests that any spin foam model which can be shown to respect spacetime diffeomorphism will contain gravity in a low energy limit. The problem is therefore to have a proper understanding of the action of spacetime diffeomorphism on spin foam models. It was argued in \(^{37}\) (and exemplified in the context of 3d gravity) that diffeomorphism symmetry should act as a gauge symmetry on the spin foam amplitudes. This means that the initial spin foam amplitudes \(^{2}\) which are not gauge fixed with respect to this symmetry and which do not break diffeomorphism symmetry should possess divergences coming from the ungauged integration over the diffeomorphism gauge group. Diffeomorphism symmetry is due to the Bianchi identity which is a three form on space time and then couples to the bubbles of spin foams. Therefore diffeomorphism symmetry should manifest itself in the bubble divergences.

This analysis leads to the conclusion that the Barret-Crane GFT model proposed in \(^{34,38}\) which has no bubble divergences is not a satisfactory model\(^{12}\). From the GFT point of view the bubble divergences is analogous to the loop divergences in usual field theory. We know that such divergences are the manifestation of a non trivial renormalization group acting on the parameter space of field theory.

Since, as we have seen in this note, any relevant property of spin foam model admits its dual formulation in the GFT, this strongly suggest that a proper understanding of the action of the diffeomorphism group on spin foam models is related to a proper understanding of the GFT renormalization group that needs to be further developed \(^{39}\).

\(^{11}\) Of course this integral is well defined only if we introduce a cutoff on the ‘momentum’ (spin) of the field. So more work is needed in each particular model to prove that such a statement is valid in the unregularised theory.

\(^{12}\) The prescription of \(^{34,38}\) differs from the original prescription \(^{28}\) (which possess bubble divergences) by the choice of the kinetic term of the GFT. This kinetic term controls the way different vertex amplitude are glued together. There is a large consensus and good understanding of the Barrett-Crane vertex amplitude but so far, no general agreement on the choice of the kinetic term has been reached. Different choices leads to different properties with respect to the bubble divergences. Of course this argumentation is not yet conclusive since it contain hypothesis and unresolved issues.
In this review we have seen that the GFT is a universal structure hidden behind the attempt of dealing with quantum gravity in a background independent manner. This attempt leads naturally to loop quantum gravity whose dynamics is governed by local spin foam models. Such spin foam models are all expressed as Feynman graph evaluation of a GFT. In analogy with usual field theory whose Feynman graphs describe the dynamics of interacting relativistic particles, this leads to the point of view that GFT is a third quantization of gravity. If one takes this property seriously this forces us to a change of point of view in which GFT is more fundamental than spin foam models. This has various consequences, first the classical equation of motion of group field theory is related to the Wheeler-DeWitt equation expressed in loop variables and a natural proposal for a triangulation independent physical scalar product is obtained. Second, the quantization of the GFT leads to a proposal for quantum gravity amplitudes including sum over topologies.

We will like this letter to be an invitation for the reader to look more closely and further develop the GFT's as a third quantized version of gravity. As we have argued, in order to insure that these theories effectively encode the dynamics of General relativity one needs to gain an understanding on the action of diffeomorphisms on spin foams model and its counterpart in GFT, presumably implemented as a renormalisation group. We have discussed so far pure gravity models and a consistent inclusion of matter fields and particles in the GFT framework is clearly needed. Finally, an understanding of the physical meaning and properties of GFT instantons will provides us a window into the non perturbative physics of these theories.

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