A Refinement of Carlson’s Theorem

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Abstract

Carlson’s theorem estimates the growth of an analytic function along the imaginary axis, provided that the function is zero at non-negative integers. We refine this theorem and describe not only the function’s growth but also necessary and sufficient conditions in terms of its spectral measure.

Keywords: Carlson’s theorem, Paley-Wiener type theorems, sweeping measures, Hardy spaces, Luzin-Privalov’s boundary uniqueness principle, completeness of exponentials.

MSC Codes: 30C15 (Primary), 30A10, 30B50, 30D10, 30E05 (Secondary).

1. Formulation

1.1. Carlson’s theorem (see [1]) estimates the growth of an analytic function along the imaginary axis, provided that the function is zero at non-negative integers. Namely,

**Theorem 1.1.1 (Carlson).** Let $g$ be holomorphic in the closed right half plane,

$$g \in \text{Hol}(\Re(z) \geq 0),$$

and let $g$ vanish at non-negative integers,

$$g(n) = 0, \quad \text{for all } n \in \mathbb{Z}_+.$$  

(1.1.1) (1.1.2)

Then the following dichotomy holds: either $g \equiv 0$ or $g$ must grow substantially fast on the imaginary axis,

$$\limsup_{y \in \mathbb{R}, |y| \to +\infty} \frac{\ln |g(iy)|}{|y|} \geq \pi.$$  

(1.1.3)

In other words, under assumptions (1.1.1) and (1.1.2), Carlson’s theorem provides a necessary condition (1.1.3). In this paper, under weaker assumptions (1.1.4) and (1.1.5), we find the necessary and sufficient condition (1.1.7). That is, instead of estimating the function’s growth, we describe its spectral measure. Namely, we prove:

**Theorem 1.1.2.** Let the function $g$ be holomorphic and of (at most) exponential type in the open right half plane, that is, for some $a \in \mathbb{R}, b \geq 0$ we have

$$\sup_{x \geq 0, \delta \in \mathbb{R}} \frac{\ln |g(x + iy)|}{ax + b|y| + \epsilon|z|} < +\infty, \quad \text{for all } \epsilon, \delta > 0.$$  

(1.1.4)

Then the function $g$ vanishes at non-negative integers,

$$g(n) = 0, \quad \text{for all } n \in \mathbb{N},$$  

(1.1.5)

if and only if it can be represented as an integral of exponents via a spectral measure $\mu$,

$$g(z) = \int_{\mathbb{C}} e^{\omega} d\mu(\omega)$$

with the additional restriction on $\mu$ that, if we introduce the circular-periodized equivalent of $\mu$ by

$$\nu(e^E) = \mu(E + \cup_{k \in \mathbb{Z}} [i \cdot 2\pi k]), \quad \text{for all } E \subset \mathbb{R} + i[-\pi, \pi)$$

(1.1.6)

then the measure $\nu$ has bounded support, and the Cauchy type integral $H$ of $\nu$’s sweeping measure belongs to the (complex-valued) Hardy space $\mathbb{H}_1$ and is zero at the origin,

$$H \in \mathbb{H}_1, \quad H(0) = 0.$$  

(1.1.7)
1.2. The paper is composed as follows: section 2 contains a brief history of the subject, section 3 is devoted to the proof of theorem 1.1.2 by using Brown-Shields-Zeller theorem about balayage and some basic facts about Hardy spaces in order to specialize Morimoto’s integral representation to assumption (1.1.5). Section 4 contains a derivation Carlson’s theorem from theorem 1.1.2 by referring to Luzin-Privalov’s boundary uniqueness principle. Section 5 discusses the analogue of our proof of Carlson’s theorem for the polynomial case.

2. Brief history

Carlson’s theorem was first proved in [1]. See [2], p.330 for an alternate proof by a certain summation formula. See [3] for a proof by Plana’s summation formula that uses Morimoto’s integral representation indirectly.

An active area of research constitutes substituting the condition (1.1.2) to the assumption that $g$ vanishes on a different set. See e.g. [4], [5], [6]. A general condition for when $g$ vanishes on a sequence is provided in [7].

Some of the applications of Carlson’s theorem are: the uniqueness of analytic continuation, Dyson conjecture in statistical mechanics, calculation of Selberg integral (as exposed in [8]). It is also closely related to completeness of systems of exponentials (see [9]).

3. Proof of the Main Theorem 1.1.2

3.1. We define the classes $\text{Exp}_{a,b}$ of exponential functions in the right half plane and then describe a Paley-Wiener type theorem for them. For $-\infty < a < +\infty$, $0 \leq b < +\infty$, define $\text{Exp}_{a,b}$ to be the space of functions $g$ holomorphic in the open right half plane, whose growth is bounded by the constants $a, b$ as follows:

$$\text{Exp}_{a,b} = \left\{ g(x + iy) \in \text{Hol}(x > 0) : \forall \epsilon, \delta > 0 \left\{ \sup_{x \geq \delta y \in \mathbb{R}} \frac{\ln |g(x + iy)|}{a x + b |y| + |\epsilon|} < +\infty \right\} \right\}$$

**Theorem 3.1.1 (Morimoto).** For any $g \in \text{Exp}_{a,b}$ and $\epsilon > 0$ there exists a finite (complex-valued) spectral measure $\mu$: $\mathbb{C} \to \mathbb{C}$ with (possibly unbounded) support

$$\text{supp} (\mu) \subset (-\infty, a + \epsilon] + i[-b - \epsilon, b + \epsilon]$$

such that the function $g$ may be expressed as the following integral of exponents

$$g(z) = \int_{\omega \in \mathbb{C}} e^{z \omega} d\mu(\omega).$$

**Proof.** For a proof, see theorem 5.1 in [12] together with proposition 2.1.1 in [13]. Also, to recover the exact phrasing that we used, insert $\delta = 0$, $\epsilon' = 1$, $e^{\omega \Delta} \mu(\omega) = d\mu(\omega)$ in theorem 1.1 and proposition 2.4 in [14]. □

3.2. While keeping the notations introduced in [3.1], define the measure $\nu$ as the circular-periodized equivalent of the measure $\mu$. Namely, relate the measures $\mu$ and $\nu$ by (1.1.6). We now translate the properties of $\mu$ into those of $\nu$. Firstly, it follows from condition (3.1.1) that the measure $\nu$ has bounded support. Also, in view of integral representation (3.1.2) of the function $g$ and definition (1.1.6) of the measure $\nu$, the conditions (1.1.5) are equivalent to the following condition on the measure $\nu$:

$$\int_{\mathbb{C}} z^n \nu = 0, \quad \text{for all } n \in \mathbb{Z}_+.$$  

(3.2.1)

3.3. Assume that the measure $\nu$ is non-trivial, $\nu \neq 0$. Then the number

$$r = \sup_{z \in \text{supp} (\nu)} |z|$$

(3.3.1)

is well-defined. The measure $\nu$ has bounded support, so that $r$ is finite. Consider the open disc

$$D = \{ \zeta \in \mathbb{C} : |\zeta| < r \}.$$  

(3.3.2)

We will use a “balayage” type theorem [3.3.1] to sweep the measure $\nu$ (whose support lies in the closed disc $\overline{D}$) into the boundary $\partial D$ of that disc. By sweeping, we will reduce the case of the planar measure $\nu$ to the case of a linear measure. Indeed, introduce two measures associated with the measure $\nu$: the internal measure $\nu_{\text{int}}$ - the planar measure which is the restriction of $\nu$ to $D$, and the external measure $\nu_{\text{ext}}$ - the linear measure which is induced by $\nu$ on $\partial D$.

**Theorem 3.3.1 (Brown, Shields, Zeller).** For a bounded measure $\nu_{\text{int}}$ on the open disc $D$, there exists a function $h_{\text{int}} \in L_1(\partial D)$ such that for all bounded analytic functions $f$ on $D$ we have

$$\int_{D} f \nu_{\text{int}} = \int_{\partial D} f \cdot h_{\text{int}},$$

(3.3.3)

**Proof.** For a proof see [15]. Also, to recover the exact phrasing that we used, insert $\mu = \nu$ into formula (4.23.1) of [16]. □
3.4. As a consequence of theorem 3.3.1 we may write
\[ 0 = \int_{D} \xi^\alpha d\nu = \int_{D} \xi^\alpha d\nu = \int_{\partial D} \xi^\alpha \cdot (h_{\text{int}} + d\nu_{\text{ext}}), \quad \text{for all } n \in \mathbb{Z}_+, \] (3.4.1)

where the first equality is condition (3.2.1), the second equality is implied from the definition of $D$ (see (3.3.2), (3.3.1)), and the third equality comes from the sweeping argument (3.3.3). By the F. and M. Riesz theorem (see e.g. [17]), equality (3.4.1) implies that the measure $\nu_{\text{ext}}$ is absolutely continuous on $\partial D$. Denote its Radon-Nikodym derivative (with respect to the linear Lebesgue measure on $\partial D$) by $h_{\text{ext}} \in L_1(\partial D)$. For brevity, denote $h = h_{\text{int}} + h_{\text{ext}} \in L_1(\partial D)$ and rewrite (3.4.1) as
\[ \int_{\partial D} \xi^\alpha \cdot h = 0, \quad \text{for all } n \in \mathbb{Z}_+. \] (3.4.2)

Denote by $H$ the Cauchy type integral of the function $h$, or equivalently
\[ H(\zeta) = \sum_{n=0}^{\infty} a_n \xi^n, \quad a_n = \int_{\partial D} \xi^n \cdot h. \]

By theorem 3.12 in [11], the condition (3.4.2) is equivalent to claiming that $H$ is in the (complex-valued) Hardy space $\mathbb{H}_1$ and is zero at the origin, that is condition (1.1.7) holds. Moreover, a review of section 3 shows that conditions (1.1.5) imposed on the function $g$ are in fact equivalent to the condition (1.1.7).

4. Derivation of Carlson’s theorem

4.1. Let $g$ be holomorphic in the closed right half plane, $g \in \text{Hol}(\text{Re}(z) \geq 0)$. Recall that the indicator of $g$ is defined to be the following function:
\[ h_i(\theta) = \limsup_{r \to \infty} \frac{\ln |g(re^{i\theta})|}{r}, \quad -\pi/2 \leq \theta \leq \pi/2. \]

For brevity denote $I^* = h_i(\pi/2)$, $I_0 = h_i(-\pi/2)$. Assume that $I^*, I_0 < +\infty$. From trigonometric convexity of the indicator, it follows that for some $a \in \mathbb{R}$ we have $g \in \text{Exp}_{a, \text{max}(I^*, I_0)}$. Pick any $\epsilon > 0$. Let $\mu$ be the corresponding spectral measure provided by theorem 3.1.1. Denote the effective endpoints of the integral in formula (3.1.2) to be
\[ b^* = \sup_{\omega \in \text{supp}(\mu)} \text{Im}(\omega), \quad b_0 = \inf_{\omega \in \text{supp}(\mu)} \text{Im}(\omega). \]

From the description (3.1.1) of $\mu$’s support, we estimate
\[ \max(|b_0|, |b^*|) \leq \max(I^*, I_0), \] (4.1.1)

4.2. We now deduce Carlson’s theorem 1.1.1 from theorem 1.1.2. Indeed, let the inequality (1.1.3) claimed by Carlson’s theorem not hold, that is
\[ \max(I_0, I^*) < \pi. \] (4.2.1)

By (4.1.1), the restriction (4.2.1) implies
\[ -\pi < b_0 \leq b^* < \pi. \] (4.2.2)

We now reduce the case of two-dimensional spectral measure $\mu$ to that of one-dimensional, as claimed in (4.2.4). Specifically, assume $g \not\equiv 0$, so that the disc $D$ is well-defined by (3.3.2). We may write
\[ g(z) = \int_{\mathbb{C}} e^{z\omega} d\mu(\omega) = \int_{D} \xi^\alpha d\nu(\xi) = \int_{\partial D} \xi^\alpha \cdot h(\xi) = \int_{\text{Int}(r)+i(-\pi,\pi)} e^{\omega\xi} \cdot h(\omega), \quad \text{Re}(z) > 0. \] (4.2.3)

Here, the first equality is the integral representation (3.1.2). The second equality doesn’t hold for arbitrary measures $\mu$, as the change of variables involved in passing from integration over $\mu$ to integration over $\nu$ may not be one-to-one, or, in other words, the function $\zeta \to \xi^\alpha$ is, generally speaking, multivariate. However, due to restriction (4.2.2) on the support of measure $\mu$, the mapping $\zeta = \omega^\alpha$ is one-to-one on $\omega \in \text{supp}(\mu)$. Also, due to restriction (4.2.2), while integrating a function over the measure $\nu$, the values of that function on the ray $\arg(\zeta) = -\pi$ are not taken into account. Thus, while talking about the integral of the function $\zeta \to \zeta^\omega$ over the measure $\nu$, we can think of the branch of that function, generated by splitting the complex plane by the ray $\arg(\zeta) = -\pi$. The third equality follows from applying the theorem 3.3.1 about balayage to the bounded analytic function $\zeta \to \zeta^\omega$: here, the fact that it is indeed a bounded analytic function follows from the restriction $\text{Re}(z) > 0$. The last equality is obtained by a change of variable. For brevity, rewrite (4.2.4) as an integral representation of the function $g$ by a one-dimensional spectral measure,
\[ g(z) = \int_{\text{Int}(r)+i(-\pi,\pi)} e^{z\omega} \cdot h(\omega), \quad \text{Re}(z) > 0. \] (4.2.4)
By \cite{13}, p.323 (or by corollary 6.9.4 in \cite{19}, p.108), the restriction (4.2.1) implies
\[-\pi < \inf_{\zeta \in \text{supp}(h)} \arg(\zeta) \leq \sup_{\zeta \in \text{supp}(h)} \arg(\zeta) < \pi\] (4.2.5)

Recall Luzin-Privalov’s boundary uniqueness principle for the (complex-valued) Hardy space $\mathbb{H}_1$ (see e.g. \cite{10}, pp. 73-74, or \cite{11}, p.102 for our exact formulation):

**Theorem 4.2.1** (Luzin-Privalov). Let $f \in \mathbb{H}_1$. Then the following dichotomy holds: either $f \equiv 0$ or $f$ can vanish only on a subset of the unit circle $|z| = 1$ of measure zero.

Remarkably, Luzin-Privalov’s principle is phrased as a dichotomy, just as Carlson’s theorem does. By theorem 4.2.1, the restriction (4.2.5) implies $h \equiv 0$. By (4.2.4), we have $g \equiv 0$, yet we assumed the converse. The proof of Carlson’s theorem is now complete.

5. Relation to the polynomial case

5.1. We remark that if $p$ is a certain polynomial with (at least) $n$ zeros, then the following dichotomy holds: either $p \equiv 0$ or $p$ must grow substantially fast, namely
\[\lim_{x \to \infty} \frac{\ln |p(x)|}{\ln |x|} \geq n.\]

Carlson’s theorem is the analogue of this remark for analytic functions.

5.2. Remark 5.1 about polynomials may be proved by polynomial remainder (little Bezout) theorem. Similarly, Carlson’s theorem may be proved by using the condition (1.1.2) to express $g$ as an infinite product, and by estimating the growth of $g$ from this expression (see e.g. \cite{2}).

However, there is another way to prove remark 5.1 about polynomials. Namely, the polynomial $p$ is a sum of monomials; further, the rate of growth of that sum coincides with that of its main term; finally, the proof will follow from the maximum principle - if the polynomial $p$ is non-zero, then the polynomial $p$ and its main term have the same number zeros in $\mathbb{C}$. Hence, remark 5.1 follows. In 5.2, we proved Carlson’s theorem in a similar way. Namely, the holomorphic function $g$ turned out to be an integral of exponents; further, the rate of growth of that integral was determined by its effective endpoints; finally, the proof followed from Luzin-Privalov’s boundary uniqueness principle, an analogue of maximum principle.

5.3. We also note that, if (some of the) zeros of a non-zero polynomial are given explicitly, say,
\[p(0) = p(1) = \cdots = p(n - 1) = 0,\] (5.3.1)

then not only we can estimate $p$’s growth from below, but additionally, by invoking Vieta’s formulas, we can claim that certain relations between coefficients of $p$’s expansion into monomials are equivalent to (5.3.1). Our main theorem 1.1.2 being a refinement of Carlson’s theorem, may be viewed as a generalization of this fact for analytic functions. For example, 5.4 claims certain relations between coefficients of an analytic function’s expansion into exponents.

5.4. Note that the only discrete measure, that is also absolutely continuous, is the zero measure. Hence, by specializing theorem 1.1.2 to discrete measures, we obtain the following remark: a finite sum of exponents
\[g(z) = \sum_{k=1}^{N} c_k e^{i\omega_k z}, \quad \{\omega_k\}_{k=1}^{N} \subset \mathbb{R}\]

equals zero at non-negative integers if and only if
\[\sum_{k: \omega_k \mod 2\pi = \omega} c_k = 0, \quad \text{for all } \omega \in [0, 2\pi).\]

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