LOCALLY HOMOGENEOUS CONNECTIONS ON PRINCIPAL BUNDLES OVER HYPERBOLIC RIEMANN SURFACES

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Abstract. Let $g$ be locally homogeneous (LH) Riemannian metric on a differentiable compact manifold $M$, and $K$ be a compact Lie group endowed with an ad-invariant inner product on its Lie algebra $\mathfrak{k}$. A connection $A$ on a principal $K$-bundle $p : P \rightarrow M$ on $M$ is locally homogeneous if for any two points $x_1, x_2 \in M$ there exists an isometry $\varphi : U_1 \rightarrow U_2$ between open neighborhoods $U_i \ni x_i$ which sends $x_1$ to $x_2$ and admits a $\varphi$-covering bundle isomorphism preserving the connection $A$.

This condition is invariant under the action of the automorphism group (gauge group) of the bundle, so the classification problem for LH connections leads to an interesting moduli problem: for fixed objects $(M, g, K)$ as above describe geometrically the moduli space of all LH connections on principal $K$-bundles on $M$ (up to bundle isomorphisms).

Note that if $A$ is LH, then the associated connection metric $g_A$ on $P$ is locally homogenous, so it defines a geometric structure (in the sense of Thurston) on the total space of the bundle. Therefore this moduli problem is related to the classification of LH (geometric) Riemannian manifolds which admit a Riemannian submersion onto the given manifold $M$.

Omitting the details, our moduli problem concerns the classification of geometric fibre bundles over a given geometric base.

We develop a general method for describing moduli spaces of LH connections on a given base. Using our method we give explicit descriptions of these moduli spaces when the base manifold is a hyperbolic Riemann surface $(M, g)$ and $K \in \{S^1, PU(2)\}$. The case $K = S^1$ leads to a new construction of the moduli spaces of Yang-Mills $S^1$-connections on hyperbolic Riemann surfaces, and the case $K = PU(2)$ leads to a one-parameter family of compact, 5-dimensional geometric manifolds, which we study in detail.

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1. Introduction

1.1. From locally homogeneous to homogeneous. Let \((M,g)\) be a Riemannian manifold, \(K\) be a compact Lie group, \(p: P \to M\) be a principal \(K\) bundle on \(M\), and \(A\) connection on \(P\). Fix an ad-invariant inner product on the Lie algebra \(\mathfrak{k}\) of \(K\). Using these objects we obtain a Riemannian metric \(g_A\) on \(P\) uniquely determined by the conditions:

1. The restrictions of the differential \(p_*\) to the \(A\)-horizontal spaces are isometries, in particular \(p\) is a Riemannian submersion.
2. The \(A\)-horizontal and vertical distributions are orthogonal.
3. For any \(y \in P\) the infinitesimal action \(\mathfrak{k} \to T_y(P)\) is an isometric embedding.

This class of metrics on principal bundles, called connection metrics by some authors [FZ], have been intensively studied in the literature. For instance in [Le], [WZ] the authors study classes of metrics \(g_A\) which are Einstein. In [HK] the authors study classes of Einstein connection metrics on \(S^1\)-bundles and \(S^3\)-bundles over homogeneous Einstein manifolds.

The main problem studied in this article is the classification of locally homogeneous \((LH)\) connection metrics. Recall that a Riemannian metric \(g\) on a differentiable manifold \(M\) is locally homogeneous if, for every two points \(x_1, x_2 \in M\) there exists an isometry \(\varphi: U_1 \to U_2\) between open neighborhoods \(U_i\) of \(x_i\) such that \(\varphi(x_1) = x_2\).

Let \((M, g)\) be a connected, compact, LH Riemannian manifold, \(K\) be a connected, compact Lie group, and \(p: P \to M\) be a principal \(K\)-bundle on \(M\).

**Definition 1.1.** A connection \(A\) on \(P\) will be called locally homogeneous (LH) if for every two points \(x_1, x_2 \in M\) there exists open neighborhoods \(U_i\) of \(x_i\) respectively, an isometry \(\varphi: U_1 \to U_2\) such that \(\varphi(x_1) = x_2\), and a \(\varphi\)-covering bundle isomorphism \(\Phi: P_{U_1} \to P_{U_2}\) such that \(\Phi^*(AV_2) = AV_1\).

In this definition the subscript \(\mathfrak{v}\) denotes “restriction to \(U\)”. A connection \(A\) as above is LH on \((M, g)\) if and only if \((g, P \xrightarrow{\mathfrak{k}}, M, A)\) is a locally homogeneous triple on \(M\) in the sense of [Ba1]. The classification of LH connections on a given connected, compact LH Riemannian manifold \((M, g)\) is related to the classification of compact LH manifolds. Indeed, the connection metric \(g_A\) associated with any LH connection \(A \in \mathcal{A}(P)\) is locally homogeneous, hence it defines a geometric structure in the sense of Thurston [Th, p. 358] on the total space \(P\). Therefore the classification of LH connections on \((M, g)\) is related to the classification of compact, geometric fiber bundles over \(M\). This provides a strong motivation for studying and classifying locally homogeneous connections.

Moreover the classification of 3-dimensional and 4-dimensional geometries ([Th, Sc, Wa, Ef]) shows that such “fibered” geometries are quite abundant. In dimension 3 the following geometries are fibered in our sense (can be defined by connection metrics associated with LH connections): \(E^3, S^3, S^2 \times E^1, H^2 \times E^1, SL_2, Nil^3\). In dimension 4 we have many interesting examples of fibered geometric manifolds, some of them possessing a complex structure compatible with the geometric structure [Wa]. For instance the geometric structure of a topologically non-trivial principal elliptic bundle over a Riemann surface \(Y\) can be defined by an LH connection metric on a \(T^2\)-bundle over \(Y\). The corresponding geometries are \(Nil^3 \times E^1, S^3 \times E^1, SL_2 \times E^1\). A large class of fibered geometric 4-manifolds is studied in the articles [Ue1], [Ue2] which deal with the classification of geometric Seifert 4-manifolds.
Our strategy for the classification of LH connections on principal bundles is inspired by a fundamental theorem of Singer [Si], which states that any LH, complete, simply connected Riemannian manifold is homogeneous. Let $(M,g)$ be a complete Riemannian manifold, and $\pi : \tilde{M} \to M$ be the universal cover of $M$. Singer’s theorem shows that the metric $\tilde{g} := \pi^* g$ is homogeneous. Therefore any LH metric on $M$ is the quotient of a homogeneous metric on the universal cover $\tilde{M}$. A similar classification theorem holds for LH connections [Bi1, Bi2]:

**Theorem 1.2.** Let $(M,g)$ be a compact LH Riemannian manifold, $K$ be a compact Lie group, and $p : P \to M$ be a principal $K$-bundle on $M$. Let $\pi : \tilde{M} \to M$ be the universal cover of $M$, $\Gamma$ be the corresponding covering transformation group. Then, for any LH connection $\mathcal{A}$ on $P$ there exists

1. A connection $\mathcal{B}$ on the pull-back bundle $Q := \pi^*(P)$.
2. A closed subgroup $G \subset \text{Iso}(\tilde{M}, \tilde{g})$ acting transitively on $\tilde{M}$ which contains $\Gamma$ and leaves invariant the gauge class $[\mathcal{B}] \in \mathcal{B}(Q)$.
3. A lift $\iota : \Gamma \to G^B(Q)$ of the inclusion monomorphism $\iota : \Gamma \to G$, where $G^B(Q)$ stands for the group of automorphisms of the pair $(Q,B)$ which lift transformations in $G$.
4. An isomorphism between the $\Gamma$-quotient of $(Q,B)$ and the initial pair $(P,A)$.

We can interpret Theorem 1.2 as follows: any pair $(P,A)$ consisting of a principal $K$-bundle on $M$ and a LH connection on $P$ can be obtained as the $\Gamma$-quotient of a homogeneous pair $(Q,B)$ on the universal cover $\tilde{M}$.

We will denote by $\hat{G}$ the group $G^B(Q)$ given by Theorem 1.2 to save on notations; it is a Lie group and its definition yields the short exact sequence

$$\{1\} \to G^B(Q) \to \hat{G} \xrightarrow{p_\pi} G \to \{1\},$$

whose left hand term $G^B(Q)$ is the stabilizer of the connection $B$ with respect to the action of the gauge group $G(Q)$ on the space of connections $\mathcal{A}(Q)$. Fixing a point $y_0 \in Q$ gives an isomorphism between $G^B(Q)$ and a closed subgroup $L$ of $K$, namely the centraliser in $K$ of the holonomy group of $B$ [DK Lemma 4.2.8]. The conjugacy class of $L$ is independent of $y_0$. Note that $B$ is irreducible if and only if its stabilizer coincides with center $Z(K)$ of $K$.

The obtained pair $(Q,B)$ is $\hat{G}$-homogeneous [Bi1, Bi2]. Therefore one can use [Bi1, Bi2] Theorem 2.10 which gives, in full generality, an explicit description of the moduli space of homogeneous connections on a given homogeneous space.

In this way we obtain a general, effective method for the classification of LH connections on given LH connected, Riemannian manifold $(M,g)$:

1. Determine all closed subgroups $G \subset \text{Iso}(\tilde{g}, \tilde{M})$ which contain $\Gamma$ and act transitively on the universal cover $\tilde{M}$. Note that, if $G$ acts transitively on $\tilde{M}$, also does its identity component $G_0$. Moreover, in many interesting situations $G_0$ contains $\Gamma$, so one can assume that $G$ is connected.
2. For a subgroup $G \subset \text{Iso}(\tilde{g}, \tilde{M})$ as above classify the pairs $(\hat{G},\iota)$ consisting of a Lie group extension $\hat{G}$ of $G$ by a closed subgroup $L$ of $K$, and a lift $\iota : \Gamma \to G$ of the monomorphism $\iota : \Gamma \to G$. For such a pair, classify the $\hat{G}$-homogeneous pairs $(Q,B)$ on $\tilde{M}$ using [Bi1, Bi2] Theorem 2.10]. Select the $\hat{G}$-homogeneous pairs $(Q,B)$ whose stabilizer is $L$, and such that $\hat{G}$ acts effectively on $Q$.

Note that, in fact, only very few conjugacy classes of subgroups $L \subset K$ may occur. Indeed, if $L$ is the stabilizer of a $K$-connection, then it coincides with the centralizer of a subgroup $H \subset K$. This is a very restrictive condition (see [Bi1, Ch. 1, Section 3]). For any Lie group extension $\hat{G}$ as above we obtain a space of triples...
(Q, B, j), where (Q, B) is a $\hat{G}$-homogeneous pair, and $j : \Gamma \to \hat{G}$ is a lift of $\iota$ to $\hat{G}$. Any such triple yields a pair $(P, A)$ consisting of a principal $K$-bundle on $M$ and a LH connection on $P$ (obtained as the $\Gamma$-quotient of $(Q, B)$).

By Theorem 1.2 we know that in this way we obtain all LH connections on $K$-bundles on $M$.

1.2. Classification Theorems. We will use the general method explained above to classify the locally homogeneous connections on $S^1$ and PU(2)-bundles on hyperbolic Riemann surfaces.

Consider first the case $K = S^1$. An $S^1$-connection on a surface endowed with a constant curvature metric is LH if and only if it is Yang-Mills [Ba2, Ch. 4, Proposition 2.1]. For any $\Gamma$-invariant connection $A$ on the associated principal $S^1$-bundle $\hat{G}_t \times_{\chi_t} S^1 \to \mathbb{H}^2$.

Using these objects we will prove the following result, which is stated without proof in [Ba2, Ch. 4, Proposition 2.2]:

**Theorem 1.3.** Let $M := \mathbb{H}^2/\Gamma$ be a compact, hyperbolic Riemann surface, where $\Gamma$ is a discrete subgroup of $\text{PSL}(2, \mathbb{R})$. For a principal $S^1$-bundle $P$ on $M$ put $t := c_1(P) \cdot 1/g$. For any LH connection $A$ on $P$ there exists a unique lift $j : \Gamma \to \hat{G}_t$ of the monomorphism $\Gamma \hookrightarrow \text{PSL}(2, \mathbb{R})$ such that

$$(P, A) \simeq (\hat{G}_t \times_{\chi_t} S^1, A_t)/\Gamma,$$

where $\Gamma$ acts on $\hat{G}_t \times_{\chi_t} S^1$ via $j$.

Moreover, varying the lift $j$ in the space $\mathfrak{J}$ of $\hat{G}_t$-valued lifts of $\Gamma \hookrightarrow \text{PSL}(2, \mathbb{R})$, we obtain a parametrization of the torus of gauge classes of Yang-Mills $S^1$-connections on $P$. This gives a new and explicit construction of this Yang-Mills moduli space.

The classification of locally homogeneous PU(2)-connections on a hyperbolic Riemann surface $(M, g)$ is obtained taking into account the stabilizer of the pull-back connection on the universal cover $\mathbb{H}^2$. The most interesting case is the one of LH connections with irreducible pull-back to $\mathbb{H}^2$. Let $H$ be the stabilizer of the point $x_0 = i \in \mathbb{H}^2$ with respect to the standard $\text{PSL}(2, \mathbb{R})$-action on $\mathbb{H}^2$. We will define a Lie group monomorphism $\chi_1 : H \to \text{PU}(2)$ and for any $z \in \mathbb{C}$, a $\text{PSL}(2, \mathbb{R})$-invariant connection $A_z$ on the associated bundle $P_{x_1} := \text{PSL}(2, \mathbb{R}) \times_{\chi_1} \text{PU}(2)$. We will prove (see Proposition 2.3 [Ba2] and Theorem 5.4 in this article):

**Theorem 1.4.** Let $(M, g)$ be the hyperbolic compact Riemann surface, where $M = \mathbb{H}^2/\Gamma$ for a discrete subgroup $\Gamma \subset \text{PSL}(2, \mathbb{R})$ acting properly discontinuously on $\mathbb{H}^2$. Let $P$ be a principal PU(2)-bundle on $M$, and $A$ be a LH connection on $P$ whose pull-back to $\mathbb{H}^2$ is irreducible. There exists a unique $r > 0$ for which the pair $(P, A)$ is isomorphic to the $\Gamma$-quotient of $(P_{x_1}, A_z)$ by $\Gamma$.

Therefore the moduli space of locally homogeneous PU(2)-connections on $(M, g)$ (with irreducible pull-back to $\mathbb{H}$) can be naturally identified with $(0, \infty)$. This result yields a family of compact geometric 5-manifolds which are SO(3)-bundles over hyperbolic Riemann surfaces.
Remark 1.5. For any $r \in (0, \infty)$ we obtain a PU(2)-invariant, LH Riemannian metric $g_r$ on the quotient bundle $Q = P_{X_1}/T$, which makes the bundle projection $Q \to M$ a Riemannian submersion.

Our one parameter family of LH connections can be obtained explicitly in a geometric way using a foliation by umbilic surfaces of the cylinder $M \times \mathbb{R}$ endowed with a hyperbolic metric (see Proposition 3.6):

Proposition 1.6. There exists a hyperbolic metric on $M := \mathbb{R} \times M$ such that, putting $M_t := \{t\} \times M$ and denoting by $B \in A(SO(M))$ the Levi-Civita connection of $M$, one has

1. $M_0$ is totally geodesic and hyperbolic.
2. For any $x \in M$ the path $t \mapsto (t, x)$ is a geodesic of $M$.
3. For any $t \in \mathbb{R}$ the surface $M_t$ is umbilic in $M$ and the restriction

$$(SO(M)|_{M_t}, B|_{M_t})$$

is isomorphic to the $\Gamma$-quotient of $(P_{X_1}, A_{\sinh(t)})$.

Therefore all locally homogeneous PU(2)-connections on $M$ with irreducible pull-back to $\mathbb{H}$ can be obtained, up to equivalence, by restricting the Levi-Civita connection of the hyperbolic cylinder $\mathbb{H}$ to the umbilic leaves $M_t$ for $t \in (0, \infty)$.

We obtain a clear geometric interpretation of our LH Riemannian manifolds: $(Q, g_r)$ can be identified with the restriction of the frame bundle $SO(M)$ (endowed with its Levi-Civita connection and the associated bundle metric) to the umbilic leaf $M_t$.

This result illustrates our general principle explained above: studying moduli spaces of LH connections on principal bundles leads to new classes of compact LH Riemannian manifolds. We conclude with several general interesting problems concerning these manifolds:

1. Classify the LH Riemannian manifolds obtained in this way which are Einstein.
2. What is the behavior of the Ricci flow with initial condition $g_0 = g_A$.
3. Give alternative geometric interpretations of these Riemannian manifolds and their moduli spaces (using appropriate generalizations of Proposition 1.5).

2. HOMOGENEOUS CONNECTIONS ON PRINCIPAL BUNDLES

2.1. General theory. In this section we present briefly the main result of [BiTe] on the classification of $G$-homogeneous connections on a $G$-manifold. We begin with

Definition 2.1. [Ya, p. 30] [SaWa, Example 4, p. 165] A pair $(G, H)$ consisting of a Lie group $G$ and a closed subgroup $H \subset G$ is called reductive if $\mathfrak{h}$ has an $\text{ad}_{H}$-invariant complement in $\mathfrak{g}$.

Note that if $H$ is compact in $G$, then $(G, H)$ is automatically reductive.

Fix now a reductive pair $(G, H)$ and an $\text{ad}_{H}$-invariant complement $\mathfrak{s}$ of $\mathfrak{h}$ in $\mathfrak{g}$. Applying left translations to $\mathfrak{s}$ we obtain a left invariant connection $A$ on the principal $H$-bundle $G \to G/H$.

Let $\alpha : G \times X \to X$ be a transitive smooth action of a connected Lie group $G$ on a connected differentiable $n$-manifold $X$. Fixing a point $x_0 \in X$ defines an identification $G/H \cong \tilde{X}$, where $H$ is the stabiliser of $x_0$ with respect to $\alpha$. The map $\pi_{x_0} : G \to X$ given by $\pi_{x_0}(g) = gx_0$ is a principal $H$-bundle on $X$. We assume that the pair $(G, H)$ is reductive, and let $\mathfrak{z}$ be an $\text{ad}_{H}$-invariant complement of $\mathfrak{h}$ in...
Let now $K$ be a Lie group. Our problem is the classification of triples $(P, A, \beta)$, where $P$ is principal $K$-bundle on $X$, $A$ is a connection on $P$, and $\beta : G \times P \to P$ an $\alpha$-covering $G$-action on $P$ by bundle isomorphisms which preserve $A$ (see \cite{BTE}, \cite{B2} for details). In the terminology of \cite{BTE} a triple $(P, A, \beta)$ as above is called a $G$-homogeneous $K$-connection on $X$.

Note that $G$ acts in a natural way on the set of all gauge classes of $K$-connections (on principal $K$-bundles) on $X$ \cite{BTE}. The gauge class $[A]$ corresponding to a $G$-homogeneous $K$-connection $(P, A, \beta)$ is obviously $G$-invariant. Unfortunately, in general, a $G$-invariant gauge class $[A]$ is not necessarily associated with a $G$-homogeneous $K$-connection.

Let $\Phi_{\alpha,K}$ be the set of isomorphism classes of $G$-homogeneous $K$-connections on $X$. Following \cite{BTE} we put

$$A(G, H, K) := \{ (\chi, \mu) \in \text{Hom}(H, K) \times \text{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{t}) : \mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu, \forall h \in H \},$$

$$\mathcal{M}(G, H, K) := \frac{A(G, H, K)}{K}. $$

In the second definition the $K$-action on $\text{Hom}(H, K) \times \text{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{t})$ is given by

$$k \cdot (\chi, \mu) = (\iota_k \circ \chi; \text{ad}_k \circ \mu).$$

Therefore the moduli space $\mathcal{M}(G, H, K)$ is the $K$-quotient of the $K$-invariant real algebraic variety $A(G, H, K) \subset \text{Hom}(H, K) \times \text{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{t})$.

**Theorem 2.2.** \cite{BTE} Let $\alpha : G \times X \to X$ be a transitive action of $G$ on $X$. Let $H \subset G$ be the stabiliser of a fixed point $x_0 \in X$, and suppose that the pair $(G, H)$ is reductive. One has a natural identification $\mathcal{M}(G, H, K) \to \Phi_{\alpha,K}$.

The identification $\mathcal{M}(G, H, K) \to \Phi_{\alpha,K}$ mentioned Theorem 2.2 can be described explicitly as follows. Given $(\chi, \mu) \in A(G, H, K)$ we construct a principal $K$-bundle $P_\chi$ with a natural $\alpha$-covering $G$-action by bundle isomorphisms, and a $G$-invariant connection $A_{\chi,\mu}$ on $P_\chi$. The pair $(P_\chi, A_{\chi,\mu})$ is constructed as follows:

- $P_\chi$ is just the principal bundle
  $$\pi_\chi^X : G \times \chi K \to X$$
  associated with the pair $(\pi_{x_0}, \chi)$. This bundle comes with
  - a distinguished point $y_0 := [e_G, e_K] \in (P_\chi)_{x_0}$.
  - an obvious bundle morphism

  \[
  \begin{array}{ccc}
  G & \xrightarrow{\rho_\chi} & P_\chi \\
  \pi_{x_0} \downarrow & & \downarrow \pi_\chi^X \\
  X & \xleftarrow{\pi_0^X} & \end{array}
  \]

  of type $\chi$ mapping $e_G$ to $y_0$.

- The connection $A_{\chi,\mu}$ is defined by

  $$A_{\chi,\mu} := (\rho_\chi)_\ast (A^\beta) + a_\mu, \tag{2}$$

  where $a_\mu$ is the unique $G$-invariant tensorial $\mathfrak{t}$-valued 1-form on $P_\chi$ satisfying

  $$\rho_\chi^\ast (a_\mu)_{e_G} = \mu. \tag{3}$$

  The curvature $F_{A_{\chi,\mu}}$ of the connection $A_{\chi,\mu}$ is the unique $G$-invariant tensorial $\mathfrak{t}$-valued 2-form on $P_\chi$ satisfying the identity

  $$\rho_\chi^\ast (F_{A_{\chi,\mu}})_{e_G} (u, v) = -\chi([u, v]_\mathfrak{h}) + [\mu(u), \mu(v)] - \mu([u, v]). \tag{4}$$
The right hand side of (1) defines a skew-symmetric bilinear 2-form
\[ \Phi_{\chi,\mu} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{k} \]
which has an interesting interpretation: Let \( \lambda_{\chi,\mu} : \mathfrak{g} \rightarrow \mathfrak{k} \) be the linear form which coincides with \( \chi \) on \( \mathfrak{h} \) and with \( \mu \) on \( \mathfrak{s} \). One has the identity
\[ \Phi_{\chi,\mu}(u, v) = [\lambda_{\chi,\mu}(u), \lambda_{\chi,\mu}(v)] - \lambda_{\chi,\mu}([u, v]). \]
Therefore \( \Phi_{\chi,\mu} \) measures the failure of \( \lambda_{\chi,\mu} \) to be a Lie algebra morphism. Formula (1) gives the curvature form at the point \( y_0 \in (P_\chi)_{x_0} \).

2.2. PSL(2, \mathbb{R})-homogeneous connections on the hyperbolic plane. Homogeneous connections on the hyperbolic plane have been studied in [BI], [BTe]. In this section we present briefly the main results.

2.2.1. An explicit description of the moduli space. The case Abelian. Let
\[ \mathbb{H}^2 = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \} \]
be the hyperbolic plane endowed with its standard hyperbolic metric \( g_{\mathbb{H}^2} \). We recall that the group of orientation preserving isometries of \( (\mathbb{H}^2, g_{\mathbb{H}^2}) \) can be identified with \( G := \text{PSL}(2, \mathbb{R}) \). The stabiliser \( H = G_{x_0} \) of \( x_0 = i \in \mathbb{H}^2 \) with respect to the standard action
\[ \alpha : \text{PSL}(2, \mathbb{R}) \times \mathbb{H}^2 \rightarrow \mathbb{H}^2 \]
coincides with the image of \( \text{SO}(2) \) in \( \text{PSL}(2, \mathbb{R}) \). Therefore
\[ H = \left\{ h_t := \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix} \mid t \in [0, 2\pi] \right\} \subset \text{PSL}(2, \mathbb{R}). \tag{5} \]
Let \( \sigma : H \rightarrow S^1 \) be the isomorphism \( h_t \mapsto e^{it} \). Our (non-standard) way to parameterize the stabilizer \( H \) is justified by the following remark, which shows that \( G \), regarded as a principal \( S^1 \)-bundle over \( \mathbb{H}^2 \) via the isomorphism \( \sigma : H \rightarrow S^1 \), can be naturally identified with the frame bundle of the oriented Riemannian surface \( \mathbb{H}^2 \).

Remark 2.3. Let \( e_1^{x_0} \in T_{x_0} \) be the tangent vector defined by the first element of the standard basis of \( \mathbb{H}^2 \), and let \( S(T_{x_0}) \) be the circle bundle of \( \mathbb{H}^2 \), endowed with the \( S^1 \)-action defined by the complex orientation of \( \mathbb{H}^2 \). The map
\[ \delta : G \rightarrow S(T_{x_0}), \delta(\phi) := \phi_{x_0}(e_1^{x_0}) \]
is a \( \sigma \)-equivariant diffeomorphism over \( \mathbb{H}^2 \).

In other words, \( \delta \) induces an isomorphism between the principal bundle \( G \times_{\sigma} S^1 \) and the principal \( S^1 \)-bundle \( S(T_{x_0}) \). The latter can be obviously identified with the frame bundle \( \text{SO}(\mathbb{H}^2) \) of the oriented Riemannian surface \( \mathbb{H}^2 \).

Identifying \( \mathfrak{g} \) with \( \text{sl}(2, \mathbb{R}) \), we have \( \mathfrak{h} = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix} \), and an \( \text{ad}_H \)-invariant complement of \( \mathfrak{h} \) in \( \mathfrak{g} \) is
\[ \mathfrak{s} := \left\langle \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle. \]

Put
\[ b_1 := \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b_2 := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b_3 := \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
We have
\[ [b_1, b_2] = -b_3, \quad [b_2, b_3] = b_1, \quad [b_3, b_1] = b_2. \]
The action \( \text{ad}_{b_1} \) on \( \mathfrak{s} \) is given by
\[ \text{ad}_{b_1}(u_1 b_1 + u_2 b_2) = (b_1, b_2) \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \]
so $\text{ad}_{h_i}$ acts on the oriented plane $s = \langle b_1, b_2 \rangle$ by a rotation of angle $t$.

For a Lie group $K$ we obtain an identification

$$A(\text{PSL}(2, \mathbb{R}), H, K) = \{ (\chi, \mu) \in \text{Hom}(H, K) \times \text{Hom}_{\mathbb{R}}(s, t) \mid \mu \in \text{Hom}_{\mathbb{R}}^{\chi \circ \sigma'}(s, t) \}, \quad (6)$$

where $\sigma' := \sigma^{-1}$, and the subspace $\text{Hom}_{\mathbb{R}}^{\chi \circ \sigma'}(s, t) \subset \text{Hom}(s, t)$ is defined by

$$\text{Hom}_{\mathbb{R}}^{\chi \circ \sigma'}(s, t) := \{ \mu \in \text{Hom}_{\mathbb{R}}(s, t) \mid \mu \circ \zeta = \text{ad}_{\chi \circ \sigma'(\zeta)} \mu \quad \forall \zeta \in S^1 \}.$$

In this formula we used the notation

$$R_{\mu'}(u_1 b_1 + u_2 b_2) = \text{ad}_{h_i}(u_1 b_1 + u_2 b_2) = (h_1, b_2)
\begin{pmatrix}
\cos(t) & -\sin(t) \\
\sin(t) & \cos(t)
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}.$$

We can identify $\text{Hom}_{\mathbb{R}}(s, t)$ with the complexification $\mathfrak{t}^C$ of $\mathfrak{t}$ using the isomorphism

$$I : \text{Hom}_{\mathbb{R}}(s, t) \to \mathfrak{t}^C, \quad I(\mu) = \mu(b_1) - i\mu(b_2).$$

Using the identity $I(\mu \circ R_\zeta) = \zeta I(\mu)$ for $\zeta \in S^1$, and putting

$$\mathfrak{t}_{\chi \circ \sigma'}^C := \{ Z \in \mathfrak{t}^C \mid \text{ad}_{\chi \circ \sigma'(\zeta)} Z = \zeta Z, \quad \forall \zeta \in S^1 \},$$

we obtain a further identification (see [41]):

$$A(\text{PSL}(2, \mathbb{R}), H, K) = \{ (\chi, Z) \in \text{Hom}(H, K) \times \mathfrak{t}^C \mid Z \in \mathfrak{t}_{\chi \circ \sigma'}^C \}. \quad (7)$$

Note that $\mathfrak{t}_{\chi \circ \sigma'}^C \subset \mathfrak{t}^C$ is just the weight space associated with the weight $\text{id}_{S^1}$ of the $S^1$-representation $\zeta \mapsto \text{ad}_{\chi \circ \sigma'(\zeta)} \in \text{GL}(\mathfrak{t}^C)$. Formula (7) combined with Theorem 2.2 yields a simple description of the moduli space of isomorphism classes of $G$-homogeneous $K$-connections on the hyperbolic space for an Abelian Lie group $K$:

**Remark 2.4.** If $K$ is Abelian, then $\mathfrak{t}_{\chi \circ \sigma'}^C = 0$, so one has identifications

$$\text{Hom}(H, K) \times \{ 0 \} = A(\text{PSL}(2, \mathbb{R}), H, K),$$

$$\text{Hom}(H, K) \xrightarrow{\cong} M(\text{PSL}(2, \mathbb{R}), H, K) \cong \Phi_{\alpha, K}.$$

In particular, for $K = S^1$ one has natural identifications

$$Z \xrightarrow{\cong} M(\text{PSL}(2, \mathbb{R}), H, S^1) \xrightarrow{\cong} \Phi_{\alpha, S^1} \quad (8)$$

given explicitly by

$$k \mapsto [\pi^k \circ \sigma, 0] \mapsto [A_{\pi^k \circ \sigma, 0}],$$

where $\pi^k : S^1 \to S^1$ is the Lie group morphism $\zeta \mapsto \zeta^k$.

**Corollary 2.5.** The direct image $\delta_* (A^g)$ coincides with the Levi-Civita connection $A_{LC}$ on the $S^1$-bundle $S(T_{\mathbb{H}^2}) = \text{SO}(\mathbb{H}^2)$.

**Proof.** Indeed, the Levi-Civita connection on the bundle $S(T_{\mathbb{H}^2})$ (endowed with the obvious $G$-action) is obviously $G$-homogeneous. On the other hand Remark 2.4 shows that this bundle has a unique $G$-homogeneous connection. □

**Remark 2.6.** The Levi-Civita connection $A_{LC}$ on the $S^1$-bundle $\text{SO}(\mathbb{H}^2)$ corresponds to the integer $k = 1$ under the identification given by Remark 2.4. Therefore, the identification $Z \xrightarrow{\cong} \Phi_{\alpha, S^1}$ given by Remark 2.4 has a geometric interpretation, and is also given by

$$k \mapsto [A_{\pi^k \circ \sigma, 1}]. \quad (9)$$
In formula (10) we used the following notation: for a principal $S^1$-bundle $P \times \pi \rightarrow S^1$ and any monomorphism $S^1 \rightarrow S^1$ where $[\ ]$ means class modulo $\{\pm I_2\}$. The image of $\theta$ is a maximal torus of $\text{PU}(2)$, and any monomorphism $S^1 \rightarrow \text{PU}(2)$ is equivalent with $\theta$ modulo an interior automorphism of $\text{PU}(2)$. Moreover, any morphism $\chi : S^1 \rightarrow \text{PU}(2)$ is equivalent (modulo an interior automorphism) with a morphism of the form $\theta_k := \theta \circ \pi^k$ with $k \in \mathbb{N}$. Similarly, any morphism $\chi : H \rightarrow \text{PU}(2)$ is equivalent with a morphism of the form $\theta_k \circ \sigma$ with $k \in \mathbb{N}$.

The set of weights of the representation

$$S^1 \ni \zeta \mapsto \text{ad}_{\theta_k(\zeta)} \in \text{GL}(\text{su}(2) \mathbb{C}) = \text{GL}(\text{sl}(2, \mathbb{C}))$$

is $\{\pi^l | l \in \{0, \pm k\}\}$, so $\text{su}(2) \mathbb{C}_{\theta_k} = \{0\}$ for any $k \in \mathbb{N} \setminus \{1\}$. For any such $k$ formula (7) gives $\text{su}(2) \mathbb{C}_{\theta_k} = \{0\}$. The case $k = 1$ is more interesting. First note that, by Remark 2.3, we have

**Remark 2.7.** The bundle $P_{\theta_1 \circ \sigma}$ can be identified with the $\text{PU}(2)$-extension

$$\text{SO}(\mathbb{H}^2) \times_{S^1} \text{PU}(2)$$

of the frame bundle $\text{SO}(\mathbb{H}^2)$.

The space $\text{su}(2) \mathbb{C}_{\theta_1}$ is the complex line $C I(\mu_0)$, where $\mu_0 \in \text{Hom}^{\theta_1}_{S^1}(s, \text{su}(2))$ is defined by

$$\mu_0(u_1b_1 + u_2b_2) := \frac{1}{2} \begin{pmatrix} 0 & -u_1 - iu_2 \\ u_1 - iu_2 & 0 \end{pmatrix} = u_1a_2 - u_2a_1,$$

and $(a_1, a_2, a_3)$ is the standard basis of $\text{su}(2)$:

$$a_1 := \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad a_2 := \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad a_3 := \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$
where \((e^1_{x_0}, e^2_{x_0})\) is the dual basis of the standard basis \((e^1_{x_0}, e^2_{x_0})\) of \(T_{x_0}\). Using the \(G\)-invariance of the connection \(A_{\theta, \sigma, z\mu_0}\), this formula determines the curvature form \(F_{A_{\theta, \sigma, z\mu_0}}\) at any point.

It is interesting to have explicit geometric interpretations of the oriented Euclidean rank 3 vector bundle \(E = G \times_H \mathfrak{su}(2) \rightarrow \mathbb{H}^2\) associated with the principal bundle \(P_{\theta, \sigma}\) via the standard isomorphism \(\text{ad} : PU(2) \rightarrow \text{SO}(su(2)) = \text{SO}(3)\), and of the linear connection \(\nabla^z\) on \(E\) which corresponds to \(A_{\theta, \sigma, z\mu_0}\). The morphism \(\theta_1 \circ \sigma\) leaves invariant the direct sum decomposition \(\mathfrak{su}(2) = \langle a_1, a_2 \rangle \oplus \mathfrak{u}_3\); the subgroup \(H\) acts on the plane \(\langle a_1, a_2 \rangle\) via the isomorphism \(\sigma\), and acts trivially on the line \(\mathbb{R}\). Therefore we get a direct sum decomposition

\[
E = (G \times_H \langle a_1, a_2 \rangle) \oplus \mathfrak{u}_3
\]

where \(\mathfrak{u}_3\) stands for the trivial line bundle of fibre \(\mathfrak{u}_3\). Taking into account Remark 2.7 the first summand can be identified with the tangent bundle \(T_{x_0}\) via a \(G\)-invariant isomorphism mapping \([e_G, a_i]\) onto \(e_i^{x_0}\) for \(i \in \{1, 2\}\). Taking into account Corollary 2.25 and formulæ (2), (3) we obtain the following matrix decomposition of the linear connection \(\nabla^z\):

\[
\nabla^z = \begin{pmatrix} \nabla^{LC} & B^z \\ -B^z & \nabla^0 \end{pmatrix},
\]

(10)

where \(\nabla^{LC}\) is the Levi-Civita connection on \(T_{x_0}\), \(\nabla^0\) is the trivial connection on the trivial line bundle \(\mathfrak{u}_3\), and the second fundamental form \(B^z \in A^1(\mathbb{H}^2, T_{x_0})\) is the \(G\)-invariant \(T_{x_0}\)-valued 1-form on \(\mathbb{H}^2\) which is given at the point \(x_0\) by

\[
B^z_{x_0} = r\left(e_{x_0}^1(\cos(t)e_{x_0}^1 + \sin(t)e_{x_0}^2) + e_{x_0}^2(-\sin(t)e_{x_0}^1 + \cos(t)e_{x_0}^2)\right).
\]

This shows that

**Remark 2.8.** Identifying \(A^1(\mathbb{H}^2, T_{x_0})\) with \(A^0(\mathbb{H}^2, \text{End}_\mathbb{R}(T_{x_0}))\), and endowing \(T_{x_0}\) with its natural complex structure, one has \(B^z = \text{id}_{T_{x_0}}\).

We can now give an explicit description of the moduli space

\[
\mathcal{M}(\text{PSL}(2, \mathbb{R}), H, PU(2)) \simeq \Phi_{\alpha, PU(2)}
\]

classifying isomorphism classes of \(G\)-homogeneous \(PU(2)\)-connections on the hyperbolic plane. The centraliser of the morphism \(\theta_1 \circ \sigma : H \rightarrow PU(2)\) with respect to the adjoint action of \(PU(2)\) is \(\text{im}(\theta_1) \simeq S^1\), and it acts with weight 1 on the complex line \(\text{Hom}_{\mathbb{C}}(\mathfrak{su}_2, \mathfrak{su}_2)\). This shows that

\[
\mathcal{M}(\text{PSL}(2, \mathbb{R}), H, PU(2)) = \{[\theta_k \circ \sigma, 0] \mid k \in \mathbb{N} \setminus \{1\}\} \cup \mathcal{M}_1,
\]

where

\[
\mathcal{M}_1 = \{[\theta \circ \sigma, r\mu_0] \mid r \in [0, \infty)\} \simeq [0, \infty).
\]

2.2.3. **Homogeneous SO(3)-connections on the hyperbolic plane, and umbilical foliations on the hyperbolic space.** Let

\[
\mathbb{H}^3 := \{x \in \mathbb{R}^3 | x_3 > 0\}
\]

be the hyperbolic space endowed with the standard hyperbolic metric

\[
g = \frac{1}{x_3^2} \left(1 + \sum_i dx_i^2\right).
\]

For a point \(x \in \mathbb{H}^3\) the \(g\)-normalized vectors \(\vec{e}_i^x = x_3 e_i^x\) give an orthonormal basis of the Euclidean space \((T_x \mathbb{H}^3, g_x)\). We denote by \(\vec{e}_i \in \mathcal{X}(\mathbb{H}^3)\) the vector field \(x \rightarrow \vec{e}_i^x\).
For $t \in \mathbb{R}$ let $f^t : \mathbb{H}^3 \to \mathbb{H}^3$ be the diffeomorphism of $\mathbb{H}^3$ induced by linear isomorphism associated with the matrix

$$F^t := \begin{pmatrix} 1 & 0 & \text{tanh}(t) \\ 0 & 1 & 0 \\ 0 & 0 & \text{sech}(t) \end{pmatrix}.$$ 

The family $(f_t)_{t \in \mathbb{R}}$ has an important interpretation:

**Remark 2.9.** For any $x \in \mathbb{H}^3$, the curve $t \mapsto f^t(x)$ coincides with the geodesic path $\gamma^x : \mathbb{R} \to \mathbb{H}^3$ determined by the initial conditions $\gamma_x(0) = x$, $\dot{\gamma}_x(0) = e^t_1$.

Put $H_0 := \{ x \in \mathbb{H}^3 | x_1 = 0 \}$, $H_t := f_t(H_0)$. The surface $H_t$ is the intersection of $\mathbb{H}^3$ with the plane defined by the equation $x_1 = \text{sinh}(t)x_3$. It is well known that $H_0$ is a totally geodesic hypersurface of $\mathbb{H}^3$, whereas $H_t$ is an umbilic hypersurface of $\mathbb{H}^3$ for any $t \in \mathbb{R}$. The family $(H_t)_{t \in \mathbb{R}}$ defines a foliation $\mathcal{F}$ of $\mathbb{H}^3$ by umbilic hypersurfaces. Note that there is an interesting literature dedicated to the classification of foliations by umbilic submanifolds (see for instance [Wals]). The classification theorem [Wals, Theorem 2.6] states that any isoparametric 1-codimensional umbilic foliations of a Riemannian manifold of negative curvature is either a foliation by horospheres, or a foliation by hypersurfaces equidistant from a totally geodesic hypersurface. Our foliation $\mathcal{F}$ belongs to the second class.

The following proposition gives a geometric interpretation of the $G$-homogeneous connections $A_{\theta_1,\sigma,\mu_0}$ obtained with formal methods:

**Proposition 2.10.** Let $B_{\text{LC}}$ be the Levi-Civita connection on $\text{SO}(\mathbb{H}^3)$, and let $\varphi_1 : \mathbb{H}^2 \to \mathbb{H}^3$ be the embedding $(x_1, x_2) \mapsto f_t(0, x_1, x_2)$ whose image is the umbilic surface $H_t$. There exists a natural isomorphism of pairs

$$(\varphi_1^*(\text{SO}(\mathbb{H}^3)), \varphi_1^*(B_{\text{LC}})) \simeq (P_{\theta_1,\sigma, A_{\theta_1,\sigma,\text{sinh}(t)\mu_0}}).$$

**Proof.** Using the orthogonal direct sum decomposition $T_{\mathbb{H}^3}|_{H_t} = T_{H_t} \oplus N_{H_t}$, we see that the restriction $\text{SO}(\mathbb{H}^3)|_{H_t}$ can be identified with the $\text{SO}(3)$-extension of the $\text{SO}(2)$-bundle $\text{SO}(H_t)$. Taking into account our geometric description of the bundle $P_{\theta_1,\sigma}$ (see Remark 2.7), we obtain an obvious identification $\varphi_1^*(\text{SO}(\mathbb{H}^3)) \simeq P_{\theta_1,\sigma}$. To get the claimed identification $\varphi_1^*(B_{\text{LC}}) = A_{\theta_1,\sigma,\text{sinh}(t)\mu_0}$ one uses formula (10) and the similar matrix decomposition of the restriction $B_{\text{LC}}|_{H_t}$. It suffices to identify the second fundamental forms of the two connections. The equality follows using Remark 2.8 and a direct computation of the second fundamental form of $H_t$.

**Proposition 2.10** states that:

**Remark 2.11.** Via the diffeomorphism

$$\mathbb{H}^2 \xrightarrow{\varphi_{\text{argsh}}} H_{\text{argsh(s)}}$$

the $G$-homogeneous connection $A_{\theta_1,\sigma,\mu_0}$ can be identified with the restriction of the Levi-Civita connection $B_{\text{LC}} \in \mathcal{A}($SO($\mathbb{H}^3)$) to the leaf $H_{\text{argsh(s)}}$ of the umbilic foliation $\mathcal{F}$.

This also gives a geometric interpretation of the moduli space $\mathcal{M}_1$: it can be identified with the space of isomorphism classes of restrictions

$$B_{\text{LC}}|_{H_t}, \ t \in [0, \infty).$$
3. Classification theorems for LH connections

3.1. The case \( K = S^1 \). LH and Yang-Mills connections. The classification of locally homogeneous \( S^1 \)-connections on hyperbolic Riemann surfaces is more interesting than expected. The natural idea is to write \( M \) as a quotient of \( \mathbb{H}^2/\Gamma \), where \( \Gamma \subset PSL(2, \mathbb{R}) \) acts properly discontinuously on \( \mathbb{H}^2 \), and to consider \( \Gamma \)-quotients of \( PSL(2, \mathbb{R}) \)-homogeneous \( S^1 \)-connections on \( \mathbb{H}^2 \). Unfortunately, in this way one obtains a very small class of locally homogeneous \( S^1 \)-connections on \( M \). Indeed, Remark 2.6 shows that this method yields only the tensor powers \( (A_{LC}^M)^{\otimes k} \) of the Levi-Civita connection \( A_{LC}^M \) of \( M \). The tensor power \( (A_{LC}^M)^{\otimes k} \) is a connection on \( SO(M)^{\otimes k} \), whose Chern class is \( 2k(1-g(M)) \). This shows that a principal \( S^1 \)-bundle \( P \) admits a connection obtained in this way if and only if \( c_1(P) \in 2\mathbb{Z}(1-g(M)) \) and, if this is the case, \( P \) admits a unique gauge class of such a connection. On the other hand one has the following general result concerning the classification of LH \( S^1 \)-connections on Riemann surfaces:

**Proposition 3.1.** [10,2] Let \( (M, g) \) be a connected, oriented, compact Riemann surface endowed with a Riemannian metric with constant curvature, let \( P \) be a principal \( S^1 \)-bundle on \( M \) and \( A \in \mathcal{A}(P) \) be a connection on \( P \). Then \( A \) is LH if and only if \( A \) is Yang-Mills.

This shows that, for any fixed integer \( k \in \mathbb{Z} \), the set of isomorphism classes of locally homogeneous pairs \((P, A)\) with \( c_1(P) = k \) can be identified with the moduli space of Yang-Mills connections on a Hermitian line bundle of Chern class \( k \), which is a torus of dimension \( 2g \) (see for instance [10]).

We will see that, using the classification Theorem [1,2] and the method explained in section 2.3, any locally homogeneous (or, equivalently, any Yang-Mills) \( S^1 \)-connection on \( M \) can be obtained explicitly as the quotient of a homogeneous connections on \( \mathbb{H}^2 \), but we will have to replace the group \( PSL(2, \mathbb{R}) \) by an \( S^1 \)-extension of it (which depends on the Chern class of the underlying \( S^1 \)-bundle).

We will need the universal cover \( \pi : \widetilde{PSL(2, \mathbb{R})} \to PSL(2, \mathbb{R}) \) of the Lie group \( PSL(2, \mathbb{R}) \)

\[
e^{i\tau} \mapsto h_{\tau} := \begin{bmatrix} \cos(\tau/2) & \sin(\tau/2) \\ -\sin(\tau/2) & \cos(\tau/2) \end{bmatrix},
\]

defines a generator \( \gamma \) of \( \ker(c) = \pi_1(PSL(2, \mathbb{R})) \). The Lie group \( \hat{G}_t \) defined by

\[
\hat{G}_t := PSL(2, \mathbb{R}) \times S^1/\{ (\gamma^k, e^{-2\pi tk}) \mid k \in \mathbb{Z} \}.
\]

fits in the short exact sequence

\[
1 \to S^1 \to \hat{G}_t \to PSL(2, \mathbb{R}) \to 1,
\]

where the morphisms \( j_t, c_t \) are given by \( j_t(z) := [e, z] \), \( c_t([y, \zeta]) := c(y) \).

The restriction of \( c \) to \( \tilde{H} := e^{-1}(H) \subset PSL(2, \mathbb{R}) \) is a universal cover of \( H \). Note that \( \tilde{H} \) is the image of a 1-parameter subgroup \( \mathbb{R} \to \hat{h}_\tau \), where \( \hat{h}_\tau \) is a lift of \( h_\tau \), and \( \gamma = \hat{h}_{2\pi} \). The subgroup \( \tilde{H}_t := c_t^{-1}(H) \subset \hat{G}_t \) can be written as

\[
\tilde{H} \times S^1/\{ (\gamma^k, e^{-2\pi tk}) \mid k \in \mathbb{Z} \},
\]

and is abelian. We have a short exact sequence

\[
1 \to S^1 \to \tilde{H}_t \to H \to 1,
\]

(11)
where the morphisms $i_t$, $p_1$ are defined similarly to $j_t$, $c_t$. The Lie algebra $\hat{g}_t$ of $\hat{G}_t$ decomposes as $\hat{g}_t = \text{sl}(2, \mathbb{R}) \times i\mathbb{R}$; the subspace $\hat{s}_t := \mathbb{R} \subset \hat{g}_t$ is a $\hat{H}_t$-invariant complement of $\hat{h}_t$.

The group morphism $\chi_t : \hat{H}_t \to S^1$, $\chi_t([\hat{h}_t, \zeta]) := e^{i\tau t} \zeta$ is a left splitting of the short exact sequence (11), so it defines an isomorphism $\hat{H}_t \cong S^1 \times S^1$.

The group $G_t$ acts on $\mathbb{H}^2$ via $c_t$ and the stabiliser of $x_0$ with respect to this action is $c_t^{-1}(H) = H_t$. Using the general construction method explained in section 2.4 we obtain a $\hat{G}_t$-homogeneous connection $(P_{\chi_t}, A_{\chi_t, 0})$ associated with the triple $(G_t, \hat{H}_t, \hat{s}_t)$ and the pair $(\chi_t, 0)$.

Using this construction we obtain the following theorem, which describes all Yang-Mills $S^1$-connections (hence all LH $S^1$-connections) on hyperbolic Riemann surfaces as quotients of homogeneous connections on $\mathbb{H}^2$. This result is stated without proof in [Ba2].

**Theorem 3.2.** Let $M := \mathbb{H}^2/\Gamma$ be a compact, hyperbolic Riemann surface, where $\Gamma$ is discrete subgroup of $\text{PSL}(2, \mathbb{R})$ acting properly discontinuously on $\mathbb{H}^2$. For a principal $S^1$-bundle $P$ on $M$ put $t := \frac{\chi_t(P)}{2(1-g)}$. For any LH connection $A$ on $P$ there exists a unique lift $\tilde{j} : \Gamma \to \hat{G}_t$ of the monomorphism $\Gamma \hookrightarrow \text{PSL}(2, \mathbb{R})$ such that

$$(P, A) \cong (\hat{G}_t \times_{\chi_t} S^1, A_{\chi_t, 0})/\Gamma,$$

where $\Gamma$ acts on $\hat{G}_t \times_{\chi_t} S^1$ via $j$.

**Proof.** The curvature formula (4) shows that

$$F_{A_{\chi_t, 0}} = t F_{A_{\chi_t, 0}} = t F_{A_{\epsilon, 0}}.$$

On the other hand, by Remark 2.6 we have $F_{A_{\epsilon, 0}} = F_{A_{\epsilon, C}}$. Note that the Chern form $c_1(A_{\chi_t, 0}) = \frac{1}{2\pi} F_{A_{\chi_t, 0}}$ is a $\Gamma$-invariant 2-form on $\mathbb{H}^2$, so it descends to a 2-form $\tilde{c}_1(A_{\chi_t, 0})$ on the closed surface $M$. Putting $c := c_1(P) \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}, t := \frac{c}{2(1-g)}$ we obtain

$$[\tilde{c}_1(A_{\chi_t, 0})]_{\text{DR}} = c. \tag{12}$$

On the other hand the obstruction to the existence of a lift $j : \Gamma \to \hat{G}_t$ of the embedding monomorphism $\epsilon : \Gamma \hookrightarrow \text{PSL}(2, \mathbb{R})$ is a cohomology class $\epsilon \in H^2(\Gamma, \mathbb{Z})^1 = H^2(M, \mathbb{Z})$. Using a standard Čech-de Rham double complex argument, it follows that $\epsilon$ is precisely the image of $[\tilde{c}_1(A_{\chi_t, 0})]_{\text{DR}} \in H^2(M, \mathbb{R})$ in the quotient $H^2(M, \mathbb{Z}) = H^2(M, \mathbb{Z})/H^2(M, \mathbb{Z})$. Therefore (12) shows that $\epsilon$ vanishes, so the set $\tilde{j}$ of lifts $j : \Gamma \to \hat{G}_t$ of the embedding monomorphism $\epsilon : \Gamma \hookrightarrow \text{PSL}(2, \mathbb{R})$ is non-empty. For any $j \in \tilde{j}$ the Chern class of the corresponding quotient bundle

$$P_j = P_{\chi_t}/j \Gamma$$

is $[\tilde{c}_1(A_{\chi_t, 0})]_{\text{DR}} = c$, so $P_j \cong P$. Therefore for any $j \in \tilde{j}$ we obtain a quotient connection $A_j$ induced by $A_{\chi_t, 0}$ on an $S^1$-bundle $P_j \cong P$. The set $\tilde{j}$ has an obvious structure of a $\text{Hom}(\Gamma, S^1)$-torsor, and it is easy to see that, for any $\rho \in \text{Hom}(\Gamma, S^1)$, the connection $A_{\rho, j}$ can be identified with the tensor product of $A_j$ by the flat connection associated with $\rho$. Therefore the set of isomorphism classes $\{[A_j] \mid j \in \tilde{j}\}$ coincides with the whole torus of gauge classes of Yang-Mills connections on $P$, and the map $j \mapsto [A_j]$ is bijective.

### 3.2. The case $K = \text{PU}(2)$

Let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a discrete subgroup acting properly discontinuously on $\mathbb{H}^2$ with compact quotient, and let $(M, g_{M})$ be the hyperbolic Riemann surface $M := \mathbb{H}^2/\Gamma$. The classification of locally homogeneous $\text{PU}(2)$-connections on $M$ is obtained by taking into account the stabilizer of the pull-back connection on $\mathbb{H}^2$. Let $P$ be a principal $\text{PU}(2)$-bundle on $M$, and $A$ be
a locally homogeneous connection on $P$. Let $B$ be the pull-back connection on the pull-back bundle $Q$ on $\mathbb{H}^2$.

3.2.1. **Locally homogeneous connections with irreducible pull-back.** Suppose that $B$ is irreducible, i.e. it has trivial stabiliser. In this case the classification Theorem [1,2] applies with $\mathcal{G}^B(Q) = \{\text{id}\}$, and gives

1. A closed subgroup $G \subset \text{Iso}(\mathbb{H}^2, g_{\text{BH}})$ acting transitively on $\mathbb{H}^2$ which contains $\Gamma$ and leaves invariant the gauge class $[B] \in \mathcal{B}(Q)$.
2. A lift $j : \Gamma \to G$ of the inclusion monomorphism $\iota_{\Gamma} : \Gamma \to G$.
3. An isomorphism between the $\Gamma$-quotient of $(Q, B)$ and the initial pair $(P, A)$.

The connected component $G_0 \subset G$ of id $\in G$ still acts transitively on $\hat{M}$ but it is not clear if it still contains $\Gamma$.

**Remark.** With the notations introduced in Theorem [1,2] suppose that $(\hat{M}, \hat{g}) = (\mathbb{H}^2, g_{\text{BH}})$, and put $\Gamma_0 := \Gamma \cap G_0$. The quotient $\Gamma_0 \backslash G_0$ can be identified with the connected component of $\Gamma$ in $\Gamma \backslash G$ in this quotient, therefore the compactness of $\Gamma \backslash G$ implies the compactness of $\Gamma_0 \backslash G_0$. This shows that $G_0$ is a connected Lie subgroup of $\text{PSL}(2, \mathbb{R})$ which acts transitively on $\mathbb{H}^2$ and is unimodular. This implies $G_0 = \text{PSL}(2, \mathbb{R})$, in particular $G_0$ contains $\Gamma$.

Therefore we may suppose $G = G_0 = \text{PSL}(2, \mathbb{R})$, so we reduced our problem to the classification of $\text{PSL}(2, \mathbb{R})$-homogeneous $\text{PU}(2)$-connections on $\mathbb{H}^2$ studied in section 2.2.2. We obtain:

**Theorem 3.4.** Let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a discrete subgroup acting properly discontinuously on $\mathbb{H}^2$ with compact quotient, and let $(M, g_M)$ be the hyperbolic Riemann surface $M := \mathbb{H}^2 / \Gamma$. Let $P$ be a principal $\text{PU}(2)$-bundle on $M$, and $A$ be a locally homogeneous connection on $P$ whose pull-back connection to $\mathbb{H}^2$ is irreducible. There exists a unique $r > 0$ for which the pair $(P, A)$ is isomorphic to the $\Gamma$-quotient of $(P_{\theta_{1,\sigma}}, A_{\theta_{1,\sigma}, r \sigma_0})$.

Therefore

**Corollary 3.5.** The moduli space of locally homogeneous $\text{PU}(2)$-connections on $M$ with irreducible pull-back to $\mathbb{H}^2$ can be naturally identified to the half-line $(0, \infty)$.

Using the results proved in section 2.2.3 we obtain a geometric interpretation of this 1-parameter family of locally homogeneous $\text{PU}(2)$-connections on $M$ in terms of umbilic foliations on hyperbolic 3-manifolds:

**Proposition 3.6.** There exists a hyperbolic metric $g_M$ on $M := \mathbb{R} \times M$ such that, putting $M_t := \{t\} \times M$ and denoting by $\mathcal{B} \in \mathcal{A}(\text{SO}(M))$ the Levi-Civita connection of $M$, one has

1. $M_0$ is totally geodesic and hyperbolic.
2. For any $x \in M$ the path $t \mapsto (t, x)$ is a geodesic of $M$.
3. For any $t \in \mathbb{R}$ the surface $M_t$ is umbilic in $M$ and the restriction $(\text{SO}(M)|_{M_t}, \mathcal{B}|_{M_t})$

is isomorphic to the $\Gamma$-quotient of $(P_{\theta_{1,\sigma}}, A_{\theta_{1,\sigma}, \sigma_0, \sinh(t) \sigma_0})$.

**Proof.** Using the notations introduced in section 2.2.3 note that for any orientation preserving isometry $\psi$ of $\mathbb{H}^2$ there exists an orientation preserving isometry $\Psi$ of $\mathbb{H}^3$ such that $\Psi \circ \varphi_t = \varphi_t \circ \psi$ for any $t$. The isometry $\Psi$ is obtained by extending $\psi$ in the obvious way along the geodesics $\gamma^x$, $x \in \mathbb{H}^2$. The obtained map $\psi \mapsto \Psi$ is
a Lie group monomorphism. Let \( \hat{\Gamma} \) be the image of \( \Gamma \) under this monomorphism, and put \( \hat{M} := \mathbb{H}^3/\hat{\Gamma} \). We obtain a diffeomorphism

\[
\hat{\varphi} : \mathbb{R} \times \hat{M} \to \hat{M}
\]

induced by the diffeomorphism \( \mathbb{R} \times \mathbb{H}^2 \ni (t, x) \to \varphi_t(x) \in \mathbb{H}^3 \). Putting \( g_M := \hat{\varphi}^*(g_{\mathbb{H}^3}) \) the claim follows. \( \blacksquare \)

Proposition \( 3.6 \) shows that

**Remark 3.7.** All locally homogeneous PU(2)-connections on \( M \) with irreducible pull-back to \( \mathbb{H}^2 \) can be obtained, up to equivalence, by restricting the Levi-Civita connection of the hyperbolic cylinder \( M \) to the umbilic leaves \( M_t \) for \( t \in (0, \infty) \).

3.2.2. **Locally homogeneous connections with Abelian, non-flat pull-back.** Suppose now that the stabilizer of \( B \) is \( S^1 \). Therefore the holonomy of \( B \) at a point \( y \in Q \) is contained in the centralizer of \( S^1 \) in \( SU(2) \) (which is \( S^1 \)), so \( Q \) admits a \( B \)-invariant \( S^1 \)-reduction \( Q_0 \subset Q \). The direct sum decomposition \( \mathfrak{su}(2) = i\mathbb{R} \oplus \mathbb{C} \) is \( S^1 \)-invariant; the subgroup \( S^1 \subset PU(2) \) acts trivially on the first summand, and with weight 1 on the second. Therefore, we obtain a \( B \)-invariant direct sum decomposition

\[
\text{ad}(Q) = Q_0 \times_{S^1} \mathfrak{su}(2) = i\mathbb{R} \oplus L
\]

where \( L \) is a Hermitian line bundle on \( \mathbb{H}^2 \). The curvature form \( F_B \) takes values in the first summand, so \( F_B = \text{vol}_{\mathbb{H}^2} \eta_B \) with \( \eta_B \in C^\infty(\mathbb{H}^2, i\mathbb{R}) \).

The local homogeneity condition of \( A \) shows that \( F_B \) has constant norm, so \( \eta_B \) is constant.

Since \( B \) is not flat, it follows that \( \eta_B \) is a non-vanishing constant. The decomposition \( \mathfrak{su}(2) \) is \( B \)-parallel, so \( \eta_B \) is also a \( B \)-parallel section of \( \text{ad}(Q) \). The group \( \Gamma \) acts by \( B \)-preserving bundle isomorphisms of \( Q \) and by orientation preserving isometries on \( \mathbb{H}^2 \), so it leaves invariant \( F_B \) and \( \text{vol}_{\mathbb{H}^2} \); it follows that it also leaves invariant the section \( \eta_B \in A^0(\mathbb{H}^2, \text{ad}(Q)) \). Therefore \( \eta_B \) descends to a non-trivial \( A \)-parallel section of \( \text{ad}(P) \), which implies that \( P \) admits an \( A \)-parallel \( S^1 \)-reduction \( P_0 \subset P \). The obtained \( S^1 \)-connection \( A_0 \in \mathcal{A}(P_0) \) is still locally homogeneous, so the classification of locally homogeneous PU(2)-connections of this type on \( \hat{M} \) reduces to the classification of locally homogeneous \( S^1 \)-connections, which has been studied in section \( 3.2.3 \).

3.2.3. **Locally homogeneous connections with flat pull-back.** If the stabiliser of \( B \) is \( PU(2) \), then \( A \) will be flat, so the pair \( (P, A) \) will be defined by a representation \( \pi_1(M) = \Gamma \to PU(2) \). The moduli space of isomorphism classes of flat PU(2)-connections on \( M \) is just the quotient \( \text{Hom}(\pi_1(M), SU(2))/SU(2) \) modulo conjugation. The statement generalizes for an arbitrary locally homogeneous Riemannian manifold \( (M, g) \) and Lie group \( K \): any flat \( K \)-connection is LH, and the moduli space of isomorphism classes of flat \( K \)-connections on \( M \) can be identified with the quotient \( \text{Hom}(\pi_1(M), K)/K \). The case when \( (M, g) \) is a Riemann surface and \( K = PU(r) \) is especially interesting because of the classical Narasimhan–Seshadri theorem \( [NaSe], [Do] \) which states that a holomorphic rank \( r \)-bundle on a Riemann surface (regarded as a complex projective curve) is polystable if and only if it admits a projectively flat Hermitian metric. In the special case of bundles of degree 0, we get an isomorphism of moduli spaces

\[
\text{Hom}(\pi_1(M), SU(r))/SU(r) \cong \mathcal{M}^{\text{pol}}(r, 0)
\]

onto the moduli space of polystable holomorphic rank \( r \)-bundles of degree 0.
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