Exact vortex solutions in a $CP^N$ Skyrme-Faddeev type model

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Abstract

We consider a four dimensional field theory with target space being $CP^N$ which constitutes a generalization of the usual Skyrme-Faddeev model defined on $CP^1$. We show that it possesses an integrable sector presenting an infinite number of local conservation laws, which are associated to the hidden symmetries of the zero curvature representation of the theory in loop space. We construct an infinite class of exact solutions for that integrable submodel where the fields are meromorphic functions of the combinations $(x^1 + i x^2)$ and $(x^3 + x^0)$ of the Cartesian coordinates of four dimensional Minkowski space-time. Among those solutions we have static vortices and also vortices with waves traveling along them with the speed of light. The energy per unity of length of the vortices show an interesting and intricate interaction among the vortices and waves.

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1 Introduction

The development of non-perturbative methods is of crucial importance for the study of strong coupling phenomena in Physics, specially in field theories. It has become clear that solitons and hidden symmetries play an special role in that context. In many theories the solitons are the suitable normal modes to describe the strong coupling regime. That is true for instance in the two dimensional sine-Gordon model [1] and the four dimensional supersymmetric gauge theories [2], where a duality exists between the weak and strong coupling sectors where fundamental particles and solitons exchange roles. In addition, the appearance of solitons is associated to a high degree of symmetries and conservation laws. An important aspect is that in general those are not symmetries of the Lagrangian or of the equations of motion, and for that reason are called hidden symmetries. They appear in special and deep structures of the theory, and in the case of integrable two dimensional field theories they are the symmetries of the zero curvature condition or the Lax-Zakharov-Shabat equation [3]. Of course, it would be very important to discover the counterpart of those structures in realistic four dimensional field theories like gauge theories. A proposal [4] to approach that problem uses the concept flat connections on loop spaces to construct the generalization of the zero curvature condition for theories in dimensions higher than two. Such approach has obtained success in several models leading to infinite number of conservation laws and exact solutions [5].

In this paper we explore the ideas of [4, 5] to study a non-linear four dimensional field theory which is in fact an extension of the Skyrme-Faddeev model on the target space $\mathbb{CP}^N$. We show that such model possesses an integrable sector that presents an infinite number of conserved currents which are not associated to any symmetry of the Lagrangian or equations of motion. They are hidden symmetries of the representation of the theory in terms of the generalized zero curvature in loop space. The integrable submodel is obtained by restricting the theory by some constraints and a relation among the coupling constants. In addition, we show that such integrable sector possesses an infinite class of exact solutions where the $N$ complex fields of the model are meromorphic functions of the combinations $x^1 + ix^2$ and $x^3 + x^0$, of the Cartesian coordinates $x^\mu$, $\mu = 0, 1, 2, 3$, of four dimensional Minkowski space-time. Among those solutions we have static vortices and also vortices with waves traveling along them with the speed of light. The static vortices are Bogomolny type solutions and their energy per unity of length come from boundary terms in the energy functional. They correspond in fact to the Bogomolny solutions of the $\mathbb{CP}^N$ model in two dimensions [6, 7, 8], but not to the static non-Bogomolny solutions of that model. Our results constitute a generalization to $\mathbb{CP}^N$ of the those obtained in [9] for an extension of the usual Skyrme-Faddeev model [10].
defined on $CP^1$ (or equivalently $S^2$). Vortex type solutions of the usual $CP^1$ Skyrme-Faddeev model were also considered in [11, 12], and vortex with waves were considered in supersymmetric gauge theories in [13]. It is worth mentioning that the Skyrme-Faddeev model on the coset space $SU(N+1)/U(1)^N$ has been conjectured to correspond to a low energy effective theory for pure $SU(N+1)$ Yang-Mills theory [14]. In [15] it has been argued that for $N \geq 2$, the relevant low energy degrees of freedom may also be described by the coset space $SU(N+1)/SU(N) \otimes U(1)$, or $CP^N$, which the authors call the minimum case. Therefore, the results of this paper may play some role in that approach proposed in [15].

The paper is organized as follows: in section 2 we define the model exploring the fact that $CP^N$ is the symmetric space $SU(N+1)/SU(N) \otimes U(1)$. In section 3 we present the integrable sector possessing an infinite number of conservation laws, and the exact solution are constructed in subsection 3.1. The energies per unity of length of the vortex solutions are calculated in section 4. The spectrum is quite interesting showing an intricate interaction among the vortices and waves. The energy of the static vortices comes from boundary terms, as it is common in Bogomolny type solutions. The case of $CP^2$ is discussed in more detail in subsection 4.1.

2 The model

We shall explore the fact that $CP^N$ is a symmetric space [16]. Indeed, it is a coset space $CP^N = SU(N+1)/SU(N) \otimes U(1)$ with the subgroup $SU(N) \otimes U(1)$ being invariant under the involutive automorphism ($\sigma^2 = 1$)

$$\sigma (T) = \Omega T \Omega^{-1}; \quad \Omega = e^{i\pi \Lambda}; \quad \Lambda = \frac{2\lambda_N \cdot H}{\alpha_N^2}$$

(2.1)

where $\lambda_N$ is the last fundamental weight of $SU(N+1)$, i.e. the highest weight of the $\tilde{N}$ representation, and $\alpha_N$ is the simple root of $SU(N+1)$ associated to that fundamental weight, i.e. $2\lambda_N \cdot \alpha_N/\alpha_N^2 = 1$. The invariant subgroup $SU(N) \otimes U(1)$ is generated by the Cartan subalgebra generators $H_i$, $i = 1, 2, \ldots N$, and the step operators associated to roots not containing $\alpha_N$ in its expansion in terms of simple roots. The space $CP^N = SU(N+1)/SU(N) \otimes U(1)$ has a nice parametrization in terms of the so-called principal variable [17, 18], defined by

$$X (g) \equiv g \sigma(g)^{-1} \quad g \in SU(N+1)$$

(2.2)

Indeed, if $k \in SU(N) \otimes U(1)$ then $X (g k) = X (g)$, since $\sigma (k) = k$, and so we have just one matrix $X (g)$ for each coset in $SU(N+1)/SU(N) \otimes U(1)$.  

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We now introduce, in 3 + 1 dimensions, a field theory with target space being \( \mathbb{C}P^N \) and defined by the Lagrangian
\[
\mathcal{L} = -\frac{M^2}{2} \text{Tr} \left( X^{-1} \partial_\mu X \right)^2 + \frac{1}{e^2} \text{Tr} \left( \left[ X^{-1} \partial_\mu X, X^{-1} \partial_\nu X \right] \right)^2 + \frac{\beta}{2} \left[ \text{Tr} \left( X^{-1} \partial_\mu X \right)^2 \right]^2 + \gamma \left[ \text{Tr} \left( X^{-1} \partial_\mu X X^{-1} \partial_\nu X \right) \right]^2 \tag{2.3}
\]

The coupling constant \( M^2 \) has dimension of mass, and \( e^2, \beta \) and \( \gamma \) are dimensionless coupling constants. The derivatives \( \partial_\mu \) are with respect to the Cartesian coordinates \( x^\mu, \mu = 0, 1, 2, 3 \). Notice that it consists of a generalization to \( \mathbb{C}P^N \) of the Skyrme-Faddeev model \([10]\), and we shall refer to it as the \( \mathbb{C}P^N \) extended Skyrme-Faddeev model (CPNSF). In the case of \( N = 1 \) it coincides with an extension of the Skyrme-Faddeev model considered in \([9]\). The presence of terms which are quadratic and quartic in derivatives of the \( X \) field imply, according to Derrick’s theorem, that one can have stable static solutions in three spatial dimensions. However, we will be concerned in this paper with exact static and time-dependent vortex solutions of (2.3). The theory (2.3) has a global left \( SU(N+1) \) symmetry such that \( g \rightarrow \bar{g} g \), with \( \bar{g}, g \in SU(N+1) \), which implies that \( X \rightarrow \bar{g} X \sigma (\bar{g})^{-1} \), and so \( X^{-1} \partial_\mu X \rightarrow \sigma (\bar{g}) X^{-1} \partial_\mu X \sigma (\bar{g})^{-1} \). In addition it has a right local \( SU(N) \otimes U(1) \) symmetry such that \( g \rightarrow g k \), with \( k \in SU(N) \otimes U(1) \) and \( g \in SU(N+1) \), which implies that \( X \) is invariant. That symmetry will play a role when we work with the group elements \( g \) instead of \( X \), since \( g^{-1} \partial_\mu g \rightarrow k^{-1} g^{-1} \partial_\mu g k + k^{-1} \partial_\mu k \).

The Euler-Lagrange equations associated to (2.3) are given by
\[
\partial^\mu J_\mu = 0 \tag{2.4}
\]
with
\[
J_\mu \equiv \left[ M^2 \eta_{\mu\nu} - 2 \beta \text{Tr} \left( X^{-1} \partial_\mu X \right)^2 \eta_{\mu\nu} - 4 \gamma \text{Tr} \left( X^{-1} \partial_\mu X X^{-1} \partial_\nu X \right) \right] X^{-1} \partial^\nu X + \frac{4}{e^2} \left[ \left[ X^{-1} \partial_\mu X, X^{-1} \partial_\nu X \right], X^{-1} \partial^\nu X \right] \tag{2.5}
\]
where \( \eta_{\mu\nu} = \text{diag} (1, -1, -1, -1) \), is the Minkowski metric. In order to obtain (2.4) we have used the fact that for any vector quantity \( Y_\mu \) we have the identities \([Y^\mu, Y_\mu] = 0\) and \([Y_\mu, [Y_\nu, [Y^\mu, Y^\nu]]] = 0\). The current \( J_\mu \) lies in the algebra of \( SU(N+1) \) and so the number of independent conserved currents is equal to \( \text{dim} \left[ SU(N+1) \right] = N^2 + 2N \). They are the Noether currents associated to left global \( SU(N+1) \) symmetry of (2.3) mentioned above.

Using (2.2) one gets that
\[
X^{-1} \partial_\mu X = \sigma (g) P_\mu \sigma (g)^{-1} \tag{2.6}
\]
with \( P_\mu \) being the odd part, under \( \sigma \), of the Maurer-Cartan form, i.e.

\[
P_\mu = g^{-1} \partial_\mu g - \sigma \left(g^{-1} \partial_\mu g\right)
\]  
(2.7)

Therefore, one gets that

\[
J_\mu = \sigma(g) B_\mu \sigma(g)^{-1}
\]  
(2.8)

with

\[
B_\mu \equiv \left[M^2 \eta_{\mu\nu} - 2 \beta \text{Tr} (P_\rho)^2 \eta_{\mu\nu} - 4 \gamma \text{Tr} (P_\mu P_\nu)\right] P^\nu + \frac{4}{e^2} \left[P_\mu, P_\nu\right] P^\nu
\]  
(2.9)

Since \( P_\mu \) is odd it turns out that so is \( B_\mu \), i.e.

\[
\sigma(B_\mu) = -B_\mu
\]  
(2.10)

Then it follows that the equation of motion (2.4) can be written as

\[
\partial^\mu B_\mu + [A^\mu, B_\mu] = 0 \quad \text{with} \quad A_\mu = g^{-1} \partial_\mu g
\]  
(2.11)

Therefore, the equations of motion of (2.3) takes the form of the generalized zero curvature conditions proposed in \[4, 5\], in terms of a vector \( B_\mu \) and a flat connection \( A_\mu \). We discuss below how to use the methods of \[4, 5\] to construct an infinite number of conserved currents for a submodel of (2.3). It turns out in fact that the equations of motion (2.3) depend only on the even part of the flat connection \( A_\mu \). Indeed, subtracting (2.11) from its image under \( \sigma \) one gets that

\[
\partial^\mu B_\mu + [a^\mu, B_\mu] = 0 \quad \text{with} \quad a_\mu \equiv \frac{(1 + \sigma)}{2} A_\mu
\]  
(2.12)

The odd part of \( A_\mu \) is \( P_\mu \) and that commutes with \( B_\mu \) thanks to the identities mentioned below (2.5). Therefore, the equations of motion (2.12) depend only on the representation of the subgroup \( SU(N) \otimes U(1) \) under which \( B_\mu \) transforms, under the action of the (not-flat) connection \( a_\mu \). But \( B_\mu \) belongs to the odd subspace of the algebra of \( SU(N + 1) \) and that transforms under the \( N(1) + \bar{N}(-1) \) representation of \( SU(N) \otimes U(1) \). In order to see that, notice that the generators of \( SU(N + 1) \) which are odd under the automorphism \( \sigma \) defined in (2.1), are the step operators \( E_\alpha \) associated to roots \( \alpha \) that contain the simple root \( \alpha_N \) in their expansion in terms of simple roots. Indeed, we have that

\[
S_{\pm i} \equiv E_{\pm(\alpha_i + \alpha_{i+1} + \ldots + \alpha_N)} \quad \sigma(S_{\pm i}) = -S_{\pm i} \quad i = 1, 2, \ldots N
\]  
(2.13)

In the \((N + 1)\)-dimensional defining representation of \( SU(N + 1) \) those generators are given by the matrices

\[
(S_i)_{rs} = \delta_{r, i} \delta_{s, N+1} \quad S_{-i} = S_i^\dagger \quad r, s = 1, 2, \ldots N + 1
\]  
(2.14)
A basis for the $N^2$ generators of $SU(N) \otimes U(1)$ can be taken as $[S_i, S_{-j}]$, with $i, j = 1, 2, \ldots N$, and one can easily check that

$$
\begin{align*}
\left[[S_i, S_{-j}], S_k\right] &= \delta_{ij} S_k + \delta_{jk} S_i \\
\left[[S_i, S_{-j}], S_{-k}\right] &= -\delta_{ij} S_{-k} - \delta_{ik} S_{-j}
\end{align*}
$$

(2.15)

which establishes that the subspaces generated by $S_i$ and $S_{-i}$ correspond indeed to the $N(1)$ and $\bar{N}(-1)$ representations respectively of $SU(N) \otimes U(1)$. The $U(1)$ generator of $SU(N) \otimes U(1)$ corresponds to the $\Lambda$ operator defined in (2.1), and in such basis is given by

$$
\Lambda = \frac{1}{N+1} \sum_{i=1}^{N} [S_i, S_{-i}]
$$

(2.16)

In addition, such subspaces are abelian

$$
[S_i, S_j] = [S_{-i}, S_{-j}] = 0
$$

(2.17)

and satisfy

$$
\text{Tr} (S_i S_{-j}) = \delta_{ij} \quad \text{Tr} (S_i S_j) = \text{Tr} (S_{-i} S_{-j}) = 0
$$

(2.18)

We now introduce coordinates in $CP^N$ by providing a suitable parametrization of the group elements $g \in SU(N+1)$. We follow the results of section 8 of [19], and introduce complex fields $u_i$ as

$$
g = e^{iu_i S_i} e^{u_i u_j^* S_{-j}} e^{i u_i^* S_{-i}} \quad \varphi \equiv \frac{\log \sqrt{1 + u^\dagger \cdot u}}{u^\dagger \cdot u}
$$

(2.19)

In the $(N+1)$-dimensional defining representation of $SU(N+1)$ we have that $g$ is given by the matrices

$$
g \equiv \frac{1}{\vartheta} \begin{pmatrix} \Delta & iu \\ iu^\dagger & 1 \end{pmatrix} \quad \vartheta = \sqrt{1 + u^\dagger \cdot u}
$$

(2.20)

where $\Delta$ is the hermitian $N \times N$-matrix

$$
\Delta_{ij} = \vartheta \delta_{ij} - \frac{u_i u_j^*}{1 + \vartheta} \quad \text{which satisfies} \quad \Delta \cdot u = u; \quad u^\dagger \cdot \Delta = u^\dagger
$$

(2.21)

In such representation the group element $\Omega$ introduced in (2.1) can be written as

$$
\Omega = e^{i \pi/(N+1)} \begin{pmatrix} I_{N \times N} & 0 \\ 0 & -1 \end{pmatrix}
$$

(2.22)
Therefore, for $g$ given by (2.20), one has $\sigma(g) = g^{-1}$, and so the principal variable introduced in (2.2) becomes

$$X(g) = g^2 = \begin{pmatrix} 1_{N \times N} & 0 \\ 0 & -1 \end{pmatrix} + \frac{2}{\vartheta^2} \begin{pmatrix} -u \otimes u^\dagger & i u \\ i u^\dagger & 1 \end{pmatrix}$$

(2.23)

Therefore, under a left translation $X \rightarrow U X \sigma(U)^{-1}$, with $U$ being an element of $SU(N) \otimes U(1)$, i.e.

$$U = e^{i \theta/(N+1)} \begin{pmatrix} \hat{U} & 0 \\ 0 & e^{-i \theta} \end{pmatrix} \quad \text{with} \quad \hat{U} \hat{U}^\dagger = 1; \quad \det \hat{U} = 1$$

(2.24)

we have that $u$ transforms under the defining $N$-dimensional representation of $SU(N) \otimes U(1)$, i.e. $u \rightarrow e^{i \theta} \hat{U} u$.

The odd part of the Maurer-Cartan form introduced in (2.7) can be written as $P_\mu = P^{(+)}_\mu + P^{(-)}_\mu$, where

$$P^{(+)}_\mu = \frac{2i}{\vartheta^2} \sum_{i=1}^N (\Delta \cdot \partial_\mu u)_i S_i \quad \quad P^{(-)}_\mu = \frac{2i}{\vartheta^2} \sum_{i=1}^N (\partial_\mu u^\dagger \cdot \Delta)_i S_{-i}$$

(2.25)

which satisfies $(P^{(+)}_\mu)^\dagger = -P^{(-)}_\mu$. In addition, the even part of the same one-form, namely the connection $a_\mu$ introduced in (2.12), becomes

$$a_\mu = (1 + \sigma) \frac{g^{-1}}{2} \partial_\mu g = \sum_{i,j=1}^N \kappa_{ij} \frac{\vartheta}{\vartheta^2} [S_i, S_{-j}]$$

(2.26)

with

$$\kappa_{ij} = \frac{1}{2} \left( u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u \right) \frac{u_i u_j^*}{(1 + \vartheta)^2} + \frac{\vartheta}{1 + \vartheta} \left( u_i \partial_\mu u_j^* - \partial_\mu u_i u_j^* \right)$$

(2.27)

Using (2.15) and (2.25) one can check that

$$[[ P_\mu , P_\nu ] , P^\nu] = \tau^{(\nu)}_\nu P_\mu + (\tau^{(\mu)}_\nu - 2 \tau^{(\nu)}_\nu) P^{(+)}_\mu + (\tau^{(\nu)}_\mu - 2 \tau^{(\mu)}_\nu) P^{(-)}_\mu$$

(2.28)

where we have introduced

$$\tau^{(\nu)}_\mu \equiv \text{Tr} \left( P^{(+)}_\mu P^{(-)}_\nu \right) = \frac{4}{\vartheta^2} \partial_\mu u^\dagger \cdot \Delta^2 \cdot \partial_\mu u$$

(2.29)

with $\Delta$ being given in (2.21), and so $(\Delta^2)_{ij} = \vartheta^2 \delta_{ij} - u_i u_j^*$. Notice that $\tau^{*\mu}_\nu = \tau^{(\nu)}_\mu$, since $\Delta$ is hermitian. One can then write the operator $B_\mu$ introduced in (2.9) as

$$B_\mu = B^{(+)}_\mu + B^{(-)}_\mu \quad \text{with} \quad B^{(+)}_\mu \equiv C_{\mu\nu} P^{(\nu)} \quad B^{(-)}_\mu \equiv P^{(\nu)} C_{\mu\nu}$$

(2.30)
and where we have introduced the quantity

\[ C_{\mu\nu} \equiv M^2 \eta_{\mu\nu} - \frac{4}{e^2} \left( (\beta e^2 - 1) \tau^\rho_{\mu} \eta_{\nu\rho} + (\gamma e^2 - 1) \tau_{\mu\nu} + (\gamma e^2 + 2) \tau_{\nu\mu} \right) \]  

(2.31)

which satisfies \( C^*_{\mu\nu} = C_{\nu\mu} \). Therefore, we have \( (B^{(+)}_{\mu})^\dagger = -B^{(-)}_{\mu} \).

From (2.15) one observes that \( B^{(+)}_{\mu} \) and \( B^{(-)}_{\mu} \) transform under different irreducible representations of \( SU(N) \otimes U(1) \), namely \( N(1) \) and \( \bar{N}(-1) \). Therefore, the equation (2.12) split into two components, one for each one of those two irreducible representations. One can check using (2.15) and (2.25)-(2.27) that

\[ \partial_{\mu} B^{(+)}_{\mu} + \left[ a^\mu, B^{(+)}_{\mu} \right] = 2 i S_i \frac{\Delta_{ij}}{\partial^2} \left[ \partial^\mu \left( C_{\mu\nu} \partial^\nu u_j \right) - \frac{1}{\partial^2} (u^*_j \delta_{jk} + u^*_k \delta_{ji}) C_{\mu\nu} \partial^\mu u_l \partial^\nu u_k \right] \]

with the equation for \( B^{(-)}_{\mu} \) being obtained from that by complex conjugation. The equations of motion are then given by

\[ (1 + u^\dagger \cdot u) \partial^\mu \left( C_{\mu\nu} \partial^\nu u_i \right) - C_{\mu\nu} \left[ (u^\dagger \cdot \partial^\mu u) \partial^\nu u_i + (u^\dagger \cdot \partial^\nu u) \partial^\mu u_i \right] = 0 \]  

(2.32)

together with their complex conjugates. We then have \( 2N \) equations of motion corresponding to the \( 2N \) fields \( u_i \) and \( u^*_i \), \( i = 1, 2, \ldots N \). Notice that (2.32) resembles the \( CP^N \) equations of motions, and in fact they reduce to it when \( C_{\mu\nu} \rightarrow \eta_{\mu\nu} \).

### 3 The integrable sector

We now use the concepts of [4, 5] of generalized integrability to construct an infinite number of conserved currents for a sector of the theory (2.31). The vector \( B_{\mu} \) appearing in (2.11) and (2.12) lies in the adjoint representation of \( SU(N + 1) \) and transforms under the \( N(1) + \bar{N}(-1) \) representation of \( SU(N) \otimes U(1) \). Therefore, the number of conserved currents one gets in (2.8) is equal to the dimension of the adjoint of \( SU(N + 1) \). The basic idea is to look for conditions on the fields that will make the relations (2.11) and (2.12) still valid when \( B_{\mu} \) lives in higher (possibly infinite) representations. A detailed account of such procedure is given in [4, 5]. For the model under consideration the relevant conditions are given by

\[ \partial_{\mu} u_i \partial^\mu u_j = 0 \quad \text{for any } i, j = 1, 2, \ldots N \]  

(3.1)

Therefore the integrable sector is selected by those solutions of (2.32) which also satisfy (3.1). That is a generalization to \( CP^N \) of the constraints used in the models with target space being \( CP^1 \), or equivalently the two dimensional sphere \( S^2 \) [4, 5]. Such constraints
have already been considered in [19] in the context of the pure \( CP^N \) model, leading to an infinite number of conserved currents. A further study of integrable sectors of pure \( CP^N \) model is given in [20]. It then follows from (2.31) that, when (3.1) holds true, one has

\[
C_{\mu \nu} \partial^\mu u_i \partial^\nu u_j = 0 \tag{3.2}
\]

and the equations of motion (2.32) reduce to

\[
\partial^\mu (C_{\mu \nu} \partial^\nu u_i) = 0 \tag{3.3}
\]

As a consequence of that one can then check that the currents

\[
J^G_\mu \equiv \sum_{i=1}^{N} \left[ \frac{\delta G}{\delta u_i} C_{\mu \nu} \partial^\nu u_i - \frac{\delta G}{\delta u_i^*} \partial^\nu u_i^* C_{\nu \mu} \right] \tag{3.4}
\]

are conserved, i.e. \( \partial^\mu J^G_\mu = 0 \), where \( G \) is a functional of \( u_i \) and \( u_i^* \), but not of their derivatives. The conservation of the currents follows directly from (3.2) and (3.3). Notice that such currents were not obtained as Noether currents associated to symmetries of the submodel defined by the constraints (3.1). In fact, it is not even certain that such submodel possesses a regular Lagrangian. Such currents are related to hidden symmetries of the generalized zero curvature based on infinite dimensional representations of \( SU(N+1) \) (see [19, 4, 5] for details).

### 3.1 Exact solutions

Notice that due to the constraints (3.1) we have that \( \tau_{\nu \mu} \partial^\nu u_i = 0 \). Therefore the last term in \( C_{\mu \nu} \), given in (2.31), drops out when contracted with \( \partial^\nu u_i \). Therefore the reduced equations of motion (3.3) can be written as, using (2.31) and (2.29),

\[
M^2 \partial^2 u_i + \frac{16}{e^2} \partial^\mu \left[ \left( \partial^\nu u_i^* \cdot \frac{\Delta^2}{\partial^4} \right)_j \left[ (\beta e^2 - 1) \partial_{\nu} u_j \partial_{\mu} u_i + (\gamma e^2 - 1) \partial_{\mu} u_j \partial_{\nu} u_i \right] \right] = 0 \tag{3.5}
\]

One can check that

\[
\partial^\mu \left( \partial^\nu u_i^* \cdot \frac{\Delta^2}{\partial^4} \right)_j = R^\mu_{\nu j} + N^\mu_{\nu j} \tag{3.6}
\]

with

\[
R^\mu_{\nu j} \equiv \partial^2 \partial^\mu \partial^\nu u_k^* - \left[ \left( \partial^\nu u_i^\dagger \cdot u \right) \partial^\mu u_k^* + \left( \partial^\mu u_i^\dagger \cdot u \right) \partial^\nu u_k^* \right] \frac{\Delta^2}{\partial^4} \tag{3.7}
\]

and

\[
N^\mu_{\nu j} \equiv - \left[ \left( \Delta^2 \right)_{ij} u_k^* + \left( \Delta^2 \right)_{lk} u_j^* \right] \frac{\partial^\mu u_k \partial^\nu u_j^*}{\partial^4} \tag{3.8}
\]
Notice that $R_{j}^{\mu \nu} = R_{j}^{\nu \mu}$, and $R_{j \mu}^{ \nu}$ is proportional to the $C P^{N}$ equations of motion. In addition, due to the constraint (3.1) we have that $N_{j}^{\mu \nu} \partial_{\mu} u_{l} = 0$. Therefore, (3.5) becomes

$$M^{2} \partial^{2} u_{i} + \frac{16}{e^{2}} \left( \partial^{\nu} u^{\dagger} : \frac{\Lambda^{2}}{\partial^{4}} \right)_{j} \partial_{j} \left[ (\beta e^{2} - 1) \partial_{\nu} u_{j} \partial_{\mu} u_{i} + (\gamma e^{2} - 1) \partial_{\mu} u_{j} \partial_{\nu} u_{i} \right]$$

$$+ \frac{8}{e^{2}} \left( \beta e^{2} + \gamma e^{2} - 2 \right) R_{j}^{\mu \nu} \left( \partial_{\mu} u_{j} \partial_{\nu} u_{i} + \partial_{\mu} u_{i} \partial_{\nu} u_{j} \right) = 0 \quad (3.9)$$

Let us now introduce the coordinates

$$\begin{align*}
&z = x^{1} + i \varepsilon_{1} x^{2} ; \quad \bar{z} = x^{1} - i \varepsilon_{1} x^{2} ; \\
y_{+} = x^{3} + \varepsilon_{2} x^{0} ; \quad y_{-} = x^{3} - \varepsilon_{2} x^{0} \end{align*} \quad (3.10)$$

with $\varepsilon_{a} = \pm 1, a = 1, 2$. Then the metric is

$$d s^{2} = \eta_{\mu \nu} dx^{\mu} dx^{\nu} = -dz d\bar{z} - dy_{+} dy_{-} \quad (3.11)$$

If we now assume that all the $u_{i}$’s are functions of $z$ and $y_{+}$ only, i.e.

$$u_{i} = u_{i} (z, y_{+}) \quad \text{and} \quad u_{i}^{*} = u_{i}^{*} (\bar{z}, y_{+}) \quad (3.12)$$

then it follows that

$$\partial^{2} u_{i} = 0 ; \quad \partial^{\mu} u_{i} \partial_{\mu} u_{j} = 0 ; \quad \partial^{\mu} [\partial_{\nu} u_{i} \partial_{\mu} u_{j}] = 0 \quad (3.13)$$

If in addition we choose the coupling constants such that

$$\beta e^{2} + \gamma e^{2} - 2 = 0 \quad (3.14)$$

Then the equations (3.9) as well the constraints (3.1) are satisfied. We then get an infinite class of exact solutions for the theory (2.3), given by the configurations (3.12).

4 The energy

Using (2.6), (2.28), (2.29) and (2.31) one can write the Lagrangian (2.3) as

$$\mathcal{L} = -\frac{1}{2} \left[ M^{2} \eta_{\mu \nu} + C_{\mu \nu} \right] \tau^{\mu \nu} \quad (4.1)$$

Therefore, any variation of $\mathcal{L}$ w.r.t. the fields leads to

$$\delta \mathcal{L} = -C_{\mu \nu} \delta \tau^{\mu \nu} \quad (4.2)$$
The Hamiltonian density is given by

\[ \mathcal{H} = \frac{\delta L}{\delta \partial_0 u_i} \partial_0 u_i + \frac{\delta L}{\delta \partial_0 u_i^*} \partial_0 u_i^* - L \]

\[ = \frac{1}{2} \left[ M^2 \eta_{\mu \nu} + C_{\mu \nu} \right] \tau^{\mu \nu} - C^{\rho \theta} \tau_{0 \rho} - C^{\rho \theta} \tau_{\rho 0} \]

\[ = -M^2 (\tau_{00} + \tau_{aa}) + \frac{2}{e^2} \left[ (\beta e^2 - 1) \tau_\rho^\rho (3 \tau_{00} + \tau_{aa}) + (\gamma e^2 - 1) (4 \tau^{\rho \theta} \tau_{0 \rho} - \tau^{\rho \nu} \tau_{\nu \rho}) \right] \]

\[ + \left( \gamma e^2 + 2 \right) \left( 2 \tau^{\rho \theta} \tau_{0 \rho} + 2 \tau^{\rho \nu} \tau_{\rho \nu} \right) \]

(4.3)

where \( a = 1, 2, 3 \), stands for the space coordinates \( x^a \). Notice that the last term, proportional to \( (\gamma e^2 + 2) \), vanishes when the constraints (3.1) are imposed. In addition, when the condition (3.14), among the coupling constants, are taken into account we have that the Hamiltonian reduces to

\[ \mathcal{H}_c = 4 M^2 \frac{\left( \partial_0 u^\dagger \cdot \Delta^2 \cdot \partial_0 u + \partial_a u^\dagger \cdot \Delta^2 \cdot \partial_a u \right)}{(1 + u^\dagger \cdot u)^2} \]

\[ + 16 (\beta - \gamma) \frac{\left( \Delta^2 \right)_{ij} \left( \Delta^2 \right)_{kl}}{(1 + u^\dagger \cdot u)^4} \left[ (\partial_0 u_j \partial_a u_l - \partial_0 u_l \partial_a u_j) (\partial_a u^*_i \partial_0 u^*_k - \partial_0 u^*_i \partial_a u^*_k) \right] \]

\[ + \sum_{a < b} (\partial_a u_j \partial_b u_l - \partial_a u_l \partial_b u_j) (\partial_b u^*_i \partial_a u^*_k - \partial_a u^*_i \partial_b u^*_k) \]

(4.4)

with \( a, b = 1, 2, 3 \), and \( i, j, k, l = 1, 2, \ldots N \). Notice that the case \( \beta = \gamma \) is special, since the reduced Hamiltonian becomes positive definite. For the solutions of the type we are considering, namely (3.12), we get

\[ \mathcal{H}_c = 8 M^2 \frac{\left( \partial_z u^\dagger \cdot \Delta^2 \cdot \partial_z u + \partial_y^+ u^\dagger \cdot \Delta^2 \cdot \partial_y^+ u \right)}{(1 + u^\dagger \cdot u)^2} \]

\[ + 64 (\beta - \gamma) \frac{\left( \Delta^2 \right)_{ij} \left( \Delta^2 \right)_{kl}}{(1 + u^\dagger \cdot u)^4} \left[ (\partial_y^+ u_j \partial_z u_l - \partial_y^+ u_l \partial_z u_j) (\partial_z u^*_i \partial_y^+ u^*_k - \partial_y^+ u^*_i \partial_z u^*_k) \right] \]

(4.5)

Let us first consider the static solutions. If a given solution of the type (3.12) does not depend upon the time, then it does not depend upon \( x^3 \) as well. Therefore, the contribution to the energy comes only from the first term in (4.5). Notice however that for the solutions (3.12) (static or not) one has the identity

\[ \partial_z \partial_z \ln (1 + u^\dagger \cdot u) = \frac{\partial_z u^\dagger \cdot \Delta^2 \cdot \partial_z u}{(1 + u^\dagger \cdot u)^2} \]

(4.6)

Therefore, the energy per unit of length for the static solutions is given by

\[ \mathcal{E}_{\text{static}} = \int dx^1 dx^2 \mathcal{H}_c = 8 M^2 \int dx^1 dx^2 \partial_z \partial_z \ln (1 + u^\dagger \cdot u) \]

(4.7)
So, the problem reduces to that of the $\text{CP}^n$ model in two Euclidean dimensions, i.e. our static vortex solutions have an energy per unit of length which equals the Euclidean action of the $\text{CP}^n$ lumps. As shown by [21] the finite energy (action) solutions are those where the $u$ fields are rational functions, i.e.

$$u_i = \frac{p_i(z)}{q_i(z)} \quad (4.8)$$

where $p_i(z)$ and $q_i(z)$ are polynomials in the $z$ variable. Following the arguments of [21, 6] (see also [22]) one finds that (4.7) is essentially equal to the number of poles of $u_i$’s including those at infinity, i.e.

$$E_{\text{static}} = 8 \pi M^2 \left[ d_{\text{max}} + \sum_{s_{\text{max}}} h_i^{(s)} \right] \quad (4.9)$$

where $d_{\text{max}}$ is the highest degree of the polynomials $p_i$’s, and the sum is over the zeroes of the polynomials $q_i$’s, with $h_i^{(s)}$ being the highest order of the zeroes of $q_i$’s at $z = z_0^{(s)}$. Notice therefore that the energy per unit of length of the static vortex does not depend upon their relative position on the $x^1 x^2$ plane, as long as they are all parallel to the $x^3$-axis. As an example consider the solutions of the form

$$u_i = c_i \left( \frac{z}{r_0} \right)^{n_i} = c_i \left( \frac{\rho}{r_0} \right)^{n_i} e^{i \epsilon_1 n_i \varphi} \quad i = 1, 2, \ldots N \quad (4.10)$$

with $c_i$ being complex constants, $n_i$ being integers in order for the solution to be single valued, $r_0$ being a length scale, and where we have introduced polar coordinates on the $x^1 x^2$ plane, i.e. $z = x^1 + i \epsilon_1 x^2 = \rho e^{i \epsilon_1 \varphi}$, $(\epsilon_1 = \pm 1)$. Therefore, for such static vortices we have

$$E_{\text{static}} = 8 \pi M^2 \left( n_{\text{max}} + | n_{\text{min}} | \right) \quad (4.11)$$

where $n_{\text{max}}$ is the highest positive integer in the set $n_i$, $i = 1, 2, \ldots N$ (which corresponds to the highest degree $d_{\text{max}}$ of the polynomials $p_i$ in (4.8)), and $n_{\text{min}}$ is the lowest negative integer in the same set, such that the corresponding complex constants $c_i$ are non-vanishing. We have in such case zeroes at $z = 0$ only, and so $(-n_{\text{min}})$ corresponds to $h_i^{(1)}$ with $z_0^{(1)} = 0$. Notice that such result is independent of the number of $n_i$’s equal to $n_{\text{max}}$ or $n_{\text{min}}$.

Again following the reasoning of [21] we can associated topological charges to those vortex solutions. The fields $u_i$ provide a mapping from the $x^1 x^2$ plane into $\text{CP}^n$. However, in order to have a finite energy per unit length one needs the fields to go to a constant at infinity on that plane. Therefore, as long as topological properties are concerned we can consider the $x^1 x^2$ plane compactified into the sphere $S^2$. Then the finite energy solutions
define maps from $S^2$ to $CP^N$, and they can be classified into the homotopy classes of $\pi_2(\text{CP}^N)$. There exists however a theorem [21, 23, 24] stating that $\pi_2(G/H) = \pi_1/(H)_G$, where $\pi_1(H)_G$ is the subset of $\pi_1(H)$ formed by closed paths in $H$ which can be contracted to a point in $G$. Since $CP^N = SU(N + 1)/SU(N) \otimes U(1)$, the topological charges are given by $\pi_1/(SU(N) \otimes U(1))_{SU(N+1)}$. According to [21] the topological charges of the configurations (4.8) are equal to the number of poles of $u_i$, including those at infinity.

Therefore, the energy per unit length of the vortex solutions (4.8), given by (4.9), is proportional to the topological charge, as it is usual in Bogomolny type solutions.

We now show that the energy of vortices dependent upon $y_+$ is related to some Noether charges. The Lagrangian (4.1) (or equivalently (2.3)) is invariant under the phase transformations $u_i \rightarrow e^{i\alpha_i}u_i$, $i = 1, 2, \ldots, N$, with $\alpha_i$ being constant parameters. Those transformations correspond in fact to a $U(1)^N$ subgroup of $SU(N) \otimes U(1)$, and so of $SU(N+1)$. The Noether currents associated to such symmetry are given by

$$J^{(i)}_\mu = -i\frac{2}{\partial^2} \sum_{j=1}^N u_i^* \left( \Delta^2 \right)_{ij} C_{\mu\nu} \partial^\nu u_j - \partial^\nu u_j^* C_{\nu\mu} \left( \Delta^2 \right)_{ji} u_i$$  \hspace{1cm} (4.12)

Notice that the submodel defined by the constraints (3.1) is also invariant by those phase transformations, and in fact the currents (4.12) can be obtained from (3.4) by taking the functions $G$ as $G^{(i)} = u_i u_i^* / (1 + u_i^\dagger \cdot u_i)$.

If one imposes the constraints (3.1) and the condition (3.14) on the coupling constants, then the currents (4.12) become (the upper index $c$ stands for constrained currents)

$$J^{(i)}_\mu^{(c)} = -i\frac{2}{\partial^2} M^2 \sum_{j=1}^N u_i^* \left( \Delta^2 \right)_{ij} \partial_\mu u_j - \partial_\mu u_j^* \left( \Delta^2 \right)_{ji} u_i$$

$$- \frac{32 i \beta - \gamma}{\partial^8} \sum_{j,k,l=1}^N \left( \left( \Delta^2 \right)_{ij} \left( \Delta^2 \right)_{kl} u_i^* \partial^\nu u_k^* \partial_\nu u_j \partial_\nu u_l - \partial_\nu u_j \partial_\nu u_l \partial_\nu u_i \right)$$

$$- \left( \Delta^2 \right)_{ji} \left( \Delta^2 \right)_{lk} u_i \partial^\nu u_k \left( \partial_\mu u_j^* \partial_\nu u_l - \partial_\nu u_j^* \partial_\nu u_l \partial_\mu u_i \right)$$  \hspace{1cm} (4.13)

If one now considers solutions of the class (3.12) of the form

$$u_i = v_i(z) e^{i k_i y_+}$$  \hspace{1cm} (4.14)

with $k_i$ being the inverse of a wavelength, then the energy density (4.5) can be written as

$$\mathcal{H}_c = 8 M^2 \partial z \partial \bar{z} \ln \left( 1 + u^\dagger \cdot u \right) + \varepsilon_2 \sum_{i=1}^N k_i J^{(i)}_0$$  \hspace{1cm} (4.15)

where we have used (4.6), and $\varepsilon_2$ is defined in (3.10).
Therefore, using the results leading to (4.11), one obtains that for vortex solutions of the form
\[ u_i = c_i \left( \frac{z}{r_0} \right)^{n_i} e^{i k_i y} = c_i \left( \frac{\rho}{r_0} \right)^{n_i} e^{i \varepsilon_1 n_i \varphi} e^{i k_i (x^3 + \varepsilon_2 x^5)} \] (4.16)
the energy per unit length is given by
\[ \mathcal{E}_{\text{vortex/wave}} = \int dx \, dx^2 \, \mathcal{H}_c = 8 \pi M^2 (n_{\text{max}} + | n_{\text{min}} |) + \varepsilon_2 \sum_{i=1}^{N} k_i Q^{(i)} \] (4.17)
where \( Q^{(i)} \) are the Noether charges per unit length associated to the phase transformations \( u_i \rightarrow e^{i \alpha_i} u_i \), i.e.
\[ Q^{(i)} = \int dx \, dx^2 \, J^{(i)}_0 \] (4.18)

For the solutions of the type (4.16) one obtains that
\[ Q^{(i)} = 8 \pi M^2 \varepsilon_2 r_0^2 \left[ k_i | c_i |^2 \mathcal{I}_{n_i,2 \tilde{n}, \tilde{c}} + \sum_{j=1}^{N} (k_i - k_j) | c_i |^2 | c_j |^2 \mathcal{I}_{n_i,n_j,2 \tilde{n}, \tilde{c}} \right] \\
- 128 \pi \beta \gamma \varepsilon_2 \left[ \sum_{j=1}^{N} n_j (k_i n_j - k_j n_i) | c_i |^2 | c_j |^2 \mathcal{I}_{n_i+n_j-1,2 \tilde{n}, \tilde{c}} \right] \\
- \sum_{j,k=1}^{N} n_k [k_i n_j - k_j n_i + k_j n_k - k_k n_j] | c_i |^2 | c_j |^2 | c_k |^2 \mathcal{I}_{n_i,n_j+n_k-1,3 \tilde{n}, \tilde{c}} \]
where we have introduced the integrals
\[ \mathcal{I}_{a,b,\tilde{n}, \tilde{c}} \equiv \int_0^\infty d\zeta \frac{\zeta^a}{\left[ 1 + \sum_{k=1}^{N} | c_k |^2 \zeta^{n_k} \right]^{b}} \] (4.19)
where the integration variable is given by \( \zeta \equiv \rho^2/r_0^2 \), and \( \tilde{n} \) and \( \tilde{c} \) stand for the set of integers \( n_i \) and constants \( c_i \) respectively, i.e. \( \tilde{n} = (n_1, n_2, \ldots, n_N) \), and \( \tilde{c} = (c_1, c_2, \ldots, c_N) \).

Therefore the second term in (4.17) becomes
\[ \varepsilon_2 \sum_{i=1}^{N} k_i Q^{(i)} = 8 \pi M^2 r_0^2 \left[ \sum_{i=1}^{N} k_i^2 | c_i |^2 \mathcal{I}_{n_i,2 \tilde{n}, \tilde{c}} + \sum_{i<j}^{N} (k_i - k_j)^2 | c_i |^2 | c_j |^2 \mathcal{I}_{n_i,n_j,2 \tilde{n}, \tilde{c}} \right] \\
- 64 \pi \beta \gamma \left[ \sum_{i,j=1}^{N} (k_i n_j - k_j n_i)^2 | c_i |^2 | c_j |^2 \mathcal{I}_{n_i+n_j-1,2 \tilde{n}, \tilde{c}} \right] \] (4.20)
\[ - 2 \sum_{i,j,k=1}^{N} (k_i n_j - k_j n_i) (k_i n_k - k_k n_i) | c_i |^2 | c_j |^2 | c_k |^2 \mathcal{I}_{n_i,n_j+n_k-1,3 \tilde{n}, \tilde{c}} \]
Notice that in the double sums in (4.20), whenever the two indices are equal the corresponding coefficients vanish. In the last term, involving a triple sum, the coefficients vanish whenever the indices in two pairs are equal (namely \((i, j)\) and \((i, k)\)) but not when the indices in the third pair are equal (namely \((j, k)\)). Therefore, following the analysis of the appendix A we conclude that the contribution to the energy per unity of length, given by (4.20), is finite if

\[ 2 n_{\text{max}} > 1 + n_i \quad \text{and} \quad 2 n_{\text{max}} > 1 + n_i + n_j \quad (i \neq j) \quad \text{when} \quad \beta = \gamma \quad (4.21) \]

where \(n_{\text{max}}\) is the highest positive integer in the set \(\vec{n} = (n_1, \ldots n_N)\), such that the corresponding constant \(c_i\) is non-vanishing. Now, if \(\beta \neq \gamma\) we need in addition the following conditions

\[ 3 n_{\text{max}} > n_i + n_j + n_k ; \quad 2 |n_{\text{min}}| > -n_i - n_j ; \quad 3 |n_{\text{min}}| > -n_i - n_j - n_k \quad (4.22) \]

with \(i \neq j\) and \(i \neq k\), and where \(n_{\text{min}}\) is the lowest negative integer in the set \(\vec{n} = (n_1, \ldots n_N)\), such that the corresponding constant \(c_i\) is non-vanishing.

Let us make some comments about the structure of the energy per unit length of the vortices as given by (4.17) and (4.20). Consider the case where all the integers \(n_i\) and wave vectors \(k_i\) are equal, i.e. \(n_i \equiv n\) and \(k_i \equiv k\), for \(i = 1, 2, \ldots N\). Then (4.16) becomes

\[ \vec{u} = \vec{c} \left( \frac{z}{r_0} \right)^n e^{i k y} \quad (4.23) \]

Therefore we have that \(n_{\text{max}} = n\), if \(n > 0\), or \(n_{\text{min}} = n\) if \(n < 0\). In addition, all the terms in (4.20) vanish except for the first one. We have in fact a \(CP^1\) vortex pointing in a given fixed direction in \(CP^N\), and the energy density (4.17) reduces to the case \(CP^1\) discussed in [9]. Indeed, we have that (4.17) becomes

\[ \mathcal{E}_{\text{vortex/wave}} = 8\pi M^2 \left[ \left| n \right| + \left| \vec{c} \right|^2 |n| \Gamma \left( \frac{|n| + 1}{|n|} \right) \right] \quad (4.24) \]

where \(|\vec{c}|^2 = \sum_{i=1}^{N} |c_i|^2\), and where we have rescaled \(\zeta \rightarrow |\zeta| / |\vec{c}|^{2/n}\), and used the fact that \(\int_0^\infty d\zeta \frac{e^{\zeta^n}}{(1+\zeta^n)} = \int_0^\infty d\zeta \frac{e^{-\zeta^n}}{(1+\zeta^n)} = \frac{1}{|n|} \Gamma \left( \frac{|n|+1}{|n|} \right) \Gamma \left( \frac{|n|-1}{|n|} \right)\). Notice that an equivalent result would have been obtained by setting all the \(c_i\)’s to zero except for one of them. The second term in (4.21) is the energy coming from the coupling of the wave with the vortex. It grows with \(k^2\) which accounts for the kinetic energy of the wave. However, as \(|n| \rightarrow \infty\) that term behaves as \(1 / |n|\), and so the interaction between the wave and vortex decreases as \(|n| \) increases. In addition, we notice that the factor involving \(|\vec{c}|\) depends on the sign of \(n\). Therefore, if \(n > 0\) we see that the energy from the interaction between wave and vortex decreases with the increase of \(|\vec{c}|\), and that behavior reverts if
\( n < 0 \). That result is related to the fact that the energy depends upon the length scale \( r_0 \), and from (4.23) we see that \( |\vec{c}| \) rescales \( r_0 \) differently for different signs of \( n \).

In fact the scaling effects of \( \vec{c} \) on the energy density (4.17) can be inferred by considering the case where \( c_i = \lambda \bar{n}_i e^{i\theta_i} \), with \( \lambda \) real and positive. From (4.20) and (4.19) one observes that \( \lambda \) can be absorbed into \( \zeta \) by the rescaling \( \zeta \rightarrow \lambda^{-2} \zeta \). Then all the factors \( |c_i|^2 \) disappear from (4.20) and (4.19), and everything is rescaled by \( 1/\lambda^2 \) due to the measure of the integrals \( d\zeta \). So one gets that \( \varepsilon^2 \sum_{i=1}^{N} k_i Q^{(i)} \rightarrow \frac{1}{\lambda^2} \varepsilon^2 \sum_{i=1}^{N} k_i Q^{(i)} \). Therefore, the energy density coming from the interaction between wave and vortex (second term in (4.17)), decreases as \( \lambda \) increases. At the same time, as \( \lambda \) increase we have that \( c_i \) increases for \( n_i \) positive and decrease for \( n_i \) negative.

Notice that the term proportional to \( (\beta - \gamma) \) in (4.20) drops if the vectors \( \vec{n} \) and \( \vec{k} \) are proportional, i.e. \( k_i = k n_i \). That implies that we have \( u_i = c_i v^{n_i} \) with \( v = \frac{\bar{n}}{r_0} e^{i k y} \).

4.1 The case \( N = 2 \)

In the case \( N = 2 \) we have that the expression (4.20) becomes

\[
\varepsilon^2 \sum_{i=1}^{2} k_i Q^{(i)} = 8\pi M^2 r_0^2 \left[ k_1^2 |c_1|^2 I_{(n_1,2,\bar{n},\bar{c})} + k_2^2 |c_2|^2 I_{(n_2,2,\bar{n},\bar{c})} \right]
+ (k_1 - k_2)^2 |c_1|^2 |c_2|^2 I_{(n_1+n_2,2,\bar{n},\bar{c})}
- 128\pi (\beta - \gamma) |c_1|^2 |c_2|^2 (k_1 n_2 - k_2 n_1)^2 \left[ I_{(n_1+n_2-1,2,\bar{n},\bar{c})} \right]
- |c_1|^2 I_{(2 n_1+n_2-1,3,\bar{n},\bar{c})} - |c_2|^2 I_{(2 n_2+n_1-1,3,\bar{n},\bar{c})}
\]

(4.25)

We show below the lowest non-divergent energies per unity length, as given in (4.17), for the vortices with waves traveling along them, and taking \( |c_1|^2 = |c_2|^2 = 1 \).
$E_{\text{vortex/wave}}/8\pi M^2$

| $(n_1, n_2)$ | Expression |
|--------------|------------|
| (2, 0)       | $2 + \frac{3}{16} r_0^2 (4k_1^2 - 4k_1 k_2 + 3k_2^2) - \frac{1}{M^2} k_2^2 (\beta - \gamma)$ |
| (3, 0)       | $3 + \frac{3\sqrt{3}}{8} r_0^2 (k_1^2 - k_1 k_2 + k_2^2) - \frac{6}{M^2} k_2^2 (\beta - \gamma)$ |
| (2, −1)      | $3 + r_0^2 (0.83k_1^2 - 0.48k_1 k_2 + 0.44k_2^2) - \frac{5.89}{8M^2} (\beta - \gamma)(k_1 + 2k_2)^2$ |
| (3, 1)       | $3 + r_0^2 (0.83k_1^2 - 1.18k_1 k_2 + 0.79k_2^2) - \frac{5.89}{8M^2} (\beta - \gamma)(k_1 - 3k_2)^2$ |
| (−3, −2)     | $3 + r_0^2 (0.44k_1^2 - 0.41k_1 k_2 + 0.79k_2^2) - \frac{5.89}{8M^2} (\beta - \gamma)(2k_1 - 3k_2)^2$ |
| (4, 0)       | $4 + \frac{1}{32\sqrt{2}} r_0^2 (4k_1^2 - 4k_1 k_2 + 5k_2^2) - \frac{8}{M^2} k_2^2 (\beta - \gamma)$ |
| (2, −2)      | $4 + \frac{2}{3\sqrt{7}} \frac{3\sqrt{3}}{8} r_0^2 (5k_1^2 - 2k_1 k_2 + 2k_2^2) - \frac{16}{M^2} (2\sqrt{3\pi} - 9) (\beta - \gamma)(k_1 + k_2)^2$ |
| (3, −1)      | $4 + r_0^2 (0.42k_1^2 - 0.32k_1 k_2 + 0.35k_2^2) - \frac{4.36}{8M^2} (\beta - \gamma)(k_1 + 3k_2)^2$ |
| (4, 1)       | $4 + r_0^2 (0.42k_1^2 - 0.52k_1 k_2 + 0.46k_2^2) - \frac{4.36}{8M^2} (\beta - \gamma)(k_1 - 4k_2)^2$ |
| (5, 0)       | $5 + \frac{\sqrt{2}}{8\pi} (6.17k_1^2 - 6.17k_1 k_2 + 9.26k_2^2) - \frac{10}{M^2} (\beta - \gamma) k_2^2$ |
| (−1, 4)      | $5 + r_0^2 (0.31k_1^2 - 0.24k_1 k_2 + 0.29k_2^2) - \frac{3.45}{8M^2} (\beta - \gamma)(4k_1 + k_2)^2$ |
| (3, −2)      | $5 + r_0^2 (0.38k_1^2 - 0.23k_1 k_2 + 0.26k_2^2) - \frac{3.56}{8M^2} (\beta - \gamma)(2k_1 + 3k_2)^2$ |
| (2, −3)      | $5 + r_0^2 (0.73k_1^2 - 0.22k_1 k_2 + 0.23k_2^2) - \frac{3.56}{8M^2} (\beta - \gamma)(3k_1 + 2k_2)^2$ |

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\textbf{A \ Analysis of the integrals (4.19)}

Consider the integrals (4.19) and let $n_{\text{max}}$ and $n_{\text{min}}$ be the highest positive and lowest negative integers respectively, in the set $\vec{n} = (n_1, \ldots, n_N)$, such that the corresponding constants $c_i$’s are non-vanishing. We then factor out $\zeta^{|n_{\text{min}}|}$, and rewrite (4.19) as

$$I(a,b,\vec{n},\vec{c}) \equiv \int_0^\infty d\zeta \; \zeta^{a+b|n_{\text{min}}|} \left[ \zeta^{|n_{\text{min}}|} + \sum_{k=1}^N \left| c_k \right|^2 \zeta^{n_k+n_{\text{min}}} \right]^b$$  \hspace{1cm} (A.1)

Then all the powers of $\zeta$ in the denominator are non-negative. We are interested in cases where $b$ equals 2 or 3, which are the ones appearing in (4.20). Therefore, for $\zeta \to \infty$, we have that the integrand in (A.1) behaves as $\zeta^{a-bn_{\text{max}}}$, and so in order for the integral to converge we need

$$b n_{\text{max}} - a > 1 \hspace{1cm} (A.2)$$

On the other hand, for $\zeta \to 0$, the integrand in (A.1) behaves as $\zeta^{a+b|n_{\text{min}}|}$. Therefore, in order for the integral to converge we need

$$a + b \left| n_{\text{min}} \right| > -1 \hspace{1cm} (A.3)$$

Let us now consider the integrals appearing in the expression (4.20). According to the above analysis the integrals $I(n_i,2,\vec{n},\vec{c})$ converges if $2n_{\text{max}} > 1 + n_i$ and if $2 \left| n_{\text{min}} \right| > -1 - n_i$. Notice that the second condition is always satisfied since, if $n_{\text{min}} = 0$ we have that all $n_i$’s are non-negative, and if $n_{\text{min}} \neq 0$ then the worst situation happens when $n_i = - \left| n_{\text{min}} \right|$, and the inequality is still satisfied. Therefore

$$I(n_i,2,\vec{n},\vec{c}) \hspace{1cm} \text{converges if} \hspace{1cm} 2 n_{\text{max}} > 1 + n_i \hspace{1cm} (A.4)$$

Now, the integrals $I(n_i+n_j,2,\vec{n},\vec{c})$ converges if $2n_{\text{max}} > 1 + n_i + n_j$ and if $2 \left| n_{\text{min}} \right| > -1 - n_i - n_j$. Again the second inequality is always satisfied since, if $n_{\text{min}} = 0$ we have that all $n_i$’s are non-negative, and if $n_{\text{min}} \neq 0$ then the worst situation happens when $n_i = n_j = - \left| n_{\text{min}} \right|$, and the condition is still satisfied. Therefore

$$I(n_i+n_j,2,\vec{n},\vec{c}) \hspace{1cm} \text{converges if} \hspace{1cm} 2 n_{\text{max}} > 1 + n_i + n_j \hspace{1cm} (A.5)$$

According to the analysis above we have in addition the following results

$$I(n_i+n_j-1,2,\vec{n},\vec{c}) \hspace{1cm} \text{converges if} \hspace{1cm} 2 n_{\text{max}} > n_i + n_j \hspace{1cm} \text{and} \hspace{1cm} 2 \left| n_{\text{min}} \right| > -n_i - n_j \hspace{1cm} (A.6)$$

and

$$I(n_i+n_j+n_k-1,3,\vec{n},\vec{c}) \hspace{1cm} \text{converges if} \hspace{1cm} 3n_{\text{max}} > n_i + n_j + n_k \hspace{1cm} \text{and} \hspace{1cm} 3 \left| n_{\text{min}} \right| > -n_i - n_j - n_k \hspace{1cm} (A.7)$$
References

[1] S. R. Coleman, “Quantum sine-Gordon equation as the massive Thirring model,” Phys. Rev. D 11, 2088 (1975).
S. Mandelstam, “Soliton operators for the quantized sine-Gordon equation,” Phys. Rev. D 11, 3026 (1975).

[2] C. Montonen and D. I. Olive, “Magnetic Monopoles As Gauge Particles?,” Phys. Lett. B 72, 117 (1977).
C. Vafa and E. Witten, “A Strong coupling test of S duality,” Nucl. Phys. B 431, 3 (1994) [arXiv:hep-th/9408074].
N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,” Nucl. Phys. B 426, 19 (1994) [Erratum-ibid. B 430, 485 (1994)] [arXiv:hep-th/9407087].

[3] P. D. Lax, “Integrals Of Nonlinear Equations Of Evolution And Solitary Waves,” Commun. Pure Appl. Math. 21, 467 (1968). V.E. Zakharov and A.B. Shabat, Zh. Exp. Teor. Fiz. 61 (1971) 118-134; english transl. Soviet Phys. JETP 34 (1972) 62-69.

[4] O. Alvarez, L.A. Ferreira, J. Sanchez Guillen, “A new approach to integrable theories in any dimension”, Nucl. Phys. B529 (1998) 689-736, [arXiv:hep-th/9710147]

[5] O. Alvarez, L. A. Ferreira and J. Sanchez-Guillen, “Integrable theories and loop spaces: fundamentals, applications and new developments,” Int. J. Mod. Phys. A 24, 1825 (2009) [arXiv:0901.1654 [hep-th]].

[6] W.J. Zakrzewski, Low Dimensional Sigma Models (Hilger, Bristol, 1989).

[7] A. M. Din and W. J. Zakrzewski, Nucl. Phys. B 174, 397 (1980).

[8] A. M. Grundland and W. J. Zakrzewski, “On CP^1 and CP^2 maps and Weierstrass representations for surface immersed into multi-dimensional Euclidean spaces”; Journal of Nonlinear Mathematical Physics Bf 10, Number 1, (2003), 110-135.

[9] L. A. Ferreira, “Exact vortex solutions in an extended Skyrme-Faddeev model,” Journal of High Energy Physics JHEP05(2009)001, [arXiv:0809.4303 [hep-th]].

[10] L. D. Faddeev, “Quantization of solitons”, Princeton preprint IAS Print-75-QS70 (1975).
L. D. Faddeev, in it 40 Years in Mathematical Physics, (World Scientific, 1995).
L. D. Faddeev and A. J. Niemi, “Knots and particles,” Nature \textbf{387}, 58 (1997) \texttt{arXiv:hep-th/9610193}.

P. Sutcliffe, “Knots in the Skyrme-Faddeev model,” Proc. Roy. Soc. Lond. A \textbf{463}, 3001 (2007) \texttt{arXiv:0705.1468 [hep-th]}.

J. Hietarinta and P. Salo, “Faddeev-Hopf knots: Dynamics of linked un-knots,” Phys. Lett. B \textbf{451}, 60 (1999) \texttt{arXiv:hep-th/9811053}.

J. Hietarinta and P. Salo, “Ground state in the Faddeev-Skyrme model,” Phys. Rev. D \textbf{62}, 081701 (2000).

[11] J. Hietarinta, J. Jaykka and P. Salo, “Dynamics of vortices and knots in Faddeev’s model”, in \emph{Workshop on Integrable Theories, Solitons and Duality} (2002), Proceedings of Science PoS(unesp2002)017, \url{http://pos.sissa.it/cgi-bin/reader/conf.cgi?confid=8}.

J. Hietarinta, J. Jaykka and P. Salo, “Relaxation of twisted vortices in the Faddeev-Skyrme model,” Phys. Lett. A \textbf{321} (2004) 324 \texttt{arXiv:cond-mat/0309499}.

J. Jaykka and J. Hietarinta, “Unwinding in Hopfion vortex bunches,” \texttt{arXiv:0904.1305 [hep-th]}.

[12] M. Hirayama, C. G. Shi and J. Yamashita, “Elliptic solutions of the Skyrme model,” Phys. Rev. D \textbf{67}, 105009 (2003) \texttt{arXiv:hep-th/0303092};

M. Hirayama and C. G. Shi, “A class of exact solutions of the Faddeev model,” Phys. Rev. D \textbf{69}, 045001 (2004) \texttt{arXiv:hep-th/0310042},

C. G. Shi and M. Hirayama, “Approximate vortex solution of Faddeev model,” Int. J. Mod. Phys. A \textbf{23}, 1361 (2008) \texttt{arXiv:0712.4330 [hep-th]}.

[13] M. Eto, Y. Isozumi, M. Nitta and K. Ohashi, “1/2, 1/4 and 1/8 BPS equations in SUSY Yang-Mills-Higgs systems: Field theoretical brane configurations,” Nucl. Phys. B \textbf{752}, 140 (2006) \texttt{arXiv:hep-th/0506257}. M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Solitons in the Higgs phase: The moduli matrix approach,” J. Phys. A \textbf{39}, R315 (2006) \texttt{arXiv:hep-th/0602170}.

[14] L. D. Faddeev and A. J. Niemi, “Partial duality in SU(N) Yang-Mills theory,” Phys. Lett. B \textbf{449}, 214 (1999) \texttt{arXiv:hep-th/9812090}.

L. D. Faddeev and A. J. Niemi, “Decomposing the Yang-Mills field,” Phys. Lett. B \textbf{464}, 90 (1999) \texttt{arXiv:hep-th/9907180}.

[15] K. I. Kondo, T. Shinohara and T. Murakami, “Reformulating SU(N) Yang-Mills theory based on change of variables,” Prog. Theor. Phys. \textbf{120}, 1 (2008) \texttt{arXiv:0803.0176 [hep-th]}.
[16] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, New York: Academic Press 1978.

[17] H. Eichenherr and M. Forger, “More About Nonlinear Sigma Models On Symmetric Spaces,” Nucl. Phys. B 164, 528 (1980) [Erratum-ibid. B 282, 745 (1987)].

[18] L. A. Ferreira and D. I. Olive, “Noncompact Symmetric Spaces And The Toda Molecule Equations,” Commun. Math. Phys. 99, 365 (1985).

[19] L. A. Ferreira and E. E. Leite, “Integrable theories in any dimension and homogeneous spaces,” Nucl. Phys. B 547, 471 (1999) [arXiv:hep-th/9810067].

[20] C. Adam, J. Sanchez-Guillen and A. Wereszczynski, “New integrable sectors in Skyrme and 4-dimensional CP(n) model,” J. Phys. A 40, 1907 (2007) [arXiv:hep-th/0610024].

[21] A. D’Adda, M. Luscher and P. Di Vecchia, “A 1/N Expandable Series Of Nonlinear Sigma Models With Instantons,” Nucl. Phys. B 146, 63 (1978).

[22] N. S. Manton and P. Sutcliffe, “Topological solitons,” *Cambridge, UK: Univ. Pr. (2004) 493 p*

[23] P. Goddard and D. I. Olive, “New Developments In The Theory Of Magnetic Monopoles,” Rept. Prog. Phys. 41, 1357 (1978).

[24] N. D. Mermin, “The topological theory of defects in ordered media,” Rev. Mod. Phys. 51, 591 (1979).