A braided interpretation of fractional supersymmetry in higher dimensions

R.S. Dunne

Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW

Abstract

A many variable $q$-calculus is introduced using the formalism of braided covector algebras. Its properties when certain of its deformation parameters are roots of unity are discussed in detail, and related to fractional supersymmetry. The special cases of two dimensional supersymmetry and fractional supersymmetry are developed in detail.

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1 Introduction

In four recent papers \[1, 2, 3, 4\] the properties of the braided line \[5, 6\] when its deformation parameter is a root of unity were discussed. Most notably these studies led to a novel understanding of one dimensional supersymmetry and fractional supersymmetry \[7, 8, 9\]. Our aim in the present paper is to extend these results to the many variable case. In order to do this we construct a many variable, generic \( q \), generalised Grassmann algebra using the formalism of braided covector algebras \[3, 9\] with suitable \( R \) and \( R' \) matrices. Within the framework provided by this formalism the construction of the corresponding many variable left \( q \)-calculus is straightforward. After a little further work the corresponding right \( q \)-calculus is also obtained.

In this many variable case it is convenient to generalise the graded brackets used in \[4\] to a pair of braided brackets (left and right), which we introduce in section 2. This change has several, in general useful, consequences. In particular, left and right differentiation and integration become truly distinct, rather than being the same thing induced by different algebraic operators as was the case in \[2\]. There are well defined and simple commutation relations between all of these operations. Another advantage over the approach of \[2\] is that the conditions which govern the commutation relations of non-commuting constants are built into the many variable algebra, so that they are no longer additional constraints. In contrast to the situation with graded brackets, the conditions necessary in order that left and right differentiation be induced are compatible. A consequence of this is that we can now work quite generally with both left and right differentiation/integration, rather than being restricted to one or the other, as was the case for graded bracket based \( q \)-calculus.

In section 3 we take the \( q_a \to \bar{q}_a \) limit (\( \bar{q}_a \) a root of unity) of the many variable \( q \)-calculus, and obtain many variable analogues of the structures and decompositions seen in \[2, 3, 4\]. A full set of commutation relations between the different algebraic elements, derivatives and integrals is also given. At the end of this section the braided Hopf structure of both the coordinates and the derivatives is given, as well as the duality between them. Further details of this duality, as well as an alternative discussion of the braided line Hopf algebra (at generic \( q \) and at \( q \) a root of unity) are give in the appendix.

Section 4 deals with the case of two dimensional supersymmetry. This plays an important role in superstring theory \[10\], in which it corresponds to supersymmetry on the world sheet of the string. The full two dimensional supersymmetry algebra and transformations are recovered, and all of the transformation properties of the bosonic spacetime variables \( x \) and \( t \) emerge as consequences of their definition as different combinations of the \( q_a \to -1 \) limits of two braided line coordinates \( \{ \theta_a \} \) (\( a=1,2 \)). Together these two braided lines make up a braided plane, but we note that this is not the
braided/quantum plane which is usually encountered in the literature \[1, 2\]. In the limit, translations within this braided plane induce supersymmetry transformations of \(x\) and \(t\). Furthermore, once the Lorentz transformations of \(\theta_1\) and \(\theta_2\) are specified, those of \(x\) and \(t\) follow automatically. The results are in agreement with the standard version of the two dimensional super-Poincaré transformation.

Section 5 extends the results of section 4 to the case of mixed fractional supersymmetry in two dimensions. The word mixed is used to indicate that the deformation parameters of the two braided plane coordinates on which the fractional supersymmetry is based are not necessarily at the same root of unity. All of the algebraic and transformation properties are worked out, and as in the supersymmetric case, spacetime Lorentz transformations are induced by suitable transformations of the braided plane coordinates. We are thus able to introduce full two dimensional mixed fractional super-Poincaré transformations. Finally we extend the arguments of \[2\] concerning the Berezin integral to the two dimensional case.
2 \( q \)-calculus for an arbitrary number of variables

In this section we develop the \( q \)-calculus associated with \( r \) generalised Grassmannian variables. This calculus can be viewed as a particular example of the braided differential calculus described in [5, 6] and we present it from this point of view.

Given any matrix \( R_{12} \in M_r \otimes M_r \) which satisfies the quantum Yang-Baxter equation,

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} ,
\]

as well as an associated matrix \( R'_{12} \in M_r \otimes M_r \) satisfying

\[
R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12} , \quad R_{23}R_{13}R'_{12} = R'_{12}R_{13}R_{23} , \quad R_{21}R'_{12} = R'_{12}R_{21} , \quad (PR + 1)(PR' - 1) = 0 ,
\]

where \( P \) is the permutation matrix, we can define a braided covector algebra [5, 6] with elements \( \{x_i, 1\} \). This has braided Hopf algebraic structure given by

\[
x_1x_2 = x_2x_1 R'_{12} , \quad i.e. \quad x_i x_j = x_c x_b R'^{bc}_{ij} , \quad \Delta x_i = x_i \otimes 1 + 1 \otimes x_i , \quad \varepsilon(x_i) = 0 , \quad S(x_i) = -x_i , \quad \Psi_{12}(x_1 \otimes x_2) = x_2 \otimes x_1 R_{12} ,
\]

as well as \( \Delta(1) = 1 \otimes 1, \varepsilon(1) = 1, S(1) = 1, \Psi_{12}(1 \otimes x_i) = x_i \otimes 1, \Psi_{12}(x_i \otimes 1) = 1 \otimes x_i \). It is convenient to use the notation \( w = x \otimes 1, x = 1 \otimes x \) so that the \( \{w_i\} \) satisfy the same algebra as the \( \{x_i\} \). In this notation [5] the coproduct \( \Delta \) above is just

\[
\Delta x = w + x ,
\]

and the braiding \( \Psi_{12} \) is equivalent to the following braid statistics between \( x \) and \( w \)

\[
x_1w_2 = w_2x_1 R_{12} , \quad i.e. \quad x_i w_j = w_c x_b R^{bc}_{ij} .
\]

The notation (6) suggests that we regard the coproduct as a generating left translations within the braided covector space, and motivates its alternative name \textit{coaddition} [4, 6]. No additional
information is needed to construct the corresponding braided left differential calculus. The left
derivatives form a braided vector algebra, with commutation relations give by
\[
\partial_{12} = R'_{12} \partial_{21} \partial_{11} \ ,
\]
and the cross relations giving their action on the covectors are
\[
\partial_{1} x_{2} - x_{2} R_{21} \partial_{1} = \delta_{12} \ .
\]

The reason for the form of the second of these relations is that along with (7) it implies that
\([w_{1} \partial_{11}, x_{2}] = \delta_{12} w_{1},\) so that \(w_{1} \partial_{11}\) can be viewed as the generator of the left translation (6). Equation (9) can be viewed as giving the braiding between the covectors and their derivatives [11]. To see this we identify
\[
\Psi^{-1}_{12}(\partial_{11} \otimes x_{2}) = x_{2} \otimes R_{21} \partial_{11} \ ,
\]
so that we can rewrite (9) as
\[
\partial_{1} x_{2} - \Psi^{-1}_{12}(\partial_{11} \otimes x_{2}) = \delta_{12} \ .
\]

It is not difficult to extend this formalism so that it includes right derivatives, and since these play
an important role in supersymmetric and fractional supersymmetric theories, we will do so explicitly.
Among themselves the right derivatives have the same commutation relations as the left derivatives,
and thus also form a braided vector algebra so that
\[
\partial_{r1} \partial_{r2} = R'_{12} \partial_{r2} \partial_{r1} \ .
\]

To find the cross relations between these derivatives and the covectors we must first reinterpret \(\Delta\)
as the generator of a right shifts. We can do this by writing
\[
\Delta x = x + y \ ,
\]
where we have introduced the alternative notation \(x = x \otimes 1\) and \(y = 1 \otimes x\). From the braiding \(\Psi_{12}\)
given by (3) we obtain the braid statistics
\[
y_{1} x_{2} = x_{2} y_{1} R_{12} \ .
\]

In order for the right derivatives to generate the translation (13) they must satisfy \([y_{1} \partial_{r1}, x_{2}] = \delta_{12}.\) In combination with the braid statistics (14) this implies that
\[
y_{1} \partial_{r1} x_{2} - y_{1} x_{2} R_{12}^{-1} \partial_{r1} = \delta_{12} y_{1} \ ,
\]
from which it is clear that suitable cross relations are

\[ \partial_r x_2 - x_2 R^{-1}_{12} \partial_r = \delta_{12}. \]  

(16)

As in the case of left derivatives we can interpret this as giving the braiding between the right
derivatives and the covectors. Thus by identifying

\[ \Psi_{21}(\partial_r \otimes x_2) = x_2 \otimes R^{-1}_{12} \partial_r, \]  

(17)

we can rewrite (16) as

\[ \partial_r x_2 - \cdot \Psi_{21}(\partial_r \otimes x_2) = \delta_{12}. \]  

(18)

Relationships (11) and (18) motivate the introduction of bilinear left and right braided brackets,

\[ [A, B]_L := AB - \cdot \Psi^{-1}_{12}(A \otimes B), \]
\[ [A, B]_R := AB - \cdot \Psi_{21}(A \otimes B). \]  

(19)

The bilinearity follows from the bilinearity of \( \Psi \). This bracket is well defined on products as long as
we remember the expansion rule for the braiding. From [12] this is

\[ \Psi(AB \otimes C) = (1 \otimes \cdot)(\Psi \otimes 1)(A \otimes \Psi(B \otimes C)) \]
\[ \Psi(A \otimes BC) = (\cdot \otimes 1)(1 \otimes \Psi)(\Psi(A \otimes B) \otimes C). \]  

(20)

Note that (as one would expect) \( \Psi^{-1}_{12} \) and \( \Psi_{21} \) expand in the same way. Using these brackets we can
define left and right differentiation as follows,

\[ \left( \frac{d}{dx_1} \right)_L f := [\partial_l, f]_L, \]
\[ \left( \frac{d}{dx_1} \right)_R f := [\partial_r, f]_R. \]  

(21)

Here \( f = f\{x_i\} \). We will provide a specific example shortly. We are now able to introduce the
generalised Grassmann algebra, which we define as the braided covector algebra in which \( R \) and \( R' \)
are the following, \( r \) dimensional matrices,

\[ R'_{12} = W_{12}, \quad R_{12} = W_{12} + (Q_1 - 1) \delta_{12} = T_{12}, \]  

(22)

the coordinate form of which is:

\[ R'_{ab} = \omega_{ab} \delta^i_a \delta^j_b, \quad R_{ab} = (\omega_{ab} + (q_a - 1) \delta_{ab}) \delta^i_a \delta^j_b = t_{ab} \delta^i_a \delta^j_b. \]  

(23)
Here $\omega_{ba} = \omega_{ab}^{-1}$ so that $\omega_{aa} = 1$, $\omega_{ab} \neq 0$, and $q_a \neq 0$. It follows directly from the fact that $R$ and $R'$ are diagonal that (1)-(3) are satisfied. To show that (4) is also satisfied we expand it explicitly

$$(PR + 1)(PR' - 1) = R_{21}R'_{12} + R'_{21} + R_{21} - 1,$$

$$= (W_{21} + (Q_2 - 1)\delta_{21})W_{12} + W_{21} - W_{21} - (Q_2 - 1)\delta_{21} - 1,$$

$$= 1 + (Q_2 - 1)\delta_{21} - (Q_2 - 1)\delta_{21} - 1,$$

$$= 0.$$  \hfill (24)

Putting this $R'$ into (3) we obtain the defining algebra of $r$ generalised Grassmann variables.

$$\theta_a\theta_b = \theta_j\theta_i\omega_{ab}\delta_a^i\delta_b^j,$$  \hfill (25)

which is equivalent to

$$[\theta_a, \theta_b]_{\omega_{ab}} = 0.$$  \hfill (26)

For left shifts we use the notation $\theta_a = 1 \otimes \theta_a$ and $\epsilon_a = \theta_a \otimes 1$. From (4) we obtain

$$\theta_a\epsilon_b - \epsilon_j\delta_i^a\delta_i^b\delta_{ij} = 0,$$  \hfill (27)

which is equivalent to

$$[\epsilon_a, \theta_b]_{\epsilon_{ab}} = 0.$$  \hfill (28)

For the corresponding derivatives $D_{La}$ we obtain from (8) and (9) the following commutation and cross relations.

$$[D_{La}, D_{Lb}]_{\omega_{ab}} = 0,$$  \hfill (29)

$$[D_{La}, \theta_b]_{t_{ba}} = \delta_{ab}.$$  \hfill (29)

For right shifts we use the notation $\theta_a = \theta_a \otimes 1$ and $\eta_a = 1 \otimes \theta_a$. Then from (14) we obtain

$$[\eta_a, \theta_b]_{t_{ab}} = 0,$$  \hfill (30)

while (12) and (16) give us

$$[D_{Ra}, D_{Rb}]_{\omega_{ab}} = 0,$$  \hfill (31)

$$[D_{Ra}, \theta_b]_{t_{ab}} = \delta_{ab}.$$  \hfill (31)

These derivatives are generated by the left and right braided brackets, thus from (13),

$$\left( \frac{d}{d\theta_a} \right)_L \theta_b = [D_{La}, \theta_b]_L = [D_{La}, \theta_b]_{t_{ba}} = \delta_{ab},$$  \hfill (32)

$$\left( \frac{d}{d\theta_a} \right)_R \theta_b = [D_{Ra}, \theta_b]_R = [D_{Ra}, \theta_b]_{t_{ab}} = \delta_{ab}.$$
As an example of differentiation induced by the braided brackets (19) we consider the case of $r = 2$, and functions $f(\theta_1, \theta_2)$ which can be expanded as positive power series of the form

$$f(\theta_1, \theta_2) = \sum_{l,m=0}^{\infty} C_{l,m} \theta_1^l \theta_2^m.$$  \hfill (33)

Then using definitions (21) we find that

$$\left(\frac{d}{d\theta_1}\right)_L f(\theta_1, \theta_2) = [D_{L1}, f(\theta_1, \theta_2)]_L = \left[\sum_{l,m=0}^{\infty} C_{l,m} \theta_1^l \theta_2^m\right]_L \quad \Rightarrow \quad \sum_{l,m=0}^{\infty} C_{l,m} [D_{L1}, \theta_1^l \theta_2^m]_L \quad \Rightarrow \quad \sum_{l,m=0}^{\infty} [l + 1]_{q_1} C_{l+1,m} \theta_1^l \theta_2^m.$$  \hfill (34)

Similarly, for right differentiation we find

$$\left(\frac{d}{d\theta_1}\right)_R f(\theta_1, \theta_2) = [D_{R1}, f(\theta_1, \theta_2)]_R = \left[\sum_{l,m=0}^{\infty} C_{l,m} \theta_1^l \theta_2^m\right]_R \quad \Rightarrow \quad \sum_{l,m=0}^{\infty} C_{l,m} [D_{R1}, \theta_1^l \theta_2^m]_R \quad \Rightarrow \quad \sum_{l,m=0}^{\infty} [l + 1]_{q_1} C_{l+1,m} \theta_1^l \theta_2^m.$$  \hfill (35)

More generally the $C_{l,m}$ can be functions of $\theta_j$ where $j \neq 1, 2$. This does not affect the result of the above differentiation, but of course the explicit form given in the third line of (34) is no longer valid. In fact, since for $a \neq b$, $t_{ab} = t_{ba} = \omega_{ab}$, we have for $C = C(\theta_j)$ with $j \neq i$

$$[D_{Li}, C]_L = [D_{Li}, C]_{q_{ci}} = 0 \quad ,$$  \hfill (36)

$$[D_{Ri}, C]_R = [D_{Ri}, C]_{q_{ci}} = 0 \quad ,$$

which are the braided bracket analogues of (3.10) and (3.11). Note that unlike in the graded bracket case considered in [2] these conditions are not additional constraints on $C$, but instead follow directly from our definition of the many variable $q$-calculus. Note also that the conditions for induced left and right differentiation are compatible, so that in the many variable case, working with braided brackets, it is not necessary to choose between these. Another difference between the graded bracket induced derivatives of [2] and the braided bracket induced derivatives of the present paper is that in the latter case $D_{Ra}$ appears on the left of the braided bracket. One consequence of this is that here $D_{Ra}$ has a different normalisation. In the many variable case, and with the new normalisation, the number and shift operators are as follows,

$$N_a = \sum_{m=0}^{\infty} \frac{(1 - (q_a)^{m-1})}{[m]_{q_a}^0} \theta_a^m D_{La}^m = \sum_{m=0}^{\infty} \frac{1 - (q_a)^{1-m}}{[m]_{q_a}^{-1}} \theta_a^m D_{Ra}^m \quad ,$$  \hfill (37)
\[ q_a^{k N_a} = \sum_{m=0}^{\infty} \frac{1}{[m]_{q_a}} \left( \prod_{l=1}^{m-1} (q_a^k - q_a^{-l}) \right) \theta_a^m D_L^a , \]
\[ q_a^{-k N_a} = \sum_{m=0}^{\infty} \frac{1}{[m]_{q_a^{-1}}} \left( \prod_{l=1}^{m-1} (q_a^{-k} - q_a^{l-1}) \right) \theta_a^m D_R^a , \]
\[ G_L^a = \exp_{q_a^{-1}}(\epsilon_a D_L^a) , \quad G_R^a = \exp_{q_a}(\eta_a D_R^a) . \]

These satisfy
\[ [N_a, \theta_b] = \delta_{ab} \theta_a , \quad [N_a, D_Lb] = -\delta_{ab} D_Lb , \quad [N_a, D_Rb] = -\delta_{ab} D_Rb , \]
\[ G_La^b \theta_b G_L^{-1} = \delta_{ab} \epsilon_a + \theta_b , \quad G_Ra^b \theta_b G_R^{-1} = \theta_b + \delta_{ab} \eta_a . \]

Using the identity \( D_La \theta_a - \theta_a D_La = q_a^{N_a} \) which follows from (38), it is clear that with the braided bracket normalisation the relationship between the left and right algebraic derivatives is
\[ D_Ra = q_a^{-N_a} D_La . \]

It follows immediately from this and (29) or (31) that
\[ [D_Ra, D_Lb]_{\omega_{ab}} = 0 . \]

Another consequence of this change of normalisation is that the \( Q_a \) and \( D_a \) are related by
\[ Q_a = D_La , \quad D_a = D_Ra . \]

Left and right integrals [13] can also be introduced. As in the one dimensional case these are defined so as to invert the effect of the corresponding derivatives. Another important advantage of the switch to braided brackets is that the left and right integrals are truly distinct, and that there are simple and well defined commutation relations amongst these as well as between them and the derivatives. Specifically, the left integrals are defined by
\[ \int (d\theta_a)_L \theta_a^m = \frac{\theta_a^{m+1}}{[m+1]_{q_a}} , \]
and the right integrals by
\[ \int (d\theta_a)_R \theta_a^m = \frac{\theta_a^{m+1}}{[m+1]_{q_a^{-1}}} . \]

To integrate functions of many variables we also need the cross relations
\[ \left[ \int (d\theta_a)_L, \theta_b \right]_{\omega_{ab}} = \left[ \int (d\theta_a)_R, \theta_b \right]_{\omega_{ab}} = 0 . \]
which hold for $a \neq b$. It is also straightforward to show, for example by comparing

$$
\left( \frac{d}{d\theta_a} \right)_L \int (d\theta_a)_R \theta_a^m = \frac{[m+1]_q^a}{[m+1]_q^{-1}^a} \theta_a^m , \tag{48}
$$

with

$$
\int (d\theta_a)_R \left( \frac{d}{d\theta_a} \right)_L \theta_a^m = \frac{[m]_q^a}{[m]_q^{-1}^a} \theta_a^m , \tag{49}
$$

that the commutation relations between differentiation and integration are as follows,

$$
\left[ \frac{d}{d\theta_a} \right]_L , \int (d\theta_b)_L \omega_{ba} = 0 , \quad \left[ \frac{d}{d\theta_a} \right]_R , \int (d\theta_b)_R \omega_{ba} = 0 , \tag{50}
$$

By similar methods we also find

$$
\left[ \int (d\theta_a)_L , \int (d\theta_b)_L \omega_{ab} \right] = 0 , \quad \left[ \int (d\theta_a)_R , \int (d\theta_b)_R \omega_{ab} \right] = 0 , \quad \left[ \int (d\theta_a)_R , \int (d\theta_b)_L t_{ab} \right] = 0 , \tag{51}
$$

which are the integral analogues of (29), (31) and (43).

3 Generalised Grassmann calculus at $q^a$ a root of unity

One of the central results of [2], was that if $\tilde{q}^a$ is a primitive $n^a$th root of unity, and $z^a$, $\partial z^a$ are defined by

$$
z^a = \lim_{q^a \to \tilde{q}^a} \frac{(\theta_b)^{n^a}}{[n^a]_q^a} , \quad \partial z^a = D_{L a}^{n^a} = -(-1)^{n^a} D_{R a}^{n^a} , \tag{52}
$$

in which (12) has been used, and it is assumed that $(\theta^a)^{n^a} \to 0$ as $q^a \to \tilde{q}^a$ in such a way that $z^a$ is well defined in this limit, then

$$
[\partial z^a , z^a] = 1 . \tag{53}
$$

Using these definitions and the results of section 4.2 it is easy to establish the full commutation relations in the limit as $q^a \to \tilde{q}^a$ (note that this limit need not be taken for all $a$). When $q^a \to \tilde{q}^a$ and $q^b \to \tilde{q}^b$ we find from (26), (29) and (53) that

$$
[\partial z^a , z^b]_{(t_{ba})^{n^a n^b}} = \delta_{ab} , \quad [z^a , z^b]_{(\omega_{ab})^{n^a n^b}} = 0 , \quad [\partial z^a , \partial z^b]_{(\omega_{ab})^{n^a n^b}} = 0 . \tag{54}
$$

This clearly reduces to ordinary calculus if $(\omega_{ba})^{n^a n^b} = 1$. It will often be sensible to make this choice. It also follows from (29), (31) and (52) that

$$
[D_{L a} , z^b]_{(t_{ba})^{n^a n^b}} = \delta_{ab} \frac{([\theta^b]^{n^a - 1})_{[n^b - 1]_q^b}}{[n^b - 1]_q^b} , \quad [D_{R a} , z^b]_{(t_{ab})^{-n^a n^b}} = (-1)^{n^b} \delta_{ab} \frac{([\theta^b]^{n^a - 1})_{[n^b - 1]_q^{-1}^b}}{[n^b - 1]_q^{-1}^b} . \tag{55}
$$
and that
\[ [D_{La}, \partial_{z_a}]^{(n_a)} = 0, \quad [D_{Ra}, \partial_{z_a}]^{(n_a)} = 0, \quad [\partial_{z_a}, \theta_a]^{(n_b)} = 0 \quad \text{(56)}. \]

Note that (53) and (56) hold even when \( q_a \) is not a root of unity, as long as \( q_b \) is. Following [2] we can, when \( q_a \) is a root of unity, expand the algebraic total derivatives \( D_{La} \) and \( D_{Ra} \) by using the algebraic partial derivatives \( \partial_{\theta_a} \) and \( \delta_{\theta_a} \). These satisfy
\[
(\partial_{\theta_a})^{n_a} = (\delta_{\theta_a})^{n_a} = 0, \quad [\partial_{\theta_a}, \partial_{\theta_b}] = 0, \quad [\partial_{\theta_a}, \delta_{\theta_b}] = 0 \quad \text{(57)}.
\]

as well as
\[
[\partial_{\theta_a}, \delta_{\theta_b}] = \delta_{ab}, \quad [\partial_{\theta_a}, z_b]^{(t_{ab})} = 0, \quad [\delta_{\theta_a}, \partial_{\theta_b}]^{(t_{ab})} = 0, \quad [\delta_{\theta_a}, \delta_{\theta_b}]^{(t_{ab})} = 0 \quad \text{(58)}.
\]

So that if we expand \( D_{La} \) and \( D_{Ra} \) as follows
\[ D_{La} = \partial_{\theta_a} + \frac{\theta_a^{n_a}}{[n_a - 1]_q a^!} \partial_{z_a}, \]
\[ D_{Ra} = \delta_{\theta_a} - (-1)^{n_a} \frac{\theta_a^{n_a}}{[n_a - 1]_q a^!} \partial_{z_a}, \quad \text{(59)} \]
then (54) and (55) are implied by (57)-(58).

If we note the identity
\[
\lim_{q_a \to q \atop r n_a + p \atop q_a} \frac{\theta_a^{r n_a + p}}{[r n_a + p]_q a^!} = \frac{z_a^r \theta^p}{r^! [p]^!} \quad \text{(60)},
\]
then we can take the limit of (43) and (46) to obtain
\[
\int (d\theta_a)_L z_a^r \theta_p^p = (1 - \delta_{p, n - 1})\frac{z_a^r \theta_a^{p+1}}{[p + 1]_q a^!} + \delta_{p, n - 1} [n_a - 1]_q a^! \frac{z_a^{r+1} \theta_a^p}{(r + 1)} \quad \text{(61)}.
\]

In analogy with the introduction of partial derivatives, we introduce the following ‘partial’ integrals,
\[
\int d\theta_a z_a^r \theta_p^p = (1 - \delta_{p, n - 1})\frac{z_a^r \theta_a^{p+1}}{[p + 1]_q a^!} \quad \text{(62)},
\]
\[
\int \delta \theta_a z_a^r \theta_p^p = (1 - \delta_{p, n - 1})\frac{z_a^r \theta_a^{p+1}}{[p + 1]_q a^!} \quad \text{(62)},
\]
\[
\int dz_a z_a^r \theta_p^p = \frac{z_a^{r+1} \theta_a^p}{(r + 1)} \quad \text{(62)}.
\]
Using these and (61) we can write
\[
\int (d\theta_a)_L = \int d\theta_a + \frac{\partial^{n_a-1}}{\partial^{n_a-1} \theta_a} \int dz_a \quad ,
\int (d\theta_a)_R = \int \delta \theta_a - (-1)^{n_a} \frac{\partial^{n_a-1}}{\partial^{n_a-1} \theta_a} \int dz_a \quad ,
\]
which are the integral analogues of (63). We also note the identities
\[
\int (d\theta_a)_L^n = \int dz_a \quad , \quad \int (d\theta_a)_R^n = -(-1)^{n_a} \int dz_a \quad , \quad \int d\theta_a^n = \int \delta \theta_a^n = 0 \quad .
\]
We conclude this section with some comments on the braided Hopf structure of the generalised Grassmann algebra and the dual algebra of derivatives as \( q_a \to \tilde{q}_a \). For an alternative derivation of (52) as well as a derivation of the duality properties in the single variable case see the appendix. The results of this appendix are easily extended to the many variable case, and we give the results below.

For generic \( q_a \) the braided Hopf structure of \( \theta_a \), which follows directly from (5) is as follows
\[
\Delta \theta_a = \theta_a \otimes 1 + 1 \otimes \theta_a \quad ,
\varepsilon(\theta_a) = 0 \quad ,
S(\theta_a^m) = q^{\frac{m(m-1)}{2}}(-\theta_a)^m \quad .
\]

When \( q_a \to \tilde{q}_a \) it follows directly from this and (52) that in addition to (52), which holds as in the generic case, we have the following braided Hopf structure for \( z_a \),
\[
\Delta z_a = z_a \otimes 1 + 1 \otimes z_a + \sum_{m=1}^{n_a-1} \frac{\theta_a^m \otimes \theta_a^{n_a-m}}{[m]_{q_a}! [n_a-m]_{q_a}!} \quad ,
\varepsilon(z_a) = 0 \quad ,
S(z_a) = -z_a \quad .
\]

In the dual Hopf algebra with elements \( D_{L_a} \), the braided Hopf structure is as follows
\[
\Delta D_{L_a} = D_{L_a} \otimes 1 + 1 \otimes D_{L_a} \quad ,
\varepsilon(D_{L_a}) = 0 \quad ,
S(D_{L_a}) = q^{\frac{m(m-1)}{2}}(-D_{L_a})^m \quad .
\]

The duality is given by the inner product
\[
\langle D_{L_a}, \theta_b \rangle = \delta_{ab} \quad ,
\]
which satisfies/is extended to products by all of the usual identities (see appendix), equation (118). When \( q_a \to \tilde{q}_a \) the Hopf structure is extended to include

\[
\Delta \partial_{za} = \partial_{za} \otimes 1 + 1 \otimes \partial_{za} ,
\]

\[
\varepsilon(\partial_{za}) = 0 ,
\]

\[
S(\partial_{za}) = -\partial_{za} .
\]
(69)

In this case the duality is given by.

\[
\langle D_{La}, \theta_b \rangle = \delta_{ab} , \quad \langle \partial_{za}, z_b \rangle = \delta_{ab} , \quad \langle \partial_{za}, \theta_b \rangle = 0 , \quad \langle D_{La}, z_b \rangle = 0 ,
\]
(70)

which follow directly from (52) and (68). Note that we could equally well have worked with \( D_{Ra} \), the only advantage of using \( D_{La} \) being that we avoid the factors of \((-1)^{n_a+1}\) which would arise due to (52).

4 The braided interpretation of two dimensional SUSY

We can use the work in the previous sections to extend our new interpretation of supersymmetry to the two dimensional case. This is of great interest in physics since it is related to world sheet supersymmetry in superstring theory. The most interesting new feature in two dimensions is the presence of Lorentz transformations. We consider a two dimensional generalised Grassmann algebra \( \{\theta_a\}, a = 1, 2 \), and its associated calculus, examining first the case of \( q_1 = q_2 = \omega_{12} = q \). We begin by defining

\[
p_\mu = -\frac{1}{2} D_{La}(\gamma_\mu \gamma_0)_{ab} D_{Lb} ,
\]

\[
x_\mu = \lim_{q \to -1} \frac{i}{2} \frac{1}{q} \theta_a (\gamma_0 \gamma_\mu)_{ab} \theta_b .
\]
(71)

Here \( \mu = 0, 1 \) and \( \gamma_0 = \sigma_2, \gamma_1 = i \sigma_1 \), where \( \sigma_a \) are the usual Pauli matrices, so that we are working in the Majorana-Weyl basis for the Dirac gamma matrices. Note that other than those implied by the RHS, no transformation properties are assigned to \( p_\mu \) and \( x_\mu \). Since \( \gamma_0 \gamma_\mu \) is diagonal, the above can be written as

\[
x_0 = i(z_1 + z_2) , \quad p_0 = -\frac{1}{2} (\partial_{z_1} + \partial_{z_2}) ,
\]

\[
x_1 = i(z_1 - z_2) , \quad p_1 = \frac{1}{2} (\partial_{z_1} - \partial_{z_2}) ,
\]
(72)

from which, using (54) it is clear that

\[
[p_\mu, x_\nu] = -i g_{\mu \nu} , \quad [p_\mu, p_\nu] = 0 , \quad [x_\mu, x_\nu] = 0 .
\]
(73)
Here \( g_{\mu\nu} = \text{diag}\{1, -1\} \) so that \( \{p_\mu\} \) and \( \{x_\mu\} \) behave just like the quantized momenta and coordinates of two dimensional spacetime. To establish their transformation properties, we proceed as follows. Under a translation

\[
\theta_a \rightarrow \epsilon_a + \theta_a ,
\]

we find from (74) that the coordinates \( \{x_\mu\} \) transform as follows,

\[
x_\mu \rightarrow \lim_{q \to -1} i[\frac{1}{2}q] \theta_a (\gamma_0 \gamma_\mu)_{ab} \epsilon_b + \theta_b
\]

\[
= \lim_{q \to -1} \frac{i}{2} \theta_a (\gamma_0 \gamma_\mu)_{ab} \epsilon_b + \theta_b + \lim_{q \to -1} \frac{i}{2} \epsilon_a (\gamma_0 \gamma_\mu)_{ab} \epsilon_b + i \epsilon_a (\gamma_0 \gamma_\mu)_{ab} \theta_b
\]

\[
= x_\mu + x'_\mu + i \epsilon_a (\gamma_0 \gamma_\mu)_{ab} \theta_b
\]

(75)

Together (74) and (75) constitute the usual two dimensional SUSY transformation [10], only now we can see that just as the \( \{x_\mu\} \) are defined by (71) in terms of the \( \{\theta_a\} \), the \( \{x'_\mu\} \) are defined by

\[
x'_\mu \lim_{q \to -1} \frac{i}{2} \epsilon_a (\gamma_0 \gamma_\mu)_{ab} \epsilon_b
\]

(76)

which is the same as (71) but with \( \theta_a \) replaced by \( \epsilon_a \). In the notation of generalised Grassmann calculus, the infinitesimal generators of the translation (74) are \( \epsilon_a D_L a \). On the other hand, in the usual SUSY notation, this transformation is generated by the supercharge \( Q_a \), and thus (as expected) we can make the identification

\[
D_L a = Q_a
\]

(77)

Using this we can write the definition (74) of \( p_\mu \) as

\[
p_\mu = -\frac{1}{2} Q_a (\gamma_\mu \gamma_0)_{ab} Q^a
\]

(78)

which can easily be inverted to yield

\[
\{Q_a, Q_b\} = -2 (\gamma_0 \gamma_\mu)_{ab} p_\mu
\]

(79)

where \( p_\mu = g^{\mu\nu} p_\nu \). Along with \( [p_\mu, p_\nu] = 0 \) from (73), this is just the two dimensional supersymmetry algebra in its usual form. The usual superspace realization of this algebra can be obtained by using (59) and (71). We find

\[
Q_a = D_L a = \partial b a + \theta_a \partial z a
\]

(80)

\[
= \partial b a - (\gamma_0 \gamma_\mu)_{ab} \theta b p_\mu
\]

The covariant derivatives \( D_a \) from two dimensional SUSY also arise naturally in the \( q \to -1 \) limit of two dimensional \( q \)-calculus. To see this, we write down their usual superspace realization,

\[
D_a = \partial b a + (\gamma_0 \gamma_\mu)_{ab} \theta b p_\mu
\]

(81)
Then since, using (38) and (57)
\[
\partial_{\theta a} = (-1)^{N_a} \delta_{\theta a} \\
= (D_La \theta_a - \theta_a D_La) \delta_{\theta a} \\
= (\partial_{\theta a} \theta_a - \theta_a \partial_{\theta a}) \delta_{\theta a} \\
= \delta_{\theta a} ,
\]
we can write this as
\[
D_a = \delta_{\theta a} + (\gamma_0 \gamma_\mu)_{ab} \theta_b \gamma_\mu 
\]
and thus from (59) and (71) we have
\[
D_a = D_{Ra} 
\]
These satisfy
\[
\{D_a, D_b\} = 2(\gamma_0 \gamma_\mu)_{ab} \gamma_\mu
\]
The cross relations \{D_a, Q_b\} = 0 follow directly from (43). Thus the supercharges and covariant derivatives used in two dimensional supersymmetry, correspond respectively to the left and right total derivatives in the \(q_a \rightarrow -1\) limit of two dimensional \(q\)-calculus.

In two dimensional SUSY, the Grassmann variables \(\theta_a\) transform as the components of a Lorentz spinor,
\[
\theta_a \rightarrow S_{ab} \theta_b ,
\]
where
\[
S_{ab} = \begin{pmatrix}
\exp(\phi / 2) & 0 \\
0 & \exp(-\phi / 2)
\end{pmatrix}.
\]
Due to (71), the transformation properties of the coordinates \(\{x_\mu\}\) are entirely determined by those of the \(\theta_a\). To find these explicitly we first note that
\[
\gamma_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
so that
\[
S \gamma_0^2 S^T = \begin{pmatrix} \exp \phi & 0 \\ 0 & \exp(-\phi) \end{pmatrix} = \gamma_0^2 \cosh \phi + \gamma_0 \gamma_1 \sinh \phi ,
\]
\[
S \gamma_0 \gamma_1 S^T = \begin{pmatrix} \exp \phi & 0 \\ 0 & -\exp(-\phi) \end{pmatrix} = \gamma_0^2 \sinh \phi + \gamma_0 \gamma_1 \cosh \phi .
\]
Now from (71) and (86), we find that under a Lorentz transformation the coordinates \( \{x_\mu\} \) behave as follows,

\[
x_\mu \rightarrow \lim_{q \rightarrow -1} \left[ \frac{i}{2} \right] q \theta_b S_{ab} (\gamma_0 \gamma_\mu)_{ab} S_{cd} \theta_d
\]

\[
= \lim_{q \rightarrow -1} \left[ \frac{i}{2} \right] q \theta_a \Lambda_\mu^\nu (\gamma_0 \gamma_\nu)_{ab} \theta_b
\]

\[
= \Lambda_\mu^\nu x_\nu ,
\]

in which from (89) \( \Lambda_\mu^\nu \) has the form

\[
\Lambda_\mu^\nu = \left( \begin{array}{cc} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{array} \right),
\]

so we have shown that, as expected, the coordinates \( \{x_\mu\} \) transform like the components of a covariant Lorentz vector. Note that \( x_\mu := g^{\mu\nu} x_\nu \) also has the expected transformation properties, i.e. the \( \{x^\mu\} \) transform like the components of a contravariant Lorentz vector

\[
x^\mu = (\Lambda_\nu^\mu)^{-1} x_\nu ,
\]

so that the length \( x_\mu x^\mu \) is invariant. Note also that from \( \{D_{La}, \theta_b \} = \delta_{ab} \) and (86) it follows that under a Lorentz transformation \( D_{La} \rightarrow D_{Lb} S_{ba}^{-1} \), and that through definition (71), this leads to the correct transformation properties for the \( \{p_\mu\} \). By combining the translation and Lorentz transformation above, we can consider the effect on the coordinates \( \{x_\mu\} \) of a general super-Poincaré transformation of \( \theta \):

\[
\theta_a \rightarrow \epsilon_a + S_{ab} \theta_b .
\]

Under such a transformation we have, from (71)

\[
x_\mu \rightarrow \lim_{q \rightarrow -1} \left[ \frac{i}{2} \right] q \left( \epsilon_a + S_{ab} \theta_b \right)^2
\]

\[
= \lim_{q \rightarrow -1} \left[ \frac{i}{2} \right] q (\epsilon_a (\gamma_0 \gamma_\mu)_{ab} \epsilon_b + \lim_{q \rightarrow -1} \left[ \frac{i}{2} \right] q \theta_b S_{ab} (\gamma_0 \gamma_\mu)_{ac} S_{cd} \theta_d + i \epsilon_a (\gamma_0 \gamma_\mu)_{ab} S_{bc} \theta_c ) ,
\]

which by (75) and (90) reduces to

\[
x_\mu \rightarrow x_\mu' + \Lambda_\nu^\mu x_\nu + i \epsilon_a (\gamma_0 \gamma_\mu)_{ab} S_{bc} \theta_c ,
\]

which is in exact agreement with the usual super-Poincaré transformation of \( \{x_\mu\} \).

Although it seems reasonable to expect that there is an analogous interpretation of super-Poincaré transformations in higher dimensions, based on (71) or some similar relationship, and it is indeed straightforward to construct higher dimensional algebras with supersymmetric properties using our techniques, the generalisation of our work in this section to \( d > 2 \) is a nontrivial problem, and at present it remains unsolved.
5 Mixed FSUSY in two dimensions

Using (72) it is clear that in terms of \( \{ \theta_a \} \) and \( \{ z_a \} \) the general super-Poincaré transformation (73) of the coordinates \( \{ x_\mu \} \) which follows from (93) takes on the following simple form

\[
\begin{align*}
    z_1 &\rightarrow z_1 \exp \phi + z'_1 + \epsilon_1 \theta_1 \exp(\phi/2) \\
    z_2 &\rightarrow z_2 \exp(-\phi) + z'_2 + \epsilon_2 \theta_2 \exp(-\phi/2)
\end{align*}
\]

(96)

The fact that the pairs \( \{ z_1, \theta_1 \} \) and \( \{ z_2, \theta_2 \} \) are not mixed by this transformation has the consequence that in this basis the generalisation to fractional supersymmetry (FSUSY) is straightforward. To construct the most general two dimensional FSUSY, we consider a two dimensional \( q \)-calculus in the limit as \( q_i \rightarrow \tilde{q}_i \), and choose \( \omega_{12} \) so that \( \omega_{12}^{n_1} = \omega_{12}^{n_2} = 1 \). We have included the \( n_1 \neq n_2 \) case, and for this reason refer to our construction as mixed FSUSY. A suitable definition for \( S_{ab} \) in the Lorentz transformation \( \theta_a \rightarrow S_{ab} \theta_b \) of mixed anyonic spinors, such as \( \theta_a \) is

\[
S_{ab} = \begin{pmatrix} \exp \left( \frac{\phi}{n_1} \right) & 0 \\ 0 & \exp \left( -\frac{\phi}{n_2} \right) \end{pmatrix}.
\]

(97)

As we will see, this ensures that \( \{ x_\mu \} \) transform as the components of a Lorentz vector. Under a mixed anyonic Poincaré transformation

\[
\theta_a \rightarrow \epsilon_a + S_{ab} \theta_b,
\]

(98)

it follows from (72) that \( z_1 \) and \( z_2 \) transform as follows,

\[
\begin{align*}
    z_1 &\rightarrow z_1 \exp \phi + z'_1 + \sum_{m=1}^{n_1-1} \frac{\epsilon_1 \theta_1^{n_1-m}}{n_1 - m} \frac{\exp \left( \frac{(n_1 - m)\phi}{n_1} \right)}{q_1} \\
    z_2 &\rightarrow z_2 \exp(-\phi) + z'_2 + \sum_{m=1}^{n_2-1} \frac{\epsilon_2 \theta_2^{n_2-m}}{n_2 - m} \frac{\exp \left( \frac{(m - n_2)\phi}{n_2} \right)}{q_2}
\end{align*}
\]

(99)

To make contact with the usual spacetime coordinates \( \{ x_\mu \} \) we note that as in the \( n_a = n_b = 2 \) case \( z_1 z_2 \) is invariant under a pure Lorentz transformation \( (\epsilon_1 = \epsilon_2 = 0) \). Thus we have \( z_1 z_2 \propto x_0^2 - x_1^2 \).

In fact the definitions of \( x_0 \) and \( x_1 \) in terms of \( z_1 \) and \( z_2 \) are

\[
\begin{align*}
x_0 &= F(z_1 + z_2), & p_0 &= -\frac{i}{2F}(\partial_{z_1} + \partial_{z_2}) \\
x_1 &= F(z_1 - z_2), & p_1 &= \frac{i}{2F}(\partial_{z_1} - \partial_{z_2})
\end{align*}
\]

(100)

with \( F = i \) for even \( n \) as in (72) and \( F = 1 \) for odd \( n \). These factors correspond to those relating \( t \) to \( z \) in [1, 2, 3, 4] and ensure the reality of \( p_\mu \) and \( x_\mu \). From (54) it follows that the operators defined
by (100) satisfy (73) as in the supersymmetric case covered in the last section. After a little algebra we obtain the mixed anyonic transformation of the \( \{x_\mu\} \) coordinates

\[
x_\mu \rightarrow x'_\mu + \Lambda^\nu_\mu x_\nu + \sum_{a,b=1}^{2} \sum_{m=1}^{n_a-1} \sum_{n=1}^{n_b-1} \frac{F_{ea}^m (\gamma_0 \gamma_\mu)_{ab} (S_{ba} \theta_a)_{n_a-m}}{[n_a-m]_{qa}! [m]_{qa}!} . \tag{101}
\]

Here \( \Lambda^\nu_\mu \) is the same as in (91). The fractional supercharge and covariant derivative are also easy to deduce. From (59) and (100) we find

\[
Q_a = D_{La} = \partial_{\theta_a} + \frac{\theta^{n_a-1}_a}{[n_a-1]_{qa}!} \partial_{za} = \partial_{\theta_a} + \frac{iF}{[n_b-1]_{nb}!} (\gamma_0 \gamma_\mu)_{ab} \theta^{n_b-1}_b p^\mu, \tag{102}
\]

and

\[
D_a = D_{Ra} = \delta_{\theta_a} - (-1)^{n_a} \frac{\theta^{n_a-1}_a}{[n_a-1]_{qa}!} \partial_{za} = \delta_{\theta_a} - \frac{iF (-1)^{n_a}}{[n_b-1]_{qb}!} (\gamma_0 \gamma_\mu)_{ab} \theta^{n_b-1}_b p^\mu . \tag{103}
\]

The algebraic (left) integral of a function \( f(z_1, z_2, \theta_1, \theta_2) \) on 2-dimensional fractional superspace is

\[
\int (d\theta_2) L \int (d\theta_1) L \ f(z_1, z_2, \theta_1, \theta_2) . \tag{104}
\]

Note that this integral would change by an overall multiplicative factor if we reversed the order of \( \int (d\theta_2) L \) and \( \int (d\theta_1) L \), so that in writing down (104), we have made a choice of convention. This integral can be expanded using (61) to yield

\[
\int d\theta_2 \int d\theta_1 \ f + \int d\theta_2 \frac{\partial^{n_1-1}}{\partial^{n_1-1} \theta_1} \int dz_1 \ f + \frac{\partial^{n_2-1}}{\partial^{n_2-1} \theta_2} \int dz_2 \ f + \frac{\partial^{n_2-1}}{\partial^{n_2-1} \theta_2} \frac{\partial^{n_1-1}}{\partial^{n_1-1} \theta_1} \int dz_2 \int dz_1 \ f . \tag{105}
\]

To obtain a numerical measure from this algebraic integral we now make use of an argument similar to that given in [2]. \( \theta_1 \) and \( \theta_2 \) are nilpotent and thus all of their eigenvalues are zero. On the other hand, the bosonic limits denoted by \( z_1 \) and \( z_2 \) are non-nilpotent and thus do have non-zero eigenvalues. After integration, the first three terms in (105) always involve \( \theta_1 \) or \( \theta_2 \) raised to some non-zero power, whereas the last term involves \( z_1 \) and \( z_2 \) only. Any numerical measure based on the integral (105) must be based on its eigenvalues in some representation. Consequently only the last term contributes and thus the first three can be dropped. It is convenient at this point to introduce a fractional Berezin integral as in [2].

\[
\int (d\theta_a)_{Ber} = \frac{\partial^{n_a-1}}{\partial^{n_a-1} \theta_a} , \tag{106}
\]
The resulting numerical integral measure on two dimensional fractional superspace can now be written as

\[ I(f) = \int dz_2 dz_1 (d\theta_2)_{Ber} (d\theta_1)_{Ber} f(z_1, z_2, \theta_1, \theta_2). \]  

(107)

If we expand \( f \) as a power series

\[ f(z_1, z_2, \theta_1, \theta_2) = \sum_{n_1=0}^{n_1-1} \sum_{n_2=0}^{n_2-1} C_{n_1,n_2} (z_1, z_2) \frac{\theta_{1}^{m_1}}{|m_1|_{q_1}} \frac{\theta_{2}^{m_2}}{|m_2|_{q_2}}, \]

(108)

then (107) reduces to

\[ I(f) = \int dz_2 dz_1 C_{n_1-1,n_2-1}(z_1, z_2). \]

(109)

Note that up to a constant Jacobian factor this is equal to

\[ \int dx_0 dx_1 C_{n_1-1,n_2-1}(z_1, z_2), \]

(110)

which, for \( n=2 \), is just the integral which arises in supersymmetric field theories involving one space and one time dimension.

**Appendix**

The results of [1, 2, 3, 4] can also be derived from a different and in some ways mathematically nicer point of view. Our work here uses a technique similar to that employed by G.I. Lusztig in his work on the properties of deformed universal enveloping algebras with deformation parameter equal to a root of unity [14, 15]. To the best of our knowledge this is the first time such a technique has been applied to a braided object. Let us begin by introducing the braided Hopf algebra \( A \), which we define for all \( q \). This has elements \( \{ \theta(m) \} \), \( m = 0, 1, 2, \ldots, \infty \) with \( \theta(0) = 1 \), and relations

\[ \theta(m) \theta(p) = \frac{[m+p]_q!}{[m]_q! [p]_q!} \theta(m+p), \]

(111)

as well as

\[ \Delta \theta(m) = \sum_{r=0}^{m} \theta^{(m-r)} \otimes \theta^{(r)}, \]

\[ \varepsilon(\theta(m)) = \delta_{m,0}, \]

\[ S(\theta(m)) = (-1)^m q^{\frac{m(m-1)}{2}} \theta^{(m)} \]

(112)

The braiding is given by

\[ \psi(\theta^{(m)} \otimes \theta^{(s)}) = q^{ms} \theta^{(m)} \otimes \theta^{(s)}, \]

(113)
so that
\[
(\theta^{(r)} \otimes \theta^{(m)})(\theta^{(s)} \otimes \theta^{(s)}) = q^{ms} \theta^{(r)} \theta^{(s)} \otimes \theta^{(m)} \theta^{(t)}.
\] (114)

We also define $\mathcal{K}$ the braided Hopf algebra dual to $\mathcal{A}$ as follows. This has elements $\{D^{(m)}_L\}$, $m = 0, 1, 2, \ldots, \infty$ with $D^{(0)}_L = 1$, and relations
\[
D^{(m)}_L D^{(p)}_L = D^{(m+p)}_L,
\] (115)
as well as
\[
\Delta D^{(m)}_L = \sum_{r=0}^{m} \frac{[m]_q!}{[r]_q! [m-r]_q!} D^{(m-r)}_L \otimes D^{(r)}_L,
\] (116)
\[
\varepsilon(D^{(m)}_L) = \delta_{m,0},
\] (116)
\[
S(D^{(m)}_L) = (-1)^m q^{\frac{m(m-1)}{2}} D^{(m)}_L.
\] (116)
The braiding is given by
\[
\psi(D^{(m)}_L \otimes D^{(s)}_L) = q^{ms} D^{(m)}_L \otimes D^{(s)}_L.
\] (117)

These two braided Hopf algebras are dual in the sense that there is a bilinear map $\langle \ , \rangle : \mathcal{A} \otimes \mathcal{K} \mapsto \text{the complex plane}$, such that
\[
\langle a, xy \rangle = \langle \Delta a, x \otimes y \rangle,
\]
\[
\langle ab, x \rangle = \langle a \otimes b, \Delta x \rangle,
\]
\[
\langle 1, x \rangle = \varepsilon_{\mathcal{K}}(x),
\] (118)
\[
\langle a, 1 \rangle = \varepsilon_{\mathcal{A}}(a),
\]
\[
\langle S(a), x \rangle = \langle a, S(x) \rangle.
\]

Specifically, in this case we have
\[
\langle \theta^{(m)}_L, D^{(p)}_L \rangle = \delta_{mp},
\] (119)
the compatibility of which with (118) is easy to verify. We now consider the cases of generic $q$ and $q$ a root of unity separately.
i) generic $q$ or $q = 1$. From (111) it follows that

$$\theta^{(m)} = \frac{\theta^{(1)}\theta^{(m-1)}}{[m]_q} = \frac{\theta^{(1)}\theta^{(m-2)}}{[m]_q[m-1]_q} = \frac{(\theta^{(1)})^m}{[m]_q!},$$

and similarly

$$D_L^{(m)} = (D_L^{(1)})^m.$$  

Consequently, at generic $q$ the braided Hopf algebra $A$ and its dual $K$ are both finite dimensional, each containing the identity, and only one other element. If we define $\theta = \theta^{(1)}$ and $D_L = D_L^{(1)}$ then we can write the generic $q$ braided Hopf structure as follows. For $A$ we have

$$\Delta \theta = \theta \otimes 1 + 1 \otimes \theta,$$

$$\varepsilon(\theta) = 0,$$

$$S(\theta) = -\theta,$$

which recovers the braided line at generic $q$, and for $K$ we have

$$\Delta D_L = D_L \otimes 1 + 1 \otimes D_L,$$

$$\varepsilon(D_L) = 0,$$

$$S(D_L) = -D_L.$$

The duality simplifies to

$$\langle \theta, D_L \rangle = 1,$$

and is extended to products via (118). By comparing (122) and (123) with one of [5, 6, 16] we are able to identify both the braided hopf algebra $A$ and its dual $K$ with the braided line when $q$ is not a root of unity.

ii) $q$ a primitive $n$th root of unity. As in the generic $q$ case we can use (111) to obtain

$$\theta^{(p)} = \frac{(\theta^{(1)})^p}{[p]_q!}.$$
but since \([n]_q = 0\) this only works for \(p = 0, 1, \ldots, n - 1\). However we are able to write

\[
\theta^{(rn+p)} = \theta^{(rn)} \theta^{(p)} \lim_{q \to \epsilon} \frac{[rn]_q! [p]_q!}{[rn + p]_q!}
\]

\[= \theta^{(rn)} \theta^{(p)}, \tag{126}\]

where \(r \geq 0\) and \(0 \leq p \leq n - 1\). Also using (111) we find that

\[
\theta^{(rn)} = \theta^{(n)} \theta^{((r-1)n)} \lim_{q \to \epsilon} \frac{[(r-1)n]_q! [n]_q!}{[rn]_q!}
\]

\[= \frac{\theta^{(n)} \theta^{((r-1)n)}}{r}. \tag{127}\]

Iterating we finally obtain

\[
\theta^{(rn)} = \frac{(\theta^{(n)})^r}{r!}, \tag{128}\]

so that

\[
\theta^{(rn+p)} = \frac{(\theta^{(n)})^r \theta^{(p)}}{r!}. \tag{129}\]

Similarly, for the dual we find that

\[
D_L^{(rn+p)} = (D_L^{(n)})^r D_L^{(p)}. \tag{130}\]

Thus when \(q\) is a root of unity \((q \neq 1)\) the braided Hopf algebra \(A\) is finite dimensional, having two independent elements \(\theta^{(1)}\) and \(\theta^{(n)}\) besides the identity. The dual \(K\) is also finite dimensional, but it has only one independent element \(D_L^{(1)}\) besides the identity. It is convenient to define

\[
\theta = \theta^{(1)} , \quad z = \theta^{(n)} , \quad D_L = D_L^{(1)} , \quad D_z = D_L^{(n)}. \tag{131}\]

Using this notation, the algebraic relations (111) reduce to \([\theta, z] = 0\) and \(\theta^n = 0\), and the braided Hopf structure (112) reduces to

\[
\Delta \theta = \theta \otimes 1 + 1 \otimes \theta \quad ,
\]

\[\varepsilon(\theta) = 0 \quad , \tag{132}\]

\[S(\theta) = -\theta \quad ,\]
and
\[ \Delta z = z \otimes 1 + 1 \otimes z + \sum_{m=1}^{n-1} \frac{\theta^m \otimes \theta^{n-m}}{[n-m]_q! [m]_q!}, \]
\[ \varepsilon(z) = 0, \]
\[ S(z) = -z. \]  

The braided Hopf structure of the dual \( K \) is given by
\[ \Delta D_L = D_L \otimes 1 + 1 \otimes D_L, \]
\[ \varepsilon(D_L) = 0, \]
\[ S(D_L) = -D_L. \]  

which, using \( \partial_z = D_L^p \) implies the following braided Hopf structure for \( \partial_z \),
\[ \Delta \partial_z = \partial_z \otimes 1 + 1 \otimes \partial_z, \]
\[ \varepsilon(\partial_z) = 0, \]
\[ S(\partial_z) = -\partial_z. \]  

The duality (119) now takes on the form
\[ \langle z^r \theta^p, D_L^{r + r'} \rangle = \langle z^r \theta^{n}, \partial_z D_L^p \rangle = \delta_{r,r'} \delta_{p,0} r! [p]_q!, \]
so that in particular
\[ \langle \theta, D_L \rangle = 1, \quad \langle z, \partial_z \rangle = 1. \]  

Thus when \( q \) is a root of unity \( A \) coincides with the braided Hopf algebra which was associated in previous work with a limit of the braided line as its deformation parameter goes to a root of unity. Using this approach we have also obtained the braided Hopf structure of the dual and details of the duality when \( q \) is a root of unity (this is an alternative form of the braided line when \( q \) is a root of unity). Note also that the \( \theta \) part of \( A \) forms a braided sub-Hopf algebra, but that the \( z \) part does not.

The advantage of the approach adopted here is that it enables us to restrict the taking of limits to purely numerical quantities, for which they are manifestly well defined. In this appendix we have worked with left derivatives \( D_L \) only, but we could equally well have chosen right derivatives \( D_R \), for
which a closely analogous treatment exists.

The relationship between the work in this appendix and the work of [14, 15] suggests that the latter might also have a physical interpretation in terms of supersymmetry and fractional supersymmetry. This idea will be developed further in [17].

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