Quasi-inner automorphisms of Drinfeld modular groups

A. W. MASON AND ANDREAS SCHWEIZER

Abstract. Let $A$ be the set of elements in an algebraic function field $K$ over $\mathbb{F}_q$, which are integral outside a fixed place $\infty$. Let $G = GL_2(A)$ be a Drinfeld modular group. The normalizer of $G$ in $GL_2(K)$, where $K$ is the quotient field of $A$, gives rise to automorphisms of $G$, which we refer to as quasi-inner. Modulo the inner automorphisms of $G$ they form a group $\text{Quinn}(G)$ which is isomorphic to $\text{Cl}(A)_2$, the 2-torsion in the ideal class group $\text{Cl}(A)$.

The group $\text{Quinn}(G)$ acts on all kinds of objects associated with $G$. For example, it acts freely on the cusps and elliptic points of $G$. If $T$ is the associated Bruhat-Tits tree the elements of $\text{Quinn}(G)$ induce non-trivial automorphisms of the quotient graph $G \backslash T$, generalizing an earlier result of Serre. It is known that the ends of $G \backslash T$ are in one-one correspondence with the cusps of $G$. Consequently $\text{Quinn}(G)$ acts freely on the ends. In addition $\text{Quinn}(G)$ acts transitively on the those ends which are in one-one correspondence with the vertices of $G \backslash T$ whose stabilizers are isomorphic to $GL_2(\mathbb{F}_q)$.

2020 Mathematics Subject Classification: 11F06, 20E08, 20E36, 20G30

Keywords: Drinfeld modular group; quasi-inner automorphism; elliptic point; cusp; quotient graph

1. Introduction

Let $K$ be an algebraic function field of one variable with constant field $\mathbb{F}_q$, the finite field of order $q$. Let $\infty$ be a fixed place of $K$ and let $\delta$ be its degree. The ring $A$ of all those elements of $K$ which are integral outside $\infty$ is a Dedekind domain. Denote by $K_\infty$ the completion of $K$ with respect to $\infty$ and let $C_\infty$ be the $\infty$-completion of an algebraic closure of $K_\infty$. The group $GL_2(K_\infty)$ (and its subgroup $G = GL_2(A)$) act as Möbius transformations on $C_\infty$, $K_\infty$ and hence $\Omega = C_\infty \backslash K_\infty$, the Drinfeld upper halfplane. This is part of a far-reaching analogy, initiated by Drinfeld [Dr], where $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are replaced by $K, K_\infty, C_\infty$, respectively. The roles of the classical upper half plane (in $\mathbb{C}$) and the classical modular group $SL_2(\mathbb{Z})$ are assumed by $\Omega$ and $G$, respectively.

Modular curves, that is quotients of the complex upper half plane by finite index subgroups of $SL_2(\mathbb{Z})$, are an indispensable tool when proving deep theorems about elliptic curves. Of similar importance in the theory of Drinfeld $A$-modules of rank 2 are Drinfeld modular curves, which are (the “compactifications” of) the quotient spaces $H \backslash \Omega$, where $H$ is a finite index subgroup of $G$. Consequently we refer to $G$ as a Drinfeld modular group.
A complicating factor in this correspondence between $SL_2(\mathbb{Z})$ and $G$ is that, while the genus of the former is zero, for different choices of $K$ and $\infty$ the genus of $G$ can take many values. The simplest case, where $K = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$ (equivalently $g = 0$ and $\delta = 1$), has to date attracted most attention.

An element $\omega \in \Omega$ which is stabilized by a non-scalar matrix in $G$ is called elliptic. Let $E(G)$ be the set of all such elements. It is known [Ge, p.50] that $E(G) \neq \emptyset$ if and only if $\delta$ is odd. Clearly $G$ acts on $E(G)$ and the elements of the set of $G$-orbits, 

$$Ell(G) = G\setminus E(G) = \{G\omega : \omega \in E(G)\},$$

are called the elliptic points of $G$. It is known [Ge, p.50] that $Ell(G)$ is finite.

In addition $G$ acts on $\mathbb{P}^1(K) = K \cup \{\infty\}$. (Here, of course, $\infty$ refers to the one point compactification of $K$.) We refer to the elements of $\mathbb{P}^1(K)$ as rational points. For each finite index subgroup, $H$, of $G$ the elements of $\text{Cusp}(H) = H\setminus \mathbb{P}^1(K)$ are called the cusps of $H$. Since $A$ is a Dedekind domain it is well-known that $\text{Cusp}(G)$ can be identified with $\text{Cl}(A)$, the ideal class group of $A$. As M"{o}bius transformations $G$ acts without inversion on $T$, the Bruhat-Tits tree associated with $GL_2(K_\infty)$ and the ends of the quotient graph $G\setminus T$ are determined by $\text{Cusp}(G)$ [Se, Theorem 9, p.106].

Cusps and elliptic points are important for several reasons. If $H$ is a finite index subgroup of $G$, the quotient space $H\setminus \Omega$ will, after adding $\text{Cusp}(H)$, be the $C_\infty$-analog of a compact Riemann surface, which is called the Drinfeld modular curve associated with $H$. Moreover, in the covering of Drinfeld modular curves induced by the natural map $H\setminus \Omega \rightarrow G\setminus \Omega$ ramification can only occur above the cusps and elliptic points of $G$. Also, for (classical and Drinfeld) modular forms, analyticity at the cusps and elliptic points requires special care.

This paper is a continuation and extension of [MS4] which is concerned with the elliptic points of $G$. There the starting point [Ge, p.51] is the existence of a bijection between $Ell(G)$ and $\ker \overline{\Omega}$, where $\overline{\Omega} : \text{Cl}(\tilde{A}) \rightarrow \text{Cl}(A)$ is the norm map and $\tilde{A} = A.\mathbb{F}_q^\times$. It can be shown [MS4] that $\text{Cl}(\tilde{A})_2 \cap \ker \overline{\Omega}$, the 2-torsion subgroup of $\ker \overline{\Omega}$, is in bijection with $Ell(G)^{\sim} = \{G\omega : \omega \in E(G), G\omega = G\overline{\omega}\}$, where $\overline{\omega}$, the conjugate of $\omega$, is the image of $\omega$ under the Galois automorphism of $K.\mathbb{F}_q^\times/K$. (In [MS4] $Ell(G)^{\sim}$ is denoted by $Ell(G)_2$.) Here we show that, when $\delta$ is odd, $\text{Cl}(A)_2$ and the 2-torsion in $\ker \overline{\Omega}$ are isomorphic. This is the starting point for this paper where the principal focus of attention is the group $\text{Cl}(A)_2$ and its actions on various objects related to $G$. Unless otherwise stated results hold for all $\delta$.

Let $g \in N_{GL_2(K)}(G)$, the normalizer of $G$ in $GL_2(K)$. Then $g$, acting by conjugation, induces an automorphism $\iota_g$ of $G$ which we refer to as quasi-inner. If $g \in G.Z(K)$ then $\iota_g$ reduces to an inner automorphism. If $g \in N_{GL_2(K)}(G)\setminus G.Z(K)$ we call $\iota_g$ non-trivial. We denote the quotient group $N_{GL_2(K)}(G)/G.Z(K)$ by $\text{Quinn}(G)$. It is well-known [Cr] that $\text{Quinn}(G)$ is isomorphic to $\text{Cl}(A)_2$. Hence $G$ has non-trivial quasi-inner automorphisms if and only if $|\text{Cl}(A)|$ is even. Now, as an element of
GL₂(K), \( \iota_g \) acts as a Möbius transformation on the rational points and elliptic elements of \( G \), as well as \( T \). In particular \( g(\omega) = \bar{g}(\omega) \). Since all of these actions are trivial for scalar matrices they extend to actions of \( \text{Quinn}(G) \) on Cusp\( (G) \), Ell\( (G) \) and the quotient graph, \( G \setminus T \). In this paper we study of the (often surprising) properties of these actions.

**Theorem 1.1.** \( \text{Quinn}(G) \) acts freely on

(i) Cusp\( (G) \),
(ii) Ell\( (G) \) (\( \delta \) odd).

From the above it is clear that \( \text{Quinn}(G) \) can be embedded as a subgroup Ell\( (G) = \) (resp. Cl\( (A)_2 \)) of Ell\( (G) \) (resp. Cusp\( (G) \)). We show that the action of \( \text{Quinn}(G) \) is equivalent to multiplication by the elements of the subgroup. The “freeness” in this result follows immediately. Restricting to these subsets yields stronger results.

**Corollary 1.2.** \( \text{Quinn}(G) \) acts freely and transitively on

(i) Cl\( (A)_2 \),
(ii) Ell\( (G) = (\delta \text{ odd}) \).

**Corollary 1.3.** When \( \delta \) is odd \( \text{Quinn}(G) \) acts freely on Ell\( (G) \neq \{G\omega : G\omega \neq G\bar{\omega}\} \). Moreover, if ker\( \overline{N} \) has no element of order 4, then \( \text{Quinn}(G) \) acts freely on

\[ \{\{G\omega, G\bar{\omega}\} : G\omega \in \text{Ell}(G) \neq \}\} \).

**Theorem 1.4.** Every non-trivial element of \( \text{Quinn}(G) \) determines an automorphism of \( G \setminus T \) of order 2 which preserves the structure of all its vertex and edge stabilizers.

Serre [Se, Exercise 2 e), p. 117] states this result for the special case \( K = \mathbb{F}_q(t) \) with \( \delta \) even. Our result shows that in general the quotient graph has symmetries of this type provided \( |\text{Cl}(A)| \) is even. (In general this restriction is necessary.)

We now list more detailed results on the action of \( \text{Quinn}(G) \) on \( G \setminus T \). Serre [Se, Theorem 9, p.106] has described the basic structure of \( G \setminus T \). Its ends (i.e. the equivalence classes of semi-infinite paths without backtracking) are in one-one correspondence with the elements of Cl\( (A) \). To date the only cases for which the precise structures of \( G \setminus T \) are known are \( g = 0, \) [Ma2], [KMS], and \( g = \delta = 1, \) [Ta].

**Theorem 1.5.** \( \text{Quinn}(G) \) acts freely on the ends of \( G \setminus T \) and, in addition, transitively on the ends of \( G \setminus T \) corresponding to the elements of Cl\( (A)_2 \).

We show that the ends corresponding to Cl\( (A)_2 \) are in one-one correspondence with those vertices whose stabilizers are isomorphic to \( GL_2(\mathbb{F}_q) \). (Each such vertex is “attached” to the corresponding end.) It is known [MS3, Corollary 2.12] that if \( G_v \) contains a cyclic subgroup of order \( q^2 - 1 \) then \( G_v \cong \mathbb{F}_{q^2}^* \) or \( GL_2(\mathbb{F}_q) \).

The building map [Ge, p.41] extends to a map \( \lambda : \text{Ell}(G) \rightarrow \text{vert}(G \setminus T) \). This map leads to another action of \( \text{Quinn}(G) \) on the quotient graph.
Theorem 1.6. (a) Quinn(G) acts freely and transitively on

\[ \{ \bar{v} \in \text{vert}(G \setminus T) : G_v \cong GL_2(F_q) \} . \]

(b) Suppose that \( \delta \) is odd and that \( \ker N \) has no element of order 4. Then Quinn(G) acts freely on

\[ \{ \bar{v} \in \text{vert}(G \setminus T) : G_v \cong F^{*}_{q^2} \} . \]

As an illustration of our results, especially the existence of reflective symmetries as in Theorem 1.4, we conclude with diagrams of two examples of \( G \setminus T \) for each of which \( g = \delta = 1 \), the so called “elliptic” case. For these we make use of Takahashi’s paper [Ta]. Special features of these cases include the following. For part (i) see [MS4, Theorem 5.1].

Corollary 1.7. Suppose that \( \delta = 1 \).

(i) The isolated (i.e. (graph) valency 1) vertices of \( G \setminus T \) are precisely those whose stabilizers are isomorphic to \( GL_2(F_q) \) or \( F^{*}_{q^2} \).

(ii) If \( \ker N \) has no element of order 4 then Quinn(G) acts freely on the isolated vertices of \( G \setminus T \).

By looking at the stabilizers in \( G \) of the objects discussed above we obtain several statements about the action of Quinn(G) on the conjugacy classes of certain types of subgroups of \( G \). (See Sections 3 and 5.) For convenience we begin with a list of notations which will be used throughout this paper.
\(\mathbb{F}_q\) the finite field with \(q = p^n\) elements;
\(K\) an algebraic function field of one variable with constant field \(\mathbb{F}_q\);
\(g\) the genus of \(K\);
\(\infty\) a chosen place of \(K\);
\(\delta\) the degree of the place \(\infty\);
\(A\) the ring of all elements of \(K\) that are integral outside \(\infty\);
\(K_\infty\) the completion of \(K\) with respect to \(\infty\);
\(\Omega\) Drinfeld’s half-plane;
\(\mathcal{T}\) the Bruhat-Tits tree of \(\text{GL}_2(K_\infty)\);
\(G\) the Drinfeld modular group \(\text{GL}_2(A)\);
\(Gx\) the orbit of \(x\) under the action of \(G\) on the object \(x\);
\(\hat{G}\) \(\text{GL}_2(K)\);
\(Z(K)\) the set of scalar matrices in \(\hat{G}\);
\(Z\) \(Z(K) \cap G\);
\(\hat{K}\) the quadratic constant field extension \(K.\mathbb{F}_{q^2}\);
\(\hat{A}\) \(A.\mathbb{F}_{q^2}\), the integral closure of \(A\) in \(\hat{K}\);
\(\text{Cl}(R)\) the ideal class group of the Dedekind ring \(R\);
\(\text{Cl}^0(F)\) the divisor class group of degree 0 of the function field \(F\);
\(\text{Cusp}(G)\) \(G \backslash \mathbb{P}^1(K)\), the set of cusps of \(G\);
\(E(A)\) the set of elliptic elements of \(G\);
\(\text{Ell}(G)\) \(G \backslash E(A)\), the set of elliptic points of \(G\);
\(\overline{\omega}\) the image of \(\omega \in E(A)\) under the Galois automorphism of \(\hat{K}/K\);
\(\text{Ell}(G)\neq\) \(\text{Ell}(G)\backslash \text{Ell}(G)\);  
\(S(s)\) the stabilizer in a finite index subgroup \(S\) (of \(G\)) of \(s \in \mathbb{P}^1(K)\);
\(G^\omega\) the stabilizer in \(G\) of \(\omega \in C_\infty \backslash K\);
\(S_w\) the stabilizer in \(S\) of \(w \in \text{vert}(\mathcal{T}) \cup \text{edge}(\mathcal{T})\);
\(\mathcal{H}\) \(\{H \leq G : H \cong \text{GL}_2(\mathbb{F}_q)\}\);
\(\mathcal{C}\) \(\{C \leq G : C \cong \mathbb{F}_{q^2}\}\);
\(\mathcal{C}_{mf}\) \(\{C \in \mathcal{C} : C\ \text{maximally finite in} \ G\}\);
\(\mathcal{C}_{nm}\) \(\mathcal{C} \backslash \mathcal{C}_{mf}\);
\(\mathcal{V}\) \(\{\tilde{v} \in \text{vert}(G \backslash \mathcal{T}) : G_v \in \mathcal{C}\}\);  

2. **QUASI-INNER AUTOMORPHISMS**

Let \(F\) be any field containing \(A\) (and hence \(K\)) and let \(Z(F)\) denote the set of scalar matrices in \(\text{GL}_2(F)\). We are interested here in automorphisms of \(G\) arising from conjugation by a non-scalar element of \(\text{GL}_2(F)\). We first show this problem reduces to \(N_{\hat{G}}(G)\), the normalizer of \(G\) in \(\hat{G} = \text{GL}_2(K)\). For each \(x \in F\) we use \((x)\) as a shorthand for the fractional ideal \(Ax\).

**Lemma 2.1.** Let \(M_0 \in \text{GL}_2(F)\) normalize \(G\). Then  
\[ M_0 \in Z(F).N_{\hat{G}}(G). \]
Proof. Let
\[ M_0 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \]
Suppose that \( \gamma \neq 0 \). Replacing \( M_0 \) with \( \gamma^{-1}M_0 \) we may assume that \( \gamma = 1 \). Now
\[ NT(1)N^{-1} \in G, \]
where \( N = M_0^{\pm 1} \). It follows that \( \det(M_0), \alpha, \delta \in K \) and hence that \( \beta = \alpha\delta - \det(M_0) \in K \). The proof for the case where \( \gamma = 0 \) is similar.

We state a special case \((n = 2)\) of a result of Cremona \([Cr]\).

**Theorem 2.2.** Let
\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \hat{G} \]
and define
\[ q(M) := (a) + (b) + (c) + (d). \]
Then \( M \in N_{\hat{G}}(G) \) if and only if
\[ q(M)^2 = (\Delta), \]
where \( \Delta = \det(M) \).

**Corollary 2.3.** Let \( M \in N_{\hat{G}}(G) \) with \( \Delta = \det(M) \).

(i) \( \Delta^{-1}M^2 \in SL_2(A) \).

(ii) If \( \Delta \in A^* \), then \( M \in G \).

**Proof.** (i) By Theorem 2.2 every entry of \( M^2 \) is in \( q(M)^2 = (\Delta) \).

For part (ii) let \( x \) be any entry of \( M \). Then \( x^2 \in A \) by Theorem 2.2 and so \( x \in A \), since \( A \) is integrally closed.

Another important consequence \([Cr]\) of Theorem 2.2 is the following.

**Theorem 2.4.** The map \( M \mapsto q(M) \) induces an isomorphism
\[ N_{\hat{G}}(G)/Z(K).G \cong Cl(A)_2, \]
where \( Cl(A)_2 \) is the subgroup of all involutions in \( Cl(A) \).

**Proof.** This is another special case \((n = 2)\) of a result in \([Cr]\). If \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in N_{\hat{G}}(G) \),
it can be shown \([Cr]\) Remarks 2] that
\[ (a) + (b) = (a) + (c) = (d) + (b) = (d) + (c) = q(M). \]
Consequently there is a map from \( N_{\hat{G}}(G) \) to \( Cl(A)_2 \), which turns out to be an isomorphism. \( \square \)
**Definition 2.5.** An automorphism $\iota_g$ of $G$ is called **quasi-inner** if

$$\iota_g(x) = gxg^{-1} \ (x \in G),$$

for some $g \in N_{\hat{G}}(G)$. We call $\iota_g$ **non-trivial** if $g \notin Z(K).G$, i.e. if $\iota_g$ does not act like an inner automorphism. We note that

$$\iota_{g_1} = \iota_{g_2} \iff g_1g_2^{-1} \in Z(K).$$

Finally we define

$$Quinn(G) := N_{\hat{G}}(G)/Z(K).G \cong Cl(A).$$

So $Quinn(G)$ is the group of quasi-inner automorphisms modulo the inner ones. We note that in particular all quasi-inner automorphisms of $G$ act like inner automorphisms if $|Cl(A)|$ is odd.

Let $Cl^0(K)$ be the group of divisor classes of degree zero [St, p.186]. It is known [Sc, p.104] that the following exact sequence holds

$$0 \to Cl^0(K) \to Cl(A) \to \mathbb{Z}/\delta \mathbb{Z} \to 0. \quad (1)$$

Our next result is an immediate consequence of Theorem 2.4.

**Corollary 2.6.** $G$ has non-trivial quasi-inner automorphisms if and only if

$$|Cl(A)| = \delta|Cl^0(K)| \text{ is even.}$$

**Example 2.7.** We illustrate the results of this section with the simplest case $K = F_q(t)$, the rational function field over $F_q$. Then there exists a (monic) polynomial $\pi(t) \in F_q[t]$, of degree $\delta$, irreducible over $F_q$, such that

$$A = \left\{ \frac{f}{\pi^m} : f \in k[t], \ m \geq 0, \ \deg f \leq \delta m \right\}.$$

It is known [St, Theorem 5.1.15, p.193] that here $Cl^0(K)$ is trivial so that

$$Cl(A) \cong \mathbb{Z}/\delta \mathbb{Z}.$$ 

Hence $G$ has non-trivial quasi-inner automorphisms if and only if $\delta$ is even. Hence here either $Quinn(G)$ is trivial or cyclic of order 2.

For a specific illustration of Theorem 2.4 we restrict further to $\delta = 2$. In this case $\pi(t) = t^2 + \sigma t + \tau$, where $\sigma \in F_q$ and $\tau \in F_q^*$. We begin with the $A$-ideal generated by $\pi^{-1}$ and $t\pi^{-1}$ which is not principal. Let $\pi(t) = tl' + \tau$ and put

$$g_0 = \begin{bmatrix} \tau & t \\ -l' & 1 \end{bmatrix}.$$ 

Then by Theorem 2.2 $g_0 \in N_{\hat{G}}(G)$ and from Theorem 2.4 we see $g_0 \notin Z(K).G$. Hence $g_0$ provides a generator of $Cl(A)_2$. 

7
Remark 2.8. From the theory of Jacobian varieties we know that the 2-torsion in $\text{Cl}^0(K)$ is bounded by $2^{2g}$, and even by $2^g$ if the characteristic of $K$ is 2 [Ro, Theorem 11.12]. Hence by the exact sequence (1) it follows that $|\text{Quinn}(G)| = |\text{Cl}(A)_2| \leq 2^{2g+1}$ (and $\leq 2^{g+1}$, when $\text{char}(K) = 2$).

In odd characteristic we can easily find examples with $|\text{Cl}(A)_2| = 2^{2g}$, provided we are willing to accept a big constant field. Given a function field $F$ of genus $g$ with constant field $F_{p^r}$, just pick $q = p^{rn}$ such that all 2-torsion points of $\text{Jac}(F)$ are $F_{q}$-rational and consider $K = F_{q}(\sqrt{q})$. Then $\text{Cl}^0(K)_2 \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$. Choosing a place $\infty$ of $K$ of odd degree $\delta$, from the exact sequence (1), we see that $|\text{Cl}(A)_2| = 2^{2g}$.

Similarly in characteristic 2 examples for which $|\text{Cl}(A)_2| = 2^g$ can be found by choosing $F$ suitably, namely $F$ has to be ordinary.

Whether for even $\delta$ one can reach the bound $2^{2g+1}$ (resp. $2^{g+1}$) depends on whether or not the induced short exact sequence for the Sylow 2-subgroup of $\text{Cl}(A)$ splits or not.

Definition 2.9. Let $R$, $S$ be subgroups of a group $T$. We write

$$R \sim S$$

if and only if $R = S' = tSt^{-1}$, for some $t \in T$. We put

$$R^T = \{ R^t : t \in T \}.$$  

Let $S$ be a set of subgroups of $T$. We put

$$S^G = \{ S^G : S \in S \}.$$  

This paper is principally concerned with various actions of $\text{Quinn}(G)$. It is appropriate at this point to describe in detail the most important of these. Let $\iota_g$ be as above.

(i) It is clear that $GL_2(K_{\infty})$ acts on $\Omega$ as Möbius transformations and that this action is trivial for all scalar matrices. Then $\iota_g$ acts on $E(G)$ since, for all $\omega \in E(G)$,

$$G^{\iota_{g}\omega} = (G^\omega)^g(\leq G).$$

Recall that $\text{Ell}(G) = \{ G\omega : \omega \in E(G) \}$. The map

$$G\omega \mapsto Gg(\omega)$$

extends naturally to a well-defined action of $\text{Quinn}(G)$ on $\text{Ell}(G)$.

(ii) Clearly $G$ acts as Möbius transformations on $\mathbb{P}^1(K)$ and it is well-known that

$$G \backslash \mathbb{P}^1(K) \leftrightarrow \text{Cl}(A).$$

As we shall see later from the structure of the quotient graph it follows that, for all $k \in \mathbb{P}^1(K)$, $G(k)$ is infinite, metabelian. Recall that $\text{Cusp}(G) = \{ Gk : k \in \mathbb{P}^1(K) \}$. As before the map

$$Gk \mapsto Gg(k)$$

extends to a well-defined action of $\text{Quinn}(G)$ on $\text{Cusp}(G)$.

(iii) Serre [Se, Chapter II, Section 1.1, p.67] uses lattice classes as a model for the
vertices and edges of $\mathcal{T}$. It is clear that $GL_2(K_{\infty})$ acts naturally on these. In particular the scalar matrices act trivially. The map

$$Gw \mapsto Gg(w),$$

where $w \in \text{vert}(\mathcal{T}) \cup \text{edge}(\mathcal{T})$, extends to a well-defined action of $\text{Quinn}(G)$ on the quotient graph $G^{\prime} \backslash \mathcal{T}$. Note that $G_{g(w)} = ((G_w)^g)^{G} \leq G$. We will use this action to extend a result of Serre.

(iv) Suppose that

$$S = \{ H \leq G : H \cong GL_2(\mathbb{F}_q) \},$$

or $S$ is a $G$-conjugacy closed subset of $\mathcal{C} = \left\{ C \leq G : C \cong \mathbb{F}_{q^2}^* \right\}$. Then $\text{Quinn}(G)$ acts by conjugation on $S^G$. We use these to define actions of $\text{Quinn}(G)$ on significant subsets of $\text{vert}(\mathcal{T})$.

3. Action on vertex stabilizers

Almost all the results in this section hold for all $\delta$. We record the important general properties of subgroups of vertex stabilizers.

**Lemma 3.1.** (i) $G_v$ is finite, for all $v \in \text{vert}(\mathcal{T})$.

(ii) Let $S$ be a finite subgroup of $G$. Then

$$S \leq G_{v_0},$$

for some $v_0 \in \text{vert}(\mathcal{T})$.

**Proof.** See [Se, Proposition 2, p.76]. □

In this section we are concerned with subgroups of $G_v$ which contain a cyclic subgroup of order $q^2 - 1$. We record the following result.

**Lemma 3.2.** Suppose that $G_v$ contains a cyclic subgroup of order $q^2 - 1$. Then

$$G_v \cong GL_2(\mathbb{F}_q) \text{ or } G_v \cong \mathbb{F}_{q^2}^*.$$

**Proof.** See [MS3, Corollaries 2.2, 2.4, 2.12]. □

In the first part of this section we look at the action of quasi-inner automorphisms on the following set

$$\mathcal{H} = \{ H \leq G : H \cong GL_2(\mathbb{F}_q) \}.$$

**Lemma 3.3.** Let $H \in \mathcal{H}$. Then there exists $v_0 \in \text{vert}(\mathcal{T})$ for which

$$H = G_{v_0}.$$

**Proof.** Follows from Lemmas 3.1(ii) and 3.2. □
Remark 3.4. (i) Every $T$ contains a particular vertex $v_s$, usually referred to as standard (after Serre), for which

$$G_{v_s} = GL_2(\mathbb{F}_q).$$

See [Sc, Remark 3], p.97

(ii) On the other hand for the case $A = \mathbb{F}_q[t]$ (equivalently $g(K) = 0$, $\delta = 1$) it follows from Nagao’s Theorem [Sc, Corollary, p.87] that here $\text{vert}(T)$ has no stabilizer which is cyclic of order $q^2 - 1$.

Lemma 3.5. Let $H \in \mathcal{H}$. Then there exists a quasi-inner automorphism $\kappa = \iota_g$ of $G$ such that

$$H = \kappa(GL_2(\mathbb{F}_q)).$$

Proof. From the proofs of [MS3, Theorem 2.6, Corollary 2.8], as well as [MS3, Corollary 2.12] it is clear that there exists $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\tilde{K})$ such that

$$H = g(GL_2(\mathbb{F}_q))g^{-1}.$$

We denote by $\pi$ the image of $x \in \tilde{K}$ under the extension of the Galois automorphism of $\mathbb{F}_{q^2}/\mathbb{F}_q$ to $\tilde{K}$. It is clear that $gE_{ij}g^{-1} \in M_2(A)$, where $1 \leq i, j \leq 2$ and so

$$xy/\Delta = \pi y/\pi \Delta \in A,$$

for all $x, y \in \{a, b, c, d\}$, where $\Delta = \det(g)$.

Now we may assume without loss of generality that $c \neq 0$. Let $z \in \{a, b, d\}$. Then

$$c^2/\Delta = \pi^2/\pi \Delta, \quad cz/\Delta = \pi z/\pi \Delta.$$

It follows that $z/c = \pi/c$ so that $z/c \in K$. We now replace $g = M$ with $g_0 = c^{-1}M$. Then by Theorem [2.2] the map $\kappa_0 : G \rightarrow G$ defined by $\kappa_0(x) = g_0xg_0^{-1}$ is a quasi-inner automorphism of $G$. \hfill $\square$

Lemma 3.6. Let $\kappa_0 = \iota_{g_0}$ be a non-trivial quasi-inner automorphism of $G$ and let $H \in \mathcal{H}$. Then

$$\kappa_0(H) \not\sim H.$$ 

Proof. By definition $g_0 \in \hat{N}(G) \backslash G.Z(K)$. Suppose to the contrary that

$$\kappa_0(H) = gHg^{-1}$$

for some $g \in G$. Replacing $g_0$ with $g^{-1}g_0$ we may assume that $g = 1$. Now by Lemma 3.5 $H = \kappa'_0(GL_2(\mathbb{F}_q))$ for some quasi-inner $\kappa'_0 = \iota_{g'_0}$, say. It follows that

$$g_1(GL_2(\mathbb{F}_q))g_1^{-1} = GL_2(\mathbb{F}_q),$$

where $g_1 = (g'_0)^{-1}g_0g'_0$. As $\hat{N}(G)/G.Z(K)$ is abelian this implies that

$$g_1 \equiv g_0 \pmod{Z(K).G}.$$
and so we may further assume that \( g_1 = g_0 \). Let
\[
S_p = \{ T(a) = E_{12}(a) : a \in \mathbb{F}_q \}.
\]
Now \( S_p \) is a Sylow \( p \)-subgroup of \( GL_2(\mathbb{F}_q) \) and so from the above
\[
g_0(S_p)g_0^{-1} = h(S_p)h^{-1},
\]
for some \( h \in GL_2(\mathbb{F}_q) \). As above we may assume then that \( h = 1 \). It follows that \( g_0 \) “fixes” \( \infty \) and so
\[
g_0 = \begin{bmatrix} \alpha & * \\ 0 & \beta \end{bmatrix}.
\]
By Corollary 2.3 (i) we note that
\[
(\det(g_0)^{-1} \text{tr}(g_0)^2) = \gamma + \gamma^{-1} \in A,
\]
where \( \gamma = \alpha \beta^{-1} \). Since \( A \) is integrally closed it follows that \( \gamma \in A^* (= \mathbb{F}_q^*) \). Then we can replace \( g_0 \) with \( \beta^{-1}g_0 \) which belongs to \( G \) by Corollary 2.3 (ii). Thus \( g_0 \in Z(K).G \). □

**Lemma 3.7.** Let \( e \in \text{edge}(\mathcal{T}) \) be incident with \( v_s \). Then
\[
G_e \not\subseteq GL_2(\mathbb{F}_q).
\]

**Proof.** The edges attached to \( v_s \) are parametrized by \( \mathbb{P}^1(\mathbb{F}_q^\delta) \) and \( GL_2(\mathbb{F}_q) \) acts on these as Möbius transformations. See [Se, Exercise 6), p.99].

If the edge corresponds to \( f \in \mathbb{F}_q^\delta \) it is not fixed by the translations in \( GL_2(\mathbb{F}_q) \), and if it corresponds to \( \infty \), it is not fixed by \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in GL_2(\mathbb{F}_q) \). □

**Proposition 3.8.** No edge of \( \mathcal{T} \) can have a stabilizer isomorphic to \( GL_2(\mathbb{F}_q) \).

**Proof.** For odd \( \delta \) this follows from [MSS, Corollary 2.16]. We provide a proof that holds for all \( \delta \). Suppose to the contrary that there is an edge \( e \) whose stabilizer is isomorphic to \( GL_2(\mathbb{F}_q) \). Then by Lemma 3.2 the stabilizers of its terminal vertices are both \( G_e \).

By Lemma 3.5 and the action of quasi-inner automorphisms on \( \mathcal{T} \) we can assume that
\[
G_e = GL_2(\mathbb{F}_q).
\]
It follows that \( GL_2(\mathbb{F}_q) \) stabilizes the geodesic from \( v_s \) to one of the terminal vertices of \( e \) which includes \( e \) and hence an edge incident with \( v_s \). This contradicts Lemma 3.7. □

**Corollary 3.9.** Let \( H \in \mathcal{H} \). Then there exists a unique vertex \( v \in \text{vert}(\mathcal{T}) \) such that
\[
G_v = H.
\]

**Proof.** Follows from Lemma 3.8 and Proposition 3.8 □
Remark 3.10. Another interesting consequence of Lemma 3.5 and Proposition 3.8 is the following. Suppose that $G_v \in \mathcal{H}$. Then there exists $\kappa = \iota_g$ such that $\kappa(v) = v_s$. Since $\kappa$ is an automorphism of $\mathcal{T}$ the action of $G_v$ on the $q^\delta + 1$ edges of $\mathcal{T}$ incident with $v$ is identical to the action of $GL_2(\mathbb{F}_q)$ on the edges of $\mathcal{T}$ incident with $v_s$ as described in Lemma 3.7.

Definition 3.11. By definition
\[
\text{vert}(G\setminus \mathcal{T}) = \{Gv : v \in \text{vert}(\mathcal{T})\}.
\]
We put $\tilde{v} = Gv$ and define its stabilizer
\[
G_{\tilde{v}} = (G_v)^G.
\]
We refer to $G_{\tilde{v}}$ as being isomorphic to $G_v$.

Lemma 3.12. There exists a bijection
\[
\mathcal{H}^G \leftrightarrow \{\tilde{v} \in \text{vert}(G\setminus \mathcal{T}) : G_v \in \mathcal{H}\}.
\]

Proof. Follows from Corollary 3.9 and the above.

It is clear that Quinn($G$) acts on $\mathcal{H}^G$. Since $Z(K)$, represented by scalar matrices, acts trivially on $\mathcal{T}$, it is also clear that Quinn($G$) acts on $G\setminus \mathcal{T}$. We now come to the principal result in this section which follows from Lemmas 3.5, 3.6, and 3.12.

Theorem 3.13. Quinn($G$) acts freely and transitively on
(i) the conjugacy classes of subgroups of $G$ which are isomorphic to $GL_2(\mathbb{F}_q)$,
(ii) the vertices of $G\setminus \mathcal{T}$ whose stabilizers are isomorphic to $GL_2(\mathbb{F}_q)$.

A special case of this result is provided by Corollary 2.6.

Corollary 3.14. Suppose that $|\text{Cl}(A)|$ is odd. Then
(i) every subgroup $H$ of $G$ isomorphic to $GL_2(\mathbb{F}_q)$ is actually conjugate in $G$ to $GL_2(\mathbb{F}_q)$,
(ii) the only vertex in $G\setminus \mathcal{T}$ whose stabilizer is isomorphic to $GL_2(\mathbb{F}_q)$ is $\tilde{v}_s$, the image of the standard vertex $v_s$.

4. Action on Elliptic Points

Throughout this section we assume that $\delta$ is odd. Recall that
\[
\text{Ell}(G) = \{G\omega : \omega \in E(G)\}
\]
denotes the elliptic points of the Drinfeld modular curve $G\setminus \Omega$.

Definition 4.1. We define
\[
\text{Ell}(G)^\equiv = \{G\omega : G\omega = G\tilde{\omega}\} \quad \text{and} \quad \text{Ell}(G)^\not\equiv = \{G\omega : G\omega \neq G\tilde{\omega}\}.
\]
(In [MS4 Section 3] $\text{Ell}(G)^\equiv$ is denoted by $\text{Ell}(G)_2$.)

The action of an element of $GL_2(K\infty)$ on an element of $\Omega$ will always refer to its action as a M"{o}bius transformation. We record the following.
**Lemma 4.2.** Let \( g \in N_G(G) \) and \( \omega \in E(A) \). Then

(i) \( g(\omega) \in E(A) \),

(ii) \( g(\omega) = g(\overline{\omega}) \).

It is clear then that \( Quinn(G) \) acts on both \( \text{Ell}(G)^= \) and \( \text{Ell}(G)^\neq \).

In this section our approach is based on [MS4, Sections 3, 4]. We recall some details.

**Definition 4.3.** Let \( I \) be an \( A \)-ideal (resp. \( \tilde{A} \)-ideal). Then \([I]\) denotes the image of \( I \) in \( \text{Cl}(A) \) (resp. \( \text{Cl}(\tilde{A}) \)).

Fix \( \epsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \). By [MS4, Theorem 2.5] any elliptic point \( \omega \) of \( G \) can be written as \( \omega = \epsilon + s \) where \( s, t \in A \) and \( t \) divides \((\epsilon + s)(\epsilon + s)\) in \( A \). Now let

\[ J_\omega = A(\epsilon + s) + At. \]

It is known [MS4, Lemmas 3.1, 3.2] that

(i) \( J_\omega \) is an \( \tilde{A} \)-ideal.

(ii) \( J_\omega \) is independent of the choice of \( \epsilon \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \).

(iii) Let \( \omega, \omega' \in E(A) \). Then

\[ G\omega = G\omega' \iff [J_\omega] = [J_{\omega'}] \quad (\text{in } \text{Cl}(\tilde{A})). \]

Let \( \alpha \) be the Galois automorphism of \( \overline{K}/K \) (which extends that of \( \mathbb{F}_{q^2}/\mathbb{F}_q \)). Let \( k \in \overline{K} \). Then the norm of \( k \) is \( k\overline{k} \), where \( \overline{k} = \alpha(k) \). Now \( \alpha \) restricts to \( \tilde{A} \) and so acts on its ideals and hence its ideal class group. For each \( \tilde{A} \)-ideal, \( J \), the norm of \( J \), \( N(J) = A \cap (J\overline{J}) \), which is an \( A \)-ideal. We now come to the norm map

\[ \overline{N} : \text{Cl}(\tilde{A}) \to \text{Cl}(A), \]

where \( \overline{N}([I]) = [(I\overline{I}) \cap A] \). Then

\[ [I] \in \ker \overline{N} \iff (I\overline{I}) \cap A \text{ is a principal } A-\text{ideal}. \]

We restate [MS4, Theorem 3.4].

**Theorem 4.4.** The map \( \omega \mapsto [J_\omega] \) induces a one-one correspondence

\[ \text{Ell}(G) \leftrightarrow \ker \overline{N}. \]

For each \( \omega \) it is known that

(i) \( \overline{J_\omega} = J_{\overline{\omega}} \),

(ii) \( J_\omega J_{\overline{\omega}} \) is a principal \( A \)-ideal.

It follows that

\[ \ker \overline{N} = \{ [J_\omega] : [J_{\overline{\omega}}] = [J_\omega]^{-1} \}. \]

We recall from Theorem 2.4 that \( Quinn(G) \) can be identified with \( \text{Cl}(A)_2 \). From this and Theorem 4.4 we are able to study the action of \( Quinn(G) \) on \( \text{Ell}(G) \). For this purpose we require two further lemmas.

**Lemma 4.5.** Let \( \iota : \text{Cl}(A) \to \text{Cl}(\tilde{A}) \) be the canonical map, where \( \iota([I]) = [I\tilde{A}] \), \( (I \subseteq A) \). Then

\[ \iota ([I]) \in \ker \overline{N} \iff (I\tilde{I}) \cap A \text{ is a principal } A-\text{ideal}. \]

We restate [MS4, Theorem 3.4].
(i) \( \iota \) is injective.
(ii) \( \{ [I] \in \text{Cl}(\tilde{A}); [I] = [\mathfrak{I}] \} = \iota(\text{Cl}(A)). \)

**Proof.** The analogous statements are known to hold for the canonical map from \( \text{Cl}^0(K) \to \text{Cl}^0(\tilde{K}) \). See [Ro, Corollary to Proposition 11.10]. The results follow from the exact sequence in Section 2, since \( \delta \) is odd and the infinite place is inert in \( \tilde{K} \). □

**Lemma 4.6.** With the above notation, the 2-torsion in \( \text{Cl}(A) \),
\[
\text{Cl}(A)_2 \cong \iota(\text{Cl}(A)_2) = (\ker N)_2,
\]
the 2-torsion in \( \ker N \).

**Proof.** Let \( [I] \in \text{Cl}(A)_2 \). Then \( \iota([I]) \) has order 2 in \( \text{Cl}(\tilde{A}) \) by Lemma 4.5. Now
\[
\overline{N}(\iota([I])) = \iota([I])\overline{\iota([I])} = (\iota([I])^2) = 1,
\]
by Lemma 4.5 (ii). Hence \( \iota([I]) \in \ker N \). Conversely let \( [J] \in \text{Cl}(\tilde{A}) \) have order 2 and lie in \( \ker N \). Then \( [J]^2 = 1 \) and \( [J][\mathfrak{I}] = \overline{\overline{N}(J)} = 1. \) Hence \( [J] = [\mathfrak{I}] \) and so \( [J] \in \iota(\text{Cl}(A)_2) \) again by Lemma 4.5 (ii).

Any element of \( \overline{N}_G(G) \) can be represented by a matrix
\[
M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \hat{G}.
\]
By multiplying \( M \) by a suitable scalar matrix we may assume that \( a, b, c, d \in A \). As before let
\[
q(M) := (a) + (b) + (c) + (d).
\]
Then
(i) \( q(M)^2 = (\Delta) \).
(ii) \( (a) + (b) = (a) + (c) = (d) + (b) = (d) + (c) = q(M) \).

See Theorem 2.2 and [Cr, Remarks 2]. In this way \( q \) induces an isomorphism from \( \text{Quinn}(G) \) onto \( \text{Cl}(A)_2 \) and so \( \iota \circ q \) provides an embedding of \( \text{Quinn}(G) \) into \( \text{Cl}(\tilde{A}) \).

As before each \( \omega \in E(G) \) can be represented as \( \omega = \frac{s+t}{t} \) where \( s, t \in A \) and \( t \) divides \( (\varepsilon^s + s)(\varepsilon + s) \) in \( A \). The element \( M \) acts as a Möbius transformation on \( \omega \) by multiplying the column vector \( (\varepsilon^s + s) \) on the left by the matrix \( M \). It follows that \( J_{M(\omega)} \) is the \( \tilde{A} \)-ideal generated by \( a(\varepsilon + s) + bt \) and \( c(\varepsilon + s) + dt \). Our next result, the most important in this section, shows that the action of \( \text{Quinn}(G) \) on \( \text{Ell}(G) \) is equivalent to group multiplication in \( \ker N \).

**Theorem 4.7.** With the above notation,
\[
[J_{M(\omega)}] = [\iota(q(M))J_\omega] = [\iota(q(M))]J_\omega \quad \text{(in } \ker N \text{)}.
\]

**Proof.** From the above it is clear that \( J_{M(\omega)} \leq q(M)J_\omega \). Since \( \tilde{A} \) is a Dedekind domain, there is an integral ideal \( I_1 \) of \( \tilde{A} \) such that
\[
J_{M(\omega)} = q(M)J_\omega I_1.
\]
By the same argument there exists an integral ideal \(I_2\) of \(\tilde{A}\) with
\[
J_{M_2(\omega)} = q(M)J_{M(\omega)}I_2 = q(M)^2J_\omega I_1I_2 = \Delta J_\omega I_1I_2.
\]
On the other hand, from part (i) of Corollary 2.3 we see that \(J_{M_2(\omega)} = \Delta J_\omega\). Hence \(I_1 = I_2 = \tilde{A}\) and the result follows.

An immediate consequence is the following.

**Corollary 4.8.** Quinn\((G)\) acts freely on \(\text{Ell}(G)\). More precisely, a quasi-inner automorphism that fixes an elliptic point in \(G \setminus \Omega\) must necessarily be inner.

Since
\[
G_\omega = G\overline{\omega} \iff [J_\omega] = [J_\omega]^{-1}
\]
we can identify \(\text{Ell}(G)^{-}\) with \(\iota(\text{Cl}(A)_2) \cong \text{Quinn}(G)\). Combining Lemma 4.6 and Corollary 4.8 we obtain the following result.

**Theorem 4.9.** Quinn\((G)\) acts freely and transitively on \(\text{Ell}(G)^{-}\).

Theorem 3.13 (ii), which holds for all \(\delta\), provides an alternative proof of Theorem 4.9. Applying the former for the case of odd \(\delta\) the latter then follows from the existence of a Quinn\((G)\)-invariant one-one correspondence between \(\text{Ell}(G)^{-}\) and \(\{\tilde{v} \in \text{vert}(G\setminus T) : G_v \cong GL_2(F_q)\}\).

From the above it is clear that \(|\text{Ell}(G)| = n_E|\text{Ell}(G)^{-}|\), where
\[
n_E = |\ker \overline{N} : \iota(\text{Cl}(A)_2)|.
\]
It follows that \(|\text{Ell}(G)^{\neq}| = (n_E - 1)|\text{Ell}(G)^{-}|\).

We recall that the building map \([Ge\text{, p.41}]\) restricts to a map
\[
\lambda : E(G) \to \text{vert}(T),
\]
for which \(G_\omega \leq G_{\lambda(\omega)}\). Let \(\kappa\) be a quasi-inner automorphism. Then by \([Ge\text{, (iii), p.44}]\)
\[
\lambda(\kappa(\omega)) = \kappa(\lambda(\omega)).
\]
Then \(\lambda\) induces a map
\[
\text{Ell}(G) \leftrightarrow \text{vert}T.
\]
By Lemma 4.2 (ii), Theorem 3.13 and \([MS4\text{, Proposition 3.4}]\) this leads to two Quinn\((G)\)-invariant one-one correspondences.

\[
\text{Ell}(G)^{\neq} \leftrightarrow \{\tilde{v} \in \text{vert}(G\setminus T) : G_v \cong GL_2(F_q)\},
\]
\[
G = \{G_\omega, G\overline{\omega} : G_\omega \neq G\overline{\omega}\} \leftrightarrow V = \{\tilde{v} \in \text{vert}(G\setminus T) : G_v \cong F_{q^2}^*\}.
\]

Note that \(|G| = \frac{1}{2}|\text{Ell}(G)^{\neq}|\).

**Lemma 4.10.** Let \(G_\omega \in \text{Ell}(G)^{\neq}\) and \(\kappa\) be a quasi-inner automorphism represented by \(M \in N_G(G)\). Then \(\kappa(G_\omega) = G\overline{\omega}\) if and only if \([J_\omega]\) has order 4 in \(\ker \overline{N}\) and \(\iota(q(M))] = [J_\omega]^2\).
Proof. Let \( n > 2 \) be the order of \([J_\omega]\) in \( \ker \overline{N} \). If \( \kappa(G\omega) = G\overline{\omega} \) then by Theorem 4.7

\[
[\kappa(q(M))[J_\omega] = [J_\omega]^{n-1}.
\]

Hence \([\kappa(q(M))] = [J_\omega]^{n-2} = [J_\omega]^{-2} \) and so \( n = 4 \). The converse is straightforward. □

The following is an immediate consequence.

Lemma 4.11. Let \( \overline{v} \in V \) and let \( \{G\omega, G\overline{\omega}\} \) be the corresponding elliptic element of \( \overline{v} \). Then the length of the orbit of \( \overline{v} \) under the action of Quinn(G) is \( \frac{1}{2} |\text{Quinn}(G)| \) if \([J_\omega]\) has order 4 in \( \ker \overline{N} \) and \(|\text{Quinn}(G)| \) otherwise.

Proposition 4.12. Suppose that \(|\text{Ell}(G)^\circ| < |\text{Ell}(G)|\). Then

(a) Quinn(G) acts transitively on \( \text{Ell}(G)^\circ \) if and only if \( n_E = 2 \).

(b) Quinn(G) acts transitively on \( V \) if and only if \( n_E \in \{2, 3\} \).

(c) Quinn(G) acts freely on \( V \) if and only if \( n_E \) is odd.

(d) Quinn(G) acts freely and transitively on \( V \) if and only if \( n_E = 3 \).

Proof. (a) Since Quinn(G) acts freely on \( \text{Ell}(G)^\circ \) the action is transitive if and only if \(|\text{Quinn}(G)| = |\text{Ell}(G)^\circ| = |\text{Ell}(G)^\circ| \) that is if \( n_E = 2 \).

(b) If Quinn(G) acts transitively on \( V \) then \(|G| \leq |\text{Ell}(G)^\circ| \) and so \( n_E \in \{2, 3\} \). When \( n_E = 2 \) (a) applies. When \( n_E = 3 \) the two Quinn(G)-orbits represented by \( G\omega \) and \( G\overline{\omega} \) are identified in \( G \).

(c) By Lemma 4.10 the action of Quinn(G) on \( G \) is not free if and only if there exists \([J_\omega]\) of order 4 and such an element exists if and only if \( n_E \) is even.

(d) follows from (b) and (c). □

Remark 4.13. Suppose that \( g(K) = g > 0 \). The 2-torsion rank of an abelian variety of dimension \( g \) is bounded by \( 2^g \). Applying this to \( \text{Cl}^0(\overline{K}) \) or \( \text{Cl}(\overline{A}) \) (and using the fact that \( \delta \) is odd) it follows that

\[
|\text{Ell}(G)^\circ| \leq 2^{2g}.
\]

See [Ro] Chapter 11]. On the other hand by the Riemann Hypothesis for function fields [SI] Theorems 5.1.15(e), 5.2.1]

\[
|\text{Ell}(G)| = L_K(-1) \geq (\sqrt{q} - 1)^{2g}.
\]

If \( n_E = 2 \) then

\[
2^{2g+1} \geq (\sqrt{q} - 1)^{2g}.
\]

(a) If \( q \geq 16 \) (and \( g > 0 \)), then Quinn(G) cannot act transitively on \( \text{Ell}(G)^\circ \).

Another consequence follows using an identical argument.

(b) If \( q \geq 23 \) (and \( g > 0 \)), then Quinn(G) cannot act transitively on \( V \).

Remark 4.14. It is known [MS3] Corollary 2.12, Theorem 5.1] that a vertex \( \overline{v} \) of \( G \setminus \mathcal{T} \) is isolated if and only if \( \delta = 1 \) and \( G_v \cong GL_2(\mathbb{F}_q) \) or \( \mathbb{F}_q^* \). Hence when \( \delta = 1 \) therefore Theorem 4.9, Proposition 4.12 and Remarks 4.13 can be interpreted as statements about the action of Quinn(G) on the isolated vertices of \( G \setminus \mathcal{T} \).
5. Action on cyclic subgroups

Our focus of attention in this section are the subgroups of $G$ which are cyclic of order $q^2 - 1$. As distinct from Section 3 some of the results require $\delta$ to be odd.

**Definition 5.1.** A finite subgroup $S$ of $G$ is **maximally finite** if every subgroup of $G$ which properly contains it is infinite.

**Lemma 5.2.** Let $C$ be a cyclic subgroup of $G$ of order $q^2 - 1$ which is not maximally finite. Then there exists $H \in \mathcal{H}$ which contains $C$. Moreover $H$ is unique if $\delta$ is odd.

**Proof.** By Lemma 3.1(ii) there exists $G_v$ which properly contains $C$. Hence $G_v \in \mathcal{H}$ by Lemma 3.2.

Suppose now that $\delta$ is odd. If $H$ is not unique then

$$C \leq G_{v_1} \cap G_{v_2},$$

where $v_1 \neq v_2$. It follows that $C$ fixes the geodesic in $\mathcal{T}$ joining $v_1$ and $v_2$, including all its edges. This contradicts [MS3, Corollary 2.16]. □

**Lemma 5.3.** Let $C, C_0$ be cyclic subgroups of order $q^2 - 1$ contained in some $H \in \mathcal{H}$. Then $C, C_0$ are conjugate in $H$.

**Proof.** By Lemma 3.5 we may assume that $H = GL_2(\mathbb{F}_q)$. This then becomes a well-known result. In the absence of a suitable reference we sketch a proof which lies within the context of this paper.

By the proof of [MS3, Theorem 2.6] (based on [MS3, Lemma 1.4]) it follows that

$$C = F^\mu = \{g \in GL_2(\mathbb{F}_q) : g(\mu) = \mu\},$$

for some $\mu \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Let $C_0 = F^{\mu_0}$.

Now $\mu_0 = \alpha \mu + \beta$ for some $\alpha, \beta \in \mathbb{F}_q$, where $\alpha \neq 0$. Then $C_0 = g_0 C g_0^{-1}$, where

$$g_0 = \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}.$$ □

**Definition 5.4.** Let

$$C = \{C \leq G : C, \text{cyclic of order } q^2 - 1\},$$

$$C_{mf} = \{C \in C : C, \text{maximally finite}\},$$

$$C_{nm} = C \setminus C_{mf}.$$ Clearly every automorphism of $G$ acts on both $C_{mf}$ and $C_{nm}$.

**Proposition 5.5.** The quasi-inner automorphisms act transitively on all cyclic subgroups of $G$ of order $q^2 - 1$ that are not maximally finite.
Proof. Let $C \in C_{nm}$. Then by Lemmas 3.5 and 5.2 there exists $g_0 \in N_{\tilde{G}}(G)$ such that

$$C^{g_0} \in GL_2(\mathbb{F}_q).$$

The rest follows from Lemma 5.3. □

The next result follows from Proposition 5.5 and Theorem 3.13.

**Proposition 5.6.** If $\delta$ is odd, Quinn($G$) acts freely and transitively on the conjugacy classes (in $G$) of cyclic subgroups of $G$ of order $q^2 - 1$ that are not maximally finite.

The restrictions on $\delta$ in Lemma 5.2 and Proposition 5.6 are necessary.

**Example 5.7.** Consider the case where $g(K) = 0$, $\delta = 2$. This case is studied in detail in [MS1, Section 3]. By the exact sequence in Section 2 it is known that here

$$\text{Cl}(A) = \text{Cl}(A)_2 \cong \text{Quinn}(G) \cong \mathbb{Z}/2\mathbb{Z}.$$ 

There exists a vertex $v_0$ adjacent to the standard vertex $v_s$ and $g_0 \in N_{\tilde{G}}(G) \setminus G$ such that

$$G_{v_0} = GL_2(\mathbb{F}_q)^{g_0} \text{ and } G_{v_s} \cap G_{v_0} \in C_{nm}.$$ 

Hence the restriction on $\delta$ in part of Lemma 5.2 is necessary.

It is known [MS1, Theorem 3.3] that in this case

$$G = GL_2(\mathbb{F}_q) * C GL_2(\mathbb{F}_q)^{g_0},$$

where $C(= GL_2(\mathbb{F}_q) \cap GL_2(\mathbb{F}_q)^{g_0}) \in C_{nm}$. It follows by Lemma 5.3 that there exists $g \in GL_2(\mathbb{F}_q)$ for which

$$C^g = C^{g_0}.$$ 

In this case therefore Quinn($G$), which is non-trivial, fixes $C^G$. The restriction on $\delta$ in Proposition 5.6 is therefore necessary.

We conclude this section with some remarks about $C_{mf}$.

**Lemma 5.8.** Suppose that $\delta$ is odd. Then

$$C \in C_{mf} \iff C = G_v \cong \mathbb{F}_{q^2}^*.$$ 

**Proof.** Suppose that $C = G_v \cong \mathbb{F}_{q^2}^*$ and that $C \in C_{nm}$. Then by Lemmas 3.1 and 5.3 it follows that $C \leq G_v \cap G_{v_0}$ for some $v_0 \neq v$, which contradicts [MS3, Corollary 2.16]. The rest follows from Lemma 5.1. □

When $\delta$ is odd there is therefore a one-one correspondence

$$(C_{mf})^G \longleftrightarrow \mathcal{Y}.$$ 

For the case where $\delta$ is odd this shows that the results in Proposition 4.12 apply to the action of Quinn($G$) on $(C_{mf})^G$. 

18
Remark 5.9. As a Möbius transformation every member of $G$ fixes an element of $C_\infty$. Suppose now that $\delta$ is even and that $C$ is a cyclic subgroup of order $q^2 - 1$ (maximally finite or not). Then from the proof of [MS4, Proposition 2.3] it follows that $C$ fixes $\mu \in K_{\mathbb{F}_q^2 \setminus K}$. In this case however $\mu \in K_{\infty}$ as $\delta$ is even. So $\mu$, which is not in $\Omega$ and not in $K$, can neither be an inner point nor a cusp of the Drinfeld modular curve $G\setminus \Omega$. We refer to $\mu$ as pseudo-elliptic.

On the other hand suppose that $\delta$ is odd. Let $g$ be any element of infinite order in $G$ and let $g$ fix $\lambda$. Then $\lambda \in K_{\mathbb{F}_q^2 \setminus K}$.

6. Action on cusps

As distinct from Section 4 the results here hold for all $\delta$. Any element of $\hat{G}$ acts on $\mathbb{P}^1(K) = K \cup \{\infty\}$ as a Möbius transformation. In this way Quinn($G$) acts on $G\setminus \mathbb{P}^1(K) = \text{Cusp}(G)$. Every element of $\text{Cusp}(G)$ can be represented in the form $(a : b)$, where $a, b \in A$. Since $A$ is a Dedekind ring this gives rise to a one-one correspondence

$$\text{Cusp}(G) \longleftrightarrow \text{Cl}(A).$$

Hence the action of Quinn($G$) on $\text{Cusp}(G)$ translates to an action of $\text{Cl}(A)_2$ on $\text{Cl}(A)$. The principal result in this section is similar to but simpler than Theorem 4.7 It translates this action into multiplication in the group $\text{Cl}(A)$. We sketch a proof.

We can represent any cusp, $c$, by an element $(x : y) \in \mathbb{P}^1(K)$, where $x, y \in A$. Let $J_c = xA + yA$,

and let $[J_c]$ be its image in $\text{Cl}(A)$.

Now let $\kappa$ be a non-trivial element of Quinn($G$). Then as before by Theorem 2.2 $\kappa$ can be represented by a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \hat{G},$$

where we may assume that $a, b, c, d \in A$. Let $q(M)$ be the $A$-ideal generated by $a, b, c, d$.

The action of $\kappa$ on $c$ is given by the action of $M$ multiplying the column vector $(x)$ on the left by $M$. In this way

$$J_{\kappa(c)} = J_{M(c)} = (ax + by)A + (cx + dy)A.$$

Theorem 6.1. Under the identification of $\text{Cusp}(G)$ with $\text{Cl}(A)$ and Quinn($G$) with $\text{Cl}(A)_2$ the action of Quinn($G$) on the cusps translates into multiplication in the group $\text{Cl}(A)$. More precisely

$$[J_{\kappa(c)}] = [q(M)J_c] = [q(M)][J_c] \ (\text{in } \text{Cl}(A)),$$

Proof. Since $A$ is a Dedekind domain there exists an $A$-ideal $I_1$ such that

$$J_{M(c)} = q(M)J_cI_1.$$
By Corollary 2.3 (i) there exists an $A$-ideal $I_2$ with
\[ \Delta J_c = J_{MP(c)} = q(M)J_{M(c)}I_2 = q(M)^2 J_c I_1 I_2 = \Delta J_c I_1 I_2, \]
where $\Delta = \det(M)$. Hence $I_1 = I_2 = A$ and the result follows. \qed

As in the previous section we have the following immediate consequence.

**Corollary 6.2.** If a non-trivial quasi-inner automorphism $\kappa$ fixes any cusp, then $\kappa$ reduces to an inner automorphism. In particular, Quinn$(G)$ acts freely on Cusp$(G)$.

**Remark 6.3.** Quinn$(G)$ acts transitively on Cusp$(G)$ if and only if Cl$(A)_2 = \text{Cl}(A)$. From the exact sequence in Section 2 a necessary condition for this is $\delta \in \{1, 2\}$. If $g(K) = 0$, this condition is also sufficient, as then Cl$(A) \cong \mathbb{Z}/\delta\mathbb{Z}$.

But if $g(K) = g > 0$, the action cannot be transitive for $q > 9$ by an argument very similar to that used in Remark 4.11. The inequality
\[ \frac{|\text{Cl}^0(K)|}{|\text{Cl}^0(K)_2|} \geq \frac{(\sqrt{q} - 1)^{2g}}{2^{2g}} \]
shows that for fixed $q > 9$ the number of orbits of Quinn$(G)$ on Cusp$(G)$ tends to $\infty$ with $g(K)$.

The cusp $\infty(= \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ corresponds to the principal $A$-ideals. Its orbit under Quinn$(G)$ corresponds to the 2-torsion in Cl$(A)$ and in the sense of Theorem 6.1 the action of Quinn$(G)$ on it translates into Cl$(A)_2$ acting on itself by multiplication.

For every cusp $c$ represented by the ideal class $[J_c]$ in Cl$(A)$ there corresponds its (group) inverse $[J_c]^{-1}$ in Cl$(A)$. We can partition Cl$(A)$ thus
\[ \text{Quinn}(G) \leftrightarrow \text{Cl}(A)_2 = \{ [J_c] : [J_c] = [J_c]^{-1} \}, \]
\[ \text{Cl}(A) \setminus \text{Cl}(A)_2 = \{ [J_c] : [J_c] \neq [J_c]^{-1} \}. \]

Our next result follows from Theorem 6.1 analogous to the way Lemma 4.10 follows from Theorem 4.7.

**Lemma 6.4.** A quasi-inner automorphism $\kappa$, represented by $M \in \hat{N}_G(G)$, maps the cusp $c$ corresponding to $[J_c]$ in Cl$(A) \setminus \text{Cl}(A)_2$, to the cusp corresponding to $[J_c]^{-1}$ if and only if $[J_c]$ has order 4 and $[J_c]^2 = q(M)$.

In the next section we will use the results in Sections 5 and 6, together with Theorem 3.13 (ii), to examine in detail the action of Quinn$(G)$ on $G \setminus \mathcal{T}$.

7. **Action on the Quotient Graph**

The model used by Serre for $\mathcal{T}$ [Se Chapter II, Section 1.1] is based on two-dimensional so called lattice classes. Since every quasi-inner automorphism, $t_g$, can be represented by a matrix in $\hat{G}$ it acts on $\mathcal{T}$ and hence Quinn$(G)$ acts on $G \setminus \mathcal{T}$.

In this section we investigate the action of a quasi-inner automorphism on the quotient graph $H \setminus \mathcal{T}$, where $H$ is a finite index subgroup of $G$. In the process we extend
a result of Serre [Se, Exercise 2(e), p.117] which motivated our interest in this question. We begin with a detailed account of Serre’s classical description of \(G/T\). Serre’s original proof [Se, Theorem 9, p.106] is based on the theory of vector bundles. For a more detailed version which refers explicitly to matrices see [Ma1]. In addition we use the results the previous sections to shed new light on the structure of \(G/T\).

**Definition 7.1.** A **ray** \(R\) in a graph \(G\) is an infinite half-line, without backtracking. In accordance with Serre’s terminology [Se, p.104] we call \(R\) cuspidal if all its non-terminal vertices have valency 2 (in \(G\)).

Let \(\{g_1, \cdots, g_s\} \subseteq \hat{G}\), where \(s \geq 1\), be a complete system of representatives for \(\text{Cl}(A)_2(\sim = N_{\hat{G}}(G)/G.Z(K))\). Let \(c_i = g_i(\infty)\) (1 \(\leq i\) \(\leq s\). We will assume that \(c_1 = \infty\). If \(\text{Cl}(A) = \text{Cl}(A)_2\), then \(\{c_1, \cdots, c_s\}\) is a complete system of representatives for \(\text{Cl}(A)\). If \(\text{Cl}(A) \neq \text{Cl}(A)_2\) we can find further elements \(h_1, \cdots, h_t \in \hat{G}\), where \(t \geq 1\) so that

\[S = \{c_1, \cdots, c_s, d_1, \cdots, d_t\}\]

is a complete set of representatives for \(\text{Cl}(A)\), where \(d_j = h_j(\infty)\) (1 \(\leq j\) \(\leq t\).

**Theorem 7.2.** There exists a complete system of representatives \(C(\subseteq \mathbb{P}^1(K))\) for Cusp(\(G\)) (equivalently, Cl(\(A\))) of the above type such that

\[G/T = X \cup \left( \bigcup_{1 \leq i \leq s} R(c_i) \right) \left( \bigcup_{1 \leq j \leq t} R(d_j) \right),\]

where

(i) \(X\) is finite,

(ii) each \(R(c_i), R(d_j)\) is a cuspidal ray (in \(G/T\)), whose only intersection with \(X\) consists of a single vertex,

(iii) the \(|\text{Cl}(A)|\) cuspidal rays are pairwise disjoint.

Moreover if \(R(e)\) is any of these cuspidal rays then it has a lift, \(\overline{R}(e)\), to \(T\) with the following properties. Let \(\text{vert}(\overline{R}(c)) = \{v_1, v_2 \cdots\}\). Then

(i) \(G_{v_i} \leq G_{v_{i+1}},\) (\(i \geq 1\),

(ii) \(\bigcup_{i \geq 1} G_{v_i} = G(c),\)

where \(G(c)\) is the stabilizer (in \(G\)) of the cusp \(c\).

For each \(j\) let \(\tilde{d}_j\) be the element of \(\{d_1, \cdots, d_t\}\) corresponding to \(h_j^{-1}(\infty)\). We may relabel the latter set as \(\{d_1, \tilde{d}_1, \cdots, d_{t'}, \tilde{d}_{t'}\}\), where \(t' = t/2\). We can use the results in Section 3 to elaborate on the structure of the above cuspidal rays. We recall that

\(\mathcal{H} = \{H \leq G : H \cong GL_2(\mathbb{F}_q)\}\).
Corollary 7.3. For the above set of $|\text{Cl}(A)|$ cuspidal rays

(i) $\mathcal{R}_1 = \{\mathcal{R}(c_1), \ldots, \mathcal{R}(c_s)\} \leftrightarrow \{\tilde{v} \in \text{vert}(G\backslash \mathcal{T}) : G_v \in \mathcal{H}\} \leftrightarrow \text{Cl}(A)_2$.

(ii) $\mathcal{R}_2 = \{\mathcal{R}(d_j), \mathcal{R}(\tilde{d}_j) : 1 \leq j \leq t^\prime\} \leftrightarrow \text{Cl}(A) \setminus \text{Cl}(A)_2$.

Proof. Let $\tilde{v} \in \text{vert}(G\backslash \mathcal{T})$, where $G_v \in \mathcal{H}$, and let $H \in \mathcal{H}$ be any representative of its stabilizer. Then, for some unique $i$,

$$H = gg_i(GL_2(\mathbb{F}_q))(gg_i)^{-1},$$

where $g \in G$, by Lemmas 3.5 and 3.6. Now let $u$ be any unipotent element of $H$. Then $u$ fixes $gg_ih(\infty)$, for some $h \in GL_2(\mathbb{F}_q)$. It follows that $u \in G(c) \iff c = g'c_i$, where $g' \in G$. The rest follows from Corollary 3.9 together with Theorem 3.13. □

Remark 7.4. Let $\tilde{v} \in \text{vert}(G\backslash \mathcal{T})$, where $G_v \in \mathcal{H}$. Then it is shown in Corollary 7.3 that $\tilde{v}$ is adjacent in $G\backslash \mathcal{T}$ to a vertex whose stabilizer (up to conjugacy in $G$) is contained in $G(c_i)$, for some unique $i$. In this way $\tilde{v}$ can be thought of as closer in $G\backslash \mathcal{T}$ to $\mathcal{R}(c_i)$ than to any other cuspidal ray. For the case $\delta = 1$ (and only for this case) $\tilde{v}$ is isolated in $G\backslash \mathcal{T}$ by [MS3] Theorem 5.1. As in Takahashi's example [Ta] such a $\tilde{v}$ then appears as a "spike" next to its associated cuspidal ray.

For each subgroup $H$ of $G$ we recall that the elements of $H\backslash \mathcal{T}$ are

$$\text{vert}(H\backslash \mathcal{T}) = \{Hv : v \in \text{vert}(\mathcal{T})\} \quad \text{and} \quad \text{edge}(H\backslash \mathcal{T}) = \{He : e \in \text{edge}(\mathcal{T})\}.$$

Definition 7.5. Let $H, H^\ast$ be isomorphic subgroups of $G$. An isomorphism of graphs

$$\phi : H\backslash \mathcal{T} \to H^\ast\backslash \mathcal{T},$$

is said to be stabilizer invariant if the following condition holds.

For any $w \in \text{vert}(\mathcal{T}) \cup \text{edge}(\mathcal{T})$ let

$$\phi(Hw) = H^\ast w^\ast,$$

(where $w^\ast \in \text{vert}(\mathcal{T})$ if and only if $w \in \text{vert}(\mathcal{T})$). Then, for all $u \in H_w$ and $u^\ast \in H^\ast w^\ast$

$$H_u \cong H^\ast_{u^\ast}.$$ As we shall see it is easy to find examples of isomorphisms of quotient graphs which are not stabilizer invariant.

Theorem 7.6. Let $\kappa = \iota_g$, where $g \in N_G(G)$ and let $H$ be a subgroup of $G$. Then the map

$$\pi_H : H\backslash \mathcal{T} \to \kappa(H)\backslash \mathcal{T},$$

defined by

$$\pi_H(Hw) = H'w',$$
where $H' = H^g = gHg^{-1}$, $w' = g(w)$ and $w \in \text{vert}(\mathcal{T}) \cup \text{edge}(\mathcal{T})$, defines a stabilizer invariant isomorphism of the quotient graphs

$$\kappa(H) \backslash \mathcal{T} \cong H \backslash \mathcal{T}.$$  

**Proof.** Note that $\kappa_H$ is well defined since if $\kappa(x) = g_1 x g_1^{-1}$, where $g_1 \in N_G(G)$ then $gg_1^{-1} \in Z(K)$ and $Z_\infty$, the set of scalar matrices in $GL_2(K_\infty)$, stabilizes every $w$. The rest is obvious (since $g$ acts on $\mathcal{T}$). $\Box$

Let $H$ be any finite index subgroup of $G$ and let $M$ be the largest normal subgroup of $G$ contained in $H$. Then $N = M \cap M^g$ is the largest (finite index) subgroup of $G$, contained in $H$, which is normalized by $G$, $Z(K)$ and $g$. (See Section 2.)

**Corollary 7.7.** Suppose that $\kappa$ is non-trivial (i.e. $g \notin G.Z(K)$). Let $N$ be a finite index normal subgroup of $G$ normalized by $\kappa$. Then the map

$$\kappa_N : N \backslash \mathcal{T} \to N \backslash \mathcal{T},$$

defined as above, is a non-trivial stabilizer invariant automorphism whose order $n$ is even. Moreover, if $Z \leq N$, then $n = 2m$, where $m$ divides $|G : N|$. $\Box$

**Proof.** To prove that $\kappa_N$ is non-trivial it suffices to prove that $\kappa_G$ is not the identity map. There exists $v_0 \in \text{vert}(\mathcal{T})$ for which (non-central) $Gv_0 \leq G(\infty)$ [Ma1, Lemma 3.2]. Suppose to the contrary that $\kappa_G$ fixes $Gv_0$. Then there exists $g_0 \in G$ such that $g' = gg_0 \in G(\infty)$ which implies that

$$g' = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$ 

We may assume that $a, b, c \in A$. By Theorem 2.4 together with an argument used in the proof of Theorem 2.1 it follows that

$$a^2A = c^2A = acA.$$ 

Hence $a, c \in \mathbb{F}_q$. Thus $g' \in G$ and so $g \in G.Z(K)$. For the second part $n$ is the smallest $n(> 0)$ such that $g^n \in N.Z(K)$. Now $g^2 \in G.Z(K)$ by Corollary 2.3(i). If $n$ is odd then $g \in G.Z(K)$. Hence $n = 2m$ is even. In addition when $Z \leq N$ $m$ divides $|G.Z(K) : N.Z(K)| = |G : N|$. $\Box$

A special case of Corollary 7.7 combined with Corollary 2.4 is the following.

**Corollary 7.8.** Suppose that $|\text{Cl}(A)| = |\text{Cusp}(G)|$ is even. Then there exists a stabilizer invariant automorphism of $G \backslash \mathcal{T}$ of order 2.

Serre [Sc] Exercise 2(e), p.117] states this result for the case $g(K) = 0$ (i.e. $K = \mathbb{F}_q(t)$) and $\delta$ even. The restriction here is necessary. For the case $g(K) = 0$, $\delta = 1$, in which case $A = \mathbb{F}_q[t]$ and $|\text{Cl}(A)| = 1$, it is known by Nagao’s Theorem [Sc Corollary, p.87] that $G \backslash \mathcal{T}$ is a cuspidal ray whose terminal vertex is isolated. Here then the only (graph) automorphism is trivial.

Corollary 7.8 shows that Quinn($G$) acts non-trivially on $G \backslash \mathcal{T}$. This extends to an action on its cuspidal rays which we now describe. We use the notation of Theorem 7.2.
Definition 7.9. Let $\mathcal{R}_1, \mathcal{R}_2$ be rays in a graph $\mathcal{G}$. We write  
\[ \mathcal{R}_1 \sim \mathcal{R}_2 \]
if and only if $|\mathcal{R}_i \setminus \mathcal{R}_1 \cap \mathcal{R}_2| < \infty \ (i = 1, 2)$. This a well-known equivalence relation. The equivalence class containing the ray $\mathcal{R}$ is called the end (of $\mathcal{G}$) determined by $\mathcal{R}$. In the notation of Theorem 7.2 we denote by $\mathcal{E}(e)$ the end (in $G \setminus T$) determined by $\mathcal{R}(e)$, where $e = c_i, d_j$.

Now let $\kappa = \iota_g$, where $g \in N_G(G \setminus Z(K))$, be a non-trivial quasi-inner automorphism and let $\hat{\kappa}$ be the corresponding (non-trivial) element of $Quinn(G)$. Now fix $e \in S$. Let $e^* = \kappa(e)$. Then by Corollary 6.2 $e \neq e^*$ and we may assume that $e^* \in S$. As in Theorem 7.2
\[ \text{vert}(\mathcal{R}(e)) = \{\tilde{v}_1, \tilde{v}_2, \cdots\} \quad \text{and} \quad \text{vert}(\mathcal{R}(e^*)) = \{\tilde{v}_1^*, \tilde{v}_2^*, \cdots\} . \]
Recall that
\[ \bigcup_{i \geq 1} G_{v_i} = G(e), \]
and that $G_{v_i} \leq G_{v_{i+1}} \ (i \geq 1)$. In addition it is known [MS3, Theorem 2.1 (a)] that there exists a normal subgroup $N_i$ of $G_{v_i}$ such that
\[ G_{v_i}/N_i \cong \mathbb{F}_q^* \times \mathbb{F}_q^* , \]
where $N_i \cong V^+_i$, the additive group of a $\mathbb{F}_q$-vector space of dimension $n_i$. It is also known that $n_i < n_{i+1}$. Corresponding results hold for $\mathcal{R}(e^*)$. Now let
\[ m_X = \max\{|G_v| : v \in \text{vert}(X)\} . \]
(Recall that $X$ is finite.) Now choose any $m > m_X$. By the definition of graph automorphism $\overline{\kappa}_G$ (determined by the non-trivial element $\hat{\kappa}$ of $Quinn(G)$, together with Theorem 7.2) there exists $n > m_X$ such that
\[ \overline{\kappa}_G : \tilde{v}_{m+i} \mapsto \tilde{v}_{n+i}^* , \]
for all $i \geq 0$. This gives rise to a map
\[ \hat{\kappa} : \mathcal{E}(e) \mapsto \mathcal{E}(e^*) , \]
which in turn defines a $Quinn(G)$-action on the ends defined by the cuspidal rays in $G \setminus T$. (Theorem 7.2) Since this action coincides precisely with the action of $Quinn(G)$ on Cusp$(G)$ the following result is an immediate consequence of Theorem 3.13(ii), Corollary 6.2 and Lemma 6.4.

**Corollary 7.10.** With the notation of Theorem 7.2

(i) $Quinn(G)$ acts (simultaneously) freely and transitively on
\[ \{\mathcal{E}(c_1), \cdots , \mathcal{E}(c_s)\} \quad \text{and} \quad \{\tilde{v} \in \text{vert}(G \setminus T) : G_v \cong GL_2(\mathbb{F}_q)\} , \]

(ii) $Quinn(G)$ acts freely on
\[ \left\{ \mathcal{E}(d_j), \mathcal{E}(\hat{d}_j) : 1 \leq j \leq t' \right\} , \]
(iii) Quinn($G$) acts on
$$\left\{ \{E(d_j), E(\tilde{d}_j)\} : 1 \leq j \leq t' \right\}.$$  

(iv) Under the action of Quinn($G$) some $E(d_j)$ is mapped to $E(\tilde{d}_j)$ if and only if $d_j$ has order 4 in $\text{Cl}(A)$.

We recall from Proposition 4.12 that when $\delta$ is odd Quinn($G$) also acts on $\{\tilde{v} \in \text{vert}(G \setminus T) : G_v \cong \mathbb{F}_{q^2}^*\}$.

Our final result in this section concerns the action of $N_{\hat{G}}(G)$ on $T$. It is known [Se, Corollary, p.75] that $G$ acts without inversion (on the edges) of $T$. We now state this result:

**Proposition 7.11.** Suppose that $\delta$ is odd. Then every $\iota_g$ acts without inversion on $T$ and hence on every quotient graph $H \setminus T$.

**Proof.** As in Theorem 2.2 we can represent $\iota_g$ with a matrix $M$ in $\hat{G}$ and we can assume that all its entries lie in $A$. Let $\Delta = \det(M)$. Then the $A$-ideal generated by $\Delta$ is the square of an ideal in $A$, again by Theorem 2.2. It follows that, for all places $v \neq v_{\infty}$, $v(\Delta)$ is even.

By the product formula then $\delta v_{\infty}(\Delta)$ and hence $v_{\infty}(\Delta)$ is even. The result follows from [Se, Corollary, p.75].

**Example 7.12.** To conclude this section we consider the case where $g(K) = 0$ and $\delta(K) = 2$. We recall that there exists a quadratic polynomial $\pi \in \mathbb{F}_q[t]$, irreducible over $\mathbb{F}_q$, such that

$$A = \left\{ \frac{f}{\pi^m} : f \in \mathbb{F}_q[t], m \geq 0, \deg f \leq 2m \right\}.$$ 

In this case it is known that $\text{Cl}(A)_2 = \text{Cl}(A) \cong \text{Quinn}(G) \cong \mathbb{Z}/2\mathbb{Z}$. It is well-known that $G \setminus T$ is a doubly infinite line, without backtracking. See [Se, p.113] and, for a more detailed description, [MS1, Section 3]. It is known that $G \setminus T$ lifts to a doubly infinite line $D$ in $T$ which we now describe in detail. For some $g_0 \in N_{\hat{G}}(G) \setminus G.Z(K)$, $\text{vert}(D) = \{v_0, v_0^*, v_1, v_1^*, \ldots\}$, where

(i) $v_i^* = g_0(v_i)$ $(i \geq 0)$,
(ii) $G_{v_i^*} = (G_{v_i})^{g_0}$ $(i \geq 0))$,
(iii) $G_{v_0} = GL_2(\mathbb{F}_q)$,
(iv) for each $i \geq 1$

$$G_{v_i} = \left\{ \begin{bmatrix} \alpha & ce^{\pi^{-i}} \\
0 & \beta \end{bmatrix} : \alpha, \beta \in \mathbb{F}_q^*, \deg c \leq 2i \right\}.$$ 

Then $D$ maps onto (and is isomorphic to) $G \setminus T$ which has the following structure.

\[ \cdots \circ v_2 \circ v_1 \circ v_0 \circ v_0^* \circ v_1^* \circ v_2 \cdots \]
The action of the (essentially only) non-trivial quasi-inner automorphism of $G \setminus \mathcal{T}$ (represented by $g_0$) is given by

$$v_i \leftrightarrow v_i^* \quad (i \geq 0).$$

We note two features of $D$ which are of interest relevant to this section.

(i) From the structure of $D$ it is clear that the non-trivial quasi-inner automorphism determined by $g_0$ inverts the edge joining $v_0$ and $v_0^*$, which shows that the restriction on $\delta$ in Proposition 7.11 is necessary.

(ii) For this case there is only one stabilizer invariant involution. However the graph $G \setminus \mathcal{T}$ has many automorphisms. Infinitely many examples include translations (which have infinite order) and reflections in any vertex (which are involutions).

8. Two instructive examples

We conclude with two examples which demonstrate how our results apply to the structure of the quotient graph $G \setminus \mathcal{T}$. Both are elliptic function fields $K/F_q$. We record some of their basic properties.

**Definition 8.1.** A function field $K/F_q$ is elliptic [St, p.217] if $g(K) = 1$ and $K$ has a place $\infty$ of degree 1.

**Theorem 8.2.** Suppose that $K/F_q$ is elliptic. Then

(i) $$K = F_q(x, y),$$

where $x, y$ satisfy a (smooth) Weierstrass equation $F(x, y) = 0$ with

$$F(x, y) = y^2 + a_1 xy + a_3 y - x^3 - a_2 x^2 - a_4 x - a_6 \in F_q[x, y].$$

(ii) $\text{Cl}^0(K)(\cong \text{Cl}(A))$ is isomorphic to $E(F_q)$, the group of $F_q$-rational points,

$$\{(\alpha, \beta) \in F_q \times F_q : F(\alpha, \beta) = 0\} \cup \{(-\infty, \infty)\}.$$ Here the group operation is point addition $\oplus$ according to the chord-tangent law.

**Proof.** For (i) see [St, Proposition 6.1.2]. For (ii) see [St, Propositions 6.1.6, 6.1.7].

Here a rational point $(a, b) \in E(F_q)$ corresponds to the ideal class of $A(x-a) + A(y-b)$. We also require some "elliptic" properties of $\bar{K} = K.F_q^2$ (which is a constant field extension of $K$).

**Corollary 8.3.** Suppose that $K/F_q$ is elliptic. Then $\bar{K}/F_q^2$ is also elliptic and defined by the same Weierstrass equation.

**Proof.** From the above $\bar{K} = F_q^2(x, y)$, where $F(x, y) = 0$. The rest follows from [St, Proposition 6.1.3].

With our choice of infinite place we have

$$A = F_q[x, y] \quad \text{and} \quad \bar{A} = F_q^2[x, y].$$
where $x$ and $y$ satisfy the Weierstrass equation $F(x, y) = 0$. In an analogous way
\[ \text{Cl}(\tilde{A}) \cong \text{Cl}^0(\tilde{K}) \cong E(\mathbb{F}_{\varphi}). \]
We recall that the image of any $\alpha \in \mathbb{F}_{q^2}$ under the Galois automorphism of $\mathbb{F}_{q^2}/\mathbb{F}_q$ is denoted by $\overline{\alpha}$. For each rational point $P = (\alpha, \beta) \in E(\mathbb{F}_{q^2})$ we put $\overline{P} = (\overline{\alpha}, \overline{\beta})$.

**Corollary 8.4.** Suppose that $K/\mathbb{F}_q$ is elliptic. Under the identifications of $\text{Cl}^0(\overline{K})$ (resp. $\text{Cl}^0(K)$) with $E(\mathbb{F}_{q^2})$ (resp. $E(\mathbb{F}_q)$) the norm map $N : \text{Cl}^0(\overline{K}) \to \text{Cl}^0(K)$ translates to a map $N_E : E(\mathbb{F}_{q^2}) \to E(\mathbb{F}_q)$ defined by
\[ N_E(P) = P \oplus \overline{P}, \]
so that
\[ P \in \ker N_E \iff \overline{P} = -P. \]

Takahashi [Ta] has described in detail the quotient graph for an elliptic function field over any field of constants. In all cases $G\setminus \mathcal{T}$ is a tree. Since $\delta = 1$, for the case of a finite field of constants, the isolated vertices of $G\setminus \mathcal{T}$ are precisely those whose stabilizer is isomorphic to $GL_2(\mathbb{F}_q)$ or $\mathbb{F}_{q^2}^*$ by [MS3, Theorem 5.1]. For each cuspidal ray $\mathcal{R}(c)$ in $G\setminus \mathcal{T}$ has attached to its terminal vertex (appearing as a “spike”) an isolated vertex with stabilizer isomorphic to $GL_2(\mathbb{F}_q)$. The remaining cuspidal rays consist of $\frac{1}{2}|\text{Cl}(A)\setminus \text{Cl}(A)_2|$ inverse pairs $\{\mathcal{R}(c), \mathcal{R}(c')\}$ which share a terminal vertex (appearing in $G\setminus \mathcal{T}$ as the “prongs” of a “fork”).

In both our examples $q = 7$ in which case the Weierstrass equation can be assumed to take the short form
\[ y^2 = f(x) = x^3 + ax + b, \]
where $a, b \in \mathbb{F}_q$ and $f(x)$ has no repeated roots.

**Example 8.5.** $K = \mathbb{F}_7(x, y)$, $A = \mathbb{F}_7[x, y]$ with $y^2 = x^3 - 3x$

It can be easily shown that $E(\mathbb{F}_7) = \{(\infty, \infty), (0, 0), (2, 3), (3, 2), (6, -3)\}$. Since $E$ is in short Weierstrass form the 8 points are listed as (additive) inverse pairs. In particular $(0, 0)$ is the only such 2-torsion point. It follows that $\text{Quinn}(G) \cong \text{Cl}(A)_2 \cong \mathbb{Z}/2\mathbb{Z}$ and hence that $\text{Cl}(A) \cong \mathbb{Z}/8\mathbb{Z}$. Let $\kappa$ be a non-trivial quasi-inner automorphism of $G$ representing the non-trivial element of $\text{Quinn}(G)$. In $E(\mathbb{F}_7)$ $\kappa$ is represented by $(0, 0)$ and, by Theorem 6.1 its action on $\text{Cusp}(G)$ is determined by its action (via point addition $\oplus$) in $E(\mathbb{F}_7)$. In a diagram of $G\setminus \mathcal{T}$ as described in [Ta] we wish to ensure that its involution provided by $\kappa$, Corollary 7.8, is given by the reflection in the vertical axis. We begin by labelling appropriately its 8 cuspidal rays (corresponding to $E(\mathbb{F}_7)$). By Corollary 6.2 $\kappa$ acts freely on these. By Corollary 7.3 (i) it is clear that $\kappa$ interchanges the cusps $(\infty, \infty)$ and $(0, 0)$. Attached to each of these is a “spike” consisting of an isolated vertex whose stabilizer is isomorphic to $GL_2(\mathbb{F}_7)$. Since $\kappa$ is a graph automorphism it interchanges these vertices, namely, $g_1$ and $g_2$. By means of the duplication formula [S], p.53 it is easily checked that the rational 4-torsion points are $(2, 3)$ and $(2, -3)$ by Lemma 6.4. For
the remaining cusps $\kappa$ interchanges $(3, \pm 2)$ and $(6, \pm 3)$. To make this more precise we use the addition formulae [Si, p.53] which show that $(0, 0) \oplus (3, 2) = (6, 3)$. Hence $\kappa$ interchanges $(3, 2)$ and $(6, 3)$ by Theorem 6.1.

There remain the isolated vertices 1, 4 and 5 each of whose stabilizer is isomorphic to $\mathbb{F}_{49}$. We deal with these via their connection with elliptic points. We recall from Theorem 4.4 and the above that there exists a one-one correspondence

$$\text{Ell}(G) \leftrightarrow \ker N_E = \{(\alpha, \beta) \in E(\mathbb{F}_{49}) : (\bar{\alpha}, \bar{\beta}) = (\alpha, -\beta)\};$$

since the Weierstrass equation is in short form. Now let $i$ denote one of the 2 square roots of $-1$ in $\mathbb{F}_{q^2}$. Then

$$N_E = \{(\rho, \epsilon i) \in E(\mathbb{F}_{49}) : \rho, \epsilon \in \mathbb{F}_q\}.$$

We conclude then that

$$\text{Ell}(G) \leftrightarrow \{(\infty, \infty), (0, 0), (1, \pm 3i), (4, \pm 2i), (5, \pm 3i)\}.$$

Here Ell$(G)$ is identified with a subgroup of $E(\mathbb{F}_{49})$ listed as (additive) inverse pairs. Since there is only one 2-torsion point $\text{Ell}(G) \cong \mathbb{Z}/8\mathbb{Z}$. (In this case $|\text{Cl}(A)| = |\text{Ell}(G)|$. However this not a general feature. For this particular $K$ its $L$-polynomial is $L_K(t) = 1 + 7t^2$ so that $L_K(1) = L_K(-1).$) As with Cusp$(G)$ the free action (Corollary 4.8) of Quinn$(G)$ on Ell$(G)$ is represented by the action of $(0, 0)$ in $N_E$ (by point addition).

By identifications in Section 4 the pairs $(1, \pm 3i), (4, \pm 2i), (5, \pm 3i)$ correspond with the vertices 1, 4 and 5, respectively. By means of the duplication formula it is readily verified that the two points of order 4 in Ell$(G)$ are $5 \pm 3i$. By Lemma 4.10 it follows that $\kappa$ fixes vertex 5 and that $\kappa$ interchanges vertices 1, 4. For a more precise version of the latter statement we note that $(0, 0) \oplus (1, 3i) = (4, 2i)$ and so $(0, 0) \oplus (1, -3i) = (4, -2i)$.

It is of interest to use Theorem 2.2 to construct a matrix $M$ which represents $\kappa$. We begin with the $A$-ideal, $Ax + Ay$ whose square is $Ax$. In determining a possible $M$ we recall from the proof of Theorem 2.2 the observation of Cremona [Cr] that every row and column of $M$ generates $q(M)$. Two possibilities which arise are

$$M = \begin{bmatrix} y & x^2 \\ x & y \end{bmatrix} \text{ and } \begin{bmatrix} y & -x^2 \\ x & -y \end{bmatrix}.$$  

The latter is simpler since its square is a scalar matrix.
$\mathbb{R}^2$
Example 8.6. \( K = \mathbb{F}_7(x, y) \), \( A = \mathbb{F}_7[x, y] \) with \( y^2 = x^3 - x \)

It is easily verified that

\[
E(\mathbb{F}_7) = \{ (\infty, \infty), (0, 0), (1, 0), (6, 0), (4, \pm 2), (5, \pm 1) \},
\]

listed as (additive) inverse pairs. The 2-torsion points are \((0, 0), (1, 0), (6, 0)\) and so \(\text{Quinn}(G) \cong \text{Cl}(A)_2 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) and \(\text{Cusp}(G) \cong \text{Cl}(A) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}\).

Let the non-trivial quasi-inner automorphisms \(\kappa_0, \kappa_1, \kappa_6\) represent \((0, 0), (1, 0), (6, 0)\), respectively, where \(\kappa_0 = \kappa_1\kappa_6\). In the diagram representing \(G \setminus T\) we label the 8 cusps with the above rational points in such a way that (i) the action of \(\kappa_0\) is the reflection about the vertical axis (ii) the action of \(\kappa_1\) is reflection about the horizontal axis and (iii) (consequently) the action of \(\kappa_6\) is a rotation of 180 degrees about the “central” vertex \(c\).

There are 4 vertices whose stabilizers are isomorphic to \(GL_2(\mathbb{F}_7)\) which appear as “spikes” attached to the 4 cusps given by the 2-torsion points in \(E(\mathbb{F}_7)\) and so \(\kappa_6, \kappa_1\) and \(\kappa_0\) interchange the vertex pairs \(\{ g_1, g_2 \}, \{ g_1, g_4 \}\) and \(\{ g_1, g_3 \}\), respectively.

In \(\text{Cl}(A)\) there are 4 points of order 4, namely \((4, \pm 2)\) and \((5, \pm 1)\), and it is easily verified that the square of each is \((1, 0)\). By Lemma 6.4 it follows that \(\kappa_1\) interchanges the cusps \((4, 2), (4, -2)\) as well as \((5, 1), (5, -1)\). On the other hand \(\kappa_6\) interchanges the pairs \((4, \pm 2)\) and \((5, \pm 1)\). In more detail \(\kappa_6\) maps \((5, 1)\) to \((4, -2)\), since \((6, 0) \oplus (5, 1) = (4, -2)\).

There remain two vertices 2 and 3 whose stabilizers are cyclic order \(q^2 - 1\). As in the previous example we consider the elliptic function field \(\bar{K} = K.F_{49} = F_{49}(x, y) : y^2 = x^3 - x\). As before let \(i\) denote one of the square roots of \(-1\) in \(F_{49}\). It can be verified that

\[
\text{Ell}(G) \leftrightarrow N_E = \{ (\infty, \infty), ((0, 0), ((1, 0), (6, 0), (2, \pm i), (3, \pm 2i)) \},
\]

listed as additive inverse pairs in \(E(\mathbb{F}_{49})\). As before \(|\text{Cl}(A)| = |\text{Ell}(G)| = 8\). (Again this is purely coincidental because \(L_K(t) = 1 + 7t^2\).) By correspondences discussed in Section 4 the 2 vertices of interest here correspond to the pairs \((2, \pm i)\) and \((3, \pm 2i)\).

It is easily verified that the squares of all 4 of these points are \((6, 0)\). It follows from Lemma 1.10 that \(\kappa_6\) fixes 2 and 3. On the other hand \((1, 0) \oplus (2, i) = (3, -2i)\) and so \(\kappa_1\) interchanges 2 and 3. Finally using Theorem 2.2 the following matrices \(M_0, M_1, M_6 = M_0M_1\) represent \(\kappa_0, \kappa_1, \kappa_6\), respectively,

\[
M_0 = \begin{bmatrix} y & -x^2 \\ x & -y \end{bmatrix} \quad \text{and} \quad M_1 = \begin{bmatrix} y & -(x - 1)(x + 2) \\ x - 1 & -y \end{bmatrix}.
\]
References

[Cr] J. E. Cremona: On GL(n) of a Dedekind domain, Quart. J. Math. Oxford Ser. (2) 39 (1988), 423-426
[Dr] V. G. Drinfeld: Elliptic Modules, Math. USSR-Sbornik 23 (1976), 561-592
[Ge] E.-U. Gekeler: Drinfeld Modular Curves, Springer LNM 1231, Berlin Heidelberg New York, 1986
[KMS] R. Köhl, B. Mühlherr and K. Struyve: Quotients of trees for arithmetic subgroups of PGL_2 over a rational function field, J. Group Theory 18 (2015), 61-74
[Ma1] A. W. Mason: Serre’s generalization of Nagao’s theorem: an elementary approach, Trans. Amer. Math. Soc. 353 (2001), 749-767
[Ma2] A. W. Mason: The generalization of Nagao’s theorem to other subrings of the rational function field, Comm. Algebra 31 (2003), 5199-5242
[MS1] A. W. Mason and A. Schweizer: The minimum index of a non-congruence subgroup of SL_2 over an arithmetic domain II: the rank zero cases, J. London Math. Soc. (2) 71 (2005), 53-68
[MS2] A. W. Mason and A. Schweizer: Non-standard automorphisms and non-congruence subgroups of SL_2 over Dedekind domains contained in function fields, J. Pure Appl. Algebra 205 (2006), 189-209
[MS3] A. W. Mason and A. Schweizer: The stabilizers in a Drinfeld modular group of the vertices of its Bruhat-Tits tree: an elementary approach, Int. J. Algebra Comp. 23 (2013), 1653-1683
[MS4] A. W. Mason and A. Schweizer: Elliptic points of the Drinfeld modular groups, Math. Z. 279 (2015), 1007-1028
[Ro] M. Rosen: Number Theory in Function Fields, Springer GTM 210, New York, 2002
[Se] J.-P. Serre: Trees, Springer, Berlin Heidelberg New York, 1980
[Si] J. H. Silverman: The Arithmetic of Elliptic Curves, Second Edition, Springer GTM 106, Dordrecht, 2009
[St] H. Stichtenoth: Algebraic Function Fields and Codes, Second Edition, Springer GTM 254, Berlin Heidelberg, 2009
[Ta] S. Takahashi: The fundamental domain of the tree of GL(2) over the function field of an elliptic curve, Duke Math. J. 72 (1993), 85-97

A. W. Mason, Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Scotland, U.K.
Email address: awm@maths.gla.ac.uk

Andreas Schweizer, Am Felsenkeller 61, 78713 Schramberg, Germany