Abelian $L$-Functions at $s = 1$ and Explicit Reciprocity for Rubin-Stark Elements

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Abstract

Given an abelian, CM extension $K$ of any totally real number field $k$, we restate and generalise two conjectures ‘of Stark type’ made in [So5]. The Integrality Conjecture concerns the image of a $p$-adic map $\mathcal{E}_{K/k,S}$ determined by the minus-part of the $S$-truncated equivariant $L$-function for $K/k$ at $s = 1$. It is connected to the Equivariant Tamagawa Number Conjecture of Burns and Flach. The Congruence Conjecture says that $\mathcal{E}_{K/k,S}$ gives an explicit reciprocity law for the element predicted by the corresponding Rubin-Stark Conjecture for $K^+/k$. We then study the general properties of these conjectures and prove one or both of them under various hypotheses, notably when $p \nmid [K : k]$, when $k = \mathbb{Q}$ or when $K$ is absolutely abelian.

1 Introduction

Throughout this paper $k$ will be a number field of finite degree $d$ over $\mathbb{Q}$ and $K$ will be a finite, Galois extension of $k$ such that the group $G := \text{Gal}(K/k)$ is abelian. We denote by $S_\infty = S_\infty(k)$ and $S_{\text{ram}} = S_{\text{ram}}(K/k)$ the sets consisting respectively of the infinite places of $k$ and those which are finite and ramify in $K$, and we set $S^0 = S^0(k/k) = S_{\text{ram}} \cup S_\infty$. If $S$ is any finite set of places containing $S^0$ and $s$ a complex number with $\text{Re}(s) > 1$, we define a convergent Euler product in the complex group ring of $G$ (denoted simply $\mathbb{C}G$) by

$$\Theta_{K/k,S}(s) := \prod_{q \notin S} (1 - Nq^{-s} \sigma_q^{-1})^{-1}$$  \hspace{1cm} (1)

The product ranges over those places $q$ of $k$ which are not in $S$. (Here and henceforth, finite places are identified with prime ideals.) $\sigma_q = \sigma_{q,k}$ denotes the Frobenius element of $G$ for $q$. If $k$ is totally real and $K$ a CM field with complex conjugation $c \in G$, it
can be shown that the ‘minus part’ \( \Theta_{K/k,S}^- \) extends to an entire function \( \mathbb{C} \to \mathbb{C}G \). This paper concerns two conjectures of a p-adic nature about the element \( a_{K/k,S}^- := (i/\pi)^d \Theta_{K/k,S}^-(1) \) (whose coefficients turn out to be algebraic). For any number \( p \) we denote by \( U^1(K_p) \) the p-semilocal principal units of \( K \) and define a p-adic regulator on the exterior power \( \bigwedge_{\mathbb{Z}_p G}^d U^1(K_p) \). By combining this with \( a_{K/k,S}^- \) we obtain a map \( s_{K/k,S} : \bigwedge_{\mathbb{Z}_p G}^d U^1(K_p) \to \mathbb{Q}_p G \). Assuming for the rest of this Introduction that \( p \neq 2 \) and \( S \) contains all the places above \( p \) in \( k \), our first conjecture (the ‘Integrality Conjecture’ or ‘IC’) states simply that the image \( \mathcal{G}_{K/k,S} \) of \( s_{K/k,S} \) is contained in \( \mathbb{Z}_p G \). Recall now that if \( K^+ \) denotes the maximal totally real subfield of \( K \) then (the meromorphic continuation of) \( \Theta_{K^+/k,S}(s) \) has a zero of order at least \( d \) at \( s = 0 \). Furthermore, a well known conjecture of Stark (reformulated and refined by Rubin) states that the coefficient of \( s^d \) in the Taylor expansion of \( \Theta_{K^+/k,S}(s) \) is given by evaluating an \((\mathbb{R}G\text{-valued})\) regulator map on an element of a certain exterior power of the (global) \( S \)-units of \( K^+ \). Imposing further natural conditions makes this element the unique ‘Rubin-Stark Element’ of the title, here denoted \( \eta_{K^+/k,S} \). Our second conjecture (the ‘Congruence Conjecture’ or ‘CC’) assumes and refines the IC, and in so doing links the minus part of \( \Theta_{K/k,S}(s) \) at \( s = 1 \) to its plus part at \( s = 0 \). It says that if \( K \) contains the \( p^{n+1} \)-th roots of unity for some \( n \geq 0 \), then, very roughly speaking, the reduction of \( s_{K/k,S} \) modulo \( p^{n+1} \) gives the explicit reciprocity law for \( \eta_{K^+/k,S} \).

The idea for these conjectures came from the results of [So4]. We shall not elaborate on the precise connection in the present paper beyond saying that if \( p \) splits in \( k \) then certain rather strong hypotheses considered in [So4] imply a weak form of the CC at each level in a cyclotomic \( \mathbb{Z}_p \)-tower containing \( K \). The IC and the CC first appeared explicitly as Conjectures 5.2 and 5.4 at the end of [So5] in a form less general and more awkward than the present versions. Conjectures 5.2 and 5.4 also used twisted zeta-functions at \( s = 0 \) where the IC and the CC use \( \Theta_{K/k,S}(1) \) and so will, perhaps, prove more accessible.

The remainder of this paper is organised as follows. Section 2 contains the precise definitions and basic properties of the main players: the elements \( a_{K/k,S}^- \) and \( \eta_{K^+/k,S} \), the map \( s_{K/k,S} \) and the pairing \( H_{K/k,n} \) (a determinant of additive, equivariant Hilbert symbols in terms of which our conjectural reciprocity law is couched). Section 3 contains the precise statements of the two conjectures. Section 4 surveys the current evidence in their favour – now quite considerable – and includes the statements of the three main results of this paper which were announced in [So5]: Firstly, in the case \( p \nmid |G| \), we give a complete characterisation of \( \mathcal{G}_{K/k,S} \) in terms of \( L \)-functions of odd characters of \( G \) at \( s = 0 \). In this case the IC then follows, thanks to a result of Deligne-Ribet and Pi. Cassou-Noguès. Secondly, we prove the conjectures in the case \( k = \mathbb{Q} \), using an explicit reciprocity law due to Coleman. Thirdly we prove the conjectures when \( K/\mathbb{Q} \) is abelian (but \( k \) is not necessarily \( \mathbb{Q} \)) by ‘base-change’ from the previous result. In this case, we require a relatively mild technical hypothesis on \( K/k, S \) and \( p \). We also discuss briefly A. Jones’ recent results on a rather different refinement of the IC which follows from a special case of the Equivariant Tamagawa Number Conjecture (ETNC) of Burns and Flach. (On the other hand, we should mention that the CC currently has no known connection with the ETNC.) Section 5 examines the
behaviour of the conjectures as $S$, $K$ and $n$ vary. Sections 6, 7 and 8 contain the proofs of the three main results referred to above.

Jones’ refinement of the IC mentioned above states that $\mathfrak{S}_{K/k,S}$ is contained in the Fitting ideal (as $\mathbb{Z}_pG$-module) of the minus part of the $p$-part of a certain ray-class group of $K$. This creates the possibility of links between $\mathfrak{S}_{K/k,S}$ and recent work of Greither on Fitting ideals of duals of class groups (see [Gr]). Another perspective comes from recent work of the author showing that, in the case $k = \mathbb{Q}$ at least, the CC creates a link between the map $\mathfrak{s}_{K/k,S}$ and some new maps and ideals in Iwasawa Theory. The latter has connections with the Main Conjecture and applications to the plus part of the class group of abelian fields.

In addition to those introduced above, we use the following basic notations and conventions. If $\mathcal{R}$ is a commutative ring and $H$ a finite abelian group, we write $\mathcal{R}H$ for the group-ring and if $M$ is a $\mathbb{Z}G$-module we shall sometimes abbreviate $\mathcal{R} \otimes_{\mathbb{Z}} M$ to $\mathcal{R}M$ (considered as a $\mathcal{R}H$-module in the obvious way). For any subgroup $D \subset H$, we write $N_D$ for the norm element $\sum_{d \in D} d \in \mathcal{R}H$. If $m$ is a positive integer, we denote by $\mu_m(\mathcal{R})$ the group of all $m$th roots of unity in $\mathcal{R}$ and for any prime number $p$ we set $\mu_p^\infty(\mathcal{R}) = \bigcup_{i=0}^{\infty} \mu_p^i(\mathcal{R})$. All number fields in this paper are supposed of degree over $\mathbb{Q}$ and are considered as subfields of $\mathbb{Q}$ which is the algebraic closure of $\mathbb{Q}$ within $\mathbb{C}$. We shall write $\xi_m$ for the particular generator $\exp(2\pi i/m)$ of $\mu_m(\mathbb{Q})$. For any number field $F$ and any integer $r$ we shall write $S_r(F)$ for the set of places (prime ideals) of $F$ dividing $r$. If $S$ is a set of places of $F$ and $L$ is any finite extension of $F$ we shall write $S(L)$ for the set of places of $L$ lying above those in $S$. If $S$ contains $S_\infty(F)$ (see above) then the group $U_S(F)$ of $S$-units of $F$ consists of those elements of $F^\times$ which are local units at every place not in $S$ and we shall often write simply $U_S(L)$ in place of $U_S(L)(L)$. (Caution: $U_S$ and related modules will sometimes be written additively.) If $H$ is abelian and $v$ is any place of $F$ we shall write $D_v(L/F)$ for the decomposition subgroup of $H$ at any prime dividing $v$ in $L$ and similarly $T_v(L/F)$ for the inertia subgroup (if $v$ is finite). Suppose $L \supset F \supset M$ are three number fields such that $L/M$ and $F/M$ are Galois extensions. Then the restriction map $\text{Gal}(L/M) \to \text{Gal}(F/M)$ will be denoted $\pi_{L/F}$ and extended $\mathcal{R}$-linearly to a ring homomorphism $\mathcal{R}\text{Gal}(L/M) \to \mathcal{R}\text{Gal}(F/M)$ for any commutative ring $\mathcal{R}$. We also write $\nu_{L/M}$ for the $\mathcal{R}$-linear ‘corestriction’ map $\mathcal{R}\text{Gal}(F/M) \to \mathcal{R}\text{Gal}(L/M)$ which sends $g \in \text{Gal}(F/M)$ to the sum of its preimages under $\pi_{L/F}$ in $\text{Gal}(L/M)$.

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2 Dramatis Personæ

2.1 The Function $\Theta_{K/k,S}$ and the Element $a_{K/k,S}$

Let $\hat{G}$ denote the dual group of $G$, namely the group of all (irreducible) complex characters $\chi : G \to \mathbb{C}^\times$ with identity element $\chi_0$, the trivial character. For any $\chi \in \hat{G}$ we write $e_{\chi,G}$ for the associated idempotent in the complex group-ring $\mathbb{C}G$, namely $e_{\chi,G} := \frac{1}{|\mathbb{C}|} \sum_{g \in G} \chi(g)g^{-1}$. 

3
Expanding the Euler product (1), we get
\[
\Theta_{K/k,S}(s) = \sum_{g \in G} \zeta_{K/k,S}(s; g)g^{-1} = \sum_{\chi \in \hat{G}} L_{K/k,S}(s, \chi)e_{\chi^{-1}, G} \tag{2}
\]
for \(\text{Re}(s) > 1\). Here, \(\zeta_{K/k,S}(s; g)\) and \(L_{K/k,S}(s, \chi)\) denote respectively the ‘\(S\)-truncations’ of the partial zeta-function attached to \(G\) and the \(L\) function attached to \(\chi\). In particular
\[
L_{K/k,S}(s, \chi) = \prod_{q \notin S} \left(1 - Nq^{-s}\chi(\sigma_q)\right)^{-1} = \prod_{\substack{q \in S \setminus S_\infty \atop q \notin \hat{f}_\chi}} \left(1 - Nq^{-s}\hat{\chi}([q])\right) L(s, \hat{\chi}) \tag{3}
\]
where \(f_\chi\) and \(L(s, \hat{\chi})\) denote respectively the conductor of \(\chi\) and the \(L\)-function of its associated primitive ray-class character \(\hat{\chi}\) modulo \(f_\chi\).

**Remark 2.1** The second (but not the first) expression for \(L_{K/k,S}(s, \chi)\) in (3) makes sense when \(S\) is any finite set of places of \(k\), containing \(S_\infty(k)\) but not necessarily \(S_{\text{ram}}(K/k)\). In fact it agrees with the definition of the \(S\)-truncated Artin \(L\)-function attached to \(\chi\) considered as a character of \(G\) (see for example [Ta, Théorème I.1.1]).

The analytic behaviour of \(L(s, \hat{\chi})\) is well-known. Its (in general) meromorphic continuation means that we may use equations (2) and (3) to continue \(\Theta_{K/k,S}(s)\) to a meromorphic, \(\mathbb{C}G\)-valued function on \(\mathbb{C}\). These equations then hold as identities between meromorphic functions on \(\mathbb{C}\). Similarly, if \(S \supset S' \supset S^0\), then the obvious identity
\[
\Theta_{K/k,S}(s) = \prod_{q \in S \setminus S'} \left(1 - Nq^{-s}\sigma_q^{-1}\right) \Theta_{K/k,S'}(s) \tag{4}
\]
for \(\text{Re}(s) > 1\) also holds for all \(s\). In fact, the function \(L(s, \hat{\chi})\), hence also the function \(\chi(\Theta_{K/k,S}(s)) = L_{K/k,S}(s, \chi^{-1})\), is analytic on \(\mathbb{C} \setminus \{1\}\) and
\[
\text{ord}_{s=1} \chi(\Theta_{K/k,S}(s)) = \begin{cases} 0 & \text{if } \chi \neq \chi_0 \\ -1 & \text{if } \chi = \chi_0 \end{cases} \tag{5}
\]
Moreover, the residue of \(\chi_0(\Theta_{K/k,S}(s)) = \prod_{q \in S \setminus S_\infty} (1 - Nq^{-s}) \zeta_k(s)\) at \(s = 1\) is well-known (see e.g. [Ta Théorème I.1.1]).

Using the well known functional equation relating the \textit{primitive} \(L\)-function \(L(s, \hat{\chi}^{-1})\) to \(L(1-s, \hat{\chi})\) one might expect to derive a natural relation between \(\Theta_{K/k,S}(s)\) and \(\Theta_{K/k,S}(1-s)\) by means of (2) and (3). There are however two obstacles to this: firstly, the dependence on \(f_\chi\) of the second product in (3) and secondly, the presence of (Galois) Gauss sums in the functional equations. Instead, in [So5] we used these functional equations to give a precise relation between \(\Theta_{K/k,S^0}(s)\) (there denoted \(\Theta_{K/k}(s)\)) and \(\Phi_{K/k}(1-s)\), where the function \(\Phi_{K/k} : \mathbb{C} \to \mathbb{C}G\) was defined by means of twisted zeta-functions and studied, together with its \(p\)-adic analogues, in [So2, So3, So4, So5]. For each \(v \in S_\infty\), we write \(c_v\) for the unique generator of \(D_v(K/k)\) so that \(c_v = 1\) unless \(v\) is real and one (hence every) place \(w\) of \(K\)
above $v$ is complex, in which case $c_v$ is the complex conjugation associated to any such $w$. We define an entire, $\mathbb{C}D_v(K/k)$-valued function

$$C_v(s) = \begin{cases} \exp(i\pi s).1 - \exp(-i\pi s).c_v = 2i\sin(\pi s).1 & \text{if } v \text{ is complex} \\ \exp(i\pi s/2).1 + \exp(-i\pi s/2).c_v & \text{if } v \text{ is real} \end{cases}$$

Then Theorem 2.1 of [So5], combined with (4) for $S' = S_0$, gives

$$i^{r_2(k)}\sqrt{|d_k|} \prod_{q \in S \setminus S_0} (1 - Nq^{-s}\sigma_q^{-1}) \Phi_{K/k}(1 - s) = ((2\pi)^{-s}\Gamma(s))^d \left( \prod_{v \in S'_{\infty}} C_v(s) \right) \Theta_{K/k,S}(s) \tag{6}$$

where $r_2(k)$ denotes the number of complex places of $k$ and $d_k$ its absolute discriminant. Let $\Phi_{K/k,S}(s)$ be the function $(1 - e_{\chi_0,G})\Theta_{K/k,S}(s)$, which is regular at $s = 1$ by (5). So (6) gives

$$\sqrt{|d_k|} \prod_{q \in S \setminus S_0} (1 - Nq^{-1}\sigma_q^{-1}) (1 - e_{\chi_0,G})\Phi_{K/k}(0) = (2\pi)^{-d_1|S_{\infty}|} \left( \prod_{v \in S'_{\infty}} (1 - c_v) \right) \Theta_{K/k,S}^{n.t.}(1) \tag{7}$$

from which it follows that $(1 - e_{\chi_0,G})\Phi_{K/k}(0)$ vanishes unless $k$ is totally real and $K$ is totally complex. On the other hand, multiplying (3) by $e_{\chi_0,G}$ and letting $s \to 1$, we see that $e_{\chi_0,G}\Phi_{K/k}(0)$ vanishes unless $|S_{\infty}| = 1$, i.e. $k$ is $\mathbb{Q}$ or an imaginary quadratic field, in which case it may easily be calculated from $\text{res}_{s=1}\Phi_{K/k}(s)$. Thus $\Phi_{K/k}(0)$ has little interest unless $k$ is totally real and $K$ is totally complex. Even then, $\prod_{v \in S_{\infty}}(1 - c_v)$ vanishes unless there is a (unique) CM-subfield $K^-$ of $K$ containing $k$, in which case we lose little but complication by replacing $K$ by $K^-$. (See Remark 3.1(i) of [So5] for further explanations). For these reasons we shall henceforth make the

**Hypothesis 1** $k$ is totally real and $K$ is a CM field.

This means that $d_k$ is a positive integer and $c_v = c$, the unique complex conjugation in $G$, for all $v \in S_{\infty}$. Let $e^\pm$ denote the two idempotents $\frac{1}{2}(1 \pm c)$ of $\mathbb{C}G$ and let $\Theta_{K/k,S}^-(s)$ be the entire function $e^-\Theta_{K/k,S}(s) = e^-\Theta_{K/k,S}^{n.t.}(s)$. The above remarks, together with a simple calculation of $e_{\chi_0,G}\Phi_{K/k}(0)$ when $k = \mathbb{Q}$, show that equation (7) may be rewritten as

$$a_{K/k,S}^- := \left( \frac{i}{\pi} \right)^d \Theta_{K/k,S}^-(1) = \begin{cases} \prod_{q \in S \setminus S_0} (1 - Nq^{-1}\sigma_q^{-1}) \sqrt{d_k}\Phi_{K/k}(0) & \text{if } k \neq \mathbb{Q} \\ \prod_{q \in S \setminus S_0} (1 - q^{-1}\sigma_q^{-1}) \Phi_{K/Q}(0) + \frac{1}{2} \prod_{q \in S \setminus \{\infty\}} (1 - q^{-1}) e_{\chi_0,G} & \text{if } k = \mathbb{Q} \end{cases} \tag{8}$$

If $R$ is a commutative ring in which 2 is invertible and $M$ any $R(c)$-module then we shall write $M^+$ (resp. $M^-$) for the $R$-submodule $e^-M$ (resp. $e^-M$), so that $M = M^+ \oplus M^-$. In this notation, $a_{K/k,S}^-$ clearly lies in $\mathbb{C}G^-$ and multiplying (8) by $e^-$ gives

$$a_{K/k,S}^- = e^-a_{K/k,S}^- = e^- \prod_{q \in S \setminus S_0} (1 - Nq^{-1}\sigma_q^{-1}) \sqrt{d_k}\Phi_{K/k}(0) \tag{9}$$
whether or not \( k = Q \), but if \( k \neq Q \) then the term \( e^- \) may be omitted on the R.H.S. In fact, \( a_{K/k,S}^- \) has algebraic coefficients: Let \( f(K) \) be the integral ideal of \( O_k \) which is the conductor of \( K/k \) in the sense of class-field theory and let \( f(K) \) be the positive generator of the ideal \( f(K) \cap \mathbb{Z} \). The product in (9) lies in \( QG^\times \), then (9) and [So5, Prop. 3.1] show that \( a_{K/k,S}^- \) has coefficients in \( \sqrt{d_k}Q(\mu_{f(K)}) \) and that

\[
a_{K/k,S}^- Q(\mu_{f(K)})G = \sqrt{d_k}Q(\mu_{f(K)})G^- \tag{10}
\]

Integrality properties of the coefficients of \( a_{K/k,S}^- \) are given in [R-S2] where it is shown that they also lie in the Galois closure of \( K \) over \( Q \). (See ibid., Proposition 2 and Remark 6).

### 2.2 Rubin-Stark Elements for \( K^+/k \)

Let us write \( \tilde{G} \) for \( \text{Gal}(K^+/k) \cong G/\langle c \rangle \) so that \( \pi_{K/K^+} : CG \to C\tilde{G} \) induces a ring isomorphism \( CG^+ \to C\tilde{G} \) sending \( e^+\Theta_{K/k,S}(s) \) onto \( \Theta_{K^+/k,S}(s) \). To study \( \Theta_{K^+/k,S}(s) \) at \( s = 0 \), we define an integer \( r_S(\phi) \) for each \( \phi \in \hat{\tilde{G}} \) by

\[
r_S(\phi) := \begin{cases} 
  d + |\{q \in S \backslash S_\infty : \phi(D_q(K^+/k)) = \{1\}\}| & \text{if } \phi \text{ is non-trivial} \\
  d + |S \backslash S_\infty| - 1 = |S| - 1 & \text{if } \phi \text{ is trivial}
\end{cases} \tag{11}
\]

Since \( k \) and \( K^+ \) are totally real, the functional equation of \( L(s, \hat{\phi}) \) for \( \phi \in \tilde{G} \) shows that, for any such \( \phi \) we have

\[
\text{ord}_{s=0} \phi(\Theta_{K^+/k,S}(s)) = \text{ord}_{s=0} L_{K^+/k,S}(s, \phi) = r_S(\phi) \tag{12}
\]

(see e.g. [Ta, Ch. I, §3]). We shall assume until further notice

**Hypothesis 2** \( |S| \geq d + 1 \) (i.e. \( S \) contains at least one finite place.)

This implies that \( r_S(\phi) \geq d \) for \( \phi \) trivial hence for every \( \phi \in \tilde{G} \), so we may define

\[
\Theta_{K^+/k,S}^{(d)}(0) := \lim_{s \to 0} s^{-d} \Theta_{K^+/k,S}(s)
\]

(an element of \( CG \) which is easily seen to lie in \( \mathbb{R}\tilde{G} \)). Conjectures of Stark, as refined by Rubin [Ru], predict that \( \Theta_{K^+/k,S}^{(d)}(0) \) is given by a certain \( \mathbb{R}\tilde{G} \)-valued regulator of \( S \)-units of \( K^+ \) defined as follows. We fix once and for all a set \( \tau_1, \ldots, \tau_d \) of left coset representatives for \( \text{Gal}(\bar{Q}/k) \) in \( \text{Gal}(\bar{Q}/Q) \) and we define \( \mathbb{Q}\tilde{G} \)-linear, real logarithmic maps for \( i = 1, \ldots, d \):

\[
\lambda_{K^+/k,i} : \mathbb{Q}_{U_S}(K^+) \to \mathbb{R}\tilde{G}
\]

\[
\lambda_{K^+/k,i}(a \otimes \varepsilon) \mapsto a \sum_{g \in \tilde{G}} \log |\tau_i(g\varepsilon)|^{-1} \in \mathbb{R}\tilde{G}
\]
The above-mentioned regulator is the $Q\bar{G}$-linear map uniquely defined by:

$$R_{K^+/k} : \bigwedge^d_{Q\bar{G}} QU_S(K^+) \rightarrow \mathbb{R}\bar{G}$$

$$x_1 \wedge \ldots \wedge x_d \mapsto \det(\lambda_{K^+/k,i}(x_i))_{i,t=1}^d$$

The following definition generalises the above construction and will be useful later.

**Proposition/Definition 1**  
(i). Suppose $S$ is a commutative ring, $R$ a commutative $S$-algebra and that $M$ is any (left) $RH$ module for a finite group $H$. There is an isomorphism from $\text{Hom}_R(M, S)$ to $\text{Hom}_{RH}(M, SH)$ given by $f \mapsto f^H$ where $f^H$ is defined to be the map $m \mapsto \sum_{h \in H} f(h^{-1}m)h$.

(ii). Suppose $H$ is abelian and $l \in \mathbb{N}$. Then for every $l$-tuple $(f_1, \ldots, f_l) \in \text{Hom}_R(M, S)^l$ there is a $RH$-linear determinantal map $\Delta_{f_1, \ldots, f_l}$ uniquely defined by

$$\Delta_{f_1, \ldots, f_l} : \bigwedge^l_{RH} M \rightarrow SH$$

$$m_1 \wedge \ldots \wedge m_l \mapsto \det(f^H_l(m_i))_{i,t=1}^l$$

$\Delta_{f_1, \ldots, f_l}$ is $S$-multilinear and alternating as a function of $(f_1, \ldots, f_d)$. Moreover for each $i = 1, \ldots, l$ and $h \in H$ we have $\Delta_{f_1, \ldots, f_i, h, \ldots, f_l}(\mu) = \Delta_{f_1, \ldots, f_i}(\mu)h$ for all $\mu \in \bigwedge^l_{RH} M$. $\square$

For instance, taking $R = Q$, $S = \mathbb{R}$, $M = QU_S(K^+)$ and $H = \bar{G}$ gives $R_{K^+/k} = \Delta_{f_1, \ldots, f_d}$ where $f_i$ is the map sending $a \otimes \varepsilon \in QU_S(K^+)$ to its logarithmic embedding $a \log |\tau_i(\varepsilon)|$ in $\mathbb{R}$. If instead we take $R = S = Q$, then any $d$ elements $f_1, \ldots, f_d$ of $\text{Hom}_Q(QU_S(K^+), Q)$ give rise to a $Q\bar{G}$-linear map $\Delta_{f_1, \ldots, f_d} : \bigwedge^d_{Q\bar{G}} QU_S(K^+) \rightarrow QG$. Let us identify $\text{Hom}_Z(US(K^+), Z)$ with the lattice in $\text{Hom}_Z(US(K^+), Q)$ which is its image under the map $f \mapsto 1 \otimes f$. We can then define a $Z\bar{G}$ submodule $\Lambda_{0, S} = \Lambda_{0, S}(K^+/k)$ of $\bigwedge^d_{Q\bar{G}} QU_S(K^+)$ by

$$\Lambda_{0, S}(K^+/k) := \left\{ \eta \in \bigwedge^d_{Q\bar{G}} QU_S(K^+) : \Delta_{f_1, \ldots, f_d}(\eta) \in Z\bar{G} \forall f_1, \ldots, f_d \in \text{Hom}_Z(US(K^+), Z) \right\}$$

This coincides with ‘$\Lambda^d_0 US(K^+)$’ as defined by Rubin’s ‘double dual’ construction in [Ru §1]. It is clear that $\Lambda_{0, S}$ contains the lattice which is the natural image of $\bigwedge^d_{Z\bar{G}} US(K^+)$ in $\bigwedge^d_{Q\bar{G}} QU_S(K^+)$ (we denote this $\overline{\bigwedge^d_{Z\bar{G}} US(K^+)}$) but the two are not necessarily equal. In fact, Prop. 1.2 of [Ru] implies

**Proposition 1**  If $d = 1$ (i.e. $k = Q$) then $\Lambda_{0, S} = \overline{\bigwedge^1_{Z\bar{G}} US(K^+)} = US(K^+)$. In general, The index $|\Lambda_{0, S} : \bigwedge^d_{Z\bar{G}} US(K^+)|$ is finite and supported on primes dividing $|\bar{G}|$. $\square$

Let us define an idempotent $e_{S,d,\bar{G}}$, a priori in $C\bar{G}$, by setting $e_{S,d,\bar{G}} := \sum_{r_S(\phi) = d} e_{\phi, \bar{G}}$. This is the unique element element $x$ of $C\bar{G}$ such that $\phi(x) = 1$ or 0 according as $r_S(\phi) = d$ or

$$e_{S,d,\bar{G}} := \sum_{r_S(\phi) = d} e_{\phi, \bar{G}}$$
It follows easily from this description and the formula (11) that
\[
e_{S,d,G} = \begin{cases} 
\prod_{q \in S \setminus S_\infty} \left(1 - \frac{1}{|D_q(K^+/k)|} N_{D_q(K^+/k)} \right) & \text{if } |S| > d + 1 \\
\left(1 - \frac{1}{|D_q(K^+/k)|} N_{D_q(K^+/k)} \right) + e_{\chi_S,G} & \text{if } |S| = d + 1, \text{ i.e. } S = \{q\} \cup S_\infty
\end{cases}
\]
(13)

Thus \(e_{S,d,G}\) is an idempotent of \(\mathbb{Q}\bar{G}\), so lies in \(|G|^{-1}\mathbb{Z}\bar{G}\). We also deduce easily:

**Proposition 2** Let \(M\) be any \(\mathbb{Q}\bar{G}\)-module and \(m \in M\). The following are equivalent

(i). \(m \in e_{S,d,G}M\)

(ii). \(m = e_{S,d,G}m\)

(iii). For all \(q \in S \setminus S_\infty\),
\[
N_{D_q(K^+/k)}m \in \begin{cases} 
\{0\} & \text{if } |S| > d + 1 \\
M^G & \text{if } |S| = d + 1, \text{ i.e. } S = \{q\} \cup S_\infty
\end{cases}
\]
(14)

(iv). \(e_{\phi,G}(1 \otimes m) = 0\) in \(\mathbb{C} \otimes_{\mathbb{Q}} M\) for all \(\phi \in \widehat{\bar{G}}\) such that \(r_S(\phi) > d\)

For brevity, we shall sometimes refer to any of these conditions as the *eigenspace condition* on \(m\) w.r.t. \((S, d, \bar{G})\). Now, given any subring \(\mathcal{R}\) of \(\mathbb{Q}\), we formulate a version of the Rubin-Stark conjecture ‘over \(\mathcal{R}\)’:

**Conjecture** \(RSC(K^+/k, S; \mathcal{R})\)

Let \(K/k\), \(S\) and be as above, satisfying Hypotheses \(\Box\) and \(\Box\). Then there exists an element \(\eta \in \bigwedge_{\mathcal{Q}\bar{G}} \mathbb{Q}U_S(K^+)\) satisfying the eigenspace condition w.r.t. \((S, d, \bar{G})\) and such that
\[
\Theta^{(d)}_{K^+/k,S}(0) = R_{K^+/k}(\eta)
\]
(15)

and
\[
\eta \in \frac{1}{2}\mathcal{R}\Lambda_0(S(K^+/k))
\]
(16)

Notice that \(\Theta^{(d)}_{K^+/k,S}(0)\) lies in the ideal \(e_{S,d,G}\mathcal{R}\bar{G}\) (and in fact generates it) by equation \(\Box\).

Thus if \(\eta \in \bigwedge_{\mathcal{Q}\bar{G}} \mathbb{Q}U_S(K^+)\) is any solution of \(\Box\) then \(e_{S,d,G}\eta\) is a solution satisfying the eigenspace condition. On the other hand, it can be shown that \(R_{K^+/k}\) is injective on \(e_{S,d,G} \bigwedge_{\mathcal{Q}\bar{G}} \mathbb{Q}U_S(K^+)\) (this follows from \([\text{Ru}, \text{Lemma 2.7}]\)) so a solution of \(\Box\) satisfying the eigenspace condition is unique. For this reason, we call such an element ‘the Rubin-Stark element for \(K^+/k\) and \(S\)’ and denote it \(\eta_{K^+/k,S}\) since it is independent of \(\mathcal{R}\). Of course, Condition \(\Box\) is redundant if \(\mathcal{R} = \mathbb{Q}\) and for any prime number \(p\) we have
\[
RSC(K^+/k, S; \mathbb{Z}) \Rightarrow RSC(K^+/k, S; \mathbb{Z}(p)) \Rightarrow RSC(K^+/k, S; \mathbb{Q})
\]
(where $\mathbb{Z}_{(p)}$ denotes the localisation $\{a/b \in \mathbb{Q} : p \nmid b\}$). Moreover $RSC(K^+/k, S; \mathbb{Z})$ is equivalent to the conjunction of $RSC(K^+/k, S; \mathbb{Z}_{(p)})$ for all primes $p$. We shall mainly be interested in $RSC(K^+/k, S; \mathbb{Z}_{(p)})$ when $p \neq 2$, in which case (10) reduces to $\eta \in \mathbb{Z}_{(p)}\Lambda_0$.\vspace{1ex}

**Remark 2.2** Since $R_{K^+/k}$ depends on the choice (and ordering) of the $\tau_i$’s, so will $\eta_{K^+/k, S}$, but in a simple way. For example, if one $\tau_i$ is replaced by $\tau_i \tau^{-1}$ for some $\tau \in \text{Gal}(\mathbb{Q}/k)$ then we must replace $\eta_{K^+/k, S}$ by $\tau|_{K^+} \eta_{K^+/k, S}$ where $\tau|_{K^+} \in \widetilde{G}$.

**Remark 2.3** $RSC(K^+/k, S; \mathbb{Q})$ and $RSC(K^+/k, S; \mathbb{Z})$ follow from certain special cases of Conjectures $A'$ and $B'$ of [Ru] respectively. Indeed, if we choose the extension ‘$K/k$’ of Rubin’s paper to be our $K^+/k$, his ‘$S$’ to be ours, his ‘$r$’ to be $d$ and his chosen places ‘$w_1, \ldots, w_r$’ to be the real places of $K^+$ defined by $\tau_1, \ldots, \tau_d$. Then Rubin’s Hypotheses 2.1.1-2.1.4 are satisfied. His conjectures also require an auxiliary set $T$ of finite places of $k$ satisfying certain conditions, although for Conjecture $A'$ the precise choice of such $T$ does not affect the truth of the conjecture. For simplicity we take $T = \{q\}$ for some prime $q \not\in S$ not dividing 2 and splitting in $K^+$ (infinitely many of these exist by Chebotarev’s theorem). Then Rubin’s Hypothesis 2.1.5 certainly holds since $U_S(K^+)_{tor} = \{\pm 1\}$. Moreover his ‘$\Theta_S^{(r)}(0)$’ is our $(1 - Nq)\Theta_{K^+/k, S}(0)$ and his ‘$\Lambda_0 U_{S,T}$’ is a sublattice of our $\Lambda_0 S(T)$ which also spans $\bigwedge_{S \in S}^d \mathbb{Q}U_S(K^+)$ over $\mathbb{Q}$. It follows easily that $RSC(K^+/k, S; \mathbb{Q})$ is equivalent to Rubin’s Conjecture $A'$ with these choices and this (hence any) $T$. Moreover, if both hold then Rubin’s ‘$\epsilon_{S,T}$’ equals our $(1 - Nq)\eta_{K^+/k, S}$, by uniqueness. It follows that Rubin’s Conjecture $B'$ with these choices amounts to the further condition that $(1 - Nq)\eta_{K^+/k, S}$ lie in his $\Lambda_0 U_{S,T}$ hence in our $\Lambda_0 S(T)$. But as $q$ varies subject to the above conditions, Lemma IV.1.1 of [Ta] says that the g.c.d. of the corresponding integers $1 - Nq$ is $|\mu(K^+)| = 2$. Thus the corresponding cases of Rubin’s Conjecture $B'$ together imply $RSC(K^+/k, S; \mathbb{Z})$.

The connection with Stark’s original conjecture in terms of characters (see [Ta Conjecture I.5.1]) is as follows. Propositions 2.3 and 2.4 of [Ru] show that it holds for $K^+/k$, $S$ and every character $\phi \in \widetilde{G}$ satisfying $r_S(\phi) = d$ if and only if Rubin’s Conjecture $A'$ holds (for any $T$) which is equivalent to $RSC(K^+/k, S; \mathbb{Q})$, by the above.

In the next section we shall be interested in determinantal maps obtained from a $d$-tuple $(f_1, \ldots, f_d) \in \text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z}/p^{n+1}\mathbb{Z})$ for some prime $p$ and $n \geq 0$. Taking $\mathcal{R} = \mathbb{Z}$, $S = \mathbb{Z}/p^{n+1}\mathbb{Z}$ and $M = U_S(K^+)$ in Proposition/Definition 1 gives such a map $\Delta_{f_1, \ldots, f_d} : \bigwedge_{S \in S}^d U_S(K^+) \to (\mathbb{Z}/p^{n+1}\mathbb{Z})\widetilde{G}$. We shall now show that provided $p$ is odd, this map ‘extends’ naturally to $\mathbb{Z}(p)\Lambda_0 S$ in a sense to be explained below. First, we have

**Lemma 1** If $p$ is odd then the following sequence is exact.

$$0 \to \text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z})^p \to \text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z}) \to \text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z}/p^{n+1}\mathbb{Z}) \to 0$$

**Proof** $K^+$ is totally real so $U_S(K^+)/\{\pm 1\}$ is $\mathbb{Z}$-free. Thus the sequence is exact if $U_S(K^+)$ is replaced by $U_S(K^+)/(\{\pm 1\})$. But since $\mathbb{Z}$ and $\mathbb{Z}/p^{n+1}\mathbb{Z}$ have no 2-torsion, we may identify $\text{Hom}_\mathbb{Z}(U_S(K^+)/(\{\pm 1\}), \mathbb{Z})$ with $\text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z})$ and $\text{Hom}_\mathbb{Z}(U_S(K^+)/(\{\pm 1\}, \mathbb{Z}/p^{n+1}\mathbb{Z})$ with $\text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z}/p^{n+1}\mathbb{Z})$. \square
Thus given $f_1, \ldots, f_d$ in $\text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z}/p^{n+1}\mathbb{Z})$ for $p$ odd, we can choose lifts $\tilde{f}_1, \ldots, \tilde{f}_d$ in $\text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z})$. As previously, we may regard these as elements of $\text{Hom}_\mathbb{Q}(\mathcal{Q}U_S(K^+), \mathbb{Q})$ and use Proposition/Definition 1 to construct $\Delta_{\tilde{f}_1, \ldots, \tilde{f}_d}: \wedge_{\mathcal{Q}\mathcal{G}}^d \mathcal{Q}U_S(K^+) \to \mathcal{Q}\mathcal{G}$. If $\eta \in \Lambda_{0,S}$ then $\Delta_{\tilde{f}_1, \ldots, \tilde{f}_d}(\eta)$ lies in $\mathcal{G}\mathcal{B}$ by definition of $\Lambda_{0,S}$ and we write $\tilde{\Delta}_{\tilde{f}_1, \ldots, \tilde{f}_d}(\eta)$ for its image in $(\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{\mathcal{G}}$. The latter is independent of the choice of each lift $\tilde{f}_i$, as one easily checks using Lemma 1, the linearity of $\Delta_{\tilde{f}_1, \ldots, \tilde{f}_d}$ in $\tilde{f}_i$ and the fact that $\eta \in \Lambda_{0,S}$. Consequently we have a well-defined map $\tilde{\Delta}_{\tilde{f}_1, \ldots, \tilde{f}_d}: \Lambda_{0,S} \to (\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{\mathcal{G}}$ which is linear and so extends uniquely to $\mathbb{Z}(p)\Lambda_{0,S}$. It is now an easy exercise to check the following properties of $\tilde{\Delta}_{\tilde{f}_1, \ldots, \tilde{f}_d}$:

**Proposition 3** Let $p$ be odd and choose $f_1, \ldots, f_d \in \text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z}/p^{n+1}\mathbb{Z})$

(i). The map $\tilde{\Delta}_{\tilde{f}_1, \ldots, \tilde{f}_d}: \mathbb{Z}(p)\Lambda_{0,S}(K^+/k) \to (\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{\mathcal{G}}$ is $\mathcal{G}\mathcal{B}$-linear.

(ii). It is also $(\mathbb{Z}/p^{n+1}\mathbb{Z})$-multilinear and alternating as a function of $(f_1, \ldots, f_d)$ and for each $i = 1, \ldots, d$ we have $\tilde{\Delta}_{\tilde{f}_1, \ldots, \tilde{f}_i, g, \ldots, \tilde{f}_d}(\eta) = \tilde{\Delta}_{\tilde{f}_1, \ldots, \tilde{f}_d}(\eta)g$ for all $g \in \mathcal{G}$ and $\eta \in \mathbb{Z}(p)\Lambda_{0,S}(K^+/k)$.

(iii). The following diagram commutes

\[
\begin{array}{ccc}
\wedge_{\mathcal{Q}\mathcal{G}}^d U_S(K^+) & \xrightarrow{\Delta_{\tilde{f}_1, \ldots, \tilde{f}_d}} & (\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{\mathcal{G}} \\
\alpha \downarrow & & \\
\mathbb{Z}(p)\Lambda_{0,S}(K^+/k) & \xrightarrow{\Delta_{\tilde{f}_1, \ldots, \tilde{f}_d}} & (\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{\mathcal{G}}
\end{array}
\]

where $\alpha$ is the natural map $\wedge_{\mathcal{Q}\mathcal{G}}^d U_S(K^+) \to \wedge_{\mathcal{Q}\mathcal{G}}^d \mathcal{Q}U_S(K^+)$ with restricted range.

\[\square\]

**Remark 2.4** This shows in particular that $\Delta_{\tilde{f}_1, \ldots, \tilde{f}_d}$ vanishes on the kernel of $\alpha$ in the above diagram. One can show that $\ker(\alpha)$ is always finite and supported on primes dividing $2|\tilde{\mathcal{G}}| = |\mathcal{G}|$. Also, Proposition 1 implies that $\text{im}(\alpha)$ spans $\mathbb{Z}(p)\Lambda_{0,S}$ over $\mathbb{Z}(p)$ whenever $p \nmid |\mathcal{G}|$. So if $p \nmid |\mathcal{G}|$ then any $\mathcal{G}\mathcal{B}$-linear map $F: \wedge_{\mathcal{Q}\mathcal{G}}^d U_S(K^+) \to (\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{\mathcal{G}}$ vanishes on $\ker(\alpha)$ and has a unique ‘extension’ $\tilde{F}: \mathbb{Z}(p)\Lambda_{0,S} \to (\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{\mathcal{G}}$ satisfying $F = \tilde{F} \circ \alpha$.

### 2.3 Hilbert Symbols and the Pairing $H_{K/k,n}$

Suppose that $L$ is a local field containing $\mu_m$ for some positive integer $m$ coprime to the characteristic of $L$. We recall that the Hilbert symbol is the map

\[(\cdot, \cdot)_{L,m} : L^\times \times L^\times \to \mu_m \subset L^\times \]

\[(\alpha, \beta) \mapsto (\beta^{1/m})^{\sigma_{\alpha,L}-1}\]
where $\beta^{1/m}$ is any $m$th root of $\beta$ in any abelian closure $L^{ab}$ of $L$ and $\sigma_{\alpha,L}$ denotes the image of $\alpha$ under the reciprocity homomorphism $(\cdot, L)$ of local class field theory from $L^\times$ to $\text{Gal}(L^{ab}/L)$. The Hilbert symbol is bilinear and skew-symmetric. For the general theory, see [A-T, Ch. 12], [Ne V.3] or [Se, Ch. XIV]. (Note that our notation $(\alpha, \beta)^{L,m}$ is compatible with that of [A-T] and [Ne] but represents the element denoted $(\beta, \alpha)$ see [A-T, Ch. 12], [Ne, V.3] or [Se, Ch. XIV].)

Let $p$ be a prime number and $n \geq 0$ an integer. We shall assume until further notice that $K$ contains $\mu_p^{n+1}$ for some $n \geq 0$. Let $\kappa_n : \text{Gal}(\bar{Q}/Q) \to (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ be the cyclotomic character modulo $p^{n+1}$, determined by $\tau(\zeta) = \zeta^{\kappa_n(\tau)} \forall \zeta \in \mu_{p^{n+1}}$, $\tau \in \text{Gal}(\bar{Q}/Q)$. Since $K$ contains $\mu_p^{n+1}$, the restriction of $\kappa_n$ to $\text{Gal}(\bar{Q}/K)$ factors through a homomorphism $G \to (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times$ which we denote by the same symbol. We also use the shorthand $\zeta_n$ for $\zeta_{p^{n+1}} \in K$. If $K_\mathfrak{P}$ denotes the completion of $K$ at some prime ideal $\mathfrak{P}$, we may define a bilinear pairing $[\cdot, \cdot]_{\mathfrak{P}, n} : K_\mathfrak{P}^\times \times K_\mathfrak{P}^\times \to \mathbb{Z}/p^{n+1}\mathbb{Z}$ by setting

$$\iota_{\mathfrak{P}}(\zeta_n)^{[\alpha, \beta]_{\mathfrak{P}, n}} = (\alpha, \beta)_{K_\mathfrak{P}, p^{n+1}}$$

for all $\alpha, \beta \in K_\mathfrak{P}^\times$ where $\iota_{\mathfrak{P}} : K \to K_\mathfrak{P}$ is the natural embedding. Every $g \in G$ induces an isomorphism $K_\mathfrak{P} \to K_\mathfrak{P}$, also denoted $g$ and such that $g \circ \iota_{\mathfrak{P}} = \iota_{\mathfrak{P}} \circ g$. Standard facts from Local Class Field theory imply that $(g\alpha, g\beta)_{K_\mathfrak{P}, p^{n+1}} = g(\alpha, \beta)_{K_\mathfrak{P}, p^{n+1}}$ in $K_\mathfrak{P}^\times$ for any $\alpha, \beta \in K_\mathfrak{P}^\times$. It follows easily that

$$[g\alpha, g\beta]_{\mathfrak{P}, n} = \kappa_n(g)[\alpha, \beta]_{\mathfrak{P}, n} \quad \text{for all } \alpha, \beta \in K_\mathfrak{P}^\times, \ g \in G \quad (17)$$

If $\mathfrak{P}$ divides $p$ then $\iota_{\mathfrak{P}}$ extends to a $\mathbb{Q}_p$-algebra map from $K_p := K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ to $K_\mathfrak{P}$. Thus we obtain a pairing

$$[\cdot, \cdot]_{K, n} : K_p^\times \times K_p^\times \to \mathbb{Z}/p^{n+1}\mathbb{Z}$$

$$(\alpha, \beta) \mapsto \sum_{\mathfrak{P}|p} [\iota_{\mathfrak{P}}(\alpha), \iota_{\mathfrak{P}}(\beta)]_{\mathfrak{P}, n}$$

Letting $G$ act on $K_p$ through $K$, we still have $\iota_{\mathfrak{P}} \circ g = g \circ \iota_{\mathfrak{P}}$ for any $g \in G$ and $\mathfrak{P}|p$, so $(17)$ implies

$$[g\alpha, g\beta]_{K, n} = \kappa_n(g)[\alpha, \beta]_{K, n} \quad \text{for all } \alpha, \beta \in K_p^\times, \ g \in G \quad (18)$$

The product map $\prod_{\mathfrak{P}|p} \iota_{\mathfrak{P}} : K_p \to \prod_{\mathfrak{P}|p} K_{\mathfrak{P}}$ is a $G$-equivariant ring isomorphism (where $g((x_\mathfrak{P})_{\mathfrak{P}}) = (g x_\mathfrak{P}^{-1})_{\mathfrak{P}}$ in $\prod_{\mathfrak{P}|p} K_\mathfrak{P}$). We shall regard this as an identification so that $\iota_{\mathfrak{P}}$ identifies with the projection $\prod_{\mathfrak{P}|p} K_{\mathfrak{P}} \to K_{\mathfrak{P}}$. Thus we identify the principal semilocal units $\prod_{\mathfrak{P}|p} U^1(K_{\mathfrak{P}})$ with a $\mathbb{Z}G$-submodule of $K_p^\times$ and denote it $U^1(K_p)$. Regarding each $U^1(K_{\mathfrak{P}})$ as a finitely generated $\mathbb{Z}_p$-module, $U^1(K_p)$ becomes a finitely generated $\mathbb{Z}_p G$-module.

From now on we assume that $p$ is odd. Consider the unique ring automorphism of $(\mathbb{Z}/p^{n+1}\mathbb{Z})G$ sending $g \in G$ to $\kappa_n(g)g^{-1}$. Since $\kappa_n(c) = -1$, this restricts to a ring isomorphism from $(\mathbb{Z}/p^{n+1}\mathbb{Z})G^+ \to (\mathbb{Z}/p^{n+1}\mathbb{Z})G^-$. Composing with $\bar{2}^{-1} \nu_{K/K^+} : (\mathbb{Z}/p^{n+1}\mathbb{Z})G \to$
(\mathbb{Z}/p^{n+1}\mathbb{Z})G$, we obtain a ring isomorphism \( \tilde{\kappa}^*_n = \tilde{\kappa}^*_{K,n} : (\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{G} \rightarrow (\mathbb{Z}/p^{n+1}\mathbb{Z})G^- \).

Explicitly, if \( h \in \tilde{G} \) and \( g \in G \) then

\[
\tilde{\kappa}^*_n(h) = 2^{-1} \sum_{h \in \tilde{G}} \kappa_n(\bar{h})h^{-1} \quad \text{and hence} \quad \tilde{\kappa}^*_n(\pi_{K/K^+}(g)) = e^{-\kappa_n(g)}g^{-1}
\]  

(19)

Given a set \( S \supseteq S^0 \) as in previous sections, any \( u \in U^1(K_p) \) defines homomorphism \( f_u \in \text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z}/p^{n+1}\mathbb{Z}) \) by setting \( f_u(\varepsilon) = [\varepsilon, u]_{K,n} \) (by abuse, we write \( \varepsilon \) for \( \varepsilon \otimes 1 \in K^*_p \)).

Using the ‘\( \tilde{\Delta} \)’ notation of the last section we may now define a map

\[
H_{K/k,S,n} : \mathbb{Z}(\eta)_0S(K^+/k) \times U^1(K_p)^d \rightarrow (\mathbb{Z}/p^{n+1}\mathbb{Z})G^-
\]

\[
(\eta; u_1, \ldots, u_d) \mapsto 2^d\tilde{\kappa}^*_n(\tilde{\Delta}_{u_1 \cdots u_d}(\eta))
\]

**Proposition 4** Suppose \( \eta \in \mathbb{Z}(\eta)_0S(K^+/k) \) and \( u_1, \ldots, u_d \in U^1(K_p)^d \)

(i). For any \( x \in \tilde{G} \) we have \( H_{K/k,S,n}(x\eta; u_1, \ldots, u_d) = \tilde{\kappa}^*_n(x)H_{K/k,S,n}(\eta; u_1, \ldots, u_d) \) where \( \bar{x} \) denotes the image of \( x \) in \((\mathbb{Z}/p^{n+1}\mathbb{Z})\tilde{G}\)

(ii). \( H_{K/k,S,n} \) is \( \mathbb{Z}G \)-multilinear (hence \( \mathbb{Z}pG \)-multilinear) and alternating as a function of \( u_1, \ldots, u_d \)

**Proof** Part (i) follows from part (i) of Proposition 3. The \( \mathbb{Z} \)-multilinearity in part (ii) follows from part (ii) of Proposition 3 so it suffices to prove that replacing \( u_i \) by \( gu_i \) (for \( g \in G \)) multiplies \( H_{K/k,S,n}(\eta; u_1, \ldots, u_d) \) by \( g \), or indeed by \( e^{-g} \) since it lies in the minus part. But if we write \( h \) for \( \pi_{K/K^+}(g) \in \tilde{G} \) then Equation (18) and Proposition 3 part (ii) give

\[
\tilde{\kappa}^*_n(\tilde{\Delta}_{u_1 \cdots u_i \cdots u_d}(\eta)) = \tilde{\kappa}^*_n(\tilde{\Delta}_{u_1 \cdots u_i \cdots u_d}(\eta)) = e^{-\kappa_n(g)}g^{-1}
\]

by (19) and the result follows.

By part (ii) of the Proposition, \( H_{K/k,S,n} \) defines a unique pairing (also denoted \( H_{K/k,S,n} \)) from \( \mathbb{Z}(\eta)_0S \times \bigwedge_{\mathbb{Z}G}^d U^1(K_p) \) to \((\mathbb{Z}/p^{n+1}\mathbb{Z})G^- \). By \( \mathbb{Z}pG \)-linearity in the second variable, it factors through the projection on \( \bigwedge_{\mathbb{Z}G}^d U^1(K_p)^- \). An important and simple special case is when \( \eta \) equals \((1 \otimes \varepsilon_1) \wedge \ldots \wedge (1 \otimes \varepsilon_d) \in \bigwedge_{\mathbb{Z}G}^d U_S(K^+) \). Using Proposition 3 (iii) and equation (19)
and tracing through the definitions, we find that for all \( u_1, \ldots, u_d \) in \( U^1(K_p) \)
\[
H_{K/k,S,n}(1 \otimes \varepsilon_1) \wedge \cdots \wedge (1 \otimes \varepsilon_d), u_1 \wedge \cdots \wedge u_d \) =
\[
2^d \delta_n^*(\Delta_{f_{u_1}, \ldots, f_{u_d}}(\varepsilon_1 \wedge \cdots \wedge \varepsilon_d))
\]
\[
= \delta_n^*(\det \left( \sum_{h \in G} [h^{-1} \varepsilon_i, u_i]_{K,n} h_{i,t=1} \right)^d)
\]
\[
= \det \left( \delta_n^*(\sum_{g \in G} [g^{-1} \varepsilon_i, u_i]_{K,n} \pi_{K/K^\circ}(g))_{i,t=1} \right)^d
\]
\[
= \det \left( \pi_n^*(\sum_{g \in G} [\kappa_n(g)[g^{-1} \varepsilon_i, u_i]_{K,n} g^{-1})_{i,t=1} \right)^d
\]

But \( \sum_{g \in G} \kappa_n(g)[g^{-1} \varepsilon_i, u_i]_{K,n} g^{-1} \) clearly lies in the minus part and Equation (18) allows us to rewrite it as \( \sum_{g \in G} [\varepsilon_i, gu_i]_{K,n} g^{-1} \). Thus we obtain simply
\[
H_{K/k,S,n}(1 \otimes \varepsilon_1) \wedge \cdots \wedge (1 \otimes \varepsilon_d), u_1 \wedge \cdots \wedge u_d = \det \left( \sum_{g \in G} [\varepsilon_i, gu_i]_{K,n} g^{-1} \right)_{i,t=1}^d
\] (20)

This shows in particular that on \( \bigwedge_{Z^{0}_G} U_S(K^+) \times \bigwedge_{Z^{0}_G} U^1(K_p) \), the pairing \( H_{K,n}(\alpha(\cdot), \cdot) \) agrees with that defined by the pairing \( H_{K,n}(\cdot, \cdot) \) of [S05].

**Remark 2.5** If \( S \supset S' \supset S^0 \) we shall always view the natural injection \( \bigwedge_{Z^0_G} QU_S(K^+) \to \bigwedge_{Z^0_G} QU_S(K^+) \) as an inclusion. It is then a simple exercise to check ‘compatibility of the pairings as \( S \) varies’ in the sense that \( \Lambda_{0,S} \) contains \( \Lambda_{0,S'} \) and \( H_{K/k,S,n} \) agrees with \( H_{K/k,S',n} \) on \( Z(p) \Lambda_{0,S'} \times \bigwedge_{Z^0_G} U^1(K_p) \). For this reason, we shall usually omit the reference to \( S \) and write simply \( H_{K/k,n} \).

### 2.4 The Map \( \mathfrak{s}_{K/k,S} \)

For the time being we drop Hypothesis 2 and the assumption that \( K \) contains \( \mu_{p^n+1} \). We use the element \( a_{K/k,S} \) to define a generalisation of the map \( \mathfrak{s}_{K/k} \) of [S05] (slightly modified). Let \( j \) be any embedding of \( \bar{Q} \) into a fixed algebraic closure \( \bar{Q}_p \) of \( Q_p \). For each \( i = 1, \ldots, d \), the composite \( j_\tau_i : \bar{Q} \to \bar{Q}_p \) defines a prime ideal \( \mathfrak{p}_i \) of \( \mathcal{O}_K \) dividing \( p \), namely \( \mathfrak{p}_i = \{ a \in \mathcal{O}_K : |j_\tau_i(a)|_p < 1 \} \). (Of course the ideals \( \mathfrak{p}_1, \ldots, \mathfrak{p}_d \) are not in general distinct.) So \( j_\tau_i \) gives rise to an isometric embedding \( K_{Q_p} \to \bar{Q}_p \) (with the appropriately normalised \( \mathfrak{p}_i \)-adic metric on \( K_{Q_p} \)) whose image is the topological closure \( \bar{j_\tau_i(K)} \). This embedding is also denoted \( j_\tau_i \), by abuse. There is a composite homomorphism of \( Q_p \)-algebras
\[
\delta_i = \delta_i^{(j)} := j_\tau_i \circ \iota_{Q_p} : K_p \to \bar{Q}_p
\]

where \( \iota_{Q_p} : K_p \to K_{Q_p} \) is as in the previous section. It follows in particular that if \( u \) lies in \( U^1(K_p) \subset K_p \) then \( |\delta_i^{(j)}(u)| - 1 |_p < 1 \) for all \( i \), hence the element \( \log_p(\delta_i^{(j)}(u)) \) of \( \bar{j_\tau_i(K)} \) is
given by the usual logarithmic series. In Proposition/Definition 1 we take \( R = \mathbb{Z}_p \), \( S = \bar{\mathbb{Q}}_p \), \( M = U^1(K_p) \), \( H = G \), \( l = d \) and set \( f_i(u) := \log_p(\delta_i^{(j)}(u)) \) \( \forall u \in U^1 \), \( i \in \{1, \ldots, d\} \) to get a \( p \)-adic regulator map \( R_{K/k,p}^{(j)} := \Delta_{f_1,\ldots,f_d} : \bigwedge^d \mathbb{Z}_p U^1(K_p) \to \bar{\mathbb{Q}}_p G \). (We will denote it \( R_{K/k,p}^{(j)} \) or \( R_{K/k,p}^{(j_1,\ldots,j_d)} \) if we need to indicate the dependence on \( j \) and/or \( \tau_1, \ldots, \tau_d \).) For any abelian group \( H \) and commutative ring \( \mathcal{R} \) we define an involutive automorphism \( * \) of \( \mathcal{R} H \) by setting \( (\sum a_h h)^* = \sum a_h h^{-1} \). The element \( a_{K/k,S} \) lies in \( \mathbb{Q}_pG^- \) by (10), hence so does \( a_{K/k,S}^{-*} \) and applying \( j \) to the coefficients we obtain an element \( j(a_{K/k,S}^{-*}) \) of \( \mathbb{Q}_pG^- \).

**Definition 1** For any \( \theta \in \bigwedge^d \mathbb{Z}_p U^1(K_p) \) we define \( s_{K/k,S}(\theta) = s_{K/k,S,p}(\theta) \) to be the product \( j(a_{K/k,S}^{-*})R_{K/k,p}^{(j)}(\theta) \) in \( \bar{\mathbb{Q}}_p G \).

**Remark 2.6** It is easy to see that permuting the \( \tau_i \) can only change the sign of the regulator \( R_{K/k,p}^{(j)} \) and hence of the map \( s_{K/k,S} \) and that if \( \tau_1 \) is replaced by \( \tau_1 \tau \) for some \( \tau \in \text{Gal}(\mathbb{Q}/k) \) then both are multiplied by \( \tau|_K \in G \). If clarity demands it we shall indicate this (simple) dependence on the \( \tau_i \) by writing \( s_{K/k,S}^{\tau_1,\ldots,\tau_d} \) instead of \( s_{K/k,S} \).

If \( s_{K/k} \) denotes the map introduced in Definition 3.1 of [So5] then (9) gives
\[
s_{K/k,S}(\theta) = e^{-} \prod_{q \in S',S^0} (1 - Nq^{-1}\sigma_q) j(\sqrt{d_k}\Phi_{K/k}(0)^*)R_{K/k,p}^{(j)}(\theta) = e^{-} \prod_{q \in S',S^0} (1 - Nq^{-1}\sigma_q) s_{K/k}(\theta)
\]
and if \( k \neq \mathbb{Q} \) then we can even drop the factor \( e^{-} \). Equation (21) and Proposition 3.4 of ibid. imply the important

**Proposition 5** \( s_{K/k,S}(\theta) \) lies in \( \mathbb{Q}_pG^- \) for every \( \theta \in \bigwedge^d \mathbb{Z}_p U^1(K_p) \). Moreover it is independent of the choice of \( j \).

In [So5], \( s_{K/k} \) was considered as a \( (\mathbb{Z}_p G) \)-linear map from \( \bigwedge^d \mathbb{Z}_p U^1(K_p) \) to \( \mathbb{Q}_p G \). But because of the factor \( e^{-} \) in (21), we now have \( s_{K/k,S}(e^{-}\theta) = e^{-} s_{K/k,S}(\theta) = s_{K/k,S}(\theta) \) for this reason, we prefer to consider \( s_{K/k,S} \) as a \( \mathbb{Z}_p G \)-linear map from \( \bigwedge^d \mathbb{Z}_p U^1(K_p)^- \) to \( \mathbb{Q}_p G^- \).

**Proposition 6** The kernel of \( s_{K/k,S} \) is precisely the \( (\mathbb{Z}_p^-) \)-torsion submodule of \( \bigwedge^d \mathbb{Z}_p U^1(K_p)^- \) which is finite. The image of \( s_{K/k,S} \) is a fractional ideal of \( \mathbb{Q}_p G^- \) (i.e. a finitely generated \( \mathbb{Z}_p G \)-submodule of \( \mathbb{Q}_p G^- \) which spans it over \( \mathbb{Q}_p \)).

**Proof** In Remark 3.2 of [So5] it was shown that \( \ker(R_{K/k,p}^{(j)}) \) is finite and that \( \text{im}(R_{K/k,p}^{(j)}) \) spans \( \mathbb{Q}_p G \) over \( \mathbb{Q}_p \). Also, Equation (10) implies that \( j(a_{K/k,S}^{-*}) \) is a unit of the ring \( \mathbb{Q}_p G^- \).

It follows that \( \ker(s_{K/k,S}) \) lies in \( \ker(R_{K/k,p}^{(j)}) \) and hence in \( \left( \bigwedge^d \mathbb{Z}_p U^1(K_p)^- \right)_\text{tor} \). The reverse inclusion is clear, since \( \mathbb{Q}_p G^- \) is torsion-free. For the second statement, finite-generation follows from that of \( U^1(K_p) \) and we have \( \mathbb{Q}_p\text{im}(s_{K/k,S}) = \mathbb{Q}_p G^-\text{im}(s_{K/k,S}) = \mathbb{Q}_p G^-\text{im}(R_{K/k,p}^{(j)}) = \mathbb{Q}_p G^- \) Since \( \mathbb{Q}_p G^- \) contains a \( \mathbb{Q}_p \)-basis of \( \mathbb{Q}_p G^- \), it follows that \( \mathbb{Q}_p\text{im}(s_{K/k,S}) = \mathbb{Q}_p G^- \). \( \square \)
Definition 2 We set \( S_{K/k,S} = \text{im}(s_{K/k,S,p}) \subset \mathbb{Q}_p G^- \). (Proposition \( \ref{Proposition} \) and Remark \( \ref{Remark} \) show that \( S_{K/k,S} \) is independent of \( j \) and the choice and ordering of the \( \tau_i \)'s.)

Thus
\[
S_{K/k,S} = e^{-\prod_{q \in S \setminus S^0} (1 - Nq^{-1} \sigma_q)} S_{K/k} \tag{22}
\]

where \( S_{K/k} = \text{im}(s_{K/k}) \) as in \([So5]\), and if \( k \neq Q \) then we can drop the factor \( e^- \). Finally, the dependence of \( s_{K/k,S} \) and \( S_{K/k,S} \) on \( S \) is clear: if \( S \supset S' \supset S^0 \) then (4) and the definition of \( a_{K/k,S} \) give
\[
s_{K/k,S} = \prod_{q \in S \setminus S'} (1 - Nq^{-1} \sigma_q) s_{K/k,S'} \quad \text{and} \quad S_{K/k,S} = \prod_{q \in S \setminus S'} (1 - Nq^{-1} \sigma_q) S_{K/k,S'} \tag{23}
\]

3 Statements of the Conjectures

Let us write \( S_p \) for \( S_p(k) \) and \( S^1 = S^1(K/k) \) for \( S_p \cup S^{0} = S_p \cup S_{\text{ram}}(K/k) \cup S_{\infty} \).

Hypothesis 3 \( S \) contains \( S^1 \)

Henceforth, the three conditions \( p \neq 2 \), Hypothesis \( \ref{Hypothesis} \) and Hypothesis \( \ref{Hypothesis} \) will be referred to as ‘the standard hypotheses’ and will be assumed to hold unless it is explicitly stated otherwise. Our ‘Integrality Conjecture’ (IC) reads:

**Conjecture** \( IC(K/k, S, p) \) \( S_{K/k,S} \subset \mathbb{Z}_p G^- \).

Remark 3.1 By using \([So5]\ Cor. 2.1\) and estimates of \( \log_p \) one can find explicit values of \( N \) such that \( S_{K/k,S} \subset p^{-N} \mathbb{Z}_p G^- \) (cf. the proof of Prop. 4.2, ibid.). The conjecture says we can take \( N = 0 \). Fixing \( K/k \) but letting \( p \) (hence \( S^1 \)) vary, one can also show that \( S_{K/k,S^1,p} = \mathbb{Z}_p G^- \) for all but finitely many \( p \neq 2 \). In fact, this follows easily from Theorem \( \ref{Theorem} \).

Remark 3.2 Equation (22) gives
\[
S_{K/k,S^1} = e^{-\prod_{p \in S_p \setminus S_{\text{ram}}} (1 - Np^{-1} \sigma_p)} S_{K/k} = e^{-\left( \prod_{p \in S_p \setminus S_{\text{ram}}} Np \right)^{-1}} S_{K/k}
\]

(For the second equality, observe that if \( p \) lies in \( S_p \setminus S_{\text{ram}} \) then \( Np - \sigma_p \) is a unit of \( \mathbb{Z}_p G^- \).)

If \( k \neq Q \) we may, as usual, drop the factor \( e^- \) in the last equation. It follows in particular that if \( k \neq Q \) then \( IC(K/k, S^1, p) \) is equivalent to Conjecture 5.2 of \([So3]\ § 5.2\). If \( k = Q \) the latter conjecture was proven in ibid.. \( IC(K/Q, S^1, p) \) follows on applying \( e^- \) and will be re-proven in Theorem \( \ref{Theorem} \).

Hypothesis \( \ref{Hypothesis} \) implies Hypothesis \( \ref{Hypothesis} \) so that the conditions of Conjecture \( RSC(K^+/k, S; \mathbb{Z}(p)) \) are met. Our ‘Congruence Conjecture’ (CC) reads:
Conjecture $CC(K/k, S, p, n)$ (Congruence Conjecture)
Suppose that Conjecture $IC(K/k, S, p)$ holds and that $RSC(K^+/k, S; \mathbb{Z}(p))$ holds with solution $\eta_{K^+/k, S}$. If also $K \supset \mu_{p^{n+1}}$ for some $n \geq 0$ then, for all $\theta \in \bigwedge^d_{\mathcal{Z}G} U^1(K_p)^-$ we have

$$\mathcal{S}_{K/k,S}(\theta) = \kappa_n(\tau_1 \ldots \tau_d) H_{K/k,n}(\eta_{K^+/k, S}, \theta) \quad (\text{in } \mathbb{Z}/p^{n+1}\mathbb{Z})$$

(24)

Remark 3.3 The factor $\kappa_n(\tau_1 \ldots \tau_d)$ means that the Congruence Conjecture is independent of the choice of $\tau_1, \ldots, \tau_d$. For example, if we replace $\tau_i$ by $\tau_i^{-1}$ for some $\tau \in \text{Gal}(\bar{\mathbb{Q}}/k)$ then Remark 2.2, Proposition 4 (i) and (19) show that the R.H.S. is multiplied by $\tau|_K^{-1} \in G$ and the same is true for the L.H.S. by Remark 2.6.

Remark 3.4 $CC(K/k, S, p, n)$ replaces Conjecture 5.4 of \cite{So5}. The latter is essentially the special case of the CC in which $S = S^1$ and $p$ splits in $k$ (so that $\mu_p \subset K$ forces $S^0 = S^1$).

In fact, it is a direct consequence of this case provided one assumes (with no significant loss of generality) that $K$ is CM, $k \neq \mathbb{Q}$ and one replaces $\bigwedge^d_{\mathcal{Z}G} K^+$ in Conjecture 5.4 with $\bigwedge^d_{\mathcal{Z}G} U_S(K^+)$ as here. The awkwardness in the formulation of Conjecture 5.4 (using $\mathcal{I}(\eta_{K/k}^+)$, $\tilde{\eta}$ etc.) has been avoided in the CC thanks to our ‘extension’ of $H_{K,n}$ to $\mathbb{Z}(p)\Lambda_0,S$.

4 Evidence for the IC and the CC

4.1 The Results of \cite{So5}

Conjecture 5.2 of \cite{So5} implies $IC(K/k, S, p)$ for $S = S^1$ (see Remark 3.2) and hence for all $S$ by Prop.9 Therefore Proposition 4.2 of \cite{So5} translates to

Theorem 1 $IC(K/k, S, p)$ holds whenever $p$ is unramified in $K/\mathbb{Q}$.

Similarly, the main result, Theorem 4.1, of \cite{So5} gives

Theorem 2 $IC(K/k, S, p)$ holds whenever $p$ is splits completely in $k/\mathbb{Q}$ and either $S_{\text{ram}}(K/k) \not\subset S_p(k)$ or $\mu_p(K) = \{1\}$.

4.2 The IC and the ETNC

Working on the original version of the IC in \cite{So5}, Andrew Jones has shown that a certain refinement would follow from the ETNC (see the Introduction). Let $\text{Cl}_m(K)$ be the ray-class group of $K$ corresponding to the cycle which is the formal product of the finite places of $K$ above those in $S^1$ and write $\text{Fitt}_{\mathcal{Z}G}(\text{Cl}_m(K))$ for its (initial) Fitting ideal as a $\mathbb{Z}G$-module. In our notation, the first part of \cite{Jo} Theorem 4.1.1 then says that the relevant case of the ETNC (namely \cite{Bu} Conj. 4.(iv) for the pair $(h^{\text{Spec}}(K)(1), e^-\mathcal{Z}G)$) implies

$$\mathcal{S}_{K/k,S^1} \subset (Z_p \text{Fitt}_{\mathcal{Z}G}(\text{Cl}_m(K)))^- = (\text{Fitt}_{Z_pG^-}((\text{Cl}_m(K) \otimes Z_p)^-)$$

(25)
for all odd primes \( p \). The inclusion (25), hence the ETNC, clearly implies \( IC(K/k, S, p) \) for \( S = S^1 \), hence for all \( S \). Of course, it implies considerably more (for instance, that \( \mathfrak{S}_{K/k, S^1} \) annihilates the \( p \)-part of \( Cl_m(K) \)) and in this sense refines the IC in a different direction from the CC. We note that the relevant case of the ETNC has been proven in our set-up only when \( K \) is an absolutely abelian field (see below).

### 4.3 Strengthenings of the IC in the Case \( p \nmid |G| \)

The second part of Jones’ Theorem 4.1.1 states that if the above case of the ETNC holds and also \( p \nmid |G| \), then we have the following strengthening of (25)

\[
\mathfrak{S}_{K/k, S^1} = \begin{cases} 
\text{Fitt}_{\mathbb{Z}_p G^-}((\mu_p \infty(K) / \mu_{p^\infty}(K)) \text{ Fitt}_{\mathbb{Z}_p G^-}((Cl_m(K) \otimes \mathbb{Z}_p)^-) & \text{if } S_{\text{ram}}(K/k) \subset S_p \\
\text{Fitt}_{\mathbb{Z}_p G^-}((\mu_p \infty(K) / \mu_{p^\infty}(K)) \text{ Fitt}_{\mathbb{Z}_p G^-}((Cl_m(K) \otimes \mathbb{Z}_p)^-) & \text{if } S_{\text{ram}}(K/k) \not\subset S_p
\end{cases}
\]

Corollary 4.1.8 of [Jo] also establishes (26) when \( p \nmid |G| \) without assuming the ETNC but imposes a mild condition on the characters of \( G \). (The proof uses results of [Wi] and work of Bley, Burns and others on, roughly speaking, the compatibility of the ETNC with the functional equations of \( L \)-functions.)

Independently, we used the functional equations themselves and more elementary, index-type arguments to give a different (and unconditional) formula for \( \mathfrak{S}_{K/k, S} \) whenever \( p \nmid |G| \). This is presented as Theorem 6 in Section 6. Corollary 1 shows how one may quickly deduce \( IC(K/k, S, p) \) in this case. Of course, it would also follow immediately from Jones’ formula (26). In fact, there is a direct link between the two formulae, explained in Remark 6.1.

### 4.4 The Case \( k = \mathbb{Q} \)

When \( k = \mathbb{Q} \), the IC follows from Corollary 4.1 of [So5] (or indeed from the work of Jones, see below). In Section 7 we shall prove the CC in this case, re-proving the IC along the way:

**Theorem 3**

(i). Conjecture \( IC(K/\mathbb{Q}, S, p) \) holds and

(ii). If \( K \) contains \( \mu_{p^{n+1}} \) for some \( n \geq 0 \), then Conjecture \( CC(K/\mathbb{Q}, S, p, n) \) holds.

### 4.5 The Case of Absolutely Abelian \( K \)

As noted in [Jo] Cor. 4.1.7, the relevant case of the ETNC follows from [B-F] Cor. 1.2 whenever \( K \) is absolutely abelian and \( k \) is any totally real subfield (possibly but not necessarily equal to \( \mathbb{Q} \)). Thus the inclusion (25) holds and in particular

**Theorem 4** If \( K \) is an abelian extension of \( \mathbb{Q} \), then Conjecture \( IC(K/k, S, p) \) holds. \( \square \)

To state our result for the CC, in this case, we first define a set of rational primes \( \text{Bad}(S) := \{ q \in S_{\text{ram}}(k/\mathbb{Q}) : S_q(k) \not\subset S \} \) and formulate
Hypothesis 4  \( p \nmid e_q(k/Q) \) for all \( q \in \text{Bad}(S) \)

In Section 8 we shall show

Theorem 5  If \( K \) is an abelian extension of \( Q \) containing \( \mu_{p^{n+1}} \) for some \( n \geq 0 \) and Hypothesis 4 is satisfied, then Conjecture CC\((K/Q,S,p,n)\) holds.

(At the same time we shall obtain a second proof of Theorem 4 which assumes Hypothesis 4 but is independent of the ETNC.) The proof of Theorem 5 uses induction formulae for \( L \)-functions to relate the situation for \( K/k \) to that of \( F/Q \) for various CM subfields \( F \) of \( K \), and this in two parallel applications. The first concerns \( s_{K/k,S} \) and works at \( s = 1 \). The second concerns \( RSC(K^+/k;Z_{(p)}) \) and works at \( s = 0 \). Popescu introduced the latter application in [Po2]. (In fact, he applied it to his own variant of Rubin’s conjecture \( B' \) which also implies \( RSC(K^+/k,S;Z) \).) He worked under a hypothesis which implies \( \text{Bad}(S) = \emptyset \). This simplifies matters (we only need consider \( F = K \)) but is rather restrictive (e.g. \( \text{Bad}(S^1(K/k)) \neq \emptyset \) whenever a rational prime \( q \neq p \) ramifies in \( k/Q \) but not in \( K/k \)). The elaboration of Popescu’s techniques which allows us to conclude under our weaker Hypothesis 4 is one ingredient of Cooper’s work on Popescu’s Conjecture in [Coo].

Hypothesis 4 holds, for example, whenever \( p \nmid [k : Q] \) (e.g. \( [k : Q] \) is a power of 2). Alternatively, suppose \( K = Q(\xi_f) \) and \( k = K^+ \) where \( f = p^{n+1}f' \neq 2 \pmod{4}, n \geq 0 \) and \( p \nmid f' \). If we take \( S = S^1(K/k) = S_\infty \cup S_p \) then \( \text{Bad}(S) = \emptyset \Leftrightarrow f' = 1 \) but Hypothesis 4 holds provided only \( p \nmid q - 1 \) for all \( q \mid f' \).

4.6 Two ‘Trivial’ Cases of the Congruence (24)

Suppose \( K \supseteq \mu_{p^{n+1}} \) for some \( n \geq 0 \) and \( S \) contains at least \( d + 2 \) places and at least one finite place \( q \) that splits completely in \( K^+ \). Equations (11) and (12) imply \( \Theta_{K^+/k,S}^{(d)} = 0 \) so that \( RSC(K^+/k,S;k) \) holds with \( \eta_{K^+/k,S} = 0 \). The congruence (24) is thus equivalent to \( s_{K/k,S}(\theta) \in p^{n+1}Z_pG^- \). The extension \( K/K^+ = K^+(\mu_{p^{n+1}})/K^+ \) is unramified outside \( p \), so if \( q \) does not divide \( p \) then it cannot lie in \( S^1 \) (which forces \( |S| \geq d + 2 \)). We can then apply the following result. (For a case with \( q \mid p \), see the next subsection.)

Proposition 7  Suppose \( K \supseteq \mu_{p^{n+1}} \) and \( q \in S \setminus S^1(K/k) \) splits in \( K^+ \). If \( IC(K/k,S\setminus\{q\},p) \) holds (e.g. if \( p \nmid |G| \)) then \( \mathcal{S}_{K/k,S} \subset p^{n+1}Z_pG^- \). In particular, \( CC(K/k,S,p,n) \) holds.

Proof  By equation (23) it clearly suffices to show that \( p^{n+1} \) divides \((Nq-\sigma_q)e^-\). Since \( q \) splits in \( K^+ \), \( \sigma_q \) is either 1 or \( c \) and since \( q \nmid p \), it acts on \( \mu_{p^{n+1}} \) by \( Nq \). If \( \sigma_q = 1 \), it also acts trivially, so \( p^{n+1} \) divides \( Nq - 1 \) hence also \((Nq - 1)e^- = (Nq - \sigma_q)e^-\). If \( \sigma_q = c \), it also acts by \(-1 \), so \( p^{n+1} \) divides \( Nq + 1 \) hence also \((Nq + 1)e^- = (Nq - \sigma_q)e^-\).

Next, suppose \( \theta \) is a \( Z_p \)-torsion element in \( \bigwedge^d_{Z_pG} U^1(K_p)^- \). Proposition 3 implies that the L.H.S. of (24) vanishes, so, assuming \( RSC(K^+/k,S;Z_{(p)}) \), this congruence is equivalent to \( H_{K/k,S}^1(\eta_{K^+/k,S},\theta) = 0 \). If also \( p \nmid |G| \), then this is an immediate consequence of the following result, to be proved in Section 6. (The verification seems harder if \( p \mid |G| \) (and \( d \geq 2 \), not least because \( \left( \bigwedge^d_{Z_pG} U^1(K_p) \right)_{\text{tor}} \) is then harder to characterise.)
Proposition 8 Suppose \( p \nmid |G| \), \( K \supset \mu_{p^{n+1}} \) and \( \eta \) is any element of \( \mathbb{Z}_p \Lambda_{0,S}(K^+/k) \) satisfying the eigenspace condition w.r.t. \( (S,d,G) \). Then \( H_{K/k,n}(\eta,\theta) = 0 \) for all \( \theta \in \left( \bigwedge^d_{\mathbb{Z}_pG} U^1(K_p) \right)_{\text{tor}} \).

4.7 The Case \( k = K^+ \)

In this case \( G = \{e, c\} \) so \( p \nmid |G| \) and the IC holds for all admissible \( S \). For the CC, we assume \( K \supset \mu_{p^{n+1}} \) with \( n \geq 0 \) so that \( K = k(\mu_{p^{n+1}}) \) and \( S^1 = S_\infty \cup S_p \). All places of \( k \) split in \( K^+ \) so if \( S \neq S^1 \) then \( CC(K/k, S, p, n) \) holds by Proposition 7. Also, if \( |S_p(k)| \geq 2 \) then \( |S| \geq r + 2 \) so once again \( CC(K/k, S^1, p, n) \) is equivalent to \( \mathfrak{S}_{K/k}\mathcal{S} \subset p^{n+1}\mathbb{Z}_p G^- \) (see above). But this will follow from equation (32) in Section 6 (for the unique odd character \( \phi \). Indeed, the first term on the RHS of (32) is clearly divisible by \( (p^{n+1})^{S_p(-1)} \). This leaves only the case \( S = S_\infty \cup S_p \) with \( |S_p(k)| = 1 \). Then \( \eta_{k/k,S^1} \) is non-zero and can be written explicitly in terms of a \( \mathbb{Z} \)-basis \( \varepsilon_1, \ldots, \varepsilon_d \) for \( U_{S_\infty}(k)/\{\pm 1\} \) and the \( S_p \)-classnumber of \( k \). In this case \( CC(K/k, S^1, p, n) \) reduces to an apparently novel identity in \( \mathbb{Z}/p^{n+1}\mathbb{Z} \), relating a \( p \)-adic regulator of elements of \( U^1(K_p)^- \) to a determinant of their Hilbert symbols with the \( \varepsilon_i \). In ongoing work, Malcolm Bovey has done extensive computations verifying this identity and established some partial results supporting it.

4.8 Other Computational Results

\( RSC(K^+/k, S; \mathbb{Q}) \) is not currently known to hold non-trivially for any \( S \) unless either \( K^+ \) is absolutely abelian or all the characters \( \chi \in \hat{G} \) satisfying \( \text{ord}_{s=0} L_{K^+/k,S}(s, \chi) = d \) are of order 1 or 2. However, if \( d \) is not too large, high-precision computation can identify \( \eta_{K^+/k,S} \) with virtual certainty as the unique solution of (15) in \( e_{S,d,G} \bigwedge^d_{\mathbb{Q}G} \mathbb{Q}U_S(K^+) \). (This was done in [R-S1].) This makes it possible to check the CC (and simultaneously the IC) on a computer. In [R-S2] we give details of such numerical verifications for more than 40 cases of \( CC(K/k, S^1, p, n) \) with \( k \) real quadratic, \( n = 0 \) or 1 and varying \( K \) and \( p \).

5 Changing \( S, K \) and \( n \)

If \( q \) is a prime ideal of \( k \) not in \( S_p \) then \( (1 - Nq^{-1}\sigma_q) \) lies in \( \mathbb{Z}_p G \). Hence equation (23) gives

Proposition 9 If \( S \supset S' \supset S^1 \) then IC\((K/k, S', p)\) implies IC\((K/k, S, p)\). \( \square \)

Remark 5.1 For the converse implication one would need \( e^{-}(1 - Nq^{-1}\sigma_q) \) to be invertible in \( \mathbb{Z}_p G^- \) for each \( q \in S \setminus S' \). But for any such \( q \) one has an isomorphism of \( \mathbb{Z}_p G \)-modules

\[
\mathbb{Z}_p G / (1 - Nq^{-1}\sigma_q) \cong (\mathcal{O}_K/q\mathcal{O}_K)^x \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \bigoplus_q (\mathbb{F}_q^x \otimes_{\mathbb{Z}} \mathbb{Z}_p)
\] (27)
where \( \Omega \) runs through the primes dividing \( q \) in \( K \) and \( \mathbb{F}_\Omega \) denotes \( \mathcal{O}_K/\Omega \). We deduce that if \( c \notin D_q(K/k) \) (respectively, \( c \in D_q(K/k) \)) then \( e^{-} (1 - Nq^{-1}\sigma_q) \) lies in \((\mathbb{Z}_q G^-)^{\times}\) if and only if \( p \nmid |\mathbb{F}_\Omega^*| \) (respectively, \( p \nmid |\mathbb{F}_\Omega^*/(1 + c)\mathbb{F}_\Omega^*| \)) for one, hence any \( \Omega \). This fails in particular if \( \mu_p \subset K \), in which case \( IC(K/k, S, p) \) does not by itself imply \( IC(K/k, S', p) \) for any \( S \supseteq S' \).

**Proposition 10** Suppose \( S \supseteq S' \supseteq S^1 \) and \( \text{RSC}(K^+/k, S'; \mathbb{Z}(p)) \) holds with solution \( \eta_{K^+/k,S'} \). Then \( \text{RSC}(K^+/k, S; \mathbb{Z}(p)) \) holds with solution \( \eta_{K^+/k,S} = \prod_{q \in S \setminus S'} (1 - \sigma_q^{-1}) \eta_{K^+/k,S'} \).

**Proof** It follows easily from (14) and characterisation (iii) of the eigenspace condition that \( \prod_q (1 - \sigma_q^{-1}) \eta_{K^+/k,S'} \) is a solution of \( \text{RSC}(K^+/k, S; \mathbb{Q}) \). The result follows since \( \mathbb{Z}(p)\Lambda_{0,S'} \) is a \( \mathbb{Z}G \)-submodule of \( \mathbb{Z}(p)\Lambda_{0,S} \).

**Proposition 11** If \( K \supseteq \mu_{p^{n+1}} \) for some \( n \geq 0 \) and \( S \supseteq S' \supseteq S^1 \) then \( \text{CC}(K/k, S', p, n) \) implies \( \text{CC}(K/k, S, p, n) \).

**Proof** We assume that \( \text{CC}(K/k, S', p, n) \) holds so also \( IC(K/k, S', p) \) and \( \text{RSC}(K^+/k, S'; \mathbb{Z}(p)) \). Thus \( IC(K/k, S', p) \) and \( \text{RSC}(K^+/k, S'; \mathbb{Z}(p)) \) hold by Props. 9 and 10. Using the latter and Proposition 4, we find that for any \( \theta \in \bigwedge_{\mathbb{Z}_p G}^d U^1(K_p)^- \)

\[
\kappa_n(\tau_1 \ldots \tau_d)H_{K/k,n}(\eta_{K^+/k,S}, \theta) = \prod_{q \in S \setminus S'} (1 - \bar{\kappa}_n^*\sigma_q^{-1})\kappa_n(\tau_1 \ldots \tau_d)H_{K/k,n}(\eta_{K^+/k,S'}, \theta)
\]

\[
= \prod_{q \in S \setminus S'} (1 - \bar{\kappa}_n^*\sigma_q^{-1})s_{K/k,S'}(\theta) \text{ in } (\mathbb{Z}/p^{n+1}\mathbb{Z})G^-
\] (28)

For each \( q \in S \setminus S' \), equation (19) with \( g = \sigma_q^{-1} = \sigma_q^{-1} \) gives \( \bar{\kappa}_n^*\sigma_q^{-1} = e^{-}\kappa_n(\sigma_q^{-1})^{-1}q \) and since \( q \nmid p \) it follows that \( \kappa_n(\sigma_q^{-1}) = \bar{\kappa}_n(\sigma_q^{-1})^{-1}q \) on \( (\mathbb{Z}/p^{n+1}\mathbb{Z})G^- \) and combining (28) with (23) gives (24), as required.

Now suppose that \( F \) is any CM subfield of \( K \) containing \( k \). Then, \( p, F \) and \( S \) satisfy the standard hypotheses. We write \( G_F \) for \( \text{Gal}(F/k) \) and \( N_{K/F} \) for the norm map \( K_p \to F_p \). (If we identify \( K_p \) with \( \prod_q K_q \) and \( F_p \) with \( \prod_p F_p \), then \( N_{K/F} \) sends \( (x \varphi) \) to \( (y \varphi) \), where \( y_p = \prod_{q \mid p} N_{K_q/F_q} x \varphi \). We shall also write \( N_{K/F} \) for the \( \mathbb{Z}_p \)-linear map \( \bigwedge_{\mathbb{Z}_p G}^d U^1(K_p) \to \bigwedge_{\mathbb{Z}_p G}^d U^1(F_p) \) sending \( u_1 \wedge \ldots \wedge u_d \) to \( N_{K/F} u_1 \wedge \ldots \wedge N_{K/F} u_d \). One checks easily that \( \pi_{K/F} \circ R_{K/k,p}^{(j)} = R_{F/k,p}^{(j)} \circ N_{K/F} \) and also that \( \pi_{K/F} \circ \Theta_{K/k,S} = \Theta_{F/k,S} \) (as meromorphic functions \( \mathbb{C} \to \mathbb{C}G_F \)) so that \( \pi_{K/F}(a \bar{\kappa}_{K/k,S}) = a \bar{\kappa}_{F/k,S} \) We deduce easily

**Proposition 12** If \( K \supseteq F \supseteq k \) as above then \( \pi_{K/F} \circ \bar{\kappa}_{K/k,S} = \bar{\kappa}_{F/k,S} \circ N_{K/F} \). In particular, if \( N_{K/F} : \bigwedge_{\mathbb{Z}_p G} U(F_p) \to \bigwedge_{\mathbb{Z}_p G} U(F_p) \) is surjective then \( \pi_{K/F}(\bar{\kappa}_{K/k,S}) = \bar{\kappa}_{F/k,S} \), so \( IC(K/k, S, p) \) implies \( IC(F/k, S, p) \).

\[\square\]
Remark 5.2 The surjectivity condition is certainly satisfied whenever \( N_K/F(U^1(K_p)) = U^1(F_p) \). The latter condition holds iff \( K/F \) is at most tamely ramified at each prime in \( S_p(F) \) (by local class-field theory). Of course, it actually suffices that \( N_K/F(U^1(K_p)^-) = U^1(F_p)^- \) which can be shown to be equivalent to the statement ‘\( K/F^+ \) is at most tamely ramified at each prime in \( S_p(F^+) \) which splits in \( F^+ \)’. One can also show that \( v_K/F \circ g_{F/k,S} = |G|^{-d_S} \eta \circ i_K/F \) where \( i_K/F \) is the natural map \( \bigwedge^d_{\mathbb{Z}_p G_F} U^1(F_p)^- \to \bigwedge^d_{\mathbb{Z}_p G_F} U^1(K_p)^- \), but for present purposes this is only helpful when \( p \nmid |G| \) or \( d = 1 \).

Let \( \tilde{G}_F = \text{Gal}(F^+/k) \). The norm \( N_{K/F^+} \) maps \( U_S(K^+) \) into \( U_S(F^+) \). We also use the symbol ‘\( N_{K/F^+} \)’ to denote the map \( 1 \otimes N_{K/F^+} : \mathbb{Q} U_S(K^+) \to \mathbb{Q} U_S(F^+) \) and the \( \mathbb{Q} \)-linear map \( \bigwedge^d_{\mathbb{Q} G} \mathbb{Q} U_S(K^+) \to \bigwedge^d_{\mathbb{Q} G} \mathbb{Q} U_S(F^+) \) sending \( x_1 \land \ldots \land x_d \) to \( N_{K/F^+} x_1 \land \ldots \land N_{K/F^+} x_d \).

Proposition 13 Suppose \( K \supset F \supset k \) as above and \( \text{RSC}(K^+/k,S; \mathbb{Q}) \) holds with solution \( \eta_{K^+/k,S} \). Then \( \text{RSC}(F^+/k,S; \mathbb{Q}) \) holds with solution \( \eta_{F^+/k,S} = N_{K/F^+} \eta_{K^+/k,S} \).

Proof \( \pi_{K/F^+}(N_{Dq(K^+/k)}) \) is a \( \mathbb{Z} \)-multiple of \( N_{Dq(F^+/k)} \) for all \( q \in S \setminus S_\infty \). It follows easily from this that form [iii] of the eigenspace condition on \( \eta_{K^+/k,S} \) (w.r.t. \( (S,d,G) \)) implies the same on \( N_{K/F^+} \eta_{K^+/k,S} \) (w.r.t. \( (S,d,G_F) \)). Similarly, since \( \pi_{K/F^+} \circ \Theta_{K^+/k,S} = \Theta_{F^+/k,S} \) and \( \pi_{K/F^+} \circ \tilde{R}_{K^+/k} = \tilde{R}_{F^+/k} \circ N_{K/F^+} \), if we apply \( \pi_{K/F^+} \) to condition [15] for \( \eta_{K^+/k,S} \) then we get the equivalent condition on \( N_{K/F^+} \eta_{K^+/k,S} \).

Before attacking the Congruence Conjecture in this context, we need two Lemmas.

Lemma 2 If \( d = 1 \) then \( N_{K/F^+}(\Lambda_{0,S}(K^+/k)) \subset \Lambda_{0,S}(F^+/k) \). If \( d > 1 \), then \( N_{K/F^+}(\Lambda_{0,S}(K^+/k)) \) is contained in \( e^{-d} \Lambda_{0,S}(F^+/k) \) where \( e = \exp((U_S(K^+))_\text{tor}(F^+)/(U_S(F^+))_\text{tor}) = 1 \) or 2.

Proof The first statement follows from that of Prop. [1]. Next, \( (U_S(K^+)/(U_S(F^+))_\text{tor} \) injects into \( \text{Hom}(\text{Gal}(K^+/F^+), \mu(K^+)) = \text{Hom}(\text{Gal}(K^+/F^+), \{\pm 1\}) \) by sending \( \varepsilon \) to the map \( g \mapsto g(\varepsilon)/\varepsilon \) so \( e = 1 \) or 2. Let \( (U_S(K^+)/(U_S(F^+))_\text{tor} = V/U_S(F^+) \) where \( U_S(K^+) \supset V \supset U_S(F^+) \). Since \( U_S(K^+)/V \) is torsionfree, \( U_S(K^+) \) splits over \( \mathbb{Z} \) as \( V \oplus V' \). The sum \( U_S(F^+) \) is also direct and contains \( U_S(F^+) \). Therefore, any \( f_1, \ldots, f_d \) lying in \( \text{Hom}_Z(U_S(F^+), \mathbb{Z}) \) (considered as a subset of \( \text{Hom}_\mathbb{Q}(\mathbb{Q} U_S(F^+), \mathbb{Q}) \) extend to \( \tilde{f}_1, \ldots, \tilde{f}_d \) in \( \text{Hom}_\mathbb{Z}(U_S(F^+) \oplus V', \mathbb{Z}) \) considered as a subset of \( \text{Hom}_\mathbb{Q}(\mathbb{Q} U_S(K^+), \mathbb{Q}) \) and \( e \tilde{f}_i \in \text{Hom}_\mathbb{Z}(U_S(K^+), \mathbb{Z}) \) for all \( i \). It is easy to see from the definitions that

\[
\pi_{K/F^+}(\Delta_{f_1, \ldots, f_d}(\eta)) = \Delta_{f_1, \ldots, f_d}(N_{K/F^+} \eta) \quad \text{for all } \eta \in \bigwedge^d_{\mathbb{Q} G} \mathbb{Q} U_S(K^+) \tag{29}
\]

Hence, if \( \eta \in \Lambda_{0,S}(K^+/k) \) then \( \Delta_{f_1, \ldots, f_d}(e^d N_{K/F^+} \eta) = \pi_{K/F^+}(\Delta_{e f_1, \ldots, e f_d}(\eta)) \) lies in \( Z G_F \). Letting the \( f_i \) vary, it follows that \( N_{K/F^+} \eta \) lies in \( e^{-d} \Lambda_{0,S}(F^+/k) \). (The proof shows that \( e = 1 \) if, for instance, \( |\text{Gal}(K^+/F^+)| = [K : F] \) is odd.) Suppose now that \( \mu_{F^+,1} \subset F \) for some \( n \geq 0 \) and that \( \Phi \in S_p(K) \) lies above \( p \in S_p(F) \), so we may regard \( F_p \) as a subfield of \( K_\Phi \). Basic properties of the Hilbert symbol show that
Lemma 3 Let $\eta \in \mathbb{Z}_p\Lambda_{0,S}(K^+/k)$ and $\theta \in \bigwedge_{\mathbb{Z}_p G}^d U^1(K_p)$. Then $N_{K^+/F^+}\eta$ lies in $\mathbb{Z}_p\Lambda_{0,S}(F^+/k)$ and $\pi_{K/F}(H_{K/k,n}(\eta, \theta)) = H_{F/k,n}(N_{K^+/F^+}\eta, N_{K/k}\theta)$.

By $\mathbb{Z}_p$-linearity in $\eta$ and the fact that $N_{K^+/F^+}(\eta)$ lies in $\Lambda_{0,S}(F^+/k) \subset \mathbb{Z}_p\Lambda_{0,S}(F^+/k)$. Similarly, we may assume that $\theta = u_1 \wedge \ldots \wedge u_d$ with $u_i \in U^1(K_p)$ for each $i$. If $\tilde{g}_i$ denotes the restriction of $f_i$ to $U_S(F^+)$ then equation (30) says that $\tilde{g}_i$ lifts the map $g_i := [\cdot, N_{K/F}u_i]_{F,n} \in \text{Hom}_\mathbb{Z}(U_S(F^+), \mathbb{Z}/p^{n+1}\mathbb{Z})$. Just as for (29) we find $\pi_{K^+/F^+}(\Delta_{f_1, \ldots, f_d}(\eta)) = \Delta_{\tilde{g}_1, \ldots, \tilde{g}_d}(N_{K^+/F^+}\eta)$ and since both sides lie in $Z_G F$, we can reduce modulo $p^{n+1}$ to get $\pi_{K^+/F^+}(\Delta_{f_1, \ldots, f_d}(\eta)) = \Delta_{\tilde{g}_1, \ldots, \tilde{g}_d}(N_{K^+/F^+}\eta)$. We conclude by applying $2^d\kappa_{F,n}$ to both sides and using $\kappa_{F,n} \circ \pi_{K^+/F^+} = \pi_{K/F} \circ \kappa_{K,n}$.

Proposition 14 Suppose $K \supset F \supset k$ as above and $N_{K/F} : \bigwedge_{\mathbb{Z}_p G}^d U^1(K_p)^- \to \bigwedge_{\mathbb{Z}_p G}^d U^1(F_p)^-$ is surjective. If $F \supset \mu_{p^{n+1}}$ for some $n \geq 0$ then $CC(K/k, S, p, n)$ implies $CC(F/k, S, p, n)$.

Proof We assume that $CC(K/k, S, p, n)$ holds, so also $IC(F/k, S, p)$ holds and $RSC(F^+/k, S; \mathbb{Z}_p)$ holds with solution $\eta_{K^+/k,S}$, say. Prop. 12 implies $IC(F/k, S, p)$. Moreover, Prop. 13 and Lemma 3 imply $RSC(F^+/k, S; \mathbb{Z}_p)$ and that for any $\theta \in \bigwedge_{\mathbb{Z}_p G}^d U^1(K_p)^-$ we have

$\kappa_n(\tau_1 \ldots \tau_d)_{F/k,n}(\eta_{F^+/k,S}, N_{K/F}\theta) = \kappa_n(\tau_1 \ldots \tau_d)_{F/k,n}(N_{K^+/F^+}\eta_{K^+/k,S}, N_{K/F}\theta) = \pi_{K/F}(\kappa_n(\tau_1 \ldots \tau_d)_{K/k,n}(\eta_{K^+/k,S})) = \pi_{K/F}(\kappa_n(\eta_{K^+/k,S})) = \kappa_n(\eta_{K/F,S})$

The result now follows from the surjectivity condition.

Finally, if $n \geq n' \geq 0$ then $H_{K,n}(\eta, \theta) \equiv H_{K,n'}(\eta, \theta) \mod p^{n'+1}$ for all $\eta \in \mathbb{Z}_p\Lambda_{0,S}(K^+/k)$ and $\theta \in \bigwedge_{\mathbb{Z}_p G}^d U^1(K_p)^-$. (The proof is an exercise using the definitions of the Hilbert symbol, $[\cdot, \cdot]_{K,n}, \tilde{\Delta}, H_{k,n}, \kappa_n$ etc. and the fact that $\zeta_n^{p^{n-n'}-1} = \zeta_{n'}^1$). One deduces easily

Proposition 15 If $K \supset \mu_{p^{n+1}}$ for some $n \geq 0$ then $CC(K/k, S, p, n)$ implies $CC(K/k, S, p, n')$ for all $n'$ with $n \geq n' \geq 0$. \hfill \Box
6 The Case $p \nmid |G|$

Let $X_{Q_p}$ denote the set of irreducible $Q_p$-valued characters of $G$ which is in natural bijection with $\text{Gal}(Q_p/Q_p)$-conjugacy classes of absolutely irreducible characters $\phi \in \text{Hom}(G, Q_p^\times)$. (Precisely, if $\Phi$ lies in $X_{Q_p}$ then its idempotent $e_\Phi \in Q_p G$ splits in $\hat{Q}_p G$ as the sum of the idempotents $e_\phi$ where $\phi$ runs once through the conjugacy class corresponding to $\Phi$). We shall say that the characters $\phi$ in this conjugacy class belong to $\Phi$ and we shall call $\Phi$ odd iff one – hence any – such $\phi$ is odd (i.e. $\phi(c) = -1$). Henceforth we set $a := \mathbb{Z}_p G$ and $a_\Phi := e_\Phi \mathbb{Z}_p G$.

Any $\phi$ belonging to $\Phi$ extends $\mathbb{Q}_p$-linearly to $\Theta := \mathbb{Q}_p G$ to a homomorphism $\mathbb{Q}_p G \to F_\phi := \mathbb{Q}_p(\phi)$ which in turn restricts to $\mathbb{Q}_p G \to F_\phi$ and from $a_\Phi$ to $O_\phi := \mathbb{Z}_p[\phi]$, the ring of valuation integers of $F_\phi$. In particular, $a_\Phi$ is a complete d.v.r., hence a p.i.d.

For the rest of this section we suppose that the prime $p$ does not divide $|G|$. This means that the idempotent $e_\Phi$ lies in $\mathbb{Z}_p G$ for each $\Phi \in X_{Q_p}$ so that $a$ is a product $\prod_{\Phi \in X_{Q_p}} a_\Phi$. Any $a$-module $M$ splits as a corresponding direct sum $\bigoplus_{\Phi \in X_{Q_p}} M_\Phi$, where $M_\Phi$ denotes the $a_\Phi$-module $e_\Phi M$, and $M \to M_\Phi$ is an exact functor. Since any $\phi$ belonging to $\Phi$ has order prime to $p$, a uniformiser of $O_\phi$ – hence of $a_\Phi$ – is given by $p$. The $a_\Phi$-order ideal $[N]_{a_\Phi}$ of any finite (=finite length) $a_\Phi$-module $N$ is therefore $p^l a_\Phi$ where $l$ is the length of any $a_\Phi$-composition series for $N$. We shall assume the usual properties of the order ideal, such as multiplicity in exact sequences. Each $p$-adic-valued character $\phi \in \text{Hom}(G, Q_p^\times)$ corresponds to a unique complex character $\chi \in G$ such that $\phi = j \circ \chi$ where $j$ is the fixed embedding $\mathbb{Q} \to \mathbb{Q}_p$. We write $\hat{\chi}$ and $\hat{\phi} = j \circ \hat{\chi}$ respectively for the associated complex and $p$-adic primitive ray-class characters, $f_\phi$ for $f_{\hat{\chi}}$ and $K^\phi$ for the field $K^\ker(\phi) = K^\ker(\chi)$ cut out by $\phi$, so that $\chi$ and $\phi$ factor through $G_\phi := \text{Gal}(K^{\phi}/k)$.

Work of Siegel [81] and Klingen (see also Shintani [Sh Cor. to Thm. 1]) implies that $\Theta K^{\phi/k, S_{\infty}}(0)$ lies in $\mathbb{Q}G_\phi$ so that $L(0, \hat{\chi}^{-1}) = \chi(\Theta K^{\phi/k, S_{0}(K^{\phi}/k)}(0))$ lies in $\mathbb{Q}(\chi)$. Thus $j(L(0, \hat{\chi}^{-1})) = \hat{\phi}(\Theta K^{\phi/k, S_{0}(K^{\phi}/k)}(0))$ lies in $F_\phi$ and is independent of $j$ so, by a slight abuse of notation, we write it simply as $L(0, \hat{\phi}^{-1})$.

**Theorem 6** If $p \nmid |G|$ then, for any odd $\Phi \in X_{Q_p}$ and $\phi \in \text{Hom}(G, Q_p^\times)$ belonging to $\Phi$, we have

$$\phi(\mathfrak{S}_{K/k,S}) = \phi([U^1(K_p)_{\text{tor}}]_{a_\Phi}) \prod_{q \in \mathfrak{S} \setminus \mathfrak{S}_{\infty}} \left(1 - Nq^{-1} \hat{\phi}([q])\right) L(0, \hat{\phi}^{-1})$$

(31)

(an equality of fractional ideals of $F_\phi$) where $L(0, \hat{\phi}^{-1})$ is as defined above.

Equation (31) for each $\Phi$ clearly determines $\mathfrak{S}_{K/k,S}$. Before giving the proof, we reformulate it and deduce some consequences. Firstly, $U^1(K_p)_{\text{tor}}$ is nothing but $\mu_{p^\infty}(K_p) = \prod_{q \mid p} \mu_{p^\infty}(K_q)$. Next, for given $\phi$ as above we define a $\mathbb{Z}_p G_\phi$ submodule of $\mathbb{Q}_p G_\phi$ by

$$J_\phi := \text{ann}_{\mathbb{Z}_p G_\phi}(\mu_{p^\infty}(K^{\phi})) \Theta K^{\phi/k, S_{0}(K^{\phi}/k)}(0)$$

Since $p \nmid [K : K^{\phi}]$, we have

$$\phi(\nu_{K/K^{\phi}}(J_\phi)) = \phi(\text{ann}_{a}(\mu_{p^\infty}(K^{\phi}))\nu_{K/K^{\phi}}(\Theta K^{\phi/k, S_{0}(K^{\phi}/k)}(0)))$$

23
Remark 6.1 (the last equation because \( \mu_p(\phi) \) is cyclic over \( \mathbb{Z} \), so over \( \mathfrak{a}_\phi \)). Thus we may reformulate (31) as

\[
\phi(\mathfrak{S}_{K/k,S}) = \phi((\mu_p(\phi)(K^\phi)[K : K^\phi])_{\mathfrak{a}_\phi}) \prod_{q \in S_1 \setminus S_\infty} \left( 1 - Nq^{-1} \hat{\phi}([q]) \right) \phi(\nu_{K/K^\phi}(J_\phi)) \quad (32)
\]

But \( J_\phi \) is spanned over \( \mathbb{Z}_p \) by \( \text{ann}_{\mathbb{Z}G_\phi}(\mu(\phi))\Theta_{K^\phi/k,S(S^0(\nu_{K^\phi})^0)(0)} \) which lies in \( \mathbb{Z}G_\phi \) by the well-known result of Deligne-Ribet and (independently) Pi. Cassou-Noguès (see Théorème 6.1 of [1a, p. 107]). Hence \( J_\phi \subset \mathbb{Z}_pG_\phi \) and so (32) implies that \( \phi(\mathfrak{S}_{K/k,S}) \subset \mathcal{O}_\phi \) for all odd \( \phi \in \text{Hom}(G, \mathbb{Q}_p^\times) \). Consequently,

**Corollary 1** If \( p \nmid |G| \) then \( IC(K/k, S, p) \) holds.

Remark 6.1 We explain the relation between Jones’ formula (26) and our Theorem 6, recast as equation (32) for all odd \( \phi \): The ray-class group \( \text{Cl}_m(K) \) appearing in (26) fits into an exact sequence of \( \mathbb{Z}G \)-modules:

\[
0 \to \mathbb{O}_K^\times \to \prod_{q \in S_1 \setminus S_\infty} \prod \mathcal{O}_{K/\mathfrak{q}}^\times \to \text{Cl}_m(K) \to \text{Cl}(K) \to 0
\]

(where the first non-zero term is simply the image of \( \mathcal{O}_K^\times \) in the second). Now tensor this sequence with \( \mathbb{Z}_p \) and take minus parts. Using the fact \( (\mathcal{O}_K^\times \otimes \mathbb{Z}_p)^- = \mu_p(\phi)(K) \) and isomorphisms similar to (27) one finds with a little work that (26) is equivalent to the following for each odd \( \phi \) as in Theorem 6

\[
\phi(\mathfrak{S}_{K/k,S^1}) = \phi((\mu_p(\phi)(K^\phi)/(\mu_p(\phi)(K)))_{\mathfrak{a}_\phi}) \prod_{q \in S_1 \setminus S_\infty} \left( 1 - Nq^{-1} \hat{\phi}([q]) \right) \phi((\text{Cl}(K) \otimes \mathbb{Z}_p)_{\mathfrak{a}_\phi})
\]

Since \( p \nmid [K : K^\phi] \), one sees that this in turn is equivalent to our (32) (with \( S = S^1 \)) if and only if \( \phi(J_\phi) = \phi((\text{Cl}(K^\phi) \otimes \mathbb{Z}_p)_{\mathfrak{a}_\phi}) \) (where \( \Phi \) and \( \phi \) are now considered as odd characters of \( G_\phi \)). But Theorem 3 of [1b] establishes the latter equality subject to a rather mild condition (‘\( S_{\phi,p} = 0 \)’) on the character \( \phi \).

**Proof of Theorem 6** For each \( i = 1, \ldots, d \), we write \( \mathfrak{p}_i \) for \( \mathfrak{p}_i \cap k \) (namely the prime ideal in \( S_p(k) \) which is defined by the embedding \( j\tau_i : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p \)). The map \( \{1, \ldots, d\} \to S_p(k) \) sending \( i \) to \( \mathfrak{p}_i \) is clearly surjective so for any \( \mathfrak{p} \in S_p(k) \) we write \( I(\mathfrak{p}) \) for its fibre over \( \mathfrak{p} \) and choose an element \( i(\mathfrak{p}) \in I(\mathfrak{p}) \). Thus \( \mathfrak{p}_{i(\mathfrak{p})} \cap k = \mathfrak{p}_{i(\mathfrak{p})} \mathfrak{p}, \forall \mathfrak{p} \in S_p(k) \) and the extension \( K_{\mathfrak{p}_{i(\mathfrak{p})}}/k \) is Galois with group \( D_p(K/k) \) of order prime to \( p \). It follows (\textit{e.g.} by a theorem
of E. Noether, since $K_{\mathfrak{p}(p)}/K_p$ is tame that we may choose an element $b_p \in O_{K_{\mathfrak{p}(p)}}$ freely generating $O_{K_{\mathfrak{p}(p)}}$ over $O_{K_p} D_p(K/k)$. Let $b$ be the element of $O_{K_p} := \prod_{\mathfrak{p} \in S_p(K)} O_{K_{\mathfrak{p}}}$ whose component in $O_{K_p}$ is $b_p$ whenever $\mathfrak{p} = \mathfrak{p}(p)$ for some $p \in S_p(k)$ and is 0 otherwise. Then $b$ is a free generator for $O_{K_p}$ over $O_{K_p} G$, where $O_{K_p}$ denotes the ring $\prod_{\mathfrak{p} \in S_p(k)} O_{K_{\mathfrak{p}}}$ which we identify with $O_k \otimes \mathbb{Z}_p$. So if $c_1, \ldots, c_d$ is a $\mathbb{Z}$-basis of $O_k$ then $c_1 \otimes 1, \ldots, c_d \otimes 1$ is a $\mathbb{Z}_p$-basis of $O_{K_p}$ and $a_1 := b(c_1 \otimes 1), \ldots, a_d := b(c_d \otimes 1)$ is a free basis for $O_{K_p}$ over $\mathbb{Z}_p G = a$.

For any $\mathfrak{p} \in S_p(K)$ let $e_\mathfrak{p}$ and $w_\mathfrak{p}$ denote respectively the maximal ideal and the ramification index of $K_{\mathfrak{p}}/\mathbb{Q}_p$. Clearly, $e_\mathfrak{p}$ depends only on $p$, the prime lying below $\mathfrak{p}$ in $K$. The exponential series converges on $pO_{K_{\mathfrak{p}}} = \mathfrak{p}^{e_\mathfrak{p}}$ for each $\mathfrak{p} \in S_p(K)$ and defines a $\mathbb{Z}_p D_p(K/k)$-isomorphism to $U_{e_\mathfrak{p}}(K_{\mathfrak{p}})$. To shorten notation, we write $U^1$ for $U^1(K_p)$ and $U^2$ for $\prod_{\mathfrak{p} \in S_p(K)} U_{e_\mathfrak{p}}(K_{\mathfrak{p}}) \subset U^1$. It follows from the above that the map $\text{Exp}_p = \left( \prod_{\mathfrak{p} \in S_p(K)} \text{exp}_p \right) : pO_{K_p} \to U^2$ is an $a$-isomorphism and hence that $U^2$ is free over $a$ with basis $w_1 := \text{Exp}_p(pa_1), \ldots, w_d := \text{Exp}_p(pa_d)$. It is also of finite index in $U^1$ and since $a$ is a product of the p.i.d.'s $a_\Phi$, it follows that $U^1/U^1_{\text{tor}}$ must also be $a$-free of rank $d$, so, in an additive notation, we get

$$U^1 = U^1_{\text{tor}} \oplus \bigoplus_{i=1}^d a u_i \quad \text{where } u_1, \ldots, u_d \text{ is any free basis of } U^1/U^1_{\text{tor}} \quad (33)$$

Now let $\phi$ and $\Phi$ be as in the statement of the Theorem and let $M \in M_d(a_\Phi)$ be the matrix representing $e_\phi \bar{w}_1, \ldots, e_\phi \bar{w}_d$ in terms of the $a_\Phi$-basis $e_\Phi \bar{u}_1, \ldots, e_\Phi \bar{u}_d$ of $(U^1/U^1_{\text{tor}})_\Phi$. The determinant of $M$ has two different interpretations. On the one hand, if we write $U^2$ for the isomorphic image of $U^2$ in $U^1/U^1_{\text{tor}}$ then the general theory of p.i.d.'s and order ideals gives

$$\det(M) a_\Phi = [(U^1/U^1_{\text{tor}})_\Phi/(U^2) a_\Phi] a_\Phi = [U^1_{\Phi}/U^2_{\Phi}] a_\Phi [(U^1_{\text{tor}})_\Phi] a_\Phi^{-1}$$

Now, for each $p \in S_p(k)$, $\mathfrak{p} \in S_p(K)$ above $p$ and $l \geq 1$, there is a well-known $\mathbb{Z}_p D_p(K/k)$-isomorphism $U^l(K_{\mathfrak{p}})/U^{l+1}(K_{\mathfrak{p}}) \to \mathfrak{p}^l/\mathfrak{p}^{l+1}$ induced by $x \mapsto x - 1$. This gives an $a$-isomorphism after taking products of both sides over the $\mathfrak{p}$ above $p$. Applying $e_\phi$ and letting $p$ and $l$ vary, a simple argument with exact sequences shows that $U^l_{\Phi}/U^2_{\Phi}$ has the same $a_\Phi$-order ideal as $\mathfrak{m}_{\Phi}/(pO_{K_p})_\Phi$ where $\mathfrak{m}$ denotes $\prod_{\mathfrak{p} \in S_p(K)} \mathfrak{p} \subset K_p$. Therefore

$$\det(M) a_\Phi = [\mathfrak{m}_{\Phi}/(pO_{K_p})_\Phi] a_\Phi [(U^1_{\text{tor}})_\Phi] a_\Phi^{-1} = [(O_{K_{\mathfrak{p}}})_\Phi/(pO_{K_{\mathfrak{p}}})_\Phi] a_\Phi [(O_{K_p})_\Phi/\mathfrak{m}_{\Phi}] a_\Phi^{-1} [(U^1_{\text{tor}})_\Phi] a_\Phi^{-1} = p^d [(O_{K_{\mathfrak{p}}})_\Phi/\mathfrak{m}_{\Phi}] a_\Phi^{-1} [(U^1_{\text{tor}})_\Phi] a_\Phi^{-1}$$

(34) since $O_{K_p}$ is free of rank $d$ over $a$. On the other hand, Proposition 6 Equation (33) and the definition of $s_{K/k,S}$ give

$$\phi(\det(M)) \phi(s_{K/k,S}) = \phi(\det(M)) \phi(e_\Phi s_{K/k,S}(u_1 \wedge \ldots \wedge u_d) a) = \phi(\det(M) s_{K/k,S}(e_\Phi u_1 \wedge \ldots \wedge e_\Phi u_d)) \phi(a) = \phi(s_{K/k,S}(e_\Phi u_1 \wedge \ldots \wedge e_\Phi u_d)) \phi(a) = j(x(a_{K/k,S}^* \Phi)) \phi(R_{K/k,p}(w_1 \wedge \ldots \wedge w_d)) \phi(a)$$

(35)
where \( \phi = j \circ \chi \). But tracing through the definitions we have

\[
R^{(j)}_{K/k,p}(w_1 \wedge \ldots \wedge w_d) = \det \left( \sum_{g \in G} \log_p(\delta_i^{(j)}(g^{-1}\text{Exp}_p(pa_t)))g \right)_{i,t=1}^d
\]

and

\[
\log_p(\delta_i^{(j)}(g^{-1}\text{Exp}_p(pa_t))) = \log_p(j\tau_i \circ i\varphi_i \text{Exp}_p(g^{-1}pa_t)) = \log_p(j\tau_i \exp_p(\varphi_i g^{-1}pa_t)) = \delta_i^{(j)}(g^{-1}pa_t) = p\delta_i^{(j)}(g^{-1}b)j\tau_i(c_t)
\]

so that

\[
R^{(j)}_{K/k,p}(w_1 \wedge \ldots \wedge w_d) = p^d \prod_{i=1}^d \left( \sum_{g \in G} \delta_i^{(j)}(g^{-1}b)g \right) \det(j\tau_i(c_t))_{i,t=1}^d = \pm p^d j(\sqrt{d_k}) \prod_{i=1}^d \delta_i^{(j,G)}(b)
\]

Applying \( \phi \) to Equations (34) and (36) and combining them with (35) gives

\[
\phi(\mathcal{S}_{K/k,S}) = \phi \left( \left[ (\mathcal{O}_{K_p})_\Phi / \mathcal{M}_\Phi \right]_{\alpha_\Phi} \right) \phi \left( \left[ (U_{\text{tor}}^1)_\Phi \right]_{\alpha_\Phi} \right) j(\sqrt{d_k} \chi(a_{K/k,S}^{-s} \cdot \cdot \cdot)) \prod_{i=1}^d \phi(\delta_i^{(j,G)}(b))
\]

(37)

Now fix \( p \in S_p(k) \) and write \( D_p \) for \( D_p(K/k) \) and \( T_p \) for \( T_p(K/k) \). Considering \( \prod_{\mathfrak{p} \nmid p}(\mathcal{O}_{K_p}/\mathcal{P}) \) as an \( \mathfrak{a} \)-submodule of \( \mathcal{O}_{K_p}/\mathcal{M} \), we have natural \( \mathfrak{a} \)-isomorphisms:

\[
\prod_{\mathfrak{p} \nmid p}(\mathcal{O}_{K_p}/\mathcal{P}) \cong \mathfrak{a} \otimes_{\mathcal{Z}_p D_p}(\mathcal{O}_{K_p,\mathfrak{p}}/\mathcal{P}_{\mathfrak{p},\mathfrak{p}}) \cong \mathfrak{a} \otimes_{\mathcal{Z}_p D_p}(\mathcal{Z}_p D_p \otimes \mathcal{Z}_p T_p)(\mathcal{O}_k/\mathfrak{p}) \cong \mathfrak{a} \otimes_{\mathcal{Z}_p T_p} \mathcal{O}_{k/\mathfrak{p}}
\]

(where the action on \( \mathcal{O}_k/\mathfrak{p} \) is trivial and the second isomorphism is from the normal basis theorem in the residue field extension of \( K_{\mathfrak{p},\mathfrak{p}}/k_p \)). It follows easily that \( \prod_{\mathfrak{p} \nmid p}(\mathcal{O}_{K_p}/\mathcal{P}) \Phi \) is trivial unless \( T_p \subseteq \ker(\phi) \) (i.e. \( p | f_\phi \)) in which case it has order ideal \((N\mathfrak{p})\alpha_{\Phi} \). Taking the product over all \( p \in S_p(k) \), it follows that

\[
\phi \left( \left[ (\mathcal{O}_{K_p})_\Phi / \mathcal{M}_\Phi \right]_{\alpha_\Phi} \right) = \left( \prod_{p \in S_p(k) \atop p \nmid f_\phi} N\mathfrak{p} \right) \mathcal{O}_\phi
\]

(38)

Furthermore, equations (8), (2) and (3) give:

\[
\sqrt{d_k} \chi(a_{K/k,S}^{-s}) = \prod_{q \in \mathfrak{q} \setminus \mathfrak{q}_{\infty} \atop \mathfrak{q} \nmid f_\chi} \left( 1 - Nq^{-1}\check{\chi}([q]) \right) \sqrt{d_k(i/\pi)^dL(1,\check{\chi})} = \prod_{q \in \mathfrak{q} \setminus \mathfrak{q}_{\infty} \atop \mathfrak{q} \nmid f_\chi} \left( 1 - Nq^{-1}\check{\chi}([q]) \right) (-1)^d \tau(\chi)^{-1}L(0,\check{\chi}^{-1})
\]

(39)
The second equality follows from Hecke’s functional equation for the $L$-function. (To be perfectly precise, we are using the version stated on p. 36 of [Fr], taking $s = 0$ and Fröhlich’s complex character $\hat{\theta}$ on $\text{Id}(k)$ – the idèle group of $k$ – to be the one obtained by composing $\chi$ with the map $\text{Id}(k) \to G$ coming from class-field theory.) Applying $j$ to and (39) and combining with Equations (37) and (38) gives

$$
\phi(\mathfrak{S}_{K/k, S}) = \phi \left( \left[ (U_{tor}^1)_{\Phi} \right]_{a_b} \right) \prod_{p \in \mathfrak{P}_p(k)} Np \prod_{q \in S \setminus S_\infty \setminus \mathfrak{M}_f \setminus \mathfrak{M}_p} \left( 1 - Nq^{-1} \hat{\phi}([q]) \right) L(0, \hat{\phi}^{-1}) j(\tau(\chi))^{-1} \prod_{i=1}^{d} \phi(\delta_i^{(j), G}(b))
$$

where we have used the facts that every prime ideal $p$ in $\mathfrak{S}_p(k)$ is contained in $S$ and that if, in addition, it does not divide $f_\phi$ then $Np(1 - Np^{-1} \hat{\phi}([q])) = (Np - \hat{\phi}([q]))$ lies in $\mathcal{O}_\phi^\times$. The argument so far shows that $j(\tau(\chi))^{-1} \prod_{i=1}^{d} \phi(\delta_i^{(j), G}(b))$ lies in $F_\phi$. The theorem will follow if we can prove that it too lies in $\mathcal{O}_\phi^\times$, i.e. that

$$
\phi(\mathfrak{S}_{K/k, S}) \sim \prod_{i=1}^{d} \phi(\delta_i^{(j), G}(b))
$$

where ‘$a \sim b$’ means that $a, b \in \mathbb{Q}_p^\times$ have the same $p$-adic absolute value. Recall that Fröhlich defines $\tau(\chi)$ as the product $\prod_{q \notin S_\infty} \tau(\chi_q)$ where $\chi_q : k_q^\times \to \mathbb{Q}_p^\times$ is the $q$-component of the complex idèle character associated to $\chi$ and $\tau(\chi_q)$ is the ‘local Gauss sum’ (which equals 1 unless the $q|f_\chi$, so the product is finite). For definitions and basic properties of the algebraic integers $\tau(\chi_q)$ see see [Fr], p. 34-35 or [Ma, II-$\S$2]. In particular, Eq. (5.7) on [Fr], p. 34] shows that $\phi(\mathfrak{S}_{K/k, S}) \sim \prod_{i \in \mathcal{L}(p)} \phi(\delta_i^{(j), G}(b))$ it suffices to show that for any $p$ in $\mathfrak{S}_p(k)$

$$
\phi(\mathfrak{S}_{K/k, S}) \sim \prod_{i \in \mathcal{L}(p)} \phi(\delta_i^{(j), G}(b))
$$

But this is essentially (a special case of) Theorem 23 of [Fr]: Take $F := j_{\tau_i(p)}(k)$, $L := j_{\tau_i(p)}(K)$ as subfields of $\mathbb{Q}_p$, isomorphic via $j_{\tau_i(p)}$ to $K_p$ and $K_{\Phi_i(p)}$, respectively. The extension $L/F$ is thus abelian with Galois group $\Gamma$ which we identify via $j_{\tau_i(p)}$ with $D_p$. We take Fröhlich’s character ‘$\chi$’ to be our $\chi_p : k_p^\times \to \mathbb{Q}_p^\times$ which factors through the local reciprocity map $k_p^\times \to D_p$ and so may also be regarded as $\chi$ restricted to $D_p = \Gamma$. Thus Fröhlich’s ‘$\chi'$’ may similarly be identified with our $\phi$ restricted to $\Gamma$. Since $\Gamma$ of order prime to $p$, $L/F$ is tame so Theorem 23 applies to give (with these identifications)

$$
\phi(\mathfrak{S}_{K/k, S}) \sim \mathcal{N}_{F/\mathbb{Q}_p}(j_{\tau_i(p)}(b_p)|\phi)
$$
where the R.H.S. is the norm resolvent (see below) associated to the free generator $j\tau_i(p)(b_p)$ of $\mathcal{O}_L$ over $\mathcal{O}_{F}\Gamma$. Thus (40) and hence our Theorem will follow from

$$
\prod_{i\in I(p)} \phi(\delta_i^{(j)}G(b)) \sim N_{F/Q_p}(j\tau_i(p)(b_p)|\phi)
$$

(41)

The proof of (41) is largely a matter of unravelling our definitions and comparing with Fröhlich’s, so we only sketch it. For any $i \in I(p)$ we can choose $g_i \in G$ such that $g_i, \mathfrak{P}_i = \mathfrak{P}_{i(p)}$ and then $\sigma_i \in \text{Gal}(\bar{\mathbb{Q}}_p/Q_p)$ such that $\sigma_i j\tau_i(p)(x) = j\tau_i g_i^{-1}(x)$ for any $x \in K_{\mathfrak{P}_{i(p)}}$. Then

$$
\phi(\delta_i^{(j)}G(b)) = \sum_{g \in G} j\tau_i g_i^{-1}(g^{-1}b)\phi(g) \\
= \sum_{h \in D_p} j\tau_i h^{-1}(b_p)\phi(h_i) \\
= \phi(g_i)\sigma_i \left( \sum_{\gamma \in \Gamma} \gamma^{-1}(j\tau_i(p)(b_p)\sigma_i^{-1}(\phi(\gamma)) \right) \\
\sim \sigma_i (j\tau_i(p)(b_p)|\sigma_i^{-1} \circ \phi)
$$

where $(j\tau_i(p)(b_p)|\sigma_i^{-1} \circ \phi)$ denotes the resolvent defined for example in [Fr] Eq. (4.4), p. 29]. Equation (41) now follows on taking the product over $i \in I(p)$, using the definition of the norm resolvent in [Fr] Eq. (1.4), p. 107] and the fact (which the reader can easily check) that as $i$ runs through $I(p)$, so $\sigma_i$ runs once through a set of left coset representatives for $\text{Gal}(\bar{\mathbb{Q}}_p/F)$ in $\text{Gal}(\bar{\mathbb{Q}}_p/Q_p)$. (Fröhlich uses right cosets because of his exponential notation for Galois action). This completes the proof of Theorem 3.

Some of the facts used in the above proof will also be useful in the

**Proof of Proposition 8**

Since $p \nmid |G|$, we can use equation (33) to show that any $\theta \in \bigwedge^d \mathbb{Z}_p U^1$ may be expressed as the sum of $xu_1 \wedge \ldots \wedge u_d$ (for some $x \in \mathfrak{a}$) and finitely many terms of form $z \wedge v_2 \wedge \ldots \wedge v_{d}$ with $z \in U_1^\text{tor}$ and $v_i \in U^1$ for $i = 2, \ldots, d$. Since we are assuming that $\theta$ is $\mathbb{Z}_p$-torsion, so also is its image $x(\bar{u}_1 \wedge \ldots \wedge \bar{u}_d)$ in $\bigwedge^d (U^1/ U_1^\text{tor})$ and since $\bar{u}_1 \wedge \ldots \wedge \bar{u}_d$ freely generates the latter over $\mathfrak{a}$, it follows that $x = 0$. Thus, by linearity, we may assume that $\theta = z \wedge v_2 \wedge \ldots \wedge v_d$. On the other hand, $p \nmid |G|$ also implies $\mathbb{Z}_p \Lambda_{0,S} = \mathbb{Z}_p \bigwedge^d \mathcal{U}_S(K^+)$ by Prop. 1. If we write $\bar{e}$ for $|G|e_{S,d,G} \in \mathbb{Z}\bar{G}$ then it follows from the eigenspace condition on $\eta$ that it equals $|G|^{-d}(2\bar{e})^d \eta$ and so may be written as a $\mathbb{Z}_p$-linear combination of terms of form $(1 \otimes \bar{e}_1) \wedge \ldots \wedge (1 \otimes \bar{e}_d)$ with $\varepsilon_i \in \mathcal{U}_S(K^+)$ for $i \in I(p)$. By $\mathbb{Z}_p$-linearity and equation (20), it therefore suffices to show that $[\bar{e}, z]_{K,n} = 0$ for any $z \in \mathcal{U}_S(K^+)$ and any $z \in U^1_1 \wedge \mu_p\infty(K_p)$, say $z = (z\mathfrak{P})_{p\mathfrak{P}}$ with $z\mathfrak{P} \in \mu_p\infty(K_p)$ for each $\mathfrak{P} \in S_p(K)$. By the definitions of $[\cdot, \cdot]_{K,n}$, $[\cdot, \cdot]_{\mathfrak{P},n}$ and $(\cdot, \cdot)_{K_p, p, n}$ this reduces further to the statement that $\sigma_{\mathfrak{P}}(\bar{e})_{K_p} = 0$ for each $\mathfrak{P}$, where $\bar{e} := \bar{e}/p+1$ is a $p$-power root of unity in $(K_{\mathfrak{P}})^{ab}$. But $\bar{e}$ actually lies in $\mathbb{Q}_p^{ab}$, so local class field theory tells us that $\sigma_{\mathfrak{P}}(\bar{e})_{K_p} = \sigma_{\mathfrak{P}, \mathbb{Q}_p}(\bar{e})_{\mathfrak{P}}$ where $a_{\mathfrak{P}} := N_{K_p/\mathbb{Q}_p} t_{\mathfrak{P}}(\bar{e}) = N_{K_p/\mathbb{Q}_p} t_{\mathfrak{P}}(N_{D_p(K)/E}\bar{e})$ and $p \in S_p(k)$ lies below $\mathfrak{P}$. But the
image of \( N_{p(K/k)} \) in \( \mathbb{Z}G \) is \( N_{p(K/k)} + 2N_{p(K^+/k)} \). If \(|S| > d + 1\) then, since \( p \) lies in \( S \), the formula \( (13) \) shows that \( N_{p(K^+/k)} \) vanishes in \( \mathbb{Z}G \), so \( a_{p} = 1 \) for all \( \mathfrak{p} \) and the result follows. Finally, if \(|S| = d + 1\) then we must have \( S = S_{\infty}(k) \cup S_{p}(k) = S_{\infty}(k) \cup \{p\} \) and \( (13) \) now implies \( N_{p(K^+/k)} \) vanishes in \( \mathbb{Z}G \). Hence \( a_{p} \) is a power of \( N_{k'/Q'}(N_{K^+/Q'}) \) which equals \( N_{p(K/k)}N_{K^+/Q} \) since \( S_{p}(k) = \{p\} \). But \( \epsilon \in U_{S}(K^+) \) implies that \( N_{K^+/Q} \), hence also \( a_{Q} \), is a power of \( p \) and the result follows from the well known fact that \( \sigma_{p, Q_{p}}(\zeta_{Q}) = \zeta_{Q} \) (Indeed, \( p = N_{Q_{p}(Q_{p})/Q_{p}}(1 - \zeta_{Q}) \) implies that \( \sigma_{p, Q_{p}} \) restricts to the identity on \( Q_{p}(Q_{p}) \)). □

7 The Case \( k = \mathbb{Q} \)

The following lemmas will be used in the proof of Theorem \( 3 \). Let \( p \) be an odd prime and \( f \) a positive integer. We write \( f \) as \( f'p^{m+1} \) for some \( m \geq -1 \) and \( f' \) prime to \( p \). We shall abbreviate \( \mathbb{Q}(\xi_{f}) \) to \( K_{f} \), \( \operatorname{Gal}(\mathbb{Q}(\xi_{f})/\mathbb{Q}) \) to \( G_{f} \) and \( \operatorname{Gal}(\mathbb{Q}(\xi_{f})^{+}/\mathbb{Q}) \) to \( G_{f} \). For any \( \tilde{a} \in (\mathbb{Z}/f\mathbb{Z})^{\times} \) we write \( \sigma_{a} \) for the element of \( G_{f} \) sending \( \xi_{f} \) to \( \xi_{f}^{a} \).

**Lemma 4** Let \( S = \{\infty\} \cup S_{f}(\mathbb{Q}) \) which contains \( S^{0}(K_{f}/\mathbb{Q}) \). Then, with the above notations,

(i.) \( \Theta_{K_{f}/Q}(s)(0) = -\frac{1}{2} \sum_{\tilde{g} \in G_{f}} \log |\tilde{g}((1 - \xi_{f})(1 - \xi_{f}^{-1}))|^{-1} \)

(ii.) \( a_{K_{f}/Q,S} = e^{-\frac{1}{f}} \sum_{g \in G_{f}} g((\xi_{f}^{1})(1 - \xi_{f})) g^{-1} \)

**Proof** For part (i), see e.g. [11] p. 203. A rather indirect proof of the equation in (ii) uses Prop. 1 of [11] to calculate \( \Phi_{K_{f}/Q}(0) \) as outlined in [10] Example 3.1 and returns to \( s = 1 \) with \( (9) \). In principle one can also work ‘\( \chi \)-by-\( \chi \)’, calculating \( \chi(LH.S.) \) in (ii) from the usual formula for for \( L(1, \phi) \) when \( \phi \) is an odd primitive Dirichlet character. (See e.g. in [21] Theorem 67 (b)). However, the imprimitivity of our \( \chi \) and presence of a Gauss sum in the formula make the relation to \( \chi(R.H.S.) \) surprisingly difficult. We therefore sketch a direct and very elementary proof of (ii), similar in some respects to that of [21] Theorem 67: Equation \( (2) \) shows that \( \Theta_{K_{f}/Q,S}(1) = \sum_{a=1, (a,f)=1}^{f-1} t_{a} \sigma_{a}^{-1} \) where \( t_{a} = \frac{1}{2} \lim_{s \to 1} (\zeta_{K_{f}/Q,S}(s, \sigma_{a}) - \zeta_{K_{f}/Q,S}(s, \sigma_{a}^{-1})) \) and \( \zeta_{K_{f}/Q,S}(s, \sigma_{a}) = \sum_{n \geq 1, n=a \pmod{f}} n^{-s} \) for \( \Re(s) > 1 \). For any \( 1 \leq c \leq f - 1 \) the function \( Z(s, c) := \sum_{n \geq 1} \xi_{f}^{cn} n^{-s} \) converges (conditionally) to a continuous function of \( s \in (0, \infty] \). For each \( 1 \leq a \leq f - 1 \) with \( (a, f) = 1 \) and any \( s \in (1, \infty] \) we find easily that

\[
\zeta_{K_{f}/Q,S}(s, \sigma_{a}) - \zeta_{K_{f}/Q,S}(s, \sigma_{a}^{-1}) = \frac{1}{f} \sum_{b=1}^{f-1} (\xi_{f}^{-ab} - \xi_{f}^{ab}) Z(s, b)
\]

\[
= \frac{1}{2f} \sum_{b=1}^{f-1} (\xi_{f}^{-ab} - \xi_{f}^{ab})(Z(s, b) - Z(s, f - b))
\]

(42)

But if \( \log \) denotes the principal branch of logarithm, then Abel’s Lemma and some Euclidean geometry show that \( Z(1, c) = -\log(1 - \xi_{f}) = -\log|1 - \xi_{f}| + i\pi(\frac{1}{2} - \frac{c}{f}) \). So, letting \( s \to 1+ \)
Let us write the following cyclotomic explicit reciprocity law due to Coleman. (The case \( Z \in (42) \), substituting for \( \hat{\sigma} \), makes sense because \( \frac{x}{1-x} \). This follows from Corollary 15 of [Col]. We first write \( \hat{\sigma} \).

**Proof** Let \( K_f \) for the field \( \mathbb{Q}_p(\mu_f) \subset \mathbb{Q}_p \). The proof of Theorem 3 depends crucially on the following cyclotomic explicit reciprocity law due to Coleman. (The case \( f' = 1 \) was proved much earlier by Artin and Hasse in [A-H]).

**Lemma 5** Let \( \hat{\xi}_f \) be any primitive \( f \)th root of unity in \( K_f \) and let \( v \in U^{1}(\hat{K}_f) \). Then \( b(\hat{\xi}_f, v) := \frac{1}{f} \text{Tr}_{K_f/\mathbb{Q}_p}(\langle \hat{\xi}_f/(1 - \hat{\xi}_f) \rangle) \log_p(v) \) lies in \( \mathbb{Z}_p \). Furthermore

\[
(1 - \hat{\xi}_f, v)_{\hat{K}_f, p^{m+1}} = (\hat{\xi}_f')^{-b(\hat{\xi}_f, v)}
\]

(The R.H.S. makes sense because \( \hat{\xi}_f' \) is a primitive \( p^{m+1} \)th root of unity.)

**Proof** This follows from Corollary 15 of [Col]. We first write \( \hat{\xi}_f \) uniquely as \( \hat{\xi}_f = \hat{\xi}_{p^{m+1}} \hat{\xi}_{f'} \) where \( \hat{\xi}_{p^{m+1}} \) and \( \hat{\xi}_{f'} \) are generators respectively of \( \mu_{p^{m+1}} \) and \( \mu_{f'} \) in \( K_f \). We also write \( H \) for \( \hat{K}_f \), an unramified extension of \( \mathbb{Q}_p \), and \( \mathcal{O}_H \) for its ring of integers. Now \( 1 - \hat{\xi}_f = h(u_{m}) \) where \( h(T) \) denotes the linear polynomial \( 1 - \hat{\xi}_{f'}(1 - T) \in \mathcal{O}_H[T] \) and \( u_{m} := 1 - \hat{\xi}_{p^{m+1}} \). The Frobenius element \( \varphi \) of \( \text{Gal}(H/\mathbb{Q}_p) \) may be extended to an automorphism of \( \mathcal{O}_H[T] \) (resp. of \( H(\mu_{p^{m}}) \subset \mathbb{Q}_p \)) by acting trivially on \( T \) (resp. on \( \mu_{p^{m}} \)). Suppose \( l \geq 1 \) and \( l \geq i \geq 0 \). Since \( \varphi(\hat{\xi}_{f'}) = \hat{\xi}_{f'} \), one verifies easily that

\[
\varphi^{l-1}h(1 - (1 - T)^{p^{l-i}}) = \prod_{\hat{\xi} \in \mu_{p^{l-i}}} (1 - \hat{\xi}_{f'} \hat{\xi}(1 - T))
\]

Substituting \( T = u_{l} = (1 - \hat{\xi}_{p^{l+1}}) \) for any generator \( \hat{\xi}_{p^{l+1}} \) of \( \mu_{p^{l+1}} \), it is easy to see that the R.H.S. becomes the norm from \( H(\mu_{p^{l+1}}) \) to \( H(\mu_{p^{l+1}}) \) of \( h(u_{l}) \). Thus \( h(T) \) lies in the subgroup of \( \mathcal{O}_H((T))^\times \) denoted \( \mathcal{M}(l) \) by Coleman, and this for any \( l \geq 1 \). Indeed, this follows from the equation at the foot of p. 376 of [Col] after correcting the misprint ‘\( \phi^{n} \)’ to read ‘\( \phi^{n-1} \)’ (which is necessary for consistency with Coleman’s equation (1) on p. 377). Now we can apply Coleman’s Corollary 15, p. 396, after first correcting another obvious misprint: the meaningless ‘\( \lambda(\alpha) \)’ in the main equation should be replaced by \( \lambda(1 - \alpha) \) (\( = -\log_p(\alpha) \)). If we take Coleman’s ‘\( n \)’ to be our \( m \), his ‘\( u \)’ to be our \( u_{m} \) (so that his ‘\( H_{n} \)’ is our \( \hat{K}_{f} \)) his ‘\( \alpha \)’ to be our \( v \) and his \( g \) to be our \( h \) (so that \( \delta h(T) = (1 - T)h(T)^{-1}dh(T)/dT = \hat{\xi}_{f'}(1 - T)/(1 - \hat{\xi}_{f'}(1 - T)) \)) then the R.H.S. of the main equation in his Corollary 15 equals \( f'\log_p(\hat{\xi}_f, v) \). The Corollary implies that this lies in \( \mathbb{Z}_p \) and (taking into account Coleman’s definitions of ‘\( \text{Ind}_{u_{m}} \)’ and of ‘\( (x, y)_{m} \)’), the latter agreeing with our \( (y, x)_{\hat{K}_f, p^{m+1}} \) that \( (1 - \hat{\xi}_f, v)_{\hat{K}_f, p^{m+1}} = (1 - u_{m})^{-f'\log_p(\hat{\xi}_f, v)} \), from which our Lemma follows immediately.

**Proof of Theorem 3** Let \( K \) be an absolutely abelian CM field and suppose that \( f = f'p^{m+1} \) is the conductor of \( K \) i.e. the smallest positive integer such that \( K \subset K_{f} \). Then \( S_{\text{ram}}(K/Q) = S_{\text{ram}}(K_f/Q) = S_f(Q) \). Since \( p \) is odd, \( \mu_{p^{m+1}} \subset K \) implies (e.g. by ramification)
that \( n \leq m \). Therefore, if \( m = -1 \) then the Congruence Conjecture doesn’t apply and \( IC(K/\mathbb{Q}, S, p) \) follows from Theorem 1. So we may assume \( m \geq 0 \). By Props. 9 and 11 we may further assume that \( S = S^1(K/\mathbb{Q}) = \{\infty\} \cup S_f(\mathbb{Q}) \) (which is also equal to \( S^0(K/\mathbb{Q}) \)) and to \( S^1 = S^0(K_f/\mathbb{Q}) \). If \( m = 0 \) then \( K_f/\mathbb{Q} \) is tamely ramified at \( p \). If \( m \geq 1 \) then (since \( p \neq 2 \)) the ramification group \( T_p(K_f/\mathbb{Q}) = \text{Gal}(K_f/\mathbb{Q}(\xi_f)) \) has a unique minimal subgroup of order \( p \), namely \( \text{Gal}(K_f/\mathbb{Q}(\xi_{f/p})) \). This cannot be contained in \( \text{Gal}(K_f/\mathbb{Q}) \) by minimality of the conductor \( f \). Thus, in any case, \( K_f/\mathbb{Q} \) is at most tamely ramified above \( p \). So by Remark 5.2 it suffices to prove \( CC(K_f/\mathbb{Q}, S^1, p, m) \) and apply Props. 17 and 18.

We start with \( RSC(K_f^+/\mathbb{Q}, S^1; \mathbb{Z}_p) \) (see also [14] p. 79). The algebraic integer \((1 - \xi_f)(1 - \xi_f^{-1}) = (1 - \xi_f)^{1+c} \) lies in \( K_f^+ \) and the norm relations for cyclotomic numbers (see for example [So1, Lemma 2.1]) show that, for any \( q \in S_f(\mathbb{Q}) \), the number \( N_{Dq(K_f/\mathbb{Q})}(1 - \xi_f) \) equals \( p \) or \( 1 \) according as \( f' = 1 \) or \( f' \neq 1 \). It follows firstly that \((1 - \xi_f)^{1+c} \) lies in \( U_{S_f}(K_f^+) \) and even in \( U_{S_f}(K_f^+) \) for \( f' \neq 1 \) and secondly, using Proposition 2 that \( \eta_f := -\frac{1}{2} \otimes (1 - \xi_f)^{1+c} \in \frac{1}{2}\Lambda_{\mathbb{Z}, G} U_{S_f}(K_f^+) = \frac{1}{2} U_{S_f}(K_f^+) \) satisfies the eigenspace condition w.r.t. \( (S^1, 1, G_f) \). Moreover \( R_{K_f^+/\mathbb{Q}} = \lambda_{K_f^+/\mathbb{Q},1} \) so taking \( \tau_1 \) to be the identity, Lemma 1(ii) shows that \( \eta_f \) is the unique solution \( \eta_{K_f^+/\mathbb{Q},S^1} \) of \( RSC(K_f^+/\mathbb{Q}, S^1; \mathbb{Z}) \). For any \( u \in \Lambda_{\mathbb{Z}, G_f} U(1)(K_f)^- = U(1)(K_f)^- \), it follows from (20) and (18) that

\[
H_{K_f/\mathbb{Q}, m}(\eta_f, u) = -2^{-1} \sum_{g \in G_f} [(1 - \xi_f)^{(1+c)}, gu]_{K_f, m} g^{-1}
\]

\[
= -2^{-1} \sum_{g \in G_f} [1 - \xi_f, gu]_{K_f, m} g^{-1}
\]

\[
= - \sum_{g \in G_f} [1 - \xi_f, gu]_{K_f, m} g^{-1}
\]

\[
= \sum_{g \in G_f} \left( \sum_{q \in S_p(K_f)} -[1 - \xi_f, q \eta_f]_{q, m} \right) g^{-1}
\]

(43)

(\( (1 - \xi_f) \) is identified with its natural images \((1 - \xi_f) \otimes 1 \) and \( \eta_f(1 - \xi_f) \) in \( K_p \) and \( K_p \) respectively). Next we need to calculate the map \( s_{K_f/\mathbb{Q}, S^1} \). Fix a choice of \( j : \bar{\mathbb{Q}} \to \mathbb{Q}_p \). It follows easily from the definitions of \( R_{K_f/\mathbb{Q}, p}^{(j)} \) and \( s_{K_f/\mathbb{Q}, S^1} \) and from Lemma 1(ii) that for any \( u \in U(1)(K_f)^- \)

\[
s_{K_f/\mathbb{Q}, S^1}(u) = \sum_{g \in G_f} a_g(u) g^{-1} \quad \text{where} \quad a_g(u) = \frac{1}{f} \sum_{h \in G_f} j(h)(\xi_f/(1 - \xi_f)) \log_p(\delta_{1}^{(j)}(hg))
\]

(44)

Let \( D \) denote \( D_{\mathbb{Q}}(K_f/\mathbb{Q}) \), identified with \( \text{Gal}(K_f, \mathbb{Q}_p) \), for one hence any \( \mathfrak{P} \in S_p(K_f) \). Recall that \( \mathfrak{P}_1 \in S_p(K_f) \) is the ideal defined by the embedding \( j = j_{\mathfrak{P}_1} \) which therefore gives rise to an isomorphism (also denoted \( j \) from \( K_f, \mathfrak{P}_1 \) to \( j(K_f) = \hat{K}_f \). This in turn induces an isomorphism from \( D \) to \( \hat{D} := \text{Gal}(\hat{K}_f/\mathbb{Q}_p) \) sending \( d \in D \) to \( \hat{d} \) say, where \( j \hat{d} = \hat{d} \).
For each \( \mathfrak{P} \in S_p(K_f) \) we choose \( h_{\mathfrak{P}} \in G_f \) such that \( h_{\mathfrak{P}}(\mathfrak{P}) = \mathfrak{P}_1 \) so that \( h_{\mathfrak{P}} \) extends to an isomorphism from \( K_{f,\mathfrak{P}} \) to \( K_{f,\mathfrak{P}_1} \). Thus \( G = \bigcup_{\mathfrak{P}} h_{\mathfrak{P}}D \) and for any \( d \in D \), \( jh_{\mathfrak{P}d} = \hat{d}h_{\mathfrak{P}} \) defines an isomorphism from \( K_{f,\mathfrak{P}} \) to \( \hat{K}_f \). It follows that if \( u \in U^1(K_{f,p})^- \) and \( g \in G \), then \( \log_p(\delta^{(j)}(h_{\mathfrak{P}dg})) = \log_p(jh_{\mathfrak{P}_1}(h_{\mathfrak{P}dg})) = \log_p(jh_{\mathfrak{P}d\mathfrak{P}_1}(g)) = \hat{d}\log_p(jh_{\mathfrak{P}d\mathfrak{P}_1}(g)). \) Consequently, we find

\[
a_g(u) = \sum_{\mathfrak{P} \in S_p(K_f)} \frac{1}{f} \sum_{d \in D} \log_p(jh_{\mathfrak{P}d}(\xi_f/(1 - \xi_f))) \log_p(\delta^{(j)}(h_{\mathfrak{P}dg}))
= \sum_{\mathfrak{P} \in S_p(K_f)} \frac{1}{f} \log_p(jh_{\mathfrak{P}_1}(\xi_f/(1 - \xi_f))) \log_p(jh_{\mathfrak{P}d\mathfrak{P}_1}(g))
= \sum_{\mathfrak{P} \in S_p(K_f)} b(\hat{\xi}_f, v_{g,\mathfrak{P}})
\] (45)

where, for each \( \mathfrak{P} \in S_p(K_f) \), we have set \( \hat{\xi}_f, v_{g,\mathfrak{P}} \) and \( v_{g,\mathfrak{P}} := jh_{\mathfrak{P}d\mathfrak{P}_1}(g) \) and where \( b(\hat{\xi}_f, v_{g,\mathfrak{P}}) \) is as defined in Lemma 5. The first statement of this Lemma therefore shows that \( a_g(u) \in \mathbb{Z}_p \) for all \( u \in U^1(K_{f,p})^- \) and \( g \in G \), i.e. that \( IC(K_f/\mathbb{Q}, S^1, p) \) holds. Also, the definition of the pairing \( [, .]_{\mathfrak{P}, m+1} \) gives

\[
\iota_{\mathfrak{P}}(\hat{\xi}_f^r) |_{1 - \xi_f, v_{\mathfrak{P}}(g)} = (1 - \xi_f, \iota_{\mathfrak{P}}(g))_{\mathfrak{P}, m+1}
\]

Applying \( jh_{\mathfrak{P}} \) to both sides and using the second statement of Lemma 5 we get

\[
(\hat{\xi}_f, v_{\mathfrak{P}}(g))_{\mathfrak{P}, m+1} = (1 - \hat{\xi}_f, \iota_{\mathfrak{P}}(g))_{\mathfrak{P}, m+1} \equiv (\hat{\xi}_f, v_{\mathfrak{P}}(g))_{\mathfrak{P}, m+1} \mod p^{m+1}.
\]

which implies that \( b(\hat{\xi}_f, v_{g,\mathfrak{P}}) \equiv -[1 - \xi_f, \iota_{\mathfrak{P}}(g)]_{\mathfrak{P}, m+1} \mod p^{m+1} \). Summing this congruence over all \( \mathfrak{P} \in S_p(K_f) \) and combining with equations (15), (14) and (33), we obtain

\[
\varepsilon_{K_f/\mathbb{Q}, S^1}(u) \equiv H_{K_f/\mathbb{Q}, m}(\eta_f, u) \mod p^{m+1} \quad \text{for any } u \in U^1(K_f)^- \text{ giving } CC(K_f/\mathbb{Q}, S^1, p, m).
\]

8 The Case of \( K \) Absolutely Abelian

If \( L/M \) is any Galois extension of number fields and \( \phi \) any complex character of Gal\((L/M)\), the \( T \)-truncated Artin \( L \)-function \( L_{L/M,T}(s, \phi) \) is defined for any finite set \( T \) of places of \( M \) containing \( \mathcal{S}_\infty(M) \) but not necessarily \( \mathcal{S}_{\text{ram}}(L/M) \). If Gal\((L/M)\) is abelian and \( \phi \) is irreducible (i.e. \( \phi \in \text{Gal}(L/M) \)) then, as noted in Remark 2.1, the definition agrees with the third member in (3). In particular, there is no conflict with previous notation in the case \( T \supset \mathcal{S}_{\text{ram}}(L/M) \) and we always have

\[
L_{L/M,T}(s, \phi) = \prod_{q \in T \cap \mathcal{S}_\infty(M)} (1 - Nq^{-s}\hat{\phi}([q])) L(s, \hat{\phi}) = \prod_{q \not\in T \cup \mathcal{S}_{\text{ram}}(L^\phi/M)} (1 - Nq^{-s}\hat{\phi}(q))^{-1}
\] (46)

where \( \hat{\phi} \) denotes the associated primitive ray-class character modulo \( f_\phi \), \( L^\phi = L^{\ker(\phi)} \) and the infinite product converges only for \( \text{Re}(s) > 1 \).
Lemma 6 Suppose $L/M$ and $T$ are as above, with $\text{Gal}(L/M)$ abelian, and suppose $l$ is any intermediate field, $L \supset l \supset M$. Then for any $\chi \in \text{Gal}(L/l)$, we have an identity of meromorphic functions on $\mathbb{C}$:
\[
L_{L/l,T(l)}(s, \chi) = \prod_{\phi \in \text{Gal}(L/M), \phi|_{\text{Gal}(L/l)} = \chi} L_{L/M,T}(s, \phi)
\]

Proof This follows from the usual induction and ‘additivity’ properties for Artin $L$-functions (see [13, p. 15]) and the fact (e.g. by Frobenius Reciprocity) that $\text{Ind}_{\text{Gal}(L/l)}^{\text{Gal}(L/M)} \chi = \sum_{\phi \in \text{Gal}(L/M), \phi|_{\text{Gal}(L/l)} = \chi} \phi$. $\square$

Lemma 7 Let $B$ be a finite abelian group, $C$ any subgroup of $B$ and $x$ any element of $\mathbb{C}B$. We write $x|\mathbb{C}B$ for the endomorphism of $\mathbb{C}B$, considered as a free $\mathbb{C}C$-module, determined by multiplication by $x$. For any $\chi \in \hat{C}$, we have
\[
\chi(\det_{\mathbb{C}C}(x|\mathbb{C}B)) = \prod_{\phi \in \hat{B}, \phi|_{\mathbb{C}C} = \chi} \phi(x)
\]
(All characters extended linearly to homomorphisms from the complex group-rings to $\mathbb{C}$).

Proof Choose any $\mathbb{C}C$-basis $\mathcal{B} = \{y_1, \ldots, y_n\}$ for $\mathbb{C}B$ (where $n = |B : C|$) and let $T = (t_{ij})_{i,j} \in M_n(\mathbb{C})$ be the matrix of $x|\mathbb{C}B$ w.r.t. $\mathcal{B}$. If $e_{\chi,C}$ denotes the idempotent attached to $\chi$ in $\mathbb{C}C$, then $x|\mathbb{C}B$ acts on the submodule $e_{\chi,C}\mathbb{C}B$ and its matrix w.r.t. the $\mathbb{C}$-basis $\{e_{\chi,C}y_1, \ldots, e_{\chi,C}y_n\}$ of the latter is clearly $\chi(T) := (\chi(t_{ij}))_{i,j} \in M_n(\mathbb{C})$. Hence $\chi(\det_{\mathbb{C}C}(x|\mathbb{C}B)) = \chi(\det(T)) = \det(\chi(T)) = \det_{\mathbb{C}C}(x|e_{\chi,C}\mathbb{C}B)$. On the other hand, $e_{\chi,C}\mathbb{C}B$ also has a $\mathbb{C}$-basis consisting of the $\mathbb{C}B$-idempotents $e_{\phi,B}$ for the characters $\phi \in \hat{B}$ such that $\phi|_{\mathbb{C}C} = \chi$. (This follows easily from the fact that $e_{\chi,C}$ is the sum of the corresponding $e_{\phi,B}$'s). The result follows, since $xe_{\phi,B} = \phi(x)e_{\phi,B}$. $\square$

For the rest of this section, we fix $K/k$, $S$ and $p$ satisfying the standard hypotheses with $K$ absolutely abelian. Thus $G = \text{Gal}(K/k)$ is a subgroup of the abelian group $\Gamma := \text{Gal}(K/Q)$. We define a set of places $S_Q$ of $Q$ by
\[
S_Q = \{\infty\} \cup \{q \text{ prime such that } S_q(k) \subset S\}
\]
Thus $p \in S_Q$ and $S_Q(k)$ is the maximal $\text{Gal}(k/Q)$-stable subset of $S$. The definition of $\text{Bad}(S)$ in Subsection 4.3 gives
\[
S_{\text{ram}}(K/Q) = \text{Bad}(S) \cup (S_{\text{ram}}(K/Q) \cap S_Q)
\]
(disjoint union). Let us write $A$ for the subgroup $\prod_{q \in \text{Bad}(S)} T_q(K/Q)$ of $\Gamma$ (trivial if $\text{Bad}(S) = \emptyset$). If $F$ is any subfield of $K$, it follows that
\[
F \subset K^A \iff \text{all primes } q \in \text{Bad}(S) \text{ are unramified in } F \iff S_{\text{ram}}(F/Q) \subset S_Q \quad (47)
\]
We denote by $\mathcal{X}_Q(A)$ the set of irreducible $Q$-valued characters of $A$. Each $A \in \mathcal{X}_Q(A)$ corresponds to a Gal($Q/Q$)-conjugacy class of characters $\alpha \in \hat{A}$ which ‘belong’ to $A$. We set $\ker(A) := \ker(\alpha)$ for one (hence any) such $\alpha$ and $K^A := K^{\ker(A)} \supset K^A$. We define

$$S_A = S_Q \cup S_{\text{ram}}(K^A/Q) \supset S^1(K^A/Q)$$

and write $\tilde{\nu}_A$ for the ‘averaged corestriction’ map $|\ker(A)|^{-1}\nu_{K/K^A}$ which is a (non-unital) homomorphism from $\mathbb{C}\text{Gal}(K^A/Q)$ to $\mathbb{C}\Gamma$. Finally, let $e_A$ denote the idempotent of $QA$ corresponding to $A$. With these notations, we define a meromorphic function

$$x_{K/k,S} : C \rightarrow \mathbb{C}\Gamma$$

$$s \mapsto \sum_{A \in \mathcal{X}_Q(A)} e_A\tilde{\nu}_A(\Theta_{K^A/Q,S_A}(s))$$

**Proposition 16** With the above hypotheses and notations

$$\Theta_{K/k,S_Q(k)}(s) = \det_{CG}(x_{K/k,S}(s) | \mathbb{C}\Gamma)$$

(as $\mathbb{C}G$-valued meromorphic functions of $s \in \mathbb{C}$).

**Proof** By meromorphic continuation, it suffices to prove $\chi(\text{L.H.S}) = \chi(\text{R.H.S})$ in (48), for $\text{Re}(s) > 1$ and for all $\chi \in \hat{G}$. Equation (2) and Lemma 7 give

$$\chi(\text{L.H.S of (48)}) = L_{K/k,S_Q(k)}(s, \chi^{-1}) = \prod_{\phi \in \Gamma} L_{K/Q,S_Q}(s, \phi^{-1})$$

and evaluating $\chi(\text{R.H.S of (48)})$ via Lemma 7 it suffices to show that $L_{K/Q,S_Q}(s, \phi^{-1}) = \phi(x_{K/k,S}(s))$ for any $\phi \in \hat{\Gamma}$. Suppose $\alpha_\phi := \phi|_A$ belongs to $A_\phi \in \mathcal{X}_Q(A)$ so that $\ker(\alpha_\phi) = A \cap \ker(\phi)$ and $K^{A_\phi} = K^A \phi$. On the one hand, this means that $\phi$ factors through $\text{Gal}(K^{A_\phi}/Q)$ and $\phi(e_{A_\phi} \tilde{\nu}_{A_\phi}(y)) = \phi(y)$ for all $y \in \mathbb{C}\text{Gal}(K^{A_\phi}/Q)$, while $\phi(e_A) = 0$ for any $A \neq A_\phi$. On the other, it implies that $S_{\text{ram}}(K^{A_\phi}/Q) = S_{\text{ram}}(K^A/Q) \cup S_{\text{ram}}(K^\phi/Q)$. Now, crucially for our argument, (17) implies that $S_{\text{ram}}(K^A/Q) \subset S_Q$ so $S_{A_\phi} = S_Q \cup S_{\text{ram}}(K^\phi/Q)$. Putting this together, (2), (3) and (16) give (for $\text{Re}(s) > 1$):

$$\phi(x_{K/k,S}(s)) = \phi(e_{A_\phi} \tilde{\nu}_{A_\phi}(\Theta_{K^{A_\phi}/Q,S_{A_\phi}}(s))) = L_{K^{A_\phi}/Q,S_{A_\phi}}(s, \phi^{-1}) = \prod_{q \notin S_Q \cup S_{\text{ram}}(K^\phi/Q)} (1 - q^{-s} \phi^{-1}(q)) = L_{K/Q,S_Q}(s, \phi^{-1})$$

Let us write $\mathcal{X}_Q^- (A)$ for the set $\{ A \in \mathcal{X}_Q(A) : c \notin \ker(A) \} (= \mathcal{X}_Q(A) \text{ if } c \notin A)$ and $x_{K/k,S}^-$ for the function $e^{-x_{K/k,S}} : C \rightarrow \mathbb{C}\Gamma^-$. If $A \in \mathcal{X}_Q(A)$ lies in $\mathcal{X}_Q^-(A)$ then $K^A$ is CM.
Otherwise $e^{-e_A} = 0$. Therefore $x^-_{K/k,S}(s)$ equals $\sum_{A \in \chi^-_Q(A)} e_A \bar{\nu}_A(\Theta^-_{K/A,Q,S_A}(s))$ and is entire as a function of $s$. Now take $s = 1$, multiply by $i/\pi$ and apply the involution $\gamma^* : \Gamma \to \Gamma$ (which fixes each $e$ in $\Gamma$, hence also a basis for $G$)

$$\left(\frac{i}{\pi}\right) x^-_{K/k,S}(1)^* = \sum_{A \in \chi^-_Q(A)} e_A \bar{\nu}_A(a^-_{K/A,Q,S_A})$$

(49)

which lies in $\bar{\mathbb{Q}}\Gamma$ by (10). On the other hand, multiplying $\Theta^-_{K/k,S}(s)$ by $(i/\pi)^d e^- = ((i/\pi)e^-)^{|\Gamma:G|}$ in the previous Proposition and letting $s \to 1$ implies that $a^-_{K/k,S}(k)$ is the $CG$-determinant of $(i/\pi)x^-_{K/k,S}(1)$ acting on $\mathbb{C}\Gamma$. It follows easily from this that

$$a^-_{K/k,S}(1) = \det_{\bar{\mathbb{Q}}} \left(\frac{i}{\pi}\right) x^-_{K/k,S}(1)^* \bar{\mathbb{Q}}\Gamma$$

(50)

For each $A \in \chi^-_Q(A)$ the data $K^A/Q$, $S_A$ and $p$ satisfy the standard hypotheses. In particular we have a well-defined $\mathbb{Z}_p\text{Gal}(K/A/Q)$-linear map $s^{id}_{K,A,Q,S_A}$ from $U^1(K^-_p)$ to $\mathbb{Q}_p\text{Gal}(K^A/Q)^-$ (where ‘id’ denotes the identity element of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$) Both the norm map $N_{K/K^A} : U^1(K_p) \to U^1(K^-_p)$ and the averaged corestriction $\bar{\nu}_A : \mathbb{Q}_p\text{Gal}(K^A/Q) \to \mathbb{Q}_p\Gamma$ take minus parts to minus parts. The automorphism $\tau_i \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$ restricts to an element $\gamma_i := \tau_i|_K$ of $\Gamma$ for $i = 1, \ldots, d$ such that $\{\gamma_1, \ldots, \gamma_d\}$ is a set of coset representatives for $G$ in $\Gamma$, hence also a basis for $\mathcal{R}\Gamma$ over $\mathcal{R}G$, for any commutative ring $\mathcal{R}$. We can now state:

**Theorem 7** With the above hypotheses and notations, suppose that $u_1, \ldots, u_d$ are any elements of $U^1(K^-_p)$. Then

$$s^\tau_{1, \ldots, \tau_d}_{K/k,S_p}(k)(u_1 \wedge \ldots \wedge u_d) = \det(c_{i,l})^{d}_{i=1, l=1}$$

where $c_{i,l} \in \mathbb{Q}_pG^-$ is the coefficient of $\gamma_i$ when the element $\sum_{A \in \chi^-_Q(A)} e_A \bar{\nu}_A(s^{id}_{K,A,Q,S_A}(N_{K/K^A}u_i))$ of $\mathbb{Q}_p\Gamma$ is expressed in the $\mathbb{Q}_pG$-basis $\{\gamma_1, \ldots, \gamma_d\}$ of $\mathbb{Q}_p\Gamma$.

**Proof** Choose an embedding $j : \mathbb{Q} \to \mathbb{Q}_p$ inducing a prime ideal $\mathfrak{P} \in \mathfrak{S}_p(K)$, say, and write $\lambda_p$ for the $(1 \times 1)$ regulator $R^{(1, \mathfrak{p})}_{K/\mathbb{Q},p} : U^1(K_p) \to \mathbb{Q}_p\Gamma$. If $u \in U^1(K_p)$ then, by definition,

$$\lambda_p(u) = \sum_{i=1}^d \sum_{g \in G} \log_g(j \circ \iota_{\mathfrak{P}}(g_i g^{-1} u))(g \gamma_i^{-1}) = \sum_{i=1}^d \sum_{g \in G} \log_g(j \tau_i \circ \iota_{\mathfrak{P}}(g^{-1} u)g) \gamma_i^{-1}$$

where $j \tau_i$ induces $\mathfrak{P}_i \in \mathfrak{S}_p(K)$. Now $\bigwedge^d_{\mathbb{Q}_pG, \mathfrak{P}_i} \mathbb{Q}_p\Gamma$ is $\mathbb{Q}_pG$-free of rank one on $\gamma_1^{-1} \wedge \ldots \wedge \gamma_d^{-1}$ and it follows easily from the last equation and the definition of $R^{(1, \mathfrak{p})}_{K/\mathbb{Q},p}$ that

$$\lambda_p(u_1) \wedge \ldots \wedge \lambda_p(u_d) = R^{(1, \mathfrak{p})}_{K/\mathbb{Q},p}(u_1 \wedge \ldots \wedge u_d) \gamma_1^{-1} \wedge \ldots \wedge \gamma_d^{-1} \quad \text{in} \quad \bigwedge^d_{\mathbb{Q}_pG, \mathfrak{P}_i} \mathbb{Q}_p\Gamma$$

(51)
On the other hand, \( R_{K/A}^{(j, l)} \circ N_{K/K} = \pi_{K/K} \circ \lambda_p \) for each \( A \in \mathcal{X}_Q^- (A) \) so that \( s_{K/A, l}^{(j)} (N_{K/K} u_l) = j(a_{K/A, l}^{(j)}) \pi_{K/K} (\lambda_p (u_l)) \) for each \( A \) and \( l \). It follows that

\[
\sum_{i=1}^{d} c_i \gamma_i^{-1} = \left( \sum_{A \in \mathcal{X}_Q^- (A)} e_A \bar{\nu}_A (j(a_{K/A, l}^{(j)})) \right) \lambda_p (u_l) = j((i/\pi) x_{K/K}^{(1)} \lambda_p (u_l))
\]

by equation (49). Using equation (50), we deduce easily that

\[
\det(c_i) \gamma_i^{-1} = (j((i/\pi) x_{K/K}^{(1)} \lambda_p (u_l)) \wedge \ldots \wedge (j((i/\pi) x_{K/K}^{(1)} \lambda_p (u_l))) = j(a_{K/K, l}^{(j)}) \lambda_p (u_l) \wedge \ldots \wedge \lambda_p (u_l)
\]

and combining this with equation (51), the result follows from the definition of \( s_{K/k, S_0(k)}^{(j)} \).

**Proof of Theorem 4 under Hypothesis 4** By Prop. 9 it suffices to prove \( IC(K/k, S_0(k), p) \), i.e. that \( s_{K/k, S_0(k)} (u_1 \wedge \ldots \wedge u_d) \) lies in \( Z_p G \) for all \( u_1, \ldots, u_d \in U^1(K_p)^- \) and this will clearly follow from Theorem 7 provided we can show

\[
e_A \bar{\nu}_A (s_{K/A, l}^{(j)} (N_{K/K} u_l)) \in Z_p \Gamma \quad \forall l, \forall A \in \mathcal{X}_Q^- (A)
\]

But Theorem 3 (i) implies that \( s_{K/A, l}^{(j)} (N_{K/K} u_l) \) lies in \( Z_p \text{Gal}(K/A/Q) \). Furthermore, if \( q \in \text{Bad}(S) \) then \( |T_q (K/Q)| = e_q (k/Q) \) and Hypothesis 4 implies that this is prime to \( p \) for all such \( q \), hence that \( p \nmid |A| \). It follows that \( p \nmid [K : K] \) for every \( A \) so that \( e_A \in Z_p \) and \( \bar{\nu}_A (s_{K/A, l}^{(j)} (N_{K/K} u_l)) \in Z_p \Gamma \), establishing (52).

Turning to the Congruence Conjecture, we suppose from now on that \( K \) contains \( \mu_{p^{n+1}} \) for some \( n \geq 0 \). Since \( S_{\text{ram}} (Q(\mu_{p^{n+1}})/Q) = \{ p \} \subset S_Q \), it follows from (47) that \( K+A \) contains \( Q(\mu_{p^{n+1}}) \) so is CM and \( \mathcal{X}_Q^- (A) = \mathcal{X}_Q (A) \). We write \( \Gamma \) for \( \text{Gal}(K^+/Q) \).
Now $RSC(K^{A,+}/\mathbb{Q}, S_A; \mathbb{Q})$ holds for each $A \in \mathcal{X}_Q(A)$. Indeed, let us write $f_A$ for the conductor of $K^A$ so that $p^{n+1}|f_A$ and $S^1(K^A/\mathbb{Q}) = S^1(\mathbb{Q}(\xi_{f_A})/\mathbb{Q}) = \{\infty\} \cup S_{f_A}(\mathbb{Q})$. Then, the determination of $\eta_{\xi_{f_A}}(\mathcal{S})$, $\mathcal{S} \subset \mathcal{S}^1(\mathbb{Q}(\xi_{f_A})/\mathbb{Q})$ in the proof of Theorem 3 together with Props. 13 and 14 imply that the solution $\eta_{K^{A,+}/\mathbb{Q}, S_A}$ of $RSC(K^{A,+}/\mathbb{Q}, S_A; \mathbb{Q})$ (with $\tau_1 = \text{id}$) is

$$\eta_A := \prod_{g \in S_A \setminus S^1(K^A/\mathbb{Q})} \left(1 - \sigma_{q,K^{A,+}/\mathbb{Q}}^{-1}\right) N_{\mathbb{Q}(\xi_{f_A})/\mathbb{Q}}(\eta_{\xi_{f_A}}) \left(\mathcal{S}^1(\mathbb{Q}(\xi_{f_A})/\mathbb{Q})\right)$$

$$= \prod_{g \in S_A \setminus S^1(K^A/\mathbb{Q})} \left(1 - \sigma_{q,K^{A,+}/\mathbb{Q}}^{-1}\right) N_{\mathbb{Q}(\xi_{f_A})/\mathbb{Q}}\left(\frac{1}{2} \otimes (1 - \xi_{f_A})^{1/\mathbb{Q}}\right)$$

In fact, $(1 - \xi_{f_A})$ lies in $U_{\{\infty\}}(\mathbb{Q}(\xi_{f_A}))$ unless $f_A$ is a power of a prime, necessarily $p$, in which case it lies in $U_{\{\infty,p\}}(\mathbb{Q}(\xi_{f_A}))$. Thus for all $A \in \mathcal{X}_Q(A)$, the element $\eta_A$ lies in $\frac{1}{2}U_{\{\infty,p\}}(K^{A,+}) \subset \mathbb{Q}U_{\{\infty,p\}}(K^{A,+})$. Let us write $i_A$ for the natural injection $QU_{\{\infty,p\}}(K^{A,+}) \rightarrow QU_{\{\infty,p\}}(K^\prime)$ and $i_A$ for $|\text{ker}(A)|^{-1/\mathbb{Q}}$. We define

$$\alpha_{S_0(k)} := \sum_{A \in X_0(A)} e_A i_A(\eta_A) \quad \text{and} \quad \eta_{S_0(k)} := \gamma_1^{-1} \alpha_{S_0(k)} \wedge \ldots \wedge \gamma_d^{-1} \alpha_{S_0(k)}$$

from which it is clear that $\alpha_{S_0(k)}$ lies in $|A|^{-2\frac{1}{2}}U_{\{\infty,p\}}(K^\prime)$ and $\eta_{S_0(k)}$ in $(|A|^{-2\frac{1}{2}}dA)^{d/\mathbb{Z}_G} U_{\{\infty,p\}}(K^\prime)$.

**Proposition 17** With the above hypotheses, $RSC(K^+/k, S_Q(k); \mathbb{Q})$ holds with solution $\eta_{K^+/k,S_0(k)} = \eta_{S_0(k)}$.

We defer the proof. The final ingredient in the Proof of Theorem 3 is

**Lemma 8** Suppose $\alpha$ is an element of $U_{\{\infty,p\}}(K^\prime)$ so that $\gamma_1^{-1}\alpha \wedge \ldots \wedge \gamma_d^{-1}\alpha$ lies in the subset $\bigcap G U_{S_0(k)}(K^\prime)$ of $\bigwedge^d G U_{S_0(k)}(K^\prime)$. Then, for any $u_1, \ldots, u_d \in U^1(K^\prime)$ we have

$$\kappa_n(\tau_1 \ldots \tau_d) H_{K/k,n}(\gamma_1^{-1}\alpha \wedge \ldots \wedge \gamma_d^{-1}\alpha, u_1 \wedge \ldots \wedge u_d) = \text{det}(d_{i,1})_{i=1}^n$$

where $d_{i,1} \in (\mathbb{Z}/p^{n+1}\mathbb{Z})G^\prime$ is the coefficient of $\gamma_i^{-1}$ when the element $H_{K/k,n}(\alpha, u_i)$ of $(\mathbb{Z}/p^{n+1}\mathbb{Z})\Gamma^\prime$ is expressed in the $(\mathbb{Z}/p^{n+1}\mathbb{Z})G$-basis $\{\gamma_1^{-1}, \ldots, \gamma_d^{-1}\}$ of $(\mathbb{Z}/p^{n+1}\mathbb{Z})\Gamma^\prime$.

**Proof** If $\alpha = 1 \wedge \varepsilon$ for some $\varepsilon \in U_{\{\infty,p\}}(K^\prime)$ then $\gamma_i^{-1}\alpha = 1 \wedge \gamma_i^{-1}\varepsilon$ with $\gamma_i^{-1}\varepsilon \in U_{\{\infty,p\}}(K^\prime) \subset U_S(K^\prime)$ for all $i$. Equations (20) and (18) applied to $K/\mathbb{Q}$ give

$$H_{K/k,n}(\alpha, u_i) = \sum_{i=1}^d \left(\sum_{g \in G} \varepsilon_i, g \gamma_i g^{-1} \right) \gamma_i^{-1} = \sum_{i=1}^d \left(\kappa_n(\tau_i) \sum_{g \in G} \left[\gamma_i^{-1}\varepsilon_i, g u_i \right]_{K,n} g^{-1} \right) \gamma_i^{-1}$$

since $\gamma_i = \tau_i|_K$. Thus $d_{i,1} = \kappa_n(\tau_i) \sum_{g \in G} \gamma_i^{-1}\varepsilon_i, g u_i |_{K,n} g^{-1}$. Now use (20) for $K/k$.

**Proof of Theorem 3** By Prop. 11 it suffices to establish $CC(K/k, S_Q(k), p, n)$ under Hypothesis 4. But the latter has already been shown to imply $IC(K/k, S_Q(k), p)$ and that
Therefore, \( \eta_{S_0(k)} \) lies in \( \mathbb{Z}_p \setminus \bigwedge_{G} U_{\{\infty,p\}}(K^+) \subset \mathbb{Z}_p \lambda_{0,S_0(k)} \) and so is the solution of \( RSC(K^+/k,S_0(k);\mathbb{Z}_p) \) by Proposition [17]. It remains to prove the congruence (24) holds with \( \eta_{K^+/k,S} = \eta_{S_0(k)} \) and \( \theta = u_1 \land \ldots \land u_d \) with \( u_i \in U_1(K_p) \) for \( i \). (Such \( \theta \) generate \( \bigwedge_{G} U(K_p)^- \).) For each \( \mathcal{A} \in X_Q(\mathcal{A}) \) we may write \( 2\eta_A \) as \( 1 \otimes \varepsilon_\mathcal{A} \) where \( \varepsilon_\mathcal{A} \) lies in \( U_{\{\infty,p\}}(K_\mathcal{A},+) \). From equation (30) with \( F = K^\mathcal{A} \) and (20) (with \( d = 1! \)) it follows easily that

\[
H_{K/Q,n}(i_A(2\eta_A), u_i) = \nu_{K/K^\mathcal{A}}(H_{K^\mathcal{A}/Q,n}(2\eta_A, N_{K/K^\mathcal{A}}u_i))
\]

Therefore, using the \( \mathbb{Z}\Gamma \)-linearity of \( H_{K/Q,n}(\cdot, \cdot) \) in the first variable and the fact \( |A|e_\mathcal{A} \in \mathbb{Z}\Gamma \), we have, for each \( l \):

\[
H_{K/Q,n}(2|A|^2\alpha_{S_0(k)}, u_l) = \sum_{A \in X_Q(\mathcal{A})} (|A||A : \ker(\mathcal{A})|e_\mathcal{A})H_{K/Q,n}(i_A(2\eta_A), u_l)
\]

\[
= \sum_{A \in X_Q(\mathcal{A})} (|A||A : \ker(\mathcal{A})|e_\mathcal{A})\nu_{K/K^\mathcal{A}}(H_{K^\mathcal{A}/Q,n}(2\eta_A, N_{K/K^\mathcal{A}}u_l))
\]

\[
= \sum_{A \in X_Q(\mathcal{A})} 2(|A||A : \ker(\mathcal{A})|e_\mathcal{A})\nu_{K/K^\mathcal{A}}(S_{K^\mathcal{A}/Q,S_A}(N_{K/K^\mathcal{A}}u_l))
\]

\[
= \sum_{A \in X_Q(\mathcal{A})} 2|A|^2e_\mathcal{A}\tilde{\mathcal{A}}(S_{K^\mathcal{A}/Q,S_A}(N_{K/K^\mathcal{A}}u_l))
\]

\[
= \sum_{i=1}^d 2|A|^2c_{i,l}\gamma_i^{-1} \pmod{p^{n+1}}
\]

where \( c_{i,l} \) is precisely as defined in Theorem [7]. Note that the first congruence above comes from Theorem [3] which also shows that the last three expressions lie in \( \mathbb{Z}_p \Gamma \). It follows from Proposition [17] and Lemma [8] that

\[
(2|A|^2)^d\kappa_n(\gamma_1 \ldots \gamma_d)H_{K/k,n}(\eta_{S_0(k)}, u_1 \land \ldots \land u_d)
\]

\[
= \kappa_n(\gamma_1 \ldots \gamma_d)H_{K/k,n}(\gamma_1^{-1}(2|A|^2\alpha_{S_0(k)}) \land \ldots \land \gamma_d^{-1}(2|A|^2\alpha_{S_0(k)}), u_1 \land \ldots \land u_d)
\]

\[
= \text{det}(2|A|^2c_{i,l})_{i,l}
\]

in \( (\mathbb{Z}/p^{n+1}\mathbb{Z})G \). Since \( p \nmid 2|A| \), we may cancel the factor \( (2|A|^2)^d \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \) on both sides above and combining with Theorem [7] we obtain

\[
S_{K/k,S_0(k)}^{\gamma_1 \ldots \gamma_d}(u_1 \land \ldots \land u_d) = \text{det}(c_{i,l})_{i,l} = \kappa_n(\gamma_1 \ldots \gamma_d)H_{K/k,n}(\eta_{S_0(k), u_1 \land \ldots \land u_d})
\]

as required.

**Remark 8.1** Burns has proven Conjecture B' of [Rn] whenever \( K \) is absolutely abelian (see [Bn2, Theorem A]). It follows from Remark 2.3 that \( \eta_{S_0(k)} = \eta_{K^+/k,S_0(k)} \) must also lie in \( \frac{1}{2}\lambda_{0,S_0(k)}(K^+/k) \), although this is not obvious from our expression for \( \eta_{S_0(k)} \) and Burns’ results do not appear to provide an explicit expression. On the other hand Cooper obtains
essentially our expression $\gamma_1^{-1} \alpha S_0(k) \land \ldots \land \gamma_d^{-1} \alpha S_0(k)$ in [Coo]. (Indeed, we adapt his methods in the proof of Proposition [L7] below.) By manipulating it cleverly and using the norm relations for cyclotomic numbers, he shows explicitly that if $A$ is cyclic and of odd order, then $\eta S_0(k)$ lies in $2^{-d} \bigcap_{\mathbf{Z}G} U_{\infty,p}(k^+)$. (This follows from [Coo Theorem 5.2.2]).

**Proof of Proposition [L7]** The arguments are mostly familiar by now: Applying $e^+$ to (18) one deduces that $\Theta_{K^+/k,S_0(k)}(s)$ is the determinant of $\sum_A e_A| \ker(A)|^{-1} \nu_{K^+/K^{A+}}(\Theta_{K^{A+},Q,S_A}^{(1)}(s))$ acting on $\mathbb{C} \Gamma$, where $\Gamma := \text{Gal}(K^+/\mathbb{Q})$ and we are identifying $A$ with $\text{Gal}(K^+/K^{A+})$ by restriction. Now $\Theta_{K^{A+},Q,S_A}(0) = 0$ and $\Theta_{K^{A+},Q,S_A}^{(1)}(0)$ has real coefficients for each $A$, so

$$\Theta_{K^+/k,S_0(k)}^{(c)}(0) = \det_{\mathbb{R} G} \left( \sum_A e_A| \ker(A)|^{-1} \nu_{K^+/K^{A+}}(\Theta_{K^{A+},Q,S_A}^{(1)}(0)) | \mathbb{R} \Gamma \right) = \det(e_{i,l})_{i,l=1}^d$$

say, where $(e_{i,l})_{i,l=1}^d$ is the matrix of multiplication by $\sum_A \ldots$ on $\mathbb{R} \Gamma$ w.r.t. the $\mathbb{R} G$-basis

$$\gamma_i^{-1} : i = 1, \ldots, d \quad (\text{where } \gamma_i := \gamma_i|K^+ = \tau_i|K^+).$$

Fix $l$ and take $\tau_1 = \text{id}$. Using (15) for each extension $K^{A+}/\mathbb{Q}$ and the relation $\nu_{K^+/K^{A+}} \circ \lambda_{K^{A+},Q,1} = \lambda_{K^+/Q,1} \circ \iota_{K^+/K^{A+}}$, we get

$$\sum_A e_A| \ker(A)|^{-1} \nu_{K^+/K^{A+}}(\Theta_{K^{A+},Q,S_A}^{(1)}(0)) \gamma_i^{-1} = \sum_A e_A| \ker(A)|^{-1} \nu_{K^+/K^{A+}}(\lambda_{K^{A+},Q,1}(\eta_A)) \gamma_i^{-1} = \lambda_{K^+/Q,1}(\alpha S_0(k)) \gamma_i^{-1}$$

But for any element $\alpha = a \otimes \varepsilon$ of $\mathbb{Q}U_S(K^+)$ (with $a \in \mathbb{Q}$) we have

$$\lambda_{K^+/Q,1}(\alpha) \gamma_i^{-1} = \gamma_i^{-1} \lambda_{K^+/Q,1}(\gamma_i^{-1} \alpha S_0(k)).$$

Substituting this in (53), it follows that $\eta S_0(k)$ satisfies condition (17) for $K/k^+$ and $S_0(k)$.

To show that $\eta S_0(k)$ satisfies the eigenspace condition w.r.t. $(S_0(k),d,G)$, one could adapt the argument of [Coo] (based on [P62 Prop. 3.1.2]) using condition (iv) of Prop. 2. We sketch a more ‘algebraic’ argument based on the equivalent condition (iii). Suppose $q \in S Q(k) \setminus S_\infty(k)$ lies above $q \in S_\mathbb{Q}\setminus \{\infty\}$, write $D$ for $D_q(K/\mathbb{Q})$ and $\mathfrak{D}$ for $D_q(K/k) = D \cap G$. Let $\rho_1, \ldots, \rho_t$ be a set of representatives for $D$ mod $\mathfrak{D}$, hence for $DG$ mod $G$, and let $\sigma_1, \ldots, \sigma_m$ be a set of representatives for $\Gamma$ mod $DG$. Then $d = mt$ and both $\{\sigma_a \rho_b\}_{a,b}$ and $\{\gamma_i^{-1}\}_i$ are sets of representatives for $\Gamma$ mod $G$. Writing also $\eta$ and $\alpha$ for $\eta S_0(k)$ and $\alpha S_0(k)$ respectively, it follows that $\eta = \pm g \bigwedge_{a=1}^m \bigwedge_{b=1}^t \sigma_a \rho_b \alpha$ for some $g \in G$. (The unordered ‘wedge product’ (over $\mathbb{Q} \mathfrak{D}$) on the RHS is defined only up to sign.) Since $N_{D_q(K^+/k)} \eta$ equals $\frac{1}{2} N_{\mathfrak{D}} \eta$ or $N_{\mathfrak{D}} \eta$, condition (iii) for $m = \eta$ and $S = S_0(k)$ will follow if we can show that $N_{\mathfrak{D}} \eta$ is fixed by $G$ (hence by $\mathfrak{D}$) and is zero if $|S_0(k)| > d + 1$. But

$$N_{\mathfrak{D}} \eta = \pm |\mathfrak{D}|^{1-d} \sum_{a=1}^m \sigma_a N_{\mathfrak{D}} \rho_b \alpha = \pm |\mathfrak{D}|^{1-d} \sum_{a=1}^m (\sigma_a N_{\mathfrak{D}} \alpha \land \sigma_a N_{\mathfrak{D}} \rho_2 \alpha \ldots \land \sigma_a N_{\mathfrak{D}} \rho_t \alpha)$$

(54)
(the second equality since $\sigma_a N_D \alpha = \sum_{t=1}^t \sigma_a N_D \rho_t \alpha$ for each $a$). If $|S_Q| > 2$ then $|S_A| > 2$ for each $A \in X_Q(A)$ so the eigenspace condition on $\eta_A$ as a solution of $RSC(K^A+/Q, S_A; Q)$ implies that it is annihilated by $N_{D_q(K^A+/Q)}$, hence by $N_D$. It follows that $N_D \alpha = 0$ hence $N_D \eta = 0$ by (54). Otherwise, $|S_Q| = 2$, $S_Q = \{\infty, q\}$ (so $q = p$) and $|S_Q(k)|$ is precisely $d + m$. In this case, the eigenspace condition on $\eta_A$ still shows that $N_{D_q(K^A+/Q)} \eta_A$ is fixed by $\text{Gal}(K^A+/Q)$ for all $A$ and it follows as above that $N_D \alpha$ is fixed by $G$. So equation (54) implies that $N_D \eta$ is fixed by $G$ and, if $m > 1$, that it is zero, since $\sigma_1 N_D \alpha = \sigma_2 N_D \alpha$. \hfill \Box

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