LIE SUPERALGEBRAS OF SUPERMATRICES OF COMPLEX SIZE.
THEIR GENERALIZATIONS AND RELATED INTEGRABLE SYSTEMS

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Abstract. We distinguish a class of simple filtered Lie algebras $LU_g(\lambda)$ of polynomial growth with increasing filtration and whose associated graded Lie algebras are not simple. We describe presentations of such algebras. The Lie algebras $LU_g(\lambda)$, where $\lambda$ runs over the projective space of dimension equal to the rank of $g$, are quantizations of the Lie algebras of functions on the orbits of the coadjoint representation of $g$.

The Lie algebra $gl(\lambda)$ of matrices of complex size is the simplest example: it is $LU_{sl(2)}(\lambda)$. The dynamical systems associated with it in the space of pseudodifferential operators in the same way as the KdV hierarchy is associated with $sl(n)$ are those studied by Gelfand–Dickey and Khesin–Malikov. For $g \neq sl(2)$ we get generalizations of $gl(\lambda)$ and the corresponding dynamical systems, in particular, their superized versions. The algebras $LU_{sl(2)}(\lambda)$ possess a trace and an invariant symmetric bilinear form, hence, with these Lie algebras associated are analogs of the Yang-Baxter equation, KdV, etc.

Our presentation of $LU_s(\lambda)$ for a simple $s$ is related to presentation of $s$ in terms of a certain pair of generators. For $s = sl(n)$ there are just 9 such relations.

This is our paper published in: by E. Ramírez de Arellano, et. al. (eds.) Proc. Internatnl. Symp. Complex Analysis and related topics, Mexico, 1996, Birkhauser Verlag, 1999, 73–105. We just wish to make it more accessible. Here we made minory corrections, e.g., replaced $pgl(\lambda)$ with $sl(\lambda)$: to write $pgl(\lambda)$ is correct, but notation $sl(\lambda)$ is closer to its finite dimensional particular case.

§0. Introduction

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0.0. History. About 1966, V. Kac and B. Weisfeiler began the study of simple filtered Lie algebras of polynomial growth. Kac first considered the $Z$-graded Lie algebras associated with the filtered ones and classified simple graded Lie algebras of polynomial growth under a technical assumption and conjectured the inessential nature of the assumption. It took more than 20 years to get rid of the assumption: see very complicated papers by O. Mathieu, cf. [K] and references therein. For a similar list of simple $Z$-graded Lie superalgebras of polynomial growth see [KS], [LS].

The Lie algebras Kac distinguished (or rather the algebras of derivations of their nontrivial central extensions, the Kac–Moody algebras) proved very interesting in applications. These algebras aroused such interest that the study of filtered algebras was arrested for two decades.
Little by little, however, the simplest representative of the new class of simple filtered Lie superalgebras (of polynomial growth), namely, the Lie algebra \( \mathfrak{gl}(\lambda) \) of matrices of complex size, and its projectivization, i.e., the quotient modulo the constants, \( \mathfrak{pgl}(\lambda) \), drew its share of attention \([4],[5],[6]\).

While we typed this paper, Shoikhet \[7\] published a description of representations of \( \mathfrak{gl}(\lambda) \); we are thankful to M. Vasiliev who informed us of still other applications of generalizations of \( \mathfrak{gl}(\lambda) \), see \([8],[9],[10]\).

This paper begins a systematic study of a new class of Lie algebras: simple filtered Lie algebras of polynomial growth (SFLAPG) for which the graded Lie algebras associated with the filtration considered are not simple; \( \mathfrak{sl}(\lambda) \) is our first example. Actually, an example of a Lie algebra of class SFLAPG was known even before the notion of Lie algebras was introduced. Indeed, the only deformation (physicists call it quantization) \( Q \) of the Poisson Lie algebra \( \mathfrak{po}(2n) \) sends \( \mathfrak{po}(2n) \) into \( \mathfrak{diff}(n) \), the Lie algebra of differential operators with polynomial coefficients; the restriction of \( Q \) to \( \mathfrak{h}(2n) = \mathfrak{po}(2n)/\text{center} \), the Lie algebra of Hamiltonian vector fields, sends \( \mathfrak{h}(2n) \) to the projectivization \( \mathfrak{pdiff}(n) = \mathfrak{diff}(n)/\mathbb{C} \cdot 1 \) of \( \mathfrak{diff}(n) \). The Lie algebra \( \mathfrak{pdiff}(n) \) escaped Kac’s classification, though it is the deform of an algebra from his list, because its intrinsically natural filtration given by \( \deg q_i = -\deg \partial q_i = 1 \) is not of polynomial growth while the graded Lie algebra associated with the filtration of polynomial growth (given by \( \deg q_i = \deg \partial q_i = 1 \)) is not simple.

Observe that from the point of view of dynamical systems the Lie algebra \( \mathfrak{diff}(n) \) is not very interesting: it does not possesses a nondegenerate bilinear symmetric form; we will consider its subalgebras that do.

In what follows we will usually denote the associative (super)algebras by Latin letters; the Lie (super)algebras associated with them by Gothic letters; e.g., \( \mathfrak{gl}(n) = L(\text{Mat}(n)) \), \( \mathfrak{diff}(n) = L(\text{Diff}(n)) \), where the functor \( L \) replaces the dot product with the bracket.

0.1. The construction. Problems related. Each of our Lie algebras (and Lie superalgebras) \( LU_g(\lambda) \) is realized as a quotient of the Lie algebra of global sections of the sheaf of twisted \( D \)-modules on the flag variety, cf. \([11],[12],[13]\). The general construction consists of the preparatory step 0), the main steps 1) and 2) and two extra steps 3) and 4).

We distinguish two cases: A) \( \dim g < \infty \) and \( g \) possesses a Cartan matrix and B) \( g \) is a simple vectorial Lie (super)algebra.

Let \( g = g_- \oplus h \oplus g_+ \), where \( g_+ = \bigoplus_{\alpha > 0} g_{\alpha} \) and \( g_- = \bigoplus_{\alpha < 0} g_{\alpha} \), be one of the simple \( \mathbb{Z} \)-graded Lie algebras of polynomial growth, either finite dimensional or of vector fields, represented as the sum of its maximal torus (usually identical with the Cartan subalgebra) \( h \) and the root subspaces \( g_{\alpha} \) corresponding to an order in the set \( R \) of roots.

Observe that each order of \( R \) is in one-to-one correspondence with a system of simple roots. For the finite dimensional Lie algebras \( g \) all systems of simple roots are equivalent, the equivalence is established by the Weyl group. For Lie superalgebras and infinite dimensional Lie algebras of vector fields there are inequivalent systems of simple roots; nevertheless, there is an analog of the Weyl group and the passage from system to system is described in \([14]\).

For vectorial Lie algebras and Lie superalgebras, even the dimension of the superspaces \( X = (g_-)^* \) associated with systems of simple roots can vary. It is not clear if only essential (see \([15]\)) systems of simple roots are essential in the construction of Verma modules (roughly speaking, each Verma module is isomorphic to the space of functions on \( X \)) in which we will realize \( LU_g(\lambda) \), but hopefully not all.

Step 0): From \( g \) to \( \tilde{g} \). From representation theory it is clear that there exists a realization of the elements of \( g \) by differential operators of degree \( \leq 1 \) on the space \( X = (g_-)^* \). The
realization has rank $\mathfrak{g}$ parameters (coordinates $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathfrak{h}^*$ of the highest weight of the $\mathfrak{g}$-module $M^\lambda$). For the algorithms of construction and its execution in some cases see [BMP], [B], [BGLS].

Let $\tilde{\mathfrak{g}}$ be the image of $\mathfrak{g}$ with respect to this realization. Let $S^*(\mathfrak{g})$ be the associative subalgebra generated by $\tilde{\mathfrak{g}}$. Clearly, $S^*(\mathfrak{g})$ is equal to $S^*(\tilde{\mathfrak{g}})$. Set

$$U_\mathfrak{g}(\lambda) = S^*(\tilde{\mathfrak{g}})/J(\lambda),$$

where $J(\lambda) = \tilde{\mathfrak{g}}^\lambda$ is the maxim ideal.

Observe that $J(\lambda) = 0$ for generic $\lambda$.

Roughly speaking, $U_\mathfrak{g}(\lambda)$ is “Mat”($L^\lambda$), where $L^\lambda$ is the quotient of $M^\lambda$ modulo the maximal submodule $I(\lambda)$ (it can be determined and described with the help of the Shapovalov form, see [K]) and $S^*(\mathfrak{g})$ is the subalgebra generated by $\tilde{\mathfrak{g}}$ in the symmetric algebra of $\mathfrak{g}$ modulo the relations between differential operators. Clearly, $S^*(\mathfrak{g})$ is smaller than $S^*(\mathfrak{h})$ due to the relations between the differential operators that span $\mathfrak{g}$.

To explicitly describe the generators of $J(\lambda)$ is a main technical problem. We solve it in this paper for $rk\mathfrak{g} = 1$. The general case will be considered elsewhere.

Step 1) From $U_\mathfrak{g}(\lambda)$ to $LU_\mathfrak{g}(\lambda)$ Recall that $LU_\mathfrak{g}(\lambda)$ is the Lie algebra whose space is the same as that of $U_\mathfrak{g}(\lambda)$ and the bracket is the commutator.

Step 2) Montgomery’s functor $S.$ Montgomery suggested [M] a construction of simple Lie superalgebras:

$$Mo: \text{ a central simple } \mathbb{Z} \text{-graded algebra } \mapsto \text{ a simple Lie superalgebra.} \quad (Mo)$$

Observe that the associative algebras $U_\mathfrak{g}(\lambda)$ constructed from simple Lie algebras $\mathfrak{g}$ are central simple. In [LM] we intend to consider Montgomery superalgebras $Mo(U_\mathfrak{g}(\lambda))$ and compare them with the Lie superalgebras $LU_\mathfrak{s}(\lambda)$ constructed from Lie superalgebras $\mathfrak{s}$. Montgomery functor often produces new Lie superalgebras, e.g., if $\mathfrak{g}$ is equal to $\mathfrak{f}_4$ or $\mathfrak{c}_4$, though not always:

$$Mo(U_{\mathfrak{s}f(2)}(\lambda)) \cong LU_{osp(1|2)}(\lambda).$$

Step 3) Twisted versions An outer automorphism $a$ of $\mathfrak{g} = LU_\mathfrak{g}(\lambda)$ or $Mo(U_\mathfrak{g}(\lambda))$ might single out a new simple Lie subsuperalgebra $a_0(\mathfrak{g})$, the set of fixed points of $\mathfrak{g}$ under $a$.

For example, the intersection of $LU_{\mathfrak{s}f(2)}(\lambda)$ with the set of skew-adjoint differential operators is a new Lie algebra $\mathfrak{o}/\mathfrak{sp}(\lambda)$ while the intersection of $Mo(U_{\mathfrak{s}f(2)}(\lambda)) = LU_{osp(1|2)}(\lambda)$ with the set of superskew-adjoint operators is the Lie superalgebra $\mathfrak{osp}(\lambda + 1|\lambda)$. For the description of the outer automorphisms of $\mathfrak{g}l(\lambda)$ see [LAS]. In general even the definition is unclear.

Step 4) Deformations The deformations of Lie algebras and Lie superalgebras obtained via steps 1) – 3) may lead to new algebras of class SFLAPG, cf. [G]. A. Sergeev posed the following interesting problem:

what Lie algebras and Lie superalgebras can we get by applying the above constructions 1) – 3) to the quantum deformation $U_{\rho}(\mathfrak{g})$ of $U(\mathfrak{g})$?

Remark . The above procedure can be also applied to (twisted) loop algebras $\mathfrak{g} = \mathfrak{h}^k$ and the stringy algebras; the result will be realized with differential operators of infinitely many indeterminates; they remind vertex operators. The algebra $LU_{\mathfrak{h}(k)}(\lambda)$ is a polynomial one but not of polynomial growth.

0.2. Another description of $U_\mathfrak{g}(\lambda)$. For the finite dimensional simple $\mathfrak{g}$ there is an alternative description of $U_\mathfrak{g}(\lambda)$ as the quotient of $U(\mathfrak{g})$ modulo the central character, i.e., modulo the ideal $C_\lambda$ generated by rank $\mathfrak{g}$ elements $C_i - k_i(\lambda)$, where the $C_i$ is the $i$-th Casimir element and the $k_i(\lambda)$ is the (computed by Harish-Chandra and Berezin) value of $C_i$ on $M^\lambda$. This description of $U_\mathfrak{g}(\lambda)$ goes back, perhaps, to Kostant, cf. [K]. From this description it is clear that, after the shift by $\rho$, the half sum of positive roots, we get

$$LU_\mathfrak{g}(\sigma(\lambda)) \cong LU_\mathfrak{g}(\lambda) \quad \text{for any } \sigma \in W(\mathfrak{g}).$$
A similar isomorphism holds for \( M_0(U_g(\lambda)) \). In particular, over \( \mathbb{R} \), it suffices to consider the \( \lambda \) that belong to one Weyl chamber only.

For vectorial Lie algebras the description of \( U_g(\lambda) \) as \( U(g)/C(\lambda) \) is inapplicable. For example, let \( g = \text{vect}(n) \). The highest weight Verma modules are (for the standard filtration of \( g \)) identical with Verma modules over \( sl(n+1) \), but the center of \( U(\text{vect}(n)) \) consists of constants only. It is a research problem to describe the generators of \( C(\lambda) \) in such cases.

Though the center of \( U(g) \) is completely described by A. Sergeev for all simple finite dimensional Lie superalgebras \([S]\), the problem describe the generators of the ideal \( C(\lambda) \) is open for Lie superalgebras even if \( g \) is of the form \( g(A) \) (i.e., if \( g \) has Cartan matrix \( A \)) different from \( \mathfrak{osp}(1|2n) \): for them the center of \( U(g) \) is not noetherian and it is a priori unclear if \( C(\lambda) \) has infinitely or finitely many generators. (As we will show elsewhere, \( C(\lambda) \) is generated for Lie superalgebras \( g \) of the form \( g(A) \) by the first \( \text{rk} g \) Casimir operators and finitely many extra elements. For algebras \( g \) of other types we do not even have a conjecture.)

0.3. Our result. The main result is the statement of the fact that the above constructions 1) – 4) yield a new class of simple Lie (super) algebras of polynomial growth (some of which have nice properties).

Observe that our Lie algebras \( LU_g(\lambda) \) are quantizations of the Lie algebras considered in \([DGS]\) which are also of class SFLAPG and are contractions of our algebras. Indeed, Donin, Gurevich and Shnider consider the Lie algebras of functions on the orbits of the coajoint representation of \( g \) with respect to the Poisson bracket. These DGS Lie algebras are naturally realized as the quotients of the polynomial algebra modulo an inhomogeneous ideal that singles out the orbit; we realize the result of quantization of DGS Lie algebras by differential operators.

In this paper we consider the simplest case of the superization of this construction: replace \( sl(2) \) with \( \mathfrak{osp}(1|2) \). The cases of higher ranks will be considered elsewhere. The Khesin–Malikov construction \([KM]\) can be applied almost literally to the Lie (super)algebras \( LU_g(\lambda) \) such that \( g \) admits a (super)principal embedding, see, e.g., \([GL2]\).

Our main theorems: 2.6 and 4.3. The structure of the algebras \( LU_g(\lambda) \) (real forms, automorphisms, root systems) will be described elsewhere, see e.g., \([LAS]\).

Observe that while the polynomial Poisson Lie algebra has only one class of nontrivial deformations and all the deformed algebras are isomorphic, cf. \([LSc]\), the dimension of the space of parameters of deformations of Lie algebras of Donin, Gurevich and Shnider is equal to the rank of \( g \) and all of the deform are pairwise nonisomorphic, generally.

0.4. The defining relations. The notion of defining relations is clear for a nilpotent Lie algebra. This is one of the reasons why the most conventional way to present a simple Lie algebra \( g \) is to split it into the direct sum of a (commutative) Cartan subalgebra and 2 maximal nilpotent subalgebras \( g_{\pm} \) (positive and negative). There are about \( (2 \cdot \text{rk} g)^2 \) relations between the \( 2 \cdot \text{rk} g \) generators of \( g_{\pm} \). The generators of \( g_+ \) together with the generators of \( g_- \) generate \( g \) as well. In \( g \), there are about \( (3 \cdot \text{rk} g)^2 \) relations between these generators; the relations additional to those in \( g_+ \) or \( g_- \), i.e., between the positive and the negative generators, are easy to grasp. Though numerous, all these relations — called Serre relations — are neat and this is another reason for their popularity. These relations are good to deal with not only for humans but for computers as well, cf. sec. 7.3.

Nevertheless, it so happens that the Chevalley-type generators and, therefore, the Serre relations are not always available. Besides, as we will see, there are problems in which other generators and relations naturally appear, cf. \([GL2]\).
Though not so transparent as for nilpotent algebras, the notion of generators and relations makes sense in the general case. For instance, with the principal embeddings of $\mathfrak{sl}(2)$ into $\mathfrak{g}$ one can associate only two elements that generate $\mathfrak{g}$; we call them Jacobson’s generators, see [GL]. We explicitly describe the associated with the principal embeddings of $\mathfrak{sl}(2)$ presentations of simple Lie algebras, finite dimensional and certain infinite dimensional; namely, the Lie algebra “of matrices of complex size” realized as a subalgebra of the Lie algebra $\mathfrak{diff}(1)$ of differential operators in 1 indeterminate or of $\mathfrak{gl}_\infty$, see §2.

The relations obtained are rather simple, especially for nonexceptional algebras. In contradistinction with the conventional presentation there are just 9 relations between Jacobson’s generators for $\mathfrak{sl}(\lambda)$ series (actually, 8 if $\lambda \in \mathbb{C} \setminus \mathbb{Z}$) and not many more for the other algebras.

It is convenient to present $\mathfrak{sl}(\lambda)$ as the Lie algebra generated by two differential operators: $X^+ = u^2 \frac{d}{du} - (\lambda - 1)u$ and $Z_{\mathfrak{sl}} = \frac{d^2}{du^2}$; its Lie subalgebra $\mathfrak{o}/\mathfrak{sp}(\lambda)$ of skew-adjoint operators — a hybrid of Lie algebras of series $\mathfrak{o}$ and $\mathfrak{sp}$ (do not confuse with the Lie superalgebra of $\mathfrak{osp}$ type!) — is generated by the same $X^+$ and $Z_{\mathfrak{o}/\mathfrak{sp}} = \frac{d^2}{du^2}$; to make relations simpler, we always add the third generator $X^- = -\frac{d}{du}$. For integer $\lambda$ each of these algebras has an ideal of finite codimension and the quotient modulo the ideal is the conventional $\mathfrak{sl}(n)$ (for $n = \lambda$ and $\mathfrak{gl}(\lambda)$) and either $\mathfrak{o}(2n + 1)$ (for $\lambda = 2n + 1$) or $\mathfrak{sp}(2n)$ (for $\lambda = 2n$), respectively, for $\mathfrak{o}/\mathfrak{sp}(\lambda)$.

In this paper we superize [GL]: replace $\mathfrak{sl}(2)$ with its closest relative, $\mathfrak{osp}(1|2)$. We denote by $\mathfrak{sl}(\lambda|\lambda + 1)$ the Lie superalgebra generated by $\nabla^+ = x \partial_x + x\theta \partial_x - \lambda \theta$, $Z = \partial_x \partial_\theta - \theta \partial_x^2$ and $U = \partial_\theta - \theta \partial_x$, where $x$ is an even indeterminate and $\theta$ is an odd one. We define $\mathfrak{osp}(\lambda + 1|\lambda)$ as the Lie subsuperalgebra of $\mathfrak{sl}(\lambda|\lambda + 1)$ generated by $\nabla^+$ and $Z$. The presentations of $\mathfrak{sl}(\lambda|\lambda + 1)$ and $\mathfrak{osp}(\lambda + 1|\lambda)$ are associated with the superprincipal embeddings of $\mathfrak{osp}(1|2)$. For $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ these algebras are simple. For integer $\lambda = n$ each of these algebras has an ideal of finite codimension and the quotient modulo the ideal is the conventional $\mathfrak{sl}(n|n + 1)$ and $\mathfrak{osp}(2n + 1|2n)$, respectively.

0.5. Some applications. (1) Integrable systems like continuous Toda lattice or a generalization of the Drinfeld–Sokolov construction are based on the superprincipal embeddings in the same way as the Khesin–Malikov construction [KM] is based on the principal embedding, cf. [GL2].

(2) To $q$-quantize the Lie algebras of type $\mathfrak{sl}(\lambda)$ à la Drinfeld, using only Chevalley generators, is impossible; our generators indicate a way to do it.

0.6. Related topics. We would like to draw attention of the reader to several other classes of Lie algebras. One of the reasons is that, though some of these classes have empty intersections with the class of Lie algebras we consider here, they naturally spring to mind and are, perhaps, deformations of our algebras in some, yet unknown, sense.

- Krichever–Novikov algebras, see [SH] and refs. therein. The KN-algebras are neither graded, nor filtered (at least, wrt the degree considered usually). Observe that so are our algebras $LU_\theta(\lambda)$ with respect to the degree induced from $U(\mathfrak{g})$, so a search for a better grading is a tempting problem.
- Odessky or Sklyanin algebras, see [FO] and refs. therein.
- Continuum algebras, see [SV] and refs. therein. In particular cases these algebras coincide with Kac–Moody or loop algebras, i.e., have a continuum analog of the Cartan matrix. But to suspect that $\mathfrak{gl}(\lambda)$ has a Cartan matrix is wrong, see sec. 2.2. Nevertheless, in the simplest cases, if $\text{rk} \mathfrak{g} = 1$, the algebras $LU_\theta(\lambda)$ and their “relatives” obtained in steps...
1) – 3) (and, perhaps, 4)) of sec 0.1 do possess Saveliev-Vershik’s nonlinear Cartan operator which replaces the Cartan matrix.

§1. Recapitulation: finite dimensional simple Lie algebras

This section is a continuation of [LP], where the case of the simplest base (system of simple roots) is considered and where non-Serre relations for simple Lie algebras first appear, though in a different setting. This paper is also the direct superization of [GL1]; we recall its results. For presentations of Lie superalgebras with Cartan matrix via Chevalley generators, see [LS], [GL3].

What are “natural” generators and relations for a simple finite dimensional Lie algebra? The answer is important in questions when it is needed to identify an algebra \( \mathfrak{g} \) given its generators and relations. (Examples of such problems are connected with Estabrook–Wahlquist prolongations, Drinfeld’s quantum algebras, symmetries of differential equations, integrable systems, etc.).

1.0. Defining relations. If \( \mathfrak{g} \) is nilpotent, the problem of its presentation has a natural and unambiguous solution: representatives of the homology \( H_1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \) are the generators of \( \mathfrak{g} \) and the elements from \( H_2(\mathfrak{g}) \) correspond to relations.

On the other hand, if \( \mathfrak{g} \) is simple, then \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \) and there is no “most natural” way to select generators of \( \mathfrak{g} \). The choice of generators is not unique.

Still, among algebras with the property \( \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \) the simple ones are distinguished by the fact that their structure is very well known. By trial and error people discovered that for finite dimensional simple Lie algebras, there are certain “first among equal” sets of generators:

1) Chevalley generators corresponding to positive and negative simple roots;
2) a pair of generators that generate any finite dimensional simple Lie algebra associated with the principal \( \mathfrak{sl}(2) \)-subalgebra (considered below).

The relations associated with Chevalley generators are well-known, see e.g., [OV], [K]. These relations are called Serre relations.

The possibility to generate any simple finite dimensional Lie algebra by two elements was first claimed by N. Jacobson; for the first (as far as we know) proof see [BO]. We do not know what generators Jacobson had in mind; [BO] take for them linear combinations of positive and negative root vectors with generic coefficients; nothing like a “natural” choice that we suggest to refer to as Jacobson’s generators was ever proposed.

To generate a simple algebra with only two elements is tempting but nobody yet had explicitly described relations between such generators, perhaps, because to check whether the relations between these elements are nice-looking is impossible without a modern computer (cf. an implicit description in [F]). As far as we could test, the relations for any other pair of generators chosen in a way distinct from ours are too complicated. There seem to be, however, one exception cf. [GL2].

1.1. The principal embeddings. There exists only one (up to equivalence) embedding \( r : \mathfrak{sl}(2) \rightarrow \mathfrak{g} \) such that \( \mathfrak{g} \), considered as \( \mathfrak{sl}(2) \)-module, splits into \( \text{rk}\mathfrak{g} \) irreducible modules, cf. [LP] or [OV]. This embedding is called principal and, sometimes, minimal because for the other embeddings (there are plenty of them) the number of irreducible \( \mathfrak{sl}(2) \)-modules is > \( \text{rk}\mathfrak{g} \). Example: for \( \mathfrak{g} = \mathfrak{sl}(n) \), \( \mathfrak{sp}(2n) \) or \( \mathfrak{o}(2n+1) \) the principal embedding is the one corresponding to the irreducible representation of \( \mathfrak{sl}(2) \) of dimension \( n \), \( 2n \), \( 2n+1 \), respectively.

For completeness, let us recall how the irreducible \( \mathfrak{sl}(2) \)-modules with highest weight look like. (They are all of the form \( L\mu \), where \( L\mu = M\mu \) if \( \mu \not\in \mathbb{Z}_+ \), and \( L\mu = M\mu/M_{-n-2} \) if
The following basis in \( \mathfrak{sl}(2) \):

\[
X^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

The \( \mathfrak{sl}(2) \)-module \( M^n \) is illustrated with a graph whose nodes correspond to the eigenvectors \( l_{\mu-2i} \) of \( H \) with the weight indicated;

\[
\ldots \mu-\overset{2i-2}{\underset{i}{\circ}} - \mu-\overset{2i}{\underset{i}{\circ}} - \ldots - \mu-\overset{2}{\underset{i}{\circ}} - \mu
\]

the edges depict the action of \( X^\pm \) (the action of \( X^+ \) is directed to the right, that of \( X^- \) to the left: \( X^-l_{\mu-2i} = l_{\mu-2i-2} \) and

\[
X^+l_{\mu-2i} = X^+(X^-l_{\mu}) = i(\mu - i + 1)l_{\mu-2i+2}; \quad X^+(l_{\mu}) = 0.
\]

As follows from (1.1), the module \( M^n \) for \( n \in \mathbb{Z}_+ \) has an irreducible submodule isomorphic to \( M^{-n-2} \); the quotient, obviously irreducible, as follows from the same (1.1), will be denoted by \( L^n \).

There are principal \( \mathfrak{sl}(2) \)-subalgebras in every finite dimensional simple Lie algebra, though, generally, not in infinite dimensional ones, e.g., not in affine Kac-Moody algebras. The construction is as follows. Let \( X_1^\pm, \ldots, X_r^\pm \) be Chevalley generators of \( \mathfrak{g} \), i.e., the generators corresponding to simple roots. Let the images of \( X^\pm \in \mathfrak{sl}(2) \) in \( \mathfrak{g} \) be

\[
X^- \mapsto \sum X_i^-; \quad X^+ \mapsto \sum a_i X_i^+
\]

and select the \( a_i \) from the relations \([X^+, X^-], X^\pm] = \pm 2X^\pm \) true in \( \mathfrak{sl}(2) \). For \( \mathfrak{g} \) constructed from a Cartan matrix \( A \), there is a solution for \( a_i \) if and only if \( A \) is invertible.

In Table 1.1 a simple finite dimensional Lie algebra \( \mathfrak{g} \) is described as the \( \mathfrak{sl}(2) \)-module corresponding to the principal embedding (cf. [OV], Table 4). The table introduces the number \( 2k_2 \) used in relations. We set \( k_1 = 1 \).

**Table 1.1. \( \mathfrak{g} \) as the \( \mathfrak{sl}(2) \)-module.**

| \( \mathfrak{g} \) | the \( \mathfrak{sl}(2) \)-spectrum of \( \mathfrak{g} = L^2 \oplus L^{2k_2} \oplus L^{2k_3} \ldots \) | \( 2k_2 \) |
|------------------|-------------------------------------------------|-----|
| \( \mathfrak{sl}(n) \) | \( L^2 \oplus L^4 \oplus L^6 \ldots \oplus L^{2n-2} \) | 4 |
| \( \mathfrak{o}(2n+1), \mathfrak{sp}(2n) \) | \( L^2 \oplus L^6 \oplus L^{10} \ldots \oplus L^{4n-2} \) | 6 |
| \( \mathfrak{o}(2n) \) | \( L^2 \oplus L^6 \oplus L^{10} \ldots \oplus L^{4n-6} \oplus L^{2n-2} \) | 6 |
| \( \mathfrak{g}_2 \) | \( L^2 \oplus L^{10} \) | 10 |
| \( \mathfrak{f}_4 \) | \( L^2 \oplus L^{10} \oplus L^{14} \oplus L^{22} \) | 10 |
| \( \mathfrak{e}_6 \) | \( L^2 \oplus L^8 \oplus L^{10} \oplus L^{14} \oplus L^{16} \oplus L^{22} \) | 8 |
| \( \mathfrak{e}_7 \) | \( L^2 \oplus L^{10} \oplus L^{14} \oplus L^{18} \oplus L^{22} \oplus L^{26} \oplus L^{34} \) | 10 |
| \( \mathfrak{e}_8 \) | \( L^2 \oplus L^{14} \oplus L^{22} \oplus L^{26} \oplus L^{34} \oplus L^{38} \oplus L^{46} \oplus L^{58} \) | 14 |

One can show that \( \mathfrak{g} \) can be generated by two elements: \( x := X^+ \in L^2 = \mathfrak{sl}(2) \) and a lowest weight vector \( z := l_- \) from an appropriate module \( L^r \) other than \( L^2 \) from Table 1.1. For the role of this \( L^r \) we take either \( L^{2k_2} \) if \( \mathfrak{g} = \mathfrak{o}(2n) \) or the last module \( L^{2n-2} \) in the above table if \( \mathfrak{g} = \mathfrak{o}(2n) \). (Clearly, \( z \) is defined up to proportionality; we will assume that a basis of \( L^r \) is fixed and denote \( z = t \cdot l_- \) for some \( t \in \mathbb{C} \) that can be fixed at will, cf. §3.)

The exceptional choice for \( \mathfrak{o}(2n) \) is occasioned by the fact that by choosing \( z \in L^r \) for \( r \neq 2n-2 \) instead, we generate \( \mathfrak{o}(2n-1) \).
We call the above \( x \) and \( z \), together with \( y := X^- \in L^2 \) taken for good measure, *Jacobson's generators*. The presence of \( y \) considerably simplifies the form of the relations, though slightly increases their number. (One might think that taking the symmetric to \( z \) element \( l_r \) will improve the relations even more but in reality just the opposite happens.)

Concerning \( \mathfrak{g} = \mathfrak{o}(2n) \) see sec. 7.2.

### 1.2. Relations between Jacobson's generators

First, observe that if an ideal of a free Lie algebra is homogeneous (with respect to the degrees of the generators of the algebra), then the number and the degrees of the defining relations (i.e., the generators of the ideal) is uniquely defined provided the relations are homogeneous. This is obvious.

A simple Lie algebra \( \mathfrak{g} \), however, is the quotient of a free Lie algebra \( \mathfrak{f} \) modulo a inhomogeneous ideal, \( \mathfrak{j} \), the ideal without homogeneous generators. Therefore, we can speak about the number and the degrees of relations only conditionally. Our condition is the possibility to express any element \( x \in \mathfrak{j} \) via the generators \( g_1, \ldots \) of \( \mathfrak{f} \) by a formula of the form

\[
x = \sum [c_i, g_i], \text{ where } c_i \in \mathfrak{f} \text{ and } \deg c_i + \deg g_i \leq \deg x \text{ for all } i.
\]

Under condition (*) the number of relations and their degrees are uniquely determined. Now we can explain why do we need an extra generator \( y \): without \( y \) the weight relations would have been of very high degree.

We divide the relations between the Jacobson generators into the types corresponding to the number of occurrences of \( z \) in them: 0. Relations in \( L^2 = \mathfrak{sl}(2) \); 1. Relations coming from the \( \mathfrak{sl}(2) \)-action on \( L^{2k_2} \); 2. Relations coming from \( L^{2k_1} \wedge L^{2k_2} \geq 3 \). Relations coming from \( L^{2k_2} \wedge L^{2k_2} \wedge L^{2k_2} \wedge \ldots \) with \( \geq 3 \) factors; among the latter relations we distinguish one — of type “\( \infty \)” — the relation that shears the dimension. (For small rank \( \mathfrak{g} \) the relation of type \( \infty \) can be of the above types.)

Observe that, apart from relations of type \( \infty \), the relations of type \( \geq 3 \) are those of type 3 except for \( \mathfrak{e}_7 \) which satisfies stray relations of types 4 and 5, cf. [GL1].

The relations of type 0 are the well-known relations in \( \mathfrak{sl}(2) \)

\[
\begin{align*}
0.1. \quad & [[x, y], x] = 2x, & 0.2. \quad & [[x, y], y] = -2y.
\end{align*}
\]

The relations of type 1 mirror the fact that the space \( L^{2k_2} \) is the \((2k_2 + 1)\)-dimensional \( \mathfrak{sl}(2) \)-module. To simplify notations we denote: \( z_i = (ad z)^i z \). Then the type 1 relations are:

\[
\begin{align*}
1.1. \quad & [y, z] = 0, & 1.2. \quad & [[x, y], z] = -2k_2 z, & 1.3. \quad & z_{2k_1+1} = 0 \text{ with } 2k_2 \text{ from Table 1.1.} \tag{Rel1}
\end{align*}
\]

### 1.3. Theorem

*For the simple finite dimensional Lie algebras all the relations between the Jacobson generators are the above relations (Rel0), (Rel1) and the relations from [GL1]*.

In §3 these relations from [GL1] are reproduced for the classical Lie algebras.

### §2. The Lie algebra \( \mathfrak{sl}(\lambda) \) as a quotient algebra of \( \text{diff}(1) \) and a subalgebra of \( \mathfrak{sl}_+(\infty) \)

#### 2.1. \( \mathfrak{gl}(\lambda) \) is endowed with a trace

The Poincaré-Birkhoff-Witt theorem states that, as spaces, \( U(\mathfrak{gl}(2)) \cong \mathbb{C}[X^-, H, X^+] \). We also know that to study representations of \( \mathfrak{g} \) is the same as to study representations of \( U(\mathfrak{g}) \). Still, if we are interested in irreducible representations, we do not need the whole of \( U(\mathfrak{g}) \) and can do with a smaller algebra, easier to study.
This observation is used now and again; Feigin applied it in [F] writing, actually, (as deciphered in [PH], [GL], [Sh]) that setting
\[ X^- = \frac{d}{du}, \quad H = 2u \frac{d}{du} - (\lambda - 1), \quad X^+ = u^2 \frac{d}{du} - (\lambda - 1)u \]
we obtain a morphism of \( \mathfrak{sl}(2) \)-modules and, moreover, of associative algebras: \( U(\mathfrak{sl}(2)) \to \mathbb{C}[u, \frac{d}{du}] \). The kernel of this morphism is the ideal generated by \( \Delta - \lambda^2 + 1 \), where \( \Delta = 2(X^+X^- + X^-X^+) + H^2 \). Observe, that this morphism is not an epimorphism, either. The image of this morphism is our Lie algebra of matrices of “complex size”.

Remark. In their proof of certain statements from [F] that we will recall, [PH] made use of the well-known fact that the Casimir operator \( \Delta \) acts on the irreducible \( \mathfrak{sl}(2) \)-module \( L^\mu \) (see sec 1.1) as the scalar operator of multiplication by \( \mu^2 + 2\mu \). The passage from [PH]'s \( \lambda \) to [F]'s \( \mu \) is done with the help of a shift by the weight \( \rho \), a half sum of positive roots, which for \( \mathfrak{sl}(2) \) can be identified with 1, i.e., \( (\lambda - 1)^2 + 2(\lambda - 1) = \lambda^2 - 1 \) for \( \lambda = \mu + 1 \).

Consider the Lie algebra \( LU(\mathfrak{sl}(2)) \) associated with the associative algebra \( U(\mathfrak{sl}(2)) \). Set
\[ U_\lambda = U(\mathfrak{sl}(2))/(\Delta - \lambda^2 + 1). \]
The definition directly implies that \( \mathfrak{gl}(-\lambda) \cong \mathfrak{gl}(\lambda) \), so speaking about real values of \( \lambda \) we can confine ourselves to the nonnegative values, cf. sec. 0.2. It is easy to see that, as \( \mathfrak{sl}(2) \)-module,
\[ LU_\lambda = L^0 \oplus L^2 \oplus L^4 \oplus \cdots \oplus L^{2n} \oplus \cdots \]
It is not difficult to show (see [PH] for details) that the Lie algebra \( LU_n \) for \( n \in \mathbb{Z} \setminus \{0\} \) contains an ideal \( J_n \) and the quotient \( LU_n/J_n \) is the conventional \( \mathfrak{gl}(n) \). In [PH] it is proved that for \( \lambda \neq \mathbb{Z} \setminus \{0\} \) the Lie algebra \( LU_\lambda \) has only two ideals — the space \( L^0 \) of constants and its complement. Set
\[ \mathfrak{pgl}(\lambda) = \mathfrak{gl}(\lambda)/L^0, \quad \text{where} \quad \mathfrak{gl}(\lambda) = \begin{cases} LU_\lambda & \text{for } \lambda \not\in \mathbb{Z} \setminus \{0\} \\ LU_n/J_n & \text{for } n \in \mathbb{Z} \setminus \{0\}. \end{cases} \]

Observe, that \( \mathfrak{gl}(\lambda) \) is endowed with a trace. This follows directly from (2.3) and the fact that
\[ \mathfrak{gl}(\lambda) \cong L^0 \oplus [\mathfrak{gl}(\lambda), \mathfrak{gl}(\lambda)]. \]
Therefore, \( \mathfrak{pgl}(\lambda) \) can be identified with \( \mathfrak{sl}(\lambda) \), the subalgebra of the traceless matrices in \( \mathfrak{gl}(\lambda) \). We can normalize the trace at will, for example, if we set \( \text{tr}(id) = \lambda \), then the trace that our trace induces on the quotient of \( LU_{\mathfrak{sl}(2)}(n) \) modulo \( J(n) \) coincides with the usual trace on \( \mathfrak{gl}(n) \) for \( n \in \mathbb{N} \).

Another way to introduce the trace was suggested by J. Bernstein. We decipher its description in [KM] as follows. Look at the image of \( H \in \mathfrak{sl}(2) \) in \( \mathfrak{gl}(M^\lambda) \). Bernstein observed that though the trace of the image is an infinite sum, the sum of the first \( D + 1 \) summands is a polynomial in \( D \), call it \( \text{tr}(H) \). It is easy to see that \( \text{tr}(H) \) vanishes if \( D = \lambda \).

Similarly, for any \( x \in LU_\mathfrak{g}(\lambda) \) considered as an element of \( \mathfrak{gl}(M^\lambda) \) set
\[ \text{tr}(x; D) = \sum_{i=1}^{D} x_{ii}. \]
Let \( D(\lambda) \) be the value of the dimension of the irreducible finite dimensional \( \mathfrak{g} \)-module with highest weight \( \lambda \), for an exact formula see [D], [OV]. Set \( \text{tr}(x) = \text{tr}(x; D(\lambda)) \); as is easy to see, this formula determines the trace on \( LU_\mathfrak{g}(\lambda) \) for arbitrary values of \( \lambda \).
Observe that whereas for any irreducible finite dimensional module over the simple Lie algebra \( \mathfrak{g} \) there is just one formula for \( D(\lambda) \) (H. Weyl dimension formulas) for Lie superalgebra there are several distinct formulas depending on how “typical” \( \lambda \) is.

2.2. There is no Cartan matrix for \( \mathfrak{sl}(\lambda) \). What replaces it? Are there Chevalley generators in \( \mathfrak{sl}(\lambda) \)? In other words are there elements \( X_i^\pm \) of degree \( \pm 2 \) and \( H_i \) of degree 0 (the degree is the weight with respect to the \( \mathfrak{sl}(2) = L^2 \subset \mathfrak{sl}(\lambda) \)) such that

\[
[X_i^+, X_j^-] = \delta_{ij} H_i, \quad [H_i, H_j] = 0 \quad \text{and} \quad [H_i, X_j^\pm] = \pm A_{ij} X_j^\pm.
\]

The answer is NO: \( \mathfrak{sl}(\lambda) \) is too small. To see what is the problem, consider the following elements of degree \( \pm 2 \) from \( L^4 \) and \( L^6 \) of \( \mathfrak{gl}(\lambda) \):

\[
\begin{align*}
\text{deg} = -2 & : -4uD^2 - 2(\lambda - 2)D \\
\text{deg} = 2 & : -4u^2D^2 + 6(\lambda - 2)uD - 2(\lambda - 1)(\lambda - 2)u
\end{align*}
\]

\[
\begin{align*}
\text{deg} = -2 & : 15u^2D^3 - 15(\lambda - 3)uD^2 + 3(\lambda - 2)(\lambda - 3)D \\
\text{deg} = 2 & : 15u^4D^3 - 30(\lambda - 3)u^3D^2 + 18(\lambda - 2)(\lambda - 3)u^2D - 3(\lambda - 1)(\lambda - 2)(\lambda - 3)u
\end{align*}
\]

To satisfy (2.5), we can complete \( \mathfrak{gl}(\lambda) \) by considering infinite sums of its elements, but the completion erases the difference between different \( \lambda \)'s:

Proposition. For \( \lambda \neq \rho \) the completion of \( \mathfrak{sl}(\lambda) \) generated by Jacobson’s generators (see Tables) is isomorphic to \( \mathfrak{dij}f(1) \), the quotient of the Lie algebra of differential operators with formal coefficients modulo constants.

Though there is no Cartan matrix, Saveliev and Vershik [SV] suggested an operator \( K \) which replaces Cartan matrix. For further details see paper by Shihoth and Vershik [ShV].

2.3. The outer automorphism of \( L_{\mathfrak{g}} \mathfrak{u}(\lambda) \). The invariants of the mapping

\[
X \mapsto -SX' S \quad \text{for} \quad X \in \mathfrak{gl}(n), \quad \text{where} \quad S = \text{antidiag}(1, -1, 1, -1 \ldots)
\]

constitute \( \mathfrak{o}(n) \) if \( n \in 2\mathbb{N} + 1 \) and \( \mathfrak{sp}(n) \) if \( n \in 2\mathbb{N} \). By analogy, Feigin defined \( \mathfrak{o}(\lambda) \) and \( \mathfrak{sp}(\lambda) \) as subalgebras of \( \mathfrak{gl}(\lambda) = \bigoplus_{k \geq 0} L^{2k} \) invariant with respect to the involution analogous to (2.6):

\[
X \mapsto \begin{cases} -X & \text{if} \ X \in L^{4k} \\ X & \text{if} \ X \in L^{4k+2}. \end{cases}
\]

Since \( \mathfrak{o}(\lambda) \) and \( \mathfrak{sp}(\lambda) \) — the subalgebras of \( \mathfrak{gl}(\lambda) \) singled out by the involution (2.7) — differ by a shift of the parameter \( \lambda \), it is natural to denote them uniformly (but so as not to confuse with the Lie superalgebras of series \( \mathfrak{osp} \)), namely, by \( \mathfrak{o}/\mathfrak{sp}(\lambda) \). For integer values of the parameter it is clear that

\[
\mathfrak{o}/\mathfrak{sp}(\lambda) = \begin{cases} \mathfrak{o}(\lambda) \ni I_\lambda & \text{if} \ \lambda \in 2\mathbb{N} + 1, \\ \mathfrak{sp}(\lambda) \ni I_\lambda & \text{if} \ \lambda \in 2\mathbb{N}, \end{cases}
\]

where \( I_\lambda \) is an ideal.

In the realization of \( \mathfrak{sl}(\lambda) \) by differential operators the transposition is the passage to the adjoint operator; hence, \( \mathfrak{o}/\mathfrak{sp}(\lambda) \) is a subalgebra of \( \mathfrak{sl}(\lambda) \) consisting of self-skew-adjoint operators with respect to the involution

\[
a(u) \frac{d^k}{du^k} \mapsto (-1)^k \frac{d^k}{du^k} a(u)^*.
\]

The superization of this formula is straightforward: via Sign Rule.
2.4. The Lie algebra \( \mathfrak{gl}(\lambda) \) as a subalgebra of \( \mathfrak{gl}_+(\infty) \). Recall that \( \mathfrak{gl}_+(\infty) \) often denotes the Lie algebra of infinite (in one direction; index + indicates that) matrices with nonzero elements inside a (depending on the matrix) strip along the main diagonal and containing it. The subalgebras \( \mathfrak{o}(-\infty) \) and \( \mathfrak{sp}(\infty) \) are naturally defined, while \( \mathfrak{sl}(\infty) \) is, by abuse of language, sometimes used to denote \( \mathfrak{pgl}(\infty) \).

When it comes to superization, one shall be very careful selecting an appropriate candidate for \( \mathfrak{sl}(\infty|\infty) \) and its subalgebra, cf. [E].

The realization (2.1) provides with an embedding \( \mathfrak{sl}(\lambda) \subset \mathfrak{gl}_+(\infty) = \"\mathfrak{sl}(M^{\lambda})\" \), so for \( \lambda \neq \mathbb{N} \) the Verma module \( M^{\lambda} \) with highest weight \( \mu \) is an irreducible \( \mathfrak{sl}(\lambda) \)-module.

**Proposition.** The completion of \( \mathfrak{gl}(\lambda) \) (generated by the elements of degree \( \pm 2 \) with respect to \( H \in \mathfrak{sl}(2) \subset \mathfrak{gl}(\lambda) \)) is isomorphic for any noninteger \( \lambda \) to \( \mathfrak{gl}_+(\infty) = \"\mathfrak{gl}(M^{\lambda})\" \).

2.5. The Lie algebras \( \mathfrak{sl}(\ast) \) and \( \mathfrak{o}/\mathfrak{sp}(\ast) \) for \( \ast \in \mathbb{C}P^1 = \mathbb{C} \cup \{\ast\} \). The “dequantization” of the relations for \( \mathfrak{sl}(\lambda) \) and \( \mathfrak{o}/\mathfrak{sp}(\lambda) \) (see §3) is performed by passage to the limit as \( \lambda \to \infty \) under the change:

\[
t \mapsto \begin{cases} \frac{t}{\lambda} & \text{for } \mathfrak{sl}(\lambda) \\ \frac{1}{\lambda t} & \text{for } \mathfrak{o}/\mathfrak{sp}(\lambda). \end{cases}
\]

So the parameter \( \lambda \) above can actually run over \( \mathbb{C}P^1 = \mathbb{C} \cup \{\ast\} \), not just \( \mathbb{C} \). In the realization with the help of deformation, cf. 2.7 below, this is obvious. Denote the limit algebras by \( \mathfrak{sl}(\ast) \) and \( \mathfrak{o}/\mathfrak{sp}(\ast) \) in order to distinguish them from \( \mathfrak{sl}(\infty) \) and \( \mathfrak{o}(\infty) \) or \( \mathfrak{sp}(\infty) \) from sec. 2.4.

It is clear that it is impossible to embed \( \mathfrak{sl}(\ast) \) and \( \mathfrak{o}/\mathfrak{sp}(\ast) \) into the “quadrant” algebra \( \mathfrak{sl}_+(\infty) \): indeed, \( \mathfrak{sl}(\ast) \) and \( \mathfrak{o}/\mathfrak{sp}(\ast) \) are subalgebras of the whole “plane” algebras \( \mathfrak{sl}(\infty) \) and \( \mathfrak{o}(\infty) \) or \( \mathfrak{sp}(\infty) \).

2.6. **Theorem.** For Lie algebras \( \mathfrak{sl}(\lambda) \) and \( \mathfrak{o}/\mathfrak{sp}(\lambda) \), \( \lambda \in \mathbb{C}P^1 \), all the relations between the Jacobson generators are the relations of types 0, 1 with \( 2k_2 \) found from Table 1.1 and the borrowed from [GL1] relations from §3.

### §3. Jacobson’s generators and relations between them

In what follows the \( E_{ij} \) are the matrix units; \( X_{i}^\pm \) stand for the conventional Chavalley generators of \( \mathfrak{g} \). For \( \mathfrak{sl}(\lambda) \) and \( \mathfrak{o}/\mathfrak{sp}(\lambda) \) the generators \( x = u^2 \frac{du}{\lambda u} - (\lambda - 1)u \) and \( y = -\frac{du}{\lambda u} \) are the same; \( z_{\ast} = t \frac{dt}{\lambda u} \) while \( z_{\mathfrak{o}/\mathfrak{sp}} = t \frac{dt}{\lambda u} \). For \( n \in \mathbb{C} \setminus \mathbb{Z} \) there is no shearing relation of type \( \infty \); for \( n = \ast \in \mathbb{C}P^1 \) the relations are obtained with the substitution 2.5. The parameter \( t \) can be taken equal to 1; we kept it explicit to clarify how to “dequantize” the relations as \( \lambda \to \infty \).

**\( \mathfrak{sl}(\ast) \).**

2.1. \( 3[z_1, z_2] - 2[z, z_3] = 24y \),
2.2. \( 9[z_2, z_3] - 5[z_1, z_4] = 216z_2 - 432y \),
2.3. \( z, [z, z_1] = 0 \).

**\( \mathfrak{o}/\mathfrak{sp}(\ast) \).**

2.1. \( 2[z_1, z_2] - [z, z_3] = 72z \),
2.2. \( 9[z_2, z_3] - 5[z_1, z_4] = 216z_2 - 432y \),
2.3. \( 7([z, z_1], z_3) + 6[z_2, [z, z_2]] = -720[z, z_1] \).
\(\mathfrak{sl}(n)\) for \(n \geq 3\). Generators:
\[
x = \sum_{1 \leq i \leq n-1} i(n - i)E_{i,i+1}, \quad y = \sum_{1 \leq i \leq n-1} E_{i+1,i}, \quad z = t \sum_{1 \leq i \leq n-2} E_{i+2,i}.
\]
Relations:
\[
\begin{align*}
2.1. & \quad 3[z_1, z_2] - 2[z, z_3] = 24t^2(n^2 - 4)y, \\
3.1. & \quad [z, [z, z_1]] = 0, \\
3.2. & \quad 4[z_3, [z, z_1]] - 3[z_2, [z, z_2]] = 576t^2(n^2 - 9)z, \\
\infty = n - 1. & \quad (\text{ad}z_1)^{n-2}z = 0.
\end{align*}
\]
For \(n = 3, 4\) the degree of the last relation is lower than the degree of some other relations, this yields simplifications.
\(\mathfrak{so}(2n + 1)\) for \(n \geq 3\). Generators:
\[
x = n(n + 1)(E_{n+1,2n+1} - E_{n,n+1}) + \sum_{1 \leq i \leq n-1} i(2n + 1 - i)(E_{i,i+1} - E_{n+i+2,n+i+1}), \\
y = (E_{2n+1,n+1} - E_{n+1,n}) + \sum_{1 \leq i \leq n-1} (E_{i+1,i} - E_{n+i+1,n+i+2}), \\
z = t((E_{2n-1,n+1} - E_{n+1,n-2}) - (E_{2n+1,n-1} - E_{2n,n}) + \sum_{1 \leq i \leq n-3} (E_{i+3,i} - E_{n+i+1,n+i+4})).
\]
Relations:
\[
\begin{align*}
2.1. & \quad 2[z_1, z_2] - [z, z_3] = 144t(2n^2 + 2n - 9)z, \\
2.2. & \quad 9[z_2, z_3] - 5[z_1, z_4] = 432t(2n^2 + 2n - 9)z_2 + 1728t^2(n - 1)(n + 2)(2n - 1)(2n + 3)y, \\
3.1. & \quad [z, [z, z_1]] = 0, \\
3.2. & \quad 7[z_3, [z, z_1]] - 6[z_2, [z, z_2]] = 2880t(n - 3)(n + 4)[z, z_1], \\
\infty = n. & \quad (\text{ad}z_1)^{n-2}z = 0.
\end{align*}
\]
\(\mathfrak{sp}(2n)\) for \(n \geq 3\). Generators:
\[
x = n^2E_{n,2n} + \sum_{1 \leq i \leq n-1} i(2n - i)(E_{i,i+1} - E_{n+i+1,n+i}), \\
y = E_{2n,n} + \sum_{1 \leq i \leq n-1} (E_{i+1,i} - E_{n+i,n+i+1}), \\
z = t((E_{2n,n-2} + E_{2n-2,n}) - E_{2n-1,n-1} + \sum_{1 \leq i \leq n-3} (E_{i+3,i} - E_{n+i,n+i+3})).
\]
Relations:
\[
\begin{align*}
2.1. & \quad 2[z_1, z_2] - [z, z_3] = 72t(4n^2 - 19)z, \\
2.2. & \quad 9[z_2, z_3] - 5[z_1, z_4] = 216t(4n^2 - 19)z_2 + 1728t^2(n^2 - 1)(4n^2 - 9)y, \\
3.1. & \quad [z, [z, z_1]] = 0, \\
3.2. & \quad 7[z_3, [z, z_1]] - 6[z_2, [z, z_2]] = 720t(4n^2 - 49)[z, z_1], \\
\infty = n. & \quad (\text{ad}z_1)^{n-2}z = 0.
\end{align*}
\]
For Jacobson generators and corresponding defining relations for the exceptional Lie algebras see [GL1].
§4. Lie superalgebras

4.0. Linear algebra in superspaces. Superization has certain subtleties, often disregarded or expressed too briefly. We will dwell on them a bit, see [2].

A superspace is a \( \mathbb{Z}/2 \)-graded space; for a superspace \( V = V_0 \oplus V_1 \) denote by \( \Pi(V) \) another copy of the same superspace: with the shifted parity, i.e., \( (\Pi(V))_i = V_{i+1} \).

A superspace structure in \( V \) induces that in the space \( \text{End}(V) \). A basis of a superspace is always a basis consisting of homogeneous vectors; let \( \text{Par} = (p_1, \ldots, p_{\dim V}) \) be an ordered collection of their parities, called the \text{format} of \( V \). A square supermatrix of format (size) \( \text{Par} \) is a \( \dim V \times \dim V \) matrix whose \( i \)th row and \( i \)th column are said to be of parity \( p_i \). The matrix unit \( E_{ij} \) is supposed to be of parity \( p_i + p_j \) and the bracket of supermatrices (of the same format) is defined via Sign Rule: if \text{something of parity} \( p \) \text{moves past something of parity} \( q \) \text{the sign} \((-1)^{pq} \text{ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity. For example:} \[ [X,Y] = XY - (-1)^{p(X)p(Y)} YX; \]

the sign \( \wedge \) in what follows is also understood in superense, etc.

Usually, \( \text{Par} \) is considered to be of the form \( (0, \ldots, 0, 1, \ldots, 1) \). Such a format is called standard. The Lie superalgebra of supermatrices of size \( \text{Par} \) is denoted by \( \mathfrak{gl}(\text{Par}) \), usually \( \mathfrak{gl}(0, \ldots, 0, 1, \ldots, 1) \) is abbreviated to \( \mathfrak{gl}(\dim V_0 | \dim V_1) \).

For \( \dim V_0 = \dim V_1 = 1 \) we will often use another format, the alternating one, \( \text{Par}_{\text{alt}} = (0, 1, 0, 1, \ldots, 1) \).

The supertrace is the map \( \mathfrak{gl}(\text{Par}) \rightarrow \mathbb{C}, (A_{ij}) \mapsto \sum (-1)^{p_i} A_{ii} \). The supertraceless matrices constitute a Lie subsuperalgebra, \( \mathfrak{s}(\text{Par}) \).

To the linear map \( F \) of superspaces there corresponds the dual map \( F^* \) between the dual superspaces; if \( A \) is the supermatrix corresponding to \( F \) in a format \( \text{Par} \), then to \( F^* \) the supertransposed matrix \( A^{st} \) corresponds:

\[
(A^{st})_{ij} = (-1)^{(p_i + p_j)(p_i + p(A))} A_{ji}.
\]

The supermatrices \( X \in \mathfrak{gl}(\text{Par}) \) such that

\[
X^{st} B + (-1)^{p(X)p(B)} B X = 0 \quad \text{for a homogeneous matrix} \quad B \in \mathfrak{gl}(\text{Par})
\]

constitute the Lie superalgebra \( \text{aut}(B) \) that preserves the bilinear form on \( V \) with matrix \( B \).

The superspace of bilinear forms is denoted by \( \text{Bil}_C(M,N) \) or \( \text{Bil}_C(M) \) if \( M = N \). The \text{upsetting of forms} \( u_f : \text{Bil}_C(M,N) \rightarrow \text{Bil}_C(N,M) \), is defined by the formula

\[
B^{u_f}(n,m) = (-1)^{p(n)p(m)} B(m,n).
\]

A form \( B \in \text{Bil}_C(M) \) is called \text{supersymmetric} if \( B^{u_f} = B \) and \text{superskew-symmetric} if \( B^{u_f} = -B \).

Given bases \( \{m_i\} \) and \( \{n_j\} \) of \( C \)-modules \( M \) and \( N \) and a bilinear form \( B : M \otimes N \rightarrow C \), we assign to \( B \) the matrix

\[
(m_f B)_{ij} = (-1)^{p(m_i)p(B)} B(m_i, n_j).
\]

For a nondegenerate supersymmetric form whose matrix in the standard format is

\[
B_{m,2n} = \begin{pmatrix}
1_m & 0 \\
0 & J_{2n}
\end{pmatrix}, \quad \text{where} \quad J_{2n} = \begin{pmatrix}
0 & 1_n \\
-1_n & 0
\end{pmatrix}.
\]

The usual notation for \( \text{aut}(B_{m,2n}) \) is \( \mathfrak{osp}^{sg}(m|2n) \) or just \( \mathfrak{osp}(m|2n) \). (Observe that the passage from \( V \) to \( \Pi(V) \) sends the supersymmetric forms to superskew-symmetric ones, preserved by \( \mathfrak{osp}^{sk}(m|2n) \) which is isomorphic to \( \mathfrak{osp}(m|2n) \) but has a different matrix realization.)
We will need the orthosymplectic supermatrices in the alternating format; in this format we take the matrix $B_{m,2n}(\text{alt}) = \text{antidiag}(1, \ldots, 1, -1, \ldots, -1)$ with the only nonzero entries on the side diagonal, the last $n$ being $-1$’s. The Lie superalgebra of such supermatrices will be denoted by $\mathfrak{osp}(\text{alt}_{m|2n})$, where, as is easy to see, either $m = 2n \pm 1$ or $m = 2n$.

There is a 1-parameter family of deformations $\mathfrak{osp}_\alpha(4|2)$ of the Lie superalgebra $\mathfrak{osp}(4|2)$; its only explicit description we know (apart from [BGLS], of course) is in terms of Cartan matrix [GL3].

4.1. The superprincipal embeddings. Not every simple Lie superalgebra, even a finite dimensional one, hosts a superprincipal $\mathfrak{osp}(1|2)$-subsuperalgebra. Let us describe those that do. (Aside: an interesting problem is to describe semiprincipal embeddings into $\mathfrak{g}$, defined as the ones with the least possible number of irreducible components.)

We select the following basis in $\mathfrak{osp}(1|2) \subset \mathfrak{sl}(\overline{0}|\overline{1}|\overline{0})$:

- $X^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $X^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

- $\nabla^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\nabla^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

The highest weight $\mathfrak{osp}(1|2)$-module $M^\mu$ is illustrated with a graph whose nodes correspond to the eigenvectors $l_i$ of $H$ with the weight indicated; the horizontal edges depict the $X^\pm$-action (the $X^+$-action is directed to the right, that of $X^-$ to the left; each horizontal string is an irreducible $\mathfrak{sl}(2)$-submodule; two such submodules are glued together into an $\mathfrak{osp}(1|2)$-module by the action of $\nabla^\pm$ (we set $\nabla^+(l_n) = 0$ and $\nabla^-(l_i) = l_{i-1}$; the corresponding edges are not depicted below); we additionally assume that $p(l_\mu) = 0$:

As follows from the relations of type 0 below in sec 4.2, the module $M^n$ for $n \in \mathbb{Z}_+$ has an irreducible submodule isomorphic to $\Pi(M^{-n-1})$; the quotient, obviously irreducible as follows from the same formulas, will be denoted by $L^n$.

Serganova completely described superprincipal embeddings of $\mathfrak{osp}(1|2)$ into a simple finite dimensional Lie superalgebra [LSS] (the main part of her result was independently obtained in [WJ]).

As the $\mathfrak{osp}(1|2)$-module corresponding to the superprincipal embedding, a simple finite dimensional Lie superalgebra $\mathfrak{g}$ is as follows (the missing simple algebras $\mathfrak{g}$ do not contain a superprincipal $\mathfrak{osp}(1|2)$):
Table 4.1. $\mathfrak{g}$ that admits a superprincipal subalgebra: as the $\mathfrak{osp}(1|2)$-module.

| $\mathfrak{g}$                        | $\mathfrak{g} = \mathcal{L}^2 \oplus (\bigoplus_{i \geq 2} \mathcal{L}^{2k_i})$ for $i \geq 2 \oplus (\bigoplus_j \Pi(\mathcal{L}^{n_j}))$ for $j \geq 1$ |
|--------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\mathfrak{sl}(n|n+1)$              | $\mathfrak{g} = \mathcal{L}^2 \oplus \mathcal{L}^4 \oplus \mathcal{L}^6 \cdots \oplus \mathcal{L}^{2n-2}$                                                                                             |
| $\mathfrak{osp}(2n-1|2n)$           | $\mathfrak{g} = \mathcal{L}^2 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{10} \cdots \oplus \mathcal{L}^{4n-6}$                                                                                       |
| $(n > 1)$                            |                                                                                                                                                                                                  |
| $\mathfrak{osp}(2n+1|2n)$           | $\mathfrak{g} = \mathcal{L}^2 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{10} \cdots \oplus \mathcal{L}^{4n-2}$                                                                                       |
| $\mathfrak{osp}(2|2) \cong \mathfrak{sl}(1|2)$ | $\mathfrak{g} = \mathcal{L}^2$                                                                                                                                                                     |
| $\mathfrak{osp}(4|4)$               | $\mathfrak{g} = \mathcal{L}^2 \oplus \mathcal{L}^6$                                                                                                                                              |
| $\mathfrak{osp}(2n|2n)$             | $\mathfrak{g} = \mathcal{L}^2 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{10} \cdots \oplus \mathcal{L}^{4n-2} \oplus \mathcal{L}^{2n-2}$                                                             |
| $\mathfrak{osp}(2n+2|2n)$           | $\mathfrak{g} = \mathcal{L}^2 \oplus \mathcal{L}^6 \oplus \mathcal{L}^{10} \cdots \oplus \mathcal{L}^{4n+2} \oplus \mathcal{L}^{2n}$                                                             |
| $\mathfrak{osp}_\alpha(4|2)$        | $\mathfrak{g} = \mathcal{L}^2 \oplus \mathcal{L}^6$                                                                                                                                              |

The Lie superalgebra $\mathfrak{g}$ of type $\mathfrak{osp}$ that contains a superprincipal subalgebra $\mathfrak{osp}(1|2)$ can be generated by two elements. For such elements we can take $X := \nabla^+ \in \mathcal{L}^2 = \mathfrak{osp}(1|2)$ and a lowest weight vector $Z := \ell_\cdot \ell$ from the module $M = \mathcal{L}^r$ or $\Pi(\mathcal{L}^r)$, where for $M$ we take $\Pi(\mathcal{L}^3)$ if $\mathfrak{g} \neq \mathfrak{osp}(2n|2m)$ or the last module with the even highest weight vector in the above table (i.e., $\mathcal{L}^{2n-2}$ if $\mathfrak{g} = \mathfrak{osp}(2n|2n)$ and $\mathcal{L}^{2n}$ if $\mathfrak{g} = \mathfrak{osp}(2n+2|2n)$).

To generate $\mathfrak{sl}(n|n+1)$ we have to add to the above $X$ and $Z$ a lowest weight vector $U$ from $\Pi(\mathcal{L}^1)$. (Clearly, $Z$ and $U$ are defined up to factors that we can select at our convenience; we will assume that a basis of $L^r$ is fixed and denote $Z = t \cdot \ell_\cdot \ell$ and $U = s \cdot \ell^{-1}$ for $t, s \in \mathbb{C}$.)

We call the above $X$ and $Z$, together with $U$, and fortified by $Y := X^- \in \mathcal{L}^2$ the Jacobson’s generators. The presence of $Y$ considerably simplifies the form of the relations, though slightly increases the number of them.

4.2. Relations between Jacobson’s generators. We repeat the arguments from sec. 1.2. Since we obtain the relations recurrently, it could happen that a relation of higher degree implies a relation of a lower degree. This did not happen when we studied $\mathfrak{sl}(\lambda)$, but does happen in what follows, namely, relation 1.2 implies 1.1.

We divide the relations between Jacobson’s generators into the types corresponding the number of occurrence of $z$ in them: 0. Relations in $\mathfrak{sl}(1|2)$ or $\mathfrak{osp}(1|2)$; 1. Relations coming from the $\mathfrak{osp}(1|2)$-action on $\mathcal{L}^{2k_2}$; 2. Relations coming from $\mathcal{L}^{2k_1} \wedge \mathcal{L}^{2k_2}$; 3. Relations coming from $\mathcal{L}^{2k_2}$. Relation that shear the dimension.

The relations of type 0 are the well-known relations in $\mathfrak{sl}(1|2)$, those of them that do not involve $U$ (marked with an *) are the relations for $\mathfrak{osp}(1|2)$. The relations of type 1 that do not involve $U$ express that the space $\mathcal{L}^{2k_2}$ is the $\mathfrak{osp}(1|2)$-module with highest weight $2k_2$.

To simplify notations we denote: $Z_i = \text{ad}X^i Z$ and $Y_i = \text{ad}X^i Y$.

- **0.1**. $[Y, Y_1] = 0$, **0.2**. $[Y_2, Y] = 2Y$, **0.3**. $[Y_2, X] = -X$,
- **0.4**. $[Y, U] = 0$, **0.5**. $[U, U] = -2Y$; **0.6**. $[U, Y_1] = 0$,
- **0.7**. $[[X, X], [X, U]] = 0$, **0.8**. $[Y_2, U] = U$.

1.1. $[Y, Z] = 0 \iff \mathbf{1.2}$. $[[X, Y], Z] = 0$, 1.3. $Z_{4k_1} = 0$, 1.4. $[Y_2, Z] = 3Z$.

4.3. **Theorem.** For the Lie superalgebras indicated, all the relations between Jacobson’s generators are the above relations of types 0, 1 and the relations from §6.
§5. THE LIE SUPERALGEBRA $\mathfrak{gl}(\lambda|\lambda+1)$ AS THE QUOTIENT OF $\mathfrak{diff}(1|1)$ AND A SUBALGEBRA OF $\mathfrak{sl}_+(\infty|\infty)$

There are several ways to superize $\mathfrak{sl}_+(\infty|\infty)$. For a description of “the best” one from a certain point of view see [4]. For our purposes any version of $\mathfrak{sl}_+(\infty|\infty)$ will do.

5.1. The Poincaré-Birkhoff-Witt theorem states that $U(\mathfrak{osp}(1|2)) \cong \mathbb{C}[X^-, \nabla^-, H, \nabla^+, X^+]$, as superspaces. Set $U_\lambda = U(\mathfrak{osp}(1|2))/(\Delta - \lambda^2 + \frac{\partial}{\partial \theta})$. Denote: $\partial_x = \frac{\partial}{\partial x}$, $\partial_{\theta} = \frac{\partial}{\partial \theta}$ and set $X^- = -\partial_x$, $\nabla^- = \partial_{\theta} - \theta \partial_x$, $H = 2x\partial_x + \theta \partial_{\theta}(\lambda - 1)$, $\nabla^+ = x\partial_x + x\theta \partial_{\theta} - \lambda \theta$, $X^+ = x^2\partial_x - (\lambda - 1)x$.

These formulas establish a morphism of $\mathfrak{osp}(1|2)$-modules and, moreover, of associative superalgebras: $U_\lambda \longrightarrow \mathbb{C}[x, \theta, \partial_x, \partial_{\theta}]$.

In what follows we will need a well-known fact: the Casimir operator

$$\Delta = 2(X^+X^- + X^-X^+) + \nabla^+\nabla^- - \nabla^-\nabla^+ + H^2$$

acts on the irreducible $\mathfrak{osp}(1|2)$-module $\mathcal{L}^\mu$ as the scalar operator of multiplication by $\mu^2 + 3\mu$. (The passage from $\mu$ to $\lambda$ is done with the help of a shift by $\frac{3}{2}$.)

Consider the Lie superalgebra $LU(\mathfrak{osp}(1|2))$ associated with the associative superalgebra $U_\lambda$. It is easy to see that, as $\mathfrak{osp}(1|2)$-module,

$$LU_\lambda = \mathcal{L}^0 \oplus \mathcal{L}^2 \oplus \cdots \oplus \mathcal{L}^{2n} \oplus \cdots \oplus \Pi(\mathcal{L}^1 \oplus \mathcal{L}^3 \oplus \ldots) \quad (5.1)$$

In the same way as for Lie algebras we show that $LU_n$ contains an ideal $I_n$ for $n \in \mathbb{N} \setminus \{0\}$ and the quotient $LU_n/I_n$ is the conventional $\mathfrak{sl}(n|n+1)$. It is clear that for $\lambda \neq \mathbb{Z}$ the Lie algebra $LU_\lambda$ has only one ideal — the space $\mathcal{L}^0$ of constants and $LU_\lambda = \mathcal{L}^0 \oplus [LU_\lambda, LU_\lambda]$; hence, there is a supertrace on $LU_\lambda$. This justifies the following notations

$$\mathfrak{sl}(\lambda|\lambda+1) = \mathfrak{gl}(\lambda|\lambda+1)/\mathcal{L}^0, \quad \text{where } \mathfrak{gl}(\lambda|\lambda+1) = \begin{cases} U_\lambda & \text{for } \lambda \neq \mathbb{N} \setminus \{0\} \\ LU_n/I_n & \text{otherwise.} \end{cases} \quad (5.2)$$

The definition directly implies that $\mathfrak{gl}(-\lambda|\lambda+1) \cong \mathfrak{gl}(\lambda|\lambda+1)$, so speaking about real values of $\lambda$ we can confine ourselves to the nonnegative values.

Define $\mathfrak{osp}(\lambda+1|\lambda)$ as the Lie subsuperalgebra of $\mathfrak{sl}(\lambda|\lambda+1)$ invariant with respect to the involution

$$X \rightarrow \begin{cases} -X & \text{if } X \in \mathcal{L}^{4k} \text{ or } X \in \Pi(\mathcal{L}^{4k\pm1}) \\ X & \text{if } X \in \mathcal{L}^{4k\pm2} \text{ or } X \in \Pi(\mathcal{L}^{4k\pm3}), \end{cases} \quad (5.3)$$

which is the analogue of the map

$$X \rightarrow -X^{st} \quad \text{for } X \in \mathfrak{gl}(m|n). \quad (5.4)$$

5.2. The Lie superalgebras $\mathfrak{sl}(**+1)$ and $\mathfrak{osp}(2**+1)$, for $** \in \mathbb{CP}^1 = \mathbb{C} \cup \{\ast\}$. The “dequantization” of the relations for $\mathfrak{sl}(\lambda|\lambda+1)$ and $\mathfrak{osp}(\lambda+1|\lambda)$ is performed by passage to the limit as $\lambda \longrightarrow \infty$ under the change $t \mapsto \frac{1}{t}$. We denote the limit algebras by $\mathfrak{sl}(**+1)$ and $\mathfrak{osp}(**+1)$ in order not to mix them with $\mathfrak{sl}(\infty|\infty+1)$ and $\mathfrak{osp}(\infty|\infty+1)$, respectively.

§6. Tables. The Jacobson generators and relations between them

**Table 6.1. Infinite dimensional case.**

| Generator | Relations |
|-----------|-----------|
| $X = x\partial_\theta + x\theta \partial_x - \lambda \theta$, $Y = \partial_x$, $Z = t(\partial_x \partial_\theta - \theta \partial_x^2)$ | **2.1.** $3[Z, Z_3] + 2[Z_1, Z_2] = 6t(2\lambda + 1)Z$, **2.2.** $[Z_1, Z_3] = 2t^2(\lambda - 1)(\lambda + 2)Y + 2t(2\lambda + 1)Z_1$, **3.1.** $[Z_1, [Z, Z]] = 0$. |
• \( \mathfrak{osp}(\star \star +1) \). Relations: the same as in sec 4.2 plus the following relations:

\[
\begin{align*}
2.1. & \quad 3[Z, Z_3] + 2[Z_1, Z_2] = 12t Z, \\
2.2. & \quad [Z_1, Z_3] = 2t^2 Y + 4t Z_1.
\end{align*}
\]

• \( \mathfrak{sl}(\lambda + 1|\lambda) \) for \( \lambda \in \mathbb{CP}^1 \). Generators (for \( \lambda \in \mathbb{C} \)): the same as for \( \mathfrak{osp}(\lambda|\lambda + 1) \) and \( U = \partial_\theta - \theta \partial_x \).

Relations: the same as for \( \mathfrak{osp}(\lambda|\lambda + 1) \) plus the following

\[
\begin{align*}
1.5. & \quad 3[Z, [X, U]] - [U, Z_1] = 0, \quad 2.3. & \quad [Z, [U, Z]] = 0, \\
1.6. & \quad [[X, U], Z_1] = 0, \quad 2.4. & \quad [Z_1, [U, Z]] = 0.
\end{align*}
\]

### Table 6.2. Finite dimensional algebras

In this table \( E_{ij} \) are the matrix units; \( X_i^\pm \) stand for the conventional Chevalley generators of \( \mathfrak{g} \).

- **\( \mathfrak{sl}(n + 1|n) \) for \( n \geq 3 \).** Generators:

\[
\begin{align*}
X &= \sum_{1 \leq i \leq n} \left( (n-i+1)E_{2i-1,2i} - iE_{2i,2i+1} \right), \\
Y &= \sum_{1 \leq i \leq n} (-1)^{i+1} E_{i+1,i}, \\
Z &= \sum_{1 \leq i \leq 2n-1} (-1)^i E_{i+1,i}.
\end{align*}
\]

Relations: those for \( \mathfrak{sl}(\lambda + 1|\lambda) \) with \( \lambda = n \) and an extra relation to shear the dimension:

\[(\text{ad} Z)^n([X, X]) = 0.\]

For \( n = 1 \) the relations degenerate in relations of type 0.

- **\( \mathfrak{osp}(2n + 1|2n) \).** Generators:

\[
\begin{align*}
X &= \sum_{1 \leq i \leq n} \left( (2n-i+1)E_{2i-1,2i} + E_{4n+2-2i,4n+3-2i} - i(E_{2i,2i+1} - E_{4n+1-2i,4n+2-2i}) \right), \\
Y &= E_{2n+2,2n} + \sum_{1 \leq i \leq 2n-1} (E_{i+1,i} - E_{4n+2-i,4n-i}); \\
Z &= -E_{2n+2,2n-1} - E_{2n+3,2n} + \sum_{1 \leq i \leq 2n-2} (-1)^i E_{i+3,i} + E_{4n+2-i,4n-1-i}.
\end{align*}
\]

Relations: those for \( \mathfrak{osp}(2\lambda + 1|2\lambda) \) with \( \lambda = n \) and an extra relation to shear the dimension (the form of the relation is identical to that for \( \mathfrak{sl}(n + 1|n) \)).

- **\( \mathfrak{osp}(4|2) \).** Generators: As \( \mathfrak{osp}(1|2) \)-module, the algebra \( \mathfrak{osp}_{\alpha}(4|2) \) has 2 isomorphic submodules. The generators \( X \) and \( Y \) belong to one of them. It so happens that we can select \( Z \) from either of the remaining submodules and still generate the whole Lie superalgebra. The choice (a) is from \( \Pi(\mathcal{L}^3) \); it is unique (up to a factor). The choices (b) and (c) are from \( \mathcal{L}^2 \); one of them seem to give simpler relations.

\[
\begin{align*}
X &= \frac{-\alpha+1}{\alpha} X_1^+ + \frac{\alpha+1}{\alpha+1} X_2^+ + \frac{1}{\alpha(\alpha+1)} X_3^+, \quad Y = [X_1^-, X_2^-] + [X_1^-, X_3^-] + [X_2^-, X_3^-], \\
Z &= \begin{cases}
(a) & -[X_1^-, X_2^-], [X_3^-, X_3^-]; \\
(b) & -(1 + 2\alpha)[X_1^-, X_2^-] + \alpha^2(2 + \alpha)[X_1^-, X_3^-] + (\alpha - 1)(1 + \alpha)^2[X_2^-, X_3^-]; \\
(c) & -[X_2^-, X_3^-] + (\alpha + 1)[X_1^-, X_2^-].
\end{cases}
\end{align*}
\]

Relations of type 0 are common for cases a) – c):

\[
\begin{align*}
0.1 & \quad [Y, [Y, [X, X]]] = 4Y; \quad 0.2 & \quad [Y_1[X, X]] = -2X;
\end{align*}
\]

The other relations are as follows.
Relations a): 

1.1 \([Y_1, Z_1] = 3Z\), \quad 1.2 (ad\([X, X])^3 Z_1 = 0;\)

2.1 \([Z, Z] = 0;\) \quad 2.2 \([Z_1, [[X, X], Z]] = -4\alpha^2+\alpha+1\alpha Z,\)

3.1 \([ad[X, X](Z_1), [Z_1, ad[X, X](Z_1)]] = -\frac{16}{\alpha(\alpha+1)}Y + 8\frac{\alpha^2+\alpha+1}{\alpha(\alpha+1)}[Z_1, ad[X, X](Z_1)] + 16\frac{(\alpha^2+\alpha+1)^2}{\alpha^2(\alpha+1)^2}Z_1.\)

Relations b): 

1.1 \([Y_1, Z_1] = 2Z;\) \quad 1.2 (ad\([X, X])^2 Z_1 = 0;\)

2.1* \([Z_1, Z_1] = 2[Z, [Z, [X, X]]] - 18\alpha^2(1+\alpha)^2Y + 4(1-\alpha)(2+\alpha)(1+2\alpha)Z;\)

3.1 (ad\([Z])^3 X = 0,\)

3.2* \([[[Z, Z], (ad[X, X])^2 Z_1] = (-1+\alpha)(2+\alpha)(1+2\alpha)[Z, [Z, [X, X]]] + 12(1-\alpha)(2+\alpha)(1+2\alpha)^2(1+\alpha)^2Y + 8(1-3\alpha^2-3\alpha)(-1-3\alpha+\alpha^3)Z.\)

Relations c): same as for b) except that the relations marked in b) by an * should be replaced with the following ones 

2.1 \([Z_1, Z_1] = 2[Z, [Z, [X, X]]] - 2\alpha^2 Y + 4(2+\alpha)Z;\)

2.2 (ad\([X, X])^2 Z_1 = (-2 - \alpha)[Z, [Z, [X, X]]] - 8(1+\alpha) Z + 4\alpha^2(2+\alpha)Y.\)

§7. REMARKS AND PROBLEMS

7.1. On proof. For the exceptional Lie algebras and superalgebras \(osp_\alpha(4|2)\) the proof is direct: the quotient of the free Lie algebra generated by \(x, y\) and \(z\) modulo our relations is the needed finite dimensional one. For rank \(g\) \leq 12 we similarly computed relations for \(g = sl(n)\), \(o(2n + 1)\) and \(sp(2n)\); as Post pointed out, together with the result of \([PH]\) on deformation (cf. 2.7) this completes the proof for Lie algebras. The results of \([PH]\) on deformations can be directly extended for the case of \(sl(2)\) replaced with \(osp(1|2)\); this proves Theorem 4.3.

Our Theorem 2.6 elucidates Proposition 2 of \([F]\); we just wrote relations explicitly. Feigin claimed \([F]\) that for \(sl(\lambda)\) the relations of type 3 follow from the decomposition of \(L^{2k_2} \land L^{2k_2} \subset L^{2k_2} \land L^{2k_2} \land L^{2k_2}\). We verified that this is so not only in Feigin’s case but for all the above-considered algebras except \(e_6, e_7\) and \(e_8\): for the latter one should consider the whole \(L^{2k_2} \land L^{2k_2} \land L^{2k_2}\), cf. \([GL1]\). Theorem 4.3 is a direct superization of Theorem 2.6.

7.2. Problems. 1) How to present \(o(2n)\) and \(osp(2m|2n)\)? One can select \(z\) as suggested in sec. 1.1. Clearly, the form of \(z\) (hence, relations of type 1) and the number of relations of type 3 depend on \(n\) in contradistinction with the algebras considered above. Besides, the relations are not as neat as for the above algebras. We should, perhaps, have taken the generators as for \(o(2n - 1)\) and add a generator from \(L^{2n-2}\). We have no guiding idea; to try at random is frustrating, cf. the relations we got for \(osp_\alpha(4|2)\).

2) We could have similarly realized the Lie algebra \(sl(\lambda)\) as the quotient of \(U(\text{vect}(1))\), where \(\text{vect}(1) = \text{derv}\mathbb{C}[u]\). However, \(U(\text{vect}(1))\) has no center except the constants. What are the generators of the ideal — the analog of (2.0) — modulo which we should factorize \(U(\text{vect}(1))\) in order to get \(sl(\lambda)?\) (Observe that in case \(U(g)\), where \(g\) is a simple finite dimensional Lie superalgebra such that \(Z(U(g))\) is not noetherian, the ideal — the analog of (5.0) — is, nevertheless, finitely generated, cf. \([GL2]\).)

3) Feigin realized \(sl(*)\) on the space of functions on the open cell of \(CP^1\), a hyperboloid, see \([F]\). Examples of \([DGS]\) are similarly realized. Give any realization of \(o/\text{sp}(*\) and its superanalog.

4) Other problems are listed in sec. 8.1–8.3 below.
7.3. Serre relations are more convenient than our ones. The following Table represents results of V. Kornyak’s computations. $N_{GB}$ is the number of relations in Groebner basis, $N_{comm}$ is the number of non-zero commutators in the multiplication table, $D_{GB}$ is a maximum degree of relations in $GB$. Space is measured in in bytes. The corresponding values for Chevalley generators/Serre relations are given in brackets.

| alg | $N_{GB}$ | $N_{comm}$ | $D_{GB}$ | Space | Time          |
|-----|----------|------------|---------|-------|--------------|
| $\mathfrak{sl}(3)$ | 23 (24)  | 21 (21)    | 9 (4)   | 1300 (1188) | < 1sec (< 1sec) |
| $\mathfrak{sl}(4)$ | 69 (84)  | 70 (60)    | 17 (6)  | 3888 (3612)  | < 1sec (< 1sec) |
| $\mathfrak{sl}(5)$ | 193 (218) | 220 (126)  | 25 (8)  | 13556 (8716) | < 1sec (< 1sec) |
| $\mathfrak{sl}(6)$ | 444 (473) | 476 (225)  | 33 (10) | 34692 (18088) | < 1sec (< 1sec) |
| $\mathfrak{sl}(7)$ | 893 (908) | 937 (363)  | 41 (12) | 80272 (33700) | < 1sec (< 1sec) |
| $\mathfrak{sl}(8)$ | 1615 (1594) | 1632 (546) | 49 (14) | 162128 (57908) | 34sec (3sec) |
| $\mathfrak{sl}(9)$ | 2705 (218) | 2714 (1071) | 65 (18) | 534684 (143456) | 109sec (6sec) |
| $\mathfrak{sl}(10)$ | 4263 (4063) | 4132 (546) | 49 (14) | 162128 (57908) | 34sec (3sec) |
| $\mathfrak{sl}(11)$ | 6405 (6048) | 6224 (1245) | 73 (20) | 921972 (211428) | 1058sec (19sec) |

For the other Lie algebras, especially exceptional ones, the comparison is even more unfavourable. Nevertheless, for $\mathfrak{sl}(\lambda)$ with noninteger $\lambda$ there are only the Jacobson generators and we have to use them.

§8. Lie algebras of higher ranks. The analogs of the exponents and $W$-algebras

The following Tables 8.1 and 8.2 introduce the generators for the Lie algebras $U_\lambda(g)$ and the analogues of the exponents that index the generalized $W$-algebras (for their definition in the simplest cases from different points of view see [FFr] and [KM]; we will follow the lines of [KM]).

Recall that (see 0.1) one of the definitions of $U_\lambda(g)$ is as the associative algebra generated by $\tilde{g}$; we loosely denote it by $\tilde{S}(\tilde{g})$. For the generators of $LU_\lambda(g)$ we take the Chevalley generators of $g$ (since by 7.3 they are more convenient) and the lowest weight vectors of the irreducible $g$-modules that constitute $\tilde{S}(\tilde{g})$.

8.1. The exponents. This section is just part of Table 1 from [OV] reproduced here for the convenience of the reader. Recall that if $g$ is a simple (finite dimensional) Lie algebra, $W = W_\lambda(g)$ is its Weyl group, $l = \text{rk } g$, $\alpha_1, \ldots, \alpha_l$ the simple roots, $\alpha_0$ the lowest root; the $n_i$ the coefficients of linear relation among the $\alpha_i$ normed so that $n_0 = 1$; let $c = r_1 \cdots r_l$, where $r_i$ are the reflections from $W$ associated with the simple roots, be the Killing–Coxeter element. The order $h$ of $c$ (the Coxeter number) is equal to $\sum_{i>0} n_i$. The eigenvalues of $c$ are $\varepsilon^{k_1}, \ldots, \varepsilon^{k_l}$, where $\varepsilon$ is a primitive $h$-th root of unity. The numbers $k_i$ are called the exponents. Then

The exponents $k_i$ are the respective numbers $k_i$ from Table 1.1, e.g., $k_1 = 1$. The number of roots of $g$ is equal to $l \sum_{i>0} n_i = 2 \sum_{i>0} k_i$. The order of $W$ is equal to

$$z l! \prod_{i>0} n_i = \prod_{i>0} (k_i + 1),$$

where $z$ is the number of 1’s among the $n_i$’s for $i > 0$ (the number $z$ is also equal to the order of the centrum $Z(G)$ of the simply connected Lie group $G$ with the Lie algebra $g$). The algebra of $W$-invariant polynomials on the maximal diagonalizable (Cartan) subalgebra of $g$ is freely generated by homogeneous polynomials of degrees $k_i + 1$. 


We will use the following notations:

For a finite dimensional irreducible representations of finite dimensional simple Lie algebras \( R(\lambda) \) denotes the irreducible representation with highest weight \( \lambda \) and \( V(\lambda) \) the space of this representation;

\[
\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \text{ or } \rho \text{ is a weight such that } \rho(\alpha_i) = A_{ii} \text{ for each simple root } \alpha_i.
\]

The weights of the Lie algebras \( \mathfrak{o}(2l) \) and \( \mathfrak{o}(2l + 1), \mathfrak{sp}(2l) \) and \( \mathfrak{f}_4 \) \((l = 4)\) are expressed in terms of an orthonormal basis \( \varepsilon_1, \ldots, \varepsilon_l \) of the space \( \mathfrak{h}^* \) over \( \mathbb{Q} \). The weights of the Lie algebras \( \mathfrak{sl}(l + 1) \) as well as \( \mathfrak{e}_7, \mathfrak{e}_8 \) and \( \mathfrak{g}_2 \) \((l = 7, 8 \text{ and } 2, \text{ respectively})\) are expressed in terms of vectors \( \varepsilon_1, \ldots, \varepsilon_{l+1} \) of the space \( \mathfrak{h}^* \) over \( \mathbb{Q} \) such that \( \sum \varepsilon_i = 0 \). For these vectors \( (\varepsilon_i, \varepsilon_i) = \frac{i}{l+1} \) and \( (\varepsilon_i, \varepsilon_j) = \frac{1}{l+1} \) for \( i \neq j \). The indices in the expression of any weight are assumed to be different.

The analogues of the exponents for \( LU_\mathfrak{g}(\lambda) \) are the highest weights of the representations that constitute \( \tilde{S}^k(\mathfrak{g}) \).

**Problem.** Interpret these exponents in terms of the analog of the Weyl group of \( LU_\mathfrak{g}(\lambda) \) in the sense of [PS] and invariant polynomials on \( LU_\mathfrak{g}(\lambda) \).

### 8.2. Table. The Lie algebras \( U_\mathfrak{g}(\lambda) \) as \( \mathfrak{g} \)-modules.

Columns 2 and 3 of this Table are derived from Table 5 in [OV]. Columns 4 and 5 are results of a computer-aided study. To fill in the gaps is a research problem, cf. [GL2] for the Lie algebras different from \( \mathfrak{sl} \) type.

| \( \mathfrak{g} \) | \( \text{ad} \) | \( \tilde{S}^2(\mathfrak{g}) \) | \( \tilde{S}^3(\mathfrak{g}) \) | \( \tilde{S}^k(\mathfrak{g}) \) |
|---|---|---|---|---|
| \( \mathfrak{sl}(2) \) | \( R(2\pi) \) | \( R(4\pi) \) | \( R(6\pi) \) | \( R(2k\pi) \) |
| \( \mathfrak{sl}(3) \) | \( R(\pi_1 + \pi_2) \) | \( R(2\pi_1 + 2\pi_2) \) | \( R(3\pi_1 + 3\pi_2) \) | \( R(k\pi_1 + k\pi_2) \) |
| & | \( R(\pi_1 + 2\pi_2) \) | \( R(2\pi_1 + 2\pi_2) \) | \( R((k-1)\pi_1 + (k-1)\pi_2) \) | |
| \( \mathfrak{sl}(4) \) | \( R(\pi_1 + \pi_3) \) | \( R(2\pi_1 + 2\pi_3) \) | \( R(3\pi_1 + 3\pi_3) \) | |
| & | | \( R(\pi_1 + \pi_3) \) | | |
| & | | \( R(2\pi_2) \) | | |
| \( \mathfrak{sl}(n+1) \) | \( n \geq 4 \) | \( R(\pi_1 + \pi_n) \) | \( R(2\pi_1 + 2\pi_n) \) | \( R(3\pi_1 + 3\pi_n) \) |
| | | \( R(\pi_1 + \pi_n) \) | \( R(2\pi_1 + 2\pi_n) \) | \( R(2\pi_1 + 2\pi_n) \) |
| | | \( R(\pi_1 + \pi_n) \) | \( R(2\pi_1 + 2\pi_n) \) | \( R(\pi_1 + \pi_n) \) |
| | | \( R(\pi_2 + \pi_n) \) | \( R(\pi_2 + \pi_n) \) | \( R(\pi_2 + \pi_n) \) |
| | | \( R(\pi_1 + \pi_n) \) | \( R(\pi_1 + \pi_n) \) | \( R(\pi_1 + \pi_n) \) |

The generators of \( LU_\mathfrak{g}(\lambda) \) are the Chevalley generators \( X_i^\pm \) of \( \mathfrak{g} \) AND the lowest weight vectors from \( \tilde{S}^2 \). Denote the latter by \( z_1, z_2 \) (sometimes there is a third one, \( z_3 \)). Then the relations are (recall that \( h_i = [X_i^+, X_i^-] \)):

(type 0) the Serre relations in \( \mathfrak{g} \)
The relations between \(X_i^+\) and \(z_j\), namely:

\[ X_i^-(z_j) = 0; \quad h_i(z_j) = \text{weight}_i(z_j); \quad (adX_i^+)^n \text{ the power determined by the weight of } z_j(z_j) = 0. \]

**Problem.** Give an explicit form of the relations of higher types.

### 8.3. Tougher problems.

Even if the explicit realization of the exceptional Lie algebras by differential operators on the base affine space were known at the moment, it is, nevertheless, a difficult computer problem to fill in the blank spaces in the above table and similar tables for Lie superalgebras. To make plausible conjectures we have to compute \(\tilde{S}_k(\mathfrak{g})\) to, at least, \(k = 4\).

Observe that for simple Lie algebras \(\mathfrak{g}\) we have a remarkable theorem by Kostant which states that \(U_{\mathfrak{g}}(\lambda)\) contains every finite dimensional irreducible \(\mathfrak{g}\)-module \(V\) with multiplicity equal to the multiplicity of the zero weight in \(V\); in view of which only the \(\mathfrak{sl}(2)\)-line is complete.

## §9. A connection with integrable dynamical systems

We recall the basic steps of the Khesin–Malikov construction and then superize them.

### 9.1. The Hamilton reduction.

Let \((M^{2n},\omega)\) be a symplectic manifold with an action act of a Lie group \(G\) on \(M\) by symplectomorphisms (i.e., \(G\) preserves \(\omega\)). The derivative of the \(G\)-action gives rise to a Lie algebra homomorphism \(a : \mathfrak{g} = \text{Lie}(G) \rightarrow \mathfrak{h}(2n)\). The action act, or rather, \(a\) is called a Poisson one, if \(a\) can be lifted to a Lie algebra homomorphism \(\tilde{a} : \mathfrak{g} \rightarrow \mathfrak{po}(2n)\), where the Poisson algebra \(\mathfrak{po}(2n)\) is the nontrivial central extension of \(\mathfrak{h}(2n)\).

For any Poisson \(G\)-action on \(M\) there arises a \(G\)-equivariant map \(p : M \rightarrow \mathfrak{g}^*\), called the moment map, given by the formula

\[ \langle p(x), g \rangle = \tilde{a}(g)(x) \quad \text{for any} \quad x \in M, g \in \mathfrak{g}. \]

Fix \(b \in \mathfrak{g}^*\); let \(G_b \subset G\) be the stabilizer of \(b\). Under certain regularity conditions (see \([AT]\)) \(p^{-1}(b)/G_b\) is a manifold. This manifold is endowed with the symplectic form

\[ \omega(\bar{v}, \bar{w}) = \omega(v, w) \quad \text{for arbitrary preimages} \quad v, w \text{ of } \bar{v}, \bar{w}, \text{ respectively} \]

wrt the natural projection \(T(p^{-1}(b)) \rightarrow T(p^{-1}(b)/G_b)\).

The passage from \(M\) to \(p^{-1}(b)/G_b\) is called Hamilton reduction. In the above picture \(M\) can be the Poisson manifold, i.e., \(\omega\) is allowed to be nondegenerate not on the whole \(M\); the submanifolds on which \(\omega\) is nondegenerate are called symplectic leaves.

**Example.** Let \(\mathfrak{g} = \mathfrak{sl}(n)\) and \(M = \mathfrak{g}^*\), let \(G\) be the group \(N\) of upper triangular matrices with 1 on the diagonal. The coadjoint \(N\)-action on \(\mathfrak{g}^*\) is a Poisson one, the moment map is the natural projection \(\mathfrak{g}^* \rightarrow \mathfrak{n}^*\) and \(\mathfrak{g}^*/\mathfrak{n}\) is a Poisson manifold.

### 9.2. The Drinfeld–Sokolov reduction.

Let \(\mathfrak{g} = \mathfrak{a}^{(1)}\), where \(\mathfrak{a}\) is a simple finite dimensional Lie algebra (the case \(\mathfrak{a} = \mathfrak{sl}(n)\) is the one considered by Gelfand and Dickii), hat denotes the Kac–Moody central extension. The elements of \(M = \mathfrak{g}^*\), can be identified with the \(\mathfrak{a}\)-valued differential operators:

\[ (f(t)dt, az^*) \mapsto (tf(t) + at \frac{dz^*}{dt}) \frac{dt}{t}. \]

Let \(N\) be the loop group with values in the group generated by positive roots of \(\mathfrak{a}\). For the point \(b\) above take the element \(y \in \mathfrak{a} \subset \mathfrak{g}^*\) described in §3. If \(\mathfrak{a} = \mathfrak{sl}(n)\), we can represent
every element of $p^{-1}(b)/N$ in the form

$$t \frac{d}{dt} + y + \begin{pmatrix} b_1(t) & \ldots & b_n(t) \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \end{pmatrix} \leftrightarrow \frac{d^n}{d\varphi^n} + \tilde{b}_1(\varphi) \frac{d^{n-1}}{d\varphi^{n-1}} + \ldots + \tilde{b}_n(\varphi),$$

To generalize the above to $\mathfrak{sl}(\lambda)$, Khesin and Zakharevich described the Poisson–Lie structure on symbols of pseudodifferential operators, see [KM] and refs therein. Let us recall the main formulas.

9.3.1. The Poisson bracket on symbols of $\Psi DO$. Set $D = \frac{d}{dx}$; define

$$D^\lambda \circ f = f D^\lambda + \sum_{k \geq 1} \binom{\lambda}{k} f^{(k)} D^{(\lambda-k)}, \quad \text{where} \quad \binom{\lambda}{k} = \frac{\lambda(\lambda-1)\ldots(\lambda-k+1)}{k!}.$$

Set

$$G_\lambda = \left\{ D^\lambda(1 + \sum_{k \geq 1} u_k(x) D^{(-k)}) \right\}$$

and

$$TG_\lambda = \left\{ \sum_{k \geq 1} v_k(x) D^{(-k)} \right\} \circ D^\lambda, \quad T^*G_\lambda = D^{-\lambda} \circ DO.$$

For $X = D^{-\lambda} \circ \sum_{k \geq 0} u_k(x) D^{(k)} \in T^*G_\lambda$ and $L = \left( \sum_{k \geq 1} v_k(x) D^{(-k)} \right) \circ D^\lambda \in TG_\lambda$ define the pairing $\langle X, L \rangle$ to be

$$\langle X, L \rangle = Tr(L \circ X), \quad \text{where} \quad Tr(\sum w_k(x) D^{(k)}) = \text{Res}_{x=0} w_{-1}.$$

The Poisson bracket on the space of pseudodifferential symbols $\Psi DS_\lambda$ is defined on linear functionals by the formula

$$\{X, Y\}(L) = X(H_Y(L)), \quad \text{where} \quad H_Y(L) = (LY)_+ L - L(YL)_+.$$

**Theorem.** (Khesin–Malikov) For $\mathfrak{a} = \mathfrak{sl}(\lambda)$ in the Drinfeld–Sokolov picture, the Poisson manifolds $p^{-1}(b)/N_b$ and $\Psi DS_\lambda$ are isomorphic. Each element of the Poisson leaf has a representative in the form

$$t \frac{d}{dt} + y + \begin{pmatrix} b_1(t) & \ldots & b_n(t) & \ldots \\ 0 & \ldots & 0 & \ldots \\ 0 & \ldots & 0 & \ldots \end{pmatrix} \leftrightarrow D^\lambda \left( 1 + \sum_{k \geq 1} \tilde{b}_k(\varphi) D^{(-k)} \right).$$

The Drinfeld–Sokolov construction [DS], as well as its generalization to $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$ ([KM]), hinges on a certain element that can be identified with the image of $X^+ \in \mathfrak{sl}(2)$ under the principal embedding. For the case of higher ranks this is the image in $U_{\mathfrak{sl}}(\lambda)$ of the element $y \in \mathfrak{g}$ described in §3 for Lie algebras. In $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$ this image is just $\frac{d}{dx}$ (or the matrix whose only nonzero entries are the 1’s under the main diagonal in the realization of $\mathfrak{sl}(\lambda)$ and $\mathfrak{o}/\mathfrak{sp}(\lambda)$ by matrices).

9.4. Superization.
9.4.1 Basics. Further facts from Linear Algebra in Superspaces.

The tensor algebra $T(V)$ of the superspace $V$ is naturally defined: $T(V) = \bigoplus_{n \geq 0} T^n(V)$, where $T^0(V) = k$ and $T^n(V) = V \otimes \cdots \otimes V$ ($n$ factors) for $n > 0$.

The symmetric algebra of the superspace $V$ is $S(V) = T(V)/I$, where $I$ is the two-sided ideal generated by $v_1 \otimes v_2 - (-1)^{p(v_1)p(v_2)}v_2 \otimes v_1$ for $v_1, v_2 \in V$.

The exterior algebra of the superspace $V$ is $E(V) = S(\Pi(V))$. Clearly, both the exterior and symmetric algebras of the superspace $V$ are supercommutative superalgebras. It is worthwhile to mention that if $V_0 \neq 0$ and $V_1 \neq 0$, then both $E(V)$ and $S(V)$ are infinite dimensional.

A Lie superalgebra is defined with Sign Rule applied to the definition of a Lie algebra. Its multiplication is called bracket and is usually denoted by $[\cdot, \cdot]$ or $\{\cdot, \cdot\}$. If, however, we try to use this definition in attempts to apply the standard group-theoretical methods to differential equations on supermanifolds we will find ourselves at a loss: the supergroups and their modules are objects from different categories! Accordingly, the following (equivalent to the conventional, “naive” one, see [LI]) definition becomes useful: a Lie superalgebra is a superalgebra $g$ (defined over a field or, more generally, a supercommutative superalgebra $k$); the bracket should satisfy the following conditions: $[X, X] = 0$ and $[Y, [Y, Y]] = 0$ for any $X \in (C \otimes g)_0$ and $Y \in (C \otimes g)_1$ and any supercommutative superalgebra $C$ (the bracket in $C \otimes g$ is defined via $C$-linearity and Sign Rule).

With an associative (super)algebra $A$ we associate Lie (super)algebras (1) $A_L$ with the same (super)space $A$ and the multiplication $(a, b) \mapsto [a, b]$ and (2) $\text{der}. A$, the algebra of derivations of $A$, defined via Sign and Leibniz Rules.

From a Lie superalgebra $g$ we construct an associative superalgebra $U(g)$, called the universal enveloping algebra of the Lie superalgebra $g$ by setting $U(g) = T(g)/I$, where $I$ is the two-sided ideal generated by the elements $x \otimes y - (-1)^{p(x)p(y)}y \otimes x - [x, y]$ for $x, y \in g$.

The Poincaré–Birkhoff–Witt theorem for Lie algebras extends to Lie superalgebras with the same proof (beware Sign Rule) and reads as follows:

If $\{X_i\}$ is a basis in $g_0$ and $\{Y_j\}$ is a basis in $g_1$, then the monomials $X_{i_1}^{n_1} \cdots X_{i_r}^{n_r} Y_{j_1}^{\varepsilon_1} \cdots Y_{j_s}^{\varepsilon_s}$, where $n_i \in \mathbb{Z}^+$ and $\varepsilon_j = 0, 1$, constitute a basis in the space $U(g)$.

A superspace $M$ is called a left module over a superalgebra $A$ (or a left $A$-module) if there is given an even map act: $A \otimes M \rightarrow M$ such that if $A$ is an associative superalgebra with unit, then $(ab)m = a(bm)$ and $1m = m$ and if $A$ is a Lie superalgebra, then $[a, b]m = a(bm) - (1)^{p(a)p(b)}b(am)$ for any $a, b \in A$ and $m \in M$. The definition of a right $A$-module is similar.

Convention. We endow every module $M$ over a supercommutative superalgebra $C$ with a two-sided module structure: the left module structure is recovered from the right module one and vice versa according to the formula $cm = (-1)^{p(m)p(c)}mc$ for any $m \in M$ and $c \in C$. Such modules will be called $C$-modules. (Over $C$, there are two canonical ways to define a two-sided module structure, see [LI]; the meaning of such an abundance is obscure.)

The functor $\Pi$ is, actually, tensoring by $\Pi(\mathbb{Z})$. So there are two ways to apply $\Pi$ to $C$-modules: to get $\Pi(M) = \Pi(\mathbb{Z}) \otimes_\mathbb{Z} M$ and $(M)\Pi = M \otimes_\mathbb{Z} \Pi(\mathbb{Z})$. The two-sided module structures on $\Pi(M)$ and $(M)\Pi$ are given via Sign Rule.

Sometimes, instead of the map act a morphism $\rho: A \rightarrow \text{End} M$ is defined if $A$ is an associative superalgebra (or $\rho: A \rightarrow \text{End} M)_L$ if $A$ is a Lie superalgebra); $\rho$ is called a representation of $A$ in $M$.

The simplest (in a sense) modules are those which are irreducible. We distinguish irreducible modules of $G$-type (general); these do not contain invariant subspaces different from 0 and the whole module; and their “odd” counterparts, irreducible modules of $Q$-type, which
do contain an invariant subspace which, however, is not a subsuperspace. Consequently, Schur’s lemma states that over \( \mathbb{C} \) the centralizer of a set of irreducible operators is either \( \mathbb{C} \) or \( \mathbb{C} \otimes \mathbb{C}^* = Q(1; \mathbb{C}) \), see the definition of the superalgebras \( Q \) below.

The next in terms of complexity are indecomposable modules, which cannot be split into the direct sum of invariant submodules.

A \( C \)-module is called free if it is isomorphic to a module of the form \( C \oplus \cdots \oplus C \oplus \Pi(C) \oplus \cdots \oplus \Pi(C) \) (\( C \) occurs \( r \) times, \( \Pi(C) \) occurs \( s \) times).

The rank of a free \( C \)-module \( M \) is the element \( \text{rk} M = r + s \) from the ring \( \mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1) \). Over a field, \( C = k \), we usually write just \( \dim M = (r, s) \) or \( r|s \) and call this pair the superdimension of \( M \).

The module \( M^* = \text{Hom}_C(M, C) \) is called dual to a \( C \)-module \( M \). If \( (\cdot, \cdot) \) is the pairing of modules \( M^* \) and \( M \), then to each operator \( F \in \text{Hom}_C(M, N) \), where \( M \) and \( N \) are \( C \)-modules, there corresponds the dual operator \( F^* \in \text{Hom}_C(N^*, M^*) \) defined by the formula

\[
(F(m), n^*) = (-1)^{p(F)p(m)}(m, F^*(n^*)) \quad \text{for any} \; m \in M, \; n^* \in N^*.
\]

Over a supercommutative superalgebra \( C \) a supermatrix is a supermatrix with entries from \( C \), the parity of the matrix with only \((i, j)\)-th nonzero element \( c \) is equal to \( p(i) + p(j) + p(c) \). Denote by Mat(Par; \( C \)) the set of Par \times Par matrices with entries from a supercommutative superalgebra \( C \).

The even invertible elements from Mat(Par; \( C \)) constitute the general linear group \( GL(Par; C) \).

Put \( GQ(Par; C) = Q(Par; C) \cap GL(Par; C) \).

On the group \( GL(Par; C) \) an analogue of the determinant is defined; it is called the Berezinian (in honour of F. A. Berezin who discovered it). In the standard format the explicit formula for the Berezinian is:

\[
\text{Ber} \begin{pmatrix} A & B \\ D & E \end{pmatrix} = \det(A - BE^{-1}D) \det E^{-1}.
\]

For the matrices from \( GL(Par; C) \) the identity \( \text{Ber} XY = \text{Ber} X \cdot \text{Ber} Y \) holds, i.e., \( \text{Ber} : GL(Par; C) \to GL(1|0; C) = GL(0|1; C) \) is a group homomorphism. Set \( SL(Par; C) = \{X \in GL(Par; C) : \text{Ber} X = 1 \} \). The orthosymplectic group of automorphisms of the bilinear form with the even canonical matrix is denoted (in the standard format) by \( Osp(n|2m; C) \).

### 9.4.2. Pseudodifferential operators on the supercircle. Residues.

Let \( V \) be a superspace. For \( \theta = (\theta_1, \ldots, \theta_n) \) set

\[
V[x, \theta] = V \otimes \mathbb{K}[x, \theta]; \quad V[x^{-1}, x, \theta] = V \otimes \mathbb{K}[x^{-1}, x, \theta];
\]

\[
V[[x^{-1}, \theta]] = V \otimes \mathbb{K}[[x^{-1}, \theta]];
\]

\[
V((x, \theta)) = V \otimes \mathbb{K}[[x^{-1}]][x, \theta].
\]

We call \( V((x, \theta))x^\lambda \) the space of pseudodifferential symbols. Usually, \( V \) is a Lie (super)algebra. Such symbols correspond to pseudodifferential operators (pdo) of the form

\[
\sum_{i=-\infty}^{n} \sum_{k_0 + \cdots + k_n = i} a_i(\partial_x)^{k_0} \theta_1^{k_1} \cdots \theta_n^{k_n},
\]

Here \( k_i = 0 \) or \( 1 \) for \( i > 0 \) and \( a_i(x, \theta) \in V \). This is clear.

For any \( P = \sum_{i \leq m} P_i x^i \theta_0^k \theta^j \in V((x, \theta)) \) we call \( P_+ = \sum_{i,j, k \geq 0} P_i x^i \theta_0^k \theta^j \) the differential part of \( P \) and \( P_- = \sum_{i, k < 0} P_i x^i \theta_0^k \theta^j \) the integral part of \( P \).

The space \( \Psi DO \) of pdos is, clearly, the left module over the algebra \( \mathcal{F} \) of functions. Define the left \( \Psi DO \)-action on \( \mathcal{F} \) from the Leibniz formula thus making \( \Psi DO \) into a superalgebra.
Define the involution in the superalgebra $\Psi DO$ setting

$$(a(t, \theta)D^i D^j)^* = (-1)^{ijp(D)p(D)} D^j D^i a^*(x, \theta).$$

The following fact is somewhat unexpected. If $D$ is an odd differential operator, then $D^2$ is well-defined as $\frac{1}{2}[D, D]$. Hence, we can consider the set $V((x, \theta, \theta) x^\lambda$ for an odd $x$! Therefore, there are two types of pdos: contact ones, when $D^2 \neq 0$ for odd $D$’s and general ones, when all odd $D$’s are nilpotent.

For the definition of distinguished stringy superalgebgras crucial in what follows see [GLS1].

Conjecture. There exists a residue for all distinguished dimensions, i.e. for the contact type pdos in dimensions $1|n$ for $n \leq 4$ and for the general pdos in dimensions $1|n$ for $n \leq 2$.

So far, however, the residue was defined only for contact type pdos, of $t^L$ type, and only for $n = 1$ at that.

Let us extend [MR] and define the residue of $P = \sum_{i \leq m} P_i x^i \theta^i \theta^j \in V((x, \theta_0, \theta))$ for $n = 1$.

We can do it thanks to the following exceptional property of $t^L(1|1)$ and $t^M(1|1)$. Indeed, over $t^L(1|1)$, the volume form “$dt^L_{\theta^\alpha}$” is, more or less, $d\theta$: consider the quotient $\Omega^1/F\alpha$, where $\alpha$ is the contact form preserved by $t^L(1|1)$; similarly, over $t^M(1|1)$, the transformation rules of $dt^M_{\theta^\alpha}$ and $\tilde{\alpha}$, where $\tilde{\alpha}$ is the contact form preserved by $t^M(1|1)$, are identical. Therefore, define the residue by the formula

$$\text{Res } P = \text{coefficient of } \frac{\theta}{x} \text{ in the expansion of } P_{-1}.$$  

Remark. Manin and Radul [MR] considered the Kadomtsev–Petviashvili hierarchy associated with $\mathfrak{ns}$, i.e., for $D = K_\theta$. The formula for the residue allows one to directly generalize their result and construct a simialr hierarchy associated with $\mathfrak{r}$, i.e., for $D = K_\theta$.

This new phenomenon — an invertible odd symbol — doubles the old picture: let $\theta_0$ be the symbol of $D$, and let $x$ be the symbol of the differential operator $D^2$. We see that the case of the odd $D$ reduces to either $V((x, \theta_0, \theta)) x^\lambda$ or $V((x, \theta_0^{-1}, \theta)) x_\lambda$.

9.5. Continuous Toda lattices. Khesin and Malikov [KM] considered straightforward generalizations of the Toda lattices — the dynamical systems on the orbits of the coadjoint representation of a simple finite dimensional Lie group $G$ defined as follows. Let $\mathcal{X}$ be the image of $X^+ \in \mathfrak{sl}(2)$ in $\mathfrak{g} = \text{Lie}(G)$ under the principal embedding. Having identified $\mathfrak{g}$ with $\mathfrak{g}^*$ with the help of the invariant nondegenerate form, consider the orbit $\mathcal{O}_X$. On $\mathcal{O}_X$, the traces $H_i(A) = \text{tr}(A + X)^i$ are the commuting Hamiltonians.

In our constructions we only have to consider in $LU_\mathfrak{g}(\lambda)$ either (for Lie algebras) the image of $\mathcal{X}$ or (for Lie superalgebras) the image of $\nabla^+ \in \mathfrak{osp}(1|2)$ under the superprincipal embedding of $\mathfrak{osp}(1|2)$. For superalgebras we also have to replace trace with the supertrace.

For the general description of dynamical systems on the orbits of the coadjoint representations of Lie supergroups see [LST]. A possibility of odd mechanics is pointed out in [LST] and in the subsequent paper by R. Yu. Kirillova in the same Proceedings. To take such a possibility into account, we have to consider analogs of the principal embeddings for $\mathfrak{sq}(2)$. This is a full-time job; its results will be considered elsewhere.

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