New Unitarity Triangles of Quark Mixing and CP Violation

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Abstract

Unitarity of the Cabibbo-Kobayashi-Maskawa matrix gives rise to nine new unitarity triangles in the standard model.
Huge samples of $B$ mesons produced at the $B$ factories and recent developments in the experiments with kaons and charmed mesons boost the studies of CP violation and quark mixing. According to the six-quark scheme of Kobayashi and Maskawa [1], CP symmetry is violated because one complex phase remains in the coupling constants of the charged current weak interactions after taking into account unitarity and rephasing quark fields. In the standard model CP violation and quark mixing are intimately related and described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1,2]

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{1}$$

The CKM matrix $V$ originating from the diagonalization of quark mass matrices must be unitary: $V V^\dagger = V^\dagger V = 1$. Unitarity of the CKM matrix is at the basis of the GIM mechanism [3]. Unitarity strongly constrains and correlates the magnitudes and phases of the nine elements of the CKM matrix, although their values are not predicted in detail. Consequently, CP-conserving observables and CP-violating observables are connected with each other. Confrontation of unitarity relations of the CKM matrix with experimental measurements is an important test of the standard model. A deviation from expectations would signal new physics.

Unitarity of the CKM matrix is conveniently displayed in the well-known “unitarity triangle” [4]. There are six unitarity triangles resulting from the orthogonality between each pair of rows, or each pair of columns, in the $3 \times 3$ CKM matrix.

The orthogonality condition between two different rows reads

$$V_{ad}^* V_{\beta d} + V_{as}^* V_{\beta s} + V_{ab}^* V_{\beta b} = 0. \tag{2}$$

Here and hereafter, Greek subscripts $(\alpha, \beta, \gamma)$ run over the up-type quarks $u, c, t$, while Latin ones $(i, j, k)$ run over the down-type quarks $d, s, b$. $\alpha \beta \gamma$ is some cyclic permutation of $uct$, and $ijk$ of $dsb$. The relation (2) can be geometrically represented as a triangle in the complex plane, formed by the three sides $|V_{ad} V_{\beta d}|$, $|V_{as} V_{\beta s}|$, and $|V_{ab} V_{\beta b}|$. A physical observable must be independent of phase convention of quark fields. Let

$$\omega_{\gamma k} \equiv \text{arg}(-V_{\alpha i}^* V_{\beta j}^* V_{\alpha j} V_{\beta i}) \tag{3}$$

be the phase of the product of the matrix elements $V_{\alpha i}^* V_{\beta j}^* V_{\alpha j} V_{\beta i}$. Since $V_{\alpha i}^* V_{\beta j}^* V_{\alpha j} V_{\beta i}$ is invariant under rephasing of quark fields, $\omega_{\gamma k}$ is an observable phase. Then for the three interior angles of the unitarity triangle, the angle between the sides $|V_{as} V_{\beta s}|$ and $|V_{ab} V_{\beta b}|$ is $\omega_{\gamma d}$, the angle between the sides $|V_{ab} V_{\beta b}|$ and $|V_{ad} V_{\beta d}|$ is $\omega_{\gamma s}$, the angle between the sides $|V_{ad} V_{\beta d}|$ and $|V_{as} V_{\beta s}|$ is $\omega_{\gamma b}$. By definition,

$$\omega_{\gamma d} + \omega_{\gamma s} + \omega_{\gamma b} = \pi, \tag{4}$$

since

$$(-V_{as}^* V_{\beta b} V_{ab} V_{\beta s})(-V_{ab}^* V_{\beta d} V_{ad} V_{\beta b})(-V_{ad}^* V_{\beta s} V_{as} V_{\beta d}) = -|V_{ad} V_{as} V_{ab} V_{\beta d} V_{\beta s} V_{\beta b}|^2. \tag{5}$$
The relation (2) leads to three unitarity triangles. We label them by their \( \alpha \beta \) indices (with \( \alpha \beta \) being \( uc, ct, \) and \( tu \), respectively).

Similarly, the orthogonality condition between two different columns reads

\[
V_{ui}^* V_{uj} + V_{ci}^* V_{cj} + V_{ti}^* V_{tj} = 0. \tag{6}
\]

This relation can also be geometrically represented as a triangle in the complex plane, with three sides \( |V_{ui} V_{uj}| \), \( |V_{ci} V_{cj}| \), and \( |V_{ti} V_{tj}| \). For the three interior angles, the angle between the sides \( |V_{ci} V_{cj}| \) and \( |V_{ti} V_{tj}| \) is \( \omega_{uk} \); the angle between the sides \( |V_{ti} V_{tj}| \) and \( |V_{ui} V_{uj}| \) is \( \omega_{ck} \); the angle between the sides \( |V_{ui} V_{uj}| \) and \( |V_{ci} V_{cj}| \) is \( \omega_{tk} \). By definition,

\[
\omega_{uk} + \omega_{ck} + \omega_{tk} = \pi. \tag{7}
\]

The relation (3) also leads to three unitarity triangles. We label these three triangles by their \( ij \) indices (with \( ij \) being \( ds, sb, \) and \( bd \), respectively). Among them is the \( bd \) unitarity triangle which is particularly useful for the study of \( B \) physics, with its sides determined by semileptonic \( B \) decays and \( B^0 - \bar{B}^0 \) mixing and its angles \( \omega_{us} = \arg(-V_{tb}^* V_{ud}^* V_{tb} V_{ub}) = \beta \), \( \omega_{cs} = \arg(-V_{tb}^* V_{ud}^* V_{cd} V_{ub}) = \alpha \), and \( \omega_{ts} = \arg(-V_{tb}^* V_{ud}^* V_{cd} V_{cb}) = \gamma \) determined by CP-violating asymmetries in \( B \) decays.

Although different in shapes, all the six unitarity triangles have the same area being half the Jarlskog invariant \[ J_{\text{CP}} \equiv |\text{Im}(V_{\alpha i}^* V_{\beta j}^* V_{\beta i} V_{\beta j})| \]. The Jarlskog invariant \( J_{\text{CP}} \) is a universal rephasing invariant quantity which is independent of the \( \alpha \beta \) and \( ij \) indices due to unitarity. CP-violating observables are all proportional to \( J_{\text{CP}} \).

The above six conventional unitarity triangles picture six orthogonality conditions of the CKM matrix. Other different relationships can be obtained from unitarity of the CKM matrix. Indeed, we find nine new unitarity relations following from a combination of orthogonality and normalization conditions. These unitarity relations can also be pictured as triangles with the interesting feature that all the nine new unitarity triangles also have the same area equal to \( J_{\text{CP}}/2 \). Let us demonstrate these unitarity relations. From the orthogonality condition Eq. (2) between two different rows, we have

\[
V_{\alpha i}^* V_{\beta i} + V_{\alpha j}^* V_{\beta j} = -V_{\alpha k}^* V_{\beta k}. \tag{8}
\]

Multiplying Eq. (5) by its complex conjugation yields

\[
2\text{Re}(V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*) = -|V_{\alpha i} V_{\beta i}|^2 - |V_{\alpha j} V_{\beta j}|^2 + |V_{\alpha k} V_{\beta k}|^2. \tag{9}
\]

By applying the normalization condition

\[
|V_{\alpha i}|^2 + |V_{\alpha j}|^2 + |V_{\alpha k}|^2 = 1 \tag{10}
\]

to Eq. (6), we find

\[
2\text{Re}(V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*) = 2|V_{\alpha i} V_{\beta j} V_{\alpha j} V_{\beta i}| \cos(\pi - \omega_{\gamma k}) = |V_{\alpha i} V_{\beta j}|^2 + |V_{\alpha j} V_{\beta i}|^2 - |V_{\gamma k}|^2. \tag{11}
\]

If we start from the column orthogonality, we arrive at the same relation (11) by using the normalization condition. The unitarity relation (11), satisfying the Cosine rule of plane trigonometry, defines a triangle with three sides \( |V_{\alpha i} V_{\beta j}|, |V_{\alpha j} V_{\beta i}|, \) and \( |V_{\gamma k}| \). Its interior angle
between the sides $|V_{\alpha i}V_{\beta j}|$ and $|V_{\alpha j}V_{\beta i}|$ is $\pi - \omega_{\gamma k}$. This is the only angle among its three interior angles, which is of direct physical interest in the description of CP violation. The unitarity relation (11) leads to nine unitarity triangles. We refer to each of these triangles by the $\gamma k$ indices corresponding to its side $|V_{\gamma k}|$. Each of these unitarity triangles are formed by five magnitudes of the CKM matrix elements, rather than six ones forming the conventional unitarity triangle. However, it is easy to show that all these nine new unitarity triangles have the same area equal to $J_{CP}/2$ as the conventional ones. A nonzero area implies CP violation.

The angles of the unitarity triangles are also correlated. First, the six conventional unitarity triangles have 9 rather than 18 different angles, since any triangle representing the row orthogonality (2) and any triangle representing the column orthogonality (6) contain a common angle. This can be understood since one can construct only nine different phases $\omega_{\gamma k}$ from the rephasing invariants $V_{\alpha i}^*V_{\beta j}^*V_{\alpha j}V_{\beta i}$. On the other hand, there is a one-to-one correspondence between these nine different angles and the nine new unitarity triangles. Second, only four out of the total of the nine observable phases $\omega_{\gamma k}$ are independent, since Eqs. (4) and (7) impose five independent constraints on them. The four independent phases form a complete set of weak phases for the description of CP violation, and fully determine the entire CKM matrix [6]. We can choose the four independent phases to be $\omega_{us} = \beta$, $\omega_{ts} = \gamma$, $\omega_{ud} = \arg(-V_{cs}^*V_{tb}^*V_{cb}) = \chi$, and $\omega_{tb} = \arg(-V_{us}^*V_{cs}^*V_{us}V_{cd}) = \chi'$. Using Eqs. (4) and (7), the remaining phases can then be expressed in terms of them:

$$
\begin{align*}
\omega_{ub} &= \pi - \beta - \chi, \\
\omega_{cd} &= \gamma + \chi' - \chi, \\
\omega_{cs} &= \pi - \beta - \gamma, \\
\omega_{cb} &= \beta + \chi - \chi', \\
\omega_{td} &= \pi - \gamma - \chi',
\end{align*}
$$

which are essentially determined by the two angles $\beta$ and $\gamma$ (in leading order of $\lambda$, except for $\omega_{cs}$), since empirically $\beta$ and $\gamma$ are large (i.e., far from 0 and $\pi$), while $\chi$ is small, of order $\lambda^2$, and $\chi'$ is much smaller, of order $\lambda^4$, where $\lambda \equiv |V_{us}| \approx 0.22$. Note that Eqs. (4), (7) and (12) are simply a consequence of the definition (3) of the phase $\omega_{\gamma k}$; they have nothing to do with unitarity of the CKM matrix. Equations (4), (7) and (12) still hold, even if the three family quark mixing matrix is not unitary and the unitarity triangle is not closed. Therefore, comparing the relations (4), (7) and (12) with experiments does not test unitarity. Deviations from unitarity will show up in a failure of unitarity relations involving both angles and magnitudes [7].

From Eq. (11) it follows that

$$
\begin{align*}
\sin \frac{\omega_{\gamma k}}{2} &= \frac{s_{\gamma k}(s_{\gamma k} - |V_{\gamma k}|)}{|V_{\alpha i}V_{\beta j}V_{\alpha j}V_{\beta i}|}, \\
\cos \frac{\omega_{\gamma k}}{2} &= \frac{(s_{\gamma k} - |V_{\alpha i}V_{\beta j}|)(s_{\gamma k} - |V_{\alpha j}V_{\beta i}|)}{|V_{\alpha i}V_{\beta j}V_{\alpha j}V_{\beta i}|},
\end{align*}
$$

where $s_{\gamma k} = (|V_{\alpha i}V_{\beta j}| + |V_{\alpha j}V_{\beta i}| + |V_{\gamma k}|)/2$ is half the perimeter of the $\gamma k$ triangle. When

TABLE I. Qualitative comparison of the sides of the unitarity triangles

| Relative length | Length               | Conventional triangle | New triangle       |
|-----------------|----------------------|-----------------------|-------------------|
| 1/1/1           | $\lambda^4, \lambda^3, \lambda^2$ | $tu, bd$             | $ub, td$          |
| $1/1/\lambda^2$ | $1, 1, \lambda^2, \lambda^2, \lambda^4$ | $ct, sb$             | $cb, ts$          |
| $1/1/\lambda^4$ | $1, 1, \lambda^4, \lambda, \lambda, \lambda^5$ | $uc, ds$             | $us, cd$          |
| $1/1/\lambda^6$ | $1, 1, \lambda^6$ |                       | $cs$              |

The three sides of a unitarity triangle are given, these formulas may be used in order to find the angle.

The unitarity relations (11), (13) and (14) can serve as a good test of the standard model. These tests require precise direct measurements, without assuming unitarity of the CKM matrix, of the sides and angles of the unitarity triangles. By measuring both the sides and angles, the unitarity triangles will be overconstrained. Present knowledge of the magnitudes of the elements in the third row of the CKM matrix involving the top quark in Eq. (1), but also of $|V_{cs}|, |V_{cd}|$ and $|V_{ub}|$ is still rather imprecise [8]. Among the four independent phase angles $\beta, \gamma, \chi, \chi'$, only direct measurement of $\sin(2\beta)$ has recently been made [3] from the time-dependent CP-violating asymmetry in $B^0 \rightarrow J/\psi K^0_S$ decays [10]. The current experimental error on $\sin(2\beta)$ is large, but is expected to be improved in the near future by the $B$ factories. A determination of the angle $\beta$ from the measured $\sin(2\beta)$ still has a four-fold ambiguity: $\beta, \pi/2 - \beta, \pi + \beta, 3\pi/2 - \beta$ are all allowed solutions. Additional measurements are required to resolve the discrete ambiguities [11–16]. A major experimental effort will be made to determine the angles of the unitarity triangles [17]. It is also important to precisely measure the magnitudes of the CKM matrix elements, since measuring the phase angles alone cannot test unitarity of the CKM matrix as discussed above.

We can analyze the shapes of the unitarity triangles by using the Wolfenstein parametrization [13] of the CKM matrix. The sides of the unitarity triangles are compared in Table I. The unitarity triangles can be divided into two categories: fat and flat. The $ub$ and $td$ triangles, like the conventional $tu$ and $bd$ triangles, have a “fat shape” — all three sides of each of them are of comparable lengths. The other (conventional and new) triangles are flat — they almost collapse to a line, with two sides having comparable lengths and the other side being suppressed relative to them by $O(\lambda^2), O(\lambda^4)$, and $O(\lambda^6)$, respectively. Among the new unitarity triangles, the $ub$ and $td$ triangles may therefore be particularly useful for testing the standard model. Since all three sides of each of them have comparable lengths, of order $\lambda^3$, measurements of the sides with only moderate precision would be sufficient to test the unitarity relations. In contrast, the other unitarity triangles require high precision measurements of the sides to make a test. Neglecting $\chi$ and $\chi'$, we can deduce from Eqs. (11) and (12) the unitarity relations

$$\cos \beta \simeq \cos(\pi - \omega_{ub}) = \frac{|V_{cd}V_{ts}|^2 + |V_{cs}V_{td}|^2 - |V_{ub}|^2}{2|V_{cd}V_{ts}V_{cs}V_{td}|},$$

(15)
\[
\cos \gamma \simeq \cos(\pi - \omega_{td}) = \frac{|V_{us}V_{cb}|^2 + |V_{ub}V_{cs}|^2 - |V_{td}|^2}{2|V_{us}V_{cb}|V_{ub}V_{cs}|}, \tag{16}
\]

for the \(ub\) and \(td\) triangles, respectively. Similarly, from Eqs. (12), (13) and (14) we can write explicitly for the \(ub\) triangle

\[
\cos \frac{\beta}{2} \simeq \sin \frac{\omega_{ub}}{2} = \sqrt{s_{ub} (s_{ub} - |V_{ub}|)} / |V_{cd}V_{ts}V_{cs}V_{td}|, \tag{17}
\]

\[
\sin \frac{\beta}{2} \simeq \cos \frac{\omega_{ub}}{2} = \sqrt{(s_{ub} - |V_{cd}V_{ts}|)(s_{ub} - |V_{cs}V_{td}|)} / |V_{cd}V_{ts}V_{cs}V_{td}|, \tag{18}
\]

where \(s_{ub} = (|V_{cd}V_{ts}| + |V_{cs}V_{td}| + |V_{ub}|) / 2\); for the \(td\) triangle

\[
\cos \frac{\gamma}{2} \simeq \sin \frac{\omega_{td}}{2} = \sqrt{s_{td} (s_{td} - |V_{td}|)} / |V_{us}V_{cb}V_{ub}V_{cs}|, \tag{19}
\]

\[
\sin \frac{\gamma}{2} \simeq \cos \frac{\omega_{td}}{2} = \sqrt{(s_{td} - |V_{us}V_{cb}|)(s_{td} - |V_{ub}V_{cs}|)} / |V_{us}V_{cb}V_{ub}V_{cs}|, \tag{20}
\]

where \(s_{td} = (|V_{us}V_{cb}| + |V_{ub}V_{cs}| + |V_{td}|) / 2\). A comparison of these determinations of \(\beta\) and \(\gamma\) with the measured ones would offer an excellent test of the standard model.

It is useful to have as many different tests of the CKM mechanism for quark mixing and CP violation as possible, since that allows many potential new physics effects to be explored and the features of new physics to be identified. The nine new unitarity triangles complement the conventional ones, providing more ways of testing the CKM mechanism and probing new physics. These can be done by testing each of the individual unitarity relations represented as a unitarity triangle alone and by comparing the areas of the unitarity triangles with one another which are all equal in the standard model. These consistency checks may give us some clue to develop some principles which might enable us to understand quark mixing and CP violation beyond the phenomenological level.

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