ZARISKI PAIRS ON SEXTICS II

MUTSUO OKA

Dedicated to Professor Kyoji Saito for his 60th birthday

1. Introduction

We continue to study Zariski pairs in sextics. In this paper, we study Zariski pairs of sextics which are not irreducible. The idea of the construction of Zariski partner sextic for reducible cases is quite different from the irreducible case. It is crucial to take the geometry of the components and their mutual intersection data into account. When there is a line component, flex geometry (i.e., linear geometry) is concerned to the geometry of sextics of torus type and non-torus type. When there is no linear components, the geometry is more difficult to distinguish sextics of torus type. For this reason, we introduce the notion of conical flexes.

We have observed in [9] that the case \( \rho(C, 5) = 6 \) is critical in the sense that the Alexander polynomial \( \Delta_C(t) \) can be either trivial or non-trivial for sextics. If \( \rho(C, 5) > 6 \) (resp. \( \rho(C, 5) < 6 \)), the Alexander polynomial is not trivial (resp. trivial) ([9]). For the definition of \( \rho(C, 5) \)-invariant, see [9]. Thus we concentrate ourselves in this paper the case \( \rho(C, 5) = 6 \). In [10], we have classified the possible configurations for reducible sextics of torus type. In particular, the configurations with \( \rho(C, 5) = 6 \) are given as in Theorem 1 below. Hereafter we use the same notations as [9] for denoting component types. For example, \( C = B_1 + B_5 \) implies that \( C \) has a linear component \( B_1 \) and a quintic component \( B_5 \). We denote the configuration of the singularities of \( C \) by \( \Sigma(C) \).

**Theorem 1.** ([10]) Assume that \( C \) is a reducible sextic of torus type with \( \rho(C, 5) = 6 \) and only simple singularities. Let \( \Sigma_{in} \) be the inner singularities. Then the possible configurations of simple singularities are as follows.

1. \( \Sigma_{in} = [A_5, 4A_2] : C = B_5 + B_1 \) and \( \Sigma(C) = [A_5, 4A_2, 2A_1], [A_5, 4A_2, 3A_1], [A_5, 4A_2, 4A_1] \).
2. \( \Sigma_{in} = [2A_5, 2A_2] \):
   a. \( C = B_1 + B_5 \) : \( \Sigma(C) = [2A_5, 2A_2, 2A_1], [2A_5, 2A_2, 3A_1] \).
   b. \( C = B_1 + B_1' + B_4 \) : \( \Sigma(C) = [2A_5, 2A_2, 3A_1], [2A_5, 2A_2, 4A_1] \).
   c. \( C = B_2 + B_4 \) : \( \Sigma(C) = [2A_5, 2A_2, 2A_1], [2A_5, 2A_2, 3A_1] \).
   d. \( C = B_3 + B_3' \) : \( \Sigma(C) = [2A_5, 2A_2, 3A_1] \).
3. \( \Sigma_{in} = [E_6, A_5, 2A_2] \) : \( C = B_1 + B_5 \), \( \Sigma(C) = [E_6, A_5, 2A_2, 2A_1], [E_6, A_5, 2A_2, 3A_1] \).
4. \( \Sigma_{in} = [3A_5] \).

Key words and phrases. torus type, Zariski pair, flex points, conical flex points.
In this paper, we only studied possible Zariski pairs of reducible sextics

(b) \( C = B_2 + B_4: \Sigma(C) = [3A_5, 2A_1] \).

d) \( C = B_3 + B_3: \Sigma(C) = [3A_5, 3A_1, 3A_5, 2A_1] \).

e) \( C = B_1 + B_2 + B_3: \Sigma(C) = [3A_5, 2A_1, 3A_5, 3A_1] \).

(f) \( C = B_1 + B'_1 + B''_1 + B_3: \Sigma(C) = [3A_5, 3A_1, 3A_5, 4A_1] \).

g) \( C = B_2 + B'_2 + B''_2: \Sigma(C) = [3A_5, 3A_1] \).

\( \Sigma_{in} = [2A_5, E_6] \):
(a) \( C = B_1 + B_5: \Sigma(C) = [E_6, 2A_5, 2A_1] \).
(b) \( C = B_2 + B_4: \Sigma(C) = [E_6, 2A_5, 2A_1] \).
(c) \( C = B_1 + B'_1 + B_4: \Sigma(C) = [E_6, 2A_5, 3A_1] \).

\( \Sigma_{in} = [A_8, A_5, A_2]: C = B_1 + B_5, \Sigma(C) = [A_8, A_5, A_2, 2A_1], [A_8, A_5, A_2, 3A_1] \).

\( \Sigma_{in} = [A_{11}, 2A_2]: \)
(a) \( C = B_2 + B_4: \Sigma(C) = [A_{11}, 2A_2, 2A_1], [A_{11}, 2A_2, 3A_1] \).
(b) \( C = B_3 + B'_3: \Sigma(C) = [A_{11}, 2A_2, 3A_1] \).

\( \Sigma_{in} = [A_{11}, A_5]: \)
(a) \( C = B_1 + B_5: \Sigma(C) = [A_{11}, A_5, 2A_1] \).
(b) \( C = B_2 + B_4: \Sigma(C) = [A_{11}, A_5, 2A_1] \).
(c) \( C = B_3 + B'_3: \Sigma(C) = [A_{11}, A_5, [A_{11}, A_5, A_1], [A_{11}, A_5, 2A_1] \).

(2) \( \Sigma(C) = [2A_5, 2A_2, 4A_1] \) with \( C = B_1 + B'_1 + B_4 \).

(3) \( \Sigma(C) = [E_6, A_5, 2A_2, 3A_1] \) with \( C = B_5 + B_1 \).

(4) \( \Sigma(C) = [3A_5, 4A_1] \) with \( C = B_3 + B_1 + B'_1 + B''_1 \).

(5) \( \Sigma(C) = [E_6, 2A_5, 3A_1] \) with \( C = B_4 + B_1 + B'_1 \).

Our main result in this paper is:

**Theorem 2.** There are Zariski partner sextics with the above configurations with the following exceptions:

1. \( \Sigma(C) = [A_5, 4A_2, 4A_1] \) with \( C = B_5 + B_1 \).
2. \( \Sigma(C) = [2A_5, 2A_2, 4A_1] \) with \( C = B_1 + B'_1 + B_4 \).
3. \( \Sigma(C) = [E_6, A_5, 2A_2, 3A_1] \) with \( C = B_5 + B_1 \).
4. \( \Sigma(C) = [3A_5, 4A_1] \) with \( C = B_3 + B_1 + B'_1 + B''_1 \).
5. \( \Sigma(C) = [E_6, 2A_5, 3A_1] \) with \( C = B_4 + B_1 + B'_1 \).

The non-existence of sextics of non-torus type with the above exceptional configurations will be explained by flex geometry. The existence will be also explained by the flex geometry for those which has a line components and by conical flex geometry for the component type \( B_4 + B_2, B_2 + B'_2 + B''_2 \).

**Remark 3.** (1) In the list of Theorem [2] there are certainly several cases which are already known. For example, the configuration \( C = B_3 + B'_3 \) with one singularity \( A_{17} \) is given by Artal [17].

(2) In this paper, we only studied possible Zariski pairs of reducible sextics \( (C, C') \) where \( C \) is of torus type and \( C' \) is not of torus type. On the other hand, the possibility of Zariski pairs among reducible sextics of the same class is not discussed here. Several examples are known.
among reducible sextics of non-torus type. For such cases, Alexander polynomials can not
distinguish the difference. See papers [2, 5, 4, 3].

2. REDUCIBLE Sextics of non-torus type

To compute explicit polynomials defining reducible sextics, it is not usually easy to look
for special degenerations into several irreducible components starting from the generic sextics
\[ \sum_{i+j \leq 6} a_{ij} x^i y^j \]. Recall that we have classified all possible reducible simple configurations in
[10] and it is easier to start from a fixed reducible decomposition. In fact, the geometry of the
configuration of a reducible sextic depends very much on the geometry of each components.

A smooth point \( P \in \mathbb{C}^2 \) is called a flex point if the intersection multiplicity of the tangent line
and \( C \) at \( P \) is strictly greater than 2. First we recall the following fact for flex points ([6, 8]).

**Lemma 4.** Let \( C : F(X, Y, Z) = 0 \) be an irreducible plane curve of degree \( n \) with singularities
\( \{P_1, \ldots, P_k\} \). Then the number of flexes \( \iota(C) \) is given by
\[
\iota(C) = 3n(n-2) - \sum_{i=1}^k \varepsilon(P_i; C)
\]
where the second term \( \varepsilon(P_i; C) \) is the flex defect and given by the local intersection number of
\( C \) and the hessian curve of \( C \) at \( P_i \).

Generic flex defect of simple singularities we use are
\[
\varepsilon(A_1) = 6, \quad \varepsilon(A_2) = 8, \quad \varepsilon(A_{3r-1}) = 9r, \quad (r \geq 2), \quad \varepsilon(E_6) = 22
\]

Recall that flex points of a curve are described by the hessian of the defining homogeneous
equation. When we have an affine equation \( C : f(x, y) = 0 \), flex points in \( \mathbb{C}^2 \) are described
by \( f(x, y) = \text{flex}_f(x, y) = 0 \) ([8]) where
\[
\text{flex}_f(x, y) := f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2
\]
This is an easy way to check flex points from the affine equation.

A sextic \( C \) is of \((2,3)\)-torus type if we can take a defining polynomial of the form \( f_2(x, y)^3 + f_3(x, y)^2 = 0 \) where degree \( f_j = j \). The intersections \( f_2 = f_3 = 0 \) are singular points of \( C \) and
we call them inner singularities. For a given sextic \( C \) of torus type whose singularities are
simple, the possible inner singularities are

\[
(\sharp) : \{A_2, A_5, A_8, A_{11}, A_{14}, A_{17}, E_6\}.
\]

A convenient criterion for \( C \) to be of torus type is the existence a certain conic \( C_2 \) such that
\( C_2 \cap C \subset \Sigma(C) \) (Tokunaga’s criterion [11], Lemma 3, [7]).

A sextic of torus type \( C \) is called of linear torus type if the conic polynomial \( f_2 \) can be
written as \( f_2(x, y) = \ell(x, y)^2 \) for some linear form \( \ell(x, y) \) ([9]). A sextic of linear torus type
can have only \( A_5, A_{11}, A_{17} \) as inner singularities and the location of these singularities are
colinear.

The proof of Theorem 2 is done by giving explicit examples. For the better understanding
of the existence or non-existence of the Zariski pairs, we divide the above configurations into
the following classes.
(1) $C$ has a quintic component. The corresponding component type is $B_5 + B_1$ and $B_1$ is a flex tangent line.

(2) $C$ has a quartic component. There are two subcases.
   (a) $C = B_1 + B_1 + B_1'$. In this case, two line components are flex tangent lines.
   (b) $C = B_1 + B_2$.

(3) $C$ has a cubic component. There are two subcases.
   (a) Sextics of linear torus type.
   (b) Sextics, not of linear torus type.

(4) $C = B_2 + B'_2 + B''_2$.

3. Configuration coming from quintic flex geometry

Let $B_5$ be an irreducible quintic and let $P$ be a flex point of $B_5$. We denote the tangent line at $P$ by $L_P$. We say that $P$ is a flex of torus type (respectively a flex of non-torus type) if $B_5 \cup L_P$ is a sextic of torus type (resp. of non-torus type). The following configurations are mainly related to the flex geometry of certain quintics. (By 'flex geometry', we mean the geometry of the tangent lines at the flex points and the curve.) Recall that $\Sigma(B_5)$ is the configuration of the singularities of $B_5$. Let $\iota$ be the number of flex points on $B_5$.

(1) $C = B_5 + B_1$ with $\Sigma(C) = [A_5, 4A_2, kA_1]$, $k = 2, 3, 4$. Then $\Sigma(B_5) = [4A_2, (k - 2)A_1]$ for $k = 2, 3, 4$ and $\iota = 13, 7, 1$ respectively.

(2) $C = B_5 + B_1$ with $\Sigma(C) = [2A_5, 2A_2, kA_1]$, $k = 2, 3$. Then $\Sigma(B_5) = [A_5, 2A_2, (k - 2)A_1]$ and $\iota = 11, 5$ respectively.

(3) $C = B_5 + B_1$ with $\Sigma(C) = [E_6, A_5, 2A_2, kA_1]$, $k = 2, 3$. Then $\Sigma(B_5) = [E_6, 2A_2, (k - 2)A_1]$ and $\iota = 7, 1$ respectively for $k = 2, 3$. The case $k = 3$ corresponds to sextics of torus type.

(4) $C = B_5 + B_1$ with $\Sigma(C) = [E_6, 2A_5, 2A_1]$. The quintic $B_5$ has $\Sigma(C) = [E_6, A_5]$ and $\iota = 5$.

(5) $C = B_5 + B_1$ with $\Sigma(C) = [3A_5, 2A_1]$. Then $\Sigma(B_5) = [2A_5]$ and $\iota = 9$.

(6) $C = B_5 + B_1$ with $\Sigma(C) = [A_8, A_5, A_2, kA_1]$, $k = 2, 3$. Then $\Sigma(B_5) = [A_8, A_2, (k - 2)A_1]$ and $\iota = 10, 4$ for $k = 2, 3$.

(7) $C = B_5 + B_1$ with $\Sigma(C) = [A_{11}, A_5, 2A_1]$. Then $\Sigma(B_5) = [A_{11}]$ and $\iota = 9$.

We are going to show the stronger assertion for the above configurations: the Zariski partner sextic of non-torus type are simply given by replacing the flex line components of the above cases, if $B_5$ has at least two flex points.

Let $\Xi$ be a configuration of singularities on $B_5$, which is one of the above list. Let $\mathcal{M}(\Xi; 5)$ be the configuration space of quintics $B_5$ such that $\Sigma(B_5) = \Xi$. We consider it as a topological subspace of the space of quintics. For our purpose, it is enough to consider the marked configuration subspace $\mathcal{M}(\Xi; 5)'$ which consists of the pair $(B_5, P)$, where $B_5 \in \mathcal{M}(\Xi; 5)$ and $P$ is a flex point of torus type. The following describes the existence of sextics of non-torus type with the above configurations.
Theorem 5. Let $\Xi$ be a configuration of singularities on $B_5$, which is one of the above list. 
1. The configuration subspace $M(\Xi; 5)'$ is connected for each $\Xi$.
2. For each $\Xi \neq [4A_2, 2A_1]$, $[E_6, 2A_2, A_1]$ and $B_5 \in M(\Xi; 5)'$, a Zariski pair sextics are given as $\{B_5 \cup L_P, B_5 \cup L_Q\}$ where $P$ and $Q$ are flex points of torus-type and of non-torus type respectively.
3. For these two exceptional cases, we have the equality $M(\Xi; 5)' = M(\Xi; 5)$ and a quintic $B_5 \in M(\Xi; 5)$ does not contain any flexes of non-torus type.

Remark 6. Let $\iota_t$, $\iota_{nt}$ be the respective number of flex points of torus type and of non-torus type on a generic $B_5 \in M(\Xi; 5)'$. We do not need the precise number $\iota_t, \iota_{nt}$ for our purpose. The sum $\iota = \iota_t + \iota_{nt}$ is described by Lemma 3. By an explicit computation, we have the next table which describes the distributions of number of flex points. The second line is the configuration of singularity and the last line is the pair of flex numbers $(\iota_t, \iota_{nt})$.

|    | 2    | 3    | 4    | 5    | 6    | 7    |
|----|------|------|------|------|------|------|
| $4A_2$ | $A_5 + 2A_2$ | $E_6 + 2A_2$ | $E_6 + A_5$ | $2A_5$ | $A_8 + A_2 + A_1$ | $A_{11}$ |
| $4A_2 + A_1$ | $A_5 + 2A_2 + A_1$ | $E_6 + 2A_2 + A_1$ | $E_6 + A_5$ | $2A_5$ | $A_8 + A_2 + A_1$ | $A_{11}$ |
| $(1,12)$ | $(1,10)$ | $(1,6)$ | $(1,4)$ | $(1,8)$ | $(1,9)$ | $(1,8)$ |
| $(1,6)$ | $(1,4)$ | $(1,0)$ | $(1,0)$ | $0$ | $(1,3)$ | $0$ |

Proof. First recall that the topology of the complement of the sextics $B_5 \cup L_P$ for a flex point $P$ of torus type and non-torus type are different. They can be distinguished by Alexander polynomial (9). Therefore to show the assertion about the positivity $\iota_{nt} > 0$, it is enough to check the assertion by some quintic $B_5$. Examples will be given in the next subsection. Secondly, the irreducibility of the configuration space $M(\Xi; 5)$ of quintics $f_5(x, y) = 0$ with singularities $\Xi = [4A_2, 2A_1], [E_6, 2A_2, A_1]$ are easily proves as follows. For $\Xi = [4A_2, 2A_1]$, the dual curves of quintics in this configuration space are quartics with configuration $[A_2, 2A_1]$. As the irreducibility of the configuration space $M([A_2, 2A_1]; 4)$ is easy to be checked, the irreducibility of $M([4A_2, 2A_1]; 5)$ follows. Take $\Xi = [E_6, 2A_2, A_1]$. For a quintic $B_5$ with $\Sigma(B_5) = \Xi$, the dual curve $B_5^*$ is again a quartic with $[A_2; 2A_1]$ (thus mapped into the same configuration space with the dual of quintics with $[4A_2, 2A_1]$). Hoewer we can not apply the same argument. The reason is that the dual curve $B_5^*$ is not generic in the configuration space $M([A_2, 2A_1]; 4)$: the quartic $B_5^*$ has not 4 flexes but three flexes, one flex of flex order 4 (=dual of $E_6$) and 2 flexes of flex order 3 (i.e., dual of $2A_2$). Thus we need another argument. Note that any three singular points can not be colinear on $B_5$ by Bézout theorem. We can consider the slice condition:

$(\ast)$: $E_6$ is at $(-1, 0)$ and two $A_2$ are at $(0, 1), (0, -1)$ and one $A_1$ at $(1, 0)$.  


It is easy to compute that a Zariski open subset of this slice has the normal form:

\[ h := e_1^3 - 4 y^3 x^2 - 2 e_1^3 x^3 + x^4 e_1^3 + x e_1^3 + 3 y^5 e_1^2 + e_1^3 y^4 + 3 y e_1^2 + 10 e_1^3 y^2 x^3 
+ 18 e_1^3 y^2 x^2 - 12 e_1 y^2 x^2 + 6 e_1^3 y^2 x - 12 e_1 y^2 x - 9 y x^4 e_1^2 - 12 y e_1^2 x^3 
+ 12 y x e_1^2 + 6 y x^2 e_1^2 - 6 y^3 x^2 e_1^2 - 6 y^3 e_1^2 - 2 e_1^3 y^2 + e_1^3 x^5 - 7 e_1^3 y^4 x 
+ 12 e_1 y^4 x - 12 y^3 x e_1^2 - 2 x^2 e_1^2 + 8 y^3 - 4 y - 4 y^5 - 8 y x + 8 y x^3 + 4 y x^4 
+ 8 y^3 x, \quad e_1 \neq 0, \pm 2/\sqrt{3} \]

Thus the irreducibility of \( M([E_6, 2A_2, A_1]; 5) \) follows. For each of them we know that the number of flex points is 1 and \( M(\Xi; 5)' \neq \emptyset \). On the other hand, the topology of sextics \( B_5 \cup L_P \), of torus type and non-torus type, are distinguished by the Alexander polynomials \((t^2 - t + 1)(t - 1)\) and \((t - 1)\) respectively. This implies \( M(\Xi; 5) = M(\Xi; 5) \).

\[ \square \]

3.1. Example for sextics with quintic components. We gives examples of sextics with a quintic components.

1. \( C = B_5 + B_1, \Sigma = [A_5, 4A_2, k A_1], k = 2, 3 \): First we consider the case \( k = 2 \). The quintic has \( 4A_2 \) and 13 flex points.

\[
C : \left( \frac{28}{153} x + \frac{8}{153} \right) y^4 + \left( \frac{52}{51} x^3 - \frac{20}{51} x^2 - \frac{152}{153} x - \frac{16}{153} \right) y^2 \right) \right) x^3 + \frac{2}{17} x^4 - \frac{193}{51} x^3 
+ \frac{116}{51} x^2 + \frac{124}{153} x + \frac{8}{153} \left( \frac{80}{17} x + \frac{880}{51} - \frac{320}{51} y \right) \]

The sextics of torus type is obtained by replacing the line component by \( \frac{28}{153} x + \frac{8}{153} \).

Next, we consider the case \( k = 3 \). Th equintic \( B_5 \) has \( 4A_2 + A_1 \). The flex which gives a sextic of torus type is \((1, 0)\).

\[
f_5 := -\frac{385}{16} x^4 y + \frac{3885}{16} x y^2 + \frac{1897}{128} x y^4 + \frac{345}{16} y^3 x - \frac{441}{4} y^3 x + \frac{529}{4} y x^2 + 73 y + 72 y^3 
+ \frac{403}{128} x^3 y^2 - \frac{16783}{128} x^2 y^2 - \frac{811}{64} x y - \frac{869}{64} y^4 + \frac{7087}{128} x - \frac{3675}{32} y^2 + \frac{3201}{512} x^5 
+ \frac{601}{32} x^4 - y^5 + \frac{313}{8} x^2 y^3 - \frac{11511}{256} x^2 - \frac{997}{64} - \frac{10167}{512} x^3 \]

\( B_5 \) has two obvious flex points: \( P := (1, 0) \) and \( Q := (-1520/293, -287/293) \), where \( P \) is a flex of torus type and \( Q \) is a flex of non-torus type. There are 5 other flex points whose \( x \)-coordinates are the solution of

\[
R_1 := 5926214587003 x^5 - 32698277751050 x^4 + 69779834665700 x^3 
- 7291858361100 x^2 + 37638730560000 x - 7728486400000 = 0 \]

We can check that the roots of \( R_1 = 0 \) corresponds to flexes of non-torus type as follows. (The same argument applies to other cases.) Note that any conics which is passing through \( 4 A_2 \) of \( B_5 \) are given by

\[
h_2 := y^2 - \frac{1}{2} d_{01} y x + d_{01} y - \frac{1}{2} x^2 + \frac{1}{19} x^2 d_{01} - \frac{5}{2} x - \frac{10}{19} d_{01} x + 3 + \frac{16}{19} d_{01} \]
Thus if there is a flex $P(a,b)$ of torus type (so $R_1(a) = 0$), there is a cubic form $h_3(x,y)$ such that the sextic $C = B_5 \cup L_P$ is described as $C := \{h_3^2 + h_3 = 0\}$. On the other hand, put $S_2(x,d_01)$ be the polynomial of degree 2 in $x$ defined by $S_2(x,d_01) = R(h_2,f_5,y)/P(x)^2$ where $R(h_2,f_5,y)$ is the resultant of $h_2$ and $f_5$ in $y$ and $P(x) = 0$ is the defining polynomial for the $x$-coordinates of 4 $A_2$. Then $S_2$ must be $c(x - a)^2$ for some $c \neq 0$. Let $b_1(d_01)$ be the discriminant polynomial of $S_2$ in $x$ and let $b_2(d_01)$ be the resultant of $S_2(x,d_01)$ and $R_1(x)$ in $x$. Thus we obtain two polynomials $b_1(d_01)$, $b_2(d_01)$ of the parameter $d_01$ which must have a common root: We can check that $b_1(d_01) = b_2(d_01) = 0$ has no common root in $d_01$.

2. $C = B_5 + B_1$, $\Sigma = [2A_5,2A_2,j A_1]$, $j = 2, 3$. The quintic $B_5$ has $A_5 + 2A_2$.

$$j = 2, [2A_5,2A_2,2A_1]: \left(-\frac{16145}{1024}y^3 - \frac{93}{64}y^5 - \frac{877727}{8192}y^2 - \frac{110055}{1024}y - \frac{329525}{16384}x\right)$$
$$+ 11025x^3 - \frac{543975}{16384}x^2 + \frac{1751733}{16384}y^2x^2 + \frac{79625}{4096}x^5 - \frac{235529}{16384}y^4 + \frac{1999101}{16384}y^2x^3$$
$$- \frac{100809}{8192}y^3x + \frac{1199495}{16384}y^3x^2 + \frac{289995}{4096}y^4x^2 - \frac{1199495}{16384}yx + \frac{150225}{4096}y^4x$$
$$- \frac{498845}{4096}y^2x - \frac{275703}{16384}y^4x + \frac{23889}{8192}y^3x^2 + \frac{18625}{512}x^4 - \frac{52025}{16384}(y + 1)$$

A sextic of torus type is give by replacing the line component by $x - 1 = 0$.

$$j = 3, [2A_5,2A_2,3A_1]: \left(\frac{2}{7} + \frac{5}{7}x^5 - \frac{4}{7}x^2 - 2x^3 + \frac{2}{7}x^4 + x - \frac{2}{7}y^2x^3 + \frac{12}{7}x^2y^2 - \frac{1}{7}xy^4\right)$$
$$- \frac{6}{7}y^2x - \frac{4}{7}y^2 + \frac{2}{7}y^4\right)\left(\frac{44064}{34157767}y\sqrt{-963 + 1182\sqrt{6}} + \frac{16521840}{34157767}\frac{7198560}{34157767}\sqrt{6}\right)$$
$$- \frac{28320}{34157767}y\sqrt{-963 + 1182\sqrt{6}} + \frac{7328592}{34157767}\sqrt{6}x - \frac{2704104}{4879681}x$$

The quintic $B_5$ has $A_5 + 2A_2 + A_1$ and 5 flex points and among them, there exists a unique flex of torus type. The tangent line at this flex of torus type is given by $2 - x = 0$.

3. A sextic $C = B_5 + B_1$ with $\Sigma(C) = [E_6, A_5,2A_2,2A_1]$ is given by

$$f := \left(\frac{53}{141}x + \frac{3}{47}y + y^5 - \frac{50}{47}y^3 + \frac{4}{47}y^2x - \frac{769}{141}y^4x - \frac{614}{141}y^2x^3 + 2y^2 + \frac{53}{141}x^5 + \frac{56}{141}y^x\right)$$
$$- \frac{10}{3}yx + \frac{1174}{141}y^3x + \frac{1256}{141}y^3x^2 + \frac{10}{3}x^3y - \frac{69}{47}y^4 - \frac{25}{47}x^4 + \frac{50}{47}x^2 - \frac{106}{141}x^3$$
$$- \frac{1462}{141}y^2x^2 - \frac{65}{141}yx^4 - \frac{25}{47}(y + 1 - \frac{8}{3}x)$$

The quintic has 7 flex points and there is a unique one among them which is of torus type at $\left(\frac{2400}{1357}, \frac{357}{1357}\right)$. 
4. $C = B_5 + B_1$ with $[E_6, 2A_5, 2A_1]$. The quintic $B_5$ has $E_6 + A_5$ and it has 5 flexes. Among them, there is a unique flex of torus type. A sextic of non-torus type:

$$f := (4451 + 9742 y^2 x + 4639 y^4 x - 9501 y - 14381 x - 423 y^5 \sqrt{33} - 351 \sqrt{33}$$
$$- 16343 x^3 + 6546 y^3 - 8005 y^4 + 3554 y^2 + 19373 x^2 - 19373 y^2 x^2 + 9836 x^4$$
$$+ 2955 y^5 + 10266 y x^3 - 2936 x^5 - 19020 y^3 x + 19020 y x - 14661 y x^2$$
$$+ 14661 y^3 x^2 + 1521 \sqrt{33} y - 1098 y^3 \sqrt{33} + 1593 y^4 \sqrt{33} + 756 x^4 \sqrt{33}$$
$$+ 1215 x^2 - 1242 y^2 \sqrt{33} - 1917 x^3 \sqrt{33} + 297 x \sqrt{33} + 999 y^2 x^3 \sqrt{33}$$
$$+ 36 y^4 \sqrt{33} + 1458 y^2 x \sqrt{33} + 918 y x^3 \sqrt{33} + 2052 y^3 x \sqrt{33} - 423 y x^2 \sqrt{33}$$
$$- 2052 y x \sqrt{33} + 423 y^3 x^2 \sqrt{33} - 1215 y^2 x^2 \sqrt{33} - 1755 y^4 x \sqrt{33} + 6077 y^2 x^3$$
$$- 5124 y x^4)(y + 1)$$

A sextic torus type is given by replacing $y + 1$ by the flex tangent at $(\alpha, \beta)$ where

$$\alpha := \frac{1476423}{6805087} + \frac{176748}{6805087} \sqrt{33}, \quad \beta := \frac{1469468}{6805087} - \frac{931392}{6805087} \sqrt{33}$$

5. $C = B_5 + B_1$ with $[3A_5, 2A_1]$.

$$- \frac{12}{6279955}(-2516 + 27 \sqrt{69})(-2516 - 10064 x^4 - 45 x y^4 \sqrt{69} + 90 x y^2 \sqrt{69}$$
$$+ 108 \sqrt{69} y^3 x^2 - 108 \sqrt{69} y x^2 + 180 x^5 \sqrt{69} + 54 y^2 \sqrt{69} - 45 x \sqrt{69} - 27 y^4 \sqrt{69}$$
$$- 108 x^4 \sqrt{69} - 27 \sqrt{69} + 10060 y x^4 - 5030 y^3 - 10064 y x^2 - 2516 y^4$$
$$+ 10064 y^3 x^2 + 2515 y^5 + 10060 x^2 - 840 x y^4 + 1680 x y^2 + 5032 y^2 - 10060 y^2 x^2$$
$$+ 3360 x^5 + 2515 y - 840 x)$$

$$\left(\frac{31104}{2497} - \frac{1620}{2497} \sqrt{69}\right) y + \frac{108}{2497} (52 \sqrt{69} - 499) (x - 1)$$

The flex point which gives a sextic of torus type is

$$(\alpha, \beta), \quad \alpha := \frac{957138004}{22902646825} + \frac{2339358408}{22902646825} \sqrt{69}, \quad \beta := \frac{540908244}{916105873} - \frac{52210443}{916105873} \sqrt{69}$$

6. a. A sextic of torus type with $[A_8, A_5, A_2, 2A_1]$ with line component is given by:

$$f := (-60 y^2 + 60 y - x^2)^3$$
$$+ \left( - \frac{81}{25} y^3 + (- \frac{6849}{25} x - \frac{9639}{25}) y^2 + \left( \frac{162}{25} x^2 + \frac{6849}{25} x + \frac{1944}{5} \right) y + x^3 - \frac{162}{25} x^2 \right)^2$$
and the line component is defined by $y - 1 = 0$. It has 10 flexes and the flex at $(0,1)$ gives a flex of torus type. Other flex tangent lines give a sextic of non-torus type. For example

$$f_6 := \frac{324}{25}x^5 - \frac{26244}{625}x^4 - \frac{2223126}{625}x^3 + \frac{40132557}{625}y^2x^2 - \frac{428706}{625}yx^4 - \frac{43281837}{625}y^2x^2 - \frac{134993439}{625}y^5 + \frac{271568079}{625}y^4 - \frac{8419248}{125}y^3 - \frac{3779136}{25}y^2 + \frac{629856}{125}y^2x^2 + \frac{1733076}{625}y^3x + \frac{132035022}{625}y^3x + \frac{1109538}{625}y^4x$$

b. A sextic of torus type with $[A_8, A_5, A_2, 3A_1]$ and with component type $B_5 + B_1$ is given by

$$f := \left(\frac{15}{2}y^2 - \frac{15}{2}y - 16x^2\right)^3 + \left(\frac{455}{24}y^3 + \left(-\frac{80}{3}x + \frac{245}{6}\right)y^2 + \left(\frac{140}{3}x^2 + \frac{80}{3}x - \frac{175}{8}\right)y + 64x^3 - \frac{140}{3}x^2\right)^2$$

It has the line component $y - 1 = 0$ and the quintic has 4 flex points among which only the flex $(0,1)$ is of torus type. An example of sextic of non-torus type is given by

$$f_6 := \left(\frac{17920}{3}x^5 - \frac{19600}{9}x^4 + \frac{3500}{3}y^2x - \frac{6125}{3}yx^2 + \frac{47600}{9}x^3y - \frac{30625}{64}y^2 + \frac{332125}{192}y^3 - \frac{11275}{3}y^3x^2 + 5800y^2x^2 - \frac{19600}{9}xy^3 + \frac{9100}{9}xy^4 - \frac{44240}{9}x^3y^2 - \frac{40720}{9}y^4x\right)$$

$$f_6 := \left(-\frac{1170775}{576}y^4 + \frac{450025}{576}y^5\right)\left(-\frac{2222000000}{255584169}y + \frac{2156000000}{255584169}y + \frac{492800000}{85194723}x\right)$$

7. A quintic with $A_{11}$ has 9 flex points, among which there exists a unique flex of torus type. In the following example, our quintic has a flex of torus type at $(0,1)$ so the the sextic of torus type is given by

$$f := \left(-\frac{28}{25}y^2 + \frac{28}{25}y - \frac{25}{x^2}\right)^3 + \left(\frac{511}{100}y^3 + \left(-\frac{28}{25}x - \frac{7}{250}\right)y^2 + \left(-\frac{91}{20}x^2 + \frac{28}{25}x - \frac{637}{125}\right)y - \frac{x^3 + \frac{91}{20}x^2}{2}\right)^2$$

and the line component is $y - 1 = 0$. $B_5$ has 8 flex of non-torus type. We can take one at $(1,-1)$ so that a sextic of non-torus type is given by

$$f_6 := \left(\frac{6176793}{2500000}y^5 - \frac{7154}{625}y^4x + \frac{7194719}{250000}y^4 - \frac{245049}{5000}y^3x^2 - \frac{859901}{3125}y^3 + \frac{98}{3125}y^3x\right)$$

$$+ \frac{35672}{3125}y^2x - \frac{405769}{15625}y^2 + \frac{13181}{5000}y^2x^2 - \frac{7}{250}y^2x^2 - \frac{2548}{125}y^3x^3 + \frac{57967}{1250}y^3x^2$$

$$+ \frac{7833}{400}y^4x - \frac{8281}{400}x^4 + \frac{91}{10}x^5\right)\left(-\frac{3306744}{15625}y + \frac{1928934}{15625}y + \frac{275562}{3125}x\right)$$
4. Configuration coming from quartic geometry

4.1. Configuration coming from quartic flex geometry. We consider the sextics $C$ with component type $B_4 + B_1 + B_1'$. The corresponding possible configurations are

(a) $\Sigma(C) = [3A_5 + 3A_1]$ and $\Sigma(B_4) = [A_5]$ or (b) $\Sigma(C) = [2A_5 + 2A_2 + kA_1]$, $k = 3, 4$ and $\Sigma(B_4) = [2A_2 + (k - 3)A_1]$ or (c) $\Sigma(C) = [E_6 + 2A_5 + 3A_1]$ and $\Sigma(B_4) = [E_6]$.

Let $P, Q$ be two flex points on $B_4$ and let $L_P, L_Q$ be the flex tangents. We say that a pair of flex points $\{P, Q\}$ are a flex pair of torus type if the sextic $B_4 \cup L_P \cup L_Q$ is a sextic of torus type.

Theorem 7. Case (a) $\Sigma(C) = [3A_5 + 3A_1]$. The quartic $B_4$ has one $A_5$ and 6 flex points and two line components are flex tangent lines. There exist two flex pairs of torus type. The other choices give sextics of non-torus type.

Case (b) $\Sigma(C) = [2A_5 + 2A_2 + kA_1]$, $k = 3, 4$. The quartic $B_4$ has $2A_2$ or $2A_2 + A_1$ according to $k = 3$ or 4 and $B_4$ has 8 or 2 flex points respectively. For the case, $k = 3$, there are both flex pairs of torus type and of non-torus type. For $k = 4$, the choice of $\{P, Q\}$ is unique and it is a pair of torus type.

Case (c) $\Sigma(C) = [E_6 + 2A_5 + 3A_1]$. $B_4$ has two flexes and they gives a pair of torus type. Thus there is no sextic of non-torus type with $E_6 + 2A_5 + 3A_2$ with component type $B_4 + B_1 + B_1'$.

Proof. As the configuration spaces of quartics with one $A_5$, or $2A_2$ or $2A_2 + A_1$ or $E_6$ are connected, it is enough to check the assertion by an example.

For the non-existence, note that a quartic with $\Sigma(B_4) = 2A_2 + A_1$ or $\Sigma(B_4) = [E_6]$ has exactly 2 flexes. Thus the existence of sextic of torus type $B_4 + B_1 + B_1'$ with $[2A_2 + 2A_2 + 4A_1]$ or $[E_6 + 2A_5 + 3A_1]$ implies that there does not exist sextic of non-torus type with these two configurations $\square$

Example I. We consider the quartic $B_4 := \{g_4 = 0\}$ with one $A_5$:

\[
g_4 := 639 - 1350x^2 + 351x^4 + 468x^3 - 108x + 288y + 1608Iy^2x^2\sqrt{3} + 1452Iy^3x\sqrt{3} + 676Iy^3x\sqrt{3} - 1032Iy^2x\sqrt{3} - 1452Iy^3x\sqrt{3} - 288y^3 - 918y^2 + 279y^4
\]
\[
- 648y^3x + 1350y^2x^2 + 108y^2x - 936y^3x^2 + 648yx - 432y^2x\sqrt{3} + 776Iy^3x\sqrt{3} - 1608Iy^2x\sqrt{3} - 152Iy^2x\sqrt{3} - 776Iy^3x\sqrt{3} + 1032Iy^2x\sqrt{3} + 584Iy^3x\sqrt{3} + 728Iy^3x\sqrt{3}
\]
$B_4$ has an $A_5$ singularity at $(1,0)$ and 6 flexes at

$$P_1 = (0, -1), P_2 = (0, 1), P_3 = (-\frac{130}{1069} + \frac{370}{1069} I\sqrt{3}, \frac{263}{1069} + \frac{156}{1069} I\sqrt{3})$$

$$P_4 := \left(\frac{2110}{13333} - \frac{4790}{39999} I\sqrt{3}, -\frac{12671}{13333} - \frac{4492}{39999} I\sqrt{3}\right),$$

$$P_5 = \left(\frac{2116}{6841} I\sqrt{3} + \frac{632}{20160427} \sqrt{90988874 - 65462588 I\sqrt{3}} + \frac{4086}{6841}\right)$$

$$+ \frac{498}{20160427} I\sqrt{90988874 - 65462588 I\sqrt{3}},$$

$$- \frac{9056}{47887} - \frac{6133}{47887} I\sqrt{3} + \frac{4}{47887} \sqrt{90988874 - 65462588 I\sqrt{3}}$$

$$P_6 = \left(-\frac{632}{20160427} \sqrt{90988874 - 65462588 I\sqrt{3}} + \frac{4086}{6841}\right)$$

$$- \frac{498}{20160427} I\sqrt{90988874 - 65462588 I\sqrt{3}} + \frac{2116}{6841} I\sqrt{3},$$

$$- \frac{9056}{47887} - \frac{6133}{47887} I\sqrt{3} - \frac{4}{47887} \sqrt{90988874 - 65462588 I\sqrt{3}}$$

It is easy to check that $\{P_1, P_3\}, \{P_2, P_4\}, \{P_5, P_6\}$ give sextics of torus type. The other cases give sextics of non-torus type. For example, a nice sextic of non-torus type is given by taking tangent lines at $A$ and $B$. Sextics of torus type are given by taking tangent lines at $\{x, y\}$. For example, a sextic of torus type is given by adding two lines ($A$ and $B$ type).

Now we consider the quartic with $2A_2$ (case (b) with $k = 3$).

$$f_4(x, y) := \frac{254143}{4096} x^4 - \frac{251}{16} x^3 + \frac{11}{32} y^3 x^3 - \frac{5893}{2048} y^2 x^2 - \frac{2761}{1024} y x^2 - \frac{126093}{2048} y^2 - \frac{11}{32} x + \frac{1}{32} y^3 x$$

$$+ \frac{251}{16} x + \frac{5893}{2048} y^2 - \frac{1957}{4096} + \frac{211}{4096} y + \frac{2761}{1024} y - \frac{251}{1024} y^3$$

It has 8 flex points and four flexes are explicitly written as

$$P_1 = (1, 0), P_2 = (-1, 0), P_3 = (0, -1), P_4 = \left(\frac{16064}{64025}, \frac{61977}{64025}\right)$$

Sextics of torus type are given by taking tangent lines at $\{P_1, P_2\}, \{P_3, P_4\}$. As a sextic of non-torus type, we can take the tangent lines at $P_1, P_3$ so that the sextics is given by adding two lines ($x - 1)(-4y - 4 + 16x) = 0$. The configuration space of quartic with $2A_2 + A_1$ is connected. Each quartic has two flex points and with two tangent lines $B_1, B_1'$, $B_4 \cup B_1 \cup B_1'$ gives a sextic of torus type with $[2A_5, 2A_2, 4A_1]$. Thus there is no sextic of non-torus type $C = B_4 + B_1 + B_1'$ with configuration $[2A_5, 2A_2, 4A_1]$.

4.2. **Conical geometry of quartic.** Now we consider the configuration with component type $B_4 + B_2$ will be considered here. The corresponding configurations are

1. $\Sigma(C) = [2A_5, 2A_2, 2A_1], [2A_5, 2A_2, 3A_1]$.
2. $\Sigma(C) = [3A_5, 2A_1]$.
3. $\Sigma(C) = [E_6, 2A_5, 2A_1]$.
4. $\Sigma(C) = [A_{11}, 2A_2, 2A_1], [A_{11}, 2A_2, 3A_1]$.
5. $\Sigma(C) = [A_{11}, A_5, 2A_1]$. 
Zariski pairs with the above configurations with fixed component type $B_4 + B_2$ can not be explained by the flex geometry.

We have to generalize the notion of flex points. Let $B$ be a given irreducible plane curve of degree $d$. Let $\Phi$ be a linear system of conics and let $\alpha$ be the dimension of $\Phi$. For a general smooth point $P \in B$, the maximal intersection number of $I(B, B_2; P)$ for $B_2 \in \Phi$ is $\alpha$. We say $P$ is a conical flex point with respect to $\Phi$ if the intersection number $I(B, B_2; P) \geq \alpha + 1$. If $\dim \Phi = 5$ (so $\Phi$ is the family of all conics), we say simply that $P$ is a conical flex point.

1. Let us consider the case $\Sigma(C) = [2A_5, 2A_2, 2A_1], [2A_5, 2A_2, 3A_1]$. We consider first a sextic of torus type $C = \{f_2^3 + f_3^2 = 0\}$ which decomposes into a quartic $B_4$ and a quartic $B_2$:

$$f(x, y) := (y^2 - 2 + 2x^2)^3 +$$
$$(-25y^3 + (13x - 23)y^2 + (-26x^2 + 26)y + 13x^3 - 23x^2 - 13x + 23)^2$$
$$= (y^2 + x^2 - 1)(177x^4 - 598x^3 - 676yx^3 + 344x^2y^2 + 849x^2y^2 + 1196yx^2 - 650y^3x$$
$$+ 598x + 676yx - 598yx^2 - 521 - 151y^2 + 1150y^3 - 1196y + 626y^4)$$

Our quartic is defined by

$$g_4(x, y) = (177x^4 - 598x^3 - 676yx^3 + 344x^2y^2 + 849x^2y^2 + 1196yx^2 - 650y^3x$$
$$+ 598x + 676yx - 598yx^2 - 521 - 151y^2 + 1150y^3 - 1196y + 626y^4) = 0$$

Note that the singularities $B_4$ are $2A_2$. The intersection $B_2 \cap B_4$ makes two $A_5$ at $P := (1, 0)$ and $Q := (-1, 0)$. We consider the linear system $\Phi$ of conics of dimension 2 which are defined by the conics $C_2 := \{h_2(x, y) = 0\}$ such that $I(C_2, B_4; P) \geq 3$. Then we consider the conical flex points $R = (a, b) \in B_4$ with respect to $\Phi$, which is described by the condition $\exists h_2 \in \Phi$ such that $I(h_2, B_4; R) \geq 3$. We found that there are 11 conical flex points. Two of them can be explicitly given as $S_1 = Q$ and $S_2 := (0, -1)$. The corresponding conics are given as

$$h_2(x, y) = x^2 + y^2 - 1, \{h_2 = 0\} \cap B_4 \supset \{P, Q\}$$
$$g_2(x, y) = (y^2 + 2yx - 2y - x^2 + 4x - 3), \{g_2 = 0\} \cap B_4 \supset \{P, S_2\}$$

We can easily check that the sextic

$$(y^2 + 2yx - 2y - x^2 + 4x - 3)(177x^4 - 598x^3 - 676yx^3 + 344x^2y^2 + 849x^2y^2 + 1196yx^2 - 650y^3x$$
$$+ 1196yx^2 - 598yx^2 + 598x + 676yx - 598yx^2 - 521 - 151y^2 + 1150y^3$$
$$- 1196y + 626y^4) = 0$$

is not of torus type. Thus $B_4 \cup \{h_2(x, y) = 0\}$ and $B_4 \cup \{g_2(x, y) = 0\}$ is a Zariski pair.
Similarly the case \([2A_5, 2A_2, 3A_1]\) can be treated in the same way. We start from a sextic of torus type:

\[
f(x, y) = \left( -\frac{49}{64} y^2 - \frac{15}{64} + \frac{15}{64} x^2 \right)^3 + \left( -\frac{131}{256} y^3 + \frac{729}{512} x - \frac{297}{256} y^2 + \left( -\frac{387}{256} x^2 + \frac{387}{256} y + \frac{729}{512} x^3 - \frac{297}{256} x^2 - \frac{729}{512} x + \frac{297}{256} \right)^2 \right)
\]

\[
= \frac{27}{262144} (y^2 + x^2 - 1) (19808 x^4 - 41796 y x^3 - 32076 x^3 + 40521 y^2 x^2 - 6865 x^2 + 34056 x y^2 - 32076 x y^2 + 32076 x + 41796 x y - 14148 x y^3 - 7770 y^2 - 34056 y - 1815 y^4 + 11528 y^3 - 12943)
\]

The quartic \(B_4\) has two \(2A_2 + A_1\) and we consider the linear system \(\Phi\) of conics intersecting \(B_4\) at \(P = (1, 0)\) with intersection number 3. We find that there exist 5 conical flex points with respect to \(\Phi\), and among them we have two explicit ones: \(Q = (-1, 0)\) and \((0, -1)\). We see that the conic corresponding to \((0, -1)\) gives a Zariski partner sextic \(f_6 = 0\) to \(C = \{ f = 0 \}\).

\[
f_6 := (5 y^2 - 64 x y + 64 y + 69 x^2 - 128 x + 59) (19808 x^4 - 41796 y x^3 - 32076 x^3 + 40521 y^2 x^2 - 6865 x^2 + 34056 x y^2 - 32076 x y^2 + 32076 x + 41796 x y - 14148 x y^3 - 7770 y^2 - 34056 y - 1815 y^4 + 11528 y^3 - 12943)
\]

**Remark 8.** The calculation of conical flex points are usually very heavy. We used maple 7 to compute in the following recipe. a. First compute the normal form of \(h_2 \in \Phi\). It contains two parameters. b. Assume \((u, v) \in B_4\). Put \(gg_4(x, y) := (x + u, y + v)\) and \(hh_2(x, y) := h_2(x + u, y + v)\). Consider the maximal contact coordinate at \((u, v)\): \(\Phi(x) = a_1 x + a_2 x^2 + a_3 x^3\) and put \(GG_4(x) := gg_4(x, \Phi(x))\) and \(HH_2(x) := hh_2(x, \Phi(x))\). Our assumption implies that \(Coeff(GG_4, x, 1) = Coeff(GG_2, x, 2) = 0\) and \(Coeff(HH_2, x, 0) = Coeff(HH_2, x, 1) = Coeff(HH_2, x, 2) = 0\). Solve the equations \(Coeff(GG_4, x, 1) = Coeff(GG_2, x, 2) = 0\) in \(a_1, a_2\). Then solve the equations \(Coeff(HH_2, x, 0) = Coeff(HH_2, x, 1) = 0\) in the remaining parameters of the linear system. Then we get two equations in \(u, v\):

\[
g_4(u, v) = Coeff(HH_2, x, 2) = 0
\]

b. Use the resultant computation to solve the above equations to obtain the possibility of conical flex points.

c. Use the resultant computation to solve the above equations to obtain the possibility of conical flex points.

2. \(\Sigma(C) = [3A_5, 2A_1]\) with \(B_4 + B_2\):

\[
B_4 : g_4(x, y) = (6 y x + \frac{1710}{91} y x^2 - \frac{1466}{91} x y^3 - \frac{1992}{91} y^2 x^2 - 6 y^3 x + \frac{790}{91} y^3 y + \frac{4904}{91} x^3 + \frac{1992}{91})
\]

\[
- \frac{790}{91} y + \frac{939}{91} y^2 x^2 + \frac{1161}{91} y^2 + \frac{1698}{91} x^2 - \frac{1252}{91} + \frac{2196}{91} x^4
\]

It has an \(A_5\) at \(P := (1, 0)\). We consider the linear system \(\Phi\) of conics of dimension 2 whose conic are intersecting with \(B_4\) at \((0, 1)\) with intersection number 3. We find 14 conical flex points with respect to \(\Phi\) in which two are explicit: \(R = (\frac{498727}{900817}, \frac{266266}{900817})\) and \(Q = (0, -1)\).
The corresponding conics \( f_2 = 0, \ k_2 = 0 \) intersecting \( B_4 \) at \( P, Q \) or \( P, R \) are given by the following and they gives sextics of non-torus type and of torus type respectively.

\[
f_2(x, y) := y^2 - 1 + \frac{171}{79} x^2
\]

\[
k_2(x, y) = (y^2 + (\frac{268}{759} x + \frac{4424}{2277}) y + \frac{85291}{19987} x^2 - \frac{268}{759} x - \frac{6701}{2277})
\]

3. \( \Sigma(C) = [E_6, 2A_5, 2A_1] \): We start the next quartic

\[
B_4 : y^4 + (-\frac{195}{64} x + \frac{169}{64}) y^3 + (\frac{105}{32} x^2 - \frac{33}{8} x + \frac{27}{32}) y^2 + (-\frac{143}{64} x^3 + \frac{117}{64} x^2 + \frac{195}{64} x - \frac{169}{64}) y
\]

\[
+ \frac{45}{32} x^4 - \frac{19}{8} x^3 - \frac{21}{16} x^2 + \frac{33}{8} x - \frac{59}{32} = 0
\]

Sextics of non-torus type and of torus type are given by the conics \( B_2, B'_2 \):

\[
B_2 : y^2 - 1 + \frac{9}{13} x^2 = 0
\]

\[
B'_2 : y^2 + (-\frac{770}{1147} x + \frac{156}{1147}) y + \frac{11025}{14911} x^2 + \frac{770}{1147} x - \frac{1303}{1147} = 0
\]

4. We consider the configurations \([A_{11}, 2A_2, 2A_1], [A_{11}, 2A_2, 3A_1]\). First we consider two cuspidal quartics with \([2A_2]\):

\[
f_4 := 5805 x^4 - 2916 I x^3 \sqrt{2} + 3888 I x^3 y \sqrt{2} - 1269 x^2 y^2 - 729 x^2 - 3834 x^2 y
\]

\[
- 3888 I x y^2 \sqrt{2} + 108 I x \sqrt{2} y^3 + 2916 I x y \sqrt{2} + 1323 y^3 - 1971 y^2 - 81 y^4
\]

\[
+ 729 y = 0
\]

It can makes sextics of torus type and non-torus type with configuration \([A_{11}, 2A_2, 2A_1]\) with respective conics:

\[
f_2(x, y) = y - x^2
\]

\[
h_2(x, y) = (\frac{1}{8} I \sqrt{2} y^2 + x - \frac{3}{4} I y \sqrt{2} - \frac{5}{4} I \sqrt{2} x^2 - 3 x + \frac{5}{8} I \sqrt{2})
\]

They correspond to the conical flex points \((0, 0)\) and \((0, 1)\). The other conical flexes are very heavy to be computed.

Next we consider the configuration \([A_{11}, 2A_2, 3A_1]\) which is produced by a quartic \( B_4 \) with \( 2A_2 + A_1 \) and a conic \( B_2 \) with a single tangent at a conical flex.

\[
B_4 : \frac{1}{16} y^4 + \frac{3}{4} x y^3 + \frac{59}{8} y^2 x^2 - y^2 x - \frac{1}{8} y^2 + \frac{27}{4} y x^3 - 6 y x^2 - \frac{3}{4} y x + \frac{17}{16} x^4 - x^3 - \frac{9}{8} x^2 + x + \frac{1}{16} = 0
\]

We find three conical flex points

\[
P1 := (-1/19, 12/19), \quad P2 := (-\frac{4}{13}, \frac{15}{13}), \quad P3 := (-1, 0)
\]
The corresponding conic which are tangent at the respective conical flex point \( P_i, i = 1, 2, 3 \) are given by

\[
g_{21} := y^2 + \left( \frac{1270}{141} x + \frac{10}{141} \right) y + \frac{38711}{423} x^2 - \frac{5486}{423} x - \frac{457}{423} \]
\[
g_{22} := y^2 + \left( \frac{7462}{2517} x - \frac{5462}{2517} \right) y + \frac{13007}{7551} x^2 - \frac{21902}{7551} x + \frac{8831}{7551} \]
\[
g_{23} := -\frac{3}{22} y^2 - \frac{61}{11} y x + \frac{73}{66} x^2 - \frac{1}{33} x + \frac{71}{66} \]

The corresponding sextic \( f_4(x, y) g_{2i} \) is of torus type for \( i = 1 \) and of non-torus type for \( i=2,3 \). The torus decomposition of \( f_4 g_{21} \) is given by

\[
\frac{1}{423} \sqrt{2(2/3)(3y^2 + 24yx + 293x^2 - 34x - 3)}
\]
\[
\frac{1}{678} I(5 - 189x - 3477x^2 + 20045x^3 - 3y - 54yx + 2361y x^2 - 5y^2 + 247y^2 x + 3y^3) \sqrt{6768}
\]

5. Lastly, we consider the configuration \([A_{11}, A_5, 2A_1]\) which is associated to a quartic \(B_4\) with an \(A_5\) singularity and a conic \(B_2\) tangent at a conical flex point with intersection number 6.

As a quartic, we take:

\[
f_4 := y^4 + xy^3 + \frac{7}{15} x^2 y^2 - 2xy^2 - 3y^2 + \frac{2}{15} x^3 y - \frac{2}{15} x^2 y + xy + 2y + \frac{4}{75} x^4 - \frac{2}{15} x^3 - \frac{1}{3} x^2
\]

\(B_4\) has apparently 26 conical flex points. (The calculation is very heavy.) We take four explicit conical flex points:

\[
P_1 := (0, 0), P_2 := \left(-\frac{540}{493}, \frac{250}{493}\right), P_3 := \left(-\frac{270}{301}, \frac{58}{301}\right), P_4 := \left(-\frac{270}{193}, -\frac{50}{193}\right)
\]

After an easy computation, the respective conics are given as

\[
n_{21} := y^2 + \left( -\frac{5}{59} x - \frac{50}{59} \right) y + \frac{25}{177} x^2
\]
\[
n_{22} := y^2 + \left( -\frac{10845}{262699} x - \frac{273650}{262699} \right) y + \frac{135675}{262699} x^2 - \frac{153000}{262699} x + \frac{70000}{262699}
\]
\[
n_{23} := y^2 + \left( -\frac{16681}{32607} x - \frac{17318}{10869} \right) y + \frac{30251}{32607} x^2 - \frac{1544}{10869} x - \frac{112}{10869}
\]
\[
n_{24} := y^2 + \left( -\frac{5225}{1633} x + \frac{350}{71} \right) y + \frac{4225}{1633} x^2 + \frac{13000}{1633} x + \frac{10000}{1633}
\]

Put \(f_{6j}(x, y) := f_4(x, y) n_{2j}(x, y)\) and \(C^{(i)} = \{f_{6j} = 0\}\). It is also easy to see that \(C^{(1)}, C^{(2)}\) are of non-torus type and \(C^{(3)}, C^{(4)}\) are of torus type.

## 5. Flex Geometry of Cubic Curves

### 5.1. Configurations coming from cubic flex geometry: a cubic component and a line component

Let us consider first configurations which occurs in sextics which have at least a cubic component \(B_3\). We divide into the following cases.
(1) $C = B_3 + B_3'$.
   (a) $\Sigma(C) = [A_{17}],[A_{17},A_1],[A_{17},2A_1]$.
   (b) $\Sigma(C) = [A_{11},A_5],[A_{11},A_5,A_1],[A_{11},A_5,2A_1]$.
   (c) $\Sigma(C) = [A_{11},2A_2,3A_1]$
   (d) $\Sigma(C) = [3A_5],[3A_5+A_1],[3A_5,2A_1]$.

(2) $C = B_3 + B_2 + B_1$.
   (a) $\Sigma(C) = [A_{11}+A_5+2A_1],[A_{11}+A_5+3A_1]$.
   (b) $\Sigma(C) = [3A_5+2A_1],[3A_5+3A_1]$.

(3) $C = B_3 + B_1 + B_1' + B_1''$ with configuration $[3A_5+3A_1],[3A_5+4A_1]$.

We first consider the cases (2) and (3). In these cases, there are one cubic component $B_3$ and at least one line component $B_1$. Recall that the configurations in (2) and (3) occurs as sextics of linear torus type. For a reducible sextic $C$ which is classified in either (2) or (3), the necessary and sufficient condition for $C$ to be of torus type is there exists a line $L$ containing inner singularities. In the case of $\Sigma(C) = [A_{11},A_5]$, $L$ is also tangent to the tangent cone of $A_{11}$. We first recall the following basic geometry for cubic curves.

**Proposition 9.** 1. A smooth cubic $C$ has 9 flex points. Among 84 choices of three flex points, 12 colinear triples of flexes.
   
   2. A nodal cubic has 3 flex points, and they are colinear.
   
   3. A cuspidal cubic has one flex point.

For the proof of the assertion 1, see Example below.

**Corollary 10.** The configuration $[3A_5+4A_1]$ with components type $B_3 + B_1 + B_1' + B_1''$ does not exist as a sextic of non-torus type.

**Proof.** The cubic has a node and three line components are flex tangent lines at three flex points. We know that such configuration exists as a sextics of linear torus type \([\square] \). As the configuration space of one nodal cubics is connected, every sextics $B_3 + B_1 + B_1' + B_1''$ is of torus type.

\( \Box \)

(2) $C = B_3 + B_2 + B_1$ with $\Sigma(C) = [3A_5,kA_1]$, $k = 2,3$. In this case, $B_3$ is either smooth or nodal and two intersection points $B_3 \cap B_2$ generates $2A_5$. The third $A_5$ is generated by a flex tangent line $B_1$.

**Proposition 11.** Assume that a cubic $B_3$ and a conic $B_2$ are intersecting at two points $P,Q$ with respective intersection number 3, producing $2A_5$-singularities. Then the line passing through $P, Q$ intersects $B_3$ at another point, say $R$, and $R$ is a flex point of $B_3$.

This Proposition describes sextics of torus type and non-torus type with configuration $[3A_5,jA_1]$, $j = 0,1,2$ ((2-b)).

**Proof of Proposition \([\square] \).** Assume that $P = (0,1)$ and $Q = (0, -1)$ with the tangent lines $y = \pm1$ respectively. Then by an easy computation, the cubic $B_3$ is defined by a polynomial $f_3 := y^3 + y^2 a_{12} x - y^2 a_{00} + y a_{21} x^2 - y + a_{30} x^3 - a_{00} a_{21} x^2 - a_{12} x + a_{00} = 0$. \( \Box \)
and the conic is given as \( y^2 + a_{21} x^2 - 1 = 0 \). Then \( R \) is given as \( (0, a_{00}) \) and we can easily see that \( R \) is a flex of \( B_3 \).

By the same calculation, we see that

**Proposition 12.** Assume that a cubic \( B_3 \) and a conic \( B_2 \) are intersecting at one points \( P \) with intersection multiplicity 6 producing an \( A_{11} \)-singularity. Then the tangent line passing at \( P \) intersects another point \( R \in B_3 \) and \( R \) is a flex point of \( B_3 \).

Proof is similar. Putting \( P = (0,0) \) and assuming \( y = 0 \) as the tangent line, the cubic is written as

\[
\begin{align*}
f_3 := (y^3 t_2^4 - y^2 x t_3 a_{01} t_4 t_2 + y^2 x a_{21} t_3 t_2^2 + 2 y^2 x a_{11} t_3^2 t_2 + 2 y^2 x a_{01} t_3^3 \\
- y^2 x a_{11} t_4 t_2^2 - y^2 t_2^2 a_{01} t_4 - y^2 t_2^3 a_{21} - y^2 t_2^2 a_{11} t_3 + y t_2^4 a_{21} x^2 \\
+ y t_2^4 a_{11} x + y t_2^4 a_{01} - x^3 t_2^4 a_{01} t_3 - x^3 t_2^5 a_{11} - a_0 t_2^5 (x^2)/t_2^4
\end{align*}
\]

and \( R = (0, x t_2 a_{11} + a_0 t_2 + x a_{01} t_3) \).

**Lemma 13.** Assume that a conic \( B_2 \) is tangent to an irreducible curve \( C \) of degree \( d \geq 3 \) at a smooth point \( P \in C \) so that \( I(B_2, C; P) \geq 3 \). Then \( P \) is not a flex point of \( C \).

Proof. Let \( h_2(x, y) = 0 \) be a conic equation which defines \( B_2 \). In fact, if \( P = (a, b) \) is a flex point of \( C \), \( I(C, B_2; P) \geq 3 \) implies that \( C \) is locally parametrized as \( y_1(x) = t_1 x_1 + t_3 x_3^2 + (\text{higher terms}) \) where \( (x_1, y_1) = (x - a, y - b) \) assuming the tangent line is not \( x - a = 0 \). As \( B_2 \) does not have any flex, the equation \( h_2(x, y) = 0 \) is solved as \( y_1 = s_1 x_1 + s_2 x_3^2 + (\text{higher terms}) \) with \( s_2 \neq 0 \). Thus \( I(C, B_2; P) = 1 \) or \( 2 \) according to \( s_1 \neq t_1 \) or \( s_1 = t_1 \).

First, we consider the case (2-b). Then the cubic is either smooth or have a node. Thus it has at least 3 flexes. As the intersection \( B_3 \cap B_2 \) are not flex points, we can find another flex point \( S \in B_3, S \neq R \). Taking flex tangent \( L := T_S B_3 \) as the line component, the corresponding sextic is not of torus type. As a cuspidal cubic has a unique flex points, we see that a sextic \( C = B_3 + B_2 + B_1 \) with \( [3A_5 + A_2 + 2A_1] \) does not appear as a sextic of non-torus type.

Now we consider (3). Assume that the cubic is smooth. Then there are 9 flex points and \( B_1, B'_1, B''_1 \) are flex tangents. The sextic is of torus type if and only if three flexes are colinear \([11]\). This case, the configuration is \([3A_5 + 3A_1] \).

Finally we consider the case (2-a). In this case, \( B_3 \cap B_2 \) is a single point \( P \) and \( I(B_3, B_2; P) = 6 \) and the intersection singularity is \( A_{11} \). \( B_3 \) has at most a node and so it has at least 3 flex points. Taking a line component which is the flex tangent at \( S \) other than \( R \), we get a sextic \( C = B_3 + B_2 + B_1 \) with \([A_{11} + A_5 + 2A_1] \) or \([A_{11} + A_5 + 3A_1] \).

We omit explicit examples for (2-a) and (2-b) as they can be easily obtained from sextic of torus type with the same configuration \([10]\) and replacing the flex line \( B_1 \). We only gives an example of (3).

**Example** \( B_3 + B_1 + B'_1 + B''_1 \), with configuration \([3A_5 + 3A_1] \). In this case, the cubic \( B_3 \) is smooth and has 9 flexes and three lines \( B_1, B'_1, B''_1 \) are the tangent lines at flexes \( P_1, P_2, P_3 \).
(see below) of $B_3$. Three $A_1$ are the intersections of lines. We know that $B_3 \cup L_P \cup L_Q \cup L_R$ is of torus type if and only if $P, Q, R$ are colinear, where $L_P$ is the tangent line at the flex point $P$. Let $B_3 : f_3(x, y) = 0$. An Example of such a sextic of non-torus type is given by

$$f(x, y) = (y^3 + (-3 + \frac{1}{2} I \sqrt{3}) x - 2) y^2 - y + (1 - \frac{1}{2} I \sqrt{3}) x^3 + (-3 + \frac{1}{2} I \sqrt{3}) x + 2. $$

We can moreover explicitly compute 9 flex points $P_1, \ldots, P_9$ as follows.

$$P_1 := (1, 0, 1), P_2 := (0, 1, 1), P_3 := (0, -1, 1), P_4 := (0, 2, 1), P_5 := (\frac{3}{5}, \frac{4}{5}, 1), P_6 := \frac{1}{2} + \frac{1}{6} I \sqrt{3}, \frac{1}{2} - \frac{1}{6} I \sqrt{3}, 1), P_7 := (\frac{15}{14} + \frac{3}{14} I \sqrt{3}, \frac{1}{14} + \frac{3}{14} I \sqrt{3}, 1), P_8 := (\frac{21}{38} + \frac{9}{38} I \sqrt{3}, \frac{31}{38} + \frac{3}{38} I \sqrt{3}, 1), P_9 := (\frac{33}{62} + \frac{3}{62} I \sqrt{3}, \frac{37}{62} + \frac{9}{62} I \sqrt{3}, 1)$$

Thus by a direct checking, we find the following 12 triples which are colinear.

$${C_1} := [P_1, P_2, P_6], {C_2} := [P_1, P_3, P_7], {C_3} := [P_1, P_4, P_5]$$

$${C_4} := [P_1, P_8, P_9], {C_5} := [P_2, P_3, P_4], {C_6} := [P_2, P_5, P_8]$$

$${C_7} := [P_3, P_5, P_9], {C_8} := [P_3, P_6, P_5], {C_9} := [P_4, P_6, P_3]$$

$${C_{10}} := [P_4, P_7, P_8], {C_{11}} := [P_5, P_6, P_7], {C_{12}} := [P_2, P_7, P_9]$$

5.2. Sextics with two cubic components: $C = B_3 + B'_3$. Now we consider sextics with two cubic curves $B_3, B'_3$. The possible configurations are

1. $C = B_3 + B'_3$.  
   (a) $\Sigma(C) = [A_{17}], [A_{17}, A_1], [A_{17}, 2A_1]$.  
   (b) $\Sigma(C) = [A_{11}, A_5], [A_{11}, A_5, A_1], [A_{11}, A_5, 2A_1]$.  
   (c) $\Sigma(C) = [A_{11}, 2A_2, 3A_1]$  
   (d) $\Sigma(C) = [3A_5], [3A_5 + A_1], [3A_5, 2A_1]$.

First we consider two cubics $B_3, B'_3$ which are tangent at the origin with intersection number 9. Let $f(x, y) = 0$ and $f'(x, y) = 0$ be the defining polynomials of $B_3$ and $B'_3$ respectively and we may assume that the tangent line of $B_3$ is given by $y = 0$. Let $y = \sum_{i=2}^{\infty} t_i x^i$ be the solution of $f(x, y) = 0$ at $O$. Then by the assumption $I(B_3, B'_3; O) = 9$, we must have $\text{ord}_x f'(x, \sum_{i=2}^{\infty} t_i x^i) = 9$.

**Lemma 14.** The sextic $C := B_3 \cup B'_3$ is of torus type if and only if $t_2 = 0$. This implies that $O$ is a flex of $B_3$.

**Proof.** This assertion is given by Artal in [1]. Our proof is computational. In fact, if $t_2 = 0$, $O$ is a flex for both $B_3, B'_3$ and we see that $y = 0$ is a flex tangent line for $B_3, B'_3$. Thus by Tokunaga’s criterion, $y^2 = 0$ is the conic which gives a linear torus decomposition. For the detail about linear torus decomposition, we refer [10].
Assume that $t_2 \neq 0$ and we prove that any such $C$ is of non-torus type. In fact, supposing $C$ to be a sextic of torus type, take a torus decomposition $f(x,y)f'(x,y) = f_2(x,y)^3 + f_3(x,y)^2$. Put $y_1 := y - \sum_{i=2}^{\infty} t_i x^i$. So the assumption implies that

$$f(x,y_1 + \phi(x)) \times f'(x,y_1 + \phi(x)) = y_1(y_1 + cx^9) + \text{(higher terms)}, \quad c \neq 0$$

and thus $y'_1 := y - \sum_{i=2}^{8} t_i x^i$ is the maximal contact coordinate and it is also the solution of $f_3(x,y) = 0$ in $y \mod x^9$ and $f_2(x,\sum_{i=2}^{8} t_i x^i) \equiv 0 \modulo (x^6)$. For the existence of a non-trivial conic $f_2(x,y)$, we see that the coefficient must satisfy:

$$(2) \quad J_0 := -3t_4 t_3 t_2 + 2t_3^3 + t_5 t_2^2 = 0$$

(conics are five dimensional but we have 6 equation $coeff(f_2(x,\sum_{i=2}^{8} t_i x^i), x,j) = 0$ for $j = 0, \ldots, 5$). Then we examine the other equations

$$(\sharp) \quad \text{Coeff}(g(x,\sum_{i=2}^{8} t_i x^i), x,j) = 0, \quad j = 0, \ldots, 8$$

where $g(x,y)$ is a cubic which corresponds to either $f(x,y)$ or $f'(x,y)$. Write $g(x,y)$ as a generic cubic form with 10 coefficients (but by scalar multiplication, one coefficient can be normalized to be 1, say that of $x^6$ is 1, and we have 9 free coefficients), we solve the equations $(\sharp)$ from $j = 0$ to $j = 0$ to $j = 8$ to express the coefficients in rational functions of $t_2, \ldots, t_7$. At the last step, we have one free coefficient undetermined, say the coefficient $c$ of $x^a y^b$ and a linear equation $\text{Coeff}(g(x,\sum_{i=2}^{8} t_i x^i), x, 8) = 0$. This is written as $K_1 c + K_0 = 0$ where $K_1, K_0$ are rational functions of $t_2, \ldots, t_7$. Thus to have two non-trivial cubic solutions, we need to have $K_1 = K_0 = 0$. Now we can easily check that there are no solutions if we assume that $J_0(t_2, \ldots, t_5) = 0$. The other assertion will be checked in the explicit construction of examples. \hfill \Box

If $t_2 \neq 0$, there is no line $L$ such that $L$ intersects only at $O$. Thus the sextic can not be of torus type by the classification in [10]. However the above argument is useful for the explicit computation of non-torus sextics.

**1-a** Let us consider the case $[A_{17}], [A_{17}, A_1], [A_{17}, 2A_1]$. The sextics of non-torus type with above configurations are obtained using above computation $(t_2 \neq 0)$. Their Zariski partners are cubics intersecting at flex points.

(a-1) $[A_{17}], C = B_3 + B_3', C := \{f_3(x,y)g_3(x,y) = 0\}$ where

$$f_3 := -y^3 - y^2 + (-x^2 + x) y - x^3 + x$$
$$g_3 := x - \frac{10}{9} y^3 - x^2 y + \frac{10}{9} x y - y^2 - \frac{10}{9} x^3$$

(a-2) $[A_{17} + A_1], C = B_3 + B_3'$ with $B_3$ has a node: $C := \{f_3(x,y)g_3(x,y) = 0\}$ where

$$f_3 := x - 3 xy + 3 x^2 y - 2 x^3 - y^2 + 2 y^3 + 4 y^2 x + x^3$$
$$g_3 := x + xy + 11 x^2 y - 10 x^2 - y^2 - 2 y^3 + 8 y^2 x + 5 x^3$$
5.3. Exceptional configuration: \([A_{11}, 2A_2, 3A_1]\) with two cubic components. In this case, we do the similar computation. We compute sextics \(C = B_3 \cup B_3'\) such that \(B_3\) and \(B_3'\) have two \(A_2\) at \((0,1)\) and \((1,0)\) respectively and they intersect at \((0,-1)\) with intersection number 6 to make \(A_{11}\). We have the following sextics of non-torus type.

\[
\begin{align*}
\text{(a-3) } [A_{17} + 2A_1], \quad C = B_3 + B_3' &= \{ f_3(x,y)g_3(x,y) = 0 \}, \text{ where cubics are nodal and} \\
\quad f_3 := & \frac{1}{48}I(48x - 96x^2 - 264xy + 124y^3 - 48y^2 + 411y^2x + 264x^2y + 48x^3 \nonumber \\
& - 104Ix \sqrt{3} + 181Iy^2x \sqrt{3} + 104Ix^2y \sqrt{3} + 16Ix \sqrt{3} - 32Ix^2 \sqrt{3} \nonumber \\
& + 16Ix^3 \sqrt{3} - 16Iy^2 \sqrt{3} + 52Iy^3 \sqrt{3}) \sqrt{3}/(-1 + I \sqrt{3}) \\
\quad g_3 := & \frac{1}{8}(48x - 72x^2 + 10Ix^2y \sqrt{3} - 25Iy^2x \sqrt{3} + 56Ix \sqrt{3} - 216xy + 68y^3 - 48y^2 \\
& + 231y^2x + 138x^2y + 23x^3 + 7Ix^3 \sqrt{3} + 8Ix^2 \sqrt{3} - 16Ix \sqrt{3} - 12Iy^3 \sqrt{3} \\
& + 16Iy^2 \sqrt{3})/((3 + I \sqrt{3})(-1 + I \sqrt{3}))
\end{align*}
\]

5.4. Examples of (b) and (d). As the corresponding sextics of torus type are linear, we only need to check the singularities are not colinear. (b) \(\Sigma(C) \supset \{A_{11}, A_5\}\). We put \(A_{11}\) at \((0,0)\) with tangent line \(x = 0\) and \(A_5\) at \((1,0)\).

(b-1) \(\Sigma(C) = [A_{11}, A_5]\) :

\[
\begin{align*}
\quad f(x,y) &= (-y^3 + (9x - 1)y^2 + 7x^3 - 8x^2 + x) \times \\
& (-2y^3 + (5x - 1)y^2 + (-x^2 + x)y + 4x^3 - 5x^2 + x)
\end{align*}
\]
(b-2) \( \Sigma(C) = [A_{11}, A_5, A_1] : \)

\[
f(x, y) = \frac{1}{53}(-16xy - 2y^2 + 16x^2 + 14y^2 + 8x^2 + 5x^3 + 3x)
= (11xy + 4y^2 - 11x^2y + 98y^2x - 5x^2 + 11x^3 + 14y^3 - 6x)
\]

(b-3) \( \Sigma(C) = [A_{11}, A_5, 2A_1] : \)

\[
f(x, y) := (-175y^3 + 11x^2\sqrt{3} - 88x^2 - y^3\sqrt{3} - 30y^2x\sqrt{3} - 18y^2x\sqrt{3} + 27x + 61x^3 + 48y^2
- 94yx + 83y^2x + 126y^2x + 1x^2\sqrt{3} + 5Iy^2\sqrt{3} - 17Iy^2x - Ix\sqrt{3} - 3Iyx
+ 11I\sqrt{3}y^3 - 6y^2\sqrt{3} - 11x\sqrt{3} - 8Ix^2 + 21Iy^2 + 8Ix - 27Iy^3 + 25Iy^2x\sqrt{3}
+ 8Iyx\sqrt{3} + 15Iy^2x\sqrt{3} + 2Iyx^2 + 27x\sqrt{3} + 175y^3 + 11x^2\sqrt{3} - 88x^2
+ y^3\sqrt{3} - 30y^2x\sqrt{3} + 18y^2x\sqrt{3} + 27x + 61x^3 + 48y^2 + 94yx - 83y^2 + 11I\sqrt{3}y^3 - 6y^2\sqrt{3}
- 11x\sqrt{3} - 27Iy^3 + 8Iyx\sqrt{3}
+ 15Iyx^2\sqrt{3} + 2Iyx^2 - 27yx\sqrt{3} - 25Iy^2x\sqrt{3} + 8Ix^2 - 21Iy^2 - 8Ix
+ 1Ix\sqrt{3} - Ix^2\sqrt{3} - 5Iy^2\sqrt{3} + 17Iy^2x)
\]

(d) \( \Sigma(C) \supset \{3A_5\}, C = B_3 + B'_3 \). We put three \( A_5 \) at \((0,1), (0,-1) \) and \((1,0)\).

\[
[3A_5]: f(x, y) := \left( \frac{3}{7} + x - y + y^3 - \frac{3}{7}x^3 - x^2 - \frac{1}{7}y^2x - y^2x - \frac{3}{7}y^2 \right)
\]

\[
\quad \quad \quad \quad = (-\frac{4}{5}x - y + 1 + \frac{4}{5}y^2x - \frac{3}{5}x^2 + \frac{3}{5}y^2x^2 + y^3 - y^2 + \frac{2}{5}x^3)
\]

\[
[3A_5, A_1]: f(x, y) := \left( y^3 + \frac{57}{4}y^2x + \frac{1}{4}y^2x + yx + \frac{1}{4}x^3 + \frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{4} \right)
\]

\[
\quad \quad \quad \quad = (y^3 + \frac{71}{5}y^2x - y^2 + \frac{49}{5}y^2x^2 - 20yx - y^1 + \frac{67}{5}x^3 - \frac{101}{5}x^2 + \frac{29}{5}x + 1)
\]

\[
[3A_5 + 2A_1]: f(x, y) := \left( 21y^2x - 12I\sqrt{3}y^2x + 12y^2x - I\sqrt{3}y^2x^2 + 12yx - 2I\sqrt{3}y^2x - Iy\sqrt{3}
+ Iy^3\sqrt{3} - 3 + 3y^2 + 3x^2 - 3x + 3x^2)(-1 + 49x^3 - 17y^2x + 28y^2x^2 + 63x^2
- 12yx + y^2 + 15x - 7I\sqrt{3}y^2x - 6I\sqrt{3}y^2x + I\sqrt{3}y^2x + Iy\sqrt{3} - y^3\sqrt{3})
\]

6. Three conics

In this section, we study the last case \( C = B_3 + B'_3 + B''_3 \) with the configuration of the singularities \([3A_5, 3A_1]\). Such a sextic is given when each pair of conics are intersecting at two points: at one point, with intersection multiplicity 3 and at another point, transversely.

We can understand Zariski pairs in this situation using conical flex points. Assume that the respective defining polynomials of \( B_2, B'_2, B''_2 \) are \( f_2(x, y), g_2(x, y), h_2(x, y) \) and the location of two \( A_5 \)'s are \( P_1 = (0,1), P_2 = (0,-1) \) with respective tangent cones are \( y \parallel 1 = 0 \). We assume further \( P_1 \in B_2 \cap B'_2 \) and \( P_2 \in B_2 \cap B''_2 \). We fix \( B_2, B'_2 \) generically and consider a
linear system $\Phi$ of conics $B''_2$ of dimension 2 such that $B''_2$ and $B_2$ are tangent at $P_2$. Under this situation we assert that

**Proposition 15.** There exist 5 conical flex points $Q_i$, $i = 1, \ldots , 5$ on $B''_2$ with respect to $\Phi$ so that $Q_1$ is a conical flex of torus type and the other are of non-torus type.

**Proof.** To avoid the complexity of the equation, we choose a generic $B_2, B'_2$ so that

$$f_2(x, y) = (y^2 - x^2)$$
$$g_2(x, y) = (y^2 + (-\frac{2}{15} \sqrt{130} x - \frac{2}{3} y + \frac{2}{3} x^2 + \frac{2}{15} \sqrt{130} x - \frac{1}{3})$$

We find 5 conical flex points on $B'_2$:

$$Q_1 = (-\frac{1}{9} \sqrt{130}, -\frac{17}{9})$$
$$Q_2 = (\frac{7}{3} I \sqrt{10} \sqrt{3} - \frac{2}{3} I \sqrt{10} \sqrt{3}, -4 + \sqrt{13} + \frac{5}{3} I \sqrt{10} \sqrt{3} - \frac{19}{3} I \sqrt{10})$$
$$Q_3 = (-\frac{7}{3} I \sqrt{10} \sqrt{3} + \frac{2}{3} I \sqrt{10} \sqrt{3}, -4 + \frac{19}{3} I \sqrt{10} + \frac{5}{3} I \sqrt{10} \sqrt{3})$$
$$Q_4 = (-\frac{7}{3} I \sqrt{10} \sqrt{3} - \frac{2}{3} I \sqrt{10} \sqrt{3}, -4 - \frac{5}{3} I \sqrt{10} \sqrt{3} - \frac{19}{3} I \sqrt{10} - \sqrt{13}),$$
$$Q_5 = (\frac{7}{3} I \sqrt{10} \sqrt{3} + \frac{2}{3} I \sqrt{10} \sqrt{3}, -4 + \frac{19}{3} I \sqrt{10} - \sqrt{13} + \frac{5}{3} I \sqrt{10} \sqrt{3})$$

$Q_1$ gives a sextic of torus type so that $B''_2$ is given by

$$h_2(x, y) = (\frac{338}{201} y - \frac{104}{1005} \sqrt{130} x - \frac{104}{1005} \sqrt{130} x + \frac{137}{201} + y^2 + \frac{32}{201} x^2)$$

The other conical flex points give sextics of non-torus type. For example, $Q_2$ gives $B''_2$ described as:

$$h_2(x, y) = 75 + 150 I \sqrt{3} + 50 \sqrt{13} - 45 I \sqrt{13} \sqrt{3} - 104 \sqrt{13} \sqrt{10} x - 80 \sqrt{10} x$$
$$+ 72 \sqrt{10} x \sqrt{3} + 300 I y \sqrt{3} + 790 y - 90 I y \sqrt{13} \sqrt{3} + 100 y \sqrt{13}$$
$$- 104 y \sqrt{13} \sqrt{10} x - 80 x y \sqrt{10} + 72 I \sqrt{10} x y \sqrt{13} - 45 I y^2 \sqrt{13} \sqrt{3} + 50 y^2 \sqrt{13}$$
$$+ 150 I y^2 \sqrt{3} + 715 y^2 + 320 x^2$$

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Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku
Tokyo 162-8601
E-mail: oka@ma.kagu.tus.ac.jp