ON SOME FRACTAL SETS ONLY CONTAINING IRRATIONALS

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Abstract. In this paper, we prove that many fractal sets generated by the associated dynamical systems only contain irrationals. We give two applications. Firstly, we explicitly construct some overlapping self-similar sets which only consist of irrationals. Secondly, we prove that some scaled univoque sets only contain irrationals.

1. Introduction

Kurt Mahler [16] posed the following famous conjecture:

Conjecture 1.1. Any number from the ternary Cantor set is either a rational or a transcendental number.

This conjecture is elegant. It is a classical problem in Diophantine approximation. To the best of our knowledge, this conjecture is still far from being solved. However, it motivates us to consider the numbers in fractal sets. For instance, it is natural to ask when some fractal sets only consist of irrationals or transcendental numbers. We introduce some related work on this topic. Jia, Li and Jiang [11] proved the following result:

Proposition 1.2. Let $E \subset \mathbb{R}$ be a set with zero Lebesgue measure. Then for Lebesgue almost every $t$, we have that

$$E + t = \{x + t : x \in E\}$$

is a set only containing irrationals or transcendental numbers.

Moreover, they considered some special Cantor sets, and constructed explicitly $t$’s such that $E + t$ only contains irrationals. In this paper, we shall analyze some more general fractal sets, including invariant sets of some dynamical systems, self-similar sets with overlaps and so forth. We prove that these fractal sets only contain irrational numbers. Before introducing the main results of this paper, we give some definitions and notation. Let $m \in \mathbb{N}_{\geq 2}$ and $t \in (0,1)$. We say that $t$ has an $m$-adic expansion if there...
exists some \((t_k) \in \{0, 1, 2, \ldots, m-1\}^\mathbb{N}\) such that
\[
t = \sum_{k=1}^{\infty} \frac{t_k}{m^k}.
\]
Given any non-empty word \((x_1x_2\cdots x_k) \in \{0, 1, 2, \ldots, m-1\}^k, k \geq 1\). If there exists some \(k_0 \in \mathbb{N}\) such that
\[
t_{k_0+1}t_{k_0+2}\cdots t_{k_0+k} = x_1x_2\cdots x_k,
\]
then we call \((t_k)\) a universal expansion of \(t\).

We say that an expansion \((t_k)\) is normal if for any finite nonempty word \(a\) taken from \(\{0, 1, 2, \ldots, m-1\}^* = \bigcup_{k=1}^{\infty} \{0, 1, 2, \ldots, m-1\}^k\) we have that the limiting frequency of the appearances of \(a\) as a subword of \((t_k)\) is
\[
\frac{1}{m^{|a|}},
\]
where \(|a|\) denotes the length of \(a\). If \(x\) has a normal expansion, then we call \(x\) a normal number.

We define the following sets.
\[
A_{\pm t} = \{ x \pm t : x \in A \}, tA = \{ tx : x \in A \setminus \{0\} \}, t^{-1}A = \{ t^{-1}x : x \in A \setminus \{0\} \}.
\]
In what follows, we will use similar notation. Now we state the main result of this paper.

**Theorem 1.3.** Let \(m \in \mathbb{N}_{\geq 2}\). We define
\[
T : [0, 1) \to [0, 1)
\]
by
\[
T(x) = mx \mod 1.
\]
Let \(A\) be any nowhere dense invariant set of \(T\). For any \(t \in (0, 1)\) such that \(t\) has a universal \(m\)-adic expansion, then we simultaneously have
\[
A - t \subset \mathbb{Q}^c, A + t \subset \mathbb{Q}^c, A t^{-1}A \subset \mathbb{Q}^c.
\]
Moreover, if \(t \in (1, \infty)\) and \((1/t)\) has a universal \(m\)-adic expansion, then
\[
tA \subset \mathbb{Q}^c.
\]
In particular, for Lebesgue almost every \(t \in (0, 1)\), we have the above inclusions.

**Remark 1.4.** A set \(A\) is called an invariant set of \(T\) if \(T(A) \subset A\). There are many invariant sets with respect to \(T\). Therefore, our result gives a uniform \(t\) such that \(A \pm t, A/t\) and \(tA\) only contain irrationals. Dynamically, for any \(t \in (0, 1)\) with an \(m\)-adic universal expansion, its orbit
\[
\{ T^n(t) : n \geq 0 \}\]
is dense in $[0,1]$. In fact, our condition can be weaken. We only need to assume that $(t_k)$ is “local” universal, i.e. it suffices to assume that

$$(t_k) = x_1 x_2 \cdots x_i s_1 s_2 s_3 \cdots,$$

where $(s_j)$ is a universal expansion and $x_1 x_2 \cdots x_i$ is any word from

$$\{0, 1, 2, \cdots, m-1\}^i.$$

Here $i$ can be any positive integer. Roughly speaking, “local” universal means the dynamical orbit of $t$ is locally dense in some subset of $[0,1]$. The last statement is trivial as it is well-known that for Lebesgue almost every $t \in (0,1)$, $t$ has an $m$-adic universal expansion. This is due to the famous Borel’s normal number theorem.

If $A$ is a special set, for instance, the survival set of some open dynamical system [9], then we still have the above result. For simplicity, we only consider one hole. For multiple holes, our result is still correct.

**Corollary 1.5.** Given a hole $(a, b) \subset [0,1)$. Let $A$ be the survival set of the following open dynamics, i.e.

$$A = \{ x \in [0,1) : T^n(x) \notin (a, b) \text{ for any } n \geq 0 \}.$$  

Then for any $t \in (0,1)$ such that $t$ has a universal $m$-adic expansion, we have

$$A - t \subset Q^c, A + t \subset Q^c, \frac{A}{t} \subset Q^c.$$  

Moreover, if $t \in (1,\infty)$ and $(1/t)$ has a universal $m$-adic expansion, then

$$tA \subset Q^c.$$  

With similar discussion, we are allowed to prove the following result.

**Corollary 1.6.** Given any $m \in \mathbb{N}_{\geq 2}$. Let $\Sigma$ be a subshift of finite type [14] over $\{0, 1, 2, \cdots, m-1\}^\mathbb{N}$. Let

$$A = \left\{ \sum_{i=1}^{\infty} \frac{x_i}{m^i} : (x_i) \in \Sigma \right\}.$$  

If $A$ is nowhere dense in $[0,1]$, then for any $t \in (0,1)$ such that $t$ has a universal $m$-adic expansion, we have

$$A - t \subset Q^c, A + t \subset Q^c, \frac{A}{t} \subset Q^c.$$  

Moreover, if $t \in (1,\infty)$ and $(1/t)$ has a universal $m$-adic expansion, then

$$tA \subset Q^c.$$  

In Theorem [13] if we choose $t$ with stronger property, then we are able to prove the following result. We still use the notation defined in Theorem [13].
**Theorem 1.7.** Let $A$ be any nowhere dense invariant set of $T$. Then for any normal number $t \in (0, 1)$,

$$A + it \subset \mathbb{Q}^c, \ \frac{A}{it} \subset \mathbb{Q}^c, \ i \in \mathbb{Z} \setminus \{0\}.$$

In the above theorem, we let

$$A + it = \{x + it : x \in A\}, \ \frac{A}{it} = \left\{\frac{x}{it} : x \in A \setminus \{0\}\right\}, \ i \neq 0.$$

As an application of the idea of Theorem 1.3, we explicitly construct some overlapping self-similar sets which only consist of irrationals.

**Theorem 1.8.** Let $K$ be the attractor of the IFS

$$\left\{f_i(x) = \frac{x + a_i}{q}\right\}_{i=1}^n,$$

where

$$q \in \mathbb{N}_{\geq 3}, a_i \in \mathbb{Q} \cap [0, q-1].$$

Suppose that $K$ is nowhere dense in $[0, 1]$. Then for any $t \in (0, 1)$ such that $t$ has a universal $q$-adic expansion, we have

$$K - t \subset \mathbb{Q}^c, \ K + t \subset \mathbb{Q}^c, \ K \cdot t \subset \mathbb{Q}^c.$$

Moreover, for any $t \in (1, +\infty)$ such that $1/t$ has a universal $q$-adic expansion in $[0, 1]$, then we have

$$tK \subset \mathbb{Q}^c.$$

The main idea of Theorem 1.3 is still useful for some non-integer expanding maps. We give another application to the univoque set. Let $\beta \in (1, 2)$. Given any $x \in [0, 1/(\beta - 1)]$. Then there exists an expansion $(x_n) \in \{0, 1\}^\mathbb{N}$ such that

$$x = \sum_{n=1}^{\infty} \frac{x_n}{\beta^n}.$$

If $(x_n)$ is unique, then we call $x$ a univoque point. Usually, for a generic point, it has uncountably many different expansions [2, 3, 4, 8]. We denote by $U_\beta$ all the univoque points in base $\beta$. It is well-known that for any $\beta \in (\beta_s, 2)$, $U_\beta$ is a uncountable set, where $\beta_s$ is the Komornik-Loreti constant [1, 8, 12].

Now, we state the final result of this paper.

**Theorem 1.9.** Let $1 < \beta < 2$. Let $G : [0, 1) \rightarrow [0, 1)$ be a map defined by $G(x) = \beta x \mod 1$. Suppose that $A$ is a nowhere dense invariant set of $G$. Then for any $t \in (0, 1)$ such that

$$\{\beta^k t \mod 1 : k \geq 0\}$$

is dense in $[0, 1)$, we have

$$tA \subset \mathbb{Q}^c.$$
Moreover, for any \( t \in (1, +\infty) \) such that
\[
\{ \beta^k(1/t) \mod 1 : k \geq 0 \}
\]
is dense in \([0, 1)\), then \((1/t)A \subset \mathbb{Q}^c\).

In particular, if \( A = U_\beta \cap \left[ \frac{2 - \beta}{\beta - 1}, 1 \right] \) and \( \beta_* < \beta < 2 \), then for some appropriate \( t \)'s, \( tA \subset \mathbb{Q}^c \) or \((1/t)A \subset \mathbb{Q}^c\).

This paper is arranged as follows. In Section 2, we give the proofs of main results. In Section 3, we give some problems.

2. Proofs of main results

Proof of Theorem 1.3. We only prove \( A + t, t^{-1}A \subset \mathbb{Q}^c \). The other two inclusions can be proved analogously. First, we show \( A + t \subset \mathbb{Q}^c \). Suppose that there exists some \( x \in A \) such that
\[ x + t = r \in \mathbb{Q}. \]
Then we consider
\[
\{ m^k(r - x) \mod 1 : k \geq 0 \} = \{ m^k t \mod 1 : k \geq 0 \}.
\]
Note that the right side of the above equation is dense in \([0, 1]\). This is because \( t \) has a universal expansion. For the left side, it is not dense. Therefore, we obtain a contradiction and finish the proof.

Now, we prove the left side of the above equation is not dense. We let
\[ r = \kappa_1/\kappa_2, \kappa_i \in \mathbb{Z}. \]
Therefore,
\[ m^k(r - x) \mod 1 = (m^k r - m^k x) \mod 1. \]
Since \( x \in A \), there exists an \( m \)-adic expansion
\[ (x_i) \in \{0, 1, 2, \ldots, m - 1\}^\mathbb{N} \]
such that
\[ x = \sum_{i=1}^{\infty} \frac{x_i}{m^i}. \]
Since \( A \) is invariant, it follows that \( T(x) \in A \). Therefore,
\[ x' := m^k x \mod 1 = T^k(x) = \sum_{i=1}^{\infty} \frac{x_{k+i}}{m^i} \in A. \]
Now, we have
\[ m^k(r - x) \mod 1 = (m^k r - m^k x) \mod 1 = m^{k \kappa_1/\kappa_2} - x' \mod 1. \]
Therefore,
\[
\{ m^k(r - x) \mod 1 : k \geq 0 \} \subset \bigcup_{j=1}^{\kappa_2} \bigcup_{i=0}^{2\kappa_2} \left( \frac{i}{\kappa_2} + j - A \right) \mod 1.
\]
Here \( a - F = \{ a - x : x \in F \} \), \( a \in \mathbb{R} \).

It is well-known that any finite union of nowhere dense sets is still nowhere dense. Hence, \( \{ m^k(r - x) \mod 1 : k \geq 0 \} \) is not dense in \([0, 1)\). We finish the proof of \( A + t \subset \mathbb{Q}^c \).

Next, we prove that \( t^{-1}A \subset \mathbb{Q}^c \). We still use some notation when we prove \( A + t \subset \mathbb{Q}^c \). Suppose that there exists some \( x \in A \) such that
\[
t^{-1}x = r \in \mathbb{Q}.
\]
We let
\[
r = \frac{\kappa_3}{\kappa_4},
\]
where \( \kappa_i \in \mathbb{Z} \). Note that
\[
\{ m^k(r^{-1}x) \mod 1 : k \geq 0 \} = \{ m^k(t) \mod 1 : k \geq 0 \}.
\]
The right side of the above equation is dense. Now, we prove that the left side is not dense. Notice that
\[
m^k(r^{-1}x) \mod 1 = r^{-1}m^kx \mod 1 = r^{-1}\left( m^kx \right) \mod 1
\]
\[
= r^{-1}\left( \sum_{i=1}^{k} m^{k-i}x_i + T^k(x) \right) \mod 1
\]
\[
= r^{-1}\left( \sum_{i=1}^{k} m^{k-i}x_i + x' \right) \mod 1
\]
\[
\subset \left( r^{-1}A + \frac{\kappa_4}{\kappa_3}M_k \right) \mod 1,
\]
where \( M_k := \sum_{i=1}^{k} m^{k-i}x_i \in \mathbb{N}^+ \). Note that
\[
\left\{ \left( r^{-1}A + \frac{\kappa_4}{\kappa_3}M_k \right) \mod 1 : k \geq 0 \right\} \subset \left( \left\{ r^{-1}A + \frac{i}{\kappa_3} \mod 1 \right\} \right)_{i=0}^{\kappa_3}.
\]
We observe that for each \( i \),
\[
r^{-1}A + \frac{i}{\kappa_3} \mod 1
\]
is a nowhere dense set as \( A \) is nowhere dense. Therefore,
\[
\{ m^k(r^{-1}x) \mod 1 : k \geq 0 \}
\]
is contained in a union of some nowhere dense sets. Hence, we have proved that
\[
\{ m^k(r^{-1}x) \mod 1 : k \geq 0 \}
\]
is not dense in \([0, 1)\). \( \square \)

Proof of Corollary 1.5. We need to check that the survival set is invariant. By the definition of survival set, we have that for any \( x \in A \), \( T(x) \in A \). Therefore, \( T(A) \subset A \). By the definition of open dynamics, the survival set \( A \) is closed. Therefore, to prove \( A \) is nowhere dense, it suffices to prove that
A does not have interior. Otherwise, if $A$ contains some interval, we denote it by $(\delta_1, \delta_2)$. It is well-known by the ergodic theorem that for Lebesgue almost every $x \in (\delta_1, \delta_2)$, the orbit of $x$ under the map $T$ is dense in $[0,1]$. This is clearly a contradiction as the the orbits of points from the survival set never hit the hole $(a, b)$. We finish the proof.

Proof of Corollary 1.6: The proof is similar to that of Theorem 1.3. We leave it to the reader.

Proof of Theorem 1.7. We recall two well-known results on the sequences of uniform distribution [13].

Claim 1: Let $q \in \mathbb{N}_{\geq 2}, t \in (0,1)$. Then
\[
\{(q^k t) : k \geq 0\}
\]
is of uniform distribution in $[0,1]$ if and only if $t$ is a normal number in base $q$, where $(q^k t)$ denotes the fractional part of $q^k t$.

Claim 2: If
\[
\{(q^k t) : k \geq 0\}
\]
is of uniform distribution in $[0,1]$, then for any $s \in \mathbb{Z} \setminus \{0\}$, we have
\[
\{(q^k ts) : k \geq 0\}
\]
is of uniform distribution in $[0,1]$.

We only prove $A + it \subset \mathbb{Q}^c, i \neq 0$.

If there is some $x \in A$ such that
\[
x + it = r \in \mathbb{Q}.
\]
Then we consider
\[
m^k (r - x) \mod 1 = m^k it \mod 1, k \geq 0.
\]
Since $t$ is normal, it follows by Claim 1 and Claim 2 that
\[
\{m^k it : k \geq 0\}
\]
is uniformly distributed modulo 1. However, the left side of equation (2.1) is not dense in $[0,1]$ (the reader may refer to the proof of Theorem 1.3). We find a contradiction.

Proof of Theorem 1.8. We only prove $K + t \subset \mathbb{Q}^c$. The other two cases can be proved in a similar way. Suppose on contrary that there exists some $x \in K$ such that
\[
x + t = r \in \mathbb{Q}.
\]
Then we have
\[
q^k (r - x) \mod 1 = q^k t \mod 1.
\]
Since $x \in K$, it follows that there exists a greedy expansion (roughly speaking, a greedy expansion is largest in the sense of lexicographical order, the
reader may refer to [3] for more information) \((b_i) \in \{a_1, a_2, \ldots, a_n\}^\mathbb{N}\) such that

\[ x = \sum_{k=1}^{\infty} \frac{b_k}{q^k}. \]

Therefore,

\[ \{q^k(r-x) \mod 1 : k \geq 0\} \subset (F - K \mod 1) \cap [0,1], \]

where

\[ F = \left( \bigcup_{t_i, k \in \mathbb{N}, 0 \leq i \leq n} (q^k r - (t_1 a_1 + t_2 a_2 + \cdots t_n a_n)) \right), \]

and \(F - K \mod 1 = \{y - x \mod 1 : x \in K, y \in F\}\). However, we can only take finite values from \(F \mod 1 = \{x \mod 1 : x \in F\}\). This is because \(a_i, r \in \mathbb{Q}, 1 \leq i \leq n\). We may assume

\[ r = \frac{C}{D}, a_i = \frac{B_i}{A_i}, \]

where

\[ C, D, A_i, B, B_i \in \mathbb{Z}^+. \]

Note that \(K \subset [0,1]\). Hence, if we take some \(y \in F\) which is bigger than 2, then

\[ y - K \mod 1 = y - 1 - K \mod 1. \]

In other words, when we consider \(F - K \mod 1\), we can take only finitely many values from \(F\). More precisely, we may take at most

\[ 2D(\prod_{i=1}^{n} A_i) + 1 \]

numbers from \(F\).

Hence, \(F - K \mod 1\) is contained in a union of finitely many nowhere dense sets. Therefore, \(\{q^k(r-x) \mod 1 : k \geq 0\}\) is also nowhere dense. This contradicts with the fact

\[ \{q^k t \mod 1 : k \geq 0\} \]

is dense in \([0,1]\).

\[ \square \]

**Proof of Theorem 1.9.** We only prove \(t^{-1}A \subset \mathbb{Q}^c\). The other one can be proved in a similar way. Suppose that there exists some \(x \in A\) such that

\[ t^{-1}x = r \in \mathbb{Q}. \]

We let

\[ r = \frac{\kappa_5}{\kappa_6}, \]

where \(\kappa_i \in \mathbb{Z}\). Note that

\[ \{\beta^k(r^{-1}x) \mod 1 : k \geq 0\} = \{\beta^k t \mod 1 : k \geq 0\}. \]
The right side of the above equation is dense. The left side, however, is not. Now, we prove that the left side is not dense. Notice that
\[
\beta^k (r^{-1} x) \mod 1 = r^{-1} \beta^k x \mod 1 = r^{-1} \left( \beta^k x \right) \mod 1 = r^{-1} \left( \sum_{i=1}^{k} \beta^{k-i} x_i + G^k(x) \right) \mod 1
\]
\[
\subset \left( r^{-1} A + \frac{\kappa_6}{\kappa_5} M_k \right) \mod 1,
\]
where \( x = \sum_{i=1}^{\infty} \frac{x_i}{\beta^i} \) has a greedy expansion \((x_i)\) in base \( \beta \) \((\text{3})\), and
\[
M_k := \sum_{i=1}^{k} m^{k-i} x_i \in \mathbb{N}^+.
\]
Note that
\[
\left\{ \left( r^{-1} A + \frac{\kappa_6}{\kappa_5} M_k \right) \mod 1 : k \geq 0 \right\} \subset \left( \left\{ r^{-1} A + \frac{i}{\kappa_5} \mod 1 \right\}^{\kappa_5}_{i=0} \right).
\]
We observe that for each \( i \),
\[
r^{-1} A + \frac{i}{\kappa_5} \mod 1
\]
is a nowhere dense set as \( A \) is nowhere dense. Therefore,
\[
\{ \beta^k (r^{-1} x) \mod 1 : k \geq 0 \}
\]
is contained in a union of some nowhere dense sets. Hence, we have proved that
\[
\{ \beta^k (r^{-1} x) \mod 1 : k \geq 0 \}
\]
is not dense in \([0,1]\).

For the univoque set, we have the following simple properties. First, we always have
\[
G \left( U_\beta \cap \left[ \frac{2-\beta}{\beta-1}, 1 \right] \right) \subset U_\beta \cap \left[ \frac{2-\beta}{\beta-1}, 1 \right].
\]
Next, for any \( x \in U_\beta \), there exists some \( k \in \mathbb{N} \) such that for any \( i \geq k \)
\[
G^i(x) \in \left[ \frac{2-\beta}{\beta-1}, 1 \right].
\]
In terms of the above two properties, we have the desired results for the univoque sets. \( \Box \)
3. SOME QUESTIONS

We pose some problems.

(1) How can we find a $t$ such that $C + t$ only consists of transcendental numbers, where $C$ is the ternary Cantor set.

(2) For any two non-trivial invariant sets, denoted by $A$ and $B$, under the maps $T_2$ and $T_3$, respectively, then how can we find a concrete uniform $t$ such that

$$A + t, B + t \subset \mathbb{Q}^c,$$

where $T_2(x) = 2x \mod 1$ and $T_3(x) = 3x \mod 1$ are defined on $[0, 1)$.

(3) In Theorem 1.3 we do not know whether for any $t \in (0, 1)$ such that $t$ has a universal $m$-adic expansion, we always have

$$tA = \{ta : a \in A \setminus \{0\}\} \subset \mathbb{Q}^c.$$

For the last question, the only thing we know is the following fact from uniform distribution modulo one. Let $q \in \mathbb{N}_{\geq 2}$. Then for Lebesgue almost every $a$,

$$\{q^k a \mod 1 : k \geq 0\}$$

is of uniform distribution in the unit interval. Therefore, for Lebesgue almost every $a$, we simultaneously have that

$$\{q^k a \mod 1 : k \geq 0\}$$

and

$$\{q^k a^{-1} \mod 1 : k \geq 0\}$$

is of uniform distribution in $[0, 1]$. By the above analysis, we claim that for Lebesgue almost every $t \in (0, 1)$,

$$A - t \subset \mathbb{Q}^c, A + t \subset \mathbb{Q}^c, tA \subset \mathbb{Q}^c, \frac{A}{t} \subset \mathbb{Q}^c.$$

Throughout the paper, we essentially analyze the following problem. Let

$$\beta \in (1, \infty)$$

be a real, and $r$ be a rational number. Then when do we have that the following set

$$\{\beta^k r \mod 1 : k \geq 0\}$$

takes only finitely many values. More generally, how can we find the non-trivial lower and upper bounds of the above set. This is related to the theory of sequences modulo one, see [5, 7] and references therein. For the above problem, it is natural to consider some Pisot numbers and rational numbers. In [5], Dubickas proved that when $\beta$ is a Pisot number (an algebraic number is called a Pisot number if all of whose other Galois conjugates have modulus strictly less than 1) and $r$ is rational, then

$$\{\beta^k r \mod 1 : k \geq 0\}$$
has finitely many limit points.

In [7], Flatto, Lagarias, and Pollington considered when $\beta$ is a rational number, they give the lower bound of the length of

$$\{\beta^k r \mod 1 : k \geq 0\}.$$

All these results are useful to our analysis on the translations of the invariant sets and self-similar sets.

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