TOWARDS A DICHOTOMY FOR THE REIDEMEISTER ZETA FUNCTION

WOJCIECH BONDAREWICZ, ALEXANDER FEL’SHTYN AND MALWINA ZIETEK

ABSTRACT. We prove a dichotomy between rationality and a natural boundary for the analytic behavior of the Reidemeister zeta function for automorphisms of non-finitely generated torsion abelian groups and for endomorphisms of groups $\mathbb{Z}_p^d$, where $\mathbb{Z}_p$ the group of p-adic integers. As a consequence, we obtain a dichotomy for the Reidemeister zeta function of a continuous map of a topological space with fundamental group that is non-finitely generated torsion abelian group. We also prove the rationality of the coincidence Reidemeister zeta function for tame endomorphisms pairs of finitely generated torsion-free nilpotent groups, based on a weak commutativity condition.

0. INTRODUCTION

Let $G$ be a group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \beta \in G$ are said to be $\phi$-conjugate or twisted conjugate, iff there exists $g \in G$ with $\beta = g \alpha \phi(g^{-1})$. We shall write $\{x\}_\phi$ for the $\phi$-conjugacy or twisted conjugacy class of the element $x \in G$. The number of $\phi$-conjugacy classes is called the Reidemeister number of an endomorphism $\phi$ and is denoted by $R(\phi)$. If $\phi$ is the identity map then the $\phi$-conjugacy classes are the usual conjugacy classes in the group $G$. We call the endomorphisms $\phi$ tame if the Reidemeister numbers $R(\phi^n)$ are finite for all $n \in \mathbb{N}$. Taking a dynamical point of view, we consider the iterates of a tame endomorphism $\phi$, and we may define following [11] a Reidemeister zeta function of $\phi$ as a power series:

$$R_\phi(z) = \exp \left( \sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n \right),$$

2010 Mathematics Subject Classification. Primary 37C25; 37C30; 22D10; Secondary 20E45; 54H20; 55M20.

Key words and phrases. Twisted conjugacy class; Reidemeister number; Reidemeister zeta function; unitary dual.

The work is funded by the Narodowe Centrum Nauki of Poland (NCN) (grant No. 2016/23/G/ST1/04280 (Beethoven 2)).
where \( z \) denotes a complex variable. The following problem was investigated [13]: for which groups and endomorphisms is the Reidemeister zeta function a rational function? Is this zeta function an algebraic function?

In [11, 13, 23, 14, 12], the rationality of the Reidemeister zeta function \( R_{\phi}(z) \) was proven in the following cases: the group is finitely generated and an endomorphism is eventually commutative; the group is finite; the group is a direct sum of a finite group and a finitely generated free abelian group; the group is finitely generated, nilpotent and torsion-free. In [38] the rationality of the Reidemeister zeta function was proven for endomorphisms of fundamental groups of infra-nilmanifolds under some sufficient conditions. Recently, the rationality of the Reidemeister zeta function was proven for endomorphisms of fundamental groups of infra-nilmanifolds [6]; for endomorphisms of fundamental groups of infra-solvmanifolds of type (R) [16]; for automorphisms of crystallographic groups with diagonal holonomy \( \mathbb{Z}_2 \) and for automorphisms of almost-crystallographic groups up to dimension 3 [7]; for the right shifts of a non-finitely generated, non-abelian torsion groups \( G = \bigoplus_{i \in \mathbb{Z}} F_i, F_i \cong F \) and \( F \) is a finite non-abelian group [36].

Let \( G \) be a group and \( \phi, \psi : G \to G \) two endomorphisms. Two elements \( \alpha, \beta \in G \) are said to be \( (\phi, \psi) - \text{conjugate} \) iff there exists \( g \in G \) with

\[
\beta = \psi(g)\alpha\phi(g^{-1}).
\]

The number of \( (\phi, \psi) \)-conjugacy classes is called the Reidemeister coincidence number of an endomorphisms \( \phi \) and \( \psi \), denoted by \( R(\phi, \psi) \). If \( \psi \) is the identity map then the \( (\phi, id) \)-conjugacy classes are the \( \phi \)-conjugacy classes in the group \( G \) and \( R(\phi, id) = R(\phi) \). The Reidemeister coincidence number \( R(\phi, \psi) \) has useful applications in Nielsen coincidence theory. We call the pair \( (\phi, \psi) \) of endomorphisms tame if the Reidemeister numbers \( R(\phi^n, \psi^n) \) are finite for all \( n \in \mathbb{N} \). For such a tame pair of endomorphisms we define the coincidence Reidemeister zeta function

\[
R_{\phi,\psi}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{R(\phi^n, \psi^n)}{n} z^n \right).
\]

If \( \psi \) is the identity map then \( R_{\phi,\text{id}}(z) = R_{\phi}(z) \). In the theory of dynamical systems, the coincidence Reidemeister zeta function counts the synchronisation points of two maps, i.e. the points whose orbits intersect under simultaneous iteration of two endomorphisms; see [29], for instance.

In [20], in analogy to works of Bell, Miles, Ward [1] and Byszewski, Cornelissen [2, §5] about Artin-Mazur zeta function, the Pólya–Carlson dichotomy between rationality and a natural boundary for analytic behavior of the coincidence Reidemeister zeta function was proven for tame pair of
commuting automorphisms of non-finitely generated torsion-free abelian groups that are subgroups of $\mathbb{Q}^d, d \geq 1$.

In [15] Pólya–Carlson dichotomy was proven for coincidence Reidemeister zeta function of tame pair of endomorphisms of non-finitely generated torsion-free nilpotent groups of finite Prüfer rank by means of profinite completion techniques.

In this paper we present results in support of a dichotomy between rationality and a natural boundary for the Reidemeister zeta functions of endomorphisms of new broad classes of abelian groups.

We prove a dichotomy between rationality and a natural boundary for the Reidemeister zeta function of automorphisms of non-finitely generated torsion abelian groups and of endomorphisms of the groups $\mathbb{Z}_p^d, d \geq 1$, where $\mathbb{Z}_p$, $p$-prime, additive group of $p$-adic integers.

As a consequence, we obtain a dichotomy for Reidemeister zeta function of continuous maps of topological spaces those fundamental group is non-finitely generated torsion abelian group or non-finitely generated torsion-free abelian group that is subgroup of $\mathbb{Q}^d, d \geq 1$.

We also prove the rationality of the coincidence Reidemeister zeta function for tame endomorphisms pairs of finitely generated torsion-free nilpotent groups, based on weak commutativity condition.

Acknowledgments. This work was supported by the grant Beethoven 2 of the NCN of Poland( grant No. 2016/23/G/ST1/04280). The second author is indebted to the Max-Planck-Institute for Mathematics(Bonn) for the support and hospitality and the possibility of the present research during his visit there.

1. Pólya-Carlson dichotomy for Reidemeister zeta function of automorphisms of torsion abelian non-finitely generated groups

Let $\phi : G \to G$ be an endomorphism of a countable discrete abelian group $G$. Let $R = \mathbb{Z}[t]$ be a polynomial ring. Then the abelian group $G$ naturally carries the structure of a $R$-module over the ring $R = \mathbb{Z}[t]$ where multiplication by $t$ corresponds to application of the endomorphism: $tg = \phi(g)$ and extending this in a natural way to polynomials. That is, for $g \in G$ and $f = \sum_{n \in \mathbb{Z}} c_n t^n \in R = \mathbb{Z}[t]$ set

$$fg = \sum_{n \in \mathbb{Z}} c_n t^n g = \sum_{n \in \mathbb{Z}} c_n \phi^n(g),$$

where all but finitely many $c_n \in \mathbb{Z}$ are zero. This is a standard procedure for the study of dual automorphisms of compact abelian groups, see Schmidt [33] for an overview.
1.1. **Main Theorem.** We will repeatedly apply the following results to calculate the Reidemeister numbers of iterations.

**Lemma 1.1.** [28] Let \( L \subset N \) be \( R \)-modules and \( g \in R \). Then

1. \[
\frac{|N|}{|gN|} = \frac{|N/L|}{|L \cap gN|}
\]

2. If \( N/L \) is finite and the map \( x \to gx \) is a monomorphism of \( N \) then \[
\frac{|N|}{|gN|} = \frac{|L|}{|gL|}.
\]

**Lemma 1.2.** [28] Let \( N \) be an \( R \)-module for which \( \text{Ass}(N) \) consists of finitely many non-trivial principal ideals and suppose 

\[
m(p) = \dim_{\mathbb{K}(p)} N_p < \infty,
\]

where \( \mathbb{K}(p) \) denotes the field of fractions of \( R/p \) and \( N_p = N \otimes_R \mathbb{K}(p) \) is the localization of the module \( N \) at \( p \). If \( g \in R \) is such that the map \( x \to gx \) is a monomorphism of \( N \), then \( N/gN \) is finite.

If \( p \subset R \) is a principal prime ideal, \( \mathbb{K}(p) \) above is a global field.

Additionally, if every element of a module \( N \) has finite additive order then every associated prime ideal \( p \in \text{Ass}(N) \) contains a rational prime. The coheight of a prime ideal \( p \subset R \), denoted \( \text{coht}(p) \), coincides with the Krull dimension of the domain \( R/p \) and \( \text{coht}(p) \leq 1 \) for all \( p \in \text{Ass}(N) \).

Our results are phrased in terms of global fields \( \mathbb{K} \) hence if every element of a \( R \)-module \( N \) has finite additive order (this is the case when the group \( G \) above is a torsion abelian non-finitely generated group) then global field \( \mathbb{K}(p) \) will be a function field of transcendence degree one (i.e. the Krull dimension 1) over a finite field.

Recall that a **global field** \( \mathbb{K} \) of characteristic \( p > 0 \) is a finite extension of the rational function field \( \mathbb{F}_p(t) \), where \( t \) is an indeterminate. The **places** of \( \mathbb{K} \) are the equivalence classes of absolute values on \( \mathbb{K} \), which are all non-archimedean. For example, the **infinite place** of \( \mathbb{F}_p(t) \) is given by \( |f/g|_\infty = p^{\deg(f)-\deg(g)} \) and all other places of \( \mathbb{F}_p(t) \) correspond, in the usual way, to valuation rings obtained by localizing the domain \( \mathbb{F}_p[t] \) at its non-trivial prime ideals (generated by irreducible polynomials). The places of \( \mathbb{K} \) are extensions of those just described, and the set of all such places is denoted \( \mathcal{P}(\mathbb{K}) \). Given a finite place of \( \mathbb{K} \), there corresponds a unique discrete valuation \( v \) whose precise value group is \( \mathbb{Z} \). The corresponding normalised absolute value \( | \cdot |_v = |\mathcal{R}_v|^{-v(\cdot)} \), where \( \mathcal{R}_v \) is the (necessarily finite) residue class field of \( v \). For any set of places \( S \), we write \( |x|_S = \prod_{v \in S} |x|_v \).
Such sets of places provide a foundation for the formulas for the Reidemeister numbers of iterations of an automorphism of a torsion abelian non-finitely generated group presented here.

The main results of this section are the formulas for the Reidemeister numbers and a dichotomy between rationality and a natural boundary for the analytic behaviour of the Reidemeister zeta function. We use methods of Bell, Miles and Ward to study the Artin-Mazur zeta function of compact abelian groups automorphisms in [1, Theorem 15],[30, Theorem 1.1]

**Theorem 1.3.** Let $\phi : G \to G$ be an automorphism of a torsion abelian non-finitely generated group $G$. Suppose that the group $G$ as $R = \mathbb{Z}[t]$-module satisfies the following conditions:

1. $G$ is a Noetherian $R = \mathbb{Z}[t]$-module and its finite set of associated primes $\text{Ass}(G)$ consists entirely of non-zero principal ideals of the polynomial ring $R = \mathbb{Z}[t]$,

2. the map $g \to (t^j - 1)g$ is a monomorphism of $G$ for all $j \in \mathbb{N}$ (equivalently, $t^j - 1 \notin p$ for all $p \in \text{Ass}(G)$ and all $j \in \mathbb{N}$),

3. for each $p \in \text{Ass}(G)$, $\text{dim}_{\mathbb{K}(p)} G_p < \infty$.

Then there exist function fields $\mathbb{K}_1, \ldots, \mathbb{K}_n$ of the form $\mathbb{K}_i = \mathbb{F}_{p(i)}(t)$, where each $p(i)$ is a rational prime, sets of finite places $P_i \subset \mathcal{P}(\mathbb{K}_i)$, sets of infinite places $P_i^\infty \subset \mathcal{P}(\mathbb{K}_i)$, sets of finite places $S_i = \mathcal{P}(\mathbb{K}_i) \setminus (P_i^\infty \cup P_i)$, such that

$$R(\phi^j) = \prod_{i=1}^n \prod_{v \in P_i} |t^j - 1|_v^{-1} = \prod_{i=1}^n |t^j - 1|_{P_i}^{-1} = \prod_{i=1}^n |t^j - 1|_{P_i^\infty \cup S_i}$$

for all $j \in \mathbb{N}$.

**Proof.** The Reidemeister number of an endomorphism $\phi$ of an Abelian group $G$ coincides with the cardinality of the quotient group $\text{Coker}(\phi - \text{Id}_G) = G/\text{Im}(\phi - \text{Id}_G)$ (or $\text{Coker}(\text{Id}_G - \phi) = G/\text{Im}(\text{Id}_G - \phi)$).

The multiplicative set $U = \bigcap_{p \in \text{Ass}(G)} R - p$ has $U \cap \text{ann}(a) = \emptyset$ for all non-zero $a \in G$, so the natural map $G \to U^{-1}G$ is a monomorphism. Identifying localizations of $R$ with subrings of $\mathbb{Q}(t)$, the domain $\mathfrak{R} = U^{-1}R = \bigcap_{p \in \text{Ass}(G)} R_p$ is a finite intersection of discrete valuation rings and is therefore a principal ideal domain [26]. The assumptions (1) - (3) force $U^{-1}G$ to be a Noetherian $\mathfrak{R}$-module. Hence, there is a prime filtration

$$\{0\} = G_0 \subset G_1 \subset \cdots \subset G_n = U^{-1}G$$

in which $G_i/G_{i-1} \cong \mathfrak{R}/q_i$ for non-trivial primes $q_i \subset \mathfrak{R}, 1 \leq i \leq n$. Moreover, $p_i = q_i \cap R \in \text{Ass}(G)$ for all $1 \leq i \leq n$. Identifying $G$ with its image in $U^{-1}G$ and intersecting the chain above with $G$ gives a chain
\[ \{0\} = L_0 \subset L_1 \subset \cdots \subset L_n = G. \]

Considering this chain of \( R \)-modules, for each \( 1 \leq i \leq n \) there is an induced inclusion

\[
\frac{L_i}{L_{i-1}} \hookrightarrow \frac{G_i}{G_{i-1}} \cong \mathfrak{R}_{q_i} \cong \mathbb{K}(p_i) = K_i
\]

where field \( \mathbb{K}(p_i) \) is a function field of transcendence degree one (i.e. the Krull dimension 1) over a finite field and \( N_i = L_i/L_{i-1} \) may be considered as a fractional ideal of \( E_i = R/p_i \). Using Lemma 1.1(1),

\[
\left| \frac{L_i}{(t^j - 1)L_i} \right| = \left| \frac{N_i}{(t^j - 1)N_i} \left| \frac{L_{i-1}}{L_{i-1} \cap (t^j - 1)L_{i-1}} \right|, \right.
\]

where \( 1 \leq i \leq n \). Let \( y \in L_i \), let \( \eta \) denote the image of \( y \) in \( N_i \) and let \( \xi_i \) denote the image of \( t \) in \( E_i \). If \( (t^j - 1)y \in L_{i-1} \) then \( (\xi_i^j - 1)\eta = 0 \). An assumption (2) implies \( t^j - 1 \notin p_i \) so \( (\xi_i^j - 1) \neq 0 \). Therefore, \( \eta = 0 \) and \( y \in L_{i-1} \). It follows that \( L_{i-1} \cap (t^j - 1)L_i = (t^j - 1)L_{i-1} \) and hence,

\[
\left| \frac{L_i}{(t^j - 1)L_i} \right| = \left| \frac{N_i}{(t^j - 1)N_i} \left| \frac{L_{i-1}}{(t^j - 1)L_{i-1}} \right|, \right.
\]

Successively applying this formula to each of the modules \( L_i, 1 \leq i \leq n \), gives,

\[
|G/(t^j - 1)G| = \prod_{i=1}^{n} |N_i/(t^j - 1)N_i|
\]

Consider now an individual term \( |N_i/(t^j - 1)N_i| \). Since \( \text{char} \ (E_i) > 0 \), \( E_i \cong \mathbb{F}_p[t] \) for some rational prime \( p \) and \( E_i \) is a finitely generated Dedekind domain. We may consider \( I_i = E_i \otimes_{E_i} N_i \) as a fractional ideal of \( E_i \). Lemma 1.2 and Lemma 1.1(2) imply that \( |N_i/(\xi_i^j - 1)N_i| = |I_i/(\xi_i^j - 1)I_i| \) and are finite (see [28]). By considering \( I_i/(\xi_i^j - 1)I_i \) as an \( E_i \)-module, finding a composition series for this module and successively localizing at each of its associated primes to obtain multiplicities, it follows that

\[
|I_i/(\xi_i^j - 1)I_i| = \prod_{m \in \text{Ass}(I_i/(\xi_i^j - 1)I_i)} q_m^{\delta_m(\xi_i, I_i)},
\]

where \( q_m = |E_i/m| \) and \( \delta_m(\xi_i, I_i) = \dim_{E_i/m}(I_i/(\xi_i^j - 1)I_i)_m \). Let

\[
P_i = \{m \in \text{Spec}(E_i) : I_m \neq K_i\}.
\]

It follows that the product above may be taken over all \( m \in P_i \) to yield the same result. Each localization \( (E_i)_m \) is a distinct valuation ring of \( K_i \) and
$P_i$ may be identified with a set of finite places of the global field $K_i$. Hence, since $\delta_m(\xi_i, E_i) = v_m(\xi_i^{j} - 1)$, finally we have

$$|I_i/(\xi_i^{j} - 1)I_i| = \prod_{m \in P_i} \delta_m(\xi_i, E_i) = \prod_{m \in P_i} v_m(\xi_i^{j} - 1) = \prod_{m \in P_i} |\xi_i^{j} - 1|_{m}^{-1},$$

where $|.|_m$ is the normalised absolute value arising from $E_m$. This concludes the proof of the formula $R(\phi^j) = \prod_{i=1}^n \prod_{v \in P_i} |\xi_i^{j} - 1|_{P_i}^{-1} = \prod_{i=1}^n |\xi_i^{j} - 1|_{P_i}$.

Applying the Artin product formula [37] gives

$$R(\phi^j) = \prod_{i=1}^n |\xi_i^{j} - 1|_{P_i}^{-1} = \prod_{i=1}^n |\xi_i^{j} - 1|_{P_i \cup S_i},$$

\[\square\]

We remind the definition of a natural boundary.

**Definition 1.4.** Suppose that an analytic function $F$ is defined somehow in a region $D$ of the complex plane. If there is no point of the boundary $\partial D$ of $D$ over which $F$ can be analytically continued, then $\partial D$ is called a **natural boundary** for $F$.

**Theorem 1.5.** Let $\phi : G \to G$ be an automorphism of a torsion abelian non-finitely generated group $G$ and define a number $h(\phi) = \sum_{i=1}^r \log p(i)$, where the rational primes $p(i)$ are the same as in Theorem 1.3. Suppose that the group $G$ as an $R = \mathbb{Z}[t]$-module satisfies the conditions (1)–(3) in Theorem 1.3. If the Reidemeister zeta function $R_{\phi}(z)$ has a radius of convergence $R = e^{-h(\phi)}$, then it is either a rational function, namely

$$R_{\phi}(z) = (1 - e^{h(\phi)}z)^{-1}$$

or the circle $|z| = e^{-h(\phi)}$ is a natural boundary for the function $R_{\phi}(z)$.

**Proof.** By Theorem 1.3, there exist function fields $K_1, ..., K_n$ of the form $K_i = \mathbb{F}_{p(i)}(t)$, where each $p(i)$ is a rational prime and sets of finite places $P_i \subset \mathcal{P}(K_i)$ for $i = 1, ..., n$, such that:

$$R(\phi^j) = \prod_{i=1}^n \prod_{v \in P_i} |t^j - 1|_{v}^{-1}$$

for all $j \in \mathbb{N}$. If we denote for each $i = 1, ..., n$ by $P_i^{\infty} \subset \mathcal{P}(K_i)$ the infinite place and by $S_i = \mathcal{P}(K_i) \setminus (P_i^{\infty} \cup P_i)$ set of remaining finite places then the above formula can be also expressed as

$$R(\phi^j) = \prod_{i=1}^n \prod_{v \in P_i^{\infty} \cup S_i} |t^j - 1|_{v}.$$
Since 
\[ \prod_{i=1}^{n} |t^j - 1|_{p^i} = \prod_{i=1}^{n} p(i)^{\deg(t^j - 1)} = \prod_{i=1}^{n} p(i)^j = e^{\sum_{i=1}^{n} \log p(i)j} = e^{h(\phi)j}, \]
then 
\[ R(\phi^j) = \prod_{t \in P^\infty \cup S_i} |t^j - 1|_v = e^{h(\phi)j} \prod_{t \in P^\infty \cup S_i} |t^j - 1|_v. \]
Set \( a_j = \prod_{t \in S_i} |t^j - 1|_v, \) define \( F(z) = \sum_{j=1}^{\infty} a_j z^j \) and note that 
\[ F(e^{h(\phi)}z) = zR_{\phi}'(z)/R_{\phi}(z). \]
By hypothesis \( R_{\phi}(z) \) has a radius of convergence \( e^{h(\phi)} \), so it follows that the radius of convergence of \( F(z) \) is 1. If \( S_i = \{(t)\} \) for all \( i = 1, \ldots, n \), then \( a_j = 1 \) and it follows immediately that \( R_{\phi}(z) = (1 - e^{h(\phi)}z)^{-1}. \)
If not, there exists \( 1 \leq m \leq n \) such that \( S_m \) contains a place \( w \) corresponding to a polynomial of degree \( d_w \geq 1 \) such that \( w \neq t \), so that means that \( w \nmid t \), equivalently \( \text{ord}_w(t) = 0 \) and thus \( |t|_w = 1 \). Set \( p = p(i) \) and \( j_k = l_w p^k \), where \( l_w \) denotes the multiplicative order of the image of \( t \) in the finite residue field at \( w \), i.e. \( t^{l_w} = 1 \). From the binomial expansion and the fact that \( p = p(i) = \text{char}(\mathbb{F}_k) > 0 \) we obtain 
\[ t^{l_w p^k} - 1 = (t^{l_w} - 1)^{p^k} = (w \cdot f(t))^{p^k} = w^{p^k} \cdot f(t)^{p^k}. \]
Since \( w \nmid f(t) \), then \( \text{ord}_w(t^{l_w p^k} - 1) = p^k \). Let us denote the degree of \( w \) as \( d_w \). Then 
\[ |t^{j_k} - 1|_w = p^{-\text{ord}_w(t^{j_k - 1}) \cdot \deg w} = p^{-\text{ord}_w(t^{l_w p^k} - 1) \cdot d_w} = p^{-p^k \cdot d_w} < 1. \]
For all \( 1 \leq j \) we have that \( a_j \leq |t^j - 1|_w \), hence 
\[ \limsup_{j_k \to \infty} a_{j_k}^{1/j_k} \leq p^{-d_w/l_w}. \]
Therefore \( F \) contains a sequence of partial sums that is uniformly convergent for \( |z| < p^{d_w/l_w} \). Hence, the series \( F \) is overconvergent, so series may be written as a sum of a series convergent on \( z < |p| \) and a lacunary series and hence the unit circle is a natural boundary for \( F \) (see [34], sec. 6.2). It follows that the circle \( |z| = e^{-h(\phi)} \) is a natural boundary for \( R_{\phi}(z) \). \( \square \)

1.2. Examples. To give an example of the dichotomy for analytic behavior of the Reidemeister zeta function we use the calculations of Miles and Ward in [30] for the number of periodic points of the dual compact abelian group endomorphism \( \hat{\phi} \). Let us consider a ring \( M = \mathbb{F}_p[t^{\pm 1}, (t - 1)^{\pm 1}] \), and an endomorphism \( \phi : g \to tg \) which is the multiplication by \( t \) on \( M \). It follows from [30], Example 4.1 that \( M \) has a structure of a \( \mathbb{Z}[t] \)-module. Then using
the calculations of Miles and Ward in [30], Example 3.1 and the formula 1 for the Reidemeister numbers

$$R(\phi^j) = \prod_{i=1}^{n} |t^j - 1|_{p_i^{\infty} \cup S_i},$$

we obtain

$$R(\phi^j) = |t^j - 1|_{\infty} |t^j - 1|_{(t)} |t^j - 1|_{(t-1)}.$$ 

Since $|t^j - 1|_{\infty} = p^j$, $|t^j - 1|_{(t)} = p^{-\text{ord}_t(t^j-1)\cdot \deg t} = 1$,

$$|t^j - 1|_{(t-1)} = p^{-\text{ord}_{t-1}(t^j-1)\cdot \deg(t-1)} = p^{-p^{\text{ord}_p(j)}} = p^{-|j|_{p}^{-1}},$$

then finally $R(\phi^j) = p^j - v(j)$, where $v(j) = |j|_{p}^{-1}$.

By the Cauchy – Hadamard formula, the radius of convergence $R$ of the Reidemeister zeta function $R_\phi(z)$ is equal to

$$R = \left( \limsup_{j \to \infty} \sqrt[j]{p^j - v(j)} \right)^{-1} = p^{-1}.$$ 

Let us define $F(z) = \sum_{j=1}^{\infty} a_j z^j$, where $a_j = p^{-v(j)}$. Then the radius $R_F$ of convergence of $F$ is equal to

$$R_F = \left( \limsup_{j \to \infty} \sqrt[j]{p^{-v(j)}} \right)^{-1} = 1.$$ 

Moreover,

$$F(pz) = z R'_\phi(z) / R_\phi(z),$$

so if the Reidemeister zeta function $R_\phi(z)$ has the analytic continuation beyond the circle $|z| = p^{-1}$, then $F$ has the analytic continuation beyond the unit circle. However, $F$ contains a sequence of partial sums that is uniformly convergent for $|z| < p$, since for $j_k = p^k$, we have

$$\limsup_{j_k \to \infty} a_{j_k}^{1/j_k} = p^{-1}.$$ 

It follows that the series $F$ is overconvergent, so it may be written as a sum of a series convergent on $|z| < p$ and a lacunary series, and hence the unit circle is a natural boundary for $F$. Therefore $|z| = p^{-1}$ is a natural boundary for $R_\phi(z)$.

2. Pólya-Carlson dichotomy for the Reidemeister zeta function of endomorphisms of the groups $\mathbb{Z}_p^d$

In this section we prove a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of the Reidemeister zeta function for endomorphisms of groups $\mathbb{Z}_p^d$, $d \geq 1$, where $\mathbb{Z}_p$, p-prime, denotes additive group of p-adic integers. The group $\mathbb{Z}_p$ is the most basic infinite pro-p group, it is totally disconnected, compact, abelian, torsion-free
The field of \( p \)-adic numbers is denoted by \( \mathbb{Q}_p \) and the \( p \)-adic absolute value (as well as its unique extension to the algebraic closure \( \overline{\mathbb{Q}_p} \)) by \( | \cdot |_p \).

We need the following statement

**Lemma 2.1.** (cf. [1]) Let \( Z(z) = \sum_{n=1}^{\infty} R(\varphi^n)z^n \). If \( R(\varphi) \) is rational then \( Z(z) \) is rational. If \( R(\varphi) \) has an analytic continuation beyond its circle of convergence, then so does \( Z(z) \) too. In particular, the existence of a natural boundary at the circle of convergence for \( Z(z) \) implies the existence of a natural boundary for \( R(\varphi) \).

**Proof.** This follows from the fact that \( Z(z) = z \cdot R(\varphi(z)) / R(\varphi(z)) \). □

One of the important links between the arithmetic properties of the coefficients of a complex power series and its analytic behaviour is given by the Pólya–Carlson theorem [34].

**Pólya–Carlson Theorem.** A power series with integer coefficients and radius of convergence 1 is either rational or has the unit circle as a natural boundary.

**Lemma 2.2.** \( \text{End}(\mathbb{Z}_p) = \mathbb{Z}_p \) for abelian group \( \mathbb{Z}_p \).

**Proof.** Let \( \phi \in \text{End}(\mathbb{Z}_p) \). We have \( p^n\phi(x) = \phi(p^nx) \). Then \( \phi(p^n\mathbb{Z}_p) \subset p^n\mathbb{Z}_p \), so \( \phi \) is continuous. For every \( x \in \mathbb{Z}_p \) there exists a sequence of integers \( x_n \) converging to \( x \). Then

\[
\phi(x) = \lim \phi(x_n) = \lim x_n\phi(1) = \phi(1)x,
\]

so \( \phi \) is a multiplication by \( \phi(1) \). □

Let \( \phi \in \text{End}(\mathbb{Z}_p) \), then \( \phi(x) = ax \), where \( a \in \mathbb{Z}_p \). We have \( \phi^n(x) = a^n x \).

By definition,

\[
y \sim_{\phi} x \iff \exists b \in \mathbb{Z}_p : y = b + x - ab = x + b(1-a) \iff y \equiv x \pmod{(1-a)}.\]

This implies that \( R(\phi) = |\mathbb{Z}_p/(1-a)\mathbb{Z}_p| \). But

\[
(1-a)\mathbb{Z}_p = p^{\nu_p(1-a)}\mathbb{Z}_p = |1-a|_p^{-1}\mathbb{Z}_p,
\]

so we can write \( R(\phi) = |1-a|_p^{-1} |a - 1|_p^{-1} \) and, more generally,

\[
R(\phi^n) = |1-a^n|_p^{-1} = |a^n - 1|_p^{-1}, \text{ for all } n \in \mathbb{N}.
\]

Now consider a group \( \mathbb{Z}_p^d, d \geq 2 \). It follows easily from Lemma 2.2, that \( \text{End}(\mathbb{Z}_p^d) = M_d(\mathbb{Z}_p) \). For any matrix \( A \in M_d(\mathbb{Z}_p) \) there exists a diagonal matrix \( D \in M_d(\mathbb{Z}_p) \) and unimodular matrices \( E, F \in M_d(\mathbb{Z}_p) \) such that \( D = EAF \).

**Lemma 2.3.** For endomorphism \( \phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d \) we have

\[
R(\phi_p) = \#\text{Coker}(1 - \phi_p) = |\det(\Phi_p - \text{Id})|_p^{-1},
\]

where \( \Phi_p \) is a matrix of \( \phi_p \).
Proof. Let matrices $D, E, F \in M_d(\mathbb{Z}_p)$ be such that $D = E(\text{Id} - \Phi_p)F$, where $D = (a_i)$ is diagonal matrix, $a_i \in \mathbb{Z}_p$, $1 \leq i \leq d$, and matrices $E, F$ are unimodular. Then we have

$$R(\phi_p) = \#\text{Coker}(1 - \phi_p) = |Z^d_p : (\text{Id} - \Phi_p)Z^d_p| = |Z^d_p : DZ^d_p| =$$

$$= |Z_p : a_1Z_p| \cdot |Z_p : a_2Z_p| \cdot \ldots \cdot |Z_p : a_dZ_p| = |a_1|^{-1} \cdot \ldots \cdot |a_d|^{-1} =$$

$$= |\det(D)|^{-1} = |\det(\text{Id} - \Phi_p)|^{-1} = |\det(\Phi_p - \text{Id})|^{-1}.$$

\[ \square \]

In order to handle the sequence $R(\phi^n) = |a^n - 1|_p^{-1}, n \in \mathbb{N}$ more easily, we need a way to evaluate expressions of the form $|a^n - 1|_p$ when $|a|_p = 1$. The following technical lemma is useful.

**Lemma 2.4.** ([27, Lemma 4.9]) Let $K_v$ be a non-archimedean local field and suppose $x \in K_v$ has $|x|_v = 1$ and infinite multiplicative order. Let $p > 0$ be the characteristic of the residue field $F_v$ and $\gamma \in \mathbb{N}$ the multiplicative order of the image of $x$ in $F_v$. Then $|x^n - 1|_v = 1$ whenever $(\gamma, n) = 1$ and $\gamma \neq 1$. Furthermore, there are constants $0 < C < 1$ and $r_0 > 0$ such that whenever $n = k\gamma p^r$ with $(p, k) = 1$ and $r > r_0$, then $|x^n - 1|_v = C|p|^r_v$ if $\text{char}(K_v) = 0$.

Now we prove a Pólya–Carlson dichotomy between rationality and a natural boundary for the analytic behaviour of the Reidemeister zeta function for endomorphisms of groups $\mathbb{Z}_p^d, d \geq 1$.

**Theorem 2.5.** Let $\phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ be a tame endomorphism and $\lambda_1, \ldots, \lambda_d \in \overline{\mathbb{Q}_p}$ be the eigenvalues of $\Phi_p$, including multiplicities. Then the Reidemeister zeta function $R_{\phi_p}(z)$ is either a rational function or it has the unit circle as a natural boundary. Furthermore, the latter occurs if and only if $|\lambda|_{1,p} = 1$ for some $i \in \{1, \ldots, d\}$.

**Proof.** Firstly, we consider separately the case of the group $\mathbb{Z}_p$ as it illustrates some of ideas needed for the proof of the dichotomy in general case when $d \geq 1$. Lemma 2.2 yields $\phi_p(x) = ax$, where $a \in \mathbb{Z}_p$. Hence $|a|_p \leq 1$. Then the Reidemeister numbers $R(\phi^n_p) = |a^n - 1|_p^{-1}$, for all $n \in \mathbb{N}$. If $|a|_p < 1$, then $R(\phi^n_p) = |a^n - 1|_p^{-1} = 1$, for all $n \in \mathbb{N}$. Hence the radius of convergence of $R_{\phi_p}(z)$ equals 1 and the Reidemeister zeta function $R_{\phi_p}(z) = \frac{1}{1 - z}$ is a rational function.

From now on, we shall write $a(n) \ll b(n)$ if there is a constant $c$ independent of $n$ for which $a(n) < c \cdot b(n)$. When $|a|_p = 1$, we show that the radius of convergence of $R_{\phi_p}(z)$ equals 1 by deriving the bound

$$\frac{1}{n} \ll |a^n - 1|_p \leq 1$$
Upper bound in (3) follows from the definition of the p-adic norm. We may suppose that $|a^n - 1|_p < 1$. Let $F$ denote the smallest field which contains $\mathbb{Q}_p$ and is both algebraically closed and complete with respect to $| \cdot |_p$. The p-adic logarithm $\log_p$ is defined as

$$\log_p(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n,$$

and converges for all $z \in F$ such that $|z|_p < 1$. Setting $z = a^n - 1$ we get

$$\log_p(a^n) = (a^n - 1) - \frac{(a^n - 1)^2}{2} + \frac{(a^n - 1)^3}{3} - \ldots$$

and so $|\log_p(a^n)|_p \leq |a^n - 1|_p$. We always have

$$\frac{1}{n} \ll |n \log_p(a)|_p = |\log_p(a^n)|_p,$$

so this establishes (3).

The bound (3) implies by Cauchy - Hadamard formula that the radius of convergence of $R_{\phi_p}(z)$ equals 1. Hence it remains to show that the Reidemeister zeta function $R_{\phi_p}(z)$ is irrational if $|a|_p = 1$. Then $R_{\phi_p}(z)$ has the unit circle as a natural boundary by the Lemma 2.1 and by the Pólya–Carlson Theorem. For a contradiction, assume that Reidemeister zeta function $R_{\phi_p}(z)$ is rational. Then Lemma 2.1 implies that the function $Z_p(z) = \sum_{n=1}^{\infty} R(\phi^n_p) z^n$ is rational also. Hence the sequence $R(\phi^n_p)$ satisfies a linear recurrence relation.

Let $q \neq p$ be a rational prime, and define

$$n(e) = q^e \gamma p^r,$$

where integer constant $r \geq 0$ and $e \geq 1$. Applying Lemma 2.4, we see that

$$R(\phi_p^{kn(e)}) = R(\phi_p^{n(e)})$$

whenever $k$ is coprime to $n(e)$. Hence the sequence $R(\phi^n_p)$ assumes infinitely many values infinitely often, and so it cannot satisfy a linear recurrence by a result of Myerson and van der Poorten [31, Prop. 2], giving a contradiction.

Now we consider the general case of a tame endomorphism $\phi_p : \mathbb{Z}_p^d \to \mathbb{Z}_p^d$, $d \geq 1$. According to the Lemma 2.3 we have

$$R(\phi^n_p) = \#\text{Coker}(1 - \phi^n_p) = |\det(\Phi^n_p - \text{Id})|^{-1} = \prod_{i=1}^{d} |\lambda_i^n - 1|_p^{-1},$$

where $\Phi_p$ is a matrix of $\phi_p$ and $\lambda_1, \lambda_2, \ldots \lambda_d \in \overline{\mathbb{Q}}_p$ are the eigenvalues of $\Phi_p$, including multiplicities. The polynomial $\prod_{i=1}^{d} (X - \lambda_i)$ has coefficients in $\mathbb{Z}_p$, in particular, $|\lambda_i|_p \leq 1$ for every $i \in \{1, \ldots, d\}$ (see [15]). If $|\lambda_i|_p < 1$,
for every \( i \in \{1, \ldots, d\} \) then 
\[
R(\phi_p^n) = \prod_{i=1}^d |\lambda_i^n - 1|_p^{-1} = 1, \text{ for all } n \in \mathbb{N}.
\]
Hence the radius of convergence of \( R_{\phi_p}(z) \) equals 1 and the Reidemeister zeta function \( R_{\phi_p}(z) = \frac{1}{1-z} \) is a rational function. If \( |\lambda_i|_p = 1 \), for some \( i \in \{1, \ldots, d\} \) then the bound (3) implies the bound
\[
\frac{1}{n^d} << R(\phi_p^n) = \prod_{i=1}^d |\lambda_i^n - 1|_p^{-1} \leq 1.
\]
Hence the radius of convergence of the Reidemeister zeta function \( R_{\phi_p}(z) \) equals 1 by the Cauchy–Hadamard formula and the bound (4). Now for the proof of the theorem it remains to show that the Reidemeister zeta satisfies a linear recurrence relation. Let \( q \neq p \) be a rational prime, and define \( n(e) = q^e \gamma_p^e \), where integer constant \( r \geq 0 \) and \( e \geq 1 \). Applying Lemma 2.4, we see that \( R(\phi_p^{kn(e)}) = R(\phi_p^{n(e)}) \) whenever \( k \) is coprime to \( n(e) \). Hence the sequence \( R(\phi_p^n) \) assumes infinitely many values infinitely often, and so it cannot satisfy a linear recurrence by a result of Myerson and van der Poorten [31, Prop. 2], giving a contradiction.

\[\Box\]

3. Pólya–Carlson dichotomy for the Reidemeister zeta function of continuous map

We assume \( X \) to be a path-connected topological space admitting a universal cover \( p: \tilde{X} \to X \) and \( f: X \to X \) to be a continuous map. Let \( \tilde{f}: \tilde{X} \to \tilde{X} \) a lifting of \( f \), ie. \( p \circ \tilde{f} = f \circ p \). Two liftings \( \tilde{f} \) and \( \tilde{f}' \) are called conjugate if there is a \( \gamma \in \Gamma \cong \pi_1(X) \) such that \( \tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1} \). The subset \( p(Fix(\tilde{f})) \subset Fix(f) \) is called the fixed point class of \( f \) determined by the lifting class \([\tilde{f}]\). The number of lifting classes of \( f \) (and hence the number of fixed point classes, empty or not) is called the Reidemeister Number of \( f \), denoted \( R(f) \). This is a positive integer or infinity. \( R(f) \) is homotopy invariant.

Let a specific lifting \( \tilde{f}: \tilde{X} \to \tilde{X} \) be chosen as reference. Let \( \Gamma \) be the group of covering translations of \( \tilde{X} \) over \( X \). Then every lifting of \( f \) can be written uniquely as \( \gamma \circ \tilde{f} \), with \( \gamma \in \Gamma \). So elements of \( \Gamma \) serve as coordinates of liftings with respect to the reference \( \tilde{f} \). Now for every \( \gamma \in \Gamma \) the composition \( \tilde{f} \circ \gamma \) is a lifting of \( f \) so there is a unique \( \gamma' \in \Gamma \) such that
This correspondence $\gamma \rightarrow \gamma'$ is determined by the reference $f$, and is obviously a homomorphism.

**Definition 3.1.** The endomorphism $\tilde{f}_*: \Gamma \rightarrow \Gamma$ determined by the lifting $\tilde{f}$ of $f$ is defined by

$$\tilde{f}_*(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma.$$ 

It is well known that $\Gamma \cong \pi_1(X)$. We shall identify $\pi = \pi_1(X, x_0)$ and $\Gamma$ in the following way. Pick base points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{p}^{-1}(x_0) \subset \tilde{X}$ once and for all. Now points of $\tilde{X}$ are in 1-1 correspondence with homotopy classes of paths in $X$ which start at $x_0$: for $\tilde{x} \in \tilde{X}$ take any path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$ and project it onto $X$; conversely for a path $c$ starting at $x_0$, lift it to a path in $\tilde{X}$ which starts at $\tilde{x}_0$, and then take its endpoint. In this way, we identify a point of $\tilde{X}$ with a path class $<c>$ in $X$ starting from $x_0$. Under this identification, $\tilde{x}_0 = <<e>>$ is the unit element in $\pi_1(X, x_0)$. The action of the loop class $\alpha = <a> \in \pi_1(X, x_0)$ on $\tilde{X}$ is then given by

$$\alpha = <a>:<c> \mapsto \alpha.c = <a.c>.$$ 

Now we have the following relationship between $\tilde{f}_*: \pi \rightarrow \pi$ and

$$f_*: \pi_1(X, x_0) \rightarrow \pi_1(X, f(x_0)).$$

**Lemma 3.2.** Suppose $\tilde{f}(\tilde{x}_0) = <w>$. Then the following diagram commutes:

$$\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(X, f(x_0)) \\
\tilde{f}_* \downarrow & & \downarrow w_* \\
\pi_1(X, x_0) & \xrightarrow{w_*} & \pi_1(X, x_0)
\end{array}$$

**Lemma 3.3.** [22] Lifting classes of $f$ are in 1-1 correspondence with $\tilde{f}_*$-conjugacy classes in $\pi$, the lifting class $[\gamma \circ \tilde{f}]$ corresponding to the $\tilde{f}_*$-conjugacy class of $\gamma$. We therefore have $R(f) = R(\tilde{f}_*)$.

Taking a dynamical point of view, we consider the iterates of $f$ assume that $R(f^n) < \infty$ for all $n > 0$. The Reidemeister zeta function of $f$ is defined [11] as power series:

$$R_f(z) := \exp \left( \sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n \right).$$

Using Lemma 3.3 we may apply the theorems of previous section about the Reidemeister zeta function for automorphisms to the Reidemeister zeta function of continuous maps.

**Theorem 3.4.** Suppose that the fundamental group $\Gamma$ of the topological space $X$ is a torsion abelian non-finitely generated group and $\tilde{f}_* : \Gamma \rightarrow \Gamma$ is
an automorphism. Suppose that the group $\Gamma$ as $R = \mathbb{Z}[t]$-module satisfies the following conditions:

(1) the set of associated primes $\text{Ass}(\Gamma)$ is finite and consists entirely of non-zero principal ideals of the polynomial ring $R = \mathbb{Z}[t]$,

(2) the map $g \to (t^j - 1)g$ is a monomorphism of $\Gamma$ for all $j \in \mathbb{N}$ (equivalently, $t^j - 1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(\Gamma)$ and all $j \in \mathbb{N}$),

(3) for each $\mathfrak{p} \in \text{Ass}(\Gamma)$, $m(\mathfrak{p}) = \dim_{\mathbb{K}(\mathfrak{p})} G_{\mathfrak{p}} < \infty$.

Then there exist function fields $\mathbb{K}_1, \ldots, \mathbb{K}_n$ of the form $\mathbb{K}_i = \mathbb{F}_{p(i)}(t)$, where each $p(i)$ is a rational prime, sets of finite places $P_i \subset \mathcal{P}(\mathbb{K}_i)$, sets of infinite places $P_i^\infty \subset \mathcal{P}(\mathbb{K}_i)$, sets of finite places $S_i = \mathcal{P}(\mathbb{K}_i) \setminus (P_i^\infty \cup P_i)$ such that

\begin{equation}
R(f^j) = \prod_{i=1}^n \prod_{v \in P_i} |t^j - 1|_v^{-1} = \prod_{i=1}^n |t^j - 1|_{P_i}^{-1} = \prod_{i=1}^n |t^j - 1|_{P_i^\infty \cup S_i}
\end{equation}

for all $j \in \mathbb{N}$. If the Reidemeister zeta function $R_f(z) = R_{\hat{f}_s}(z)$ has radius of convergence $R = e^{-h(f)}$, where $h(f) = \sum_{i=1}^r \log p(i)$, then it is either a rational function, namely

$$R_f(z) = R_{\hat{f}_s}(z) = (1 - e^{h(f)}z)^{-1}$$

or the circle $|z| = e^{-h(f)}$ is a natural boundary for the function $R_f(z) = R_{\hat{f}_s}(z)$.

Similarly, theorem 5.8 in [19] implies the following Pólya-Carlson dichotomy between rationality and a natural boundary for the Reidemeister zeta function of continuous map.

**Theorem 3.5.** Suppose that the fundamental group $\Gamma$ of the topological space $X$ is a countable abelian group that is a subgroup of $\mathbb{Q}^d$, where $d \geq 1$ and $\hat{f}_s : \Gamma \to \Gamma$ is an automorphism. Suppose that the group $\Gamma$ as $R = \mathbb{Z}[t]$-module satisfies the following conditions:

(1) the set of associated primes $\text{Ass}(\Gamma)$ is finite and consists entirely of non-zero principal ideals of the polynomial ring $R = \mathbb{Z}[t]$,

(2) the map $g \to (t^j - 1)g$ is a monomorphism of $\Gamma$ for all $j \in \mathbb{N}$ (equivalently, $t^j - 1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}(\Gamma)$ and all $j \in \mathbb{N}$),

(3) for each $\mathfrak{p} \in \text{Ass}(\Gamma)$, $m(\mathfrak{p}) = \dim_{\mathbb{K}(\mathfrak{p})} G_{\mathfrak{p}} < \infty$.

Then there exist algebraic number fields $\mathbb{K}_1, \ldots, \mathbb{K}_n$,

sets of finite places $P_i \subset \mathcal{P}(\mathbb{K}_i)$, $S_i = \mathcal{P}(\mathbb{K}_i) \setminus P_i$, and elements $\xi_i \in \mathbb{K}_i$, no one of which is a root of unity for $i = 1, \ldots, n$, such that

\begin{equation}
R(f^j) = \prod_{i=1}^n \prod_{v \in P_i} |\xi_i^j - 1|_v^{-1} = \prod_{i=1}^n |\xi_i^j - 1|_{P_i}^{-1} = \prod_{i=1}^n |\xi_i^j - 1|_{P_i^\infty \cup S_i}
\end{equation}

for all $j \in \mathbb{N}$. 
Suppose that the last product in (6) only involves finitely many places and that \(|\xi_i|_v \neq 1\) for all \(v\) in the set of infinite places \(P_i^\infty\) of \(K\) and all \(i = 1, \ldots, n\).

Then the Reidemeister zeta function \(R_f(z) = R_{\tilde{f}_i}(z)\) is either rational function or has a natural boundary at its circle of convergence, and the latter occurs if and only if \(|\xi_i|_v = 1\) for some \(v \in S_i, 1 \leq i \leq n\).

4. The rationality of the coincidence Reidemeister zeta function for endomorphisms of finitely generated torsion-free nilpotent groups

Example 4.1. ([15], Example 1.3) Let \(G = \mathbb{Z}\) be the infinite cyclic group, written additively, and let

\[
\phi : \mathbb{Z} \to \mathbb{Z}, \quad x \mapsto d_\phi x \quad \text{and} \quad \psi : \mathbb{Z} \to \mathbb{Z}, \quad x \mapsto d_\psi x
\]

for \(d_\phi, d_\psi \in \mathbb{Z}\). The coincidence Reidemeister number \(R(\phi, \psi)\) of endomorphisms \(\phi, \psi\) of an Abelian group \(G\) coincides with the cardinality of the quotient group \(\text{Coker}(\phi - \psi) = G/\text{Im}(\phi - \psi)\) (or \(\text{Coker}(\psi - \phi) = G/\text{Im}(\psi - \phi)\)). Hence we have

\[
R(\phi^n, \psi^n) = \begin{cases} 
|d_\psi^n - d_\phi^n| & \text{if } d_\phi^n \neq d_\psi^n, \\
\infty & \text{otherwise.}
\end{cases}
\]

Consequently, \((\phi, \psi)\) is tame precisely when \(|d_\phi| \neq |d_\psi|\) and, in this case,

\[
R_{\phi,\psi}(z) = \frac{1 - d_2 z}{1 - d_1 z} \quad \text{where } d_1 = \max\{|d_\phi|, |d_\psi|\} \text{ and } d_2 = \frac{d_\phi d_\psi}{d_1}.
\]

This simple example (or at least special cases of it) are known. The aim of the current section is to generalise this example to finitely generated torsion-free nilpotent groups. Let \(G\) be a finitely generated group and \(\phi, \psi : G \to G\) two endomorphisms.

Lemma 4.2. Let \(\phi, \psi : G \to G\) are two automorphisms. Two elements \(x, y\) of \(G\) are \(\psi^{-1}\phi\)-conjugate if and only if elements \(\psi(x)\) and \(\psi(y)\) are \((\psi, \phi)\)-conjugate. Therefore the Reidemeister number \(R(\psi^{-1}\phi)\) is equal to \(R(\phi, \psi)\). For a tame pair of commuting automorphisms \(\phi, \psi : G \to G\) the coincidence Reidemeister zeta function \(R_{\phi,\psi}(z)\) is equal to the Reidemeister zeta function \(R_{\psi^{-1}\phi}(z)\).

Proof. If \(x\) and \(y\) are \(\psi^{-1}\phi\)-conjugate, then there is a \(g \in G\) such that \(x = g \psi \psi^{-1}\phi(g^{-1})\). This implies \(\psi(x) = \psi(g)\psi(y)\phi(g^{-1})\). So \(\psi(x)\) and \(\psi(y)\) are \((\phi, \psi)\)-conjugate. The converse statement follows if we move in opposite direction in previous implications. \(\Box\)
We assume \( X \) to be a connected, compact polyhedron and \( f : X \to X \) to be a continuous map. The Lefschetz zeta function of a discrete dynamical system \( f^n \) is defined as 
\[
L(f^n) := \exp \left( \sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n \right),
\]
where
\[
L(f^n) := \dim X \sum_{k=0}^{\dim X} (-1)^k \text{tr} \left( f^n_{*k} : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q}) \right)
\]
is the Lefschetz number of the iterate \( f^n \) of \( f \). The Lefschetz zeta function is a rational function of \( z \) and is given by the formula:
\[
L(f)(z) = \prod_{k=0}^{\dim X} \det \left( I - f_{*k} \cdot z \right)^{(-1)^k+1}.
\]

In this section we consider finitely generated torsion-free nilpotent group \( \Gamma \). It is well known [25] that such group \( \Gamma \) is a uniform discrete subgroup of a simply connected nilpotent Lie group \( G \) (uniform means that the coset space \( G/\Gamma \) is compact). The coset space \( M = G/\Gamma \) is called a nilmanifold. Since \( \Gamma = \pi_1(M) \) and \( M \) is a \( K(\Gamma, 1) \), every endomorphism \( \phi : \Gamma \to \Gamma \) can be realized by a selfmap \( f : M \to M \) such that \( f_{*} = \phi \) and thus \( R(f) = R(\phi) \). Any endomorphism \( \phi : \Gamma \to \Gamma \) can be uniquely extended to an endomorphism \( F : G \to G \). Let \( \bar{F} : \bar{G} \to \bar{G} \) be the corresponding Lie algebra endomorphism induced from \( F \).

**Lemma 4.3.** (cf. Theorem 23 of [12] and Theorem 5 of [14]) Let \( \phi : \Gamma \to \Gamma \) be a tame endomorphism of a finitely generated torsion free nilpotent group. Then the Reidemeister zeta function \( R_\phi(z) = R_f(z) \) is a rational function and is equal to
\[
R_\phi(z) = R_f(z) = L_f((-1)^p z)^{(-1)^r},
\]
where \( p \) the number of \( \mu \in \text{Spectr}(\bar{F}) \) such that \( \mu < -1 \), and \( r \) the number of real eigenvalues of \( \bar{F} \) whose absolute value is \( > 1 \).

Every pair of automorphisms \( \phi, \psi : \Gamma \to \Gamma \) of finitely generated torsion free nilpotent group \( \Gamma \) can be realized by a pair of homeomorphisms \( f, g : M \to M \) such that \( f_{*} = \phi \), \( g_{*} = \psi \) and thus \( R(g^{-1}f) = R(\psi^{-1}\phi) = R(\phi, \psi) \). An automorphism \( \psi^{-1}\phi : \Gamma \to \Gamma \) can be uniquely extended to an automorphism \( L : G \to G \). Let \( \bar{L} : \bar{G} \to \bar{G} \) be the corresponding Lie algebra automorphism induced from \( L \).

Lemma 4.2 and Lemma 4.3 imply the following

**Theorem 4.4.** Let \( \phi, \psi : \Gamma \to \Gamma \) be a tame pair of commuting automorphisms of a finitely generated torsion-free nilpotent group \( \Gamma \). Then the coincidence Reidemeister zeta function \( R_{\phi,\psi}(z) \) is a rational function and is
equal to

\[ R_{\phi,\psi}(z) = R_{\psi^{-1}\phi}(z) = R_{g^{-1}f}(z) = L_{g^{-1}f}((-1)^{\mu}z)^{(-1)^r}, \]

where \( p \) the number of \( \mu \in \text{Spectr}(\tilde{L}) \) such that \( \mu < -1 \), and \( r \) the number of real eigenvalues of \( \tilde{L} \) whose absolute value is \( > 1 \).

For an arbitrary group \( G \), we can define the \( k \)-fold commutator group \( \gamma_k(G) \) inductively as \( \gamma_1(G) := G \) and \( \gamma_{k+1}(G) := [G, \gamma_k(G)] \). Let \( G \) be a group. For a subgroup \( H \leq G \), we define the isolator \( \sqrt[\gamma_k]{H} \) of \( H \) in \( G \) as:

\[ \sqrt[\gamma_k]{H} = \{ g \in G \mid g^n \in H \text{ for some } n \in \mathbb{N} \}. \]

Note that the isolator of a subgroup \( H \leq G \) doesn’t have to be a subgroup in general. For example, the isolator of the trivial group is the set of torsion elements of \( G \).

**Lemma 4.5.** (see [5], Lemma 1.1.2 and Lemma 1.1.4) Let \( G \) be a group. Then

(i) for all \( k \in \mathbb{N} \), \( \sqrt[\gamma_k]{G} \) is a fully characteristic subgroup of \( G \),
(ii) for all \( k \in \mathbb{N} \), the factor \( G/\sqrt[\gamma_k]{G} \) is torsion-free,
(iii) for all \( k, l \in \mathbb{N} \), the commutator \( \sqrt[\gamma_k]{G}, \sqrt[\gamma_l]{G} \) \( \leq \sqrt[\gamma_{k+l}]{G} \),
(iv) for all \( k, l \in \mathbb{N} \) such that \( k \geq l \) if \( M := \sqrt[\gamma_l]{G} \), then

\[ \frac{G/M}{\sqrt[\gamma_k]{G/M}} = \frac{\sqrt[\gamma_l]{G}}{M}. \]

We define the adapted lower central series of a group \( G \) as

\[ G = \sqrt[\gamma_1]{G} \geq \sqrt[\gamma_2]{G} \geq \ldots \geq \sqrt[\gamma_k]{G} \geq \ldots, \]

where \( \gamma_k(G) \) is the \( k \)-th commutator of \( G \).

The adapted lower central series will terminate if and only if \( G \) is a torsion-free, nilpotent group. Moreover, all factors \( \sqrt[\gamma_k]{G}/\sqrt[\gamma_{k+1}]{G} \) are torsion-free.

We are particularly interested in the case where \( G \) is a finitely generated, torsion-free, nilpotent group. In this case the factors of the adapted lower central series are finitely generated, torsion-free, abelian groups, i.e. for all \( k \in \mathbb{N} \) we have that

\[ \frac{\sqrt[\gamma_k]{G}}{\sqrt[\gamma_{k+1}]{G}} \cong \mathbb{Z}^{d_k}, \text{ for some } d_k \in \mathbb{N}. \]

Let \( N \) be a normal subgroup of a group \( G \) and \( \phi, \psi \in \text{End}(G) \) with \( \phi(N) \subseteq N, \psi(N) \subseteq N \). We denote the restriction of \( \phi \) to \( N \) by \( \phi|_N \), \( \psi \) to \( N \) by \( \psi|_N \) and the induced endomorphisms on the quotient \( G/N \) by \( \phi', \psi' \) respectively. We then get the following commutative diagrams with exact rows:
Theorem 4.7. Let $N$ be a finitely generated, torsion-free, nilpotent group and

$$N = \sqrt[γ_1(N)]{\sqrt[γ_2(N)]{\cdots \sqrt[γ_c(N)]{\sqrt[γ_{c+1}(N)]{1}}} = 1}$$

be an adapted lower central series of $N$. Suppose that $R(φ, ψ) < \infty$ and $R(φ_k, ψ_k) < \infty$ for a pair $φ, ψ$ of endomorphisms of $N$ and for every pair $φ_k, ψ_k$, of induced endomorphisms on the finitely generated torsion-free abelian factors

$$\sqrt[γ_k(N)]{\sqrt[γ_{k+1}(N)]{Z^{d_k}}} , d_k \in \mathbb{N}, 1 \leq k \leq c,$$
then

\[ R(\phi, \psi) = \prod_{k=1}^{c} R(\phi_k, \psi_k). \]

**Proof.** We will prove the product formula for the coincidence Reidemeister numbers by induction on the length of an adapted lower central series. Let us denote \( \sqrt[k]{\gamma_k(N)} \) as \( N_k \). If \( c = 1 \), the result follows trivially. Let \( c > 1 \) and assume the product formula holds for a central series of length \( c - 1 \).

Let \( \phi, \psi \in \text{End}(N) \), then \( \phi(N_c) \subseteq N_c, \psi(N_c) \subseteq N_c \) and hence we have the following commutative diagram of short exact sequences:

\[
\begin{array}{cccccccc}
1 & \longrightarrow & N_c & \xrightarrow{i} & N & \xrightarrow{p} & N/N_c & \longrightarrow & 1 \\
& & \phi_c, \psi_c & \downarrow & \phi , \psi & \downarrow & \phi', \psi' & \quad & \\
1 & \longrightarrow & N_c & \xrightarrow{i} & N & \xrightarrow{p} & N/N_c & \longrightarrow & 1 \end{array}
\]

where \( \phi_c, \psi_c \) are induced endomorphisms on the \( N_c \). The quotient \( N/N_c \) is a finitely generated, nilpotent group with a central series \( N/N_c = N_1/N_c \geq N_2/N_c \geq ... \geq N_{c-1}/N_c \geq N_c/N_c = 1 \) of length \( c - 1 \).

Every factor of this series is of the form \( (N_k/N_c)/(N_{k+1}/N_c) \approx N_k/N_{k+1} \) by the third isomorphism theorem, hence it is also torsion-free. Moreover, because of this natural isomorphism we know that for every induced pair of endomorphisms \( (\phi'_k, \psi'_k) \) on \( (N_k/N_c)/(N_{k+1}/N_c) \) it is true that \( R(\phi'_k, \psi'_k) = R(\phi_k, \psi_k) \).

The assumptions of the theorem imply, that \( R(\phi', \psi') < \infty \) and that \( R(\phi_c, \psi_c) < \infty \).

Moreover, let \( [g_1 N_c]_{\phi', \psi'}, ..., [g_n N_c]_{\phi', \psi'} \) be the \( (\phi', \psi') \) – Reidemeister classes and \( [c_1]_{\phi_c, \psi_c}, ..., [c_m]_{\phi_c, \psi_c} \) – the \( (\phi_c', \psi'_c) \)-Reidemeister classes. Since \( M_c \subseteq Z(N) \), by Lemma 4.6 we obtain that \( R(\phi, \psi) \leq R(\phi_c, \psi_c) R(\phi', \psi') \).

To prove the opposite inequality it suffices to prove that every Reidemeister class \( [c_i g_j]_{\phi, \psi} \) represents a different \( (\phi, \psi) \)-Reidemeister class. Then we obtain

\[ R(\phi, \psi) = R(\phi_c, \psi_c) R(\phi', \psi') \]

and then the theorem follows from the induction hypothesis.

Suppose, that there exists some \( h \in N \) such that \( c_i g_j = \psi(h) c_l g_m \phi(h)^{-1} \).
Then by taking the projection to $N/N_c$ we find that
\[ g_jN_c = p(c_i g_j) = p(\psi(h)c_a g_j \phi(h)^{-1}) = \psi(hN_c)(g_b N_c)\phi(hN_c)^{-1}. \]

Hence $[g_j N_c]_{\phi', \psi'} = [g_b N_c]_{\phi', \psi'}$. Assume that $c_i g_j = \psi(h) c_a g_j \phi(h)^{-1}$. If $h \in N_c \subseteq Z(N)$, then $c_i g_j = \psi(h) c_a \phi(h)^{-1} g_j$ and consequently $[c_i]_{\phi_c, \psi_c} = [c_a]_{\phi_c, \psi_c}$, so let us assume that $h \notin N_c$ and that $N_k$ is the smallest group in the central series which contains $h$. Then
\[ c_i g_j = \psi(h) c_a g_j \phi(h)^{-1} \iff g_j c_i = \psi(h) c_a g_j \phi(h)^{-1} \iff c_i = g_j^{-1} \psi(h) c_a g_j \phi(h)^{-1} \iff c_i = g_j^{-1} \psi(h) g_j \phi(h)^{-1} c_a \iff c_i c_a^{-1} = g_j^{-1} \psi(h) g_j \phi(h)^{-1} \]
and therefore
\[ c_i c_a^{-1} N_{k+1} = g_j^{-1} \psi(h) g_j \phi(h)^{-1} N_{k+1} = [g_j, \psi(h)^{-1}] (\psi(h) \phi(h)^{-1}) N_{k+1}. \]

As $c_i c_a^{-1} \subseteq N_c \subseteq N_{k+1}$ and $[g_j, \psi(h)^{-1}] \subseteq N_{k+1}$, we find that
\[ (\psi_k)(hN_{k+1}) = (\psi_k)(hN_{k+1}). \]

That means that the set of coincidence points $\text{Coin}(\phi'_k, \psi'_k) \neq \{1\}$, which implies that $R(\phi', \psi') = \infty$ and this contradicts assumption.

**Theorem 4.8.** Let $\phi, \psi: N \to N$ be a tame pair of endomorphisms of a finitely generated torsion-free nilpotent group $N$. Let $c$ denote the nilpotency class of $N$ and, for $1 \leq k \leq c$, let $\phi_k, \psi_k: G_k \to G_k$, $1 \leq k \leq c$, denote the tame pairs of induced endomorphisms of the finitely generated torsion-free abelian factor groups

\[ G_k = N_k / N_{k+1} = \sqrt[\gamma_k(N)]{ \sqrt[\gamma_{k+1}(N)]{N_k}} \cong \mathbb{Z}^{d_k}, \]

for some $d_k \in \mathbb{N}$, that arise from an adapted lower central series of $N$. Then the following hold.

(1) For each $n \in \mathbb{N}$,
\[ R(\phi^n, \psi^n) = \prod_{k=1}^{c} R(\phi_k^n, \psi_k^n) \quad \text{for } n \in \mathbb{N}. \]

(2) For $1 \leq k \leq c$, let
\[ \phi_{k, Q}, \psi_{k, Q}: G_k \to G_k, \]

where $Q$ is a finite set of primes.
denote the extensions of $\phi_k, \psi_k$ to the divisible hull $G_{k,\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} G_k \cong \mathbb{Q}^{d_k}$ of $G_k$. Suppose that each pair of endomorphisms $\phi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}}$ is simultaneously triangularisable. Let $\xi_{k,1}, \ldots, \xi_{k,d_k}$ and $\eta_{k,1}, \ldots, \eta_{k,d_k}$ be the eigenvalues of $\phi_{k,\mathbb{Q}}$ and $\psi_{k,\mathbb{Q}}$ in the field $\mathbb{C}$, including multiplicities, ordered so that, for $n \in \mathbb{N}$, the eigenvalues of $\phi_{k,\mathbb{Q}}^n - \psi_{k,\mathbb{Q}}^n$ are $\xi_{k,1}^n - \eta_{k,1}^n, \ldots, \xi_{k,d_k}^n - \eta_{k,d_k}^n$. Then for each $n \in \mathbb{N}$,

$$R(\phi_k^n, \psi_k^n) = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|;$$

(3) Moreover, suppose that $|\xi_{k,i}| \neq |\eta_{k,i}|$ for $1 \leq k \leq c, 1 \leq i \leq d_k$. If $\phi, \psi$ is a tame pair of endomorphisms of $N$ and $\phi_{k,\mathbb{Q}}, \psi_{k,\mathbb{Q}}, 1 \leq k \leq c$, are simultaneously triangularisable pairs of endomorphisms of $G_{k,\mathbb{Q}}$, then the coincidence Reidemeister number $R(\phi, \psi)$ is a rational function.

**Proof.** The coincidence Reidemeister number $R(\phi, \psi)$ of automorphisms $\phi, \psi$ of an Abelian group $G$ coincides with the cardinality of the quotient group $\text{Coker} (\phi - \psi) = G/\text{Im}(\phi - \psi)$ (or $\text{Coker} (\psi - \phi) = G/\text{Im}(\psi - \phi)$).

For $1 \leq k \leq c$, let tame pairs $\phi_k, \psi_k: G_k \to G_k$ of induced endomorphisms of the finitely generated torsion-free abelian factor groups $G_k = N_k/N_{k+1} \cong \mathbb{Z}^{d_k}$, are represented by integer matrices $A_k, B_k \in M_{d_k}(\mathbb{Z})$ associated to them respectively. There is a diagonal matrix $C_k = \text{diag}(c_1, \ldots, c_{d_k})$ such that $C_k = M_k(A_k - B_k)N_k$; where $M_k$ and $N_k$ are unimodular matrices. Now we have $\det C_k = \det(A_k - B_k)$ and the order of the cokernel of $\phi_k - \psi_k$ is the order of the group $\mathbb{Z}/c_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/c_{d_k}\mathbb{Z}$. Thus the order of the cokernel of $\phi_k - \psi_k$ is $|\text{Coker} (\phi_k - \psi_k)| = |c_1\cdots c_{d_k}| = |\det C_k| = |\det(\phi_k - \psi_k)|$.

Then for each $n \in \mathbb{N}$ and $1 \leq k \leq c$, $R(\phi_k^n, \psi_k^n) = |\text{Coker} (\phi_k - \psi_k)| = |
\det(\phi_k - \psi_k)| = |\det(\phi_k,\mathbb{Q} - \psi_k,\mathbb{Q})| = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|.$

Now we will prove the rationality of $R(\phi,\psi)(z)$. We open up the absolute values in the product $R(\phi_k^n, \psi_k^n) = \prod_{i=1}^{d_k} |\xi_{k,i}^n - \eta_{k,i}^n|, 1 \leq k \leq c$. Complex eigenvalues $\xi_{k,i}$ in the spectrum of $\phi_{k,\mathbb{Q}}$, respectively $\eta_{k,i}$ in the spectrum of $\psi_{k,\mathbb{Q}}$, appear in pairs with their complex conjugate $\overline{\xi_{k,i}}$, respectively $\overline{\eta_{k,i}}$.

Moreover, such pairs can be lined up with one another in a simultaneous triangularisation as follows. Write $\phi_{k,\mathbb{C}}, \psi_{k,\mathbb{C}}$ for the induced endomorphisms of the $\mathbb{C}$-vector space $V = \mathbb{C} \otimes_{\mathbb{Q}} G \cong \mathbb{C}^{d_k}$. If $v \in V$ is, at the same time, an eigenvector of $\phi_{k,\mathbb{C}}$ with complex eigenvalue $\xi_{k,d_k}$ and an eigenvector of $\psi_{k,\mathbb{C}}$ with eigenvalue $\eta_{k,d_k}$, then there is $w \in V$ such that $w$ is, at the same time, an eigenvector of $\phi_{k,\mathbb{C}}$ with eigenvalue $\xi_{k,d_k}$ and an eigenvector of $\psi_{k,\mathbb{C}}$ with eigenvalue $\eta_{k,d_k}$, possibly equal to $\eta_{k,d_k}$. Thus we can start our complete flag of $\{\phi, \psi\}$-invariant subspaces of $V$ with $\{0\} \subset \langle v \rangle \subset \langle v, w \rangle$, and proceed with $V/\langle v, w \rangle$ by induction to produce the
rest of the flag in the same way, treating complex eigenvalues of $\psi_{k,i}$ in the same way as they appear. If at least one of $\xi_{k,i}, \eta_{k,i}$ is complex so that these eigenvalues of $\phi_{Q}$ and $\psi_{Q}$ are paired with eigenvalues $\xi_{k,j}, \eta_{k,j}$ for suitable $j \neq i$, as discussed above, we see that

$$|\xi_{k,i}^{n} - \eta_{k,i}^{n}| = |\xi_{k,j}^{n} - \eta_{k,j}^{n}| = (\xi_{k,i}^{n} - \eta_{k,i}^{n}) \cdot (\xi_{k,j}^{n} - \eta_{k,j}^{n}).$$

If $\xi_{k,i}$ and $\eta_{k,i}$ are both real eigenvalues of $\phi_{Q}$ and $\psi_{Q}$, not paired up with another pair of eigenvalues, then exactly as in Example 4.1 above we have

$$|\xi_{k,i}^{n} - \eta_{k,i}^{n}| = |\xi_{k,j}^{n} - \eta_{k,j}^{n}| = \delta_{1,k,i}, \quad \delta_{2,k,i}, \quad \text{where } \delta_{1,k,i} = \max\{|\xi_{k,i}|, |\eta_{k,i}|\} \quad \text{and } \delta_{2,k,i} = \frac{\xi_{k,i} \cdot \eta_{k,i}}{\delta_{1,k,i}}.$$

Hence we can expand each product $R(\phi_{k,i}^{n}, \psi_{k,i}^{n})$, $1 \leq k \leq c$ using an appropriate symmetric polynomial, to obtain for the Reidemeister numbers $R(\phi^{n}, \psi^{n})$ an expression of the form

$$R(\phi^{n}, \psi^{n}) = \prod_{k=1}^{c} \prod_{i=1}^{d_{k}} |\xi_{k,i}^{n} - \eta_{k,i}^{n}| = \sum_{j \in J} c_{j} w_{j}^{n},$$

where $J$ is a finite index set, $c_{j} \in \{-1, 1\}$ and $\{w_{j} \mid j \in J\} \subseteq \mathbb{C} \setminus \{0\}$. Consequently, the coincidence Reidemeister zeta function can be written as

$$R_{\phi, \psi}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{R(\phi^{n}, \psi^{n})}{n} z^{n} \right) = \exp \left( \sum_{j \in J} c_{j} \sum_{n=1}^{\infty} \frac{(w_{j} z)^{n}}{n} \right).$$

and it follows immediately that $R_{\phi, \psi}(z) = \prod_{j \in J} (1 - w_{j} z)^{-c_{j}}$ is a rational function.

□

REFERENCES

[1] J. Bell, R. Miles, T. Ward, Towards a Pólya–Carlson dichotomy for algebraic dynam-ics, Indag. Math.(N.S. 25 (2014), no. 4, 652-668.
[2] J. Byyszewski and G. Cornelissen, Dynamics on abelian varieties in positive charac-teristic, with an appendix by R. Royals and T. Ward, Algebra Number Theory 12 (2018), 2185–2235.
[3] F. Carlson, ‘Über ganzwertige Funktionen’, Math. Z. 11 (1921), no. 1-2, 1–23.
[4] V. Chothi, G. Everest, and T. Ward, S-integer dynamical systems: periodic points, J. Reine Angew. Math. 489 (1997), 99–132.
[5] Dekimpe K. Almost-Bieberbach groups: affine and polynomial structures. Vol. 1639. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1996, pp. x+259
[6] Dekimpe, K. and Dugartein, G.-J. Nielsen zeta functions for maps on infra-nilmanifolds are rational, J. Fixed Point Theory Appl., 17(2)(2015), 355–370.
[7] Karel Dekimpe, Sam Tertooy, and Iris Van den Bussche, Reidemeister zeta functions of low-dimensional almost-crystallographic groups are rational, Communications in Algebra, 46 (9)(2018), 4090–4103.
[8] D. Eisenbud, Commutative algebra, volume 150 of Graduate Texts in Mathematics, Springer - Verlag, New York, 1995. With a view toward algebraic geometry.
[9] G. Everest, V. Stangoe, and T. Ward, ‘Orbit counting with an isometric direction’, in Algebraic and topological dynamics, in Contemp. Math. 385 (2005), pp. 293–302, Amer. Math. Soc., Providence, RI.

[10] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, Recurrence sequences, in Mathematical Surveys and Monographs 104, Amer. Math. Soc., Providence, RI, 2003.

[11] A. L. Fel’shtyn, The Reidemeister zeta function and the computation of the Nielsen zeta function, Colloq. Math., 62, (1991), 153–166.

[12] A. Fel’shtyn, Dynamical zeta functions, Nielsen theory and Reidemeister torsion, Mem. Amer. Math. Soc., 699, Amer. Math. Soc., Providence, R.I. 2000.

[13] A. L. Fel’shtyn and R. Hill, The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion, K-theory, 8 (1994), 367–393.

[14] A. L. Fel’shtyn, R. Hill and P. Wong, Reidemeister numbers of equivariant maps, Topology Appl., 67 (1995), 119–131.

[15] Alexander Fel’shtyn and Benjamin Klopsch, Pólya–Carlson dichotomy for coincidence Reidemeister zeta functions via profinite completions. e-print, 2021, arXiv:2102.10900(to appear in Indagationes Mathematicae, 2022).

[16] Alexander Fel’shtyn and Jong Bum Lee, The Nielsen and Reidemeister numbers of maps on infra-solvmanifolds of type (R). Topology Appl. 181 (2015), 62–103.

[17] A. Fel’shtyn, E. Troitsky, Pólya–Carlson dichotomy for dynamical zeta functions and twisted Burnside-Frobenius theorem. Russ. J. Math. Phys. 28 (2021), No. 4, 455–463.

[18] A. Fel’shtyn, E. Troitsky, and M. Zietek. New Zeta Functions of Reidemeister Type and the Twisted Burnside-Frobenius Theory. Russ. J. Math. Phys. 27(2020), No. 2, 199–211.

[19] A. Fel’shtyn and M. Zietek, Dynamical zeta functions of Reidemeister type and representations spaces, 57–81, in: Contemp. Math. 744(2020), 57–81, Amer. Math. Soc., Providence, R.I.

[20] A. Fel’shtyn and M. Zietek, Dynamical zeta functions of Reidemeister type, Topological Methods Nonlinear Anal. 56 (2020), 433-455.

[21] Daciiberg L. Gonçalves, Peter N.-S. Wong: Homogenous spaces in coincidence theory. Forum Math. 17 (2005), 297–313.

[22] B. Jiang, Nielsen Fixed Point Theory, Contemp. Math. 14, Birkhäuser, 1983.

[23] L. Li, On the rationality of the Nielsen zeta function, Adv. in Math. (China), 23 (1994) no. 3, 251–256.

[24] D. A. Lind and T. Ward, Automorphisms of solenoids and p-adic entropy, Ergodic Theory Dynam. Systems 8 (1988), no. 3, 411–419.

[25] A. Mal’cev, On a class of homogeneous spaces. Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya, 13 (1949), 9-32.

[26] H. Matsumura, Commutative ring theory, volume 8 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[27] R. Miles, Zeta functions for elements of entropy rank-one actions, Ergodic Theory Dynam. Systems 27 (2007), no. 2, 567–582.

[28] R. Miles, Periodic points of endomorphisms on solenoids and related groups, Bull. Lond. Math. Soc. 40 (2008), no. 4, 696–704.

[29] R. Miles, Synchronization points and associated dynamical invariants, Trans. Amer. Math. Soc. 365 (2013), 5503–5524.
[30] R. Miles and T. Ward, The dynamical zeta function for commuting automorphisms of zero-dimensional groups, Ergodic Theory Dynam. Systems 38 (2018), no. 4, 1564–1587.

[31] G. Myerson and A. J. van der Poorten, Some problems concerning recurrence sequences, Amer. Math. Monthly 102 (1995), no. 8, 698–705.

[32] G. Pólya, ‘Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe’, Math. Ann. 99 (1928), no. 1, 687–706.

[33] K. Schmidt, Dynamical systems of algebraic origin, in Progress in Mathematics 128, Birkhäuser Verlag, Basel, 1995.

[34] S. L. Segal. Nine introductions in complex analysis, volume 208 of North-Holland Mathematics Studies. Elsevier Science B.V., Amsterdam, revised edition, 2008.

[35] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., 73 (1967), 747–817.

[36] Evgenij Troitsky, Two examples related to the twisted Burnside-Frobenius theory for infinitely generated groups, Fundam. Appl. Math. 21(2016), No. 5, 231–239.

[37] A. Weil, Basic number theory, in Die Grundlehren der mathematischen Wissenschaften, Band 144, Springer-Verlag New York, Inc., New York, 1967.

[38] P. Wong, Reidemeister zeta function for group extensions, J. Korean Math. Soc., 38 (2001), 1107–1116.

WOJCIECH BONDAREWICZ, INSTYTUT MATematyki, Uniwersytet Szczeciński, ul. Wielkopolska 15, 70-451 Szczecin, Poland
Email address: wojciech.bondarewicz@usz.edu.pl

ALEXANDER FEL’SHTYN, INSTYTUT Matematyki, Uniwersytet Szczeciński, ul. Wielkopolska 15, 70-451 Szczecin, Poland
Email address: alexander.felshyn@usz.edu.pl

MALWINA ZIETEK, INSTYTUT Matematyki, Uniwersytet Szczeciński, ul. Wielkopolska 15, 70-451 Szczecin, Poland
Email address: malwina.zietek@gmail.com