A Numerical Study for Solving Mathematical Fuzzy Population Model

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Abstract. Population models are used to determine maximum harvest for agriculturists and to understand the dynamics of biological invasions and for environmental conservation. Also, population models are used to understand the spread of parasites, viruses and disease. However, the mathematical model of population can be used to make future prediction of the growth of population. In this paper, numerical study for solving mathematical fuzzy population model has been introduced. The development of direct explicit numerical integrators of RK-type to be consistent with solving fuzzy ordinary differential equations. The computational efficiency of the numerical method has been developed. The advantage and the efficiency of the proposed method have been shown clearly using the numerical results which are agree well with analytical solutions due to the proposed method is more efficient and accurate method.

Keyword: Fuzzy; Fuzzy Differential Equations; Mathematical Model; Population Model; Mathematical Fuzzy; Model; Mathematical Fuzzy; Population Model; ODE; Numerical Method.

1. Introduction
Differential equations (DEs) have significant rule the history of mathematics and sciences due to some laws of nature are expressed through DEs, so we can say that DEs are the means by which scientists describe and understand the world. The DE is an equation where the unknown and its derivative appear in the equation and it is necessary to mathematically describe nature, for example there are Newton's equations, Maxwell's rates for magnetism, Lagrange's rates for classical mechanics, Einstein's equation for the ratio of general gravity, and the Schrödinger equation for quantum mechanics [1]. Fuzzy differential equations (FDEs) have taken an important role in many fields, and the ideal is mathematics, economics, engineering, and science [2]. DEs have been used by some experts in these fields to make some of the problems under study more understandable in our day. Nowadays, there are many statistical, physical and dynamic processes that use mathematical modeling for the problems of initial value and dietary limits. FDEs are used to refer to DEs with initial values of noise or boundary values as well as ambiguous, meaning that the fuzzy equations are differential equations but deal with fuzzy numbers. Zadeh was first developed the fuzzy theory in 1965 [3]. Chang and Zadeh 1972 first submitted the notion of a fuzzy derivative followed by Dubois and Prade 1982. In 1987 the term of FDEs was launched by Kandel and Byatt [4]. The First order FDEs appear in many applications. However the form of such an equation is very simple. Kaleva [5], Seikkala [6] submited the FDE with the initial value problems (IVPs) (Cauchy problem). He and Yi [7], formed the numerical methods for solving FDEs are introduced. Buckley and Feuring [8] introduced two analytical methods for solving nth-order linear FDEs with fuzzy initial conditions (ICs). In 2009, Nieto et al. [9] indicated that any numerical method using to
solve ordinary differential equations (ODEs) can be used to solve numerically FDEs under generalized differentiability. Allahviranloo et al. [10] solved FDEs by using the generalized differentiability and applied differential transformation method. In 2011, Kha\stan and et al. [11] approaching the general form to solve first order linear FDEs. Recently, Ghazanfari and et al [12] solving first order FDE, by using the Runge-Kutta-like formulae of order 4.

2. Preliminary

In this section, we introduce background, which included some definitions and the necessary notations, which will be used throughout this paper.

Definition 2.1 [13]
Let \( \mathbb{R}^{g_{1+50}} \) be a universal set, \( \mathbb{R}^{g_{1+50}} \neq \emptyset \). A fuzzy set \( \mathbb{R}^{g_{1+73}} \) in \( \mathbb{R}^{g_{1+50}} \), \( \mathbb{R}^{g_{1+73}} \subseteq \mathbb{R}^{g_{1+50}} \), is characterized by membership function \( \mathbb{R}^{g_{2020}g_{1+73}} : \mathbb{R}^{g_{1+73}} \rightarrow [0,1] \). Then, \( \mathbb{R}^{g_{1+73}}(\mathbb{R}^{g_{1+76}}) \) assigns \( \mathbb{R}^{g_{1+76}} \in \mathbb{R}^{g_{1+73}} \) by a positive degree less than one.

Let us denote by \( \mathbb{R}^{g_{3007}} \) the class of fuzzy subsets of the real axis \((\mathbb{R}^{g_{1+61}}: \mathbb{R}^{g_{1+57}}: \mathbb{R}^{g_{1+73}}: \mathbb{R} \rightarrow [0,1])\) satisfying the following properties:

I. \( \mathbb{R}^{g_{1+73}} \) is upper semi continuous on \( \mathbb{R} \),

II. \( \mathbb{R}^{g_{1+73}} \) is normal fuzzy set, i.e., there exists such that \( s_0 \in \mathbb{R} \),

III. \( \mathbb{R}^{g_{1+73}} \) is a convex fuzzy set \((\mathbb{R}^{g_{1+61}}. \mathbb{R}^{g_{1+57}}. \mathbb{R}^{g_{1+73}}(\mathbb{R}^{g_{1+72}} \mathbb{R}^{g_{1+71}} + (1 - \mathbb{R}^{g_{201}})\mathbb{R}^{g_{1+70}}) \geq \mathbb{R}^{g_{1+65}}\mathbb{R}^{g_{1+61}}\mathbb{R}^{g_{1+66}}(\mathbb{R}^{g_{1+73}}(\mathbb{R}^{g_{1+71}}), \mathbb{R}^{g_{1+73}}(\mathbb{R}^{g_{1+77}})) \); \( \forall \mathbb{R}^{g_{201}} \in [0,1] \); \( \mathbb{R}^{g_{1+71}}, \mathbb{R}^{g_{1+70}} \in \mathbb{R} \).

Definition 2.2 [13]
Let \( F : I \rightarrow \mathbb{R}^{g_{3007}} \) we say F is 1-differentiable function on the interval I if F is differentiable function and its first derivative is denoted by \( \mathbb{R}^{g_{1+301}}\mathbb{R}^{g_{1+32}} \), and similarly for 2-differentiability we have \( \mathbb{R}^{g_{1+302}}\mathbb{R}^{g_{1+32}} \).

Definition 2.3 [13]
Let \( \mathbb{R}^{g_{1+50}} \) be a nonempty universal set, \( \mathbb{R}^{g_{1+50}} \neq \emptyset \) and \( \mathbb{R}^{g_{1+73}} \subseteq \mathbb{R}^{g_{1+50}} \). A fuzzy set \( \mathbb{R}^{g_{1+73}} \) is defined by a membership function \( \mathbb{R}^{g_{2020}g_{1+73}} \) that marks every element in \( \mathbb{R}^{g_{1+73}} \) to the unit interval \( \mathbb{R}^{g_{1+35}} = [0, 1] \).

A fuzzy set \( \mathbb{R}^{g_{1+73}} \) in \( \mathbb{R}^{g_{1+50}} \) may be written as a set of ordered pairs of a grades element \( \mathbb{R}^{g_{1+72}} \) and its membership value, as shown in the following equation \( \mathbb{R}^{g_{1+73}} = \{(\mathbb{R}^{g_{1+72}}, \mathbb{R}^{g_{1+73}}(\mathbb{R}^{g_{1+72}})) \mid \mathbb{R}^{g_{1+72}} \in \mathbb{R}^{g_{1+50}} \} \).

Definition 2.4 [13]
Let \( \mathbb{R}^{g_{1+73}} \) be a fuzzy set defined in the universal set \( \mathbb{R}^{g_{1+50}} \). The support of \( \mathbb{R}^{g_{1+73}} \) is the crisp set of all elements in \( \mathbb{R}^{g_{1+50}} \) such that the membership function of \( \mathbb{R}^{g_{1+73}} \) is non-zero, that is, \( \mathbb{R}^{g_{1+55}}\mathbb{R}^{g_{1+67}}\mathbb{R}^{g_{1+70}}\mathbb{R}^{g_{1+57}}(\mathbb{R}^{g_{1+73}}.) \) = \( \{\mathbb{R}^{g_{1+72}} \in \mathbb{R}^{g_{1+50}} \mid \mathbb{R}^{g_{1+73}}. (\mathbb{R}^{g_{1+76}}.) > 0 \} \).

Definition 2.5 [13]
Let \( \mathbb{R}^{g_{1+73}} \) be a fuzzy set defined in \( \mathbb{R}^{g_{1+50}} \). The core of \( \mathbb{R}^{g_{1+73}} \) is the crisp set of all elements in \( \mathbb{R}^{g_{1+50}} \) such that the membership value of \( \mathbb{R}^{g_{1+73}} \) is 1, that is, \( \mathbb{R}^{g_{1+55}}\mathbb{R}^{g_{1+67}}\mathbb{R}^{g_{1+70}}\mathbb{R}^{g_{1+57}}(\mathbb{R}^{g_{1+73}}.) \) = \( \{\mathbb{R}^{g_{1+72}} \in \mathbb{R}^{g_{1+50}} \mid \mathbb{R}^{g_{1+73}}. (\mathbb{R}^{g_{1+72}}.) = 1 \} \).

Definition 2.6 [13]
Let \( \mathbb{R}^{g_{1+73}} \) be a fuzzy set defined in the universal set \( \mathbb{R}^{g_{1+50}} \). \( \mathbb{R}^{g_{1+73}} \) is called a fuzzy interval if

- \( \mathbb{R}^{g_{1+73}} \) is normal, that is there exists \( x_0 \in \mathbb{R} \) such that \( \mathbb{R}^{g_{1+73}}(x_0) = 1 \);
- \( \mathbb{R}^{g_{1+73}} \) is convex, that is for all \( x, y \in \mathbb{R} \) and \( 0 \leq \lambda \leq 1 \), it holds that \( \mathbb{R}^{g_{1+73}}(\lambda x + (1 - \lambda)y) \geq \min(\mathbb{R}^{g_{1+73}}(x), \mathbb{R}^{g_{1+73}}(y)) \);
- \( \mathbb{R}^{g_{1+73}} \) is upper semi-continuous, that is for any \( x_0 \in \mathbb{R} \), it holds that \( \mathbb{R}^{g_{1+73}}(x_0) \geq \lim_{x \to x_0^+} \mathbb{R}^{g_{1+73}}(x) \) \( \{\mathbb{R}^{g_{1+73}} \}^0 = \{\mathbb{R}^{g_{1+73}} \mid \mathbb{R}^{g_{1+73}}(\mathbb{R}^{g_{1+70}}) \geq \alpha \} \) is a compact subset of \( \mathbb{R}^{g_{1+50}} \).

Definition 2.7 [13]
Let \( \mathbb{R}^{g_{1+73}} \) be a fuzzy interval defined in the universal set \( \mathbb{R}^{g_{1+50}} \). The \( \alpha \) – cut of \( \mathbb{R}^{g_{1+73}} \) is the crisp set \( \{\mathbb{R}^{g_{1+73}} \}^\alpha \) that contains all elements in \( \mathbb{R}^{g_{1+50}} \) such that the membership values of the subset \( \mathbb{R}^{g_{1+73}} \) is greater than or equal
to \( \alpha \), that is \([u]^\alpha = \{ t \in \mathcal{R} \mid u(t) \geq \alpha \}, \alpha \in (0,1] \). For a fuzzy interval \( u \), its \( \alpha \)-cuts are closed intervals in \( \mathcal{R} \) and we denote them by \([u]^\alpha = [a^\alpha, b^\alpha] \), where \( \alpha \in (0,1] \).

**Definition 2.8 [13]**
A fuzzy interval \( u \) is called a triangular fuzzy interval if its membership function has the following form:

\[
\begin{align*}
    u(x) &= \begin{cases} 
        0, & \text{if } x < a \\
        \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\
        \frac{c-x}{c-b}, & \text{if } b \leq x \leq c \\
        0, & \text{if } x > c,
    \end{cases}
\end{align*}
\]

And it is \( \alpha \)-cuts are simply:

\([u]^\alpha = [a + \alpha(b-a), c - \alpha(c-b)] \), for any \( \alpha \in (0,1] \). The set of all fuzzy intervals is denoted by \( F(\mathcal{R}) \).

![Figure 1. The triangular fuzzy interval \( u \)](image)

**Theorem 2.1.**
Let \( f : I \rightarrow \mathcal{R} \), and put \([f(t)]^\alpha = [f_\alpha(t); h_\alpha(t)] \) for each \( \alpha \in [0,1] \).

(i) If \( f \) is 1- differentiable then \( f_\alpha \) and \( h_\alpha \) are differentiable functions and

\[ [D1 \ f(t)]^\alpha = [f'_\alpha(t); h(t)] \]

(ii) If \( f \) is 2- differentiable then \( f \) and \( h \) are differentiable functions and we have

\[ [D_2 f(t)]^\alpha = [f''_\alpha(t); h'(t)] \]

**2.2. Fuzzy differential Equations**
Consider the quasi linear \( n \)th order FDE

\[
\begin{align*}
    \bar{y}^{(n)}(t) &= f(t, \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-2)}(t), \bar{y}^{(n-1)}(t)), \tag{1}
\end{align*}
\]

where, \( \bar{y}(t) \) is a fuzzy function of \( t \) and \( f(t, \bar{y}(t)) \) is a fuzzy function of crisp variable \( t \) and fuzzy variable \( \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-2)}(t), \bar{y}^{(n-1)}(t) \) are Hukuhara fuzzy derivative of \( \bar{y}(t) \). If an ICs \( \bar{y}^{(j)}(t_0) = \bar{y}^{(j)}(t_0) \in \mathcal{R} \) are given for \( j=0,1,\ldots,n-1 \) a fuzzy Cauchy problem of first-order.

Using theorem 2 in [1], we may replace the fuzzy Cauchy problem in Equation (1) by the following equivalent system

\[
\begin{align*}
    \bar{y}^{(n)}(t) &= \bar{f}(t, \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-2)}(t), \bar{y}^{(n-1)}(t)), \\
    \bar{f}(t, \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-2)}(t), \bar{y}^{(n-1)}(t) = F_3(t, \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-2)}(t), \bar{y}^{(n-1)}(t), \bar{y}^{(n-1)}(t), \bar{y}^{(n-1)}(t)) \tag{2}
\end{align*}
\]

and

\[
\begin{align*}
    \bar{y}^{(n)}(t) &= \bar{f}(t, \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-2)}(t), \bar{y}^{(n-1)}(t)), \\
    \bar{f}(t, \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-2)}(t), \bar{y}^{(n-1)}(t), \bar{y}^{(n-1)}(t), \bar{y}^{(n-1)}(t) = F_2(t, \bar{y}(t), \bar{y}'(t), \bar{y}''(t), ..., \bar{y}^{(n-2)}(t), \bar{y}^{(n-1)}(t), \bar{y}^{(n-1)}(t), \bar{y}^{(n-1)}(t)) \tag{3}
\end{align*}
\]

with the ICs

\[
\begin{align*}
    \bar{y}(0) = \left[ \bar{y}_0, \bar{y}_0, \bar{y}_1, \bar{y}_1, \bar{y}_2, \bar{y}_2, ..., \bar{y}_{n-1}, \bar{y}_{n-1} \right].
\end{align*}
\]
Where,
\[ \lim_{h \to 0} \tau(t; h; \alpha) = \tau(t; \alpha). \]

**Definition 2.9** [13] *Initial Value Problem with Fuzzy Differential Equations*

Consider the following initial value problem:
\[
x'(t) = f(t, X(t)); \quad x(t_0) = x_0. \tag{1}
\]
where \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) is a real-valued function and \( x_0 \in \mathbb{R}^n \). Assume that the initial value in (1) is uncertain and modeled by a fuzzy interval, then we have the following fuzzy initial value problem:
\[
x'(t) = f(t, X(t)) \tag{4}
\]
where \( f : [0, T] \times F(\mathbb{R}^n) \to F(\mathbb{R}^n) \) is a fuzzy interval-valued function and \( x_0 \in F(\mathbb{R}^n) \). Suppose that the problem (1) has the solution \( x(t) = x(t, x_0) \). Then by using Zadeh’s extension principle to \( x(t, x_0) \) in relation to the initial value \( x_0 \), we get the solution \( X(t) = x(t, x_0) \), which is a fuzzy solution of the problem (2). To illustrate this we take the following example:
Assume the following fuzzy IVP:
\[
x'(t) = -ax(t) \tag{5}
\]
where the initial value \( x_0 \) is any fuzzy interval.

**Definition 2.10** [2] *Mathematical model*

Mathematical model is a set of equations which define evolution of the state variable over the dependent variables.

The general idea is to observe the phenomenology of a real system order to extract its main features and to provide a model suitable to describe the evolution in time and space of its relevant aspect.

3. **Mathematical Population Models**

A population growth is a dynamic process that can be better described using differential equations. One of the population growth models developed by economists and physicists is the Mathusian growth model [13], which was carried out in the year 1798. The simplest growth model was proposed by the British scientist Thomas Robert Malthus [13].

3.1 **Some Population Models for Individual Species** [1]

Population models are used to determine maximum harvest for agriculturists, to understand the dynamics of biological invasions, and for environmental conservation. Population models are also used to understand the spread of parasites, viruses, and disease. The most famous models of population growth developed by economists and physicists is the Malthusian Growth Model. This model shows the exponential growth of the population, as will be explained later.

3.1.1 **Exponential Growth Model** [14]

This model shows the exponential growth of the population and was described by the following DE:
\[
\frac{dp}{dt} = ap, \quad \text{with } P(0) = P_0 \tag{6}
\]
where \( a \) is a positive constant for growth. Hence, \( p(t) = P_0 e^{at} \).

Where \( A \) derives from the constant of integration and is calculated using the IC. [14] Suppose that we can determine the population at some point in time, and let it be \( t = t_0 \). we want to find a function of population \( P(t) \) which achieves \( p(t_0) = p_0, \quad t_0 \leq t \leq t_1 \).
Let’s begin by assuming the new population \((\mathcal{R}_g + \Delta \mathcal{R}_g)\) is the old population \(p(t)\) plus the number of births minus the number of deaths
\[
\mathcal{R}_g + \Delta \mathcal{R}_g = \mathcal{R}_g + \Delta \mathcal{R}_g, \quad \mathcal{R}_g + \Delta \mathcal{R}_g.
\]
Thus we are really assuming that the average rate of change of the population over an interval of time is proportional to the size of the population. Using the instantaneous rate of change to approximate the average rate of change, we have the following model
\[
\frac{dp}{dt} = kp, \quad p(t_0) = p_0, \quad t_0 \leq t \leq t_1
\]
where \(k\) is a positive constant
\[
p(t) = p_0e^{kt}
\]
This equation is known as the Malthusian model of population growth.

Now, we will applying this equation to predict the population for United State for several years, specifically in 1970 the population was 203,211,929, and in 1950 it was 150,697,000. Substituting these values into equation (2) and dividing the first result and the second gives
So:
\[
k = \frac{1}{20}\log_{10}\frac{203,211,929}{150,697,000} = 0.015
\]
That is from 1950 to 1970, population in the United States was increasing at the average rate of 1.5% per year we will using this information together with Equation (2) to predict the population for 1980. In this case
\[
t_0 = 1970, \quad p_0 = 203,211,926,
\]
Yields:
\[
p(1980) = 203,211,926e^{0.015(1980-1970)} = 236,098,574
\]
Here, a model expects that the population of the USA will reach 28,688, billion a number that exceeds the current estimates of the maximum population, so the population model was improved for Malthus and it was considered that the proportionality factor \(k\) is not constant in Equation (3) but is indicative for the population as it increases the population and approaches the maximum population \(M\), the rate \(k\) decreases and a simple model for \(K\) was developed
\[
k = (M - P), r > 0
\]
where \(r\) is constant. Substitution into Equation (1):
\[
\frac{dp}{dt} = r(M - P)P
\]
We suppose the IC \(p(t_0) = p_0\) Form Equation (4) was first introduced by the Dutch biologist Pierre François Verhulst (1804-1849)

3.1.2 Logistic Growth Model [14]

It follows from elementary algebra that
\[
\frac{1}{p(M-P)} = \frac{1}{M} + \frac{1}{M-P}
\]
Thus, Equation (5) we will rewriting as:
\[
\frac{dp}{p} + \frac{dp}{M-P} = r Mt + C
\]
for some arbitrary constant \(C\). To evaluate \(C\) by using the IC in this case:
\[ P < M: \]
\[ C = \ln \frac{P_0}{M - P_0} - rM t_0 \]

Substituting into Equation (6) and simplifying gives:
\[ \ln \frac{P}{M - P} - \ln \frac{P_0}{P_0} = rM(t - t_0) \]

Then, \[ P(t) = \frac{P_0 M e^{rM(t - t_0)}}{M - P_0 + P_0 e^{rM(t - t_0)}} \] (11)

We can rewrite the last equation as
\[ P(t) = \frac{P_0 M}{[P_0 + (M - P_0) e^{rM(t - t_0)}]} \] (12)

From this equation we get:
\[ r'' = rM \dot{p} - 2rp \dot{p} = r\dot{p}(M - 2p) \] (13)

So that: \[ P^n = 0 \] when \[ p = \frac{M}{2} \], the population \( p \) reaches half the limiting population \( M \), the growth \( \frac{dp}{dt} \) is most rapid, and then it starts to diminish toward zero.

### 3.2. Mathematical Fuzzy Population Models

The aim of this subsectio is to analyze the behavior of models which describe the population dynamics taking into account the subjectivity in the state variables or in the parameters. The models in this work have demographic and environmental fuzziness. The environmental fuzziness is presented using a life expectancy model where the fuzziness of parameters is considered. The demographic fuzziness is presented using the continuous Malthus and logistic discrete models. An outstanding result in this case is the emergence of new fixed points and bifurcation values to the discrete logistic model with subjective state variables in form of fuzzy sets. An interpretation is offered for this fact which differs from the deterministic one.

#### 3.2.1 Biological population model

Mathematical and mathematical models are widely used in the interpretation of biomedical data. Models are applied to enable the simulation of the complex biological processes that generate the required hypotheses.

#### 3.2.2 Numerical Method [13]

We have a systems of two independent fuzzy differential equation:

\[ X'(t) = f(X, Y), \]
\[ Y'(t) = g(X, Y), \] (14)

with the ICs
\[ X(t_0) = X_0, \]
\[ Y(t_0) = Y_0 \quad \text{where } 0 \leq t \leq 1 \]

To approximate the solution to previous system, we will using the Euler method to solve system of two independent DEs:
\[ x_{i+1} = x_i + hf(x_i, y_i) \]  
\[ y_{i+1} = y_i + hg(x_i, y_i) \] 

For \( i=0,1,2,\ldots,N \), \( h \) is a step amount. Suppose that:

\[ F(h, x_i, y_i) = x_i + hf(x_i, y_i), \]

And

\[ G(h, x_i, y_i) = x_i + hf(x_i, y_i). \]

The equation 2 becomes

\[ x_{i+1} = F(h, x_i, y_i), \]

\[ y_{i+1} = G(h, x_i, y_i). \]

Such that \( F \) and \( G \) are real valued functions and from Zadeh’s extension principle we get:

\[ X_{i+1} = F(h, X_i, Y_i), \]

\[ Y_{i+1} = G(h, X_i, Y_i). \]

Here \( F,G \) are fuzzy interval valued functions. The functions of membership of \( F \) and \( G \) can be written as

\[ F(h, X_i, Y_i)(z) = \begin{cases} \sup_{x \in \text{range}(F)} \min \{X_i(x), Y_i(y)\}, & \text{if } z \in \text{range}(F), \\ 0, & \text{if } z \notin \text{range}(F), \end{cases} \]

And

\[ G(h, X_i, Y_i)(z) = \begin{cases} \sup_{x \in \text{range}(G)} \min \{X_i(x), Y_i(y)\}, & \text{if } z \in \text{range}(G), \\ 0, & \text{if } z \notin \text{range}(G), \end{cases} \]

In addition to:

\[ F(h, [X_i]^a, [Y_i]^a) = \min \{F(h, u, v) | u \in [x_i^a, x_i^{a+}] \land v \in [y_i^a, y_i^{a+}]\}, \]

\[ \max \{F(h, u, v) | u \in [x_i^a, x_i^{a+}] \land v \in [y_i^a, y_i^{a+}]\}, \]

And

\[ G(h, [X_i]^a, [Y_i]^a) = \min \{G(h, u, v) | u \in [x_i^a, x_i^{a+}] \land v \in [y_i^a, y_i^{a+}]\}, \]

\[ \max \{G(h, u, v) | u \in [x_i^a, x_i^{a+}] \land v \in [y_i^a, y_i^{a+}]\}. \]

Let \([X_i]^a = [x_i^a, x_i^{a+}], \) \([Y_i]^a = [y_i^a, y_i^{a+}]\).  

\[
\begin{align*}
    x_i^{a+1} & = \min \{F(h, u, v) | u \in [x_i^a, x_i^{a+}] \land v \in [y_i^a, y_i^{a+}]\}, \\
    x_i^{a+2} & = \max \{F(h, u, v) | u \in [x_i^a, x_i^{a+}] \land v \in [y_i^a, y_i^{a+}]\}, \\
    y_i^{a+1} & = \min \{G(h, u, v) | u \in [x_i^a, x_i^{a+}] \land v \in [y_i^a, y_i^{a+}]\}, \\
    y_i^{a+2} & = \max \{G(h, u, v) | u \in [x_i^a, x_i^{a+}] \land v \in [y_i^a, y_i^{a+}]\}.
\end{align*}
\]

Finally, we will making a partition of the form \( t_0 < t_1 < t_2 \ldots < t_{N+1} = T \).  

On the interval \([t_0,T]\), to approximate the solution of (7) at each \( \alpha - \)cut , and the partition with equal space between each such that \( t_\ell = t_0 + \ell h, \) \( \ell = 0, 1, 2 \ldots, N \), and the partition \( h = \frac{T-t_0}{N} > 0 \), is very small.

### 3.2.3 A Fuzzy Predator-Prey Model[13]

We will use some results of classic predator–prey model[22]. Based on this form, we will present a template fuzzy predator–prey model. Let treat the prey population at time \( t \) is given by \( X \), and the predator population by \( Y \). obscurity when the predators absent, the prey will growing exponentially. As
stated by  \( x' = ax \) for \( a > 0 \). Furthermore we suppose that the death rate of the prey is proportionate to \( xy \), with appositive constant of proportion \( b \). so we have:

\[
X'(t) = aX - bXY
\]

Then without prey, predators will die exponentially as follows: \( y' = -cy \) for \( c > 0 \). Their birth strongly depend on both population sizes, finally we find for \( d > 0 \). And we have:

\[
Y'(t) = -cY + dXY
\]

We suppose that the initial populations of prey and predator are fuzzy and the parameters \( a, b, c \) and \( d \) are brash numbers, and we will denote \( X_0 \) and \( Y_0 \) are fuzzy initial populations of prey and predator at to, respectively. Combining equations (1) and (2) with fuzzy initial population we get fuzzy predator-prey model:

\[
\begin{align*}
X'(t) &= aX - bXY, \quad X(t_0) = X_0 \\
Y'(t) &= -cY + dXY, \quad Y(t_0) = Y_0
\end{align*}
\]

Example 3.2.1.4 suppose the following of fuzzy predator-prey model:

\[
\begin{align*}
X'(t) &= X - 0.03X(t)Y(t), \\
Y'(t) &= -0.4Y + 0.01X(t)Y(t)
\end{align*}
\]

So the initial population of prey and predator are determine:

\[
X_0 = \begin{cases} 
0, & \text{if } x < 14, \\
x - 14, & \text{if } 14 \leq x \leq 1, \\
-x + 16, & \text{if } 15 \leq x \leq 16, \\
0, & \text{if } x > 16,
\end{cases}
\]

and

\[
Y_0 = \begin{cases} 
0, & \text{if } y < 14, \\
y - 14, & \text{if } 14 \leq y \leq 1, \\
-y + 16, & \text{if } 15 \leq y \leq 16, \\
0, & \text{if } y > 16,
\end{cases}
\]

Mizukosh [21] et al. has been shown if \( x^* \) is a point of an equilibrium for a classical dynamical system then \( \chi_{\{x^*\}} \) is the characteristics function of \( x^* \). From this result, one can note that the fuzzy predator-prey model has two fuzzy equilibrium points: \([0,0]\) and \( \chi [40,33.33] \). In order to determine the stability of these points, we start fuzzy initial populations near them. In this case, we have the following three possibilities:

1. if the fuzzy initial populations start sufficiently close to the fuzzy equilibrium points and stay close when \( t \) increases, then the fuzzy equilibrium points are said fuzzy stable.
2. if the fuzzy initial populations start sufficiently close to the fuzzy equilibrium points and approach to them when \( t \) approaches to infinity, then the fuzzy equilibrium points are said asymptotically fuzzy stable.
3. if the fuzzy initial populations start sufficiently close to the fuzzy equilibrium points and move away from them when \( t \) increases, then the fuzzy equilibrium points are said fuzzy unstable.
3.2.4. **Fuzzy Growth and Decay Model** [16-20]

In population growth model, we assume that rate increase of population. That is:

\[
\frac{dx}{dt} = kx(t) , x(t) \text{ is a population at time } t , k \text{ is a constant represents growth rate. For population of bacteria. That we will suppose initially number of bacteria is approximately 20. Therefor this approximate number can be represented using fuzzy number. When we solving growth model, we cannot get exact amount of population. so we can described it by using fuzzy concept.}
\]

Fuzzy growth and decay models are studied by Buckley and Feuring in [25] as an application of fuzzy differential equations. S. P. Mondal et al. [26] have also studied solution of fuzzy growth and decay model. They have not considered any differentiability concept to study the model. In this paper, solution of fuzzy growth and decay model is studied using Seikkala differentiability of fuzzy-valued function.

Suppose a fuzzy initial value problem (FIVP)

\[
\frac{dy}{dt} = f(t,y) = k \times y, y(0) = c,
\]

where \(y: I \rightarrow F(R)\) is a unknown fuzzy-valued function, for \(t \in I\), \(k\) and \(c\) are fuzzy numbers and \(f: I \times F(R) \rightarrow F(R)\) is a fuzzy-valued function. We assume that fuzzy-valued function \(y\) is Seikkala differentiable on \(I\).

Using fuzzy arithmetic, the (FIVP) can be written as system of crisp parametric differential equations:

\[
\frac{dy_1}{dt} = f_1(t, y_1, y_2, \alpha) = \min\{k_1(\alpha)y_1(t, \alpha), k_1(\alpha)y_2(t, \alpha), k_2(\alpha)y_1(t, \alpha), k_2(\alpha)y_2(t, \alpha)\} \quad (1)
\]

\[
\frac{dy_2}{dt} = f_2(t, y_1, y_2, \alpha) = \max\{k_1(\alpha)y_1(t, \alpha), k_1(\alpha)y_2(t, \alpha), k_2(\alpha)y_1(t, \alpha), k_2(\alpha)y_2(t, \alpha)\} \quad (2)
\]

**Solution**

There are two case:

**Case one:** \(k_1(\alpha), k_2(\alpha), y_1(t, \alpha), y_2(t, \alpha) \geq 0\) then we get

\[
\frac{dy_1}{dt} = f_1(t, y_1, y_2, \alpha) = k_1(\alpha)y_1(t, \alpha), \quad (3)
\]

\[
\frac{dy_2}{dt} = f_2(t, y_1, y_2, \alpha) = k_2(\alpha)y_2(t, \alpha), y_1(0, \alpha) = c_1(\alpha) \text{ and } y_2(0, \alpha) = c_2(\alpha), \text{ for } t \in I \text{ and } \alpha.
\]

By solving this:

\[
y_1(t, \alpha) = c_1(\alpha)e^{k_1(\alpha, t)} \quad (25)
\]

\[
y_2(t, \alpha) = c_2(\alpha)e^{k_2(\alpha, t)}, \quad t \in I \quad (26)
\]

We suppose \(c_1(\alpha), c_2(\alpha) \geq 0\). to check \([y_1(t, \alpha), y_2(t, \alpha)]\) defines fuzzy number or not
for each \( t \in I \) and we must check \([y'_1(t, \alpha), y'_2(t, \alpha)]\)
defines fuzzy number for each \( t \in I \) so that we say that
solution \( y \) is Seikkala differentiable.

That is, we want to check that \( \partial y_1(t, \alpha) / \alpha > 0 \) and \( \partial y_2(t, \alpha) / \alpha > 0 \)
So:

\[
\frac{\partial y_1(t, \alpha)}{\partial \alpha} = c_1'(\alpha)e^{k_1(\alpha)t} + c_1(\alpha)k_1(\alpha)e^{k_1(\alpha)t}k_1'(\alpha) > 0
\]  
(27)

As \( c_1(\alpha) > 0 \) and \( k_1'(\alpha) > 0 \)

\[
\frac{\partial y_2(t, \alpha)}{\partial \alpha} = c_2'(\alpha)e^{k_2(\alpha)t} + c_2(\alpha)k_2(\alpha)e^{k_2(\alpha)t}k_2'(\alpha) < 0
\]  
(28)

As \( c_2(\alpha) < 0 \) and \( k_2'(\alpha) < 0 \)

Therefore: \([y_1(t, \alpha), y_2(t, \alpha)]\) defines fuzzy number for each \( t \in I \). We also need to
check \([y'_1(t, \alpha), y'_2(t, \alpha)]\) defines fuzzy number for each \( t \in I \).

Since \( y'_1(t, \alpha) = c_1(\alpha)k_1(\alpha)e^{k_1(\alpha)t} \) and \( y'_2(t, \alpha) = c_2(\alpha)k_2(\alpha)e^{k_2(\alpha)t} \), it is easy to notice \( \frac{\partial y'_1(t, \alpha)}{\partial \alpha} > 0 \) and \( \frac{\partial y'_2(t, \alpha)}{\partial \alpha} < 0 \), therefore, \( y \) is Seikkala differentiable on \( I \), \( y = ce^{kt} \), where \( c \) and \( k \) are fuzzy numbers.

**Case II:** \( k_1(\alpha), k_2(\alpha) < 0 \).

We suppose that \( y_1(t, \alpha) \) and \( y_2(t, \alpha) \) are positive. Equations (1) and (2) are writing as

\[
\frac{dy_1}{dt} = f_1(t, y_1, y_2, \alpha) = k_1(\alpha)y_2(t, \alpha),
\]  
(29)

\[
\frac{dy_2}{dt} = f_2(t, y_1, y_2, \alpha) = k_2(\alpha)y_1(t, \alpha),
\]  
(30)

Where

\( y_1(0, \alpha) = c_1(\alpha) \) and \( y_2(0, \alpha) = c_2(\alpha) \), \( t \in I \) and \( \alpha \)

We suppose that \( c_1(\alpha), c_2(\alpha) \geq 0. \)

And \( y_1(t, \alpha) = A_{11}(\alpha)e^{pt} + A_{12}(\alpha)e^{-pt} \); \n(31)

\( y_2(t, \alpha) = A_{21}(\alpha)e^{pt} - A_{22}(\alpha)e^{-pt} \), \n(32)

Where \( p = \sqrt{k_1(\alpha)k_2(\alpha)} \) and \( q = \sqrt{k_1(\alpha)k_2(\alpha)} \), \n(33)

\( A_{11}(\alpha) = 0.5(c_1(\alpha) + q c_2(\alpha)) \)
\( A_{12}(\alpha) = 0.5(c_1(\alpha) - q c_2(\alpha)); \)
\( A_{21}(\alpha) = \frac{0.5(c_1(\alpha))}{q} + c_2(\alpha) \)
\[ A_{22}(\alpha) = \frac{0.5(c_1(\alpha) - c_2(\alpha))}{q} \text{ for } t \in I \text{ and } \alpha. \]

We want to check \([y_1(t, \alpha), y_2(t, \alpha)]\) defines fuzzy number or not for each \(t \in I\). so we want to check \([y_1'(t, \alpha), y_2'(t, \alpha)]\) defines fuzzy number for each \(t \in I\) so that we say that solution \(y\) is Seikkala differentiable.

For this we need to check sufficient conditions for existence of fuzzy numbers. That is, we need to check that \(\frac{\partial y_1(t, \alpha)}{\partial \alpha} > 0\) and \(\frac{\partial y_2(t, \alpha)}{\partial \alpha} < 0\)

We need too to check that \([y_1'(t, \alpha), y_2'(t, \alpha)]\) defines fuzzy number for each \(t \in I\).That is, \(\frac{\partial y_1'(t, \alpha)}{\partial \alpha} > 0\) and \(\frac{\partial y_2'(t, \alpha)}{\partial \alpha} < 0\) then that \(y\) is Seikkala differentiable on \(I\).

So \(y\) is a fuzzy solution of fuzzy decay model. To solve this model analytically:

Suppose \(k = -1\) and \(c = (2, 4, 6)\) be a triangular fuzzy number. Then we get :

\[
\begin{align*}
\frac{dy_1}{dt} &= f_1(t, y_1, y_2, \alpha) = -y_2(t, \alpha), \\
\frac{dy_2}{dt} &= f_2(t, y_1, y_2, \alpha) = -y_1(t, \alpha),
\end{align*}
\]

when \(y_1(0, \alpha) = (2 + 2\alpha)\) and \(y_2(0, \alpha) = (6 - 2\alpha)\), for \(t \in I\) and \(\alpha\).

We take \(k_1(\alpha) = k_2(\alpha) = -1\). Then the solution of the system is:

\[
\begin{align*}
y_1(t, \alpha) &= A_{11}(\alpha)e^{pt} + A_{12}(\alpha)e^{-pt}; \\
y_2(t, \alpha) &= A_{21}(\alpha)e^{pt} - A_{22}(\alpha)e^{-pt},
\end{align*}
\]

where \(p = \sqrt{k_1(\alpha)k_2(\alpha)} = 1, q = \sqrt{k_1(\alpha)k_2(\alpha)} = 1\)

\[
\begin{align*}
A_{11}(\alpha) &= 0.5(c_1(\alpha) - c_2(\alpha)), \\
A_{12}(\alpha) &= 0.5(c_1(\alpha) - c_2(\alpha)); \\
A_{21}(\alpha) &= 0.5(c_1(\alpha) + c_2(\alpha)) \\
A_{22}(\alpha) &= 0.5(c_1(\alpha) + c_2(\alpha))
\end{align*}
\]

for \(t \in I\) and \(\alpha\).

We need to check \([y_1(t, \alpha), y_2(t, \alpha)]\) defines fuzzy number or not for each \(t \in I\). That is, we need to check that \(\frac{\partial y_1(t, \alpha)}{\partial \alpha} > 0\) and \(\frac{\partial y_2(t, \alpha)}{\partial \alpha} < 0\)

\[
\frac{\partial y_1(t, \alpha)}{\partial \alpha} = A_{11}'(\alpha)e^{pt} + A_{11}t(\alpha)e^{pt}p'(\alpha) + A_{12}'(\alpha)e^{-pt} + A_{12}(\alpha)(-t)e^{-pt}p'(\alpha). \quad (38)
\]
But $p'(\alpha) = 0$ as $p(\alpha) = 1$, we get :

$$\frac{\partial y_i(t, \alpha)}{\partial \alpha} = A_{i1}(\alpha)e^{pt} + A'_{i2}(\alpha)e^{-pt}$$

$$= 0.5(c'_1(\alpha) + c'_2(\alpha))e^{pt} + 0.5c'_1(\alpha) - c'_2(\alpha)e^{-pt} \quad (39)$$

Using $c'_1(\alpha) = 2$ and $c'_2(\alpha) = -2$

$$= 0.5(2 - 2)e^{pt} + 0.5(2 - (-2))e^{-pt} \quad (40)$$

$$= 2e^{-t} = \frac{2}{e^t} > 0 \quad \text{for } t > 0 \text{ in } I$$

Now,

$$\frac{\partial y_1(t, \alpha)}{\partial \alpha} = A'_{11}(\alpha)e^{pt} - A'_{22}(\alpha)e^{-pt} \quad (42)$$

$$= 0.5(c'_1(\alpha) + c'_2(\alpha))e^{pt} - 0.5c'_1(\alpha) - c'_2(\alpha)e^{-pt} \quad (43)$$

$$= 0.5(2 - 2)e^{pt} - 0.5(2 - (-2))e^{-pt} \quad (44)$$

$$= -2e^{-t} = \frac{2}{e^t} < 0 \quad \text{for } t > 0 \text{ in } I$$

Hence $[y_1(t, \alpha), y_2(t, \alpha)]$ defined fuzzy number for each $t \in I$.

Furthermore, we need to check $[y'_1(t, \alpha), y'_2(t, \alpha)]$ defines fuzzy number for each $t$ in $I$, so that we say that solution $y$ is Seikkala differentiable. That is, we need to check

$$\frac{\partial y'_1(t, \alpha)}{\partial \alpha} > 0 \text{ and } \frac{\partial y'_2(t, \alpha)}{\partial \alpha} < 0$$

Consider,

$$\partial y'_1(t, \alpha) = A_{11}(\alpha)e^{pt}p - A_{12}(\alpha)e^{-pt}p \quad (45)$$

$$\partial y'_2(t, \alpha) = A_{21}(\alpha)e^{pt}p + A_{22}(\alpha)e^{-pt}p \quad (46)$$

Now,

$$\frac{\partial y'_1(t, \alpha)}{\partial \alpha} = A'_{11}(\alpha)e^{pt} - A'_{12}(\alpha)e^{-pt} \quad (47)$$

$$= 0.5(c'_1(\alpha) + c'_2(\alpha))e^{pt} - 0.5(c'_1(\alpha) - c'_2(\alpha))e^{-pt} \quad (48)$$

But $c'_1(\alpha) = 2$ and $c'_2(\alpha) = -2,$

$$= 0.5(2 - 2)e^{pt} - 0.5(2 - (-2))e^{-pt} \quad (49)$$
\[ -2e^{-t} = \frac{2}{e^t} < 0 \text{ for } t > 0 \text{ in } I \]

In the same way, we have

\[ \frac{\partial y'(t, \alpha)}{\partial \alpha} > 0 \text{ for } t > 0 \text{ in } I \]

Therefore, \([y_1'(t, \alpha), y_2'(t, \alpha)]\) does not define fuzzy number for \(t\) in \(I\) and hence we say that solution \(y\) is not Seikkala differentiable. Therefore, we conclude that fuzzy decay model has no solution under Seikkala differentiability of fuzzy-valued function. So we note that

When the coefficient \(k = -1\) the solution is called fuzzy decay model and does not exist under Seikkala differentiability, while solution of fuzzy growth model exists with fuzzy coefficient \(k\) and fuzzy initial conditions.

4. Discussion and Conclusion

In this paper, the solution to the fuzzy growth and the decay model was studied by using the function with the fuzzy values (seikkala), where different values of the \(k\) factor were taken and we found that when \(k = -1\) is the solution for the fuzzy model is not present and when \(k = 1\) is the solution is present with fuzzy initial conditions. In addition, the FST method was applied to solve the fuzzy differential equation with the number \(Z\).

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