Non-Hermitian $d$-dimensional Hamiltonians with position dependent mass and their $\eta$-pseudo-Hermiticity generators

Omar Mustafa$^1$ and S.Habib Mazharimousavi$^2$
Department of Physics, Eastern Mediterranean University, G Magusa, North Cyprus, Mersin 10,Turkey
$^1$E-mail: omar.mustafa@emu.edu.tr
$^2$E-mail: habib.mazhari@emu.edu.tr

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Abstract
A class of non-Hermitian $d$-dimensional Hamiltonians with position dependent mass and their $\eta$-pseudo-Hermiticity generators is presented. Illustrative examples are given in 1D, 2D, and 3D for different position dependent mass settings.

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1 Introduction
The proposal of the non-Hermitian $\mathcal{PT}$ -symmetric Hamiltonians by Bender and Boettcher in [1] has relaxed the Hermiticity of a Hamiltonian as a necessary condition for the reality of the spectrum [1-7]. In the $\mathcal{PT}$ -symmetric setting, the hermiticity assumption $H = H^\dagger$ is replaced by the mere $\mathcal{PT}$ -symmetric one $\mathcal{PT}H(\mathcal{PT})^{-1} = \mathcal{PT}H\mathcal{PT} = H$, where $\mathcal{P}$ denotes the parity ($\mathcal{P}x\mathcal{P} = -x$) and the anti-linear operator $\mathcal{T}$ mimics the time reflection ($\mathcal{T}i\mathcal{T} = -i$). Intensive attention was paid to the potentials $V(x)$ which are analytic on the full axis $x \in (-\infty, \infty)$ in one-dimension (1D).

However, in the transition to more dimensions with singularities manifested by, say, the central repulsive/attractive $d$-dimensional core, $\ell_d(\ell_d + 1)/r^2$ with $r \in (0, \infty)$, there still exist models unfortunate in methodical considerations. Nevertheless, a pioneering $\mathcal{PT}$ -symmetric model with physically acceptable impact has been the Buslaev and Grecchi [2] quartic anharmonic oscillator, where a simple constant downward shift of the radial coordinate (i.e., $r \rightarrow x - ic$; $x \in (-\infty, \infty)$ and $\text{Im } r = -c < 0$) is employed (cf.,e.g., Znojil and Lévai [3] for illustrative examples).
In a broader class (where $\mathcal{PT}$-symmetric Hamiltonians constitute a subclass among others) of non-Hermitian pseudo-Hermitian Hamiltonians [8-14] (a generalization of $\mathcal{PT}$-symmetry, therefore), it is concreted that the eigenvalues of a pseudo-Hermitian Hamiltonian $H$ are either real or come in complex-conjugate pairs. In this case, a Hamiltonian $H$ is pseudo-Hermitian if it obeys the similarity transformation:

$$\eta H \eta^{-1} = H^\dagger$$

Where $\eta$ is a Hermitian (and so is $\eta H$) invertible linear operator and $(\dagger)$ denotes the adjoint. However, the reality of the spectrum is secured if the pseudo-Hermitian Hamiltonian is an $\eta$-pseudo-Hermitian with respect to

$$\eta = O^\dagger O$$

for some linear invertible operator $O : \mathcal{H} \to \mathcal{H}$ (where $\mathcal{H}$ is the Hilbert space of the quantum system with a Hamiltonian $H$) and satisfies the intertwining relation (cf., e.g., [12,14] and references therein)

$$\eta H = H^\dagger \eta. \quad (2)$$

In a recent study [14], we have presented a class of spherically symmetric non-Hermitian Hamiltonians and their $\eta$-pseudo-Hermiticity generators. Therein, a generalization beyond the nodeless 1D states is proposed and illustrative examples are presented, including an exactly solvable non-Hermitian $\eta$-pseudo-Hermitian Morse model.

On the other hand, a position-dependent effective mass, $M (r) = m_\circ m (r)$, associated with a quantum mechanical particle constitutes a useful model for the study of many physical problems [15-28]. They are used, for example, in the energy density many-body problem [15], in the determination of the electronic properties of semiconductors [16] and quantum dots [17], in quantum liquids [18], in $^3He$ cluster [19], etc.

In this work, we present (in section 2) a $d$-dimensional recipe for $\eta$-pseudo-Hermiticity generators for a class of non-Hermitian Hamiltonians with position-dependent masses, $M (r) = m_\circ m (r)$. An immediate recovery of our generalized $\eta$-pseudo-Hermiticity generators for Hamiltonians with radial symmetry [14] is obvious through the substitution $m (r) = 1$. Our illustrative examples are given in section 3. Section 4 is devoted for our concluding remarks.

### 2 Non-Hermitian $d$-dimensional Hamiltonians with position dependent mass and their $\eta$-pseudo-Hermiticity generators

Following the symmetry ordering recipe of the momentum and position-dependent effective mass, $M (\vec{r}) = m_\circ m (\vec{r})$, a Schrödinger Hamiltonian with a complex potential field $V (\vec{r}) + iW (\vec{r})$ would read

$$H = \frac{1}{2} \left( \vec{p} - \frac{1}{M (\vec{r})} \right) \cdot \vec{p} + V (\vec{r}) = -\frac{\hbar}{2m_\circ} \left( \vec{\nabla} \cdot \frac{1}{m (\vec{r})} \right) \cdot \vec{\nabla} + V (\vec{r}) + iW (\vec{r}). \quad (3)$$
Using the atomic units ($\hbar = m_o = 1$), and assuming the $d$-dimensional spherical symmetric recipe (cf, e.g., Mustafa and Znojil in [6]), with

$$\Psi (\vec{r}) = r^{-(d-1)/2} R_{\ell_d, m_d} (r) Y_{\ell_d, m_d} (\theta, \varphi) ,$$

Hamiltonian (3) would result in the following $d$-dimensional non-Hermitian Hamiltonian with a position-dependent mass $M (r) = m_o m (r)$ (in $\hbar = m_o = 1$ units)

$$H = -\frac{1}{2m (r)} \frac{\ell_d (\ell_d + 1)}{2m (r)} r^2 - \frac{m' (r)}{2m (r)} \left( \frac{d-1}{2r} - \partial_r \right) + V (r) + i W (r) .$$

(5)

Where $\ell_d = \ell + (d-3)/2$ for $d \geq 2$, $\ell$ is the regular angular momentum quantum number, $n_r = 0, 1, 2, \cdots$ is the radial quantum number, and $m' (r) = dm (r) / dr$. Moreover, the $d = 1$ can be obtained through $\ell_d = -1$ and $\ell_d = 0$ for even and odd parity, $\mathcal{P} = (-1)^{\ell_d+1}$, respectively.

Then $H$ has a real spectrum if and only if there is an invertible linear operator $O : \mathcal{H} \rightarrow \mathcal{H}$ such that $H$ is $\eta$-pseudo-Hermitian with the linear invertible operator

$$O = \mu (r) \partial_r + Z (r) \implies O^\dagger = -\mu (r) \partial_r - \mu' (r) + Z^* (r)$$

(6)

where

$$Z (r) = F (r) + i G (r);$$

$$F (r) = \left[-\frac{\ell_d + 1}{r} + f (r) \right] \mu (r), \quad G (r) = g (r) \mu (r)$$

(7)

and $\mu (r) = \sqrt{1/2m (r)}$, $f (r)$, $g (r)$ are real-valued functions and $\mathbb{R} \ni r \in (0, \infty)$. Equation (1), in turn, implies

$$\eta = -\mu (r)^2 \partial_r^2 - [2\mu (r) \mu' (r) + 2i \mu (r) G (r)] \partial_r$$

$$- \left[ \mu (r) Z' (r) + \mu' (r) Z (r) - F (r)^2 - G (r)^2 \right] ,$$

(8)

where primes denote derivatives with respect to $r$. Herein, it should be noted that the operators $O$ and $O^\dagger$ are two intertwining operators and the Hermitian operator $\eta$ only plays the role of a certain auxiliary transformation of the dual Hilbert space and leads to the intertwining relation (2) (cf, e.g., [12]). Hence, considering relation (2) along with the eigenvalue equation for the Hamiltonian, $H/E_i \rangle = E_i \langle E_i \rangle$, and its adjoint, $H^\dagger / E_i \rangle = E_i^* \langle E_i \rangle$, one can show that any two eigenvectors of $H$ satisfy

$$\langle E_i / H^\dagger \eta - \eta H / E_j \rangle = 0 \implies (E_i^* - E_j) \langle \langle E_i / E_j \rangle \rangle_{\eta} = 0 .$$

(9)

Which implies that if $E_i^* \neq E_j$ then $\langle \langle E_i / E_j \rangle \rangle_{\eta} = 0$. Therefore, the $\eta$- orthogonality of the eigenvectors suggests that if $\psi$ is an eigenvector (of eigenvalue $E = E_1 + i E_2, \forall E_1, E_2 \in \mathbb{R}$) related to $H$ then

$$\eta \psi = 0 \implies O^\dagger O \psi = 0 \implies O \psi = 0 ,$$

(10)
\[
\frac{Z(r)}{\mu(r)} = -\frac{\psi'(r)}{\psi(r)} = -\partial_r \ln \psi(r) \implies \psi(r) = \exp \left( - \int \frac{Z(z)}{\mu(z)} dz \right).
\] (11)

The intertwining relation (2) would lead to
\[
W(r) = -2\mu(r) G'(r)
\] (12)
\[
V(r) = F(r)^2 - G(r)^2 - \mu'(r) F(r) - \mu(r) F'(r) + \beta
\] (13)
\[
F(r)^2 - \mu(r) F(r) - \mu(r) F'(r) = \frac{1}{2} \left[ \frac{\mu(r)^2 G''(r)}{G(r)} - \mu(r) \mu''(r) \right] - \frac{\mu(r)^2}{4G(r)^2} \left[ \left( \frac{G(r)}{\mu(r)} \right)^2 + \frac{\alpha}{4G(r)^2} \right]
\] (14)

where \(\alpha, \beta \in \mathbb{R}\) are integration constant.

On the other hand, with \(E = E_1 + iE_2\), \(H\) in (5), and \(\psi(r)\) in (11) the eigenvalue problem \(H\psi(r) = E\psi(r)\) implies
\[
\beta = E_1
\] (15)
\[
F(r) = \frac{\mu'(r) G(r) - \mu(r) G'(r) - E_2}{2G(r)}
\] (16)

and
\[
\alpha = E_2^2
\] (17)

Obviously, one would accept \(\mathbb{R} \ni \alpha \geq 0 \implies \mathbb{R} \ni E_2 = \pm \sqrt{\alpha}\), and negate \(\alpha < 0 \implies E_2 \in \mathbb{C}\) since \(\mathbb{R} \ni E_2 \notin \mathbb{C}\), a requirement of pseudo-Hermiticity mentioned early on. Yet \(E_2 \in \mathbb{C}\) contradicts with the real/imaginary descendants, (12) to (17). However, the reality of the spectrum is secured by our \(\eta\)-pseudo-Hermiticity generators. This in turn acquires \(\alpha = 0\) in the forthcoming developments.

The Hamiltonian in (5) may now be recast as
\[
H = -\frac{1}{2m(r)} \partial_r^2 + \frac{m'(r)}{2m(r)^2} \partial_r + \tilde{V}(r) + iW(r)
\] (18)

where
\[
\tilde{V}(r) = \ell_d (\ell_d + 1) - \frac{m'(r)}{2m(r)^2} \left( \frac{d - 1}{2r} \right) + V(r).
\] (19)

We may now summarize our results in terms of our \(\eta\)-pseudo-Hermiticity generators \(g(r)\) and \(f(r)\) as
\[
W(r) = -2\mu(r) [g(r) \mu(r)]'
\] (20)
\[ V(r) = \frac{\ell_d (\ell_d + 1)}{2m(r)r^2} + \frac{2\mu(r)\mu'(r)[\ell_d + 1]}{r} + \mu(r)^2 \left[ f(r)^2 - g(r)^2 \right] \]
\[ - \frac{2(\ell_d + 1)}{r} f(r) \mu(r)^2 - 2\mu'(r)\mu(r)f(r) - \mu(r)^2 f'(r) + \beta \quad (21) \]

\[ g(r) = r^{2(\ell_d + 1)} \exp \left( -2 \int f(z) dz \right) \quad (22) \]

and

\[ \psi(r) = r^{\ell_d + 1} \exp \left( - \int [f(z) + ig(z)] dz \right) \quad (23) \]

Hence, \( f(r) \) and/or \( g(r) \) are our generating function(s) for the \( \eta \)-pseudo-Hermiticity of the class of non-Hermitian Hamiltonians in (5) with real spectra and \( \psi(r) \) in (25) as an eigenfunction (not necessarily normalizable). Nevertheless, it should be reported here that our results cover the 1D Fityo’s ones [13] through the substitutions \( \ell_d = -1, m(r) = 1/2, \) and \( r \in (0, \infty) \rightarrow x \in (-\infty, \infty) \). Moreover, they also collapse into our recent results on \( \eta \)-pseudo-Hermiticity generators in [14] by the substitutions \( \ell_d = \ell, \) and \( m(r) = 1/2 \) (where we considered constant mass settings).

### 3 Illustrative examples

In this section, we construct \( \eta \)-pseudo-Hermiticity of some non-Hermitian Hamiltonians with position-dependent mass using the above mentioned procedure through the following illustrative examples:

**Example 1:** For a quantum particle endowed with a position-dependent mass \( m(r) = \frac{r^2}{2} \) and with the generating function \( f(r) = r \) we consider the following cases:

**A)** For \( d = 3, \ell = 0, \) and \( \mathbb{R} \ni r \in (0, \infty) \), one finds

\[ g(r) = r^2 e^{-r^2} \quad (24) \]

\[ \tilde{V}(r) = -\frac{1}{r^2} - \frac{2}{r^4} - r^2 e^{-2r^2} + 1 + \beta \quad (25) \]

\[ W(r) = -\frac{2}{r} e^{-r^2} + 4re^{-r^2} \quad (26) \]

\[ \psi(r) = \left( \frac{4}{\sqrt{\pi}} \right)^{1/2} r \exp \left( -\frac{r^2}{2} - i \left( \frac{-r^2}{2} e^{-r^2} + \frac{\sqrt{\pi}}{4} \text{erf}(r) \right) \right) \quad (27) \]
B) For \( d = 3, \ell = 1, \) and \( \mathbb{R} \ni r \in (0, \infty): \)

\[
g(r) = r^4 e^{-r^2} \tag{28}
\]

\[
\tilde{V}(r) = -\frac{3}{r^2} - \frac{2}{r^4} - r^6 e^{-2r^2} + 1 + \beta \tag{29}
\]

\[
W(r) = 2r (-3 + 2r^2) e^{-r^2} \tag{30}
\]

\[
\psi(r) = \left(\frac{8}{3\sqrt{\pi}}\right)^{1/2} r^2 e^{-r^2/2} \exp \left(-i \left(\frac{r^3}{2} e^{-r^2} - \frac{3r}{4} e^{-r^2} + \frac{3\sqrt{\pi}}{8} \text{erf}(r)\right)\right) \tag{31}
\]

C) For \( d = 2, \ell = 0 \) and \( \mathbb{R} \ni r \in (0, \infty) \rightarrow \mathbb{R} \ni \rho \in (0, \infty): \)

\[
g(\rho) = \rho e^{(-\rho^2)} \tag{32}
\]

\[
\tilde{V}(\rho) = -\frac{5}{4} \frac{1}{\rho^4} - e^{-2\rho^2} + 1 + \beta \tag{33}
\]

\[
W(\rho) = 4e^{-\rho^2} \tag{34}
\]

\[
\psi(\rho) = \sqrt{2} \sqrt{\rho} \exp \left(-\frac{\rho^2}{2} - i \left(\rho^2 e^{-\rho^2}\right)\right) \tag{35}
\]

D) For \( d = 1, \ell = 0 \) and \( \mathbb{R} \ni r \in (0, \infty) \rightarrow \mathbb{R} \ni x \in (-\infty, \infty): \)

\[
g(x) = e^{(-x^2)} \tag{36}
\]

\[
\tilde{V}(x) = \frac{1}{x^2} - \frac{e^{-2x^2}}{x^2} + 1 + \beta \tag{37}
\]

\[
W(x) = 4e^{-x^2} + \frac{2e^{-x^2}}{x^3} \tag{38}
\]

\[
\psi(x) = \left(\frac{1}{\sqrt{\pi}}\right)^{1/2} \exp \left(-\frac{x^2}{2} - i \left(\frac{x}{2} \sqrt{\pi} \text{erf}(x)\right)\right) \tag{39}
\]

**Example 2:** For a quantum particle endowed with a position-dependent mass \( m(r) = 1/ \left[2 \cosh^2 (r)\right]\) and with the generating function \( f(r) = \tanh (r) / 2, \) we consider the following cases:
i) For \(d = 1, \ell = 0\) and \(\mathbb{R} \ni r \in (0, \infty) \rightarrow \mathbb{R} \ni x \in (-\infty, \infty)\) one gets

\[
g(x) = \frac{1}{\cosh(x)} \quad (40)
\]

\[
\tilde{V}(x) = -\frac{3 \cosh^2(x)}{4} - \frac{3}{4} + \beta \quad (41)
\]

\[
W(x) = 0 \quad (42)
\]

\[
\psi(r) = \sqrt{\frac{1}{\sqrt{\pi}}} \frac{1}{\sqrt{\cosh(x)}} \exp(-2i \tanh^{-1}(e^x)) \quad (43)
\]

ii) For \(d = 2, \ell = 0\) and \(\mathbb{R} \ni r \in (0, \infty) \rightarrow \mathbb{R} \ni \rho \in (0, \infty)\):

\[
g(\rho) = \frac{\rho}{\cosh(\rho)} \quad (44)
\]

\[
\tilde{V}(\rho) = -\rho^2 - \frac{\cosh^2(\rho)}{4\rho^2} - \frac{3}{4} \cosh^2(\rho) + \frac{\cosh(\rho)\sinh(\rho)}{2\rho} + \frac{1}{4} + \beta \quad (45)
\]

\[
W(\rho) = -2 \cosh(\rho) \quad (46)
\]

\[
\psi(\rho) = (2/\sqrt{\pi})^{1/2} \sqrt{\frac{\rho}{\cosh(\rho)}} \exp\left(-i \int^{\rho} \frac{z}{\cosh(z)} dz\right) \quad (47)
\]

iii) For \(d = 3, \ell = 0\) and \(\mathbb{R} \ni r \in (0, \infty)\):

\[
g(r) = \frac{r^2}{\cosh(r)} \quad (48)
\]

\[
\tilde{V}(r) = -r^4 - \frac{3}{4} \cosh^2(r) + \frac{\cosh(r)\sinh(r)}{r} + \frac{1}{4} + \beta \quad (49)
\]

\[
W(r) = -4 \cosh(r) \quad (50)
\]

\[
\psi(r) = (0.5079) \sqrt{\frac{r^2}{\cosh(r)}} \exp\left(-i \int^{r} \frac{z^2}{\cosh(z)} dz\right) \quad (51)
\]

iv) For \(d = 3, \ell = 2\) and \(\mathbb{R} \ni r \in (0, \infty)\):

\[
g(r) = \frac{r^6}{\cosh(r)} \quad (52)
\]
\[ V(r) = -r^{12} - \frac{3}{4} \cosh^2(r) + \frac{3 \sinh(2r)}{2r} + \frac{6 \cosh^2(r)}{r^2} + \frac{1}{4} + \beta \] (53)

\[ W(r) = -12r^5 \cosh(r) \] (54)

\[ \psi(r) = (0.02636) r^3 \sqrt{\frac{1}{\cosh(r)}} \exp \left( -i \int^r \frac{z^6}{\cosh(z)} dz \right) \] (55)

4 Concluding Remarks

In this paper a class of non-Hermitian \( d \)-dimensional Hamiltonians for particles endowed with position-dependent mass and their \( \eta \)-pseudo-Hermiticity generators is introduced. Our illustrative examples include the 1D, 2D, and 3D Hamiltonians at different position-dependent mass settings. The results are presented in such a way that they reproduce our generalized \( \eta \)-pseudo-Hermiticity generators reported in [14] for node-less states with \( m(r) = 1 \) and \( \ell_d = \ell \). The 1D Fityo’s [13] results are also reproducible through the substitutions \( \ell_d = -1 \), \( m(r) = 1 \), and \( r \in (0, \infty) \rightarrow x \in (-\infty, \infty) \). It should be noted, moreover, that example 2-i, with the imaginary part of the effective potential \( W(x) = 0 \), documents that Hermiticity is a possible by-product of \( \eta \)-pseudo-Hermiticity.

It is anticipated that with some luck one would be able to obtain some position-dependent mass non-Hermitian \( \eta \)-pseudo-Hermitian Hamiltonians that are exactly solvable. However, this issue already lies beyond our current methodical proposal. Moreover, a point canonical transformation could be invested in the process to serve for exact solvability, an issue we shall deal with in the near future.
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