Randomised Buffer Management with Bounded Delay against Adaptive Adversary

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1 Introduction

We study the Buffer Management with Bounded Delay problem, introduced by Kesselman et al. [4], or, in the standard scheduling terminology, the problem of online scheduling of unit jobs to maximise weighted throughput. The adaptive-online adversary model for this problem has recently been studied by Bieńkowski et al. [2], who proved a lower bound of \( \frac{4}{3} \) on the competitive ratio and provided a matching upper bound for 2-bounded sequences. In particular, the authors of [2] claim that the algorithm RMIX [3] is \( e^{-1} \)-competitive against an adaptive-online adversary. However, the original proof of Chin et al. [3] holds only in the oblivious adversary model. The reason is as follows. First, the potential function used in the proof depends on the adversary’s future schedule, and second, it assumes that the adversary follows the earliest-deadline-first policy. Both of these cannot be assumed in adaptive-online adversary model, as the whole schedule of such adversary depends on the random choices of the algorithm. We give an alternative proof that RMIX indeed is \( e^{-1} \)-competitive against an adaptive-online adversary.

Similar claim about RMIX was made in another paper by Bieńkowski et al. [1] studying a slightly more general problem. It assumes that the algorithm does not know exact deadlines of the packets, and instead knows only the order of their expirations. However, any prefix of the deadline-ordered sequence of packets can expire in every step. The new proof that we provide holds even in this more general model, as both the algorithm and its analysis rely only on the relative order of packets’ deadlines.

2 RMIX and its new analysis

The algorithm RMIX works as follows. In each step, let \( b \) be the heaviest pending job. Select a real \( x \in [-1, 0] \) uniformly at random. Transmit \( f \), the earliest-deadline pending packet with \( w_f \geq e^x \cdot w_h \).

We write \( a \triangleleft b \) (\( a \sqsubseteq b \)) to denote that the deadline of packet \( a \) is earlier (not later) that the deadline of packet \( b \). This is consistent with the convention of [1] for the more general problem studied therein.

Theorem 1. RMIX is \( e/(e - 1) \)-competitive against an adaptive-online adversary.
Proof. We use the paradigm of modifying the adversary’s buffer used in the paper of Li et al. [3]. Namely, in each time step we assume that RMIX and the adversary Adv have the same buffers. Both RMIX and Adv transmit a packet. If after doing so, the contents of their buffers become different, we modify the adversary’s buffer to make it identical with that of RMIX. To do so, we may have to let the adversary transmit another packet and keep the one originally transmitted in the buffer, or upgrade one of the packets in its buffer by increasing its weight and deadline. We show that in each step the expected gain of RMIX is at least \( \frac{1}{e} \) times the expected amortized gain of the adversary, denoted \( \text{Adv}' \).

The latter is defined as the sum of weights of the packets that Adv eventually transmitted in the step. Both expected values are taken over possible random choices of RMIX.

First, we compute the expected gain of RMIX in a single step.

\[
\mathbb{E}[\text{RMIX}] = \mathbb{E}[w_f] = \int_{-1}^{0} w_f \, dx .
\]

Assume now that Adv transmits a packet \( j \). Without loss of generality, we may assume that for each packet \( k \) from the buffer, either \( w_j \geq w_k \) or \( j \preceq k \). We call it a greediness property. We consider two cases.

1. \( f \prec j \). By the greediness property, \( w_j \geq w_f \). After both Adv and RMIX transmit their packets, we replace \( f \) in the buffer of Adv by \( j \).
2. \( j \preceq f \). After both Adv and RMIX transmit their packets, we let Adv transmit additionally \( f \) in this round and we reinsert \( j \) into its buffer.

Therefore the amortized gain of Adv is \( w_j \) and additionally \( w_f \) if \( j \preceq f \). By the definition of the algorithm, \( j \preceq f \) only if \( w_f \geq w_j \). Let \( y = \ln(w_j/w_h) \).

\[
\mathbb{E}[\text{Adv}'] = w_j + \mathbb{E}[w_f | w_f \geq w_j] = w_j + \int_{y}^{0} w_f \, dx .
\]

Finally, we compare the gains, obtaining

\[
\frac{\mathbb{E}[\text{RMIX}]}{\mathbb{E}[\text{Adv}']} = \frac{\int_{-1}^{0} w_f \, dx + \int_{y}^{0} w_f \, dx}{w_j + \int_{y}^{0} w_f \, dx} \geq \frac{\int_{-1}^{y} e^x w_h \, dx + \int_{y}^{0} e^x w_h \, dx}{w_j + \int_{y}^{0} e^x w_h \, dx}
\]

\[
= \frac{\int_{-1}^{0} e^x w_h \, dx}{w_j + \int_{y}^{0} e^x w_h \, dx} = \frac{w_h \cdot (1 - 1/e)}{w_j + w_h \cdot (1 - w_j/w_h)}
\]

\[
= 1 - 1/e ,
\]

which concludes the proof. \( \square \)

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