Quantum Capacity of Partially Corrupted Quantum Network

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We discuss a quantum network, in which the sender has $m_0$ outgoing channels, each channel is of capacity $d$, each intermediate node applies invertible unitary, only $m_1$ channels are corrupted, and other non-corrupted channels are noiseless. As our result, we show that the quantum capacity is not smaller than $(m_0 - 2m_1 + 1)\log d$ under the following two settings. In the first case, the unitaries on intermediate nodes are arbitrary and the corruptions on the $m_1$ channels are individual. In the second case, the unitaries on intermediate nodes are restricted to Clifford operations and the corruptions on the $m_1$ channels are adaptive, i.e., the attacker is allowed to have a quantum memory. Further, our code in the second case realizes the noiseless communication even with the single-shot setting and is constructed dependently only on the network topology and the places of the $m_1$ corrupted channels while this result holds regardless of the network topology and the places.

I. INTRODUCTION

When two distant players communicate their quantum states via a single channel, their communication can be disturbed by the corruption of the channel. In the classical information theory, to resolve this problem, they employ information transmission over a network, which is composed of nodes and channels \cite{1,2}. Combining a network code, they realized a reliable communication even when a part of nodes are corrupted because a network code realizes the diversification of risk. Many existing papers for the quantum network addressed the multiple-unicast network \cite{3,4,5,6,7}. The coding scheme proposed in \cite{5} was already implemented experimentally \cite{8}. However, they did not discuss a reliable quantum network code over corruptions on the quantum network even in the unicast network. Since a corruption on a node can be propagated to the entire network due to the network structure, it is desired to discuss a reliable quantum network code for the unicast network even in the presence of the corruption. Indeed, while the preceding paper \cite{5} constructed such a network code, it did not discuss the optimality of the transmission rate. Also, the code constructed in \cite{5} works only when the unitary operations on the network nodes belong to a limited class.

This paper discusses the quantum capacity of a partially corrupted quantum unicast network, i.e., the optimal value of the reliable transmission rate under the knowledge of the form of the corruption for a more general class of node operations on the network. Our problem setting is slightly different from those of the preceding papers \cite{13,15} because \cite{13,15} discussed quantum network coding under limited knowledge of the corruption. Whereas conventional network coding considers the optimization of node operations given a directed graph of the network, we consider the quantum capacity when node operations are given as well because it is often quite difficult to control node operations. Specifically, we address the worst-case capacity among a certain condition, which is formulated as follows.

Every quantum channel transmits a $d$-dimensional system by one use of the network. The sender has $m_0$ outgoing quantum channels. Each intermediate node has the same number of incoming quantum channels and outgoing quantum channels. The node applies a fixed unitary across the incoming quantum systems and outputs them to the outgoing quantum channels. Finally, the receiver receives $m_0$ quantum systems via $m_0$ incoming quantum channels. We assume that only $m_1$ quantum channels are corrupted at most. Other channels are assumed to be noiseless in the same way as secure classical network coding \cite{9,10,11,12} because the errors of these normal channels can be corrected by quantum error correcting code. Also, the network is assumed to have no cycle and to be well synchronized, i.e., to have no delayed transmission. Only the sender and the receiver are allowed to optimize their coding operation due to the difficulty of node operation control. Although the paper \cite{13} assumed that the sender and the receiver do not know the places of corrupted channels, the places are assumed to be known to them in our setting.

There are two types of quantum settings. The first one is the individual corruption, in which the corruption on each corrupted quantum channel is done individually. The other is the adaptive corruption, in which the corruptions on respective corrupted quantum channels are done adaptively. That is, the attacker has a quantum memory, and the quantum memory interacts the corrupted quantum channel on each corruption. For an adversar-
eral corruption, we need to consider such a malicious case. This kind of corruption can be written by quantum comb [16, 17]. The adaptive setting is more general than the individual setting, and the adaptive setting often cannot be reduced to the individual setting in general. For example, adaptive strategies cannot be reduced to individual strategies quantum channel discrimination [18].

In the classical case of this setting, we can show that the capacity, i.e., the maximum transmission rate is not smaller than \((m_0-m_1) \log d\). In contrast, when our quantum channel has only individual corruptions, we find that the quantum capacity, i.e., the maximum transmission rate of the quantum state is not smaller than \((m_0-2m_1+1) \log d\). This fact is shown by the analysis of coherent information on the quantum network. Further, when the unitaries on our network are limited to Clifford operations, the quantum capacity is not smaller than \((m_0-2m_1+1) \log d\) even when the corruptions are adaptive. In this case, our code can be constructed in the single-shot setting by using Clifford operations. This construction depends only on the applied Clifford operations and the places of the corrupted channels, and is independent of Eve’s operation to the corrupted channels. This phenomena can be intuitively explained for the case of Clifford operations as follows. The sender can identify the place of the first corruption. However, at the other corruptions, the corrupted computation bases and the corrupted Fourier bases split in general. For this characterization, the behavior of the error is described by the the symplectic structure. In particular, the symplectic diagonalization plays an essential role in the code construction. Hence, in the worst case, totally \(2m_1-1\) quantum systems are corrupted.

II. CLASSICAL NETWORK MODEL

When all the node operations are invertible linear on a finite field of order \(q (= d)\), the receiver can find a linear subspace for corrupted information, as discussed in [2]. The dimension of the subspace is bounded by \(m_1\). Hence, the capacity is not smaller than \((m_0-m_1) \log d\). However, when node operations are not necessarily linear but are invertible, we cannot apply the above discussion. Even in this case, we can show that the capacity is not smaller than \((m_0-m_1) \log d\) as shown in Appendix A.

III. GENERAL UNITARY NETWORK MODEL

The general unitary network model is described as follows. In this model, we assume that the places of the channels to be corrupted are known. Since our network is composed of unitary operations and partial corruptions, our network model of the adaptive corruption is given as the general form with Fig. 1 whose reason is illustrated in Fig. 2. The input and output systems are the \(m_0\)-tensor product system \(\mathcal{H}^\otimes m_0\) of the same system \(\mathcal{H}\) of dimension \(d\), and \(m_1+1\) unitaries \(U = (U_0, U_1, \ldots, U_{m_1})\) are applied between the input and output systems, which has \(m_1\) intervals. Eve can access only the first system on each interval, and has her memory so that the corruption in the \(i\)-th interval is given as the unitary \(\tilde{U}_i\) between her memory and the corrupted system, i.e., the first system on the \(i\)-th interval.

Our first result is on the minimum capacity of the general unitary network with individual corruption, in which
The minimum quantum capacity is given as follows.

$$\min C(\Lambda(U, \Gamma)) = (m_0 - 2m_1 + 1) \log d. \quad (1)$$

Here, the minimum is taken over all channels $\Lambda(U, \Gamma)$ under the individual corruption.

To show the theorem, we employ the coherent information $I_c(\rho, \Lambda)$ for an input state $\rho$ and a quantum channel $\Lambda$ with an output system $A$. By using the environment $E$ of $A$, the coherent information $I_c(\rho, \Lambda)$ is written as $H(A) - H(E)$, where $H(A)$ and $H(E)$ are the von Neumann entropy of the respective system when the input state is $\rho$ [19, 20, (8.37)]. It is known that the quantum capacity $C(\Lambda)$ is given as the maximum

$$\lim_{n \to \infty} \max_{\rho} \frac{1}{n} I_c(\rho, \Lambda^{\otimes n}), \quad (2)$$

where the maximum is taken over all the input densities on the $n$-tensor system of the input system of $\Lambda$ [21, 22, 23, Theorem 9.10].

Lemma 1. A individual corruption $\Lambda(U, \Gamma)$ satisfies

$$\max_{\rho} I_c(\rho, \Lambda(U, \Gamma)) \geq (m_0 - 2m_1 + 1) \log d. \quad (3)$$

The proof of Lemma 1 is given in Appendix B. Due to the additivity property $I_c(\rho^{\otimes n}, \Lambda^{\otimes n}) = nI_c(\rho, \Lambda)$, the lemma and the above capacity formula [2] guarantee the inequality $\geq$ in (1). As shown later, Theorem 2 guarantees the existence of a channel $\Lambda(U, \Gamma)$ to satisfy the equality, which completes the proof of Theorem 1.

IV. CLIFFORD NETWORK MODEL

To extend Theorem 1 to the adaptive corruption described in Fig. 4, we introduce Clifford network model.

In this model, our code can achieve the capacity even with the single-shot setting. For this aim, we prepare several notations. Given a prime power $q = p^k$, our Hilbert space $\mathcal{H}$ is assumed to be spanned by the computational basis $\{|x\rangle\}_{x \in \mathbb{F}_q^d}$, where $\mathbb{F}_q$ is the algebraic extension of the finite field $\mathbb{F}_p$ with degree $d_q$. That is, the dimension of the Hilbert space $\mathcal{H}$ is assumed to be $q$. Then, for $s, t \in \mathbb{F}_q$, we define the generalized Pauli operators $X(s)$ and $Z(t)$ on the $n$-fold tensor product system $\mathcal{H}^{\otimes n}$ as $X(s) := \sum_{x \in \mathbb{F}_q} |x + s\rangle \langle x|$ and $Z(t) := \sum_{x \in \mathbb{F}_q} \omega^{tr_x x} |x\rangle \langle x|$, where $\omega := e^{2\pi i/n}$. Here, for an element $z \in \mathbb{F}_q$, $tr_z$ expresses the element $\text{Tr} M_z \in \mathbb{F}_p$, where $M_z$ denotes the matrix representation of the multiplication map $x \mapsto zx$ with identifying the finite field $\mathbb{F}_q$ with the vector space $\mathbb{F}_p^d$. We define the Fourier basis $\{|y\rangle \in \mathcal{H}\}_{y \in \mathbb{F}_q}$ of the computational basis $\{|x\rangle\}_{x \in \mathbb{F}_q} \subset \mathcal{H}$ as

$$|y\rangle := \frac{1}{\sqrt{q}} \sum_{x \in \mathbb{F}_q} \omega^{tr_x y} |x\rangle.$$

To consider our network model, for vectors $s = (s_1, \ldots, s_n), t = (t_1, \ldots, t_n) \in \mathbb{F}_q^n$, we define the operators $X(s)$ and $Z(t)$ on the $n$-fold tensor product system $\mathcal{H}^{\otimes n}$ as $X(s) := X(s_1) \otimes \cdots \otimes X(s_n)$ and $Z(t) := Z(t_1) \otimes \cdots \otimes Z(t_n)$. Then, the discrete Weyl operator is defined as $W(s, t) := X(s)Z(t)$. Then, for $(s, t), (s', t') \in \mathbb{F}_q^{2n}$, we define the skew symmetric matrix $J$ on $\mathbb{F}_q^{2n}$ and the inner product $\langle (s, t), (s', t') \rangle \in \mathbb{F}_p^d$ as $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ with $\langle (s, t), (s', t') \rangle := \sum_{i=1}^{n} \text{tr}(s_i s'_i + t_i t'_i)$. Then, the commutation relation

$$W(s, t)W(s', t') = \omega^{\langle (s, t), J(s', t') \rangle} W(s', t')W(s, t) \quad (4)$$

holds, and a square matrix $g$ on $\mathbb{F}_q^{2n}$ is called a symplectic matrix when $\langle (s, t), J(s', t') \rangle = \langle g(s, t), Jg(s', t') \rangle$ for $(s, t), (s', t') \in \mathbb{F}_q^{2n}$.

Next, we introduce Clifford group as a subset of the set $U(\mathcal{H}^{\otimes n})$ of unitaries on $\mathcal{H}^{\otimes n}$. Using the set $\mathcal{W} := \{cW(s, t)\}_{|c| = 1}$, we define the Clifford group $\mathcal{C}$ as $\mathcal{C} := \{U \in U(\mathcal{H}^{\otimes n})|UWU^{-1} = W\}$. An element of $\mathcal{C}$ is called a Clifford unitary.

For any element $U \in \mathcal{C}$, there exists a symplectic matrix $g$ such that

$$UW(s, t)U^{-1} = cW(g(s, t)) \quad (5)$$

for $(s, t) \in \mathbb{F}_q^{2n}$ with a complex number $c$ satisfying $|c| = 1$. Conversely, for any symplectic matrix $g$, there exists a unitary $U$ to satisfy (5). A typical construction of such a unitary $U$ is given in [24, Section 8.3]. This construction is called metaplectic representation and is denoted by $U(q)$ in this paper.

Now, the input and output systems are assumed to be $\mathcal{H}^{\otimes m_o}$, and the unitary $U_i$ is to be an element of Clifford group. Such a network is called Clifford network. We choose a symplectic matrix $g_i$ as $U(g_i) = U_i$. 

FIG. 3: Unitary network with individual corruption. After the application of the unitary $U_{i-1}$, the noisy operation $\Gamma_i$ is applied.
Theorem 2. For Clifford network, the minimum quantum capacity is \((m_0 - 2m_1 + 1)\log q\) in the adaptive corruption, i.e., the case when Eve has a memory to perform adaptive attacks.

To show Theorem 2 we describe the behaviors of the errors in the terms of the symplectic structure. That is, the errors can be described by vectors in \(F_q^{2m_0}\). For this aim, we introduce notations and parameters of the network as follows. Let \(e_i\) be the vector in \(F_q^{2m_0}\) that has only one nonzero element \(e_i = 1\) in the \(i\)-th entry. Using \(e_1\) and \(e_{m_0+1}\) we define \(2m_1\) vectors \(v_1, \ldots, v_{2m_1} \in F_q^{2m_0}\) as \(v_i := g_0^{-1} \cdots g_{i-1}^{-1} e_1\) and \(v_{m_1+i} := g_0^{-1} \cdots g_{i-1}^{-1} e_{m_0+1}\) for \(i = 1, \ldots, m_1\). Since \(e_1\) and \(e_{m_0+1}\) describe the directions of errors in the respective interval, all the directions in the linear space \(V\) spanned by \(v_1, \ldots, v_{2m_1}\) are corrupted in this whole network. However, since the direction \(v_i\) is corrupted, the direction \(Jv_i\) is also corrupted. Therefore, the set of all corrupted directions is given by \(V + JV\).

In this paper, when a matrix \(P\) satisfies \(P^2 = P\) and \(\text{Im}P = \mathcal{V}\), it is called a projection onto \(\mathcal{V}\). Then, we choose a projection \(P_0\) onto \(\mathcal{V}\). Since \(P_0^\dagger J P_0\) is also an anti-symmetric matrix, the rank of \(P_0^\dagger J P_0\) is an even number. The rank of the matrix \(P_0^\dagger J P_0\) equals the rank of matrix \(\langle (v_i, Jv_i) \rangle.\) Hence, the rank of \(P_0^\dagger J P_0\) does not depend on the choice of the projection \(P_0\) onto \(\mathcal{V}\) while the choice of the projection \(P_0\) onto \(\mathcal{V}\) is not unique. With these observations, we define the integers \(m_+\) and \(m_*\) as \(m_+ = (\text{rank} \ P_0^\dagger J P_0)/2\) and \(m_* = \dim \mathcal{V} - m_+\).

Since the rank of the submatrix \(\langle (v_i, Jv_j) \rangle_{i,j=1}^{2m_1}\) is \(2\), the rank of \(\langle (v_i, Jv_j) \rangle_{i,j=1}^{2m_1}\) is at least \(2\). As the rank of \(\langle (v_i, Jv_j) \rangle_{i,j=1}^{2m_1}\) equals the rank of \(P_0^\dagger J P_0\), the inequality
\[
m_* \geq 1
\]
holds. Thus, the inequality \(\dim \mathcal{V} \geq \text{rank} \ P_0^\dagger J P_0\) implies
\[
m_* \geq m_+.
\]
So, since \(2m_1 \geq m_* + m_*\), we have
\[
m_* \leq 2m_1 - 1.
\]

The quantum capacity is characterized as follows.

**Lemma 2.** The capacity \(C\) is lower bounded as \(C \geq (m_0 - m_*)\log q\) in the adaptive case, i.e., the case when Eve has a memory to perform her attack.

It is a key point for the construction of a code achieving the rate \((m_0 - m_*)\log q\) to avoid the space \(V + JV\) from the encoded space. As shown in Appendix D, we can choose \(2m_0\) independent vectors \(w_1, \ldots, w_{m_0}\) and \(w_1', \ldots, w_{m_0}'\) satisfying the following conditions. (i) \(\langle w_i, Jw_i \rangle = \delta_{i,j}\) and \(\langle w_i, Jw_j \rangle = \langle w_i', Jw_j' \rangle = 0\) for \(i, j = 1, \ldots, m_0\). (ii) The space \(V + JV\) is spanned by \(w_0, -w_0, \ldots, w_{m_0}, -w_{m_0}\) and \(w_0', -w_0', \ldots, w_{m_0}', -w_{m_0}'\). The condition (i) enables us to choose a symplectic matrix \(g_*\) such that \(g_* e_i = w_i\) and \(g_* e_{m_0+i} = w_i'\) for \(i = 1, \ldots, m_0\) because the vectors \(w_i\) and \(w_i'\) with \(i = 1, \ldots, m_0\) have the same symplectic structure as the vectors \(e_1, \ldots, e_{2m_0}\).

Then, we define the encoding unitary \(U_e := U(g_*)\) and the decoding unitary \(U_d := U(g_*)^{-1} U_{m_1}^{-1}\). The message space is set to \(H^{\otimes m_0 - m_*}\). The encoder is given as follows. We fix an arbitrary density \(\rho_0\) on \(H^{\otimes m_0}\). For any input density \(\rho_0\) on \(H^{\otimes m_0-m_*}\), the encoder is given as \(\rho \mapsto U_e(\rho \otimes \rho_0)U_e^{-1}\). The decoder is given as \(\rho \mapsto \text{Tr}_{m_*}(U_d\rho U_d^{-1})\), where \(\text{Tr}_{m_*}\) is the partial trace with respect to the system \(H^{\otimes m_*}\).

To analyze this code, we remind the following facts. First, the set of all corrupted directions is given by \(V + JV\). Second, the information on the computational basis and the information on the Fourier basis are decoded perfectly, if and only if the transmitted quantum state is decoded perfectly [25, 21, Section 8.15]. It is clear from the first fact and the construction of \(U_e\) and \(U_d\) that the information on the computation basis and the information on the Fourier basis is decoded perfectly even when Eve makes an adaptive corruption. Therefore, even for any adaptive corruption, the pair of the above encoder and the above decoder can decode the original state \(\rho_0\) on \(H^{\otimes m_0-m_*}\). This discussion shows Lemma 2. Our code construction depends only on \(g_0, \ldots, g_{m_1}\). That is, it is independent of the remaining unitaries \(U_1, \ldots, U_{m_1}\) of Eve’s corruption. The tightness of the evaluation in Lemma 2 is guaranteed as follows.

**Lemma 3.** When Eve changes the state on the corrupted edge to the completely mixed state, the capacity \(C\) equals \((m_0 - m_*)\log q\).

Lemma 3 is shown in Appendix C. In this way, the capacity is characterized by rank \(P_0^\dagger J P_0\) and \(\dim \mathcal{V}\). The following lemma clarifies the possible range of these two numbers.

**Lemma 4.** The following conditions are equivalent for two integers \(l_*\) and \(l_{**}\).

(1): \(2m_1 - 1 \geq l_* \geq l_{**} \geq 1\) and \(m_0 \geq l_*\).

(2): There exists a sequence of Clifford unitaries \(U(g_0), \ldots, U(g_{m_1})\) such that \(2l_* = \text{rank} P_0^\dagger J P_0\) and \(l_* + l_{**} = \dim \mathcal{V}\).

The relation (2)\(\Rightarrow\) (1) is shown as follows. The inequality \(l_* \geq l_{**}\) follows from (2), and the inequality \(l_* \geq 1\) follows from (2). Since \(\text{rank} P_0^\dagger J P_0 = 2l_*\), there exist \(l_*\) independent vectors \(x_1, \ldots, x_{l_*} \in \mathcal{V}\) such that \(\langle x_i, Jx_j \rangle = 0\) for \(i, j = 1, \ldots, l_*\). Since the number of such vectors is upper bounded by \(m_0\), we have \(m_0 \geq l_*\). The relation \(2m_1 - 1 \geq l_{**}\) follows from (2). The opposite direction (1)\(\Rightarrow\) (2) will be shown later by using after Lemma 5. Lemmas 2, 3 and 4 imply that the worst quantum capacity is \((m_0 - 2m_1 + 1)\log q\), which shows Theorem 2.

**V. BASIS-LINEAR NETWORK MODEL**

To construct a concrete network model to satisfy Condition (2) given in Lemma 4, we consider a special class of
Clifford networks, called basis-linear networks. In basis-linear networks, we assume that each Clifford unitary $U_i$ is characterized as the basis exchange caused by an invertible matrix $\bar{g}_i$ on $\mathbb{F}_q^{m_0}$, which is similar to the case of CSS (Calderbank-Shor-Steane) code [24, 27]. That is, the Clifford unitary $U_i$ is given as the unitary $\bar{U}(\bar{g}_i)$ defined by $\bar{U}(\bar{g})|x⟩ = |\bar{g}x⟩$. Its action on the Fourier basis $\{ |y⟩_F \}_{y \in \mathbb{F}_q^{m_0}}$ is characterized as $\bar{U}(\bar{g})|y⟩_F = |[\bar{g}]_F y⟩_F$, where $|[\bar{g}]_F⟩$ is defined as the transpose $(\bar{g}^{-1})^\top$ of the inverse matrix $\bar{g}^{-1}$. Appendix A. Hence, we have

$$\bar{U}(\bar{g}) = U \left( \begin{pmatrix} \bar{g} & 0 \\ 0 & [\bar{g}]_F \end{pmatrix} \right).$$

Let $\bar{e}_i$ be the vector in $\mathbb{F}_q^{m_0}$ that has only one nonzero element 1 in the $i$-th entry. By using the vector $\bar{v}_i = (1,0,\ldots,0) \in \mathbb{F}_q^{m_0}$, the vectors $v_1, \ldots, v_{2m_1}$ are written as $v_i = (\bar{v}_i, 0)$ and $v_{m_1+i} = (0, \bar{v}'_i)$ with $\bar{v}_i := \bar{g}_0 \cdots \bar{g}_{i-1} \bar{e}_1$ and $\bar{v}'_i := \bar{g}_0 \cdots \bar{g}_{i-1} \bar{e}_1$ for $i = 1, \ldots, m_1$. We define the matrices $V$ and $V'$ as $(\bar{v}_1, \ldots, \bar{v}_{m_1})$ and $(\bar{v}'_1, \ldots, \bar{v}'_{m_1})$. Then, we have

$$m_\ast = \text{rank}(V')^\top V, \quad m_{\ast\ast} = \text{rank} V + \text{rank} V' - m_\ast.$$  \hspace{1cm} (10)

**Lemma 5.** The following conditions are equivalent for three integers $l_1, l_2$, and $l_3$.

1. $m_1 \geq l_1 \geq l_3 \geq 1$, $m_1 \geq l_2 \geq l_3 \geq 1$, and $m_0 \geq l_1 + l_2 - l_3$.

2. There exists a sequence of invertible matrices $\bar{g}_0, \ldots, \bar{g}_{m_1}$ over finite field $\mathbb{F}_q$ such that rank $V = l_1$, rank $V' = l_2$, and rank $(V')^\top V = l_3$.

Lemma 5 is shown in Appendix E with the concrete construction of a basis-linear network to satisfy Condition (2) in Lemma 5.

When Condition (1) of Lemma 4 holds, Condition (1) of Lemma 5 holds with the condition $l_\ast = l_3$ and $l_{\ast\ast} = l_1 + l_2 - l_3$. Since Condition (2) in Lemma 5 implies Condition (2) of Lemma 4 with this condition, we obtain the direction (1)$\Rightarrow$(2) in Lemma 4 which completes the proof of Theorem 2.

VI. DISCUSSION

We have shown that the quantum capacity is not smaller than $(m_0 - 2m_1 + 1) \log d$ when the sender has $m_0$ outgoing channels, the receiver has $m_0$ incoming channels, each intermediate node applies invertible unitary, only $m_1$ channels are corrupted in our quantum network model, and other non-corrupted channels are noiseless. Our result holds with the following two cases. In the first case, the unitaries on intermediate nodes are arbitrary and the corruptions on the $m_1$ channels are individual. In the second case, the unitaries on intermediate nodes are restricted to Clifford operations and the corruptions on the $m_1$ channels are adaptive, i.e., the attacker is allowed to have a quantum memory. Further, our code in the second case realizes the noiseless communication even with the single-shot setting, and depends only on the node operations, the network topology, and the places of the $m_1$ corrupted channels. That is, it is independent of Eve’s operation on the $m_1$ corrupted channels. This code utilizes the following structure of this model. The error in the first corrupted channel can be concentrated to one quantum system. However, the errors of the computation basis and the Fourier basis in another corrupted channel split to two quantum systems in general. Hence, $2m_1 - 1$ quantum systems are corrupted in the worst case. As explained in Appendix, the first case has been shown by the analysis of the coherent information, and symplectic structure including symplectic diagonalization on the discrete system plays a key role in the second case. It is an interesting remaining problem to derive the quantum capacity when the operations on intermediate nodes are arbitrary unitaries and the corruptions on the $m_1$ channels are adaptive.

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APPENDIX

The appendix is organized as follows. Appendix A discusses the classical case. Appendix B proves Lemma 4. Appendix C proves Lemma 5. Appendix D gives the choice of vectors $w_1, \ldots, w_{m_0}$ and $w'_1, \ldots, w'_{m_0}$ that are used in code to achieve the rate $C \geq (m_0 - m_{\ast\ast}) \log q$. Since it employs symplectic diagonalization, Appendix D2 summarizes a fundamental knowledge of symplectic diagonalization. Appendix E proves Lemma 5.

Appendix A: Classical network model

We consider a classical network model as follows. Every channel transmits a system whose number of elements is $d$ by one use of the network. The sender has $m_0$ outgoing channels. Node operations are not necessarily linear but are invertible. $m_1$ channels are corrupted. Hence, we can assume that $m_1$ corruptions are done sequentially. Let $X'_i$ and $X_i$ be the whole information before and after the $i$-th corruption, respectively. Then, $X'_i$ is written as $f_i(X_i)$ by using an invertible function $f_i$. Here, we denote the input and output information of this network by $X_0$ and $X'_{m_1}$, respectively. Since the channel capacity of classical communication is given by the maximum
mutual information between the input information and the output information, it is sufficient to show that
\[ I(X_0; X_{m_1}') = I(X_0; X_{m_1}) \geq (m_0 - m_1) \log d \]  
(A1) with a certain distribution \( P_{X_0} \) of \( X_0 \).

Now, we set the distribution \( P_{X_0} \) of \( X_0 \) to be the uniform distribution, which implies that
\[ H(X_0) = m_0 \log d. \]  
(A2)

From the network structure, we find the relation \( H(X_i|X_{i+1}) \leq \log d \). The chain rule of conditional entropy implies that
\[
H(X_0|X_{m_1}) \leq H(X_0 \ldots X_{m_1-1}|X_{m_1}) \leq \sum_{l=0}^{m_1-1} H(X_l|X_{l+1}X_{l+2} \ldots X_{m_1}) \leq m_1 \log d. \]  
(A3)

The combination of (A2) and (A3) yields (A1).

Appendix B: Proof of Lemma 1

It is sufficient to show the case when \( U_0 \) is the identity matrix. Let \( \rho_{\text{mix},m_0-1} \) be the completely mixed state on \( \mathcal{H}^{\otimes m_0-1} \). We set the initial state to be \( |0\rangle \otimes \rho_{\text{mix},m_0-1} \).

Consider the time after the unitary \( U_{i-1} \) is applied but \( \Gamma_i \) is not applied yet. At this time, we denote the system to be attacked and the remaining system by \( A_i \) and \( B_i \), respectively. After the application of \( \Gamma_i \), we denote the system to be attacked and the remaining system by \( A_i' \) and \( B_i' \), respectively. We consider Steinspring representation \( \tilde{U}_i \) of \( \Gamma_i \), in which the output of the environment is \( E_i \). Fig. 4 summarizes the relation among the systems \( A_{i-1}', B_{i-1}, A_i, B_i, A_i', B_i', \) and \( E_i \).

![Fig. 4](image)

FIG. 4: Relation among systems \( A_{i-1}', B_{i-1}, A_i, B_i, A_i', B_i', \) and \( E_i \). After the application of the unitary \( U_{i-1} \), the operation \( \tilde{U}_i \) is applied.

Since the state on \( A_i' E_i \) is pure, we have \( H(A_i') = H(E_i) \). Hence, we have
\[
H(A_i' B_i') - H(E_i) = H(A_i') + H(B_i') - H(E_i) = H(B_i') = (m_0 - 1) \log d. \]  
(B1)

As shown later, for \( i = 2, \ldots, m_1 \), we have
\[
H(A_i'B_i') - H(E_i) \geq H(A_{i-1}' B_{i-1}') - 2 \log d. \]  
(B2)

Combining (B1) and (B2), we have
\[
I_c(|0\rangle \otimes \rho_{\text{mix}},m_0-1, \Lambda(U_0, \Gamma_1, U_1, \ldots, \Gamma_{m_1}, U_{m_1})) = H(A_{m_1}', B_{m_1}') - H(E_1, \ldots, E_{m_1}) \geq (m_0 - 2m_1 + 1) \log d. \]  
(B3)

Next, we show (B2). Consider the purification of \( \rho_{A_i'E_i} \) by using the reference system \( R \). Then, \( H(R) = H(A_i'E_i) = H(A_i) \). Since \( H(A_i) + H(A_i') - H(E_i) = H(R) + H(A_i') - H(RA_i') = I(R; A_i') \geq 0 \), we have
\[
H(E_i) \leq H(A_i) + H(A_i'). \]  
(B4)

Thus,
\[
\begin{align*}
(a) & \quad H(A_i'B_i') - H(E_i) \geq H(A_i'B_i') - H(A_i) - H(A_i') \\
& = H(B_i') + H(A_i') - H(A_i) - H(A_i') - I(A_i'B_i') \\
& = H(A_i'B_i'E_i) - H(A_i'E_i) + I(A_i'E_i; B_i') \\
& \quad + H(A_i') - H(A_i) - H(A_i') - I(A_i'B_i') \\
(b) & \quad = H(A_iB_i) - H(A_i) + I(E_i; B_i'|A_i') - H(A_i) \\
& \geq H(A_iB_i) - 2H(A_i) = H(A_{i-1}' B_{i-1}') - 2H(A_i) \\
& \geq H(A_{i-1}' B_{i-1}') - 2 \log d. \quad (B5)
\end{align*}
\]

Here, (a) follows from (B3), (b) follows from \( H(A_i'B_i'E_i) = H(A_iB_i) \) and \( H(A_i'E_i) = H(A_i) \), and (c) follows from \( I(E_i; B_i'|A_i') \geq 0 \).

Appendix C: Proof of Lemma 3

To show Lemma 3, we prepare the following lemma, which will be shown in the end of this Appendix.

**Lemma 6.** When a channel \( \Lambda_A \) is entanglement-breaking, a channel \( \Lambda_B \) satisfies the condition
\[
\max_{\rho} I_c(\rho, \Lambda_A \otimes \Lambda_B) = \max_{\tau} I_c(\tau, \Lambda_B). \]  
(C1)

For the preparation of Lemma 3, we define the notation for the generalized Pauli channel as follows. Given a distribution \( P \) on \( \mathbb{F}_q^2 \), we define the channel \( \Lambda_P \) as
\[
\Lambda_P(\rho) := \sum_{(s,t) \in \mathbb{F}_q^2} P(s,t)W(s,t)\rho W(s,t)^{-1}. \]  
(C2)

Then, we denote the uniform distribution over \( \mathbb{F}_q^{2m} \) and the uniform distribution over the subset \( \{(s,0)\}_{s \in \mathbb{F}_q^2} \) by \( P_{\text{mix},m} \) and \( P_{\text{mix}Z,m} \), respectively. Hence, the channels on the attacked edges are the Pauli channel \( \Lambda_{P_{\text{mix},1}} \).

We assume that the sender applies the encoding unitary \( U_c \) before transmission and the receiver applies the
decoding unitary $U_d$ after the reception. Considering the output behaviors on the computation basis and on the Fourier basis, we find that the channel from the sender to the receiver is given as $Id_{m_0-m_+} \otimes \Lambda_{\text{min},m_+} \otimes \Lambda_{\text{p},m_+ \cdots -m_+}$. Since $P_{\text{p},m_+ \cdots -m_+}$ is the pinching channel with respect to the measurement on the Fourier basis, we find that the channel $\Lambda_{\text{p},m_+ \cdots -m_+} \otimes \Lambda_{\text{min},m_+}$ is entanglement-breaking.

Since the channel $\Lambda_{\text{p},m_+ \cdots -m_+} \otimes \Lambda_{\text{min},m_+}$ is entanglement-breaking, Lemma [D] guarantees that

$$\max_\rho I_c(\rho, (Id_{m_0-m_+} \otimes \Lambda_{\text{min},m_+} \otimes \Lambda_{\text{p},m_+ \cdots -m_+})^\otimes n)$$

$$= \max_\rho I_c(\rho', (Id_{m_0-m_+}^\otimes n) = n(m_0 - m_+) \log q, \quad \text{(C3)}$$

where $I_c(\rho, \Lambda)$ is the coherent information.

Since the maximum transmission rate is upper bounded by the maximum coherent information, we obtain the converse part.

Proof of Lemma [E] Let $A$ and $B$ ($A'$ and $B'$) be the input (output) systems of $\Lambda_A$ and $\Lambda_B$, respectively. We choose a state $\rho_{AB}$ on $AB$. Let $C$ be the reference system of the state $\rho_{ABC}$ so that $\rho_{AB|C}$ is the purification of $\rho_{AB}$. Let $\rho'$ be the output system on the whole system of $A'$, $B'$ and $C$.

Since $\Lambda_A$ is entanglement-breaking, it is written as $\Lambda_A(\sigma) = \sum_a \rho_{A=a}\text{Tr} M_a \sigma$, where $\{M_a\}$ is a POVM and rank $M_a = 1$. Hence, $\Lambda_A(\rho_{ABC})$ is written as $\sum_a P_{A'=a} \rho_{ABC|A'=a} \otimes \rho_{A=a}$, where $P_{A'=a} := \text{Tr} M_a \rho_{ABC}$ and $\rho_{ABC|A'=a} := \text{Tr}_A M_a \rho_{ABC} / P_{A'=a}$. Hence, rank $\rho_{ABC|A'=a} = 1$.

Then, we denote $\Lambda_B(\rho_{BC|A'=a})$ by $\rho_{B'C'|A'=a}$. The coherent information $I_c(\rho_{AB}, \Lambda_A \otimes \Lambda_B)$ equals $D(\rho_{A'B'C'}|\rho_{A'B'} \otimes I_C)$, which is evaluated as

$$D(\rho_{A'B'C'}|\rho_{A'B'} \otimes I_C)$$

$$= D\left( \sum_a P_{A'=a} \rho_{B'C'|A'=a} \otimes \rho_{A'=a} \right)$$

$$= D\left( \sum_a P_{A'=a} \rho_{B'|A'=a} \otimes \rho_{A'=a} \right)$$

$$\leq D\left( \sum_a P_{A'=a} \rho_{B'C'|A'=a} \otimes |a\rangle \langle a| \right)$$

$$= \sum_a P_{A'=a} D(\rho_{B'C'|A'=a} | \rho_{B'|A'=a} \otimes I_C). \quad \text{(C4)}$$

The inequality follows from the information processing inequality for the map $|a\rangle \langle a|$ to $\rho_{A'=a}$.

Since rank $\rho_{BC|A'=a} = 1$, $\rho_{BC|A'=a}$ is a purification of $\rho_{B|A'=a}$. $D(\rho_{B'C'|A'=a} | \rho_{B'|A'=a} \otimes I_C)$ equals the coherent information $I_c(\rho_{B|A'=a}, \Lambda_B)$. Hence, we have

$$I_c(\rho_{AB}, \Lambda_A \otimes \Lambda_B)$$

$$\leq \sum_a P_{A'=a} I_c(\rho_{B|A'=a}, \Lambda_B), \quad \text{(C5)}$$

which implies that

$$\max_\rho I_c(\rho, \Lambda_A \otimes \Lambda_B) \leq \max_\tau I_c(\tau, \Lambda_B). \quad \text{(C6)}$$

Next, we show the converse inequality

$$\max_\rho I_c(\rho, \Lambda_A \otimes \Lambda_B) \geq \max_\tau I_c(\tau, \Lambda_B). \quad \text{(C7)}$$

For any state $\tau$ on the system $B$, define $\rho_{AB} = \rho_A \otimes \tau$ where $\rho_A$ is a pure state. Then, we have

$$I_c(\rho_{AB}, \Lambda_A \otimes \Lambda_B) = I_c(\tau, \Lambda_B). \quad \text{(C8)}$$

Therefore, we obtain (C7).

Appendix D: Choice of vectors $w_1, \ldots, w_{m_0}$ and $\tilde{w}_1, \ldots, \tilde{w}_{m_0}$

1. Construction except for $w_{m_0-m_++m_++1}, \ldots, w_{m_0}$

Now, we choose we can choose $2m_0$ independent vectors $w_1, \ldots, w_{m_0}$ and $\tilde{w}_1, \ldots, \tilde{w}_{m_0}$ satisfying the following conditions. (i) $\langle w_i, Jw_j \rangle = \delta_{i,j}$ and $\langle v_i, Jw_j \rangle = \langle w_i, Jw_j \rangle = 0$ for $i, j = 1, \ldots, m_0$. (ii) The space $V + JV$ is spanned by $w_{m_0-m_++m_++1}, \ldots, w_{m_0}$.

Define $V_2 := \text{Ker} P_{V} J P_{V} \cap V$. We choose another subspace $V_1$ of $V$ such that $V_1 \oplus V_2 = V$. Since $P_{V} J P_{V}$ is non-degenerate on $V_1$, we can choose $m_0$ independent vectors $w_{m_0-m_++m_++1}, \ldots, w_{m_0-m_++m_++m_0} \in V_1 \subset \mathbb{F}_{q^{m_0}}$ and other $m_0$ independent vectors $w_{m_0-m_++m_++1}, \ldots, w_{m_0-m_++m_++m_0} \in V_1 \subset \mathbb{F}_{q^{m_0}}$ such that $\langle w_i, Jw_j \rangle = \delta_{i,j}$, $\langle w_i, Jw_j \rangle = \langle w_i, Jw_j \rangle = 0$ for $i, j = m_0 - m_++1, \ldots, m_0 - m_++m_0$.

Let $w_{m_0-m_++m_++1}, \ldots, w_{m_0}$ be a basis of $V_2$. We define $V_3 := \text{Ker} P_{V} J P_{V}$. We have the direct sum $V_1 \oplus V_3 = \mathbb{F}_{q^{m_0}}$. Based on them, as shown in Subsection [D3] we can choose independent vectors $w_{m_0-m_++m_++1}, \ldots, w_{m_0} \in V_3 \subset \mathbb{F}_{q^{m_0}}$ to satisfy the conditions $\langle w_i, Jw_j \rangle = \delta_{i,j}$ and $\langle w_i, Jw_j \rangle = \langle w_i, Jw_j \rangle = 0$ for $i, j = m_0 - m_++1, \ldots, m_0 - m_++m_0$.

The subspace $V_4$ is defined as the space spanned by $w_{m_0-m_++m_++1}, \ldots, w_{m_0}$ and $w_{m_0-m_++m_++1}, \ldots, w_{m_0}$. Choosing a projection $P_{V_3}$ onto $V_4$, we define the subspace $V_5 := \text{Ker} P_{V_3} J$. Since the dimension of the image of $P_{V_3} J$ is $2m_+$, that of $V_5$ is $2(m_0 - m_+)$. Also, there is no cross term in $J$ between $V_4$ and $V_5$ because $\langle v, Jv' \rangle = \langle P_{V_3} v, Jv' \rangle = \langle v, P_{V_3} Jv' \rangle = 0$ for $v \in V_4, v' \in V_5$. Thus, the rank of $P_{V_3} J P_{V_3}$ is $2(m_0 - m_+)$, where $P_{V_3}$ is a projection to $V_5$. Hence, we choose $2(m_0 - m_+)$ independent vectors $w_1, \ldots, w_{m_0-m_+} \in V_5$ and $w_1, \ldots, w_{m_0-m_+} \in V_3$ such that $\langle w_i, Jw_j \rangle = \delta_{i,j}$ and $\langle w_i, Jw_j \rangle = \langle w_i, Jw_j \rangle = 0$ for $i, j = 1, \ldots, m_0 - m_+$.
Since there is no cross term in $J$ between $\mathcal{V}_3$ and $\mathcal{V}_2$, the chosen vector $w_1, \ldots, w_m \in \mathbb{F}_q^{2m_0}$ and $w'_1, \ldots, w'_m \in \mathbb{F}_q^{2m_0}$ satisfy the conditions $\langle w'_i, Jw'_j \rangle = \delta_{i,j}$ and $\langle w_i, Jw_j \rangle = 0$ for $i, j = 1, \ldots, m$. Therefore, we can choose a symplectic matrix $g_z$ such that $g_z e_1 = w_i$ and $g_z e_{m+1} = w'_i$ for $i = 1, \ldots, m$ because the vectors $w_i$ and $w'_i$ with $i = 1, \ldots, m$ have the same symplectic structure as the vectors $e_1, \ldots, e_{2m}$. 

2. Symplectic diagonalization

For the choice of $w'_m, w_{m-1}, \ldots, w'_1$, we prepare fundamental knowledge for symplectic diagonalization in the finite dimensional system. Assume that $\mathcal{V}$ is a finite-dimensional vector space over a finite field $\mathbb{F}_q$. We consider a bilinear form $Q$ from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{F}_q$. An element $v \in \mathcal{V}$, $Q(v, \cdot)$ can be regarded as an element of the dual space $\mathcal{V}^\ast$ of $\mathcal{V}$. In this sense, $Q$ can be regarded as a linear map from $\mathcal{V}$ to $\mathcal{V}^\ast$. A bilinear form $Q$ from $\mathcal{V} \times \mathcal{V}$ to $\mathbb{F}_q$ is called anti-symmetric when $Q(v_1, v_2) = -Q(v_2, v_1)$ and $Q(v_1, v_1) = 0$ for $v_1, v_2 \in \mathcal{V}$.

Lemma 7. Assume that an anti-symmetric bilinear form $Q$ is surjective, i.e., Ker $Q = \{0\}$. Then, the dimension of $\mathcal{V}$ is an even number 2k. There exists a basis $w_1, w_2, w'_1, \ldots, w'_k \in \mathcal{V}$ such that $Q(w'_i, w_j) = \delta_{i,j}$ and $Q(w_i, w_j) = Q(w'_i, w'_j) = 0$ for $i, j = 1, \ldots, k$.

Proof of Lemma 7. Such a basis can be chosen inductively. We choose a non-zero vector $w_1 \in \mathcal{V}$. Since $Q$ is surjective, we can choose another non-zero vector $w'_1 \in \mathcal{V}$ such that $Q(w'_1, w_1) = 1$.

Due to the assumption of induction, we have vectors $w_1, \ldots, w_i, w'_1, \ldots, w'_i \in \mathcal{V}$ such that $Q(w'_i, w_j) = \delta_{i,j}$ and $Q(w_i, w'_j) = Q(w'_i, w'_j) = 0$ for $i, j = 1, \ldots, l$. Then, we define the subspace $\mathcal{V}_i := \{v \mid Q(w_i, v) = \cdots = Q(w'_i, v) = \cdots = Q(w'_i, v) = 0\}$. Also, we define the subspace $\mathcal{V}'_i$ spanned by $w_1, w_2, w'_1, \ldots, w'_i \in \mathcal{V}$. Since $\mathcal{V}_i \cap \mathcal{V}'_i = \{0\}$, we have the direct sum $\mathcal{V} = \mathcal{V}_i \oplus \mathcal{V}'_i$. We choose a non-zero vector $w_{i+1} \in \mathcal{V}_i$. Since $Q$ is surjective, we can choose another non-zero vector $w'_{i+1} \in \mathcal{V}'_i$ such that $Q(w_{i+1}, w'_{i+1}) = 1$. Also, we have $Q(w_i, w_{i+1}) = Q(w'_i, w'_{i+1}) = 0$ for $i = 1, \ldots, l$. Based on the direct sum $\mathcal{V} = \mathcal{V}_i \oplus \mathcal{V}'_i$, we have the decomposition $w_{i+1} = w'_{i+1} + w^\ast_{i+1}$ with $w^\ast_{i+1} \in \mathcal{V}'_i$ and $w'_{i+1} \in \mathcal{V}_i$. Since $w'_{i+1} \in \mathcal{V}_i$, we have $Q(w_i, w^\ast_{i+1}) = Q(w'_i, w^\ast_{i+1}) = 0$ for $i = 1, \ldots, l$. Since $Q(w_{i+1}, w_{i+1}) = 0$, we have $Q(w'_{i+1}, w'_{i+1}) = 1$. Therefore, we obtain a desired basis.

3. Choice of $w'_m, w_{m-1}, \ldots, w'_1$

Using symplectic diagonalization, we show that we can choose independent vectors $w'_m, w_{m-1}, \ldots, w'_1$ in $\mathcal{V}_3 \subset \mathbb{F}_q^{2m_0}$ inductively. First, we choose $w_m, w_{m-1}, \ldots, w'_1$ to satisfy the condition $\langle w_m, w_{m-1}, \ldots, w'_1, Jw_j \rangle = \delta_{m0-m, m+1, j}$ for $j = m_0 - m, \ldots, m$. Based on the direct sum $\mathcal{V}_1 \oplus \mathcal{V}_3 = \mathbb{F}_q^{2m_0}$, we decompose $u_{m_0-m, m+1} = u'_m + w'_m, m+1$ such that $u'_m, w'_m, m+1 \in \mathcal{V}_3$. Since $\langle u'_m, w'_m, m+1, Jw_j \rangle = 0$, we have $\langle u'_m, w'_m, m+1, Jw_j \rangle = \delta_{m0-m, m+1, j}$ for $j = m_0 - m, m+1, \ldots, m_0$.

Next, from vectors $w'_m, w_{m-1}, \ldots, w'_1 \in \mathcal{V}_3 \subset \mathbb{F}_q^{2m_0}$, we choose $u_{i+1}$ to satisfy the conditions

$$\langle u_{i+1}, Jw_j \rangle = \delta_{i,j}, \langle u_{i+1}, Jw'_j \rangle = 0$$

for $j = m_0 - m, m + 1, \ldots, m_0$ and $i = m_0 - m, m + 1, \ldots, l$. Based on the direct sum $\mathcal{V}_1 \oplus \mathcal{V}_3 = \mathbb{F}_q^{2m_0}$, we decompose $u_{i+1} = \bar{u}'_{i+1} + \hat{w}'_{i+1}$ such that $\bar{u}'_{i+1} \in \mathcal{V}_1, \hat{w}'_{i+1} \in \mathcal{V}_3$. Then, we define $u_{i+1}' := \bar{u}'_{i+1} + \sum_{l=m_0-m, \ldots, m+1}^{i} \langle \bar{u}'_{i+1}, Jw_j \rangle w_j$. Hence, we have

$$u_{i+1} = \bar{u}'_{i+1} - \sum_{l=m_0-m, \ldots, m+1}^{i} \langle \bar{u}'_{i+1}, Jw'_j \rangle \hat{w}'_{i+1} + \hat{w}'_{i+1}$$

Since $u_{i+1} = \sum_{i=m_0-m, \ldots, m+1}^{i} \langle \bar{u}'_{i+1}, Jw_j \rangle w_j \in \mathcal{V}$ and $w_j \in \mathcal{V}_2 = \text{Ker} P_j^\dagger J P_j^T \mathcal{V}$ for $j = m_0 - m, \ldots, m_0, m + 1$, we have the relation

$$\langle u_{i+1}' - \sum_{l=m_0-m, \ldots, m+1}^{i} \langle \bar{u}'_{i+1}, Jw'_j \rangle \hat{w}'_{i+1} \rangle, J P_j^T \mathcal{V} = 0.$$  

(D2)

For $i' = m_0 - m, m + 1, \ldots, l$, we have

$$\langle u_{i+1}' - \sum_{l=m_0-m, \ldots, m+1}^{i} \langle \bar{u}'_{i+1}, Jw'_j \rangle \hat{w}'_{i+1} \rangle, J P_j^T \mathcal{V} = 0.$$  

(D3)

The combination of (D1), (D2), and (D3), implies the relations $\langle w_{i+1}' \rangle Jw_j = \delta_{i+1, j}$ and $\langle w_{i+1}', Jw'_j \rangle = 0$.

Notice that $\langle w_{i+1}' \rangle Jw'_i = 0$ holds since $J$ is anti-symmetric.

Appendix E: Proof of Lemma 5

Now, we show (2)⇒(1). Since the relations $m_1 \geq l_1 \geq l_3$, $m_1 \geq l_2 \geq l_3$ are trivial, it is sufficient to show $m_0 \geq l_1 \geq l_3 - l_3$, $l_3 \leq 1$. Since the matrix $(V')^\dagger V$ is not zero, we have $l_3 \geq 1$. The relation $m_0 \geq l_1 \geq l_2 - l_3$ follows from (8) and (10). Thus, we obtain the relation (2)⇒(1).

Now, we show (1)⇒(2). We assume that $l_1 \geq l_2$. Otherwise, we can exchange the computation basis and the
The matrix \([A_{i+1}]_F\) is characterized for \(i = 1, \ldots, l_2 - l_3\) as follows. For the 1-st, \(l_3 + 2i - 1\)-th, and \(l_3 + 2i\)-th entries, it is given as \(\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\). For other indices \(j, j'\), \(([A_{i+1}]_F)_{j,j'}\) is given as \(\delta_{j,j'}\).

The matrix \([A_{i+1}]_F\) is characterized for \(i = 1, \ldots, l_1 - l_2\) as follows. For the 1-st and 2\(l_2 - l_3\) + i-th entries, it is given as \(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\). For other indices \(j, j'\), the matrix component \(([A_{i+1}]_F)_{j,j'}\) is given as \(\delta_{j,j'}\). Then, for \(i = 1, \ldots, l_3\), we have \(\bar{v}_i = \bar{e}_i\). For \(i = 1, \ldots, l_2 - l_3\), we have \(\bar{v}_{i+l} = \bar{e}_i + \bar{e}_{l_3+2i-1}\). For \(i = 1, \ldots, l_1 - l_2\), we have \(\bar{v}_{i+l} = \bar{e}_i + \bar{e}_{2l_2 - l_3+i}\). For \(i = l_1 + 1, \ldots, m_1\), we have \(\bar{v}_i = \bar{e}_i\). Therefore, we have \(V = \bar{l}_1\) and rank\(V' = l_2\).

Also, when \(j = 2, \ldots, l_2\) or \(j' = 2, \ldots, l_3\), we have \(((V')^\top V)_{j,j'} = \delta_{j,j'}\). When \(j = 1, l_3 + 1, \ldots, m_3\) or \(j' = 1, l_3 + 1, \ldots, m_1\), we have \(((V')^\top V)_{j,j'} = 1\). Hence, rank\((V')^\top V = l_3\). Thus, we obtain the relation (1)⇒(2).

[1] N. Cai and R. W. Yeung, “Network error correction, Part 2: Lower bounds,” Commun. Inf. and Syst., vol. 6, no. 1, 37-54, (2006).

[2] S. Jaggi, M. Langberg, S. Katti, T. Ho, D. Katabi, M. Medard, and M. Effros, “Resilient Network Coding in the Presence of Byzantine Adversaries,” IEEE Trans. Inform. Theory, vol. 54, no. 6, 2596–2603 (2008).

[3] H. Yao, D. Silva, S. Jaggi, and M. Langberg, “Network Codes Resilient to Jamming and Eavesdropping,” IEEE/ACM Transactions on Networking, vol. 22, no. 6, 1978–1987 (2014).

[4] M. Hayashi, K. Iwama, H. Nishimura, R. Raymond, and S. Yamashita, “Quantum Network Coding,” in STACS 2007 SE - 52 (W. Thomas and P. Weil, eds.), vol. 4393 of Lecture Notes in Computer Science, pp. 610–621, Springer Berlin Heidelberg, 2007.

[5] M. Hayashi, “Prior entanglement between senders enables perfect quantum network coding with modification,” Phys. Rev. A, vol. 76, no. 4, 40301, 2007.

[6] H. Kobayashi, F. Le Gall, H. Nishimura, and M. Rötteler, “General Scheme for Perfect Quantum Network Coding with Free Classical Communication,” in Automata, Languages and Programming SE - 52 (S. Albers, A. Marchetti-Spaccamela, Y. Matias, N. Nikoletseas, and W. Thomas, eds.), vol. 5555 of Lecture Notes in Computer Science, pp. 622–633, Springer Berlin Heidelberg, 2009.

[7] D. Leung, J. Oppenheim, and A. Winter, “Quantum Network Communication: The Butterfly and Beyond,” IEEE Trans. Inform. Theory, vol. 56, no. 7, 3478–3490, 2010.

[8] H. Kobayashi, F. Le Gall, H. Nishimura, and M. Rötteler, “Perfect quantum network communication protocol based on classical network coding,” in Proceedings of 2010 IEEE International Symposium on Information Theory (ISIT), pp. 2686–2690, 2010.

[9] H. Kobayashi, F. Le Gall, H. Nishimura, and M. Rötteler, “Constructing quantum network coding schemes from classical nonlinear protocols,” in Proceedings of 2011 IEEE International Symposium on Information Theory (ISIT), pp. 109–113, 2011.

[10] A. Jain, M. Franceschetti, and D. A. Meyer, “On quantum network coding,” J. Math. Phys., vol. 52, 032201, 2011.

[11] M. Owari, G. Kato, and M. Hayashi, “Secure Quantum Network Coding on Butterfly Network,” Quantum Sci. and Technology, vol. 3, 014001 (2017).

[12] G. Kato, M. Owari, and M. Hayashi, “Single-Shot Secure Quantum Network Coding for General Multiple Unicast Network with Free Public Communication,” In: Shikata J. (eds) 10th International Conference on Information Theoretic Security (ICITS2017). Lecture Notes in Computer Science, vol 10681. Springer, pp. 166-187.

[13] S. Song and M. Hayashi, “Quantum Network Code for Multiple-Unicast Network with Quantum Invertible Linear Operations,” In: S. Jeffery (eds) 13th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2018). Leibniz International Proceedings in Informatics (LIPIcs), vol 111. pp. 10:1–10:20. Centre for Quantum Software and Information (QSI), University of Technology Sydney, July 16 – 18, 2018.

[14] H. Lu, Z. Li, X. Yin, R. Zhang, X. Fang, L. Li, N. Liu, F. Xu, Y. Chen, and J. Pan, npj Quantum Inf. vol. 5, 89 (2019).

[15] S. Song and M. Hayashi, “Secure Quantum Network Code without Classical Communication,” IEEE Trans. Inform. Theory (In press).

[16] G. Chiribella, G. M. D’Ariano, and P. Perinotti, “Quantum circuit architecture,” Phys. Rev. Lett., vol. 101, 060401, 2008.
[17] G. Chiribella, G. M. D’Ariano, and P. Perinotti, “Theoretical framework for quantum networks,” Phys. Rev. A, vol. 80, 022339, 2009.

[18] A. W. Harrow, A. Hassidim, D. W. Leung, and J. Watrous, “Adaptive versus nonadaptive strategies for quantum channel discrimination,” Phys. Rev. A vol. 81, 032339 (2010).

[19] B. Schumacher and M. A. Nielsen, “Quantum data processing and error correction,” Phys. Rev. A, vol. 54, 2629 (1996).

[20] M. Hayashi, Quantum Information Theory, Graduate Texts in Physics, Springer (2017).

[21] S. Lloyd, “The capacity of the noisy quantum channel,” Phys. Rev. A, vol. 56, 1613 (1997).

[22] H. Barnum, M. A. Nielsen, and B. Schumacher, “Information transmission through a noisy quantum channel.” Phys. Rev. A, vol. 57, 4153–4175 (1997).

[23] P. W. Shor, The quantum channel capacity and coherent information, in Lecture Notes, MSRI Workshop on Quantum Computation (2002). http://www.msri.org/publications/ln/msri/2002/quantumcrypto/shor/1/

[24] I. Devetak, The private classical capacity and quantum capacity of a quantum channel. IEEE Trans. Inform. Theory, vol. 51, 44–55 (2005)

[25] J.M. Renes, “Duality of privacy amplification against quantum adversaries and data compression with quantum side information,” Proc. Roy. Soc. A, vol. 467(2130), 1604–1623 (2011)

[26] A. R. Calderbank and P. W. Shor, “Good quantum error-correcting codes exist,” Phys. Rev. A, vol. 54, pp. 1098-1105 (1996).

[27] A. M. Steane, “Error correcting codes in quantum theory,” Phys. Rev. Lett., vol. 77, pp. 793-767 (1996).