DIFFUSION PHENOMENA FOR THE WAVE EQUATION WITH
SPACE-DEPENDENT DAMPING TERM GROWING AT
INFINITY

MOTOHIRO SOBAJIMA AND YUTA WAKASUGI

Abstract. In this paper, we study the asymptotic behavior of solutions to
the wave equation with damping depending on the space variable and growing
at the spatial infinity. We prove that the solution is approximated by that of
the corresponding heat equation as time tends to infinity. The proof is based
on semigroup estimates for the corresponding heat equation and weighted en-
ergy estimates for the damped wave equation. To construct a suitable weight
function for the energy estimates, we study a certain elliptic problem.

1. Introduction

Let $N \geq 2$ and let $\Omega$ be an exterior domain in $\mathbb{R}^N$ with a smooth boundary $\partial \Omega$. We assume that $0 \notin \bar{\Omega}$ without loss of generality. We study the initial-boundary
value problem for the damped wave equation

$$
\begin{aligned}
  u_{tt} - \Delta u + a(x)u_t &= 0, \quad x \in \Omega, t > 0, \\
  u(x, t) &= 0, \quad x \in \partial \Omega, t > 0, \\
  u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad x \in \Omega.
\end{aligned}
$$

Here $u = u(x, t)$ is a real-valued unknown function. We assume that $a(x) \in C^2(\bar{\Omega})$, $a(x) > 0$ on $\bar{\Omega}$ and, there exist constants $\alpha > 0$ and $a_0 > 0$ such that

$$
\lim_{|x| \to \infty} |x|^{-\alpha} a(x) = a_0,
$$

that is, $a(x)$ is unbounded and diverges at the spatial infinity. The initial data $(u_0, u_1)$ is compactly supported, let us say supp $(u_0, u_1) \subset B(0, R_0)$ with some $R_0 > 0$, and satisfies the compatibility condition of order 1, namely,

$$
(u_0, u_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega).
$$

Noting the compactness of the support of the initial data and the finite propagation property, we can show that there exists a unique strong solution

$$
u \in \bigcap_{i=0}^2 C^i([0, \infty); H^{2-i}(\Omega))
$$

in a standard way (see Ikawa [6, Theorem 2]).

The term $a(x)u_t$ describes the damping effect, which plays a role in reducing the energy of the wave. Our aim is to clarify how the strength of the damping affects the behavior of the solution. Particularly, in this paper we study the case where the

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strength of the damping increases at the spatial infinity, and as a typical example, we consider the damping satisfying (1.2). We investigate the asymptotic profile of the solution and how the exponent $\alpha$ is related to the decay rate of the energy of the solution.

For this purpose, we also consider the corresponding parabolic problem

$$\begin{cases}
u_t - a(x)^{-1} \Delta \nu = 0, & x \in \Omega, t > 0, \\
v(x, t) = 0, & x \in \partial \Omega, t > 0, \\
v(x, 0) = v_0(x), & x \in \Omega.
\end{cases}$$

This equation is formally obtained by dropping the term $u_{tt}$ from the equation (1.1) and dividing it by $a(x)$.

To observe the relation between the equations (1.1) and (1.5), we formally consider the case $a(x) = |x|^\alpha$ and $\Omega = \mathbb{R}^N$, namely, we drop the boundary condition and consider an initial value problem. For the equation (1.1), we change the variables as

$$u(x, t) = \phi(\lambda^{1/(2+\alpha)}x, \lambda^{1/2}t), \quad y = \lambda^{1/(2+\alpha)}x, \quad s = \lambda^{1/2}t$$

with a parameter $\lambda > 0$. Then, the function $\phi = \phi(y, s)$ satisfies

$$\lambda^2(1+\alpha)/(2+\alpha) \phi_{ss} - \Delta y \phi + |y|^\alpha \phi_s = 0.$$

Thus, letting $\lambda \to 0$, we have the heat equation

$$|y|^\alpha \phi_s - \Delta y \phi = 0.$$

We note that $\lambda \to 0$ with fixing $s$ is corresponding to $t \to \infty$.

The above observation suggests that the solution of (1.1) has the so-called diffusion phenomena, that is, the solution of the damped wave equation (1.1) is approximated by a solution of the heat equation (1.5) as time tends to infinity. Moreover, in the constant damping case $\alpha = 0$, there is a derivation of the damped wave equation from the heat balance law and the time-delayed Fourier’s law (Cattaneo-Vernotte law). This derivation also indicates the diffusion phenomena. For the detail, see for example, [1, 18, 51].

Indeed, for the damped wave equation with constant damping in the whole space

$$u_{tt} - \Delta u + u_t = 0, \quad x \in \mathbb{R}^N, t > 0,$$

many mathematicians studied the asymptotic behavior of solutions and verified the diffusion phenomena. We refer the reader to [22, 4, 31, 32, 33, 14, 32, 31, 5, 30, 44]. For an exterior domain $\Omega \subset \mathbb{R}^N$, namely, in the case $a(x) \equiv 1$ in our problem (1.1), the diffusion phenomena was proved by [8, 11, 2, 40].

Asymptotic behavior of solutions to the wave equation with variable coefficient damping

$$\begin{cases}
u_{tt} - \Delta \nu + b(x, t) \nu_t = 0, & x \in \Omega, t > 0, \\
u(x, t) = 0, & x \in \partial \Omega, \\
u(x, 0) = \nu_0(x), \quad \nu_t(x, 0) = \nu_1(x), & x \in \Omega
\end{cases}$$

has been also studied for a long time. Concerning the behavior of the total energy

$$E(t; u) = \frac{1}{2} \int_\Omega (u_t^2 + |\nabla u|^2) dx,$$

Mochizuki [25] considered the case $\Omega = \mathbb{R}^N$ with $N \neq 2$ and proved that if

$$0 \leq b(x, t) \leq C(1 + |x|)^{-1-\delta}$$

then

$$E(t; u) \leq C(t\delta)^{-1/\delta}.$$
with some $\delta > 0$ and $|b_t(x,t)| \leq C$, then the Møller wave operator exists and it is not identically zero. Namely, there exists initial data $(u_0, u_1)$ with finite energy and for the associated solution $u$, the total energy $E(t; u)$ does not decay to zero as $t \to \infty$, and there also exists a solution $w$ of the free wave equation

$$w_{tt} - \Delta w = 0$$

such that $\lim_{t \to \infty} E(t; u - w) = 0$ holds (see [42] for the case $b = b(x) \in C_0^{\infty}(\mathbb{R}^N)$ and [27, 24, 29, 15, 35] for further improvements).

On the other hand, Matsumura [23] and Uesaka [50] studied the case $b(x, t) \geq C(1 + t)^{-1}$ and proved that the total energy decays to zero. Mochizuki and Nakazawa [27] and Ueda [49] gave a logarithmic improvement of the assumption on the damping. Also, Nakao [28], Ikehata [9] and Mochizuki and Nakao [26] proved the total energy decay for the damping localized near infinity.

As for sharp decay estimates, for the problem (1.1) with $\Omega = \mathbb{R}^N$, Todorova and Yordanov [38] proved that if the damping $a = a(x)$ is radially symmetric and satisfies (1.2) with $\alpha \in (-1, 0]$, then the solution is estimated as

$$E(t; u) = O(t^{-\frac{N + \alpha}{2 + \alpha}},)$$

$$\|\sqrt{a}u(t)\|_{L^2(\Omega)} = O(t^{-\frac{N + \alpha}{2 + \alpha}}),$$

as $t \to \infty$. Here $\delta$ is an arbitrary small number (see also [38, 39] for higher order energy estimates and see [13] for the case $\alpha = -1$). For the asymptotic profile of solutions, the second author [52] proved that if $\Omega = \mathbb{R}^N$ and $a(x) = (1 + |x|^2)^{\alpha/2}$ with $\alpha \in (-1, 0]$, then the asymptotic profile of the solution is given by that of the corresponding parabolic problem. After that, in our previous results [45, 46], we extended the result of [52] to exterior domains and more general (but bounded) damping satisfying the condition (1.2) with $\alpha \in (-1, 0]$. Radu, Todorova and Yordanov [11] and Nishiyama [36] proved the diffusion phenomena in an abstract setting.

However, for the damping term increasing at the spatial infinity, there is no result about the diffusion phenomena, while Khader [16, 17] studied energy estimates and global existence of small solutions for some nonlinear problems.

In this paper, we establish sharp semigroup estimates for the heat equation (1.5) and prove the almost sharp weighted energy estimates for the damped wave equation (1.1). As their application, we show that the solution of (1.1) actually has the diffusion phenomena in a suitable weighted $L^2$ space.

Our main result reads as follows:

**Theorem 1.1.** Let $u$ be the solution to (1.1) and let $v$ be the solution to (1.5) with $v_0(x) = u_0(x) + a(x)^{-1}u_1(x)$. Then, for any $\delta > 0$, there exists $C = C(N, \alpha, a_0, R_0, \delta) > 0$ such that we have for $t \geq 1$

$$\left\|\sqrt{a}((u(\cdot,t) - v(\cdot,t))\right\|_{L^2} \leq C(1 + t)^{-\frac{N + \alpha}{2 + \alpha} - \frac{1 + \alpha}{2 + \alpha} + \delta}\|u_0, u_1\|_{H^2_0 \times H^1}.

As a byproduct of Theorem 1.1, we obtain the almost optimal $L^2$-estimate for the solution.

**Corollary 1.2.** The solution $u$ of (1.1) satisfies

$$\left\|\sqrt{a}u(\cdot,t)\right\|_{L^2} \leq C(1 + t)^{-\frac{N + \alpha}{2 + \alpha} + \delta}\|u_0, u_1\|_{H^2_0 \times H^1},$$
where \( \delta > 0 \) is an arbitrary small number and \( C = C(N, \alpha, a_0, R_0, \delta) > 0 \). When \( N = 2 \), we may take \( \delta = 0 \).

Our strategy for the proof of Theorem 1.1 is the following. First, we treat the term \( u_{tt} \) as a perturbation and write the equation (1.1) as

\[
  u_t - a(x)^{-1} \Delta u = -a(x)^{-1} u_{tt}.
\]

This is natural because we expect the diffusion phenomena, and for the solution \( v \) to the parabolic problem (1.5), the term \( v_{tt} \) decays faster than \( v_t \) and \( a(x)^{-1} \Delta v \). Then, by Duhamel principle, the above equation can be formally transformed into the integral equation

\[
  u(t) = e^{ta(x)^{-1} \Delta} u_0 - \int_0^t e^{(t-s)a(x)^{-1} \Delta}[a(x)^{-1} u_{ss}(s)] \, ds.
\]

We further apply the integration by parts to obtain

\[
  u(t) = e^{ta(x)^{-1} \Delta}(u_0 + a(x)^{-1} u_1) - \int_{t/2}^t e^{(t-s)a(x)^{-1} \Delta}[a(x)^{-1} u_{ss}(s)] \, ds
  \quad - e^{ta(x)^{-1} \Delta}[a(x)^{-1} u_t(t/2)]
  \quad - \int_0^{t/2} a(x)^{-1} \Delta e^{(t-s)a(x)^{-1} \Delta}[a(x)^{-1} u_t(s)] \, ds
\]

By putting \( v(t) = e^{ta(x)^{-1} \Delta}(u_0 + a(x)^{-1} u_1) \), it suffices to estimate the each term in the right-hand side.

To this end, in Section 2, we first investigate the heat semigroup \( e^{ta(x)^{-1} \Delta} \). We let it make sense in an appropriate weighted space via a suitable changing variable. Moreover, by the Beurling-Deny criterion and the Gagliardo-Nirenberg inequality, we derive the following optimal estimate for the semigroup \( e^{ta(x)^{-1} \Delta} \), which is crucial for our argument:

\[
  \|e^{ta(x)^{-1} \Delta} v_0\|_{L^2(\Omega, d\mu)} \leq C t^{-(N/2)(1/q-1/2)} \left( \int_{\Omega} |v_0(x)|^q |x|^{(N-2)\alpha(2-q)/4} \, d\mu \right)^{1/q},
\]

where \( d\mu = a(x) \, dx \) and \( q \in [1, 2] \) \((N = 2)\), \( q \in (p_\alpha', 2] \) \((N \geq 3)\) with \( p_\alpha = 2N(2 + \alpha)/\alpha(N - 2) \). In contrast with the decaying damping cases studied by [45] [46] in which \( q \) can be taken freely in \([1, 2]\), we cannot have \( L^2-L^1 \) estimate for \( N \geq 3 \).

After that, to estimate the terms including \( u \) itself, we apply energy estimates for the damped wave equation (1.1) with a Ikehata-Todorova-Yordanov type weight function \( \Phi = \exp(\beta A(x)/(1 + t)) \), where \( \beta \) is a positive constant and \( A(x) \) is a solution of the elliptic equation

\[
  \Delta A(x) = a(x).
\]

Roughly speaking, the weight function \( \Phi \) comes from the dual problem of the heat equation

\[
  a(x) v_t + \Delta v = 0.
\]

We remark that Lions and Masmoudi [19] [20] firstly introduced this type of weight functions to study the uniqueness for the Navier-Stokes equations.
In Section 3, we discuss the construction of the auxiliary function $A(x)$. When $a(x)$ is decaying at spatial infinity and radially symmetric, Todorova and Yordanov [48] solved the equation (1.8) by reducing the problem to an ordinary differential equation, and applied it to obtain energy estimates for damped wave equations. However, in our problem, we assume $a(x)$ is unbounded. In this case we cannot directly construct a solution by the Newton potential. Moreover, $a(x)$ is not radially symmetric as in [48] and we cannot reduce the problem to an ordinary differential equation. Indeed, in [45, Remark 3.1], we show an example of non-radial $a(x)$ satisfies $a(x) \to 1 (|x| \to \infty)$ but the corresponding $A(x)$ has bad behavior at $|x| \to \infty$.

To overcome these difficulties, we follow our previous results [46] and weaken the problem (1.8) to the inequality

$$(1 - \varepsilon)a(x) \leq \Delta A_\varepsilon(x) \leq (1 + \varepsilon)a(x)$$

with arbitrary small $\varepsilon > 0$, namely, we shall find a function $A_\varepsilon(x)$ satisfying the above inequality and having good behavior as $|x| \to \infty$.

In Section 4, using this auxiliary function $A_\varepsilon(x)$, we apply weighted energy methods developed by [47, 10, 12, 38, 34, 52, 45, 46] and prove almost sharp higher order energy estimates of solutions.

Finally, in Section 5, combining the heat semigroup estimates and the weighted energy estimates for solutions to the damped wave equation (1.1), we have the desired estimate and finish the proof of Theorem 1.1.

We end up this section with introducing the notations and the terminologies used throughout this paper. We shall denote by $C$ various constants, which may change from line to line. In particular, we sometimes write $C = C(\ast, \ldots, \ast)$ to emphasize the dependence on the parameters appearing in parentheses.

For $x_0 \in \mathbb{R}^N$ and $R > 0$, we denote by $B(x_0, R)$ the open ball centered at $x_0$ with the radius $R$. Also, $\overline{B}(x_0, R)$ stands for the closure of $B(x_0, R)$.

We denote by $L^p(\Omega)$ ($1 \leq p \leq \infty$), $H^k(\Omega)$ ($k \in \mathbb{Z}_{\geq 0}$) the usual Lebesgue space and Sobolev space with the norms

$$
\| f \|_{L^p} = \begin{cases} 
\left( \int_\Omega |f(x)|^p \, dx \right)^{1/p} & (1 \leq p < \infty), \\
\text{ess sup}_{x \in \Omega} |f(x)| & (p = \infty),
\end{cases}
$$

$$
\| f \|_{H^k} = \left( \sum_{|\gamma| \leq k} \| \partial_\gamma^2 f \|_{L^2}^2 \right)^{1/2},
$$

respectively. For an interval $I$ and a Banach space $X$, we define $C^r(I; X)$ as the space of $r$-times continuously differentiable mapping from $I$ to $X$ with respect to the topology in $X$.

2. $L^p$-$L^q$ ESTIMATES FOR THE PARABOLIC PROBLEM

In this section, we consider the asymptotic behavior of solutions to the parabolic problem (1.5). We note that the assumption (1.2) on $a(x)$ implies that there exist constants $0 < a_1 \leq a_2$ such that

$$
a_1 |x|^\alpha \leq a(x) \leq a_2 |x|^\alpha
$$

holds for $x \in \overline{\Omega}$. We remark that the results of this section requires only (2.1) and we do not use the property (1.2). For $1 \leq p < \infty$, we define the weighted Lebesgue
spaces
\[ L^p_{d\mu}(\Omega) := \left\{ f \in L^p_{\text{loc}}(\Omega); \| f \|_{L^p_{d\mu}} = \left( \int_{\Omega} |f(x)|^p a(x) \, dx \right)^{1/p} < \infty \right\}. \]

We deal with the operator
\[ L = -a(x)^{-1} \Delta \]
in \( L^2_{d\mu}(\Omega) \). The operator \( L \) is regarded as an associated operator of the symmetric sesquilinear form
\[ a(u,v) := \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx \]
with the domain
\[ \mathcal{D} := D(a) = \{ u \in C^\infty_c(\bar{\Omega}); u|_{\partial\Omega} = 0 \}. \]

The basic property of \( L \) is given in the following.

**Lemma 2.1.** The sesquilinear form \( a \) is densely defined, continuous, accretive, symmetric and closable in \( L^2_{d\mu}(\Omega) \). Therefore, there exists a realization \( L_* \) of \( L \) such that \( L_* \) is nonnegative and self-adjoint. Moreover, \( \mathcal{D} \) is a core for \( L_* \).

While this lemma was already discussed in [45], we give a sketch of the proof for the reader’s convenience.

**Proof of Lemma 2.1.** It is obvious that \( a \) is densely defined, continuous, accretive, symmetric in \( L^2_{d\mu}(\Omega) \). Moreover, by [37, Proposition 1.3], we easily see that \( a \) is closable in \( L^2_{d\mu}(\Omega) \). Let \( a_* \) be the closure of \( a \). Then, [45, Lemma 2.1] shows that the bilinear form \( a_* \) is characterized as
\[ D(a_*) = \left\{ u \in L^2_{d\mu}(\Omega) \cap \dot{H}^1(\Omega); \int_{\Omega} \frac{\partial u}{\partial x_j} \varphi \, dx = -\int_{\Omega} u \frac{\partial \varphi}{\partial x_j} \, dx \right. \]
for \( \varphi \in C^\infty_c(\mathbb{R}^N), \; j = 1, \ldots, N \}, \]
\[ a_*(u,v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx. \]

We define the Friedrichs extension \( L_* \) of \( L \) by
\[ D(L_*) := \left\{ u \in D(a_*); \text{there exists } f \in L^2_{d\mu}(\Omega) \text{ such that} \right. \]
\[ a_*(u,v) = (f,v)_{L^2_{d\mu}} \text{ for any } v \in D(a_*) \}, \]
\[ L_* u = f. \]

Then, by [43, Theorem X.23], we see that \( L_* \) is a nonnegative self-adjoint operator on \( L^2_{d\mu}(\Omega) \). Moreover, we have for \( u, v \in \mathcal{D} \),
\[ (-Lu,v)_{L^2_{d\mu}} = \int_{\Omega} (-Lu)v \, d\mu = \int_{\Omega} (\Delta u)v \, dx = a_*(u,v). \]

By a density argument, we can take \( v \in D(a_*) \) in the above equalities. This implies \( u \in D(L_*) \) and \( Lu = L_* u \), and hence, \( L_* \) is an extension of \( L \).

\[ \square \]
2.1. Transformation into a usual Lebesgue spaces. We introduce a diffeomorphism \( \Psi \in C^\infty(\Omega; \Omega_\Psi) \) as
\[
\Psi(x) := |x|^{\alpha/2}x, \quad \Omega_\Psi := \{ y \in \mathbb{R}^N; |y|^{-\alpha/(2+\alpha)}y \in \Omega \}.
\]
Since we assume that \( \partial \Omega \) is smooth, so is \( \partial \Omega_\Psi \).

The following lemma is a fundamental fact on changes of variables.

**Lemma 2.2.**

(i) One has
\[
\frac{\partial \Psi}{\partial x}(x) = |x|^{\alpha/2} \left( I + \frac{\alpha}{2} Q(x) \right), \quad Q(x) = \left( \frac{x_j x_k}{|x|^2} \right)_{j,k=1,...,N},
\]
\[
\det \left( \frac{\partial \Psi}{\partial x}(x) \right) = 2 + \frac{\alpha}{2} |x|^{N\alpha/2}.
\]
Here we denote by \( I \) the identity matrix of the order \( N \).

(ii) Define \( m(x) := |x|^{-\alpha}a(x) \) for \( x \in \Omega \), \( \tilde{m}(y) := m(\Psi^{-1}(y)) \) for \( y \in \Omega_\Psi \) and \( d\nu = \tilde{m}(y)dy \). Then, the norm of \( L^2(\Omega_\Psi, d\nu) \) is equivalent to \( L^2 \)-norm with usual Lebesgue measure on \( \Omega_\Psi \). Moreover,
\[
\int_{\Omega_\Psi} |v(y)|^2 \, d\nu = \int_\Omega |v(\Psi(x))|^2 |x|^{(N-2)\alpha/2} \, d\mu,
\]
holds for \( v \in C^\infty_c(\Omega_\Psi) \).

**Proof.** (i) is straightforward. For (ii), we note that the assumption (2.1) implies
\[
a_1 \leq m(x) \leq a_2
\]
for any \( x \in \bar{\Omega} \). From this, we obtain \( a_1 \leq \tilde{m}(y) \leq a_2 \) for any \( y \in \overline{\Omega_\Psi} \), which leads to the equivalence between \( L^2(\Omega_\Psi, d\nu) \) and \( L^2(\Omega_\Psi) \). Moreover, we have
\[
\int_{\Omega_\Psi} |v(y)|^2 \, d\nu = \int_{\Omega} |v(y)|^2 m(\Psi^{-1}(y)) \, dy
\]\
\[= \int_\Omega |v(\Psi(x))|^2 m(x) \left| \det \left( \frac{\partial \Psi}{\partial x} \right) \right| \, dx
\]
\[= \frac{2 + \alpha}{2} \int_\Omega |v(\Psi(x))|^2 |x|^{(N-2)\alpha/2} \, d\mu,
\]
which completes the proof. \( \square \)

In view of the above lemma, we introduce an isometry from \( L^2(\Omega_\Psi, d\nu) \) to \( L^2(\Omega, d\mu) \) as
\[
Jv(x) := \sqrt{\frac{2 + \alpha}{2} |x|^{(N-2)\alpha/4} v(\Psi(x))}
\]
for \( v \in L^2(\Omega_\Psi, d\nu) \). It is easy to see that the inverse of \( J \) is given by
\[
J^{-1}u(y) = \sqrt{\frac{2}{2 + \alpha}} |y|^{-(N-2)\alpha/(2(2+\alpha))} u \left( |y|^{-\alpha/(2+\alpha)}y \right)
\]
for \( u \in L^2(\Omega, d\mu) \). Then, the operator \( L \) is transformed into the following uniformly elliptic operator on \( L^2(\Omega_\Psi, d\nu) \).
Lemma 2.3. The operator $B = J^{-1}LJ$ with the domain $\mathcal{D}_\psi := \{u \circ \psi^{-1} : u \in \mathcal{D}\}$ is given by

$$
Bv(y) = \tilde{m}(y)^{-1} \left( -\text{div} (b(y) \nabla v(y)) - \frac{(N - 2)^2 \alpha (4 + \alpha)}{16|y|^2} v(y) \right)
$$

for $y \in \Omega_\psi$ and $v \in \mathcal{D}_\psi$, where

$$
b(y) = I + \alpha \left(1 + \frac{\alpha}{4}\right) Q(\Psi^{-1}(y)).
$$

Moreover, $B$ is a uniformly elliptic operator in $\Omega_\psi$ with bounded coefficients.

Proof. Let $\varphi \in \mathcal{D}_\psi$ be arbitrary fixed and let $v \in \mathcal{D}_\psi$. Then, we obtain

\[
\int_{\Omega_\psi} (Bv(y)) \varphi(y) \, dv = \frac{2 + \alpha}{2} \int_{\Omega} (J^{-1}LJv(\Psi(x))) \varphi(\Psi(x)) |x|^{(N-2)\alpha/2} \, d\mu
\]

\[
= \int_{\Omega} (AJv(x)) J \varphi(x) \, d\mu
\]

\[
= \int_{\Omega} \nabla Jv(x) \cdot \nabla J \varphi(x) \, dx.
\]

Noting that

$$
\sqrt{\frac{2}{2 + \alpha}} \nabla J \varphi(x) = \frac{(N - 2)\alpha}{4} |x|^{(N-2)\alpha/4 - 2} x \varphi(\Psi(x))
$$

and putting $\beta = (N - 2)\alpha/4$, we have

\[
\int_{\Omega} \nabla Jv(x) \cdot \nabla J \varphi(x) \, dx
\]

\[
= \int_{\Omega} \left( \beta \frac{y}{|y|^2} v(y) + \left( I + \frac{\alpha}{2} Q(\Psi^{-1}(y)) \right) \nabla v(y) \right)
\]

\[
\cdot \left( \beta \frac{y}{|y|^2} \varphi(y) + \left( I + \frac{\alpha}{2} Q(\Psi^{-1}(y)) \right) \nabla \varphi(y) \right) \, dy
\]

\[
= \int_{\Omega} \left( I + \frac{\alpha}{2} Q(\Psi^{-1}(y)) \right)^2 \nabla v(y) \cdot \nabla \varphi(y) \, dy + \beta \left(1 + \frac{\alpha}{2}\right) \int_{\Omega} \frac{y}{|y|^2} \cdot \nabla (v(y) \varphi(y)) \, dy
\]

\[
+ \beta^2 \int_{\Omega} \frac{1}{|y|^2} v(y) \varphi(y) \, dy
\]

\[
= - \int_{\Omega} \text{div} \left( (b(y) \nabla v(y)) \varphi(y) \right) \, dy + \left( \beta^2 - (N - 2)\beta \left(1 + \frac{\alpha}{2}\right) \right) \int_{\Omega} \frac{1}{|y|^2} v(y) \varphi(y) \, dy.
\]

Finally, the uniform ellipticity of $B$ follows from $Q \geq 0$ as a symmetric matrix. The boundedness of the coefficients immediately shown by the assumption $0 \not\in \Omega$ and the definition of $Q$. \hfill \Box

From the above lemma, by the same procedure as [37], pp.100–101], we can associate an operator $B_2$ on $L^2(\Omega_\psi, dv)$ with the sesquilinear form

\[
b(u, v) = \int_{\Omega_\psi} \left( (b(y) \nabla u(y)) \cdot \nabla v(y) - \frac{(N - 2)^2 \alpha (4 + \alpha)}{16|y|^2} u(y)v(y) \right) \, dy,
\]

$$
D(b) = \{u \circ \Psi^{-1} : u \in D(a_\ast)\},
$$
and \(-B_2\) is the generator of an analytic semigroup \(T_2(t) = e^{-tB_2}\) on \(L^2(\Omega, d\nu)\). Therefore, it follows from [37, Theorem 1.52] that \(-B_2\) generates an analytic semigroup on \(L^2(\Omega_\psi, d\nu)\).

Furthermore, applying [37, Theorem 4.28], we can extend it to an analytic semigroup generated by \(-B\) in \(L^p(\Omega_\psi, d\nu)\). Summarizing the above, we have the following:

**Lemma 2.4.** For every \(1 < p < \infty\), the operator \(-B_p\) defined by (2.5) in \(L^p(\Omega_\psi, d\nu)\) endowed with the domain \(D(B_p) = W^{2,p}(\Omega_\psi, d\nu) \cap W^{1,p}_0(\Omega_\psi, d\nu)\) generates an analytic semigroup \((T_p(t))_{t \geq 0}\) and \(D_p\) is a core for \(B_p\). Moreover, \(T_p(t)\) is consistent for all \(1 < p < \infty\).

By virtue of Lemma 2.4, we introduce \((T(t))_{t \geq 0}\) as a semigroup generated by \(-B\) with Dirichlet boundary condition.

Next, we consider \(L^2-L^q\) estimates for the semigroup \((T(t))_{t \geq 0}\). As we said in the introduction, this is crucial to prove the diffusion phenomena for the damped wave equation (1.1).

**Proposition 2.5.** Let \(N \geq 2\) and \(\alpha > 0\). Then we have the following \(L^p-L^2\) estimates:

(i) If \(N = 2\), then the the semigroup \((T(t))_{t \geq 0}\) is sub-Markovian and satisfies the estimate

\[
\|T(t)f\|_{L^\infty(\Omega_\psi, d\nu)} \leq Ct^{-1/2}\|f\|_{L^2(\Omega_\psi, d\nu)};
\]

(ii) If \(N \geq 3\), then for every \(2 < p < p_\alpha := 2N(2 + \alpha)/(\alpha(N - 2))\), then the semigroup \((T(t))_{t \geq 0}\) satisfies the estimate

\[
\|T(t)f\|_{L^p(\Omega_\psi, d\nu)} \leq Ct^{-(N/2)(1/2 - 1/p)}\|f\|_{L^2(\Omega_\psi, d\nu)}.
\]

**Remark 2.1.** Let \(N \geq 3\), \(\Omega = \mathbb{R}^N \setminus B(0,1)\) and \(\varepsilon > 0\). Then, the function

\[
\psi_0(y) = |y|^{-(N-2)\alpha/4} (1 - |y|^{-N+2})
\]

belongs to \(L^{p_\alpha+\varepsilon}(\Omega_\psi, d\nu)\) and satisfies \(B\psi_0 = 0\) in \(\Omega_\psi\). This means that \(\psi_0\) is a stationary solution of the equation

\[
\begin{aligned}
&\psi_t + B_{p_\alpha+\varepsilon}\psi = 0, \quad y \in \Omega_\psi, \quad t > 0, \\
&\psi(y, t) = 0, \quad y \in \partial\Omega_\psi, \quad t > 0, \\
&\psi(0, y) = \psi_0(y), \quad y \in \Omega_\psi.
\end{aligned}
\]

By virtue of \(\psi_0\), any decay estimate for the semigroup \((T(t))_{t \geq 0}\) in \(L^{p_\alpha+\varepsilon}(\Omega_\psi, d\nu)\) cannot be expected. We would expect that for \(N \geq 3\) the \(L^2-L^{p_\alpha}\) estimate is given by

\[
\|T(t)f\|_{L^{p_\alpha}(\Omega_\psi, d\nu)} \leq Ct^{-(N/2)(1/2 - 1/p_\alpha)}(1 + \log(1 + t))^{1/p_\alpha}\|f\|_{L^2(\Omega_\psi, d\nu)}.
\]

We postpone the proof of Proposition 2.5 in the following subsections, and here we give a practical version of it. For the original variable \(x\), Proposition 2.5 and the identity

\[
\left(\frac{2+\alpha}{2}\right)^{1-p/2} \int_\Omega |Jv(x)|^p |x|^{-(N-2)\alpha(p-2)/4} a(x) dx = \int_{\Omega_\psi} |v(y)|^p d\nu
\]

imply the following estimate.

**Proposition 2.6.** Let \(N \geq 2\) and \(\alpha > 0\) and let \(v\) be a solution to (1.1). Then, we have the following.
(i) If \( N = 2 \), then we have
\[
\| v(t) \|_{L^2(\Omega, d\mu)} \leq C t^{-1/2} \| v_0 \|_{L^1(\Omega, d\mu)};
\]
(ii) If \( N \geq 3 \), then for every \((p, \alpha)'<q \leq 2\), we have
\[
\| v(t) \|_{L^2(\Omega, d\mu)} \leq C t^{-(N/2)(1/q-1/2)} \left( \int_{\Omega} |v_0(x)|^q |x|^{(N-2)(2-q)/4} d\mu \right)^{1/q}.
\]

2.2. Proof of Proposition 2.5 for \( N = 2 \). In this subsection we use the following Gagliardo–Nirenberg inequality with \( N = 2 \).

**Lemma 2.7** (Gagliardo–Nirenberg inequality). Let \( N = 2 \). For every \( 2 \leq q < \infty \), there exists a constant \( C_q \) such that
\[
\| w \|_{L^q(\Omega)} \leq C_q \| \nabla w \|_{L^2(\Omega)}^{1-2/q} \| w \|_{L^2(\Omega)}^{2/q}
\]
holds for any \( w \in W^{1,2}_0(\Omega) \).

For the proof, see for example [3]. From this, we obtain the following estimate.

**Lemma 2.8.** For every \( 2 \leq q < \infty \), we have
\[
\| \varphi \|_{L^q(\Omega \Psi, d\nu)} \leq C_q \left( \int_{\Omega \Psi} |\varphi(y)|^q \right)^{1/q}
\]
for any \( \varphi \in L^2(\Omega \Psi, d\nu) \).

**Proof.** Since \( D \Psi \) is a core for \( B_2 \), it suffices to show (2.6) for \( \varphi \in D \Psi \). Let \( \varphi \in D \Psi \). Noting that
\[
b(y) \geq \min \left\{ 1, 1 + \alpha + \frac{\alpha^2}{4} \right\} I = I
\]
in the sense of symmetric matrix, we have
\[
(B_2 \varphi, \varphi)_{dv} = \int_{\Omega \Psi} (b(y) \nabla \varphi) \cdot \nabla \varphi \, dy \geq \| \nabla \varphi \|_{L^2(\Omega \Psi)}^2.
\]
Combining the above estimate with \( a_1 \leq \tilde{m} \leq a_2 \) and Lemma 2.7, we obtain
\[
\| \varphi \|_{L^q(\Omega \Psi, dv)} \leq a_2^{1/q} \left( \int_{\Omega \Psi} |\varphi(y)|^q \right)^{1/q}
\]
\[
\leq a_2^{1/q} C_q \left( \int_{\Omega \Psi} |\nabla \varphi(y)|^2 \, dy \right)^{1/2} \left( \int_{\Omega \Psi} |\varphi(y)|^2 \, dy \right)^{1/q}
\]
\[
\leq a_2^{1/q} a_1^{-1/2} C_q (B_2 \varphi, \varphi)_{dv}^{1/2} \| \varphi \|_{L^2(\Omega \Psi, dv)},
\]
which implies (2.6).

**Lemma 2.9.** The semigroup \((T(t))_{t \geq 0}\) is sub-Markovian, that is, \( T(t) \) is positively preserving:
\[
f \in L^2(\Omega \Psi, dv), \quad f \geq 0 \implies T(t)f \geq 0
\]
and \( L^\infty \)-contractive:
\[
\| T(t)f \|_{L^\infty(\Omega \Psi, dv)} \leq \| f \|_{L^\infty(\Omega \Psi, dv)} \quad \text{for} \quad f \in L^2(\Omega \Psi, dv) \cap L^\infty(\Omega \Psi, dv).
\]


Proof. Let \( v \in D(b) \). Using the characterization (2.2) of \( D(a^*) \), we easily see that \( |v| \in D(b) \). Moreover, we have
\[
b(|v|, |v|) = \int_{\Omega} \left( b(y) \frac{v}{|v|} \nabla v(y) \right) \cdot \frac{v}{|v|} \nabla v(y) \, dy = b(v, v).
\]
Thus, applying [37, Theorem 2.7], we have the positively preserving property of \( T(t) \). Next, we show the \( L^\infty \)-contractive property of \( T(t) \). By using the characterization (2.2) of \( D(a^*) \) again, we also see that \( v \in D(b) \) implies \( Pv = (1 \wedge |v|) \text{sign } v \in D(b) \). Furthermore, we have
\[
b(Pv, v - Pv) = \int_{\Omega} \left( b(y) \chi_{|v|<1} \nabla v(y) \right) \cdot \left((1 - \chi_{|v|<1}) \nabla v(y) \right) \, dy = 0.
\]
Therefore, applying [37, Theorem 2.13], we prove the \( L^\infty \)-contractive property of \( T(t) \). □

Proof of Proposition 2.5 for \( N = 2 \). By Lemma 2.9, we see that the semigroup \( T(t) \) is \( L^\infty \)-contractive. Moreover, by Lemma 2.8, \( T(t) \) satisfies the second condition of [37, Theorem 6.2] with \( d = 2 \). Thus, applying [37, Theorem 6.2], we conclude
\[
\|T(t)f\|_{L^\infty(\Omega, \bar{\nu})} \leq Ct^{1/2}\|f\|_{L^2(\Omega, \nu)},
\]
which gives the assertion of Proposition 2.5 for \( N = 2 \). □

2.3. Proof of Proposition 2.5 for \( N \geq 3 \). In this case we need to prepare some result for analytic semigroup generated by \( -B_p \) via \( L^p \)-theory. First we prove that \( -B_p \) generates an analytic contraction semigroup when
\[
2 - \frac{4}{4 + \alpha} < p < 2 + \frac{4}{\alpha}.
\]

Lemma 2.10. Let \( N \geq 3 \). Assume that (2.7) is satisfied. Then, there exists a constant \( \ell_{N,p,\alpha} \geq 0 \) depending only on \( N, p \) and \( \alpha \) such that \( B_p \) is m-sectorial of type \( S(\ell_{N,p,\alpha}) \) in \( L^p(\Omega, \nu) \), that is,
\[
\text{Im} \int_{\Omega} (B_p\varphi)\bar{\varphi}|\varphi|^{p-2} \, d\nu \leq \ell_{N,p,\alpha} \left( \text{Re} \int_{\Omega} (B_p\varphi)\bar{\varphi}|\varphi|^{p-2} \, d\nu \right)
\]
holds for \( \varphi \in D(B_p) \). Moreover, there exists a constant \( M_{N,p,\alpha} \geq 1 \) depending only on \( N, p \) and \( \alpha \) such that
\[
\|B_p T(t)f\|_{L^p(\Omega, \nu)} \leq \frac{M_{N,p,\alpha}}{t}\|f\|_{L^p(\Omega, \nu)}
\]
holds for \( f \in L^p(\Omega, \nu) \) and \( t > 0 \).

Proof. Since \( B_p' \) is the adjoint operator of \( B_p \) and \( D_\phi \) is a core for \( B_p \) by Lemma 2.4, it suffices to prove (2.8) for \( p \geq 2 \) and \( \varphi \in D_\phi \). Indeed, if (2.8) is true for \( p \geq 2 \)

and \( \varphi \in \mathcal{D}_\Psi \), we deduce for \( p \leq 2 \) and \( \varphi \in \mathcal{D}_\Psi \) that

\[
\left| \operatorname{Im} \int_{\Omega_{\Psi}} (B_{\rho'} \varphi) \bar{\varphi} |\varphi|^{p-2} \, d\nu \right| = \left| \operatorname{Im} \int_{\Omega_{\Psi}} \varphi (B_{\rho'} (\bar{\varphi} |\varphi|^{p-2})) \, d\nu \right|
= \left| \operatorname{Im} \int_{\Omega_{\Psi}} \bar{f} |\varphi|^{p-2} (B_{\rho'} (f)) \, d\nu \right|
\leq \ell_{N,p,\alpha} \left( \operatorname{Re} \int_{\Omega_{\Psi}} \bar{f} |\varphi|^{p-2} (B_{\rho'} (f)) \, d\nu \right)
= \ell_{N,p,\alpha} \left( \operatorname{Re} \int_{\Omega_{\Psi}} \varphi (B_{\rho'} (\bar{\varphi} |\varphi|^{p-2})) \, d\nu \right)
= \ell_{N,p,\alpha} \left( \operatorname{Re} \int_{\Omega_{\Psi}} (B_{\rho'} \varphi) \bar{\varphi} |\varphi|^{p-2} \, d\nu \right),
\]

where \( f = \bar{\varphi} |\varphi|^{p-2} \).

Therefore, in what follows we assume \( p \geq 2 \) and \( \varphi \in \mathcal{D}_\Psi \). Setting \( u = J \varphi \in \mathcal{D} \), we have

\[
\int_{\Omega_{\Psi}} (B_{\rho'} \varphi) \bar{\varphi} |\varphi|^{p-2} \, d\nu = \int_{\Omega_{\Psi}} (J^{-1} Lu(y)) \bar{J^{-1} u(y)} |J^{-1} u(y)|^{p-2} \, d\nu
= \left( \frac{2}{2 + \alpha} \right)^{(p-2)/2} \int_{\Omega} (-\Delta u(x)) \bar{u(x)} |u(x)|^{p-2} |x|^{-\beta} \, dx,
\]

where \( \beta = (N-2)\alpha(p-2)/4 \). Moreover, integration by parts yields

\[
(2.10) \quad \int_{\Omega} (-\Delta u) \bar{u} |u|^{p-2} |x|^{-\beta} \, dx = \int_{\Omega} (\nabla u \cdot \nabla (\bar{\bar{u}} |u|^{p-2})) |x|^{-\beta} \, dx
+ \int_{\Omega} (\nabla u \cdot \nabla (|x|^{-\beta})) \bar{\bar{u}} |u|^{p-2} \, dx.
\]

By integration by parts again and taking the real-part, we see that

\[
\operatorname{Re} \int_{\Omega} (\nabla u \cdot \nabla (\bar{\bar{u}} |u|^{p-2})) |x|^{-\beta} \, dx
= \operatorname{Re} \int_{\Omega} (|\nabla u|^2 |u|^{p-2} + (p-2) \nabla u \cdot (\bar{\bar{u}} |u|^{p-4} \operatorname{Re} (\bar{\bar{u}} \nabla u)))) |x|^{-\beta} \, dx
= \operatorname{Re} \int_{\Omega} (|\nabla u|^2 + (p-2)(\bar{\bar{u}} \nabla u) \operatorname{Re} (\bar{\bar{u}} \nabla u)) |u|^{p-4} |x|^{-\beta} \, dx
= (p-1) \int_{\Omega} |\operatorname{Re} (\bar{\bar{u}} \nabla u)|^2 |u|^{p-4} |x|^{-\beta} \, dx + \int_{\Omega} |\operatorname{Im} (\bar{\bar{u}} \nabla u)|^2 |u|^{p-4} |x|^{-\beta} \, dx,
\]

\[
\operatorname{Re} \int_{\Omega} (\nabla u \cdot \nabla (|x|^{-\beta})) \bar{\bar{u}} |u|^{p-2} \, dx
= \frac{1}{p} \int_{\Omega} \nabla(|u|^p) \cdot \nabla (|x|^{-\beta}) \, dx
= -\frac{1}{p} \int_{\Omega} |u|^p \Delta (|x|^{-\beta}) \, dx
= \frac{\beta(N-2-\beta)}{p} \int_{\Omega} |u|^p |x|^{-\beta-2} \, dx.
\]
and hence, we eventually have

\begin{equation}
(2.11) \quad \text{Re} \int_{\Omega} (B_p \varphi) \bar{\varphi} |\varphi|^{p-2} \, dv = (p-1) \left( \frac{2}{2 + \alpha} \right)^{(p-2)/2} \int_{\Omega} |\text{Re} (\bar{u} \nabla u)|^2 |u|^{p-4} |x|^{-\beta} \, dx \\
+ \left( \frac{2}{2 + \alpha} \right)^{(p-2)/2} \int_{\Omega} |\text{Im} (\bar{u} \nabla u)|^2 |u|^{p-4} |x|^{-\beta} \, dx \\
+ \frac{\beta(N-2-\beta)}{p} \left( \frac{2}{2 + \alpha} \right)^{(p-2)/2} \int_{\Omega} |u|^{p} |x|^{-\beta-2} \, dx.
\end{equation}

Therefore, \( B_p \) is accretive if \( \beta \leq N-2 \), that is, \( p \leq 2 + 4/\alpha \). On the other hand, taking the imaginary part of (2.10), we have

\[ |\text{Im} \int_{\Omega} (B_p \varphi) \bar{\varphi} |\varphi|^{p-2} \, dv | \leq |p-2| \left( \frac{2}{2 + \alpha} \right)^{(p-2)/2} \int_{\Omega} |x|^{-\beta} |\text{Re} (\bar{u} \nabla u)||\text{Im} (\bar{u} \nabla u)||u|^{p-4} \, dx \\
+ \beta \left( \frac{2}{2 + \alpha} \right)^{(p-2)/2} \int_{\Omega} |x|^{-\beta-1} |\text{Im} (\bar{u} \nabla u)||u|^{p-2} \, dx. \]

Therefore, choosing \( 2 \leq p < 2 + 4/\alpha \) and

\[ \ell_{N,p,\alpha} = \sqrt{\frac{(p-2)^2}{4(p-1)}} + \frac{\beta p}{4(N-2-\beta)} \\
= \sqrt{\frac{(p-2)^2}{4(p-1)}} + \frac{p(p-2)}{4} \left( 2 + \frac{4}{\alpha} - p \right)^{-1}, \]

we obtain

\[ |\text{Im} \int_{\Omega} (B_p \varphi) \bar{\varphi} |\varphi|^{p-2} \, dv | \leq \ell_{N,p,\alpha} \left( \text{Re} \int_{\Omega} (B_p \varphi) \bar{\varphi} |\varphi|^{p-2} \, dv \right). \]

This gives the \( m \)-sectoriality of \( B_p \) and hence \( -B_p \) generates an analytic contraction semigroup on \( L^p(\Omega, dv) \). The estimate (2.9) is an immediate consequence of a property of an analytic semigroup.

\[ \square \]

To deduce \( L^p - L^q \) estimates for \( T(t) \), we use the following Gagliardo–Nirenberg inequality (see for example [3]).

**Lemma 2.11.** Let \( N \geq 3 \). Then, there exists a constant \( C_N > 0 \) depending only on \( N \) such that for every \( w \in W^{1,2}_0(\Omega) \), we have

\[ \|w\|_{L^{2N/(N-2)}(\Omega)} \leq C_N \|\nabla w\|_{L^2(\Omega)}. \]

From this, we obtain the following estimate.

**Lemma 2.12.** Let \( N \geq 3 \) and let \( 2 \leq p < 2 + 4/\alpha \). Then, there exists a constant \( C_{N,p} > 0 \) depending only on \( N \) and \( p \) such that for every \( \varphi \in D_\Psi \), we have

\begin{equation}
(2.12) \quad \|\varphi\|_{L^{pN/(N-2)}(\Omega)} \leq C_{N,p} \left( \text{Re} \int_{\Omega} (B_p \varphi) \bar{\varphi} |\varphi|^{p-2} \, dv \right)^{1/p}. \end{equation}
Proof. Here we use the differential expression of $B$.

\[
\text{Re} \int_{\Omega_{\theta}} (B_{\theta} \varphi) \overline{\varphi} |\varphi|^{p-2} \, d\nu = \text{Re} \int_{\Omega_{\theta}} b(y) \nabla \varphi \cdot \nabla (\overline{\varphi} |\varphi|^{p-2}) \, dy \\
- \frac{(N-2)^{2} \alpha (4 + \alpha)}{16} \int_{\Omega_{\theta}} |\varphi(y)|^{p} \, dy \\
\geq (p-1) \left( \frac{2 + \alpha}{2} \right)^{2} \int_{\Omega_{\theta}} |\text{Re} (\overline{\varphi} \varphi_{\theta})| |\varphi|^{p-4} \, dy \\
- \left( \frac{N-2}{2} \right)^{2} \left( \left( \frac{2 + \alpha}{2} \right)^{2} - 1 \right) \int_{\Omega_{\theta}} \frac{|\varphi(y)|^{p}}{|y|^{2}} \, dy,
\]

where $\varphi_{\theta}$ stands for the derivative of $\varphi$ with respect to the radial direction. The above inequality can be proved by recalling $b(y) = I + \alpha (1 + \frac{\alpha}{2}) Q(\Psi^{-1}(y))$, $Q(\Psi^{-1}(y)) = (\frac{b(y)}{|y|})_{i,j=1,\ldots,N}$, and noting that

\[
\text{Re} \left[ b(y) \nabla \varphi \cdot \nabla (\overline{\varphi} |\varphi|^{p-2}) \right] \\
= |\varphi|^{p-4} \left( (p-1) |\text{Re} (\overline{\varphi} \nabla \varphi)|^{2} + |\text{Im} (\overline{\varphi} \nabla \varphi)|^{2} \right) + \alpha \left( 1 + \frac{\alpha}{4} \right) |\varphi|^{p-4} \left( (p-1) \left| \text{Re} \left( \frac{y}{|y|} \cdot (\overline{\varphi} \nabla \varphi) \right) \right|^{2} \right) \\
\geq \left( 1 + \alpha \left( 1 + \frac{\alpha}{4} \right) \right) (p-1) |\varphi|^{p-4} \left| \text{Re} \left( \frac{y}{|y|} \cdot (\overline{\varphi} \nabla \varphi) \right) \right|^{2} \\
= (p-1) \left( \frac{2 + \alpha}{2} \right)^{2} |\text{Re} (\overline{\varphi} \varphi_{\theta})| |\varphi|^{p-4}
\]

Applying the Hardy’s inequality

\[
\int_{\Omega_{\theta}} |\text{Re} (\overline{\varphi} \varphi_{\theta})| |\varphi|^{p-4} \, dy \geq \left( \frac{N-2}{2} \right)^{2} \frac{4}{p^{2}} \int_{\Omega_{\theta}} |\varphi(y)|^{p} \, dy
\]

and

\[
(p-1) \left( \frac{2 + \alpha}{2} \right)^{2} \left( \frac{N-2}{2} \right)^{2} \frac{4}{p^{2}} - \left( \frac{N-2}{2} \right)^{2} \left( \left( \frac{2 + \alpha}{2} \right)^{2} - 1 \right) > 0,
\]

we see that there exists $\delta \in (0, 1)$ such that

\[
\text{Re} \int_{\Omega_{\theta}} (B_{\theta} \varphi) \overline{\varphi} |\varphi|^{p-2} \, d\nu \geq \delta^{2} \int_{\Omega_{\theta}} |\overline{\varphi} \varphi_{\theta}|^{2} |\varphi|^{p-4} \, dy \\
= \frac{4\delta^{2}}{p^{2}} \int_{\Omega_{\theta}} |\nabla |\varphi|^{p/2}|^{2} \, dx.
\]

By Lemma \ref{lem:8} with $w = |\varphi|^{p/2}$, we obtain

\[
\|\varphi\|_{L^{p^{N/(N-2)}(\Omega_{\theta}, d\nu)}} \geq \left\| |\varphi|^{p/2} \right\|_{L^{2N/(N-2)}(\Omega_{\theta}, d\nu)}^{2/p} \\
\leq C_{N} \left\| |\nabla |\varphi|^{p/2}|^{2} \right\|_{L^{2}(\Omega_{\theta}, d\nu)}^{2/p} \\
\leq C_{N} \left( \frac{2\delta}{p} \right)^{-2/p} \left( \text{Re} \int_{\Omega_{\theta}} (B_{\theta} \varphi) \overline{\varphi} |\varphi|^{p-2} \, d\nu \right)^{1/p}.
\]

Since $a_{1} \leq m$, the proof is finished. \hfill \Box
Proof of Proposition \[2.5\] for \( N \geq 3 \). Let \( 2 \leq r < 1 + 4/\alpha \) and \( f \in L^r(\Omega_\Psi) \). Applying \[2.9\] and Lemma \[2.12\] we have

\[
\|T(t)f\|_{L^{rN/(N-2)}(\Omega_\Psi, d\nu)} \leq C_{N,r,\alpha} \left( \Re \int_{\Omega_\Psi} (B_r T(t)f) \overline{T(t)f} |T(t)f|^{r-2} \, d\nu \right)^{1/r} \\
\leq C_{N,r,\alpha} \|B_r T(t)f\|_{L^r(\Omega_\Psi, d\nu)}^{1/r} \|T(t)f\|_{L^r(\Omega_\Psi, d\nu)}^{1-1/r} \|f\|_{L^r(\Omega_\Psi, d\nu)}^{1/r}.
\]

Fix \( 2 < p < (N/(N-2))(2 + 4/\alpha) \). Then the case \( 2 < p \leq 2N/(N-2) \) is already verified by the above result with \( r = 2 \) and \( q = p \).

In the case \( 2N/(N-2) < p < (N/(N-2))(2 + 4/\alpha) \), we set \( q_0 = 2, \quad q_m = \frac{N-2}{N} p, \quad q_j = \left( \frac{q_m}{q_0} \right)^{j/m} q_0 \) \((j = 1, \ldots, m-1)\), where \( m \) is an integer satisfying

\[
\left( 1 + \frac{2}{\alpha} \right)^{1/m} < \frac{N}{N-2}.
\]

Since

\[
2 = q_0 < q_1 < \cdots < q_m < 2 + \frac{4}{\alpha}
\]

and

\[
\frac{q_j}{q_{j-1}} = \left( \frac{q_m}{q_0} \right)^{1/m} = \left( \frac{N-2}{2N} p \right)^{1/m} < \left( 1 + \frac{2}{\alpha} \right)^{1/m} < \frac{N}{N-2},
\]

it follows from the previous discussion that \( L^{q_j-1} - L^{q_j} \) estimates are true for \( j = 1, \ldots, m \) and

\[
\|T(t/2)f\|_{L^{q_m}(\Omega_\Psi, d\nu)} \leq \left( \prod_{j=1}^{m} \|T(t/2m)\|_{L^{q_j-1} \to L^{q_j}} \right) \|f\|_{L^2(\Omega_\Psi, d\nu)} \\
\leq C_{2,q_m} t^{-(N/2)(1/2-1/q_m)} \|f\|_{L^2(\Omega_\Psi, d\nu)}.
\]

Since we have \( \|T(t/2)f\|_{L^{q_m} \to L^p} \leq C_{q_m,p} t^{-1/p} \) in the previous step with \( r = (N-2)p/N \), we obtain the desired assertion. \( \square \)

3. Related elliptic problem: construction of a weight function

In this section, we construct a weight function, which will be used for the weighted energy estimate of the damped wave equation \[1.1\]. As we mentioned in the introduction, in \[45\] Remark 3.1, it is shown that the solution \( A(x) \) of the Poisson equation \( \Delta A(x) = a(x) \) does not have the desired property in general (in particular, the condition \[3.2\] fails). Therefore, following \[46\], we weaken the problem to the inequality

\[
(1 - \varepsilon)a(x) \leq \Delta A(x) \leq (1 + \varepsilon)a(x)
\]
for \( x \in \Omega \), where \( \varepsilon \in (0,1) \) is a parameter. We construct a function \( A_\varepsilon \) satisfying (3.1) and

(3.2) \[ A_{1\varepsilon}(x)^{2+\alpha} \leq A_\varepsilon(x) \leq A_{2\varepsilon}(x)^{2+\alpha}, \]

(3.3) \[ \frac{|\nabla A_\varepsilon(x)|^2}{a(x)A(x)} \leq \frac{2 + \alpha}{N + \alpha} + \varepsilon \]

for some \( A_{1\varepsilon}, A_{2\varepsilon} > 0 \).

**Lemma 3.1.** For every \( \varepsilon \in (0,1) \), there exists \( A_\varepsilon \in C^2(\mathbb{R}^N) \) such that (3.1) – (3.3).

*Proof.* While the proof is the same as that of [46, Lemma 2.1], we give a proof for the reader’s convenience.

First, noting \( a(x) \in C^2(\Omega) \) and \( a(x) > 0 \), we extend \( a(x) \) as a positive function on the whole space \( \mathbb{R}^N \) and denote it by \( a(x) \) again. We define

\[ a_\varepsilon(x) := b_1(x) + \eta_\varepsilon b_2(x), \]

where

\[
\begin{align*}
b_1(x) &= \Delta \left( \frac{a_0}{(N + \alpha)(2 + \alpha)} \langle x \rangle^{2+\alpha} \right) \\
&= a_0(x)^{\alpha} - \frac{a_0\alpha}{N + \alpha} \langle x \rangle^{\alpha - 2},
\end{align*}
\]

\[ b_2(x) = a(x) - b_1(x), \]

and \( \eta_\varepsilon \in C^\infty_c(\mathbb{R}^N, [0,1]) \) is a cut-off function such that \( \eta_\varepsilon \equiv 1 \) on \( B(0, R_\varepsilon) \) with some \( R_\varepsilon > 0 \). By the assumption (1.2), for any \( \varepsilon > 0 \), we obtain

(3.4) \[ (1 - \varepsilon)a(x) \leq a_\varepsilon(x) \leq (1 + \varepsilon)a(x), \]

provided that \( R_\varepsilon \) is sufficiently large.

We take \( A_\varepsilon \) as an appropriate solution of the Poisson equation

(3.5) \[ \Delta A_\varepsilon(x) = a_\varepsilon(x), \]

that is, we define

\[ A_\varepsilon(x) := \frac{a_0}{(N + \alpha)(2 + \alpha)} \langle x \rangle^{2+\alpha} - \int_{\mathbb{R}^N} \mathcal{N}(x-y)\eta_\varepsilon(y)b_2(y)\,dy + \lambda_\varepsilon, \]

where \( \lambda_\varepsilon \) is a large constant determined later, and \( \mathcal{N} \) is the Newton potential given by

\[
\mathcal{N}(x) := \begin{cases} 
\frac{1}{2\pi} \log \frac{1}{|x|} & \text{if } N = 2, \\
\frac{\Gamma(N/2 + 1)}{N(N - 2)\pi^{N/2}} |x|^{2-N} & \text{if } N \geq 3.
\end{cases}
\]

Then, clearly we have (3.5) and the property (3.1) is verified. Since the leading term of \( A_\varepsilon(x) \) for large \( |x| \) is \( \frac{a_0}{(N + \alpha)(2 + \alpha)} \langle x \rangle^{2+\alpha} \), we have (3.2), provided that \( \lambda_\varepsilon > 0 \) is sufficiently large. Moreover, we easily compute

\[
\begin{align*}
\lim_{|x| \to \infty} \left( \frac{|\nabla A_\varepsilon(x)|^2}{a(x)A_\varepsilon(x)} \right) \\
= \lim_{|x| \to \infty} \left( \frac{\langle x \rangle^\alpha}{a(x)} \cdot \frac{1}{\langle x \rangle^{\alpha - 2} A_\varepsilon(x)} \right) \\
= \frac{2 + \alpha}{N + \alpha}.
\end{align*}
\]
Thus, retaking $\lambda_\epsilon > 0$ sufficiently large, we have (3.3). □

4. Weighted energy estimates for damped wave equations

In this section, we give weighted energy estimates for solutions to the damped wave equation (1.1). As we described in the introduction, we use the auxiliary function $A_\epsilon$ constructed in Section 3.

We first recall the finite propagation speed property for the equation (1.1). For the proof, see for example [7].

**Lemma 4.1** (Finite speed of the propagation). Let $u$ be a solution of the equation (1.1) with initial data $(u_0, u_1)$ satisfying $\text{supp} (u_0, u_1) \subset \bar{\Omega} \cap B(0, R_0)$. Then, we have

$$\text{supp} u(\cdot, t) \subset \bar{\Omega} \cap B(0, R_0 + t)$$

for $t \geq 0$.

**Remark 4.1.** From Lemma 4.1, we see that $\langle x \rangle \leq R_0 + 1 + t$ for $x \in \text{supp} u(\cdot, t)$. We shall frequently use this inequality.

We start with the following weighted energy identities. In this step we do not need any special property for the weight function.

**Lemma 4.2** ([45, Lemma 3.7]). Let $\Phi \in C^2(\overline{\Omega} \times [0, \infty))$ satisfy $\Phi > 0$ and $\partial_t \Phi < 0$ and let $u$ be a solution of (1.1). Then, we have

$$\frac{d}{dt} \left[ \int_\Omega \left( |\nabla u|^2 + |u_t|^2 \right) \Phi \, dx \right] = \int_\Omega (\partial_t \Phi)^{-1} |\partial_t \Phi \nabla u - u_t \nabla \Phi|^2 \, dx + \int_\Omega \left( -2a(x)\Phi + \partial_t \Phi - (\partial_t \Phi)^{-1} |\nabla \Phi|^2 \right) |u_t|^2 \, dx.$$

**Lemma 4.3** ([45, Lemma 3.9]). Let $\Phi \in C^2(\overline{\Omega} \times [0, \infty))$ satisfy $\Phi > 0$ and $\partial_t \Phi < 0$ and let $u$ be a solution to (1.1). Then, we have

$$\frac{d}{dt} \left[ \int_\Omega \left( 2uu_t + a(x)|u|^2 \right) \Phi \, dx \right] = 2 \int_\Omega uu_t (\partial_t \Phi) \, dx + \int_\Omega |u_t|^2 \Phi \, dx - 2 \int_\Omega |\nabla u|^2 \Phi \, dx + \int_\Omega (a(x)\partial_t \Phi + \Delta \Phi) |u|^2 \, dx.$$

Using the function $A_\epsilon(x)$ constructed in the previous section, we introduce our weight function.

**Definition 4.4.** Let $h := \frac{2+n}{N+n}$ and $\epsilon \in (0, 1)$. We define

$$\Phi_\epsilon(x, t) = \exp \left( \frac{1}{h + 2\epsilon} \frac{A_\epsilon(x)}{1 + t} \right),$$

where $A_\epsilon$ is given in Lemma 3.1. For $t \geq 0$, we also define the following energy

$$E_{\partial t}(t; u) := \int_\Omega |\nabla u|^2 \Phi_\epsilon \, dx, \quad E_{u_t}(t; u) := \int_\Omega |u_t|^2 \Phi_\epsilon \, dx,$$

$$E_a(t; u) := \int_\Omega a(x)|u|^2 \Phi_\epsilon \, dx, \quad E_\epsilon(t; u) := 2 \int_\Omega uu_t \Phi_\epsilon \, dx.$$
and $E_1(t; u) := E_{\partial x}(t; u) + E_{\partial t}(t; u)$ and $E_2(t; u) := E_x(t; u) + E_a(t; u)$.

The main result of this section is the following. We give weighted energy estimates for solutions of (1.1).

**Proposition 4.5.** Assume that $(u_0, u_1)$ satisfies supp $(u_0, u_1) \subset \overline{B}(0, R_0)$ and the compatibility condition of order $k_0 \geq 1$. Let $u$ be a solution of the problem (1.1).

For every $\delta > 0$ and $0 \leq k \leq k_0 - 1$, there exist $\varepsilon > 0$ and $M_{\delta, k, R_0} > 0$ such that for every $t \geq 0$,

$$(1 + t)^{\frac{\delta}{h + 2\varepsilon} + 2k + 1 - \delta} \left( E_{\partial x}(t; \partial^k u) + E_{\partial t}(t; \partial^k u) \right) + (1 + t)^{\frac{\delta}{h + 2\varepsilon} + 2k - \delta} E_a(t; \partial^k u)$$

$$\leq M_{\delta, k, R_0} \|(u_0, u_1)\|_{H^{k+1} \times H^k(\Omega)}^2.$$

To prove this, we need to prepare the following lemmas. First, we calculate derivatives of the weight function $\Phi_\varepsilon$. 

**Lemma 4.6.** We have

$$\partial_t \Phi_\varepsilon(x, t) = -\frac{1}{h + 2\varepsilon} \frac{A_\varepsilon(x)}{(1 + t)^2} \Phi_\varepsilon(x, t),$$

$$\nabla \Phi_\varepsilon(x, t) = \frac{1}{h + 2\varepsilon} \frac{\nabla A_\varepsilon(x)}{1 + t} \Phi_\varepsilon(x, t),$$

$$\Delta \Phi_\varepsilon(x, t) = \frac{1}{h + 2\varepsilon} \frac{\Delta A_\varepsilon(x)}{1 + t} \Phi_\varepsilon(x, t) + \left| \frac{1}{h + 2\varepsilon} \frac{\nabla A_\varepsilon(x)}{1 + t} \right|^2 \Phi_\varepsilon(x, t).$$

In particular, we obtain

$$-\Delta \Phi_\varepsilon(x, t) + \frac{\left| \nabla \Phi_\varepsilon(x, t) \right|^2}{\Phi_\varepsilon(x, t)} = -\frac{1}{h + 2\varepsilon} \frac{\Delta A_\varepsilon(x)}{1 + t} \Phi_\varepsilon(x, t),$$

$$\frac{\left| \nabla \Phi_\varepsilon(x, t) \right|^2}{\partial_t \Phi_\varepsilon(x, t)} = -\frac{1}{h + 2\varepsilon} \frac{\left| \nabla A_\varepsilon(x) \right|^2}{A_\varepsilon(x)} \Phi_\varepsilon(x, t).$$

The proof is straightforward and we omit it. The second tool is a Hardy type inequality.

**Lemma 4.7.** For $t \geq 0$, we have

$$(4.6) \quad \frac{1 - \varepsilon}{h + 2\varepsilon} \frac{1}{1 + t} E_a(t; u) \leq E_{\partial x}(t; u).$$

**Proof.** As in the proof of [45, Lemma 3.6], by integration by parts we have

$$\int_{\Omega} \Delta(\log \Phi_\varepsilon)|u|^2 \Phi_\varepsilon \, dx = \int_{\Omega} \left( \Delta \Phi_\varepsilon \Phi_\varepsilon - \frac{|\nabla \Phi_\varepsilon|^2}{\Phi_\varepsilon} \right) |u|^2 \, dx$$

$$= \int_{\Omega} \left| \nabla u \right|^2 \Phi_\varepsilon \, dx - \int_{\Omega} \Phi_\varepsilon^{-1} |\nabla(\Phi_\varepsilon u)|^2 \, dx$$

$$\leq \int_{\Omega} \left| \nabla u \right|^2 \Phi_\varepsilon \, dx.$$

Using Lemma 4.6 and (3.1), we see that

$$\Delta(\log \Phi_\varepsilon(x)) = \frac{1}{h + 2\varepsilon} \frac{\Delta A_\varepsilon(x)}{1 + t} \geq \frac{1 - \varepsilon}{h + 2\varepsilon} \frac{a(x)}{1 + t},$$

which leads to (4.6).
Next, in order to clarify the effect of the finite propagation speed property, we put

\[ a_1 := \inf_{x \in \Omega} \langle x \rangle^{-\alpha} a(x). \]

Then we have the following.

**Lemma 4.8.** For \( t \geq 0 \), we have

\[ E_\partial(t; u) \leq \frac{1}{a_1} E_a(t; \partial_t u). \]  

\[ \int_\Omega A_x(x) \frac{u_t}{a(x)} |u_t|^2 \Phi_\varepsilon \, dx \leq \frac{A_2}{a_1} (R_0 + 1 + t)^2 E_\partial(t; u), \]  

\[ |E_a(t; u)| \leq \frac{2}{\sqrt{a_1}} \sqrt{E_a(t; u) E_\partial(t; u)}. \]

**Proof.** By \( a(x)/a_1 \geq \langle x \rangle^\alpha \geq 1 \), we have

\[ \int_\Omega |u_t|^2 \Phi_\varepsilon \, dx \leq \int_\Omega a(x) \frac{a_1}{a_1} |u_t|^2 \Phi_\varepsilon \, dx \leq \frac{1}{a_1} E_a(t; \partial_t u), \]

which shows (4.7). By (3.2), \( 1/a(x) \leq (a_1 \langle x \rangle^\alpha)^{-1} \) and Lemma 4.1, we have

\[ \frac{A_x(x)}{a(x)} \leq \frac{A_2 (\langle x \rangle^\alpha)^{2+\alpha}}{a_1 \langle x \rangle^\alpha} \leq \frac{A_2}{a_1} (R + t + 1)^2, \]

which implies (4.8). Finally, using the Cauchy-Schwarz inequality and the inequality \( a(x)/a_1 \geq \langle x \rangle^\alpha \geq 1 \), we see that

\[ \left| \int_\Omega u u_t \Phi_\varepsilon \, dx \right|^2 \leq \frac{1}{a_1} \left( \int_\Omega a(x) |u_t|^2 \Phi_\varepsilon \, dx \right) E_\partial(t; u) \leq \frac{1}{a_1} E_a(t; u) E_\partial(t; u), \]

which yields (4.9). The proof of (4.9) is similar and we omit it. \( \square \)

Applying lemmas proved above, we have the following differential inequality for \( E_1(t; u) \) and \( E_2(t; u) \).

**Lemma 4.9.** (i) For every \( t \geq 0 \), we have

\[ \frac{d}{dt} E_1(t; u) \leq -E_a(t; \partial_t u). \]  

\[ \frac{d}{dt} E_2(t; u) \leq -\frac{1 - 3\varepsilon}{1 - \varepsilon} E_\partial(t; u) \]

\[ + \left( \frac{2}{a_1} + \frac{A_2 (R_0 + 1)^2}{\varepsilon a_1^2} \right) E_a(t; \partial_t u). \]
Proof: To prove (4.10), we use Lemma 4.2 and estimate the last term of (4.1). By virtue of Lemma 4.6, (3.3) and that \( A \geq 0 \), we have

\[
-2a(x) \Phi_\varepsilon + \partial_t \Phi_\varepsilon - (\partial_t \Phi_\varepsilon)^{-1} |\nabla \Phi_\varepsilon|^2 \\
= \left( -2a(x) - \frac{A_\varepsilon(x)}{(h + 2\varepsilon)(1 + t)^2} + \frac{1}{h + 2\varepsilon} \frac{|\nabla A_\varepsilon(x)|^2}{A_\varepsilon(x)} \right) \Phi_\varepsilon \\
\leq \left( -2a(x) + \frac{h + \varepsilon}{h + 2\varepsilon} a(x) \right) \Phi_\varepsilon \\
\leq -a(x) \Phi_\varepsilon,
\]

which implies (4.10). Next, we use Lemma 4.3 to verify (4.11) and estimate the last term of (4.2). It follows from Lemma 4.6 and (3.1) that

\[
a(x) \partial_t \Phi_\varepsilon + \Delta \Phi_\varepsilon = \frac{1}{h + 2\varepsilon} \left( -a(x) A_\varepsilon(x) \frac{1}{(1 + t)^2} + \frac{|\nabla A_\varepsilon(x)|^2}{(h + 2\varepsilon)(1 + t)^2} + \frac{\Delta A_\varepsilon(x)}{1 + t} \right) \Phi_\varepsilon \\
\leq \left( -\frac{\varepsilon}{h + 2\varepsilon} a(x) A_\varepsilon(x) \frac{1}{(1 + t)^2} + \frac{1 + \varepsilon}{h + 2\varepsilon} a(x) \frac{1}{1 + t} \right) \Phi_\varepsilon.
\]

and hence, with Lemma 4.7 we have

\[
\int_\Omega (a(x) \partial_t \Phi_\varepsilon + \Delta \Phi_\varepsilon) |u|^2 \, dx \\
\leq \frac{1 + \varepsilon}{1 - \varepsilon} \int_\Omega |\nabla u|^2 \Phi_\varepsilon \, dx - \frac{\varepsilon}{(h + 2\varepsilon)^2 (1 + t)^2} \int_\Omega a(x) A_\varepsilon(x) |u|^2 \Phi_\varepsilon \, dx.
\]

By using (4.8) and (4.7), the first term of the right-hand side of (4.2) is also estimated as

\[
2 \int_\Omega uu_t (\partial_t \Phi_\varepsilon) \, dx \\
= -\frac{2}{h + 2\varepsilon} \frac{1}{(1 + t)^2} \int_\Omega uu_t A_\varepsilon(x) \Phi_\varepsilon \, dx \\
\leq \frac{2}{h + 2\varepsilon} \frac{1}{(1 + t)^2} \left( \int_\Omega a(x) A_\varepsilon(x) |u|^2 \Phi_\varepsilon \, dx \right)^{\frac{1}{2}} \left( \int_\Omega \frac{A_\varepsilon(x)}{a(x)} |u_t|^2 \Phi_\varepsilon \, dx \right)^{\frac{1}{2}} \\
\leq \frac{\varepsilon}{(h + 2\varepsilon)^2} \frac{1}{(1 + t)^2} \int_\Omega a(x) A_\varepsilon(x) |u|^2 \Phi_\varepsilon \, dx + \frac{A_\varepsilon(R_0 + 1)^2}{\varepsilon a_1^2} E_a(t; \partial_t u).
\]

Combining them with (4.2), we reach (4.11).

Furthermore, multiplying \( E_1(t; u) \) and \( E_2(t; u) \) by the time weight functions \( (t_1 + t)^m \) and \( (t_2 + t)^h \), respectively, we have the following time-weighted differential inequalities.

Lemma 4.10. The following assertions hold:

(i) Set \( t_*(\alpha, m) := 2m/a_1 \). Then for every \( t, m \geq 0 \) and \( t_1 \geq t_*(\alpha, m) \),

\[
(4.12) \quad \frac{d}{dt} \left( (t_1 + t)^m E_1(t; u) \right) \leq m(t_1 + t)^{m-1} E_{\partial x}(t; u) - \frac{1}{2}(t_1 + t)^m E_a(t; \partial_t u).
\]
(ii) For every \( t, \lambda \geq 0 \) and \( t_2 \geq 1 \),

\[
\frac{d}{dt} \left( (t_2 + t)^\lambda E_2(t; u) \right) \\
\leq \lambda (1 + \varepsilon)(t_2 + t)^{\lambda-1} E_\alpha(t; u) - \frac{1 - 3\varepsilon}{1 - \varepsilon} (t_2 + t)^\lambda E_{\partial x}(t; u) \\
+ \left( \frac{2}{a_1} + \frac{A_2e(0 + 1)^2}{\varepsilon a_1^2} \right) (t_2 + t)^\lambda E_\alpha(t; \partial_t u).
\]

(iii) In particular, setting

\[
\nu := \frac{4}{a_1} + \frac{2A_2e(R_0 + 1)^2}{\varepsilon a_1^2} + \frac{1}{\varepsilon a_1},
\]

\[
t_*(\varepsilon, \alpha, \lambda) := \max \left\{ \left( \frac{1 - \varepsilon}\nu \right) \frac{1}{\varepsilon} \frac{\lambda}{\varepsilon}, \lambda, 1, t_*(\alpha, m) \right\},
\]

one has that for \( t, \lambda \geq 0 \) and \( t_3 \geq t_*(\varepsilon, \alpha, \lambda) \),

\[
\frac{d}{dt} \left( \nu(t_3 + t)^\lambda E_1(t; u) + (t_3 + t)^\lambda E_2(t; u) \right) \\
\leq -\frac{1 - 4\varepsilon}{1 - \varepsilon} (t_3 + t)^\lambda E_{\partial x}(t; u) + \lambda (1 + \varepsilon)(t_3 + t)^{\lambda-1} E_\alpha(t; u).
\]

Proof. (i) Let \( m \geq 0 \) and \( t_1 \geq t_*(\alpha, m) \). Using (4.10) and (4.7), we have

\[
(t_1 + t)^{-m} \frac{d}{dt} \left( (t_1 + t)^m E_1(t; u) \right) \\
= \frac{m}{t_1 + t} E_{\partial x}(t; u) + \frac{m}{t_1 + t} E_{\partial t}(t; u) + \frac{d}{dt} E_1(t; u) \\
\leq \frac{m}{t_1 + t} E_{\partial x}(t; u) + \frac{m}{t_1 + t} E_{\partial t}(t; u) - E_\alpha(t; \partial_t u) \\
\leq \frac{m}{t_1 + t} E_{\partial x}(t; u) + \left( \frac{m}{a_1(t_1 + t)} - 1 \right) E_\alpha(t; \partial_t u).
\]

Since \( t_* = 2m/a_1 \), we see that

\[
\frac{m}{a_1(t_1 + t)} - 1 \leq -\frac{1}{2}
\]

holds for \( t_1 \geq t_* \) and \( t \geq 0 \). This implies (4.12).

(ii) By using (4.13) and (4.11), \( t \geq 0 \) and \( t_2 \geq 1 \), we have

\[
(t_2 + t)^{-\lambda} \frac{d}{dt} \left( (t_2 + t)^\lambda E_2(t; u) \right) \\
\leq \frac{\lambda}{t_2 + t} E_{\partial x}(t; u) + \frac{\lambda}{t_2 + t} E_\alpha(t; u) + \frac{d}{dt} E_2(t; u) \\
\leq \frac{\lambda}{t_2 + t} E_{\partial x}(t; u) + \frac{\lambda}{t_2 + t} E_\alpha(t; u) - \frac{1 - 3\varepsilon}{1 - \varepsilon} E_{\partial x}(t; u) \\
+ \left( \frac{2}{a_1} + \frac{A_2e(0 + 1)^2}{\varepsilon a_1^2} \right) E_\alpha(t; \partial_t u).
\]
Noting that (4.9) and (4.7) lead to
\[
\frac{\lambda}{t_2 + t} E_\ast(t; u) \leq \frac{2\lambda}{\sqrt{a_1 (t_2 + t)}} \sqrt{E_a(t; u) E_{\partial t}(t; u)} \\
\leq \frac{2\lambda}{a_1 (t_2 + t)} \sqrt{E_a(t; u) E_\partial(t; u)} \\
\leq \frac{\lambda \varepsilon}{t_2 + t} E_a(t; u) + \frac{\lambda}{\varepsilon a_1^2 (t_2 + t)} E_a(t; \partial u),
\]
we deduce (4.13).

(iii) Combining (4.12) with \( m = \lambda \) and (4.11), we have for \( t_3 \geq t_\ast(\varepsilon, \alpha, \lambda) \) and \( t \geq 0 \),
\[
\frac{d}{dt} \left( \nu (t_3 + t)^\lambda E_1(t; u) + (t_3 + t)^\lambda E_2(t; u) \right) \\
\leq \left( \frac{\nu \lambda}{t_3 + t} \frac{1 - 3\varepsilon}{1 - \varepsilon} \right) (t_3 + t)^\lambda E_{\partial x}(t; u) + \lambda (1 + \varepsilon)(t_3 + t)^\lambda - 1 E_a(t; u) \\
+ \left( \frac{2}{a_1} + \frac{A_2 (R_0 + 1)^2}{\varepsilon a_1^2} + \frac{\lambda}{2 \varepsilon a_1^2 t_3} - \frac{\nu'}{2} \right) (t_3 + t)^\lambda E_a(t; \partial u) \\
\leq - \frac{1 - 4\varepsilon}{1 - \varepsilon} (t_3 + t)^\lambda E_{\partial x}(t; u) + \lambda (1 + \varepsilon)(t_3 + t)^\lambda - 1 E_a(t; u),
\]
which gives the assertion. \( \square \)

**Proof of Proposition 4.5.** By (4.9), we easily have
\[
(4.15) \quad \nu E_1(t; u) + E_2(t; u) \geq \frac{3}{4} E_a(t; u).
\]

By using the above estimate, we prove the assertion via mathematical induction. **Step 1 (\( k = 0 \)).** From (4.14) and Lemma 4.7, we deduce that
\[
\frac{d}{dt} \left( \nu (t_3 + t)^\lambda E_1(t; u) + (t_3 + t)^\lambda E_2(t; u) \right) \\
\leq \left( \frac{1 - 4\varepsilon}{1 - \varepsilon} + \frac{\lambda (1 + \varepsilon)(h + 2\varepsilon)}{1 - \varepsilon} \right) (t_3 + t)^\lambda E_{\partial x}(t; u).
\]

Taking \( \lambda_0 = \frac{(1 - \varepsilon)(1 - 4\varepsilon)}{(1 + \varepsilon)(h + 2\varepsilon)} \), integrating over \((0, t)\) with respect to \( t \) and using (4.15), we have
\[
\frac{3}{4} (t_3 + t)^{\lambda_0} E_a(t; u) + \frac{\varepsilon (1 - 4\varepsilon)}{1 - \varepsilon} \int_0^t (t_3 + s)^{\lambda_0} E_{\partial x}(s; u) \, ds \\
\leq \nu t_3^{\lambda_0} E_1(0; u) + t_3^{\lambda_0} E_2(0; u).
\]

This gives the desired estimate for \( E_a(t; u) \). Also, as a byproduct, we obtain
\[
\frac{\varepsilon (1 - 4\varepsilon)}{1 - \varepsilon} \int_0^t (t_3 + s)^{\lambda_0} E_{\partial x}(s; u) \, ds \leq \nu t_3^{\lambda_0} E_1(0; u) + t_3^{\lambda_0} E_2(0; u).
\]
Using this inequality and (4.12) with $m = \lambda_0 + 1$, and integrating over $(0, t)$, we obtain

\[
(t_3 + t)^{\lambda_0 + 1} E_1(t; u) + \frac{1}{2} \int_0^t (t_3 + s)^{\lambda_0 + 1} E_a(s; \partial_t u) \, ds
\]

\[
\leq t_3^{\lambda_0 + 1} E_1(0; u) + (\lambda_0 + 1) \int_0^t (t_3 + s)^{\lambda_0} E_{\partial_x}(s; u) \, ds
\]

\[
\leq t_3^{\lambda_0 + 1} E_1(0; u) + \frac{(\lambda_0 + 1)(1 - \varepsilon)}{\varepsilon(1 - 4\varepsilon)} \left( \nu t_3^{\lambda_0} E_1(0; u) + t_3^{\lambda_0} E_2(0; u) \right) .
\]

Finally, putting $\delta = 1/h - \lambda_0$, we have the desired assertion with $k = 0$. Moreover, we also have

\[
\frac{1}{2} \int_0^t (t_3 + s)^{\lambda_0 + 1} E_a(s; \partial_t u) \, ds
\]

\[
\leq t_3^{\lambda_0 + 1} E_1(0; u) + \frac{(\lambda_0 + 1)(1 - \varepsilon)}{\varepsilon(1 - 4\varepsilon)} \left( \nu t_3^{\lambda_0} E_1(0; u) + t_3^{\lambda_0} E_2(0; u) \right),
\]

which will be used in the next step.

**Step 2** ($1 < k \leq k_0 - 1$). Suppose that for every $t \geq 0$,

\[(1+t)^{\lambda_0+2k-1} E_1(t; \partial^k \partial_t u) + (1+t)^{\lambda_0+2k-2} E_a(t; \partial^k \partial_t u) \leq M_{\varepsilon,k-1} \| (u_0, u_1) \|_{H^k \times H^{k-1}(\Omega)}^2
\]

and additionally,

\[
\int_0^t (1+s)^{\lambda_0+2k-1} E_a(s; \partial^k \partial_t u) \, ds \leq M'_{\varepsilon,k-1} \| (u_0, u_1) \|_{H^k \times H^{k-1}(\Omega)}^2
\]

are valid. Since the initial value $(u_0, u_1)$ satisfies the compatibility condition of order $k$, $\partial^k u$ is also a solution of (1.1) with the initial data $(\partial^k u, \partial^{k+1} u)(x, 0) = (u_{k-1}, u_k)(x)$. Applying (4.14) with $\lambda = \lambda_0 + 2k$, putting $t_{3k} = t_{4k}(\varepsilon, R_0, \alpha, \lambda_0 + 2k)$ (see Lemma 4.10 (iii)), integrating over $(0, t)$, and noting (4.15), we have

\[
\frac{3}{4} \left( t_{3k} + t \right)^{\lambda_0+2k} E_a(t; \partial^k \partial_t u) + \frac{1 - 4\varepsilon}{1 - \varepsilon} \int_0^t (t_3k + s)^{\lambda_0+2k} E_{\partial_x}(s; \partial^k \partial_t u) \, ds
\]

\[
\leq \nu t_{3k}^{\lambda_0+2k} E_1(0; \partial^k \partial_t u) + t_{3k}^{\lambda_0+2k-1} E_2(0; \partial^k \partial_t u)
\]

\[
+ (\lambda_0 + 2k)(1 + \varepsilon) M'_{\varepsilon,k-1} \| (u_0, u_1) \|_{H^k \times H^{k-1}(\Omega)}^2.
\]

In particular, we obtain the boundedness of the second term of the left-hand side. From this and (4.12) with $m = \lambda_0 + 2k + 1$ we conclude

\[
(t_{3k} + t)^{\lambda_0+2k+1} E_1(t; \partial^k \partial_t u) + \frac{1}{2} \int_0^t (t_{3k} + s)^{\lambda_0+2k+1} E_a(s; \partial^{k+1} \partial_t u) \, ds
\]

\[
\leq M_{\varepsilon,k}'' \left( E_1(0; \partial^k \partial_t u) + E_2(0; \partial^k \partial_t u) + \| (u_0, u_1) \|_{H^k \times H^{k-1}(\Omega)}^2 \right)
\]

with some constant $M_{\varepsilon,k}'' > 0$. Therefore, by induction, we obtain the desired inequalities for all $k \leq k_0 - 1$. \qed

5. Proof of the diffusion phenomena

In this section, we give a proof of Theorem 1.1. We use the following lemma stated in [45, Section 4].
Lemma 5.1. Assume that \((u_0, u_1) \in (H^2 \cap H^1_0(\Omega)) \times H^1_0(\Omega)\) and suppose that \(\text{supp } (u_0, u_1) \subseteq \{ x \in \Omega; |x| \leq R_0 \}\). Then for every \(t \geq 0\),

\[
\begin{align*}
    u(x, t) - e^{-tL^*} [u_0 + a(\cdot)^{-1}u_1] &= - \int_{t/2}^{t} e^{-(t-s)L^*} [a(\cdot)^{-1}u_{tt}(\cdot, s)] ds \\
    & \quad - e^{-\frac{1}{2}L^*} [a(\cdot)^{-1}u_t(\cdot, t/2)] \\
    & \quad + \int_{0}^{t/2} L^* e^{-(t-s)L^*} [a(\cdot)^{-1}u_t(\cdot, s)] ds,
\end{align*}
\]

(5.1)

where \(L^*\) is the Friedrichs extension of \(L = -a(x)^{-1} \Delta\) in \(L^2_{\mu}\).

Proof of Theorem 1.1. First we show the assertion for \((u_0, u_1)\) satisfying the compatibility condition of order 2. Taking \(L^2_{\mu}\)-norm of both side, we have

\[
\left\| u(x, \cdot) - e^{-tL^*} [u_0 + a(\cdot)^{-1}u_1] \right\|_{L^2_{\mu}} \leq \mathcal{J}_1(t) + \mathcal{J}_2(t) + \mathcal{J}_3(t),
\]

where

\[
\begin{align*}
    \mathcal{J}_1(t) &= \int_{t/2}^{t} \left\| e^{-(t-s)L^*} [a(\cdot)^{-1}u_{tt}(\cdot, s)] \right\|_{L^2_{\mu}} ds, \\
    \mathcal{J}_2(t) &= \left\| e^{-\frac{1}{2}L^*} [a(\cdot)^{-1}u_t(\cdot, t/2)] \right\|_{L^2_{\mu}}, \\
    \mathcal{J}_3(t) &= \int_{0}^{t/2} \left\| L^* e^{-(t-s)L^*} [a(\cdot)^{-1}u_t(\cdot, s)] \right\|_{L^2_{\mu}} ds.
\end{align*}
\]

Noting that \(a(x) \geq a_1\) and \(\Phi(x, t) \geq 1\), and applying Proposition 4.5 with \(k = 1\) and \(k = 0\) we see that

\[
\left\| a(\cdot)^{-1}\partial_x^{k+1} u(\cdot, s) \right\|^2_{L^2_{\mu}} = \int_\Omega a(x)^{-1} |\partial_x^{k+1} u(\cdot, s)|^2 dx \\
\leq \frac{1}{a_1} \int_\Omega |\partial_x^{k+1} u(\cdot, s)|^2 \Phi_x dx \\
\leq \frac{1}{a_1} E_{\partial_t}(t, \partial_x u) \\
\leq \frac{M_{\delta,k}}{a_1} (1 + t)^{-\frac{N+\alpha}{2} - 2k + 1 + \delta} \| (u_0, u_1) \|_{H^{k+1} \times H^k}^2.
\]

Therefore, we have

\[
\mathcal{J}_1(t) \leq \int_{t/2}^{t} \left\| a(\cdot)^{-1}u_{tt}(\cdot, s) \right\|_{L^2_{\mu}} ds \\
\leq \frac{M_{\delta,1}}{a_1} \|(u_0, u_1)\|_{H^2 \times H^1} \int_{t/2}^{t} (1 + s)^{-\frac{N+\alpha}{2} - \frac{3}{2} + \frac{\delta}{2}} ds \\
\leq C_{1, \delta} (1 + t)^{-\frac{N+\alpha}{2} - \frac{3}{2} + \frac{\delta}{2}} \|(u_0, u_1)\|_{H^2 \times H^1},
\]

where \(C_{1, \delta}\) is a positive constant depending on \(\delta\). In the same way, we deduce

\[
\mathcal{J}_2(t) \leq \left\| a(\cdot)^{-1}u_t(\cdot, t/2) \right\|_{L^2_{\mu}} \leq \frac{M_{\delta,0}}{a_1} (1 + t)^{-\frac{N+\alpha}{2} - \frac{3}{2} + \frac{\delta}{2}} \|(u_0, u_1)\|_{H^1 \times L^2}.
\]
Next, we estimate the term \( J_3(t) \). First, we treat the case \( N = 2 \). In this case, by Proposition 2.6 we have

\[
J_3(t) \leq C \int_0^{t/2} (t-s)^{-\frac{3}{2}} \|a(\cdot)^{-1}u_t(\cdot, s)\|_{L^2} \, ds \\
\leq C t^{-\frac{3}{2}} \int_0^{t/2} \sqrt{\|\Phi^{-1}_e(\cdot, s)\|_{L^1(\Omega)}} E_H(s; u) \, ds.
\]

We compute

\[
\|\Phi^{-1}_e(\cdot, t)\|_{L^1(\Omega)} \leq \int_{\mathbb{R}^2} \exp \left( -\frac{A_1\alpha}{h + 2\varepsilon} |x|^{2+\varepsilon} \frac{1}{1+t} \right) dx \\
= (1 + t)^{\frac{2+\alpha}{h + 2\varepsilon}} \int_{\mathbb{R}^2} \exp \left( -\frac{A_1\alpha}{h + 2\varepsilon} |y|^{2+\varepsilon} \right) dy.
\]

From this and Proposition 4.3 we conclude

\[
J_3(t) \leq C t^{-\frac{3}{2}} \int_0^{t/2} (1 + s)^{-\frac{2+\alpha}{h + 2\varepsilon} - \frac{3}{2} + \frac{3}{2} + \frac{3}{2}} ds \cdot \|(u_0, u_1)\|_{H^1 \times L^2} \leq C t^{-\frac{3}{2}} \int_0^{t/2} (1 + s)^{-\frac{2+\alpha}{h + 2\varepsilon}} ds \cdot \|(u_0, u_1)\|_{H^1 \times L^2}.
\]

When \( N \geq 3 \), Proposition 2.6 leads to

\[
J_3(t) \leq C \int_0^{t/2} (t-s)^{-\frac{N}{2(N-2)}} \left( \int_{\Omega} |a(x)^{-1}u_t(x, s)|^q |x|^{(N-2)\alpha(2-q)/4} d\mu(x) \right)^{1/q} ds \\
\leq C t^{-\frac{N}{2(N-2)}} \int_0^{t/2} \left( \int_{\Omega} |u_t(x, s)|^q a(x)^{(q-1)} |x|^{(N-2)\alpha(2-q)/4} dx \right)^{1/q} ds,
\]

where \( q \in ((p_\alpha)', 2] \) and sufficiently close to \((p_\alpha)'\). Using \( a(x) \geq a_1(x)^\alpha \) and the Cauchy–Schwarz inequality, we further calculate

\[
J_3(t) \leq C t^{-\frac{N}{2(N-2)}} \int_0^{t/2} \left\| u_t(s) |x|^\alpha [\Phi^{-1}_e(\cdot, s) - \frac{1}{2}] \right\|_{L^q} ds \\
\leq C t^{-\frac{N}{2(N-2)}} \int_0^{t/2} \left\| u_t(s) \Phi^{-1/2}_e \right\|_{L^2} \left\| \Phi^{-1/2}_e |x|^\alpha [\Phi^{-1}_e(\cdot, s) - \frac{1}{2}] \right\|_{L^{2q/(2-q)}} ds.
\]

We also notice that if \( q \) is close to \((p_\alpha)'\), then \( \frac{N}{2(N-2)} \left( \frac{1}{q} - \frac{1}{2} \right) \leq 0 \) holds. Therefore, by changing variables, we estimate

\[
\left\| \Phi^{-1/2}_e |x|^\alpha [\Phi^{-1}_e(\cdot, s) - \frac{1}{2}] \right\|_{L^{2q/(2-q)}} \leq C (1 + t)^{\frac{N}{2(N-2)} \left( \frac{1}{q} - \frac{1}{2} \right) + \frac{N}{2(N-2)}} \left\| (u_0, u_1) \right\|_{H^1 \times L^2} \\
\leq C (1 + t)^{\frac{N}{2(N-2)} \left( \frac{1}{q} - \frac{1}{2} \right) + \frac{N}{2(N-2)}} \left\| (u_0, u_1) \right\|_{H^1 \times L^2}.
\]

From this estimate and Proposition 4.3 we obtain

\[
J_3(t) \leq C (1 + t)^{-\frac{N}{2(N-2)}} \int_0^{t/2} (1 + s)^{-\frac{N}{2(N-2)} \left( \frac{1}{q} - \frac{1}{2} \right) - \frac{N}{2(N-2)}} (1 + s)^{-\frac{N}{2(N-2)}} ds \cdot \|(u_0, u_1)\|_{H^1 \times L^2} \\
\leq C (1 + t)^{-\frac{N}{2(N-2)}} \max \left( 0, \frac{N}{2(N-2)} - \frac{N}{2(N-2)} \right) \left\| (u_0, u_1) \right\|_{H^1 \times L^2},
\]

with arbitrary small \( \delta' > 0 \). Taking \( q \) sufficiently close to \((p_\alpha)'\), we conclude

\[
J_3(t) \leq C (1 + t)^{-\frac{N}{2(N-2)}} \left\| (u_0, u_1) \right\|_{H^1 \times L^2},
\]

where \( \delta'' > 0 \) depends on \( \delta, \delta' \) and it can be taken arbitrary small.
Finally, combining the estimates for $\mathcal{J}_1(t), \mathcal{J}_2(t), \mathcal{J}_3(t)$ and noting $1/2 < (1 + \alpha)/(2 + \alpha)$, we have the desired estimate.

Next we show the assertion for $(u_0, u_1)$ satisfying $(u_0, u_1) \in (H^2 \times H^1_0(\Omega)) \times H^1_0(\Omega)$ (the compatibility condition of order 1) via an approximation argument. Fix $\phi \in C_c^\infty(\mathbb{R}^N, [0, 1])$ such that $\phi \equiv 1$ on $\overline{B}(0, R_0)$ and $\phi \equiv 0$ on $\mathbb{R}^N \setminus B(0, R_0 + 1)$ and define for $n \in \mathbb{N}$,

\[
\left( \begin{array}{c} u_{0n} \\ u_{1n} \end{array} \right) = \left( \begin{array}{c} \phi \tilde{u}_{0n} \\ \phi \tilde{u}_{1n} \end{array} \right), \quad \left( \begin{array}{c} \tilde{u}_{0n} \\ \tilde{u}_{1n} \end{array} \right) = \left( 1 + \frac{1}{n} A \right)^{-1} \left( \begin{array}{c} u_0 \\ u_1 \end{array} \right),
\]

where $A$ is an quasi-$m$-accretive operator in $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega)$ associated with \eqref{1.1}, that is,

\[
A = \left( \begin{array}{cc} 0 & -1 \\ -\Delta & a(x) \end{array} \right)
\]

endowed with domain $D(A) = (H^2 \cap H^1_0(\Omega)) \times H^1_0(\Omega)$. Then $(u_{0n}, u_{1n})$ satisfies $\text{supp}(u_{0n}, u_{1n}) \subset \overline{B}(0, R_0 + 1)$ and the compatibility condition of order 2. Let $v_n$ be a solution of \eqref{1.1} with $(u_{0n}, u_{1n})$. Observe that

\[
\| (u_{0n}, u_{1n}) \|^2_{H^2 \times H^1} \leq C^2 \| \phi \|^2_{W^{2, \infty} \times H^2 \times H^1} + C'^2 \| u_0, u_1 \|^2_{H^1} + C'' \| (u_{0n}, u_{1n}) \|^2_{H^1}
\]

with suitable constants $C, C', C'' > 0$, and

\[
\left( \begin{array}{c} u_{0n} \\ u_{1n} \end{array} \right) \to \left( \begin{array}{c} \phi u_0 \\ \phi u_1 \end{array} \right) \quad \text{in} \quad \mathcal{H}
\]

as $n \to \infty$ and also $u_{0n} + a^{-1} u_{1n} \to u_0 + a^{-1} u_1$ in $L^2_{dp}$ as $n \to \infty$. Using the result of the previous step, we deduce

\[
\left\| v_n(\cdot, t) - e^{tA} [u_0 + a(\cdot)^{-1} u_1] \right\|_{L^2_{dp}} \leq \tilde{C} (1 + t)^{-\frac{N+\alpha}{2+\alpha}} \| (u_0, u_1) \|^2_{H^2 \times H^1}
\]

with some constant $\tilde{C} > 0$. Letting $n \to \infty$, by continuity of the $C_0$-semigroup $e^{-tA}$ in $\mathcal{H}$ we also obtain diffusion phenomena for initial data in $(H^2 \cap H^1_0(\Omega)) \cap H^1_0(\Omega)$.

\[ \Box \]

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(M. Sobajima) Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda-shi, Chiba, 278-8510, Japan
E-mail address: msobajima1984@gmail.com

(Y. Wakasugi) Department of Engineering for Production and Environment, Graduate School of Science and Engineering, Ehime University, 3 Bunkyo-cho, Matsuyama, Ehime, 790-8577, Japan
E-mail address: wakasugi.yuta.vi@ehime-u.ac.jp