A generalized Viéte’s-like formula for pi with rapid convergence

S. M. Abrarov∗ and B. M. Quine†

October 25, 2016

Abstract

We present a generalized Viéte’s-like formula for pi with rapid convergence. This formula is based on the arctangent function identity with argument $x = \sqrt{2} - a_{K-1}/a_K$, where

$$a_K = \sqrt{\frac{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}{K \text{ square roots}}}$$

is a radical consisting of $K$ nested square roots of twos. The computational test we performed reveals that the generalized Viéte’s-like formula provides a significant improvement in accuracy as the integer $K$ increases.

Keywords: Viéte’s formula, constant pi, arctangent function

1 Introduction

In 1593 the French lawyer and amateur mathematician François Viéte discovered a classical formula for the constant pi that can be expressed elegantly in radicals consisting of nested square roots and twos [1 2 3]

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2 + \sqrt{2}}}{2} \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots . \quad (1)$$

∗Dept. Earth and Space Science and Engineering, York University, Toronto, Canada, M3J 1P3.
†Dept. Physics and Astronomy, York University, Toronto, Canada, M3J 1P3.
He found this formula for pi geometrically by considering a regular polygon enclosed inside the circle with unit radius. It is convenient to define

\[ a_k = \sqrt{2 + a_{k-1}} \quad \text{and} \quad a_1 = \sqrt{2} \]

to represent the Viète’s formula (1) for pi in a more compact form as

\[ \frac{\pi}{2} = \lim_{K \to \infty} \prod_{k=1}^{K} \frac{a_k}{2}. \]

There is a simple derivation of the formula (1) that utilizes the sinc function (also known as the cardinal sine function) [4, 5]

\[ \text{sinc} (t) = \begin{cases} \frac{\sin (t)}{t}, & t \neq 0 \\ 1, & t = 0. \end{cases} \]

The sinc function can be expressed as an infinite product of cosines [4, 5]

\[ \text{sinc} (t) = \prod_{k=1}^{\infty} \cos \left( \frac{t}{2^k} \right). \tag{2} \]

Using the identity for the cosine double-angle

\[ \cos \left( \frac{\pi/2}{2^k} \right) = 2\cos^2 \left( \frac{\pi/2}{2^{k+1}} \right) - 1 \]

and taking into consideration that for the largest angle

\[ \cos \left( \frac{\pi/2}{2^{21}} \right) = \frac{\sqrt{2}}{2}, \]

we can find each cosine multiplier in equation (2) in form

\[ \cos \left( \frac{\pi/2}{2^k} \right) = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}. \tag{3} \]

\( k \) square roots

Consequently, at \( t = \pi/2 \) the cosine infinite product (2) yields

\[ \text{sinc} \left( \frac{\pi}{2} \right) = \frac{2}{\pi} = \cos \left( \frac{\pi/2}{2^{21}} \right) \cos \left( \frac{\pi/2}{2^{22}} \right) \cos \left( \frac{\pi/2}{2^{23}} \right) \cdots. \]
Substituting the identity (3) into this equation, we obtain the Viète’s formula (1) for pi.

In this paper we derive a generalized Viète’s-like formula for pi with rapid convergence. Our approach is based on the arctangent function identity with argument \( x = \sqrt{2 - a_{K-1}/a_K} \), where

\[
a_K = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}.
\]

The computational test reveals that accuracy of the constant pi can be considerably improved by consecutive increment of the integer \( K \).

2 Derivation

As it has been shown in our recent paper [6], any function \( f(t) \) differentiable within the interval \( t \in [0, 1] \) can be integrated numerically by truncating the parameters \( L \) or \( M \) in the following limits (see equations (9) and (6) in [6])

\[
\int_0^1 f(t) \, dt = \lim_{L \to \infty} \sum_{\ell=1}^L \sum_{m=0}^M \frac{(-1)^m + 1}{(2L)^{m+1}} \frac{(m+1)!}{(m)!} f^{(m)}(t) \bigg|_{t = \ell^{-1/2}} \quad (4a)
\]

and

\[
\int_0^1 f(t) \, dt = \lim_{M \to \infty} \sum_{\ell=1}^L \sum_{m=0}^M \frac{(-1)^m + 1}{(2M)^{m+1}} \frac{(m+1)!}{(m)!} f^{(m)}(t) \bigg|_{t = \ell^{-1/2}} \quad (4b)
\]

respectively.

Since

\[
\int_0^1 \frac{x}{1 + x^2 t^2} dt = \arctan(x)
\]

substituting the integrand

\[
f(t) = \frac{x}{1 + x^2 t^2}
\]
into the equation (4a) results in (see [7] for more details in derivation)

\[
\arctan(x) = \lim_{L \to \infty} i \sum_{\ell=1}^{L} \sum_{m=1}^{\lfloor \frac{M}{2} \rfloor + 1} \frac{1}{2m-1} \left( \frac{1}{((2\ell - 1) + 2iL/x)^{2m-1}} - \frac{1}{((2\ell - 1) - 2iL/x)^{2m-1}} \right).
\]

(5a)

According to equations (4a) and (4b) the parameters \(L\) and \(M\) under the limit notation are interchangeable (compare equations (9) and (6) from the paper [6]). Consequently, using absolutely same derivation procedure as described in [7] for the equation (5a) above, we can also write

\[
\arctan(x) = \lim_{M \to \infty} i \sum_{\ell=1}^{L} \sum_{m=1}^{\lfloor \frac{M}{2} \rfloor + 1} \frac{1}{2m-1} \left( \frac{1}{((2\ell - 1) + 2iL/x)^{2m-1}} - \frac{1}{((2\ell - 1) - 2iL/x)^{2m-1}} \right).
\]

(5b)

Comparing equations (5a) and (5b) we can see that at least one of the parameters \(L\) or \(M\) must be large enough in truncation for high-accuracy approximation. However, as the parameter \(M\) is more important for rapid convergence, the limit (5b) is preferable for numerical analysis. Since in the limit (5b) the integer \(L\) may not be necessarily large, in order to simplify it we can choose any small value, say \(L = 1\). This leads to the equation

\[
\arctan(x) = \lim_{M \to \infty} i \sum_{m=1}^{\lfloor \frac{M}{2} \rfloor + 1} \frac{1}{2m-1} \left( \frac{1}{(1 + 2i/x)^{2m-1}} - \frac{1}{(1 - 2i/x)^{2m-1}} \right).
\]

(6)

Consider the following relation

\[
\frac{\pi}{2K+1} = \arctan \left( \tan \left( \frac{\pi}{2\sqrt{K}} \right) \right) = \arctan \left( \frac{\sin \left( \frac{\pi}{2\sqrt{K}} \right)}{\cos \left( \frac{\pi}{2\sqrt{K}} \right)} \right),
\]

where

\[
\sin \left( \frac{\pi}{2\sqrt{K}} \right) = \sqrt{1 - \cos^2 \left( \frac{\pi}{2\sqrt{K}} \right)}.
\]

(8)

Substituting the equation (3) into the identity (8) leads to

\[
\sin \left( \frac{\pi}{2\sqrt{K}} \right) = \frac{1}{2} \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}.
\]

(9)
Consequently, from the equations (3), (7) and (9) we obtain a simple Viète’s-like formula for the constant π

\[
\frac{\pi}{2^{K+1}} = \arctan \left( \sqrt{2 - \frac{\sqrt{2}}{2 + \sqrt{2 + \cdots + \sqrt{2}}}} \right)
\]

or

\[
\frac{\pi}{2^{K+1}} = \arctan \left( \frac{\sqrt{2} - a_{K-1}}{a_K} \right).
\]  

(10)

Lastly, combining equations (6) and (10) together results in

\[
\pi = 2^{K+1} \lim_{M \to \infty} \sum_{m=1}^{\left[\frac{M}{2}\right]+1} \frac{1}{2m-1} \times \left( \frac{1}{(1 + 2i a_K/\sqrt{2 - a_{K-1}})^{2m-1}} - \frac{1}{(1 - 2i a_K/\sqrt{2 - a_{K-1}})^{2m-1}} \right)
\]

(11)

It will be shown later that the equation (11) represents a generalized Viète’s-like formula for π.

\section{Algorithmic implementation}

\subsection{Methodology description}

As it has been reported previously in the paper [7], the decrease of the argument \(x\) in the limit (5a) improves significantly the accuracy in computing pi. Therefore, we may also expect a considerable improvement in accuracy of the arctangent function identity (6) when its argument \(x\) decreases. In fact, the limit (11) is based on the arctangent function identity (6) when its argument \(x\) is equal to \(\sqrt{2 - a_{K-1}/a_K}\). The increment of the integer \(K\) by one decreases the argument \(x = \sqrt{2 - a_{K-1}/a_K}\) by a factor that tends to two as \(K \to \infty\). Therefore, the value of argument \(x = \sqrt{2 - a_{K-1}/a_K}\) decreases
very rapidly in a geometric progression as the integer $K$ increases. As a consequence, this approach leads to a significant improvement in accuracy of the constant $\pi$.

### 3.2 Computational results

The computational test reveals that even at smallest values of the integer $K$ the truncated limit (11) can be quite rapid in convergence. In particular, at $M = 100$ and $K$ equals only to 1, 2 and 3 we can observe a relatively large overlap in digits coinciding with actual value of the constant $\pi$ as given by

\[
3.141592653\ldots 279502884\ldots 361003237\ldots
\]

38 coinciding digits

\[
3.141592653\ldots 92307816405927185386\ldots
\]

73 coinciding digits

and

\[
3.141592653\ldots 17067982140598570306\ldots
\]

105 coinciding digits

respectively.

Further, in order to estimate the convergence rate, we performed sample computations of the constant $\pi$ by using a rearranged form of the equation (11) as follows

\[
\pi = 2^{K+1} i \sum_{m=1}^{\lfloor \frac{M}{2} \rfloor + 1} \frac{1}{2m-1} \left( \frac{1}{1 + 2i a_K/\sqrt{2 - a_K}^{2m-1}} - \frac{1}{1 - 2i a_K/\sqrt{2 - a_K}^{2m-1}} \right)
\]

\[+ \varepsilon,\]

where $\varepsilon$ is the error term. In algorithmic implementation we incremented $K$ by one at each consecutive step while keeping value of the parameter $M$ fixed and equal to 336. Thus, the computational test we performed shows that at $K$ equals to 15, 16, 17, 18, 19 and 20, the error term $\varepsilon$ becomes equal to $1.92858 \times 10^{-1564}$, $3.44437 \times 10^{-1666}$, $6.15140 \times 10^{-1768}$, $1.098591 \times 10^{-1869}$, $1.96199 \times 10^{-1971}$ and $3.50396 \times 10^{-2073}$, respectively. As we can see, with only $\lfloor 336/2 \rfloor + 1 = 169$ summation terms each increment of the integer $K$ just by one contributes for more than 100 additional decimal digits of $\pi$. 

6
4 Theoretical analysis

Proposition

According to computational test that has been shown in the previous section, even if the parameter $M$ is truncated and fixed at 336, the accuracy of pi, nevertheless, improves continuously while the integer $K$ increases. Therefore, relying on these experimental results we assume that the limit (11) can be modified as

$$\pi = i \lim_{K \to \infty} 2^{K+1} \sum_{m=1}^{\left\lfloor \frac{M}{2} \right\rfloor +1} \frac{1}{2m-1} \times \left( \frac{1}{(1 + 2i \frac{a_K}{\sqrt{2 - a_K - 1}})^{2m-1}} - \frac{1}{(1 - 2i \frac{a_K}{\sqrt{2 - a_K}})^{2m-1}} \right).$$  \quad (12)

Proof

The proof is not difficult. Consider the following integral

$$\int_0^1 2^{K+1} \frac{\sqrt{2 - a_{K-1}/a_K}}{1 - (\sqrt{2 - a_{K-1}/a_K})^2 t^2} \, dt = 2^{K+1} \arctan \left( \frac{\sqrt{2 - a_{K-1}}}{a_K} \right) = \pi. \quad (13)$$

The integrand from integral (13)

$$g_K(t) = 2^{K+1} \frac{\sqrt{2 - a_{K-1}/a_K}}{1 - (\sqrt{2 - a_{K-1}/a_K})^2 t^2}$$

at the limit when $K \to \infty$ becomes

$$\lim_{K \to \infty} g_K(t) = g_\infty(t) = \pi.$$

Since the function $g_\infty(t)$ is just a constant, only its zeroth order of the derivative $g_\infty^{(0)}(t)$ is not equal to zero. This signifies that if the function $g_\infty(t)$ is substituted into equation (14b), then it is no longer necessary to tend the integer $M$ to infinity because for any other than zeroth order of the derivative we always get

$$g_\infty^{(m)}(t) = 0, \quad m > 0.$$
Consequently, we can infer

\[
\pi = \lim_{K \to \infty} \int_0^1 g_K(t) \, dt \\
= \lim_{M \to \infty} \lim_{K \to \infty} \sum_{\ell=1}^L \sum_{m=0}^M \frac{(-1)^m + 1}{(2L)^{m+1}(m+1)!} g_K^{(m)}(t) \bigg|_{t=\frac{\ell-1/2}{L}} \quad (14)
\]

and from the equations (13), (14) and (5b) it follows now that

\[
\pi = i \lim_{K \to \infty} 2^{K+1} \sum_{\ell=1}^L \sum_{m=0}^M \frac{1}{2m-1} \frac{1}{(2\ell - 1) + 2i L a_K/\sqrt{2-a_K-1})^{2m-1}} - \frac{1}{(2\ell - 1) - 2i L a_K/\sqrt{2-a_K-1})^{2m-1}} \quad (15)
\]

At \( L = 1 \) this limit is reduced to equation (12). This completes the proof.

**Corollary**

At \( L = 1 \) and \( M = 1 \) the limit (15) is simplified as

\[
\pi = i \lim_{K \to \infty} 2^{K+1} \left( \frac{1}{1 + 2i a_K/\sqrt{2-a_K-1}} - \frac{1}{1 - 2i a_K/\sqrt{2-a_K-1}} \right) \quad (16)
\]

and since (see equation (3))

\[
\lim_{K \to \infty} a_K = 2 \lim_{K \to \infty} \cos \left( \frac{\pi/2}{2^K} \right) = 2
\]

the limit (16) provides

\[
\pi = i \lim_{K \to \infty} 2^{K+1} \left( \frac{1}{1 + 4i / \sqrt{2-a_K-1}} - \frac{1}{1 - 4i / \sqrt{2-a_K-1}} \right) \\
= \lim_{K \to \infty} 2^{K+1} \frac{8/\sqrt{2-a_K-1}}{1 + 16/(2-a_K-1)} \\
= \lim_{K \to \infty} 2^{K+1} \frac{8}{\sqrt{2-a_K-1} + 16/\sqrt{2-a_K-1}} \quad (17)
\]
From the equation (9) it immediately follows that

$$\lim_{K \to \infty} \sqrt{2 - a_{K-1}} = 2 \lim_{K \to \infty} \sin \left( \frac{\pi/2}{2K} \right) = 0.$$  

Consequently, the limit (17) can be simplified further as

$$\pi = \lim_{K \to \infty} 2^{K+1} \frac{8}{16/\sqrt{2} - a_{K-1}} = \lim_{K \to \infty} 2^K \sqrt{2 - a_{K-1}}$$

or

$$\pi = \lim_{K \to \infty} 2^K \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}. \tag{18}$$

This is a well-known Viète’s-like formula for pi [1].

Since the equation (18) is just a simplest case that appears trivially at $L = M = 1$ and $K \to \infty$, the limits (12) and (15) can be regarded as generalized Viète’s-like formulas for pi. Thus, we can see that being a variation of equation (12), the limit (11) is also a generalization of the Viète’s-like formula for pi.

5 Conclusion

The generalized Viète’s-like formula for pi based on the arctangent function identity with argument $x = a_K/\sqrt{2 - a_{K-1}}$, where

$$a_K = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}},$$

is presented. This approach demonstrates high efficiency in computation due to a rapid convergence. Specifically, the computational test reveals that with only 169 summation terms the increment of integer $K$ just by one provides more than 100 additional decimal digits of the constant pi.

Acknowledgments

This work is supported by National Research Council Canada, Thoth Technology Inc. and York University.
References

[1] L.D. Servi, Nested square roots of 2, Amer. Math. Monthly, 110 (4) (2003) 326-330.
   http://dx.doi.org/10.2307/3647881

[2] A. Levin, A new class of infinite products generalizing Viéte’s product formula for \( \pi \), Ramanujan J. 10 (3) (2005) 305-324.
   http://dx.doi.org/10.1007/s11139-005-4852-z

[3] R. Kreminski, \( \pi \) to thousands of digits from Vieta’s formula, Math. Magazine, 81 (3) (2008) 201-207.
   http://www.jstor.org/stable/27643107

[4] W.B. Gearhart and H.S. Shultz, The function \( \frac{\sin(x)}{x} \), College Math. J. 21 (1990) 90-99.
   http://www.jstor.org/stable/2686748

[5] M. Kac, Statistical independence in probability, analysis and number theory, Carus Monographs, 12, Mathematical Association of America, Washington DC, 1959.

[6] S.M. Abrarov and B.M. Quine, Identities for the arctangent function by enhanced midpoint integration and the high-accuracy computation of pi, arXiv:1604.03752, 2016.

[7] S.M. Abrarov and B.M. Quine, A simple identity for derivatives of the arctangent function, arXiv:1605.02843, 2016.