DEWITT-SCHWINGER RENORMALIZATION OF $\langle \phi^2 \rangle$ IN $d$ DIMENSIONS

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A compact expression for the DeWitt-Schwinger renormalization terms suitable for use in even-dimensional space-times is derived. This formula should be useful for calculations of $\langle \phi^2(x) \rangle$ and $\langle T_{\mu\nu}(x) \rangle$ in even dimensions.

1. Introduction

A major impediment to using semi-classical general relativity is calculating the renormalized expectation value of the stress tensor. Properly renormalized values for $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ provide information on particle production and spontaneous symmetry breaking, and are also required to calculate backreaction. Since in general relativity energy density is itself a source of curvature, great care must be taken in deciding what may be dismissed as ‘unphysical’. Fortunately, there are several generally accepted renormalization schemes for curved space-times. Our purpose is to present a compact formula for the renormalization terms that may be applied to $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ calculations in arbitrary black hole space-times of even dimension.

2. Connection to Green’s Functions

Calculating $\langle T_{\mu\nu} \rangle$ for a general d-dimensional black hole space-time is difficult. For a scalar field, $T_{\mu\nu} \propto \phi^2$ and its derivatives, so we start with the simpler problem of calculating $\langle \phi^2 \rangle = \langle H\mid \phi^2 \mid H \rangle$, where $|H\rangle$ is the Hartle-Hawking vacuum. Note that $\langle \phi^2 \rangle$ is the coincidence limit of the two point function $\langle \phi^2 \rangle = \lim_{x \to x'} \langle \phi(x)\phi(x') \rangle$, and so may be expressed in terms of Green’s functions. In particular, the Feynman Green’s function is related to the time ordered propagator, $iG_F(x, x') = \langle T(\phi(x)\phi(x')) \rangle$. A Wick rotation allows us to work in Euclidean space where $G_F(t, x; t', x') = -iG_E(it, x; it', x')$. The Euclidean Green’s function, $G_E$, now obeys

$$\left(\Box_E - m^2 - \xi R(x)\right)G_E(x, x') = -|g(x)|^{-1/2}\delta^d(x - x'),$$

where $\Box_E$ is the Laplace-Beltrami operator in d-dimensional curved Euclidean space. To solve for $G_E$, start with the Euclidean metric for a static space-time in d dimensions with line element

$$ds^2 = f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2.$$

Here $\tau$ is the Euclidean time, $\tau = -it$, $r$ is a radial coordinate, and $\Omega$ represents a $(d - 2)$-dimensional angular space. The only restriction for this method is that the metric must be diagonal. If the scalar field is at temperature $T$, then the Green’s
where function \( \chi \) is periodic in \( \tau - \tau' \) with period \( T^{-1} \). Assuming a separation of variables, standard Green’s function techniques lead to the formal solution

\[
G_E(x, x') = \frac{\kappa}{2\pi} \sum_{n=-\infty}^{\infty} \sum_{\mu} e^{i\mu n} Y_{\ell}(\mu_j)(\Omega) Y_{\ell}(\mu_j)^* (\Omega') \chi_n(r, r'),
\]

where \( \kappa = 2\pi T \), and \( Y_{\ell}(\mu_j)(\Omega) \) are eigenfunctions of the Helmholtz equation obtained from the from the angular part of Eq. \( 1 \). For black holes with spherical topology these are equivalent to the set of hyperspherical harmonics. The radial function \( \chi_n(r, r') \) obeys a complicated differential equation obtained by putting the above expression into Eq. \( 1 \). This expression is divergent in the sum over \( n \).

### 3. DeWitt-Schwinger Renormalization

The \( \langle \phi^2 \rangle \) computation has been reduced to computing the coincidence limit of the Green’s function – a divergent quantity. To assign physical meaning to \( \langle \phi^2 \rangle \) it must be rendered finite via some renormalization process, and the standard approach is to renormalize the expression for \( G_E(x, x') \) via Christensen’s point splitting method applied to the DeWitt-Schwinger expansion of the propagator. In \( d \) dimensions, the adiabatic DeWitt-Schwinger expansion of the Euclidean propagator is

\[
G_{DS}^{E}(x, x') = \frac{\pi\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} a_k(x, x') \left(-\frac{\partial}{\partial m^2}\right)^k \left(-\frac{z}{2im^2}\right)^{1-d/2} H_{d/2-1}^{(2)}(z).
\]

Equation \( 4 \) introduces several new variables. Let \( s(x, x') \) be the geodesic distance between \( x \) and \( x' \), then define \( 2\sigma(x, x') = s^2(x, x') \) and \( z^2 = -2m^2\sigma(x, x') \). The \( a_k(x, x') \) are called DeWitt coefficients, and \( H_{\nu}^{(2)}(z) \) is a Hankel function of the second kind. Lastly, \( \Delta(x, x') = \sqrt{g(x)D(x, x')} \sqrt{g(x')} \) is the Van Vleck–Morette determinant, where \( g(x) = \det(g_{\mu\nu}(x)) \) and \( D(x, x') = \det(-\sigma_{\mu\nu}) \). Using the derivative properties of Bessel functions, noting that \( z = i|z| \) is purely imaginary in Euclidean space, and defining \( \nu = d/2 - 1 - k \), Eq. \( 4 \) can be written as

\[
G_{DS}^{E}(x, x') = \frac{-2i\Delta^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{\infty} a_k(x, x')(2m^2)^{\nu}|z|^{-\nu} \left[-1\right]^{\nu} I_{\nu}(|z|) + iK_{\nu}(|z|).
\]

The DeWitt-Schwinger expansion is a WKB expansion of the Euclidean propagator for a generic space-time when the point separation is small. For a particular space-time, this procedure does not give the correct results for the Green’s function with finite point separation because it ignores global space-time properties that determine the Green’s function – such as the effective potential around a black hole – but it should reproduce the same divergent terms in the coincidence limit. Therefore, if the divergent terms of the DeWitt-Schwinger expansion can be isolated, then subtracting these terms from \( G_E(x, x') \) will make it finite as \( x \to x' \). Since we are working in Euclideanized space the physical renormalization terms come from the real part of Eq. \( 4 \). The asymptotic behavior of \( K_{\nu}(|z|) \) as \( z \to 0 \) implies that only
terms with $\nu \geq 0$ contribute divergences in the coincidence limit, so
\[
G_{\text{div}}(x, x') = \frac{2^{1/2}}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} a_k(x, x')(2m^2)^\nu |z|^{-\nu} K_{\nu}(|z|). \tag{6}
\]

To renormalize $G_E$, Eq. (6) must be made commensurate with Eq. (3). We have shown that an integral representation of $K_{\nu}(z)$ for small $z$ and integer-valued $\nu$ is
\[
K_{\nu}(z) = \frac{(-1)^\nu \sqrt{\pi}}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty dt \cos(zt)(t^2 + 1)^{-\nu - 1/2}. \tag{7}
\]

Changing of variables and using the Plana sum formula to convert the integral to a sum, the renormalization terms for the $d$-dimensional space-time of Eq. (2) are
\[
G_{\text{div}}(x, x') = \frac{2}{(4\pi)^{d/2}} \sum_{k=0}^{k_d} \left\{ \frac{a_k}{\sqrt{\pi}} \left[ \Gamma(\nu + \frac{1}{2}) \sum_{n=1}^{\infty} \cos(\kappa zn) \left( (\kappa^2 n^2 + m^2 f)^{\nu - \frac{1}{2}} - \frac{1}{2} (\kappa^2 + m^2 f)^{\nu - \frac{1}{2}} \right) \right] - i \int_0^\infty \frac{dt}{e^{\nu t} - \frac{1}{2}} \left\{ \left( 1 + it \right)^{\nu - 1} - \left( 1 - it \right)^{\nu - 1} \right\} 
\]
\[
-(m^2 f)^{\nu - \frac{1}{2}} \sum_{j=0}^{\infty} \left( \frac{1}{2}, 1, \frac{3}{2}, -\frac{\kappa^2}{m^2 f} \right) \right\} + \frac{\nu}{a_k} E_{\nu} + \sum_{n=1}^{2n} \sum_{p=1}^{2n-1} \frac{2^{-n-1}(m^2)^{\nu - n} \Gamma(n)}{\Gamma(\nu - n + 1)} \left( \frac{\Delta_{p-j}^{1/2}}{\sigma_{\rho}} \right) \right\} \tag{8}
\]

for a scalar field at nonzero temperature $T > 0$. In this expression the $E_{\nu}$ are terms depending on the metric function $f$ and have been tabulated elsewhere\(^5\) while $a_k^n$ and $\Delta_{m}^{1/2}$ represent the $m^{th}$ term in an expansion in powers of $\sigma^\nu$. This expression generalizes previously known four-dimensional results\(^6\) The corresponding renormalization terms for a scalar field at zero temperature $T = 0$ are similarly found\(^6\)

4. Discussion

Semi-classical general relativity requires calculation of $\langle T_{\mu\nu}\rangle_{\text{ren}}$ in complicated – possibly higher dimensional – space-times. The first step in calculating $\langle T_{\mu\nu}\rangle_{\text{ren}}$ for a scalar field is calculating $\langle \phi^2\rangle_{\text{ren}}$. We have presented a compact expression for the renormalization terms for $\langle \phi^2\rangle$ in even dimensional, static, black hole space-times.

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