The set of hyperbolic equilibria and of invertible zeros on the unit ball is computable

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Abstract

In this note, we construct an algorithm that, on input of a description of a structurally stable planar dynamical flow \( f \) defined on the closed unit disk, outputs the exact number of the (hyperbolic) equilibrium points and their locations with arbitrary accuracy. By arbitrary accuracy it is meant that any accuracy required by the input can be achieved. The algorithm can be further extended to a root-finding algorithm that computes the exact number of zeros as well the location of each zero of a continuously differentiable function \( f \) defined on the closed unit ball of \( \mathbb{R}^d \), provided that the Jacobian of \( f \) is invertible at each zero of \( f \); moreover, the computation is uniform in \( f \).

1 Introduction

Consider the autonomous system of ordinary differential equations

\[
\dot{x} = f(x)
\]

for \( f \in C^1(K) \), where \( C^1(K) \) is the set of all continuously differentiable functions in an open subset of \( \mathbb{R}^d \) containing \( K \) with values in \( \mathbb{R}^d \), and \( K = \overline{B}(0, 1) \) is the closed unit ball centered at the origin in \( \mathbb{R}^d \). An equilibrium (or equilibrium point) of \( \Box \) is a solution that does not change with time; that is, the system \( \Box \) has an equilibrium solution \( x(t) = x_0 \) if and only if \( f(x_0) = 0 \), where \( x_0 \in K \) and \( 0 \in \mathbb{R}^d \).

An equilibrium is the simplest possible solution of a dynamical system. Nevertheless, it is fundamentally important because the equilibria form a basis for analyzing more complicated behavior. Yet, finding an equilibrium or, equivalently, solving the equation \( f(x) = 0 \) is easy only in a few special cases and, in
general, the equilibria of (1) cannot be located exactly but only to be approximated by numerical root-finding algorithms. There are numerous root-finding algorithms such as Newton’s method, Bisection method, Secant method, and Inverse Interpolation method, to mention just a few. A numerical algorithm is usually efficient when it is applied to some special classes of elementary functions such as polynomials or augmented with additional information such as good initial guesses or stronger smoothness of the function.

There are also topological algorithms for computing zeros (or, equivalently, fixed points) of a $C^k$ function defined on a compact subset of $\mathbb{R}^d$ under certain conditions (see, for example, [Mil02], [Col08], [BRMP19, Proposition 4.7], [Nen19, Section 5.1] and references therein). The topological algorithms usually compute the set of all zeros as a subset of $\mathbb{R}^d$ and, for this reason, are often unable to exhibit the position of each zero and to provide the exact number of the zeros, if the set of zeros is finite. In other words, a topological algorithm can provide arbitrarily good adumbrations of the set of zeros but may not be able to do so for each zero individually. On the other hand, from the perspective of dynamical systems, what matters is the nature and the location of each individual equilibrium; for example, the Hartman-Grobman theorem, an important theorem in dynamical systems, shows that the nonlinear vector field $f(x)$ is conjugate to its linearization $Df(x_0)x$ in a neighborhood of a hyperbolic equilibrium $x_0$, where the neighborhood does not contain any other equilibrium point (see e.g. [Per01]).

In this note, we construct an algorithm that computes the exact number and the locations of the equilibrium points of a structurally stable planar dynamical system, for that is our motivation. Intuitively, a dynamical system is structurally stable if the qualitative behavior of its trajectories persists under small perturbations. More formally (see e.g. [Per01, Definition 1 in p. 317]), the system (1) is structurally stable if there is some $\varepsilon > 0$ such that for any $g \in C^1(K)$, if $\|f - g\|_{1, \infty} < \varepsilon$ on $K$, then there exists a homeomorphism $h : K \to K$ such that $h$ maps every trajectory of (1) onto a trajectory of $\dot{y} = g(y)$ preserving time orientation, where $\|f\|_{1, \infty} = \max_{x \in K} \|f(x)\| + \max_{x \in K} \|Df(x)\|$.

If the system (1) is structurally stable on $K$, then the number of its equilibria won’t change when $f$ is slightly perturbed. Moreover, the location of every equilibrium depends continuously on the perturbation (see e.g. [Per01, Theorems 1 and 2 in p. 321]). The equilibrium points which resist the small perturbations are called hyperbolic equilibria. More precisely, an equilibrium $x_0$ of (1) is said to be hyperbolic if all eigenvalues of the Jacobian matrix of $f$ at $x_0$ have non-zero real parts. This implies that the Jacobian of $f$ is invertible at each hyperbolic equilibrium. Hyperbolic equilibria are as robust as expected: small perturbations on $f$ do not alter the topological character of the phase portrait near a hyperbolic equilibrium, but only distort the trajectories near it by a
small amount according to the Hartman-Grobman theorem. Furthermore, if \( K \) is a structurally stable planar dynamical system defined on \( K \) (with \( d = 2 \)), then the system has only finitely many equilibrium points and all of them are hyperbolic according to the celebrated Peixoto theorem \cite{Pei59}. The Peixoto theorem also shows that being structurally stable is a generic property in the sense that the set of all structurally stable planar vector fields is an open set of \( C^1(K) \) and it is dense in \( C^1(K) \). Peixoto’s theorem however does not contain quantitative information such as the exact number of the equilibrium points or their locations inside the open unit disk. Our algorithm, on the other hand, will supply such quantitative information. The following theorem is the main result of the note; its proof is given in Section 3.

Let \( K_2 \) be the closed unit ball when \( d = 2 \), i.e. \( K_2 = B(0, 1) \subseteq \mathbb{R}^2 \). Take \( S(K_2) = \{ f \in C^1(K_2) : \text{the system defined by } f \text{ is structurally stable} \} \) and let \( \text{Zero}(f) \) be the set of zeros of \( f \) in \( K_2 \); i.e., the set of the equilibrium points of \( f \). We recall that a \( C^1 \)-name of a real-valued function \( f \) defined on a compact \( \Omega \subseteq \mathbb{R} \) is a sequence \( \{ P_k \} \) of polynomials with rational coefficients satisfying \( ||f - P_k||_{1, \infty} < 2^{-k} \) on \( \Omega \); \( f \) is computable if there is a Turing machine that outputs the coefficients of \( P_k \) on input \( k \). A name for a vector-valued function defined on \( \Omega \subseteq \mathbb{R}^d \) consists of the names of its component (real-valued) functions. A vector-valued function is computable if each of its components is computable. (We use \( K \) to denote the closed unit ball in \( \mathbb{R}^d \) for different dimension \( d \); the dimension may not be specified if the context is clear.)

**Theorem 1.1 (Main result)** The operator that assigns to each \( f \in S(K_2) \) the set \( \text{Zero}(f) \) and the exact size of \( \text{Zero}(f) \) is computable. More precisely, there is an algorithm \( \mathcal{E} \), when given any \( C^1 \)-name of \( f \in S(K_2) \) and \( n_0 \in \mathbb{N} \) as input, \( \mathcal{E} \) produces an integer \( n \geq n_0 \), a nonnegative integer \( \#_0(f) \), and a list \( C \) of finitely many squares with rational vertices as output, such that

1. \( \#_0(f) \) is the exact number of the equilibrium points of \( f \);
2. each square in \( C \) contains exactly one equilibrium point and has side of length \( 1/n \). Furthermore, any zero of \( f \) (in \( K_2 \)) is contained in a square of \( C \). In particular, this implies that \( d_H(\text{Zero}(f), \cup C) \leq 1/n \), where \( d_H(\text{Zero}(f), \cup C) \) is the Hausdorff distance between \( \text{Zero}(f) \) and \( \cup C \).

We note that each square in \( C \) can be viewed as a pixel. This result is restricted to planar systems, where the dimension is \( d = 2 \), which is the usual case when considering structurally stable systems following Peixoto’s theorem. Nevertheless, Theorem 1.1 can be generalized to a zero-finding algorithm for functions in \( Z(K) \), which works in all dimensions \( d \geq 1 \) (i.e. for all \( C^1 \) functions defined on \( K = B(0, 1) \subseteq \mathbb{R}^d \), \( d \geq 1 \)) and is more general in the sense that it can also compute non-hyperbolic zeros \( x \) of \( f \) for which \( Df(x) \) might have eigenvalues with zero real part, as long as \( \det Df(x) \neq 0 \) (hyperbolic points satisfy this condition), where

\[
Z(K) = \{ f \in C^1(K) : \det Df(\alpha) \neq 0 \text{ and } \alpha \notin \partial K \text{ whenever } \alpha \text{ is a zero of } f \},
\]
\( \partial K \) denotes the boundary of \( K \), and \( K \subseteq \mathbb{R}^d \) is the closed unit ball, \( d \geq 1 \). The formal description of the zero-finding algorithm and its proof are presented in Section 4. We note that if (1) is structurally stable, then there is no equilibrium on \( \partial K \).

As we mentioned before, the continuity of \( f \) alone is not strong enough for being able to compute the zeros of \( f \). Some additional conditions are needed according to the following well-known facts in computable analysis (see, for example, [BHW08]): (1) there is a computable function \( f : [0, 1] \to \mathbb{R} \) having infinitely many zeros but none is computable; (2) the zero set of a function \( f \) is not uniformly computable in \( f \); i.e. there is no algorithm that outputs an approximation of the zero set of \( f \) with error less than \( 1/n \) (or \( 2^{-n} \)) in some measurements when taking as input a description of \( f \) and a natural number \( n \); and (3) given that \( f \) has only finitely many zeros, the number of zeros of \( f \) is not uniformly computable in \( f \). In fact, (2) and (3) are true even for families of elementary functions.

2 Definitions

Let \( \| \cdot \|_2 \) and \( \| \cdot \|_\infty \) denote the Euclidean norm and the maximum norm of \( \mathbb{R}^d \), respectively; let \( K \) be the closure of the open unit ball \( B(0, 1) \) of \( \mathbb{R}^d \) in \( \| \cdot \|_\infty \) norm; let \( C^1(K) \) be the set of continuously differentiable functions defined on some open subset of \( \mathbb{R}^d \) containing \( K \) with values in \( \mathbb{R}^d \); and let \( Df(x) \) denote the (Fréchet) derivative of \( f \) at \( x \) for \( x \in K \), which is a linear transformation from \( \mathbb{R}^d \) to \( \mathbb{R}^d \). Since \( f \) is \( C^1 \), it follows that, for each \( x \in K \), the linear transformation \( Df(x) \) is continuous (thus bounded) and is the same as the linear transformation induced by the \( d \times d \) Jacobian matrix of the partial derivatives of \( f \) at \( x \); moreover, \( Df(x) \) is continuous in \( x \).

Let \( A = (a_{ij}) \) be a \( d \times d \) square matrix (of real entries). Then the following matrix norms are all equivalent: \( \| A \|_2 = \sup_{y \in \mathbb{R}^d, \| y \|_2 = 1} \| Ay \|_2 = \sqrt{\rho(AA^T)} \), \( \| A \|_\infty = \sup_{y \in \mathbb{R}^d, \| y \|_\infty = 1} \| Ay \|_\infty = \max_i \sum_{j=1}^d |a_{ij}| \), and \( \| A \|_{HS} = \sqrt{\sum_{i,j=1}^d a_{ij}^2} \), where \( \rho(A) = \max_{1 \leq j \leq d} |\lambda_j| \), \( \lambda_j \) is an eigenvalue of \( A \). The first two norms are called operator norms while the third is called the Hilbert-Schmidt norm.

Convention Since the norms \( \| \cdot \|_2 \) and \( \| \cdot \|_\infty \) imposed on \( \mathbb{R}^d \) are equivalent, and the three norms defined on the square \( d \times d \) matrices \( A \) are also equivalent, in what follows we use \( \| \cdot \| \) to denote either of the five norms when there is no confusion in the context.

Next we recall several notions from computable analysis. For more details and rigorous definitions the reader is referred to [Wei00] and references therein. Let \( X \) be a set and let \( \delta : \Sigma^\omega \to X \) be a surjective map, where \( \Sigma \) is a finite set (the alphabet) and \( \Sigma^\omega = \{ w | w : \mathbb{N} \to \Sigma \} \) is the set of all one-way infinite sequences over \( \Sigma \). In this case, \( \delta \) is called a representation of the set \( X \). If \( \delta(w) = x \), then \( w \) is called a (\( \delta \)-)name of \( x \); an element \( x \in X \) is called (\( \delta \)-)computable if it has a computable name.
3 Proof of the main result

Before giving a precise description of the algorithm $\mathcal{E}$ and proving Theorem \ref{thm:main_result}, we present a number of lemmas and auxiliary results. In what follows we assume that $f \in \mathcal{S}(K_2)$.

**Lemma 3.1** Let $M = \max_{x \in K_2} \{|Df(x)|, |\det(Df(x))|, 1\}$, where $\| \cdot \|$ is an operator norm. Then $M$ is computable from $f$.

**Proof.** Let $M = \max_{x \in K_2} \{|Df(x)|, |\det(Df(x))|, 1\}$, where $\| \cdot \|$ is an operator norm. Then it follows from the discussion in the first paragraph of the previous section that $Df(x)$ is a bounded linear operator for each $x \in K_2$ and is continuous in $x$. Moreover, since a $C^1$-name of $f$ is given as a part of the input to the algorithm to be constructed, the Jacobian matrix of the partial derivatives of $f$ is computable uniformly in $x$ on the compact set $K_2$; thus the map $\|Df\| : K_2 \to \mathbb{R}^+$, $x \to \|Df(x)\|$, is computable from $x$ and $f$ and, as a consequence, $M$ is computable from $f$ (and $K_2$). \hfill $\blacksquare$

**Lemma 3.2** Let $D$ be a small disk of $\mathbb{R}^2$ containing the origin such that $x+h$ is in the domain of $f$ for every $x \in K_2$ and $h \in D$. Then there is a decreasing sequence $\{r_m\}_{m \geq 1}$, computable from $f$, of positive numbers $r_m \leq \min\{1, \text{the radius of } D\}$ such that whenever $\|h\| \leq r_m$, one has

$$\|f(x+h) - f(x) - (Df(x))(h)\| \leq 2^{-m-1}\|h\| \quad (2)$$

**Proof.** Let $g(t) = f(x + th) - f(x) - tDf(x)h$, where $t \in [0, 1]$. Then $g'(t) = Df(x + th)h - Df(x)h = (Df(x + th) - Df(x))h$ and $g(0) = 0$. Subsequently,

$$\|f(x+h) - f(x) - Df(x)h\| = \|g(1)\| \leq \sup_{t \in [0, 1]} \|g(t) - g(t)(1 - 0)\| \leq \sup_{t \in [0, 1]} \|Df(x + th) - Df(x)\| \cdot \|h\|. $$

(the first inequality is obtained by applying the mean value theorem componentwise to $g$). Since $Df$ is computable and admits a computable modulus of continuity, there is a sequence $\{r_m\}_{m \geq 1}$, computable from $f$, of positive numbers $r_m \leq \min\{1, \text{the radius of } D\}$ such that $\|Df(x + th) - Df(x)\| \leq 2^{-m-1}$ whenever $\|h\| \leq r_m$ and $t \in [0, 1]$. Thus $\sup_{t \in [0, 1]} \|Df(x + th) - Df(x)\| \leq 2^{-m-1}$ whenever $\|h\| \leq r_m$, which implies that for all $x \in K_2$ and all $\|h\| \leq r_m$, $\|h\| \neq 0$. Without loss of generality we may also assume that $r_m \geq r_{m+1}$. This concludes the proof of the lemma. \hfill $\blacksquare$

**Lemma 3.3** Let $K^m_{Df} = \{x \in K_2 : 2^{-m} \leq \min\{|Df(x)|, |\det(Df(x))|\} \leq M\}$, where $m \in \mathbb{N}$ is arbitrary and $M$ is given in Lemma 3.1. Let $\{r_m\}_{m \geq 1}$ be the sequence defined in Lemma 3.2. Then for each $x \in K^m_{Df}$ and every $\|h\| \leq r_m$, $\|h\| \neq 0$,

$$\|f(x+h) - f(x)\| \geq 2^{-m-1}\|h\| \quad (3)$$
Proof. The estimate (3) is obtained from the following calculation: since \( x \in K_{Df}^m \), it follows that \( \|Df(x)\| \geq 2^{-m} \), which implies that
\[
2^{-m}\|h\| \leq \|(Df(x))(h)\| \\
= \|f(x+h) - f(x) - (Df(x))(h) + f(x) - f(x+h)\| \\
\leq \|f(x+h) - f(x) - (Df(x))(h)\| + \|f(x) - f(x+h)\| \\
\leq 2^{-m-1}\|h\| + \|f(x) - f(x+h)\|
\]
Consequently
\[
\|f(x) - f(x+h)\| \geq 2^{-m}\|h\| - 2^{-m-1}\|h\| \\
= 2^{-m-1}\|h\|
\]

We note that if \( \det Df(x) \neq 0 \) for some \( x \in K_2 \), then \( \|Df(x)\|_{HS} \neq 0 \) which would then imply that \( \|Df(x)\|_\infty \neq 0 \). Therefore, if \( Df(x) \) is invertible at some \( x \in K_2 \), then \( \det Df(x) \neq 0 \) and \( \|Df(x)\| \neq 0 \).

The lemma below is also needed for the construction of the algorithm \( \mathcal{E} \). Its proof can be found in [Rud06].

Lemma 3.4 Let \( g : A \to \mathbb{R}^d \) be a \( C^1 \) function defined on an open subset of \( \mathbb{R}^d \) containing the closed ball \( A = \overline{B}(x_0, r) \) with \( r > 0 \); let \( M = \max_{x \in A} \|Dg(x)\|_\infty \). Then for all \( x, y \in A \) we have \( \|g(y) - g(x)\|_\infty \leq M \|y - x\|_\infty \).

Some notations are called upon before presenting the algorithm \( \mathcal{E} \). Let \( C_n \) be a set of small squares of side length \( 1/n \), called \( n \)-squares which covers exactly \( K_2 \). More precisely, let \( G_n = ((\mathbb{Z} + 1/2)/n)^2 \cap K_2 \), where \( (\mathbb{Z} + 1/2)/n = \{ z + 1/2/n \in \mathbb{R} : z \in \mathbb{Z} \} = \{ \ldots, -3/2n, -1/2n, 1/2n, 3/2n, \ldots \} \); let \( C_n = \{ \overline{B}(x, 1/(2n)) : x \in G_n \} \), where \( \overline{B}(x, 1/(2n)) = \{ y \in \mathbb{R}^d : \|y - x\|_\infty \leq 1/(2n) \} \).

We now define the algorithm \( \mathcal{E} \). We will explain why the work algorithm works after presentation of the algorithm. Note that when we can compute a quantity such as \( d(0, f(s)) \) in Step (3-1), we can compute it with accuracy \( 2^{-(n+1)} \), obtaining an approximation \( r \). Therefore, if \( r \leq 3/2 \cdot 2^{-n} \), then we conclude that \( d(0, f(s)) \leq 2^{-(n-1)} \). Otherwise, we have for sure that \( d(0, f(s)) > 2^{-n} \). We write this procedure as “decide whether \( d(0, f(s)) > 2^{-n} \) or \( d(0, f(s)) \leq 2^{-(n-1)} \).

On input of (a \( C^1 \)-name of) \( f \) and \( n_0 \) \((n_0 \in \mathbb{N} \) defines the accuracy \( 1/n_0 \) of the result), the algorithm \( \mathcal{E} \) runs as follows:

1. Set \( n = 3n_0 \).
2. For each \( n \)-square \( s \in C_n \), set result\((s) = \) Undefined and counter\((s) = 0 \).
3. For each \( n \)-square \( s \in C_n \), do the following:
   
   (3-1) \( \mathcal{E} \) computes \( d(0, f(s)) \) until it decides whether \( d(0, f(s)) > 2^{-n} \) or \( d(0, f(s)) \leq 2^{-(n-1)} \). If \( d(0, f(s)) > 2^{-n} \), then \( \mathcal{E} \) sets result\((s) = \) False and go to step (3-3). Otherwise, compute \( d(s, \partial K_2) \), increment \( n \), and go to step 2 if \( d(s, \partial K_2) < 5/n \).
(3-2) If $d(s, \partial K_2) \geq \frac{1}{4}$, then let $\mathcal{N}(s) = \{x \in K_2 : d(x, s) \leq 1/n\} \subseteq K_2$ i.e. $\mathcal{N}(s)$ consists of $s$ and the $n$-squares adjacent to $s$ and let $\mathcal{M}(s) = \{x \in K_2 : d(x, s) \leq 3/n\} \subseteq K_2$. Clearly $s \subseteq \mathcal{N}(s) \subseteq \mathcal{M}(s) \subseteq K_2$. $\mathcal{E}$ now computes $\min_{x \in \mathcal{M}(s)}||Df(x)||, |\det(Df(x))|)$ until it decides whether $\min_{x \in \mathcal{M}(s)}||Df(x)||, |\det(Df(x))| \leq 2^{-n+1}$ or $\min_{x \in \mathcal{M}(s)}||Df(x)||, |\det(Df(x))| > 2^{-n}$. In the case the condition $\min_{x \in \mathcal{M}(s)}||Df(x)||, |\det(Df(x))| \leq 2^{-n+1}$ holds, then increment $n$ and go to step 2.

(3-3) Repeat step (3-1) with a new $n$-square from $C_n$ or proceed to step 4 if no $n$-square is left in $C_n$.

4. For each $n$-square $s \in C_n$, do the following:

(4-1) If result($s$) = False, then go to step (4-3).

(4-2) Pick $\tilde{n} \geq n$ such that for every $\tilde{n}$-square $s_j \subseteq s$, it holds true that $\mathcal{M}(s_j) \subset B(x_{s_j}, r_n)$, where $x_{s_j}$ is the center of $s_j$ and $B(x_{s_j}, r_n)$ is the open ball centered at $x_{s_j}$ with radius $r_n$, where $r_n$ is given by Lemma 3.2. Note that it follows from Lemma 3.3 and 3.4 that $f$ is injective on $\mathcal{M}(s_j)$. Let $s_j$, $1 \leq j \leq J(s)$, be $n$-squares such that $s = \bigcup_{j=1}^{J(s)} s_j$ and any two distinct squares are either disjoint or intersect only in their boundaries. Note that this condition holds only when $\tilde{n} = jn$ for some $j \in \mathbb{N}$. We assume without loss of generality that this requirement is satisfied. Set $l = \tilde{n}$. For each $s_j$, $1 \leq j \leq J(s)$, do the following:

(4-2-1) $\mathcal{E}$ computes $d(0, f(s_j))$. Set result($j, s$) = False if $d(0, f(s_j)) > 2^{-l}$; or go to (4-2-2) if $d(0, f(s_j)) \leq 2^{-l+1}$.

(4-2-2) $\mathcal{E}$ computes rational points $x_1, x_2, \ldots, x_{e(j,l)}$ in the interior of $s_j$ such that $s_j \subseteq \bigcup_{i=1}^{e(j,l)} B(x_i, 2^{-l-1})$; afterwards, $\mathcal{E}$ computes the numbers $\theta_i = d(x_i, \partial N(s_j)), 1 \leq i \leq e(j,l)$. We note that $\theta_i \leq r_n$ because $N(s_j) \subset \mathcal{M}(s_j) \subset B(x_{s_j}, r_n)$ and $\theta_i \geq 1/n$ because $x_i \in s_j$ and $d(x_i, \partial N(s_j)) \geq 1/n$. Let $B_i = B(x_i, \theta_i), 1 \leq i \leq e(j,l)$. Then $B_i \subset N(s_j)$. Next, $\mathcal{E}$ computes $d_i = \min_{x \in \partial B_i} ||f(x) - f(x_i)||$ for each $1 \leq i \leq e(j,l)$. Since $\theta_i \leq r_n$ and $\min_{x \in \mathcal{M}(s_j)}||Df(x)||, |\det(Df(x))| > 2^{-n}$, it follows from (3.4) that $||f(x) - f(x_i)|| \geq 2^{-n-1}\theta_i$ for every $x \in \partial B_i$; thus $d_i \geq 2^{-n-1}\theta_i \geq 2^{-n-1}/n$ for all $1 \leq i \leq e(j,l)$.

(4-2-3) Let $D_i = B(f(x_i), d_i/2)$.

(a) If $0 \notin \bigcup_{i=1}^{e(j,l)} D_i$ and $s_j$ is adjacent to an $\tilde{n}$-square $s_k^*$ (note that this $\tilde{n}$-square may belong to an $n$-square $s' \neq s$ if $s_j$ is adjacent to the boundary of $s$) for which result($k, s^*$) is defined and equal to True, then $\mathcal{E}$ sets result($j, s$) = False and moves to the next $\tilde{n}$-square $s_{j+1}$.

(b) If $0 \notin \bigcup_{i=1}^{e(j,l)} D_i$ and $s_j$ is not adjacent to an $\tilde{n}$-square $s_k^*$ for which result($k, s^*$) is defined and equal to True, then $\mathcal{E}$ sets result($j, s$) = True and moves to the next $\tilde{n}$-square $s_{j+1}$. 

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(c) If \( 0 \not\in \bigcup_{i=1}^{r(j)} D_i \), then \( \mathcal{E} \) increments \( l \) by 1 and returns to (4-2-1).

(4-2-4) Repeat step (4-2-1) with the next \( n\)-square \( s_{j+1} \) or proceed to step (4-2-5) if no more \( n\)-square is left in \( \{s_1, s_2, \ldots, s_J(s)\} \).

(4-2-5) Set result\((s) = False \) if result\((j, s) = False \) for all \( 1 \leq j \leq J(s) \). Otherwise, set result\((s) = True \); set \( \widetilde{N}(s) = \bigcup\{N(s_j) : \) result\((j, s) = True\}; and set counter\((s) = \) the cardinality of the set \( \{j : \) result\((j, s) = True, 1 \leq j \leq J(s)\} \).

(4-3) Repeat step (4-1) with a new \( n\)-square from \( C_n \) or proceed to step 5 if there is no more \( n\)-square left in \( C_n \).

5. Let \( C = \emptyset \).

6. For all \( s \in C_n \) do:

   (a) If result\((s) = True \), then \( C = C \cup \widetilde{N}(s) \).

   (b) Set \( \#_0(f) = \sum_{s \in C_n} \) counter\((s) \).

7. Output \( C \) and \( \#_0(f) \).

**Proof of Theorem 1.1**

We now show that the algorithm \( \mathcal{E} \) works. In step 1, the accuracy is increased because \( \mathcal{E} \) may not be able to tell whether a zero is in an \( n\)-square \( s \) but, nevertheless, it is capable of determining if a zero is in a \((1/n)\)-neighborhood of \( s \). Therefore, in order to get the requested accuracy bound of \( 1/n_0 \), we will have to use smaller squares whose side lengths are at most diameter \( 1/(3n_0) \). This is why \( \mathcal{E} \) starts at \( n = 3n_0 \).

After step 1, \( \mathcal{E} \) studies whether or not \( s \) contains a zero of \( f \) for every \( s \in C_n \). The result of this investigation is saved into result\((s) \). This is why result\((s) \) is initially set to be undefined in step 2.

In step (3-1) \( \mathcal{E} \) performs a first test (with accuracy \( 2^{-n} \)) to check whether \( f(s) \) contains a zero. Clearly, a False output at step (3-1) indicates that \( s \) contains no zeros of \( f \). If this is the case \( \mathcal{E} \) proceeds to another \( n\)-square. If we instead obtain \( d(0, f(s)) \leq 2^{-(n-1)} \), then \( s \) may or may not have a zero, and so further investigations are needed. Meanwhile \( \mathcal{E} \) runs a test to determine whether \( d(s, \partial K_2) < 5/n \) or \( d(s, \partial K_2) \geq 4/n \) for the purpose of getting rid of those \( n \) squares which are too close to the boundary \( \partial K_2 \) of \( K_2 \) because the further investigations are to be performed on some \((3/n)\)-neighborhood of \( s \). In the case that \( d(s, \partial K_2) < 5/n \), \( \mathcal{E} \) increments \( n \) and then repeats step (3-1). It follows from Lemma 4.1 that \( f \) has only finitely many zeros and none is on \( \partial K_2 \); therefore, when \( n \) is large enough, \( \mathcal{E} \) will output \( d(0, f(s)) \geq 2^{-n+1} \) whenever \( d(s, \partial K_2) < 5/n \) for every \( s \in C_n \). In other words, \( \mathcal{E} \) will not go into a loop; it will either output result\((s) = False \) or it will move on to the next step for sufficiently large \( n \); for example, for \( n \) that satisfies the condition: \( 2^{-n+1} \leq \min\{\|f(x)\| : x \in K_2 \text{ and } d(x, \partial K_2) \leq \sigma/2\} \), where \( \sigma = \min\{d(x, \partial K_2) : f(x) = 0\} \).
In step (3-2) $\mathcal{E}$ tests whether the jacobian $Df$ is invertible on $\mathcal{M}(s)$ for those $s \in \mathcal{C}_n$ satisfying $d(s, \partial K_2) \geq 4/n$; i.e., those $n$-squares whose final status result(s) are in need to be updated from Undefined to True or False. In case that $\min_{x \in \mathcal{M}(s)} |\det(Df(x))| > 2^{-n}$, the inverse function theorem can be applied to determine whether or not $s$ contains a zero of $f$. And so we wish to put each $n$-square $s$ satisfying $d(s, \partial K_2) \geq 4/n$ into one of the two groups: either $d(0, f(s)) > 2^{-n}$ or $\min_{x \in \mathcal{M}(s)} \min\{|Df(x)|, |\det(Df(x))|\} > 2^{-n}$. This is achievable because, once again due to Lemma 4.1, $f$ has only finitely many zeros, say $\xi_1, \ldots, \xi_k$, with the property that $|\det(Df(\xi_i))| > 0$ and $|\det(Df(\xi_i))| > 0$ as well (recall that $|\det(Df(x))|$ is equivalent to $|\det(Df(x)|_{HS}$). By continuity of $Df$, there exist $\rho_1, \ldots, \rho_k > 0$ such that $\min\{|Df(y)|, |\det(Df(y))|\} > 0$ for all $y$ in the closed ball $\overline{B}(\xi_i, \rho_i)$, $1 \leq i \leq k$. Then $\min\{|Df(y)|, |\det(Df(y))|\} > 0$ for all $y \in \overline{B}(\xi_i, \rho)$, $1 \leq i \leq k$, where $0 < \rho = \min_{1 \leq i \leq k} \rho_i$. Thus when $n$ is large enough meeting the conditions $4/n \leq \rho$ and

$$\min_{y \in \bigcup_{i=1}^{k} \overline{B}(\xi_i, \rho)} \min\{|Df(y)|, |\det(Df(y))|\} \geq 2^{-n+1},$$

the test of step (3-2) is guaranteed to succeed for all squares $s$ which reach step (3-2).

In step 4 we fix the first $n$ which successfully led us to this step. We observe that if result$(j, s) = True$, then $\mathcal{N}(s_j)$ contains and only one equilibrium because for every $1 \leq i \leq e(j, s), B_i \subset \mathcal{N}(s_j) \subset \mathcal{M}(s_j)$, $f : B_i \bigcap f^{-1}(D_i) \rightarrow D_i$ is a homeomorphism (see, for example, [5,8]), and $f$ is injective on $\mathcal{M}(s_j)$. However there remains a potential problem of a zero of $f$ being counted multiple times. This may happen when $\mathcal{N}(s_j)$ contains a zero of $f$ and $s_j$ is adjacent to a portion of a common side shared by $s$ and another $n$-square $\hat{s}$. If $\mathcal{E}$ acts on $\hat{s}$ before it picks up $s$, then the zero of $f$ contained in $\mathcal{N}(s_j)$ may already be detected and counted by $\mathcal{E}$ at the time when $\mathcal{E}$ was working on $\hat{s}$. The potential multiple-counting may also happen when the interior of $\mathcal{N}(s_j)$ intersects the interior of $\mathcal{N}(s_{j'})$ for some $\hat{n}$-square $s_{j'}$ (contained in the $n$-squares) that is adjacent to $s_j$. If the intersection contains an equilibrium and if $\mathcal{E}$ acts on $s_{j'}$ prior to picking up $s_j$, then the equilibrium would have been counted by $\mathcal{E}$ at the time while working with $s_{j'}$, if not earlier. The step (4-2-3) is a preventive mechanism designed to ensure that an equilibrium is counted exactly once. It is also true that, for each $n$-square $s$ satisfying $\min_{x \in \mathcal{M}(s)} |Df(x)| > 2^{-n}$, $\mathcal{E}$ will halt and produce either result$(s) = False$ or result$(s) = True$. To see this let us fix an $s_j$ with $1 \leq j \leq J(s)$. If $s_j$ does not contain an equilibrium, then the inequality $d(0, f(s_j)) > 2^{-l}$ would appear for $l$ large enough. On the other hand, assume that $s_j$ contains an equilibrium $x_0$. Let us pick some $l$ such that $2^{-l-1} < 2^{-n-2}/\hat{n}$. Then there is some $x_i$ in the interior of $s_j$, $1 \leq i \leq e(j, l)$, such that $|x_i - x_0| \leq \frac{2^{-l-1}}{M}$. It now follows from Lemma 3.4 that $|f(x_0) - f(x_i)| \leq 4M|x_0 - x_i| \leq 2^{-l-1}$; in other words, $0 \in B(f(x_i), 2^{-l-1}) \subseteq B(f(x_i), d_i/2) = D_i$ (recall that $d_i \geq 2^{-n-1} \theta_1 \geq 2^{-n-1}/\hat{n}$ for all $1 \leq i \leq e(j, l)$).
4 Computing invertible zeros

It is shown in [Wei00, Theorem 6.3.2] that the multi-valued function \( f \in C[0,1] \to \{(f,x) : f(x) = 0\} \) is not continuous and thus not computable. This raises the question: are there topological/regularity conditions which we can impose on the family of functions to ensure the uniform computability of the zero sets and the cardinalities of the zero sets. Here the answer is yes. The problem is to find the right conditions.

In this section, we consider the compact set \( K = \overline{B}(0,1) \subseteq \mathbb{R}^d, \ d \geq 1 \). The algorithm \( E \) suggests that the following conditions can be imposed on the family of continuously differentiable functions which will ensure the uniform computability of the zero sets and their sizes: let

\[ Z(K) = \{ f \in C^1(K) : \det Df(\alpha) \neq 0 \text{ and } \alpha \not\in \partial K \text{ whenever } \alpha \text{ is a zero of } f \} \]

where \( \partial K \) denotes the boundary of \( K \). We note that if \( [1] \) is structurally stable, then there is no equilibrium on \( \partial K \). And if we dismiss the condition that \( \alpha \not\in \partial K \), the uniform computability may no longer be guaranteed. For example, let \( A = \{ f_a(x) : a \in \mathbb{R} \} \), where \( f_a(x) = x - a, \ x \in K = [0,1] \). Then \( f_a \in C^1(K) \) and \( f'_a = 1 \) on \( K \) for all \( a \); but the operator \( A \to \{0,1\}, f_a \mapsto \# \text{ of zeros of } f_a \), is not uniformly computable because it is not even continuous. The problem is caused by two “bad” functions \( f_0 \) and \( f_1 \) - the zero of \( f_0 \) and the zero of \( f_1 \) lie on \( \partial K \). If we get rid of these two functions, then we can compute the number of zeros of \( f_a \) uniformly on \( A \setminus \{f_0,f_1\} \).

The algorithm \( E \) described in the previous section can be adapted in a straightforward manner to \( K \subseteq \mathbb{R}^d \) for \( d \geq 1 \) and be applied, together with the lemma below, to compute the exact number and the locations of the zeros of \( f \in Z(K) \), uniformly on \( Z(K) \), thus showing Theorem 4.2.

**Lemma 4.1** Let \( f \in Z(K) \). Then \( f \) has at most a finite number of zeros in \( K \).

**Proof.** Assume otherwise that \( f \) has infinitely many zeros \( \alpha_n, \ n \in \mathbb{N} \), in \( K \). Then since \( K \) is compact, it follows that \( \{\alpha_n\} \) has a convergent subsequence, say \( \{\alpha_{n_k}\} \), that converges to \( \alpha \in K \), which results in the following limit

\[ 0 = \lim_{k \to \infty} f(\alpha_{n_k}) = f(\lim_{k \to \infty} \alpha_{n_k}) = f(\alpha). \]

Note that the limit can be taken into \( f \) is due to the fact that \( f \) is continuous on a compact set. Thus \( \alpha \) is a zero of \( f \). Moreover, since \( f \in Z(K) \), it follows that \( Df(\alpha) \) is invertible, which is equivalent to the condition that the Jacobian determinant of the matrix \( Df(\alpha) \) is nonzero. Therefore \( f \) itself is invertible in a neighborhood of \( \alpha \). But this is a contradiction because in any neighborhood \( B \) of \( \alpha \) there is an \( \alpha_{n_k} \in B \) and thus \( f(\alpha_{n_k}) = f(\alpha) = 0 \), which implies that \( f \) cannot be injective in any neighborhood of \( \alpha \), no matter how small it is. \( \square \)

The construction of the algorithm \( E' \) is the same as that of the algorithm \( E \).

**Theorem 4.2 (Computing the set of invertible zeros)** The operator that assigns to each \( f \in Z(K) \) its zero set and the cardinality of the zero set is
computable. More precisely, there is an algorithm $E'$, when given any $C^1$-name of $f$ and $n_0 \in \mathbb{N}$ as input, $E'$ produces an integer $n \geq n_0$, a nonnegative integer $\#_0(f)$, and a list $C$ of finitely many squares with rational vertices (or hypercubes if $d \geq 3$) as output, such that

1. $\#_0(f)$ is the exact number of zeros of $f$;

2. each square in $C$ has side length $1/n$ and contains exactly one zero of $f$. Furthermore, $\text{Zero}(f) = \{x \in K : f(x) = 0\} \subseteq \cup C$, which implies that $d_H(\text{Zero}(f), \cup C) \leq 1/n$, where $d_H(\text{Zero}(f), \cup C)$ is the Hausdorff distance between $\text{Zero}(f)$ and $\cup C$.

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