We review some aspects of Nambu mechanics on the basis of the works previously published separately by the present author. Main focuses are on three themes, its various symmetry structures, their possible relevance to string/M theory, and a Hamilton-Jacobi like reformulation. We try to elucidate the basic ideas, most of which were rooted in more or less the same ground, and to explain motivations behind these works from a unified and vantage viewpoint. Various unsolved questions are mentioned. We also include some historical account on the genesis of the Nambu mechanics, and discuss (in the Appendix) some parallelism of various ideas behind the Nambu’s paper with Dirac’s old works which are related to the description of vortical flows in terms of gauge potentials.

Subject Index A0, A1, A6, B0, B2, B8

1. Introduction

I would like to start this review\(^1\) first by presenting a brief comment on the historical genesis of our subject. It seems to me that Nambu’s paper “Generalized Hamiltonian Dynamics” (GHD)[1], published almost five decades ago in 1973, is a true ‘singularity’, occupying a special position in the history of theoretical physics. His other seminal works such as the ones on a dynamical model of elementary particles based on an analogy with the BCS theory of superconductivity, the discovery of the string interpretation of the Veneziano amplitude, and many other notable works have all been generated under close interactions with the environment of the contemporary developments of physics at those periods. This is evidenced by the fact that in these cases more or less similar works by other authors have been appearing independently and almost simultaneously.

The case of the GHD, in contrast, seems entirely different. As far as I know, he himself had never mentioned this paper in his later research papers, except for some expository accounts or reminiscences. However, we can clearly see from his Acknowledgement in the paper that the generalization of Hamiltonian dynamics attempted in this work had been a theme which

\(^1\)Written version of an invited talk in the workshop “Space-time topology behind formation of micro-macro magneto-vortical structure by Nambu mechanics”, Osaka City University, Sept. 28–Oct.1, 2020.

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he had been devoting himself for more than twenty years to take shape into, since his
early years at the Osaka City University. He expressed his gratitude to K. Fusimi, a well-
known expert on statistical mechanics, for encouragement. In fact, we can also see a similar
acknowledgement to Fusimi in another [2] of his earlier papers that discussed the Onsagar’s
rigorous solution of the two-dimensional Ising model. Nambu’s method of simplifying the
Onsager solution using creation-annihilation operators in this work was quite close to the
Fusimi’s independent work [3] on honeycomb lattice. Apparently, Nambu’s keen interest on
statistical mechanics seems to motivate the singular GHD work. We can also envisage from
his later lectures, belonging to last few talks [4] in his whole life, that his special interest on
fluid mechanics had also been closely connected with his ideas leading to the GHD work.

However, apart from all his personal motivations, it seems difficult to imagine any interac-
tion between the theme of the GHD paper and the contemporary developments of physics in
the 1960s and the 70s. That is why I have described the GHD paper as a ‘singularity’. There
does not seem to be any necessity for the appearance of the GHD paper in 1973. It would
be no surprise if we could find the same Nambu equations of motion in some old literature
written even a century ago. But perhaps unexpectedly even to Nambu himself, the GHD
paper has gradually turned out to be relevant to some aspects of string/M theory and also
to other disciplines of physics, as the readers of this special section will recognize from the
wide range of the contributors with different backgrounds. The ‘singularity’ became a small
but very impressive star shining in the sky of theoretical physics.

Now I outline the contents of this paper. In section 2, I will first discuss some salient
features of the Nambu mechanics, focusing on its symmetry properties that are important
to applications to string/M theory and to its possible schemes of quantization. We shall also
mention some unsolved problems. Then in section 3, I will discuss the possible relevance
of Nambu-type structures to string and membrane theories. In its main part (subsection
3.3), I will review our old attempt toward the discretized version of the Nambu bracket.
In section 4, I will review my previous attempt toward a ‘covariantization’ of the M-theory
from the standpoint of the so-called discrete light-cone quantization (DLCQ). Formally, this
attempt has a similar flavor with the regularization of (super) membranes discussed in the
previous section. But the physical meaning will be quite remote. Only the main crux of the
construction will be presented succinctly, since giving a full discussion would become too
intricate and long for this review. Then in the final section, I return to the original intension
of Nambu in the GHD which aimed at a new quantum mechanical formulation of dynamics.
I will summarize the main points of a possible path, a generalized Hamilton-Jacobi-like
reformulation of the Nambu mechanics, toward its “wave-mechanical” quantization, on the
basis of my previous work. A general discussion on the nature of quantization of the Nambu
mechanics will also be given.

The purpose of the whole discussions in this review is not to repeat the previous publica-
tions, but to elucidate various ideas related to the Nambu mechanics, which may be rather
foreign to most of readers and have been scattered in different papers, from a unified stand-
point of the symmetry structure of the Nambu mechanics. Hopefully, that would be useful
for interested readers before going to the original papers directly. In Appendix, I will give a
historical account on an interesting parallelism between Dirac’s old attempts and Nambu’s
ideas, for the purpose of stimulating interests in these almost forgotten works.
2. Symmetry structure of the Nambu mechanics

The usual Hamilton equations of motion describe an incompressible flow in phase space,

\[
\frac{dX}{dt} = \{H, X\} \equiv D^i(H)\partial_i X \quad (1)
\]

where \(D^i(H)\partial_i\) is the operator corresponding to the vector flow governed by a Hamiltonian \(H = H(\xi)\),

\[
D^i(H) \equiv \epsilon^{ij} \partial_j H, \quad \partial_i D^i = 0. \quad (2)
\]

Here, for simplicity, the phase space is taken to be two dimensions \((\xi^1, \xi^2) = (q, p)\) and the flow is area-preserving. Throughout the present paper, we assume the summation convention for repeated indices of the components of the phase-space coordinates, unless stated otherwise explicitly. The Nambu equations of motion are simply a natural extension of this structure to a 3-dimensional phase space \((\xi^1, \xi^2, \xi^3)\), by introducing two Hamiltonians \(H, G\) such that

\[
\frac{dX}{dt} = \{H, G, X\} \equiv D^i(H, G)\partial_i X, \quad (3)
\]

\[
D^i(H, G) \equiv \epsilon^{ijk} \partial_j H \partial_k G, \quad \partial_i D^i = 0. \quad (4)
\]

Thus the area-preserving flow is now replaced by the three dimensional volume-preserving flow, the Poisson bracket being extended to the 3-dimensional Jacobian,

\[
\{K, L, M\} = \frac{\partial(K, L, M)}{\partial(\xi^1, \xi^2, \xi^3)} = \epsilon^{ijk} \partial_i K \partial_j L \partial_k M. \quad (5)
\]

In the following, we call this expression “Nambu bracket”. As alluded to in the previous section, Nambu’s original motivation for this generalization was statistical mechanics, where Liouville theorem (2) plays a critical role under the assumption of ergodicity. Accordingly, he suggested a canonical ensemble characterized by a generalized Boltzmann distribution with a weight factor \(e^{-\beta H - \gamma G}\) with two temperature parameters \(1/\beta\) and \(1/\gamma\), corresponding to the two conserved quantities \(H\) and \(G\).

It is obvious that the Nambu bracket can be extended to arbitrary \(n\)-dimensional phase space as

\[
\frac{dX}{dt} = \{H_1, H_2, \ldots, H_{n-1}, X\} = D^i(H_1, H_2, \ldots, H_{n-1})\partial_i X, \quad \partial_i D^i = 0, \quad (6)
\]

using the \(n\) dimensional Jacobian, with \(n - 1\) conserved quantities \((H_1, H_2, \ldots, H_{n-1})\). In the present paper, we treat only the case \(n = 3\), unless stated otherwise.

As an example of physical systems possessing the above structure, Nambu mentioned the Euler equation for a rigid rotator,

\[
\frac{d\xi^1}{dt} = \frac{(I_2 - I_3)\xi^2\xi^3}{I_2I_3}, \quad \frac{d\xi^2}{dt} = \frac{(I_3 - I_1)\xi^3\xi^1}{I_3I_1}, \quad \frac{d\xi^3}{dt} = \frac{(I_1 - I_2)\xi^1\xi^2}{I_1I_2}, \quad (7)
\]

\[
\frac{d\xi^i}{dt} = \{H, G, \xi^i\}, \quad H = \frac{1}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2), \quad G = \frac{\xi_1^2}{2I_1} + \frac{\xi_2^2}{2I_2} + \frac{\xi_3^2}{2I_3}. \quad (8)
\]

According to him, this example is enough as a justification for exploring the proposed idea. Indeed, it is remarkable that the phase space coordinates are identified directly with the
components \((\xi^i = L_i)\) of the angular momentum with respect to the principal axes of a rigid body and \(L_i\)'s are the corresponding moments of inertia. Thus the components of the angular momentum themselves are now the canonical coordinates satisfying the canonical Nambu bracket relation

\[
\{L_i, L_j, L_k\} = \{\xi^i, \xi^j, \xi^k\} = \epsilon^{ijk},
\]

instead of the standard Poisson bracket relation and the equations of motion

\[
\{L_i, L_j\} = \epsilon_{ijk} L_k, \quad \frac{dL_i}{dt} = \{G, L_i\}
\]

with the single Hamiltonian \(G\). Note that, in terms of the Nambu bracket formulation, the conservation of both \(H\) and \(G\) is manifest. They play completely symmetrical roles. By contrast, in the standard Poisson bracket formulation, \(H\) of course plays the role of a Casimir invariant with respect to the algebra of the components of the angular momentum, while \(G\) is the kinetic energy of rotation. The relation between these two different canonical structures will be clarified in section 5.

2.1. The fundamental identity

With these preparations, let us now turn to the symmetry properties of this system. First we examine whether the canonical Nambu bracket relation \(\{\xi^i, \xi^j, \xi^k\} = \epsilon^{ijk}\) is preserved under the flow (3). It requires that

\[
0 = \{(H, G, \xi^i), \xi^j, \xi^k\} + \{\xi^i, \{H, G, \xi^j\}, \xi^k\} + \{\xi^i, \xi^j, \{H, G, \xi^k\}\}.
\]

This is guaranteed by the so-called “fundamental identity” (FI) identity [5]

\[
\{F_1, F_2, \{F_3, F_4, F_5\}\} = \{(F_1, F_2, F_3), F_4, F_5\}
+ \{F_3, \{F_1, F_2, F_4\}, F_5\} + \{F_3, F_4, \{F_1, F_2, F_5\}\},
\]

that is valid by definition for arbitrary five functions, \(F_1, F_2, \ldots, F_5\) of the coordinates \(\xi^i\)'s. By choosing \(F_1 = H, F_2 = G\) and \(F_3 = \xi^i, F_4 = \xi^j, F_5 = \xi^k\) in this identity, (11) immediately follows. Apparently, Nambu himself was not aware of the identity (12). Clearly, the FI replaces the Jacobi identity of the Poisson bracket.

In connection with the FI, it is important to notice that a seemingly natural extension of a single triple coordinates \((\xi^1, \xi^2, \xi^3)\) to the multiple sets of \(3N\) coordinates \((\xi_a^1, \xi_a^2, \xi_a^3)\) with \(a = 1, 2, \ldots, N\), by defining

\[
\{H, G, F\}^{(N)} \equiv \sum_{a=1}^{N} \frac{\partial(H, G, F)}{\partial(\xi_a^1, \xi_a^2, \xi_a^3)},
\]

violate the FI in general, except for completely decoupled systems where the functions \((H, G, F)\) are linear sums of the contributions depending only on a single triplet. This is a crucial difference of the Nambu dynamics from the ordinary Hamiltonian dynamics. In this sense, the universality of the former seems to be largely diminished in comparison with the latter. However, we may equally take a different standpoint that the Nambu mechanics could be useful for some special cases in constraining applicable systems by higher symmetries, subject to which we next turn.
2.2. N-gauge symmetry

Let us examine whether there are any freedom in the choice of the pair of two Hamiltonians \((H,G)\). By rewriting the equations of motion explicitly for the coordinate \(\xi^i\) using two-dimensional Jacobian as

\[
\frac{d\xi^i}{dt} = \frac{1}{2} \epsilon^{ijk} \frac{\partial (H,G)}{\partial (\xi^j, \xi^k)},
\]

we find that, for a given set \((H,G)\), any different set of the new Hamiltonians \((H',G')\) satisfying

\[
\frac{\partial (H',G')}{\partial (H,G)} = 1 \quad \text{(or} \quad \epsilon^{ijk} \partial_j H \partial_k G = \epsilon^{ijk} \partial_j H' \partial_k G' \text{)}
\]

(15)

gives the same equations of motion. This condition can equivalently be reformulated as

\[
H \partial_i G - H' \partial_i G' = \partial_i \Lambda
\]

(16)

for an arbitrary “generating” function \(\Lambda\), which means that \(\Lambda\) can be regarded as a function of \((G,G')\) and

\[
\frac{\partial \Lambda}{\partial G} = H, \quad \frac{\partial \Lambda}{\partial G'} = -H'.
\]

(17)

This is reminiscent of the ordinary canonical transformations in the standard Hamiltonian formalism, at least in a formal sense. As noted originally by Nambu, this symmetry can alternatively be rephrased as a kind of gauge transformation. From a general viewpoint of volume-preserving flow, the vector field appearing \(D^i(H,G)\) in (4) can always be expressed by defining the gauge field \(A_i\) through

\[
D^i = \frac{1}{2} \epsilon^{ijk} F_{jk}, \quad F_{jk} = \partial_j A_k - \partial_k A_j.
\]

(18)

The expression (4) corresponds to the particular form (so-called Clebsch representation that is familiar in fluid mechanics)

\[
A_i = H \partial_i G + \partial_i \psi,
\]

(19)

where \(\psi\) is an arbitrary undetermined function. The transformation \((H,G) \rightarrow (H',G')\) defined by (16) is nothing but a gauge transformation,

\[
\psi \rightarrow \psi - \Lambda,
\]

(20)

that keeps the above particular form. Throughout the present paper, we call the transformation \((H,G) \rightarrow (H',G')\) characterized by (16) the “N-gauge” transformation.

Nambu considered a further generalization of the equations of motion to

\[
\frac{dX}{dt} = \sum_a \{H_a, G_a, X\}
\]

(21)

by introducing an arbitrary number of the pairs \((H_a, G_a)\) of Hamiltonians. Then the N-gauge transformation is generalized to

\[
\sum_a \left( H_a \partial_i G_a - H'_a \partial_i G'_a \right) = \partial_i \Lambda
\]

(22)
Hence, the N-gauge transformation becomes akin further to the ordinary canonical transformation for many variables as
\[
\frac{\partial \Lambda}{\partial G_a} = H_a, \quad \frac{\partial \Lambda}{\partial G'_a} = -H'_a.
\] (23)

Then, however, none of $H_a$ or $G_a$ is conserved for a general choice of them unless $\{H_a, G_a, H_b\} = 0 = \{H_a, G_a, G_b\}$ for all different sets with $a \neq b$. It would diminish the possible merit in adopting the Nambu bracket notation at least in the sense of a dynamical system, while from the standpoint of utilizing the Nambu flow toward a generalization of symmetry to higher ones in a given system it may still be useful. This viewpoint will be useful later.

2.3. Genuine canonical transformation

From the general viewpoint of the canonical structure of the equations of motion, the genuine canonical transformation of the Nambu equations of motion should be defined to be the coordinate transformations preserving the canonical bracket relation (9). Consequently, the flows described by the equations of motion are a special case of such general canonical transformations. Then the infinitesimal form of the canonical transformation with arbitrary two functions $(F,G)$ is given by
\[
\delta \xi^i = \{F,G,\xi^i\} \equiv D(F,G)\xi^i
\] (24)
where
\[
D(F,G) \equiv D^i(F,G)\partial_i.
\] (25)

The FI guarantees that the canonical bracket relation is preserved. We can then define a finite transformation by
\[
X(s) \equiv \exp(sD(F,G))
\]
\[
= \sum_{n=0}^{\infty} \frac{s^n}{n!} \{F,G,\{F,G,\{\cdots,\{F,G,\{F,G,X\}\}\}\}\}\}
\] (26)
that satisfies the Nambu flow equation
\[
\frac{dX(s)}{ds} = \{F,G,X(s)\}.
\] (27)

The group property of these transformations was discussed in our previous work [6]. First, using the FI, we find identically for any $X$,
\[
\{D(F_1,G_1),D(F_2,G_2)\}X = \{F_1,G_1,\{F_2,G_2,X\}\} - \{F_2,G_2,\{F_1,G_1,X\}\}
\]
\[
= \{\{F_1,G_1,F_2\},G_2,X\} + \{F_2,\{F_1,G_1,F_2\},X\}
\]
\[
= D(\{F_1,G_1,F_2\},G_2)X + D(F_2,\{F_1,G_1,F_2\})X.
\] (28)

---

2 Nambu himself considered such canonical transformations which are linear with respect to $\xi^i$’s, and came to notice the difficulties mentioned above for the case with $3N$ variables related to (13).
But the last line must be antisymmetric with respect to 1 and 2 by definition. This allows us to conclude\(^3\)

\[
[D(F_1, G_1), D(F_2, G_2)] = \frac{1}{2} \left( D(\{F_1, G_1, F_2\}, G_2) + D(F_2, \{F_1, G_1, G_2\}) - D(\{F_2, G_2, F_1\}, G_1) - D(F_1, \{F_2, G_2, G_1\}) \right).
\tag{29}
\]

Note that the right-hand side does not match with the naive expectation

\[
[D(F_1, G_1), D(F_2, G_2)] = D(F_3, G_3)
\]

for some appropriate \(F_3(F_1, G_1; F_2, G_2)\) and \(G_3(F_1, G_1; F_2, G_2)\). The reason behind this phenomena seems to be the fact that the Clebsch form for the gauge field is not the most general form for volume-preserving flows. This is in contrast to the area-preserving flows where the composition law is simply expressed as

\[
[D(H), D(G)] = D(K), \quad K = \{H, G\},
\tag{30}
\]

as a consequence of the Jacobi identity.

Instead, we have pointed out in [6] that the following skew-symmetric triple commutator relation are valid:

\[
D(A_{[1}, A_{2})D(A_{3]}\cmp, B) = 2D(\{A_{1}, A_{2}, A_{3}\}, B),
\tag{31}
\]

\[
D(B_{[1}, B_{2})D(A_{[1}, A_{2})D(A_{3]}\cmp, B_{3]}\cmp) = 4D(\{A_{1}, A_{2}, A_{3}\}, \{B_{1}, B_{2}, B_{3}\}),
\tag{32}
\]

where

\[
[A, B, C]_{\cmp} \equiv ABC - ACB + BCA - BAC + CAB - CBA.
\tag{33}
\]

This triple commutator satisfying a complete skew symmetry was originally defined by Nambu as a candidate for quantum version of the Nambu bracket. However, (33) does not satisfy the FI. It is not clear how to interpret this structure. It might be possible that the group-like property for the finite canonical transformations could be understood in terms of a new composition law which would lead to the above triple commutation law, as a special subset of the group of all possible volume-preserving flows. To my knowledge, this has never been realized in any closed form. More about the problems of finite canonical transformations will be discussed in section 4 in connection with a Hamilton-Jacobi like approach to the Nambu mechanics.

3. String/membrane theories and the Nambu bracket

From the viewpoint of string theory, possible physical applications of the Nambu bracket and its associated symmetry arise in the theories of relativistic membranes that are expected to be relevant to the putative M-theory, which has been supposed to be the description of the strong coupling region of the type-IIA string theory. Before going to membranes, let us first recall the case of relativistic strings, where the Poisson bracket naturally plays a similar role.

\(^3\)There is a typo in eq. (2.19) in ref.[6].
3.1. The case of string and some generalizations

A general form [7] for the classical action integral of the world sheet, parametrized by \((\xi^1, \xi^2)\) of a single string whose space-time coordinates are \(X^\mu(\xi)\), takes the following form,

\[
S_k = \int d^2\xi e^k \left\{ \frac{1}{e^k} \left[ -\frac{1}{2} (\sigma^{\mu\nu} \sigma_{\mu\nu})^2 \right]^{k/2} + k - 1 \right\},
\]

(34)

where \(\sigma^{\mu\nu} \equiv \epsilon^{ij} \partial_i X^\mu \partial_j X^\nu\) is the induced surface elements of the world sheet and \(e = e(\xi)\) is an auxiliary variable ("Ein-bein") defined on the world sheet whose role is to make the action parametrization invariant. Note that we use the unit where the slope parameter \(\alpha'\) is equal to \(1/4\pi\). For any positive integer \(k\), this gives the same equations of motion upon eliminating the auxiliary variable \(e\). The familiar Nambu-Goto action corresponds to the simplest case \(k = 1\). The next simplest case \(k = 2\) was proposed first by Schild [8], though he has not considered the re-parametrization invariance. This amounts to setting the gauge \(e = 1\) from the outset. For a demonstration of quantum-mechanical equivalence of this case to the so-called Polyakov action that uses the world-sheet metric tensor \(h_{ij}\) as auxiliary fields

\[
S_P = -\frac{1}{2} \int d^2\xi \sqrt{-h} h^{ij} \partial_i X^\mu \partial_j X^\mu,
\]

(35)

we refer readers to [7].

Now the surface element \(\sigma^{\mu\nu}\) can be regarded as the Poisson bracket of \(X^\mu\) and \(X^\nu\) if we treat the world-sheet as a two-dimensional phase space as used in (1) of the previous section:

\[
\sigma^{\mu\nu} = \{X^\mu, X^\nu\},
\]

(36)

in terms of which the \(n = 2\) action takes the form

\[
S_2 = -\int d^2\xi \left[ \frac{1}{2e} \{X^\mu, X^\nu\}\{X_\mu, X_\nu\} - e \right].
\]

(37)

The equations of motion and the constraint associated with the \(e\) are, respectively,

\[
\{X_\mu, e^{-1}\{X^\mu, X^\nu\}\} = 0, \quad \frac{1}{2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\} = -e^2.
\]

(38)

It should be noted that the constraint condition is actually equivalent to the standard Virasoro condition, conforming to the world-sheet conformal symmetry,

\[
P^2 + \partial_\sigma X^2 = 0, \quad P \cdot \partial_\sigma X = 0,
\]

(39)

when we define the canonical momentum by

\[
P^\mu = \frac{1}{2e} \left( \partial_\tau X^\mu (\partial_\sigma X)^2 - \partial_\sigma X^\mu (\partial_\tau X \cdot \partial_\sigma X) \right),
\]

(40)

where the time-like and the space-like coordinates in the sense of world sheet are denoted by \(\tau\) and \(\sigma\), respectively. This guarantees that the canonical formalism is equivalent to the ordinary one as derived from the Nambu-Goto action or the Polyakov action.

With the gauge choice \(e = 1\), the re-parametrization symmetry is reduced to area-preserving re-parametrizations. For this case, Nambu [9] has suggested an extension of the string theory by replacing the Poisson bracket \(\sigma^{\mu\nu}\) by a commutator of covariant derivatives \(D_\mu = -i(\partial_\mu - iA_\mu)\) of the non-Abelian gauge theory, \(\{X_\mu, X_\nu\} \rightarrow -i[D_\mu, D_\nu]\). To quote his own words, “the string may be a special realization of gauge fields in which some dynamical
degrees of freedom are frozen while the other have become classical in a sense.” This seems to prophesy of various modern matrix models that are obtained by dimensional reductions from the maximal super Yang-Mills theory in 10 dimensions, and also of the discretized light-like gauge action for a relativistic membrane which was investigated first by Hoppe [10] in his thesis (1982).

On the other hand, in ref.[7], the constraint equation in the form (38) was interpreted as the classical form for an uncertainty relation of space-time $\Delta X \Delta T \gtrsim 1$, which the present author has been advocating as the qualitative but intrinsic characterization [12] of the short-distance space-time structure of (critical) string theory. For extensive discussions of this subject, I would like to refer the reader to [13] and other earlier references therein.

From the viewpoint of the emergence of the higher-dimensional phase space as in (6), it is also useful to rewrite the action integral for $D$-dimensional extended objects as

$$S^{(d)} = \int d^D \xi \left[ p_{\mu_1 \mu_2 \cdots \mu_D} \{ X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_D} \} + \frac{e}{2} ( p_{\mu_1 \mu_2 \cdots \mu_D} p^{\mu_1 \mu_2 \cdots \mu_D} + 2 ) \right],$$

(41)

where $p_{\mu_1 \mu_2 \cdots \mu_D}$ is a new auxiliary “momentum” variable. The first term suggests the $D$-dimensional form $d \omega^{(D)} = p_{\mu_1 \mu_2 \cdots \mu_D} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_D}$. However, the second term of the action integral is nothing but the Hamiltonian in the sense of the ordinary Hamiltonian dynamics with a single Hamiltonian, resulting in the Hamiltonian constraint

$$\frac{1}{2} p_{\mu_1 \mu_2 \cdots \mu_D} p^{\mu_1 \mu_2 \cdots \mu_D} = -1,$$

(42)

which requires that the $D$-form $p_{\mu_1 \mu_2 \cdots \mu_D}$ must be time-like. The latter condition is again consistent with the interpretation of the space-time uncertainty relation which essentially governs the dynamics of these systems. This aspect of the space-time uncertainty relation (with $D = 2$) has been discussed in detail in [13] as a possible realization of non-commutative space-time geometry. The flow equation (6) with $n-1 ( = D-1)$ Hamiltonians should here be interpreted as describing symmetries that characterize this system, rather than the dynamical flow directly which should correspond to the genuine time variable $\tau$.

A lesson we should learn from this form of the action is that the appearance of the higher-dimensional phase-space structure does not necessarily imply the relevance of the Nambu mechanics. Conversely, the Nambu-type mechanics equipped with higher-dimensional phase space does not directly imply the higher-dimensional objects. For instance, the action proposed by Takhtajan in his remarkable paper [5] is formulated as if a one-dimensional string-like object with $(\xi^i = \xi^i(\sigma, \tau))$ is treated. The presence of two Hamiltonians of Nambu mechanics corresponds to the Clebsch form $H \partial_\sigma G \sigma \wedge d\tau$, instead the second term in the above action (41) with $D = 2$ that involves no derivative. Because of this difference, the $\sigma$ direction in such a formulation merely parametrizes a continuous family of ordinary one-dimensional trajectories in three dimensional phase space $(\xi^1, \xi^2, \xi^3)$. Although the system has a re-parametrization symmetry with respect to $\sigma$ in a kinematical sense due to the $\sigma$-derivative, the $\sigma$ direction has no dynamical significance, since there is no Hamiltonian constraint for combining the $\sigma$ direction with the $\tau$ direction: the Hamiltonian constraint, if existed, would be responsible for assigning a dynamical role to $\sigma$ of defining a potential energy.

This crucial feature will become relevant and be elucidated further from a slightly different perspective to motivate an attempt toward a generalized Hamilton-Jacobi formalism of Nambu mechanics in section 5.
3.2. Nambu bracket for the covariant regularization of a relativistic membrane

Now let us focus our attention on the \((k = 2, n = 3)\) case of the generalized Schild-type action (37),

\[
S_{\text{mem}} = - \int d^3 \xi \left( \frac{1}{2e} \{X^\mu, X^\nu, X^\sigma\} \{X_\mu, X_\nu, X_\sigma\} - e \right). \tag{43}
\]

With the gauge choice \(e = 1\), the re-parametrization symmetry reduce to the volume-preserving re-parametrizations. Unlike the case \((k = 2, n = 2)\) of the string, the covariant quantization of this system turned out to be extremely difficult. Only tractable way even to this day is to adopt the light-like gauge condition \(X^\tau = X^{10} + X^0 = \tau\). But still the system is non-linear: the Hamiltonian is

\[
H = \int d^2 \sigma \frac{1}{p^+} \left( p_i^2 + \frac{1}{2}(X^i, X^j)^2 \right) + \cdots \tag{44}
\]

where the indices \(i, j, \ldots\) are SO(9) transverse directions and the \(p_i\)'s are the momentum variable conjugate to \(X_i\)'s. In the above, we have already alluded to the matrix regularization that was applied to this system. In [11], its supersymmetric generalization was studied in detail. The so-called BFSS model can be regarded as a re-interpretation of this system toward a possible non-perturbative formulation of M-theory in a special light-like frame.

One of the hopes in the late 90s, perhaps pursued by many researchers, was to find appropriate discretized regularizations directly to (43) without assuming the special light-like frame, such that it gives a tractable covariant formulation of the dynamics of (super) membranes. If we denote such a “quantized version” of the Nambu bracket by \([X^\mu, X^\nu, X^\sigma]\) that could replace the Nambu bracket, the action would essentially be (in the \(e = 1\) gauge)

\[
S = \frac{1}{2} \text{Tr} \left( [X^\mu, X^\nu, X^\sigma] [X_\mu, X_\nu, X_\sigma] \right) + \cdots \tag{45}
\]

where \(X^\mu, \ldots\) are now the discretized counterpart of the world-volume coordinates for which the symbol Tr replaces the integral over the world volume. However, no one has achieved satisfactory progress in such attempts. For example, our work [6] was a by-product of an unsuccessful attempt along this line.

3.3. Discretized versions of the Nambu bracket

Here we briefly review some of the main results of [6]. If we restrict to the usual square matrices \(A, B, C, \ldots\), we define

\[
[A, B, C] \equiv (\text{Tr}A)[B, C] + (\text{Tr}B)[C, A] + (\text{Tr}C)[A, B]. \tag{46}
\]

Note that, when one of the matrices is the unit matrix, this reduces to the usual commutator for the remaining two matrices, and is obviously totally skew symmetric and satisfies \(\text{Tr}[A, B, C] = 0\) for arbitrary three matrices.

Also it is not difficult to prove that the FI,

\[
[F, G, [A, B, C]] = [[F, G, A], B, C] + [A, [F, G, B], C] + [A, B, [F, G, C]], \tag{47}
\]

is satisfied. First we calculate the l.h.s,

\[
[F, G, [A, B, C]] = (\text{Tr}A)[F, G, [B, C]] + (\text{c.p.})
\]

\[
= (\text{Tr}A)(\text{Tr}F)[G, [B, C]] - (\text{Tr}A)(\text{Tr}G)[F, [B, C]] + (\text{c.p.}) \tag{48}
\]
where the symbol (c.p.) abbreviates the contributions obtained by the cyclic permutation of $A, B, C$ from the expressions appearing prior to it. The r.h.s of the FI is

$$\text{(Tr} F)[[G, A], B, C] + (\text{Tr} G)[[A, F], B, C] + (\text{Tr} A)[[F, G], B, C] + (\text{c.p.}) . \quad (49)$$

The third term cancels after the corresponding contribution from (c.p.) are added. It is a consequence of the following identity,

$$(\text{Tr} A)[[F, G], B, C] + (\text{Tr} B)[[F, G], C, A] + (\text{Tr} C)[[F, G], A, B] = 0. \quad (50)$$

Thus the r.h.s of the FI reduces to

$$(\text{Tr} F)[[G, A], B, C] + (\text{Tr} G)[[A, F], B, C] + (\text{c.p.})$$

$$= (\text{Tr} F)\left((\text{Tr} B)[C, [G, A]] + (\text{Tr} C)[[G, A], B]\right) + (\text{Tr} G)\left((\text{Tr} B)[C, [A, F]] + (\text{Tr} C)[[A, F], B]\right) + (\text{c.p.}). \quad (51)$$

Now by comparing this result with (48), it is equal to the latter owing to the Jacobi identity. For example, the term $(\text{Tr} A)(\text{Tr} F)[G, B, C]$ in (48) have the corresponding contribution $(\text{Tr} F)(\text{Tr} A)(C, [G, B])$ in (51), which indeed reduces to the former due to the Jacobi identity.

Now let us examine the structure of the flow given by this realization of the discretized 3-bracket with two bosonic matrices $(F, G)$:

$$\delta A \equiv i[F, G, A] = i[(\text{Tr} F)G - (\text{Tr} G)F, A] + i(\text{Tr} A)[F, G] \quad (52)$$

Note that the second term takes a very peculiar form, while the first one is nothing but the standard form of a gauge transformation. To the extent of being a matrix commutator, the second term allows us to make an unusual shift of the matrix independently of the original matrix $A$, provided that its trace is non-vanishing. Since $[F, G]$ with suitable choice of $(F, G)$ can be any element of the Cartan algebra of $su(N)$, we can shift any Hermitian matrices with non-zero trace to the unit matrix: $A \rightarrow N^{-1}\text{Tr} A$.

On the other hand, the derivation law is not satisfied for generic products of matrices in the sense that

$$[F, G, A]B + A[F, G, B] \neq [F, G, AB], \quad (53)$$

owing to the presence of the second term in (52). However, provided that $\text{Tr} A = \text{Tr} B = 0$,

$$\delta \text{Tr}(AB) \equiv \text{Tr}\left((\delta A)B + A(\delta B)\right) \quad (54)$$

$$= \text{Tr}\left(i[(\text{Tr} F)G - (\text{Tr} G)F, AB]\right) = 0, \quad (55)$$

which is valid for the trace of an arbitrary number of the products of traceless matrices. This result has a nontrivial significance for the symmetry of the action of the type (45), since the FI guarantees that

$$\delta[X^\mu, X^\nu, X^\sigma] \equiv [\delta X^\mu, X^\nu, X^\sigma] + [X^\mu, \delta X^\nu, X^\sigma] + [X^\mu, X^\nu, \delta X^\sigma]$$

$$= i[F, G, [X^\mu, X^\nu, X^\sigma]] = i[(\text{Tr} F)G - (\text{Tr} G)F, [X^\mu, X^\nu, X^\sigma]] \quad (56)$$

and $\text{Tr}[X^\mu, X^\nu, X^\sigma] = 0$. Thus the action (45) is invariant under the generalized gauge transformation (52). These results show that the requirement of the derivation law in its most
general form is not quite important from the viewpoint of symmetries, if one assumes that the physical observables are restricted exclusively to the objects that can be constructed as the polynomials of the ‘bracketed’ objects. We can still have a useful set of invariants when the basic entities are subjected to the generalized gauge transformations.

In ref.[6], the possible extensions of these structures to the objects with three or many indices are also investigated. Here we only briefly mention the case of the cubic matrices $A_{pqm}, \ldots$ with three indices. We define the generalization of trace operations as

$$\langle A \rangle \equiv \sum_{pm} A_{pmp}, \quad \langle AB \rangle \equiv \sum_{pqm} A_{pqm}B_{qmp}, \quad \langle ABC \rangle \equiv \sum_{pqmr} A_{pqm}B_{qmr}C_{rmp}$$

(57)

that satisfy

$$\langle AB \rangle = \langle BA \rangle, \quad \langle ABC \rangle = \langle BCA \rangle = \langle CAB \rangle.$$  

(58)

A triple product is then defined as

$$(ABC)_{ijk} = \sum_p A_{ijp} \langle B \rangle_{pjk} = \sum_{pqm} A_{ijp}B_{qmq}C_{pjk}.$$  

(59)

In terms of them, the Nambu triple-bracket is

$$[A,B,C]^{(3)} \equiv (ABC) + (BCA) + (CAB) - (CBA) - (ACB) - (BAC)$$

(60)

that is skew-symmetric and satisfies the FI. Note that the middle index $j$ of $A_{ijk}$ in these definitions plays the role of internal index, while the two outer indices $i$ and $k$ behave like the indices of square matrices. Thus it is indeed natural to expect that the FI is satisfied by essentially the same mechanism as in the case of the square matrices with the definition (46). For details of proof, we refer the reader to [6], where the possibilities other than the above cubic products leading to the FI were mentioned. It is also straightforward to extend the cubic case to more indices.

As in the case of the representation in terms of square matrices, the above triple brackets reduce to the commutator if at least one the three cubic matrices is a generalized unit cubic matrices defined by

$$I_{ijk} \equiv \delta_{ij}^{(ij)}_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$$

(61)

for any $j$. Then we have

$$(AIB)_{ijk} = \langle I \rangle \sum_p A_{ijp}B_{pjk}, \quad (IAB) = (BAI) = \langle A \rangle B,$$

(62)

which leads to

$$[A,I,B]^{(3)}_{ijk} = \langle I \rangle \sum_p (A_{ijp}B_{pjk} - B_{ijp}A_{pjk}).$$

(63)

Note that this can indeed be regarded as a commutator for any fixed middle index $j$.

Furthermore, the trace for cubic case satisfies

$$\langle [F,G,A]B \rangle + \langle A[F,G,B] \rangle = 0$$

(64)

provided that $\langle A \rangle = \langle B \rangle = 0$, which implies that the trace of the product of Nambu brackets $\langle [A,B,C]X,Y,Z \rangle$ is invariant under the gauge transformation generated by the Nambu
bracket, since \((A, B, C) = 0\) for arbitrary choices of cubic matrices \(A, B, C\). The same is true for the generalized trace for the product of \(n\) cubic matrices of the form

\[
(AB \cdots Z) \equiv \sum_{p_1, p_2, \cdots, p_n, m} A_{p_1 m p_2} B_{p_2 m p_3} \cdots Z_{p_n m p_1}.
\]

Hence the trace of the products of arbitrary number of Nambu brackets are also gauge invariants.

4. An attempt toward a covariantized M(atrix) theory

The properties reviewed in the previous section might be useful if one pursues the construction of covariant theories for membranes in a regularized form. In particular, in the case of the cubic matrices, the middle index is expected to play the role of the discretized time variable of the world volume, while the first and third indices would correspond to the two-dimensional discretized spatial coordinates. However, there is no known concrete attempt along this line. One among various apparent difficulties in achieving such a goal is the problem of extracting causal dynamical development with respect to this would-be discrete time. For example, a crucial problem is how to connect it to the classical action (43), where all the three parameters \(\xi^i\) of the world-volume coordinates appear completely symmetrical manner, in the continuum limit. It seems rather difficult to imagine how such a symmetrical structure could naturally emerges from the asymmetrical roles of the three indices. From this viewpoint, it might be worthwhile to investigate the other possible definitions of triple products of cubic matrices suggested in [6].

In the present section, however, I will review a more modest approach which is trying to ‘covariantize’ the M(atrix) theory of BFSS [14], rather than insisting further on the possibilities of direct regularized theory of membranes. This attempt [15] is based on a slight generalization of the definition of the 3-bracket in terms of the usual square matrices, for the purpose of implementing a new type of gauge symmetry, as has been already exhibited above in (52) that is higher than the usual SU(\(N\))-type gauge symmetries. Since all the details have been published five years ago in the above reference, we will restrict ourselves only to main conceptual issues. I hope that the following concise summary would be useful for raising readers’ attention to this old problem which seems to be almost forgotten for two decades up to now, in spite of its potential importance. I also wish to refer readers to [15] for a fuller bibliography related to this subject.

4.1. DLCQ approach to the M(atrix) theory and preparations toward fully covariant generalization

Let us first briefly summarize what the DLCQ approach [16] is. It has been proposed as a variant of the original M(atrix)-theory conjecture. According to the latter, the type IIA string theory in 10 space-time dimensions is interpreted as being compactified along a circle of radius \(R_{11} = g_s \ell_s\) of the 10-th spatial direction \((x^{10} \sim x^{10} + 2\pi R_{11})\), where \(g_s\) is the string coupling and \(\ell_s\) \((\sim \sqrt{\alpha'})\) is the fundamental string length constant. The gravitational length in 11 dimensions is \(\ell_{11} = g_s^{1/3} \ell_s\). The unit of 10-dimensional momentum is then equal to \(1/R_{11}\) and is interpreted as the mass of a single D0-brane (or ‘D-particle’). Now let us suppose that the total momentum \(P^\mu\) of this system satisfies the mass-shell condition

\[
P^\mu P_\mu + M_{\text{eff}}^2 = 0,
\]

where \(M_{\text{eff}}\) is the total effective mass, and consider the limit of large total
10-th momentum $P_{10} = N/R_{11}$ ($N \to \infty$). This is the so-called infinite momentum frame (IMF) where we have the non-relativistic approximation,

$$P^0 = P_{10} + \sqrt{(P^i)^2 + (P_{10})^2 + M_{\text{eff}}^2 - P_{10}} \sim \frac{(P^i)^2 + M_{\text{eff}}^2}{2P_{10}} + \ldots$$

(66)

where the indices $i = 1, 2, \ldots, 9$ run only over the S0(9) directions. However, If we define the light-like momenta by $P^\pm = P_{10} \pm P^0$, the mass-shell condition is expressed exactly as

$$-P^- = \frac{(P^i)^2 + M_{\text{eff}}^2}{P^+}.$$  

(67)

for arbitrary value of $P^+$. In the IMF, we can make identification $P^+ \sim 2P_{10} \sim 2N/R_{11}$. The BFSS M(atrix) model was originally proposed on the basis of analogy of this phenomena with the membrane Hamiltonian (44) in the light-like gauge. The $M_{\text{eff}}^2$ is replaced with the Hamiltonian for the effective (super) Yang-Mills theory for D0-branes,

$$H = N\text{Tr}\left(\hat{P}_i^2 - \frac{1}{2\ell_{11}^4} [X_i, X_j]^2 + \cdots \right)$$

(68)

where $\hat{P}_i$ ($i \in \text{S0}(9)$) are the traceless components of the matrix momenta that are canonical conjugates to the traceless parts to the Higgs (hermitian $N \times N$) matrix fields $X_i$: the diagonal parts of $X_i$ represent the coordinates of $N$ D-particles while the off-diagonal parts correspond to short open strings connecting them. Intuitively, D-particles are nothing but ‘partons’ as the constituents for membranes and, hopefully, other objects in M-theory. As a one-dimensional gauge theory with only time direction, the gauge field degrees of freedom do not appear and is signified only by the existence of a gauge constraint which is preserved by the time development described by the above Hamiltonian.

Now the DLCQ approach is essentially a proposal of reinterpreting it by assuming the compactification along the light-like direction $x^- (\equiv x^{10} - x^0) \sim x^- + 2\pi R$ directly, which implies

$$P^+ = 2N/R$$

(69)

for an arbitrary finite $N$ and $R$, instead of the compactification along the 10th spatial direction. In the large $N$ limit, both are formally equivalent to each other. There is, however, a crucial difference for the interpretation of Lorenz transformations. In the former, the original BFSS conjecture, a Lorentz boost along the 10th spatial direction induces a discrete change of the quantum number $N$ with fixed and Lorentz invariant $R_{11}$. By contrast, in the DLCQ interpretation, a Lorentz boost is a continuous change of the radius parameter $R$ itself ($R \to e^\rho R, P^+ \to e^{-\rho} P^+$), while by definition $N$ is Lorentz invariant and fixed. Thus, the longitudinal momentum $P^+$ is a genuine continuous dynamical variable. This also implies that the weak coupling limit $g_s \to 0$ is meaningful in the DLCQ interpretation and can be connected to the strong coupling limit by a large Lorentz boost. Of course, the limit of uncompactified 11-dimensional theory requires to take both $R$ and $N$ infinitely large.

From these discussions, it seems clear that there must exist the Lorentz invariant formulation for $M_{\text{eff}}^2$ as the generalization of (68) even for finite $N$, as a prerequisite for the feasibility of the DLCQ approach. Since now the theory must be meaningful as an exact formulation of, at least, one corner of M-theory even with finite and fixed $N$, the standpoint of interpreting the M(atrix) theory as a regularization of membrane should be abandoned if
one seriously adopt the DLCQ point of view. This was the basic motivation for my previous
work [15]. Namely, we should be able to extend the Hamiltonian to a Lorentz invariant $M^2_{\text{eff}}$
in such a way that it reduces to (68) once we take the light-like frame. In order to realize
this expectation, we have to require the followings:

(1) Since the would-be Hamiltonian (as the mass-square operator) is itself Lorentz invari-
ant, the time variable must also be Lorentz invariant as in the case of proper-time
formalism for a relativistic particle quantum mechanics.
(2) All of the 11-dimensional space-time directions must be treated on an equal footing
as matrix degrees of freedom, since the transverse directions are already appearing as
the matrix fields in the light-like gauge.
(3) It must be equipped with some higher-symmetries that encompass the ordinary SU($N$)
gauge symmetry and enable us to eliminate the additional longitudinal matrix degrees
of freedom by appropriate gauge fixing procedure.

Concerning the last point (3), the structure exhibited in (52) seems promising, since it
indeed encompasses the usual SU($N$) transformation by the presence of a non-standard
shift term that can eliminate the traceless part of any single Hermitian matrix with non-
vanishing trace. In fact, however, it is possible even to eliminate the restriction to a matrix
with non-vanishing trace. We associate an auxiliary non-matrix variable $X_M$ to a hermitian
$(N \times N)$ matrix $X$, and denote the pair of them by $X = (X_M, X)$. Then we define the
3-bracket as

$$\left[ X, Y, Z \right] \equiv (0, X_M[Y, Z] + Y_M[Z, X] + Z_M[X, Y]).$$

Note that in this slight generalization of (46), the role of the traces of the matrices are played
by the ‘$M$’-components which are treated as new independent dynamical variables. It is easy
to check that the FI is still satisfied in the same way as we have discussed in subsection 3.2,
due essentially to the Jacobi identity for matrices. For example, the absence (i.e. $[X, Y, Z]_M =
0$) of the $M$-component for the 3-bracket is guaranteed by the cancellation of the contributions
involving the commutator $[F, G]$ without performing any trace operations for the three
elements $(X, Y, Z)$ in the r.h.s of the FI,

$$[F, G, [X, Y, Z]] = [[F, G, X], Y, Z] + [X, [F, G, Y], Z] + [X, Y, [F, G, Z]].$$

Otherwise we would have a contribution of the form $[X, Y, Z]_M[F, G]$ which would ruin the
consistency of the above definition.

With this slight extension, the generalized gauge transformation takes the form

$$\delta X = i[F, G, X] = (0, i[F_M G - G_M F, X] + i[I, G, X_M]).$$

In this form, however, the SU($N$) part and the shift part are not yet completely independent
to each other. This is remedied by introducing an arbitrary number of independent gauge
variables $(F^a, G^a)$ discriminated by the indices $a = 1, 2, \ldots$, following Nambu’s generalization
(21):

$$\delta X = i \sum_a [F^a, G^a, X] = \sum_a (0, i[F_M G^a - G_M F^a, X] + i[F^a, G^a]X_M),$$
which can be redefined as
\[ \delta_{HL} X \equiv \delta_H X + \delta_L X = (0, i[H, X]) + (0, L X_M), \] (74)
\[ H \equiv \sum_a (F^a_M G^a - G^a_M F^a), \quad L \equiv \sum a [F^a, G^a]. \] (75)
In this form, the matrix gauge variables \( H \) and \( G \), both of which are assumed to be traceless without losing generality, can be treated as two different Hermitian matrices that are completely independent of each other. Note that the distribution law for the 3-bracket under this gauge transformation,
\[ \delta_{HL}[X,Y,Z] = [\delta_{HL}X,Y,Z] + [X,\delta_{HL}Y,Z] + [X,Y,\delta_{HL}Z], \] (76)
is still satisfied after this extension, since each term with different \( a \) of the r.h.s of (73) satisfies (71), separately. In terms of this new notation, the original 3-bracket form (72) is no more meaningful at least for the gauge transformation law, as has also been the case for the extended classical Nambu flow equations mentioned in the end of the subsection 2.2 with the multiple pairs \( (H_a, G_a) \) instead of a single-pair of two-Hamiltonians. However, the 3-bracket notation will still be useful for describing the invariants (hence the action) under these generalized gauge transformations. For example, it is manifest, as an extension of (55), that
\[ \delta_{HL} \langle \prod_{i=1}^n [X_i, Y_i, Z_i] \rangle = 0 \] (77)
where the symbol \( \langle \prod_{i=1}^n A^{(i)} \rangle \equiv \text{Tr}(A^{(i)} \cdots) \) is used for the products of arbitrary number of the objects \( A^{(i)} (i = 1, 2, \ldots) \) whose M-components vanish, \( A_M^{(i)} = 0 \).

4.2. Covariantized M(atrix) theory

Now I present my proposal for the covariantized M(atrix) theory satisfying all of the three requirements (1)~(3) above. To make the following account reasonably short, only the final results for the bosonic part will be discussed. The further details, including the fermionic part, the reader should refer to the original paper [15]. The basic degrees of freedom are the 11-dimensinal coordinate vectors \( X^\mu = (X^\mu_M, X^\mu) \) and the conjugate momentum vector \( P^\mu = (P^\mu_M, P^\mu) \), both of which are functions of the single Lorentz-invariant proper time \( \tau \). They are canonical conjugates to each other classically, in the sense of Lorentz-invariant canonical formalism with respect to \( \tau \), satisfying the usual canonical (equal-time) Poisson bracket relations:
\[ \{X^\mu_M, P^\nu_M\}_P = \eta^{\mu\nu}, \] (78)
\[ \{X^\mu_{ab}, P^\nu_{cd}\}_P = \eta^{\mu\nu} \delta_{ad} \delta_{bc}, \] (79)
with all the other Poisson brackets being zero. In the second line, the indices \( a, b, \text{etc} \) represent the matrix elements of the matrix part of the variables. The meaning and the role of the M-components \( (X^\mu_M, P^\mu_M) \) will be elucidated later. It should be kept in mind that all these variables are assumed to be scalar under the re-parametrization \( \tau \to \tau' \) with respect to the proper time \( \tau \). In order to take into account the re-parametrization symmetry, we need to introduce the auxiliary ein-bein \( e(\tau) \) satisfying \( e(\tau) = (d\tau'/d\tau)e'(\tau) \).
Throughout this subsection, when it becomes necessary to separate the center-of-mass degrees and the rest, we use the following convention:

\[ X^\mu_0 \equiv \frac{1}{N} \text{Tr}(X^{\mu}), \quad X^\mu = X^\mu_0 + \hat{X}^\mu, \quad (80) \]

\[ P^\mu_0 \equiv \text{Tr}(P^\mu), \quad P^\mu = \frac{1}{N} P^\mu_0 + \hat{P}^\mu. \quad (81) \]

Thus any matrix with a hat is traceless by definition. The gauge transformation law is

\[ \delta_{HL} \hat{P}^\mu = i[H, \hat{P}^\mu] = \delta_H \hat{P}^\mu, \quad \delta_{HL} P^\mu_M = -\text{Tr}(LP^\mu) = \delta_L P^\mu, \quad (82) \]

with other components being inert. Together with the corresponding law for the coordinate variables \( X^\mu = (X^\mu_M, X^\mu) \) defined in the previous subsection, the Poisson bracket relations are invariant.

The total Poincaré integral corresponding to the canonical structure exhibited by the above Poisson bracket relations is given by

\[
\int d\tau \left[ P_M \frac{dX^\mu_M}{d\tau} + \text{Tr}\left( P_\mu \frac{dX^\mu}{d\tau} \right) \right] \equiv \int d\tau \left[ P_M \frac{dX^\mu_M}{d\tau} + P_{\nu} \frac{dX^\mu_{\nu}}{d\tau} + \text{Tr}\left( \hat{P}_\mu \frac{d\hat{X}^\mu}{d\tau} \right) \right], \quad (83)
\]

where the derivative symbols with upper-case \( D \) are the covariant derivatives, defined as follows.

\[
\frac{DX^\mu}{D\tau} = \frac{dX^\mu}{d\tau} - eB_0 X^\mu_M + e\text{Tr}(Z \hat{X}^\mu), \quad (84)
\]

\[
\frac{D\hat{X}^\mu}{D\tau} = \frac{d\hat{X}^\mu}{d\tau} + ie[A, X^\mu] - eBX^\mu_M, \quad (85)
\]

\[
\frac{DP^\mu_M}{D\tau} = \frac{dP^\mu_M}{d\tau} + e\text{Tr}((B + B_0)P^\mu) = \frac{dP^\mu_M}{d\tau} + e\text{Tr}(BP^\mu) + eB_0 P^\mu, \quad (86)
\]

\[
\frac{DP^\mu}{D\tau} = \frac{dP^\mu}{d\tau} + ie[A, P^\mu] - eZP^\mu, \quad (87)
\]

It should be noted here that the \( e \) is the ein-bein (not coupling constant), which is necessary to keep the reparametrization symmetry for the covariant derivatives. There are four gauge fields (the three traceless Hermitian matrices: \( A, B, Z \), and a scalar \( B_0 \)) whose gauge transformations are, for \( \delta_{HL} \),

\[
\delta_{HL} A = i[H, A] - \frac{1}{e} \frac{d}{d\tau} H \equiv -\frac{1}{e} DH, \quad (88)
\]

\[
\delta_{HL} B = i[H, B] - i[L, A] + \frac{1}{e} \frac{dL}{d\tau}, \quad (89)
\]

\[
\delta_{HL} Z = i[H, Z], \quad (90)
\]

\[
\delta_{HL} B_0 = \text{Tr}(LZ). \quad (91)
\]

We have included the additional new gauge symmetry \( \delta_w \) and \( \delta_Y \),

\[
\delta_w B_0 = \frac{1}{e} \frac{dw}{d\tau}, \quad \delta_w Z = 0, \quad (92)
\]

\[
\delta_Y B_0 = -\text{Tr}(YB), \quad \delta_Y Z = i[A, Z] + \frac{1}{e} \frac{dY}{d\tau} \equiv \frac{1}{e} \frac{DY}{d\tau}. \quad (93)
\]
for which the gauge transformation law of the canonical variables is given as

\[ \delta w X^\mu = w X^\mu, \quad \delta w P^\mu = 0, \quad \delta_{\dot{w}} X^\mu_M = 0, \quad \delta_{\dot{w}} P^\mu_M = -w P^\mu, \]

(94)

\[ \delta Y \hat{X}^\mu = 0, \quad \delta Y \hat{P}^\mu = P^\mu Y, \quad \delta Y X^\mu_M = -\text{Tr}(Y \hat{X}^\mu), \quad \delta Y P^\mu = 0, \]

(95)

where \( w \) and \( Y \) are an arbitrary function and an arbitrary traceless matrix functions, respectively, as parameters for the new gauge transformations. The canonical Poisson bracket relations are kept invariant under their actions. The other variables not shown explicitly are all inert. Note that the \( P^\mu \) and \( X^\mu_M \) are themselves completely gauge invariant. The role of the covariant derivatives with these gauge fields is of course to make the transformations laws the same after acting these covariant derivative to the canonical variables. All of these gauge transformations are summarized by defining the canonical generator function as

\[ C = w P^\mu \cdot X_M + \text{Tr} \left( -(P^\mu \cdot X) Y + iP^\mu [H, X^\mu] + (X_M \cdot P) L \right), \]

(96)

such that the transformation law are expressed by taking Poisson brackets between the \( C \) and the canonical variables. Here for brevity, some of the scalar products are represented by using dot symbols. In Fig. 1, the structure of the gauge symmetries as a whole is summarized.

![Fig. 1 Schematic structure of the higher gauge symmetries: the different shapes of the objects indicate different scaling dimensions (see below in the text) of canonical variables. The directions of arrows indicate how the variables are mixed into others (or into themselves) by gauge transformations. The row in the middle represents conserved vectors, while the top row represents the corresponding cyclic variables.]

We assume as usual that no \( \tau \)-derivatives are allowed in the Hamiltonian in terms of the canonical variables. This essentially amounts to assuming that the equations of motion do not involve higher derivatives than the second derivative. Then the above Poincaré invariant tells us the following four Gauss constraints that are obtained by the variational principle with respect to the gauge fields.

\[ \delta A : \quad [P^\mu, X^\mu] + \ldots \approx 0, \]

(97)

\[ \delta B : \quad \hat{P}^\mu X^\mu_M \approx 0, \]

(98)

\[ \delta Z : \quad P^\mu \dot{X}^\mu_M \approx 0, \]

(99)

\[ \delta B : \quad P^\mu_M X_M^\mu \approx 0, \]

(100)

where the notation \( \approx \) symbolizes that these constraints are regarded as weak equations before gauge fixing. This set of constraints is closed in the sense of Poisson bracket algebra and hence is of first-class. The first one with the ellipsis being the contribution from the fermionic part is the requirement of the SU(\( N \)) invariance, that exists already in the original M(atrix) theory which is supposed to be obtained from our system by an appropriate gauge
fixing in the light-like frame. The remaining three constraints are consequences from the 
higher gauge symmetries: the second corresponds to the shift symmetry originated from our 
definition of the 3-bracket, the third does to the $\delta_Y$ and the last does to the $\delta_w$.

We make a natural assumption that the center-of-mass momentum $P^\mu_0$ and the M-variables $X^\mu_M$ are conserved, $\frac{dP^\mu_0}{d\tau} = \frac{dX^\mu_M}{d\tau} = 0$, as being associated with translation symmetries with respect to the center-of-mass coordinate $X^\mu_0$ and the M-component $P^\mu_M$ of the canonical momentum variables, respectively; these are cyclic coordinates in our system. Since the $P^\mu_0$ is a time-like constant vector, we can assume by the last constraint that the $X^\mu_M$ is a space-like constant vector. The third and the second constraints then imply that the time-component of the traceless coordinate degrees of freedom $\hat{X}^\mu$ and one of the space-components of the traceless momentum degrees of freedom $\hat{P}^\mu$ become unphysical, respectively. This fits our requirement of reducibility of our system to the M(atrix) theory with SO(9) degrees of freedom alone for the traceless matrix canonical variables. We will see that the coordinate M-variable $X^\mu_M$ determines the gauge-coupling constant, namely the expectation value of the dilaton in the sense of string theory.

Now let us proceed to the full bosonic action of this system. The simplest possible gauge-
invariant that is consistent with the M(atrix) theory is given as

$$A_{boson} = \int d\tau \left[ P^\mu_M \frac{dX^\mu_M}{d\tau} + P^\mu_0 \frac{DX^\mu_0}{D\tau} + \text{Tr} \left( \hat{P}^\mu \frac{D\hat{X}^\mu}{D\tau} \right) \right. $$

$$\left. - \frac{e}{2N} P^2_0 - \frac{e}{2} \text{Tr}(\hat{P} - P_0 K)^2 + \frac{e}{12} \langle [X^\mu, X^\nu, X^\sigma][X_\mu, X_\nu, X_\sigma] \rangle \right],$$

(101)

where the $K$ is an auxiliary traceless matrix variable which is introduced for the purpose of recovering the $\delta_Y$ gauge symmetry for the quadratic kinetic term $\hat{P}^2$ in the Hamiltonian with the transformation law $\delta_Y K$. The situation is analogous to the Stueckelberg formalism of the well-known gauge invariant formulation of a massive Abelian vector field: after a gauge choice $K = 0$ for the gauge transformation $\delta_Y$, the equations of motion for $K$ become the Gauss-type constraint,

$$P^\mu_0 \hat{P}^\mu = 0,$$

(102)

which, together with (99), eliminates completely the time component of the canonical pair of the traceless matrix variables.

The variation of the ein-bein $e$ gives the mass-shell condition

$$P^2_0 + M^2_{boson} \approx 0$$

(103)

where the effective mass-square is

$$M^2_{boson} = N \text{Tr} \hat{P}^2 - \frac{N}{6} \langle [X^\mu, X^\nu, X^\sigma][X_\mu, X_\nu, X_\sigma] \rangle$$

(104)

where we have assumed the gauge condition $K = 0$ and all of the Gauss-type constraints (97) $\sim$ (100) and (102). The second term, ‘potential energy’, in this expression takes the form

$$V \equiv - \frac{N}{2} \text{Tr} \left( X^2_M [X^\mu, X^\sigma][X_\nu, X_\sigma] - 2[X_M \cdot X, X^\nu][X_M \cdot X, X_\nu] \right),$$

(105)

in terms of the matrix notation.
It is easy to check that the equations of motion are consistent with these constraints. It should be noted that the cyclic coordinates $X_\mu^i$ and $P_\mu^i$ do not participate in the dynamics and, as such, are determined passively by other independent dynamical variables. For instance, the $P_\mu^i$ obeys $\frac{DP_\mu^i}{d\tau} = \frac{\partial V}{\partial X_\mu^i}$. As a matter of course, the center-of-mass momentum is fixed by the Hamiltonian constraint (i.e. the mass-shell condition above), which in turn determines the center-of-mass coordinate through the equations of motion, $\frac{dX_\mu^i}{d\tau} = \frac{1}{\ell^6_{11}} P_\mu^i$.

One of the remarkable properties of the above action is that it is invariant under the following scale transformation:

$$X_\mu \to \lambda X_\mu, \quad P_\mu \to \lambda^{-1} P_\mu, \quad X^i_\mu \to \lambda^{-3} X^i_\mu, \quad P^i_\mu \to \lambda^3 P^i_\mu,$$

$$A \to \lambda^{-2} A, \quad B \to \lambda^2 B, \quad H \to H, \quad L \to \lambda^4 L,$$

$$B_0 \to \lambda^2 B_0, \quad Z \to \lambda^{-2} Z, \quad K \to K.$$

The emergence of such a scale symmetry is not surprising if we recall that the effective Yang-Mills gauge theory can also be regarded as being invariant under the “generalized conformal symmetry” which has been playing some useful roles in extending the AdS/CFT correspondence for “non-dilatonic” branes such as D3-branes to the so-called “dilatonic” D-branes in the sense of the GKPW relation. See e.g. [17] and [18] and references therein. We now clearly see also that a role of the M-variable $X_\mu^i_M$ as a conserved space-like vector is to set the length scale in our system by fixing its absolute value, namely, the coupling constant for the potential $V$ as we have already alluded to. Furthermore, the choice of its space-like direction in the ambient 11 dimensional space-time, together with the time-like direction of another conserved vector $P_\mu^0$, fixes a particular two-dimensional plane embedded in the 11 dimensions, to which the transverse SO(9) directions ($\mu \to i = 1, 2, \ldots, 9$) are orthogonal. We call this plane “M-plane”.

Let us check explicitly how we can obtain the BFSS M(atrix) theory in the light-like gauge. The M-plane is foliated by the light-front coordinates $P^\pm_0 \equiv P_0^{10} \pm P_0^0$, $X^\pm_M \equiv X_M^{10} \pm X_M^0$, satisfying $P^+_0 X^-_M + P^-_0 X^+_M \approx 0$. Since $X^i_M$ and $P^0_0$ are space-like and time-like, respectively, we can assume that $X^+_M$ and $P^+_0$ are not zero. Then, by the $\delta_L$ gauge transformation, we can choose the gauge such that $\dot{X}^+ = 0$, which allows us to eliminate the $\dot{X}^-$ by the Gauss constraint (99) that can now be regarded as a strong equation:

$$0 = P^+_0 \dot{X}^+ + P^-_0 \dot{X}^- = P^+_0 \dot{X}^- \quad \Rightarrow \quad \dot{X}^- = 0. \tag{109}$$

For the momentum variables, the first-order equations of motion

$$\dot{P}^\pm = \frac{1}{e} \frac{d\dot{X}^\pm}{d\tau} + i[A, \dot{X}^\pm] - B X^\pm_M \quad \Rightarrow \quad -B X^\pm_M = 0 \tag{110}$$

allows us to conclude $B = \dot{P}^\pm = 0$ by using the Gauss constraint (98). Thus the light-like components of the traceless dynamical matrix degrees of freedom are completely eliminated. The effective mass square then takes the form

$$\mathcal{M}_{\text{boson}}^2 = N \text{Tr} \left( \dot{P}^2 - \frac{1}{2} X^2_M [X_i, X_j]^2 \right). \tag{111}$$

Thus the conserved Lorentz invariant $X^2_M$ determines the 11 dimensional gravitation length as

$$X^2_M = \frac{1}{\ell^6_{11}}. \tag{112}$$
The scaling symmetry is spontaneously broken by this identification. It seems natural to interpret this emergence of the fundamental scale of M-theory as a super-selection rule in the sense that we do not allow superposition of states with different values of the Lorentz invariant $X^2_M$ in the Hilbert space after quantization.

It is easy to check that the equations of motion for the center-of-mass variables are reduced to the usual ones $P^\pm_0 = N \frac{dX^\pm_0}{dx}$ under the gauge choice $B_\sigma = 0$ for the $\delta_\omega$-gauge transformation, where we defined the reparametrization invariant time parameter $s$ by $ds = ed\tau$. To summarize all, the effective action for the physical transverse variables can be expressed as

$$A_{\text{light-front}} = \int dx^+ \frac{1}{2R} \text{Tr} \left[ \left( \frac{D\hat{X}_i}{Dx^+} \right)^2 + \frac{R^2}{2\ell_{11}^6} [\hat{X}_i, \hat{X}_j]^2 \right].$$ (113)

Here, we have redefined the time parameter by $s = 2N x^+/P^+_0$ (or equivalently $X^+_0 = 2x^+$).

This reproduces the identification of the continuous parameter $R$ by (69) in subsection 4.1.

Finally, we would like to add a remark on some peculiar nature of the higher gauge symmetry. The fact that we can eliminate both of the traceless parts, $\hat{X}^\pm$, of the matrix degrees of freedom means that the space-time directions corresponding to the M-plane are locally unobservable with respect to the dynamics of M-theory partons. In contrast to this, another light-like component $X^-$ of a single string (or of single membrane) is non-vanishing in the light-front gauge $\partial_\sigma X^+ = 0$: it is expressed in terms of the transverse components and behaves as a passive variable that does not participate in the dynamics. In our case, if $X^+_M \hat{X}^-$ in the potential $V$ were not eliminated, we would have a term $-(X^+_M X^-_{ab}(x^a_i - x^b_i))^2$ giving non-zero potential of wrong sign for purely diagonal configurations of the transverse directions. The absence of this term is consistent of a remarkable aspect of general-relativistic interactions of M-theory partons that the bundles of parallel trajectories of partons are exact classical solutions. It should be remembered here that the parallel pencil-like trajectories of massless particles are non-interacting to each other in the classical 11 dimensional general relativity. This corresponds to the well known property that for the metric of the form

$$ds^2 = dx^\mu dx_\mu + h_{--}(dx^-)^2,$$ (114)

with the coordinate condition $\partial_+ h_{--} = 0$, the vacuum Einstein equation reduces to the linear Laplace equation $\partial^2 h_{--} = 0$ in the transverse space. These properties make possible the interpretation of states with higher quantized momenta $P^+_0$ as composite states consisting of constituent states with unit momentum $1/R$ along the compactified spatial direction.

There are many remaining problems left to the future. Most importantly, although 11-dimensional Lorentz covariance is realized at least kinematically, it is not clear whether this reformulation of the M(atrix) theory may lead to further insight on the nature of the M-theory conjecture, especially its non-perturbative dynamical aspects of string/M theory. I can only hope that the present discussion would be an intermediate step toward such a goal that has not been attained for more than two decades since its first inception occurred during various explosive developments in the 1990s. In connection with this, the generalization of the present formalism to matrix-string theory seems to be relevant. For example, we may try to make covariant the procedure adopted in [19] (to which I would like to refer readers for a bibliography) for deriving matrix-string theory directly from the classical theory of (super) membranes. The emergence of the Nambu-type brackets may also be useful in extending the system to various non-trivial backgrounds (see, e.g. [20]).
5. Generalized Hamilton-Jacobi formalism of Nambu mechanics

We now return to Nambu’s original motivation of devising a new possible canonical formulation of dynamical development and its quantization. Concerning quantization, there have been a number of further discussions continuing and improving Nambu’s attempt in various directions. However, it seems to be a remarkable fact that there has been no serious discussion in the spirit of wave-mechanical quantization, until the attempt [21] of the present author in which a Hamiltonian-Jacobi like reformulation of the Nambu mechanics was proposed as a plausible prerequisite for quantization. That formulation then suggested a natural approach to quantum theory. As in the previous section, the purpose of this section is to concentrate on my motivation toward such a direction and explain the basic ideas of this approach, leaving the details to this reference.

In fact, a naive thinking along the traditional way of formulating the usual Hamilton-Jacobi (HJ) formalism in the ordinary analytical dynamics suggests the following obstacles against such a direction:

(i) The canonical triplet $\xi_i$ ($i = 1, 2, 3$) does not lend natural decomposition of the phase space into pairs of generalized coordinates and momenta.

(ii) There is no known explicit formulation of finite canonical transformation, as opposed to the infinitesimal one. This is closely related to what we have discussed in subsection 2.3. Remember that the usual text-book formulation of the HJ formalism rests upon the generating function for finite canonical transformation.

(iii) There is no action integral defined for each one-dimensional trajectory in the phase space, as opposed to a known action [5] function for a continuous family, forming a two-dimensional surface, of such trajectories.

In connection with (iii), it should be emphasized again that the action principle of [5] suffers from an infinite redundancy from the viewpoint of dynamics since a continuous family of trajectories has only kinematical meaning: the string-like world surface as the continuous collection of one-dimensional trajectories has nothing to do with the physical force or tension along the ‘$\sigma$’ direction. This feature perhaps correlates with the difficulty (ii).

A similar infinite redundancy also appears when we try to quantize the local dynamics of Yang-Mills field in terms of the set of non-local gauge invariant Wilson loops. In that case, however, we can express the local dynamics of quantum Yang-Mills fields in the form of a non-local field theory by deriving appropriate set of constraints for the string fields corresponding to the Wilson loops, such that the result gives a theory of string fields which is in principle equivalent with the usual formulation in terms of the original local fields. For an explicit construction of such a dynamical system of non-local string fields that is guaranteed to be equivalent with the usual local formulation in the context of lattice gauge theory, see my early work [22] and references therein. In this case, there is also an important physical motivation for such a direction, since in the long-distance strong-coupling regime the string-like objects of colored electric fluxes become the dominant physical excitations. Furthermore, that would also be relevant for weak-coupling regime in the large $N$ limit from the viewpoint of the planar expansion of Feynman graphs.

By constrast, in the case of the Nambu mechanics, we have to keep in mind that no definite and universally acceptable quantum formulation in terms of the local dynamical variables has been established. There do not seem sound physical motivations analogous to the flux
strings or the large $N$ limit, either. Indeed, what we have discussed in section 4 is the application of the Nambu equations of motion not for realizing the dynamics of membranes, but only for a clue toward higher symmetries within the framework of usual Hamiltonian dynamics. In my viewpoint, it is important to discriminate symmetry and dynamics. In the present section, the main focus is on the latter, though the former will also play an essential role.

Historically, there have been several attempts at generalizing HJ formalism to systems with higher symplectic forms. For a convenient review, the reader is recommended to consult [23]. Indeed, any field theory, including world-volume theories of extended objects can formally have a higher symplectic form whose rank coincides with the dimensions of the object, as we dealt with the action (41). However, all those works have concerned only about the various cases with a single Hamiltonian. As a notable example, we can mention Nambu’s attempt [24] toward a novel HJ-like formulation of strings which appeared in 1980. That seems to be in fact equivalent with the earlier De Donder-Weyl formalism as reviewed in [23].

5.1. Preliminaries

Now we can proceed to our main theme of this section. Fortunately, there is an interesting reformulation of the HJ formalism that does not presuppose any knowledge of an action functional nor of a concrete form for finite canonical transformations, starting from scratch with only the equations of motion. This was suggested by Einstein hundred years ago in an almost unknown short paper [25] where he emphasized that his method was “free of surprising tricks of trades”. His arguments arose from his attempt [26] at generalizing the Sommerfeld-Epstein quantization to non-separable cases and giving a coordinate-independent formulation of semi-classical quantum theory. In this subsection, a derivation of the standard HJ formalism à la Einstein will be reviewed first for the usual Hamiltonian formalism. After this preparation, generalization to the Nambu mechanics will be given in the next subsection.

The basic idea is that the trajectories $q^i = q^i(t), p_i = p_i(t)$ obeying the Hamilton equations of motion in the usual phase space

\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}
\]  

are reformulated to describe the time developments of the momentum $p_i$ as the fields $p_i = p_i(q,t)$ defined on the configuration space of the the coordinates $q^i$’s. Namely, using the modern terminology, the phase space is interpreted as a fiber bundle (called the ‘cotangent bundle’) over the configuration space, $p_i$’s and $q_i$’s being the parameters describing the fiber and base space, respectively. The momentum part of the Hamilton equations of motion is then rewritten, using the coordinate part, as

\[
\frac{dp_i}{dt} = \frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial p_i}{\partial q^j} = -\frac{\partial H}{\partial q^i}
\]

The second equality

\[
\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial p_j} \frac{\partial p_i}{\partial q^j} + \frac{\partial H}{\partial q^i} = 0
\]  

(116)

can be interpreted as the partial differential equations of first order for the vector fields $p_i(q,t)$. Let us call this type of equations an “Euler-Einstein” (EE) equation, since it is analogous to Euler equations of motion in fluid mechanics. In order to avoid possible confusions,
we introduce the notation
\[
\bar{H} = \bar{H}(q, t) = H(p(q, t), q),
\] (117)
in terms of which the EE equation is \(\frac{\partial p_i}{\partial t} + \frac{\partial \bar{H}}{\partial q_i} = 0\). The next step is to require that the flow described by the EE equation has no vorticity and hence can be expressed as a gradient flow associated with a scalar potential field \(J = J(q, t)\):
\[
\frac{\partial p_j}{\partial q_i} - \frac{\partial p_i}{\partial q_j} = 0 \quad \Rightarrow \quad p_i = \frac{\partial J}{\partial q_i}.
\] (118)
This requirement replaces the demand for the existence of the generating function arising from the action in the textbook formulation of the HJ formalism. Einstein’s motivation for this requirement is that the action integral \(\int p_i dq_i\) along a curve in the configuration space should take the same value for all deformations that can be continuously connected to each other, at least locally, under the condition of fixed end points. That would assign a special invariant meaning to the integral \(\oint p_i dq_i\) which is subjected to the Sommerfeld-Epstein condition for quantization. Then, the EE equation takes the form
\[
\frac{\partial}{\partial q_i} \left( \frac{\partial J}{\partial t} + \bar{H} \right) = 0 \quad \Rightarrow \quad \frac{\partial J}{\partial t} + \bar{H} = f(t)
\]
where \(f\) is an arbitrary function of time only. But we can eliminate this arbitrariness by redefining \(J\) with \(\partial_t S = \partial_t J - f\). Thus we arrive at the HJ equation
\[
\frac{\partial S}{\partial t} + \bar{H} = 0, \quad p_i = \frac{\partial S}{\partial q_i}.
\] (119)
It seems fairly obvious that this process does not depend on the dimensionality of the phase space and the configuration space, nor on the existence of the action integral, at least explicitly. The dimensions of the fiber and base space can have different dimensions.

It is a standard matter to reverse this process, obtaining the Hamilton equations of motion for trajectories \((q^i(t), p_i(t))\) starting from the HJ equation. For the sake of reminder, let us recall its essence here. First consider the momentum vector fields obtained from a solution of the HJ equation. They satisfy
\[
\frac{dp_i}{dt} = \frac{\partial p_i}{\partial t} + \frac{\partial p_i}{\partial q^j} dq^j = \frac{\partial^2 S}{\partial q^i \partial t} + \frac{\partial^2 S}{\partial q^j \partial q^i} dq^j = -\frac{\partial H}{\partial q^i} - \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial p_j \partial q^i} + \frac{\partial^2 S}{\partial q^j \partial q^i} dq^j = -\frac{\partial H}{\partial q^i}
\]
provided we know that the \(q^i\)'s satisfy the coordinate part of the Hamilton equations of motion. Therefore our problem is reduced to deriving the latter. Given the ‘complete’ solution of the HJ equation, this is achieved by imposing the so-called Jacobi condition. Remember that a ‘complete’ solution is a solution of the HJ equation with \(n\) independent integration constants \(Q_i\) when the dimensions of the configuration space is \(n\). Then the \(S\) can be regarded as a function \(S = S(q, Q; t)\) of the \(2n\) independent variables \((q_i, Q_i)\) and \(t\). In the usual textbook derivation of the HJ equation, \(q^i\)'s and \(Q^i\)'s actually arise as the set of initial and final coordinates, respectively, for general trajectories satisfying the equations of motion. The
The Jacobi condition is the following restrictions imposed upon a complete solution

\[ \frac{\partial S}{\partial Q^i} = -P_i, \]  

(120)

with an additional set of \( n \) integration constants \( P_i \). Under the solvability condition

\[ \det \left( \frac{\partial^2 S}{\partial Q^i \partial q^j} \right) \neq 0, \]  

(121)

they implicitly determine the coordinates \( q^i \)'s as the functions of time, satisfying the coordinate part of the Hamilton equations of motion: by performing a total differentiation of the Jacobi condition with respect to time \( t \), we obtain

\[
0 = \frac{\partial^2 S}{\partial Q^i \partial t} + \frac{\partial^2 S}{\partial Q^i \partial q^j} \frac{dq^j}{dt} = -\frac{\partial H}{\partial Q^i} + \frac{\partial^2 S}{\partial Q^i \partial q^j} \frac{dq^j}{dt} = -\frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial q^j \partial Q^i} + \frac{\partial^2 S}{\partial Q^i \partial q^j} \frac{dq^j}{dt},
\]

which indeed leads to the desired equations under (121). The Jacobi condition is the clue to the quantization.

In the language of differential forms, the above process of arriving at the HJ equation is summarized as follows: define a closed (and exact) 2-form in the phase space adjoined by a time variable,

\[ \omega^{(2)} = dp_i \wedge dq^i - dH \wedge dt = d\omega^{(1)}, \quad \omega^{(1)} = p_i dq^i - H dt. \]  

(122)

Then, the demand that the 2-form \( \omega^{(2)} \) vanishes when it is evaluated after making projection to the configuration space by assuming \( p_i = p_i(q,t) \) reproduces the EE equation.

\[ \tilde{\omega}^{(2)} \equiv \omega^{(2)}|_{(q,t)} = \frac{1}{2} \left( \frac{\partial p_j}{\partial q^i} - \frac{\partial p_i}{\partial q^j} \right) dq^i \wedge dq^j - \left( \frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q^i} \right) dq^i \wedge dt = 0. \]  

(123)

Then, the 1-form \( \omega^{(1)} \) in the phase space must be an exact form

\[ \tilde{\omega}^{(1)} \equiv \omega^{(1)}|_{(q,t)} = d\tilde{\omega}^{(0)} = \frac{\partial S}{\partial q^i} dq^i + \frac{\partial S}{\partial t} dt \]  

(124)

in terms of a scalar function \( \tilde{\omega}^{(0)} \equiv S(q,t) \) after the projection. This is nothing but the HJ equation. As a matter of fact, this is the well known modern mathematical form of the HJ formalism [27] (see also [23]), which is usually explained on the basis of the existence of the action integral, since the function \( S \) as the generator of finite canonical transformation describing the equations of motion is originated from the action.

5.2. Generalized HJ formalism for Nambu mechanics

For generalizing the HJ formalism to the Nambu mechanics along the foregoing procedure, it is convenient to set three steps as follows.

(I) The EE equations from the equations of motion: the phase space \( (\xi^1, \xi^2, \xi^3) \) is decomposed into fiber and base space.

(II) The HJ-like equations from the EE equations: we have to find appropriate vanishing condition under the projection to the base space, in the sense explained above.

(III) The equations of motion from the HJ equation: we have to find appropriate Jacobi-like condition.
The steps (I) and (II) are relatively straightforward, but the last step (III) that is most crucial for quantization will turn out to be rather difficult, since at least naively the solvability condition (121) associated with (120) seems to require the same dimensions for the configuration space and momentum space as the fiber directions.

There are two possibilities for the decomposition,

\begin{align*}
(1/2)\text{decomposition} : (\xi^1,\xi^2,\xi^3) & \rightarrow (\xi^1,\xi^2,\xi^3(\xi^1,\xi^2,t)), \\
(2/1)\text{decomposition} : (\xi^1,\xi^2,\xi^3) & \rightarrow (\xi^1,\xi^2(\xi^1,t),\xi^3(\xi^1,t)).
\end{align*}

In this review, only the first case will be treated in some detail. For the second case, the final results will be described briefly.

5.2.1. Step (I). We define

\begin{align*}
\bar{H} &= \bar{H}(\xi^1,\xi^2,t) = H(\xi^1,\xi^2,\xi^3(\xi^1,\xi^2,t)), \\
\bar{G} &= \bar{G}(\xi^1,\xi^2,t) = G(\xi^1,\xi^2,\xi^3(\xi^1,\xi^2,t)).
\end{align*}

The EE equation for the field \(\xi^3 = \xi^3(\xi^1,\xi^2,t)\) is derived by the same method as in the usual Hamilton equations of motion. For notational brevity, the partial derivatives with respect to the base space coordinates and time will be abbreviated as \(\partial_i\) and \(\partial_t\). Using the Jacobian form (14) of the Nambu equation, we obtain

\[
\frac{\partial(H,G)}{\partial(\xi^1,\xi^2)} = \partial_1\xi^3 + \partial_1\xi^3 \frac{\partial(H,G)}{\partial(\xi^2,\xi^3)} + \partial_2\xi^3 \frac{\partial(H,G)}{\partial(\xi^3,\xi^1)}. \tag{127}
\]

On the other hand, the Jacobians are in terms of \(\bar{H}\) and \(\bar{G}\) are rewritten, using \(\partial_1\bar{H} = \partial_i\bar{H} + \partial_3\bar{H}\partial_i\xi^3, \partial_1\bar{G} = \partial_i\bar{G} + \partial_3\bar{G}\partial_i\xi^3\) as

\[
\begin{align*}
\frac{\partial(H,G)}{\partial(\xi^2,\xi^3)} &= \partial_2\bar{H}\partial_3\bar{G} - \partial_2\bar{G}\partial_3\bar{H}, \\
\frac{\partial(H,G)}{\partial(\xi^3,\xi^1)} &= \partial_3\bar{H}\partial_1\bar{G} - \partial_3\bar{G}\partial_1\bar{H}, \\
\frac{\partial(H,G)}{\partial(\xi^1,\xi^2)} &= \partial_1\bar{H}\partial_2\bar{G} - \partial_1\bar{G}\partial_2\bar{H} - \partial_1\bar{G}\partial_3\bar{H} - \partial_3\bar{G}\partial_2\bar{H} - \partial_2\xi^3(\partial_3\bar{H}\partial_2\bar{G} - \partial_3\bar{G}\partial_2\bar{H}) - \partial_2\xi^3(\partial_1\bar{H}\partial_3\bar{G} - \partial_1\bar{G}\partial_3\bar{H}).
\end{align*}
\]

By substituting these expressions to (127), we finally obtain

\[
\partial_1\xi^3 = \partial_1(\bar{H}\partial_2\bar{G}) - \partial_2(\bar{H}\partial_1\bar{G}), \tag{128}
\]

which is indeed the desired EE equation, since the r.h.side is a known algebraic function of \(\xi^1,\xi^2,\xi^3(\xi^1,\xi^2,t)\) and the derivatives \(\partial_3\xi^3(\xi^1,\xi^2,t)\). This finishes the step (I).

5.2.2. Step (II). The result of the step (I) naturally suggests the answer to step (II). (128) shows that the field \(\xi^3\) is expressed as the vorticity of a two-component vector field \((S_1,S_2)\) as

\[
\xi^3 = \epsilon^{3ij}\partial_j S_j = \partial_1 S_2 - \partial_2 S_1. \tag{129}
\]

The EE equation then takes the form of a set of paired partial differential equations for \(S_i:\)

\[
\partial_i S_i = \bar{H}\partial_i\bar{G} + \partial_i S_0, \tag{130}
\]

\[\text{(Please note that there are trivial typos in eq. (25) of [21].)}\]
where $S_0$ is an arbitrary scalar field and
\[
\bar{H} = H(\xi^1, \xi^2, \partial_1 S_2 - \partial_2 S_1), \quad \bar{G} = G(\xi^1, \xi^2, \partial_1 S_2 - \partial_2 S_1).
\]

In this formalism, the N-gauge symmetry is effectively absorbed as a degree of freedom of making a redefinition $S_0 \to S_0 - \Lambda$, keeping the r.h.side of (130) in the transformation of the pair of Hamiltonians $(H, G)$ into a new pair $(H', G')$. Independently of the N-gauge symmetry, this system is characterized by a new gauge symmetry of its own,
\[
S_\mu \to S_\mu + \partial_\mu \lambda
\]
where $S_\mu \ (\mu = 1, 2, 0)$ is the three-dimensional vector field $(S_1, S_2, S_0)$ on the base `space-time' $\xi^\mu = (\xi^1, \xi^2, t)$. We call this extended gauge symmetry ”S”-gauge symmetry. The S-gauge symmetry is nothing to do with the gauge symmetry associated with the gauge field $A_i$ of (18). Note that the latter is now connected to the field strength $F_i^S \equiv \partial_i S_0 - \partial_0 S_i$ which is invariant under the S-gauge transformations. As a matter of course, this gauge symmetry suggests the Wilson lines $e^{\int S_\mu d\xi^\mu}$ as a natural set of gauge-invariant observables. But, we have already emphasized that use of them as basic dynamical variables would annoy us with too much and unnecessary redundancy.

The above result can be recast in the language of differential forms as follows. Obviously, we have already a natural 1-form on the base space-time
\[
\bar{\Omega}^{(1)} \equiv S_\mu d\xi^\mu.
\]
The generalized HJ equation is then expressed by the condition
\[
\bar{\Omega}^{(2)} \equiv d\bar{\Omega}^{(1)} = (\partial_1 S_2 - \partial_2 S_1)d\xi^1 \land d\xi^2 + (\partial_1 S_0 - \partial_0 S_1)d\xi^i \land dt
\]
\[= \xi^3 d\xi^1 \land d\xi^2 - \bar{H} \partial_1 \bar{G} d\xi^i \land dt. \tag{134}\]
The EE equation (128) is nothing but the vanishing condition for the 3-form $\bar{\Omega}^{(3)} = d\bar{\Omega}^{(2)}$.
\[
0 = \bar{\Omega}^{(3)} = \partial_0 \xi^3 d\xi^1 \land d\xi^2 \land dt - (\partial_1 \bar{H} \partial_2 \bar{G} - \partial_2 \bar{H} \partial_1 \bar{G})d\xi^1 \land d\xi^2 \land dt. \tag{135}\]

Unlike the standard HJ formalism, the fiber is not directly related to the tangent planes of the base space. Instead, the one-dimensional fiber parametrized by $\xi^3$ is the measure of vorticity in the base space. We may call our 3 dimensional phase space a ”vorticity bundle”.

We can also rephrase the structure of the vorticity bundle before making the Einstein projection. We are led to the 2-form on the total 4-dimensional space-time $(\xi^1, \xi^2, \xi^3, t)$
\[
\Omega^{(2)} \equiv \xi^3 d\xi^1 \land d\xi^2 - H dG \land dt \tag{136}\]
and the closed and exact 3-form
\[
\Omega^{(3)} \equiv d\Omega^{(2)} = d\xi^1 \land d\xi^2 \land d\xi^3 - dH \land dG \land dt, \tag{137}\]
such that they reduce to $\bar{\Omega}^{(2)}, \bar{\Omega}^{(3)}$ after the Einstein projection. The 2-form $\Omega^{(2)}$ coincides with the one that was used for defining the action integral [5] for 1-dimensional family of the Nambu-flow trajectories, as previously mentioned in subsection 3.1. That action principle is essentially equivalent to the null condition for the 3-form $\Omega^{(3)}$,
\[
i_L(\Omega^{(3)}) = 0 \quad \text{with} \quad L \equiv X^i \partial_i + \partial_t, \quad X^i \equiv \epsilon^{ijk} \partial_j H \partial_k G \tag{138}\]
where the symbol $i_L$ is interior multiplication with respect to the vector field operator $L$ acting on the differential form defined on the 4-dimensional space-time. This is the analog
of the similar property of $\omega^{(2)}$ of the ordinary Hamilton dynamics where the Hamiltonian flow with operator $V \equiv e^{ij} \partial_j H \partial_i + \partial_t$ satisfies the null condition $i_V(\omega^{(2)}) = 0$. Thus, the structure of the generalized HJ equation associated with the EE equation is precisely parallel to that in the standard Hamiltonian dynamics, provided the dimensions of the corresponding differential forms are up-graded by one: $(\bar{\omega}^{(1)}, \bar{\omega}^{(0)}) \rightarrow (\bar{\Omega}^{(2)}, \bar{\Omega}^{(1)}), (\omega^{(2)}, \omega^{(1)}) \rightarrow (\Omega^{(3)}, \Omega^{(2)})$.

5.2.3. Step (III). A prerequisite for this step is that, given the trajectory $(\xi^1(t), \xi^2(t))$ satisfying the Nambu equations of motion, $\xi^3$ determined by the EE equation automatically satisfy the last of the Nambu equation. As in the ordinary HJ formalism, this is easily checked by reversing the process of obtaining the EE equation from the Nambu equations. The real problem in step (III) is to show how the first and the second components (the ‘coordinate’ part) of the Nambu equations of motion are derived from the generalized HJ equations (130) by imposing some analog of the Jacobi condition.

Here a fundamental difficulty arises when we try to follow the usual procedure. In the standard case, the rationale for the Jacobi condition (120) is that we can unfold the dynamics to the vanishing Hamiltonian by finite canonical transformation whose generating function $S$ satisfies, after projection,

$$p_i dq^i - \bar{H} dt = dS + P_i dQ^i.$$  \hfill (139)

Namely, the $S$ can be treated as a function of $(q^i, Q^i)$ obeying

$$p_i = \frac{\partial S}{\partial q^i}, \quad P_i = -\frac{\partial S}{\partial Q^i}. \hfill (140)$$

Since the corresponding form in the case of the Nambu mechanics is $\bar{\Omega}^{(2)}$, one would expect a natural extension to be something like

$$\xi^3 d\xi^1 \wedge d\xi^2 - \bar{H} d\bar{G} \wedge dt = d\Sigma^{(1)} + Q_3 dQ_1 \wedge dQ_2$$

for an unfolding transformation to the vanishing Hamiltonian with three integration constants denoted by $(Q_1, Q_2, Q_3)$ and a possible generating 1-form $\Sigma^{(1)}$. But this amounts to treating the generating 1-form $\Sigma^{(1)}$ as a function of four variables $(\xi^1, \xi^2; Q_1, Q_2)$, which is obviously wrong since we have only three independent canonical variables. For a more detailed discussion on this problem, readers are referred to the original paper [21]. In the last reference, we have suggested two ways to resolve this difficulty. In the present review, only the first one which is somewhat restricted but simpler than the second will be treated to some extent for the purpose of showing an attitude to the present problem concretely.

The first resolution is to simplify the generalized HJ equation by assuming an axial gauge condition with respect to the N-gauge symmetry (or by an appropriate choice of canonical coordinates) for the choice of the set of Hamiltonians,

$$\partial_3 G = 0 \quad \rightarrow \bar{G} = G(\xi^1, \xi^2), \hfill (141)$$

and to go to the $S_0 = 0$ gauge with respect to the S-gauge symmetry. It may not be always justified to make this simplification. However, as we will see later explicitly, this is possible.

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5. Because of the difference of the ranks between $\omega^{(2)}$ and $\Omega^{(3)}$, the action principle along one-dimensional trajectories is impossible for the Nambu mechanics.
in an important class of systems including the case of the Euler top (7) mentioned in section 2. Then, we first set

\[ S_i = -S \partial_i G + g_i \]

where the single function \( S \) is defined by a solution to the ordinary-looking HJ equation

\[ \partial_t S = -\bar{H}, \]

by which we have

\[ \partial_t g_i = 0, \quad \xi^3 = \partial_1 G \partial_2 S - \partial_2 G \partial_1 S + \partial_1 g_2 - \partial_2 g_1. \]

Using the residual time-independent \( S \)-gauge transformation, we can set \( g_i = \epsilon^{ij} \partial_j g \) with a time-independent scalar function \( g \). Then, it is possible to absorb the \( g \) into \( S \) by making a redefinition \( S \rightarrow S + F \) where \( F \) is defined by

\[ F \partial_i G = \epsilon^{ij} \partial_j g, \]

which is indeed solved as

\[ d_G F = \partial_i \partial^i g, \quad d_G \equiv \epsilon^{ij} \partial_i G \partial_j \]

by assuming at least one \( \partial_i G \) is not zero. Therefore, without losing generality within the present gauge condition, we can set

\[ \xi^3 = d_G S. \]

The generalized HJ equation is now reduced to (142) adjoined with (145). This is almost the usual form except for the difference that there is only one momentum defined by the latter equation (in spite of the two-dimensional configuration space \( (\xi^1, \xi^2) \)), which is still a sort of vorticity composed by an outer product of two vectors \( \partial_i G \) and \( \partial_i S \), instead of the gradient of the usual HJ. Step (III) can now be achieved.

Suppose we have a complete solution to (142) with an integration constant \( Q_1 \). Since the value of \( G \) is preserved against the only differential operation involved in (142) due to \( d_G G = 0 \), we can assume the value of \( G = \bar{G} \) as the second integration constant \( Q_2 \). As the Jacobi condition, we impose

\[ \frac{\partial S}{\partial Q_1} = Q_3, \]

by introducing the third integration constant \( Q_3 \). These two conditions implicitly determine \( \xi^1, \xi^2 \) as functions of time, \( \xi^i(t; Q_1, Q_2, Q_3) \), under the condition that (146) is not invariant against \( d_G \), namely,

\[ \frac{\partial^2 S}{\partial \xi^1 \partial Q_1} \partial_2 G - \frac{\partial^2 S}{\partial \xi^2 \partial Q} \partial_1 G \neq 0. \]

It is also possible to derive the condition (146) by considering the canonical transformation under the condition \( \partial_3 G = 0 \), as discussed in [21].
Let us confirm how the above Jacobi condition leads to the Nambu equations of motion for $\xi^1, \xi^2$. Taking a total time derivative of (146) and $G = Q_2$ yields

\[
\frac{\partial^2 S}{\partial Q_1 \partial t} + \frac{\partial^2 S}{\partial \xi_1 \partial Q_1} \frac{d\xi_1}{dt} + \frac{\partial^2 S}{\partial \xi_2 \partial Q_1} \frac{d\xi_2}{dt} = 0,
\]

(148)

\[
\partial_t G \frac{d\xi_1}{dt} + \partial_2 G \frac{d\xi_2}{dt} = 0.
\]

(149)

Using (142), the first term of the first equation is rewritten as

\[
\frac{\partial^2 S}{\partial Q_1 \partial t} = -\frac{\partial H}{\partial \xi_3} \left( \partial_t G \frac{\partial^2 S}{\partial \xi_2 \partial Q_1} - \partial_2 G \frac{\partial^2 S}{\partial \xi_1 \partial Q_1} \right).
\]

(150)

Then, using the second equation, we can conclude, due to (147),

\[
\frac{d\xi_1}{dt} = -\partial_3 H \partial_2 G, \quad \frac{d\xi_2}{dt} = \partial_3 H \partial_1 G,
\]

(151)

which are indeed the Nambu equations of motion under the condition $\partial_3 G = 0$, as desired.

The second resolution of the difficulty in step (III) is essentially to decompose the two-dimensional base space ($\xi^1, \xi^2$) further into fiber and base space (1/1) ($\xi^1, \xi^2(\xi^1, t)$) in which $\xi^2$ is now regarded as a new fiber direction and $\xi^3$ is implicitly determined by setting the condition

\[
\bar{H}(\xi^1, \xi^2, \xi^3; \xi^1, \xi^2; E) = E\]

(152)

where $E$ is a constant, as in the time-independent HJ formalism in the ordinary Hamiltonian dynamics. Thus the decomposition of the phase space is made in two steps $(3) \rightarrow (1/2) \rightarrow (1/1/1)$. This method can in principle be applied to arbitrary choices of two Hamiltonians $(H, G)$ without using the N-gauge symmetry. In this case again, the generalized HJ equations are reducible to an almost usual HJ formalism, but with more complicated relation between the momentum variable and the coordinate $\xi^2$ along the fiber direction.

As for the details of the other case of decompositions of the phase space into fiber and base space, namely, the (2/1) decomposition ($\xi^1, \xi^2(\xi^1, t), \xi^3(\xi^1, t)$), the reader is recommended to consult the original paper [21]. Here we only mention that, in the language of differential forms, the generalized HJ system is characterized by the vanishing condition under projection for two independent 2-forms, instead of a single 3-form $\Omega^{(3)}$ of the (1/2) formalism,

\[
\Omega^{(2)}_2 = d\xi^2 \wedge d\xi^1 + (\partial_3 H dG - \partial_3 G dH) \wedge dt,
\]

(153)

\[
\Omega^{(2)}_3 = d\xi^3 \wedge d\xi^1 - (\partial_2 H dG - \partial_2 G dH) \wedge dt,
\]

(154)

which are related to the $\Omega^{(3)}$ by

\[
\Omega^{(3)} = -\Omega^{(2)}_2 \wedge (d\xi^3 - X^3 dt) = \Omega^{(2)}_3 \wedge (d\xi^2 - X^2 dt).
\]

(155)

It seems that this pair of two 2-forms has never been mentioned in the literature. Geometrically, they are related to the Nambu equations of motion by the null conditions in the same

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\(6\) (155) has not been explicitly mentioned in [21]. We also warn the reader that there are trivial typos: $d\xi^2 \rightarrow d\xi^1$ in eq. (79), and $\partial_1 \bar{H} = \partial_1 \bar{G} = 0 \rightarrow \partial_1 \bar{H} = 0$ in the first line in page 21.
sense as (138) for \(\Omega^{(3)}\),
\[ i_L(\Omega^{(2)}_2) = i_L(\Omega^{(2)}_3) = 0. \tag{156} \]

Using the language of the Einstein projection, these are equivalent with the vanishing
conditions (=EE equations)
\[ \hat{\Omega}^{(2)}_2 \equiv -\partial_t \xi^2 d\xi^1 \wedge dt + (\partial_3 H \partial_1 \hat{G} - \partial_3 G \partial_1 \hat{H}) d\xi^1 \wedge dt = 0, \tag{157} \]
\[ \hat{\Omega}^{(2)}_3 \equiv -\partial_\xi \xi^3 d\xi^1 \wedge dt - (\partial_2 H \partial_1 \hat{G} - \partial_2 G \partial_1 \hat{H}) d\xi^1 \wedge dt = 0, \tag{158} \]
where \(\hat{H}(\xi^1, t) \equiv H(\xi^1, \xi^2(\xi^1, t), \xi^3(\xi^1, t))\) etc. Actually, the final result of this case is
equivalent with the one we obtain through the second of the two methods in the
(1/2) formulation above, as is expected since in that method the base coordinate is the same
one-dimensional direction of \(\xi^1\) in the end, corresponding to the final decomposition \((1/1)/1\).

Of course, our discussion cannot exclude other possibilities for resolving the problem. It
might also be of some interest, quite independently of Nambu mechanics proper, to extend
the above structure of the ‘vorticity bundle’ to new higher canonical structures with multiple
vorticity components for configuration spaces of arbitrary dimensions.

5.3. An example: the Euler top
We exhibit an application of the formalism of the previous subsection, rather than pursuing
the general theory further. It would be useful for interested readers to get a concrete picture
on what the generalized HJ theory for the Nambu mechanics is. Here again we restrict
ourselves to the \((1/2)\) formalism.

In this case, we have \(H = \frac{1}{2}((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2), \quad G = \frac{1}{2}((\xi^1)^2/I_1 + (\xi^2)^2/I_2 + (\xi^3)^2/I_3).\)

By performing an N-gauge transformation with the generator \(\Lambda = H^2/(2I_3)\), we have a new
set of Hamiltonians satisfying the gauge condition \(\partial_3 G = 0,\)
\[ H \rightarrow H = \frac{1}{2}((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2), \quad G - H/I_3 \rightarrow G = \frac{\alpha}{2}(\xi^1)^2 + \frac{\beta}{2}(\xi^2)^2, \tag{159} \]
where
\[ \alpha \equiv \frac{I_3 - I_1}{I_3 I_1}, \quad \beta \equiv \frac{I_3 - I_2}{I_3 I_2}. \tag{160} \]

The generalized HJ equations are now reduced to
\[ \partial_t S = -\frac{1}{2}((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2), \tag{161} \]
\[ \xi^3 = \partial_1 G \partial_2 S - \partial_2 G \partial_1 S = \alpha \xi^1 \partial_2 S - \beta \xi^2 \partial_1 S. \tag{162} \]

It is convenient to change the variables \((\xi^1, \xi^2)\) to the elliptic coordinate \((G, u)\), using Jacobi’s
elliptic functions.
\[ \xi^1 = \left(\frac{2G}{\alpha}\right)^{1/2} \text{sn} u, \quad \xi^2 = \left(\frac{2G}{\beta}\right)^{1/2} \text{cn} u. \tag{163} \]

The modulus parameter \(k\) of the elliptic function will be fixed later such that the above HJ
equation takes the simplest form. Note that, though originally \(G\) is the radial coordinate,

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7It is possible to extend the previous discussions to the Nambu equations of motion in general
\(n\)-dimensional phase space, as discussed in the Appendix of [21].

8Our notations are such that \(\text{sn}^2 u + \text{cn}^2 u = 1, k \text{sn}^2 u + \text{dn}^2 u = 1, \text{sn}' u = \text{cn} u \text{dn} u, \text{cn}' u = -\text{sn} u \text{dn} u, \text{dn}' u = -k^2 \text{sn} u \text{cn} u\), with abbreviation \(\text{sn} u = \text{sn}(u; k)\ etc\ suppressing\ the\ modulus\ parameter\ \(k\).\)
the HJ equation is a differential equation with respect only to the coordinate $u$ and hence $G$ can be treated as a constant parameter. Then

\[ \xi^3 = - \frac{(\alpha\beta)^{1/2}}{\text{dn} u} \partial_u S. \]

By defining the reduced time-independent function $\bar{S}$ as

\[ S = -E(t - u_0) + \bar{S}(\xi^1, \xi^2) \]

where $E$ and $t_0$ are constants, the HJ equation becomes

\[ \frac{\alpha\beta}{2\text{dn}^2 u} (\partial_u \bar{S})^2 = E + \frac{G(\alpha - \beta)}{\alpha\beta} \text{sn}^2 u - \frac{G}{\beta}. \]

Then we choose the modulus parameter $k$ as

\[ k^2 = \frac{G(\beta - \alpha)}{\alpha\beta E_0}, \quad E_0 = E - \frac{G}{\beta}, \]

and the HJ equation can be integrated,

\[ \bar{S} = \frac{A}{\alpha\beta} \mathcal{E}(u), \quad \xi^3 = -\sqrt{2E_0} \text{dn} u, \quad A \equiv (2\alpha\beta E_0)^{1/2}, \]

where the last expression $\mathcal{E}(u)$ is known as the fundamental elliptic integral of second kind, obeying the differential equation,

\[ \frac{d\mathcal{E}}{du} = \text{dn}^2 u = \text{cn} u \text{dn} u \frac{\text{dn} u}{\text{cn} u} = \left( \frac{1 - k^2 \text{sn}^2 u}{1 - u^2} \right)^{1/2} \frac{d\text{sn} u}{du}. \]

In terms of the original independent base-space coordinates $(\xi^1, \xi^2)$, the final result for a complete solution with two integration constants $E$ and $t_0$ is

\[ S = -E(t - t_0) + \frac{A}{\alpha\beta} \int_0^{\xi^1/\sqrt{(\xi^1)^2 + \frac{2}{\alpha} (\xi^2)^2}} \left( \frac{1 - k^2 x^2}{1 - x^2} \right)^{1/2} dx. \]

Let us check whether this solution yields the original Nambu equations of motion following the prescription derived in the previous subsection. The Jacobi condition is

\[ \frac{\partial S}{\partial E} = \text{constant} \]

together with the condition $\frac{dG}{dt} = 0$. Since we have included $t_0$, the constant on the r.h.s. can be absorbed into the definition of $\bar{S}$. Now taking the total time derivative, the Jacobi
condition leads to
\[ \frac{d}{dt} \left( \frac{\partial S}{\partial E} \right) = \frac{du}{dt} \frac{\partial S}{\partial E} = 1. \] (173)

By some straightforward calculation using various properties of the elliptic functions (see [21]), we find
\[ u = At, \] (174)
up to the an arbitrary choice of the origin of time \( t \). This immediately gives the familiar general solution for the Euler top:
\[ \xi^1 = \sqrt{\frac{2G}{\alpha}} \text{sn} \, At, \quad \xi^2 = \sqrt{\frac{2G}{\beta}} \text{cn} \, At, \quad \xi^3 = -\sqrt{2(E - G/\beta)} \text{dn} \, At. \] (175)

If we compare this derivation with the one using the ordinary HJ treatments, our method seems much more elegant even from a practical viewpoint putting aside the question of principle, since the components of angular momentum themselves are directly the canonical variables: the process of rewriting the system in terms of the Euler angles as canonical coordinates is completely circumvented. Perhaps for other systems that can be succinctly expressed in the framework of the Nambu mechanics, it seems natural to expect the same merit.

5.4. Implication to quantization
The \((1/2)\) formulation outlined in subsection 5.3 with the axial gauge condition \( \partial_3 G = 0 \) suggests the following quantum version, at least semi-classically. The Hilbert space consists of functions \( \langle \xi^1, \xi^2 | \psi_1(t) \rangle \) and \( \xi^3 \) is interpreted as a first-order differential operator, ‘vorticity operator’
\[ \xi^3 \to -i\hbar (\partial_1 G \partial_2 - \partial_2 G \partial_1) = -i\hbar d_G. \] (176)
The Schrödinger equation is
\[ i\hbar \langle \xi^1, \xi^2 | \psi_1(t) \rangle = H(\xi^1, \xi^2, -i\hbar d_G) \langle \xi^1, \xi^2 | \psi_1(t) \rangle, \] (177)
which automatically preserves \( G \). In the WKB approximation, \( \langle \xi^1, \xi^2 | \psi_1(t) \rangle \sim e^{iS(\xi^1, \xi^2, t)}/\hbar \) this reduces to the HJ equation (142) with (145), as it should be.

One might wonder whether and how this quantization is understood from the viewpoint of the canonical structure of Nambu bracket emphasized in section 2. That is easily seen if we recall, as has been already pointed out in [5], the usual Poisson bracket structure is buried or subordinated in the Nambu bracket in the following sense. Define a 2-bracket
\[ \{ A, B \}_G \equiv \{ A, G, B \} \] (178)
by treating one, say \( G \), of the two Hamiltonians. The Jacobi identity for this 2-bracket is an automatic consequence of the FI identity.
\[ \{ \{ A, B \}_G, C \}_G + \{ \{ B, C \}_G, A \}_G + \{ \{ C, A \}_G, B \}_G = 0. \] (179)
If we assume the axial gauge condition \( \partial_3 G = 0 \), the canonical Nambu bracket is rewritten using this 2-bracket into
\[ \{ \xi^1, \xi^2 \}_G = 0, \quad \{ \xi^3, \xi^1 \}_G = -\partial_2 G, \quad \{ \xi^3, \xi^2 \}_G = \partial_1 G, \] (180)
and correspondingly the Nambu equations of motion to
\[ \frac{d\xi^i}{dt} = \{H, \xi^i\}_G. \]  \hspace{1cm} (181)
It is clear that by replacing the above 2-bracket by a commutator
\[-i\hbar \{ , \}_G \to [ , ],\]
we are naturally led to the above wave-mechanical quantization.

Although we have not presented in this review the details of the alternative treatment of the case of 1-dimensional base space \(\xi^1\) in terms of the \((1/1/1)\) or \((2/1)\) formalism, the final results in this case can also be explained on the basis of a subordinate 2-bracket in a similar way. The Hilbert space consists of functions \(\langle \xi^1 | \psi_2(t) \rangle\). We treat \(H\) as a conserved quantity \(H = E\) from the beginning and define \(\{ , \}_H = \{H, A, B\}\) assuming \(\partial_3 H \neq 0\). Then the canonical Nambu bracket takes the form
\[ \{\xi^1, \xi^2\}_H = \partial_3 H, \quad \{\xi^3, \xi^1\}_H = \partial_2 H, \quad \{\xi^3, \xi^2\}_H = -\partial_1 H, \]  \hspace{1cm} (182)
and the Nambu equations of motion is
\[ \frac{d\xi^i}{dt} = \{G, \xi^i\}_H. \]  \hspace{1cm} (183)
Under the constraint
\[ H(\xi^1, \xi^2(\xi^1, t), \xi^3(\xi^1, t)) = E, \]  \hspace{1cm} (184)
we also have
\[ 0 = \partial_2 H + \partial_3 H \partial_2 \xi^3, \quad 0 = \partial_1 H + \partial_3 H \partial_1 \xi^3, \]  \hspace{1cm} (185)
which are equivalent with the last two 2-bracket relation exhibited in (182). The first one in (182) is, after quantization, interpreted as giving the expression of the operator \(\hat{\xi}^2\) in terms of the differential operator \(\partial_1\)
\[ -i\hbar \partial_1 = -\int_{\hat{\xi}^2} dx \frac{dx}{\partial_3 H(\xi^1, x)} \]  \hspace{1cm} (186)
such that \([\xi^1, \hat{\xi}^2] = -i\hbar \partial_3 H(\xi^1, \hat{\xi}^2)\). The Schrödinger equation is now
\[ i\hbar \langle \xi^1 | \psi_2(t) \rangle = \hat{G}(\xi^1, \xi^2) \langle \xi^1 | \psi_2(t) \rangle, \]  \hspace{1cm} (187)
where the symbol \(\hat{G}\) means that \(G\) is regarded as a function of the base-space coordinate and the differential operator \(\hat{\xi}^2\) by eliminating \(\xi^3\) implicitly through the constraint (184). The complexity of this operator is a price we have to pay, since we do not assume any special gauge condition with respect to the N-gauge symmetry.

These two approaches to the quantization look quite different: even the dimensions of the base space are different from each other, the first being of two dimensions while the second of one dimension. And yet in the classical limit we must have the same Nambu equations of motion. Some of the readers may feel that our reliance on the 2-bracket structure is a backward step if one takes the viewpoint that two Hamiltonians \(H, G\) should appear on an equal footing. However, the presence of the N-gauge symmetry of the Nambu equations of

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10 If we consider the Euler top, this 2-bracket gives the usual algebra (10) of angular momenta.
motion means that such a description with manifestly symmetrical appearance of both $H$ and $G$ necessarily has a large degree of arbitrariness which is usually foreign to a definite form of quantum theory, as we have learnt in the gauge theory or the quantum gravity: to have a definite formulation we always need to fix such gauge degrees of freedom to some extent. Otherwise we have to attain a description only in terms of completely gauge-invariant observables. That would be a dauntingly difficult task either, if one is not allowed to make any approximation, such as lattice approximation for the Yang-Mills theory as we have alluded to in the introductory part of this section.

From these considerations, I emphasize that the real question of quantization in pursuing Nambu mechanics is how to make certain the invariance (or, more appropriately, covariance) of the dynamics and its physical interpretation under the N-gauge symmetry. It seems that, in most of past attempts, a sufficient attention has not been paid to this question. Our result for the generalized HJ equation (130) given in subsection 5.2 encompasses the N-gauge symmetry and exhibits the new S-gauge symmetry. Keeping the latter symmetry in the same sense as the N-gauge symmetry without introducing too much redundancy must be important from this viewpoint.

6. Concluding remarks

My motivation for studying the Nambu mechanics has originally been a hope that it might be useful for finding clues in exploring possible new methods of expressing dynamics at the most fundamental microscopic level in string/M theory or quantum gravity. In this review, I have discussed three works which I have done along this line from a unified and, as far as possible, elementary (or pedagogical) viewpoint. I have focused on the streams of basic ideas: in section 3 and 4, I have explained on how an aspect of the Nambu mechanics can be useful for thinking about possible new symmetries higher than the usual gauge symmetries with which we are familiar in formulating fundamental interactions. Then in section 5, a possible new canonical structure behind the Nambu mechanics is discussed from a different perspective, the Hamilton-Jacobi theory, which has been scarcely taken up previously. Although, at the present level of development, I could not provide any compelling reason for the relevance of our results from the viewpoint of my original motivation, I hope that the questions which I have been asking here would become relevant in the near future. An old truth may connect to a new truth, or “an old sake in a new cup” (a favorite saying of Nambu [4]) may sometimes help us, like the case where Hamilton-Jacobi theory in the 19th century connected to quantum mechanics in the 20th century. Perhaps, we should also learn more about Nambu’s passion and imagination for physics!

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A. Dirac’s attempt in the 1950s: a vortical stream in terms of the gauge potential

In this Appendix, I briefly review Dirac’s old works related in some broad sense to the ideas discussed in the present text, especially to the vortical structure associated with the Nambu mechanics which is the main theme of this workshop. In the workshop, this addendum was provided as a topic for one of brainstorming sessions, in the hope of casting a different light on our subject from a historical perspective.

From the viewpoint of the general theory of volume preserving flows, the Nambu equations of motion correspond to the particular (Clebsch) form for the potential function in (18),

\[ A_i - \partial_i \psi = H \partial_i G. \]  

(A1)

Essentially the same form was studied in a quite different context by Dirac in the 1950s. Dirac was pursuing a new classical theory of electrons, aiming at, as a final goal, a formulation of quantum electrodynamics that is completely free from ultraviolet infinities. He was thinking that the trouble of the usual QED was not a fault in the general principles of quantization, but should be ascribed to a wrong classical theory on which the usual formulation of QED was based. In one of his many attempts at remedying the situation, he tried to give exact classical equations without requiring any assumptions about the structure of electrons. In [28], he proposed to describe electric charges with no dynamical variables explicitly corresponding to them. As a first example of such a possibility, he proposed to study the Maxwell theory in terms of only the electromagnetic field under the gauge condition,

\[ A_\mu A^\mu = -k^2 \]  

(A2)

with the Lagrangian density

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \lambda (A_\mu A^\mu + k^2), \]  

(A3)

where \( k \) and \( \lambda \) are undetermined constant. The Maxwell equations lead to the current

\[ j_\mu = -\lambda (\partial_\mu S + A_\mu), \]  

(A4)

where \( S \) appears by taking into account the gauge degrees of freedom \( A_\mu \rightarrow A_\mu + \partial_\mu S \). Thus the above gauge condition is now

\[ (\partial_\mu S + A_\mu)(\partial^\mu S + A^\mu) = -k^2, \]  

(A5)

which can be interpreted as the Hamilton-Jacobi equation for an electron of mass \( m = ke \) in the field of the gauge potential \( A_\mu \). Interestingly, Nambu came across to the same result in 1968 [29]: he also discussed the QED with the same gauge condition (A2) and drew the same analogy with the HJ equation, without knowing Dirac’s old work at the time of his writing. He mentioned [28] only as a postscript to the paper at the end. Nambu’s original intention was to utilize the above gauge condition in order to interpret photon as a Goldstone boson arising as a result of (superficial) spontaneous breakdown of Lorentz invariance, providing the existence of a non-vanishing expectation value \( \langle j_\mu \rangle \neq 0 \).

Now let us go back to Dirac’s works in the 50s. He noticed a problem with his result: (A4) implies that the velocity field of the current \( j_\mu \) is equal to

\[ v_\mu = k^{-1}(\partial_\mu S + A_\mu) \]  

(A6)

and thus the vector \( kv_\mu - A_\mu \) is irrotational. Since in practice there are situations where this vector can be vortical, he generalized his argument in his next work [30] on this subject as follows.
First he replaces the usual classical (Lorentz) equations of motion for an electron moving with velocity, $k \frac{dv}{ds} = v_\nu F^{\mu\nu}$, to a stream of electrons, by looking upon $v^\mu$ as a continuous field as functions of space-time coordinates:

$$kv^\nu \partial_\nu v^\mu = v_\nu F^{\mu\nu}. \tag{A7}$$

Since by definition $v_\nu v^\nu = 1$, it follows that

$$v_\nu \partial_\mu v^\nu = 0 \rightarrow v_\nu f^{\mu\nu} = 0 \tag{A8}$$

where

$$f_{\mu\nu} = F_{\mu\nu} + k(\partial_\nu v_\mu - \partial_\mu v_\nu) = \partial_\mu (A_\nu - kv_\nu) - \partial_\nu (A_\mu - kv_\mu). \tag{A9}$$

(A8) allows us to conclude that there exists a vector $u^\mu$ such that

$$\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} = u^\sigma v^\nu - v^\sigma u^\nu. \tag{A10}$$

In this four-dimensional context, the identity

$$\epsilon^{\mu\nu\rho\sigma} \partial_\rho f_{\mu\nu} = 0 \tag{A11}$$

corresponds to the condition of incompressibility $\partial_i D^i = 0$, when the argument is adapted to the gauge potential $A_i$ of the 3-dimensional flow. Using this condition, Dirac shows that the two vectors $u_\mu$ and $v_\nu$ lie in integrable two-dimensional surfaces, and hence the stream lines of the electricity obeying $dx^\mu/ds = v^\mu$ are contained in these surfaces. He next introduces two independent scalar functions $\xi$ and $\eta$ which are constant on these surfaces,

$$v^\sigma \partial_\sigma \xi = u^\sigma \partial_\sigma \xi = v^\sigma \partial_\sigma \eta = u^\sigma \partial_\sigma \eta = 0, \tag{A12}$$

and hence satisfy

$$\epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} \partial_\rho \xi = 0, \quad \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} \partial_\rho \eta = 0. \tag{A13}$$

This ensures that

$$f_{\mu\nu} = \alpha (\partial_\mu \xi \partial_\nu \eta - \partial_\nu \xi \partial_\mu \eta), \tag{A14}$$

where $\alpha$ is some scalar function, satisfying

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu \xi \partial_\rho \eta \partial_\sigma \alpha = 0, \tag{A15}$$

because of the identity (A11). This implies that three vectors $\partial_\mu \xi, \partial_\nu \eta, \partial_\rho \alpha$ are coplanar, and thus $\alpha$ is a function of $\xi$ and $\eta$. Then we can redefine a new pair of scalar functions $h$ and $g$ such that the Jacobian is equal to $\alpha$,

$$\frac{\partial (h, g)}{\partial (\xi, \eta)} = \alpha, \tag{A16}$$

which allows us to express

$$f_{\mu\nu} = \partial_\mu h \partial_\nu g - \partial_\nu h \partial_\mu g = \partial_\mu (h \partial_\nu g) - \partial_\nu (h \partial_\mu g). \tag{A17}$$

This is equivalent to the promised representation with a suitable choice of gauge,

$$A_\mu = kv_\mu + h \partial_\mu g, \tag{A18}$$

which can involve a vortical component as desired.
The reader must be aware of interesting parallelisms between Dirac’s arguments and Nambu’s discussions for the gauge potential for volume-preserving flows. Dirac considered a stream of charges, while Nambu discussed a stream in 3-dimensional phase space, which is in general vortical. However, once we assume that there exist a field $v^j$ satisfying $F_{ij}v^j = 0$ for the field strength $F_{ij}$, Dirac’s argument from (A8) applies equally well to the 3-dimensional incompressible flows. That this condition is guaranteed can be convinced if we reverse his arguments. In the Dirac case, a stream line lies on two-dimensional planes in four dimensions characterized by two vectors $v^\mu$ and $u^\mu$. A stream line in the Nambu case lies in the intersections of two surfaces with constant $H$ and $G$.

We can also mention that due to the nature of vortical flows, the Hamilton-Jacobi theory suggested from their common analogy can not take the usual form and must somehow be generalized. There is thus some flavor of parallelism between Dirac’s discussion and our attempt of a generalized HJ theory discussed in section 5.

Actually Dirac himself abandoned this approach later, because of the difficulty of quantization, and proceeded to yet another new idea [31] which may be regarded as a precursor to modern string theory, or perhaps more appropriately to the string picture that emerges in the strong coupling regime of lattice gauge theory. In this work, he tried to formulate QED in a manifestly gauge invariant fashion. But I must stop here, since that would bring us to a subject which is too far from this special section.

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