The gonality and the Clifford index of curves on a toric surface

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Abstract

We determine the gonality and the Clifford index for curves on a compact smooth toric surface. Moreover, it is shown that their gonality are computed by pencils on the ambient surface. From the geometrical view point, this means that the gonality can be read off from the lattice polygon associated to the curve.

1 Introduction

In this paper, a curve always means a smooth irreducible projective curve over the complex number field unless otherwise mentioned. The gonality and the Clifford index are significant invariants in the study of linear systems on curves, which are defined by

\[
\text{gon}(C) = \min\{\deg f \mid f : C \to \mathbb{P}^1 \text{ surjective morphism}\} = \min\{k \mid C \text{ has } g_k \}
\]

\[
\text{Cliff}(C) = \min\{\deg D - 2h^0(D) + 2 \mid D \text{ : divisor on } C, h^0(D) \geq 2, h^1(D) \geq 2\}
\]

for a curve \(C\). A curve of gonality \(k\) is said to be \(k\)-gonal, and a pencil on a curve is called a gonality pencil if its degree is equal to the gonality. We cite several basic facts about the gonality and the Clifford index. Clearly, \(\text{gon}(C) = 1\) means that \(C\) is rational. The three statements \(\text{gon}(C) = 2, \text{Cliff}(C) = 0\) and \(C\) is elliptic or hyperelliptic are equivalent. Besides, \(\text{Cliff}(C) = 1\) holds if and only if \(C\) is trigonal or a smooth plane quintic curve. On the other hand, Brill-Noether theory gives us upper bounds \(\text{gon}(C) \leq \left\lfloor \frac{g + 3}{2} \right\rfloor\) and \(\text{Cliff}(C) \leq \left\lfloor \frac{g - 1}{2} \right\rfloor\) for a curve of genus \(g\), and equalities hold if the curve is general in moduli (cf. [1]). Lastly, we mention a close relation between these two invariants: \(\text{gon}(C) - 3 \leq \text{Cliff}(C) \leq \text{gon}(C) - 2\) (cf. [5]).

Although a considerable amount of work has revealed properties of the gonality and the Clifford index, it is still not easy to determine them for a given curve. Ideally, we would also like to see what kind of gonality pencils does a curve have. In fact, however, it is difficult even to know whether the number of gonality pencils is finite or infinite. There are only two systematic results giving satisfactory answers for these questions: the cases of plane curves and curves on Hirzebruch surfaces (Theorem 1.1 and 1.2). In this paper, we will study more general cases. Concretely, we consider curves on a compact smooth toric surface, and compute the gonality and the Clifford index (Theorem 1.3). From the point of view of the geometry of convex bodies, our result states that the gonality of such a curve coincides with...
the lattice width (see Definition 3.2) of the lattice polygon associated to the curve. Namely, we can read off the gonality by observing the shape of the lattice polygon. This fact has been conjectured by Castryck and Cools in [2], and our result gives an affirmative answer for it. In addition, Theorem 1.3 tells us that apart from a few exceptional cases, a curve on a toric surface has a finite number of gonality pencils, and moreover, they become restrictions of preassigned $\mathbb{P}^1$-fibrations of the surface called toric fibrations. On the other hand, in the process to prove the main result, we also obtain the lower bound for the self-intersection number of a curve on a toric surface (Corollary 3.8). This formula by itself is suggestive and of wide application, although which is just a tool in this paper.

Before we state the main theorem, let us review the cases of curves on the projective plane and Hirzebruch surfaces. First, the gonality and the Clifford index of plane curves are of wide application, although which is just a tool in this paper.

**Theorem 1.1** ([12, 6]). Let $C$ be a smooth plane curve of degree $d$. If $d \geq 2$, then $\text{gon}(C) = d - 1$ and any gonality pencil is cut out by lines passing through a fixed point on $C$. Furthermore, if $d \geq 5$, then $\text{Cliff}(C) = d - 4$.

Next, let $\Sigma_e$ be a Hirzebruch surface of degree $e$, and $\pi : \Sigma_e \to \mathbb{P}^1$ the ruling of $\Sigma_e$. Note that if $e = 0$, then $\Sigma_0$ has another ruling $\pi'$ to $\mathbb{P}^1$ whose fiber is a section of $\pi$. In this case, we may assume that $\deg\pi|_C \leq \deg\pi'|_C$. For curves on Hirzebruch surfaces, Martens has computed the gonality.

**Theorem 1.2** ([11]). Let $C$ be a smooth curve on $\Sigma_e$ which is not isomorphic to a smooth plane curve. Then $\text{gon}(C) = \deg\pi|_C$. In the case where $e \geq 1$, or $e = 0$ and $\deg\pi|_C < \deg\pi'|_C$, $\pi|_C$ is a unique gonality pencil on $C$. In the case where $e = 0$ and $\deg\pi|_C = \deg\pi'|_C$, $C$ has exactly two gonality pencils $\pi|_C$ and $\pi'|_C$.

In the case of Theorem 1.2, since the set of gonality pencils is finite, we obtain $\text{Cliff}(C) = \text{gon}(C) - 2$ (cf. [5]). Here we recall that the projective plane and Hirzebruch surfaces are simplest examples of toric surfaces. Hence, as a natural continuation of the above results, we expect to determine the gonality and gonality pencils of a curve on a toric surface. In order to give a precise statement, we recall some terminology. Let $S$ be a compact smooth toric surface. Then $S$ contains an algebraic torus $T$ as a nonempty Zariski open set together with an action of $T$ on $S$, which is a natural extension of the torus action of $T$ on itself. A prime divisor on $S$ is called a $T$-invariant divisor if it is invariant with respect to the above action. Any $T$-invariant divisor is isomorphic to the projective line. A blowing-down of a $T$-invariant divisor gives a morphism from $S$ to another toric surface. We call a composition of such morphisms an equivariant morphism. It is known that if $S$ is not a projective plane, there exist a finite number of equivariant morphisms from $S$ to Hirzebruch surfaces. Hence, by composing such equivariant morphisms and the rulings of Hirzebruch surfaces, we obtain a finite number of $\mathbb{P}^1$-fibrations of $S$. We call them toric fibrations. Now, we state the main theorem of this paper.

**Theorem 1.3.** Let $S$ be a compact smooth toric surface, and $C$ a $k$-gonal nef curve of genus $g \geq 2$ on $S$ which is not isomorphic to a smooth plane curve. Put $q = \min\{\deg\varphi|_C \mid \varphi : \text{toric fibration of } S\}$. Then $q$ is equal to the lattice width (see Definition 3.2) of the lattice polygon associated to $C$, and the followings hold.

(i) If $(g, q) \neq (4, 4), (5, 4), (10, 6)$, then any gonality pencil on $C$ is the restriction of a toric fibration of $S$. 


(ii) The equalities \[ k = \begin{cases} q & \text{if } (g, q) \neq (4, 4) \\ q - 1 & \text{if } (g, q) = (4, 4) \end{cases} \]
hold.

(iii) If \((g, q) \neq (10, 6)\), then \(\text{Cliff}(C) = k - 2\).

(iv) If \((g, q) = (10, 6)\), then \(C\) is a complete intersection of two hypercubics in \(\mathbb{P}^3\).

In the case \((g, q) = (4, 4)\), since \(C\) is trigonal by (ii), we see that \(C\) has one or two gonality pencils. In Section 5, we will show that both cases can occur. In the case \((g, q) = (5, 4)\), the gonality of \(C\) achieves the maximum of the upper bound \(\text{gon}(C) \leq \lfloor \frac{g + 3}{2} \rfloor\). It follows that \(C\) has infinitely many gonality pencils. If \((g, q) = (10, 6)\), by virtue of (iv) and Martens’ work [10], we see that \(\text{gon}(C) = 6\), \(\text{Cliff}(C) = 3\) and \(C\) has infinitely many gonality pencils. By a simple consideration, we can rewrite Theorem 1.3 as follows:

**Corollary 1.4.** Let \(S\) and \(C\) be as in Theorem 1.3.

(i) If \((g, k) \neq (4, 3), (5, 4), (10, 6)\), then any gonality pencil on \(C\) is the restriction of a toric fibration of \(S\).

(ii) If \((g, k) \neq (4, 3)\), then \(k = q\).

(iii) If \((g, k) \neq (10, 6)\), then \(\text{Cliff}(C) = k - 2\).

(iv) If \((g, k) = (10, 6)\), then \(C\) is a complete intersection of two hypercubics in \(\mathbb{P}^3\).

In Section 2 we review the theory of toric surfaces, which is the main stage of our study. The aim of Section 3 is to reveal several properties of the self-intersection number of a curve on a toric surface, which will be utilized to prove the key proposition (Proposition 4.3) and Theorem 1.3 in Section 4. Most proofs in Section 3, however, are just elementary and tedious computations for convex polygons in the affine plane. Hence the reader can skip them without losing the continuity of the paper. In Section 5, as already mentioned after Theorem 1.3, we investigate trigonal curves of genus four. Finally, as an application of our results, we compute Weierstrass gap sequences at ramification points of a trigonal covering of \(\mathbb{P}^1\) in Section 6. In fact, gap sequences at such points are well studied and the classification of them has been already completed ([3], [4], [7], [9]). However, by combining Corollary 1.4 with results in [8], we can compute gap sequences in a completely different way and propose a novel geometric interpretation of the reason why the difference between types of gap sequences occurs. This approach can be adapted not only to trigonal curves but also to curves of higher gonality. Hence, as a generalization of the results in Section 5, it is expected that we can classify Weierstrass gap sequences at ramification points of gonality pencils on a curve on a toric surface in the future.

### 2 Fans and lattice polygons

In this section, we briefly review basic notions in the theory of toric surfaces. For further background and applications of them, we refer the reader to [13] without explicit mention. We henceforth assume that a surface is always compact and smooth.

For a toric surface \(S\), there exists a fan \(\Delta_S\), which is the division of \(\mathbb{R}^2\) consisting of a finite number of half-lines starting from the origin called cones (see Fig. 1). Each cone \(\sigma(D_i)\) corresponds to a \(T\)-invariant divisor \(D_i\), and a lattice point on \(\sigma(D_i)\) is called a primitive element if it is closest to the origin. We denote by \((x_i, y_i)\) the primitive element of \(\sigma(D_i)\), and by \(\text{Pr}(S)\) the set of primitive elements of cones in \(\Delta_S\). We assume that \((x_1, y_1) = (0, 1)\).
The assumption that $S$ is smooth means that $x_{i+1}y_i - y_{i+1}x_i = 1$ for $i = 1, \ldots, d$, where we formally set $D_{d+1} = D_1$. The Picard group of $S$ is generated by the classes of $D_1, \ldots, D_d$. For instance, the canonical divisor of $S$ has the relation $K_S \sim -\sum_{i=1}^d D_i$. We next define a lattice polygon associated to a divisor on $S$, which is the essential notion in the study of curves on a toric surface.

**Definition 2.1.** For a divisor $D = \sum_{i=1}^d n_iD_i$ on $S$, a lattice polygon associated to $D$ is defined by $\square_D = \{(z, w) \in \mathbb{R}^2 \mid x_i z + y_i w \leq n_i \text{ for } 1 \leq i \leq d\}$.

Lastly, we mention the structures of fibers of toric fibrations. We define $\Pr^* (S) = \{(x, y) \in \Pr(S) \mid (-x, -y) \in \Pr(S)\}$ and $M(u, v) = \{(x, y) \in \Pr(S) \mid uy - vx < 0\}$ for integers $u$ and $v$.

**Fact 2.2.** For any primitive elements $(x_i, y_i)$ and $(x_j, y_j)$, we can uniquely write $(x_j, y_j) = \alpha_j(x_i, y_i) + \beta_j(x_{i+1}, y_{i+1})$ with some integers $\alpha_j$ and $\beta_j$. We can describe fibers of toric fibrations as follows:

$$\{\text{fibers of toric fibrations of } S\} = \left\{ \sum_{(x_j, y_j) \in M(x_i, y_i)} |\beta_j|D_j \mid (x_i, y_i) \in \Pr^*(S) \right\}.$$

## 3 The lower bound for the self-intersection number

Let $S$ be a toric surface. In this section, we will find the evaluation formula for the self-intersection number of a curve on $S$ for later use in the proof the key proposition (Proposition 4.3). First, we extend the notion of coprime.

**Definition 3.1.** For non-negative integers $x$ and $y$, we write $(x, y) = 1$ if they satisfy the following property: If either $x$ or $y$ is zero, then the other one is one. If both $x$ and $y$ are positive, then they are coprime.

**Definition 3.2.** Let $D$ be a divisor on $S$, and $x$ and $y$ integers with $(|x|, |y|) = 1$. We denote by $n(x, y)$ the minimal integer satisfying $\{(z, w) \mid xz + yw \leq n(x, y)\} \supset \square_D$. For $x$, $y$ and $n(x, y)$, we define

$$l(D, (x, y)) = \{(z, w) \in \mathbb{R}^2 \mid xz + yw = n(x, y)\},$$

$$L(D, (x, y)) = \{(z, w) \in \mathbb{R}^2 \mid xz + yw \leq n(x, y)\},$$

$$d(D, (x, y)) = n(x, y) + n(-x, -y).$$

In particular, we call $\min\{d(D, (x, y)) \mid x, y : \text{integers with } (|x|, |y|) = 1\}$ the lattice width of $\square_D$. 

![Figure 1](image-url)
Remark 3.3. By definition, if \((z_1, w_1) \in L(D, (x, y))\) and \((z_2, w_2) \in L(D, (x, y))\), then 
\[d(D, (x, y)) \geq x(z_1 - z_2) + y(w_1 - w_2).\]
In addition, an easy computation gives 
\[d(D, (x, y)) = \sum_{(x, y) \in M} (x \geq 0) D.\]

Let \(C\) be a curve on \(S\). By Fact [22] and Remark [3.3] we see that 
\(q = \min\{d(C, (x, y)) \mid (x, y) \in \Pr^*(S)\}\). Without loss of generality, we can assume that \((0, 1) \in \Pr^*(S)\) and 
\(d(C, (0, 1)) = q\). In the case where \(\sharp \Pr^*(S) = 2\), that is, \(\Delta_S\) contains only one line passing 
through the origin, we can assume that \((1, 0) \in \Pr(S)\) and \((x, y) \notin \Pr(S)\) if \(y < \max\{0, x\}\).

On the other hand, in the case where \(\sharp \Pr^*(S) \geq 4\), we can assume that \((1, 0) \in \Pr^*(S)\) and 
\(d(C, (1, 0)) \geq d(C, (1, 0))\) for any \((1, y) \in \Pr^*(S)\). In this paper, we will keep the above 
assumptions for \(C\) and \(\Delta_S\), and put \(q' = d(C, (1, 0))\).

Lemma 3.4. Let \(C\) be a nef curve on \(S\). For integers \(x \geq 1\) and \(y\) with \(|x|, |y| = 1\), the 
inequality \(d(C, (x, y)) \geq q'\) holds.

**Proof.** (i) Consider the case where \(\sharp \Pr^*(S) = 2\). We denote by \(O = (0, 0)\) (resp. \(P\)) the 
vertex of \(\square C\) on \(l(C, (0, -1))\) whose \(z\)-coordinate is minimal (resp. maximal). By a simple 
consideration, we see that \(P = (q', 0)\). Then by Remark [3.3] we have 
\(d(C, (x, y)) \geq xyq'\). (ii) In the case where \(\sharp \Pr^*(S) \geq 4\), we will prove only the case where both \(x\) and \(y\) are 
positive. One can show other cases by a similar method. We denote by \(n\) the maximal 
integer such that \((1, n) \in \Pr^*(S)\) and \(nx - y \leq 0\), and define 
\[P_1 = (z_1, w_1) = \begin{cases} l(C, (1, n)) \cap l(C, (1, 0)) & (n \geq 1), \\ l(C, (1, 0)) \cap l(C, (0, -1)) & (n = 0), \end{cases}\]
\[P_2 = (z_2, w_2) = \begin{cases} l(C, (-1, -n)) \cap l(C, (-1, 0)) & (n \geq 1), \\ l(C, (-1, 0)) \cap l(C, (0, 1)) & (n = 0). \end{cases}\]
In the case where \(n \geq 1\), since 
\[d(C, (1, n)) = z_1 + nw_1 - z_2 - nw_2 = q' + n(w_1 - w_2) \geq q',\]
we have \(w_1 \geq w_2\). Accordingly, we have 
\(d(C, (x, y)) \geq \alpha xw_1 - zw_2 - yw_2 \geq xyq'\). Assume 
that \(n = 0\). Note that \(x > y\) in this case. Since it is clear that 
\(P_1 \in L(C, (x, y))\) and 
\(P_2 = L(C, (-x, -y))\), by Remark [3.3] we have 
\(d(C, (x, y)) \geq xyq' - xy \geq (x - y)q'\). \(\square\)

Lemma 3.5. For a nef curve \(C\) on \(S\), there exist a compact smooth toric surface \(S_0\) and a 
curve \(C_0\) on \(S_0\) satisfying the following properties:

(i) \(d(C_0, (0, 1)) = q, d(C_0, (1, 0)) = q'\) and \(d(C_0, (1, \pm 1)) \geq q'\).

(ii) \(C_0 \leq C^2\).

(iii) The lattice polygon \(\square C_0\) has three or four vertices, and moreover, each of them is on 
one of the four lines 
\(l(C_0, (0, \pm 1)), l(C_0, (\pm 1, 0))\).

(iv) If \(l(C_0, (0, 1))\) contains two distinct vertices, then they are \(l(C_0, (0, 1)) \cap l(C_0, (1, 0))\) 
and \(l(C_0, (0, 1)) \cap l(C_0, (-1, 0))\). A similar property holds for \(l(C_0, (0, -1))\).

(v) If \(l(C_0, (1, 0))\) contains two distinct vertices, then one of them is \(l(C_0, (1, 0)) \cap l(C_0, 
(0, 1))\) or \(l(C_0, (1, 0)) \cap l(C_0, (0, -1))\). A similar property holds for \(l(C_0, (-1, 0))\).

**Proof.** In this proof, we will gradually deform the polygon \(\square C\) toward \(\square C_0\). In the 
process of the deformation, we construct five polygons \(\square C_i, (i = 1, \ldots, 5)\). For simplicity, we 
abuse notation “the properties (i) and (ii)” for these curves \(C_i\).
We assume that the vertex of \( \square_C \) on \( l(C, (0, 1)) \) whose \( z \)-coordinate is minimal is the
origin \( O \), and denote by \( P_1 = (z_1, w_1) \) the vertex of \( \square_C \) on \( l(C, (1, 0)) \) whose \( w \)-coordinate
is minimal. If either \( z_1 \) or \( w_1 \) is zero, we define \( C_1 = C \). In the case where neither \( z_1 \) nor \( w_1 \) is zero, we define a polygon \( C_1 \) by the following procedure. We first construct a polygon
\( \square_{E_1} \) from \( \square_C \) by connecting \( O \) and \( P_1 \). We put \( P_2 = (z_1, w_2) = l(C, (1, 1)) \cap l(C, (1, 0)) \) and
\( P_3 = (z_3, 0) = l(C, (1, 1)) \cap l(C, (0, 1)) \) (see Fig. 2). If \( z_1 + w_1 \geq 0 \) (resp. \( z_1 + w_1 < 0 \)),
we denote by \( \square_{C_1} \) the convex hull of \( \square_{E_1} \cup \{ P_2 \} \) (resp. \( \square_{E_1} \cup \{ P_3 \} \)). By definition, \( C_0 \)
clearly satisfies the property (i). Let us show the inequality \( C_1^2 \leq C^2 \). Since this is obvious
if \( C_1 = C \), we consider the case where neither \( z_1 \) nor \( w_1 \) is equal to zero. If \( z_1 + w_1 \geq 0 \), we
can take a non-negative integer \( a \) such that the lattice point \( P_4 = P_2 + a(-1, 1) \) is contained
in \( l(C, (1, 1)) \cap \square_C \). We denote by \( \square_{E_2} \) the convex hull of \( \square_{E_1} \cup \{ P_3 \} \). Since \( E_2^2 \leq C^2 \), it is
sufficient to verify \( C_1^2 \leq E_2^2 \). Note that the difference between \( C_1^2 \) and \( E_2^2 \) is caused only by
the two sides \( P_2P_1 \) and \( P_3P_1 \). Hence we obtain
\[
C_1^2 - E_2^2 = (w_2 - w_1)z_1 - (w_2 + a - w_1)(z_1 - a) + a(-w_2 - a) = -a(z_1 + w_1) \leq 0.
\]
Similarly we can show \( C_1^2 \leq C^2 \) in the case where \( z_1 + w_1 < 0 \). The shape of upper right
corner of \( \square_{C_1} \) is one of the three types as in Fig. 3 where we put \( Q_1 = l(C, (0, 1)) \cap \)
\( l(C, (1, 0)) \). By adapting a similar operation to other three corners of \( \square_{C_1} \), we construct
a polygon \( \square_{C_2} \). It is obvious that \( C_2 \) satisfies the properties (i) and (ii). Besides, every
vertices of \( \square_{C_2} \) is on one of the four lines \( l(C_2, (0, \pm 1)) \), \( l(C_2, (\pm 1, 0)) \). We put \( Q_4 = l(C_2, (-1, 0)) \cap l(C, (0, 1)) \). Let us show that if \( l(C_2, (0, 1)) \) contains two distinct vertices of
\( \square_{C_2} \) and one of them is \( Q_1 \) (resp. \( Q_4 \)), then the other one is \( Q_4 \) (resp. \( Q_1 \)). Assume that
\( \square_{C_2} \) contains \( Q_1 \) but not \( Q_4 \). Considering the construction method of \( \square_{C_2} \), we deduce that
\( Q_1 \) is contained in \( \square_{C_1} \) also (see Fig. 4). We denote by \( Q_0 \) the vertex of \( \square_{C_1} \) on \( l(C_1, (-1, 0)) \)
whose \( w \)-coordinate is minimal. Since the slant of the segment \( Q_1Q_0 \) is at most one, in the

![Figure 2](image2.png)

![Figure 3](image3.png)

![Figure 4](image4.png)
first two cases, $\Box_{c_2}$ has only one vertex $Q_1$ on $l(C_2, (0, 1))$. In the third case, it is obvious that $\Box_{c_2}$ has two vertices $Q_1$ and $Q_4$ on $l(C_2, (0, 1))$. We next consider the case where $\Box_{c_2}$ contains $Q_4$ but not $Q_1$. Note that $Q_1$ is not contained in $\Box_{c_1}$. Hence we obtain the four possibilities for the upper shape of $\Box_{c_1}$ as in Fig. 5. By the assumption $Q_4 \in \Box_{c_2}$, the

\begin{align*}
Q_4 \quad \Box_{c_1} \\
\text{or} \\
Q_4 \quad \Box_{c_1} \\
\text{or} \\
Q_4 \quad \Box_{c_1} \\
\text{or}
\end{align*}

Figure 5:

first two cases can be excluded. The third case does not occur. Indeed, since $Q_4$ must be contained in $\Box_{c_i}$ in this case, we have $z_1 + w_1 \geq 0$. It follows that $\Box_{c_i}$ does not have vertices on $l(C_1, (0, 1))$ except for $Q_4$. Thus only the last case remains, in which $\Box_{c_2}$ has one vertex $Q_4$ on $l(C_2, (0, 1))$. Similarly, with respect to the points $Q_2 = l(C, (1, 0)) \cap l(C, (0, -1))$ and $Q_3 = l(C, (0, -1)) \cap l(C, (-1, 0))$, we can show that if $l(C_2, (0, -1))$ contains two distinct vertices of $\Box_{c_2}$ and one of them is $Q_2$ (resp. $Q_3$), then the other one is $Q_3$ (resp. $Q_2$).

(a) Consider the case where $Q_1$ and $Q_4$ are contained in $\Box_{c_2}$. In this case, $\Box_{c_2}$ has at most six vertices (see Fig. 6). The right and left vertical sides and the lower horizontal one

\begin{align*}
Q_4 & \qquad \quad \Box_{c_2} \\
\text{or} & \qquad \quad \Box_{c_2} \\
\text{or} & \qquad \quad \Box_{c_2} \\
\text{or} & \qquad \quad \Box_{c_2}
\end{align*}

Figure 6:

may not exist. We denote by $Q$ the vertex of $\Box_{c_2}$ on $l(C_2, (0, -1))$ whose $z$-coordinate is minimal. Then we can finish the proof by defining $\Box_{c_0}$ as a triangle $Q_1Q_4Q$. Indeed, a simple computation shows that $C_0$ satisfies the property (i).

(b) An argument similar to that in (a) goes through for the case where $Q_2$ and $Q_3$ are contained in $\Box_{c_2}$.

(c) We put $L_1 = l(C_2, (0, 1)) \cap \Box_{c_2}$ and $L_2 = l(C_2, (0, -1)) \cap \Box_{c_2}$, and consider the case where $L_1$ and $L_2$ are not the segments $Q_1Q_4$ and $Q_2Q_3$, respectively. In this case, the polygon $\Box_{c_2}$ is as in Fig. 7 (I). We construct a polygon $\Box_{c_3}$ by the following procedure.

\begin{align*}
Q_1 \quad \Box_{c_2} \\
R_1 = (u_1, v_1) \\
R_2 = (u_1, v_2) \\
R_3 = (u_3, -q)
\end{align*}

(I)

\begin{align*}
\Box_{c_2} \\
Q_2 \\
\Box_{c_2}
\end{align*}

(II)

\begin{align*}
L_1 = \{Q_1\} \\
L_2 = \{Q_2\} \\
R_1 = R_2
\end{align*}

Figure 7:

If one of the equalities $L_1 = \{Q_1\}$, $L_2 = \{Q_2\}$ and $R_1 = R_2$ holds, we define $\Box_{c_3} = \Box_{c_2}$ (see Fig. 7 (II)). Assume that $L_1 \neq \{Q_1\}$, $L_2 \neq \{Q_2\}$ and $R_1 \neq R_2$. If $u_1 + v_1 \leq 0$ (resp. $u_1 + v_1 > 0$ and $u_1 - u_3 \leq v_2 + q$), we construct $\Box_{c_3}$ from $\Box_{c_2}$ by connecting $O$ and $R_2$ (resp. $R_1$ and $R_3$) as in Fig. 8. On the other hand, in the case where $u_1 + v_1 > 0$ and
Lemma 3.5. Then the polygon □ if and only if inequality C

Remark 3.6. Let C

Proposition 3.7. Let C be a nef curve on S, and C0 a curve as in Lemma 3.5. Then the inequality \( C_0^2 \geq \frac{4}{9} q^2 \) (in particular, \( C^2 \geq \frac{4}{9} q^2 \)) holds.

Proof. Recall that when we construct \( C_0 \), we divided the situation into the three cases (a), (b) and (c) in the proof of Lemma 3.5. In cases (a) and (b), an easy computation shows that \( C_0^2 = qq' \geq q^2 \). Let us consider case (c), that is, we assume that both \( l(\emptyset, (0,1)) \) and \( l(C_0(0,-1)) \) contain only one vertex. We keep the notation \( Q_1, \ldots, Q_4 \) in the proof of Lemma 3.5. Then the polygon □ is drawn as in Fig. 11 where we define

\( \square_{C_4} \) in the first two cases, vertical or horizontal sides may not exist. If □ is a triangle or a square, we can finish the proof by putting \( C_0 = C_4 \). If □ has more than four vertices, we construct a polygon □ by the following procedure (see Fig. 10). If \( O = S_1 \), we define

\( \square_{C_4} = \square_{C_4} \cdot \). Assume that \( O \neq S_1 \). If \( t_2 \geq t_3 \) (resp. \( t_2 < t_3 \)), we construct □ by connecting \( O \) and \( S_2 \) (resp. \( S_1 \) and \( S_3 \)). Then an easy computation shows that \( \square_{C_4} \) satisfies the properties (i) and (ii). By applying a similar operation to the lower side of □, we obtain the desired lattice polygon □.

Using Lemma 3.5, we can find the lower bound of the self-intersection number of \( C \).

Remark 3.6. As it is apparent from the construction method, the equality \( C^2 = C_0^2 \) holds if and only if □ = □.

Proof. Recall that when we construct \( C_0 \), we divided the situation into the three cases (a), (b) and (c) in the proof of Lemma 3.5. In cases (a) and (b), an easy computation shows that \( C_0^2 = qq' \geq q^2 \). Let us consider case (c), that is, we assume that both \( l(C_0, (0,1)) \) and \( l(C_0(0,-1)) \) contain only one vertex. We keep the notation \( Q_1, \ldots, Q_4 \) in the proof of Lemma 3.5. Then the polygon □ is drawn as in Fig. 11 where we define

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passes through the point $P_a$ and $P_e$.

(ii) Assume that $(b,e) \neq (q,0),$ $(0,q).$ By computing, we obtain the formula

$$C_0^2 = qq' + (a + c - q')(b + e - q).$$

Without loss of generality, we can assume that $b + e \geq q$ and $(q' - a)b \geq (q' - c)e$.

(i) In the case where $a + e \geq q'$, we have $b + c \geq q'$. Indeed, if not, the inequality $eb \geq (q' - a)b \geq (q' - c)e$ gives that $e = 0$ and $b = q$, a contradiction. Since the line $l(C_0, (1, -1))$ (resp. $l(C_0, (-1, 1))$) passes through the point $P_3$ (resp. $P_1$), the condition $d(C_0, (1, -1)) \geq q'$ implies that $a + q - (q' - c) \geq q'$. Hence we have $C_0^2 \geq qq' + (q' - q)(b + e - q) \geq qq'$.

(ii) Assume that $a + e < q'$ and $b + c \leq q'$. Since the line $l(C_0, (1, -1))$ (resp. $l(C_0, (-1, 1))$) passes through the point $P_2$ (resp. $P_4$), we have $q' - e + q - b \geq q'$. It follows that $b + e = q$ and $C_0^2 = qq'$.

(iii) Assume that $a + e < q'$ and $b + c > q'$. Since the line $l(C_0, (1, -1))$ (resp. $l(C_0, (-1, 1))$) passes through the point $P_3$ (resp. $P_2$), we have $q' - e + q - (q' - c) \geq q'$. Hence we have

$$C_0^2 \geq qq' + (a + e - q)(b + e - q) = qq' + \left(e + \frac{a + b - 2q}{2}\right)^2 - \left(\frac{a - b}{2}\right)^2, \tag{1}$$

where the equality holds if and only if $c - e = q' - q$ or $b + e = q$. If $a \geq b$ and $b + e = q$, we have $C_0^2 = qq'$. On the other hand, if $a \geq b$ and $b + e > q$, we have $a + e > q$ and $C_0^2 > qq'$ by the first inequality of (1). Lastly, if $a < b$, we have $0 < b - a \leq q$ and

$$C_0^2 \geq qq' - \frac{q^2}{4} \geq \frac{3}{4}q^2. \tag{2}$$

Proposition \ref{prop} yields the following interesting corollary, though, which has no direct relation to the subject of this paper.

**Corollary 3.8.** Let $S$ be a compact smooth toric surface. For a $k$-gonal nef curve $C$ on $S$, the inequality $C_0^2 \geq \frac{3}{4}k^2$ holds.
Considering an irredundant embedding of $C$, we can obtain a more precise lower bound for $C^2$ when $q$ takes a small value. Here, the ‘irredundancy’ has the following meaning: If there exists a $T$-invariant divisor $D_i$ on $S$ such that $D_i^2 = -1$ and $C.D_i \leq 1$, then by blowing it down, we can embed $C$ in another compact smooth toric surface. By carrying out such operation repeatedly, we obtain an embedding satisfying the following condition.

**Definition 3.9.** Let $C$ be a smooth curve on $S$. The pair $(S, C)$ (or simply the curve $C$) is said to be relatively minimal if $C.D_i \geq 2$ for any $T$-invariant divisor $D_i$ on $S$ with self-intersection number $-1$.

In the remaining part of this section, we set $O = l(C, (0, 1)) \cap l(C, (1, 0))$. Note that it is equal to the point $l(C_0, (0, 1)) \cap l(C_0, (1, 0))$.

**Proposition 3.10.** Let $C$ be a curve as in Theorem 1.3 and assume that $(S, C)$ is relatively minimal. If $q = 2$ (resp. 3), then $C^2 \geq 12$ (resp. 18).

**Proof.** We shall prove only for the case $q = 3$. Considering the relative minimality of $C$ and the reflection of $\Box_C$ about $z$-axis, we can assume that $O$ is contained in $\Box_C$. Moreover, we see that the right shape of $\Box_C$ must be a segment connecting $O$ and $(-3m, -3)$, where $m$ is a non-negative integer. If $m \neq 0$, then by the condition $d(C, (1, -1)) \geq q'$, we see that $(-q', 0) \in \Box_C$ and the left shape of $\Box_C$ is a segment connecting $(-q', 0)$ and $(-q' + 3n, -3)$, where $n$ is non-negative integer. In the case $m = 0$, by the relative minimality of $C$, the left shape of $\Box_C$ is a segment connecting $(-q', -3)$ and $(-q' + 3l, 0)$, or $(-q', 0)$ and $(-q' + 3l, -3)$, where $l$ is non-negative integer. Consequently, without loss of generality, we can assume that $\Box_C$ is a trapezium (possibly a triangle) as in Fig. 12 where the inequalities $m \geq n \geq 0$

![Figure 12](image-url)

hold. The inequality $C^2 \geq 18$ is obvious if $q' \geq 6$. If $q' = 5$, then there exist two possibilities $(m, n) = (0, 0), (1, 0)$. In the cases $q' = 3, 4$, since $C$ is not isomorphic to a plane curve, $(m, n)$ must be $(0, 0)$. In each case, we obtain $C^2 \geq 18$ by computing. □

**Proposition 3.11.** Let $C$ be a curve as in Theorem 1.3 and assume that $(S, C)$ is relatively minimal. If $q = 4$ and $C^2 \leq 16$, the shape of $\Box_C$ is one of the six types in Fig. 13 provided that we ignore congruence relations.

![Figure 13](image-url)

**Proof.** We take a curve $C_0$ as in Lemma 3.3. Recall the three cases (a), (b) and (c) in the proof of Lemma 3.3. In case (a) (that is, the upper side of $\Box_{C_0}$ is a horizontal line of length $q'$), an easy computation gives $C_0^2 \geq 16$. Suppose that $C^2 = 16$. Then, since $C^2 = C_0^2 = 16$, we obtain the five possibilities for the shape of $\Box_{C_0}$ as in Fig. 14. Note
In case (iii), by the inequality (2), we have $q \in \Box$.

We divide the situation into the three cases (i), (ii) and (iii). In cases (i) and (ii), we have $(a, b, c, e) = (0, 4, 1, 0)$, $(0, 4, 2, 0)$ and $(0, 4, 3, 0)$ by Remark 3.6. This contradicts the relative minimality. Hence it is sufficient to consider the case $q' = 4$.

Let us examine the possibility of the shape of $\Box_C$ satisfying $q = q' = 4$ and $C^2 \leq 16$. We denote by $P$ the vertex of $\Box_C$ on $l(C, (0, 1))$. Note that $P \neq (-1, 0), (-3, 0)$. Indeed, if $(-1, 0)$ (resp. $(-3, 0)$) is contained in $\Box_C$, then also $(0, 0)$ (resp. $(-4, 0)$) is contained in $\Box_C$ by the relative minimality, which contradicts to the assumption in (c). Since the case $P = (-4, 0)$ is essentially equivalent to the case $P = O$, it is sufficient to consider the two cases $P = O, (-2, 0)$. Assume that $P = O$. In this case, we can assume that none of $(-3, 0), (-4, 0)$ and $(-4, -1)$ is contained in $\Box_C$. Indeed, if not, $(-4, 0)$ is contained in $\Box_C$ by the relative minimality, a contradiction. We can take a unique integer $a$ with $-4 \leq a \leq 0$ such that the line $l(C, (1, -1))$ passes through $(0, a)$. Since $d(C, (1, -1)) \geq 4$, the cases $a = 0, -1$ do not occur. If $a = -3$ or $-4$, then by the relative minimality, the point $(0, -4)$ must be contained in $\Box_C$. Hence, by an argument similar to that in case (a), we see that $\Box_C$ is a triangle with vertices $(0, 0), (0, -4)$ and $(-4, -2)$. In the case $a = -2$, since $d(C, (1, -1)) \geq 4$, at least one of the points $(-2, 0), (-3, -1)$ and $(-4, -2)$ is contained in $\Box_C$. We put $R_1 = (-2, -4)$ and $R_2 = (-4, -4)$. Assume that $R_1, R_2 \in \Box_C$. Then, by computing, we obtain the three types of $\Box_C$ satisfying $C^2 \leq 16$ as in Fig. 16. By the relative minimality, the second type is excluded. Assume that $R_1 \notin \Box_C$ and $R_2 \in \Box_C$. In this case, by the relative minimality, the lower side of $\Box_C$ must be the segment connecting two points $(0, -2)$ and $R_2$. Then, by computing, we obtain the three types of $\Box_C$ satisfying $C^2 \leq 16$ as in Fig. 16.
Let $C$ in Fig. 18. In the first case, we have $C^2 = 20$. Lastly, we consider the case $P = (-2,0)$.

![Figure 17:](image)

In order to avoid the duplication, we assume that $\square C$ contains none of the four corner points $O, (0,-4), (-4,-4)$ and $(-4,0)$. Then only one possibility remains: $\square C$ is a square with vertices $(-2,0), (0,-2), (-2,-4)$ and $(-4,-2)$.

**Proposition 3.12.** Let $C$ be a curve as in Theorem I, and assume that $(S,C)$ is relatively minimal. If $q = 5$, then $C^2 \geq 25$.

**Proof.** We take a curve $C_0$ as in Lemma 3.3. By the same argument as that in the proof of Proposition 3.11 we have $C_0^2 \geq 25$ except for case (iii) in the proof of Proposition 3.7. Note that $q' \leq 6$, $c - e \geq q' - 5$ and $a < b$ if $C_0^2 \leq 24$. In the case $q' = 6$, we obtain the four possibilities $(a,b,c,e) = (0,5,3,2),(0,5,4,3)$ by computing. In both cases, we have $C_0^2 = 24$. On the other hand, the relative minimality of $C$ implies that $\square C$ is not equal to $\square C_0$, which means that $C^2 > C_0^2$.

We next consider the case $q' = 5$. If none of $O, (0,-5), (-5,-5)$ and $(-5,0)$ is contained in $\square C$, then also the eight points $(-1,0), (0,-1), (0,-5), (-1,-5), (-4,-5), (-5,-4), (-5,-1)$ and $(-4,0)$ are not contained in $\square C$. Hence the left (resp. right) shape of $\square C$ is one of three types in Fig. 19 (I) (resp. (II)). By noting the condition $d(C,(1,\pm 1)) \geq 5$, we have $C^2 \geq 25$ in any case. Therefore, considering the reflection, we see that it is sufficient to verify our lemma under the assumption $O \in \square C$. Since the inequality $C^2 \geq 25$ is obvious if $(0,-5)$ or $(-5,0)$ is contained in $\square C$, we assume that $(0,-5),(-5,0) \notin \square C$. It follows that also the four points $(0,-4), (-1,-5), (-5,-1)$ and $(-4,0)$ are not contained in $\square C$. We denote by $P$ the vertex of $\square C$ on $l(C,(0,-1))$ whose $z$-coordinate is maximal. Let us consider the case $P = (-2,-5)$. Then we have the four possibilities as in Fig. 20. In case (I), by the relative minimality, we see that there exist integers $m_1$ and $m_2$ with

$C^2 \leq 16$ as in Fig. 17. By the relative minimality, the second type is excluded. Assume that $R_1 \in \square C$ and $R_2 \notin \square C$. The relative minimality implies that neither $(-3,-4)$ nor $(-4,-3)$ is contained in $\square C$. Hence there exist the four possibilities for the shape of $\square C$ as in Fig. 18. In the first case, we have $C^2 = 20$. Lastly, we consider the case $P = (-2,0)$.

![Figure 18:](image)

$\times$ : the point which is not contained in $\square C$

![Figure 19:](image)

$\times$ : the point which is not contained in $\square C$
Lemma 4.1. Let $C$ be a curve as in Theorem 1.3. Assume that $(S, C)$ is relatively minimal and $q \geq 3$. Let $V$ be an effective divisor on $S$.

(i) If $h^0(S, V) \geq 2$ and $h^0(S, V + K_S) \geq 1$, then $C.V \geq q + 2$.
(ii) If $h^0(S, V) \geq 2$ and $h^0(S, V + K_S) \geq 2$, then $C.V \geq q + 3$.
(iii) If $h^0(S, V) \geq 3$ and $h^0(S, V + K_S) \geq 1$, then $C.V \geq q + 3$. 

4 Proof of the main theorem

To prove Theorem 1.3, we first aim to show that any gonality pencil on $C$ can be extended to a morphism from $S$. Let us prove several lemmas needed later. Also in this section, we use the notion of coprime in the wide sense (see Definition 3.1).

-3 \leq m_1, m_2 \leq -1 such that $(0, m_1)$ and $(m_2, 0)$ is contained in $\Box C$. An easy computation gives $C^2 \geq 25$ in any case. Consider case (II). By the relative minimality, we see that either $(0, -1)$ or $(0, -3)$ is contained in $\Box C$, and likewise either $(-3, -5)$ or $(-5, -5)$ is contained in $\Box C$. Then it is obvious that the minimum value of $C^2$ is attained when the lower shape of $\Box C$ is the polygonal line connecting $O, (0, -1), (-2, -5), (-3, -5)$ and $(-5, -3)$. Then we see that $C^2$ achieves its minimum 25 when the upper shape of $\Box C$ is the polygonal line connecting $(-5, -3), (-4, -2)$ and $O$. Consider cases (III) and (IV). We note that the point $(-5, -5)$ is contained in $\Box C$ in case (III) also. By the condition $d(C, (1, -1)) \geq 5$, $\Box C$ has a lattice point in the domain $A$ in Fig. 21. When $\Box C$ is a square with vertices $O, (-2, -5)$,

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure}
\caption{Figure 21:}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure}
\caption{Figure 20:}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure}
\caption{Figure 21:}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure}
\caption{Figure 20:}
\end{figure}

$(-5, -5)$ and $Q$, the self-intersection number $C^2$ achieves its minimum 25, where $Q$ is either $(-5, -3), (-4, -2), (-3, -1)$ or $(-2, 0)$. In the case $P = (-3, -5)$, we assume that $(-5, -2)$ is not contained in $\Box C$ in order to avoid the duplication. By the relative minimality and the condition $d(C, (1, -1)) \geq 5$, we see that $l(C, (1, -1))$ passes through $P$ and $(-3, 0)$ is contained in $\Box C$. Then, since the upper shape of $\Box C$ must be the polygonal line connecting $(-5, -4), (-3, 0)$ and $O$, we obtain $C^2 \geq 25$. We next consider the case $P = (-4, -5)$. By the relative minimality and the condition $d(C, (1, -1)) \geq 5$, we see that $(-5, -5)$ is contained in $\Box C$, and moreover, either $(0, -2)$ or $(0, -3)$ is contained in $\Box C$. Then, considering the reflection and the rotation, this case can be reduced to the case where $P = (-2, -5)$ or $(-3, -5)$. The same argument goes through for the case $P = (-5, -5)$.

\qed
Proof. We write \( V = \sum_{i=1}^{d} n_i D_i \) with non-negative integer coefficients. We denote by \( \sigma(D_{i_0}) \) the cone in \( \Delta_S \) whose primitive element is \((0, -1)\).

(i) By assumption, we can assume that the origin \( O \) is contained in \( \square_V \) and there exists another lattice point \( P = (z, w) \) contained in the interior of \( \square_V \). Without loss of generality, we can assume that \( z \geq 0, w \leq 0 \) and \((z, -w) = 1\). We denote by \( A_1 \) the domain drawn in Fig. 22(I). Since \( P \) is contained in the interior of \( \square_V \), the inequality \( x_i z + y_i w < n_i \) holds for any \((x_i, y_i) \in A_1 \cap \text{Pr}(S)\). We thus obtain

\[
C.V = \sum_{i=1}^{d} n_i C.D_i \geq \sum_{\sigma(D_i) \subset A_1} (x_i z + y_i w + 1) C.D_i \\
= d(C, (-w, z)) + \sum_{\sigma(D_i) \subset A_1} C.D_i \geq q + \sum_{\sigma(D_i) \subset A_1} C.D_i.
\]

Thus it is sufficient to verify \( \sum_{\sigma(D_i) \subset A_1} C.D_i \geq 2 \). This inequality is true if there exists a cone \( \sigma(D_i) \subset A_1 \) such that \( D_i^2 = -1 \). Hence we suppose that there does not exist such a cone (we call this the ‘nonexistence condition’) and \( \sum_{\sigma(D_i) \subset A_1} C.D_i = 1 \). We can take a cone \( \sigma(D_{i_0}) \subset A_1 \) such that

\[
C.D_i = \begin{cases} 
1 & (i = i_0), \\
0 & (i \neq i_0, \sigma(D_i) \subset A_1).
\end{cases}
\]

If there exists only one cone \( \sigma(D_i) \) included in \( A_1 \setminus \mathbb{R}(-w, z) \), then \( d(C, (x_j -1, y_j -1)) \) is equal to one, a contradiction. We thus have \( N = \# \{ \sigma(D_i) \in \Delta_S \mid \sigma(D_i) \subset A_1 \setminus \mathbb{R}(-w, z) \} \geq 2 \).

Then, by the nonexistence condition, we deduce that neither \( z \) nor \( w \) is equal to zero. We denote by \( m \) the maximal integer satisfying \( z + mw \geq 0 \). By the nonexistence condition, there does not exist a cone included in the domain \( A_2 \) except for \( \sigma(D_{i_0}) \) (see Fig. 22(II)). On the other hand, since \( N \geq 2 \), there exists a cone \( \sigma(D_l) \subset A_1 \) such that \( x_l \neq 0 \). Consider the case where \( x_l \) is positive. Since \( \sigma(D_l) \subset A_1 \setminus A_2 \), we have \((1, m) \in \text{Pr}(S)\). Thus, it follows from \( D_{i_0}^2 \neq -1 \) that there does not exist a cone in the domain \( A_3 \) (see Fig. 22(II I)). We deduce that

\[
\{(x_j, y_j) \in M(x_{i_0} - 1, y_{i_0} - 1) \mid C.D_j \geq 1\} = \{(x_{i_0}, y_{i_0})\},
\]

which implies the contradiction \( d(C, (x_{i_0} - 1, y_{i_0} - 1)) = 1 \). In the case where \( x_l \) is negative, one can obtain a similar contradiction.

(ii) In this case, we can assume that \( \square_V \) has two distinct lattice points \((0, 0)\) and \((z, w)\)
in its interior, where \( z \geq 0 \), \( w \leq 0 \) and \( (z, -w) = 1 \). As we saw in (i), the inequality 
\[
\sum_{\sigma(D_i) \subset A_1} n_i C.D_i \geq q + 2 \text{ holds.}
\]
on the other hand, since \((0, 0)\) is contained in the interior of \( \square V \), the coefficient \( n_i \) is positive for every \( T \)-invariant divisor \( D_i \) \((i = 1, \ldots, d)\). If \( C.D_i = 0 \) for any \( \sigma(D_i) \not\subset A_1 \), then we have \( d(C, (-w, z)) = 0 \), a contradiction. We thus have \( C.V \geq q + 2 + \sum_{\sigma(D_i) \not\subset A_1} n_i C.D_i \geq q + 3 \).

(iii) In this case, there exist three distinct lattice points \((0, 0)\), \((z, w)\) and \((z', w')\) in \( \square V \), especially \((z, w)\) is contained in the interior of \( \square V \). We can assume that \( z \geq 0 \), \( w \leq 0 \), \((z, -w) = 1\) and \((|z'|, |w'|) = 1\). Suppose that \( C.V = q + 2\), and denote by \( i_1 \) (resp. \( i_2 \)) the minimal (resp. maximal) integer in \( \{i \mid \sigma(D_i) \subset A_1, C.D_i \geq 1\} \). By a computation similar to that in [3], we obtain 
\[
\sum_{\sigma(D_i) \subset A_1} C.D_i = 2 \quad \text{and} \quad \sum_{\sigma(D_i) \not\subset A_1} n_i C.D_i = 0.
\]
It follows that \( C.D_i = 0 \) for \( \sigma(D_i) \subset A_1 \) except for \( i = i_1, i_2 \). Moreover, we see that \( n_i = n_{+1} = C.D_{i_1} = C.D_{i_2} = 1 \) (resp. \( n_i = 1 \) and \( C.D_i = 2 \)) if \( i_1 \neq i_2 \) (resp. \( i_1 = i_2 \)).

Let us consider the case where \( zw' - wz' \geq 0 \).

\[
\begin{align*}
\sigma(D_j) \quad \text{be a cone included in the domain} \quad B_1 \quad \text{drawn in Fig. 23.} \\
\text{Since} \quad (z', w') \quad \text{is contained} \qquad \text{in} \quad L(V,(x_j, y_j)), \quad \text{we have} \quad \text{the inequalities} \quad n_j \geq x_j z' + y_j w' > 0.
\end{align*}
\]

Hence \( C.D_j = 0 \). Noting that \( (z', w') \in L(V,(x_i, y_i)) \cap L(V,(x_{i2}, y_{i2})) \), we have
\[
d(C, (-w', z')) = \sum_{\sigma(D_i) \subset B_2} (x_i z' + y_i w') C.D_i
\]
\[
= \begin{cases} 
(w' y_i + z' x_i) C.D_{i_1} + (w' y_{+1} + z' x_{+1}) C.D_{i_2} \leq n_i C.D_{i_1} + n_{+1} C.D_{i_2} = 2 \quad (i_1 \neq i_2), \\
(w' y_i + z' x_i) C.D_{i_1} \leq n_i C.D_{i_1} = 2 \quad (i_1 = i_2).
\end{cases}
\]
This contradicts the assumption \( q \geq 3 \). A similar argument can be carried out for the case where \( zw' - wz' \leq 0 \).

\[
\square
\]

We are now in a position to show the extension of a gonality pencil. In the proof, the following Serrano’s result plays an essential role.

**Theorem 4.2** [1]. Let \( X \) be a smooth curve on a smooth surface \( Y \), and \( f : X \rightarrow \mathbb{P}^1 \) a surjective morphism of degree \( p \). Assume that \( X^2 > 4p \). If there does not exist an effective divisor \( V \) on \( Y \) satisfying the following properties (a) and (b), then there exists a morphism from \( Y \) to \( \mathbb{P}^1 \) whose restriction to \( X \) is \( f \).

\[
\begin{align*}
\text{(a)} & \quad 1 \leq V^2 < (X - V).V \leq p, \quad \text{(b)} \quad X^2 \leq \frac{(p + V^2)^2}{V^2}.
\end{align*}
\]

**Proposition 4.3.** Let \( C \) be a curve as in Theorem 4.2 and \( f \) a gonality pencil on \( C \). If \((g, q) \neq (4, 4), (5, 4), (10, 6)\), then there exists a morphism from \( S \) to \( \mathbb{P}^1 \) whose restriction to \( C \) is \( f \).

---

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Proof. If \((S, C)\) is not relatively minimal, by the method explained before Definition 3.9 we can obtain an equivariant morphism \(\psi\) from \(S\) to another compact smooth toric surface \(S'\) such that \((S', C)\) is relatively minimal. Clearly, for a morphism \(\varphi\) from \(S'\) to \(\mathbb{P}^1\), the composite \((\varphi \circ \psi)|_C\) coincides with \(\varphi|_C\). Hence, it is sufficient to consider the case where \((S, C)\) is relatively minimal.

By the condition \(q \geq 2\), we have \(q \geq 2\). If \(q = 2\), then \(k = 2\) and \(C\) has only one gonality pencil. Thus our lemma is obvious in this case. Let us consider the case where \(q \geq 3\). Suppose that, for \(C, S\) and \(f\), there exists an effective divisor \(V\) satisfying the two properties (a) and (b) in Theorem 1.2. We put \(s = V^2\). By the inequality \(q \geq k\) and Proposition 3.7, we have \(1 \leq s < k\) and \(\frac{3}{4}k^2 \leq C^2 \leq \frac{(k + s)^2}{s}\). It follows that

\[
\begin{align*}
  s &= 1 \quad (k \geq 9), \\
  1 \leq s &\leq 2 \quad (k = 7, 8), \\
  1 \leq s &\leq 3 \quad (k = 6), \\
  1 \leq s &\leq 4 \quad (k = 5), \\
  1 \leq s &\leq 3 \quad (k = 4), \\
  1 \leq s &\leq 2 \quad (k = 3), \\
  s &= 1 \quad (k = 2).
\end{align*}
\]

We first consider the case where \(s \leq 2\). By Riemann-Roch theorem, we have

\[
\begin{align*}
  0 &\leq h^1(S, V + K_S) = h^0(S, V + K_S) + h^0(S, -V) - \frac{1}{2}(V + K_S)\cdot V - \chi(O_S) \\
  \frac{1}{2}V\cdot K_S &\leq h^0(S, V + K_S) - \frac{1}{2}, \\
  0 &\leq h^1(S, V) = h^0(S, V) + h^0(S, K_S - V) - \frac{1}{2}V\cdot (V - K_S) - \chi(O_S) \\
  &\leq h^0(S, V) + h^0(S, V + K_S) - s - 2.
\end{align*}
\]

Since \(h^0(S, V) \geq h^0(S, V + K_S)\), by (4), we obtain \(2h^0(S, V) \geq s + 2\). Assume that \(s = 1\). Then we have \(h^0(S, V) \geq 2\) and \(C\cdot V \leq k + 1 \leq q + 1\) by the property (a). Hence we have \(h^0(S, V + K_S) = 0\) by Lemma 4.1 (i). It follows from (4) that \(h^0(S, V) \geq s + 2\). In the case \(s = 2\), since \(h^0(S, V) \geq 2\) and \(C\cdot V \leq q + 2\), we have \(h^0(S, V + K_S) \leq 1\) by Lemma 4.1 (ii). We thus have \(h^0(S, V) \geq 3\) by (4), and \(h^0(S, V + K_S) = 0\) by Lemma 4.1 (iii). It follows from (4) that \(h^0(S, V) \geq s + 2\). On the other hand, since

\[
C\cdot (V - C) = C\cdot V - C^2 \leq \begin{cases} 
  k + s - \frac{3}{4}q^2 \leq q + 2 - \frac{3}{4}q^2 < 0 & (q \geq 3), \\
  k + s - 12 < 0 & (q = 2)
\end{cases}
\]

by Proposition 3.10 and Proposition 5.11, we obtain \(h^0(S, V - C) = 0\). Therefore, in the case where \(s \leq 2\), the cohomology exact sequence

\[
0 \to H^0(S, V - C) \to H^0(S, V) \to H^0(C, V|_C) \to \cdots
\]

gives the inequality \(h^0(C, V|_C) \geq s + 2\). If we write \(g^r_l = |V|_C|\), then \(r \geq s + 1\) and \(l \leq k + s\).

We obtain a net \(g^3_l\) by subtracting \(r - 2\) general points of \(C\) from it. Since \(C\) is not isomorphic to a plane curve, \(g^{3}_{l-r+2}\) is not very ample. Then we obtain a pencil \(g^3_{l-r+2}\) such that \(l - r \leq k + s - (s + 1) = k - 1\), a contradiction.

Let us show that the cases where \((k, s) = (4, 3), (5, 3), (5, 4)\) do not occur. Assume that \(k = 4\). If \(q \geq 5\), then we have \(C^2 \geq \frac{3}{4}q^2 > 18\). If \(q = 4\), then we deduce \(C^2 \geq 17\) by the assumption \((g, q) \neq (4, 4), (5, 4)\) and Proposition 5.11. We conclude that \(s\) must be at most 16.
two by the property (b). In the case \( k = 5 \), since \( q \geq 5 \), we have \( C^2 \geq 25 \) by Proposition 3.12 and Proposition 5.1. Hence \( s \) is one in this case.

The case \( (k, s) = (6, 3) \) remains to consider. Assume that \( k = 6 \) and \( s = 3 \). We take a curve \( C_0 \) as in Lemma 3.5. By Proposition 5.7,

\[
27 = \frac{3}{4} k^2 \leq \frac{3}{4} q^2 \leq C^2 \leq \frac{(k + s)^2}{s} = 27,
\]

which yields \( q = 6 \) and \( C_0^2 = C^2 = 27 \). Hence we have \( \square_C = \square_{C_0} \) by Remark 3.9. By the argument in the proof of Proposition 3.7 we see that \( C_0 \) is a curve of type (iii) in it. Then the inequality ④ gives \( q' = 6 \). Moreover, by the inequality ③, we deduce that \( c = e = 6 - (a + b)/2 \) and \( a - b = -6 \). We thus have \( (a, b, c, e) = (0, 6, 3, 3) \) and \( g = 10 \) (see Fig. 24).

\[

\text{Figure 24:}

\]

**Lemma 4.4.** Let \( C \) be a curve as in Theorem 1.3 and assume that \( (S,C) \) is relatively minimal. If \( q = 6 \) and \( g = 10 \), then \( \square_C \) is a triangle as in Fig. 24.

**Proof.** In this proof, we often use the relative minimality of \( C \) and the property \( d(C, (1, -1)) \geq q' \) without further mention. We denote by \( \text{Int} \square_C \) the interior of \( \square_C \), and by \( l((a_1, b_1), (a_2, b_2)) \) the segment connecting two points \( (a_1, b_1) \) and \( (a_2, b_2) \). We assume that the point \( l(C, (0,1)) \cap l(C, (1,0)) \) is the origin \( O \). Now we suppose that none of \( O, (0, -6), (-q', -6) \) and \( (-q', 0) \) is contained in \( \square_C \). It follows that also the eight points \( (-1, 0), (0, -1), (0, -5), (1, -6), (-q' + 1, -6), (-q', -5), (-q', -1), (-q' + 1, 0) \) are not contained in \( \square_C \). Assume that \( (0, -3) \) is contained in \( \square_C \). We can take a lattice point \( P \in l(-1,0) \cap \square_C \). We define \( A \) as a domain surrounded by the four segments \( l((0, -3), (-q' + 2, -6)), l((-2, -6), P), l(P, (-2, 0)) \) and \( l((-q' + 2, 0), (0, -3)) \). In any case, we see that there exist more than ten lattice points in the interior of \( A \) (see Fig. 25). Since

\[
\square_C \text{ includes } A, \text{ we obtain } g \geq 11.
\]

Next we assume that \( (0, -4) \) is contained in \( \square_C \). Similarly to the previous case, we obtain \( g \geq 11 \) except for the two cases where \( q' = 6 \) and \( P = (-6, -2), (-6, -4) \) (see Fig. 26). If \( q' = 6 \) and \( P = (-6, -2) \), then either \( (-3, -1) \) or \( (-4, -1) \) is contained in \( \text{Int} \square_C \). Besides, either \( (-2, -5) \) or \( (-3, -5) \) is contained in \( \text{Int} \square_C \). On the other hand, if \( q' = 6 \) and \( P = (-6, -4) \), then either \( (-2, -5) \) or \( (-4, -5) \) is contained in \( \text{Int} \square_C \). Hence, in each case, we obtain \( g \geq 11 \).
By the above consideration, we can assume that $O$ is contained in $\Box_C$. There exists an integer $a$ with $-6 \leq a \leq 0$ such that $l(C, (1, -1))$ passes through $(0, a)$. We first remark that the cases $a = -1, -5$ do not occur by the relative minimality. If $a = 0$, then by the assumption $g = 10$, $\Box_C$ must be a triangle with vertices $O$, $(-6, -6)$ and $(-6, 0)$. This contradicts the assumption that $C$ is not a plane curve. We obtain the same contradiction if $a = -6$. Let us consider the case $a = -2$. Then either $(-q', -2)$ is contained in $\Box_C$. We define $B$ as a domain surrounded by the five segments $l((0, 0), (-4, -6)), l((0, -2), (-q', -6)), l((-4, -6), (-q', 0)), l((-q', -6), (-q' + 2, 0))$ and $l((-q', -2), (0, 0))$. Since $\Box_C$ includes $B$, we obtain $g \geq 11$ if $q' \geq 7$ (see Fig. 27). In the case $q' = 6$, we can observe that there are at least eight lattice points in Int $\Box_C$. Note that either $(-2, -3)$ or $(-3, -4)$ is contained in Int $\Box_C$. Moreover, if $(-4, 0) \in \Box_C$ (resp. $(-6,-2) \in \Box_C$), then $(-3, -1)$ and $(-4, -1)$ (resp. $(-5,-2)$ and $(-5, -3)$) are contained in Int $\Box_C$. Hence, we have $g \geq 11$ in this case also.

We next consider the case $a = -4$. We can take a lattice point $Q \in l(C, (-1, 0)) \cap \Box_C$. In the cases $Q = (-q', 0), (-q', -1), (-q', -2), (-q', -3), (-q', -4), (-q', -5)$, we define $B_1$ as a domain surrounded by the four segments $l(O, (-2, -6)), l((0, -4), (-q', -6)), l((-2, -6), Q)$ and $l(Q, O)$. Then we see that there exist more than ten lattice points in the interior of $B_1$ (that is, $g \geq 11$) except for the case where $q' = 6$ and $Q = (-6, -4)$ (see Fig. 28). In this exceptional case, since either $(-3, -5)$ or $(-4, -5)$ is contained in Int $\Box_C$, we obtain $g \geq 11$. When $Q$ is $(-q', -5)$ or $(-q', -6)$, we define $B_2$ as a domain surrounded by the five segments $l(O, (-2, -6)), l((0, -4), (-q', -6)), l((-2, -6), Q), l(Q, (-q', -4))$ and $l((-q', -4), O)$. Then we obtain $g \geq 11$.

Lastly, we consider the case $a = -3$. We define $E_1$ as a domain surrounded by the five seg-
Lemma 4.5. \(q\) is contained in \(\text{Int} \square_C\). We shall show that \(\phi\) and \(z\) implies that \(\phi - O\) connecting two lattice points. We denote these points by \(\phi - v\). In what follows, by reembedding if necessary, we may assume that \((\phi, \text{Int} \square_C)\) is a triangle with vertices \(\phi - v\), \(\phi - w\), and \(\phi - z\). Since \((\phi, \text{Int} \square_C)\) is relatively \(g\)-minimal, we see that also \((\phi, \text{Int} \square_C)\) is a triangle with vertices \(\phi - v\), \(\phi - w\), and \(\phi - z\). Moreover, if \((\phi, \text{Int} \square_C)\) is contained in \(\text{Int} \square_C\), then we see that also \((-5, -2)\) is contained in \(\text{Int} \square_C\). This means that \(g \geq 11\). Let us consider the case \(g = 11\). If \((\phi, \text{Int} \square_C)\) is contained in \(\text{Int} \square_C\), then we see that also \((\phi, \text{Int} \square_C)\) is contained in \(\text{Int} \square_C\). Moreover, if \((\phi, \text{Int} \square_C)\) is contained in \(\text{Int} \square_C\), then we see that also \((-5, -2)\) is contained in \(\text{Int} \square_C\). This means that \(g \geq 11\). Let us consider the case \(g = 11\). We denote by \(R = (-6, b)\) the vertex of \(\text{Int} \square_C\) on \((l(C, (-1, 0)))\) whose \(w\)-coordinate is maximal. In the cases \(b = -4, -5, -6\), a domain \(E_2\) surrounded by the five segments \(l(O, (-3, -6)), l((0, -3), (-6, -6)), l((-3, -6), R), l(R, (-3, 0))\) and \(l((-6, -3), O)\) is included in \(\text{Int} \square_C\) (see Fig. 29). Note that, in each case, either \((-1, -2)\) or \((-4, -5)\) is contained in \(\text{Int} \square_C\). Moreover, in the case \(b = -6\), either \((-2, -1)\) or \((-5, -4)\) is contained in \(\text{Int} \square_C\). We thus obtain \(g \geq 11\). If \(-3 \leq b \leq 0\), we define \(E_3\) as a domain surrounded by the four segments \(l(O, (-3, -6)), l((0, -3), (-6, -6)), l((-3, -6), R)\) and \(l(R, O)\). If \(-2 \leq b \leq 0\), then there exist more than eleven lattice points in the interior of \(E_3\). Assume that \(b = -3\). If \((-3, -5)\) is not contained in \(\text{Int} \square_C\), then by a simple consideration, we see that \((-1, -2)\) are contained in \(\text{Int} \square_C\). If \((-3, -5)\) is contained in \(\text{Int} \square_C\), it is clear that the equality \(g = 10\) holds if and only if \(R = (-6, -3)\) and \(\square_C\) is a triangle with vertices \(O, (-3, -6)\) and \((-6, -3)\) \(\square_C\).

Similarly to Lemma 4.4 we obtain the following lemma.

**Lemma 4.5.** Let \(C\) be a curve as in Theorem 1.3 and assume that \((S, C)\) is relatively minimal. If \(q = 4\) and \(g = 4\), then \(\square_C\) is the first triangle in Fig. 15.

We are now ready to prove the main theorem.

**Proof of Theorem 1.3.** As mentioned at the beginning of the proof of Proposition 4.3, the statements (i) and (ii) are obvious if \(q = 2\). Hence we assume that \(q \geq 3\).

(i) Let \(\varphi\) be a morphism from \(S\) to \(\mathbb{P}^1\) of \(\text{deg} \varphi|_C = k\) whose existence is guaranteed by Proposition 4.3. We shall show that \(\varphi\) is a toric fibration of \(S\). We denote by \(F\) the fiber of \(\varphi\). Since \(C(F - C) \leq k - \frac{3}{2}k^2 < 0\), we have \(h^0(C, F - C) = 0\). Hence the cohomology exact sequence

\[
0 \to H^0(S, F - C) \to H^0(S, F) \to H^0(C, F|_C) \to \cdots
\]

implies that \(h^0(C, F|_C) \geq h^0(S, F)\). Hence we have \(h^0(S, F) \leq 2\). Namely, \(\square_F\) is a segment connecting two lattice points. We denote these points by \(O = (0, 0)\) and \(P = (z, w)\), where \(z\) and \(w\) are integers such that \((|z|, |w|) = 1\). Then the point \((-w, z)\) must be contained in \(\text{Pr}^*(S)\). Therefore, by Fact 2.2, we see that \(F\) is a fiber of some toric fibration.

In what follows, by reembedding if necessary, we may assume that \((S, C)\) is relatively minimal.

(ii) If \((g, q) \neq (4, 4), (5, 4), (10, 6)\), we have \(k = q\) by (i). Assume that \(g = q = 4\). In this case, we have \(k \leq \lfloor \frac{2 + 3}{2} \rfloor = 3\) and \(C^2 \geq 12\) by Proposition 3.11. Suppose that \(k = 2\). Then by Theorem 4.2 there exists an effective divisor \(V\) on \(S\) satisfying the properties (a) and (b)
The case where $(g, q) = (4, 4)$

In this section, we focus on the exceptional curve in Theorem 1.3 (ii), and exhibit its structure. Let $S$, $C$ and $q$ be as in Theorem 1.3, and assume $q = q = 4$. By Lemma 4.5, the fan $\Delta_S$ and the lattice polygon $\square_C$ are as in Fig. 30. Considering the shape of $\square_C$, we can take a plane model

$$C' : x^4y^2 + x^2y^4 + ax^2y^2z^2 = z^6 \quad (a \in \mathbb{C})$$

of $C$. We shall denote the pull-backs on $S$ of functions $x$, $y$ and $z$ by same symbols. Since $K_S \sim -\sum_{i=1}^3 D_i$ and $C \sim -2K_S$, we obtain $h^0(S, K_S) = h^1(S, K_S) = 0$, which implies that $H^0(C, K_C) \simeq H^0(S, -K_S) = (x^2y, xy^2, xyz, z^3)$. Hence the restriction of the rational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$$

$$(x : y : z) \mapsto (x^2y : xy^2 : xyz : z^3)$$

for $X = C$ and $p = 2$. Recall that, as we saw at the beginning of the proof of Proposition 4.3, $V^2$ must be more than two. Since this contradicts the property (a), we can conclude that $k = 3$. In the case where $g = 5$ and $q = 4$, similarly to the previous case, we can show that the supposition $k \leq 3$ yields a contradiction. If $g = 10$ and $q = 6$, then we obtain $k = 6$ by (iv) which is proved below.

(iii) By [1], Clifdim$(C) = 2$ if and only if $C$ is isomorphic to a plane curve of degree $\geq 5$. Besides, if Cliffdim$(C) \geq 3$, then $g \geq 10$ holds. Hence, we have Cliffdim$(C) = 1$ in the cases $(g, q) = (4, 4), (5, 4)$. In other cases (except for the case $(g, q) = (10, 6)$), we see that the number of gonality pencils on $C$ is finite by (i). It follows that Cliffdim$(C) = 1$.

(iv) By Lemma 4.4 and Fig. 24, in this case, we can see the explicit structures of $C$. Besides, if Cliffdim$(C) = 3$, then we obtain $\deg\Phi \geq \Phi \geq 10$ holds. Hence, we have Cliffdim$(C) = 1$.

First, $| - K_S |$ has no base points, $h^0(S, -K_S) = 4$ and $(-K_S)^2 = 3$. Besides, we can write $C \sim -3K_S$, that is, $C, (-K_S) = 9$. We consider the morphism $\Phi_{| - K_S|} : S \rightarrow \mathbb{P}^3$, and put $T = \Phi_{| - K_S|}(S)$. Then, by the equality

$$\deg \Phi_{| - K_S|} \cdot \deg T = (-K_S)^2 = 3,$$

we obtain $\deg \Phi_{| - K_S|} = 1$ and $\deg T = 3$. We denote by $H$ a hyper plane of $T$. The short exact sequence of sheaves $0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3}(3H) \rightarrow \mathcal{O}_T(3H) \rightarrow 0$ induces the surjection $H^0(\mathbb{P}^3, 3H) \rightarrow H^0(T, 3H|_T) = H^0(T, C)$, where we abuse notation to denote the image of $C$ under $\Phi_{| - K_S|}$ by the same symbol. Hence we see that $C$ is an irreducible component of $T \cap T'$, where $T'$ is a cubic surface in $\mathbb{P}^3$. Since

$$9 = \deg T \cap T' \geq \deg C = C, (-K_S) = 9,$$

we can conclude that $C = T \cap T'$. \hfill \Box

5 The case where $(g, q) = (4, 4)$

In this section, we focus on the exceptional curve in Theorem 1.3 (ii), and exhibit its structure. Let $S$, $C$ and $q$ be as in Theorem 1.3, and assume $q = q = 4$. By Lemma 4.5, the fan $\Delta_S$ and the lattice polygon $\square_C$ are as in Fig. 30. Considering the shape of $\square_C$, we can take

\[
\begin{array}{c}
\sigma(D_0) \\
\sigma(D_1) \\
\sigma(D_2)
\end{array}
\]

\[
\Delta_S
\]

\[
\begin{array}{c}
\sigma(D_0) \\
\sigma(D_1) \\
\sigma(D_2)
\end{array}
\]

\[
\square_C
\]

\[
\begin{array}{c}
\sigma(D_0) \\
\sigma(D_1) \\
\sigma(D_2)
\end{array}
\]

\[
\Delta_S
\]

\[
\begin{array}{c}
\sigma(D_0) \\
\sigma(D_1) \\
\sigma(D_2)
\end{array}
\]

\[
\square_C
\]

\[
\begin{array}{c}
\sigma(D_0) \\
\sigma(D_1) \\
\sigma(D_2)
\end{array}
\]

\[
\Delta_S
\]

\[
\begin{array}{c}
\sigma(D_0) \\
\sigma(D_1) \\
\sigma(D_2)
\end{array}
\]

\[
\square_C
\]
to $C'$ gives the canonical embedding of $C$. Let $(t : u : v : w)$ be a homogeneous coordinate system in $\mathbb{P}^3$. Then the canonical curve of $C$ lies on a quadric surface $T : t^2 + u^2 + a v^2 = w^2$. Thus if $a \neq 0$, then two families of lines on $T$ cut out two distinct pencils $g_{3}^{1}$ and $h_{3}^{1}$ on $C$. On the other hand, if $a = 0$, then $T$ is a quadric cone, and one family of lines cuts out a unique $g_{3}^{1}$ on $C$. In sum, we can conclude that

1. If $a \neq 0$, $C$ is a curve of bidegree $(3, 3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.
2. If $a = 0$, $C$ is linearly equivalent to $3\Delta_0 + 6F$ on $\Sigma_2$, where $\Delta_0$ and $F$ denote the minimal section and the fiber of the ruling of $\Sigma_2$, respectively.

Unfortunately, however, we can not distinguish the above difference from the information of the lattice polygon.

### 6 Application

By combining Theorem 6.1 with results in [8], we can compute Weierstrass gap sequences at ramification points (with high ramification indexes) of a gonality pencil. For example, in this section, we consider trigonal curves and provide a geometric interpretation of the structure of gap sequences at ramification points. Let us review the preliminary results.

Firstly, it is known that a trigonal covering of $\mathbb{P}^1$ has four types of gap sequences.

**Theorem 6.1** ([3][4]). Let $C$ be a smooth trigonal curve of genus $g$ and Maroni invariant $m$, and $P$ a ramification point of a trigonal covering from $C$ to $\mathbb{P}^1$. Then the Weierstrass gap sequence at $P$ is one of the following types.

In the case where $P$ is a total ramification point:

- type I $\{1, 2, 4, 5, \ldots, 3m + 1, 3m + 2, 3m + 4, 3m + 7, \ldots, 3(g - m) - 5\}$,
- type II $\{1, 2, 4, 5, \ldots, 3m + 1, 3m + 2, 3m + 5, 3m + 8, \ldots, 3(g - m) - 4\}$.

In the case where $P$ is an ordinary ramification point:

- type I $\{1, 2, 3, \ldots, 2m + 1, 2m + 2, 2m + 3, 2m + 5, \ldots, 2(g - m) - 3\}$,
- type II $\{1, 2, 3, \ldots, 2m + 1, 2m + 2, 2m + 4, 2m + 6, \ldots, 2(g - m) - 2\}$.

Besides, Kato and Horiuchi presented the following criterion for distinguishing the above types.

**Theorem 6.2** ([7]). Let $C$ be a trigonal curve of genus $g \geq 5$ and Maroni invariant $m$. Then $C$ has a plane model defined by

$$y^3 + x^\mu A(x)y + x^\nu B(x) = 0,$$

where $\deg A(x) + \mu = 2m + 4$, $\deg B(x) + \nu = 3m + 6$ and $A(0)B(0) \neq 0$.

(i) If $\mu \geq \nu = 1$, there exists a total ramification point of type I over $x = 0$.
(ii) If $\mu \geq \nu = 2$, there exists a total ramification point of type II over $x = 0$.
(iii) If $\mu = \nu = 0$ and the order of zero of $4A(x)^3 + 27B(x)^2$ at $x = 0$ is odd, there exists an ordinary ramification point of type I over $x = 0$.
(iv) If $\nu > \mu = 1$, there exists an ordinary ramification point of type II over $x = 0$.
(v) Otherwise, there exist no ramification points over $x = 0$. 

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In order to interpret Theorem 6.2 in terms of lattice polygons, we transform the defining equation \([\text{5}]\) in the case (iii). Note that we need the condition \(m < (g-2)/2\) in this case, since \(2(g-m) - 3\) must be at least \(g\). We can write two polynomials as \(A(x) = \sum_{i=1}^{2m+4} a_ix^i - 3k^2\) and \(B(x) = \sum_{i=1}^{3m+6} b_ix^i + 2k^3\), where \(k \neq 0\). We formally set \(a_{2m+5} = \cdots = a_{3m+6} = 0\), and define \(\alpha = \min\{i \mid a_i \neq 0\}\), \(\beta = \min\{i \mid kA_i + b_i \neq 0\}\), \(A_1(x) = \sum_{i=1}^{3m+6} a_ix^i\), \(B_1(x) = \sum_{i=1}^{3m+6} b_ix^i\) and \(E(x) = A(x) - A_1(x) + 3k^2\). Then we have \(\text{mindeg}(kA_1(x) + B_1(x)) = \beta\), where the notation ‘mindeg’ denotes the minimal degree of a polynomial. Since \(m > g/2\), we see that \(\beta\) is less than \(2m + 4\). Let us check that \(\text{mindeg}(A(x)^3 + 27B(x)^2) = \beta < 2\alpha\).

By a simple computation, we have
\[
4A(x)^3 + 27B(x)^2 = 4(E(x) + A_1(x))^3 + 27(-kE(x) + B_1(x))^2 - 36k^2(E(x) + A_1(x))^2 \tag{6}
+ 108k^3(kA_1(x) + B_1(x)).
\]

In the case where \(E(x) = 0\), \(\beta\) is equal to \(\alpha\) by definition, and the equality \(\text{mindeg}(A(x)^3 + 27B(x)^2) = \beta\) follows from \([\text{4}]\). On the other hand, if \(E(x) \neq 0\), we have \(\text{mindeg}(A(x)^3 + 27B(x)^2) = \min\{2\text{mindeg}(E(x)) = \beta\} \) by \([\text{4}]\) and its oddness. Note that \(\text{mindeg}(E(x)) = \alpha\) in this case. Consequently, if we perform a coordinate transformation \(y' = y - k\), and put \(y' = y\) again, then the defining equation \([\text{5}]\) is rewritten as
\[
y^3 + 3ky^2 + x^\alpha C(x)y + x^\beta D(x) = 0, \tag{7}
\]
where \(\alpha\) and \(\beta\) are positive integers such that \(\beta\) is odd and \(\beta < \min\{2\alpha, 2m + 4\}\), \(\deg C(x) = 2m + 4\), \(\deg D(x) + \beta = 3m + 6\) and \(C(0)D(0) \neq 0\).

Now we are in a position to translate Theorem 6.2 in terms of the geometry of lattice polygons. We embed \(C\) in a toric surface by blowing up repeatedly. Then the lattice polygon \(\square_C\) associated to \(C\) is drawn as in Fig. 31. In the case (i), the fan \(\Delta_S\) associated to \(S\) is as in Fig. 32.

Considering the process of blowing-ups, we see that an intersection point \(P = C \cap D_0\) is a unique point over the origin of the plane model \([\text{4}]\). On the other hand, Theorem 1.3 and Fact 2.2 show that the fiber \(F\) of a trigonal covering from \(C\) to \(\mathbb{P}^1\) is \(F \sim D_4 + 2D_5 + 3D_6 + D_7\), which implies that \(P\) is a total ramification point. By applying Corollary 1.6 in \([\text{8}]\), we can to determine the gap sequence at \(P\) as
\[
\{j \mid \text{the line } 3X + Y = 3 + j \text{ has a lattice point in } \text{Int} \square_C\}.
\]
For better understanding, we attach to each lattice point an integer \(j\) such that the line \(3X + Y = 3 + j\) passes through it (see Fig. 33). Then we can find the gap sequence...
as a set of integers contained in Int $\square_C$. Since the genus of $C$ is equal to the number of lattice points contained in Int $\square_C$, we have $g = 3m + 4$ in this case. Hence the above gap sequence at a total ramification point $P$ is truly of type I in Theorem 6.1. Similarly, for the remaining cases in Fig. 31, we obtain the gap sequence at a ramification point and the genus of $C$ as follows.

(ii) \( \{ j \mid \text{the line } 3X + 2Y = 6 + j \text{ has a lattice point in Int } \square_C \}, \ g = 3m + 3 \),

(iii) \( \{ j \mid \text{the line } 2X + 3Y = 2\beta + j \text{ has a lattice point in Int } \square_C \}, \ g = 3m - \frac{\beta - 9}{2} \),

(iv) \( \{ j \mid \text{the line } 2X + Y = 3 + j \text{ has a lattice point in Int } \square_C \}, \ g = 3m + 3 \).

Consequently, in each case, the result of Theorem 6.1 can be visualized in a similar way as in Fig. 33.

This idea is applicable for the cases of higher gonality. By Theorem 1.3, a lattice polygon associated to a $k$-gonal curve $C$ can be drawn as a polygon with height $k$ and sufficiently large width. Assume that there exists an oblique side which has no lattice points except for two end points, and denote by $D$ the $T$-invariant divisor corresponds to this side. In this case, a point $P = C \cap D$ is a total ramification point of a gonality pencil on $C$, and moreover, $P$ satisfies the assumption in Corollary 1.6 in [8]. Hence one can determine the Weierstrass gap sequence at $P$ by moving the oblique side similarly to Fig. 33. This fact suggests the possibility of the classification of gap sequences at total ramification points of a curve on a toric surface. We will deal with this prospective problem in future work.

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References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, Geometry of Algebraic Curves Vol. I, Springer-Verlag, New York, 1985.

[2] W. Castryck and F. Cools, Newton polygons and curve gonality, J. Algebraic Combin. 35 (2012), 367–372.

[3] M. Coppens, The Weierstrass gap sequences of the total ramification points of trigonal coverings of $\mathbb{P}^1$, Indag. Math. 43 (1985), 245–276.

[4] M. Coppens, The Weierstrass gap sequence of the ordinary ramification points of trigonal coverings of $\mathbb{P}^1$; existence of a kind of Weierstrass gap sequence, J. Pure Appl. Algebra 43 (1986), 11–25.

[5] M. Coppens and G. Martens, Secant spaces and Clifford’s theorem, Compositio Math. 78 (1991), 193–212.

[6] D. Eisenbud, H. Lange, G. Martens and F.-O. Schreyer, The Clifford dimension of a projective curve, Compositio Math. 72 (1989), 173–204.

[7] T. Kato and R. Horiochi, Weierstrass gap sequences at the ramification points of trigonal Riemann surfaces, J. Pure Appl. Algebra 50 (1988), 271–285.

[8] R. Kawaguchi, Weierstrass gap sequences on curves on toric surfaces, Kodai. Math. J. 30 (2010), 63–86.

[9] S. J. Kim, On the existence of Weierstrass gap sequences on trigonal curves, J. Pure Appl. Algebra 63 (1990), 171–180.

[10] G. Martens, Über den Clifford-Index algebraischer Kurven, J. Reine Angew. Math. 336 (1982), 83–90.

[11] G. Martens, The gonality of curves on a Hirzebruch surface, Arch. Math. 64 (1996), 349–452.

[12] M. Namba, Geometry of projective algebraic curves, Monographs and Textbooks in Pure and Appl. Math. 88, Marcel Dekker, Inc., New York, 1984.

[13] T. Oda, Convex bodies and algebraic geometry, Springer-Verlag, Berlin, 1988.

[14] F. Serrano, Extension of morphisms defined on a divisor, Math. Ann. 277 (1987), 395–413.

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