Heat kernels, Bergman kernels, and cusp forms

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Abstract

In this article, we describe a geometric method to study cusp forms, which relies on heat kernel and Bergman kernel analysis. This new approach of applying techniques coming from analytic geometry is based on the micro-local analysis of the heat kernel and the Bergman kernel from [3] and [2], respectively, using which we derive sup-norm bounds for cusp forms of integral weight, half-integral weight, and real weight associated to a Fuchsian subgroup of first kind.

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1 Introduction

This is both a survey and a research article elucidating heat kernel and Bergman kernel methods for studying cusp forms. We describe a geometric approach of Bouche and Berman to study sup-norm bounds for sections of a positive line bundle defined over a compact complex manifold. We then apply these methods to study cusp forms associated to a Fuchsian subgroup of first kind, which yields optimal results when the Fuchsian subgroup is cocompact. However, this approach does not give optimal results when the Fuchsian subgroup is cofinite. But an extension of the methods of Bouche to cuspidal neighborhoods should allow one to derive optimal bounds for cusp forms, even when the Fuchsian subgroup is cofinite.

1.1 Notation

Let \( \mathbb{C} \) denote the complex plane. For \( z \in \mathbb{C} \), let \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \) denote the real and imaginary parts of \( z \), respectively. Let

\[
\mathbb{H} = \{ z \in \mathbb{C} | y = \text{Im}(z) > 0 \}
\]

be the upper half-plane. Let \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) be a Fuchsian subgroup of the first kind acting by fractional linear transformations on \( \mathbb{H} \). We assume that \( \Gamma \) admits no elliptic elements.

Let \( X \) be the quotient space \( \Gamma \backslash \mathbb{H} \) of genus \( g > 1 \). The quotient space \( X \) admits the structure of a hyperbolic Riemann surface of finite volume. We allow \( X \) to have genus \( g = 1 \), if \( X \) is not compact.

We denote the \((1, 1)\)-form corresponding to the hyperbolic metric of \( X \), which is compatible with the complex structure on \( X \) and has constant negative curvature equal to minus one, by \( \mu_{\text{hyp}}(z) \). Locally, for \( z \in X \), it is given by

\[
\mu_{\text{hyp}}(z) = \frac{i}{2} \frac{dz \wedge d\overline{z}}{\text{Im}(z)^2}.
\]
Let $\mu_{\text{shyp}}(z)$ denote the rescaled hyperbolic metric $\mu_{\text{hyp}}(z)/\text{vol}_{\text{hyp}}(X)$, which measures the volume of $X$ to be one.

For $k \in \mathbb{R}_{>0}$, let $\nu$ denote the factor of automorphy of weight $k$ with the associated character being unitary. Let $S^k(\Gamma, \nu)$ denote the complex vector space of weight-$k$ cusp forms with respect to $\Gamma$ and $\nu$. Let $\{f_1, \ldots, f_j\}$ denote an orthonormal basis of $S^k(\Gamma, \nu)$ with respect to the Petersson inner product. Then, for $z \in X$, put

$$B_{X}^{k,\nu}(z) := \sum_{i=1}^{j} y^k |f_i(z)|^2.$$ 

When the associated character is trivial, we put $B_{X}^{k,\nu}(z) = B_{X}^{k}(z)$.

### 1.2 Sup norm bounds for the function $B_{X}^{k,\nu}(z)$

Let the Fuchsian subgroup $\Gamma$ be cocompact, i.e., $X$ is a compact Riemann surface. With notation as above, for $k \in \frac{1}{2} \mathbb{Z}$ (or $2\mathbb{Z}$), we have the following estimate

$$\lim_{k} \sup_{z \in X} \frac{1}{k} B_{X}^{k}(z) = O(1),$$

where the implied constant is independent of $\Gamma$.

Furthermore, for a fixed $k \in \mathbb{R}_{>0}$, let $\nu$ denote the factor of automorphy of weight $k$ with the associated character being unitary. Then, with notation as above, we have the following estimate

$$\lim_{n} \sup_{z \in X} \frac{1}{nk} B_{X}^{nk,\nu}(z) = O(1),$$

where $n \in \mathbb{Z}$, and the implied constant is independent of $\Gamma$.

Let $\Gamma$ now be a cofinite subgroup, i.e., $X$ is a noncompact hyperbolic Riemann surface of finite volume, and let $A$ be a compact subset of $X$. Then, with notation as above, for $k \in \frac{1}{2} \mathbb{Z}$ (or $2\mathbb{Z}$) and $z \in A$, we have the following estimate

$$\lim_{k} \frac{1}{k} B_{X}^{k}(z) = O_A(1),$$

where the implied constant depends on $A$.

Furthermore, for a fixed $k \in \mathbb{R}_{>0}$, let $\nu$ denote the factor of automorphy of weight $k$ with the associated character being unitary. Let $A$ be any compact subset of $X$. Then, with notation as above, for any $z \in A$, we have the following estimate

$$\lim_{n} \frac{1}{nk} B_{X}^{nk,\nu}(z) = O_A(1),$$

where $n \in \mathbb{Z}$ and the implied constant depends on $A$.

Our estimates (1), (2), (3), and (4) are optimal. However, when $X$ is noncompact we cannot extend estimates (3) and (4) to the entire Riemann surface, i.e., our method does not yield optimal estimates when $A$ is equal to $X$. However, an extension of Bouche’s methods to cuspidal neighborhoods will enable the extension of estimates (3) and (4) to $X$.

Lastly, our methods extend with notational changes to higher dimensions, namely to Hilbert modular cusp forms and Siegel modular cusp forms.
1.3 Existing results on sup-norm bounds for the function $B_{X}^{k,\nu}(z)$

In [6], using heat kernel analysis, Jorgenson and Kramer derived sup-norm bounds for the Bergman kernel $B_{X}^{2}(z)$, associated to any hyperbolic Riemann surface $X$ (compact or noncompact of finite volume). The bounds of Jorgenson and Kramer are optimal. Especially for the case $X = Y_{0}(N)$, they derived

$$\sup_{z \in Y_{0}(N)} B_{Y_{0}(N)}^{2}(z) = O(1),$$

where the implied constant does not depend on the modular curve $Y_{0}(N)$.

In [5], extending their method from [6], Jorgenson, Kramer, and Friedman derived sup-norm bounds for the Bergman kernel $B_{X}^{k}(z)$, associated to any hyperbolic Riemann surface $X$ (compact or noncompact of finite volume). When $X$ is a compact hyperbolic Riemann surface, they showed that

$$\sup_{z \in X} B_{X}^{k}(z) = O(k),$$

where the implied constant is independent of the Riemann surface $X$. When $X$ is a noncompact hyperbolic Riemann surface of finite volume, they showed that

$$\sup_{z \in X} B_{X}^{k}(z) = O(k^{\frac{3}{2}}),$$

where the implied constant is independent of the Riemann surface $X$. The estimates of Jorgenson and Kramer are optimal, as shown in [6].

It is possible to extend the heat kernel analysis of Jorgenson and Kramer to higher dimensions, namely to Hilbert modular cusp forms and Siegel modular cusp forms of both integral and half-integral weight. However, one has to address certain non trivial convergence issues, while doing so.

For $k \in \frac{1}{2} \mathbb{Z}$ and $N \in \mathbb{N}$, let $f$ be any weight-$k$ cusp form with respect to the arithmetic subgroup $\Gamma_{0}(4N)$. Furthermore, let $f$ be normalized with respect to the Petersson inner-product. Then, in [4], Kiral has derived the following estimate

$$\sup_{z \in Y_{0}(N)} |y^{k}|f(z)|^{2} = O_{k,\varepsilon}(N^{\frac{1}{2} - \frac{1}{18} + \varepsilon}),$$

for any $\varepsilon > 0$. Using above estimate, one can derive

$$\sup_{z \in Y_{0}(N)} B_{Y_{0}(N)}^{k}(z) = O_{k,\varepsilon}(N^{\frac{1}{2} - \frac{1}{18} + \varepsilon}),$$

for any $\varepsilon > 0$.

For $k \in \mathbb{R}_{>0}$ with $k > 2$, the Bergman kernel $B_{X}^{k}(z)$ can be represented by an infinite series, which is uniformly convergent in $z \in X$. Using which, Steiner has extended the bounds of Jorgenson and Kramer to real weights.

Let $\Gamma$ any subgroup of finite index in $\text{SL}_{2}(\mathbb{R})$, and for a fixed $k \in \mathbb{R}_{>0}$ with $k \gg 1$, let $\nu$ denote the factor of automorphy of weight $k$ with the associated character being unitary. Furthermore, let $A$ be a compact subset of $X$. Then, in [7], Steiner has derived the following estimates

$$\sup_{z \in A} B_{X}^{k,\nu}(z) = O_{A}(k),$$

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where the implied constant depends on the compact subset $A$; and
\[ \sup_{z \in X} B_{X}^{k^n}(z) = O_X(k^{\frac{1}{2}}), \]
where the implied constant depends on $X$.

## 2 Heat kernels and Bergman kernels on compact complex manifolds

In this section, we recall the main results from [3] and [2], which we use in the next section.

Let $(M, \omega)$ be a compact complex manifold of dimension $n$ with a Hermitian metric $\omega$. Let $\mathcal{L}$ be a positive Hermitian holomorphic line bundle on $M$ with the Hermitian metric given by $\|s(z)\|_{\mathcal{L}}^2 := e^{-\phi(z)}|s(z)|^2$, where $s \in \mathcal{L}$ is any section, and $\phi(z)$ is a real-valued function defined on $M$.

For any $k \in \mathbb{N}$, let $\square_k := (\overline{\partial} + \partial)^2$ denote the $\overline{\partial}$-Laplacian acting on smooth sections of the line bundle $\mathcal{L}^\otimes k$. Let $K_{M, \mathcal{L}}^{k}(t; z, w)$ denote the smooth kernel of the operator $e^{-\frac{2k}{t} \square_k}$.

We refer the reader to p. 2 in [3], for the details regarding the properties which uniquely characterize the heat kernel $K_{M, \mathcal{L}}^{k}(t; z, w)$. When $z = w$, the heat kernel $K_{M, \mathcal{L}}^{k}(t; z, w)$ admits the following spectral expansion
\[ K_{M, \mathcal{L}}^{k}(t; z, w) = \sum_{n \geq 0} e^{-\frac{2k}{t} \lambda^k_n} \phi_n(z) \otimes \phi_n^*(w), \tag{5} \]
where $\{\lambda^k_n\}_{n \geq 0}$ denotes the set of eigenvalues of $\square_k$ (counted with multiplicities), and $\{\varphi_n\}_{n \geq 0}$ denotes a set of associated orthonormal eigenfunctions.

Let $\{s_i\}$ denote an orthonormal basis of $H^0(M, \mathcal{L}^\otimes k)$. For any $z \in M$, the Bergman kernel is given by
\[ B_{M, \mathcal{L}}^{k}(z) := \sum_i \|s_i(z)\|_{\mathcal{L}^\otimes k}^2. \tag{6} \]

For any $z \in M$ and $t \in \mathbb{R}_{>0}$, from the spectral expansion of the heat kernel $K_{M, \mathcal{L}}^{k}(t; z, w)$ described in equation (5), it is easy to see that
\[ B_{M, \mathcal{L}}^{k}(t; z, z) \leq K_{M, \mathcal{L}}^{k}(t; z, z) \quad \text{and} \quad \lim_{t \to 0} K_{M, \mathcal{L}}^{k}(t; z, z) = B_{M, \mathcal{L}}^{k}(t; z). \tag{7} \]

For $z \in M$, let $c_{1}(\mathcal{L})(z) := \frac{i}{2\pi} \overline{\partial} \partial \phi(z)$ denote the first Chern form of the line bundle $\mathcal{L}$. Let $\alpha_1, \ldots, \alpha_n$ denote the eigenvalues of $\overline{\partial} \partial \phi(z)$ at the point $z \in M$. Then, with notation as above, from Theorem 1.1 in [3], for any $z \in M$ and $t \in (0, k^\varepsilon)$, and for a given $\varepsilon > 0$ not depending on $k$, we have
\[ \lim_{k \to \infty} \frac{1}{k^n} K_{M, \mathcal{L}}^{k}(t; z, z) = \prod_{j=1}^{n} \frac{\alpha_j}{(4\pi)^{n} \sinh(\alpha_j t)}. \tag{8} \]
and the convergence of the above limit is uniform in $z$.

Using equations (7) and (8), in Theorem 2.1 in [3], Bouche derived the following asymptotic estimate
\[ \lim_{k \to \infty} \frac{1}{k^n} B_{M, \mathcal{L}}^{k}(z) = O(\det_{\omega}(c_{1}(\mathcal{L})(z))), \tag{9} \]
where the implied constant does not depend on $X$, and the convergence of the above limit is uniform in $z \in X$.

When $M$ is a noncompact complex manifold, using micro-local analysis of the Bergman kernel, in [2], Berman derived the following estimate

$$\limsup_k \frac{1}{k^n} B^k_M(z) \leq \det(\omega(c_1(L)(z))).$$

Furthermore, let $A$ be any compact subset of $M$. Then, for any $z \in A$, from the proof of Corollary 3.3 in [2], we have

$$\lim \frac{1}{k^n} B^k_M(z) = O_A\left(\det(\omega(c_1(L)(z)))\right),$$

where the implied constant depends on the compact subset $A$.

3 Estimates of cusp forms

In this section, using results from previous section, we prove estimates (1), (2), (3), and (4).

Let notation be as in Section 1. Let $\Omega_X$ denote the cotangent bundle over $X$. Then, for any $k \in 2\mathbb{Z}$, cusp forms of weight $k$ with respect to $\Gamma$ are global section of the bundle $\Omega_X \otimes k/2$. Furthermore, recall that for any $f \in \Omega_X$, i.e., $f$ a weight-2 cusp form, the Petersson metric on the line bundle $\Omega_X$ is given by

$$\|f(z)\|^2_{\Omega_X} := y^2 |f(z)|^2.$$ (11)

Let $\omega_X$ denote the line bundle of cusp forms of weight $1/2$ over $X$. Then, for any $k \in 1/2\mathbb{Z}$, cusp forms of weight-$k$ with respect to $\Gamma$ are global section of the line bundle $\omega_X \otimes 2k$. Furthermore, recall that for any $f \in \omega_X$, i.e., $f$ a weight-$1/2$ cusp form, the Petersson metric on the line bundle $\omega_X$ is given by

$$\|f(z)\|^2_{\omega_X} := y^{1/2} |f(z)|^2.$$ (12)

**Remark 1.** For any $z \in X$ and $k \in 2\mathbb{Z}$, from the definition of the Bergman kernel $B^k_{X,\Omega_X}(z)$ for the line bundle $\Omega_X \otimes k/2$ from equation (6), we have

$$B^k_{X,\Omega_X}(z) = B^k_X(z).$$

Similarly, for any $z \in X$ and $k \in 1/2\mathbb{Z}$, from the definition of the Bergman kernel $B^{2k}_{X,\omega_X}(z)$ for the line bundle $\omega_X \otimes 2k$ from equation (6), we have

$$B^{2k}_{X,\omega_X}(z) = B^{k}_X(z).$$ (13)

**Theorem 2.** Let $\Gamma$ be cocompact, i.e., $X$ is a compact hyperbolic Riemann surface. Then, with notation as above, for $k \in 1/2\mathbb{Z}$ (or $2\mathbb{Z}$), we have the following estimate

$$\limsup_k \frac{1}{k^n} B^k_X(z) = O(1),$$

where the implied constant is independent of $\Gamma$. 5
Proof. We refer the reader to Theorem 2 in [1] for the proof of the theorem. We briefly describe the proof of the theorem for \( k \in \frac{1}{2}\mathbb{Z} \), and the case for \( k \in 2\mathbb{Z} \), follows automatically with notational changes. For any \( z \in X \), observe that

\[
c_1(\omega_X^{\otimes 2})(z) = \frac{1}{4\pi} \mu_{\text{hyp}}(z),
\]

which shows that the line bundle \( \omega_X^{\otimes 2} \) is positive, and \( \det \mu_{\text{hyp}}(c_1(\omega_X^{\otimes 2})(z)) = \frac{1}{4\pi} \). Using equation (13), and applying estimate (9) to the complex manifold \( X \) with its natural Hermitian metric \( \mu_{\text{hyp}} \) and the line bundle \( \omega_k^{\otimes 2k} \), we find

\[
\lim_{k \to 1} \frac{1}{k} B^k_X(z) = \lim_{k \to 1} \frac{1}{k} B^k_{X,\omega_X}(z) = O\left(\det \mu_{\text{hyp}}(c_1(\omega_X^{\otimes 2})(z))\right) = O(1).
\]

As the above limit convergences uniformly in \( z \in X \), and as \( X \) is compact, we have

\[
\sup_{z \in X} \lim_{k \to 1} \frac{1}{k} B^k_X(z) = \lim_{k \to 1} \sup_{z \in X} \frac{1}{k} B^k_X(z) = O(1),
\]

which completes the proof of the theorem. \( \square \)

Corollary 3. Let \( \Gamma \) be cofinite, i.e., \( X \) is a noncompact hyperbolic Riemann surface of finite volume, and let \( A \) be a compact subset of \( X \). Then, with notation as above, for \( k \in \frac{1}{2}\mathbb{Z} \) (or \( 2\mathbb{Z} \)), and \( z \in A \), we have the following estimate

\[
\lim_{k \to 1} \frac{1}{k} B^k_X(z) = O_A(1),
\]

where the implied constant depends on \( A \).

Proof. The proof of the theorem follows from estimate (10), and from similar arguments as in Theorem 2. \( \square \)

Remark 4. For a fixed \( k \in \mathbb{R}_{>0} \), and let \( \omega_{X,k,\nu} \) denote the line bundle of weight-\( k \) cusp forms with the factor of automorphy \( \nu \), and associated character being unitary. Then, for any \( n \in \mathbb{Z} \), cusp forms of weight-\( nk \) with respect to \( \Gamma \) and \( \nu^n \) are global sections of the line bundle \( \omega_X^{\otimes n} \). Furthermore, recall that for any \( f \in \omega_{X,k,\nu} \), the Petersson metric on the line bundle \( \omega_{X,k,\nu} \) is given by

\[
\|f(z)\|^2_{\omega_{X,k,\nu}} := y^k |f(z)|^2.
\]

(14)

For any \( z \in X \) and \( n \in \mathbb{Z} \), from the definition of the Bergman kernel \( B^n_{X,\omega_X,k,\nu,n}(z) \) for the line bundle \( \omega_X^{\otimes n} \), from equation (6), we have

\[
B^n_{X,\omega_X,k,\nu,n}(z) = B^{nk,\nu^n}_X(z).
\]

(15)

Theorem 5. Let \( \Gamma \) be cocompact, i.e., \( X \) is a compact hyperbolic Riemann surface. For a fixed \( k \in \mathbb{R}_{>0} \), let \( \nu \) denote the factor of automorphy of weight \( k \) with the associated character being unitary. Then, with notation as above, we have the following estimate

\[
\lim_{n \to \infty} \sup_{z \in X} \frac{1}{nk} B^{nk,\nu^n}_X(z) = O(1),
\]

where \( n \in \mathbb{Z} \), and the implied constant is independent of \( \Gamma \).
Proof. From equation (14), for any $z \in X$, observe that
\begin{align*}
c_1(\omega_{X,k,\nu})(z) &= -\frac{i}{2\pi \partial \partial \log (y_k|f(z)|^2) = \frac{k}{4\pi} \mu_{hyp}(z),}
\end{align*}
which shows that the line bundle $\omega_{X,k,\nu}$ is positive, and $\det_{\mu_{hyp}}(c_1(\omega_{X,k,\nu})(z)) = \frac{k}{4\pi}$. Using equation (15), and applying estimate (9) to the complex manifold $X$ with its natural Hermitian metric $\mu_{hyp}$ and the line bundle $\omega_{X,k,\nu} \otimes n^X$, we find
\begin{align*}
\lim_{n \to \infty} \frac{1}{nk} B_{nk,\nu}^X(z) &= \lim_{n \to \infty} \frac{1}{nk} B_{X,\omega_{X,k,\nu}}^n(z) = O \left( \frac{1}{k} \det_{\mu_{hyp}} \left( c_1(\omega_{X,k,\nu})(z) \right) \right) = O(1).
\end{align*}
As the above limit converges uniformly in $z \in X$, and as $X$ is compact, we have
\begin{align*}
\sup_{z \in X} \lim_{n \to \infty} \frac{1}{nk} B_{nk,\nu}^X(z) = \lim_{n \to \infty} \sup_{z \in X} \frac{1}{nk} B_{nk,\nu}^X(z) = O(1),
\end{align*}
which completes the proof of the theorem. \hfill \square

Corollary 6. Let $\Gamma$ be cofinite, i.e., $X$ is a noncompact hyperbolic Riemann surface of finite volume. For a fixed $k \in \mathbb{R}_{>0}$, let $\nu$ denote the factor of automorphy of weight $k$ with the associated character being unitary. Let $A$ be a compact subset of $X$. Then, with notation as above, for any $z \in A$, we have the following estimate
\begin{align*}
\lim_{n \to \infty} \frac{1}{nk} B_{nk,\nu}^X(z) &= O_A(1),
\end{align*}
where $n \in \mathbb{N}$, and the implied constant depends on $A$.

Proof. The proof of the theorem follows from estimate (10), and from similar arguments as in Theorem 5. \hfill \square

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