Some bounds on the eigenvalues of uniform hypergraphs

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Abstract

Let \( H \) be a uniform hypergraph. Let \( A(H) \) and \( Q(H) \) be the adjacency tensor and the signless Laplacian tensor of \( H \), respectively. In this note we prove several bounds for the spectral radius of \( A(H) \) and \( Q(H) \) in terms of the degrees of vertices of \( H \).

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1 Introduction

We denote the set \( \{1, 2, \ldots, n\} \) by \( [n] \). Hypergraph is a natural generalization of simple graph (see [1]). A hypergraph \( H = (V(H), E(H)) \) on \( n \) vertices is a set of vertices, say \( V(H) = \{1, 2, \ldots, n\} \) and a set of edges, say \( E(H) = \{e_1, e_2, \ldots, e_m\} \), where \( e_i = \{i_1, i_2, \ldots, i_l\}, i_j \in [n], j = 1, 2, \ldots, l. \) If \( |e_i| = k \) for any \( i = 1, 2, \ldots, m \), then \( H \) is called a \( k \)-uniform hypergraph. The degree \( d_i \) of vertex \( i \) is defined as \( d_i = |\{e_j : i \in e_j \in E(H)\}|. \) If \( d_i = d \) for any vertex \( i \) of hypergraph \( H \), then \( H \) is called a \( d \)-regular hypergraph. An order \( k \) dimension \( n \) tensor \( T = (T_{i_1, i_2, \ldots, i_k}) \in \mathbb{C}^{n \times n \times \cdots \times n} \) is a multidimensional array with \( n^k \) entries, where \( i_j \in [n] \) for each \( j = 1, 2, \ldots, k. \) To study the properties of uniform hypergraphs by algebraic methods, adjacency matrix and signless Laplacian matrix of graph are generalized to adjacency tensor and signless Laplacian tensor of uniform hypergraph.

Definition 1 [2] [10]. Let \( H = (V(H), E(H)) \) be a \( k \)-uniform hypergraph on \( n \) vertices. The adjacency tensor of \( H \) is defined as the \( k \)-th order \( n \)-dimensional tensor \( A(H) \) whose \( (i_1 \cdots i_k) \)-entry is:

\[
(A(H))_{i_1, i_2, \ldots, i_k} = \begin{cases} \frac{1}{(k-1)!} & \{i_1, i_2, \ldots, i_k\} \in E(H) \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( D(H) \) be a \( k \)-th order \( n \)-dimensional diagonal tensor, with its diagonal entry \( D_{ii \cdots i} \) being \( d_i \), the degree of vertex \( i \), for all \( i \in [n] \). Then

\[
Q(H) = D(H) + A(H)
\]

is the signless Laplacian tensor of the hypergraph \( H \).

The following general product of tensors, is defined in [11] by Shao, which is a generalization of the matrix case.

Definition 2 Let \( A \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2} \) and \( B \in \mathbb{C}^{n_2 \times n_3 \times \cdots \times n_{k+1}} \) be order \( m \geq 2 \) and \( k \geq 1 \) tensors, respectively. The product \( AB \) is the following tensor \( C \) of order \((m - 1)(k - 1) + 1\) with entries:

\[
C_{i_1, \ldots, i_{m-1}} = \sum_{i_2, \ldots, i_m \in [n_2]} A_{i_1, \ldots, i_m} B_{i_2, \ldots, i_m, i_{m-1}} \quad (1)
\]

Where \( i \in [n], \alpha_1, \ldots, \alpha_{m-1} \in [n_3] \times \cdots \times [n_{k+1}] \).

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Let $\mathcal{T}$ be an order $k$ dimension $n$ tensor, let $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension $n$. Then by (1) $\mathcal{T}x$ is a vector in $\mathbb{C}^n$ whose $i$th component is as the following

$$(\mathcal{T}x)_i = \sum_{i_2, \cdots, i_k=1}^n \mathcal{T}_{i_2 \cdots i_k} x_{i_2} \cdots x_{i_k}. \quad (2)$$

Let $x^{[k]} = (x_1^k, \cdots, x_n^k)^T$. Then (see [10]) a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $\mathcal{T}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequations

$$\mathcal{T}x = \lambda x^{[k-1]}, \quad (3)$$

and in this case, $x$ is called an eigenvector of $\mathcal{T}$ corresponding to eigenvalue $\lambda$.

An eigenvalue of $\mathcal{T}$ is called an H-eigenvalue, if there exists a real eigenvector corresponding to it ([10]). The maximal absolute value of eigenvalues of $\mathcal{T}$ is called the spectral radius of $\mathcal{T}$ denoted by $\rho(\mathcal{T})$ (see [12]).

In [3], the weak irreducibility of nonnegative tensors was defined. It was proved in [4] and [15] that a $k$-uniform hypergraph $\mathcal{H}$ is connected if and only if its adjacency tensor $\mathcal{A}(\mathcal{H})$ (and so $\mathcal{Q}(\mathcal{H})$) is weakly irreducible. They furthered proved the following results, which implies that $\rho(\mathcal{T})$ is an H-eigenvalue of $\mathcal{T}$ under some conditions.

**Lemma 3** [4][15] Let $\mathcal{T}$ be a nonnegative tensor. Then $\rho(\mathcal{T})$ is an H-eigenvalue of $\mathcal{T}$ with a nonnegative eigenvector. Furthermore, if $\mathcal{T}$ is weakly irreducible, then $\rho(\mathcal{T})$ has a positive eigenvector.

Let $\mathcal{T}$ be a tensor of order $k$ and dimension $n$. For $i = 1, 2, \cdots, n$, denote by

$$r_i(\mathcal{T}) = \sum_{i_2, \cdots, i_k \in [n]} \mathcal{T}_{i_2 \cdots i_k}. \quad (4)$$

For a nonnegative tensor $\mathcal{T}$ the following bound for $\rho(\mathcal{T})$ in terms of $r_i(\mathcal{T})$ was proposed in [13], and the conditions for the equal cases were studied in [7].

**Lemma 4** [7][13] Let $\mathcal{T}$ be a nonnegative tensor of dimension $n$. We have

$$\min_{1 \leq i \leq n} r_i(\mathcal{T}) \leq \rho(\mathcal{T}) \leq \max_{1 \leq i \leq n} r_i(\mathcal{T}). \quad (5)$$

Moreover, if $\mathcal{T}$ is weakly irreducible, then the equality in (5) holds if and only if $r_1(\mathcal{T}) = \cdots = r_n(\mathcal{T})$.

For adjacency tensor $\mathcal{A}(\mathcal{H})$ of $k$-uniform hypergraph $\mathcal{H}$ we have $r_i(\mathcal{A}(\mathcal{H})) = d_i$, where $d_i$ is the degree of vertex $i$. Hence Lemma 4 implies the following result, which is an analog of a classical theorem in spectral graph theory.

**Corollary 5** [3] Let $\mathcal{H}$ be a $k$-uniform hypergraph with maximum degree $\Delta$. Then $\rho(\mathcal{A}(\mathcal{H})) \leq \Delta$.

In this note we first give a bound on $\rho(\mathcal{A}(\mathcal{H}))$ in terms of degrees of vertices, which improves the bound as shown in Corollary 5. Some bounds on $\rho(\mathcal{Q}(\mathcal{H}))$ are also proved.

## 2 Several bounds on $\rho(\mathcal{A}(\mathcal{H}))$ and $\rho(\mathcal{Q}(\mathcal{H}))$

Some techniques of this note are based on the facts that diagonal similar tensors have the same spectra (see [11]).

**Definition 6** [11][13] Let $\mathcal{A}$ and $\mathcal{B}$ be two order $k$ dimension $n$ tensors. Suppose that there exists a nonsingular diagonal matrix $D$ of order $n$ such that $\mathcal{B} = D^{k-1} \mathcal{A}D$, then $\mathcal{A}$ and $\mathcal{B}$ are called diagonal similar.
Theorem 2.1 of [11] implied the following result for similar tensors, thus for diagonal similar tensors.

**Lemma 7** [11] Let $\mathcal{A}$ and $\mathcal{B}$ be two order $k$ dimension $n$ similar tensors. Then $\mathcal{A}$ and $\mathcal{B}$ have the same spectra.

Now we introduce a special class of hypergraphs, whose spectral radius of the adjacency tensor can be determined by Theorem 8. Let $\mathcal{G}_0$ be a $d$-regular $(k-1)$-uniform hypergraph on $n-1$ vertices. If $\mathcal{G}$ is obtained from $\mathcal{G}_0$ by adding a new vertex $v$ to each edge of $\mathcal{G}_0$, then we may call that $\mathcal{G}$ is a blow-up of $\mathcal{G}_0$ and write $\mathcal{G} = \mathcal{G}_0(v)$. Obviously, $\mathcal{G}$ is a $k$-uniform hypergraph on $n$ vertices with $d_v = |E(\mathcal{G})| = |E(\mathcal{G}_0)|$ and $d_u = d$ for any $u \in (V(\mathcal{G}) \setminus \{v\})$. Let $K^{k-1}_{k-1}$ be the $(k-1)$-uniform hypergraph on $k-1$ vertices, and $tK^{k-1}_{k-1}$ be $t$ disjoint unions of $K^{k-1}_{k-1}$. For example, the hyperstar $S_{(k-1)+1, k}$ (see [5]) is a blow-up of $tK^{k-1}_{k-1}$.

**Theorem 8** Let $\mathcal{H}$ be a $k$-uniform hypergraph on $n$ vertices with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Let $\mathcal{A}(\mathcal{H})$ be the adjacency tensor of $\mathcal{H}$. Then

$$\rho(\mathcal{A}(\mathcal{H})) \leq d_1^k d_2^{1-k}.$$ 

Equality holds if and only if $\mathcal{H}$ is a regular hypergraph, or $\mathcal{H}$ is a blow-up of some regular hypergraph.

**Proof** Write $\mathcal{A} = \mathcal{A}(\mathcal{H})$ for short.

1. If $d_1 = d_2$, by Lemma 4 we have

$$\rho(\mathcal{A}) \leq \max_{1 \leq i \leq n} r_i(\mathcal{A}) = \max_{1 \leq i \leq n} d_i = d_1 = d_1^k d_2^{1-k}.$$ 

Equality holds if and only if $r_i(\mathcal{A})$ is a constant. So $\mathcal{H}$ is a regular hypergraph.

2. Now we suppose that $d_1 > d_2$ holds. If $\mathcal{P}$ is a diagonal matrix, then by (1), we have

$$(P^{-(k-1)} \mathcal{A})_{i_1i_2\cdots i_k} = P^{-(k-1)}_{i_1i_2} \mathcal{A}_{i_3i_4\cdots i_k} \mathcal{P}_{i_3i_4} \cdots \mathcal{P}_{i_ki_k}.$$ 

Now take $\mathcal{P} = \text{diag}(x, 1, \cdots, 1)$ with $x > 1$. Then we have

$$r_1(P^{-(k-1)} \mathcal{A}) = \sum_{i_2, \cdots, i_k \in [n]} (P^{-(k-1)} \mathcal{A})_{i_1i_2\cdots i_k}$$

$$= \sum_{i_2, \cdots, i_k \in [n]} P^{-(k-1)}_{1i_2} \mathcal{A}_{i_3i_4\cdots i_k} \mathcal{P}_{i_3i_4} \cdots \mathcal{P}_{i_ki_k}$$

$$= \frac{1}{x^{k-1}} \sum_{i_2, \cdots, i_k \in [n]} \mathcal{A}_{i_2\cdots i_k}$$

$$= \frac{d_1}{x^{k-1}}.$$ 

Denote by $d_{\{1,i\}}$ the number of edges, which contain vertices both 1 and $i$, i.e.,

$$d_{\{1,i\}} = |\{e_j : \{1,i\} \subset e_j \in E(\mathcal{H})\}|.$$ 

For $2 \leq i \leq n$, we have
\[ r_i(P^{-(k-1)}AP) = \sum_{i_2, \ldots, i_k \in [n]} (P^{-(k-1)}AP)_{i_2 \cdots i_k} \]
\[ = \sum_{i_2, \ldots, i_k \in [n]} P_{ii}^{-(k-1)}A_{i_2 \cdots i_k}P_{i_2} \cdots P_{i_k} \]
\[ = \sum_{i_2, \ldots, i_k \in [n]} P_{ii}^{-(k-1)}A_{i_2 \cdots i_k}P_{i_2} \cdots P_{i_k} + \sum_{i_2, \ldots, i_k \in [n]} P_{ii}^{-(k-1)}A_{i_2 \cdots i_k}P_{i_2} \cdots P_{i_k} \]
\[ = xd_{\{1,i\}} + d_i - d_{\{1,i\}} \]
\[ \leq xd_i \]
\[ \leq xd_2. \]

Noting that \( d_1 > d_2 \), if we take
\[ x = \left( \frac{d_1}{d_2} \right)^{\frac{1}{k}}, \]
then \( x > 1 \), and
\[ r_i(P^{-(k-1)}AP) = d_i^{\frac{1}{k}}d_2^{1 - \frac{1}{k}}, \]
for \( 2 \leq i \leq n \),
\[ r_i(P^{-(k-1)}AP) \leq xd_2 = d_i^{\frac{1}{k}}d_2^{1 - \frac{1}{k}}. \]
Thus for each \( 1 \leq i \leq n \), we have
\[ r_i(P^{-(k-1)}AP) \leq d_i^{\frac{1}{k}}d_2^{1 - \frac{1}{k}}. \]

Then by Lemma 4,
\[ \rho(P^{-(k-1)}AP) \leq \max_{1 \leq i \leq n} r_i(P^{-(k-1)}AP) = d_i^{\frac{1}{k}}d_2^{1 - \frac{1}{k}}. \]

Furthermore, by Lemma 7 we have
\[ \rho(A) = \rho(P^{-(k-1)}AP) \leq d_1^{\frac{1}{k}}d_2^{1 - \frac{1}{k}}. \]
(6)

If the equality in (6) holds we have \( d_{\{1,i\}} = d_i \), and \( d_2 = d_3 = \cdots = d_n \). The condition \( d_{\{1,i\}} = d_i \) implies that any edge containing vertex \( i \) contains vertex 1, so \( d_1 \) equals to the number of edges of \( \mathcal{H} \).
Concerning that \( d_2 = d_3 = \cdots = d_n \), then \( \mathcal{H} \) is a blow-up of a \( d_2 \)-regular and \((k-1)\)-uniform hypergraph.

On the other hand, if \( \mathcal{H} = \mathcal{H}_0(v) \), where \( \mathcal{H}_0 \) is a \( d_2 \)-regular and \((k-1)\)-uniform hypergraph, we take
\[ P = \text{diag}(\left( \frac{d_1}{d_2} \right)^{\frac{1}{k}}, 1, \cdots, 1). \]

Then for each \( 1 \leq i \leq n \), we have
\[ r_i(P^{-(k-1)}AP) = d_i^{\frac{1}{k}}d_2^{1 - \frac{1}{k}}, \]
and Lemma 4 and Lemma 7 implies that
\[ \rho(A) = \rho(P^{-(k-1)}AP) = d_1^{\frac{1}{k}}d_2^{1 - \frac{1}{k}}. \]

\[ \square \]

**Lemma 9** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two order \( k \) dimension \( n \) tensors satisfying \( |\mathcal{A}| \leq \mathcal{B} \), where \( \mathcal{B} \) is weakly irreducible. Let \( \lambda \) be an eigenvalue of \( \mathcal{A} \). Then \( |\lambda| \leq \rho(\mathcal{B}) \).

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Corollary 10 Let $H$ be a connected $k$-uniform hypergraph on $n$ vertices with degree sequence $d_1 \geq \cdots \geq d_n$. Let $Q(H)$ be the signless Laplacian tensor of $H$. Then

(1) $\rho(Q(H)) \geq d_1$;

(2) $\rho(Q(H)) \leq d_1 + d_1^{1/2} d_2^{1/2}$, equality holds if and only if $H$ is a regular hypergraph.

Proof (1) Noting that $\mathcal{D}(H) \leq Q(H)$, Lemma 9 implies that $d_1 = \rho(\mathcal{D}(H)) \leq \rho(Q(H))$.

(2) Let $I$ be the unit tensor and $D = d_1 I$. Then $|Q(H)| \leq D' + A(H)$, and so by Lemma 9 we have $\rho(Q(H)) \leq \rho(D' + A(H))$. It is not difficult to see that $\rho(Q(H)) = \rho(D' + A(H))$ if and only if $d_1 = d_n$. Thus

$$\rho(Q(H)) \leq \rho(D' + A(H)) = \rho(D') + \rho(A(H)) = d_1 + \rho(A(H)) \leq d_1 + d_1^{1/2} d_2^{1/2}.$$ 

Equality holds if and only if $d_1 = d_n$, namely, $H$ is a regular hypergraph.

Theorem 11 Let $H$ be a connected $k$-uniform hypergraph on $n$ vertices, and $b_i > 0$ for each $1 \leq i \leq n$. Then,

$$\rho(Q(H)) \leq \max_{e \in E(H)} \max_{\{i, j\} \subseteq e} \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4b_i b_j}}{2},$$

where $b_p' = b_p^{(k-1)} \sum_{\{p, p_2, \cdots, p_k\} \in E(H)} b_{p_2} \cdots b_{p_k}$ for any $1 \leq p \leq n$.

Proof Write $Q(H) = Q$ and $\rho(Q(H)) = \rho$ for short. Let $B = \text{diag}(b_1, b_2, \cdots, b_n)$ and $b_i > 0$ for any $1 \leq i \leq n$. By Lemma 7 we know that $\rho(B^{-(k-1)}QB) = \rho$. By (1) we have have

$$B^{-(k-1)}QB_{i_1i_2\cdots i_k} = B^{-(k-1)}_{i_1i_2\cdots i_k} B_{i_2i_3} \cdots B_{i_ki_k}.$$ 

Since $H$ is connected, the tensor $Q$ and so the tensor $B^{-(k-1)}QB$ is weakly irreducible. By Lemma 3 we know that $\rho$ is an H-eigenvalue of $B^{-(k-1)}QB$ and there exists a positive eigenvector corresponding to $\rho$, denoted by $x$. We may suppose that $x_i = 1, x_p = 1$ for any vertex $p$ different from $i$. Let

$$x_j = \max\{x_p : \{i, p\} \subseteq e \in E(H)\}.$$ 

From the definitions of eigenvalue and eigenvector (see (3)), we have

$$(B^{-(k-1)}QB)x = \rho x^{[k-1]}.$$ 

For any vertex $p$ we have

$$(B^{-(k-1)}QB)_x = \rho x^{k-1}.$$ 

By (1) we have

$$B^{-(k-1)}QB_{pp_2\cdots p_k} x_p x_{p_2} \cdots x_{p_k} = \rho x^{k-1}_p,$$

then,

$$d_p x^{k-1}_p + \sum_{\{p, p_2, \cdots, p_k\} \in E(H)} b_p^{-(k-1)} b_{p_2} \cdots b_{p_k} x_{p_2} \cdots x_{p_k} = \rho x^{k-1}_p.$$ 

Hence we have

$$(\rho - d_p)x^{k-1}_p = b_p^{-(k-1)} \sum_{\{p, p_2, \cdots, p_k\} \in E(H)} b_{p_2} \cdots b_{p_k} x_{p_2} \cdots x_{p_k}. (7)$$

Recall that for any vertex $p$

$$b_p' = b_p^{-(k-1)} \sum_{\{p, p_2, \cdots, p_k\} \in E(H)} b_{p_2} \cdots b_{p_k}.$$
Now take $p = i$ in (7), then we obtain
\[
\rho - d_i = b_i^{-1} \sum_{\{i, i_2, \ldots, i_k\} \in E(H)} b_{i_2} \cdots b_{i_k} x_{i_2} \cdots x_{i_k} \\
\leq b_i^{-1} \sum_{\{i, i_2, \ldots, i_k\} \in E(H)} b_{i_2} \cdots b_{i_k} x_{i_2}^{k-1},
\]
\[
= b'_ix_{i}^{k-1}.
\]
And take $p = j$ in (7), then we have
\[
(\rho - d_j)x_j^{k-1} = b_j^{-1} \sum_{\{j, j_2, \ldots, j_k\} \in E(H)} b_{j_2} \cdots b_{j_k} x_{j_2} \cdots x_{j_k} \\
\leq b_j^{-1} \sum_{\{j, j_2, \ldots, j_k\} \in E(H)} b_{j_2} \cdots b_{j_k},
\]
\[
= b'_jx_{j}^{k-1}.
\]
Now we obtain
\[
\rho - d_i \leq b'_ix_{i}^{k-1} \text{ and } (\rho - d_j)x_j^{k-1} \leq b'_j.
\]
Noting that $\rho \geq d_p$, (see Corollary 13), multiplying the left and right sides of the two inequalities, respectively, we have,
\[
(\rho - d_i)(\rho - d_j)x_j^{k-1} \leq b'_ib'_jx_j^{k-1}.
\]
Thus we have,
\[
\rho^2 - (d_i + d_j)\rho + d_id_j - b'_ib'_j \leq 0,
\]
and then
\[
\rho \leq \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4b'_ib'_j}}{2}.
\]
So we have proved that
\[
\rho(Q(H)) \leq \max_{e \in E(H)} \max_{\{i, j\} \subseteq e} \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4b'_ib'_j}}{2}.
\]
The proof is completed. \qed

For each $i$, if we take $b_i = 1$, then $b'_i = d_i$ in Theorem 11 we may obtain the following result.

**Corollary 12** Let $H$ be a $k$-uniform hypergraph. Then we have
\[
\rho(Q(H)) \leq \max_{e \in E(H)} \max_{\{i, j\} \subseteq e} (d_i + d_j).
\]
For a vertex $i$ of the $k$-uniform hypergraph $H$, denote by
\[
m_i = \frac{\sum_{\{i, i_2, \ldots, i_k\} \in E(H)} d_{i_2} \cdots d_{i_k}}{d_i^{k-1}},
\]
which is a generalization of the average of degrees of vertices adjacent to $i$ of the simple graph. For each $i$, if we take $b_i = d_i$, then $b'_i = m_i$ in Theorem 11 we may obtain the following result.

**Corollary 13** Let $H$ be a $k$-uniform hypergraph. Then we have
\[
\rho(Q(H)) \leq \max_{e \in E(H)} \max_{\{i, j\} \subseteq e} \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_im_j}}{2}.
\]
Remark 14 Take \( B = \text{diag}(1, 1, \cdots, 1) \) and use the similar arguments as that in Theorem \([11]\) for the tensor \( B^{-(k-1)} A(H) B \), then we may obtain that

\[
\rho(A(H)) \leq \max_{e \in E(H)} \max_{\{i, j\} \subseteq e} \sqrt{d_id_j}.
\]

If take \( B = \text{diag}(d_1, d_2, \cdots, d_n) \), then we may prove that

\[
\rho(A(H)) \leq \max_{e \in E(H)} \max_{\{i, j\} \subseteq e} \sqrt{m_im_j}.
\]

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