Fermi Statistics of Weakly Excited Granular Materials in a Vibrating Bed II: One Dimensional Experiment

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(March 24, 2022)

Abstract

A one dimensional experiment in granular dynamics is carried out to test the thermodynamic theory of weakly excited granular systems [Hayakawa and Hong, Phys. Rev. Lett. 78, 2764(1997)] where granular particles are treated as spinless Fermions. The density profile is measured and then fit to the Fermi distribution function, from which the global temperature of the system, $T$, is determined. Then the center of mass, $< z(T) >$, and its fluctuations, $< (\Delta z(T))^2 >$, are measured and plotted as functions of $T$. The Fermi function fits the density profile fairly well, with the value of $T$ being fairly close to the predicted value. The scaling behavior of $< z(T) >$ and $< (\Delta z(T))^2 >$ is in excellent agreement with the theory.

PACS numbers: 81.05Rm, 05

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I. The Experimental Setup

This paper is a sequel to the preceding paper [1], where the thermodynamic theory of Hayakawa and Hong [HH] [2] was tested with extensive Molecular Dynamics simulations. The purpose of this paper is to test the theory of HH experimentally for a one dimensional vibrating granular system. A one dimensional system is peculiar in the sense that randomness associated with collisions is suppressed in contrast with what happens in higher dimensions. Nevertheless, the Fermi statistics arising from hard core repulsion still apply to this simple one dimensional system. As in the preceding paper, we determine the configurational statistics of a one dimensional vibrating granular system by properly taking the ensemble average of the steady state. Then we measure the following quantities and compare them to those predicted by HH. First, we measure the density profile and determine the dimensionless Fermi temperature $T = T_f/mgD$ ($T_f$ is the Fermi temperature) by fitting the profile to the Fermi function. Second, we compare the measured Fermi temperature $T$ to those predicted by the theory (Eq.(6a) of the preceding paper.) Third, we measure the center of mass and the fluctuations of the vibrating bed and test the scaling predictions (Eqs. (5c) and (7) of the preceding paper.) Since the detailed theoretical aspect can be found in the preceding paper, we only discuss here the experimental aspect and the analysis of the experimental data as well as its comparison with the theoretical predictions.

We now explain the experimental setup in some detail. The one dimensional column of particles used in this experiment was 30 plastic beads. The beads, having holes through their centers, were strung through a thin piece of copper wire. The wire was stretched extremely tight between two horizontal rods clamped to a ring stand. Each of the beads could freely move up and down the copper wire. The beads were all close to the same size, with an average diameter of $D=5.74$ mm. Differences in the diameter of the beads were not noticeable to the naked eye. The bottom of the copper wire ran through a small metal bar connected horizontally to a mechanical vibrator, very similar to an audio speaker. One of the plastic beads was glued to the metal bar, acting as the bottom wall when the vibration was turned on. A schematic diagram of the set up is shown in Fig.1.
The mechanical vibrator was then connected to a function generator, which allowed one to control the type, amplitude, and frequency of the mechanical vibration. For this experiment, the only type of vibration used was a cosine wave, $A \cos(\omega t)$, where $A$ is the amplitude and $\omega = 2\pi f$ is the angular frequency. The frequency of the vibration and the voltage which set the amplitude were both controlled with the function generator. The relationship between the voltage and the vibration amplitude may be nonlinear and complex. Hence, instead of deriving the mathematical formula, for a given applied voltage, we measured directly the corresponding amplitude of the vibration by putting one bead on the vibrating plate and measured its maximum height. We find that the voltage with a range of 0.00 V to 9.50 V corresponds to an amplitude of 0.000 cm to 0.584 cm. In this experiment, the frequency $f$ was kept at $f=40\text{Hz}$ while the voltage was varied by 2 volts from 1.00 V to 9.00 V. The corresponding amplitudes, $A$, and the vibration strength ($\Gamma = A\omega^2/g$) with $g$ the gravitational acceleration are listed in Table I. This set of control parameters satisfies the criterion of the weakly excited granular system used to test the theory of HH (Eq.(3) of the preceding paper), namely:

$$R = \mu_0 D / \Gamma A >> 1$$

with $\mu_0$ being the initial number of layers. To analyze the system, pictures were taken with a digital camera while the beads were subjected to vibration. The camera was placed on a stand in front of the beads, and a large convex lens was placed in between to focus the image properly. This is schematically illustrated in Fig.2. A set of 32 pictures was taken for each value of $\Gamma$ used in this experiment, including a picture of the beads at rest in the beginning and the end of each run. Then, the images were scanned into the computer so that the center of each bead in every picture could be obtained. Once the individual bead coordinates were known, they were fed into various FORTRAN programs and used to calculate the density profile, the center of mass, and the fluctuations of the center of mass for each voltage setting.

To find the density profile, equally sized boxes with a height equal to the diameter of a bead were constructed from the grain at the bottom, so that each box contained, one bead
in the ground state. Note that the reference point in the density measurement is not the vibrating plate but the particle at the bottom, which is fluidized. Before proceeding, we want to clarify two points regarding the density measurement: First, on the density plot presented in this paper, the density could exceed one for certain data points. This is because box sizes were assumed to all have a length and width equal to the average bead radius. In reality, however, this was not true, leading to a density value that can be greater than 1. Second, since the bottom layer is the reference point, one may expect that the density of the first data point should remain the same, but one may notice that the values of the density of the bottom data changes slightly. This is due to the erosion of the balls that occurred due to the repeated motion on the copper wire. This erosion of the center hole caused the configuration of the bottom few balls to be more dense as more trials were completed.

For this experiment, 50 boxes were chosen because the motion of the column of beads never reached that height, even for the largest voltage. Therefore, every bead was included in the calculations of all the density profiles. Since this is a one dimensional experiment, each bead was considered to be a circle instead of a sphere. Hence, the maximum possible area of a bead in a box, $a_o$, is $\pi r^2$ where $r = \langle D \rangle / 2 = \sum_i D_i / 2N$ is the average radius of a bead with $D_i$ the diameter of each bead and $N$ the total number of beads, i.e, $N=30$. In the ground state, when the system is at rest, each box up to the Fermi surface contains one bead with an area of $a_o$. However, when the beads are subjected to vibration, the system expands, and the time averaged position of each grain rises. The area of the grains in each box is computed from the pictures, which is denoted as $a$. Then, the density of each box, having a value between zero and one, was computed using

$$\rho = \frac{a}{a_o}.$$  

This process was done for all 50 boxes and in all 32 pictures for each voltage used. An average density for each box was then found by summing up the 32 densities per box and dividing by 32. The density profile for a particular voltage was obtained by plotting the average density of each box against the box number. The center of the first box is chosen
as the origin. The profile is then fit with the Fermi function as discussed in the previous section and a global temperature, $T$, is obtained. The fitting of the density profile was done with a non-linear least squares program as well as the eye.

FORTRAN programs were used to obtain the center of mass and its fluctuations. The center of mass is computed with the following formula:

$$z = \frac{\sum z_i m_i}{\sum m_i},$$

where $z_i$ is the vertical position in centimeters of the $i$th bead and $m_i$ is the mass in grams of the $i$th bead. These summations were carried out over all 30 beads. This formula was used to get the center of mass for each picture, and then the average was taken over all 32 pictures to get the average center of mass, $<z>$. The average fluctuations of the center of mass are computed using the following standard formula for the deviations:

$$< (\Delta z)^2 > = \frac{\sum_{j}(z(j) - <z>)^2}{32},$$

where $z(j)$ is the center of mass of an individual picture, and $<z>$ is the average center of mass found previously for a particular setting of the voltage. In this case, the summation is carried out over 32, the number of pictures taken per run. Both the center of mass and the fluctuations were computed for each voltage setting.

II. Data and Results

**Density Profile**: This experiment was carried out for 30 particles using a cosine wave vibration for voltages of 1.00 V, 3.00 V, 5.00 V, 7.00 V, and 9.00 V at a constant frequency of $f = 40$ Hz. The change in $V$ is synonymous with a change in the vibration strength, $\Gamma$. Density profiles, $\rho$, as a function of vertical position $z$, for the different voltages were fit to the Fermi profile and can be seen in Fig.3a-e. Note that for an electron gas, the Fermi profile is given by:

$$\rho(\epsilon_i) = 1/(1 + exp(\epsilon_i - \bar{\mu})/T_f)$$

(2a)
where $\epsilon_i$ is the i-th energy level of the electrons, and $\bar{\mu} \equiv mg\mu(T)D$ is the Fermi energy and $T_f$ is the Fermi temperature. As discussed in [1], for granular systems, $\mu(T)$ is independent of the temperature because the density of the state is constant. We denote this constant as $\mu_0 \equiv \mu(T = 0)$, which is the initial number of layers. Note that the dimensionless Fermi energy at T=0 is the initial number of layers, i.e., $\mu(T = 0) \equiv \mu_0$. For the configurational statistics of granular materials in a vibrating bed, the energy level is given by the average position of the particle. Thus, the energy level is given by the gravitational energy, $\epsilon_i = mgDz_i \equiv mgD\bar{z}$ where $z_i$ is the dimensionless vertical position of the i-th grain measured in units of D. Note that $\bar{z}$ runs from 0 to 50 because the position of the first grain is taken as the origin. The relationship between the dimensionless fitting Fermi temperature $T$ in this paper and the Fermi temperature $T_f$ in (2a) is given by $T = T_f/mgD$:

$$\rho(\bar{z}) = 1/(1 + \exp([\epsilon_i(\bar{z}) - \bar{\mu})/T_f]) \equiv 1/(1 + \exp((\bar{z} - \mu)/T))$$  \hspace{1cm} (2b)$$

All the density profiles plotted together can be seen in Fig.4. Note that the Fermi statistics are valid when the parameter $R$ defined by (Eq.(3)) in the preceding paper [1] is greater than 1, namely: $R = \mu D/\Gamma A$. In our experiment, $R$ changes from 131.07 (voltage = 1.00 V) to .9327 (voltage = 9.00 V) (See Table II).

**Fermi temperature fitting:** When fitting the experimental data with the Fermi function, the following two adjustable parameters are used: the Fermi energy, $\mu$, and the temperature, $T$. The parameter $\mu$ shifts the location of the Fermi energy horizontally, and the temperature, $T$, controls the curvature around the Fermi energy. As discussed above, the Fermi energy is expected to remain constant regardless of the temperature $T$ (If the density of states is constant, i.e, $\mu(T) = \mu_o$). However, the experimental fitting shows that there is some slight variation in $\mu$. In this experiment, $\mu$ varies from 29.6 to 32.0, a relative increase of about $\Delta \mu/\mu = 0.08$. The scaling behavior of the center of mass and its fluctuations don’t seem to be affected by the change in $\mu$. However, there is a large affect in the amplitude as we have experienced in the Molecular Dynamics simulations [1]. The fitting values for $\mu$ and $T$ are listed in Table II along with the ratio $R$. When the temperature, $T$, obtained by the
Fermi fitting is compared to the to the theoretical prediction, $T_o$, (Eq.6a) of the preceeding paper, they seem to match fairly well. For the sake of completeness, we present this formula again:

$$T_o = \frac{1}{D mg} \frac{T_f}{T_f} = \frac{1}{D} \sqrt{\frac{6D(gH_o(\Gamma)/\omega^2)}{\pi^2\alpha}}$$

(3)

A best fit value was $\alpha = 1.5$. Translating this into a physical picture, we notice that the relationship between the actual amount of expansion in the center of mass in the one dimensional vibrating bed, $\Delta h$, and the jump height, $\bar{h} = gH_o/\omega^2$, of a single ball fired gently (meaning the relative velocity and relative acceleration between the ball and the plate are both zero) to the free surface by the vibrating plate is given by the relation: $\Delta h = \bar{h}/\alpha$.

Note that the last point in Table II is off, for which $R \approx 0.93 < 1$. This is expected since the Fermi analysis is no longer valid at such a high voltage. Table III shows the values of both the predicted temperature, $T_o$, and the fitting temperature $T$. They agree fairly well except for the last one.

**The center of mass:** The relative increase in the center of mass of the one dimensional column is denoted as $< \Delta z(T) >$, which is the difference in the actual position of the center of mass, $z(T)$, and that of the ground state, i.e., $\Delta z(T) = z(T) - z(T = 0)$ is given by (Eq. (5) of ref.[1]):

$$< \Delta z(T) > = \frac{\pi^2 D T^2}{6\mu}$$

(4)

where $\mu$ is the dimensionless Fermi energy (the initial number of layers.) Note the appearance of the factor D in (4). It is because, in ref. [1], the temperature $T$ has the dimension of length, while here it is dimensionless. The center of mass is plotted in Fig.5 as a function of $T^2$ in a graph of five different values of $\Gamma$. By using the solid line as a guide for the eye, one can see that the graph seems to confirm the scaling predictions of the Fermi statistics, $\Delta z \propto T^2$. Notice that the last point deviates drastically from the best fit line. Once again, this is because the conditions of the system violate the inequality (1) and make the Fermi analysis invalid. Hence, this point was not included in obtaining the best fit line. There is a discrepancy in the proportionality constant C. Theory predicts that one should get
The experimental results yield, \( C \approx .45 \), a difference of about a factor 14 or more when compared to the theory. It has been shown in a preceding paper [1] that this discrepancy is due to the extreme sensitivity of the center of mass, to the Fermi energy, \( \mu \). When the density of states is independent of the energy, the Fermi energy must remain constant. However, in this one dimensional experiment, it changes for each value of \( \Gamma \). This small change does not seem to change the Fermi fitting in Fig.3a-e, but it does effect the amplitude of the scaling relationships.

**Fluctuations of the center of mass:** The fluctuation of the center of mass, \( <(\Delta z)^2> \), is given by (Eq.(7) of ref.[1]):

\[
< (\Delta z)^2 > = \frac{\pi^2 D^2}{3 \mu^2} T^3
\]

(5)

The fluctuations of the center of mass were also measured and plotted as a function of \( T^3 \) in Fig.4. Using the guide line, the graph once again seems to confirm the validity of the Fermi statistics which imply that \( <(\Delta z)^2> \propto T^3 \). The proportionality constant is way off from the theoretical value, \( D^2 \pi^2 / 3 \mu^2 \approx 10^{-3} \). From the graph, the slope is approximately 0.15. The last point was again eliminated when obtaining the best fit line, just as it was for the center of mass for the reason explained above. Considering that the amplitude of the center of mass is off by a factor of 14, one should expect that the fluctuations would be off by a factor of at least \( 14^2 \). This is, again, due to the sensitivity of the Fermi integral to \( \mu \), where small changes in \( \mu \) are greatly magnified in the fluctuations, resulting in the factor of order \( 10^2 \) difference. We have found a similar trend in the MD simulations [1]. Another source of error is due to the fact that the fluctuations in the center of mass are quite large in the vibrating column, and all the particles, not just those near the surface, fluctuate in the continuum space of the experiment. This is different from the Fermi model which makes all particles below the Fermi surface, inactive. So the positions of the grains, on average, may obey the Fermi distribution function well, but the magnitude of the fluctuations may not. It is very surprising, though, that the \( T^3 \) prediction of the Fermi statistics still seems to hold.
III. Conclusion

We now summarize the main results of this experiment as follows. First, the configurational statistics of grains in a one dimensional system subject to vibration seem to obey the Fermi statistics of spinless particles for weakly excited systems as was predicted in the paper by HH[2]. Second, the temperature determined by fitting the Fermi profile is fairly close to the theoretical prediction. Third, the scaling relations of the center of mass and its fluctuations obey the Fermi statistics, but there are discrepancies in the amplitudes or proportionality constants. Note that another source of error in this experiment results from friction. The beads were confined to vibration in a single column by the copper that ran through their centers. After many trials, filings from the inside of the plastic beads were found at the base of the setup. The motion of the beads on the wire eroded away some of the hole in the center of each bead. This friction could be responsible for some of the error in the results. Nevertheless, we find that the essence of the Fermi statistics survives in this one dimensional experiment.

Acknowledgement

This research was supported by NSF as a part of Research Experiences for Undergraduate Students program at Lehigh University.
References

[1] P. Quinn and D. C. Hong, preceding paper and references therein.

[2] H. Hayakawa and D. C. Hong, Phys. Rev. Lett. 78, 2764 (1997).
**Figure Captions**

Fig.1. The schematic picture of the experimental setup used to vibrate a column of beads at a controlled amplitude and frequency.

Fig.2. The schematic picture of the experimental setup used to take pictures of the vibrating column of beads with the digital camera.

Fig.3. Density profiles of the one dimensional vibrating beads and the fitting to the Fermi function. The voltages used in each run are: (a)1.00 V, (b)3.00 V, (c)5.00 V, (d)7.00 V, and (e)9.00 V. A plot of all the profiles at once is given in (f).

Fig.4 The center of mass, \( \langle z(T) \rangle \), is plotted as a function of \( T^2 \). The straight line is a guide to the eye.

Fig.5. Fluctuations in the center of mass, \( \langle \Delta z(T)^2 \rangle \), are plotted as a function of \( T^3 \). The straight line is a guide to the eye. In both Figs. 4 and 5, the last point was not included in the estimation of the best fit line, because it violates the criterion given by Eq.(2).
Table Captions

Table I. The values of the vibrational amplitude $A$, and the vibration strength $\Gamma$ for each corresponding voltage $V$.

Table II. A set of parameters used in the experiment for various voltages. Here, $\mu$ is the dimensionless Fermi energy, $T$ is the dimensionless Fermi temperature, and $R = \mu D/\Gamma A$ is the parameter that determines the validity of the Fermi statistics. (Eq.1)

Table III. The comparison between the theoretical Fermi temperature, $T_o$ and the experimentally measured Fermi temperature, $T$, for different vibrations at voltage $V$. 

Mechanical Vibrator
(Acos(wt))

30 Beads On A Copper Wire

Ring Stand

Function Generator
Digital Camera

Convex Lens

Beads
Density Profile for 1.00 Volt

$\rho(z)$ vs. $z$

- Experimental Density Profile
- Theoretical Density Profile
Density Profile for 3.00 Volts

$\rho(z) \text{ vs. } z$

- **Experimental Density Profile**
- **Theoretical Density Profile**
Density Profile for 5.00 Volts

$\rho(z)$ vs. $z$

- **Experimental Density Profile**
- **Theoretical Density Profile**
Density Profile for 7.00 Volts

ρ(z) vs. z

- Experimental Density Profile
- Theoretical Density Profile
Density Profile for 9.00 Volts

ρ(z) vs. z

- Experimental Density Profile
- Theoretical Density Profile

Z

ρ(z)
Density Profiles for 1.00, 3.00, 5.00, 7.00, and 9.00 Volts

\[ \rho(z) \text{ vs. } z \]

- 1.00 Volt
- 3.00 Volts
- 5.00 Volts
- 7.00 Volts
- 9.00 Volts
One Column of Thirty Particles

$< z >$ vs. $T^2$

Experimental Data Points

Best Fit Line: $y = 0.45x + 8.46$
One Column of Thirty Particles

$\langle (\Delta z)^2 \rangle$ vs. $T^3$

Experimental Data Points

Best Fit Line: $y = 0.15x$
| $V$  | $A$    | $\Gamma$ |
|------|--------|----------|
| 1.00 V | 0.045 cm | 2.90     |
| 3.00 V | 0.174 cm | 11.27    |
| 5.00 V | 0.300 cm | 19.32    |
| 7.00 V | 0.410 cm | 26.40    |
| 9.00 V | 0.553 cm | 35.61    |
| $V$ | Parameters | $R = \frac{\mu D}{\Gamma A}$ |
|-----|------------|-----------------------------|
|     | $\mu$    | $T$                         |
| 1.00 V | 29.800 | 0.26800 | 131.07 |
| 3.00 V | 29.612 | 0.77000 | 8.62  |
| 5.00 V | 31.182 | 1.07299 | 3.09  |
| 7.00 V | 31.800 | 1.27999 | 1.69  |
| 9.00 V | 32.000 | 2.03750 | 0.93  |
| $V$  | $T$    | $T_o$  |
|------|--------|--------|
| 1.00 V | 0.26800 | 0.26899 |
| 3.00 V | 0.6759  | 0.77865 |
| 5.00 V | 1.0821  | 1.07246 |
| 7.00 V | 1.4859  | 1.27794 |
| 9.00 V | 2.0375  | 1.50445 |