Parallel Algorithms for Small Subgraph Counting

Amartya Shankha Biswas∗† Talya Eden∗‡
Quanquan C. Liu∗ Slobodan Mitrović∗§ Ronitt Rubinfeld¶
February 25, 2020

Abstract

With the prevalence of large graphs, it is becoming increasingly important to design scalable algorithms. Over the last two decades, frameworks for parallel computation, such as MapReduce, Hadoop, Spark and Dryad, gained significant popularity. The Massively Parallel Computation (MPC) model is a de-facto standard for studying these frameworks theoretically. Subgraph counting is a fundamental problem in analyzing massive graphs, often studied in the context of social and complex networks. There is a rich literature on designing scalable algorithms for this problem, with the main challenge to design methods which are both efficient and accurate. In this work, we tackle this challenge and design several new algorithms for subgraph counting in MPC.

Given a graph $G$ over $n$ vertices, $m$ edges and $T$ triangles, our first main result is an algorithm that, with high probability, outputs a $(1 + \varepsilon)$-approximation to $T$, with asymptotically optimal round and total space complexity provided any $S \geq \max(\sqrt{m}, \frac{n^2}{m})$ space per machine and assuming $T = \Omega(\sqrt{m/n})$.

Our second main result is an $O(\delta(\log \log n))$-rounds algorithm for exactly counting the number of triangles, parametrized by the arboricity $\alpha$ of the input graph. The space per machine is $O(n^\delta)$ for any constant $\delta$, and the total space is $O(n \alpha)$, which matches the time complexity of (combinatorial) triangle counting in the sequential model. We also prove that this result can be extended to exactly counting $k$-cliques for any constant $k$, with the same round complexity and total space $O(n \alpha^{k-2})$. Alternatively, allowing $O(\alpha^2)$ space per machine, the total space requirement reduces to $O(n \alpha^2)$.

Finally, we prove that a recent result of Bera, Pashanasangi and Seshadhri (ITCS 2020) for exactly counting all subgraphs of size at most 5, can be implemented in the MPC model in $\tilde{O}(\sqrt{\log n})$ rounds, $O(n^3)$ space per machine and $O(n \alpha^3)$ total space. Therefore, this result also exhibits the phenomenon that a time bound in the sequential model translates to a space bound in the MPC model.

∗CSAIL, MIT, Cambridge, MA, USA, \{asbiswas,teden,quanquan,slobo\}@mit.edu, ronitt@csail.mit.edu
†Amartya Shankha Biswas was supported by NSF Award Numbers: IIS-1741137 and CCF-1733808 and MIT-IBM Watson AI Lab and Research Collaboration Agreement No. W1771646.
‡Talya Eden was supported by NSF TRIPODS grant No. 1740751, and by the generosity of Eric and Wendy Schmidt by recommendation of the Schmidt Futures program.
§Slobodan Mitrović was supported by the Swiss NSF grant P2ELP2_181772, MIT-IBM Watson AI Lab and Research Collaboration Agreement No. W1771646, and FinTech@CSAIL.
¶Ronitt Rubinfeld was supported by NSF Award Numbers: IIS-1741137 and CCF-1733808 and MIT-IBM Watson AI Lab and Research Collaboration Agreement No. W1771646.
1 Introduction

Estimating the number of small subgraphs, cliques in particular, is a fundamental problem in computer science, and has been extensively studied both theoretically and from an applied perspective. Given its importance, the task of counting subgraphs has been explored in various computational settings, e.g., sequential [AYZ97, Vas09, BHKK09], distributed and parallel [SV11, PT12, KPP+14, PSKP14, LQLC15], streaming [BYKS02, KMSS12, BC17, MVV16], and sublinear-time [ELRS17, ABG+18, AKK19, ERS20]. There are usually two perspectives from which subgraph counting is studied: first, optimizing the running time (especially relevant in the sequential and sublinear-time settings) and, second, optimizing the space or query requirement (relevant in the streaming, parallel, and distributed settings). In each of these perspectives, there are two, somewhat orthogonal, directions that one can take. The first is exact counting. However, in most scenarios, algorithms that perform exact counting are prohibitive, e.g., they require too much space or parallel rounds to be implementable in practice.

Hence, the second direction of obtaining an estimate/approximation on the number of small subgraphs is both an interesting theoretical problem and of practical importance. If $H_\#$ is the number of subgraphs isomorphic to $H$, the main question in approximate counting is whether we can design algorithms that, under given resource constraints, provide approximations that concentrate well. This concentration is usually parametrized by $H_\#$ (and potentially some other parameters). In particular, most known results do not provide a strong approximation guarantee when $H_\#$ is very small, e.g., $H_\# \in O(1)$. So, the main attempts in this line of work is to provide an estimation that concentrates well while imposing as small a lower bound on $H_\#$ as possible.

On the applied side, counting small subgraphs has a wide range of applications. It has been used in fraud detection [BRDR05], link recommendation [TDM+11], and bioinformatics [MSOI+02]. This task is also an important metric in analyzing structures of complex networks [MSOI+02, UBK13].

Due to ever increasing sizes of data stores, there has been an increasing interest in designing scalable algorithms. The Massively Parallel Computation (MPC) model is a theoretical abstraction of popular frameworks for large-scale computation, such as MapReduce [DG08], Hadoop [Whi12], Spark [ZCF+10] and Dryad [IBY+07]. MPC gained a significant interest recently, most prominently in building algorithmic toolkit for graph processing [GSZ11, LMSV11, BKS13, ANOY14, BKS14, HP15, AG15, RVW16, IMS17, CLM+18, Ass17, Abb+19a, GGK+18, HLL18, BU18, ASW18, BEG+18, ASS+18c, BDH+19, BBD+19, BHH19, BDE+19a, ASS+18b, ASZ19, Abb+19b, ASW19, GLM19, GKMS19, GU19, LMOS19, ILMP19, CFG+19, GUK19, GNT20]. Efficiency of an algorithm in MPC is characterized by three parameters: round complexity, the space per machine in the system, and the number of machines/total memory used. Our work aims to design efficient algorithms with respect to all three parameters and is guided by the following question:

*How does one design efficient, massively parallel algorithms for small subgraph counting?*

1.1 The MPC Model

In this paper, we are working in the Massively Parallel Computation (MPC) model introduced by [KSV10, GSZ11, BKS13]. The model operates as follows. There exist $\mathcal{M}$ machines that communicate with each other in synchronous rounds. The graph input is initially distributed across the machines in some organized way such that machines know how to access the relevant information via communication with other machines. During each round, the machines first perform computation locally without communicating with other machines. The computation done locally can be unbounded (although the machines have limited space so any reasonable program will not
do an absurdly large amount of computation). At the end of the round, the machines exchange
texts to inform the computation for the next round. The total size of all messages that can be
received by a machine is upper bounded by the size of its local memory, and each machine outputs
messages of sufficiently small size that can fit into its memory.

If \( N \) is the total size of the data and each machine has \( S \) words of space, we are interested in
the settings when \( S \) is sublinear in \( N \). We use total space to refer to \( M \cdot S \), which represent the
joined available space across all the machines.

### 1.2 Our Contribution

We start by studying the question of triangle counting, both approximately, and exactly. In all
that follows let \( G \) be a graph over \( n \) vertices, \( m \) edges and \( T \) triangles.

First we study the question of approximately counting the number of triangles under the re-
striction that the round and total space complexities are essentially optimal, i.e., \( O(1) \) and \( \tilde{O}(m) \),
respectively.

**Theorem 1.1.** Let \( G = (V,E) \) be a graph over \( n \) vertices, \( m \) edges, and let \( T \) be the number of
triangles in \( G \). Assuming

\[
(i) \quad T = \tilde{\Omega} \left( \sqrt{\frac{m}{S}} \right), \quad \quad (ii) \quad S = \tilde{\Omega} \left( \max \left\{ \frac{\sqrt{m}}{\varepsilon}, \frac{n^2}{m} \right\} \right),
\]

there exists an MPC algorithm, using \( M \) machines, each with local space \( S \), and total space
\( MS = \tilde{O}_\varepsilon(m) \), that outputs a \((1 \pm \varepsilon)\)-approximation of \( T \), with high probability, in \( O(1) \) rounds.

For \( S = \Theta(n \log n) \) (specifically, \( S > 100n \log n \)) in Theorem 1.1, we derive the following
corollary.

**Corollary 1.2.** Let \( G \) be a graph and \( T \) be the number of triangles it contains. If \( T \geq \sqrt{d_{avg}} \),
then there exists an MPC algorithm that in \( O(1) \) rounds with high probability outputs a \((1 + \varepsilon)\)-
approximation of \( T \). This algorithm uses a total space of \( \tilde{O}(m) \) and space \( \tilde{\Theta}(n) \) per machine.

There is a long line of work on computing approximate triangle counting in parallel computa-
tion [Coh09, TKMF09, SV11, YK11, PT12, KMPT12, PC13, SPK13, AKM13, PSKP14, KPP+14, JS17] and references therein. Despite this progress, and to the best of our knowledge, on one hand,
each MPC algorithm for exact triangle counting either requires strictly super-polynomial in \( m \) total space, or the number of rounds is super-constant. On the other hand, each algorithm
for approximate triangle counting requires \( T \geq n \) even when the space per machine is \( \Theta(n) \). We
design an algorithm that has essentially optimal total space and round complexity, while at least
quadratically improving the requirement on \( T \).

Furthermore, since the amount of messages sent and received by each machine is bounded by
\( O(n) \), by [BDH18], our algorithm directly implies an \( O(1) \)-rounds algorithm in the CONGESTED-
CLIQUE model under the same restriction \( T = \Omega(\sqrt{m/n}) \). The best known (to our knowledge)
triangle approximation algorithm for general graphs in this model, is an \( O(n^{2/3}/t^{1/3}) \)-rounds algo-
rithm by [DLP12]. This bound only results in a constant round complexity when regime \( T = \Omega(\sqrt{n}) \).

The second question we consider is the question of exact counting, for which we present an
algorithm whose space depends on the arboricity of the input graph.

The arboricity of a graph (roughly) equals the average degree of its densest subgraph. The class
of graphs with bounded arboricity includes many important graph families such as planar graphs,
bounded degree graphs and randomly generated preferential attachment graphs. Also many real-world graphs exhibit bounded arboricity [GG06, ELS13, SERF18], making this property important also in practical settings. For many problems a bound on the arboricity of the graph allows for much more efficient algorithms and/or better approximation ratios [AG09, ELS13].

Specifically for the task of subgraph counting, in a seminal paper, Chiba and Nishizeki [CN85] prove that triangle enumeration can be performed in \(O(m\alpha)\) time, and assuming 3SUM-hardness this result is optimal up to dependencies in \(\log n\) [Pat10, KPP16]. Many applied algorithms also rely on the property of having bounded arboricity in order to achieve better space and time bounds, e.g., [SW05, CC11, Lat08]. Our main theorem with respect to this question is the following.

**Theorem 1.3.** Let \(G = (V,E)\) be a graph over \(n\) vertices, \(m\) edges and arboricity \(\alpha\). \(\text{Count-Triangles}(G)\) takes \(O_\delta (\log \log n)\) rounds, \(O(n^\delta)\) space per machine for any \(\delta > 0\), and \(O(m\alpha)\) total space.

It is interesting to note that our total space complexity matches the time complexity (both upper and conditional lower bounds) of combinatorial\(^1\) triangle counting algorithms graphs in the sequential model [CN85, Pat10, KPP16].

We prove the above theorem in Section 5, and in Section 6, we prove that this result can be extended to exactly counting \(k\)-cliques for any constant \(k\):

**Theorem 1.4.** Let \(G = (V,E)\) be a graph over \(n\) vertices, \(m\) edges and arboricity \(\alpha\). \(\text{Count-Cliques}(G)\) takes \(O_\delta (\log \log n)\) rounds, \(O(n^\delta)\) space per machine for any \(\delta > 0\), and \(O(m\alpha^{k-2})\) total space.

We can improve on the total space usage if we are given machines where the memory for each individual machine satisfies \(\alpha < n^{\delta'/2}\) where \(\delta' < \delta\). In this case, we obtain an algorithm that counts the number of \(k\)-cliques in \(G\) using \(O(n\alpha^2)\) total space and \(O_\delta(\log \log n)\) communication rounds.

Finally, in Section 7, we consider the problem of exactly counting subgraphs of size at most 5, and show that the recent result of [BPS20] for this question in the sequential model, can be implemented in the MPC model. Here too, our total space complexity matches the time complexity of the sequential model algorithm.

**Theorem 1.5.** Let \(G = (V,E)\) be a graph over \(n\) vertices, \(m\) edges, and arboricity \(\alpha\). The algorithm of BPS for counting the number of occurrences of a subgraph \(H\) over \(k \leq 5\) vertices in \(G\) can be implemented in the MPC model with high probability and round complexity \(O_\delta(\sqrt{\log n \log \log m})\). The space requirement per machine is \(O(n^{2\delta})\) and the total space is \(O(m\alpha^3)\).

2 Preliminaries

**Counting Duplicates** We make use of interval trees for certain parts of our paper to count the number of repeating elements in a sorted list, given bounded space per machine. We use the interval tree implementation given by [GSZ11] to obtain our interval tree implementation in the MPC model. We prove the following theorem in the MPC model using the interval tree implementation provided by [GSZ11]. The proofs of the following claims are given in Appendix A.

**Theorem 2.1.** Given a sorted list of \(N\) elements implemented on processors where the space per processor is \(S\) and the total space among all processors is \(O(N)\), for each unique element in the list, we can compute the number of times it repeats in \(O(\log S N)\) communication rounds.

\(^1\)Combinatorial algorithms, usually, refers to algorithms that do not rely on fast matrix multiplication.
Lemma 2.2. Given two sets of tuples $Q$ and $C$ (both of which may contain duplicates), for each tuple $q \in Q$, we return whether $q \in C$ in $O(|Q \cup C|)$ total space and $O_\delta(1)$ rounds given machines with space $O(n^\delta)$ for $\delta > 0$.

Lemma 2.3. Given a machine $M$ that has space $O(n^{2\delta})$ for $\delta > 0$ and contains data of $O(n^\delta)$ words, we can generate $x$ copies of $M$, each holding the same data as $M$, using $O(M \cdot x)$ machines with $O(n^\delta)$ space each in $O(\log n^\delta x)$ rounds.

Relevant Concentration Bounds. We will use the following well-known variant of the Chernoff bound.

Theorem 2.4 (Chernoff bound). Let $X_1, \ldots, X_k$ be independent random variables taking values in $[0, 1]$. Let $X \overset{\text{def}}{=} \sum_{i=1}^{k} X_i$ and $\mu \overset{\text{def}}{=} \mathbb{E}[X]$. Then, or any $\delta \in [0, 1]$ it holds $\mathbb{P}[X \leq (1-\delta)\mu] \leq \exp\left(-\delta^2 \mu / 2\right)$.

3 Overview of Our Techniques

3.1 Approximate Triangle Counting

Our work reduces approximate triangle counting to exact triangle counting in multiple induced subgraphs of the original graph. Vertex-based partitioning was previously used for triangle counting, e.g., [PT12], but the partitions are disjoint. In our work, the induced subgraphs on different machines might overlap. This allows us to obtain better concentration bounds compared to prior work, but also brings certain challenges in the analysis and MPC implementation.

The high level idea is that each machine $M_i$ samples a subset of vertices $V_i$ by including each vertex in $V_i$ with probability $\hat{p}$. Then each machine computes the induced subgraph $G[V_i]$ and the number of triangles in that subgraph. The average number of triangles across the machines is an unbiased estimator to the number of triangles in $G$. This approach gives rise to several challenges:

Limited space per machine. First, we need to make sure that the induced subgraphs do not exceed the space per machine $O(S)$. This limit implies an upper bound on our sampling probability, which in turn implies a lower bound on the number of triangles in the graph from which we can guarantee a good approximation with high probability.

Computing the induced subgraph. Second, it is unclear how to compute the induced subgraph on each machine without exceeding the total allowed space of $\tilde{O}(m)$. We tackle this challenge by using a globally known hash function $h : V \times [\mathcal{M}] \rightarrow [0, 1]$, to indicate whether vertex $v$ is sampled in the $i$th machine. By requiring that the hash function will be known to all machines, we can efficiently compute which edges to send to each machine, i.e., which edges belong to the subgraph $G[V_i]$. However, in order for all machines to be able to compute the hash function, it has to be of limited space. Hence we cannot hope for a fully independent function, rather we can only use an $(S / \log n)$-wise independent hash function. Still, we manage to show, that we are able to handle the dependencies introduced by the above, even if we allow as little as $O(\log n)$-independence.

High probability bound on the size of the induced subgraphs. The next challenge is in proving that with high probability, the sizes of the induced subgraphs do not exceed the allowed space per machine. To achieve this, we deal separately with light and heavy vertices. For light vertices it is fairly straightforward to bound their contribution to the number of edges in $G[V_i]$,
since their total number of edges is small to begin with. For heavy vertices, we use a variation of Chernoff’s inequality that is suitable for bounded dependence variables, and prove that their “sampled” degree behaves as expected.

The number of triangles across the machines concentrates. Our final challenge is in proving that the sum of triangles over the induced subgraphs is close to its expected value. Here too we have to take into account the dependencies introduced by the hash function.

Exploiting the properties of MPC The strategy described above takes advantage of the properties of the MPC model in several ways. First, we use each individual machine to perform a single “experiment,” and then view the collection of machines as repeating this experiment \( M \) times in parallel. More importantly, our approach exploits the computational power of MPC, even before counting any triangles. Namely, to compute which machines an edge \( e = (u, w) \) should be sent to, we evaluate the hash function for \( u \) and for \( w \) w.r.t. each machine. Then, the intersection of these two lists represent the number of machines \( e \) should be sent to. We also show that on average each edge is replicated \( O(1) \) times, while \( O(M) \) computation is performed to evaluate the hash function. This effectively means that we reduce round and communication costs at the expense of computation. We hope that this technique will find applications in other problems as well. It is an intriguing question to obtain a method for induced subgraph partitioning with overlaps whose computation does not significantly exceed the total size of the induced subgraphs.

3.2 Exact Triangle Counting

Let \( G \) be a graph over \( n \) vertices, \( m \) edges and with arboricity at most \( \alpha \). We tackle the task of exactly counting the number of triangles in \( G \) in \( O(\delta \log \log n) \) rounds using two main ideas. In each round \( i \), we partition the vertices into low-degree vertices \( A_i \) and high-degree vertices, according to a degree threshold \( \gamma_i \), which grows doubly exponentially in the number of rounds. We then count the number of triangles incident to the set of low degree vertices \( A_i \): Any neighbor \( u \) of \( v \) that detects a common neighbor \( w \) to \( u \) and \( v \), adds the triangle \((u, v, w)\) to the list of discovered triangles. Once all triangles incident to \( A_i \) are processed, we remove this set from the graph and continue with the now smaller graph.

Finding common neighbors is not trivial, as vertices might receive multiple neighbors lists from multiple neighbors in \( A_i \), which might exceed the space limit per machine.

Moreover, even if it would fit into the memory of a single machine, a vertex of degree \( d \) might receive \( d^2 \) messages in this way. If not done carefully, this can significantly exceed the total space. To combat that, we gradually process vertices of larger and larger degrees, initially starting with vertices of degree roughly \( \alpha \). We can show then that this treatment of degrees reduces the number of edges in the graph by an increasing factor (determined by \( \gamma_i \)) after each step. This in turn allows us to handle larger and larger degrees from step to step, while using a total space of \( O(m\alpha) \). This behavior also leads to the \( O(\delta \log \log n) \) round complexity, as after this many rounds all vertices are processed.

Ideas similar to some of the above were used in [ASS+18a, BDE+19b].

3.3 Counting \( k \)-cliques and 5-subgraphs.

We use a similar technique for both problems of exactly counting the number of \( k \)-cliques and of subgraphs up to size 5. See Section 6.1 for details on the former task, and Section 7 for details on the latter. Let \( H \) denote the subgraph of interest. We say that a subgraph that can be mapped to
a subset of \( H \) of size \( i \) is a \( i \)-subcopy of \( H \). Our main contribution in this section is a procedure that in each round, tries to extend \( i \)-subcopies of \( H \) to \((i + 1)\)-subcopies of \( H \), by increasing the total space by a factor of at most \( \alpha \). This is possible by ordering the vertices in \( H \) such that each vertex has at most \( O(\alpha) \) outgoing neighbors, so that in each iteration only \( \alpha \) possible extensions should be considered per each previous discovered subcopy.

4 Approximate Triangle Counting in General Graphs

In this section we provide our algorithm for estimating the number of triangles in general graphs (see Algorithms 1 and 3) and hence prove Theorem 1.1.

Theorem 1.1. Let \( G = (V, E) \) be a graph over \( n \) vertices, \( m \) edges, and let \( T \) be the number of triangles in \( G \). Assuming

\[
\begin{align*}
(i) \quad T &= \tilde{\Omega}\left(\sqrt{\frac{m}{S}}\right), \\
(ii) \quad S &= \tilde{\Omega}\left(\max\left\{\frac{\sqrt{m}}{\epsilon}, \frac{n^2}{m}\right\}\right),
\end{align*}
\]

there exists an MPC algorithm, using \( M \) machines, each with local space \( S \), and total space \( MS = \tilde{O}_\epsilon(m) \), that outputs a \((1 \pm \epsilon)\)-approximation of \( T \), with high probability, in \( O(1) \) rounds.

The rational behind the lower bound constraints in Theorem 1.1 will become clear when we will discuss the challenges and analysis (formally presented in Sections 4.2 and 4.3).

4.1 Overview of the Algorithm and Challenges

Our approach is to use the collection of machines to repeat the following experiment multiple times in parallel. Each machine samples a subset of vertices \( V_i \) on each machine \( M_i \), and then count the number of triangles \( \hat{T}_i \) seen in each induced graph \( G[V_i] \). We then use the sum \( \hat{T} \) of all \( \hat{T}_i \)'s as an unbiased estimator (after appropriate scaling) for the number of triangles \( T \) in the original graph.

Algorithm 1. Approximate Triangle Counting

1: function APPROX-TRIANGLES-SUBROUTINE\((G = (V, E))\)
2: \( R \leftarrow 0 \)
3: for \( i \leftarrow 1 \ldots M \) do
4: \( \quad \) Let \( V_i \) be a random subset of \( V \) \( \triangleright \) See Section 4.1.1 for details about the sampling
5: \( \quad \) if size of \( G[V_i] \) exceeds \( S \) then
6: \( \quad \quad \) Ignore this sample and set \( \hat{T}_i \leftarrow 0 \)
7: \( \quad \) else
8: \( \quad \quad \) Let \( \hat{T}_i \) be the number of triangles in \( G[V_i] \)
9: \( \quad \quad R \leftarrow R + 1 \)
10: \( \quad \) Let \( \hat{T} = \sum_{i=1}^{M} \hat{T}_i \)
11: return \( \frac{\hat{T}}{MR} \)

Moving forwards, for the most part, we will focus on a specific machine \( M_i \) containing \( V_i \) (a single experiment). We list the main challenges in the analysis of this algorithm, along with the section that describes them.

1. Section 4.1.1: The induced subgraph \( G[V_i] \) can fit into the memory \( S \) of \( M_i \) (thus allowing us to count the number of triangles in \( G[V_i] \) in one round).
2. **Section 4.1.2:** We can efficiently (in one round) collect all the edges in the induced subgraph \(G[V_i]\). This involves presenting an MPC protocol such that the number of messages sent and received by any machine is at most the space per machine \(S\).

3. **Section 4.1.3:** With high probability, the sum of triangles across all machines, \(\hat{T}\), is close to its expected value.

### 4.1.1 Challenge (1): Ensuring That \(G[V_i]\) Fits on a Single Machine

**Ensuring that edges fit on a machine:** Our algorithm constructs \(V_i\) by including each \(v \in V\) with probability \(\hat{p}\), which implies that the expected number of edges in \(G[V_i]\) is \(\hat{p}^2 m\). Since we have to ensure that each induced subgraph \(G[V_i]\) fits on a single machine, we obtain the constraint \(\hat{p}^2 m = O(S)\). Concretely, we achieve this by defining:

\[
\hat{p} \overset{\text{def}}{=} \frac{1}{10} \sqrt{\frac{S}{mk}}, \tag{1}
\]

where the parameter \(k = O(\log n)\) will be exactly determined later (See Section 4.1.2).

**Ensuring that vertices fit on a machine:** In certain regimes of values of \(n\) and \(m\), the expected number of vertices ending up in an induced subgraph – \(\hat{p}n\), may exceed the space limit \(S\). Avoiding this scenario introduces an additional constraint \(\hat{p}n = O(S) \iff S = \Omega(kn^2/m)\).

**Getting a high probability guarantee:** As discussed above, the value of \(\hat{p} = \tilde{\Theta}(\sqrt{S/m})\) is chosen specifically so that, the expected number of edges in the induced subgraphs \(G[V_i]\) is \(\hat{p}^2 m \leq \Theta(S)\), thus using all the available space (asymptotically). In order to guarantee that this bound holds with high probability (see Section 4.2.1), we require an additional constraints on the space per machine \(S = \tilde{\Omega}(\sqrt{m})\). We remark that this lower bound \(S = \tilde{\Omega}(\sqrt{m})\) is essentially saying that \(\mathcal{M} = \tilde{O}(\sqrt{m})\), i.e. the space per machine is much larger than the number of machines. This is a realistic assumption as in practice we can have machines with \(10^{11}\) words of local random access memory, however, it is unlikely that we also have as many machines in our cluster.

**Lower Bound on space per machine:** Combining the above two constraints, we get:

\[
S > \max \left\{ 15 \frac{\sqrt{mk}}{\varepsilon}, \frac{100kn^2}{m} \right\} \implies S = \tilde{\Omega}_\varepsilon \left( \max \left\{ \sqrt{m}, \frac{n^2}{m} \right\} \right) \tag{2}
\]

Note that Eq. (2) always allows linear space per machine, as long as \(m = \omega(n)\). Sections 4.2.1 and 4.2.2 present a detailed analysis, showing that the number of vertices and edges in each subgraph is at most \(S\) with high probability.

### 4.1.2 Challenge (2): Using \(k\)-wise Independence to Compute the Induced Subgraph \(G[V_i]\) in MPC

For each sub-sampled set of vertices \(V_i\), we need to compute \(G[V_i]\), i.e. we need to send all the edges in the induced subgraph \(G[V_i]\) to the machine \(M_i\). Let \(Q_u\) denote the set of all machines containing \(u\). Each edge \((u, w)\) then needs to be sent to all machines that contain both \(u\) and \(w\), \(Q_u \cap Q_w\). Naively, one could try to send the sets \(Q_u\) and \(Q_w\) to the edge \(e = (u, w)\), for all \(e \in E\). However, this strategy could result in \(Q_v\) being replicated \(d(v)\) times. Since the expected size of \(Q_v\) is \(|Q_v| = \hat{p}\mathcal{M}\)
the total expected memory usage of this strategy would be \( \sum_{v \in V} |Q_v| \cdot d(v) = \tilde{\Theta}(m \cdot \tilde{p}M) = \tilde{\omega}(m) \), since \( \tilde{p} = \Theta(1/\sqrt{M}) \). This defies our goal of optimal total memory.

Instead, we address this challenge by using globally known hash function to sample the vertices on each machine. That is, we let \( h : V \times [M] \to \{0, 1\} \) (formally presented in Definition 4.1) be a hash function known globally to all the machines. This allows us to compute the induced subgraphs \( G[V_i] \) using the following procedure.

**Algorithm 2. Finding Induced Subgraphs**

1: \textbf{function} Send-Edge\((e = (v, w))\)
2: \hspace{1cm} \(Q_v \leftarrow \{i \in [M] \mid h(v, i) = 1\}\).
3: \hspace{1cm} \(Q_w \leftarrow \{i \in [M] \mid h(w, i) = 1\}\).
4: \hspace{1cm} \text{for } i \in Q_v \cap Q_w \text{ do}
5: \hspace{2cm} Send \(e\) to machine \(M_i\), containing \(V_i\).

**Definition 4.1.** The hash function \( h(v, i) \) indicates whether vertex \( v \) is sampled in \( V_i \) or not. Specifically, \( h : V \times [M] \to \{0, 1\} \) such that \( \mathbb{P}[h(v, i) = 1] = \tilde{p} \) for all \( v \in V \) and \( i \in [M] \). Recall that \( M \) is the number of machines, and \( \tilde{p} = \frac{1}{10} \cdot \sqrt{\frac{S}{600m \log n}} \) is the sampling probability set in Eq. (1).

Using limited independence: Ideally, we would want a perfect hash function, which would allow us to sample the \( V_i \)'s i.i.d. from the uniform distribution on \( V \). However, since the hash function needs to be known globally, it must fit into each of the machines. This implies that we cannot use a fully independent perfect hash function. Rather, we can use one that has a high level of independence. Specifically, given that the space per machine is \( S \), we can have a globally known hash function \( h \) that is \( k \)-wise independent\(^2\) for any \( k < \Theta(S/\log n) \). In fact, we can get away with as little as \((6 \log n)\)-wise independence (i.e. \( k = 6 \log n \)). Recalling Eq. (1), this also fixes the sampling probability to be \( \tilde{p} = \sqrt{S/600m \log n} \).

**Sub-challenge: Showing that Protocol Succeeds with probability at least \( \frac{9}{10} \)**

We need to show that the number of edges sent and received by any machine \( M_i \) is at most \( S \) with high constant probability. To this end, we partition the vertex set \( V \) into \( V_{\text{light}} \) and \( V_{\text{heavy}} \) by picking a threshold degree \( \tau \) for the vertices. Following this, we define \( \text{light edges} \) as ones that have both end-points in \( V_{\text{light}} \), and conversely, any edge with at least one end-point in \( V_{\text{heavy}} \) is designated as \( \text{heavy} \). In order for the protocol to succeed, the following must hold:

(A) The number of \( \text{light edges} \) concentrates (see Section 4.2.1).

(B) The number of \( \text{heavy edges} \) concentrates (see Section 4.2.2).

(C) The number of sent messages is at most \( S \) (see Section 4.2.3).

The first two items ensure that each machine \( M_i \) receives at most \( S \) messages, and the last item ensures that each machine sends at most \( S \) messages. Given the above, we proceed to address the last challenge.

\(^2\) A \( k \)-wise independent hash function is such that the hashes of any \( k \) distinct keys are guaranteed to be independent random variables (see [WC81]).
4.1.3 Challenge (3): \( \hat{T} \) is close to its expected value

In this section, we provide merely a brief discussion of this challenge for intuition, and we fully analyze the approximation guarantees of our algorithm in Section 4.3.1. That analysis also makes clear the source of our advertised lower-bound on \( T \) for which an estimated count concentrates well.

**Lower Bound on Number of Triangles:** In order to output any approximation (note that we are ignoring factors of \( \varepsilon \) and \( \log n \) here) to the triangle count, we must see \( \Omega(1) \) triangles amongst all of the induced subgraphs on all the machines. The expected number of triangles in a specific induced \( G[V_i] \) is \( \hat{p}^3 T \), and therefore, the expected number of triangles overall is \( \hat{p}^3 T M = \Omega(1) \).

Since, from the previous paragraph, we set \( \hat{p}^2 m = \Theta(S) \Rightarrow \hat{p}^2 M = \Theta(1) \), this implies that \( \hat{p} \cdot T = \Omega(1) \). Specifically, we will show (see Lemma 4.10) that when \( T > \frac{1}{\hat{p}} \), we can obtain a \((1 \pm \varepsilon)\)-approximation. To get some intuition for this lower bound on \( T \), note that, in the linear memory regime, when \( S = \Theta(n) \), this translates to \( T > \sqrt{d_{avg}} \), where \( d_{avg} \) is the average degree of \( G \).

\[
\hat{p} \cdot T = \Omega(1) \\
\Rightarrow \hat{p} \cdot T = \Theta(S) \Rightarrow \hat{p} \cdot T = \Theta(1).
\]

4.2 Approximate Triangle Counting: Bounding the number of Messages Sent/Received by a Machine

This section analyzes the estimation algorithm from Section 4. Recall, from Section 4.1.2, that we use a \( k \)-wise independent hash function, to compute the induced sub-graphs \( G[V_i] \), where \( k = 6 \log n \).

In the subsequent proofs, we will use the following assumptions from within Theorem 1.1 (note that we added specific constants).

\[
T \geq 10 \sqrt{\frac{mk}{S}} \quad S \geq \max \left\{ \frac{15 \sqrt{mk}}{\varepsilon}, \frac{100 kn^2}{m} \right\} \quad \mathcal{M} = \frac{2000 mk \varepsilon^2}{S} \tag{3}
\]

Note that we set the number of machines to a specific value, instead of lower bounding it. This is acceptable, because we can just ignore some of the machines. We will now bound the probability that any of the induced subgraphs does not fit on a machine. To that end, we set a degree threshold \( \tau = \frac{k}{\hat{p}} \), and define the set of light vertices \( V_{light} \) to be the ones with degree less than \( \tau \). All other vertices are heavy, and we let them comprise the set \( V_{heavy} \).

Fix a machine \( M_i \). We prove that, with probability at least \( 9/10 \), the number of edges in \( G[V_i] \) is upper bounded by \( S \).

We start with analyzing the contribution of the light vertices to the induced subgraphs.

4.2.1 Bounding the Number of Light Edges Received by a Machine

Fix a machine \( M_i \). We first consider the simpler case of bounding the number of edges in \( G[V_i] \) that have both end-points in \( V_{light} \). We refer to such edges as light edges and denote them by \( E_{light} \).

For every edge \( e \in E_{light} \), we define a random variable \( Z_e^{(i)} \) as follows.

\[
Z_e^{(i)} = \begin{cases} 
1 & \text{if } e \in G[V_i], \\
0 & \text{otherwise}.
\end{cases}
\]

We let \( Z^{(i)} \) be the sum over all random variables \( Z_e, Z^i = \sum_{e \in E_{light}} Z_e \), and we let \( m_{\ell} \) denote the total number of edges with light endpoints in the original graph \( G \), i.e., \( m_{\ell} = |E_{light}| \).

We prove the following lemma.
Lemma 4.2. With probability at least $9/10$, for every $i \in \mathcal{M}$, $G[V_i]$ contains at most $\frac{1}{4}S$ light edges.

Proof. Fix a machine $M_i$, and let $Z = Z^i$ be as defined in the previous paragraph.

$$E[Z] = E \left[ \sum_{e \in E_{light}} Z_e \right] = m_\ell \hat{p}^2 \leq m \cdot \frac{S}{100mk} = \frac{S}{100} \leq \frac{S}{100}.$$

As $Z_e$ are $\{0, 1\}$ random variables, we also have $E[Z] = E \left[ \sum_{e \in E_{light}} Z_e^2 \right]$. Now we upper-bound the variance.

$$\text{Var}[Z] = E \left[ \left( \sum_{e \in E_{light}} Z_e \right)^2 \right] - E \left[ \sum_{e \in E_{light}} Z_e \right]^2$$

$$\leq \sum_{e \in E_{light}} E[Z_e^2] + \sum_{e_1, e_2 \in E_{light}, e_1 \neq e_2} 2 \cdot E[Z_{e_1} Z_{e_2}] - \sum_{e_1, e_2 \in E_{light}, e_1 \neq e_2} 2 \cdot E[Z_{e_1}]E[Z_{e_2}]$$

$$= m_\ell \cdot \hat{p}^2 + \sum_{e_1, e_2 \in E_{light}, e_1 \neq e_2} 2 \cdot E[Z_{e_1} Z_{e_2}] = m_\ell \cdot \hat{p}^2 + \sum_{e_1, e_2 \in E_{light}, e_1 \neq e_2} 2 \cdot E[Z_{e_1}]E[Z_{e_2}]$$

$$\leq m_\ell \cdot \hat{p}^2 + \sum_{e_1 \text{ and } e_2 \text{ intersect}} 2 \cdot E[Z_{e_1} Z_{e_2}]$$

$$\leq m_\ell \cdot \hat{p}^2 + \left( \sum_{v \in V_{light}} d(v)^2 \right) \cdot \hat{p}^3$$

$$\leq m_\ell \cdot \hat{p}^2 + \left( \sum_{v \in V_{light}} d(v) \right) \cdot \frac{k}{\hat{p}} \cdot \hat{p}^3$$

$$\leq 3m_\ell \cdot \hat{p}^2 \cdot k \leq 3m \cdot \frac{S}{100mk} \cdot k < \frac{S}{30}$$

We can now use Chebyshev’s inequality to conclude that

$$P \left[ |Z^{(i)} - E[Z^{(i)}]| > \frac{S}{\sqrt{3}} \right] \leq \frac{\text{Var}[Z^{(i)}]}{S^2/3} \leq \frac{3}{30S} = \frac{1}{10S}.$$

Finally, we can use union bound over all $\mathcal{M}$ machines to upper bound the probability that, any of the $Z^{(i)}$ values exceeds $3S/4$ (using the the constraints described in Eq. (3) to simplify).

$$\frac{\mathcal{M}}{10S} = \frac{2000mk}{\varepsilon^2S} \cdot \frac{1}{10S} \leq \frac{200mk}{\varepsilon^2} \cdot \frac{1}{(15\sqrt{mk}/\varepsilon)^2} = \frac{200mk}{\varepsilon^2 S^2},$$

Therefore, with probability at least $9/10$, none of the induced subgraphs $G[V_i]$ will contain more than $3S/4$ light edges.
4.2.2 Bounding the Number of Heavy Edges Received by a Machine

Next, we turn our attention to the edges that have at least one endpoint in \( V_{\text{heavy}} \) (we call such edges heavy). We will show that for each \( v \in V_{\text{heavy}} \cap V_i \), the number of edges contributed by \( v \) concentrates around its expectation.\(^3\) In this section, we will use \( 2m_h \) to denote the total degree of all the heavy vertices i.e. \( 2m_h = \sum_{v \in V_{\text{heavy}}} d(v) \).

Let \( Z_w^{(v)} \) be the \( \{0, 1\} \) indicator random variable for \( w \in V_i \) conditioned on the event that \( v \in V_i \cap V_{\text{heavy}} \). We use this conditioning on \( v \) being present, because, in it’s absence, the number of edges contributed by \( v \), can be zero with probability \((1 - \hat{p})\), i.e. this naive estimator would not concentrate around its expectation.

Let \( Z^{(v)} \) be the sum of all \( Z_w^{(v)} \) for \( w \in N(v) \). For a particular \( v \), the \( Z_w^{(v)} \) variables are \( k \)-wise independent, which allows us to use the following lemma to bound \( Z^{(v)} \). In what follows, we will omit the super-script \( (v) \) for the sake of convenience.

**Lemma 4.3.** If \( Z_1, Z_2, \ldots, Z_n \) are \( k \)-wise independent \( \{0, 1\} \) random variables with \( \mathbb{E}[Z_i] = p \) and \( k \leq np \), then for \( Z = \sum_i Z_i \) we have

\[
\mathbb{P}[Z > 3np] \leq 2^{-k}.
\]

**Proof.** To prove the claim, we will re-write \( \mathbb{P}[\sum_i Z_i > 3np] \), as the probability that the number of size \( k \) subsets of \( \{Z_1, Z_2, \ldots, Z_n\} \) that are all equal to 1 is larger than \( \binom{3np}{k} \).

\[
\mathbb{P}[Z > 3np] = \mathbb{P}\left[ \left| \left\{ T : T \subseteq [n], |T| = k, \text{ and } Z_i = 1 \text{ \forall } i \in T \right\} \right| > \binom{3np}{k} \right] \\
\leq \mathbb{E}\left[ \left| \left\{ T : T \subseteq [n], |T| = k, \text{ and } Z_i = 1 \text{ \forall } i \in T \right\} \right| \right] \\
= \frac{n^k \cdot p^k}{\binom{3np}{k}} \leq \left( \frac{n}{3np - k} \cdot p \right)^k \leq \left( \frac{np}{2np} \right)^k = 2^{-k}
\]

where to obtain \( 3np - k \geq 2np \) we used our assumption that \( k \leq np \). \( \square \)

Since \( v \) is heavy, there are at least \( \tau \) variables in the sum \( Z^{(v)} = \sum_{w \in N(v)} Z_w^{(v)} \). Additionally, we know that \( \mathbb{E}[Z_w^{(v)}] = \hat{p} \) and \( k \leq \tau \hat{p} \). Thus, we obtain the following corollary from Lemma 4.3:

**Corollary 4.4.** For any vertex \( v \in V_{\text{heavy}} \cap V_i \), we get \( \mathbb{P}[Z^{(v)} > 3d(v) \cdot \hat{p}] < 2^{-k} \), or explicitly

\[
\mathbb{P}[|N(v) \cap V_i| > 3d(v) \hat{p} \mid v \in V_i \text{ and } d(v) > \tau] < 2^{-k} = \frac{1}{n^6}
\]

**Corollary 4.5.** With high probability \( 1 - \frac{1}{n^6} \), we ensure that for all \( v \in V_{\text{heavy}} \), \( Z^{(v)} \leq 3 \cdot \mathbb{E}[Z^{(v)}] \)

The important point is that the sum of \( Z^{(v)} \) (over all \( v \in V_i \)) is an upper bound on \( m_h \) – the number of heavy edges in \( G[V_i] \). In order to bound this sum, we define random variables \( W_v \) for each \( v \in V_{\text{heavy}} \) as follows:

\[
W_v = \begin{cases} 
\frac{d(v)}{n} & \text{if } v \in V_i \\
0 & \text{otherwise}
\end{cases}
\]

We also define \( W \) to be the sum of all \( W_v \), thus implying \( \mu = \mathbb{E}[W] = \sum_{v \in V_{\text{heavy}}} \hat{p} \cdot \frac{d(v)}{n} \leq \frac{2\hat{p}m_h}{n} \).

\(^3\) Intuitively, this is because \( v \) has high degree, and therefore the number of it’s sampled neighbors (\(|N(v) \cap V_i|\)) will concentrate.
Theorem 4.6. (Theorem 5 from [SSS95]) If $W$ is the sum of $k$-wise independent random variables, each of which takes values in the interval $[0, 1]$, and $\delta \geq 1$, then:

$$k < \lceil \delta \mu e^{-1/3} \rceil \implies \Pr[|W - \mu| > \delta \mu] \leq e^{\lceil k/2 \rceil}$$

Corollary 4.7. $\Pr[W > 4\hat{p}mkn] \leq e^{-\lceil k/2 \rceil}$

Proof. We can use the fact the random variables $W_v$ are $k$-wise independent to apply Theorem 4.6. First, we ensure that $k < \lceil \delta \mu e^{-1/3} \rceil$, that we achieve by setting $\delta = \frac{mk}{m_h}$.

Recall that $m_h$ is the number of heavy edges (ones with at least one heavy end-point), and $m$ is the total number of edges in the original graph $G$.

$$\delta = \frac{mk}{m_h} \implies \delta \mu e^{-1/3} = \frac{mk \cdot 2\hat{p}m_h}{m_h \cdot n} \cdot e^{-1/3} > \frac{\hat{p}mk}{n} \implies \delta \mu e^{-1/3} > k$$

In the last step, we used the fact that $S > 100kn^2/m$ from Eq. (3), to imply that $\hat{p}m/n > 1$. Therefore, we can now apply Theorem 4.6 to conclude:

$$\Pr[|W - \mu| > \delta \mu] \leq e^{\lceil k/2 \rceil}$$

$$\implies \Pr[W > \mu + \frac{2\hat{p}mk}{n}] \leq e^{\lceil k/2 \rceil}$$

$$\implies \Pr[W > \frac{4\hat{p}mk}{n}] \leq e^{\lceil k/2 \rceil}$$

In the second step, we used the fact that $\mu = \mathbb{E}[W] = \sum_{v \in V_{\text{heavy}}} \hat{p} \cdot \frac{d(v)}{n} \leq \frac{2m\hat{p}}{n}$. \hfill \square

Now we are finally ready to upper bound the number of heavy edges in $G[V_i]$. With high probability (using Corollary 4.4), the following holds:

$$\# \text{ (heavy edges in } G[V_i]) \leq \sum_{v \in V_{\text{heavy}}} \Pr[v \in V_i] \cdot (3d(v)\hat{p})$$

$$\leq \sum_{v \in V_{\text{heavy}}} W_v \cdot n \cdot (3\hat{p}) = 3n\hat{p} \cdot W$$

$$\leq 12\hat{p}^2 mk = \frac{12S}{100} < \frac{S}{8}$$

Theorem 4.8 (Heavy edges). With high probability, the number of edges in $G[V_i]$ that have some endpoint with degree larger than $\tau$ is at most $S/8$.

Combining this result with Theorem 4.8, we conclude the following:

Theorem 4.9. With probability at least $9/10$, the maximum number of edges in any of the $G[V_i]$s (where $i \in [R]$) does not exceed $S$, and hence Algorithm 1 does not terminate on Line 5.

4.2.3 Upper-Bounding the Number of Messages Sent by any Machine

Recalling Algorithm 2, we note that the number of messages received by the machine containing $V_i$, is equal to the number of edges in $G[V_i]$. Therefore, the last section essentially proved that the number of messages (edges) received by a particular machine is upper-bounded by $S$. Conversely,
in this section, we will justify that the number of messages sent by any machine is $O(S)$. Since the number of edges stored in a machine is $\leq S$, it suffices to show that for each edge $e$, Algorithm 2 sends only $O(1)$ messages (each message is a copy of the edge $e$).

Let $Z^{(e)}_i$ be the $(0,1)$ indicator random variable for $e \in G[V_i]$, and let $Z^{(e)}$ be the sum of $Z^{(e)}_i$ for all $i \in [M]$. Here, $Z^{(e)}$ represents the number of messages that are created by edge $e$. Additionally we make $r = SM/m = O_e(\log n)$ copies of each edge $e$, and ensure that all replicates reside on the same machine. We distribute the $Z^{(e)}$ messages evenly amongst the replicates, so that each replica is only responsible for $Z^{(e)}/r$ messages.

Since all replicates are on the same machine, this last step is purely conceptual, but it will simplify our argument, by allowing us to charge the outgoing messages to each replicate (as opposed to each edge). Our goal will be show that each replicate is responsible for only $O(1)$ messages, which is the same as showing that w.h.p. $Z^{(e)}/r = O(1)$.

Clearly $\mu = \mathbb{E}[Z^{(e)}] = \bar{p}^2 \cdot M = \frac{SM}{100mk}$. This allows us to apply Lemma 4.3 with $\delta = \frac{100e^{1/3}mk^2}{SM}$

$$
\mathbb{P}\left[Z^{(e)} > \delta \mu\right] \leq e^{-k/2} = \frac{1}{n^3} \implies \mathbb{P}\left[\frac{Z^{(e)}}{r} > \frac{e^{1/3}k}{r}\right] \leq \frac{1}{n^3}
$$

Using the assumption (from Eq. (3)) that $M > 2000mk/S \implies r > 2000k$, we see that with high probability, the number of messages sent by any replicate is bounded above by $e^{1/3}/2000 \leq 1$. So, the number of messages sent from any machine is bounded by $S$ with high probability.

### 4.3 Approximate Triangle Counting: Showing that the Estimate Concentrates

#### 4.3.1 Showing Concentration for the Triangle Count

Algorithm 1 outputs an estimate on the number of triangles in $G$ (Line 11). It is not hard to show that in expectation this output equals $T$. The main challenge is to show that this output also concentrates well around its expectation. Specifically, we show the following claim.

**Lemma 4.10.** Ignore Line 5 of Algorithm 1. Let $\hat{T}$ be as defined on Line 10 and $M = \frac{20}{\epsilon^2 \bar{p}^3}$ be as defined in Eq. (3), and assume that $T \geq 1/\bar{p}$. Then, the following hold:

(A) $\mathbb{E}\left[\hat{T}\right] = \bar{p}^3 \cdot R \cdot T$, and  

(B) $\mathbb{P}\left[|\hat{T} - \mathbb{E}\left[\hat{T}\right]| > \epsilon \mathbb{E}\left[\hat{T}\right]\right] < \frac{1}{10}$.

We will prove Property (B) of the claim by applying Chebyshev’s inequality, for which we need to compute $\text{Var}\left[\hat{T}\right]$. Let $\Delta(G)$ be the set of all triangles in $G$. For a triangle $t \in \Delta(G)$, let $\hat{T}_{i,t} = 1$ if $t \in V[G_i]$, and $\hat{T}_{i,t} = 0$ otherwise. Hence, $\hat{T}_i = \sum_{t \in \Delta(G)} \hat{T}_{i,t}$. We begin by deriving $\mathbb{E}\left[\hat{T}\right]$ and then proceed to showing that $\text{Var}\left[\hat{T}\right] = \sum_{i=1}^R \text{Var}\left[\hat{T}_i\right]$. After that we upper-bound $\text{Var}\left[\hat{T}_i\right]$ and conclude the proof by applying Chebyshev’s inequality.

**Deriving $\mathbb{E}\left[\hat{T}\right]$.** Let $t$ be a triangle in $G$. Let $\hat{T}_t$ be a random variable denoting the total number of times $t$ appears in $G[V_i]$, for all $i = 1 \ldots R$. Given that $\mathbb{P}\left[u \in V_i\right] = \bar{p}$, we have that $\mathbb{P}\left[t \in G[V_i]\right] = \bar{p}^3$. Therefore, $\mathbb{E}\left[\hat{T}_t\right] = R \cdot \bar{p}^3$.
Since $\hat{T} = \sum_{t \in \Delta(G)} \hat{T}_t$, we have
\[
\mathbb{E} \left[ \hat{T} \right] = \sum_{t \in \Delta(G)} \mathbb{E} \left[ \hat{T}_t \right] = \hat{p}^3 \cdot R \cdot T. \tag{4}
\]
This proves Property (A) of this claim.

**Decoupling** $\text{Var} \left[ \hat{T} \right]$. To compute variance, one considers the second moment of a given random variable. So, to compute $\text{Var} \left[ \hat{T} \right]$, we will consider products $\hat{T}_{i,t_1} \cdot \hat{T}_{j,t_2}$. Each of those products depend on at most 6 vertices. Now, given that we used a 6-wise independent function (see Section 4.1.2) to sample vertices in each $V_i$, one could expect that $\text{Var} \left[ \hat{T}_i \right]$ and $\text{Var} \left[ \hat{T}_j \right]$ for $i \neq j$ behave like they are independent, i.e., one could expect that it holds $\text{Var} \left[ \hat{T} \right] = \sum_{i=1}^{R} \text{Var} \left[ \hat{T}_i \right]$. As we show next, it is indeed the case. We have
\[
\text{Var} \left[ \hat{T} \right] = \mathbb{E} \left[ \hat{T}^2 \right] - \mathbb{E} \left[ \hat{T} \right]^2
= \mathbb{E} \left[ \left( \sum_{i=1}^{R} \sum_{t \in \Delta(G)} \hat{T}_{i,t} \right)^2 \right] - \left( \sum_{i=1}^{R} \sum_{t \in \Delta(G)} \mathbb{E} \left[ \hat{T}_{i,t} \right] \right)^2 \tag{5}
\]
Consider now $\hat{T}_{i,t_1}$ and $\hat{T}_{j,t_2}$ for $i \neq j$ and some $t_1, t_2 \in \Delta(G)$ not necessarily distinct. In the first summand of (5), we will have $\mathbb{E} \left[ 2 \hat{T}_{i,t_1} \cdot \hat{T}_{j,t_2} \right]$. The vertices constituting $t_1$ and $t_2$ are 6 distinct copies of some (not necessarily all distinct) vertices of $V$. Since they are chosen by applying a 6-wise independent function, we have $\mathbb{E} \left[ 2 \hat{T}_{i,t_1} \cdot \hat{T}_{j,t_2} \right] = 2 \mathbb{E} \left[ \hat{T}_{i,t_1} \right] \cdot \mathbb{E} \left[ \hat{T}_{j,t_2} \right]$. On the other hand, the second summand of (5) also contains $2 \mathbb{E} \left[ \hat{T}_{i,t_1} \right] \cdot \mathbb{E} \left[ \hat{T}_{j,t_2} \right]$, which follows by direct expansion of the sum. Therefore, all the terms $\mathbb{E} \left[ 2 \hat{T}_{i,t_1} \cdot \hat{T}_{j,t_2} \right]$ in $\text{Var} \left[ \hat{T} \right]$ for $i \neq j$ cancel each other. So, we can also write $\text{Var} \left[ \hat{T} \right]$ as
\[
\text{Var} \left[ \hat{T} \right] = \sum_{i=1}^{R} \mathbb{E} \left[ \left( \sum_{t \in \Delta(G)} \hat{T}_{i,t} \right)^2 \right] - \sum_{i=1}^{R} \left( \sum_{t \in \Delta(G)} \mathbb{E} \left[ \hat{T}_{i,t} \right] \right)^2
= \sum_{i=1}^{R} \text{Var} \left[ \hat{T}_i \right] \tag{6}
\]
Therefore, to upper-bound $\text{Var} \left[ \hat{T} \right]$ it suffices to upper-bound $\text{Var} \left[ \hat{T}_i \right]$. 

14
Upper-bounding \( \text{Var} [ \hat{T}_i ] \). We have

\[
\text{Var} [ \hat{T}_i ] = \mathbb{E} \left[ \left( \sum_{t \in \Delta(G)} \hat{T}_{i,t} \right)^2 \right] - \left( \sum_{t \in \Delta(G)} \mathbb{E} [ \hat{T}_{i,t} ] \right)^2 \leq \mathbb{E} \left[ \left( \sum_{t \in \Delta(G)} \hat{T}_{i,t} \right)^2 \right] + \mathbb{E} \left[ \sum_{t_1, t_2 \in \Delta(G); t_1 \neq t_2} \hat{T}_{i,t_1} \cdot \hat{T}_{i,t_2} \right].
\]

(7)

Since each \( \hat{T}_{i,t} \) is a 0/1 random variables, \( \hat{T}_{i,t}^2 = \hat{T}_{i,t} \). Let \( t_1 \neq t_2 \) be two triangles in \( \Delta(G) \). Let \( k \) be the number of distinct vertices they are consisted of, which implies \( 4 \leq k \leq 6 \). Then, observe that \( \mathbb{E} [ \hat{T}_{i,t_1} \cdot \hat{T}_{i,t_2} ] = \hat{p}^k \leq \hat{p}^4 \). We now have all ingredients to upper-bound \( \text{Var} [ \hat{T}_i ] \). From (7) and our discussion it follows

\[
\text{Var} [ \hat{T}_i ] \leq T \hat{p}^3 + T^2 \hat{p}^4 \leq 2T^2 \hat{p}^4,
\]

(8)

where we used our assumption that \( T \geq 1/\hat{p} \).

**Finalizing the proof.** From (6) and (8) we have

\[
\text{Var} [ \hat{T} ] \leq 2RT^2 \hat{p}^4.
\]

So, from Chebyshev’s inequality and (4) we derive

\[
\mathbb{P} \left[ |\hat{T} - \mathbb{E} [ \hat{T} ]| > \varepsilon \mathbb{E} [ \hat{T} ] \right] < \frac{\text{Var} [ \hat{T} ]}{\varepsilon^2 \mathbb{E} [ \hat{T} ]^2} \leq \frac{2RT^2 \hat{p}^4}{\varepsilon^2 \hat{p}^6 R^2 T^2} = \frac{2}{\varepsilon^2 \hat{p}^2 R}.
\]

Hence, for \( R \geq \frac{20}{\varepsilon^2 \hat{p}^2} \) we get the desired bound.

### 4.3.2 Getting the High Probability Bound

By building on Lemma 4.10 and Algorithm 1, we design Algorithm 3 that outputs an approximate triangle counting with high probability, as opposed with only constant success probability.

**Algorithm 3. Approximate Triangle Counting**

```plaintext
1: function APPROX-TRIANGLES-MAIN(G = (V, E))
2:     Let \( I \leftarrow 100 \cdot \log n \).
3:     for \( i \leftarrow 1 \ldots I \) do
4:         Let \( Y_i \) be the output of Algorithm 1 invoked on \( G \). We assume that each invocation
```
of Algorithm 1 uses fresh randomness compared to previous runs.

5: Let $Y$ be the list of all $Y_i$, for $i = 1 \ldots I$.
6: Sort $Y$ in non-decreasing order.
7: return the median of $Y$

We have the following guarantee for Algorithm 3.

**Theorem 4.11.** Let $Y$ be the output of Algorithm 3. Then, with high probability it holds

$$|Y - T| \leq \varepsilon T.$$ 

**Proof.** The proof of this theorem is essentially the so-called “Median trick”. We provide full proof here for completeness.

Let $Y_i$ be as defined on Line 4 of Algorithm 3. By Theorem 4.9, with probability at most $1/10$ Algorithm 1 terminates due to creating too big subgraphs. If we ignore Line 5 of Algorithm 1, then by Property (A) of Lemma 4.10 we have $E[Y_i] = T$. $Y_i$ significantly deviates from its expectation if Algorithm 1 terminates on Line 5 or if the estimate $Y_i$ is simply off. Define a 0/1 variable $Z_i$ which equals 1 iff $|Y_i - T| \leq \varepsilon T$. By union bound on Property (B) of Lemma 4.10 and Theorem 4.9, we have $P[Z_i = 1] \geq 1 - 1/10 - 1/10 = 4/5$. Also, following Line 4 of Algorithm 3 we have that all $Z_i$ are independent.

Let $Z = \sum_{i=1}^{I} Z_i$. We have that $E[Z] \geq \frac{4}{5} I$, implying that in expectation at least 4/5 fraction on $Z$-variables are 1. We now bound the probability that at least 2/5 of these variables equal 0, i.e., at most 3/5 of them equal 1. Since $Z$-variables are independent, for this we can use Chernoff bound for bounding this probability, e.g., Theorem 2.4, obtaining

$$P[Z \leq \frac{3}{5} I] \leq P[Z \leq \left(1 - \frac{1}{5}\right)E[Z]] \leq \exp \left(-E[Z]/50\right).$$

Given that $I = 100 \cdot \log n$ (see Line 2 of Algorithm 3), we derive that $P[Z \leq \frac{3}{5} I] < n^{-1}$. This now implies that with probability at least $n^{-1}$ the output of Algorithm 3 is some $Y_j$ such that $Z_j = 1$. This completes the analysis.

5 Exact Triangle Counting in $O(m\alpha)$ Total Space

In this section we describe our algorithm for (exactly) counting the number of triangles in graphs $G = (V, E)$ of arboricity $\alpha$ and prove Theorem 1.3, restated here, in Section 5.2.

**Theorem 5.1.** Let $G = (V, E)$ be a graph over $n$ vertices, $m$ edges and arboricity $\alpha$. Count-Triangles$(G)$ takes $O_5 (\log \log n)$ rounds, $O(n^\delta)$ space per machine for some constant $0 < \delta < 1$, and $O (ma)$ total space.

We note that we assume that the algorithm is given a bound on the arboricity of the input graph $G$, as otherwise we can use Theorem 2 of [GLM19] to get a constant approximation of $\alpha(G)$ in $O_5(\sqrt{\log n} \log \log n)$ rounds, $O(n^\delta)$ per machine and $O(\max\{m, n^{1+\delta}\})$ running time.

In this section, we assume that individual machines have space $\Theta(n^\delta)$ where $\delta$ is some constant $0 < \delta < 1$. Given this setting, there are several challenges associated with this problem.

**Challenge 5.2.** The entire subgraph neighborhood of a vertex may not fit on a single machine. This means that all triangles incident to a particular vertex cannot be counted on one machine. Even if we are considering vertices with degree at most $\alpha$, it is possible that $\alpha > n^\delta$. Thus, we need to have a way to count triangles efficiently when the neighborhood of a vertex is spread across multiple machines.
The second challenge deals with calculating the exact number of triangles.

**Challenge 5.3.** When counting triangles among different machines, over-counting the triangles might occur if, for example, two different machines count the same triangle. We need some way to deal with duplicate counting of the triangles to obtain the exact count of the triangles.

We deal with the above challenges in our procedures below. We assume in our algorithm that each vertex can access its neighbors in $O(1)$ rounds of communication; such can be ensured via standard MPC techniques. Let $d_Q(v)$ be the degree of $v$ in the subgraph induced by vertex set $Q$, i.e., in $G[Q]$. Our main algorithm consists of the following Count-Triangles($G$) procedure.

**Algorithm 4. Count-Triangles**

1: function Count-Triangles($G = (V,E)$)
2: Let $Q_i$ be the set of vertices that have not yet been processed by iteration $i$. Initially set $Q_0 ← V$.
3: Let $T$ be the current count of triangles. Set $T ← 0$.
4: for $i = 0$ to $i = \lceil \log_3/2(\log_2(n)) \rceil$ do
5:   $\gamma_i ← 2^{(3/2)^i} \cdot 2\alpha$.
6:   Let $A_i$ be the list of vertices $v ∈ Q_i$ where $d_{Q_i}(v) ≤ \gamma_i$. Set $Q_{i+1} ← Q_i \setminus A_i$.
7:   for $v ∈ A_i$ do
8:     Retrieve all neighbors of $v$. Let this list of $v$’s neighbors be $L_v$.
9:     Send each of $v$’s neighbors a copy of $L_v$.
10:   parfor $w ∈ Q_i$ do
11:     Let $L_w = \bigcup_{v \in (N(w) \cap A_i)} L_v$ be the union of neighbor lists received by $w$.
12:     Set $T ← T + \text{Find-Triangles}(w, L_w)$. $\triangleright$ Find-Triangles$(w, L_w)$ returns count of all triangles incident to $w$ and $A_i$.
13:   end parfor
14: Return $T$.

**5.1 MPC Implementation Details**

In order to implement Count-Triangles($G$) in the MPC model, we define our Find-Triangles($A_i, L$) procedure and provide additional details on sending and storing neighbor lists across different machines. We define high-degree vertices to be the set of vertices whose degree is $> \gamma$ and low-degree vertices to be ones whose degree is $\leq \gamma$ (for some $\gamma$ defined in our algorithm). We now define the function Find-Triangles($A_i, L$) which we use in our above procedure.

**Algorithm 5. Find-Triangles**

1: function Find-Triangles($w, L_w$)
2: Set the number of triangles $T_i ← 0$.
3: Sort all elements in $(L_w ∪ N(w))$ lexicographically where $N(w) ∈ Q_i$ using the procedure given in Lemma 4.3 of [GSZ11]. Let this sorted list of all elements be $S$.
4: Count the duplicates in $S$ using Theorem 2.1 as described below. $^a$
5: Return the total count of duplicates as the number of triangles $T_i$.

$^a$Some care must be taken here to account for over-counting of triangles (since a distinct triangle can show
up as several counted duplicates). We describe our MPC implementation in Algorithm 6 that corrects for the over-counting.

Allocating machines for sorting Since each \( v \in Q_i \) could have multiple neighbors whose degrees are \( \leq \gamma \), the total size of all neighbor lists \( v \) receives could exceed their allowed space \( \Theta(n^\delta) \). Thus, we allocate \( \tilde{O}(\gamma d Q_i(v)n^\delta) \) machines for each vertex \( v \in Q_i \) to store all neighbor lists that \( v \) receives.

Details about finding duplicate elements using Theorem 2.1 \begin{algorithm}
\begin{algorithmic}[1]
\Function{Find-Triangles-Exact}{\( w, \mathcal{L}_w \)}
\State Set the number of triangles \( T_i \leftarrow 0 \).
\State Sort all elements in \((\mathcal{L}_w \cup N(w))\) lexicographically where \( N(w) \in Q_i \) using the procedure given in Lemma 4.3 of \[GSZ11\]. Let this sorted list of all elements be \( S \).
\State Count the duplicates in \( S \) using Theorem 2.1.
\ParFor{all \( v \in N(w) \)}
\State Let \( R \) be the number of duplicates of \( v \) returned by Theorem 2.1.
\If{\( d_{Q_i}(v) > \gamma_i \) and \( d_{Q_i}(w) > \gamma_i \)}
\State Increment \( T_i \leftarrow T_i + \frac{R-1}{2} \).
\ElsIf{\( d_{Q_i}(v) > \gamma_i \) and \( d_{Q_i}(w) \leq \gamma_i \) or \( d_{Q_i}(v) \leq \gamma_i \) and \( d_{Q_i}(w) > \gamma_i \)}
\State Increment \( T_i \leftarrow T_i + \frac{R-1}{4} \).
\Else
\State Increment \( T_i \leftarrow T_i + \frac{R-1}{6} \).
\EndIf
\EndParFor
\State Return \( T_i \).
\EndFunction
\end{algorithmic}
\end{algorithm}

\begin{algorithm}
\caption{Find-Triangles-Exact}
\begin{algorithmic}
\State Find-Triangles\( (w, \mathcal{L}_w) \) finds triangles by counting the number of duplicates that occur between elements in lists. Theorem 2.1 provides a MPC implementation for finding the count of all occurrences of every element in a sorted list. Provided a sorted list of neighbors of \( v \in Q_i \) and neighbor lists in \( \mathcal{L}_v \), this function counts the number of intersections between a neighbor list sent to \( v \) and the neighbors of \( v \). Every intersection indicates the existence of a triangle. As given, \( \text{Find-Triangles}(w, \mathcal{L}_w) \) returns a \( 6 \)-approximation of the number of triangles in any graph. We provide a detailed and somewhat more complicated algorithm \( \text{Find-Triangles-Exact}(w, \mathcal{L}_w) \) that accounts for over-counting of triangles and returns the exact number of triangles.

Since Theorem 2.1 returns the total count of each element, we subtract the value returned by 1 to obtain the number of intersections. Finally, each triangle containing one low-degree vertex will be counted twice, each containing two low-degree vertices will be counted 4 times, and each containing three low-degree vertices will be counted 6 times. Thus, we need to divide the counts by 2, 4, and 6, respectively, to obtain the exact count of unique triangles.

Substituting \( \text{Find-Triangles-Exact} \) in \( \text{Count-Triangles} \) finds the exact count of triangles in graphs with arboricity \( \alpha \) using \( O(m\alpha) \) total space. The complete analysis that proves Theorem 5.1 is given in Section 5.2.

We provide two additional extensions of our triangle counting algorithm to counting \( k \)-cliques:
Theorem 5.4. Given a graph $G = (V, E)$ with arboricity $\alpha$, we can count all $k$-cliques in $O(m\alpha^{k-2})$ total space, $O_\delta(\log \log n)$ rounds, on machines with $O(n^2\delta)$ space for any $0 < \delta < 1$.

We can prove a stronger result when we have some bound on the arboricity of our input graph. Namely, if $\alpha = O(n^{\delta'/2})$ for any $\delta' < \delta$, then we obtain the following result:

Theorem 5.5. Given a graph $G = (V, E)$ with arboricity $\alpha$ where $\alpha = O(n^{\delta'/2})$ for any $\delta' < \delta$, we can count all $k$-cliques in $O(n\alpha^2)$ total space and $O_\delta(\log \log n)$ rounds, on machines with $O(n\delta)$ space for any $0 < \delta < 1$.

The proofs of these theorems are provided in Section 6.

5.2 Exact Triangle Counting in $O(m\alpha)$ Total Space Analysis

We start by observing that the value of $\delta$ during iteration $i$ of Algorithm 4 is $\gamma_i = 2^{(3/2)i} \cdot (2\alpha)$. This allows us to bound the number of vertices remaining at the beginning of the $i$th iteration.

Lemma 5.6. At the beginning of iteration $i$ of Count-Triangles, given $\gamma_i$, the number of remaining vertices $N_i = |Q_i|$ is at most $n \cdot 2^{2((3/2)^i - 1)}$.

Proof. Let $N_i$ be the number of vertices in $Q_i$ at the beginning of iteration $i$. Since the subgraph induced by $Q_i$ must have arboricity bounded by $\alpha$, we can bound the total degree of $Q_i$,

$$\sum_{v \in Q_i} d_{Q_i}(v) < 2\alpha|Q_i| = 2N_i\alpha.$$

At the end of the iteration, we only keep the vertices in $Q_{i+1} = \{v \in Q_i \mid d_{Q_i}(v) > \gamma_i\}$. If we assume that $|Q_{i+1}| > \frac{N_i}{\gamma_i/(2\alpha)}$, then we obtain a contradiction since this implies that

$$\sum_{v \in Q_{i+1}} d_{Q_i}(v) > |Q_{i+1}| \cdot \gamma_i > 2N_i\alpha > \sum_{v \in Q_i} d_{Q_i}(v).$$

Then, the number of remaining vertices follows directly from the above by induction on $i$ with base case $N_1 = n$,

$$N_i \leq \frac{N_{i-1}}{\gamma_i/(2\alpha)} = \frac{N_{i-1}}{2^{(3/2)i-1}} = \frac{n}{\prod_{j=0}^{i-1} 2^{(3/2)^j}} = \frac{n}{2^{2((3/2)^i - 1)}}, \quad \Box$$

We can show a similar statement for the number of edges that remain at the start of the $i$th iteration.

Lemma 5.7. At the beginning of iteration $i$ of Count-Triangles, given $\gamma_i$, the number of remaining edges $m_i$ is at most $m_i \leq \frac{m}{2^{2((3/2)^i - 1)}}$.

Proof. The number of vertices remaining at the beginning of iteration $i$ is given by $|Q_i|$. Thus, because the arboricity of our graph is $\alpha$, we can upper bound $m_i$ by

$$m_i \leq |Q_i|\alpha.$$
Then, we can also lower bound the number of edges at the beginning of iteration \( i - 1 \) since the vertices that remain at the beginning of round \( i \) are ones which have greater than \( \gamma_{i-1} \) degree,

\[
m_{i-1} \geq \frac{1}{2} \sum_{v \in Q_{i-1}} d_{Q_{i-1}}(v) \geq \frac{1}{2} |Q_i| \gamma_{i-1}.
\]

Thus, we conclude that \( m_i \leq \frac{2m_{i-1}}{\gamma_{i-1}} \). By induction on \( i \) with base case \( m_0 = m \), we obtain,

\[
m_i \leq 2\alpha \left( \frac{m_{i-1}}{\gamma_{i-1}} \right) = \frac{m}{\prod_{j=0}^{i-2} 2^{(3/2)^j}} = \frac{m}{2^{2((3/2)^{i-1}-1)}}.
\]

**Lemma 5.8.** \( \text{Count-Triangles}(G) \) uses \( O(m \alpha) \) total space when run on a graph \( G \) with arboricity \( \alpha \).

**Proof.** The total space the algorithm requires is the sum of the space necessary for storing the neighbor lists sent by all vertices with degree \( \leq \gamma_i \) and the space necessary for all vertices to store their own neighbor lists. The total space necessary for each vertex to store its own neighbor list is \( O(m) \).

Now we compute the total space used by the algorithm during iteration \( i \). The number of vertices in \( Q_i \) at the beginning of this iteration is at most \( N_i = \frac{n}{2^{2((3/2)^i-1)}} \) by Lemma 5.6. Each vertex \( v \) with \( d_{Q_i}(v) \leq \gamma_i \), makes \( d_{Q_i}(v) \) copies of its neighbor list \( (N(v) \cap Q_i) \) and sends each neighbor in \( N(v) \cap Q_i \) a copy of the list. Thus, the total space required by the messages sent by \( v \) is \( d_{Q_i}(v)^2 \leq \gamma_i^2 \). \( v \) sends at most one message of size \( d_{Q_i}(v) \leq \gamma_i \) along each edge \( (v,w) \) for \( w \in N(v) \cap Q_i \). Then, by Lemma 5.7 the total space required by all the low-degree vertices in round \( i \) is at most (assuming at most two messages are sent along each edge):

\[
2m_i \cdot \gamma_i < \frac{m}{2^{2((3/2)^i-1)-1}} \cdot \left[ 2^{(3/2)^i} (2\alpha) \right] = 16m\alpha.
\]

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** By Lemma 5.6, the number of vertices remaining in \( Q_i \) at the beginning of iteration \( i \) is \( \frac{n}{2^{2((3/2)^i-1)}} \). This means that the procedure runs for \( O(\log \log n) \) iterations before there will be no vertices. For each of the \( O(\log \log n) \) iterations, \( \text{Count-Triangles}(G) \) uses \( O(1) \) rounds of communication for the low-degree vertices to send their neighbor lists to their neighbors. The algorithm then calls \( \text{Find-Triangles-Exact}(w, L_w) \) on each vertex \( w \in Q_i \) (in parallel) to find the number of triangles incident to \( w \) and vertices in \( A_i \subseteq Q_i \). \( \text{Find-Triangles-Exact}(w, L_w) \) requires \( O(\log \alpha(m\alpha)) = O(1/\delta) \) rounds by Lemma 4.3 of [GSZ11] and Theorem 2.1. Therefore, the total number of rounds required by \( \text{Count-Triangles}(G) \) is \( O\left( \log \log n \right) = O(\delta \log \log n) \).

6 Extensions to Exact \( k \)-Clique Counting in Graphs with Arboricity \( \alpha \)

In this section, we briefly provide two algorithms for exact counting of \( k \)-cliques (where \( k \) is constant) in graphs with arboricity \( \alpha \). The first is an extension of our exact triangle counting result given
in Section 5. The second is a query-based algorithm where the neighborhood of a low-degree vertex is constructed on a single machine via edge queries. In this case, the triangles incident to any given low-degree vertex can be counted on the same machine.

6.1 Exact \(k\)-Clique Counting

**Exact \(k\)-Clique Counting in \(O(m\alpha^{k-2})\) Total Space and \(O(\delta \log \log n)\) Rounds** We extend our algorithm given in Section 5 to exactly count \(k\)-cliques (where \(k\) is constant) in \(O(m\alpha^{k-2})\) total space and \(O(\delta \log \log n)\) rounds. Given a graph \(G = (V, E)\) with arboricity \(\alpha\), the idea behind the algorithm is the following: let \(G_k = (V_k \cup V, E_k \cup E)\) represent the graph where the set of vertices \(V_k\) represents each \(k\)-clique in \(G\). Let \(K(u)\) represent the \(K_k \in G\) represented by \(u \in V_k\). \(E_k\) consists of edges between vertices in \(V_k\) and vertices in \(V\). An edge \(e(u, v)\) exists in \(E_k\) iff \(K(u) \cup v\) is a \((k + 1)\)-clique in \(G\). We begin by constructing \(G_2\) using our exact triangle counting algorithm. Then, we recursively construct \(G_i\) by using our exact triangle counting algorithm and \(G_{i-1}\). Once we construct \(G_{k-2}\), we obtain our final count of the number of \(k\)-cliques by running our exact triangle counting algorithm one last time. The total space used is dominated by running the triangle counting algorithm on \(G_{k-2}\), which uses \(O(m\alpha^{k-2})\) total space. Since we run the triangle counting algorithm \(O(k)\) times and \(k\) is a constant, the total number of rounds of communication necessary is \(O(\delta \log \log n)\) rounds. This detailed algorithm is given below.

Below, we describe our \(O(\alpha^{k-1})\) total space, \(O(\log \log n)\) rounds exact \(k\)-clique counting algorithm that can be run on machines with space \(O(\delta)\). Calling COUNT-\(k\)-CLIQUE\((G, k, k)\) for any given graph \(G = (V, E)\) returns the number of \(k\)-cliques in \(G\).

**Algorithm 7. \(k\)-Clique-Counting**

```
1: function COUNT-k-CLIQUE\((G = (V, E), k, k')\)
2:     if \(k \leq 1\) then
3:         Return \((|N|, G)\)
4:     else
5:         \((x, G_{k-1}) \leftarrow COUNT-k-CLIQUE\((G, k - 1, k')\)\)
6:         \(T \leftarrow ENUMERATE-TRIANGLES\((G_{k-1})\)\). Let \(T\) be the set of all enumerated triangles.
7:         Initialize sets \(V_k \leftarrow \emptyset\) and \(E_k \leftarrow \emptyset\).
8:         parfor \(t \in T\) do
9:             Let \(K(t)\) represent the set of vertices in \(V\) composing the clique represented by \(t \in T\).
10:            parfor \(v \in K(t)\) do
11:                Let \(v'(S)\) be a vertex \(v\) representing a set of vertices \(S\). In other words, \(K(v) = K(v'(S)) = S\).
12:                   \(V_k \leftarrow V_k \cup v'(K(t) \setminus v)\).
13:                   \(E_k \leftarrow E_k \cup (v, v'(K(t) \setminus v))\).
14:            end parfor
15:        end parfor
16:     if \(k = k' - 2\) then
17:         Return \(|T|\).
18:     else
19:         Return \((|V_k|, G_k(V \cup V_k, E \cup E_k))\).
```
6.2 MPC Implementation

To implement COUNT-$k$-CLIQUES in the MPC model, we must be able to create the graph $G_2, \ldots, G_{k-1}$ efficiently in our given space and rounds. The crux of this algorithm is the procedure for enumerating all triangles given a set of vertices in $G$ where $d(v) \leq \gamma$ for all $v \in A$. To do the triangle enumeration, we prove Lemma 6.1 which can enumerate all such triangles incident to $A$ in $O(m\gamma)$ total space, $O_{\delta}(1)$ rounds given machines with space $O(n^{2\delta})$.

**Lemma 6.1.** Given a graph $G$, a constant integer $k \geq 2$, and a subset $A \subseteq G$ of vertices such that for every $v \in A$, $d(v) \leq \gamma$, we can generate all triangles in $G$ that are incident to vertices in $A$ in $O_{\delta}(1)$ rounds, $O(n^{2\delta})$ space per machine, and $O(m\gamma)$ total space.

**Proof.** Let $R$ be the set of machines holding the edges incident to $A$. Here too, similarly to the proof of Lemma 7.5, it will be easier to think of each machine $M$ as a set of $n^{\alpha}$ parts, so that each edge incident to a vertex in $A$, resides on a single part. We duplicate each such part, holding some neighbor of $A$, $\alpha$ times, using Lemma 2.3. (We will actually duplicate machines, but, again, think of the duplicated machines as a collection of duplicated parts.) By Lemma 2.3, this takes $O(\log_{n^{\alpha}} n) = O_{\delta}(1)$ rounds. Fix some vertex $v \in A$ and assume that $u \in N(v)$ resides on part $P_i(v)$. After the duplication step, there are $\alpha$ copies of each part. We denote these copies $P_{i,1}(v), \ldots, P_{i,\alpha}(v)$. All parts $P_{i,j}(v)$ where $j \in [\alpha]$ and $v \in A$ then asks for $v$’s $i$-th neighbor in $O(1)$ rounds of communication. Now, each part $P_{i,j}(v)$ creates $O(1)$ edge queries to check whether its vertices form a triangle. All of the queries generated by all parts can be answered in parallel using Lemma 2.2 in $O_{\delta}(1)$ rounds. Then each part that discovered a triangle incident to $v$ adds it to a list $K$. Now we sort the list $K$ and remove any duplicated triangles, so that the list only holds a single copy of every clique incident to some vertex in $A$. The total round complexity is $O_{\delta}(1)$ due to the duplications, sorting, and answering the queries. The space per machine is $O(n^{2\delta})$ and the total memory is $O(m\alpha)$ as each machine was duplicated $\alpha$ times.

Using Lemma 6.1, we can now prove the space usage and round complexity of ENUMERATE-TRIANGLES.

**Lemma 6.2.** Given a graph $G = (V, E)$ with arboricity $\alpha$, ENUMERATE-TRIANGLES($G$) uses $O(m\alpha^2)$ total space, $O_{\delta}(1)$ rounds on machines with $O(n^{2\delta})$ space.

**Proof.** By Lemma 2.6, the number of vertices remaining in $Q_i$ at the beginning of the $i$-th iteration of ENUMERATE-TRIANGLES is at most $n / 2^{((3/2)^i - 1)}$. By Lemma 6.1, the total space usage of
Count-

Thus, by Lemma 6.2 and Lemma 6.3, \(i\) vertices and at least \(3(\alpha + 1)\cdot 2\alpha\) edges. This implies, by the definition of arboricity, that the arboricity of \(G_i\) is greater than \(3(i + 1)\cdot 2\alpha\) edges.

The number of rounds required by this algorithm is \(O(\log \log n) \cdot O_\delta(1) = O_\delta(\log \log n)\).

Given the total space usage and number of rounds required by \textsc{Enumerate-Triangles}, we can now prove the total space usage and number of rounds required by \textsc{Count-k-Cliques}. But first, we show that for any graph \(G = (V, E)\) with arboricity \(\alpha\), all graphs \(G_1, \ldots, G_{k-1}\) created by \textsc{Count-k-Cliques} has arboricity \(O(\alpha)\) for constant \(k\).

\textbf{Lemma 6.3.} Given a graph \(G = (V, E)\) with arboricity \(\alpha\) as input to \textsc{Enumerate-Triangles}, all graphs \(G_1, \ldots, G_{k-1}\) generated by the procedure have arboricity \(O(\alpha)\) for constant \(k\).

\textbf{Proof.} We prove this lemma via induction. In the base case, \(G_1 = G\) and so \(G_1\) has arboricity \(\alpha\). Now we assume that \(G_i\) for \(i \in [k-1]\) has arboricity \(O(\alpha)\) (for constant \(i\)) and show that \(G_{i+1}\) has \(O(\alpha)\) arboricity. Suppose that \(G_i\) has arboricity \(c\alpha\) for some constant \(c\). We prove via contradiction that the arboricity of \(G_{i+1}\) is upper bounded by \(3(i + 1)c\alpha\). Suppose for the sake of contradiction that the arboricity of \(G_{i+1}\) is greater than \(3(i + 1)c\alpha\). Then, there must exist a subgraph, \(G_{i+1}[V']\) for some vertex set, \(V'\), of \(G_{i+1}\) that contains greater than \(3(i + 1)c\alpha|V'|\) edges (by definition of arboricity). We now convert this subgraph \(G_{i+1}[V']\) to a subgraph in \(G_i\). Every vertex in \(V'\) maps to at most \(i\) pairs of vertices in \(G_i\) connected by an edge. Every edge in \(G_{i+1}[V']\) maps to at least 1 edge. Thus, the subgraph in \(G_i\) that \(G_{i+1}[V']\) maps to contains at most \(2i|V'|\) vertices and at least \(3(i + 1)c\alpha|V'|\) edges. This implies, by the definition of arboricity, that the arboricity of \(G_i\) is \(\geq \frac{3(i + 1)c\alpha|V'|}{2i|V'|} > c\alpha\), a contradiction. Hence, the arboricity of \(G_{i+1}\) is at most \(3(i + 1)c\alpha\) and we have proven that the arboricity of \(G_{i+1}\) is \(O(\alpha)\) for constant \(k\). By induction, all graphs \(G_1, \ldots, G_{k-1}\) have arboricity \(O(\alpha)\).

Now we prove our final theorem of the space and round complexity of \textsc{Count-k-Cliques}.

\textbf{Proof of Theorem 5.4.} The number of \(i\)-cliques in a graph with arboricity \(\alpha\) is at most \(O(m\alpha^{i-2})\). Thus, by Lemma 6.2 and Lemma 6.3, \textsc{Count-k-Cliques} during the \(i\)-th call uses \(O(m\alpha^i)\) total space, \(O_\delta(\log \log n)\) rounds. Thus, \textsc{Count-k-Cliques} uses \(O(m\alpha^{k-2})\) space, \(O_\delta(\log \log n)\) rounds given machines with \(O(n^2\delta)\) space to count \(k\)-cliques given that the procedure terminates on the \((k-2)\)-th iteration. \hfill \Box

\subsection{Exact \(k\)-Clique Counting in \(O(n\alpha^2)\) Total Space and \(O_\delta(\log \log n)\) Rounds}

We can improve on the total space usage if we are given machines where the memory for each individual machine satisfies \(\alpha < n^{\delta'/2}\) where \(\delta' < \delta\). In this case, we obtain an algorithm that counts the number of \(k\)-cliques in \(G\) using \(O(n\alpha^2)\) total space and \(O_\delta(\log \log n)\) communication rounds.

The entire neighborhood of any vertex with degree \(\leq n^{\delta'/2}\) can fit on one machine. Suppose that \(\alpha < n^{\delta'/2}\) where \(\delta' < \delta\), then, there will always exist vertices that have degree \(\leq n^{\delta'/2}\). Our algorithm proceeds as follows:

\begin{itemize}
    \item enumerating all triangles incident to \(A_i\) is \(O(m\gamma_i) = O\left(m \cdot \left(2^{(3/2)i} \cdot 2\alpha\right)\right)\).
    \item The summation of the space used for all \(i\) is then:
      \[
      \sum_{i=0}^{\lfloor \log_2\left(\log_2(n)\right) \rfloor} \left(\frac{n}{2^{2((3/2)i-1)}}\right) \cdot \left(2^{\left(3/2\right)i} \cdot 2\alpha^2\right) = O(n\alpha^2).
      \]
    \item The number of rounds required by this algorithm is \(O(\log \log n) \cdot O_\delta(1) = O_\delta(\log \log n)\).
\end{itemize}
Algorithm 9. Count-Cliques

1: function Count-Cliques(G = (V,E))
2: Let Q_i be the set of vertices that have not yet been processed by iteration i. Initially set $Q_0 ← V$.
3: Let $C$ be the current count of cliques. Set $C ← 0$.
4: for $i = 0$ to $i = \left\lceil \log_{3/2}(\log_2(n)) \right\rceil$ do
5: \hfill $γ_i ← 2^{(3/2)^i \cdot 2α}$.
6: Let $A_i$ be the list of vertices where $d_{Q_i}(v) ≤ \min(cnδ/2, γ_i)$ for some constant $c$.
7: Set $Q_{i+1} ← Q_i \setminus A_i$.
8: parfor $v ∈ A_i$ do
9: \hfill Retrieve all neighbors of $v$. Let this list of $v$’s neighbors be $L_v$.
10: \hfill Query for all pairs $u,v ∈ L_v$ to determine whether edge $(u,v)$ exist. Retrieve all edges that exist.
11: \hfill Count the number of triangles $T_v$ incident to $v$, accounting for duplicates.
12: \hfill $T ← T + T_v$.
13: end parfor

6.4 MPC Implementation Details

Accounting for Duplicates  We account for duplicates by counting for each iteration $i$ how many triangles on each machine contains 1, 2 or 3 vertices which have degree $≤ \min(cnδ/2, γ_i)$ (again we call these vertices low-degree). We multiply the count of triangles which have $t ≥ 2$ low-degree vertices by $\frac{1}{t}$ to correct for over-counting due to multiple low-degree vertices performing the count on the same triangle. Each machine can retrieve the degrees of vertices in it in $O_δ(1)$ rounds and such information can be stored on the machine given sufficiently small constant $c$ in COUNT-Clique.

Proof of Theorem 5.5. Since we are considering vertices with degree at most $\min(cnδ/2, γ_i)$, by Lemma 5.8, the total space used by our algorithm during any iteration $i$ is

$$N_i \cdot \left( \min(cnδ/2, γ_i) \right)^2 < 16na^2.$$

By Lemma 2.2, we query for whether each of the $\min(cnδ/2, γ_i)^2$ potential edges on each machine is an edge in $G$ in parallel using $O(na^2)$ total space and $O_δ(1)$ rounds.

If $γ_i < cnδ/2$ for all iterations $i$, then by Theorem 5.1, the number of communication rounds required by COUNT-Clique is $O_δ(\log \log n)$. If, on the other hand, $cnδ/2 < γ_i$, then the number of vertices remaining in $Q_i$ decreases by a factor of $cnδ/2$ every round. Thus, the number of rounds required in this case is $O(\frac{2+δ'}{δ})$. Since we assume $δ'$ and $δ$ are constants, the number of communication rounds needed by this algorithm is $O_δ(\log \log n)$.

7 Counting subgraphs of size at most 5 in bounded arboricity graphs

In this section, we present a procedure that for every subgraph $H$ for $|H| ≤ 5$, counts, with high probability, the exact number of occurrences of $H$ in $G$ in $O(\sqrt{\log n})$ rounds and $O(ma^3)$ total
memory, where as before, $\alpha$ is an upper bound on the arboricity of $G$ (strictly speaking, we will have $\alpha \leq 5\alpha(G)$ but as this does not affect the asymptotic bounds, it is easier to just relate to it as the exact arboricity). The procedure is based on a recent paper by Bera, Pashanasangi and Seshadhri [BPS20] (henceforth BPS) which presented an $O(m\alpha^3)$ time and space algorithm for the same task in the sequential model. We will start by a short description of the BPS result, and then continue to explain how to implement it in the MPC model.

**Challenges** The major challenge we face here in implementing BPS in the MPC model is dealing with finding and storing copies of small (constant-sized) subgraphs. This is a challenge due to the fact that an entire neighborhood of a vertex $v$ may not fit on one machine (recall that we have no restrictions on the $\delta$ in $O(n^\delta)$ machine size). Thus, we cannot compute all such small subgraphs on one machine. However, if not done carefully, computing small subgraphs across many machines could potentially result in many rounds of computation (since we potentially have to try all combinations of vertices in a neighborhood). We solve this issue in Section 7.2 by formulating a procedure in which we carefully duplicate neighborhoods of vertices across machines.

### 7.1 The BPS algorithm

BPS generalize the ideas of Chiba and Nisheziki [CN85] for counting constant-size-cliques and 4-cycles in the classical sequential model to counting all subgraphs of up to 5 nodes in $O(n + ma^3)$ time. Let $H$ be the subgraph in question. The main idea of BPS is as follows. First, they consider a degeneracy ordering of $G$. The degeneracy ordering of a graph is an acyclic orientation of $G$, denoted $\overset{\rightarrow}{G}$, where each vertex has at most $O(\alpha)$ outgoing neighbors (see Definition 7.2 for a formal definition). Then they consider all acyclic orientations of $H$ (up to isomorphisms), and for each such acyclic orientation $\overset{\rightarrow}{H}$, they count the number of occurrences of $\overset{\rightarrow}{H}$ in $\overset{\rightarrow}{G}$, as described next. They compute what they refer to as a largest directed rooted tree subgraph of $\overset{\rightarrow}{H}$, denoted DRTS $\overset{\rightarrow}{T}$. That is, the DRTS $\overset{\rightarrow}{T}$ is a largest (in number of vertices) tree that is contained in $\overset{\rightarrow}{H}$ such that all of the edges are directed away from the root of $\overset{\rightarrow}{T}$. Given a DRTS $\overset{\rightarrow}{T}$, they proceed by looking for all copies of $\overset{\rightarrow}{T}$ in $\overset{\rightarrow}{G}$. Once a copy of $\overset{\rightarrow}{T}$ is found, it needs to be verified whether it can be extended to a copy of $\overset{\rightarrow}{H}$ in $\overset{\rightarrow}{G}$. This verification is based on the observation that for any directed subgraph $\overset{\rightarrow}{H}$ on at most 5 vertices, and for every largest directed rooted tree $\overset{\rightarrow}{T}$ of $\overset{\rightarrow}{H}$, the complement of $\overset{\rightarrow}{T}$ in $\overset{\rightarrow}{H}$ is a collection of rooted paths and stars (this does not hold for $H$ that are stars, but stars can be dealt with differently). Therefore, all potential completions of a copy of $\overset{\rightarrow}{T}$ to $\overset{\rightarrow}{H}$ in $\overset{\rightarrow}{G}$ can be computed and hashed in time $O(m \cdot \text{poly}(\alpha))$. See figure below for an illustration of a possible $\overset{\rightarrow}{H}$ and its DRTS $\overset{\rightarrow}{T}$ (adapted from [BPS20]). Hence, whenever a copy of $\overset{\rightarrow}{T}$ is discovered in $\overset{\rightarrow}{G}$, it can be verified in $O(1/\delta)$ whether this copy can be extended to $\overset{\rightarrow}{H}$. Since all copies of $\overset{\rightarrow}{T}$ can be enumerated in $O(ma^3)$ time, the overall algorithm takes $O(ma^3)$ time.

### 7.2 Implementation in the MPC model

We start with some definitions and notations.

**Notation 7.1** (Outgoing neighbors and out-degree). Let $\overset{\rightarrow}{G} = (V, \overset{\rightarrow}{E})$ be a directed graph. For a vertex $v \in V$, We denote by $N^+(v)$ its set of outgoing neighbors, and by $d^+(v) = |N^+(v)|$ its outgoing degree or out-degree.

**Definition 7.2** (Degeneracy and degeneracy ordering.). A degeneracy ordering of a graph $G$, is an ordering obtained by repeatedly removing the minimal degree vertex and all the edges incident
A vertex $u$ precedes a vertex $v$ in this ordering, $u < v$, if $u$ was removed before $v$.

The degeneracy of a graph $G$ is then the maximal outgoing degree over all vertices in a degeneracy ordering of $G$.

The following is a folklore result stating the relation between the arboricity of a graph and its degeneracy.

**Theorem 7.3.** For every graph $G$ with degeneracy $\kappa(G)$ and arboricity $\alpha(G)$,

$$\frac{\kappa(G)}{2} \leq \alpha(G) \leq \kappa(G).$$

We rely on the following theorems.

**Theorem 7.4** (Thm 2 in [GLM19].) Given a graph $G$ with degeneracy value $\kappa$, it outputs a with high probability an $O(\kappa)$-orientation of $G$. That is, an orientation of $G$ where each vertex has out-degree at most $O(\kappa)$. The algorithm performs $O(\sqrt{\log n \cdot \log \log n})$ rounds, uses $\tilde{O}(n^\delta)$ space per machine, for an arbitrary constant $\delta \in (0, 1)$, and the total memory is $O(\max\{m, n^{1+\delta}\})$.

We are now ready to present our key lemma in this section.

**Lemma 7.5.** Let $\vec{T}$ be a directed graph over $m$ edges such that each vertex has out-degree at most $\alpha$. Let $\vec{T}_t$ be a directed rooted tree (henceforth DRT) of size $t \geq 2$. We can list all copies of $\vec{T}_t$ in $G$ in $O(1/\delta)$ rounds, $O(n^2\delta)$ space per machine, and $O(m \cdot \alpha t - 2)$ total memory.

**Proof.** Let $a_1, \ldots, a_t$ denote the vertices of $\vec{T}_t$, where $a_1$ is the root, and $a_i$ is the $i$th vertex with respect to the BFS ordering of $\vec{T}_t$. Let $\vec{T}_i$ denote $\vec{T}[\{a_0, \ldots, a_i\}]$.

We prove the claim by induction on $t$. For $t = 2$, all edges in $G$ are copies of $\vec{T}^2$, so the claim holds trivially.

Assume that the claim holds for $i$, and we now prove it for $i + 1$. By the assumption, in $O(1/\delta)$ rounds and $O(m\alpha^{i-2})$ total memory, all copies of $\vec{T}^i$ can be listed. We will show that we can use these copies to find all copies of $\vec{T}^{i+1}$ in $O(1/\delta)$ rounds and $O(m\alpha^{i-1})$ memory. Recall that we have machines with $O(n^{2\delta})$ memory. We will divide the copies among the machines, so that each machine only holds $O(n^\delta)$ copies. Let $M$ be some machine containing copies $\tau_1, \ldots, \tau_{n^\delta}$ of $\vec{T}_i$. It will be easier to think of $M$ as a collection of $n^\delta$ constant memory parts, each holding a single copy of $\vec{T}_i$. Consider a specific copy $\tau$ of $\vec{T}_i$ and let $P_{\tau}$ denote the part storing that copy. Let $a_p$ denote the vertex in $\vec{T}$ that is the parent of $a_{i+1}$, and let $u$ denote the vertex in $\tau$ that is mapped to $a_p$. We would like to create all tuples $(\tau, w)$, where $w \in \alpha^+(u) \setminus \tau$ and $w$ can be mapped to $a_{i+1}$. In order to achieve this we duplicate $P_{\tau} \alpha$ times, to get copies $P_{\tau,1}, \ldots, P_{\tau,\alpha}$. Each part $P_{\tau,i}$ then...

Figure 1: From left to right: A subgraph $H$; a possible directed copy of $H$; the DRTS in green, and its complement with respect to $H$ in red. Based on a figure from BPS [BPS20].
asks \( u \) for its \( i \)th neighbor \( w \). \( P_{(\tau, i)} \) then checks if \( \tau \) can be extended to \( \vec{T}_{t_1 + 1} \) using \( w \). If \((\tau, w)\) is a copy of \( \vec{T}_{t_1 + 1} \), then the part creates the tuple \((\tau, w)\). All the the duplications above can be done in parallel to all copies of \( \vec{T} \) residing on a single machine, so that in total each machine is duplicated \( \alpha \) time. Since each machine has \( O(n^\delta) \) information, and \( O(n^{2\delta}) \) space, by Lemma 2.3, this process takes \( O(\log n \cdot \alpha) = O(1/\delta) \) rounds. Furthermore, as each machine is duplicated \( \alpha \) times, the amount of total memory increases by a factor of \( \alpha \).

Hence, at the end of the process, all copies of \( \vec{T} \) are generated, the total round complexity is \( O(1/\delta) \), and the total memory is \( O(m\alpha^{t-2}) \).

For a directed graph \( \vec{G} \), we consider the following lists of key-value pairs, as described in Lemma 15 in [BPS20].

- \( \mathcal{HM}_1 \) : \((((u, v), 1)) for all \((u, v) \in E(\vec{G})\).
- \( \mathcal{HM}_2 \) : \((S, r) \forall S \subseteq V(\vec{G}) \) such that \( 1 \leq |S| \leq 4 \) and \( r \) is the number of vertices \( u \) such that \( S \subseteq N^+(u) \).
- \( \mathcal{HM}_3 \) : \(((S_1, S_2, \ell)) \forall S_1, S_2 \subseteq V(\vec{G}) \), where \( 1 \leq |S_1 \cup S_2| \leq 3 \), and \( \ell \) is the number of edges \( e = (u, v) \in E(\vec{G}) \) such that \( S_1 \subseteq N^+(u) \) and \( S_2 \subseteq N^+(v) \).

**Lemma 7.6.** Let \( \vec{G} \) be a directed graph with \( m \) edges, such that for every \( v \in V(\vec{G}) \), \( d^+(v) \leq \alpha \). The lists \( \mathcal{HM}_1, \mathcal{HM}_2 \) and \( \mathcal{HM}_3 \) can be computed in \( O(1/\delta) \) rounds and \( O(m\alpha^3) \) total memory.

**Proof.** In order to create \( \mathcal{HM}_1 \), each vertex \( u \) simply adds for each \( v \in \alpha^+(u) \) the pair \(((u, v), 1)) \) to the list. Clearly this can be done in \( O(1) \) rounds, and \( O(m) \) total memory.

We now consider \( \mathcal{HM}_2 \). Fix \( S \), and let \( \vec{T} \) be a DRT which consists of a root and \( |S| \) outgoing neighbors. By Lemma 7.5, we can generate all copies of \( \vec{T} \) in \( O(1/\delta) \) rounds, and \( O(m \cdot \alpha^{S-2}) = O(m \cdot \alpha^2) \) total memory. From each copy \((v, u_1, \ldots, u_{|S|})\) of \( \vec{T} \), we create a tuple \(((u_1, \ldots, u_{|S|}), 1)) \) and add it to a temporary list \( \mathcal{HM}_2' \). Finally, we use Theorem 2.1 to sort this list and aggregate the counts of each set \( \{u_1, \ldots, u_{|S|}\} \), so that for every set \( S \) we create the tuple \((S, \ell)) \) and add it to \( \mathcal{HM}_2 \), where \( \ell \) is the number of occurrences of the tuple \((S, 1)) \) in \( \mathcal{HM}_2' \). By Theorem 2.1, this takes \( O(\log m \cdot m \cdot \alpha^2) = O(1/\delta) \) rounds.

\( \mathcal{HM}_3 \) is constructed similarly. Fix some \( s_1 \) and \( s_2 \) such that \( 1 \leq s_1 + s_2 \leq 3 \), and consider the corresponding DRT \( \vec{T} \). That is, \( \vec{T} \) is a DRT with a vertex \( u \) with \( s_1 \) outgoing neighbors, where one of the neighbors has \( s_2 \) additional outgoing neighbors. This is a DRT over \( |S_1| + |S_2| + 1 \leq 4 \) vertices, so by Lemma 7.5, we can generate all copies in \( O(1/\delta) \) rounds, and \( O(m \cdot \alpha^2) \) total memory. From the list of all copies we can generate \( \mathcal{HM}_3 \), similarly to as described for \( \mathcal{HM}_2 \), in \( O(1/\delta) \) rounds.

We are now ready to prove the main theorem of this section.

**Theorem 7.7.** Let \( G \) be a graph over \( n \) vertices and \( m \) edges. The algorithm of BPS for counting the number of occurrences of a subgraph \( H \) over \( k \leq 5 \) vertices in \( G \) can be implemented in the MPC model with high probability and round complexity \( O(\sqrt{\log n} + 1/\delta) \). The memory requirement per machine is \( O(n^{2\delta}) \) and the total memory is \( O(m\alpha^3) \).

**Proof.** If \( H \) is a star of size \( k \), then the number of occurrence of \( H \) in \( G \) is simply \( \sum_{v \in V} \binom{d(v)}{k} \) where \( \binom{d(v)}{k} = 0 \) for \( k > d(v) \). Therefore, we can generate the degree sequence of \( G \), and then compute the above value in \( O(1) \) rounds. Therefore, we assume that \( H \) is not a star.
The first step in the algorithm of BPS is to direct the graph $G$ according to the degeneracy ordering (see Definition 7.2). We achieve this using the algorithm of [GLM19] described in Theorem 7.4. Note that the algorithm of [GLM19] returns an approximate degeneracy ordering, but as the degeneracy of a graph is at most twice the arboricity, it holds that each vertex has out-degree $O(\alpha)$.

Given the ordering of $\overrightarrow{G}$, the algorithm continues by considering all orientations $\overrightarrow{H}$ of $H$ (up to isomorphisms). For each $\overrightarrow{H}$ it computes the maximal rooted directed tree, DRT, of $\overrightarrow{H}$, denoted $\overrightarrow{T}$. As $H$ is of constant size, this can be computed in $O(1)$ rounds on a single machine.

The next step is to find all copies of $\overrightarrow{T}$ in $\overrightarrow{G}$. By Lemma 7.5, this can be implemented in $O(1/\delta)$ rounds, $O(n^{2\delta})$ space per machine, and $O(m\alpha^2)$ total memory.

Now, for each copy of $\overrightarrow{T}$ in $\overrightarrow{G}$ it needs to be verified if the copy can be completed to a copy of $\overrightarrow{H}$ in $\overrightarrow{G}$. By Lemma 16 in [BPS20], this can be computed in if given query access to $HM_1, HM_2$ and $HM_3$, as defined in Section 7.2. That is, it can be determined if a copy $\tau$ of $\overrightarrow{T}$ using $O(|H|^2) = O(1)$ queries to the lists $HM_1, HM_2$ and $HM_3$. By Lemma 7.6, these lists can be generated in $O(1/\delta)$ rounds, and $O(m\alpha^2)$ total memory. For $i \in [1..3]$, let $Q_i$ denote the set of all queries to list $HM_i$. By [GSZ11], all queries $Q_i$ to $HM_i$ can be answered in time $O(1/\delta)$.

Finally, by Lemma 16 in [BPS20], each $v$ can use the answers to its queries, to compute the number of copies of $\overrightarrow{H}$ it can be extended to. Therefore, by summing over all vertices and over all possible orientations of $H$, and taking into account isomorphisms, we can compute the number of occurrences of $H$ in $\overrightarrow{G}$. The total round complexity is dominated by computing the approximate arboricity orientation of $G$ and the sorting operations. Therefore the round complexity is $O(\sqrt{\log n \log \log n + 1/\delta})$. The space per machine is $O(n^{2\delta})$, and the total memory over all machines is $O(m\alpha^3)$.

\[\square\]

References

[ABB+19a] Sepehr Assadi, MohammadHossein Bateni, Aaron Bernstein, Vahab Mirrokni, and Cliff Stein. Coresets meet edcs: algorithms for matching and vertex cover on massive graphs. In Proceedings 30th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 2019.

[ABB+19b] Sepehr Assadi, MohammadHossein Bateni, Aaron Bernstein, Vahab S. Mirrokni, and Cliff Stein. Coresets meet EDCS: algorithms for matching and vertex cover on massive graphs. In Timothy M. Chan, editor, Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019, pages 1616–1635. SIAM, 2019.

[ABG+18] Maryam Aliakbarpour, Amartya Shankha Biswas, Themis Gouleakis, John Peebles, Ronitt Rubinfeld, and Anak Yodpinyanee. Sublinear-time algorithms for counting star subgraphs via edge sampling. Algorithmica, 80(2):668–697, 2018.

[AG09] Noga Alon and Shai Gutner. Linear time algorithms for finding a dominating set of fixed size in degenerated graphs. Algorithmica, 54(4):544–556, 2009.

[AG15] Kook Jin Ahn and Sudipto Guha. Access to data and number of iterations: Dual primal algorithms for maximum matching under resource constraints. pages 202–211, 2015.
[AKK19] Sepehr Assadi, Michael Kapralov, and Sanjeev Khanna. A simple sublinear-time algorithm for counting arbitrary subgraphs via edge sampling. In ITCS, volume 124 of LIPIcs, pages 6:1–6:20. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2019.

[AKM13] Shaikh Arifuzzaman, Maleq Khan, and Madhav Marathe. Patric: A parallel algorithm for counting triangles in massive networks. In Proceedings of the 22nd ACM international conference on Information & Knowledge Management, pages 529–538, 2013.

[ANOY14] Alexandr Andoni, Aleksandar Nikolov, Krzysztof Onak, and Grigory Yaroslavtsev. Parallel algorithms for geometric graph problems. pages 574–583, 2014.

[Ass17] Sepehr Assadi. Simple round compression for parallel vertex cover. arXiv preprint arXiv:1709.04599, 2017.

[ASS+18a] A. Andoni, Z. Song, C. Stein, Z. Wang, and P. Zhong. Parallel graph connectivity in log diameter rounds. In 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS), pages 674–685, Oct 2018.

[ASS+18b] Alexandr Andoni, Zhao Song, Clifford Stein, Zhengyu Wang, and Peilin Zhong. Parallel graph connectivity in log diameter rounds. In Mikkel Thorup, editor, 59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018, pages 674–685. IEEE Computer Society, 2018.

[ASS+18c] Alexandr Andoni, Clifford Stein, Zhao Song, Zhengyu Wang, and Peilin Zhong. Parallel graph connectivity in log diameter rounds. In Proceedings 59th IEEE Symposium on Foundations of Computer Science (FOCS), pages 674–685, 2018.

[ASW18] Sepehr Assadi, Xiaorui Sun, and Omri Weinstein. Massively parallel algorithms for finding well-connected components in sparse graphs. arXiv preprint arXiv:1805.02974, 2018.

[ASW19] Sepehr Assadi, Xiaorui Sun, and Omri Weinstein. Massively parallel algorithms for finding well-connected components in sparse graphs. In Peter Robinson and Faith Ellen, editors, Proceedings of the 2019 ACM Symposium on Principles of Distributed Computing, PODC 2019, Toronto, ON, Canada, July 29 - August 2, 2019, pages 461–470. ACM, 2019.

[ASZ19] Alexandr Andoni, Clifford Stein, and Peilin Zhong. Log diameter rounds algorithms for 2-vertex and 2-edge connectivity. In Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano Leonardi, editors, 46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece, volume 132 of LIPIcs, pages 14:1–14:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[AYZ97] Noga Alon, Raphael Yuster, and Uri Zwick. Finding and counting given length cycles. Algorithmica, 17(3):209–223, 1997.

[BBD+19] Soheil Behnezhad, Sebastian Brandt, Mahsa Derakhshlan, Manuela Fischer, MohammadTaghi Hajiaghayi, Richard M. Karp, and Jara Uitto. Massively parallel computation of matching and MIS in sparse graphs. In Peter Robinson and Faith Ellen,
[BC17] Suman K Bera and Amit Chakrabarti. Towards tighter space bounds for counting triangles and other substructures in graph streams. In *34th Symposium on Theoretical Aspects of Computer Science*, 2017.

[BDE+19a] Soheil Behnezhad, Laxman Dhulipala, Hossein Esfandiari, Jakub Lacki, and Vahab S. Mirrokni. Near-optimal massively parallel graph connectivity. In David Zuckerman, editor, *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 1615–1636. IEEE Computer Society, 2019.

[BDE+19b] Soheil Behnezhad, Laxman Dhulipala, Hossein Esfandiari, Jakub Lacki, and Vahab S. Mirrokni. Near-optimal massively parallel graph connectivity. In *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 1615–1636, 2019.

[BDH18] Soheil Behnezhad, Mahsa Derakhshan, and MohammadTaghi Hajiaghayi. Semimapreduce meets congested clique. *arXiv preprint arXiv:1802.10297*, 2018.

[BDH+19] Soheil Behnezhad, Mahsa Derakhshan, MohammadTaghi Hajiaghayi, Marina Knittel, and Hamed Saleh. Streaming and massively parallel algorithms for edge coloring. In Michael A. Bender, Ola Svensson, and Grzegorz Herman, editors, *27th Annual European Symposium on Algorithms, ESA 2019, September 9-11, 2019, Munich/Garching, Germany*, volume 144 of *LIPIcs*, pages 15:1–15:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[BEG+18] Mahdi Boroujeni, Soheil Ehsani, Mohammad Ghodsi, MohammadTaghi HajiAghayi, and Saeed Seddighin. Approximating edit distance in truly subquadratic time: quantum and MapReduce. pages 1170–1189, 2018.

[BFU18] Sebastian Brandt, Manuela Fischer, and Jara Uitto. Breaking the linear-memory barrier in MPC: Fast MIS on trees with $n^\epsilon$ memory per machine. *arXiv preprint arXiv:1802.06748*, 2018.

[BHH19] Soheil Behnezhad, MohammadTaghi Hajiaghayi, and David G. Harris. Exponentially faster massively parallel maximal matching. In David Zuckerman, editor, *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 1637–1649. IEEE Computer Society, 2019.

[BHKK09] Andreas Björklund, Thore Husfeldt, Petteri Kaski, and Mikko Koivisto. Counting paths and packings in halves. *Algorithms - ESA 2009*, page 578–586, 2009.

[BKS13] Paul Beame, Paraschos Koutris, and Dan Suciu. Communication steps for parallel query processing. In *Proceedings of the 32Nd ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems (PODS)*, pages 273–284, 2013.

[BKS14] Paul Beame, Paraschos Koutris, and Dan Suciu. Skew in parallel query processing. In *Proceedings of the 33rd ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS)*, pages 212–223, 2014.
[BPS20] Suman K. Bera, Noujan Pashanasangi, and C. Seshadhri. Linear Time Subgraph Counting, Graph Degeneracy, and the Chasm at Size Six. In Thomas Vidick, editor, 11th Innovations in Theoretical Computer Science Conference (ITCS 2020), volume 151 of Leibniz International Proceedings in Informatics (LIPIcs), pages 38:1–38:20, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[BRDR05] Eric Bloedorn, Neal J Rothleder, David DeBarr, and Lowell Rosen. Relational graph analysis with real-world constraints: An application in irs tax fraud detection. Grobelnik et al.[63], 2005.

[BYKS02] Ziv Bar-Yossef, Ravi Kumar, and D Sivakumar. Reductions in streaming algorithms, with an application to counting triangles in graphs. In Proceedings of the thirteenth annual ACM-SIAM symposium on Discrete algorithms, pages 623–632. Society for Industrial and Applied Mathematics, 2002.

[CC11] Shumo Chu and James Cheng. Triangle listing in massive networks and its applications. In Proceedings of the 17th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 672–680, 2011.

[CFG+19] Yi-Jun Chang, Manuela Fischer, Mohsen Ghaffari, Jara Uitto, and Yufan Zheng. The complexity of (Δ+1) coloring in congested clique, massively parallel computation, and centralized local computation. In Peter Robinson and Faith Ellen, editors, Proceedings of the 2019 ACM Symposium on Principles of Distributed Computing, PODC 2019, Toronto, ON, Canada, July 29 - August 2, 2019, pages 471–480. ACM, 2019.

[CLM+18] Artur Czumaj, Jakub Lacki, Aleksander Madry, Slobodan Mitrovic, Krzysztof Onak, and Piotr Sankowski. Round compression for parallel matching algorithms. pages 471–484, 2018.

[CN85] Norishige Chiba and Takao Nishizeki. Arboricity and subgraph listing algorithms. SIAM Journal on computing, 14(1):210–223, 1985.

[Coh09] Jonathan Cohen. Graph twiddling in a mapreduce world. Computing in Science & Engineering, 11(4):29–41, 2009.

[DG08] Jeffrey Dean and Sanjay Ghemawat. Mapreduce: simplified data processing on large clusters. Communications of the ACM, 51(1):107–113, 2008.

[DLP12] Danny Dolev, Christoph Lenzen, and Shir Peled. “tri, tri again”: Finding triangles and small subgraphs in a distributed setting. Distributed Computing, page 195–209, 2012.

[ELRS17] Talya Eden, Amit Levi, Dana Ron, and C Seshadhri. Approximately counting triangles in sublinear time. SIAM Journal on Computing, 46(5):1603–1646, 2017.

[ELS13] David Eppstein, Maarten Löffler, and Darren Strash. Listing all maximal cliques in large sparse real-world graphs. ACM Journal of Experimental Algorithms, 18(3):364–375, 2013.

[ERS20] Talya Eden, Dana Ron, and C. Seshadhri. Faster sublinear approximation of the number of k-cliques in low-arboricity graphs. In Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020, pages 1467–1478, 2020.
[GG06] Gaurav Goel and Jens Gustedt. Bounded arboricity to determine the local structure of sparse graphs. In *Graph-Theoretic Concepts in Computer Science, 32nd International Workshop, WG 2006, Bergen, Norway, June 22-24, 2006, Revised Papers*, pages 159–167, 2006.

[GGK+18] Mohsen Ghaffari, Themis Gouleakis, Christian Konrad, Slobodan Mitrović, and Ronitt Rubinfeld. Improved massively parallel computation algorithms for mis, matching, and vertex cover. arXiv:1802.08237, 2018.

[GKMS19] Buddhima Gamlath, Sagar Kale, Slobodan Mitrovic, and Ola Svensson. Weighted matchings via unweighted augmentations. In Peter Robinson and Faith Ellen, editors, *Proceedings of the 2019 ACM Symposium on Principles of Distributed Computing, PODC 2019, Toronto, ON, Canada, July 29 - August 2, 2019*, pages 491–500. ACM, 2019.

[GKU19] Mohsen Ghaffari, Fabian Kuhn, and Jara Uitto. Conditional hardness results for massively parallel computation from distributed lower bounds. In David Zuckerman, editor, *60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 9-12, 2019*, pages 1650–1663. IEEE Computer Society, 2019.

[GLM19] Mohsen Ghaffari, Silvio Lattanzi, and Slobodan Mitrović. Improved parallel algorithms for density-based network clustering. In Kamalika Chaudhuri and Ruslan Salakhutdinov, editors, *Proceedings of the 36th International Conference on Machine Learning*, volume 97 of *Proceedings of Machine Learning Research*, pages 2201–2210, Long Beach, California, USA, 09–15 Jun 2019. PMLR.

[GNT20] Mohsen Ghaffari, Krzysztof Nowicki, and Mikkel Thorup. Faster algorithms for edge connectivity via random 2-out contractions. In Shuchi Chawla, editor, *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020*, pages 1260–1279. SIAM, 2020.

[GSZ11] Michael T. Goodrich, Nodari Sitchinava, and Qin Zhang. Sorting, searching, and simulation in the mapreduce framework. In *Proceedings of the 22Nd International Conference on Algorithms and Computation*, ISAAC’11, pages 374–383, Berlin, Heidelberg, 2011. Springer-Verlag.

[GU19] Mohsen Ghaffari and Jara Uitto. Sparsifying distributed algorithms with ramifications in massively parallel computation and centralized local computation. In Timothy M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, San Diego, California, USA, January 6-9, 2019*, pages 1636–1653. SIAM, 2019.

[HLL18] Nicholas J. A. Harvey, Christopher Liaw, and Paul Liu. Greedy and local ratio algorithms in the MapReduce model. In *Proceedings of the 30th on Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 43–52, New York, NY, USA, 2018. ACM.

[HP15] James W Hegeman and Sriram V Pemmaraju. Lessons from the congested clique applied to MapReduce. *Theoretical Computer Science*, 608:268–281, 2015.
[IBY+07] Michael Isard, Mihai Budiu, Yuan Yu, Andrew Birrell, and Dennis Fetterly. Dryad: distributed data-parallel programs from sequential building blocks. In ACM SIGOPS operating systems review, volume 41, pages 59–72. ACM, 2007.

[ILMP19] Giuseppe F. Italiano, Silvio Lattanzi, Vahab S. Mirrokni, and Nikos Parotsidis. Dynamic algorithms for the massively parallel computation model. In Christian Scheideler and Petra Berenbrink, editors, The 31st ACM on Symposium on Parallelism in Algorithms and Architectures, SPAA 2019, Phoenix, AZ, USA, June 22-24, 2019, pages 49–58. ACM, 2019.

[IMS17] Sungjin Im, Benjamin Moseley, and Xiaorui Sun. Efficient massively parallel methods for dynamic programming. pages 798–811, 2017.

[JS17] Shweta Jain and C Seshadhri. A fast and provable method for estimating clique counts using turán’s theorem. In Proceedings of the 26th International Conference on World Wide Web, pages 441–449, 2017.

[KMPT12] Mihail N Kolountzakis, Gary L Miller, Richard Peng, and Charalampos E Tsourakakis. Efficient triangle counting in large graphs via degree-based vertex partitioning. Internet Mathematics, 8(1-2):161–185, 2012.

[KMSS12] Daniel M Kane, Kurt Mehlhorn, Thomas Sauerwald, and He Sun. Counting arbitrary subgraphs in data streams. In International Colloquium on Automata, Languages, and Programming, pages 598–609. Springer, 2012.

[KPP+14] Tamara G Kolda, Ali Pinar, Todd Plantenga, C Seshadhri, and Christine Task. Counting triangles in massive graphs with mapreduce. SIAM Journal on Scientific Computing, 36(5):S48–S77, 2014.

[KPP16] Tsvi Kopelowitz, Seth Pettie, and Ely Porat. Higher lower bounds from the 3sum conjecture. In Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms, pages 1272–1287. SIAM, 2016.

[KSV10] Howard Karloff, Siddharth Suri, and Sergei Vassilvitskii. A model of computation for MapReduce. pages 938–948, 2010.

[Lat08] Matthieu Latapy. Main-memory triangle computations for very large (sparse (power-law)) graphs. Theoretical computer science, 407(1-3):458–473, 2008.

[LMOS19] Jakub Lacki, Slobodan Mitrovic, Krzysztof Onak, and Piotr Sankowski. Walking randomly, massively, and efficiently. CoRR, abs/1907.05391, 2019.

[LMSV11] Silvio Lattanzi, Benjamin Moseley, Siddharth Suri, and Sergei Vassilvitskii. Filtering: a method for solving graph problems in MapReduce. pages 85–94, 2011.

[LQLC15] Longbin Lai, Lu Qin, Xuemin Lin, and Lijun Chang. Scalable subgraph enumeration in mapreduce. Proceedings of the VLDB Endowment, 8(10):974–985, 2015.

[MSOI+02] Ron Milo, Shai Shen-Orr, Shalev Itzkovitz, Nadav Kashtan, Dmitri Chklovskii, and Uri Alon. Network motifs: simple building blocks of complex networks. Science, 298(5594):824–827, 2002.
[MVV16] Andrew McGregor, Sofya Vorotnikova, and Hoa T Vu. Better algorithms for counting triangles in data streams. In Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, pages 401–411, 2016.

[Pat10] Mihai Patrascu. Towards polynomial lower bounds for dynamic problems. In Proceedings of the 42nd ACM Symposium on Theory of Computing, STOC 2010, Cambridge, Massachusetts, USA, 5-8 June 2010, pages 603–610, 2010.

[PC13] Ha-Myung Park and Chin-Wan Chung. An efficient mapreduce algorithm for counting triangles in a very large graph. In Proceedings of the 22nd ACM international conference on Information & Knowledge Management, pages 539–548, 2013.

[PSKP14] Ha-Myung Park, Francesco Silvestri, U Kang, and Rasmus Pagh. Mapreduce triangle enumeration with guarantees. In Proceedings of the 23rd ACM International Conference on Conference on Information and Knowledge Management, pages 1739–1748, 2014.

[PT12] Rasmus Pagh and Charalampos E Tsourakakis. Colorful triangle counting and a mapreduce implementation. Information Processing Letters, 112(7):277–281, 2012.

[RVW16] Tim Roughgarden, Sergei Vassilvitskii, and Joshua R. Wang. Shuffles and circuits: (on lower bounds for modern parallel computation). pages 1–12, 2016.

[SERF18] Kijung Shin, Tina Eliassi-Rad, and Christos Faloutsos. Patterns and anomalies in k-cores of real-world graphs with applications. Knowledge and Information Systems, 54(3):677–710, 2018.

[SPK13] Comandur Seshadhri, Ali Pinar, and Tamara G Kolda. Triadic measures on graphs: The power of wedge sampling. In Proceedings of the 2013 SIAM International Conference on Data Mining, pages 10–18. SIAM, 2013.

[SSS95] Jeanette P Schmidt, Alan Siegel, and Aravind Srinivasan. Chernoff–hoeffding bounds for applications with limited independence. SIAM Journal on Discrete Mathematics, 8(2):223–250, 1995.

[SV11] Siddharth Suri and Sergei Vassilvitskii. Counting triangles and the curse of the last reducer. In Proceedings of the 20th international conference on World wide web, pages 607–614, 2011.

[SW05] Thomas Schank and Dorothea Wagner. Finding, counting and listing all triangles in large graphs, an experimental study. In Experimental and Efficient Algorithms, pages 606–609. 2005.

[TDM+11] Charalampos E Tsourakakis, Petros Drineas, Eirinaios Michelakis, Ioannis Koutis, and Christos Faloutsos. Spectral counting of triangles via element-wise sparsification and triangle-based link recommendation. Social Network Analysis and Mining, 1(2):75–81, 2011.

[TKMF09] Charalampos E Tsourakakis, U Kang, Gary L Miller, and Christos Faloutsos. Doulion: counting triangles in massive graphs with a coin. In Proceedings of the 15th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 837–846, 2009.
[UBK13] Johan Ugander, Lars Backstrom, and Jon Kleinberg. Subgraph frequencies: Mapping the empirical and extremal geography of large graph collections. In Proceedings of the 22nd international conference on World Wide Web, pages 1307–1318, 2013.

[Vas09] Virginia Vassilevska. Efficient algorithms for clique problems. Information Processing Letters, 109(4):254–257, 2009.

[WC81] Mark N Wegman and J Lawrence Carter. New hash functions and their use in authentication and set equality. Journal of computer and system sciences, 22(3):265–279, 1981.

[Whi12] Tom White. Hadoop: The definitive guide. O’Reilly Media, Inc., 2012.

[YK11] Jin-Hyun Yoon and Sung-Ryul Kim. Improved sampling for triangle counting with mapreduce. In International Conference on Hybrid Information Technology, pages 685–689. Springer, 2011.

[ZCF+10] Matei Zaharia, Mosharaf Chowdhury, Michael J Franklin, Scott Shenker, and Ion Stoica. Spark: Cluster computing with working sets. HotCloud, 10(10-10):95, 2010.

A Preliminaries

A.1 Proof of Theorem 2.1

Using the construction of the interval tree defined in [GSZ11] that has branching factor \( d = M/2 \) we perform the following to count the number of times each element repeats in our sorted list of \( N \) elements. To initialize the tree, each leaf of the tree contains exactly one of the elements in the sorted list of elements where leaf \( v_l \) contains element \( x_i \) of the list. Let the height of the tree be \( L \), the leaves of the tree be at level \( L−1 \) and the root be at level 0. Then, the rest of the algorithm proceeds in two phases:

1. **Bottom-up phase:** For each level \( \ell = L−1 \) up to 0:
   (a) For each node \( v \) on level \( \ell \):
      i. If \( v \) is a leaf, it sends its value \( x_i \) to its parent \( p(v) \).
      ii. If \( v \) is a vertex in level \( L−2 \), let \( (x_i, x_{i+1}, \ldots, x_{i+j}) \) where \( j < d \) be values obtained from its leaf children from left to right. Let \( c(x_i) \) be the count of element \( x_i \) among the values obtained from the children of \( v \). The counts are computed locally on the machine storing \( v \). Then, \( v \) sends \( x_i, c(x_i), x_{i+j}, c(x_{i+j}) \) to its parent \( p(v) \).
      iii. If \( v \) is a non-leaf node on level \( \ell < L−2 \), let \( x_a, c(a), x_b, c(b), \ldots \) be the values of elements obtained from its children and their counts. \( v \) updates the counts of all elements received. For example, if \( x_a = x_b \), \( v \) updates \( c(a) \) and \( c(b) \) to be \( c(a) + c(b) \). Let \( x_{left} \) be the first element received from \( v \)’s leftmost leaf and \( x_{right} \) be the second element received from \( v \)’s rightmost leaf. Then, send these elements and their updated counts, \( x_{left}, c(x_{left}), x_{right}, c(x_{right}) \), to its parent \( p(v) \).

2. **Top-down phase:** For each level \( \ell = 0 \) down to \( \ell = L−1 \):
   (a) For each node \( v \) at level \( \ell \):
i. If $v$ is the root, then it computed and stored in its memory new repeating counts for the values it received from its children: $x_a, c(x_a), x_b, c(x_b), \ldots$. It sends the new counts and values to its respective child that sent it the value originally (e.g. $x_{left}, c(x_{left})$ to $v_{left}$). Intuitively, this updates the child’s count of values with values that are not in its subtree.

ii. If $v$ is not the root and is a non-leaf node, it receives the values from its parents for its leftmost and rightmost child counts. Given the set of values it stored from its children it updates the counts with counts of values received from its parents. This allows for the counts to reflect values not in its subtree. Then, it sends the updated counts to its children.

iii. If $v$ is a leaf, it receives values $x_i, c(x_i)$ from its parent. $c(x_i)$ is then the number of times $x_i$ occurs in the sorted list.

The above procedure uses $O(d)$ space per processor and $O(L)$ rounds of communication. Since $L = O(\log d(N))$ and $d = \mathcal{M}/2$, the number of rounds of communication that is necessary is $O(\log \mathcal{M} N)$.

A.2 Proof of Lemma 2.2

We first create the following tuples in parallel to represent tuples in $Q$ and $C$, respectively. For each tuple $q \in Q$, we create the tuple $(q, 1)$. For each tuple $c \in C$, we create the tuple $(c, 0)$. Let $F$ denote the set of tuples $(c, 0)$ and $(q, 1)$. First, we sort the tuples in $F$ lexicographically (where 0 comes before 1) [GSZ11]. Then, we use the predecessor primitive given in (e.g. [GSZ11, ASS+18a], Appendix A of [BDE+19b]) to determine the queries $q \in Q$ that are in $C$. Given the sorted $F$, we use the predecessor algorithm of [BDE+19b] to determine for each $(q, 1)$ tuple, the first tuple that appears before it that has value 0. Suppose this tuple is $(c, 0)$. Then, if $q = c$, then the queried tuple $q$ is in $C$. For all tuples $q \in Q$, we can then return in parallel whether $q \in C$ also. Both the sorting and the predecessor queries take $O(|Q \cup C|)$ total space and $O_\delta(1)$ rounds.

A.3 Proof of Lemma 2.3

Let $M$ be some machine with $n^\delta$ information and $O(n^{2\delta})$ space. We create the $x$ duplicates by repeatedly duplicating each machine $M^i_j$ to $n^\delta$ machines $M^{i+1}_j, \ldots, M^{i+1, n^\delta, j+n^\delta-1}_j$, starting with $M^0_0 = M$. Therefore, after $\ell = \log_x n^\delta$ rounds this process terminates, and the required duplicates is the set of machines $M^\ell_1$ to $M^\ell_x$. 