MINIMAL GENERATORS OF HALL ALGEBRAS OF 1-CYCLIC PERFECT COMPLEXES

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Abstract. Let $A$ be the path algebra of a Dynkin quiver over a finite field, and let $C_1(\mathcal{P})$ be the category of 1-cyclic complexes of projective $A$-modules. In the present paper, we give a PBW-basis and a minimal set of generators for the Hall algebra $\mathcal{H}(C_1(\mathcal{P}))$ of $C_1(\mathcal{P})$. Using this PBW-basis, we firstly prove the degenerate Hall algebra of $C_1(\mathcal{P})$ is the universal enveloping algebra of the Lie algebra spanned by all indecomposable objects. Secondly, we calculate the relations in the generators in $\mathcal{H}(C_1(\mathcal{P}))$, and obtain quantum Serre relations in a quotient of certain twisted version of $\mathcal{H}(C_1(\mathcal{P}))$. Moreover, we establish relations between the degenerate Hall algebra, twisted Hall algebra of $A$ and those of $C_1(\mathcal{P})$, respectively.

1. Introduction

Let $A$ be always a finite dimensional hereditary algebra over a finite field. In what follows, all modules are assumed to be finite dimensional. In 2011, Bridgeland [2] considered the Hall algebra of 2-cyclic complexes of projective $A$-modules. By taking some localization and reduction, he achieved an algebra called the Bridgeland Hall algebra of $A$. He proved that the quantum enveloping algebra associated to $A$ is embedded into the Bridgeland Hall algebra of $A$. This provides a beautiful realization of the full quantum enveloping algebra by Hall algebras. Bridgeland [2] made a statement without proofs that the Bridgeland Hall algebra of $A$ is isomorphic to the Drinfeld double of its extended Ringel–Hall algebra, which is later proved by Yanagida in [19] (see also [20]). Inspired by Bridgeland’s work, for each positive integer $m \geq 2$, Chen and Deng [4] considered the Hall algebra of $m$-cyclic complexes of projective $A$-modules. For a representation-finite hereditary algebra $A$ they proved the existence of Hall polynomials in the category of $m$-cyclic complexes of projective $A$-modules; using the Hall polynomials, they defined the generic Bridgeland Hall algebra of 2-cyclic complexes, and showed that it contains a subalgebra isomorphic to the integral form of the quantum enveloping algebra associated to $A$; in particular, for the degenerate case, this provides a realization of the simple Lie algebra associated to $A$. For $m > 2$, the algebra structure of the Bridgeland Hall algebra

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\(DH_m(A)\) of \(m\)-cyclic complexes of projective \(A\)-modules has a characterization in \([21]\), in particular, it is proved that there exist Heisenberg double structures in \(DH_m(A)\).

On the other hand, the most difficult case is the 1-cyclic complex case. Let \(C_1(\mathcal{P})\) be the category of 1-cyclic complexes of projective \(A\)-modules, and for each \(A\)-module \(M\) the corresponding 1-cyclic complex of projective \(A\)-modules is denoted by \(C_M\). Then for any two \(A\)-modules \(M, N\),

\[
\text{Ext}^1_{C_1(\mathcal{P})}(C_M, C_N) \cong \text{Hom}_A(M, N) \oplus \text{Ext}^1_A(M, N) \quad \text{(see Lemma 2.3)}.
\]

That is, the exact structure of \(C_1(\mathcal{P})\) is more complicated than that of the \(A\)-module category. Actually, let \(A = kQ\) be the path algebra of a finite acyclic quiver \(Q\), and let \(\Lambda = k[x]/[x^2]\) be the algebra of dual numbers, then the category of 1-cyclic complexes of \(A\)-modules is equivalent to the category of modules over the path algebra \(AQ\), and \(C_1(\mathcal{P})\) is exactly the category of Gorenstein projective \(AQ\)-modules (cf. \([17]\)). Ringel and Zhang pointed out in \([17]\) that “The class of 1-Gorenstein algebras is a class of algebras which includes both the hereditary and the self-injective algebras—two classes of algebras whose representations have been investigated very thoroughly and have been shown to be strongly related to Lie theory”. Thus, one might hope that all the 1-Gorenstein algebras should have a close relation with Lie theory. As a representative of the class of 1-Gorenstein algebras, the path algebra \(AQ\) is 1-Gorenstein. Ringel and Zhang \([17]\) have showed that the Kac theorem yields a correspondence between the isoclasses (isomorphism classes) of indecomposable objects in the stable category of \(C_1(\mathcal{P})\) and the positive roots of the Kac–Moody algebra associated to \(A\).

Inspired by the above correspondence given by Ringel and Zhang, for a hereditary algebra \(A\) of Dynkin type, some research on the Lie theory of \(C_1(\mathcal{P})\) has been undertaken in \([18]\). They gave a minimal set of generators for the Lie algebra \(\tilde{n}^+\) spanned by the isoclasses of indecomposable non-projective objects in \(C_1(\mathcal{P})\), and calculated some relations in these generators. In particular, for a bipartite quiver, the Lie algebra \(\tilde{n}^+\) provides a realization of the positive part of the corresponding simple Lie algebra; for a linearly oriented quiver of type \(A\), it provides a realization of free 2-step nilpotent Lie algebra. Moreover, for all the quivers of type \(A\), the generators and generating relations for \(\tilde{n}^+\) have been determined. This achieves the desire that \(C_1(\mathcal{P})\) should be related to Lie theory.

As mentioned above, compared with the \(A\)-module category, the exact structure of \(C_1(\mathcal{P})\) is “twisted” severely. In other words, the algebra structure of the Hall algebra of \(C_1(\mathcal{P})\) becomes more complicated. In the present paper, we will give some characterizations on the Hall algebra \(H(C_1(\mathcal{P}))\) of \(C_1(\mathcal{P})\). Explicitly, we give a PBW-basis for the Hall algebra of \(C_1(\mathcal{P})\). Then for a hereditary algebra of Dynkin type, using this PBW-basis, we firstly prove the degenerate Hall algebra of \(C_1(\mathcal{P})\) is the universal enveloping algebra of the Lie algebra spanned by all isoclasses of indecomposable objects; secondly,
we give a minimal set of generators for \( \mathcal{H}(C_1(\mathcal{P})) \), calculate the relations in these generators in \( \mathcal{H}(C_1(\mathcal{P})) \), and obtain some fundamental relations in a quotient of \( \mathcal{H}(C_1(\mathcal{P})) \). In order to get quantum Serre relations, we define a twisted version of \( \mathcal{H}(C_1(\mathcal{P})) \). Moreover, we establish relations between the degenerate Hall algebra, twisted Hall algebra of \( A \) and those of \( C_1(\mathcal{P}) \), respectively.

Let us fix some notations used throughout the paper. \( k = F_q \) is always a finite field with \( q \) elements, \( v = \sqrt{q} \), and \( Q(v) \) is the rational function field of \( v \). Let \( Q \) be a finite acyclic quiver with \( n \) vertices, and \( A \) be the path algebra of \( Q \) over \( k \). Denote by \( \text{mod} \ A \) the category of finite dimensional (left) \( A \)-modules, and by \( \mathcal{P} \subset \text{mod} \ A \) the subcategory of projective \( A \)-modules. The bounded derived category and Grothendieck group of \( \text{mod} \ A \) are denoted by \( D^b(A) \) and \( K(A) \), respectively; and for any \( M \in \text{mod} \ A \) we denote by \( \hat{M} \) the image of \( M \) in \( K(A) \). For each vertex \( i \) of \( Q \), we denote by \( S_i \) the corresponding simple \( A \)-module, and by \( P_i \) the projective cover of \( S_i \). For an \( A \)-module \( M \) and a positive integer \( m \), \( mM \) stands for the direct sum of \( m \) copies of \( M \). We denote by \([X]\) the isoclass of an object \( X \) in an additive category. For a finite set \( S \), we denote by \(|S|\) its cardinality. For a Lie algebra \( \mathfrak{g} \), we denote by \( U(\mathfrak{g}) \) its universal enveloping algebra (refer to [10] for Lie theory).

2. Preliminaries

In this section, we collect some definitions and properties of 1-cyclic complexes and Hall algebras.

2.1. 1-cyclic complexes. A 1-cyclic complex of \( A \)-modules is by definition a pair \( M^\cdot = (M, d) \) where \( M \) is an \( A \)-module and \( d \) is an endomorphism of \( M \) satisfying \( d^2 = 0 \). Let \( (M, d) \) and \( (M', d') \) be two 1-cyclic complexes of \( A \)-modules, a morphism \( f : (M, d) \to (M', d') \) is given by a homomorphism \( f : M \to M' \) of \( A \)-modules such that \( d'f = fd \). Two morphisms \( f, g : (M, d) \to (M', d') \) are said to be homotopic provided there exists a homomorphism \( s : M \to M' \) of \( A \)-modules such that \( f - g = sd + ds \). For each 1-cyclic complex \( M^\cdot = (M, d) \) of \( A \)-modules, its homology \( H_0(M^\cdot) := \text{Ker} d/\text{Im} d \). We denote by \( C_1(\text{mod} \ A) \) the category of 1-cyclic complexes of \( A \)-modules. Let \( C_1(\mathcal{P}) \) be the subcategory of \( C_1(\text{mod} \ A) \) consisting of 1-cyclic complexes of projective \( A \)-modules, and denote by \( K_1(\mathcal{P}) \) the homotopy category obtained from \( C_1(\mathcal{P}) \) by identifying homotopic morphisms. It is similar to the ordinary bounded complexes that we have a shift functor \([1] : C_1(\text{mod} \ A) \to C_1(\text{mod} \ A) \) defined by \( M^\cdot[1] := (M, -d) \), where \( M^\cdot = (M, d) \). It is well to be reminded that \( C_1(\mathcal{P}) \) is a Frobenius exact category, whose stable category coincides with the homotopy category \( K_1(\mathcal{P}) \).

The following lemma is significant to the calculation of extension groups in \( C_1(\mathcal{P}) \) (cf. [8] [4] [22]).
Lemma 2.1. If \( X^\cdot, Y^\cdot \in C_1(\mathcal{P}) \), then
\[
\text{Ext}^1_{C_1(\mathcal{P})}(X^\cdot, Y^\cdot) \cong \text{Hom}_{K_1(\mathcal{P})}(X^\cdot, Y^\cdot[1]).
\]

Given a morphism \( f : \Omega \to P \) of projective \( A \)-modules, one defines a 1-cyclic complex
\[
C_f = \left( P \oplus \Omega, \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \right) \in C_1(\mathcal{P}).
\]
Hence, for each projective \( A \)-module \( P \), we have a 1-cyclic complex \( K_P := C_{\text{Id}_P} \in C_1(\mathcal{P}) \).

For each \( A \)-module \( M \), we fix a minimal projective resolution of \( M \):
\[
0 \longrightarrow \Omega_M \overset{\delta_M}{\longrightarrow} P_M \overset{e_M}{\longrightarrow} M \longrightarrow 0.
\]
(2.1)

Then we set \( C_M := C_{\delta_M} \). Since the minimal projective resolution is unique up to isomorphism, \( C_M \) is well-defined up to isomorphism.

The following lemma gives characterizations of the classification of indecomposable objects in \( C_1(\mathcal{P}) \) and the structure of its homotopy category.

Lemma 2.2. ([17, Theorem 1])

(1) The objects \( C_M \) and \( K_P \), where \( M \) is an indecomposable \( A \)-module, and \( P \) is an indecomposable projective \( A \)-module, provide a complete set of indecomposable objects in \( C_1(\mathcal{P}) \). Moreover, all \( K_P \) are exactly the whole indecomposable projective-injective objects in \( C_1(\mathcal{P}) \).

(2) The homotopy category \( K_1(\mathcal{P}) \) is equivalent to the orbit category \( D^b(A)/[1] \) as triangulated categories.

Combining Lemma 2.1 with Lemma 2.2 we obtain the following

Lemma 2.3. For any \( X^\cdot, Y^\cdot \in C_1(\mathcal{P}) \),
\[
\text{Ext}^1_{C_1(\mathcal{P})}(X^\cdot, Y^\cdot) \cong \text{Hom}_A(H_0(X^\cdot), H_0(Y^\cdot)) \oplus \text{Ext}^1_A(H_0(X^\cdot), H_0(Y^\cdot)).
\]

Proof. By Lemma 2.2 we write \( X^\cdot = C_M \oplus K_P \) and \( Y^\cdot = C_N \oplus K_\Omega \) for some \( M, N \in \text{mod} A \), and \( P, \Omega \in \mathcal{P} \). Then
\[
\text{Ext}^1_{C_1(\mathcal{P})}(X^\cdot, Y^\cdot) = \text{Ext}^1_{C_1(\mathcal{P})}(C_M \oplus K_P, C_N \oplus K_\Omega)
\cong \text{Ext}^1_{C_1(\mathcal{P})}(C_M, C_N)
\cong \text{Hom}_{K_1(\mathcal{P})}(C_M, C_N[1])
\cong \text{Hom}_{D^b(A)[1]}(M, N[1])
\cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(A)}(M, N[i + 1])
\cong \text{Hom}_A(M, N) \oplus \text{Ext}^1_A(M, N)
\cong \text{Hom}_A(H_0(X^\cdot), H_0(Y^\cdot)) \oplus \text{Ext}^1_A(H_0(X^\cdot), H_0(Y^\cdot)).
\]
\[\square\]
The following well-known result is also needed in the sequel.

**Lemma 2.4.** For each short exact sequence of $A$-modules

$$\xi : 0 \to M \to L \to N \to 0,$$

we have that $\dim k \text{End}_A L \leq \dim k \text{End}_A (M \oplus N)$, and "=" holds if and only if $\xi$ is splitting.

In order to have an intuitive cognition of the structure of $C_1(\mathcal{P})$, we give an example of the Auslander–Reiten quiver of $C_1(\mathcal{P})$. We recommend [17] for the Auslander–Reiten theory of $C_1(\mathcal{P})$.

**Example 2.5.** Let $Q$ be the quiver of type $A_3$

$$1 \to 2 \to 3.$$

The Auslander–Reiten quiver of $C_1(\mathcal{P})$ is as follows:

\[
\begin{array}{ccc}
\text{K}_P_1 & \to & \text{C}_S_1 \\
\downarrow & & \downarrow \\
\text{C}_P_1 & \to & \text{C}_P_3 \\
\downarrow & & \downarrow \\
\text{C}_P_2 & \to & \text{C}_I_2 \\
\downarrow & & \downarrow \\
\text{C}_P_3 & \to & \text{C}_S_2 \\
\downarrow & & \downarrow \\
\text{K}_P_3 & \to & \text{K}_P_2 \\
\end{array}
\]

where the horizontal dashed lines denote different $\tau$-orbits, and we should join the vertical dotted lines such that their directions match, that is, the Auslander–Reiten quiver of $C_1(\mathcal{P})$ is like a Mobius strip. All $K_P_i$ are projective and injective in $C_1(\mathcal{P})$, and thus their orbits are themselves.

2.2. **Hall algebras.** Let $\mathcal{A}$ be a finitary and skeletally small exact $k$-category and let $W_{XY}^Z$ denote the set $\{(\varphi, \psi) \mid 0 \to Y \xrightarrow{\varphi} Z \xrightarrow{\psi} X \to 0 \text{ is exact in } \mathcal{A}\}$. The group $G := \text{Aut } X \times \text{Aut } Y$ acts on $W_{XY}^Z$ via

\[
\begin{array}{ccc}
0 & \to & Y \\
g & \downarrow & \varphi \downarrow \\
0 & \to & Y \\
\end{array}
\quad \text{ and } \quad
\begin{array}{ccc}
0 & \to & Z \\
\psi & \downarrow & \varphi \downarrow \\
0 & \to & Z \\
\end{array}
\quad \text{ and } \quad
\begin{array}{ccc}
0 & \to & X \\
f & \downarrow & \psi \downarrow \\
0 & \to & X \\
\end{array}
\]

That is, for any $(\varphi, \psi) \in W_{XY}^Z$ and $(f, g) \in G$, $(f, g) \cdot (\varphi, \psi) = (\varphi g^{-1}, f \psi)$. We denote the set of $G$-orbits by $V_{XY}^Z$. Since $\varphi$ is monic and $\psi$ epic this action is free, and we define

$$F_{XY}^Z := |V_{XY}^Z| = \frac{|W_{XY}^Z|}{|\text{Aut } X| \cdot |\text{Aut } Y|}.$$
By the Riedtmann–Peng formula \[12, 11\], we know that for any objects \( X, Y, Z \) in \( \mathcal{A} \)
\[ F^Z_{XY} = \frac{|\text{Ext}^1_{\mathcal{A}}(X, Y)_Z|}{|\text{Hom}_\mathcal{A}(X, Y)|} \cdot \frac{|\text{Aut} Z|}{|\text{Aut} X| \cdot |\text{Aut} Y|}, \]
where \( \text{Ext}^1_{\mathcal{A}}(X, Y)_Z \) denotes the subset of \( \text{Ext}^1_{\mathcal{A}}(X, Y) \) consisting of equivalence classes of exact sequences of the form \( 0 \to Y \to Z \to X \to 0 \).

**Definition 2.6.** Let \( \mathcal{A} \) be a finitary and skeletally small exact \( k \)-category. The Hall algebra \( \mathcal{H}(\mathcal{A}) \) of \( \mathcal{A} \) is the vector space over \( \mathbb{C} \) with basis the isoclasses \([X]\) of objects in \( \mathcal{A} \), and with multiplication defined by
\[ [X] \cdot [Y] = \sum_{[Z]} F^Z_{XY}[Z]. \]

In particular, if \( \mathcal{A} = \text{mod} A \), we obtain the Hall algebra \( \mathcal{H}(\text{mod} A) \) of \( A \), which is also denoted by \( \mathcal{H}(A) \); if \( \mathcal{A} = C_1(\mathcal{P}) \), we obtain the Hall algebra \( \mathcal{H}(C_1(\mathcal{P})) \) of \( C_1(\mathcal{P}) \).

2.3. **Degenerate Hall algebras and Lie algebras associated to** \( C_1(\mathcal{P}) \). In this subsection, let \( Q \) be a Dynkin quiver, that is, \( Q \) is of type \( ADE \), and let \( \Gamma \) be the underlying graph of \( Q \). For each prime power \( q \) (\( \neq 1 \) by convention), we denote by \( A = A(q) \) the path algebra of \( Q \) over \( k = \mathbb{F}_q \).

By the well-known theorem of Gabriel \[6, 7\], the correspondence \( M \mapsto \text{dim } M \) induces a bijection between the set of isoclasses of indecomposable \( A \)-modules and the set of positive roots \( \Phi^+ \) of the simple Lie algebra \( \mathfrak{g} \) associated with \( \Gamma \). For each \( \alpha \in \Phi^+ \), let \( M_q(\alpha) \) denote a representative of the corresponding indecomposable \( A \)-modules. For each \( 1 \leq i \leq n \), let \( \beta_i \) be the root in \( \Phi^+ \) such that \( M_q(\beta_i) \cong P_i \).

By Lemma 2.2, the set
\[ \{C_{M_q(\alpha)}, K_{M_q(\beta_i)} \mid \alpha \in \Phi^+, 1 \leq i \leq n\} \]
is a complete set of indecomposable objects in \( C_1(\mathcal{P}) \). Set \( I = \{1, \cdots, n\} \), \( I_1(\Gamma) = \Phi^+ \cup I \), and define \( \mathfrak{P}_1(\Gamma) = \{\lambda : I_1(\Gamma) \to \mathbb{N}\} \). By the Krull–Schmidt theorem, the correspondence sending \( \lambda \in \mathfrak{P}_1(\Gamma) \) to
\[ C(\lambda) = C_q(\lambda) = (\bigoplus_{\alpha \in \Phi^+} \lambda(\alpha) C_{M_q(\alpha)}) \bigoplus (\bigoplus_{1 \leq i \leq n} \lambda(i) K_{M_q(\beta_i)}) \]
induces a bijection from \( \mathfrak{P}_1(\Gamma) \) to the set of isoclasses of objects in \( C_1(\mathcal{P}) \). An element \( \lambda \in \mathfrak{P}_1(\Gamma) \) is said to be **indecomposable** if \( C_q(\lambda) \) is indecomposable and **decomposable** otherwise.

**Remark 2.7.** There is a bijection from the set of functions \( \lambda : \Phi^+ \to \mathbb{N} \) to the set of isoclasses of \( A \)-modules by sending \( \lambda \mapsto [M_q(\lambda)] \), where
\[ M_q(\lambda) = \bigoplus_{\alpha \in \Phi^+} \lambda(\alpha) M_q(\alpha) \in \text{mod } A. \]
Proposition 2.8. ([18] Theorem 3.6) For any $\lambda, \mu, \nu \in \mathcal{P}_1(\Gamma)$, there exists a polynomial $\psi^\lambda_{\mu\nu}(x) \in \mathbb{Z}[x]$ such that for each prime power $q$,

$$
\psi^\lambda_{\mu\nu}(q) = F^C_q(\lambda)_{C_q(\mu)C_q(\nu)}.
$$

The polynomials $\psi^\lambda_{\mu\nu}$ in Proposition 2.8 are called Hall polynomials.

The degenerate Hall algebra $\mathcal{H}_1(C_1(\mathcal{P}))$ of $C_1(\mathcal{P})$ is the same as $\mathcal{H}(C_1(\mathcal{P}))$ as vector spaces, but with multiplication defined by

$$
[C_q(\mu)][C_q(\nu)] = \sum_{\lambda \in \mathcal{P}_1(\Gamma)} \psi^\lambda_{\mu\nu}(1)[C_q(\lambda)].
$$

Remark 2.9. By the well-known result of Ringel [15], the Hall polynomials exist in $\text{mod } A$. Hence, we can similarly define the degenerate Hall algebra $\mathcal{H}_1(A)$ of $A$.

Let $\tilde{n}$ be the vector space over $\mathbb{C}$ spanned by the isoclasses of indecomposable objects in $C_1(\mathcal{P})$, and let $\tilde{n}^+$ (resp. $\mathfrak{h}$) be the subspace of $\tilde{n}$ spanned by the isoclasses of indecomposable and non-projective (resp. projective) objects in $C_1(\mathcal{P})$.

Proposition 2.10. ([18] Proposition 4.1) $\tilde{n}$ is a Lie algebra with Lie bracket defined by

$$
[[C_q(\mu)], [C_q(\nu)]] = \sum_{\lambda \in \mathcal{P}_1(\Gamma)} (\psi^\lambda_{\mu\nu}(1) - \psi^\lambda_{\nu\mu}(1))[C_q(\lambda)]
$$

for any indecomposable $\mu, \nu \in \mathcal{P}_1(\Gamma)$, and $\tilde{n}^+, \mathfrak{h}$ are two Lie subalgebras (ideals) of $\tilde{n}$.

We remark that as Lie algebras $\tilde{n} \cong \tilde{n}^+ \oplus \mathfrak{h}$, and $[\tilde{n}, \mathfrak{h}] = 0$. Thus by the PBW-basis theorem of universal enveloping algebras, we have the decomposition of the universal enveloping algebra $U(\tilde{n}) \cong U(\tilde{n}^+) \otimes U(\mathfrak{h})$, and $U(\mathfrak{h}) \cong k[x_1, x_2, \ldots, x_n]$. Actually, since $[\tilde{n}^+, \mathfrak{h}] = 0$, it is easy to see that

$$
g : U(\tilde{n}^+) \otimes k[x_1, x_2, \ldots, x_n] \longrightarrow U(\tilde{n}), \quad [C_M] \mapsto [C_M], \quad x_i \mapsto K_{P_i},
$$

(2.3)

where $M \in \text{mod } A$ is indecomposable and $1 \leq i \leq n$, is an isomorphism of algebras.

In order to state the following theorem, which is the main result of [18], we introduce the path matrix $E = (a_{ij})_{n \times n}$ of the Dynkin quiver $Q$: if there is a path between $i$ and $j$ in $Q$, say from $i$ to $j$, then $a_{ij} = 1$ and $a_{ji} = -1$; otherwise, $a_{ij} = a_{ji} = 0$.

Theorem 2.11. ([18] Theorem 4.3) The Lie algebra $\tilde{n}^+$ is generated by $\{[C_{P_i}] \mid 1 \leq i \leq n\}$, and these generators satisfy the following relations:

(a) If $|a_{ij}| = 1$, (ad $\epsilon_i)(\epsilon_j) = (\text{ad} \epsilon_j)(\epsilon_i) = 0$;

(b) If $a_{ij}a_{jk} = 1$, $[\epsilon_i, [\epsilon_j, \epsilon_k]] = [\epsilon_k, [\epsilon_i, \epsilon_j]] = 0$;

(c) If $a_{ij} = 0$, $[\epsilon_i, \epsilon_j] = 0$,

where $\epsilon_i := [C_{P_i}]$ for each $1 \leq i \leq n$. 
3. PBW-basis of $\mathcal{H}(C_1(\mathcal{P}))$

In this section, we will use the intrinsic filtered structure of $\mathcal{H}(C_1(\mathcal{P}))$ to prove its PBW-basis theorem. The PBW-basis theorem of the Hall algebra of an algebra is proved in [9], and it is generalized to the Hall algebra of a finitary exact category in [1].

Let us state the PBW-basis theorem of $\mathcal{H}(C_1(\mathcal{P}))$ as follows:

**Theorem 3.1.** $S := \{ [X^i] \mid X^i \in C_1(\mathcal{P}) \text{ is indecomposable} \}$ is a universal PBW-basis of the Hall algebra $\mathcal{H}(C_1(\mathcal{P}))$. That is, for any total order $\leq$ on $S$, the monomials $[X_1]^{a_1}[X_2]^{a_2} \cdots [X_N]^{a_N}$, where $[X_1] \leq [X_2] \leq \cdots \leq [X_N]$ are the whole isoclasses of pairwise non-isomorphic indecomposable objects and all $a_i \in \mathbb{N}$, together with the unit $[0]$, form a basis for $\mathcal{H}(C_1(\mathcal{P}))$.

**Proof.** For any short exact sequence in $C_1(\mathcal{P})$

$$0 \rightarrow Y^i \rightarrow Z^i \rightarrow X^i \rightarrow 0,$$

it induces a long exact sequence in homology

$$0 \rightarrow K \rightarrow H_0(Y^i) \xrightarrow{\varphi} H_0(Z^i) \xrightarrow{\psi} H_0(X^i) \rightarrow Q \rightarrow 0 \quad (3.1)$$

Since $\ker \varphi \cong \text{Im} f \cong H_0(X^i)/\ker f \cong H_0(X^i)/\text{Im} \psi \cong \text{coker} \psi$, we obtain that $K \cong Q$ and $\dim H_0(Z^i) = \dim H_0(X^i) + \dim H_0(Y^i) - 2\dim Q$.

For any $X^i \in C_1(\mathcal{P})$, define $\dim X^i := \dim H_0(X^i)$. For each $\alpha \in \mathbb{N}^n$, let $\mathcal{H}_{\leq \alpha}(C_1(\mathcal{P}))$ be the subspace of $\mathcal{H}(C_1(\mathcal{P}))$ spanned by all $[X]$ with $\dim X^i \leq \alpha$ (in the sense that each component of $\dim X^i$ is equal or less than the corresponding component of $\alpha$). Then $\mathcal{H}_{\leq \alpha}(C_1(\mathcal{P})) \ast \mathcal{H}_{\leq \beta}(C_1(\mathcal{P})) \subseteq \mathcal{H}_{\leq \alpha + \beta}(C_1(\mathcal{P}))$, and thus $\mathcal{H}(C_1(\mathcal{P}))$ is an $N^n$-filtered algebra.

By Lemma 2.23

$$\text{Ext}^1_{C_1(\mathcal{P})}(X^i, Y^i) \cong \text{Hom}_A(H_0(X^i), H_0(Y^i)) \oplus \text{Ext}^1_A(H_0(X^i), H_0(Y^i)).$$

If $f$ in (3.1) is nonzero, then $\dim H_0(Z^i) < \dim H_0(X^i) + \dim H_0(Y^i)$; if $f$ in (3.1) is zero, then $K = Q = 0$, thus we obtain the short exact sequence

$$\xi : 0 \rightarrow H_0(Y^i) \rightarrow H_0(Z^i) \rightarrow H_0(X^i) \rightarrow 0,$$

in this case, $\dim H_0(Z^i) = \dim H_0(X^i) + \dim H_0(Y^i)$; and if $\xi$ is not splitting, by Lemma 2.4

$$\dim_k \text{End}_A(H_0(Z^i)) < \dim_k \text{End}_A(H_0(X^i) \oplus H_0(Y^i)).$$
Let us give an order on \( \mathbb{N}^n \times \mathbb{N} \): 
\[
(\alpha_1,d_1) \leq (\alpha_2,d_2) \iff "\alpha_1 < \alpha_2" \text{ or } "\alpha_1 = \alpha_2, d_1 \leq d_2".
\]

For any \( X' \in C_1(\mathcal{P}) \), set \( \deg X' := (\dim H_0(X'),\dim \text{End}_A(H_0(X'))) \). Then for any \( X',Y' \in C_1(\mathcal{P}) \),
\[
[X'][Y'] = a_{X' \oplus Y'} [X' \oplus Y'] + \sum_{[Z] : \deg Z < \deg X' \oplus Y'} a_Z [Z].
\]

For each fixed total order \( \leq \) on the set \( S \) such that \( [X_1] \leq [X_2] \leq \cdots \leq [X_N] \) are all pairwise non-isomorphic indecomposable objects in \( C_1(\mathcal{P}) \). Let \( X' = \bigoplus_{i=1}^N a_i X_i \), \( a_i \in \mathbb{N} \). Then
\[
[X_1]^{a_1} [X_2]^{a_2} \cdots [X_N]^{a_N} = a_X [X'] + \sum_{[Z] : \deg Z < \deg X'} a_Z [Z].
\]
Clearly, \( a_X \neq 0 \). By induction on \( \deg X' \), we complete the proof. \( \square \)

**Remark 3.2.** Let \( Q \) be a Dynkin quiver. Let \( V_0 \) be the direct sum of one copy of each indecomposable \( A \)-module. For each \( A \)-module \( M \), set \( d(M) := \dim_k \text{Hom}_A(V_0,M) \). Then for each short exact sequence of \( A \)-modules \( 0 \to M \to L \to N \to 0 \), \( d(L) \leq d(M) + d(N) \) and "=" holds if and only if \( \xi \) is splitting (cf. \([\ref{corollary:PBW}])

For any \( X' \in C_1(\mathcal{P}) \), define \( \deg X' = (\dim H_0(X'),d(H_0(X'))) \). Then we can obtain that \( \mathcal{H}(C_1(\mathcal{P})) \) is \( \mathbb{N}^{n+1} \)-filtered with \( \mathcal{H}_{\leq \alpha}(C_1(\mathcal{P})) \) being the subspace of \( \mathcal{H}(C_1(\mathcal{P})) \) spanned by all \( [X'] \) with \( \deg X' \leq \alpha \), where \( \alpha \in \mathbb{N}^{n+1} \).

**Corollary 3.3.** Let \( Q \) be a Dynkin quiver. Then the degenerate Hall algebra \( \mathcal{H}_1(C_1(\mathcal{P})) \) is isomorphic to the universal enveloping algebra \( U(\mathfrak{n}) \).

**Proof.** Since \( \mathcal{H}_1(C_1(\mathcal{P})) \) is an associative algebra with Lie subalgebra \( \mathfrak{n} \), there is a unique homomorphism \( f \) of algebras from the universal enveloping algebra \( U(\mathfrak{n}) \) to \( \mathcal{H}_1(C_1(\mathcal{P})) \) such that the diagram
\[
\begin{array}{ccc}
\mathfrak{n} & \xrightarrow{\mathfrak{n}} & U(\mathfrak{n}) \\
\downarrow f & & \downarrow f \\
\mathcal{H}_1(C_1(\mathcal{P})) & \xrightarrow{\mathcal{H}_1(C_1(\mathcal{P}))} &
\end{array}
\]
commutes. By Theorem \([\ref{theorem:PBW}]\) and the PBW-basis theorem of universal enveloping algebra, we conclude that \( f \) is an isomorphism. \( \square \)

4. **Minimal generators of \( \mathcal{H}(C_1(\mathcal{P})) \)**

From now onwards until the end of the paper, let \( Q \) be always a Dynkin quiver. In this section, we will give a minimal set of generators for the Hall algebra \( \mathcal{H}(C_1(\mathcal{P})) \). Then
combining with [18] Theorem 4.3, we establish a relation between the degenerate Hall algebras $H_1(A)$ and $H_1(C_1(\mathcal{P}))$.

**Theorem 4.1.** \{[(CP)_i], [KP] \mid 1 \leq i \leq n\} is a minimal set of generators for the Hall algebra $H(C_1(\mathcal{P}))$.

By Theorem [3.1] we only prove that every $[C_M]$ corresponding to indecomposable non-projective $A$-module $M$ can be generated by the elements in \{[(CP)_i], [KP] \mid 1 \leq i \leq n\}.

Let $M$ be an indecomposable non-projective $A$-module, and fix the minimal projective resolution of $M$:

$$0 \rightarrow \Omega \rightarrow P \rightarrow M \rightarrow 0.$$  

Before proving Theorem 4.1, we first give the following Lemmas.

**Lemma 4.2.** Let $P$ be an indecomposable projective $A$-module. Then for any positive integer $m$,

$$[C_{mP}][CP] = \left( \frac{q^{m+1} - 1}{q-1} \right)[C_{(m+1)P}] + \frac{1}{q^{m-1}}[C_{(m-1)P}][KP].$$

**Proof.**

By Lemma 2.3, $\text{Ext}^1_{C_1(\mathcal{P})}(C_{mP}, CP) \cong \text{Hom}_A(mP, P) \cong km$.

Let $f = (f_1, \ldots, f_m) \in \text{Hom}_A(mP, P)$ be nonzero. We assume that $f_i \neq 0$ for some $1 \leq i \leq m$, then $f_i$ is an isomorphism. Thus, $f$ is a splitting epimorphism, and $\text{Ker } f \cong (m-1)P$. Consider the extension corresponding to $f$: $0 \rightarrow C_P \rightarrow X' \rightarrow C_{mP} \rightarrow 0$, then $X'$ must be isomorphic to $C_{(m-1)P} \oplus KP$. Hence,

$$[C_{mP}][CP] = F_{C_{mP}CP}^{C_{(m+1)P}}[C_{(m+1)P}] + F_{C_{mP}CP}^{C_{(m-1)P} \oplus KP}[C_{(m-1)P} \oplus KP].$$

By Riedtmann–Peng formula, it is easy to see that

$$F_{C_{mP}CP}^{C_{(m+1)P}} = \frac{q^{m+1} - 1}{q-1} \text{ and } F_{C_{mP}CP}^{C_{(m-1)P} \oplus KP} = 1.$$

Since $[C_{(m-1)P}][KP] = F_{C_{(m-1)P}KP}^{C_{(m-1)P} \oplus KP}[C_{(m-1)P} \oplus KP] = q^{m-1}[C_{(m-1)P} \oplus KP]$, we complete the proof. \hfill \Box

Let $(i_1, i_2, \cdots, i_n)$ be a permutation of $(1, 2, \cdots, n)$ such that $\text{Hom}_A(P_{i_k}, P_{i_l}) \neq 0$ implies that $s \geq t$.

**Lemma 4.3.** Let $a_i \in \mathbb{N}, 1 \leq i \leq n$. Then for any projective $A$-module $P \cong \oplus_{i=1}^n a_iP_i$, we have that

$$[CP] = \sum_{s_{i_1}, s_{i_2}, \ldots, s_{i_n} \geq 1, 1 \leq i \leq n} a_{s_{i_1}, s_{i_2}, \ldots, s_{i_n}} \prod_{l=1}^n [CP_i]^{s_{i_l}} [KP_i]^{t_{i_l}}$$

for some $a_{s_{i_1}, s_{i_2}, \ldots, s_{i_n}} \in \mathbb{Q}$.
Proof. Using Lemma 4.2 by induction on \( m \), we obtain that
\[
[C_mP] = \sum_{s,t: s+2t=m} a^s_t [C_P]^s[K_P]^t, \quad \text{for some } a^s_t \in \mathbb{Q}.
\]
By the fact that \([C_P] = [C_{a_{i_1}P_{i_1}} \oplus C_{a_{i_2}P_{i_2}} \oplus \cdots \oplus C_{a_{i_n}P_{i_n}}] \) and \( \text{Hom}_A(a_i P_{i}, a_i P_{i}) = 0 \) for any \( s < t \), we complete the proof. \( \square \)

Lemma 4.4. Let \( 0 \rightarrow C_P \rightarrow Z' \rightarrow C_\Omega \rightarrow 0 \) be a short exact sequence with \( Z' \in C_1(\mathcal{P}) \) indecomposable. Then there exists a monomorphism \( f: \Omega \rightarrow P \) such that \( \text{Coker } f \cong H_0(Z') \). Moreover, \( Z' \cong C_M \).

Proof. Clearly, \( Z' \) is not projective, since \( P \) is not isomorphic to \( \Omega \). Hence, we assume that \( Z' = C_L \) for some indecomposable \( A \)-module \( L \). Considering the long exact sequence in homology
\[
\begin{align*}
P & \longrightarrow L \longrightarrow \Omega \xrightarrow{f} P \longrightarrow L \longrightarrow \Omega, \\
\text{Ker } f & \longrightarrow \text{Coker } f
\end{align*}
\]
we obtain the short exact sequence \( 0 \rightarrow \text{Coker } f \rightarrow L \rightarrow \text{Ker } f \rightarrow 0 \), and \( L \cong \text{Ker } f \oplus \text{Coker } f \), since \( \text{Ker } f \leq \Omega \) is projective. We conclude that \( \text{Ker } f = 0 \) or \( \text{Coker } f = 0 \), since \( L \) is indecomposable.

Suppose that \( \text{Coker } f = 0 \), that is, \( f \) is an epimorphism. Then we have the short exact sequence \( 0 \rightarrow L \rightarrow \Omega \rightarrow P \rightarrow 0 \), and \( P \oplus L \cong \Omega \), thus \( L \) is projective. By the short exact sequence \( 0 \rightarrow C_P \rightarrow Z' \rightarrow C_\Omega \rightarrow 0 \), we obtain that \( L \cong P \oplus \Omega \). This is a contradiction, since \( P, \Omega \) are nonzero, and \( L \) is indecomposable. Hence, \( \text{Ker } f = 0 \) and \( \text{Coker } f \cong L \cong H_0(Z') \). Since \( \dim L = \dim H_0(Z') = \dim P - \dim \Omega = \dim M \), and \( L, M \) are both indecomposable, we obtain that \( L \cong M \), and thus \( Z' = C_L \cong C_M \).

\( \square \)

Proof of Theorem 4.7: By Lemma 2.3 we have that
\[\text{Ext}^1_{C_1(\mathcal{P})}(C_\Omega, C_P) \cong \text{Hom}_A(\Omega, P).\]
For any \( f \in \text{Hom}_A(\Omega, P) \), we consider the corresponding extension:
\[
0 \rightarrow C_P \rightarrow Z'(f) \rightarrow C_\Omega \rightarrow 0.
\]
(4.1)
As before, we know that \( \dim H_0(Z'(f)) = \dim P + \dim \Omega - 2(\dim \Omega - \dim \text{Ker } f) = \dim M + 2\dim \text{Ker } f \geq \dim M \). Hence, if \( f \) is not injective, then \( \dim H_0(Z'(f)) > \dim M \); if \( f \) is injective and \( Z'(f) \not\cong C_M \), then by Lemma 4.3 \( Z'(f) \) is decomposable.

For any \( X' \in C_1(\mathcal{P}) \), set \( \deg X' := (\dim H_0(X'), m(X')) \), where \( m(X') \) is the number of indecomposable direct summands of \( X' \).
Consider the opposite of the lexicographical order on $\mathbb{N}^n \times \mathbb{N}$:

$$(\alpha_1, d_1) \leq (\alpha_2, d_2) \iff \alpha_1 > \alpha_2 \quad \text{or} \quad \alpha_1 = \alpha_2, d_1 \geq d_2.$$  \hspace{1cm} (4.2)

Hence, for any injective $f \in \text{Hom}_A(\Omega, P)$ such that $Z(f) \nsubseteq C_M$, or any non-injective $f \in \text{Hom}_A(\Omega, P)$, we have that $\deg Z(f) < \deg C_M$.

Let $s$ and $t$ be the number of indecomposable direct summands of $P$ and $\Omega$, respectively. Then for any $0 \to C_P \to Z(f) \to C_\Omega \to 0$ as in (4.1), $Z(f)$ has at most $s + t$ indecomposable direct summands. If $f = 0$, then $Z(f) \cong C_P \oplus C_\Omega$, in this case, $\dim H_0(Z(f))$ and $m(Z(f))$ are both maximal, and thus $\deg Z(f)$ is minimal under the order defined in (4.2). That is, this case is the starting point of our induction.

$$[C_\Omega][C_P] = a[C_P \oplus C_\Omega] + b[C_M] + \sum_{[Z]: \deg Z < \deg C_M} c_Z [Z].$$

Clearly, $b \neq 0$. By induction on $\deg C_M$ together with Lemma 4.3, we complete the proof.

5. Relations between degenerate Hall algebras $H_1(A)$ and $H_1(C_1(\mathcal{P}))$

In this section, as a first application of Theorem 4.1, we establish a relation between degenerate Hall algebras $H_1(A)$ and $H_1(C_1(\mathcal{P}))$. For any $1 \leq i \neq j \leq n$, we set

$$L^2_{ij} := \{\alpha_{ij} \mid \alpha_{ij} \text{ is a path between } i \text{ and } j, \text{ which is of length at least two}\},$$

and define the ideal of $H_1(C_1(\mathcal{P}))$

$$\mathcal{J}_0 := \langle [C_P][C_P] - [C_P][C_P] \mid L^2_{ij} \neq \emptyset, 1 \leq i \neq j \leq n \rangle.$$

We remark that for any $1 \leq i \neq j \leq n$ either $L^2_{ij} = \emptyset$ or $|L^2_{ij}| = 1$, since $Q$ is a Dynkin quiver.

**Theorem 5.1.** There exists an epimorphism of algebras

$$\varphi_0 : H_1(A) \otimes k[x_1, x_2, \ldots, x_n] \longrightarrow H_1(C_1(\mathcal{P}))/\mathcal{J}_0$$

defined by $[S_i] \mapsto [C_P]$, and $x_i \mapsto [K_P]$.

**Proof.** It is easy to see that for any $M \in \text{mod} A$ and $P, Q \in \mathcal{P}$ we have that $[K_P][C_M] = [C_M][K_P]$ and $[K_P][K_Q] = [K_Q][K_P]$ in $H_1(C_1(\mathcal{P}))$. Then by Theorem 2.11 we obtain that $\varphi_0$ is a homomorphism of algebras, since it is well-known that $H_1(A)$ is generated by all $[S_i]$ and the Serre relations (cf. [14, 16]). It follows that $\varphi_0$ is an epimorphism from Theorem 4.1.

If each vertex of the quiver $Q$ is either a sink or a source, then $Q$ is said to be bipartite.

**Corollary 5.2.** Let $Q$ be a bipartite Dynkin quiver. Then there exists an isomorphism of algebras $\psi_0 : H_1(A) \otimes k[x_1, x_2, \ldots, x_n] \to H_1(C_1(\mathcal{P}))$ defined by $[S_i] \mapsto [C_P]$, and $x_i \mapsto [K_P]$. 
Proof. Since $Q$ is bipartite, we obtain that $L_{ij}^{>2} = \emptyset$ for any $1 \leq i \neq j \leq n$. By Theorem 5.1, $\psi_0$ is an epimorphism of algebras.

Let $\mathfrak{n}^+$ be the positive part of the simple Lie algebra associated to $Q$, whose canonical generators are denoted by $e_i, 1 \leq i \leq n$. Since $Q$ is bipartite, using Theorem 2.11, we can prove that there exists an isomorphism of Lie algebras $\tilde{\gamma} : U(\mathfrak{n}^+) \to U(\tilde{\mathfrak{n}}^+)$ defined by $e_i \mapsto [C_{P_i}]$ (cf. [18, Corollary 4.6]). Hence, it induces an isomorphism of algebras $\tilde{\gamma} : U(\mathfrak{n}^+) \to U(\tilde{\mathfrak{n}}^+)$ defined by $e_i \mapsto [C_{P_i}]$. It is well-known that there exists an isomorphism of algebras $\eta : \mathcal{H}_1(A) \to U(\mathfrak{n}^+)$ defined by $[S_i] \mapsto e_i$ (cf. [13, 15]). By the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{H}_1(A) \otimes k[x_1, x_2, \ldots, x_n] & \xrightarrow{\psi_0} & \mathcal{H}_1(C_1(\mathcal{P})) \\
\eta \otimes 1 \cong & & \\
U(\mathfrak{n}^+) \otimes k[x_1, x_2, \ldots, x_n] & \xrightarrow{\gamma} & U(\tilde{\mathfrak{n}}^+) \cong f \\
\gamma \otimes 1 \cong & & \\
U(\tilde{\mathfrak{n}}^+) \otimes k[x_1, x_2, \ldots, x_n] & \xrightarrow{\cong} & U(\tilde{\mathfrak{n}}) \\
\end{array}
$$

where $g$ and $f$ are from (2.3) and (3.2), respectively, we conclude that $\psi_0$ is an isomorphism. \qed

6. Fundamental relations associated to $\mathcal{H}(C_1(\mathcal{P}))$

Recall that $Q$ is always a Dynkin quiver. In this section, we calculate certain relations in the generators given in Theorem 4.1 and obtain some fundamental relations, which have appeared in the Hall algebra of $A$ (cf. [14, 3]), in a quotient of $\mathcal{H}(C_1(\mathcal{P}))$.

For any $1 \leq i \neq j \leq n$, if $a_{ij} = 1$, i.e., there is a path from $i$ to $j$, then $\text{Hom}_A(P_j, P_i) = 0$ and $\text{Hom}_A(P_j, P_i) \cong k$. Note that each nonzero morphism from $P_j$ to $P_i$ is a monomorphism. Take an arbitrary $0 \neq f \in \text{Hom}_A(P_j, P_i)$, and set $M_{ij} : = \text{Coker} f$. Then we have a short exact sequence of $A$-modules

$$
0 \longrightarrow P_j \xrightarrow{f} P_i \longrightarrow M_{ij} \longrightarrow 0. \tag{6.1}
$$

Clearly, $M_{ij}$ is indecomposable, thus $M_{ij}$ is uniquely determined up to isomorphism, which does not depend on the nonzero $f$. Actually, the sequence (6.1) is the minimal projective resolution of $M_{ij}$.

Proposition 6.1. For any $1 \leq i \neq j \leq n$,

1. if $a_{ij} = 0$, then $[C_{P_i}][C_{P_j}] = [C_{P_j}][C_{P_i}] = [C_{P_i \oplus P_j}]$ in $\mathcal{H}(C_1(\mathcal{P}))$;
(2) if \( a_{ij} = 1 \), then the following relations

\[
\begin{align*}
[C_{P_j}]^2[C_{P_i}] & = (q + 1)[C_{P_i} \oplus 2P_{P_j}] + (q + 1)[C_{P_i} \oplus C_{M_{ij}}] + [C_{P_i} \oplus K_{P_j}] \\
[C_{P_i}][C_{P_j}] & = q(q + 1)[C_{P_i} \oplus 2P_{P_j}] + q[C_{P_i} \oplus C_{M_{ij}}] + q[C_{P_i} \oplus K_{P_j}] \\
[C_{P_j}][C_{P_j}] & = q^2(q + 1)[C_{P_i} \oplus 2P_{P_j}] + q[C_{P_i} \oplus K_{P_j}]
\end{align*}
\]

and

\[
\begin{align*}
[C_{P_i}]^2[C_{P_j}] & = q^2(q + 1)[C_{P_i} \oplus 2P_{P_j}] + q[C_{P_i} \oplus K_{P_j}] \\
[C_{P_i}][C_{P_j}][C_{P_i}] & = q(q + 1)[C_{P_i} \oplus 2P_{P_j}] + q[C_{P_i} \oplus C_{M_{ij}}] + q[C_{P_i} \oplus K_{P_j}] \\
[C_{P_j}][C_{P_j}]^2 & = (q + 1)[C_{P_i} \oplus 2P_{P_j}] + (q + 1)[C_{P_i} \oplus C_{M_{ij}}] + [C_{P_i} \oplus K_{P_j}]
\end{align*}
\]

hold in \( \mathcal{H}(C_1(\mathcal{P})) \).

**Proof.** In the whole proof, we will use the Riedtmann–Peng formula together with Proposition 2.4 in [18] to calculate Hall numbers.

(1) Since \( a_{ij} = 0 \), that is, there is no path between \( i \) and \( j \), we obtain that

\[
\text{Hom}_A(P_i, P_j) = \text{Hom}_A(P_j, P_i) = 0,
\]

thus \( \text{Ext}^{1}_{C_1(\mathcal{P})}(C_{P_i}, C_{P_j}) = \text{Ext}^{1}_{C_1(\mathcal{P})}(C_{P_j}, C_{P_i}) = 0 \). Then it is easy to see that

\[
F_{C_{P_j} \oplus C_{P_j}}^{C_{P_j} \oplus C_{P_j}} = F_{C_{P_j} \oplus C_{P_j}}^{C_{P_j} \oplus C_{P_j}} = 1.
\]

(2) Applying \( \text{Hom}_A(P_j, -) \) to the sequence (6.1), we obtain that \( \text{Hom}_A(P_j, M_{ij}) = 0 \), thus \( \text{Ext}^{1}_{C_1(\mathcal{P})}(C_{P_j}, C_{M_{ij}}) \cong \text{Hom}_A(P_j, M_{ij}) = 0 \). Similarly, applying \( \text{Hom}_A(-, P_j) \) to the sequence (6.1), we get that \( \text{Ext}^{1}_{C_1(\mathcal{P})}(C_{M_{ij}}, C_{P_j}) \cong \text{Ext}^{1}_{A}(M_{ij}, P_j) \) is one-dimensional.

\[
[C_{P_j}]^2 = F_{C_{P_j} \oplus C_{P_j}}^{C_{P_j} \oplus C_{P_j}}[C_{2P_j}] + F_{C_{P_j} \oplus C_{P_j}}^{K_{P_j}}[K_{P_j}]
= \frac{1}{q} \cdot \frac{(q^2 - q)(q^2 - 1)}{(q - 1)^2} [C_{2P_j}] + \frac{q - 1}{q} \cdot \frac{(q - 1)q}{(q - 1)^2} [K_{P_j}]
(6.4)
\]

\[
[C_{P_j}][C_{P_j}] = [C_{P_j}][(q + 1)[C_{2P_j}] + [K_{P_j}])
= (q + 1)[C_{P_j}][C_{2P_j}] + [C_{P_j}][K_{P_j}]
= (q + 1) \cdot \frac{(q - 1)(q^2 - q)(q^2 - 1)q^2}{(q - 1)(q^2 - q)(q^2 - 1)} [C_{P_j} \oplus 2P_{P_j}] + q[C_{P_j} \oplus K_{P_j}]
(6.5)
\]

\[
[C_{P_j}][C_{P_j}] = F_{C_{P_j} \oplus C_{P_j}}^{C_{P_j} \oplus C_{P_j}}[C_{P_j} \oplus C_{P_j}] + F_{C_{P_j} \oplus C_{P_j}}^{C_{M_{ij}}}[C_{M_{ij}}]
= \frac{1}{q} \cdot \frac{(q - 1)^2q}{(q - 1)^2} [C_{P_j} \oplus C_{P_j}] + \frac{q - 1}{q} \cdot \frac{(q - 1)q}{(q - 1)^2} [C_{M_{ij}}]
(6.6)
\]

\[
= [C_{P_j} \oplus C_{P_j}] + [C_{M_{ij}}];
\]
Hence, we complete the proof of the relations in (6.12).

\[
[C_P][C_P] = ([C_{P \otimes P_P}] + [C_{M_{ij}}])[C_P] \\
= [C_{P \otimes P_P}][C_P] + [C_{M_{ij}}][C_P] \\
= FC_{P \otimes 2P_P}C_P[C_{P \otimes 2P_P}] + FC_{P \otimes K_P}C_P[C_{P \otimes K_P}] \\
+ FC_{M_{ij}}C_P[C_{P \otimes M_{ij}}] + FC_{K_P}C_{M_{ij}}[C_{P \otimes K_P}] \\
= \frac{1}{q} \cdot \frac{(q-1)(q^2-q)(q^2-1)q^3}{(q-1)^3q}[C_{P \otimes 2P_P}] + \frac{q-1}{q} \cdot \frac{(q-1)^2q^3}{(q-1)^3q}[C_{P \otimes K_P}] \\
+ q[C_{P \otimes M_{ij}}] + \frac{q-1}{q} \cdot \frac{(q-1)^2q^3}{(q-1)^3q}[C_{P \otimes K_P}] \\
= q(q + 1)[C_{P \otimes 2P_P}] + q[C_{P \otimes M_{ij}}] + q[C_{P \otimes K_P}]; \\
(6.7)
\]

\[
[C_P]^2[C_P] = [C_P][C_P \otimes B] + [C_P][C_{M_{ij}}] \\
= FC_{P \otimes 2P_P}C_P[C_{P \otimes 2P_P}] + FC_{P \otimes K_P}C_P[C_{P \otimes K_P}] \\
+ FC_{P \otimes M_{ij}}C_P[C_{P \otimes M_{ij}}] + FC_{P \otimes K_P}C_{M_{ij}}[C_{P \otimes K_P}] \\
= \frac{1}{q^2} \cdot \frac{(q-1)(q^2-q)(q^2-1)q^2}{(q-1)^3q}[C_{P \otimes 2P_P}] + \frac{(q-1)q}{q^2} \cdot \frac{(q-1)^2q^2}{(q-1)^3q}[C_{P \otimes K_P}] \\
+ \frac{q-1}{q^2} \cdot \frac{(q-1)^2q^2}{(q-1)^3q}[C_{P \otimes M_{ij}}] + \frac{1}{q} \cdot \frac{(q-1)^2q^3}{(q-1)^2q^2}[C_{P \otimes K_P}] \\
= (q + 1)[C_{P \otimes 2P_P}] + (q + 1)[C_{P \otimes M_{ij}}] + [C_{P \otimes K_P}]. \\
(6.8)
\]

Hence, we complete the proof of the relations in (6.2).

\[
[C_P][C_P] = FC_{P \otimes P_P}C_P[C_{P \otimes P_P}] = \frac{(q-1)^2q}{(q-1)^2}[C_{P \otimes P_P}] = q[C_{P \otimes P_P}]; \\
(6.9)
\]

\[
[C_P]^2[C_P] = q[C_P][C_{P \otimes P_P}] \\
= q\left(\frac{1}{q} \cdot \frac{(q-1)(q^2-q)(q^2-1)q^2}{(q-1)^3q}[C_{2P \otimes P_P}] + \frac{q-1}{q} \cdot \frac{(q-1)^2q^2}{(q-1)^3q}[C_{P \otimes K_P}]\right) \\
= q^2(q + 1)[C_{2P \otimes P_P}] + q[C_{P \otimes K_P}]; \\
(6.10)
\]

\[
[C_P][C_P][C_P] = [C_P][C_{P \otimes P_P}] + [C_P][C_{M_{ij}}] \\
= q(q + 1)[C_{2P \otimes P_P}] + [C_{P \otimes K_P}] + q[C_{P \otimes M_{ij}}] + (q - 1)[C_{P \otimes K_P}] \\
= q(q + 1)[C_{2P \otimes P_P}] + q[C_{P \otimes K_P}] + q[C_{P \otimes M_{ij}}]; \\
(6.11)
\]

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\[ [C_{P_j}][C_{P_i}] = (q + 1)[C_{P_j}][C_{P_i}] + [C_{P_j}][K_{P_i}] \]
\[ = (q + 1) \left( \frac{1}{q^2} : \frac{q^3(q - 1)^3(q + 1)}{q(q - 1)^3(q + 1)} \right) [C_{2P_j \oplus P_i}] + \]
\[ q^2 \cdot \frac{q^3(q - 1)^2}{q(q - 1)^3(q + 1)} [C_{P_j} \oplus C_{M_{ij}}] + [C_{P_j} \oplus K_{P_i}] \]
\[ = (q + 1)[C_{2P_j \oplus P_i}] + (q + 1)[C_{P_j} \oplus C_{M_{ij}}] + [C_{P_j} \oplus K_{P_i}] \]

Hence, we complete the proof of the relations in (6.13). \( \square \)

**Remark 6.2.**

(1) As a corollary of Proposition 6.1, taking \( q = 1 \), we obtain Relation (a) in Theorem 2.

(2) There do not exist \( x, y, z \in Q(v) \) such that

\[ x[C_{P_j}]^2[C_{P_i}] + y[C_{P_j}][C_{P_i}] + z[C_{P_j}][C_{P_i}] = 0 \]

or

\[ x[C_{P_j}]^2[C_{P_i}] + y[C_{P_j}][C_{P_i}] + z[C_{P_j}][C_{P_i}] = 0. \]

That is, we fail to obtain an analogue of quantum Serre relations in \( \mathcal{H}(C_1(\mathcal{P})) \).

Let \( \mathfrak{h}_0 \) be the ideal of \( \mathcal{H}(C_1(\mathcal{P})) \) generated by all \( [K_{P_i}] \), \( 1 \leq i \leq n \). Set

\[ \mathcal{H}(C_1(\mathcal{P})) := \mathcal{H}(C_1(\mathcal{P}))/\mathfrak{h}_0. \]

By abuse of notation, in what follows, the image of each \( [C_{P_j}] \) under the canonical projection is also denoted by \( [C_{P_j}] \).

**Proposition 6.3. (Fundamental relations)**

For any \( 1 \leq i \neq j \leq n \), we have in \( \mathcal{H}(C_1(\mathcal{P})) \) that

(1) if \( a_{ij} = 0 \), then \( [C_{P_i}][C_{P_j}] = [C_{P_j}][C_{P_i}] \);

(2) if \( a_{ij} = 1 \), then

\[ [C_{P_i}][C_{P_j}]^2 - (q + 1)[C_{P_i}][C_{P_j}] + q[C_{P_j}]^2[C_{P_i}] = 0 \]

and

\[ q[C_{P_j}][C_{P_i}]^2 - (q + 1)[C_{P_i}][C_{P_j}] + [C_{P_i}]^2[C_{P_j}] = 0. \]

**Proof.**

(1) This is clear.

(2) By Proposition 6.1, noting that \( [C_{P_i} \oplus K_{P_j}] = [K_{P_j}][C_{P_i}] \in \mathfrak{h}_0 \) and \( [C_{P_i} \oplus K_{P_j}] = [C_{P_i}][K_{P_j}] \in \mathfrak{h}_0 \), we obtain in \( \mathcal{H}(C_1(\mathcal{P})) \) that

\[
\begin{align*}
[C_{P_i}]^2[C_{P_j}] &= (q + 1)[C_{P_i} \oplus 2P_j] + (q + 1)[C_{P_i} \oplus C_{M_{ij}}] \\
[C_{P_i}][C_{P_i}][C_{P_j}] &= q(q + 1)[C_{P_i} \oplus 2P_j] + q[C_{P_i} \oplus C_{M_{ij}}] \\
[C_{P_i}][C_{P_j}]^2 &= q^2(q + 1)[C_{P_i} \oplus 2P_j]
\end{align*}
\] (6.13)
and
\[
\begin{align*}
[C_P_i]^2[C_P_j] &= q^2(q + 1)[C_P_i \oplus 2P_i] \\
[C_P_i][C_P_j][C_P_n] &= q(q + 1)[C_P_i \oplus 2P_i] + q[C_P_i \oplus C_M_{i,j}] \\
[C_P_j][C_P_i]^2 &= (q + 1)[C_P_j \oplus 2P_i] + (q + 1)[C_P_i \oplus C_M_{i,j}].
\end{align*}
\] (6.14)

Then it is easy to obtain the desired relations. \qed

7. Quantum groups and Hall algebras

In this section, in order to acquire the quantum Serre relations via \( H(C_1(\mathcal{P})) \), we need to consider the twisted versions of Hall algebras.

For any \( M, N \in \text{mod} \ A \), define
\[
\langle M, N \rangle := \dim_k \text{Hom}_A(M, N) - \dim_k \text{Ext}_A^1(M, N).
\]
It induces a bilinear form
\[
\langle \cdot, \cdot \rangle : K(A) \times K(A) \rightarrow \mathbb{Z},
\]
known as the Euler form. The Ringel–Hall algebra \( H_{tw}(A) \) of \( A \) is the same vector space as \( H(A) \) but with twisted multiplication defined by
\[
[M] \ast [N] = v^{(M, N)} \sum_{[L]} F_{MN}^L [L].
\]

Let \( U_v(n^+) \) be the positive part of the quantum group associated to the Dynkin quiver \( Q \), whose canonical generators are denoted by \( E_i, 1 \leq i \leq n \). The following theorem is well-known.

**Theorem 7.1.** There exists an isomorphism of algebras
\[
R : U_v(n^+) \rightarrow H_{tw}(A)
\]
defined on generators by \( R(E_i) = [S_i] \).

Let us give a twisted version of the Hall algebra \( H(C_1(\mathcal{P})) \).

**Definition 7.2.** The twisted Hall algebra \( H_{tw}(C_1(\mathcal{P})) \) of \( C_1(\mathcal{P}) \) is the same vector space as \( H(C_1(\mathcal{P})) \) but with twisted multiplication defined by
\[
[X] \ast [Y] = v^{-\langle Y_0, X_0 \rangle} \sum_{[Z]} F_{X Y}^Z [Z],
\]
where \( X^* = (X_0, c), Y^* = (Y_0, d) \in C_1(\mathcal{P}). \)

It is easy to see that \( H_{tw}(C_1(\mathcal{P})) \) is still an associative algebra.
Remark 7.3. Let $\mathfrak{g}$ be the subalgebra of $\mathcal{H}_{\text{tw}}(C_1(\mathcal{P}))$ generated by all $[K_P]$, $1 \leq i \leq n$. Then $\mathfrak{g} \cong k[x_1, x_2, \cdots, x_n]$, and moreover it is contained in the center of $\mathcal{H}_{\text{tw}}(C_1(\mathcal{P}))$. That is, for any $P \in \mathcal{P}$ and $X \in C_1(\mathcal{P})$, we have that $[K_P] * [X] = [X] * [K_P]$ (This can be easily verified by Riedtmann–Peng formula together with Proposition 2.4 in [18]).

Let $\mathfrak{R}_1$ be the ideal of $\mathcal{H}_{\text{tw}}(C_1(\mathcal{P}))$ generated by all $[K_P]$, $1 \leq i \leq n$. Set

\[ \mathcal{H}_{\text{tw}}(C_1(\mathcal{P})) := \mathcal{H}_{\text{tw}}(C_1(\mathcal{P}))/\mathfrak{R}_1. \]

Let us reformulate Propositions 6.1 and 6.3

Proposition 7.4. For any $1 \leq i \neq j \leq n$,

1. if $a_{ij} = 0$, then $[C_P_1] * [C_P_2] = [C_P_2] * [C_P_1] = [C_{P_i \oplus P_j}]$ in $\mathcal{H}_{\text{tw}}(C_1(\mathcal{P}))$;

2. if $a_{ij} = 1$, then the following relations

\[
\begin{align*}
[C_P_1]^{*2} & * [C_P_1] = (v + v^{-1})[C_{P_1 \oplus 2P_1}] + (v + v^{-1})[C_{P_1} \oplus M_{M_1}] + v^{-1}[C_{P_1} \oplus K_{P_1}] \\
[C_P_1] * [C_P_1] * [C_P_1] & = (q + 1)[C_{P_{1 \oplus 2P_1}}] + [C_{P_1} \oplus M_{M_1}] + [C_{P_1} \oplus K_{P_1}] \\
[C_P_1] * [C_P_1]^{*2} & = (v + v^{-1})[C_{P_{1 \oplus 2P_1}}] + (v + v^{-1})[C_{P_1} \oplus M_{M_1}] + v^{-1}[C_{P_1} \oplus K_{P_1}]
\end{align*}
\]

and

\[
\begin{align*}
[C_P_1]^{*2} & * [C_P_1] = v(q + 1)[C_{P_1 \oplus 2P_1}] + v^{-1}[C_{P_1} \oplus K_{P_1}] \\
[C_P_1] * [C_P_1] * [C_P_1] & = (q + 1)[C_{P_{1 \oplus 2P_1}}] + [C_{P_1} \oplus M_{M_1}] + [C_{P_1} \oplus K_{P_1}] \\
[C_P_1] * [C_P_1]^{*2} & = (v + v^{-1})[C_{P_{1 \oplus 2P_1}}] + (v + v^{-1})[C_{P_1} \oplus M_{M_1}] + v^{-1}[C_{P_1} \oplus K_{P_1}]
\end{align*}
\]

hold in $\mathcal{H}_{\text{tw}}(C_1(\mathcal{P}))$.

Proof. (1) $[C_P_1] * [C_P_1] = v^{-\langle P_1, P_1 \rangle}[C_{P_1}][C_P_1] = [C_{P_1}][C_P_1]$. Similarly, $[C_P_1] * [C_{P_1}] = [C_{P_1}][C_P_1]$;

(2) $[C_P_1]^{*2} * [C_P_1] = v^{-\langle P_1, P_1 \rangle + \langle P_1, P_1 \rangle}[C_{P_1}]^{*2}[C_{P_1}] = v^{-1}[C_{P_1}]^{*2}[C_{P_1}]$. Similarly, we can prove the other. □

Proposition 7.5. (Quantum Serre relations) For any $1 \leq i \neq j \leq n$, we have in $\mathcal{H}_{\text{tw}}(C_1(\mathcal{P}))$ that

1. if $a_{ij} = 0$, then $[C_P_1] * [C_P_2] = [C_P_2] * [C_P_1]$;

2. if $a_{ij} = 1$, then

\[
[C_P_1] * [C_P_1]^{*2} - (v + v^{-1})[C_P_1] * [C_P_1] + [C_P_1]^{*2} * [C_P_1] = 0
\]

and

\[
[C_P_1] * [C_P_1]^{*2} - (v + v^{-1})[C_P_1] * [C_P_1] + [C_P_1]^{*2} * [C_P_1] = 0.
\]

Proof. (2) We only need to note that (7.4) and (7.3) are symmetric with respect to $i$ and $j$, then by Proposition 7.4(2), we complete the proof. □

Define the ideal of $\mathcal{H}_{\text{tw}}(C_1(\mathcal{P}))$

\[ \mathfrak{I}_1 := \langle [C_P_1] * [C_P_2] - [C_P_2] * [C_P_1] | L_{ij}^2 \neq \emptyset, 1 \leq i \neq j \leq n \rangle. \]
Theorem 7.6. There exists an epimorphism of algebras
\[ \varphi : \mathcal{H}_{tw}(A) \longrightarrow \mathcal{H}_{tw}(C_1(\mathcal{P}))/I_1 \]
defined by \([S_i] \mapsto [C_{P_i}]\).

Proof. We only need to note that \(\mathcal{H}_{tw}(A)\) is generated by all \([S_i]\) and the quantum Serre relations. Then combining Proposition 7.5 with Theorem 4.1, we complete the proof. \(\square\)

Corollary 7.7. Let \(Q\) be a bipartite Dynkin quiver. Then there exists an isomorphism of algebras \(\psi : \mathcal{H}_{tw}(A) \rightarrow \mathcal{H}_{tw}(C_1(\mathcal{P}))\) defined by \([S_i] \mapsto [C_{P_i}]\).

Proof. Consider
\[ \psi' : \mathcal{H}_{tw}(C_1(\mathcal{P})) \rightarrow \mathcal{H}_{tw}(A), [C_M] \mapsto [\text{top } M], \]
then it is easy to see that \(\psi'\psi = 1\), thus \(\psi\) is injective. \(\square\)

Combining Corollary 7.7 with Theorem 7.1, we obtain the following

Corollary 7.8. Let \(Q\) be a bipartite Dynkin quiver. Then there exists an isomorphism of algebras \(\tilde{\psi} : U_v(n^+) \rightarrow \mathcal{H}_{tw}(C_1(\mathcal{P}))\) defined by \(E_i \mapsto [C_{P_i}]\).

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