Vortex oscillations in confined Bose-Einstein condensate interacting with 1D optical lattice

Tomoya ISOSHIMA

Institute of Physics, Academia Sinica, Nankang, Taipei, Taiwan 11529

We study Bose-Einstein condensate of atomic Boson gases trapped in a composite potential of a harmonic potential and an optical lattice potential. We found a series of collective excitations that induces localized vortex oscillations with a characteristic wavelength. The oscillations might be observed experimentally when radial confinement is tight. We present the excitation spectra of the vortex oscillation modes and propose a way to experimentally excite the modes.

KEYWORDS: vortex, optical lattice, Bose-Einstein condensate

1. Introduction

Since the realization of vortex state in Bose-Einstein condensates (BEC) in traps,\(^1,2\) various kinds of vortex configurations are experimentally created. A clear advantage of studying vortices in the condensates of atomic gases is that the vortex cores are directly observable. Using the imaging of atom density in planes perpendicular to the vortex axis, vortex lattices,\(^3\) giant vortex,\(^4\) coreless vortex,\(^5\) and doubly quantized vortices\(^6,7\) have been observed. By the imaging parallel to the vortex lines the bending\(^8\) and the oscillation\(^9\) of the vortex line are found.

Recently dilute atomic gases interacting with the optical lattice\(^10\) are actively studied. The Boson gases in the optical lattice behave in two different ways, as a superfluid or a Mott-insulator.\(^11\) The condition of the phase transition between these two phases is determined by various parameters, such as the particle number and the intensity and the dimension of the lattice potential. Vortices are also an interesting phenomena here. Vortices may exist in presence of the optical lattice potentials, and the properties of the vortices are affected by the lattice potential. There also are researches on several kinds of the vortices on the BEC in the optical lattices near the Mott-insulator transition\(^12\) and in the superfluid region.\(^13,15\)

The vortices whose cores are perpendicular to the 2D optical lattices are studied in ref. 13. The dynamics of vortices and a new kind of vortex that is unique to the optical lattices are reported. On 1D optical lattice, the vortices parallel to the optical lattice are studied\(^13,15\) and they predicted various vortex oscillations utilizing the analytical formulations. The formulation assumes periodic system along the lattice and neglects the effects of the axial confinement potential.

In this paper, we treat the condensate in 1D optical lattice confined along the principal axis, to find features unique to the confined systems. The whole confined system is treated within the Gross-Pitaevskii (GP) and Bogoliubov equations, and we study structures of the spectra that are unique to the confined system.

In our previous work, we found\(^16\) characteristic excitations in the condensate confined in 1D optical lattice in absence of the vortex. The branch of these characteristic excitations was identified by a breaking down of phase count, but the detailed origin and the properties were not clarified. This paper studies corresponding modes, especially within the Kelvin modes that accompanies vortices. And identify the characteristic excitations as the shortest wavelength modes both in the vortex systems and the vortexfree systems.

2. Formulation

We study the excitation spectra of BEC confined in a composite potential

\[
V(r, z) = \frac{m}{2} (2\pi \nu_r)^2 r^2 + V_z(z) + V_{opt}(z),
\]

\[
V_z(z) = \frac{m}{2} (2\pi \nu_z)^2 z^2,
\]

\[
V_{opt}(z) = s E_R \sin^2 \left( \frac{\pi z}{d} \right)
\]

that consists of an optical lattice in the principal direction of consideration and harmonic confinement potentials along the lateral dimensions. Here \(s\) is an intensity parameter of the periodic optical potential, \(E_R = \frac{k^2}{2m} (\frac{\lambda}{d})^2\) is a recoil energy, \(m\) is the mass of a \(^{87}\)Rb atom, \(\nu_r\) and \(\nu_z\) are the frequencies of the harmonic potentials, and \(d\) is the period of the potential. We employ \(d = 0.395 \mu m\) which is one-half of the typical\(^17\) wavelength \(\lambda\) for the laser beams, \(\nu_r = 100 s^{-1}\) and \(\nu_z = 30 s^{-1}\) which are also within the typical ranges in experiments.

The condensate may have finite angular momentum which is essential to treat the systems having vortices.\(^14\) For simplicity, we assume the condensate rotational symmetry

\[
\phi(r, z, \theta) = \phi(r, z) e^{i\omega \theta},
\]
where \( w = 0 \) for vortexfree system and \( w = 1 \) for systems with a vortex. The wavefunction \( \phi(r, z) \) is a real function. If this function is complex and has vorticity in \( rz \) plane, the condensate will have parallel vortex rings\(^{18}\) which is also an interesting subject.

The BEC in an optical lattice with the lower value of the intensity \( s \) is well described by the Gross-Pitaevskii equation\(^{19}\)

\[
\left(-C\nabla^2 + V(r, z) + g|\phi(r)|^2\right) \phi(r) = \mu \phi(r)
\]

(5)

where \( \phi(r) \) is the wavefunction of the condensate, \( \mu \) is the chemical potential, \( g = \frac{4\pi\hbar^2a}{m} \) is an interaction parameter, and \( C = \hbar^2/(2m) \) is a constant. We denote the components of \( r \) to be \( (r, z, \theta) \). We treat a condensate with \( 5 \times 10^4 \) atoms of \(^{87}\)Rb. The scattering length \( a = 5.4 \text{ nm} \) is employed for the \(^{87}\)Rb atoms. The period of the optical lattice is \( d = 0.395 \text{ \mu m} \) and the condensate occupies about 80 sites. The centered lattice site \((-d/2 < z < d/2)\) is occupied by about 1290 (1110) atoms when \( s = 0 \) (\( s = 10 \)).

The condensate supports excitation spectra within the Bogoliubov framework. The Bogoliubov equations\(^{19}\)

\[
\left(-C\nabla^2 + V(r, z) + 2g|\phi(r)|^2 - \mu\right) u_q - g\phi^*v_q = \varepsilon_q u_q,
\]

(6)

\[
\left(-C\nabla^2 + V(r, z) + 2g|\phi(r)|^2 - \mu\right) v_q - g\phi^*u_q = -\varepsilon_q v_q
\]

(7)

yield the excitation energies \( \varepsilon_q \) and the corresponding wavefunctions, \( u_q \) and \( v_q \). The symmetry of system enables us to rewrite the wavefunctions

\[
u_q(r) = u_q(r, z)e^{i(w+q_\theta)\theta},
\]

(8)

\[
u_q(r) = u_q(r, z)e^{i(w-q_\theta)\theta},
\]

(9)

where \( u_q(r, z) \) and \( v_q(r, z) \) are real functions and \( q_\theta \) is an integer representing the angular momentum. The excitations with angular momentum \( q_\theta = -1 \) is related to the oscillation of vortex core and are called Kelvin modes.

3. Vortex Oscillations

We consider the small perturbation around the stationary state

\[
\phi'(r, t) = \phi(r) + C_1 u_q(r)e^{-i\omega t}/\hbar - v_q(r)e^{i\omega t}/\hbar
\]

(10)

where \( C_1 \) is a constant. At \( t = 0 \), the perturbation by the Kelvin modes \((q_\theta = -1)\) is

\[
\phi'(r, 0) = \phi(r, z)e^{i\theta} + C_1 (u_q(r, z) - v_q(r, z)e^{-2i\theta})
\]

(11)

and the corresponding modified density \( n' \equiv |\phi'|^2 \) at \( t = 0 \) is

\[
n'(r, z, \theta) \simeq n(r, z) + 2C_1 \phi(r, z) \times \{u_q(r, z)\cos\theta - v_q(r, z)\cos 3\theta\}
\]

(12)

where \( n \equiv |\phi|^2 \). The principal direction of perturbation of the density \( n'(r, z, \theta) \) above is along the \( x \)-axis \((\theta = 0\text{ and }\pi)\). We write the displacement of the vortex core in the \( xz \) plane \( \Delta x(z) \), which is defined at bottoms of the lattice sites \( z = jd \) \((j \text{ is an integer})\). Figures 1(a) - (g) show several patterns of the displacement \( \Delta x(z) \) of the vortex core.

To balance the original density and the perturbation, we employ a value

\[
C_1 = 0.1 \frac{\max_{r} n(r)}{\max_{r, q} ||\phi(r) \{u_q(r) - v_q(r)\}||}.
\]

(13)

in the figures. Here \( q \) covers the Kelvin modes \((q_\theta = -1)\).

On unconfined periodic system, a wavenumber, e.g. \( k \), denotes the phase of the wavefunction through a factor \( \exp(i k z) \). But explicitly, this definition of \( k \) cannot be used in confined systems. The exception is the condensate interacting with moving optical lattices. The velocity of the optical lattice and the wavenumber are closely related, and the wavenumber remains good approximation to denote an excitation.\(^{10,20,21}\) Because the moving optical lattice is not interest in this article, we employ another index similar to the wavenumber, but better related to an observable property of the excitation.

We define a number \( c \) that counts the change of sign of \( \Delta x \). Because \( v_q(r, z) = 0 \) at \( r = 0 \), the sign and the amplitude of \( \Delta x \) corresponds to those of \( u_q(r, z) \) at \( r = 0 \). Figure 2(a) plots displacement count \( c \) vs. excitation energy \( \varepsilon \) for system with (bullets, circles) and without (crosses) the optical lattice potential. As the optical lattice potential becomes stronger \((s \text{ increases})\), the excitation energy for certain \( c \) becomes lower. It means that vortex oscillations with many nodes are more likely to be excited at stronger lattice potential.

4. Shortest wavelength modes

The displacement count \( c \) of the Kelvin modes smoothly increase as shown in Fig. 2(a). But this increase breaks down and the \( c \) decreases at higher energy, see Fig. 2(b). The largest displacement count is around
Accompanying this shrink, the displacement count \( c \) increases to, for example, 36 [Fig. 1(f) - (g)]. These plots show that the displacement \( \Delta r \) in this decreasing aspect has opposite sign between the neighboring sites. It means that the wavefunctions \( u \) at the vortex core \((r = 0)\) have opposite phase between the neighboring sites. Therefore, the wavelength of these excitations are \( 2d \). This is the shortest wavelength possible within the range of excitation energy \( \varepsilon \ll sE_R \) we consider. We call these excitations the shortest wavelength excitations hereafter.

We pointed out the shortest wavelength excitations within the Kelvin modes. In addition to these, excitations of this kind commonly exist in the confined condensates interacting with an optical lattice. The condensate with and without the vortex supports shortest wavelength excitations within the breathing \((q_\theta = 0)\), dipole \((q_\theta = \pm 1)\), quadrupole \((q_\theta = \pm 2)\) modes, and modes with several higher angular momenta. Figures 3(c) and 3(d) are those within the quadrupole modes of \( q_\theta = \pm 2 \). One-dimensional system with 1D optical lattice and harmonic confinement potential also has corresponding excitations. The set of these excitations are mentioned as “branch B” in ref. 16.

4.1 Energy of the shortest wavelength modes

In an unconfined 1-dimensional periodic system with period \( d \), the largest wavenumber is \( \pi/d \), and the corresponding phase difference between the neighboring sites is \( \pi \) at the upper end of the so-called first band. The shortest wavelength modes in our confined systems also have the phase differences \( \pi \). And there is always an energy gap above the shortest wavelength excitations if we ignore the modes with radial nodes. So it is reasonable to think that these excitations also belong to the upper end of the first band.

Figure 4(a) compares ranges of the energy of the shortest wavelength excitations and the upper end \( E_{1D}(s) \) of the first band of an ideal (noninteracting, unconfined) 1-dimensional periodic system.\(^{10,16,21}\) It is seen that the energy ranges of the shortest wavelength excitations

\[ \varepsilon \ll \frac{sE_R}{E} \]

were within the Kelvin modes. In addition to these, excitations of this kind commonly exist in the confined condensates interacting with an optical lattice. The condensate with and without the vortex supports shortest wavelength excitations within the breathing \((q_\theta = 0)\), dipole \((q_\theta = \pm 1)\), quadrupole \((q_\theta = \pm 2)\) modes, and modes with several higher angular momenta. Figures 3(c) and 3(d) are those within the quadrupole modes of \( q_\theta = \pm 2 \). One-dimensional system with 1D optical lattice and harmonic confinement potential also has corresponding excitations.

The set of these excitations are mentioned as “branch B” in ref. 16.

\[ \varepsilon \ll \frac{sE_R}{E} \]
4.2 Origin of the spatial dependence

While other modes are elongated whole the condensate, the shortest wavelength excitations are localized near the center \( z = 0 \) of the system. This feature is clearer for modes with the highest energies, e.g. Figs. 1(g) and 3(d).

Within the GP equation and the Bogoliubov equations, the depth of the optical lattice \( s \) is changed due to the meanfield energy \( g|\phi|^2 \) in eqs. (5), (6), and (7). The effective depth becomes smallest near the center \( z = 0 \) of the system. In 1D periodic system, the smaller \( s \) becomes, the higher the upper edge \( E_{1D}(s) \) of the first band becomes. This agrees with the localization of the highest shortest wavelength mode around \( z = 0 \) at which the effective \( s \) becomes the smallest.

To make this discussion clear, we define effective amplitude

\[
s_{\text{eff}}(z) = s - \max \left( g|\phi(r, z)|^2 \right) / E_R
\]

at bottoms of the lattice sites \( z = jd \) (\( j \) is an integer). Within the ‘max’ above, the radius \( r \) is chosen to have the maximum. The function \( s_{\text{eff}}(z) \) is smallest near the center \( z = 0 \) of the system.

Spatial (along \( z \)-axis) dependence of the effective depth \( s_{\text{eff}}(z) \) and the energy may be related by a function \( E_{1D}(s_{\text{eff}}(z)) \). This function and the energy and spatial localization of the Kelvin modes are compared in Fig. 4(b). The excitations at the left hand side above \( \varepsilon > 0.27 E_R \) are the wavefunctions of the shortest wavelength modes. The correspondence between the \( s_{\text{eff}} \) and the axial \( (z-) \) range of the wavefunctions is clear.

5. How to observe the shortest wavelength modes

On trapped BEC of atomic gases, the quadrupole modes may be excited experimentally by deforming the confining potential. And both the experiment and the following theoretical study insist that Kelvin modes \( (q_0 = -1) \) are excited by the decay from the counter-rotating quadrupole mode \( (q_0 = -2) \). For the condensates interacting with the periodic potentials, a resonance condition between the quadrupole mode and the
Kelvin modes is given in § V of ref. 14. The criteria given in the references above may be simplified to this: The excitation energy $\varepsilon_{-1}$ (or frequency $\varepsilon_{-1}/\hbar$) of the excited Kelvin modes satisfy

$$\varepsilon_{-1} = \varepsilon_{-2}/2,$$  \hspace{1cm} (15)

where $\varepsilon_{-2}$ is the energy of the counter-rotating quadrupole mode at $q_0 = -2$. If one of the shortest wavelength modes satisfies eq. (15), the mode will be excited as an oscillation of the vortex core. The oscillation will be experimentally observable.

The corresponding excitations in this paper is the lowest centered counter-rotating quadrupole mode $q_0 = -2$ in Fig. 3(f) and the Kelvin mode whose energy satisfies eq. (15). The excitation energy $\varepsilon_{-2}$ varies only little (stays between 0.0343$E_R$ to 0.0351$E_R$) for $s = 0$ to 10. And half of the energy $\varepsilon_{-2}/2$ is indicated in Fig. 2(a). The figure shows that Kelvin modes with $c = 13$ ($c = 22$) at $s = 5$ ($s = 22$) are on the dotted line and satisfying the criterion.

Then how can we make the shortest wavelength modes satisfy eq. (15)? The right hand side, the energy of the modes has common wave lengths satisfies eq. (15), the mode will be excited and one of the shortest wavelength modes satisfies eq. (15). Here approximate a function of radial confinement potential $\nu_r$, this two may be varied independently.

We increased $\nu_r$ while keeping $d$. At $\nu_r = 4000$ Hz, the shortest wavelength mode satisfies eq. (15). Here axial trap frequency $\nu_z$ particle number $N$ are changed to $\nu_z = 300$ Hz and $N = 10^5$. Figure 5 presents spectra of the Kelvin modes and the energy $\varepsilon_{-2}/2$. In this tight radial trapping potential, the shortest wavelength modes satisfies eq. (15) at $s > 5$ and localized vortex oscillation of period $2d$ will be observed by excitations of the counter-rotating quadrupole mode.

In writing eq. (15), we neglected the width of the resonances whose evaluation requires huge computational resource. Explicitly speaking, the condition is satisfied at only discrete values of $s$. The allowed width of $s$ is a function of various parameters, including the temperature. This discreteness can be avoided by smoothly varying $s$.

6. Discussion

In BEC of atom gases in 1D optical lattice, existence of the spatially localized modes was reported in ref. 16. But the nature of the modes was not clarified there. In this paper, we find that the modes have common wavelength $2d$ and commonly exist at various angular momenta, and exists both at the vortexfree condensate and the condensate with a vortex.

Among the modes, we report on the Kelvin modes and the corresponding vortex oscillations in systems with vortex. Provided that the vortex oscillation was observed in systems without the optical lattice, the same observation will be possible with optical lattice. By choosing the trapping potentials and the intensity of the optical lattice, we can make the vortex oscillation modes of wavelength $2d$ satisfy the resonance condition eq. (15). Then the corresponding oscillation, whose directions are the opposite between the neighboring sites, will be observed.

I thank S. K. Yip and T. Mizushima for useful discussions.