Equilibrium and potential in coalitional congestion games

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Abstract The model of congestion games is widely used to analyze games related to traffic and communication. A central property of these games is that they are potential games and hence posses a pure Nash equilibrium. In reality, it is often the case that some players cooperatively decide on their joint action in order to maximize the coalition’s total utility. This is modeled by Coalitional Congestion Games. Typical settings include truck drivers who work for the same shipping company, or routers that belong to the same ISP. The formation of coalitions will typically imply that the resulting coalitional congestion game will no longer posses a pure Nash equilibrium. In this paper, we provide conditions under which such games are potential games and posses a pure Nash equilibrium.

Keywords Congestion games · Equilibrium · Potential · Coalitions

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1 Introduction

Congestion games, introduced by Rosenthal (1973), form a very natural model for studying many real-life strategic settings: traffic problems, load balancing, routing, network planning, facility locations and more. In a congestion game players must
choose some subset of resources from a given set of resources (e.g., a subset of edges leading from the source to the target on a graph). The congestion of a resource is a function of the number of players choosing it and each player seeks to minimize his total congestion across all chosen resources. In many modeling instances players and the decision making entity have been thought of as one and the same. However, in a variety of settings this may not be the case.

Consider, for example, a traffic routing game where each driver chooses his route in order to minimize his travel time, while accounting for congestion along the route caused by other drivers. Now, in many cases drivers are actually employees in shipping firms, and in fact it is in the interest of the shipping firm to minimize the total travel time of its fleet. Similarly, routers in a communication network participate in a congestion game. However, as various routers may belong to the same ISP we are again in a setting where coalitions naturally form. This motivated Fotakis et al. (2006) and Hayrapetyan et al. (2006) to introduce the notion of Coalitional Congestion Games (CCG). In a CCG, we think about the coalitions as players and each coalition maximizes its total utility. The coalitional congestion game inherits its structure from the original game, once the coalitions of players from the original congestion games (now, becoming the players of the coalitional congestion game) have been identified.

A most notable property of congestion games is that the set of pure Nash equilibria is nonempty. This has been shown by Rosenthal in (1973). Later, Monderer and Shapley (1996) formally introduce potential games and show the equivalence of these two classes. The fact that potential games posses a pure Nash equilibrium is straightforward. Unfortunately, the statement that a CCG is a potential game or that it possesses a pure Nash equilibrium is generally false. In this paper, we investigate conditions under which this statement is true. We focus on a subset of congestion games called simple congestion games, where each player is restricted to choose a single resource.

Our main contributions are:

1. Whenever each coalition contains at most two players a CCG induced from a simple congestion game possesses a pure-strategy Nash equilibrium (Theorem 1).
2. If some coalition contains three players, then there need not exist a pure-strategy Nash equilibrium (Example 1).
3. If a the congestion game is not simple then there need not exist a pure-strategy Nash equilibrium, even if the largest coalition is a pair (Example 2); and
4. Suppose there exists at least one singleton coalition and at least one coalition composed of two players, then a coalitional congestion game induced from a simple congestion game is a potential game if and only if cost functions are linear (Theorem 2).

Our results extend and complement the results in Fotakis et al. (2006) and Hayrapetyan et al. (2006). For example, Fotakis et al. (2006) show that if the resource cost functions are linear then the coalitional congestion game is a potential game. We show that, with some additional mild conditions on the partition structure, this is also a necessary condition. Hayrapetyan et al. (2006) shows that if the underlying congestion game is simple and costs are weakly convex then the game possesses a pure Nash equilibrium. We demonstrate additional settings where this holds.
The structure of the paper is as follows: Sect. 2 provides a model of coalitional congestion games, Sect. 3 discusses the conditions for the existence of a pure Nash equilibrium in such games, and Sect. 4 discusses the conditions for the existence of a potential function. All proofs are relegated to an appendix.

2 Model

Let \( G = (N, (S_i)_{i=1}^n, (U_i)_{i=1}^n) \) be a non-cooperative game in strategic form, where \( N \) is a finite set of \( n \) players, \( S_i \) is the set of strategies of player \( i \), and \( U_i : S \to \mathbb{R} \) is player \( i \)’s utility function (with \( S = \bigtimes_{i=1}^n S_i \)). Let \( C = \{C_1, \ldots, C_{nC}\} \) be a partition of \( N \) into \( nC \) nonempty sets. Hence: \( \bigcup_{k=1}^{nC} C_k = N \) and \( C_k \cap C_l = \emptyset \ \forall k \neq l \in \{1, \ldots, nC\} \).

The game \( G \) and the partition \( C \) induce a Coalitional Non-Cooperative Game, \( G^C = (N^C, S^C, U^C) \), defined as follows:

- \( N^C \) is the set of agents which are the elements of \( C \).
- The strategy space is \( S^C = \times_{k \in C} S^C_k \) where \( S^C_k = \times_{i \in C_k} S_i \).

Note that \( S^C = \times_{k \in C} S^C_k = \times_{i \in C} \times_{i \in C_k} S_i \) is isomorphic to \( S = \times_{i=1}^n S_i \), since we only changed the order of the coordinates. Thus, an arbitrary element \( s^C \in S^C \) can also be viewed as an agent vector in \( S \).

- The utility function is defined as follows: \( \forall s^C \in S^C \ \ U^C_k(s^C) = \sum_{i \in C_k} U_i(s^C) \) and \( U^C = \{U^C_k\}_{k \in C} \).

In the context of \( G^C \), \( G \) is the Underlying Game and a player in \( G \) is referred to as a sub-agent. As always, a pure Nash Equilibrium of the game \( G^C \) is a strategy profile \( s \in S^C \) such that \( \forall k \in N^C, U^C_k(s) \geq U^C_k(s_{-k}, t_{-k}) \ \forall t_{-k} \in S^C_{-k} \). Let \( NE(G^C) \) denote the (possibly empty) set of pure Nash equilibrium strategy profiles in \( G^C \).

A congestion game is a game \( G = (N, R, \Sigma, P) \) where \( N \) is the finite set of agents, \( R \) is the finite set of resources, \( P = \{P_r\}_{r \in R} \) are the resource costs functions, where \( P_r : \{1, \ldots, n\} \to \mathbb{R} \) and \( \Sigma = \times_{i \in N} \Sigma_i \), where \( \Sigma_i \subseteq 2^R \), is \( i \)’s strategy space. Agent \( i \) selects \( s_i \in \Sigma_i \) and pays \( \sum_{r \in s_i} P_r(c(s)_r) \), where \( c(s)_r = \sum_{j \in N} 1_{[\{r \in s_j\}]} \) is the number of agents who select \( r \) in \( s \). In utility terms, the utility of agent \( i \) is \( U_i(s) = -\sum_{r \in s_i} P_r(c(s)_r) \). If \( \Sigma_i = R \ \forall i \in N \) then \( G = (N, R, \Sigma, P) \) is called a Simple Congestion Game. We make the natural assumption that the functions \( P_r \) are non-negative and increasing.

Fix a set of resources \( R \). A congestion vector is an element of \( \mathbb{N}^R \) whose elements sum up to \( n \). A simple congestion game \( G \) and a strategy profile \( s \) induce a congestion vector \( c(s) = \{c(s)_r\}_{r \in R} \).

A strategy profile \( s^C \) of a coalitional congestion game \( G^C \) induces a private congestion vector, denoted \( c_k \), for each of the agents in \( G^C \). Formally, \( c_k(s^C)_r = |\{i \in C_k : s_i = r\}|, \forall r \in R \). Thus, \( c_k = (c_k)_r \in R \) is an element of \( \mathbb{N}^R \) whose entries sum up to \( |C_k| \).

Let \( X \subseteq S \) be a subset of strategy profiles. We denote by \( c(X) \) the corresponding set of congestion vectors: \( c(X) = \{c(s) | s \in X\} \).
3 CCG and pure Nash equilibria

The following preliminary result asserts that if a Nash equilibrium of the underlying game is composed of strategies such that all sub-agents of any agent choose different resources, then it is also an equilibrium of the coalitional game.

**Proposition 1** Let \( G \) be a simple congestion game, \( C \) a partition of \( N \) and \( G^C \) be the induced CCG. Let \( s^C \) be a strategy profile of \( G^C \) such that for all \( k \) and for all \( i \neq j \in C_k \), \( s_i \neq s_j \). If \( c(s^C) \in c(N E(G)) \) then \( s^C \in N E(G^C) \).

This is key to proving our central result about the existence of a Nash equilibrium in CCGs.

**Theorem 1** Let \( G \) be a Simple Congestion Game, \( C \) a partition where the largest element is of size 2, and \( G^C \) the CCG with the underlying game \( G \) and the partition \( C \). Then \( N E(G^C) \neq \emptyset \). That is, if the largest coalition is a pair, a pure Nash equilibrium always exists.

Does this existence result extend to other partition forms, where the maximal element has more than two sub-agents? The following example demonstrates that this is not true in general:

**Example 1** Consider a game with two identical resources A and B and four sub-agents, with the following payment functions:

| Resource/agents #: | 1 | 2 | 3 | 4 |
|--------------------|---|---|---|---|
| A                  | 0 | 12| 16| 18|
| B                  | 0 | 12| 16| 18|

When \( C = \{\{1, 2, 3\}, \{4\}\} \) the matrix form of the resulting 2-player CCG is

\[
\begin{array}{ccc}
G^C & A & B \\
A, A, A & -54, -18 & -48, 0 \\
A, A, B & -32, -16 & -36, -12 \\
A, B, B & -36, -12 & -32, -16 \\
B, B, B & -48, 0 & -54, -18 \\
\end{array}
\]

Whereas the underlying simple congestion game has a pure Nash equilibrium this CCG has none. To verify this note that for the compound agent (made up of 3 sub-agents) the strategies \( AAA \) and \( BBB \) are dominated. Following their deletion the remaining game is one of the matching pennies and hence has no pure Nash equilibrium.

Can the result of Theorem 1 be extended to CCGs with small coalition size, but with an underlying congestion game that is not simple? Again, the answer is negative.
Example 2 Let $G$ be a congestion game with three identical resources (A, B, and C) and three agents. Each agent of $G$ chooses two of the three resources. The cost of each resource is $P(n) = 6 - \frac{6}{n}$.

Let $C = \{\{1, 2\}\{3\}\}$, and $G^C$ is a CCG with underlying game $G$ and partition $C$. After omitting identical strategies due to sub-agents symmetry $G^C$ looks as follows:

| $G^C$ | AB  | AC  | BC  |
|-------|-----|-----|-----|
| AB, AB | -16, -8 | -14, -8 | -14, -4 |
| AC, AC | -14, -4 | -16, -4 | -14, -4 |
| BC, BC | -14, -4 | -14, -4 | -16, -8 |
| AB, AC | -11, -7 | -11, -7 | -12, -6 |
| AB, BC | -11, -7 | -12, -6 | -11, -7 |
| AC, BC | -12, -6 | -11, -7 | -11, -7 |

Note that compound agent’s strategies (AB,AB), (AC,AC), and (BC,BC) are dominated. Note that the remaining game has no pure Nash equilibrium.

4 CCG and potential

An Exact Potential is a function $P: S \rightarrow \mathbb{R}$ satisfying

$$P(s) - P(s_{-i}, t_i) = U_i(s) - U_i(s_{-i}, t_i) \quad \forall i \in N, \quad \forall t_i \in S_i, \quad \forall s \in S_1 \times S_2 \cdots \times S_n$$ (1)

Games with a potential function are called Potential Games. It is well known that potential games have a pure Nash equilibrium [see Monderer and Shapley (1996)]. In particular Congestion Games are potential games. Fotakis et al. (2006) prove that a CCG, where the cost functions of the resources are linear, is a potential game.

Removing the linearity assumption is problematic. In fact, even in the case of a CCG with a maximal coalition of size 2, which guarantees the existence of a pure Nash equilibrium (Theorem 1), the existence of Exact Potential is not guaranteed. In the following example, we show that the existence of a potential function implies linearity of the cost functions.

Example 3 Consider a congestion games with two resources, A and B, with costs $a_1, a_2, a_3$ and $b_1, b_2, b_3$, correspondingly. Assume there are three players and set the coalition structure to $C = \{\{1, 2\}\{3\}\}$. This induces the following two player CCG, given in matrix form:
Assume this game has an exact potential with the following values:

| $G^C$ | A    | B    |
|-------|------|------|
| A, A  | $2a_3, a_3$ | $2a_2, b_1$ |
| A, B  | $a_2 + b_1, a_2$ | $a_1 + b_2, b_2$ |
| B, B  | $2b_2, a_1$ | $2b_3, b_3$ |

From the definition of exact potential the following must hold [see, in addition, Theorem 2.9. in Monderer and Shapley (1996)]:

\[
(a_2 + b_1 - 2a_3) + (a_3 - b_1) + (2a_2 - a_1 - b_2) + (b_2 - a_2) = (P_3 - P_1) + (P_1 - P_2) + (P_2 - P_4) + (P_4 - P_3) = 0.
\]

Similarly

\[
a_2 + b_1 - 2b_2 + a_1 - b_3 + 2b_3 - a_1 - b_2 + b_2 - a_2 = 0
\]

\[
2a_3 - 2b_2 + a_1 - b_3 + 2b_3 - 2a_2 + b_1 - a_3 = 0.
\]

Manipulating these equalities leads to

\[
2a_2 = a_1 + a_3
\]

\[
2b_2 = b_1 + b_3,
\]

which implies that the cost functions are linear.

Using this example, we can now prove our final result.

**Theorem 2** Let $G$ be a Simple Congestion Game and let $C$ be a partition that has at least one element of size 1 and at least one element of size 2. The induced CCG, $G^C$, will have an Exact Potential iff the $P$ functions are linear.

### 5 Summary

In this work, we discuss a setting similar to congestion games but we assume individual agents are in fact orchestrated jointly and so a player in such a congestion game is actually a coalition of agents in the original game. This model describes traffic settings where shipping companies form routing decisions for the individual vessels or ISPs that form routing decisions for various packets. We refer to such games as CCG.
Our benchmark for analysis is the state of knowledge for congestion games. The most illuminating observation for congestion games is that they are Potential Games. This in turn implies that such games possess the “Finite Improvement Property” (FIP) and consequently a pure Nash equilibrium. In our view, these observations provide a sound foundation for Nash equilibrium as a practical solution concept and hence the importance of such results.

When we study these properties in the context of Coalitional Congestions Games we encounter some disappointments—these properties fail to hold in general. Thus, the equilibrium analysis of coalitional congestions games is not as robust. On the other hand, the paper points out to domains where some of these desired properties do hold. For example, whenever the coalitions are small (1 or 2 players) then a pure Nash equilibrium does exist (Theorem 1).

Proposition 1 suggests that if one ignores the coalitional setting and considers the underlying congestion games then in some cases one can retrieve a pure Nash equilibrium of the actual coalitional congestion game. This happens when agents within a coalition actually choose different resources. Intuitively this means that when coalitions are small and the number of resources is high then one could hope for such equilibria to prevail. In fact, Theorem 1 proves this for the case of coalitions of size 2 or less.

We furthermore recall a result of Fotakis which argues that linear costs functions ensure such games are potential games for general coalitional structure and hence the other two properties (FIP and pure NE) hold as well. We go on and prove that one cannot go beyond linear functions with this observation (Theorem 2).

Our work suggests questions for future research involving general structural limitations on the congestion game structure which are necessary and sufficient for the aforementioned properties to hold in the coalitional version.

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Appendix: Omitted proofs

We begin with the definition of an auxiliary game. Let $G$ be a Simple Congestion Game and let $C$ be a partition. The **Restricted Coalitional Congestion Game**, denoted $G^C$, is a CCG where coalitions are restricted to strategies such that distinct sub-agents choose distinct resources. More formally.

**Definition 1** Let $G$ be a Simple Congestion Game and let $C$ be a partition. The **Restricted Coalitional Congestion Game**, denoted $G^C$, is the game $G^C = \{N^C, S^C, U^C\}$, where $N^C$ and $U^C$ are as before and $S^C = \times \{S_k^C\}_{k \in C}$ where $S_k^C = \{s_k \in S_k^C : s_{k,i} \neq s_{k,j} \forall i \neq j \in C_k\}$.

The following result about pure Nash equilibria in restricted CCG will be useful for proving our main result.
Lemma 1 Let $G$ be a Simple Congestion Game and $C$ a partition of $N$. Let $G^C$ be the restricted CCG with the underlying game $G$ and the partition $C$. Let $s$ be a strategy profile of $G^C$ (where $s_{k,i} \neq s_{k,j}$ $\forall k \in N^C$, $\forall i, j \in C_k$, where $i \neq j$). If $c(s) \in c(NE(G))$ then $s \in NE(G^C)$.

Proof For two congestion vectors $u, v$, let $d(u, v) = \frac{\sum_{r \in R} |u_r - v_r|}{2}$, denote the distance between these two vectors.

Let $s$ be a strategy profile satisfying the condition of the lemma, let $k$ be an arbitrary agent in the game $G^C$ and denote by $s_{-k}$ be the strategy profile of all players except $k$. We denote by $BR(s_{-k})$ the set of $k$‘s best reply strategies to $s_{-k}$. Assume, by way of contradiction, that $s_k \not\in BR(s_{-k})$ and let $t_k \in BR(s_{-k})$ be a best reply to $s_{-k}$ which corresponding congestion vector has a minimal distance to $c(s_k)$. Namely, $d(c(t_k), c(s_k)) \leq d(c(t'_k), c(s_k)) \forall t'_k \in BR(s_{-k})$.

In $G^C$ each agent selects each resource at most once. As $c_k(t_k) \neq c_k(s_k)$ this implies that there are two resources, say $r$ and $x$, such that: $(c_k(t_k)_r, c_k(t_k)_x) = (1, 0)$ and $(c_k(s_k)_r, c_k(s_k)_x) = (0, 1)$ and, in addition, that $c(s)_r = c(t_k, s_{-k})_r - 1$ and $c(s)_x = c(t_k, s_{-k})_x + 1$.

Let $i$ be the sub-agent of $k$ that chooses $r$ in $t_k$ but chooses a resource different than $r$ in $s_k$. Let $t'_k$ be a strategy for agent $k$, derived from $t_k$ by moving sub-agent $i$ from $r$ to $x$. This results in $c(t'_k, s_{-k})_r = c(s)_r$ and $c(t'_k, s_{-k})_x = c(s)_x$. By assumption, $c(s) \in NE(G)$. Therefore $P_x(c(s)_x) \leq P_r(c(s)_r + 1)$ and so

$$P_x(c(t'_k, s_{-k})_x) = P_x(c(s)_x) \leq P_r(c(s)_r + 1) = P_r(c(t_k, s_{-k})_r).$$

Thus, the contribution of sub-agent $i$ to agent $k$’s payment in $(t'_k, s_{-k})$ is less or equal its contribution to $k$’s payment in $(t_k, s_{-k})$. As the only change in $k$’s strategy between $t_k$ and $t'_k$ is $i$’s choice, and no other sub-agent of $k$ chooses $r$ or $x$, we conclude that $k$’s payment in $(t'_k, s_{-k})$ is less or equal its payment in $(t_k, s_{-k})$, and so $t'_k \in BR(s_{-k})$.

However, by construction $d(c(t'_k), c(s_k)) \leq d(c(t_k), c(s_k))$, contradicting the way $t_k$ was chosen. \hfill $\square$

Proof of Proposition 1 Let $s$ be a profile as described in the Proposition. Let $t_k$ be a best reply strategy for agent $k$ to $s_{-k}$. We show that $c_k(t_k)_r \leq 1$ $\forall k \in N^C$ and $\forall r \in R$.

Assume this is not true and there exist $i$, $j \in C_k$ with $i \neq j$, such that $t_{k,i} = t_{k,j} = r$ for some resource $r$. Since $c_k(s_k)_r \leq 1$ we know that $c(s_{-k}, t_k)_r > c(s)_r$. Therefore there must exist some resource $x$ such that $c(s_{-k}, t_k)_x < c(s)_x$.

Let $t'_k$ be a strategy profile derived from $t_k$ by moving sub agent $i$ from $r$ to $x$. In the strategy profile $(s_{-k}, t'_k)$ agent $i$ pays $P_x(c(s_{-k}, t'_k)_x)$. By construction

$$c(s_{-k}, t'_k)_x = c(s_{-k}, t_k)_x + 1 \leq c(s)_x.$$  

From Eq. 3 and the fact that $c(s) \in NE(G)$, we get that

$$P_x(c(s_{-k}, t'_k)_x) = P_x(c(s_{-k}, t_k)_x + 1) \leq P_x(c(s)_x) \leq P_r(c(s)_r + 1).$$
Using monotonicity of the cost functions and the fact that \( c(s)_r + 1 \leq c(s_{-k}, t_k)_r \), we get

\[
P_r(c(s)_r + 1) \leq P_r(c(s_{-k}, t_k)_r).
\]

Combining Eqs. 4 and 5, we get that

\[
P_x(c(s_{-k}, t'_k)_x) \leq P_r(c(s_{-k}, t_k)_r).
\]

We will now show that \( k \) is better off in the strategy profile \((s_{-k}, t'_k)\) than in \((s_{-k}, t_k)\), thus contradicting the fact that \( t_k \) is a best response to \( s_{-k} \). We do this analysis for each of \( k \)'s sub-agents:

- Sub-agent \( i \) pays in \((s_{-k}, t'_k)\), where he chose \( x \), no more than in \((s_{-k}, t_k)\), where he chose \( r \) (Eq. 6).
- From definition of \( x \), \( c_k(t_k)_x < c_k(s_k)_x \). Since \( c_k(s_k)_x = 1 \) and \( c_k(s_k)_x > c_k(t_k)_x \), we get that \( c_k(t_k)_x = 0 \). Thus, apart from agent \( i \) no other sub-agent of \( k \) chose \( x \) in \( t_k \).
- Sub-agent \( j \), who selects \( r \) both in \((s_{-k}, t'_k)\) and \((s_{-k}, t_k)\), pays strictly less in \((s_{-k}, t'_k)\) than in \((s_{-k}, t_k)\), because \( c(s_{-k}, t'_k)_r < c(s_{-k}, t_k)_r \). This inequality holds for any other sub-agent of \( k \) who chose \( r \) in \( t_k \).
- All sub-agents who choose a resource in the set \( R \setminus \{r, x\} \) pay the same in \((s_{-k}, t_k)\) and \((s_{-k}, t'_k)\).

To conclude, agent \( k \) pays strictly less in \((s_{-k}, t'_k)\) than in \((s_{-k}, t_k)\). This contradicts the fact that \( t_k \) is a best reply to \( s_{-k} \). Therefore, any best reply of \( k \) to \( s_{-k} \) must be such that all the sub-agents choose different resources.

Thus, agent \( k \)'s best reply strategy to \( s_{-k} \) is a strategy that is allowed also in \( G^C \).

We couple this observation with the observation that \( s \) is a Nash equilibrium of \( G^C \) (follows from Lemma 1) and the fact that \( k \) is arbitrary to conclude that \( s \in NE(G^C) \).

\( \square \)

**Proof of Theorem 1** Let \( s \) be an arbitrary Nash equilibrium of \( G \)

Case 1 Assume that \( c(s)_r \leq |N^C| \) for all resources \( r \in R \). In this case, we can re-arrange the players over the resources so that the result will be a strategy profile with the same congestion vector, and furthermore, for any \( k \in N^C \) its two sub-agents, \( i, j \), choose different resources. The resulting vector is also in \( NE(G) \) and complies with the conditions in Proposition 1. Therefore that proposition implies that \( s \) is a Nash equilibrium of \( G^C \).

Case 2 Let us denote by \( r \) the resource with the highest congestion in \( s \) and assume \( c(s)_r > |N^C| \). Since \( 2|N^C| > N \), Such \( r \) is unique. We argue that without loss of generality (by rearranging the players) \( s \) has the following two properties:

(a) if agent \( k \) has its two sub-agents on the same resource then it must be the case that the corresponding resource is \( r \), that is, \( \forall k \in N^C \ c(s_k)_r > 1 \) implies \( r' = r \); and (b) all agents have at least one sub-agent choose \( r \).
Note some properties of the strategy tuple $s$:

1. Let $k$ be an agent with a single sub-agent. Then this sub-agent must be on $r$ and it has no profitable deviation.
2. Let $k$ be an agent with a two sub-agents, $i$ and $j$. Assume $i$ is on $r$ and $j$ is on some $r' \neq r$. Moving a single sub-agent cannot be profitable.
3. Let $k$ be an agent with two sub-agents, $i$ and $j$. Assume $i$ is on $r$ and $j$ is on some $r' \neq r$. Moving both sub-agents simultaneously cannot be profitable as at least one of these moves makes $k$ worse off, while the other cannot improve $k$ payoff.
4. Let $k$ be an agent with two sub-agents, $i$ and $j$, both on $r$. Moving both sub-agents cannot be profitable.

Thus, if $s$ is not a NE of $G^C$, the only profitable deviation possible is for an agent $k$ with two sub-agents, $i$ and $j$, both on $r$, to move one sub-agent, say $j$, to another resource, say $r'$. Furthermore, let us assume that this is the most profitable deviation for $k$. That is $P_r(c(s_r) + 1) \leq P_{r'}(c(s_{r'}) + 1)$ for all $r'' \neq r$. We denote the resulting strategy profile by $s'$. Note the properties of $s'$:

1. All agents with a single sub-agent choose the resource $r$.
2. All agents with two sub-agents, have at least one sub-agent in $r$.
3. The payment of all sub-agents in $r$ is lower compared with $s$, while the payment of all sub-agents in $R \setminus \{r\}$ is at least as large compared with the payment in $s$.
4. Any agent $k$ with two sub-agents, one on $r$ and one on some $r' \neq r$ pays (weakly) less than what $k$ paid in $s$.

Assume $s'$ is not a Nash equilibrium, then there must be some profitable deviation. What are the possible profitable deviations?

1. Let $k'$ be an agent with a single sub-agent. Then this sub-agent must be on $r$ and it has no profitable deviation. Recall that the payment of $k$ is $s'$ is lower than the payment of $k'$ is $s$.
2. Let $k'$ be an agent with a two sub-agents, $i$ and $j$. Assume $i$ is on $r$ and $j$ is on some $r' \neq r$. Moving $i$ cannot be profitable for the same argument as above. Moving $j$ to another resource in $R \setminus \{r\}$ cannot be profitable, so the only profitable deviation might be moving $j$ back to $r$. However, this would result $k'$ paying the same payment that $k$ paid in $s$, which by construction of $s'$ is higher than what $k$ pays in $s'$. Implies $k'$ had a profitable deviation in $s$, thus contradicting what we already know.
3. Let $k$ be an agent with two sub-agents, $i$ and $j$. Assume $i$ is on $r$ and $j$ is on some $r' \neq r$. Moving both sub-agents simultaneously cannot be profitable as at least one of these moves makes $k$ worse off, while the other cannot improve $k$ payoff.
4. Let $k$ be an agent with two sub-agents, $i$ and $j$, both on $r$. Moving both sub-agents cannot be profitable.

Once again, the only profitable deviation possible is for an agent $k'$ with two sub-agents both on $r$, to move one sub-agent. The resulting strategy profile $s''$, once more, only allows for profitable deviations of the same form. Namely, for an agent $k'$ with two sub-agents both on $r$, to move one sub-agent. We continue iteratively in the same manner. As the process is bounded by the number of agents selecting $r$ with both
sub-agents in $s$, it must end in finitely many steps. The final strategy vector has no profitable deviations and is, therefore, a Nash equilibrium of the game.

Proof of Theorem 2  Sufficiency: This has been obtained by Fotakis et al. (2006) (Theorem 6).

Necessity: Assume some resource, denoted $A$, has a nonlinear cost function. In particular, there exists an integer $k : 1 < k < n$ such that

$$2P_A(k) \neq P_A(k - 1) + P_A(k + 1).$$  \hfill (7)

Let $B \neq A$ be an additional resource. Let $i$ denote the agent with two sub-agents and $j$ the one with a single sub-agent. Consider a strategy tuple of all agents except $i, j$ denoted $s^{\neg\{i,j\}}$, such that $k - 2$ sub-agents select $A$ and the remaining $n - 3 - (k - 2) = n - 1 - k$ sub-agents select $B$, where $n$ is the total number of sub-agents in the CCG. This profile induces a 2-player CCG for players $i$ and $j$, where only resources $A$ and $B$ are available.

In this game the resource cost functions, denoted $\hat{P}_A, \hat{P}_B$, are $a_m = \hat{P}_A(m) = P_A(k - 2 + m)$ and $b_m = \hat{P}_B(m) = P_B(n - 1 - k + m)$. Writing the game in matrix form yields the exact game of Example 3. However, Eq. 7 implies that $2a_2 \neq a_1 + a_3$. Therefore, this game has no potential, which in turn implies that the original game has no potential, as required.

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