Dark matter and non-Newtonian gravity from General Relativity on a stringy background

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Abstract: An exact solution of Einstein’s field equations for a static spherically symmetric medium with a radially boost invariant energy-momentum tensor is presented. In the limit of an equation of state corresponding to a distribution of radially directed strings there is a $1/r$ correction to Newton’s force law. At large distances and small accelerations this law coincides with the phenomenological force law invented by Milgrom in order to explain the flat rotation curves of galaxies without introducing dark matter. The present model explains why the critical acceleration of Milgrom is of the same order of magnitude as the Hubble parameter.
1 Introduction

Einstein’s theory has had remarkable success in explaining observed and inferred gravitational phenomena. There seems to be only one serious problem — the missing mass problem. On large scales, the scales of galaxies and beyond, the Einstein/Newton dynamics seems to imply that there is much more mass than we can observe directly. From observations of the rotational velocities of the gaseous component of galaxies it is found that the velocity approaches a constant at large distances (Sancisi & van Albada 1987), and from the relation \( v^2/r = g \), one finds that the gravitational acceleration decreases as \( 1/r \) here. If luminous matter were a good tracer of mass and Newton’s law were valid at these scales one should find \( g \sim 1/r^2 \).

There are two explanations to this discrepancy. (a) Newtonian dynamics is wrong at these scales, or (b) there is a lot of unseen “dark matter” in the galaxies. Many authors have advocated the first explanation and modifications of Newton’s gravitational dynamics have been proposed (Bekenstein & Milgrom 1984; Finzi 1963; Milgrom 1983a; Kuhn & Kruglyak 1987; Liboff 1992; Sanders 1990). Milgrom’s theory (for reviews see: Milgrom 1987, 1989; Milgrom & Bekenstein, 1987) has worked impressingly well both for galaxies (Milgrom 1983b, 1984, 1986) and galaxy systems (Milgrom 1983c), and has recently been found to be the best phenomenological description of the systematics of the mass discrepancy in galaxies (Begeman, Broeils, & Sanders 1991; see however Gerhard & Spergel 1992). Another argument in favor of an effective \( 1/r \) correction to the force law at large distances, is that such a term could stabilize a cold stellar disk in a numerical galaxy model (Tohline 1983).

In spite of the phenomenological success of non-Newtonian dynamics the scientific community has been reluctant to abolish Newton’s theory of gravitation, partly because Newton’s theory is far more aesthetically attractive than any of the modified theories, and since the interactions appear to be more fundamental than matter, one would rather introduce new matter than new forces, but above all it is objected that none of the modified theories have a viable relativistic counterpart (Lindley 1992; Milgrom 1989). Bekenstein’s (1988a, b) phase coupling gravitation is consistent with both extragalactic systematics and solar system tests, but at least in the version with a sextic scalar potential (Bekenstein 1988a) it has tachyonic propagation of scalar waves, and it is unclear whether the model can be saved (Sanders 1990). In general if a viable scalar-tensor theory that mimics Milgrom’s exists, it will be very complicated and contain many new fundamental con-
stants. Therefore, the most widely accepted explanation is that the Universe is filled with huge amounts of dark matter.

Recently it has been suggested that quantum gravity effects may account for some of the missing mass (Goldman et al. 1992) through a large scale variability of the effective $G$. Here the idea of attributing dark matter to a very specific and simple underlying principle is followed up, but instead of a variable $G$ from quantum gravity, we propose a stringy aether as a specific form of dark matter. In the case of spherical symmetry this energy-momentum tensor reproduces the phenomenological force law of Milgrom (1983a) within the framework of classical General Relativity. The appearance of the Hubble constant in the modified Newtonian force law is explained as a consequence of the stringy nature of the dark matter.

2 Radial boost invariance

Newton’s inverse square law follows from the weak field limit of Einstein’s theory when one assumes that the energy-density of empty space is exactly zero. Einstein (1917) was the first to allow for different large scale dynamics when he introduced the cosmological constant, $\Lambda$, in order to get a long range repulsive force to balance the Newtonian attraction and produce a static cosmological model. Later the $\Lambda$-term has been understood as equivalent to the energy-density of a maximally symmetric vacuum (Gliner 1966). In fact, $\Lambda g_{\mu\nu}$ is the unique form of the energy-momentum tensor if one assumes that the energy momentum tensor is boost invariant in all directions.

In the empty space outside a star or a galaxy one expects that isotropy, translational invariance, and part of the boost invariance of the vacuum is broken. However, in order to retain as much as possible of the spirit of the Strong Equivalence Principle one would desire that boost invariance is preserved at least in the radial direction, for only in this case would the energy density be independent of the infall velocity of the observer. Then a freely falling observer would in principle be unable to measure his radial velocity relative to the vacuum. It is felt that this symmetry is the closest one can get to the Strong Equivalence Principle in the case of an anisotropic vacuum energy. Accordingly, we will here take as our fundamental assumption that the medium outside a point mass has an effective energy-momentum tensor

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1 The Strong Equivalence Principle states that the result of any local test experiment, gravitational and nongravitational, in a freely falling inertial frame does not depend on where and when it is performed and not on the velocity of the inertial system.
which is invariant under radial boosts. Thus, we will assume $T_t^t = T_r^r$. No assumptions will be made concerning the translational invariance of the medium outside a point mass. In the next section the exact, static, spherically symmetric solution of Einstein’s field equations for radially boost invariant energy-momentum tensors with angular components proportional to the energy density is found.

3 Exact solution

Consider a static, spherically symmetric space-time. Up to coordinate transformations, we may write the metric as

$$ds^2 = -e^{2\mu}dt^2 + e^{2\lambda}dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2$$  

(1)

where $\mu$ and $\lambda$ depends on the radial coordinate $r$ only. With this metric the Ricci tensor takes the form

$$R_t^t = \left[-\mu'' + \lambda'\mu' - \mu'^2 - 2\frac{\mu'}{r}\right]e^{-2\lambda}$$  

(2)

$$R_r^r = \left[-\mu'' + \lambda'\mu' + \mu'^2 + 2\frac{\lambda'}{r}\right]e^{-2\lambda}$$  

(3)

$$R_{\Omega\Omega} = \left[-\frac{\mu'}{r} + \frac{\lambda'}{r} - \frac{1}{r^2}\right]e^{-2\lambda} + \frac{1}{r^2}$$  

(4)

where $\Omega$ stands for both $\theta$ and $\phi$.

With radial boost invariance and spherical symmetry, the gravitational field is uniquely determined once an “equation of state” specifies the relation between $T_t^t$ and $T_{\Omega\Omega}$. Here we will assume that $T_t^t$ is proportional to $T_{\Omega\Omega}$. Thus we get an energy-momentum tensor of the form

$$T_t^t = T_r^r = -\alpha T_{\Omega\Omega}^\Omega$$  

(5)

where $\alpha$ is a finite dimensionless constant. With this Ansatz Einstein’s field equations

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}g^\mu_\nu R = 8\pi T^\mu_\nu$$  

(6)

imply

$$G_t^t = G_r^r \quad \text{and} \quad G_t^t = -\alpha G_{\Omega\Omega}^\Omega$$  

(7)

In this paper geometrical units are used, i.e. $G = c = 1$. 

4
From the first equation we find $\lambda = -\mu$. Then the second equation becomes

$$\frac{1}{r^2} \left( (re^{2\mu})' - 1 \right) = -\frac{\alpha}{2r} \left( (re^{2\mu})' - 1 \right)' .$$  \hspace{1cm} (8)

The case $\alpha = 0$ corresponds to the Schwarzschild solution. For $\alpha \neq 0$, integration yields

$$(re^{2\mu})' - 1 = -\epsilon \left( \frac{\ell}{r} \right)^{2/\alpha}$$  \hspace{1cm} (9)

where $\ell$ is a positive integration constant of dimension length and $\epsilon = \pm 1$ is a sign factor which determines the sign of the energy-density of the anisotropic vacuum ($\epsilon = 1$ corresponds to a positive energy density). Integrating once more, we find

$$e^{2\mu} = 1 - 2M/r - \left\{ \begin{array}{cc}
\epsilon \ell^{-1} \ln (\lambda r) & \text{for } \alpha = 2 \\
\epsilon \alpha (\alpha - 2)^{-1} \ell^{2/\alpha - 2/\alpha} & \text{for } \alpha \neq 2
\end{array} \right.$$  \hspace{1cm} (10)

The following special cases are singled out: $\alpha = -1$ corresponds to the Schwarzschild–de Sitter solution, $\alpha = 0$ corresponds to the Schwarzschild solution, and $\alpha = 1$ corresponds to the Reissner-Nordström solution.

In the generic case, $\alpha \notin \{0, 2\}$, the classical gravitational acceleration is

$$g = \frac{M}{r^2} + \epsilon \frac{\ell^{-1}}{(\alpha - 2)^{1+2/\alpha}} .$$  \hspace{1cm} (11)

The energy-density, $\rho_s$, corresponding to these solutions are found from equations (8) and (9). Hence, using that $8\pi \rho_s = -8\pi T^t_t = -G^t_t$, one finds

$$8\pi \rho_s = \frac{\epsilon}{r^2} \left( \frac{\ell}{r} \right)^{2/\alpha} .$$  \hspace{1cm} (12)

4 String-like background and dark matter

From equation (11) one gets an effective attractive $1/r$ correction to Newton’s force if $\epsilon \alpha \gg 1$. In the present context a large $\alpha$ means that $|T^t_t| = |T^r_r| \gg |T^\Omega_\Omega|$. This may be understood as the energy-momentum tensor of a cloud of radially directed strings at low but nonzero temperature.

A straight string has vanishing gravitational mass, because the gravitational effect of tension exactly cancels the effect of its mass. Note that if we let $\alpha \to \infty$ the correction term in the metric coefficient of Equation (10)
becomes a constant, and thus in the zero temperature limit the strings do not produce any gravitational forces. Assume that the string has a mass \( m_s \).

If this mass is evenly distributed along the string which is stretching across the whole universe, the mass per length is \( \mu = m_s H_0 \). Thus the energy gained by falling into a galaxy is

\[
\Delta E = \frac{M \mu r}{r} = m_s M H_0.
\]  

This energy is available to produce transverse motion of the string. Hence, there will be a transverse pressure given by

\[
(pV)^2 = (m_s + \Delta E)^2 - m_s^2.
\]

The transverse pressure will contribute with a pressure per energy density \( pV/m_s \approx (M H_0)^{1/2} \). This determines the dimensionless constant in the Ansatz (5) as follows

\[
\alpha \approx \epsilon (M H_0)^{-1/2}.
\]

Hence, for positive \( \rho_s \) (\( \epsilon = 1 \)), the angular pressure is also positive. This agrees with the general lore that pressure contributes to gravitational attraction. The predicted value of \( |\alpha| \) is so large that \( (\ell/r)^2/\alpha \approx 1 \) for all reasonable values of \( \ell \). Hence, with this information, Equation (11) implies that Newton’s force law is changed to

\[
g = \frac{M}{r^2} + k_0 (M H_0)^{1/2} \frac{1}{r}
\]

where the constant \( k_0 \approx 1 \). The presence of the Hubble parameter in the local force law signals that the translational invariance of the background aether is broken not only in spatial directions but also in the time-direction. This is what one would expect by including strings of cosmic extension in the background aether.

Note that the force law of Equation (16) agrees with the Tully-Fisher law (1977) which relates the rotational velocity, \( v \), to the luminosity, \( L \), by \( v \propto L^{1/4} \) if the luminosity is proportional to the Newtonian mass. This is a reasonable assumption if the ratio of dark and luminous matter densities is a constant. Also the mysterious coincidence that Milgrom’s critical acceleration is equal to the Hubble parameter (Milgrom 1983a, 1989), is explained as a result of having dark matter of cosmic extension.

It could be objected that most observations seem to imply that the mass density of the universe is smaller than the closure density, and that the
concept of strings extending across the universe therefore is meaningless. Note, however, that an isotropic stringy background has an energy-density of the form (Vilenkin 1985)

\[ 8\pi \rho_s = \frac{3w}{R^2} \]  

where \( R \) is the cosmic scale factor and \( w \) is a constant. Accordingly, the first Friedmann equation takes the form

\[ H^2 = \frac{8\pi}{3} \rho + \frac{w - k}{R^2}. \]  

Hence, if one neglects the stringy background the effective curvature is \( k_{\text{eff}} = k - w \) rather than \( k \). Thus a closed universe with a geometric \( k \geq 0 \) as predicted by inflation could be in agreement with the observed \( k_{\text{eff}} < 0 \) if the Universe has a stringy background with \( w > 0 \) (Vilenkin 1985).

A string dominated universe requires that the intercommuting probability of the strings is very small, i.e. the strings pass freely through one another. The strings of a string dominated universe could have a very small mass per length \( G \mu \sim 10^{-30} \), and even when passing through the observer they would be difficult to detect (Vilenkin 1984).

5 Conclusion

Up to now the missing mass problem has been resolved by assuming that dark matter is present in whatever quantities and distributions that are needed to explain away all mass discrepancies. The main problem with this approach is that the dark matter hypothesis in this form is too flexible to give any unavoidable predictions (Milgrom 1989), and it is in principle not testable before one specifies the nature of the dark matter. In contrast, the approach of Milgrom is testable, and it gives specific predictions which are in good agreement with observations. The main problem has been that the modified dynamics has no viable relativistic counterpart.

Here it has been shown that Einstein’s General Relativity coupled to a stringy aether reproduces the force law of Milgrom (1983a). In a closed universe this model explains the Machian character of Milgrom’s acceleration, \( a_0 \approx H_0 \), as a consequence of a stringy aether extending over the whole universe. The dark matter model presented here is a phenomenological model, but it can no longer be objected that the \( 1/r \) modification of Newton’s force
law is "an orphan in the classical world" (Lindley 1992). Instead it follows from General Relativity with a relativistic energy-momentum tensor corresponding to a stringy aether.

Despite the success of Milgrom’s force law and the fact that the present model reproduces it, there are many reasons why it is premature to identify a stringy aether as the solution to the missing mass problem. First, there is no field theoretic realization of this particular model. Second, the proposal has only been studied in a static, spherically symmetric model. In contrast, the real universe is non-static and it contains many galaxies so both of the symmetry assumptions are broken. It is not clear how deviations from spherical symmetry will affect the solution, and especially how strings passing through more than one galaxy will affect the model. Finally one expects realistic string models to predict that strings will intercommute and produce closed loops.

However, from a general relativistic perspective it is very interesting that such a simplistic model can reproduce the non-Newtonian force law. The generally accepted solution — dark matter — need therefore not be a very complicated system of epicycles as have been argued by the proponents of non-Einsteinian gravitation (e.g. Sanders 1990).

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