The averaged tensors of the relative energy-momentum and angular momentum in general relativity and some of their applications

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(March 24, 2022)

Abstract

There exist different kinds of averaging of the differences of the energy-momentum and angular momentum in normal coordinates \( \text{NC}(\mathbf{P}) \) which give tensorial quantities. The obtained averaged quantities are equivalent mathematically because they differ only by constant scalar dimensional factors. One of these averaging was used in our papers [1-8] giving the canonical superenergy and angular supermomentum tensors.

In this paper we present another averaging of the differences of the energy-momentum and angular momentum which gives tensorial quantities with proper dimensions of the energy-momentum and angular momentum densities. But these averaged relative energy-momentum and angular momentum tensors, closely related to the canonical superenergy and angular supermomentum tensors, depend on some fundamental length \( L > 0 \).

The averaged relative energy-momentum and angular momentum tensors of the gravitational field obtained in the paper can be applied, like the canonical superenergy and angular supermomentum tensors, to coordinate indepen-

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dent analysis (local and in special cases also global) of this field.

We have applied the averaged relative energy-momentum tensors to analyze vacuum gravitational energy and momentum and to analyze energy and momentum of the Friedman (and also more general) universes. The obtained results are interesting, e.g., the averaged relative energy density is positive definite for the all Friedman universes.

04.20.Me.0430.+x
I. THE AVERAGED RELATIVE ENERGY-MOMENTUM AND ANGULAR
MOMENTUM TENSORS IN GENERAL RELATIVITY

In the papers [1-8] we have defined the canonical superenergy and angular supermomentum tensors, matter and gravitation, in general relativity (GR) and studied their properties and physical applications. In the case of the gravitational field these tensors gave us some substitutes of the non-existing gravitational energy-momentum and gravitational angular momentum tensors.

The canonical superenergy and angular supermomentum tensors were obtained pointwise as a result of some special averaging of the differences of the energy-momentum and angular momentum in normal coordinates $\text{NC}(P)$. The role of the normal coordinates $\text{NC}(P)$ is, of course, auxilliary, only to extract tensorial quantities even from pseudotensorial ones.

The dimensions of the components of the canonical superenergy and angular supermomentum tensors can be written down as: [the dimensions of the components of an energy-momentum or angular momentum tensor (or pseudotensor)] $\times m^{-2}$.

In this paper we propose a new averaging of the energy-momentum and angular momentum differences in $\text{NC}(P)$ which is very like to the averaging used in [1-8] and which gives the averaged quantities with proper dimensionality of the energy-momentum and angular momentum densities.

Namely, we propose the following general definition of the averaged tensor (or pseudotensor) $T_a^b$

\[
<T_a^b(P)> := \lim_{\varepsilon \to 0} \frac{\int_\Omega \left[ T_{(a)}^{(b)}(y) - T_{(a)}^{(b)}(P) \right] d\Omega}{\varepsilon^2/2 \int_\Omega d\Omega},
\]

where

\[
T_{(a)}^{(b)}(y) := T_k^i(y) e^i_{(a)}(y) e_k^{(b)}(y),
\]

\[
T_{(a)}^{(b)}(P) := T_k^i(P) e^i_{(a)}(P) e_k^{(b)}(P) = T_a^b(P)
\]
are the tetrad (or physical) components of a tensor or a pseudotensor $T^i_\,{}^k(y)$ which describes an energy-momentum distribution, $y$ is the collection of normal coordinates $\textbf{NC}(\mathbf{P})$ at a given point $\mathbf{P}$, $e^i_{\,(a)}(y)$, $e^i_{\,(b)}(y)$ denote an orthonormal tetrad field and its dual, respectively,

$$e^i_{\,(a)}(P) = \delta^i_a, \quad e^i_{\,(a)}(y) = \delta^a_i, \quad e^i_{\,(a)}(y) e^i_{\,(b)}(y) = \delta^a_b,$$

and they are parallelly propagated along geodesics through $\mathbf{P}$.

For a sufficiently small domain $\Omega$ which surrounds $\mathbf{P}$ we require

$$\int_\Omega y^i d\Omega = 0, \quad \int_\Omega y^i y^k d\Omega = \delta^{ik} M,$$

where

$$M = \int_\Omega (y^0)^2 d\Omega = \int_\Omega (y^1)^2 d\Omega = \int_\Omega (y^2)^2 d\Omega = \int_\Omega (y^3)^2 d\Omega,$$

is a common value of the moments of inertia of the domain $\Omega$ with respect to the subspaces $y^i = 0, \quad (i = 0, 1, 2, 3)$.

The procedure of averaging of an energy-momentum tensor or an energy-momentum pseudotensor given in (1) is a four-dimensional modification of the proposition by Mashhoon [9-12].

Let us choose $\Omega$ as a small analytic ball defined by

$$(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \leq R^2 = \varepsilon^2 L^2,$$

which can be described in a covariant way in terms of the auxiliary positive-definite metric $h^{ik} := 2v^i v^j - g^{ik}$, where $v^i$ are the components of the four-velocity of an observer $\mathbf{O}$ at rest at $\mathbf{P}$ (see, e.g., [1-8]). $\varepsilon$ means a small parameter: $\varepsilon \in (0; 1)$ and $L > 0$ is a fundamental length.

Since at $\mathbf{P}$ the tetrad and normal components are equal, from now on we will write the components of any quantity at $\mathbf{P}$ without (tetrad) brackets, e.g., $T^a_\,{}^b(P)$ instead of $T_{(a)}^\,(b)(P)$ and so on.

Let us now make the following expansions for the energy-momentum tensor of matter $T^i_\,{}^k(y)$ and for $e^i_{\,(a)}(y), e^i_{\,(b)}(y)$ [13]
\[ T^k_i(y) = \hat{T}^k_i + \nabla_i \hat{T}^k_i y^l + 1/2 \hat{T}^k_i,lm y^l y^m + R_3 \]
\[ = \hat{T}^k_i + \nabla_i \hat{T}^k_i y^l + 1/2 \left[ \nabla(l \nabla_m) \hat{T}^k_i \right. \]
\[ \left. - \frac{1}{3} \hat{R}^c_{(li|jm)} \hat{T}^k_i + 1/3 \hat{R}^k_{(li|jm)} \hat{T}^i_c \right] y^l y^m + R_3, \]
(8)

\[ e^i_{(a)}(y) = \hat{e}^i_{(a)} + 1/6 \hat{R}^i_{lkmc} \hat{e}^k_{(a)} y^l y^m + R_3, \]
(9)

\[ e^k_{(b)}(y) = \hat{e}^k_{(b)} - 1/6 \hat{R}^p_{lkmc} \hat{e}^p_{(b)} y^l y^m + R_3, \]
(10)

which give (1) in the form

\[ <_m T^b_a(P) = \lim_{\varepsilon \to 0} \frac{\int \left[ \nabla_i \hat{T}^b_{a} y^l + 1/2 \nabla(l \nabla_m) \hat{T}^b_{a} y^l y^m + \text{THO} \right] d\Omega}{\varepsilon^2 / 2 \int d\Omega}, \]
(11)

where \( \text{THO} \) means the terms of higher order in the expansion of the differences \( T^k_i(y) - T^k_i(P) \) = \( T_{(b)}(y) - T_{(b)}(P) \); \( R_3 \) is the remainder of the third order and \( \nabla \) denotes covariant differentiation. Hat denotes the value of an object at \( P \) and the round brackets denote symmetrization from which the indices inside vertical lines, e.g., \((a|c|b)\) are excluded.

The first and \( \text{THO} \) terms in the numerator of (11) do not contribute to \( <_m T^b_a(P) > \). Hence, we finally get from (11)

\[ <_m T^b_a(P) =_m S^b_a(P) \frac{L^2}{6}, \]
(12)

where

\[ _m S^b_a(P) := \delta^{lm} \nabla(l \nabla_m) \hat{T}^b_a \]
(13)

is the canonical superenergy tensor of matter [1-8].

By introducing the four velocity \( \hat{v}^l \equiv \delta^l_0 \), \( v^l v_l = 1 \) of an observer \( \mathbf{O} \) at rest at \( P \) and the local metric \( \hat{g}^{ab} \equiv \eta^{ab} \), where \( \eta^{ab} \) is the inverse Minkowski metric, one can write (13) in a covariant way as

\[ _m S^b_a(P; \hat{v}^l) = (2 \hat{v}^l \hat{v}^m - \hat{g}^{lm}) \nabla(l \nabla_m) \hat{T}^b_a. \]
(14)
The sign $\doteq$ means that an equality is valid only in some special coordinates.

The matter superenergy tensor $mS^b_a(P; v^l)$ is symmetric.

As a result of an averaging the tensor $mS^b_a(P; v^l)$, and in consequence the averaged tensor $\langle mT^b_a(P; v^l) \rangle$, do not satisfy any local conservation laws in general relativity. However, these tensors satisfy trivial local conservation laws in special relativity (see, e.g., [1-8]).

Now let us take the gravitational field and make the expansion

\[ E^i_k(y) = \frac{\alpha}{9} \left[ \hat{B}^k_{ilm} + \hat{P}^k_{ilm} - \frac{\delta^k_i}{2} \hat{R}^{abc}_{i} (\hat{R}_{abc} + \hat{R}_{acbm}) + 2\beta^2 \delta^k_i \hat{E}_{(l|} \hat{E}^{g}_{|m)} 
- 3\beta^2 \hat{E}_{i(l|} \hat{E}^{k}_{|m)} + 2\beta \hat{R}^k_{(gi)(l|} \hat{E}^{g}_{|m)} \right] y^l y^m + R_3. \] (15)

Here $E^i_k$ mean the components of the canonical Einstein energy-momentum pseudotensor of the gravitational field.

In a holonomic frame we have

\[ E^i_k = \alpha \left\{ \delta^k_i g^{ms} (\Gamma^l_m g^{rp} - \Gamma^r_ms \Gamma^l_r) + g^{ms} \Gamma^l_m - 2/3 (\Gamma^k_l g^{tp} - \Gamma^l_g^{kt} g^{ms}) \right\}. \] (16)

\[ \alpha = \frac{c^4}{16\pi G} = \frac{1}{2\beta^3} E^i_k := T^i_k - 1/2\delta^k_i T, \] (17)

and, in any frame,

\[ B^b_{alm} := 2R^b_{ik} (l| R_{ak|m} - 1/2\delta^b_i R^j_{ik} R_{jkm}, \] (18)

is the Bel-Robinson tensor, while the tensor

\[ \text{Trivial local conservation laws because the integral superenergetic quantities or, equivalently, integral averaged relative energy-momentum calculated from them for a closed system in special relativity vanish}. \]
\[ P_{\text{alm}}^b := 2R^{bik}_{(l|R_{akj|m})} - 1/2\delta_{a}^{b}R^{ijk}_{(l|R_{akjm})} \]  

(19)

is very closely related to the former.\footnote{Very closely related because this tensor has almost the same analytic form as the Bel-Robinson tensor and the same symmetry properties.}

The expansion (15) with the help of (9) and (10) gives the following averaged gravitational relative energy-momentum tensor

\[ \langle g T_a^b(P; v^l) \rangle \rangle = g S_a^b(P; v^l) \frac{L^2}{6}, \]  

(20)

where the tensor \( g S_a^b(P; v^l) \) is the \textit{canonical superenergy tensor} for the gravitational field \([1-8]\).

We have \([1-8]\)

\[ g S_a^b(P; v^l) = \frac{2\alpha}{9} (2\hat{v}^l\hat{v}^m - \hat{g}^{lm}) \left[ \hat{B}^b_{\text{alm}} + \hat{P}^b_{\text{alm}} ight] \]

\[ - \frac{1}{2\delta_{a}^{b}}\hat{R}^{ijk}_{m}(\hat{R}_{ijkl} + \hat{R}_{ikjl}) + 2\beta 2\delta_{a}^{b}\hat{E}_{(l|g} \hat{E}_{|m)} \]

\[ - 3\beta 2\hat{E}_{a(l|} \hat{E}_{b|m)} + 2\beta \hat{R}_{(ag)(l|} \hat{E}_{b|m)}, \]  

(21)

In vacuum the tensor \( g S_a^b(P; v^l) \) reduces to the simpler form

\[ g S_a^b(P; v^l) = \frac{8\alpha}{9} (2\hat{v}^l\hat{v}^m - \hat{g}^{lm}) \left[ \hat{R}^{b(lk)}_{(l|} \hat{R}_{aik|m)} - 1/2\delta_{a}^{b}\hat{R}^{l(kp)}_{(l|} \hat{R}_{ikp|m)} \right], \]  

(22)

which is symmetric and the quadratic form \( g S_{ab}(P; v^l)\hat{v}^a\hat{v}^b \) is \textit{positive-definite}.

In vacuum we also have the \textit{local conservation laws}

\[ \nabla_b g \hat{S}_a^b = 0. \]  

(23)

and the analogous laws satisfied by the averaged tensor \( \langle g T_a^b(P; v^l) \rangle \rangle \).

The averaged energy-momentum tensors \( \langle m T_a^b(P; v^l) \rangle \rangle \) and \( \langle g T_a^b(P; v^l) \rangle \rangle \) can be considered as the \textit{averaged tensors of the relative energy-momentum}. They can also be interpreted as the fluxes of the appropriate canonical superenergy. It is easily seen from the formulas (12) and (20).
Now let us consider the *averaged angular momentum tensors* in GR. The constructive definition of these tensors, in analogy to the definition of the averaged energy-momentum tensors, is as follows.

In normal coordinates NC(P) we define

\[
< M^{(a)(b)(c)}(P) > = < M^{abc}(P) > := \lim_{\varepsilon \to 0} \frac{\int_{\Omega} [M^{(a)(b)(c)}(y) - M^{(a)(b)(c)}(P)] d\Omega}{\varepsilon^2/2 \int_{\Omega} d\Omega},
\]

(24)

where

\[
M^{(a)(b)(c)}(y) := M^{ikl}(y) e^a_i(y) e^b_k(y) e^c_l(y),
\]

(25)

\[
M^{(a)(b)(c)}(P) := M^{ikl}(P) e^a_i(P) e^b_k(P) e^c_l(P) = M^{ikl}(P) \delta^a_i \delta^b_k \delta^c_l = M^{abc}(P),
\]

(26)

are the *physical* (or tetrad) components of the field \( M^{ikl}(y) = (-)M^{kil}(y) \) which describes the angular momentum densities. As in (2) and (3), \( e^i_{(a)}(y), e^k_{(b)}(y) \) denote mutually dual orthonormal tetrads parallelly propagated along geodesics through \( P \) such that \( e^i_{(a)}(P) = \delta^i_a, e^k_{(b)}(P) = \delta^k_b \). The compact four-dimensional domain \( \Omega \) is defined in the same way as in the formula (1) and we will again take \( \Omega \) as a sufficiently small four-dimensional ball with centre at \( P \) and with radius \( R = \varepsilon L \).

At \( P \) the tetrad and normal components of an object are equal. We apply this once more and omit tetrad brackets for the indices of any quantity attached to the point \( P \); for example, we write \( M^{abc}(P) \) instead of \( M^{(a)(b)(c)}(P) \) and so on.

For matter as \( M^{ikl}(y) \) we take

\[
mM^{ikl}(y) = \sqrt{|g|[y^j T^{kli}(y) - y^k T^{jli}(y)]},
\]

(27)

where \( T^{ik}(y) = T^{ki}(y) \) are the components of a symmetric energy-momentum tensor of matter and \( y^i \) denote the normal coordinates NC(P).

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3 Of course, \( M^{abc}(P) = 0 \), but we leave \( M^{abc}(P) \) in our formulas.
The formula (27) gives us the total angular momentum densities, orbital and spinorial, because the symmetric energy-momentum tensor of matter $T^{ik} = T^{ki}$ comes from the canonical one by using the Belinfante-Rosenfeld symmetrization procedure and, therefore, includes the canonical spin of matter [14].

For the gravitational field we take the gravitational angular momentum pseudotensor proposed by Bergmann and Thomson [14,18] which in a NC(P) (and in any other holonomic frame) reads

$$ gM^{ikl}(y) = F U^[i[kl]}(y) - F U^[k[i}]q(y) + \sqrt{|g|} \left[ y^i_{BT} t^{kl} - y^k_{BT} t^{il} \right], \quad (28) $$

where, in a holonomic frame,

$$ F U^[i[kl]} := g^{im} F U^[m[kl]} = \alpha g^{im} g^{ma} \left[ (-g) (g^{ka} g^{lb} - g^{la} g^{kb}) \right] $$

and

$$ BT t^{kl} := g^{ki} E^l = g^{mk} F U^[m[l]q] $$

are Freud’s superpotentials with the first index raised and

are the components of the Bergmann-Thomson gravitational energy-momentum pseudotensor [14,18]. $E^l_i$ mean the components of the Einstein canonical gravitational energy-momentum pseudotensor of the gravitational field.

The Bergmann-Thomson gravitational angular pseudotensor is most closely related to the Einstein canonical energy-momentum complex $E K^k_i := \sqrt{|g|} (T_i^k + E t_i^k)$, matter and gravitation, and it has better physical and transformational properties than the famous gravitational angular momentum pseudotensor proposed by Landau and Lifschitz [15-17]. This is why we apply it here.

One can interpret the Bergmann-Thomson gravitational angular momentum pseudotensor as the sum of the spinorial part

$$ S^{ikl} := F U^[i[kl]} - F U^[k[i]q] $$

and the orbital part
of the gravitational angular momentum “densities”.

Substitution of (27) and (28) (expanded up to third order), (9),(10) and the expansion

\[ \sqrt{|g|} = 1 - 1/6 \hat{R}_{ab} y^a y^b + R_3 = 1 - 1/6 \hat{\beta} \hat{E}_{ab} y^a y^b + R_3, \]

into (24) gives us the following averaged angular momentum tensors for matter and gravitation respectively

\[ <_m M^{abc}(P; v^l) > _m S^{abc}(P; v^l) \frac{L^2}{6}, \]

\[ <_g M^{abc}(P; v^l) > _g S^{abc}(P; v^l) \frac{L^2}{6}. \]

Here

\[ mS^{abc}(P; v^l) = 2[(2 \hat{v}^a \hat{v}^p - \hat{g}^{ap}) \nabla_p \hat{T}^{bc} - (2 \hat{v}^b \hat{v}^p - \hat{g}^{bp}) \nabla_p \hat{T}^{ac}], \]

and

\[ gS^{abc}(P; v^l) = \alpha(2 \hat{v}^p \hat{v}^t - \hat{g}^{pt}) \left[ \beta(\hat{g}^{ac} \hat{g}^{br} - \hat{g}^{bc} \hat{g}^{ar}) \nabla_p \hat{E}_{pr} \right. \]

\[ + 2 \hat{g}^{ar} \nabla_p (\hat{R}_{(b}^{(c} \hat{r}_{p \ r)} - 2 \hat{g}^{br} \nabla_p (\hat{R}_{(a}^{(c} \hat{r}_{p \ r)}) \]

\[ + 2/3 \hat{g}^{bc}(\nabla_r \hat{R}_{(t \ p)}^{(a} \hat{r}_{(t \ p)}) - \beta \nabla_p \hat{E}_{(t \ p)}^{(a} \hat{r}_{p \ (t \ p)}) - 2/3 \hat{g}^{ac}(\nabla_r \hat{R}_{(t \ p)}^{b \ r} - \beta \nabla_p \hat{E}_{(t \ p)}^{b \ r)}) \left. \right] \]

are the components of the canonical angular supermomentum tensors for matter and gravitation, respectively [4,6,8].

In special relativity the averaged tensor \( <_m M^{abc}(P; v^l) > \), and the canonical angular supermomentum tensors for matter \( mS^{abc}(P; v^l) \) satisfy trivial conservation laws [1-8]. In the framework of the GR only the tensors \( gS^{abc}(P; v^l) \) and \( <_g M^{abc}(P; v^l) > \) satisfy local conservation laws in vacuum.

In vacuum, when \( T_{ik} = 0 \iff E_{ik} := T_{ik} - 1/2 g_{ik} T = 0 \), the canonical gravitational angular supermomentum tensor \( gS^{abc}(P; v^l) \) given by (37) simplifies to

\[ gS^{abc}(P; v^l) = 2\alpha(2 \hat{v}^p \hat{v}^t - \hat{g}^{pt}) \left[ \hat{g}^{ar} \nabla_p (\hat{R}_{(b}^{(c} \hat{r}_{(t \ p)}) - \hat{g}^{br} \nabla_p (\hat{R}_{(a}^{(c} \hat{r}_{(t \ p)}) \right]. \]

Some remarks are in order:
1. The orbital part $O^{ikl} = \sqrt{|g|} (y^{BT}_{ik} - y^{BT}_{kl})$ of the $gM^{ikl}$ does not contribute to the tensor $gS^{abc}(P; v^l)$ and, therefore, also to the tensor $<g M^{abc}(P; v^l)>$. Only the spinorial part $S^{ikl} = F_{[kl]} - F_{il}$ gives nonzero contribution to these tensors.

2. The averaged angular momentum tensors $<g M^{abc}(P; v^l) >$, $<m M^{abc}(P; v^l) >$, like as the canonical angular supermomentum tensors, do not need any radius-vector for existing.

The averaged tensors $<m M^{abc}(P; v^l) >$, $<g M^{abc}(P; v^l) >$, likely as the averaged relative energy-momentum tensors, can be interpreted as the averaged tensors of the relative angular momentum and also as the fluxes of the appropriate angular supermomentum.

The formulas (12), (20), (34) and (35) give the direct link between the canonical superenergy and angular supermomentum tensors

$$gS_a^b(P; v^l), \ mS_a^b(P; v^l), \ gS^{abc}(P; v^l), \ mS^{abc}(P; v^l)$$

and the averaged relative energy-momentum and angular momentum tensors

$$<g t_a^b(P; v^l) >, <m T_a^b(P; v^l) >, <g M^{abc}(P; v^l) >, <m S^{abc}(P; v^l) > .$$

Namely, it is easily seen from these formulas that the averaged relative energy-momentum and angular momentum tensors differ from the canonical superenergy and angular supermomentum tensors only by the constant scalar multiplicator $L^2 > 0$, where $L > 0$ means some fundamental length. Thus, from the mathematical point of view, these two kind of tensors are equivalent. Physically they are not because their components have different dimension. Moreover the averaged energy-momentum and angular momentum tensors depend on a fundamental length $L > 0$, i.e., they need introduction a supplementary element into GR.\(^4\)

\(^4\)Of course, the angular momentum is always relative quantity, in principle. Despite that we will keep the term relative angular momentum tensors.

\(^5\)The fundamental length $L > 0$ must be infinitesimally small because its existence violates local Lorentz invariance. It is generally believed that a fundamental length exists in Nature.
Owing to the last fact and the formulas (12), (20), (34), (35) it seems that the canonical superenergy and angular supermomentum tensors are more fundamental than the averaged energy-momentum and angular momentum tensors. But the averaged energy-momentum and angular momentum tensors have one important superiority over the canonical superenergy and angular supermomentum tensors: their components possess proper dimensions of the energy-momentum and angular momentum densities.

The averaged tensors

\[ <g t_a^b(P; v^l)>, <m T_a^b(P; v^l)>, <g M^{abc}(P; v^l)>, <m M^{abc}(P; v^l)> \tag{41} \]

depend on the four-velocity \( \vec{v} \) of a fiducial observer \( O \) which is at rest at the beginning \( P \) of the normal coordinates \( \text{NC}(P) \) used for averaging and on some fundamental length \( L > 0 \). After fixing the fundamental length \( L \) one can determine univocally these tensors along the world line of an observer \( O \).

In general one can unambiguously determine these tensors (after fixing \( L \)) in the whole spacetime or in some domain \( \Omega \) if in the spacetime or in the domain \( \Omega \) a geometrically distinguished timelike unit vector field \( \vec{v} \) exists. An example of such a kind of the spacetime is given by Friedman universes.

One can try to fix\(^6\) the fundamental length \( L \), e.g., by using loop quantum gravity. Namely, one can take as \( L \) the smallest length \( l \) over which the classical model of the spacetime is admissible.

Following loop quantum gravity [19-29] one can say about continuous classical differential geometry already just a few orders of magnitude above the Planck scale, e.g., for distances \( l \geq 100L_P = 100\sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33} \text{ m} \). So, one can take as the fundamental length \( L \) the value \( L = 100L_P \approx 10^{-33} \text{ m} \).

\(^6\)But this is not necessary. One can effectively use the averaged energy-momentum and angular momentum tensors without fixing \( L \) explicitly.

\(^7\)Concerning other propositions fixing of \( L \) see, e.g., [9–12].
After fixing the fundamental length $L$ one has the averaged relative energy-momentum and angular momentum tensors as precisely defined as the canonical superenergy and angular supermomentum tensors are.

The averaged tensors (with $L$ fixed or no)

\[
<_{m} T_{a}^{b}(P;v^{I}), <_{g} T_{a}^{b}(P;v^{I}), <_{m} M^{abc}(P;v^{I}), <_{g} M^{abc}(P;v^{I})>
\]

(42)
give us as good tool to a local analysis (and also to global analysis iff in spacetime a privileged global unit timelike vector field exists) of the gravitational and matter fields as the canonical superenergy and angular supermomentum tensors

\[
<_{m} S_{a}^{b}(P;v^{I}), <_{g} S_{a}^{b}(P;v^{I}), <_{m} M^{abc}(P;v^{I}), <_{g} M^{abc}(P;v^{I})>
\]

(43)
give. For example, one can apply the averaged energy-momentum and angular momentum tensors to the all problems which have been analyzed in the papers [1-8] by using the canonical superenergy and angular supermomentum tensors.

II. SOME APPLICATIONS OF THE AVERAGED RELATIVE ENERGY-MOMENTUM TENSORS

In this paper we apply the averaged gravitational relative energy-momentum tensor

\[
<_{g} T_{a}^{b}(P;v^{I})>
\]

only to decide if free vacuum gravitational field has energy-momentum; especially, if gravitational waves carry any energy-momentum, and the averaged gravitational and matter relative energy-momentum tensors to analyze the energy and momentum of the Friedman universes.

Albrow and Tryon were the first who assumed that the net energy of the closed Friedman universes may be equal to zero [30-31]. We will show in this paper that this assumption is, most probably, incorrect.

Let us begin from the vacuum gravitational energy and momentum. The problem was
revived recently because some authors conjectured [32-36], by using coordinate dependent\(^8\) pseudotensors and double index complexes, that the energy and momentum in general relativity are confined only to the regions of non-vanishing energy-momentum tensor of matter and that the gravitational waves carry no energy and momentum. The argumentation is the following. For some solutions to the Einstein equations and in some special coordinates, e.g., in Bonnor’s spacetime [37] in Bonnor’s or in Kerr-Schild coordinates, the Einstein canonical gravitational energy-momentum pseudotensor (and other most frequently used gravitational energy-momentum pseudotensors also) globally vanishes outside of the domain in which \( T^{ik} \neq 0 \). The analogous global vanishing of the canonical pseudotensor \( E^b_t a \) we have for the plane and for the plane-fronted gravitational waves in, e.g., null coframe [3,38]. But one should emphasize that all these results are coordinate dependent [3,7,38], i.e., in other coordinates the used gravitational energy-momentum pseudotensors do not vanish in vacuum. Moreover, one should interpret physically the global vanishing of the canonical pseudotensor (and other pseudotensors also) in some coordinates in vacuum as a global cancellation of the energy-momentum of the real gravitational field which has \( R_{iklm} \neq 0 \) with energy-momentum of the inertial forces field which has \( R_{iklm} = 0 \); not as a proof of vanishing of the energy-momentum of the real gravitational field. It is because the all used pseudotensors were entirely constructed from the Levi-Civita’s connection \( \Gamma^i_{kl} = \Gamma^i_{lk} \) and from the metric \( g_{ik} \) which describe a mixture of the real gravitational field (\( R_{iklm} \neq 0 \)) and an inertial forces field (\( R_{iklm} = 0 \)).

In order to get the coordinate independent results about energy-momentum of the real gravitational field one must use tensorial expressions which depend on curvature tensor, like the averaged gravitational relative energy-momentum tensor \( \langle g t^b_a (P; v^l) \rangle \). This tensor

\(^8\)By “coordinate dependent” quantity we mean a quantity which is not a tensor (in general—which is not a tensor valued p-form). By “coordinate independent” quantity we mean a tensor quantity (in general – a tensor valued p-form).
vanishes iff $R_{iklm} = 0$, i.e., iff the spacetime is flat and we have no real gravitational field.

When calculated, the averaged gravitational relative energy-momentum tensor
\[ <g\ t_a^b(P; v^l) > \]
always gives the *positive-definite* averaged free relative gravitational energy density and, in the case of a gravitational wave, its non-zero flux. It is easily seen from the our papers [1-8,38] in which we have used the canonical gravitational superenergy tensor and from the formula (20) of this paper which gives the direct connection between the averaged relative gravitational energy-momentum tensor and the canonical gravitational superenergy tensor.

Thus, the conjecture about localization of the gravitational energy only to the regions of the non-vanishing energy-momentum tensor of matter is *incorrect* for the real gravitational field which has $R_{iklm} \neq 0$.

It is interesting that the gravitational angular momentum pseudotensor (28) *does not vanish* in Bonnor’s spacetime and in Bonnor’s coordinates *outside* of the domain in which $T^{ik} \neq 0$. This important fact which, as I think, is unknown for the authors of the conjecture, gives other *direct proof* that this conjecture is *incorrect*. If the conjecture were correct, then we would have an absurd situation: the energy-momentum density–free vacuum gravitational field has non-vanishing “densities” of the angular momentum.

In a similar way as above one can use the averaged gravitational relative angular momentum tensor $<g\ M^{abc}(P; v^l) >$ to coordinate independent analysis of the angular momentum of the real gravitational field.

Now, let us pass to the problem of the energy and momentum of the Friedman universes. Of course, the problem of the global energy and global linear (or angular) momentum for Friedman universes (and also for more general universes) is not *well-posed* from the physical point of view because these universes are not asymptotically flat spacetimes [39]. Despite this important fact recently many authors concluded [40-50] that the energy and momentum of the Friedman universes, flat and closed, are equal to zero locally and globally (flat universes) or only globally (closed universes). Such conclusion, which has a mathematical sense, originated from calculations performed in special comoving coordinates called “Carte-
sian coordinates” by using *coordinate dependent* double index energy-momentum complexes, matter and gravitation.

One can introduce in GR many different energy-momentum complexes. The six of them are most frequently used: Einstein’s canonical complex, Landau-Lifshitz complex, Bergmann-Thomson complex, Møller complex, Papapetrou complex and Weinberg energy-momentum complex. These all energy-momentum complexes *are neither geometrical objects nor coordinate independent objects*, e.g., they can vanish in some coordinates locally or globally and in other coordinates they can be different from zero. It results that the double index energy–momentum complexes and the gravitational energy-momentum pseudotensors *have no physical meaning* to a local analysis of the gravitational field, e.g., to study gravitational energy-density distribution. They can be reasonably used *only to calculate the global quantities* for the very precisely defined asymptotically flat spacetimes (in spatial or in null direction).

The general opinion is that the best one of the all possible double index energy-momentum complexes from physical and geometrical points of view is the canonical Einstein’s double index energy-momentum complex $E K_i^k = \sqrt{|g|}(T_i^k + E t_i^k)$. The global results obtained by use of this canonical energy-momentum complex are usually treated as correct and giving some pattern. In fact, the other double index energy-momentum complexes were constructed following the instruction: they should give the same global results as the Einstein energy-momentum complex gives at least in the simplest cases, e.g., in the case of a closed system. That is why we have confined in the paper (and also in the all our previous papers) only to this double index energy-momentum complex.

So, let us consider the results of the formal calculations of the global energy and momentum for Friedman universes in the standard comoving coordinates by using canonical Einstein’s double index energy-momentum complex. Any other sensible double index energy-momentum complex gives equivalent results.
1. In the “Cartesian coordinates” \((t, x, y, z)\) in which the line element has the form\(^9\)

\[
ds^2 = dt^2 - R^2(t) \frac{(dx^2 + dy^2 + dz^2)}{[1 + k/4(x^2 + y^2 + z^2)]^2}, \quad k = 0, \pm 1, \tag{44}
\]

we obtain after simple calculations [1,5] that for flat universes the global quantities \(P_i (i = 0, 1, 2, 3)\), where \(P_i\) mean the components of the energy-momentum contained inside of a slice \(t = \text{const}\), are equal to zero. In this case the all integrands (energy and momentum “densities”) in the integrals on \(P_i (i = 0, 1, 2, 3)\) identically vanish because they are multiplied by the curvature index \(k\). So, one can say that for flat Friedman universes the integral quantities \(P_i (i = 0, 1, 2, 3)\) vanish locally and globally in the “Cartesian” coordinate.\(^{10}\)

For closed Friedman universes we also get \(P_i = 0, \quad (i = 0, 1, 2, 3)\), but this time the integrands do not vanish. Only after integration one gets that the integrals representing \(P_i, (i = 0, 1, 2, 3)\) are equal to zero. In the case of the open Friedman universes one gets \(E = P_0 = (-)\infty, \quad P_1 = P_2 = P_3 = 0\). The integrands also do not vanish in this case.

2. In the coordinates \((t, \chi, \vartheta, \varphi)\) in which the line element reads

\[
ds^2 = dt^2 - R^2(t)[d\chi^2 + S^2(\chi)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \tag{45}
\]

where

\[
S(\chi) = \{ \sin \chi \quad \text{if} \quad k = 1, \quad \chi \quad \text{if} \quad k = 0, \quad \text{sh} \chi \quad \text{if} \quad k = -1\}, \tag{46}
\]

\(^9\)From now on we will use geometrized units in which \(G = c = 1\).

\(^{10}\)It is interesting that the angular momentum “densities” when calculated, e.g., by using Bergmann–Thomson angular momentum complex (28) do not vanish in the case even for flat Friedman universes.
one gets drastically different results: $E = P_0 = (-)\infty$, $P_1 = (-)\infty$, $P_2 = P_3 = 0$ for flat universes; $E = P_0 = \frac{\pi}{2} R(t)$, $P_1 = P_2 = P_3 = 0$ for closed universes and $E = P_0 = (-)\infty$, $P_1 = (-)\infty$, $P_2 = P_3 = 0$ for open universes.

3. Finally, in the coordinates $(t, r, \vartheta, \varphi)$ in which the line element has the form

$$ds^2 = dt^2 - R^2(t)[\frac{dr^2}{1 - kr^2} + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \quad k = 0, \pm 1,$$

we obtain the following results: $E = P_0 = (-)\infty$, $P_1 = (-)\infty$, $P_2 = P_3 = 0$ for flat universes; $E = P_0 = \frac{\pi}{2} R(t)$, $P_1 = (-)\infty$, $P_2 = P_3 = 0$ for closed universes and $E = P_0 = (-)\infty$, $P_1 = (-)\infty$, $P_2 = P_3 = 0$ for open Friedman universes.

In the all cases in which the integrands (=“densities” of the calculated four-momentum) do not vanish, these integrands go to zero if $R(t) \to 0$. So, these integrands (“densities” of the energy-momentum) are not suitable for analysis of the Big-Bang singularity.

The authors which assert that the energy and momentum of the Friedman universes, flat and closed, are equal to zero have performed their calculations only in the “Cartesian” comoving coordinates $(t, x, y, z)$ by using coordinate dependent double index energy-momentum complexes and have got zero results. But in the case of the Friedman universes the “Cartesian”coordinates are by no means better than the comoving coordinates $(t, \chi, \vartheta, \varphi)$ or $(t, r, \vartheta, \varphi)$ in which we have obtained non-zero results. Only in a flat and in an asymptotically flat spacetimes one can distinguish in some reasonable way the Cartesian coordinates; but not in the case of the Friedman universes. So, the conclusion of these authors about vanishing of the energy and linear momentum of the Friedman universes, flat and closed, cannot be correct.

By using double index energy-momentum complexes one rather should conclude that the energy and momentum of the Friedman universes explicite depend on the used comoving coordinates and, therefore, that they are undetermined locally and globally. This last conclusion is very sensible because one cannot measure the global energy and global momentum of the Friedman (and more general) universes. One can do this only in the case of an isolated
system [39]. On the other hand the former conclusion directly follows from the coordinate
dependence of the energy-momentum complexes.

May be one would try to support the \textit{mathematically sensible} hypothesis which states
that energy and momentum of the Friedman universes, flat and closed, disappear by using
coordinate independent expressions, like Pirani’s expression on global energy, matter and
gravitation, or like single index Komar’s expression (Komar’s single index complex) on global
energy-momentum and global angular momentum, matter and gravitation \textsuperscript{11}.

The Pirani’s expression (for the energy only, see, eg., [51]) is unique and can be applied in
a spacetime having a privileged set of observers whose world-lines form a normal congruence.
In such spacetime there exists a family of spatial hypersurfaces which are orthogonal to the
four-velocities of this set of observers.

The Pirani’s expression is coordinate independent but it has two defects: calculated
total energy density, matter and gravitation, \textit{is not positive-definite}, and, if the congruence
is geodesic, then the total energy-density \textit{is identically zero}, and, in consequence, the global
energy \textit{trivially vanishes in the case}. However, this zero values \textit{are not a property of the
gravitational and matter fields}. They are only a property of the geodesics congruence.

In Friedman universes does exist privileged set of observers called \textit{fundamental or
isotropic} observers. For these observers the four-velocity $\vec{v}$ has components $v^k = \delta^k_0$
in a comoving coordinates and the family of the spatial hypersurfaces orthogonal to $\vec{v}$ is given
by $t = \text{const}$. But, unfortunately, the congruence of the isotropic observers in Friedman
universes is geodesic and, therefore, the Pirani’s expression \textit{fails} in the case \textit{giving trivially
zero}.

On the other hand, coordinate independent Komar’s expression (see, e.g., [51-53]) \textit{needs
\textsuperscript{11}We would like to remark that the Pirani’s and Komar’s expressions, though coordinate inde-
pendent, depend (like double index energy-momentum complexes) not only on real gravitational
field ($R_{iklm} \neq 0$) but also on inertial forces field ($R_{iklm} = 0$).
Killing vector fields: translational timelike Killing vector field as energy descriptor, translational spatial Killing vector fields as descriptors of the linear momentum and rotational spatial Killing vector fields as descriptors of the angular momentum.

Friedman universes admit only six linearly independent spatial Killing vector fields, three translational Killing vector fields and three rotational Killing vector fields (see, e.g., [54]). So, one can consider in Friedman universes six coordinate independent integrals (scalars) which correctly represent (from mathematical point of view) the components of the global linear momentum and the components of the global angular momentum (see, eg., [54]). These integrals trivially vanish for Friedman universes, i.e., integrands in these integrals identically vanish, independently of the curvature index $k = 0, \pm 1$. This is very sensible result and it can be interpreted as a mathematically correct proof that the linear and angular momentum for Friedman universes disappear in a comoving coordinates.

But we still have a problem with energy of the Friedman universes because we have no energy descriptor, i.e., translational timelike Killing vector field, in these universes. Therefore, one cannot use the coordinate independent Komar’s expression in order to calculate correctly from the mathematical point of view the energy of the Friedman universes.

If one formally uses in Komar’s expression the four-velocity of the privileged set of the isotropic observers as the energy descriptor, then one will get identically zero because for a geodesic timelike congruence the integrand in this expression, like integrand in Pirani’s expression, identically vanishes. But this vanishing is also only a property of the geodesics congruence. It is not a property of the gravitational and matter fields.

Resuming, one cannot use the coordinate independent Pirani’s and Komar’s expressions in order to correctly prove the statement that the energy of the Friedman universes disappears, i.e., that these universes are complete energetic nonentity.

For this purpose one cannot also use the coordinate independent KBL bimetric approach

\footnote{Correctly from the mathematical point of view.}
because the results obtained in this approach depend not only on the used background but also on mapping of the real spacetime onto this background.

Therefore, the mathematically sensible statement that the closed and flat Friedman universes have no energetic content is still not satisfactory proved.

It is interesting that the using of the coordinate independent averaged relative energy-momentum tensors to analyze the energetic content of the Friedman universes lead us to positive-definite results for the all Friedman universes.

Namely, let us apply the averaged relative energy-momentum tensors for gravitation \( <_g t_{i}^{k}(P; v^j) > \) and for matter \( <_m T_{i}^{k}(P; v^j) > \) to calculate the averaged relative energy density for Friedman (and more general) universes. With this aim let us define

\[
g \epsilon := <_g t_{a}^{b}(P; v^j) > v^a v_b \tag{48}
\]

— the averaged relative gravitational energy density,

\[
m \epsilon := <_m T_{a}^{b}(P; v^j) > v^a v_b \tag{49}
\]

— the averaged relative matter energy density, and

\[
\epsilon := g \epsilon + m \epsilon \tag{50}
\]

— the averaged relative total energy density.

Here \( v^a \) are the components of the four-velocity of an observer \( O \) which is studying gravitational and matter fields.

In Friedman universes, if we take as the observers \( O \) the globally defined set of the fundamental observers, then we can also define the global averaged total relative energy \( E \) of a Friedman universe

\[
E := \int_{t=\text{const}} \epsilon \sqrt{|g|} d^3v = \int_{t=\text{const}} [<_g t_{i}^{0} > + <_m T_{i}^{0} >]v^i \sqrt{|g|} d^3v, \tag{51}
\]

and, in analogous way, the global averaged relative energy for matter and for gravitation.
Here $d^3v$ means the product of the differentials of the coordinates which parametrize slices $t = \text{const}$ of the Friedman universes, e.g., $d^3v = dx dy dz$ in the Cartesian comoving coordinates $(t, x, y, z)$.

After something tedious but very simple calculations we will obtain for Friedman universes $[1,2,5]$:

1. $g \epsilon$, $m \epsilon$ and, in consequence $\epsilon$, are positive definite for the all Friedman universes.

2. $\lim_{R \to 0} g \epsilon = \lim_{R \to 0} m \epsilon = \lim_{R \to 0} \epsilon = +\infty$, $(k = 0, \pm 1)$.

   It follows from this that one can use the averaged relative energy densities to study the Big-Bang singularity.

3. $\lim_{R \to \infty} g \epsilon = \lim_{R \to \infty} m \epsilon = \lim_{R \to \infty} \epsilon = 0$, $(k = 0, -1)$.

4. The global averaged relative energies, gravitation, matter and total, are infinite ($+\infty$) for flat and for open Friedman universes and they are finite and positive for closed Friedman universes.

Also the other three invariant integrals which formally represent the components $P_{(\alpha)}$ ($\alpha = 1, 2, 3$) of the global averaged relative linear momentum for Friedman universes

\[
P_{(\alpha)} := \int_{t = \text{const}} \{ <g t_i^0 > + <m T_i^0 > \} e_i^{(\alpha)} \sqrt{|g|} d^3v, \quad (\alpha = 1, 2, 3),
\]

vanish trivially in a comoving coordinates [1,2,5] because the integrands in these integrals (densities of the averaged relative linear momentum components) identically vanish [1,2,5].

Here $e_i^{(\alpha)}$, ($\alpha = 1, 2, 3$) mean the components of the three translational spatial Killing vector fields (descriptors of the linear momentum) which exist in the Friedman universes (see, e.g., [54]).

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13 The results given below are easily seen from the our previous papers [1,2,5] and from the formulas (12) and (20) which connect the canonical superenergy tensors used in the papers [1,2,5] with the averaged relative energy and momentum tensors which we are using in this paper.
We would like to emphasize that the integrals (51) and (52) do not depend on the used coordinates. They depend only on a slice $t = \text{const}$.

The all above results are very sensible and satisfactory from the physical point of view.

We will finish this Section with remark that the analogous situation as for flat Friedman universes one has also for the more general, only homogeneous, Kasner vacuum universes [15] and Bianchi–type I universes filled with stiff matter (see, e.g., [44-50, 56-59]). Namely, the most frequently used double-index energy-momentum complexes, when used in Cartesian comoving coordinates to analyze of these universes, give zero results locally and globally.

Of course, in other comoving coordinates, e.g., in the $t, r, \vartheta, \varphi$ comoving coordinates, we have non-zero and globally divergent results.

If one applies the averaged relative energy-momentum tensors $<_{g} t_{a}^{b}(P; v^{l}) >$, $<_{m} T_{a}^{b}(P; v^{j}) >$ to analyze of a vacuum Kasner universe and a Bianchi–type I universe filled with stiff matter, then one gets the following, coordinate independent results:

1. The averaged relative gravitational energy of a vacuum Kasner universe has positive-definite density and the same limits when $t \rightarrow 0$ or when $t \rightarrow +\infty$ as it was in the case of a flat Friedman universe. Also the suitable integral global quantity defined in analogous way as in the case of the Friedman universes is divergent to $+\infty$.

2. For an expanding Bianchi–type I universe filled with stiff matter the averaged relative gravitational energy density and the averaged relative energy-density for matter are still positive-definite and lead to divergent to $+\infty$ global energies.

Thus, one can conclude that these two more general, only homogeneous universes, like Friedman flat universes, also are not energetic nonentity.

Concerning of the components of the linear momentum for Kasner vacuum universes and for Bianchi–type I universes filled with stiff matter one can easily check that these components, defined in analogous way as in the case of the Friedman universes, identically vanish locally and globally in a comoving coordinates.
III. CONCLUSION

We have introduced in the paper the averaged tensors of the relative energy-momentum and the averaged tensors of the relative angular momentum, for matter and for gravitation. These tensors are very closely related to the canonical superenergy and angular supermomentum tensors and they can be used to analyze the same problems which we have analyzed in the our papers [1-8] with the help of the canonical superenergy and angular supermomentum tensors. The superiority of the averaged relative energy-momentum and angular momentum tensors in comparison with the canonical superenergy and angular supermomentum tensors is the following: the averaged tensors have proper dimensionality of the energy-momentum and angular momentum densities.

The averaged relative energy-momentum and relative angular momentum tensors of the gravitational field refer to the energy-momentum and angular momentum of the real gravitational field for which we have $R_{iklm} \neq 0$. These tensors vanish iff $R_{iklm} = 0$, i.e., iff we have no real gravitational field.

In our opinion the all existing (and projected in near future) detectors of the gravitational waves will measure the averaged relative gravitational energy density and its flux; not the gravitational energy defined by pseudotensors. It is easily seen from the fact that the acting of these detectors relies on the equations of the geodesics deviation which explicitly depend on the curvature tensor.

In this paper we have applied the averaged relative gravitational energy-momentum tensor to decide if free vacuum gravitational field has energy and momentum and the averaged gravitational and matter relative energy-momentum tensors to analyze energy and momentum of the Friedman universes and also to analyze the Kasner and Bianchi–type I universes. The latter problem is recently very popular despite the fact that the problem of the global quantities for Friedman universes (and for more general cosmological models also) is not well-posed from the physical point of view. The global energy and momentum have physical meaning only when spacetime is asymptotically flat either in spatial or null direction. Of
course, this is not a case of the Friedman and Kasner or Bianchi–type I cosmological models.

We have obtained the following results:

1. The real vacuum gravitational field for which we have $R_{iklm} \neq 0$ always possesses his own positive-definite averaged relative energy density and in the cases in which the gravitating system is not at rest, the gravitational field possesses also the non-zero averaged relative linear momentum.

2. The coordinate independent averaged relative energy-momentum tensors, gravitation and matter, give positive-definite densities of the averaged relative energy, matter and gravitation, for the all Friedman universes. Therefore, these tensors indicate that the Friedman universes are not energetic nonentity. They are not energetic nonentity in the following sense: one can construct from the canonical energy-momentum complex, matter and gravitation, non-local tensorial, i.e., coordinate-independent expressions with correct dimensions which give positive-definite energy densities for the all Friedman universes.

The averaged relative energy-momentum tensors tensors give also zero values of the averaged relative linear momenta for these universes in a comoving coordinates.

The above results directly follow from the results obtained in the our previous papers [1-5] in which we have used the canonical superenergy (and angular supermomentum) tensors, gravitation and matter, and from the formulas (12) and (20) of this paper which connect the averaged relative energy-momentum tensors with the canonical superenergy tensors.

The coordinate independent results presented in this paper for the Friedman universes are very satisfactory from the physical point of view. Much more satisfactory than the strange, coordinate dependent results which one obtains by using gravitational energy-momentum pseudotensors and double index energy-momentum complexes, matter and gravitation. By using of these objects one can only conclude that the energy and momentum of the Friedman universes are undetermined locally and globally.
The analogous conclusion as given above for Friedman universes is also correct for the more general Kasner and Bianchi–type I universes.

We are planning to use in a future the averaged relative energy-momentum tensors, and also the averaged tensors of the relative angular momentum, to analyze much more general homogeneous universes, like the universes which have been considered in the papers [44-50, 56-60].
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