Bounds on the Automata Size for Presburger Arithmetic

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Automata provide a decision procedure for Presburger arithmetic. However, until now only crude lower and upper bounds were known on the sizes of the automata produced by this approach. In this paper, we prove an upper bound on the number of states of the minimal deterministic automaton for a Presburger arithmetic formula. This bound depends on the length of the formula and the quantifiers occurring in the formula. The upper bound is established by comparing the automata for Presburger arithmetic formulas with the formulas produced by a quantifier elimination method. We also show that our bound is tight, even for nondeterministic automata. Moreover, we provide optimal automata constructions for linear equations and inequations.

Categories and Subject Descriptors: F.1.1 [Computation by Abstract Devices]: Models of Computation—automata; F.4.1 [Mathematical Logic and Formal Languages]: Mathematical Logic—computational logic

General Terms: Algorithms, Theory

Additional Key Words and Phrases: Automata-based Decision Procedures, Presburger Arithmetic, Quantifier Elimination, Complexity

1. INTRODUCTION

Presburger arithmetic (PA) is the first-order theory with addition and the ordering relation over the integers. A number of decision problems can be expressed in it, such as solvability of systems of linear Diophantine equations, integer programming, and various problems in system verification. The decidability of PA was established around 1930 independently by Presburger [1930; 1984] and Skolem [1931; 1970] using the method of quantifier elimination.

Due to the applicability of PA in various domains, its complexity and the complexity of decision problems for fragments of it have been investigated intensively. For example, Fischer and Rabin [1974; 1998] gave a double exponential nondeterministic time lower bound on any decision procedure for PA. Later, Berman [1980] showed that the decision problem for PA is complete in the complexity class...
LATIME$\left(2^{2^{O(n)}}\right)$, i.e., the class of problems solvable by alternating Turing machines in time $2^{2^{O(n)}}$ with a linear number of alternations. The upper bound for PA is established by a result from Ferrante and Rackoff [1979] showing that quantified variables need only to range over a restricted finite domain of integers. Grädel [1988] and Schöning [1997] investigated the complexity of decision problems of fragments of PA.

The complexity of different decision procedures for PA has also been studied, e.g., in [Oppen 1978; Reddy and Loveland 1978; Ferrante and Rackoff 1975; 1979]. For instance, Oppen [1978] showed that Cooper’s quantifier elimination decision procedure for PA [Cooper 1972] has a triple exponential worst case complexity in deterministic time. Reddy and Loveland [1978] improved Cooper’s quantifier elimination and used it for obtaining space and deterministic time upper bounds for checking the satisfiability of PA formulas in which the number of quantifier alternations is bounded.

Another approach for deciding PA or fragments of it that has recently become popular is to use automata; a point that was already made by Büchi [1960]. The idea is simple: Integers are represented as words, e.g., using the 2’s complement representation, and the word automaton (WA) for a formula accepts precisely the words that represent the integers making the formula true. The WA can be recursively constructed from the formula, where automata constructions handle the logical connectives and quantifiers. This automata-based approach for PA led to deep theoretical insights, e.g., the languages that are regular in any base are exactly the sets definable in PA [Cobham 1969; Semenov 1977; Bruyère et al. 1994]. More recently, the use of automata has been proposed for mechanizing decision procedures for PA and for manipulating sets definable in PA [Boudet and Comon 1996; Wolper and Boigelot 1995]. Roughly speaking, this applied use of WAs for PA is similar to the use of binary decision diagrams (BDDs) for propositional logic. For example, the automata library LASH [LASH ] provides tool support for manipulating PA definable sets using automata to represented these sets, and it has been successfully used to verify systems with variables ranging over the integers. Other model checkers that use WAs for computing the potential infinite sets of reachable states of systems with integer variables are, e.g., FAST [Bardin et al. 2003] and ALV [Yavuz-Kahveci et al. 2005].

A crude complexity analysis of automata-based decision procedures for PA leads to a non-elementary worst case complexity. Namely, for every quantifier alternation there is a potential exponential blow-up. However, experimental comparisons [Shiple et al. 1998; Bartzis and Bultan 2003; Ganesh et al. 2002] illustrate that automata-based decision procedures for PA often perform well in comparison with other methods. In [Boudet and Comon 1996], the authors claimed that the minimal deterministic WA for a PA formula has at most a triple exponential number of states in the length of the formula. Unfortunately, as explained by Wolper and Boigelot [2000], the argument used in [Boudet and Comon 1996] to substantiate this claim is incorrect. Wolper and Boigelot [2000] gave an argument why there must be an elementary upper bound on the size of the minimal deterministic WA for a PA formula. However, their argumentation is rather sketchy and only indicates that there has to be an elementary upper bound.
In this paper, we rigorously prove an upper bound on the size of the minimal deterministic WA for PA formulas and thus, answer a long open question. Namely, for a PA formula in prenex normal form, we show that the minimal deterministic WA has at most $2^{n(2^a+b+4)}$ states, where $n$ is the formula length, $a$ is the number of quantifier alternations, and $b$ is the maximal length of the quantifier blocks. A similar upper bound holds for arbitrary PA formulas. This bound on the automata size for PA contrasts with the upper bound on the automata size for the monadic second-order logic WS1S, or even WS1S with the ordering relation “$<$” as a primitive but without quantification over monadic second-order variables. There, the number of states of the minimal WA for a formula can be non-elementary larger than the formula’s length [Stockmeyer 1974; Reinhardt 2002]. In order to establish the upper bound on the automata size for PA, we give a detailed analysis of the deterministic WAs for formulas by comparing the constructed WAs with the quantifier-free formulas produced by using Reddy and Loveland’s quantifier elimination method. From this analysis, we obtain the upper bound on the size of the minimal deterministic WA for PA formulas.

We also show that the upper bound on the size of deterministic WAs for formulas is tight. In fact, we show a stronger result. Namely, we give a family of Presburger arithmetic formulas for which even a nondeterministic WA must have at least triple exponentially many states.

Furthermore, we investigate the automata constructed from atomic formulas. Specific algorithms for constructing WAs for linear (in)equations have been developed in [Boudet and Comon 1996; Boigelot 1999; Wolper and Boigelot 2000; Bartzis and Bultan 2003; Ganesh et al. 2002]. We give upper and lower bounds on the automata size for linear (in)equations and we improve the automata constructions in [Boigelot 1999; Wolper and Boigelot 2000; Ganesh et al. 2002] for linear (in)equations. We prove that our automata constructions are optimal in the sense that the constructed deterministic WAs are minimal.

We proceed as follows. In §2, we give background. In §3, we investigate the WAs for quantifier-free formulas. In §4, we prove the upper bound on the size of the minimal deterministic WA for PA formulas and in §5, we give a worst case example. Finally, in §6, we draw conclusions.

2. PRELIMINARIES

2.1 Presburger Arithmetic

Presburger arithmetic (PA) is the first-order logic over the structure $\mathcal{3} := (\mathbb{Z}, <, +)$. We use standard notation. For instance, we write $\mathcal{3} \models \varphi[a_1, \ldots, a_r]$ for a formula $\varphi(x_1, \ldots, x_r)$ and $a_1, \ldots, a_r \in \mathbb{Z}$ if $\varphi$ is true in $\mathcal{3}$ when the variable $x_i$ is interpreted as the integer $a_i$, for $1 \leq i \leq r$. Analogously, $t[a_1, \ldots, a_r]$ denotes the integer when the $x_i$s are interpreted as the $a_i$s in the term $t(x_1, \ldots, x_r)$. For a formula $\varphi(x_1, \ldots, x_r)$, we define $[\varphi] := \{(a_1, \ldots, a_r) \in \mathbb{Z}^r : \mathcal{3} \models \varphi[a_1, \ldots, a_r]\}$.

2.1.1 Extended Logical Language. We extend the logical language of PA by (i) constants for the integers 0 and 1, (ii) the unary operation “$-$” for integer negation, and (iii) the unary predicates “$d|$” for the relation “divisible by $d$,” for each $d \geq 2$. These constructs are definable in PA, e.g., the formula $\exists x (x + \cdots + x = t)$.
defines \(d|t\), where \(x\) occurs \(d\) times in the term \(x + \cdots + x\) and \(x\) does not appear in the term \(t\). The reason for the extended logical language, where (i), (ii), and (iii) are treated as primitives, is that it admits quantifier elimination, i.e., for a formula \(\exists x \varphi(x, y)\), where \(\varphi\) is quantifier-free, we can construct a logically equivalent quantifier-free formula \(\psi(y)\).

Additionally, we allow the relation symbols \(\leq, >, \geq, \neq\) with their standard meanings. In the following, we assume that terms and formulas are defined in terms of the extended logical language for PA. We denote by PA the set of all Presburger arithmetic formulas over the extended logical language and QF denotes the set of quantifier-free formulas.

For convenience, we use standard symbols when writing terms. For instance, \(c\) stands for \(1 + \cdots + 1\) (repeated \(c\) times) if \(c > 0\), and \(-(1 + \cdots + 1)\) if \(c < 0\). We call the term \(c\) a constant and identify the term \(c\) with the integer that it represents. Analogously, we write \(k \cdot x\) for \(x + \cdots + x\) (repeated \(k\) times) if \(k > 0\), and \(- (x + \cdots + x)\) if \(k < 0\). Moreover, if \(k = 0\) then \(k \cdot x\) abbreviates \(x + (-x)\). We say that \(k\) is a coefficient. For a term \(t\) and \(k \in \mathbb{Z}\), \(k \cdot t\) denotes the term where the constant and the coefficients in \(t\) are multiplied by \(k\).

A term \(t\) is homogeneous if it is either 0 or of the form \(k_1 \cdot x_1 + \cdots + k_r \cdot x_r\), for some \(r \geq 1\), where the variables \(x_1, \ldots, x_r\) are pairwise distinct and \(k_1, \ldots, k_r \in \mathbb{Z} \setminus \{0\}\). The normalized form of \(t_1 \otimes t_2\), with \(\otimes \in \{=, \neq, \leq, >, \geq\}\), is the logically equivalent (in)equation \(t \otimes c\), where summands of the form \(k \cdot x\) in \(t_1\) and \(t_2\) are collected on the left-hand side \(t\) and constants in \(t_1\) and \(t_2\) are collected on the right-hand side \(c\) according to standard calculation rules. The normalized form of \(d|t\) is the formula \(d|t' + c\), where \(c \in \mathbb{Z}\) is the sum of the constants in \(t\) and \(t'\) is the homogeneous term in which the coefficients of the summands of the form \(k \cdot x\) in \(t\) are collected. We use \(A(\varphi)\) to denote the set of atomic formulas occurring in \(\varphi \in \text{PA}\) in their normalized forms.

### 2.1.2 Formula Length

The length of a formula is the number of letters used in writing the formula. Note that the length of a formula depends significantly on how we define the coefficients and constants. For instance, \(x = 10 \cdot y\) contains 6 letters, namely, \(x, =, 1, 0, \cdot, y\). The “expanded version” has \(2 + 19\) letters since \(10 \cdot y\) abbreviates the term \(y + y + y + y + y + y + y + y + y\). We use the same definition of the length of a formula as in [Oppen 1978; Fischer and Rabin 1974; Reddy and Loveland 1978]. In particular, the length of a coefficient or constant is the number of letters of the expanded version. However, it is possible to express \(k \cdot x\) by a formula of length \(O(\log |k|)\). The idea is illustrated by \(x = 10 \cdot y\): the formula is logically equivalent to \(\exists z(x = z + z \land \exists x(z = x + x + y \land x = y + y))\). Note that we only need a fixed number of variables for any \(k\) (see Fischer and Rabin 1974]). For the sake of uniformity, we define the length of the formula \(d|t\) as the length of the term \(t\) plus \(d + 1\). Again, there is a logically equivalent formula of length \(O(\log d)\) plus the length of \(t\). For the results in this paper it does not matter if we define the length of an integer \(k\) as \(O(\log |k|)\) or as \(O(|k|)\).

### 2.1.3 Nesting of Quantifiers

It is well-known that we obtain coarse complexity bounds for checking satisfiability if we only take into account the formula length. We obtain more precise complexity bounds when we additionally for account the
number of quantifiers and the number of quantifier alternations.

The quantifier number of \( \varphi \in \text{PA} \) is the number of quantifiers occurring in \( \varphi \), i.e.,

\[
\text{qn}(\varphi) := \begin{cases} 
\text{qn}(\psi) & \text{if } \varphi = \neg \psi, \\
\text{qn}(\psi_1) + \text{qn}(\psi_2) & \text{if } \varphi = \psi_1 \oplus \psi_2 \text{ with } \oplus \in \{\land, \lor, \rightarrow, \leftrightarrow\}, \\
1 + \text{qn}(\psi) & \text{if } \varphi = Q x \psi \text{ with } Q \in \{\exists, \forall\}, \\
0 & \text{otherwise}.
\end{cases}
\]

For a quantifier \( Q \in \{\exists, \forall\} \), \( \overline{Q} \) denotes its dual, i.e., \( \overline{Q} := \forall \) if \( Q = \exists \), and \( \overline{Q} := \exists \) if \( Q = \forall \). The number of quantifier alternations of \( \varphi \in \text{PA} \) is

\[
\text{qa}(\varphi) := \min\{\text{qa}_\exists(\varphi), \text{qa}_\forall(\varphi)\},
\]

where

\[
\text{qa}_Q(\varphi) := \begin{cases} 
\text{qa}_{\overline{Q}}(\psi) & \text{if } \varphi = \neg \psi, \\
\max\{\text{qa}_Q(\psi_1), \text{qa}_Q(\psi_2)\} & \text{if } \varphi = \psi_1 \oplus \psi_2 \text{ with } \oplus \in \{\lor, \land\}, \\
\text{qa}_Q(\psi_1 \lor \psi_2) & \text{if } \varphi = \psi_1 \rightarrow \psi_2, \\
1 + \text{qa}_{\overline{Q}}(\psi) & \text{if } \varphi = Q x \psi, \\
\max\{1, \text{qa}_Q(\psi)\} & \text{if } \varphi = Q x \psi, \\
0 & \text{otherwise},
\end{cases}
\]

for \( Q \in \{\exists, \forall\} \).

2.2 Automata over Finite Words

The set of all words over an alphabet \( \Sigma \) is denoted by \( \Sigma^* \), \( \Sigma^+ \) denotes the set of all non-empty words over \( \Sigma^* \), and \( \lambda \) denotes the empty word. The length of the word \( w \in \Sigma^* \) is denoted by \(|w|\).

A deterministic word automaton (DWA) is a tuple \( A = (Q, \Sigma, \delta, q_1, F) \), where \( Q \) is a finite set of states, \( \Sigma \) is a finite alphabet, \( \delta : Q \times \Sigma \to Q \) is the transition function, \( q_1 \in Q \) is the initial state, and \( F \subseteq Q \) is the set of accepting states. The size of \( A \) is the cardinality of \( Q \). The language of \( A \) is \( L(A) := \{w \in \Sigma^* : \delta(q_1, w) \in F\} \), where \( \hat{\delta}(q, \lambda) := q \) and \( \hat{\delta}(q, wb) := \delta(\delta(q, w), b) \), for \( q \in Q \), \( b \in \Sigma \), and \( w \in \Sigma^* \). A state \( q \in Q \) is reachable from \( p \in Q \) if there is a word \( w \in \Sigma^* \) such that \( \delta(p, w) = q \).

Let \( A = (Q, \Sigma, \delta, q_1, F) \) be a DWA, where we assume that every state is reachable from \( q_1 \). Note that the states that are not reachable from \( q_1 \) have no affect on the language of the DWA and can be eliminated. The states \( p, q \in Q \) are equivalent, \( p \sim_A q \) for short, if for all \( w \in \Sigma^* \), we have that \( \hat{\delta}(p, w) \in F \) iff \( \hat{\delta}(q, w) \in F \). We omit the subscript in the relation \( \sim \) if \( A \) is clear from the context. Note that \( \sim \subseteq Q \times Q \) is an equivalence relation. We denote the equivalence class of \( q \in Q \) by \( \overline{q} \). Since we assume that all states are reachable from \( q_1 \), the states \( p, q \in Q \) can be merged iff \( p \sim \overline{q} \). We obtain the DWA \( \overline{A} := (\{q : q \in Q\}, \Sigma, \overline{\delta}, \overline{q_1}, \{q : q \in F\}) \) with \( \delta(\overline{q}, b) := \delta(q, b) \), for \( q \in Q \) and \( b \in \Sigma \). We have that \( L(\overline{A}) = L(A) \) and \( A \) is minimal, i.e., for every DWA \( B \) with \( L(B) = L(A) \), either \( B \) has more states than \( \overline{A} \) or \( \overline{B} \) is isomorphic to \( \overline{A} \).
3. AUTOMATA CONSTRUCTIONS

In this section, we investigate the automata for quantifier-free PA formulas. In §3.1, we define how DWAs recognize sets of integers, in §3.2, we provide optimal automata constructions for linear (in)equality, in §3.3, we give an automata construction for the divisibility relation, and finally, in §3.4, we give an upper bound on the size of the minimal DWA for a quantifier-free formula.

3.1 Representing Sets of Integers with Automata

We use an idea that goes back at least to Büchi [1960] for using automata to recognize tuples of numbers by mapping words to tuples of numbers. There are many possibilities to represent integers as words. We use an encoding similar to [Boigelot 1999; Wolper and Boigelot 2000], which is based on the $g$’s complement representation of integers, where $g \geq 2$ and the most significant bit is the first digit. For the remainder of the paper, we fix $g \geq 2$ and let $\Sigma$ be the alphabet $\{0, \ldots, g - 1\}$.

**Definition 3.1.** For $b_{n-1} \ldots b_0 \in \Sigma^*$, we define $\langle b_{n-1} \ldots b_0 \rangle_N := \sum_{0 \leq i < n} g^i b_i$. We generalize this encoding to integers as follows. For $b_n b_{n-1} \ldots b_0 \in \Sigma^+$, we define

$$\langle b_n b_{n-1} \ldots b_0 \rangle_Z := \langle b_{n-1} \ldots b_0 \rangle_N - \begin{cases} 0 & \text{if } b_n = 0, \\ g^n & \text{if } b_n \neq 0. \end{cases}$$

We call the first letter $b_n$ the **sign letter**, since it determines whether the word represents a positive or a negative number.

Note that the empty word $\lambda$ does not represent an integer. This requirement saves us from considering some special cases in §3.2.2 and §3.2.2 where we optimize the automata constructions for (in)equality. However, for the natural numbers, it holds that $\langle \lambda \rangle_N = 0$. Furthermore, note that the encoding of an integer is not unique. First, we have that $\langle bu \rangle_Z = \langle bcu \rangle_Z$, where $b, c \in \Sigma$ and $u \in \Sigma^*$ with $c = 0$ if $b = 0$ and $c = g - 1$, otherwise. Second, it holds that $\langle bu \rangle_Z = \langle b' u \rangle_Z$, for all $u \in \Sigma^*$ and $b, b' \in \Sigma \setminus \{0\}$, i.e., the sign letter $b \neq 0$ can be replaced by any other letter $b' \neq 0$. The motivation for allowing any letter to be the sign letter is that we do not have to deal with words in $\Sigma^+$ that do not represent an integer. This eliminates case distinctions of the automata constructions in the next subsections.

We extend the encoding to tuples of natural numbers and integers as follows: A word $w := b_{n-1} \ldots b_0 \in (\Sigma')^*$ represents the tuple $\pi := (a_1, \ldots, a_r) \in \mathbb{N}^r$ of integers, where the $i$th “track” of the word $w$ encodes the natural number $a_i$. That is, for all $1 \leq i \leq r$, we have that $a_i = \langle b_{n-1,i} \ldots b_0,i \rangle_Z$, where $\overbracket{b_{j,i}}{j=1}^{n}$ is defined analogously for a word $w = b_{n-1} \ldots b_0 \in (\Sigma')^+$. The first letter $\overbracket{b_n}{n}$ of $w$ is the **sign letter** since it determines the signs of the integers $z_1, \ldots, z_r$. We define $\sigma (\overbracket{b_n}) := (c_1, \ldots, c_r)$, where $c_i = 0$ if the $i$th coordinate of $\overbracket{b_n}$ is $0$ and $c_i = -1$, otherwise, for each $1 \leq i \leq r$. We abuse notation and write $\langle w \rangle_N$ to denote the tuple $\pi \in \mathbb{N}^r$ and $\langle w \rangle_Z$ to denote the integer tuple $\pi$.

Moreover, we write $\langle (\pi) \rangle_N$ for the shortest word in $(\Sigma')^*$ that represents $\pi \in \mathbb{N}^r$. Note that $\langle (\pi) \rangle_N$ is well-defined since (1) there is a word $w \in (\Sigma')^*$ with $\langle w \rangle_Z = \pi$, and (2) if $\langle u \rangle_N = \langle v' \rangle_N$ for $v, v' \in (\Sigma')^*$, then $v$ and $v'$ have a common suffix $u \in (\Sigma')^*$ with $\langle u \rangle_N = \langle v \rangle_N$. Similar to $\langle \pi \rangle_N$ for $\pi \in \mathbb{N}^r$, we define $\langle \pi \rangle_Z$, for...
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Fig. 1. DWA over the alphabet \(\{0,1\}^2\) representing the set \(\{(x, y) \in \mathbb{Z}^2 : y = 2x\}\).

\(\mathfrak{z} \in \mathbb{Z}^r\), as the shortest word \(w \in (\Sigma^r)^+\) with \(\mathfrak{z} = \langle w \rangle\) and the first letter of \(w\) is in \(\{0, q - 1\}^r\).

**Definition 3.2.** Let \(U \subseteq \mathbb{Z}^r\). The language \(L \subseteq (\Sigma^r)^*\) represents \(U\) if \(L = \{w \in (\Sigma^r)^+ : \langle w \rangle \in U\}\). A DWA \(A\) represents \(U\) if \(L(A)\) represents \(U\).

Note that by this definition not every language over \(\Sigma^r\) represents a set of tuples of integers, and not every DWA with alphabet \(\Sigma^r\) represents a subset of \(\mathbb{Z}^r\).

**Example 3.3.** The set of pairs \((x, y) \in \mathbb{Z}^2\) where \(y\) equals \(2x\) is represented by the DWA depicted in Figure 1 by using the base \(\bar{\sigma} = 2\) for representing integers as words, i.e., the alphabet of the DWA is \(\{0, 1\}^2\). In the figure, we use abbreviations like \((0, –)\) to denote the letters \((0, 0)\) and \((0, 1)\).

3.2 Linear Equations and Inequations

In this subsection, we first recall the automata constructions given in [Boigelot et al. 1998; Boigelot 1999; Wolper and Boigelot 2000; Ganesh et al. 2002] for linear (in)equations. Then, we improve these constructions such that they are optimal, i.e., the constructed DWAs are minimal. Assume that the (in)equation \(t \succ c\) is given in normalized form, i.e., \(t(x_1, \ldots, x_r)\) is a homogeneous term, \(\succ \in \{=, \neq, <, \leq, >, \geq\}\), and \(c \in \mathbb{Z}\).

First, we make the following observation for a word \(u \in (\Sigma^r)^+\) and \(\overline{b} \in \Sigma^r\). If \(u \neq \lambda\) then \(\langle \overline{u} \rangle = \sigma(u) + \overline{b}\). For \(u = \lambda\), we have that \(\langle \overline{b} \rangle = \sigma(\overline{b})\). Given this, it is relatively straightforward to obtain an analog of a DWA with infinitely many states for \(t \succ c\). The set of states is \(\{q_l\} \cup \mathbb{Z}\), where \(q_l\) is the initial state. Note that we identify integers with states. The idea is to keep track of the value of \(t\) as successive bits are read. Thus, except for the special initial state, a state in \(\mathbb{Z}\) represents the current value of \(t\). Lemma 3.4 below justifies this intuition. The transition function \(\eta : (\{q_l\} \cup \mathbb{Z}) \times \Sigma^r \rightarrow (\{q_l\} \cup \mathbb{Z})\) is defined as follows for a letter \(\overline{b} \in \Sigma^r\). For the initial state, we define \(\eta(q, \overline{b}) := \sigma(q) + t[\sigma(\overline{b})]\). For \(q \in \mathbb{Z}\), we define \(\eta(q, \overline{b}) := q + t[\overline{b}]\).

**Lemma 3.4.** For \(u \in (\Sigma^r)^+\) of length \(n \geq 0\) we have that

(a) \(\hat{\eta}(q, u) = \sigma^u q + t[{\langle \overline{u} \rangle}]\), for \(q \in \mathbb{Z}\), and
(b) \(\hat{\eta}(q, \overline{bu}) = t[{\langle \overline{bu} \rangle}]\), for \(\overline{b} \in \Sigma^r\).

**Proof.** (a) is easily proved by induction over \(n\), and (b) follows from (a) and the definition of \(\eta\).
Later we make use of the following lemma, which translates the question whether \( q \in \mathbb{Z} \) is reachable from \( p \in \mathbb{Z} \) via \( \eta \) to a number-theoretic problem.

**Lemma 3.5.** Let \( p, q \in \mathbb{Z} \). There are \( N, a_1, \ldots, a_r \geq 0 \) such that \( N \geq \lceil \log_q (1 + \max\{a_1, \ldots, a_r\}) \rceil \) and \( q^N p + t[a_1, \ldots, a_r] = q \) iff there is a word \( w \in \langle \Sigma^* \rangle^* \) such that \( \eta(p, w) = q \).

**Proof.** (\( \Rightarrow \)) Assume that \( \langle a_1, \ldots, a_r \rangle \in \mathbb{N} \) has length \( \ell \). Note that \( \ell \leq N \). This follows from the fact that for every \( a \in \mathbb{N} \), there is a word \( u \in \Sigma^* \) of length \( \lceil \log_q (1 + a) \rceil \) such that \( \langle u \rangle \in a \). By Lemma 3.4(a), we have that

\[
\eta(p, 0^N - \ell \langle a_1, \ldots, a_r \rangle) = q^N p + t[a_1, \ldots, a_r] = q.
\]

(\( \Leftarrow \)) Assume that \( \eta(p, w) = q \), for some \( w \in \langle \Sigma^* \rangle^* \). Let \( N \) be the length of \( w \). We have that \( N \geq \lceil \log_q (1 + a) \rceil \), where \( a \) is the largest number in the tuple \( \langle w \rangle \). It follows from Lemma 3.4(a) that \( \eta(p, w) = q^N p + t[\langle w \rangle] \).

The automata constructions in [Wolper and Boigelot 2000; Ganesh et al. 2002] are based on the observation that the states \( q, q' \in \mathbb{Z} \) can be merged if, intuitively speaking, \( q \) and \( q' \) are both small or both large. Here, the meaning of “small” and “large” depends on the coefficients of \( t \) and on the constant \( c \). More precisely, we say that \( q \in \mathbb{Z} \) is small if \( q < \min(c, -\|t\|_+ \rangle \), and large if \( q > \max(c, \|t\|_-) \), where

\[
\|t\|_- := \sum_{1 \leq j \leq r \text{ and } k_j < 0} |k_j| \quad \text{and} \quad \|t\|_+ := \sum_{1 \leq j \leq r \text{ and } k_j > 0} k_j
\]

assuming that \( t \) is of the form \( k_1 x_1 + \cdots + k_r x_r \). Note that from a small value we can only obtain smaller values and from a large value we can only obtain larger values by \( \eta \), i.e., for all \( \overline{b} \in \Sigma^r \), if \( q > \|t\|_- \rangle \text{ then } \eta(q, \overline{b}) = q \cdot t[\overline{b}] > q \), and if \( q < -\|t\|_+ \rangle \text{ then } \eta(q, \overline{b}) = q \cdot t[\overline{b}] < q \). A difference between the constructions in [Wolper and Boigelot 2000] and [Ganesh et al. 2002] are the bounds that determine the meaning of “small” and “large”.

For \( m < n \), we define \( A_{(m, n)}^{t \leq c} := (Q, \Sigma^r, \delta, q_0, F) \), where \( Q := \{q_1 \} \cup \{q \in \mathbb{Z} : m \leq q \leq n \} \) and

\[
\delta(q, \overline{b}) := \begin{cases} m & \text{if } \eta(q, \overline{b}) \leq m, \\ n & \text{if } \eta(q, \overline{b}) \geq n, \\ \eta(q, \overline{b}) & \text{otherwise,} \end{cases}
\]

for \( q \in Q \) and \( \overline{b} \in \Sigma^r \). Moreover, let \( F := \{q \in Q \cap \mathbb{Z} : q \geq c \} \).

**Lemma 3.6.** The DWA \( A_{(m, n)}^{t \leq c} \) represents \( \|t \leq c \rangle \) if \( m \) is small and \( n \) is large. Moreover, \( A_{(m, n)}^{t \leq c} \) has \( 2 + n - m \) states.

**Proof.** The fact that \( A_{(m, n)}^{t \leq c} \) represents \( \|t \leq c \rangle \) follows from Lemma 3.4, and \( A_{(m, n)}^{t \leq c} \) has \( 2 + n - m \) states by definition.

In the following, we optimize the constructions such that the produced DWA for an (in)equality is minimal. Moreover, we give a lower bound on the minimal DWA for an (in)equality. However, these results are not needed for the upper
bound on the minimal DWA for a PA formula. In the remainder of this subsection, let $A^{t \geq c}_{(m,n)} = (Q, \Sigma^r, \delta, q_1, F)$ for the (in)equation $t \geq c$ with $m = \max\{q \in \mathbb{Z} : q \text{ is small}\}$ and $n = \min\{q \in \mathbb{Z} : q \text{ is large}\}$. We restrict ourselves to the cases where $\exists \in \{=, <, >\}$. The cases with $\exists \in \{\neq, \leq, \geq\}$ reduce to the cases for $\geq, <, >$ and complementation of DWAs, since $t \neq c$ is logically equivalent to $\neg t = c$, $t \leq c$ is logically equivalent to $\neg t > c$, and $t \geq c$ is logically equivalent to $\neg t < c$. Note that complementation of a DWA can be done by flipping accepting and non-accepting states. After complementation we have to make the initial state of the DWA non-accepting since the empty word does not represent any integer tuple. The resulting DWA is minimal iff the original DWA is minimal.

3.2.1 Eliminating Unreachable States. An obvious optimization is to eliminate the states in $Q \cap \mathbb{Z}$ that are not a multiple of the greatest common divisor of the absolute values of the coefficients in the term $t$, since they are not reachable from the initial state $q_i$. We define the greatest common divisor of the term $t(x_1, \ldots, x_r)$ as $\gcd(t) := \gcd(|k_1|, \ldots, |k_r|)$, where $k_i$ is the coefficient of the variable $x_i$, for $1 \leq i \leq r$.

**Lemma 3.7.** The state $q \in Q \cap \mathbb{Z}$ is reachable from the initial state $q_1$ iff $q$ is a multiple of $\gcd(t)$.

**Proof.** ($\Rightarrow$) This direction is easy to prove by induction on the length of $w \in (\Sigma^r)^*$ with $\delta(q, w) \in \mathbb{Z}$: for all $\delta_1 \in \Sigma$, it holds that (i) $\delta(q, \delta_1) = t[\sigma(\delta_1)]$ is a multiple of $\gcd(t)$, and (ii) if $\delta(q, w) \in \mathbb{Z}$ is a multiple of $\gcd(t)$ then $\delta(q, w) + t[\delta_1]$ is a multiple of $\gcd(t)$.

($\Leftarrow$) Assume that $q$ is a multiple of $\gcd(t)$. There are $v_1, \ldots, v_r \in \mathbb{Z}$ such that $t[v_1, \ldots, v_r] = q$. With Lemma 3.4(b) we conclude that $\delta(q, \langle v_1, \ldots, v_r \rangle_2) = t[v_1, \ldots, v_r]$. \qed

Alternatively, instead of filtering out the states $q \in \mathbb{Z}$ that are not a multiple of $\gcd(t)$ we can rewrite the (in)equation $t \geq c$ to the logically equivalent atomic formula $\alpha$ and then construct the DWA for $\alpha$, where $\alpha$ is defined as

$$
\alpha := \begin{cases} 
\alpha' \geq \left\lceil \frac{t}{\gcd(t)} \right\rceil & \text{if } \geq \text{ is }, \\
\alpha' \geq \left\lceil \frac{t}{\gcd(t)} \right\rceil & \text{if } \geq \text{ is }, \\
\alpha' \geq \left\lceil \frac{t}{\gcd(t)} \right\rceil & \text{if } \geq \text{ is } \text{ and } c \text{ is a multiple of } \gcd(t), \\
1 < 0 & \text{otherwise},
\end{cases}
$$

where the coefficients in $\alpha'$ are the coefficients of $t$ divided by $\gcd(t)$. In the remainder of this subsection we assume that $\gcd(t) = 1$.

3.2.2 Optimal Construction for Inequations. In the following, we assume that the inequation is of the form $t > c$, with $c \geq 0$. The cases where $\geq$ is $<$ or $c \geq 0$ are analogous. The following example illustrates that many states of $A^{t \geq c}_{(m,n)}$ can be merged if $c$ is significantly larger than $\|t\|_\infty$.

**Example 3.8.** The automata construction described above for the inequation $x - y > 32$ produces a DWA with the set of states $Q = \{q_1, -2, -1, 0, \ldots, 32, 33\}$; but...
Assume that equivalent states, we define the following strictly monotonically decreasing sequence \( \gcd(t) = 1 \). We use the notation \([d,d']\) for the set \(\{d,d' - 1\}\) if \(d,d' \in \mathbb{Z}\), and if \(d \in \mathbb{Z}\) and \(d' = \infty\) then \([d,d'] := \{z \in \mathbb{Z} : z \geq d\}\). In order to identify the equivalent states, we define the following strictly monotonically decreasing sequence \(d_0 > d_1 > \cdots > d_\ell\), for some \(\ell \geq 1\). Let \(d_0 := \infty\) and \(d_1 := \max\{c + 1, ||t||\}\). Assume that \(d_0 > d_1 > \cdots > d_i\) are already defined, for some \(i \geq 1\).

- If \(d_i = ||t||\) then we are done, i.e., \(\ell = i\).
- If \(d_i > ||t||\) then let \(d_{i+1} < d_i\) be the smallest integer greater than \(||t||\) such that for all \(b \in \Sigma^r\), there is an index \(j\) with \(1 \leq j \leq i\) and

\[
g_{d_{i+1} + t[b]} \neq g(d_i - 1) + t[b] \in [d_j, d_{j-1}) .
\]  

(1)

Note that \(d_{i+1}\) is well-defined since \(d_i - 1\) satisfies (1), for all \(b \in \Sigma^r\).

The following lemma characterizes the equivalent states. In particular, it shows that we can merge the states in \(R := \{-||t||, ||t|| + 1\}\), and for each \(i \leq i \leq \ell\), the states in \([d_i, d_{i-1})\) can be collapsed to one state.

**Lemma 3.9.** For all \(p,q \in Q\), it holds that \(p \sim q\) iff \(p = q\) or \(p,q \in R\) or \(p,q \in [d_i, d_{i-1})\), for \(1 \leq i \leq \ell\).

**Proof.** (\(\Rightarrow\)) If \(p = q\) then it is obvious that \(p \sim q\). If \(p,q \in R\) then we also have that \(p \sim q\), since both states are non-accepting and all transitions from these states either go to \(-||t||\) or to \(-||t|| + 1\). It remains to prove that for \(1 \leq i \leq \ell\), if \(p,q \in [d_i, d_{i-1})\) then \(p \sim q\). We prove this claim by induction over \(i\). For \(i = 1\), there...
is nothing to prove, since \([d_1, d_0] \cap Q\) is a singleton. For the induction step, assume that \(i > 1\) and let \(p, q \in [d_i, d_{i-1}]\). Without loss of generality we assume that \(p \leq q\). By the definition of the transition function \(\delta\) and the sequence \(d_0 > d_1 > \cdots > d_\ell\), we have that 

\[
qd_i + t[\overline{b}] \leq \delta(p, \overline{b}) \leq \delta(q, \overline{b}) \leq q(d_{i-1} - 1) + t[\overline{b}],
\]

for all \(\overline{b} \in \Sigma^r\). Since there is a \(1 \leq j < i\) with \(qd_i + t[\overline{b}], qd_{i-1} + t[\overline{b}] \in [d_j, d_{j-1})\) we conclude that \(\delta(p, \overline{b}), \delta(q, \overline{b}) \in [d_j, d_{j-1})\). The claim now follows from the induction hypothesis.

\((\Rightarrow)\) We prove the claim by contraposition, i.e., \(p \neq q\) is implied by the three conditions (i) \(p \neq q\), (ii) \(p \in R \Rightarrow q \notin R\), and (iii) for all \(1 \leq i \leq \ell\), \(p \in [d_i, d_{i-1}) \Rightarrow q \notin [d_i, d_{i-1})\). Assume \(p \neq q\). It suffices to distinguish the following three cases.

Case 1: \(p \in R\) and \(q \notin R\). Since we can reach an accepting state from \(q\), we have that \(p \neq q\).

Case 2: \(p \in [d_i, d_{i-1})\) and \(q \notin [d_i, d_{i-1})\), for some \(1 \leq i \leq \ell\). It is straightforward to prove by induction over \(i\) that \(p \neq q\).

Case 3: \(p \notin R \cup \bigcup_{1 \leq i \leq \ell} [d_i, d_{i-1})\). Note that the conditions (ii) and (iii) are satisfied. We have that either \(p = q_1\) or \(p \in S\), where \(S := \{s \in Q \cap \mathbb{Z} : -\|t\|_+ < s < \|t\|_+\}\).

If \(p = q_1\) and \(q \in R\) then we conclude similar to Case 1 that \(p \neq q\). Assume that \(p = q_1\) and \(q \notin R\). Let \(\overline{b} \in \Sigma^r\) be the letter that has a 0 in its \(i\)th coordinate iff the \(i\)th coefficient of \(t\) is negative, and otherwise the \(i\)th coordinate is \(q - 1\). It holds that \(q_1 \neq q\), since \(\delta(q_1, \overline{b}) = -t[\overline{b}] \in R\) and \(\delta(q, \overline{b}) = qq + q\|t\|_+ \geq q\). From Case 1, it follows that \(p \neq q\).

Assume that \(p \in S\). Note that for every \(s \in S\) there is a \(\overline{b} \in \Sigma^r\) such that \(\delta(s, \overline{b}) \in S\). It follows that for every \(n \geq 0\) there is a word \(u \in (\Sigma^r)^\ast\) of length \(n\) such that \(\delta(p, u) \in S\). We conclude that there is a word \(u \in (\Sigma^r)^\ast\) such that \(\delta(p, u) \in S\) and \(\delta(q, u) \in R \cup \bigcup_{1 \leq i \leq \ell} [d_i, d_{i-1})\), since \(\delta(s, \overline{b}) - \delta(s', \overline{b}) = q(s - s')\), for all \(s, s' \in S\) and all \(\overline{b} \in \Sigma^r\). Analogously to the Cases 1 and 2 we conclude that \(p \neq q\).

From Lemma 3.9, it follows that the minimal DWA representing \([t > c]\) has at least \(\|t\|_+ + \|t\|_\ast\) states. Note that this is in contrast to the number of symbols we need to write the inequation \(t > c\) if coefficients are represented as binary numbers. For instance, we need \(22 + 7\) letters for \(1025 \cdot x - 1024 \cdot y > 0\), since each of the two coefficients can be represented with 11 digits. The same lower bound on the minimal DWA size holds for \(t < c\). In the following, we show that a similar lower bound holds for equations.

3.2.3 Optimal Construction for Equations. For an equation \(t = c\), we can collapse the states in \(A_{\mathbb{Q}}^t\) from which we cannot reach the accepting state \(c \in Q\) to a single non-accepting state. These optimizations produce the minimal DWA for \(t = c\). For instance, the case for \(p \in Q \cap \mathbb{Z}\) is proved as follows. Assume that we can reach the state \(c\) from \(p \in Q \cap \mathbb{Z}\), i.e., there is a \(u \in (\Sigma^r)^\ast\), with \(\tilde{\delta}(p, u) = c\). Any other states \(q \in Q \cap \mathbb{Z}\) with \(q \neq p\) from which we can reach \(c\) cannot be merged with \(p\),
Step 1: proceed in two steps: where \( t \) order to show the existence of a solution we assume that \( \gcd(t) = 1 \). In particular, there are positive and negative coefficients in \( t \) since \( \gamma \geq 1 \) by Dixmier [1990]. It is straightforward to show that for all \( \gamma \geq 1 \), \( G(c_1, \ldots, c_N) < c_1 + \cdots + c_N \). In the following, we will prove the lemma, i.e., for \( p \in S \) and \( q \in Q \cap \mathbb{Z} \) there is a word \( u \in (\Sigma^c)^* \) such that \( \delta(p, u) = q \). Note that if \( r = 0 \) and \( r = 1 \) then \( S = \emptyset \) and the claim is trivially true. Assume that \( r \geq 2 \). By Lemma 3.5, it suffices to show that the equation

\[
g^N p + t(x_1, \ldots, x_r) = q
\]

has a solution \( a_1, \ldots, a_r \geq 0 \) with \( N \geq \lceil \log_q (1 + \max\{a_1, \ldots, a_r\}) \rceil \). We distinguish four cases depending on \( p \) and \( q \).

**Case 1:** \( p = 0 \). Equation (3) simplifies to

\[
t(x_1, \ldots, x_r) = q.
\]

There are positive and negative coefficients in \( t \), since \( p \in S \). It follows that equation (4) has infinitely many solutions in the natural numbers. Recall that we assume that \( \gcd(t) = 1 \). In particular, there are \( a_1, \ldots, a_r \geq 0 \) with \( g^N p + t[a_1, \ldots, a_r] = q \), for some appropriate large enough \( N \).

**Case 2:** \( p > 0 \) and \( q \geq 0 \). Let \( k_{i_1}, \ldots, k_{i_\mu} \) be the positive coefficients in \( t \), and let \( k_{j_1}, \ldots, k_{j_\nu} \) be the negative coefficients in \( t \). Let \( N \) be the size of the DWA \( A_{(m,n)}^{c<} \), i.e., \( N = 3 + \max\{|c|, \|t\|_+\} + \max\{c, \|t\|_-\} \). We rewrite equation (3) to

\[
g^N p - q + t_1(x_{i_1}, \ldots, x_{i_\mu}) = t_2(x_{j_1}, \ldots, x_{j_\nu}),
\]

where \( t_1 \) is the term \( k_{i_1} x_{i_1} + \cdots + k_{i_\mu} x_{i_\mu} \), and \( t_2 \) is the term \( |k_{j_1}| x_{j_1} + \cdots + |k_{j_\nu}| x_{j_\nu} \). Note that \( g^N p - q \geq 0 \) since \( p > 0 \) and \( g^N \geq q \). Let \( D := \gcd(|k_{j_1}|, \ldots, |k_{j_\nu}|) \). In order to show the existence of a solution \( a_{i_1}, \ldots, a_{i_\mu} \in [g^N] \) of equation (5), we proceed in two steps:

**Step 1:** There are \( a_{i_1}, \ldots, a_{i_\mu} \in [D] \) such that

\[
D \mid q^N p - q + t_1[a_{i_1}, \ldots, a_{i_\mu}].
\]
Step 2: There are \(a_{j_1}, \ldots, a_{j_\nu} \in \mathbb{N}^\nu\) such that
\[
g^N p - q + t_1[a_{i_1}, \ldots, a_{i_\mu}] = t_2[a_{j_1}, \ldots, a_{j_\nu}] .
\]

Proof of Step 1: If \(\mu = 0\) then there is nothing to prove. Assume that \(\mu > 0\).
There are \(K, R \geq 0\) such that \(g^N p - q = DK + R\) with \(R < D\).
It suffices to show that there are \(a_{i_1}, \ldots, a_{i_\mu}\) with \(0 \leq a_{i_1}, \ldots, a_{i_\mu} < D\), and \(K' \geq 0\), such that
\[
DK' = R + t_1[a_{i_1}, \ldots, a_{i_\mu}],
\]
and thus, \(D[g^N p - q + t_1[a_{i_1}, \ldots, a_{i_\mu}]\).

First, assume the existence of \(a_{i_1}, \ldots, a_{i_\mu} \geq 0\) with \(D|R + t_1[a_{i_1}, \ldots, a_{i_\mu}]\), where \(a_{i_\xi} \geq D\), for some \(1 \leq \xi \leq \mu\). To simplify matters, we assume without loss of generality that \(\xi = 1\). There is an \(a \geq 0\) with \(a_{i_1} = D + a\). Further, assume that there is no \(b < a_{i_1}\) with \(D|R + t_1[b, a_{i_2}, \ldots, a_{i_\mu}]\).

It remains to show the existence of \(a_{i_1}, \ldots, a_{i_\mu} \geq 0\) with \(D|R + t_1[a_{i_1}, \ldots, a_{i_\mu}]\).
The existence reduces to the problem of whether the equation
\[
D \cdot y - k_{i_1} \cdot x_{i_1} - \cdots - k_{i_\nu} \cdot x_{i_\nu} = R
\]
has a solution in the natural numbers. This is the case since \(\gcd(D, k_{i_1}, \ldots, k_{i_\nu}) = 1\), by assumption.

Proof of Step 2: Assume that there are \(\gamma \geq 1\) distinct coefficients in \(t_2\) of equation (5). Without loss of generality, assume that \(0 < |k_{j_1}| < \cdots < |k_{j_\gamma}|\). Let
\[
W := \frac{g^N p - q + t_1[a_{i_1}, \ldots, a_{i_\mu}]}{D}
\]
and \(\xi := \frac{|k_{j_1}|}{D}\), for \(1 \leq \xi \leq \nu\). Note that \(\xi_1 < \cdots < \xi_\gamma\) and that \(\gcd(\xi_1, \ldots, \xi_\gamma) = 1\). Equation (5) simplifies with the \(a_i\)s from Step 1 to
\[
W = \xi_1 \cdot x_{j_1} + \cdots + \xi_\gamma \cdot x_{j_\gamma} .
\]

An upper bound on \(W\) is
\[
W \leq \frac{g^N p - q + \max\{\xi_1, |\xi_\gamma|\} \cdot D}{D} \leq \frac{g^N (|\xi_\gamma| - 1) + (p-1) \cdot |\xi_\gamma| + |\xi_\gamma|}{D}
\]
and a lower bound on \(W\) is
\[
W \geq \frac{g^N - q}{D} \geq \frac{g^N - \max\{\xi_1, |\xi_\gamma|\} \cdot D}{D} \geq \frac{D(\ell_1 + \cdots + \ell_\nu) - D(\ell_1 + \cdots + \ell_\gamma)}{D} \geq \frac{g^N - q}{D} \geq \frac{g^N - \max\{\xi_1, |\xi_\gamma|\} \cdot D}{D} \geq \frac{D(\ell_1 + \cdots + \ell_\gamma)}{D} \geq \frac{g^N - q}{D} \geq \frac{g^N - \max\{\xi_1, |\xi_\gamma|\} \cdot D}{D} \geq \frac{D(\ell_1 + \cdots + \ell_\gamma)}{D} .
\]

From the lower bound on \(W\) and the upper bound on Frobenius numbers (2), it follows that equation (6) has a solution in the natural numbers. Let \(\kappa \geq 0\) be maximal such that there are \(a_{i_1}, \ldots, a_{i_\gamma} \geq 0\) with
\[
W = \ell_1 a_{i_1} + \cdots + \ell_\gamma a_{i_\gamma} + \kappa L .
\]
where $L := \frac{W}{D}$. By contradiction, we obtain that $a_1, \ldots, a_\gamma < L$. Assume that there is a $\xi$, $1 \leq \xi \leq \gamma$ with $a_\xi = L + a$, for some $a \geq 0$. Without loss of generality, assume that $\xi = 1$. This contradicts the assumption that $\kappa$ is maximal:

$$W = \kappa L + \ell_1(L + a) + \ell_2a_2 + \cdots + \ell_\gamma a_\gamma = (\kappa + \ell_1)L + \ell_1 a + \ell_2a_2 + \cdots + \ell_\gamma a_\gamma.$$  

From $\kappa$ and $a_1, \ldots, a_\gamma$, we obtain a solution for equation (6) in the natural numbers, namely

$$W = \kappa L + \ell_1 a_1 + \cdots + \ell_\gamma a_\gamma = \kappa(\ell_1 + \cdots + \ell_\nu) + \ell_1 a_1 + \cdots + \ell_\gamma a_\gamma = \ell_1(\kappa + a_1) + \cdots + \ell_\gamma(\kappa + a_\gamma) + \ell_{\gamma+1}\kappa + \cdots + \ell_{\nu}\kappa.$$  

It suffices to show that $\kappa < \theta^N - \max\{a_1, \ldots, a_\gamma\}$. An upper bound on $\kappa$ is

$$\kappa \leq \frac{W}{L} - \frac{\max\{a_1, \ldots, a_\gamma\}}{L} \leq \frac{\theta^N}{DL} - \frac{\theta^N}{DL} + \frac{(\theta^N-1)\|t\|_+ - \max\{a_1, \ldots, a_\gamma\}}{L} \leq \theta^N - \frac{\theta^N}{DL} + \frac{(\theta^N-1)\|t\|_+ - \max\{a_1, \ldots, a_\gamma\}}{L}.$$  

It remains to check whether the inequality

$$\theta^N - \frac{\theta^N}{DL} + \frac{(\theta^N-1)\|t\|_+ - \max\{a_1, \ldots, a_\gamma\}}{L} < \theta^N - \max\{a_1, \ldots, a_\gamma\}$$  

is valid. The previous inequality simplifies to

$$\frac{(\theta^N-1)\|t\|_+ - \max\{a_1, \ldots, a_\gamma\}}{L} < \frac{\theta^N}{DL}.$$  

Multiplying with the common denominator $DL$, the inequality simplifies further to

$$D(\theta^N - 1)\|t\|_+ + D\max\{a_1, \ldots, a_\gamma\}(L-1) < \theta^N.$$  

Since $\max\{a_1, \ldots, a_\gamma\} \leq L - 1$ and $N \geq \|t\|_- + \|t\|_+ = DL + \|t\|_+$, it suffices to show the validity of the inequality

$$D(\theta^N - 1)\|t\|_+ + D(L-1)^2 < \theta^{DL+\|t\|_+}. \quad (9)$$  

It is straightforward to show that the inequality (9) is true for all $D, L \geq 1$ and $\|t\|_+, \geq 0$.

**Case 3:** $p < 0$ and $q \leq 0$. It suffices to prove that there is a solution $a_1, \ldots, a_r \in [\theta^N]$ for the equation

$$t_1(x_{i_1}, \ldots, x_{i_r}) = \theta^N|p| - |q| + t_2(x_{j_1}, \ldots, x_{j_r}),$$  

where $t_1$ and $t_2$ are defined as in Case 2. This equation is similar to equation (5) except $t_1$ and $t_2$ are swapped. We can use a similar argumentation as in Case 2 for showing the existence of $a_1, \ldots, a_r \in [\theta^N].$

**Case 4:** $p > 0$ and $q < 0$. This case can be solved with Case 1 and Case 2. Since $p > 0$ and $q < 0$, we have that $0 \in S$. By Case 2, the state $0$ is reachable from $p$, and by Case 1, $q$ is reachable from state $0$. 

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Case 5: $p < 0$ and $q > 0$. Analogously, this case can be solved by Case 3 and Case 1.

With Lemma 3.10 at hand, it is straightforward to prove for $A_{t(m,n)}^{\le n}$ that $p \sim q$ iff $p = q$, for all $p, q \in S$. Therefore, we have that the minimal automaton representing $[t = c]$ has at least $|S|$ states.

Another consequence of Lemma 3.10 is that $S$ is a strongly connected component in $A_{t(m,n)}^{\le n}$. By Lemma 3.10, every state $q \in S$ is reachable from every $p \in S$, and it is easy to show that the initial state $q_0$ is not reachable from a state in $S$ and that a state in $S$ cannot be reached from any state that is not in $S \cup \{q_0\}$.

3.3 Divisibility Relation

In this subsection, we give an upper bound of the size of the minimal DWA for a formula $d|t + c$, where $d \ge 2$, $t(x_1, \ldots, x_r)$ is a homogeneous term, and $c \in \mathbb{Z}$.

Let $A_{d|t+c}$ be the DWA with the set of states $Q := \{q_1, 0, 1, \ldots, d - 1\}$. A state $q \in Q \cap \mathbb{Z}$ has an intuitive interpretation: if we reach the state $q$ with a word $w \in (\Sigma^r)^*$ then the remainder of the division of $t([w]_2)$ by $d$ equals $q$. We denote by $\text{rem}(q, d)$ the remainder of $q \in \mathbb{Z}$ divided by $d$. Let $A_{d|t+c} := (Q, \Sigma^r, \delta, q_0, F)$ with

$$
\delta(q, \overline{b}) := \begin{cases} 
\text{rem}(t[\sigma(\overline{b})], d) & \text{if } q = q_0, \\
\text{rem}(\overline{b}q + t[\overline{b}], d) & \text{otherwise}, 
\end{cases}
$$

for $q \in Q$ and $\overline{b} \in \Sigma^r$, and $F := \{q \in Q \cap \mathbb{Z} : d|q + c\}$. Note that there is exactly one $q \in Q \cap \mathbb{Z}$ with $d|q + c$.

The correctness of our construction follows from two facts:

(a) For $n \in \mathbb{Z}$, $d|n + c$ iff $d|\text{rem}(n, d) + c$.
(b) For $w \in (\Sigma^r)^+$, $\hat{\delta}(q_0, w) = \text{rem}(t([w]_2), d)$.

The proof of (a) is straightforward. There are $p, q \in \mathbb{Z}$ such that $pd + q = n$ and $0 \le q < d$. Note that $q = \text{rem}(n, d)$. By definition, $d|n + c$ iff there is a $k \in \mathbb{Z}$ with $dk = n + c = pd + q + c$. The equality can be rewritten to $d(k - p) = q + c$, i.e., $d|\text{rem}(n, d) + c$.

We prove (b) by induction over the length of $w$. For the base case, let $w = \overline{b} \in \Sigma^r$. Since we represent integers using $\overline{b}$'s complement, we have that $t([\overline{b}]_2) = t[\sigma(\overline{b})]$. By definition, $\hat{\delta}(q_0, \overline{b}) = \text{rem}(t([\overline{b}]_2), d)$. For the step case, assume $\hat{\delta}(q_0, w) = \text{rem}(t([w]_2), d)$ and let $\overline{b} \in \Sigma^r$. There are $p, q \in \mathbb{Z}$ with $t([w]_2) = pd + q$ and $0 \le q < d$. Note that $q = \text{rem}(t([w]_2), d)$ and $t([w\overline{b}]_2) = g\text{rem}(t([w]_2), d) = pd + q + t[\overline{b}]$. We have that

$$
\text{rem}(t([w\overline{b}]_2), d) = \text{rem}(gpd + qg + t[\overline{b}], d) = \text{rem}(\overline{b}q + t[\overline{b}], d) = \delta(q, \overline{b}).
$$

Lemma 3.11. The DWA $A_{d|t+c}$ represents $[d|t + c]$ and has $d + 1$ states.

An optimization of the construction is to filter out the states that are not a multiple of $\gcd(\gcd(t), d)$. These states are not reachable from the initial state since $\text{rem}(t[\overline{b}], d)$ is a multiple of $\gcd(\gcd(t), d)$, for every $\overline{b} \in \Sigma^r$. 

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3.4 Quantifier-free Formulas

In this subsection, we give an upper bound on the size of the minimal DWA for a quantifier-free PA formula. This upper bound depends on the maximal absolute value of the constants occurring in the (in)equations of the formula, the homogeneous terms, and the divisibility relations. The upper bound does not depend on the Boolean combination of the atomic formulas. This is not obvious since Boolean connectives are handled by the product construction if we construct the DWA recursively over the structure of the quantifier-free formula. The size of the resultant DWA using the product construction is in the worst case the product of the number of states of the two DWAs.

Let $T$ be a finite nonempty set of homogeneous terms and let $D$ be a finite set of atomic formulas of the form $d|t$, where $d \geq 1$ and $t$ is a homogeneous term. Moreover, let $\ell > \max\{||t||_+ : t \in T\} \cup \{||t||_- : t \in T\}$ and $\ell' > \max\{d : d|t \in D\}$.

**Theorem 3.12.** Let $\psi$ be a Boolean combination of atomic formulas $t \cong c$ and $d|t + c'$, with $t \in T$, $d|t \in D$, $-\ell < c < \ell$, $c' \in \mathbb{Z}$, and $\cong \in \{\neq, <, \leq, >, \geq\}$. The size of the minimal DWA for $\psi$ is at most $(2 + 2\ell)|T| \cdot \ell'|D|.$

**Proof.** Without loss of generality, we assume that the variables occurring in terms in $T$ are $y_1, \ldots, y_r$. Let $C$ be the product automaton of all the $A_{(-\ell,\ell)}$s and $A_d|t$s, for $t \in T$ and $d|t \in D$. To simplify notation we omit the subscripts $(-\ell,\ell)$ and we assume that $T = \{t_1, \ldots, t_m\}$ and $D = \{d_1|t_1, \ldots, d_n|t_n\}$. Note that the states of $C$ are tuples $(p_1, \ldots, p_m, q_1, \ldots, q_n)$, where $p_i$ is a state of $A_{t_i=0}$ and $q_j$ is a state of $A_{d_j|t_j}$. By Lemma 3.6, $A_{t_i=0}$ has $2 + 2\ell$ states, and by Lemma 3.11, $A_{d_j|t_j}$ has $1 + d_j \leq \ell'$ states. It follows that the size of $C$ is at most

$$\prod_{t \in T} (2 + 2\ell) \cdot \prod_{d|t \in D} (1 + d) \leq (2 + 2\ell)|T| \cdot \ell'|D|.$$ 

It remains to define the set of accepting states of $C$ according to $\psi$. We define the DWA $D$ as $C$ except the set $E$ of accepting states is defined as follows. A state $q = (p_1, \ldots, p_m, q_1, \ldots, q_n) \in \mathbb{Z}^{m+n}$ of $D$ is in $E$ if $3 \models \psi_q$, where $\psi_q$ is the formula obtained by substituting

---

-the integer $p_i$ for the term $t_i$ in the atomic formulas of the form $t_i \cong c$, and

-the integer $q_j$ for the term $t_j$ in the atomic formulas of the form $d_j|t_j + c$.

Note that $\psi_q$ is either true or not in $3$ since it is a sentence.

It remains to prove that $D$ represents $[\psi]$. Let $w \in (\Sigma^*)^+$ be a word representing $\overline{a} \in \mathbb{Z}^r$. For a term $t \in T$, the value $t[\overline{a}]$ can be replaced by $\ell$ if $t[\overline{a}] \geq \ell$ and by $-\ell$ if $t[\overline{a}] \leq -\ell$ in every atomic formula of the form $t \cong c$ without changing its truth value since $-\ell < c < \ell$. This modified value corresponds to the state reached by $A_{t=0}$ after reading the word $w$. For an atomic formula of the form $d|t + c$, with $d|t \in D$, we can replace $t[\overline{a}] + c$ by $\text{rem}(t[\overline{a}] + c, d)$ without changing the truth value. This adjusted value corresponds to the state reached by $A_{d|t}$ after reading the word $w$. From the definition of $E$, it follows that $w \in L(D)$ iff $3 \models \psi[\overline{a}]$. 

\[\square\]
4. AN UPPER BOUND ON THE AUTOMATA SIZE

In this section, we give an upper bound on the size of the minimal DWA for PA formulas. We obtain this bound by examining the quantifier-free formulas constructed by applying Reddy and Loveland’s quantifier elimination method [Reddy and Loveland 1978], which improves Cooper’s quantifier elimination method [Cooper 1972]. We use Reddy and Loveland’s quantifier elimination method since the produced formulas are “small” with respect to the following parameters on which the upper bound of the minimal DWA in Theorem 3.12 depends.

Definition 4.1. For \( \varphi \in \text{PA} \), we define

\[
T(\varphi) := \{ t : t \gg c \in A(\varphi) \},
\]

\[
D(\varphi) := \{ d | t : d | t + c \in A(\varphi) \},
\]

and

\[
\max_{\text{coeff}}(\varphi) := \max\{1\} \cup \{ |k| : k \text{ is a coefficient in } t \gg c \in A(\varphi) \},
\]

\[
\max_{\text{const}}(\varphi) := \max\{1\} \cup \{ |c| : t \gg c \in A(\varphi) \},
\]

\[
\max_{\text{div}}(\varphi) := \max\{1\} \cup \{ d : d | t + c \in A(\varphi) \}.
\]

4.1 Eliminating a Quantifier

For the sake of completeness, we briefly recall Reddy and Loveland’s quantifier elimination method. Consider the formula \( \exists x \varphi \) with \( \varphi(x,y) \in \text{QF} \). The construction of \( \psi(y) \in \text{QF} \) proceeds in 2 steps.

Step 1: First, eliminate the connectives \( \rightarrow \) and \( \leftrightarrow \) in \( \varphi \) using standard rules, e.g., a subformula \( \chi \rightarrow \chi' \) is replaced by \( \neg \chi \lor \chi' \). Second, push all negation symbols in \( \varphi \) inward (using De Morgan’s laws, etc.) until they only occur directly in front of the atomic formulas. Third, rewrite all atomic formulas and negated atomic formulas in which \( x \) occurs such that they are all of one of the forms

\[ k \cdot x < t(y_1, \ldots, y_n), \]

(\text{A})

\[ t(y_1, \ldots, y_n) < k \cdot x, \]

(\text{B})

or

\[ d \mid t(x, y_1, \ldots, y_n) \]

(\text{C})

with \( k > 0 \). For instance, the negated inequation \( \neg 2 \cdot x + 9 \cdot y < 5 \) is rewritten to \( -9 \cdot y + 5 - 1 < 2 \cdot x \), and the negated equation \( \neg 2 \cdot x + 9 \cdot y = 5 \) is replaced by the disjunction \( -9 \cdot y + 5 < 2 \cdot x \lor 2 \cdot x < -9 \cdot y + 5 \). Let \( \varphi'(x, y) \) be the resulting formula.

Step 2: Let \( \psi_{-\infty} \) be the formula where all the atomic formulas of type (A) in \( \varphi' \) are replaced by “true”, i.e., \( 0 < 1 \), and all atomic formulas of type (B) are replaced by “false”, i.e., \( 1 < 0 \). We assume in the following, without loss of generality, that \( 0 < 1 \) and \( 1 < 0 \) do not occur as proper subformulas. Note that by propositional reasoning, we can always eliminate such subformulas, e.g., \( \alpha \land 0 < 1 \) can be simplified to \( \alpha \). Let \( B(x, y) \) be the set of the atomic formulas in \( \varphi' \) of type (B), and let \( \text{lcm}(x, \varphi) \) be the least common multiple of the \( d \)s in the atomic formulas of type (C) and of
the coefficients of the variable \( x \) in the atomic formulas of type (B). Let \( \psi \) be the formula

\[
\bigvee_{1 \leq j \leq \text{lcm}(x,\varphi)} \psi_{-\infty}[j/x] \lor \bigvee_{1 \leq j \leq \text{lcm}(x,\varphi)} (k \mid t + c + j \land \varphi'[t + c + j/k \cdot x]),
\]

where \( \varphi'[t + c + j/k \cdot x] \) means that every atomic formula \( \alpha \) in \( \varphi' \) in which \( x \) occurs is first multiplied by \( k \) and then \( k \cdot x \) is substituted by \( t + c + j \). Formally, for an atomic formula \( \alpha \), a term \( t \), and \( k \in \mathbb{Z} \setminus \{0\} \), we define

\[
\alpha[t/k \cdot x] := \begin{cases} 
  k' \cdot t < k \cdot t' & \text{if } \alpha = k' \cdot x < t', \\
  k \cdot t' < k' \cdot t & \text{if } \alpha = t' < k' \cdot x, \\
  kd|k' \cdot t + k \cdot t' & \text{if } \alpha = d|k' \cdot x + t', \\
  \alpha & \text{otherwise.}
\end{cases}
\]

**Fact 4.2.** The formula \( \psi \) is logically equivalent to \( \exists x \varphi \).

### 4.2 Analysis

We can construct from an arbitrary formula a logically equivalent quantifier-free formula by successively replacing subformulas of the form \( Qx \varphi \), where \( \varphi \in \text{QF} \) and \( Q \in \{\exists, \forall\} \), with the logically equivalent quantifier-free formulas that are produced by the quantifier elimination method. Oppen [1978] analyzed the length of the formulas that are produced by iteratively applying Cooper’s quantifier elimination method. Oppen proved a triple exponential upper bound on the formula length by relating the growth in the number of atomic formulas, the maximum of the absolute values of constants and coefficients appearing in these atomic formulas, and the number of distinct coefficients and divisibility predicates that may appear. Similar analysis of improved versions of Cooper’s quantifier elimination method are in [Reddy and Loveland 1978; Grädel 1988].

Reddy and Loveland [1978] observed that they obtain shorter formulas when pushing quantifiers inward before applying their quantifier elimination method. For example, using the quantifier elimination method to eliminate the quantified variable \( x_2 \) in \( \exists x_1 \exists x_2 \varphi \) with \( \varphi \in \text{QF} \), we obtain a formula of the form \( \exists x_1(\varphi_1 \lor \ldots \lor \varphi_n) \). Instead of applying the quantifier elimination method to \( \exists x_1(\varphi_1 \lor \ldots \lor \varphi_n) \), rewriting the formula first to \( (\exists x_1 \varphi_1) \lor \ldots \lor (\exists x_1 \varphi_n) \) and then applying the quantifier elimination method to each of the disjuncts separately produces shorter formulas due to the following reasons. First, we avoid using \( \text{lcm}(x_1, \varphi_1 \lor \ldots \lor \varphi_n) \) in Step 2 of the quantifier elimination method; instead we determine \( \text{lcm}(x_1, \varphi_i) \), for each disjunct \( \varphi_i \) separately. Second, we use an inequation \( k \cdot x_1 < t \) of type (B) occurring in a disjunct \( \varphi_i \) only for eliminating \( x_1 \) in \( \varphi_i \). We do not use this inequation \( k \cdot x_1 < t \) for eliminating \( x_1 \) in disjuncts \( \varphi_j \) in which the inequation \( k \cdot x_1 < t \) does not occur. However, if the variable \( x_1 \) is universally quantified, then we cannot push the quantifier inward. Note that in order to apply the quantifier elimination method, we have to rewrite the formula \( \forall x_1(\varphi_1 \lor \ldots \lor \varphi_n) \) to \( \neg \exists x_1(\neg(\varphi_1 \lor \ldots \lor \varphi_n)) \). To eliminate \( x_1 \), we have to use in Step 2 \( \text{lcm}(x_1, \neg(\varphi_1 \lor \ldots \lor \varphi_n)) \) and the set \( B \) of the inequations of type (B) occurring in the formula produced by Step 1 normalizing \( \neg(\varphi_1 \lor \ldots \lor \varphi_n) \).

Reddy and Loveland analyzed the quantifier-free formulas produced by success-
sively applying their quantifier elimination method to formulas in prenex normal form. We refine and extend their analysis to arbitrary formulas. However, before launching into the analysis, we need the following definitions. For $\varphi \in \text{PA}$, we define

$$T_+ (\varphi) := \{ t \in T(\varphi) : \text{in } t \text{ there occurs a variable that is bound in } \varphi \}$$

and

$$D_+ (\varphi) := \{ d | t \in D(\varphi) : \text{in } t \text{ there occurs a variable that is bound in } \varphi \}.$$ 

Furthermore, let $T_-( \varphi) := T(\varphi) \setminus T_+ (\varphi)$ and $D_-( \varphi) := D(\varphi) \setminus D_+ (\varphi)$.

**Lemma 4.3.** For every $\varphi \in \text{PA}$ of the form $Qx_1 \ldots Qx_s \vartheta$, with $Q \in \{\exists, \forall\}$ and $\vartheta \in \text{QF}$, there is a logically equivalent $\psi \in \text{QF}$ such that

$$|T(\psi) \setminus T_-(\varphi)| \leq |T_+(\varphi)|^{s+1},$$

$$|D(\psi) \setminus D_-(\varphi)| \leq (|T_+(\varphi)| + 1)^s \cdot (|D_+(\varphi)| + s),$$

and

$$\max_{\text{coeff}} (\psi) < a^{2^{2s}},$$

$$\max_{\text{div}} (\psi) < a^{2^{2s}},$$

$$\max_{\text{const}} (\psi) < ba^{2^{2s}}(|T_+(\varphi)| + |D_+(\varphi)| + s),$$

where $a > \max\{2, \max_{\text{coeff}} (\varphi), \max_{\text{div}} (\varphi)\}$ and $b > \max\{2, \max_{\text{const}} (\varphi)\}$.

**Proof.** We first describe how we construct the quantifier-free formula $\psi$, where we assume that $Q = \exists$. For $Q = \forall$, we rewrite $\varphi$ to $\neg \exists x_1 \ldots \exists x_s \neg \vartheta$ and eliminate the quantified variables in $\exists x_1 \ldots \exists x_s \neg \vartheta$ as described below.

By a preprocessing step we rewrite $\vartheta$ to negation norm form (i.e., we eliminate the connectives $\rightarrow$ and $\leftrightarrow$, and we push the negation symbols inward such that the connective $\neg$ only occurs directly in front of atomic formulas) and we rewrite (in)equations so that we only have inequations of the form $t < t'$ or $t > t'$ and no negation occurs in front of an inequation. For instance, $t \leq t'$ is rewritten to $t < t' + 1$ and $\neg t \leq t'$ is rewritten to $t > t'$. Let $\vartheta_0$ be the formula that we obtain by the rewriting. The only parameter that is changed by this rewriting is the maximal absolute value of a constant, which increases by at most 1. Observe that this special form of a formula is preserved when we apply the quantifier elimination method: In Step 1 we only rewrite the inequations such that they are of type (A) or (B). Such rewriting does not alter the parameters. Step 2 also preserves this special form.

After the preprocessing step, we construct the quantifier-free formula $\psi$ iteratively in $s$ steps by constructing intermediate formulas $\varphi_0, \ldots, \varphi_s$, where $\psi$ will be $\varphi_s$. Let $\varphi_0 := \exists x_1 \ldots \exists x_s \vartheta_0$. In the $\ell$th step we eliminate the variable $x_{s-\ell+1}$, where $1 \leq \ell \leq s$. This is done as follows. Assume that $\varphi_{\ell-1} = \exists x_1 \ldots \exists x_{s-\ell+1} \vartheta_{\ell-1}$, where $\vartheta_{\ell-1} = \vartheta_{\ell-1,1} \lor \ldots \lor \vartheta_{\ell-1,n_{\ell-1}}$. We push the existential quantification of $x_{s-\ell+1}$ inward in $\vartheta_{\ell-1}$ as far as possible. For every $1 \leq i \leq n_{\ell-1}$, we apply the quantifier elimination method to $\exists x_{s-\ell+1} \vartheta_{\ell-1,i}$. The $n_{\ell-1}$ applications of the quantifier elimination method, we obtain for some $n_{\ell} \geq 1$, a formula $\vartheta_{\ell} := \vartheta_{\ell,1} \lor \ldots \lor \vartheta_{\ell,n_{\ell}}$ that is logically equivalent to $\exists x_{s-\ell} \vartheta_{\ell-1}$. Let $\varphi_{\ell} := \exists x_1 \ldots \exists x_{s-\ell} \vartheta_{\ell}$.
We now prove the upper bounds on the parameters of \( \psi \). Let \( n_0 := 1 \) and \( \vartheta_{0,1} := \vartheta_0 \). It is straightforward to prove by induction over \( 0 \leq \ell \leq s \):

(i) There are indices \( 1 \leq i_1, \ldots, i_k \leq n_\ell \) such that
\[
T(\varphi_\ell) = T(\vartheta_{\ell,i_1}) \cup \cdots \cup T(\vartheta_{\ell,i_k}),
\]
where \( k \leq |T_+(\varphi)|^\ell \).

(ii) There are indices \( 1 \leq i_1, \ldots, i_k \leq n_\ell \) such that
\[
D(\varphi_\ell) = D(\vartheta_{\ell,i_1}) \cup \cdots \cup D(\vartheta_{\ell,i_k}),
\]
where \( k \leq (|T_+(\varphi)| + 1)^\ell \).

The upper bounds on \( |T(\psi)| \setminus T_-(\varphi) \) and \( |D(\psi)| \setminus D_-(\varphi) \) follow immediately from (i) and (ii), respectively, since \( |T(\vartheta_{\ell,1}) \setminus T_-(\varphi)| \leq |T_+(\varphi)| \) and \( |D(\vartheta_{\ell,1}) \setminus D_-(\varphi)| \leq |D_+(\varphi)| + \ell \), for every \( 0 \leq \ell \leq s \) and \( 1 \leq i \leq n_\ell \).

We establish upper bounds on \( \max\text{\,coeff}(\psi) \), \( \max\text{\,div}(\psi) \), and \( \max\text{\,const}(\psi) \): We prove by induction over \( \ell \) that
\[
\max\text{\,coeff}(\varphi_\ell), \max\text{\,div}(\varphi_\ell) < a^{2\ell} \quad \text{and} \quad \max\text{\,const}(\varphi_\ell) < ba^{2\ell(|T_+(\varphi)| + |D_+(\varphi)| + \ell)}.
\]
For \( \ell = 0 \), these upper bounds are obviously true. Assume that \( \ell > 0 \). For \( 1 \leq i \leq n_{\ell-1} \), we examine at the formula produced by the quantifier elimination method applied to \( \exists x_{s-\ell+1} \vartheta_{\ell-1,i} \). Note that Step 1 of the quantifier elimination method does not alter the absolute values of the coefficients and constants, and the \( d_s \) in the divisibility predicate because of our preprocessing step by rewriting \( \vartheta \) to \( \vartheta_0 \). It suffices to look at the substitutions \( \alpha[t + c + j/k \cdot x] \) carried out in Step 2, where \( \alpha \) is an atomic formula in \( \vartheta_{\ell-1,i} \), \( t + c < k \cdot x \) is an inequation of type (B) in \( \vartheta_{\ell-1,i} \), and \( 1 \leq j \leq \text{lcm}(x_{s-\ell+1}, \vartheta_{\ell-1,i}) \).

—Assume that \( \alpha = d[t] \), for some \( d \geq 1 \) and some term \( t \). By the induction hypothesis, we have that
\[
k d < a^{2\ell(t-1)} \cdot a^{2\ell(t-1)} = a^{2\ell-1} \leq a^{2\ell}.
\]
It follows that \( \max\text{\,div}(\varphi_\ell) < a^{2\ell} \).

—Assume that \( \alpha = k' \cdot x < t' \) or \( \alpha = t' < k' \cdot x \), for some \( k' > 0 \) and some term \( t' \). By the induction hypothesis, we have that \( k \), \( k' \), and the absolute values of the coefficients occurring in \( t \) and \( t' \) are smaller than \( a^{2\ell(t-1)} \). It follows that the absolute values of the coefficients in the normalized inequations of \( k' \cdot (t + c + j) < k \cdot t' \) and \( k \cdot t' < k' \cdot (t + c + j) \) are smaller than
\[
a^{2\ell(t-1)} \cdot a^{2\ell(t-1)} + a^{2\ell(t-1)} \cdot a^{2\ell(t-1)} = 2a^{2\ell-1} \leq a^{2\ell}. \]
Hence, \( \max\text{\,coeff}(\varphi_\ell) < a^{2\ell} \).

The absolute values of the constants in the normalized inequations \( k' \cdot (t + c + j) < k \cdot t' \) and \( k \cdot t' < k' \cdot (t + c + j) \) is bounded by
\[
\max\text{\,coeff}(\varphi_{\ell-1}) \cdot (\max\text{\,const}(\varphi_{\ell-1}) + \text{lcm}(x_{s-\ell+1}, \vartheta_{\ell-1,i})) + \max\text{\,coeff}(\varphi_{\ell-1}) \cdot \max\text{\,const}(\varphi_{\ell-1}),
\]
which rewrites to
\[
\max_{\text{coef}}(\varphi_{\ell-1}) \cdot (2 \max_{\text{const}}(\varphi_{\ell-1}) + \text{lcm}(x_{s-\ell+1}, \vartheta_{\ell-1,i})).
\]
(10)

An upper bound on \(\text{lcm}(x_{s-\ell+1}, \vartheta_{\ell-1,i})\) is
\[
(a^{2(\ell-1)})^{T_+} + |D_+(\varphi)| + \ell - 1 = a^{2\ell - 1} - (|T_+| + |D_+(\varphi)|) + \ell - 1
\]
since we determine the least common multiple of at most \(|T_+| + |D_+(\varphi)|\) + \(\ell - 1\) numbers and all these numbers are bounded by \(a^{2\ell - 1}\). By the induction hypothesis, we have that \(|\varphi|\) and the absolute value of the constant in \(\ell^i\) is smaller than \(ba^{2\ell - 1} - (|T_+| + |D_+(\varphi)|) + \ell - 1\). Therefore, (10) is smaller than
\[
a^{2\ell - 1} - 2ba^{2\ell - 1} - (|T_+| + |D_+(\varphi)|) + \ell - 1
\]
\[
\leq ba^{2\ell - 1} - (|T_+| + |D_+(\varphi)|) + \ell.
\]
It follows that \(\max_{\text{const}}(\varphi_{\ell}) < ba^{2\ell - 1} - (|T_+| + |D_+(\varphi)|) + \ell\).

By iteratively applying Lemma 4.3 we obtain the following upper bounds for formulas in prenex normal form.

**Lemma 4.4.** For every \(\varphi \in \mathsf{PA}\) of the form \(Q_1x_1 \ldots Q_rx_r \varphi_0\) with \(\psi_0 \in \mathsf{QF}\) there is logically equivalent \(\psi \in \mathsf{QF}\) such that
\[
|T(\psi)| \leq T^{(\ell + 1)^{m(\varphi)}}\quad \text{and} \quad |D(\psi)| \leq DT^{(\ell + 1)^{m(\varphi)} + 2},
\]
where \(T = \max\{2, |T(\varphi)|\}, D = \max\{1, |D(\varphi)|\}\), and \(\ell\) is the maximal length of a quantifier block in \(\varphi\). Furthermore, it holds that
\[
\max_{\text{coef}}(\psi) < a^{2^m(\varphi)}\quad \text{and} \quad \max_{\text{div}}(\psi) < a^{2^m(\varphi)},
\]
and
\[
\max_{\text{const}}(\psi) < ba^{2^m(\varphi) + DT^{(\ell + 1)^{m(\varphi)} + 2}},
\]
where \(a > \max\{2, \max_{\text{coef}}(\varphi), \max_{\text{div}}(\varphi)\}\) and \(b > \max\{2, \max_{\text{const}}(\varphi)\}\).

**Proof.** We construct the quantifier-free formula \(\psi\) by successively eliminating the quantifier blocks in \(\varphi\), starting from the innermost block. Assume that after the \(k\)th step, where \(0 \leq k < \text{qa}(\varphi)\), we have produced the formula
\[
Q_1x_1 \ldots Q_jx_j Q_{i+1} \ldots Q_x \psi_k,
\]
where \(1 \leq i < j \leq r\), \(Q_1, \ldots, Q_r, Q \in \{\exists, \forall\}\) with \(Q_i \neq Q\), and \(\psi_k \in \mathsf{QF}\). Let \(\psi_{k+1} \in \mathsf{QF}\) be the formula from Lemma 4.3 that is logically equivalent to \(\varphi_k := Qx_{i+1} \ldots Qx \psi_k\). We define \(\psi := \psi_{\text{qa}(\varphi)}\).

For \(1 \leq i \leq \text{qa}(\varphi)\), let \(\ell_i\) be the length of the \(i\)th quantifier block. We prove by induction over \(0 \leq k \leq \text{qa}(\varphi)\) that
\[
|T(\psi_k)| \leq T^{(\ell + 1)^k}\quad \text{and} \quad |D(\psi_k)| \leq DT^{(\ell + 1)^k + 2},
\]
\[
\max_{\text{coef}}(\psi_k) < a^{2^{(\ell_1 + \ldots + \ell_k)}}\quad \text{and} \quad \max_{\text{div}}(\psi_k) < a^{2^{(\ell_1 + \ldots + \ell_k)}},
\]
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and

$$\max_{\text{const}}(\psi_k) < ba^{2^{i_1+\cdots+i_k}}DT^{(\ell+1)^{k+2}}.$$  

The base cases for $k = 0$ are trivial. For the step cases, let $k > 0$.

1. By Lemma 4.3, we have that

$$|T(\psi_k) \setminus T_-(\varphi_{k-1})| \leq |T_+(\varphi_{k-1})|^{\ell+1}$$

and

$$|D(\psi_k) \setminus D_-(\varphi_{k-1})| \leq (|T_+(\varphi_{k-1})| + 1)^{\ell} \cdot (|D_+(\varphi_{k-1})| + \ell)$$

Note that $T \geq 2$ and $D \geq 1$.

2. By Lemma 4.3, we have that

$$\max_{\text{coeff}}(\psi_k) \leq \left( \max \left\{ 2, \max_{\text{coeff}}(\psi_{k-1}) \right\} \right)^{2^{i_k}}$$

Analogously, we obtain the upper bound for $\max_{\text{div}}(\psi_k)$.

3. By Lemma 4.3, we have that

$$\max_{\text{const}}(\psi_k) \leq \max_{\text{const}}(\psi_{k-1}) \cdot (a^{2^{(i_1+\cdots+i_{k-1})}})^{2^{i_k}(|T_+(\varphi_{k-1})| + |D_+(\varphi_{k-1})| + i_k)}$$

Before we generalize Lemma 4.4 to arbitrary formulas, we want to point out that transforming a formula first into prenex normal form and then eliminating the quantifiers is not a good thing to do. The formula size can increase because of the following reasons.
First, a transformation into prenex normal form can increase the number of quantifier alternations. For instance, any transformation of \((\forall x \varphi) \land (\exists y \psi)\) into prenex normal form will introduce at least one additional alternation of quantifiers.

Second, when transforming a formula into prenex normal form we have to introduce fresh variables when pushing quantifiers to the front. As an example, consider the formula in prenex normal form

\[
\exists z_{n-1} \ldots \exists z_2 \exists z_1 (x = z_{n-1} + z_{n-1} \land z_{n-1} = z_{n-2} + z_{n-2} \land \ldots \land z_2 = z_1 + z_1 \land z_1 = y + y),
\]

for some \(n \geq 1\). It consists of \(n\) distinct equations. A logically equivalent formula that consists of at most 4 distinct equations is

\[
\exists z(x = z + z \land \exists z'(z = z' + z' \land \ldots \land \exists z'(z' = z + z \land z = y + y)) \ldots )).
\]

Furthermore, the formula length decreases by a factor of \(O(\log n)\) since we use a fixed number of variables, i.e., we use \(x, y, z, z'\) instead of \(x, y, z_1, \ldots, z_{n-1}\).

The third reason why a transformation into prenex normal form is not a good idea is illustrated by the formula \((\forall x \varphi) \leftrightarrow \psi\). Quantifiers do in general not distribute over \(\rightarrow\) and \(\leftrightarrow\). Therefore, we eliminate the connective \(\leftrightarrow\) and obtain \(((\forall x \varphi) \rightarrow \psi) \land (\psi \rightarrow \forall x \varphi)\). Eliminating \(\rightarrow\) yields \(((\neg \forall x \varphi) \land (\neg \psi \lor \forall x \varphi)\). To move the quantifiers to the front, we have to push the first negation inward. Finally, we obtain \(\exists x \forall x'((\neg \varphi \lor \psi) \land (\neg \psi \lor \varphi[x'/x]))\) assuming that \(x\) does not occur free in \(\psi\), and \(x'\) does not occur free in \(\varphi\) and \(\psi\). We have not only doubled the length of the formula but we have also doubled the number of quantifiers. We want to eliminate quantifiers and have ended up doubling our work.

In analogy to the maximum of the lengths of the quantifier blocks of a formula in prenex normal form, we define the quantifier block length of the formula \(\varphi\) as

\[
qbl(\varphi) := \max\{qbl_Q(\psi) : Q \in \{\exists, \forall\} \text{ and } \psi \text{ is a subformula of } \varphi\},
\]

where

\[
qbl_Q(\varphi) := \begin{cases} 
qbl_Q(\psi) & \text{if } \varphi = \neg \psi, \\
qbl_Q(\psi_1) + qbl_Q(\psi_2) & \text{if } \varphi = \psi_1 \lor \psi_2 \text{ with } \oplus \in \{\land, \lor\}, \\
qbl_Q(\neg \psi_1 \lor \psi_2) & \text{if } \varphi = \psi_1 \rightarrow \psi_2, \\
qbl_Q((\psi_1 \rightarrow \psi_2) \lor (\psi_2 \rightarrow \psi_1)) & \text{if } \varphi = \psi_1 \leftrightarrow \psi_2, \\
1 + qbl_Q(\psi) & \text{if } \varphi = Qx \psi, \\
0 & \text{otherwise},
\end{cases}
\]

for \(Q \in \{\exists, \forall\}\).

**Theorem 4.5.** For every \(\varphi \in \text{PA}\) of length \(n\), there is a logically equivalent \(\psi \in \text{QF}\) such that

\[
|T(\psi)| \leq n^{qbl(\varphi) + 1} \text{ and } |D(\psi)| \leq n^{1 + qbl(\varphi) + 1} \text{ and } \max_{\text{coef}}(\psi) < a^{2^{qbl(\varphi)}} \text{ and } \max_{\text{div}}(\psi) < a^{2^{qbl(\varphi)}},
\]

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and
\[ \max_{\text{const}}(\psi) < ba^{2^{\varphi + q_{\text{const}}(\psi)} + \varphi(q_{\text{bl}}(\varphi) + 1) + q_{\text{div}}(\varphi) + 2}, \]
where \( a > \max\{2, \max_{\text{coeff}}(\varphi), \max_{\text{div}}(\varphi)\} \) and \( b > \max\{2, \max_{\text{const}}(\varphi)\} \).

**Proof.** We require that variables are not reused in \( \varphi \), i.e., the set of free variables of \( \varphi \) is disjoint from the set of bound variables and the bound variables are pairwise distinct. Note that this can be achieved by replacing quantified variables by fresh variables. Such a variable renaming can increase the number of distinct atomic formulas. However, the number of atomic formulas after such a renaming still is less than or equal to the length of the original formula. Note that \( n \geq \max\{2, |T(\varphi)|, |D(\varphi)|\} \).

We construct the formula \( \psi \in \text{QF} \) in \( qa(\varphi) \) steps. Let \( \varphi_0 := \varphi \). Let \( 0 < k \leq qa(\varphi) \) and assume that after the \((k - 1)\)st step we have produced the formula \( \varphi_{k-1} \). Let \( \Phi \) be the set of maximal subformulas \( \vartheta \) of \( \varphi_{k-1} \) with \( qa(\vartheta) \leq 1 \) and where variables are either only existentially quantified or universally quantified. We can assume without loss of generality that every formula in \( \Phi \) is in prenex normal form and that \( \Phi = \{ \vartheta_1, \ldots, \vartheta_m \} \). For \( 1 \leq i \leq m \), let \( \xi_i \in \text{QF} \) be the logically equivalent formula to \( \vartheta_i \) from Lemma 4.3. We replace in \( \varphi_{k-1} \) every \( \vartheta_i \) by \( \xi_i \). We obtain the formula \( \varphi_k \) that is logically equivalent to \( \varphi \) and \( qa(\varphi_k) = qa(\varphi) - k \). For \( k = qa(\varphi) \), we define \( \psi := \varphi_k \).

For the formula \( \varphi_k \), we have that
\[ T(\varphi_k) \subseteq T(\varphi_{k-1}) \setminus \left( \bigcup_{1 \leq i \leq m} T_+(\vartheta_i) \right) \cup \bigcup_{1 \leq i \leq m} (T(\xi_i) \setminus T_-(\vartheta_i)). \]
Since variables are not reused in \( \varphi \), it follows that
\[ |T(\varphi_k)| \leq |T(\varphi_{k-1})| - \sum_{1 \leq i \leq m} |T_+(\vartheta_i)| + \sum_{1 \leq i \leq m} |T_+(\vartheta_i)|^{q_{\text{div}}(\vartheta_i) + 1}. \]
It is straightforward to show that the left hand side has its maximum when \( m = 1 \) and \( |T_+(\vartheta_1)| = |T(\varphi_{k-1})| \). Analogously to the step case in the proof of Lemma 4.4 for formulas in prenex normal form, it follows that \( |T(\varphi_k)| \leq n(q_{\text{bl}}(\varphi) + k + 1) \) under the assumption that \( |T(\varphi_{k-1})| \leq n(q_{\text{bl}}(\varphi) + k) \).

We can argue similarly for \( |D(\varphi_k)| \). Similar as in the proof of Lemma 4.4 for formulas in prenex normal form we obtain the upper bounds for \( \max_{\text{coeff}}(\varphi_k) \), \( \max_{\text{div}}(\varphi_k) \), and \( \max_{\text{const}}(\varphi_k) \).

**4.3 Main Result**

We now prove our main result: The upper bound on the automata size of the minimal DWA for Presburger arithmetic formulas.

**Theorem 4.6.** The size of the minimal DWA for a formula \( \varphi \in \text{PA} \) of length \( n \) is at most \( 2^{n(q_{\text{bl}}(\varphi) + 1) + q_{\text{div}}(\varphi) + 4} \).

**Proof.** Since we measure the length of integers linearly, we have that the absolute value of every integer occurring in \( \varphi \) is bounded by \( n \). It holds that \( n > \max_{\text{const}}(\varphi), n > \max_{\text{coeff}}(\varphi), \) and \( n > \max_{\text{div}}(\varphi) \).
For \( q_n(\varphi) = 0 \), we have that the size of the minimal DWA is at most \( 2^n \). For every atomic formula \( \alpha_i \) of length \( n_i \) in \( \varphi \), we can build a DWA of size at most \( n_i \) by using the constructions in §3.2 and §3.3. Applying the product construct yields a DWA of size at most \( \prod_{1 \leq i \leq m} n_i \leq \sum_{1 \leq i \leq m} n_i \leq 2^n \), where \( m \) is the number of atomic formulas in \( \varphi \).

In the following, assume that \( q_n(\varphi) \geq 1 \) and, therefore, we have that \( qa(\varphi) \geq 1 \) and \( qbl(\varphi) \geq 1 \). For the sake of readability, we define \( a := qa(\varphi) \) and \( \ell := qbl(\varphi) \).

From Theorem 4.5 it follows that there is a logically equivalent \( \psi \in \mathbb{QF} \) with

\[
|T(\psi)| \leq n^{1+\ell+1} \quad \text{and} \quad |D(\psi)| \leq n^{1+\ell+1}.
\]

Upper bounds on \( \max_{\text{coeff}}(\psi) \), \( \max_{\text{div}}(\psi) \), and \( \max_{\text{const}}(\psi) \) are

\[
\max_{\text{coeff}}(\psi), \max_{\text{div}}(\psi) < n^{2q_n(\varphi)} \leq 2^{2^{2a}\ell n} \leq 2^{n^{1+2a}}.
\]

and

\[
\max_{\text{const}}(\psi) < n^{1+3\ell} n^{1+1+a} \leq 2^{n^{3+3a} + (\ell+1)n^2} \leq 2^{n^{(\ell+1)n^2+1} + (\ell+1)^n+2}.
\]

Note that \( n \geq 2, a, \ell \geq q_n(\varphi) \), and \( x^y = 2^{y \log_2 x} \), for \( x \geq 1 \) and \( y \geq 0 \).

Assume that there are \( r \leq n \) free variables in \( \varphi \). Since every term in \( \psi \) contains at most the free variables of \( \varphi \), the sum of the absolute values of the coefficients in a term is bounded by \( n \cdot n^{2q_n(\varphi)} \leq 2^{n^{2+2a}} < 2^{n^{3+2a}} \). With Theorem 3.12 at hand, we know that the size of the minimal DWA for \( \psi \) is at most

\[
\left( 2 + 2 \cdot 2^{n^{(\ell+1)n^2+1} + (\ell+1)^n} \right) |T(\psi)| \cdot \max_{\text{div}}(\psi)|D(\psi)|.
\]

From

\[
\left( 2 + 2 \cdot 2^{n^{(\ell+1)n^2+1} + (\ell+1)^n} \right)^\ell |T(\psi)| \leq 2^{n^{(\ell+1)n^2+1} + (\ell+1)^n} \leq 2^{n^{(\ell+1)n^3}}
\]

and

\[
\max_{\text{div}}(\psi)|D(\psi)| \leq 2^{n^{2+2a\ell + (\ell+1)n^2+2}} \leq 2^{n^{2(\ell+1)n^2 + (\ell+1)^n+2}} \leq 2^{n^{(\ell+1)n^3+4}}
\]

we conclude that the size of the minimal DWA for \( \varphi \) is at most \( 2^{n^{(\ell+1)n^4}} \). \( \square \)

Theorem 4.6 does not change if we measure the length of integers logarithmically and not linearly. The only change is that the maximal absolute integer in \( \varphi \) is now smaller than \( 2^n \). We have to adjust the bounds on \( \max_{\text{coeff}}(\psi) \), \( \max_{\text{div}}(\psi) \), and \( \max_{\text{const}}(\psi) \). For instance, we still have that

\[
\max_{\text{coeff}}(\psi) < (2^n)^{2q_n(\varphi)} = 2^{n^{2q_n(\varphi)}} \leq 2^{n^{1+2qa(\varphi)qbl(\varphi)}}.
\]

We argue analogously for \( \max_{\text{div}}(\psi) \) and \( \max_{\text{const}}(\psi) \).

**Corollary 4.7.** Let \( \text{PA}_c \) be the set of \( \text{PA} \) formulas with at most \( c \geq 0 \) quantifiers. The size of the minimal DWA for each \( \varphi \in \text{PA}_c \) is at most \( 2^{n^{O(1)}} \), where \( n \) is the length of \( \varphi \).

**Proof.** If \( q_n(\varphi) \leq c \) then \( qa(\varphi) \leq c \) and \( qbl(\varphi) \leq c \). Since \( c \) is fixed the claim follows directly from Theorem 4.6. \( \square \)
We want to remark that Theorem 4.6 and Corollary 4.7 only give upper bounds on the sizes of the minimal DWAs for PA formulas. If the Boolean connectives and the quantifiers are handled by standard automata constructions, like complementation and subset construction, and the DWAs are minimized after every automata construction step, it may be the case that the whole construction uses one exponent more space. The reason is that an exponential blow-up can occur each time the subset construction is applied. It is an open question whether the standard automata constructions already suffice to construct a DWA in $2^{n(qbl(\varphi) + 1)qbl(\varphi) + 4}$ space or time, for a given $\varphi \in \text{PA}$ of length $n$. It is also open if there are more efficient automata constructions than the standard ones for constructing DWAs for PA formulas.

5. A WORST CASE EXAMPLE

We give a worst case example that shows that our upper bound on the automata size is tight. We use the formulas $\text{Prod}_n(x, y, z)$ defined by Fischer and Rabin [1974], for $n \geq 0$. It holds that

$$[\text{Prod}_n] = \{(a, b, c) \in \mathbb{N} : ab = c \text{ and } a, b, c < \prod_{p \text{ is prime and } p < f(n+2)} p\},$$

where $f(n) := 2^{2^n}$. Note that it follows from the Prime Number Theorem that

$$\prod_{p \text{ is prime and } p < f(n+2)} p \geq 2^{f(n+1)}.$$

Fischer and Rabin looked at the structure $(\mathbb{N}, +)$ and not at $\mathbb{Z}$, but it is straightforward to adapt the definition of $\text{Prod}_n(x, y, z)$ to $\mathbb{Z}$. For $n \geq 0$, the length of $\text{Prod}_n$ and the number of quantifier alternations is linear in $n$. The quantifier block length is constant, i.e., there is a $c \geq 0$ such that for all $n \geq 0$, $qbl(\text{Prod}_n) = c$. By Theorem 4.6 we know that the minimal DWA for $\text{Prod}_n$ has at most $2^{2^{2^{O(n)}}}$ states.

Before we prove the lower bound on the automata size for the formulas $\text{Prod}_n$, we need the following lemma.

**Lemma 5.1.** Let $\ell \geq 1$. For all $z \in \mathbb{N}$ with $\varrho^x \leq z \leq \varrho^x - 2$, there are $x, y, z' \in [\varrho^x]$ such that $xy = \varrho^x z + z'$.

**Proof.** Assume that $\varrho^{x-1} \leq z \leq \varrho^x - 2$. Let $x, y \in [\varrho^x]$ with $xy \geq \varrho^x z$ and $xy - \varrho^x z$ is minimal. Note that it is always possible to find $x, y \in [\varrho^x]$ with $xy \geq \varrho^x z$ since for $x = y = \varrho^x - 1$, we have that

$$xy = (\varrho^x - 1)^2 = \varrho^{2x} - 2\varrho^x + 1 \geq \varrho^x(\varrho^x - 2) \geq \varrho^x z.$$

Let $z' := xy - \varrho^x z$. We have to show that $z' \in [\varrho^x]$. Since $xy \geq \varrho^x z$ we have that $z' \geq 0$. For the sake of absurdity, assume that $z' \geq \varrho^x$. It follows that

$$(x - 1)y = xy - y = \varrho^x z + z' - y \geq \varrho^x z$$

since $y < \varrho^x$ and $z' \geq \varrho^x$. This contradicts the minimality of $xy - \varrho^x z$ since $xy > (x - 1)y \geq \varrho^x z$. □
Our proof for the lower bound on the automata size for a formula $\text{Prod}_n$ is based on the following lemma about the set
\[
\text{MULT}_m := \{(a,b,c) \in \mathbb{Z}^3 : a, b \in [q^m] \text{ and } ab = c\},
\]
for $m \geq 0$.

**Lemma 5.2.** Let $m \geq 0$. Every DWA representing $\text{MULT}_m$ has at least $q^m$ states.

**Proof.** For $m = 0$, the claim is trivial. In the following, assume that $m > 0$ and that $A = (Q, \Sigma^3, \delta, q_0, F)$ is a DWA representing $\text{MULT}_m$. Let $K$ be the set of words of the form $(0,0,0)(0,0,b_{m-1}) \ldots (0,0,b_0) \in (\Sigma^3)^*$ with $b_{m-1} \neq 0$ and $b_0 \leq q - 2$. Let $w \in K$ and let $z$ be the integer that is encoded by the third track of $w$. It holds that
\[
q^{m-1} = z \leq q^m - 2.
\]
From Lemma 5.1 it follows that there are $x, y, z' \in [q^m]$ such that
\[
xy = q^m z + z'.
\]
We conclude that for every prefix $u$ of a word in $K$ there is a word $w \in (\Sigma^3)^*$ such that $\langle wu \rangle \notin \text{MULT}_m$.

Now, let $L$ be the set of all prefixes of $K$. Let $u, u' \in L \setminus \{\lambda\}$ with $u \neq u'$. Moreover, let $v \in (\Sigma^3)^*$ with $\langle uv \rangle \notin \text{MULT}_m$. The first and second tracks of $uv$ and $u'v$ encode both the pair $(x,y)$. The third tracks of $uv$ and $u'v$ are different.

It follows that $\langle u'v \rangle \notin \text{MULT}_m$ and hence, $\delta(q_l, u) \neq \delta(q_l, u')$. We conclude that the DWA $A$ must have a distinct state for every word in $L$.

In the following, we determine the cardinality of $L$. For $0 \leq i \leq m + 1$, let
\[
L_i := \{w \in L : |w| = i\}. \quad \text{We have that } L_0 = \{\lambda\}, L_1 = \{(0,0,0)\}, L_2 = \{(0,0,0)b : b \in \Sigma \setminus \{0\}\}, L_i = \{wb : w \in L_{i-1} \text{ and } b \in \Sigma\}, \text{ for } 3 \leq i \leq m, \text{ and } L_{m+1} = K. \text{ It holds that}
\]
\[
|L| = |L_0| + |L_1| + |L_2| + |L_3| + \cdots + |L_m| + |L_{m+1}|
= 1 + 1 + (q - 1) + (q - 1)q + \cdots + (q - 1)q^{m-2} + (q - 1)q^{m-1} - 2
=q^m. \quad \Box
\]

**Theorem 5.3.** Let $n \geq 0$. The size of every DWA representing $[\text{Prod}_n]$ is at most least $2^\left\lfloor \frac{f(n+1)}{\log_2 q} \right\rfloor$.

**Proof.** Assume that for $n \geq 0$, there is a DWA $B$ with less than $2^\left\lfloor \frac{f(n+1)}{\log_2 q} \right\rfloor$ states representing the set $[\text{Prod}_n]$. Let $m := \left\lfloor \frac{f(n+1)}{\log_2 q} \right\rfloor$. It holds that $\text{MULT}_m \subseteq [\text{Prod}_n]$ since $(q^n - 1)^2 < q^{2m} = q^{f(n+1)} \leq 2^{f(n+1)}$. It is straightforward to construct from $B$ a DWA representing the set $\text{MULT}_m$ that has as many states as $B$ by making some of the accepting states in $B$ non-accepting. This contradicts Lemma 5.2. \quad \Box

**Remark 5.4.** We make the following remarks on nondeterministic word automata and alternating word automata [Brzozowski and Leiss 1980; Chandra et al. 1981].

(i) The proof of Theorem 5.3 carries over to nondeterministic word automata. That means, that we obtain the same lower bound for nondeterministic word
automata as for DWAs although nondeterministic word automata can sometimes be exponentially more succinct than DWAs.

(ii) A lower bound for the number of states of alternating word automata for the formula \( \text{Prod}_n \) is at least \( \left\lfloor \frac{f(n+1)}{2 \log_2 \phi} \right\rfloor \). This lower bound follows by contradiction from the remark (i) above and the fact that an alternating word automaton can be translated to an equivalent nondeterministic word automaton with exponentially more states.

6. CONCLUSION

We analyzed the automata-theoretic approach for deciding Presburger arithmetic and established a tight upper bound on the automata size. Moreover, we improved the automata constructions in [Boigelot 1999; Wolper and Boigelot 2000; Ganesh et al. 2002] for equations and inequations and proved that our automata constructions are optimal.

The main technique to prove the upper bound on the automata size was to relate deterministic word automata with the formulas constructed by a quantifier elimination method. This technique can also be used to prove upper bounds on the sizes of minimal automata for other logics that admit quantifier elimination and where the structures are automata representable [Khoussainov and Nerode 1995; Blumensath and Grädel 2000; Rubin 2004], i.e., these structures are provided with automata for deciding equality on the domain and the atomic relations of the structure. Prominent examples are the mixed first-order theory over the structure \((\mathbb{R}, \mathbb{Z}, <, +)\) [Boigelot et al. 2001; Weispfenning 1999] and the first-order theory of queues [Rybina and Voronkov 2001; 2003].

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Received Month Year; revised Month Year; accepted Month Year