The sympathetic sceptic’s guide to semigroup representations

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Abstract. This is an elementary, examples driven, introduction to the representation theory of finite semigroups. We illustrate the Clifford-Munn correspondence between the representations of a semigroup and the representations of its maximal subgroups. The emphasis throughout is on naturally occurring examples.

Introduction

This is an elementary introduction to the representation theory of finite semigroups. As the title suggests, it is not necessarily intended for semigroup theorists.

We start with a quick primer on the semigroups that will interest us – the inverse and regular monoids – and spend a certain amount of energy selling these objects to the general mathematical public. Section 2 introduces from scratch linear actions of semigroups on vector spaces, where the emphasis is on those aspects of the theory that are in common with group representations. By this point the symmetric group $S_n$ will have appeared a number of times, so we divert to describe its “atomic” representations. There are two fundamental constructions, reduction and induction, that connect group theory and semigroup theory, at least when it comes to representations. These are described in Sections 3-4. Section 5 contains, what is, from the point of view of these notes, the fundamental theorem of semigroup representation theory: the Clifford-Munn correspondence. It gives a mechanism for producing the atomic representations of semigroups using only knowledge from group theory. The last section is essentially a gratuitous excuse to draw pictures of our favourite polytope, the permutohedron, dressed up as a worked example of the representations of an interesting Renner monoid.

Throughout, three running examples, $S_n$ (a group), $I_n$ (an inverse monoid) and $T_n$ (a regular monoid) are used as illustration. By the end of Section 4 the emphasis will have completely moved to inverse monoids. We also start with actions on vector spaces over an arbitrary field $k$, but in later sections we retreat to the relative safety of representations over $\mathbb{C}$. We borrow heavily from a number of sources – full attributions are given in the Notes and References section at the end.

1. Semigroups

A semigroup is a set equipped with an associative multiplication. This leaves us with quite a bit of scope! In this section we feel our way towards a manageable class of semigroups to study. Our guiding principle will be the role of inverses in semigroup theory.

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We start with three finite examples that are the most typical of their type. Throughout, we write \([n]\) for the set \(\{1, 2, \ldots, n\}\. 

- **The symmetric group \(S_n\):** consisting of all bijections \(g : [n] \to [n]\) with multiplication the usual composition of maps.
- **The symmetric inverse monoid \(I_n\):** consisting of all partial bijections \(s : [n] \to [n]\), i.e. bijections \(s : X \to Y\) where \(X, Y \subseteq [n]\). The multiplication is composition of partial maps as shown in Figure 1. All our functions, actions, etc, will be on the left, so the partial map \(st\) has domain the \(t\)-preimage of \(\text{im} t \cap \text{dom} s\) and image the \(s\)-image of \(\text{im} t \cap \text{dom} s\), and is the usual composition of \(t\) followed by \(s\) between these two sets. If \(\text{im} t \cap \text{dom} s\) is empty, then \(st\) is the unique bijection \(\emptyset \to \emptyset\), which we will call the zero map \(0\).
- **The full transformation monoid \(T_n\):** consisting of all mappings \(s : [n] \to [n]\) with multiplication the usual composition of maps.

**Inverses in semigroup theory.** Naively, a semigroup is a group, except without inverses. But to completely rule out inverses in a semigroup is unnecessarily defeatist. The elements of \(S_n\) are “global” symmetries of the set \([n]\) – with global inverses – while the elements of \(I_n\) are “local” symmetries of \([n]\), with local inverses to match. Our three running examples motivate three ways in which inverses arise:

(i). The symmetric group \(S_n\) is a group, obviously, so for every \(g \in S_n\) there is a unique \(h \in S_n\) with \(gh = id = hg\). Write \(h = g^{-1}\) as usual.

(ii). If \(s : X \to Y\) is an element of \(I_n\) then there is a unique \(X \leftarrow Y : t\), that is the inverse of \(s\), but only defined on the set \(Y\). Indeed, \(st = id_Y\) and \(ts = id_X\), where \(id_X : X \to X\) and \(id_Y : Y \to Y\) are partial identities, and in particular, idempotents (i.e: \(id_Yid_X = id_X\) and \(id_Yid_Y = id_Y\)).

As a working definition of the local inverse of \(s\), we could take it to be an element \(t\) such that \(st\) and \(ts\) are idempotents, but not necessarily the global idempotent \(id\). It turns out that this isn’t quite satisfactory, as any map defined on some subset of the image of \(s\), and equal to the inverse of \(s\) on this subset, also has this property.
Instead, we have $s \text{id}_X = s = s \text{id}_Y t$. Together with $st = s \text{id}_Y$ and $ts = t \text{id}_X$ we get that $t$ satisfies $sts = s$; similarly $tst = t$.

A semigroup with the property that for every $s$ there is a unique $s^*$ satisfying

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*$$  

is called an inverse semigroup; a semigroup with an identity $\text{id}$ is a monoid, and an inverse semigroup with an identity $\text{id}$ is an inverse monoid. $I_n$ is thus the most inverse monoid-like of the inverse monoids. We will sometimes call $s^*$ an inverse “in the sense of semigroup theory” and reserve the notation $s^{-1}$ for inverses in a group.

(iii). Definition (1) of inverses opens up new possibilities. The element $s \in T_n$ shown on the left of Figure 2 has kernel the partition of $[n]$ whose blocks are the fibers of $s$: the $s$-preimages of a point in the image of $s$. Now construct an element $t \in T_n$ in the following way:

- partition the image copy of $[n]$ so that each block of the partition contains exactly one element of the image of $s$; this partition will be the kernel of $t$.
- For each block in this new kernel choose an $x$ in the domain copy of $[n]$ such that the block contains the point $sx$; then define $t$ so that it maps this block to $x$; see the right of Figure 2.

The $t$ just constructed satisfies $sts = s$ and $tst = t$; conversely, any $t$ satisfying these relations must come about in this way. But this $t$ is clearly not unique – there is choice in the partition of the image $[n]$ and for each block of this partition, choice in the $x$ so that the block is labelled by $sx$.

A monoid with the property that for every $s$ there is some $t$ satisfying $sts = s$ and $tst = t$ is called a regular monoid.

From now on: $S$ will be a finite regular monoid.

The structure of semigroups: Green’s relations. These allow us to draw strategic pictures of semigroups. Define an equivalence relation $\mathcal{L}$ on $S$ by $s\mathcal{L}t$ when $Ss = St$, where $Ss = \{rs : r \in S\}$ is a left ideal (hence the “$\mathcal{L}$”). Dually, define $s\mathcal{R}t$ when $sS = tS$.

In $I_n$, and indeed any group, these relations are trivial: any two elements are $\mathcal{L}$ and $\mathcal{R}$-related. In $I_n$ and $T_n$ they take a particularly simple form:

- $s\mathcal{L}t$ when the fibers of $s$ are equal to the fibers of $t$ (or $s$ and $t$ have the same kernel). In $I_n$ this is equivalent to $\text{dom } s = \text{dom } t$.
- $s\mathcal{R}t$ when $im s = im t$.

If we consider the equivalence relation $(\mathcal{L}, \mathcal{R})$ generated by $\mathcal{L}$ and $\mathcal{R}$, then something very nice happens. The $\mathcal{L}$-class of any element $t$ that is $\mathcal{R}$-related to $s$ intersects the $\mathcal{R}$-class of any element $r$ that is $\mathcal{L}$-related to $s$. It is particularly easy to see for $I_n$ as on the left of Figure 3.
In the symmetric inverse monoid $I_n$, the $\mathcal{L}$-class $\mathcal{L}_t$ of any element $t$ that is $\mathcal{R}$-related to $s$ intersects the $\mathcal{R}$-class $\mathcal{R}_s$ of any element $r$ that is $\mathcal{L}$-related to $s$ (left) and an eggbox grid of a $(\mathcal{L}, \mathcal{R})$-class (right) partitioned into mutually intersecting $\mathcal{L}$ and $\mathcal{R}$-classes.

The $(\mathcal{L}, \mathcal{R})$-classes are thus partitioned into $\mathcal{L}$-classes and partitioned into $\mathcal{R}$-classes, with any $\mathcal{L}$-class intersecting any $\mathcal{R}$-class and vice-versa. Semigroup theorists call this grid an “eggbox” – see the right of Figure 3.

Moreover, pursuing the ideal theme, define a relation $\mathcal{J}$ on $S$ by $s\mathcal{J}t$ when $StS = StS$. Again, this has a simple form in $I_n$ and $T_n$, with $s\mathcal{J}t$ when $\text{im } s$ and $\text{im } t$ are sets of the same size. But the $(\mathcal{L}, \mathcal{R})$-class of $I_n$ in Figure 3 consists precisely of those partial bijections whose image has the fixed size $|\text{im } s| = |Y|$. Thus $\mathcal{J} = (\mathcal{L}, \mathcal{R})$ in $I_n$, and in general for any finite $S$.

Our final relation is $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, so that $s\mathcal{H}t$ when they are both $\mathcal{L}$ and $\mathcal{R}$-related. In $I_n$ and $T_n$ a pair $s\mathcal{H}t$ means that $s$ and $t$ have the same fibers (or domains in $I_n$) and the same images. The $\mathcal{H}$-classes are thus the small boxes in the eggbox grid with one marked on the right of Figure 3.

Write $\mathcal{L}_s$, $\mathcal{R}_s$, $\mathcal{J}_s$, and $\mathcal{H}_s$ for the equivalence class of $s \in S$ under these relations.

The $\mathcal{J}$-classes are not just floating around in the ether in a disembodied fashion. They can be compared to each other; in other words, they form a poset. This is what we mean by “strategic picture”.

Again we can see this quite naturally by looking at $I_n$ and $T_n$, where the $\mathcal{J}$-classes are parametrised by the possible sizes of the images: by $\{0, \ldots, n\}$ in $I_n$ and by $\{1, \ldots, n\}$ in $T_n$. Indeed, $SsS$ consists of the maps having image size $\leq |\text{im } s|$, so that $SsS \subseteq StS$ exactly when $|\text{im } s| \leq |\text{im } t|$. We will write $\mathcal{J}_m$ for the $\mathcal{J}$-class consisting of those maps with image size $m$.

In general, define a partial order on the $\mathcal{J}$-classes of a semigroup $S$ by $\mathcal{J}_i \leq \mathcal{J}_f$ whenever $StS \subseteq StS$.

**Idempotents.** An idempotent is an element $e$ with the property that $e^2 = e$. In a group there is precisely one: the identity $\text{id}$. But in $I_n$ there are others, and in $T_n$ even more again.

The idempotents in $I_n$ are the maps $\text{id}_X : X \to X$ that are the identity on some $X \subseteq [n]$; they are the *partial identities*. In the eggbox on the right of Figure 3, $\text{id}_X$ lives on the diagonal in the row and column labelled by $X$. Moreover, $\text{id}_X$ is the only idempotent in its row and column.
In fact, this is true for any inverse semigroup: each $\mathcal{R}$-class and each $\mathcal{L}$-class contains a unique idempotent.

To find idempotents in $T_n$, fix any partition of $[n]$; this will be the fibers/kernel of $e$. In each fiber fix a point, and then define $e$ to map each fiber to the point chosen in it – see Figure 5. (A slicker way to say it is that $e$ restricts to the identity on its image.) If the fibers – and hence the $\mathcal{L}$-class – are fixed, there is still wiggle-room in the choice of point in each one. So a given $\mathcal{L}$-class may contain several idempotents. Dually, fixing some image points (and hence the $\mathcal{R}$-class) there are many partitions of $[n]$ with a unique image point in each block of the partition, and so several idempotents in a given $\mathcal{R}$-class. This behaviour is typical of regular, non-inverse semigroups. Figure 4 compares the $\mathcal{H}$-classes containing idempotents in $I_5$ and $T_5$.

In any case, in both $I_n$ and $T_n$ – and in a regular monoid in general – each $\mathcal{H}$-class contains at most one idempotent.

Subgroups. In any monoid the units are the elements that have inverses in the sense of group theory, and these form a subgroup. In our three examples $S_n$, $I_n$ and $T_n$, these are the bijections
Fig. 6. Strategic picture of $I_n$. The $\beta$-class poset (left), the stacked eggboxes (middle) and the eggbox picture of the $\beta$-class $\beta_m$ (right); there are $\binom{n}{m}$ rows and columns with the maximal subgroups $\cong S_m$ down the diagonal.

$[n] \to [n]$, so the group of units is $S_n$ with identity id : $[n] \to [n]$. In $S_n$ this is the whole story, but in $I_n$ there are other subgroups, disjoint from the units, and in $T_n$ even more again.

For $X \subseteq [n]$ fixed, the bijections $X \to X$ form a subgroup of $I_n$ isomorphic to $S_m$, where $m = |X|$. This subgroup is precisely the diagonal $H$-class containing the idempotent $\text{id}_X : X \to X$.

In general, if $H_e$ is the $H$-class containing the idempotent $e$, then this is a subgroup of $S$ with identity $e$. Moreover, any subgroup of $S$ is a subgroup of an $H_e$ for some $e$, hence these are maximal subgroups of $S$. We write $G_e$ for $H_e$ from now on, to stress its group structure.

$T_n$ has many more $H$-classes containing idempotents, hence many more maximal subgroups. The $H$-class containing the idempotent $e$ on the left of Figure 7 consists of the maps with fibers $X_1, \ldots, X_m$ and image points $y_1, \ldots, y_m$, and where these maps give a bijection $\{X_1, \ldots, X_m\} \to \{y_1, \ldots, y_m\}$. The maximal subgroups of $T_n$ are thus symmetric groups $S_m$ as well, but in a slightly different way to $I_n$. Figure 4 therefore also shows these $S_m$ subgroups (shaded), for $0 \leq m \leq 5$, in $I_5$ and $T_5$.

The strategic picture for $I_n$: is given in Figure 6. The $\beta$-class poset is on the left – the image sizes $\{0, \ldots, n\}$ with their usual total order – and the stacked eggboxes are in the middle. The
maximal \( \beta \)-class consists of all the bijections with image size \( n \), so is the symmetric group \( S_n \), and the minimal \( \beta \)-class has single element the zero map \( 0 : \emptyset \to \emptyset \). The class \( \beta_m \) has rows and columns indexed by the \( \binom{n}{m} \) subsets of size \( m \), with the blue box in Figure 6 containing the bijections \( s : X \to Y \). The idempotents \( \text{id}_X : X \to X \) lie down the diagonal, with the maximal subgroups consisting of all the bijections \( X \to X \) for fixed \(|X| = m \), and thus \( \cong S_m \).

**Green’s lemma.** The maximal subgroups can be used to parametrise the \( \mathcal{L} \) and \( \mathcal{R} \) classes containing them – indeed, the \( \mathcal{H} \)-classes in \( \mathcal{R} \) are like right cosets of the subgroup \( G_e \) and the \( \mathcal{H} \)-classes in \( \mathcal{L} \) are like left cosets.

It is easy to see in \( \mathcal{L} \): let \( e \) be the idempotent \( \text{id}_X : X \to X \), contained in the maximal subgroup \( G_e \) of all bijections \( X \to X \) (see Figure 8). An \( s \in \mathcal{L}_e \) is a bijection \( s : X \to Y \). For any \( g \in G_e \), the composition

\[
X \xrightarrow{g} X \xrightarrow{s} Y
\]

is a bijection \( X \to Y \), and all such bijections arise in this way via some \( g \). Put another way, left multiplication by \( s \) is a bijection \( s(-) : G_e \to \mathcal{H}_s \). Thus:

\[
\text{every element of the } \mathcal{H} \text{-class } \mathcal{H}_s \text{ can be uniquely expressed as } sg \text{ for some } g \in G_e \quad (2)
\]

(so \( \mathcal{H}_s = sG_e \) is the left coset in \( \mathcal{L}_e \) of the maximal subgroup \( G_e \)). Dually, if \( t \in \mathcal{R}_e \) is some bijection \( t : Z \to X \) then every element of \( \mathcal{H}_t \) has a unique expression as \( gt \) for some \( g \in G_e \), and so \( \mathcal{H}_t \) is the right coset \( G_e t \) of \( G_e \) in \( \mathcal{R}_e \). These observations are called Green’s lemma.

An important consequence is that the maximal subgroups in a fixed \( \beta \)-class are isomorphic. Again we see it in \( \mathcal{R}_e \): let \( e = \text{id}_X \) and \( f = \text{id}_Y \) be idempotents in the \( \beta \)-class \( \beta_e = \beta_f \) and let \( G_e, G_f \) be the corresponding maximal subgroups – see Figure 9. Somewhat incidentally, \( G_e \cong S_m \cong G_f \) with \(|X| = m = |Y| \), but this isomorphism also arises naturally as follows. “Complete the square” of \( \mathcal{H} \)-classes that has the maximal subgroups at its diagonal corners, and fix a representative \( s : X \to Y \) of the \( \mathcal{H} \)-class lying the the same column as \( G_e \) and row as \( G_f \). The inverse \( s^* : Y \to X \) then lies in the diagonally opposite \( \mathcal{H} \)-class.

Any \( h : Y \to Y \) in the group \( G_f \) can now be decomposed as:

\[
Y \xrightarrow{h} Y = Y \xrightarrow{s^*} X \xrightarrow{g} X \xrightarrow{s} Y
\]
for some \( g \in G_e \), and the map \( g \mapsto sgs^* \) is a homomorphism \( G_e \rightarrow G_f \) with inverse the map \( h \mapsto s^*hs \).

**The inverse monoids \( S(G,L) \).** Mathematics contains many examples of a group acting on a poset or a lattice – we now describe an inverse monoid that wraps up the group, the lattice and the action into a single object. It turns out that many naturally occurring inverse monoids arise this way. It is also a particularly useful format for understanding their representations – we will find it essential for the examples of §6.

Let \( G \) be a finite group and \( L \) a finite lattice – a poset in which every pair of elements \( a, b \) has a greatest lower bound, or *meet* \( a \land b \), and a least upper bound, or *join* \( a \lor b \). Suppose that \( G \) acts on \( L \): each \( g \in G \) gives rise to a poset map \( a \mapsto g \cdot a \), so that if \( a \leq b \) in \( L \) then \( g \cdot a \leq g \cdot b \). As this must also be true for \( g^{-1} \), we have \( a \leq b \) iff \( g \cdot a \leq g \cdot b \).

We form a semigroup \( S(G,L) \) out of this input data: the elements have *expressions* of the form \( ga \) where \( g \in G \) and \( a \in L \). Two different expressions can represent the same element:

\[
g_a = h_b \text{ in } S(G,L) \text{ iff } a = b \text{ and } g^{-1}h \cdot c = c \text{ for all } c \leq a
\]

Finally, the product is given by

\[
g_a h_b = (gh)_{h^{-1} \land a \land b}
\]

where \( gh \) is the product of \( g \) and \( h \) in \( G \). If it seems a little mysterious, you can think of \( g_a \) as the element of the symmetric inverse monoid on the set \( L \) and having domain the interval \( L_{\leq a} = \{ c \in L : c \leq a \} \) with effect the restriction of \( g \) to this interval. Then (4) is just the composition of partial bijections for \( I_L \) and (3) warns us that different elements of \( G \) can restrict to the same partial bijection in \( S(G,L) \).

As \( L \) is a finite lattice it has a maximum \( 1 = \bigvee_{a \in L} a \) and a minimum \( 0 = \bigwedge_{a \in L} a \), hence \( S(G,L) \) has an identity \( id_1 \), where \( id \) is the identity in \( G \), and a zero \( g_0 \), for any \( g \in G \) (as \( g \cdot 0 = 0 = h \cdot 0 \) for any other \( h \), we have by (3) that \( g_0 = h_0 \)). More significantly, \( g_a \) has the semigroup inverse \( g^*_a = g_{g^*_a}^{-1} \), so that \( S(G,L) \) is an inverse monoid.
The Green’s relation structure follows the dictates of the symmetric inverse monoid on $L$: we have $g_a L h_b$ exactly when $a = b$ and $g_a R h_b$ when $g \cdot a = h \cdot b$. In particular the $L$-class of $g_a$ consists of all the $h_a$ as $h \in G$ varies, and the $R$-class of all the $h_b^{-1} a$. The $J$-class of $g_a$ consists of the $h_a$ for those $h \in G$ such that $h \cdot a = g \cdot a$. The $J$-classes correspond to the orbits of the $G$-action on $L$: if $\{a_1, \ldots, a_m\}$ is an orbit, then the eggbox decomposition of the corresponding $J$-class has rows and columns indexed by the $a_i$ and the $J$-class consists of all the $g_a$, where $g \in G$ and $a$ is one of the $a_i$.

To complete the strategic picture, let $J_1$ and $J_2$ be $J$-classes corresponding to the $G$-orbits $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_l\}$. Then $J_1 \leq J_2$ exactly when $a_i \leq b_j$ in $L$ for some $a_i$ and some $b_j$ (or equivalently $a_i \leq b_j$ for any $a_i$ and some $b_j$, or, any $b_j$ and some $a_i$).

The idempotents of $S(G, L)$ are the $id_a$ for $id$ the identity of $G$, and the units are the $g_1$ with $1$ the maximum of $L$. By (3)–(4) the units form an isomorphic copy of $G$ in $S(G, L)$.

The maximal subgroup $G_a$ containing the idempotent $id_a$ consists of the $g_a$ with $g \cdot a = a$, subject to our running ambiguity (3). It turns out that the ambiguity can be easily ironed out: let $G^a$ be those elements of $G$ with $g \cdot a = a$ and let $G^{\leq a}$ be those elements of $G$ with $g \cdot c = c$ for all $c \leq a$. Then $G^{\leq a}$ is a normal subgroup of $G^a$ and there is an isomorphism

$$G_a \equiv G^a / G^{\leq a}$$

from the maximal subgroup $G_a$ to this sub-quotient of the group of units $G$.

Let $s = g_a$ be an element in the $L$-class of the maximal subgroup $G_a$ and $s(\cdot) : G_a \to \mathcal{H}_s$ the bijection promised by Green’s lemma. We can make the decomposition (2) quite explicit: if $t = h_a$ is another element of $\mathcal{H}_s$, then

$$t \equiv s \cdot g_b^{-1} h_a$$

(6)

where $b = g \cdot a$ and $g_b^{-1} h_a \in G_a$.

Example 1. We can shoehorn $I_n$ into this setting: $G = S_n$ and $L$ is the lattice of subsets of $[n]$ ordered by inclusion with $S_n$ acting on $L$ in the obvious way. We leave the reader to show that $S(S_n, L) \equiv I_n$ via the map that sends $g_a$ to the partial permutation obtained by restricting $g$ to the subset $a$.

Exercise 1. If $[G : H]$ is the index of the subgroup $H$ in $G$, show that $|S(G, L)| = \sum_{a \in L} |G : G^{\leq a}|$. Hence $|I_n| = \sum_{X \subseteq [n]} |S_n : S_X|$, where $S_X$ is the symmetric group on the set $X$.

Example 2. An important lattice in combinatorics is the partition lattice $\Pi(n)$, having elements the partitions $\lambda = \{A_1, \ldots, A_p\}$ of $[n]$ ordered by $\{A_1, \ldots, A_p\} \leq \{A_1, \ldots, A_q\}$ iff each $A_i$ is a subset of some $A_j$. The symmetric group $S_n$ acts on $\Pi(n)$ via $g \cdot \{A_1, \ldots, A_p\} = \{gA_1, \ldots, gA_p\}$. The resulting $S(G, L)$ is called the monoid of uniform block permutations and the strategic picture, when $n = 4$, is in Figure 10. The $J$-class poset is the poset of partitions $\lambda = \{A_1, \ldots, A_p\}$ of the integer $n$ (see the beginning of the Interlude), and the corresponding maximal subgroup is isomorphic to the Young subgroup $S_{A_1} \times \cdots \times S_{A_p}$. In particular, the order of the monoid of uniform block permutations is $\sum_{\lambda \in \Pi(n)} |S_n : S_{A_1} \times \cdots \times S_{A_p}|$.

2. Representations

This section contains the basics of representation theory that are common to all finite regular monoids. The theme is the extent to which representations can be decomposed into “atomic”
Fig. 10. Strategic picture of $S(G,L)$ when $G = S_4$ and $L = \Pi(4)$ from Example 2: (a). the partition lattice $\Pi(4)$; (b). the poset of $J$-classes labelled by the type of partitions; (c). the strategic picture.

pieces. These can then be reassembled to get a handle on the sociology of the representations of a semigroup. It turns out that this is almost always possible for groups and inverse monoids, but less so for regular, non-inverse monoids.

Throughout, $k$ is a field and $V$ a finite dimensional vector space over $k$. Let $\text{End}(V)$ be the monoid, under composition, of all vector space homomorphisms (or linear maps) $V \to V$.

An $S$-action on $V$ or linear representation of $S$ is a monoid homomorphism

$$\varphi : S \to \text{End}(V).$$

We adopt the convention that all monoid homomorphisms send $1$’s to $1$’s, so that $\varphi(1_S)$ is the identity homomorphism $\text{id} : V \to V$. In particular $\text{im} \varphi \neq \{0\}$, and so our representations are not null. If $S$ is a group then necessarily $\text{im} \varphi \subset GL(V)$, the group of vector space isomorphisms (or invertible linear maps) $V \to V$. The notion of a semigroup representation is thus a straight generalisation of that of a group representation.
We will identify \( s \in S \) and \( \varphi(s) \in \text{End}(V) \), so that if \( v \in V \), we just write \( s \cdot v \), or even \( sv \), for the effect of the linear map \( \varphi(s) \) on the vector \( v \). Mostly we will just write \( V \) for an \( S \)-representation without explicit reference to the action.

The following representation of our three running examples \( S_n, I_n \) and \( T_n \) will turn out to display the full range of possible behaviours:

**Example 3 (mapping representations).** Fix a basis \( \{v_1, \ldots, v_n\} \) for the space \( V \) and for \( s \in S_n, I_n \) or \( T_n \), define

\[
  s \cdot v_i = v_{\sigma(i)} \quad (s \in S_n, T_n) \quad \text{or} \quad s \cdot v_i = \begin{cases} v_{\sigma(i)}, & \text{if } i \in \text{dom } s \\ 0, & \text{else.} \end{cases} \quad (s \in I_n)
\]

and then extend linearly. To analyse the structure of the mapping representations, we need to know how to decompose representations in general.

**Sub-representations and reducibility.** These allow us to understand representations in the large. If \( V \) is a representation and \( U \) is a subspace left invariant by the \( S \)-action, i.e. \( SU = U \), then we call \( U \) an \( (S-)subrepresentation \) of \( V \). The quotient space \( V/U \) then carries an \( S \)-action via \( s \cdot (v + U) = sv + U \), well-defined, as \( sU = U \). There is then a 1-1 correspondence between the subrepresentations of \( V/U \) and the subrepresentations \( W \) of \( V \) such that \( U \subseteq W \subseteq V \).

If \( V \) has a proper, non-zero subrepresentation \( U \), then call \( V \) reducible; \( V \) is irreducible if the only subrepresentations are \( \{0\} \) and \( V \).

A subrepresentation \( U \) of \( V \) is maximal when \( U \nsubseteq V \) but for any subrepresentation \( W \) of \( V \) with \( U \subseteq W \subseteq V \) we have either \( W = U \) or \( W = V \). Because of the 1-1 correspondence mentioned above, \( U \) is maximal exactly when \( V/U \) is irreducible.

If \( U, W \) are subrepresentations of \( V \) such that \( V = U \oplus W \) as vector spaces, then the representation \( V \) is the (internal) direct sum of \( U \) and \( W \). Externally, if \( U \) and \( W \) are arbitrary \( S \)-representations then the vector space direct sum \( U \oplus W \) carries an \( S \)-action via \( s \cdot (u+w) = sv + sw \), and \( U \oplus W \) is the (external) direct sum of \( U \) and \( W \).

**Example 4 (the permuting coordinates and reflectional representations of \( S_n \)).** The \( S_n \)-action in (7) is by “permuting coordinates” (or more accurately, permuting basis vectors). In particular, for \((i, j) \in S_n\) the resulting isomorphism \( V \rightarrow V \) is the reflection in the hyperplane with equation \( x_i - x_j = 0 \). As \( S_n \) is generated by the transpositions, its image in \( GL(V) \) is generated by reflections, i.e. is a reflection group.
The vector \( u = v_1 + \cdots + v_n \) is fixed by any permutation in \( S_n \), so that if \( U \) is the \( k \)-span of \( u \) then \( S_n U = U \), a subrepresentation. As each vector in \( U \) is fixed by every element of \( S_n \), this is the trivial representation of \( S_n \) (see Figure 11).

Thus if \( n > 1 \) then the permuting coordinates representation \( V \) is reducible. Moreover, as \( U \) is 1-dimensional it has only the two subspaces, \( S_n \)-invariant or otherwise, namely \( \{0\} \) and \( U \). Hence \( U \) is irreducible. When \( n = 1 \) we have \( V = U \) is irreducible.

Now let \( W \) be the hyperplane with equation \( x_1 + x_2 + \cdots + x_n = 0 \), that is, the set of points whose coordinates with respect to the \( v_i \) sum to 0. Permuting the coordinates of such a vector doesn’t change the fact that they add to 0, hence \( W \) is also a subrepresentation of \( V \). Figure 11 has the plane \( W \) when \( n = 3 \), shifted off the origin to make it easier to see. For reasons that are maybe a little obscure at the moment, \( W \) is called the reflectional representation of \( S_n \).

Moreover, if the characteristic \( \text{char}(k) \) of the field does not divide \( \dim V = n \), then \( W \) is irreducible. For suppose that \( X \neq \{0\} \) is a \( S_n \)-invariant subspace of \( W \) and let \( v \in X \) with \( v \neq 0 \). If all the coordinates of \( v \) are equal to some \( \lambda \in k \), then these sum to 0 to give \( n \lambda = 0 \), hence – by the restriction on the characteristic – we must have \( \lambda = 0 \), hence \( v = 0 \), a contradiction. The vector \( v \) must therefore have two coordinates that are different. For each \( 1 \leq i < n \) we can engineer a \( g_i \in S_n \) such that in the vector \( g_i \cdot v \) it is the \( i \)-th and \((i+1)\)-st coordinates that are different. Then \((i, i+1)g_i \cdot v - g_i \cdot v \) is a non-zero multiple of \( v_i - v_{i+1} \). As \( X \) is \( S_n \)-invariant we conclude that for each \( 1 \leq i < n \) the vector \( v_i - v_{i+1} \) is an element of \( X \). But these vectors form a basis for \( W \), so \( X = W \), and \( W \) is irreducible as claimed.

The permuting coordinates representation of \( S_n \) can thus be decomposed \( V = U \oplus W \) into the trivial and reflectional representations, with both of these irreducible.

**Example 5 (the partial reflectional representation of \( I_n \)).** The \( I_n \)-action (7) is by partial permutations of coordinates and the image of \( I_n \) in \( \text{End}(V) \) is a reflection monoid.

Assume that \( n > 1 \). The line \( \mathcal{U} \) spanned by \( u = v_1 + \cdots + v_n \) is no longer \( I_n \)-invariant: if \( s \in I_n \) is the partial identity with domain \( \{1\} \) then \( s \cdot u = v_1 \notin \mathcal{U} \). Similarly \( \mathcal{U} \) is not \( I_n \)-invariant.

In fact, \( \mathcal{U} \) itself is irreducible: for suppose \( U \) is an \( I_n \)-invariant subspace containing \( \{0\} \neq v \in U \) with \( v = \sum \lambda_i v_i \) and \( \lambda_i \neq 0 \) for some \( i \). For each \( 1 \leq i \leq n \) let \( s_i \in I_n \) be the partial permutation shown in Figure 12. Then \( s_i \cdot v = \lambda_i v_i \), hence \( v_i \in U \) for all \( i \), and so \( U = V \).

![Fig. 12. \( s_i \in I_n \) for \( 1 \leq i \leq n \); the notation \([j, i]\) is explained in the Notes and References section.](image-url)

**Example 6 (the mapping representation of \( T_n \)).** Again suppose that \( n > 1 \). The hyperplane \( W \) goes back to being a subrepresentation of \( V \); if \( w \in W \) with \( w = \sum \lambda_i v_i \) where \( \lambda_i = 0 \), then \( s w \) for \( s \in T_n \) is shown in Figure 13. In particular the non-zero coordinates of \( s w \) are sums of the coordinates of \( w \), and so still sum to 0; i.e. \( s w = \sum \mu_i v_i \) where \( \mu_i = \sum \lambda_{ij} \), the sum over the \( j \) in the fiber of \( i \). As \( \sum \mu_i = \sum \lambda_i = 0 \), we get \( s w \in W \).

If \( S \) is any monoid, \( V \) an \( S \)-representation and \( T \) a submonoid of \( S \), then it is easy to see that restricting the \( S \)-action to \( T \) makes \( V \) into a \( T \)-representation. This observation gives that
\[ w = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6 \lambda_7 \]

\[ s \cdot w = \lambda_4 + \lambda_5 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_6 + \lambda_7 \]

Fig. 13. The hyperplane \( W \) is a \( T_n \)-subrepresentation.

\( W \) is irreducible: if \( X \) is a subrepresentation of \( W \) then it is an \( S_n \)-subrepresentation of the \( S_n \)-representation \( W \). The irreducibility of this – when \( \text{char}(k) \) does not divide \( n \) – then gives \( X = \{0\} \) or \( W \).

Just as for \( I_n \), the line \( U \) spanned by \( v_1 + \cdots + v_n \) is not a subrepresentation: for example when \( s \) is the constant map in \( T_n \) that sends all of \([n]\) to 1, then \( su = nv \not\in U \).

In fact, when \( n > 2 \) we claim that there are no 1-dimensional subrepresentations of the \( T_n \)-mapping representation \( V \). For, suppose that \( v \neq 0 \) so that \( v = \sum \lambda_i v_i \) with \( \lambda_j \neq 0 \) for some \( j \). If \( s_1, s_2 \in T_n \) are given by Figure 14, where everything apart from \( j \) is sent to \( n \), with \( s_j \) sending \( j \) to \( i \), then

\[ s_1(v) = \lambda_j v_1 + \left( \sum_{i \neq j} \lambda_i \right) v_n \quad \text{and} \quad s_2(v) = \lambda_j v_2 + \left( \sum_{i \neq j} \lambda_i \right) v_n. \]

As these are independent, any non-trivial \( T_n \)-invariant subspace must be at least 2-dimensional, and the claim follows.

Fig. 14. The \( s_1 \) and \( s_2 \) in \( T_n \) for Example 6.

**Semisimplicity.** If \( S \) is a finite regular monoid and \( k \) a field, then the pair \((S,k)\) is semisimple when every \( S \)-representation \( V \) over \( k \) can be decomposed

\[ V = \bigoplus V_i \]

with the \( V_i \) irreducible subrepresentations of \( V \).

Such a decomposition is then unique in the following sense: a morphism \( q : V \to U \) of \( S \)-representations is a linear map that commutes with the \( S \)-actions on \( V \) and \( U \), i.e. for all \( s \in S \)
the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{s(-)} & V \\
\downarrow q & & \downarrow q \\
U & \xrightarrow{s(-)} & U
\end{array}
\]

commutes, where the top \( s \cdot (-) \) is the \( S \)-action on \( V \) and the bottom is the \( S \)-action on \( U \). Call \( q \) an isomorphism if it is a bijective morphism. Uniqueness then means:

**Theorem 1 (Jordan-Hölder).** Let \( V \) be an \( S \)-representation and \( V = \bigoplus V_i \) with the subrepresentations \( V_i \) irreducible. If \( W \) is an irreducible subrepresentation of \( V \) then \( W \) is isomorphic to one of the \( V_i \).

We saw at the end of Example 4 that the reflectional representation of \( S_n \) can be so decomposed, and indeed:

**Theorem 2 (Mashke).** If \( S \) is a finite group then \((S, k)\) is semisimple if and only if the characteristic \( \text{char}(k) \) does not divide the order of \( S \).

In particular, \((S_n, k)\) is semisimple exactly when \( \text{char}(k) \) doesn’t divide \( n! \), so that characteristic 0 representations can always be decomposed. The situation for inverse monoids is almost as good:

**Theorem 3 (Munn-Oganesyan).** If \( S \) is a finite inverse monoid then \((S, k)\) is semisimple if and only if \( \text{char}(k) \) does not divide the order of any subgroup of \( S \).

As any subgroup of \( S \) is in turn a subgroup of a maximal subgroup \( G_e \) for some idempotent \( e \), it suffices that the characteristic does not divide the order of any \( G_e \).

For our model inverse monoid \( I_n \), we have already seen that the maximal subgroups are isomorphic to \( S_m \) for \( 1 \leq m \leq n \). The pair \((I_n, k)\) is thus semisimple when the characteristic of \( k \) does not divide \( m! \) for any \( m \leq n \), i.e. when it does not divide \( n! \). We therefore get the same condition for the semisimplicity of \( I_n \) and \( S_n \) representations.

For \( T_n \) things are not so good. If the mapping representation \( V \) of \( T_n \) is decomposable \( V = \bigoplus V_i \) with the \( V_i \) irreducible, then by Theorem 1, one of the \( V_i \) is isomorphic to the representation on the hyperplane \( W \) with equation \( x_1 + \cdots + x_n = 0 \) that we saw above. The decomposition of \( V \) must then be \( V \cong W \oplus W' \) with \( W' \) a 1-dimensional subrepresentation. But we have seen for \( n > 2 \) that there are no 1-dimensional subrepresentations of \( V \), and so no such decomposition of \( V \) can exist when \( n > 2 \).

Thus the pair \((T_n, k)\) is not semisimple, when \( n > 2 \), for any \( k \) whose characteristic does not divide \( n \). In particular, not even for \( k \) of characteristic 0.

**Here then is what we have learned from the three examples:** in characteristic 0 the partial permuting coordinates (or reflectional) representation of \( I_n \) is “atomic”; the permuting coordinates representation of \( S_n \) is not atomic but can be decomposed into pieces that are; the mapping representation of \( T_n \) is not atomic and cannot even be decomposed into atomic pieces.
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\[ \lambda_1 \quad \lambda_2 \quad \vdots \quad \lambda_p \]

\[
\begin{array}{cccc}
2 & 3 & 12 & 7 \\
5 & 1 & 10 & 14 \\
4 & 9 & 11 & 16 \\
6 & 8
\end{array}
\quad
\begin{array}{cccc}
2 & 7 & 12 & 3 \\
1 & 10 & 5 & 14 \\
16 & 11 & 9 & 4 \\
8 & 6
\end{array}
\]

Fig. 15. A Young diagram (left), a tableau \( T \) (middle) and the resulting tabloid \( \{ T \} \) (right) corresponding to a partition \( \lambda = \{ \lambda_1, \ldots, \lambda_p \} \vdash n \).

Interlude: the symmetric group

The moral of §§3-5 is that the representations of a (finite regular) monoid \( S \) are largely driven by the representations of its maximal subgroups. In every example that we have seen so far these maximal subgroups have been symmetric groups, or products of symmetric groups. It seems reasonable then to understand better the representations of the symmetric group, at least when \( k = \mathbb{C} \).

We do this in a completely self-contained-tailored-to-\( S_n \) way, without any reference to the general theory of representations of finite groups. This will make it seem a little like pulling a rabbit out of a hat; the reader who is interested in the broader context of these facts should consult the Notes and References at the end.

By Theorem 2, any \( S_n \)-representation over \( \mathbb{C} \) is a direct sum of irreducible representations; we will thus content ourselves with describing just these. Despite the comments in the previous paragraph, we allow ourselves one general fact: the irreducible representations over \( \mathbb{C} \) of a finite group are in 1-1 correspondence with the conjugacy classes of the group. For \( S_n \), these in turn are in 1-1 correspondence with the possible cycle structures of permutations of degree \( n \) and these in turn with the partitions of the integer \( n \): the integer sequences \( \lambda_1 \geq \ldots \geq \lambda_p > 0 \) with \( \sum \lambda_i = n \). Write \( \lambda = \{ \lambda_1, \ldots, \lambda_p \} \vdash n \).

Fix \( \lambda = \{ \lambda_1, \ldots, \lambda_p \} \vdash n \) a partition of \( n \). A Young diagram of shape \( \lambda \) illustrates the structure of \( \lambda \), as on the left of Figure 15 and a tableau \( T \) is a Young diagram filled with entries from \( [n] \) with no repeats allowed – as in the middle of Figure 15. The tableau is standard, or \( T \) is a standard tableau, when the entries increase along the rows and down the columns. Finally, a tableau \( T \) yields a tabloid \( \{ T \} \), which is just a tableau where we no longer care about the ordering in the rows – see the right of Figure 15.

The symmetric group \( S_n \) acts on the set of tableau of shape \( \lambda \), via \( g : T \mapsto gT \) for \( g \in S_n \), where \( gT \) is the tableau that has \( g(i) \) in the box in which \( T \) has \( i \). This action extends to the set of tabloids of shape \( \lambda \) via \( g \cdot \{ T \} = \{ gT \} \). For a tableau \( T \), the column group \( c_T \) is defined

\[ c_T = \{ g \in S_n : g \text{ preserves each column of } T \} \subseteq S_n \]

Let \( M^d \) be the \( \mathbb{C} \)-vector space with basis the tabloids \( \{ T \} \) of the fixed shape \( \lambda \). Via the action above, \( S_n \) acts on \( M^d \) by permuting the basis vectors. For the partition \( \lambda = \{ n - 1, 1 \} \) we will see below that \( M^d \) is the permuting coordinates representation of \( S_n \). Now we have other representations of \( S_n \).

In any case, the \( M^d \) are in general reducible – much like the permuting coordinates representation – and we will pass to a particular subrepresentation. If \( T \) is a tableau then let \( v_T \in M^d \) be the vector

\[ v_T = \sum_{h \in c_T} \text{sign}(h) h \cdot \{ T \} \]
where \( \text{sign}(h) = 1 \) or \(-1\) depending on whether \( h \) is an even or odd permutation. We will see below that in general the vectors \( v_T \), as \( T \) ranges over the tableau of shape \( \lambda \), are not independent. (The \( v_T \) for \( T \) standard are an independent subset, although we won’t need this fact here.) In any case, let

\[
S^A := \text{the subspace of } M^A \text{ spanned by the } v_T.
\] (9)

It turns out that \( S^A \) is an irreducible subrepresentation of \( M^A \), and as \( \lambda \) varies over the partitions of \( n \), the \( S^A \), called Specht representations, give a complete and non-redundant list of the irreducible \( S_n \)-representations over \( \mathbb{C} \).

**Example 7.** If \( \lambda = \square \square \square \) then there is a single tabloid:

\[
T = \begin{array}{cccc}
1 & 2 & 3 & \cdots & n
\end{array}
\]

and the column group \( c_T \) is trivial. There is thus just one vector \( v_T = \{ T \} \) in the 1-dimensional space \( M^A \), with \( g \cdot \{ T \} = \{ T \} \) for all \( g \in S_n \), so that \( M^A = S^A \) is the trivial 1-dimensional representation of \( S_n \).

**Example 8.** At the other extreme we have:

\[
\lambda = \square \square \square \square
\]

For any tableau of this shape the column group \( c_T \) is the full symmetric group \( S_n \), and up to sign, there is just one of the vectors

\[
v_T = \sum_{h \in S_n} \text{sign}(h) h \cdot \{ T \} = A - B,
\]

where \( A \) is the sum of those terms with \( h \) even and \( B \) the sum involving those with \( h \) odd. An even permutation \( g \in S_n \) preserves both summands and an odd one swaps them over, so that

\[
g \cdot v_T = \begin{cases} 
A - B = v_T, & g \text{ even,} \\
B - A = -v_T, & g \text{ odd.} 
\end{cases} = \text{sign}(g) \cdot v_T
\]

The resulting Specht representation \( S^A \) is thus 1-dimensional (hence irreducible) but not the trivial representation; it is called the sign representation of \( S_n \).

**Example 9.** If now \( \lambda = \square \square \square \square \square \) then \( M^A \) is an \( n \)-dimensional space with basis the tabloids:

\[
v_1 = \begin{array}{cccc}
2 & 3 & \cdots & n
\end{array} \quad \cdots \quad v_n = \begin{array}{cccc}
2 & 3 & \cdots & n - 1
\end{array}
\]

\[
1 \quad \cdots \quad n
\]
and the $S_n$-action is $g \cdot v_i = v_{g \cdot i}$; it is thus the permuting coordinates representation of Example 3. If $T = i \ j \ \ldots \ \ldots$ then $v_T = j \ i \ \ldots \ \ldots - i \ j \ \ldots \ \ldots$

i.e. $v_T = v_i - v_j \in M^4$, and the $v_T$, as $T$ ranges over the tableau of shape $\lambda$, give the vectors \(\{v_i - v_j\}_{1 \leq i \leq n} \) which span the hyperplane in $M^4$ having equation $\sum x_i = 0$ (and as promised, the $v_T$, as $T$ ranges over all tableau, form a dependent set). The restriction of the $S_n$-action on $M^4$ to this hyperplane then gives that $S^4$, the reflectional representation of Example 4.

When $n = 3$ the possible $\lambda$ are $\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \hline \end{array}$, and the $S^4$ are the three examples in Figure 16.

**Exercise 2.** Show that the exterior power $\bigwedge^p S \cong S^n \cong S_n$, where there are $n - p$ boxes in the first row of the Young diagram of the second Specht representation. (The exterior powers of the reflectional representation are thus also irreducible).

### 3. Reduction

This section and the next give two constructions for shuttling back and forth between representations of a finite regular monoid $S$ and representations of the maximal subgroups $G_e$ of $S$. The first of these – reduction – squashes $S$-representations down to $G_e$-representations; the second, induction, blows up $G_e$-representations into $S$-representations. In Section 5 we will see that with a little care in the choice of $e$, these constructions turn out to be inverses of each other. Throughout, the underlying field $k = \mathbb{C}$.

We start with two examples that illustrate all the key features:

**Example 10.** Let $S = I_n$, the symmetric inverse monoid, and let $V$ be the partial reflectional representation of Example 5 we saw in that example that $V$ is irreducible with basis $\{v_1, \ldots, v_n\}$.

Now let $e$ be an idempotent in the $J$-class $J_m$ in the strategic picture for $I_n$ of Figure 6. This idempotent is the identity map $\text{id}_X : X \to X$ on some subset $X \subseteq [n]$ of size $m$ and is the identity of the maximal subgroup $G_e$ consisting of all bijections $X \to X$, which in turn is a copy of the symmetric group $S_m$. 
To squash $V$ down to a representation of $G_e \cong S_m$, we take its image under $e$: let $eV := e \cdot V = \{ev : v \in V\}$. Then, as $e \cdot v_i \neq 0$ exactly when $i \in \text{dom}(e) = X$, in which case $e \cdot v_i = v_i$, the space $eV$ has basis the $v_i$ for $i \in X$. Define an action of $G_e$ on $eV$ by:

$$g \cdot (ev) = (ge) \cdot v, \quad (g \in G_e)$$

observing, as $e$ is an identity for $G_e$, that $(ge) \cdot v = (eg) \cdot v = e \cdot (gv) \in eV$. Indeed, $eV$ with this action is just the permuting coordinates representation of $S_m$ given in Example 4. We saw there that this representation is reducible when $m \geq 2$, irreducible when $m = 1$, and if $e \in \mathfrak{J}_0$ is the zero map then $eV = 0$.

Upto isomorphism of representations, $eV$ doesn’t depend on the choice of the idempotent $e$ in $\mathfrak{J}_m$. For suppose that $f = \text{id}_Y : Y \to Y$ is another idempotent in $\mathfrak{J}_m$, with $Y$ a subset of size $m$, and $f$ the identity of the maximal subgroup $G_f$. We know from Figure 9 that $G_f \cong S_f \cong S_X \cong G_e$ via the map $h \mapsto s^h s$, where $s$ is some bijection $X \to Y$ and $s^h$ is its semigroup inverse. Defining a $G_f$-action on $fV$ as in (10) gives a representation of $G_f$.

The spaces $fV$ and $eV$ are incidentally isomorphic as they both have dimension $m$; but the map $f \cdot v \mapsto (s^h) \cdot v$ naturally gives an isomorphism $fV \to eV$ that respects the actions of $G_f$ and $G_e$: firstly $s^h f = e s^h$, so that $(s^h) \cdot v = (e s^h) \cdot v = e \cdot (s^h v) \in eV$. Then

$$fV \xrightarrow{h(-)} fV \cong eV \xrightarrow{s^h ht(-)} eV$$

commutes, and so the representations $fV$ and $eV$ are indeed isomorphic as claimed. The results of the example are summarised on the left of Figure 17.

**Example 11.** The calculations, if not necessarily the results, are similar for the mapping representation of $T_n$ from Example 6. Now $V$ is reducible, so we start instead with the hyperplane $W$ consisting of the $w = \sum \lambda_i v_i$ with $\sum \lambda_i = 0$; this is an irreducible representation of $T_n$.

Let $e$ be the idempotent in $\mathfrak{J}_m$ (the maps $[n] \to [n]$ having image size $m$) given in Figure 18. Then the maximal subgroup $G_e$ consists of all bijections from the fibres of $e$ to the image of
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1 2 \quad m 1 2 m n

Fig. 18. \( e \in J_m \subset T_n \) for Example 11.

\( e \) – again, isomorphic to the symmetric group \( S_m \). The space \( eV \) has basis \{\( v_1, \ldots, v_m \)\} and the subspace \( eW \) is the hyperplane in \( eV \) whose coordinates add to 0 with respect to this basis. Via the action \( (10) \) the space \( eV \) is again the permuting coordinates representation of \( S_m \) and \( eW \) is now the reflectional representation – see Figure 17 (right).

The general picture is as follows: let \( S \) be a finite regular monoid and \( V \) an irreducible representation of \( S \). Choose an idempotent in each \( J \)-class of \( S \); the spaces \( eV \), equipped with the action \( (10) \), are then representations of the maximal subgroups \( G_e \) for the various choices of \( e \). It doesn’t matter, up to isomorphism of the resulting representations, which idempotent in a given \( J \)-class is chosen.

In particular, whether \( eV = 0 \), or not, is a property of the \( J \)-class containing \( e \). The \( J \)-classes for which \( eV \neq 0 \) form an interval in the poset of \( J \)-classes: there is a \( J \)-class \( J_V \) such that

\[
eV \neq 0 \text{ exactly when } e \in J_s \text{ with } J_s \geq J_V.
\]

The \( J \)-class \( J_V \) is called the apex of the representation \( V \), although “trough” would probably be a better name. Figure 19 shows the idea for the \( J \)-class poset of Figure 10 (left and middle) and generically (right).

In Example 10, the partial reflectional representation has a peak the \( J \)-class \( J_1 \) consisting of the partial bijections on sets of size 1; the interval of \( J \)-classes \( J_1 \leq J \) is marked on the left in Figure 17. For the irreducible representation \( W \) of \( T_n \) in Example 11, the apex is the \( J \)-class \( J_2 \) of maps with image size 2 – see the right of Figure 17.

In both examples, starting with an irreducible \( S \)-representation \( V \), we get an irreducible \( G_e \)-representation when \( e \) is in the apex \( J \)-class \( J_V \). If \( f \) is an idempotent lying in a \( J \)-class strictly greater than \( J_V \), then the resulting \( G_f \)-representation may or may not be irreducible.

Definition. (reduced representations) Let \( V \) be an irreducible representation of \( S \) with apex \( J_V \) and \( e \in J_V \) an idempotent. Then the reduced \( G_e \)-representation is given by

\[
V \downarrow G_e := eV \quad (11)
\]

together with the \( G_e \)-action \( (10) \).

Exercise 3. We can verify the general picture for \( I_n \) and then for inverse monoids of the form \( S = S(G, L) \). The reader might want to leave this exercise until after they have read §5.

1. Let \( S = I_n \) and let \( V \) be a representation of \( S \). Let \( I \) be the set of \( J \)-classes \( J_e \) such that \( eV \neq 0 \). Show that \( I \neq \emptyset \) and if \( J_e \in I \) and \( J_e \leq J_f \) in the \( J \)-class poset, then \( J_f \in I \). Hence \( I \) forms a (closed) interval in the \( J \)-class poset. Denote the minimum element by \( J_v \).
2. Let $e$ be an idempotent in this minimal $\mathcal{J}$-class $\mathcal{J}_V$ and let $T = \{s_i\}$ be a collection of representatives for the $\mathcal{H}$-classes in the $\mathcal{L}$-class $\mathcal{L}_e$. Suppose that we have $0 \neq eW \neq eV$ with $eW$ a $G_e \cong S_m$-subrepresentation of $eV$, and consider the subspace

$$U = \sum_{s_i \in T} s_i eW$$

of $V$. Show that $U$ is an $I_e$-subrepresentation of $V$.

3. If now $V$ is an irreducible $I_e$-representation, then use the arguments of §5 (about the composition $\text{Irr}_m(I_n) \to \text{Irr}(S_m) \to \text{Irr}_m(I_n)$) to show that $V = \bigoplus_{s_i \in T} s_i eV$. Deduce that $0 \neq U \neq V$ for the $U$ of (12) and hence that $eV$ is irreducible as a $S_m$-representation.

4. Repeat the whole thing for an inverse monoid of the form $S = S(G, L)$.

4. **Induction**

Induction is the opposite of reduction: it takes as input a representation of a maximal subgroup and spits out a representation of the whole semigroup.

We start with the general construction when $k = \mathbb{C}$. Let $e$ be an idempotent in the semigroup $S$, with $G_e$ the maximal subgroup having identity $e$, and let $V$ be a representation of the group $G_e$.

The induction of $V$ to an $S$-representation is controlled by the $\mathcal{H}$-classes that are in the $\mathcal{L}$-class $\mathcal{L}_e$ containing $G_e$. Choose a transversal for these $\mathcal{H}$-classes, i.e. a set $T = \{s_i\}$ with exactly one $s_i$ in each $\mathcal{H}$-class of $\mathcal{L}_e$; choose $e$ itself as the representative in the $\mathcal{H}$-class $G_e$. The transversal is just scaffolding for the construction – the resulting $S$-representation is independent of the choice of $T$.

For each $s_i \in T$, let $V_i$ be an isomorphic copy of the space $V$ having the elements

$$V_i = \{s_i \otimes v : v \in V\}$$

and vector space operations given by $\lambda(s_i \otimes v) + \mu(s_i \otimes u) = s_i \otimes (\lambda v + \mu u)$. The “$s_i \otimes$” notation is just a device to tell us which particular copy of $V$ we are working in; other than that it serves no purpose and just comes along for the ride in the vector space operation on $V_i$ (although see the comments in the Note and References section).
Let $U$ be the space

$$U = \bigoplus_{s_i \in T} V_i \quad (13)$$

and define an $S$-action on $U$ by

$$t \cdot (s_i \otimes v) = \begin{cases} s_j \otimes g \cdot v, & \text{if } ts_i \in \mathcal{L}_e, \text{ hence } ts_i = s_jg \\ 0, & \text{if } ts_i \notin \mathcal{L}_e, \end{cases} \quad (14)$$

for $t \in S$. The action of $t$ thus kills the vector $s_i \otimes v$, unless $ts_i$ is also in the $\mathcal{L}$-class $\mathcal{L}_e$, in which case by (2) there are unique $s_j \in T$ and $g \in G_e$ with $ts_i = s_jg$. The vector $v$ is then moved in $V$ by the action of $g$, with the resulting image transferred to the corresponding element of $V_j$—see Figure 20.

**Example 12 (trivial $S_1$ to trivial $I_n$).** Let $S = I_n$ and $e$ be the zero map $0 : \emptyset \to \emptyset$. The subgroup $G_e$ is the trivial group (or $S_1$!) with the single element 0. If $V$ is the trivial representation of $S_1$ then (slightly confusingly) $0 \cdot v = v$ for all $v \in V$, and the transversal $T$ consists of the one element $\{0\}$, so $U = V_0$ is a single copy of $V$. Finally, for any $t \in I_n$ we have $t \cdot (0 \otimes v) = 0 \otimes 0 \cdot v = 0 \otimes v$, and so $U$ is the trivial 1-dimensional representation of $I_n$.

**Example 13.** At the opposite end of the strategic picture for $I_n$ we have the idempotent $e = \text{id} : [n] \to [n]$, the identity of the group of units $S_n$. If $V$ is any representation of $S_n$, then again we have a transversal $T$ containing the single element $\{e\}$ and so $U = V_e$. Moreover $t \cdot U = 0$ unless $t \in S_n$ is also a unit, in which case it has effect that of the representation $V$.

Every $S_n$-representation is thus also an $I_n$-representation, just by making that part of $I_n$ not in $S_n$ (i.e. the stacked $\mathcal{J}$-classes $0, \mathcal{J}_1, \ldots, \mathcal{J}_{n-1}$ in the middle of Figure 6) act as the zero map. It is easy to check that in general any $G$-representation is also an $S$-representation in this way, when $G$ is the group of units of $S$. 

![Figure 20. Schematic of the first step of induction.](image-url)
Example 14 (trivial \( S_1 \) to partial reflectional \( I_n \)). Moving up one rung from the bottom in the strategic picture for \( I_n \) in Figure 6 let \( e \) be the partial identity \( 1 \mapsto 1 \) with domain and image \( \{1\} \), so that \( G_e \) is again the trivial group \( \{e\} \). Also again, let \( V \) be the trivial 1-dimensional representation of \( G_e \), with basis vector \( v \), and action \( e \cdot v = v \).

The \( \beta \)-class containing the subgroup \( G_e \) consists of all the bijections with domain and image of size 1 (Figure 21) and \( \mathcal{L}_e \) is the column of all the maps with domain 1. There is no choice for the representatives \( T \): they are the partial bijections \( s_j : 1 \mapsto i \). The copy \( V_i \) of \( V \) has basis the vector \( s_i \otimes v \), and so the space \( U \) of (13) is \( n \)-dimensional with basis \( \{s_i \otimes v\}_{1 \leq i \leq n} \).

For the \( I_n \)-action, we have \( ts_j \in \mathcal{L}_e \) when it has domain 1, and this is exactly when \( i \) lies in the domain of \( t \), in which case \( ts_j = s_{t(i)} = s_{t(i)}e \). Thus

\[
(t \cdot (s_j \otimes v)) = \begin{cases} s_{t(i)} \otimes v, & \text{if } i \in \text{dom}(t), \\ 0, & \text{else}. \end{cases}
\]

Replacing \( s_j \otimes v \) by \( v_i \) we get the formula (7), and so \( U \) is the partial reflectional representation of \( I_n \).

In Examples 12, 14 an irreducible \( G_e \)-representation \( V \) becomes an irreducible \( S \)-representation \( U \). We are not always so lucky:

Example 15 (trivial \( S_1 \) to mapping \( T_n \)). This is very similar to Example 14. Let \( S \) be the full transformation monoid \( T_n \), with strategic picture on the left of Figure 22.

The \( \beta \)-class at the bottom consists of the maps with image size 1 – the constant maps. There is a single \( \mathcal{L} \)-class and \( n \) \( \mathcal{R} \)-classes, each containing the single constant map \( e_i : [n] \mapsto i \) for \( 1 \leq i \leq n \). These are all idempotents, so that every \( \mathcal{R} \)-class in this \( \beta \)-class is a maximal subgroup. (You can see this in Figure 4 where every box of \( \mathcal{R}_1 \) is a maximal subgroup). In any case, there is no choice once again for \( T \), which must be the \( \{e_i\}_{1 \leq i \leq n} \).

Let \( e_1 \) be the constant map \( e : [n] \mapsto 1 \) and \( V \) be the trivial \( G_{e_1} \)-representation with basis the vector \( v \). For each \( i \), the space \( V_i \) has basis the vector \( e_i \otimes v \) and the \( U \) of (13) is \( n \)-dimensional with basis \( \{e_i \otimes v\}_{1 \leq i \leq n} \).

For any \( t \in T_n \) we have \( t \cdot e_1 = e_{t(i)} = e_{t(i)}e_1 \), so that (14) becomes \( t \cdot (e_i \otimes v) = e_{t(i)} \otimes v \), and once again we have the formula (7), after replacing \( e_i \otimes v \) with \( v_i \). The space \( U \) thus carries the
mapping representation of $T_n$, which is reducible, even though the seeding representation $V$ of the subgroup $G_{e_1} \cong S_1$ is irreducible.

Before leaving the example we observe something for later on: the $R$-class $R_{e_1}$ contains just the single element $\{e_1\}$ with $e_1 \cdot (e_i \otimes v) = e_1 \otimes v$ for all $i$. Suppose that $u \in U$ is a vector that is annihilated by $e_1$, i.e.

$$ u = \sum \lambda_i (e_i \otimes v) \text{ with } e_1 \cdot u = 0. $$

Then

$$ e_1 \cdot u = 0 \iff (\sum \lambda_i)(e_1 \otimes v) = 0 \iff \sum \lambda_i = 0, $$

so that, after replacing $e_i \otimes v$ with $v_i$, the set of such annihilated vectors is the hyperplane $W$ consisting of the $w = \sum \lambda_i v_i$ where $\sum \lambda_i = 0$.

These annihilated vectors thus form a subrepresentation of $U$. Even more is true: $U$ is the $n$-dimensional mapping representation and $W \subset U$ is $(n-1)$-dimensional, so the quotient representation $U/W$ is 1-dimensional, hence irreducible. In particular, $W$ is a maximal subrepresentation of $U$.

**Returning to generalities.** Let $V$ be a representation of the maximal subgroup $G_e$ and $U$ the space given in (13). As in the example just done, consider the vectors in $U$ that are annihilated by the elements of the $R$-class $R_{e_1}$:

$$ \text{Ann}_{e_1}(U) := \{ u \in U : s \cdot u = 0 \text{ for all } s \in R_{e_1} \}. \quad (15) $$

**Definition. (induced representations)** Let $V$ be a representation of the maximal subgroup $G_e$ of $S$ and $U$ be the $S$-representation given by (13) and (14). Then the $S$-representation induced by $V$ is the quotient

$$ V \uparrow S := U/\text{Ann}_{e_1}(U) \quad (16) $$

As in Example 15, if $V$ is irreducible, then $\text{Ann}_{e_1}(U)$ is a maximal subrepresentation of $U$, and $V \uparrow S$ is irreducible. The construction depends only on the $J$-class of $e$: if $e$ and $f$ lie in the same $J$-class then the resulting induced representations are isomorphic.
Example 16. We can verify some of these general claims in the setting of \( I_n \). In the Exercise following, we do this for an inverse monoid of the form \( S(G, L) \).

Suppose that \(|X| = m\) and \( V \) is a representation of the maximal subgroup \( G_e \cong S_m \), with \( e \) the partial identity \( X \to X \), and let \( U \) be the \( I_n \)-representation described in (13)–(14). We show first that the annihilator \( \text{Ann}_e(U) \) is trivial.

To see this, let \( T = \{ s_j \} \) be the transversal used for the induction and \( u = \sum_i s_i \otimes v_i \in \text{Ann}_e(U) \). Fix an \( s_j : X \to Y \in T \) with \( s_j^* : Y \to X \) the semigroup inverse of \( s_j \) – see Figure 23. Then for any \( i \) we have:

\[
s_j^* s_i \in \mathcal{L}_e \iff \text{dom}(s_j^* s_i) = X \iff \text{im}(s_j) = \text{im}(s_i) = Y \iff s_i = s_j.
\]

Thus, on the one hand, by (14):

\[
s_j^* u = s_j^* (s_j \otimes v_j) = e \otimes v_j,
\]

while on the other, \( s_j^* \in \mathcal{R}_e \) and \( u \in \text{Ann}_e(U) \) gives \( s_j^* \cdot u = 0 \). The conclusion is that \( v_j = 0 \) and hence \( s_j \otimes v_j = 0 \). Letting \( j \) vary we see that \( u = 0 \) and hence \( \text{Ann}_e(U) = 0 \) as claimed.

The annihilator is then certainly an \( I_n \)-subrepresentation of \( U \), albeit for trivial reasons! Suppose now that \( V \) is an irreducible \( S_m \)-representation. We claim that the annihilator is a maximal subrepresentation, or equivalently, that \( U \) is an irreducible \( I_n \)-representation. Let \( u \) be a non-zero vector in \( U = \bigoplus_T s_i \otimes V \) with \( u = \sum_i s_i \otimes v_i \) and \( s_j \otimes v_j \neq 0 \) for some \( j \) (so that in particular, \( v_j \neq 0 \)). Let \( W = I_n \cdot u \subset U \) be the set (hence subspace – Exercise) of all images of \( u \) under the elements of \( I_n \).

We claim that \( W = U \). We have \( s_j^* \cdot u \in W \), where as above

\[
s_j^* u = e \otimes v_j
\]

with \( 0 \neq e \otimes v_j \in V \). On the one hand, we have \( S_m \cdot (e \otimes v_j) \subseteq W \) (as \( S_m \subseteq I_n \)), while on the other \( S_m \cdot (e \otimes v_j) \) is a non-zero \( S_m \)-subrepresentation of the irreducible \( S_m \)-representation \( V \). The conclusion is that \( S_m \cdot (e \otimes v_j) = V \), and hence \( V \subseteq W \). For any \( s_i \in T \) we have by the definition of the action in (14) that

\[
s_i \cdot (e \otimes v) = s_i \otimes v
\]

and so \( s_i \cdot V = s_i \otimes V \). Thus, as soon as we have \( V \subseteq W \) then we have \( s_i \otimes V \subseteq W \) for each \( s_i \), and thus \( U \subseteq W \). This proves the claim, and so the \( I_n \)-representation \( U \) is irreducible.
To summarise: if $V$ is an irreducible $S_n$-representation then the induced representation $V \uparrow I_n$ is the space $U$, and this in turn is an irreducible $I_n$-representation.

**Exercise 4.** Let $S$ be an inverse monoid of the form $S = S(G, L)$ and $V$ an irreducible representation of the maximal subgroup $G_e$. Let $U$ be the $S$-representation given by (13)-(14).

1. Show that $\text{Ann}_n(U) = 0$ (hint: prove the following fact first: if the finite group $G$ acts on the lattice $L$, and if $a, b$ lie in the same $G$-orbit with $a \leq b$, then $a = b$; in other words, distinct elements of $L$ in the same $G$-orbit are not comparable.)
2. Mimic the argument above for $I_n$ to show that $U$ is irreducible.

Here is how induction works for an inverse monoid of the form $S = S(G, L)$. Let $e = \text{id}_a$ be an idempotent and $V$ a representation of the maximal subgroup $G_a$ given in (5).

The $\mathcal{H}$-classes in $L_e$ are parametrised by the $G$-orbit of $a$, say $G \cdot a = \{a, b, \ldots\}$; let $\alpha = \text{id}, \beta, \ldots$ be elements of $G$ such that

$$\alpha : a \mapsto a, \beta : a \mapsto b, \ldots$$

We then take our transversal $T$ to be $\alpha_a, \beta_a, \ldots$ In light of Exercise 4, the induced representation is carried by the space

$$V \uparrow S = \bigoplus_{\beta_a \in T} b \otimes V$$

where (after simplifying notation a little) $b \otimes V = \{b \otimes v : v \in V\}$ and, as usual, the vector space operations happen in the “$v$” coordinate, with the “$b \otimes$” just a label for the copy of $V$.

Suppose that $s = g_e$ is some element of $S$. To understand the action in (14) we need to compute products like $s \beta_a$: as $s \beta_a = g_e \beta_a = (g \beta)_{\beta^{-1} \cdot c \cdot a}$, we have that $s \beta_a$ lies in $L_e$ exactly when

$$\beta^{-1} \cdot c \cdot a = a \iff a \leq \beta^{-1} \cdot c \iff b \leq c$$

(recall that $\beta \in G$ sends $a$ to $b$). Moreover, if $b \leq c$, then $s \beta_a$ lies in the $\mathcal{H}$-class $\mathcal{H}_d$, where $d = g \cdot b$. The description (6) gives the element $\delta_a^{-1} s \beta_a = (\delta^{-1} g \beta)_a$ of $G_a$ and

$$s \beta_a = \delta_a \cdot (\delta^{-1} g \beta)_a$$

Thus, for $b \otimes v$ an element of $V \uparrow S$ and for $g_e$ an element of $S = S(G, L)$ we have the action:

$$g_e \cdot (b \otimes v) = \begin{cases} d \otimes h \cdot v, & \text{if } b \leq c \\ 0, & \text{if } b \not\leq c \end{cases}$$

(17)

where $d = g \cdot b$ and $h = (\delta^{-1} g \beta)_a$ with $\delta : a \mapsto d$ one of the elements of $G$ chosen above.

We can say more. If $s$ lies in the $\mathcal{H}$-class $\mathcal{B}_c$ with $\mathcal{B}_a \not\subseteq \mathcal{B}_c$, then for any $b$ in the $G$-orbit of $a$ we have $b \not\leq c$ in $L$, hence $s \cdot (b \otimes v) = 0$, and so $s \cdot V \uparrow S = 0$. On the other hand, if $s = \text{id}_e \in \mathcal{B}_e$ with $a \leq c$ and if $a \otimes v \neq 0$ in $V \uparrow S$, then $s \cdot (a \otimes v) = a \otimes v$, and so $s \cdot V \uparrow S \neq 0$. The conclusion is that $s \cdot V \uparrow S \neq 0$ precisely for those $s$ lying in the $\mathcal{H}$-classes that are $\subseteq \mathcal{B}_a$ in the $\mathcal{H}$-class poset. In particular, the apex of $V \uparrow S$, for $V$ a representation of $G_e$, is $\mathcal{B}_a$.

**Example 17.** We return to $T_n$ and Example 15 where $e_1 : [n] \mapsto 1$ is our idempotent, $V$ is the trivial representation of $G_{e_1} \cong S_1$ and $U$ the mapping representation of $T_n$. We saw at the end of Example 15 that $\text{Ann}_n(U) = \text{the hyperplane in } U$ consisting of the $w = \sum_{i} \lambda_i v_i$ with $\sum \lambda_i = 0$.

The induced representation $V \uparrow T_n = U/W$ is thus 1-dimensional, and as $v_i + W = v_j + W$ for all $i$ and $j$. Taking $v_1 + W$ to be the basis vector for $V \uparrow T_n$, we have for any $t \in T_n$ that:

$$t \cdot (v_1 + W) = t(v_1) + W = v_{t(1)} + W = v_1 + W,$$

so that $V \uparrow T_n$ is the trivial representation.
5. The Clifford-Munn correspondence

Induction creates irreducible representations of a (finite regular) monoid out of irreducible representations of its maximal subgroups. With a little care in the accounting, this process gives a 1-1 correspondence between the irreducible $S$-representations and the irreducibles of a certain collection of maximal subgroups. This bijection is called the Clifford-Munn correspondence.

The bijection comes about by showing that reduction is the inverse of induction; for us, this is the principal purpose of reduction. The apex of an $S$-representation $V$ tells us the “right” maximal subgroup to reduce to.

Figure 24 illustrates the correspondence, where as usual, the strategic picture of $S$ drives the whole process. Let $\text{Irr}(S)$ be the set of isomorphism classes of irreducible $S$-representations and $E = \{e_j\}$ be a set of idempotents in 1-1 correspondence with the $J$-classes of $S$. For $e \in E$ let

$$\text{Irr}_e(S) = \{V \in \text{Irr}(S) : J_V = J_e\},$$

be the irreducible $S$-representations $V$ whose apex $J_V$ is the $J$-class $J_e$ containing $e$. Every irreducible $V$ has a uniquely determined apex – the set of $S$-irreducibles $\text{Irr}(S)$ is thus partitioned into the $\text{Irr}_e(S)$ as $e$ ranges over $E$. Finally, let $\text{Irr}(G_e)$ be the irreducible representations of the maximal subgroup $G_e$.

**Theorem 4 (Clifford-Munn correspondence).** For a fixed $e \in E$, the maps:

$$\text{Irr}_e(S) \xrightarrow{V \mapsto V \uparrow_{G_e}} \text{Irr}(G_e),$$

$$\text{Irr}(G_e) \xleftarrow{V \mapsto V \downarrow_{G_e}} \text{Irr}_e(S)$$

are mutual inverses, inducing a bijection $\text{Irr}(S) \xleftrightarrow{e \in E} \bigcup_{e \in E} \text{Irr}(G_e)$.

We will prove the correspondence in the context of the symmetric inverse monoid $I_n$ when $k = \mathbb{C}$. Exercise [5] at the end of the section asks for a proof for an inverse monoid of the form $S = S(G, L)$.

Fix then, in $I_n$, the $J$-class $J_m$ for some $0 \leq m \leq n$ and the idempotent $e = \text{id} : [m] \to [m]$. The maximal subgroup $G_e$ is isomorphic to $S_m$ and consists of all partial bijections $[m] \to [m]$. As we have a nice total order on the $J$-classes, we write $\text{Irr}_m(I_n)$ for $\text{Irr}_e(I_n)$. 

**Fig. 24.** Schematic of the Clifford-Munn correspondence: the irreducibles of $S$ (left) partitioned into their various apexes (middle) which in turn are in 1-1 correspondence with the irreducibles of the corresponding maximal subgroups (right).
The map $\text{Irr}_m(I_n) \to \text{Irr}(S_m)$ given by $V \mapsto V \downarrow S_m$: as $V \in \text{Irr}_m(I_n)$, it is irreducible with apex $\mathcal{J}_m$, and hence $V \downarrow S_m = eV$ is an irreducible $S_m$-representation by §3. Thus $V \downarrow S_m \in \text{Irr}(S_m)$.

The map $\text{Irr}(S_m) \to \text{Irr}_m(I_n)$ given by $V \mapsto V \uparrow I_n$: first, we show that this is indeed a map. For $V \in \text{Irr}(S_m)$, we saw in Example 16 that $\text{Ann}_m(U) = 0$ where $U$ is the $I_n$-representation given in (13), (14) and that $V \uparrow I_n = U$ is irreducible. Thus $V \uparrow I_n \in \text{Irr}(I_n)$; we need it to be in $\text{Irr}_m(I_n)$, i.e. to have apex the $\mathcal{J}$-class $\mathcal{J}_m$. The following essentially repeats the more general arguments immediately preceding Example 17 but in a concrete setting.

The $\mathcal{L}$-class $\mathcal{L}_c$ consists of all the partial bijections with domain $[m]$. If $Y = \{i_1, \ldots, i_m\}$ is some subset of size $m$, then let $s_Y : [m] \to Y$ be the map $s_Y : j \mapsto i_j$ given on the left of Figure 25. We take $T = \{s_Y\}$ to be the transversal used in the induction process, for $Y$ ranging over all $m$-subsets of $[n]$. Thus

$$V \uparrow I_n = U = \bigoplus_{s_Y} V_Y$$

where $V_Y$ is the vector space consisting of the vectors $s_Y \otimes v$ for $v \in V$. The $I_n$-action on $U$ is given by (14).

We claim the following: if $f$ is an idempotent, then $f(V \uparrow I_n) \neq 0$ exactly when $f$ lies in a $\mathcal{J}$-class $\mathcal{J}_\ell$ with $\mathcal{J}_\ell \leq \mathcal{J}_m$. Moreover, $e(V \uparrow I_n)$ is itself isomorphic, as an $S_m$-representation, to $V$.

We choose $f$ conveniently in its $\mathcal{J}$-class $\mathcal{J}_\ell$: $f = \text{id} : [\ell] \to [\ell]$. We have $\mathcal{J}_\ell \leq \mathcal{J}_m$ exactly when $\ell < m$, in which case the right part of Figure 25 shows that $m \not\in \text{dom}(f s_Y)$ for any $m$-subset $Y$. Hence $f s_Y \not\in \mathcal{L}_c$ (the partial bijections with domain $[m]$) for any $Y$, and so by (14)

$$f \cdot (s_Y \otimes v) = 0$$

for all $Y$ and all $v$. Thus $f(V \uparrow I_n) = 0$ when $\mathcal{J}_\ell < \mathcal{J}_m$. If now $f = e$ then we have

$$e s_Y \in \mathcal{L}_c \iff \text{dom}(e s_Y) = [m] \iff Y = [m] \iff s_Y = e$$

in which case

$$e \cdot (s_Y \otimes v) \neq 0 \iff s_Y \otimes v = e \otimes v.$$
commutes. Thus, the $S_m$-representations $e(V \uparrow I_n)$ and $V$ are isomorphic as claimed. Finally, Exercise 3 part 1 gives that $f(V \uparrow I_n) \neq 0$ when $\beta_r > \beta_m$. This establishes all our claims.

In particular, $\beta_m$ is the apex of the $I_n$-representation $V \uparrow I_n$, and so $V \uparrow I_n$ is indeed in $\text{Irr}_m(I_n)$.

The composition $\text{Irr}(S_m) \to \text{Irr}_m(I_n) \to \text{Irr}(S_m)$: We have just seen, for $V$ an irreducible $S_m$-representation, that $e(V \uparrow I_n) \cong V$. Thus $(V \uparrow I_n) \downarrow S_m \cong V$, and the composition is the identity.

The composition $\text{Irr}_m(I_n) \to \text{Irr}(S_m) \to \text{Irr}_m(I_n)$: we now show that $(V \downarrow S_m) \uparrow I_n \cong V$ when $V$ is an irreducible $I_n$-representation with apex $\beta_V = \beta_m$. The strategy is to reconstruct the representation $(V \downarrow S_m) \uparrow I_n$ inside $V$.

We have already the idempotent $e = \text{id} : [m] \to [m]$ and the transversal $T = \{s_t\}$ in Figure 25 for the $m$-sized subsets $Y$ of $[n]$.

Consider now the subspaces $(s_Y e)V$ of $V$ for the various $Y$. Then:

- Each vector space $(s_Y e)V$ is isomorphic to $eV$: the linear map $eV \to (s_Y e)V$ given by $ve \mapsto (s_Y e)V$ has inverse the map $(s_Y e)V \mapsto s_Y^{-1}(s_Y e)V = e^2 V = eV$, and so is an isomorphism.

- The sum $\sum_Y (s_Y e)V$ of these spaces is direct: for which we need to show that for a fixed subset $Y$, the intersection

$$s_Y eV \cap \sum_{Z \neq Y} s_Z eV$$

is the zero space. We have just seen that $s_Y^e$ gives an isomorphism $s_Y eV \to eV$, hence maps the subspace $s_Y eV \cap \sum_{Z \neq Y} s_Z eV$ of $s_Y eV$ isomorphically onto its image in $eV$. But

$$s_Y^e \left( s_Y eV \cap \sum_{Z \neq Y} s_Z eV \right) \subseteq s_Y^e s_Y eV \cap s_Y^e \sum_{Z \neq Y} s_Z eV = eV \cap \sum_{Z \neq Y} s_Y^e s_Z eV$$

where $Z \neq Y$ gives that the domain of $s_Y^e s_Z e$ has size strictly less than $m$, and so $s_Y^e s_Z e$ lies in a $\beta$-class lower down the strategic picture than $s_Y^e$ does. As $s_Y^e$ is the apex of $V$ we have $s_Y^e s_Z e V = 0$ for all $Z$, so that the right hand side of $19$ is $0$, and hence $18$ is too.

- Restricting the $S$-action on $V$ to the subspace $\bigoplus_Y s_Y eV$: if $t \in I_n$ then there are two possibilities for the product $t s_Y$. Either:

(i). $t s_Y \in \mathcal{L}_e$, in which case by (2), there is a $g \in G_e$ and an $m$-subset $Z$ such that $t s_Y = s_Z g$; or

(ii). $t s_Y \notin \mathcal{L}_e$, and since this $\mathcal{L}$-class consists of all partial bijections with domain $[m]$, and $\text{dom}(t s_Y) \subseteq [m]$, we have that $\text{dom}(t s_Y)$ is a proper subset of $[m]$. In particular $t s_Y$ lies in a $\beta$-class lower down the strategic picture than $s_Y^e$.

The $S$-action on $\bigoplus_Y s_Y eV$ is therefore given by

$$t \cdot (s_Y e) \cdot v = \begin{cases} (s_Y e) \cdot (g \cdot v), & \text{if } t s_Y \in \mathcal{L}_e, \\
0, & \text{else}. \end{cases}$$

We conclude, first of all, that the subspace $\bigoplus_Y s_Y eV$ is in fact a subrepresentation of $V$; moreover $\bigoplus_Y s_Y eV$ contains, by taking $Y = [m]$, the subspace $eV \neq 0$. Thus $\bigoplus_Y s_Y eV$ is a non-zero subrepresentation of the irreducible representation $V$, hence

$$\bigoplus_Y s_Y eV = V.$$
Finally, if \( tsy \in L_e \) then the diagram
\[
\begin{align*}
(sye) \cdot v & \xrightarrow{\kappa^-} (sZe) \cdot (g \cdot v) \\
& \Downarrow s_y \otimes e \cdot v \xrightarrow{\kappa^-} s_Z \otimes (eg) \cdot v
\end{align*}
\]
commutes (it trivially commutes if \( tsy \notin L_e \)). Thus \( (V \downarrow S_m) \uparrow I_n \cong V \) as \( I_n \)-representations, and the composition \( \text{Irr}_m(I_n) \to \text{Irr}(S_m) \to \text{Irr}_m(I_n) \) is the identity map.

This completes the proof of the Clifford-Munn correspondence when \( S = I_n \).

Exercise 5. Mimic the proof above for an inverse monoid \( S \) of the form \( S = S(G, L) \) (hint: much of the proof can be found scattered among what we have already said).

Example 18 (The irreducibles of \( I_n \)). We are finally in a position to describe the irreducible representations over \( C \) of the symmetric inverse monoid \( I_n \). By Theorem 3, every \( I_n \)-representation over \( C \) is a direct sum of these. As we will be doing things this way in \( \S6 \) – and this is sort of a dry run at it – we will use the \( S(S_n, L) \) description of \( I_n \) that we saw at the end of \( \S1 \) where \( L \) is the lattice of subsets of \([n]\). This allows us to follow the recipe for induction given at the end of \( \S4 \).

Fix an \( m \) in the range \( 0 \leq m \leq n \), hence a \( \beta \)-class corresponding to the \( S_n \)-orbit on \( L \) consisting of the subsets of \([n]\) having size \( m \). Let \( a = \{1, 2, \ldots, m\} \) and \( G_a \) the maximal subgroup containing the idempotent \( id_a \). The elements of \( G_a \) are the \( g_a \) where \( g \in S_n \) is such that \( g \cdot a = a \) (rather than being the bijections \( a \to a \) as they would be in the “usual” way of describing \( I_n \)). Finally, let \( \lambda \) be a partition of \( m \) and \( S^\lambda \) be the Specht representation spanned by the \( v_T \) in \( \S3 \) as \( T \) ranges over the tableau of shape \( \lambda \).

We will describe the representation \( S^\lambda \uparrow I_n \). The Clifford-Munn correspondence tells us that the \( S^\lambda \uparrow I_n \), as both \( \lambda \) and \( m \) vary in \( \lambda + m \), form a complete and non-redundant list of the \( I_n \)-irreducibles over \( C \).

If \( b = \{i_1, \ldots, i_m\} \) is a subset of \([n]\) of size \( m \), then let \( \beta \) be an element of \( S_n \) that sends \( j \in a \) to \( i_j \in b \). We then take the transversal \( T \) needed for induction to be the resulting \( \beta \cdot a \) as \( T \) ranges over the subsets of size \( m \).

If \( T \) is a tableau of shape \( \lambda \) filled with entries from \( a \), then \( \beta \cdot T \) is a tableau of shape \( \lambda \) filled with entries from \( b \). Let \( S^{\lambda, b} \) be a copy of \( S^\lambda \), spanned by the
\[
b \otimes v_T = \sum_{h \in c_{\beta,T}} \text{sign}(h) h \cdot \{\beta \cdot T\},
\]
as \( T \) varies over the tableau (on \( a \)), and where \( c_{\beta,T} \) are those elements of the symmetric group on the set \( b \) preserving the columns of \( T \). The vector \( b \otimes v_T \) is just the vector \( v_T \), but with every occurrence of \( j \in a \) in a tableau replaced by \( i_j \in b \), and \( S^{\lambda, b} \) is the space spanned by the \( b \otimes v_T \).

The representation \( S^\lambda \uparrow I_n \) acts on the space
\[
S^\lambda \uparrow I_n = \bigoplus_{|h|=m} S^{\lambda, b}
\]
To see how, fix an \( s = g_c \in I_n \). We saw at the end of \( \S4 \) that the apex of \( S^\lambda \uparrow I_n \) is the \( \beta \)-class containing the maximal subgroup \( G_a \) that we started with, so if \( |c| < m \) we get \( s \cdot S^\lambda \uparrow I_n = 0 \).
On the other hand, by (11), if $c$ has size at least $m$, then it will not kill those summands $S^{λ,b}$ for which $b \subseteq c$. In this case $sβ_d$ lies in the $[n]$-class labelled by the subset $d = g \cdot b$, so that for $b \otimes v_T$ spanning $S^{λ,b}$ we get

$$s \cdot (b \otimes v_T) = d \otimes h \cdot v_T$$

where $h = (δ^{-1}gβ)_a \in G_a$.

6. A sexy example

For the purposes of these notes, “sexy” will mean a certain family of Renner monoids. These encode much of the structure of algebraic monoids, and are ubiquitous in nature.

We first set the examples up in the form $S(G, L)$ from Section 11. As usual $G$ is the symmetric group $S_n$, but the lattice is one we haven’t seen before. Let $L_0$ consist of the ordered partitions of $[n]$, i.e. the tuples $Λ = (Λ_1, . . . , Λ_p)$ with $\{Λ_1, . . . , Λ_p\}$ a partition of $[n]$. Partially order the ordered partitions via $(Λ_1, . . . , Λ_p) \leq (Λ_1, . . . , Λ_q)$ if and only if

- each $Λ_i \subseteq$ some $Δ_j$, and
- if $Λ_i \subseteq Δ_j$ and for $i < k$ we have $Λ_k \subseteq Δ_ℓ$, then $j \leq ℓ$.

$L_0$ then has maximum element the ordered partition $([n])$ with a single block and minimal elements the ordered partitions where every block has size one; these minima are in 1-1 correspondence with the permutations of $[n]$.

Formally adjoin a minimum $0$ to $L_0$ to get the lattice $L$. The $S_n$-action on $L$ is the usual $g \cdot (Λ_1, . . . , Λ_p) = (g \cdot Λ_1, . . . , g \cdot Λ_p)$ together with $g \cdot 0 = 0$.

A short diversion on where the example comes from. A linear algebraic group $G$, over an algebraically closed field $k$, is an affine algebraic variety over $k$, together with a morphism $ϕ : G \times G \to G$ of varieties, such that the product $gh := ϕ(g, h)$ gives $G$ the structure of a group. Generalising this idea, a linear algebraic monoid $M$ arises when $ϕ : M \times M \to M$ gives $M$ the structure of a monoid.

The canonical examples are $G = GL_nk$, the group of invertible matrices over $k$, and $M = M_nk$, the monoid of all $n \times n$ matrices over $k$ (both under multiplication). In fact $GL_nk$ is the group of units of the monoid $M_nk$, and indeed for sensible $M$ the group of units $G$ is an algebraic group with Zariski closure $G = M$.

There is a standard construction of algebraic monoids that starts with a sensible algebraic group $G_0$ and a sensible representation $f : G_0 \to GL(V)$. The resulting algebraic monoid is then $M = k^x f(G_0) \subseteq M_nk$ with group of units $G = k^x f(G_0)$. For example, if $G_0 = SL_n$, $SO_n$ and $Sp_n$ and $f$ is the natural representation of $G_0$, then the resulting $M$ are the classical monoids: the general linear monoids $M_n = k^x SL_n$, the orthogonal monoids $MSO_n = k^x SO_n$ and the symplectic monoids $MSp_n = k^x Sp_n$.

Associated to a (reductive) algebraic group $G$ is a finite group – called the Weyl group – that encodes much of the structure of $G$; for a (reductive) algebraic monoid $M$ there is a finite inverse monoid $R$ – called the Renner monoid – that plays an analogous role. For example, the Weyl group of $GL_nk$ is the symmetric group $S_n$ and the Renner monoid of $M_nk$ is the symmetric inverse monoid $I_n$. In general the group of units of the Renner monoid $R$ of $M$ is the Weyl group $W$ of the algebraic group of units $G$ of $M$. 
If $R$ is the Renner monoid of the algebraic monoid $\mathbb{M}$ having group of units the algebraic group $\mathbb{G}$, then $R$ is an inverse monoid of the form $S(W, L)$: the group $W$ is the Weyl group of $\mathbb{G}$ and the lattice $L$ turns out to be the face lattice of a convex polytope.

To see what this means, a polytope $P$ in $\mathbb{R}^m$ is the convex hull of a finite set of points. It has $r$-dimensional faces, for $-1 \leq r \leq m$, with the $0$-dimensional faces being the vertices, $1$-dimensional faces the edges, and so on, with $P$ itself the unique $m$-dimensional face; for formal reasons (mainly so that we get a lattice below) we take the empty set $\emptyset$ to be the unique face of dimension $-1$. The face lattice of $P$ consists of the faces ordered by inclusion; it is a lattice with meet $\sigma \wedge \tau$ the intersection and join $\sigma \vee \tau$ the smallest face containing both $\sigma$ and $\tau$.

The Renner monoid of $M_n$ has the form $S(W, L)$ where $W$ is the symmetric group $S_n$ and $L$ is the face lattice of an $(n-1)$-dimensional simplex. If $[n] = \{1, \ldots, n\}$ are the labels of the vertices of the simplex, then $L$ is the lattice of subsets of $[n]$ ordered by inclusion, and the $S_n$-action on $L$ is the usual one. This is the description of $I_n$ we gave at the end of §1.

Now to the example we are interested in: let $\mathbb{G}_0 = \text{SL}_n$ and $V_0$ be the natural module for $\mathbb{G}_0$. Let $\wedge^p V_0$ be the $p$-th exterior power of $V_0$ and finally

$$V = \bigotimes_{p=1}^{n-1} \wedge^p V_0, \text{ with } \dim V := m = \prod_{p=1}^{n-1} \binom{n}{p}$$

If $f : \mathbb{G}_0 \to GL(V)$ is the corresponding representation then let $\mathbb{M} = k[G(\mathbb{G}_0)]$ and let $R$ be the Renner monoid of $\mathbb{M}$. Then $R \cong S(W, L)$ with $W$ the symmetric group $S_n$ and $L$ the face lattice of the $(n-1)$-dimensional permutohedron. This is the polytope in $\mathbb{R}^n$ obtained by taking the convex hull of the $n!$ points arising from all permutations of the coordinates of the point $(1, 2, \ldots, n)$. As all these points lie in the hyperplane with equation $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$, the polytope is actually $(n-1)$-dimensional.

The face lattice of the permutohedron is isomorphic to the lattice $L$ of ordered partitions, with $\emptyset$ adjoined, described at the beginning of the section. Figure 26 shows the $n = 4$ case.
To describe the irreducible representations over \(\mathbb{C}\) of our Renner monoid, we use the \(S(G, L)\) description from the beginning of the section, and start by drilling down a little more into the structure of the monoid, following \(\S 1\).

First, we have our usual ambiguity with the elements of \(S(G, L)\), where \(g_a = h_b\) when \(a = b\) and \(g^{-1} \cdot c = c\) for all \(c \leq a\). In this case it turns out to disappear. If \(a\) is the ordered partition \((A_1, \ldots, A_p)\) and \(c\) is a minimal element \(\neq\emptyset\) with the property that \(c \leq a\), then \(c\) has the form

\[
c = ([x_{11}, \ldots, x_{1q_1}], \ldots, [x_{p1}, \ldots, x_{pq_p}])
\]

where \(A_1 = \{x_{11}, \ldots, x_{1q_1}\}, \ldots, A_p = \{x_{p1}, \ldots, x_{pq_p}\}\). If \(k\) is an element of \(S_n\) with \(k \cdot c = c\) then \(k = \text{id}\). Thus \(g_a = h_b\) iff \(a = b\) and \(g = h\).

If \((A_1, \ldots, A_p)\) is an ordered partition with \(\lambda_i = |A_i|\), then the ordered tuple \((A_1, \ldots, A_p)\) is called a composition of \(n\): namely, the \(\lambda_i\) are totally ordered with \(\sum \lambda_i = n\). Call the composition the type of the ordered partition. Two ordered partitions are then in the same \(S_n\)-orbit when they have the same type, and the \(\beta\)-class poset has elements the compositions ordered by \((A_1, \ldots, A_p) \leq (\mu_1, \ldots, \mu_q)\) whenever

\[
(A_1, \ldots, A_p) = (A_{11}, \ldots, A_{1m_1}, \ldots, A_{p1}, \ldots, A_{p,m_p})
\]

with \(\mu_i = \lambda_{i1} + \ldots + \lambda_{i,p_i}\). Figure 27 shows this poset when \(n = 4\).

Fix a composition \((\lambda_1, \ldots, \lambda_p)\) and consider the \(\beta\)-class of ordered partitions of this type. If \(a\) is one of them, then the maximal subgroup is \(G_a = \{g_a : g \cdot a = a\}\), and this is isomorphic to the Young subgroup \(S_{\lambda_1} \times \cdots \times S_{\lambda_p}\) of \(S_n\). We saw above that our usual ambiguity in expressing elements vanishes in \(R\); this is why there is no need to form a quotient when describing \(G_a\).

Let \(a = (A_1, \ldots, A_p)\) be the ordered partition of type \((\lambda_1, \ldots, \lambda_p)\) given by:

\[
A_1 = \{1, \ldots, A_1\}, \ldots, A_p = \{n - A_p + 1, \ldots, n\}, \quad \text{(20)}
\]

We now describe the irreducible representations of \(R\) that arise by inducing up those of the maximal subgroup \(G_a \cong S_{\lambda_1} \times \cdots \times S_{\lambda_p}\). Varying the composition produces a complete list of the irreducibles over \(\mathbb{C}\) of the Renner monoid \(R\). We borrow one more fact from the representation theory of finite groups: if \(\{V_i\}_{i \in I}\) and \(\{U_j\}_{j \in J}\) are the irreducibles of the groups \(G\) and \(H\), then the \(\{V_i \otimes U_j\}_{i \in I, j \in J}\) are the irreducibles of \(G \times H\), with \(V_i \otimes U_j\) a \((G \times H)\)-representation via the action \((g, h) \cdot v \otimes u = g \cdot v \otimes h \cdot u\).

Now fix partitions \(\mu_1 \vdash \lambda_1, \ldots, \mu_p \vdash \lambda_p\) and consider the irreducible \((S_{\lambda_1} \times \cdots \times S_{\lambda_p})\)-representation

\[
S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \quad \text{(21)}
\]

where \(S^{\mu_i}\) is the Specht representation of \(S_{\lambda_i}\) corresponding to the partition \(\mu_i \vdash \lambda_i\). The representation \(\text{(21)}\) is spanned by the vectors

\[
v_{T_1} \otimes \cdots \otimes v_{T_p}
\]

defined in \(\S 3\) and as the \(T_i\) range over the tableau of shape \(\mu_i\) filled with the numbers \(\lambda_i\) in \(\text{(20)}\). To describe \(S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \uparrow R\), let \(b = (A_1, \ldots, A_p)\) be another ordered partition of type \((\lambda_1, \ldots, \lambda_p)\) and let \(\beta \in S_n\) be such that \(\beta : A_i \mapsto A_i\) in an order preserving way, i.e. if \(x < y \in A_i\) then \(\beta(x) < \beta(y) \in A_i\). Let \(S^{\mu_1;\beta} \otimes \cdots \otimes S^{\mu_p;\beta}\) be a copy of \(\text{(21)}\) with spanning vectors of the form:

\[
\beta \otimes v_{T_1} \otimes \cdots \otimes v_{T_p},
\]
Fig. 27. The $\beta$-class poset of our Renner monoid $R$ when $n = 4$, corresponding to the poset of compositions of 4.

defined to be $v_{T_1} \otimes \cdots \otimes v_{T_p}$, but with every occurrence of $j$ in a tabloid replaced by $\beta(j)$. (Warning: this vector is linear in the $v_{T_j}$ coordinates only; the “$\beta \otimes$”, as usual, is just notation that comes along for the ride).

The representation $S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \uparrow R$ is carried by the space

$$S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \uparrow R = \bigoplus_b S^{\mu_1,\beta} \otimes \cdots \otimes S^{\mu_p,\beta}$$

with the direct sum over the ordered partitions of type $(\Delta_1, \ldots, \Delta_p)$. Let $s = g_c \in R$ with $g \in S_n$ and $c$ the ordered partition $(\Omega_1, \ldots, \Omega_q)$ of type $(\omega_1, \ldots, \omega_q)$. If $(\lambda_1, \ldots, \lambda_p) \not\leq (\omega_1, \ldots, \omega_q)$ then

$$s \cdot (S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \uparrow R) = 0.$$

Otherwise, when $(\lambda_1, \ldots, \lambda_p) \leq (\omega_1, \ldots, \omega_q)$ we have $s \cdot (S^{\mu_1,\beta} \otimes \cdots \otimes S^{\mu_p,\beta}) \neq 0$ when $b = (\lambda_1, \ldots, \lambda_p) \leq (\Omega_1, \ldots, \Omega_q)$ and in this case

$$s \cdot (\beta \otimes v_{T_1} \otimes \cdots \otimes v_{T_p}) = \delta \otimes h \cdot (v_{T_1} \otimes \cdots \otimes v_{T_p})$$

where $d = g \cdot b$ and $h = (\delta^{-1} g \beta)_a \in G_a \cong S_{\lambda_1} \times \cdots \times S_{\lambda_p}$.

Notes and References

There are numerous books that deal with semigroup representations, starting with the classic [CP61, Chapter 5]; more modern sources are [GM09, Ste16]. The reader who has got this far will see large overlap with [GM09], making [Ste16] a good next step. The original papers of Clifford [Cli42] and Munn [Mun55, Mun64] are still very readable, as is the later reworking by Rhodes [RZ91].

Semigroups. The standard reference on semigroups is [How95], where we have followed Chapters 1, 2 and 5; see also [CP61, Law98, GM09]. The three running examples are very much in the style of [GM09]. For the reader who is wondering about the “full” transformation semigroup, there is a partial version $PT_n$, which is a sort-of-amalgam of $I_n$ and $T_n$; see [GM09, Chapter 2].
The restriction to finite regular monoids is purely to make things cleaner. An expert (who shouldn’t be reading this anyway) can make the appropriate adjustments, especially in Section 5. One convenience that results is that the relation \( (L, R) \), usually called \( D \) by semigroup theorists, coincides with \( J \). So all mention of \( D \) (which gives the eggbox pictures) has been merged with \( J \) (which gives the partial order on the eggboxes). Figure 5 is adapted from a picture by James East.

The inverse monoids \( S(G, L) \) appear in [EF10, Section 9.2] as monoids of partial permutations, although they are implicit in the literature. Their purpose in [EF10] is to shed light on the factorisable inverse monoids: these are monoids \( S \) with the property that \( S = EG = GE \), where \( G \) is the group of units of \( S \) and \( E \) the idempotents – see [CH74, Fit10]. Exercise 1 can be done by counting the \( g_a \), but bearing in mind the ambiguity; another way, more natural in this context, is to count up the entries in the boxes in the strategic picture. The monoid of uniform block permutations of Example 2 first appears in [Fit03]. The picture of the Hasse diagram for the partition lattice \( P(4) \) in Figure 10 is based on one by Tilman Piesk [JB].

Representations. There are many books on group representation theory; we have followed the notation and style of [FH91, Part I]. In particular the approach is elementary, aka “module-free”. In this section the representations are over an arbitrary field \( k \); one moral to be extracted at the end is that in dealing with semigroup representations in characteristic \( p > 0 \), one needs to be just as careful, if not more careful, than one does in group representation theory. The emphasis thus moves to \( k = \mathbb{C} \) in later sections. Another omission is the theory of semigroup characters, which is well developed for the running examples.

The restriction to monoids (rather than semigroups) and monoid homomorphisms removes null representations from consideration – this makes a number of statements less cluttered.

The standard reference on reflection groups is [Hum90]. A finite reflection group (acting on a real vector space) can be boiled down to a very concise piece of combinatorial data called a Coxeter symbol. Starting from a Coxeter symbol one can construct a representation of the reflection group, called the reflectional representation; a fundamental result in the theory of reflection groups is that the reflectional representation is irreducible. Starting from the type \( A \) Coxeter symbol:

\[
\begin{align*}
    & s_1 \quad s_2 \quad s_{n-2} \quad s_{n-1} \\
    \quad & \quad \quad \quad \cdots \\
\end{align*}
\]

this process gives the reflectional representation of Example 4. The elementary argument showing that this is irreducible was supplied by Michael Bate.

Munn [Mun57b] extends the cycle notation for permutations in \( S_n \) to elements \( s \in I_n \) in the following neat way: for \( x \in [n] \), repeated application of \( s \) either results in a cycle: \( x, s(x), s^2(x), \ldots, s^{k+1}(x) = x \), in which case we write \( (x, s(x), \ldots, s^k(x)) \) as usual; or, \( s^k(x) \) is the first iteration of \( S \) that does not lie in the domain of \( s \), so that no more applications of \( s \) can be made. In this case we have a link \( [x, s(x), \ldots, s^k(x)] \). Any \( s \in I_n \) can then be written uniquely as a juxtaposition of disjoint cycles and links; the element \( [1, 2, 3] \in I_3 \) on the right of Figure 11 is an example. Reflection monoids appear in [EF10], where \( I_n \) is a Boolean monoid of type \( A \).

The formulation of semisimplicity suffers a little from the module-free approach, where it is cleaner to talk in terms of the semisimplicity of the semigroup algebra \( kS \). We have also avoided the notion of decomposability: the mapping representation of \( T_n \) is thus indecomposable but not irreducible, even in characteristic 0. One imagines that this is the aspect of the whole thing that group theorists find most distressing. Theorem 1 is standard – we have followed [Wei03].
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Theorem 6.1.15: Theorem similarly (see e.g. [Wei03, Theorem 3.1.14]); Theorem 3 is less well known, except to the cognoscenti; see [CP61, Ste16]. For \( I_n \) and \( T_n \) see also [GM09, Section 11.5].

**Interlude: the symmetric group.** A standard introductory text to all aspects of the representations of \( S_n \) is [Sag01]; for the Young tableau of this section we have followed [Ful97, Section 7.2]; see also [FH91, Chapter 4]. The irreducibles of \( S_n \) are more commonly called Specht modules rather than representations; as we are not mentioning modules, we hope the change of nomenclature is not too discombobulating. The representation \( S^t \) in Figure 16 is \( S_3 \) as the symmetries – obtained by permuting its three vertices – of the equilateral triangle. In general \( S^t \) is the representation of \( S_n \) acting as the symmetries of the regular \((n - 1)\)-simplex; it is another incarnation of the reflectional representation of \( S_n \) mentioned in Example 4 and in the notes to the previous section. The number of irreducible representations of \( S_n \) over \( \mathbb{C} \) is equal to the number \( p(n) \) of partitions \( \lambda \vdash n \); there is no known closed formula for \( p(n) \), but many weird and wonderful properties are known. To choose just one, there is the generating function

\[
\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}
\]

Exercise 2 is [FH91, Exercise 4.6].

**Reduction.** We have generally followed [GMS09]. The philosophy of the Clifford-Munn correspondence described in Section 5 is that knowledge of group representations yields knowledge of semigroup representations. The passage from groups to semigroups is the induction construction of Section 4. The current section is thus a little more perfunctory, as reduction – for us – is merely the inverse construction, its principal purpose being to establish the Clifford-Munn bijection. The usual terminology is “restriction” in much of the literature, but we have gone for reduction on two counts: it is first of all a double restriction – in that an action of \( S \) on \( V \) is being restricted to both a subgroup of \( S \) and a subspace of \( V \) – and secondly, reduction seems a more satisfying counterpoint to induction.

**Induction.** The section is based on [GM09, Chapter 11]. Like there we adopt an elementary approach; for example, in module-theoretic terms the representation \( U \) is \( kS \otimes_{kG} V \); the notation \( s_i \otimes v \) for the elements of the copy \( V_i \) of \( V \) is a nod to this. That the construction is independent of the transversal \( T \) is [GM09, Theorem 11.3.1(ii)]. The general picture is from [GMS09, Theorem 7]. The justification in Example 16 that \( U \) is an irreducible \( I_n \)-representation closely follows [GM09, Theorem 11.3.1].

**The Clifford-Munn correspondence.** Again we have followed [GM09, GMS09] for the general picture. The irreducibles of the symmetric inverse monoid in Example 18 are a venerable topic. Munn [Mun57b] took a character-theoretic approach while Grood [Gro02] constructed the “Specht” representations for \( I_n \) from scratch, and seemingly without reference to the Clifford-Munn correspondence. Our approach follows [Alb], where this and representations of other Boolean reflection monoids are described.

**A sexy example.** For algebraic groups and Weyl groups see [Hum75] and for algebraic monoids and Renner monoids, the books of Putcha and Renner [Put88, Ren05]. A beautiful expository article is [Sol95]; the example in this section is taken from [Sol95, Example 5.7].
The meaning of “sensible”, when talking about algebraic groups and monoids, depends on the context. If $\mathbb{M}$ is irreducible, meaning its underlying variety is irreducible, then the units $G$ are a connected algebraic group with $G = \mathbb{M}$. If $G_0$ is connected semisimple and the representation $f : G_0 \to GL(V)$ is rational with finite kernel, then we have the construction for $\mathbb{M} = \bar{k}^* f(G_0)$ described.

The Renner monoid of $M_n, k$ is isomorphic to the symmetric inverse monoid; in this incarnation, $I_n$ is called the Rook monoid and consists of the $n \times n$ matrices, with 0, 1-entries, such that each row and column contains at most one 1. The name comes about as the matrices can be identified with $n \times n$ chessboards, with rooks in the positions occupied by 1’s, and with the property that no two rooks are attacking each other. Warning: the Renner monoid is not in general a submonoid of $\mathbb{M}$, much as the Weyl group is not in general a subgroup of $G$; both $GL_n, k$ and $M_n, k$ are a little special in this way.

Good references for polytopes are [Grü03, Zie95], where one can also find the combinatorial description of the face polytope of a permutohedron in terms of ordered partitions.

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