GLOBAL EXISTENCE FOR A TWO-COMPONENT CAMASSA–HOLM SYSTEM WITH AN ARBITRARY SMOOTH FUNCTION

ZENG ZHANG
School of Science, Wuhan University of Technology
Wuhan 430070, China

ZHAOYANG YIN*
Department of Mathematics, Sun Yat-sen University
Guangzhou 510275, China
and
Faculty of Information Technology
Macau University of Science and Technology, Macau, China

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Abstract. This paper is concerned with a two-component integrable Camassa-Holm type system with arbitrary smooth function \( H \). If the function \( H \) belongs to a set \( \mathcal{H} \) (defined in Section 4), then we obtain the existence and uniqueness of global strong solutions and global weak solutions to the system. Our obtained results generalize and improve considerably recent results in [38, 39].

1. Introduction. In this paper, we consider the following two-component system proposed by Xia, Qiao and Zhou in [33]:

\[
\begin{align*}
\frac{\partial m}{\partial t} &= (mH)_x + mH - \frac{1}{2}m(u - u_x)(v + v_x), \\
\frac{\partial n}{\partial t} &= (nH)_x - nH + \frac{1}{2}n(u - u_x)(v + v_x), \\
m &= u - u_{xx}, \quad n = v - v_{xx},
\end{align*}
\]

(1)

where \( H \) is an arbitrary function of \( u, v \) and their derivatives. They proved that this system is integrable through its Lax pair and infinitely many conservation laws. By choosing suitable \( H \), they also investigated the bi-Hamiltonian structure and the interaction of multi-peakons.

Since \( H \) is an arbitrary function of \( u, v \) and their derivatives, (1) contains a large number of equations. For example, as \( v = 2 \) and \( H = -u \), (1) is reduced to the standard Camassa-Holm (CH) equation

\[
m_t + um_x + 2u_xm = 0, \quad m = u - u_{xx},
\]

(2)

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* Corresponding author: Zhaoyang Yin.
which was derived by Camassa and Holm [3] as a shallow water model. The CH equation has a bi-Hamiltonian structure [26, 21] and is completely integrable [3, 5]. The CH equation possesses peakon solutions of the form $Ce^{-|x-Ct|}$, which are orbitally stable [14, 15]. It is worth pointing out that the peakons, as solitary waves with a peak at the crest, model accurately the famous wave of greatest height pattern of the free-boundary incompressible Euler equations [6, 9, 7, 32]. The Cauchy problem and initial boundary problem of the CH equation has been established in [13, 34], the global conservative and dissipative conditions were discussed in [4, 8, 11, 17, 18], and blow-up solutions in finite time was studied in [16]. Global existence of weak solutions of the CH equation was established in [13, 34], the global conservative and dissipative solutions were studied in [1, 2, 24].

As $v = 2u$, $H = -(u^2 - u_x^2)$, (1) is reduced to the following cubic Camassa-Holm equation

$$m_t + (m(u^2 - u_x^2))_x = 0, \quad m = u - u_{xx},$$

which was proposed independently by Fokas [19], Fuchssteiner [22], Olver and Rosenau [29], and Qiao [30] as an integrable peakon equations with cubic nonlinearity. One can refer to [30, 20, 23] for the Lax pair, peakon and soliton solutions, local well-posedness and blow-up phenomena of (3).

If $H$ is any an arbitrary polynomial in $u, v$ and their derivatives, Zhang and Yin [37] proved the local well-posedness for (1) in Besov spaces; Hu and Qiao [25] studied the analyticity and the Gevrey regularity for (1). Recently, for some special choices of $H : H = a(uv_x - u_xv) + b(uv - u_xv_x), H = f(v + v_x) - g(u - u_x)$ with $f, g$ a one-order polynomial of $(u, v, u_x, v_x)$, blow-up phenomena were studied in [35, 36, 37, 39]. If $H = ax^2 + bx + c$, under some sign conditions, global strong solutions and global weak solutions were obtained under some sign conditions on the initial data [38]. It is not clear whether or not there exists global solutions to (1) with more general functions $H$.

In this paper, $H$ is a polynomial function of $(u, u_x, v, v_x)$. We aim to find out a function set $\mathcal{H}$, with $H \in \mathcal{H}$, the system (1) may have global strong solutions and global weak solutions. For the strong solutions, by taking advantage of the sign-preserving property and two conservation laws of (1), we observe that with a large class of $H$, defined by $\mathcal{H}$, we can control the $H^1$-norm of $(u, v)$ and the $L^\infty$-norm of $H$, and can get upper bounds for $H_x$ in finite time, under some suitable sign condition assumption on the initial data. Therefore, the $L^\infty$-norm of $(m, n)$ is bounded in finite time, and the solution $(m, n)$ exists globally in time. For the weak solutions, the main idea is based on the approximation of smooth global solutions $(m^k, n^k)$. Weak convergence for the high-order nonlinear terms, such as $m^kH$, can be obtained on account of the strong convergence for $u^k, u^k_x, v^k, v^k_x$. By a regularization technique, we then show that the obtained solution depends continuously on time in some sense. The uniqueness is proved by estimating the $W^{1, q}$-norm of the difference of two solutions.

The rest of our paper is then organized as follows. In Section 2, we study some properties about the strong solutions of (1). In Section 3, we establish the global existence of strong solutions of (1). In Section 4, we introduce the definition of the
2. Preliminaries. $H$ is a polynomial function of $(u, u_x, v, v_x)$. We begin with recalling the local well-posedness result and a blow-up criteria, a conservation law and the sign-preserving property for (1).

**Lemma 2.1.** [37] Let $(m_0, n_0) \in H^s$ with $s > \frac{1}{2}$, then there exists a time $T_1 > 0$ such that Eq.(1) has a unique solution $(m, n) \in C([0, T_1]; H^s) \cap C^1([0, T_1], H^{s-1})$. Moreover, the solution $(m, n)$ blows up in finite time $T$ if and only if

$$\limsup_{t \to T} (\|m(t)\|_{L^\infty} + \|n(t)\|_{L^\infty}) = \infty.$$  

**Lemma 2.2.** [38] Let $(m_0, n_0) \in H^s$ with $s > \frac{1}{2}$, and let $T$ be the maximal existence time of the corresponding solution $(m, n)$ to (1). Then for $t \in [0, T)$, we have

$$\int_{\mathbb{R}} m(v + v_x)dx = \int_{\mathbb{R}} n_0(0 + v_0_x)dx, \int_{\mathbb{R}} n(u - u_x)dx = \int_{\mathbb{R}} n_0(u_0 - u_0_x)dx.$$  

Consider the following initial value problem

$$\begin{cases}
q_t(t, x) = -H(q, t), & t \in [0, T), \\
q(0, x) = x, & x \in \mathbb{R}.
\end{cases}$$

Then it has a unique solution $q \in C^1([0, T] \times \mathbb{R}; \mathbb{R})$. Moreover, the mapping $q(t, \cdot)$ $(t \in [0, T])$ is an increasing diffeomorphism of $\mathbb{R}$, with $q_x(t, x) = \exp \left(-\int_0^t (H_x(\tau, q(\tau, x))d\tau \right) > 0$.

Furthermore, for any $t \in [0, T)$, we have

$$m(t, q(t, x))q_x(t, x) = m_0(x) \exp \left(\int_0^t [H - \frac{1}{2}(u - u_x)(v - v_x)](\tau, q(\tau, x))d\tau \right),$$  

(4)

$$n(t, q(t, x))q_x(t, x) = n_0(x) \exp \left(\int_0^t [-H + \frac{1}{2}(u - u_x)(v - v_x)](\tau, q(\tau, x))d\tau \right).$$  

(5)

We then recall a partial integration result for Bochner spaces.

**Lemma 2.3.** [28] Let $T > 0$. If $f, g \in L^2((0, T); H^1(\mathbb{R}))$ and $\frac{df}{dt}, \frac{dg}{dt} \in L^2((0, T); H^{-1}(\mathbb{R}))$,

then $f, g$ are a.e. equal to a function from $[0, T]$ into $L^2(\mathbb{R})$ and

$$\langle f(t), g(t) \rangle - \langle f(s), g(s) \rangle = \int_s^t \langle \frac{df(\tau)}{d\tau}, g(\tau) \rangle d\tau + \int_s^t \langle \frac{dg(\tau)}{d\tau}, f(\tau) \rangle d\tau$$

for all $s, t \in [0, T]$, where $\langle \cdot, \cdot \rangle$ is the $H^{-1}$ and $H^1$ duality bracket.

3. Global strong solutions. For simplicity, denote $G(t, x) = \frac{1}{2}(u - u_x)(v + v_x)$.

**Proposition 1.** Let $(m_0, n_0) \in H^s$ with $s > \frac{1}{2}$, and let $T$ be the maximal existence time of the corresponding solution $(m, n)$ to (1). If the polynomial $H(u, u_x, v, v_x)$ satisfies

(1) there exists a monotone increasing function $g(t)$ on $\mathbb{R}_+$, such that $t \in [0, T)$,

$$\|(H - G)(t)\|_{L^\infty} \leq g(t),$$

(2) there exists a monotone increasing function $f(t)$ on $\mathbb{R}_+$, such that $t \in [0, T)$,
\[ \sup_{x \in \mathbb{R}} H_x(t, x) \leq f(t), \]
then \( T = \infty \) and the solution \((m, n)\) exists globally in time.

**Proof.** If \( H \) satisfies the conditions (1) and (2), then we deduce from (4) in Lemma 2.2 that

\[
|m(t, q(t, x))| = |m_0(x)| \exp \left( \int_0^t (H_x(\tau, q(\tau, x)))d\tau \right) \exp \left( \int_0^t (H - G)(\tau, q(\tau, x)))d\tau \right) 
\leq \|m_0\|_{L^\infty} e^{f(t) t} e^{g(t) t}.
\]

Since the mapping \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \), we get

\[
\|m(t, \cdot)\|_{L^\infty} = \|m(t, q(t, \cdot))\|_{L^\infty} \leq \|m_0\|_{L^\infty} e^{f(t) t} e^{g(t) t}.
\]

Similar arguments show that \( \|n(t)\|_{L^\infty} \leq \|n_0\|_{L^\infty} e^{f(t) t} e^{g(t) t} \). According to the continuation criterion in Lemma 2.1, \((m, n)\) exists globally in time. \( \square \)

According to (4)-(5) in Lemma 2.2, the solution \((m, n)\) has sign-preserving property. The following lemma will be frequently used throughout the paper.

**Lemma 3.1.** Let \((m_0, n_0) \in H^s\) with \( s > \frac{1}{2} \), and \( m_0, n_0 \) do not change sign. Then for any \( t \in [0, T) \), \( m(t, x), u(t, x), (u \pm u_x)(t, x), n(t, x), v(t, x), (v \pm v_x)(x, t) \) do not change sign, and

\[
|u_x(t, x)| \leq |u(x, t)|, |v_x(t, x)| \leq |v(x, t)|
\]

\[
\|(u - u_x)(v + v_x)\|_{L^\infty} \leq \|m_0(v_0 + v_0x)\|_{L^1} + \|n_0(u_0 - u_0x)\|_{L^1}.
\]

**Proof.** We only consider the case \( m_0 \geq 0 \) since the others are similar. Applying (4), we have \( m(x, t) \geq 0 \). Noticing the following relations

\[
u(t, x) = (1 - \partial_x^2)^{-1} m(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} m(t, y) dy,
\]

\[
u_x(t, x) = -\frac{1}{2} \int_{\mathbb{R}} \text{sign}(x-y) e^{-|x-y|} m(t, y) dy,
\]

we have \( |u_x(t, x)| \leq u(x, t) \). Thus, \( (u \pm u_x)(t, x) \geq 0 \). By Lemma 2.2, we have

\[
|\left(u - u_x\right)(v + v_x)(t, x)| = \left| \int_{-\infty}^x \left|m(v + v_x) - n(u - u_x)\right|(t, y) dy \right|
\]

\[
\leq \left| \int_{-\infty}^x m(v + v_x) dy \right| + \left| \int_{-\infty}^x n(u - u_x) dy \right|
\]

\[
\leq \|m_0(v_0 + v_0x)\|_{L^1} + \|n_0(u_0 - u_0x)\|_{L^1}.
\]

\( \square \)

Next we present another conservation law for (1).

**Lemma 3.2.** Let \((m_0, n_0) \in H^s\) with \( s > \frac{1}{2} \). Then for \( t \in [0, T) \), we have

\[
\int_{\mathbb{R}} m(v + v_x)^2(u - u_x) dx = \int_{\mathbb{R}} m_0(v_0 + v_0x)^2(u_0 - u_0x) dx,
\]

\[
\int_{\mathbb{R}} n(u - u_x)^2(v + v_x) dx = \int_{\mathbb{R}} n_0(u_0 - u_0x)^2(v_0 + v_0x) dx.
\]
Proof. From (1), we obtain
\[ \partial_t(u - u_x) = Hm - (1 - \partial_x)(1 - \partial_{xx})^{-1}(Gm), \]
\[ \partial_t(v + v_x) = -Hn + (1 + \partial_x)(1 - \partial_{xx})^{-1}(Gn). \]
Integration by parts, we deduce that
\[ \frac{d}{dt} \int_R m(v + v_x)(u - u_x)(v + v_x)dx = \frac{d}{dt} \int_R n(u - u_x)(v + v_x)dx \]
\[ = 2 \int_R [(u - u_x)(u - u_x)(v + v_x)n + (v + v_x)n(u - u_x)(v + v_x)m]dx \]
\[ = 2 \int_R \left( HmGn - 2[(1 - \partial_x)(1 - \partial_{xx})^{-1}(Gm)]Gn \right. \]
\[ \left. - HnGm + 2[(1 + \partial_x)(1 - \partial_{xx})^{-1}(Gn)]Gm \right) dx = 0. \]

For simplicity, denote \( 2K_1 = ||m_0(v_0 + v_{0x})||_{L^1} + ||n_0(u_0 - u_{0x})||_{L^1}, K_2 = ||m_0(v_0 + v_{0x})^2(u_0 - u_{0x})||_{L^1} + ||n_0(u_0 - u_{0x})^2(v_0 + v_{0x})||_{L^1}, K_3 = ||u_0||_{H^1} + ||v_0||_{H^1}. \) If the initial data \((m_0, n_0)\) do not change sign, then we have the following global well-posedness result.

**Theorem 3.3.** Let \((m_0, n_0) \in H^s \) with \( s > \frac{1}{2} \), and \( m_0, n_0 \) do not change sign. Let \( T \) be the maximal existence time of the corresponding solution \((m, n)\) to (1). If for any \( t \in [0, T) \), the \( l \)-order polynomial \( H(u, u_x, v, v_x) \) satisfies
\[ (1') \sup_{x \in R} |H(t, x)| \leq C(t)(||u||_{H^1} + ||v||_{H^1} + 1); \]
\[ (2) \sup_{x \in R} |H(t, x)| \leq C(t), \]
where \( C(t) = C(t, K_1, K_2, K_3) \) depends on \( K_1, K_2, K_3 \) and is increasing with \( t \). Then \( ||(H - G)(t)||_{L^\infty} \leq C(C(t), t, l), \) and the solution exists globally in time. Moreover, if \((m_0, n_0) \in L^p, 1 \leq p \leq \infty, \) then for \( t \in [0, \infty), \)
\[ ||m(t)||_{L^p} + ||n(t)||_{L^p} \leq (||m_0||_{L^p} + ||n_0(t)||_{L^p})C(C(t), t, l). \]

Proof. From (1), we obtain
\[ \frac{1}{2} \frac{d}{dt} ||u||_{H^1}^2 = \int_R m_xudx = \int_R |m(u - u_x)H - Gmu|dx. \]
\[ \frac{1}{2} \frac{d}{dt} ||v||_{H^1}^2 = \int_R n_xvdx = \int_R [-n(v + v_x)H + Gnv]dx. \]
Since \( m_0, n_0 \) do not change sign, applying Lemma 3.1, we get
\[ \int_R -Gmu \leq ||G(t)||_{L^\infty} \int_R mudx \leq K_1 ||u||_{H^1}^2, \]
\[ \int_R Gnv \leq ||G(t)||_{L^\infty} \int_R nvdx \leq K_1 ||v||_{H^1}^2. \]
If \( H \) satisfies the condition \((1')\), then direct calculation gives
\[ \frac{1}{2} \frac{d}{dt} (||u||_{H^1}^2 + ||v||_{H^1}^2) \leq C(t)(||u||_{H^1}^2 + ||v||_{H^1}^2 + 1). \]
Then, for all \( t \in [0, T), \)
\[ ||u||_{H^1}^2 + ||v(t)||_{H^1}^2 \leq (||u_0||_{H^1}^2 + ||v_0||_{H^1}^2) e^{\int_0^t C(r)dr} = C(C(t), t). \]
Due to Lemma 3.1,
\[
\| u_x(t) \|_{L^\infty} + \| v_x(t) \|_{L^\infty} \leq \| u(t) \|_{L^\infty} + \| v(t) \|_{L^\infty}
\]
\[
\leq \| u(t) \|_{H^1} + \| v(t) \|_{H^1} \leq C(\alpha, t).
\]
Since \( H \) is \( l \)-order polynomial, we get
\[
\| H(t) \|_{L^\infty} \leq C(1 + \| u \|_{L^\infty} + \| u_x \|_{L^\infty} + \| v \|_{L^\infty} + \| v_x \|_{L^\infty})^l \leq C(\alpha, t, l).
\]
Hence, \( \|(H - G)(t)\|_{L^\infty} \leq C(\alpha, t, l) \). Combining this inequality with the condition (2) in Theorem 3.3, and by virtue of Proposition 1, we see that the solution \((m, n)\) exists globally.

Next, if \( p = \infty \), by the estimates for \( H_x \) and \( \| H - G \|_{L^\infty} \), we readily obtain from (4)-(5),
\[
\| m \|_{L^\infty} + \| n(t) \|_{L^\infty} \leq (\| m_0 \|_{L^\infty} + \| n_0(t) \|_{L^\infty}) C(\alpha, t, l).
\]
If \( 1 \leq p < \infty \), then by virtue of (1) and integration by parts, we have
\[
\frac{1}{p} \int \frac{d}{dt} \int f \, dx = \frac{1}{p} \int \int m^{p-1} H_x \, dx = \int \int [(1 - \frac{1}{p})m^p H_x \, dx + m^p(H - G)] \, dx
\]
\[
\leq C(\alpha, t, l) \int f \, dx.
\]
Applying Gronwall’s inequality then gives \( \| m \|_{L^p} \leq \| m_0 \|_{L^p} C(\alpha, t, l) \). Similar arguments shows \( \| n \|_{L^p} \leq \| n_0 \|_{L^p} C(\alpha, t, l) \). This completes the proof of the theorem. 

\[ \square \]

4. Examples. Define \( \mathcal{H} = \{ H \mid H \) is a polynomial of \((u, u_x, v, v_x)\) and satisfies (1') and (2) in Theorem 3.3 \}.

We mention that in Theorem 3.3, \( m_0, n_0 \) do not change sign, Lemma 3.1 can be used. We also mention that for some \( H \in \mathcal{H} \), additional initial condition such as \( m_0n_0 \leq 0 \) should also be added in Theorem 3.3. Then we claim that \( \mathcal{H} \) is not empty. In fact, it contains many functions. Here we list a few examples.

Case 1. \( H(v) = v^2 \)

Obviously, \( \int \! \! - n(v + v_x) \, v^2 \, dx \leq 0 \), we then deduce from (8) that \( \frac{d}{dt} \| v \|_{H^1}^2 \leq K_1 \| v \|_{H^1}^2 \). Hence, \( \| v_x \|_{L^\infty} \leq \| v \|_{L^\infty} \leq \| v(t) \|_{H^1} \leq C(t) \). We deduce that
\[
(1) \int \mathbb{R} \left[ m(u - u_x) - n(v + v_x) \right] v^2(t, x) \, dx \leq \| v \|_{L^\infty}^2 \int \mathbb{R} \left[ m(u - u_x) \right] (t, x) \, dx
\]
\[
\leq C(t) \| u \|_{H^1}^2;
\]
\[
(2)(v^2)_x(t, x) = 2vv_x(t, x) \leq C(t).
\]

Some deformations and generalizations of Case 1, for example, \( a(v + b)^2 + c \ (a, b, c \in \mathbb{R}, a > 0), v^{2k} \ (k = 0, 1, 2, \cdots), v^{2k+1} \ (k = 0, 1, 2, \cdots) \) with \( n_0 \geq 0, -u^2 \), also belong to the set \( \mathcal{H} \).

Case 2. \( H(v) = (v + v_x)^2 \)

First, similar arguments as that in Case 1, we get that \( \| v_x \|_{L^\infty} \leq \| v \|_{L^\infty} \leq \| v(t) \|_{H^1} \leq C(t) \). Then we have
\[
(1) \int \mathbb{R} \left[ m(u - u_x) - n(v + v_x) \right] (v + v_x)^2(t, x) \, dx \leq \| v - v_x \|_{L^\infty}^2 \int \mathbb{R} \left[ m(u - u_x) \right] (t, x) \, dx
\]
\[
\leq C(t) \| u \|_{H^1}^2;
\]
and
\[(2)((v + v_x)^2)_x = 2(v - v_x)(v_x + v - v + v_{xx}) \]
\[= 2(v - v_x)(v_x + v) - 2(v - v_x)n \leq 2(v - v_x)(v_x + v) \leq C(t).\]

Some deformations and generalizations of Case 2, for example, \(a(v + v_x + b)^2 + c\) \((a, b, c \in \mathbb{R}, a > 0), (v + v_x)^2k\) \((k = 0, 1, 2, \cdots), (v + v_x)^{2k+1}\) \((k = 0, 1, 2, \cdots)\) with \(n_0 \geq 0, -(u - u_x)^2\), also belong to the set \(H\).

**Case 3.** \(H(v) = u(v + v_x)\) with \(m_0n_0 \geq 0\)

Applying the conservation law in Lemma 2.2, we get
\[\int_{\mathbb{R}} [m(u - u_x) - n(v + v_x)]u(v + v_x)(t, x)\,dx \leq \int_{\mathbb{R}} [m(v + v_x)]\|u - u_x\|_{L^\infty}\|u\|_{L^\infty} \leq 2K_1\|u\|_{H^1}^2.\]

Thus, applying Gronwall’s lemma to (7)-(8) shows
\[\left(\|u_x\|_{L^\infty} + \|v\|_{L^\infty}\right)^2 \leq (\|u\|_{L^\infty} + \|v\|_{L^\infty})^2 \leq \|u\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \leq C(C(t), t).\]

Then we have
\[\int_{\mathbb{R}} [m(u - u_x)](v + v_x) - u_x(v - v_x) + u(v_x + v - n) \leq u_x(v - v_x) + u(v_x + v) \leq C(t).\]

Some deformations and generalizations of Case 3, for example, \((u + u_x)v, u(v + v_x)^2\) (use the conservation law in Lemma 2.2), \((u + u_x)(v + v_x)^2, u^2(v + v_x)^2, (u + u_x)^2(v + v_x)^2, (u + u_x)(v + v_x)^3\), with \(m_0n_0 \geq 0\) also belong to the set \(H\).

**Case 4.** \(H = (u - u_x)^{k_1}(v + v_x)^{k_2}\) (for any positive integer \(k_1, k_2\)) with \(\text{supp } m_0 \in [b, +\infty), \text{ supp } n_0 \in (-\infty, a], a \leq b\).

We infer from 4.5 in Lemma 2.2 that
\[m(t, q(t, x)) = m_0(x) \exp\left(\int_0^t [H - \frac{1}{2}(u - u_x)(v - v_x) + H_x](\tau, q(\tau, x))\,d\tau\right),\]
\[n(t, q(t, x)) = n_0(x) \exp\left(\int_0^t [-H - \frac{1}{2}(u - u_x)(v - v_x) + H_x](\tau, q(\tau, x))\,d\tau\right).\]

Since \(\text{supp } m_0 \in [b, +\infty), \text{ supp } n_0 \in (-\infty, a]\), and noticing the mapping \(q(t, \cdot)\) is an increasing diffeomorphism of \(\mathbb{R}\), we get
\[m(t, y) = 0, \text{ if } y < q(t, b), \text{ } n(t, y) = 0, \text{ if } y > q(t, a).\]

Hence,
\[u(t, x) - u_x(t, x) = e^{-x} \int_{-\infty}^x e^y m(t, y)\,dy = 0, \text{ if } x \leq q(t, b),\]
\[v(t, x) + v_x(t, x) = e^{x} \int_{-\infty}^x e^{-y} n(t, y)\,dy = 0, \text{ if } x \geq q(t, a).\]

Since \(a \leq b\), we readily get \(q(t, a) \leq q(t, b)\). We then deduce that \(H = (u - u_x)^{k_1}(v + v_x)^{k_2} = 0\) on \(\mathbb{R}\). (1’) and (2) is obviously satisfied. Similarly, \(H = (u + u_x)^{k_1}(v - v_x)^{k_2}\) (for any positive integer \(k_1, k_2\)) with \(\text{supp } m_0 \in (-\infty, a), \text{ supp } n_0 \in [b, +\infty), a \leq b\), also belongs to the set \(H\).

**Remark 1.** We can verify that (i) \(H = av^2 + bv + c\) with \(a > 0\), and (ii) \(H = bv + c\) with \(b \geq 0, n_0 \geq 0\) or \(b \leq 0, n_0 \leq 0\), which were studied in [37], belong to the set \(H\) (refer to Case 1). Therefore, our obtained global result in Theorem 3.3, to a large extent, generalizes and covers the recent global result of Theorem 3.8 in [37].
5. Global weak solutions.

**Theorem 5.1.** Let \((m_0, n_0) \in L^p\) for some \(1 < p \leq 2\), \(m_0, n_0\) do not change sign, and \(H \in \mathcal{H}\) (See Section 3). Then the system (1) has a global solution \((m, n) \in C_w(0, T; L^p)\) for arbitrary time \(T\). Moreover \((u, v) = (1 - \partial_{xx})^{-1}(m, n) \in C_w(0, T; W^{1, s}) \cap C^1(0, T; L^s) \cap C(0, T; H^1)\) with \(p \leq s \leq \infty\) for arbitrary time \(T\).

**Proof.** **Step 1.** Denote \(\rho(x)\) as a mollification kernel, and \(\rho_k(x) = k \rho(kx)\). Define \((m^k_0, n^k_0) = (\rho_k \ast m_0, \rho_k \ast n_0)\). Then \((m^k_0, n^k_0) \in H^\infty\) do not change sign, and \((m^k_0, n^k_0) \to (m_0, n_0)\) in \(L^p\) as \(k \to \infty\). By Theorem 1, there exists a global strong solution \((m^k, n^k)\) of (1) with the initial data \((m^k_0, n^k_0)\). Moreover, \((m^k, n^k) \in C^1([0, \infty); H^\infty(\mathbb{R}))\).

**Step 2.** Denote \(p' : \frac{1}{p} + \frac{1}{p'} = 1\). Note that \((1 - \partial_{xx})^{-1}f = \frac{1}{2}e^{-|x|} \ast f\). Applying Young’s inequality for the convolution of two functions, we deduce that, for \(s \in [p, \infty]\), \(1 + \frac{1}{s} = \frac{1}{p} + \frac{1}{p'}\),

\[
\|u^k_0\|_{L^s} + \|u^k_{0x}\|_{L^s} = \left\|\left(\frac{1}{2}e^{-|x|} + \frac{1}{2}\text{sign}(x)e^{-|x|}\right) \ast m^k_0\right\|_{L^s} \leq C\left(\|e^{-|x|}\|_{L^s} + \|\text{sign}(x)e^{-|x|}\|_{L^s}\right)\|m^k_0\|_{L^p}.
\]

Hence, if \(1 < p \leq 2\), then \(p' \geq 2\), we have

\[
2K^k_1 = \|m^k_0(v^k \ast v^k_{0x})\|_{L^1} + \|n^k_0(u^k_0 \ast u^k_{0x})\|_{L^1} \leq (\|m^k_0\|_{L^p} + \|n^k_0\|_{L^p})(\|v^k \ast v^k_{0x}\|_{L^{p'}} + \|u^k_0 \ast u^k_{0x}\|_{L^{p'}}) \leq C(\|m_0\|_{L^p} + \|n_0\|_{L^p})^2 \leq C(\|m_0\|_{L^p}, \|n_0\|_{L^p}).
\]

Similarly, we have \(K^k_2, K^k_3 \leq C(\|m_0\|_{L^p}, \|n_0\|_{L^p})\). Then, for fixed \(T > 0\), By (6) in Theorem 1, we find that \((m^k, n^k)\) is bounded in the space \(L^\infty(0, T; L^p(\mathbb{R}))\). Note that

\[
\begin{align*}
(u^k, v^k) &= (1 - \partial_{xx})^{-1}(m^k, n^k), \\
(u^k_x, v^k_x) &= \partial_x(1 - \partial_{xx})^{-1}(m^k, n^k), \\
m^k_t &= (m^k H^k) + n^k(H^k - G^k), \\
n^k_t &= (n^k H^k) - n^k(H^k - G^k), \\
u^k_t &= \partial_x(1 - \partial_{xx})^{-1}(m^k H^k) + (1 - \partial_{xx})^{-1}(m^k(H^k - G^k)), \\
v^k_t &= \partial_x(1 - \partial_{xx})^{-1}(m^k H^k) - (1 - \partial_{xx})^{-1}(m^k(H^k - G^k)),
\end{align*}
\]

where \(H^k = H(u^k, u^k_x, v^k, v^k_x)\), \(G^k = \frac{1}{2}(u^k - u^k_x)(v^k + v^k_x)\). Applying Youngs inequality for the convolution of two functions, we obtain

(*) \((u^k, v^k) \to (m, n)\) a.e. on \((0, T) \times \mathbb{R}\), as \(k \to \infty\),

and \((n^k_t, m^k_t)\) are bounded in \(L^\infty(0, T; W^{-1, p})\). Thus, it has a subsequence still denote by \(k\), such that \((m^k, n^k) \to (m, n)\) in \(C_w([0, T], L^p)\). Taking \(\frac{1}{2}e^{-|x|} \in L^p(\mathbb{R})\) and \(\frac{1}{2}\text{sign}(x - \cdot)e^{-|x|} \in L^p(\mathbb{R})\) as a test function, we obtain

\[
\begin{align*}
(u^k, v^k) \to (1 - \partial_{xx})^{-1}(m, n) = (u, v) \text{ a.e. on } (0, T) \times \mathbb{R}, \text{as } k \to \infty, \\
(u^k_x, v^k_x) \to \partial_x(1 - \partial_{xx})^{-1}(m, n) = (u_x, v_x) \text{ a.e. on } (0, T) \times \mathbb{R}, \text{as } k \to \infty.
\end{align*}
\]

By (*), we obtain \((u, v) \in C_w(0, T; W^{1, s}) \cap C^1(0, T; L^s)\) with \(p \leq s \leq \infty\) for arbitrary time \(T\). Further more, by using the monotone convergence theorem, we have for all \(p \leq s < \infty\),

\[
(u^k, v^k) \to (u, v) \text{ in } L^{s_1}_{loc}(\mathbb{R}^+ \times \mathbb{R}), (u^k_x, v^k_x) \to (u, v) \text{ in } L^{s_1}_{loc}(\mathbb{R}^+ \times \mathbb{R}).
\]
Step 3. Since $H = H(u, u_x, v, v_x)$ and $G = G(u, u_x, v, v_x)$ are polynomials of $(u, u_x, v, v_x)$, we readily obtain $H^k \to H \in L^1_{lo} (\mathbb{R}^+ \times \mathbb{R})$, $G^k \to G \in L^1_{lo} (\mathbb{R}^+ \times \mathbb{R})$ for all $s_1 \in [p, \infty)$. Thus, product items, such as $M^k H^k$, can be well handled, we then claim that
\begin{align*}
\begin{cases}
m_t = (mH)_x + m(H - G), \\
n_t = (nH)_x - n(H - G), \\
u_t = \partial_x(1 - \partial_x)^{-1}(mH) + (1 - \partial_x)^{-1}(m(H - G)), \\
u_v = \partial_x(1 - \partial_x)^{-1}(nH) - (1 - \partial_x)^{-1}(n(H - G)), \\
u_{xt} = -mH + (1 - \partial_x)^{-1}(mH) + \partial_x(1 - \partial_x)^{-1}(m(H - G)), \\
v_{xt} = -nH + (1 - \partial_x)^{-1}(nH) - \partial_x(1 - \partial_x)^{-1}(n(H - G)),
\end{cases}
\end{align*}
(9)
are satisfied in the sense of distributions.

Step 4. We show that $(u, v) \in C([0, T], H^1)$. Since $(u, v) \in C_w([0, T], H^1)$, it is enough to show $(\|u\|_{H^1}, \|v\|_{H^1})$ is continuous on $\mathbb{R}^+$. As $(m, n)$ solves (9) in the sense of distribution, we see for a.e. $t \in \mathbb{R}^+$,
\begin{align*}
\rho_k * u_t &= \rho_k * \partial_x(1 - \partial_x)^{-1}(mH) + \rho_k * (1 - \partial_x)^{-1}(m(H - G)), \\
\rho_k * u_{xt} &= -\rho_k * (mH) + \rho_k * (1 - \partial_x)^{-1}(mH) + \rho_k * \partial_x(1 - \partial_x)^{-1}(m(H - G)), \\
\text{with } \rho_k (x) &= k p(kx) \text{ the mollifier defined in Step 1. Multiplying with } \rho_k * u \text{ and } \\
\rho_k * u_x \text{ respectively, and in view of Lemma 2.3, we have for a.e. } t \in \mathbb{R}^+,
\end{align*}
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\rho_k * u)^2 dx &= \int_{\mathbb{R}} (\rho_k * u)(\rho_k * \partial_x(1 - \partial_x)^{-1}(mH)) \\
&\quad + (\rho_k * u)(\rho_k * (1 - \partial_x)^{-1}(m(H - G))) dx, \\
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\rho_k * u_x)^2 dx &= \int_{\mathbb{R}} -(\rho_k * u)(\rho_k * (mH)) + (\rho_k * u_x)(\rho_k * (1 - \partial_x)^{-1}(mH)) \\
&\quad + (\rho_k * u_x)(\rho_k * \partial_x(1 - \partial_x)^{-1}(m(H - G))) dx.
\end{align*}
Note that $(m, n) \in L^\infty(0, T; L^p) \cap L^\infty(0, T; L^1)$ with $p \leq s \leq \infty$, and $\|\rho_k * f\|_{L^s} \to \|f\|_{L^s}$ as $k \to \infty$ with $1 \leq q < \infty$. So letting $k \to \infty$, we have
\begin{align*}
(\|u(t)\|_{L^2} + \|u_x(t)\|_{L^2}) - (\|u(0)\|_{L^2} + \|u_x(0)\|_{L^2}) = 2 \int_0^t F(\tau) d\tau,
\end{align*}
where
\begin{align*}
F(\tau) &= \int_{\mathbb{R}} u_\tau(1 - \partial_x)^{-1}(mH) + u(1 - \partial_x)^{-1}(m(H - G)) \\
&\quad - u_x mH + u_x(1 - \partial_x)^{-1}(mH) + u_x \partial_x(1 - \partial_x)^{-1}(m(H - G)) dx \in L^\infty([0, T].
\end{align*}
This proves $u \in C([0, T]; H^1)$. Likewise, $v \in C([0, T]; H^1)$. This completes the proof of the theorem.

For the uniqueness, we estimate the $W^{1,q}$ $(q \leq p < \infty)$ norm of the difference of two solutions. We have the following theorem.

**Theorem 5.2.** Let $(m_0, n_0) \in L^p \cap L^\infty$ for some $1 < p \leq 2$, $m_0, n_0$ do not change sign, and $H \in \mathcal{H}$ (See Section 4). Then the system (1) has a unique global solution $(m, n) \in C_w(0, T; L^p) \cap L^\infty(0, T; L^\infty)$ for arbitrary time $T$. Moreover $(u, v) = (1 - \partial_x)^{-1}(m, n) \in C_w(0, T; W^{1,s}) \cap C^1(0, T; L^s) \cap C(0, T; H^1)$ with $p \leq s \leq \infty$ for arbitrary time $T$. 


Proof. Following exactly the same line as in Theorem 5.1, and by using (6) in Theorem 3.3 for the approximation of smooth solutions \((m^k, n^k)\), the global weak solution \((m, n)\) obtained in Theorem 5.1 also belongs to the space \(L^\infty(0, T; L^\infty)\). This completes the existence part.

For the uniqueness part, assume \((m_i, n_i)\) \((i = 1, 2)\) be two weak solutions of (1) with \((m_i, n_i) \in C_w(0, T; L^p) \cap L^\infty(0, T; L^\infty)\) and \((u_i, v_i) = (1 - \partial_{xx})^{-1}(m_i, n_i) \in C_w(0, T; W^{1,s}) \cap C^1(0, T; L^s) \cap C(0, T; H^1)\) with \(p \leq s \leq \infty\) for arbitrary time \(T\). For fixed time \(T > 0\), let

\[
B(T) = \sup_{i=1,2} \|m_i\|_{L^\infty(0, T; L^p \cap L^\infty)} + \|n_i\|_{L^\infty(0, T; L^p \cap L^\infty)}.
\]

(10)

By Young’s inequality, we have,

\[
\|u_t\|_{L^\infty(0, T; W^{1, p} \cap W^{1, \infty})} + \|u_t\|_{L^\infty(0, T; W^{1, s} \cap W^{1, \infty})} \leq C\left(\|m_i\|_{L^\infty(0, T; L^p \cap L^\infty)} + \|n_i\|_{L^\infty(0, T; L^p \cap L^\infty)}\right) \leq CB(T).
\]

(11)

Define \(H_i = H(u_i, v_i, u_{ix}, v_{ix})\), \(G_i = \frac{1}{2}(u_i - u_{ix})(v_i + v_{ix})\), \(\hat{m} = m_1 - m_2\), \(\hat{n} = n_1 - n_2\), \(\hat{u} = u_1 - u_2\), \(\hat{v} = v_1 - v_2\), \(\hat{H} = H_1 - H_2\), \(\hat{G} = G_1 - G_2\). Define \(P(x) = e^{-|x|}\), we have \((1 - \partial_{xx})^{-1}f = P \ast f\), and \(\partial_x (1 - \partial_{xx})^{-1}f = P_x \ast f\). From (1), we find that

\[
\rho_k \ast \hat{u}_t = P \ast P \ast (\hat{m}H_1 + m_2\hat{H}) + P \ast P \ast (\hat{n}H_1 - G_1) + P \ast P \ast (\hat{u}_xH_1 - G_1)
\]

In the same way, we get

\[
\rho_k \ast \hat{u}_{xt} = P \ast P \ast (\hat{m}H_1 + m_2\hat{H}) + P \ast P \ast (\hat{n}H_1 + m_2\hat{H})
\]

\[
\rho_k \ast \hat{u}_x = P \ast P \ast (\hat{m}H_1 + m_2\hat{H}) + P \ast P \ast (\hat{n}H_1 - G_1)
\]

\[
\rho_k \ast \hat{u} = P \ast P \ast (\hat{m}H_1 + m_2\hat{H}) + P \ast P \ast (\hat{n}H_1 - G_1)
\]

Applying Lemma 2.3 to \(\rho_k \ast \hat{u}\) and \(\rho_k \ast \hat{u}_x\), we find, for \(p \leq q < \infty\), and for a.e. \(t \in (0, T)\),

\[
\frac{1}{q} \frac{d}{dt} \int \rho_k \ast \hat{u}^q dx = \int \rho_k \ast \hat{u}_t |\rho_k \ast \hat{u}|^{q-1} \text{sign}(\rho_k \ast \hat{u}) dx,
\]

\[
\frac{1}{q} \frac{d}{dt} \int \rho_k \ast \hat{u}_x^q dx = \int \rho_k \ast \hat{u}_{xt} |\rho_k \ast \hat{u}_x|^{q-1} \text{sign}(\rho_k \ast \hat{u}_x) dx.
\]
Similarly,
\[
\frac{1}{q} \frac{d}{dt} \int_R |\rho_k * \tilde{v}|^q dx = \int_R \rho_k * \tilde{v}_t |\rho_k * \tilde{v}|^{q-1} \text{sign}(\rho_k * \tilde{v}) dx,
\]
\[
\frac{1}{q} \frac{d}{dt} \int_R |\rho_k * \tilde{v}_x|^q dx = \int_R \rho_k * \tilde{v}_xt |\rho_k * \tilde{v}_x|^{q-1} \text{sign}(\rho_k * \tilde{v}_x) dx.
\]
Denote \(X_k(t) = \|\rho_k * \tilde{u}(t)\|_{L^q}^q + \|\rho_k * \tilde{u}_x(t)\|_{L^q}^q + \|\rho_k * \tilde{v}(t)\|_{L^q}^q + \|\rho_k * \tilde{v}_x(t)\|_{L^q}^q\). We claim that
\[
\frac{d}{dt} X_k(t) \leq C(T) X_k(t) + R_k(t),
\]
where
\[
R_k(t) \to 0 \text{ as } k \to \infty, \text{ and } |R_k(t)| \leq C(T) \text{ for all } k \geq 1.
\]
In order to prove (12), we only give details below for special two terms which are underlined above, the other terms can be dealt with analogously.
\[
\int_R P_x * \rho_k * (m_2 \tilde{H}) |\rho_k * \tilde{u}|^{q-1} \text{sign}(\rho_k * \tilde{u}) dx
\]
\[
= \int_R P_x * ((\rho_k * m_2)(\rho_k * \tilde{H})) |\rho_k * \tilde{u}|^{q-1} \text{sign}(\rho_k * \tilde{u}) dx + R_{1k}(t),
\]
where
\[
|R_{1k}(t)| = \int_R P_x * ((\rho_k * (m_2 \tilde{H}) - (\rho_k * m_2)(\rho_k * \tilde{H})) |\rho_k * \tilde{u}_x|^{q-1} \text{sign}(\rho_k * \tilde{u}_x) dx|
\]
\[
\leq C \|(\rho_k * (m_2 \tilde{H}) - (\rho_k * m_2)(\rho_k * \tilde{H})\|_{L^s} \|\rho_k * \tilde{u}_x\|_{L^q}^{q-1}
\]
\[
\leq \|m_2\|_{L^2} \|\tilde{H}\|_{L^2} \|\tilde{u}_x\|_{L^s}^{q-1} \leq C(T).
\]
Above we have used (10)-(11). Since \(\|\rho_k * f - f\|_{L^{s}} \to 0\) as \(k \to \infty\) with \(1 \leq s < \infty\), we have
\[
\|(\rho_k * (m_2 \tilde{H}) - (\rho_k * m_2)(\rho_k * \tilde{H})\|_{L^s}
\]
\[
\leq \|(\rho_k * (m_2 \tilde{H}) - m_2 \tilde{H})\|_{L^s} + \|m_2 \tilde{H} - (\rho_k * m_2)(\rho_k * \tilde{H})\|_{L^s}
\]
\[
+ \|(\rho_k * m_2)\tilde{H} - (\rho_k * m_2)(\rho_k * \tilde{H})\|_{L^s} \to 0 \text{ as } k \to \infty.
\]
We then readily have \(R_{1k}(t) \to 0\) as \(k \to \infty\). Thus, \(R_k(t)\) belongs to the class (13).
We now deal with the first term on the right hand side of (14). For simplicity, we concentrate on the case \(H = v^2\), then \(\tilde{H} = (v_1 + v_2)\tilde{v}\). For the other choice of \(H\) in the set \(\mathcal{H}\), one can follow the similar steps to do with and can have the same conclusions.
\[
\int_R P_x * ((\rho_k * m_2)(\rho_k * \tilde{H})) |\rho_k * \tilde{u}|^{q-1} \text{sign}(\rho_k * \tilde{u}) dx
\]
\[
= \int_R P_x * ((\rho_k * m_2)(\rho_k * (v_1 + v_2))(\rho_k * \tilde{v})) |\rho_k * \tilde{u}|^{q-1} \text{sign}(\rho_k * \tilde{u}) dx + R_{2k}(t)
\]
\[
\leq C \|m_2\|_{L^\infty(0,T;L^\infty)} \|v_1 + v_2\|_{L^\infty(0,T;L^\infty)} \|\rho_k * \tilde{u}\|_{L^s} \|\rho_k * \tilde{u}_x\|_{L^q}^{q-1} + R_{2k}(t)
\]
\[
\leq C(T) X_k(t) + R_{2k}(t),
\]
where
\[
R_{2k}(t) = \int_R P_x * [(\rho_k * m_2)(\rho_k * ((v_1 + v_2)\tilde{v}))
\]
\[- ((\rho_k * m_2)(\rho_k * (v_1 + v_2))(\rho_k * \tilde{v})) |\rho_k * \tilde{u}_x|^{q-1} \text{sign}(\rho_k * \tilde{u}_x) dx
\]
belongs to the class (13) (the same argument as for $R_{1k}(t)$). From the previous estimate, we infer that
\[
\int_{\mathbb{R}} P_x \ast \rho_k \ast (m_2 \bar{H}|\rho_k \ast \tilde{u}|^{q-1}\text{sign}(\rho_k \ast \tilde{u})dx \leq C(T)X_k(t) + R_{1k}(t) + R_{2k}(t)
\]
with $R_{1k}(t) + R_{2k}(t)$ belonging the class (13). For the term $\rho_k \ast (\tilde{u}_{xx}H_1)$, we have
\[
\int_{\mathbb{R}} \rho_k^*(\tilde{u}_{xx}H_1)|\rho_k \ast \tilde{u}_x|^{q-1}\text{sign}(\rho_k \ast \tilde{u}_x)dx
\]
\[
= \int_{\mathbb{R}} (\rho_k \ast \tilde{u}_{xx})(\rho_k \ast H_1)|\rho_k \ast \tilde{u}_x|^{q-1}\text{sign}(\rho_k \ast \tilde{u}_x)dx + R_{3k}(t)
\]
\[
= \frac{1}{q} \int_{\mathbb{R}} \frac{d}{dx}|\rho_k \ast \tilde{u}_x|^{q}(\rho_k \ast H_1)dx + R_{3k}(t)
\]
\[
= -\frac{1}{q} \int_{\mathbb{R}} |\rho_k \ast \tilde{u}_x|^{q}(\rho_k \ast H_{1x})dx + R_{3k}(t)
\]
\[
\leq C\|H_{1x}\|_{L^{\infty}(0,T;L^\infty)} \|\rho_k \ast \tilde{u}\|_{L^\infty}^q + R_{3k}(t)
\]
\[
\leq C(T)X_k(t) + R_{3k}(t),
\]
where we have used
\[
\|H_{1x}\|_{L^\infty} \leq C\left(\|u_1\|_{L^\infty}, \|u_{1x}\|_{L^\infty}, \|m_1\|_{L^\infty}, \|v_1\|_{L^\infty}, \|v_{1x}\|_{L^\infty}, \|m_2\|_{L^\infty}\right),
\]
and the estimates (10)-(11). Above,
\[
R_{3k}(t) = \int_{\mathbb{R}} (\rho_k \ast (\tilde{u}_{xx}H_1) - (\rho_k \ast \tilde{u}_{xx})(\rho_k \ast H_1))|\rho_k \ast \tilde{u}_x|^{q-1}\text{sign}(\rho_k \ast \tilde{u}_x)dx
\]
begins to the class (13). Hence, (12)-(13) hold true. Applying Gronwall’s inequality, we have for $0 \leq t \leq T$,
\[
X_k(t) \leq e^{C(T)\tau}X_k(0) + \int_0^T e^{-C(T)\tau}R_k(\tau)d\tau.
\]
Letting $k \to \infty$, we obtain by the Lebesgue’s dominates convergence theorem that
\[
X(t) = \|	ilde{u}(t)\|_{L^q} + \|	ilde{u}_x(t)\|_{L^q} + \|\tilde{v}(t)\|_{L^q} + \|\tilde{v}_x(t)\|_{L^q}.
\]
Uniqueness is thus obtained. \(\square\)

**Remark 2.** For the uniqueness, the addition initial condition $(m_0, n_0) \in L^\infty$ was required in Theorem 5.2. Under this condition, we get $(m, n) \in L^\infty(0,T;L^\infty)$, therefore, for a general polynomial function $H \in \mathcal{H}$, $\|H_{1x}\|_{L^\infty(0,T;L^\infty)}$ can be bounded in (15). This addition condition is technical.

**Remark 3.** It is worth mentioning that if $H = H(v)$ (or $H = H(u)$) $\in \mathcal{H}$, the addition initial condition $(m_0, n_0) \in L^\infty$ is unnecessary to obtain the uniqueness of weak solutions. It is because $\|H_{1x}\|_{L^\infty(0,T;L^\infty)}$ can be controlled by $\|v_1\|_{L^\infty(0,T;L^\infty)} + \|v_{1x}\|_{L^\infty(0,T;L^\infty)}$, which is already bounded according to (11).

**Remark 4.** For the case, (i) $H = av^2 + bv + c$ with $a \geq 0$, and (ii) $H = bv + c$ with $b \geq 0, n_0 \geq 0$ or $b \leq 0, n_0 \leq 0$, $(m_0, n_0) \in L^\infty$ is unnecessary. Moreover, $(m_0, n_0) \in L^1 \cap L^p$ for some $1 < p < \infty$ implies $(m_0, n_0) \in L^p$ for some $1 < p \leq 2$, due to the embedding inequality $L^1 \cap L^p \hookrightarrow L^p$ if $1 < p \leq 2$, and $L^1 \cap L^p \hookrightarrow L^q$, $1 < q \leq 2$ if $p > 2$. Therefore, our obtained results in Theorem 5.1-Theorem 5.2,
to a large extent, generalize and cover the recent result-Theorem 3.1 in [38], which requires \((m_0, n_0) \in L^1 \cap L^p\) for some \(1 < p < \infty\).

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E-mail address: zhangzeng5345340163.com
E-mail address: mcsyzy@mail.sysu.edu.cn