ON FOUR-DIMENSIONAL STEADY GRADIENT RICCI SOLITONS THAT DIMENSION REDUCE

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Abstract. In this paper, we will study the asymptotic geometry of 4-dimensional steady gradient Ricci solitons under the condition that they dimension reduce to 3-manifolds. We will show that such solitons either strongly dimension reduce to a spherical space form $S^3/\Gamma$ or weakly dimension reduce to the 3-dimensional Bryant soliton. We also show that 4-dimensional steady gradient Ricci soliton singularity models with nonnegative Ricci curvature outside a compact set either are Ricci-flat ALE 4-manifolds or dimension reduce to 3-dimensional manifolds. As a further application, we prove that any steady gradient Kähler-Ricci soliton singularity models on complex surfaces with nonnegative Ricci curvature outside a compact set must be hyperkähler ALE 4-manifolds.

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1. Introduction

Steady gradient Ricci solitons arise as singularity models in the case of Type II singular solutions of the Ricci flow \[37\]. In dimension 4, Ricci flow singularity formation may be quite complicated, with a given forming singularity possibly having different associated singularity models at different curvature scales.

For example, Appleton’s work \[4\] proves the existence of 4-dimensional Ricci flow singularity formation where the associated singularity model at the highest curvature scale is the Eguchi–Hanson Ricci flat ALE, which is a steady soliton with trivial potential function, and where at a lower scale the model is either flat \(\mathbb{R}^4/\mathbb{Z}_2\) or the \(\mathbb{Z}_2\)-quotient of the Bryant soliton, which is a steady soliton with nontrivial potential function.

By Bamler’s recent works \[5, 6, 7\], which solved a version of a conjecture of Hamilton, there is a definite sense in which most singularity models in all dimensions are shrinking gradient Ricci solitons. In particular, Bamler’s theory proves that for any forming singularity one always has that at the appropriate (parabolic) scale and approach to the singularity the associated singularity model is a shrinking gradient Ricci soliton or a Ricci flat cone.

In dimension 4, steady Ricci solitons may also be relevant to the study of shrinking gradient Ricci solitons with quadratic curvature growth (which is the maximum possible growth) via a limit argument; see Proposition 10 in \[26\]. The reason why steady and shrinking gradient solitons may be related to each other in dimension 4 is that the shrinking soliton equation \(\text{Ric} - \text{Hess} f = \frac{1}{2} g\) degenerates to \(\text{Ric} = \text{Hess} f\) as a limit of dilations provided convergence holds, which is true in the quadratic curvature growth case. Bamler has asked the question of whether 4-dimensional quadratic curvature growth shrinking solitons analogous to flying wings may exist.

Throughout this paper, we will use the following notations. A triple \((M^n, g, f)\) of a smooth manifold, a complete Riemannian metric, and a function is an \(n\)-dimensional steady gradient Ricci soliton, \(\{\phi_t\}_{t \in (-\infty, \infty)}\) is the 1-parameter group of diffeomorphisms generated by \(-\nabla f\), and \(g(t) = \phi_t^* g\). By definition, the Ricci curvature \(\text{Ric}\) of \(g\) on \(M\) satisfies

\[
\text{Ric} = \text{Hess} f. \tag{1.1}
\]

Defining \(f(t) = f \circ \phi_t\), we have an eternal solution to the Ricci flow:

\[
\frac{\partial}{\partial t} g(t) = -2 \text{Ric}_g(t) = -2 \text{Hess}_g(t) f(t). \tag{1.2}
\]

Regarding the qualitative study of steady gradient Ricci solitons with relatively mild conditions on curvature, save positivity, there are a number
of important results, including [12], [13], [21], [27], [31], [33], [38], [46], [47], [48], and [49].

In this paper we will study the asymptotic geometry of 4-dimensional steady gradient Ricci solitons under the assumption that they “dimension reduce” to 3-manifolds (see below for definitions). We also prove dimension reduction must hold only assuming that the Ricci curvature is nonnegative outside a compact set. We first introduce the definition of dimension reduction on steady gradient Ricci solitons.

Definition 1.1. We say that \((M^n, g, f)\) **dimension reduces to** \((n - 1)\)-**manifolds** if for any sequence \(\{p_i\}_{i \in \mathbb{N}}\) tending to infinity, a subsequence of \((M, K_i g(K_i^{-1} t), p_i)\) converges to \((N^{n-1} \times \mathbb{R}, g_N(t) + ds^2, p_{\infty})\) in the \(C^\infty\) pointed Cheeger–Gromov sense, where \((N, g_N(t)), t \in (-\infty, 0]\), is an \((n - 1)\)-dimensional complete ancient Ricci flow with bounded curvature and where \(K_i = |\text{Rm}(p_i)| > 0\). In this definition, \((N, g_N(t))\) may depend on the choice of the base points \(\{p_i\}\) and the subsequence. We call any such \((N, g_N(t))\) a **dimension reduction** of \((M, g, f)\).

We say \((M^n, g, f)\) **strongly dimension reduces to** \((N^{n-1}, g_N(t))\) provided that \((M, g, f)\) dimension reduces to \((n - 1)\)-manifolds, where the dimension reduction of \((M, g, f)\) is always \((N, g_N(t))\) and hence is independent of the choice of \(\{p_i\}\) and subsequence.

Definition 1.2. We say that \((M^n, g, f)\) **weakly dimension reduces to** \((N^{n-1}, g_N(t))\) if there exists a sequence of points \(\{p_i\}_{i \in \mathbb{N}}\) tending to infinity such that \((M, K_i g(K_i^{-1} t), p_i)\) converges to \((N \times \mathbb{R}, g_N(t) + ds^2, p_{\infty})\), where \(K_i = |\text{Rm}(p_i)| > 0\).

Observe that in the definitions above, \((N, g_N(t))\) is always nonflat since \(\text{Rm}(p_{\infty}, 0) = 1\).

From now on we assume that \((M^4, g, f)\) is **\(\kappa\)-noncollapsed**. By this we mean that if \(|\text{Rm}| \leq r^{-2}\) in \(B(p, r)\), where \(p \in M\) and \(r \in (0, \infty)\), then \(\text{vol}(B(p, r)) \geq \kappa r^4\). If \((M^4, g)\) is a singularity model of the Ricci flow (see Definition 1.4), then it satisfies the stronger property that the above holds with \(|\text{Rm}|\) replaced by the scalar curvature \(R\) (this is a theorem of Perelman [25, Theorems 28.6 and 28.9]); in this case we say that \((M^4, g)\) is **strongly \(\kappa\)-noncollapsed**.

According to the works of Hamilton and Perelman on ancient solutions, Brendle’s remarkable solution to Perelman’s conjecture on the classification of noncompact 3-dimensional \(\kappa\)-solutions [14], and the recent classification of compact 3-dimensional \(\kappa\)-solutions by Brendle, Daskalopoulos, and Sesum [15] (see Bamler and Kleiner [9] for a later, alternative treatment stemming from their fundamental proof of the generalized Smale conjecture [10] using
the strong stability of the 3-dimensional Ricci flow), for any given sequence of points \( \{p_i\}_{i \in \mathbb{N}}, (N^3, g_N(t)) \) in Definition 1.1 is one the following solutions:

1. quotients of \( k \)-solutions on \( S^3 \) (such as shrinking spherical space forms \( S^3/\Gamma \) and Perelman’s ancient solutions on \( S^3 \) and \( \mathbb{R}P^3 \));
2. the ancient Ricci flow generated by the 3-dimensional Bryant soliton (which we henceforth abbreviate as the Bryant 3-soliton);
3. quotients of shrinking round cylinders on \( S^2 \times \mathbb{R} \) (the only orientable \( k \)-noncollapsed quotient is by \( \mathbb{Z}_2 \) acting diagonally).

In this paper, we obtain further restrictions on the possible asymptotic geometries of 4-dimensional steady gradient Ricci solitons that dimension reduce to 3-manifolds. Precisely, we prove the following result.

**Theorem 1.3.** If \((M, g, f)\) is a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds, then either it strongly dimension reduces to \( S^3/\Gamma \) or it weakly dimension reduces to the Bryant 3-soliton.¹

Observe that if \((M^4, g, f)\) weakly dimension reduces to the Bryant 3-soliton, then it also weakly dimension reduces to \( S^2 \times \mathbb{R} \). A key content of Theorem 1.3 is that, under its hypothesis, if dimension reduction to \( S^2 \times \mathbb{R} \) occurs for a given steady soliton, then conversely dimension reduction to the Bryant 3-soliton must occur for that steady soliton. Although this agrees with one’s intuitive picture of the landscape of 4-dimensional steady solitons, this is technically difficult to prove.

Examples of steady gradient Ricci solitons that strongly dimension reduce to \( S^3/\Gamma \) are the Bryant 4-soliton (to \( S^3 \)) and Appleton’s cohomogeneity one examples on real plane bundles over \( S^2 \) (to \( S^3/\mathbb{Z}_k, k \geq 3 \)); see [3]. The first named author also conjectured that there exist similar steady gradient Ricci solitons on plane bundles over \( \mathbb{R}P^2 \). Among these solitons, only the Bryant 4-soliton has positive Ricci curvature (it in fact has positive curvature operator). We note that Hamilton has conjectured that there exists a family of 4-dimensional \( k \)-noncollapsed steady gradient Ricci solitons, called flying wings, with positive curvature operator that weakly dimension reduce to the Bryant 3-soliton. Lai [41] recently confirmed Hamilton’s conjecture in dimension 3 by proving the existence of 3-dimensional flying wings, which reduce to the cigar soliton.

When \((M^4, g, f)\) weakly dimension reduces to the Bryant 3-soliton and its scalar curvature has no decay, one may conjecture that \((M, g, f)\) either is the product of the Bryant 3-soliton and a line or is a flying wing. It is also unknown whether there exists a steady gradient Ricci soliton which

¹More precisely, we mean it strongly dimension reduces to \( (S^3/\Gamma, g_{S^3/\Gamma}(t)) \), where \( (S^3/\Gamma, g_{S^3/\Gamma}(t)) \) is a group of shrinking spherical space forms.
weakly dimension reduces to the Bryant 3-soliton and has scalar curvature uniformly decaying to zero. If such a steady gradient Ricci soliton exists, we expect that the asymptotic behavior of its level set flow\(^2\) should be similar to that of the 3-dimensional \(\kappa\)-solution constructed by Perelman.

A key idea of Theorem 1.3 is that (3.1) holds if the steady Ricci soliton doesn’t reduce to a 3-dimensional steady gradient Ricci soliton. Condition (3.1) actually implies that the steady Ricci soliton has linear curvature decay and hence strongly dimension reduces to \(S^3/\Gamma\). Although Brendle, Daskalopoulos and Sesum [15] have obtained the classification of 3-dimensional compact \(\kappa\)-solutions, which was based on the asymptotic behavior of 3-dimensional compact \(\kappa\)-solutions studied in [2], our proof does not rely on their results.

In most cases, one is interested in Ricci solitons which are singularity models. The definition of a singularity model is as follows.

**Definition 1.4.** Let \((M^n, g(t)), t \in [0, T)\), be a finite time singular solution to Ricci flow on a closed oriented manifold such that \(\sup_{M \times [0, T)} |Rm| = \infty\) and \(T < \infty\). An associated **singularity model** \((M_\infty^n, g_\infty(t)), t \in (-\infty, 0]\), is a complete ancient solution which is a limit of pointed rescalings. More precisely, there exists a sequence of space-time points \((x_i, t_i)\) in \(M \times [0, T)\) with \(K_i \cong |Rm|(x_i, t_i) \to \infty\) such that the sequence of pointed solutions \((M, g_i(t), x_i)\), where \(g_i(t) = K_i g(K_i^{-1} t + t_i)\) and \(t \in [-K_i t_i, 0]\), converges in the Cheeger-Gromov sense to the complete ancient solution \((M_\infty, g_\infty(t), x_\infty), t \in (-\infty, 0]\).

As an application of Theorem 1.3, we can give a description of the asymptotic behavior of any 4-dimensional steady gradient Ricci soliton singularity model \((M, g, f)\) whose Ricci curvature is nonnegative outside a compact set \(K\), i.e., its Ricci curvature \(\text{Ric}(x)\) satisfies
\[
(1.3) \quad \text{Ric}(x) \geq 0, \quad \forall x \in M \setminus K.
\]

Here, dimension reduction is not one of the hypotheses.

**Theorem 1.5.** Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton singularity model satisfying (1.3). Then, one of the following holds:

1. \((M, g, f)\) is a Ricci flat ALE 4-manifold;
2. \((M, g, f)\) strongly dimension reduces to \(S^3/\Gamma\);
3. \((M, g, f)\) weakly dimension reduces to the Bryant 3-soliton.

The aforementioned Appleton’s cohomogeneity-one steady soliton examples have positive curvature operator outside a compact set. Hamilton’s

\(^2\)One may see Section 3 in [31] for the definition of the level set flow of a steady gradient Ricci soliton.
conjectured flying wings are expected to have at least positive Ricci curvature outside a compact set. At the present time, all known (non-splitting) noncollapsed 4-dimensional steady solitons have positive curvature outside a compact set.

ALE 4-manifolds are defined as follows.

**Definition 1.6.** A complete noncompact Riemannian 4-manifold \((M^4, g)\) is an **asymptotically locally Euclidean (ALE) space of order** \(\tau > 0\) if there exist a finite subgroup \(\Gamma\) of \(SO(4)\), a compact subset \(K\) of \(M\), and a diffeomorphism

\[
\Phi : (\mathbb{R}^4 - B_1(0))/\Gamma \rightarrow M \setminus K
\]

(1.4)

such that \(\tilde{g}_{ij} = \pi^* \Phi^* g\), where \(\pi : \mathbb{R}^4 - B_1(0) \rightarrow (\mathbb{R}^4 - B_1(0))/\Gamma\) is the projection, which satisfies \(|\tilde{g}_{ij} - \delta_{ij}| \leq O(r^{-\tau})\) and \(|\partial^k g_{ij}| \leq O(r^{-\tau - |k|})\) for multi-indices \(k\).

When the steady Ricci solitons in Theorem 1.5 are Kähler-Ricci solitons, we can classify any steady gradient Kähler-Ricci soliton singularity model \((M, g, f)\) of complex dimension 2, whose Ricci curvature is nonnegative outside a compact set. Precisely, we have:

**Theorem 1.7.** Steady gradient Kähler-Ricci soliton singularity models on complex surfaces must be hyperkähler ALE Ricci-flat 4-manifolds if they satisfy condition (1.3).

By Bando, Kasue, and Nakajima [11], for any 4-dimensional ALE, there exists \(\Phi\) so that the order of the ALE is 4. It is conjectured that any ALE Ricci flat 4-manifold must be hyperkähler. The conjecture is true when the 4-manifold is a Kähler manifold by [39, 40]. Simply-connected hyperkähler ALE 4-manifolds have been classified by Kronheimer [39, 40]. The non-simply-connected hyperkähler ALE 4-manifolds have also been classified by Suvaina [52] and Wright [54]. We remark that Kähler-Ricci flat non-ALE spaces may have infinite type; see [1]. Note also that Appleton’s cohomogeneity-one examples are non-Kähler.

Now, we explain the ideas in proving Theorem 1.5 and Theorem 1.7. When the scalar curvature has no uniform decay, we will prove a dimension reduction theorem for steady Ricci solitons. In [30], Xiaohua Zhu and the second named author proved the dimension reduction for steady gradient Kähler-Ricci solitons with nonnegative bisectional curvature. Under (1.3), we can find a geodesic line in the limit of a sequence of steady gradient Ricci solitons. In this paper we will prove the following dimension reduction theorem.
Theorem 1.8. Let \((M, g, f)\) be an \(n\)-dimensional \(\kappa\)-noncollapsed steady gradient Ricci soliton with bounded curvature. If \((M, g, f)\) satisfies (1.3) and does not have uniform scalar curvature decay, then it weakly dimension reduces to an \((n - 1)\)-dimensional steady gradient Ricci soliton.

Recall that in dimension 4, bounded curvature holds in the case that the steady gradient Ricci soliton is a singularity model.

By Theorem 1.8, to prove Theorem 1.5 it suffices to deal with 4-dimensional \(\kappa\)-noncollapsed steady gradient Ricci solitons satisfying (1.3) and uniform scalar curvature decay, i.e., \(R(x) \to 0\) as \(x \to \infty\). We prove the following.

Theorem 1.9. Let \((M, g, f)\) be a 4-dimensional \(\kappa\)-noncollapsed steady gradient Ricci soliton with uniform scalar curvature decay. If it satisfies condition (1.3), then one of the following holds:

1. \((M, g, f)\) is Ricci flat;
2. \((M, g, f)\) strongly dimension reduces to \(S^3/\Gamma\);
3. \((M, g, f)\) weakly dimension reduces to the Bryant 3-soliton.

It turns out that it is important to study steady gradient Ricci solitons with maximal volume growth in order to prove Theorem 1.9. We say that an \(n\)-dimensional steady gradient Ricci soliton \((M, g, f)\) satisfying condition (1.3) has maximal volume growth if the asymptotic volume ratio (AVR)

\[
\text{AVR}(g) := \lim_{r \to \infty} \frac{V_x(r)}{r^n} > 0,
\]

where \(V_x(r) = \text{vol} B(x, r)\). We will show in Lemma A.2 in the appendix that the limit exists and does not depend on the basepoint \(x\). For such steady gradient Ricci solitons with uniform curvature decay, we will prove in Proposition A.7 that if \(\text{AVR}(g) > 0\), then there is a uniform constant \(c > 0\) depending only on the Ricci lower bound over \(K\) such that

\[
\frac{V_y(r)}{r^n} \geq c \text{AVR}(g), \quad \forall y \in M \setminus K, \; r > 0.
\]

For 4-dimensional steady gradient Ricci solitons with maximal volume growth, we have the following rigidity theorem.

Theorem 1.10. Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton with uniform scalar curvature decay. If \((M, g, f)\) satisfies condition (1.3) and has maximal volume growth, then \((M, g)\) must be Ricci flat.

It is interesting to know whether \(n\)-dimensional steady gradient Ricci solitons should be Ricci flat if their volume growth satisfies

\[
\text{vol} B(x_0, r_i) \geq cr_i^n,
\]
where \( c \) is a positive constant and \( x_0 \) is a fixed point on \( M \). Moreover, \( r_i \) is a sequence of positive constants such that \( r_i \to \infty \) as \( i \to \infty \).

With the help of Theorem 1.10, we are able to show the following convergence result.

**Theorem 1.11.** Let \((M^4, g, f)\) be a steady gradient Ricci soliton which is not Ricci flat and satisfies the hypotheses of Theorem 1.9. Then, for any \( p_i \) tending to infinity, \((M, R(p_i)g(R(p_i)^{-1}t), p_i)\) converges subsequentially to a complete limit \((M_\infty, g_\infty(t), p_\infty)\). Moreover, \((M_\infty, g_\infty(t), p_\infty)\) has uniformly bounded curvature and splits off a line.

Theorem 1.9 is then a corollary of Theorem 1.3 and Theorem 1.11. Combining Theorem 1.9 with Theorem 1.11, we obtain:

**Theorem 1.12.** Let \((M, g, f)\) be a 4-dimensional \( \kappa \)-noncollapsed steady gradient Ricci soliton with bounded curvature. If it satisfies (1.3), then one of the following holds:

1. \((M, g, f)\) is Ricci flat;
2. \((M, g, f)\) strongly dimension reduces to \( S^3/\Gamma \);
3. \((M, g, f)\) weakly dimension reduces to the Bryant 3-soliton.

When the steady gradient Ricci solitons in Theorem 1.12 are Kähler-Ricci solitons, we have:

**Theorem 1.13.** \( \kappa \)-noncollapsed steady gradient Kähler-Ricci solitons with bounded curvature on complex surfaces must be Ricci flat if they satisfy (1.3).

Theorem 1.5 is in fact a direct corollary of Theorem 1.12 by Theorem 1 in [26], Perelman’s no local collapsing theorem and the work of Cheeger and Naber [23, Corollary 8.86]. Similarly, Theorem 1.7 is a direct corollary of Theorem 1.13.

**Conjecture 1.14.** If \((M, g, f)\) is a 4-dimensional steady gradient Ricci soliton singularity model, then it dimension reduces to 3-manifolds.

Regarding Ricci flow analysis, compactness theory, and singularity models, particularly striking are the recent breakthroughs of Bämler [5, 6, 7]. In [8] by Bämler, Zhang, and the three authors, Bämler’s theory is applied to obtain results regarding the tangent flows at infinity of 4-dimensional steady soliton singularity models.

This paper is organized as follows. In Section 2, we study the linear growth of the potential function. In Section 3, we study the linear curvature decay of steady gradient Ricci solitons. Sections 4, 5, 6 are devoted to proving Theorem 1.3. Theorem 1.10 is proved in Section 7. In Section 8, we prove Theorem 1.11. Theorem 1.8 is proved in Section 9. Theorem 1.5, Theorem 1.13 and Theorem 1.7 will be proved in Section 10.
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2. Linear growth of the potential function

The growth of the potential function \( f \) is important for studying the rotational symmetry of steady gradient Ricci solitons (see [12, 13], [31]). It is known that \( f \) grows linearly when the Ricci curvature is nonnegative and there exists an equilibrium point \( o \) of \( f \) on \( M \), i.e., \( \nabla f(o) = 0 \) (see [19], [20]). Let \( \phi_t \) be the 1-parameter group of diffeomorphisms generated by \( -\nabla f \). Then the nonnegativity of the Ricci curvature implies that

\[
\frac{d}{dt} R(\phi_t(p)) \geq 0.
\]

In this section we will show that \( f \) grows linearly under the assumption that \( \frac{d}{dt} R(\phi_t(p)) \geq 0 \) outside some compact set \( K \) and that the scalar curvature decays uniformly. Precisely, we have:

Theorem 2.1. Let \((M^n, g, f)\) be a non-Ricci-flat steady gradient Ricci soliton. Suppose that the scalar curvature decays uniformly and

\[
\frac{d}{dt} \bigg|_{t=0} R(\phi_t(p)) \geq 0 \quad \text{for all } p \in M \setminus K,
\]

where \( K \) is a compact subset of \( M \). Then there exist positive constants \( r_0, C_1 \) and \( C_2 \) such that

\[
C_1 \rho(x) \leq f(x) \leq C_2 \rho(x) \quad \text{for all } x \in M \text{ such that } \rho(x) \geq r_0,
\]

where \( \rho(x) \) is the distance function from a fixed point \( x_0 \in M \). That is, the potential function is uniformly equivalent to the distance (to a fixed point) function.

Remark. Condition (2.1) is equivalent to \( \langle \nabla R, \nabla f \rangle \leq 0 \) on \( M \setminus K \), which in turn is equivalent to \( \text{Ric}(\nabla f, \nabla f) \geq 0 \) on \( M \setminus K \).

Now we fix some notations in this section. Let \((M^n, g, f)\) be a non-Ricci-flat steady gradient Ricci soliton with scalar curvature decaying uniformly. It is well known that the following identity holds:

\[
R(x) + |\nabla f|^2(x) = C,
\]

where \( C \) is a positive constant. Since \((M, g, f)\) is non-Ricci-flat, \( R(x) \) must be positive by B.-L. Chen [24]. In particular, (2.3) implies that

\[
|\nabla f|^2(x) \leq C \quad \text{for all } x \in M.
\]

Let \( R_{\text{max}} = \sup_{x \in M} R(x) \) and define

\[
S(\varepsilon) = \{ x \in M : R(x) \geq R_{\text{max}} - \varepsilon \}.
\]
Note that $R_{\text{max}} \leq C$ and $S(\varepsilon) = \{ x \in M : |\nabla f|^2 \leq \varepsilon + C - R_{\text{max}} \}$, where $C$ is the constant in (2.3). Since the scalar curvature decays uniformly, $S(\varepsilon)$ is compact for each $\varepsilon \in [0, R_{\text{max}})$. Moreover, $M$ is exhausted by the family of sets $\{ S(\varepsilon) \}_{\varepsilon \in (0, R_{\text{max}}]}$. Hence, there exists a positive constant $\varepsilon_0 < R_{\text{max}}$ such that
\begin{equation}
K \subseteq S(\varepsilon) \text{ for all } \varepsilon \in [\varepsilon_0, R_{\text{max}}),
\end{equation}
where $K$ is as in the hypothesis of Theorem 2.1.

**Lemma 2.2.** Under the hypotheses of Theorem 2.1, for any $p \in M \setminus S(\varepsilon_0)$, we have
\begin{equation}
\phi_t(p) \in M \setminus S(\varepsilon_0) \text{ for all } t \in (-\infty, 0].
\end{equation}
Hence,
\begin{equation}
\frac{d}{dt} R(\phi_t(p)) \geq 0 \text{ for all } (p, t) \in (M \setminus S(\varepsilon_0)) \times (-\infty, 0).
\end{equation}

**Proof.** Let $p \in M \setminus S(\varepsilon_0)$. By the definition of $S(\varepsilon_0)$, we have that $|\nabla f|^2(p) > \varepsilon_0 + C - R_{\text{max}}$. Therefore, for $t_0 < 0$ sufficiently small, we have that $|\nabla f|^2(\phi_t(p)) > \varepsilon_0 + C - R_{\text{max}}$ for all $t \in [t_0, 0]$. Let
\begin{equation}
T = \inf \{ t \leq 0 : \phi_t(p) \in M \setminus S(\varepsilon_0) \}.
\end{equation}
If $T$ is finite, then $|\nabla f|^2(\phi_t(p)) > \varepsilon_0 + C - R_{\text{max}}$ for $T < t \leq 0$ and $|\nabla f|^2(\phi_T(p)) = \varepsilon_0 + C - R_{\text{max}}$. Note that
\begin{equation}
\frac{d}{dt} |\nabla f|^2(\phi_t(p)) = -\frac{d}{dt} R(\phi_t(p)) \leq 0 \text{ for all } (p, t) \in (M \setminus S(\varepsilon_0)) \times (T, 0].
\end{equation}
Hence,
\begin{equation}
|\nabla f|^2(\phi_T(p)) \geq |\nabla f|^2(\phi_t(p)) > \varepsilon_0 + C - R_{\text{max}}
\end{equation}
for $t \in (T, 0]$. This contradicts the fact that $\phi_T(p) \in S(\varepsilon_0)$. Hence $T = -\infty$, i.e., $\phi_t(p) \in M \setminus S(\varepsilon_0)$ for all $t \leq 0$. We have completed the proof. \hfill \Box

When $t \rightarrow +\infty$, we have the following lemma.

**Lemma 2.3.** Under the hypotheses of Theorem 2.1, for any $p \in M \setminus S(\varepsilon_0)$, there exists $t_p \in (0, \infty)$ such that $\phi_{t_p}(p) \in S(\varepsilon_0)$ and $\phi_t(p) \in M \setminus S(\varepsilon_0)$ for all $t \in (-\infty, t_p)$.

**Proof.** The proof is by contradiction. If the lemma is not true, then there exists a point $p \in M \setminus S(\varepsilon_0)$ such that $\phi_t(p) \in M \setminus S(\varepsilon_0)$ for all $t \geq 0$. Then
\begin{equation}
|\nabla f|^2(\phi_t(p)) \geq \varepsilon_0 \text{ for all } t \geq 0.
\end{equation}
It follows that
\begin{equation}
f(p) - f(\phi_t(p)) = \int_0^t |\nabla f|^2(\phi_s(p))ds \geq \varepsilon_0 t.
\end{equation}
For any fixed \( t \geq 0 \), let \( \gamma_t : [0, L_t] \to M \) be a minimal geodesic with \( \gamma_t(0) = p \) and \( \gamma_t(L_t) = \phi_t(p) \), where \( L_t = d(p, \phi_t(p)) \), and let \( s \) be the arc length parameter. Then we have

\[
|f(p) - f(\phi_t(p))| = \left| \int_{0}^{L_t} \langle \nabla f(\gamma_t(s)), \gamma_t'(s) \rangle ds \right|
\leq \int_{0}^{L_t} |\nabla f| ds
\leq \sqrt{C} d(p, \phi_t(p)),
\]

where we have used the identity (2.3). Combining (2.10) with (2.11), we see that \( \phi_t(p) \) tends to infinity as \( t \to +\infty \).

On the other hand, \( \frac{d}{dt} R(\phi_t(p)) \geq 0 \) since \( \phi_t(p) \in M \setminus S(\varepsilon_0) \) for all \( t \geq 0 \). It follows that \( \phi_t(p) \in S(R_{\max} - R(p)) \) for \( t \geq 0 \). Note that \( S(R_{\max} - R(p)) \) is compact. Hence, \( \phi_t(p) \) cannot tend to infinity. We obtain a contradiction.

Hence the proof of the lemma is complete. \( \square \)

Now, we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** For any \( x \in M \setminus S(\varepsilon_0) \), by Lemma 2.3 there exists a positive constant \( t_x > 0 \) such that \( \phi_{t_x}(x) \in S(\varepsilon_0) \) and \( \phi_t(x) \in M \setminus S(\varepsilon_0) \) for all \( t \in (-\infty, t_x) \). Note that

\[
f(x) - f(\phi_{t_x}(x)) = \int_{0}^{t_x} |\nabla f|^2(\phi_s(x)) ds
\]

and

\[
d(x, \phi_{t_x}(x)) \leq \text{Length}(\phi_s(x)|_{s \in [0, t_x]}, g) = \int_{0}^{t_x} |\nabla f|(\phi_s(x)) ds.
\]

By (2.3), we have

\[
\frac{d}{dt} |\nabla f|^2(\phi_t(x)) = -\frac{d}{dt} R(\phi_t(x)) \leq 0 \quad \text{for all } t \in [0, t_x],
\]

\[
|\nabla f|(\phi_s(x)) \geq |\nabla f|(\phi_{t_x}(x)) \quad \text{for all } s \in [0, t_x].
\]

Consequently,

\[
f(x) - f(\phi_{t_x}(x)) = \int_{0}^{t_x} |\nabla f|^2(\phi_s(x)) ds
\geq |\nabla f|(\phi_{t_x}(x)) \int_{0}^{t_x} |\nabla f|(\phi_s(x)) ds
\geq \varepsilon_0 d(x, \phi_{t_x}(x)).
\]
Now we fix a point \( x_0 \in S(\epsilon_0) \). We then obtain for all \( x \in M \setminus S(\epsilon_0) \),

\[
f(x) - f(x_0) \geq \epsilon_0 d(x, \phi_t(x)) + f(\phi_t(x)) - f(x_0) \\
\geq \epsilon_0 d(x, x_0) - (2 + \epsilon_0)A,
\]

where

\[
A = \sup_{x \in S(\epsilon_0)} |f(x)| + \text{diam}(S(\epsilon_0))
\]

and \( \text{diam}(S(\epsilon_0)) = \sup_{x,y \in S(\epsilon_0)} d(x, y) \).

On the other hand, for any \( x \in M \) we have the following. Let \( \gamma : [0, L] \to M \) be a minimal geodesic with \( \gamma(0) = x_0 \) and \( \gamma(L) = x \), where \( L = d(x, x_0) \), and let \( s \) be the arc length parameter. Then we have

\[
|f(x) - f(x_0)| = \left| \int_0^L \langle \nabla f(\gamma(s)), \gamma'(s) \rangle ds \right| \leq \int_0^L |\nabla f| ds \leq \sqrt{C}d(x_0, x).
\]

Combining (2.12) with (2.13) completes the proof of Theorem 2.1. \( \square \)

**Remark 2.4.** Under the hypotheses of Theorem 2.1, it is easy to see that the constant \( C \) in (2.3) satisfies \( C = R_{\max} = \sup_{x \in M} R(x) \). Moreover, the linear growth estimate of \( f \) can be improved to:

\[
\frac{f(x)}{\rho(x)} \to \sqrt{R_{\max}} \quad \text{as} \quad \rho(x) \to \infty.
\]

The details can be found in the proof of Lemma 2.2 in [32].

### 3. Linear curvature decay of steady GRS

Linear curvature decay is an important condition in the study of the asymptotic geometry of steady gradient Ricci solitons (see [12, 13], [28, 29, 30, 31, 32]). Originally, linear curvature decay of steady gradient Ricci solitons were obtained on positively curved steady gradient Ricci solitons in dimension 3 by Guo (see [34]).

In this section, we prove linear curvature decay under conditions stronger than that of Theorem 2.1. Let \( \rho(x) \) denote the distance function from a fixed point \( x_0 \in M \). Let \( S(\epsilon_0) \) be the set defined as in Section 2.

**Theorem 3.1.** Let \((M^n, g, f)\) be a non-Ricci-flat steady gradient Ricci soliton. Suppose that the scalar curvature decays uniformly and that

\[
\frac{1}{R^2(p)} \cdot \left. \frac{d}{dt} \right|_{t=0} R(\phi_t(p)) \geq \epsilon > 0 \quad \text{for all} \quad p \in M \setminus K,
\]

where \( K \) is a compact subset of \( M \), \( \phi_t \) is the 1-parameter group of diffeomorphisms generated by \(-\nabla f\), and \( \epsilon \) is independent of \( p, t \). Then there exist
constants $r_0$ and $c$ such that

$$R(x) \leq \frac{c}{\rho(x)} \quad \text{for all } x \in M \text{ such that } \rho(x) \geq r_0. \quad (3.2)$$

That is, the scalar curvature decays linearly.

**Proof.** Note that $(M, g, f)$ satisfies the hypotheses of Theorem 2.1. By Lemma 2.2, we see that

$$\frac{1}{R^2(p)} \cdot \frac{d}{dt} R(\phi_t(p)) \geq \epsilon > 0 \quad \text{for all } p \in (M \setminus S(\varepsilon_0)) \times (-\infty, 0]. \quad (3.3)$$

By Lemma 2.3, for any $x \in M \setminus S(\varepsilon_0)$ there exists $t_x > 0$ such that $\phi_{t_x}(x) \in S(\varepsilon_0)$ and $\phi_t(x) \in M \setminus S(\varepsilon_0)$ for all $t \in (-\infty, t_x)$. By (3.3), we have

$$-\frac{d}{dt}[R^{-1}(\phi_t(x))] \geq \epsilon \quad \text{for all } t \in [0, t_x]. \quad (3.4)$$

Integrating this formula over $t \in [0, t_x]$, we obtain

$$R(x) \leq \frac{1}{\epsilon t_x + R^{-1}(\phi_{t_x}(x))}. \quad (3.5)$$

On the other hand,

$$f(x) - f(\phi_{t_x}(x)) = \int_0^{t_x} |\nabla f|^2(\phi_s(x)) \, ds \leq C t_x. \quad (3.6)$$

By (3.5) and (3.6), we have

$$R(x) \leq \frac{C}{\epsilon(f(x) - A) + C(R_{\text{max}})^{-1}}, \quad (3.7)$$

where $A = \sup_{x \in S(\varepsilon_0)} f(x)$. Hence the curvature decay estimate (3.2) follows from (3.7) and Theorem 2.1. \qed

Similarly, we have the same linear decay estimate under an upper bound rather than a lower bound for $-\frac{d}{dt}(R^{-1} \circ \phi_t)$.

**Theorem 3.2.** Let $(M^n, g, f)$ be a non-Ricci-flat steady gradient Ricci soliton. Suppose that the scalar curvature decays uniformly and

$$0 \leq \frac{1}{R^2(p)} \cdot \frac{d}{dt} \bigg|_{t=0} R(\phi_t(p)) \leq C' \quad \text{for all } p \in M \setminus K, \quad (3.8)$$

where $K$ is a compact subset of $M$ and $\epsilon$ is independent of $p, t$. Then there exist positive constants $r_0$ and $c'$ such that

$$R(x) \geq \frac{c'}{\rho(x)} \quad \text{for all } \rho(x) \geq r_0. \quad (3.9)$$
Proof. Note that \((M, g, f)\) satisfies the hypotheses of Theorem 2.1. By Lemma 2.2, we have
\[
0 \leq \frac{1}{R^2(p)} \frac{d}{dt}R(\phi_t(p)) \leq C' \quad \text{for all } (p, t) \in (M \setminus S(\varepsilon_0)) \times (-\infty, 0].
\]
Let \(x \in M \setminus S(\varepsilon_0)\). By Lemma 2.3, there exists a constant \(t_x > 0\) such that \(\phi_{t_x}(x) \in S(\varepsilon_0)\) and \(\phi_t(x) \in M \setminus S(\varepsilon_0)\) for all \(t \in (-\infty, t_x)\). By (3.10), we have
\[
-\frac{d}{dt}\left[R^{-1}(\phi_t(x))\right] \leq C' \quad \text{for all } t \in [0, t_x].
\]
Integrating the formula above over \(t \in [0, t_x]\), we obtain
\[
R(x) \geq \frac{1}{C't_x + R^{-1}(\phi_{t_x}(x))}.
\]
On the other hand,
\[
f(x) - f(\phi_{t_x}(x)) = \int_0^{t_x} |\nabla f|^2(\phi_s(x)) ds \geq \varepsilon_0 t_x.
\]
By (3.12) and (3.13), we have
\[
\frac{\varepsilon_0}{C'f(x) + \varepsilon_0 B},
\]
where \(B = \sup_{x \in S(\varepsilon_0)} R^{-1}(x)\). Hence (3.9) follows from (3.14) and (2.1). □

4. Asymptotic Geometry

By the methods in [31, 32], one can prove Theorem 1.3 with the help of Theorem 3.1 and Theorem 3.2.

Lemma 4.1. Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds. Then there exists a positive constant \(C_3\) such that
\[
\frac{|Rm|(x)}{R(x)} \leq C_3 \quad \text{for all } x \in M^4 \setminus K.
\]
If no dimension reduction of \((M^4, g, f)\) is a steady gradient Ricci 3-soliton, then there exist positive constants \(\epsilon\) and \(C_4\) such that
\[
\epsilon \leq \frac{\Delta R(x) + 2|\text{Ric}|^2(x)}{R^2(x)} \leq C_4 \quad \text{for all } x \in M \setminus K.
\]
Proof. We first show that (4.1) is true. If it is not true, then there exists a sequence of points \(\{p_i\}\) tending to infinity such that \(\frac{R(p_i)}{|Rm|(p_i)} \to 0\). Let \(K_i = |Rm|(p_i)\). Then \((M, K_i g(K_i^{-1}t), p_i)\) converges to \((M^4, g_\infty(t), p_\infty)\), where \((M^4, g_\infty(t))\) is the product of a 3-dimensional ancient Ricci flow and a line. By Chen [24], \((M^4, g_\infty(t))\) has nonnegative sectional curvature.
Since $\frac{R(p_i)}{|Rm|(p_i)} \to 0$, the scalar curvature of $(M^\infty, g^\infty(t))$ is zero for all $t$. Hence, $(M^\infty, g^\infty(t))$ must be flat. However, we have $|Rm_{g^\infty(0)}|(p^\infty) = 1$ by the definition of the sequence. Hence, we obtain a contradiction. Thus (4.1) is proved.

Next, we show that the second inequality in (4.2) is true. By the convergence assumption, it is easy to see by a contradiction argument that

$$\Delta R(x) + 2|\text{Ric}|^2(x) \leq C_4 \text{ for all } x \in M \setminus K.$$  

Then, the inequality on the right-hand side of (4.2) follows from (4.1) and (4.3).

Finally, we prove the first inequality in (4.2). If the inequality is not true, then there exists a sequence of points $p_i$ tending to infinity such that

$$\frac{\Delta R(p_i) + 2|\text{Ric}|^2(p_i)}{R^2(p_i)} \to 0 \text{ as } i \to \infty$$

(by (4.1), $|Rm|$ and $R$ are comparable). By hypothesis, we may assume that

$$(M, R(p_i)g(R^{-1}(p_i)t), p_i) \to (M^4_\infty, g^\infty(t), p^\infty).$$

Also, by hypothesis, we have that $(M^\infty, g^\infty(t))$ is a product of a line and a complete ancient Ricci flow with bounded nonnegative curvature operator. We also note that (4.4) implies that

$$\frac{\partial}{\partial t} \bigg|_{t=0} R_{g^\infty(t)}(p^\infty) = \frac{\Delta_{g^\infty(0)} R_{g^\infty(0)}(p^\infty) + 2|Ric_{g^\infty(0)}|^2_{g^\infty(0)}(p^\infty)}{R^2_{g^\infty(0)}(p^\infty)} = 0.$$  

Hence, by Hamilton’s eternal solutions result [37], we have that $(M^\infty, g^\infty(t))$ is a product of a line and a steady gradient Ricci 3-soliton. This contradicts our assumption. Hence, we have completed the proof of (4.2).

As a corollary, we have:

**Corollary 4.2.** Let $(M, g, f)$ be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds. If the scalar curvature does not have uniform decay, then $(M^4, g, f)$ weakly dimension reduces to a steady gradient Ricci 3-soliton.

**Proof.** Let $A = \lim_{r \to \infty} \sup_{x \in M \setminus B(x_0, r)} R(x)$, where $x_0$ is a fixed point. If the scalar curvature does not have uniform decay, then $A > 0$. We can choose a sequence of points $\{p_i\}$ tending to infinity such that $R(p_i) \to A$ as

---

3Hamilton’s eternal solutions result holds for complete ancient solutions to the Ricci flow with bounded nonnegative curvature operator and with $\frac{\partial R}{\partial t} = 0$ at a point.
$i \to \infty$. By Definition 1.1 and by (4.1), there exists a constant $C(A)$ such that

$$|\text{Rm}|_{g_i(0)}(x, t) \leq C(A) \quad \forall x \in M, \ t \in (-\infty, +\infty),$$

where $g_i(t) = R(p_i)g(R^{-1}(p_i)t)$. Hence, $(M^4, g_i(t), p_i)$ subconverges to a limit $(N^3 \times \mathbb{R}, g_N(t) + ds^2, p_\infty)$, where $g_N(t)$ is an eternal flow. By our choice of the sequence $\{p_i\}$, we have that

$$R_N(p_\infty, 0) = 1 = \sup_{(x, t) \in N^3 \times (-\infty, +\infty)} R_N(x, t),$$

i.e., $R_N(x, t)$ attains its maximum in space-time at the space-time point $(p_\infty, 0)$. Hence, $(N, g_N(t))$ must admit a steady gradient Ricci soliton structure (see [37]).

The steady gradient Ricci 3-soliton in the corollary above must in fact be the Bryant 3-soliton; see Theorem 6.1 below.

Note that

$$(4.7) \quad \frac{1}{R^2(p)} \cdot \left. \frac{d}{dt} \right|_{t=0} R(\phi_t(p)) = \frac{\Delta R(p) + 2|\text{Ric}|^2(p)}{R^2(p)}.$$

By Lemma 4.1, Corollary 4.2, Theorem 2.1, Theorem 3.1 and Theorem 3.2, we have:

**Proposition 4.3.** Let $(M, g, f)$ be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds. If $(M, g, f)$ does not weakly dimension reduce to a steady gradient Ricci 3-soliton, then there exist positive constants $r_0$, $C_1$, $C_2$, $C_3$, $c_1$ and $c_2$ such that

$$(4.8) \quad C_1 \rho(x) \leq f(x) \leq C_2 \rho(x) \quad \text{for all } \rho(x) \geq r_0,$$

$$(4.9) \quad \frac{c_1}{\rho(x)} \leq R(x) \leq \frac{c_2}{\rho(x)} \quad \text{for all } \rho(x) \geq r_0,$$

$$(4.10) \quad |\text{Rm}|(x) \leq C_3 R(x) \quad \text{for all } x \in M.$$

Proposition 4.3 implies that the level set of potential function $f(x)$ is compact and $|\nabla f|^2(x) > 0$ when $\rho(x) \geq r_0$. Hence, we are able to use the level set flow (see Section 3 in [31]) to give an estimate of the diameter of the level set $\{x \in M : f(x) = r\}$ when $r$ is large enough. In [31], the nonnegativity of the sectional curvatures is only used to get the following estimate

$$|\text{Rm}(\frac{\nabla f}{|\nabla f|}, Y, Y, \frac{\nabla f}{|\nabla f|})| \leq \text{Ric}(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}) = \frac{|\langle \nabla R, \nabla f \rangle|}{|\nabla f|^2} \leq C_0 R^2,$$

where $Y$ is a unit vector tangent to the level set. By Lemma 3.1 in [32], we have

$$(4.11) \quad \text{Rm}(\nabla f, e_j, e_k, \nabla f) = -\frac{1}{2}(\text{Hess} R)_{jk} - R_{jl}R_{kl} + \Delta R_{jk} + 2R_{ijkl}R_{il}.$$
Under the hypotheses of Theorem 1.3, it is easy to see by contradiction argument that

\[(4.12) \quad \frac{\|\nabla^k \text{Rm}\|}{R^{k+\frac{2}{n}}(x)} \leq C(k) \text{ for all } \rho(x) \geq \rho_0.\]

Hence, we have

\[(4.13) \quad \left| \text{Rm} \left( \frac{\nabla f}{|\nabla f|}, Y, Y, \frac{\nabla f}{|\nabla f|} \right) \right| \leq \frac{|\text{Hess} R| + |\Delta \text{Ric}| + |\text{Ric}|^2 + 2|\text{Rm}| \cdot |\text{Ric}|}{|\nabla f|^2} \leq C_0 R^2,\]

where \(Y\) is a unit vector tangent to the level set.

Hence, under the hypotheses of Theorem 1.3, we can replace (4.11) with (4.13) to get the following diameter estimate for the level sets.

**Proposition 4.4.** Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds. If \((M, g, f)\) does not weakly dimension reduce to a steady gradient Ricci 3-soliton, then there exists a constant \(C_5\) independent of \(r\) such that

\[(4.14) \quad \text{diam}(\Sigma_r, g) \leq C_5 \sqrt{r} \text{ for all } r \geq r_0,\]

where \(\Sigma_r = \{x \in M : f(x) = r\}\).

With the help of Proposition 4.3, Proposition 4.4 and (4.12), we can use the argument in Sections 2 through 4 of [31] to prove a weak version of Theorem 1.3 in [31].

**Theorem 4.5.** Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds. Suppose that \((M, g, f)\) does not weakly dimension reduce to a steady gradient Ricci 3-soliton. Then for any \(p_i \to \infty\) the rescaled Ricci flows \((M, \text{R}(p_i)g(\text{R}^{-1}(p_i)t), p_i)\) converge subsequentially to \((\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t)), t \in (-\infty, 0]\), in the Cheeger–Gromov topology, where \(\Sigma\) is diffeomorphic to a level set \(\Sigma_{r_0}\) and \((\Sigma, g_\Sigma(t))\) is a 3-dimensional compact ancient solution to the Ricci flow. Moreover, the scalar curvature \(R_\Sigma(x, t)\) of \((\Sigma, g_\Sigma(t))\) satisfies

\[(4.15) \quad R_\Sigma(x, t) \leq \frac{C}{|t|} \text{ for all } x \in \Sigma, \ t < 0,\]

where \(C\) is a constant.

We note that \(\Sigma\) in Theorem 4.5 is independent of the choice of the sequence \(\{p_i\}\) and the subsequence since it is diffeomorphic to the level set \(\Sigma_{r_0}\) of the potential \(f\) for some \(r_0\). We also note that \(\Sigma\) is connected. This is due to the following theorem in [48].
Theorem 4.6 (Munteanu and Wang). A complete noncompact steady gradient Ricci soliton is either connected at infinity (i.e., has exactly one end) or splits as the product of $\mathbb{R}$ with a compact Ricci flat manifold. Hence, a complete noncompact non-Ricci-flat steady gradient Ricci soliton must be connected at infinity.

By Theorem 2.1, we know that $M = S(\varepsilon_0) \cup M \setminus S(\varepsilon_0)$ and $M \setminus S(\varepsilon_0)$ is diffeomorphic to $\Sigma \times (A, +\infty)$. Since $(M, g)$ has only one end, $\Sigma$ must be connected.

To prove Theorem 1.3, we are left to show that $\Sigma$ in Theorem 4.5 is diffeomorphic to $\mathbb{S}^3/\Gamma$ and that $g_\Sigma(t)$ is a family of round metrics. Moreover, we need to show that the limit steady gradient Ricci soliton must be the product of a line and the Bryant 3-soliton if it weakly dimension reduces to a steady gradient Ricci 3-soliton. These results will be proved in the next two sections (see Theorem 5.1 and Theorem 6.1).

5. Uniqueness of the limit ancient Ricci flow

Theorem 5.1. The 3-manifold $\Sigma$ in Theorem 4.5 is diffeomorphic to $\mathbb{S}^3/\Gamma$ and $g_\Sigma(t)$ is a family of shrinking round metrics.

We first note that the following lemma holds because of the dimension reduction assumption.

Lemma 5.2. Let $(M, g, f)$ be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds. Then there exist positive constants $\kappa$ and $r_0$ such that (noncollapsing at the curvature scale of a point)

$$\text{vol} B(p, 1; R(p)g) \geq \kappa$$

for all $p \in M$ such that $\rho(p) \geq r_0$. (5.1)

For any positive number $\bar{r}$, there exists a positive constant $C(\bar{r})$ such that (bounded curvature at bounded distance)

$$\frac{R(x)}{R(p)} \leq C(\bar{r})$$

for all $x \in B(p, \bar{r}; R(p)g)$ and $p$ such that $\rho(p) \geq r_0$. (5.2)

Proof. The proof is by contradiction. Suppose that the lemma is not true. Then, there exists a sequence of points $p_i$ tending to infinity such that

$$\text{vol} B(p_i, 1; R(p_i)g) \to 0 \text{ as } p_i \to \infty.$$ (5.3)

On the other hand, we may assume that $(M, R(p_i)g(R^{-1}(p_i)t), p_i)$ subconverges to $(M_\infty, g_\infty(t), p_\infty)$. By taking $t = 0$, $(M, R(p_i)g, p_i)$ converges to $(M_\infty, g_\infty(0), p_\infty)$. Therefore, for $i$ large, we have

$$\text{vol} B(p_i, 1; R(p_i)g) \geq \frac{1}{2}\text{vol} B(p_\infty, 1; g_\infty(0)) > 0.$$ (5.4)

This contradicts (5.3).
Similarly, one can prove (5.2) by a contradiction argument. This completes the proof.

\[\square\]

In [31], X.-H. Zhu and the second author used the noncollapsing condition to obtain (5.1). By Lemma 5.2, we obtain (5.1) without assuming the noncollapsing condition. Therefore, we can follow the argument of Lemma 4.2 and Lemma 4.3 in [31] to obtain the following volume estimate for level sets.

**Lemma 5.3.** Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds, but does not weakly dimension reduce to a steady gradient Ricci 3-soliton. Then there exists a positive constant \(\kappa_1\) independent of \(r\) such that

\[
\text{vol}(\Sigma_r, g) \geq \kappa_1 (\sqrt{r})^3 \quad \text{for all} \quad r \geq r_0,
\]

where \(\Sigma_r = \{ x \in M : f(x) = r \}\).

As a corollary, we have the following lemma.

**Lemma 5.4.** Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds, but does not weakly dimension reduce to a steady gradient Ricci 3-soliton. Then there exist positive constants \(\kappa_2\) and \(C_6\) such that

\[
\text{vol}(\Sigma, g_\Sigma(t)) \geq \kappa_2 (-t)^{\frac{3}{2}}.
\]

and

\[
\text{diam}(\Sigma, g_\Sigma(t)) \leq C_6 \sqrt{1 - t}.
\]

**Proof.** Fix \(t < 0\). Under the hypotheses of Theorem 4.5, for \(p_i\) tending to infinity, we assume the following convergence

\[(5.8)\quad (M, R(p_i)g(R^{-1}(p_i)t), p_i) \to (\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t), p_\infty).\]

Therefore, we have

\[(5.9)\quad (M, R(p_i)g(\phi_{R^{-1}(p_i)t})(p_i)) \to (\mathbb{R} \times \Sigma, ds^2 + g_\Sigma(t), p_\infty).\]

Let \(q_i = \phi_{R^{-1}(p_i)t}(p_i)\). Similar to Corollary 3.4 in [31], there exists a positive constant \(C'_0\) such that

\[(5.10)\quad \Sigma_{f(q_i)} \subseteq B(q_i, C'_0; R(q_i)g).\]

By (5.2), we get

\[(5.11)\quad \frac{R(x)}{R(q_i)} \leq C(C'_0) \quad \text{for all} \quad x \in \Sigma_{f(q_i)}.\]
Note that
\[ R(p_i) f(q_i) = R(p_i) \left( f(p_i) + \int_0^{R^{-1}(p_i)|t|} |\nabla f|^2(\phi_s(p_i)) ds \right) \geq c_1 C_1 + \varepsilon_0|t|. \]
Hence,
\[ \frac{R(x)}{R(p_i)} = \frac{R(x) R(q_i) f(q_i)}{R(q_i) R(p_i) f(q_i)} \leq \frac{C'_1}{c_1 C_1 + \varepsilon_0|t|} \]
\[ \leq \frac{C'_2}{1 + |t|} \quad \text{for all } x \in \Sigma_{f(q_i)}. \]
\[ (5.13) \]
By Lemma 5.3, we have the following volume estimate:
\[ \text{vol} \left( \Sigma_{f(q_i)}, R(p_i) g \right) = \text{vol} \left( \Sigma_{f(q_i)}, f^{-1}(q_i) g \right) \cdot \left( R(q_i) f(q_i) \right)^{\frac{3}{2}} \cdot \left( \frac{R(p_i)}{R(q_i)} \right)^{\frac{3}{2}} \]
\[ \geq \kappa_1 \cdot (c_1 C_1)^{\frac{3}{2}} \cdot \left( \frac{1 + |t|}{C'_2} \right)^{\frac{3}{2}} \]
\[ \geq \kappa_2 (-t)^{\frac{3}{2}}. \]  
\[ (5.14) \]
Moreover, by Proposition 4.4, we have the diameter estimate for the level set:
\[ \text{diam} \left( \Sigma_{f(q_i)}, g \right) \leq C_5 \sqrt{f(q_i)}. \]
Therefore,
\[ \text{diam} \left( \Sigma_{f(q_i)}, R(p_i) g \right) \leq C_5 \sqrt{R(p_i) f(q_i)} \]
\[ = C_5 \sqrt{R(p_i) \left( f(p_i) + \int_0^{R^{-1}(p_i)|t|} |\nabla f|^2(\phi_s(p_i)) ds \right)} \]
\[ \leq C_5 \sqrt{c_2 C_2 + R_{\text{max}}(-t)}. \]
\[ (5.15) \]
Similar to [31], we can use (5.13), (5.14) and (5.15) to show that
\[ (5.16) \]
\[ \left( \Sigma_{f(q_i)}, R(p_i) g \right) \to (\Sigma, g_{\Sigma}(t)). \]
Hence, we get the following estimates by (5.14), (5.15) and the convergence (5.16):
\[ \text{vol}(\Sigma, g_{\Sigma}(t)) = \lim_{i \to \infty} \text{vol}(\Sigma_{f(q_i)}, R(p_i) g) \geq \kappa_2 (-t)^{\frac{3}{2}} \]
and
\[ \text{diam}(\Sigma, g_{\Sigma}(t)) = \lim_{i \to \infty} \text{diam}(\Sigma_{f(q_i)}, R(p_i) g) \leq C_5 \sqrt{c_2 C_2 + R_{\text{max}}(-t)}. \]
Now, we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. For fixed \( q \in M \), by the curvature estimate (4.15) and the volume estimate (5.6), we can use the argument in [49] to show that there exists \( \tau_i \to +\infty \) such that

\[
(\Sigma, \tau_i^{-1} g(\tau_i t), q) \to (\Sigma, h(t), q), \quad t < 0.
\]

Moreover, \((\Sigma, h(t), q)\) is a gradient shrinking Ricci soliton. Hence, \(\Sigma\) must be a finite quotient of \(\mathbb{R}^3, \mathbb{S}^2 \times \mathbb{R}\) or \(\mathbb{S}^3\). By the diameter estimate (5.7), we know that \(\Sigma\) is diffeomorphic to \(\Sigma\). Hence, \(\Sigma\) is compact. Therefore, \(\Sigma\) must be a finite quotient of \(\mathbb{S}^3\) and \(h(t)\) is a round metric. Hence, \((\Sigma, g_{\Sigma}(t))\) is a three-dimensional compact gradient shrinking Ricci soliton.

We have completed the proof. \(\Box\)

6. The limit soliton is the Bryant 3-soliton

Theorem 6.1. Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton that dimension reduces to 3-manifolds. Suppose that \((M, g, f)\) weakly dimension reduces to a steady gradient Ricci 3-soliton \((N^3_3, h)\). Then \((N_\infty, h)\) must be isometric to the Bryant 3-soliton.

Lemma 6.2. Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton which dimension reduces to 3-manifolds. Suppose that \((M, g, f)\) weakly dimension reduces to a steady gradient Ricci 3-soliton \((N^3_3, g_N, f_N)\). Then \((N_\infty, g_N, f_N)\) dimension reduces to 2-manifolds.

Proof. We may assume that \((M, g, f)\) dimension reduces to \((N, g_N, f_N)\) along points \(p_i\) tending to infinity. Precisely, we have that

\[
(M, R(p_i) g(R^{-1}(p_i) t), p_i) \to (M_\infty, g_\infty(t), p_\infty),
\]

where \((M_\infty, g_\infty(t)) = (N \times \mathbb{R}, g_N(t) + ds^2)\).

Claim 6.3. For any \(q_k \in N\) tending to infinity, by taking a subsequence, we have the convergence

\[
(N, R_N(q_k, 0) g(R_N^{-1}(q_k, 0) t), q_k) \to (N_\infty, \bar{g}_\infty(t), q_\infty),
\]

where \((N_\infty, \bar{g}_\infty(t))\) is a 3-dimensional complete ancient flow.
Let \( \tilde{q}_k = (q_k, 0) \in N \times \mathbb{R} \) for \( k \in \mathbb{N} \). By convergence (6.1), for any fixed \( k \in \mathbb{N} \), there exists a sequence of points \( q_{k,i} \in M \) such that

\[
(6.3) \quad (M, R(p_i)g(R^{-1}(p_i)t), q_{k,i}) \to (M_\infty, g_\infty(t), \tilde{q}_k).
\]

The convergence above implies that

\[
(6.4) \quad \frac{R(q_{k,i})}{R(p_i)} \to R_\infty(\tilde{q}_k, 0), \text{ as } i \to \infty.
\]

By (6.3) and (6.4), we have

\[
(6.5) \quad (M, R(q_{k,i})g(R^{-1}(q_{k,i})t), q_{k,i}) \to (M_\infty, R_\infty(\tilde{q}_k, 0)g(R_\infty^{-1}(\tilde{q}_k, 0)t), \tilde{q}_k).
\]

By Lemma 5.2, for fixed \( \tilde{r} > 0 \), we have

\[
(6.6) \quad \frac{R(x)}{R(q_{k,i})} \leq C(2\tilde{r}) \quad \text{for all } x \in B(q_{k,i}, 2\tilde{r}; R(q_{k,i})g), \quad \rho(q_{k,i}) \geq r_0,
\]

and

\[
(6.7) \quad \text{vol } B(q_{k,i}, 1; R(q_{k,i})g) \geq \kappa.
\]

By (6.5), (6.6) and (6.7), we have

\[
(6.8) \quad R_\infty(x) \leq C(2\tilde{r}) \quad \text{for all } x \in B(\tilde{q}_k, \tilde{r}; R_\infty(\tilde{q}_k, 0)g_\infty(0)).
\]

and

\[
(6.9) \quad \text{vol } B(\tilde{q}_k, 1; R_\infty(\tilde{q}_k, 0)g_\infty(0)) \geq \kappa.
\]

It follows that

\[
(6.10) \quad R_N(x, 0) \leq C(2\tilde{r}) \quad \text{for all } x \in B(q_k, \tilde{r}; R_N(q_k, 0)g_N(0)),
\]

and

\[
(6.11) \quad \text{vol } B(\tilde{q}_k, 1; R_\infty(\tilde{q}_k, 0)g_\infty(0)) \geq \frac{\kappa}{2}.
\]

Note that \((N, g_N, f_N)\) is a 3-dimensional steady gradient Ricci soliton. Then, \((N, g_N)\) has nonnegative sectional curvature by [24]. Note that \(g_N(t)\) is a Ricci flow generated by \((N, g_N, f_N)\). Therefore,

\[
(6.12) \quad \frac{\partial R_N(x, t)}{\partial t} = 2\text{Ric}_N(\nabla f_N, \nabla f_N)(\varphi_t(x)) \geq 0,
\]

where \(\varphi_t\) is generated by \(-\nabla f_N\).

By (6.10) and (6.12), we get

\[
(6.13) \quad |\text{Rm}_N|(x, t) \leq C(n)R_N(x, t) \leq C(n)C(2\tilde{r}) \quad \forall x \in B(q_k, \tilde{r}; R_N(q_k, 0)g_N(0)).
\]

Let \(h_k(t) = R_N(q_k, 0)g_N(R_N^{-1}(q_k, 0)t)\). Then, \(h_k(t)\) satisfies the Ricci flow equation. Note that the sectional curvature of \(h_k(t)\) is nonnegative. Hence,

\[
(6.14) \quad h_k(x, t) \geq h_k(x, 0) \quad \forall t \leq 0, \quad x \in N.
\]
Therefore,
\[ B(q_k, \bar{r}; h_k(t)) \subset B(q_k, \bar{r}; h_k(0)), \quad \forall \ t \leq 0. \]

Finally, for \( t \leq 0 \), (6.13) implies
\[ |\text{Rm}_N|(x, t) \leq C(n)C(2\bar{r}) \text{ for all } x \in B(q_k, \bar{r}; h_k(t)). \] (6.14)

For any \( \bar{r} > 0 \), we can find constant \( C(2\bar{r}) \) such that (6.14) holds. By Theorem 1.7 in [53], Claim 6.3 follows from (6.11) and (6.14).

Let \( (N_\infty, \bar{g}_\infty(t)) \) be the ancient Ricci flow in Claim 6.3. We are left to show the following claim.

**Claim 6.4.** \( (N_\infty, \bar{g}_\infty(t)) = (S \times \mathbb{R}, g_S(t) + ds^2), \) where \( (S, g_S(t)) \) is a two-dimensional ancient flow with bounded curvature.

For any sequence \( q_k \) tending to infinity, we may assume that \( R_N(q_k, 0) \to A \) by taking a subsequence. If \( A > 0 \), then \( R_N(q_k, 0)d_{g_N(0)}^2(q_k, q_0) \to \infty \) as \( k \to \infty \). By Theorem 5.35 in [45], we get \( (N_\infty, \bar{g}_\infty(t)) = (S \times \mathbb{R}, g_S(t) + ds^2) \), where \( (S, g_S(t)) \) is a two-dimensional ancient flow. By (6.12), \( (S, g_S(t)) \) also satisfies
\[ \frac{\partial R_S(x, t)}{\partial t} \geq 0 \text{ for all } x \in S. \] (6.15)

Now, we need to show that \( R_S(x, t) \) has bounded curvature for all \( x \in S \) and \( t \leq 0 \). It is sufficient to show \( R_S(x, 0) \) is bounded for \( x \in S \). The idea is similar to the proof of Lemma 4.4 in [28]. If it is not true, then there exists \( x_i \to \infty \) such that \( R_S(x_i, 0) \to \infty \). By Claim 6.3 and the argument in the proof of Claim 6.3, we have the following convergence by taking a subsequence
\[ (S, R_S(x_i, 0)g(R_S^{-1}(x_i, 0)t), x_i) \to (S_\infty, g_\infty'(t), x_\infty). \] (6.16)

Note that \( R_\infty'(x_\infty, 0) = 1 \).

On the other hand, we note that \( R_S(x_i, 0)d_{g_S(0)}^2(x_i, x_0) \to \infty \) and therefore \( (S_\infty, g_\infty'(t)) \) splits off a line by Theorem 5.35 of Chapter 5 in [45]. Since \( S_\infty \) is a two-dimensional manifold, \( (S_\infty, g_\infty'(t)) \) must be flat. Then, \( R_\infty'(x_\infty, 0) = 0 \). This is impossible. Hence, we have shown that \( R_S(x, t) \) has bounded curvature.

We are left to deal with the case \( A = 0 \). We follow the notation in Claim 6.3 and assume the following convergence for \( q_k \in N \) tending to infinity
\[ (N, R_N(q_k, 0)g(R_N^{-1}(q_k, 0)t), q_k) \to (N_\infty, \bar{g}_\infty(t), q_\infty), \] (6.17)
Let $g_k(t) = R_N(q_k,0)g(R_N^{-1}(q_k,0)t)$ and $X_{(k)} = R_N(q_k,0)^{-\frac{1}{2}}\nabla f_N$. For any fixed $\bar{r} > 0$, we have

$$\sup_{B(q_k,\bar{r};g_k(0))} |\nabla X_{(k)}|_{g_k(0)} = \sup_{B(q_k,\bar{r};g_k(0))} \frac{|\text{Ric}_N|_{g_k}}{\sqrt{R_N(q_k)}} \leq C \sqrt{R_N(q_k)} \to 0.$$ 

Similarly,

$$\sup_{B(q_k,\bar{r};g_k(0))} |\nabla^m X_{(k)}|_{g_k(0)} \leq C(n) \sup_{B(q_k,\bar{r};g_k(0))} |\nabla^{m-1}\text{Ric}_{g_k(0)}|_{g_k(0)} \leq C_1.$$ 

Thus $X_{(k)}$ converges subsequentially to a parallel vector field $X_{(\infty)}$ on $(M_\infty, g_\infty(0))$. Moreover,

$$|X_{(i)}|_{g_k(0)}(x) = |\nabla f_N|_{g_k}(x) = \sqrt{R_{\text{max}}} + o(1) > 0 \text{ for all } x \in B(p_i, \bar{r}; g_i),$$

as long as $f(p_i)$ is large enough. This implies that $X_{(\infty)}$ is non-trivial. Hence, $(N_\infty, \bar{g}_\infty(t))$ locally splits off a line along $X_{(\infty)}$. It is not hard to show that the integral curve of $X_{(\infty)}$ is a line. So $(M_\infty, g_\infty(t))$ splits off a line globally.

Now, we already have $(N_\infty, \bar{g}_\infty(t)) = (S \times \mathbb{R}, g_S(t) + ds^2)$, where $(S, g_S(t))$ is a two-dimensional ancient flow. We can use the argument in the case $A > 0$ to show that $(S, g_S(t))$ has uniformly bounded curvature.

\[\square\]

**Lemma 6.5.** $(N, g_N, f_N)$ has a uniform curvature decay.

**Proof.** We prove this by contradiction. If the lemma is not true, then there exists a sequence $q_i$ tending to infinity such that $R_N(q_i) \geq C_0$ for some positive constant $C_0$ and all $i \in \mathbb{N}$. Since $R_N(x)$ is bounded, we may assume that $R_N(q_i) \to C_0$ as $i \to \infty$. By a delicate choice of $q_i$, we can even make sure that $C_0 = \lim_{r \to \infty} \sup_{x \in N \setminus B(q_0, \bar{r}; g_N)} R_N(x)$. By Lemma 6.2, we may assume the following convergence

$$\tag{6.18} (N, R_N(q_i,0)g(R_N^{-1}(q_i,0)t), q_i) \to (N_\infty, \bar{g}_\infty(t), q_\infty),$$

where $(N_\infty, \bar{g}_\infty(t)) = (S \times \mathbb{R}, g_S(t) + ds^2)$ and $(S, g_S(t))$ is a two-dimensional ancient flow with bounded curvature.

Hence, $R_{N_\infty}(q_\infty,0) = 1$ and $R_{N_\infty}(x,t) \leq 1$ for all $x \in N_\infty$ and $t \in (-\infty, +\infty)$. Note that $R_{N_\infty}(\bar{x},t) = R_S(x,t)$ for $\bar{x} = (x,0) \in S \times \mathbb{R}$. Therefore, $(S, g_S(t))$ must be a Ricci flow generated by the cigar soliton. Since the cigar soliton is collapsed, for any $x_i \in S$ tending to infinity, we have

$$\tag{6.19} \lim_{i \to \infty} \text{vol} B(x_i, 1; R_{S}(x_i,0)g_S(0)) = 0$$

Let $\bar{x}_i = (x_i, 0) \in S \times \mathbb{R}$. It follows that

$$\tag{6.20} \lim_{i \to \infty} \text{vol} B(\bar{x}_i, 1; R_{N_\infty}(\bar{x}_i,0)g_{N_\infty}(0)) = 0$$
On the other hand, by Claim 6.3 as well as the argument in the proof of Claim 6.3, by taking a subsequence, we have

\begin{equation}
\text{vol } B(\tilde{x}_i, 1; R_{N,\infty}(\tilde{x}_i, 0)g_{N,\infty}(0)) \geq c_0,
\end{equation}

for some positive constant $c_0$. This is impossible. Hence, we have completed the proof.

\begin{lemma}
There exists a constant $\epsilon$ such that

\begin{equation}
\epsilon \leq \frac{\Delta_N R_N(x) + 2|\text{Ric}_N|^2_N(x)}{R^2_N(x)} \leq C_4 \text{ for all } x \in N \setminus K,
\end{equation}

where $K$ is a compact set.

\end{lemma}

\begin{proof}
Claim 6.3 implies the inequality on the right-hand side of the lemma. Similar to the proof of Lemma 4.1, $(N, g_N, f_N)$ dimension reduces to a two-dimensional steady gradient Ricci soliton if the left-hand side of (6.22) is not true. However, by the argument in Lemma 6.5, $(N, g_N, f_N)$ cannot dimension reduce to a two-dimensional steady gradient Ricci soliton. Hence, the inequality on the left-hand side of (6.22) holds.
\end{proof}

Now, we are ready to prove Theorem 6.1.

\begin{proof}[Proof of Theorem 6.1]
By Lemma 6.5, Lemma 6.6, Theorem 3.1 and Theorem 3.2, we have

\begin{equation}
\frac{c}{\rho_N(x)} \leq R_N(x) \leq \frac{c}{\rho_N(x)} \text{ for all } \rho(x) \geq r_0,
\end{equation}

for some positive constants $c'$, $c$ and $r_0$. By the result in [29], $(N, g_N, f_N)$ must be isometric to the Bryant 3-soliton.
\end{proof}

7. Volume growth of steady GRS

In this section, we prove Theorem 1.10.

\begin{lemma}
Let $(M, g, f)$ be a steady gradient Ricci soliton. For any $p \in M$ and number $k > 0$, we have

\begin{equation}
B \left( p, \frac{k}{\sqrt{R_{\max}}}; M(p)g \right) \subset M_{p,k},
\end{equation}

where $M_{p,k} = \left\{ x \in M : f(p) - \frac{k}{\sqrt{M(p)}} \leq f(x) \leq f(p) + \frac{k}{\sqrt{M(p)}} \right\}$.
\end{lemma}
Proof. For any \( q \in M \), let \( \gamma(s) \) be any curve connecting \( p \) and \( q \) such that \( \gamma(s_1) = q \) and \( \gamma(s_2) = p \). Then,
\[
\mathcal{L}(q, p) = \int_{s_1}^{s_2} \sqrt{\langle \gamma'(s), \gamma'(s) \rangle} \, ds \\
\geq \int_{s_1}^{s_2} \frac{|\langle \gamma'(s), \nabla f \rangle|}{|\nabla f|} \, ds \\
\geq \frac{1}{\sqrt{R_{\text{max}}}} \int_{s_1}^{s_2} |\langle \gamma'(s), \nabla f \rangle| \, ds \\
= \frac{1}{\sqrt{R_{\text{max}}}} |f(p) - f(q)|,
\]
where we have used \( |\nabla f|^2(x) = R_{\text{max}} - R(x) \leq R_{\text{max}} \forall x \in M \).

It follows that
\[
d(q, p) \geq \frac{1}{\sqrt{R_{\text{max}}}} |f(p) - f(q)|.
\]
In particular, for \( q \in M \setminus M_{p,k} \), we get
\[
d(q, p) \geq \frac{1}{\sqrt{R_{\text{max}}}} \cdot \frac{k}{\sqrt{M(p)}}.
\]
Hence
\[
(7.2) \quad B \left( p, \frac{k}{\sqrt{R_{\text{max}}}} \cdot M(p) g \right) \subset M_{p,k}.
\]

\[\Box\]

Lemma 7.2. Let \((M, g, f)\) be a steady gradient Ricci soliton with uniform scalar curvature decay which satisfies condition (1.3). If \( f(x) \geq r \), then
\[
R(x) \leq \sup_{y \in \Sigma_r} R(y) \quad \forall \ r \geq r_0,
\]
where \( r_0 \) is a positive constant.

Proof. By Theorem 2.1, we may assume that there exist constants \( C_1 \) and \( C_2 \) such that
\[
(7.4) \quad C_1 \rho(x) \leq f(x) \leq C_2 \rho(x), \forall \ f(x) \geq r_0.
\]
Let \( S(\varepsilon_0) \) be the compact set in Lemma 2.2. We may also assume that \( S(\varepsilon_0) \subseteq \{x \in M : f(x) \geq r_0\} \). Similar to the proof of Lemma 2.3, we can find \( t_x \geq 0 \) such that \( f(\phi_{t_x}(x)) = r \), where \( \phi_t \) is generated by \( -\nabla f \). Since the Ricci curvature is nonnegative, we have
\[
(7.5) \quad R(x) \leq R(\phi_{t_x}(x)) \leq \sup_{y \in \Sigma_r} R(y).
\]
We have completed the proof. \[\Box\]
Now, we let \( M(p) = \sup_{x \in \Sigma_{t(p)}} R(x) \). By Lemma 7.1, we can prove:

**Lemma 7.3.** Let \((M, g, f)\) be a steady gradient Ricci soliton with uniform scalar curvature decay which satisfies condition (1.3). Fix \( \epsilon > 0 \). Then, for any \( p \in M \) with \( M(p) \geq \frac{\epsilon}{f(p)} \) and number \( k > 0 \), there exist constants \( r_0(k, \epsilon) \) and \( C(m) \) such that

\[
\frac{\left| \nabla^m Rm \right|(q)}{M^{m+2}(p)} \leq C(m) \quad \forall \ q \in M_{p,k}, \ f(p) \geq r_0(k, \epsilon).
\]

**Proof.** Fix any \( q \in M_{p,k} \) with \( f(p) \geq r_0 \gg 1 \). Similar to the proof of Lemma 7.1, we have

\[
B \left( q, \frac{1}{\sqrt{R_{\text{max}}}}; M(p)g \right) \subseteq \left\{ x \in M : f(q) - \frac{1}{\sqrt{M(p)}} \leq f(x) \leq f(q) + \frac{1}{\sqrt{M(p)}} \right\} \\
\subseteq M_{p,k+1}.
\]

Since \( M(p) \geq \frac{\epsilon}{f(p)} \), for \( r_0(k, \epsilon) \) large enough, we have

\[
f(x) \geq \frac{f(p)}{2} \quad \forall \ x \in M_{p,k+1}.
\]

Hence, by Lemma 7.2, we get

\[
R(x) \leq M(p) \quad \forall \ x \in B \left( q, \frac{1}{\sqrt{R_{\text{max}}}}; M(p)g \right).
\]

Let \( \phi_t \) be generated by \(-\nabla f\). Then \( g(t) = \phi_t^* g \) satisfies the Ricci flow,

\[
\frac{\partial g(t)}{\partial t} = -2 \text{Ric}(g(t)).
\]

Also, the rescaled flow \( g_p(t) = M(p)g(M^{-1}(p)t) \) satisfies (7.8). Since the Ricci curvature is nonnegative,

\[
B \left( q, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(t) \right) \subseteq B \left( q, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(0) \right), \ t \in [-1, 0].
\]

Combining this with (7.7) and the estimate in [21], we get

(7.9) \[
\left| \nabla^m Rm \right|_{g_p(t)}(x) \leq CR_{g_p(t)}(x) \leq C \quad \forall \ x \in B \left( q, \frac{1}{\sqrt{R_{\text{max}}}}; g_p(0) \right), \ t \in [-1, 0].
\]

Thus, by Shi’s higher order local derivative of curvature estimates, we obtain

\[
\left| \nabla^m \left[ g_p(t) \operatorname{Rm} \right]_{g_p(t)}(x) \right| \leq C(m) \quad \forall \ x \in B \left( q, \frac{1}{2\sqrt{R_{\text{max}}}}; g_p(-1) \right), \ t \in \left[ -\frac{1}{2}, 0 \right].
\]

It follows that

\[
\left| \nabla^m Rm \right|(x) \leq C(m)M^{m+2}(p) \quad \forall \ x \in B \left( q, \frac{1}{2\sqrt{R_{\text{max}}}}; g_p(-1) \right).
\]

In particular, we have

\[
\left| \nabla^m Rm \right|(q) \leq C(m)M^{m+2}(p).
\]
Lemma 7.4. Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton with uniform scalar curvature decay which satisfies condition (1.3). Suppose \((M, g)\) has maximal volume growth. Suppose \(\{p_i\}_{i \in \mathbb{N}_+} \subset M\) is a sequence of points with the property \(M(p_i) \geq \frac{\epsilon}{f(p_i)}\), where \(\epsilon\) is a given constant. If \(p_i\) tends to infinity, then for any \(\epsilon > 0\), there exists a constant \(r_0(\epsilon, \epsilon)\) such that
\[
R(p_i) \leq \epsilon M(p_i) \quad \forall \quad f(p_i) \geq r_0(\epsilon, \epsilon).
\]
(7.10)

Proof. We prove this by contradiction. We only need to consider the case that \((M, g, f)\) is non-Ricci-flat. If the lemma is not true, then by taking a subsequence, we may assume that there exists a constant \(\epsilon > 0\) such that
\[
R(p_i) > \epsilon M(p_i) \quad \text{as} \quad i \to \infty.
\]
(7.11)

Now we consider the sequence of Ricci flows \((M, M(p_i)g(M^{-1}(p_i)t), p_i)\). Let \(g(t) = M(p_i)g(M^{-1}(p_i)t)\). Since we assume that \(M(p_i) \geq \frac{\epsilon}{f(p_i)}\), by Lemma 7.3, we have
\[
\frac{|\nabla^m \text{Rm}|(x)}{M^{m+2}(p_i)} \leq C(m) \quad \forall \quad x \in M_{p_i,k}, \quad f(p_i) \geq r_0(k, \epsilon).
\]
(7.12)

By Lemma 7.1, we also have
\[
B\left(p_i, \frac{k}{\sqrt{R_{\text{max}}}}; g_i(0)\right) \subseteq M_{p_i,k}.
\]
(7.13)

By (7.12) and (7.13), we have
\[
\frac{|\nabla^m \text{Rm}|(x)}{M^{m+2}(p_i)} \leq C(m) \quad \forall \quad x \in B\left(p_i, \frac{k}{\sqrt{R_{\text{max}}}}; g_i(0)\right), \quad f(p_i) \geq r_0(k, \epsilon).
\]
(7.14)

By Proposition A.7, \((M, g_i(t), p_i)\) is \(\kappa\)-noncollapsed. By the estimate (7.14) and Proposition A.7, by taking a subsequence, \((M, g_i(t), p_i)\) converges to a limit \((M_{\infty}, g_{\infty}(t), p_{\infty})\) with maximal volume growth. Since the scalar curvature of \((M, g, f)\) decays uniformly, by the same argument as in the proof of Claim 6.3, we can show that \((M_{\infty}, g_{\infty}(t), p_{\infty})\) splits off a line. We assume that \((M_{\infty}, g_{\infty}(t)) = (N \times \mathbb{R}, g_N(t) + ds^2)\). Note that (7.14) also implies that \((M_{\infty}, g_{\infty}(t))\) has bounded curvature. Hence, \(g_N(t)\) is a three-dimensional \(\kappa\)-solution with bounded curvature. We also note that \(g_N(t)\) has maximal volume growth. Hence, \((N, g_N(t))\) must be flat. It follows that \(R_{\infty}(p_{\infty}, 0) = 0\). Hence,
\[
\lim_{i \to \infty} \frac{R(p_i)}{M(p_i)} = R_{\infty}(p_{\infty}, 0) = 0.
\]
(7.15)

This contradicts (7.11). Hence, we have completed the proof. □
**Lemma 7.5.** Let \((M, g, f)\) be a 4-dimensional steady gradient Ricci soliton with uniform scalar curvature decay which satisfies condition (1.3). If \((M, g, f)\) has maximal volume growth, then there exists a constant \(r_0(\epsilon)\) such that

\[
R(x) \leq \frac{4\epsilon}{f(x)} \quad \forall \ f(x) \geq r_0(\epsilon).
\]  

(7.16)

**Proof.** We only need to consider the case that \((M, g, f)\) is non-Ricci-flat. Let \(\{p_i\}_{i \in \mathbb{N}_+}\) be a sequence of points such that 

\[
f(p_i) = 2^i \quad \text{and} \quad R(p_i) = \sup_{x \in \Sigma_{f(p_i)}} R(x).
\]

(7.17)

We first show that

\[
R(p_i) \leq \frac{2\epsilon}{f(p_i)} \quad \forall \ i \geq i_0,
\]

(7.18)

for some constant \(i_0 > 0\). We need the following claim.

**Claim 7.6.** There are infinitely many \(i\) such that

\[
R(p_i) \leq \frac{\epsilon}{f(p_i)}.
\]

(7.19)

**Proof.** We prove the claim by contradiction. If the claim is not true, then there exists a constant \(i_0 > 0\) such that

\[
R(p_i) \geq \frac{\epsilon}{f(p_i)} \quad \forall \ i \geq i_0.
\]

(7.20)

Let \(\epsilon = \frac{\epsilon}{4}\). We can make \(i_0\) large enough such that

\[
f(p_i) \geq r_0(\epsilon, \epsilon) \quad \forall \ i \geq i_0,
\]

where \(r_0(\epsilon, \epsilon)\) is the constant in Lemma 7.4. By Lemma 7.4, we have

\[
\frac{R(p_{i+1})}{R(p_i)} \leq \epsilon \quad \forall \ i \geq i_0.
\]

Hence,

\[
R(p_i) \leq \epsilon^{i-i_0} R(p_{i_0}) \quad \forall \ i > i_0.
\]

(7.21)

On the hand, by our assumption, we have

\[
R(p_i) \geq \frac{\epsilon}{f(p_i)} = \frac{\epsilon}{2^i} \quad \forall \ i > i_0.
\]

(7.22)

Note that \(\epsilon = \frac{\epsilon}{4} \leq \frac{1}{4}\). When \(i\) is large enough, we get a contradiction by combining (7.21) and (7.22). \(\square\)
By Claim 7.6, there exists a constant \( i_0 \) such that
\[
R(p_{i_0}) \leq \frac{\epsilon}{f(p_{i_0})}
\]
and
\[
f(p_{i_0}) \geq r_0(\epsilon, \epsilon),
\]
where \( r_0(\epsilon, \epsilon) \) is the constant in Lemma 7.4 and we take \( \epsilon = \frac{\epsilon}{4} \leq \frac{1}{4} \). If there are only finitely many \( p_i \) such that
\[
R(p_i) > \epsilon f(p_i),
\]
then (7.18) holds for \( i \geq i_0 + 1 \), where \( i_0 \) is the largest number such that \( p_i \) satisfies (7.25). Now, we assume that there are infinitely many \( N_i \).

We need the following claim.

**Claim 7.7.** Suppose \( N > i_0 \) and
\[
R(p_{N-1}) \leq \frac{\epsilon}{f(p_{N-1})}, \quad R(p_N) > \frac{\epsilon}{f(p_N)}.
\]
Then, we have
\[
R(p_N) \leq \frac{2\epsilon}{f(p_N)}
\]
and
\[
R(p_{N+1}) \leq \frac{\epsilon}{f(p_{N+1})}.
\]

**Proof.** By Lemma 7.2, we have
\[
R(p_N) \leq R(p_{N-1}) \leq \frac{2\epsilon}{f(p_{N-1})}.
\]
We have proved (7.27).

By Lemma 7.4 and (7.27), we have
\[
R(p_{N+1}) \leq \epsilon R(p_N) \leq \frac{2\epsilon \epsilon}{f(p_N)} = \frac{\epsilon}{f(p_{N+1})}.
\]
This completes the proof of the claim. \( \square \)

Now, we let \( N_i \) be the number such that \( p_{N_i} \) is the \( i \)-th point such that \( N_i > i_0 \) and
\[
R(p_{N_i}) > \frac{\epsilon}{f(p_{N_i})}.
\]
By Claim 7.7 and an induction argument, it is easy to see that \( N_{i+1} \geq N_i + 1 \) and therefore
\[
R(p_{N_i}) \leq \frac{2\epsilon}{f(p_{N_i})} \quad \forall \ i \geq 1.
\]
By our definition of \( N_i \) and the estimate (7.32), we have completed the proof of (7.18).

For any \( x \in M \) such that \( f(x) \geq 2^i \), we assume that \( f(x) \in [2^i, 2^{i+1}) \).
By Lemma 7.2 and (7.18), we get
\[
R(x) \leq R(p_i) \leq \frac{2e}{f(p_i)} = \frac{2e}{2^i} < \frac{4e}{f(x)} \quad \forall f(x) \geq 2^i.
\]

By the curvature estimate in [21], we see that \(|Rm|(x) \leq C R(x)|\), for all \( x \in M \) and some constant \( C \). By Theorem 6.1 in [32] and Theorem 2.1, we have the following theorem.

**Theorem 7.8.** Let \((M, g, f)\) be a 4-dimensional \( \kappa \)-noncollapsed steady gradient Ricci soliton which satisfies condition (1.3) and the scalar curvature \( R(x) \) satisfies
\[
R(x) \leq \frac{C}{\rho(x)} \quad \forall \rho(x) \geq r_0.
\]
If \((M, g, f)\) is non-Ricci-flat, then there exist constants \( c_1, r_0 > 0 \) such that
\[
R(x) \geq \frac{c_1}{\rho(x)} \quad \forall \rho(x) \geq r_0.
\]

Now, we prove Theorem 1.10.

**Proof of Theorem 1.10.** We prove this by contradiction. Suppose \((M, g, f)\) is not Ricci-flat.

**Case 1:** If the scalar curvature does not have uniform decay, then \((M, g, f)\) dimension reduces to a 3-dimensional \( \kappa \)-noncollapsed and non-flat steady Ricci soliton with maximal volume growth by Theorem 1.8 (which is proved in Section 9). However, 3-dimensional non-flat ancient \( \kappa \)-solutions cannot have maximal volume growth (see [50]). Hence, the steady Ricci soliton must be Ricci flat.

**Case 2:** In this case, we assume that the scalar curvature has uniform decay. By Lemma 7.5, there exists a constant \( r_0 \) such that
\[
R(x) \leq \frac{1}{f(x)} \quad \forall f(x) \geq r_0.
\]
By Proposition A.7, we see that \((M, g, f)\) is \( \kappa \)-noncollapsed. Hence, \((M, g, f)\) is a \( \kappa \)-noncollapsed steady gradient Ricci soliton with linear curvature decay.
By Theorem 7.8 and Theorem 2.1, there exist constants \( c_1, r_1 > 0 \) such that
\[
R(x) \geq \frac{c_1}{f(x)} \quad \forall f(x) \geq r_1.
\]
On the other hand, by taking $\epsilon = \frac{c_1}{2}$ in Lemma 7.5, we have

$$R(x) \leq \frac{c_1}{2f(x)} \quad \forall \ f(x) \geq r_0 \left( \frac{c_1}{2} \right).$$

Hence, we get a contradiction. We have completed the proof.

8. Noncollapsed GRS with nonnegative Ricci curvature

In this section, we will deal with 4-dimensional $\kappa$-noncollapsed steady gradient Ricci solitons with uniform scalar curvature decay and satisfying condition (1.3).

Proposition 8.1. Let $(M, g, f)$ be a 4-dimensional steady gradient Ricci soliton which is not Ricci flat and satisfies the hypotheses of Theorem 1.9. Given any $\gamma > 0$, if $p_i \in M$ and $r_i > 0$ satisfy $\text{Vol} B(p_i, r_i; g) \geq \gamma r_i^4$, then there exists a constant $C(\gamma)$ such that $r_i^2 R(q) \leq C(\gamma)$ for all $q \in B(p_i, r_i; g)$, where $C(\gamma)$ is independent of $p_i$ and $r_i$.

Proof. We prove the proposition by contradiction. If the proposition is not true, then by taking a subsequence, we may assume that

$$(8.1) \quad \text{Vol} B(p_i, r_i; g) \geq \gamma r_i^4,$$

$$(8.2) \quad r_i^2 Q_i \to \infty, \quad \text{as} \quad i \to \infty,$$

where $Q_i = R(q_i)$ and $q_i \in B(p_i, r_i; g)$.

By Lemma 9.37 in [45], we can find points $q'_i \in B(p_i, 2r_i; g)$ and constants $s_i \leq r_i$ such that

$$(8.3) \quad R(q'_i)s_i^2 = Q_i r_i^2,$$

and

$$(8.4) \quad R(q) \leq 4R(q'_i) \quad \text{for all} \quad q \in B(q'_i, s_i; g).$$

We set $Q'_i = R(q'_i)$. Then, (8.2) and (8.3) imply that

$$(8.5) \quad s_i^2 Q'_i \to \infty, \quad \text{as} \quad i \to \infty.$$

We also note that

$$B(p_i, r_i; g) \subseteq B(q'_i, 3r_i; g).$$

It follows that

$$(8.6) \quad \text{Vol} B(q'_i, 3r_i; g) \geq \text{Vol} B(p_i, r_i; g) \geq \gamma r_i^4 = \frac{\gamma}{81} \cdot (3r_i)^4.$$

Step 1: We show that $\{q'_i\}_{i \in \mathbb{N}}$ tends to infinity. If it is not true, then there exists a subsequence of $\{q'_i\}_{i \in \mathbb{N}}$ that stays in a bounded subset of $M$. 
By taking a subsequence, we may assume that \( q'_i \to q'_\infty \) as \( i \to \infty \). Now, we consider the volume growth of \((M, g)\). Let \( l_i = 3r_i \). By (8.5), we have

\[
R(q'_i) \cdot \frac{s_i^2}{r_i} \cdot \left( \frac{l_i}{3} \right)^2 = Q'_i s_i^2 \to \infty.
\]

Note that \( R(q'_i) \leq R_{\text{max}} \) and \( s_i/r_i \leq 1 \). Hence, we get \( l_i \to \infty \). Let \( \epsilon_i = d_g(q'_i, q'_\infty) \). Then, \( \epsilon_i \to 0 \). For \( i \) large, by (8.6), we have

\[
\text{vol} B(q'_\infty, l_i + \epsilon_i; g) \geq \text{vol} B(q'_i, l_i; g) \geq \frac{\gamma}{162} \cdot (l_i + \epsilon_i)^4.
\]

As \( l_i + \epsilon_i \) tends to infinity, the inequality above implies that \((M, g)\) has maximal volume growth. Therefore, \((M, g)\) is Ricci flat by Theorem 1.10. However, \((M, g)\) is not Ricci flat by our assumption. Hence, \( \{q'_i\}_{i \in \mathbb{N}} \) tends to infinity.

**Step 2:** We show that \( B(q'_i, 3r_i; g) \cap K = \emptyset \) when \( i \) large. If it is not true, we may assume that \( B(q'_i, 3r_i; g) \cap K \neq \emptyset \) by taking a subsequence. Let \( d = \text{Diam}(K, g) \) and fix a point \( o \in K \). Let \( l_i = 3r_i \) as in **Step 1**. Hence, \( d_g(q'_i, o) \leq d + l_i \). Then, \( B(q'_i, l_i; g) \subseteq B(o, 2(d + l_i); g) \). By (8.6), for \( i \) large, we have

\[
\frac{\text{vol} B(o, 2(d + l_i); g)}{(2(l_i + d))^4} \geq \gamma \cdot \frac{l_i^4}{81} \cdot \frac{1}{16(l_i + d)^4} \geq \frac{\gamma}{2 \cdot 6^4}.
\]

As in **Step 1**, \( l_i \to \infty \). Hence, (8.8) implies that \((M, g)\) has maximal volume growth. Hence, \((M, g)\) is Ricci flat. This contradicts our assumption. Hence, \( B(q'_i, 3r_i; g) \cap K = \emptyset \) when \( i \) is sufficiently large.

**Step 3:** Let \( \phi_i \) be generated by \(-\nabla f\) and \( g(t) = \phi^*_i g \). Let \( g_i(t) = Q'_i g((Q'_i)^{-1} t) \). Since \( B(q'_i, 3r_i; g) \cap K = \emptyset \) for large \( i \), we have \( \phi_i(x) \in M \setminus K \), for all \( x \in B(q'_i, 3r_i; g) \). Hence, \( \text{Ric}(x, t) \geq 0 \) for all \( x \in B(q'_i, 3r_i; g) \).

By the Bishop-Gromov volume comparison theorem and (8.6),

\[
\text{vol} B(q'_i, s, g) \geq \frac{\gamma}{81} \cdot s^4 \text{ for all } s \leq s_i.
\]

Since the Ricci curvature is nonnegative, we have \( \frac{\partial}{\partial t} R(x, t) \geq 0 \), for all \( x \in B(q'_i, 3r_i; g) \). Then, by (8.4), we have

\[
R_{g_i(t)}(q) \leq R_{g_i(0)}(q) \leq 4 \text{ for all } t \leq 0, q \in B(q'_i, s_i; g).
\]

By [21], there exists a constant \( C \) such that

\[
|R_{g_i(t)}(q)| \leq CR_{g_i(0)}(q) \leq 4C \text{ for all } t \leq 0, q \in B(q'_i, s_i; g).
\]

Note that \( B(q'_i, s_i; g_i(0)) = B(q'_i, s_i; g) \). By (8.5), (8.9) and (8.10), we obtain that \((M, g_i(t), q'_i)\) converges subsequentially to a complete limit \((M_\infty, g_\infty(t), g_\infty)\) for \( t \leq 0 \). Moreover, (8.5) and (8.9) implies that the asymptotic volume ratio of \((M_\infty, g_\infty(0))\) is greater than \( \frac{\gamma}{81} \).
Note that \( q_i' \) tends to infinity. Then, \( R(q_i') \to 0 \) as \( i \to \infty \) by assumption. Hence, we see that the limit \( (M_\infty, g_\infty(t), q_\infty) \) splits off a line. Hence, \( (M_\infty, g_\infty(t), q_\infty) \) is the product of a line and a 3-dimensional \( \kappa \)-solution. Since the asymptotic volume ratio of any 3-dimensional \( \kappa \)-solution is zero, the asymptotic volume ratio of \( (M_\infty, g_\infty(t), q_\infty) \) must be zero, too. This contradicts the volume growth of \( (M_\infty, g_\infty(0)) \) that we have obtained.

The proof of the proposition is complete. \( \square \)

The following lemma is similar to Corollary 2.4 in [28].

**Lemma 8.2.** Let \((M, g, f)\) be a steady gradient Ricci soliton satisfying the condition in Proposition 8.1. Then, its asymptotic scalar curvature ratio \( R(M, g) = \limsup_{x \to \infty} R(x) d(x, x_0)^2 = \infty. \)

**Proof.** We prove the corollary by contradiction. Suppose \( R(M, g) < A \) for some positive constant \( A > 1 \). For a fixed point \( p \in M \), we have \( R(x) \leq Ar^{-2} \) for all \( x \in M \setminus B(p, r) \) when \( r > r_0 \). Fix any \( q \in B(p, 3\sqrt{A}r) \setminus B(p, 2\sqrt{A}r) \). Then, we have \( R(x) \leq r^{-2} \) for all \( x \in B(q, r) \). Since \((M, g)\) is \( \kappa \)-noncollapsed, we get \( \text{vol} B(q, r) \geq kr^4 \). Hence,

\[
\text{vol} B(p, (3\sqrt{A} + 1)r) \geq \text{vol} B(q, r)
\]

\[
\geq \kappa(3\sqrt{A} + 1)^{-n}(3\sqrt{A} + 1)r^n \quad \text{for all} \quad r > r_0.
\]

It follows that

\[
\mathcal{V}(M, g) \geq \kappa(3\sqrt{A} + 1)^{-n}.
\]

This contradicts our assumption that \( \mathcal{V}(M, g) = 0 \). \( \square \)

Now, we begin to prove Theorem 1.11. The proof of Theorem 1.11 is similar to the proof of the compactness theorem of 3-dimensional \( \kappa \)-solutions in [50] (see also [45]). Under the hypotheses of Theorem 1.11, we let \( g_i(t) = R(p_i)gR^{-1}(p_i)t \). By Lemma 8.2, we can always find \( q_i \) such that

\[
d_{g_i(0)}(p_i, q_i)^2 R_{g_i(0)}(q_i) = 1.
\]

We first note that the following lemma holds.

**Lemma 8.3.** \( q_i \) tends to infinity.

**Proof.** If the lemma is not true, then we may assume that the \( q_i \) converge to a point \( q_\infty \). Since \((M, g)\) is not Ricci flat, \( R(q_\infty) > 0 \). Note that (8.11) implies that

\[
d_{g_i(0)}(p_i, q_i) R(q_i) = 1.
\]

By assumption, \( p_i \) tends to infinity. Therefore, \( d_g(p_i, q_i) \to \infty \) as \( i \to \infty \). Hence, \( R(q_i) \to 0 \) as \( i \to \infty \) by (8.12). This contradicts the fact that the \( q_i \) converge to \( q_\infty \) and \( R(q_\infty) > 0 \). \( \square \)
Let $d_i = d_{g_i(0)}(p_i, q_i)$. Then, we have the following curvature estimate.

**Lemma 8.4.** There is a uniform constant $C > 0$ such that $R_{g_i(0)}(x) \leq CR_{g_i(0)}(q_i)$ for all $x \in B(q_i, 2d_i; g_i(0))$.

**Proof.** Suppose that the lemma is not true. By taking a subsequence, we may assume that there exist points $q_i' \in B(q_i, 2d_i; g_i(0))$ such that

$$\lim_{i \to \infty} (2d_i)^2 R(q_i', 0) = \infty.$$  

By Proposition 8.1, for any $\gamma > 0$, there is an $i(\gamma)$ such that

$$\text{vol} B(q_i', 0) < \gamma (2d_i)^4$$

for all $i > i(\gamma)$. Hence, by applying the diagonal method, we may assume that

$$(8.13) \quad \lim_{i \to \infty} \text{vol} B(q_i, 2d_i, g_i(0))/(2d_i)^4 = 0.$$  

In particular,

$$\text{vol} B(q_i, 2d_i; g_i(0)) < (\omega/2)(2d_i)^4$$  

for all $i \geq i_0$, where $\omega$ is the volume of unit ball in $\mathbb{R}^4$ and $i_0$ is a constant.

Let $F_i(s) = \frac{\text{vol} B(q_i, s, g_i(0))}{s^4}$, for $s \in (0, 2d_i]$. Note that $F_i(s)$ is continuous. Moreover,

$$\lim_{s \to 0} F_i(s) = \omega\quad \text{and}\quad F_i(2d_i) < \frac{\omega}{2}.$$  

Therefore, there exists an $r_i < 2d_i$ for each $i \in \mathbb{N}$ such that $F_i(r_i) = \frac{\omega}{2}$, i.e.,

$$(8.14) \quad \text{vol} B(q_i, r_i, g_i(0)) = (\omega/2)r_i^4.$$  

By (8.13) and (8.14) we have

$$(8.15) \quad \lim_{i \to \infty} r_i/d_i = 0.$$  

Next we consider the sequence of rescaled ancient flows $(M_i, g_i'(t), q_i)$, where $g_i'(t) = r_i^{-2} g_i(r_i^2 t)$. Since we want to use Proposition 8.1 to get the curvature estimates on geodesic balls of $(M_i, g_i'(t))$, we need to show that $B(q_i, A; g_i'(0)) \cap K = \emptyset$ for any fixed $A \gg 1$. It suffices to exclude the following two cases.

**Case 1:** There exist infinite many $i$ such that $B(q_i, A; g_i'(0)) \cap K \neq \emptyset$ and $r_i \sqrt{R^{-1}(p_i)}$ is uniformly bounded. In this case, note that

$$(8.16) \quad B(q_i, Ar_i \sqrt{R^{-1}(p_i)}; g) = B(q_i, A; g_i'(0)).$$  

Then,

$$(8.17) \quad B(q_i, Ar_i \sqrt{R^{-1}(p_i)}; g) \cap K \neq \emptyset.$$
Suppose \( r_i \sqrt{R^{-1}(p_i)} \leq C \) for all \( i \). Then, (8.17) implies that \( d_g(q_i, K) \leq AC \). Hence, \( q_i \) stays in a bounded set of \( M \) as \( i \to \infty \). This contradicts Lemma 8.3. So this case is impossible.

**Case 2:** There exist infinitely many \( i \) such that \( B(q_i, A; g'_i(0)) \cap K \neq \emptyset \) and \( r_i \sqrt{R^{-1}(p_i)} \to \infty \) for \( i \to \infty \). In this case, let \( l_i = r_i \sqrt{R^{-1}(p_i)} \). Note that (8.16) and (8.17) still hold. (8.17) implies that \( d_g(q_i, K) \leq Al_i \). Fix a point \( o \in K \). We have

\[
B(q_i, Al_i; g) \subseteq B(o, Al_i + d; g),
\]

where \( d = \text{Diam}(K, g) \). By (8.18) and (8.14), we have

\[
\text{vol} B(o, Al_i + d; g) \geq \text{vol} B(q_i, l_i; g) = \frac{\omega}{2} \cdot l_i^4.
\]

Since \( l_i \to \infty \) and \( A, d \) are constants, (8.19) implies that \((M, g)\) has maximal volume growth. This is impossible by Theorem 1.10.

**Case 1** and **Case 2** imply that for any \( A \gg 1 \), there exists an \( i(A) \) such that for any \( i \geq i(A) \), we have

\[
B(q_i, A; g'_i(0)) \cap K = \emptyset.
\]

Hence, \( \text{Ric}(x, t) \geq 0 \) for all \( x \in B(q_i, A; g'_i(0)) \) and \( t \leq 0 \) when \( i \geq i(A) \).

By (8.14), we have

\[
\text{vol} B(q_i, A; g'_i(0)) \geq \text{vol} B(q_i, 1, g'_i(0)) = \frac{\omega}{2A^4} \cdot A^4,
\]

where \( A > 0 \) is any fixed constant. It follows that

\[
\text{vol} B(q_i, Al_i; g) \geq \frac{\omega}{2A^4} \cdot (Al_i)^4,
\]

where \( l_i = r_i \sqrt{R^{-1}(p_i)} \) and \( i \geq i(A) \). Note that

\[
B(q_i, A; g'_i(0)) = B(q_i, Al_i; g).
\]

By applying Proposition 8.1 to the ball \( B(q_i, Al_i; g) \), there is a constant \( K(A) \) independent of \( i \) such that

\[
A^2 R_{g'_i(0)}(q) = (Al_i)^2 R(q) \leq K(A), \forall q \in B(q_i, A; g'_i(0)).
\]

Since the flow \( g(t) \) is generated by a steady gradient Ricci soliton and the Ricci curvature is nonnegative on \( B(q_i, A; g'_i(0)) \), its scalar curvature is non-decreasing in \( t \). Hence, the scalar curvature on \( B(q_i, A; g'_i(0)) \times (-\infty, 0] \) is uniformly bounded by \( K(A)/A^2 \). By [21], there exists a constant \( C \) such that

\[
|\text{Rm}| \leq CR(x) \text{ for all } x \in M.
\]

It follows that

\[
|\text{Rm}_{g_i(t)}|_{g_i(t)}(x) \leq CR_{g_i(t)}(x) \leq CK(A) \text{ for all } (x, t) \in B(q_i, A; g'_i(0)) \times (-\infty, 0].
\]
By Hamilton’s Cheeger–Gromov compactness theorem, \((M_i, g'_i(t), q_i)\) converges to a limit flow \((M_\infty, g_\infty(t), q_\infty)\). Note by (8.15) that

\[
R(q_\infty, g_\infty(0)) = \lim_{i \to \infty} R(q_i, g'_i(0)) = \lim_{i \to \infty} \frac{(r_i)^2}{d_i^2} = 0.
\]

Therefore, the strong maximum principle implies that \((M_\infty, g_\infty(t))\) is a Ricci-flat flow. (8.22) implies that

\[
(8.23) \quad |\text{Rm}_{g_\infty(t)}|_{g_\infty(t)}(x) \leq CR_{g_\infty(t)}(x) \quad \text{for all} \quad x \in M_\infty.
\]

Hence, \((M_\infty, g_\infty(t))\) is flat.

At last, we prove that \((M_\infty, g_\infty(t))\) is isometric to Euclidean space for any \(t \leq 0\). Fix any \(r > 0\). Obviously,

\[
\sup_{x \in B(q_\infty, r; g_\infty(0))} |\text{Rm}(x)| = 0 \leq \varepsilon,
\]

where \(\varepsilon\) can be chosen so that \(\frac{\varepsilon}{\sqrt{\varepsilon}} > 2r\). Note that \((M_\infty, g_\infty(t))\) is \(\kappa\)-noncollapsed for each \(t \leq 0\). Thus we have

\[
\text{vol} B(q_\infty, r; g_\infty(0)) \geq \kappa r^4.
\]

It follows from the estimate of Cheeger, Gromov, and Taylor [22] that

\[
\text{inj}(q_\infty) \geq \frac{\pi}{2\sqrt{\varepsilon}} + \frac{1}{\omega(r/4)^4 \text{vol}(B(q_\infty, r; g_\infty(0)))} \geq \frac{\kappa}{\kappa + \omega} \cdot r.
\]

Hence \(B(q_\infty, \frac{\kappa}{\kappa + \omega} \cdot r; g_\infty(0))\) is simply connected for all \(r > 0\). Therefore, \(M_\infty\) is simply connected, and consequently \(g_\infty(t)\) are all isometric to the Euclidean metric.

Since \((M_\infty, g_\infty(t))\) is isometric to 4-dimensional Euclidean space, we obtain that \(\text{vol}(B(q_\infty, 1; g_\infty(0))) = \omega\). On the other hand, by the convergence of \((M_i, g'_i(t); p_i)\) and the relation (8.14), we get

\[
\text{vol}(B(q_\infty, 1; g_\infty(0))) = \omega/2.
\]

This is a contradiction. \(\square\)

We still need to show that the \(B(q_i, 2d_i; g_i(0))\) stay outside of \(K\) as \(i \to \infty\).

**Lemma 8.5.** For \(A \gg 2\), there exists a constant \(i(A) > 0\) such that \(B(q_i, Ad_i; g_i(0)) \cap K = \emptyset\) for \(i \geq i(A)\).

**Proof.** Let \(h_i(t) = d_i^{-2} g_i(d_i^2 t)\). Then, \(h_i(0) = R(q_i)g\). So, we only need to show that \(B(q_i, A\sqrt{R^{-1}(q_i)}; g) \cap K = \emptyset\) when \(i \geq i(A)\) for some \(i(A) > 0\).

If it is not true, then we may assume that \(B(q_i, A\sqrt{R^{-1}(q_i)}; g) \cap K \neq \emptyset\) as \(i \to \infty\) by taking a subsequence. Let \(l_i = A\sqrt{R^{-1}(q_i)}\). By Lemma 8.3 and
the curvature uniform decay assumption, we have \( l_i \to \infty \) as \( i \to \infty \). Since \( h_i(t) \) is \( \kappa \)-noncollapsing, by Lemma 8.4 and [21], we have
\[
\text{vol } B(q_i, l_i; g) \geq c l_i^{4},
\]
where \( c \) is a positive constant. Let \( d = \text{Diam}(K, g) \). Fixing \( o \in K \), we have
\[
\text{vol } B(o, 2(A l_i + d); g) \geq \text{vol } B(q_i, l_i; g) \geq c l_i^{4},
\]
Note that \( l_i \to \infty \) as \( i \to \infty \). Since \( A \) and \( d \) are constants, \( (M, g) \) has maximal volume growth. This contradicts Theorem 1.10. □

Compared with 3-dimensional ancient \( \kappa \)-solutions, we do not have the Harnack inequality (see [36]) in our case. Fortunately, we have the following lemma.

**Lemma 8.6.** Let \((M_i, g_i(t), p_i)\) be a sequence of \( \kappa \)-noncollapsed ancient and complete Ricci flows. Suppose there exist a constant \( C \) such that
\[
|\text{Rm}_{g_i}(t)|_{g_i(t)}(x) \leq CR_{g_i(t)}(x) \text{ for all } x \in M_i, \ t \leq 0
\]
and
\[
\frac{\partial}{\partial t} R_{g_i(t)}(x) \geq 0 \text{ for all } x \in M_i, \ t \leq 0.
\]
We also assume that \( R_{g_i(0)}(p_i) = 1 \) and
\[
R_{g_i(0)}(x) \leq C_1 \text{ for all } x \in B(p_i, 2; g_i(0)).
\]
Then, there exists a constant \( \delta > 0 \) independent of \( i \) and \( x \) such that
\[
R_{g_i(0)}(x) \geq \delta \text{ for all } i \in \mathbb{N} \text{ and } d_{g_i(0)}(x, p_i) = 1.
\]

*Proof.* We prove this by contradiction. If the lemma is not true, we can find \( x_i \in M_i \) such that \( d_{g_i(0)}(x_i, p_i) = 1 \) and
\[
R_{g_i(0)}(x_i) \to 0, \text{ as } i \to \infty.
\]
By our assumption, it is easy to see that \((B(p_i, 2; g_i(0)), g_i(t), p_i)\) converge subsequentially to a limit \((B_\infty, g_\infty(t), p_\infty)\). Note that \( R_{g_\infty(0)}(p_\infty) = 1 \). Since \( g_i(t) \) is ancient and complete for each \( i \), \( R_{g_i(t)}(x) \geq 0 \) (See [24]). Hence, \( R_{g_\infty(t)}(x) \geq 0 \) for all \( x \in B_\infty \) and \( t \leq 0 \). By (8.28), we have \( R_{g_\infty(0)}(x_\infty) = 0 \), where \( x_\infty \) is the limit of \( x_i \). By the maximum principle, we see that \( R_{g_\infty(t)}(x) \) is flat for all \( x \in B_\infty \) and \( t \leq 0 \). This contradicts the fact that \( R_{g_\infty(0)}(p_\infty) = 1 \). Hence, we have completed the proof. □

Now, we complete the proof of Theorem 1.11.
Proof of Theorem 1.11. By Lemma 8.4, there is a uniform constant $C > 0$ such that $R_{g_i(0)}(x) \leq CR_{g_i(0)}(q_i)$ for all $x \in B(q_i, 2d_i; g_i(0))$, where $d_i^2R_{g_i(0)}(q_i) = 1$. Let $h_i(t) = d_i^2g_i(d_i^2t)$, for $t \leq 0$. Then, $R_{h_i(0)}(q_i) = 1$ and 
\begin{equation}
R_{h_i(0)}(x) \leq C \quad \text{for all } \quad x \in B(q_i, 2; h_i(0)) = B(q_i, 2d_i; g_i(0)).
\end{equation}
We also note that $d_{h_i(0)}(p_i, q_i) = 1$. By Lemma 8.5, $\text{Ric}_{h_i(t)}(x) \geq 0$ for all $x \in B(q_i, 2; h_i(0))$ and $t \leq 0$. Hence, $R_{h_i(t)}(x)$ is increasing in $t$ for all $x \in B(q_i, 2; h_i(0))$. Applying Lemma 8.6 to $(M, h_i(t), q_i)$ and $p_i$, there exists a positive constant $\delta > 0$ such that
\begin{equation}
R_{h_i(0)}(p_i) \geq \delta.
\end{equation}
It follows that
\begin{equation}
d_i^2 = \frac{1}{R_{g_i(0)}(q_i)} = \frac{R_{g_i(0)}(p_i)}{R_{g_i(0)}(q_i)} = R_{h_i(0)}(p_i) \geq \delta.
\end{equation}
Combining the above result and Lemma 8.4, we have proved the following estimate
\begin{equation}
R_{g_i(0)}(x) \leq C\delta^{-1} \quad \text{for all } \quad x \in B(p_i, \sqrt{\delta}; g_i(0)).
\end{equation}
We take $\varepsilon = \frac{1}{\sqrt{C+1}}$. By the $\kappa$-noncollapsing property of $g_i(t)$, we get
\[ \text{vol} \, B(p_i, \varepsilon; g_i(0)) \geq \kappa \varepsilon^4. \]
For any $r > 0$ large enough, $B(p_i, \varepsilon + r; g_i(0)) \subseteq B(q_i, 2r\delta^{-\frac{1}{2}}d_i; g_i(0))$. By Lemma 8.5, $B(p_i, \varepsilon + r; g_i(0)) \cap K = \emptyset$ for $i \geq i(r\delta^{-\frac{1}{2}})$, where $i(r\delta^{-\frac{1}{2}})$ is a constant. Hence, $(M, g_i(t))$ has nonnegative Ricci curvature on $B(p_i, \varepsilon + r; g_i(0))$ when $i \geq i(r\delta^{-\frac{1}{2}})$. Therefore, we have
\[ \text{vol} \, B(p_i, \varepsilon + r; g_i(0)) \geq \text{vol} \, B(p_i, \varepsilon; g_i(0)) \geq \frac{\kappa}{(1 + (r/\varepsilon))^4} (\varepsilon + r)^4. \]
Hence,
\[ \text{vol} \, B(p_i, (\varepsilon + r)\sqrt{R^{-1}(p_i); g}) \geq \frac{\kappa}{(1 + (r/\varepsilon))^4} \cdot [(\varepsilon + r)\sqrt{R^{-1}(p_i)}]^{4}. \]
Applying Proposition 8.1 to each ball $\text{vol} \, B(p_i, (\varepsilon + r)\sqrt{R^{-1}(p_i); g})$, we see that there is a $C(r)$ independent of $i$ such that
\[ R_{g_i(0)}(q) \leq C(r)(r + \varepsilon)^{-2} \quad \text{for all } \quad q \in B(p_i, \varepsilon + r; g_i(0)). \]
Since the scalar curvature is non-decreasing, we also get
\begin{equation}
R_{g_i(0)}(q) \leq C(r)(r + \varepsilon)^{-2} \quad \text{for all } \quad q \in B(p_i, \varepsilon + r; g_i(0)).
\end{equation}
Recall that $(M, g_i(t))$ has nonnegative Ricci curvature on $B(p_i, \varepsilon + r; g_i(0))$ and $g_i(t)$ satisfies the Ricci flow equation. Then,
\[ g_i(q, t) \geq g_i(q, 0) \quad \forall \, t \leq 0, \quad q \in B(p_i, \varepsilon + r; g_i(0)). \]
Therefore,
\[ B(p_i, \varepsilon + r; g_i(t)) \subset B(p_i, \varepsilon + r; g_i(0)) \quad \forall \, t \leq 0. \]

By (8.33) and (8.21), for \( t \leq 0 \), we have
\[ |Rm|_{g_i(t)}(q) \leq C(n)C(r)(r + \varepsilon)^{-2} \quad \text{for all} \quad q \in B(p_i, \varepsilon + r; g_i(t)). \]

For any \( r > 0 \) large enough, we can find constant \( C(r) \) such that (8.34) holds. Hence, Theorem 1.7 in [53] implies that \((M, g_i(t), p_i)\) subsequentially converges to a complete Ricci flow \((M_\infty, g_\infty(t))\) for any \( t \leq 0 \). The splitting follows from the scalar curvature decay as proved in Claim 6.4. Note that the limit flow is a 3-dimensional ancient flow which has monotonic scalar curvature in \( t \) and distance-curvature estimate. Hence, the flow has uniformly bounded curvature at each time slice.

\[ \square \]

9. Dimension reduction for steady GRS without curvature decay

In this section, we prove Theorem 1.8. We first introduce a lemma.

**Lemma 9.1.** Let \((M, g, f)\) be an \( n \)-dimensional \( \kappa \)-noncollapsed steady gradient Ricci soliton with bounded curvature. Suppose \( \{p_i\}_{i \in \mathbb{N}} \) is a sequence of points tending to infinity. Then, \((M, g, p_i)\) subsequentially converges to a gradient steady Ricci soliton \((M_\infty, g_\infty, p_\infty)\).

**Proof.** Since \((M, g, f)\) is \( \kappa \)-noncollapsed and has bounded curvature, it is easy to see that \((M, g, p_i)\) subsequentially converges in the Cheeger-Gromov sense to a limit \((M_\infty, g_\infty, p_\infty)\). Let \( f_i(x) = f(x) - f(p_i) \), for \( i \in \mathbb{N} \) and \( R_{\text{max}} = \sup_{x \in M} R(x) \). By integrating \( f(x) \) along a minimal geodesic connecting \( p_i \) and \( x \), we get
\[ |f(x) - f(p_i)| \leq \sqrt{R_{\text{max}} d(x, p_i)}. \]

Hence, for any fixed \( r > 0 \), we have
\[ |f_i(x)| \leq r \quad \forall \, x \in B(p_i, r; g). \]

Since the curvature is bounded, we also note that
\[ |\nabla^k f_i|(x) = |\nabla^{k-2}\text{Ric}|(x) \leq C \quad \forall \, x \in M. \]

By (9.1) and (9.2), we see that \( f_i(x) \) converges to a smooth function \( f_\infty(x) \) defined on \( M_\infty \). Note that
\[ \nabla \nabla f_i(x) = \nabla \nabla f(x) = \text{Ric} \quad \forall \, x \in M. \]

By the convergence of \((M, g, p_i)\) and \( f_i(x) \), we have
\[ \nabla \nabla f_\infty(x) = \text{Ric}_\infty(x) \quad \forall \, x \in M_\infty. \]
We have completed the proof. □

Next, we prove a special case of Theorem 1.8.

Lemma 9.2. Let \((M, g, f)\) be an \(n\)-dimensional \(\kappa\)-noncollapsed steady gradient Ricci soliton with bounded curvature and nonnegative Ricci curvature. Suppose the scalar curvature \(R(x)\) attain its maximum at \(o \in M\) and \(|\nabla f|_o = 1\). Then, \((M, g, f)\) weakly dimension reduces to an \((n - 1)\)-dimensional steady gradient Ricci soliton.

Proof. Let \(\phi_t\) be a group of diffeomorphisms generated by \(-\nabla f\) and \(g(t) = \phi_t^* g\). Let \(\gamma(s)\) be the integral curve of \(\nabla f\) passing through \(o\) such that \(\gamma(0) = o\). Note that \(\gamma(s) = \phi_{-s}(o)\). We first show that \(\gamma(s)\) is a geodesic with respect to \(g\).

We may assume that \(R(\gamma(s)) = R_{\max}\) for all \(s \in \mathbb{R}\), where \(R_{\max} = \sup_{x \in M} R(x)\). Hence,

\[
\text{Ric}(\nabla f, \nabla f)(\gamma(s)) = -\frac{1}{2} \cdot \frac{dR(\gamma(s))}{ds} = 0.
\]

(9.3)

Therefore, \(\nabla f(\gamma(s))\) is a zero eigenvector of \(\text{Ric}(\gamma(s))\). Hence,

\[
\text{Ric}(\nabla f, Y)(\gamma(s)) = 0 \quad \forall Y \in T_{\gamma(s)} M.
\]

It follows that

\[
\langle \nabla \gamma'(s), \gamma'(s), Y \rangle = \text{Ric}(\nabla f, Y) = 0 \quad \forall Y \in T_{\gamma(s)} M.
\]

Hence, \(\gamma(s)\) is a geodesic.

Let \(p_i = \gamma(t_i)\) for \(t_i \to +\infty\). By Lemma 9.1, \((M, g(t), p_i)\) converges subsequentially to \((M, g_\infty, p_\infty)\). Moreover, there exists a smooth function \(f_\infty\) such that \(\text{Ric}_\infty = \text{Hess} f_\infty\)

and

\[
\nabla f \to \nabla f_\infty, \text{ as } i \to \infty.
\]

Let \(\gamma_\infty(s)\) be the integral curve of \(\nabla f_\infty\) passing through \(p_\infty\) such that \(\gamma_\infty(0) = p_\infty\). Similar to \(\gamma(s)\), we can show that \(\gamma_\infty(s)\) is a geodesic with respect to \(g_\infty(0)\). Actually, we want to show that \(\gamma_\infty(s)\) is a geodesic line. We need the following claim.

---

4It is easy to check that \(R(\gamma(s)) = R_{\max}\) for all \(s \leq 0\). Then, \((M, g, \gamma(-i))\) converges to a limit \((M, g_\infty, p_\infty)\) with potential \(f_\infty\) when \(i \to \infty\). Let \(\varphi_t\) be generated by \(-\nabla f_\infty\). Then, \(R_{\infty}(\varphi_t(p_\infty)) = R_{\max}\) for all \(t \in \mathbb{R}\). Hence, one may replace \((M, g, f, o)\) by \((M_\infty, g_\infty, f_\infty, p_\infty)\).
Claim 9.3. Suppose $a < b$ and $a,b \in \mathbb{R}$. Then,

$$d_\infty(\gamma_\infty(a), \gamma_\infty(b)) = \sup_{r \in \mathbb{R}} d(\gamma(t), \gamma(b - a + t)),$$

where $d$ and $d_\infty$ are the distance functions with respect to $g$ and $g_\infty(0)$, respectively.

Proof. By the convergence of $(M, g(t), p_i)$ and $\nabla f$, we have

$$d_\infty(\gamma_\infty(a), \gamma_\infty(b)) = \lim_{t_i \to +\infty} d(\gamma(t_i), \gamma(b - a + t_i)).$$

So, it suffices to show that $d(\gamma(t), \gamma(b - a + t))$ is increasing in $t$. More precisely, we only need to show that

$$d(\gamma(c), \gamma(d)) \leq d(\gamma(c + t), \gamma(d + t)) \quad \forall \, c, d \in \mathbb{R}, \; t > 0.$$

Let $l(\sigma)$ be a minimal geodesic connecting $\gamma(c)$ and $\gamma(d)$ with respect to $g(-t)$. Suppose $l(0) = \gamma(t)$ and $l(L) = \gamma(d)$. Since the Ricci curvature is nonnegative, for $t \geq 0$, we have

$$d_{g(-t)}(\gamma(c), \gamma(d)) = \int_0^L \sqrt{\langle l'(s), l'(s) \rangle_{g(-t)}} ds \geq \int_0^L \sqrt{\langle l'(s), l'(s) \rangle_g} ds \geq d(\gamma(c), \gamma(d)).$$

It follows that

$$d(\gamma(c + t), \gamma(d + t)) = d_{g(-t)}(\gamma(c), \gamma(d)) \geq d(\gamma(c), \gamma(d)) \quad \forall \, t \geq 0.$$

We have completed the proof of the claim. \qed

As a corollary of Claim 9.3, we have

$$(9.4) \quad d_\infty(\gamma_\infty(a), \gamma_\infty(b)) = d_\infty(\gamma_\infty(a + t), \gamma_\infty(b + t)) \quad \forall \, a, b, t \in \mathbb{R}. $$

Now, we show that $\gamma_\infty(s)$ is a minimal geodesic connecting $\gamma_\infty(a)$ and $\gamma_\infty(b)$ with respect to $g_\infty(0)$. Suppose $l(s)$ is a minimal geodesic connecting $\gamma_\infty(a)$ and $\gamma_\infty(b)$ with respect to $g_\infty(0)$. Suppose $l(0) = \gamma_\infty(a)$ and $l(L) = \gamma_\infty(b)$. We want to show that

$$(9.5) \quad \text{Ric}_\infty(l'(s), l'(s)) = 0 \quad \forall \, s \in [0, L].$$
For $\delta > 0$, we have
\[
\begin{align*}
&d_\infty(\gamma_\infty(a), \gamma_\infty(b)) - d_\infty(\gamma_\infty(a - \delta), \gamma_\infty(b - \delta)) \\
= &d_\infty(\gamma_\infty(a), \gamma_\infty(b)) - d_\infty(\gamma_\infty(a), \gamma_\infty(b)) \\
\geq &\int_0^L \sqrt{\langle l'(s), l'(s) \rangle_{g_\infty(0)}} ds - \int_0^L \sqrt{\langle l'(s), l'(s) \rangle_{g_\infty(\delta)}} ds \\
= &\int_0^\delta \int_0^L \frac{2 \text{Ric}_{g_\infty(\sigma)}(l'(s), l'(s))}{\sqrt{\langle l'(s), l'(s) \rangle_{g_\infty(\sigma)}}} ds d\sigma.
\end{align*}
\]
(9.6)

By (9.4) and (9.6), we get
\[
\int_0^\delta \int_0^L \frac{2 \text{Ric}_{g_\infty(\sigma)}(l'(s), l'(s))}{\sqrt{\langle l'(s), l'(s) \rangle_{g_\infty(\sigma)}}} ds d\sigma = 0.
\]
(9.7)

Since the Ricci curvature is nonnegative, (9.7) implies (9.5). Note that (9.5) implies that $l'(s)$ is a zero eigenvector of $\text{Ric}_\infty(l(s))$. Hence,
\[
\text{Ric}_\infty(l'(s), Y)(l(s)) = 0 \quad \forall \ Y \in T_{l(s)}M_\infty.
\]

Therefore,
\[
\frac{dR_\infty(l(s))}{ds} = -2 \text{Ric}_\infty(\nabla f_\infty(l(s)), l'(s)) = 0 \quad \forall \ s \in [0, L].
\]

Hence,
\[
R_\infty(l(s)) = R_\infty(\gamma_\infty(s)) = R_\infty(p_\infty).
\]

By the convergence of $(M, g(t), p_t)$, we have
\[
|\nabla f_\infty|_{g_\infty(0)}(p_\infty) = 1.
\]

Note that
\[
|\nabla f_\infty|^2(x) + R_\infty(x) \equiv C.
\]

We conclude that
\[
|\nabla f_\infty|_{g_\infty(0)}(l(s)) = 1 \quad \forall \ s \in [0, L].
\]

For any $b > a$, we have
\[
\begin{align*}
f_\infty(\gamma_\infty(b)) - f_\infty(\gamma_\infty(a)) &= \int_0^L \langle l'(s), \nabla f_\infty(l(s)) \rangle_{g_\infty(0)} ds \\
&\leq \int_0^L |l'(s)|_{g_\infty(0)} \cdot |\nabla f_\infty(l(s))|_{g_\infty(0)} ds \\
& \leq L.
\end{align*}
\]
(9.8)
Note that \( l(s) \) is a minimal geodesic. Since \( \gamma_\infty(s) \) is a curve connecting \( \gamma_\infty(a) \) and \( \gamma_\infty(b) \) and the length of \( \gamma_\infty(s)|_{[a,b]} = b - a \), we get
\[
(9.9) \quad f_\infty(\gamma_\infty(b)) - f_\infty(\gamma_\infty(a)) \leq L \leq b - a.
\]
On the other hand, since \( \gamma_\infty(s) \) is the integral curve of \( \nabla f \) and \( |\nabla f|(\gamma(s)) = 1 \), we have
\[
(9.10) \quad f_\infty(\gamma_\infty(b)) - f_\infty(\gamma_\infty(a)) = b - a.
\]
By (9.9) and (9.10), we have \( L = b - a \).

Finally, we get \( \gamma_\infty(s) \) is a minimal geodesic connecting \( \gamma_\infty(a) \) and \( \gamma_\infty(b) \) for any \( a, b \in \mathbb{R} \). Hence, \( \gamma_\infty(s) \) is a geodesic line. Since \( (M_\infty, g_\infty(0)) \) has nonnegative Ricci curvature, it splits off a line by the Cheeger-Gromoll splitting theorem. We have completed the proof.

\[ \square \]

Now, we can prove the dimension reduction result in a more general case.

**Lemma 9.4.** Let \((M, g, f)\) be an \( n \)-dimensional \( \kappa \)-noncollapsed steady gradient Ricci soliton with bounded curvature and nonnegative Ricci curvature. Suppose there exists a sequence of points \( p_i \) tending to infinity such that \( R(p_i) \) attains the maximum of \( R(x) \) at each \( p_i \). Then, \((M, g, f)\) weakly dimension reduces to an \((n - 1)\)-dimensional steady gradient Ricci soliton.

**Proof.** If there exists a point \( x_0 \in M \) such that \( R(x_0) \) attains the maximum of \( R(x) \) and \( |\nabla f|(x_0) > 0 \), then the lemma holds according to Lemma 9.2. Now, it suffices to consider the case that \( |\nabla f|(p) = 0 \) if \( R(p) = R_{\text{max}} \), where \( R_{\text{max}} = \sup_{x \in M} R(x) \). Hence, \( |\nabla f|(p_i) = 0 \) for all \( i \in \mathbb{N} \).

Let \( \phi_t \) be a group of diffeomorphisms generated by \( -\nabla f \) and \( g(t) = \phi_t^* g \). Let \( \gamma_i(s) \) be a minimal geodesic connecting \( p_0 \) and \( p_i \) with respect to \( g \). Since \( \phi_t \) is an isomorphism and \( |\nabla f|(p_i) = 0 \), we have
\[
d_{g(t)}(p_1, p_0) = d(p_1, p_0) \quad \forall \ t \in \mathbb{R}.
\]

Let \( d_i = d(p_i, p_0) \). Let \( L(t) \) be the length of \( \gamma_i(s)|_{[0,d_i]} \) and \( s \) be the arc-parameter with respect to \( g(t) \). By the nonnegativity of the Ricci curvature, we have
\[
\frac{dL(t)}{dt} = -\int_0^{d_i} 2\text{Ric}_{g(t)}(\gamma'_i(s), \gamma'_i(s)) \langle \gamma'_i(s), \gamma'_i(s) \rangle_g(t) ds \leq 0.
\]
Hence,
\[
L(t) \leq d_i \quad \forall \ t \geq 0.
\]

---

\(^5\)By rescaling, this is equivalent to the case that \( |\nabla f|(x_0) = 1 \).
On the other hand,

\[ L(t) \geq d_{g(t)}(p_0, p_i) = d_i. \]

Therefore,

\[ L(t) \equiv d_i. \]

It follows that

\[ \text{Ric}_{g(t)}(\gamma_i'(s), \gamma_i'(s)) \equiv 0 \quad \forall \, s \in [0, d_i], \, t \in \mathbb{R}. \]

Then,

\[ \text{Ric}(\gamma_i'(s), Y) = 0 \quad \forall \, Y \in T_{\gamma_i(s)}M. \]

Hence,

\[ \frac{dR(\gamma_i(s))}{ds} = 2\text{Ric}(\nabla f(\gamma_i(s)), \gamma_i(s)) = 0. \]

It follows that

\[ R(\gamma_i(s)) = R(\gamma_i(0)) = R(p_0) = R_{\text{max}}. \]  \hfill (9.11)

Let \( q_i = \gamma_i(d_i^2) \). Note that \( R(q_i) = R_{\text{max}} \) and \( q_i \) tends to infinity. Moreover, \( \gamma_i(s)|_{[0,d_i]} \) is a minimal geodesic passing through \( q_i \), and its length \( d_i \) tends to infinity. Now, we consider \((M, g(t), q_i)\). By taking a subsequence, \((M, g(t), p_i)\) converges to \((M_\infty, g_\infty(t), p_\infty)\). This means that there exist diffeomorphisms \( \Phi_i : U_i(\subseteq M_\infty) \to \Phi_i(U_i)(\subseteq M_i) \) such that \( \Phi_i^*(g(t)) \) converges to \( g_\infty(t) \) and \( \Phi_i(p_\infty) = p_i \), where the \( U_i \) exhaust \( M_\infty \). Let

\[ V_i = \left. \frac{d}{ds} \right|_{s=d_i^2} \Phi_i^{-1}(\gamma_i(s)) \quad \forall \, i \in \mathbb{N}. \]  \hfill (9.12)

Since \( s \) is the arc-parameter, we get \( |V_i| \Phi_i^*(g) = 1 \) and \( V_i \in T_{p_\infty}M_\infty \). By taking a subsequence, we may assume that \( V_i \to V_\infty \) as \( i \to \infty \). Therefore, by the convergence of \((M, g(t), q_i)\) and \( V_i \), we get that \( \gamma_i(s)|_{[0,d_i]} \) converges subsequentially to a geodesic line passing through \( p_\infty \). Note that \((M_\infty, g_\infty(t))\) has nonnegative Ricci curvature. We conclude that \((M_\infty, g_\infty(t))\) splits off a line. Similar to the proof of Lemma 9.2, \((M_\infty, g_\infty(t))\) is also a steady gradient Ricci soliton. \[ \square \]

We also need the following lemma.

**Lemma 9.5.** Let \((M, g, f)\) be a non-Ricci-flat gradient steady Ricci soliton with nonnegative Ricci curvature. Let \( S = \{ x \in M : \nabla f(x) = 0 \} \). Suppose \( S \) is not empty. Then, for any \( p \in M \), we have

\[ \lim_{t \to +\infty} d(\phi_t(p), S) = 0, \]

where \( \phi_t \) is generated by \(-\nabla f\).
Proof. Let $g(t) = \phi^*_t g$. Then, $g(t)$ satisfies the Ricci flow equation. Note that $\phi_t$ is an isomorphism and the Ricci curvature of $(M, g(t))$ is nonnegative. For any $q \in M$, we have $d(\phi_t(q), \phi_t(p)) = d_t(p, q)$ is decreasing in $t$.

Suppose $d = d(p, S)$. Since $S$ is closed, we can find $q \in S$ such that $d(p, q) = d$. Hence, we have

$$d(\phi_t(p), S) \leq d(\phi_t(p), q) = d_t(p, q) \leq d, \text{ for } t \geq 0.$$ 

Therefore, there exists a sequence of times $t_i \to +\infty$ such that $\phi_{t_i}(p)$ converges to some point $p_\infty$.

Now, we show that $p_\infty \in S$. If $p_\infty \notin S$, then we let $c_0 = |\nabla f|^2(p_\infty) > 0$. Since the Ricci curvature is nonnegative, we have

$$\frac{d}{dt} |\nabla f|^2(\phi_t(p)) = -\frac{d}{dt} R(\phi_t(p)) \leq 0. \tag{9.13}$$

By the convergence of $\phi_{t_i}(p)$ and (9.13), we have

$$|\nabla f|^2(\phi_{t_i}(p)) \geq |\nabla f|^2(p_\infty) = c_0 \forall t_i \geq 0. \tag{9.14}$$

By (9.13) and (9.14), we have

$$|\nabla f|^2(\phi_t(p)) \geq c_0, \text{ for } t \in \mathbb{R}.$$ 

Then,

$$f(p) - f(p_\infty) = \lim_{t_i \to +\infty} \int_0^{t_i} |\nabla f|^2(\phi_t(p))dt \geq \lim_{t_i \to +\infty} c_0 t_i = +\infty.$$

This is impossible. Hence, we have proved that $p_\infty \in S$. Hence,

$$\lim_{t_i \to +\infty} d(\phi_{t_i}(p), p_\infty) = 0.$$ 

Note that

$$d(\phi_t(p), S) \leq d(\phi_t(p), p_\infty) \leq d(\phi_{t_i}(p), p_\infty) \forall t \geq t_i.$$ 

Hence, we have completed the proof. \hfill \square

Now, we prove Theorem 1.8 by assuming the Ricci curvature is nonnegative.

**Lemma 9.6.** Theorem 1.8 holds when the Ricci curvature is nonnegative.

**Proof.** We first note that Lemma 9.6 can be reduced to the case that there exists a point $p_0 \in M$ such that $R(p_0) = R_{\text{max}}$, where $R_{\text{max}} = \sup_{x \in M} R(x)$.

If $R(x) < R_{\text{max}}$ for any $x \in M$, then there exists a sequence of points $p_i$ tending to infinity such that $R(p_i) \to R_{\text{max}}$. By taking a subsequence, we see that $(M, g(t), p_i)$ converges to $(M_\infty, g_\infty(t), p_\infty)$. As in the proof of Lemma 9.2, $(M_\infty, g_\infty(t), p_\infty)$ is a gradient steady Ricci soliton with nonnegative Ricci curvature and bounded curvature. Moreover, $R_\infty(x, 0)$ attains its maximum at $p_\infty$. If $(M_\infty, g_\infty(0), p_\infty)$ weakly dimension reduces to an $(n -$
1)-dimensional steady Ricci soliton, then \((M, g, f)\) also weakly dimension reduces to the same steady Ricci soliton.

Hence, we may assume there exists a point \(p_0 \in M\) such that \(R(p_0) = R_{\max}\). Let \(S' = \{x \in M : R(x) = R_{\max}\}\). If \(S'\) is unbounded, then \((M, g, f)\) weakly dimension reduces to an \((n - 1)\)-dimensional steady Ricci soliton by Lemma 9.4. Therefore, we may assume that \(S'\) is a non-empty and bounded set. By Lemma 9.2, if there exists a point \(x_0 \in S'\) such that \(|\nabla f(x_0)| > 0\), then \((M, g, f)\) weakly dimension reduces to an \((n - 1)\)-dimensional steady Ricci soliton.

Finally, we only need to consider the case that \(S'\) is a non-empty and bounded set and \(|\nabla f(x)| = 0\), for all \(x \in S'\). Let \(S = \{x \in M : \nabla f(x) = 0\}\). In this case, \(S' = S\). By Lemma 9.5, for any \(p \in M \setminus S\), we have

\[
\phi_t(p) \to S, \quad \text{as } t \to +\infty.
\]  

By (9.15), for any \(x \in M\) such that \(d(x, S) \geq 1\), there exists a constant \(t_x \geq 0\) such that \(d(\phi_{t_x}(x), S) = 1\). Since the Ricci curvature is nonnegative, \(R(\phi_t(x))\) is increasing in \(t\). Hence,

\[
R(x) \leq R(\phi_{t_x}(x)) \leq \sup_{d(y, S) = 1} R(y) \quad \forall d(x, S) \geq 1.
\]

Let \(C = \sup_{d(y, S) = 1} R(y)\). Obviously, \(C < R_{\max}\). Hence, we get

\[
R(x) \leq C < R_{\max} \quad \forall d(x, S) \geq 1.
\]

Let \(A = \lim_{r \to \infty} \sup_{x \in M \setminus B(x_0, r)} R(x)\), where \(x_0\) is a fixed point. Since we have assumed that the scalar curvature does not have uniform decay, we get \(A > 0\). We also note that \(A \leq C < R_{\max}\). We can choose a sequence of points \(\{p_i\}\) tending to infinity such that \(R(p_i) \to A\) as \(i \to \infty\). It is easy to see that \((M, g(t), p_i)\) converges subsequentially to a limit \((M_\infty, g_\infty(t), p_\infty)\). Then, \((M_\infty, g_\infty(t), p_\infty)\) is a steady gradient Ricci soliton with nonnegative Ricci curvature and bounded curvature. We also have \(R_{\infty}(p_\infty, 0)\) attains the maximum of \(R_{\infty}(x, 0)\) at the point \(p_\infty\). By the convergence, we also have

\[
|\nabla f_\infty|^2(p_\infty) + R_{\infty}(p_\infty, 0) = \lim_{i \to \infty} (|\nabla f|^2(p_i) + R(p_i)) = R_{\max}
\]

and

\[
R_{\infty}(p_\infty, 0) = \lim_{i \to \infty} R(p_i) = A < R_{\max}.
\]

Hence,

\[
|\nabla f_\infty|^2(p_\infty) = R_{\max} - A > 0.
\]
By Lemma 9.2, \((M_\infty, g_\infty(0), p_\infty)\) weakly dimension reduces to an \((n-1)\)-dimensional steady Ricci soliton. Therefore, \((M, g, f)\) also dimension reduces to an \((n-1)\)-dimensional steady Ricci soliton. We have completed the proof.

\[
\square
\]

To prove Theorem 1.8, we need to introduce the following lemma.

**Lemma 9.7.** Let \((M, g)\) be a complete Riemannian manifold and let \(\{p_j\}_{j \in \mathbb{N}_+}\) be a sequence of points tending to infinity. Then, for any given compact set \(K\), there exist infinitely many \(i, j \in \mathbb{N}_+\) such that any minimal geodesic connecting \(p_i\) and \(p_j\) is away from \(K\).

**Proof.** Let \(p = p_0\). Suppose \(K \subset B(p, r - 1)\) for some \(r > 0\). Let \(\gamma_j(t)\) be a minimal geodesic connecting \(p\) and \(p_j\), where \(t\) is the arc-parameter and \(\gamma_j(0) = p\). Suppose \(\gamma_j(t) = \exp_p(tv_j)\) for some unit vector \(v_j \in T_p M\). By passing to a subsequence, we may assume that \(v_j \to v\) for some \(v \in T_p M\).

Let \(\gamma(t) = \exp_p(tv)\). Fix \(l > 2r\). By the convergence of \(v_j\), we have

\[
\gamma_j(t) \to \gamma(t) \quad \text{as} \quad j \to \infty \quad \forall \ t \in [0, l].
\]

Now, we claim that there exists a constant \(j_0 \in \mathbb{N}_+\) such that any minimal geodesic connecting \(p_i\) and \(p_j\) is away from \(B(p, r)\) if \(i, j \geq j_0\). The lemma follows from this claim immediately.

We prove the claim by contradiction. Let \(\sigma_{ij}(s)\) be a minimal geodesic connecting \(p_i\) and \(p_j\), where \(s\) is the arc-parameter and \(\sigma_{ij}(0) = p_j\). If the claim is not true, we may assume that \(\sigma_{ij}(s_0) \in B(p, r)\) for some \(s_0 \in (0, d(p_i, p_j))\). Then,

\[
d(p_i, \sigma_{ij}(s_0)) \geq d(p_i, p) - d(\sigma_{ij}(s_0), p) \geq d(p_i, p) - r,
\]

\[
d(p_j, \sigma_{ij}(s_0)) \geq d(p_j, p) - d(\sigma_{ij}(s_0), p) \geq d(p_j, p) - r.
\]

Therefore,

\[
d(p_i, p_j) = d(p_i, \sigma_{ij}(s_0)) + d(p_j, \sigma_{ij}(s_0)) \geq d(p_i, p) + d(p_j, p) - 2r.
\]

On the other hand, by the definition of \(\gamma_i(t)\) and \(\gamma_j(t)\), we have

\[
d(p_i, p_j) \leq d(p_i, \gamma_i(l)) + d(\gamma_i(l), \gamma_j(l)) + d(p_j, \gamma_j(l))
\]

\[
= d(\gamma_i(l), \gamma_j(l)) + d(p_i, p) + d(p_j, p) - 2l.
\]

By (9.17) and (9.18), we get

\[
d(\gamma_i(l), \gamma_j(l)) \geq 2(l - r) > 2r > 0.
\]

However, by (9.16)

\[
d(\gamma_i(l), \gamma_j(l)) \to 0 \quad \text{as} \quad i, j \to \infty.
\]

This contradicts (9.19). We have completed the proof.  

\[
\square
\]
Now, we are ready to prove Theorem 1.8.

**Proof of Theorem 1.8.** Let

\[
R_{\max} = \sup_{x \in M} R(x), \quad A = \lim_{r \to \infty} \sup_{x \in M \setminus B(x_0, r)} R(x),
\]

where \(x_0\) is a fixed point. Since we have assumed that the scalar curvature does not have uniform decay, we get \(A > 0\). We can choose a sequence of points \(\{p_i\}\) tending to infinity such that \(R(p_i) \to A\) as \(i \to \infty\). Then, \((M, g, p_i)\) converges subsequentially to a limit \((M_\infty, g_\infty, p_\infty)\), where \((M_\infty, g_\infty, p_\infty)\) is a \(\kappa\)-noncollapsed steady gradient Ricci soliton with nonnegative Ricci curvature and bounded curvature. We also have \(R_\infty(p_\infty)\) attains the maximum of \(R_\infty(x)\) at the point \(p_\infty\).

**Case 1:** \((M_\infty, g_\infty)\) does not have uniform scalar curvature decay. We apply Lemma 9.6 to \((M_\infty, g_\infty, p_\infty)\). Then, \((M_\infty, g_\infty, p_\infty)\) weakly dimension reduces to an \((n-1)\)-dimensional steady Ricci soliton. Therefore, \((M, g, f)\) also dimension reduces to an \((n-1)\)-dimensional steady Ricci soliton.

**Case 2:** \((M_\infty, g_\infty)\) has uniform scalar curvature decay. We will exclude this case. Note that

\[
R_\infty(p_\infty) = A > 0.
\]

By the curvature decay, we can choose \(r_0 > 0\) such that

\[
R_\infty(x) \leq \frac{A}{2}, \quad \forall x \in M_\infty \setminus B(p_\infty, r_0; g_\infty(0)).
\]

By the convergence of \((M, g, p_i)\), there exists a constant \(i_0 > 0\) such that

\[
(9.20) \quad R(x) < \frac{3A}{4} < R(p_i) \quad \forall i \geq i_0, x \in \partial B(p_i, r_0; g).
\]

We also assume that \(K \cap B(p_i, r_0; g) = \emptyset\) for \(i \geq i_0\).

Let \(\phi_t\) be a group of diffeomorphisms generated by \(-\nabla f\). We first claim that \(\phi_t(p_i) \in B(p_i, r_0; g)\) for all \(t \geq 0\) and \(i \geq i_0\).

If the claim is not true, then there exists \(T > 0\) such that \(\phi_T(p_i) \in \partial B(p_i, r_0; g)\) and \(\phi_t(p_i) \in B(p_i, r_0; g) \forall t \in (0, T)\). Hence, \(R(\phi_T(p_i)) < R(p_i)\) by \((9.20)\). Since \(\phi_t(p_i) \in B(p_i, r_0; g) \forall t \in (0, T)\) and the Ricci curvature is nonnegative on \(B(p_i, r_0; g)\) for \(i \geq i_0\), we know \(R(\phi_t(p_i))\) is increasing for \(t \in (0, T)\). Therefore, \(R(\phi_T(p_i)) \geq R(p_i)\). However, we have shown that \(R(\phi_T(p_i)) < R(p_i)\). Hence, \(\phi_t(p_i) \in B(p_i, r_0; g) \forall t \geq 0\).

Next, we claim that there exists a point \(q_i \in B(p_i, r_0; g)\) such that \(\nabla f(q_i) = 0\) for all \(i \geq i_0\).

We prove the claim. Since \(\phi_t(p_i)\) stays in \(B(p_i, r_0; g)\) and the Ricci curvature is nonnegative on \(B(p_i, r_0; g)\), \(R(\phi_t(p_i))\) is increasing for \(t \geq 0\), i.e., \(|\nabla f|^2(\phi_t(p_i)) = C - R(\phi_t(p_i))\) is decreasing for \(t \geq 0\). We assume that
\[ |\nabla f|^2(\phi_t(p_i)) \to c \] as \( t \to +\infty \). Let
\[
\Delta = \sup_{x \in B(p_i, \rho_0; g)} f(x) - \inf_{x \in B(p_i, \rho_0; g)} f(x).
\]
It is obvious that \( \Delta > 0 \) is finite. Note that
\[
\Delta \geq f(p_i) - f(\phi_t(p_i)) = \int_0^t |\nabla f|^2(\phi_s(p_i))ds \geq c^2 t \quad \forall \ t \geq 0.
\]
Note that \( \Delta \) is independent of \( t \). By taking \( t \to +\infty \), we get \( c = 0 \). Hence, \( |\nabla f|^2(\phi_t(p_i)) \to 0 \) as \( t \to +\infty \). Since \( \phi_t(p_i) \) stays in \( B(p_i, \rho_0; g) \), we may assume \( \phi_{t_k}(p_i) \) converges to some point \( q_i \) as \( i_k \to +\infty \). Hence, \( |\nabla f|(q_i) = 0 \) by the convergence of \( \phi_{t_k}(p_i) \). We have completed the proof the claim.

By the claim, there exists a point \( q_i \in B(p_i, \rho_0; g) \) such that \( \nabla f(q_i) = 0 \) and \( R(q_i) = R_{\max} \) for all \( i \geq i_0 \). By taking a subsequence, we may also assume that \( B(p_i, \rho_0; g) \cap B(p_j, \rho_0; g) = \emptyset \) and \( B(p_i, \rho_0; g) \cap K = \emptyset \) for \( i, j \geq i_0 \) and \( i \neq j \). By Lemma 9.7, we can find \( i, j \in \mathbb{N}_+ \) such that there exists a minimal geodesic \( \sigma_{ij}(s) \) connecting \( q_i \) and \( q_j \) such that
\[ (9.21) \quad d(\sigma_{ij}(s), K) \geq 1 \quad \forall \ s \in [0, d(q_i, q_j)], \]
where \( s \) is the arc-parameter and \( \sigma_{ij}(0) = q_i \).

Note that \( \sigma_{ij}(s) \) is a minimal geodesic connecting \( q_i \) and \( q_j \). Moreover, \( \nabla f(q_i) = \nabla f(q_j) = 0 \) and \( \text{Ric}(\sigma_{ij}(s)) \geq 0 \) for all \( s \in [0, d(q_i, q_j)] \). By the argument in the proof of Lemma 9.4 (see the proof of (9.11)), we get
\[ (9.22) \quad R(\sigma_{ij}(s)) = R(\sigma_{ij}(0)) = R_{\max} \quad \forall \ s \in [0, d(q_i, q_j)]. \]

By the choice of \( q_i, q_j \), we get \( q_i \in B(p_i, \rho_0; g) \) and \( q_j \notin B(p_i, \rho_0; g) \). Then, there exists \( s_0 \in (0, d(q_i, q_j)) \) such that \( \sigma_{ij}(s_0) \in \partial B(p_i, \rho_0; g) \). By (9.20),
\[
R(\sigma_{ij}(s_0)) < R(p_i) \leq R_{\max}.
\]
This contradicts (9.22). Hence, the scalar curvature does not have uniform decay.

10. Proofs of Theorem 1.5, Theorem 1.13 and Theorem 1.7

As we have mentioned in the introduction, Theorem 1.9 is a corollary of Theorem 1.3 and Theorem 1.11. Combining Theorem 1.9 and Theorem 1.8, we get Theorem 1.12. Then, Theorem 1.5 is a corollary of Theorem 1.12.

Proof of Theorem 1.5. By [25, Theorems 28.6 and 28.9]), \( (M, g) \) is strongly \( \kappa \)-noncollapsed. By Theorem 1 in [26], \( (M, g) \) also has bounded curvature. Hence, Theorem 1.5 is true by Theorem 1.12 if \( (M, g) \) is not Ricci flat. Note that \( (M, g) \) is strongly \( \kappa \)-noncollapsed. If \( (M, g) \) is Ricci flat, then it has maximal volume growth. By Corollary 8.86 in [23], \( (M, g) \) must be an ALE 4-manifold. We have completed the proof.
Next, we prove Theorem 1.13.

**Proof of Theorem 1.13.** It suffices to exclude the case that \((M, g, f)\) is not Ricci flat. Suppose \((M, g, f)\) is not Ricci flat. By Theorem 1.12, \((M, g_i(t), p_i)\) converges to \((N \times \mathbb{R}, g_N(t) + ds^2, p_\infty)\) for \(p_i\) tending to infinity, where \(g_i(t) = K_i g(K_i^{-1} t)\) and \(N\) are defined as in Definition 1.1. Since \((M, g)\) is a Kähler manifold, \((N \times \mathbb{R}, g_N(t) + ds^2)\) is also a Kähler manifold. Let \(J_\infty\) be the Kähler structure of \((N \times \mathbb{R}, g_N(t) + ds^2)\). Let \(V\) be the parallel vector field parallel in the \(\mathbb{R}\) direction, i.e., \(\nabla V \equiv 0\). Then, \(\nabla J_\infty V \equiv 0\). Hence, \(J_\infty V\) is also a parallel vector field. Hence, \((N \times \mathbb{R}, g_N(t) + ds^2)\) locally splits off a complex line. Since the oriented 2-dimensional \(\kappa\)-solution must be a family of shrinking round spheres, \((N \times \mathbb{R}, g_N(t) + ds^2)\) should be a quotient of \((\mathbb{S}^2 \times \mathbb{R}^2, g_{\mathbb{S}^2}(t) + ds^2 + ds_2^2)\). On the other hand, \(N\) is either diffeomorphic to \(\mathbb{S}^3/\Gamma\) or diffeomorphic to \(\mathbb{R}^3\) by Theorem 1.12. We get a contradiction. Hence, \((M, g)\) must be Ricci flat. This completes the proof. \(\square\)

Finally, we prove Theorem 1.7.

**Proof of Theorem 1.7.** By [25, Theorems 28.6 and 28.9]), Theorem 1 in [26] and Theorem 1.13, \((M, g)\) must be Kähler-Ricci flat. Similar to the proof of Theorem 1.5, \((M, g)\) has maximal volume growth and therefore is an ALE 4-manifold. By Kronheimer [39, 40], any Kähler-Ricci flat ALE of real dimension 4 must be hyperkähler. Hence, we have completed the proof. \(\square\)

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**Appendix A. Volume Comparison Theorems**

In this appendix, we prove some volume comparison theorems for complete Riemannian manifolds with nonnegative Ricci curvature outside a compact set. These results are applicable to the setting of this paper.

Let \((M^n, g)\) be a Riemannian manifold. For points \(x, y \in M\), we denote by

\[ |xy| := |x, y| := d(x, y) \]

the distance between them. We write

\[ B_x(r) = \{ y \in M : |xy| < r \}, \quad \bar{B}_x(r) = \{ y \in M : |xy| \leq r \}, \]

\[ A_x(r_1, r_2) = B_x(r_2) \setminus \bar{B}_x(r_1). \]
For any measurable set $\Omega \subset M$, we denote by $|\Omega|$ the volume of $\Omega$ induced by $g$. We write
\[ V_x(r) := |B_x(r)|. \]
We say a pointed manifold $(M^n, g, o) \in \mathcal{R}(\rho, \Lambda)$, for some constants $\rho > 0$, $\Lambda \geq 0$, if $(M^n, g)$ is a complete noncompact manifold satisfying
\[ \text{Ric} \geq 0 \text{ on } M \setminus B_o(\rho), \]
and
\[ \text{Ric} \geq -(n-1)\Lambda/\rho^2 \text{ on } \bar{B}_o(\rho). \]
Clearly, if $\lambda > 0$, then
\[ (M^n, g, o) \in \mathcal{R}(\rho, \Lambda) \iff (M^n, \lambda^2 g, o) \in \mathcal{R}(\rho\lambda, \Lambda). \]

A.1. Asymptotic volume ratio. We shall prove that if $(M^n, g)$ has nonnegative Ricci curvature outside a compact set, we can still make sense of the notion of asymptotic volume ratio.

We first prove a variant of the Bishop–Gromov volume comparison theorem.

**Proposition A.1.** Let $(M^n, g) \in \mathcal{R}(\rho, \Lambda)$. For any $x \in M$, if $B_o(\rho) \subset B_x(a)$ for some $a > 0$, then
\[ \frac{V_x(r) - V_x(a)}{(r-a)^n} \]
is non-increasing in $r$ for $r > a$.

**Proof.** We write $d_x(y) := d(x, y)$. It is standard to prove that outside of $B_x(a)$,
\[ \Delta d_x \leq \frac{n-1}{d_x - a}, \]
in the sense of distributions. See, for example, [51, Corollary 1.2] or [42, Lemma 4.1].

Let $F(r) = \text{vol}(B_x(r) \setminus B_x(a))$. For $r > a$,
\[
F'(r) = \int_{\partial B_x(r)} dA = \frac{1}{r-a} \int_{\partial B_x(r)} (d_x - a) \frac{\partial}{\partial r} (d_x - a) dA = \frac{1}{2} \int_{B_x(r) \setminus B_x(a)} \Delta ((d_x - a)^2) = \frac{1}{r-a} \sum_{B_x(r) \setminus B_x(a)} \frac{1}{2} ((d_x - a) \Delta (d_x - a) + |\nabla (d_x - a)|^2) \leq \frac{n}{r-a} F(r). \]
Hence
\[
\frac{d}{dr} \frac{V_x(r) - V_x(a)}{(r-a)^n} \leq \frac{nF(r)}{(r-a)^{n+1}} - \frac{nF(r)}{(r-a)^{n+1}} = 0.
\]
\[\square\]

**Lemma A.2.** Let \((M^n, g, o) \in \mathcal{R}(\rho, \Lambda)\). Then
\[
\text{AVR}(g) := \lim_{r \to \infty} \frac{V_o(r)}{r^n}
\]
is well-defined and does not depend on the basepoint.

**Proof.** By the monotonicity formula above, AVR\((g)\) is well-defined.

For any \(x, y \in M\), put \(\delta = d(x, y)\). Suppose that \(B_o(\rho) \subset B_y(a)\) for some \(a > 0\). For \(r > 0\) sufficiently large,
\[
-r^n V_x(r) \leq r^n V_y(r + \delta) \leq r^n (V_y(r) - V_y(a)) \frac{(r + \delta - a)^n}{(r - a)^n} + r^n V_y(a).
\]
Hence
\[
\lim_{r \to \infty} \frac{V_x(r)}{r^n} \leq \lim_{r \to \infty} \frac{V_y(r)}{r^n}.
\]
By the symmetry of the roles of \(x, y\), AVR\((g)\) does not depend on the basepoint. \[\square\]

**A.2. Volume comparison for small radii.** In a previous version of our paper, we tried to apply Theorem 1 of Mahaman [44]. However, there is a gap in the proof of Theorem 1 therein, and in fact we provide a counterexample below. We are very grateful for the anonymous referee for pointing out a gap in the proof in [44].

In [44], Bazanfaré’s proof actually yields the following volume comparison theorem.

**Theorem A.3.** Suppose that \((M^n, g, o) \in \mathcal{R}(\rho, \Lambda)\). For any \(x \in M \setminus B_p(o)\),
\[
\frac{V_x(r)}{r^n}
\]
is decreasing for \(r < \ell - \rho\), where \(\ell := |ox|\). For \(r \geq \ell\), \(0 < s < r\), we have
\[
\frac{V_x(r)}{V_x(s)} \leq C(\Lambda, n) \left(1 + \frac{M}{\rho}\right)^{n-1} \left(\frac{r}{s}\right)^n.
\]

This result was also previously asserted by Cai in [16]. The constant in this comparison theorem depends on the distance to the origin and it is almost optimal as is seen from the example below.

**Example.** Let \(N\) be a Riemannian manifold with \(Rm > 0\), AVR > 0. Let \(M = N \# (S^{n-1}_\epsilon \times [0, \infty))\) be the connected sum of \(N\) with a thin cylinder of radius \(\epsilon\). Pick a point \(x\) on the cylinder with \(\ell = |ox| \gg 10\). Then
\[
\frac{V_x(2\ell)}{V_x(\ell/2)} \sim \frac{\epsilon^{n-1} + \ell^n}{\epsilon^{n-1} \ell} \sim (\ell/\epsilon)^{n-1}.
\]
This example has two ends. We may wonder if there is a better volume comparison theorem just assuming in addition that the manifold is connected at infinity. However, we may need the following stronger topological condition, possibly because the topology of a smooth manifold is not as rigid under Ricci curvature restrictions.

We say that a pointed Riemannian manifold \((M^n, g, o)\) has **connected annuli at distances at least** \(r_0 > 0\), if for any \(r \geq r_0\), there is an open set \(\Omega_r\) such that

\[
\text{(CA)} \quad \Omega_r \text{ is connected, and } A_o(r/2, 2r) \subset \Omega_r \subset A_o(r/3, 3r).
\]

**Theorem A.4.** Suppose that \((M^n, g, o) \in R(\rho, \Lambda)\) and satisfies \((\text{CA})\) at distances at least \(r_0 \geq 10(1 + \rho)\). Then for any \(r \geq r_0\), \(x \in \partial B_o(r)\), \(\alpha \in (0, 1/10]\),

\[
\frac{|A_o(r/2, 2r)|}{|B_z(\alpha r)|} \leq C(n, \Lambda, \alpha).
\]

**Proof.** For any \(r \geq r_0\), let \(\Omega_r\) be the connected domain as in \((\text{CA})\). By Theorem 1 in Zhong-Dong Liu’s thesis [43] (which is available online), we can pick a set of points \(\{p_1, \ldots, p_N\}\) in \(\Omega_r\) satisfying

\[
\Omega_r \subset \bigcup_{i=1}^N B_{p_i}(\alpha r),
\]

where

\[
N \leq \bar{N} = \bar{N}(n, \Lambda, \alpha).
\]

**Claim:** For any \(1 \leq a, b \leq N\), we can find a subsequence \(i_0, i_1, \ldots, i_m\) such that

\[
a = i_0, \quad b = i_m, \quad |p_{i_j}p_{i_{j+1}}| \leq 2\alpha r, \quad m \leq \bar{N}.
\]

**Proof of Claim.** This is essentially because \(\Omega_r\) is connected. c.f. [43, Corollary 2 on p. 21]. Let \(W_0 = B_{p_a}(\alpha r)\). Define \(W_k\) to be the union of \(W_{k-1}\) with those balls \(B_{p_i}(\alpha r)\) satisfying \(W_{k-1} \cap B_{p_i}(\alpha r) \neq \emptyset\). This process stops in at most \(N \leq \bar{N}\) steps. If there is any ball \(B_{p_j}(\alpha r)\) that does not intersect \(W_N\), then we can find two open sets in \(\Omega_r\) that do not intersect. This is a contradiction to the fact that \(\Omega_r\) is connected. \(\square\)

If \(y, z \in \Omega_r\), \(|yz| \leq 2\alpha r\), then

\[
V_y(\alpha r) \leq V_z(3\alpha r) \leq 3^n V_z(\alpha r),
\]

since \(B_z(3\alpha r) \subset M \setminus B_o(\rho)\) if \(\alpha \leq 1/10\). So for any indices \(1 \leq a, b \leq N\),

\[
V_{p_a}(\alpha r) \leq 3^n \bar{N} V_{p_b}(\alpha r).
\]
For any \( x \in \partial B_{\rho}(r) \), there exists \( b \leq N \) such that \( x \in B_{\rho b}(\alpha r) \). Then \( B_{\rho b}(\alpha r) \subset B_x(2\alpha r) \), and
\[
V_x(\alpha r) \geq 2^{-n}V_x(2\alpha r) \geq 2^{-n}V_{\rho b}(\alpha r).
\]
It follows that
\[
\frac{|A_x(r/2,2r)|}{|B_x(\alpha r)|} \leq \frac{|\Omega_x|}{|B_x(\alpha r)|} \leq 2^n \sum_{a=1}^{N} \frac{V_{\rho a}(\alpha r)}{V_{\rho b}(\alpha r)} \leq 2^n 3^n \bar{N}.
\]
\( \square \)

**Corollary A.5.** Suppose that \( (M^n, g, o) \in R(\rho, \Lambda) \) and satisfies (CA) at distances at least \( r_0 \geq 10(1 + \rho) \). If \( AVR(g) > 0 \), then for any \( x \notin B_\rho(r_0) \) and any \( r > 0 \),
\[
\frac{V_x(r)}{r^n} \geq c(n, \Lambda) AVR(g).
\]

**Proof.** For any \( x \in M \setminus B_\rho(r_0) \), let \( \ell = |\alpha r| \). Then, \( x \in \partial B_o(\ell) \) and \( \ell \geq r_0 \).
By Theorem A.4,
\[
(1.1) \quad \frac{V_o(2\ell) - V_o(\ell/2)}{V_x(\ell/10)} \leq C(n, \Lambda).
\]
Note that the Ricci curvature is nonnegative on \( B_x(\ell/10) \). By (1.1), for any \( r \in (0, \ell/10] \), we have
\[
\frac{V_x(r)}{r^n} \geq \frac{V_x(\ell/10)}{(\ell/10)^n} \geq \frac{V_o(2\ell) - V_o(\ell/2)}{C(n, \Lambda)(2\ell - \ell/2)^n}.
\]
For any \( s \geq 2\ell \), by the monotonicity formula Proposition A.1, it follows that
\[
\frac{V_x(r)}{r^n} \geq \frac{V_o(2\ell) - V_o(\ell/2)}{C(n, \Lambda)(2\ell - \ell/2)^n} \geq \frac{V_o(s) - V_o(\ell/2)}{C(n, \Lambda)(s - \ell/2)^n}.
\]
By taking \( s \to \infty \), for \( r \in (0, \ell/10] \),
\[
\frac{V_x(r)}{r^n} \geq c(n, \Lambda) AVR(g).
\]
For any \( r \in (\ell/10, 4\ell] \),
\[
\frac{V_x(r)}{r^n} \geq \frac{V_x(\ell/10)}{(4\ell)^n} \geq \frac{c(n, \Lambda)}{40^n} AVR(g).
\]
Note that \( B_\rho(\rho) \subset B_x(2\ell) \). For any \( r \geq 4\ell \), by Lemma A.2 and Proposition A.1, we have
\[
\frac{V_x(r)}{r^n} \geq \frac{V_x(r) - V_x(2\ell)}{(r - 2\ell)^n} \geq 2^{-n} \frac{V_x(r) - V_x(2\ell)}{(r - 2\ell)^n} \geq 2^{-n} AVR(g).
\]
\( \square \)

Now we give a criterion for (CA) that will be useful in our setting.
Lemma A.6. Suppose that \((M^n, g)\) is a complete Riemannian manifold and \(M\) is connected at infinity. Suppose that there is a smooth positive function \(\beta\) on \(M\) satisfying
\[
\lim_{x \to \infty} \frac{\beta(x)}{|\nabla \beta(x)|} = 1,
\]
for some point \(o \in M\). Suppose that there is \(r_0 > 0\) such that \(|\nabla \beta(x)| > 0\) for any \(x \in \{\beta(x) \geq r_0\}\). Then \((M^n, g, o)\) satisfies \((\text{CA})\).

Proof. By assumption, \(M = \{\beta > r_0\} \cup \{\beta \leq r_0\}\). Note that \(\{\beta \leq r_0\}\) is compact and \(\{\beta > r_0\}\) is diffeomorphic to \(\{\beta = r_0\} \times (0, +\infty)\). Since \(M\) is connected at infinity, \(\{\beta = r_0\}\) must be connected. For sufficiently large \(r\), we may choose
\[
\Omega_r := \{x : \frac{5r}{12} < \beta(x) < \frac{5r}{2}\}.
\]
Hence, \(\Omega_r\) is connected. \(\square\)

Corollary A.7. Suppose that \((M^n, g, o) \in \mathcal{R}(\rho, \Lambda)\). Suppose that either:
\begin{itemize}
  \item \(\text{sec} \geq 0\) outside a compact set and \(M\) is connected at infinity; or
  \item there is a smooth function \(f\) on \(M\) such that \((M, g, f)\) is a steady gradient Ricci soliton and the scalar curvature decays uniformly.
\end{itemize}
Then \((M^n, g, o)\) satisfies condition \((\text{CA})\) at distances at least \(r_0\) for some large \(r_0 > 0\). As a consequence of Corollary A.5, if in addition \(\text{AVR}(g) > 0\), then for \(x \notin B_o(r_0)\), and any \(r > 0\),
\[
\frac{V_x(r)}{r^n} \geq c(n, \Lambda) \text{AVR}(g).
\]

Proof. For \((M^n, g)\) with sectional curvature nonnegative outside a compact set, Li and Tam proved \((M^n, g)\) satisfies \((\text{CA})\) when \(M\) is connected at infinity (see [42, Section 2]).

If \((M, g, f)\) is a steady soliton with uniformly decaying scalar curvature, by Theorem 2.1 and Remark 2.4, \(f\) has linear growth and the level sets of \(f\) are in fact diffeomorphic to each other outside a compact set. Here, we normalized the steady soliton so that \(R + |\nabla f|^2 = 1\). It was proved by Munteanu and Wang in [48, Corollary 1.1] that \(M\) is connected at infinity. Hence the conditions in Lemma A.6 are satisfied. \(\square\)

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