A Fourier type transform on translation invariant valuations on convex sets.

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Abstract

Let $V$ be a finite dimensional real vector space. Let $Val^{sm}(V)$ be the space of translation invariant smooth valuations on convex compact subsets of $V$. Let $Dens(V)$ be the space of Lebesgue measures on $V$. The goal of the article is to construct and study an isomorphism $F_V : Val^{sm}(V) \rightarrow Val^{sm}(V^*) \otimes Dens(V)$ such that $F_V$ commutes with the natural action of the full linear group on both spaces, sends the product on the source (introduced in [5]) to the convolution on the target (introduced in [15]), and satisfies a Plancherel type formula. As an application, a version of the hard Lefschetz theorem for valuations is proved.

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0 Introduction.

0.1 An overview of the main results.

Let $V$ be a finite dimensional real vector space, dim $V = n$. The goal of the article is to construct an isomorphism between the space of translation invariant valuations on convex compact subsets of $V$ and the space of translation invariant valuations (twisted by the line of Lebesgue measures) on the dual space $V^*$. This isomorphism is analogous to the classical Fourier transform. It has various nice properties studied in detail in this article. As an application we prove a version of the hard Lefschetz theorem for translation invariant
valuations. To state the main results more precisely let us fix some notation and remind definitions.

Let $\mathcal{K}(V)$ denote the class of all convex compact subsets of $V$. Equipped with the Hausdorff metric, the space $\mathcal{K}(V)$ is a locally compact space.

0.1.1 Definition. a) A function $\phi : \mathcal{K}(V) \to \mathbb{C}$ is called a valuation if for any $K_1, K_2 \in \mathcal{K}(V)$ such that their union is also convex one has

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

b) A valuation $\phi$ is called continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}(V)$.

The notion of valuation is very classical in convexity. For the classical theory of valuations we refer to the surveys by McMullen-Schneider [36] and McMullen [35]. For the general background from convexity we refer to the book by Schneider [39]. Approximately during the last decade there was a considerable progress in the valuation theory. New classification results of special classes of valuations have been obtained [30], [40], [1]-[4], [6], [32]. Also new structures on valuations have been discovered [4], [5], [15]. Moreover some parts of the classical theory of valuations on affine spaces have been generalized to more general context of arbitrary manifolds [8]-[10], [12] (see also a survey [11] of these results).

Let us denote by $Val(V)$ the space of translation invariant continuous valuations. Equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$, $Val(V)$ is known to be a Banach space. In [5] there was introduced a dense subspace of smooth valuations $Val^{sm}(V) \subset Val(V)$. The definition is recalled in Section 2.5 below. Note that $Val^{sm}(V)$ is equipped with the natural Fréchet topology which is stronger than the topology induced from $Val(V)$.

0.1.2 Example. 1) A Lebesgue measure $vol$ on $V$ belongs to $Val^{sm}(V)$.

2) The Euler characteristic $\chi$ belongs to $Val^{sm}(V)$. (Recall that $\chi(K) = 1$ for any $K \in \mathcal{K}(V)$.)

3) Let us fix a compact strictly convex set $A \subset V$ with infinitely smooth boundary. The functional $K \mapsto vol(K + A)$ is a smooth translation invariant valuation. Here $K + A$ is the Minkowski sum $\{k + a | k \in K, a \in A\}$.

4) Let $0 \leq i < n = \text{dim } V$. Fix $A_1, \ldots, A_i$ compact strictly convex subsets of $V$ with infinitely smooth boundary. Then the mixed volume $K \mapsto V(K[n-i], A_1, \ldots, A_i)$ belongs to $Val^{sm}(V)$ (here $K[n-i]$ means that the set $K$ is taken $n-i$ times). For the notion of mixed volume see e.g. [39], especially Ch. 5,6.

The space $Val^{sm}(V)$ carries a canonical structure of commutative associative topological algebra with unit (the unit is the Euler characteristic). This structure was constructed by the author in [5]; the main properties of it are recalled in Section 2.5.

Let us denote by $Dens(V)$ the complex one dimensional space of complex valued Lebesgue measures on $V$. The space $Val^{sm}(V^*) \otimes Dens(V)$ carries a canonical structure of commutative associative topological algebra with unit. This structure was recently constructed by Bernig and Fu [15]; the main properties of it are recalled in Section 2.5.
Next observe that group $GL(V)$ of all invertible linear transformations of $V$ acts naturally on $Val(V)$ (and $Val^{sm}(V)$) as follows: $(g\phi)(K) = \phi(g^{-1}K)$ for any $g \in GL(V), K \in K(V)$, and $\phi \in Val(V)$ or $Val^{sm}(V)$.

Our first main result says that these two topological algebras with actions of $GL(V)$ are isomorphic. More precisely we prove the following result.

0.1.3 Theorem. There exists an isomorphism of linear topological spaces

$$F_V : Val^{sm}(V) \rightarrow Val^{sm}(V^*) \otimes Dens(V)$$

which satisfies the following properties:

1) $F_V$ commutes with the natural action of the group $GL(V)$ on both spaces;

2) $F_V$ is an isomorphism of algebras when the source is equipped with the product and the target with the convolution.

3)(Plancherel type formula) Consider the composition $E_V$

$$Val^{sm}(V) \xrightarrow{F_V} Val^{sm}(V^*) \otimes Dens(V) \xrightarrow{F_V^* \otimes Id_{Dens(V)}} Val^{sm}(V) \otimes Dens(V^*) \otimes Dens(V) = Val^{sm}(V).$$

This composition $E_V$ satisfies

$$(E_V \phi)(K) = \phi(-K).$$

0.1.4 Remark. 1) On even valuations, the operator $F_V$ was first introduced by the author in [4] under a different name and notation (in [4] it was denoted by $D$).

2) Part (2) of Theorem 0.1.3 was first proved for even valuations by Bernig and Fu [15].

3) The isomorphism $F_V$ from Theorem 0.1.3 is not quite canonical. One can show that in certain precise sense there exist exactly four different isomorphisms satisfying the theorem provided $n > 1$; for $n = 1$ there exist exactly two such isomorphisms (see Remark 6.4.4 below for a precise statement).

0.1.5 Example. Let us describe the isomorphism $F_V$ in dimensions 1 and 2. First assume that dim $V = 1$. In this case the space of valuations is two dimensional: $Val(V) = Val^{sm}(V) = \mathbb{C} \cdot \chi \oplus \mathbb{C} \cdot vol_V$ where $vol_V$ is a non-zero Lebesgue measure on $V$. Let $vol_V^{-1}$ be the corresponding Lebesgue measure on $V^*$ (see (2.1.9) below). Then $Val^{sm}(V^*) \otimes Dens(V) = \mathbb{C} \cdot (vol_V^{-1} \otimes vol_V) \oplus \mathbb{C} \cdot (\chi \otimes vol(V))$. Then

$$F_V(\chi) = vol_V^{-1} \otimes vol_V \quad (0.1.1)$$
$$F_V(vol_V) = \chi \otimes vol(V). \quad (0.1.2)$$

Let us assume now that dim $V = 2$. Let us fix a Euclidean metric on $V$. It induces identifications $V^* \simeq V$, $Dens(V) \simeq \mathbb{C}$. Under these identifications $F_V : Val^{sm}(V) \rightarrow Val^{sm}(V)$. One has $Val^{sm}(V) = \mathbb{C} \chi \oplus Val_1^{sm}(V) \oplus \mathbb{C} vol_V$ where $Val_1^{sm}(V)$ denotes the subspace of 1-homogeneous valuations. Let us fix also an orientation on $V$. Then

$$F_V(\chi) = vol_V,$$
$$F_V(vol_V) = \chi.$$
In order to describe the action of $F_V$ on $Val^{sm}_1(V)$ recall that by Hadwiger’s theorem [23] any valuation $\phi \in Val^{sm}_1(V)$ can be written uniquely in the form

$$\phi(K) = \int_{S^1} h(\omega) dS_1(K, \omega)$$

where $h: S^1 \to \mathbb{C}$ is a smooth function which is orthogonal on $S^1$ to the two dimensional space of linear functionals. Let us decompose $h$ to the even and odd parts:

$$h = h_+ + h_-.$$  

Let us decompose further the odd part $h_-$ to ”holomorphic” and ”anti-holomorphic” parts

$$h_- = h_-^{hol} + h_-^{anti}$$

as follows. Let us decompose $h_-$ to the usual Fourier series on the circle $S^1$:

$$h_-(\omega) = \sum_k \hat{h}_-(k) e^{ik\omega}.$$  

Then by definition

$$h_-^{hol}(\omega) := \sum_{k>0} \hat{h}_-(k) e^{ik\omega},$$

$$h_-^{anti}(\omega) := \sum_{k<0} \hat{h}_-(k) e^{ik\omega}.$$  

Then the Fourier transform of the valuation $\phi$ is equal to

$$(\mathbb{F}\phi)(K) = \int_{S^1} (h_+(J\omega) + h_-^{hol}(J\omega)) dS_1(K, \omega) - \int_{S^1} h_-^{anti}(J\omega) dS_1(K, \omega)$$

where $J$ is the rotation of $\mathbb{R}^2$ by $\frac{\pi}{2}$ counterclockwise. (Notice the minus sign before the second integral.) Observe that $\mathbb{F}$ preserves the class of real valued even valuations, but for odd real valued valuations this is not true. This phenomenon also holds in higher dimensions.

0.1.6 Remark. In dimension higher than 2 the author does not know such a simple construction of $F_V$, especially in the odd case studied in detail in the article. Note however that the Fourier transform of a Lebesgue measure and the Euler characteristic can be computed by the equalities (0.1.1) in any dimension.

The construction in the odd case is quite involved and uses various characterization theorems on valuations, in particular Klain-Schneider characterization of simple valuations [30], [40] (see Theorem 2.3.6 below), Irreducibility Theorem [3] (see Theorem 2.3.7 below), and also some additional representation theoretical computations based on the Beilinson-Bernstein localization [13] (see Section 1.4 below).

As an application of the Fourier transform (combined with Bernig-Bröcker theorem [14]) we prove a version of hard Lefschetz theorem for valuations. In order to state it let us denote
by $Val^\text{sm}_i(V) \subset Val^\text{sm}(V)$ the subspace of $i$-homogeneous valuations ($\phi \in Val^\text{sm}_i(V)$ if and only if $\phi(\lambda K) = \lambda^i \phi(K)$ for any $\lambda \geq 0, K \in K(V)$). By McMullen’s theorem \cite{33}

$$Val^\text{sm}(V) = \oplus_{i=0}^{n} Val^\text{sm}_i(V).$$

Let us fix a Euclidean metric on $V$. Let us denote by $V_1$ the first intrinsic volume (see \cite{39}, p. 210). Here we just recall that $V_1$ is the only (up to a constant) continuous isometry invariant $1$-homogeneous valuation (this characterization is due to Hadwiger \cite{24}).

**0.1.7 Theorem** (hard Lefschetz theorem). Let $0 \leq i < n/2$. Then the map

$$Val^\text{sm}_i(V) \rightarrow Val^\text{sm}_{n-i}(V)$$

given by $\phi \mapsto (V_1)^{n-2i} \cdot \phi$ is an isomorphism.

**0.1.8 Remark.** 1) Theorem 0.1.7 was proved by the author for *even* valuations in \cite{4}.

2) There is another version of the hard Lefschetz theorem for valuations (see Theorem 7.1.3 below). In the even case it was proved by the author \cite{4}, and in the general one by Bernig and Bröcker \cite{14}. Our proof of Theorem 0.1.7 (see Section 7) uses in an essential way this result of Bernig-Bröcker. Also we use the fact that the Fourier transform $F$, establishes an equivalence of two versions of the hard Lefschetz theorem (Lemma 7.1.4 below). This fact was recently observed in the even case by Bernig and Fu \cite{15}; our Lemma 7.1.4 is a straightforward generalization of their observation.

Another new construction presented in this article is a construction of pushforward under linear maps of translation invariant continuous valuations (twisted by the line of Lebesgue measures). Namely if $f: V \rightarrow W$ is a linear space of vector spaces, we define a linear map

$$f_*: Val(V) \otimes Dens(V^*) \rightarrow Val(W) \otimes Dens(W^*).$$

We refer to Section 3.2 for the details. Here we notice that this pushforward map allows to compute the convolution of valuations in sense of Bernig and Fu \cite{15} in two steps: first one takes the exterior product of valuations in sense of \cite{5}, and then the pushforward under the addition map $a: V \oplus V \rightarrow V$.

Another interesting property of the Fourier transform is that it intertwines the pullback of valuations (obviously defined, see Section 3.1) and the pushforward. With an oversimplification, one can say that the Fourier transform of the pullback of a valuation is equal to the pushforward of the Fourier transform. There are however some technical difficulties of making this statement rigorous due to the fact that the operations of pullback and pushforward do not preserve in general the class of smooth valuations. Nevertheless a rigorous result is possible though it sounds more technical: see Theorems 6.2.1, 6.2.4 below.

**0.2 Organization of the article.**

In Section 1 we summarize a necessary background from representation theory. In Section 2 we describe necessary facts mostly from valuation theory. These two sections do not contain new results.
In Section 3 we introduce operations of pullback and pushforward on translation invariant valuations. We relate them to operations of product and convolution (Sections 3.3, 3.4), and prove a version of the base change theorem (Section 3.5).

In Section 4 we prove an isomorphism of $GL(V)$-modules $Val^{-, sm}_{n-p}(V)$ and $Val^{-, sm}_p(V^*) \otimes Dens(V)$ (here $Val^{-, sm}_i(V)$ denotes the space of smooth odd translation invariant $i$-homogeneous valuations on $V$). The existence of such isomorphism and some related representation theoretical calculations will be used in the construction of the Fourier transform in Section 0.

In Section 5 we study separately the Fourier transform on valuations on a two dimensional plane. The two dimensional case will be used for higher dimensions in Section 6.

Section 6 is the main one. Here we construct the Fourier transform in full generality and prove the main properties of it.

In Section 7 a hard Lefschetz type theorem for valuations is proved.

The appendix contains a slight generalization of the construction of the exterior product of smooth valuations given in [5]: here we explain how to multiply a smooth valuation by a continuous one. This generality is necessary in this article for technical reasons.

0.3 Notation.

- $\mathcal{K}(V)$ - the family of convex compact subsets of a vector space $V$.
- $\mathcal{K}^{sm}(V)$ - the family of strictly convex compact subsets of a vector space $V$ with infinitely smooth boundary.
- $f^\vee$ - a dual map to a linear map $f$.
- $f \boxtimes g$, $f \times g$, $f \oplus g$ - operations with linear maps $f$ and $g$, see Section 2.1.
- $Gr_i(V)$ - the Grassmannian of $i$-dimensional linear subspaces of a vector space $V$.
- $\mathcal{F}_{p,p+1}(V)$ - the (real) variety of partial flags in $V \{ (E,F) \mid E \subset F, \dim E = p, \dim F = p + 1 \}$.
- $\mathcal{F}$ - the (real) variety of complete flags in a real vector space.
- $\mathcal{C} \mathcal{F}$ - the (complex) variety of complete flags in a complex vector space.
- $T_{k,V}$, $T_{k,V;i}$, $T^0_{k,V}$, $T^0_{k,V;i}$ - certain vector bundles, see Section 6.1.
- $|\omega_X|$ - the complex line bundle of densities over a manifold $X$.

1 Background from representation theory.

1.1 Some structure theory of reductive groups.

In this subsection we remind few basic definitions from the structure theory of real reductive groups. For simplicity we will do it only in the case of the group $GL_n(\mathbb{R})$. This will suffice for the purposes of this article.

Let $G_0 = GL_n(\mathbb{R})$. It acts naturally on $\mathbb{R}^n$. Let us denote by $H_0 \subset G_0$ the subgroup of diagonal invertible matrices ($H_0$ is a Cartan subgroup).

1.1.1 Definition. (i) A subgroup $P_0 \subset G_0$ is called parabolic if there exists a partial flag of linear subspaces $0 \neq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k = \mathbb{R}^n$ such that $P_0$ is the stabilizer in $G_0$ of this flag.
(ii) A parabolic subgroup \( P_0 \) is called \textit{standard} if \( P_0 \supseteq H_0 \).

\section*{1.1.2 Example.} Let us fix a positive integer \( k \) and a decomposition \( n = n_1 + \ldots + n_k \), \( n_i \in \mathbb{N} \).

Let

\[
P_0 := \left\{ \begin{bmatrix} A_1 & * & \ldots & * \\ 0 & A_2 & \ldots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & A_k \end{bmatrix} \mid A_i \in GL_{n_i}(\mathbb{R}) \right\}
\]

be the subgroup of block upper triangular invertible matrices. It is a standard parabolic.

Note that if one takes \( k = 1, n_1 = n \) one gets \( P_0 = G_0 \). If one takes \( k = n, n_1 = \cdots = n_n = 1 \) then \( P_0 \) is equal to the subgroup of upper triangular invertible matrices (it is a minimal parabolic).

For a parabolic subgroup \( P_0 \) let us denote by \( U_{P_0} \) its unipotent radical. Thus \( U_{P_0} \) is a normal subgroup of \( P_0 \). More explicitly, if \( P_0 \) is the stabilizer of a partial flag \( 0 \neq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_k = \mathbb{R}^n \) then \( U_{P_0} \) consists of transformations from \( P_0 \) inducing the identity map on all consecutive quotients \( F_i/F_{i-1} \), \( i = 0, \ldots, k \).

\section*{1.1.3 Definition.} Let \( P_0 \) be a parabolic. An algebraic subgroup \( M \subset P_0 \) is called a \textit{Levi subgroup} if the canonical homomorphism \( P_0 \to P_0/U_{P_0} \) induces an isomorphism \( M \to P_0/U_{P_0} \).

A Levi subgroup always exists but not unique. In Example 1.1.2 the unipotent radical is

\[
U_{P_0} = \left\{ \begin{bmatrix} I_{n_1} & * & \ldots & * \\ 0 & I_{n_2} & \ldots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{n_k} \end{bmatrix} \right\}.
\]

A Levi subgroup can be chosen to be equal to

\[
M = \left\{ \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & A_k \end{bmatrix} \mid A_i \in GL_{n_i}(\mathbb{R}) \right\} \cong GL_{n_1}(\mathbb{R}) \times \cdots \times GL_{n_k}(\mathbb{R}).
\]

Let us return back to a general parabolic \( P_0 \). If \( M \) is its Levi subgroup then \( P_0 = M \cdot U_{P_0} \).

\section*{1.1.4 Definition.} Let \( P_0 = M_{P_0} \cdot U_{P_0} \), \( Q_0 = M_{Q_0} \cdot U_{Q_0} \) be two parabolics, where \( M_{P_0}, M_{Q_0} \) are their Levi subgroups. Then \( P_0 \) and \( Q_0 \) are called \textit{associated} if there exists \( x \in G_0 \) such that

\[
M_{Q_0} = x^{-1} M_{P_0} x.
\]

\section*{1.2 Admissible and tempered growth representations and a theorem of Casselman-Wallach.}

\section*{1.2.1 Definition.} Let \( \pi \) be a continuous representation of a Lie group \( G_0 \) in a Fréchet space \( F \). A vector \( \xi \in F \) is called \( G_0 \)-smooth if the map \( g \mapsto \pi(g)\xi \) is an infinitely differentiable map from \( G_0 \) to \( F \).
It is well known (see e.g. [42], Section 1.6) that the subset $F^{sm}$ of smooth vectors is a $G_0$-invariant linear subspace dense in $F$. Moreover it has a natural topology of a Fréchet space (which is stronger than the topology induced from $F$), and the representation of $G_0$ in $F^{sm}$ is continuous. Moreover all vectors in $F^{sm}$ are $G_0$-smooth.

Let $G_0$ be a real reductive group. Assume that $G_0$ can be imbedded into the group $GL_N(\mathbb{R})$ for some $N$ as a closed subgroup invariant under the transposition. Let us fix such an imbedding $p : G_0 \hookrightarrow GL_N(\mathbb{R})$. (In our applications $G_0$ will be either $GL_n(\mathbb{R})$ or a direct product of several copies of $GL_n(\mathbb{R})$.) Let us introduce a norm $| \cdot |$ on $G_0$ as follows:

$$|g| := \max\{\|p(g)\|, \|p(g^{-1})\|\}$$

where $\| \cdot \|$ denotes the usual operator norm in $\mathbb{R}^N$.

1.2.2 Definition. Let $\pi$ be a smooth representation of $G_0$ in a Fréchet space $F$ (namely $F^{sm} = F$). One says that this representation has moderate growth if for each continuous semi-norm $\lambda$ on $F$ there exists a continuous semi-norm $\nu_\lambda$ on $F$ and $d_\lambda \in \mathbb{R}$ such that

$$\lambda(\pi(g)v) \leq |g|^{d_\lambda}\nu_\lambda(v)$$

for all $g \in G, v \in F$.

The proof of the next lemma can be found in [42], Lemmas 11.5.1 and 11.5.2.

1.2.3 Lemma. (i) If $(\pi, G_0, H)$ is a continuous representation of $G_0$ in a Banach space $H$, then $(\pi, G, H^{sm})$ has moderate growth. (ii) Let $(\pi, G_0, V)$ be a representation of moderate growth. Let $W$ be a closed $G_0$-invariant subspace of $V$. Then $W$ and $V/W$ have moderate growth.

Remind that a continuous Fréchet representation $(\pi, G_0, \mathcal{F})$ is said to have finite length if there exists a finite filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = F$$

by $G_0$-invariant closed subspaces such that $F_i/F_{i-1}$ is irreducible, i.e. does not have proper closed $G_0$-invariant subspaces. The sequence $F_1, F_2/F_1, \ldots, F_m/F_{m-1}$ is called the Jordan-Hölder series of the representation $\pi$. It is well known (and easy to see) that the Jordan-Hölder series of a finite length representation is unique up to a permutation.

1.2.4 Definition. A Fréchet representation $(\rho, G_0, F)$ of a real reductive group $G_0$ is called admissible if its restriction to a maximal compact subgroup $K$ of $G_0$ contains an isomorphism class of any irreducible representation of $K$ with at most finite multiplicity. (Remind that a maximal compact subgroup of $GL_n(\mathbb{R})$ is the orthogonal group $O(n)$.)

1.2.5 Theorem (Casselman-Wallach, [19]). Let $G_0$ be a real reductive group. Let $(\rho, G_0, F_1)$ and $(\pi, G_0, F_2)$ be smooth representations of moderate growth in Fréchet spaces $F_1, F_2$. Assume in addition that $F_2$ is admissible of finite length. Then any continuous morphism of $G_0$-modules $f : F_1 \to F_2$ has closed image.

The following proposition is essentially a common knowledge; the proof can be found in [8], Proposition 1.1.8.
1.2.6 Proposition. Let $G_0$ be a real reductive Lie group. Let $F_1, F_2$ be continuous Fréchet $G_0$-modules. Let $\xi : F_1 \to F_2$ be a continuous morphism of $G_0$-modules. Assume that the assumptions of the Casselman-Wallach theorem are satisfied, namely $F_1$ and $F_2$ are smooth and have moderate growth, and $F_2$ is admissible of finite length. Assume moreover that $\xi$ is surjective.

Let $X$ be a smooth manifold. Consider the map

$$\hat{\xi} : C^\infty(X, F_1) \to C^\infty(X, F_2)$$

defined by $(\hat{\xi}(f))(x) = \xi(f(x))$ for any $x \in X$.

Then $\hat{\xi}$ is surjective.

1.3 Induced representations.

Let $H \subset G_0$ be a closed subgroup. Let $\pi$ be a representation of $H$ in a Fréchet space $F$. Let us consider the space of continuous functions

$$\Phi := \{ f : G_0 \to V \mid f(x \cdot h) = \pi(h)^{-1}f(x) \text{ for any } x \in G_0, h \in H \}. $$

The group $G_0$ acts on $\Phi$ by left translation. The representation of $G_0$ in $\Phi$ is called the induced representation and denoted by $\text{Ind}_H^{G_0} \pi$. Note that the space $\Phi$ is a space of continuous sections of a $G_0$-equivariant vector bundle over $G_0/H$ with fiber $F$.

Let $P_0 \subset G_0$ be a parabolic subgroup. Let us consider the natural representation of $G_0$ in the space of (complex valued) half-densities (see e.g [22], Ch. II §6) on $G_0/P_0$. It is easy to see that there exists a character $\rho_{P_0} : P_0 \to \mathbb{C}^*$ such that this representation is isomorphic to $\text{Ind}_{P_0}^{G_0} \rho_{P_0}$.

1.3.1 Remark. It is easy to see that $U_{P_0} \subset [P_0, P_0]$. Hence any character of $P_0$ is trivial on $U_{P_0}$, and hence factorizes via $P_0/U_{P_0}$.

Let $\pi$ be a representation of $P_0/U_{P_0}$ considered as a representation of $P_0$. Let us denote by

$$\mathcal{H}(P_0, \pi) := \text{Ind}_{P_0}^{G_0}(\pi \otimes \rho_{P_0}).$$

We will need the next result which is standard in representation theory.

1.3.2 Theorem. Let $P_0 \subset G_0$ be a parabolic subgroup. Let $\pi$ be a character of $P_0/U_{P_0}$. Then $\text{Ind}_{P_0}^{G_0}(\pi)$ is an admissible representation of finite length.

Let now $P_0 = M_{P_0} \cdot U_{P_0}, Q_0 = M_{Q_0} \cdot U_{Q_0}$ be two associated parabolics (see Definition 1.1.4). Thus $M_{Q_0} = \alpha^{-1} M_{P_0} \alpha$ for some $\alpha \in G_0$. Let $\phi$ be a character of $M_{P_0}$ considered as a character of $P_0$. Let $\pi'$ be the character of $M_{Q_0}$ defined by

$$\pi'(x) = \pi(\alpha \cdot x \cdot \alpha^{-1}).$$

The following result is a special case of a theorem by Harish-Chandra (see e.g. [41], Proposition 4.1.20).
1.3.3 Theorem (Harish-Chandra). Let $P_0, Q_0 \subset G_0$ be two associated parabolics as above. Let $\pi$ be a character of $P_0/U_{P_0}$. Let $\pi'$ be the character of $Q_0/U_{Q_0}$ as above. The representations $\mathcal{H}(P_0, \pi)$ and $\mathcal{H}(Q_0, \pi')$ have the same Jordan-Hölder series.

We will use below Theorem 1.3.3 in two particular situations. To describe the first one, let us fix an integer $k = 1, 2, \ldots, n - 1$. Let

$$P_0 := \left\{ \begin{bmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & c \end{bmatrix} \middle| A \in GL_{n-k-1}(\mathbb{R}), B \in GL_k(\mathbb{R}), c \in \mathbb{R}^* \right\}.$$

Then $G_0/P_0$ is the partial flag space $\{(F, E) \mid F \in Gr_{n-k-1}(\mathbb{R}^n), E \in Gr_{n-k-1}(\mathbb{R}^n)\}$. Let $p: G_0/P_0 \to Gr_{n-k-1}(\mathbb{R}^n)$ be the natural map given by $p(F, E) = F$. Let $\mathcal{V} \to G_0/P_0$ be the line bundle whose fiber over a partial flag $(F, E)$ is equal to $\text{Dens}(E/F) \otimes \text{or}(\mathbb{R}^n/E)$. Let $|\omega_{Gr_{n-k-1}}| \to Gr_{n-k-1}(\mathbb{R}^n)$ denote the line bundle of densities. Let $\mathcal{A}$ denote the space of sections $\mathcal{A} := C(G_0/P_0, \mathcal{V} \otimes p^*(|\omega_{Gr_{n-k-1}}|))$. Clearly $\mathcal{A}$ is a $G_0$-module.

1.3.4 Lemma.

$$\mathcal{A} = \mathcal{H}(P_0, \pi)$$

where $\pi: P_0 \to \mathbb{C}^*$ is the character defined by

$$\pi\left( \begin{bmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & c \end{bmatrix} \right) = |\det A|^{\frac{k+1}{2}} |\det B|^{-\frac{n-k-1}{2}} |c|^{-\frac{n-k}{2}+k} \text{sgn}(c). \quad (1.3.1)$$

Proof is straightforward computation. Q.E.D.

Let us consider another parabolic subgroup

$$Q_0 := \left\{ \begin{bmatrix} B & * & * \\ 0 & c & * \\ 0 & 0 & A \end{bmatrix} \middle| A \in GL_{n-k-1}(\mathbb{R}), B \in GL_k(\mathbb{R}), c \in \mathbb{R}^* \right\}.$$

It is easy to see that $P_0$ and $Q_0$ are associated. Let $\xi: Q_0 \to \mathbb{C}^*$ be the character defined by

$$\xi\left( \begin{bmatrix} B & * & * \\ 0 & c & * \\ 0 & 0 & A \end{bmatrix} \right) = |\det B|^{-1} \text{sgn}(c).$$

1.3.5 Lemma.

$$\text{Ind}_{Q_0}^{G_0} \xi = \mathcal{H}(Q_0, \rho)$$

where $\rho\left( \begin{bmatrix} B & * & * \\ 0 & c & * \\ 0 & 0 & A \end{bmatrix} \right) = |\det A|^{\frac{k+1}{2}} |\det B|^{-\frac{n-k-1}{2}} |c|^{-\frac{n-k}{2}+k} \text{sgn}(c)$. 

Proof is straightforward computation. Q.E.D.

1.3.6 Corollary. The $G_0$-modules $\mathcal{A}^{sm}$ and $(\text{Ind}_{Q_0}^{G_0} \xi)^{sm}$ have the same Jordan-Hölder series.
Proof. Obviously $P_0$ and $Q_0$ are associated, indeed $M_{P_0} = \alpha \cdot M_{Q_0} \cdot \alpha^{-1}$ where $\alpha = \begin{bmatrix} 0 & I_{d_{n-k}} \\ I_{d_k} & 0 \end{bmatrix}$. Moreover $\xi(x) = \pi(a^{-1}xa)$ where $\pi$ is defined by (1.3.1). Hence the result follows from Theorem 1.3.3, Q.E.D.

Let us describe now the second situation where Theorem 1.3.3 will be used.

1.3.7 Corollary. Let us consider a parabolic subgroup

$$P'_0 = \left\{ \begin{bmatrix} A & * & * \\ 0 & c & * \\ 0 & 0 & B \end{bmatrix} \mid A \in GL_p(\mathbb{R}), c \in \mathbb{R}^*, B \in GL_{n-p-1}(\mathbb{R}) \right\}.$$

Let us denote by $\omega: P'_0 \to \mathbb{C}^*$ the character such that $\text{Ind}_{P_0}^{G_0}(\omega)$ is isomorphic to the representation of $G_0$ in densities on $G_0/P'_0$. Let $\alpha: P'_0 \to \mathbb{C}^*$ be the character defined by

$$\alpha \left( \begin{bmatrix} A & * & * \\ 0 & c & * \\ 0 & 0 & B \end{bmatrix} \right) = |\det(A)| \cdot \text{sgn}(c) \cdot \omega.$$

Let us consider another parabolic subgroup

$$Q'_0 = \left\{ \begin{bmatrix} B & * & * \\ 0 & c & * \\ 0 & 0 & A \end{bmatrix} \mid A \in GL_p(\mathbb{R}), c \in \mathbb{R}^*, B \in GL_{n-p-1}(\mathbb{R}) \right\}.$$

Let $\beta: Q'_0 \to \mathbb{C}^*$ be the character defined by

$$\beta \left( \begin{bmatrix} B & * & * \\ 0 & c & * \\ 0 & 0 & A \end{bmatrix} \right) = |\det(A)| \cdot \text{sgn}(c).$$

Then $(\text{Ind}_{P_0}^{G_0}(\alpha))^{sm}$ and $(\text{Ind}_{Q_0}^{G_0}(\beta))^{sm}$ have the same Jordan-Hölder series.

Proof is a straightforward computation similar to the proof of Corollary 1.3.6, Q.E.D.

1.4 The Beilinson-Bernstein localization.

We recall the Beilinson-Bernstein theorem on localization of g-modules following [13] (see also [16]). Then we recall the version of this result for dominant but not regular characters following [27]. We denote by capital letters the Lie groups, and by the corresponding small letters their Lie algebras.

Let $G$ be a complex reductive algebraic group. Let $T$ denote a Cartan subgroup of $G$. In our examples $G = GL_n(\mathbb{C})$. Let $B$ be a Borel subgroup of $G$ containing $T$. Let $t$ denote the Lie algebra of $T$. Let $b$ denote the Lie algebra of $B$. Let $n$ denote the nilpotent radical of $b$.

In the case when $G$ is complexification of a real reductive group $G_0$ let us denote by $K$ the complexification of a maximal compact subgroup of $G_0$. Thus if $G = GL_n(\mathbb{C})$ then $K = O(n, \mathbb{C})$ is the group of complex orthogonal matrices.
Let \( R(t) \subseteq t^* \) be the set of roots of \( t \) in \( g \). The set \( R(t) \) is naturally divided into the set of roots whose root spaces are contained in \( n \) and its complement. Let \( R^+(t) \) be the set of roots of \( t \) in \( g/b \). If \( \alpha \) is a root of \( t \) in \( g \) then the dimension of the corresponding root subspace \( g_\alpha \) is called the multiplicity of \( \alpha \). Let \( \rho_b \) be the half sum of the roots contained in \( R^+(t) \) counted with their multiplicities.

1.4.1 Definition. We say that \( \lambda \in t^* \) is dominant if for any root \( \alpha \in R^+(t_b) \) we have \( <\lambda, \alpha^V> \neq -1, -2, \ldots \). We shall say that \( \lambda \in t_b^* \) is \( B \)-regular if for any root \( \alpha \in R^+(t_b) \) we have \( <\lambda, \alpha^V> \neq 0 \).

For the definitions and basic properties of the sheaves of twisted differential operators we refer to [16], [27]. Here we will present only the explicit description of the sheaf \( D_\lambda \) in order to agree about the normalization.

Let \( ^C F \) be the complete flag variety of \( G \) (then \( ^C F = G/B \)). Let \( O_F \) denote the sheaf of regular functions on \( ^C F \). Let \( U(g) \) denote the universal enveloping algebra of \( g \). Let \( U^o \) be the sheaf \( U(g) \otimes _C O_F \), and \( g^o := g \otimes _C O_F \). Let \( T_F \) be the tangent sheaf of \( ^C F \). We have a canonical morphism \( \alpha: g^o \rightarrow T_F \). Let also \( b^o := \text{Ker} \alpha = \{ \xi \in g^o | \xi_x \in b_x \forall x \in \mathbb{C} \} \). Let \( \lambda: b \rightarrow \mathbb{C} \) be a linear functional which is trivial on \( \mathbb{C} \). We will denote by \( D_\lambda \) the sheaf of twisted differential operators corresponding to \( \lambda - \rho_b \), i.e. \( D_\lambda \) is isomorphic to \( U^o/I_\lambda \), where \( I_\lambda \) is the two sided ideal generated by the elements of the form \( \xi - (\lambda - \rho_b)^o(\xi) \) where \( \xi \) is a local section of \( b^o \).

Let \( D_\lambda := \Gamma(^C F, D_\lambda) \) denote the ring of global sections of \( D_\lambda \). We have a canonical morphism \( U(g) \rightarrow D_\lambda \). For the complete flag variety \( ^C F \) this map is onto \( [13] \). The kernel of this homomorphism was also described in [13]). To describe it, remind that we have the Harish-Chandra isomorphism \( Z(U(g)) \rightarrow (\text{Sym}^* t)^W \) where \( Z(U(g)) \) denotes the center of \( U(g) \), \( (\text{Sym}^* t)^W \) is the algebra of elements of the full symmetric algebra of \( t \) invariant under the Weyl group \( W \). Hence the element \( \lambda \in t^* \) defines a homomorphism \( Z(U(g)) \rightarrow \mathbb{C} \) called the infinitesimal character. Let \( I_\lambda \) be the two-sided ideal in \( U(g) \) generated by the kernel of this homomorphism. Then by [13], \( I_\lambda \) is equal to the kernel of the homomorphism \( U(g) \rightarrow D_\lambda \).

1.4.2 Remark. The category of \( D_\lambda \)-modules coincides with the category of \( g \)-modules with the given infinitesimal character.

In this notation one has the following result proved in [13].

1.4.3 Theorem (Beilinson-Bernstein). (1) If \( \lambda \in t^* \) is dominant then the functor \( \Gamma: D_\lambda \rightarrow U(g) - \text{mod} \) is exact.

(2) If \( \lambda \in t^* \) is dominant and regular then the functor \( \Gamma \) is also faithful.

Note also that always the functor \( \Gamma \) has a left adjoint functor (called the localization functor) \( \Delta: D_\lambda \rightarrow D_\lambda - \text{mod} \). It is defined as \( \Delta(M) = D_\lambda \otimes _D M \). \( \Delta(M) \) is called the localization of \( M \).

The proof of the next lemma can be found in [16], Proposition I.6.6.

1.4.4 Lemma. Suppose \( \Gamma: D_\lambda \rightarrow D_\lambda - \text{mod} \) is exact. Then the localization functor \( \Delta: D_\lambda \rightarrow D_\lambda - \text{mod} \) is the right inverse of \( \Gamma \):

\[
 \Gamma \circ \Delta = Id.
\]
We have the following immediate corollary (see [16], p. 24).

1.4.5 Corollary. Let $\lambda \in t^*$ is dominant and regular. Then the functor

$$\Gamma : D_\lambda - \text{mod} \to D_\lambda - \text{mod}$$

is an equivalence of categories. Moreover this equivalence holds for $K$-equivariant versions of these categories.

The following sufficient condition for $\lambda$ being regular and dominant will be useful.

1.4.6 Proposition. Let $G_0 = GL_n(\mathbb{R})$, $G = GL_n(\mathbb{C})$. Let $B_0 \subset G_0$ be the subgroup of real invertible upper triangular matrices, $B$ be its complexification. Let $\lambda \in t^*$. Let $\chi : B_0 \to \mathbb{C}^*$ be a character such that its (complexified) differential $b \mapsto \mathbb{C}$ is equal to $\lambda - \rho_b$. Assume that the representation $\text{Ind}^G_{F_0} \chi$ has a non-zero finite dimensional $G_0$-submodule. Then $\lambda$ is dominant and regular.

1.5 Some analysis on manifolds.

Let $X$ be a smooth manifold countable at infinity. Let $E \to X$ be a finite dimensional vector bundle. We denote by $C(X, E)$, $C^\infty(X, E)$ the spaces of continuous, $C^\infty$-smooth sections respectively. The space $C(X, E)$ being equipped with the topology of uniform convergence on compact subsets of $X$ is a Fréchet space; if $X$ is compact then it is a Banach space.

The space $C^\infty(X, E)$ being equipped with the topology of uniform convergence on compact subsets of $X$ of all partial derivatives is a Fréchet space.

The following result is well known (see e.g. [20]).

1.5.1 Theorem. Let $X_1$ and $X_2$ be compact smooth manifolds. Let $E_1$ and $E_2$ be smooth finite dimensional vector bundles over $X_1$ and $X_2$ respectively. Let $G$ be a Fréchet space. Let

$$B : C^\infty(X_1, E_1) \times C^\infty(X_2, E_2) \to G$$

be a continuous bilinear map. Then there exists unique continuous linear operator

$$b : C^\infty(X_1 \times X_2, E_1 \boxtimes E_2) \to G$$

such that $b(f_1 \otimes f_2) = B(f_1, f_2)$ for any $f_i \in C^\infty(X_i, E_i)$, $i = 1, 2$.

2 Background on valuation theory.

We collect in this section some necessary notation and results from the valuation theory and linear algebra.
2.1 Linear algebra.

Let $V$ be a finite dimensional real vector space of dimension $n$. We denote

$$\det V := \bigwedge^n V. \quad (2.1.1)$$

Also we denote by $Dens(V)$ the space of complex valued Lebesgue measures on $V$; thus $Dens(V)$ is a complex line. ($Dens$ stays for densities.)

Let us define $or(V)$ the orientation line of $V$ as follows. Let us denote by $Bas(V)$ the set of all basis in $V$. The group $GL(V)$ of linear invertible transformations acts naturally on $Bas(V)$ by $g((x_1, \ldots, x_n)) = (g(x_1), \ldots, g(x_n))$. Then set

$$or(V) = \{ f: Bas(V) \to \mathbb{C} | f(g(x)) = sgn(det(g)) \cdot f(x) \text{ for any } g \in GL(V), x \in Bas(V) \}.$$  

Clearly $or(V)$ is a one dimensional complex vector space. The operation of passing to the biorthogonal basis gives an identification $Bas(V) \simeq Bas(V^*)$. It induces an isomorphism of vector spaces

$$or(V) \simeq or(V^*) \quad (2.1.2)$$

which will be used throughout the article. Also one has another natural isomorphism

$$Dens(V) \simeq det(V^*) \otimes_{\mathbb{C}} or(V). \quad (2.1.3)$$

Next let

$$0 \to U \to V \to W \to 0 \quad (2.1.4)$$

be a short exact sequence of finite dimensional vector spaces. Then one has canonical isomorphisms

$$\det U \otimes \det W \to \det V, \quad (2.1.5)$$

$$or(U) \otimes or(W) \to or(V), \quad (2.1.6)$$

$$Dens(U) \otimes \Dens(W) \to \Dens(V). \quad (2.1.7)$$

The isomorphism (2.1.5) is given by $x \otimes y \mapsto x \wedge \tilde{y}$ where $\tilde{y}$ is an arbitrary lift of $y \in \det W$ to $\wedge^{\text{dim}W} V$. To describe the isomorphisms (2.1.6) and (2.1.7) let us fix an arbitrary linear splitting of the exact sequence (2.1.4), $s : W \to V$. Thus $V \simeq U \oplus W$. Then $Bas(U) \times Bas(W) \subset Bas(V)$. Thus the restriction from functions on $Bas(V)$ to functions on $Bas(U) \times Bas(W)$ defines the isomorphism (2.1.6) which is in fact independent of the choice of a splitting $s$. Next the usual product measure construction defines a map $Dens(U) \otimes Dens(W) \to Dens(V)$ which is an isomorphism independent of a choice of a splitting $s$.

Also we will use an isomorphism

$$Dens(V^*) \to Dens(V)^* \quad (2.1.8)$$

which can be described as follows. Let us fix a basis of $V$, and let $(x_1, \ldots, x_n)$ denote coordinates of a vector in $V$ in this basis. Let $(y_1, \ldots, y_n)$ denote coordinates of a vector in
$V^*$ in the biorthogonal basis. Let us choose the isomorphism \((2.1.8)\) such that the Lebesgue measure on \(|dy_1 \wedge \cdots \wedge dy_n|\) on $V^*$ goes to the linear functional on $Dens(V)$ whose value on the Lebesgue measure \(|dx_1 \wedge \cdots \wedge dx_n|\) on $V$ is equal to 1. It is easy to see that this isomorphism does not depend on a choice of a basis. In this situation we also write
\[
|dy_1 \wedge \cdots \wedge dy_n| = |dx_1 \wedge \cdots \wedge dx_n|^{-1}. \tag{2.1.9}
\]

For a linear map of vector spaces $f: V \to W$ we denote by $f^*: W^* \to V^*$ the dual space. (This notation is different from the probably more standard notation $f^*$, since we keep the symbol $f^*$ to denote the pullback of valuations, see Section 3.1.)

Let $f_i: V_i \to W_i$, $i = 1, 2$, be two linear maps of vector spaces. We denote by $f_1 \boxtimes f_2: V_1 \oplus V_2 \to W_1 \oplus W_2$ \((2.1.10)\) defined, as usual, by $(f_1 \boxtimes f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$.

Let $g_i: V \to W_i$, $i = 1, 2$, be two linear maps. We denote by $g_1 \times g_2: V \to W_1 \oplus W_2$ \((2.1.11)\) the map defined by $(g_1 \times g_2)(x) = (g_1(x), g_2(x))$.

Let $h_i: V_i \to W$, $i = 1, 2$, be two linear maps. We denote by $h_1 \oplus h_2: V_1 \oplus V_2 \to W$ \((2.1.12)\) the map defined by $(h_1 \oplus h_2)(x_1, x_2) = h_1(x_1) + h_2(x_2)$.

### 2.2 McMullen’s decomposition of valuations.

Let $V$ be an $n$-dimensional real vector space. Let $Val(V)$ denote the space of translation invariant continuous valuations on $V$. Let $\alpha$ be a complex number. We say that $\phi$ is $\alpha$-homogeneous if
\[
\phi(\lambda K) = \lambda^\alpha \phi(K) \text{ for any } \lambda > 0, K \in K(V).
\]

Let us denote by $Val_\alpha(V)$ the subspace of $Val(V)$ of $\alpha$-homogeneous convex valuations. The following result is due to P. McMullen \([33]\).

#### 2.2.1 Theorem \([33]\).
Let $n = \dim V$. Then
\[
Val(V) = \bigoplus_{i=0}^{n} Val_i(V).
\]

### 2.3 Characterization theorems on valuations.

In this section we describe several theorems on translation invariant continuous valuations which will be used in the article.

#### 2.3.1 Proposition.
(1) $Val_0(V)$ is spanned by the Euler characteristic $\chi$.
(2) (Hadwiger, \([27]\)) $Val_n(V)$ is spanned by a Lebesgue measure $\text{vol}$. 

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Note that part (1) of the proposition is obvious. Let us remind now a description of 
\((n - 1)\)-homogeneous valuations due to P. McMullen [34].

Let \(P^\vee_+(V)\) denote the manifold of cooriented linear hyperplanes in \(V\) (recall that coorientation of \(E \subset V\) is just an orientation of \(V/E\)). Let \(\mathcal{L} \to P^\vee_+(V)\) denote the complex line bundle whose fiber over \(E \in P^\vee_+(V)\) is equal to \(\text{Dens}(E)\). Let us construct a continuous linear map

\[
\Psi: C(P^\vee_+(V), \mathcal{L}) \to \text{Val}_{n-1}(V).
\] (2.3.1)

For let us fix \(\xi \in C(P^\vee_+(V), \mathcal{L})\). Let us define first a valuation \(\phi_\xi\) on convex compact polytopes in \(V\). Let \(P \subset V\) be a convex compact polytope. Any \((n - 1)\)-dimensional face of \(P\) carries a coorientation such that the exterior normal of it has a positive direction. Set

\[
\Psi(\xi)(P) := \sum_{F \in \{(n-1)\text{-faces of } P\}} \xi(F).
\] (2.3.2)

2.3.2 Lemma. \(\Psi(\xi)\) extends (uniquely) to a continuous valuation on \(K(V)\). This valuation (also denoted by \(\Psi(\xi)\)) is continuous, translation invariant, and \((n - 1)\)-homogeneous.

Proof. In fact this claim is due to Schneider [38] in a slightly different language. We are going to explain this. Let us fix a Euclidean metric on \(V\). It induces an identification of \(P^\vee_+(V)\) with the unit sphere \(S^{n-1}\), and an isomorphism of \(\mathcal{L}\) with the trivial line bundle. Then if \(\xi \in C(P^\vee_+(V), \mathcal{L}) \simeq C(S^{n-1})\) then for a polytope \(P\)

\[
\Psi(\xi)(P) = \int_{S^{n-1}} \xi(\omega)dS_{n-1}(P, \omega)
\]

where \(dS_{n-1}(P, \bullet)\) dentes the \((n-1)\)-th area measure (see e.g. [39], p.203). But the functional on \(K(V)\)

\[
K \mapsto \int_{S^{n-1}} \xi(\omega)dS_{n-1}(K, \omega)
\]

is a continuous translation invariant \((n - 1)\)-homogeneous valuation by [38]. Q.E.D.

The next theorem is due to McMullen.

2.3.3 Theorem ([34]). The map \(\Psi\) is onto. The kernel is \(n\)-dimensional.

2.3.4 Remark. The line bundle \(\mathcal{L}\) is obviously \(GL(V)\)-equivariant. The map \(\Psi\) is \(GL(V)\)-equivariant.

Let us describe explicitly the kernel of \(\Psi\). First let us construct a linear map

\[
\wedge^{n-1}V^* \otimes \text{or}(V) \to C(P^\vee_+(V), \mathcal{L}).
\]

Fix an arbitrary cooriented hyperplane \(E \in P^\vee_+(V)\). By (2.1.3), (2.1.2), (2.1.6) one has

\[
\text{Dens}(E) = \det E^* \otimes \text{or}(E) = \det E^* \otimes \text{or}(V) \otimes \text{or}(V/E).
\]

Since \(E\) is cooriented \(\text{or}(V/E) = \mathbb{C}\). Hence

\[
\text{Dens}(E) = \det E^* \otimes \text{or}(V).
\]
But one has a canonical map $\wedge^{n-1} V^* \otimes \text{or}(V) \to \det E^* \otimes \text{or}(V)$ induced by the map $V^* \to E^*$ dual to the identity imbedding $E \to V$. When $E$ varies this defines the desired map $\wedge^{n-1} V^* \otimes \text{or}(V) \to C(\mathbb{P}^n(V), \mathcal{L})$. Clearly it is $GL(\cdot)$-equivariant and injective.

The claim is that the image of the above map is equal precisely to the kernel of $\Psi$. This is a $GL(V)$-equivariant interpretation of the well known fact from convexity that the closed (in the weak topology on measures) linear span of all area measures $- \wedge \mathcal{S}$ is equal to all measures $\mu$ on $S^{n-1}$, when $K$ runs through $\mathcal{K}(V)$, that have canonical isomorphisms

$$\int_{S^{n-1}} \omega \cdot d\mu(\omega) = 0.$$ 

(This fact is an easy consequence of the Minkowski existence theorem, see e.g. [39], Theorem 7.1.2.)

Let $Val^+(V)$ denote the subspace of even valuations, i.e. such that $\phi(-K) = \phi(K)$ for any $K \in \mathcal{K}(V)$, and let $Val^-(V)$ denote the subspace of odd valuations, i.e. such that $\phi(-K) = -\phi(K)$ for any $K \in \mathcal{K}(V)$. Similarly $Val^\pm(V)$ denote the analogous subspaces in $i$-homogeneous valuations.

Let $C^+(\mathbb{P}^n(V), \mathcal{L})$ denote the subspace of $C(\mathbb{P}^n(V), \mathcal{L})$ of even section, i.e. sections which do not change when one reverses a coorientation of a hyperplane. Let $C^-(\mathbb{P}^n(V), \mathcal{L})$ denote the subspace of $C(\mathbb{P}^n(V), \mathcal{L})$ of odd section, i.e. sections which change the sign when one reverses a coorientation of a hyperplane. Let us denote $\Psi^+$ (resp. $\Psi^-$) the restriction of $\Psi$ to $C^+(\mathbb{P}^n(V), \mathcal{L})$ (resp. $C^-(\mathbb{P}^n(V), \mathcal{L})$).

The claim is that the image of the above map is equal precisely to the kernel of $\Psi$. This is an isomorphism of vector spaces. The kernel of $\Psi^-$ is equal to $\wedge^{n-1} V^* \otimes \text{or}(V)$.

Then obviously one has

$$\Psi^+: C(\mathbb{P}^n(V), \mathcal{L}^+) \to Val^+_n(V),$$

$$\Psi^-: C(\mathbb{P}^n(V), \mathcal{L}^-) \to Val^-_{n-1}(V).$$

The following claim is obvious from the previous discussion.

**2.3.5 Claim.** $\Psi^+$ is an isomorphism of vector spaces. The kernel of $\Psi^-$ is equal to $\wedge^{n-1} V^* \otimes \text{or}(V)$.

The next theorem is very useful. In the even case it was proved by Klain [30], and in the odd case by Schneider [10]. First recall that a valuation is called *simple* if it vanishes on all convex compact sets of dimension less than $n$.

**2.3.6 Theorem ([30],[10]).** A translation invariant continuous valuation is simple if and only if it is representable as a sum of a Lebesque measure and an odd $(n-1)$-homogeneous translation invariant continuous valuation.
The group $GL(V)$ acts continuously and linearly in the space $Val(V)$ as follows: $(g\phi)(K) = \phi(g^{-1}K)$ for any $g \in GL(V), \phi \in Val(V), K \in K(V)$. Obviously this action preserves degree of homogeneity and parity of valuations. The next result was proved by the author [3], it will be used in the article many times.

2.3.7 Theorem (Irreducibility Theorem, [3]). The natural representation of $GL(V)$ in $Val_i^+(V)$, $i = 0, 1, \ldots, n$, is irreducible, i.e. there is no $GL(V)$-invariant proper closed subspace.

2.4 Klain-Schneider realizations of valuations.

In this section we describe $GL(V)$-equivariant realizations of $Val_i^+(V)$ as a subspace of the space of sections of certain line bundle over the Grassmannian $Gr_i(V)$, and of $Val_i^-(V)$ as a subspace of a quotient of the space of sections of certain line bundle over the partial flag space $F_{i,i+1}(V)$. The exposition follows [2], [3]. The even case was also considered in [31] using a slightly different language. These realizations of even and smooth valuations we call respectively Klain and Schneider realizations. The reason for such terminology is that behind these constructions stays a deep Klain-Schneider theorem.

Let us start with the even case. Let us denote by $M \rightarrow Gr_i(V)$ the complex line bundle whose fiber over $E \in Gr_i(V)$ is equal to $Dens(E)$. Let $\phi \in Val_i^-(V)$. For any $E \in Gr_i(V)$ let us consider the restriction $\phi|_E$. Clearly $\phi|_E \in Val_i(E)$. But by Proposition 2.3.1(2) (due to Hadwiger) $Val_i(E) = Dens(E)$. Then $\phi$ defines a section in $C(Gr_i(V), M)$. We get a continuous $GL(V)$-equivariant map

$$Val_i^+(V) \rightarrow C(Gr_i(V), M). \quad (2.4.1)$$

The key fact is that this map is injective. This can be easily deduced by induction on the dimension from the even case (due to Klain) of the Klain-Schneider theorem (see [2], Proposition 3.1, or [31]). We call this imbedding the Klain imbedding.

Let us consider the odd case. Let $F_{i,i+1}(V)$ denote the partial flag variety $F_{i,i+1}(V) := \{(E,F)| E \subset F, \dim E = i, \dim F = i + 1\}$.

Let us denote by $X \rightarrow Gr_{i+1}(V)$ the (infinite dimensional) vector bundle whose fiber over $F \in Gr_{i+1}(V)$ is equal to $Val_i^-(F)$. Let $\phi \in Val_i^+(V)$. For any $F \in Gr_{i+1}(V)$ let us consider the restriction $\phi|_F \in Val_i^-(F)$. Thus we get a $GL(V)$-equivariant continuous map

$$Val_i^-(V) \rightarrow C(Gr_{i+1}(V), X).$$

The key point is that this map is injective. This easily follows by the induction on dimension from the odd case (due to Schneider) of the Klain-Schneider theorem (see [3], Proposition 2.6 for the details).

Let $\mathcal{N} \rightarrow F_{i,i+1}(V)$ denote the line bundle whose fiber over $(E,F) \in F_{i,i+1}(V)$ is equal to $Dens(E) \otimes or(F/E)$. Applying the map $\Psi$ (see (2.3.4)) to every subspace $F \in Gr_{i+1}(V)$ (instead of $V$) we get a continuous map

$$C(F_{i,i+1}(V), \mathcal{N}) \rightarrow C(Gr_{i+1}(V), X).$$
This map is onto. This follows from the fact that $\Psi^{-}$ is onto and has finite dimensional kernel.

Thus $Val_{i}^{-}(V)$ is realized as a subspace of a quotient of $C(\mathcal{F}_{i,i+1}(V),\mathcal{N})$. We call this realization the Schneider realization.

2.5 Product and convolution of smooth valuations.

Let us denote by $Val^{sm}(V)$ the space of smooth valuations (in sense of Definition 1.2.1) under the natural action of the group $GL(V)$ on the space $Val(V)$.

Let $K^{sm}(V)$ denote the space of strictly convex compact subsets of $V$ with $C^{\infty}$-smooth boundary. A typical example of smooth valuation is a functional $K \mapsto vol(K + A)$ where $A \in K^{sm}(V)$ is fixed.

Product on smooth translation invariant valuations was defined by the author in [5]. Let us summarize the main properties of the product in the following theorem. Let us fix on $V$ a positive Lebesgue measure $vol_{V}$. Below we denote by $vol_{V \times V}$ the product measure on $V \times V$.

2.5.1 Theorem ([5]). There exists a bilinear map

\[ Val^{sm}(V) \times Val^{sm}(V) \rightarrow Val^{sm}(V) \]

which is uniquely characterized by the following two properties:

1) continuity;

2) if $\phi(\bullet) = vol_{V}(\bullet + A)$, $\psi = vol_{V}(\bullet + B)$ then

\[ (\phi, \psi) \mapsto vol_{V \times V}(\Delta(\bullet) + (A \times B)) \]

where $\Delta: V \rightarrow V \times V$ is the diagonal imbedding.

This bilinear map defines a product making $Val^{sm}(V)$ a commutative associative algebra with unit (which is the Euler characteristic).

2.5.2 Example ([5], Proposition 2.2). Assume that $\dim V = 2$. Let $\phi(K) = V(K,A)$, $\psi(K) = V(K,B)$. Then

\[ (\phi \cdot \psi)(K) = \frac{1}{2} V(A, -B) vol(K). \]

Convolution on $Val^{sm}(V) \otimes Den^{s}(V^{*})$ was defined by Bernig and Fu in [15]. Let us summarize their result in the following theorem.

2.5.3 Theorem ([15]). There exists a bilinear map

\[ Val^{sm}(V) \otimes Den^{s}(V^{*}) \times Val^{sm}(V) \otimes Den^{s}(V^{*}) \rightarrow Val^{sm}(V) \otimes Den^{s}(V^{*}) \]

which is uniquely characterized by the following two properties:

1) continuity;

2) if $\phi(\bullet) = vol_{V}(\bullet + A) \otimes vol_{V}^{-1}$, $\psi = vol_{V}(\bullet + B) \otimes vol_{V}^{-1}$ then

\[ (\phi, \psi) \mapsto vol_{V}(\bullet + A + B) \otimes vol_{V}^{-1}. \]

This bilinear map defines a product making $Val^{sm}(V) \otimes Den^{s}(V^{*})$ a commutative associative algebra with unit (which is equal to $vol_{V} \otimes vol_{V}^{-1}$).
2.6 A technical lemma.

For a future reference we state a simple and well known lemma (see [39], p. 294, particularly equality (5.3.23)).

2.6.1 Lemma. Let \( f : V \to W \) be a linear epimorphism of finite dimensional real vector spaces. Let \( k := \dim \ker(f) \). Let \( \text{vol}_{\ker}, \text{vol}_{\text{ker}} \) be Lebesgue measures on \( \ker(f), W \) respectively. Let \( \text{vol}_{V} := \text{vol}_{\ker} \otimes \text{vol}_{W} \) be the corresponding Lebesgue measure on \( V \). Let \( A \in \mathcal{K}(V), B \in \mathcal{K}(\ker(f)) \). Then

\[
\frac{1}{k!} \frac{d^k}{d\varepsilon^k} \big|_{\varepsilon=0} \text{vol}_V(A + \varepsilon B) = \text{vol}_{\ker}(B) \cdot \text{vol}_W(f(A)).
\]

Proof. It is an application of the Fubini theorem. Q.E.D.

3 Functorial properties of translation invariant valuations.

3.1 Pullback of valuations.

3.1.1 Definition. Let \( f : V \to W \) be a linear map of vector spaces. Let us define a map, called pullback,

\[
f^* : \text{Val}(W) \to \text{Val}(V)
\]

by \((f^* \phi)(K) = \phi(f(K))\) for any \( K \in \mathcal{K}(V)\).

3.1.2 Proposition. (i) \( f^* \) is a continuous map of Banach spaces.

(ii) \( f^* \) preserves degree of homogeneity and parity of valuations.

(iii) \((f_1 \circ f_2)^* = f_2^* \circ f_1^*\).

Proof is obvious. Q.E.D.

3.2 Pushforward of valuations twisted by densities.

For a linear map \( f : V \to W \) we are going to define in this section a canonical map, called pushforward,

\[
f_* : \text{Val}(V) \otimes \text{Dens}(V^*) \to \text{Val}(W) \otimes \text{Dens}(W^*)
\]

The main result of this section is the following proposition which is also a definition.

3.2.1 Proposition. (i) Let \( f : V \to W \) be a linear map of vector spaces. Then there exists a continuous linear map, called pushforward,

\[
f_* : \text{Val}(V) \otimes \text{Dens}(V^*) \to \text{Val}(W) \otimes \text{Dens}(W^*)
\]

which is uniquely characterized by the following property. Let us fix Lebesgue measures \( \text{vol}_V \) on \( V \), \( \text{vol}_{\ker} \) on \( \ker(f) \), \( \text{vol}_{\text{coker}} \) on \( \text{coker}(f) = W/\text{im}(f) \), and let \( \text{vol}_{\text{im}} \) be the induced
Lebesgue measure on $\text{Im}(f)$ (it is obtained as the image of the measure $\frac{\text{vol}_V}{\text{vol}_K}$ on $V/k\text{er}(f)$ under the isomorphism $f: V/k\text{er}(f) \to \text{Im}(f)$). Then for any $A \in \mathcal{K}(V)$

$$f_*(\text{vol}_V(\cdot + A) \otimes \text{vol}_V^{-1}) =$$

$$\left(\int_{z \in \text{CoKer}(f)} \text{vol}_\text{Im}((\cdot + f(A)) \cap z) d\text{vol}_\text{CoKer}(z)\right) \otimes (\text{vol}_\text{Im} \otimes \text{vol}_\text{CoKer})^{-1}$$

where $\text{vol}_\text{Im} \otimes \text{vol}_\text{CoKer}$ is considered as a Lebesgue measure on $W$ under the isomorphism $\text{Dens}(W) \simeq \text{Dens}(\text{Im}(f)) \otimes \text{Dens}(\text{CoKer}(f))$.

(ii) $f_*$ preserves the parity, and $f_*(\text{Val}_*(V)) \subset \text{Val}_* + \text{dim} W - \text{dim} V(W)$.

(iii) $(f_1 \circ f_2)_* = f_1_* \circ f_2$.

3.2.2 Remark. Before we prove this proposition let us discuss two special cases of the pushforward. Let us fix Lebesgue measures $\text{vol}_V$ on $V$ and $\text{vol}_W$ on $W$. This choice induces isomorphisms $\text{Dens}(V) \to \mathbb{C}$, $\text{Dens}(W) \to \mathbb{C}$. Under these identifications, $f_*: \text{Val}(V) \to \text{Val}(W)$.

(1) Let us assume that $V$ is a subspace of $W$, and $f: V \hookrightarrow W$ is the imbedding map. Consider the Lebesgue measure $\text{vol}_{W/V} := \frac{\text{vol}_W}{\text{vol}_V}$ on $W/V$. For any $\phi \in \text{Val}(V)$

$$(f_*\phi)(K) = \int_{z \in W/V} \phi(K \cap z) d\text{vol}_{W/V}(z).$$

(2) Let us assume that $W$ is a quotient space of $V$, and $f: V \twoheadrightarrow W$ is the quotient map. Let $\phi \in \text{Val}(V)$ has the form $\phi(K) = \text{vol}_V(K + A)$ where $A \in \mathcal{K}(V)$ is fixed. Then for any $K \in \mathcal{K}(W)$

$$(f_*\phi)(K) = \text{vol}_W(K + f(A)).$$

Proof of Proposition 3.2.1 The uniqueness follows immediately from the McMullen’s conjecture. Let us prove the existence. Let us decompose $f$ as a composition of a linear surjection $p: V \to X$ followed by a linear injection $j: X \hookrightarrow W$; thus $f = j \circ p$. Such a decomposition is unique up to an isomorphism (in the obvious sense).

Let us define $j_*: \text{Val}(X) \otimes \text{Dens}(X^*) \to \text{Val}(W) \otimes \text{Dens}(W^*)$ as in Remark 3.2.21. Let us define now $p_*: \text{Val}(V) \otimes \text{Dens}(V^*) \to \text{Val}(X) \otimes \text{Dens}(X^*)$. Let us fix Lebesgue measures $\text{vol}_{\text{ker}}$ on $\text{ker}(p)$ and $\text{vol}_X$ on $X$. This induces a Lebesgue measure $\text{vol}_V := \text{vol}_{\text{ker}} \otimes \text{vol}_X$ on $V$. Let $\phi \in \text{Val}(V)$. Fix a set $L \in \mathcal{K}(V)$. Consider a valuation $\tau$ on $\text{ker}(p)$ defined by

$$\tau(S) = \frac{1}{k!} \frac{d^k}{d\varepsilon^k} \bigg|_{\varepsilon = 0} \phi(L + \varepsilon S)$$

where $S \in \mathcal{K}(\text{ker}(p))$, $k = \text{dim}(\text{ker}(p))$. Recall that by a result of McMullen [33], $\phi(L + \varepsilon S)$ is a polynomial in $\varepsilon \geq 0$ of degree at most $k$. It is easy to see that $\tau$ is a $k$-homogeneous translation invariant continuous valuation on $\text{ker}(p)$. By a result of Hadwiger [24] $\tau$ must be proportional to $\text{vol}_{\text{ker}}$ with a constant depending on $L$ and $\phi$:

$$\tau = C(L, \phi) \text{vol}_{\text{ker}}.$$
3.2.3 Claim. $\tilde{\phi}(K)$ does not depend on a choice of $\tilde{K}$ such that $p(\tilde{K}) = K$.

**Proof.** Since $C(\tilde{K}, \phi)$ is continuous in $\phi \in Val(V)$ when $\tilde{K}$ in fixed, by the McMullen’s conjecture, it is enough to prove the claim for $\phi(\bullet) = vol_V(\bullet + A)$ for $A \in \mathcal{K}(V)$. Let us fix $S \in \mathcal{K}(\ker(p))$ with $vol_{\ker(S)}(S) = 1$. Then $C(\tilde{K}, \phi) = \frac{1}{k!} \frac{d^k}{d\epsilon^k}_{|\epsilon=0} vol_V(\tilde{K} + A + \epsilon S)$. But the last expression is equal to $vol_X(p(\tilde{K} + A)) = vol_X(K + p(A))$ by Lemma 2.6.1. Q.E.D.

Next $\tilde{\phi}$ is a continuous translation invariant valuation on $X$. Translation invariance is obvious. In order to prove continuity and valuation property let us fix a linear right inverse of $p$, $s: X \to V$. For any $K \in \mathcal{K}(X)$ let us choose $\tilde{K} := s(K)$. Thus $\tilde{\phi}(K) = C(s(K), \phi)$. Clearly the last expression is a continuous valuation in $K \in \mathcal{K}(X)$. Let us define

$$p_*(\phi \otimes vol_V^{-1}) := \tilde{\phi} \otimes vol_X^{-1}. \quad (3.2.1)$$

It is easy to see that the definition of $p_*$ does not depend on a choice of Lebesgue measures $vol_V, vol_X$. Finally define

$$f_* := j_* \circ p_*.$$

It follows from the construction that $f_*$ satisfies the assumptions of the proposition.

(ii) is obvious.

(iii) Let $U \xrightarrow{f_2} V \xrightarrow{f_1} W$. We have to show

$$(f_1 \circ f_2)_* = f_{1*} \circ f_{2*}. \quad (3.2.2)$$

First we will show this in a number of special cases. If $f_1$ is an injection and $f_2$ is a surjection the equality (3.2.2) is clear by the construction of pushforward from the proof of part (i).

3.2.4 Lemma. The equality (3.2.2) holds if $f_1, f_2$ are injections.

**Proof.** We may and will assume for simplicity that $f_1, f_2$ are imbeddings of linear subspaces, thus $U \subset V \subset W$. Let us fix Lebesgue measures $vol_U, vol_{V/U}, vol_{W/V}$ on $U, V/U, W/V$ respectively. Let $vol_V := vol_U \otimes vol_{V/U}$, $vol_W := vol_V \otimes vol_{W/V}$ be Lebesgue measures on $V, W$ respectively. Let us fix $\phi \in Val(U), K \in \mathcal{K}(W)$. We have

$$(f_{2*}(\phi \otimes vol_U^{-1}))(K) = \left( \int_{z \in V/U} \phi(K \cap z) dvol_{V/U}(z) \right) \otimes vol_V^{-1},$$

$$(f_{1*}(f_{2*}(\phi \otimes vol_U^{-1}))))(K) = \left( \int_{w \in W/V} \left( f_{2*}(\phi \otimes vol_U^{-1})(K \cap w) \otimes vol_V \cdot dvol_{W/V}(w) \right) \otimes vol_W^{-1} = \right) \left( \int_{x \in W/U} \phi(K \cap x) d \left( \frac{vol_W}{vol_U} \right)(x) \right) \otimes vol_W^{-1} = (f_1 \circ f_2)_* (\phi \otimes vol_U^{-1}).$$

Lemma is proved. Q.E.D.

3.2.5 Lemma. The equality (3.2.2) holds if $f_1, f_2$ are surjections.
Proof. Let us fix Lebesgue measures $vol_1, vol_2, vol_W$ on $Ker(f_1), Ker(f_2), W$ respectively. Set $vol_U := vol_1 \otimes vol_2 \otimes vol_W \in Dens(U)$, $vol_V = vol_1 \otimes vol_W \in Dens(V)$. Assume that $\phi \in Val(U)$ has the form

$$\phi(\bullet) = vol_U(\bullet + A)$$

where $A \in K(U)$ is fixed. Then

$$(f_2*(\phi \otimes vol_U^{-1}))(K) = vol_V(K + f_2(A)) \otimes vol_V^{-1},$$

$$(f_1*(f_2*(\phi \otimes vol_U^{-1}))) (K) = vol_W(K + f_1(f_2(A))) \otimes vol_W^{-1}. \quad (3.2.3)$$

On the other hand

$$(f_1 \circ f_2) *(\phi \otimes vol_U^{-1})(K) = vol_W(K + (f_1 \circ f_2)(A)) \otimes vol_W^{-1} \quad (3.2.4)$$

Comparing (3.2.3) and (3.2.4) one concludes the lemma using the McMullen’s conjecture. Q.E.D.

3.2.6 Lemma. The equality (3.2.3) holds if $f_1$ is a surjection, $f_2$ is an injection.

Proof. We will assume for simplicity and without loss of generality that $U \subset V$ and $f_2: U \rightarrow V$ is the identity imbedding. Also we may assume that $W$ is a quotient space of $V$, and $f_1: V \rightarrow W$ is the canonical quotient map.

Step 1. We consider the case

$Ker(f_1) \subset U.$

Denote $k := \dim Ker(f_1)$. Let us fix Lebesgue measures $vol_{Ker}$ on $Ker(f_1)$, $vol_U$ on $U$, $vol_{V/U}$ on $V/U$. Let $vol_V = vol_U \otimes vol_{V/U}$, $vol_W := \frac{vol_W}{vol_{Ker}}$ be the Lebesgue measure on $W$. Let $\phi \in Val(U)$ has the form

$$\phi(\bullet) = vol_U(\bullet + A)$$

for some $A \in K(U)$. Then for any $L \in K(V)$ we have

$$(f_2*(\phi \otimes vol_U^{-1}))(L) = \left( \int_{z \in V/U} vol_U((L \cap z) + A) dv_{V/U}(z) \right) \otimes vol_V^{-1}.$$  

Now let us fix a linear right inverse of $f_1$

$$s: W \rightarrow V.$$ 

Let $M \in K(W)$. Let $\tilde{M} := s(M)$. Let us fix a set $S \in K(Ker(f_1))$ with $vol_{Ker}(S) = 1$. By the construction in the proof of part (i) we get

$$\left( f_1*(f_2*(\phi \otimes vol_U^{-1})) \right)(M) = \quad (3.2.5)$$

$$\frac{1}{k!} \left. \frac{d^k}{dz^k} \right|_{z=0} \left( \int_{z \in V/U} vol_U \left( ((\tilde{M} + \varepsilon S) \cap z) + A \right) dv_{V/U}(z) \right) \otimes vol_W^{-1} = \quad (3.2.6)$$

$$\frac{1}{k!} \left. \frac{d^k}{dz^k} \right|_{z=0} \left( \int_{z \in V/U} vol_U \left( (\tilde{M} \cap z) + \varepsilon S + A \right) dv_{V/U}(z) \right) \otimes vol_W^{-1} = \quad (3.2.7)$$

$$\left( \int_{z \in (V/Ker(f_1))/(U/Ker(f_1))} vol_W \left( (M \cap z) + f_1(A) \right) dv_{V/U}(z) \right) \otimes vol_W^{-1}. \quad (3.2.8)$$
where the equality \((3.2.6)\) follows from the fact that \((\bar{M} + \varepsilon S) \cap z = (\bar{M} \cap z) + \varepsilon S\) since \(S \subset \text{Ker}(f_1) \subset U\), and in \((3.2.7)\) we used the identification \(V/U = (V/\text{Ker}(f_1))/(U/\text{Ker}(f_1))\).

Let us compute now \((f_1 \circ f_2)_*(\phi \otimes \text{vol}_U^{-1})\). The map \(f_1 \circ f_2\) factorizes as

\[
U \xrightarrow{p} U/\text{Ker}(f_1) \xrightarrow{j} V/\text{Ker}(f_1) = W.
\]

By the construction from part (i)

\[
(f_1 \circ f_2)_* = j_* \circ p_*.
\]

We have for any \(L \in \mathcal{K}(V)\)

\[
(p_*(\phi \otimes \text{vol}_U^{-1}))(L) = \text{vol}_{U/\text{Ker}}(L + p(A)) \otimes \text{vol}_U^{-1}_{U/\text{Ker}}.
\]

Next for any \(M \in \mathcal{K}(W)\)

\[
(j_*(p_*(\phi \otimes \text{vol}_U^{-1}))(M) = \left(\int_{z \in V/U} \text{vol}_W ((M \cap z) + p(A)) \text{dvol}_{V/U}(z) \right) \otimes \text{vol}_W^{-1}.
\]

Comparing \((3.2.7)\) and \((3.2.10)\) and observing that \(p(A) = f_1(A)\), we conclude Step 1.

Step 2. At this step we will assume that \(f_1 \circ f_2\) is injective.

Let us choose \(s: W \to V\) a right inverse of \(f_1\) so that \(s(W) \supseteq f_2(U)\). Let \(H := \text{Ker}(f_1)\). To simplify the notation and without loss of generality we will identify \(U\) with \(f_2(U)\), and \(W\) with \(s(W)\). Let us fix Lebesgue measures \(\text{vol}_U\) on \(U\), \(\text{vol}_{W/U}\) on \(W/U\), \(\text{vol}_H\) on \(H\). Let \(\text{vol}_W := \text{vol}_U \otimes \text{vol}_{W/U}\), \(\text{vol}_V := \text{vol}_W \otimes \text{vol}_H\), \(\text{vol}_{V/U} := \text{vol}_{W/U} \otimes \text{vol}_H\) be the corresponding Lebesgue measures on \(W, V, V/U\) respectively.

Let \(\phi \in \text{Val}(U)\). Let us fix \(K \in \mathcal{K}(W)\). Then we have

\[
((f_1 \circ f_2)_*(\phi \otimes \text{vol}_U^{-1}))(K) = \int_{z \in W/U} \phi(K \cap z) \text{dvol}_{W/U}(z) \otimes \text{vol}_W^{-1}.
\]

Let us fix a subset \(S \subset H\) with \(\text{vol}_H(S) = 1\). Denote \(k := \dim H\). Then for any \(L \in \mathcal{K}(V)\) we have

\[
(f_2_*(\phi \otimes \text{vol}_U^{-1}))(L) = \left(\int_{x \in V/U} \phi(L \cap x) \text{dvol}_{V/U}(x) \right) \otimes \text{vol}_V^{-1}.
\]

Next for any \(K \in \mathcal{K}(W)\)

\[
\frac{1}{k!} \frac{d^k}{d\varepsilon^k} \bigg|_{\varepsilon=0} \left(\int_{x \in V/U} \phi((K + \varepsilon S) \cap x) \text{dvol}_{V/U}(x) \right) \otimes \text{vol}_W^{-1}.
\]

But since \(K \subset W\) and \(S \subset H\), \(x \in V/U\),

\[
(K + \varepsilon S) \cap x = (K \cap x) + (\varepsilon S \cap x)
\]
and $\varepsilon S \cap x$ is either a point or the empty set. Hence (3.2.13) can be continued as
\[
\frac{1}{k!} \frac{d^k}{d\varepsilon^k} \bigg|_{\varepsilon=0} \left( \phi(K \cap x) d\nu_{W/U}(x) \right) \otimes \nu_{V}^{-1} = \\
\left( \int_{x \in W/U} \phi(K \cap x) d\nu_{W/U}(x) \right) \otimes \nu_{V}^{-1} \left( f_1 \circ f_2 \right)_*(\phi \otimes \nu_{U}^{-1})(K).
\]
This completes Step 2.

Step 3. Let us consider finally the case of general surjection $f_1$ and general injection $f_2$.

We will assume again that $U$ is a subspace of $V$, and $W$ is a quotient space of $V$. Set $A := U \cap Ker(f_1)$, $X := V/A$. We can decompose uniquely $f_2 : V \to W$ as a composition of two surjections
\[
V \xrightarrow{q} X \xrightarrow{p} W
\]
where $p : V \to V/A = X$ is the canonical surjection. By Lemma 3.2.5
\[(3.2.14) \quad f_1* = p_* \circ q_*.
\]
Let us denote $Y := U/A$. Let $t : U \to Y$ be the canonical surjection, and $j : Y \hookrightarrow X$ be the natural imbedding. Note that $Ker(q) = A \subset U$. Then by Step 1 and the construction of the pushforward from part (i) we get
\[(3.2.15) \quad q_* \circ f_2* = (q \circ f_2)_* = (j \circ t)_* = j_* \circ t_*.
\]
Clearly $p \circ j : Y \to W$ is injective. Hence by Step 2
\[(3.2.16) \quad p_* \circ j_* = (p \circ j)_*.
\]
Using (3.2.14), (3.2.15), and (3.2.16) we obtain
\[
f_1* \circ f_2* = p_* \circ q_* \circ f_2* = p_* \circ j_* \circ t_* = (p \circ j)_* \circ t_* = (f_1 \circ f_2)_*
\]
where the last equality follows from the facts that $f_1 \circ f_2 = (p \circ j) \circ t$, $t$ is surjective, $p \circ j$ is injective, and from the construction of the pushforward from the proof of part (i). Lemma 3.2.6 is proved. Q.E.D.

Now let us finish the proof of Proposition 3.2.1(ii) for general $f_1, f_2$. Let us decompose
\[
f_1 = j_1 \circ p_1, \quad f_2 = j_2 \circ p_2
\]
where $j_1, j_2$ are injections, $p_1, p_2$ are surjections. By definition
\[(3.2.17) \quad f_1* \circ f_2* = j_1* \circ p_1* \circ j_2* \circ p_2*.
\]
Let us decompose $p_1 \circ j_2 = j_3 \circ p_3$ where $j_3$ is an injection, $p_3$ is a surjection. By Lemma 3.2.6 $p_1* \circ j_2* = (p_1 \circ j_2)_* = j_3* \circ p_3*$. Hence using this, (3.2.17), and Lemmas 3.2.4 3.2.5 we get
\[
f_1* \circ f_2* = j_1* \circ j_3* \circ p_3* \circ p_2* = (j_1 \circ j_3)_* \circ (p_3 \circ p_2)_* = (f_1 \circ f_2)_*
\]
where the last equality follows from the construction of the pushforward given in the proof of part (i). Proposition 3.2.1 is proved. Q.E.D.
3.3 Relations to product and convolution.

We will explain the relation of pullback and pushforward to product and convolution. But first we will have to remind the notion of exterior product of smooth translation invariant valuations. Let $V, W$ be finite dimensional real vector spaces. In [3] the author has defined a continuous linear map

$$Val^{sm}(V) \times Val^{sm}(W) \to Val(V \times W) \quad (3.3.1)$$

called the exterior product. For $\phi \in Val^{sm}(V), \psi \in Val^{sm}(W)$ their exterior product is denoted by $\phi \boxtimes \psi$. The map (3.3.1) is uniquely characterized by the following property. Let $\phi(\bullet) = vol_V(\bullet + A), \psi(\bullet) = vol_W(\bullet + B)$ where $vol_V, vol_W$ are Lebesgue measures on $V, W$ respectively, $A \in K^{sm}(V), B \in K^{sm}(W)$. Then

$$(\phi \boxtimes \psi)(K) = (vol_V \boxtimes vol_W)(K + (A \times B))$$

for any $K \in K(V \times W)$, and where $vol_V \boxtimes vol_W$ denotes the usual product measure. Note that the exterior product of two smooth valuations may not be smooth.

In Appendix of this article we show a slightly more precise statement which will be useful later for some technical reasons. Namely it is shown that the exterior product extends (uniquely) to a continuous bilinear map

$$Val(V) \times Val^{sm}(W) \to Val(V \times W),$$

i.e. the first variable may be replaced by continuous valuations instead of smooth.

The following proposition is essentially the definition of the product of smooth valuations from [3].

3.3.1 Proposition. For any $\phi, \psi \in Val^{sm}(V)$

$$\phi \cdot \psi = \Delta^*(\phi \boxtimes \psi)$$

where $\Delta: V \to V \times V$ denotes the diagonal imbedding.

Let us explain now the relation between the convolution and the pushforward. Clearly $Dens(V \times W)^* = Dens(V^*) \otimes Dens(W^*)$. Hence the exterior product (3.3.1) tensored with $Id_{Dens(V \times W)^*}$ gives a continuous bilinear map

$$(Val^{sm}(V) \otimes Dens(V^*)) \times (Val^{sm}(W) \otimes Dens(W^*)) \to Val(V \times W) \otimes Dens(V \times W)^*$$

which will also be called exterior product and denoted by $\boxtimes$. Let us denote by $a: V \times V \to V$ the addition map, i.e. $a(x, y) = x + y$.

3.3.2 Proposition. For any $\phi, \psi \in Val^{sm}(V) \otimes Dens(V^*)$ one has

$$\phi \ast \psi = a_*(\phi \boxtimes \psi).$$
Proof. Let us fix a Lebesgue measure \( \text{vol}_V \) on \( V \). By continuity and the McMullen’s conjecture it is enough to prove the proposition for \( \phi(\bullet) = \text{vol}_V(\bullet + A) \), \( \psi(\bullet) = \text{vol}_V(\bullet + B) \) with \( A, B \in K^{sm}(V) \). Then

\[
(\phi \boxtimes \psi)(\bullet) = (\text{vol}_V \boxtimes \text{vol}_V)(\bullet + (A \times B)) \otimes (\text{vol}_V \otimes \text{vol}_V)^{-1}.
\]

Next

\[
(a_*(\phi \boxtimes \psi))(\bullet) = \text{vol}_V(\bullet + a(A \times B)) \otimes \text{vol}_V^{-1} = \text{vol}_V(\bullet + A + B) \otimes \text{vol}_V^{-1} = (\phi \ast \psi)(\bullet).
\]

Q.E.D.

### 3.4 Homomorphism property of pushforward.

The main result of this section is the following proposition.

#### 3.4.1 Proposition. Let \( p: X \to Y \) be a linear epimorphism of vector spaces. Then for any \( \phi \in \text{Val}^{sm}(X) \otimes \text{Dens}(X^*) \) the pushforward \( p_* \phi \) is smooth, i.e. \( p_* \phi \in \text{Val}^{sm}(Y) \otimes \text{Dens}(Y^*) \), and

\[
p_*: \text{Val}^{sm}(X) \otimes \text{Dens}(X^*) \to \text{Val}^{sm}(Y) \otimes \text{Dens}(Y^*)
\]

is a homomorphism of algebras (when both spaces are equipped with convolution).

It is easy to see that \( p_* \phi \) is smooth if \( p \) is surjective and \( \phi \) is smooth. In order to prove the second statement of the proposition we will need another proposition.

#### 3.4.2 Proposition. Let

\[
\begin{align*}
    f_1: V_1 &\to W_1, \\
    f_2: V_2 &\to W_2
\end{align*}
\]

be linear maps. Let \( \phi_i \in \text{Val}(W_i) \otimes \text{Dens}(W_i^*), \ i = 1, 2 \). Assume that \( f_1 \) is surjective and \( \phi_1 \) is smooth. Then

\[
(f_1 \boxtimes f_2)_*(\phi_1 \boxtimes \phi_2) = f_1_* \phi_1 \boxtimes f_2_* \phi_2.
\]

Proof. Let us fix Lebesgue measures \( \text{vol}_{Ker} \) on \( \text{Ker}(f_i) \), \( \text{vol}_{W_1} \) on \( W_1 \), \( \text{vol}_{W_2} \) on \( W_2 \), \( \text{vol}_{V_2} \) on \( V_2 \). Let \( \text{vol}_{V_1} = \text{vol}_{Ker} \otimes \text{vol}_{W_1} \) be the induced Lebesgue measure on \( V_1 \). Observe that by the Appendix to this article, both sides of the last equality are continuous with respect to \( \phi_1 \in \text{Val}^{sm}(V_1) \otimes \text{Dens}(V_1^*) \), \( \phi_2 \in \text{Val}(V_2) \otimes \text{Dens}(V_2^*) \).

Hence, by the McMullen’s conjecture, we may assume that

\[
\phi_i(\bullet) = \text{vol}_{V_i}(\bullet + A_i) \otimes \text{vol}_{V_i}^{-1}, \ i = 1, 2.
\]

Then

\[
(\phi_1 \boxtimes \phi_2)(\bullet) = (\text{vol}_{V_1} \boxtimes \text{vol}_{V_2})(\bullet + (A_1 \times A_2)) \otimes (\text{vol}_{V_1} \boxtimes \text{vol}_{V_2})^{-1}. \tag{3.4.1}
\]

Then

\[
(f_1 \ast \phi_i)(\bullet) = \text{vol}_{W_1}(\bullet + f_i(A_i)) \otimes \text{vol}_{W_i}^{-1}, \ i = 1, 2.
\]
\[(f_1 \boxtimes f_2)_*(\phi_1 \boxtimes \phi_2)(\bullet) = (vol_{W_1} \boxtimes vol_{W_2})(\bullet + (f_1(A_1) \times f_2(A_2))) \otimes (vol_{W_1} \boxtimes vol_{W_2})^{-1} = (f_1 \ast \phi_1 \boxtimes f_2 \ast \phi_2)(\bullet).\]

Proposition 3.4.2 is proved. Q.E.D.

**Proof** of Proposition 3.4.1 Let

\[a_X : X \times X \rightarrow X,\]
\[a_Y : Y \times Y \rightarrow Y\]

be the addition maps. Then

\[p_*(\phi \ast \psi) = p_* (a_{X*}(\phi \boxtimes \psi)) = (p \circ a_{X*})(\phi \boxtimes \psi) = (a_Y \circ (p \times p))_* (\phi \boxtimes \psi) = a_Y*(p_\ast (\phi \boxtimes p_\ast \psi))\]

\[\text{Lemma 3.4.2}\]

\[a_Y*(p_\ast (\phi \boxtimes p_\ast \psi)) = p_\ast (p_\ast \phi \ast p_\ast \psi).\]

Q.E.D.

### 3.5 Base change theorem.

Recall that a commutative diagram of linear maps of vector spaces

\[
\begin{array}{ccc}
A & \overset{f}{\rightarrow} & B \\
g & & u \\
C & \underset{v}{\rightarrow} & D \\
\end{array}
\]

is called a *Cartesian square* if it is isomorphic to a diagram

\[
\begin{array}{ccc}
Y \times_Z X & \overset{pr_X}{\rightarrow} & X \\
pr_Y & & u \\
Y & \underset{v}{\rightarrow} & Z \\
\end{array}
\]

where \(Y \times_Z X := \{(y, x) \in Y \times X | v(y) = u(x)\}\) and \(pr_X : Y \times_Z X \rightarrow X\) and \(pr_Y : Y \times_Z X \rightarrow Y\) are the natural maps.

**3.5.1 Lemma.** (i) Let

\[
\begin{array}{ccc}
\bar{X} & \overset{\bar{f}}{\rightarrow} & \bar{Y} \\
\bar{g} & & g \\
X & \overset{f}{\rightarrow} & Y \\
\end{array}
\]
be a Cartesian square of vector spaces such that \( f \oplus g : X \oplus \tilde{Y} \to Y \) is onto. Then there exists a canonical isomorphism

\[
\frac{\text{Dens}(\tilde{X}^*)}{\text{Dens}(X^*)} \to \frac{\text{Dens}(\tilde{Y}^*)}{\text{Dens}(Y^*)}.
\]

(ii) The following transitivity property of the isomorphism from part (i) holds. Assume that we have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}_1} & \tilde{Y} & \xrightarrow{\tilde{f}_2} & \tilde{Z} \\
\tilde{g} & & & \tilde{g} & & \\
X & \xrightarrow{f_1} & Y & \xrightarrow{f_2} & Z
\end{array}
\]

If the two small squares are Cartesian then the exterior contour is Cartesian. If \( f_1 \oplus g \) and \( f_2 \oplus \hat{g} \) are onto, then \((f_2 \circ f_1) \oplus \hat{g}\) is onto.

Moreover in the last case the isomorphism \( \frac{\text{Dens}(\tilde{X})}{\text{Dens}(X)} \to \frac{\text{Dens}(\tilde{Z})}{\text{Dens}(Z)} \) corresponding to the exterior contour of the diagram by the part (i) of the lemma, is equal to the composition of the isomorphisms

\[
\frac{\text{Dens}(\tilde{X})}{\text{Dens}(X)} \to \frac{\text{Dens}(\tilde{Y})}{\text{Dens}(Y)} \to \frac{\text{Dens}(\tilde{Z})}{\text{Dens}(Z)}
\]

corresponding to the small Cartesian squares.

**Proof.** (i) We have the short exact sequence of vector spaces

\[
0 \to \tilde{X} \xrightarrow{\tilde{f}_x} X \oplus \tilde{Y} \xrightarrow{f \oplus \tilde{g}} Y \to 0.
\]

Hence

\[
\text{Dens}(X \oplus \tilde{Y}) \simeq \text{Dens}(\tilde{X}) \otimes \text{Dens}(Y).
\]

But on the other hand \(\text{Dens}(X \oplus \tilde{Y}) = \text{Dens}(X) \otimes \text{Dens}(\tilde{Y})\). Hence

\[
\text{Dens}(\tilde{X}) \otimes \text{Dens}(Y) \simeq \text{Dens}(X) \otimes \text{Dens}(\tilde{Y}).
\]

Dualization of this isomorphism implies part (i).

The proof of part (ii) we leave to the reader. Q.E.D.

We have the following result we call the base change property. Roughly put, it says that for a Cartesian square as above one has

\[
g^* \circ f_* = \tilde{f}_* \circ \tilde{g}^*.
\]
3.5.2 Theorem (Base change theorem). Let

\[ \begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\tilde{g} & & g \\
X & \xrightarrow{f} & Y
\end{array} \]

be a Cartesian square of vector spaces such that \( f \oplus g : X \oplus \tilde{Y} \to Y \) is onto. Consider the following two maps

\[ \text{Val}(X) \otimes \text{Dens}(\tilde{X}^*) \to \text{Val}(\tilde{Y}) \otimes \text{Dens}(\tilde{Y}^*), \]

the first map given is by the composition

\[
\text{Val}(X) \otimes \text{Dens}(\tilde{X}^*) = (\text{Val}(X) \otimes \text{Dens}(X^*)) \otimes \frac{\text{Dens}(\tilde{X}^*)}{\text{Dens}(X^*)} \xrightarrow{f_* \otimes \text{Id}} \\
(\text{Val}(Y) \otimes \text{Dens}(Y^*)) \otimes \frac{\text{Dens}(\tilde{X}^*)}{\text{Dens}(X^*)} = \text{Val}(Y) \otimes \text{Dens}(Y^*) \xrightarrow{g_* \otimes \text{Id}_{\text{Dens}(\tilde{Y}^*)}} \\
\text{Val}(\tilde{Y}) \otimes \text{Dens}(\tilde{Y}^*)
\]

where we have used the identification \( \text{Dens}(Y^*) \otimes \frac{\text{Dens}(\tilde{X}^*)}{\text{Dens}(X^*)} \simeq \text{Dens}(\tilde{Y}^*) \) from Lemma 3.5.1(i); and the second map is given by the composition

\[
\text{Val}(X) \otimes \text{Dens}(\tilde{X}^*) \xrightarrow{\tilde{g}^* \otimes \text{Id}_{\text{Dens}(\tilde{X}^*)}} \text{Val}(\tilde{X}) \otimes \text{Dens}(\tilde{X}^*) \xrightarrow{\tilde{f}_*} \text{Val}(\tilde{Y}) \otimes \text{Dens}(\tilde{Y}^*). 
\]

Then these two maps coincide.

Proof. We will prove this result in several steps. In the first two steps we will show that it is enough to prove the theorem under the assumption that each of \( f \) and \( g \) is either injection or surjection. Then we will prove the theorem in each of these cases.

Step 1. Transitivity with respect to \( f \).

Assume that we have a commutative diagram

\[ \begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}_1} & \tilde{Y} \\
\tilde{g} & & g \\
X & \xrightarrow{f_1} & Y \\
\tilde{f}_2 & & \tilde{Z} \\
\tilde{g} & & g
\end{array} \]

Then these two maps coincide.
such that \( f_1 \oplus g \) and \( f_2 \oplus \hat{g} \) are onto and two small squares satisfy conclusions of the theorem. Then the diagram of the exterior contour also satisfies these conclusions by Propositions 3.1.2(iii), 3.2.1(iii), and Lemma 3.5.1(ii).

**Step 2.** Transitivity with respect to \( g \).

Assume that we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow{\tilde{g}_1} & & \downarrow{g_1} \\
X & \xrightarrow{f} & Y \\
\downarrow{\tilde{g}_2} & & \downarrow{g_2} \\
\tilde{Z} & \xrightarrow{\tilde{f}} & \tilde{Z}
\end{array}
\]

such that \( f \oplus g_1 \) and \( \hat{f} \oplus g_2 \) are onto, and the small squares satisfy the conclusions of the theorem. Then, as in Step 1, the diagram of the exterior contour also satisfies these conclusions by Propositions 3.1.2(iii), 3.2.1(iii), and Lemma 3.5.1(ii).

**Step 3.** Let us assume that \( g \) is surjective.

Then we may and will assume that

\[
\tilde{Y} = Y \oplus L
\]

and \( g \) is the projection \( pr_Y : Y \oplus L \to Y \). Hence \( \tilde{X} = X \oplus L \), \( \tilde{g} \) is the projection \( pr_X : X \oplus L \to X \), and \( \tilde{f} = f \boxtimes Id_L : X \oplus L \to Y \oplus L \). Thus the diagram (3.5.1) becomes equal to the diagram

\[
\begin{array}{ccc}
X \oplus L & \xrightarrow{f \boxtimes Id_L} & Y \oplus L \\
\downarrow{pr_X} & & \downarrow{pr_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\]

(3.5.2)

Observe that we have canonical isomorphisms

\[
\frac{D(\tilde{X}^*)}{D(X^*)} \simeq \frac{D(\tilde{Y}^*)}{D(Y^*)} \simeq D(L^*).
\]

Let us denote \( l := \dim L \). Let us fix Lebesgue measures \( vol_L \) on \( L \) and \( vol_X \) on \( X \). Let us fix also \( S \in \mathcal{K}(L) \) such that \( vol_L(S) = 1 \). Abusing the notation we will denote the first map in the statement of the theorem by "\( pr_Y^* \circ f_* \)", and the second map by "\( (f \times Id_L)_* \circ pr_X^* \)". We have to show that they coincide. We may decompose \( f \) into a composition of injection and surjection and, using Step 1, prove the result separately in each case.

**Case a.** Assume that \( f \) is surjective.
Then $X$ may and will be assumed to be equal to $Y \oplus M$, and $f: Y \oplus M \rightarrow Y$ is the natural projection. Thus $\widetilde{X} = Y \oplus M \oplus L$. Then the diagram \[ (3.5.2) \] becomes equal to

\[
\begin{array}{ccc}
Y \oplus M \oplus L & \xrightarrow{f \boxtimes Id_L} & Y \oplus L \\
pr_X & & pr_Y \\
X = Y \oplus M & \xrightarrow{f} & Y
\end{array}
\]

(3.5.3)

Let us fix a Lebesgue measure $\text{vol}_M$ on $M$. Let $\text{vol}_Y = \frac{\text{vol}_X}{\text{vol}_M} \in \text{Dens}(Y)$. Let us denote $m := \dim M$. Let us fix $T \in \mathcal{K}(M)$ such that $\text{vol}_M(T) = 1$. Let us fix $\phi \in \text{Val}(Y \oplus M) \otimes \text{Dens}((Y \oplus M \oplus L)^*)$. Finally let us fix an arbitrary subset $K \in \mathcal{K}(Y \oplus M)$. Then we have

\[
"(pr_Y^* \circ f_*)"(\phi)(K) = ("f_*" \phi)(pr_Y(K)) = \int_{z \in Y/X} \phi(pr_Y(K) \cap z) d\text{vol}_{Y/X}(z) \otimes vol_{Y/X}^{-1} = \int_{z \in Y/X} \phi(pr_Y(K \cap z \times L)) d\text{vol}_{Y/X}(z) \otimes vol_{Y/X}^{-1}.
\]

(3.5.4)

On the other hand

\[
"(f \boxtimes Id_L)_* \circ pr_X^*"(\phi)(K) = \int_{z \in Y/X} \phi(pr_X(K \times \varepsilon T)) \otimes vol_{Y/X} = \int_{z \in Y/X} \phi(pr_X(K \cap z \times L)) d\text{vol}_{Y/X}(z) \otimes vol_{Y/X}^{-1}.
\]

(3.5.5)

Since \[ (3.5.6) = (3.5.9) \], Case a is proved.

Case b. Assume that $f: X \rightarrow Y$ is injective.

It suffices to consider the case when $X$ is a subspace of $Y$, and $f$ is the identity imbedding. Let us fix Lebesgue measures $\text{vol}_X$ on $X$, $\text{vol}_Y$ on $Y$. Set $\text{vol}_{Y/X} := \frac{\text{vol}_Y}{\text{vol}_X} \in \text{Dens}(Y/X)$. Then the diagram \[ (3.5.2) \] becomes equal to

\[
\begin{array}{ccc}
X \oplus L & \xrightarrow{f \boxtimes Id_L} & Y \oplus L \\
pr_X & & pr_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

(3.5.10)

We have

\[
"pr_Y^* \circ f_*"(\phi)(K) = ("f_*" \phi)(pr_Y(K)) = \int_{z \in Y/X} \phi(pr_Y(K) \cap z) d\text{vol}_{Y/X}(z) \otimes vol_{Y/X}^{-1} = \int_{z \in Y/X} \phi(pr_Y(K \cap (z \times L))) d\text{vol}_{Y/X}(z) \otimes vol_{Y/X}^{-1}.
\]

(3.5.11)

(3.5.12)

(3.5.13)
On the other hand we get
\[ "(f \boxtimes Id_L)_* \circ \text{pr}_X^*"(\phi)(K) = \int_{z \in Y/X} (\text{pr}_X^* \phi)(K \cap (z \times L))\text{dvol}_{Y/X} \otimes \text{vol}_{Y/X}^{-1} = \int_{z \in Y/X} \phi(\text{pr}_X(K \cap (z \times L)))\text{dvol}_{Y/X} \otimes \text{vol}_{Y/X}^{-1} \] (3.5.14)

Comparing (3.5.13) and (3.5.16) and making appropriate identifications of subsets of \(X\) inside \(Y\), we conclude Case b. Thus Step 3 is completed.

Step 4. Assume that \(g\) is injective.

Then we may and will assume that \(\tilde{Y} \subset Y\) and \(g : \tilde{Y} \to Y\) is the identity imbedding. Let us fix Lebesgue measures \(\text{vol}_{\tilde{Y}}\) on \(\tilde{Y}\) and \(\text{vol}_Y\) on \(Y\). Set
\[ \text{vol}_{Y/\tilde{Y}} := \frac{\text{vol}_Y}{\text{vol}_{\tilde{Y}}} \in \text{Dens}(Y/\tilde{Y}). \]

By Step 1, it suffices to prove the result in two cases: either \(f\) is surjective or injective.

Case a. Assume that \(f\) is surjective.

Then we may assume that \(X = Y \oplus M\) and \(f = \text{pr}_Y : Y \oplus M \to Y\) is the natural projection. Then the diagram (3.5.1) becomes equal to
\[ \tilde{Y} \oplus M \xrightarrow{\tilde{g}} \tilde{Y} \]
\[ \tilde{g} = g \boxtimes Id_M \]
\[ Y \oplus M \xrightarrow{\text{pr}_Y} Y. \] (3.5.17)

Let us fix a Lebesgue measure \(\text{vol}_M\) on \(M\), and \(T \in \mathcal{K}(M)\) such that \(\text{vol}_M(T) = 1\). Set \(\text{vol}_X := \text{vol}_Y \otimes \text{vol}_M \in \text{Dens}(Y \oplus M) = \text{Dens}(X)\). Let \(\phi \in \text{Val}(Y \oplus M) \otimes \text{Dens}(\tilde{Y} \oplus M)\). Let us fix also \(K \in \mathcal{K}(\tilde{Y})\).

We have
\[ ("g^* \circ \text{pr}_Y^*")(\phi)(K) = ("\text{pr}_Y^*"(\phi))(K) = \frac{1}{m!d\varepsilon^m} \int_0^\varepsilon |\phi(K \times \varepsilon T)|. \] (3.5.18)

On the other hand we have
\[ ("\text{pr}_{\tilde{Y}}^* \circ \tilde{g}^*") (\phi)(K) = \frac{1}{m!d\varepsilon^m} \int_0^\varepsilon ("\tilde{g}^*"\phi)(K \times \varepsilon T) = \frac{1}{m!d\varepsilon^m} \int_0^\varepsilon \phi(K \times \varepsilon T). \] (3.5.20)

Comparing (3.5.19) and (3.5.21) we conclude Case a.

Case b. Assume that \(f\) is injective.
We may and will assume that \( X \subset Y \) and \( f \) is the identity imbedding. Under these assumptions \( \tilde{X} = X \cap \tilde{Y} \). Let us fix decompositions
\[
X = \tilde{X} \oplus L,
\]
\[
\tilde{Y} = \tilde{X} \oplus M.
\]
Then \( Y = \tilde{X} \oplus L \oplus M \) (since we assume that \( f \oplus g : X \oplus \tilde{Y} \to Y \) is onto). Then the diagram (3.5.1) becomes the following diagram of imbeddings
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{Id_{\tilde{X}} \otimes 0_M} & \tilde{X} \oplus M \\
\downarrow{Id_{\tilde{X}} \otimes 0_L} & & \downarrow{Id_{\tilde{X} \oplus L} \otimes 0_M} \\
\tilde{X} \oplus L & \xrightarrow{Id_{\tilde{X} \oplus L} \otimes 0_M} & \tilde{X} \oplus L \oplus M
\end{array}
\]
(3.5.22)

Let us fix Lebesgue measures \( \text{vol}_{\tilde{X}}, \text{vol}_L, \text{vol}_M \) on \( \tilde{X}, L, M \) respectively. Let
\[
\begin{align*}
\text{vol}_X &:= \text{vol}_{\tilde{X}} \otimes \text{vol}_L \in \text{Dens}(X), \\
\text{vol}_{\tilde{Y}} &:= \text{vol}_{\tilde{X}} \otimes \text{vol}_M \in \text{Dens}(\tilde{Y}), \\
\text{vol}_Y &:= \text{vol}_{\tilde{X}} \otimes \text{vol}_L \otimes \text{vol}_M \in \text{Dens}(Y).
\end{align*}
\]
Fix \( \phi \in Val(X) \otimes \text{Dens}(\tilde{X}^*) \). Fix \( K \in \mathcal{K}(\tilde{Y}) \). We have
\[
(”g^* \circ f_*”)(\phi)(K) = (”f_*”)(\phi)(K) = \int_{x \in M} \phi(K \cap (x + (\tilde{X} \oplus L))) d\text{vol}_M(x) \otimes \text{vol}_M^{-1} = \int_{x \in M} \phi((K \cap (x + \tilde{X})) \times \{0_L\}) \otimes \text{vol}_M^{-1}. (3.5.23)
\]
On the other hand we have
\[
(”\tilde{f}_* \circ g^{**}”)(\phi)(K) = \int_{x \in M} (”g^{**}”)(\phi)(K \cap (x + \tilde{X})) d\text{vol}_M(x) \otimes \text{vol}_M^{-1} = \int_{x \in M} \phi((K \cap (x + \tilde{X})) \times \{0_L\}) \otimes \text{vol}_M^{-1}. \quad (3.5.24)
\]
Comparing (3.5.23) and (3.5.24) we conclude Case b. Thus Step 4 is proved.

Step 5. Let us consider the general case. By Step 2 we may assume that \( g \) is either injective or surjective. Now the theorem follows from Steps 3, 4. Q.E.D.

The next result is in fact an equivalent reformulation of the base change theorem. It will be needed later.

3.5.3 Theorem. Let
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\downarrow{\tilde{g}} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
\]
be a Cartesian square of vector spaces such that \( f \oplus g : X \oplus \tilde{Y} \rightarrow Y \) is onto. Consider the following two maps

\[
Val(\tilde{Y}) \otimes \text{Dens}(\tilde{Y}^*) \rightarrow Val(X) \otimes \text{Dens}(Y^*) .
\]

The first map is the composition

\[
Val(\tilde{Y}) \otimes \text{Dens}(\tilde{Y}^*) \xrightarrow{g^*} Val(Y) \otimes \text{Dens}(Y^*) \xrightarrow{f^* \otimes \text{Id}_{\text{Dens}(Y^*)}} Val(X) \otimes \text{Dens}(Y^*) ;
\]

and the second map is the composition

\[
Val(\tilde{Y}) \otimes \text{Dens}(\tilde{Y}^*) \xrightarrow{\tilde{f}^* \otimes \text{Id}_{\text{Dens}(\tilde{Y}^*)}} Val(X) \otimes \text{Dens}(Y^*)
\]

\[
Val(\tilde{X}) \otimes \text{Dens}(\tilde{Y}^*) = \left( Val(X) \otimes \text{Dens}(\tilde{X}^*) \right) \otimes \frac{\text{Dens}(\tilde{Y}^*)}{\text{Dens}(X^*)} \xrightarrow{\tilde{g}^* \otimes \text{Id}}
\]

\[
(Val(X) \otimes \text{Dens}(X^*)) \otimes \frac{\text{Dens}(Y^*)}{\text{Dens}(X^*)} = Val(X) \otimes \text{Dens}(Y^*)
\]

where in the last equality we have used the identification from Lemma 3.5.1(i). Then these two maps coincide.

**Proof.** This result is obtained from Theorem 3.5.2 by flipping the diagram in the latter theorem with respect to the diagonal, twisting all the spaces by \( \text{Dens}(\tilde{Y}^*) \otimes \text{Dens}(X^*) \), and using the isomorphism \( \frac{\text{Dens}(\tilde{Y}^*)}{\text{Dens}(X^*)} \otimes \text{Dens}(X^*) \simeq \text{Dens}(Y^*) \) from Lemma 3.5.1(i). Q.E.D.

### 4 An isomorphism of \( GL(V) \)-modules \( Val_{n-p}^{-,sm}(V) \) and \( Val_{p}^{-,sm}(V^*) \otimes \text{Dens}(V) \)

The main result of this section is the following proposition.

**4.1.1 Proposition.** Let \( 1 \leq p \leq n-1 \). The \( GL(V) \)-modules \( Val_{n-p}^{-,sm}(V) \) and \( Val_{p}^{-,sm}(V^*) \otimes \text{Dens}(V) \) are isomorphic.

**Proof.** Step 1. By [3] the product on valuations

\[
Val_{n-p}^{-,sm} \times Val_{p}^{-,sm}(V) \rightarrow Val_n(V) = \text{Dens}(V)
\]

is a perfect pairing. It follows that the induced map

\[
Val_{n-p}^{-,sm} \rightarrow (Val_{p}^{-,sm}(V))^{*,sm} \otimes \text{Dens}(V)
\]  

\[(4.1.1)\]

is an isomorphism of \( GL(V) \)-modules. Thus to prove the proposition we have to show that the \( GL(V) \)-modules \( (Val_{p}^{-,sm}(V))^{*,sm} \) and \( Val_{p}^{-,sm}(V^*) \) are isomorphic.

Step 2. For a vector space \( W \) let us denote by \( F_{k,k+1}(W) \) the manifold of partial flags

\[
F_{k,k+1}(W) := \{(E,F) \mid E \in Gr_k(W), F \in Gr_{k+1}(W), E \subset F\}.
\]
Let $\mathcal{M}_{k,k+1}(W) \to \mathcal{F}_{k,k+1}(W)$ be the vector bundle such that its fiber over a pair $(E \subset F) \in \mathcal{F}_{k,k+1}(W)$ is equal to $E$. Similarly let $\mathcal{N}_{k,k+1}(W) \to \mathcal{F}_{k,k+1}(W)$ be the vector bundle such that its fiber over $(E \subset F) \in \mathcal{F}_{k,k+1}(W)$ is equal to $F$. Thus $\mathcal{M}_{k,k+1}(W) \subset \mathcal{N}_{k,k+1}(W)$.

By [2], $Val_p^{-,sm}(W)$ is isomorphic to an irreducible quotient of

$$C^\infty((\mathcal{F}_{k,k+1}(W), \det \mathcal{M}_{k,k+1}^*(W) \otimes or(\mathcal{N}_{k,k+1}(W))).$$

Thus $Val_p^{-,sm}(V^*)$ is isomorphic to an irreducible quotient of

$$C^\infty((\mathcal{F}_{p,p+1}(V^*), \det \mathcal{M}_{p,p+1}^*(V^*) \otimes or(\mathcal{N}_{p,p+1}(V^*))).$$

Step 3. Now we will show that both $(Val_p^{-,sm(V)})^{*,sm}$ and $Val_p^{-,sm}(V^*)$ appear in the Jordan-Hölder series of the same degenerate principal series representation.

Step 2 implies that $(Val_p^{-,sm(V)})^{*,sm}$ is isomorphic to an irreducible quotient of

$$C^\infty((\mathcal{F}_{p,p+1}(V), \det \mathcal{M}_{p,p+1}(V) \otimes or(\mathcal{N}_{p,p+1}(V)) \otimes |\omega_{\mathcal{F}_{p,p+1}(V)}|) \quad (4.1.2)$$

where $|\omega_X|$ denotes the line bundle of densities over a manifold $X$.

Recall that by Step 2 $Val_p^{-,sm}(V^*)$ is isomorphic to an irreducible quotient of

$$C^\infty((\mathcal{F}_{p,p+1}(V^*), \det \mathcal{M}_{p,p+1}^*(V^*) \otimes or(\mathcal{N}_{p,p+1}(V^*))).$$

By taking the orthogonal complement, $\mathcal{F}_{p,p+1}(V^*)$ is identified with $\mathcal{F}_{n-p-1,n-p}(V)$. Then $\mathcal{M}_{p,p+1}^*(V^*)$ is identified with $\mathcal{V}/\mathcal{N}_{n-p-1,n-p}(V)$, and $\mathcal{N}_{p,p+1}(V^*)$ is identified with the bundle $((\mathcal{V}/\mathcal{M}_{n-p-1,n-p}(V))^*)$ where $\mathcal{V} = \mathcal{F}_{n-p-1,n-p}(V) \times V$. Of course, all the identifications are $GL(V)$-equivariant. Hence

$$C^\infty((\mathcal{F}_{p,p+1}(V^*), \det \mathcal{M}_{p,p+1}(V^*) \otimes or(\mathcal{N}_{p,p+1}(V^*))) = \quad (4.1.3)$$

$$C^\infty((\mathcal{F}_{n-p-1,n-p}(V), \det(\mathcal{V}/\mathcal{N}_{n-p-1,n-p}(V)) \otimes or(\mathcal{V}/\mathcal{M}_{n-p-1,n-p}(V))). \quad (4.1.4)$$

By Corollary 1.3.7 the natural representations of $GL(V)$ in the spaces (4.1.2) and (4.1.4) have the same Jordan-Hölder series. Hence both $(Val_p^{-,sm(V)})^{*,sm}$ and $Val_p^{-,sm}(V^*)$ appear in the Jordan-Hölder series of (4.1.4) which is isomorphic to (4.1.3), i.e. to

$$\mathcal{X} = C^\infty((\mathcal{F}_{p,p+1}(V^*), \det \mathcal{M}_{p,p+1}(V^*) \otimes or(\mathcal{N}_{p,p+1}(V^*)). \quad (4.1.5)$$

Step 4. In this last and the most technical step we will show that $(Val_p^{-,sm(V)})^{*,sm}$ cannot be isomorphic to any constituent of the Jordan-Hölder series of (4.1.3) different from $Val_p^{-,sm}(V^*)$. This will finish the proof of the proposition. Remind that to any finitely generated $U(g)$-module $M$ one can attach an algebraic subvariety of the variety of nilpotent element of $g$ which is called associated variety or Bernstein variety. We refer to [17] for the detail on this notion. It turns out that the associated variety of $(Val_p^{-,sm(V)})^{*,sm}$ is equal to the variety of complex symmetric nilpotent matrices of rank at most 1. Indeed by the Poincaré duality (4.1.1) this space is isomorphic to $Val_n^{-,sm}(V) \otimes Dens(V^*)$. Clearly the associated variety of the last space coincides with that of $Val_n^{-,sm}(V)$. But by [3], Theorem 3.1, the associated variety of $Val_n^{-,sm}(V)$ is equal to the variety of complex symmetric nilpotent matrices of rank at most 1.
Let us remind few facts about the structure of the space \((4.1.5)\). Let
\[ q: \mathcal{F}_{p,p+1}(V^*) \rightarrow Gr_{p+1}(V^*) \]
be the canonical projection. Let \(T_{p+1}(V^*) \rightarrow Gr_{p+1}(V^*)\) be the tautological bundle, i.e. the bundle whose fiber over \(F \in Gr_{p+1}(V^*)\) is equal to \(F\). It is clear that \(N_{p,p+1}(V^*) = q^*(T_{p+1}(V^*))\). Hence
\[
\wedge^p N_{p,p+1}^*(V^*) = q^* (\wedge^p T_{p+1}^*(V^*)),
\]
\[
or (N_{p,p+1}(V^*)) = q^*(or (T_{p+1}(V^*)�).
\]

Consider the map of vector bundles \(N_{p,p+1}(V^*) \rightarrow M_{p,p+1}(V^*)\) dual to the natural imbedding \(M_{p,p+1}(V^*) \hookrightarrow N_{p,p+1}(V^*)\). The \(p\)-th exterior power of this map induces a map
\[
\mathcal{Y} := C^\infty(Gr_{p+1}(V^*), \wedge^p T_{p+1}^*(V^*) \otimes or(T_{p+1}(V^*))) \rightarrow C^\infty(\mathcal{F}_{p,p+1}(V^*), det M_{p,p+1}(V^*) \otimes or(N_{p,p+1}(V^*))) = \mathcal{X}.
\]

Clearly this map is \(GL(V)\)-equivariant (and in fact injective). The Casselman-Wallach theorem implies that the image of this map is a closed subspace. We will identify \(\mathcal{Y}\) with its image in \(\mathcal{X}\) under this map. By \([2]\), \(Val^-_{p,sm}(V^*)\) imbeds \(GL(V)\)-equivariantly as a subspace in \(\mathcal{X}/\mathcal{Y}\). Moreover, by \([3]\), Section 5, \(Val^-_{p,sm}(V^*)\) is the only irreducible subquotient of \(\mathcal{X}/\mathcal{Y}\) whose associated variety consists of complex symmetric nilpotent matrices of rank at most 1.

Hence it remains to show that \((Val^-_{p,sm}(V^*)^{*,sm}\) cannot be isomorphic to any of the irreducible subquotients of \(\mathcal{Y} = C^\infty(Gr_{p+1}(V^*), \wedge^p T_{p+1}^*(V^*) \otimes or(T_{p+1}(V^*))\)).

Before we will treat the general case, let us observe now that if \(p + 1 = n\) then the last statement is trivial since \((Val^-_{p,sm}(V^*)^{*,sm}\) is infinite dimensional while \(\mathcal{Y} = \wedge^p V^* \otimes or(V^*)\) is finite dimensional (since \(Gr_{p+1}(V^*)\) is just a point). Hence for \(p = n - 1\) Proposition \([4.1.1]\) is proved. By symmetry, replacing \(V\) by \(V^*\), Proposition \([4.1.1]\) follows also for \(p = 1\).

Let us assume now that \(2 \leq p \leq n - 2\), hence \(n \geq 4\). In this case we are going to use the Beilinson-Bernstein localization theorem. First we will have to introduce more notation and remind some constructions from \([3]\).

Let us fix a Euclidean metric on \(V^*\). Let \(G_0 = GL(V^*)\). Let \(g_0 = Lie(G_0)\) be the Lie algebra of \(G_0\). Let \(g := g_0 \otimes_R \mathbb{C}\) be its complexification. Let \(K_0 \subset G_0\) be the subgroup of the orthogonal transformations of \(V^*\). Let \(K\) be the complexification of \(K_0\). Let \(G\) be the complexification of \(G_0\). Thus \(G \simeq GL_n(\mathbb{C}), Lie(G) = g\).

Let \(\mathbb{C}Gr_k\) denote the Grassmannian of complex \(k\)-dimensional subspaces of \(V^* \otimes_R \mathbb{C} =: \mathbb{C}V\). Denote
\[
\mathbb{C}F_{p,p+1} := \{(E, F)| E \subset F, E \in \mathbb{C}Gr_k, F \in \mathbb{C}Gr_{p+1}\}.
\]
Let \(\mathbb{C}F\) be the variety of complete flags in \(\mathbb{C}V^*\). We have the canonical projection
\[
\bar{q}: \mathbb{C}F \rightarrow \mathbb{C}F_{p,p+1} \quad (4.1.6).
\]
It is well known that the group \(K\) acts on \(\mathbb{C}F\) (and hence on \(\mathbb{C}F_{p,p+1}, \mathbb{C}Gr_k\)) with finitely many orbits.
Let us fix a basis $e_1, e_2, \ldots, e_n$ in $V^*$. Let $T_0 \subset G_0$ be the subgroup of diagonal transformations with respect to this basis. Let $B_0 \subset G_0$ be the subgroup of upper triangular transformations. Let $T$ and $B$ be the complexifications of $T_0$ and $B_0$ respectively. Thus $T \subset G$ is a Cartan subgroup, $B \subset G$ is a Borel subgroup.

Let $P_0 \subset G_0$ be the subgroup of transformations of preserving the flag $\text{span}_\mathbb{R}\{e_1, \ldots, e_p\} \subset \text{span}_\mathbb{R}\{e_1, \ldots, e_p, e_{p+1}\}$. Let $P \subset G$ be its complexification which is a parabolic subgroup of $G$. Then clearly $T_0 \subset B_0 \subset P_0$, $T \subset B \subset P$. By $t, b, p$ we will denote the Lie algebras of $T, B, P$ respectively.

Clearly in the basis $e_1, \ldots, e_n$ the subgroup $T$ consists of complex diagonal invertible matrices, $B$ consists of complex upper triangular invertible matrices, and

$$P = \left\{ \begin{bmatrix} A & * & * \\ 0 & b & * \\ 0 & 0 & C \end{bmatrix} \mid A \in GL_p(\mathbb{C}), b \in \mathbb{C}^*, C \in GL_{n-p-1}(\mathbb{C}) \right\}.$$  

Let us consider the character $\chi: p/[p, p] \to \mathbb{C}$ given by

$$\chi \left( \begin{bmatrix} A & * & * \\ 0 & b & * \\ 0 & 0 & C \end{bmatrix} \right) = -Tr(A). \quad (4.1.7)$$

Let $\hat{\chi}: b/[b, b] \to \mathbb{C}$ be the composition of $\chi$ with the canonical map $b/[b, b] \to p/[p, p]$. It is easy to see that

$$\hat{\chi} \left( \begin{bmatrix} x_1 & * & * \\ 0 & \ddots & * \\ 0 & \cdots & x_p & * \\ 0 & \cdots & 0 & x_{p+1} \\ \cdots & \cdots & \cdots & \ddots & * \\ 0 & \cdots & \cdots & \cdots & 0 & x_n \end{bmatrix} \right) = -(x_1 + \cdots + x_p).$$

We will be interested in $D_{\chi}$-modules on $^C\mathcal{F}_{p,p+1}$ and $D_{\hat{\chi}}$-modules on $^C\mathcal{F}$. Clearly we have the pullback functor

$$\bar{q}^*: D_{\chi}(^C\mathcal{F}_{p,p+1}) - mod \to D_{\hat{\chi}}(^C\mathcal{F}) - mod.$$  

**4.1.2 Lemma.** The character $\hat{\chi}: b/[b, b] \to \mathbb{C}$ is dominant and regular in sense of Definition 1.4.1.

**Proof.** Let $\bar{\chi}: B_0/[B_0, B_0] \to \mathbb{C}^*$ be the character of the group defined by

$$\bar{\chi} \left( \begin{bmatrix} z_1 & * & * \\ 0 & \ddots & * \\ 0 & \cdots & z_n \end{bmatrix} \right) = (z_1 \cdots z_n)^{-1}.$$  

Consider the representation $\text{Ind}^{G_0}_{B_0}\bar{\chi}$. By Proposition 1.4.6 in order to prove that $\hat{\chi}$ is dominant and regular it is enough to show that $\text{Ind}^{G_0}_{B_0}\bar{\chi}$ has a non-zero finite dimensional submodule. But we have a natural non-zero map $\wedge^n V \to \text{Ind}^{G_0}_{B_0}\bar{\chi}$. Hence lemma is proved. Q.E.D.
4.1.3 Corollary. The functor of global sections

\[ \Gamma : D_\chi(CF_{p,p+1}) \to g \mod \]

is exact and faithful.

Proof. Since the morphism \( \bar{q} \) is projective and smooth

\[ \Gamma = \hat{\Gamma} \circ \bar{q}^* \]

where \( \hat{\Gamma} \) is the functor of global sections on the category \( D_\chi(CF) \). \( \hat{\Gamma} \) is exact and faithful by Lemma 4.1.2 and the Beilinson-Bernstein theorem. The functor \( \bar{q}^* \) is exact and faithful too since \( \bar{q} \) is a smooth morphism. Hence \( \Gamma \) is also exact and faithful. Q.E.D.

Let us now remind, following [3], the construction of a \( K \)-equivariant \( D_\chi \)-module \( M \) on \( CF_{p,p+1} \) such that the space of its global sections is isomorphic, as a \((g,K)\)-module, to the Harish-Chandra module of \( X \) (defined in (4.1.5)).

Let \( B \) be the complexification of the Euclidean form on \( V^* \). Thus \( B : CV \times CV \to \mathbb{C} \) is a symmetric non-degenerate bilinear form. Let \( U \subset CF_{p,p+1} \) denote the open \( K \)-orbit of \( CF_{p,p+1} \). Let \( j : U \hookrightarrow CF_{p,p+1} \) denote the identity imbedding. Explicitly one has

\[ U = \{ (E,F) \in CF_{p,p+1} \text{ the restrictions of } B \text{ to } E \text{ and to } F \text{ are non-degenerate} \}. \]

Let us fix an element \((E_0,F_0) \in U\). The stabilizer \( S \subset K \) of this element is isomorphic to the group \( O(p,\mathbb{C}) \times O(1,\mathbb{C}) \times O(n-p-1,\mathbb{C}) \). Note that \( S \) is a reductive group, and hence \( U = K/S \) is an affine variety.

The category of \( K \)-equivariant \( D_\chi \)-modules on the orbit \( U \) is equivalent to the category of representations of the group of connected components of \( S \). Let \( M_0 \) be the \( K \)-equivariant \( D_\chi \)-module on \( U \) corresponding to the representation of \( S \simeq O(p,\mathbb{C}) \times O(1,\mathbb{C}) \times O(n-p-1,\mathbb{C}) \) given by

\[ (A,B,C) \mapsto \det A \cdot \det B. \]

Let us define \( M := j_* M_0 \). It was shown in [3] that the \((g,K)\)-module \( \Gamma(CF_{p,p+1}M) \) is isomorphic to the Harish-Chandra module of \( X \).

Let us describe now the \( K \)-equivariant sub-\( D_\chi \)-module \( N \subset M \) corresponding to the Harish-Chandra module of \( Y \) (more precisely, \( \Gamma(CF_{p,p+1},N) \) coincides with the Harish-Chandra module of \( Y \subset X \)).

Let us denote by \( V \subset CF_{p,p+1} \) the (open) subvariety \( V := \{ (E,F) \in CF_{p,p+1} \text{ s.t. } B|_F \text{ is non-degenerate} \} \).

Then \( U \subset V \). Let \( j' : U \hookrightarrow V \) and \( j'' : V \hookrightarrow CF_{p,p+1} \) denote the identity imbedding morphisms. Set

\[ N := j''(j'_* M_0) \]

where \( j'_* \) denotes the minimal (Goresky-Macpherson) extension of \( M \) under the open imbedding \( j' \). It is easy to see that the morphism \( j'' : V \to CF_{p,p+1} \) is affine. Hence the functor \( j''_* \) is exact and we have

\[ N \subset j''(j'_* M_0) = j_* M_0 = M. \]
As it was shown in [3], $\Gamma(C_{F,p,p+1}, N)$ coincides with the Harish-Chandra module of $\mathcal{Y} \subset \mathcal{X}$. Also let us define

$$K := j_* M_0.$$ 

Thus $K \subset N$.

4.1.4 Remark. Since $Val^- \supset \mathcal{Y}$, it corresponds to a sub-$D_\chi$-module of $M/N$. Note also that $\text{supp}(M/N) \neq C_{F,p,p+1}$.

4.1.5 Lemma. No irreducible subquotient of $\Gamma(N/K) = \Gamma(N)/\Gamma(K)$ has associated variety contained in the variety of complex symmetric nilpotent matrices of rank at most 1.

**Proof.** Let $L \to C_{F,p,p+1}$ be the (algebraic) line bundle whose fiber over $(E, F) \in C_{F,p,p+1}$ is equal to $\wedge^p E^*$. Then the sheaf $D_\chi$ is the sheaf of differential operators with values in $L$. Tensoring by $L^*$ establishes an equivalence of the categories $D_\chi(C_{F,p,p+1}) - mod$ and $D(C_{F,p,p+1}) - mod$ where $D(X)$ denotes the sheaf of rings of usual (untwisted) differential operators on a variety $X$. Since the line bundle $L$ is $G$-equivariant, the analogous equivalence holds for $K$-equivariant versions of these categories. This equivalence preserves singular supports of the corresponding modules.

Let $R$ be a $D_\chi$-module. Let $R' := L^* \otimes R$ be the corresponding $D$-module. The associated variety of $\Gamma(R, C_{F,p,p+1})$ (resp. $\Gamma(R', C_{F,p,p+1})$) is equal to the image under the moment map of the singular support of $R$ (resp. $R'$) (see [3] where the discussion follows [18]). Hence it follows that $\Gamma(R, C_{F,p,p+1})$ and $\Gamma(R', C_{F,p,p+1})$ have the same associated variety.

Set $M'_0 := L^* \otimes M_0$. Then $M'_0$ is a $K$-equivariant $D$-module on the open orbit $U \subset C_{F,p,p+1}$ corresponding to the same as $M_0$ representation of the group of connected components of the group $S \simeq O(p, \mathbb{C}) \times O(1, \mathbb{C}) \times O(n-p-1, \mathbb{C})$, i.e. to $(A, B, C) \mapsto \det A \cdot \det B$.

Then

$$(M' := L^* \otimes M = j_* M'_0, \quad N' := L^* \otimes N = j''_*(j'_* M'_0))$$

where now all the functors $j_*, j'_*, j''_*$ are in the category of $D$-modules rather than $D_\chi$-modules.

Let $f : C_{F,p,p+1} \to C_{Gr}_{p+1}$ be the canonical projection $(E, F) \mapsto F$. Let us denote by $O$ the open $K$-orbit in $C_{Gr}_{p+1}$. Explicitly

$$O = \{ F \in C_{Gr}_{p+1} | B|_F \text{ is non-degenerate} \}.$$ 

It is clear that $V = f^{-1}(O)$.

The stabilizer $S' \subset K$ of a point from $O$ is isomorphic to the group $O(p + 1, \mathbb{C}) \times O(n - p - 1, \mathbb{C})$. Let us denote by $A$ the $D$-module on $O$ corresponding to the representation of the group of connected components of $S' \simeq O(p + 1, \mathbb{C}) \times O(n - p - 1, \mathbb{C})$ given by $(M, N) \mapsto \det M$.

4.1.6 Claim. The $K$-equivariant $D$-modules $f^* A$ and $j'_* M'_0$ on $V = f^{-1}(O)$ are isomorphic.

**Proof.** It is clear from the definitions of $A$ and $M'_0$ that the restriction of $f^* A$ to $U$ is isomorphic to $M'_0$. Hence we have a morphism of $K$-equivariant $D$-modules $f^* A \to j'_* M'_0$.
which is an isomorphism over $U$. It is clear that $f^* \mathcal{A}$ is an irreducible $\mathcal{D}$-module, hence its image is equal to $j_{!*} \mathcal{M}'_0$. Q.E.D.

Let $l: \mathcal{O} \hookrightarrow ^{\mathbb{C}} \mathcal{G}r_{p+1}$ denote the identity imbedding morphism. Since the morphism $f$ is smooth (in particular flat) by the flat base change theorem (see e.g. [25], Proposition 9.3) we have

$$\mathcal{N}' := \mathcal{L} \otimes \mathcal{N} = j''_*(j_! \mathcal{M}'_0) \simeq j''_*(f^* \mathcal{A}) = f^*(l_* \mathcal{A}).$$

(4.1.8)

Set $\mathcal{K} := \mathcal{L} \otimes \mathcal{K} = j_! \mathcal{M}'_0$. Since $j_{!*} = j''_* \circ j_!$, we have

$$\mathcal{K}' \simeq j''_*(f^* \mathcal{A}) = f^*(l_* \mathcal{A})$$

(4.1.9)

where the last equality also follows from the smooth base change theorem for $\mathcal{D}$-modules [?].

Since the morphism $f$ is projective and smooth, one has $\Gamma(C \mathcal{F}_{p,p+1}, f^*(\bullet)) = \Gamma(C \mathcal{G}r_{p+1}, \bullet)$. Then $\Gamma(C \mathcal{F}_{p,p+1}, \mathcal{N}' / \mathcal{K}') = \Gamma(C \mathcal{F}_{p,p+1}, f^*(l_* \mathcal{A} / l_* \mathcal{A})) = \Gamma(C \mathcal{G}r_{p+1}, l_* \mathcal{A} / l_* \mathcal{A})$. Thus to finish the proof of Lemma 4.1.5 it remains to show that the associated variety of any irreducible $(\mathfrak{g}, K)$-subquotient of $\Gamma(C \mathcal{G}r_{p+1}, l_* \mathcal{A} / l_* \mathcal{A})$ is not contained in the variety of complex symmetric nilpotent matrices of rank at most 1. But this statement was proved in fact in Theorem 4.3 in [21]. Q.E.D.

To finish the proof of Proposition 4.1.1 it remains to show that the Harish-Chandra module of $(Val_{p}^{< \infty}(V))^*$ cannot be isomorphic to the $(\mathfrak{g}, K)$-module $\Gamma(C \mathcal{F}, j_{!*} \mathcal{M}_0)$.

Since $\tilde{q}: C \mathcal{F} \rightarrow C \mathcal{F}_{p,p+1}$ is a smooth projective morphism we have

$$\Gamma(C \mathcal{F}, \tilde{q}^*(\bullet)) = \Gamma(C \mathcal{F}_{p,p+1}, \bullet).$$

(4.1.10)

Clearly $\tilde{q}^*(j_{!*} \mathcal{M}_0)$ is an irreducible $\mathcal{D}_\chi$-module with support

$$\text{supp}(\tilde{q}^*(j_{!*} \mathcal{M}_0)) = C \mathcal{F}.$$  

(4.1.11)

It is clear from (4.1.10) and from the fact that $\chi$ is dominant and regular, that $\tilde{q}^*(j_{!*} \mathcal{M}_0)$ is the Beilinson-Bernstein localization of $\Gamma(C \mathcal{F}_{p,p+1}, \bullet)$. Thus it suffices to prove the following claim.

4.1.7 Claim. The support on $C \mathcal{F}$ of the Beilinson-Bernstein localization of the Harish-Chandra module of $(Val_{p}^{< \infty}(V))^*$ is not equal to $C \mathcal{F}$.

Proof. Remind that $C \mathcal{F}$ denotes the variety of complete flags in $V^* \otimes_{\mathbb{R}} \mathbb{C}$. By taking the orthogonal complement we can identify $C \mathcal{F}$ with the variety $C \mathcal{F}'$ of complete flags in $V \otimes_{\mathbb{R}} \mathbb{C}$. Also the group $G_0 = GL(V^*)$ can be identified with the group $G'_0 = GL(V)$ via the isomorphism $g \mapsto (g^*)^{-1}$. Let $G'$ denote the complexification of $G'_0$. Now we will work with $G'_0, G'$ instead of $G_0, G$, and we will consider the Beilinson-Bernstein localization on $C \mathcal{F}'$. Let us denote similarly $\mathfrak{g}' := Lie(G')$, $K' \subset G'$ is the subgroup preserving the non-degenerate form on $V \otimes_{\mathbb{R}} \mathbb{C}$. Let us denote

$$C \mathcal{F}'_{k,k+1} := \{(E, F) | E \subset F, E \in C \mathcal{G}r_k(V \otimes_{\mathbb{R}} \mathbb{C}), F \in C \mathcal{G}r_{k+1}(V \otimes_{\mathbb{R}} \mathbb{C})\}.$$  

Remind that we have the Poincaré duality isomorphism $(Val_{p}^{< \infty}(V))^* \otimes Dens(V^*)$. Thus it suffices to prove that the support in $C \mathcal{F}'$ of the Beilinson-Bernstein localization of $Val_{p-n}^{< \infty}(V)$ is not equal to $C \mathcal{F}'$.  

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By Remark 4.1.4 applied to \( V \) instead of \( V^* \) and \( n-p \) instead of \( p \), the Harish-Chandra module of \( Val_{n-p}^{-\text{sm}}(V) \) is isomorphic to the \((g',K')\)-module of global section of certain \( K'\)-equivariant \( \mathcal{D}_{\bar{\chi}} \)-module \( \tau \) on \( \mathcal{F}'_{n-p,n-p+1} \) whose support is not equal to \( C\mathcal{F}'_{n-p,n-p+1} \). Here we denote by \( \bar{\chi} \) an appropriate character of a stabilizer of a Lie algebra \( Q \) of a point from \( C\mathcal{F}'_{n-p,n-p+1} \). Note that the lifting \( \bar{\chi} \) to the Cartan algebra \( t' \subset g' \) is dominant and regular.

Let \( g: C\mathcal{F}' \to C\mathcal{F}'_{n-p,n-p+1} \) denote the canonical projection. Then we have the functor
\[
\Gamma(C\mathcal{F}', g^*(\bullet)) = \Gamma(C\mathcal{F}'_{n-p,n-p+1}, \bullet).
\]

Hence the Beilinson-Bernstein localization of the Harish-Chandra module of \( Val_{n-p}^{-\text{sm}}(V) \) is isomorphic to \( g^*\tau \). Since \( \text{supp} \, \tau \neq C\mathcal{F}'_{n-p,n-p+1} \) then \( \text{supp} \, g^*\tau \neq C\mathcal{F}' \). Thus Claim 4.1.7 is proved. Q.E.D.

Hence Proposition 4.1.4 is proved as well. Q.E.D.

4.1.8 Remark. Note that in Step 4 of the proof of Proposition 4.1.1 we have actually proven the following result: for any \( n \)-dimensional vector space \( W \) and any \( p = 1, \ldots, n-1 \) the Jordan-Hölder series of the \( GL(W) \)-module \( C^\infty(\mathcal{F}_{p,p+1}(W), \det \mathcal{M}_{p+p}^*(W) \otimes \text{or}(\mathcal{N}_{p,p+1}(W))) \) contains \( Val_{p}^{-\text{sm}}(W) \) with multiplicity one. This fact will be used below in the proof of Proposition 6.1.8.

5 The two dimensional case.

The goal of this section is to construct the Fourier transform on two-dimensional spaces and to establish its homomorphism property. Let \( V \) be a two dimensional real vector space.

5.1 A canonical isomorphism in two dimensions.

First we are going to construct a canonical isomorphism
\[
Val_1^{-\text{sm}}(V) \to Val_1^{-\text{sm}}(V^*) \otimes \text{det} V^*.
\]

Let \( \mathcal{T}(V) \to \mathbb{P}(V) \) be the tautological vector bundle, i.e. the fiber of \( \mathcal{T}(V) \) over \( E \in \mathbb{P}(V) \) is equal to \( E \). Let \( V \to \mathbb{P}(V) \) be the trivial bundle, i.e. \( V = \mathbb{P}(V) \times V \). Then we have a canonical imbedding of vector bundles \( \mathcal{T}(V) \hookrightarrow V \).

We have the canonical epimorphism
\[
C^\infty(\mathbb{P}(V), \text{Dens} \mathcal{T}(V) \otimes \text{or}(V/\mathcal{T}(V))) \twoheadrightarrow Val_1^{-\text{sm}}(V).
\]

Recall that the kernel of this map is two dimensional irreducible \( GL(V) \)-module. Observe that
\[
\text{Dens} \mathcal{T}(V) = \mathcal{T}(V)^* \otimes \text{or}(\mathcal{T}(V)),
\]
\[
\text{or}(V/\mathcal{T}(V)) = \text{or}(V) \otimes \text{or}(\mathcal{T}(V)).
\]
Using these identifications we get a canonical epimorphism

\[ C^\infty(\mathbb{P}(V), \det \mathcal{T}(V)^*) \otimes or(V) \to Val_1^{-,sm}(V). \]  

(5.1.3)

By taking the orthogonal complement, we have a natural identification \( \mathbb{P}(V) = \mathbb{P}(V^*) \). Let \( E \subset V \) be a line. Then \( E = (V^*/E^\perp)^* = (\det V^* \otimes (E^\perp)^*)^* \). Hence

\[ E = E^\perp \otimes \det V. \]  

(5.1.4)

Similarly

\[ orE = orE^\perp \otimes orV. \]  

(5.1.5)

Hence

\[ C^\infty(\mathbb{P}(V), \mathcal{T}(V^*)^*) = C^\infty(\mathbb{P}(V^*), \det \mathcal{T}(V^*)^* \otimes V^*) = C^\infty(\mathbb{P}(V^*), \mathcal{T}(V^*)^*) \otimes \det V^*. \]

Hence (5.1.3) can be rewritten as

\[ C^\infty(\mathbb{P}(V^*), \mathcal{T}(V^*)^*) \otimes Dens(V) \to Val_1^{-,sm}(V). \]  

(5.1.6)

Replacing \( V \) by \( V^* \) in (5.1.3) we get an epimorphism

\[ C^\infty(\mathbb{P}(V^*), \mathcal{T}(V^*)^*) \otimes orV^* \to Val_1^{-,sm}(V^*). \]  

(5.1.7)

Tensoring (5.1.7) by \( \det V^* \) and observing that \( orV = orV^* \) we get

\[ C^\infty(\mathbb{P}(V^*), \mathcal{T}(V^*)^*) \otimes Dens(V) \to Val_1^{-,sm}(V^*) \otimes \det V^*. \]  

(5.1.8)

Comparing (5.1.6) and (5.1.8) we conclude that there exists a unique isomorphism of \( GL(V) \)-modules

\[ \widetilde{F}_V : Val_1^{-,sm}(V) \to Val_1^{-,sm}(V^*) \otimes \det V^* = Val_1^{-,sm}(V^*) \otimes Dens(V) \otimes orV \]  

(5.1.9)

which makes the following diagram commutative:

\[ \begin{array}{ccc}
C^\infty(\mathbb{P}(V^*), \mathcal{T}(V^*)^*) & \otimes Dens(V) \\
\downarrow & & \downarrow \\
Val_1^{-,sm}(V) & \Rightarrow & Val_1^{-,sm}(V^*) \otimes \det V^*
\end{array} \]

Remind also that in the even case we have an isomorphism

\[ F_V : Val_1^{+,sm}(V) \to Val_1^{+,sm}(V^*) \otimes Dens(V). \]  

(5.1.10)

Combining (5.1.9) and (5.1.10) together we obtain a canonical isomorphism

\[ \overline{F}_V : Val^{sm}(V) \Rightarrow (Val^{+,sm}(V^*) \oplus (Val_1^{-,sm}(V^*) \otimes orV)) \otimes Dens(V). \]  

(5.1.11)
5.2 Construction of the convolution product on $\text{Val}(V^*) \otimes \text{Dens}(V)$, $\dim V = 2$.

Let us denote

$$
\mathcal{A}_0 := \text{Val}_2(V^*) \otimes \text{Dens}(V),
\mathcal{A}_2 := \text{Dens}(V),
\mathcal{A}^+_1 := \text{Val}^{+,sm}(V^*) \otimes \text{Dens}(V),
\mathcal{A}^-_1 := \text{Val}^{-,sm}(V^*) \otimes \text{Dens}(V) \otimes \text{or} V,
\mathcal{A} := (\text{Val}^{+,sm}(V^*) \oplus (\text{Val}^{-,sm}(V^*) \otimes \text{or} V)) \otimes \text{Dens}(V) = \mathcal{A}_0 \oplus \mathcal{A}_2 \oplus \mathcal{A}^+_1 \oplus \mathcal{A}^-_1.
$$

Using the isomorphism $\mathbb{F}_V$ we can define the product $\ast$ on $\mathcal{A}$ by

$$
\phi \ast \psi := \mathbb{F}_V(\mathbb{F}_V^{-1} \phi \cdot \mathbb{F}_V^{-1} \psi).
$$

Then $\mathcal{A}$ becomes a commutative graded algebra such that the graded components are $\mathcal{A}_0, \mathcal{A}^+_1 \oplus \mathcal{A}^-_1, \mathcal{A}_2$. Note that

$$
\mathcal{A}^-_1 \ast \mathcal{A}^+_1 = \mathcal{A}^-_1 \ast \mathcal{A}_2 = 0, (5.2.1)
$$

$$
\mathcal{A}^+ \ast \mathcal{A}^-_1 \subset \mathcal{A}^-_1. (5.2.2)
$$

Let us denote

$$
\mathcal{A}^+ := \mathcal{A}_0 \oplus \mathcal{A}^+_1 \oplus \mathcal{A}_2.
$$

Let us define now an algebra structure on $\mathcal{B} := \text{Val}^{sm}(V^*) \otimes \text{Dens}(V)$. First write

$$
\mathcal{B} = \mathcal{A}_0 \oplus (\mathcal{A}^+_1 \oplus (\mathcal{A}^-_1 \otimes \text{or}(V))) \oplus \mathcal{A}_2. (5.2.3)
$$

Thus $\mathcal{B} = \mathcal{A}^+ \oplus (\mathcal{A}^-_1 \otimes \text{or}(V))$. Let us define

$$
\ast : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}
$$

as follows. First we have (using the equality $\text{or} V \otimes \text{or} V = \mathbb{C}$)

$$
\mathcal{B} \otimes \mathcal{B} = (\mathcal{A}^+ \otimes \mathcal{A}^+) \oplus (\mathcal{A}^+ \otimes \mathcal{A}^-_1 \otimes \text{or} V) \oplus (\mathcal{A}^-_1 \otimes \mathcal{A}^+ \otimes \text{or} V) \oplus (\mathcal{A}^-_1 \otimes \mathcal{A}^-_1).
$$

We define $\ast$ to be equal to $\ast$ on the first summand, to $\ast \otimes \text{id}_{\text{or} V}$ on the second and the third summands, and to $-\ast$ on the fourth summand (with the minus sign!).

It is easy to see that $(\mathcal{B}, \ast)$ is a commutative associative graded algebra with unit (since $(\mathcal{A}, \ast)$ is). The decomposition of $\mathcal{B}$ into the graded components is given by (5.2.3).

5.3 Explicit computation of the convolution.

In this subsection we show that in the two dimensional case the convolution $\ast$ defined in Section 5.2 coincides with the convolution introduced by Bernig and Fu [15].

Let us fix a Lebesgue measure $\text{vol}$ on $V^*$. It gives an isomorphism $\text{Dens}(V) \rightarrow \mathbb{C}$. For $A \in \mathcal{K}^{sm}(V^*)$ let us denote $\mu_A(K) = \text{vol}(K + A) \otimes \text{vol}^{-1}$ for any $K \in \mathcal{K}(V^*)$. It is easy to see that $\mu_A \in \text{Val}^{sm}(V^*) \otimes \text{Dens}(V)$. 45
5.3.1 Proposition. Let $V$ be a two dimensional real vector space. For the convolution product $*$ on $\text{Val}^{sm}(V^*) \otimes \text{Dens}(V)$ defined in Section 5.2 one has
\[ \mu_A * \mu_B = \mu_{A+B} \] (5.3.1)
for any $A, B \in \mathcal{K}^{sm}(V^*)$.

This proposition implies that our convolution $*$ coincides with the Bernig-Fu convolution, and hence there is no abuse of notation. Before we prove this proposition we will prove another proposition.

5.3.2 Proposition. Let us fix a Euclidean metric on $V$ and an orientation. Consider the isomorphisms $V \xrightarrow{\sim} V^*$, $\text{Dens}(V) \xrightarrow{\sim} \mathbb{C}$, or $V \xrightarrow{\sim} \mathbb{C}$ induced by these choices. With these identifications consider $\mathbb{F}_V: \text{Val}^{sm}(V) \xrightarrow{\sim} \text{Val}^{sm}(V)$. Then for any $A \in \mathcal{K}^{sm}(V)$
\[ \mathbb{F}_V(V(\bullet, A)) = V(\bullet, J^{-1}A) \] (5.3.2)
where $J: V \rightarrow V$ is the rotation by $\pi$ counterclockwise.

Proof. First let us remind that for any smooth functions $h_1, h_2$ on the unit circle one can define the mixed volume $V(h_1, h_2)$ (see e.g. [21]) which is bilinear with respect to $h_1, h_2$. The idea is as follows. First if $h_1, h_2$ are supporting functionals of convex sets $A_1, A_2$ respectively, let us define $V(h_1, h_2)$ to be equal to $V(A_1, A_2)$. Since every smooth function on the circle is a difference of supporting functions of convex bodies, let us extend this expression by bilinearity. We get a well defined notion. Thus we may and will identify convex set $A$ with its supporting function. We may assume that it is smooth on $S^1 = \mathbb{P}(V)$.

Case 1. Let us assume first that $A$ is an even function. Then the value at $l \in \mathbb{P}(V)$ of the Klain imbedding of $V(\bullet, A)$ to $C^{+,\infty}(\mathbb{P}(V))$ is equal to $\kappa A(l^\perp)$ where $\kappa$ is a normalizing constant. By the definition of $\mathbb{F}_V$ on even valuations, the value at $l \in \mathbb{P}(V)$ of the Klain imbedding of $\mathbb{F}_V(V(\bullet, A))$ is equal to $\kappa A(l^\perp) = \kappa(J^{-1}A)(l)$. Thus the proposition is proved in the even case.

Case 2. Let us assume now that $A$ is an odd function. Recall that for $K \in \mathcal{K}(V)$
\[ V(K, A) = \frac{1}{2} \int_{l \in \mathbb{P}^+(V)} A(l) dS_1(K, l) \]
where $S_1(K, \cdot)$ is the first area measure of $K$ (see e.g. [39], formula (5.1.18)). By the construction of $\mathbb{F}_V$ on odd valuations
\[ \mathbb{F}_V(V(\bullet, A))(K) = \frac{1}{2} \int_{l \in \mathbb{P}^+(V)} A(Jl) dS_1(K, l) = \frac{1}{2} \int_{l \in \mathbb{P}^+(V)} (J^{-1}A)(l) dS_1(K, l). \]
Thus the proposition is proved. Q.E.D.

Proof of Proposition 5.3.1. Let us fix a Euclidean metric and an orientation on $V$. It is easy to check (see e.g. [15], Corollary 1.3) that the equality (5.3.1) for all $A, B \in \mathcal{K}^{sm}(V^*)$ is equivalent to the following two conditions:
(a) $\text{vol}$ is the unit element with respect to $*$, i.e. $\text{vol} * x = x$ for any $x$;
(b) $V(\bullet, A) * V(\bullet, B) = \frac{1}{2} V(A, B) \chi(\bullet)$ for any $A, B \in \mathcal{K}^{sm}(V^*)$. 

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The property (a) holds due to the corresponding property of the product $\star$ on $\mathcal{A}$. Let us prove (b). We will prove it in a more general form when $A$ and $B$ are smooth functions on the unit circle. Since by (5.2.1) $\mathcal{A}_1^+ \star \mathcal{A}_1^- = 0$, the proof splits into two cases:

(a) both $A$ and $B$ are even;
(b) both $A$ and $B$ are odd.

Though the even case was considered in [15] (for all dimensions), we will prove it here in two dimensions for the sake of completeness. Thus let us first consider the case (a). Since $A,B$ are even, we have

$$V(\bullet, A) \cdot V(\bullet, B) = \mathbb{F}_V(\mathbb{F}_V^{-1}(V(\bullet, A)) \cdot \mathbb{F}_V^{-1}(V(\bullet, B))) =$$

$$\mathbb{F}_V(V(\bullet, JA) \cdot V(\bullet, JB)) \quad \text{Example 2.5.2} = \mathbb{F}_V\left(\frac{1}{2}V(JA, -JB) \cdot \text{vol}(\bullet)\right) =$$

$$\frac{1}{2}V(A, -B) \cdot \chi(\bullet) = \frac{1}{2}V(A, B) \cdot \chi(\bullet).$$

Now let us consider the case (b), i.e. $A,B$ are odd. Then $V(\bullet, A), V(\bullet, B) \in \mathcal{A}_1^\times \otimes \nu V$. Hence we have by the definition of $\star$

$$V(\bullet, A) \cdot V(\bullet, B) = -\mathbb{F}_V(\mathbb{F}_V^{-1}(V(\bullet, A)) \cdot \mathbb{F}_V^{-1}(V(\bullet, B))) .$$

Similarly to the previous case the last expression is equal to

$$-\frac{1}{2}V(A, -B)\chi(\bullet) = \frac{1}{2}V(A, B)\chi(\bullet).$$

Proposition is proved. Q.E.D.

5.4 **Isomorphisms of algebras** $Val^{sm}(V)$ and $Val^{sm}(V^*) \otimes Dens(V)$.

We are going to prove the following result.

**5.4.1 Theorem.** Let $V$ be a two dimensional real vector space. There exists an isomorphism of topological vector spaces

$$\mathbb{F}_V : Val^{sm}(V) \rightarrow Val^{sm}(V^*) \otimes Dens(V)$$

which satisfies the following properties:

(1) $\mathbb{F}_V$ commutes with the natural $GL(V)$-action;
(2) $\mathbb{F}_V$ is an isomorphism of algebras, i.e. $\mathbb{F}_V(\phi \cdot \psi) = \mathbb{F}_V(\phi) \cdot \mathbb{F}_V(\psi)$ for any $\phi, \psi \in Val^{sm}(V)$;
(3) This isomorphism takes real valued valuations to real valued.

**Proof.** The isomorphism on even valuations $\mathbb{F}_V : Val^{+, sm}(V) ightarrow Val^{+, sm}(V^*) \otimes Dens(V)$ was defined by the author in [4]. Its homomorphism property was proved by Bernig and Fu in [15]. We will extend this map to odd valuations. The construction will depend on a choice of orientation of $V$. Thus let us fix an orientation of $V$.

Consider the action of the subgroup $SL(V) \subset GL(V)$ on $Val^{-, sm}(V^*) \otimes Dens(V)$. Then

$$Val^{-, sm}(V^*) \otimes Dens(V) = W_1 \oplus W_2$$

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where \( W_1, W_2 \subset Val^{-\text{sm}}_1(V^*) \otimes \text{Dens}(V) \) are closed \( SL(V) \)-invariant \( SL(V) \)-irreducible infinite dimensional subspaces. \( W_1 \) and \( W_2 \) are not isomorphic to each other as \( SL(V) \)-modules. Moreover one can choose a Cartan subalgebra and a root system of the Lie algebra of \( sl(V) \) (compatible in an appropriate way with the orientation of \( V \)) so that \( W_1 \) is a highest weight module, and \( W_2 \) is a lowest weight module. The property of being either highest or lowest module depends only on the orientation of \( V \), and not on a Cartan subalgebra and a positive root system.

5.4.2 Lemma. The map

\[
\mathbb{L} := \text{Id}_{W_1} \oplus (-\text{Id}_{W_2}) : Val^{-\text{sm}}_1(V^*) \otimes \text{Dens}(V) \rightarrow Val^{-\text{sm}}_1(V^*) \otimes \text{Dens}(V) \otimes \text{or}(V)
\]

is an isomorphism of \( GL(V) \)-modules.

Here is a warning: the notation of the lemma is a bit misleading. The target space seems to be different form the source space since it is twisted by \( \text{or}(V) \). The meaning is that we just consider the target space to be equal to the source space as a vector space, but the action of \( GL(V) \) on the target is different: it is twisted by the sign of the determinant of a matrix.

Let us postpone the proof of this lemma and finish the proof of the theorem. Let us define \( \mathbb{F}_V : Val^{-\text{sm}}_1(V) \rightarrow Val^{-\text{sm}}_1(V^*) \otimes \text{Dens}(V) \) by

\[
\mathbb{F}_V := \mathbb{L}^{-1} \circ \bar{\mathbb{F}}_V \text{ on } Val^{-\text{sm}}_1(V)
\]

where \( \bar{\mathbb{F}}_V \) was defined in (5.1.11).

In order to prove that \( \mathbb{F}_V \) is an isomorphism of algebras it remains to show that for any \( \phi, \psi \in Val^{-\text{sm}}_1(V) \) one has

\[
\mathbb{F}_V(\phi \cdot \psi) = \mathbb{F}_V(\phi) \cdot \mathbb{F}_V(\psi).
\]

In the notation of Section 5.2 one has

\[
\mathbb{F}_V(\phi \cdot \psi) = \bar{\mathbb{F}}_V \phi \cdot \mathbb{F}_V(\psi);
\]

\[
\mathbb{F}_V(\phi) \cdot \mathbb{F}_V(\psi) = -\mathbb{F}_V(\phi) \cdot \mathbb{F}_V(\psi) = -\mathbb{L}^{-1}(\mathbb{F}_V(\phi)) \cdot \mathbb{L}^{-1}(\mathbb{F}_V(\psi)).
\]

Thus we have to show that for any \( u, v \in Val^{-\text{sm}}_1(V^*) \otimes \text{Dens}(V) \) one has

\[
u \ast v = -\mathbb{L}^{-1}(u) \ast \mathbb{L}^{-1}(v). \tag{5.4.1}
\]

It is clear that if \( u \in W_1, v \in W_2 \) then (5.4.1) holds. For \( u, v \in W_i, i = 1, 2 \), the equality (5.4.1) is equivalent to \( u \ast v = 0 \). Let us prove it for \( i = 1 \); the case \( i = 2 \) is considered similarly. Let us observe that \( \ast \) induces the \( SL(V) \)-equivariant map \( W_1 \rightarrow W_1^* \otimes \text{Dens}(V) \). Since \( W_1 \) is a highest weight irreducible infinite dimensional \( SL(V) \)-module, \( W_1^* \otimes \text{Dens}(V) \) is a lowest weight irreducible infinite dimensional \( SL(V) \)-module. Hence they cannot be isomorphic. This proves that \( \mathbb{F}_V \) is a homomorphism of algebras.

It it clear from the construction of \( \mathbb{F}_V \) that \( \mathbb{F}_V \) maps real valued valuations to real valued. Hence theorem is proved modulo Lemma 5.4.2. Q.E.D.

Proof of Lemma 5.4.2. It follows from general representation theory of the group \( GL_2(\mathbb{R}) \) (see e.g. [26], Ch. I, Theorem 5.11(VI)) that the irreducible representations
Val\_1^{-}\text{sm}(V^*) \otimes Dens(V) and Val\_1^{-}\text{sm}(V^*) \otimes Dens(V) \otimes or(V) of the group \(GL(V) \simeq GL_2(\mathbb{R})\) are isomorphic. Let us fix such an isomorphism \(\mathbb{L}'\). Since \(W_1, W_2\) are irreducible non-isomorphic \(SL(V)\)-modules, \(\mathbb{L}'\) must have the form
\[
\mathbb{L}' = \alpha \text{Id}_{W_1} \oplus \beta \text{Id}_{W_2}
\]
where \(\alpha, \beta \in \mathbb{C}^*\). Dividing by \(\alpha\) we may assume that \(\alpha = 1\). The composition
\[
(\mathbb{L}' \otimes \text{Id}_{or(V)}) \circ \mathbb{L}' : Val\_1^{-}\text{sm}(V) \otimes Dens(V) \to Val\_1^{-}\text{sm}(V) \otimes Dens(V)
\]
is an automorphism of \(GL(V)\)-module, hence it must have the form \(\gamma(\text{Id}_{W_1} \oplus \beta^2 \text{Id}_{W_2})\). On the other hand this composition is equal to \(\text{Id}_{W_1} \oplus \beta \text{Id}_{W_2}\). Hence \(\beta^2 = 1\), i.e. \(\beta = \pm 1\). Since the identity map \(\text{Id}_{W_1 \oplus W_2}\) is not a morphism of \(GL(V)\)-modules \(Val\_1^{-}\text{sm}(V^*) \otimes Dens(V)\) and \(Val\_1^{-}\text{sm}(V^*) \otimes Dens(V) \otimes or(V)\), it implies that \(\beta = -1\). Lemma is proved. Q.E.D.

### 5.5 Plancherel type formula in two dimensions.

Let us consider the map
\[
\mathbb{F}_V : Val\_1^{-}\text{sm}(V^*) \otimes Dens(V) \to Val\_1^{-}\text{sm}(V).
\]
Let us denote by \(\mathcal{E}_V : Val\_1^{-}\text{sm}(V) \to Val\_1^{-}\text{sm}(V)\) the operator given by \((\mathcal{E}_V \phi)(K) = \phi(-K)\) for any \(\phi \in Val\_1^{-}\text{sm}(V), K \in K(V)\).

#### 5.5.1 Proposition. Let \(V\) be a two dimensional vector space. Then
\[
(\mathbb{F}_V \otimes \text{Id}_{\text{Dens}(V)}) \circ \mathbb{F}_V = \mathcal{E}_V.
\]

**Proof.** Obviously, on 0- and 2-homogeneous valuations both operators are obviously equal to identity. For 1-homogeneous valuations the result follows from Proposition 5.3.2. Q.E.D.

### 6 Fourier transform on valuations in higher dimensions.

The goal of this section is to construct the Fourier transform in higher dimensions and to prove its main properties.

#### 6.1 Construction of the Fourier transform.

Let \(0 \leq k \leq n\). Let us consider the (infinite dimensional) vector bundle
\[
\mathcal{T}_{k,V}^0 \to Gr_{n-k}(V)
\]
whose fiber over \( F \in Gr_{n-k}(V) \) is equal to \( Val^m(V/F) \). Similarly let \( T_{k,V;i}^0 \rightarrow Gr_{n-k}(V) \), \( T_{k,V;i}^{0,\pm} \rightarrow Gr_{n-k}(V) \) be the vector bundles whose fiber over \( F \in Gr_{n-k}(V) \) is equal to \( Val_i^m(V/F) \) or \( Val_i^{\pm,m}(V/F) \) respectively. Let

\[
T_{k,V} := T_{k,V;i}^0 \otimes |\omega_{Gr_{n-k}(V)}|, \quad (6.1.1)
\]

\[
T_{k,V;i} := T_{k,V;i}^0 \otimes |\omega_{Gr_{n-k}(V)}|, \quad (6.1.2)
\]

\[
T_{k,V;i}^{0,\pm} := T_{k,V;i}^{0,\pm} \otimes |\omega_{Gr_{n-k}(V)}|. \quad (6.1.3)
\]

Also for a subspace \( F \subset V \) let us denote by \( p_F \) the canonical map

\[
p_F : V \rightarrow V/F.
\]

Consider the natural map

\[
\Xi_{k,V} : C^\infty(Gr_{n-k}(V), T_{k,V;i}) \rightarrow Val_i^m(V)
\]

defined by \( \xi \mapsto \int_{F \in Gr_{n-k}(V)} p_F^*(\xi(F)) \). Note that the map \( \Xi_{k,V} \) is \( GL(V) \)-equivariant, hence its image is indeed contained in \( Val_i^m(V) \).

**6.1.1 Lemma.** Let \( f_i : V_i \rightarrow W_i, \ i = 1, 2 \) be two linear maps such that \( f_2 \) is injective. Then for any \( \phi_1 \in Val(W_1), \phi_2 \in Val^m(W_2) \) the pushforward \( f_2^*(\phi_2) \in Val^m(V_2) \) and

\[
(f_1 \boxtimes f_2)^*(\phi_1 \boxtimes \phi_2) = f_1^*\phi_1 \boxtimes f_2^*\phi_2. \quad (6.1.4)
\]

**6.1.2 Remark.** In the statement of the proposition we use the construction of the product of a continuous valuation and a smooth one from the appendix.

**Proof** of Lemma [6.1.1] It is clear that \( f_2^*(\phi_2) \in Val^m(V_2) \). Next both sides of (6.1.4) are continuous with respect to \( \phi_1 \in Val(W_1), \phi_2 \in Val^m(W_2) \). Hence by the McMullen’s conjecture we may assume that

\[
\phi_i(\bullet) = vol_i(\bullet + A_i), \ i = 1, 2,
\]

where \( vol_i \) is a Lebesgue measure on \( W_i \), \( A_i \in K^{sm}(W_i) \). Then one has

\[
(\phi_1 \boxtimes \phi_2)(K) = (vol_1 \boxtimes vol_2)(K + (A_1 \times A_2)), \quad (6.1.5)
\]

\[
(f_1 \boxtimes f_2)^*(\phi_1 \boxtimes \phi_2)(K) = (vol_1 \boxtimes vol_2)((f_1 \boxtimes f_2)(K) + (A_1 \times A_2)). \quad (6.1.6)
\]

On the other hand

\[
(f_2^*\phi_1)(K) = vol_i(f_1(K) + A_i), \ i = 1, 2.
\]

Note that \( (f_1 \boxtimes f_2)^* = (f_1 \boxtimes Id)^* \circ (Id \boxtimes f_2)^* \). Let us compute first \( (Id \boxtimes f_2)^*(\phi_1 \boxtimes \phi_2) \). First we will identify \( V_2 \) with its image in \( W_2 \). Let us fix Lebesgue measures \( vol_{W_2} \) on \( W_2 \) and \( vol_{W_2/V_2} \) on \( W_{2/V_2} \) such that \( vol_2 = vol_{W_2} \otimes vol_{W_2/V_2} \). We have

\[
(f_2^*\phi_2)(\bullet) = \int_{z \in W_2/V_2} vol_{W_2}(\bullet + (A_2 \cap z)) dvol_{W_2/V_2}(z). \quad (6.1.7)
\]

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Next for any $K \in \mathcal{K}(V_1 \times V_2)$ we have

$$(Id \times f_2)^*(\phi_1 \boxtimes \phi_2)(K) = (6.1.8)$$

$$(Id \times f_2)^*(\phi_1 \boxtimes \phi_2)(K) = (6.1.8)$$

$$\int_{z \in W_2/V_2} (vol_1 \boxtimes vol_2) (K + (A_1 \times A_2)) dvol_{W_2/V_2}(z) = (6.1.9)$$

$$\int_{z \in W_2/V_2} (vol_1 \boxtimes vol_2) (K + A_1 \times (A_2 \cap z)) dvol_{W_2/V_2}(z). (6.1.10)$$

Now observe that the map $W_2/V_2 \to Val(W_2)$ defined by

$$z \mapsto vol_{V_2}(\bullet + (A_2 \cap z)) =: \phi(z)(\bullet)$$

is a bounded map which is continuous almost everywhere (more precisely, this map is continuous in the interior of the image of $A_2$ in $W_2/V_2$). The expression $(6.1.11)$ is equal to

$$\int_{z \in W_2/V_2} (\phi_1 \boxtimes \phi(z))(K) dvol_{W_2/V_2}(z), (6.1.11)$$

and by the mentioned above continuity we may rewrite the expression $(6.1.11)$ as

$$\left(\phi_1 \boxtimes \int_{z \in W_2/V_2} \phi(z) dvol_{W_2/V_2}(z)\right)(K).$$

By $(6.1.7)$, $\int_{z \in W_2/V_2} \phi(z) dvol_{W_2/V_2}(z) = f_2^*\phi_2$. Thus we have proven that $(Id \times f_2)^*(\phi_1 \boxtimes \phi_2) = f_2^*\phi_1 \boxtimes \phi_2$.

Now we may assume that $f_2 = Id$, and it remains to show that

$$(f_1 \times Id)^*(\phi_1 \boxtimes \phi_2) = f_1^*\phi_1 \boxtimes \phi_2.$$

Let us decompose $f_1 = g \circ h$ where $h$ is surjection and $g$ is injection. Then $(f_1 \times Id)^* = (h \boxtimes Id)^* \circ (g \boxtimes Id)^*$. Since we have assumed that both $\phi_1$ and $\phi_2$ can be chosen smooth, by the above proven case of injections one has $(g \boxtimes Id)^*(\phi_1 \boxtimes \phi_2) = g^*\phi_1 \boxtimes \phi_2$.

Thus it remains to show that for any surjection $h: V_1 \to W_1$, and any $A_i \in \mathcal{K}^m(W_i)$, $\phi_i(\bullet) = vol_i(\bullet + A_i)$, $i = 1, 2$, one has

$$(h \boxtimes Id)^*(\phi_1 \boxtimes \phi_2) = h^*\phi_1 \boxtimes \phi_2.$$
6.1.3 Proposition. Let $0 \leq k, l \leq n$. Let 
\[ \phi = \Xi_{k,V}(\xi), \psi = \Xi_{l,W}(\eta) \]
with $\xi \in C^\infty(\text{Gr}_{n-k}(V), \mathcal{T}_{k,V})$, $\eta \in C^\infty(\text{Gr}_{n-l}(V), \mathcal{T}_{l,V})$.
Then
\[ \phi \boxtimes \psi = \int_{F \in \text{Gr}_{n-k}(V)} \int_{E \in \text{Gr}_{n-l}(W)} (p_F \boxtimes p_E)^*(\xi(F) \boxtimes \eta(E)) \]  
(6.1.13)
where $p_F \boxtimes p_E : V \times W \to V/F \times W/E$.

6.1.4 Remark. In (6.1.13) $\xi(F) \boxtimes \eta(E)$ is considered as an element of $\text{Val}(V/F \times V/E) \otimes |\omega_{\text{Gr}_{n-k}(V)}|_F \otimes |\omega_{\text{Gr}_{n-l}(V)}|_E$.

Proof of Proposition 6.1.3 We have
\[
\phi \boxtimes \psi = \int_{F \in \text{Gr}_{n-k}(V)} p_F^* (\xi(F)) \boxtimes \psi \quad \text{Lemma 6.1.1}
\]
\[
\int_{F \in \text{Gr}_{n-k}(V)} (p_F \boxtimes \text{Id}_W)^* (\xi(F) \boxtimes \psi) \quad \text{Lemma 6.1.1}
\]
\[
\int_{F \in \text{Gr}_{n-k}(V)} (p_F \boxtimes \text{Id}_W)^* \int_{E \in \text{Gr}_{n-l}(W)} (\text{Id}_F \boxtimes p_E)^* (\xi(F) \boxtimes \eta(E)) =
\int_{F \in \text{Gr}_{n-k}(V)} \int_{E \in \text{Gr}_{n-l}(W)} (p_F \boxtimes p_E)^* (\xi(F) \boxtimes \eta(E)).
\]

Proposition 6.1.3 is proved. Q.E.D.

6.1.5 Proposition. For any $0 \leq k \leq n$
\[ \Xi_{k,V} \left( C^\infty(\text{Gr}_{n-k}(V), \mathcal{T}_{k,V;k}) \right) = \text{Val}_{k,sm}^+(V), \]
\[ \Xi_{k+1,V} \left( C^\infty(\text{Gr}_{n-k-1}(V), \mathcal{T}_{k+1,V;k}) \right) = \text{Val}_{k,sm}^-(V). \]

Proof. The map $\Xi_{k,V}$ is $GL(V)$-equivariant non-zero map. Hence the result follows from the Irreducibility theorem and the Casselman-Wallach theorem. Q.E.D.

Let us remind the construction of an isomorphism
\[ \mathbb{F}_V : \text{Val}_{k,sm}^+(V) \to \text{Val}_{n-k}^+(V^*) \otimes \text{Dens}(V) \]
on even valuations from [41] (where it was denoted by $\mathbb{D}$). First let us do it for $k = n$. Then
\[ \text{Val}_n^+(V) = \text{Val}_n(V) = \text{Dens}(V). \]
On the other hand
\[ \text{Val}_{0,sm}^+(V^*) \otimes \text{Dens}(V) = \text{Val}_0(V^*) \otimes \text{Dens}(V) = \mathbb{C} \otimes \text{Dens}(V) = \text{Dens}(V). \]
Take
\[ \mathbb{F}_V : \text{Val}_n(V) \to \text{Val}_0(V) \otimes \text{Dens}(V) \]
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to be the identity isomorphism.

Let us consider the case $k < n$. Let $F \in Gr_{n-k}(V)$. We have

$$F_{V/F} : Val^{sm}_k(V/F) \to Val^{sm}_0(F^\perp) \otimes \text{Dens}(F^{\perp *}).$$

(6.1.14)

Let $p_F^\vee : F^\perp \to V^*$ be the map dual to the projection $p_F : V \to V/F$. Let us consider the map

$$S^+_k : C^\infty(Gr_{n-k}(V), T_{k,V;\cdot;k}) \to Val^{+,sm}_{n-k}(V^*) \otimes \text{Dens}(V)$$

(6.1.15)

given by

$$S^+_k(\xi) = \int_{F \in Gr_{n-k}(V)} p_F^\vee(F_{V/F}(\xi(F))).$$

6.1.6 Theorem ([4]). There exists a unique map

$$F_V : Val^{+,sm}_{k}(V) \to Val^{+,sm}_{n-k}(V^*) \otimes \text{Dens}(V)$$

which makes the following diagram commutative

$$\begin{array}{c}
C^\infty(Gr_{n-k}(V), T_{k,V;\cdot;k}) \\
\downarrow S^+_k \\
Val^{+,sm}_{n-k}(V^*) \otimes \text{Dens}(V)
\end{array}$$

$$\Xi_{k,V} \quad F_V$$

This map $F_V$ is a $GL(V)$-equivariant isomorphism of topological vector spaces.

Let us construct $F_V$ on odd $k$-homogeneous valuations. Thus let $1 \leq k \leq n-1$. First let us consider the case $k = 1$. We have

$$\Xi_{2,V} : C^\infty(Gr_{n-2}(V), T_{2,V;\cdot;1}) \to Val^{-,sm}_1(V).$$

Consider the map

$$S_2^- : C^\infty(Gr_{n-2}(V), T_{2,V;\cdot;1}) \to Val^{-,sm}_{n-1}(V)$$

given by

$$S_2^-(\xi) = \int_{F \in Gr_{n-2}(V)} p_F^\vee(F_{V/F}(\xi(F)))$$

where $F_{V/F} : Val^{-,sm}_{1}(V/F) \to Val^{-,sm}_{1}(F^\perp) \otimes \text{Dens}(F^{\perp *})$ is the Fourier transform defined in the two dimensional case by Theorem 5.4.1.

6.1.7 Proposition. There exists a unique map

$$F_V : Val^{-,sm}_1(V) \to Val^{-,sm}_{n-1}(V^*) \otimes \text{Dens}(V)$$

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Moreover the Jordan-Hölder series of making the following diagram commutative

\[
\begin{array}{ccc}
C^\infty(\text{Gr}_n^{-2}(V), \mathcal{T}_{2V;1}) & \xrightarrow{\Xi_{2,V}} & \text{Val}^{-,sm}_1(V) \\
S_2^- & \mapsto & F_V \\
\text{Val}^{-,sm}_{n-1}(V^*) \otimes \text{Dens}(V) & \mapsto & 
\end{array}
\]

This map \( F_V \) is a \( GL(V) \)-equivariant isomorphism of topological vector spaces.

Before we prove this proposition let us prove the following result.

**6.1.8 Proposition.** Let \( 1 \leq k \leq n-1 \). The \( GL(V) \)-module \( C^\infty(\text{Gr}_{n-k}, \mathcal{T}_{k+1,V;k}) \) is admissible of finite length, and the Jordan-Hölder series of it contains \( \text{Val}^{-,sm}_1(V) \) with multiplicity one.

**Proof.** Let us fix an isomorphism \( V \simeq \mathbb{R}^n \). Let \( Q_0 \) be the subgroup of \( GL_n(\mathbb{R}) \) consisting of matrices

\[
Q_0 = \left\{ \begin{bmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & c \end{bmatrix} | A \in GL_{n-k-1}(\mathbb{R}), B \in GL_k(\mathbb{R}), c \in \mathbb{R}^* \right\}.
\]

\( Q_0 \) is a parabolic subgroup of \( GL_n(\mathbb{R}) \). Let us denote by \( V \) the complex line bundle over \( F_{n-k-1,n-1}(\mathbb{R}^n) \simeq GL_n(\mathbb{R})/Q_0 \) whose fiber over \( (F,E) \in F_{n-k-1,n-1}(V) \) is equal to \( \text{Dens}(E/F) \otimes \text{or}(V/E) \). \( V \) is \( GL_n(\mathbb{R}) \)-equivariant in a natural way.

Let \( p: F_{n-k-1,n-1}(\mathbb{R}^n) \to \text{Gr}_{n-k-1}(\mathbb{R}^n) \) be the natural map, i.e. \( p(F,E) = F \). For any \( F \in \text{Gr}_{n-k-1}(\mathbb{R}^n) \) the space \( \text{Val}^{-,sm}_1(\mathbb{R}^n/F) \) is canonically a quotient of \( C^\infty(p^{-1}(F), V) \). Hence the \( GL_n(\mathbb{R}) \)-module \( C^\infty(\text{Gr}_{n-k-1}(V), \mathcal{T}_{k+1,V;k}) \) is a quotient of the \( GL_n(\mathbb{R}) \)-module \( C^\infty(GL_n(\mathbb{R})/Q_0, V \otimes p^*(|\omega_{\text{Gr}_{n-k}(V)}|)) \).

Let now \( P_0 \subset GL_n(\mathbb{R}) \) be the parabolic subgroup of matrices

\[
P_0 = \left\{ \begin{bmatrix} B & * & * \\ 0 & c & * \\ 0 & 0 & A \end{bmatrix} | A \in GL_{n-k-1}(\mathbb{R}), B \in GL_k(\mathbb{R}), c \in \mathbb{R}^* \right\}.
\]

Recall that \( \text{Val}^{-,sm}_1(\mathbb{R}^n) \) is a subquotient of \( \text{Ind}_{P_0}^{GL_n(\mathbb{R})} \xi \) where \( \xi: P_0 \to \mathbb{C} \) is the character given by

\[
\xi \left( \begin{bmatrix} B & * & * \\ 0 & c & * \\ 0 & 0 & A \end{bmatrix} \right) = sgn(c) \cdot |\det B|^{-1}.
\]

Moreover the Jordan-Hölder series of \( \text{Ind}_{P_0}^{GL_n(\mathbb{R})} \xi \) contains \( \text{Val}^{-,sm}_1(\mathbb{R}^n) \) with multiplicity one by Remark 4.1.8. The \( GL(V) \)-modules \( C^\infty(\text{Gr}_n(\mathbb{R}^n)/Q_0, V \otimes p^*(|\omega_{\text{Gr}_{n-k}(V)}|)) \) and \( \text{Ind}_{P_0}^{GL_n(\mathbb{R})} \xi \) have the same Jordan-Hölder series by Corollary 1.3.6. Hence Proposition 6.1.8 is proved. Q.E.D.
Proof of Proposition 6.1.7. It is clear that the map

$$S_2^- : C^\infty(Gr_{n-2}, \mathcal{T}_{2V;1}) \to Val_{n-1}^{-,sm}(V^*) \otimes \text{Dens}(V)$$

is non-vanishing and $GL(V)$-equivariant. By Proposition 4.1.1 the $GL(V)$-modules $Val_1^{-,sm}(V)$ and $Val_{n-1}^{-,sm}(V^*) \otimes \text{Dens}(V)$ are isomorphic. Hence by Proposition 6.1.8 all the irreducible subquotients of $C^\infty(Gr_{n-2}, \mathcal{T}_{2V;1})$ which are non-isomorphic to $Val_1^{-,sm}(V)$, are contained in $\text{Ker}(S_2^-)$. Hence $F_V$ does exist indeed. Moreover $F_V$ is continuous by the open mapping theorem for Fréchet spaces (see e.g. [37], Ch. III, §2). The rest of the statements of the theorem follow from the Irreducibility theorem and the Casselman-Wallach theorem. Q.E.D.

Now let us consider the case $k = n - 1$. By Proposition 6.1.7 we have the isomorphism

$$F_{V*} : Val_1^{-,sm}(V^*) \to Val_{n-1}^{-,sm}(V) \otimes \text{Dens}(V^*).$$

Define

$$F_V : Val_{n-1}^{-,sm}(V) \to Val_1^{-,sm}(V^*) \otimes \text{Dens}(V)$$

by

$$F_V := E_{V*} \circ (F^{-1}_{V*} \otimes \text{Id}_{\text{Dens}(V)}) \quad (6.1.16)$$

where $(E_{V*}\phi)(K) = \phi(-K)$ (thus $E_{V*}$ is just multiplication by $-1$ on $Val^{-,sm}(V^*) \otimes \text{Dens}(V)$).

6.1.9 Remark. For $n = 2, k = 1$ both definitions of $F_V : Val_1^{-,sm}(V) \to Val_1^{-,sm}(V^*) \otimes \text{Dens}(V)$ coincide with the construction discussed in Section 5.4 due to Proposition 5.5.1.

Now let us consider the general case $1 \leq k \leq n - 1$. We have the epimorphism

$$\Xi_{k+1,V} : C^\infty(Gr_{n-k-1}(V), \mathcal{T}_{k+1,V;k}) \to Val_k^{-,sm}(V).$$

Consider the map

$$S_{k+1}^- : C^\infty(Gr_{n-k-1}(V), \mathcal{T}_{k+1,V;k}) \to Val_{n-k}^{-,sm}(V^*) \otimes \text{Dens}(V)$$

defined by $S_{k+1}^-(\xi) = \int_{F \in Gr_{n-k-1}(V)} p_{V/F*}(F(V/F)(\xi))$ where $F_{V/F}$ in the right hand side is defined by (6.1.16).

6.1.10 Theorem. Let $1 \leq k \leq n - 1$. There exists a unique map $F_V : Val_k^{-,sm}(V) \to Val_{n-k}^{-,sm}(V^*) \otimes \text{Dens}(V)$ making the following diagram commutative

$$\begin{array}{ccc}
C^\infty(Gr_{n-k-1}(V), \mathcal{T}_{k+1,V;k}) & \xrightarrow{\Xi_{k+1,V}} & Val_k^{-,sm}(V) \\
| & S_{k+1}^- & | \\
Val_{n-k}^{-,sm}(V^*) \otimes \text{Dens}(V) & \xrightarrow{F_V} & \\
& Val_{n-k}^{-,sm}(V^*) \otimes \text{Dens}(V) & \\
\end{array}$$

Thus map $F_V$ is a $GL(V)$-equivariant isomorphism of linear topological spaces.
6.1.11 Remark. It is clear that the construction of $\mathbb{F}_V$ given in Theorem 6.1.10 coincides with the previous constructions for $k = 1, n - 1$.

Proof of Theorem 6.1.10. It is clear that the map

$$S_{k+1}^\ast: C^\infty(Gr_{n-k-1}; T_{k+1; V, k}) \rightarrow Val^{-, sm}_k(V^*) \otimes Dens(V)$$

is non-vanishing and $GL(V)$-equivariant. By Proposition 6.1.1, the $GL(V)$-modules $Val^{-, sm}_n(V)$ and $Val^{-, sm}_{n-k}(V^*) \otimes Dens(V)$ are isomorphic. Hence by Proposition 6.1.8 all the irreducible subquotients of $C^\infty(Gr_{n-k-1}, T_{k+1; V, k})$ which are non-isomorphic to $Val^{-, sm}_k(V)$, are contained in $Ker(S_{k+1})$. Hence $\mathbb{F}_V$ does exist indeed. Moreover $\mathbb{F}_V$ is continuous by the open mapping theorem for Fréchet spaces (see e.g. [37], Ch. III, §2). The rest of the theorem follow from the Irreducibility theorem and the Casselman-Wallach theorem. Q.E.D.

6.2 Relations of the Fourier transform to the pullback and push-forward.

6.2.1 Theorem. Let $i: L \hookrightarrow V$ be an injection of linear spaces. Let $\phi \in Val^{sm}(V)$. Then $i^*\phi \in Val^{sm}(L)$, and

$$\mathbb{F}_L(i^*\phi) = i^\vee(\mathbb{F}_V \phi). \quad (6.2.1)$$

Proof. It is clear that $i^*\phi \in Val^{sm}(L)$. Let us prove the equality (6.2.1). Obviously this equality is true if $i$ is an isomorphism.

Case 1. Assume that $\phi \in Val^{+, sm}(V)$. If $\dim L < k$ then both sides of (6.2.1) vanish.

Case 1a. Let us assume in addition that $\dim L = k$. By Proposition 6.1.5 there exists $\xi \in C^\infty(Gr_{n-k}(V), T_{k; V, k}^+)$ such that

$$\phi = \Xi_{k; V}(\xi).$$

More explicitly

$$\phi = \int_{F \in Gr_{n-k}(V)} p_F^\ast(\xi(F)).$$

Then

$$i^*\phi = \int_{F \in Gr_{n-k}(V)} (p_F \circ i)^\ast(\xi(F)) = \int_{F \in Gr_{n-k}(V), F \cap L = \{0\}} (p_F \circ i)^\ast(\xi(F)).$$

Now observe that for any $F \in Gr_{n-k}(V)$ such that $F \cap L = \{0\}$ the map $p_F \circ i: L \rightarrow V/F$ is an isomorphism. Hence

$$\mathbb{F}_L(i^*\phi) = \int_{F \in Gr_{n-k}(V), F \cap L = \{0\}} \mathbb{F}_L((p_F \circ i)^\ast(\xi(F))) =$$

$$\int_{F \in Gr_{n-k}(V), F \cap L = \{0\}} (i^\vee_s \circ p_F^\vee)(\mathbb{F}_V/F(\xi(F))) =$$

$$i^\vee_s \left( \int_{F \in Gr_{n-k}(V), F \cap L = \{0\}} p_F^\vee(\mathbb{F}_V/F(\xi(F))) \right) =$$

$$i^\vee_s \left( \int_{F \in Gr_{n-k}(V)} p_F^\vee(\mathbb{F}_V/F(\xi(F))) \right) = i^\vee(\mathbb{F}_V \phi)$$

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Case 1b. Let us assume that \( \dim L > k \). Let \( E \subseteq L \) be an arbitrary \( k \)-dimensional subspace of \( L \), and let \( j: E \hookrightarrow L \) denote the imbedding map. Using Case 1a we have

\[
\mathbb{F}_E(j^*(i^*\phi)) = \mathbb{F}_E((ij)^*\phi) = (ij)^V_*(\mathbb{F}_V\phi) = j^V_*(i^V_*(\mathbb{F}_V\phi)).
\]

On the other hand using again Case 1a, we have

\[
\mathbb{F}_E(j^*(i^*\phi)) = j^V_*(\mathbb{F}_L(i^*\phi)).
\]

Thus we get

\[
j^V_*(i^V_*(\mathbb{F}_V\phi)) = j^V_*(\mathbb{F}_L(i^*\phi)).
\]

Thus Theorem 6.2.1 in the even case follows from the following lemma.

6.2.2 Lemma. Let \( L \) be an \( l \)-dimensional vector space. Let \( 0 \leq k < l \). Let \( \phi \in \text{Val}^{+}_{l-k}(L) \otimes \text{Dens}(L^*) \). Assume that for any surjection \( p: L \rightarrow K \) of rank \( k \)
\[
p_*\psi = 0.
\]

Then \( \psi = 0 \).

Proof. Let \( \mathcal{V} \) be the subspace of \( \text{Val}^{+}_{l-k}(L) \) consisting of \( \psi \) such that \( p_*\psi = 0 \) for any surjection \( p \) of rank \( k \). It is easy to see that \( \mathcal{V} \) is \( GL(V) \)-invariant closed subspace, \( \mathcal{V} \neq \text{Val}^{+}_{l-k}(L) \). Hence by the Irreducibility theorem \( \mathcal{V} = 0 \). Q.E.D.

Theorem 6.2.1 is proved in the even case.

Case 2. Assume that \( \phi \in \text{Val}^{-}_{l-k}(V) \). The proof of this case will be similar to the proof of Case 1. First observe that if \( \dim L \leq k \) then both sides of (6.2.1) vanish.

Case 2a. Let us assume in addition that \( \dim L = k + 1 \). By Proposition 6.1.5 there exists \( \eta \in C^\infty(Gr_{n-k-1}(V), T_{k+1,V}; k) \) such that

\[
\phi = \Xi_{k+1,V}(\eta).
\]

More explicitly

\[
\phi = \int_{F \in Gr_{n-k-1}(V)} p^*_F(\eta(F)).
\]

Analogously to Case 1a we have

\[
i^*\phi = \int_{F \in Gr_{n-k-1}(V), F \cap L = \{0\}} (p_F \circ i)^*(\eta(F)).
\]

For any \( F \in Gr_{n-k-1}(V) \) such that \( F \cap L = \{0\} \) the map \( p_F \circ i: L \rightarrow V/F \) is an isomorphism. Hence

\[
\mathbb{F}_L(i^*\phi) = \int_{F \in Gr_{n-k-1}(V), F \cap L = \{0\}} \mathbb{F}_L((p_F \circ i)^*(\eta(F))) = \int_{F \in Gr_{n-k-1}(V), F \cap L = \{0\}} (p_F \circ i)^V_*(\mathbb{F}_V(\eta(F))) = i^V_*(\int_{F \in Gr_{n-k-1}(V)} p^V_{F*}(\mathbb{F}_V(\eta(F)))) = i^V_*(\mathbb{F}_V\phi)
\]

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where the last equality holds by the definition of \( \mathbb{F}_V \).

**Case 2b.** Let us assume now that \( \dim L > k + 1 \). Let \( E \subset L \) be an arbitrary subspace of dimension \( k + 1 \), and let \( j: E \hookrightarrow L \) be the imbedding map. By Case 2a we have

\[
\mathbb{F}_E(j^*(i^*\phi)) = \mathbb{F}_E((ij)^*\phi) = (ij)^*(\mathbb{F}_V \phi) = j^*(i^*\mathbb{F}_V \phi).
\]

Using again Case 2a we have

\[
\mathbb{F}_E(j^*(i^*\phi)) = j^*(\mathbb{F}_L(i^*\phi)).
\]

Thus we obtain

\[
j^*(i^*\mathbb{F}_V \phi) = j^*(\mathbb{F}_L(i^*\phi)).
\]

Then the proof of Case 2 follows immediately from the following lemma.

**6.2.3 Lemma.** Let \( L \) be an \( l \)-dimensional vector space. Let \( 1 \leq k \leq l - 1 \). Let \( \psi \in Val_{l-k}(L) \otimes Dens(L^*) \). Assume that for any surjection \( p: L \twoheadrightarrow N \) of rank \( k + 1 \)

\[
p_*\psi = 0.
\]

Then \( \psi = 0 \).

**Proof.** The proof is completely analogous to the proof of Lemma 6.2.2 and will be omitted. Q.E.D.

Thus Theorem 6.2.1 is proved. Q.E.D.

**6.2.4 Theorem.** Let \( \xi \in C^\infty(Gr_{n-k}(V), \mathcal{T}_{k,V}) \). Then

\[
\mathbb{F}_V(\Xi_{k,V}(\xi)) = \int_{F \in Gr_{n-k}(V)} p_{*F}^*(\mathbb{F}_V/F(\xi(F))).
\]

**Proof.** We may and will assume that \( \xi \in C^\infty(Gr_{n-k}(V), \mathcal{T}_{k,V;i}^+) \).

**Case 1.** Let us assume that \( \xi \in C^\infty(Gr_{n-k}(V), \mathcal{T}_{k,V;i}^+) \). If \( k < i \) then both sides of (6.2.2) vanish. If \( k = i \) then (6.2.2) is just the definition of \( \mathbb{F}_V \). Thus let us assume that \( k > i \). Let us denote by \( \mathcal{F}_{n-k,n-i}(V) \) the manifold of partial flags

\[
\mathcal{F}_{n-k,n-i}(V) := \{(F,E)| F \subset E, \dim F = n - k, \dim E = n - i\}.
\]

Let

\[
p_{n-k}: \mathcal{F}_{n-k,n-i}(V) \to Gr_{n-k}(V),
p_{n-i}: \mathcal{F}_{n-k,n-i}(V) \to Gr_{n-i}(V)
\]

be the natural projections defined by \( p_{n-k}((F,E)) = F, p_{n-i}((F,E)) = E \). Recall that we denote by \( \mathcal{T}_{k,V;i}^{0,+} \) the vector bundle over \( Gr_{n-k}(V) \) whose fiber over \( F \in Gr_{n-k}(V) \) is equal to \( Val_{l-sim}^+(V/F) \), and by

\[
\mathcal{T}_{k,V;i}^+ := \mathcal{T}_{k,V;i}^{0,+} \otimes |\omega_{Gr_{n-k}(V)}|.
\]
Note that the fiber of $T_{i,V;0}^0(V)$ over $L \in Gr_{n-i}(V)$ is equal to $Dens(V/L)$. Let us denote

$$\tilde{T}_{i,V;0}^0 := p_{n-i}^*(T_{i,V;0}^0),$$
$$\tilde{T}_{i,V;0}^0 := \tilde{T}_{i,V;0}^0 \otimes |\omega_{Gr_{n-k,n-i}(V)}|.$$

We have the canonical map

$$\Psi_{k,i}^+: C^\infty (F_{n-k,n-i}(V), \tilde{T}_{i,V;0}^0) \to C^\infty (Gr_{n-k}(V), T_{k,V;0}^+).$$

(6.2.3)

given by

$$(\Psi_{k,i}^+(\zeta))(F) = \int_{L \in Gr_{n-i}(V), L \supset F} q_{L,F}^*(\zeta(L, F))$$

where $q_{L,F} : V/F \to V/L$ is the canonical map. It is clear that $\Psi_{k,i}^+$ is $GL(V)$-equivariant continuous map.

**6.2.5 Claim.** $\Psi_{k,i}^+$ is onto.

**Proof.** This claim easily follows from Propositions 6.1.5 and 1.2.6. Q.E.D.

We have

$$\Xi_{k,V}(\Psi_{k,i}^+(\zeta)) = \int_{F \in Gr_{n-k}(V)} p_F^* \left( \int_{L \in Gr_{n-i}(V), L \supset F} q_{L,F}^*(\zeta(L, F)) \right) = \int_{F \in Gr_{n-k}(V)} \left( \int_{L \in Gr_{n-i}(V), L \supset F} p_L^*(\zeta(L, F)) \right) = \int_{L \in Gr_{n-i}(V)} p_L^*(\eta(L))$$

where $\eta(L) := \int_{F \in Gr_{n-k}(L)} \zeta(L, F)$, thus $\eta \in C^\infty (Gr_{n-i}(V), T_{k,V;0}^+)$. Hence by the definition of $F_{V}$ one has

$$F_{V} (\Xi_{k,V}(\Psi_{k,i}^+(\zeta))) = \int_{L \in Gr_{n-i}(V)} p_L^*(F_{V/L}(\eta(L))).$$

But by the continuity of the Fourier transform $F_{V/L}(\eta(L)) = \int_{F \in Gr_{n-i}(L)} F_{V/L}(\zeta(L, F))$. Also

$$p_L = q_{L,F} \circ p_F,$$
$$p_L^\vee = p_F^\vee \circ q_{L,F}^\vee.$$

Hence

$$F_{V}(\Xi_{k,V}(\Psi_{k,i}^+(\zeta))) = \int_{F_{n-k,n-i}(V)} p_{F}^\vee \left( q_{L,F}^\vee(F_{V/L}(\zeta(L))) \right).$$

(6.2.4)

By the definition of $F_{V/F}$ we have

$$\int_{L \in Gr_{n-i}(V), L \supset F} q_{L,F}^\vee(F_{V/L}(\zeta(L, F))) = F_{V/F} \left( \int_{L \in Gr_{n-i}(V), L \supset F} q_{L,F}^*(\zeta(L, F)) \right) = F_{V/F}(\Psi_{k,i}^+(\zeta)(F)).$$
Substituting this into (6.2.4) we get
\[ F_V(\Xi_{k,V}(\Psi^+_{k,V}(\zeta))) = \int_{F \in Gr_{n-k}(V)} p^*_F \left( F^{\vee}_V(\Psi^+_{k,V}(\zeta)(F)) \right). \]

Since \( \Psi^+_{k,i} \) is onto (by Claim 6.2.5), Case 1 is proved.

**Case 2.** Now we assume that \( \xi \in C^\infty(Gr_{n-k}(V), T^1_{k,V;ii}) \).

If \( k < i + 1 \) then both sides of (6.2.2) vanish. If \( k = i + 1 \) then the result is just the definition of \( F_V \). Let us assume that \( k > i + 1 \). The proof will be analogous to Case 1. Similarly to Case 1 we denote by \( F_{n-k,n-i-1}(V) \) the manifold of partial flags
\[ F_{n-k,n-i-1}(V) := \{(F, E)| F \subset E \subset V, \dim F = n-k, \dim E = n-i-1\}. \]

We denote
\[ p_{n-k}: F_{n-k,n-i-1}(V) \to Gr_{n-k}(V), \]
\[ p_{n-i-1}: F_{n-k,n-i-1}(V) \to Gr_{n-i-1}(V) \]
the natural projections. Define
\[ \tilde{T}^0_{i+1,V;ii} := p^*_{n-i-1}(T^0_{i+1,V;ii}), \]
\[ \tilde{T}^{0-}_{i+1,V;ii} := \tilde{T}^{0-}_{i+1,V;ii} |_{\omega_{F_{n-k,n-i-1}(V)}}. \]

We have the canonical map
\[ \Psi^-_{k,i}: C^\infty(F_{n-k,n-i-1}(V), \tilde{T}^{0-}_{i+1,V;ii}) \to C^\infty(Gr_{n-k}, T^-_{k,V;ii}) \quad (6.2.5) \]
given, for any \( F \in Gr_{n-k}(V) \), by
\[ (\Psi^-_{k,i}(\zeta))(F) = \int_{L \in Gr_{n-i-1}(V), L \supset F} q^*_{L,F}(\zeta(L, F)) \]
where as previously \( q_{L,F}: V/F \to V/L \) is the canonical map. It is clear that \( \Psi^-_{k,i} \) is a \( GL(V) \)-equivariant continuous map.

**6.2.6 Claim.** \( \Psi^-_{k,i} \) is onto.

**Proof** easily follows from Proposition 6.1.5. Q.E.D.

The rest of the proof is parallel to Case 1. We have
\[ \Xi_{k,V}(\Psi^-_{k,i}(\zeta)) = \int_{F \in Gr_{n-k}(V)} p^*_F \left( \int_{L \in Gr_{n-i-1}(V), L \supset F} q^*_{L,F}(\zeta(L, F)) \right) = \int_{F \in Gr_{n-k}(V)} \int_{L \in Gr_{n-i-1}(V), L \supset F} p^*_F(\zeta(L, F)) = \int_{L \in Gr_{n-i-1}(V)} \left( \int_{F \in Gr_{n-k}(L)} p^*_L(\zeta(L, F)) \right) = \int_{L \in Gr_{n-i-1}(V)} p^*_L(\eta(L)). \]

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where \( \eta(L) := \int_{F \in \text{Gr}_{n-k}(L)} \zeta(L, F) \), thus \( \eta \in C^\infty(\text{Gr}_{n-i-1}(V), \mathcal{I}_{i+1,V}) \). Hence by the definition of \( F_V \) we have

\[
\mathbb{F}_V \left( \Xi_{k,V}(\Psi_{k,i}^{-}(\zeta)) \right) = \int_{L \in \text{Gr}_{n-i-1}(V)} p_{L^*}^V (F_{V/L}(\eta(L))).
\]

But by the continuity of the Fourier transform one has \( F_{V/L}(\eta(L)) = \int_{F \in \text{Gr}_{n-k}(L)} F_{V/L}(\zeta(L, F)) \).

Hence

\[
\mathbb{F}_V \left( \Xi_{k,V}(\Psi_{k,i}^{-}(\zeta)) \right) = \int_{F \in \text{Gr}_{n-k}(V)} p_{F^*}^V (F_{V/F}(\zeta(L, F))).
\]

By the definition of \( F_{V/F} \) we have

\[
\int_{L \in \text{Gr}_{n-i-1}(V), L \supset F} q_{L,F}^* (F_{V/L}(\zeta(L))) = \mathbb{F}_{V/F} \left( \int_{L \in \text{Gr}_{n-i-1}(V), L \supset F} q_{L,F}^* (\zeta(L, F)) \right) = \mathbb{F}_{V/F}(\Psi_{k,i}^{-}(\zeta)(F)).
\]

Substituting this into (6.2.6) we get

\[
\mathbb{F}_V \left( \Xi_{k,V}(\Psi_{k,i}^{-}(\zeta)) \right) = \int_{F \in \text{Gr}_{n-k}(V)} p_{F^*}^V (F_{V/F}(\Psi_{k,i}^{-}(\zeta)(F))).
\]

Since \( \Psi_{k,i}^{-} \) is onto (by Claim 6.2.6), Case 2 is proved. Thus Theorem 6.2.4 is proved. Q.E.D.

### 6.3 A Plancherel type formula in higher dimensions

The main result of this section is Theorem 6.3.9 below.

Let us introduce more notation. Let

\[
\mathcal{S}_{k,V;i}^0 \rightarrow \text{Gr}_k(V)
\]

(6.3.1)

denote the vector bundle whose fiber over \( E \in \text{Gr}_k(V) \) is equal to \( \text{Val}^{sm}(E) \otimes \text{Dens}(E^*) \).

Let

\[
\mathcal{S}_{k,V;i}^0 := \mathcal{S}_{k,V;i}^0 \otimes |\omega_{\text{Gr}_k(V)}|.
\]

(6.3.2)

Similarly let

\[
\mathcal{S}_{k,V}^0 \rightarrow \text{Gr}_k(V)
\]

(6.3.3)

denote the vector bundle whose fiber over \( E \in \text{Gr}_k(V) \) is equal to \( \text{Val}^{sm}(E) \otimes \text{Dens}(E^*) \).

Let

\[
\mathcal{S}_{k,V} := \mathcal{S}_{k,V}^0 \otimes |\omega_{\text{Gr}_k(V)}|.
\]

(6.3.4)

For a vector space \( W \) we can consider the map

\[
\mathbb{F}_W \otimes \text{Id}_{\text{Dens}(W^*)} : \text{Val}^{sm}(W) \otimes \text{Dens}(W^*) \rightarrow (\text{Val}^{sm}(W^*) \otimes \text{Dens}(W)) \otimes \text{Dens}(W^*) = \text{Val}^{sm}(W^*).
\]
6.3.1 Lemma. Let $V$ be an $n$-dimensional vector space. Let $\eta \in C^\infty(Gr_2(V), S_{2,V}^-)$. Then

$$(\mathbb{F}_V \otimes \text{Id}_{Dens(V^*)}) \left( \int_{E \in \text{Gr}_2(V)} i_{E*}(\eta(E)) \right) = \int_{E \in \text{Gr}_2(V)} (i_E^*)^* (\mathbb{F}_E \otimes \text{Id}_{Dens(E^*)})(\eta(E))$$

(6.3.5)

where $i_E : E \to V$ denotes the imbedding map.

Proof. First note that $i_{E*}(\eta(E)) \in \text{Val}_{n-1}(V) \otimes \text{Dens}(V^*)$. Recall that by the definition of the Fourier transform $\mathbb{F}_V : \text{Val}_{n-1}^{-sm}(V) \to \text{Val}_1^{-sm}(V^*) \otimes \text{Dens}(V)$

$$\mathbb{F}_V = \mathcal{E}_V \circ (\mathbb{F}_V \otimes \text{Id}_{Dens(V)}).$$

Hence the equality (6.3.5) is equivalent to

$$\int_{E \in \text{Gr}_2(V)} i_{E*}(\eta(E)) = (\mathcal{E}_V \circ \mathbb{F}_V^*) \left( \int_{E \in \text{Gr}_2(V)} (i_E^*)^* (\mathbb{F}_E \otimes \text{Id}_{Dens(E^*)})(\eta(E)) \right).$$

(6.3.6)

The right hand side of (6.3.6) is equal to

$$\mathbb{F}_V^* \left( \int_{E \in \text{Gr}_2(V)} (i_E^*)^* (\mathcal{E}_E \circ (\mathbb{F}_E \otimes \text{Id}_{Dens(E^*)}))(\eta(E)) \right).$$

But by the Plancherel formula in dimension two (Proposition 5.5.1) we know that

$$\mathcal{E}_E \circ (\mathbb{F}_E \otimes \text{Id}_{Dens(E^*)}) = \mathbb{F}_E^{-1}.$$

Hence (6.3.6) is equivalent to

$$\int_{E \in \text{Gr}_2(V)} i_{E*}(\eta(E)) = \mathbb{F}_V^* \left( \int_{E \in \text{Gr}_2(V)} (i_E^*)^* \mathbb{F}_E^{-1}(\eta(E)) \right).$$

The last equality is a special case of Theorem 6.2.4. Q.E.D.

6.3.2 Lemma. The composition map

$$\text{Val}_1^{-sm}(V) \xrightarrow{F_V} \text{Val}_{n-1}^{-sm}(V^*) \otimes \text{Dens}(V) \xrightarrow{F_{V^*} \otimes \text{Id}_{Dens(V)}} \text{Val}_1^{-sm}(V)$$

is multiplication by $-1$.

Proof. Let $\phi \in \text{Val}_1^{-sm}(V)$. By Proposition 6.1.5 there exists $\xi \in C^\infty(Gr_2(V), T_{2,V}^-)$ such that $\phi = \Xi_{2,V}(\xi)$. Then by the definition of $\mathbb{F}_V$

$$\mathbb{F}_V(\phi) = \int_{F \in \text{Gr}_{n-2}(V)} p_{F*}(\mathbb{F}_V/F(\xi(F))) = \int_{E \in \text{Gr}_2(V^*)} i_{E*}(\mathbb{F}_{V/E^*}(\xi(E^\perp))).$$

Then applying Lemma 6.3.1 we get

$$(\mathbb{F}_V^* \otimes \text{Id}_{Dens(V)})(\mathbb{F}_V(\phi)) = \int_{E \in \text{Gr}_2(V^*)} (i_E^*)^* ((\mathbb{F}_E \otimes \text{Id}_{Dens(E^*)}) \circ \mathbb{F}_{V/E^*} \circ \mathcal{E}_{E^*})(\xi(E^\perp))$$

(6.3.7)

But $(\mathbb{F}_E \otimes \text{Id}_{Dens(E^*)}) \circ \mathbb{F}_{V/E^*} = (\mathbb{F}_E \otimes \text{Id}_{Dens(E^*)} \circ \mathcal{E}_{E^*} = \mathcal{E}_{E^*}$ by the two-dimensional case (Proposition 5.5.1). Substituting this into (6.3.7) we get

$$(\mathbb{F}_V^* \otimes \text{Id}_{Dens(V)})(\mathbb{F}_V(\phi)) = -\int_{E \in \text{Gr}_2(V^*)} (i_E^*)^* (\xi(E^\perp)) = -\int_{F \in \text{Gr}_{n-2}(V)} p_{F*}(\xi(F)) = -\phi.$$

Lemma is proved. Q.E.D.
6.3.3 Lemma. The composition map

\[ \text{Val}^{+, \text{sm}}(V) \xrightarrow{F_V} \text{Val}^{+, \text{sm}}(V^*) \otimes \text{Dens}(V) \xrightarrow{F_{V^*} \otimes \text{Id}_{\text{Dens}(V)}} \text{Val}^{+, \text{sm}}(V) \]

is equal to the identity.

Proof. Let \(0 \leq k \leq n\). Let \(\mathcal{L}_k(V) \to \text{Gr}_k(V)\) denote the line bundle whose fiber over \(F \in \text{Gr}_k(V)\) is equal to \(\text{Dens}(F)\). We have the Klain imbeddings

\[ \tau : \text{Val}^{+, \text{sm}}_k(V) \hookrightarrow C^\infty(\text{Gr}_k(V), \mathcal{L}_k(V)), \]
\[ \tau' : \text{Val}^{+, \text{sm}}_{n-k} \otimes \text{Dens}(V) \hookrightarrow C^\infty(\text{Gr}_{n-k}(V^*), \mathcal{L}_{n-k}(V^*)) \otimes \text{Dens}(V). \]

Taking the orthogonal complement we have the identification

\[ \text{Gr}_k(V) \simeq \text{Gr}_{n-k}(V^*). \]

Also for any \(F \in \text{Gr}_k(V)\) we have

\[ \text{Dens}(F) = \text{Dens}(F^\perp) \otimes \text{Dens}(V). \]

Thus we have a \(GL(V)\)-equivariant isomorphism

\[ \gamma : C^\infty(\text{Gr}_k(V), \mathcal{L}_k(V)) \rightarrow C^\infty(\text{Gr}_{n-k}(V^*), \mathcal{L}_{n-k}(V)) \otimes \text{Dens}(V). \]

Let \(\phi \in \text{Val}^{+, \text{sm}}_k(V)\). By \[4\]

\[ \tau'(F_V \phi) = \gamma(\tau \phi). \]

It is easy to see that for any \(\psi \in \text{Val}^{+, \text{sm}}_{n-k}(V^*) \otimes \text{Dens}(V)\) one has

\[ \tau((F_{V^*} \otimes \text{Id}_{\text{Dens}(V)})(\psi)) = \gamma^{-1}(\tau' \psi). \]

Hence

\[ \tau(((F_{V^*} \otimes \text{Id}_{\text{Dens}(V)}) \circ F_V)(\phi)) = \gamma^{-1}(\tau'(F_V \phi)) = \gamma^{-1}(\gamma(\tau \phi)) = \tau \phi. \]

Lemma is proved. Q.E.D.

6.3.4 Proposition. Let \(p : V \to W\) be a surjective linear map of vector spaces. Then the following diagram is commutative

\[ \begin{array}{ccc}
\text{Val}^{\text{sm}}(V) \otimes \text{Dens}(V^*) & \xrightarrow{F_V \otimes \text{Id}_{\text{Dens}(V^*)}} & \text{Val}^{\text{sm}}(V^*) \\
p_* & & p^*_V \\
\text{Val}^{\text{sm}}(W) \otimes \text{Dens}(W^*) & \xrightarrow{F_W \otimes \text{Id}_{\text{Dens}(W^*)}} & \text{Val}^{\text{sm}}(W^*)
\end{array} \] (6.3.8)
Proof. Case 1. Let us consider the case of even valuations.

By Lemma 6.3.3 on even valuations $F_V \otimes Id_{Dens(V^*)} = F_{V^*}^{-1}$, $F_W \otimes Id_{Dens(W^*)} = F_{W^*}^{-1}$. Hence the commutativity of the diagram (6.3.8) is equivalent to commutativity of the following diagram

$$
\begin{array}{ccc}
Val(V^*) & \overset{\mathbb{F}_{V^*}}{\longrightarrow} & Val(V) \otimes Dens(V^*) \\
\downarrow p^\vee & & \downarrow p \\
Val(W^*) & \overset{\mathbb{F}_{W^*}}{\longrightarrow} & Val(W) \otimes Dens(W^*)
\end{array}
$$

But the last diagram is a special case of Theorem 6.2.1. Thus the theorem is proved in the even case.

Case 2. Let us consider the odd case.

Let us denote $m := \dim W$. Let $1 \leq j \leq n - 1$. We have to show that the following diagram is commutative:

$$
\begin{array}{ccc}
Val_j^{-,sm}(V^*) \otimes Dens(V^*) & \overset{\mathbb{F}_V \otimes Id_{Dens(V^*)}}{\longrightarrow} & Val_{n-j}^{-,sm}(V^*) \\
\downarrow p_* & & \downarrow p^\vee \\
Val_j^{-,sm}(W^*) \otimes Dens(W^*) & \overset{\mathbb{F}_W \otimes Id_{Dens(W^*)}}{\longrightarrow} & Val_{n-j}^{-,sm}(W^*)
\end{array}
$$

(6.3.9)

Case 2a. Let us assume in addition that $\dim W = 2$.

Then necessarily $j = n - 1$. In this case, by Lemma 6.3.2, we have $\mathbb{F}_V \otimes Id_{Dens(V^*)} = -\mathbb{F}_{V^*}^{-1}$, $\mathbb{F}_W \otimes Id_{Dens(W^*)} = -\mathbb{F}_{W^*}^{-1}$. Hence the commutativity of the diagram (6.3.9) is equivalent to the commutativity of the diagram

$$
\begin{array}{ccc}
Val_1^{-,sm}(V^*) & \overset{\mathbb{F}_{V^*}}{\longrightarrow} & Val_n^{-,sm}(V) \otimes Dens(V^*) \\
\downarrow p^\vee & & \downarrow p \\
Val_1^{-,sm}(W^*) & \overset{\mathbb{F}_{W^*}}{\longrightarrow} & Val_n^{-,sm}(W) \otimes Dens(W^*)
\end{array}
$$

(6.3.10)

The last diagram is again a special case of Theorem 6.2.1.

Case 2b. Let us assume that $j - n + m = 1$.

In other words we have to show that the following diagram is commutative

$$
\begin{array}{ccc}
Val_j^{-,sm}(V^*) \otimes Dens(V^*) & \overset{\mathbb{F}_V \otimes Id_{Dens(V^*)}}{\longrightarrow} & Val_{n-j}^{-,sm}(V^*) \\
\downarrow p_* & & \downarrow p^\vee \\
Val_1^{-,sm}(W^*) \otimes Dens(W^*) & \overset{\mathbb{F}_W \otimes Id_{Dens(W^*)}}{\longrightarrow} & Val_{n-j}^{-,sm}(W^*)
\end{array}
$$

(6.3.11)
Let us fix an arbitrary 2-dimensional subspace \( i: E \hookrightarrow W \). Let us form a Cartesian square

\[
\begin{array}{ccc}
Z & \xrightarrow{i} & V \\
\downarrow{\tilde{i}} & & \downarrow{p} \\
E & \xrightarrow{i} & W
\end{array}
\]

Note that \( p, \tilde{p} \) are surjections, \( i, \tilde{i} \) are injections. Set \( z := \dim Z = 2 + n - m \).

Let us consider the following cube diagram where we denote \( Dens(\bullet) \) by \( D(\bullet) \) for brevity:

The following facets of this cube commute:
- the left facet, by base change Theorem 3.5.3,
- the right facet, by base change Theorem 3.5.2,
- the bottom facet, by Case 2a (since \( \dim E = 2 \));
- the back facet, by Theorem 6.2.1.
the front facet, by Theorem 6.2.1.

These properties and a straightforward diagram chasing imply that

\[(i^\vee \otimes Id_{D(W)}) \circ (p^\vee \circ (F_V \otimes Id_{D(V)}) - (F_W \circ Id_{D(W)}) \circ p_\ast)) = 0.\]

Since \(i^\vee\) is an arbitrary surjection of rank two, Lemma 6.2.3 implies that

\[p^\vee \circ (F_V \otimes Id_{D(V)}) - (F_W \circ Id_{D(W)}) \circ p_\ast) = 0.\]

This means that the diagram (6.3.11) is commutative. Thus Case 2b is proved.

**Case 2c.** Let us consider the general odd case.

Let us fix an arbitrary surjection \(q: W \twoheadrightarrow X\) with \(\dim X = n - j + 1\). Let us consider the following diagram:

\[
\begin{align*}
\text{Val}_{j}^{-,sm}(V) \otimes Dens(V^*) & \xrightarrow{F_V \otimes Id_{Dens(V^*)}} \text{Val}_{n-j}^{-,sm}(V^*) \\
\downarrow p_\ast & \\
\text{Val}_{j-n+m}^{-,sm}(W) \otimes Dens(W^*) & \xrightarrow{F_W \otimes Id_{Dens(W^*)}} \text{Val}_{n-j}^{-,sm}(W^*) \\
\downarrow q_\ast & \\
\text{Val}_{1}^{-,sm}(X) \otimes Dens(X^*) & \xrightarrow{F_X \otimes Id_{Dens(X^*)}} \text{Val}_{n-j}^{-,sm}(X^*)
\end{align*}
\]

By Case 2b, the lower square of this diagram commutes. Also by Case 2b, the big exterior contour of this diagram commutes. It follows that

\[q^\vee \circ (p^\vee \circ (F_V \otimes Id_{Dens(V^*)}) - (F_W \otimes Id_{Dens(W^*)}) \circ p_\ast) = 0.\]

Since the last equality holds for an arbitrary surjection \(q\) of rank \(n - j + 1\), \(q^\vee\) is an arbitrary injection of an \((n - j + 1)\)-dimensional subspace. But by Schneider’s theorem [40] odd \(k\)-homogeneous valuations are uniquely determined by their restrictions to all \((k + 1)\)-dimensional subspaces. Hence

\[p^\vee \circ (F_V \otimes Id_{Dens(V^*)}) - (F_W \otimes Id_{Dens(W^*)}) \circ p_\ast = 0.\]

This means that the diagram (6.3.9) commutes. Proposition is proved. Q.E.D.

The following theorem generalizes Lemma 6.3.1.

**6.3.5 Theorem.** Let \(0 \leq k \leq n\). Let \(\eta \in C^\infty(Gr_k(V), S_k, V)\) (\(S_k, V\) was defined in (6.3.3), (6.3.4)). Then

\[
(F_V \otimes Id_{Dens(V^*)}) \left( \int_{E \in Gr_k(V)} i_{E*}(\eta(E)) \right) = \int_{E \in Gr_k(V)} (i_E^\vee)^* (F_E \otimes Id_{Dens(E^*)})(\eta(E)(6.3.12)
\]

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Proof. Case 1. Let us consider the even case, i.e. $\eta(E) \in Val^+_{i,sm}(E) \otimes Dens(E^*)$ for any $E \in Gr_k(V)$, $0 \leq i \leq k$.

Case 1a. Assume in addition that $i = 0$. Thus $\eta(E) \in Dens(E^*)$ for any $E \in Gr_k(V)$.

We have the canonical identification

$$Dens(E^*) = Dens(V/E) \otimes Dens(V^*). \quad (6.3.13)$$

It is easy to see from the definitions that under this identification the map

$$i_{E*}: Dens(E^*) \to Val(V) \otimes Dens(V^*)$$

coincides with the map

$$p_E^* \otimes Id_{Dens(V^*)}: Dens(V/E) \otimes Dens(V^*) \to Val(V) \otimes Dens(V^*)$$

where $p_E: V \to V/E$ is the canonical projection.

Let us denote by $\tilde{\eta} \in C^\infty(Gr_k(V), T_{k,V;\k}) \otimes Dens(V^*)$ the section corresponding to $\eta$ under the isomorphism $(6.3.13)$. Thus

$$\int_{E \in Gr_k(V)} i_{E*}(\eta(E)) = \int_{E \in Gr_k(V)} (p_E^* \otimes Id_{Dens(V^*)})(\tilde{\eta}(E)). \quad (6.3.14)$$

Let us fix a Lebesgue measure $vol_V$ on $V$. Set

$$\hat{\eta} := \tilde{\eta} \cdot vol_V \in C^\infty(Gr_k(V), T_{k,V;\k}).$$

Then

$$\int_{E \in Gr_k(V)} i_{E*}(\eta(E)) = \left( \int_{E \in Gr_k(V)} p_E^*(\hat{\eta}(E)) \right) \otimes vol_V^{-1}.$$

Hence by the definition of the Fourier transform

$$(F_V \otimes Id_{Dens(V^*)}) \left( \int_{E \in Gr_k(V)} i_{E*}(\eta(E)) \right) = \left( \int_{E \in Gr_k(V)} p_E^{\vee*}(F_{V/E}(\hat{\eta}(E))) \right) \cdot vol_V^{-1}.$$

Hence it suffices to check that for any $E \in Gr_k(V)$

$$p_E^{\vee*}(F_{V/E}(\hat{\eta}(E))) \cdot vol_V^{-1} = (i_{E*})^*(F_E \otimes Id_{Dens(E^*)})(\eta(E)). \quad (6.3.15)$$

To prove the identity $(6.3.15)$ let us fix a Lebesgue measure $vol_E$ on $E$. Let $vol_V := \frac{vol_V}{vol_E}$ be the corresponding Lebesgue measure on $V/E$. It is sufficient to prove $(6.3.15)$ for $\eta(E) = vol_E^{-1}$. Then under the isomorphism $(6.3.13)$ we have

$$\eta(E) = vol_{V/E} \otimes vol_V^{-1}.$$

Hence

$$\hat{\eta}(E) = vol_{V/E}.$$

We have

$$(F_E \otimes Id_{Dens(E^*)})(\eta(E)) = vol_E^{-1} \in Dens(E^*).$$
Let us denote by $\hat{\text{GL}}(6.3.17)$. This map is $E$ such that for any $H$ implies (6.3.15). Hence Case 1a is proved.

Case 1b. Assume that $i > 0$.

Recall that we denote by $F_{k-i,k}(V)$ the partial flag manifold

$$F_{k-i,k}(V) := \{ (L,E) | L \subset E, L \in \text{Gr}_{k-i}(V), E \in \text{Gr}_k(V) \}.$$ 

For $(L,E) \in F_{k-i,k}(V)$ let us denote by $i_{E,L}$ the imbedding $L \hookrightarrow E$. Let us denote by $\mathcal{C}^0$ the line bundle over $F_{k-i,k}(V)$ whose fiber over $(L,E)$ is equal to $Dens(L^*)$. Let

$$\mathcal{G} := \mathcal{C}^0 \otimes |\omega_{F_{k-i,i}(V)}|.$$ 

**6.3.6 Claim.** For any $\eta \in C^\infty(\text{Gr}_k(V), \mathcal{S}^+_k, V)$, $k > i$, there exists $\xi \in C^\infty(F_{k-i,k}(V), \mathcal{G})$ such that for any $E \in \text{Gr}_k(V)$

$$\eta(E) = \int_{L \in \text{Gr}_{k-i}(E)} i_{E,L_*}(\xi(L,E)). \quad (6.3.17)$$

**Proof.** Define the map $C^\infty(F_{k-i,k}(V), \mathcal{G}) \to C^\infty(\text{Gr}_k(V), \mathcal{S}^+_k, V)$ by the right hand side of (6.3.17). This map is $GL(V)$-equivariant. Then Irreducibility Theorem, Casselman-Wallach theorem, and Proposition 6.2.6 imply the claim. Q.E.D.

Let us continue proving Case 1b. We have

$$\int_{E \in \text{Gr}_k(V)} i_{E_*}(\eta(E)) = \int_{E \in \text{Gr}_k(V)} \int_{L \in \text{Gr}_{k-i}(E)} i_{E_*}(i_{E,L_*}(\xi(L,E))) =$$

$$\int_{E \in \text{Gr}_k(V)} \int_{L \in \text{Gr}_{k-i}(E)} i_{L_*}(\xi(L,E)) = \int_{L \in \text{Gr}_{k-i}(V)} i_{L_*} \left( \int_{E \in \text{Gr}_k(V), E \supset L} \xi(L,E) \right).$$

Let us denote by $\hat{\xi}(L)$ the inner integral in the last expression. Thus $\hat{\xi} \in C^\infty(\text{Gr}_{k-i}(V), \mathcal{S}_{k-i,V,0})$. Hence by Case 1a

$$(\mathbb{F}_V \otimes I_{\text{Dens}(V^*)}) \left( \int_{E \in \text{Gr}_k(V)} i_{E_*}(\eta(E)) \right) = \quad (6.3.18)$$

$$(\mathbb{F}_V \otimes I_{\text{Dens}(V^*)}) \left( \int_{E \in \text{Gr}_k(V)} i_{L_*}(\hat{\xi}(L)) \right) = \quad (6.3.19)$$

$$\int_{L \in \text{Gr}_{k-i}(V)} (i_{L_*})^*(\mathbb{F}_L \otimes I_{\text{Dens}(L^*)})(\hat{\xi}(L)). \quad (6.3.20)$$
Using again Case 1a we obtain that the right hand side of the equality (6.3.12) is equal to
\[
\int_{E \in Gr_k(V)} (i_E^\top)(\mathbb{F}_E \otimes Id_{Dens(E^*)}) \left( \int_{L \in Gr_{k-i}(E)} i_{E,L^*}(\xi(L,E)) \right) = \\
\int_{E \in Gr_k(V)} (i_E^\top) \left( \int_{L \in Gr_{k-i}(E)} (i_L^\top)(\mathbb{F}_L \otimes Id_{Dens(L^*)})(\xi(L,E)) \right) = \\
\int_{E \in Gr_k(V)} \int_{L \in Gr_{k-i}(E)} (i_L^\top)(\mathbb{F}_L \otimes Id_{Dens(L^*)})(\xi(L,E)) = \\
\int_{L \in Gr_{k-i}(E)} (i_L^\top)(\mathbb{F}_L \otimes Id_{Dens(L^*)})(\hat{\xi}(L)) \text{ by (6.3.20)} = \\
(\mathbb{F}_E \otimes Id_{Dens(V^*)}) \left( \int_{E \in Gr_k(V)} i_{E^*}(\eta(E)) \right).
\]

Thus Case 1b is proved. Hence Case 1 is proved too.

**Case 2.** Let us consider the odd case, i.e. \( \eta(E) \in Val_i^{-sm}(E) \otimes Dens(E^*) \) for any \( E \in Gr_k(V), 1 \leq i \leq k - 1 \).

Then the equality (6.3.12) becomes equivalent to
\[
\int_{E \in Gr_2(V)} i_{E^*}(\eta(E)) = \mathbb{F}_V \left( \int_{E \in Gr_2(V)} (i_E^\top)(\mathbb{F}_E^{-1}(\eta(E))) \right).
\]

The last equality is a special case of Theorem 6.2.4. Case 2a is proved.

**Case 2a.** Assume that \( i = 1 \).

Let us fix an arbitrary surjection \( p: V \to W \) of rank \( k \). We have
\[
p^\top\left( (\mathbb{F}_W \otimes Id_{Dens(W^*)}) \left( \int_{E \in Gr_k(V)} \eta(E) \right) \right) \text{ Prop. (6.3.14)} = \quad (6.3.21)
\]
\[
(\mathbb{F}_W \otimes Id_{Dens(W^*)}) \left( p_* \left( \int_{E \in Gr_k(V)} i_{E^*}(\eta(E)) \right) \right) = \quad (6.3.22)
\]
\[
(\mathbb{F}_W \otimes Id_{Dens(W^*)}) \left( \int_{E \in Gr_k(V)} (p \circ i_E)_*(\eta(E)) \right) = \quad (6.3.23)
\]
\[
(\mathbb{F}_W \otimes Id_{Dens(W^*)}) \left( \int_{E \in Gr_k(V), E \cap Ker(p) = \{0\}} (p \circ i_E)_*(\eta(E)) \right) = \quad (6.3.24)
\]
\[
\int_{E \in Gr_k(V), E \cap Ker(p) = \{0\}} (p \circ i_E)^\top(\mathbb{F}_E \otimes Id_{Dens(E^*)})(\eta(E)) \quad (6.3.25)
\]

where the last equality follows from the fact that \( p \circ i_E: E \to W \) is an isomorphism for any \( E \in Gr_k(V) \) such that \( E \cap Ker(p) = \{0\} \). Next the expression (6.3.25) is equal to
\[
\int_{E \in Gr_k(V)} (p \circ i_E)^\top (\mathbb{F}_E \otimes Id_{Dens(E^*)})(\eta(E)) = \quad (6.3.26)
\]
\[
\int_{E \in Gr_k(V)} p^\top \left( i_E^\top(\mathbb{F}_E \otimes Id_{Dens(E^*)})(\eta(E)) \right) = \quad (6.3.27)
\]
\[
p^\top \left( \int_{E \in Gr_k(V)} i_E^\top(\mathbb{F}_E \otimes Id_{Dens(E^*)})(\eta(E)) \right). \quad (6.3.28)
\]
Thus we have shown that (6.3.21)=(6.3.28). In other words for any surjection \( p: V \to W \) of rank \( k \)
\[
p^\vee (\text{l.h.s. of (6.3.12)}) = p^\vee (\text{r.h.s. of (6.3.12)}).
\]
Since \( i = 1 \), this implies (6.3.12). Thus Case 2a is proved.

**Case 2b.** Assume now that \( i > 1 \).

Let us consider the (infinite dimensional) vector bundle \( \mathcal{H}^0 \) over the partial flag manifold \( \mathcal{F}_{k-i+1,k}(V) \) whose fiber over \((L,E) \in \mathcal{F}_{k-i+1,k}(V)\) is equal to \( \text{Val}_{1}^{-sm}(L) \otimes \text{Dens}(L^*) \).

\[
\mathcal{H} := \mathcal{H}^0 \otimes |\omega|_{\mathcal{F}_{k-i+1,k}(V)}.
\]

**6.3.7 Claim.** For any \( \eta \in C^\infty(\text{Gr}_k(V), S_{k,V;1}^-), k > l > 1 \), there exists \( \xi \in C^\infty(\mathcal{F}_{k-i+1,k}(V), \mathcal{H}) \) such that for any \( E \in \text{Gr}_k(V) \)
\[
\eta(E) = \int_{L \in \text{Gr}_k(V)} i_{E,L*}(\xi(L,E)). \tag{6.3.29}
\]

**Proof.** The map \( C^\infty(\mathcal{F}_{k-i+1,k}(V), \mathcal{H}) \) given by the right hand side of (6.3.29) is \( GL(V) \)-equivariant. Then the Irreducibility theorem, the Casselman-Wallach theorem, and Proposition 1.2.6 imply the claim. Q.E.D.

Next we have
\[
\int_{E \in \text{Gr}_k(V)} i_{E*}(\eta(E)) = \int_{E \in \text{Gr}_k(V)} \int_{L \in \text{Gr}_{k-i+1}(E)} i_{E*}(i_{E,L*}(\xi(L,E))) = \int_{E \in \text{Gr}_k(V)} \int_{L \in \text{Gr}_{k-i+1}(E)} i_{L*}(\xi(L,E)).
\]

Let us denote by \( \hat{\xi}(L) \) the inner integral in the last expression. Thus \( \hat{\xi} \in C^\infty(\text{Gr}_{k-i}(V), S_{k-i+1,V;1}^-) \). Hence by Case 2a
\[
(\mathbb{F}_V \otimes \text{Id}_{\text{Dens}(V^*)}) \left( \int_{E \in \text{Gr}_k(V)} i_{E*}(\eta(E)) \right) = \tag{6.3.30}
\]
\[
(\mathbb{F}_V \otimes \text{Id}_{\text{Dens}(V^*)}) \left( \int_{L \in \text{Gr}_{k-i+1}(V)} i_{L*}(\hat{\xi}(L)) \right) = \tag{6.3.31}
\]
\[
\int_{L \in \text{Gr}_{k-i+1}(V)} (i_{L}^\vee)^*(\mathbb{F}_L \otimes \text{Id}_{\text{Dens}(L^*)})(\hat{\xi}(L)). \tag{6.3.32}
\]
Using again Case 2a we obtain that the right hand side of the equality (6.3.12) is equal to
\[
\int_{E \in \text{Gr}_k(V)} (i^V_L)^* (\mathbb{F}_E \otimes \text{Id}_{\text{Dens}(E^*)}) \left( \int_{E \in \text{Gr}_{k-1+i}(E)} (i_{E,L})^*(\mathbb{F}_E \otimes \text{Id}_{\text{Dens}(E^*)})(\xi(L,E)) \right) = \\
\int_{E \in \text{Gr}_k(V)} (i^V_L)^* \left( \int_{E \in \text{Gr}_{k-1+i}(E)} (i_{E,L})^*(\mathbb{F}_E \otimes \text{Id}_{\text{Dens}(E^*)})(\xi(L,E)) \right) = \\
\int_{E \in \text{Gr}_k(V)} \int_{E \in \text{Gr}_{k-1+i}(E)} (i^V_L)^* (\mathbb{F}_E \otimes \text{Id}_{\text{Dens}(E^*)})(\xi(L,E)) = \\
\int_{E \in \text{Gr}_{k-1+i}(E)} (i^V_L)^* (\mathbb{F}_E \otimes \text{Id}_{\text{Dens}(E^*)})(\xi(L,E)) \text{ by (6.3.32)}
\]

\[
= (\mathbb{F}_V \otimes \text{Id}_{\text{Dens}(V^*)}) \left( \int_{E \in \text{Gr}_k(V)} i_{E,s}(\eta(E)) \right).
\]

Thus Case 2a is proved. Hence Case 2 is proved too, and Theorem 6.3.5 is proved. Q.E.D.

The following immediate reformulation of Theorem 6.3.5 will be useful in the proof of Theorem 6.4.1 below.

**6.3.8 Corollary.** Let \(0 \leq k \leq n\). Let \(\tilde{\eta} \in C^\infty(\text{Gr}_k(V), S_{k,V}) \otimes \text{Dens}(V)\). Then

\[
\mathbb{F}_V \left( \int_{E \in \text{Gr}_k(V)} (i_{E,s} \otimes \text{Id}_{\text{Dens}(V)})(\tilde{\eta}(E)) \right) =
\]

\[
\int_{E \in \text{Gr}_k(V)} (i^V_L)^* (\mathbb{F}_E \otimes \text{Id}_{\text{Dens}(V^*)})(\tilde{\eta}(E)) \tag{6.3.33}
\]

where \(i_{E,s} \otimes \text{Id}_{\text{Dens}(V)}\) in the left hand side is the map \((\text{Val}^{sm}(E) \otimes \text{Dens}(E^*)) \otimes \text{Dens}(V) \rightarrow \text{Val}^{sm}(V)\), and \(\mathbb{F}_E \otimes \text{Dens}(V/E)\) in the right hand side is the map

\[
\text{Val}^{sm}(E) \otimes \text{Dens}(E^*) \otimes \text{Dens}(V) = \text{Val}^{sm}(E) \otimes \text{Dens}(V/E) \xrightarrow{\mathbb{F}_E \otimes \text{Id}_{\text{Dens}(V/E)}} \text{Val}^{sm}(E^*) \otimes \text{Dens}(E) \otimes \text{Dens}(V/E) = \text{Val}^{sm}(E^*) \otimes \text{Dens}(V).
\]

**6.3.9 Theorem (Plancherel formula).**

\[
(\mathbb{F}_V \otimes \text{Id}_{\text{Dens}(V)}) \circ \mathbb{F}_V = \mathcal{E}_V.
\]

**Proof.** In the even case this theorem is precisely Lemma 6.3.3. Let us consider the odd \(i\)-homogeneous case

\[
(\mathbb{F}_V \otimes \text{Id}_{\text{Dens}(V)}) \circ \mathbb{F}_V : \text{Val}^{sm}_{-1}(V) \rightarrow \text{Val}^{sm}_i(V).
\]

If \(i = 1\) the result is just Lemma 6.3.2. If \(i = n - 1\) then the result follows immediately from the definition of the Fourier transform \(\mathbb{F}_V : \text{Val}^{sm}_{n-1}(V) \rightarrow \text{Val}^{sm}_{-1}(V^*) \otimes \text{Dens}(V)\) since \(\mathbb{F}_V := - (\mathbb{F}_{V^*} \otimes \text{Id}_{\text{Dens}(V)})^{-1}\).

Let us assume that \(2 \leq i \leq n - 2\). Fix \(\phi \in \text{Val}^{sm}_{i-1}(V)\). By Proposition 6.1.5 there exists \(\xi \in C^\infty(\text{Gr}_{n-i-1}(V), T_{i+1,V,i}^1)\) such that \(\phi = \Xi_{i+1,V}(\xi)\). By the definition of \(\mathbb{F}_V\) in this case,

\[
\mathbb{F}_V(\xi) = \int_{F \in \text{Gr}_{n-i-1}(V)} p_{F,i}(\mathbb{F}_{V/F}(\xi(F))).
\]
By Theorem 6.3.5 we have
\[(\mathbb{F}_V \otimes \text{Id}_{\text{Dens}(V^*)})(\mathbb{F}_V(\phi)) = \quad (6.3.35)\]
\[
\int_{F \in \text{Gr}_{n-1}(V)} p_F^F \left( (\mathbb{F}_{(V/F)^*} \otimes \text{Id}_{\text{Dens}(V/F)})(\mathbb{F}_{V/F}(\xi(F))) \right) = \quad (6.3.36)
\]
\[- \int_{F \in \text{Gr}_{n-1}(V)} p_F^F(\xi(F)) \quad (6.3.37)\]
where the last equality holds by the proved above case corresponding to \(n = i + 1\). But the expression (6.3.37) is equal to \(-\Xi_{i+1,V}(\xi) = -\phi\). Hence theorem is proved. Q.E.D.

Let us fix now a Euclidean metric on \(V\). This induces isomorphisms
\[V^* \to V, \text{Dens}(V) \to \mathbb{C}. \quad (6.3.38)\]
Under these identifications \(\mathbb{F}_V : \text{Val}^{sm}(V) \to \text{Val}^{sm}(V)\). From Theorem 6.3.9 we easily deduce

6.3.10 Corollary. Under the identifications (6.3.38) one has \(\mathbb{F}_V^4 = \text{Id}\). Moreover \(\mathbb{F}_V^2 \neq \text{Id}\) provided \(\dim V > 1\).

6.4 Homomorphism property of the Fourier transform in higher dimensions.

6.4.1 Theorem. The map
\[ \mathbb{F}_V : \text{Val}^{sm}(V) \to \text{Val}^{sm}(V^*) \otimes \text{Dens}(V) \]
is an isomorphism of algebras.

Proof. It remains to show that \(\mathbb{F}_V\) is a homomorphism of algebras, i.e.
\[ \mathbb{F}_V(\phi \cdot \psi) = \mathbb{F}_V(\phi) \ast \mathbb{F}_V(\psi) \quad (6.4.1)\]
for any \(\phi, \psi \in \text{Val}^{sm}(V)\). We will consider a number of cases.

Case 1. Assume that \(\phi\) and \(\psi\) are even.

This case was proved by Bernig and Fu [15]. We will not prove this case here though it can be proved similarly to Case 2 below, but simpler.

Case 2. Assume that \(\phi \in \text{Val}^{+\cdot sm}_i(V), \psi \in \text{Val}^{-\cdot sm}_j(V)\).

We will need a general lemma.

6.4.2 Lemma. Let \(F \subset V\) be a linear subspace. Let \(i_F : F \hookrightarrow V, p_F : V \twoheadrightarrow V/F\) be the canonical imbedding and projection maps. Let \(E \subset V\) be another linear subspace, and let \(p_E : V \twoheadrightarrow V/E\) be the canonical projection.

Let \(\mu \in \text{Dens}(V/F) \subset \text{Val}^{sm}(V/F), \zeta \in \text{Val}^{sm}(V/E)\). Then
\[ (p_F \times p_E)^*(\mu \boxtimes \zeta) = (i_F^* \otimes \text{Id}_{\text{Dens}(V)})((p_E \circ i_F)^*(\zeta) \otimes \mu) \quad (6.4.2)\]
where
\[ p_F \times p_E : V \twoheadrightarrow V/F \times V/E, \]
\[ (p_E \circ i_F)^*(\zeta) \otimes \mu \in \text{Val}(F) \otimes \text{Dens}(V/F) = (\text{Val}(F) \otimes \text{Dens}(F^*)) \otimes \text{Dens}(V), \]
\[ i_F^* \otimes \text{Id}_{\text{Dens}(V)} : (\text{Val}(F) \otimes \text{Dens}(F^*)) \otimes \text{Dens}(V) \twoheadrightarrow \text{Val}(V). \]
Proof. By the McMullen’s conjecture we may assume that
\[ \zeta(\bullet) = \text{vol}_{V/E}(\bullet + A) \]
where \( \text{vol}_{V/E} \) is a Lebesgue measure on \( V/E \), and \( A \in K^{sm}(V/E) \). Let us compute first the left hand side of (6.4.2). One has
\[
(p_F \times p_E)^*(\mu \boxtimes \zeta)(K) = (6.4.3)
\]
\[
(\mu \boxtimes \zeta)((p_F \times p_E)(K)) = (6.4.4)
\]
\[
\int_{z \in V/F} \text{vol}_{V/E}(((p_F \times p_E)(K) \cap \{z\} \times V/E) + A) d\mu(z). (6.4.5)
\]
Let us compute now the right hand side of (6.4.2). One has
\[
(i_F^* \otimes \text{Id}_{\text{Dens}(V)})(((p_E \circ i_F)^* (\zeta) \otimes \mu)(K) = (6.4.6)
\]
\[
\int_{z \in V/F} \zeta((p_E \circ i_F)(K \cap z)) d\mu(z). (6.4.7)
\]
\[
\int_{z \in V/F} \text{vol}_{V/E}(p_E(K \cap z) + A) d\mu(z). (6.4.8)
\]
But obviously
\[
\{z\} \times p_E(K \cap z) = (p_F \times p_E)(K) \cap \{z\} \times V/E.
\]
Substituting this into (6.4.8) and comparing with (6.4.5) we get that
\[
6.4.3 = 6.4.6.
\]
Thus lemma is proved. Q.E.D.

Let us return now to the proof of Case 2 of the theorem. First observe that if \( i + j > n - 1 \) then both sides of (6.4.1) vanish. Thus we will assume that \( i + j \leq n - 1 \).

Case 2a. Let us assume in addition that \( i + j = n - 1 \).

By Proposition 6.1.5 there exist \( \mu \in C^\infty(Gr_{n-i}(V), T_{i,V}^+) \) and \( \zeta \in C^\infty(Gr_i(V), T_{j+1,V,j}^-) \) such that \( \phi = \Xi_{i,V}(\mu), \psi = \Xi_{j+1,V}^{-1}(\zeta) \). In other words
\[
\phi = \int_{E \in Gr_{n-i}(V)} p_E^*(\mu(E)),
\]
\[
\psi = \int_{F \in Gr_i(V)} p_F^*(\zeta(F)).
\]
Then we have
$$
F_V(\phi \cdot \psi) = \text{Prop. 6.1.3, Lemma 6.4.2 (6.4.9)}
$$
$$
F_V \left( \int_{E \in \text{Gr}_{n-i}(V)} \int_{F \in \text{Gr}_i(V)} (p_F \times p_E)^* (\mu(F) \boxtimes \zeta(E)) \right) = \text{Lemma 6.4.2 (6.4.10)}
$$
$$
F_V \left( \int_{E \in \text{Gr}_{n-i}(V)} \int_{F \in \text{Gr}_i(V)} (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) \left( ((p_E \circ i_F)^* (\zeta(E)) \otimes \mu(F) ) \right) \right) = \text{Lemma 6.4.2 (6.4.11)}
$$

Let us denote
$$
Z := \{(E, F) \in \text{Gr}_{n-i}(V) \times \text{Gr}_i(V) \mid E \cap F = 0\}.
$$

Thus $Z \subset \text{Gr}_{n-i}(V) \times \text{Gr}_i(V)$ is an open dense subset whose complement is a (singular) subvariety of positive codimension. Then clearly
$$
\text{Lemma 6.4.11} = F_V \left( \int_Z (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) \left( ((p_E \circ i_F)^* (\zeta(E)) \otimes \mu(F) ) \right) \right).
$$

Let us denote for brevity
$$
\omega(E, F) := (p_E \circ i_F)^* (\zeta(E)) \otimes \mu(F).
$$

It is easy to see that
- $\omega$ is a continuous section of the bundle $\mathcal{T}_{i,V;i}^+ \boxtimes \mathcal{T}_{j+1,V;j}$ over $\text{Gr}_{n-i}(V) \times \text{Gr}_i(V)$;
- $\omega$ is smooth over $Z$.

Let us fix a sequence $\gamma_N : \text{Gr}_{n-i}(V) \times \text{Gr}_i(V) \to [0, 1], N \in \mathbb{N}$, of $C^\infty$-smooth functions vanishing in a neighborhood (depending on $N$) of the complement of $Z$, and converging uniformly on compact subsets of $Z$ to the function 1 with all partial derivatives.

From the mentioned properties of $\omega$ and the choice of a sequence $\{\gamma_N\}$ it easily follows that
$$
\lim_{N \to \infty} \int_Z \gamma_N \cdot (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) \omega(E, F) = \int_Z (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) \omega(E, F)
$$
where the convergence is understood in the space $Val^{sm}(V)$. Hence
$$
\text{Lemma 6.4.11} = \lim_{N \to \infty} \int_Z \gamma_N \cdot (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) \left( ((p_E \circ i_F)^* (\zeta(E)) \otimes \mu(F) ) \right) = \text{Lemma 6.4.11 (6.4.12)}
$$
$$
\lim_{N \to \infty} \int_{E \in \text{Gr}_{n-i}(V)} \int_{F \in \text{Gr}_i(V)} \gamma_N \cdot (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) \left( ((p_E \circ i_F)^* (\zeta(E)) \otimes \mu(F) ) \right) = \text{Lemma 6.4.11 (6.4.12)}
$$

We may apply Corollary 6.3.8 to the last expression; it is equal to
$$
\lim_{N \to \infty} \int_{E \in \text{Gr}_{n-i}(V)} \int_{F \in \text{Gr}_i(V)} \gamma_N \cdot (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) \left( ((p_E \circ i_F)^* (\zeta(E)) \otimes \mu(F) ) \right).
$$

The last limit is clearly equal to
$$
\int_{(E,F) \in Z} (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) \left( (p_E \circ i_F)^* (\zeta(E)) \otimes \mu(F) \right) = \text{Lemma 6.4.11 (6.4.13)}
$$
$$
\int_{(E,F) \in Z} (i_{F*} \otimes \text{Id}_{\text{Dens}(V)}) (p_E \circ i_F)^* (\zeta(E)) \otimes \mu(F). \quad \text{(6.4.14)}
$$
Since for \((E, F) \in Z\) the map \(p_E \circ i_F: F \to V/E\) is an isomorphism, we have
\[
\mathcal{F}_F \circ (p_E \circ i_F)^* = (p_E \circ i_F)^* \circ \mathcal{F}_{V/E}.
\]
Hence
\[
(6.4.14) = \int_{\mathcal{Z}} (i_F^* \otimes \text{Id}_{\text{Dens}(V)}) (i_F^* (p_E^*(\mathcal{F}_{V/E} \xi(E))) \otimes \mu(F)) = \\
\int_{F \in \text{Gr}_1(V)} (i_F^* \otimes \text{Id}_{\text{Dens}(V)}) (i_F^* (\mathcal{F}_V \psi) \otimes \mu(F)).
\]

To summarize the above computation, we have obtained the equality
\[
\mathcal{F}_V(\phi \cdot \psi) = \int_{F \in \text{Gr}_1(V)} (i_F^* \otimes \text{Id}_{\text{Dens}(V)}) (i_F^* (\mathcal{F}_V \psi) \otimes \mu(F)). \tag{6.4.15}
\]

Hence to finish the proof of Case 2a it remains to prove the following lemma.

6.4.3 Lemma. Let \(\mu \in C^\infty(\text{Gr}_i(V), \mathcal{T}_{i,V})\). Let \(\phi = \int_{F \in \text{Gr}_1(V)} p_E^*(\mu(F))\). Let \(\xi \in \text{Val}^{sm}(V^*) \otimes \text{Dens}(V)\). Then
\[
\int_{F \in \text{Gr}_1(V)} (i_F^* \otimes \text{Id}_{\text{Dens}(V)}) (i_F^* (\xi) \otimes \mu(F)) = \xi * \mathcal{F}_V(\phi). \tag{6.4.16}
\]

Proof. First recall that \(\text{Dens}(V/F) = \text{Dens}(F^*) \otimes \text{Dens}(V)\). Under this identification, for any \(F \in \text{Gr}_i(V)\) we have \(\mu(F) \in \text{Dens}(F^*) \otimes \text{Dens}(V) \otimes |\omega_{\text{Gr}_i(V)}||_F\). Next we have
\[
\mathcal{F}_V(\phi) = \int_{F \in \text{Gr}_1(V)} (i_F^* \otimes \text{Id}_{\text{Dens}(V)}) (\mu(F)).
\]
Let us fix a Lebesgue measure \(\text{vol}_V\) on \(V\). Then by Proposition 6.1.3
\[
\xi \boxtimes \mathcal{F}_V(\phi) = \int_{F \in \text{Gr}_1(V)} \xi \boxtimes (i_F^* \otimes \text{Id}_{\text{Dens}(V)}) (\mu(F)) = \\
\int_{F \in \text{Gr}_1(F)} \xi \boxtimes (i_F^* (\mu(F) \otimes \text{vol}_V^{-1}) \otimes \text{vol}_V).
\]
Hence, using Proposition 3.3.2 we get
\[
\xi * \mathcal{F}_V(\phi) = a_*(\xi \boxtimes \mathcal{F}_V(\phi)) = \\
\int_{F \in \text{Gr}_1(F)} a_* (\xi \boxtimes (i_F^* (\mu(F) \otimes \text{vol}_V^{-1}) \otimes \text{vol}_V))
\]
where \(a: V^* \times V^* \to V^*\) is the addition map. Thus in order to prove the lemma, it suffices to show that for any \(F \in \text{Gr}_i(V)\), any \(\xi \in \text{Val}^{sm}(V^*) \otimes \text{Dens}(V)\), and any \(\mu \in \text{Dens}(V/F) = \text{Dens}(F^*) \otimes \text{Dens}(V)\) one has
\[
a_*(\xi \boxtimes (i_F^* (\mu \otimes \text{vol}_V^{-1}) \otimes \text{vol}_V)) = (i_F^* \otimes \text{Id}_{\text{Dens}(V)}) (i_F^* (\xi) \otimes \mu). \tag{6.4.17}
\]
Let us fix also a Lebesgue measure \( \text{vol}_F \) on \( F \). Let us denote \( \text{vol}_{V/F} := \frac{\text{vol}_V}{\text{vol}_F} \) the corresponding Lebesgue measure on \( V/F \). It is enough to prove the equality (6.4.17) for \( \mu = \text{vol}_{V/F} \). Then \( \mu \otimes \text{vol}_V^{-1} = \text{vol}_{F}^{-1} \in \text{Dens}(F^*) \). Furthermore, by the McMullen’s conjecture we may assume that \( \xi(\bullet) = \text{vol}_{V}^{-1}(\bullet + A) \otimes \text{vol}_V \) where \( A \in \mathcal{K}^{sm}(V^*) \).

Let us fix \( K \in \mathcal{K}(V^*) \). Then we have

\[
(\text{r.h.s. of } (6.4.17))(K) = (i_F^\vee(\xi) \otimes \mu)(i_F^\vee(K)) = \tag{6.4.18}
\]

\[
(\text{vol}_F^{-1}(i_F^\vee(K) + i_F^\vee(A)) \otimes \text{vol}_F) \otimes \text{vol}_{V/F} = \tag{6.4.19}
\]

\[
\text{vol}_F^{-1}(i_F^\vee(K + A)) \otimes \text{vol}_V. \tag{6.4.20}
\]

On the other hand we have

\[
\xi \boxtimes (i_F^\vee(\mu \otimes \text{vol}_V^{-1}) \otimes \text{vol}_V) = \tag{6.4.21}
\]

\[
(\text{vol}_V^{-1}(\bullet + A) \otimes \text{vol}_V) \boxtimes (i_F^\vee(\text{vol}_F^{-1}) \otimes \text{vol}_V). \tag{6.4.22}
\]

Let us fix \( S \in \mathcal{K}(\text{Ker}(i_F^\vee)) = \mathcal{K}((V/F)^*) \) with \( \text{vol}_{V/F}(S) = 1 \). Then by Lemma 2.6.1 we have

\[
(i_F^\vee(\text{vol}_F^{-1}))(K) = \text{vol}_F^{-1}(i_F^\vee(K)) = \frac{1}{(n - i)!} \frac{d^{n-i}}{d\varepsilon^{n-i}}|_{\varepsilon=0} \text{vol}_V^{-1}(K + \varepsilon S). \tag{6.4.23}
\]

Substituting (6.4.23) into (6.4.22) and using the definition of pushforward we get

\[
(\text{l.h.s. of } (6.4.17)) = \text{vol}_V^{-1}(i_F^\vee(K + A)) \otimes \text{vol}_V. \tag{6.4.24}
\]

Comparing (6.4.24) and (6.4.20) we conclude the equality (6.4.17). Hence Lemma 6.4.3 is proved. Q.E.D.

Case 2a is proved as well.

Case 2b. Let us assume now that \( i + j + 1 < n \).

By Schneider’s theorem [40] every odd \((i + j + 1)\)-homogeneous valuation is uniquely determined by its restrictions to all \((i + j + 1)\)-dimensional subspaces. Hence it is enough to check that for any \((i + j + 1)\)-dimensional subspace \( i: L \hookrightarrow V \) one has

\[
i^*(\phi \cdot \psi) = i^*(F_V^{-1}(F_V \phi * F_V \psi)).
\]

Let us compute the right hand side of the above equality:

\[
i^*(F_V^{-1}(F_V \phi * F_V \psi)) \xrightarrow{\text{Thm. 6.2.1}} F_L^{-1}(i^\vee(i^*_L(F_L \phi * F_L \psi))) \xrightarrow{\text{Prop. 3.4.1}} F_L^{-1}(i^\vee(F_L \phi) * i^\vee(F_L \psi)) \quad \text{Case 2a}
\]

\[
i^* \phi \cdot i^* \psi = i^*(\phi \cdot \psi).
\]

Thus Case 2b is proved. Hence Case 2 is proved too.

Case 3. Assume that \( \phi, \psi \in \text{Val}_1^{-sm}(V) \).

We will use the homomorphism property of the Fourier transform in two dimensions proved in Section 5.4. By Klain’s theorem [30] it is enough to show that for any 2-dimensional space \( E \) and an imbedding \( i: E \hookrightarrow V \) one has

\[
i^*(\phi \cdot \psi) = i^*(F_V^{-1}(F_V \phi * F_V \psi)). \tag{6.4.25}
\]
Let us compute the right hand side of (6.4.25):

\[ i^* \left( \mathbb{F}^{-1}_V (\mathbb{F}_V \phi * \mathbb{F}_V \psi) \right) \text{ Thm. 6.2.1} \]
\[ = \mathbb{F}^{-1}_L (i^\vee_*(\mathbb{F}_V \phi * \mathbb{F}_V \psi)) \text{ Prop. 3.4.1} \]
\[ = \mathbb{F}^{-1}_L (i^\vee_*(\mathbb{F}_V \phi) * i^\vee_*(\mathbb{F}_V \psi)) \text{ Thm. 6.2.1} \]
\[ = \mathbb{F}^{-1}_L (\mathbb{F}^{-1}_L (i^\vee(i^* \phi) * \mathbb{F}_L (i^* \psi))) \text{ Thm. 5.4.1 2) } \]
\[ i^* \phi * i^* \psi = i^*(\phi \cdot \psi). \]

Case 3 is proved.

Case 4. Let us prove the equality (6.4.1) in general.

The only remaining case is \( \phi \in Val_{i-}^{sm}(V), \psi \in Val_{j-}^{sm}(V), \) \( \text{and either } i > 1 \text{ or } j > 1. \) For any \( k \geq 1 \) the subspace \( Val_{i-}^{sm}(V) \cdot Val_{j-}^{sm}(V) \) is dense in \( Val_{i-}^{sm}(V) \) by the Irreducibility Theorem. Hence we may assume that \( \phi = \phi^- \cdot \phi^+ \) where \( \phi^- \in Val_{i-}^{sm}(V), \phi^+ \in Val_{i-1}^{+sm}(V) \); and similarly \( \psi = \psi^- \cdot \psi^+ \) where \( \psi^- \in Val_{i-}^{sm}(V), \psi^+ \in Val_{j-1}^{+sm}(V). \) Then we have

\[ \mathbb{F}_V (\phi \cdot \psi) = \mathbb{F}_V ((\phi^+ \cdot \psi^+) \cdot (\phi^- \cdot \psi^-)) \text{ Case 1} \]
\[ \mathbb{F}_V (\phi^+ \cdot \psi^+) * \mathbb{F}_V (\phi^- \cdot \psi^-) \text{ Cases 1,3} \]
\[ = \mathbb{F}_V (\phi^+) * \mathbb{F}_V (\psi^+) * \mathbb{F}_V (\phi^-) * \mathbb{F}_V (\psi^-) \text{ Case 2} \]
\[ \mathbb{F}_V (\phi^+ \cdot \phi^-) * \mathbb{F}_V (\psi^+ \cdot \psi^-) = \mathbb{F}_V (\phi) * \mathbb{F}_V (\psi). \]

Theorem 6.4.1 is proved. Q.E.D.

6.4.4 Remark. The Fourier transform \( \mathbb{F}_V \) we have constructed is not quite canonical. More precisely, let us fix \( n > 1. \) Let \( C_n \) denote the category whose objects \( Ob(C_n) \) are \( n \)-dimensional real vector spaces, and morphisms between them are linear isomorphisms. Assume that for any object \( V \) of \( C_n \) we are given an isomorphism \( \mathbb{F}_V : Val^{sm}(V) \rightarrow Val^{sm}(V^*) \otimes Dens(V) \) of linear topological spaces such that

- for any morphism \( f : V \rightarrow W \) in \( C_n \) (i.e. \( f \) is just a linear isomorphism) the following diagram is commutative

\[
\begin{array}{ccc}
Val^{sm}(V) & \longrightarrow & Val^{sm}(W) \\
\mathbb{F}_V & \downarrow & \mathbb{F}_W \\
Val^{sm}(V^*) \otimes Dens(V) & \longrightarrow & Val^{sm}(W^*) \otimes Dens(W)
\end{array}
\]

where the horizontal arrows are obvious isomorphisms induced by the isomorphisms \( V^f \rightarrow W \) and \( W^* \xrightarrow{f^*} V^* \) where \( f^* \) is the dual of \( f; \)
- for any \( V \in Ob(C_n) \) the map \( \mathbb{F}_V \) is an isomorphism of algebras when the source is equipped with the product and the target with the convolution;
- for any \( V \in Ob(C_n) \) one has the Plancherel type formula as in Theorem 0.1.3(3).

Then one can show that there exist exactly four families of maps \( \{ \mathbb{F}_V \}_{V \in Ob(C_n)} \) satisfying the above conditions. The difficult part (which is the main subject of this article) is to prove existence of at least one such a family.
7 A hard Lefschetz type theorem for valuations.

Let $V$ be a Euclidean $n$-dimensional space. Let $V_1 \in \text{Val}_1(V)$ denote the first intrinsic volume (see e.g. [39], p. 210). This valuation is invariant under the orthogonal group and it is smooth. The main result of this section is the following theorem.

7.1.1 Theorem (hard Lefschetz type theorem). Let $0 \leq i < n/2$. Then the map

$$\text{Val}_i^{sm}(V) \to \text{Val}_n^{sm-i}(V)$$

given by $\phi \mapsto (V_1)^{n-2i} \cdot \phi$ is an isomorphism.

7.1.2 Remark. For even valuations this result was proved first by the author in [7].

Before we prove Theorem 7.1.1 we need some preparations. In [4] the author has introduced an operator

$$\Lambda: \text{Val}(V) \to \text{Val}(V)$$

defined by $(\Lambda \phi)(K) = \frac{d}{d\varepsilon}|_{\varepsilon=0} \phi(K + \varepsilon \cdot D)$ for any $\phi \in \text{Val}(V)$, $K \in \mathcal{K}(V)$. Note that $\phi(K + \varepsilon \cdot D)$ is a polynomial in $\varepsilon \geq 0$ of degree at most $n$ by a result of McMullen [33]. The operator $\Lambda$ decreases the degree of homogeneity by 1. We are going to use the following theorem which was proved by the author [4] for even valuations and by Bernig and Bröcker [14] in general.

7.1.3 Theorem. Let $n \geq i > n/2$. The operator

$$\Lambda^{2i-n}: \text{Val}_i^{sm}(V) \to \text{Val}_n^{sm-i}(V)$$

is an isomorphism.

Next, the Euclidean metric on $V$ induces the identifications $V \to V^*$ and $\text{Dens}(V) \to \mathbb{C}$. Under these identifications the Fourier transform acts $\mathbb{F}_V: \text{Val}^{sm}(V) \to \text{Val}^{sm}(V)$. We will need the following lemma which was observed by Bernig and Fu in [15], Corollary 1.9, in the case of even valuations.

7.1.4 Lemma. For any $\phi \in \text{Val}^{sm}(V)$ one has

$$V_1 \cdot \phi = \kappa (\mathbb{F}_V^{-1} \circ \Lambda \circ \mathbb{F}_V)(\phi)$$

where $\kappa$ is a non-zero constant depending on $n$ only.

Proof. The proof is essentially the same as in the even case [15], once one has the Fourier transform. By the homomorphism property of the Fourier transform we have

$$\mathbb{F}_V(V_1 \cdot \phi) = \mathbb{F}_V(V_1) * \mathbb{F}_V(\phi). \quad (7.1.1)$$

Observe that $\mathbb{F}_V(V_1)$ is an $O(n)$-invariant valuation homogeneous of degree $n-1$. Hence by the Hadwiger characterization theorem [24] it must be proportional to the $(n-1)$-th intrinsic
volume $V_{n-1}$, which is proportional to the valuation $K \mapsto \frac{d}{d\varepsilon}|_{\varepsilon=0} \text{vol}(K + \varepsilon D)$. Next observe that for any $A \in K^{sm}(V)$ and any $\phi \in Val^{sm}(V)$

$$vol(\bullet + A) \ast \phi = \phi(\bullet + A).$$

(7.1.2)

Indeed the equality (7.1.2) is easily checked for $\phi$ of the form $\phi(\bullet) = vol(\bullet + B)$, and the general case follows from the McMullen’s conjecture. Hence $F_{V}(V_{1}) \ast F_{V}(\phi)$ is proportional to

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}(F_{V}\phi)(\bullet + \varepsilon D) = \Lambda(F_{V}\phi).$$

(7.1.3)

Then Lemma 7.1.4 follows from (7.1.1), (7.1.3). Q.E.D.

Proof of Theorem 7.1.1. It follows immediately from Theorem 7.1.3 and Lemma 7.1.4. Q.E.D.

8 Appendix: a remark on exterior product on valuations.

In this appendix we will prove a slightly more refined statement on the exterior product of valuations than it appears in [5]. Though for the purposes of this article we need only the case of translation invariant valuations, we will prove the result in a greater generality of polynomial valuations following [5].

Let us remind the definition of a polynomial valuation introduced by Khovanskii and Pukhlikov in [28]. Let $V$ be an $n$-dimensional real vector space.

8.1.1 Definition. A valuation $\phi$ is called polynomial of degree at most $d$ if for every $K \in K(V)$ the function $x \mapsto \phi(K + x)$ is a polynomial on $V$ of degree at most $d$.

Note that valuations polynomial of degree 0 are just translation invariant valuations. Polynomial valuations have many nice combinatorial-algebraic properties ([28], [29]).

Let $PVal_{d}(V)$ denote the space of continuous valuations on $V$ which are polynomial of degree at most $d$. It is a Fréchet space (in fact a Banach space) with the topology of uniform convergence on compact subsets of $K(V)$. Let $\Omega_{d}^{n}(V)$ denote the (finite dimensional) space of $n$-densities on $V$ with polynomial coefficients of degree at most $d$ (clearly $\Omega_{d}^{n}(V)$ is canonically isomorphic to $\bigoplus_{i=0}^{d} \text{Sym}^{i}V^{*} \otimes |\wedge^{n}V^{*}|$ where $|\wedge^{n}V^{*}|$ denotes the space of Lebesgue measures on $V$).

The group $GL(V)$ acts naturally on $PVal_{d}(V)$ as usual: $(g\phi)(K) = \phi(g^{-1}K)$. This action is continuous. The subspace of $GL(V)$-smooth vectors is denoted by $PVal_{d}^{sm}(V)$.

For a vector space $U$, a smooth measure $\mu$ on $U$, and $A \in K(U)$ let us denote by $\mu_{A}$ the valuation $[K \mapsto \mu(K + A)]$. Now let us state the main result of this appendix which refines Proposition 1.10 from [5].

8.1.2 Proposition. Let $V, W$ be finite dimensional real vector spaces. There exists a continuous bilinear map

$$PVal_{d}^{sm}(V) \times PVal_{d'}^{sm}(W) \rightarrow PVal_{d+d'}^{sm}(V \times W)$$
which is uniquely characterized by the property that for any polynomial measures \( \mu, \nu \) on \( V, W \) respectively, and any \( A \in \mathcal{K}(V), B \in \mathcal{K}^m(W) \) one has
\[
(\mu_A, \nu_B) \mapsto (\mu \boxtimes \nu)_{A \times B}
\]
where \( \mu \boxtimes \nu \) denotes the usual product measure. This map is called the exterior product and is denoted by \( \boxtimes \).

8.1.3 Remark. An important difference of this proposition in comparison to \([5]\) is that now we can consider the exterior product of a smooth valuation by a continuous one (and not just a product of two smooth valuations).

Before we prove this proposition let us introduce more notation and remind some constructions from \([5]\). Let us denote by \( \mathbb{P}_+(V^*) \) the manifold of oriented lines passing through the origin in \( V^* \). Let \( L \) denote the line bundle over \( \mathbb{P}_+(V^*) \) whose fiber over an oriented line \( l \) consists of linear functionals on \( l \).

We are going to remind the construction from \([5]\) of a natural linear map
\[
\Theta_{k,d}: \Omega^d(V) \otimes C^\infty([\mathbb{P}_+(V^*)]^k, L^\otimes k) \to PVal_d(V)
\]
which commutes with the natural action of the group \( GL(V) \) on both spaces and induces an epimorphism on the subspaces of smooth vectors.

The construction is as follows. Let \( \mu \in \Omega^d_d(V), A_1, \ldots, A_k \in \mathcal{K}(V) \). Then
\[
\int_{\sum_{j=1}^k \lambda_j A_j} \mu
\]
This can be easily seen directly, but it was also proved in general for polynomial valuations by Khovanskii and Pukhlikov \([28]\). Also it easily follows that the coefficients of this polynomial depend continuously on \( (A_1, \ldots, A_k) \in \mathcal{K}(V)^k \) with respect to the Hausdorff metric. Hence we can define a continuous map \( \Theta'_{k,d}: \Omega^d_d(V) \times \mathcal{K}(V)^k \to PVal_d(V) \) given by
\[
(\Theta'_{k,d}(\mu; A_1, \ldots, A_k))(K) := \frac{\partial^k}{\partial\lambda_1 \ldots \partial\lambda_k} \bigg|_{\lambda_j = 0} \int_{K + \sum_{j=1}^k \lambda_j A_j} \mu.
\]

It is clear that \( \Theta'_{k,d} \) is Minkowski additive with respect to each \( A_j \). Namely, say for \( j = 1, a, b \geq 0 \), one has
\[
\Theta'_{k,d}(\mu; aA_1 + bA_1', A_2, \ldots, A_k) = a\Theta'_{k,d}(\mu; A_1', A_2, \ldots, A_k) + b\Theta'_{k,d}(\mu; A_1'', A_2, \ldots, A_k).
\]

Remind that for any \( A \in \mathcal{K}(V) \) one defines the supporting functional \( h_A(y) := \sup_{x \in A}(y, x) \) for any \( y \in V^* \). Thus \( h_A \in \mathcal{C}(\mathbb{P}_+(V^*), L) \). Moreover it is well known (and easy to see) that \( A_N \to A \) in the Hausdorff metric if and only if \( h_{A_N} \to h_A \) in \( \mathcal{C}(\mathbb{P}_+(V^*), L) \). Also any section \( F \in \mathcal{C}^2(\mathbb{P}_+(V^*), L) \) can be presented as a difference \( F = G - H \) where \( G, H \in \mathcal{C}^2(\mathbb{P}_+(V^*), L) \) are supporting functionals of some convex compact sets and \( \max\{|G|_{C^2}, |H|_{C^2}| \leq c|F|_{C^2} \) where \( c \) is a constant. (Indeed one can choose \( G = F + R \cdot h_D, H = R \cdot h_D \) where \( D \) is the unit Euclidean ball, and \( R \) is a large enough constant depending on \( |F|_{C^2} \).) Hence we can uniquely extend \( \Theta'_{k,d} \) to a multilinear continuous map (which we will denote by the same letter):
\[
\Theta'_{k,d}: \Omega^d_d(V) \times (\mathcal{C}^2(\mathbb{P}_+(V^*), L))^k \to PVal_d(V).
\]
By Theorem 1.5.1 it follows that this map gives rise to a continuous linear map
\[ \Theta_{k,d} : \Omega^n_d(V) \otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\otimes k}) \to PV al_d(V). \]

Since \( \Theta_{k,d} \) commutes with the action of \( GL(V) \), its image is contained in \( PV al^{sm}_d(V) \). Thus we got a continuous map
\[ \Theta_{k,d} : \Omega^n_d(V) \otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\otimes k}) \to PV al^{sm}_d(V). \]
which we wanted to construct.

We will study this map \( \Theta_{k,d} \). Note that it depends on \( k \) and \( d \) which will be fixed from now on. Let us denote by \( \Theta_d \) the sum of the maps \( \bigoplus_{k=0}^n \Theta_{k,d} \). Thus
\[ \Theta_d : \Omega^n_d(V) \otimes \left( \bigoplus_{k=0}^n C^\infty(\mathbb{P}_+(V^*)^k, L^{\otimes k}) \right) \to PV al^{sm}_d(V). \]

The following result was proved by the author in [3], Corollary 1.9.

**8.1.4 Lemma ([3]).** The map \( \Theta_d \) is onto \( PV al^{sm}_d(V) \).

Since the source and the target spaces of \( \Theta_d \) are Fréchet spaces, by the open mapping theorem (see e.g. [37], Ch. III, §2) the topology on \( PV al^{sm}_d(V) \) is the quotient topology on \( \Omega^n_d(V) \otimes \left( \bigoplus_{k=0}^n C^\infty(\mathbb{P}_+(V^*)^k, L^{\otimes k}) \right) \).

**Proof** of Proposition 8.1.2. Denote \( n := \dim V, m := \dim W \). We have the following claim whose proof is easy and is omitted.

**8.1.5 Claim.** Let \( \phi \in PV al_d(V) \). Let \( \mu \in \Omega^n_d(W) \). Then the map \( \Psi_{\phi,\mu} : \mathcal{K}(V \times W) \to \mathbb{C} \) given by
\[ \Psi_{\phi,\mu}(K) := \int_{w \in W} \phi(K \cap (V \times \{w\}))d\mu(w) \]
is a continuous valuation polynomial of degree at most \( d + d' \).

Hence by a result of Khovanskii and Pukhlikov [28], \( \Psi_{\phi,\mu}(\sum_{i=1}^s \lambda_i K_i) \) is a polynomial in \( \lambda_1, \ldots, \lambda_s \geq 0 \) of degree at most \( d + d' + n + m \) for any \( K_1, \ldots, K_s \in \mathcal{K}(V \times W) \) (for translation invariant valuations this fact was proved earlier by McMullen [33]). Hence for any \( A_1, \ldots, A_k \in \mathcal{K}(W) \), \( K \in \mathcal{K}(V \times W) \) the expression
\[ \Psi_{\phi,\mu} \left( K + \left( \{0\} \times \sum_{i=1}^k \lambda_i A_i \right) \right) = \int_{w \in W} \phi \left( \left( K + \left( \{0\} \times \sum_{i=1}^k \lambda_i A_i \right) \right) \cap (V \times \{w\}) \right) d\mu(w) \]
is a polynomial in \( \lambda_1, \ldots, \lambda_k \geq 0 \) of degree at most \( d + d' + n + m \) (in particular, there is a uniform bound on the degree).

It easily follows that
\[ \frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \bigg|_0 \Psi_{\phi,\mu} \left( K + \left( \{0\} \times \sum_{i=1}^k \lambda_i A_i \right) \right) \]
is a continuous valuation with respect to \( K \in \mathcal{K}(V \times W) \). Moreover the map
\[ PV al_d(V) \times \Omega^n_d(W) \times \mathcal{K}(W)^k \to PV al_{d+d'}(V \times W) \quad (8.1.1) \]
given by

\[(\phi, \mu; A_1, \ldots, A_k) \mapsto \left[ K \mapsto \frac{\partial^k}{\partial \lambda_1 \ldots \partial \lambda_k} \bigg|_0 \Psi_{\phi,\mu} \left( K + \left( \{0\} \times \sum_{i=1}^k \lambda_i A_i \right) \right) \right] \]

is continuous. Also this map is Minkowski additive with respect to each \(A_j \in K(W)\). By the argument used in the construction of the map \(\Theta_d\), the map (8.1.1) extends (uniquely) to a multilinear continuous map

\[ PVal_d(V) \times \Omega^m_{d'}(W) \times C^\infty(\mathbb{P}(W^*), L)^k \rightarrow PVal_{d+d'}(V \times W). \tag{8.1.2} \]

By Theorem 1.5.1 the map (8.1.2) gives rise to a bilinear continuous map

\[ PVal_d(V) \times \Omega^m_{d'}(W) \otimes C^\infty(\mathbb{P}(W^*)^k, L^{\otimes k}) \rightarrow PVal_{d+d'}(V \times W). \tag{8.1.3} \]

Summing up over \(k = 0, \ldots, m\) we obtain a bilinear continuous map

\[ PVal_d(V) \times \left( \Omega^m_{d'}(W) \otimes \left( \bigoplus_{k=0}^m C^\infty(\mathbb{P}(W^*)^k, L^{\otimes k}) \right) \right) \rightarrow PVal_{d+d'}(V \times W). \tag{8.1.4} \]

8.1.6 Lemma. The map (8.1.4) factorizes (uniquely) as

\[ PVal_d(V) \times PVal_{d+d'}^sm(W) \rightarrow PVal_{d+d'}(V \times W) \]

**Proof.** If \(PVal_d(V)\) is replaced by \(PVal_d^{sm}(V)\), the corresponding result was proved in [5], and the obtained map

\[ PVal_{d+d'}^{sm}(V) \times PVal_{d+d'}^{sm}(W) \rightarrow PVal_{d+d'}(V \times W) \]

was exactly the exterior product. Our lemma follows from this fact and the continuity of the map (8.1.4) because \(PVal_{d+d'}^{sm}(V) \subset PVal_d(V)\) is a dense subspace. Lemma is proved. Q.E.D.

The map

\[ PVal_d(V) \times PVal_{d+d'}^sm(W) \rightarrow PVal_{d+d'}(V \times W) \]

from Lemma 8.1.6 is the map we need. Proposition 8.1.2 is proved. Q.E.D.
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