Consistency Decision

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April 2014
last revised May 16, 2014

Abstract

The consistency formula for set theory can be stated in terms of the free-variables theory of primitive recursive maps. Free-variable p.r. predicates are decidable by set theory, main result here, built on recursive evaluation of p.r. map codes and soundness of that evaluation in set theoretical frame: internal p.r. map code equality is evaluated into set theoretical equality. So the free-variable consistency predicate of set theory is decided by set theory, \( \omega \)-consistency assumed. By Gödel’s second incompleteness theorem on undecidability of set theory’s consistency formula by set theory under assumption of this \( \omega \)-consistency, classical set theory turns out to be \( \omega \)-inconsistent.

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1 Primitive recursive maps

Define the theory $\text{PR}$ of objects and p.r. maps as follows recursively as a subsystem of set theory $\text{T}$:

- the objects
  \[ 1 = \{0\}, \mathbb{N}, \mathbb{N} \times \mathbb{N}, \ldots, A, \ldots, B, A \times B \text{ etc.} \]
- the map constants
  \[ 0 : 1 \to \mathbb{N} \text{ (zero), } s = s(n) = n + 1 \text{ (successor), } \text{id}_A : A \to A \text{ (identities), } \Pi : A \to 1 \text{ (terminal maps), } l = l(a, b) = a : A \times B \to A, r = r(a, b) = b : A \times B \to B \text{ (left and right projections)}; \]
- closure against (associative) map composition,
  \[ g \circ f = (g \circ f)(a) = g(f(a)) : A \to B \to C; \]
- closure against forming the induced map \((f, g) = (f, g)(c) = (f(c), g(c)) : C \to A \times B\) into a product, for given components \(f : C \to A, g : C \to B, l \circ (f, g) = f, r \circ (f, g) = g; \]
• closure against forming the *iterated* map

\[ f^\$ = f^\$ (a, n) = f^n (a) : A \times \mathbb{N} \to A, \]
\[ f^0 (a) = \text{id}_A (a) = a, \]
\[ f^{sn} (a) = f^\$ (a, sn) = (f \circ f^\$)(a, n) = f (f^n (a, n)). \]

Furthermore \( \textbf{PR} \) is to inherit from \( T \) uniqueness of the *initialised iterated*, in order to inherit uniqueness in the following *full schema of primitive recursion*:

\[ g = g(a) : A \to B \text{ (initialisation),} \]
\[ h = h((a, n), b) : (A \times \mathbb{N}) \times B \to B \text{ (step)} \]
\[ f = f(a, n) : A \times \mathbb{N} \to B, \]
\[ f(a, 0) = g(a) \]
\[ f(a, sn) = h((a, n), f(a)) \]
\[ + \text{uniqueness of such p.r. defined map } f. \]

This schema allows in particular construction of *for* loops,

\[ \text{ for } i := 1 \text{ to } n \text{ do } \ldots \text{ od } \]

as for verification if a given text (code) is an (arithmetised) proof of a given coded assertion, Gödel’s p.r. formula 45. \( xBy \), \( x \) ist *Beweis von* \( y. \)

(Formel 46. \( \text{Bew } y = \exists xBy, x \) is provable, is not p.r.)
2 PR code sets and evaluation

The map code set—set of gödel numbers—we want to evaluate is 

\[ \text{PR} = \bigcup_{A,B} [A, B] \subset \mathbb{N} \text{ in } T, \quad [A, B] = [A, B]_{\text{PR}} \]

the set of p.r. map codes from A to B.

Together with evaluation on suitable arguments it is recursively defined as follows:

- Basic map constants ba in \text{PR}:
  - \[ \top 0 \in [1, \mathbb{N}] \subset \text{PR} \text{ (zero),} \]
    \[ ev(\top 0, 0) = 0, \]
  - \[ \top s \in [\mathbb{N}, \mathbb{N}] \text{ (successor),} \]
    \[ ev(\top s, n) = s(n) = n + 1, \]
  - For an object A \[ \top \text{id}_A \in [A, A] \text{ (identity),} \]
    \[ ev(\top \text{id}_A, a) = \text{id}_A(a) = a, \]
  - \[ \top \Pi_A \in [A, 1] \text{ (terminal map),} \]
    \[ ev(\top \Pi_A, a) = \Pi_A(a) = 0. \]
  - for objects A, B \[ \top l_{A,B} \in [A \times B, A] \text{ (left projection),} \]
    \[ ev(\top l_{A,B}, (a, b)) = l_{A,B}(a, b) = a, \]
  - \[ \top r_{A,B} \in [A \times B, B] \text{ (right projection),} \]
    \[ ev(\top r_{A,B}, (a, b)) = r_{A,B}(a, b) = b. \]
• For \( u \in [A, B], v \in [B, C] \) : \( v \odot u \in [A, C] \) (internal composition),
  \[
  ev(v \odot u, a) = ev(v, ev(u, a)).
  \]

• For \( u \in [C, A], v \in [C, B] \) : \( \langle u; v \rangle \in [C, A \times B] \) (induced map code into a product),
  \[
  ev(\langle u; v \rangle, c) = (ev(u, c), ev(v, c)).
  \]

• For \( u \in [A, A] \) : \( u^g \in [A \times \mathbb{N}, A] \) (iterated map code),
  \[
  ev(u^g, 0) = id_A(a) = a,
  ev(u^g, sn) = ev(u, ev(u^g, n)) \quad \text{(double recursion)}
  \]

This recursion terminates in set theory \( T \), with correct results:

**Objectivity Theorem:** Evaluation \( ev \) is objective, i.e. for \( f : A \to B \) in \( PR \) we have

\[
ev(\mathbf{\ulcorner} f \mathbf{\urcorner}, a) = f(a).
\]

**Proof** by substitution of codes of \( PR \) maps into code variables \( u, v \in PR \subset \mathbb{N} \) in the above double recursive definition of evaluation, in particular:

• composition
  \[
  ev(\mathbf{\ulcorner} g \mathbf{\urcorner} \odot \mathbf{\ulcorner} f \mathbf{\urcorner}, a) = ev(\mathbf{\ulcorner} g \mathbf{\urcorner}, ev(\mathbf{\ulcorner} f \mathbf{\urcorner}, a)),
  = g(f(a)) = (g \circ f)(a)
  \]
  recursively, and
iteration

\[ ev(\Gamma f^{-\$}, \langle a; sn \rangle) = ev(\Gamma f^{-}, ev(\Gamma f^{-\$}, \langle a; n \rangle)) \]
\[ = f(f^{\frac{n}{k}}(a, n)) = f(f^{n}(a)) = f^{sn}(a) \]

recursively.

3 \ PR soundness within set theory

Notion \( f =_{PR} g \) of p. r. maps is externally p. r. enumerated, by complexity of (binary) deduction trees.

Internalising—formalising—gives an internal notion of PR equality,

\[ u \equiv_k v \in PR \times PR \]

coming by \( k \)th internal equation proved by \( k \)th internal deduction tree \( dtree_k \).

**PR evaluation soundness theorem framed by set theory T**: For p. r. theory \( PR \) with its internal notion of equality \( \equiv \) we have:

(i) PR to T evaluation soundness:

\[ T \vdash u \equiv v \implies ev(u, x) = ev(v, x) \quad (\bullet) \]

Substituting in the above “concrete” \( PR \) codes into \( u \) resp. \( v \), we get, by objectivity of evaluation \( ev \):

(ii) \( T \)-framed objective soundness of \( PR \):

For p. r. maps \( f, g : A \to B \)

\[ T \vdash \Gamma f^{-} \equiv \Gamma g^{-} \implies f(a) = g(a). \]
(iii) Specialising to case $f := \chi : A \to 2 = \{0, 1\}$ a p.r. predicate, and to $g := \text{true}$, we get

$T$-framed logical soundness of $\text{PR}$:

$$T \vdash \exists k \text{Prov}_\text{PR}(k, \neg \chi ^\uparrow) \implies \forall x \chi(x) :$$

If a p.r. predicate is—within $T$—$\text{PR}$-internally provable, then it holds in $T$ for all of its arguments.

Proof by primitive recursion on $k$, $\text{dtree}_k$ the $k$th deduction tree of the theory, proving its root equation $u \equiv_k v$. These (argument-free) deduction trees are counted in lexicographical order.

Super Case of equational internal axioms, in particular

- associativity of (internal) composition:

$$\langle \langle w \odot v \rangle \odot u \rangle \equiv \langle w \odot \langle v \odot u \rangle \rangle \implies$$

$$\begin{align*}
ev(\langle w \odot v \rangle \odot u, a) &= \ev(\langle w \odot v \rangle, \ev(u, a))
= \ev(w, \ev(\langle v \odot u \rangle, a))
= \ev(w, \ev(\langle v \odot u \rangle, a)) = \ev(w \odot \langle v \odot u \rangle, a).
\end{align*}$$

This proves assertion (●) in present associativity-of-composition case.

- Analogous proof for the other flat, equational cases, namely reflexivity of equality, left and right neutrality of identities, all substitution equations for the map constants, Godement’s equations for the induced map:

$$l \odot \langle u; v \rangle \equiv u, \ r \odot \langle u; v \rangle \equiv v,$$
as well as surjective pairing

\[ \langle l \circ w; r \circ w \rangle \cong w \]

and distributivity equation

\[ \langle u; v \rangle \circ w \cong \langle u \circ w; v \circ w \rangle \]

for composition with an induced.

- **proof** of (●) for the last equational case, the

*Iteration step*, case of genuine iteration equation

\[ u^\# \circ \langle \neg \text{id} \# \neg s \cong u \circ u^\# \rangle, \# \text{ the internal cartesian product of map codes:} \]

\[
\begin{align*}
T \vdash & \ ev(u^\# \circ \langle \neg \text{id} \# \neg s \cong u \circ u^\# \rangle, \langle a; n \rangle) \\
& = ev(u^\#, ev(\langle \neg \text{id} \# \neg s \cong u \circ u^\# \rangle, \langle a; n \rangle)) \\
& = ev(u^\#, \langle a; s n \rangle) \\
& = ev(u, ev(u^\#, \langle a; n \rangle)) \\
& = ev(u \circ u^\#, \langle a; n \rangle).
\end{align*}
\]

(2)

**Proof** of termination-conditioned inner soundness for the remaining genuine HORN case axioms, of form

\[ u \cong_i v \land u' \cong_j v' \Rightarrow w \cong_k w', \ i, j < k : \]

**Transitivity-of-equality** case

\[ u \cong_i v \land v \cong_j w \Rightarrow u \cong_k w : \]
Evaluate at argument \( a \in A \) and get in fact

\[
T \vdash u \cong_k w \\
\quad \rightarrow ev(u, a) = ev(v, a) \land ev(v, a) = ev(w, a)
\]

(by hypothesis on \( u, v \))

\[
\quad \rightarrow ev(u, a) = ev(w, a) :
\]

transitivity export q.e.d. in this case.

**Compatibility case** of composition with equality,

\[
u \cong u' \implies \langle v \circ u \rangle \cong \langle v \circ u' \rangle :
\]

\[
ev(v \circ u, a) = ev(v, ev(u, a)) = ev(v, ev(u', a))
\]

\[
= ev(v \circ u', x),
\]

by hypothesis on \( u \cong u' \) and by Leibniz’ substitutivity in \( T \), q.e.d. in this first compatibility case.

**Case** of composition with equality in second composition factor,

\[
v \cong_i v' \implies \langle v \circ u \rangle \cong_k \langle v' \circ u \rangle :
\]

\[
ev(\langle v \circ u \rangle, x) = ev(v, ev(u, x)) = ev(v', ev(u, x)) \quad (*)
\]

\[
= ev(\langle v' \circ u \rangle, x).
\]



(*) holds by \( v \cong v' \), induction hypothesis on \( v, v' \), and Leibniz’ substitutivity: same argument put into equal maps.

This proves soundness assertion (●) in this 2nd compatibility case.

(Redundant) Case of **compatibility** of forming the induced map, with equality, is analogous to compatibilities above,
even easier, since the two map codes concerned are independent from each other what concerns their domains.

**Final** Case of Freyd’s (internal) uniqueness of the initialised iterated, is case

\[
\langle w \circ \langle \text{id} \rangle ; r \circ \langle \Pi \rangle \rangle \equiv_i u \\
\land \langle w \circ \langle \text{id} \rangle \# r \rangle \equiv_j \langle v \circ w \rangle \\
\implies w \equiv_k v^* \circ \langle u \# \text{id} \rangle
\]

**Comment:** \(w\) is here an internal comparison candidate fullfilling the same internal p. r. equations as the initialised iterated \(\langle v^* \circ \langle u \# \text{id} \rangle \rangle\). It should be – **is:** soundness – evaluated equal to the latter, on \(A \times N\).

Soundness **assertion** (●) for the present Freyd’s uniqueness case recurs on \(\equiv_i, \equiv_j\) turned into predicative equations ‘=’, these being already deduced, by hypothesis on \(i, j < k\). Further ingredients are transitivity of ‘=’ and established properties of evaluation \(ev\).

So here is the remaining – inductive – **proof**, prepared by

\[
T \vdash ev(w, \langle a ; 0 \rangle ) = ev(u; a)
\]  
\((\bar{0})\)
as well as

\[
ev(w, \langle a ; sn \rangle ) = ev \left( w \circ \langle \text{id} \rangle \# r \langle s \rangle \right), \langle a ; n \rangle \\
= ev \left( v \circ w, \langle a ; n \rangle \right), \quad (\bar{s})
\]

the same being true for \(w' : = v^* \circ \langle u \# \text{id} \rangle\) in place of \(w\), once more by (characteristic) double recursive equations for \(ev\), this time with respect to the initialised internal iterated itself.
(0) and (s) put together for both then show, by induction on iteration count \( n \in \mathbb{N} \)—all other free variables \( u, v, w, a \) together form the passive parameter for this induction—soundness assertion (●) for this Freyd’s uniqueness case, namely

\[
T \vdash ev(w, \langle a; n \rangle) = ev(v^\$ \circ \langle u \# \text{id} \rangle, \langle a; n \rangle).
\]

**Induction** runs as follows:

**Anchor** \( n = 0 \):

\[
ev(w, \langle a; 0 \rangle) = ev(u, a) = ev(w', \langle a; 0 \rangle),
\]

**step:**

\[
\begin{align*}
&ev(w, \langle a; n \rangle) = ev(w', \langle a; n \rangle) \implies \varepsilon \\
&ev(w, \langle a; sn \rangle) = ev(v, ev(w, \langle a; n \rangle)) \\
&= ev(v, ev(w', \langle a; n \rangle)) = ev(w', \langle a; sn \rangle),
\end{align*}
\]

q. e. d.

4 PR-predicate decision

We consider here PR predicates for decidability by set theo-
rie(s) \( T \). Basic tool is \( T \)-framed soundness of PR just above, namely

\[
\chi = \chi(a) : A \to 2 \ \text{PR predicate}
\]

\[
T \vdash \exists k \ \text{Prov}_{PR}(k, \Gamma \chi^\gamma) \implies \forall a \chi(a).
\]

11
Within $\mathbf{T}$ define for $\chi : A \to 2$ out of $\mathbf{PR}$ a partially defined (alleged, individual) $\mu$-recursive decision $\nabla \chi : 1 \to 2$ by first fixing decision domain

$$ D = D_\chi := \{ k \in \mathbb{N} : \neg \chi(\text{ct}_A(k)) \lor \text{Prov}_{\mathbf{PR}}(k, \neg \chi) \}, $$

c $\text{ct}_A : \mathbb{N} \to A$ (retractive) Cantor count of $A$; and then, with (partial) recursive $\mu D : 1 \to \mathbb{N}$ within $\mathbf{T}$:

$$ \nabla \chi = \begin{cases} 
\text{false if } \neg \chi(\text{ct}_A(\mu D)) \\
(\text{counterexample}), \\
\text{true if } \text{Prov}_{\mathbf{PR}}(\mu D, \neg \chi) \\
(\text{internal proof}), \\
\perp (\text{undefined}) \text{ otherwise, i.e.} \\
\text{if } \forall a \chi(a) \land \forall k \neg \text{Prov}_{\mathbf{PR}}(k, \neg \chi).
\end{cases} $$

[This (alleged) decision is apparently $\mu$-recursive within $\mathbf{T}$, even if apriori only partially defined.]

There is a first consistency problem with this definition: are the defined cases disjoint?

Yes, within frame theory $\mathbf{T}$ which soundly frames theory $\mathbf{PR}$:

$$ \mathbf{T} \vdash (\exists k \in \mathbb{N}) \text{Prov}_{\mathbf{PR}}(k, \neg \chi) \implies \forall a \chi(a). $$

We show now, that decision $\nabla \chi$ is totally defined, the undefined case does not arise, this for $\mathbf{T}$ $\omega$-consistent in Gödel’s sense.

We have the following complete – metamathematical – case distinction on $D = D_\chi \subseteq \mathbb{N}$:
• **1st case**, termination: $D$ has at least one ("total") PR point $\mathbb{1} \rightarrow D \subseteq \mathbb{N}$, and hence

$$ t = t_{\nabla \chi} = \text{by def} \, \mu D = \min D : \mathbb{1} \rightarrow D $$

is a (total) p.r. point.

**Subcases:**

- **1.1**, negative (total) subcase:
  $$ \neg \chi \text{ct}_A(t) = \text{true.} $$
  [Then $T \vdash \nabla \chi$ = false.]

- **1.2**, positive (total) subcase:
  $$ \text{Prov}_\text{PR}(t, \lceil \chi \rceil) = \text{true.} $$
  [Then $T \vdash \nabla \chi$ = true,
  by $T$-framed objective soundness of $\text{PR}$.
  These two subcases are **disjoint**, disjoint here by $T$-framed soundness of theory $\text{PR}$ which reads

  $$ T \vdash \text{Prov}_\text{PR}(k, \lceil \chi \rceil) \Rightarrow \forall a \chi(a), \ k \text{ free,} $$

  here in particular – substitute $t : \mathbb{1} \rightarrow \mathbb{N}$ into $k$ free:

  $$ \pi R \vdash \text{Prov}_\text{PR}(t, \lceil \chi \rceil) \Rightarrow \forall a \chi(a). $$

  So furthermore, by this framed soundness, in present subcase:

  $$ T \vdash \forall a \chi(a) \land \text{Prov}_\text{PR}(t, \lceil \chi \rceil). \ (\bullet) $$

• **2nd case**, derived non-termination:

  $$ T \vdash D = \emptyset \equiv \{ \mathbb{N} : \text{false}_\mathbb{N} \} \subset \mathbb{N} $$

  [then in particular $T \vdash \forall a \neg \chi(a) = \text{false},$}
so $T \vdash \forall a \chi(a)$ in this case],
and furthermore

$$T \vdash \forall k \neg \text{Prov}_{\text{PR}}(k, \ulcorner \chi \urcorner), \text{ so }$$

$$T \vdash \forall a \chi(a) \land \forall k \neg \text{Prov}_{\text{PR}}(k, \ulcorner \chi \urcorner) \ (\ast)$$
in this case.

- **3rd**, remaining, *ill case* is:

  $D$ (metamathematically) *has no (total) points* $1 \to D$,
  *but is nevertheless not empty*.

  Take in the above the (disjoint) union of 2nd subcase of 1st case, ($\bullet$), and of 2nd case, ($\ast$), as new case. And formalise last, remaining case. Arrive at the following

  **Quasi-Decidability Theorem**: each p.r. predicate $\chi : A \to 2$ gives rise within set theory $T$ to the following complete (metamathematical) case distinction:

  (a) $T \vdash \forall a \chi(a)$ or else

  (b) $T \vdash \neg \chi \text{ct}_{At} : 1 \to D_{\chi} \to 2$

  *(defined counterexample)*, or else

  (c) $D = D_{\chi}$ *non-empty, pointless*, formally: in this case we would have within $T$ :

  $$T \vdash \exists \dot{a} \in D,$$

  and “nevertheless” for each p.r. point $p : 1 \to \mathbb{N}$

  $$T \vdash p \notin D.$$

  We **rule out** the latter – general – possibility of a *non-empty* predicate without p.r. points, for frame theory $T$ by
gödelian assumption of ω-consistency. In fact it rules out above instance of ω-inconsistency: all numerals 0, 1, 2, ... are p.r. points. Hence it rules out – in quasi-decidability above – possibility (c) for decision domain $D = D_\chi \subseteq \mathbb{N}$ of decision operator $\nabla$ for predicate $\chi : A \rightarrow \mathbb{2}$, and we get

**Decidability theorem:** Each free-variable p.r. predicate $\chi : A \rightarrow \mathbb{2}$ gives rise to the following complete case distinction by set theory $T$:

Under assumption of ω-consistency for $T$:

- $T \vdash \forall a \chi(a)$ (theorem) or
- $T \vdash (\exists a \in A) \neg \chi(a)$. (counterexample)

Now take here for predicate $\chi$, $T$’s own free-variable p.r. consistency formula

$$\text{Con}_T = \neg \text{Prov}_T(k, \lceil \text{false} \rceil) : \mathbb{N} \rightarrow \mathbb{2},$$

and get, under assumption of ω-consistency for $T$, a consistency decision $\nabla_{\text{Con}_T}$ for $T$ by $T$.

This contradiction to (the postcedent of) Gödel’s 2nd Incompleteness theorem shows that the assumption of ω-consistency for set theories $T$ must fail:

Set theories $T$ are ω-inconsistent.

This concerns all classical set theories as in particular PM, ZF, and NGB. The reason is ubiquity of formal quantification within these (arithmetical) theories.

**Problem:** Does it concern Peano Arithmetic either?
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