THE MODIFIED KÄHLER-RICCI FLOW AND
SOLITONS

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Abstract

We investigate the Kähler-Ricci flow modified by a holomorphic vector field. We find equivalent analytic criteria for the convergence of the flow to a Kähler-Ricci soliton. In addition, we relate the asymptotic behavior of the scalar curvature along the flow to the lower boundedness of the modified Mabuchi energy.

1 Introduction

Let $M$ be a compact Kähler manifold of complex dimension $n$ with $c_1(M) > 0$. A Kähler-Ricci soliton on $M$ is a Kähler metric $\omega = \frac{i}{2} g_{kj} dz^j \wedge \overline{dz}^k$ in the cohomology class $\pi c_1(M)$ together with a holomorphic vector field $X$ such that

$$\text{Ric}(\omega) - \omega = \mathcal{L}_X \omega. \quad (1.1)$$

Alternatively, in coordinate notation, writing $X_k = g_{k\ell} X^\ell$,

$$R_{kj} - g_{kj} = \nabla_j X_k. \quad (1.2)$$

Let $\Phi_t$ be the 1-parameter group of automorphisms of $M$ generated by $\text{Re} X$. The family of metrics $g_{kj}(t) \equiv \Phi_{-t}^*(g_{kj})$ provides then a solution of the Kähler-Ricci flow,

$$\dot{g}_{kj}(t) = -R_{kj} + g_{kj} \quad (1.3)$$

where the evolution in time is just by reparametrization.

If $X$ is the zero vector field then (1.1) reduces to the Kähler-Einstein equation. Kähler-Ricci solitons are in many ways similar to extremal metrics, which generalize constant scalar curvature Kähler metrics and are characterized by the condition that the vector field $\nabla^i R$ is holomorphic. Extremal metrics are the critical points of the Calabi functional $C(g_{kj}) = \| R - \bar{R} \|^2_{L^2}$, where $\bar{R}$ is the average of the scalar curvature. A classic conjecture of Yau [Y2] asserts that the existence of constant scalar curvature metrics in a given integral Kähler class should be equivalent to the stability of the polarization in the sense of geometric invariant theory. Notions of K-stability for constant scalar curvature metrics

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have been proposed by Tian [T] and Donaldson [D1], and extended to the case of extremal metrics by Szekelyhidi [S1] (see also [M]). Similarly, the existence of Kähler-Ricci solitons is expected to be equivalent to a suitable notion of stability.

Kähler-Ricci solitons can also be viewed as the stationary points of the modified Kähler-Ricci flow, that is,

$$\dot{g}_{kj} = -R_{kj} + g_{kj} + \nabla_j X_k$$

(1.4)

which is the flow (1.3) reparametrized by the automorphisms $\Phi_t$ generated by the real part $\text{Re} X$ of the holomorphic vector field $X$. Similar reparametrizations, in the context of Hamilton’s original Ricci flow [H], had been introduced by DeTurck [DeT] to give a simpler proof of the short-time existence of the flow.

The modified Kähler-Ricci flow appears in the work of Tian-Zhu [TZ2] as part of their study of the Kähler-Ricci flow assuming a priori the presence of a Kähler-Ricci soliton. They make use of a Moser-Trudinger type inequality from [CTZ] to deduce convergence of the flow in the sense of Cheeger-Gromov. (In the special case where there are no nontrivial holomorphic vector fields, it is known by the work of [P2], [TZ2], [PSSW2] that the existence of a Kähler-Einstein metric implies the exponential convergence of the Kähler-Ricci flow to that metric.)

In this paper, we study the long-time behavior of the modified Kähler-Ricci flow without assuming the existence of a Kähler-Ricci soliton. We give analytic conditions which are both necessary and sufficient for the convergence of the flow to a Kähler-Ricci soliton. These conditions are analogous to the ones given in [PSSW1] for the convergence of the Kähler-Ricci flow. As explained in [PS1] and [PSSW1] they can be interpreted as stability conditions in an infinite-dimensional geometric invariant theory, where the orbits are those of the diffeomorphism group acting on the space of almost-complex structures.

We provide now a description of our results. We will assume always that $M$ is a compact Kähler manifold with $c_1(M) > 0$ and $X$ is a holomorphic vector field whose imaginary part $\text{Im} X$ induces an $S^1$ action on $M$. Note that once a maximal compact subgroup $G$ of $\text{Aut}^0(M)$ is fixed, there is a natural choice of such a vector field $X$ (for more details see Section §2.1 below).

First, we define the notion of the Hamiltonian $\theta_{X,\omega}$ and modified Ricci potential $u_{X,\omega}$. Write $\mathcal{K}_X$ for the space of Kähler metrics in $\pi c_1(M)$ which are invariant under $\text{Im} X$. Given $\omega = \frac{i}{2} \sigma_j dz^j \wedge d\bar{z}^k \in \mathcal{K}_X$ we define a real-valued function $\theta_{X,\omega}$ by

$$X^j \sigma_j = \partial_k \theta_{X,\omega}, \quad \int_M e^{\theta_{X,\omega}} \omega^n = \int_M \omega^n =: V.$$  

(1.5)

The Ricci potential $f = f(\omega)$ is given by

$$\sigma_{kj} - R_{kj} = \partial_k \partial_j f, \quad \int_M e^{-f} \omega^n = V,$$  

(1.6)
and we define a modified Ricci potential $u_{X, \omega}$ by

$$u_{X, \omega} = f + \theta_{X, \omega}. \quad (1.7)$$

If $M$ admits a Kähler-Ricci soliton $\omega \in \pi c_1(M)$ with respect to the vector field $X$, then $\omega$ is necessarily in $\mathcal{K}_X$ and $u_{X, \omega} = 0$. Let $g_{kj}(t)$ evolve by the modified Kähler-Ricci flow and set

$$Y_X(t) = \int_M |\nabla u_{X, \omega}|^2 e^{\theta_{X, \omega}} \omega^n. \quad (1.8)$$

The modified Kähler-Ricci flow starting at $\omega_0 \in \mathcal{K}_X$ preserves the Kähler class, and can be expressed as a flow of Kähler potentials. Define

$$\mathcal{P}_X(M, \omega_0) = \{ \varphi \in C^\infty(M) \mid \omega = \omega_0 + i\frac{1}{2} \partial \bar{\partial} \varphi > 0, \text{ Im}(X(\varphi)) = 0 \}, \quad (1.9)$$

which, modulo constants, can be identified with $\mathcal{K}_X$. Let $\varphi = \varphi(t) \in \mathcal{P}_X(M, \omega_0)$ be the solution of the equation

$$\dot{\varphi} = \log \frac{\omega^n}{\omega_0^n} + \varphi + \theta_{X, \omega} + f(\omega_0),$$

$$\varphi(0) = c_0. \quad (1.10)$$

Then the Kähler metrics $\omega = \omega_0 + i\frac{1}{2} \partial \bar{\partial} \varphi$ evolve by the modified Kähler-Ricci flow (1.4). The initial constant $c_0$ can affect the growth of $\varphi$ for large time, and has to be chosen with some care. We choose it to be given by the value (2.34) described in Section §2 below.

Our first main theorem is a general characterization of the convergence of the modified Kähler-Ricci flow, which shows in particular that if convergence occurs, it always does so at an exponential rate:

**Theorem 1** Let $\omega_0 \in \mathcal{K}_X$, $\omega_0 := \frac{i}{2} g_{kj}^0 dz^j \wedge d\bar{z}^k$, and consider the modified Kähler-Ricci flow (1.4) with initial metric $\omega_0$. Then the following conditions are equivalent:

(i) The modified Kähler-Ricci flow $g_{kj}(t)$ converges in $C^\infty$ to a Kähler-Ricci soliton $g_{kj}(\infty)$ with respect to $X$.

(ii) The function $\| R - n - \nabla_j X^j \|_{C^0}$ is integrable, i.e.,

$$\int_0^\infty \| R - n - \nabla_j X^j \|_{C^0} dt < \infty. \quad (1.11)$$

(iii) Let the potential $\varphi(t)$ evolve according to (1.10), with initial value $c_0$ as specified in the equation (2.34) below. Then we have

$$\sup_{t \geq 0} \| \varphi(t) \|_{C^0} < \infty. \quad (1.12)$$

(iv) Let the function $Y_X(t)$ be defined by (1.8). Then there exist constants $\kappa, C$ with $\kappa$ strictly positive so that

$$Y_X(t) \leq C e^{-\kappa t}. \quad (1.13)$$

(v) The modified Kähler-Ricci flow $g_{kj}(t)$ converges exponentially fast in $C^\infty$ to a Kähler-Ricci soliton $g_{kj}(\infty)$ with respect to $X$. 

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The preceding theorem shows that the growth of \( Y_X(t) \), or alternatively, of the function \( \|R-n-\nabla_j X^j\|_{C^0}(t) \) is key to the problem of convergence of the modified Kähler-Ricci flow. Our next result addresses the behavior of these quantities under a stability assumption. Following [TZ1], we define the modified Mabuchi K-energy \( \mu_X : \mathcal{P}_X(M, \omega_0) \to \mathbb{R} \) by

\[
\delta \mu_X(\varphi) = -\frac{1}{V} \int_M \delta \varphi \left( R - n - \nabla_j X^j - X u_{X,\omega} \right) e^{\theta_{X,\omega} n}, \quad \mu_X(0) = 0. \tag{1.14}
\]

To clarify this definition, since \( R - n - \nabla_j X^j - X u = -\left( \Delta + \text{Re}X \right) u_{X,\omega} \), the integrand is real and thus \( \mu_X \) does indeed map into \( \mathbb{R} \). For a proof that \( \mu_X(\varphi) \) is independent of choice of path in \( \mathcal{P}_X(M, \omega_0) \), see [TZ1].

We consider the condition:

\[(A_X) \quad \mu_X \text{ is bounded from below on } \mathcal{P}_X(M, \omega_0)\]

In [TZ1] it is shown that \((A_X)\) is a necessary condition for the existence of a Kähler-Ricci soliton \( \omega \) with respect to \( X \). Here we shall establish the following theorem:

**Theorem 2** Assume that Condition \((A_X)\) holds, and let \( \omega_0 \in \mathcal{K}_X \). Then we have, along the modified Kähler-Ricci flow (1.4) starting at \( \omega_0 \),

\[
Y_X(t) \to 0 \quad \text{and} \quad \|R-n-\nabla_j X^j\|_{C^0} \to 0, \quad \text{as } t \to \infty. \tag{1.15}
\]

Furthermore, for any \( p > 2 \), we have

\[
\int_0^\infty \|R-n-\nabla_j X^j\|_{C^0}^p dt < \infty. \tag{1.16}
\]

Note that a metric \( \omega \in \mathcal{K}_X \) satisfies \( R-n-\nabla_j X^j = 0 \) if and only if \( \omega \) is a Kähler-Ricci soliton with respect to \( X \). However, the convergence of \( \|R-n-\nabla_j X^j\|_{C^0} \) to zero is of course weaker than the convergence of the metrics \( g_{kj}(t) \) themselves to a Kähler-Ricci soliton, and this is to be expected, since the condition \((A_X)\) is only a semi-stability type of condition.

Next, we describe another consequence of the condition \((A_X)\). Associated to the modified Mabuchi K-energy is the modified Futaki invariant \( F_X \) (see [TZ1]), given by

\[
F_X(Z) = -\int_M (Z u_{X,\omega}) e^{\theta_{X,\omega} n}, \tag{1.17}
\]

for a holomorphic vector field \( Z \). The modified Futaki invariant \( F_X \) is independent of the choice of \( \omega \in \mathcal{K}_X \). It follows immediately that \( F_X \equiv 0 \) is a necessary condition for the existence of a Kähler-Ricci soliton in \( \mathcal{K}_X \).

In the unmodified case, corresponding to \( X = 0 \), the condition \((A_X)\) reduces to the condition \((A)\) of lower boundedness of the Mabuchi K-energy. It is then easy to show that \((A)\) implies that the unmodified Futaki invariant \( F_{X=0}(Z) \) vanishes for all holomorphic vector fields \( Z \in H^0(M, T^{1,0}) \), by differentiating the functional along the integral paths of \( Z \). We show how to rework this argument to prove the analogous statement when \( X \neq 0 \) (to our knowledge, this result is not in the literature).
Proposition 1 If \((A_X)\) holds then \(F_X(Z) = 0\) for all holomorphic vector fields \(Z\).

Our third theorem shows that \((A_X)\) together with one additional assumption give necessary and sufficient conditions for the convergence of the metrics \(g_{kj}^\bar{\omega}(t)\) themselves. Let \(\lambda(t)\) be the first positive eigenvalue of the operator 
\[ -g^{k\bar{p}}\nabla_j \nabla_k \] acting on smooth \(T^{1,0}\) vector fields. Namely,
\[
\lambda(t) = \inf_{V \perp H^0(M, T^{1,0})} \frac{\|\bar{\partial}V\|^2}{\|V\|^2},
\]
where \(H^0(M, T^{1,0})\) is the space of holomorphic vector fields on \(M\) and we are using the natural \(L^2\) inner product induced by \(g_{kj}^\omega(t)\) on the spaces \(T^{1,0}\) and \(T^{1,-1}\). This quantity was first introduced in the context of the Kähler-Ricci flow in [PS1]. Recall the following condition from [PSSW1]:

\[(S)\] \(\inf_{t \geq 0} \lambda(t) > 0\).

Then we have:

**Theorem 3** The modified Kähler-Ricci flow (1.4), starting at an arbitrary metric \(\omega_0 \in \mathcal{K}_X\), converges exponentially fast in \(C^\infty\) to a Kähler-Ricci soliton with respect to the holomorphic vector field \(X\) if and only if the conditions \((A_X)\) and \((S)\) are satisfied.

Since condition \((S)\) is invariant under automorphisms, an immediate consequence of Theorem 3 is that convergence modulo automorphisms implies full convergence. More precisely, suppose that \(g_{kj}(t)\) is a solution of the modified Kähler-Ricci flow starting at \(\omega_0 \in \mathcal{K}_X\) and assume there exists a family of automorphisms \(\{\Psi_t\}_{t \in [0,\infty)}\) such that \(\Psi_t^*(g_{kj})\) converges to a Kähler-Ricci soliton with respect to \(X\). Then \(g_{kj}(t)\) converges exponentially fast to a Kähler-Ricci soliton with respect to \(X\).

Finally we discuss in more detail the behavior of \(Y_X(t)\) which, as can be seen from Theorem 1, is key to the convergence of the Kähler-Ricci flow. The next result provides information on the growth of \(Y_X(t)\) for the completely general modified Kähler-Ricci flow and brings to light the obstructions to the convergence of the flow.

It is convenient to introduce a quantity \(\lambda_X\) which is uniformly equivalent to the eigenvalue \(\lambda\) described above (see Lemma 12 below). Equip the spaces \(T^{1,0}\) and \(T^{1,-1}\) with the norms
\[
\|V\|^2 = \int_M g_{kj} V^j \nabla^k e^{\theta_x} \omega^n,
\]
\[
\|W\|^2 = \int_M g_{kj} W^q \nabla^q \omega^n,
\]
with respect to which they can be completed into Hilbert spaces. Define the eigenvalue \(\lambda_X(t)\) by
\[
\lambda_X(t) = \inf_{V \perp H^0(M, T^{1,0})} \frac{\|\bar{\partial}V\|^2}{\|V\|^2},
\]
(1.19)
where the notion of perpendicularity is taken with respect to the norm $\| \cdot \|_{\theta}$. Then we have the following:

**Theorem 4** Consider the modified Kähler-Ricci flow (1.4) with initial metric $\omega_0 \in \mathcal{K}_X$. Then there exist $N, \delta_j \geq 0, 0 \leq j \leq N$, depending only on $n$ and satisfying the condition

$$\frac{1}{2} \sum_{j=0}^{N} \delta_j > 1 \quad (1.21)$$

and a constant $C > 0$, depending only on the initial metric $\omega_0$ so that, for all $t \geq 2N$, the following difference-differential inequality holds

$$\dot{Y}_X(t) \leq -2 \lambda_X(t) Y_X(t) - 2 \lambda_X(t) F_X(\pi(\nabla(u_X,\omega))) + C \prod_{j=0}^{N} Y_X(t - 2j) \delta_j. \quad (1.22)$$

Here $\nabla u_{X,\omega}$ is the vector field $g^{jk} \partial_k u_{X,\omega}$, and $\pi$ is the orthogonal projection, with respect to the norm $\| \cdot \|_{\theta}$, of the space $T^{1,0}$ of vector fields onto its subspace $H^0(M,T^{1,0})$ of holomorphic vector fields.

We stress that the estimate (1.22) holds in full generality, without any stability assumptions.

The main point of the estimate (1.22) is that certain difference-differential inequalities, while weaker than the standard inequality $\dot{Y} \leq -\lambda Y$, can still guarantee the exponential decay of $Y(t)$. A key example for our purposes is Lemma 13 below. In view of the estimate (1.22) for $\dot{Y}_X(t)$, we see that the convergence of the modified Kähler-Ricci flow is related to three issues:

(a) The vanishing of the modified Futaki invariant $F_X$;
(b) Whether $Y_X(t)$ tends to 0 as $t \to \infty$;
(c) The existence of a strictly positive uniform lower bound for $\lambda_X(t)$ (or $\lambda(t)$).

The arguments and viewpoint in this paper run closely in parallel to those of [PSSW1], which deals with the case $X = 0$. In what follows, we have emphasized only the main changes due to the modified Kähler-Ricci flow and the holomorphic vector field $X$. Nevertheless, we have tried to make the discussion reasonably self-contained, and taken the opportunity to bring out some estimates for both the Kähler-Ricci flow and the modified Kähler-Ricci flow which hold in all generality, independent of any stability assumption. That is particularly the case for Theorems 1 and 4 above.

In addition, we mention one result used in the proofs of the main theorems which may be of independent interest. Assuming only that the initial metric $g_{\mathcal{F}j}$ is invariant under $\text{Im} X$, we have (see Proposition 2 below)

$$g_{\mathcal{F}j} X^j \overline{X}^k = |X|^2 \leq C, \quad (1.23)$$
for all time, where \( g_{\bar{\nabla}j} \) is a solution of either the Kähler-Ricci flow or the modified flow.

We give a brief outline of the paper. In Section §2 we give some background on the modified Mabuchi K-energy and Futaki invariant, prove Proposition 1 and describe the Kähler-Ricci and modified Kähler-Ricci flows. In Section §3 we prove some estimates for the modified Kähler-Ricci flow, which hold in a general setting. The proof of the difference-differential inequality, Theorem 4, is given in Section §4. Theorems 2, 1 and 3 are then proved in Sections §5, §6 and §7 respectively. Finally, in Section §8 we mention a few further remarks and questions.

2 The modified Mabuchi K-energy, Futaki invariant and Kähler-Ricci flow

2.1 The modified Mabuchi K-energy and Futaki invariant

We discuss in this subsection the modified Mabuchi K-energy and modified Futaki invariant and mention two results of Tian-Zhu.

From the definition of the modified Mabuchi K-energy,
\[
\delta \mu_X(\varphi) = -\frac{1}{\sqrt{V}} \int_M \delta \varphi \left( R - n - \nabla_j X_j - Xu_{X,\omega} \right) e^{\theta_{X,\omega} \omega_n}, \quad \mu_X(0) = 0,
\]
we see that its critical points satisfy
\[
(\Delta + X)u_{X,\omega} = 0,
\]
and hence are Kähler-Ricci solitons with respect to \( X \). Note that after integrating by parts the variation of \( \mu_X \) can be rewritten as
\[
\delta \mu_X(\varphi) = -\frac{n}{\sqrt{V}} \int_M \frac{i}{2} \partial u_{X,\omega} \wedge \overline{\partial} \varphi \wedge e^{\theta_{X,\omega} \omega_n - 1},
\]
a formulation that will be useful later.

A key result that we will use in this paper is as follows:

**Theorem 5** ([TZ1]) If there exists a Kähler-Ricci soliton with respect to \( X \), then \( \mu_X \) is bounded below on \( \mathcal{P}_X(M,\omega_0) \).

Next, the modified Futaki invariant \( F_X : H^0(M,T^{1,0}) \to \mathbb{C} \) is given by
\[
F_X(Z) = -\int_M (Z u_{X,\omega}) e^{\theta_{X,\omega} \omega_n}.
\]

An important property of this invariant is that \( F_X \) vanishes on the reductive part of \( H^0(M,T^{1,0}) \) for a particular choice of \( X \). More precisely, fix a maximal compact subgroup \( G \) of the identity component \( \text{Aut}^0(M) \) of the automorphism group of \( M \). Write \( \text{Lie}(G)^\mathbb{C} \) for the complexification of the Lie algebra of \( G \), a reductive Lie subalgebra of the space of holomorphic vector fields. Then:
Theorem 6 ([TZ1]) There exists a unique holomorphic vector field $X'$ in $\text{Lie}(G)^\mathbb{C} \subset H^0(M, T^{1,0})$ satisfying $\text{Im}X' \in \text{Lie}G$ and

$$F_{X'}(Z) = 0, \quad \text{for all } Z \in \text{Lie}(G)^\mathbb{C}. \quad (2.5)$$

Moreover, $X'$ lies in the center of $\text{Lie}(G)^\mathbb{C}$.

Note that the condition $\text{Im}X' \in \text{Lie}G$ ensures that the imaginary part of $X'$ induces an $S^1$ action on $M$. In this paper, we do not explicitly take our holomorphic vector field $X$ to be this choice $X'$. On the other hand, it follows from Proposition 1 and the uniqueness part of Theorem 6 that once we fix a maximal compact subgroup $G$, if condition $(A_X)$ holds and $\text{Im}X$ is in $\text{Lie}G$ then $X = X'$.

Theorem 6 is related to the uniqueness result of [TZ1] for Kähler-Ricci solitons, which we state now for the reader’s convenience. If $\omega, \omega'$ are Kähler-Ricci solitons with respect to holomorphic vector fields $X, X'$ then there exists $\sigma$ in $\text{Aut}^0(M)$ such that

$$\omega = \sigma^* \omega', \quad \text{and} \quad X' = \sigma_*(X). \quad (2.6)$$

2.2 $(A_X)$ and the modified Futaki invariant $F_X$

In this subsection, we give the proof of Proposition 1.

We will first show that $F_X(Z) = 0$ for all holomorphic vector fields $Z$ satisfying

$$\mathcal{L}_{\text{Im}X} Z = 0. \quad (2.7)$$

Fix a Kähler metric $\omega_0 \in \mathcal{K}_X$. Write $\Psi_t$ for the 1-parameter family of automorphisms of $M$ induced by $\text{Re}Z$. Define $\omega_t$ and $\psi_t$ by

$$\Psi_t^* \omega_0 = \omega_t = \omega_0 + \frac{i}{2} \partial\bar{\partial} \psi_t. \quad (2.8)$$

Note that $\psi_t$ is defined only up to the addition of a constant, but this will not affect our calculations. Since by assumption $Z$ is invariant under $\text{Im}X$ we see that $\psi_t \in \mathcal{P}_X(M, \omega_0)$ and hence $\mu_X(\psi_t)$ is well-defined. Also,

$$\frac{i}{2} \partial\bar{\partial} \psi = \mathcal{L}_{\text{Re}Z} \omega, \quad (2.9)$$

where here, and henceforth, we are dropping the $t$ subscript.

On the other hand, there exists a complex-valued function $\theta_{Z,\omega}$, invariant under $\text{Im}X$, such that

$$i_Z \omega = \frac{i}{2} \partial\bar{\partial} \theta_{Z,\omega}. \quad (2.10)$$
Indeed, all manifolds with positive first Chern class are simply connected so the \( \overline{\partial} \)-closed \((0,1)\)-form \( Z_j^g k \) must be \( \overline{\partial} \)-exact. Since

\[
\frac{i}{2} \partial \overline{\partial} \psi = d_{\text{Re}Z} \omega = \text{Re}(d \omega) = \frac{i}{2} \partial \overline{\partial} \text{Re} \theta_{Z,\omega},
\]

we can assume without loss of generality that \( \dot{\psi} = \text{Re} \theta_{Z,\omega} \). Calculate

\[
\mathcal{L}_Z(\text{Re} \theta_{X,\omega} \omega^n) = d(\text{e}^{\text{Re} \theta_{X,\omega} \omega} u_{\text{Re}Z} \omega^n)
= \text{e}^{\text{Re} \theta_{X,\omega}} \left( \Delta \dot{\psi} \omega^n + n d \theta_{X,\omega} \wedge u_{\text{Re}Z} \wedge \omega^{n-1} \right)
= \text{e}^{\theta_{X,\omega}} \left( \Delta \dot{\psi} \omega^n + n \text{Re} \left( \frac{i}{2} \partial \overline{\partial} \text{Re} \theta_{X,\omega} \wedge \overline{\partial} \dot{\psi} \wedge \omega^n \right) \right),
\]

(2.12)

Then, by (2.3),

\[
\frac{d}{dt} \mu_X(\psi) = -\text{Re} \left( \frac{n}{V} \int_M \frac{i}{2} \partial u_{X,\omega} \wedge \overline{\partial} \dot{\psi} \wedge \text{e}^{\theta_{X,\omega} \omega} \omega^{n-1} \right)
= \frac{n}{V} \int_M u_{X,\omega} \text{e}^{\theta_{X,\omega}} \left( \frac{i}{2} \partial \overline{\partial} \dot{\psi} + \text{Re} \left( \frac{i}{2} \partial \theta_{X,\omega} \wedge \overline{\partial} \dot{\psi} \right) \right) \wedge \omega^n
= \frac{1}{V} \int_M u_{X,\omega} \mathcal{L}_Z(\text{e}^{\theta_{X,\omega} \omega} \omega^n)
= -\frac{1}{V} \int_M (\text{Re}Z)(u_{X,\omega}) \text{e}^{\theta_{X,\omega} \omega} \omega^n
= \frac{1}{V} \text{Re} \left( F_X(Z) \right).
\]

(2.13)

To go from the 2nd to 3rd lines, we have used the fact that \( \dot{\psi} = \theta_{Z,\omega} - i \text{Im} \theta_{Z,\omega} \) and

\[
n\text{Re} \left( \frac{1}{2} \partial \theta_{X,\omega} \wedge \overline{\partial} \text{Im} \theta_{Z,\omega} \wedge \omega^{n-1} \right) = -n \text{Re} \left( \frac{1}{2} \partial \text{Im} \theta_{Z,\omega} \wedge \overline{\partial} \theta_{X,\omega} \wedge \omega^n \right)
= -(\text{Im} X)(\text{Im} \theta_{Z,\omega}) \omega^n = 0,
\]

since by the assumption (2.7), \( \text{Im} \theta_{Z,\omega} \) is invariant under \( \text{Im} X \).

Condition \((A_X)\) implies from (2.13) that \( \text{Re}(F_X(Z)) = 0 \). Replacing \( Z \) by \( iZ \) shows that \( F_X(Z) = 0 \) for all holomorphic vector fields \( Z \) invariant under \( \text{Im} X \).

Now let \( Z \) be an arbitrary holomorphic vector field. Denote by \( \{\sigma_t\}_{t \in [0,2\pi]} \) the 1-parameter family of automorphisms induced by \( \text{Im} X \) and define a holomorphic vector field \( \hat{Z} \) by

\[
\hat{Z} = \frac{1}{2\pi} \int_0^{2\pi} \sigma_t^* Z \ dt.
\]

(2.14)

By the argument above, since \( \hat{Z} \) is invariant under \( \text{Im} X \), we have \( F_X(\hat{Z}) = 0 \). Then for each \( t \in [0,2\pi] \),

\[
F_X(Z) = -\int_M (Zu_{X,\omega}) \omega^n_0 = -\int_M (\sigma_t^* Z)(\sigma_t^* u_{X,\omega}) \sigma_t^* \omega^n_0 = F_X(\sigma_t^* Z),
\]

(2.15)
using the fact that $\omega_0$ and $u_{X,\omega_0}$ are invariant under $\text{Im} \, X$. Thus
\[ F_X(Z) = \frac{1}{2\pi} \int_0^{2\pi} F_X(\sigma^*_t Z) \, dt = F_X(\hat{Z}) = 0, \tag{2.16} \]
completing the proof of Proposition 1.

### 2.3 The modified Kähler-Ricci flow

We list now some basic properties of the modified Kähler-Ricci flow. Some result from the fact that the modified Kähler-Ricci flow is a reparametrization of the Kähler-Ricci flow, and the most important is a normalization for the modified Kähler potential $\varphi$ which ensures that $\sup_{t \geq 0} \| \varphi \|_{C^0} < \infty$. We remark that this bound is proved in [TZ2] assuming the existence of a Kähler-Ricci soliton. In our case, we only require the invariance of the initial metric $\omega_0$ under $\text{Im} \, X$.

Fix $\omega_0 \in \mathcal{K}_X$, $\omega_0 = \frac{i}{2} g_{k\bar{j}}^0 \, dz^j \wedge d\bar{z}^k$. Then the Kähler-Ricci flow and modified Kähler-Ricci flow starting at $\omega_0$ are given by
\[ \frac{\partial}{\partial t} g_{k\bar{j}}(t) = -R_{k\bar{j}} + g_{k\bar{j}}, \quad g_{k\bar{j}}(0) = g_{k\bar{j}}^0 \tag{2.17} \]
\[ \frac{\partial}{\partial t} \tilde{g}_{k\bar{j}}(t) = -\tilde{R}_{k\bar{j}} + \tilde{g}_{k\bar{j}}, \quad \tilde{g}_{k\bar{j}}(0) = \tilde{g}_{k\bar{j}}^0 \tag{2.18} \]
respectively. Note that if $\{ \Phi_t \}_{t \in [0, \infty)}$, $\Phi_0 = \text{id}$, is the 1-parameter family of automorphisms of $M$ generated by $\text{Re} \, X$, then the solutions to (2.17) and (2.18) are related by $g_{k\bar{j}}(t) = \Phi_t^*(\tilde{g}_{k\bar{j}})$. The Kähler-Ricci flow preserves the $S^1$ action induced by $\text{Im} \, X$ and so the Kähler forms $\tilde{\omega}(t)$ and $\omega(t)$ lie in $\mathcal{K}_X$. In the sequel, we will often drop the $t$. Also, we will denote by $\tilde{f}$, $\tilde{\nabla}$ and $\tilde{\Delta}$ the Ricci potential (see (1.6)), covariant derivative and Laplacian with respect to $\tilde{g}_{k\bar{j}}$. Since we are using different notation for solutions of (2.17) and (2.18), and each of these flows can easily be obtained from the other, we will sometimes refer simply to ‘the flow’ rather than the specific equation (2.17) or (2.18).

Before continuing our discussion of these two flows, we list without proof some known estimates for the Kähler-Ricci flow. The first three statements are due to Perelman [P2] (see [ST]) and the fourth is due to Zhang [Zha] and Ye [Ye].

**Theorem 7** In the following, all norms are taken with respect to the metric $\tilde{g}_{k\bar{j}}(t)$.

(i) There exists a constant $C$ depending only on $\omega_0$ such that the Ricci potential $\tilde{f}$ along the flow satisfies
\[ \| \tilde{f} \|_{C^0} + \| \tilde{\nabla} \tilde{f} \|_{C^0} + \| \tilde{\Delta} \tilde{f} \|_{C^0} \leq C. \tag{2.19} \]
(ii) The diameters $\text{diam}_{\tilde{g}(t)} M$ are uniformly bounded along the flow by a constant depending only on $\omega_0$. 

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(iii) Let $\rho > 0$ be fixed. Then there exists a constant $c > 0$ depending only on $\omega_0$ and $\rho$ such that for all $x \in M$ and all $r$ with $0 < r \leq \rho$ we have

$$
\int_{B_r(x)} \tilde{\omega}^n(t) \geq c r^{2n},
$$

(2.20)

where $B_r(x)$ is the geodesic ball centered at $x$ of radius $r$ with respect to $\tilde{g}_{kj}(t)$.

(iv) There exists a constant $C_S$ depending only on $\omega_0$ such that the Sobolev inequality

$$
\|\eta\|_{L^{2n/(n-1)}} \leq C_S (\|\tilde{\nabla} \eta\|_{L^2} + \|\eta\|_{L^2}),
$$

for all $\eta \in C^\infty(M)$ (2.21)

holds.

We remark that this theorem makes no reference to the vector field $X$ and indeed does not require the initial metric $\omega_0$ to be invariant under $\text{Im}X$. Moreover, all of the above statements are invariant under automorphisms and hence the analogous statements hold also for the metrics $g_{kj}$.

We now describe (2.17) and (2.18) in terms of potentials. Define $\tilde{\phi} = \tilde{\phi}(t)$ and $\phi = \phi(t)$ in $P_X(M,\omega_0)$ by

$$
\frac{\partial \tilde{\phi}}{\partial t} = \log \frac{\tilde{\omega}^n}{\omega_0^n} + \tilde{\phi} + f(\omega_0), \quad \tilde{\phi}(0) = \tilde{c}_0,
$$

(2.22)

$$
\frac{\partial \phi}{\partial t} = \log \frac{\omega^n}{\omega_0^n} + \phi + \theta_{X,\omega} + f(\omega_0), \quad \phi(0) = c_0,
$$

(2.23)

where the constants $\tilde{c}_0$ and $c_0$ will be defined shortly. One can check that $\tilde{\omega} = \omega_0 + \frac{i}{2} \frac{\partial}{\partial \tilde{\phi}} \tilde{\phi}$ and $\omega = \omega_0 + \frac{i}{2} \frac{\partial}{\partial \phi} \phi$ satisfy (2.17) and (2.18) respectively.

Recall that $\theta_{X,\omega}$ is the Hamiltonian function defined by (1.5). It is well-known that $\theta_{X,\omega}$ is well-defined. Indeed, by the same argument as for (2.10), there is a complex-valued function $\theta_{X,\omega}$ solving $X^j g_{kj} = \partial_k \theta_{X,\omega}$. The equalities

$$
0 = \mathcal{L}_{\text{Im}X\omega} = \text{d}_t \text{Im}X\omega = \frac{i}{2} \frac{\partial}{\partial \phi} \text{Im} \theta_{X,\omega}
$$

(2.24)

ensure the existence of a real-valued function $\theta_{X,\omega}$ satisfying (1.5).

Following the conventions of [PSS] (cf. [CT]) we define

$$
\tilde{c}_0 := \frac{1}{V} \int_0^\infty e^{-t} \int_M |\tilde{\nabla} \tilde{f}|^2 \tilde{\omega}^n dt - \frac{1}{V} \int_M f(\omega_0) \omega_0^n.
$$

(2.25)

Note that by Theorem 7 we have $|\tilde{\nabla} \tilde{f}| \leq C$ and so $\tilde{c}_0$ is finite. With this choice of $\tilde{c}_0$, there exists a uniform $C$ such that

$$
\|\partial \tilde{\phi}/\partial t\|_{C^0} \leq C.
$$

(2.26)

Before we define $c_0$, we will need two lemmas. The first lemma implies that $\theta_{X,\omega}$ is uniformly bounded along the flow.
Lemma 1 For all $\omega' \in K_X$, we have
\[ \|\theta_{X,\omega'}\|_{C^0} = \|\theta_{X,\omega_0}\|_{C^0}. \] (2.27)

Proof of Lemma 1: This result is well-known (see [FM] or [Zhu1], for example), but for the reader’s convenience we outline here a proof. We use the following version of Moser’s theorem (which can be easily derived from [CdS], p.43-44, for example).

Theorem 8 Let $M$ be a compact complex manifold with Kähler forms $\omega_0$ and $\omega_1$ which are invariant under an $S^1$ action induced by a real vector field $W$. Assume in addition that $[\omega_0] = [\omega_1]$. Then there exists a diffeomorphism $\Psi$ of $M$ satisfying
\[ \Psi^*\omega_1 = \omega_0 \quad \text{and} \quad \Psi^*W = W. \] (2.28)

Write $\omega_1 := \omega' \in K_X$. By definition, $\theta_0 := \theta_{X,\omega_0}$ and $\theta_1 := \theta_{X,\omega_1}$ satisfy
\[ \iota_{\text{Im} X} \omega_0 = \frac{1}{4} d\theta_0, \quad \iota_{\text{Im} X} \omega_1 = \frac{1}{4} d\theta_1, \] (2.29)
with
\[ \int_M e^{\theta_0} \omega_0^n = V = \int_M e^{\theta_1} \omega_1^n. \] (2.30)

From Theorem 8 we obtain a diffeomorphism $\Psi$ of $M$ with $\Psi^*\omega_1 = \omega_0$ and $\Psi^*\text{Im} X = \text{Im} X$. Applying $\Psi^*$ to the equation $\iota_{\text{Im} X} \omega_1 = \frac{1}{4} d\theta_1$ we obtain $d\Psi^*\theta_1 = d\theta_0$, and hence $\Psi^*\theta_1 = \theta_0 + c$ for some constant $c$. But from (2.30) we see that $c = 0$. This implies that $\theta_0, \theta_1 : M \to \mathbb{R}$ have the same image in $\mathbb{R}$. Q.E.D.

Given this lemma we can now prove the following.

Lemma 2 Along the modified Kähler-Ricci flow we have
\[ \int_M |X|^2 e^{\theta_{X,\omega}} \omega^n \leq C. \] (2.31)

Proof of Lemma 2: From the definition of the modified Futaki invariant and the definition of $\theta_{X,\omega}$ we see that
\[ \int_M |X|^2 e^{\theta_{X,\omega}} \omega^n = - \int_M (Xf) e^{\theta_{X,\omega}} \omega^n - F_X(X). \] (2.32)

Hence, since $F_X(X)$ is independent of choice of metric, we have
\[ \int_M |X|^2 e^{\theta_{X,\omega}} \omega^n \leq \int_M |X||\nabla f| e^{\theta_{X,\omega}} \omega^n + C \]
\[ \leq \frac{1}{2} \int_M |X|^2 e^{\theta_{X,\omega}} \omega^n + \frac{1}{2} \int_M |\nabla f|^2 e^{\theta_{X,\omega}} \omega^n + C, \] (2.33)

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and the lemma follows from Theorem 7, part (i) and Lemma 1. Q.E.D.

We now define the constant $c_0$ as follows:

$$c_0 := \frac{1}{V} \int_0^\infty e^{-t} \int_M |\nabla u_{X,\omega}|^2 e^{\theta_{X,\omega} \omega} dt - \frac{1}{V} \int_M u_{X,\omega_0} e^{\theta_{X,\omega_0} \omega_0^n},$$

(2.34)

where we recall that $u_{X,\omega} = f + \theta_{X,\omega}$. To see that $c_0$ is finite, observe that

$$|\nabla u_{X,\omega}|^2 \leq 2(|\nabla f|^2 + |X|^2) \leq C + 2|X|^2;$$

(2.35)

and hence by Lemma 1 and Lemma 2,

$$\int_M |\nabla u_{X,\omega}|^2 e^{\theta_{X,\omega} \omega^n} \leq C.$$  

(2.36)

We will end this section by proving a uniform bound on $\dot{\phi}$. First, we have another general lemma on Hamiltonian functions:

**Lemma 3** For any $\omega' \in K_X$ with $\omega' = \omega_0 + \frac{i}{2} \partial \bar{\partial} \phi'$, the Hamiltonian functions $\theta_{X,\omega'}$ and $\theta_{X,\omega_0}$ are related by:

$$\theta_{X,\omega'} = \theta_{X,\omega_0} + X(\phi'),$$

(2.37)

**Proof of Lemma 3:** This is proved in [TZ1], p.301. Q.E.D.

We can now prove:

**Lemma 4** There exists a uniform constant $C$ such that along the flow,

$$\|\dot{\phi}\|_{C^0} \leq C.$$  

(2.38)

**Proof of Lemma 4:** From (2.17) and (2.18) we obtain

$$\frac{\partial \phi}{\partial t} = \Phi_t \frac{\partial \phi}{\partial t} + \theta_{X,\omega} + m(t),$$

(2.39)

for some constant $m(t)$. Define

$$\alpha(t) = \frac{1}{V} \int_M \dot{\phi} e^{\theta_{X,\omega} \omega^n}.$$  

(2.40)

Using Lemma 3, we have

$$\frac{\partial \dot{\phi}}{\partial t} = (\Delta + X)\dot{\phi} + \dot{\phi}.$$  

(2.41)

Since

$$\dot{\phi} = u_{X,\omega} + c,$$  

(2.42)
for a constant $c$ depending only on time, we have

$$\frac{d}{dt} \alpha = \alpha - \frac{1}{V} \int_M |\nabla u_{X,\omega}|^2 e^{\theta_{X,\omega}} \omega^n. \quad (2.43)$$

Integrating this ODE (cf. the argument in [PSS]) and applying (2.36) shows that

$$0 \leq \alpha(t) = \frac{1}{V} \int_t^\infty e^{-(s-t)} \int_M |\nabla u_{X,\omega}|^2(s) e^{\theta_{X,\omega}(s)} \omega^n(s) \, ds \leq C, \quad (2.44)$$

or in other words, the average of $\dot{\varphi}$ with respect to the measure $e^{\theta_{X,\omega}}$ is bounded along the flow. The lemma now follows immediately from the formula (2.39) together with (2.26), (2.44) and Lemma 1. Q.E.D.

### 3 Estimates for the modified Kähler-Ricci flow

In this section, we establish the key estimates for the modified Kähler-Ricci flow needed in the sequel. They include the analogues of Perelman’s estimates for the Ricci potential and the scalar curvature for the Kähler-Ricci flow, the estimates for the Laplacian of the Hamiltonian function $\theta_{X,\omega}$, and a smoothing lemma.

#### 3.1 Estimates for the modified Ricci potential

Define

$$v := -\dot{\varphi}. \quad (3.1)$$

Recall from (2.42) that $\dot{\varphi}$ differs from $f + \theta_{X,\omega} = u_{X,\omega}$ by a constant depending on time. Thus, when computing time evolutions, we have to distinguish between $-v$ and $u_{X,\omega}$, but the difference disappears whenever $\nabla$ is applied. In the last section we established the bound

$$\|v\|_{C^0} \leq C, \quad (3.2)$$

and in this section we will prove the following further estimates for the modified Kähler-Ricci flow.

**Proposition 2** Along the flow, the quantities

$$\|\nabla u_{X,\omega}\|_{C^0}, \|\Delta u_{X,\omega}\|_{C^0}, \|X\|_{C^0}, \text{ and } \|\Delta \theta_{X,\omega}\|_{C^0} \quad (3.3)$$

are uniformly bounded by a constant depending only on the initial data. Here, all norms, covariant derivatives and Laplacians are taken with respect to the evolving metric $g_{kj}(t)$.

These bounds will be obtained using the maximum principle. The proof of this proposition is contained in the following three lemmas.
Lemma 5. We have the following identities along the modified Kähler-Ricci flow:

(i) \[ \frac{\partial v}{\partial t} = (\Delta + X)v + v \]

(ii) \[ \frac{\partial}{\partial t} |\nabla v|^2 = (\Delta + X)|\nabla v|^2 - |\nabla \nabla v|^2 + |\nabla v|^2. \]

(iii) \[ \frac{\partial}{\partial t} (\Delta + X)v = (\Delta + X)(\Delta + X)v + (\Delta + X)v + |\nabla \nabla v|^2. \]

Proof of Lemma 5: This is a straightforward calculation (cf. [CTZ]). Q.E.D.

Note that although \( X \) is not a real operator on functions on \( M \), the quantities \( v, |\nabla v|^2 \) and \( \Delta v \) above are all invariant under \( \text{Im} X \), and so \( \Delta + X \) can be replaced by the real operator \( \Delta + \text{Re} X \).

Using these evolution equations we can give a proof of the following lemma.

Lemma 6. There exists a uniform constant \( C \) depending only on the initial data such that along the flow,

\[ \| \nabla v \|_{C^0} \leq C. \quad (3.4) \]

Proof of Lemma 6: This is a straightforward modification of Perelman’s maximum principle argument for the bound of the gradient of the Ricci potential (see [ST], Proposition 6). Since \( v \) is uniformly bounded along the flow by Lemma 4, we may choose a constant \( B \) such that \( v + B \geq 0 \). Define

\[ H = \frac{|\nabla v|^2}{v + 2B}. \]

Compute, using Lemma 5,

\[ (\Delta + X - \partial_t)H = \frac{H(H - 2B)}{v + 2B} - \frac{2\text{Re} (g^{jk}\partial_j H \partial_k v)}{v + 2B} + \frac{|\nabla \nabla v|^2 + |\nabla v|^2}{v + 2B}. \quad (3.5) \]

Fix \( T > 0 \). At a maximum point of \( H(x,t) \) for \((x,t) \in M \times (0,T]\), the middle term on the right side of (3.5) vanishes and the left hand side of (3.5) is nonpositive. Then

\[ \sup_{(x,t) \in M \times [0,T]} H(x,t) \leq \max (2B, \sup_{x \in M} H(x,0)), \quad (3.6) \]

and since \( v \) is uniformly bounded, the result follows. Q.E.D.

We can now prove:

Lemma 7. There exists a uniform constant \( C \) depending only on the initial data such that along the flow,

\[ \| \nabla \theta_{X,\omega} \|_{C^0} = \| X \|_{C^0} \leq C. \quad (3.7) \]
Proof of Lemma 7: Since
\[ \nabla \Phi_t \frac{\partial \tilde{\phi}}{\partial t} = \nabla f \]
and \(|\nabla f|\) is bounded by Theorem 7, part (i), the result follows from (2.39) and Lemma 6. Q.E.D.

Lemma 8 There exists a uniform constant \(C\) such that along the flow we have
\[ \|\Delta v\|_{C^0} \leq C, \quad \|\Delta \theta_{X,\omega}\|_{C^0} \leq C. \] (3.8)

Proof of Lemma 8: First note that \(|Xv| \leq |X||\nabla v| \leq C\) by Lemma 6 and Lemma 7. From Lemma 5, part (iii) we have
\[ (\Delta + X - \partial_t)((\Delta + X)v) = -\Delta v - Xv - |\nabla \tilde{\nabla} v|^2 \leq -\Delta v \left(1 + \frac{\Delta v}{n}\right) + C, \] (3.9)
where we have used the elementary inequality \(|\Delta v|^2 \leq n|\nabla \nabla v|^2\). Fix an arbitrary \(T > 0\). At a minimum point of \((\Delta + X)v\) on \(M \times (0, T]\) the left hand side of (3.9) is nonnegative and hence \(\Delta v\) is bounded uniformly from below at this point. This gives the lower bound of \((\Delta + X)v\) along the flow, depending only on the initial data.

To estimate \(\|\Delta v\|_{C^0}\), it suffices to prove a uniform upper bound for \((\Delta + X)v\). This argument is similar to Perelman’s estimate of the scalar curvature (see [ST]). Define
\[ G = \frac{(\Delta + X)v + 2|\nabla v|^2}{v + 2B} \] (3.10)
where \(B\) is chosen as in the proof of Lemma 6. Compute
\[ (\Delta + X - \partial_t)G = -2Re \left( \frac{\nabla G \cdot \nabla v}{v + 2B} \right) + \frac{|\nabla \nabla v|^2 + 2|\nabla v|^2}{v + 2B} - \frac{2BG}{(v + 2B)}. \] (3.11)

Since \(1/(v + 2B)\), \(|Xv|\) and \(|\nabla v|\) are uniformly bounded, we have
\[ (\Delta + X - \partial_t)G \geq -2Re \left( \frac{\nabla G \cdot \nabla v}{v + 2B} \right) + C_1|\nabla \nabla v|^2 - C_2|\Delta v| + C_3, \] (3.12)
for uniform constants \(C_1, C_2, C_3 > 0\) with \(C_1\) uniformly bounded from below away from 0. By the maximum principle and a similar argument to the one above, we have \((\Delta + X)v \leq C\) for some uniform constant \(C\). This gives the estimate for \(\Delta v\). Notice that
\[ \Delta(v + \theta_{X,\omega}) = -\Delta f, \] (3.13)
which is uniformly bounded by Theorem 7, part (i). It follows that \(\Delta \theta_{X,\omega}\) is uniformly bounded. Q.E.D.
3.2 An $L^2/C^0$ Poincaré inequality

Recall that we have the following Poincaré-type inequality on Kähler manifolds $(M, \omega)$ with $\omega$ in $\pi_{\omega} C_1(M)$ (see [F], or Lemma 2 of [PSSW1])

$$\frac{1}{V} \int_M \eta^2 e^{-f} \omega^n \leq \frac{1}{V} \int |\nabla \eta|^2 e^{-f} \omega^n,$$  \hspace{1cm} (3.14)

for all $\eta \in C^\infty(M)$ with $\int_M \eta e^{-f} \omega^n = 0$. Define $b = b(t)$ by

$$b = \frac{1}{V} \int_M u_{X, \omega} e^{-f} \omega^n.$$  \hspace{1cm} (3.15)

Making use of (3.14) together with Theorem 7, parts (i) and (iii) we can prove the following:

**Lemma 9** There exists a uniform constant $C$ such that

$$\|u_{X, \omega} - b\|^\prime_{C^0} \leq C \|\nabla u_{X, \omega}\|_{L^2} \|\nabla u_{X, \omega}\|_{C^0}.$$  \hspace{1cm} (3.16)

**Proof of Lemma 9:** See Lemma 3 in [PSSW1]. Q.E.D.

3.3 A smoothing lemma

The following is an analogue of the smoothing lemma from [PSSW1] (see [B], [CTZ] for related results.)

**Lemma 10** There exist positive constants $\delta$ and $K$ depending only on $n$ and the constant $C_X = \sup_{t \in [0, \infty)} \|X\|_{C^0}(t)$ with the following property. For any $\varepsilon$ with $0 < \varepsilon \leq \delta$ and any $t_0 \geq 0$, if

$$\|(u_{X, \omega} - b)(t_0)\|_{C^0} \leq \varepsilon,$$ \hspace{1cm} (3.17)

then

$$\|\nabla u_{X, \omega}(t_0 + 2)\|_{C^0} + \|(\Delta + X)u_{X, \omega}(t_0 + 2)\|_{C^0} \leq K \varepsilon.$$ \hspace{1cm} (3.18)

**Proof of Lemma 10:** The modified Ricci potential $u_{X, \omega}$ evolves by

$$\dot{u}_{X, \omega} = (\Delta + X)u_{X, \omega} + u_{X, \omega} - b(t).$$ \hspace{1cm} (3.19)

Indeed, observe that $u_{X, \omega}$ differs from $-v$ by a constant depending only on time, so applying part (i) of Lemma 5 gives (3.19) modulo a constant. To determine the constant, differentiate the equation $\int_M e^{-f} \omega^n = V$ in time and use the relation $\dot{f} = \dot{u}_{X, \omega} - Xu_{X, \omega}$.

Define constants $c = c(t)$ by

$$\dot{c} = \dot{b} + c, \quad c(t_0) = 0,$$ \hspace{1cm} (3.20)
so that \( w = -u_{X,\omega} + b - c \) satisfies
\[
\frac{\partial w}{\partial t} = (\Delta + X)w + w. \tag{3.21}
\]
Moreover, \( \|w(t_0)\|_{C^0} \leq \varepsilon \).

Now since \( v \) and \( w \) differ only by a constant, we have from Lemma 5 that
\[
\frac{\partial}{\partial t} |\nabla w|^2 = (\Delta + X)|\nabla w|^2 - |\nabla \nabla w|^2 + |\nabla w|^2, \tag{3.22}
\]
and
\[
\frac{\partial}{\partial t} (\Delta + X)w = (\Delta + X)(\Delta + X)w + (\Delta + X)w + |\nabla \nabla w|^2. \tag{3.23}
\]

We can then modify the argument of Lemma 1 in [PSSW1] to obtain the result. Assume without loss of generality that \( t_0 = 0 \). By (3.21), the maximum principle gives
\[
\|w(t)\|_{C^0} \leq e^{2\varepsilon} \text{ for } t \in [0,2].
\]
From
\[
\frac{\partial}{\partial t} (e^{-2t}(w^2 + t|\nabla w|^2)) \leq (\Delta + X)(e^{-2t}(w^2 + t|\nabla w|^2)), \tag{3.24}
\]
we have \( \|\nabla w\|_{C^0}(t) \leq e^{4\varepsilon^2} \) and \( \|Xw\|_{C^0}(t) \leq C_X e^{2\varepsilon} \) for \( t \in [1,2] \). Define
\[
L = e^{-(t-1)}(|\nabla w|^2 - \varepsilon n^{-1}(t - 1)(\Delta + X)w), \tag{3.25}
\]
and estimate, using the inequality \( (\Delta w)^2 \leq n|\nabla \nabla w|^2 \),
\[
\frac{\partial}{\partial t} L \leq (\Delta + X)L + e^{-(t-1)}n^{-1}(-\Delta w)(\varepsilon + \Delta w) - e^{-(t-1)}\varepsilon n^{-1} Xw. \tag{3.26}
\]
We claim that \( L < 2(1 + C_X)e^{4\varepsilon^2} \) for \( t \in [1,2] \). If the claim is false then, at some point \((x',t') \in M \times [1,2] \), when the inequality first fails, we have \(- (\Delta + X)w \geq (1 + 2C_X)e^{4\varepsilon} \). In particular, at this point, \( \varepsilon + \Delta w \leq -\varepsilon \). It then follows from (3.26) that \(-\Delta w \leq |Xw| \leq C_X e^{2\varepsilon} \), a contradiction, thus proving the claim. Hence, at \( t = 2 \), the inequality
\[
(\Delta + X)w > -2n(1 + C_X)e^{5\varepsilon}, \tag{3.27}
\]
holds on all of \( M \). To obtain the upper bound of \( (\Delta + X)w \) we evolve the quantity
\[
U = e^{-(t-1)}(|\nabla w|^2 + \varepsilon n^{-1}(t - 1)(\Delta + X)w), \tag{3.28}
\]
and conclude by a similar argument, after choosing \( \varepsilon \) sufficiently small, that \( (\Delta + X)w < 2n(1 + C_X)e^{5\varepsilon} \) at \( t = 2 \). Q.E.D.

**Remark** Note that by the uniform bound of \( |X| \) along the flow, we can replace (3.18) in the conclusion of Lemma 10 with
\[
\|\nabla u_{X,\omega}(t_0 + 2)\|_{C^0} + \|\Delta u_{X,\omega}(t_0 + 2)\|_{C^0} \leq K\varepsilon. \tag{3.29}
\]
4 Proof of Theorem 4

We begin by deriving the following analogue for the modified Kähler-Ricci flow of an identity in [PS1],
\[
\dot{Y}_X = -2\|\nabla \nabla u\|^2_\theta + \int_M (X u) |\nabla u|^2 \theta^n
- \int_M (R_{kj} - g_{kj} - \nabla_j X_k) \nabla^j u \nabla^k \theta^n - \int_M (R - n - \nabla_j X^j) |\nabla u|^2 \theta^n. \tag{4.1}
\]

Here and henceforth, we have denoted \(u_{X, \theta}\) and \(\theta_{X, \omega}\) just by \(u\) and \(\theta\) to simplify the notation. Note that, as in (1.19), the notation \(|\cdot|_\theta\) refers to \(L^2\) norms with the volume form \(\omega^n\) replaced by \(e^\theta \omega^n\). To establish the above identity, we use the flow for \(\theta\), as can be easily checked by integration by parts, using the relation \(X_k = \partial_k \theta\). Next, we have the following formula of Bochner-Kodaira type, if \(X^j\) is a holomorphic vector field and \(u\) is a function invariant under \(\text{Im} X\),
\[
\|\nabla \nabla u\|^2_\theta = \|\nabla \nabla u\|^2_\theta + \int_M R_{kj} \nabla^j u \nabla^k u e^\theta \omega^n - \int_M \nabla_j X_k \nabla^j u \nabla^k u e^\theta \omega^n. \tag{4.4}
\]

To establish this, we integrate by parts,
\[
\int_M e^\theta \omega^n (\Delta + X) \eta = 0 \tag{4.3}
\]
for any smooth function \(\eta\), as can be easily checked by integration by parts, using the relation \(X_k = \partial_k \theta\). To establish this, we use the flow for \(\theta\), as can be easily checked by integration by parts, using the relation \(X_k = \partial_k \theta\).
where we have used the fact that $X^j$ is a holomorphic vector field. But

$$X^p \nabla_p u - X^j \nabla_j u = \bar{X}u - Xu = 0,$$

(4.7)
since $u$ is invariant under $\text{Im} X$ and thus we are left with the desired formula (4.4). Substituting this formula and the relation (4.3) in the earlier identity (4.2) gives the identity (4.1).

Once the identity (4.1) is available, the arguments of [PSSW1] apply to give the proof of Theorem 4, with suitable modifications. Write $\pi(\nabla u)$ for the orthogonal projection with respect to the norm $\| \cdot \|_\theta$ of the $T^{1,0}$ vector field $\bar{\nabla} u$ onto the space of holomorphic vector fields. Then

$$\| \bar{\nabla} \nabla u \|_\theta^2 \geq \lambda_X(t) \| \nabla u - \pi(\nabla u) \|_\theta^2$$

$$= \lambda_X(t) \left( \| \nabla u \|_\theta^2 - \| \pi(\nabla u) \|_\theta^2 \right),$$

(4.8)

where $\lambda_X(t)$ is the eigenvalue introduced in (1.20). Since $\| \pi(\nabla u) \|_\theta^2 = \int_M \pi(\nabla u)^j \partial_j u e^\theta \omega^n = -F_X(\pi \bar{\nabla} u)$, we obtain the inequality

$$\tilde{Y}_X(t) \leq -2\lambda_X(t) Y_X(t) - 2\lambda_X(t) F_X(\pi \nabla u) + \int_M |\nabla u|^2(X u) e^\theta \omega^n$$

$$- \int_M (R_k^j - g_k^j - \nabla_j X_k) \nabla^j u \bar{\nabla} \bar{u} e^\theta \omega^n - \int_M (R - n - \nabla_j X^j)|\nabla u|^2 e^\theta \omega^n.$$  

(4.9)

We come now to the proof of the difference-differential inequality for $Y_X(t)$ in the statement of Theorem 4. First, observe that

$$\| R_k^j - g_k^j - \nabla_j X_k \|_{L^2} = \| R - n - \nabla_j X^j \|_{L^2}.$$  

(4.10)

This is because the left hand side equals $\| \bar{\nabla} \nabla u \|_{L^2}$ and the right hand side equals $\| \Delta u \|_{L^2}$, which are readily seen to be equal by an integration by parts. Next, we claim that the last three terms on the right hand side of (4.9) can all be bounded by a constant multiple of

$$\|\nabla u\|_{L^2}^2 (u - b)(t - 2)\|_{C^0}.$$  

(4.11)

Indeed, since $\theta$ is bounded, we can write

$$\left| \int_M (R_k^j - g_k^j - \nabla_j X_k) \nabla^j u \bar{\nabla} \bar{u} e^\theta \omega^n \right| \leq C \|\nabla u\|_{C^0} \|\nabla u\|_{L^2} \| R_k^j - g_k^j - \nabla_j X_k \|_{L^2}$$

$$= C \|\nabla u\|_{C^0} \|\nabla u\|_{L^2} \| R - n - \nabla_j X^j \|_{L^2}$$

$$\leq C \|\nabla u\|_{L^2}^2 (u - b)(t - 2)\|_{C^0},$$  

(4.12)

where we have applied Lemma 10 to obtain the last line. Note that if $||(u-b)(t-2)||_{C^0} > \varepsilon$, for $\varepsilon$ as in Lemma 10, then we can still obtain the bound

$$\|\nabla u\|_{C^0} \| R - n - \nabla_j X^j \|_{L^2} \leq C \| (u - b)(t - 2) \|_{C^0}^2,$$  

(4.13)
using the uniform estimates of $\|\nabla u\|_{C^0}$ and $\|\Delta u\|_{C^0}$. Similarly,

$$\left| \int_M (R - n - \nabla_j X^j)|\nabla u|^2 e^\theta \omega^n \right| \leq C \|\nabla u\|_{L^2}^2 \|(u - b)(t - 2)\|_{C^0}^2,$$  \hspace{1cm} (4.14)

while the estimate for the remaining term,

$$\left| \int_M |\nabla u|^2 (X u) e^\theta \omega^n \right| \leq \|\nabla u\|_{L^2} \|(u - b)(t - 2)\|_{C^0},$$  \hspace{1cm} (4.15)

is even easier, since $|X u| \leq |X| \cdot |\nabla u| \leq C \|(u - b)(t - 2)\|_{C^0}$.

Let $0 < \rho := 1/(n + 1) < 1$. By the $L^2/C^0$ Poincaré inequality and Lemma 10, we can write

$$\|(u - b)(t - 2)\|_{C^0}^2 \leq C \|\nabla u(t - 2)\|_{L^2}^{2\rho} \|\nabla u(t - 2)\|_{C^0}^{2(1 - \rho)}$$

$$\leq CY_X(t - 2)\rho \|(u - b)(t - 4)\|_{C^0}^{2(1 - \rho)}.$$  \hspace{1cm} (4.16)

We note that these inequalities are homogeneous, in the sense that the sum of the exponents on either side always match. We can thus iterate, and obtain

$$\|(u - b)(t - 2)\|_{C^0}^2 \leq C Y_X(t - 2)\rho \|(u - b)(t - 4)\|_{C^0}^{2(1 - \rho)}$$

$$\leq C Y_X(t - 2)\rho Y_X(t - 4)\rho \|(u - b)(t - 6)\|_{C^0}^{2(1 - \rho)^2}$$

$$\leq \cdots$$

$$\leq C Y_X(t - 2)\frac{\delta_1}{\rho} Y_X(t - 4)\frac{\delta_2}{\rho} \cdots Y_X(t - 2N)\frac{\delta_N}{\rho} \|(u - b)(t - 2(N + 1))\|_{C^0}^{2(1 - \rho)^N},$$  \hspace{1cm} (4.17)

with $\sum_{j=1}^N \delta_j + 2(1 - \rho)^N = 2$. Fix $N$ with $2(1 - \rho)^N < 1$ and set $\delta_0 = 1$. Since the quantity $\|(u - b)(t - 2(N + 1))\|_{C^0}$ is bounded by Lemma 4, the statement of Theorem 4 follows from the inequalities (4.9), (4.12), (4.14), (4.15) and (4.17).

## 5 Proof of Theorem 2

Theorem 2 follows easily from what we have proved above. Indeed, by (2.3) and (2.42), the variation of the modified Mabuchi energy along the modified Kähler-Ricci flow is given by

$$\dot{\mu}_X = -\frac{1}{V} \int_M |\nabla u_{X,\omega}|^2 e^{\theta_{X,\omega}} \omega^n = -\frac{1}{V} Y_X(t).$$  \hspace{1cm} (5.1)

Integrating in $t$, we see that condition $(A_X)$ implies:

$$\int_0^\infty Y_X(t) dt < \infty.$$  \hspace{1cm} (5.2)

On the other hand, from (4.2) and the uniform bounds of $\theta, X u_{X,\omega}, R$ and $\nabla_j X^j$ we obtain

$$\dot{Y}_X \leq CY_X.$$  \hspace{1cm} (5.3)
The inequalities (5.2) and (5.3) imply (as in Section § 2 of [PS1]) that $Y_X(t) \to 0$ as $t \to \infty$.

By the uniform bound of $\|\nabla u_{X,\omega}\|_{C^0}$ and Lemma 9 we have

$$\|u_{X,\omega} - b\|_{C^0} \to 0, \quad \text{as } t \to \infty. \quad (5.4)$$

Then from Lemma 10 we see that

$$\|\Delta u_{X,\omega}\|_{C^0} \to 0, \quad \text{as } t \to \infty. \quad (5.5)$$

Since $\Delta u_{X,\omega} = R - n - \nabla_j X^j$, the first part of Theorem 2 is established. The $L^p$ integrability of $\|R - n - \nabla_j X^j\|_{C^0}$ on $[0, \infty)$ is established in the same way as part (ii) of Theorem 1 in [PSSW1]. The proof of Theorem 2 is complete.

6 Proof of Theorem 1

It is convenient to introduce the following fifth condition:

(o) For each $k = 0, 1, 2, \ldots$, there exists a finite constant $A_k$ so that

$$\sup_{t \geq 0} \|\varphi\|_{C^k} \leq A_k. \quad (6.1)$$

We shall prove the following implications

$$(o) \iff (iii)$$

$$(o) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (iii)$$

$$(iv) \Rightarrow (v) \Rightarrow (i) \Rightarrow (iii) \quad (6.2)$$

from which the equivalence of all five conditions (i)-(v), and hence Theorem 1, all follow at once.

6.1 (o) $\iff$ (iii)

This is the extension to the case of the modified Kähler-Ricci flow of the classical fact that a $C^0$ estimate for the complex Monge-Ampère equation implies $C^k$ estimates to all orders. We present it here, with emphasis only on those points that require additional arguments. We note that in [TZ2], a different method is used to obtain higher order estimates, involving a modification of the potential $\varphi$ along the flow. We give here a direct proof of the higher order estimates for solutions of (2.23).

The first step is to show that $C^0$ estimates for $\varphi$ imply second order estimates for $\varphi$. In this section, for ease of notation, we will use $\hat{g}_{kj}$ to denote the original metric $g^0_{kj}$, and $\hat{\Delta}$ for the Laplacian with respect to this metric. As in the original approach of Yau [Y1] and Aubin [A], we apply the maximum principle to the flow of $\log (n + \hat{\Delta} \varphi) - A \varphi$, where
A is a large constant to be chosen later. It is convenient to use the formulas obtained in [PSS] for general flows. For this, we introduce the endomorphism
\[ h^{\alpha \beta} = \hat{g}^{\alpha \gamma} g_{\gamma \beta}. \] (6.3)

Then \( n + \hat{\Delta} \varphi = \text{Tr} h \), and we have (see e.g. [PSS], eq. (2.27))
\[ (\Delta - \partial_t) \log \text{Tr} h = \frac{1}{\text{Tr} h} \left( \hat{\Delta} \left( \log \frac{\omega^n}{\omega^n_0} - \hat{\varphi} \right) - \hat{R} \right) \]
\[ + \frac{1}{\text{Tr} h} h^r_j (h^{-1})^{ps} \hat{g}^{sq} \hat{R}_{qp}^r \]
\[ + \left\{ \frac{g^{jk} \text{Tr} (\nabla_j h^{-1} \nabla_k h)}{\text{Tr} h} - \frac{g^{jk} \nabla_j \text{Tr} h \nabla_k \text{Tr} h}{(\text{Tr} h)^2} \right\}, \] (6.4)

where \( \hat{R}, \hat{R}_{kj}^{\alpha \beta} \) are the scalar and the Riemann curvature tensor of the metric \( \hat{g}_{kj}, \hat{g}^{ik} \) is its inverse, and otherwise all indices are raised and lowered using the metric \( g_{kj} \). The last line is non-negative (see [Y1]), and the second line is bounded below by \(-C_1 \text{Tr} h^{-1}\). Since \( 0 < (\text{Tr} h)^{-1} < \text{Tr} h^{-1}\), we obtain
\[ (\Delta - \partial_t) \log \text{Tr} h \geq \frac{1}{\text{Tr} h} \hat{\Delta} \left( \log \frac{\omega^n}{\omega^n_0} - \hat{\varphi} \right) - C_1 \text{Tr} h^{-1}. \] (6.5)

For the modified Kähler-Ricci flow, we have
\[ \hat{\Delta} \left( \log \frac{\omega^n}{\omega^n_0} - \hat{\varphi} \right) = -\hat{\Delta} \varphi - \hat{\Delta} \theta - \hat{\Delta} f(\omega_0) = -\text{Tr} h + n - \hat{\Delta} \theta - \hat{\Delta} f(\omega_0). \] (6.6)

The new term compared to the Kähler-Ricci flow is \(-\hat{\Delta} \theta\), which is not yet known to be bounded. To eliminate it, we consider instead the expression \((\Delta + X - \partial_t) \log \text{Tr} h\). The main observation is that:
\[ X \text{Tr} h = X(\hat{\Delta} \varphi) = \hat{\Delta} \theta - \hat{\Delta} \hat{\theta} + (\hat{\nabla}_j X^m) h^j_m - \hat{\nabla}_m X^m, \] (6.7)
where \( \hat{\theta} = \theta_{X,\omega_0} \). To see this, observe that using the fact that \( X \) is holomorphic,
\[ \hat{\Delta} (X \varphi) = \hat{g}^{jk} \hat{\nabla}_j \hat{\nabla}_k (X^m \nabla_m \varphi) = g^{jk} \hat{\nabla}_j X^m (g_{km} - \hat{g}_{km}) + X(\hat{\Delta} \varphi). \] (6.8)

Then (6.7) follows from the identity \( X \varphi = \theta - \hat{\theta} \).

Hence for the modified Kähler-Ricci flow,
\[ (\Delta + X - \partial_t) \log \text{Tr} h \geq \frac{1}{\text{Tr} h} (-\text{Tr} h + n - \hat{\Delta} \hat{\theta} + (\hat{\nabla}_j X^m) h^j_m - \hat{\nabla}_m X^m) - C_1 \text{Tr} h^{-1} \]
\[ \geq -C_2 - C_3 \text{Tr} h^{-1}. \] (6.9)

From here, the proof can proceed as before. Set \( A = C_3 + 1 \). Since \( \Delta \varphi = -\text{Tr} h^{-1} + n \), and \( \hat{\varphi} \) and \( X \varphi = \theta - \hat{\theta} \) are bounded by Lemmas 4 and 1, we have
\[ (\Delta + X - \partial_t)(\log \text{Tr} h - A \varphi) \geq -C_4 + \text{Tr} h^{-1}. \] (6.10)
Fix $T > 0$. Then at a maximum point $(x_0, t_0)$ of the function $\log \text{Tr} h - A\varphi$ on $M \times (0, T]$, the quantity $\text{Tr} h^{-1}$ is bounded from above. But from the modified Kähler-Ricci flow equation (2.23) and the fact that $\varphi, \theta$ and $\varphi$ are bounded, we see that the logarithm of the product of the eigenvalues of $h$ is bounded, giving an upper bound of $\text{Tr} h(x_0, t_0)$. Thus, using again the estimate of $\|\varphi\|_{C^0}$, we obtain a uniform upper bound of $\text{Tr} h$ along the modified Kähler-Ricci flow. The positivity of the metric $g_kj = \hat{g}_kj + \partial_j \partial_k \varphi$ ensures uniform bounds of $\partial_j \partial_k \varphi$ along the flow. In addition, from the lower bound of $\log (\omega^n / \hat{\omega}^n)$, we have an estimate $g_kj \geq C_5 \hat{g}_kj$, for $C_5 > 0$, showing that $g$ is uniformly equivalent to $\hat{g}$ along the flow.

We now give the third order estimates. As in [Y1], set $\varphi_{jkm} = \hat{\nabla}_m \partial_k \partial_j \varphi$ and $S \equiv g^{i\bar{j}} g^{k\bar{l}} \varphi_{jkm} \varphi_{\bar{l}st}$. Again, it is convenient to compute instead in terms of the connection $\nabla h h^{-1}$, in terms of which we have

$$S = g^{\gamma\delta} g_{\mu\beta} g^\ell (\nabla_m h h^{-1})^\beta (\nabla_\gamma h h^{-1})^\mu = |\nabla h h^{-1}|^2.$$  

(6.11)

From [PSS] eq. (2.48), under any flow, we have the general formula

$$(\Delta - \partial_t) S = |\hat{\nabla}(\nabla h h^{-1})|^2 + |\nabla h h^{-1}|^2$$

$$+ g^{\mu\gamma}(\Delta - \partial_t)(\nabla_m h h^{-1}), \nabla_\gamma h h^{-1}) + g^{\mu\gamma}(\nabla_m h h^{-1}, (\Delta - \partial_t)(\nabla_\gamma h h^{-1}))$$

$$+ \left((h^{-1} \dot{h} + \text{Ric})^{\mu\gamma} g_{\mu\beta} g^\ell + g^{\mu\gamma}(h^{-1} \dot{\text{Ric}})_{\mu\beta} g^\ell + g^{\mu\gamma} g_{\mu\beta}(h^{-1} \dot{\text{Ric}})_{\mu\beta} \right)$$

$$\times (\nabla h h h^{-1})^\beta (\nabla_\gamma h h^{-1})^\mu \alpha.$$  

(6.12)

Next, we specialize to the modified Kähler-Ricci flow. We always have the following formula relating the curvatures of two metrics $g_kj$ and $\hat{g}_kj$

$$\Delta (\nabla_m h h^{-1})^\alpha_{\beta} = \nabla^\delta \hat{R}_{\alpha\mu\beta} \beta - \nabla_m R^\alpha_{\beta}. $$  

(6.13)

For the modified Kähler-Ricci flow, we have

$$\partial_t (\nabla_m h h^{-1})^\alpha_{\beta} = \nabla_m (h^{-1} \dot{h})^\alpha_{\beta} = -\nabla_m R^\alpha_{\beta} + \nabla_m \nabla_\beta X^\alpha.$$  

(6.14)

Thus we obtain

$$(\Delta - \partial_t)(\nabla_m h h^{-1})^\alpha_{\beta} = \nabla^\delta \hat{R}_{\alpha\mu\beta} \beta - \nabla_m \nabla_\beta X^\alpha.$$  

(6.15)

Similarly

$$(h^{-1} \dot{h})_{kj} + R_{kj} = g_{kj} + \nabla_j X_k.$$  

(6.16)

Clearly, the terms $\nabla_m \nabla_\beta X^\alpha$ and $\nabla_j X_k$ on the right hand side of the previous two equations are the only changes due to the modified Kähler-Ricci flow. Using then the equation (2.51) for the Kähler-Ricci flow in [PSS], we obtain immediately the following formula

$$(\Delta - \partial_t) S = |\hat{\nabla}(\nabla h h^{-1})|^2 + |\nabla h h^{-1}|^2 + |\nabla h h^{-1}|^2$$

$$+ g^{\mu\gamma}\nabla^\delta \hat{R}_{\alpha\beta} \gamma (\nabla_\gamma h h^{-1})^\mu \alpha$$

$$+ (I) + (II) + (III) + (IV) + (V), $$  

(6.17)
where the terms (I)-(V) are due to the modifications arising from the holomorphic vector field $X$, and given explicitly by

\[
\begin{align*}
(I) &= \nabla^\gamma X^m g_{\mu\beta} g^{\delta\alpha} (\nabla_m h h^{-1})^{\beta} (\nabla_\gamma h h^{-1})^{\mu} \\
(II) &= -g^{m\gamma} g_{\mu\beta} g^{\delta\alpha} \nabla_m \nabla_\ell X^\beta (\nabla_\gamma h h^{-1})^{\mu} \\
(III) &= g^{m\gamma} g_{\mu\beta} \nabla^\alpha X^{\ell} (\nabla_m h h^{-1})^{\beta} (\nabla_\gamma h h^{-1})^{\mu} \\
(IV) &= -g^{m\gamma} g_{\mu\beta} g^{\delta\alpha} (\nabla_m h h^{-1})^{\beta} (\nabla_\gamma h h^{-1})^{\mu} \\
(V) &= -g^{m\gamma} \nabla_\beta X_{\mu} g^{\delta\alpha} (\nabla_m h h^{-1})^{\beta} (\nabla_\gamma h h^{-1})^{\mu}.
\end{align*}
\]

(6.18)

Because of the presence of the connection in e.g. $\nabla^\gamma X^m = g^{\alpha\gamma} \nabla_\alpha X^m$, the first covariant derivatives of $X^m$ are of order $O(S^{1/2})$, and hence

\[
|(I)| + |(III)| + |(V)| \leq C_6 S |\nabla X|. \tag{6.19}
\]

The second covariant derivatives of $X^m$ can be expressed as follows

\[
\nabla_\gamma \nabla_\alpha X^\mu = \nabla_\gamma (\nabla_\alpha X^m) + (\nabla_\alpha h h^{-1})^{\mu} X^\nu
\]

\[
= \nabla_\gamma \nabla_\alpha X^m + \nabla_\gamma (\nabla_\alpha h h^{-1})^{\mu} X^\nu + (\nabla_\alpha h h^{-1})^{\mu} \nabla_\gamma X^\nu. \tag{6.20}
\]

The first term on the right hand side is again of order $O(S^{1/2})$. The second term can be bounded by $|\nabla(\nabla h h^{-1})|$ since $|X|$ is bounded. Thus we can write

\[
|(II)| + |(IV)| \leq C_7 (S + 1) + |X| \cdot |\nabla(\nabla h h^{-1})| \cdot |\nabla h h^{-1}| + |\nabla h h^{-1}|^2 \cdot |\nabla X| \leq C_8 S + \frac{1}{2} |\nabla(\nabla h h^{-1})|^2 + S |\nabla X| + C_9. \tag{6.21}
\]

Putting this all together, we obtain the following estimate for the flow of $S$ in the modified Kähler-Ricci flow,

\[
(\Delta - \partial_t) S \geq \frac{1}{2} |\nabla(\nabla h h^{-1})|^2 + |\nabla(\nabla h h^{-1})|^2 - C_9 S |\nabla X| - C_{10} (1 + S). \tag{6.22}
\]

By the method of [Y1], we can control terms of order $O(S)$ using the evolution equation for $\text{Tr} \, h$. However, we will need an additional argument to deal with the quantity $S |\nabla X|$ which is of the order $O(S^{3/2})$. Since $|X|$ is uniformly bounded along the flow, we have

\[
(\Delta - \partial_t) |X|^2 = |\nabla X|^2 - |X|^2 - \partial_t \partial_\ell^2 \theta X^\ell X^\ell \geq \frac{1}{2} |\nabla X|^2 - C_{11}. \tag{6.23}
\]

We define a constant $K = 65 \sup_{M \times [0, \infty]} (|X|^2 + 1)$ and compute the evolution of the quantity $S/(K - |X|^2)$. Combining (6.22) and (6.23) we have

\[
(\Delta - \partial_t) \left( \frac{S}{K - |X|^2} \right) \geq \frac{|\nabla(\nabla h h^{-1})|^2 - |\nabla(\nabla h h^{-1})|^2}{2(K - |X|^2)} + \frac{S |\nabla X|^2}{2(K - |X|^2)^2}
\]

1There are actually some partial cancellations between the terms I-V. We shall not need this fact, and won’t exhibit it more explicitly.
We will use the good first and second terms on the right hand side of this inequality to deal with the bad third and fifth terms. We estimate the third term as follows:

\[
\left| 2g^{ij}\partial_i S \partial_j X \right|^2 \leq \frac{S|\nabla X|^2}{4(K - |X|^2)^2} + \frac{32|X|^2 \left( |\nabla (\nabla h \cdot h^{-1})|^2 + |\nabla (\nabla h \cdot h^{-1})|^2 \right)}{(K - |X|^2)^2} \leq \frac{S|\nabla X|^2}{4(K - |X|^2)^2} + \frac{\left( |\nabla (\nabla h \cdot h^{-1})|^2 + |\nabla (\nabla h \cdot h^{-1})|^2 \right)}{2(K - |X|^2)}. \tag{6.25}
\]

For the fifth term, observe that:

\[
\frac{C_9 S|\nabla X|}{K - |X|^2} \leq \frac{S|\nabla X|^2}{4(K - |X|^2)^2} + C_9^2 S. \tag{6.26}
\]

Combining all of the above, we obtain

\[
(\Delta - \partial_t) \left( \frac{S}{K - |X|^2} \right) \geq -C_{13}(1 + S). \tag{6.27}
\]

But from the computation for the second order estimate, we have

\[
(\Delta - \partial_t) \text{Tr} \, h \geq \frac{1}{2} S - C_{14}, \tag{6.28}
\]

and so applying the maximum principle to the quantity \( S/(K - |X|^2) + 3C_{13} \text{Tr} \, h \) it follows that \( S/(K - |X|^2) \) and hence \( S \) is bounded.

**Remark** Instead of computing the evolution of \( S/(K - |X|^2) \), an alternative is to compute the evolution of the tensor \( T_j^k = (\nabla_j h^{-1})_\ell^k X^\ell \). Indeed one can prove that:

\[
(\Delta - \partial_t)|T|^2 \geq -B_1 S - B_2, \tag{6.29}
\]

for uniform constants \( B_1 \) and \( B_2 \). Combining this with the evolution of \( \text{Tr} \, h \) gives an upper bound of \( |T| \) and hence \( |\nabla X| \) along the flow. This implies that the term \( S|\nabla X| \) is of order \( O(S) \) and one can proceed in the usual way to bound \( S \).

In order to apply the standard parabolic estimates to obtain the higher order estimates, we require a derivative bound of \( g_{\ell j} \) in the \( t \)-direction (cf. \([\text{Ch}]\), for example). Given the estimates proved so far, it is sufficient to bound \( |\text{Ric}(g)| \). The evolution of the Ricci curvature along the modified Kähler-Ricci flow is given by

\[
(g^{\ell \eta} \nabla_\ell \nabla_\eta - \partial_t) R_{\ell j} = R_{\ell \eta}^{\ell \eta} R_{\ell j} - R_{\ell j}^{\ell \eta} R_{\eta \eta} - X^\ell \nabla_\ell R_{\ell j} + R_{\ell \eta} \nabla_j X^\ell . \tag{6.30}
\]
Then given the estimates on $T \, r$ and $S$, we have

$$
(\Delta - \partial_t + X)|\text{Ric}(g)| = \frac{1}{|\text{Ric}(g)|} \left\{ |\nabla \text{Ric}(g)|^2 - |\nabla |\text{Ric}||^2 + |\text{Ric}(g)|^2
- R_{\xi_j}^{\tau \rho} R_{\tau \rho} R_{\xi_j}^\tau + \nabla^\tau X^p g^{\rho \tau} R_{\xi_j} R_{\tau \rho} - R_{\xi_{\ell}} \nabla_j X^\ell R_{j \xi_j} \right\}
\geq -C_{15}(|\text{Rm}|^2 + 1). \tag{6.31}
$$

But from the computations above for the evolution of $S$, there exist uniform constants $C_{16}, C_{17}$ with $C_{16} > 0$ such that

$$
(\Delta - \partial_t + X)S \geq C_{16} |\text{Rm}|^2 - C_{17}. \tag{6.32}
$$

Then by applying the maximum principle to the quantity $|\text{Ric}(g)| + \frac{1}{C_{16}} (C_{15} + 1)S$ we obtain the desired upper bound on $|\text{Ric}(g)|$.

We have now established uniform parabolic $C^1$ estimates for the metric $g_{\xi_j}$ along the flow. One can now obtain the higher order estimates in the usual way. We differentiate the equation (2.23) in space, making use of Lemma 3, and then apply the standard parabolic estimates (see [L], for example) together with a bootstrapping argument.

### 6.2 (o) $\Rightarrow$ (iv)

The remaining implications are all straightforward adaptations of arguments in [PSSW1], so we shall be brief. It is convenient to formulate the following lemmas:

**Lemma 11** Let $W(t)$ be a non-negative $C^\infty$ function of $t \in [0, \infty)$ with $W(t) \leq K_0$ satisfying the difference-differential inequality

$$
\dot{W}(t) \leq -2\lambda W(t) + \lambda \prod_{j=0}^{N} W(t - 2j)^{\nu_j}, \quad \text{for } t \geq K_1 \geq 2N, \tag{6.33}
$$

where $\lambda$ is a strictly positive constant, and $\nu_j \geq 0$ satisfy $\frac{1}{2} \sum_{j=0}^{N} \nu_j = 1$. Then there exist constants $C, \kappa$ with $\kappa > 0$ depending only on $K_0, K_1, \lambda, N, \nu_j$ so that

$$
W(t) \leq C e^{-\kappa t}. \tag{6.34}
$$

**Proof of Lemma 11**: See Section §5 of [PSSW1]. Q.E.D.

**Lemma 12** There exist constants $c_1, c_2 > 0$ depending only on the complex manifold $M$ and the holomorphic vector field $X$ such that for all $\omega \in \mathcal{K}_X$,

$$
c_1 \lambda(\omega) \leq \lambda_X(\omega) \leq c_2 \lambda(\omega). \tag{6.35}
$$
Proof of Lemma 12: This follows immediately from the fact that $\theta_{X,\omega}$ is uniformly bounded (by a constant independent of choice of metric in $\mathcal{K}_X$) and the argument of Lemma 1 in [PSSW2]. Q.E.D.

Lemma 13 Let $Y_X(t)$ be given as in (1.8) for the modified Kähler-Ricci flow. Assume that the following three conditions are satisfied

(a) $F_X \equiv 0$
(b) $Y_X(t) \to 0$ as $t \to \infty$
(c) $\inf_{t \geq 0} \lambda(t) > 0$.

Then there exists constants $C, \kappa$ with $\kappa > 0$ so that $Y_X(t) \leq Ce^{-\kappa t}$.

Proof of Lemma 13: We apply Theorem 4. Under conditions (a)-(c), the difference-differential inequality given there, together with Lemma 12, implies that $Y_X(t)$ satisfies a difference-differential inequality exactly of the type formulated in Lemma 11. The desired inequality follows then from this lemma. Q.E.D.

Returning to the proof of (o) $\Rightarrow$ (iv), assume that (o) holds, that is, all norms $\|\varphi(t)\|_C^k$ are uniformly bounded in time for each $k$. Then there exists a sequence of times $t_m \to \infty$ such that $\varphi(t_m)$ converges in $C^\infty$ to an element $\varphi(\infty)$ of $\mathcal{P}_X(M,\omega_0)$. Since the modified Mabuchi K-energy $\mu_X$ is decreasing along the modified flow, it follows that for any time $t$,

$$\mu_X(\varphi(t)) \geq \mu_X(\varphi(\infty)), \quad (6.36)$$

and hence $\mu_X$ is bounded below along the modified flow. Using this, one can see that the limit metric $g_{kj}(\infty)$ must be a Kähler-Ricci soliton with respect to $X$ (c.f. the proof of Theorem 2). Theorem 5 then establishes condition $(A_X)$. Next, we claim that Condition (S) is also satisfied, that is, the eigenvalues $\lambda(t)$ are bounded below by a strictly positive constant $\lambda$. Otherwise, let $\varphi(t_m)$ be a subsequence with $\lambda(t_m) \to 0$. It contains a subsequence $\varphi(t_\ell)$ such that the corresponding metrics $g_{kj}(t_\ell)$ converge in $C^\infty$ to a Kähler-Ricci soliton $g_{kj}(\infty)$ with respect to $X$. In [PS1], it was shown that $\lambda(t_\ell) \to \lambda(\infty)$ if $g_{kj}(t_\ell) \to g_{kj}(\infty)$ and the dimensions of the holomorphic vector fields of the complex structures for $g_{kj}(t_\ell)$ and $g_{kj}(\infty)$ are the same. In the present case, the complex structures of $g_{kj}(t_\ell)$ and $g_{kj}(\infty)$ are the same, so we do have $\lambda(t_\ell) \to \lambda(\infty)$. Since $\lambda(\infty) > 0$ by definition, we obtain a contradiction. Condition (S) is established.

The existence of a Kähler-Ricci soliton with respect to $X$ implies that (a) in Lemma 13 holds and condition $(A_X)$ gives (b) by Theorem 2. Since (c) in this Lemma is the same as (S), Lemma 13 applies, and (iv) is established.

6.3 (iv) $\Rightarrow$ (ii)

Assume that (iv) is satisfied, and thus $Y_X(t)$ is rapidly decreasing. Then Lemma 9 implies that

$$\|u_{X,\omega} - b\|_{C^0} \leq C e^{-\frac{1}{2(n+1)} \kappa t} \quad (6.37)$$
But Lemma 10 implies then that
\[ \| R - n - \nabla_j X^j \|_{C^0} \leq C' e^{-\frac{1}{2(n+1)} t} \]  
(6.38)
which gives (ii).

### 6.4 (ii) $\Rightarrow$ (iii)
Assume (ii). Since
\[ \partial_t \log \left( \omega^n / \omega^n_0 \right) = g^{j\bar{k}} \tilde{g}_{kj} = -(R - n - \nabla_j X^j), \]
we obtain immediately, uniformly in $t \in [0, \infty)$
\[ | \log \left( \omega^n / \omega^n_0 \right) | \leq \int_0^\infty \| R - n - \nabla_j X^j \|_{C^0} dt < C. \]  
(6.39)
Next, from the modified Kähler-Ricci flow and the uniform bound for $\| \dot{\varphi} \|_{C^0}$ (Lemma 4), it follows that
\[ \| \varphi \|_{C^0} = \| \dot{\varphi} - \log \left( \omega^n / \omega^n_0 \right) - \theta + f(\omega_0) \|_{C^0} \leq C. \]  
(6.40)

### 6.5 (iv) $\Rightarrow$ (v)
Assume (iv). We have already seen that (iv) implies (ii), which implies in turn (iii), which is equivalent to (o). Thus all metrics $g_{kj}(t)$ are equivalent with uniform bounds in $t$. The same arguments as in [PS1], applied with straightforward modifications to $u_{X,\omega} = f + \theta_{X,\omega}$ instead of the Ricci potential there, show that $\| u_{X,\omega} \|_{(s)}$ converges exponentially fast to 0 with respect to any Sobolev norm $s$. It follows easily from there that $g_{kj}(t)$ converges exponentially fast to a Kähler-Ricci soliton $g_{kj}(\infty)$.

Since all the remaining implications in (6.2) are trivial, the proof of Theorem 1 is complete.

### 7 Proof of Theorem 3
It is now easy to prove Theorem 3. If the modified Kähler-Ricci flow converges to a Kähler-Ricci soliton with respect to $X$ then, by Theorem 5, Condition $(A_X)$ is satisfied. Furthermore, as part of the proof of the step $(o) \Rightarrow (iv)$, we have seen that the uniform boundedness of $\| \varphi(t) \|_{C^k}$ for each $k$ implies that Condition $(S)$ is satisfied. Thus it remains only to establish the sufficiency of $(A_X)$ and $(S)$ for the exponential convergence of the Kähler-Ricci flow.

The arguments are now practically the same as in the proof of the implication $(o) \Rightarrow (iv)$ earlier. By Proposition 1 and Theorem 2, $(A_X)$ implies (a) and (b) of Lemma 13. In addition, $(S)$ gives condition (c). Thus we obtain the exponential decay of $Y_X(t)$, that is, Condition (iv) of Theorem 1 is satisfied. But Theorem 1 implies then the exponential convergence of the modified Kähler-Ricci flow to a Kähler-Ricci soliton. Q.E.D.
8 Further remarks and questions

(1) As mentioned in the introduction, our results imply that if a metric $\omega_0$ is invariant under the $S^1$ action induced by the imaginary part of a holomorphic vector field $X$ then along the Kähler-Ricci flow starting at $\omega_0$, the quantity $|X|^2$ is uniformly bounded. We remark that an immediate consequence of this is that if $M$ and the initial metric are toric then $g_{kj}(t)$ is bounded from above on compact subsets of the interior of the polytope $\Delta \subset \mathbb{R}^n$. Of course, the case of the Kähler-Ricci flow on toric varieties is by now already well-understood by the work of Zhu [Zhu2] (see also [WZ], [TZ2]).

(2) An obvious question is: what other notions of stability correspond to the existence of a Kähler-Ricci soliton on $M$? By Theorem 2 we see that, assuming condition $(A_X)$, the $L^2$ norm of the quantity $(R - n - \nabla_j X^j)$ tends to zero along the Kähler-Ricci flow. Is there a natural analogue of the Calabi energy and a version of Donaldson’s result [D2] giving ‘semi-stability’ in this case?

(3) If one assumes uniform curvature bounds along the modified Kähler-Ricci flow then, in a similar vein to [PSSW2] and [S2], one would expect convergence of the flow to a soliton if the modified Futaki invariant $F_X$ vanishes and an addition stability assumption holds, such as: condition (S), condition (B) of [PS1], or a modified version of K-stability.

(4) Does the stronger conclusion

$$\|R_{kj} - g_{kj} - \nabla_j X_k\|_{C^0} \to 0, \text{ as } t \to \infty$$

(8.1)

follow from condition $(A_X)$? Observe that $L^2$ convergence to zero does hold assuming $(A_X)$, since by Theorem 2,

$$\int_M |\nabla \nabla u_{X,\omega}|^2 \omega^n = \int_M |\Delta u_{X,\omega}|^2 \omega^n \to 0.$$ 

(8.2)

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