Nonstandard coproducts and the Izergin-Korepin open spin chain

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Abstract

Corresponding to the Izergin-Korepin ($A_2^{(2)}$) $R$ matrix, there are three diagonal solutions ("$K$ matrices") of the boundary Yang-Baxter equation. Using these $R$ and $K$ matrices, one can construct transfer matrices for open integrable quantum spin chains. The transfer matrix corresponding to the identity matrix $K = I$ is known to have $U_q(o(3))$ symmetry. We argue here that the transfer matrices corresponding to the other two $K$ matrices also have $U_q(o(3))$ symmetry, but with a nonstandard coproduct. We briefly explore some of the consequences of this symmetry.
1 Introduction and summary

The notion of coproduct is of fundamental importance in the theory of representations of algebras. Given a representation of an algebra on a vector space $V$, the coproduct $\Delta$ determines the representation on the tensor product space $V \otimes V$. For a classical Lie algebra, the coproduct is trivial: for any generator $x$, the coproduct is $\Delta(x) = x \otimes I + I \otimes x$, where $I$ is the identity matrix. For quantum algebras, the situation is more interesting. Indeed, consider the case $U_q(su(2))$, with a set of three generators $\{j_\pm, h\}$ obeying

$$[h, j_\pm] = \pm j_\pm. \quad (1)$$

As is well known, the “standard” coproduct

$$\Delta(h) = h \otimes I + I \otimes h,$$

$$\Delta(j_\pm) = j_\pm \otimes q^h + q^{-h} \otimes j_\pm, \quad (2)$$

is compatible with the commutation relation

$$[j_+, j_-] = \frac{q^{2h} - q^{-2h}}{q - q^{-1}}. \quad (3)$$

Perhaps less well-known is the fact that there is also a “nonstandard” coproduct

$$\Delta(h) = h \otimes I + I \otimes h,$$

$$\Delta(j_\pm) = j_\pm \otimes I + q^h \otimes j_\pm, \quad (4)$$

which is compatible instead with the $q$-commutation relation

$$j_+ j_- - q^{-1} j_- j_+ = \frac{I - q^{2h}}{1 - q^2}. \quad (5)$$

Remarkably, both of these types of coproducts can be realized in the open integrable quantum spin chain constructed with the $A_2^{(2)}$ $R$ matrix [1] by choosing appropriate boundary conditions. Let us briefly recall the history of this model. Sklyanin [2] pioneered the generalization of the Quantum Inverse Scattering Method [3] to systems with boundaries, and showed that integrable boundary conditions can be obtained from solutions $K(u)$ of the boundary Yang-Baxter equation [4], [5]. This approach was then generalized [6] to spin chains associated with general affine Lie algebras [7], [8]. In particular, for the $A_2^{(2)}$ case, it was found [9] that there are only three diagonal solutions of the boundary Yang-Baxter equation:

$$K^{(0)}(u) = I = \text{diag}(1, 1, 1),$$

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\[ K^{(1)}(u) = \text{diag}(e^{-u}, \frac{\sinh(\frac{i}{2}(3\eta - \frac{i\pi}{2} + u))}{\sinh(\frac{i}{2}(3\eta - \frac{i\pi}{2} - u))}, e^{u}), \]
\[ K^{(2)}(u) = \text{diag}(e^{-u}, \frac{\cosh(\frac{i}{2}(3\eta - \frac{i\pi}{2} + u))}{\cosh(\frac{i}{2}(3\eta - \frac{i\pi}{2} - u))}, e^{u}), \]

(6)

where \( u \) is the spectral parameter, and \( \eta \) is the anisotropy parameter. Let us denote the corresponding transfer matrices for open quantum spin chains with \( N \) sites by \( t^{(i)}(u), i = 0, 1, 2. \) (The construction of these transfer matrices is described below in Section 2.) It was shown in [10], [11] that the transfer matrix \( t^{(0)}(u) \) constructed with the identity matrix \( K^{(0)} \) has \( U_q(o(3)) \) symmetry:

\[ [t^{(0)}(u), S^\pm] = 0, \quad [t^{(0)}(u), S^3] = 0, \quad (7) \]

where the generators obey

\[ [S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \frac{q^{2S^3} - q^{-2S^3}}{q - q^{-1}}, \quad (8) \]

and

\[ S^\pm = \sum_{k=1}^{N} q^{\frac{1}{4}(s_{N-k+1}^3 + s_{k+1}^3)} s_k^\pm q^{-(s_{N-k+1}^3 + s_{k+1}^3)}, \quad S^3 = \sum_{k=1}^{N} s_k^3, \quad (9) \]

where

\[ s^+ = \sqrt{2 \cosh \eta} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad s^- = \sqrt{2 \cosh \eta} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (10) \]

and \( q = e^\eta. \) That is, the transfer matrix has quantum algebra symmetry with the “standard” coproduct (2). This is a generalization of the observation [12], [13] of \( U_q(su(2)) \) symmetry for the \( A^{(1)}_1 \) case. Batchelor and Yung [14] later showed that the open \( A^{(2)}_2 \) spin chain can be mapped to the problem of polymers at surfaces, and that the above three solutions \( K^{(i)}(u) \) correspond to three distinct surface critical behaviors.

There has remained the question: what symmetry – if any – do the transfer matrices constructed with \( K^{(1)} \) and \( K^{(2)} \) have? Naively, one expects that since \( K \neq I, \) there is less symmetry. However, this is not the case. We argue here that the transfer matrices \( t^{(1)}(u) \) and \( t^{(2)}(u) \) also have \( U_q(o(3)) \) symmetry, but with a “nonstandard” coproduct (4):

\[ [t^{(i)}(u), S^\pm] = 0, \quad [t^{(i)}(u), S^3] = 0, \quad i = 1, 2, \quad (11) \]

\footnote{This expectation holds true for the \( A^{(1)}_1 \) case [15]. Indeed, there the diagonal \( K \) matrices contain an additional continuous parameter \( \xi; \) and \( K = I \) is a point \( (\xi \to \infty) \) of enhanced symmetry.}
where the generators obey

\[ [S^3, S^\pm] = \pm 2S^\pm, \quad S^+S^- - q^{-2}S^-S^+ = \frac{\mathbb{1} - q^{2S^3}}{1 - q^2}, \quad (12) \]

and

\[ S^\pm = \sum_{k=1}^{N} s_k^\pm q^{s_{k+1}^1 + \cdots + s_1^1}, \quad S^3 = \sum_{k=1}^{N} s_k^3, \quad (13) \]

where

\[ s^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad s^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad (14) \]

and \( q = e^{4\eta} \). Knowledge of such symmetry is essential for understanding important features of the models such as degeneracies of the spectrum and the Bethe Ansatz solution. Eqs. (11) - (14) are the main results of this Letter. In Section 2 we provide some pertinent details about the construction and symmetry of the models, and we conclude in Section 3 with a brief discussion.

## 2 Some details

In this Section, we briefly review the construction of the transfer matrices, and outline the argument for their symmetry. The solution \( R(u) \) of the Yang-Baxter equation found by Izergin and Korepin \([1]\), which corresponds \([7], [8]\) to the case \( A_2^{(2)} \), can be written in the following form \([10], [11]\)

\[ R(u) = \begin{pmatrix} \begin{array}{ccc|ccc} c & b & e & f \\ d & e & g \\ \hline \bar{e} & \bar{g} & a & g \\ \bar{d} & \bar{g} & b & \bar{e} \\ \hline \end{array} \end{pmatrix} \quad (15) \]

where

\[ a = \sinh(u - 3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta, \quad b = \sinh(u - 3\eta) + \sinh 3\eta, \]
\[ c = \sinh(u - 5\eta) + \sinh \eta, \quad d = \sinh(u - \eta) + \sinh \eta, \]
\[ e = -2e^{-\frac{u}{2}} \sinh 2\eta \cosh(u - 3\eta), \quad \bar{e} = -2e^{\frac{u}{2}} \sinh 2\eta \cosh(u - 3\eta), \]
\[ f = -2e^{-u+2\eta} \sinh \eta \sinh 2\eta - e^{-\eta} \sinh 4\eta, \quad \bar{f} = 2e^{u-2\eta} \sinh \eta \sinh 2\eta - e^{\eta} \sinh 4\eta, \]
\[ g = 2e^{-\frac{u}{2}+2\eta} \sinh \frac{u}{2} \sinh 2\eta, \quad \bar{g} = -2e^{\frac{u}{2}-2\eta} \sinh \frac{u}{2} \sinh 2\eta. \]

It has the regularity property \( R(0) \propto \mathcal{P} \), where \( \mathcal{P} \) is the permutation matrix, as well as unitarity, \( PT \) symmetry, and crossing symmetry

\[ R_{12}(u) = V_1 \, R_{12}(-u - \rho)^{t_2} \, V_1^* = V_2^{t_2} \, R_{12}(-u - \rho)^{t_1} \, V_2^{t_2}, \tag{16} \]

where the crossing matrix \( V \) is given by

\[ V = \begin{pmatrix} -e^{-\eta} & 0 \\ 0 & 1 \\ -e^{\eta} & 0 \end{pmatrix}, \tag{17} \]

and \( \rho = -6\eta + i\pi \).

Given a solution \( K(u) \) of the boundary Yang-Baxter equation, a corresponding transfer matrix \( t(u) \) for an open integrable quantum spin chain with \( N \) sites is given by \([2], [3], [15]\)

\[ t(u) = \text{tr}_0 \, M_0 K_0(-u - \rho)^{t_0} T_0(u), \tag{18} \]

where

\[ T_0(u) = T_0(u) \, K_0(u) \, \hat{T}_0(u), \tag{19} \]

with

\[ T_0(u) = R_{0N}(u) \cdots R_{01}(u), \quad \hat{T}_0(u) = R_{10}(u) \cdots R_{N0}(u), \tag{20} \]

and

\[ M = V^t V = \text{diag}(e^{2\eta}, 1, e^{-2\eta}). \tag{21} \]

Indeed, the transfer matrix forms a one-parameter commutative family \([t(u), t(v)] = 0\), which contains the Hamiltonian \( \mathcal{H} \),

\[ \mathcal{H} \propto \left. \frac{d}{du} t(u) \right|_{u=0}. \tag{22} \]

For the three \( K \) matrices \( K^{(i)}(u) \) given in Eq. (3), we denote by \( t^{(i)}(u) \) the corresponding transfer matrices, and by \( \mathcal{H}^{(i)} \) the corresponding Hamiltonians. We now restrict our attention
to the cases \( i = 1, 2 \). For 2 sites \((N = 2)\), we have checked the \( U_q(o(3)) \) symmetry of the transfer matrix by direct computation. In particular, Eq. (22) implies that the 2-site Hamiltonian also has this symmetry. For general \( N \), the Hamiltonian is given by a sum of 2-site Hamiltonians plus boundary terms. It follows that, for general \( N \), the Hamiltonian \( H^{(i)} \) has \( U_q(o(3)) \) symmetry

\[
[H^{(i)}, S^\pm] = 0, \quad [H^{(i)}, S^3] = 0, \quad i = 1, 2,
\]

where the symmetry generators obey (12) - (14). We have also checked the symmetry (11) of the transfer matrix for 3 sites \((N = 3)\) by direct computation, and we conjecture that it holds for general \( N \).

We remark that the symmetry generators \( S^\pm, S^3 \) defined in (13), (14) lie in the fundamental algebraic structures of QISM. Indeed, note the asymptotic behavior of the \( R \) and \( K \) matrices for \( u \to \infty \):

\[
R(u) \sim e^u R^+ + R^{++} + O(e^{-u}),
\]

\[
K^{(i)}(u) \sim e^u K^{(i)+} + K^{(i)++} + O(e^{-u}), \quad i = 1, 2,
\]

where \( R^+, R^{++}, K^{(i)+}, K^{(i)++} \) are independent of \( u \). It follows that the quantity \( T^{(i)}(u) \) defined as in Eq. (19) has the asymptotic behavior for \( u \to \infty \)

\[
T^{(i)}(u) \sim e^{(2N+1)u} T^{(i)+} + e^{2Nu} T^{(i)++} + \ldots,
\]

where \( T^{(i)+}, T^{(i)++} \) are independent of \( u \). The basic observation is that the generators \( S^\pm \) lie in the antidiagonal corners of \( T^{(i)++} \) (viewed as a \( 3 \times 3 \) auxiliary-space matrix, with operator-valued entries):

\[
T^{(i)++} = \begin{pmatrix}
0 & 0 & S^- \\
0 & * & * \\
S^+ & * & *
\end{pmatrix}.
\]

We expect that this observation will be useful for formulating a QISM proof of the symmetry (11).

3 Discussion

One immediate consequence of the symmetry which we have uncovered is the explanation of degeneracies in the spectrum for finite \( N \). For instance, consider the pseudovacuum vector
\[ \omega = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right)^{\otimes N}, \]

\[ i^{(i)}(u) \, \omega = \Lambda^{(i)}(u) \, \omega, \quad i = 1, 2, \]

(28)

where \( \Lambda^{(i)}(u) \) is the corresponding pseudovacuum eigenvalue. Commutativity of the transfer matrix with \( S^- \) implies that the vectors \((S^-)^n \omega\) for \( n = 1, 2, \ldots, N \) are also eigenvectors of the transfer matrix with the same eigenvalue. Moreover, we observe that each site carries a reducible representation of the \( U_q(o(3)) \) algebra, namely \( 2 \oplus 1 \) (instead of \( 3 \)), implying the degeneracy pattern \( (2 \oplus 1)^{\otimes N} \).

Note that the pseudovacuum vector \( \omega \) is annihilated by \( S^+ \); that is, \( S^+ \omega = 0 \). We expect that all Bethe Ansatz states (which can presumably be constructed by applying appropriate creation-like operators to \( \omega \)) are such highest-weight states. (See, e.g., [17], [10], [18], [19].)

Finally, we remark that we have considered here only the first of the infinite family of models \( A_{2n}^{(2)}, n = 1, 2, \ldots \). For these \( R \) matrices [7], [8], there are again only three distinct diagonal solutions of the boundary Yang-Baxter equation: \( K^{(0)} = I \) [1], and \( K^{(1)}, K^{(2)} \) given in [20]. The transfer matrix constructed with \( K^{(0)} \) has [10] the symmetry \( U_q(o(2n + 1)) \) with the standard coproduct. We expect that the transfer matrices constructed with \( K^{(1)} \) and \( K^{(2)} \) also have \( U_q(o(2n + 1)) \) symmetry, but with a nonstandard coproduct. We hope to report on this and related matters in a future publication.

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