Entanglement sudden death and sudden birth in two uncoupled spins

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Abstract
We investigate the entanglement evolution of two qubits interacting with a common environment through a Heisenberg XX mechanism. We reveal the possibility of realizing the phenomenon of entanglement sudden death as well as the entanglement sudden birth acting on the environment. Such analysis is of maximal interest in the light of the large applications that spin systems have in quantum information theory.

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1. Introduction
Interest toward spin systems, and more in particular toward spin dynamics in semiconductor structures, has remarkably increased in the last few years, and also in connection with new emerging areas of physics, such as quantum information and computation. In this framework, it becomes a relevant subject to analyze the entanglement behavior in spin systems in order to assess the performance of applications, for example, in quantum information processing. Quite recently, it has been shown that entanglement in two qubit systems can experience sudden death and sudden birth. This phenomenon [1, 2] deserves great attention also in applicative contexts from quantum optical to condensed matter systems, and has been observed in the laboratory in experiments with entangled photon pairs [3] and atomic ensembles [4]. In this paper, we demonstrate the possibility of realizing such a behavior in a system of two uncoupled spins in a common environment.

2. Physical system
Let us consider a bipartite system constituted by two spins $A$ and $B$, hereafter called central spins, that interact with the same coupling constants $\alpha$, with a system of $N$ uncoupled spins. The Hamiltonian model that describes such a physical situation is

$$H = H_0 + H_1$$

with

$$H_0 = \omega (S_z + J_z), \quad H_1 = \alpha (S_+ J_+ + S_- J_-),$$

where $S_z = \frac{1}{2} (\sigma^A_z + \sigma^B_z)$ and $S_{\pm} = (\sigma^A_{\pm} + \sigma^B_{\pm})$ are spin operators acting on the Hilbert space of the central system and $J_z = \frac{1}{2} \sum_{i=1}^{N} \sigma^i_z$ and $J_{\pm} = \sum_{i=1}^{N} \sigma^i_{\pm}$ are the collective operators describing the other $N$ spins. In solid-state physics, for example, this model can effectively describe many physical systems such as quantum dots [5], two-dimensional electron gases [6] and optical lattices [7]. The time-dependent Schrödinger equation has already been solved for an arbitrary initial condition [8, 9]. In what follows, we analyze the dynamics of the entanglement in the central system when the surrounding spins are prepared in specific initial conditions.

3. Collapses and revivals in the entanglement evolution

3.1. Binomial initial state
Suppose that the $N$ uncoupled spins around the central system are prepared in a linear superposition, with binomial weight, of eigenstates $|J, M\rangle$ of $J^2$ and $J_z$ with $M = 0$. The two central spins $A$ and $B$ are instead prepared in the state $|S = 1, M_S = 0\rangle$ that is a maximally entangled state. The initial condition we are considering can thus be written as

$$|\psi(0)\rangle = \sum_{J=0}^{N/2} B^J_0 |1, 0\rangle |J, 0\rangle$$

$$\equiv \sum_{J=0}^{N/2} B^J_0 |1, 0, J, 0\rangle,$$

where $S_z = \frac{1}{2} (\sigma^A_z + \sigma^B_z)$ and $S_{\pm} = (\sigma^A_{\pm} + \sigma^B_{\pm})$ are spin operators acting on the Hilbert space of the central system and $J_z = \frac{1}{2} \sum_{i=1}^{N} \sigma^i_z$ and $J_{\pm} = \sum_{i=1}^{N} \sigma^i_{\pm}$ are the collective operators describing the other $N$ spins. In solid-state physics, for example, this model can effectively describe many physical systems such as quantum dots [5], two-dimensional electron gases [6] and optical lattices [7]. The time-dependent Schrödinger equation has already been solved for an arbitrary initial condition [8, 9]. In what follows, we analyze the dynamics of the entanglement in the central system when the surrounding spins are prepared in specific initial conditions.
displays the behavior of the exact concurrence structure:
\[
\left| \uparrow \uparrow \right\rangle
\]
possible to prove that at any time instant it is
possible to find the following closed form of \( C(t) \): \( C(t) \approx \max \left[ 0, \cos^{N/2}(2\alpha t) \cos((N + 2)\alpha t) \right] \). Figure 1 displays the behavior of the exact concurrence function \( C(t) \) as given by equation (9) (continuous line) and its approximation, equation (10), (dotted line). As expected the agreement is excellent at least for \( at \ll N \). Moreover, the figure puts into light an interesting behavior in the time evolution of the entanglement present in the central spins. Starting, indeed, by construction from \( C(0) = 1 \), the concurrence function evolves showing collapse and revival phenomena. On the other hand, it is possible to prove that during the plateau of \( C(t) \) (that is when \( C(t) \) maintains the zero value) the two spins are in a separable state described by the following density matrix:
\[
\rho_{AB}(t) = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
that describes a system characterized by an equal probability of finding all the states \( | \uparrow \uparrow \rangle, | \uparrow \downarrow \rangle, | \downarrow \uparrow \rangle, | \downarrow \downarrow \rangle \). After the plateau of the entanglement the concurrence function suddenly grows up reaching values near to 1. This behavior periodically appears. The collapses and revivals of \( C(t) \) shown in figure 1 recall those we have in the dynamical behavior of the two-photon Jaynes–Cummings (J–C) model [11, 12] described by the following Hamiltonian model:
\[
H_{JC} = \hbar \omega_0 \sigma_+ a + a^\dagger \sigma_- \hbar \lambda \left( \sigma_+ a^2 + \sigma_- a^2 \right),
\]
where \( a \) is the annihilation operator of the single cavity mode. The analogy between our spin star system and the two-photon J–C model can be better brought to light following the suggestion of [13], that is,
\[
J_+ = \frac{a^2}{2}, \quad J_- = \frac{a^2}{2}.
\]
Exploiting indeed such a correspondence the interaction Hamiltonian (2) assumes the form
\[
H_I = \frac{\alpha}{2} \left( S_+ a^2 + S_- a^2 \right).
\]
3.2. Atomic coherent initial state

In this section we analyze a different initial condition for the $N$ spins around the central system that is the well-known atomic coherent state, introduced in 1972 by Arecchi in analogy with the coherent states of the radiation [14]. The central system is instead once again in the state $\ket{1,0}$. A coherent state of $N$ spins is a linear superposition of states $\ket{J,M}$ obtained fixing $J$ and varying $M$. In particular, putting $J = N/2$, the initial state of the global system is the following:

$$|\psi(0)\rangle = \sum_{M=-N/2}^{N/2} B_{M}^{N/2} \left( \frac{N}{2}, M \right) |1, 0\rangle$$

$$= \sum_{M=-N/2}^{N/2} B_{M}^{N/2} |1, 0, \frac{N}{2}, M\rangle,$$

where

$$B_{M}^{N/2} = \left[ \left( \frac{N}{M + N/2} \right) p^{M+N/2} (1 - p)^{(N/2) - M} \right]^{1/2},$$

$$p \in [0, 1].$$

Starting from $|\psi(0)\rangle$ at time instant $t$ we have [8]

$$|\psi(t)\rangle = \sum_{M=-N/2}^{N/2} B_{M}^{N/2} \left( A_{M}(t) |1, 0, \frac{N}{2}, M\rangle - iB_{M}(t) |1, -1, \frac{N}{2}, M+1\rangle - iC_{M}(t) |1, 1, \frac{N}{2}, M-1\rangle \right)$$

with

$$A_{M}(t) = \cos \left( \sqrt{2(q_{M}^{2} + r_{M}^{2})} at \right),$$

$$B_{M}(t) = \frac{r_{M}}{\sqrt{q_{M}^{2} + r_{M}^{2}}} \sin \left( \sqrt{2(q_{M}^{2} + r_{M}^{2})} at \right),$$

$$C_{M}(t) = \frac{q_{M}}{\sqrt{q_{M}^{2} + r_{M}^{2}}} \sin \left( \sqrt{2(q_{M}^{2} + r_{M}^{2})} at \right),$$

where

$$q_{M} = \sqrt{\frac{N}{2} \left( \frac{N}{2} + 1 \right) - M(M - 1)},$$

$$r_{M} = \sqrt{\frac{N}{2} \left( \frac{N}{2} + 1 \right) - M(M + 1)}.$$

In this case the concurrence function becomes

$$C(t) = \max \left[ 0, 2 \sum_{M=-N/2}^{N/2} \left( B_{M}^{N/2} \right)^{2} \times \left( \frac{1}{2} A_{M}(t)^{2} - \sqrt{B_{M}(t)^{2} C_{M}(t)^{2}} \right) \right].$$

The dynamical evolution of $C(t)$ against $at$ is shown in figure 2. We observe that in this case the situation is quite different from the situation previously examined: the entanglement initially present in the central system suddenly dies after some oscillations and, after a period of time in which it is absent, lives again. However in this case, the concurrence function does not reach values near 1 assuming values less than $\frac{1}{2}$. Once again it is possible to make a parallel between the J–C model and the spin star system exploiting the Holstein–Primakoff transformations [15]

$$J_{+} = \sqrt{2} J a^{\dagger}\sqrt{1 - \frac{a\dagger a}{2J}},$$

$$J_{-} = \sqrt{2} J \sqrt{1 - \frac{a\dagger a}{2J}} a,$$

that are valid in a subspace with $J$ fixed. Operating such a transformation the interaction Hamiltonian (2) becomes

$$H_{I} = \alpha \sqrt{N} \left[ S_{+}\sqrt{1 - \frac{a\dagger a}{N}} + S_{-} a^{\dagger} \sqrt{1 - \frac{a\dagger a}{N}} \right],$$

that in the limit of a large number $N$ of spins in the environment, that is $\sqrt{1 - \frac{a\dagger a}{N}} \simeq 1$, reduces to a Hamiltonian of the J–C type.

4. Conclusion

Summarizing in this paper we have focused our attention on a system constituted by two uncoupled spins embedded in a common environment composed by $N$ spins. We have proved the possibility of realizing periodic sudden death and sudden birth of the entanglement in the two not interacting spins appropriately choosing the initial condition of the environment. Generally speaking, sudden death and sudden birth of the entanglement provides an interesting resource for creation on demand of entanglement between two qubits.

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