Stability features of steady-state solutions for a diode with electron and ion counter-streams

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Abstract. Stability features of steady-state solutions for a diode with counter-streaming electron and ion flows are studied. For this purpose, the time-dependent problem for an exponential potential perturbation with complex frequency is considered. By linearization of the Poisson equation and electron and ion densities integrodifferential equation for the potential perturbation amplitude is derived. In the case of uniform unperturbed potential distribution an explicit solution of this equation is obtained. Eigen modes of the perturbation are studied. The limiting value of the diode length above which steady state solutions in question are unstable is found. The obtained analytical Eigen modes coincide with the result of numerical simulation of the potential perturbation evolution.

1. Introduction
In order to choose the operating modes of many plasma devices, the stability features of their steady-state solutions should be examined. In the case when the steady states of such devices are unstable, nonlinear oscillations can develop in the plasma [1], [2]. A similar phenomenon was discovered earlier when studying the stability of steady-state solutions in a collisionless plasma diode with counter-streaming electron and ion flows [3]: at electron current densities exceeding a certain threshold value, solutions with nonlinear oscillations were implemented in the diode instead of steady-state solutions [4]. This result was obtained numerically using a modified E, K-code [5]. In this paper, we propose an analytical approach for studying the stability features of these solutions. The method is a development of the stability theory proposed earlier in [6].

Steady-state solutions for a collisionless plasma diode of plane geometry with counter-streams of electrons and ions were studied in [3]. It was found that for given values of the inter-electrode gap \( L \) and the potential difference between the electrodes \( U \), there can be several stationary solutions. They are characterized by different values of the electric field strength at the emitter. A diagram of such solutions on the \((\delta, \varepsilon_0)\) plane, where \( \delta \) is the dimensionless value of the inter-electrode gap, and \( \varepsilon_0 \) is the dimensionless electric field strength at the emitter, is shown in Figure 1. Among the steady-state solutions, we can distinguish several classes corresponding to potential distributions with different characteristic features. In particular, there are solutions for which neither electrons nor ions, when moving in the inter-electrode gap, experience reflection from the corresponding potential extrema. Such solutions are the subject of our study.

2. Problem statement
To study the stability features of steady-state solutions, a non-stationary problem was considered in the following formulation. It was assumed that a monoenergetic flows of electrons and ions enter the
inter-electrode gap from the left and right electrodes, respectively. Let \( v_{e,0} \) be the velocity and \( n_{e,0} \) be the density of electrons, whereas the same values for ions are \( n_{i,0} \) and \( -v_{i,0} \). The particles move in the inter-electrode gap without collisions in a self-consistent electric field. When any electrode is reached, the particle is absorbed. We study the processes in which the electrons cross the inter-electrode gap in less time than the ions move a distance of the order of the Debye-Hückel length. This is true if \( L/\lambda_0 \ll (M/m)^{1/2} \), where \( m \) (\( M \)) is the electron (ion) mass. In this case, the electrons at each moment adjust "instantly" to the existing distributions of the electric field and ion density, and it is sufficient to take into account only the effects associated with the time-dependent ion movement.

Let us introduce dimensionless quantities. As the units of energy and length, we choose the electron energy \( W_{e,0} = m v_{e,0}^2 / 2 \) and the Debye length \( \lambda_0 = \left[ 2\tilde{\varepsilon}_0 W_{e,0} / e^2 n_{e,0} \right]^{1/2} \) at the left electrode, where \( e \) is the electron charge, \( \tilde{\varepsilon}_0 \) is the permittivity of the vacuum. The particle densities are measured in units of \( n_{e,0} \), the electron and ion velocities are measured in \( v_{e,0} \) and \( v_{i,0} = v_{e,0} \sqrt{m/M} \), respectively.

We investigate the stability features of steady-state solutions. For the density of electrons moving without reflection in a stationary electric field, we have

\[
n_e(\eta, \tau) = (1 + 2\eta)^{1/2}.
\]

(1)

Here \( \eta \) and \( \tau \) are the dimensionless potential and time. The formula for the density of ions emitted at the right electrode and moving in a time-dependent field without reflection can be obtained by methods similar to those used in [7]:

\[
n_i(\eta, \tau) = \left[ -2G_i(\tau, \tau_0) - Q_i(\tau, \tau_0) \right]^{1/4}.
\]

(2)

Here \( V \) is the collector potential, whereas the functions \( G_i \) and \( Q_i \) are determined by the relations

\[
G_i(\tau, \tau_0) = -\int_{\tau_0}^{\tau} d\tau' \frac{\partial}{\partial \tau'} \left( \frac{\partial}{\partial \tau'} \left( \int_{\tau_0}^{\tau} d\tau' \right) \right),
\]

(3)

\[
Q_i(\tau, \tau_0) = \int_{\tau_0}^{\tau} d\tau' \int_{\tau_0}^{\tau} d\tau'' \left( \frac{\partial}{\partial \tau''} \right) \left( \frac{\partial}{\partial \tau''} \right) \left( \int_{\tau_0}^{\tau''} d\tau''' \right) \left( \int_{\tau_0}^{\tau''} d\tau'''ight).
\]

(4)

One can also get another expression for the function \( Q_i(\tau, \tau_0) \):
\[ Q_i(t, \tau_0) = \frac{d}{d\tau_0} \zeta_i(t' + \tau_0, \tau_0) \bigg|_{t' = t - \tau_0}. \]  

(5)

In formulae (3), (5), the function \( \zeta_i(t, \tau_0) \) determines the trajectory of an ion that has left the right boundary at time \( \tau_0 \) with velocity equal to \(-1\).

The potential distribution at each time moment is found from the second-order ordinary differential equation, which is obtained from the Poisson equation after substituting the electron (1) and ion (2) densities into it:

\[ \frac{d^2\eta}{d\zeta^2} = (1 + 2\eta)^{1/2} - \left\{ \left| -2(\eta - V) - 2G_i \right|^{1/2} - Q_i \right\}^{-1}. \]  

(6)

To study the stability features of stationary solutions, we consider evolution of small perturbation of the potential distribution (PD). To do this, the PD is represented as following

\[ \eta(\zeta, \tau) = \eta_0(\zeta) + \tilde{\eta}(\zeta) \exp(-i\Omega \tau), \quad \left| \tilde{\eta}(\zeta) \right| << \left| \eta_0(\zeta) \right|. \]  

(7)

Here \( \eta_0(\zeta) \) is the unperturbed PD, \( \tilde{\eta}(\zeta) \) is the potential perturbation amplitude, and \( \Omega \) is the complex frequency. We will assume that the derivative \( d\tilde{\eta}(\zeta)/d\zeta \) is also a quantity of the order of \( \tilde{\eta}(\zeta) \). Now let us perform a linearization of the Poisson equation. For the electron density we have from (1)

\[ n_e = \frac{1}{u_e} - \frac{\tilde{\eta} \exp(-i\Omega \tau)}{u_e^2} + O(\tilde{\eta}^2). \]  

(8)

Here \( u_e = (1 + 2\eta_0)^{1/2} \) is the velocity of an electron moving in an undisturbed electric field.

We can show in the same way as in [7], that \( G_i = \tilde{G}_i(\zeta) \exp(-i\Omega \tau) \), \( Q_i = \tilde{Q}_i(\zeta) \exp(-i\Omega \tau) \) for a small potential perturbation, and functions \( \tilde{G}_i(\zeta) \) and \( \tilde{Q}_i(\zeta) \) are quantities of the order not exceeding \( \tilde{\eta} \). Let \( u_i(\zeta) = (1 - 2(\eta_0 - V))^{1/2} \) be the modulus of the ion velocity when moving in the unperturbed field at a point \( \zeta \). Then the time of flight of the ions \( \tau_i(\zeta) \) moving in an undisturbed field from the right electrode \( \zeta = \delta \) to the point \( \zeta \) is equal to

\[ \tau_i(\zeta) = \int_{\delta}^{\zeta} dx(u_i(x))^{-1}. \]  

(9)

To calculate the function \( \tilde{G}_i(\zeta) \), we substitute PD (7) in (3):

\[ \tilde{G}_i(t, \tau_0) = i\Omega \int_{\tau_0}^{t} d\tau' \tilde{\eta}(\zeta') \exp(i\Omega(\tau - \tau')) = i\Omega \int_{\delta}^{\zeta} \frac{dx}{u_i(x)} \tilde{\eta}(x) \exp(i\Omega(\tau_i(\zeta) - \tau_i(x))) . \]  

(10)

Here we have changed integration variables from time \( \tau' \) to coordinate \( x \) using equality \( d\tau'/dx = -u_i(x) \). Integrating in parts, we find

\[ \tilde{G}_i = -\tilde{\eta}(\zeta) + \int_{\delta}^{\zeta} \tilde{\eta}'(x) \exp(i\Omega(\tau(\zeta) - \tau(x))) dx . \]  

(11)

For the function \( \tilde{Q}_i(\zeta) \), using (5) and the energy conservation law for ions moving in an undisturbed electric field, we obtain the Volterra integral equation of the second kind
\[
\tilde{Q}_i(z) = \exp(i\Omega\tau) \int_0^{\tau - \tau_0} \exp(-i\Omega(x + r_0)) \left( i\Omega \tilde{\eta} + (u'_{ii}) \tilde{Q}_i(z + r_0) \right) (r - r' - \tau_0) dr' = \\
\int_{r_0}^{\tau} \exp(i\Omega(r_0(x) - r_0(x))) \left( i\Omega \tilde{\eta} + (u'_{ii}) \tilde{Q}_i(x) \right) (r_0(x) - r_0(x)) dr_0.
\]

(12)

We solve this equation using the method similar to that proposed in [7]. This gives:

\[
\tilde{Q}_i = \int_{r_0}^{\tau} \exp(i\Omega(r_0(x) - r_0(x))) \left( i\Omega \tilde{\eta} + (u'_{ii}) \tilde{Q}_i(x) \right) (r_0(x) - r_0(x)) dr_0.
\]

(13)

After linearizing the expression for the ion density (2) and substituting the found expressions for \(\tilde{G}_i(z)\) and \(\tilde{Q}_i(z)\) in (6), we obtain the integrodifferential equation for the potential perturbation amplitude

\[
\tilde{\eta}''(z) = -u^3(z)\tilde{\eta}(z) + u^3(z) \int_{r_0}^{\tau} \exp(i\Omega(r_0(x) - r_0(x))) \tilde{\eta}'(x) + \\
i\Omega u^3(z) \int_{r_0}^{\tau} dx \exp(i\Omega(r_0(x) - r_0(x))) \tilde{\eta}'(y)
\]

(14)

with boundary condition

\[
\tilde{\eta}(\delta) = 0.
\]

(15)

The value of the derivative \(\tilde{\eta}'(\delta)\) on the right boundary is an arbitrary parameter:

\[
\tilde{\eta}'(\delta) = A.
\]

(16)

Let's consider a special case \(u_0(z) = 0\). In this case \(u(z) = 1\), \(u_1(z) = 1\), \(r_1(z) = \delta - z\), and equation (14) takes the form

\[
\tilde{\eta}''(z) = -\tilde{\eta}(z) + i\Omega u^3(z) \int_{r_0}^{\tau} dx \exp(i\Omega(x - z)) \tilde{\eta}'(x) + i\Omega u^3(z) \int_{r_0}^{\tau} dy \exp(i\Omega(y - z)) \tilde{\eta}'(y).
\]

(17)

3. Results

Let us first analyze the steady state solutions in the special case of small \(\varepsilon_0\). It can be shown that at zero potential difference between the electrodes, the PD has the form

\[
\eta = \varepsilon_0 \sin \left( \sqrt{2z} \right) / \sqrt{2}.
\]

(18)

Since the boundary condition \(\eta(\delta) = 0\) must be satisfied, the solution is nonzero only for \(\delta = \delta_k = nk/\sqrt{2} \), \(k = 1, 2, \ldots\). This conclusion corresponds to the situation shown in Figure 1, where non-zero steady-state solutions without particle reflection (designated by \(n_i\) in the vicinity of zero values occur at the points \(\delta_k\).

Now we find the solution of the equation (17). It can be turned into a differential equation. To do this, we multiply both parts of (17) by \(\exp(i\Omega \tilde{\eta})\) and successively differentiate the resulting equation moving the non-integral terms to the left part. This procedure leads to a fourth-order equation with constant coefficients

\[
\tilde{\eta}'' + 2i\Omega \tilde{\eta}'' + (2 - \Omega^2) \tilde{\eta}'' + 2i\Omega \tilde{\eta} - \Omega^2 \tilde{\eta} = 0.
\]

(19)

This induces additional conditions on the right boundary

\[
\tilde{\eta}'(\delta) = 0, \quad \tilde{\eta}''(\delta) = -2\tilde{\eta}'(\delta) = -2A.
\]

(20)

The characteristic equation for (19) has the form
\[ \alpha^4 + 2i\Omega\alpha^3 + \left(2 - \Omega^2\right)\alpha^2 + 2i\Omega\alpha - \Omega^2 = 0. \]  

(21)

Its roots are

\[ \alpha_i = \beta_i - i\Omega/2, \quad \beta_{1,2} = \pm \sqrt{1 - \Omega^2/4} \pm \sqrt{1 + \Omega^2}, \quad \beta_{3,4} = -\sqrt{1 - \Omega^2/4} \pm \sqrt{1 + \Omega^2}. \]  

(22)

Therefore, the solution of equation (19) can be written as follows

\[ \tilde{\eta}(\zeta) = A\sum_{i=1}^{4} C_i \exp(\alpha_i(\zeta - \delta)). \]  

(23)

Coefficients \( C_i, \ i = 1, 2, 3, 4 \) can be found from the boundary conditions (15), (16) with \( A = 1 \) and (20) and satisfy a linear equations system

\[ \begin{align*} 
C_1 + C_2 + C_3 + C_4 &= 0, \\
C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_3 + C_4\alpha_4 &= 1, \\
C_1\alpha_1^2 + C_2\alpha_2^2 + C_3\alpha_3^2 + C_4\alpha_4^2 &= 0, \\
C_1\alpha_1^3 + C_2\alpha_2^3 + C_3\alpha_3^3 + C_4\alpha_4^3 &= -2. 
\end{align*} \]  

(24)

Using the condition of the potential perturbation vanishing on the left boundary, we obtain the dispersion equation

\[ F(\Omega, \delta) = \sum_{i=1}^{4} C_i \exp(-\alpha_i\delta) = 0. \]  

(25)

This equation provides relations between \( \Omega = \omega + i\Gamma \) and \( \delta \), which are dispersion curves. For each fixed value of the parameter \( \omega \) there are a countable number of modes. The set of modes includes one aperiodic branch (i.e., the dependence \( \Gamma(\delta) \) at \( \omega = 0 \)) and a set of oscillatory branches. Figure 2 shows the aperiodic branch of the dispersion curve obtained by solving equation (25). Note that the points where the sign of \( \Gamma(\delta) \) changes can be found analytically. To do this, put \( \Omega = \omega + i\Gamma = 0 \) in the characteristic equation. Its roots are thus equal to \( \alpha_1 = 0, \ \alpha_2 = i\sqrt{2}, \ \alpha_3 = 0, \ \alpha_4 = -i\sqrt{2} \). Using the boundary conditions, we find that the amplitude of the potential perturbation in this case has the form

\[ \tilde{\eta}(\zeta) = D\sin\left(\sqrt{2}(\zeta - \delta)\right), \]  

where \( D \) is an arbitrary constant. Since the amplitude of the perturbation potential turns to zero at \( \zeta = 0 \), one can conclude that a solution other than zero is possible only at \( \delta = \delta_k \), i.e., at the values \( \delta \) that were obtained in the analysis of stationary solutions. These values coincide with the vanishing points of the calculated dependence \( \Gamma(\delta) \) found in the solution of equation (25). Thus, we have found the boundaries of the stability domains of solutions with respect to the parameter \( \delta \).

To validate the obtained solutions, we investigated the development of a small perturbation using the numerical method described in [4] and being a modified E, K-code [5]. We calculated the potential evolution in the vicinity of the value \( \delta_1 = \pi/\sqrt{2} \approx 2.221 \) where the growth rate \( \Gamma(\delta) \) changes sign. Calculations were performed for \( \delta = 2 \) and for \( \delta = 2.5 \). Over time interval \( 0 < \tau \leq \tau_p \), a perturbation of the form \( \eta(\zeta, \tau) = 0.02\sin(\pi\zeta/\delta)\exp(10(\tau - \tau_p)) \) was introduced into the steady-state PD, which is initially equal to zero. Further, at \( \tau > \tau_p \), the self-consistent evolution of the potential distribution and particle densities was calculated. In the calculations we chose \( \tau_p = 12 \). As expected, after finishing a certain transition process, the shape of the PD stabilizes, and its maximum absolute value decreases (at \( \delta = 2 \)) or increases (at \( \delta = 2.5 \)) according to the exponential law, and the growth rate values are \( \Gamma_{\text{cal}}(2) = -0.246, \ \Gamma_{\text{cal}}(2.5) = 0.255 \), respectively. These values coincide with the growth rate values obtained from the dispersion curve \( \Gamma_{\text{theo}}(2) = -0.246, \ \Gamma_{\text{theo}}(2.5) = 0.253 \) with good accuracy. In addition, numerical analysis shows that the aperiodic mode turns out to be the main unstable mode,
and the amplitude of the potential perturbation takes the form of the mode obtained analytically (Figure 3).

Figure 2. Aperiodic branch of the dispersion curve.  

Figure 3. Analytical $\eta(\zeta)$ (black solid line) and $\eta(\zeta, \tau)$ obtained numerically at $\tau = \tau_p = 5$ (red dotted line) normalized by extrema values.

4. Conclusion
A stability theory of steady-state solutions for a diode with electron and ion counter streams is developed for the regime without reflection of ions from the potential maximum. The instability of the ion beam neutralized by electrons that we study, is very similar to the electron Bursian-Pierce instability. Just like the latter, it develops due to the positive feedback via an external circuit and not a plasma, as its threshold by the inter-electrode gap is of the order of $(2 - 3)\lambda_D$. However, the electron instability develops over average electron travel time between the electrodes, and the ions do not participate in this process. During the development of ion instability electrons don’t stay stationary at any ion displacement, and they almost instantly adjust to ion charge distribution, i.e. they seems to be “attached” to ions. The regions of the inter-electrode gap values are found, within which the solutions are unstable. The results obtained are confirmed by numerical calculations of the instability development using a high-precision numerical code. On the agenda is the development of a stability theory of solutions for the regime when there is a potential barrier in the plasma that reflects some of the ions back to the electrode from which they flew. This will allow us to analyze the stability features of all steady state solutions for a diode with counter-steaming electron and ion flows.

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