Orthogonal and symplectic Yangians: linear and quadratic evaluations
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Abstract

Orthogonal or symplectic Yangians are defined by the Yang-Baxter $RLL$ relation involving the fundamental $R$ matrix with $so(n)$ or $sp(2m)$ symmetry. Simple $L$ operators with linear or quadratic dependence on the spectral parameter exist under restrictive conditions. These conditions are investigated in general form.
1 Yangian symmetries

Yang-Baxter relations appeared historically in the study of idealized models of physical problems of particle scattering and thermodynamics. They define infinite dimensional algebras related to an underlying Lie algebra \( \mathcal{G} \). The related physical models have an extended symmetry appearing in a large number of conservation laws beyond the ones related to the Lie algebra. These relations are well known as the basis of the treatment of quantum integrable models [1, 2, 3, 4]. In recent years the range of applications of the extended symmetries has been broadened essentially, in particular to the study of gauge field theories, and the related methods have attracted increasing interest [5, 6, 7, 8].

The formulation of the Yangian algebra of type \( \mathcal{G}, \mathcal{Y}(\mathcal{G}) \), can be based on the Yang-Baxter \( R \) matrix acting on the tensor product of the fundamental representation space \( V \) of the Lie algebra \( \mathcal{G} \) and obeying the Yang-Baxter relation of the form

\[
R_{a_1b_2}(u)R_{b_1c_3}(u + v)R_{c_2a_3}(v) = R_{b_2b_3}(v)R_{b_1c_3}(u + v)R_{c_1a_2}(u).
\]

(1.1)

The generators \( (L^{(k)})^{a}_{b} \) of the extended Yangian algebra \( \mathcal{Y}(\mathcal{G}) \) appear in the expansion of the \( L \) operator

\[
L_{b}^{a}(u) = \sum_{k=0}^{\infty} \frac{(L^{(k)})^{a}_{b}}{u^k}, \quad L^{(0)} = I,
\]

(1.2)

which satisfies the Yang-Baxter RLL-relations

\[
R_{b_1b_2}^{a_1a_2}(u - v)L_{c_1}^{b_1}(u)L_{c_2}^{b_2}(v) = L_{b_2}^{a_2}(v)L_{b_1}^{a_1}(u)R_{c_1c_2}^{b_1b_2}(u - v).
\]

(1.3)

\( L(u) \) is an algebra valued matrix, \( L(u) \in \text{End} \, V \otimes \mathcal{Y}(\mathcal{G}) \), depending on the spectral parameter \( u \).

The Yangian algebra as originally defined by Drinfeld [9, 10] is obtained from the extended Yangian by factorizing central elements. The Yangians of orthogonal and symplectic types have been considered in [21] and their algebraic structure and representation theory have been considered in [22]. In the case \( \mathcal{G} = g\ell(n) \) the center is contained in the quantum determinant of \( L(u) \). In the cases \( \mathcal{G} = so(n) \) and \( \mathcal{G} = sp(n) \) the center has been analyzed in [22]. Proofs of the equivalence of the Yangian definition via (1.3) and the center factorization to the definitions by Drinfeld are given in [23, 24].

It is well known that the twofold matrix product \( L(u) = L^{(1)}(u)L^{(2)}(u + \delta) \) (here \( \delta \) is an arbitrary shift of the spectral parameter \( u \)) obeys (1.3), if both factors obey this RLL relation. The transfer matrix constructed as the trace of the \( N \)-fold product of such \( L \) operators called monodromy matrix plays a central role in the treatment of quantum integrable models. The relevant case is the one where the factors entering the monodromy matrix have a simple form related to the underlying Lie algebra.

If the Lie algebra is the general linear, \( \mathcal{G} = g\ell(n) \), the simple form \( L(u) = Iu + G \) can be chosen without restrictions, where \( I \) is the identity matrix and the matrix elements of \( G \) are the Lie algebra generators. However, in the case of the orthogonal or symplectic Lie algebras, \( \mathcal{G} = so(n) \) or \( \mathcal{G} = sp(n) \) (\( n \) even), this ansatz linear in \( u \) works with essential restrictions only. In this situation the next-to-simplest case of an ansatz quadratic in \( u \) is of importance,

\[
L(u) = IU^2 + uG + H.
\]

(1.4)

In particular the fundamental \( R \) matrix appearing in (1.1) in the case of the orthogonal or symplectic symmetry (compare (2.1) below) can be written in the form (1.4) and (1.1) can
be regarded as a particular case of (1.3). The Jordan-Schwinger class of representations, where the generators are composed of a set of Heisenberg pairs following the pattern of Quantum mechanical angular momentum provide an example of (1.4) [14, 18].

In general, the truncation of the expansion (1.2) of \( L(u) \) obeying (1.3) by imposing \( (L^{(k)})_b^a = 0, k > p \), results in conditions defining the order \( p \) evaluation of the Yangian algebra, \( Y^{(p)}(\mathcal{G}) \). The analysis of those conditions becomes complicated with increasing \( p \). It is rather involved already for \( p = 2 \). Because of its physical relevance this case deserves the effort of a complete analysis in full generality, which is the central point of this paper.

Starting from the classical results on the fundamental \( R \) matrix with orthogonal or symplectic symmetry [11, 12, 14] and on \( L \) operators [13, 14] the linear and quadratic evaluations of Yangians of such types have been studied in recent papers [17, 18, 19, 20]. Examples of \( L \) operators obeying the condition of the linear or the quadratic evaluation have been considered in these papers. The spinorial representation as a case of the linear evaluation has been studied in [22].

In the present paper we go beyond particular examples and derive from the \( RLL \) relations (1.3) with the quadratic ansatz (1.4) the defining relation of the second order evaluation algebra \( Y^{(2)}(\mathcal{G}) \). We resolve the structure of the terms in (1.4) and express them by the matrices of independent generators \( \bar{G} \) and \( \bar{H} \). We formulate the algebra relations of \( Y^{(2)}(\mathcal{G}) \) as expressions in products of the latter algebra valued matrices \( \bar{G} \) and \( \bar{H} \).

The constraints implied by the quadratic ansatz are extracted from the \( RLL \) relation in a concise tensor product formulation [16], avoiding an accumulation of indices. In sect. 3 the simpler case of the linear evaluation \( Y^{(1)}(\mathcal{G}) \) is reconsidered. We illustrate in this case how the tensor product formulation is used to formulate the constraints and also to rewrite them in convenient forms in order to understand their meaning. There we have only one extra condition on the matrix of Lie algebra generators. We add some results on the spinorial representation beyond the recent paper [18].

In sect. 4 the quadratic truncation constraints are extracted from (1.3) as eight relations in the tensor product formulation. The decomposition in the spectral parameters leads to the natural ordering of increasing complexity. In the first step of transformations the constraints are separated into parts symmetric or anti-symmetric in the tensor factors for further detailed analysis. Further, the involved algebra valued matrices \( G \) and \( H \) are decomposed into parts graded anti-symmetric and symmetric in matrix indices. The anti-symmetric constraints imply commutation relations, in particular the Lie algebra relations. All symmetric constraints can be written in a standard form and they imply relations on the graded symmetric part of the matrix expressions. Relations for the graded anti-symmetric parts are derived from the anti-symmetric constraints. Combining both we obtain the structure of the quadratic \( L \) operator and the final form of the \( Y^{(2)}(\mathcal{G}) \) algebra relations.

In sect. 5 we consider the reduced case where the second algebra valued matrix \( H \) is expressed completely in terms of the first \( G \), i.e. all Yangian generators are obtained from the ones obeying the Lie algebra relations.

2 Yangians of the orthogonal and symplectic types

The fundamental Yang-Baxter equation has been written above, (1.1). It has a solution symmetric with respect to \( \text{so}(n) \) or \( \text{sp}(n) \) [11, 12, 14, 15]

\[
R_{b_1 b_2}^{a_1 a_2}(u) = u(u + \beta)I_{b_1 b_2}^{a_1 a_2} + (u + \beta)P_{b_1 b_2}^{a_1 a_2} - \epsilon u K_{b_1 b_2}^{a_1 a_2},
\]  
\[(2.1)\]
where

$$I_{b_1^a b_2^a}^{a_1 a_2} = \delta_{b_1^a}^{a_1} \delta_{b_2^a}^{a_2} , \quad P_{b_1^a b_2^a}^{a_1 a_2} = \delta_{b_2^a}^{a_1} \delta_{b_1^a}^{a_2} , \quad K_{b_1^a b_2^a}^{a_1 a_2} = \varepsilon_{a_1 a_2} \varepsilon_{b_1 b_2} , \quad \beta = \frac{n}{2} - \epsilon . \quad (2.2)$$

Here $\varepsilon_{ab}$ is a non-degenerate invariant tensor, defining the scalar product in $V$,

$$\varepsilon_{ab} = \epsilon \varepsilon_{ba} , \quad \varepsilon_{ab} \varepsilon^{bd} = \delta_{a}^{d} . \quad (2.3)$$

The sign factor $\epsilon$ allows a uniform treatment of both orthogonal and symplectic cases, $\epsilon = +1$ in the orthogonal and $\epsilon = -1$ in the symplectic case. We shall call an expression graded symmetric if it is symmetric in the orthogonal ($\epsilon = +1$) case and anti-symmetric in the symplectic ($\epsilon = -1$) case.

The existence of the invariant tensor $\varepsilon_{ab}$ leads to the third term in the corresponding expression of the $R$-matrices and to the quadratic dependence on the spectral parameter $u$.

We consider now the RLL relation (1.3) with $R_{b_1^a b_2^a}^{a_1 a_2}(u - v)$ substituted by the Yang-Baxter $R$-matrix (2.1). Further, substituting (1.2) results in the relations between the extended Yangian generators $(L^{(k)})_{b}^{a}$. We shall investigate these relations in the case of truncation.

The fundamental Yang-Baxter equation has been written above, (1.1), in the index form referring to a basis in the tensor product of three copies of the $n$ dimensional linear space $V$, $V_1 \otimes V_2 \otimes V_3$. The labels 1, 2, 3 at the indices refer to the action on the corresponding tensor factor. Alternatively, (1.1) can be written in the standard auxiliary space notation [16] in terms of operators acting in this tensor product space,

$$R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u) . \quad (2.4)$$

where e.g. $R_{12}$ acts trivially on the third tensor factor and its matrix elements with respect to a basis in $V_1 \otimes V_2$ are denoted by $R_{b_1^a b_2}^{a_1 a_2}$ as in (1.1). The expression (2.1) for the $R$ matrix is written in this notation in terms of $P_{12}, K_{12}$, the matrix elements in a basis of $V_1 \otimes V_2$ of which are as in (2.2). They obey the important relations

$$P_{12} K_{12} = \epsilon K_{12} = K_{12} P_{12} , \quad K_{12}^2 = n \varepsilon K_{12} . \quad (2.5)$$

In analogy, in this notation the RLL relation (1.3) is formulated in the product composed of the algebras of $\text{End} \ V_1 \otimes \text{End} \ V_2$ and the Yangian algebra $\mathcal{Y}(\mathcal{G})$,

$$R_{12}(u - v) L_1(u) L_2(v) = L_2(v) L_1(u) R_{12}(u - v) . \quad (2.6)$$

Here $R_{12}$ has the unit element in the $\mathcal{Y}(\mathcal{G})$ factor, $L_1$ has the unit element in the factor $\text{End} \ V_2$ and $L_2$ has the unit element in the factor $\text{End} \ V_1$. The index form (1.3) is reconstructed from (2.6) with the matrix elements of $L_1, L_2$ with respect to a basis in $V_1 \otimes V_2$ written as

$$L_{b_1^a b_2^a}^{a_1 a_2} = L_{b_1^a}^{a_1} \delta_{b_2^a}^{a_2} , \quad (L_2)^{a_1 a_2}_{b_1^a b_2^a} = \delta_{b_1^a}^{a_1} L_{b_2^a}^{a_2} . \quad (2.7)$$

The auxiliary space notation allows not only to write the basic relations in a concise form but also to do effectively the transformations of the truncated Yangian algebra relations. In the following we apply this notation to transform the algebra relations into a form convenient to understand how they constrain the representations.
The constraints to be obtained from the RLL relation (1.3) will be denoted by \( C^{(i,k)} \) with the superscripts \( i = 1, k = 1, 2, 3 \) in the case of linear evaluation and \( i = 2, k = 1, \ldots, 8 \) in the case of quadratic evaluation. \( C^{(i,k)}_1, C^{(i,k)}_2 \) or \( \bar{C}^{(i,k)}_1, \bar{C}^{(i,k)}_2 \) will be used to abbreviate expressions related to the corresponding constraints \( C^{(i,k)} \). In the auxiliary space notation these expressions are algebra valued matrices in \( \text{End} \, V_1 \otimes \text{End} \, V_2 \). As in the case of \( L_1, L_2 \) the subscripts 1 or 2 mean that the expression has the unity in \( \text{End} \, V_2 \) or \( \text{End} \, V_1 \) correspondingly. \( C^{(i,k)} \) (no subscript 1, 2) is to be understood as an algebra valued element of \( \text{End} \, V \). We shall decompose it with respect to index symmetry and have in particular trace contributions denoted by \( c^{(i,k)}_I \), where \( I \) is the unity in \( \text{End} \, V \).

3 Linear evaluation

We put all generators \( (L^{(k)})^a_b \in \mathcal{Y}(\mathcal{G}) \) with \( k > 1 \) equal to zero and substitute in (1.3) the \( L \)-operator linear in the spectral parameter:

\[
L^a_b(u) = u \delta^a_b + G^a_b. \tag{3.1}
\]

We formulate the known result which appeared already in [13, 14] and has been considered in [22, 17, 18].

**Proposition 1.** The Yangian linear evaluation \( \mathcal{Y}^{(1)}(\mathcal{G}) \) for \( \mathcal{G} = \text{so}(n) \) and \( \mathcal{G} = \text{sp}(n) \) is defined by the following algebra relations in terms of the algebra valued matrix \( G \) appearing in (3.1).

1. There is no traceless graded symmetric contribution in the matrix \( G \), i.e.

\[
G = gI + \bar{G}, \quad \text{tr} \, G = n \cdot g, \quad \bar{G}_{ba} = -\varepsilon \bar{G}_{ab}.
\]

\( g \) is central and \( \bar{G} \) is the graded antisymmetric part of the matrix \( G \).

2. The matrix elements of \( \bar{G} \) obey the Lie algebra relations of \( \text{so}(n) \) (\( \varepsilon = +1 \)) or \( \text{sp}(n) \) (\( \varepsilon = -1 \)), which can be written in the auxiliary space notation as

\[
[\bar{G}_1 + P_{12} - \varepsilon K_{12}, \bar{G}_2] = 0.
\]

3. The algebra relations are more restrictive compared to the Lie algebra relations by the additional condition on the matrix \( \bar{G} \),

\[
\bar{G}^2 + \beta \bar{G} = m_2 I, \quad m_2 = \frac{1}{n} \text{tr} \, \bar{G}^2.
\]

The proof extends over the next two subsections. The purpose is to illustrate the methods in the simpler case of the linear evaluation as a preparation to the application in the case of the quadratic evaluation.

3.1 The conditions of truncation

Here we start with the proof of Proposition 1 by obtaining the condition on \( G \) for the RLL relation (1.3) to hold. With \( L(u) \) substituted in the form (3.1) the defining RLL-relation (1.3) takes the form:

\[
\left( u(u + \beta)I_{12} + (u + \beta)P_{12} - u\varepsilon K_{12} \right)(u + v + G_1)(v + G_2) = \]

\[
\]

\[
\]
\[ (v + G_2)(u + v + G_1) \left( u(u + \beta)I_{12} + (u + \beta)P_{12} - u\epsilon K_{12} \right). \]

The index form of \( I_{12}, P_{12}, K_{12} \) is written above in (2.2). The index form of \( G_1 \) and \( G_2 \) is according to (2.7)

\[ (G_1)^{a_1,a_2}_{b_1,b_2} = G^{a_1}_{b_1}\delta^{a_2}_{b_2}, \quad (G_2)^{a_1,a_2}_{b_1,b_2} = \delta^{a_1}_{b_1}G_{a_2,b_2}. \]

The defining relation can be further rewritten as

\[ (u + \beta) \left( [G_1,G_2] + (G_1 - G_2)P_{12} - \epsilon[K_{12},G_2] \right) - \epsilon v[K_{12},G_1 + G_2] - \epsilon K_{12}(G_1 - \beta)G_2 + \epsilon G_2(G_1 - \beta)K_{12} = 0, \]

and has to hold identically in \( u \) and \( v \), implying three restrictions on the matrix of generators \( G \),

\[ -v\mathcal{C}^{(1,1)} = -\epsilon v[K_{12},G_1 + G_2] = 0, \tag{3.2} \]

\[ (u + \beta)\mathcal{C}^{(1,2)} = (u + \beta) \left( [G_1,G_2] + (G_1 - G_2)P_{12} - \epsilon[K_{12},G_2] \right) = 0, \tag{3.3} \]

\[ -\mathcal{C}^{(1,3)} = -\epsilon \left( K_{12}(G_1 - \beta)G_2 - G_2(G_1 - \beta)K_{12} \right) = 0. \tag{3.4} \]

We decompose the constraints with respect to the \( 1 \leftrightarrow 2 \) permutation parity.

\[ \mathcal{C} = \mathcal{C}_s + \mathcal{C}_a, \quad P_{12}\mathcal{C}_sP_{12} = \mathcal{C}_s, \quad P_{12}\mathcal{C}_aP_{12} = -\mathcal{C}_a. \]

Note that the first constraint is symmetric \( \mathcal{C}^{(1,1)} = \mathcal{C}^{(1,1)}_s \), while the second is antisymmetric \( \mathcal{C}^{(1,2)} = \mathcal{C}^{(1,2)}_a \). The antisymmetric part of the third constraint is obtained by right and left multiplication by the projector \( \frac{1}{2}(1 - \epsilon P_{12}) \).

\[ \mathcal{C}^{(1,3)}_{a,L} = K_{12} \left( [G_1,G_2] + \beta(G_1 - G_2) \right) = 0 = \left( [G_1,G_2] - \beta(G_1 - G_2) \right) K_{12} = \mathcal{C}^{(1,3)}_{a,R}. \tag{3.5} \]

However these relations do not imply a new constraint because they follow from the second constraint upon multiplication by \( K_{12} \).

The substantial part of the third relation, containing the new restriction is obtained by symmetrization:

\[ \mathcal{C}^{(1,3)}_s = [K_{12},G_1^2 + G_2^2] = 0. \tag{3.6} \]

### 3.2 Analysis of the constraints

We continue the proof of Proposition 1.

Multiplying the first constraint \( \mathcal{C}^{(1,1)} \) by \( n\epsilon - K_{12} \) from the left and right, one obtains:

\[ \epsilon K_{12}(G_1 + G_2)K_{12} = nK_{12}(G_1 + G_2) = n(G_1 + G_2)K_{12} = 2K_{12}\text{tr}G, \]

because by (2.5) \( K_{12}G_1K_{12} = K_{12}G_2K_{12} = \epsilon K_{12}\text{tr}G. \)

In any case \( G \) can be separated into a traceless part \( \tilde{G} \) and the trace part denoted by \( g \):\n
\[ G = gI + \tilde{G}. \tag{3.7} \]

Then the traceless part obeys

\[ K_{12}(\tilde{G}_1 + \tilde{G}_2) = 0 = (\tilde{G}_1 + \tilde{G}_2)K_{12}, \quad \Rightarrow \quad \text{tr}\tilde{G} \cdot K_{12} = 0. \]
The latter relation means that the traceless graded-symmetric part of the algebra-valued matrix $G$ vanishes. This can be checked by rewriting it in the index notation. Thus the first constraint (3.2) implies that $G$ consists of an arbitrary traceless, graded-antisymmetric part $\tilde{G}$ and the trace part, proportional to the metric tensor. In the index notation this means:
\[
(gI)^a_b = g\delta^a_b, \quad (gI)_{ab} = g\varepsilon_{ab},
\]
\[
\tilde{G}^a_a = 0 = \tilde{G}_a^a, \quad \tilde{G}_{c_1c_2} + \varepsilon \tilde{G}_{c_2c_1} = 0 = G^{a_1a_2} + \varepsilon G^{a_2a_1}.
\] (3.8)
Here repeated indices abbreviate the sum with the index running over the index range and $G^{a_1a_2} = \varepsilon_{a_1a}G^{a_2a_2}$.

Substituting the solution (3.7) into the relations one concludes that the first constraint $\mathcal{C}^{(1,1)}$ in terms of the traceless part $\tilde{G}$ is reduced to two:
\[
\mathcal{C}^{(1,1)}_L = K_{12}(\tilde{G}_1 + \tilde{G}_2) = 0, \quad \mathcal{C}^{(1,1)}_R = (\tilde{G}_1 + \tilde{G}_2)K_{12} = 0.
\] (3.9)

The second constraint is graded-antisymmetric (it has a symmetry opposite to the one of the metric tensor). It expresses the symmetry of $G$ with respect to the simultaneous rotation in the auxiliary ($\text{End} V_1$ or $\text{End} V_2$) and the quantum ($\mathcal{Y}^{(1)}$ representation) spaces and can be rewritten as $\mathcal{C}^{(1,2)}$ in terms of $G$ as
\[
\mathcal{C}^{(1,2)} = [\tilde{G}_1 + P_{12} - \epsilon K_{12}, \tilde{G}_2] = [\tilde{G}_1, \tilde{G}_2 + P_{12} - \epsilon K_{12}].
\] (3.10)
$\mathcal{C}^{(1,2)} = 0$ is equivalent to the Lie algebra commutation relations conventionally written as
\[
[G_{ab}, G_{cd}] = -\varepsilon_{cb}G_{ad} + \varepsilon_{ad}G_{cb} + \varepsilon_{ac}G_{bd} - \varepsilon_{db}G_{ca}.
\] (3.11)
(3.10) also shows that $g$ is central.

Multiplying $\mathcal{C}^{(1,2)}$ by $K_{12}$ from the left and right and using the identities $P_{12}K_{12} = \epsilon K_{12} = K_{12}P_{12}$, $K_{12}^2 = n\epsilon K_{12}$ one obtains a relation between the second and the third constraint,
\[
\mathcal{C}^{(1,2)}_L = K_{12}([\tilde{G}_1, \tilde{G}_2] - 2\beta \tilde{G}_2) = \mathcal{C}^{(1,3)}_{a,L}, \quad \mathcal{C}^{(1,2)}_R = (\tilde{G}_1, \tilde{G}_2 + 2\beta \tilde{G}_2)K_{12} = \mathcal{C}^{(1,3)}_{a,R}.
\] (3.12)
These relations can be obtained by multiplication of (3.4) by $1 - \epsilon P_{12}$ from the left and from the right correspondingly. Using (3.9) one can rewrite (3.12) as
\[
\mathcal{C}^{(1,2)}_L = K_{12}((\tilde{G}_1^2 + \beta \tilde{G}_1) - (\tilde{G}_2^2 + \beta \tilde{G}_2)) = 0,
\] (3.13)
\[
\mathcal{C}^{(1,2)}_R = ((\tilde{G}_1^2 + \beta \tilde{G}_1) - (\tilde{G}_2^2 + \beta \tilde{G}_2))K_{12} = 0.
\] These consequences of the second constraint tell that $\tilde{G}^2 + \beta \tilde{G}$ is graded-symmetric. This can be checked also in the index notation on the basis of (3.11) From (3.8) we see that $\tilde{G}_{ad}^2$ equals $\varepsilon \tilde{G}_{da}^2$ up to a commutator term, which can be obtained from (3.11) by multiplying with $\varepsilon \tilde{G}^{bc}$ and summing over repeated indices. We obtain $\tilde{G}_{ad}^2 = \epsilon \tilde{G}_{da}^2 - 2\beta G_{ad}$ and this implies the index symmetry relation $\tilde{G}_{ad}^2 + \beta G_{ad} = \epsilon (\tilde{G}_{da}^2 + \beta G_{da})$.

The last symmetric constraint (3.6) can be written as
\[
\mathcal{C}^{(1,3)} = [\epsilon K_{12}, (\tilde{G}_1^2 + \beta \tilde{G}_1) + (\tilde{G}_2^2 + \beta \tilde{G}_2)] = 0,
\] (3.14)
and allows us to deduce in analogy to the arguments leading to (3.7) and with the just established symmetry property of $\bar{G}^2 + \beta \bar{G}$ that the latter term is just a trace contribution

$$G^2 + \beta \bar{G} = c^{(1.3)} I = m_2 \cdot I. \quad (3.15)$$

Here $\bar{G}^2 + \beta \bar{G}$ is the graded-symmetric part of $G^2$ and $m_2 = \frac{1}{n} G^a{}_b G^b{}_a$ is the quadratic Casimir element.

The relation (3.15) restricts the Lie algebra representation, which provides the linear evaluation of the Yangian $\mathcal{Y}^{(1)}(\mathcal{G})$.

This completes the proof of Proposition 1.

In this way, the Yangian linear evaluation $\mathcal{Y}^{(1)}(\mathcal{G})$ is defined by the following algebra relations: the first constraint fixes the form of generator matrix (3.7), the next one defines Lie algebra relations (3.10), while the last one imposes the additional condition (3.15). Besides of the central element $g$ the generators of $\mathcal{G}$ are more restrictive by the additional condition (3.15), which is not fulfilled identically in the universal enveloping algebra $U(\mathcal{G})$. It can be fulfilled in distinguished $so(n)$ and $sp(n)$ representations only. For example the generators in the fundamental representation do not satisfy this restriction.

Repeating the above analysis for the case of $g\ell(n)$ is much easier and we recover the known statement that the linear evaluation of $\mathcal{Y}^{(1)}(g\ell(n))$ always exists. Indeed, in that case the fundamental $R$ matrix is simpler. It can be obtained from (2.1) by deleting the third term involving $K$. As a consequence the additional constraint (3.15) disappears.

### 3.3 Spinorial Yang-Baxter operators

We consider the case that the generators are composed in terms of an underlying algebra $\mathcal{C}$, which in turn is generated by the elements $c^a$, obeying the commutation relations of the oscillator algebra or of the Clifford algebra,

$$c^a c^b + \epsilon c^b c^a = \delta^{ab}, \quad c_a = \epsilon_{ab} c^b, \quad \Rightarrow \quad c_a c^b + \epsilon c^b c_a = \delta^b_a, \quad (3.16)$$

$$c_a c_b + \epsilon c_b c_a = [c_a, c_b] = \epsilon c_{ab}, \quad c_a c^a = \frac{n}{2} = \epsilon c^b c_b.$$

We may consider $c^a$ as operators in the spinor space ($so(n)$ case) or in the Fock space of $\frac{1}{2}n$ fermions ($so(n)$ case) or bosons ($sp(n)$ case).

Consider the linear map $\rho: \mathcal{Y}^{(1)}(\mathcal{G}) \to \mathcal{C}$,

$$\rho(G^a{}_b) = \bar{G}^a{}_b = \frac{\epsilon}{2} \delta^a_b - c^a c_b. \quad (3.17)$$

We see that the image of $G$ is graded-antisymmetric, i.e. $\bar{G} = \rho(\bar{G})$, $\rho(g) = 0$, and check that the Lie algebra relations (3.10) are fulfilled. Further,

$$(\bar{G}^2 + \beta \bar{G})^a{}_b = \frac{\epsilon}{4} (n - \epsilon) \delta^a_b = \frac{\epsilon}{2} (\beta + \frac{\epsilon}{2}) \delta^a_b.$$ 

Note that r.h.s. here fixes the value of the quadratic Casimir $m_2$ in this particular representation. We see that the composite generators (3.17) fulfill the additional condition (3.15). Thus the spinor representation (3.17) provides an example of the linear evaluation of the Yangian $\mathcal{Y}^{(1)}(\mathcal{G})$.

The Yang-Baxter operator $\hat{\mathcal{R}}$ intertwining two spinor representations, i.e. obeying

$$\hat{\mathcal{R}}_{12}(u) L_1(u + v) L_2(v) = L_1(v) L_2(u + v) \hat{\mathcal{R}}_{12}(u),$$
with
\[(L_1(u))^n_b = \delta^n_b (u + \frac{\epsilon}{2}) - c^n_1 c_1 b \]

is known explicitly [13, 17, 18]. In this Yang-Baxter relation the meaning of the indices 1, 2 is not literally the one of the auxiliary space notations used everywhere else in the paper. They refer here to two copies of the above oscillator/ Clifford algebras, i.e. \(V\) is replaced by the related Fock space, and \(L_1 L_2\) means multiplication of fundamental representation matrices.

Consider the more general linear Yang-Baxter operator
\[L_1(u) = u I - \frac{1}{2} c^{|a| b} c^{|a| b} \otimes G_{ab}\]

where the fundamental representation has been replaced by some representation of the Lie algebra with generators \(G_{ab}\). In [17] it has been established that this form obeys the Yang-Baxter relation, obtained from the above by replacing \(L_1, L_2\) by \(L_1, L_2, \tilde{R}_{12}(u)\) \(L_2(v) = L_1(v) L_2(u + v) \tilde{R}_{12}(u),\)

with the same spinorial \(R\) operator if the additional condition
\[\{G_{a_1 | a_2}, G_{b_1 | b_2}\} = 0\]

holds, where \([...]\) means the graded anti-symmetrization of indices and \{,...,\} means anticommutator. Here the indices 1, 2 refer to two copies of the above oscillator/ Clifford algebras as above, but \(L_1L_2\) means multiplication in the algebra generated by the matrix elements of \(G\). Note that in (3.19) only the graded antisymmetric part of \(G\) contributes. Therefore we identify \(G\) with \(\tilde{G}\) in the remaining part of this subsection.

**Proposition 2.** The additional condition (3.21) for a linear spinorial Yang-Baxter operator \(L\) (3.19) to obey (3.20) can be formulated in three other equivalent forms in terms of the graded-antisymmetric matrix \(G\) of generators obeying the \(so(n)\) or \(sp(n)\) Lie algebra relations.

1. \[\{G_{a_1 | a_2}, G_{b_1 | b_2}\} + \{G_{a_1 | b_1}, G_{b_2 | a_2}\} + \{G_{a_1 | b_2}, G_{a_2 | b_1}\} = 0.\] \(3.22\)

2. \[W_{12} = 0, \quad W_{12} = -(G_2 + \epsilon) \left[(P_{12} - \epsilon K_{12}) G_2 - \epsilon G_1\right].\] \(3.23\)

\(W_{12}\) is an algebra-valued element in \(End V_1 \otimes End V_2\) with matrix elements labeled by two index pairs,
\[(W_{12})_{a_1 b_1, a_2 b_2} = G_{a_1 b_1} G_{a_2 b_2}.\] \(3.24\)

3. \[\hat{G}^2 + \beta \hat{G} = c I, \quad \text{where} \quad \hat{G} = -\frac{1}{2} c^{[a} c^{b]} \otimes G_{ab}, \quad c = \frac{1}{8} \epsilon m_2 n.\] \(3.25\)

Further, the condition implies that the following cubic polynomial in the matrix \(G\) vanishes.
\[\chi(G) = 0, \quad \chi(z) = z^3 + (2\beta + \epsilon) z^2 + \epsilon (2\beta - \frac{m_2^2}{2}) z - \frac{m_2^2}{2}.\] \(3.26\)
Proof.

The second form (3.22) is proved by writing the first form (3.21) explicitly. The third form (3.23) is proved by writing the matrix elements of $W_{12}$ using (2.2). The fourth form is obtained by the expansion of the product \[ c[a, b, c, d, e] = c[a, b, c, e] \epsilon_{bd} + c[b, c, d, e] \epsilon_{ea} - c[b, c, e, d] \epsilon_{ad} - c[a, c, d, e] \epsilon_{be} + \frac{1}{4}(\epsilon_{ae} \epsilon_{bd} - \epsilon_{eb} \epsilon_{ad}). \]

The expression on l.h.s of (3.22) can be written as the graded-antisymmetric in indices sum of $G_{a_1 b_1} G_{a_2 b_2}$. This means, in $\hat{G}$ the contribution of the contraction with $c[a_1 c_1 b_1 c_2]$ vanishes.

The statement about the cubic polynomial can be obtained by contraction with a third factor of $G$, e.g. starting with the form (3.23) in terms of $W_{12}$,

\[ K_{12} G_{2} W_{12} = -2\epsilon K_{12} \left( G_{2}^{3} + (2\beta + \epsilon) G_{2}^{2} + \epsilon(2\beta - \frac{m_{2}}{2}) G_{2} - \frac{m_{2}}{2} \right) = -2\epsilon K_{12} \chi_{2}. \]

\[ \chi_{2} \] denotes the expression in the bracket being the cubic polynomial (3.26) with the argument substituted by $G_{2}$. Similarly one calculates:

\[ W_{12} G_{2} K_{12} = -2\epsilon \chi_{1} K_{12}. \]

The notion of $W_{12}$ will be useful in the analysis of the quadratic evaluation below. For this we notice the following properties: $W_{12}$ is annihilated by $K_{12}$,

\[ W_{12} K_{12} = 0 = K_{12} W_{12}. \]

In the index notation it means that the contraction of any pair of indices of $W^{a_2 a_1} c_1 c_2$ vanishes. Similarly, $W_{12}$ is annihilated by $P_{12} + \epsilon I_{12}$,

\[ W_{12} P_{12} = P_{12} W_{12} = -\epsilon W_{12}, \quad \Rightarrow \quad W_{21} = P_{12} W_{12} P_{12} = W_{12}. \]

The fourth form (3.25) written in terms of the spinorial matrix $\hat{G}$ is reminiscent (but having an essentially different meaning) to the additional condition of the Yangian linear evaluation (3.15) written in the matrix $G$ in $\text{End} \ V$, acting on the fundamental representation space $V$. However, the condition that the linear spinorial $L$ (3.19) obeys the spinorial Yang-Baxter relation (3.20) results in the condition on the latter matrix $G$ expressed instead in terms of the cubic polynomial $\chi(G)$ (3.26). We shall see below that such $G$, obeying (3.21-3.25), are appearing in a particular case of the quadratic evaluation.

4 Quadratic evaluation

4.1 The conditions of truncation

Now we are going to consider the case where $(L^{(k)})^{a}_{b} \in \mathcal{Y}(\mathcal{G})$ with $k > 2$ are constrained to vanish, i.e. we start from the quadratic ansatz (1.4), $L(u) = u^{2} I + uG + H$, and consider the conditions arising from (1.3) on the algebra-valued matrices $G$ and $H$.

Note that in the resulting expressions for the constraints commutators and anti-commutators will appear not graded by the dependence on $\epsilon$, therefore the notations $[\ldots, \ldots]$ and $\{\ldots, \ldots\}$ (no subscript) will be used, respectively.
Proposition 3. The RLL relations imply 8 constraints. Four of them are symmetric and four are anti-symmetric with respect to the permutation of the first two tensor factors. The four symmetric are $c_s^{(2,1)}$, $c_s^{(2,3)}$, $c_s^{(2,6)}$ and $c_s^{(2,8)}$ and the four anti-symmetric are $c_a^{(2,2)}$, $c_a^{(2,4)} = P_{12}c_a^{(2,5)}P_{12}$ and $c_a^{(2,7)}$, where

$$c_s^{(2,1)} = c^{(2,1)} = [P_{12} - \epsilon K_{12}, G_1 + G_2] = 0, \quad (4.1)$$
$$c_s^{(2,2)} = [G_1, G_2] - \frac{1}{2}[P_{12} - \epsilon K_{12}, G_1 - G_2] = 0, \quad (4.2)$$
$$c_s^{(2,3)} = [P_{12} - \epsilon K_{12}, H_1 + H_2 - \frac{1}{2}(G_1^2 + G_2^2)] = 0, \quad (4.3)$$
$$c_s^{(2,4)} = [G_1, H_2] - [G_2, H_1] - [P_{12} - \epsilon K_{12}, H_1 - H_2] = 0, \quad (4.4)$$
$$c_s^{(2,6)} = [P_{12} - \epsilon K_{12}, \{H_1, G_2\} + \{G_1, H_2\}], \quad (4.5)$$
$$c_s^{(2,7)} = [H_1, H_2] + \frac{1}{4}[P_{12} - \epsilon K_{12}, \{G_1, H_2\} - \{G_2, H_1\}], \quad (4.6)$$
$$c_s^{(2,8)} = [P_{12} - \epsilon K_{12}, \{H_1, H_2\} - \beta \epsilon (H_1 + H_2)]. \quad (4.7)$$

The proof extends over this and the next subsections. In this subsection we start analyzing the RLL relation (1.3) with the substitution (1.4) by decomposition in the spectral parameters. We shall complete the proof in the next subsection by separation of the parts symmetric and anti-symmetric in the labels 1, 2.

The defining relation (1.3) has the form:

$$[u(u + \beta)I_{12} + (u + \beta)P_{12} - \epsilon u K_{12}] \left((u + v)^2 + (u + v)G_1 + H_1\right) \left(v^2 + vG_2 + H_2\right) - (v^2 - vG_2 + H_2) \left((u + v)^2 - (u + v)G_1 + H_1\right) \left[u(u + \beta)I_{12} + (u + \beta)P_{12} - \epsilon u K_{12}\right] = 0. \quad (4.8)$$

This relation must hold at arbitrary values of the spectral parameters $u$ and $v$, i.e. the coefficients at independent monomials $u^k v^r$ must vanish. The l.h.s. of (4.8) can be represented as a sum of the following eight expressions.

$$uv^2(u + v)c^{(2,1)} = \epsilon uv^2(u + v)[K_{12}, G_1 + G_2], \quad (4.9)$$
$$(u + \beta)uv(u + v)c^{(2,2)} = (u + \beta)uv(u + v)\left([G_1, G_2] + (G_1 - G_2)P_{12} - \epsilon [K_{12}, G_2]\right), \quad (4.10)$$
$$-uv(u + v)c^{(2,3)} =$$
$$= -\epsilon uv(u + v)\left(K_{12}(H_1 + H_2 + (G_1 - G_2)) - (H_1 + H_2 + G_2(G_1 - \beta))K_{12}\right), \quad (4.11)$$
$$-(u + \beta)u(u + v)c^{(2,4)} = -(u + \beta)u(u + v)\left([G_1, H_2] + (H_1 - H_2)P_{12} - \epsilon [K_{12}, H_2]\right), \quad (4.12)$$
$$-(u + \beta)v(u + v)c^{(2,5)} = -(u + \beta)v(u + v)\left([H_1, G_2] + (H_1 - H_2)P_{12} + \epsilon [K_{12}, H_1]\right), \quad (4.13)$$
$$uv c^{(2,6)} = cuv\left(K_{12}(H_1(G_2 + \beta) + (G_1 - \beta)H_2) - (H_2(G_1 - \beta) + (G_2 + \beta)H_1)K_{12}\right), \quad (4.14)$$
$$-u(u + \beta)c^{(2,7)} =$$
$$= -u(u + \beta)\left([H_1, H_2] + (G_2H_1 - H_2G_1)P_{12} - \epsilon K_{12}(G_1 - \beta)H_2 + \epsilon H_2(G_1 - \beta)K_{12}\right), \quad (4.15)$$
$$-uc^{(2,8)} = -cu\left(K_{12}(H_1 - \beta G_1 + \beta^2)H_2 - H_2(H_1 - \beta G_1 + \beta^2)K_{12}\right). \quad (4.16)$$
They are obtained as the result of the following arguments: only two terms of (4.8) are proportional to \( uv \). After extraction of (4.9), the only terms proportional to \( u^3v \) are given by (4.10). Further, after extraction of (4.9) and (4.10), the expression (4.8) is at most cubic in the spectral parameters. Moreover, it contains \( u^3 \) only in the combination (4.12) and \( uv^2 \) only in the combination (4.13). Extracting these combinations one can write (4.8) in the form quadratic in the spectral parameter \( u \), which is given by the sum of (4.14), (4.15) and (4.16).

In other words, the spectral parameter dependence of the fundamental \( R \) matrix and the quadratic \( L \)-operators obeying (1.3) can be translated into these eight algebraic restrictions on the generators included in the algebra-valued matrices \( G, H \).

The first two constraints coincide with the ones, appearing in the linear evaluation case, sect. 3. The additional generators \( H \) enter now the third relation \( \mathcal{C}^{(2,3)} \) and modify the symmetric part of \( \mathcal{C}^{(1,3)} \) lifting the restriction (3.15). We shall see that the quadratic expression on l.h.s. of (3.15) determines the graded-symmetric part of \( H \).

### 4.2 Permutation and index symmetry

The permutation operator \( P_{12} \) interchanging the order of the tensor factors plays a crucial role. As in the linear evaluation case we separate the symmetric and antisymmetric parts of the constraints:

\[
\mathcal{C}^{(2,k)} = \mathcal{C}^{(2,k)}_s + \mathcal{C}^{(2,k)}_a, \quad k = 1, \ldots, 8, \tag{4.17}
\]

where \( \mathcal{C}^{(2,k)}_s = \frac{1}{2}(\mathcal{C}^{(2,k)} + P_{12}\mathcal{C}^{(2,k)}P_{12}) \) and \( \mathcal{C}^{(2,k)}_a = \frac{1}{2}(\mathcal{C}^{(2,k)} - P_{12}\mathcal{C}^{(2,k)}P_{12}) \). As the result of the truncation at second order we obtain \( 2p = 4 \) symmetric constraints and \( p^2 = 4 \) antisymmetric ones; the situation is similar for the general case \( \gamma^{(p)}(G) \) with the truncation at the order \( p \).

The set of defining equations (4.9-4.16) is equivalent to the following set of equations with definite symmetry with respect to \( 1 \leftrightarrow 2 \):

\[
\mathcal{C}^{(2,1)}_a = 0, \quad \mathcal{C}^{(2,1)}_s = \mathcal{C}^{(2,1)} = [P_{12} - \epsilon K_{12}, G_1 + G_2] = 0, \tag{4.18}
\]

\[
\mathcal{C}^{(2,2)}_s = \mathcal{C}^{(2,2)}_a = [G_1, G_2] - \frac{1}{2}[P_{12} - \epsilon K_{12}, G_1 - G_2] = 0, \tag{4.19}
\]

\[
\mathcal{C}^{(2,3)}_{a,L} = (1 - \epsilon P_{12})\mathcal{C}^{(2,3)} = \left([G_1, G_2] - \beta(G_1 - G_2)\right)K_{12} = \mathcal{C}^{(2,2)}_a K_{12}, \tag{4.20}
\]

\[
\mathcal{C}^{(2,3)}_{a,R} = \mathcal{C}^{(2,3)}(1 - \epsilon P_{12}) = K_{12}\left([G_1, G_2] + \beta(G_1 - G_2)\right) = K_{12}\mathcal{C}^{(2,2)}_a, \tag{4.21}
\]

\[
\mathcal{C}^{(2,3)} = [P_{12} - \epsilon K_{12}, H_1 + H_2 - \frac{1}{2}(G_1^2 + G_2^2)] + \{\mathcal{C}^{(2,1)}, G_1 + G_2\} = 0, \tag{4.22}
\]

\[
\mathcal{C}^{(2,4)} = [G_1, H_2 - \frac{1}{2}G_2^2] + [G_2, H_1 - \frac{1}{2}G_1^2] - \mathcal{C}^{(2,3)} - \frac{1}{2}[G_1 - G_2, \mathcal{C}^{(2,2)}], \tag{4.23}
\]

\[
\mathcal{C}^{(2,4)}_a = [G_1, H_2] - [G_2, H_1] - [P_{12} - \epsilon K_{12}, H_1 - H_2] = 0, \tag{4.24}
\]

\[
\mathcal{C}^{(2,6)} = \mathcal{C}^{(2,6)}_L(1 - \epsilon P_{12}) = K_{12}\mathcal{C}^{(2,4)}_a, \quad \mathcal{C}^{(2,6)}_R = (1 - \epsilon P_{12})\mathcal{C}^{(2,6)} = \mathcal{C}^{(2,4)}_a K_{12}, \tag{4.25}
\]

\[
\mathcal{C}^{(2,6)} = \frac{1}{2}(1 + \epsilon P_{12})\mathcal{C}^{(2,6)}(1 + \epsilon P_{12}) = [P_{12} - \epsilon K_{12}, \{H_1, G_2\} + \{G_1, H_2\}], \tag{4.26}
\]

\[
\mathcal{C}^{(2,7)} = \mathcal{C}^{(2,4)}_a P_{12} - \frac{\epsilon}{2}\{K_{12}, \mathcal{C}^{(2,4)}_s\} - \frac{\epsilon}{2}\mathcal{C}^{(2,6)}_s, \tag{4.27}
\]
\[
\mathcal{C}_a^{(2,7)} = [H_1, H_2] + \frac{1}{4}[P_{12} - \epsilon K_{12}, \{G_1, H_2\} - \{G_2, H_1\}] - \frac{\epsilon}{4}\{K_{12}, \mathcal{C}_a^{(2,4)}\},
\]
\[
\mathcal{C}_a^{(2,8)} = \mathcal{C}_a^{(2,8)}(1 - \epsilon P_{12}) = \epsilon K_{12}(\mathcal{C}_a^{(2,7)} + \frac{\epsilon - n}{4}\mathcal{C}_a^{(2,4)}),
\]
\[
\mathcal{C}_s^{(2,8)} = [K_{12}, \{H_1, H_2\} - \beta \epsilon (H_1 + H_2)] - \frac{\beta}{2}\{K_{12}, \mathcal{C}_s^{(2,4)}\} + \frac{\beta \epsilon}{2}\mathcal{C}_s^{(2,6)},
\]

The calculation is straightforward using the identities (2.5).

We observe that the constraints \(\mathcal{C}_s^{(2,k)}\) with the labels \(k = 1, 3, 6, 8\) contain the corresponding symmetric constraint as their main parts while their anti-symmetric parts appear as a consequence of the constraints \(\mathcal{C}_s^{(2,l)}\) with \(l < k\). In the constraints with the labels \(k = 2, 4, 5, 7\) the anti-symmetric parts are the leading contributions and the symmetric ones appear as a consequence of the constraints \(\mathcal{C}_s^{(2, l)}\) with \(l < k\). The anti-symmetric constraints involve commutators.

In this way we come to the independent constraints formulated in Proposition 3 and thus we have completed the proof.

In the remaining part of this subsection we analyze the simpler constraints with \(k = 1, 2, 3, 4, 5\) and prove the following proposition.

**Proposition 4.** The \(L\) operator representing the quadratic evaluation \(\mathcal{Z}^{(2)}(G)\),

\[
L(u) = Iu^2 + uG + H,
\]

has the following structure in the decomposition of the algebra-valued matrices into trace contributions (proportional to \(I\)), graded-antisymmetric parts \((\bar{G}, \bar{H})\) and graded-symmetric parts. The traceless graded-symmetric part \(G\) vanishes. The traceless graded-symmetric part of \(H\) equals the traceless graded-symmetric part of \(\bar{G}^2\).

\[
G = gI + \bar{G}, \quad H = hI + \bar{H} + \frac{1}{2}(\bar{G}^2 + \beta \bar{G}).
\]

The matrix elements of \(\bar{G}\) obey the Lie algebra relation of so\((n)\) or sp\((n)\) and the further Yangian algebra generators in \(\bar{H}\) transform as the adjoint representation of the latter.

**Proof.**

It is convenient to decompose \(G\) and \(H\) with respect to the index symmetry. In sect. 3 we have seen this decomposition to appear for \(G\) in analyzing the first constraint in the first order evaluation case. In the case of second order evaluation the first constraint implies like in the first order evaluation case that there is no traceless graded-symmetric part in \(G\),

\[
G = gI + \bar{G}.
\]

As in sect. 3 the trace contribution \(g\) is central and \(\bar{G}\) is graded-antisymmetric. Then the second constraint implies

\[
\mathcal{C}_a^{(2,2)} = [\bar{G}_1 + P_{12} - \epsilon K_{12}, \bar{G}_2] = 0.
\]

It encodes the Lie algebra relations. Multiplying (4.31) by \(K_{12}\) from the left and right, one deduces useful relations expressing the symmetry of \(\bar{G}^2 + \beta \bar{G}\):

\[
K_{12}(\bar{G}_1^2 + \beta \bar{G}_1) = K_{12}(\bar{G}_2^2 + \beta \bar{G}_2), \quad (\bar{G}_1^2 + \beta \bar{G}_1)K_{12} = (\bar{G}_2^2 + \beta \bar{G}_2)K_{12} = 0.
\]
It tells that $\bar{G}^2 + \beta \bar{G}$ is graded-symmetric. In the index notation this is obtained from the graded-antisymmetry of $\bar{G}$ and the Lie algebra commutation relations, which are contained in (4.31).

The symmetric constraints $k = 1, 3$ have the standard form

$$C_s^{(2,k)} = [K_{12}, C_1^{(2,k)} + C_2^{(2,k)}] = 0, \quad k = 1, 3. \tag{4.33}$$

The other symmetric constraints ($k = 6, 8$, not relevant in this proof) can be written in this form too as will be shown in the next subsection.

This implies (by the same argument as in sect.3) that the expressions $C^{(2,k)}$ have no traceless graded-symmetric contributions, i.e. decompose into a trace contribution denoted by $c^{(2,k)}$ and a graded antisymmetric matrix $\bar{C}^{(2,k)}$,

$$C^{(2,k)} = c^{(2,k)} I + \bar{C}^{(2,k)} \quad \text{and} \quad K_{12}(\bar{C}_1^{(2,k)} + \bar{C}_2^{(2,k)}) = 0 = (\bar{C}_1^{(2,k)} + \bar{C}_2^{(2,k)}) K_{12}. \tag{4.34}$$

The second relation in (4.34) does not fix the graded antisymmetric part. In the cases of $k = 1$ or $k = 3$ this part is $\bar{G}$ or $\bar{H}$ containing independent algebra generators.

The first constraint ($k = 1$) is analyzed completely above and the third ($k = 3$) implies by the latter argument the decomposition

$$H = h I + \frac{1}{2}(G^2 + \beta \bar{G}) + \bar{H}, \quad K_{12}(\bar{H}_1 + \bar{H}_2) = 0 = (\bar{H}_1 + \bar{H}_2) K_{12} \Leftrightarrow \bar{H}_{ab} = -\epsilon \bar{H}_{ba}, \tag{4.35}$$

$h$ is the trace contribution proportional to unity matrix, $h = c^{(2,3)}$. The graded-symmetric part of $H$ is fixed to be half of the graded symmetric part of $G^2$.

In terms of $G, \bar{H}$ the graded anti-symmetric constraints with $k = 4, 5$ read

$$\bar{C}^{(2,4)} = [\bar{G}_1 + P_{12} - \epsilon K_{12}, \bar{H}_2] = 0, \tag{4.36}$$

$$\bar{C}^{(2,5)} = -P_{12} \bar{C}^{(2,4)} P_{12} = [\bar{H}_1, \bar{G}_2 + P_{12} - \epsilon K_{12}] = 0. \tag{4.37}$$

These relations tell that the Yangian generators $\bar{H}$ transform under the adjoint representation of the Lie algebra.

### 4.3 Symmetric constraints

We turn to the analysis of the more involved constraints, the symmetric ones $k = 6, 8$ in this subsection and the antisymmetric $k = 7$ in the the following two subsections. This results in the algebra relations for products of $\bar{G}$ and $\bar{H}$ formulated in the following

**Proposition 5.** The quadratic evaluation conditions result in the structure formulated in Proposition 4 and further constrain the products of the algebra-valued matrices $G, \bar{H}$ as

$$\{\bar{H}, \bar{G}\} + 2\beta \bar{H} - g(\bar{G}^2 + \beta \bar{G}) = c^{(2,6)} I, \tag{4.38}$$

$$[\bar{H}_1, \bar{H}_2] + \frac{1}{8}[W_{12}, \bar{G}_1 - \bar{G}_2] + \frac{1}{8}[P_{12} - \epsilon K_{12}, \chi_1 - \chi_2 - 4g(\bar{H}_1 - \bar{H}_2)] + \frac{1}{8}\alpha [P_{12} - \epsilon K_{12}, \bar{G}_1 - \bar{G}_2] = 0, \tag{4.39}$$

$$\alpha = 4h + \beta^2 + 1 - 2\epsilon \beta + \frac{\beta^3}{2}. \tag{4.40}$$
The proof extends over this and the next two subsections. Here we analyze the symmetric constraints \( k = 6, 8 \). In the next subsection the needed information from the anti-symmetric constraints is derived resulting in particular in a relation for the commutator of the generators in \( \bar{H} \). This commutation relation will be reformulated in terms of the graded anti-symmetrized product of the generators \( \bar{G} \) in the 5th subsection.

We start the proof by recalling that the symmetric constraints (\( k = 1, 3, 6, 8 \)) can be written in the standard form (4.33), which is to be show below for \( k = 6, 8 \). This implies (by the same argument as in sect.3) that the expressions \( C^{(2,k)} \) have no traceless graded-symmetric contributions, i.e. decompose into a trace contribution denoted by \( c^{(2,k)} \) and a graded antisymmetric matrix \( C^{(2,k)} \), (4.34).

The second relation in (4.34) does not fix the graded antisymmetric part of \( C^{(2,k)} \). In the case of \( k = 1 \) or \( k = 3 \) this part is \( \bar{G} \) or \( H \) containing independent algebra generators, i.e. the constraints do not imply relations expressing them in terms of a smaller set of elements. In the other cases \( k = 6, 8 \) it takes to derive from the anti-symmetric constraints a relation of the form

\[
K_{12}(\tilde{C}_1^{(2,k)} - \tilde{C}_2^{(2,k)}) = 0 = (\tilde{C}_1^{(2,k)} - \tilde{C}_2^{(2,k)})K_{12}
\]

to fix it. This will be done in the next subsection.

We rewrite now all relations in terms of \( \bar{G}, \bar{H} \) and the trace contributions \( g, h \). In particular, the sixth constraint takes the form:

\[
-2\epsilon \bar{C}^{(2,6)} = [K_{12}, \{\bar{H}_1, \bar{G}_1\} + \{\bar{H}_2, \bar{G}_2\} - g(\bar{G}_1^2 + \bar{G}_2^2)] = 0.
\]

This relation has now the standard form (4.33) with

\[
C_1^{(2,6)} = \{\bar{H}_1, \bar{G}_1\} - g(\bar{G}_1^2 + \beta \bar{G}_1) + 2\beta \bar{H}_1.
\]

We have added the graded anti-symmetric matrix \( \beta(2\bar{H} - (\beta + g)\bar{G}) \).

Let us transform finally the 8th constraint (4.7) into the standard form (4.33).

\[
\mathcal{C}_s^{(2,8)} = [K_{12}, \{\bar{H}_1, \bar{H}_2\}] + \frac{1}{2}[K_{12}, \{\bar{H}_1, \bar{G}_2^2 + \beta \bar{G}_2\} + \{\bar{G}_1^2 + \beta \bar{G}_1, \bar{H}_2\}] +
\]

\[
+ \frac{1}{4}[K_{12}, \{\bar{G}_1^2, \bar{G}_2^2\}] + \frac{\beta}{2}[K_{12}, \{\bar{G}_1^2, \bar{G}_2\} + \{\bar{G}_2^2, \bar{G}_1\}] + \frac{\beta^2}{4}[K_{12}, \{\bar{G}_1, \bar{G}_2\}] + (h - \frac{\beta \epsilon}{2})[K_{12}, \bar{G}_1^2 + \bar{G}_2^2].
\]

We transform the terms containing \( \bar{H} \):

\[
[K_{12}, \{\bar{H}_1, \bar{H}_2\}] = [K_{12}, -\bar{H}_1^2 - \bar{H}_2^2].
\]

We show that terms linear in \( \bar{H} \) cancel. Indeed,

\[
K_{12}(\{\bar{G}_1^2, \bar{H}_2\} + \{\bar{G}_2^2, \bar{H}_1\}) = K_{12}\left( (\bar{G}_1^2 - \bar{G}_2^2)(\bar{H}_2 - \bar{H}_1) + [\bar{G}_1, \{\bar{G}_1, \bar{H}_1\}] + [\bar{G}_2, \{\bar{G}_2, \bar{H}_2\}] \right) =
\]

\[
= K_{12}\left( \beta(\bar{G}_1 - \bar{G}_2)(\bar{H}_1 - \bar{H}_2) - 2\beta([\bar{G}_1, \bar{H}_1] + [\bar{G}_2, \bar{H}_2]) \right) = 2\beta K_{12}\left( \bar{H}_1 \bar{G}_1 + \bar{H}_2 \bar{G}_2 \right),
\]

Adding the other term containing \( \bar{H} \) one obtains

\[
-\beta K_{12}(\{\bar{G}_1, \bar{H}_1\} + \{\bar{G}_2, \bar{H}_2\}) = \beta K_{12}(\{\bar{G}_1, \bar{H}_2\} + \{\bar{G}_2, \bar{H}_1\}) = 0,
\]

and similarly for terms right-multiplied by \( K_{12} \).
The terms containing only $\bar{G}$’s are

$$\frac{1}{4}[K_{12}, (\bar{G}_1^2 \bar{G}_2^2 + \bar{G}_2^2 \bar{G}_1^2)] - \frac{1}{4}[K_{12}, \bar{G}_1^4 + \bar{G}_2^4] = -\frac{1}{4}[K_{12}, (\bar{G}_1^2 - \bar{G}_2^2)^2] = \frac{\beta}{2}[K_{12}, \bar{G}_1^3 + \bar{G}_2^3],$$

and

$$\frac{\beta}{4}[K_{12}\{\bar{G}_1, \bar{G}_2^2\} + \{\bar{G}_2, \bar{G}_1^2\}] + \frac{\beta^2}{4}[K_{12}\{\bar{G}_1, \bar{G}_2\}] = \frac{\beta}{4}K_{12}(\bar{G}_1^2 - \bar{G}_2^2)(\bar{G}_2 - \bar{G}_1) - \frac{\beta}{4}(\bar{G}_2 - \bar{G}_1)(\bar{G}_1^2 - \bar{G}_2^2)K_{12} - \frac{\beta^2}{4}[K_{12}, \bar{G}_1^2 + \bar{G}_2^2] = \frac{\beta^2}{4}[K_{12}, \bar{G}_1^3 + \bar{G}_2^3].$$

Adding all contributions one obtains the standard form (4.33) with the consequences

$$K_{12}\left(-\bar{H}_1^2 - \bar{H}_2^2 + \frac{1}{4}(\bar{G}_1^4 + \bar{G}_2^4) + \frac{\beta}{2}(\bar{G}_1^3 + \bar{G}_2^3) + (h + \frac{\beta^2}{4} - \frac{\beta\epsilon}{2})(\bar{G}_1^2 + \bar{G}_2^2)\right) = K_{12}(C_1^{(2,8)} + C_2^{(2,8)}) = 0,$$

(4.43)

$$= K_{12}\left(-\bar{H}_1^2 - \bar{H}_2^2 + \frac{1}{4}(\bar{G}_1^4 + \bar{G}_2^4) + \beta(\bar{G}_1^3 + \bar{G}_2^3) + (h + \frac{5\beta^2}{4})(\bar{G}_1^2 + \bar{G}_2^2)\right).$$

$C_1^{(2,8)} + C_2^{(2,8)}$ abbreviates the expression in the bracket multiplying $K_{12}$. Here we took into account the identity

$$[K_{12}, \bar{G}_1^3 + \bar{G}_2^3 + (\beta + \frac{n}{2})(\bar{G}_1^2 + \bar{G}_2^2)] = 0.$$

### 4.4 Anti-symmetric constraints

The graded anti-symmetric constraints $k = 2, 4, 5$ have been analyzed above for the purpose of the proof of Proposition 4.

We intend to fix the graded anti-symmetric part of $C^{(2,6)}$, (4.42). The constraints $k = 2, 4$ provide the needed information. Using the projections of the antisymmetric combination

$$\mathcal{C}_{a,L}^{(2,4)} = K_{12}\left(\{\bar{H}_1, \bar{G}_1\} + 2\bar{\beta}\bar{H}_1 - (\{\bar{H}_2, \bar{G}_2\} + 2\bar{\beta}\bar{H}_2)\right) = 0,$$

(4.44)

and

$$\mathcal{C}_{a,R}^{(2,4)} = \left(\{\bar{H}_1, \bar{G}_1\} + 2\bar{\beta}\bar{H}_1 - (\{\bar{H}_2, \bar{G}_2\} + 2\bar{\beta}\bar{H}_2)\right)K_{12} = 0,$$

(4.45)

as well as

$$\mathcal{C}_{a,L}^{(2,2)} = K_{12}\left((\bar{G}_1^2 + \beta\bar{G}_1) - (\bar{G}_2^2 + \beta\bar{G}_2)\right) = 0,$$

(4.46)

and

$$\mathcal{C}_{a,R}^{(2,2)} = \left((\bar{G}_1^2 + \beta\bar{G}_1) - (\bar{G}_2^2 + \beta\bar{G}_2)\right)K_{12} = 0,$$

(4.47)

one can rewrite this as

$$K_{12}(C_1^{(2,6)} - C_2^{(2,6)}) = 0 = (C_1^{(2,6)} - C_2^{(2,6)})K_{12},$$

(4.48)

due to (3.13). Combining (4.33), (4.42) and (4.48), one deduces $[K_{12}, C_1^{(2,6)}] = 0 = [K_{12}, C_2^{(2,6)}]$. This means that $C^{(2,6)}$, (4.42), has a trace contribution proportional to $I$ only, denoted by $c^{(2,6)}$,

$$\{\bar{H}, \bar{G}\} + 2\bar{\beta}\bar{H} - g(\bar{G}^2 + \beta\bar{G}) = c^{(2,6)}I.$$

(4.49)
Using this relation one can simplify the constraints (4.6) and (4.7). We analyze the remaining graded antisymmetric constraint \( k = 7 \).

\[
\mathcal{C}_{a}^{(2,7)} = [\bar{H}_1, \bar{H}_2] + \frac{1}{2}([\bar{H}_1, G_2^2] + [G_1^2, \bar{H}_2]) + \frac{\beta}{2}([\bar{H}_1, G_2] + [G_1, \bar{H}_2]) + \frac{\beta}{4}([\bar{G}_1, G_2^2] + [G_1^2, \bar{G}_2]) + \\
+ \frac{1}{4}[G_1^2, G_2^2] + \frac{\beta}{4}([G_1, G_2] + \frac{1}{4}[P_{12} - \epsilon K_{12}, (2h - g\beta)(G_1 - G_2) - 2g\bar{H}_1 - \bar{H}_2) - g(G_1^2 - G_2^2)] + \\
+ \frac{1}{4}[P_{12} - \epsilon K_{12}, \{G_1, \bar{H}_2\} - \{G_2, \bar{H}_1\} + \frac{1}{2}([\bar{G}_1, G_2^2] - \{G_2, \bar{G}_1^2\})].
\]

First consider terms linear in \( \bar{H} \). Due to (4.36-4.37) we have

\[
[G_1, \bar{H}_2] = \frac{1}{2}[P_{12} - \epsilon K_{12}, \bar{H}_1 - \bar{H}_2] = [\bar{H}_1, \bar{G}_2],
\]

We rewrite also the last term in the form antisymmetric in 1 \( \leftrightarrow \) 2 and obtain

\[
\frac{1}{2}([\bar{H}_1, G_2^2] + [G_1, \bar{H}_2]) + \frac{1}{4}[P_{12} - \epsilon K_{12}, \{G_1, \bar{H}_2\} - \{G_2, \bar{H}_1\}] = \frac{1}{4}[P_{12} - \epsilon K_{12}, \{G_1, \bar{H}_1\} - \{G_2, \bar{H}_2\}] = \\
= \frac{1}{4}[P_{12} - \epsilon K_{12}, -2\beta(\bar{H}_1 - \bar{H}_2) + g(G_1^2 - G_2^2) + g\beta(G_1 - G_2)].
\]

The contribution of the two other terms linear in \( \bar{H} \) is

\[
\frac{\beta}{2}([\bar{H}_1, G_2] + [G_1, \bar{H}_2]) - \frac{g}{2}[P_{12} - \epsilon K_{12}, (\bar{H}_1 - \bar{H}_2)] = \frac{\beta - g}{2}[P_{12} - \epsilon K_{12}, (\bar{H}_1 - \bar{H}_2)].
\]

Then we calculate the contribution in (4.50) cubic in \( \bar{G} \),

\[
\frac{1}{4}[G_1^2, G_2^2] + \frac{1}{8}[P_{12} - \epsilon K_{12}, \{G_1, G_2^2\} - \{G_2, G_1^2\}] = \frac{1}{16}(G_1^2 + G_2^2, [P_{12} - \epsilon K_{12}, G_1 - G_2]).
\]

The next term in (4.50) simplifies as

\[
\frac{\beta}{4}([G_1^2, G_2^2] + [G_1, G_2^2]) = \frac{\beta}{4}[P_{12} - \epsilon K_{12}, G_1^2 - G_2^2],
\]

and the remaining contributions in (4.50) are trivial. Collecting all terms one obtains

\[
\mathcal{C}_{a}^{(2,7)} = \bar{\mathcal{C}}^{(2,7)} = [\bar{H}_1, \bar{H}_2] + \frac{1}{16}\{\bar{G}_1^2 + \bar{G}_2^2, [P_{12} - \epsilon K_{12}, \bar{G}_1 - \bar{G}_2]\} + \\
+ \frac{1}{4}[P_{12} - \epsilon K_{12}, (2h + \frac{\beta^2}{2})(\bar{G}_1 - \bar{G}_2) - 2g(\bar{H}_1 - \bar{H}_2) + \beta(G_1^2 - G_2^2)].
\]

The constraint \( \mathcal{C}_{a}^{(2,7)} = 0 \) results in the condition (4.39) of the proposition. In subsection 4.5 the expression will be transformed in terms of \( W_{12} \) (3.24).

In the last subsection we have derived from the 8th constraint a relation for \( \bar{H}_2 \) leaving its graded anti-symmetric part undetermined. To obtain this part of \( \bar{H}_2 \) we multiply (4.51) by \( K_{12} \):

\[
K_{12}\mathcal{C}_{a}^{(2,7)} = K_{12}\left(\bar{H}_1^2 - \bar{H}_2^2 + \frac{\epsilon - \beta}{4}(G_1^3 - G_2^3) + \beta g(\bar{H}_1 - \bar{H}_2) - \\
- \left(\frac{\epsilon \beta^2}{4} + \frac{m_2}{8} + \beta h\right)(\bar{G}_1 - \bar{G}_2)\right) = K_{12}(C_{1}^{(2,7)} - C_{2}^{(2,7)}) = 0.
\]
$C^{(2,7)}_1 - C^{(2,7)}_2$ denotes the expression in the bracket multiplying $K_{12}$. Similarly one obtains:

$$C_n^{(2,7)} K_{12} = -K_{12}(C^{(2,7)}_1 - C^{(2,7)}_2) = 0.$$  

Combining these relations and using the identities

$$K_{12} \tilde{G}_2 = -K_{12} \tilde{G}_1, \quad K_{12} \tilde{H}_2 = K_{12} \tilde{H}_1, \quad K_{12} (G_2^2 + \beta \tilde{G}_2) = K_{12} (G_1^2 + \beta \tilde{G}_1),$$  

one can rewrite (4.52) in a form convenient for comparison with (4.43) as

$$K_{12} \left( \tilde{H}_1^2 - \tilde{H}_2^2 - \frac{1}{4} (G_1^4 - G_2^4) + g \beta (\tilde{H}_1 - \tilde{H}_2) - \beta (G_1^3 - G_2^3) - \frac{5}{4} \beta^2 + h \tilde{G}_1 - G_2 - (\frac{\beta^3}{2} + 2 h \beta) (G_1 - G_2) \right) = 0,$$

In this way we determine $\tilde{H}^2$ up to a trace part involving $c^{(2,8)}$.

$$\tilde{H}^2 = c^{(2,8)} I + \frac{1}{4} G^4 - g \beta \tilde{H} + \beta \tilde{G}^3 + (\frac{5}{4} \beta^2 + \frac{\beta g}{2} + h) G^2 + (\frac{\beta^3}{2} + \frac{\beta^2 g}{2} + 2 h \beta) \tilde{G}.$$  

### 4.5 The seventh constraint in terms of $W_{12}$

We recall $W_{12}$ which is the graded-antisymmetrization in indices of $G_1 G_2$ (3.23),

$$W_{12} = -\left( G_2 + \epsilon \right) \left( P_{12} - \epsilon K_{12} \right) G_2 - \epsilon \tilde{G}_1,$$

and its relations (3.27 - 3.29).

We start with the second term in the seventh constraint (4.51) and use the notation as in (4.52):

$$C_7 = \frac{1}{16} \left\{ G_1^2 + G_2^2, [P_{12} - \epsilon K_{12}, G_1 - G_2] \right\} = \frac{1}{8} \left[ P_{12} - \epsilon K_{12}, G_1^2 \right] - \frac{1}{2} \epsilon [G_1 - \epsilon K_{12}, G_1] - 1 \leftrightarrow 2.$$  

Consider the commutator

$$[\tilde{G}_2 + \epsilon, W_{12}] = (\tilde{G}_2 + \epsilon) [P_{12} - \epsilon K_{12}, G_2] (\tilde{G}_2 + \epsilon),$$  

Multiplying it by $P_{12}$ from both sides we have also

$$[\tilde{G}_1 + \epsilon, W_{12}] = (\tilde{G}_1 + \epsilon) [P_{12} - \epsilon K_{12}, \tilde{G}_1] (\tilde{G}_1 + \epsilon),$$

due to the symmetry $W_{21} = W_{12}$ (3.30). This implies

$$[\tilde{G}_1 - \tilde{G}_2, W_{12}] = \tilde{G}_1 [P_{12} - \epsilon K_{12}, \tilde{G}_1] \tilde{G}_1 + \epsilon [P_{12} - \epsilon K_{12}, \tilde{G}_1^2] + \left[ P_{12} - \epsilon K_{12}, \tilde{G}_1 \right] - 1 \leftrightarrow 2.$$  

We see that some terms of (4.57) appear here. Multiplying this expression by $K_{12}$ one obtains using $K_{12} W_{12} = 0$

$$K_{12} (\tilde{G}_1 - \tilde{G}_2) W_{12} = K_{12} \left( \epsilon (2 \tilde{G}_1^3 - \tilde{G}_1^3 - \beta (\tilde{G}_1^2 - \tilde{G}_2^2) + [\tilde{G}_1^2, \tilde{G}_2] + [\tilde{G}_1, \tilde{G}_2^2]) + (m_2 - 2 \beta) (\tilde{G}_1 - \tilde{G}_2) \right) = \epsilon (2 \tilde{G}_1^3 - \tilde{G}_1^3 - \beta (\tilde{G}_1^2 - \tilde{G}_2^2) + [\tilde{G}_1^2, \tilde{G}_2] + [\tilde{G}_1, \tilde{G}_2^2]) + (m_2 - 2 \beta) (\tilde{G}_1 - \tilde{G}_2).$$
One proves that here in the last step we have used (4.64).

\[
\begin{align*}
&= \frac{1}{8}[W_{12}, \bar{G}_1 - \bar{G}_2] + \frac{1}{8}[P_{12} - \epsilon K_{12}, \chi_1 - \chi_2 - 2\beta(\bar{G}_1^2 - \bar{G}_2^2) + (1 - 2\epsilon\beta + \frac{m_2}{2}\epsilon)(\bar{G}_1 - \bar{G}_2)], \\
&= -2\epsilon K_{12}(\bar{G}_1 - \bar{G}_2) \equiv -2\epsilon K_{12}(\chi_1 - \chi_2).
\end{align*}
\]

Similarly, the multiplication by \( K_{12} \) from the right leads to

\[
W_{12}(\bar{G}_1 - \bar{G}_2)K_{12} = -2\epsilon \left( \bar{G}_1 - \bar{G}_2 \right) + (1 - 2\epsilon\beta + \frac{m_2}{2}\epsilon)(\bar{G}_1 - \bar{G}_2)),
\]

Here \( \chi \) is given by (3.26). Then

\[
C_7 = \frac{1}{8}[W_{12}, \bar{G}_1 - \bar{G}_2] + \frac{1}{8}[P_{12} - \epsilon K_{12}, \chi_1 - \chi_2 - 2\beta(\bar{G}_1^2 - \bar{G}_2^2) + (1 - 2\epsilon\beta + \frac{m_2}{2}\epsilon)(\bar{G}_1 - \bar{G}_2)],
\]

and hence according to (4.51)

\[
C_a^{(2,7)} = [\bar{H}_1, \bar{H}_2] + \frac{1}{8}[W_{12}, \bar{G}_1 - \bar{G}_2] + \frac{1}{8}[P_{12} - \epsilon K_{12}, \chi_1 - \chi_2 - 4g(\bar{H}_1 - \bar{H}_2)] + \\
+ \alpha\frac{1}{8}[P_{12} - \epsilon K_{12}, \bar{G}_1 - \bar{G}_2] = 0.
\]

\[
\alpha = 4h + \beta^2 + 1 - 2\epsilon\beta + \frac{m_2}{2}\epsilon.
\]

This completes the proof of Proposition 5.

### 4.6 The center of the truncated Yangian \( \mathcal{Y}^{(p)}(\mathcal{G}) \)

The center of the extended Yangian algebra of the orthogonal and symplectic types has been analyzed in [22] and formulated in terms of the reproducing function. In order to define it we consider again the RLL-relation (1.3) with the fundamental \( R \)-matrix given by (2.1) at \( u = v = \beta \). We arrive at

\[
K_{12}L_1(v - \beta)L_2(v) = L_2(v)L_1(v - \beta)K_{12}.
\]

In the index notation this reads

\[
\varepsilon^{a_1a_2}\varepsilon_{b_1b_2}L^{b_1}_{c_1}(u - \beta)L^{b_2}_{c_2}(u) = L^{a_2}_{b_2}(u)L^{a_1}_{b_1}(u - \beta)\varepsilon^{b_1b_2}\varepsilon_{c_1c_2}.
\]

After multiplication by \( \varepsilon_{a_2a_1} \) we obtain

\[
C_{ab}(u) \equiv L_{ca}(u - \beta)L^{c}_{d}(u) = \frac{1}{n}\varepsilon_{ab}L^{d}_{c}(u)L^{c}_{d}(u - \beta) \equiv \varepsilon_{abcd}(u),
\]

where according to [22] the center reproducing function is defined as

\[
C(u) = L^I(u - \beta)L(u) = c(u)I.
\]

One proves that \( C(u) \) contains central elements by showing that it commutes with \( L(v) \),

\[
C(u)L_2(v) = L^I_1(u - \beta)L_1(u)L_2(v) = L^I_1(u - \beta)R_{12}^{-1}(u - v)L_2(v)L_1(u)R_{12}(u - v) = \\
= L_2(v)R_{12}^{-1}(u - v)L^I_1(u - \beta)L_1(u)R_{12}(u - v) = L_2(v)R_{12}^{-1}(u - v)C(u)R_{12}(u - v) = L_2(v)C(u),
\]

here in the last step we have used (4.64).

In the case of the linear ansatz \( L(u) = uI + G \) the reproducing function looks like:

\[
C_{ab}^{(1)} = \varepsilon_{ad}C^{(1)}d_{b}(u) = \varepsilon_{ad}\varepsilon_{ce}((u - \beta)\delta^e_f + G^e_f)\varepsilon^{df}(u\delta^e_b + G^e_b) =
\]

\[
(4.65)
\]
\[
\varepsilon_{ce}(u + g - \beta)\delta^c_{\alpha} + \bar{G}^e_{\alpha})(u + g)\delta^e_{\beta} + \bar{G}^c_{\beta}) = \\
= \varepsilon((u + g)(u + g - \beta)\varepsilon_{ab} + (u + g - \beta)\bar{G}_{ab} - (u + g)\bar{G}_{ab} - \bar{G}_{ac}\bar{G}^c_{b}) = \\
= \varepsilon((u + g)(u + g - \beta)\varepsilon_{ab} - (\bar{G}^2 + \beta\bar{G})_{ab}) = \varepsilon((u + g)(u + g - \beta) - C^{(1,3)})\varepsilon_{ab}.
\]

Here we have used the notation (3.15) for \( C^{(1,3)} = g \) to emphasize the analogy with the quadratic evaluation case below. Note that all graded antisymmetric terms (like \( \bar{c} \)) obeying the symmetric properties:

\[
C^{(2)}(u) = \varepsilon_{ad}C^{(2)}_{b}(u) = \varepsilon_{ad}\varepsilon_{ce}((u - \beta)^2\delta^c_f + (u - \beta)G^e_f + H^e_f)\varepsilon_{df}(u^2\delta^c_b + uG^e_b + H^c_b) = \\
= \varepsilon((u - \beta)^2\varepsilon_{ac} + (u - \beta)(g\varepsilon_{ac} - \bar{G}_{ac}) + (h\varepsilon_{ac} + \frac{1}{2}(\bar{G}^2 + \beta\bar{G})_{ac} - \bar{H}_{ac})\varepsilon_{df}(u^2\delta^c_b + \frac{1}{2}(\bar{G}^2 + \beta\bar{G})_{b} + \bar{H}^c_b).
\]

Here at the first step we have used \( \varepsilon_{ad}\varepsilon_{df} = \delta^c_d \), lowered the index \( f \) by using the metric tensor. We have taken into account the symmetry properties: \( \varepsilon_{ca} = \varepsilon_{ac}, \bar{G}_{ca} = -\varepsilon\bar{G}_{ac}, \bar{H}_{ca} = -\varepsilon\bar{H}_{ac} \) and \( (\bar{G}^2 + \beta\bar{G})_{ca} = \varepsilon(\bar{G}^2 + \beta\bar{G})_{ac} \). Now we omit the matrix indices and collect similar terms:

\[
\varepsilon C^{(2)}(u) = ((u - \beta)^2 + (u - \beta)g + h)(u^2 + u\varepsilon + h) + (g(u - \beta) + \frac{\beta^2 + g\beta}{2} + h)(\bar{G}^2 + \beta\bar{G}) + \\
+ \beta h\bar{G} + (\beta^2 - 2u\beta - g\beta)\bar{H} + \frac{1}{4}(\bar{G}^4 + 4\beta\bar{G}^3 + 3\beta^2\bar{G}^2) - u\bar{H}\bar{G} - (u - \beta)\bar{G}\bar{H} + \frac{1}{2}(\bar{G}^2 + \beta\bar{G}, \bar{H}) - \bar{H}^2.
\]

We take into account that \( [\bar{G}, \bar{H}] = \beta\bar{H} \) and obtain

\[
\varepsilon C^{(2)}(u) = (h + (u - \beta)^2 + (u - \beta)g)(h + u^2 + u\varepsilon) + (\beta - u)\left([\bar{G}, \bar{H}] + 2\beta\bar{H} - g(\bar{G}^2 + \beta\bar{G})\right) + \\
+ \left(-\bar{H}^2 - g\beta\bar{H} + \frac{1}{4}\bar{G}^4 + \beta\bar{G}^3 + \frac{5}{4}\beta^2 + g\beta^2 + h\bar{G}^2 + (\frac{\beta^3}{2} + g\beta^2 + 2h\beta)\bar{G}\right) = \varepsilon_{df}(u^2 + u\varepsilon + h)(h + (u - \beta)^2 + (u - \beta)g) + (\beta - u)c^{(2,6)} - c^{(2,8)}.
\]

At the last step we have noticed that in the large-bracket terms the expressions of the conditions (4.49) and (4.56) appear.

\section{The Lie algebra resolution}

We consider the case where the second non-trivial term in the quadratic \( L \) (1.4) is completely expressed in terms of the generators in \( G \) obeying the Lie algebra relations. This restriction results in the Lie algebra resolution of the Yangian second order evaluation. Formally this can be understood as a map \( \rho \) of \( \mathcal{Y}^{(2)}(G) \) to the associative algebra generated by the matrix elements of \( G \),

\[
\rho(\bar{H}) = a\bar{G}, \quad \rho(G) = gI + \bar{G}, \quad (5.1)
\]
where $a$ is a central element.

$$\rho(H) = hI + \frac{1}{2}\tilde{G}^2 + (a + \frac{\beta}{2})\tilde{G}. \quad (5.2)$$

We shall see that in this case the algebra relations are fulfilled if a simple condition on the product of the generators in $G$ holds. Then $a$ and also $g$ and $h$ are fixed. This condition is related to the vanishing of a third order polynomial in $\tilde{G}$, in analogy to the linear evaluation case, where a condition is the vanishing of a second order polynomial in $\tilde{G}$. Higher order evaluations are related to such polynomials of corresponding higher order. In the first subsection we discuss, how polynomial conditions in $\tilde{G}$ constrain the algebra, assuming that $\tilde{G}$ obeys the Lie algebra relations.

5.1 Representations specified by characteristic polynomials

Let us consider representations of the orthogonal or symplectic Lie algebra $G$ constrained by the vanishing of a polynomial in $\tilde{G}$.

If this constraint is of first order in $\tilde{G}$ it does not allow non-trivial representations. Consider the quadratic case.

$$\chi^{(2)} = \tilde{G}^2 + A\tilde{G} + B.$$ 

and discuss which values of the coefficients $A, B$ may be chosen.

This expression can be represented as a sum of graded symmetric and antisymmetric matrices,

$$\chi_{ab}^{(2)} = \chi_s_{ab}^{(2)} + \chi_a_{ab}^{(2)}, \quad \chi_s_{ab}^{(2)} = \frac{1}{2}(\chi_{ab}^{(2)} + \epsilon\chi_{ab}^{(2)}), \quad \chi_a_{ab}^{(2)} = \frac{1}{2}(\chi_{ab}^{(2)} - \epsilon\chi_{ab}^{(2)}),$$

and the condition $\chi^{(2)} = 0$ implies both parts to vanish, $\chi_s^{(2)} = 0$ and $\chi_a^{(2)} = 0$. Let us rewrite the two conditions in terms of the tensor product notation,

$$K_{12}(\chi_s^{(2)} + \chi_s^{(2)}) = 0, \quad K_{12}(\chi_a^{(2)} - \chi_a^{(2)}) = 0.$$

Due to the Lie algebra relations the antisymmetric part $\chi_a^{(2)}$ is given by the polynomial of lower order, because

$$K_{12}(\tilde{G}_1^2 - \tilde{G}_2^2) = -\beta K_{12}(\tilde{G}_1 - \tilde{G}_2).$$

In order to avoid a restriction of the first order in $\tilde{G}$ we have to specify the coefficients such that the graded anti-symmetric part vanishes.

$$\chi_s^{(2)} = \tilde{G}^2 + \beta\tilde{G} - m_2 I,$$

where $m_2$ stands for the quadratic Casimir and $I$ is the unit matrix.

Higher order polynomial constraints are to be analyzed analogously. One finds that the order $p$ constraint should be graded symmetric if $p$ is even and graded anti-symmetric if $p$ is odd. Otherwise a constraint of order $p - 1$ would be involved.

Let us address the case of current interest and consider an arbitrary cubic polynomial in $\tilde{G}$

$$\chi^{(3)} = \tilde{G}^3 + D\tilde{G}^2 + E\tilde{G} + F.$$ 

We decompose again into graded symmetric and antisymmetric parts,

$$K_{12}\chi_1^{(3)} = \frac{1}{2}K_{12}(\chi_1^{(3)} - \chi_2^{(3)}) + \frac{1}{2}K_{12}(\chi_1^{(3)} + \chi_2^{(3)}).$$
The symmetric part is reduced to the second order polynomial,
\[ \frac{1}{2} K_{12} (\chi_1^{(3)} + \chi_2^{(3)}) = \frac{1}{2} K_{12} \left( [\bar{G}_1, \bar{G}_2] (\bar{G}_1 - \bar{G}_2) + \bar{G}_1 [\bar{G}_1, \bar{G}_2] + D (\bar{G}_1^2 + \bar{G}_2^2) + 2F \right) = \]
\[ = \frac{1}{2} K_{12} \left( (D - 2\beta - \epsilon)(\bar{G}_1^2 + \bar{G}_2^2) + 2F + m_2 \right). \]

In turn the antisymmetric part contains only one free parameter,
\[ \frac{1}{2} K_{12} (\chi_1^{(3)} - \chi_2^{(3)}) = \frac{1}{2} K_{12} \left( \bar{G}_3^3 - \bar{G}_2^3 + (E - D\beta)(\bar{G}_1 - \bar{G}_2) \right). \]

We should not allow \( \chi^{(3)} \) to include a graded symmetric part, this means \( 2F = -m_2 \) and \( D = \epsilon + 2\beta \). In this way one deduces that the cubic polynomial appropriate for a constraint has the form
\[ \chi^{(3)} = \bar{G}^3 + (\epsilon + 2\beta)\bar{G}^2 + E\bar{G} - \frac{m_2}{2}, \tag{5.3} \]
with one free parameter \( E \). It means that the graded anti-symmetric part of the third power \( \bar{G}^3 \) is constrained to be proportional to the first power \( \bar{G} \).

Now the comparison with (3.26) shows that the free parameter \( E \) in our polynomial \( \chi \) related to \( W_{12} \) is given by
\[ E = \beta D + 2\beta^2 + \epsilon \beta - \frac{m_2}{2} = \epsilon(2\beta - \frac{m_2}{2}), \]
and
\[ \chi = \bar{G}^3 + (\epsilon + 2\beta)\bar{G}^2 + (2\epsilon\beta - \frac{\epsilon m_2}{2})\bar{G} - \frac{m_2}{2}. \tag{5.4} \]

### 5.2 The \( W_{12} \) condition

**Proposition 6.** The conditions of the Lie algebra resolution of the second order Yangian evaluation, where \( L(u) \) has the form
\[ L(u) = u^2 + u(g + \bar{G}) + h + \frac{1}{2}(\bar{G}^2 + (\beta + g)\bar{G}), \tag{5.5} \]
are fulfilled if the matrix \( \bar{G} \) obeying the Lie algebra relation obeys additionally the condition of vanishing of its graded anti-symmetrized product,
\[ (W_{12})_{a_1 b_1 a_2 b_2} = G_{[a_1 b_1} G_{a_2 b_2]} = 0, \tag{5.6} \]
and the central elements are all expressed in terms of \( m_2 = \frac{1}{8} tr(\bar{G}^2) \) as
\[ g^2 = -\beta^2 - \frac{m_2}{8}, \quad 4h = 2\beta^2 - 1 + 2\beta \epsilon - \frac{m_2}{8}. \tag{5.7} \]

**Proof.**

With the above restrictions (5.1) the first five constraints (4.1-4.4) hold and we have to check the remaining three. After the substitution of (5.1) and (5.2) into (4.5) the latter takes the form
\[ [K_{12}, \frac{1}{2}\{\bar{G}_1^2, \bar{G}_2\} + \frac{1}{2}\{\bar{G}_2^2, \bar{G}_1\} + g(\bar{G}_1^2 + \bar{G}_2^2) + (2a + \beta)\{\bar{G}_1, \bar{G}_2\}] = 0. \tag{5.8} \]
We consider the first term in the commutator.
\[
\frac{1}{2}K_{12}\left(\{\tilde{G}_1^2,\tilde{G}_2\} + \{\tilde{G}_2^2,\tilde{G}_1\}\right) = K_{12}(\tilde{G}_1^2 - \tilde{G}_2^2) + \frac{\beta}{2}K_{12}(\tilde{G}_1 - \tilde{G}_2)^2 = \beta K_{12}(\tilde{G}_1^2 + \tilde{G}_2^2),
\]
\[
\frac{1}{2}\left(\{\tilde{G}_1^2,\tilde{G}_2\} + \{\tilde{G}_2^2,\tilde{G}_1\}\right)K_{12} = (\tilde{G}_1 - \tilde{G}_2)[\tilde{G}_1,\tilde{G}_2]K_{12} = \frac{\beta}{2}(\tilde{G}_1^2 - \tilde{G}_2^2)K_{12} = \beta(\tilde{G}_1^2 + \tilde{G}_2^2)K_{12}.
\]
Thus (5.8) acquires the form
\[
(g - 2a)[\tilde{G}_1^2 + \tilde{G}_2^2] = 0. \tag{5.9}
\]
The relation (4.41) then tells that
\[
2a = g, \quad e^{(2,6)} = 0.
\]
Substituting \(\tilde{H} = \frac{2}{g}\tilde{G}\) into (4.39) results in
\[
\mathcal{C}_a^{(2,7)} = \frac{1}{8}[W_{12},\tilde{G}_1 - \tilde{G}_2] + \frac{1}{8}[P_{12} - \epsilon K_{12},\chi_1 - \chi_2] + \frac{\alpha'}{8}[P_{12} - \epsilon K_{12},\tilde{G}_1 - \tilde{G}_2], \tag{5.10}
\]
where
\[
\alpha' = \alpha + 3g^2 = 4h + \beta^2 + 1 - 2\epsilon\beta + \frac{27}{8}\epsilon + 3g^2.
\]
Because \(\chi(\tilde{G})\) is a contraction of \(W_{12}\) with \(\tilde{G}\) (see proof of proposition 1) this constraint is fulfilled if
\[
W_{12} = 0, \quad \alpha' = 0. \tag{5.11}
\]
Finally, we turn to the eighth constraint. Substituting \(\tilde{H} = \frac{1}{2g}\tilde{G}\) in (4.56) we obtain the fourth order polynomial condition in the algebra valued matrix \(\tilde{G}\).
\[
G^4 + 4G^3\beta + (5\beta^2 + 4h - g^2)\tilde{G}^2 + 2\beta(\beta^2 + 4h - g^2)\tilde{G} + 4c^{(2,8)} = 0. \tag{5.12}
\]
This is compatible with the sufficient condition for solving the seventh reduced constraint (5.11) only if
\[
4h - g^2 = 3\beta^2 + 2\beta\epsilon - 1, \quad 4c^{(2,8)} = \frac{m_2}{2}(\epsilon - 2\beta).
\]
Indeed, with these relations between central elements the 4th order polynomial on l.h.s of (5.12) is expressed in terms of \(\chi(\tilde{G})\) as
\[
(\tilde{G} + 2\beta - \epsilon)\chi(\tilde{G}).
\]
This means that the 8th constraint is fulfilled if (5.6) holds. ■

The condition of the vanishing \(W_{12}\) appeared in subsect. 3.3 and has been shown to be equivalent to the one found in [17] for the linear spinorial Yang-Baxter operator \(\mathcal{L}\) (3.19) to obey the spinorial RLL relation (3.20).

We remind the Jordan-Schwinger example of second order evaluation which first appeared in [14] and has been discussed in [17, 18].

Consider the algebra \(\mathcal{H}\) generated by \(n\) Heisenberg canonical pairs \(x_a, \partial_a, a = 1, \ldots, n,\)
\[
x_a\partial_b - \varepsilon\partial_x x_a = [x_a, \partial_b] - \varepsilon = \varepsilon_{ab}, [x_a, x_b] - \varepsilon = [\partial_a, \partial_b] - \varepsilon = 0,
\]
and the map \(\rho: \mathcal{Y}^{(2)}(\tilde{G}) \to \mathcal{H}\)
\[
\rho(G_{ab}) = \tilde{G}_{ab} = x_a\partial_b - \varepsilon x_b\partial_a. \tag{5.13}
\]
Then we find that $\tilde{G}$ is graded antisymmetric and that the Lie algebra relations are fulfilled. Note that the Heisenberg pairs are bosonic in the orthogonal and fermionic in the symplectic case.

The condition $W_{12} = 0$ holds and by Proposition 6 all second order evaluation conditions are fulfilled. The easy way to check the vanishing of $W_{12}$ is to recall the fact that it is the graded-antisymmetrization of the product $G_{a_1 b_1} G_{a_2 b_2}$ (see Proposition 1) and use the Heisenberg algebra relations. It is not difficult to check the RLL relations (1.3) directly with $L(u)$ being of the form (5.5) and with the substitution (5.13).

6 Discussion

Solutions of the Yang-Baxter relations with orthogonal or symplectic symmetry and with a simple linear or quadratic dependence on the spectral parameter exist under restrictions going beyond the Lie algebra relations. In this paper we have investigated the constraints arising from the truncation of the expansion in the spectral parameter in general form.

In the linear case, the Yangian algebra $\mathcal{Y}^{(1)}(G)$ is generated by $G_{a b}$, obeying the underlying Lie algebra as well as the condition of vanishing of the graded-symmetric traceless part of the square, $G^2$.

In the quadratic case, the Yangian algebra $\mathcal{Y}^{(2)}(G)$ is generated by the matrix elements of $\tilde{G}$ obeying the Lie algebra relations and the matrix elements of $\tilde{H}$. The latter transform as the adjoint representation of the Lie algebra. $\tilde{G}$ and $\tilde{H}$ are related by further conditions. The commutation relations of the generators contained in the matrix $\tilde{H}$ are given by an expression in terms of the graded anti-symmetrized product of the generators $\tilde{G}$, $(W_{12})_{abcd} = G_{[ab}\tilde{G}_{cd]}$. The anti-commutator of $\tilde{G}$ and $\tilde{H}$ is expressed by the graded symmetric part of $G^2$. The square $\tilde{H}^2$ is expressed in terms of a fourth order polynomial in $\tilde{G}$.

The second order evaluation $\mathcal{Y}^{(2)}(G)$ can be further restricted to the Lie algebra resolution, where $\tilde{H}$ is proportional to $\tilde{G}$. Then the constraints are fulfilled by imposing, besides of relations on the central elements, the single condition of vanishing of the graded anti-symmetrized product of the generators $\tilde{G}$, $(W_{12})_{abcd} = G_{[ab}\tilde{G}_{cd]}$.

The known example of a linear $L$ operator, representing $\mathcal{Y}^{(1)}(G)$, is based on an underlying Clifford (orthogonal case) or oscillator (symplectic case) algebra. The known example of a quadratic $L$ operator, representing the Lie algebra resolution of $\mathcal{Y}^{(2)}(G)$, is based on the underlying Heisenberg algebra of $n$ canonical pairs, bosonic in the orthogonal case and fermionic in the symplectic case.

It is instructive to see how the set of constraints is fulfilled by these constructions in terms of the underlying algebras. This helps to understand better the distinguished role of these examples.

The general form of the algebra conditions given here allows now to investigate the set of all representations of the truncated Yangians $\mathcal{Y}^{(1)}(G)$ and $\mathcal{Y}^{(2)}(G)$. It is of physical relevance to see whether there are simple orthogonal or symplectic $R$ operators essentially different from the ones known so far.

Acknowledgments.

We thank A.P. Isaev for discussions. Our collaboration has been supported by JINR (Dubna) via a Heisenberg-Landau grant (R.K.) and a Smorodinski-Ter-Antonyan grant (D.K.). The work of D.K. was partially supported by the Armenian State Committee of Science grant 18RF-002 and by Regional Training Network on Theoretical Physics.
References.

[1] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, *The Quantum Inverse Problem Method. 1*, Theor. Math. Phys. 40 (1980) 688 [Teor. Mat. Fiz. 40 (1979) 194]

[2] V.O. Tarasov, L.A. Takhtajan and L.D. Faddeev, *Local Hamiltonians for integrable quantum models on a lattice*, Theor. Math. Phys. 57 (1983) 163

[3] P.P. Kulish and E.K. Sklyanin, *Quantum spectral transform method. Recent developments*, Lect. Notes in Physics, **151**, (1982), 61-119.

[4] L.D. Faddeev, *How Algebraic Bethe Ansatz works for integrable model*, In: Quantum Symmetries/Symetries Quantiques, Proc.Les-Houches summer school, LXIV. Eds. A.Connes,K.Kawedzki, J.Zinn-Justin. North-Holland, 1998, 149-211. [hep-th/9605187]

[5] L. N. Lipatov, “High-energy asymptotics of muticolor QCD and exactly solvable lattice models”, JETP Lett. 59 (1994) 596. Padua preprint DFPD/93/TH/70, hep-th/9311037.

[6] L. D. Faddeev and G. P. Korchemsky, “High-energy QCD as a completely integrable model,” Phys. Lett. B **342** (1995) 311 doi:10.1016/0370-2693(94)01363-H [hep-th/9404173].

[7] N. Beisert and M. Staudacher, “The N=4 SYM integrable super spin chain,” Nucl. Phys. B **670** (2003) 439 doi:10.1016/j.nuclphysb.2003.08.015 [hep-th/0307042];

N. Beisert, Phys. Reports **405** (2005) 1, hep-th/0407277.

[8] J. M. Drummond, J. M. Henn and J. Plefka, “Yangian symmetry of scattering amplitudes in N=4 super Yang-Mills theory,” JHEP **0905** (2009) 046 [arXiv:0902.2987 [hep-th]];

[9] V.G. Drinfeld, Hopf algebras and quantum Yang-Baxter equations, Sov. Math. Dokl. 32 (1985) 254-258.

[10] V.G. Drinfeld, Quantum Groups, in Proceedings of the Intern. Congress of Mathematics, Vol. 1 (Berkeley, 1986), p. 798.

[11] A.B. Zamolodchikov and Al.B. Zamolodchikov, “Factorized S Matrices in Two-Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Models,” Annals Phys. 120 (1979) 253.

Al.B. Zamolodchikov, "Factorizable Scattering in Assymptotically Free 2-dimensional Models of Quantum Field Theory", PhD Thesis, Dubna (1979), unpublished

[12] B. Berg, M. Karowski, P. Weisz and V. Kurak, “Factorized U(n) Symmetric s Matrices in Two-Dimensions,” Nucl. Phys. B **134** (1978) 125.

[13] R. Shankar and E. Witten, “The S Matrix of the Kinks of the (ψ−bar ψ)**2 Model,” Nucl. Phys. B **141** (1978) 349 [Erratum-ibid. B **148** (1979) 538].

[14] N.Yu. Reshetikhin, Integrable models of quantum one-dimensional models with O(n) and Sp(2k) symmetry, Theor. Math. Fiz. **63** (1985) 347-366.

[15] A.P. Isaev, Quantum groups and Yang-Baxter equations, preprint MPIM (Bonn), MPI 2004-132, [http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2004/132.pdf](http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2004/132.pdf).

[16] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, Quantization of Lie groups and Lie algebras, (Russian) Algebra i Analiz **1** (1989) no. 1, 178–206. English translation in: Leningrad Math. J. **1** (1990) no. 1, 193–225.

[17] D. Chicherin, S. Derkachov and A. P. Isaev, “Conformal group: R-matrix and star-triangle relation,” JHEP **04** (2013), 020 arXiv:1206.4150 [math-ph].

“The spinorial R-matrix,” J. Phys. A **46** (2013) 485201. arXiv:1303.4929 [math-ph].

[18] A. P. Isaev, D. Karakhanyan, R. Kirschner, “Orthogonal and symplectic Yangians and Yang-Baxter R operators,” Nucl. Phys. **B904** (2016) 124147, arXiv:1511.06152.
[19] J. Fuksa, A. P. Isaev, D. Karakhanyan and R. Kirschner, “Yangians and Yang-Baxter R-operators for ortho-symplectic superalgebras,” Nucl. Phys. B 917 (2017) 44 doi:10.1016/j.nuclphysb.2017.01.029 [arXiv:1612.04713 [math-ph]].

[20] D. Karakhanyan and R. Kirschner, “Yang-Baxter relations with orthogonal or symplectic symmetry,” J. Phys. Conf. Ser. 670 (2016) no 1, 012029 doi:10.1088/1742-6596/670/1/012029

[21] D. Arnaudon, J. Avan, N. Crampe, L. Frappat, E. Ragoucy, ”R-matrix presentation for super-Yangians Y (osp(m—2n))”, J. Math. Phys. 44 (2003), no. 1, 302-308. arXiv:math/0111325.

[22] D. Arnaudon, A. Molev, E. Ragoucy, ”On the R-matrix realization of Yangians and their representations”, Ann. Henri Poincaré 7 (2006), no. 7-8, 1269-1325. arXiv:math/0511481.

[23] N. Jing, M. Liu, A. Molev, “Isomorphism between the R-Matrix and Drinfeld presentations of Yangian in types B,C and D,” arXiv:1705.08166

[24] N. Guay, V. Regelskis, C. Wendlandt, ”Equivalences between three presentations of orthogonal and symplectic Yangians”, arXiv:1706.05176.