Turbo Codes Construction for Robust Hybrid Multitransmission Schemes

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Abstract—In this paper, a robust and efficient hybrid multitransmission scheme, previously introduced by the authors, is further investigated from the viewpoint of the encoder design, on the basis of its average distance spectrum. This scheme uses a channel coding system based on punctured turbo codes. Since the computation of the transfer function and, consequently, the union bound on the Bit or Frame Error Rate (BER or FER) of a punctured turbo code becomes highly intensive as the interleaver size and the puncturing period increase, a rapid method to calculate the most significant terms of the transfer function of a punctured turbo code is proposed and validated. The adopted channel coding system is based on the use of punctured turbo codes. This new scheme performs particularly well when the user has to cope with some random packet erasures due, e.g., to deep fading conditions on wireless links, or to congestions on wired networks. In this sense, it can be used for many different applications, e.g., in Automatic Repeat Request (ARQ) schemes, in Space-Time coding systems, in Multicast transmission of scalable sources using receiver driven hierarchical Forward Error Correction (FEC), and many others.

I. INTRODUCTION

Certain applications require transmission schemes performing well both at the presence of packet erasures and packet fading. A typical example may be a non-stationary time-varying fading channel, where errors tend to be bursty. During a long deep fade, a packet is received severely corrupted (packet fading) or completely lost (packet erasure) due to bursty errors or packet header errors in the network. A kind of error control in these environments can be performed by adopting an ARQ scheme alone. However, although ARQ systems are simple, easy to implement and provide high system reliability, they suffer a rapid decreasing of the throughput with increased channel error rates. On the other hand, forward error correction (FEC) systems maintain constant the throughput, irrespective of the channel error rates. However, FEC systems have two major drawbacks. First, when a received sequence is detected in error, the sequence has to be decoded and the decoder output has to be delivered to the user regardless of whether it is correct or incorrect. Since the probability of a decoding error is usually greater than the probability of an undetected error, FEC systems are not highly reliable. Second, in order to achieve high system reliability, a long powerful code must be used, which can correct a large number of error patterns. This makes the decoder hard to implement and expensive.

The advantage, typical of ARQ systems, of obtaining high reliability can be coupled with the advantage of FEC systems to provide constant throughput even in poor channel conditions. Such a system, which is a combination of the two basic error control schemes FEC and ARQ, is called a hybrid ARQ system. Hybrid ARQ schemes are of great interest for transmission systems that use time-varying channels, such as the mobile channel, and erasure channels, such as the heterogeneous networks, due to their intrinsic capability to adapt to different channel conditions.

Hybrid ARQ schemes can be classified into two great categories, namely into memoryless and memory ARQ (MARQ) [1]. In conventional memoryless hybrid ARQ schemes the received erroneous frames are discarded and their retransmission is requested. Type-I hybrid ARQ schemes fall into this category [2] and are best suited for channels with a fairly stationary level of noise and interference. In memory ARQ schemes the received erroneous frames are retained and then combined with different strategies by the receiver to reconstruct the original error free frame. It is evident that memory ARQ schemes are more suitable over non-stationary time-varying channels since they allow a dynamic degree of FEC encoding. However, the memory is not the only property that guarantees a good adaptation to the channel conditions. Another important property of ARQ schemes is the self-decodability of additional transmissions, which makes the difference in channels where a frame can be severely damaged [1]. To this end, to guarantee a good behaviour on the erasure channel the hybrid ARQ scheme should be designed so that its performance depends simply from the number of received packets, and not from the particular packets that are received; moreover, to guarantee a good performance also on the pure wireless link, the hybrid ARQ scheme should be designed so that it guarantees a rapid convergence of the repetition algorithm.

In this paper we investigate further on the encoder design for a new robust and efficient hybrid multitransmission scheme presented in [3]. This scheme uses a channel coding system based on punctured turbo codes. Since the computation of the transfer function and, consequently, the union bound on the Bit or Frame Error Rate (BER or FER) of a punctured turbo code becomes highly intensive as the interleaver size and the puncturing period increase, a rapid method to calculate the most significant terms of the transfer function of a punctured turbo code is proposed and validated.

The performance of the proposed scheme has been compared in [3] with that obtained applying two classical hybrid ARQ schemes. The novel scheme has been shown to perform...
particularly well on block erasure channels, i.e., when the number of erasures due, e.g., to deep fading conditions on wireless links, or to congestions on wired networks, is hard to predict. However, it has been also shown to behave quite well when the packets may be received seriously corrupted, but they are never erased, as on the pure wireless link.

The paper is organized as follows. In Section II we recall, for self-consistency sake, the novel hybrid multitransmission scheme introduced in [3]. In Section III we recall the classical union bound approach to calculate the transfer function of a punctured turbo code. In Section IV a rapid method to calculate the most significant terms of the transfer function of a punctured turbo code is introduced. This rapid method is shown to give the same results of the classical more complex method using the union bound paradigm we have used in [3]. Finally, in Section V the main conclusions are summarized.

II. THE NOVEL HYBRID MULTITRANSMISSION SCHEME

Hybrid repetition/multitransmission schemes may be classified depending on the content of different transmission attempts and on their features. Define as self-decodable a scheme for which user data may be recovered from every single transmission attempt and balanced an ARQ scheme for which all transmissions include the same amount of bits.

A novel robust and efficient hybrid multitranmission scheme has been presented in [3]. This scheme is called robust, efficient and balanced (REB), since it is designed adopting efficient encoding schemes with carefully chosen puncturing patterns. The design ensures that all the transmission attempts are self-decodable and include the same amount of bits. To validate the REB scheme efficiency, its performance has been compared in [3] with that obtained applying two classical ARQ schemes, namely the incremental type II ARQ algorithms [4], [5], named incremental schemes for short, and the type III ARQ algorithms [6], named complementary schemes. In all schemes, it is assumed that the receiver stores all the received bits in a buffer, so that they can be combined with the subsequently received ones.

Incremental schemes are very efficient, being based on an optimized, rate-compatible, puncturing design. The first transmission includes all the information bits, while the subsequent attempts include parity bits only and are not self-decodable. Thus, the performance depends on the information available at the receiver at the first transmission attempt, since a severe lack of information at this attempt determines a performance degradation which cannot be improved, whatever is going to happen in the successive transmission attempts. Complementary schemes, on the contrary, are self-decodable, but are less efficient, given that every transmission attempt includes the systematic bits. Only the parity bits are complemented at the different transmission attempts. Thus, the overall code rate decreases slowly by increasing the number of received transmissions.

The novel REB scheme is designed to include in itself the advantages of incremental and complementary schemes, without the corresponding drawbacks. These properties of the REB scheme guarantee,

1) that the performance of the repetition algorithm depends simply from the number of received packets, and not from the particular packets that are received;
2) a rapid convergence of the repetition algorithm itself.

The REB scheme is meant to be incremental but symmetrical, so that to guarantee the same usefulness of each transmission attempt, thanks to its symmetry, and also the rapid convergence of the overall repetition process, thanks to its incremental redundancy, constructed so that the overall code rate is lessened as much as possible at each repetition attempt. It is designed so that information can be recovered from every transmission, as in type III schemes, and a very low number of retransmissions may be necessary to receive the packet correctly, as in type II schemes. These features make REB schemes attractive in erasure channels, where entire packets may be lost. In these channels, their performance has been shown in [3] to be better than incremental schemes’ performance, since they are self-decodable, and better than complementary schemes’ performance, since a lower number of retransmissions may be necessary to receive the packet correctly (lower than the number of retransmissions needed by complementary schemes). For the same reasons, on fading channels, the REB schemes’ performance has been shown in [3] to be better than complementary schemes’ performance and comparable to incremental schemes’ performance, since, although REB schemes may need a higher number of retransmissions to receive the packet correctly (higher than the number of retransmissions needed by incremental schemes) they are self-decodable, and thus the information can be recovered even in case of a deep fade condition (this is not possible when using incremental schemes).

The FECs used in this scheme are a set of punctured turbo codes obtained by puncturing a 1/3 eight state turbo mother code, given by the concatenation of an upper and a lower rate 1/2 constituent convolutional codes, with generators $(g_1, g_2) = (13, 17)$ in octal form. REB adopts a non systematic fully complimentary puncturing scheme. The puncturing is designed so that information can be recovered from every retransmission, as in complementary schemes, i.e., the punctured FEC scheme used at each attempt is non catastrophic and decodable. Moreover, it is designed so that at the end of a given number of transmission attempts (in the paper the number is set to three) all the systematic bits are available to the receiver, together with all the parity check bits of the upper constituent code and most of the check bits of the lower constituent code.

III. WEIGHT ENUMERATORS AND THE UNION BOUND

We follow the approach of Benedetto and Montorsi [7] for parallel concatenated convolutional codes (PCCCs). The component codes of the parallel concatenated codes are connected through uniform random interleavers. The important property of the uniform interleaver is that its output depends only on the input word $w$, not on the distribution of the weight within the input word. A uniform interleaver of length $K$ maps an input word of weight $w$ into all of its $\binom{K}{w}$ possible permutations with equal probability. As a consequence, the use of the uniform
interleaver drastically simplifies the performance evaluation of turbo codes.

Denote by \( w_m \) the minimum weight of an input sequence generating an error event of the parallel concatenated code \( C \), and by \( h_m \) and \( h_M \) the minimum and maximum weight, respectively, of the codewords of \( C \). Also, let \( A_{w,h}^{C} \) denote the Input-Output Weight Enumerating Function (IOWEF), i.e., the average number of codewords in code \( C \) with input weight \( w \) and output weight \( h \). Similarly, we define \( A_{w,h}^{C_U} \) and \( A_{w,h}^{C_L} \) for the upper constituent code \( C_U \) and the lower constituent code \( C_L \), respectively. The bit error probability of a PCCC over an additive white Gaussian noise channel can be upper bounded by [7]

\[
P_b(e) \leq \frac{1}{2} \sum_{h=h_{m}}^{h=h_{M}} \sum_{w=w_{m}}^{w=w_{M}} A_{w,h}^{C} \text{erfc}\left( \sqrt{\frac{h R E_b}{N_0}} \right)
\]

where \( N_0/2 \) is the two-sided noise power spectral density and \( E_b \) is the energy per information bit.

Likewise, the frame error probability is upper bounded by

\[
P_f(e) \leq \frac{1}{2} \sum_{h=h_{m}}^{h=h_{M}} A_{w,h}^{C} \text{erfc}\left( \sqrt{\frac{h R E_b}{N_0}} \right)
\]

where \( A_{w,h}^{C} = \sum_{h=w}^{h=0} A_{w,h}^{C} \cdot A_{w,h}^{C_L} \).

\( A_{w,h}^{C} \) can be calculated by replacing the actual interleaver with the uniform interleaver [7] and exploiting its properties. The uniform interleaver of length \( K \) transforms an input sequence of weight \( w \) at the input of the upper constituent encoder into all its distinct \( \binom{K}{w} \) permutations. As a consequence, each input sequence of the upper code of weight \( w \), through the action of the uniform interleaver, enters the lower constituent encoder generating \( \binom{K}{w} \) codewords of the lower code. The IOWEF of the overall PCCC can then be evaluated from the knowledge of the IOWEFs of \( C_U \) and \( C_L \) [7]:

\[
A_{w,h}^{C} = A_{w,h}^{C_U} \cdot A_{w,h}^{C_L} \cdot \binom{K}{w}
\]

where \( h, h_U, \) and \( h_L \) are related by the equation \( h = h_U + h_L \).

IV. PUNCTURING DESIGN THROUGH A SIMPLIFIED DISTANCE SPECTRUM OF THE RESULTING TURBO CODE

The puncturing has been designed in [3] to obtain the minimum residual Bit Error Rate (BER) and Frame Error Rate (FER) in the waterfall region, i.e., beneath the cutoff rate of the code, since a good performance in this region is important for the applications considered in this paper.

The computation of the union bounds (1) and (2) on the BER and FER, respectively, of a punctured turbo code becomes more and more heavy as the interleaver size \( K \) and the puncturing period increase [8]. However, a rapid method to calculate the most significant terms of the distance spectrum of a punctured turbo code, which can be used as a close approximation of the union bound, can be applied.

In particular, the following procedure has been applied. Given the IOWEF \( A_{w,h}^{C} \) of a punctured parallel concatenated code, calculated employing a uniform interleaver approach [7], its residual BER and FER can be calculated by (1) and (2), respectively. However, it was shown in [7] that minimum information weight codewords are the principal contributors to residual BER and FER, as the size \( K \) of the uniform interleaver increases, and that, when Recursive Systematic Convolutional (RSC) constituent encoders are used, this minimum information weight is equal to two.

Thus, for large interleaver sizes, it follows that the input weight \( w = 2 \) is the dominant contributor to the union bounds (1) and (2) on the BER and FER, respectively. Therefore, Eqs. (1) and (2) can be rewritten as:

\[
P_b(e) \leq \frac{1}{2} \sum_{h=h_{m}}^{h=h_{M}} \frac{2}{h} A_{2,h}^{C} \text{erfc}\left( \sqrt{\frac{h R E_b}{N_0}} \right)
\]

\[
P_f(e) \leq \frac{1}{2} \sum_{h=h_{m}}^{h=h_{M}} A_{2,h}^{C} \text{erfc}\left( \sqrt{\frac{h R E_b}{N_0}} \right)
\]

One very widely used approximation of the \( \text{erfc}(x) \) function is the Chernoff bound:

\[
\text{erfc}(x) \leq 2 \exp(-x^2)
\]

This bound can be tightened as:

\[
\text{erfc}(x) \leq \exp(-x^2)
\]

Using the approximation (7) and switching to the natural logarithmic expression, the puncturing pattern can be selected as the one minimizing the following:

\[
\max_h \left[ \ln A_{2,h}^{C} - \frac{h R E_b}{N_0} \right]
\]

with \( \frac{R}{N_0} \) given by \( 2^{R-1} \), which is the cutoff rate of the code, having assumed a BPSK modulation.

In [3] a similar, but computationally heavier, criterion was used to perform the puncturing design. In particular, it was observed that, since the higher distance error events have a nontrivial contribution to error performance, the cumulative function \( \sum_{h=h_{m}}^{h=h_{M}} A_{2,h}^{C} \) of the input-output distance spectrum (IOWEF) of the parallel concatenated code with input weight \( w = 2 \) should be minimized, where \( h_m = d_{\text{eff}} \) is the minimum output weight given by a weight-two input and \( A_{2,h}^{C} \) is the average number of turbo codewords with input weight 2 and output weight \( h \). In fact, the codes for which the \( d_{\text{eff}} \) is maximum perform better in the error floor region. However, also output distances \( h > d_{\text{eff}} \) should be considered and the cumulative function \( \sum_{h=h_{m}}^{h=h_{M}} A_{2,h}^{C} \) minimized to obtain a better performance at lower SNRs.

The two criteria (i.e., the one proposed in this paper and the one proposed in [3]) can be shown to be equivalent. In fact, given two puncturing patterns presenting the same output weight \( h \) distribution with input weight \( w = 2 \), the minimization of (4) and (5) leads to the minimization of \( \sum_{h=h_{m}}^{h=h_{M}} A_{2,h}^{C} \). However, considering the minimization of (8), which is computationally less cumbersome, in place of the minimization of (4) and (5), the same results can be obtained as shown in the following example.
A. Example

To make an example on how the codes should be chosen for the REB scheme, consider Fig. 1(c) in [3], referring to this scheme. The code chosen for the first transmission attempt has the following puncturing matrix:

\[
a(10) = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(9)

\[
a(10) = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(10)

\[
a(10) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(11)

It was observed in [3], and we recall this observation here for self consistency sake, that we shall focus on the maximum number of information bits that can be maintained to preserve the maximum output weight \(d_{\text{eff}}\). It is well known, in fact, that lowering the number of information bits (and incrementing, correspondingly, the number of parity check bits) to obtain a code of a given rate, \(d_{\text{eff}}\) will be maximized, as shown in Fig. 1, thus lowering the error floor. However, we should remind that information bits make iterative decoding easier, yielding better convergence abscissa of the iterative decoding algorithm, which are fundamental in the region of low SNRs. Thus, the best code design should adopt a compromise between the maximization of \(d_{\text{eff}}\), obtained when the number of surviving parity bits (after puncturing) is maximized, and the minimization of the convergence abscissa, obtained when the number of surviving information bits (always after puncturing) is maximized. In this sense, the choice we have done of selecting the number of surviving information bits equal to the number of surviving parity bits (the number of ones in the first row of (9) is equal to the sum of the numbers of ones in the second and third row) represents a good compromise between the two above mentioned goals.

Once we have fixed the number of information bits that should be preserved to obtain a rate 4/5 code, we have many choices to do on the positions that the information and the parity check bits should have inside the puncturing period (i.e., on the positions of the ones inside the puncturing matrix). For instance, other codes of this type (i.e., with 5 ones in the first row) could have a puncturing matrix like this:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

or like this:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

In order to perform a choice among these codes, the cumulative function \(\sum_{h=hM}^{h} A_{2,h}^C\) mentioned before has been used in [3]. In Fig. 2 in [3] we have reported \(\sum_{h=hM}^{h} A_{2,h}^C\) for the code (9) (solid line), for the code (10) (dotted line) and for the catastrophic code (11) (dashed line), which has the best cumulative function since it is minimal. Of course, the code having the minimal \(\sum_{h=hM}^{h} A_{2,h}^C\) must be chosen among the invertible and non-catastrophic codes, and thus, in this sense, it was observed in [3] that the code (9) is the best.

We can come up to the same result by using the design criterion presented in this paper, i.e., through the minimization of (8). In Fig. 2 we report the output spectrum \(A_{2,h}^C\) vs. the output distance \(h\) of three rate 4/5 codes. Solid line: code (9); dotted line: code (10); dashed line: code (11).

\[
\ln A_{2,h}^C - hRE_b/N_0 = \ln A_{2,h}^C - h(2^R - 1)
\]

(12)

being \(E_b/N_0\) given by \(2^{R-1}\), which is the cutoff rate of the code and \(R = 4/5\). These values are shown for the code (9) (solid line), for the code (10) (dotted line) and for the catastrophic code (11) (dashed line).

Of course, following the criterion introduced in this paper, the best puncturing pattern can be selected as the one minimizing the following quantities:

\[
\max_h [A_{2,h}^C - h(2^R - 1)] = \min_h [A_{2,h}^C - h(2^R - 1)]
\]

(13)
and thus, in this sense, as it was observed in [3], the code (9) is the best.

Although not rigorously proved [9], it is observed that for capacity approaching codes such as turbo codes, there is a tradeoff between low pinch-off thresholds, i.e., good waterfall performance, and high effective free distance values, determining good error-floor performance [7]. Actually, in all the cases examined, the distance spectrum of the codes with minimum residual FER in the waterfall region, presented low values of the effective free distance $d_{f,\text{eff}}$, defined in [7] as the minimum weight of code sequences generated by input sequences of weight 2. Thus, a design guideline may be that of restricting the puncturing pattern search to the patterns giving codes with low values of $d_{f,\text{eff}}$, since the obtained codes have a bad error-floor performance, but, as a tradeoff, a very good waterfall performance, as shown in Fig. 4, where the simulated residual FER performance of three rate 8/9 codes with different values of $d_{f,\text{eff}}$ is shown on the AWGN channel. The thick solid curve in the figure shows the sphere-packing bound limiting performance [10], [11].

V. CONCLUSIONS

In this paper we have investigated further on the encoder design for a new robust and efficient hybrid multitransmission scheme presented in [3]. This scheme uses a channel coding system based on punctured turbo codes. Since the computation of the transfer function and, consequently, the union bound on the Bit or Frame Error Rate (BER or FER) of a punctured turbo code becomes highly intensive as the interleaver size and the puncturing period increase, a rapid method to calculate the most significant terms of the transfer function of a punctured turbo code is proposed and validated.

The two criteria, namely the one proposed in this paper and the one proposed in [3], have been shown to be equivalent. In fact, considering the minimization of (8), which is computationally less cumbersome, in place of the minimization of (5), the same results, in terms of puncturing pattern choice, have been obtained with a significant gain in computation complexity.

Comparisons with other design tools, such as the EXIT charts [12] which may accurately model the code behavior in the waterfall region, are left for further investigation.

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