Applications of Grassmannian and graph flows to coagulation systems

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Abstract We demonstrate how many classes of Smoluchowski-type coagulation models can be realised as Grassmannian or nonlinear graph flows and are therefore linearisable, and/or integrable in this sense. We first prove that a general Smoluchowski-type equation with a constant frequency kernel, that encompasses a large class of such models, is realisable as a multiplicative Grassmannian flow, and then go on to establish that several other related constant kernel models can be realised as such. These include: the Gallay–Mielke coarsening model; the Derrida–Retaux depinning transition model and a general multiple merger coagulation model. We then introduce and explore the notion of nonlinear graph flows, which are related to the notion of characteristics for partial differential equations. These generalise flows on a Grassmann manifold from sets of graphs of linear maps to sets of graphs of nonlinear maps. We demonstrate that Smoluchowski’s coagulation equation in the additive and multiplicative frequency kernel cases, are realisable as nonlinear graph flows, and are thus integrable provided we can uniquely retrace the initial data map along characteristics. The additive and multiplicative frequency kernel cases correspond to inviscid Burgers flow. We explore further applications of such nonlinear graph flows, for example, to the stochastic viscous Burgers equation. Lastly we consider an example stochastic partial differential equation with a nonlocal nonlinearity that generalises the convolution form associated with nonlinear coagulation interaction, and demonstrate it can be realised as an infinite dimensional Grassmannian flow. In our companion paper, Doikou et al. [23], we consider the application of such infinite dimensional Grassmannian flows to classical noncommutative integrable systems such as the Korteweg–de Vries and nonlinear Schrödinger equations.

Keywords Grassmannian flows · graph flows · Smoluchowski coagulation · Burgers equation

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1 Introduction

Our goal herein is to demonstrate how many classes of partial differential equations with nonlocal coagulation interaction nonlinearities are integrable, either as Grassmannian flows, or as nonlinear graph flows. We think of the Grassmann manifold as the set of all compatible graphs of linear maps. The underlying structure in the case of nonlinear graph flows, is the set of all compatible graphs of nonlinear maps. First, we establish the integrability of a general Smoluchowski-type equation, by establishing that it can be decomposed into a multiplicative Grassmannian flow. By this we mean that we can generate the solution by solving the corresponding linearised version of the general Smoluchowski-type equation, in fact the Laplace transform form of this linear equation, and then solving a linear algebraic relation for the Laplace transform of the solution. We then show that several related applications, including the Gallay–Mielke coarsening model; the Derrida–Retaux depinning transition model and a general multiple merger coagulation model, which do not quite fit into the general Smoluchowski-type class just mentioned, are also realisable as multiplicative Grassmannian flows. All the models mentioned thusfar correspond to the case of constant coagulation frequency kernel. Second, to establish the (already known) integrability of the Smoluchowski coagulation model for the additive or multiplicative coagulation frequency kernels, we introduce the nonlinear graph flows just mentioned. Smoluchowski’s coagulation equation in the additive and multiplicative frequency kernel cases corresponds to inviscid Burgers flow in Laplace transform space. We prove that it is realisable as a nonlinear graph flow, which is thus naturally related to the notion of characteristics for partial differential equations. We also apply nonlinear graph flows to the viscous Burgers equation as well as the stochastic viscous Burgers equation. Third, we consider a stochastic partial differential equation with a nonlocal nonlinearity that generalises the nonlinear convolution-type coagulation term, and with a specific multiplicative space-time white noise term. This model generalises the class of flows considered in Beck et al. [2,3] to the stochastic nonlocally nonlinear partial differential equation setting. We formally establish, via an explicit solution form, that its flow can be realised as an infinite dimensional Grassmannian flow. In a companion paper Doikou et al. [23], we explore in detail the connection between infinite dimensional Fredholm Grassmannian flows and classical noncommutative integrable systems, including the potential Korteweg–de Vries, nonlinear Schrödinger and modified Korteweg–de Vries equations, as well as nonlocal variants of the latter two. Lastly, in our Appendix we also explore the application of the Grassmannian approach to nonlinear elliptic systems.

Explicitly, though formally for the moment, let us summarise the essential ideas underlying the Grassmannian flow programme we advocate here, as well as in Doikou et al. [23]. Consider the following canonical system of linear operator equations.

**Definition 1 (Canonical linear system)** Suppose the time-dependent maps $Q = Q(t)$, $P = P(t)$ and $G = G(t)$ satisfy the following linear system of operator equations:

\[
\begin{align*}
\partial_t Q &= A(id + Q) + BP, \\
\partial_t P &= C(id + Q) + DP, \\
P &= G(id + Q).
\end{align*}
\]

Here $A$, $B$, $C$ and $D$ are known operators which depend on $t$ only.
Suppose $Q(0) = O$ and $P(0) = G_0$ for some given operator-valued data $G_0$. Assume there exist operator-valued solutions $Q = Q(t)$ and $P = P(t)$ to the coupled pair of linear evolutionary differential equations shown in Definition [1] for $t \in [0, T]$ for some $T > 0$. Assume further, there exists an operator-valued solution $G = G(t)$ to the final linear relation $P = G(id + Q)$ in Definition [1] at least for $t \in [0, T']$ for some $0 < T' \leq T$. Then a straightforward calculation shows that if $Q = Q(t)$ and $P = P(t)$ satisfy the coupled pair of linear equations shown in Definition [1] then then $G = G(t)$ satisfies the following operator-valued evolutionary Riccati differential equation for $t \in [0, T']$,

$$\partial_t G = C + DG - G(A + BG).$$

Let us now formally consider some examples. Typically the sense in which we require the operator valued solutions to the linear canonical system in Definition [1] to exists, is that $Q = Q(t)$, $P = P(t)$ and $G = G(t)$ are Hilbert–Schmidt valued operators for $t \in [0, T']$. Further since $Q(0) = O$, the null operator, the time $T' > 0$ is determined as the length time for which $\det_2(id + Q(t)) \neq 0$, where $\det_2$ is the regularised Fredholm determinant which is well-defined for operators which are Hilbert–Schmidt valued perturbations of the identity. If $Q = Q(t)$, $P = P(t)$ and $G = G(t)$ are Hilbert–Schmidt valued operators, then they have representations in terms of square-integrable kernel functions, say, respectively, $q = q(x, y; t)$, $p = p(x, y; t)$ and $g = g(x, y; t)$. In the context of the integral kernel functions, the linear Fredholm relation $P = G(id + Q)$ has the form,

$$p(x, y; t) = g(x, y; t) + \int g(x, z; t)q(z, y; t)\, dz.$$

The interval of integration depends on the application at hand. For example, we might assume $A = C = O$, $B = -\frac{1}{2}id$ and $D = d(\partial_x) + \hat{D}$, where $\hat{D}$ is a multiplicative operator in the sense that $\hat{D} \colon \phi(x) \mapsto -M(t)\phi(x)$, where $M = M(t)$ is a scalar function of time. The operator $d = d(\partial)$ denotes a constant coefficient polynomial function of $\partial = \partial_x$.

Further, suppose the interval of integration is over $\mathbb{R}$. With these choices, in terms of the integral kernel function $g = g(x, y; t)$, the Riccati evolution equation for $G = G(t)$ above becomes,

$$\partial_t g(x, y; t) = d(\partial_x)g(x, y; t) + \frac{1}{2}\int_{\mathbb{R}} g(x, z; t)g(z, y; t)\, dz - M(t)g(x, y; t).$$

Such equations with the nonlocal nonlinearity shown, are typical of the nonlinear equations considered in Beck et al. [2][3], whose solutions can be generated from the corresponding canonical linear system in Definition [1] which here has the form

$$\partial_t p = d(\partial_x)p - Mp,$$

$$\partial_t q = \frac{1}{2}p,$$

$$p = g + g \ast q,$$

where we define the ‘big matrix’ product ‘$\ast$’ for any two functions $f$ and $g$, for which it makes sense, by,

$$(f \ast g)(x, y) := \int_{\mathbb{R}} f(x, z)g(z, y)\, dz.$$

This simply represents the kernel associated with the composition of the corresponding operators $F$ and $G$. Summarising, to generate the solution $g = g(x, y; t)$ to the nonlocal nonlinear evolution equation above, we solve the linear equation governing the evolution
of \( p = p(x, y; t) \) shown above, with \( p(x, y; 0) = g_0(x, y) \), integrate the linear equation for the evolution of \( q = q(x, y; t) \) shown, with \( q(x, y; 0) = 0 \), and then solve the linear integral equation \( p = g + g * q \) for \( g = g(x, y; t) \). It is in this sense that we say the nonlocal nonlinear evolution equation above for \( g = g(x, y; t) \) is integrable. Note, in Section 2[23], we formally extend the class of nonlocal nonlinear systems considered in Beck et al.

Then the general Smoluchowksi-type equation we consider herein has the following form for \( g = g(x, y; t) \) shown above, driven by a Brownian sheet.

Let us relate our general discussion thusfar to coagulation phenomena. Consider a subclass of flows to those just considered, in which the linear relation \( P = G(id + Q) \) has a convolution form, and for which we also restrict the domain to \([0, \infty)\), so that,

\[
p(x, y; t) = g(x - y; t) + \int_0^x g(x - z; t)q(z, y; t)\, dz.
\]

If \( q = q(x, y; t) \) and \( p = p(x, y; t) \) satisfy the linear system just above, then \( g = g(x - y; t) \) satisfies the nonlocal nonlinear evolution equation,

\[
\partial_t g(x - y; t) = d(\partial_x)g(x - y; t) + \frac{1}{2} \int_0^x g(x - z; t)g(z - y; t)\, dz - M(t)g(x - y; t).
\]

If we simply set \( y = 0 \), we find that \( g = g(x; t) \) satisfies,

\[
\partial_t g(x; t) = d(\partial_x)g(x; t) + \frac{1}{2} \int_0^x g(x - z; t)g(z; t)\, dz - M(t)g(x; t).
\]

Indeed, in the Smoluchowski/convolution setting, as we demonstrate in Remark 8, the \( y \)-dependence in the evolution equations above for \( p \) and \( q \) is redundant. If \( d = 0 \), then we observe that this nonlocal nonlinear evolution equation corresponds to the constant frequency kernel Smoluchowski equation—assuming we have derived the evolution equation for the total number of clusters \( M(t) := \int_0^\infty g(x; t)\, dx \). The case when \( d \) is non-trivial corresponds to one of the generalisations we include in the general Smoluchowski-type equation we consider herein. Define the spatial convolution product ‘\(*\)’ of two functions \( f = f(x; t) \) and \( g = g(x; t) \) on \([0, \infty)\) by,

\[
(f * g)(x; t) := \int_0^x f(x - y; t)g(y; t)\, dy.
\]

Then the general Smoluchowski-type equation we consider herein has the following form for \( g = g(x; t) \), namely,

\[
\partial_t g = d(\partial)g + B_0g * g - M g + g * a - g * b_0 * \beta \, g * (\partial^m g),
\]

where \( B_0 \in (0, 1) \) and \( \beta \in \mathbb{R} \) are constants, \( m \leq \deg(d) \), and \( a \) and \( b_0 \) are given non-negative smooth functions. Actually, we set \( d(\partial_x) = -D_0 - d_0(\partial_x^m) \) for some constant \( D_0 > 0 \), and fix the sign of the constant \( d_0 \) to ensure the corresponding differential operators generates either a dispersive or diffusion effect. This is the general Smoluchowski-type equation we show can be linearised. To achieve this, we transport the equations to their corresponding Laplace transform setting. Indeed if \( q = q(s; t) \) and \( p = p(s; t) \) are the respective Laplace transforms of \( q = q(x; t) \) and \( p = p(x; t) \), then we generate the corresponding Laplace transform versions of the linear evolutionary equations for \( q = q(x; t) \) and \( p = p(x; t) \) that we require for this more general form,
see Definition 5 for their explicit form. And if \( g = g(s; t) \) is the Laplace transform of \( g = g(x; t) \), then at the kernel level in Laplace transform space, the convolution linear relation between \( p, q \) and \( g \) corresponding to \( P = G(\text{id} + Q) \) shown above, becomes,

\[
p = g(1 + q).
\]

In other words, we have a linear multiplicative functional relation for \( g = g(s; t) \), given \( q = q(s; t) \) and \( p = p(s; t) \). This linear multiplicative relation underlies the general Smoluchowski-type equation and the other examples we consider in Section 2. Indeed in Section 2 we prove that, if \( q = q(s; t) \) and \( p = p(s; t) \) satisfy the aforementioned linear evolutionary equations, then \( g = g(s; t) \) satisfies the Laplace transform version of the general Smoluchowski-type evolution equation for \( g = g(x; t) \) shown just above. Note we set \( q(s; 0) = 0 \) and \( p(s; 0) = g_0(s) \), the Laplace transform of the initial data \( g_0 = g_0(x) \) for \( g \). We observe that we can determine \( g = g(s; t) \) provided \( 1 + q(s; t) \neq 0 \), and thus for a short time at least, we expect to be able to uniquely determine \( g = g(s; t) \). Indeed we can determine an explicit solution form.

The linear flow equations for \( q = q(s; t) \) and \( p = p(s; t) \), and the linear multiplicative relation \( p = g(1 + q) \), constitute a special Grassmannian flow, we call a multiplicative Grassmannian flow. Indeed, in Laplace transform space, suppose \( Q = Q(t) \) and \( P = P(t) \) are the time-dependent multiplicative linear operators given by,

\[
Q(t) : \varphi(s) \mapsto q(s; t)\varphi(s) \quad \text{and} \quad P(t) : \varphi(s) \mapsto p(s; t)\varphi(s).
\]

Such multiplicative operators are bounded linear operators on a vector space \( \mathbb{V} \) provided \( q = q(\cdot; t) \) and \( p = p(\cdot; t) \) are bounded functions since, for example, \( \|q(t)\varphi\|_y \leq \|q(t)\|_\mathcal{X}\|\varphi\|_y \). See for example, Richard [23] or Potts [30]. In particular, provided the essential infimum of \( 1 + q(\cdot; t) \) is strictly positive, then we know \( (1 + q(\cdot; t))^{-1} \) exists and \( (\text{id} + Q(t))^{-1} \) exist. In our companion paper Doikou et al. [23] we outline the structure of Grassmannian flows and, in particular, Fredholm Grassmannian flows. We refer the reader there for more details.

The multiplicative operator setting we have here, simplifies the flow structure considerably. Such multiplicative operators correspond to general diagonal operators. Thus, for example, if we were to represent \( Q = Q(t) \) by an integral kernel—suppose that \( \mathbb{V} = L^2([0, \infty); \mathbb{R}) \) and \( Q \) is a Hilbert–Schmidt operator on \( \mathbb{V} \)—then the integral kernel would be \( q(s; t)\delta(s - \sigma) \). Herein, we can, and do, proceed at the multiplicative function level—we find explicit solution forms for \( q = q(s; t) \) and \( p = p(s; t) \) in all our main examples. Thus, here, we have a linear (Stiefel) flow parameterised by \( (1 + q(s; t), p(s; t)) \). Assuming \( (1 + q(s; t))^{-1} \) exists, then multiplicity by \( (1 + q(s; t))^{-1} \) projects the flow \( (1 + q(s; t), p(s; t)) \) onto \( (1, g(s; t)) \) where \( g(s; t) = p(s; t)(1 + q(s; t))^{-1} \). The flow \( (1, g(s; t)) \) represents a flow in the coordinate patch corresponding to the top cell of the base space Grassmannian. The form \( (1, g) \) represents a graph of a given linear map \( g \). If \( 1 + q(s; t) \) were zero, so we could not multiply by \( (1 + q(s; t))^{-1} \), we would choose a different coordinate chart. We think of the Grassmannian as the set of all such compatible graphs of linear maps. See Doikou et al. [23] for more details. In our applications in Section 2, we restrict ourselves to the top cell graph stated. However, when blow-up in \( g = g(s; t) \) does occur, investigating the continuation of solutions via a different coordinate patch is very much of interest.

Thusfar the Smoluchowski-type models we have mentioned have corresponded to the case when the frequency kernel \( K = K(\cdot, \cdot) \) is constant and equal to one. In general, the Smoluchowski coagulation equation has the form,

\[
\partial_t g(x; t) = \frac{1}{2} \int_0^x K(x, z)g(x - z; t)g(z; t)dz - g(x; t)\int_0^\infty K(x, z)g(z; t)dz,
\]
where the form of the frequency kernel $K = K(x, z)$ depends on the application at hand. Two specific forms that are of particular interest are the additive, $K = x + z$, and multiplicative, $K = xz$, kernel cases. Smoluchowski’s equation can be solved explicitly for these two cases, though in the multiplicative kernel case only locally in time, up until the time of gelation. See Menon and Pego [56] for more details. Thusfar, we have not managed to realise the flow above in the additive and multiplicative frequency kernel cases, as a Fredholm Grassmannian flow. However, the Laplace transform $g = g(s; t)$ of the solution $g = g(x(t); t)$ satisfies an inviscid Burgers flow; see Menon and Pego [56, Sec. 2]. With this in mind, let us introduce the notion of nonlinear graph flows. Recall the canonical linear system for the operators $\pi$ corresponding to a linear graph (as we outlined above), by the nonlinear relation, $P = \pi(Q, t)$,

where $\pi$ is a map which, in general, is nonlinear. A straightforward calculation reveals if $Q$ and $P$ satisfy the pair of linear differential equations in Definition 1 (incorporating the replacement of $id + Q$ by $Q$), then $\pi = \pi(Q, t)$ satisfies the nonlinear advective partial differential equation,

$$\partial_t \pi = CQ + D\pi - (\nabla \pi)(AQ + B\pi).$$

We call such flows nonlinear graph flows, as we have replaced the linear relation $P = GQ$, corresponding to a linear graph (as we outlined above), by the nonlinear relation $P = \pi(Q, t)$, corresponding to a nonlinear graph. Naturally the equation for $\pi = \pi(Q, t)$ just above, collapses to the Riccati equation for $G = G(t)$, when we assume $\pi$ is the linear time-dependent map $\pi(Q, t) = G(t)Q(t)$.

We now take inspiration from Byrnes [9] and Byrnes and Jhemi [10]. Suppose we are given an nonlinear advective evolutionary partial differential equation for $\pi = \pi(x, t)$ of the form,

$$\partial_t \pi = Cx + D\pi - (\nabla \pi)(Ax + B\pi),$$

where for simplicity we assume for the moment $A, B, C$ and $D$ are suitably commensurate matrices, all of which we have identified explicitly. In this case by characteristics, the problem is, given initial data $\pi_0 = \pi_0(\cdot)$ and a position $x$ and time $t \in [0, T]$, for some $T > 0$, at which we wish to evaluate the solution $\pi = \pi(x, t)$, to find where the characteristic through $(x, t)$ emanated from at time $t = 0$. Assume that $Q$ and $P$ are time-dependent, finite-dimensional vectors, satisfying the the pair of linear ordinary differential equations indicated in Definition 1. Assume we have solved for $Q$ and $P$ so that we know the functions $Q = Q(t, \pi_0, P_0)$ and $P = P(t, \pi_0, P_0)$ explicitly. To find where the characteristic through $(x, t)$ emanated from we must solve the nonlinear algebraic equation,

$$x = Q(t, \pi_0, \pi_0(\pi_0)),$$

for $Q_0$. Then upon substituting this value for $Q_0$, the solution to the advective partial differential equation above is given by $\pi(x, t) = P(t, \pi_0, \pi_0(\pi_0))$. In particular, the inviscid Burgers equation, which as mentioned above represents the desingularised Laplace transform of the Smoluchowski coagulation equation in the additive and multiplicative frequency cases, can be solved in this manner. In Section 3 we explore this case in detail and discuss immediate generalisations. We also formally propose the notion of
a nonlinear graph manifold as the generalisation of a Grassmann manifold. As already mentioned, the Grassmann manifold can be thought of as consisting of all collections of graphs of compatible linear maps. The nonlinear graph manifold we propose consists of all collections of graphs of all compatible maps, whether linear or nonlinear. The category of nonlinear systems for which a nonlinear functional relation is utilised, are unified as nonlinear graph flows.

Classically, Smoluchowski’s coagulation equation has been employed in many applications including the modelling of polymerisation, aerosols, clouds/smog, clustering of stars and galaxies as well as schooling and flocking; see for example Aldous [1] for more details. However more recently Smoluchowski’s equation has been used to model an extremely wide variety of coagulation/aggregation phenomena, each characterised by given kernels $K$. For example these include: genealogy, see Lambert and Scherzer [17]; coarsening, see Gallay and Mielke [32]; nanostructures on substrates including ripening or ‘island coarsening’, see Stoldt et al. [69]; growth and morphology of nanocrystals at atomic and nanoparticle scales, see Woehl et al. [74] and Kaganer et al. [10], epitaxial Si$_{1-x}$Ge$_x$/Si(001) islands, see Budiman and Ruda [7], growth of graphene on Ir(111), see Coraux et al. [18]; and gold nanoparticles on a silicon substrate, see Winkler et al. [73]; blood clotting including coagulation and formation of fibrin gel, see Guy et al. [81] and Rouleau formation where blood cells aggregate to form cylindrical clusters or ‘Rouleaux’, see Samsel and Perelson [64,65] and polymer growth of proteins in biopharmaceuticals, see Galina et al. [31] or Zidar et al. [76].

It is important to mention the central role Riccati equations and Grassmann manifolds play in optimal control theory. In optimal linear-quadratic control the optimal continuous feedback operator satisfies an analogous Riccati equation to that above. See for example Bittanti, Laub and Willems [5], Brockett and Byrnes [6], Hermann [35,36], Hermann and Martin [37], and Zelikin [75] for more details. A comprehensive list of these and other applications of Grassmannian flows can be found in Ledoux, Malham and Thümmler [48]. Byrnes [9] and Byrnes and Jhemi [10] extended the linear-quadratic optimal control theory to nonlinear systems and cost functions and derived a Riccati partial differential equation for the optimal continuous feedback map. Our advective evolutionary partial differential equation for $\pi$ above is just a special case of a Riccati partial differential equation. As for the linear-quadratic case the idea is to find the solution to the Riccati partial differential equation offline and then use it to mediate the optimal feedback in situ. See Section 5 for more details.

To summarise, what is new in this paper is that we:

1. Prove the solution flow of a general Smoluchowski-type equation with a constant frequency kernel, that encompasses a large class of models with nonlocal convolution nonlinearity, is realisable as a multiplicative Grassmannian flow. The flow is thus linearisable and integrable in this sense. We thus extend the class of flows with nonlocal nonlinearities considered in Beck et al. [24] that are linearisable;

2. Use the Grassmannian flow techniques from (1) to prove that three related constant kernel models including, the Gallay–Mielke coarsening model, the Derrida–Retaux depinning transition model and a general multiple merger coagulation model are also integrable in an analogous sense;

3. Introduce the notion of nonlinear graph flows. In such flows, the graphs of linear maps that underlie Grassmannian flows are generalised to graphs of nonlinear maps. We prove Smoluchowski’s coagulation equation in the additive and multiplicative frequency kernel cases, which correspond to inviscid Burgers flow in Laplace trans-
form space, are realisable as a nonlinear graph flows, and thus integrable in that sense (corresponding to the notion of characteristics). We also apply the nonlinear graph flow method to the stochastic viscous Burgers equation;

(4) Show, formally, that an example stochastic partial differential equation with a nonlocal nonlinearity that generalises the convolution form associated with nonlinear coagulation interaction, can be realised as an infinite dimensional Grassmannian flow and use this representation to construct a numerical simulation method.

Our paper is structured as follows. We show that the general Smoluchowski-type equation mentioned above can be realised as a Grassmannian flow, in Section 2. We then also consider applications of Grassmannian flows to related application models including the Gallay–Mielke coarsening model and the Derrida–Retaux coarsening model. In Section 3 we consider the inviscid Burgers equation, which is related to the additive and multiplicative frequency kernel cases in Smoluchowski coagulation. We introduce the notion of nonlinear graph flows to solve such cases therein and also consider further applications of such nonlinear graph flows. We consider the stochastic partial differential equation with a more general nonlocal nonlinearity in Section 4, formally demonstrating that it is realisable as an infinite dimensional Grassmannian flow, and indeed provide some numerical simulations based on its Grassmannian flow characterisation. Finally in Section 5 we formally consider further possible extensions and applications of the Grassmannian and nonlinear graph flows. Some further extensions can also be found in Appendices B and C.

2 Smoluchowski’s coagulation and related equations

We extend the programme for Grassmannian flows we developed in Beck et al. [2, 3], Doikou et al. [22] and our companion paper Doikou et al. [23], to Smoluchowski’s coagulation equations and some related models. Smoluchowski’s coagulation equation has the form

\[
\frac{\partial}{\partial t} g(x; t) = \frac{1}{2} \int_0^x K(y, x - y) g(y; t) g(x - y; t) \, dy - \int_0^x K(x, y) g(y; t) \, dy,
\]

where \( g(x; t) \) denotes the density of molecular clusters of mass \( x \) and \( K = K(x, y) \) is a given frequency kernel. Briefly, the rationale for this model is as follows. Let us focus on a cluster of mass \( x \). The underlying assumption is that the rate at which a cluster of mass \( x \) merges with a cluster of mass \( y \) is proportional to the product of the density of the clusters, i.e. proportional to \( g(x; t) g(y; t) \). The constant of proportionality is the frequency kernel \( K = K(x, y) \) which takes different forms depending on the application; see Aldous [1] for more details. Only clusters of mass less than \( x \) can coalesce to produce clusters of mass \( x \) and thus the rate at which clusters of mass \( y \leq x \) and \( x - y \) coalesce is \( \frac{1}{2} K(y, x - y) g(y; t) g(x - y; t) \), where the half factor accounts for the labelling symmetry \( x - y \leftrightarrow y \). The total contribution to the rate of increase of clusters of density \( g(x; t) \) is thus the total coalescence gain term shown above. In the coalescence process clusters of mass \( x \) are being lost at all levels at the rate \( K(x, y) g(x; t) g(y; t) \) and this results in the total loss term (second term on the right) shown above. Smoluchowski’s coagulation equation has many applications, as we outlined in the introduction.

For any frequency kernel \( K \) the total mass \( \int_{[0, \infty)} x g(x; t) \, dx \) of clusters is constant. This can be established by multiplying the equation above by \( x \), integrating over \( x \in \]
[0, \infty), swapping the order of integration in the first term and making a simple change of variables. There are three special kernels $K(x, y) = 1$, $K(x, y) = x + y$ and $K(x, y) = xy$ which are notable because in each case Smoluchowski’s coagulation equation is explicitly solvable; see for example Menon and Pego \[56\]. Our goals herein are as follows. We explore the constant kernel case in detail and show how it fits naturally into our context of Grassmannian flows. We show how more general Smoluchowski-type equations also naturally fit into this context as well. At the end of this section, we briefly discuss the additive and multiplicative kernel cases, before going on to explore them in detail in Section 3. Thus assume ‘$K(x, y) = 1$’ throughout this section apart from where indicated otherwise.

Before we begin, we need to define the Laplace transform and its inverse. For any function $g = g(x)$ such that $g \in L^1_{\text{loc}}([0, \infty); \mathbb{R})$, i.e. it is locally integrable on $[0, \infty)$, and which is of exponential order for large $x$, i.e. $g(x) = O(e^{cx})$ as $x \to +\infty$ for some constant $c \geq 0$, its Laplace transform $\hat{g} = \hat{g}(s)$ is defined for $\text{Re}(s) > c$ and given by

$$\hat{g}(s) := \int_0^\infty e^{-sx} g(x) \, dx.$$ 

Further, $g$ is analytic in $s \in \mathbb{C}$ in the region where $\text{Re}(s) > c$. The assumptions on $g$ above can be replaced by the stronger assumption $g \in L^1([0, \infty); \mathbb{R})$. The inverse Laplace transform is given by the Bromwich contour integral,

$$g(x) := \int_{\gamma-i\infty}^{\gamma+i\infty} \hat{g}(s) e^{sx} \, ds,$$

where the constant $\gamma \geq 0$ is chosen such that $\gamma > c$. For both integrals, the limits shown are understood in the obvious sense.

Formally, the evolution of the total number of clusters in Smoluchowski’s coagulation equation,

$$M(t) := \int_0^\infty g(x; t) \, dx,$$

is given as follows. Integrating the equation above over $x \in [0, \infty)$, swapping the order of integration in the first term reveals $dM/dt = -\frac{1}{2}M^2$ and so

$$M(t) = \frac{2M_0}{2 + tM_0},$$

where $M(0) = M_0$, with $M_0$ representing the initial total number of clusters. With $M = M(t)$ determined, we observe $g = g(x; t)$ satisfies

$$\partial_t g(x; t) = \frac{1}{2} \int_0^x g(y; t)g(x - y; t) \, dy - g(x; t)M(t).$$

As is well-known, the natural context for the Smoluchowski equation is Laplace transform space, see Menon and Pego \[56\]. By direct computation, if $g = g(x; t)$ satisfies Smoluchowski’s equation with $K = 1$, then its Laplace transform $\hat{g} = \hat{g}(s; t)$ satisfies the Riccati equation,

$$\partial_t \hat{g} = \frac{1}{2} \hat{g}^2 - M\hat{g}.$$ 

The natural prescription for this Riccati equation is given by the following linear system. Assume we are given arbitrary data $\hat{g}_0 = \hat{g}_0(s)$ to the Riccati equation above so $\hat{g}(s; 0) = \hat{g}_0(s)$. 
Definition 2 (Prescription: Smoluchowski constant kernel equation) Assume the real-valued functions $q = q(s; t)$, $p = p(s; t)$ and $g = g(s; t)$ satisfy the following linear system of equations:

\[
\begin{align*}
\partial_t p &= -Mp, \\
\partial_t q &= -\frac{1}{2}p, \\
p &= g(1 + q),
\end{align*}
\]

where $q(s; 0) = 0$ and $p(s; 0) = g_0(s)$.

A straightforward calculation reveals $g$ satisfies the Riccati equation above provided that $1 + q \neq 0$. Since $q(s; 0) = 0$, this is guaranteed for $t \in [0, T]$ for some $T > 0$. We show that we can take $T = \infty$, presently, in Example 1.

Remark 1 The corresponding prescription in mass cluster space in this case is as follows. Assume the scalar functions $q = q(x; t)$, $p = p(x; t)$ and $g = g(x; t)$ satisfy:

\[
\begin{align*}
\partial_t p &= -Mp, \\
\partial_t q &= -\frac{1}{2}p, \\
p(x; t) &= g(x; t) + \int_0^x g(y; t) q(x - y; t) dy,
\end{align*}
\]

where $q(x; 0) = 0$ and $p(x; 0) = g_0(x)$.

Example 1 (Explicit solution: constant kernel case) We use the linear system in Definition 2 to derive the explicit solution to the Smoluchowski equation in the constant kernel case which, for example, can be found in Scott [66]. We observe that, by direct integration of the formula for the total number of clusters $M = M(t)$ above, we have,

\[
\int_0^t M(\tau) d\tau = \log\left(\frac{2 + tM_0}{2}\right)^2.
\]

Solving the differential equation for $p = p(s; t)$ given in Definition 2 we observe that

\[
p(s, t) = \exp\left(\int_0^t M(\tau) d\tau\right) g_0(s) = \frac{4g_0(s)}{(2 + tM_0)^2},
\]

where we have used that $p(s, 0) = g_0(s)$. Next, integrating the equation for $q = q(s; t)$ given in Definition 2 and using that $q(s; 0) = 0$, we observe

\[
1 + q(s; t) = 1 - \frac{1}{2} \int_0^t \frac{4g_0(s)}{(2 + \tau M_0)^2} d\tau = 1 - \frac{g_0(s)}{M_0} + \frac{2g_0(s)}{M_0(2 + tM_0)}.
\]

Hence from Definition 2 since $g(s; t) = p(s; t)/(1 + q(s; t))$, we observe that,

\[
g(s; t) = \frac{4g_0(s)}{(2 + tM_0)(2 + tM_0 - t g_0(s))}.
\]

The solution $g = g(x; t)$ to the Smoluchowski equation in the constant kernel $K = 1$ case, is thus given by inverse Laplace transform integral of this expression for $g = g(s; t)$. Once we have accounted for the normalization of $g = g(x; t)$ by $M_0$, this exactly matches the solution to the Smoluchowski equation in the constant kernel $K = 1$ case.
in Scott [66] eq. (4.3)]. Further note that $1 + q(s; t) = 0$ if and only if $t = 2/(g_0(s) - M_0)$. From the definition of $g_0 = g_0(s)$ as the Laplace transform of the initial data $g_0 = g_0(x)$, we observe that for all $s$ with $\text{Re}(s) > 0$, $g_0(s) \leq g_0(0) \equiv M_0$. In particular, as expected, by taking the large $s$ limit we observe that $1 + q(s; t)$ is non-zero and there is no blow-up in the solution $g = g(x; t)$ for $t \in (-2/M_0, \infty)$.

**Remark 2 (Measure-valued solutions)** Menon and Pego [56] establish the existence and uniqueness of weak solutions to Smoluchowski’s equation in the constant kernel $K = 1$ case, in the sense of positive Radon measures on $(0, \infty)$. We can invoke the theory and results they establish here to affirm that, as a simple addendum, the prescription of linear equations given in Definition 2 determines such solutions. Briefly, the desingularised Laplace transform,

$$H(s) := \int_0^\infty (e^{-sx} - 1) \nu_t(dx),$$

of a weak, positive, time-dependent, measure-valued solution $\nu_t$ to the constant kernel Smoluchowski equation, satisfies $\partial_t H = \frac{1}{2} \pi^2$. This is a renormalised version of the Riccati equation for $g = g(s; t)$ above, with the total mass term involving $M = M(t)$ ‘knocked out’. The solution to this Riccati equation for $\pi = \pi(s; t)$ is naturally the solution for $g = g(s; t)$ shown in Example 1, but with $M_0$ set equal to zero. In other words, $\pi = 2\nu_0(s)/(2 - t\nu_0(s))$. Menon and Pego [56] show that if $\nu_0$ is an initial positive radon measure, then this solution for $\pi = \pi(s; t)$ determines a weakly continuous map $t \mapsto \nu_t$ into the set of positive Radon measures, for all times $t \in [0, \infty)$, with $\nu_t$ the aforementioned weak solution of constant kernel Smoluchowski equation. Since our prescription in Definition 2 with $M = M(t)$ set to zero, determines $\pi(s; t)$, which in this context replaces $g(s; t)$, and $\pi = \pi(s; t)$ determines the measure valued solutions just mentioned, our statement ‘†’ just above follows.

**Remark 3 (Rescaling)** We can always rescale the solution $g$ so that $g = H\tilde{g}$ for some function $H = H(x)$. For example in the case of Smoluchowski’s coagulation equation with $K = 1$ this would result in the equation

$$\partial_t \tilde{g}(x; t) = \frac{1}{2} \int_0^x \frac{H(y)H(x - y)}{H(x)} \tilde{g}(y; t)\tilde{g}(x - y; t) dy - \tilde{g}(x; t) \int_0^x H(y)\tilde{g}(y; t) dy.$$ 

Any coagulation-type equations of this form can be solved as outlined above as they can be converted to the $K = 1$ case by rescaling.

**Remark 4 (Exponential kernels: gain only equations)** Some not-so-obvious example frequency kernel cases can be solved in the manner outlined in Remark 3. Consider the equation shown therein, but only involving the gain term on the right-hand side. Suppose $H(x) = \exp(\alpha x^2)$ for some constant $\alpha \in \mathbb{C}$. Then the ratio $H(y)H(x-y)/H(x) = \exp(-2\alpha y(x-y))$. Thus we could solve a gain-only Smoluchowski coagulation equation with frequency $K = K(x-y) = \exp(-2\alpha y(x-y))$ by reversing the rescaling process outlined above. Indeed there is a whole family of exponential coagulation frequency kernels generated by exponential functions of monomials of $x$ of any degree, for which the gain-only Smoluchowski coagulation equation can be solved in this manner. See Stylianidis [70] for more details.
Our main goal in this section is to derive the solution to the following general Smoluchowski-type equation with constant coagulation frequency kernel $K = 1$. We achieve this via a Grassmann flow using a prescription for a linear system of equations, in a similar manner to the case we just considered. For some $n \in \mathbb{N}$, we define the operators $d = d(\partial_x)$ and $b = b(\partial_x)$ by,

$$d(\partial_x) := -D_0 - d_0 \partial_x^n \quad \text{and} \quad b(\partial_x) := B_0 + \beta \partial_x^m,$$

where $D_0 > 0$, $B_0 \in (0, 1)$ and $\beta \in \mathbb{R}$ are constants, and $m$ is a positive integer such that $m \leq n$. Further we assume $d_0 > 0$, unless $n = 2(2k - 1)$ for $k \in \mathbb{N}$, in which case we assume $d_0 < 0$. The form of $d = d(\partial)$ assumed, generates either a diffusive or dispersive effect in the equation we now present.

**Definition 3 (General Smoluchowski-type equation)** The general Smoluchowski-type equation we consider has the following form for $(x, t) \in [0, \infty)^2$:

$$\partial_t g(x; t) = \int_0^x g(x - y; t) b(\partial_y) g(y; t) \, dy - g(x; t) \int_0^\infty g(y; t) \, dy$$

$$+ d(\partial_x) g(x; t) - \int_0^x g(x - y; t) a(y; t) \, dy$$

$$- \int_0^x \int_0^y g(z; t) b_0(y - z) \, dz \, g(x - y; t) \, dy,$$

together with the initial condition $g(x; 0) = g_0(x)$ for a given function $g_0(x) \geq 0$ which is strictly positive on some finite subinterval of $[0, \infty)$, and the boundary conditions $\partial_\ell^k g(0; t) = 0$, for $\ell = 0, 1, 2, \ldots, n - 1$. Note, all the boundary conditions are fixed at $x = 0$. Further, in the equation, $a$ is a given smooth, non-negative, integrable function on $[0, \infty)^2$ and $b_0$ is a smooth, non-negative, integrable function of $x \in [0, \infty)$. We think of this equation as a general version of the constant kernel Smoluchowski equation with a similar non-local nonlinearity and additional terms, including a diffusion term. However, with the inclusion of the diffusion term, we no longer strictly associate the equation with a coagulation process and its solution $g = g(x; t)$ with the density of molecular clusters of mass $x$. In particular for example, we do not expect there to be dissipation of the form ‘$d(\partial_x) g(x; t)$’ of mass clusters in the original coagulation context. However, the other local and nonlocal terms shown could be interpreted as special fragmentation terms. We define the spatial convolution product ‘$*$’ of two functions $f = f(x; t)$ and $g = g(x; t)$ on $[0, \infty)$ by,

$$(f * g)(x; t) := \int_0^x f(x - y; t) g(y; t) \, dy.$$

The general Smoluchowski-type equation above has a natural form when expressed in terms of this convolution product as follows, with $\partial = \partial_x$, $g = g(x; t)$ satisfies:

$$\partial_t g = B_0 g * g + \beta g * (\partial^m g) - (M + D_0) g - d_0 \partial^n g - g * a - g * b_0 * g,$$

where as before $M = M(t)$ is the integral of $g = g(x, t)$ over $x \in [0, \infty)$. Formally, by integrating the Smoluchowski-type evolution equation above over $x \in [0, \infty)$, using that for any two functions $f$ and $g$ on $[0, \infty)$ we have,

$$\int_0^\infty (f * g)(x) \, dx = \left(\int_0^\infty f(x) \, dx\right) \left(\int_0^\infty g(y) \, dy\right).$$
and the boundary conditions at $x = 0$, as well as assuming $g$ decays as $x \to \infty$, we find
\[
\frac{dM}{dt} = -(D_0 + \overline{\pi}) M + (B_0 - 1 - \overline{c}_0) M^2,
\]
where $\overline{\pi} = \overline{\pi}(t)$ and $\overline{c}_0$ are given by,
\[
\overline{\pi}(t) := \int_0^\infty a(x; t) dx \quad \text{and} \quad \overline{c}_0 := \int_0^\infty b_0(x) dx.
\]

We can solve this Riccati ordinary differential equation for $M = M(t)$ by the standard linearisation approach. We suppose the functions $Q = Q(t)$ and $P = P(t)$ satisfying the following system of ordinary differential equations, together with the linear algebraic equation for $M = M(t)$ shown:
\[
\begin{align*}
\dot{Q} &= (1 + \overline{c}_0 - B_0) P, \\
\dot{P} &= -(D_0 + \overline{\pi}) P, \\
P &= MQ,
\end{align*}
\]
with $P(0) = M_0$ and $Q(0) = 1$. Solving the equation for $P = P(t)$ we find,
\[
P(t) = \exp\left(-\int_0^t (D_0 + \overline{\pi}(\tau)) d\tau\right) M_0.
\]
Substituting this into the equation for $Q = Q(t)$ and integrating that, we find,
\[
Q(t) = 1 + (1 + \overline{c}_0 - B_0) \int_0^t P(\tau) d\tau
\]\[
\Rightarrow \quad Q(t) = 1 + (1 + \overline{c}_0 - B_0) \int_0^t \exp\left(-\int_0^\tau (D_0 + \overline{\pi}(\upsilon)) d\upsilon\right) d\tau M_0.
\]

Hence we observe that,
\[
M(t) = \frac{P(t)}{1 + (1 + \overline{c}_0 - B_0) \int_0^t P(\tau) d\tau}
\]\[
\Rightarrow \quad M(t) = (1 + \overline{c}_0 - B_0)^{-1} \partial_t \log \left(1 + (1 + \overline{c}_0 - B_0) \int_0^t P(\tau) d\tau\right).
\]

Naturally we can substitute for $P = P(t)$, from the solution above, into this last formula to obtain an explicit solution for $M = M(t)$. Since $g_0(x) \geq 0$, but strictly positive on some finite subinterval of $[0, \infty)$, we know $M_0 > 0$. Further, since we assumed $a(x; t) \geq 0$ and $b_0(x) \geq 0$, we know $\overline{\pi}(t) \geq 0$ and $\overline{c}_0 \geq 0$. Thus since $D_0 > 0$ and $B_0 \in (0, 1)$, we deduce that $D_0 + \overline{\pi}(t) > 0$ and $1 + \overline{c}_0 - B_0 > 0$. This means that $M = M(t)$ strictly decreases from its initial value $M_0 > 0$ as time progresses. Furthermore $M = M(t)$ remains positive throughout, though this may not be true for $g = g(x, t)$ itself. Thus, this a-priori estimate has established that $M = M(t)$ is a bounded positive quantity on $[0, \infty)$.

Our goal now is to establish and explicit solution to the general Smoluchowski-type equation given above. To this end we define the class of Laplace Transform solutions. We assume all the properties on $D_0$, $d_0$, $B_0$, $\beta$, $g_0$, $b_0$ and $a$ outlined above, hereafter.
Definition 4 (Laplace transform solutions) We say \( g = g(x; t) \) is a Laplace transform solution to the general Smoluchowski-type equation given above, if the function \( g = g(s; t) \) satisfies the differential equation,

\[
\partial_t g = (B_0 - \beta s^m - b_0)g^2 - (M + a + D_0 + d_0 s^n)g,
\]

and the initial condition \( g(s; 0) = g_0(s) \). In these equations, \( g_0 = g_0(s) \), \( b_0 = b_0(s) \) and \( a = a(s; t) \) are the respective Laplace transforms of \( g_0(x) \), \( b_0(x) \) and \( a(x, t) \). We assume they all exist, and are thus analytic, for \( \text{Re}(s) \gtrless 0 \). Note that \( M(t) \equiv g(0; t) \).

Remark 5 If we take the limit \( s \to 0 \) in the Laplace transform equation in Definition 4, then we observe that, since \( M(t) \equiv g(0; t), \varphi(t) \equiv a(0; t) \) and \( \varphi_0 \equiv b_0(0) \), the evolution equation for \( g(0; t) \) is exactly the same as that for \( M = M(t) \) above. Hence for such Laplace transform solutions, we know that \( M(t) \equiv g(0; t) \) satisfies the properties we derived a-priori above, and in particular, has the solution formula shown.

There is a simple prescription of linear equations for the Laplace transform equations above as follows; also see Remark 6 below.

Definition 5 (Prescription: general Smoluchowski-type equation) Suppose that the functions \( q = q(s; t), \ p = p(s; t) \) and \( g = g(s; t) \) satisfy the following system of linear equations:

\[
\begin{align*}
\partial_t p &= -(M + a + D_0 + d_0 s^n)p, \\
\partial_t q &= (b_0 + \beta s^m - B_0)p, \\
p &= g(1 + q),
\end{align*}
\]

together with the initial conditions \( q(s; 0) = 0 \) and \( p(s; 0) = g_0(s) \).

Remark 6 In the scalar commutative context here, there is some choice in how to distribute the coefficients, of the linear terms in the evolution equation for \( g = g(s; t) \) in Definition 4 between the equations for \( q = q(s; t) \) and \( p = p(s; t) \) above. We have chosen to make the linear evolution equation for \( q = q(s; t) \) as simple as possible.

We now proceed to explicitly solve the linear evolution equations in Definition 5 for \( p = p(s; t) \) and then \( q = q(s; t) \). The linear evolution equation for \( p = p(s; t) \) can be solved by standard ordinary differential methods, and the unique solution is given by,

\[
p(s; t) = \exp\left(-t(D_0 + d_0 s^n) - \int_0^t \left(M(\tau) + a(s; \tau)\right) d\tau\right)g_0(s).
\]

The evolution equation for \( q = q(s; t) \) in Definition 5 integrates to reveal,

\[
q(s; t) = (b_0 + \beta s^m - B_0)\int_0^t p(s; \tau) d\tau.
\]

Hence we can solve for \( g = g(s; t) \) to find,

\[
g(s; t) = \frac{p(s; t)}{1 + q(s; t)}
\]

\[
\Rightarrow g(s; t) = \frac{p(s; t)}{1 + (b_0 + \beta s^m - B_0)\int_0^t p(s; \tau) d\tau}
\]
Naturally we can substitute the solution above for \( p = p(s; t) \) into this last expression to obtain an explicit solution formula for \( g = g(s; t) \). Let us remark on the properties of this solution. Since \( g_0 = g_0(s) \) exists and is analytic for \( \Re(s) \geq 0 \), from their forms above, we observe that both \( p(s; t) \) and \( 1 + q(s; t) \) exist for all \( t \geq 0 \), are unique, and are analytic with respect to \( s \in \mathbb{C} \) for \( \Re(s) \geq 0 \). Indeed both solutions are smooth with respect to time. Further, we also have the following.

**Lemma 1 (Reciprocal existence)** For some \( T > 0 \), we know that for \( t \in [0, T] \) we have: \( |q(s; t)| < 1 \) and \( 1 + q(s; t) \neq 0 \), and there exists a \( q_* = q_*(s; t) \) such that,

\[
(1 + q(s; t))^{-1} = 1 + q_*(s; t),
\]

with \( q_* = q_*(s; t) \) analytic with respect to \( s \in \mathbb{C} \) for \( \Re(s) \geq 0 \) and smooth in time for \( t \in [0, T] \). Naturally we have \( q_*(s; 0) = 0 \).

**Proof** The form of \( q = q(s; t) \) above reveals that it is the product of the Laplace transform function \((b_0 + \beta s^m - B_0)\) and the time integral of \( p = p(s; t) \), both of which are analytic with respect to \( s \in \mathbb{C} \) for \( \Re(s) \geq 0 \), with the latter smooth in time for \( t \in [0, T] \). Since \( q(s; 0) = 0 \), for a short time at least, \( |q(s; t)| < 1 \) and \( 1 + q(s; t) \neq 0 \). Since \( |q(s; t)| < 1 \) for that short time, \((1 + q(s; t))^{-1}\) exists and has a convergent power series expansion in \( q = q(s; t) \). Since \( q = q(s; t) \) is analytic with respect to \( s \in \mathbb{C} \) for \( \Re(s) \geq 0 \) and smooth in time, \( q_*(s; t) = (1 + q(s; t))^{-1} - 1 \) is also analytic for all \( s \in \mathbb{C} \) with \( \Re(s) \geq 0 \), and smooth in time. \( \square \)

Putting these results together, we have thus established the following.

**Theorem 1 (Local existence)** If the initial data \( g_0 = g_0(s) \) exists and is analytic for \( \Re(s) \geq 0 \), then for some \( T > 0 \), there exists a unique solution \( g = g(s; t) \) to the nonlinear equation in Definition 4 for \( t \in [0, T] \). While it exists, this solution is analytic with respect to \( s \in \mathbb{C} \) for \( \Re(s) \geq 0 \) and smooth in time.

**Remark 7 (Global solutions)** If we restrict the parameters we consider in the general Smoluchowski-type equation, we can establish global in time solutions \( g = g(s; t) \) to the nonlinear equation in Definition 4 as we did for the standard constant frequency kernel Smoluchowski equation in Example 1. For example, suppose \( B_0 = \frac{1}{2} \), \( \beta = d_0 = 0 \) and \( a \) and \( b_0 \) are identically zero. Then we observe that,

\[
p(s; t) = \exp\left(-tD_0 - \int_0^t M(\tau) \, d\tau\right)g_0(s).
\]

Using the explicit ratio formula for \( M = M(t) \) just preceding Definition 4 we see that,

\[
\exp\left(-\int_0^t M(\tau) \, d\tau\right) = \left(1 + \frac{M_0}{2D_0} (1 - e^{-tD_0})\right)^{-2}.
\]

Substituting this form into that for \( p = p(s; t) \) above, and then substituting that result into the formula for \( 1 + q(s; t) \) above, by direct computation we find,

\[
p(s; t) = e^{-tD_0}g_0(s)\left(1 + \frac{M_0}{2D_0} (1 - e^{-tD_0})\right)^{-2},
\]

where \( M_0 \) and \( D_0 \) are, respectively, the maximum and mean values of the Laplace transform function \((b_0 + \beta s^m - B_0)\) over the interval \([0, T]\).
1 + q(s; t) = 1 + \frac{g_0(s)}{M_0} \left(1 + \frac{M_0}{2D_0} (1 - e^{-tD_0})\right)^{-1} - \frac{g_0(s)}{M_0}.

Thus we observe that,
\[
g(s; t) = \frac{e^{-tD_0}g_0(s)}{(1 + \frac{M_0}{2D_0} (1 - e^{-tD_0}))(1 + \frac{1}{2\pi i} (M_0 - g_0(s)) (1 - e^{-tD_0}))}.
\]

Since for all \( s \in \mathbb{C} \) with Re(s) \( \geq 0 \) we know \(|g_0(s)| \leq |g_0(0)| \equiv M_0\), we conclude there is no blow-up for \( g = g(s; t) \) for \( t \in [0, \infty) \).

With explicit knowledge of the Laplace transform solution \( q = g(s; t) \) in hand, we now explore what we can say about the solution \( q = q(x; t) \) to the original general Smoluchowski-type equation given in Definition 3. We begin by considering the solution \( p = p(s; t) \) to the first linear equation in the Prescription in Definition 5 which is stated directly afterwards. Since \( g_0 = g_0(s) \) and \( a(s; t) \) are analytic in \( s \in \mathbb{C} \) for Re(s) \( \geq 0 \), as is \( s^n \) for \( n \in \mathbb{N} \), the inverse Laplace transform \( p = p(x; t) \) exists and is uniquely given for \( x \in [0, \infty) \) by,
\[
p(x; t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp \left(-t(D_0 + d_0 s^n) - \int_0^t (M(\tau) + a(s; \tau)) d\tau\right) g_0(s) e^{sx} ds,
\]
where the integrand, modulo the factor ‘\( e^{sx} \)’, is \( p(s; t) \). Using the Laplace transform Initial Value Theorem, see Theorem 2 below, observe that the limit as \( s \to \infty \) when \( n \neq 2(2k - 1) \), or the limit as \( s \to \infty \) when \( n = 2(2k - 1) \), of \( 's p(s; t) \)' goes to zero for every \( t \geq 0 \), and so \( p(0^+; t) = 0 \) for all \( t \geq 0 \). Similar limits for \( s^\ell p(s; t) \) tend to zero for all \( \ell = 2, 3, \ldots, n \) and \( t \geq 0 \), revealing that \( \partial^\ell p(0^+; t) = 0 \) for all \( \ell = 1, 2, \ldots, n - 1 \) and \( t \geq 0 \). Hence \( p = p(x; t) \) satisfies the boundary conditions. Now observe, by directly differentiating the expression above for \( p = p(x; t) \) with respect to time, that,
\[
\partial_t p = -D_0 p - d_0 \partial^n p - M p - a * p,
\]
with \( \partial = \partial_x \), \( M = M(t) \) and \( a = a(x; t) \). Hence the inverse Laplace transform of \( p = p(s; t) \) satisfies the linearised form of the general Smoluchowski-type equation in Definition 3. Now consider the inverse Laplace transform of \( q(s; t) \) which is given by,
\[
q(x; t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left((b_0(s) + \beta s^n - B_0) \int_0^t p(x; \tau) d\tau\right) e^{sx} ds.
\]
Note, since \( b_0 = b_0(s) \) is analytic in \( s \in \mathbb{C} \) for Re(s) \( \geq 0 \), we know this integral is well-defined from the properties we know for \( p = p(s; t) \). Since \( m \leq n \), we see that,
\[
q(x; t) = \int_0^t \left(b_0(s) + \beta s^n - B_0\right) p(x; \tau) d\tau.
\]
The inverse Laplace transforms \( q = q(x; t) \) and \( p = p(x; t) \) are thus well-defined, and satisfy the linear equations we could have anticipated they would. Let us now focus on the linear relation \( p(s; t) = g(s; t)(1 + q(s; t)) \). Given \( q = q(x; t) \) and \( p = p(x; t) \) as the unique inverse Laplace transforms of \( q = q(s; t) \) and \( p = p(s; t) \), respectively, there is thus a unique function \( g = g(x; t) \) satisfying,
\[
p(x; t) = g(x; t) + \int_0^x g(x - y; t) q(y; t) dy,
\]
for all \( x \in [0, \infty) \) and \( t \in [0, T] \), where the time \( T > 0 \) is that in Theorem \( 1 \). This result follows from the results preceding the Local Existence Theorem \( 1 \) including the Reciprocal Lemma \( 1 \) which establish the Theorem. We simply observe, \( g = g(x; t) \) is the inverse Laplace transform of \( \hat{g}(s; t) = p(s; t)/(1 + q(s; t)) \) which is well-defined and analytic for all \( s \in \mathbb{C} \) with \( \text{Re}(s) \geq 0 \) and smooth in time for all \( t \in [0, T] \) by the Reciprocal Lemma \( 1 \). Further, from that lemma, since for \( t \in [0, T] \),

\[
g(s; t) = p(s; t)(1 + q_*(s; t)),
\]

with \( q_* = q_*(s; t) \) analytic with respect to \( s \in \mathbb{C} \) for \( \text{Re}(s) \geq 0 \) and smooth in time for \( t \in [0, T] \), the regularity of \( g = g(x; t) \) is entirely determined by the regularity of \( p = p(x; t) \) for \( t \in [0, T] \). The regularity of \( p = p(x; t) \) is entirely determined by the initial data \( p_0 \equiv g_0 \). This follows, either from the form of the exponential factor in the explicit expression for \( p = p(s; t) \) given directly after Definition \( 5 \) or from the explicit partial differential equation \( p = p(x; t) \) satisfies, shown just above. Indeed for the dissipative cases, the regularity of \( p = p(x; t) \) for \( t > 0 \) is strictly better than that of the initial data. In any case, we assume \( g_0 \) and all its derivatives up to and including order \( n \) lie in \( L^1(0, \infty; [0, \infty]) \cap C(0, \infty; [0, \infty]) \), and that \( g_0 \) satisfies the zero boundary conditions at \( x = 0 \), so that \( \partial_\ell g_0(0) = 0 \), for all \( \ell = 0, 1, \ldots, n - 1 \). The relation for \( g \) in terms of \( p \) and \( q_* \) just above is equivalent to,

\[
g(x; t) = p(x; t) + \int_0^x p(x - y; t)q_*(y; t) \, dy.
\]

Taking the limit \( x \to 0 \) we see that \( g(0; t) = p(0; t) \) which equals zero. Further, we can differentiate this linear convolution relation so that we have,

\[
\partial_\ell g(x; t) = \partial_\ell p(x; t) + p(0; t)q_*(x; t) + \int_0^x \partial_\ell p(x - y; t)q_*(y; t) \, dy.
\]

Taking the limit \( x \to 0 \) we find that \( \partial_\ell g(0; t) = \partial_\ell p(0; t) + p(0; t)q_*(x; t) \), which is also zero. Taking further derivatives up to and including order \( n - 1 \) and taking the limit \( x \to 0 \), reveals that \( \partial_\ell^2 g(0; t) = \partial_\ell^2 p(0; t) \) for all \( \ell = 0, 1, 2, \ldots, n - 1 \), and \( t \in [0, T] \). and thus the function \( g = g(x; t) \), defined in this manner, satisfies all of the boundary conditions; as well as the initial condition \( g(x; 0) = g_0(x) \). Together with the Local Existence Theorem \( 1 \) the result of all this is the following, which is the main result of this section.

**Corollary 1 (Explicit solution)** For some \( T > 0 \), the solution \( g = g(x; t) \) to the general Smoluchowski-type equation given in Definition \( 1 \) is explicitly given by,

\[
g(x; t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( \frac{p(s; t)}{1 + q(s; t)} \right) e^{sx} \, ds,
\]

for \( t \in [0, T] \), where \( p = p(s; t) \) and \( q = q(s; t) \) satisfy the linear equations given in Definition \( 5 \) (with their explicit solution formulae stated directly afterwards). This solution satisfies the boundary conditions stated in Definition \( 3 \). That it satisfies the general Smoluchowski-type equation in Definition \( 5 \) can be checked by direct substitution.

We record the following ancillary result here, which is a direct consequence of the Reciprocal Lemma \( 1 \) as it will be useful to us in some of our subsequent results.
Lemma 2 (Inverse kernel) Let \( q = q(x; t) \) denote the inverse Laplace transform of \( f = f(s; t) \), and suppose that \( q(s; 0) = 0 \) and that \( f = f(s; t) \) is analytic in \( s \in \mathbb{C} \) for \( \text{Re}(s) \geq 0 \) and smooth in time. Then for some \( T > 0 \), the function \( \tau(s; t) := (1 + q(s; t))^{-1} \) exists for \( t \in [0, T] \) and is analytic in \( s \in \mathbb{C} \) for \( \text{Re}(s) \geq 0 \) and smooth in time. The inverse Laplace transform \( r = r(x; t) \) of \( \tau = \tau(s; t) \) exists for \( t \in [0, T] \), and for any \( f \in C(0, T; L^1([0, \infty]; \mathbb{R})) \) we have,

\[
\int_0^x \left( f(y; t) + \int_0^y f(z; t)q(y - z; t)\,dz \right) r(x - y; t)\,dy = f(x; t).
\]

Proof. All the statements preceding the final one, follow from Lemma 1. The relation \( f(s; t)(1 + q(s; t))\tau(s; t) = f(s; t) \) holds for the Laplace transform \( f = f(s; t) \) of the function \( f = f(x; t) \). Taking the inverse Laplace transform generates the result. \( \square \)

In our arguments above we used the following generalisation of the Laplace transform Initial Value Theorem, which relies on the generalised Riemann–Lebesgue Lemma. For completeness we give a proof in Appendix [A].

Theorem 2 (Initial Value Theorem) Suppose the function \( f = f(x) \) and its derivative \( f' = f'(x) \) are locally integrable on \([0, \infty)\), i.e. \( f, f' \in L^1_{\text{loc}}([0, \infty); \mathbb{R}) \), and are both of exponential order. Then if \( \mathcal{L} \) is the Laplace transform of \( f \), we have,

\[
f(0^+) = \lim_{s \to \infty} sf(s),
\]

where the limit can be taken along any ray in the half plane \( \text{Re}(s) \geq 0 \), not including the positive or negative imaginary \( s \)-axes. If we assume \( f, f' \in L^1([0, \infty); \mathbb{R}) \), then we can also take the limit \( s \to \pm \infty \), i.e. along the rays coinciding with the positive or negative imaginary \( s \)-axes. More generally suppose \( f : [0, \infty) \to \mathbb{R} \), and all of its derivatives up to and including order \( n \in \mathbb{N} \), are locally integrable on \([0, \infty)\) and of exponential order. Then for \( \ell \in \{0, 1, \ldots, n - 1\} \), the \( \ell \)th order initial derivatives \( f^{(\ell)}(0^+) \), are given by

\[
f^{(\ell)}(0^+) = \lim_{s \to \infty} \left( s^\ell f(s) - s^{\ell-1}f'(0^-) - \ldots - sf^{(\ell-1)}(0^-) \right),
\]

where the limit can be taken along any ray in the half plane as above, not including the imaginary axes. If \( f^{(\ell)} \in L^1([0, \infty); \mathbb{R}) \) for all \( \ell \in \{0, 1, \ldots, n\} \), then we can also take the limit along the positive or negative imaginary \( s \)-axes.

Remark 8. Let us now make a connection to the Fredholm Grassmannian flow theory we developed in our companion paper Doikou et al. [22], as well as in Beck et al. [2] and Doikou et al. [23]. If we follow the general approach advocated therein, see in particular for example Beck et al. [2] Example 1], we would construct a prescription of linear equations for the kernel functions \( p = p(x; y; t) \) and \( q = q(x; y; t) \) as follows. We would suppose the Laplace transforms \( \mathcal{L} = \mathcal{L}_p(s, y; t) \) and \( \mathcal{L} = \mathcal{L}_q(s, y; t) \) of \( p = p(x; y; t) \) and \( q = q(x; y; t) \), respectively, with respect to \( x \in [0, \infty) \), satisfy \( \partial_t p = -(M + \alpha D_0 + D_0 s^n)p \) and \( \partial_t q = (b_0 + \beta s^n - B_0)q \). We assume that \( q = q(x; y; t) \) and its Laplace transform \( \mathcal{L} = \mathcal{L}_p(s; t) \) satisfy:

\[
p(x; y; t) = g(x - y; t) + \int_0^x g(x - z; t)q(z; y; t)\,dz\quad\leftrightarrow\quad p(s; y; t) = g(s; t)e^{-sy} + g(s; t)q(s; y; t).
\]
Note, we retain the convolution form in the original $x$-space due to the form of the convolution nonlinearity present in the equation. We assume for the initial data $q(s, y; 0) = 0$ and thus from the relation for $g = g(s; t)$ just above, necessarily $p(s, y; 0) = \mathfrak{g}_0(s)e^{-sy}$. We can explicitly solve for $p = p(s, y; t)$ and $q = q(s, y; t)$ exactly in the same manner as we did above. We carry the $y$-dependence along as a passive parameter, so that the initial data in the solution formula for $p = p(s, y; t)$ is $\mathfrak{g}_0(s)e^{-sy}$, and this is the sole source of $y$-dependence in $p = p(s, y; t)$. Similarly, since the explicit solution formula for $q = q(s, y; t)$ involves a time-independent function of $s$ times the time integral of $p = p(s, y; t)$, $y$-dependence is solely through the factor $e^{-sy}$. Hence, since both $p = p(s, y; t)$ and $q = q(s, y; t)$ only depend on $y$ through the linear factor $e^{-sy}$, and $g(s, y; t) = p(s, y; t) / (e^{-sy} + q(s, y; t))$, we see that the $y$-dependence completely cancels out in this ratio. Hence the $y$-dependence in the prescription of linear equations is redundant. Thus, in the Laplace transform setting as we have seen, the operators involved are multiplicative operators, making the Grassmannian flow and prescription of linear equations far simpler than that in our companion paper Doikou et al. [23], or in Beck et al. [2,3] or Doikou et al. [22].

We now consider a sequence of five examples, each of which does not fit exactly into the format of the general Smoluchowski-type equation in Definition[3] However, in each case, we can utilise the Grassmannian flow ideas we developed above to either explicitly solve them, which applies to the first two cases, develop a useful numerical approach, which applies in the third case, generalise our approach to coagulation systems with multiple mergers, corresponding to the fourth case, as well as make the connection to the viscous Burgers equation and the Cole–Hopf transformation, which is the last case.

**Example 2 (Coarsening model)** Gallay and Mielke [32] analysed the following coarsening model for $g = g(x; t)$:

$$
\partial_t g(x; t) = g(t; t) \int_0^{x-t} g(y; t)g(x-y-t; t) \, dy,
$$

for $x > t$ and with $g(x; t) = 0$ for $x < t$. The initial data is prescribed at time $t = 1$ so that $g(x; 1) = g_1(x)$ for a suitable given function $g_1 = g_1(x)$. Also see Pego [59, p. 16–7] for more details. Gallay and Mielke used a global linearisation transform to solve this model. In our context here the equation above is a special case of a Grassmannian flow. Indeed, consider the Laplace transform $\mathfrak{g} = \mathfrak{g}(s; t)$ of $g = g(x; t)$ which is given by

$$
\mathfrak{g}(s; t) := \int_0^\infty e^{-sx} g(x; t) \, dx = \int_t^\infty e^{-sx} g(x; t) \, dx,
$$

using that $g(x; t) = 0$ for $x < t$. Taking the Laplace transform of the coarsening equation above reveals that $\mathfrak{g} = \mathfrak{g}(s; t)$ satisfies the following Riccati equation:

$$
\partial_t \mathfrak{g} = g(t, t) e^{-st} (\mathfrak{g}^2 - 1),
$$

with the initial condition $\mathfrak{g}(s; 1) = \mathfrak{g}_1(s)$, where $\mathfrak{g}_1$ is the Laplace transform of $g_1$. Now consider the following prescription which can be thought of as a modification of the Laplace transform of the general Smoluchowski-type equation prescription:

$$
\partial_t p = c(s, t) q,
$$
\[ \partial_t q = b(s, t) p, \]
\[ p = g q, \]

where \( c \) and \( b \) are given coefficient functions. Note here, compared to the notation we used for the general Smoluchowski-type equation above, we replace \( 1 + q(s; t) \) by \( q = q(s; t) \). We thus augment this prescription with the initial conditions \( q(s; 1) = 1 \) and \( p(s; 1) = g_1(s) \). A straightforward computation reveals \( g = g(s; t) \) given in this prescription satisfies,

\[ \partial_t g = c(s, t) - g b(s, t) g. \]

This matches the Riccati equation for the Laplace transform of the coarsening model above once we identify \( c(s, t) = b(s, t) = -g(t; t) e^{-st}. \) For convenience we set,

\[ G(s; t) := - \int_1^t g(\tau; \tau) e^{-s\tau} d\tau. \]

Note we have \( G(s; 1) = 0. \) Now we observe the two linear evolutionary equations for \( p = p(s; t) \) and \( q = q(s; t) \) above, with the identifications for \( b \) and \( c \) indicated just above, satisfy the system:

\[ \frac{\partial}{\partial t} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \]

where we have used the initial conditions for \( q \) and \( p \) from the prescription above. A straightforward calculation reveals that \( q = q(s; t) \) and \( p = p(s; t) \) are given by

\[ q(s; t) = \cosh G(s; t) + g_1(s) \sinh G(s; t), \]
\[ p(s; t) = \sinh G(s; t) + g_1(s) \cosh G(s; t). \]

Since \( g = p/q \) we deduce that \( g = g(s; t) \) is given by

\[ g(s; t) = \frac{\tanh G(s; t) + g_1(s)}{1 + g_1(s) \tanh G(s; t)}. \]

We now observe that since \( s > 0 \) we have

\[ g(s; t) := \int_t^\infty e^{-sx} g(x; t) dx \leq e^{-st} \int_t^\infty g(x; t) dx = e^{-st} M(t), \]

where \( M = M(t) \) is the total number of clusters at time \( t \)—as in Pego [59, p. 18] we assume this is normalised to be one. Since the right-hand side in this last inequality decays to zero as \( t \to \infty \), we deduce that

\[ g_1(s) = - \tanh G(s; \infty) = \tanh \int_1^\infty g(\tau; \tau) e^{-s\tau} d\tau. \]

Consequently, by a standard identity for hyperbolic functions we see that,

\[ g(s; t) = \frac{\tanh G(s; t) + \tanh(-G(s; \infty))}{1 + \tanh(-G(s; \infty)) \tanh G(s; t)} \equiv \tanh(G(s; t) - G(s; \infty)) \]
= \tanh \int_0^\infty g(\tau; \tau) \, d\tau.

This matches the solution presented in Pego [59] eq. (32)].

Example 3 (Derrida–Retaux depinning model) Derrida and Retaux [21] consider the following depinning transition model for the field \( r = r(x; t) \):

\[
\partial_t r = a_0 \partial_x r + B_0 r * r,
\]

together with the condition \( r(x; 0) = r_0(x) \), for a given function \( r_0 = r_0(x) \). This does not match the general Smoluchowski-type equation in Definition 3 as \( a_0 \), which corresponds to \( -d_0 \), is assumed to be strictly positive and we assume \( B_0 > 0 \) (only). Derrida and Retaux look for underlying solutions to their depinning transition model of the form,

\[ r(x; t) = R(t)e^{-D(t)x}, \]

where the functions \( R = R(t) \) and \( D = D(t) \) are determined as follows. Substituting this solution ansatz into the depinning transition model above, we observe that necessarily \( R = R(t) \) and \( D = D(t) \) must satisfy,

\[
\frac{dR}{dt} = -a_0 DR \quad \text{and} \quad \frac{dD}{dt} = -B_0 R.
\]

Derrida and Retaux integrate this pair of ordinary differential equations to obtain explicit solutions, revealing three different characteristic behaviours depending on whether:

(i) \( 2R(0)B_0 < a_0(D(0))^2 \) which corresponds to the unpinned phase; (ii) \( 2R(0)B_0 = a_0(D(0))^2 \) which corresponds to the critical case; or (iii) \( 2R(0)B_0 > a_0(D(0))^2 \) which corresponds to the pinned phase. See Derrida and Retaux [21] for more details, in particular the critical time of divergence of \( R = R(t) \) and \( D = D(t) \) in case (iii). With these in hand, our goal herein is to use the Grassmannian flow approach to derived closed form expressions for more general solutions, indeed perturbative solutions, as follows. Suppose we define the perturbation \( g = g(x; t) \) via the relation,

\[ r(x; t) = R(t)e^{-D(t)x} + g(x; t), \]

where \( R = R(t) \) and \( D = D(t) \) satisfy the ordinary differential equations above. If \( \varepsilon = \varepsilon(x; t) \) represents the function \( \varepsilon(x; t) = e^{-D(t)x} \), then substituting this solution ansatz for \( r = r(x; t) \) into the depinning transition model reveals that the perturbation \( g = g(x; t) \) necessarily satisfies,

\[ \partial_t g = a_0 \partial_x g + 2B_0 R \varepsilon * g + B_0 g * g. \]

Let us consider the Laplace transform version of this equation, which, if \( g = g(s; t) \) represents the Laplace transform of \( g = g(x; t) \), takes the form,

\[
\partial_s g = \left( a_0 s + \frac{2RB_0}{s+D} \right) g - a_0 C + B_0 g^2,
\]

where we used the explicit form for the Laplace transform of \( \varepsilon \). Consider the following prescription of linear equations for the functions \( p = p(s; t) \), \( q = q(s; t) \) and \( g = g(s; t) \):

\[
\partial_s p = \left( a_0 s + \frac{2RB_0}{s+D} \right) p,
\]
\[ \partial_t q = -B_0 p, \]
\[ p = g q. \]

Here, as in the last example, we replace \( 1 + q(s; t) \) by \( q = q(s; t) \). We augment this system of linear equations with the initial conditions \( q(s; 0) = 1 \) and \( p(s; 0) = g_0(s) \), where \( g_0 = g_0(s) \) is the Laplace transform of \( g_0(x) = \tau_0(x) - R(0)x(x; 0) \). If we differentiate the linear relation \( p = g q \) with respect to time using the product rule, we observe that \( \partial_t q = \partial_t g \) satisfies the Laplace transform evolution equation, corresponding to the depinning transition model just above. Let us now solve for \( q = q(s; t) \) and \( p = p(s; t) \). By direct integration we observe that,
\[
p(s; t) = \exp \left( a_0 s t + 2B_0 \int_0^t \frac{R(\tau)}{\mathcal{s} + D(\tau)} d\tau \right) g_0(s),
\]
\[
q(s; t) = 1 - B_0 \int_0^t p(s; \tau) d\tau.
\]

Hence the solution \( g = g(s; t) \) has the closed form \( g(s; t) = p(s; t)/q(s; t) \), which exists locally around \( t = 0 \) due to the form of \( q = q(s; t) \). We note that, by direct computation using the explicit forms for \( p = p(s; t) \) and \( q = q(s; t) \) above, for any finite small \( t \) we have \( sg(s; t) \sim -\left( a_0/B_0 \right) s^2 \) as \( s \to \infty \), so that within an infinitesimally small time interval, \( g = g(x; t) \) is divergent at the boundary \( x = 0 \). Inserting the explicit forms for \( R = R(t) \) and \( D = D(t) \) into the solutions for \( p = p(s; t) \) and \( q = q(s; t) \) above, for each of the cases (i)—(iii), and pursuing the analysis of the perturbative solution \( g = g(x; t) \) to extend the analysis in Derrida and Retaux \cite{21} is very much of interest, though it exceeds our goals here.

**Example 4 (Lambert–Scherter genealogy model)** The genealogy model of Lambert and Schertzer \cite{37} for the field \( g = g(x, t) \) has the form
\[
\partial_t g = \partial_x (\alpha g) - D_0 g + \frac{1}{2} g * g,
\]
where \( \alpha = \alpha(x) \) has the specific form \( \alpha = cx^2 \), where \( c \) is a constant, and \( D_0 = D_0(t) \) is a given function. This model does not fit directly into the format of the general Smoluchowski-type equation in Definition \cite{3}. This is because the first linear vector field \( V_1(g) := \partial_x (\alpha g) \), involves the non-constant coefficient term \( \partial_x \partial_x g \), and the second vector field \( V_2(g) := -D_0 g + \frac{1}{2} g * g \) involves the term \( \partial_x \partial_x g \) in which \( D_0 = D_0(t) \) is not constant. However the vector field \( V_1 \) is integrable on its own, as is the second vector field \( V_2 \) which constitute a Grassmannian flow. This suggests we can construct a numerical Strang splitting method \( \exp \left( \frac{1}{2} \Delta t V_1 \right) \circ \exp \left( \Delta t V_2 \right) \circ \exp \left( \frac{1}{2} \Delta t V_1 \right) \) for which the individual flows are exact. Such a numerical method is second order in time in general, and the spatial error is merely due to the discrete implementation representation we choose. It is thus useful to classify integrable Smoluchowski vector fields to improve numerical complexity.

**Example 5 (Multiple mergers)** There are Smoluchowski models with multiple mergers. The \( n \)-step descendent distribution of a Galton–Watson process satisfies a discrete Smoluchowski flow, and the Lévy jump measure of certain continuous state branching processes satisfy a multiple coalescence version of continuous Smoluchowski flow. The spatial Laplace exponent \( \Phi = \Phi(s; t) \) associated with the Lamperti transformation of the underlying continuous state branching process satisfies
\[
\partial_t \Phi + \Psi(\Phi) = 0.
\]
with \( g_0(s) = s \), where \( \Psi \) is the given branching mechanism, see Iyer et al. \cite{39}. For example, for the Smoluchowski coagulation case with constant kernel \( K = 1 \) we have considered hitherto, the function \( \Psi = \Psi(g) \) has the form \( \Psi(g) = -\frac{1}{2}g^2 \); see for example Menon and Pego \cite{56}. Formally, the evolution equation for \( g \) can be solved by a slightly more general Grassmannian flow. Suppose the scalar functions \( p = p(s; t), \ q = q(s; t) \) and \( g = g(s; t) \) satisfy the system of equations:

\[
\begin{align*}
\partial_t p &= 0, \\
\partial_t q &= p, \\
p &= \Phi(g)q.
\end{align*}
\]

Here the nonlinear function \( \Phi = \Phi(g) \) is determined as follows. If we use the product rule to compute the time derivative of the third equation we find, after dividing through by \( q \), that \( g \) necessarily satisfies \( \partial_t g + (\Phi'(g))^{-1}\Phi^2(g) = 0 \). To match this equation to that for \( g \) above we require \( \Phi = \Phi(g) \) to satisfy:

\[
\Phi' = (\Psi)^{-1}\Phi^2.
\]

This is a Riccati equation which is a Grassmannian flow. Indeed suppose the scalar fields \( P = P(g), \ Q = Q(g) \) and \( \Gamma = \Gamma(g) \) satisfy the linear system of equations:

\[
\begin{align*}
\partial_t P &= 0, \\
\partial_t Q &= -(\Psi)^{-1}P, \\
P &= \Gamma Q.
\end{align*}
\]

Taking the derivative of the third equation with respect to \( g \) and using the product rule we observe that \( \Gamma = \Gamma(g) \) satisfies the Riccati equation \( \partial_t \Gamma = (\Psi)^{-1}\Gamma^2 \) matching the Riccati equation \( \Phi' = (\Psi)^{-1}\Phi^2 \) for \( \Phi = \Phi(g) \). Hence we deduce that \( \Phi = \Phi(g) \) is given by \( \Phi = \Phi(g) \) to satisfy the linear equations shown. Since \( P(g) = P_0 \), a constant, we deduce \( \Phi = P_0Q^{-1} \). Integrating the second equation we find:

\[
Q = Q_0 - \int_{g_0}^{g(s; t)} (\Psi(\tilde{g}))^{-1}d\tilde{g} P_0,
\]

\[
\Rightarrow \Phi(g(s; t)) = \left( \left(\Phi(g_0(s))\right)^{-1} - \int_{g_0}^{g(s; t)} (\Psi(\tilde{g}))^{-1}d\tilde{g} \right)^{-1}.
\]

We can re-write the formula for \( \Phi \) shown on the right in the form

\[
\frac{1}{\Phi(g(s; t))} - \frac{1}{\Phi(g_0(s))} = -\int_{g_0}^{g(s; t)} \frac{1}{\Psi(\tilde{g})} d\tilde{g}.
\]

This suggests we take the defining relation for \( \Phi = \Phi(g) \) to be

\[
\frac{1}{\Phi(g)} = -\int_{\Psi(g)}^{g(s; t)} \frac{1}{\Psi(\tilde{g})} d\tilde{g},
\]

i.e. in terms of the indefinite integral shown, taking the additive constant to be zero. Note that in principle we could have used this form for \( \Phi = \Phi(g) \) in the relation between \( p \) and \( q \) from the outset. In any case the relation above demonstrates how we can recover the unknown \( \Phi = \Phi(g) \) from the known function \( \Psi = \Psi(g) \). Then the linear equations
for \( p = p(s,t) \) and \( q = q(s,t) \) reveal \( p(s,t) = p_0(s) \) and \( q(s,t) = q_0(s) + p_0(s)t \) where \( p_0 = p_0(s) \) and \( q_0 = q_0(s) \) are functions of ‘s’ only. We find,

\[
\Phi(g(s,t)) = \frac{p(s,t)}{q(s,t)} = \frac{1}{(\Phi(g_0(s)))^{-1} + t}.
\]

Hence, provided we can compute the inverse function \( \Phi^{-1} \), we can determine \( g = g(s,t) \) using the formula

\[
g(s,t) = \Phi^{-1} \left( \left( \Phi(g_0(s)) \right)^{-1} + t \right).
\]

For example, consider the Smoluchowski case we considered in Example 1 with \( 1 - q/2 = g_0(s) \), i.e. assume we have subsumed the coagulation loss term \( -Mg \) via the desingularised Laplace transform. Then the formula for \( \Phi = \Phi(g) \) involving the indefinite integral above reveals \( \Phi(g) = -\frac{1}{2}g \). The formula for \( g = g(s,t) \) just above reveals

\[
g(s,t) = \frac{2g_0(s)}{2 - tg_0(s)},
\]
in this special case. This matches the explicit solution given in Example 1 (ignoring the initial total number of clusters \( M_0 \)). See Remark 2 and also Menon and Pego [56].

**Example 6 (Pre-Laplace Burgers equation and Cole–Hopf transformation)** For the general Smoluchowski-type equation in Definition 3, we imposed the condition that \( d = d(\partial_x) \) and \( b = b(\partial_x) \) have constant coefficients. However there is at least one exceptional case as follows. Consider the following nonlinear differential equation for \( g = g(x;t) \) given for \( x \in [0, \infty) \) by,

\[
\partial_t g(x;t) = \nu x^2 g(x;t) + \frac{1}{2} x \int_0^x g(y)g(x-y)\,dy,
\]

where \( \nu > 0 \) is a constant parameter. Suppose that initially \( g(x;0) = g_0(x) \) for a given function \( g_0 = g_0(x) \). This corresponds to the case where \( D_0 = -\nu x^2 \) and \( B_0 = \frac{1}{2}x \), with all other parameters zero, in the general Smoluchowski-type equation. The \( x \)-dependence in \( B_0 \) and \( D_0 \) is outwith the conditions on \( d \) and \( b \) mentioned. Taking the Laplace transform of this equation, we see that the Laplace transform \( g = g(s;t) \) of \( g = g(x;t) \) satisfies the partial differential equation,

\[
\partial_s g = \nu \partial_s^2 g + \frac{1}{2} \partial_s (g^2),
\]

with \( g(s;0) = g_0(s) \), where \( g_0 = g_0(s) \) the Laplace transform of \( g_0 = g_0(x) \). This is of course the viscous Burgers equation. Consider the following prescription of linear equations for the functions \( p = p(s,t) \), \( q = q(s,t) \) and \( g = g(s;t) \):

\[
\begin{align*}
\partial_t p &= \nu \partial_s^2 p, \\
q &= -\frac{1}{2\nu} \int_s^\infty p(\sigma;t)\,d\sigma, \\
p &= qg.
\end{align*}
\]

Note here, as in Examples 2 and 3, we replace \( 1 + q(s,t) \) by \( q = q(s,t) \). We observe that with this prescription, the function \( q = q(s,t) \) also satisfies the heat equation \( \partial_t q = \nu \partial_s^2 q \), and further, the definition of \( q \) is equivalent to \( p = 2\nu \partial_s q \). By direct
computation, taking the time derivative of the third linear relation above and utilising these relations, the function $g = g(s; t)$ can be shown to satisfy the viscous Burgers partial differential equation above provided $q(s; t) \neq 0$. We augment the prescription with the initial conditions $p(s; 0) = g_0(s)$ and,

$$q(s; 0) = -(1/2\nu) \int_s^\infty g_0(\sigma) \, d\sigma.$$ 

Hence, provided $g_0$ is not identically zero, then for a short time at least $q = q(s; t)$ will also be non-zero. However, we can say more. The third linear, algebraic equation for $g$ implies $g = 2\nu \log q$, which is the Cole–Hopf transformation. Hence a unique solution exists globally in time since $q = q(s; t)$ is a solution to the heat equation—say assuming initial data $g_0 = g_0(x)$ for its Laplace transform $g_0 = g_0(s)$ is analytic for $\text{Re}(s) \geq 0$. Consequently, there is a unique global solution to the original pre-Laplace equation we considered. This result is the analogue of that in Beck et al. [2] Remark 8.

**Remark 9 (Madelung transform)** This transform takes solutions $w = w(x, t)$ to the Schrödinger equation

$$i\partial_t w + \frac{1}{2}\Delta w = (f(|w|^2) + W)w,$$

for some given potential $W = W(x, t)$, and transforms them to solutions of the following inviscid system of compressible fluid equations (and vice-versa)

$$\partial_t v + (\nabla v)v + \nabla (f(\rho) + W) = \frac{1}{2}\nabla (\Delta(\sqrt{\rho})/\sqrt{\rho}),$$

$$\partial_t \rho + \nabla T(\rho v) = 0,$$

for potential flows $v = \nabla \phi$. The transform is given by $w = \sqrt{\rho}e^{i\phi}$ and represents the complex form of the Cole–Hopf transformation (mentioned above). To see this we set $v = \nabla \phi$ and $\rho = e^{\phi}$. We set $f \equiv 0$ for simplicity. In these variables the Madelung transform takes the form $w = e^{\phi + i\phi}$ or $w = e^{i\phi}$. The pair of equations above is then equivalent to $i\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + \frac{1}{2}\Delta \phi = W$. As is well-known, if we make the transform $w = e^{\phi}$ then $w$ satisfies the linear Schrödinger equation $i\partial_t w + \frac{1}{2}\Delta w = Ww$.

For more details on the Madelung transform see Carles, Danchin and Saut [12] or Khesin, Misolek and Modin [13].

**Remark 10 (Cole–Hopf and Cameron–Martin–Girsanov Theorem)** There is a natural connection between the Cole–Hopf transformation for potential solutions of the viscous Burgers equation and the Cameron–Martin–Girsanov Theorem. See Lejay [25] and Doikou, Malham and Wiese [26].

Thusfar we have not considered Smoluchowski’s equation in the cases of the additive $K = x + y$ and multiplicative $K = xy$ coagulation frequency kernels. These two cases are intimately related, see Deaconu and Tanré [21] Section 3.3. The natural contexts for these two cases are the desingularised and modified-desingularised Laplace transform spaces, see Menon and Pego [56]. We defined the desingularised Laplace transform $\pi = \pi(s)$ for any function $g \in L^1(0, \infty; \mathbb{R})$ in Remark 2. The desingularised Laplace transform of the solution to Smoluchowski’s equation in the additive $K = x + y$ case satisfies the inviscid Burgers equation with linear decay:

$$\partial_t \pi + \pi \partial_s \pi = -\pi.$$ 

The modified-desingularised Laplace transform $\tilde{\pi} = \tilde{\pi}(s)$, for any function $g$ integrable on $\mathbb{R}_+$ with respect to any linear weight function, is defined by
\[ \dot{\pi}(s) := \int_0^\infty (e^{-sx} - 1) x g(x) \, dz. \] The modified-desingularised Laplace transform of the solution to Smoluchowski's equation in the multiplicative \( K = xy \) case satisfies the inviscid Burgers equation: \( \partial_t \pi + \partial_x \pi = 0 \). See Menon and Pego [56] for more details. These equations are examples of graph flows and we consider them in detail next.

3 Nonlinear graph flows and Burgers equation

We consider the inviscid Burgers equation in light of its connection to Smoluchowski's coagulation equation in Section 2. Our results herein were inspired by the work of Byrnes [9] and Byrnes and Jhemi [10], who consider the inviscid Burgers equation as a very special case of an optimal nonlinear control problem, as follows. They consider a control system of the form \( \dot{q} = b(q) + \sigma(q)u \) for the system state \( q = q(t) \), where \( b \) is a nonlinear vector field as are each of the columns of the matrix \( \sigma \). The function \( u = u(t) \) is a control which must be chosen to minimise a general nonlinear cost function \( \int_0^T L(q, u) \, dt + Q(q(T)) \), where \( L \) is a nonlinear Lagrangian density and \( Q \) a nonlinear final state realisation cost at time \( T > 0 \). See Byrnes [9] for more details.

For the Bolza problem, every extremal control \( u^* \) for any initial state \( q_0 \) gives rise to a canonical pair satisfying \( \dot{q} = \nabla_p H_\pi \), \( \dot{\pi} = -\nabla_q H_\pi \), for some Hamiltonian \( H_\pi \), and \( p(T) = -\nabla_q Q(q(T)) \). In addition \( u^* = u^*(q, p) \) and the goal is to find a map \( \pi \) such that \( p = \pi(q, t) \) so that the optimal control \( u^* \) can be expressed in terms of the current state \( q \). Indeed a further goal is to solve the equation for the current state feedback map \( \pi \) backwards in time from time \( T \) offline, and then use it in the dynamical forward optimal control problem. This is analogous to the case in optimal linear-quadratic control theory when \( \pi \) satisfies a Riccati ordinary differential system. Here \( \pi \) satisfies the Riccati partial differential equation \( \partial_t \pi = \nabla_q H_\pi(q, \pi) + \langle \nabla_q \pi, \nabla_p H_\pi(q, \pi) \rangle \) with \( \pi(0, t) = 0 \) and \( \pi(q, T) = \nabla_q Q(q) \). In the case of the Bolza problem for the quadratic cost function \( L(q, u) = |u|^2 \) where \( p = u \), the evolution of \( q \) and \( p \) is given by the pair of linear ordinary differential equations \( \dot{q} = p \) and \( \dot{p} = 0 \). The Riccati partial differential equation in this case corresponds to the inviscid Burgers equation.

With this knowledge in hand, we begin here with a reinterpretation of the inviscid Burgers equation as follows. On the one hand we have the classical result that the inviscid Burgers equation \( \partial_t \pi + (\nabla \pi) \pi = 0 \) for \( \pi = \pi(q, t) \) with initial data \( \pi_0 \) can be solved via characteristics by defining space-time curves \( q = q(a, t) \) by \( \dot{q} = \pi(q, t) \) with \( q(a, 0) = a \) for all \( a \in \mathbb{R}^N \). Here \( N \in \mathbb{N} \) is any fixed dimension. Hence if \( \pi \) satisfies the inviscid Burgers equation and \( \pi \) evolves according to the characteristic curves, we deduce \( \dot{\pi}(q; t) = 0 \) along the characteristics curves. Hence we find \( \pi(q(a, t); t) = \pi_0(a) \) along the characteristics which are given by \( q(a, t) = a + t\pi_0(a) \). Consequently to determine \( \pi = \pi(x; t) \) at any point \( (x, t) \in \mathbb{R}^N \times \mathbb{R}_+ \) we must solve the algebraic equation \( x = a + t\pi_0(a) \) for \( a \in \mathbb{R}^N \) and substitute the result into \( \pi(x; t) = \pi_0(a) \) so that \( \pi(x; t) = \pi_0((id + t\pi_0)^{-1}(x)) \).

On the other hand, as indicated by Byrnes [9] and Byrnes and Jhemi [10], we can consider the prescription given by the following system of equations.

**Definition 6 (Prescription: Inviscid Burgers equation)** Assume for each \( a \in \mathbb{R}^N \) the \( \mathbb{R}^N \)-valued functions \( q = q(a, t) \), \( p = p(a, t) \) and \( \pi = \pi(\cdot, t) \) satisfy
\[
\dot{q}(a, t) = p(a, t), \\
\dot{p}(a, t) = 0,
\]
Remark 11

We can also view the map \( q = \pi \) with respect to \( p \) as

\[ p(a, t) = \pi(q(a, t), t), \]

where \( q(a, 0) = a \) so \( p(a, 0) = \pi_0(a) \) and \( \pi_0 : \mathbb{R}^N \to \mathbb{R}^N \) is a given function.

We think of these equations as prescribing the evolution of a continuum of particles labelled by \( a \in \mathbb{R}^N \) at time \( t = 0 \). The path of each particle is given by \( q(a, t) \in \mathbb{R}^N \).

We call \( q(a, t) \) for all \( a \in \mathbb{R}^N \) the flow while we shall call the map \( q_t : a \mapsto q(a, t) \) the flowmap. Note that the first two equations above, which are linear, assert that the evolution of each of the particles in the ensemble is characterised by zero acceleration, i.e. \( \dot{q}(a, t) = 0 \) for all \( a \in \mathbb{R}^N \). The third relation in the prescription above however represents a fundamental difference to our development in the previous section. In our current perspective we now seek a nonlinear map \( \pi : (q(a, t), t) \mapsto p(a, t) \), i.e. from the particle trajectory coordinates to the particle momentum coordinates. In other words, we propose the existence of a time-dependent nonlinear map that emulates or reproduces the momentum path from knowledge of the particle path. In phase space we have

\[
\frac{\partial}{\partial t}(q, p) = \left( \begin{array}{c} q \\ p \end{array} \right) = \left( \frac{\partial}{\partial t}\pi_t \right) \circ q,
\]

representing the graph of \( \pi_t \) at any time \( t \geq 0 \). Note we have omitted the labels ‘a’ as well as explicit \( t \)-dependence for brevity, and to be precise \( \pi_t \circ q \equiv \pi(q, t) \).

**Remark 11** We can also view the map \( \pi_t \) above as follows. Since \( \dot{p} = 0 \), as already intimated, we have \( \pi(q(a, t), t) = \pi_0(a) \). For any given fixed \( t \geq 0 \), we can re-write this as the relation \( \pi_t \circ q_t = \pi_0 \). Hence the map \( \pi_t \) is analogous to the inverse map to \( q_t \) with respect to \( \pi_0 \), or more precisely for any given fixed \( t \geq 0 \), we see \( \pi_t \circ x = \pi_0 \circ q_t^{-1} \circ x \) for all \( x \in \mathbb{R}^N \) for which \( q_t^{-1} \circ x \in \mathbb{R}^N \) exists.

We now return to the result of Byrnes [9] and Byrnes and Jhemi [10].

**Proposition 1 (Inviscid Burgers decomposition)** For any \( a \in \mathbb{R}^N \) assume that \( q = q(a, t) \) and \( p = p(a, t) \) satisfy the prescription for the inviscid Burgers equation given above. Assume for some \( T > 0 \) there exists a smooth invertible function \( \pi \) such that for all \( t \in [0, T] \) and \( a \in \mathbb{R}^N \) we have \( p(a, t) = \pi(q(a, t), t) \). Then \( \pi = \pi(q(a, t), t) \) satisfies the inviscid Burgers equation:

\[
\partial_t \pi + (\nabla \pi) = 0.
\]

**Proof** By direct computation, since \( 0 = \dot{p}(a, t) = \dot{\pi}(q(a, t), t) \), the chain rule implies

\[
0 = (\partial_t \pi)(q(a, t), t) + ((\nabla \pi \pi)(q(a, t), t), t).
\]

\[ \square \]

We can now establish the well-known explicit solution as follows.

**Corollary 2 (Inviscid Burgers solution)** Given a smooth function \( \pi_0 : \mathbb{R}^N \to \mathbb{R}^N \), there exists a \( T > 0 \) such that for all \( t \in [0, T] \) and \( x \in \mathbb{R}^N \) the solution \( \pi = \pi(x, t) \) to the inviscid Burgers equation is given by:

\[
\pi(x, t) = \pi_0((\text{id} + t \pi_0)^{-1}(x)).
\]

**Proof** Using \( \dot{q}(a, t) = p(a, t) \) and \( p(a, t) = p(a, 0) = \pi_0(a) \) we see \( q(a, t) = a + t \pi_0(a) \). Via the standard argument, given \( x \in \mathbb{R}^N \), provided we can solve \( x = a + t \pi_0(a) \), which we can do for sufficiently small times when \( \pi_0 \) is smooth, then \( \pi(x, t) = \pi(q(a, t), t) = p(a, t) = \pi_0(a) \) where \( a = (\text{id} + t \pi_0)^{-1}(x) \). \[ \square \]
We note the essential ingredient to finding an explicit solution is the ability to determine the solution ‘a’ to the nonlinear algebraic relation \( x = a + i\pi_0(a) \).

**Remark 12 (Nonlinear graph manifold and coordinate patches)** In the inviscid Burgers prescription in Definition 6 the underlying linear flow corresponds to the phase space for \( (q_t, pt) \) shown. The prescription also posits the time-dependent nonlinear map \( \pi_t: q_t \rightarrow pt \), from the particle to the momentum coordinates. As we have just seen, the map \( \pi_t \) satisfies the inviscid Burgers equation and the prescription provides the mechanism for solution as a realisation of the method of characteristics. Recall our discussion preceding Remark 11. We can view this procedure as the projection of the mechanism for solution as a realisation of the method of characteristics. Recall

\[
\text{id}_{\mathbb{R}^N} \quad \pi_t \quad \text{id}_{\mathbb{R}^N}
\]

seen, the map \( \pi_t \) is nonlinear, this represents a coordinate patch of a nonlinear graph manifold. We provide a formal description of such an object here. We can think of the Grassmannian \( \text{Gr}(\mathbb{R}^{2N}, \mathbb{R}^N) \) as the collection of all possible suitably compatible graphs

\[
\left( \text{id}_{\mathbb{R}^N} \right)_{\pi_t^{(i)}}
\]

of linear maps \( \pi_{\pi_t^{(i)}}: \mathbb{R}^N \rightarrow \mathbb{R}^N \) where \( \mathcal{S} = \{i_1, \ldots, i_N\} \) with \( i_1 < i_2 < \cdots < i_N \) represents a subset of \( \{1, \ldots, 2N\} \). We think of the nonlinear graph manifold as the collection of all possible suitably compatible graphs of linear or nonlinear maps \( \pi_{\pi_t^{(i)}}: \mathbb{R}^N \rightarrow \mathbb{R}^N \) of the form just above. Let us consider one particular different coordinate patch, to the canonical choice where \( \mathcal{S} = \{1, \ldots, N\} \), the coordinate patch with \( \mathcal{S} = \{N+1, \ldots, 2N\} \). Hence, given the same underlying phase space \( (q_t, pt) \) from the prescription with \( \dot{q}_t = pt \) and \( \dot{p}_t = 0 \), suppose now instead we seek the nonlinear map \( \pi_{\pi_t^{(i)}}: pt \rightarrow qt \) and to determine its evolution. Thus in phase space we now have

\[
\left( \begin{array}{c} q \\ p \end{array} \right) = \left( \begin{array}{c} \pi_t^{(i)} \circ p \\ p \end{array} \right) = \left( \begin{array}{c} \pi_t^{(i)} \left( \begin{array}{c} q \\ p \end{array} \right) \end{array} \right) = \left( \begin{array}{c} \pi_t^{(i)} \end{array} \right) \circ p,
\]

which corresponds to \( \mathcal{S} = \{N+1, \ldots, 2N\} \) and represents another coordinate patch of the nonlinear graph manifold. Naturally in this case \( \pi_{\pi_t^{(i)}} = \pi_{\pi_t^{(i)}}^{-1} \). Using the underlying linear evolution equations for \( qt \) and \( pt \) we observe

\[
\dot{p}_t = p_t = \dot{q}_t = (\partial_t \pi_t^{(i)}) \circ p_t = (\partial_t \pi_t^{(i)}) \circ p_0.
\]

In other words, if we set \( y = p_0 \) we find \( \pi_t^{(i)} \circ y = p_t \circ y + ty \). For a well defined \( \pi_0^{(i)} \) this solution is always well-defined. The question is, can we use such coordinate patches to define patches in space-time using which we can construct a generalised notion of solution to the inviscid Burgers equation for all \( t > 0 \)? We intend to investigate this question in detail elsewhere.

**Remark 13** We make the following further formal observations:

(i) **Generalised inviscid Burgers models.** The evolution equations for \( q = q(a, t) \) and \( p = p(a, t) \) can be generalised as follows. Assume that

\[
\dot{q} = A(q, p, t)q + B(q, p, t)p,
\]
\[ \dot{p} = C(q, p, t)q + D(q, p, t)p, \]

where \( A, B, C \) and \( D \) are commensurate matrix-valued nonlinear functions of \( q \) and \( p \). Then if \( p = \pi(q, t) \) we observe that \( Cq + Dp = \dot{\pi} = \partial_t \pi + (\nabla_\pi \pi)(Aq + Bp) \). Hence if \( x = q(a, t) \) then \( \pi = \pi(x, t) \) satisfies the first order nonlinear equation

\[ \partial_t \pi + (\nabla_\pi)(A(x, \pi, t)x + B(x, \pi, t)\pi) = C(x, \pi, t)x + D(x, \pi, t)\pi. \]

Being able to find an explicit solution relies on being able to determine \( q \) and then \( q^{-1} \) for fixed \( t \geq 0 \) explicitly. For example assume \( A, C \) and \( D \) are all zero but \( B = f(|\pi|) \) for some smooth scalar-valued function \( f \). In this instance \( \dot{p} = 0 \) and the evolution of \( q \) is given by \( q(a, t) = a + tf(|\pi(a, t)|^2)\pi(a) \). Thus provided we can solve the nonlinear algebraic relation \( x = q(a, t) \) for \( a \), thus establishing \( q^{-1} \), then \( \pi(x, t) = \pi_0(q^{-1}(x)) \) is the solution to the equation

\[ \partial_t \pi + (\nabla_\pi)f(|\pi|) = 0. \]

(ii) **Linear Riccati maps.** Suppose in (i) above for general coefficients \( A, B, C \) and \( D \), the Riccati map \( \pi \) is linear so \( p = \pi q \), i.e. \( \pi \) is a suitably commensurate multiplicative matrix. Then in this case \( \dot{p} = \pi q + \pi \dot{q} \) and the evolution of \( q \) and \( p \) can be replaced by the pair of equations for \( q \) and \( \pi \) given by \( \pi = A(q, \pi, t)q + B(q, \pi, t)\pi q \) and the Riccati equation \( \dot{\pi} = C(q, \pi, t) + D(q, \pi, t)\pi - \pi(A(q, \pi, t) + B(q, \pi, t)\pi) \). Note if \( A = A(t), B = B(t), C = C(t), D = D(t) \) then we obtain a closed Riccati ordinary differential system for \( \pi \), namely \( \dot{\pi} = C + D\pi - \pi(A + B\pi) \). Once we solve this, we can in principle use the solution to the corresponding ordinary differential equation for \( q \); also see Ledoux, Malham and Thümmler [48].

(iii) **Riccati flow is a linear inviscid Burgers subflow.** Let us examine a special case of (ii) in detail. Suppose \( \pi_R = \pi_R(t; y) \) is a Riccati flow satisfying the ordinary differential equation \( \partial_t \pi_R + \pi_R^2 = 0 \), with \( y \in \mathbb{R}^N \) a parameter. The flow \( \pi(x, t; y) = \pi_R(t; y)x \) is a linear subflow to the inviscid Burgers equation generated by the data \( \pi_0(a; y) = \pi_R(0; y)a \). This is established as follows. The inviscid Burgers solution corresponding to such initial data requires \( x = a + \tau \pi_R(0; y)a, \) i.e. \( a = (I_n + \tau \pi_R(0; y))^{-1}x \) where the inverse is a matrix inverse. Hence we see \( \pi(x, t; y) = \pi_R(t; y)a = \pi_R(t; y)(I_n + \tau \pi_R(0; y))^{-1}x \). We note \( \pi(x, t; y) \) is linear in \( x \) so we set \( \pi_R(t; y) := \pi_R(t; y) + \tau \pi_R(0; y) \) which is the solution to the Riccati equation mentioned. Indeed explicitly we find \( \partial_t \pi(x, t; y) = -\pi_R^2(t; y)x \) and naturally \( (\nabla_x \pi)\pi = \pi_R^2(t; y)x \).

(iv) **Generalised Riccati equations.** In Beck et al. [39] we assumed \( q \) and \( p \) were in general the integral kernels associated with Hilbert–Schmidt operators, say \( Q \) and \( P \) respectively, with the operators \( A, B, C \) and \( D \), see (i) or (ii) above, as either bounded operators or with \( B \) and \( D \) possibly differential operators. Further we assumed the Riccati relation was linear, indeed the operators associated with the kernels \( q \) and \( p \) were related via a Fredholm equation. More generally though we could also consider a nonlinear Fredholm equation relating the operators associated with the kernels \( q \) and \( p \), for example of the form \( P = \pi(Q, t) \), where the Hilbert–Schmidt valued kernel of the solution operator \( \pi \) satisfies a generalised Riccati evolution equation in the sense suggested by Byrnes [38].

A natural question is how the graph flows we considered for the inviscid Burgers equation might extend to the viscous Burgers equation. Briefly, suppose we consider particle paths labelled by \( a \in \mathbb{R}^N \) generated by the stochastic processes

\[ Q_t(a) = a + \int_0^t P_s(a) \, ds + \sqrt{2}\gamma B_t, \]
where \( \nu > 0 \) is a constant, \( B_t \) is an \( \mathbb{R}^N \)-valued standard Brownian motion and the precise form for the drift \( P_s = P_s(a) \) will be given presently. A representation for the solution \( \pi = \pi(x,t) \) with \( x \in \mathbb{R}^N \) can be obtained via the Itô chain rule as follows; see Constantin and Iyer [16].

**Lemma 3 (Burgers solution representation)** Assume for some \( T > 0 \) that \( \pi : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N \) is a smooth solution \( \pi = \pi(x,t) \) to the Burgers equation

\[
\partial_t \pi + (\nabla \pi) \pi + \nu \Delta \pi = 0,
\]

for \( t \in [0,T] \) with \( \pi(x,0) = \pi_0(x) \) for all \( x \in \mathbb{R}^N \) where \( \pi_0 \in C^\infty(\mathbb{R}^N;\mathbb{R}^N) \cap H^1(\mathbb{R}^N;\mathbb{R}^N) \). Further assume that \( Q_t = Q_t(a) \) is the stochastic process given by

\[
Q_t(a) = a + \int_0^t P_s(a) \, ds + \sqrt{2\nu} B_t,
\]

with the drift function given by \( P_t(a) = \pi(Q_t(a),t) \). Then the solution \( \pi = \pi(x,t) \) to the Burgers equation has the representation:

\[
\pi(x,t) = \mathbb{E}[\pi_0(Q_t^{-1}(x))].
\]

**Proof** By direct computation using the Itô chain rule, see Øksendal [58], we find

\[
\pi(Q_t(a),t) = \pi_0(a) + \int_0^t (\partial_s \pi + (\nabla \pi) \pi + \nu \Delta \pi)(Q_s(a),s) \, ds + \sqrt{2\nu} \int_0^t (\nabla \pi)(Q_s(a),s) \, dB_s.
\]

Setting \( x = Q_t(a) \), using \( \pi \) is a smooth solution to the Burgers equation and taking the expectation of both sides, we obtain the stated representation.

This last result is reminiscent of the method of characteristics for the inviscid Burgers equation, and indeed represents the natural extension of the method to this context. For a comprehensive account of this, see Busnello, Flandoli and Romito [8] and Constantin and Iyer [16][17].

**Remark 14 (Generalised graph flows)** In general we have complete freedom in the prescriptions we have given. For example, a theme we can identify is a functional relation between three dynamic variables with the dynamics of two of the variables prescribed. Hence, as an explicit example, suppose we choose the functional relation to be \( p = q(\pi) \) where \( p = p(x,t) \), \( \pi = \pi(x,t) \) and \( q = q(\pi) \) only, with all variables scalar and real. Further assume \( p \) satisfies the linear partial differential equation \( \partial_t p = \partial_2^2 p \). We will prescribe the constraint on the function variable \( q \) presently. Our goal is to determine the potential dynamics satisfied by \( \pi \). By direct computation we observe that \( \partial_t p = (\partial_\pi q) \partial_\pi \) and \( \partial_2 p = (\partial_\pi q)(\partial_2^2 \pi) + (\partial_2^2 q)(\partial_2 \pi)^2 \). Equating these two quantities and assuming \( \partial_\pi q \neq 0 \) we find that \( \partial_\pi \pi = \partial_2^2 \pi + (\partial_2^2 q)/(\partial_\pi q)(\partial_2 \pi)^2 \). Suppose we now prescribe \( q \) so that it satisfies the linear differential equation \( \partial_2^2 q = -c\partial_2 q \) for some real constant \( c \). This completes our prescription. One solution is \( q(\pi) = \exp(-c\pi) \). This represents the Cole–Hopf transformation for the Hamilton–Jacobi equation \( \partial_t \pi + c(\partial_2 \pi)^2 = \partial_2^2 \pi \) and thus potential solutions of Burgers’ equation; see Evans [29] p. 194 and Remark[10].
Remark 15 (Stochastic Burgers equation) In Lemma 3 we derived a Feynman–Kac solution representation for the Burgers equation solution as originally considered by Constantin and Iyer. There we assumed the smooth function \( \pi \) satisfied the backwards Burgers equation. We now show how to construct solutions to a stochastic Burgers equation using the Grassmann flow approach. The idea is to consider a backwards Burgers equation. We now show how to construct solutions to a stochastic partial differential equation. This approach also more closely matches the spirit of Constantin and Iyer. Assume \( N \in \mathbb{N} \) and consider the following prescription. Assume for each \( a \in \mathbb{R}^N \) the \( \mathbb{R}^N \)-valued stochastic processes \( Q = Q(a), P = P(a) \) and \( \pi = \pi(\cdot, t) \) satisfy

\[
Q_t(a) = a + \int_0^t P_s(a) \, ds + \sqrt{2} \nu B_t,
\]

\[
P_t(a) = P_0(a),
\]

\[
P_t(a) = \pi(Q_t(a), t),
\]

where \( P_0(a) = \pi_0(a) \) and \( \pi_0 : \mathbb{R}^N \to \mathbb{R}^N \) is a given function. Naturally \( \nu > 0 \) is a given diffusion constant and \( B_t \) is a standard \( \mathbb{R}^N \)-valued Brownian motion. We remark that the generalised momentum \( P_t(a) = P_0(a) \) is constant. Given this, we see from our prescription above that for a given \( a \in \mathbb{R}^N \) the generalised coordinate \( Q_t = Q_t(a) \) is a stochastic process satisfying \( Q_t(a) = a + t \pi_0(a) + \sqrt{2} \nu B_t \). In the prescription, the nonlinear map \( \pi = \pi(\cdot, t) \) is assumed to be a stochastic process that is twice continuously differentiable with respect to its first argument and a \( C^1 \)-semimartingale; see Kunita for more details. From Kunita pg. 54 or Constantin and Iyer we have the generalised Itô formula: Given a continuous semimartingale \( Q_t \), then \( \pi = \pi(Q_t, t) \) satisfies the equation

\[
\pi(Q_t, t) = \pi_0(Q_0) + \int_0^t \pi(Q_s, ds) + \int_0^t (\nabla \pi)(Q_s, s) \, dQ_s + \frac{1}{2} \int_0^t \text{tr} \left( (\nabla \nabla^T \pi)(Q_s, s) \, d\|Q_s Q_s^T\|_s \right) + \left\langle Q_t, \int_0^t (\nabla \pi)(Q_s, ds) \right\rangle.
\]

Here \( \langle \cdot, \cdot \rangle \) is the quadratic covariation. We now follow the argument in Constantin and Iyer closely. As mentioned above, \( Q_t(a) = a + t \pi_0(a) + \sqrt{2} \nu B_t \), so \( \langle Q, Q^T \rangle_s = 2s \nu \) id. Since \( P_t(a) = P_0(a) \) we know \( \pi(Q_{t_2}(a), t_2) = \pi(Q_{t_1}(a), t_1) = \pi_0(a) \). Hence for any two times \( t_1 \) and \( t_2 \) we observe that

\[
\int_{t_1}^{t_2} \pi(Q_s, ds) + \int_{t_1}^{t_2} \left( \left\langle \left( \nabla \pi \right)(Q_s(a), s) \right\rangle \pi_0(a) + \left( \nu \Delta \pi \right)(Q_s(a), s) \right) ds + \left\langle Q_{t_2}(a) - Q_{t_1}(a), \int_{t_1}^{t_2} (\nabla \pi)(Q_s(a), ds) \right\rangle + \sqrt{2} \nu \int_{t_1}^{t_2} (\nabla \pi)(Q_s(a), s) \, dB_s
\]

equals zero. Keeping track of the finite variation terms, by direct computation, see Constantin and Iyer, we know

\[
\left\langle Q_{t_2}(a) - Q_{t_1}(a), \int_{t_1}^{t_2} (\nabla \pi)(Q_s(a), ds) \right\rangle = -2\nu \int_{t_1}^{t_2} (\Delta \pi)(Q_s(a), s) \, ds.
\]
Since these relations hold for all \( t_1 \) and \( t_2 \), and \( \pi_0(a) = \pi(Q_t(a), t) \), we observe if we set \( Q_t(a) = x \) then \( \pi = \pi(x, t) \) satisfies the stochastic Burgers equation:

\[
\mathrm{d} \pi + \left( (\nabla \pi) \pi - \nu \Delta \pi \right) \mathrm{d} t + \sqrt{2 \nu} (\nabla \pi) \mathrm{d} B_t = 0.
\]

How to interpret the solution to this stochastic partial differential equation in terms of the paths \( Q \) is discussed in Busnello et al. [8] and Constantin and Iyer [17]. We remark with regards to the particle paths for a given \( a \in \mathbb{R}^N \), assuming the driving paths \( B_t \) are differentiable for the moment, the dynamics of \( Q_t(a) \) and \( P_t(a) \) is Lagrangian \( L(Q_t, t) = \frac{1}{2} \| \dot{Q}_t - \sqrt{2 \nu} \dot{B}_t \|^2 \) and Hamiltonian \( H(P_t, t) = \frac{1}{2} \| P_t \|^2 - \sqrt{2 \nu} P_t \dot{B}_t \) (which is neither conserved, nor the total energy).

4 Evolutionary stochastic fields with nonlocal nonlinearities

Herein we look to generalise the Grassmannian flow approach for nonlocal nonlinear partial differential equations in Beck et al. [2,3] further, and formally extend the Grassmannian flow approach to classes of stochastic partial differential equations with nonlocal ‘big matrix’ nonlinearities, in the case of a specific multiplicative noise. Indeed we consider the following stochastic partial differential equation for \( g = g(x, y; t) \) with data \( g(x, y; 0) = g_0(x, y) \):

\[
\partial_t g = \alpha \partial_x^2 g + \gamma W * g - \epsilon g * g.
\]

We assume periodic boundary conditions in both \( x \) and \( y \), with period \( 2\pi \). Indeed suppose the periodic domain in \( \mathbb{T} := [-\pi, \pi]^2 \). The expression \( \partial_t \) denotes the partial derivative with respect to the first spatial variable argument. The parameters \( \alpha > 0 \), \( \beta \in \mathbb{R} \), \( \gamma > 0 \) and \( \epsilon \in \mathbb{R} \) are constant, while \( W = W(x, t) \) is a space-time standard Brownian sheet, periodic in \( x \), with period \( 2\pi \). The products ‘\( \star \)' and ‘\( \ast \)' are the convolution and ‘big matrix’ products, respectively, as discussed previously, though here in the context of a periodic domain. To be precise, for any two functions \( f = f(x) \) and \( g = g(x, y) \) we define,

\[
(f \ast g)(x, y) := \int_{-\pi}^{\pi} f(z) g(x - z, y) \mathrm{d} z,
\]

while for any two functions \( f = f(x, y) \) and \( g = g(x, y) \) we define,

\[
(f \ast g)(x, y) := \int_{-\pi}^{\pi} f(x, z) g(z, y) \mathrm{d} z.
\]

On the periodic domain \( \mathbb{T} \), the Fourier series representations for the Brownian sheet \( W = W(x, t) \) and solution \( g = g(x, y; t) \) are respectively as follows:

\[
W(x, t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} w_k e^{ikx} \quad \text{and} \quad g(x, y; t) = \sum_{(k, \kappa) \in \mathbb{Z}^2} g_t(k, \kappa) e^{i(kx + \kappa y)}.
\]

Here the Fourier coefficients of \( g = g(\cdot, \cdot; t) \) are \( g_t(k, \kappa) \). From standard theory, the Fourier coefficients of \( W = W(\cdot, \cdot; t) \) are, for \( k \neq 0 \), given by,

\[
w_t(k) = \begin{cases} (X_t^k - iY_t^k)/(2\sqrt{\kappa(k)}), & k \in \mathbb{N}, \\ (X_t^{-k} + iY_t^{-k})/(2\sqrt{\kappa(-k)}), & k \in \{-\mathbb{N}\}, \end{cases}
\]
where $X^k_t$ and $Y^k_t$, for $k \in \mathbb{N}$, are independent standard Brownian motions. We take $w_t(0) \equiv 0$. Hereafter we suppose the Fourier coefficients of $g$ are encoded in the infinite matrix $\mathbf{g} = \mathbf{g}_t$ parametrised by $(k, \kappa) \in \mathbb{Z}^2$ and the Fourier coefficients of $W$ are encoded in the infinite vector $\mathbf{w} = \mathbf{w}_t$ parametrised by $k \in \mathbb{Z}$. If we substitute the Fourier series expansions above for $g$ and $W$ into the stochastic partial differential equation with the ’big matrix’ nonlocal nonlinearity above, then we find that an equivalent formulation for the evolution equation is,

$$\partial_t \mathbf{g} = -\alpha K^2 \mathbf{g} + 2\pi \gamma \text{diag}(\mathbf{w}) \mathbf{g} - 2\pi \epsilon \mathbf{g}(R_1 \mathbf{g}),$$

Here, $K$ is the infinite diagonal matrix of Fourier numbers $k \in \mathbb{Z}$, and by direct calculation, the convolution multiplicative Brownian sheet term is replaced by the term $'2\pi \gamma \text{diag}(\mathbf{w})g'$ where $\text{diag}(\mathbf{w})$ is the diagonal matrix with the components of the time derivatives of $\mathbf{w}_t(k)$ down the diagonal. This is the result of the fact that the convolution product $'W \ast g'$ translates to a pointwise in $k$ product of the Fourier coefficients. The product between $\text{diag}(\mathbf{w})$ and $\mathbf{g}$ is just the (infinite) matrix product. Also observe that the ’big matrix’ product term is replaced by the term $'2\pi \epsilon \mathbf{g}(R_1 \mathbf{g})'$ where the operator $R_1$ acts on the matrix $\mathbf{g} = \mathbf{g}_t$ as follows: $R_2: \mathbf{g}(k, \kappa) \mapsto \mathbf{g}(-k, \kappa)$.

This is a result of the fact that, by direct computation, the $(k, \kappa)$th Fourier coefficient of $g \ast g$ is,

$$\sum_{k' \in \mathbb{Z}} \mathbf{g}_t(k, k') \mathbf{g}_t(-k', \kappa) \equiv \mathbf{g}(R_1 \mathbf{g}),$$

where the sum over $k' \in \mathbb{Z}$ is naturally encoded by the (infinite) matrix product between $\mathbf{g}$ and $R_1 \mathbf{g}$. Note the initial data associated with the evolution equation for $\mathbf{g} = \mathbf{g}_0$ above is $g_0$, which is the infinite matrix of Fourier coefficients of $g_0 = g_0(x, y)$.

The Grassmannian flow prescription for the Fourier form of the stochastic evolution equation above is as follows. Consider the prescription of linear equations:

$$\partial_t \mathbf{p} = -\alpha K^2 \mathbf{p} + 2\pi \gamma \text{diag}(\mathbf{w})\mathbf{p},$$
$$\partial_t \mathbf{q} = 2\pi \epsilon (R_1 \mathbf{p}),$$
$$\mathbf{p} = \mathbf{g}_0.$$  

Here $\mathbf{p} = \mathbf{p}_t$, $\mathbf{q} = \mathbf{q}_t$ and $\mathbf{g} = \mathbf{g}_t$ are all infinite matrices parametrised by $(k, \kappa) \in \mathbb{Z}^2$, the product between $\mathbf{g}$ and $\mathbf{q}$ is an (infinite) matrix product, and we assume $\mathbf{p}_0 = \mathbf{g}_0$ and $\mathbf{q}_0 = \text{id}$, the infinite identity matrix. We assume a solution $\mathbf{p} = \mathbf{p}_t$ to the linear stochastic partial differential equation shown exists for $t \in [0, T]$ for some $T > 0$. That indeed, the solution field $\mathbf{g} = \mathbf{g}_t$ to this system of linear equations satisfies the Fourier form of the stochastic evolution equation above follows by direct straightforward computation using that $R_1(\mathbf{g}_t) \equiv (R_1 \mathbf{g})_t$. Since $\mathbf{q}_0 = \text{id}$ and $\mathbf{p}_0 = \mathbf{g}_0$, the solution $\mathbf{g} = \mathbf{g}_t$ to the linear system above, satisfies the initial condition, and for $t \in [0, T']$ with $0 < T' < T$, we know $\mathbf{g} = \mathbf{p}(\mathbf{q})^{-1}$ exists.

We now demonstrate how to numerically solve this stochastic partial differential equation with the nonlocal nonlinearity shown for $g = g(x, y; t)$ using the Grassmannian flow approach. For these simulations we set $\alpha = 1$, $\gamma = 10$ and $\epsilon = 1000$. We provide two independent simulations: (i) Direct numerical simulation using an exponential spectral algorithm similar to an approximate mild formulation of the stochastic equation, advancing the solution $u_m$ in successive time steps; (ii) Generation of an
approximate solution \( g_{\text{approx}} \) using the Grassmann flow approach via the Fourier prescription given above. We first explain the direct numerical simulation approach. We chose an initial profile \( g_0 = g_0(x, y) \) to be
\[
    g_0(x, y) = \text{sech}(10(x + y - 2\pi)) \text{sech}(10(y - \pi)),
\]

Together with some initial added noise we describe presently. We computed the two-dimensional discrete Fourier transform of the initial data on \( 2^5 \) modes in each spatial direction, adding 0.001 times an independent standard normal random variable to each mode. The total number of time steps was \( 2^8 \). Rather crudely, to generate the numerical algorithm for the stochastic equation with the nonlocal nonlinearity concerned, we proceeded as follows, using the Fourier form of the stochastic evolution equation above, though with only \( 2^5 \) Fourier modes. Over a computation step \([t_m, t_m + \Delta t]\) where \( \Delta t = T/2^8 \), we used the integrating factor technique to integrate the discrete form of the diffusive term, corresponding to the first term in the evolution equation, to generate an equivalent approximate mild formulation for the equation. We used standard approximations for the time integrals for this approximate mild formulation, i.e. we froze the vector fields at the beginning of each computation step \([t_m, t_m + \Delta t]\). Thus overall, the numerical scheme we used to advance the approximate stochastic equation solution in Fourier space \( u_{m+1} = u_{m}(k, \kappa) \) forward in time is as follows:
\[
    u_{m+1} = \exp(-\Delta t K^2) (u_m + 2\pi \gamma \text{diag}(\Delta \mathbf{w}_m)u_m) - 2\pi \epsilon \Delta t \text{dexp}(-\Delta t K^2) u_m(R_1 u_m).
\]

Note that \( \text{diag}(\Delta \mathbf{w}_m) \) represents the diagonal matrix of time increments of each mode of \( \mathbf{w} \) across \([t_m, t_m + \Delta t]\). All the products in the time-stepping scheme are matrix products. The final computation time was \( T = 0.007 \). The result for a single realised Brownian sheet, translated back into physical space, is shown in the left panel in Figure 1. In the Grassmann flow approach, given the initial profile \( p_0 = g_0 \) we advance the solution \( p = p_t(k, \kappa) \) in Fourier space, using Itô’s Lemma, to
\[
    p_t = \exp(-t \alpha K^2 + 2\pi \gamma \text{diag}(\mathbf{w}_t) - t \pi \gamma^2 K^{-2}) p_0.
\]
Special care is required for the $k = 0$ mode. And note for comparison purposes we take $\mathbf{w}$ to be the same Brownian sheet we used for the direct numerical simulation above. An important issue now arises. To compute $q$ we need to time-integrate the differential equation $\partial_t q = 2\pi \epsilon R_1 p$ governing its evolution with $q_0 = id$. We can only achieve this by numerical quadrature and we thus require the evaluation of $p = p_T$ at sufficiently close intervening times in $[0, T]$ in order to evaluate $q_T$ sufficiently accurately. This is of course perfectly feasible but means that we lose the efficiency of the Grassmannian flow approach. Details on exact solvability of some stochastic partial differential equations can be found in Corwin [19]. Ploughing on, having accurately computed $p_T$ and $q_T$, we then solve the matrix equation $p_T = q_T q_T$ to generate a suitable approximation for $g_T$. The result, for the same single realised Brownian sheet as that for the direct simulation approach, when translated back into physical space, is shown in the right panel in Figure 1. The two results match up remarkably well.

5 Discussion

We have outlined how Smoluchowski’s coagulation model in the constant frequency case, is an example of a Grassmannian flow and is therefore linearisable. Indeed we demonstrated a class of generalised Smoluchowski-type equations are Grassmannian flows. These represent additional classes of Grassmannian flows to those considered in Beck et al. [2,3]. We also showed how the Grassmannian flow approach can also be used to solve (Examples 2, 5 and 6) or partially solve (Example 4) other practical coagulation flows that do not fit into either of the two classes just mentioned. For the Smoluchowskii coagulation model in the case of additive or multiplicative frequency kernel cases, we generalised the notion of Grassmannian flows to nonlinear graph flows and used the latter context to find solutions. Further, we also demonstrated how a stochastic partial differential equation with a nonlocal nonlinearity can be considered as a Grassmannian flow, and thus be linearised. There are many more classes of flows that are Grassmannian flows. In Appendix B we show how some general classes of nonlinear elliptic flows can be interpreted as stationary Riccati flows and thus fit into the Grassmannian context we have outlined. In Appendix C we demonstrate classes of partial differential systems with anisotropic diffusion and nonlocal nonlinearities are Grassmannian flows which in particular have quotient solutions. In our companion paper Doikou et al. [23], we also reveal the connection between Fredholm Grassmannian flows and classical noncommutative integrable systems such as the potential Korteweg–de Vries and nonlinear Schrödinger equations. Also see Doikou et al. [22] and Malham [52,51].

Let us now outline how we can further utilise our Grassmannian flow approach to integrate further nonlinear systems. Indeed we can extend the Grassmannian flows we consider above by building systems upon systems. By this we mean, now we know we can solve some nonlinear evolutionary partial differential equations via the Grassmannian flow approach, we can assume our base equations are such nonlinear evolution equations and construct more elaborate programmes for solving more sophisticated nonlinear equations. Recall, for example, the multiple mergers Example 5 in Section 2. Here, to demonstrate, as a first further example, consider the following prescription:

$$
\partial_t p = p \ast d \ast p,
$$
$$
\partial_t q = \Delta q,
$$
\[ p = g * q, \]

where \( p = p(x; t), q = q(x; t) \) and \( g = g(x; t) \) are assumed to be all sufficiently smooth functions in space \( x \in \mathbb{R}^N \) and time \( t \in [0, T] \) for some \( N \in \mathbb{N} \) and \( T > 0 \) for our subsequent arguments to make sense. Further, \( d \) is a function that will be chosen presently while ‘\(*\)’ is the convolution product on \( \mathbb{R}^N \). We note that if \( q \) and \( p \) denote the Fourier transforms of \( q \) and \( p \) respectively, then \( q(k; t) = \exp(i \pi k^2 t)q(k; 0) \) while \( \partial_t p = p \ast p \), which is a scalar Riccati equation. We can either solve this directly or equivalently via the Grassmannian flow approach we advocate to find \( p = p(k; t) \) explicitly. By a straightforward calculation taking the time derivative and using the product rule, integration by parts on the Laplacian term as well as the Inverse Kernel Lemma \( \text{[2]} \) we see \( g = g(x; t) \) satisfies the partial differential equation with nonlocal nonlinearity:

\[ \partial_t g = \Delta g + g \ast q \ast d \ast g. \]

Note by an appropriate choice of \( d \) we fit \( q \ast d \) to be any function including, by invoking the Inverse Kernel Lemma \( \text{[2]} \) again, the Dirac delta function.

As a second further example, consider the following prescription:

\[
\begin{align*}
\partial_t p &= \Delta_1 p + p \ast d \ast p, \\
\partial_t q &= -\Delta_1 q + b p, \\
p &= g \ast q,
\end{align*}
\]

where \( p = p(x, y; t), q = q(x, y; t) \) and \( g = g(x, y; t) \) are assumed to be sufficiently smooth functions with respect to the space \( x, y \in \mathbb{R}^N \) and time \( t \in [0, T] \) parameters for some \( N \in \mathbb{N} \) and \( T > 0 \) for our subsequent arguments to make sense. Here \( \Delta_1 \) denotes the Laplacian with respect to the first argument, \( b = b(x) \) and \( d = d(x, y) \) are functions chosen presently and the product ‘\(*\)’ represents the ‘big matrix’ product.

The evolutionary partial differential equation with nonlocal nonlinearity prescribing \( p = p(x, y; t) \) above can be linearised in the Grassmann flow sense as follows. Assume the prescription: \( \partial_t p' = \Delta_1 p', \partial_t q' = -b' \ast p' \) and \( p' = g' \ast q' \) for \( p' = p'(x, y; t), q' = q'(x, y; t) \) and \( g' = g'(x, y; t) \), and where \( b' = b'(x, y) \). Then by direct calculation we observe \( g' \) satisfies the evolutionary partial differential equation with nonlocal nonlinearity \( \partial_t g' = \Delta_1 g' + g' \ast b' \ast g' \) which is precisely the equation governing the evolution of \( p \) above. Also note now that given \( p = p(x, y; t) \) we can substitute it directly into the linear evolutionary partial differential equation prescribing \( q \) and in principle solve for \( q = q(x, y; t) \) explicitly. Again by direct calculation, differentiating \( p = g \ast q \) with respect to time using the product rule, using integration by parts for the Laplacian term emanating from the equation for \( q \) and postcomposing by the inverse operator to the operator corresponding to the kernel \( q \), we observe that \( g = g(x, y; t) \) satisfies the evolutionary partial differential equation with nonlocal nonlinearity

\[ \partial_t g = \Delta g + g \ast (q \ast d \ast g - b g). \]

In this case if we wanted to choose \( d \) such that \( q \ast d = \delta \) then we would need to go back and solve the equations prescribing the evolution of \( p \) and \( q \) as a coupled system. An important development here should be emphasised though, which is independent of \( d \): this equation with nonlocal nonlinearity now has a general isotropic Laplacian term as opposed to the anisotropic diffusion term prevalent in the analogous examples considered in Beck et al. \( \text{[2]} \). This is due to the term \( \Delta_1 g \) in the linear evolution equation for \( q \), though unfortunately, this corresponds to a ‘reverse diffusion’ effect.
We remark that the Smoluchowski coagulation models we considered in Section 2 are underpinned by stochastic processes, see Aldous [1]. For example, the nested Kingman coalescent underlies the genealogy model, see Lambert and Schertzer [47], while a McKean–Vlasov process underlies the Derrida–Retaux model, see Hu et al. [38]. Monte Carlo methods based on such processes are a natural numerical approach to simulating these equations. Such methods are flexible and allow for general kernels, for example $k(x, y) = (x^\alpha + y^\alpha)(x^{-\alpha} + y^{-\alpha})$, which arise in applications.

For many Smoluchowski coagulation models only some of the separate vector fields may be integrable. For example, in Example 4 we saw for the genealogy model of Lambert and Schertzer [47] that separately, the first vector field $V_1$ was integrable and the second vector field $V_2$ generated a Grassmannian flow and was thus integrable as well. However $V_1 + V_2$ is not obviously integrable. As we outlined in Example 4 in such cases we can construct accurate and efficient numerical splitting methods which utilise the integrability of the separate vector fields. Another instance of such a circumstance is in Smoluchowski-ripening or ‘island coarsening’ in nanotechnology, see Stoldt et al. [6]. The frequency kernel in models of this phenomenon is typically assumed to have the form $k(x, y) = k_1(x) + k_2(y)$. The loss vector field $-g(x; t) \int_0^\infty k_2(y)g(y; t)\,dy$ in this case is exactly solvable. Utilising this knowledge alongside classical numerical schemes for such coagulation equations, see for example Keck and Bortz [41] or Carr et al. [13], might improve simulation complexity for such models.

Lastly, McKean [55] proved that the solution to the Fisher–Kolmogorov–Petrovskii–Piskunov equation \( \partial_t w = \partial^2_x w + w^2 - w \) can be expressed in the form

$$w(x, t) = \mathbb{E}\left[ \prod w_0(Y_{t}^i) \right],$$

for any initial data satisfying $0 \leq w_0 \leq 1$, $x \in \mathbb{R}^N$. The product is over all individual particles alive at time $t$. The process $Y_t$ represents a branched Brownian motion where individual Brownian motions develop branches at times given by a particular distribution. See McKean [55], Dynkin [26] or Etheridge [28] for more details. This is a very natural representation of the actual underlying chemical autocatalyst reaction. Together with the results of Henry–Labordère, Oudjane, Tan, Touzi and Warin [45], Henry–Labordère and Touzi [46] and Thieullen and Vigot [71] these represent very interesting further directions of investigation for evolutionary nonlinear partial differential equations.

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A Proof of the Initial Value Theorem

For completeness, we give the proof of the Laplace transform Initial Value Theorem 2 in Section 2. The proof relies on the generalised Riemann–Lebesgue Lemma. See Lundberg et al. [50] for a detailed account. We follow their proof. Let us first prove the basic case. For any function \( f, f' \in L^1_{\text{loc}}([0, \infty); \mathbb{R}) \) and both are of exponential order, or \( f, f' \in L^1([0, \infty); \mathbb{R}) \), the following partial integration formula holds for Re(\( s \)) \( \geq 0 \),

\[
\int_0^\infty f'(x)e^{-sx} \, dx = -f(0^-) + s \int_0^\infty f(x)e^{-sx} \, dx.
\]

Hence we observe that for Re(\( s \)) \( \geq 0 \),

\[
sf(s) = \int_0^\infty f'(x)e^{-sx} \, dx = f(0^-) + s \int_0^\infty f(x)e^{-sx} \, dx.
\]

We now take the limit \( s \to \infty \) with Re(\( s \)) \( \geq 0 \) on both sides. Set \( s = R e^{i \theta} \) with \( \theta \in [-\pi/2, \pi/2] \). First suppose \( \theta \in (-\pi/2, \pi/2) \). Then focusing on the limit for the integral on the right-hand side, we observe that,

\[
\lim_{s \to \infty} \int_0^\infty f'(x)e^{-sx} \, dx = \lim_{R \to \infty} \int_0^\infty f'(x)e^{-Rx(e^{i \theta} + i \sin \theta)} \, dx
\]
\[ \lim_{R \to \infty} \int_{0^+}^{R} |f'(x)| e^{-Rx \cos \theta} \, dx, \]

which converges to zero by the Dominated Convergence Theorem since \( e^{-Rx \cos \theta} \to 0 \) as \( R \to \infty \), pointwise in \( x \) for \( x > 0 \) and \( \theta \in (-\pi/2, \pi/2) \). Now suppose \( \theta = \pm \pi/2 \). Then we observe that,

\[
\lim_{R \to \infty} \int_{0^+}^{R} f'(x) e^{-Rx \cos \theta} \, dx = \lim_{R \to \infty} \int_{0^+}^{R} f'(x) e^{\pm iRx} \, dx,
\]

which converges to zero by the Riemann–Lebesgue Lemma since in this case we assume \( f' \in L^1([0, \infty); \mathbb{R}) \). This completes the proof of the basic result. Now suppose that for all \( \ell \in \{0, 1, \ldots, n\}, f^{(\ell)} \in L^1_{loc}([0, \infty); \mathbb{R}) \) and of exponential order, or \( f^{(\ell)} \in L^1([0, \infty); \mathbb{R}) \). In either case the following partial integration formula holds for \( \text{Re}(s) > 0 \) and any \( \ell \in \{1, 2, \ldots, n\}, \)

\[
\int_{0^-}^{\infty} f^{(\ell)}(x)e^{-sx} \, dx = -f^{(\ell-1)}(0^-) + s \int_{0^-}^{\infty} f^{(\ell-1)}(x)e^{-sx} \, dx
\]

\[
\Leftrightarrow \quad s^{\ell-1} f(s) - \sum_{i=0}^{\ell-2} s^{\ell-2} f^{(l-2)}(0^-) = \int_{0^-}^{\infty} f^{(\ell)}(x)e^{-sx} \, dx + f^{(\ell-1)}(0^-).
\]

Now, from the same juncture, repeating the arguments for the basic case, but with \( f' \) replaced by \( f^{(\ell)} \), establishes the more general result.

## B Nonlinear elliptic systems

Herein we briefly explore how the Grassmannian flow prescription we developed above can be adapted to linearise and construct solutions, here rational solutions, to specific classes of nonlinear elliptic systems. Suppose for \( x \in \mathbb{R}^N \) with \( N \in \mathbb{N} \) that \( a = a(x) \) and \( d = d(x) \) are \( \mathbb{R}^N \)-valued smooth functions, while \( b = b(x) \) and \( c = c(x) \) are real-valued smooth functions. Consider the following prescription.

### Definition 7 (Prescription: Nonlinear elliptic system)

Assume the the real-valued function \( q = q(x) \), and the \( \mathbb{R}^N \)-valued functions \( p = p(x) \) and \( g = g(x) \) satisfy

\[
\nabla q = aq + bp,
\]

\[
\nabla^T p = cq + d^T p,
\]

\[
p = gq,
\]

where we also assume \( b \) is non-zero on \( \mathbb{R} \).

### Proposition 2 (Nonlinear elliptic system decomposition)

**Assuming there exist smooth solution functions \( q = q(x) \in \mathbb{R}, p = p(x) \in \mathbb{R}^N \) to the pair of first order differential equations given in the prescription in Definition 7 with \( q \) non-zero on \( \mathbb{R} \), then the function \( g = g(x) \) satisfies the partial differential equation:**

\[
\nabla^T g = c + d^T g - g^T a - g^T bq.
\]

**Furthermore the solution to this equation is given by \( g = pq \).**

**Proof** Using the product rule to compute the \( \nabla^T \) derivative of \( pq \) we find

\[
\nabla^T p = (\nabla^T p)q + g^T \nabla q \quad \Leftrightarrow \quad cq + d^T gq = (\nabla^T p)q + g^T (a + bq)q.
\]

Dividing by \( q \neq 0 \) and rearranging the terms gives the result. \( \square \)

This abstract result aligns with our goals herein when we assume \( g = g(x) \) is given by \( g = D\nabla \phi \) where \( D = D(x) \) is a non-singular \( \mathbb{R}^{d \times d} \) diffusion matrix and \( \phi = \phi(x) \) is a scalar potential function. Define the coefficient functions \( d' = D^T d(x), a' = D^T a(x) \) and \( b' = D^T b(x) \). Then we see that \( \phi = \phi(x) \) satisfies the nonlinear elliptic equation

\[
\nabla^T (D\nabla \phi) = c + (d')^T \nabla \phi - (\nabla \phi)^T a' - (\nabla \phi)^T b'(\nabla \phi).
\]
Now the question is whether our prescription above can be used to solve such a nonlinear elliptic equation? Consider the case when \( a = a' = 0, D(x) = \varepsilon(x)I_n \) for some scalar-valued function \( \varepsilon(x) \) and \( c = 0 \). In this case the nonlinear elliptic equation reduces to the following case (recall \( b \) is scalar):
\[
\nabla^T (\varepsilon \nabla \phi) = \varepsilon d^T \nabla \phi - \varepsilon b |\nabla \phi|^2.
\]

According to our prescription we are required to solve the pair of first order equations \( \nabla \Phi = b p \) and \( \nabla^T p = d^T p \), which upon substituting the former into the latter amounts to solving the linear elliptic equation \( \nabla^T (b^{-1} \nabla q) = d^T (b^{-1} \nabla q) \). Hence assuming we can solve this linear elliptic equation for \( q \), which is the linearised form of the nonlinear elliptic equation above, we set \( p = b^{-1} \nabla q \) and solutions to the nonlinear elliptic equation are given by \( \varepsilon \nabla \Phi = p/q \).

**Remark 16 (Time-dependent system)** We can pursue our strategy above one step further as follows. Let us assume for \( x \in \mathbb{R}^N \) and \( t \geq 0 \) that \( a = a(x,t), c(x,t) \) are real-valued smooth functions. Now consider the following time-dependent prescription. Assume the the real-valued function \( q = q(x,t) \), the \( 
abla \) of existence, the linear elliptic equation? Consider the case when \( \Delta N \cap \text{Dom}(D) \cap \text{Dom}(\Delta) \) represents the domain of the operator \( D_x \), : \( \Delta \) is a polynomial of \( \Delta x \). A linear advection term such as \( v(x) \cdot \nabla_x g(x,y,t) \) for some given advection velocity \( v = v(x) \) can also be incorporated in \( d; \) as well as more general linear terms of this form. However we will not pursue such generalisations here. The function \( b = b(y) \) is assumed to be bounded and smooth. The linear terms involving \( D_x \) might correspond to anisotropic diffusion or dispersion.

Let us now be more precise. Assume for any \( N, M, n \in \mathbb{N} \) we are given initial data \( g_0 \in C^\infty(\mathbb{R}^{2n}; \mathbb{C}^{N \times M} \cap \text{Dom}(D)) \). Here \( \text{Dom}(D) \) represents the domain of the operator \( D_x \) in \( \mathbb{L}^2(\mathbb{R}^{2n}; \mathbb{C}^{N \times M}); \) this operator is prescribed more explicitly below. Our goal is to use a simple version of the Grassmann flow approach to construct for some \( T > 0 \) of existence, the corresponding solution \( g = g(x,y,t) \) with \( g \in C^\infty([0, T]; C^\infty(\mathbb{R}^{2n}; \mathbb{C}^{N \times M}) \cap \text{Dom}(D)) \) to either of the partial differential equations \( \partial_t g = D_x g - g \mathfrak{F} \) or \( \partial_t g = D_x g - \mathfrak{F} \mathfrak{F} \). We use the notation \( \mathfrak{F} \) and \( \mathfrak{F} \) to denote the corresponding functions evaluated along the diagonal so that \( \mathfrak{F}(y,t) := g(y,y,t) \) and \( \mathfrak{F}(y,t) := p(y,y,t) \). The operator \( D_x = d(\sqrt{\Delta x}) \) is a constant coefficient polynomial of \( \sqrt{\Delta x} \), while \( b \in C^\infty(\mathbb{R}^{2n}; \mathbb{C}^{M \times N}) \cap \mathbb{L}^2(\mathbb{R}^{2n}; \mathbb{C}^{M \times N}) \cap \mathbb{L}^\infty(\mathbb{R}^{2n}, \mathbb{C}^{M \times N}) \). The function \( F: \mathbb{R}^{2N \times M} \rightarrow \mathbb{M} \times M \) is assumed to have a power series representation with infinite radius of convergence and takes the form \( F(a) = \sum_{m \geq 0} \alpha_m a^m \). Here we assume

C Classes of nonlinear PDEs with quotient solutions

We consider classes of partial differential equations with nonlocal nonlinearities and anisotropic linear diffusion or dispersion that admit quotient solutions. They are examples of simple Grassmannian flows and are characterised as follows. We assume the complex matrix-valued functions \( g = g(x,y,t) \) with \( x, y \in \mathbb{R}^n \) for some \( n \in \mathbb{N} \) satisfy either of the following two equations:
\[
\partial_t g(x,y,t) = D_x g(x,y,t) - g(x,y,t)b(x)g(y,y,t),
\]
\[
\text{or}
\partial_t g(x,y,t) = D_x g(x,y,t) - g(x,y,t)F(g(y,y,t)g^\dagger(y,y,t)).
\]

In the second equation, the notation \( g^\dagger \) denotes the complex conjugate transpose of the matrix \( g \) and the function \( F \) is purely imaginary valued. In both equations the operator \( D_x = d(\sqrt{\Delta x}) \) where \( d \) is a polynomial of \( \sqrt{\Delta x} \). A linear advection term such as \( v(x) \cdot \nabla_x g(x,y,t) \) for some given advection velocity \( v = v(x) \) can also be incorporated in \( d; \) as well as more general linear terms of this form. However we will not pursue such generalisations here. The function \( b = b(y) \) is assumed to be bounded and smooth. The linear terms involving \( D_x \) might correspond to anisotropic diffusion or dispersion.
Definition 8 (Prescription: PDE with anisotropic diffusion) For the linear operators \( d \) and \( b \) described above, assume the functions \( p = p(x, y; t) \in C_{N \times M}^{\infty} \), \( q = q(x; t) \in C_{M \times M}^{\infty} \) with \( q(0) = I_M \) and \( g = g(x; t) \in C_{M \times M}^{\infty} \) satisfy the system of linear equations

\[
\partial_q p = D_q p, \quad \partial_q q = b. \quad p = gq.
\]

Remark 18 (Higher odd degree nonlinearity prescription) For the higher odd degree example we replace the linear equation for \( q \) in \( C_{M \times M}^{\infty} \) by \( \partial_q q = F(q, p^q) \).

Lemma 4 (Existence and Uniqueness: Anisotropic PDE prescription) Assume that \( g_0 \in C_{N \times M}^{\infty}(\mathbb{R}^{2n}; C_{N \times M}^{\infty}) \cap \text{Dom}(D) \) is given and that there exists a time \( T > 0 \) such that \( p \in C_{N \times M}^{\infty}(0, T); C_{N \times M}^{\infty}(\mathbb{R}^{2n}; q, C_{M \times M}^{\infty}) \cap \text{Dom}(D) \) and \( \hat{q} \in C_{N \times M}^{\infty}(0, T); C_{M \times M}^{\infty}(\mathbb{R}^{2n}; q, C_{M \times M}^{\infty}) \) satisfy the system of linear differential equations given in the prescription for the PDE with anisotropic diffusion, with \( \hat{q}(0) = O. \) Then there exists a function \( q = g(x, y; t) \) such that \( g \in C_{M \times M}^{\infty}(0, T); C_{M \times M}^{\infty}(\mathbb{R}^{2n}; q, C_{M \times M}^{\infty}) \). The corresponding existence and uniqueness Lemma \([3]\) and/or Lemma 2.4 in Beck \(\text{et al.} [23]\) and/or Lemma 2.4 in Beck \(\text{et al.} [3]\).

Remark 19 (Initial data) Since we set \( q(0) = I_M \), consistent with the Riccati relation we hereafter set \( p(x, y; 0) = g_0(x, y) \).

Our main result is as follows.

Proposition 3 (PDE with anisotropic diffusion decomposition) Given the data \( g_0 \in C_{N \times M}^{\infty}(\mathbb{R}^{2n}; C_{N \times M}^{\infty}) \cap \text{Dom}(D) \), we set \( q(0) = I_M \) and \( p(x, y; 0) = g_0(x, y) \). Assume that \( p, \hat{q} \) and \( g \) satisfy the complete system of linear equations given in the prescription in Definition 8 and the assumptions of the existence and uniqueness Lemma 4. Then for some \( T > 0 \) the function \( q = g(x, y; t) \) satisfies \( g(x, y; 0) = g_0(x, y) \) and the evolution equation for every \( t \in [0, T] \):

\[
\partial_t q = D_q g - gb(q)q.
\]

Proof By direct computation, differentiating \( p(x, y; t) = q(x, y; t)q(y; t) \) with respect to time using the product rule and use the linear equations for \( p \) and \( q \) as follows:

\[
(\partial_t q)q = \partial_q p - gb(q)q = D_q p - gb(q)q.
\]

Postmultiplying by \( g^{-1} \) establishes the result.

Remark 20 (Higher odd degree nonlinearity) The corresponding existence and uniqueness result for the higher odd degree equation example follows from arguments analogous to those above combined with arguments adapted to this simpler case from Beck \(\text{et al.} [3]\). The corresponding decomposition proof for this case where \( \partial_q q = D_q q - g(F(q, p^q)) \) follows as follows.

First, we establish \( q(0; t) = I_M \) implies \( q(0; t)q(y; t) = I_M \) for all \( t \in [0, T] \). By definition \( f' = f \). Using the product rule \( \partial_t q(y; t) = (qq'')(y; t) \). Thus \( qq' = I_M \) is a fixed point of this flow and \( q(0; t) = I_M \), for all \( y \in \mathbb{R} \), implies \( q(y; t)q(y; t) = I_M \) for all \( t \in [0, T] \) and \( y \in \mathbb{R} \). Second, from the Riccati relation and \( q(y; t)q(y; t) = I_M \), we deduce that \( p^{-1} = F(q, p^q) = Tq \) for all \( t \in [0, T] \). Thus \( \partial_t q = D_q q - gb(q)q \) for all \( y \in \mathbb{R} \). Hence we find \( F(q, p^q) = Tq \) for all \( t \in [0, T] \) and \( y \in \mathbb{R} \). Then as in the proof above, by direct computation, we differentiate the Riccati relation \( p(x, y; t) = g(x, y; t)q(y; t) \) with respect to time using the product rule and use the appropriate equations for \( p \) and \( q \). We see \( (\partial_t q)q = \partial_q p - gb(q)q = D_q p - gb(q)q = F(p^q)(q) = (D_q q)(q - g(F(q, p^q)))q \). Again postmultiplying by \( q^{-1} \) establishes the result.

We can find an explicit solution for the general case as follows. We extend to \( \mathbb{R}^n \) our notation for the Fourier transform, so the Fourier transform of \( p(x, y) \) with respect to \( x \in \mathbb{R}^n \) is denoted by \( p = p(k, y) \) with \( k \in \mathbb{R}^n \).
Lemma 5  Let \( p = p(x, y; t) \) and \( q = q(y; t) \) be the solutions to the linear base equations given in the prescription in Definition 8. Assume \( q(y; 0) \equiv I_M \) and \( p(x, y; 0) = p_0(x, y) \in \mathcal{C}^{N \times M} \). Then for all \( t \geq 0 \) the functions \( p \) and \( q \) are explicitly given by

\[
p(x, y; t) = \int_{\mathbb{R}^n} e^{d(2\pi|k|)t} p_0(k, y) e^{2\pi ik \cdot x} \, dk
\]

and

\[
q(y; t) = I_M + b(y) \int_{\mathbb{R}^n} \left( \frac{e^{d(2\pi|k|)t} - 1}{d(2\pi|k|)} \right) p_0(k, y) e^{2\pi ik \cdot y} \, dk.
\]

Proof By definition the operator \( \sqrt{-\Delta_x} \) corresponds to multiplication by \( 2\pi|k| \) in Fourier space. Hence taking the Fourier transform of the base equation generates the decoupled equation

\[
\partial_t p(k, y; t) = d(2\pi|k|) p(k, \kappa; t).
\]

Solving this equation and taking the inverse Fourier transform generates the solution form \( p = p(x, y; t) \) shown above. To find \( q = q(y; t) \) we substitute the expression for \( p = p(x, y; t) \) into the equation for \( q \) and integrate in time. \( \square \)

Lemma 6 (Explicit solutions) Given \( g_0 \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{C}^{N \times M}) \cap \text{Dom}(D) \), the solution to the nonlinear partial differential equation \( \partial_t g = D_x g - gb(\gamma \sigma) \), for \( g = g(x, y; t) \) with \( \sigma = \sigma(y, y; t) \) is given by \( g(x, y; t) = p(x, y; t)(q(y; t))^{-1} \), where \( p = p(x, y; t) \) and \( q = q(y; t) \) have the explicit forms shown in Lemma 5. This fact can be verified by direct substitution.

Remark 21 In the higher odd degree case we solve the first equation to determine \( p = p(x, y; t) \) and thus \( \sigma = p(y, y; t) \). We then substitute \( \sigma \) into the second equation so \( \mathfrak{F}(\mathfrak{F}(\mathfrak{F})) \) is a given coefficient function.