Effective characterization for stochastic differential equations with tempered stable Lévy fluctuations

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Abstract

This work is about the effective characterization for stochastic dynamical systems with tempered stable Lévy process. To quantify macroscopic or effective dynamical behaviors of these stochastic systems, we examine two deterministic tools: mean exit time and probability density evolution, which are solutions of nonlocal partial differential equations (nonlocal elliptic equation and nonlocal Fokker-Planck equation) respectively. We develop accurate numerical methods, together with stability and convergence analysis, to compute the mean exit time and Fokker-Planck equations associated with these one and two dimensional stochastic systems. We further illustrate these methods with numerical experiments in several examples.

Key words: Tempered stable Lévy process; Mean exit time; nonlocal Fokker-Planck equation; Non-Gaussian stochastic dynamical system; Numerical schemes

1 Introduction

Stable Lévy processes have been used to model non-Gaussian random fluctuations in many fields like biology, hydrology, climate dynamics, physics and economics ([1–4]). As a modification of the well-known α-stable Lévy process, a tempered stable Lévy process has the jump measure differing with the usual α-stable jump measure by an exponential decaying function, and is known to better simulate certain non-Gaussian random effects. A tempered Lévy process is also a generalization of the normal inverse Gaussian and variance gamma processes. Unlike the α-stable counterpart, a tempered stable Lévy process has mean, variance, and moments of all order. Rosinski [6] considered the properties of tempered stable Lévy processes, and showed they are close to the α-stable Lévy process in a short time, while in a long time they resemble a Brownian motion. Koponen [5] also showed that a tempered stable Lévy process is in agreement with recent simulations of truncated Lévy flights.

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There are demonstrations of tempered Lévy processes in modelling market fluctuations. Kuchler et al. [7] investigated exponential stock models driven by tempered stable processes. They developed pricing formulae for European call options and performed a case study. Meerschaert et al. [8] proposed a novel tempered model to capture the slow convergence of sub-diffusion to a diffusion limit for passive tracers in heterogeneous media. The model was validated against particle concentrations from detailed numerical simulations and field measurements, at various scales and geological environments.

Some authors have developed simulation methods for deterministic macroscopic equations related to stochastic equations with tempered Lévy processes. Cont et al. [9] presented a finite difference method for solving parabolic partial integro-differential equations with possibly singular kernels, which can be used to price European and barrier options via stochastic models with tempered Lévy processes. They also discussed localization to a finite domain and provided an estimate for the localization error under an integrability condition on the Lévy jump measure. Li et al. [10,11] developed a high order difference scheme for a tempered fractional diffusion equation on a bounded domain, together with stability and convergence analysis.

In this present paper, we consider the following one or two dimensional stochastic differential equation

\[ dX_t = f(X_t)dt + dL_t, \]  

(1.1)

where \( f \) is a drift term (vector field), and \( L_t \) is a tempered stable Lévy process with triplet \((0,d,\epsilon \nu)\). This triplet means that this process has shift zero, diffusion \( d \), and Lévy jump measure \( \epsilon \nu \) (with \( \epsilon \) a positive magnitude of the non-Gaussian process, where we always takes \( \epsilon = 1 \) in our numerical experiments).

(i) One dimensional stochastic dynamical systems:

The jump measure \( \nu \) for a one dimensional tempered Lévy process is obtained by multiplying the \( \alpha \)-stable Lévy measure by an exponential decaying function, which is called exponential tilting of the Lévy measure for \( \alpha \)-stable Lévy measure. That is (see [12]),

\[
\nu(dy) = \nu_\alpha(dy) \left( 1 \{y>0\} e^{-\lambda_1 y} + 1 \{y<0\} e^{\lambda_2 y} \right) \\
= \left[ \frac{C_{\alpha,\lambda_1}}{e^{\lambda_1 |y|^{1+\alpha}}} 1 \{y>0\} + \frac{C_{\alpha,\lambda_2}}{e^{\lambda_2 |y|^{1+\alpha}}} 1 \{y<0\} \right] dy,
\]  

(1.2)

where \( C_{\alpha,\lambda_i} (i = 1,2) \) is a positive constant, and \( i = 1,2 \) is the positive tempering index (or tempering parameter). In this paper, we only consider ‘symmetric’ tempered Lévy process, i.e., \( \lambda_1 = \lambda_2 = \lambda \). Then \( C_{\alpha,\lambda_i} = C_{\alpha} = \frac{1}{2\Gamma(\alpha)} \) (see [11], P.135). Thus

\[
\nu(dy) = \frac{C_{\alpha} dy}{e^{\lambda |y|^{1+\alpha}}},
\]  

(1.3)

The parameter \( \alpha \in (0,1) \cup (1,2) \) is called the stable index. In fact, by taking negative values of \( \alpha \), we obtain compound Poisson models with rich structure. It may also be interesting to allow for different values of \( \alpha \) on the two sides of real axis. Here we only consider \( \alpha > 0 \), as for the well-known stable Lévy processes.

(ii) Two dimensional stochastic dynamical systems:

We consider the jump measure \( \nu \) for a two dimensional tempered Lévy process in two cases. The first case is when the two components of \( L_t \) are independent and then the jump measure is a linear combination of the jump measure (1.3) in (i) above. Thus the jumps in the noisy process are either horizontal or vertical. The second case if when the two components of \( L_t \) are dependent
and also the jump measure is rotationally symmetric (i.e., isotropic): 

\[ \nu(dy) = \tilde{C}_\alpha \frac{dy}{e^{\lambda|y|}|y|^{\alpha+2}}, \]  

(1.4)

with \( \lambda > 0 \) and \( \tilde{C}_\alpha = \frac{1}{2\pi|\Gamma(-\alpha)|} \).

To characterize certain dynamical behaviors of a stochastic system, we often examine mean exit time and probability density evolution [26]. The mean exit time (MET) is the expected time of a particle (i.e., ‘a solution’) initially inside a bounded domain until the particle first exits the domain (see [13–17]). We have also studied numerical methods for computing the exit problems (18, 19). We develop an accurate numerical scheme for MET for stochastic systems with tempered Lévy process and validate it. The numerical experiments are performed to confirm the effectiveness of the scheme.

The Fokker-Planck equation (FPE) is another important deterministic tool for quantifying stochastic dynamical systems. The solution of FPE describes the evolution of probability density function corresponding to a stochastic system [20]. Some researchers obtained the expressions for the FPE is very special cases [21, 22]. We develop an explicit finite difference method for the FPE for stochastic systems with tempered Lévy process. The numerical method is shown to be convergent both theoretically and numerically.

The paper is organized as follows. In section 2, we present effective numerical schemes to compute mean exit time and Fokker-Planck equations for one dimensional stochastic systems with tempered Lévy processes. The numerical experiments are performed to confirm the theoretical results and testify the effectiveness of the schemes. In section 3, we extend the results to two-dimensional stochastic systems, including horizontal-vertical and isotropic noise cases. Explicit finite difference methods are developed, and numerical experiments are conducted to confirm the theoretical results. We end the paper with a brief summary in section 4.

2 One dimensional stochastic dynamical systems

We now present numerical schemes to compute MET and FPE for a one dimensional stochastic dynamical system with a scalar tempered Lévy process.

2.1 Mean exit time

2.1.1 Numerical methods

For stochastic dynamical systems, the mean exit time is utilized to quantify dynamical behaviors of stochastic differential equations driven by tempered Lévy process. The first exit time for the solution orbit \( X_t \) in Eq. (1.1) starting at \( x \) from a bounded domain \( D \) is defined as

\[ \tau_x(\omega) := \inf\{t \geq 0 : X_t(\omega, x) \notin D, X_0 = x\}. \]  

(2.1)

The MET is denoted as

\[ u(x) := \mathbb{E}[\tau_x(\omega)], \]  

(2.2)

which satisfies the following integro-differential equation [26]

\[
\begin{cases}
\mathcal{L} u = f(x)u_x + \frac{\varepsilon}{2}u_{xx} \\
+ \varepsilon\int_{\mathbb{R}\setminus\{0\}} \left[ u(x + y) - u(x) + 1_{\{|y|<1\}}(y)y u_x \right] \nu(dy) = -1, & \text{for } x \in D, \\
\end{cases}
\]

(2.3)

\[ u(x) = 0, \quad \text{for } x \in D^c \]
with
\[
\begin{aligned}
\nu(dy) &= \frac{C_{\alpha} dy}{e^{\lambda |y| |y|^{1+\alpha}}}, \\
C_{\alpha} &= \frac{1}{2|\Gamma(-\alpha)|}.
\end{aligned}
\] (2.4)

**Remark 2.1.** The existence and uniqueness of solution for equation (2.3) can be derived by the similar method (see [28]).

In the following, we consider the standard interval \( D = (-1,1) \). Introduce the following function,
\[
\int_{-\infty}^{\infty} x^{-\nu} e^{-x} dx = u^{-\frac{\alpha}{2}} e^{-\frac{u}{2} W_{-\frac{1}{2}, -\frac{1}{2}}(u)}, \quad \text{for } u > 0,
\] (2.5)
where \( W \) is the Whittaker W function. Matlab use the ‘whittakerW’ to compute its value.

As the Lévy measure is symmetric, we have \( \int_{\mathbb{R} \setminus \{0\}} 1_{\{|y| < 1\}}(y) u \nu(dy) = 0 \). Then the integral term of equation (2.3) becomes
\[
\int_{\mathbb{R} \setminus \{0\}} [u(x + y) - u(x)] \nu(dy)
= C_{\alpha} \int_{-\infty}^{\infty} \frac{dy}{e^{\lambda |y| |y|^{1+\alpha}}} + C_{\alpha} \int_{1-x}^{1-x} \frac{dy}{e^{\lambda |y| |y|^{1+\alpha}}}
= -C_{\alpha} u(x) \left( \int_{1-x}^{\infty} \frac{dy}{e^{\lambda |y| |y|^{1+\alpha}}} + \int_{1-x}^{\infty} \frac{dy}{e^{\lambda |y| |y|^{1+\alpha}}} \right) + C_{\alpha} \int_{1-x}^{\infty} \frac{dy}{e^{\lambda |y| |y|^{1+\alpha}}}
= -C_{\alpha} u(x) [W_1(x) + W_2(x)] + C_{\alpha} \int_{1-x}^{\infty} \frac{dy}{e^{\lambda |y| |y|^{1+\alpha}}},
\] (2.6)
where
\[
W_1(x) = \int_{1-x}^{\infty} \frac{dy}{e^{\lambda |y| |y|^{1+\alpha}}} = \lambda^{\frac{a-1}{2}} (1 + x)^{-\frac{a+1}{2}} e^{-\frac{\lambda (1+x)}{2}} W_{-\frac{1}{2}, -\frac{1}{2}} (\lambda (1 + x)),
\]
\[
W_2(x) = \int_{1-x}^{\infty} \frac{dy}{e^{\lambda |y| |y|^{1+\alpha}}} = \lambda^{\frac{a-1}{2}} (1 - x)^{-\frac{a+1}{2}} e^{-\frac{\lambda (1-x)}{2}} W_{-\frac{1}{2}, -\frac{1}{2}} (\lambda (1 - x)).
\]

For the singular integral term of equation (2.6), we take \( \delta = \min \{1 - x, 1 + x\} \), then the following equalities hold in the sense of the Cauchy principal sense, i.e.,
\[
I := \int_{1-x}^{\infty} \frac{u(x + y) - u(x)}{e^{\lambda |y| |y|^{1+\alpha}}} dy
= \int_{1-x}^{1-x} \frac{u(x + y) - u(x) - 1_{\{|y| < \delta\}} y u'(x)}{e^{\lambda |y| |y|^{2}}} |y|^{1-\alpha} dy
= \int_{1-x}^{1-x} g(y) |y|^{1-\alpha} dy + \int_{0}^{1+x} \tilde{g}(y) y^{1-\alpha} dy
:= I_1 + I_2,
\] (2.7)
where
\[
g(y) = \frac{u(x + y) - u(x) - 1_{\{|y| < \delta\}} y u'(x)}{e^{\lambda |y| |y|^{2}}}, \quad \tilde{g}(y) = g(-y).
\] (2.8)
Denote \( G(y) = g(y)|y|^{1-\alpha}, \) \( \tilde{G}(y) = G(-y), \) \( \sum' \) means the term of upper index is multiplied by \( \frac{1}{2}, \) \( \sum'' \) means that the quantities corresponding to the two end summation indices are multiplied by \( 1/2. \) Using a modified trapezoidal rule for the singular terms \( I_1 \) and \( I_2, \) then we have

\[
I_1 = h \sum_{j=1}^{J_1} G(y_j) - \zeta(\alpha - 1)g(0)h^{2-\alpha} - \zeta(\alpha - 2)g'(0)h^{3-\alpha} + O(h^2),
\]

\[
I_2 = h \sum_{j=1}^{J_2} \tilde{G}(y_j) - \zeta(\alpha - 1)\tilde{g}(0)h^{2-\alpha} - \zeta(\alpha - 2)\tilde{g}'(0)h^{3-\alpha} + O(h^2),
\]

where \( J_1 \) and \( J_2 \) are the index corresponding to \( 1 - x \) and \( 1 + x, \) respectively, \( \zeta \) is the Riemann zeta function, and

\[
\begin{align*}
\begin{cases}
g(0) = \tilde{g}(0) = \frac{u''(x)}{2}, \\
g'(0) = \frac{u'''(x)}{6} - \lambda g(0), \\
g'(0) = -\frac{u'''(x)}{6} + \lambda g(0).
\end{cases}
\end{align*}
\]

Let us divide the interval \([-2, 2]\) into \( 4J \) subintervals and define \( x_j = jh \) for \(-2J \leq j \leq 2J\) integer, where \( j = \frac{1}{h}. \) Using central difference numerical scheme for the first and two derivatives and modifying the “punched-hole” trapezoidal rule in the nonlocal term, we get the discretization scheme of \( (2.3), \) i.e.,

\[
\frac{dP_j}{dt} := C_h U_{j+1} - 2U_j + U_{j-1} - f(x_j) \left( \frac{U_{j+1} - U_{j-1}}{2h} \right) - \varepsilon C_\alpha \left[ W_1(x_j) + W_2(x_j) \right] U_j + \varepsilon C_\alpha h \sum_{k=-J-j,k\neq0}^{J-j} u \left[ \frac{U_{j+k} - U_j}{\varepsilon C_\alpha h |x_k|^{1+\alpha}} \right]
\]

with

\[
C_h = \frac{d}{2} - \varepsilon C_\alpha (\alpha - 1)h^{2-\alpha}.
\]

We can rewrite the summation terms of Eq. (2.12) as multiplication form of matrix-vector \( RU, \) where \( R \) is a \((2J-1) \times (2J-1)\) matrix. Moreover, the matrix \( R \) can be decomposed as

\[
R = TR + DR,
\]

where \( TR \) is a Toeplitz matrix, i.e.,

\[
TR = 
\begin{pmatrix}
0 & \frac{\tilde{C}}{e^{\lambda h^{1+\alpha}}} & \frac{\tilde{C}}{e^{2\lambda h^{1+\alpha}}} & \cdots & \frac{\tilde{C}}{e^{(2J-2)\lambda h^{1+\alpha}}} \\
\frac{\tilde{C}}{e^{\lambda h^{1+\alpha}}} & 0 & \cdots & \frac{\tilde{C}}{e^{(2J-3)\lambda h^{1+\alpha}}} \\
\frac{\tilde{C}}{e^{2\lambda h^{1+\alpha}}} & \frac{\tilde{C}}{e^{\lambda h^{1+\alpha}}} & 0 & \cdots & \frac{\tilde{C}}{e^{(2J-4)\lambda h^{1+\alpha}}} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{\tilde{C}}{e^{(2J-3)\lambda h^{1+\alpha}}} & \frac{\tilde{C}}{e^{(2J-4)\lambda h^{1+\alpha}}} & \cdots & 0 & \frac{\tilde{C}}{e^{\lambda h^{1+\alpha}}} \\
\frac{\tilde{C}}{e^{(2J-2)\lambda h^{1+\alpha}}} & \frac{\tilde{C}}{e^{(2J-3)\lambda h^{1+\alpha}}} & \cdots & \frac{\tilde{C}}{e^{\lambda h^{1+\alpha}}} & 0
\end{pmatrix}
\]

with

\[
\tilde{C} = \varepsilon C_\alpha h
\]
and $D_R$ is a tridiagonal one, i.e.,

$$
D_R = \begin{pmatrix}
    a_{1-J} & 0 & \\
    0 & a_{2-J} & 0 \\
    \vdots & \vdots & \ddots \\
    0 & a_{J-2} & 0 \\
    & 0 & a_{J-1}
\end{pmatrix}
$$

with

$$
a_j = -\varepsilon C_\alpha h \sum_{k=1-J,j}^{J-j} \frac{1}{e^{\lambda|x_k|}|x_k|^{\gamma}} \quad j = 1, 2, \ldots, J - 1.
$$

### 2.1.2 Numerical experiments

In this section, we proceed to an example to verify our numerical method. Taking $u(x) = (1 - x^2)_+$ (i.e., $u(x) = 1 - x^2$ for $x \in (-1, 1)$, otherwise, $u(x) = 0$) and $\lambda_1 = \lambda_2 = \lambda$, $f = 0$, $d = 0$, $\varepsilon = 1$ into the left-hand side (LHS) of Eq. (2.3), we have

$$
LHS = C_\alpha \int_{-1-x}^{1-x} \frac{dy}{e^{\lambda|y|}|y|^{1+\alpha} \Gamma(1+\alpha)} - C_\alpha u(x) \left[ \int_{1-x}^{\infty} \frac{dy}{e^{\lambda|y|}|y|^{1+\alpha}} + \int_{1+x}^{\infty} \frac{dy}{e^{\lambda|y|}|y|^{1+\alpha}} \right]
$$

$$
= 2C_\alpha \int_{1-x}^{1+x} e^{-\lambda y\alpha} dy - C_\alpha \int_{0}^{1-x} e^{-\lambda y\alpha} dy - C_\alpha \int_{0}^{1+x} e^{-\lambda y\alpha} dy - C_\alpha \int_{0}^{1-x}(1 - x^2)[W_1(x) + W_2(x)]
$$

$$
= 2C_\alpha \lambda^{(1-\alpha)} \left[ P(2 - \alpha, \lambda(1 - x)) + P(2 - \alpha, \lambda(1 + x)) \right] - C_\alpha (1 - x^2)[W_1(x) + W_2(x)]
$$

where $P(a, x) = \int_{1-x}^{1+x} e^{-\lambda y\alpha} dy$ is the incomplete Gamma function, and $Q(a, x) = 1 - P(a, x)$ is the ‘upper’ incomplete Gamma function. Matlab uses the ‘gammainc’ to compute $P(a, x)$ and $Q(a, x)$.

First we take the exact solution $u(x) = (1 - x^2)_+$ of constructed equation to verify our numerical method and compute the convergence orders. Fig. 1 shows the errors between the numerical and exact solution with $\lambda = 0.01$, $f = 0$, $d = 0$, $\varepsilon = 1$ and different $\alpha$. Fig. 1(a) and (b) show our numerical solutions almost agrees with the exact solutions for different $\alpha$ ($\alpha = 0.5, \alpha = 1.5$). The numerical convergence order is equal to 2. To verify it, we plot $\log_{10}(|error|_2)$ against $\log_{10}(J)$ with different resolutions $J = 20, 40, 80, 160, 320$ in Fig. 1(c) ($\alpha = 0.5$) and Fig. 1(d)($\alpha = 1.5$), where $|error|_2$ represents the 2-norm errors. This above results imply that the errors almost reach our expected from the above analysis.

Next, we consider the effects of tempering parameter $\lambda$ on MET. Fig. 2 shows the numerical solution of MET for different $\lambda$ ($\lambda = 0, 0.01, 0.05, 0.1$) and $\alpha$ ($\alpha = 0.5, 1.5$) with $f = 0$, $d = 0$, $\varepsilon = 1$, $D = (-1, 1)$. For $\lambda = 0$, we use the method in reference [18] for comparison. For $\alpha = 0.5$ (see Fig. 2(a)), the ‘particle’ takes more time to exit the domain as $\lambda$ becomes larger, which is agree with our intuition, i.e., the Lévy measure is smaller as the $\lambda$ becomes larger, then the jump is smaller and the ‘particle’ is harder to exit the domain. Fig. 2(b) shows the similar results, but the effect of tempering parameter is small for $\alpha = 1.5$. 
It is interesting to point out the effect of domain $D$ and the drift term $f$ on MET. When the other parameters are fixed, we can see that the ‘particle’ will take more time to exit the domain as the domain becomes larger in Fig 3 (a) and (b). It is interesting to see that the MET for $\alpha = 1.5$ is bigger than $\alpha = 0.5$ near the origin for $D = (-5, 5)$, which is opposite to the small domain $D = (-1, 1)$. In Fig 3 (c) and (d), the ‘particle’ is harder to exit the domain, because the drift term $f(x) = -x$ drives it toward the origin.

2.2 Fokker-Planck equation

2.2.1 The existence and uniqueness

The Fokker-Planck equations for stochastic dynamical systems describe the time evolution for the probability density of solution paths. In this subsection, we consider stochastic differential equation (SDE) driven by the tempered Lévy process. The corresponding probability density function for the SDE (1.1) satisfies the following FPE (see [29])

\[
\begin{align*}
\frac{\partial p}{\partial t} &= -(f(x)p)_x + \frac{d}{2} p_{xx} \\
&+ \varepsilon \int_{\mathbb{R}\setminus\{0\}} \left[ p(x + y, t) - p(x, t) + 1_{(|y|<1)}(y)yp_x \right] \nu(dy).
\end{align*}
\]

Here we mainly consider the absorbing barrier condition [20]. The absorbing condition is that
Figure 2: The behaviors of mean exit time \( u(x) \) of Eq. (2.3) for different \( \lambda = 0, 0.01, 0.05, 0.1 \) and \( f = 0, d = 0, \varepsilon = 1, D = (-1, 1) \). (a) \( \alpha = 0.5 \); (b) \( \alpha = 1.5 \).

The process \( X_t \) disappears or is killed when \( X_t \) is outside a bounded domain \( D = (-1, 1) \), i.e., the probability \( p(x, t) \) of being outside of the bounded domain \( D = (-1, 1) \) is zero:

\[ p(x, t) = 0, \quad x \notin (-1, 1), \quad (2.17) \]

What is more, we also extend to the natural condition, i.e.,

\[ \int_{-\infty}^{\infty} p(x, t) \, dx = 1, \quad \forall t \geq 0. \]

In the following, we will show the existence and uniqueness of the models discussed in the above sections without drift. We introduce the linear operator \( A^{\alpha, \lambda} \) defined by

\[
A^{\alpha, \lambda} u(x, t) = \int_{\mathbb{R} \setminus \{0\}} \left[ u(x + y, t) - u(x, t) + 1_{\{|y|<1\}}(y)y u_x \right] \nu(dy)
\]

\[:= \int_{\mathbb{R} \setminus \{0\}} \left[ u(x + y, t) - u(x, t) + 1_{\{|y|<1\}}(y)y u_x \right] K(y, \lambda) (dy)\]

\[:= \text{P.V.} \int_{\mathbb{R} \setminus \{0\}} \left[ u(x + y, t) - u(x, t) \right] K(y, \lambda) \frac{|y|^{1+\alpha}}{|y|^{1+\alpha}} (dy)\]

\[:= -(-\Delta)_{\mathbb{R}}^{\frac{\alpha}{2}} K u(x, t), \quad (2.18)\]
Figure 3: The effect of domain $D$ and drift term $f$ on $\text{MET } u(x)$ of Eq. (2.3) for $\lambda = 0.01$ and $d = 0, \varepsilon = 1$. (a) the domain $D = (-1, 1)$ for different $\alpha = 0.5, 1.5$ with $f = 0$; (b) the same as (a) except $D = (-5, 5)$; (c) $\alpha = 0.5, D = (-1, 1)$ for different drift term $f$; (d) the same as (c) except $\alpha = 1.5$.

where

$$K(y) = \frac{C_{\alpha}I_{\{y>0\}}}{e^{\lambda y}} + \frac{C_{\alpha}I_{\{y<0\}}}{e^{-\lambda y}}.$$  \hfill (2.19)

**Remark 2.2.** There exist positive constants $c_1, c_2$ such that

$$c_1 \leq K(y) \leq c_2, \quad y \in \mathbb{R} \setminus \{0\}. \hfill (2.20)$$

Denote $H^s(\mathbb{R})$ the conventional Sobolev space of functions, equipped with the norm

$$|u|_{H^s}^2 = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} |u(x) - u(x + y)|^2 \frac{dy}{|y|^{1+2s}}. \hfill (2.21)$$

Let $H^s_0(D)$ be the subspace of $H^s(\mathbb{R})$ consisting of functions which are zero in $D$. It is isomorphic to the completion of $C_c^\infty(D)$ in $H^s(D)$. The dual space of $H^s_0(D)$ will be denoted by $H^{-s}(D)$. For any Banach space $M$, the space $L^2(0, T; M)$ consists of functions $u : (0, T) \to M$ such that

$$||u||_{L^2(0,T;M)} = \left( \int_0^T ||u(\cdot, t)||^2_M dt \right)^{\frac{1}{2}} < \infty. \hfill (2.22)$$

Define the following operator by

$$Bp = (-\Delta)^{\frac{s}{2}}K_p(x) - \frac{d}{2}p_{xx}. \hfill (2.23)$$
Introduce the following equation

\[
\begin{cases}
p_t + Bp = 0, & x \in D, \ t > 0, \\
p|_{D^c} = 0, \\
p(x,0) = p_0(x).
\end{cases}
\tag{2.24}
\]

In the following, we aim at deducing the existence of a unique solution of (2.24).

(i) \( B \) is monotone. For every \( u \in \mathcal{D}(B) \), we have

\[
(Bu,u)_{L^2} = \frac{1}{2} \int_{\mathbb{R}^2 \setminus (D^c \times D^c)} (u(x) - u(z))^2 K(z-x)dx dz + \frac{d}{2} \int_D |\nabla u|^2 dx \geq 0.
\tag{2.25}
\]

(ii) \( B \) is self-adjoint. In view of Proposition 7.6 (see [32]), it suffices to verify that \( B \) is symmetric. For every \( u, v \in \mathcal{D}(B) \) we have

\[
(Bu,v)_{L^2} = \frac{1}{2} \int_{\mathbb{R}^2 \setminus (D^c \times D^c)} (u(x) - u(z))(v(x) - v(z))K(z-x)dx dz + \frac{d}{2} \int_D \nabla u \cdot \nabla v dx
\tag{2.26}
\]

and

\[
(Bv,u)_{L^2} = \frac{1}{2} \int_{\mathbb{R}^2 \setminus (D^c \times D^c)} (v(x) - v(z))(u(x) - u(z))K(x-z)dx dz + \frac{d}{2} \int_D \nabla v \cdot \nabla u dx
\tag{2.27}
\]

so that

\[
(Bu,v) = (u,Bv).
\tag{2.28}
\]

(iii) \( B \) is maximal monotone. In fact \( B \) is maximal monotone \( \iff \) \( B \) is closed, \( B \) is dense, \( B^* \) are monotone. Obviously \( \mathcal{D}(B) \) is dense in \( L^2 \). Using Theorem 7.7 (see [32]), we deduce the existence of a unique solution of (2.24). Moreover, assume \( p_0 \in L^2(D) \), then we have \( p \in L^\infty((0,\infty);D) \cap L^2((0,\infty);H^2(D) \cap L^2(D)) \). By the energy estimate (see [30]) and the method of continuity (see [31]), we may get the existence of (3.19) with absorbing condition.

Remark 2.3. For the nature boundary condition, the existence of solution without drift term can be found from Theorem 1.1 (see [33]).

2.2.2 Stability and convergence for the numerical methods

Using central difference method for the second derivative and third-order WENO scheme for the advection term \((f(x)p)_x\) and modifying the "punched-hole" trapezoidal rule in the nonlocal term, we get the semi-discrete equation of (2.16), i.e.,

\[
\frac{dP_j}{dt} := Ch \left( \frac{P_{j+1} - 2P_j + P_{j-1}}{h^2} - \frac{(fP)_{+} + (fP)_{-}}{2} \right) - \varepsilon C_\alpha \left[ W_1 + W_2 \right] P_j + \varepsilon C_\alpha h \sum_{k=-J}^{J} \frac{P_{j+k} - P_{j-k}}{h} e^{\lambda|x_k|/|x_k|^1\alpha},
\tag{2.29}
\]

where \( C_h = \frac{d}{2} - \varepsilon C_\alpha \zeta(\alpha-1)h^{2-\alpha} \), \((fP)_{+} \) and \((fP)_{-} \) are defined as the global Lax-Friedrichs flux splitting [35], i.e., \((fP)_{+} = \frac{1}{2}(fP + M P)\) with \( M = \max |f(x)| \).

Proposition 2.1 (Maximum principle for the absorbing condition). For the absorbing boundary condition and forward Euler for time derivative, the scheme (2.12) satisfies the discrete maximum principle with \( f = d = 0 \), if the \( \Delta t \) and \( h \) satisfy the following condition,

\[
\frac{\Delta t}{h^\alpha} \leq \frac{1}{2\varepsilon C_\alpha [1 + \frac{1}{\alpha} - \zeta(\alpha-1)]}.
\tag{2.30}
\]
Proof. For $0 < P_j^n \leq M$, where $M$ is the maximum value of the initial probability density. As $0 < \alpha < 2$, we have $\zeta(\alpha - 1) \leq 0$. Applying the explicit Euler to the semi-discrete scheme, we get

$$P_{j+1}^n = P_j^n - \Delta t \varepsilon C_\alpha^j (\alpha - 1) h^{-\alpha} (P_{j+1}^n - 2P_j^n + P_{j-1}^n)$$

$$- \varepsilon C_\alpha^j \Delta t [W_1(x_j) + W_2(x_j)] P_j^n + \varepsilon C_\alpha^j \Delta t h \sum_{k=-J,j,k\neq 0}^{J-j} \frac{P_{j+k}^n - P_j^n}{e^{\lambda|x_k||x_k|^{1+\alpha}}}$$

$$= \left[ 1 + 2 \Delta t \varepsilon C_\alpha^j (\alpha - 1) h^{-\alpha} - \varepsilon C_\alpha^j (W_1(x_j) + W_2(x_j)) \right] \Delta t$$

$$- \varepsilon C_\alpha^j \Delta t h \sum_{k=-J,j,k\neq 0}^{J-j} \frac{P_{j+k}^n - P_j^n}{e^{\lambda|x_k||x_k|^{1+\alpha}}}$$

$$+ \varepsilon C_\alpha^j \Delta t h \sum_{k=-J,j,k\neq 0}^{J-j} \frac{P_{j+k}^n}{e^{\lambda|x_k||x_k|^{1+\alpha}}} \leq \{1 - \varepsilon C_\alpha \Delta t [W_1(x_j) + W_2(x_j)] \} M \leq M,$$ provided that

$$L : = \varepsilon C_\alpha^j (W_1(x_j) + W_2(x_j)) \Delta t + \varepsilon C_\alpha^j \Delta t h \sum_{k=-J,j,k\neq 0}^{J-j} \frac{1}{e^{\lambda|x_k||x_k|^{1+\alpha}}} (2.31)$$

In fact, we have

$$L \leq \varepsilon C_\alpha^j \Delta t \left[ \int_{-\infty}^{x_j} \frac{dy}{e^{\lambda y}|y|^{1+\alpha}} + \int_{x_j}^{\infty} \frac{dy}{e^{\lambda y}|y|^{1+\alpha}} + \frac{2h}{e^{\lambda h}|y|^{1+\alpha}} + \int_{(-1-x_j+\frac{h}{2},-1-x_j-\frac{h}{2})}^{(-h,h)} \frac{dy}{e^{\lambda y}|y|^{1+\alpha}} \right]$$

$$\leq \varepsilon C_\alpha^j \Delta t \left[ \frac{2}{e^{\lambda h}|y|^{1+\alpha}} + 2 \int_{h}^{\infty} \frac{dy}{e^{\lambda y}|y|^{1+\alpha}} \right]$$

$$\leq \frac{2\varepsilon C_\alpha \Delta t}{h^{\alpha}} \left( 1 + \frac{1}{\alpha} \right)$$

$$\leq \frac{2\varepsilon C_\alpha \Delta t}{h^{\alpha}} \left( 1 + \frac{1}{\alpha} \right),$$

where $x_j = jh$. By the inequality (2.30), we get

$$\frac{2\varepsilon C_\alpha \Delta t}{h^{\alpha}} \left( 1 + \frac{1}{\alpha} \right) \leq 1 + 2 \Delta t \varepsilon C_\alpha^j (\alpha - 1) h^{-\alpha}.$$

\[ \square \]

Remark 2.4. The third-order TVD Runge-Kutta method also satisfies the discrete maximum principle \[36].

Remark 2.5. The stability and convergence of the numerical scheme can be deduced by the the discrete maximum principle, \(2.10\) and Lax equivalence theorem.

Remark 2.6. For the natural condition, the semi-discrete equation becomes

$$R_j P := C_h \frac{P_{j+1} - 2P_j + P_{j-1}}{h^2} - [(fP)^+_{x,j} + (fP)^-_{x,j}] + \varepsilon C_\alpha \Delta t h \sum_{k=-J,j,k\neq 0}^{J-j} \frac{n}{e^{\lambda|x_k||x_k|^{1+\alpha}}},$$

where $J = \frac{L}{h}$ and $L$ is large enough so that the results are convergent. The stability and convergence for the natural condition can be deduced similar to the absorbing case.
2.2.3 Numerical experiments

Here we proceed to an example to illustrate our numerical method. We take the Gaussian initial condition \( p(x, 0) = \sqrt{\frac{40}{\pi}} e^{-40x^2} \) and \( f = 0, d = 0, \varepsilon = 1, D = (-1, 1) \) at time \( t = 0.2 \) for absorbing condition. Fig. 4 shows the effect of parameter \( \lambda \) for the solution of Fokker-Planck equation (2.16). We see that the solution becomes larger when the tempering parameter \( \lambda \) is bigger near the origin for different \( \alpha = 0.5, 1.5 \). Meanwhile, it is decays faster for \( \lambda \neq 0 \) than \( \lambda = 0 \), especially for \( \alpha = 1.5 \).

Besides, even for the small \( \lambda \), the solution is different from the \( \alpha \)-stable case.

In Fig. 5, we present the effect of stability index \( \alpha \) for the solution of Fokker-Planck equation. We take \( \lambda = 0.01 \) and \( f = 0, d = 0, \varepsilon = 1, D = (-1, 1) \) at time \( t = 0.2 \) with Gaussian initial condition. The time step and the space step satisfy the condition (2.30), here we take \( \Delta t = 0.5h^\alpha \). As the \( \alpha \) becomes larger, the solution is smaller near the origin, while it is opposite near the boundary.

To avoid the singular behavior of the delta function, we take the initial condition \( p(x, 0) = \sqrt{\frac{40}{\pi}} e^{-40x^2} \), and \( f = 0, d = 0, D = (-5, 5), \varepsilon = 1 \) for natural condition. Fig. 6 shows that the probability density function with different \( \alpha \) and \( \lambda \) at time \( t = 1 \). By using the trapezoidal rule, we compute the numerical integral \( \int_{-\infty}^{\infty} p(x, t)dx \) for \( \alpha = 0.5 \) and \( \alpha = 1.5 \), the values are 0.99985026 and 0.99995389 respectively, they are close to 1. From this figure, we find that the behavior of the solution is similar with the \( \alpha \)-stable process near the origin, while it decreases fast far away from it.
Figure 5: The effect of stability index $\alpha$ for probability density function of Eq. (2.16) with $\lambda = 0.01$ and $f = 0, d = 0, \varepsilon = 1, D = (-1, 1)$ at time $t = 0.2$, here we take the initial condition $p(x, 0) = \sqrt{40 \pi} e^{-40x^2}$ and the resolution $\Delta t = 0.5h^\alpha$.

### 3 Two dimensional stochastic dynamical systems

We now devise numerical schemes to compute MET and FPE for a two dimensional stochastic system (1.1) with a tempered Lévy process $L_t$.

In this two-dimensional case, the generator for (1.1) is

$$\mathcal{L} u(x) = f_i \partial_i u(x) + \frac{1}{2} d_i \partial_i^2 u(x) + \varepsilon \int_{\mathbb{R}^2 \setminus \{0\}} \left[ u(x + y) - u(x) + 1_{B_h(0)} y_j \partial_j u(x) \right] \nu(dy), \quad (3.1)$$

where $B_h(0) = \{ x : |x| \leq h \} (0 < h \ll 1)$ and the jump measure $\nu$ is a ‘tempered’ version of the usual $\alpha$-stable Lévy jump measure.

The usual $\alpha$-stable Lévy jump measure is expressed as (see [1][27])

$$\hat{\nu}(B) = \int_{S_2} \Gamma(\text{d}\theta) \int_0^\infty 1_{B(\theta)} r \frac{dr}{r^{1+\alpha}}, \forall B \in \mathcal{B}(\mathbb{R}^2), \quad (3.2)$$

with $S_2 = \{ x : |x| = 1 \}$ the unit circle in $\mathbb{R}^2$, and $\Gamma$ is the spectral measure on this unit circle.

In the following, we consider two cases tempered Lévy process $L_t$, which components are either independent (horizontal-vertical case) or dependent (isotropic case).
Figure 6: The solution of Eq. (2.16) for the natural condition with $\alpha = 0.5, 1.5, \lambda = 0.01, 0.1$ and $f = 0, d = 0, \varepsilon = 1, D = (-5, 5)$ at time $t = 1$, here we take the initial condition $p(x, 0.01) = \sqrt{\frac{40}{\pi}} e^{-40x^2}$ and the resolution $\Delta t = 0.5h^\alpha$.

3.1 MET for the horizontal-vertical case

3.1.1 Numerical methods

When the components of the tempered Lévy process $L_t$ are independent, the particles (or solutions) spread in either horizontal or vertical direction [11]. Then, as we know [1], the spectral measure $\Gamma$ in (3.2) concentrates on the points of intersection of unit circle $S_2$ and axes, so that the Lévy jump measure becomes

$$\nu(dy) = C_1 e^{\lambda_1 y_1} |y_1|^{1+\alpha_1} \delta(y_2) dy_1 dy_2 + C_2 e^{\lambda_2 y_2} |y_2|^{1+\alpha_2} \delta(y_1) dy_1 dy_2, \quad (3.3)$$

where $C_1 = \frac{1}{2|\Gamma(-\alpha_1)|}$ and $C_2 = \frac{1}{2|\Gamma(-\alpha_2)|}$.

For simplicity, we take $\alpha_1 = \alpha_2 = \alpha, \lambda_1 = \lambda_2 = \lambda, C_1 = C_2 = C_0$ and the square domain $D = (-1, 1)^2$, and the MET satisfies the following integro-differential equation [26]

$${\mathcal{L}}u = f_1 \partial_i u(x) + \frac{d}{2} \partial_i^2 u(x) + \varepsilon \int_{\mathbb{R}^2 \backslash \{0\}} [u(x+y) - u(x) + 1_{B_\varepsilon(0)} y_j \partial_j u(x)] \nu(dy) = -1, \quad \text{for } x \in D, \quad (3.4)$$

$$u(x) = 0, \quad \text{for } x \in D^c.$$

**Remark 3.1.** The existence of a unique solution of (3.4) can be derived by the similar method (see [28]).
Denote \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( y = (y_1, y_2) \in \mathbb{R}^2 \). Firstly, the integral terms in (3.4) can be divided into two parts, i.e.,

\[
\int_{\mathbb{R}^2 \setminus \{0\}} [u(x_1 + y_1, x_2 + y_2) - u(x_1, x_2)] \nu(dy),
= C_\alpha \int_{\mathbb{R}^2 \setminus \{0\}} \frac{[u(x_1 + y_1, x_2) - u(x_1, x_2)]}{e^{\lambda y_1} |y_1|^{1+\alpha}} dy_1 + C_\alpha \int_{\mathbb{R}^2 \setminus \{0\}} \frac{[u(x_1, x_2 + y_2) - u(x_1, x_2)]}{e^{\lambda y_2} |y_2|^{1+\alpha}} dy_2,
= -C_\alpha u(x_1, x_2)[W_1(x_1) + W_2(x_1)] + C_\alpha \int_{-1-x_1}^{1-x_1} \frac{[u(x_1 + y_1, x_2) - u(x_1, x_2)]}{e^{\lambda y_1} |y_1|^{1+\alpha}} dy_1
- C_\alpha u(x_1, x_2)[W_1(x_2) + W_2(x_2)] + C_\alpha \int_{-1-x_2}^{1-x_2} \frac{[u(x_1, x_2 + y_2) - u(x_1, x_2)]}{e^{\lambda y_2} |y_2|^{1+\alpha}} dy_2. \tag{3.5}
\]

Secondly, for the integral terms in (3.5), we get the modified trapezoidal rule similar to the one-dimensional case,

\[
\int_{-1-x_i}^{1-x_i} \tilde{G}(y_i)dy_i = h \sum_{k=-J-j, k \neq 0}^{J-j} t\tilde{G}(y_k) - \zeta(\alpha - 1)h^{2-\alpha}u_{x_i,x_i}(x_1, x_2) + O(h^2), \quad \text{for } i = 1, 2,
\]

where

\[
\begin{align*}
\tilde{G}(y_1) &= \frac{u(x_1 + y_1, x_2) - u(x_1, x_2)}{e^{\lambda y_1} |y_1|^{1+\alpha}}, \\
\tilde{G}(y_2) &= \frac{u(x_1, x_2 + y_2) - u(x_1, x_2)}{e^{\lambda y_2} |y_2|^{1+\alpha}}.
\end{align*} \tag{3.6}
\]

3.1.2 Numerical experiments

In this section, we aim to take an example to illustrate our numerical results in two-dimensional case. Fig displays the MET for two-dimensional horizontal-vertical case with different \( \lambda \) and \( \alpha \). Here we fix the factors \( f_i = 0, d_i = 0, D = (-1, 1)^2, \varepsilon = 1 \). We find that the MET becomes larger as the values of parameter \( \lambda \) are bigger for different \( \alpha = 0.5, 1.5 \). The solutions decays faster for \( \alpha = 1.5 \) than \( \alpha = 0.5 \) near the boundary, and it is also flatter for \( 0 < \alpha < 1 \).

3.2 Mean exit time for isotropic case

3.2.1 Numerical methods

When the particles spread uniformly in all directions, this case is called the isotropic Lévy process. Assume the process and the domain \( D = \{ x \in \mathbb{R}^2 : |x| < 1 \} \) is radially symmetric in this subsection, then we have \( u(x) = u(r) \), where \( x = (x_1, x_2) \) and \( r = |x| = \sqrt{x_1^2 + x_2^2} \). Set \( \frac{d_i}{2} = d(r) \) and \( f_i = f(r) \frac{d_i}{2} \), \( i = 1, 2 \), where \( f(\cdot) \) and \( d(\cdot) \) are smooth scalar functions (see [19]).

The MET satisfies the following integro-differential equation (see [26])

\[
f(r)u'(r) + d(r) \left[ u''(r) + \frac{u'(r)}{r} \right] + \varepsilon \tilde{C}_\alpha \int_{\mathbb{R}^2 \setminus \{0\}} \frac{u(x+y) - u(x) - 1_{B_\varepsilon(0)}y_j \partial_j u(x)}{e^{\lambda y_1} |y_1|^{\alpha+2}} dy = -1, \tag{3.7}
\]

where

\[
\tilde{C}_\alpha = \frac{1}{2\pi \Gamma(-\alpha)}. \tag{3.8}
\]
For the radially symmetric case, we only need the solution $u(x)$ for $x \in (0, 1)$. For simplicity, we denote $x = (r, 0)$ for $r \geq 0$. By taking $0 < h \ll 1$, we decompose the singular integral term in Eq. (3.7) into two parts

$$ I := \int_{\mathbb{R}^2 \setminus \{0\}} \frac{u(x + y) - u(x) - 1_{B_h}(0) y_j \partial_j u(x)}{e^{\lambda |y||y|^{\alpha + 2}}} dy $$

$$ = \int_{\mathbb{R}^2 \setminus B_h(0)} \frac{u(x + y) - u(x) - y_j \partial_j u(x)}{e^{\lambda |y||y|^{\alpha + 2}}} dy + \int_{B_h(0) \setminus \{0\}} \frac{u(x + y) - u(x) - y_j \partial_j u(x)}{e^{\lambda |y||y|^{\alpha + 2}}} dy $$

$$ = I_1 + I_2, \quad (3.9) $$

thanks to the following inequality

$$ \max \left( \frac{y_i^4}{e^{\lambda |y||y|^{\alpha + 2}}}, \frac{y_j^2 y_2^2}{e^{\lambda |y||y|^{\alpha + 2}}} \right) \leq |y|^{2-\alpha}, \quad i = 1, 2 \quad (3.10) $$

and the Polar transformation, we have

$$ \frac{1}{4!} \int_{B_h(0) \setminus \{0\}} \frac{\left( y_1 \frac{\partial^4 u(x)}{\partial x_1^4} + y_2 \frac{\partial^4 u(x)}{\partial x_2^4} \right)^4 u(x)}{e^{\lambda |y||y|^{\alpha + 2}}} dy $$

$$ = \frac{1}{4!} \int_{B_h(0) \setminus \{0\}} \frac{\left( y_1^4 \frac{\partial^4 u(x)}{\partial x_1^4} + y_2^4 \frac{\partial^4 u(x)}{\partial x_2^4} + C_2 \frac{\partial^4 u(x)}{\partial x_1^2 \partial x_2^2} y_2^2 \right)}{e^{\lambda |y||y|^{\alpha + 2}}} dy $$

$$ \leq \left( \frac{\partial^4 u(x)}{\partial x_1^4} + \frac{\partial^4 u(x)}{\partial x_2^4} + \frac{\partial^4 u(x)}{\partial x_1^2 \partial x_2^2} \right) \int_{B_h(0) \setminus \{0\}} |y|^{2-\alpha} dy $$

$$ \leq Ch^{4-\alpha}, \quad (3.11) $$

Figure 7: MET in two dimensional horizontal-vertical case \([3,4]\): The tempered Lévy process has various $\lambda$ ($\lambda = 0.01, 1.1$), $\alpha$ ($\alpha = 0.5, 1.5$) and $f_i = 0, d_i = 0, \varepsilon = 1, D = (-1, 1)^2$. 
where $C$ is a constant which is independent of $h$.

In the following, we use Taylor expansion to approximate $I_2$, then we have

\[
I_2 = \int_{B_h(0) \setminus \{0\}} \frac{(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2})^2 u(x)}{2e^{\lambda|y|}|y|^{\alpha+2}} dy + \mathcal{O}(h^{4-\alpha})
\]

\[
= \frac{1}{2} \int_{B_h(0) \setminus \{0\}} \frac{\partial^2 u(x)}{\partial x_1^2} e^{\lambda|y|}|y|^{\alpha+2} dy
\]

\[
+ \int_{B_h(0) \setminus \{0\}} \frac{\partial^2 u(x)}{\partial x_2^2} e^{\lambda|y|}|y|^{\alpha+2} dy + \mathcal{O}(h^{4-\alpha})
\]

where $C_0 = \pi \lambda^{2-\alpha} \Gamma(2-\alpha) P(2-\alpha, \lambda h)$, $P(a, x)$ is the incomplete Gamma function.

Taking $z = x + y$, then we have,

\[
I_1 = \int_{\mathbb{R}^2 \setminus B_h(0)} \frac{u(x + y) - u(x)}{e^{\lambda|y|}|y|^{\alpha+2}} dy = \int_{\mathbb{R}^2 \setminus B_h(0)} \frac{u(z) - u(x)}{e^{\lambda|y|}|y|^{\alpha+2}} dz
\]

\[
= \int_{B_h(0) \setminus B_h(x)} \frac{u(z) - u(x)}{e^{\lambda|y|}|y|^{\alpha+2}} dz - \int_{\mathbb{R}^2 \setminus B_h(0)} \frac{u(x)}{e^{\lambda|y|}|y|^{\alpha+2}} dz
\]

\[
= : I_{11} - I_{12}.
\]

For $I_{12}$, we have

\[
I_{12} = u(x) \int_{\mathbb{R}^2 \setminus B_h(0)} \frac{dz}{e^{\lambda|y|}|y|^{\alpha+2}}
\]

\[
= 2u(r) \int_0^\pi \left[ \int_0^\infty r^{-\alpha-1} e^{-r\lambda} dr \right] d\theta
\]

\[
= 2\lambda^{-\alpha+1} u(r) \int_0^\pi \left[ r^{-\alpha+1} e^{-\frac{\lambda r}{2}} W_{\frac{1+\alpha}{2}, -\frac{\alpha}{2}} (\lambda r) \right] d\theta
\]

with

\[
\tilde{r} = \sqrt{1 - r^2 \sin^2 \theta} - r \cos \theta.
\]

For $I_{11}$, taking the transformation $z_1 = s \cos \theta$, $z_2 = s \sin \theta$, then we have

\[
I_{11} = 2 \int_{(0,1) \setminus (-h,x+h)} s [u(s) - u(r)] F_1^2(s, r) ds + 2 \int_{(r-h,h+h)} s [u(s) - u(r)] F_1^2(s, r) ds
\]

where

\[
F_1^1(s, r) = \int_0^\pi e^{-\lambda \sqrt{s^2 + r^2 - 2sr \cos \theta}} [s^2 + r^2 - 2sr \cos \theta]^{-\frac{\alpha+2}{2}} d\theta,
\]

\[
F_1^2(s, r) = \int_0^\gamma e^{-\lambda \sqrt{s^2 + r^2 - 2sr \cos \theta}} [s^2 + r^2 - 2sr \cos \theta]^{-\frac{\alpha+2}{2}} d\theta,
\]

\[
\gamma = \arccos \frac{s^2 + r^2 - h^2}{2sr}.
\]

Combining (3.9) and (3.16), we get

\[
I = 2 \int_{(0,1) \setminus (-h,x+h)} s [u(s) - u(r)] F_1^2(s, r) ds + 2 \int_{(r-h,h+h)} s [u(s) - u(r)] F_1^2(s, r) ds
\]

\[
- 2\lambda^{-\alpha+1} u(r) \int_0^\pi \tilde{r}^{-\alpha+1} e^{-\frac{\lambda \tilde{r}}{2}} W_{\frac{1+\alpha}{2}, -\frac{\alpha}{2}} (\lambda \tilde{r}) d\theta + C_0 \left[ u''(r) + \frac{u'(r)}{r} \right] + \mathcal{O}(h^{4-\alpha}).
\]
3.2.2 Numerical experiments

We use the second-order central differences for \( u'(r) \) and \( u''(r) \), and take the trapezoidal rule for the nonsingular integral terms in (3.18). For \( r = 0 \), after a direct calculation, we get

\[
\mathcal{L} u(0) = (d(0) + \varepsilon \tilde{C}_\alpha C_0) \left[ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right]_{x=0} + f(0)u'(0) + 2\pi \varepsilon \tilde{C}_\alpha \int_{1}^{\infty} \frac{u(r)-u(0)}{r^{\lambda+1}} \, dr - 2\pi \varepsilon \tilde{C}_\alpha W_1(0)u(0).
\]

Figure 8: MET in the isotropic case (3.7): The tempered Lévy process has various \( \lambda \) (\( \lambda = 0.01, 0.05, 0.1 \)), \( \alpha \) (\( \alpha = 0.5, 1.5 \)) and \( f(r) = 0 \), \( d(r) = 0 \), \( \varepsilon = 1 \), \( D = B_1(0) \).

Here we consider the pure jump process (i.e. \( f(r) = 0 \), \( d(r) = 0 \)) and symmetric domain \( D = B_1(0) \). Fig 8 shows the effect of tempering parameter \( \lambda \) on the MET for the isotropic case. The Fig 8(b) and (d) shows the radially symmetric solution of Eq. (3.7) for the mean exit time, after rotating these two graphs along the vertical axis, we get the mean exit time \( u(x,y) \) for \((x,y) \in B_1(0)\) in Fig 8(a) and (c). When the parameter \( \lambda \) becomes larger, the ‘particle’ takes more time to exit the domain for these two cases. But the tempering parameter has more influence for \( 0 < \alpha < 1 \) (\( \alpha = 0.5 \)) than \( 1 < \alpha < 2 \) (\( \alpha = 1.5 \)).

3.3 Fokker-Planck equation for two-dimensional case

The corresponding probability density function for the SDE (1.1) in two dimensional case satisfies the following Fokker-Planck equation (see [26,29])

\[
p_t(x,t) = -\partial_t(f(x)p(x,t)) + \frac{1}{2} \partial^2_i(d_ip(x,t)) + \varepsilon \int_{\mathbb{R}^2 \setminus \{0\}} \left[ p(x+y,t) - p(x,t) + 1_{B_1(0)} y_j \partial_j p(x,t) \right] \nu(dy),
\]
where $\nu$ is either for the horizontal-vertical case or the isotropic case in section 3.1.1 and section 3.2.1, respectively.

At last, we extend the one dimensional Fokker-Planck equation to the two dimensional one, both in the horizontal-vertical and isotropic case. Fig. 9 shows the solution of Fokker-Planck equation (3.19) of horizontal-vertical case with different $\lambda$ ($\lambda = 0.01, 0.05, 0.1$), $\alpha$ ($\alpha = 0.5, 1.5$) and $f_i = 0, d_i = 0, \varepsilon = 1, D = (-1, 1)^2$. Here we take the Gaussian initial condition $p(x_1, x_2, 0) = \sqrt{\frac{40}{\pi^2}} e^{-40(x_1^2 + x_2^2)}$. The Fig. 9(b) and (d) are the cross sections corresponding to the Fig. 9(a) and (c). We see that the probability densities are ‘fatter’ and ‘shorter’ for $\alpha = 1.5$ than $\alpha = 0.5$, and they become larger as the tempering parameters $\lambda$ are bigger near the origin. The Fig. 10 shows the probability evolutions of isotropic cases for different time $t = 0.1, 0.2, 0.4$ with $\lambda = 0.01, f = 0, d = 0, \varepsilon = 1, D = B_1(0)$. We see that solutions decay as time goes on near the origin, and it decays faster for $\alpha = 1.5$ than $\alpha = 0.5$.

![Figure 9](image_url)

Figure 9: The solution of Fokker-Planck equation (3.19) with 2D Lévy process of horizontal-vertical case: various $\lambda$ ($\lambda = 0.01, 0.05, 0.1$), $\alpha$ ($\alpha = 0.5, 1.5$), $f_i = 0, d_i = 0, \varepsilon = 1, D = (-1, 1)^2$, and the initial condition $p(x_1, x_2, 0) = \sqrt{\frac{40}{\pi^2}} e^{-40(x_1^2 + x_2^2)}$.

4 Conclusion

In this paper, we have considered deterministic quantities that carry dynamical information or that can be used to quantify dynamical behaviors of stochastic differential equations with non-Gaussian tempered stable Lévy processes. These deterministic quantities include probability density functions and mean exit time. The mean exit time for a stochastic system quantifies how long, in expected sense, the system stays in a region in the state space. We devised accurate, stable and convergent numerical algorithms to simulate the nonlocal Fokker-Planck equations for probability
Figure 10: The solution of Fokker-Planck equation (3.19) with 2D Lévy process of isotropic case: various final time \( t \) (\( t = 0.1, 0.2, 0.4 \)), \( \alpha (\alpha = 0.5, 1.5) \), \( \lambda = 0.01 \), \( f_i = 0 \), \( d_i = 0 \), \( \varepsilon = 1 \), \( D = B_1(0) \), and the initial condition \( p(x_1, x_2, 0) = \sqrt{\frac{40}{\pi}} e^{-40(x_1^2+x_2^2)} \).

density functions and nonlocal elliptic type equation for mean exit time. These algorithms are validated against known exact solutions and the numerical solutions obtained by using other methods. The results established in this paper can be used to examine dynamical behaviors for complex systems under non-Gaussian fluctuations, in climate modeling, physical systems and financial markets (see [37,38]).

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