Renormalization of Orientable Non-Commutative Complex $\Phi^6_3$ Model

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Abstract

In this paper we prove that the Grosse-Wulkenhaar type non-commutative orientable complex scalar $\varphi^6_3$ theory, with two non-commutative coordinates and the third one commuting with the other two, is renormalizable to all orders in perturbation theory. Our proof relies on a multiscale analysis in $x$ space.

1 Introduction

Since the rebirth of non-commutative quantum field theory [1, 2, 3, 4], people encountered a major difficulty. A new kind of divergences appeared in non-commutative field theory [5], the UV/IR mixing. It is a kind of infrared divergence which appears after integrating the high scale variables and can’t be eliminated. It lead people to declare such theories non-renormalizable. But a real breakthrough of that deadlock came from H.Grosse and R.Wulkenhaar [6, 7]. They found that the right propagator for the scalar field theory in non-commutative space should be modified to obey the Langmann-Szabo duality [8]. In a series of paper they proved that the $\varphi^4$ scalar field theory in 4 dimensional Moyal plane, $\varphi^4_4$ for short, is renormalizable to all orders

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using Polchinski’s equation [9] in the matrix base. Rigorous estimates on the propagator required by the Grosse-Wulkenhaar analysis and a more explicit multiscale analysis were provided in [10]. Then Gurau et al. gave another proof that the noncommutative $\varphi^4_4$ is renormalizable, also with a multiscale analysis but completely in position space [11]. The corresponding parametric representation of the model was also built in [12]. Recently the model has been shown to have no Landau ghost, so that it is actually better behaved than its commutative counterpart [13, 14, 15] and can presumably be built non perturbatively.

Apart from the $\varphi^4_4$ theory, many other theories in non-commutative space have now also been proved to be renormalizable to all orders, such as the Gross-Neveu model in 2 dimensional Moyal plane [16], the LSZ model [17] and the $\varphi^{*3}$ theory in various dimensional space [18, 19, 20]. For an updated review, see [21, 22].

In this paper we prove that the orientable non-commutative complex $\varphi^{*6}_3$ field theory, $(\bar{\varphi} \star \varphi)^3$ for short, in 2 + 1 dimensional space, with two dimensions equipped with non-commutative Moyal product and the third one which commutes with the two others, is renormalizable to all orders of perturbation theory. In the first section we derive the propagator and establish the $x$-space power counting of the theory. In the second section we prove that the divergent subgraphs can be renormalized by counterterms of the form of the initial Lagrangian. Our proof, based solely on $x$ space with multiscale analysis, follows closely the strategy of [11]. For technical reasons, we restrict ourselves here to the simpler orientable case, but we plan to study the nonorientable case or real scalar $\varphi^6_3$ model as well.

We are motivated by the fact that the quantum Hall effect at finite temperature should also be described by a 2+1 dimensional field theory with two anticommuting space and one commuting imaginary time coordinates [23, 24, 25, 21]. Our model is therefore a first step towards understanding how to renormalize such theories. We plan to compute in a future publication the renormalization group flow of this model, which involves three parameters $\lambda$, $g$ and $\Omega$, instead of two in the $\varphi^4_4$ case.

## 2 Power Counting in $x$-Space

### 2.1 Model, Notations

The simplest orientable non-commutative complex $\varphi^{*6}_3$ theory is defined on $\mathbb{R}^3$ equipped with the associative and non-commutative Moyal product

$$\overline{a \star b}(x) = \int \frac{d^2 k}{(2\pi)^2} \int d^2 y \ a(x + \frac{1}{2} \theta \cdot k) \ b(x+y) \ e^{i k \cdot y} \quad (2.1)$$

$$a \star b(x) = \int \frac{d^2 k}{(2\pi)^2} \int d^2 y \ a(x + \frac{1}{2} \theta \cdot k) \ b(x+y) \ e^{i k \cdot y}$$

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The action functional is

\[ S[\varphi] = \int d^2x \, dx^0 \left( \partial_\mu \bar{\varphi} \star \partial^\mu \varphi + \partial_0 \bar{\varphi} \star \partial^0 \varphi + \Omega^2 (\bar{x}_\mu \varphi) \star (\bar{x}^\mu \varphi) + \mu^2_0 \bar{\varphi} \star \varphi \\
+ \frac{\lambda}{2} \bar{\varphi} \star \varphi \star \varphi \star \varphi \right)(x, x^0) \] (2.2)

where \( \bar{x}_\mu = 2(\theta^{-1})_{\mu\nu} x^\nu \), and \( x = (x^\mu), \mu = (1, 2) \) are the non-commutative variables and \( x^0 \) is the commutative variable, that is

\[ [x^\mu, x'^\nu] = i\theta^{\mu\nu}, \quad [x^0, x^\mu] = 0. \]

Here \( \theta^{\mu\nu} \) is a constant matrix and the Euclidean metric is used.

**Lemma 2.1** The kernel of the propagator in our \((\bar{\varphi} \star \varphi)^3\) model is

\[ C(x, x') = \frac{\Omega(2t)^{-\frac{3}{4}}}{\sqrt{2\pi} \sinh(2\Omega t)} e^{-\frac{\Omega \cosh(2\Omega t)}{2}(x^2 + x'^2) + \frac{\Omega}{\sinh(2\Omega t)} x \cdot x' - \frac{(x_0 - x'_0)^2}{4t} - \mu^2_0 t}, \] (2.3)

with \( x^2 = x_1^2 + x_2^2 \), \( x'^2 = x'_1^2 + x'_2^2 \), \( x \cdot x' = x_1 x'_1 + x_2 x'_2 \).

**Proof** The propagator of interest is expressed via the Schwinger parameter trick as:

\[ H^{-1} = \int_0^{\infty} dt e^{-tH}. \] (2.4)

Let \( H \) be

\[ H = -\partial_1^2 - \partial_2^2 - \partial_0^2 + \Omega^2 x^2 + \mu^2_0, \] (2.5)

where \( \mu_0 \) is the mass of the field.

The integral kernel of the operator \( e^{-tH} \) is:

\[ e^{-tH}(x, x') = \frac{\Omega(2t)^{-\frac{3}{4}}}{\sqrt{2\pi} \sinh(2\Omega t)} e^{-A}, \] (2.6)

\[ A = \frac{\Omega \cosh(2\Omega t)}{2 \sinh(2\Omega t)} (x^2 + x'^2) - \frac{\Omega}{\sinh(2\Omega t)} x \cdot x' + \frac{(x_0 - x'_0)^2}{4t} + \mu^2_0 t. \] (2.7)

At first we note that the kernel is correctly normalised: as \( \Omega \to 0 \), we have

\[ e^{-tH}(x, x') \to \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|x-x'|^2 + |x_0-x'_0|^2}{4t}}, \] (2.8)
which is the normalised heat kernel. Then we must check the equation
\[
\frac{d}{dt}e^{-tH} + He^{-tH} = 0. \tag{2.9}
\]
In fact
\[
\frac{d}{dt}e^{-tH} = \frac{\Omega e^{-A}(2t)^{-\frac{1}{2}}}{\sqrt{2\pi} \sinh(2\Omega t)} \left\{ -2\Omega \coth(2\Omega t) + \frac{\Omega^2}{\sinh^2(2\Omega t)}(x^2 + x'^2) \right. \\
- \left. \frac{1}{2}t^{-1} - \frac{2\Omega^2 \cosh(2\Omega t)}{\sinh(2\Omega t)}x \cdot x' + \frac{(x_0 - x_0')^2}{4t^2} - \mu_0^2 \right\}. \tag{2.10}
\]
Moreover
\[
(-\partial_1^2 - \partial_2^2)e^{-tH} = \frac{\Omega e^{-A}(2t)^{-\frac{1}{2}}}{\sqrt{2\pi} \sinh(2\Omega t)} \left\{ 2\Omega \coth(2\Omega t) - \frac{\Omega^2}{\sinh^2(2\Omega t)}(x^2 + x'^2) \right. \\
+ \left. \frac{2\Omega^2 \coth(2\Omega t)}{\sinh(2\Omega t)}x \cdot x' - \Omega^2 x^2 \right\}, \tag{2.11}
\]
\[-\partial_0^2e^{-tH} = \frac{\Omega e^{-A}(2t)^{-\frac{1}{2}}}{\sqrt{2\pi} \sinh(2\Omega t)} \left\{ \frac{1}{2}t^{-1} - \frac{(x_0 - x_0')^2}{4t^2} \right\}. \tag{2.12}\]

It is now straightforward to verify the differential equation (2.9), which proves the lemma.

Now let’s consider the interaction vertices. The non-commutative complex $\phi^6_3$ model may a priori exhibit both orientable vertices:
\[
V_o = \frac{1}{2}\phi \ast \phi \ast \phi \ast \phi \ast \phi(x) + \frac{1}{3}\phi \ast \phi \ast \phi \ast \phi \ast \phi \ast \phi(x) \tag{2.13}
\]
and non-orientable vertices:
\[
V_{no} = \frac{1}{2}\phi \ast \phi \ast \phi \ast \varphi(x) + \frac{1}{3}\phi \ast \phi \ast \phi \ast \phi \ast \phi \ast \varphi(x) \\
+ \frac{1}{3}\phi \ast \phi \ast \phi \ast \varphi \ast \varphi \ast \varphi \ast \varphi \ast \varphi \ast \varphi(x). \tag{2.14}
\]
In this paper we limit ourselves to the case of action (2.2), hence to orientable vertices.

In the two dimensional non-commutative space the interaction vertices (orientable or not) can be written as [28, 16]:
\[
V(x_1, x_2, x_3, x_4) = \delta(x_1 - x_2 + x_3 - x_4)e^{i\sum_{1 \leq i < j \leq 4}(-1)^{i+j+1}x_4\theta^{-1}x_j} \tag{2.15}
\]
for four point vertices and
\[
V(x_1, x_2, x_3, x_4, x_5, x_6) = \delta(x_1 - x_2 + x_3 - x_4 + x_5 - x_6) e^{i \sum_{1 \leq i < j \leq 6} (-1)^{i+j+1} x_i \theta^{-1} x_j}
\]
(2.16)
for six point vertices. Here we note \(x \theta^{-1} y \equiv \frac{2}{\theta}(x_1 y_2 - x_2 y_1)\). These vertices are completed to make them local in the commutative \(t\) coordinate, that is we have to multiply them by \(\delta(x_1^0 - x_2^0) \delta(x_1^0 - x_3^0) \delta(x_1^0 - x_4^0)\) or \(\delta(x_1^0 - x_2^0) \delta(x_1^0 - x_3^0) \delta(x_1^0 - x_4^0) \delta(x_1^0 - x_5^0) \delta(x_1^0 - x_6^0)\) respectively.

The main result of this paper is a proof in configuration space of

**Theorem 2.1 (BPHZ Theorem for non-commutative \((\bar{\varphi} \ast \varphi)^3\)** The theory defined by the action (2.2) is renormalizable to all orders of perturbation theory.

Let \(G\) be an arbitrary connected graph. The amplitude associated with this graph is (with self-explanatory notations):

\[
A_G = \int \prod_{v \in V_4, i = 1, \ldots, 4} dx_{v,i} dx_v^0 \prod_{v \in V_6, i = 1, \ldots, 6} dx_{v,i} dx_v^0 \prod_t dt_l
\]
\[
\prod_{V_4} [\delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4}) e^{i \sum_{i<j} (-1)^{i+j+1} x_v \theta^{-1} x_{v,j}}]
\]
\[
\prod_{V_6} [\delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4} + x_{v,5} - x_{v,6}) e^{i \sum_{i<j} (-1)^{i+j+1} x_v \theta^{-1} x_{v,j}}]
\]
\[
\prod_l \frac{\Omega(2t_l)^{-\frac{1}{2}}}{\sqrt{2\pi} \sinh(2\Omega t_l)} e^{-\frac{\Omega}{2} \cosh(2\Omega t_l)} x_{V_4, i}(l) x_{V_4, i}(l) + \frac{\Omega}{2} \tanh(\Omega t_l) x_{V_4, i}(l) x_{V_4, i}(l) - \frac{(x_{V_4, i}(l) - x_{V_4, i}(l))^2}{4t_l} - \mu^2 t_l.
\]

For each line \(l\) of the graph joining positions \(x_{v,i}(l)\) and \(x_{v',i'}(l)\), we choose an orientation (see next section) and we define the “short” variable \(u_l = x_{v,i}(l) - x_{v',i'}(l)\), \(u_0 = x_{v}(l) - x_{v'}(l)\) and the “long” variable \(v_l = x_{v,i}(l) + x_{v',i'}(l)\) just as the work of Gurau et al. [11]. With these notations, defining \(2\Omega t_l = \alpha_l\), the propagators in our graph can be written as:

\[
\int \prod_l \frac{\sqrt{\Omega} \alpha_l^{-\frac{1}{2}} d\alpha_l}{2\sqrt{2\pi} \sinh(\alpha_l)} e^{-\frac{\Omega}{2} \cosh(\Omega) \alpha_l^2 - \frac{\Omega}{2} \tanh(\Omega) \alpha_l^2 - \frac{\alpha_l^2}{2\mu^2} - \frac{\mu^2}{2\alpha_l} \alpha_l^2}.
\]
(2.18)

### 2.2 Orientation and Position Routing

We solve the \(\delta\) function at every vertex by a “position routing”, following the strategy and notations of [11]. The position routing is similar to the “momentum routing” of
the commutative case, but we have to take care of the cyclic invariance of the vertex. Consider a connected graph $G$. We choose a rooted spanning tree in $G$, then we start from an arbitrary orientation of a first field at the root and inductively climbing into the tree, at each vertex we follow the cyclic order to alternate entering and exiting lines. This is pictured in Figure 1.

![Figure 1: Orientation of a tree](image)

Let $n = n_6 + n_4$ be the number of vertices of the graph, with $n_6$ of the $\varphi^6$ and $n_4$ of the $\varphi^4$ type, $N$ the number of its external fields, and $L$ the number of internal lines of $G$. We have $L = 3n_6 + 2n_4 - N/2$.

Every line of the spanning tree by definition has one end exiting a vertex and one end entering another. This may not be true for the loop lines, which join two “loop fields”. Among these, some exit one vertex and enter another; they are called well-oriented. But others may enter or exit at both ends. These loop lines are subsequently referred to as “clashing lines” [11]. If there are no clashing lines, the graph is called orientable. This is exactly the case in this paper, because $\varphi$ variables can contract only to $\bar{\varphi}$ ones. Choosing the $\varphi$ variables as entering and the $\bar{\varphi}$ as exiting, the form of the vertices in (2.2) ensure alternance of entering and exiting lines.

We also define the set of “branches” associated to the rooted tree $T$. There are $n - 1$ such branches $b(l)$, one for each of the $n - 1$ lines $l$ of the tree. The full tree itself is called the root branch and noted $b_0$. Each branch is made of the subgraph $G_b$ containing all the vertices “above $l$” in $T$, plus the tree lines and loop lines joining
these vertices. It has also “external fields” which are the true external fields hooked to $G_b$, plus the loop fields in $G_b$ for the loops with one end (or “field”) inside and one end outside $G_b$, plus the upper end of the tree line $l$ itself to which $b$ is associated. We call $X_b$ the set of all external fields $f$ of $b$.

We can now describe the position routing associated to $T$. Here we will not limit ourselves to orientable graphs but will deal with the non-orientable graphs as well. There are $n$ $\delta$ functions in (2.17), hence $n$ linear equations for the $6n_6 + 4n_4$ positions, one for each vertex. The position routing associated to the tree $T$ solves this system by passing to another equivalent system of $n$ linear equations, one for each branch of the tree. This equivalent system is obtained by summing the arguments of the $\delta$ functions of the vertices in each branch. To do this we firstly fix a particular branch $G_b$, with its subtree $T_b$. In the branch sum we find a sum over all the $u_l$ short parameters of the lines $l$ in $T_b$ and no $v_l$ long parameters since $l$ both enters and exits the branch. This is also true for the set $L_b$ of well-oriented loops lines with both fields in the branch. For the set $L_{b,+}$ of clashing loops lines with both fields entering the branch, the short variable disappears and the long variable remains; the same is true but with a minus sign for the set $L_{b,-}$ of clashing loops lines with both fields exiting the branch. Finally we find the sum of positions of all external fields for the branch (with the signs according to entrance or exit). Obviously the Jacobian of this transformation is 1, so we simply get another equivalent set of $n$ $\delta$ functions, one for each branch.

For instance in the particular case of Figure 2, the delta function is

$$
\delta(u_{l_1} + u_{l_2} + u_{l_3} + u_{l_4} + u_{l_5} + v_{l_6} + X_1 - X_2 + X_3 - X_4 - X_5 + X_6).
$$

(2.19)

For an orientable graph, the position routing is summarised by:

**Lemma 2.2 (Position Routing)** We have, calling $I_G$ the remaining integrand in (2.17):

$$
A_G = \int \prod_{v_4} \left[ \delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4}) \right] \left( \prod_{v_6} \left[ \delta(x_{v,1} - x_{v,2} + x_{v,3} - x_{v,4} + x_{v,5} - x_{v,6}) I_G(\{x_{v,i}, x_{v,i}^0\}) \right] \right)
$$

(2.20)

$$
= \int \prod_{b} \delta \left( \sum_{l \in T_b \cup L_b} u_l + \sum_{f \in X_b} \varepsilon(f) x_f \right) I_G(\{x_{v,i}, x_{v,i}^0\})
$$

where $\varepsilon(f)$ is $\pm 1$ depending on whether the field $f$ enters or exits the branch.
Using the above equations one can at least solve all the long tree variables \( v_l \) in terms of external variables, short variables and long loop variables, using the \( n - 1 \) non-root branches. There remains then the root branch \( \delta \) function. If \( G_b \) is orientable, this \( \delta \) function of branch \( b_0 \) contains only short and external variables. Here we shouldn’t forget that each external variable can be written as linear combination of short variable and long variable. If \( G_b \) is non-orientable one can solve for an additional “clashing” long loop variable. We can summarise these observations in the following lemma just like that in [11]:

**Lemma 2.3** The position routing solves any long tree variable \( v_l \) as a function of:

- the short tree variable \( u_t \) of the line \( l \) itself,
- the short tree and loop variables with both ends in \( G_{b(l)} \),
- the short and long variables of the loop lines with one end inside \( G_{b(l)} \) and the other outside,
- the true external variables \( x \) hooked to \( G_{b(l)} \).

In the orientable case the root branch \( \delta \) function contains only short tree variables, short loop variables and external variables but no long variables, hence gives a linear
relation among the short variables and external positions. In the non-orientable case it gives a linear relation between the long variables \( w \) of all the clashing loops in the graph some short variables \( u \)'s and all the external positions.

From now on, each time we use this lemma to solve the long tree variables \( v_l \) in terms of the other variables, we shall call \( w_l \) rather than \( v_l \) the remaining \( 2n_6 + n_4 + 1 - N/2 \) independent long loop variables. Hence looking at the long variables names the reader can check whether Lemma 2.3 has been used or not.

### 2.3 Multiscale Analysis and Crude Power Counting

In this section we follow the standard procedure of multiscale analysis [27]. First the parametric integral for the propagator is sliced in the usual way:

\[
C(u, u^0, v) = C^0(u, u^0, v) + \sum_{i=1}^{\infty} C^i(u, u^0, v),
\]

with

\[
C^0(u, u^0, v) = \int_1^{\infty} \frac{\sqrt{\Omega} \alpha^{-\frac{1}{2}} d\alpha}{2\sqrt{2\pi} \sinh(\alpha)} e^{-\frac{\Omega}{4} \coth(\frac{\Omega}{2})u^2 - \frac{\Omega}{4} \tanh(\frac{\Omega}{2})v^2 - \frac{\mu^2}{2\alpha} - \frac{\Omega}{2\alpha}(u^0)^2}
\]

(2.22)

and

\[
C^i(u, u^0, v) = \int_{M^{-2(i-1)}}^{M^{-2i}} \frac{\sqrt{\Omega} \alpha^{-\frac{1}{2}} d\alpha}{2\sqrt{2\pi} \sinh(\alpha)} e^{-\frac{\Omega}{4} \coth(\frac{\Omega}{2})u^2 - \frac{\Omega}{4} \tanh(\frac{\Omega}{2})v^2 - \frac{\mu^2}{2\alpha} - \frac{\Omega}{2\alpha}(u^0)^2}.
\]

(2.23)

We have an associated decomposition of any amplitude of the theory as

\[
A_G = \sum_{\mu} A_G^{\mu}.
\]

(2.24)

**Lemma 2.4** For some constants \( K \) (large) and \( c \) (small):

\[
C^i(u, v) \leq KM^i e^{-c\left[M^i ||u|| + M^i ||u^0|| + M^{-i} ||v|| \right]}
\]

(2.25)

(which a posteriori justifies the terminology of “long” and “short” variables).

We can use the second order approximation of the hyperbolic functions near the origin to prove this lemma.
Taking absolute values, hence neglecting all oscillations, leads to the following crude bound:

$$|A_G| \leq \sum_{\mu} \int \prod_l du_l du_0^l dv_l C^{\bullet l}(u_l, u_0^l, v_l) \prod_v \delta_v,$$

(2.26)

where $\mu$ is the standard assignment of an integer index $i_l$ to each propagator of each internal line $l$ of the graph $G$, which represents its “scale”. We will consider only amputated graphs. Therefore we have only external vertices of the graph; in the renormalization group spirit, the convenient convention is to assign all external indices of these external fields to a fictitious $-1$ “background” scale.

To any assignment $\mu$ and scale $i$, are associated the standard connected components $G^i_k$, $k = 1, \ldots, k(i)$ of the subgraph $G^i$ made of all lines with scales $j \geq i$. These components are partially ordered according to their inclusion relations and the (abstract) tree describing these inclusion relations is called the Gallavotti-Nicolò tree $[30, \Pi]$; its nodes are the $G^i_k$’s and its root is the complete graph $G$.

More precisely for an arbitrary subgraph $g$ one defines:

$$i_g(\mu) = \inf_{l \in g} i_l(\mu), \quad e_g(\mu) = \sup_{l \text{ external line of } g} i_l(\mu).$$

(2.27)

The subgraph $g$ is a $G^i_k$ for a given $\mu$ if and only if $i_g(\mu) \geq i > e_g(\mu)$. Now we should choose the real tree $T$ compatible with the abstract Gallavotti-Nicolò tree to optimise the bound over spatial integrations, which means that the restriction $T^i_k$ of $T$ to any $G^i_k$ must still span $G^i_k$. This is always possible (by a simple induction from leaves to root). We pick such a compatible tree $T$ and use it both to orient the graph as in the previous section and to solve the associated branch system of $\delta$ functions according to Lemma 2.3. We obtain:

$$|A_{G, \mu}| \leq K^n \prod_l M_l^i \int \prod_l du_l du_0^l dv_l e^{-c \left[ M^i ||u_l|| + M^i ||u_0^l|| + M^{-i} ||v_l|| \right]} \prod_v \delta_v.$$

$$\leq K^n \prod_l M_l^i \int \prod_l du_l du_0^l dv_l e^{-c \left[ M^i ||u_l|| + M^i ||u_0^l|| + M^{-i} ||v_l|| \right]} \delta_v.$$  

(2.28)

Then we can find that any long variable integrated at scale $i$ costs $KM^{2i}$. The integration over the non-commutative short variable at scale $i$ brings $KM^{-2i}$, and the commutative one brings $KM^{-i}$ (there is no long variable in the commutative dimension) so the integration over each tree line at scale $i$ brings a total convergent factor $KM^{-3i}$. The variables “solved” by the $\delta$ functions bring or cost nothing.

For an orientable graph we should solve the $n - 1$ long variables $v_l$’s of the tree propagators in terms of the other variables, because this is the maximal number of...
long variables that we can solve, and they have highest possible indices because $T$ has been chosen compatible with the Gallavotti-Nicolò tree structure. We should study more carefully the commutative variable which is the 0th dimension of any tree line of $T$. While the model for the non-commutative variables is non local, it is local for the commutative variables. So we can’t integrate over all the position variables (or the equivalent line variables) but have to save one, the root (we name it $x_{\nu 0}$). We will use this point when we perform the renormalisation where the amputated amplitude of any connected component depends only on one commutative external position $x_{\nu 0}$. This point is also very important for the power counting of the non-orientable model as it implies the maximal number of commutative short variables we can integrate over is $n - 1$ not $n$. Finally we still have the last $\delta_{b_0}$ function (equivalent to the overall momentum conservation in the commutative case). It is optimal to use it to solve one external variable (if any) in terms of all the short variables and the external ones. Since external variables are typically smeared against unit scale test functions, this leaves power counting invariant.

We now define $S$ the set of long variables to be solved via the $\delta$ functions hence the set of $n - 1$ tree lines as there are only orientable graphs in our model.

Gathering all the corresponding factors together with the propagators prefactors $M^i$ leads to the following bound:

$$|A_{G,\mu}| \leq K^n \prod_l M^{i_l} \prod_{l \in S} M^{-3i_l}. \quad (2.29)$$

In the usual way of [27] we write

$$\prod_l M^{i_l} = \prod_l \prod_{i=1}^{i_l} M = \prod_i \prod_{l \in G^i_k} M = \prod_{i,k} M^{l(G^i_k)} \quad (2.30)$$

and

$$\prod_{l \in S} M^{-3i_l} = \prod_{l \in S} \prod_{i=1}^{i_l} M^{-3} = \prod_{i,k} \prod_{l \in G^i_k \cap S} M^{-3} \quad (2.31)$$

and we must now only count the number of elements in $G^i_k \cap S$.

As remarked above $G^i_k \cap S = T^i_k$, and the cardinal of $T^i_k$ is $n(G^i_k) - 1$.

Using the fact that $2l(G^i_k) - 6n_{\theta}(G^i_k) - 4n_4 = -N(G^i_k)$ we can summarise these results in the following lemma:
Lemma 2.5 The following bound holds for a connected graph of $(\varphi \star \varphi)^3$ model (with external arguments integrated against fixed smooth test functions):

$$|A_{G,\mu}| \leq K^n \prod_{i,k} M^{-\omega(G^i_k)}$$  \hspace{1cm} (2.32)

for some (large) constant $K$, with $\omega(G^i_k) = N(G^i_k)/2 + n_4 - 3$

This lemma proves the power counting for orientable graphs. But it is not yet sufficient for a renormalization theorem to all orders of perturbation. Indeed only planar graphs with a single broken face look like Moyal products when their internal indices become much higher than their external ones. So we must prove that the non-planar graphs or graphs with more than one broken face have better power counting than what Lemma 2.5 states. Vertices oscillations should be taken into account to prove that, and this is done in the next section.

2.4 Improved Power Counting

Recall that for any non-commutative Feynman graph $G$ we can define the genus of the graph, called $g$ and the number of faces “broken by external legs”, called $B$ \cite{7, 10}. For a general graph, we have $g \geq 0$ and $B \geq 1$.

In the previous section we established that

$$\omega(G) \geq N/2 + n_4 - 3, \text{ if } G \text{ orientable.}$$  \hspace{1cm} (2.33)

The subgraphs with $g = 0$ and $B = 1$ are called planar regular. We want to prove that they are the only non-vacuum graphs with $\omega \leq 0$.

It is easy to check that planar regular subgraphs are orientable, but the converse is not true. To prove that orientable non-planar subgraphs or orientable planar subgraphs with $B \geq 2$ are irrelevant requires to use a bit of the vertices oscillations to improve Lemma 2.5 and get:

Lemma 2.6 For orientable subgraphs with $g \geq 1$ we have

$$\omega(G) \geq N/2 + n_4 + 1.$$  \hspace{1cm} (2.34)

For orientable subgraphs with $g = 0$ and $B \geq 2$ we have

$$\omega(G) \geq N/2 + n_4 - 1.$$  \hspace{1cm} (2.35)
This lemma is sufficient for the purpose of this paper. It implies directly that graphs which contain only irrelevant subgraphs have finite amplitudes which are uniformly bounded by $K^n$, using the standard method of [27] to bound the assignment sum over $\mu$ in (2.26).

The rest of this subsection is essentially devoted to the proof of this Lemma. We return before solving $\delta$ functions, hence to the $v$ variables. We will need only to compute the oscillations which are quadratic in the long variables $v$’s to prove (2.34) and the linear oscillations in $v\theta^{-1}x$ to prove (2.35). Fortunately an analog problem was solved in momentum space by Filk and Chepelev-Roiban [28, 29] and adapted to position routing by Gurau et al. [11]. We just borrow from the method of [11]. As the procedures for our paper are almost the same as that for $\phi^4$ in [11], we reproduce the argument as concisely as possible, and we refer to [11, 16] for more details. The short variables are inessential in this subsection, as the integration of them always bring about convergent terms. But it is convenient to treat on the same footing the long $v$ and the external $x$ variables, so we introduce a new global notation $y$ for all these variables. Then the vertices rewrite as

$$
\prod_v \delta(y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + \varepsilon^i u_i) e^i \left( \sum_{i<j} (-1)^{i+j+1} y_i \theta^{-1} y_j + yQu + uRu \right) \tag{2.36}
$$

for some inessential signs $\varepsilon^i$ and some symplectic matrices $Q$ and $R$. As there are no oscillations for the commutative coordinates, there are no Filk moves for them. Since the precise oscillations in the short $u$ variables is not important to this problem, we will note in the sequel $E_u$ any linear combination of the $u$ variables. Let’s consider the first Filk reduction [28], which contracts tree lines of the graph. It creates progressively generalised vertices with even number of fields. At a given induction step and for a tree line joining two such generalised vertices with respectively $p$ and $q - p + 1$ fields (suppose $p$ is even and $q$ is odd), we assume by induction that the two vertices are

$$
\delta(y_1 - y_2 + y_3\ldots - y_p + E_u) \delta(y_p - y_{p+1} + \ldots - y_q + E_u) e^i \left( \sum_{i<j} (-1)^{i+j+1} y_i \theta^{-1} y_j + \sum_{p<i<j<q} (-1)^{i+j+1} y_i \theta^{-1} y_j + yQu + uRu \right). \tag{2.37}
$$

Using the second $\delta$ function we see that:

$$
y_p = y_{p+1} - y_{p+2} + \ldots + y_q - E_u . \tag{2.38}
$$

Substituting this expression in the first $\delta$ function we get:

$$
\delta(y_1 - y_2 + \ldots - y_{p+1} + \ldots - y_q + E_u) \delta(y_p - y_{p+1} + \ldots - y_q + E_u) e^i \left( \sum_{i<j} (-1)^{i+j+1} y_i \theta^{-1} y_j + \sum_{p<i<j<q} (-1)^{i+j+1} y_i \theta^{-1} y_j + yQu + uRu \right). \tag{2.39}
$$
The quadratic terms which include \( y_p \) in the exponential are (taking into account that \( p \) is an even number):

\[
\sum_{i=1}^{p-1} (-1)^{i+1} y_i \theta^{-1} y_p + \sum_{j=p+1}^q (-1)^{j+1} y_p \theta^{-1} y_j \quad (2.40)
\]

Using the expression (2.38) for \( y_p \) we see that the second term gives only terms in \( y_L u \), as \( \theta \) is antisymmetry. The first term yields:

\[
\sum_{i=1}^{p-1} \sum_{j=p+1}^q (-1)^{i+1+j+1} y_i \theta^{-1} y_j = \sum_{i=1}^{p-1} \sum_{k=p}^{q-1} (-1)^{i+k+1} y_i \theta^{-1} y_j , \quad (2.41)
\]

which reconstitutes the crossed terms, and we have recovered the inductive form of the larger generalised vertex.

After each Filk move we will have two more vertices. So by this procedure we will always treat only even vertices. We finally rewrite the product of the two vertices as:

\[
\delta(y_1 - y_2 + \ldots + y_{p-1} - y_{p+1} + \ldots - y_q + E_u) \delta(y_p - y_{p-1} + \ldots - y_q + E_u) \\
e^{i \left( \sum_{1 \leq i < j \leq q} (-1)^{i+j+1} y_i \theta^{-1} y_j + yQu + uRu \right)}, \quad (2.42)
\]

where the exponential is written in terms of the reindexed vertex variables. In this way we can contract all lines of a spanning tree \( T \) and reduce \( G \) to a single vertex with “tadpole loops” called a “rosette graph” [29]. In this rosette to keep track of cyclicity is essential so we draw the rosette as a cycle (which is the border of the former tree) bearing loops lines on it (see Figure 3). Remark that the rosette can also be considered as a big vertex, with \( r = 4n_6 + 2n_4 + 2 \) fields, on which \( N \) are external fields with external variables \( x \) and \( 4n_6 + 2n_4 + 2 - N \) are loop fields for the corresponding \( 2n_6 + n_4 + 1 - N/2 \) loops. When the graph is orientable, the long variables \( y_l \) for \( l \) in \( T \) will disappear in the rosette. Let us call \( z \) the set of remaining long loop and external variables. Then the rosette vertex factor is

\[
\delta(z_1 - z_2 + \ldots - z_r + E_u) e^{i \left( \sum_{1 \leq i < j \leq r} (-1)^{i+j+1} z_i \theta^{-1} z_j + zQu + uRu \right)}. \quad (2.43)
\]

We can go on performing inductively the first Filk move and the net effect is simply to rewrite the root branch \( \delta \) function and the combination of all vertices oscillations (using the other \( \delta \) functions) as the new big vertex or rosette factor (2.43).
The second Filk reduction [28] further simplifies the rosette factor by erasing the loops of the rosette which do not cross any other loops or arch over external fields. Putting together all the terms in the exponential which contain $z_l$ we conclude exactly as in [28] that these long $z$ variables completely disappear from the rosette oscillation factor, which simplifies as in [29] to

$$\delta(\sum_{i=1}^{r} (z_i - E_u) + E_u) e^{i(\sum_{I} z_I + \sum_{Q} u_R)}$$

(2.44)

where $I_{ij}$ is the antisymmetric “intersection matrix” of [29] (up to a different sign convention). Here $I_{ij} = +1$ if oriented loop line $i$ crosses oriented loop line $j$ coming from its right, $I_{ij} = -1$ if $i$ crosses $j$ coming from its left, and $I_{ij} = 0$ if $i$ and $j$ do not cross. These formulas are also true for $i$ external line and $j$ loop line or the converse, provided one extends the external lines from the rosette circle radially to infinity to see their crossing with the loops. Finally when $i$ and $j$ are external lines one should define $I_{ij} = (-1)^{p+q+1}$ if $p$ and $q$ are the numbering of the lines on the rosette cycle (starting from an arbitrary origin).

If a node $G_k^i$ of the Gallavotti-Nicolò tree is orientable but non-planar ($g \geq 1$), there must therefore exist at least two intersecting loop lines in the rosette corresponding to this $G_k^i$, with long variables $w_1$ and $w_2$. Moreover since $G_k^i$ is orientable, none of the long loop variables associated with these two lines belongs to the set $S$ of long variables eliminated by the $\delta$ constraints. Therefore, after integrating the variables in $S$ the basic mechanism to improve the power counting of a single non

Figure 3: A typical rosette
planar subgraph is the following:
\[
\int dw_1 dw_2 e^{-M^{-2i_1} w_1^2 - M^{-2i_2} w_2^2 - i w_1 \theta - w_1 E_1(x,u) + w_2 E_2(x,u)}
\]
\[
= \int dw'_1 dw'_2 e^{-M^{-2i_1} (w'_1)^2 - M^{-2i_2} (w'_2)^2 + i w'_1 \theta - w'_1 (u,x) Q(u,x)}
\]
\[
= K M^{2i_1} \int dw'_2 e^{-(M^{2i_1} + M^{-2i_2})(w'_2)^2} = K M^{2i_1} \frac{M^{-2i_1}}{1 + M^{-2(i_1 + i_2)}} \leq K. \quad (2.45)
\]

In these equations we used for simplicity \(M^{-2i}\) instead of the correct but more complicated factor \((\Omega/4) \tanh(\alpha/2)\) (see (2.18)) (of course this does not change the argument) and we performed a unitary linear change of variable \(s'\)

\[
w'_1 = w_1 + \ell_1(x,u), \quad w'_2 = w_2 + \ell_2(x,u)
\]
to compute the oscillating integral. The gain in (2.45) is \(M^{-2i_1-2i_2}\), which is the difference between \(O(1)\) and the normal factor \(M^{2i_1+2i_2}\) that would be generated by the integrals over \(w_1\) and \(w_2\) if there were not the oscillation term \(i w_1 \theta - w_2\).

So after the integration of the non-commutative part of the two clashing lines the gain is almost \(M^{-4i}\).

This basic argument must then be generalised to each non-planar leaf in the Gallavotti-Nicolò tree. Actually, in any orientable non-planar ‘primitive’ \(G^i_k\) node (i.e. not containing sub non-planar nodes) we can choose an arbitrary pair of crossing loop lines which will be integrated as in (2.45) using this oscillation. The corresponding improvements are independent.

This leads to an improved amplitude bound:
\[
|A_{G,\mu}| \leq K^n \prod_{i,k} M^{-\omega(G^i_k)} \quad (2.46)
\]

where now \(\omega(G^i_k) = N(G^i_k)/2 + n_4 + 1\) if \(G^i_k\) is orientable and non planar (i.e. \(g \geq 1\)). This bound proves (2.34).

Finally it remains to consider the case of nodes \(G^i_k\) which are planar orientable but with \(B \geq 2\). In that case there are no crossing loops in the rosette but there must be at least one loop line arching over a non trivial subset of external legs in the \(G^i_k\) rosette (see line 6 in Figure 3). We have then a non trivial integration over at least one external variable, called \(x\), of at least one long loop variable called \(u\). This “external” \(x\) variable without the oscillation improvement would be integrated with a test function of scale 1 (if it is a true external line of scale 1) or better (if it
is a higher long loop variable\footnote{Since the loop line arches over a non trivial (i.e. neither full nor empty) subset of external legs of the rosette, the variable $x$ cannot be the full combination of external variables in the “root” $\delta$ function.}. But we get now

$$
\int dx dw e^{-M^{-2i}w^2 - iw\theta^{-1}x + w\cdot E_1(x', u)}
= KM^{2i} \int dx e^{-M^{+2i}x^2} = K'.
$$

(2.47)

We find that a factor $M^{2i}$ in the former bound becomes $O(1)$ hence is improved by $M^{-2i}$. So the power counting is $\omega(G_k^i) = N(G_k^i)/2 - 1 + n_4$. We find that the two point graphs with $n_4 = 0$ and $N(G_k^i) = 2$ maybe logarithmically divergent. They do not appear renormalizable at first sight. But we remark that in the orientable $(\bar{\varphi} \star \varphi)^3$ model there will never be such subgraphs with $N(G_k^i) = 2$ and $B = 2$. This is the reason we limit ourselves to this case\footnote{We thank our referee for correcting an earlier version of this paper, which lead us to this important point.}. Then all graphs with $B \geq 2$ are also safe. The only divergent graphs which need renormalization are the planar regular graphs.

\section{Renormalization}

In this section we need to consider only divergent subgraphs, namely the planar two point, four point and six point subgraphs with a single external face ($g = 0$, $B = 1$, $N = 2, 4, 6$ for $n_4 = 0$, $N = 2, 4$ for $n_4 = 1$, and $N = 2$ for $n_4 = 2$). We shall prove that they can be renormalized by appropriate counterterms of the form of the initial Lagrangian. We would like to remark that for any graph, contrary to the non-commutative variables, the commutative variables of the external points are local. So there is only one integral over the commutative variable for each vertex.

\subsection{Renormalization of the Six-point Function}

Consider a 6 point subgraph which needs to be renormalized, hence is a node of the Gallavotti-Nicolò tree. This means that there is $(i, k)$ such that $N(G_k^i) = 6$. The six external positions of the amputated graph are labelled $x_1, x_2, x_3, x_4, x_5$ and $x_6$. We also define $Q, R$ and $S$ as three skew-symmetric matrices of respective sizes $6 \times l(G_k^i)$, $l(G_k^i) \times l(G_k^i)$ and $2[n_6(G_k^i) - 1] \times l(G_k^i)$, where we recall that $2(n(G)_6 - 1)$ is the
number of loops of a 6 point graph with $n_6$ vertices. The amplitude associated to the connected component $G_k^i$ is then

$$A(G_k^i)(x_1, x_2, x_3, x_4, x_5, x_6, x_0^0) = \int \prod_{\ell \in T^i_k} du_\ell du_\ell^0 C_\ell(x, u, u^0, w)$$

$$\prod_{\ell \in G_k^i, l \notin T} du_\ell du_\ell^0 dw_l C_l(u_l, u_l^0, w_l) \delta\left(x_1 - x_2 + x_3 - x_4 + x_5 - x_6 + \sum_{l \in G_k^i} u_l\right)$$

$$\times e^{i\left(\sum_{p<q}(-1)^{p+q+1}x_p\theta^{-1}x_q + XQU + URU + 1SW\right)}.$$  \hspace{1cm} (3.1)

Here the variable $x_0^0$ is the root commutative variable as discussed in section (2.3) and we will write it as $x^0$ hereafter. The exact form of the factor

$$\sum_{p<q}(-1)^{p+q+1}x_p\theta^{-1}x_q$$

is not essential for this paper and was discussed exhaustively in \[11] \[16]. The important fact is that there are no quadratic oscillations in $X$ times $W$ (because $B = 1$) nor in $W$ times $W$ (because $g = 0$). $C_l$ is the propagator of the line $l$. For loop lines $C_l$ is expressed in terms of $u_l$ and $w_l$ by formula (2.18), (with $v$ replaced by our notation $w$ for long variables of loop lines). But for tree lines $\ell \in T^i_k$ recall that the solution of the system of branch $\delta$ functions for $T$ has reexpressed the corresponding long variables $v_\ell$ in terms of the short variables $u$, and the external and long loop variables of the branch graph $G_\ell$ which lies “over” $\ell$ in the rooted tree $T$. This is the essential content of subsection 2.2. More precisely consider a line $\ell \in T^i_k$ with scale $i(\ell) \geq i$; we can write

$$v_\ell = X_\ell + W_\ell + U_\ell$$  \hspace{1cm} (3.2)

where

$$X_\ell = \sum_{e \in E(\ell)} \varepsilon_{\ell,e} x_e$$  \hspace{1cm} (3.3)

is a linear combination on the set of external variables of the branch graph $G_\ell$ with the correct alternating signs $\varepsilon_{\ell,e}$,

$$W_\ell = \sum_{l \in L(\ell)} \varepsilon_{\ell,l} w_l$$  \hspace{1cm} (3.4)

is a linear combination over the set $L(\ell)$ of long loop variables for the external lines of $G_\ell$ (and $\varepsilon_{\ell,l}$ are other signs), and

$$U_\ell = \sum_{l' \in s(\ell)} \varepsilon_{\ell,l'} u_{l'}$$  \hspace{1cm} (3.5)

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is a linear combination over a set $S_\ell$ of short variables that we do not need to know explicitly. The tree propagator for line $\ell$ then is

$$C_\ell(u_\ell, X_\ell, U_\ell, W_\ell, u^0_\ell) = \int_{M^{-2(l-1)}} \frac{\sqrt{\Omega_\ell \alpha_\ell^{-\frac{1}{2}} d\alpha_\ell e^{-\frac{\alpha_\ell}{2} \left( \coth(\alpha_\ell) u_\ell^2 + \tanh(\alpha_\ell) [X_\ell + W_\ell + U_\ell]^2 \right) - \frac{\alpha_\ell^2}{2\pi} \Omega}}}{2\sqrt{2\pi} \sinh(\alpha_\ell)}. \quad (3.6)$$

To renormalize, let us call $e = \max e_p, p = 1, \ldots, 6$ the highest external index of the subgraph $G^i_k$. We have $e < i$ since $G^i_k$ is a node of the Gallavotti-Nicolò tree. We evaluate $A(G^i_k)$ on external fields $\varphi^{\leq \epsilon}(x_p, x^0)$ and $\varphi^{\leq \epsilon}(x_B, x^0)$ as:

$$A(G^i_k) = \int \prod_{p=1}^{6} dx_p dx^0 \varphi^{\leq \epsilon}(x_1, x^0) \varphi^{\leq \epsilon}(x_2, x^0) \varphi^{\leq \epsilon}(x_3, x^0) \varphi^{\leq \epsilon}(x_4, x^0) \times \varphi^{\leq \epsilon}(x_5, x^0) \varphi^{\leq \epsilon}(x_6, x^0) A(G^i_k)(x_1, x_2, x_3, x_4, x_5, x_6, x^0)$$

$$\quad = \int \prod_{p=1}^{6} dx_p dx^0 e^{\text{Ext}} \prod_{\ell \in T^i_k} du_\ell du^0_\ell C_\ell(u_\ell, u^0_\ell, tX_\ell U_\ell, W_\ell) \right) \left. \left[ \Delta + t \sum_{\ell \in G^i_k} \right] \right|_{t=1}$$

This formula is designed so that at $t = 0$ all dependence on the external variables $x$ factorizes out of the $u, w$ integral in the desired vertex form for renormalization of the $\varphi \star \varphi \star \varphi \star \varphi \star \varphi$ interaction in the action (2.2). We now perform a Taylor expansion to first order with respect to the $t$ variable and prove that the remainder term is irrelevant. Let $\mathcal{U} = \sum_{\ell \in G^i_k} u_\ell$, and

$$\Re(t) = - \sum_{\ell \in T^i_k} \frac{\Omega}{4} \tanh(\frac{\alpha_\ell}{2}) \left\{ t^2 X_\ell^2 + 2tX_\ell \left[ W_\ell + U_\ell \right] \right\} \equiv -t^2 AX.X - 2tAX.(W + U). \quad (3.8)$$

For the external index to be exactly $e$ the external smearing factor should be in fact $\prod_p \varphi^{\leq \epsilon}(x_p) - \prod_p \varphi^{\leq \epsilon-1}(x_p)$ but this subtlety is inessential.
where $A_\ell = \frac\Omega 4 \tanh(\frac{\alpha_\ell}{2})$, and $X \cdot Y$ means $\sum_{\ell \in T_k^i} X_\ell Y_\ell$. We have

$$A(G_k^i) = \int \prod_{p=1}^6 dx_p dx^0 \varphi^{\leq e}(x_p, x^0) e^{\text{Ext}} \prod_{\ell \in T_k^i} du_\ell du^0_\ell C_\ell(u_\ell, u^0_\ell, U_\ell, W_\ell)$$

$$\left[ \prod_{\ell \in G_k^i} du_\ell du^0_\ell dw_\ell C_\ell(u_\ell, w_\ell) \right] e^{\mu RU + U SW}$$

$$\left\{ \delta(\Delta) + \int_0^1 dt \left[ U \cdot \nabla \delta(\Delta + t\Delta) + \delta(\Delta + t\Delta)[tXQU + \mathcal{R}'(t)] \right] e^{\mu XQU + \mu \mathcal{R}(t)} \right\}$$

where $C_\ell(u_\ell, U_\ell, W_\ell)$ is given by (3.6) but taken at $X_\ell = 0$.

The first term, denoted by $\tau A$, is of the desired form (2.16) times a number independent of the external variables $x$. It is asymptotically constant in the slice index $i$, hence the sum over $i$ at fixed $e$ is logarithmically divergent: this is the divergence expected for the six-point function. It remains only to check that $(1 - \tau)A$ converges as $i - e \to \infty$. But we have three types of terms in $(1 - \tau)A$, each providing a specific improvement over the regular, log-divergent power counting of $A$:

- The term $U \cdot \nabla \delta(\Delta + t\Delta)$. For this term, integrating by parts over external variables, the $\nabla$ acts on external fields $\varphi^{\leq e}$, hence brings at most $Me$ to the bound, whether the $U$ term brings at least $M^{-i}$.

- The term $XQU$. Here $X$ brings at most $Me$ and $U$ brings at least $M^{-i}$.

- The term $\mathcal{R}'(t)$. It decomposes into terms in $\mathcal{A}X \cdot X$, $\mathcal{A}X \cdot U$ and $\mathcal{A}X \cdot W$. Here the $\mathcal{A}_\ell$ brings at least $M^{-2i(\ell)}$, $X$ brings at worst $Me$, $U$ brings at least $M^{-i}$ and $X_\ell W_\ell$ brings at worst $M^{e+i(\ell)}$. This last point is the only subtle one: if $\ell \in T_k^i$, remark that because $T_k^i$ is a sub-tree within each Gallavotti-Nicolò subnode of $G_k^i$, in particular all parameters $w_\ell$ for $\ell' \in \mathcal{L}(\ell)$ which appear in $W_\ell$ must have indices lower or equal to $i(\ell)$ otherwise they would have been chosen instead of $\ell$ in $T_k^i$).

In conclusion, since $i(\ell) \geq i$, the Taylor remainder term $(1 - \tau)A$ improves the power-counting of the connected component $G_k^i$ by a factor at least $M^{-(i-e)}$. This additional $M^{-(i-e)}$ factor makes $(1 - \tau)A(G_k^i)$ convergent and irrelevant as desired.

### 3.2 Renormalization of the Four-point Function

Consider a 4 point subgraph which needs to be renormalized, hence is a node of the Gallavotti-Nicolò tree. This means that there is $(i, k)$ such that $N(G_k^i) = 4$. 


The four external positions of the amputated graph are labelled \(x_1, x_2, x_3\) and \(x_4\). We also define \(Q, R\) and \(S\) as three skew-symmetric matrices of respective sizes \(4 \times l(G_k^i)\), \(l(G_k^i) \times l(G_k^i)\) and \([2n_0(G_k^i) + n_4(G_k^i) - 1] \times l(G_k^i)\), where we recall that \([2n_0(G_k^i) + n_4(G_k^i) - 1]\) is the number of loops of a 4 point graph with \(n_0 + n_4\) vertices. The amplitude associated to the connected component \(G_k^i\) is then

\[
A(G_k^i)(x_1, x_2, x_3, x_4, x^0) = \int \prod_{t \in T_k^i} du_i du^0_i C_t(x, u, u^0, w) \prod_{l \in G_k^i, l \notin T} du_l du^0_l dw_l C_l(u_l, u^0_l, w_l) \delta(x_1 - x_2 + x_3 - x_4 + \sum_{l \in G_k^i} u_l) e^{i(\sum_{p<q} (-1)^{p+q+1} x_p \theta^{-1} x_q + XQU + URU + USW)}.
\]  

The renormalization procedure is almost the same as that for the 6 point function. Let us call \(e = \max e_p, p = 1, ..., 4\) the highest external index of the subgraph \(G_k^i\). We have \(e < i\) since \(G_k^i\) is a node of the Gallavotti-Nicolò tree. We evaluate \(A(G_k^i)\) on external fields \(\varphi^{\leq e}(x_p, x^0)\) and \(\varphi^{< e}(x_p, x^0)\) as:

\[
A(G_k^i) = \int \prod_{p=1}^4 dx_p dx^0 \varphi^{\leq e}(x_1, x^0) \varphi^{\leq e}(x_2, x^0) \varphi^{\leq e}(x_3, x^0) \varphi^{< e}(x_4, x^0) A(G_k^i)(x_1, x_2, x_3, x_4, x^0) \times A(G_k^i)(x_1, x_2, x_3, x_4, x^0)
\]

\[
= \int \prod_{p=1}^4 dx_p dx^0 \varphi^{\leq e}(x_1, x^0) \varphi^{\leq e}(x_2, x^0) \varphi^{\leq e}(x_3, x^0) \varphi^{< e}(x_4, x^0) e^{i\text{Ext}}
\]

\[
\prod_{t \in T_k^i} du_t du^0_t C_t(u_t, u^0_t, tXU U_t, W_t)
\]

\[
\prod_{l \in G_k^i} du_l du^0_l dw_l C_l(u_l, u^0_l, w_l) \delta(\Delta + t \sum_{l \in G_k^i} u_l) e^{iXQU + URU + USW} \bigg|_{t=1}
\]

with \(\Delta = x_1 - x_2 + x_3 - x_4\) and \(\text{Ext} = \sum_{p<q=1}^4 (-1)^{p+q+1} x_p \theta^{-1} x_q\). Then we have
\[ A(G^i_k) = \int \prod_{p=1}^{4} dx_p dx_0 \varphi^{\leq e}(x_p, x^0) e^{\nu \text{ext}} \prod_{\ell \in T_k^i} du_\ell d\mu_\ell^0 C_\ell(u_\ell, u_\ell^0, U_\ell, W_\ell) \]

\[ \left[ \prod_{\ell \in G^i_k} du_\ell d\mu_\ell^0 dw_\ell C_\ell(u_\ell, u_\ell^0, w_\ell) \right] e^{iU RU + iU SW} \]

\[ \left\{ \delta(\Delta) + U^\mu \cdot \nabla_\mu \delta(\Delta) - [iX QU - 2AX(W + U)] \times \delta(\Delta) \right. \]

\[ + \left. \frac{1}{2} \int_0^1 dt (1 - t) \left[ (\nabla \cdot \nabla)^2 \delta(\Delta + tU) + f''(t) \delta(\Delta + tU) \right] e^{itXQU + iR(t)} \right\} \]

where \( R, X \) and \( U \) are the same as (3.8), and again \( C_\ell(u_\ell, u_\ell^0, U_\ell, W_\ell) \) is given by (3.6) but taken at \( X_\ell = 0 \).

The first term, denoted by \( \tau A \), is of the desired form (2.15) times a number independent of the external variables \( x \). It is is linearly divergent: this is the divergence expected for the four-point function. It remains only to check that \( (1 - \tau)A \) converges as \( i - e \to \infty \). But we have three types of terms in \( (1 - \tau)A \), each providing a specific improvement over the regular, log-divergent power counting of \( A \):

- The term \( U^\mu \cdot \nabla_\mu \delta(\Delta) \) vanishes due to the parity, as it is odd integral over \( u \).
- The third term (the terms linearly proportional to \( U \) and \( W \)) on the r.h.s. is zero due to the parity, as they are also odd integrals over \( u \) and \( w \).
- In the remainder terms of tailor expansion, for the term \( U^2 \cdot \nabla^2 \delta(\Delta + tU) \) the \( \nabla^2 \) brings \( M^{2e} \) through the integral by parts and the \( U^2 \) brings \( M^{-2i} \). So it is convergent.
- for the last term, \( f''(t) = -2A.X.X + (XQU)^2 + 4[AX(W + U)]^2 \]

- \( -4iXQUAX(W + U) \) and this term is convergent.

### 3.3 Renormalization of the Two-point Function

We consider now the nodes such that \( N(G^i_k) = 2 \). We use the same notations than in the previous subsection. The two external points are labelled \( x \) and \( y \). Using the global \( \delta \) function, which is now \( \delta(x - y + \Omega) \), we remark that the external oscillation
\[ e^{ix\theta - iy} \text{ can be absorbed in a redefinition of the term } e^{itXQU}, \text{ which we do from now on. The full amplitude is} \]

\[ A(G_k) = \int dxdydx^0 \varphi^{\leq\epsilon}(x, x^0) \varphi^{\leq\epsilon}(y, x^0) \delta(x - y + \Omega) \prod_{l \in G_k, l \notin T} du_l du_l^0 dw_l \]

\[ C_l(u_l, u_l^0, w_l) \prod_{\ell \in T_k} du_l du_l^0 C_{\ell}(u_{\ell}, u_{\ell}^0, X_{\ell}, U_{\ell}, W_{\ell}) e^{iXQU + RU + SW}. \]

We first perform the Taylor expansion in the position variables of external fields:

\[
\begin{align*}
\bar{\varphi}^{\leq\epsilon}(x, x^0) \varphi^{\leq\epsilon}(y, x^0) \delta(x - y + \Omega) &= \varphi^{\leq\epsilon}(x, x^0) \varphi^{\leq\epsilon}(y, x^0) \delta(x - y + s\Omega) \\
&= \varphi^{\leq\epsilon}(x, x^0) \varphi^{\leq\epsilon}(y, x^0) \left[ \delta(x - y) + \Omega \cdot \nabla \delta(x - y) + \frac{1}{2} (\Omega \cdot \nabla)^2 \delta(x - y) + \frac{1}{2} \int_0^1 ds (1 - s)^2 (\Omega \cdot \nabla)^3 \delta(x - y + s\Omega) \right].
\end{align*}
\] (3.13)

We perform then a Taylor expansion in \( t \) at order 3 of the remaining function

\[ f(t) = e^{itXQU + \mathfrak{R}(t)}, \] (3.14)

where we recall that \( \mathfrak{R}(t) = -[t^2 AX.X + 2t AX.(W + U)] \). We get

\[ A_0 = \int dxdx^0 \varphi^{\leq\epsilon}(x, x^0) \varphi^{\leq\epsilon}(x, x^0) e^{i(URU + USW)} \prod_{l \in G_k, l \notin T} du_l du_l^0 dw_l C_l(u_l, w_l) \prod_{\ell \in T_k} du_l du_l^0 C_{\ell}(u_{\ell}, U_{\ell}, W_{\ell}) \]

\[ \left( f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{2} \int_0^1 dt (1 - t)^2 f^{(3)}(t) \right). \] (3.15)

In order to evaluate that expression, let \( A_{0,0}, A_{0,1}, A_{0,2} \) be the zeroth, first and second order terms in this Taylor expansion, and \( A_{0,R} \) be the remainder term. First,

\[ A_{0,0} = \int dxdx^0 \varphi^{\leq\epsilon}(x, x^0) \varphi^{\leq\epsilon}(x, x^0) e^{i(URU + USW)} \prod_{l \in G_k, l \notin T} du_l du_l^0 dw_l C_l(u_l, w_l) \prod_{\ell \in T_k} du_l du_l^0 C_{\ell}(u_{\ell}, U_{\ell}, W_{\ell}) \] (3.16)
is quadratically divergent and is exactly the expected form for the mass counterterm. Then

\[
A_{0,1} = \int dx^0 dx \bar{\varphi}_{\leq \varepsilon} x, x^0) \varphi_{\leq \varepsilon} (x, x^0) e^{i(U_R U + U_S W)} \prod_{l \in G_k, l \notin T} du_l du_l C_l(u_l, w_l) \\
\prod_{l \in T_k} du_l du_l^0 C_l(u_l, u_l^0, U_l, W_l) \left( i X QU + \mathfrak{R}'(0) \right) \tag{3.17}
\]

vanishes identically. Indeed all the terms are odd integrals over the \( u \) and \( w \) variables.

\[
A_{0,2} = \int dx^0 dx \bar{\varphi}_{\leq \varepsilon} x, x^0) \varphi_{\leq \varepsilon} (x, x^0) e^{i(U_R U + U_S W)} \prod_{l \in G_k, l \notin T} du_l du_l C_l(u_l, w_l) \\
\prod_{l \in T_k} du_l C_l(u_l, U_l, W_l) \left( \begin{array}{c} -(X QU)^2 \\
-4i X QU AX \cdot (W + U) - 2AX \cdot X + 4[AX \cdot (W + U)]^2 \end{array} \right) \tag{3.18}
\]

The four terms in \((X QU)^2\), \(X QU AX \cdot W\), \(AX \cdot X\) and \([AX \cdot W]^2\) are logarithmically divergent and contribute to the renormalization of the harmonic frequency term \(\Omega\) in (2.2). (The terms in \(x^\mu x^\nu\) with \(\mu \neq \nu\) do not survive by parity and the terms in \((x^\mu)^2\) have obviously the same coefficient.) The other terms in \(X QU AX \cdot U\), \((AX \cdot U)(AX \cdot W)\) and \([AX \cdot U]^2\) are irrelevant. Similarly the terms in \(A_{0,R}(x)\) are all irrelevant.

Next we have to consider the terms of the first order expansion in external variables in (3.13), for which we need to develop the \( f \) function only to second order. We have

\[
A_1 = \int dx dy dx^0 \bar{\varphi}_{\leq \varepsilon} x, x^0) \varphi_{\leq \varepsilon} (y, x^0) \left[ \mathbf{U} \cdot \nabla \delta(x - y) \right] e^{i(U_R U + U_S W)} \prod_{l \in G_k, l \notin T} du_l du_l \\
\times C_l(u_l, w_l) \prod_{l \in T_k} du_l du_l^0 C_l(u_l, u_l^0, U_l, W_l) \left( f(0) + f'(0) + \int_0^1 dt (1 - t) f''(t) dt \right) \tag{3.19}
\]
The first term is

$$A_{1,0} = \int dx dy d^0 \varphi \leq e(x, x^0) \varphi \leq e(y, x^0) \left[ \mathbf{U} \cdot \nabla \delta(x - y) \right] e^{i(U'R + U'S)}$$

\[
\times \prod_{l \in G_k} du_l dw_l \times C_l(u_l, w_l) \prod_{\ell \in T_k} du_\ell du_\ell^0 C_\ell(u_\ell, u_\ell^0, U_\ell, W_\ell),
\]  

(3.20)

which vanishes identically due to the parity.

The second term is

$$A_{1,0} = \int dx dy d^0 \varphi \leq e(x, x^0) \varphi \leq e(y, x^0) \left[ \mathbf{U} \cdot \nabla \delta(x - y) \right] e^{i(U'R + U'S)}$$

\[
\times \prod_{l \in G_k} du_l dw_l \times C_l(u_l, w_l) \prod_{\ell \in T_k} du_\ell du_\ell^0 C_\ell(u_\ell, u_\ell^0, U_\ell, W_\ell)
\]

\[
\times \left[ iXQU - 2AX(W + U) \right].
\]  

(3.21)

The first term in the r.h.s. is

$$\int dx dy d^0 \varphi \leq e(x, x^0) \mathbf{U} \cdot (-i) \nabla \varphi \leq e(x, x^0) e^{i(U'R + U'S)}$$

\[
\times \prod_{l \in G_k} du_l dw_l \times C_l(u_l, w_l) \prod_{\ell \in T_k} du_\ell du_\ell^0 C_\ell(u_\ell, u_\ell^0, U_\ell, W_\ell) \times XQU
\]  

(3.22)

It is logarithmically divergent and is proportional to the term \(\varphi(x \wedge p)\varphi\). But this term is also vanishing, as for each graph there is always a mirror graph (see [14] for details) that is the same as the former but reflected in a mirror. When we add them up, e.g. the tadpole up with the tadpole down, the result is zero.

- The term \(AXW \cdot \mathbf{U} \cdot \nabla \varphi \leq e(x, x^0)\). The operator \(\nabla\) brings a factor \(M^e\), \(A\) brings a factor \(M^{-2i(\ell)}\), \(\mathbf{U}\) brings a factor \(M^{-i}\) and \(W\) brings \(M^{i(\ell)}\). The final factor is \(M^{-i(\ell) - 2e} \cdot M^{-i(\ell) - i(0)}\). We remark that the scale of a tree line is higher than the loop line that lies over it, or the loop line would be chosen as tree line instead. So \(i(\ell) > i(l)\) and this term is irrelevant.

- The term \(AXU \cdot \mathbf{U} \cdot \nabla \varphi \leq e(x, x^0)\) is smaller as \(U\) brings \(M^{-i}\) and there is no long loop variables. So it is irrelevant.

- We can easily find that \(A_{1,R}\) is smaller hence irrelevant.

\[\text{We thank our referee for pointing out this important point.}\]
\[\text{We are very grateful to Prof. Rivasseau for explaining this.}\]
Now we consider the second order expansion in external variables in (3.13). We only have to expand $f(t)$ to first order. We have

$$A_2 = \int dxdydx^0 \varphi^{\leq e}(x, x^0)\varphi^{\leq e}(y, x^0) \frac{1}{2} \left[(U \cdot \nabla)^2 \delta(x - y)\right] e^{i(U RU + USW)}$$

$$\times \prod_{t \in G_k, l \notin T} du_t dw_l C_l(u_t, w_l) \prod_{\ell \in T_k} du_\ell du_\ell^0 C_\ell(u_\ell, u_\ell^0, U_\ell, W_\ell)$$

$$\times \left(f(0) + \int_0^1 dt f'(t) dt\right). \quad (3.23)$$

The first term is

$$A_{2,0} = \int dxdydx^0 \varphi^{\leq e}(x, x^0)\varphi^{\leq e}(y, x^0) \frac{1}{2} \left[(U \cdot \nabla)^2 \delta(x - y)\right] e^{i(U RU + USW)}$$

$$\times \prod_{t \in G_k, l \notin T} du_t dw_l C_l(u_t, w_l) \prod_{\ell \in T_k} du_\ell du_\ell^0 C_\ell(u_\ell, u_\ell^0, U_\ell, W_\ell). \quad (3.24)$$

The terms with $\mu \neq \nu$ do not survive by parity. The other ones reconstruct a counterterm proportional to the Laplacian. The power-counting of this factor $A_{2,0}$ is improved (with respect to $A$) by a factor $M^{-2(i-e)}$ which makes it only logarithmically divergent, as should be for a wave-function counterterm.

The second term is

$$A_{2,0} = \int dxdydx^0 \varphi^{\leq e}(x, x^0)\varphi^{\leq e}(y, x^0) \frac{1}{2} \left[(U \cdot \nabla)^2 \delta(x - y)\right] e^{i(U RU + USW)}$$

$$\times \prod_{t \in G_k, l \notin T} du_t dw_l C_l(u_t, w_l) \prod_{\ell \in T_k} du_\ell du_\ell^0 C_\ell(u_\ell, u_\ell^0, U_\ell, W_\ell)$$

$$\times \int_0^1 dt (iXQU - 2tAXX - 2AX(W + U)). \quad (3.25)$$

It is irrelevant as the terms in the integral bring at least a convergent factor $M^{-(i-e)}$.

Putting together the results of the two previous section, we have proved that the usual effective series which expresses any connected function of the theory in terms of an infinite set of effective couplings, related one to each other by a discretized flow have finite coefficients to all orders. Reexpressing these effective series in terms of the renormalized couplings would reintroduce in the usual way the Zimmermann’s forests of counterterms and build the standard renormalized series. The most explicit way to check finiteness of these renormalized series in order to complete the “BPHZ
"theorem" is to use the "classification of forests" which distributes Zimmermann’s forests into packets such that the sum over assignments in each packet is finite [27]. This part is identical to the commutative case. Hence the proof of Theorem 2.1 is completed.

A The non-commutative $\varphi^6_3$ Model

In this appendix we discuss briefly the non-orientable real scalar $\varphi^6$ model, and its renormalizability which is questionable.

The action functional, with the notations of (2.2) is now

$$S[\varphi] = \int d^2x\, dx^0 \left( \frac{1}{2} \partial_\mu \varphi \ast \partial^\mu \varphi + \frac{1}{2} \partial_0 \varphi \ast \partial^0 \varphi + \frac{\Omega^2}{2} \langle \bar{x}_\mu \varphi \rangle \ast \langle \bar{x}^\mu \varphi \rangle + \frac{1}{2} \mu_0^2 \varphi \ast \varphi + \frac{\lambda}{4} \varphi \ast \varphi \ast \varphi \ast \varphi + \frac{g}{6} \varphi \ast \varphi \ast \varphi \ast \varphi \ast \varphi \right)(x) . \quad (A.1)$$

In the real $\varphi^6$ model, there are only two kinds of cyclically invariant vertices, namely the $\varphi^4$ term:

$$V(x_1, x_2, x_3, x_4) = \delta(x_1 - x_2 + x_3 - x_4) e^{i \sum_{1 \leq i < j \leq 4} (-1)^{i+j+1} x_i \theta^{-1} x_j} \quad (A.2)$$

and the $\varphi^6$ term:

$$V(x_1, x_2, x_3, x_4, x_5, x_6) = \delta(x_1 - x_2 + x_3 - x_4 + x_5 - x_6) e^{i \sum_{1 \leq i < j \leq 6} (-1)^{i+j+1} x_i \theta^{-1} x_j} \quad (A.3)$$

times the local factor in the time direction. Again we note $x \theta^{-1} y \equiv \frac{2}{3} (x_1 y_2 - x_2 y_1)$.

The discussion is almost the same as that in $(\bar{\varphi} \ast \varphi)^3$ model, with a difference in the power counting of the non-orientable graph.

When several disjoint $G^i_k$ subgraphs are non-orientable it is better to solve longer clashing loop variables, essentially one per disjoint non-orientable $G^i_k$, because they spare higher costs than if tree lines were chosen instead. We define $S$ to be the set of $n$ long variables to be solved via the $\delta$ functions. First we put in $S$ all the $n - 1$ long tree variables $v_l$. Then we scan all the connected components $G^i_k$ starting from the leaves towards the root, and we add a clashing line to $S$ each time when a new non-orientable component $G^i_k$ appears. We also remove $p - 1$ tree lines from $S$ so that each time $p \geq 2$ non-orientable components merge into a single one. In the end we obtain a new set $S$ of exactly $n - 1 + p - (p - 1) = n$ long variables. So thanks to inductive use of Lemma 2.3 in each $G^i_k$, we can solve all the long variables
in the set $S$ with the branch system of $\delta$ functions associated to $T$ plus an additional loop variable. But for the commutative dimension, there are always $n - 1$ short tree variables to be integrated. So for a general non-orientable graph we will earn only a convergent factor $M^{-2i}$ and the degree of divergence given by this crude analysis becomes $\omega(G_k^i) = N(G_k^i)/2 + n_4 - 1$ (recall lemma 2.5).

Let us consider the improved analysis taking oscillations into account. From the analog of lemma 2.6 we see that graphs with $g = 0$, $n_4 = 0$, $B = 2$, and $N = 2$ remain dangerous. Such graphs can’t appear in the $(\bar{\phi} \star \phi)^3$ model as they are non-orientable. In the $\varphi^6$ model they can appear, are logarithmic divergent and don’t look like the initial quadratic terms in the Lagrangian. So the two point function of this $(\bar{\phi} \star \phi)^3$ model seems non-renormalizable, but maybe the situation can be rescued by combining all renormalizations together, as is done e.g. in [16] or maybe we can solve this problem by exploring further the vertex oscillations. The study of this problem is still in progress.

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