Band Structure of the Fractional Quantum Hall Effect

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The eigenstates of interacting electrons in the fractional quantum Hall phase typically form fairly well defined bands in the energy space. We show that the composite fermion theory gives insight into the origin of these bands and provides an accurate and complete microscopic description of the strongly correlated many-body states in the low-energy bands.
As is well known, the ground state is separated from the other states by an energy gap at some special filling factors in the fractional quantum Hall effect (FQHE) [1]. He, Xie, and Zhang (HXZ) [2] have recently pointed out that, at other filling factors, there is in general a band of low energy states which is more or less separated from the other states by a gap. Since all many-body states are degenerate in the absence of interactions, the fact that some states split off from the rest implies non-trivial interaction-induced correlations in these states. We investigate in this paper the following three questions: (i) How many states are there in the low energy band? (ii) What are the quantum numbers associated with these states? (iii) What is their microscopic structure? We show that the composite fermion (CF) theory of the FQHE [3] predicts correctly the number of states in the low energy band as well as the quantum numbers associated with these states. Most importantly, the trial wave functions of the CF theory give a very good microscopic description of these states. Furthermore, we find that the CF theory also provides an accurate description of the first excited band.

Central to the CF theory is a mapping between the non-trivial problem of strongly interacting electrons in the FQHE regime and the relatively straightforward problem of weakly interacting electrons in the integer quantum Hall effect (IQHE) regime [3]. This mapping proceeds in two essential steps: the first provides an intuitive construction relating the FQHE to the IQHE, and the second makes it precise in terms of explicit microscopic trial wave functions. The intuitive construction involves starting with an (incompressible) IQHE state with \( n \) filled Landau levels (LLs), attaching an even number \( 2p \) of flux quanta to each electron to convert it into a CF, and then performing a mean field approximation [4] in which this flux is smeared to produce an incompressible FQHE state at filling factor \( \nu = n/(2np + 1) \). This construction suggests the following (unnormalized) ansatz wave functions for the resulting FQHE state

\[
\chi_{n/(2np+1)} = \prod_{j<k}(z_j - z_k)^{2p}\Phi_n
\]  

(1)

where the subscript denotes the filling factor, and the position of an electron is denoted by \( z = x + iy \). The Jastrow factor attaches \( 2p \) flux quanta to all electrons in the state \( \Phi_n \) to convert them into CFs. The Laughlin states [5] are recovered for the special case of \( n = 1 \).
This construction is generalized straightforwardly to arbitrary filling factors as follows [3]. Consider a filling factor $\nu^*$, where $n < \nu^* < n + 1$, and assume, to begin with, non-interacting electrons. In this case the many-body system contains energy bands separated by the cyclotron energy $\hbar \omega_c$. The lowest energy band contains states in which the lowest $n$ LLs are completely filled and the $(n + 1)$th LL is partially occupied; the first excited band consists of states in which one electron has been excited by $\hbar \omega_c$; and so on. Now, start with these states, attach an even number $(2p)$ of flux quanta to each electron, and perform the mean field approximation to produce states at $\nu = \nu^*/(2p\nu^* + 1)$. In this process, the states within one band will in general mix with each other, but it is possible, due to the energy gap, that those in different bands may not mix. The existence of well defined bands at low energies in the FQHE state suggests that some of the low-energy bands are indeed not mixed by the above mean field process. The CF construction then predicts that the number of states in these bands at $\nu$ is equal to that in the corresponding bands at $\nu^*$.

We use the standard spherical geometry [6] for our numerical calculations, in which $N$ electrons move on the surface of a sphere under the influence of a radial magnetic field ($B$) produced by a magnetic monopole of suitable strength at the center. Due to the rotational symmetry, the orbital angular momentum, $L$, and its $L_3$ component are good quantum numbers. We will label the states by $S$ ($2S =$ integer), where the total flux through the surface of the sphere is $2Sh/e$, and the filling factor (in the thermodynamic limit) is $\nu = N/2S$. Guided by the above construction, we write the following trial wave functions to describe the correspondence between the states in the lowest energy IQHE and FQHE bands (generalization to higher bands is analogous):

$$\chi_{S,\beta} = \prod_{j<k} (u_iv_j - v_iu_j)^{2p}\Phi_{S^*\beta}^*, \quad (2)$$

where the position of an electron is denoted by the usual spinor components $u$ and $v$ [6]; $S$ and $S^*$ are related by

$$S = S^* + p(N - 1) \; ; \quad (3)$$

$\chi_{S,\beta}$ are the eigenstates of interacting electrons at $S$; and $\Phi_{S^*\beta}$ are orthogonal wave functions in the lowest energy band of non-interacting electrons at $S^*$, chosen to be eigenstates of $L^2$ and $L_3$. 
The problem thus reduces to determining the low-energy spectrum of non-interacting electrons at $S^*$. The number of single particle states in the $n$th LL ($n = 1, 2, ...$) is

$$D^*_n \equiv 2S^* + 2n - 1 . \tag{4}$$

When $S^*$ corresponds to $n$ filled LLs, i.e., $S^* = S^*_n \equiv (N - n^2)/2n$, the lowest-energy band consists only of one state, implying incompressibility at $S = (N - n^2)/2n + p(N - 1)$. At $S^*_n < S^* < S^*_n + 1$ the lowest $n$ LLs are filled and the $(n+1)$th LL contains

$$N^*_{n+1} \equiv N - n(2S^* + n) \tag{5}$$
electrons in $D^*_n+1$ states. This implies that there are

$$\frac{D^*_n!}{N^*_{n+1}!; (D^*_{n+1} - N^*_{n+1})!} \tag{6}$$
many-body states in the lowest energy band at $S^*$ and, by the CF construction, also at $S = S^* + p(N - 1)$. Since the angular momentum quantum numbers are identical for $\Phi$ and $\chi$ related by Eq.(2), the CF theory also predicts these quantum numbers for all the states in the lowest band at $S$.

The number of eigenstates of interacting electrons in the lowest energy band and their angular momentum quantum numbers, as seen in the numerical calculations of HXZ, are in complete agreement with the above predictions. Encouraged by this preliminary success, we proceed to compare the trial wave functions with the true Coulomb wave functions for a more direct and rigorous verification of the CF theory. Since the numerical calculations are conveniently performed at $B = \infty$, we need to adiabatically continue our trial wave functions to this limit. We assume that this can be done by simply projecting them on to the lowest LL. I.e., we choose as our $B = \infty$ (unnormalized) trial wave functions

$$\Xi_{S,\beta} \equiv P \chi_{S,\beta} = P \prod_{j<k} (u_i v_j - v_i u_j)^{2p} \Phi_{S^*,\beta} , \tag{7}$$

where $P$ is the lowest LL projector. $P$ commutes with $L$ [7], and is generally expected to perturb the wave functions $\chi$ only very gently, since they are already predominantly in the lowest LL [3,8]. It has been explicitly shown by Rezayi and MacDonald [7] for $\nu = 2/5$ that an adiabatic continuation of $\chi$ to $B = \infty$ essentially produces the projected state $\Xi$. 

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We study a system of six electrons interacting via Coulomb interactions. Since the energies are independent of $L_3$ for a given $L$ multiplet, it is sufficient to consider the $L_3 = 0$ sector, with the understanding that one eigenstate at any $L$ in this sector corresponds to a total of $2L + 1$ degenerate eigenstates. In the following, we will restrict our discussion to the $L_3 = 0$ sector. In several cases there is only one state in the lowest energy band for a given $L$ at $S^*$, so that the trial wave function for the corresponding eigenstate at $S$ is uniquely determined, with no free parameters. In other cases, there are several (up to three in our work) states at $S^*$ with the same $L$. In these cases, there is some arbitrariness in the choice of the IQHE states from which the FQHE states are derived. We choose that orthogonal set of states at $S^*$ which optimizes the overlaps. (Details will be given elsewhere [9].) Overlaps of the projected trial wave functions with the numerical states in the lowest band are shown in Table I [10]. The states are denoted by $2S-L$, and the parentheses show $K$, which is the total number of independent eigenstates at $S$ with angular momentum $L$.

At $2S = 15$ ($\nu = 1/3$), the lowest energy band contains a single state. The excited states at $L = 2, 3, 4, 5, 6$ seem to form a second band (Fig.1), which is natural to relate to the the first excited band at $2S^* = 5$ ($\nu^* = 1$). At first sight, however, there appears to be a serious counting problem: in addition to the above values of $L$, the first excited band at $2S^* = 5$ also contains a state at $L = 1$. We nevertheless construct trial wave functions $\chi$ and apply $\mathcal{P}$ to obtain $\bar{\chi}$. To our pleasant surprise, we find that the state $\chi$ at $L = 1$ is annihilated by the application of $\mathcal{P}$. Thus, quite remarkably, the CF theory knows that there is no $L = 1$ state in the first excited band at $\nu = 1/3$ [11]. The overlaps of the trial states at other values of $L$ with the true Coulomb states in the excited band are also shown in Table I.

The large overlaps in all cases show that the CF theory provides a good microscopic description of the actual FQHE states in the low energy bands. For reference, it should be kept in mind that a random trial wave function with a given $L$ has an overlap of the order of $1/\sqrt{K}$ with the corresponding true state. The relatively small overlap for the 15-2 state can be explained due to the fact that this state is not as well separated from the other higher energy states at $L = 2$ (Fig.1). It should be noted that a large overlap is in general much too stringent a condition for the validity of a trial wave function, and is by no means necessary. Even with poor overlap, a trial wave function can be adiabatically connected to the true
physical state and hence contain the relevant qualitative physics. A large overlap, however, certainly constitutes a powerful evidence in favor of a trial wave function.

By taking the electrons at $S^*$ to be non-interacting, we have neglected the possibility of a finer structure within the lowest energy band. When interactions are turned on, this band will broaden and may further split into smaller subbands, possibly translating into analogous structure at $S$. At some values of $S^*$, the interacting system can be incompressible even at non-integer fillings, due to formation of states related to the above FQHE states by particle-hole symmetry or LL addition, which will lead to FQHE at new filling factors via the CF construction [12]. While, in principle, the subdivision into finer and finer bands can continue \textit{ad infinitum}, leading to a hierarchy of states at \textit{all} odd-denominator rational fraction [13], we do not expect this to be the case for the realistic Coulomb interaction.

HXZ analyzed their numerical calculations using the framework of the quasiparticle hierarchy (QPH) theory of Haldane [6] and Halperin [14]. In this approach, the system at an intermediate filling is viewed as a collection of quasiparticles of the “parent” incompressible state. HXZ find that the QPH theory is successful in predicting the number of states in the lowest energy band as well as their quantum numbers. However, there is no direct way of comparing the pseudo-wavefunctions of the quasiparticles with the electron wave functions generated in the numerical work. We also note that while the CF approach provides a uniform treatment for all filling factors, in the QPH scheme, as discovered empirically by HXZ, the quasiholes must be treated as non-interacting whereas the quasielectrons as interacting (with a suitable hard-core potential) in order to obtain the right number of states in the lowest energy band.

Prior to the CF theory, the QHEs at integer and fractional filling factors were treated as two essentially unrelated phenomena. Before closing, we would like to discuss some conceptual issues concerning the existence of a fundamental connection between the two. Even though such a connection would seem quite natural, and virtually unavoidable from the phenomenological point of view, the following questions have been raised with regard to the IQHE-FQHE mapping of the CF theory: (i) Why should the higher LLs be invoked in order to understand the FQHE, a phenomenon that is believed to occur even in the limit of infinite $B$? (ii) How can there be a connection between the IQHE and the FQHE given
that the latter requires interactions in a fundamental manner whereas the the former can
occur even for non-interacting electrons? These questions are answered as follows. (i) There
is a large space of parameters in which the FQHE is observed. The primary objective of
any theory is to obtain a clear-cut understanding of the phenomenon at any one point in
this space, which can then, at least in principle, be adiabatically continued to the physical
point. No simple microscopic description of the FQHE has become possible within the lowest
LL, except in some special cases [5]. The CF theory asserts that the complete physics of
the FQHE becomes manifest when a small amount of hybridization with the higher LLs is
allowed (and an adiabatic continuation to $B = \infty$ simply requires projection on to the lowest
LL). (ii) The analogy between the IQHE and FQHE does not imply that interactions are
not required for the FQHE. The crucial role of interactions is to generate CFs. However,
once the CFs are generated, the residual interactions between them can be neglected (in the
simplest cases), and the FQHE of electrons occurs when the CFs assume IQHE-like structure
to produce incompressibility.

Even though there is no conceptual difficulty with a relationship between the IQHE and
the FQHE, the validity of the IQHE-FQHE mapping postulated by the CF theory (Eq. [7]) can
be established only from an extensive comparison of its consequences with experiments and
numerical studies. In this paper, we have studied a system of six electrons in the filling factor
range $2/7 \leq \nu \leq 2/5$ and found that the CF theory provides a complete and microscopically
accurate account of the entire low-energy Hilbert space of interacting electrons at arbitrary
filling factors. Thus, somewhat like in Landau’s fermi liquid theory, there is a one-to-one
correspondence between the low energy eigenstates of strongly interacting electrons in the
FQHE regime and those of weakly interacting electrons in the IQHE regime.

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[10] Some of the overlaps of Table I have been reported earlier, and are repeated here for completeness. These are: for 11-0 state in G. Dev and J.K. Jain, Phys. Rev. B45, 1223 (1992); for 15-0 state in F.D.M. Haldane and E.H. Rezayi, Phys. Rev. Lett. 54, 237 (1985). Also, the 15-0 and 16-3 trial wave functions are identical to those proposed by Laughlin [5].

[11] The generality of this result for arbitrary number of particles will be discussed elsewhere. Also, this example shows that sometimes the projector $P$ can affect the states $\chi$ drastically.

[12] For example, the interacting system at $2S^* = 9$ corresponds to $\nu^* = 2/3$ and is incompressible due to particle-hole symmetry in the lowest LL. This results in FQHE at $2S = 19$ which corresponds to $\nu = 2/7$. We have obtained a trial wave function for the $2/7$ state by diagonalizing the Coulomb Hamiltonian at $2S^* = 9$ and multiplying the ground state by the Jastrow factor. It has an overlap of 0.9787 with the Coulomb ground state at $2/7$.

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**Table Caption:**

Table I. Overlaps for all states in the lowest energy band for each value of $S$ in the range $11 \leq 2S \leq 19$, and for all states in the first excited band at $2S = 15$. The size of the system is $N = 6$ electrons so that the values $2S = 11, 15, \text{ and } 19$ correspond to $\nu = 2/5, 1/3, \text{ and } 2/7$, respectively. The states are denoted by $2S-L$ followed by $K$ in parentheses (see the text for the definition of these symbols). The states in the first excited band are marked by asterisk.

**Figure Caption:**

Fig. 1 Energy spectrum at $2S = 15$. The two lowest energy bands are delineated by horizontal lines.
| 19-0(10) | 19-2(23) | 19-3(21) | 19-4(37) | 19-5(32) | 19-6(49) | 19-7(43) | 19-8(56) | 19-9(51) |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 0.9956   | 0.9909   | 0.9945   | 0.9894   | 0.9932   | 0.9894   | 0.9865   | 0.9883   | 0.9854   |
| 0.9694   | 0.9720   | 0.9870   | 0.9874   |          | 0.9680   | 0.9735   | 0.9886   |          |
| 0.9758   | 0.9754   | 0.9947   | 0.9879   |          | 0.9914   | 0.9894   | 0.9822   | 0.9768   |
|          |          |          | 0.9868   | 0.9886   |          |          |          |          |
| 19-10(62)| 19-12(65)| 18-1(13) | 18-3(26) | 18-4(23) | 18-5(34) | 18-6(33) | 18-7(43) | 18-9(49) |
| 0.9758   | 0.9754   | 0.9947   | 0.9879   | 0.9886   | 0.9914   | 0.9894   | 0.9822   | 0.9768   |
|          |          |          |          |          |          |          |          |          |
| 17-0(8)  | 17-2(16) | 17-4(26) | 17-6(34) | 16-3(18) | 15-0(6)  | 15-2(11)* | 15-3(9)* | 15-4(17)* |
| 0.9714   | 0.9918   | 0.9896   | 0.9815   | 0.9889   | 0.9964   | 0.9484   | 0.9923   | 0.9915   |
|          |          |          |          |          |          |          |          |          |
| 15-5(13)*| 15-6(22)*| 14-3(14) | 13-0(5)  | 13-2(9)  | 13-4(14) | 12-1(4)  | 12-3(7)  | 11-0(3)  |
| 0.9977   | 0.9800   | 0.9882   | 0.9984   | 0.9841   | 0.9933   | 0.9786   | 0.9956   | 0.9998   |