CHARACTERISTIC VARIETY OF THE GAUSS-MANIN DIFFERENTIAL EQUATIONS OF A GENERIC PARALLELLY TRANSLATED ARRANGEMENT

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Abstract. We consider a weighted family of \( n \) generic parallelly translated hyperplanes in \( \mathbb{C}^k \) and describe the characteristic variety of the Gauss-Manin differential equations for associated hypergeometric integrals. The characteristic variety is given as the zero set of Laurent polynomials, whose coefficients are determined by weights and the Plücker coordinates of the associated point in the Grassmannian \( \text{Gr}(k, n) \). The Laurent polynomials are in involution.

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1. Introduction

There are three places, where a flat connection depending on a parameter appears:

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KZ equations, $\kappa \frac{\partial I}{\partial z_i}(z) = K_i(z)I(z)$, $z = (z_1, \ldots, z_n)$, $i = 1, \ldots, n$. Here $\kappa$ is a parameter, $I(z)$ a $V$-valued function, where $V$ is a vector space from representation theory, $K_i(z) : V \to V$ are linear operators, depending on $z$. The connection is flat for all $\kappa$, see for example [EFK, V2].

Quantum differential equations, $\kappa \frac{\partial I}{\partial z_i}(z) = p_i * z I(z)$, $z = (z_1, \ldots, z_n)$, $i = 1, \ldots, n$. Here $p_1, \ldots, p_n$ are generators of some commutative algebra $H$ with quantum multiplication $*_z$ depending on $z$. The connection is flat for all $\kappa$. These equations are part of the Frobenius structure on the quantum cohomology of a variety, see [D, M].

Differential equations for hypergeometric integrals associated with a family of weighted arrangements with parallelly translated hyperplanes, $\kappa \frac{\partial I}{\partial z_i}(z) = K_i(z)I(z)$, $z = (z_1, \ldots, z_n)$, $i = 1, \ldots, n$. The connection is flat for all $\kappa$, see for example [V1, OT].

If $\kappa \frac{\partial I}{\partial z_i}(z) = K_i(z)I(z)$, $i = 1, \ldots, n$, is a system of $V$-valued differential equations of one of these types, then its characteristic variety is

$$\text{Spec} = \{(z, p) \in T^*\mathbb{C}^n \mid \exists v \in V \text{ with } K_j(z)v = p_jv, \ j \in J\}.$$
of a critical point of the master function associated with a \( \mathfrak{gl}_N \) KZ equation can be identified with a suitable local Bethe algebra of the corresponding \( \mathfrak{gl}_N \) module.

In Section 2, we consider the algebra of functions on the critical set of the master function and describe it by generators and relations.

In Section 3, we show that these relations give us equations defining the Lagrangian variety of the master function. We show that the corresponding functions are in involution. We define coordinate systems \( (z_I, p_I) \) on the Lagrange variety and relate it to the Jacobian of the projection of the Lagrangian variety to the base of the family.

In Section 4, we remind the identification from [V4] of the Lagrangian variety of the master function and the characteristic variety of the Gauss-Manin differential equations.

2. Algebra of functions on the critical set

2.1. An arrangement in \( \mathbb{C}^n \times \mathbb{C}^k \). Let \( n > k \) be positive integers. Denote \( J = \{1, \ldots, n\} \). Consider \( \mathbb{C}^k \) with coordinates \( t_1, \ldots, t_k \). \( \mathbb{C}^n \) with coordinates \( z_1, \ldots, z_n \). Fix \( n \) linear functions on \( \mathbb{C}^k \), \( g_j = \sum_{m=1}^k b_{jm} t_m \), \( j \in J \), \( b_{jm} \in \mathbb{C} \). For \( i_1, \ldots, i_k \subset J \), denote \( d_{i_1, \ldots, i_k} = \det_{\ell,m=1}^k (b_{im}) \).

We assume that all the numbers \( d_{i_1, \ldots, i_k} \) are nonzero if \( i_1, \ldots, i_k \) are distinct. In other words, we assume that the collection of functions \( g_j, j \in J \), is generic. We define \( n \) linear functions on \( \mathbb{C}^n \times \mathbb{C}^k \), \( f_j = z_j + g_j, \ j \in J \). We define the arrangement of hyperplanes \( \tilde{C} = \{ \tilde{H}_j \ | \ j \in J \} \) in \( \mathbb{C}^n \times \mathbb{C}^k \), where \( \tilde{H}_j \) is the zero set of \( f_j \). Denote by \( U(\tilde{C}) = \mathbb{C}^n \times \mathbb{C}^k - \cup_{j \in J} \tilde{H}_j \) the complement.

For every \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), the arrangement \( \tilde{C} \) induces an arrangement \( C(z) \) in the fiber over \( z \) of the projection \( \pi : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n \). We identify every fiber with \( \mathbb{C}^k \). Then \( C(z) \) consists of hyperplanes \( H_j(z), j \in J \), defined in \( \mathbb{C}^k \) by the equations \( f_j = 0 \). Denote by \( U(C(z)) = \mathbb{C}^k - \cup_{j \in J} H_j(z) \) the complement.

The arrangement \( C(z) \) is with normal crossings if and only if \( z \in \mathbb{C}^n - \Delta \),

\[
\Delta = \cup_{\{i_1 < \cdots < i_{k+1}\} \subset J} H_{i_1, \ldots, i_{k+1}},
\]

where \( H_{i_1, \ldots, i_{k+1}} \) is the hyperplane in \( \mathbb{C}^n \) defined by the equation \( f_{i_1, \ldots, i_{k+1}}(z) = 0 \),

\[
f_{i_1, \ldots, i_{k+1}}(z) = \sum_{m=1}^{k+1} (-1)^{m-1} d_{i_1, \ldots, i_m, \ldots, i_{k+1}} z_{i_m},
\]

We have the following identity

\[
\sum_{m=1}^{k+1} (-1)^{m-1} d_{i_1, \ldots, i_m, \ldots, i_{k+1}} (z_{i_m} - f_{i_m}(z, t)) = 0.
\]

**Lemma 2.1.** Consider the \( \mathbb{C} \)-span \( S \) of the linear functions \( f_{i_1, \ldots, i_{k+1}} \), where \( \{i_1, \ldots, i_{k+1}\} \) runs through all \( k + 1 \)-element subsets of \( J \). Then \( \dim S = n - k \).

**Proof.** The dimension of \( S \) equals the codimension in \( \mathbb{C}^n \) of \( X_1 = \{ z \in \mathbb{C}^n \ | \ f_I(z) = 0 \ \text{for all} \ I \} \). The subspace \( X_1 \) is the image of the subspace \( X_2 = \{(z, t) \in \mathbb{C}^n \times \mathbb{C}^k \ | \ f_j(z, t) = 0 \ \text{for all} \ j \in J \} \).
0 for all \( j \in J \} \) under the projection \( \pi : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n \). Clearly the subspace \( X_2 \) is \( k \)-dimensional and the projection \( \pi |_{X_2} : X_2 \to X_1 \) is an isomorphism. Hence \( \dim X_1 = k \) and \( \dim S = n - k \).

2.2. Plücker coordinates. The matrix \( (b_j^m) \) is an \( n \times m \)-matrix of rank \( k \). The matrix defines a point in the Grassmannian \( \text{Gr}(k, n) \) of \( k \)-planes in \( \mathbb{C}^n \). The numbers \( d_{i_1, \ldots, i_k} \) are Plücker coordinates of this point. Most of objects in this paper is determined in terms of these Plücker coordinates. We will use the following Plücker relation.

**Lemma 2.2.** For arbitrary sequences \( j_1, \ldots, j_{k+1} \) and \( i_1, \ldots, i_{k-1} \) in \( J \), we have

\[
(2.4) \quad \sum_{m=1}^{k+1} (-1)^{m-1} d_{j_1, \ldots, j_{m-1}, j_{m+1}, \ldots, j_{k+1}} d_{j_m, i_1, \ldots, i_{k-1}} = 0.
\]

See this statement, for example, in [KL].

2.3. Algebra \( A_\Phi(z) \). Assume that nonzero weights \( (a_j)_{j \in J} \subset \mathbb{C}^\times \) are given. Denote \( |a| = \sum_{j \in J} a_j \). Assume that \( |a| \neq 0 \).

Each arrangement \( C(z) \) is weighted. The master function of the weighted arrangement \( C(z) \) in \( \mathbb{C}^k \) is the function

\[
(2.5) \quad \Phi(z, t) = \sum_{j \in J} a_j \log f_j(z, t).
\]

The critical point equations are

\[
(2.6) \quad \frac{\partial \Phi}{\partial t_i} = \sum_{j \in J} b_j^i a_j f_j = 0, \quad i = 1, \ldots, k.
\]

We have

\[
(2.7) \quad \frac{\partial \Phi}{\partial z_j} = a_j f_j, \quad j \in J.
\]

Denote by \( \mathcal{I}(z) \subset \mathcal{O}(U(C(z))) \) the ideal generated by the functions \( \frac{\partial \Phi}{\partial t_j}, j \in J \). The algebra of functions on the critical set is

\[
(2.8) \quad A_\Phi(z) = \mathcal{O}(U(C(z))) / \mathcal{I}(z).
\]

For a function \( g \in \mathcal{O}(U(C(z))) \), denote by \([g]\) its projection to \( A_\Phi(z) \). Denote

\[
p_j = [a_j / f_j], \quad j \in J.
\]

We introduce the following polynomials in \( z_1, \ldots, z_n, p_1, \ldots, p_n \). For every subset \( I = \{ i_1, \ldots, i_{k-1} \} \) of distinct elements in \( J \), we set

\[
(2.9) \quad F_I(p_1, \ldots, p_n) = \sum_{j \in J} d_{j, i_1, \ldots, i_{k-1}} p_j.
\]

For every subset \( I = \{ i_1, \ldots, i_{k+1} \} \) of distinct elements in \( J \), we set

\[
(2.10) \quad F_I(z_1, \ldots, z_n, p_1, \ldots, p_n) =
\]

\[
= p_{i_1} \cdots p_{i_{k+1}} f_{i_1, i_2, \ldots, i_{k+1}}(z) + \sum_{m=1}^{k+1} (-1)^m a_m d_{i_1, \ldots, i_{k+1}} p_{i_1} \cdots p_{i_m} \cdots p_{i_{k+1}}.
\]

The following lemma collects properties of the elements \( p_1, \ldots, p_n \).
Lemma 2.3. Let $z \in \mathbb{C}^n - \Delta$.

(i) The elements $p_j, j \in J$, generate the algebra $A_\Phi(z)$.

(ii) For every subset $I = \{i_1, \ldots, i_k\}$ of distinct elements in $J$, we have

$$F_I(p_1, \ldots, p_n) = 0.$$  

Relation (2.11) will be called the I-relation of first kind.

(iii) For every subset $I = \{i_1, \ldots, i_{k+1}\}$ of distinct elements in $J$, we have

$$F_I(z_1, \ldots, z_n, p_1, \ldots, p_n) = 0.$$  

Relation (2.12) will be called the I-relation of second kind.

(iv) In $A_\Phi(z)$, we have

$$1 = \frac{1}{|J|} \sum_{j \in J} z_j p_j.$$  

(v) We have $\dim A_\Phi(z) = \binom{n-1}{k}$, and for any $j_1 \in J$, the set of monomials $p_{i_1} \cdots p_{i_k}$, with $i_1 < \cdots < i_k$ and $j_1 \notin \{i_1, \ldots, i_k\}$, is a $\mathbb{C}$-basis of $A_\Phi(z)$.

Part (i) is Lemma 2.5 in [V4]. Parts (ii), (iii), (iv) are Lemmas 6.7, 6.8, 2.5 in [V5], respectively. The first statement of part (v) is [V4, Lemma 4.2] that follows from [V5, Lemma 6.5]. The second statement of part (v) is Theorem 6.11 in [V5].

Note that the polynomials $F_I$ in (2.11) and (2.12) are homogeneous if we put

$$\deg p_j = 1, \quad \deg z_j = -1 \quad \text{for all } j.$$  

2.4. Relations of second kind. For $j \in J$, denote

$$G_j(z_j, p_j) = z_j - a_j/p_j.$$  

Then the projection to $A_\Phi(z)$ of the left hand side of equation (2.3) can be written as

$$G_I(z, p) = \sum_{m=1}^{k+1} (-1)^{m-1} d_{i_1, \ldots, i_m, \ldots, i_{k+1}} G_{i_m}(z_{i_m}, p_{i_m})$$

$$= \sum_{m=1}^{k+1} (-1)^{m-1} d_{i_1, \ldots, i_m, \ldots, i_{k+1}} \left(z_{i_m} - \frac{d_{i_m}}{p_{i_m}}\right),$$

where $I = \{i_1, \ldots, i_{k+1}\}$. Hence in $A_\Phi(z)$ we have

$$G_I(z, p) = 0.$$  

Notice that $F_I(z, p) = p_{i_1} \cdots p_{i_{k+1}} G_I(z, p)$ and the functions $p_j$ are nonzero at every point of the critical set of the master function.

2.5. New presentation for $A_\Phi(z)$. Fix $z \in \mathbb{C}^n - \Delta$. Consider $(\mathbb{C}^x)^n$ with coordinates $p_1, \ldots, p_n$. Consider the polynomials $F_I(p)$ in (2.11) and polynomials $F_I(z, p)$ in (2.12) as elements of $O((\mathbb{C}^x)^n)$. Let $\tilde{I}(z) \subset O((\mathbb{C}^x)^n)$ be the ideal generated by all $F_I$ with $|I| = k - 1, k + 1$.

Notice that all polynomials $F_I(p)$, $|I| = k - 1$, in (2.11) and all functions $G_I(z, p), |I| = k + 1$, in (2.16) also generate $\tilde{I}(z)$.

Let $\tilde{A}(z) = O((\mathbb{C}^x)^n)/\tilde{I}(z)$ be the quotient algebra.
Theorem 2.4. The natural homomorphism \( \tilde{A}(z) \to A_k(z) \), \( p_j \mapsto [a_j/f_j] \), is an isomorphism.

Example. If \( k = 1 \) and \( f_j = t_1 + z_j \), then the ideal \( \mathcal{I}(z) \) is generated by the function \( \sum_{j \in J} a_j/(t_1 + z_j) \), while the ideal \( \tilde{\mathcal{I}}(z) \) is generated by the functions

\[
p_1 + \cdots + p_n, \quad (z_i - z_j)p_ip_j - a_ip_j + a_jp_i, \quad 1 \leq i < j \leq n,
\]
or by the functions

\[
p_1 + \cdots + p_n, \quad (z_i - a_ip_i) - (z_j - a_jp_j), \quad 1 \leq i < j \leq n.
\]

2.6. Proof of Theorem 2.4.

Lemma 2.5. Let \( I = \{i_1, \ldots, i_k\} \) be a subset of distinct elements. Then in \( \tilde{A}(z) \), we have

\[
\sum_{j \in J} z_jp_j = \frac{1}{d_{i_1, \ldots, i_k}} \sum_{j \in J \setminus I} f_{j, i_1, \ldots, i_k}(z) p_j.
\]

Proof. The statement easily follows from (2.11), that is, from relations of first kind. For example, if \( k = 2 \) and \( I = \{1, 2\} \), then the two relations of first kind

\[
p_1 = \frac{1}{d_{1,2}} \sum_{j > 2} d_{1,j}p_j
\]

and

\[
p_2 = \frac{1}{d_{1,2}} \sum_{j > 2} d_{2,j}p_j
\]

transform \( \sum_{j \in J} z_jp_j \) to \( \frac{1}{d_{1,2}} \sum_{j > 2} f_{1,j}(z)p_j \).

Lemma 2.6. In \( \tilde{A}(z) \), we have \( 1 = \frac{1}{a} \sum_{j \in J} z_jp_j \).

Proof. We have

\[
p_1 \cdots p_k \sum_{j \in J} z_jp_j = p_1 \cdots p_k \frac{1}{d_{1,\ldots,k}} \sum_{j > k} f_{j,1,\ldots,k}(z) p_j
\]

\[
= \sum_{j > k} [a_jp_1 \cdots p_k + \sum_{m=1}^k (-1)^m a_m \frac{d_{1,\ldots,m}}{d_{1,\ldots,k}} p_j p_1 \cdots \hat{p}_m \cdots p_k] = |a|p_1 \cdots p_k,
\]

where the first equality follows from Lemma 2.4, the second equality follows from the relations of second kind, the third equality follows from the relations of first kind. Denote by \( C(z) \subset (\mathbb{C}^\times)^n \) the zero set of the ideal \( \tilde{\mathcal{I}}(z) \). Then the function \( p_1 \cdots p_k \) is nonvanishing on \( C(z) \). The previous calculation shows that the multiplication of the invertible function \( p_1 \cdots p_k \) by \( \frac{1}{a} \sum_{j \in J} z_jp_j \) does not change the invertible function. This gives the lemma.

Lemma 2.7. Let \( s \leq k \) be a natural number and \( M = \prod_{j \in J} p_j^{s_j} \), \( \sum_{j \in J} s_j = s \), a monomial of degree \( s \). Let \( J_{k-s+1} = \{j_1, \ldots, j_{k-s+1}\} \) be any subset in \( J \) with distinct elements. Then by using the relations of first kind only, the monomial \( M \) can be represented as a \( \mathbb{C} \)-linear combination of monomials \( p_{i_1} \cdots p_{i_s} \) with \( 1 \leq i_1 < \cdots < i_s \leq n \) and \( \{i_1, \ldots, i_s\} \cap J_{k-s+1} = \emptyset \).

C.f. the proof of Lemma 6.9 in [V2].

Lemma 2.8. Let \( s \leq k \) be a natural number and \( M = \prod_{j \in J} p_j^{s_j} \) a monomial of degree \( s \). Fix and element \( j_1 \in J \). Then by using the relations of first kind and the relation \( 1 = \frac{1}{a} \sum_{j \in J} z_jp_j \) only, the monomial \( M \) can be represented as a linear combination of monomials \( p_{i_1} \cdots p_{i_s} \) with \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( j_1 \notin \{i_1, \ldots, i_s\} \), where the coefficients of the linear combination are homogeneous polynomials in \( z \) of degree \( s - k \).
Recall the deg $z_j = -1$ for all $j \in J$.

**Lemma 2.9.** Let $s > k$ be a natural number and $M = \prod_{j \in J} p_j^{s_j}$ a monomial of degree $s$. Then by using the relations of first kind and second kinds, the monomial $M$ can be represented as a linear combination of monomials $p_{i_1} \ldots p_{i_k}$ of degree $k$, where the coefficients of the linear combination are rational functions in $z$, regular on $\mathbb{C}^n - \Delta$ and homogeneous of degree $s - k$.

Let us finish the proof of Theorem 2.4. Let $P(p_1, \ldots, p_n)$ be a polynomial. Fix $j_1 \in J$. By using the relations of first and second kinds only, the polynomial can be represented as a linear combination of monomials $p_{i_1} \ldots p_{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$ and $j_1 \notin \{i_1, \ldots, i_k\}$, see Lemmas 2.7 - 2.9. Assume that $P(p_1, \ldots, p_n)$ projects to zero in $A_{\Phi}(z)$, then all coefficients of that linear combination $\tilde{P}$ must be zero, see part (v) of Lemma 2.3. This means that $P$ lies in the ideal $\mathcal{I}(z)$. Theorem 2.4 is proved.

### 3. Lagrangian variety of the master function

#### 3.1. Critical set

Recall the projection $\pi : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$. For any $z \in \mathbb{C}^n - \Delta$, the arrangement $\mathcal{C}(z)$ in $\pi^{-1}(z)$ has normal crossings. Recall the complement $U(\hat{\mathcal{C}}) \subset \mathbb{C}^n \times \mathbb{C}^k$ to the arrangement $\hat{A}$ in $\mathbb{C}^n \times \mathbb{C}^k$. Denote

$$U^0 = U(\hat{\mathcal{C}}) \cap \pi^{-1}(\mathbb{C}^n - \Delta) \subset \mathbb{C}^n \times \mathbb{C}^k.$$  

Consider the master function $\Phi(z, t)$, defined in (2.5), as a function on $U^0$. Denote by $C_\Phi$ the critical set of $\Phi$ with respect to variables $t$,

$$C_\Phi = \{(z, t) \in U^0 \mid \partial \Phi/\partial t_i(z, t) = 0, \ i = 1, \ldots, k\}.$$  

**Lemma 3.1.** The set $C_\Phi$ is a smooth $n$-dimensional subvariety of $U^0$.

**Proof.** For any subset $J = \{1 \leq i_1 < \cdots < i_k \leq n\} \subset J$, the $k \times k$-determinant

$$\det^k_{t=m=1} \left( \frac{\partial^2 \Phi}{\partial t_i \partial z_{j_m}} \right) = -d_{i_1, \ldots, i_k} \cdot \prod_{m=1}^{k} \frac{a_j}{f_{j_m}(z, t)}$$

is nonzero on $U^0$. \hfill $\Box$

Denote by $\mathcal{I} \subset \mathcal{O}(U^0)$ the ideal generated by the functions $\partial \Phi/\partial t_j$, $j \in J$. The algebra of functions on $C_\Phi$ is the quotient algebra

$$(3.3) \quad A_\Phi = \mathcal{O}(U^0)/\mathcal{I}.$$  

Consider $(\mathbb{C}^n - \Delta) \times (\mathbb{C}^x)^n$ with coordinates $z_1, \ldots, z_n, p_1, \ldots, p_n$. Consider the polynomials $F_I(p)$ in (2.11) and polynomials $F_{I}(z, p)$ in (2.12) as elements of $\mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^x)^n)$. Let $\tilde{\mathcal{I}} \subset \mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^x)^n)$ be the ideal generated by all $F_I$ with $|I| = k - 1$, $k + 1$. Notice that all polynomials $F_I(p)$, $|I| = k - 1$, in (2.11) and all functions $G_I(z, p)$, $|I| = k + 1$, in (2.16) also generate $\mathcal{I}(z)$. Let

$$(3.4) \quad \tilde{A} = \mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^x)^n)/\tilde{\mathcal{I}}$$

be the quotient algebra.

**Theorem 3.2.** The natural homomorphism $\tilde{A} \to A_\Phi$, $p_j \mapsto [a_j/f_j]$, is an isomorphism.

The proof is the same as the proof of Theorem 2.4.
3.2. Lagrangian variety. Consider the cotangent bundle $T^*(\mathbb{C}^n - \Delta)$ with dual coordinates $z_1, \ldots, z_n, p_1, \ldots, p_n$ with respect to the standard symplectic form $\omega = \sum_{j=1}^n dp_j \wedge dz_j$. Consider the open subset $(\mathbb{C}^n - \Delta) \times (\mathbb{C}^n)^n \subset T^*(\mathbb{C}^n - \Delta)$ of all points with nonzero coordinates $p_1, \ldots, p_n$. Consider the map

$$\varphi : C_\Phi \rightarrow (\mathbb{C}^n - \Delta) \times (\mathbb{C}^n)^n, \quad (z, t) \mapsto \left(z_1, \ldots, z_n, p_1 = \frac{\partial \Phi}{\partial z_1}(z, t), \ldots, p_n = \frac{\partial \Phi}{\partial z_n}(z, t)\right).$$

Denote by $\Lambda$ the image $\varphi(C_\Phi)$ of the critical set. The set $\Lambda$ is invariant with respect to the action of $\mathbb{C}^\times$ which multiplies all coordinates $p_j$ and divides all coordinates $z_j$ by the same number. Denote by $\hat{\mathcal{I}} \subset \mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^n)^n)$ the ideal of functions that equal zero on $\Lambda$.

**Theorem 3.3.** The ideal $\hat{\mathcal{I}} \subset \mathcal{O}((\mathbb{C}^n - \Delta) \times (\mathbb{C}^n)^n)$ coincides with the ideal $\hat{\mathcal{I}}$. The subset $\Lambda \subset (\mathbb{C}^n - \Delta) \times (\mathbb{C}^n)^n$ is a smooth Lagrangian subvariety.

**Proof.** It is clear that $\hat{\mathcal{I}} \subset \hat{\mathcal{I}}$. The proof of the inclusion $\hat{\mathcal{I}} \subset \hat{\mathcal{I}}$ is basically the same as the proof of Theorem 2.3. This gives the first statement of the theorem.

It is clear that $\dim \Lambda = n$. To prove that $\Lambda$ is smooth, it is enough to show that at any point of $\Lambda$, the span of the differentials of the functions $F_I(p), |I| = k - 1$, and $G_I(z, p), |I| = k + 1$ is at least $n$-dimensional. By Lemma 2.1, the span of the $z$-parts of the differentials of the functions $G_I(z, p), I = |I| = k + 1, n - k$-dimensional. It is easy to see that the span of the differentials of the functions $F_I(p), I = |I| = k + 1, n - k$-dimensional, c.f. the example in the proof of Lemma 2.3. Hence $\Lambda$ is smooth.

By the definition of $\varphi$, the set $\Lambda$ is isotropic. Hence $\Lambda$ is Lagrangian. $\square$

Let $I = \{i_1, \ldots, i_k\} \subset J$ be a $k$-element subset and $\bar{I}$ its complement. Then the functions $z_I = \{z_i | i \in I\}, p_I = \{p_j | j \in \bar{I}\}$, form a system of coordinates on $\Lambda$. Indeed, we have

$$p_{i_m} = -\frac{1}{d_{i_m, i_1, \ldots, i_{m-1}, i_k}} \sum_{j \in I} d_{j, i_1, \ldots, i_{m-1}, i_k} p_j, \quad m = 1, \ldots, k,$$

$$z_j = \frac{a_j}{p_j} + \frac{1}{d_{i_1, \ldots, i_k}} \sum_{m=1}^k (-1)^{k-m} d_{j, i_1, \ldots, i_{m-1}, i_k} \left(z_{i_m} - \frac{a_{i_m}}{p_{i_m}}\right), \quad j \in \bar{I},$$

where in the second line the functions $p_{i_m}$ must be expressed in terms of the functions $p_j, j \in \bar{I}$, by using the first line.

We order the functions of the coordinate system $z_I, p_I$ according to the increase of the low index. For example, if $k = 3, n = 6, I = \{1, 3, 6\}$, then the order is $z_1, p_2, z_3, p_4, p_5, z_6$.

**Lemma 3.4.** Let $I = \{i_1, \ldots, i_k\}$ and $I' = \{i'_1, \ldots, i'_k\}$ be two $k$-element subsets of $J$. Consider the corresponding ordered coordinate systems $z_I, p_I$ and $z_{I'}, p_{I'}$. Express the coordinates of the second system in terms of coordinates of the first system and denote by $\text{Jac}_{I, I'}(z_I, p_I)$ the Jacobian of this change. Then

$$\text{Jac}_{I, I'}(z_I, p_I) = \left(d_{i'_1, \ldots, i'_k}/d_{i_1, \ldots, i_k}\right)^2.$$

**Proof.** It is enough to check this formula for the case $I = \{1, 3, \ldots, k + 1\}$ and $I' = \{2, 3, \ldots, k + 1\}$. Then

$$p_1 = -\frac{d_{2, 3, \ldots, k+1}}{d_{1, 3, \ldots, k+1}} p_2 + \ldots, \quad z_2 = \frac{a_2}{p_2} + \frac{d_{2, 3, \ldots, k+1}}{d_{1, 3, \ldots, k+1}} z_1 + \ldots,$$
where the first dots denote the terms which do not depend on \( z_1, p_2 \) and the second dots denote the terms which do not depend on \( z_1 \). According to these formulas the \( 2 \times 2 \) Jacobian of the dependence of \( p_1, p_2 \) on \( z_1, p_2 \) equals \((d_{2,3},\ldots,k+1/d_{1,3},\ldots,k+1)^2\) and hence \( \text{Jac}_{I,I'}(z_1, p_I) = (d_{2,3},\ldots,k+1/d_{1,3},\ldots,k+1)^2 \).

### 3.3. Generating functions

Consider the function

\[
\Psi = \sum_{j \in J} a_j \ln p_j - \sum_{i \in I} z_i p_i
\]

of \( n + k \) variables \( z_j, j \in I, \ p_j, j \in J \). Express in \( \Psi \) the variables \( p_i, i \in I \), according to (3.5). Denote by \( \Psi(z_I, p_I) \) the resulting function of variables \( z_I, p_I \).

**Theorem 3.5.** The function \( \Psi(z_I, p_I) \) is a generating function of the Lagrangian variety \( \Lambda \). Namely, \( \Lambda \) lies the image of the map

\[
(z_I, p_I) \mapsto (z_I, z_I = \frac{\partial \Psi_I}{\partial p_I}(z_I, p_I), p_I = -\frac{\partial \Psi_I}{\partial z_I}(z_I, p_I), p_I).
\]

**Proof.** The proof that these formulas give (3.5) is by straightforward verification. \( \square \)

### 3.4. Integrals in involution

Consider the standard Poisson bracket on \( T^*(\mathbb{C}^n) \),

\[
\{M, N\} = \sum_{j=1}^{n} \left( \frac{\partial M}{\partial z_j} \frac{\partial N}{\partial p_j} - \frac{\partial M}{\partial p_j} \frac{\partial N}{\partial z_j} \right)
\]

for \( M, N \in \mathcal{O}(T^*(\mathbb{C}^n)) \). The functions are in involution if \( \{M, N\} = 0 \).

**Theorem 3.6.** All functions \( F_I(p) \), \( |I| = k - 1 \), and \( G_I(z, p) \), \( |I| = k + 1 \), are in involution.

**Proof.** Clearly, \( \{F_I, F_{I'}\} = 0 \), since \( F_I, F_{I'} \) depend on \( z \) only. If \( I = \{j_1, \ldots, j_{k+1}\} \) and \( I' = \{i_1, \ldots, i_{k-1}\} \), then

\[
\{G_I, F_{I'}\} = \sum_{m=1}^{k+1} (-1)^{m-1} d_{j_1, \ldots, j_{m-1}, \hat{j}_m, j_{m+1}, \ldots, j_{k+1}} d_{i_1, \ldots, i_{m-1}, \hat{i}_m, i_{m+1}, \ldots, i_{k-1}} = 0
\]
due to the Plücker relation 2.4.

Recall the function \( G_j(z_j, p_j) \) in (2.13). It is clear that \( \{G_j, G_{j'}\} = 0 \) for all \( j, j' \in J \). Now \( \{G_I, G_{I'}\} = 0 \) for all \( I, I' \) with \( |I| = |I'| = k + 1 \), since \( G_I, G_{I'} \) are linear combination of \( G_j \) with constant coefficients. \( \square \)

All the functions \( F_I, G_I \) define commuting Hamiltonian flows, preserving \( \Lambda \) and giving symmetries of \( \Lambda \). For \( I = \{i_1, \ldots, i_{k-1}\} \), the flow \( \varphi_t^I \) of the function \( F_I(p) \) has the form

\[
(z_1, \ldots, z_n, p) \mapsto (z_1 + d_{i_1, i_1, \ldots, i_{k-1}} t, \ldots, z_n + d_{n, i_1, \ldots, i_{k-1}} t, p).
\]

For \( I = \{j_1, \ldots, j_{k+1}\} \), the flow \( \varphi_t^I \) of the function \( G_I(z, p) \) does not change the pair of coordinate \( (z_j, p_j) \) of a point, if \( j \notin I \), and transforms the pair \( (z_{j_m}, p_{j_m}) \) to the pair

\[
(z_{j_m} - \frac{a_j}{p_{j_m}} + \frac{a_j}{p_{j_m}} (-1)^m d_{j_1, \ldots, j_m, \ldots, j_{k+1}} t, p_{j_m} + (-1)^m d_{j_1, \ldots, j_m, \ldots, i_{k+1}} t)
\]

for \( m = 1, \ldots, k + 1 \).
Remark. An interesting property of the Hamiltonians $F_I, G_I$ is that they are regular with respect the Plücker coordinates $d_{i_1,\ldots,i_k}$. Hence, they can be used to study the Langrange varieties of the arrangements in $\mathbb{C}^n \times \mathbb{C}^k$ associated with not necessarily generic matrices $(b_j^i)$.

3.5. Hessian as a function on the Lagrange variety. Let $z \in \mathbb{C}^n - \Delta$ and $t^0$ a critical point of the master function $\Phi(z, \cdot)$. An important characteristic of the critical point is the Hessian

$$\text{Hess } \Phi(z, t^0) = \det_{i,j=1}^k \left( \frac{\partial^2 \Phi}{\partial t_i \partial t_j}(z, t^0) \right),$$

see, for example, [AGV, MV, V2, V3].

For a subset $I = \{i_1, \ldots, i_k\} \subset J$, we denote by $d^2_I$ the number $(d_{i_1,\ldots,i_k})^2$.

Lemma 3.7. We have

$$(3.8) \quad \text{Hess } \Phi = (-1)^k \sum_{I \subset J, |I| = k} d^2_I \prod_{i \in I} \frac{p_i^2}{a_i}.$$ 

Proof. In [V3], the formula $\text{Hess } \Phi = (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} d^2_{i_1,\ldots,i_k} \prod_{m=1}^k a_{i_m} / f_{i_m}^2$ is given, which is the right hand side of (3.8). The formula itself is obvious. $\square$

3.6. Hessian and Jacobian. Let $M = \{m_1, \ldots, m_k\} \subset J$ be a $k$-element subset and $z_M, p_M$ the corresponding ordered coordinate system on $\Lambda$. The functions $z_1, \ldots, z_n$ form an ordered coordinate system on $\mathbb{C}^n - \Delta$. Consider the projection $\Lambda \mapsto \mathbb{C}^n - \Delta$, $(z, p) \mapsto z$, and the Jacobian $\text{Jac}_I(z_M, p_M)$ of the projection with respect to the chosen coordinate systems.

Theorem 3.8. As a function on $\Lambda$, the function $d^2_M \text{Jac}_M$ does not depend on $M$ and

$$(3.9) \quad d^2_M \text{Jac}_M = (-1)^{n-k} \sum_{L \subset J, |L| = n-k} d^2_L \prod_{j \in L} \frac{a_j}{p_j^2}.$$ 

Proof. The function $d^2_M \text{Jac}_M$ does not depend on $M$ by Lemma 3.4.

Consider the function $\tilde{\Psi} = \sum_{j \in J} a_j \ln p_j$ of $n$ variables $p_j$. Express in $\tilde{\Psi}$ the variables $p_M$ in terms of variables $p_M$ by formulas (3.5). Denote by $\tilde{\Psi}_M(p_M)$ the resulting function. By Theorem 3.5 $\text{Jac}_M = \det \left( \frac{\partial^2 \tilde{\Psi}_M}{\partial p_M \partial p_M} \right)$. This implies that $d^2_M \text{Jac}_M$ is a polynomial in $a_j, j \in J$, of the form

$$d^2_M \text{Jac}_M = \sum_{L \subset J, |L| = n-k} c_L \prod_{j \in L} \frac{a_j}{p_j^2},$$

where $c_L$ are numbers independent of $M$. Our goal is to show that $c_L = (-1)^{n-k} d^2_L$ but this is clear for $L = M$. This proves the theorem. $\square$

Corollary 3.9. We have

$$(3.10) \quad d^2_M \text{Jac}_M = (-1)^n \text{Hess } \Phi \prod_{j \in J} \frac{a_j}{p_j^2}.$$
4. Characteristic variety of the Gauss-Manin differential equations

4.1. Space Sing $V$. Consider the complex vector space $V$ generated by vectors $v_{i_1,\ldots,i_k}$ with $i_1,\ldots,i_k \in J$ subject to the relations $v_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} = (-1)^{\sigma} v_{i_1,\ldots,i_k}$ for any $i_1,\ldots,i_k \in J$ and $\sigma \in S_k$. The vectors $v_{i_1,\ldots,i_k}$ with $1 \leq i_1 < \cdots < i_k \leq n$ form a basis of $V$. If $v = \sum_{1 \leq i_1 < \cdots < i_k \leq n} c_{i_1,\ldots,i_k} v_{i_1,\ldots,i_k}$ is a vector of $V$, we introduce the numbers $c_{i_1,\ldots,i_k}$ for all $i_1,\ldots,i_k \in J$ by the rule: $c_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} = (-1)^{\sigma} c_{i_1,\ldots,i_k}$. We introduce the subspace Sing $V \subset V$ of singular vectors by the formula

$$\text{Sing } V = \left\{ \sum_{1 \leq i_1 < \cdots < i_k \leq n} c_{i_1,\ldots,i_k} v_{i_1,\ldots,i_k} \mid \sum_{j \in J} a_j c_{j,j_1,\ldots,j_{k-1}} = 0 \text{ for all } \{j_1,\ldots,j_{k-1}\} \subset J \right\}.$$

The symmetric bilinear contravariant form on $V$ is defined by the formulas: $S(v_{i_1,\ldots,i_k}, v_{j_1,\ldots,j_k}) = 0$, if $\{i_1,\ldots,i_k\} \neq \{j_1,\ldots,j_k\}$, and $S(v_{i_1,\ldots,i_k}, v_{i_1,\ldots,i_k}) = \prod_{m=1}^k a_{i_m}$, if $i_1,\ldots,i_k$ are distinct. Denote by $s^\perp : V \to \text{Sing } V$ the orthogonal projection with respect to the contravariant form.

4.2. Differential equations. Consider the master function $\Phi(z,t)$ as a function on $U^0 \subset \mathbb{C}^n \times \mathbb{C}^k$. Let $\kappa$ be a nonzero complex number. The function $e^{\Phi(z,t)/\kappa}$ defines a rank one local system $L_\kappa$ on $U^0$ whose horizontal sections over open subsets of $\tilde{U}$ are univalued branches of $e^{\Phi(z,t)/\kappa}$ multiplied by complex numbers. The vector bundle

$$\bigcup_{z \in \mathbb{C}^n-\Delta} H_k(U(\mathcal{C}(z)), L_\kappa|_{U(\mathcal{C}(z))}) \to \mathbb{C}^n-\Delta$$

has the canonical flat Gauss-Manin connection. For a horizontal section $\gamma(z) \in H_k(U(\mathcal{C}(z)), L_\kappa|_{U(\mathcal{C}(z))})$, consider the $V$-valued function

$$I_\gamma(z) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left( \int_{\gamma(z)} e^{\Phi(z,t)/\kappa} d\ln f_{i_1} \wedge \cdots \wedge d\ln f_{i_k} \right) v_{i_1,\ldots,i_k}.$$  

For any horizontal section $\gamma(z)$, the function $I_\gamma(z)$ takes values in Sing $V$ and satisfies the Gauss-Manin differential equations

$$(4.1) \quad \kappa \frac{\partial I_\gamma}{\partial z_j} = K_j(z) I_\gamma, \quad j \in J,$$

where $K_j(z) \in \text{End}(\text{Sing } V)$ are suitable linear operators independent of $\kappa$ and $\gamma$. Formulas for $K_j(z)$ see, for example, in [V4, Formula (5.3)].

For $z \in \mathbb{C}^n-\Delta$, the subalgebra $B(z) \subset \text{End}(\text{Sing } V)$ generated by the identity operator and the operators $K_j(z)$, $j \in J$, is called the Bethe algebra at $z$ of the Gauss-Manin differential equations. The Bethe algebra is a maximal commutative subalgebra of $\text{End}(\text{Sing } V)$, see [V4, Section 8].

We define the characteristic variety of the $\kappa$-dependent $D$-module associated with the Gauss-Manin differential equations (4.1) as

$$\text{Spec} = \{(z,p) \in T^*(\mathbb{C}^n-\Delta) \mid \exists v \in \text{Sing } V \text{ with } K_j(z)v = p_j v, j \in J\}.$$
4.3. Identification. Let $z \in \mathbb{C}^n - \Delta$. By Lemma 2.3, given $j_1 \in J$, the monomials $p_{i_1} \cdots p_{i_k}$, with $i_1 < \cdots < i_k$ and $j_1 \notin \{i_1, \ldots, i_k\}$, form a $\mathbb{C}$-basis of $A_{\Phi}(z)$. Consider the linear map

\[ \mu : A_{\Phi}(z) \to \text{Sing } V \]

which sends $d_{i_1, \ldots, i_k} p_{i_1} \cdots p_{i_k}$ to $s(v_{i_1, \ldots, i_k})$ for all $i_1 < \cdots < i_k$ with $j_1 \notin \{i_1, \ldots, i_k\}$.

**Theorem 4.1** ([V5, Corollary 6.16]). The linear map $\mu$ does not depend on $j_1$ and is an isomorphism of complex vector spaces. For any $j \in J$, the isomorphism $\mu$ identifies the operator of multiplication by $p_j$ on $A_{\Phi}(z)$ and the operator $K_j(z)$ on $\text{Sing } V$.

**Corollary 4.2.** The characteristic variety $\text{Spec}$ of the Gauss-Manin differential equations coincides with the Lagrangian variety of the master function.

Thus the statements in Section 3 give us information on the characteristic variety of the Gauss-Manin differential equations. In particular, equations in $A_{\Phi}(z)$ are satisfied in $B(z)$, for example,

\[ f_{i_1, i_2, \ldots, i_{k+1}}(z)K_{i_1}(z) \cdots K_{i_{k+1}}(z) = \sum_{m=1}^{k+1} (-1)^{m-1} a_{i_m} d_{i_1, \ldots, i_m, \ldots, i_{k+1}} K_{i_1}(z) \cdots \widehat{K}_{i_m}(z) \cdots K_{i_{k+1}}(z). \]

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