Bianchi type I models with two tilted fluids

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Abstract
In this paper we investigate expanding Bianchi type I models with two tilted fluids with linear equations of state. Individually the fluids have nonzero energy fluxes w.r.t. the symmetry surfaces, but these cancel each other because of the Codazzi constraint. Asymptotically toward the past the solutions approach Kasner states if the speeds of sound are less than that of light. If one of the fluids has a speed of sound that is less or equal to 1/3 of the speed of light (radiation) then the models isotropize toward the future, but if both fluids are stiffer than radiation then the final state is anisotropic with nonzero Hubble-normalized shear. The significance of our results is discussed in the context of more general models.

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1. Introduction

The construction of a relativistic model of gravity contains the following ingredients: (i) a four-dimensional manifold endowed with a Lorentzian metric, (ii) a matter source description, (iii) dynamical laws—Einstein’s field equations and, if needed, matter equations. A general relativistic cosmological model requires that in addition one attempts to describe the universe at a particular scale.

At the largest spatial scales present observational data suggest that the ‘standard’ ΛCDM model of cosmology provides the most simple consistent description of the universe today. This model is spatially homogeneous (SH) and isotropic with flat spatial geometry; the matter content consists of a dark energy component, modeled by a positive cosmological constant, supplemented with dark matter and atoms, described by pressureless fluids with $n^3$, the unit normal to the symmetry surfaces, as the common 4-velocity. Density fluctuations, described by linear scalar perturbations, are seeded by an almost Gaussian, adiabatic, nearly scale invariant process, see, e.g., [1, 2], and references therein.

However, this description does not hold on all scales, neither spatial nor temporal. On smaller spatial scales matter has to be described by many components, with energy fluxes in different directions, e.g., our galaxy is moving w.r.t. to the CMB. In the very early universe, and
perhaps also in the distant future, the $\Lambda$CDM model does not give a correct matter description, indeed, although radiation can be treated as a gravitational test field\(^1\) today it has been an important gravitational source in the past, and inflationary proponents suggest that matter could be extremely different in the very early universe. Clearly the matter description in the ’standard’ scenario is not exact, and even with a few matter components the associated energy fluxes cannot all be *exactly* aligned, not even on the largest spatial scales. What then happens with the matter and its associated energy fluxes in the far future and what was the situation in the distant past?

We believe that we understand how radiation and matter interacts, at least after the very early universe when we think our empirical experience holds. Presumably this interaction explains why the 4-velocities associated with radiation and matter today are fairly well aligned with each other on large spatial scales, or maybe this alignment was produced in the very early universe by some unknown process, perhaps inflation. But is it obvious that this alignment should have persisted to the extent present observations indicate after recombination, and is it going to persist in the far future? Do linear vector perturbations of SH and isotropic models suffice to determine this? In the early universe interactions presumably played an important role in aligning energy fluxes of different matter components, but it is unclear what those interactions were; is it possible to shed any light on this issue without knowing the details of these interactions?

Here we are going to consider two non-interacting perfect fluids that in general move w.r.t. each other and a non-negative cosmological constant, which includes the $\Lambda$CDM matter content as a special case. Although this may not be a good matter description at all times, it is still a useful step since it allows a comparative study of the effects of various types of interactions in possible future projects, an issue we return to in the concluding remarks.

Two matter source components with energy fluxes in different directions produce an anisotropic source which excludes the isotropic standard model and forces one to consider anisotropic geometries. There are several reasons that suggest that the natural anisotropic models to start with are the SH Bianchi type I models. One reason is their geometric simplicity since this sheds light on more general models—if type I turns out to be complicated, then more general models will be even worse. But more importantly is that they are the foundation in a hierarchy of ever more geometrically complex models.

The SH Bianchi models (models that admit simply transitive three-dimensional symmetry groups) form a crucial level in the geometric complexity hierarchy, and within this level the Bianchi type I models is the common ingredient since they can be obtained from all other Bianchi models by Lie algebra contractions. The consequences of this property are revealed when one casts Einstein’s equations into a dynamical system where the type I models appear as part of a state space boundary that describes asymptotic features of all other Bianchi models, see, e.g., [3, 4], and references therein.

Moreover, the Bianchi models themselves serve as building blocks for understanding asymptotic dynamics of more general inhomogeneous models, the primary reason being the following: in the very early universe near a generic or isotropic singularity, or in the very late universe in an inflationary epoch, horizons form and asymptotically shrink, asymptotically prohibiting communication—a phenomenon naturally referred to as asymptotic silence—generically pushing inhomogeneities outside the horizons leading to that the asymptotic evolution can locally be described by SH models—asymptotic locality, see [5–7]. In the dynamical systems approach these features are formally captured by recasting the field

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\(^1\) A gravitational test field is a field that does not affect the metric, i.e., it is a field for which we can neglect: (i) its source contribution to Einstein’s equations, (ii) its influence on gravitational sources.
equations into an infinite dimensional dynamical system that at each spatial point has a boundary—the silent boundary, which asymptotically attracts generic asymptotically local dynamics. The dynamics on the silent boundary, which determine the generic asymptotically local dynamics, is described by a finite dimensional dynamical system that is identical to that of the Bianchi models and hence type I is a common key ingredient in a very general context [5–10]. Furthermore, there are hints, analytical and numerical, that Bianchi models are important for describing future asymptotic states, even in the absence of inflation and asymptotic silence and locality, that sheds light on spatial structure formation.

This is not the first study using dynamical systems techniques for studying multi-fluid models. Bianchi models with two fluids with both 4-velocities being orthogonal to the symmetry surfaces were studied in [11]. However, it is not difficult to predict what is going to happen when the non-interacting fluids are aligned with each other, even in the general inhomogeneous case. Let us introduce a length scale $\ell$ defined by $\dot{\ell}/\ell = H$, where $H$ is the Hubble scalar and where the dot refers to the time derivative w.r.t. the proper clock time along the common fluid congruence. For simplicity we consider fluids with linear equations of state such that $p = w\rho$, where $p$, $\rho$ is the pressure and energy density, respectively, and where the constant $w$ describes the speed of sound $c_s$ according to $w = c_s^2$ when $w \geq 0$.

Interesting examples of equations of state are: $w = -1$, which corresponds to a positive cosmological constant; dust, $w = 0$; radiation, $w = \frac{1}{3}$; and a stiff fluid, $w = 1$, for which the speed of sound is equal to that of light. Local energy–momentum conservation then yields that $\rho \propto \ell^{-3(1+w)}$, see, e.g. [3, 12]. Thus if $\ell \to 0$ the energy densities of fluids with smaller $w$ become asymptotically negligible compared to those with larger $w$, while if $\ell \to \infty$ the opposite holds.

However, the situation is much more complicated when the non-interacting fluids are not aligned. In [13] Bianchi type V models with two fluids and a positive cosmological constant were studied; one of the fluids had a flow orthogonal to the symmetry surfaces while the other had a ‘tilted’ flow, i.e., its 4-velocity was not aligned with the normal to the SH surfaces. In [14], where brane-world cosmology was invoked to motivate the study of multiple fluids, Bianchi type VI\text{b} with two non-interacting tilted fluids was investigated. Perhaps because the focus in these papers was on the quite interesting late time behavior, there was no mentioning about the possibility of having two (or more) tilted fluids in Bianchi type I.

It is of interest to compare multiple and single fluid models, and in this context it is worth mentioning that there are a number of relatively recent studies of SH models with a single tilted fluid: [15–27]. Having several fluids open up new possibilities for producing anisotropies. In Bianchi type I the Codazzi constraint enforces the 4-velocity of a single fluid to be orthogonal to the symmetry surfaces, which yields an isotropic stress–energy tensor. However, as we will see, one can have two tilted fluids that generate anisotropies. The differences as regards anisotropies in the total stress–energy tensor between the single and multiple fluid cases become even more pronounced when studying more general models, and this affects the solution structure, particularly as regards asymptotics—multiple fluids sometimes generate different results than single fluids, which shed light on the structural stability properties of single fluid models. It should be pointed out that the reason for looking at multiple fluids is that this allows one to produce more physically realistic models, and it is rather surprising that there are so few multiple fluid studies in cosmology. However, Bianchi type I is only moderately physically interesting in itself; the justification for studying Bianchi type I with two fluids is rather that these models give a hint about the complexity of more general and physically realistic models, and, more importantly, that they act as building blocks for these models—an understanding of more physically realistic models requires a detailed analysis of the present type I models, and it is in this context the present work should be seen.
The paper is organized as follows. In the following section we derive a reduced dynamical system that describes Bianchi type I with two non-interacting fluids with linear equation of state and a cosmological constant. In the subsequent section we describe the associated state space and the influence of a positive cosmological constant; in addition we list the invariant subsets and fix points that are essential for understanding the present models. In section 4 we give several monotonic functions that are useful for determining the asymptotic dynamics, and we also briefly discuss some reasons why they exist since this allows one to produce monotonic functions for other models as well. Section 5 takes the results in the previous sections as the starting point for a dynamical systems analysis which yield our main results about asymptotic dynamics toward the past and toward the future, in the absence of a cosmological constant. We conclude with a discussion in section 6 about the significance of our results in a more general context. In appendix A detailed information about the fix points and their stability properties is presented. Appendix B contains an analysis of the Kasner subset. Throughout we use units such that \( c = 1 = 8\pi G \).

2. Derivation of the dynamical system

In the orthonormal frame approach one uses a tetrad of four orthogonal unit basis vector fields \( \{e_a\} \) and the associated dual one-forms \( \{\omega^a\} (a = 0, 1, 2, 3) \), which, when expressed in a local coordinate basis, take the form 
\[
e_a = e_{a\mu} dx^\mu = e_{a\mu}\partial_\mu, \quad \omega^a = e^\mu_a dx^\mu (\mu = 0, 1, 2, 3),
\]
where the tetrad components \( e_{a\mu}(x^\nu) \) and their inverse components \( e^{a\mu}(x^\nu) \) satisfy the duality relations \( e_{a\mu}e^{b\nu} = \delta^\nu_\mu \Leftrightarrow e_{a\mu}e^{b\mu} = \delta_a^b \), and where the orthogonality conditions are given by \( g_{ab} = e_a \cdot e_b = g_{\mu\nu}e^\mu_a e^\nu_b = \eta_{ab}; \ g_{\mu\nu} = \eta_{\mu\nu}e^\mu_a e^\nu_b, \eta_{ab} = \text{diag}(-1, 1, 1, 1) \). The commutator functions \( c^a_{bc}(x^\nu) \), defined as \( [e_a, e_b] = c^a_{ab} e_c \), are typically `elevated' to dependent variables satisfying the Jacobi identities, \( e_{[a\mu} c^c_{b\nu]} = c^d_{a[c} c^c_{b]} e^d = 0 \).

In the case of SH Bianchi models, i.e., models with a foliation of SH hypersurfaces invariant under a simply transitive group action \( G_3 \), one can introduce an orthonormal basis of vector fields \( \{e_a\} \) that is invariant under the group action such that the timelines are orthogonal to the SH hypersurfaces with \( e_0 = n = \partial / \partial t \), where \( t \) is the proper time along the geodesic timelines (the geodesic property follows from the symmetries). This yields the line-element: 
\[
d s^2 = -dt^2 + \delta_{\alpha\beta} \omega^\alpha \otimes \omega^\beta (\alpha, \beta = 1, 2, 3),
\]
where \( \omega^\alpha \) (with components \( \omega^\alpha_i \)) are the one-forms dual to the triad \( e_a \) (with components \( e_a^i \)), tangential to the symmetry surfaces, i.e., \( e_a^i e^j_i = \delta^a_j \ (i = 1, 2, 3) \).

A 3+1 split of the commutator equations w.r.t. \( e_0 = n \) yields
\[
[e_0, e_a] = -[H\delta_\beta^a + \sigma_{\alpha\beta} + e^\alpha_\beta, \Omega^\gamma] e_\beta, \tag{1a}
\]
\[
[e_a, e_\beta] = c^\gamma_{a\beta} e_\gamma = 2a_{(a} \delta_{\beta)}^\gamma + e_{a\beta} n^{\gamma}, \tag{1b}
\]
Here \( H \) is the Hubble variable, which is related to the expansion \( \theta \) of \( n \) according to \( H = \frac{\theta}{3} \); \( \sigma_{\alpha\beta} \) is the shear associated with \( n; \Omega^\gamma \) is the Fermi rotation which describes how the spatial triad rotates with respect to a gyroscopically fixed so-called Fermi frame; \( n^{\alpha\beta} \) and \( a_{a\beta} \) describe the Lie algebra of the three-dimensional simply transitive Lie group and determine the spatial three curvature, see, e.g., [3].

2 The sign in the definition of \( \Omega^\gamma \) is the same as in [28], but opposite of that in [3].

3 Due to the symmetries \( e_e^i \) can be written as \( e_e^i = \tilde{e}_e^\mu (t) \tilde{e}_\mu^i \), where \( \tilde{e}_e^\mu \) are functions of the spatial coordinates \( x_e \) along such that \( [a_{a\mu}, e_b^i] = \tilde{e}_a^\mu e_b^i - 2\tilde{a}_{a\nu} \tilde{e}_b^\nu \tilde{e}_\nu^i + \tilde{e}_a^\mu \tilde{a}_{\nu\tau} \tilde{e}_\tau^i - \tilde{e}_a^\nu \tilde{a}_{\tau\nu} \tilde{e}_\tau^i \), where \( \tilde{e}_a^\nu = \tilde{e}_a^\nu / \partial x_e^\nu \) and where \( \tilde{e}_a^\nu, \tilde{a}_{\mu\nu} \), parametrized by \( \tilde{a}_{\nu\tau}, \tilde{a}_{\mu\nu} \), are the structure constants of the symmetry group. The symmetries lead to that the equations for the variables \( \tilde{e}_a^\nu (t) (\partial \tilde{e}_a^\nu (t) / \partial t) = -(H\tilde{e}_a^\nu + \sigma_{\alpha\beta} + e^\alpha_\beta, \Omega^\gamma) \tilde{e}_\gamma^i \), as follows from (1b) decouple from the remaining field equations, and because of this they are not usually considered when discussing Bianchi models from an orthonormal frame perspective.
A 3+1 split of the total stress–energy tensor $T_{ab}$ w.r.t. $n^a$ yields

\[ T_{ab} = \rho n_a n_b + 2q (n_a n_b) + p h_{ab} + \pi_{ab}, \quad (2a) \]

\[ \rho = n^a n_b T_{ab}, \quad q_a = -h_a^b n^c T_{bc}, \quad p = \frac{1}{3} h^{ab} T_{ab}, \quad \pi_{ab} = h_{(a}^c h_{b)}^d T_{cd}, \quad (2b) \]

where $h_{ab} = n_a n_b + g_{ab}$; $\rho, p$ is the total energy density and total effective pressure, respectively, measured in the rest space of $n^a$; $(..)$ stands for the trace-free part of a symmetric spatial tensor, i.e. $A_{(ab)} = A_{ab} - \frac{1}{3} \delta_{ab} A$. In general $T_{ab}$ consists of several components $T_{ab}^{(i)}$ such that $T_{ab} = \sum_i T_{ab}^{(i)}$. If the components are non-interacting, then $\nabla_a T_{ab}^{(i)} = 0$.

A cosmological constant $\Lambda_1$ can be formally regarded as a component of $T_{ab}$ such that $\rho/\Lambda_1 = -p/\Lambda_1 = 0, q_{(a}/\Lambda_1 = 0, \pi_{a/b}(\Lambda_1) = 0$.

In the Hubble normalized approach one factors out the Hubble variable $H$ by means of a conformal transformation which yields dimensionless quantities [10]. In the present SH case this amounts to the following:

\[ (\Sigma_{a/b}, R^a, N^a_{|b}, A_a) = \frac{1}{H} (\sigma_{a/b}, \Omega^a, n^a_{|b}, a_a), \quad (\Omega, P, Q_a, \Pi_{a/b}) = \frac{1}{3 H^2} (\rho, p, q_a, \pi_{a/b}). \quad (3) \]

where we have chosen to normalize the stress–energy quantities with $3H^2$ rather than $H^2$ in order to conform with the usual definition of $\Omega$. In addition to this we choose a new dimensionless time variable $\tau$ according to

\[ \frac{d\tau}{dt} = H. \quad (4) \]

Since $H$ is the only variable with dimension, its evolution equation decouples from the remaining equations for dimensional reasons:

\[ H' = -(1 + q) H; \quad q = \Sigma^2 + \frac{1}{2} (\Omega + 3P), \quad \Sigma^2 = \frac{1}{2} \Sigma_{a/b} \Sigma^{a/b}, \quad (5) \]

where a prime henceforth denotes $d/d\tau$ and where $q$ is the deceleration parameter, obtained by means of one of Einstein’s equations—the Raychaudhuri equation; note that $\Omega$ and $P$ in the expression for $q$ refers to the total Hubble-normalized stress–energy content. A 3+1 split of the remaining Einstein’s field equations ($G_{ab} = T_{ab}$, where $G_{ab}$ is the Einstein tensor and $T_{ab}$ the total stress–energy tensor) and the Jacobi identities, yields the following reduced system of coupled equations for the Hubble-normalized variables:

\[ \Sigma'_{a/b} = -(2 - q) \Sigma_{a/b} + 2 e^\gamma_{a}^\beta (\Sigma_{(a/b)}+R_{|\gamma}) - 3 \mathcal{R}_{(a/b)} + 3 \Pi_{a/b}, \quad (6a) \]

\[ A'_a = [q \delta_a^\beta - \Sigma_a^\beta - \epsilon_a^\gamma R_{|\gamma}] A_{\beta}, \quad (6b) \]

\[ (N^a)^{\gamma}_a = [q \delta^\gamma_{a} + 2 \Sigma^\gamma_{|a} + 2 \epsilon_{a}^{\gamma} (R_{|\gamma}^2) N^{\beta \gamma}]. \quad (6c) \]

\[ 0 = 1 - \Sigma^2 + \frac{1}{2} 3 \mathcal{R} - \Omega, \quad (6d) \]

\[ 0 = (3 \delta^\gamma_{a} A_{\beta} + \epsilon_{a}^{\gamma} N_{\beta \gamma}) \Sigma^{\beta \gamma} - 3 Q_{a}, \quad (6e) \]

\[ 0 = A_{|\beta} N^\beta_{\gamma}. \quad (6f) \]

where $3 \mathcal{R}_{(a/b)}$ and $3 \mathcal{R}$ describe the trace-free and scalar parts of the Hubble-normalized three-curvature, respectively, according to

\[ 3 \mathcal{R}_{(a/b)} = B_{(a/b)} + 2 e^\gamma_{a}^\beta (N_{(a/b)} + A_{|\gamma}), \quad 3 \mathcal{R} = -\frac{1}{2} B^\gamma_{\gamma} - 6 A^2; \quad (7) \]

\[ B_{a/b} = 2 N_{a/\gamma} N^\gamma_{|b} - N^\gamma_{|a} N_{a/b}. \]
Equation (6a) are the Hubble-normalized spatial and trace-free Einstein equations; (6b) and (6c) are evolution equations obtained from the Jacobi identities; (6d) and (6e) are the Hubble-normalized Gauss and Codazzi constraints, respectively, while the constraint (6f) stems from the Jacobi identities. The conservation law \( \nabla_a T^{ab} = 0 \) for the total stress–energy tensor yields

\[
\Omega' = (2q - 1)\Omega - 3P + 2A_a Q^a - \Sigma_{a\beta} \Pi^{a\beta},
\]

(8a)

\[
Q'_a = -\left[2(1 - q)\delta_a^\beta + \Sigma_a^\beta + \epsilon_a^\beta \gamma R^\gamma\right] Q^\beta + (3\delta_a^\gamma A_\beta + \epsilon_a^\beta \gamma N_{\beta\gamma}) \Pi^{\beta\gamma}.
\]

(8b)

Let us now restrict ourselves to the Bianchi type I case with expansion \((H > 0)\), characterized by

\[ A_a = 0, \quad N^{a\beta} = 0. \]

(9)

In type I (8b) reduces to \( Q'_a = -\left[2(1 - q)\delta_a^\beta + \Sigma_a^\beta + \epsilon_a^\beta \gamma R^\gamma\right] Q^\beta \), and hence the reduction of the Codazzi constraint (6e) to \( Q_a = 0 \) is consistent since it is preserved during evolution. Thus there is no total energy flux in Bianchi type I, and hence the type I SH frame is an energy frame, in the nomenclature of Landau and Lifshitz [29]. Even so, a matter source can consist of several components that individually have nonzero energy fluxes, as long as they add up to zero.

Let us now specialize to a source that consists of a non-negative cosmological constant \( \Lambda \geq 0 \) and two non-interacting perfect fluids, i.e., \( T^{ab}_i = (\rho_i + p_i) u^a_i u^b_i + p_i g^{ab} \); \( \nabla_a T^{ab}_i = 0 \) \((i = 1, 2)\), where \( \rho_i, p_i \), is the energy density and pressure, respectively, in the rest frame of the \( i \)th fluid, while \( \tilde{u}^a_i \) is its 4-velocity. We assume that \( \rho_i \geq 0 \), and for simplicity also a linear equations of state, \( p_i = w_i \rho_i \), where \( w_i = \text{const} \). The most interesting equations of state are dust, \( w = 0 \), and radiation, \( w = \frac{1}{3} \), but it is useful not to restrict oneself to these values in order to study structural stability, however, we do restrict ourselves to

\[ 0 \leq w_{(1)} < w_{(2)} < 1, \]

(10)

since \( w_{(1)} = 1, w_{(2)} = w_{(2)} \) are associated with bifurcations that needs special treatment, to be dealt with elsewhere.

Making a 3+1 split with respect to \( n = e_0 \) yields

\[
\tilde{u}^a_i = \Gamma^a_i \left(n^a + v^a_i\right); \quad n_a v^a_i = 0, \quad \Gamma^a_i = 1/\sqrt{1 - v^2_i},
\]

(11)

which gives

\[
Q^a_i = (1 + w_i) (G^{a(i)})^{-1} v^{a(i)} \Omega_i, \quad P_i = w_i \Omega_i + \frac{1}{2} (1 - 3 w_i) Q^{a(i)} v^{a(i)}, \quad \Pi^{a\beta}_i = Q^{a\beta}_i v^{a(i)},
\]

(12)

where \( G^{a(i)} = 1 \pm w(v)^2_i; \quad \Omega_i = \rho_i /(3 H^2) \). The cosmological constant contributes \( \Omega_\Lambda = \Lambda/(3H^2) = - P_\Lambda \) to the total \( \Omega \) and \( P \), while \( Q^a_\Lambda = 0 = \Pi^{a\beta}_\Lambda \). Due to its definition and equation (5), \( \Omega_\Lambda \) satisfies

\[ \Omega'_\Lambda = 2(1 + q) \Omega_\Lambda. \]

(13)

The Codazzi constraint, \( Q_a = Q^{a(1)}_a + Q^{a(2)}_a = 0 \), taken in combination with (12) forces the 3-velocities of the two fluids to be anti-parallel. Kinematically the situation is similar to that of Bianchi type I with a general magnetic field studied in [31], and it is therefore natural to exploit the same mathematical structures in the present problem. We therefore choose the

\[ 0 < 1/3 < w_{(1)} < 0, \]

but the well-posedness of the Einstein equations for this range has been questioned, see [30].
spatial triad so that one of the frame vectors is aligned with the fluid velocities, which we choose to be $e_3$, i.e. $v_\alpha^{(i)} = (0, 0, v_{\alpha(i)})$. Demanding that these conditions on $v_\alpha^{(i)}$ hold for all times lead to the following conditions$^5$

$$R_1 = -\Sigma_{23}, \quad R_2 = \Sigma_{31}. \quad (14)$$

This leaves $R_3$ undetermined, however, we still have the freedom of arbitrary rotations in the 1-2-plane, which we use to set

$$R_3 = 0. \quad (15)$$

Following [31], we introduce the variables $\Sigma_+, \Sigma_A, \Sigma_B, \Sigma_C$ according to

$$\Sigma_+ = \frac{1}{2}(\Sigma_{11} + \Sigma_{22}), \quad \Sigma_{31} + i\Sigma_{23} = \sqrt{3}\Sigma_A e^{i\phi}, \quad \Sigma_- + \frac{i}{\sqrt{3}}\Sigma_{12} = (\Sigma_B + i\Sigma_C) e^{2i\phi},$$

where $\Sigma_- = (\Sigma_{11} - \Sigma_{22})/(2\sqrt{3})$, which leads to

$$\Sigma^2 = \Sigma_+^2 + \Sigma_A^2 + \Sigma_B^2 + \Sigma_C^2. \quad (16)$$

The above decomposition of $\Sigma_{\alpha\beta}$ has the advantage that the equation for $\phi, d\phi/d\tau = -\Sigma_C$, decouples from the other equations, leaving the following reduced constrained dynamical system of coupled equations for the Hubble-normalized shear variables $\Sigma_+, \Sigma_A, \Sigma_B, \Sigma_C$, the fluid 3-velocities $v_{(1)}, v_{(2)}$, the Hubble-normalized energy densities $\Omega_{(1)}, \Omega_{(2)}$, and the Hubble-normalized cosmological constant $\Omega_{\Lambda}$:

**Evolution equations:**

$$\Sigma_+ = -(2 - q)\Sigma_+ + 3\Sigma_A^2 - Q_{(1)}v_{(1)} - Q_{(2)}v_{(2)}, \quad (18a)$$

$$\Sigma_A = -(2 - q + 3\Sigma_+ + \sqrt{3}\Sigma_B)\Sigma_A, \quad (18b)$$

$$\Sigma_B = -(2 - q)\Sigma_B + \sqrt{3}\Sigma_A^2 - 2\sqrt{3}\Sigma_C^2, \quad (18c)$$

$$\Sigma_C = -(2 - q - 2\sqrt{3}\Sigma_B)\Sigma_C, \quad (18d)$$

$$v_{\alpha(i)}^{(i)} = (G^{(i)})^{-1}(1 - v_{\alpha(i)}^{(i)}) (3w_{(i)} - 1 + 2\Sigma_+)v_{(i)}, \quad (18e)$$

$$\Omega_{(i)} = (2q - 3w_{(i)})\Omega_{(i)} + (3w_{(i)} - 1 + 2\Sigma_+)Q_{(i)}v_{(i)}, \quad (18f)$$

$$\Omega_{\Lambda} = 2(1 + q)\Omega_{\Lambda}. \quad (18g)$$

**Constraint equations:**

$$0 = 1 - \Sigma^2 - \Omega_{(1)} - \Omega_{(2)} - \Omega_{\Lambda}, \quad (19a)$$

$$0 = Q_{(1)} + Q_{(2)}, \quad (19b)$$

where

$$q = 2\Sigma^2 + \frac{1}{2}(\Omega_m + 3P_m) - \Omega_{\Lambda} = 2 - \frac{1}{2}(\Omega_m - P_m) - 3\Omega_{\Lambda};$$

$$\Omega_m = \Omega_{(1)} + \Omega_{(2)}, \quad P_m = P_{(1)} + P_{(2)}. \quad (20)$$

Equations (18e) and (18f) were obtained by using that (8) takes the same form for non-interacting individual matter components, where, however, the total matter content enters

$^5$ In the case of a magnetic field aligned along $e_3$ one obtains $R_1 = \Sigma_{23}, R_2 = -\Sigma_{31}$, i.e., the signs are opposite of those of the two tilted fluid case! This dynamical result in turn leads to sign differences in the $\Sigma$-equations. Note that the kinematic results in [31] still hold, hence, e.g. fix points correspond to transitively self-similar models.
into \( q \), together with the type I conditions and the relations (12) for the individual perfect fluids. The assumption of non-negative energy densities and a non-negative cosmological constant, \( \Omega_\Lambda \geq 0 \), together with (20) and (19a), yields that \(-1 \leq q \leq 2\), and hence that \( 2-q \geq 0 \), where \( q = -1 \) only when \( \Omega_\Lambda = 1 \), \( \Omega_m = 0 \), \( \Sigma^2 = 0 \). It follows that \( \tau \in (-\infty, \infty) \) and \( H \to \infty \) when \( \tau \to -\infty \) (if \( \Omega_\Lambda \neq 1 \) initially).

The auxiliary equation,

\[
\rho_{(i)}' = -(1 + w_{(i)})(G^{(i)})^{-1}\left[3 + v_{(i)}^2 - 2\Sigma v_{(i)}^2\right]\rho_{(i)},
\]

implies that \( \rho_{(i)} \) is a monotonically decreasing function such that \( \rho_{(i)} \to \infty \) (\( \rho_{(i)} \to 0 \)) when \( \tau \to -\infty \) (\( \tau \to \infty \)); hence the models begin with an initial curvature singularity, where \( \Lambda \) becomes negligible when compared to \( \rho_{(i)} \) when \( \tau \to -\infty \), and then expand forever to a state where the ordinary matter is infinitely diluted, leading to that \( \Omega_m \) becomes negligible compared to \( \Omega_\Lambda \).

3. State space properties

3.1. The state space

The reduced state space consists of \( S = \{ \Sigma_i, \Sigma_A, \Sigma_B, \Sigma_C, v_{(1)}, v_{(2)}, \Omega_{(1)}, \Omega_{(2)}, \Omega_\Lambda \} \), subject to the two constraints (19), i.e., the state space is seven dimensional. From the definitions and the constraints (19) it follows that the state space is bounded. Our primary concern in this paper is the 'interior' state space for the case of two tilted fluids for which \( \Omega_{(1)} \Omega_{(2)} > 0 \), \( 0 < v_{(i)}^2 < 1 \), however, the solutions belonging to the interior state space asymptotically approach its boundary. To understand the interior dynamics we therefore consider the closure of the interior state space, \( S \), thus obtaining a compact state space, which is possible because of the regularity of the evolution equations. Hence \( \Sigma^2 \leq 1 \), \( 0 \leq v_{(i)}^2 \leq 1 \); \( 0 \leq \Omega_{(1)} \leq 1 \), \( 0 \leq \Omega_{(2)} \leq 1 \), \( 0 \leq \Omega_\Lambda \leq 1 \), in such a way so that the constraints (19) are satisfied; note that the Codazzi constraint (19b) leads to that \( v_{(1)}v_{(2)} \leq 0 \) when \( \Omega_{(1)} \Omega_{(2)} > 0 \), a condition on \( v_{(i)} \) that we extend to the boundary. The dynamical system (18) and (19) is invariant under the following discrete symmetries:

\[
\Sigma_A \to -\Sigma_A, \quad \Sigma_C \to -\Sigma_C; \quad (v_{(1)}, v_{(2)}) \to -(v_{(1)}, v_{(2)}).
\]

We therefore assume without loss of generality that \( \Sigma_A \in [0, 1] \), \( \Sigma_C \in [0, 1] \), \( v_{(1)} \in [0, 1] \), and \( v_{(2)} \in [-1, 0] \); the solutions in the other sectors of the state space are easily obtained by means of the discrete symmetries.

3.2. The influence of a cosmological constant

Equation (18a) implies that \( \Omega_\Lambda \) is monotonically increasing from zero to one. Hence

\[
\Omega_\Lambda \to 1, \quad \Sigma^2 \to 0, \quad \Omega_{(i)} \to 0 \quad \text{when} \quad \tau \to \infty,
\]

as follows from combining \( \Omega_\Lambda \to 1 \) with the Gauss constraint (19a), i.e., the solutions approach a de Sitter state when \( \tau \to \infty \). This result is a special case of the proof by Wald [32], which holds for Bianchi types I–VIII. In the present case the fluids behave as test fields on a de Sitter background at late times, obeying the equations: \( v_{(i)}' = (G^{(i)})^{-1}\left(1 - v_{(i)}^2\right)(3w_{(i)} - 1)v_{(i)} \).

It follows that \( v_{(i)} = \text{const} \) if \( w_{(i)} = \frac{1}{3} \); \( v_{(1)} \) is monotonically increasing (decreasing) from 0 to 1 (1 to 0) if \( w_{(1)} > \frac{1}{3} \) \( \left(w_{(1)} < \frac{1}{3}\right) \); \( v_{(2)} \) is monotonically decreasing (increasing) from 0 to \(-1 \) (\(-1 \) to 0) if \( w_{(2)} > \frac{1}{3} \) \( \left(w_{(2)} < \frac{1}{3}\right) \). Thus if one of the fluids has a soft equation of state, \( w_{(2)} < \frac{1}{3} \), and the other has a sufficiently stiff equation of state, \( w_{(1)} \geq \frac{1}{3} \), then the fluids will obtain a
relative velocity w.r.t each other (in general when \( w_{(1)} = \frac{1}{2} \) and always if \( w_{(1)} > \frac{1}{2} \)); this is an invariant statement, and it is not possible to eliminate this effect with any choice of reference congruence—if one has two fluids, one with a sufficiently soft and one with a sufficiently stiff equation of state, then it follows that the fluids will asymptotically form anisotropies on a de Sitter background irrespectively of the choice of reference congruence. We note that this result is compatible with the analysis of Bianchi type V in [13], and that it reflects a bifurcation that takes place at \( w = \frac{1}{2} \) for a fluid in any, homogeneous or inhomogeneous, forever expanding model with a cosmological constant, see [33].

At early times \( \Lambda \) has a negligible effect compared to normal matter and hence it suffices to study the \( \Omega_\Lambda = 0 \) subset (it follows from (18g) and the application of the monotonicity principle, see, e.g. [3, 34] and references therein\(^6\), that the \( \alpha \)-limit for all orbits (solutions) must reside on this subset (assuming that \( \Omega_\Lambda \neq 1 \) initially); cf also the discussion after equation (21)). Since \( \Lambda \) therefore has no effect on the past asymptotic dynamics and since it is of interest to also study late time behavior when one does not have a cosmological constant, we will from now on assume \( \Lambda = 0 \). The state space we henceforth therefore consider is given by

\[
\hat{\mathcal{S}} = \{ \Sigma_r, \Sigma_A, \Sigma_B, \Sigma_C, v_{(1)}, v_{(2)}, \Omega_{(1)}, \Omega_{(2)} \},
\]

subject to the constraints (19), i.e., the state space when one does not have a cosmological constant is six-dimensional; since the discrete symmetries (22) still hold we continue to assume that \( \Sigma_A \in [0, 1], \Sigma_C \in [0, 1], v_{(1)} \in [0, 1] \) and \( v_{(2)} \in [-1, 0] \). When \( \Lambda = 0 \) the deceleration parameter \( q \) is given by

\[
q = 2\Sigma^2 + \frac{1}{2}(\Omega_m + 3P_m) = 2 - \frac{3}{2}(\Omega_m - P_m) \Rightarrow \frac{1}{2} \leq q \leq 2. \tag{25}
\]

### 3.3 Invariant subsets

The dynamical system (18) and (19), with \( \Omega_\Lambda = 0 \), admits a number of invariant subsets, conveniently divided into three classes: (i) ‘geometric subsets’, i.e., sets associated with conditions on the shear and hence the metric since the type I models are intrinsically flat; (ii) invariant sets on the boundary of the physical state space for two tilted fluids that do not belong to (i); (iii) subsets that can be obtained by intersections of the subsets belonging to (i) and (ii). We will introduce a notation where the kernel suggests the type of subset and where a subscript, when existent, suggests the values of \( v_{(1)} \) and \( v_{(2)} \).

### Geometric subsets

- **T**: the ‘twisting’ subset, characterized by \( \Sigma_C = 0, \Sigma_A \neq 0 \), which leads to that the decoupled \( \phi \)-variable satisfies \( \phi = \text{const} \) and hence \( \Sigma_{12} \propto \Sigma_{11} - \Sigma_{22} \).
- **RD**: the constantly rotated diagonal subset, given by \( \Sigma_A = 0, \Sigma_C \neq 0 \) \( (R_\alpha = 0) \). This subset is the diagonal subset, discussed next, rotated with a constant angle around \( e_3 \).
- **D**: the diagonal subset, defined by \( \Sigma_A = \Sigma_C = 0 \); \( \Sigma_B = \Sigma_- \), and hence \( R_\alpha = 0 \).
- **LRS**: the locally rotationally symmetric subset. This plane symmetric subset of the diagonal subset is characterized by the additional condition \( \Sigma_B = \Sigma_- = 0 \). This is the simplest subset compatible with two tilted fluids.

\(^6\) The monotonicity principle gives information about the global asymptotic behavior of the dynamical system. If \( M : X \to \mathbb{R} \) is a \( C^1 \) function which is strictly decreasing along orbits (solutions) in \( X \), then

\[
\alpha(x) \subseteq \{ \xi \in \hat{X} \setminus X \mid \lim_{\xi \to \xi_0} M(\xi) \neq \inf M \}, \quad \alpha(x) \subseteq \{ \xi \in \hat{X} \setminus X \mid \lim_{\xi \to \xi_0} M(\xi) \neq \sup M \}
\]

for all \( x \in X \), where \( \alpha(x) \) is the \( \alpha \)-limit (\( \alpha \)-limit) set of a point \( x \in X \), defined as the set of all accumulation points of the future (past) orbit of \( x \); and analogously for strictly increasing monotonic functions.
an important example is:

\[ \chi_{10} = \text{Boundary subsets} \]

The four Kasner circles, \( K_{10}^0, K_{10}^0, K_{10}^0, K_{10}^0 \), and the eight Kasner lines, \( KL_{10}^m, KL_{10}^m, KL_{10}^m, KL_{10}^m \), define the Kasner subset and is described by \( \Omega_m = 0; \Sigma^2 = 1 \); it describes the Kasner solutions, but in general in a non-Fermi propagated frame, and with \( v(i) \) as test fields.

The lists above are far from complete, intersections of subsets are possible in many cases, an important example is:

\[ \mathcal{E}T_{11} = \mathcal{E}T_{12} \cap \mathcal{E}T_{11,1} : \text{the double extreme tilt subset where both fluids propagate with the speed of light, } v(1) = 1 \iff v(2) \iff \Omega(1) = \Omega(2) = 3 P(1) = 3 P(2). \]

There are also a number of fix points which we denote by a kernel that is related to a subset to which the fix point belong together with a subscript that indicates the fix point values of \( v(1) \) and \( v(2) \); sometimes we also use a superscript. The fix points and their stability properties are described in appendix A, but here is a brief list of them:

- The four Friedmann points, \( F_{10}^0, F_{10}^0, F_{10}^0, F_{10}^0 \), and the two lines of Friedmann fix points, \( FL_{10}^m, FL_{10}^m \).
- The two lines of Friedmann subsets, \( \mathcal{F}L_{10}^{01} \) and \( \mathcal{F}L_{10}^{10} \).
- The two lines of fix points, \( TW_{11}^{11} \) and \( TW_{11}^{11} \).
- The four lines of fix points, \( GL_{10}^{11} \) and \( GL_{10}^{11} \).

4. Monotone functions and their consequences

In the analysis of Bianchi type VI in [14] a monotone function is defined

\[ \chi = \frac{\beta(i)\Omega(i) - \beta(1)\Omega(1)}{\beta(i)\Omega(i) + \beta(1)\Omega(1)} \]

\[ \chi' = \frac{1}{2} (w(1) - w(2))(1 - \chi^2), \quad -1 \leq \chi \leq 1. \]
The above holds whether or not we include a cosmological constant. If \( w_1 = w_2 \), then \( \chi \) is a constant of the motion, however, here our concern is with the case \( w_1 \neq w_2 \), and then \( \chi \) is a monotonic function that increases from \(-1\) to \(1\), which leads to

\[
\lim_{\tau \to -\infty} \chi = -1 \quad \Rightarrow \quad \lim_{\tau \to -\infty} \left( \beta(2)\Omega(2)/\beta(1)\Omega(1) \right) = 0 \quad \Rightarrow \quad \text{at late times} \quad \beta(2)\Omega(2) \to 0.
\]

\[
\lim_{\tau \to -\infty} \chi = 1 \quad \Rightarrow \quad \lim_{\tau \to -\infty} \left( \beta(1)\Omega(1)/\beta(2)\Omega(2) \right) = 0 \quad \Rightarrow \quad \text{at late times} \quad \beta(1)\Omega(1) \to 0. \tag{27}
\]

Combined with the Codazzi constraint \((19b)\) this leads to the following possibilities if \( \tau \to -\infty \):

(i) \( \lim_{\tau \to -\infty} (\Omega(1), \Omega(2)) = (0, 0) \), i.e., the solutions \( \alpha \)-limits reside on \( K \),

(ii) \( \lim_{\tau \to -\infty} (\Omega(2), v(1)) = (0, 0) \), i.e., the solutions \( \alpha \)-limits reside on \( \mathcal{O}T_{0(0)}, \)

(iii) \( \lim_{\tau \to -\infty} v(2) = -1 \), \( \lim_{\tau \to -\infty} Q(1) = \Omega(2) \), i.e., the solutions \( \alpha \)-limits reside on \( \mathcal{E}T_{0(1)}, \)

or combinations/intersections thereof. If \( \tau \to \infty \) then:

(i) \( \lim_{\tau \to \infty} (\Omega(1), \Omega(2)) = (0, 0) \), i.e., the solutions \( \omega \)-limits reside on \( K \),

(ii) \( \lim_{\tau \to \infty} (\Omega(1), v(2)) = (0, 0) \), i.e., the solutions \( \omega \)-limits reside on \( \mathcal{O}T_{v(2)}, \)

(iii) \( \lim_{\tau \to \infty} v(1) = 1 \), \( \lim_{\tau \to -\infty} Q(1) = \Omega(1) \), i.e., the solutions \( \omega \)-limits reside on \( \mathcal{E}T_{v(1)}, \)

or combinations/intersections thereof.

Another monotonic function is given by

\[
V = v(1)^2 (1 - v(1)^2)^{1-w(1)} v(2)^2 (1 - v(2)^2)^{-(1-w(2))}, \quad V' = 6(w(1) - w(2))V, \tag{29}
\]

where \( V \) asymptotically increases from zero to infinity. Combining \( V \) with \( \chi \) to obtain a constant of the motion leads to \( Q(1)/Q(2) = \text{const} = -1 \), where the latter equality is imposed by the Codazzi constraint, so unfortunately we obtain nothing new. However, it follows that when \( \tau \to -\infty (\tau \to \infty) \) then \( v(1) \to 0 \) or and \( v(2) \to -1 (v(1) \to 1) \) or and \( v(2) \to 0 \), i.e., these limits also hold in the above \( K \) cases.

Before giving the next monotonic functions it is useful to give the following auxiliary equations:

\[
Q'(i) = 2(q - 1 + \Sigma)Q(i), \quad T'(i) = 2(2q - 1 - 3w(i))T(i), \tag{30a}
\]

where

\[
T(i) = Q(i)^2 \left( 1 - v(i)^2 \right)^{1-w(i)} v(i)^2 = (1 + w(i))^2 \left( \Omega(i)^2 \right)^{-2} \Omega(i) \left( 1 - v(i)^2 \right)^{1-w(i)}. \tag{30b}
\]

If \( \Sigma_A, \Sigma_C \neq 0 \) there exist two more monotonic functions

\[
M_A^{(i)} = Q(i)^{12} T(i)^5 \Sigma_A \Sigma_C^{-4} = Q(i)^{12} \left( 1 - v(i)^2 \right)^{9(1-w(i))} v(i)^{18} \Sigma_A \Sigma_C^{-4} ; \quad M_A^{(i)^2} = 6(5 - 9w(i)) M_A^{(i)} , \tag{31}
\]

where \( M_A^{(i)} \) asymptotically increases from zero to infinity if \( w_i < \frac{5}{9} \) while it decreases from infinity to zero if \( w_i > \frac{5}{9} \); at \( w_i = \frac{5}{9} \) \( M_A^{(i)} \) is a constant of the motion, reflecting that we have bifurcations when \( w_i = \frac{5}{9} \), see appendix A. The above four monotonic functions, \( \chi \), \( V = T(2)/T(1), \) \( M_A^{(i)}, M_A^{(i)^2}, \) can be combined to yield three constants of the motion, but one of these is just the Codazzi constraint, so there only are two independent ‘non-trivial’ constants of the motion; here are two possible representations of these constants of the motion

\[
C_{AC} = (M_A^{(1)})^{5-9w(1)} (M_A^{(2)^2})^{9w(1)-5} = \text{const}, \quad D_{AC} = V^{9w(1)-5} (M_A^{(2)} w(i)^{9w(1)-w(2)}) = \text{const}. \tag{32}
\]
In addition to these monotonic functions there also exist several monotonic functions on the various subsets.

The existence of monotone functions is not coincidental, a fact that will be discussed elsewhere, but let us here comment on $\chi$, which is a monotonic function for all class A models (i.e., Bianchi models for which $A_\alpha = 0$, see section 2 and e.g. [3]). Its existence is a consequence of that $\chi$ is expressible as a dimensionless ratio of the spatial volume density and the dimensional constants $\ell_{(i)}$ in class A, where $\ell_{(i)}$ is related to particle conservation of the $i$th fluid, see, e.g. [35]. Interestingly there exists one more constant of the motion for each fluid in class A, however, these constants of the motion, together with the constants of no use. Incidentally, other constants of the motion exist in class B and hence, based on the above insight, there should exist a monotonic function also in this case, again related to particle conservation, but in a more complicated way.

5. Future and past dynamics

5.1. Future dynamics

The following theorem is easy to prove, but is nevertheless of interest.

**Theorem 5.1.** If $\frac{1}{2} < w_{(2)} < w_{(1)} < 1$, and if $Q_{(1)} > 0$, $v_{(1)}^2 < 1$, $v_{(2)}^3 < 1$ initially, then no models isotropize when $\tau \to \infty$, i.e., $\Sigma^2 \neq 0$ when $\tau \to \infty$.

**Proof.** Assume that all solutions of the above type isotropize, i.e., that the $\omega$-limit set for each solution resides on a Friedmann–Lemaître subset. The equations for $v_{(1)}$ and $v_{(2)}$ then yield $(v_{(1)}, v_{(2)}) \to (1, -1)$ when $\tau \to \infty$, which is a contradiction since no Friedmann–Lemaître subset has $v_{(1)} v_{(2)} \neq 0$. Hence none of the solutions described in theorem 5.1 isotropize when $\tau \to \infty$. $\square$

The above theorem does not tell us where the solutions end up when $\frac{1}{2} < w_{(2)} < w_{(1)} < 1$. This turns out to depend on what geometric set they belong to, leading to a division of the models into three classes: (i) the $RD, D, LRS$ subsets (ii) the $TW$ subset, and (iii) the general case. Unfortunately we have not been able to prove what the global attractors are, but our local analysis in appendix A together with numerical simulations lead to the following conjecture:

**Conjecture 5.2.** If $Q_{(1)} > 0$, $v_{(1)}^2 < 1$, $v_{(2)}^3 < 1$ initially, and $\frac{1}{2} < w_{(2)} < w_{(1)} < 1$, then the $\omega$-limit for every such orbit that belongs to

- $RD, D, LRS$ is the fix point $LRS_{v_{(2)}^2}$,
- $TW (\Sigma_A \neq 0)$ is the fix point $LRS_{v_{(2)}^2}$, if $w_{(2)} \leq \frac{1}{2}$; the fix point $TW_{v_{(2)}^2}$, if $\frac{1}{2} < w_{(2)} < \frac{3}{5}$; the fix point $TW_{1}$, if $\frac{3}{5} \leq w_{(2)}$,
- the general set ($\Sigma_A \Sigma_C \neq 0$) is the fix point $LRS_{1v_{(2)}^2}$, if $w_{(2)} \leq \frac{1}{2}$; the fix point $TW_{1v_{(2)}^2}$, if $\frac{1}{2} < w_{2} < \frac{5}{6}$; the line of fix points $GL_{1v_{(2)}^2}$, if $w_{(2)} = \frac{5}{6}$; the fix point $G_{11}$, if $\frac{5}{6} < w_{(2)}$.

However, models for which $Q_{(1)} > 0$, $v_{(1)}^2 < 1$, $v_{(2)}^3 < 1$ initially and with $0 \leq w_{(2)} \leq \frac{1}{2}$ do isotropize (this is also true if $Q_{(1)} = 0$, even if the equations of state are stiffer than radiation), as shown in the following lemma:

**Lemma 5.3.** If $Q_{(1)} > 0$, $v_{(1)}^2 < 1$, $v_{(2)}^3 < 1$ initially, and if $0 \leq w_{(2)} \leq \frac{1}{4}$, then all models isotropize when $\tau \to \infty$, i.e., $\Sigma^2 \to 0$ when $\tau \to \infty$. 


In section 4 we showed that the future \( \omega \)-limit of a ‘matter’ orbit has to reside on either \( OT_{v_{(1)}} \) or \( ET_{v_{(2)}} \), and in appendix B we show that \( q < 2 \) and \( \Omega_m > 0 \). Assume that the \( \omega \)-limit of an orbit resides on \( OT_{v_{(1)}} \). The equations for the \( \Sigma, \Omega_m \)-variables on this subset are just those for a single orthogonal fluid, but in general in a non-Fermi propagated frame. However, in a Fermi frame the single orthogonal fluid case is easily solved and one finds that \( \Omega_m \to 1 \) and \( \Sigma^2 \to 0 \) when \( \tau \to \infty \). This statement is frame invariant and therefore holds for any frame, and hence it follows that the \( \omega \)-limit resides on the Friedmann–Lemaître subset \( FL_T^{v_{(1)}} \) and that \( \Sigma^2 \to 0 \). Let us now assume that the \( \omega \)-limit for a matter orbit resides on \( ET_{v_{(2)}} \). Then, since \( v_{(1)} = 1, q = 2 - \Omega_{(1)} - \frac{1}{2}(\Omega_{(2)} - P_{(2)}) = 2 - \Omega_m - \frac{1}{2}(1 - 3w_{(2)})(\Omega_{(2)} - Q_{(2)}v_{(2)}), \) and hence \( 2q - 1 - 3w_{(2)} = 2(1 - \Omega_m) + (1 - 3w_{(2)})(1 - \Omega_{(2)} + Q_{(2)}v_{(2)}) \geq 0, \) where the inequality is strict if \( \Omega_m < 1 \), which we now assume. Then \( T_{(2)} \) in (30b) is strictly monotonically increasing and grows without bounds, but this is impossible since \( T_{(2)} \) is finite, and hence \( \Omega_m \to 1 \), and thus \( \Sigma^2 \to 0 \) when \( \tau \to \infty \).

**Theorem 5.4.** The \( \omega \)-limit for every orbit with \( Q_{(1)} > 0, v^2_{(1)} < 1, v^2_{(2)} < 1 \) initially is the fix point \( F^{01}_{v_{(1)}} \) if \( 0 \leq w_{(2)} < w_{(1)} < \frac{1}{2} \); one of the fix points on the line \( F^{01}_{v_{(1)}} \) if \( 0 \leq w_{(2)} < w_{(1)} = \frac{1}{2} \); the fix point \( F^{01}_{v_{(1)}} \) if \( 0 \leq w_{(2)} < w_{(1)} \leq \frac{1}{2} < w_{(1)} < 1 \).

**Proof.** According to lemma 5.3 \( \Sigma^2 = 0 \) asymptotically toward the future. Imposing this condition on the \( ET_{v_{(2)}} \) subset yields the \( FL_T^{v_{(1)}} \) subset with \( v_{(1)} = 1 \), which is a special case of the other possibility that the \( \omega \)-limit of an arbitrary orbit with \( Q_{(1)} \neq 0 \) initially resides on the \( OT_{v_{(1)}} \) subset, and hence that the \( \omega \)-limit resides on \( FL_T^{v_{(1)}} \) with \( v_{(1)} \) so far undetermined (\( \Omega_{(2)} = 1 \)). To find the desired \( \omega \)-limit we only need to find the asymptotic limit of \( v_{(1)} \), which, according to (18e), is determined by the signature of \( 3w_{(1)} - 1 \) when \( \Sigma^2 = 0 \), immediately leading to the theorem.

The above theorems and conjectures are summarized in the global attractor bifurcation diagrams in figure 1.

### 5.2. Past dynamics

Based on the local analysis in appendix A and the analysis of the Kasner subset \( K \) in appendix B, in combination with a numerical investigation, we make the following conjecture:
Conjecture 5.5. The $\alpha$-limit for every orbit with $Q(1) > 0$, $v_{(1)}^2 < 1$, $v_{(2)}^2 < 1$ initially on the general geometric set with $\Sigma_A \Sigma_C \neq 0$ is one of the fix points on the global past attractor $A_{(\ast)}$ for the Kasner subset $K$ given in equation (56).

Remark. For the various geometric subsets other parts of the projected Kasner circle are the relevant building blocks for producing the global attractor for each subset, in a similar way as in the generic case (e.g., in the $RD$ case, with $\Sigma_A = 0$, $\Sigma_C \neq 0$, $0 \leq \hat{\Sigma}_- \leq 1$ is the restriction on $\hat{\Sigma}_-$, in contrast to the generic case where $0 \leq \hat{\Sigma}_- \leq \frac{\sqrt{3}}{2}$).

6. Concluding remarks

In this paper we have shown that the Bianchi type I models with two non-interacting tilted fluids exhibit a rich bifurcation structure, hinting at the complexity one can expect from more general models. Some of our results reflect features that hold under more general circumstances, while others are particular for the models at hand, but in this latter instance the present models yield a natural reference with which to compare results from more general settings. Moreover, even in the cases that our results are particular for Bianchi type I, these results act as building blocks for understanding more general and physically realistic models; the reason for this is that the present type I models appear as an invariant subset of the state space of any other multiple fluid model, an invariant subset one has to understand in order to control the dynamics of the model one is interested in—a detailed analysis of any other multiple fluid model will have to take into account our results, both as regards past and future asymptotics since these typically describe intermediate behavior, or part of an asymptotic structure, of the more general models.

The result that models with $\Lambda = 0$ and fluids stiffer than radiation asymptotically produce anisotropies toward the future shows that the isotropization results for a single fluid are structurally unstable within the Bianchi type I context, although from a physical point of view one would not expect such equations of states at late times. This result also suggests that tilted fluids may become as anisotropically significant as spatial curvature at late times (in the absence of inflation) when one considers more general models than Bianchi type I, leading to considerable complexity, further illustrated by the type VIo investigation in [14]. Our results about isotropization for soft equations of state may be regarded as a nonlinear Bianchi type I generalization of perturbations of flat FRW models with two fluids, a reasonable approximation before dark energy has becomes significant, and it is of interest then to point out that one again has radiation bifurcations.

Isotropic singularities have attracted considerable interest previously, see, e.g. [33, 36] and references therein. This motivates pointing out that there are no isotropic singularities when $\Sigma_\Lambda \neq 0$ initially in Bianchi type I with two fluids, the reason being that the shear completely destabilizes such singularities; thus the present models are quite misleading in an isotropic singularity context, in fact, it is the properties of Bianchi type I that make isotropic singularities special.

The asymptotically silent regimes of generic spacelike singularities and of an inflationary future share some properties: in the inflationary case all other matter fields than the inflationary field become test fields and do not influence the spacetime geometry—hence matter that is not inflationary matter does not matter for the spacetime geometry; in the case of a generic singularity fluids with speeds of sound less than that of light also become test fields, in this case gravity alone creates gravity to a larger extent than matter, and hence ‘matter does not matter’ in this case either [5, 8, 9]. However, that matter fields asymptotically become test fields does not mean that they do not matter observationally, on the contrary, today to a good
approximation the CMB can be regarded as a test field although it is the prime observational source for cosmology.

In the present case a cosmological constant has yielded a final de Sitter state—this is a typical feature in a forever expanding model, as is the bifurcation at the radiation value $w = \frac{1}{3}$. Hence if one has several asymptotic test fields, some less stiff and some as stiff or stiffer than radiation, one obtains anisotropies on a de Sitter background. However, one would perhaps not expect fields that are stiffer than radiation after an inflationary period in the early universe or in the far future, but does the bifurcation at the radiation value hint at that e.g. atomic matter and/or cold dark matter and radiation develop observationally significant relative velocities, perhaps nonlinearly?

As regards generic singularities, the Bianchi type I Kasner singularity is transformed into a singularity of ‘Mixmaster’ type when one considers geometrically more general models that admit Bianchi type II models on the silent boundary in such a combination with possible frame transitions so that the whole projected Kasner circle becomes unstable toward the past. But it is by no means uninteresting to examine the past behavior of type I, since matter sometimes lead to bifurcations such that matter sometimes does matter, as illustrated by e.g., a magnetic field [31], or by a kinematic description of matter [34] where matter mattered non-generically in a very subtle way, illustrating that it was not quite obvious that there would not be any non-generic subtle effects in the present case; the lack of such effects suggest that the past dynamics in general is structurally stable under a change from one to several fluids as long as $0 \leq w_{(i)} < 1$ in the sense that a generic singularity presumably is vacuum dominated. However, the central role of Bianchi type I for generic singularities suggests that the enforced anti-parallel motion of the fluids in this case may lead to a tendency to correlate the motion of two fluids in the generic case, although this issue needs further study.

Asymptotic scenarios where non-interacting matter components ‘do not matter’ may have interesting consequences when one introduces more realistic interacting sources. If the interactions only contribute source terms that are proportional to the non-interacting parts of the source, then the interactions presumably also become negligible for the determination of the geometry; it is only when interactions contribute more to the total stress–energy than the sources themselves that the matter does not matter property would be broken. Hence the approximation of non-interacting fields may be asymptotically less restrictive than one may originally think, and in combination with the importance of Bianchi type I for generic models this suggests that the present investigation may be of relevance for a larger context than one perhaps initially suspect.

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Appendix A. Fix points and local stability analysis

In this section we use the Gauss constraint (19a) to eliminate $\Omega_{(i)2}$ globally, however, the Codazzi constraint (19b) cannot, unfortunately, be analytically solved globally, but we can follow chapter 7 in [3] and use it to locally eliminate one variable, usually $\Omega_{(i)1}$, at each fix point when (19b) is non-singular. There are several features that are similar for many of the fix points. All fix points, except one, have $\Sigma_0 = 0$; several fix points have $\Sigma_A = 0$. Linearization of (18d) when $\Sigma_0 = 0$, and (18b) when $\Sigma_A = 0$, yield the eigenvalues

$$\lambda_{\Sigma_0} = -[2 - q_0 - 2\sqrt{3}(\Sigma_B)_0], \quad \lambda_{\Sigma_A} = -[2 - q_0 + 3(\Sigma_A)_0 + \sqrt{3}(\Sigma_B)_0].$$  \hspace{1cm} (A.1)
where \( q_0, (\Sigma_B)_0, (\Sigma_c)_0 \) are the fix point values of \( q, \Sigma_B, \Sigma_c \), respectively. For many fix points \( v_{(i)} = 0 \) or \( |v_{(i)}| = 1 \). In these cases linearization of (18e) yields

\[
\lambda_{i(vo)}^0 = 3w_{(i)} - 1 + 2(\Sigma_c)_0, \quad \lambda_{i(vo)}^1 = -2(3w_{(i)} - 1 + 2(\Sigma_c)_0)/(1 - w_{(i)}),
\]

(A.2)

where the subscript refers to the \( v_{(i)} \) variable the eigenvalue is connected with while the superscript denotes its absolute fix point value. Let us now turn to the various individual fix points; throughout kernel subscripts give an indication of the absolute fix point values for \( v_{(1)} \) and \( v_{(2)} \).

**Kasner fix points.** There are four circles of Kasner points and eight lines of fix points when \( 0 \leq w_{(2)} < w_{(1)} \). The Kasner circles are characterized by \( \Sigma_A = \tilde{\Sigma}_A, \Sigma_B = \tilde{\Sigma}_B, \Sigma_A = \Sigma_C = 0, \Omega_{(1)} = \Omega_{(2)} = 0, \Sigma^2 = 1, q = 2 \), where \( \Sigma_\perp \) are constants that satisfy \( \Sigma_\perp^2 + \Sigma_\parallel^2 = 1 \), and the following values of \( v_{(i)} \):

\[
K_0^0 : v_{(1)} = v_{(2)} = 0, \quad K_0^1 : v_{(1)} = 0, v_{(2)} = -1, \quad K_1^0 : v_{(1)} = 1, v_{(2)} = 0, \quad K_1^1 : v_{(1)} = v_{(2)} = 1.
\]

(A.3a)

The eigenvalues for the four cases are

\[
K_0^0 : \lambda_{\Sigma_A}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, 3(1 - w_{(1)}) , 3(1 - w_{(2)}),
\]

(A.4a)

\[
K_0^1 : \lambda_{\Sigma_A}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, 3(1 - w_{(2)}),
\]

(A.4b)

\[
K_1^0 : \lambda_{\Sigma_A}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, 3(1 - w_{(1)}),
\]

(A.4c)

\[
K_1^1 : \lambda_{\Sigma_A}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, \lambda_{\Sigma_c}^0, 2(1 + \tilde{\Sigma}_A),
\]

(A.4d)

where

\[
\lambda_{\Sigma_A}^0 = -(3\Sigma_\parallel + \sqrt{3}\Sigma_\perp), \quad \lambda_{\Sigma_c}^0 = 2\sqrt{3}\Sigma_\perp.
\]

(A.5a)

\[
\lambda_{\Sigma_c}^0 = 3w_{(i)} - 1 + 2\tilde{\Sigma}_A, \quad \lambda_{\Sigma_c}^0 = -2(3w_{(i)} - 1 + 2\tilde{\Sigma}_A)/(1 - w_{(i)}).
\]

(A.5b)

In the \( K_0^0 \) case the Codazzi constraint (19b) is singular and hence it cannot be locally solved; in all other cases (19b) has been used to eliminate \( \Omega_{(1)} \). The zero eigenvalue corresponds to the one which has a one-parameter set of fixed points. The eight lines of Kasner fix points are characterized by \( \Sigma_A = \Sigma_C = 0, \Omega_{(1)} = \Omega_{(2)} = 0, \Sigma^2 = 1, q = 2 \), and

\[
KL_{\Sigma_{(1)}} : \Sigma_\parallel = \pm\sqrt{1 - \Sigma_\perp^2}, \quad 0 \leq \Sigma_{(1)} \leq 1, \quad v_{(1)} = 0, \quad v_{(2)} = 0.
\]

(A.6a)

\[
KL_{\Sigma_{(2)}} : \Sigma_\parallel = \pm\sqrt{1 - \Sigma_\perp^2}, \quad 0 \leq \Sigma_{(1)} \leq 1, \quad v_{(1)} = 0, \quad v_{(2)} = 0.
\]

(A.6b)

\[
KL_{\Sigma_{(3)}} : \Sigma_\parallel = \pm\sqrt{1 - \Sigma_\perp^2}, \quad v_{(1)} = 0, \quad -1 \leq v_{(2)} \leq 0.
\]

(A.6c)

\[
KL_{\Sigma_{(4)}} : \Sigma_\parallel = \pm\sqrt{1 - \Sigma_\perp^2}, \quad v_{(1)} = 1, \quad -1 \leq v_{(2)} \leq 0.
\]

(A.6d)

where the superscript denotes the sign of \( \Sigma_B \). After eliminating \( \Omega_{(1)} \) locally by means of the Codazzi constraint (19b), the eigenvalues for the eight Kasner lines are

\[
KL_{\Sigma_{(0)}} : 0; 0; \lambda_{\Sigma_A}; \lambda_{\Sigma_c}; 3(1 - w_{(2)}); -3(w_{(1)} - w_{(2)}).
\]
The eigenvalues associated with the two lines are

\begin{align*}
\text{KL}_{v_{ij},1} & : \quad 0; \quad 0; \quad \lambda_{\Sigma_A}; \quad \lambda_{\Sigma_C}; \quad 3(1 - w_{(1)}); \quad 6 \frac{w_{(1)} - w_{(2)}}{1 - w_{(2)}}, \quad (A.7b) \\
\text{KL}_{w_{ij},1} & : \quad 0; \quad 0; \quad \lambda_{\Sigma_A}; \quad \lambda_{\Sigma_C}; \quad 3(1 - w_{(1)}); \quad 3(w_{(1)} - w_{(2)}), \quad (A.7c) \\
\text{KL}_{v_{ij},1} & : \quad 0; \quad 0; \quad \lambda_{\Sigma_A}; \quad \lambda_{\Sigma_C}; \quad 3(1 - w_{(2)}); \quad -6 \frac{w_{(1)} - w_{(2)}}{1 - w_{(1)}}, \quad (A.7d)
\end{align*}

where again \( \lambda_{\Sigma_A} = -(3\Sigma_A + \sqrt{3}\Sigma_B), \lambda_{\Sigma_C} = 2\sqrt{3}\Sigma_B \), where \( \Sigma_A, \Sigma_B \) take the fix point values for the relevant line of fix points. Here one zero eigenvalue corresponds to that one which has a line of fix points while the second is associated with the existence of a one parameter set of solutions that are anti-parallel w.r.t. each other on each side of the line of fix points.

**Friedmann fix points.** All four Friedmann fix points satisfy \( \Sigma_A = \Sigma_B = \Sigma_C = 0, \Omega_{(1)}\Omega_{(2)} = 0, \Omega_m = 1, v_{(1)}v_{(2)} = 0 \). They are distinguished by their \( \Omega_{(i)} \) and \( v_{(i)} \) values according to

\begin{align*}
F_{00}^{10} & : \quad v_{(1)} = 0, \quad v_{(2)} = 0, \quad \Omega_{(1)} = 1, \quad \Omega_{(2)} = 0, \quad (A.8a) \\
F_{01}^{10} & : \quad v_{(1)} = 0, \quad v_{(2)} = -1, \quad \Omega_{(1)} = 1, \quad \Omega_{(2)} = 0, \quad (A.8b) \\
F_{00}^{01} & : \quad v_{(1)} = 0, \quad v_{(2)} = 0, \quad \Omega_{(1)} = 0, \quad \Omega_{(2)} = 1, \quad (A.8c) \\
F_{01}^{01} & : \quad v_{(1)} = 1, \quad v_{(2)} = 0, \quad \Omega_{(1)} = 0, \quad \Omega_{(2)} = 1, \quad (A.8d)
\end{align*}

where the superscript refers to the values of \( \Omega_{(1)} \) and \( \Omega_{(2)} \), furthermore, \( q = \frac{1}{2}(1 + 3w_{(1)}) \) for \( F_{00}^{10}, F_{01}^{10} \), and \( q = \frac{1}{2}(1 + 3w_{(2)}) \) for \( F_{00}^{01}, F_{01}^{01} \). The associated eigenvalues are

\begin{align*}
F_{00}^{10} & : \quad \lambda_{1,2,3,4} = -\frac{3}{2}(1 - w_{(1)}) \quad 3w_{(1)} - 1 \quad 3(w_{(1)} - w_{(2)}) \quad v_{(1)} \quad \text{eliminated}, \quad (A.9a) \\
F_{01}^{10} & : \quad \lambda_{1,2,3,4} = -\frac{3}{2}(1 - w_{(1)}) \quad 3w_{(1)} - 1 \quad \frac{2(1 - 3w_{(2)})}{1 - w_{(2)}} \quad \Omega_{(1)} \quad \text{eliminated}, \quad (A.9b) \\
F_{00}^{01} & : \quad \lambda_{1,2,3,4} = -\frac{3}{2}(1 - w_{(2)}) \quad 3w_{(2)} - 1 \quad -3(w_{(1)} - w_{(2)}) \quad v_{(2)} \quad \text{eliminated}, \quad (A.9c) \\
F_{01}^{01} & : \quad \lambda_{1,2,3,4} = -\frac{3}{2}(1 - w_{(2)}) \quad 3w_{(2)} - 1 \quad \frac{2(1 - 3w_{(1)})}{1 - w_{(1)}} \quad \Omega_{(1)} \quad \text{eliminated}. \quad (A.9d)
\end{align*}

Here the last entry for each line of fix points refers to the variable that has been eliminated by means of the Codazzi constraint (19b). Two of the eigenvalues of \( \lambda_{1,2,3,4} \) refer to \( \lambda_{\Sigma_A} \) and \( \lambda_{\Sigma_C} \). If \( w_{(2)} = \frac{1}{2} \) there exists a line of Friedmann points with \( q = \frac{1}{2}(1 + 3w_{(1)}) \), parametrized by \( v_{(2)}, FL_{v_{ij},0}^{10} \), that connects \( F_{00}^{10} \) and \( F_{01}^{10} \). Similarly if \( w_{(1)} = \frac{1}{2} \) there exists a line of fix points, \( FL_{w_{ij},1}^{01} \) with \( q = \frac{1}{2}(1 + 3w_{(2)}) \), that connects \( F_{00}^{01} \) and \( F_{01}^{01} \). They are given by

\begin{align*}
FL_{v_{ij},0}^{10} & : \quad v_{(1)} = 0, \quad v_{(2)} = \text{const}, \quad \Omega_{(1)} = 1, \quad \Omega_{(2)} = 0, \quad w_{(2)} = \frac{1}{2}, \quad (A.10a) \\
FL_{v_{ij},0}^{10} & : \quad v_{(1)} = \text{const}, \quad v_{(2)} = 0, \quad \Omega_{(1)} = 0, \quad \Omega_{(2)} = 1, \quad w_{(1)} = \frac{1}{2}. \quad (A.10b)
\end{align*}

The eigenvalues associated with the two lines are

\begin{align*}
FL_{v_{ij},0}^{10} & : \quad \lambda_{1,2,3,4} = -\frac{3}{2}(1 - w_{(1)}) \quad 3w_{(1)} - 1 \quad 0 \quad \Omega_{(1)} \quad \text{eliminated}, \quad (A.11a) \\
FL_{v_{ij},0}^{10} & : \quad \lambda_{1,2,3,4} = -\frac{3}{2}(1 - w_{(2)}) \quad 3w_{(2)} - 1 \quad 0 \quad \Omega_{(1)} \quad \text{eliminated}. \quad (A.11b)
\end{align*}

We now turn to fix points for which \( 0 < \Sigma^2 < 1 \).
Fix points on $\mathcal{LRS} \cap \mathcal{ET}_{w_i}$ and $\mathcal{LRS} \cap \mathcal{ET}_{v_i}$. When $\frac{1}{3} < w_{(2)} < w_{(1)}$ there are two additional fix points, $\mathcal{LRS}_{v_i,1}$, with $\Sigma^2 = \frac{1}{3}(3w_{(1)} - 1)^2 + q = \frac{1}{3}(1 + 3w_{(1)})$, and $\mathcal{LRS}_{v_i,2}$, with $\Sigma^2 = \frac{1}{4}(3w_{(2)} - 1)^2 + q = \frac{1}{4}(1 + 3w_{(2)})$, which enter the physical state space via $F_{01}^{\Omega_1}$ and $F_{01}^{\Omega_2}$ when $w_{(1)} = \frac{1}{3}, w_{(2)} = \frac{1}{3}$, respectively, and move into the $\mathcal{LRS}$-subset with increasing values of $w_{(i)}$. In the stiff perfect fluid limit ($w_{(1)} = 1, w_{(2)} = 1$) the lines merge with the coalesced Kasner lines $\mathcal{K}_{v_i,1}^- = \mathcal{K}_{v_i,1}^+, \mathcal{K}_{v_i,2}^- = \mathcal{K}_{v_i,2}^+$, respectively. The two fix points are characterized by $\Sigma_A = \Sigma_B = \Sigma_C = 0$ and

\[
\begin{align*}
\mathcal{LRS}_{v_i,1} : & \quad \Sigma_+ = -\frac{1}{2}(3w_{(1)} - 1), & \quad v_{(1)} = \frac{3}{2}(1 - w_{(1)}), & \quad v_{(2)} = -1, \\
\Omega_{(1)} = & \quad \frac{3(1 - w_{(1)})(9w_{(1)} + 1)(1 + w_{(1)})}{32w_{(1)}}, & \quad \Omega_{(2)} = & \quad \frac{3(1 - w_{(1)})(5w_{(1)} + 1)(3w_{(1)} - 1)}{32w_{(1)}}, \\
\mathcal{LRS}_{v_i,2} : & \quad \Sigma_+ = -\frac{1}{2}(3w_{(2)} - 1), & \quad v_{(1)} = 1, & \quad v_{(2)} = \frac{3}{2}(1 - w_{(2)}), \\
\Omega_{(1)} = & \quad \frac{3(1 - w_{(2)})(5w_{(2)} + 1)(3w_{(2)} - 1)}{32w_{(2)}}, & \quad \Omega_{(2)} = & \quad \frac{3(1 - w_{(2)})(9w_{(2)} + 1)(1 + w_{(2)})}{32w_{(2)}}.
\end{align*}
\]  
\((A.12a)\)

After eliminating $\Omega_{(1)}$, locally the eigenvalues for the two $\mathcal{LRS}$-points are

\[
\begin{align*}
\mathcal{LRS}_{v_i,1} : & \quad \lambda_{\Sigma_+} = 3(2w_{(1)} - 1); & \quad \lambda_{\Sigma_C} = \lambda_{\Sigma_+} = \frac{3}{2}(1 - w_{(1)}); & \quad \lambda_{\Sigma_A} = \lambda_{\Sigma_B} = \sqrt{\frac{3}{5}}, & \quad 6w_{(1)} = w_{(2)}, \\
\Omega_{(1)} = & \quad \frac{3}{5}; & \quad \lambda_{\Sigma_C} = \lambda_{\Sigma_+} = \frac{3}{5}(1 \pm \sqrt{\frac{3}{5}}); & \quad 6w_{(1)} = w_{(2)}; & \quad 6w_{(1)} = w_{(2)}; \quad \lambda_{\Sigma_A} = \lambda_{\Sigma_B} = \sqrt{\frac{3}{5}}, \quad \Omega_{(1)} = \frac{3}{5}, \\
\mathcal{LRS}_{v_i,2} : & \quad \lambda_{\Sigma_+} = 3(2w_{(2)} - 1); & \quad \lambda_{\Sigma_C} = \lambda_{\Sigma_+} = \frac{3}{2}(1 - w_{(2)}); & \quad \lambda_{\Sigma_A} = \lambda_{\Sigma_B} = \sqrt{\frac{3}{5}}; & \quad 6w_{(1)} = w_{(2)}, \\
\Omega_{(1)} = & \quad \frac{3}{5}; & \quad \lambda_{\Sigma_C} = \lambda_{\Sigma_+} = \frac{3}{5}(1 \pm \sqrt{\frac{3}{5}}); & \quad 6w_{(1)} = w_{(2)}; & \quad 6w_{(1)} = w_{(2)}; \quad \lambda_{\Sigma_A} = \lambda_{\Sigma_B} = \sqrt{\frac{3}{5}}, \quad \Omega_{(1)} = \frac{3}{5}.
\end{align*}
\]  
\((A.13)\)

where $\text{Re}[A(w_{(i)})] < 1$; since the expression for $A(w_{(i)})$ is rather complicated we will refrain from giving it.

**Fix point on $\mathcal{TW} \cap \mathcal{ET}_{1}$:**

\[
\begin{align*}
\mathcal{TW}_{11} : & \quad \Sigma_+ = -\frac{3}{5}, \quad \Sigma_C = 0, \quad \Sigma_A = \Sigma_B = \sqrt{\frac{3}{5}}, & \quad v_{(1)} = 1, & \quad v_{(2)} = -1, & \quad \Omega_{(1)} = \Omega_{(2)} = \frac{3}{5},
\end{align*}
\]  
\((A.14)\)

and $\Sigma^2 = \frac{32}{25}, q = \frac{2}{5}$. Local elimination of $\Omega_{(1)}$ by means of the Codazzi constraint \((A.13)\) yields the eigenvalues

\[
\begin{align*}
\lambda_{\Sigma_C} = & \quad \frac{3}{5}; & \quad -\frac{3}{5}(1 \pm i\sqrt{\frac{3}{5}}); & \quad 6w_{(1)} = w_{(2)}; & \quad 6w_{(1)} = w_{(2)}; & \quad 6w_{(1)} = w_{(2)}; \quad \lambda_{\Sigma_A} = \lambda_{\Sigma_B} = \sqrt{\frac{3}{5}}, \quad \Omega_{(1)} = \frac{3}{5}, \quad \Omega_{(2)} = \frac{3}{5}.
\end{align*}
\]  
\((A.15)\)

**Fix points on $\mathcal{TW} \cap \mathcal{ET}_{v_i,1}$ and $\mathcal{TW} \cap \mathcal{ET}_{v_i,2}$.** When $\frac{1}{3} < w_{(1)} < \frac{1}{3}$ there exists one more fix point on $\mathcal{TW}$: $\mathcal{TW}_{v_i,1}$. This fix point comes into existence when the point $\mathcal{LRS}_{v_i,1}$ bifurcate into two points at $w_{(1)} = \frac{1}{3}$; it then wanders away from $\mathcal{LRS}$ when $w_{(1)}$ increases and eventually leaves the physical state space through $\mathcal{TW}_{11}$ when $w_{(1)} = \frac{3}{5}$. Yet another
similar fix point exists on \( \mathcal{T} \mathcal{W} \) if \( \frac{1}{2} < \omega_2 < \frac{3}{2} \): \( TW_{v_0^1} \). The fix points are characterized by \( \Sigma_C = 0 \) and

\[
TW_{v_0^1} : \quad \Sigma_+ = -\frac{1}{2}(3\omega_1 - 1), \quad \Sigma_A = \sqrt{\frac{3}{2}(1 - \omega_2)(2\omega_1 - 1)}, \\
\Sigma_B = \sqrt{3}(2\omega_1 - 1), \\
v(1) = v^*_1 = \frac{(1 - \omega_2)(15\omega_1 - 7)}{-25\omega_1^3 + 18\omega_1 - 1}, \quad v(2) = -1, \\
\Omega(1) = 1 - \frac{1}{4}(3\omega_1 - 1)(15\omega_1 - 7) - B(\omega_2), \quad \Omega(2) = B(\omega_1), \\
\tag{A.16a}
\]

\[
TW_{v_2^1} : \quad \Sigma_+ = -\frac{1}{2}(3\omega_2 - 1), \quad \Sigma_A = \sqrt{\frac{3}{2}(1 - \omega_2)(2\omega_2 - 1)}, \\
\Sigma_B = \sqrt{3}(2\omega_2 - 1), \\
v(1) = 1, \quad v(2) = v^*_2 = \frac{(1 - \omega_2)(15\omega_2 - 7)}{-25\omega_2^3 + 18\omega_2 - 1}, \\
\Omega(1) = B(\omega_2), \quad \Omega(2) = 1 - \frac{1}{4}(3\omega_2 - 1)(15\omega_2 - 7) - B(\omega_2), \\
\tag{A.16b}
\]

where

\[
B(\omega_i) = -\frac{3(1 - \omega_i)(7 - 15\omega_i)(25\omega_i^2 - 18\omega_i + 1)}{32(5\omega_i^3 - 5\omega_i + 1)}. \\
\tag{A.16c}
\]

In addition \( \Sigma^2 = \frac{1}{4}(3\omega_1 - 1)(15\omega_1 - 7) \), \( q = \frac{1}{2}(1 + 3\omega_1) \) for \( TW_{v_0^1} \), and similarly for \( TW_{v_2^1} \), but with \( \omega_1 \) replaced with \( \omega_2 \). Local elimination of \( \Omega_{(1)} \) yields the following eigenvalues:

\[
TW_{v_0^1} : \quad \lambda_{\Sigma_C} = -\frac{3}{2}(5 - 9\omega_1); \quad -6\frac{\omega_1 - \omega_2}{1 - \omega_2}; \\
\lambda_{3,4,5,6} = -\frac{3}{4}(1 - \omega_1)(1 \pm \sqrt{C_1(1) \pm D_1(1)}), \\
\tag{A.17a}
\]

\[
TW_{v_2^1} : \quad \lambda_{\Sigma_C} = -\frac{3}{2}(5 - 9\omega_2); \quad -6\frac{\omega_1 - \omega_2}{1 - \omega_1}; \\
\lambda_{3,4,5,6} = -\frac{3}{4}(1 - \omega_2)(1 \pm \sqrt{C_2(1) \pm D_2(1)}), \\
\tag{A.17b}
\]

where \( C_1(1) = C_{(1)}(\omega_1), D_1(1) = D_{(1)}(\omega_1) \) exhibit quite complicated expressions, which we therefore refrain from giving, such that real parts of the associated eigenvalues always are negative.

**Fix point in the generic geometric manifold.** There exists one fix point \( G_{11} \) for which all the off-diagonal components of the shear are nonzero. It thus exists on the generic geometric manifold, but on the ‘matter boundary’ \( ET_{11} \) where both fluids are extremely tilted. It is characterized by

\[
G_{11} : \quad \Sigma_+ = -\frac{1}{5}, \quad \Sigma_A = \frac{2}{3\sqrt{3}}, \quad \Sigma_B = \Sigma_C = \frac{1}{3\sqrt{3}}, \\
v(1) = 1, \quad v(2) = -1, \quad \Omega(1) = \Omega(2) = \frac{1}{5}, \\
\tag{A.18}
\]
and hence $\Sigma^2 = \frac{1}{4}$, $q = \frac{4}{5}$. Local elimination of $\Omega_{(1)}$ yields the eigenvalues

$$
\lambda_{1,2,3,4} = \frac{1}{3} \left( 1 \pm \sqrt[3]{23 \pm 12 \sqrt{2}} \right); \quad \frac{2(9w(2) - 5)}{3(1 - w(2))}; \quad \frac{2(9w(1) - 5)}{3(1 - w(1))}.
$$

(A.19)

At $w(2) < w(1) = \frac{5}{6}$, $w(2) = \frac{5}{6} < w(1)$ there exists a line of fix points, $GL_{v_{11}}(GL_{1v_{11}})$, connecting $TW_{v_{11}}(TW_{1v_{12}})$ with $G_1$; $GL_{v_{11}}$ and $GL_{1v_{11}}$ are given by

$$
SGL_{v_{11}}: \quad \Sigma_+ = \frac{1}{3}, \quad \Sigma_-^4 = \frac{1}{3} \sqrt{34v(1) - 6/3 + 4v(1)},
$$

$$
\Sigma_- = \frac{1}{3} \sqrt{13v(1) - 6/3 + 4v(1)},
$$

$$
6 \leq v(1) = \text{const} \leq 1, \quad v(2) = -1,
$$

$$
\Omega_{(1)} = \frac{9 + 5v(1)^2}{3(1 + v(1))(3 + 4v(1))}, \quad \Omega_{(2)} = \frac{14v(1)}{3(1 + v(1))(3 + 4v(1))},
$$

$$
SGL_{v_{12}}: \quad \Sigma_+ = \frac{1}{3}, \quad \Sigma_-^4 = \frac{1}{3} \sqrt{-34v(2) - 6/3 - 4v(2)},
$$

$$
\Sigma_- = \frac{1}{3} \sqrt{-13v(2) - 6/3 - 4v(2)},
$$

$$
v(1) = 1, \quad -1 \leq v(2) = \text{const} \leq 6/13,
$$

$$
\Omega_{(1)} = \frac{-14v(2)}{3(1 - v(2))(3 - 4v(2))}, \quad \Omega_{(2)} = \frac{9 + 5v(2)^2}{3(1 - v(2))(3 - 4v(2))},
$$

leading to $\Sigma^2 = \frac{7}{3}v(1)/(3 + 4v(1))$, $q = \frac{4}{5}$ for $GL_{v_{11}}$, while $\Sigma^2 = \frac{7}{3}|v(2)/(3 + 4v(2))|$, $q = \frac{4}{5}$ for $GL_{v_{12}}$. Local elimination of $\Omega_{(1)}$ yields the eigenvalues:

$$
SGL_{v_{11}}: \quad \lambda_{3,4,5,6} = \frac{1}{3} \left( 1 \pm \sqrt{F(1) \pm G(1)} \right),
$$

$$
SGL_{v_{12}}: \quad \lambda_{3,4,5,6} = \frac{1}{3} \left( 1 \pm \sqrt{F(2) \pm G(2)} \right), \quad \lambda_{3,4,5,6} = \frac{1}{3} \left( 1 \pm \sqrt{F(3) \pm G(3)} \right),
$$

(A.21a)

where $F(i) = F(i)(v(i))$, $G(i) = G(i)(v(i))$ exhibit quite messy expressions, which we therefore refrain from giving, such that real parts of the associated eigenvalues always are negative.

**Appendix B. The $K$ subset**

We discuss here the Kasner subset $K$ with the state space $K = \{ \Sigma_+, \Sigma_-^4, \Sigma_-^4, \Sigma_-^4, v(1), v(2) \}$, subjected to the Gauss constraint $\Sigma_+^2 + \Sigma_-^4 + \Sigma_-^4 + \Sigma_-^4 = 1$. The equations for the test fields $v(1) \in [0, 1], v(2) \in [-1, 0]$ decouple from those of the shear and from each other. The state space therefore can be written as the following Cartesian product:

$$
K = KP \times \{v(1)\} \times \{v(2)\}, \quad KP = \{ \Sigma_+, \Sigma_-^4, \Sigma_-^4, \Sigma_-^4 \}.
$$

(B.1)
Figure B1. The projected Kasner circle $K_{\Sigma}$ is divided into sectors $(i, j, k)$, defined by $\Sigma_i < \Sigma_j < \Sigma_k$, where $i, j, k$ is a permutation of 1, 2, 3, and where $\Sigma_i = \hat{\Sigma}_i + \sqrt{3} \Sigma_\alpha$, $\Sigma_2 = \hat{\Sigma}_i - \sqrt{3} \Sigma_\alpha$, $\Sigma_3 = -2 \hat{\Sigma}_i$, and the points $Q_\alpha$, corresponding to the non-flat plane symmetric Kasner solution, and $T_\alpha$, corresponding to the Taub form for the Minkowski spacetime. The Kasner subset is of relevance for the past dynamics and hence the arrows point in the past time direction in figures B1(b) and (c) which depicts single frame transitions projected onto $\Sigma_B$. The global past attractor for the general geometric set with $\Sigma_A/\Sigma_B \neq 0$ on $K_{\Sigma}$ consists of sector (213) together with $Q_2$ and $T_3$ on $K_{\Sigma}$. (a) Kasner sectors. (b) The $\Sigma_A = 0$ subset transitions. (c) The $\Sigma_C = 0$ subset transitions. (d) The past attractor segment.

where $K_{\Sigma}$ is the projected Kasner state space, which of course is subjected to $\Sigma^2 = 1$. By determining the $\alpha$- and $\omega$-limits for solutions on $K_{\Sigma}$ one can then determine the asymptotic states of $v_{i(1)}$ and $v_{i(2)}$ separately, and thus the $\alpha$- and $\omega$-limits for solutions on $K$. Let us therefore first turn to the equations on $K_{\Sigma}$

$$
\Sigma_i' = 3 \Sigma_i^2; \quad \Sigma_A' = -(3 \Sigma_\alpha + \sqrt{3} \Sigma_B) \Sigma_A; \quad \Sigma_B' = \sqrt{3} \Sigma_A^2 - 2 \sqrt{3} \Sigma_C^2; \quad \Sigma_C' = 2 \sqrt{3} \Sigma_B \Sigma_C.
$$

(B.2)

This system admits a circle of fixed points, the projected Kasner circle: $K_{\Sigma}$, see figure B1(a). It is described by $\Sigma_A = \Sigma_C = 0$, $\Sigma_\alpha = \hat{\Sigma}_i$, $\Sigma_B = \hat{\Sigma}_i$, where the constants $\hat{\Sigma}_i$ satisfy $\hat{\Sigma}_i^2 + \hat{\Sigma}_i^2 = 1$. 

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The subset $\Sigma_A = 0$ yields that $\Sigma_\ast = \hat{\Sigma}_\ast$, $\Sigma_B^2 + \Sigma_C^2 = \hat{\Sigma}_\ast^2$, where $\Sigma_B$ is monotonically decreasing. The subset $\Sigma_C = 0$ leads to $\Sigma_\ast = \sqrt{3}\Sigma_B = \hat{\Sigma}_\ast = \sqrt{3}\hat{\Sigma}_\ast$ while $\Sigma_\ast$ and $\Sigma_B$ are monotonically increasing and $\Sigma_A^2 = 1 - \Sigma_\ast^2 = \Sigma_B^2$. Projected onto the $\Sigma_\ast - \Sigma_B$-plane this yields the straight lines—single frame transitions, using the nomenclature of [7], given in figures B1(b) and (c) (a frame transition preserves a Kasner state while permuting the spatial axes). As discussed in [7], the general case can be regarded as multiple frame transitions that yield the same result as combinations of single transitions which therefore determine the general asymptotic solution structure on $\mathcal{KP}$ (the set associated with $\mathcal{KP}$), for details see [7]. From this we conclude that the $\alpha$-limits for all solutions with $\Sigma_A \Sigma_C \neq 0$ on $\mathcal{KP}$ resides on the segment $\mathcal{KP}^O$, yielding a segment on $\mathcal{KP}^O$ characterized by $-1 \leq \Sigma_\ast = \hat{\Sigma}_\ast \leq -\frac{1}{2}$, $0 \leq \hat{\Sigma}_\ast \leq \frac{\sqrt{3}}{2}$, i.e., the segment consists of sector (213) together with the fix points $Q_2$ and $T_3$ on $\mathcal{KP}^O$, see figure B1(d).

The $\alpha$-limits for solutions on $\mathcal{K}$ are determined by the $\alpha$-limits on $\mathcal{KP}$ which determine the asymptotic limits for $v_{(i)}$. The equation for $|v_{(i)}| \in [0, 1]$ on $\mathcal{KP}^O$ is given by: $|v_{(i)}|' = (G^{(i)})^{-1}(1 - v_{(i)}^2)((3v_{(i)} - 1 + 2\Sigma_\ast)v_{(i)}|v_{(i)}|)$. It follows that the $\alpha$-limits for all orbits on $\mathcal{K}$ on the general geometric set with $\Sigma_A \Sigma_C \neq 0$ resides on the global past attractor $\mathcal{A}_{(\ast \ast)}$, where the subscript denotes the range of values of $w_{(1)}$ and $w_{(2)}$, given by

$$A_{|w_{(2)} < w_{(1)} < \frac{3}{4}|} = \left\{ \mathcal{K}_{11}^O : \hat{\Sigma}_\ast \in \left[ -1, -\frac{1}{2} \right] \right\},$$

(B.3a)

$$A_{|w_{(2)} < w_{(1)} = \frac{3}{4}|} = \left\{ \mathcal{K}_{11}^O : \hat{\Sigma}_\ast \in \left[ -1, -\frac{1}{2} \right) \right\} \cup \left\{ \mathcal{KL}_{(w_{(1)}, 1)}^+ : \hat{\Sigma}_\ast = -\frac{1}{2} \right\},$$

(B.3b)

$$A_{|w_{(2)} < w_{(1)} < \frac{1}{4}|} = \left\{ \mathcal{K}_{11}^O : \hat{\Sigma}_\ast \in \left[ -1, -\frac{1}{2} \left( 3w_{(1)} - 1 \right) \right) \right\} \cup \left\{ \mathcal{KL}_{(w_{(1)}, 1)}^+ : \hat{\Sigma}_\ast = -\frac{1}{2} \left( 3w_{(1)} - 1 \right) \right\}$$

$$\cup \left\{ \mathcal{K}_{01}^O : \hat{\Sigma}_\ast \in \left( -\frac{1}{2} \left( 3w_{(1)} - 1 \right), -\frac{1}{2} \right] \right\},$$

(B.3c)

$$A_{|w_{(2)} = w_{(1)} < \frac{1}{4}|} = \left\{ \mathcal{K}_{11}^O : \hat{\Sigma}_\ast \in \left[ -1, -\frac{1}{2} \left( 3w_{(1)} - 1 \right) \right) \right\} \cup \left\{ \mathcal{KL}_{(w_{(1)}, 1)}^+ : \hat{\Sigma}_\ast = -\frac{1}{2} \left( 3w_{(1)} - 1 \right) \right\}$$

$$\cup \left\{ \mathcal{K}_{01}^O : \hat{\Sigma}_\ast \in \left( -\frac{1}{2} \left( 3w_{(1)} - 1 \right), -\frac{1}{2} \right] \right\} \cup \left\{ \mathcal{KL}_{(w_{(2)}, 1)}^+ : \hat{\Sigma}_\ast = -\frac{1}{2} \right\},$$

(B.3d)

$$A_{|w_{(2)} < w_{(1)} < \frac{1}{4}|} = \left\{ \mathcal{K}_{11}^O : \hat{\Sigma}_\ast \in \left[ -1, -\frac{1}{2} \left( 3w_{(1)} - 1 \right) \right) \right\} \cup \left\{ \mathcal{KL}_{(w_{(1)}, 1)}^+ : \hat{\Sigma}_\ast = -\frac{1}{2} \left( 3w_{(1)} - 1 \right) \right\}$$

$$\cup \left\{ \mathcal{K}_{01}^O : \hat{\Sigma}_\ast \in \left( -\frac{1}{2} \left( 3w_{(1)} - 1 \right), -\frac{1}{2} \left( 3w_{(2)} - 1 \right) \right) \right\}$$

$$\cup \left\{ \mathcal{KL}_{(w_{(2)}, 1)}^+ : \hat{\Sigma}_\ast = -\frac{1}{2} \left( 3w_{(2)} - 1 \right) \right\} \cup \left\{ \mathcal{K}_{00}^O : \hat{\Sigma}_\ast \in \left( -\frac{1}{2} \left( 3w_{(2)} - 1 \right), -\frac{1}{2} \right] \right\}.$$  

(B.3e)

As toward the past, the results in [7] implies that all orbits on $\mathcal{KP}$, on the generic geometric set as well as all the Kasner compatible geometric subsets, asymptotically also approach $\mathcal{KP}^O$ toward the future. From this it easily follows from the decoupled $v_{(i)}$ equations that the $\omega$-limit for any orbit on $\mathcal{K}$ is one of the Kasner fix points. But according to the local stability analysis in appendix A all fix points on $\mathcal{K}$ are destabilized toward the future by the matter degrees of freedom in the full state space, leading to the $\omega$-limit points on $\mathcal{K}$ becoming saddles in the full state space. No matter solutions with $Q_{(1)} > 0$, $v_{(1)}^2 < 1$, $v_{(2)}^2 < 1$ initially are attracted to any part of $\mathcal{K}$ when $\tau \to \infty$, and thus the $\omega$-limits for all “interior” matter solutions either resides on $\mathcal{OT}_{v_{(1)}, 0}$ or $\mathcal{ET}_{v_{(2)}}$, such that $q < 2$ when $\tau \to \infty$, since $q = 2$ only on $\mathcal{K}$.

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