Momentum/Complexity Duality
and the Black Hole Interior

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Abstract: We establish a version of the Momentum/Complexity (PC) duality between the rate of operator complexity growth and a radial component of bulk momentum for a test system falling into a black hole. In systems of finite entropy, our map remains valid for arbitrarily late times after scrambling. The asymptotic regime of linear complexity growth is associated to a frozen momentum in the interior of the black hole, measured with respect to a time foliation by extremal codimension-one surfaces which saturate without reaching the singularity. The detailed analysis in this paper uses the Volume-Complexity (VC) prescription and an infalling system consisting of a thin shell of dust, but the final PC duality formula should have a much wider degree of generality.
1 Introduction

Measures of operator complexity have received considerable recent attention in studies of information scrambling in many-body quantum systems [1–7]. One motivation is the characterization of quantum complexity in holographic systems. In that context, it has been proposed that the ‘size’ of an operator can be characterized by a mechanical momentum of an effective particle in the bulk (cf. [8–11]). The bulk particle is ‘injected’ by the ‘small’ operator $O$ on the boundary, acting on some reference state $O|\Psi\rangle$ at, say $t = 0$. If the resulting state is evolved in time

$$e^{-itH}O|\Psi\rangle = e^{-itH}O e^{itH} e^{-itH} |\Psi\rangle = O_{-t} |\Psi\rangle_t ,$$

any increase of complexity can be attributed partly to the increase in complexity of the time-evolved reference state $|\Psi\rangle_t$, and partly to the increase in complexity of the operator when evolved to the past, in what we usually refer to a ‘precursor’: $O_{-t} = e^{-itH}O e^{itH}$. If the increase in complexity of the reference state can be neglected or somehow subtracted, we can define the complexity of the operator $O_{-t}$ in terms of the complexity of the evolved state. The state (1.1) can be interpreted as a heavy particle
state falling through the bulk. More precisely, we may define the operator complexity in terms of the state complexity by the subtraction

\[ C_O(t) = C[O_{-t}\vert\Psi\rangle_t] - C[\vert\Psi\rangle_t], \tag{1.2} \]

with some appropriate normalization. In practice, this definition must be supplemented by some definite prescription for the state complexity such as, for example, the AC/VC definitions (cf. [12–16]).

Let us suppose that the state (1.1) can be interpreted as a heavy particle falling through the bulk. Then, the momentum/complexity duality proposal (PC duality for short) amounts to a relation of the form

\[ \frac{dC_O}{dt} = P_C, \tag{1.3} \]

where \( C_O \) is the complexity of the operator, and \( P_C \) is a suitable component of the mechanical momentum of the associated particle. A simple example of the PC duality is obtained by regarding the free fall of a particle in a Rindler near-horizon region as dual to operator growth in a fast scrambler. In this case, both terms in (1.3) grow exponentially in time, so that the qualitative behavior only establishes \( P_C \) as proportional to the complexity, or any of its higher time derivatives. A more precise matching can be obtained by testing the PC duality in near-extremal Reissner–Nordstrom horizons. In this case, there is a ‘pre-scrambling’ period corresponding to the fall through the AdS\(_2\) throat which, upon comparison with detailed calculations of operator growth in the SYK model [2, 10], leads to (1.3).

All checks of PC duality performed so far involve the particle fall towards the horizon. The scrambling time, which marks the end of the period of exponential complexity growth, corresponds to the particle reaching the stretched horizon, a timelike layer situated about one Planck length away from the horizon. In systems with finite entropy, one expects the growth of complexity to become linear at asymptotically large times, much larger than the scrambling time. Of course, in this late-time regime complexity must be regarded as essentially different from ‘operator size’. At any rate, in the definition (1.2) this follows from the asymptotic linearity of generic state complexities in either AC/VC prescriptions. Recently, a slightly different notion of operator complexity, known as K-complexity [5] was shown to also exhibit the characteristic linear growth at very late times (cf. [6]).

In the bulk interpretation, such a linear growth must be associated to some property of the infalling particle trajectory in the interior of the black hole. A primary objective of this paper is to fill this gap by exhibiting the conditions under which the PC duality (1.3) can be compatible with an asymptotic linear growth of operator complexity. The
basic idea is to define an appropriate time foliation extending beyond the horizon and having an accumulation surface in the black hole interior. Our main claim is simply that the extremal surfaces of the VC complexity prescription provide such a time foliation. At the same time, the relation (1.3) can be proven explicitly as a bulk calculation, for a particular choice of operators which inject spherical thin shells of dust near the boundary of AdS.

The paper is organized as follows: in section 2 we describe the class of operators for which we establish the PC duality. In section 3 we give a proof of (1.3) in this context. We end with conclusions and an appendix with some technical points.

2 Thin-shell operators and states

For a holographic CFT defined on a spherical spatial manifold $S^{d-1}$ of radius $L$, we consider its gravity dual on $\text{AdS}_{d+1}$, also taken to have curvature radius $L$. A thin shell of dust injected from the AdS boundary can be represented in the CFT by the action of a formal product operator

$$O_{\text{shell}} \sim \prod_{D_{\Lambda} \in P_{\Lambda}} \phi_{\Lambda,D_{\Lambda}},$$

where $P_{\Lambda}$ is a partition of the sphere in domains $D_{\Lambda}$ of size $\Lambda^{-1}$, the regularization cutoff. The operators $\phi_{\Lambda,D_{\Lambda}}$ can be seen as bulk operators, applied at radius of order $r_{\Lambda} \sim \Lambda L^2$, and smeared over the domain $D_{\Lambda}$. The idea is to use $\phi_{\Lambda,D_{\Lambda}}$ to inject a heavy bulk particle at radius $r_{\Lambda}$. Although we imagine specifying the operators in bulk effective field theory, we can always regard it as a CFT operator by a bulk-boundary reconstruction map, say using the HKKL formulation [17].

These operators are ‘big’ in the sense of the spatial structure, but are ‘simple’ in holographic terms, since they are constructed from operators near the boundary of AdS. By appropriately choosing $\phi_{\Lambda,D_{\Lambda}}$, we can generate a semiclassical state whose subsequent evolution is parametrized as the collapse of the shell of particles in the bulk geometry. In the case that the local factors $\phi_{\Lambda,D_{\Lambda}}$ are engineered with very massive bulk fields, or equivalently CFT operators with very large conformal weight, we can regard the shell as composed of classical massive particles forming a dust cloud with density $\sigma$ and four-velocity field $u^\mu$.

For the purposes of this paper, we define the operator complexity in terms of the general prescription (1.2), where the state complexity is regarded as computed with the VC prescription. For technical convenience, we shall take the high-temperature thermofield double state as the reference state on the Hilbert space of two copies of the CFT, and the shell state is injected on the Right CFT as indicated in Figure 1, at times...
much larger than the thermalization time $T^{-1}$, where $T$ is the Hawking temperature of the black hole.

\[ C[O_{\text{shell}}] = \frac{d-1}{8\pi G L} [\text{Vol}(\Sigma_{\text{bh+shell}}) - \text{Vol}(\Sigma_{\text{bh}})], \tag{2.2} \]

where $\Sigma$ denotes the extremal codimension-one hypersurface with given asymptotic boundary conditions, defined in the eternal black hole spacetime with and without the shell. The concrete prefactor in (2.2) is chosen for convenience of normalization. From now on shall measure bulk lengths in units of curvature radius, so that we set $L = 1$.

The complexity of the shell operator is defined in terms of bulk quantities as

The worldvolume of the thin shell is a codimension-one timelike manifold $W$ which divides the spacetime manifold in two regions: $\mathcal{V}^+$ is a Schwarzschild-AdS solution of mass $M_+$ which we identify as ‘exterior’ or ‘right’ region, and $\mathcal{V}^-$, a similar solution of mass $M_-$ referred to as the ‘interior’ or ‘left’ region. The ADM energy of the shell is given by $M_+ - M_-$ and is assumed to be positive. Spherical symmetry holds globally.

Figure 1. Penrose diagram of the collapsing shell geometry. The shell is injected in the bulk at late times compared with $T^{-1}$, causing the initial black hole of mass $M_-$ to grow up to the bigger mass $M_+$. The worldvolume of the matter shell is labelled $W$ and sets the boundary between the two black hole spacetimes $\mathcal{V}^\pm$. 
in the full spacetime, whereas stationarity is broken at $\mathcal{W}$. Both $\mathcal{V}^\pm$ have smooth Killing vectors which are timelike in the asymptotic regions and spacelike inside event horizons. Denoting these vectors as $\xi^\pm = \partial/\partial t^\pm$, where $t^\pm$ are adapted coordinates, we can write a standard form of the metric on both sides of $\mathcal{W}$:

$$
\text{d}s^2_\pm = -f_\pm \text{d}t^2_\pm + f^{-1}_\pm \text{d}r^2 + r^2 \text{d}\Omega^2_{d-1},
$$

where

$$
f_\pm = 1 + r^2 - \frac{16\pi GM_\pm}{(d-1)V_\Omega r^{d-2}},
$$

and $V_\Omega = \text{Vol}(S^{d-1})$. The shell dynamics follows from Einstein’s equations, which take the form of junction conditions (cf. [18, 19]). Denoting the induced metric on $\mathcal{W}$ as

$$
\text{d}s^2_{\mathcal{W}} = -d\tau^2 + R(\tau)^2 \text{d}\Omega^2_{d-1},
$$

in terms of the shell’s proper time $\tau$ and its radius $R(\tau)$, continuity of the spacetime metric across $\mathcal{W}$ implies the first junction condition,

$$
f_\pm(R) \left( \frac{d\ell^\pm}{d\tau} \right)^2 - \frac{1}{f_\pm(R)} \left( \frac{dR}{d\tau} \right)^2 = 1 .
$$

The second junction condition establishes the jump of the extrinsic curvature across $\mathcal{W}$ as proportional to the stress-energy on the shell’s world-volume. For a thin shell of dust we have

$$
T_{\mu\nu} = \sigma u^\mu u^\nu \delta(\ell) ,
$$

where $u^\mu$ is the four-velocity field of the shell and $\sigma$ is the surface density. The coordinate $\ell$ measures proper distance away from $\mathcal{W}$ in the orthogonal spacelike direction, increasing towards the exterior region; in other words, the normal unit vector $N^\mu_{\mathcal{W}} = \partial/\partial \ell$ satisfies $N^2_{\mathcal{W}} = 1$ and $u_\mu N^\mu_{\mathcal{W}} = 0$. For spherically infalling dust the density $\sigma(R)$ must be inversely proportional to the shell’s volume, that is to say, the total rest mass

$$
m = \sigma V_\Omega R^{d-1}
$$

remains constant.

The second junction condition specifies the jump in extrinsic curvature across $\mathcal{W}$,

$$
\sqrt{\left( \frac{dR}{d\tau} \right)^2 + f_-(R)} - \sqrt{\left( \frac{dR}{d\tau} \right)^2 + f_+(R)} = \frac{8\pi G}{d-1} \sigma R .
$$

The particular conditions of spherical symmetry and stationarity along $\mathcal{V}^\pm$ allow us to write the junction conditions in terms of the Killing vectors $\xi^\pm$, an expression
that will be useful later. Using that $\xi_\mu = g_{\mu \nu}$ and the explicit form of the metric (2.3) we find

$$ (u \cdot \xi)_\pm = -f_\pm \frac{dt_\pm}{d\tau} . $$  

(2.10)

Furthermore, since $\xi_\pm$ are orthogonal to the angular spheres, the normalization implies

$$ g_{\mu \nu} \xi_\mu \xi_\nu = (\xi_\pm)^2 = -(u \cdot \xi_\pm)^2 + (N_W \cdot \xi_\pm)^2 = -f_\pm , $$  

(2.11)

an expression which determines $N_W \cdot \xi_\pm$ once we know $u \cdot \xi_\pm$. Using (2.10) and (2.11) we may recast the two junction conditions as jumping rules for the Killing vectors, namely the component normal to $W$ is continuous

$$ N_W \cdot \xi_+ |_W = N_W \cdot \xi_- |_W , $$  

(2.12)

whereas the component tangential to $W$ jumps like the extrinsic curvature,

$$ (u \cdot \xi_+ - u \cdot \xi_-) |_W = \sqrt{\left(\frac{dR}{d\tau}\right)^2 + f_+(R)} - \sqrt{\left(\frac{dR}{d\tau}\right)^2 + f_-(R)} = -\frac{8\pi G}{d-1} \sigma R . $$  

(2.13)

Equivalently, we can say that both junction conditions boil down to the jump rule:

$$ (\Delta \xi_\mu)_W \equiv (\xi_+^\mu - \xi_-^\mu) |_W = -\frac{8\pi G}{d-1} \sigma R u^\mu . $$  

(2.14)

One more presentation of the shell dynamics is obtained by extracting from (2.9) the ADM mass of the shell as a constant of motion:

$$ M_{\text{shell}} = M_+ - M_- = m \sqrt{\left(\frac{dR}{d\tau}\right)^2 + f_+(R)} - \frac{4\pi G}{(d-1)V_\Omega} \frac{m^2}{R^{d-2}} . $$  

(2.15)

This can be interpreted as a kinetic contribution proportional to the shell’s rest mass $m$, corrected by a gravitational self-energy term. In fact, the constancy of $m$ suggests a natural $(1+1)$-dimensional picture in terms of an effective particle of mass $m$, moving in the two-dimensional section of the metric obtained by simply deleting the angular directions:

$$ ds_{1+1}^2 = \bar{g}_{\alpha \beta} dx^\alpha dx^\beta = -f_-(r)dt^2 + \frac{dr^2}{f_-(r)} . $$  

(2.16)

In particular, the shell energy (2.15) can be obtained as the canonical energy from the effective action of a free particle

$$ S_{\text{eff}} = \int d\lambda L_{\text{eff}} = -m \int d\lambda \sqrt{\bar{g}_{\alpha \beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}} . $$  

(2.17)

provided we can neglect the gravitational self-energy effects.
3 Proof of the PC duality

Our goal is to derive a PC duality relation by direct evaluation of the left hand side of (1.3), with $C_{\text{shell}}$ defined as in (2.2). This will allow us to identify the correct component of ‘radial momentum’. The complexity being defined through the VC prescription, we start with a preliminary discussion of extremal-volume surfaces in the relevant geometries.

3.1 Extremal volumes

Let a codimension-one spacelike surface $\Sigma$ be defined by the embedding functions $X^\mu(y^a)$, with $y^a$ coordinates along the hypersurface. The volume functional reads

$$ V[\Sigma] = \int_\Sigma d^d y \sqrt{h}, \quad (3.1) $$

where $h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)$ is the induced metric on $\Sigma$. Under a generic variation $\delta X^\mu$ the volume varies as

$$ \delta V = \int_\Sigma (\text{e.o.m.})_{\mu} \delta X^\mu + \int_{\partial \Sigma} dS^a \partial_a X^\mu \delta X^\mu. \quad (3.2) $$

where

$$(\text{e.o.m.})_{\mu} = - \frac{1}{\sqrt{h}} \partial_a \left( \sqrt{h} h^{ab} g_{\mu\nu} \partial_b X^\nu \right) + \frac{1}{2} h^{ab} \partial_a X^\rho \partial_b X^\sigma \partial_\mu g_{\rho\sigma} \quad (3.3)$$

vanishes precisely when the hypersurface $\Sigma$ is extremal. In this case, the variation reduces to a boundary term,

$$ \delta V \big|_{\text{extremal}} = \int_{\partial \Sigma} dS^a e^a_\mu \delta X^\mu, \quad (3.4) $$

where we have defined the vector fields $e^a_\mu = \partial_a X^\mu$ tangent to $\Sigma$.

For the geometry of interest here, $\Sigma$ is a cylindrical manifold of topology $\mathbb{R} \times S^{d-1}$, the boundary having two disconnected components consisting of spheres at the left and right spatial infinities. We shall use the same future-directed time variables on both boundaries and take a left-right symmetric time configuration $t_L = t_R = t$, so that we can write the following boundary conditions at the regularization surfaces $r = r_\Lambda$:

$$ \delta X^\mu_\pm \big|_{r=r_\Lambda} = \pm \delta t \xi_\pm \big|_{r=r_\Lambda}, \quad (3.5) $$

\footnote{We use latin indices for coordinates on the hypersurface $\Sigma$ and greek indices for general coordinates in the full spacetime.}
where the ± signs account for the fact that the left-side Killing vector $\xi_-$ is past-directed at large radii. Spherical symmetry allows us to parametrize the induced metric on extremal surfaces in the form

$$ds^2_\Sigma = h_{ab} dy^a dy^b + g(y) d\Omega_{d-1}^2,$$

(3.6)

where $y$ is a radial coordinate running over the real line, with $y = \pm \infty$ corresponding respectively to the left and right boundaries of $\Sigma$. In these coordinates, we can picture $e^\mu_y = \partial_y X^\mu$ as a unit-normalized, radial, spherically symmetric, right-pointing vector field. Denoting the spheres at infinity by $S_{\pm \infty}$ we can rewrite the volume variation of extremal surfaces (3.4) as

$$\delta V|_{\text{extremal}} = \delta t \left[ \int_{S_{-\infty}} e^\mu_y (\xi^+)_\mu + \int_{S_{+\infty}} e^\mu_y (\xi^-)_\mu \right],$$

(3.7)

where we have absorbed the sign assignments in (3.5) into a reversal of orientation for the left-boundary integral. Namely, both integrals in (3.7) are now written as scalar integrals over the boundary spheres.

This expression for the volume dependence with asymptotic time is useful because the featured integrals turn out to be Noether charges. If we view the volume functional
as an action on a collection of fields $X^\mu$ defined over $\Sigma$, the isometries of the $\mathcal{Y}^\pm$ portions are interpreted as ‘internal symmetries’ of the this field theory, with their corresponding Noether currents. The time-translation symmetries associated to $\xi_\pm$ induce Noether currents of the form \footnote{In order to prove conservation, we just use $\xi_\mu = g_{\mu\nu}$ and evaluate the equation of motion from (3.3).}

$$J_a = e_\mu^a \xi_\mu , \quad \nabla_a J^a = 0 \ . \tag{3.8}$$

In particular, the integral of the radial component $J_y$ over any fixed-$y$ section $S_y$ is a Noether charge which is conserved under transport in the $y$ direction:

$$\Pi[S_y] = \int_{S_y} e_\mu^y \xi_\mu , \quad \partial_y \Pi[S_y] = 0 \ . \tag{3.9}$$

### 3.2 Identification of the PC component

We have now the machinery in place to evaluate (2.2). The formula (3.7) implies

$$\frac{dV}{dt} = \Pi_+ + \Pi_- \ , \tag{3.10}$$

in terms of the Noether charges $\Pi_\pm \equiv \Pi[S_{\pm\infty}]$ on right and left boundaries (a similar result was derived in [20] for null shells). The normalization of the operator complexity requires the subtraction of the same expression, evaluated on the Noether charges $\Pi^{(0)}_\pm$ of the eternal black hole geometry without infalling shell, namely

$$\dot{C}[\mathcal{O}_{\text{shell}}] = \frac{d-1}{8\pi G} \left[ \Pi_+ - \Pi^{(0)}_+ + \Pi_- - \Pi^{(0)}_- \right] \ , \tag{3.11}$$

where the dot here denotes derivative with respect to asymptotic time.

Left-right symmetry of the eternal black hole geometry implies $\Pi^{(0)}_+ = \Pi^{(0)}_-$, whereas we can also set $\Pi_- \approx \Pi^{(0)}_-$ at the left regularization boundary because, for shells that enter the geometry at very late times, their worldvolume $W$ remains very far from the left boundary. Hence, near the left regularized boundary, the extremal surface $\Sigma$ is very well approximated by that of the eternal black hole. As we remove the regularization, in the limit $r_\Lambda \to \infty$, we must actually obtain $\Pi_- = \Pi^{(0)}_-$. This allows us to remove all explicit reference to the eternal black hole geometry and write

$$\dot{C}[\mathcal{O}_{\text{shell}}] = \frac{d-1}{8\pi G} \left[ \Pi_+ - \Pi_- \right] \ . \tag{3.12}$$

Furthermore, the conservation of Noether charges in either $\mathcal{Y}^+$ or $\mathcal{Y}^-$ allows us to bring the Noether charges to both sides of the shell’s worldvolume:

$$\dot{C}[\mathcal{O}_{\text{shell}}] = \frac{d-1}{8\pi G} (\Delta \Pi)_{\mathcal{W}} = \frac{d-1}{8\pi G} \int_{S_{\mathcal{W}}} e_\mu^y (\Delta \xi_\mu)_{\mathcal{W}} \ , \tag{3.13}$$
where \((\Delta \xi^\mu)_W = (\xi^\mu_+ - \xi^\mu_-)|_W\) is the jump of the Killing vectors across \(W\) and \(S_W\) is the sphere at the intersection \(\Sigma \cap W\). Using now the junction conditions in the form (2.14), we find

\[
\dot{C}[\Omega_{\text{shell}}] = - \int_{S_W} \sigma R e_y^\mu u_\mu .
\] (3.14)

We can now elaborate (3.14) in various ways in order to flesh out the PC-duality interpretation. First, we define a ‘complexity field’ over \(\Sigma\) as a rescaling of the \(e_y^\mu\) field:

\[
C^\mu_\Sigma \equiv - r e_y^\mu .
\] (3.15)

Second, we define a density of proper momentum along the shell’s worldvolume

\[
P^\mu \equiv \sigma u^\mu .
\] (3.16)

With these definitions we can rewrite (3.13) as

\[
\dot{C}[\Omega_{\text{shell}}] = P_\Sigma = \int_{S_W} P_\mu C^\mu_\Sigma ,
\] (3.17)

a relation which identifies the precise component of momentum which is dual to complexity growth, namely the projection of the proper momentum along the direction of the complexity vector field \(C^\mu_\Sigma\). It is a particular radial component with inward orientation and appropriate normalization.

**Figure 3.** Configuration of relevant vectors at the intersection sphere \(S_W = \Sigma \cap W\).
A second presentation of this result has the virtue of hiding some of the peculiarities of the concrete system we have considered so far. In fact, no explicit geometrical information about the shell’s worldvolume \( \mathcal{W} \) is needed in order to express the PC duality relation. To see this, let us consider the expression

\[- \int_{\Sigma} N^\mu_\Sigma \, T_{\mu\nu} \, C^\nu_\Sigma, \tag{3.18}\]

where \( N^\mu_\Sigma \) is the unit timelike normal to \( \Sigma \). It measures the flux through \( \Sigma \) of a suitably normalized momentum component along \( \Sigma \). Upon explicit evaluation for the spherical shell, using (2.7), we find

\[- \int dy \int_{S_y} \sigma \left( N^\Sigma_\Sigma \cdot u \right) \left( C^\Sigma_\Sigma \cdot u \right) \delta(\ell) \, \delta \left( y - y^W \right) \left| \frac{d\ell}{dy} \right|^{-1} \tag{3.19}\]

Furthermore, \( \delta(\ell) = \delta(y - y^W) \, |d\ell/dy|^{-1} \), where \( y^W \) is the value of the \( y \) coordinate at the shell’s intersection. From the definition of the \( \mathcal{W} \)-normal we have

\[ \frac{d\ell}{dy} = \partial_y X^\mu \partial_\mu \ell = e_y \cdot N^W, \]

which allows us to collapse the integral to the intersection sphere \( S^W \):

\[ \int_{S^W} \sigma R \frac{(N^\Sigma_\Sigma \cdot u) (e_y \cdot u)}{(e_y \cdot N^W)}, \tag{3.20}\]

where we have used (3.15). To further reduce this integral we notice that \( N^\Sigma_\Sigma \) and \( e_y \) are orthogonal and unit normalized, as well as the pair \( u \) and \( N^W \), so that we have \( N^\Sigma_\Sigma \cdot u = -N^W \cdot e_y \), where the minus sign accounts for the timelike character of both \( N^\mu_\Sigma \) and \( u^\mu \). This simplifies (3.20) and recovers (3.14). Hence, we have established the more intrinsic form of the PC relation:

\[ \dot{C}[\mathcal{O}_{\text{shell}}] = P_C = - \int_{\Sigma} N^\mu_\Sigma \, T_{\mu\nu} \, C^\nu_\Sigma. \tag{3.21}\]

In this version, all explicit reference to the details of the bulk state gets reduced to its stress-energy tensor. The vector fields \( N^\Sigma_\Sigma \) and \( C^\Sigma_\Sigma \) are defined in terms of the extremal surface, whose detailed geometry is also determined by \( T_{\mu\nu} \) through the back reaction on the geometry. Indeed, the form of (3.21) should remain valid for spherical shells with any internal equation of state, including those corresponding to branes which change the AdS radius of curvature across \( \mathcal{W} \). Furthermore, the role of the Noether charges in the derivation of (3.17) and (3.21) makes it clear that it applies as well to spherical thin shells collapsing in vacuum AdS and forming a one-sided black hole.

More generally, we expect that any spherical matter distribution can be approximated by a limit of many concentric thin shells, so that (3.21) should remain valid for any matter bulk distribution with spherical symmetry. It would be interesting to have a direct derivation of this fact, which could shed light on whether (3.21) remains true without spherical symmetry.
4 Late time limit and the black hole interior

One chief motivation behind this work is the elucidation of the very late time regime of operator complexity growth in the light of the PC duality. Any definition of operator complexity with the structure of equation (1.2) will assign a linear growth at late times. In particular, given that state complexities are expected to grow proportionally to $E_\Psi t$, where $E_\Psi$ is a characteristic energy of the state, the subtracted definition for operator complexity gives a slope proportional to $E_O t$, where $E_O$ is the additional energy injected by the operator $O$. Translated to our gravitational set up, we expect a late time behavior

$$\frac{d\hat{C} [O_{\text{shell}}]}{dt}_{\text{late}} \approx M_+ - M_- = M_{\text{shell}}.$$ (4.1)

We would like to check that our PC relation satisfies this expected asymptotic behavior. A simple check can be performed in the limit of very large AdS black holes. This coincides with the situation where the infalling shells have small gravitational self-energy at all times that are relevant for the calculation.

The key point is to notice that, at late times, the extremal surfaces $\Sigma_t$ accumulate in the interior of the black hole, exponentially converging to a limiting surface $\Sigma_\infty$ (cf. [14, 21]). For a shell that enters the black hole very late, this surface interpolates between the limiting surfaces $(\Sigma_\infty)^\pm$ associated to the early and late black holes of mass $M_\pm$ (cf. Figure 4). In terms of the interior Schwarzschild radial coordinates, let $\tilde{r}_\pm$ denote the saturation radii, defined by the local extremization of the ‘volume Lagrangian’ $r^{d-1}\sqrt{|f(r)|}$. By explicit calculation we find, in the limit of very large AdS black holes

$$\tilde{r}^d \approx \frac{8\pi GM}{(d-1)V_\Omega}.$$ (4.2)

We can now make use of the ‘movability’ of the Noether charges $\Pi_\pm$ to evaluate then away from $W$, but still inside the black hole interior, in a region where $\Sigma_t$ is well-approximated by a constant-$r$ surface. Let us denote the angular spheres at such points by $\tilde{S}_\pm$. Then, equation (3.12) can be rewritten as

$$\hat{C} [O_{\text{shell}}] \approx \frac{d-1}{8\pi G} \left( \Pi \left[ \tilde{S}_+ \right] - \Pi \left[ \tilde{S}_- \right] \right).$$ (4.3)

In computing the Noether charges, we notice that $\xi_\pm = \partial/\partial t_\pm$ are approximately tangent to $\Sigma_t$ in the saturation region. Hence, we can write $e_\mu^y \approx \xi^\mu/\sqrt{\xi^2}$ and the Noether integrals are simply

$$\Pi [\tilde{S}_\pm] \approx \int_{\tilde{S}_\pm} \sqrt{\tilde{g}} = V_\Omega \tilde{r}_\pm^{d-1}\sqrt{|f(\tilde{r})|} \approx V_\Omega \tilde{r}_\pm^d \approx \frac{8\pi GM_\pm}{d-1}.$$ (4.4)

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3See Appendix A for an quantitative discussion of this phenomenon.
Figure 4. The saturation slice $\Sigma_\infty$ interpolates between the extremal surface barrier inside $\tilde{r}_-$ and outside $\tilde{r}_+$. 

In the last equality we have made use of (4.2) and the approximation of a large AdS black hole. Therefore, upon subtraction we conclude the proof of (4.1).

An important observation regarding this result is the fact that the vector fields $C^\mu$ and $e_\nu^\mu$ do differ significantly in the interior saturation region, because the rescaling factor $\tilde{r}$ is non trivial there, and yet this rescaling is crucial to obtain the expected long-time asymptotics. Therefore, the peculiar normalization (3.15) of the momentum component along $\Sigma$ is necessary for the consistency of the results.

We can obtain further insight into the rationale behind the linear complexity growth by passing to the effective particle description. Again neglecting self-energy corrections, we can envision the dynamics of the shell as that of a probe particle of mass $m$ falling through the $(1 + 1)$-dimensional metric (2.16). The PC duality relation admits the two-dimensional representation:

$$\dot{C}[O_{\text{shell}}] = P_C = P_\alpha C^\alpha, \quad (4.5)$$

where $P^\alpha = m u^\alpha$, with $\alpha$ a two-dimensional index. Picking for example the standard $(r, t)$ coordinates, we have

$$P_C = -r \left( \frac{\partial t}{\partial y} P_t + \frac{\partial r}{\partial y} P_r \right). \quad (4.6)$$

Let us introduce an adapted coordinate for the radial ‘complexity field’ $C^\alpha = -r e_y^\alpha$, 

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namely we define a rescaled radial coordinate $\chi$ such that

$$C^\alpha = \left( \frac{\partial}{\partial \chi} \right)^\alpha = -r e_y^\alpha = -r \left( \frac{\partial}{\partial y} \right)^\alpha ,$$  \hspace{1cm} (4.7)

or, equivalently

$$\frac{\partial}{\partial \chi} = -r \frac{\partial}{\partial y} .$$  \hspace{1cm} (4.8)

Using the so-defined $\chi$ coordinate, we can simplify (4.6) so that

$$P_c = P_t \frac{\partial t}{\partial \chi} + P_r \frac{\partial r}{\partial \chi} = P_\chi .$$  \hspace{1cm} (4.9)

To the extent that we are only interested in describing the particle motion to the past of the saturation surface $\Sigma_\infty$, we may use a time slicing given by the extremal surfaces $\Sigma_t$ themselves, and coordinate the spacetime in terms of $(t, \chi, \Omega)$. In this frame, the complexity momentum coincides with the $\chi$-canonical momentum, provided we stay within the probe approximation:

$$P_c = P_\chi = \frac{\partial L_{\text{eff}}}{\partial \dot{\chi}} .$$  \hspace{1cm} (4.10)

This brings our general formalism into contact with the more elementary discussion of [8]. However, our treatment is capable of describing the late-time behavior of the complexity. In particular, the use of a time slicing adapted to the extremal surfaces leads to the phenomenon of saturation in the black-hole interior. This freezes the value of the momentum at a constant value for asymptotically large values of $t$, thereby explaining why the complexity can grow linearly.

**Figure 5.** The key to linear growth of complexity is the saturation of the time slicing in the interior of the black hole.
5 Conclusions and Outlook

In this paper we have presented a bulk derivation of a particular version of the momentum/complexity (PC) duality. By examining the VC complexity of thin spherical shells impinging on double-sided AdS black holes, we can explicitly identify the relevant momentum component. It turns out that our relation applies to arbitrarily late times, suggesting that one needs a time slicing given by the same extremal volume surfaces that feature in the VC prescription.

The form of the answer and standard heuristic arguments suggest that our result must apply to any spherical bulk configuration, so that any s-wave operator over the CFT sphere should admit a complexity growth of the form

\[
\dot{C}[O_{\text{spherical}}] = - \int_{\Sigma} N^\mu_{\Sigma} T_{\mu\nu} C^\nu_{\Sigma}. \tag{5.1}
\]

It would be interesting to find a general ab initio derivation of this relation which does not go through the thin-shell detour. Our derivation does apply to collapsing thin shells in the AdS vacuum. In this case, gravitational self-energy cannot be neglected at the saturation surface in the resulting one-sided black hole, so that one expects the probe approximation to be less efficient in the effective particle picture.

One outstanding question raised by our results is the true generality of a formula like (5.1). In particular, its validity for non-spherical situations and the elucidation of the deeper geometrical meaning of the ‘complexity vector field’ \( C^\mu_{\Sigma} \). Finally, it would be interesting to check the complexity slope (4.1) by direct evaluation of \( P_C \). This requires detailed control of the precise location of the intersection sphere \( S_W \) in the black-hole interior.

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A Late Time Accumulation of Maximal Slices

In this appendix, we show a proof of the exponentially fast accumulation of maximal slices in the black hole interior. For that matter, we will work within the benchmark case of an eternal black hole, whose metric is given in Eddington-Finkelstein coordinates by

\[ ds^2 = -f(r) \, du^2 + 2 \, du \, dr + r^2 dΩ_{d-1}^2. \]  

(A.1)

By spherical symmetry, the maximal surface can be written as a direct product \( \Sigma = \gamma \times S^{d-1} \), with \( \gamma \) a curve in the \( u-r \) plane. Exploiting this symmetry we can reduce thus the problem of volume extremalization to that of a spacelike geodesic in the effective two-dimensional spacetime

\[ ds_\gamma^2 = r^{2(d-1)} (-f(r) \, du^2 + 2 \, du \, dr), \]

(A.2)

so that the effective volume functional is given by

\[ V[\Sigma]V^{-1}_\Omega = V[\gamma] = \int d\lambda \, r^{d-1} \sqrt{-f(r) \, \dot{u}^2 + 2 \, \ddot{u} \, \dot{r}}, \]

(A.3)

where \( \lambda \) is an arbitrary spacelike parameter and the dot stands for \( d/d\lambda \). The Lagrangian in (A.3) enjoys a conserved charge associated to the static Killing

\[ \Pi = \frac{\partial L_\gamma}{\partial \dot{u}} = r^{d-1} \frac{-f(r) + \ddot{r}}{\sqrt{-f(r) + 2 \, \ddot{r}}}, \]

(A.4)

where \( \Pi \) is guaranteed to be positive by the spacelike character of the geodesic \( ds_\gamma^2 > 0 \) and we have taken the convenient gauge choice \( \lambda = u \). Feeding the conserved charge into the equations of motion for \( r(u) \) we get

\[ \ddot{r} = f(r) + \frac{\Pi^2}{r^{2(d-1)}} + \frac{\Pi}{r^{d-1}} \sqrt{\frac{\Pi^2}{r^{2(d-1)}} + f(r)}. \]  

(A.5)

Upon the imposition of reflection symmetry in our setup (\( t_L = t_R = t \)), the boundary conditions can be recasted to be \( \dot{r}(u_i) = 0 \) and \( r(u_\infty) = r_\infty \) for \( u_i = r_\gamma(r_i), u_\infty = t \) the values of the parameter at the symmetric turning point and boundary respectively. In terms of the turning point radius \( r_i \) we can get a simple expression for \( \Pi \)

\[ \Pi = r_i^{(d-1)} \sqrt{-f(r_i)}. \]  

(A.6)

An implicit relation between \( t \) and \( r_i \) can be obtained integrating (A.5)

\[ \int_{u_i}^{u_\infty} du = \int_{r_i}^{r_\infty} dr \frac{r^{2(d-1)}}{g^{1/2}(r) \left( \Pi + g^{1/2}(r) \right)}. \]  

(A.7)
where we have defined the function
\[ g(r) = r^{2(d-1)} f(r) - r_i^{2(d-1)} f(r_i), \tag{A.8} \]
which vanishes at the minimal radius \( r_i \). Breaking up the radial integral into an inner an outer piece and substituting the boundary conditions, we can obtain an expression for the boundary time
\[ t = \int_{r_i}^{r_h} dr \frac{r^{2(d-1)}}{g^{1/2}(r)} \left( \Pi + g^{1/2}(r) \right) + h(r_h, r_i, r_\infty). \tag{A.9} \]
where \( h(r_h, r_i, r_\infty) \) is a finite function for all values of its parameters. As we see from the structure of the zeros of \( g(r) \), the integral above contains a pole at \( r = r_i \). In order to approximate the integral (A.9) we may expand \( g(r) \) to second order around \( r_i \)
\[ g(r) = \alpha (\tilde{r}_i - r_i)(r - r_i) + \frac{\alpha}{2} (r - r_i) + ... . \tag{A.10} \]
where \( \alpha \) is a positive constant depending on the parameters of the black hole and \( \tilde{r}_i \) is the asymptotic limiting surface. The necessity to go up to second order in the expansion is revealed by the vanishing of the linear term in the late time limit corresponding to \( r_i \to \tilde{r}_i \). Feeding (A.10) into (A.9) and expanding the rest of the integral to zero order we get
\[ t \approx \frac{r_i^{2(d-1)}}{\Pi} \int_{r_i}^{r_h} dr \left[ \alpha (\tilde{r}_i - r_i)(r - r_i) + \frac{\alpha}{2} (r - r_i) \right]^{-1/2} + \text{finite}. \tag{A.11} \]
which can be solved exactly
\[ t \approx -\frac{r_i^{2(d-1)}}{\Pi (\alpha/2)^{1/2}} \log(r_i - \tilde{r}_i) + \text{finite}. \tag{A.12} \]
Inverting this expression we get the desired result, i.e. the exponentially fast saturation of maximal slices in the black hole interior
\[ r_i - \tilde{r}_i \approx b e^{-t/a}, \tag{A.13} \]
where \( a \) and \( b \) approach constant values in the late time limit.
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