On the nature of the plasma equilibrium

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We calculate the energy of a homogeneous one component plasma and find that the energy is lower for correlated motions of the particles as compared to uncorrelated motion. Our starting point is the conserved approximately relativistic (Darwin) energy for a system of electromagnetically interacting particles that arises from the neglect of radiation. For the idealized model of a homogeneous one component plasma the energy only depends on the particle canonical momenta and the vector potential. The vector potential is then calculated in terms of the canonical momenta using recent theoretical advances and the plasma Hamiltonian is obtained. The result can be understood either as due to the energy lowering caused by the attraction of parallel currents or, alternatively, as due to the inductive inertia associated with the flow of net current.

Our theoretical understanding of matter is largely based on the Coulomb interaction between charged particles. For small systems one can usually assume that the effects of the magnetic corrections are secondary but for larger systems this it not the case. Including magnetic interaction into the theories, however, has met with considerable difficulties, from the 1939 "magnetische Katastrophen" of Welker [1] to the 1999 rigorous proof of the instability of matter with magnetic interaction by Griesemer and Tix [2]. Their result is valid whether the magnetic interaction is mediated by the (Darwin -) Breit potential or via a quantized radiation field. In both these cases it is the attraction of parallel currents that causes the problem. This attraction, which is so fundamental that it is used in the definition of the ampere, the unit of electric current in the SI-system, indicates a long range energy lowering due to correlation of currents that seems to diverge in many systems.

Laboratory and astrophysical plasmas are observed to harbor intense currents and magnetic fields [3], but in plasma physics it is usually assumed that this is due to non-equilibrium, and that the equilibrium plasma is described by the traditional Maxwell-Boltzmann distribution (see e.g. Burma [4]). This is clearly at odds with the above findings of an instability of the energy minimum ground state to parallel current generation. Alastuey and Appel [5] claim that inclusion of the quantized radiation field removes the instability problem, in direct contradiction with the findings of Griesemer and Tix [2]. Most plasma physicists do not seem to be aware of the problem even if there certainly has been a fair amount of interest in energy minimizing states and self organization, see e.g. Wotjer [6], Taylor [7].

Here we will show that within a simple standard model, based on classical electrodynamics and relativistic Hamiltonian mechanics, the energy of a plasma is considerably reduced when the canonical momenta are correlated and thus that conclusions drawn from the traditional non-relativistic Maxwell-Boltzmann distribution of non-interacting particles, or particles interacting only via a Debye screened Coulomb potential, cannot be trusted. The model, which neglects radiation, gives a quantitative estimate of the energy reduction, but does not lead to any un-physical divergence.

Let us start from the following expression for the energy of a system of classical charged particles and electromagnetic fields,

\[ E = \sum_{a=1}^{N} \frac{m_a c^2}{\sqrt{1 - v_a^2/c^2}} + \frac{1}{8\pi} \int (E^2 + B^2) dV. \]  

(1)

If we think of the particles as confined to a finite volume one finds that energy leaves the volume due to electromagnetic radiation (Poynting’s theorem). If one neglects the radiation as a higher order process (\( \sim c^{-3} \)) one finds that,

\[ E_D = \sum_{a=1}^{N} \frac{m_a c^2}{\sqrt{1 - v_a^2/c^2}} + \frac{1}{2} \int (\phi + \frac{1}{c} A \cdot j) dV, \]

(2)

is a conserved energy [8]. Inserting here that,

\[ \varrho = \sum_{a=1}^{N} \epsilon_a \delta \left( r - r_a(t) \right), \quad j = \sum_{a=1}^{N} \epsilon_a v_a(t) \delta \left( r - r_a(t) \right), \]

(3)

one obtains,

\[ E_D = \sum_{a=1}^{N} \left[ E_m (v_a) + \frac{e_a}{2c} \phi (r_a) + \frac{e_a}{2c} v_a \cdot A (r_a) \right], \]

(4)

for the conserved (Darwin) energy, \( E_D \), of a closed non-radiating system of charged particles. Here \( \phi \) and \( A \) are the scalar and vector potentials, in the Coulomb gauge, as determined from the positions and velocities of the particles of the system. These can be found from the approximately relativistic Darwin approach [8–12]. From the Darwin Lagrangian,

\[ L_D = \sum_{a=1}^{N} \left[ L_m (v_a) - \frac{e_a}{2c} \phi (r_a) + \frac{e_a}{2c} v_a \cdot A_a (r_a) \right], \]

(5)
where,  
\[ L_m(v_a) = -m_a c^2 \sqrt{1 - v_a^2/c^2}, \]  
(6)

and,  
\[ \phi_a(r) = \sum_{b(\neq a)} \frac{e_b}{|r - r_b|}, \]  
(7)

\[ A_a(r) = \sum_{b(\neq a)} \frac{e_b [v_b \cdot \hat{r}_b] \hat{r}_b}{2c|r - r_b|}, \]  
(8)

where,  
\[ \hat{r}_b = (r - r_b)/|r - r_b|, \]  
(9)

one also finds the expression,  
\[ \frac{mv_a}{\sqrt{1 - v_a^2/c^2}} = p_a - \frac{e_a}{c} A_a(r_a), \]  
(10)

where,  
\[ p_a = \partial L_D / \partial \dot{v}_a, \] is the canonical momentum.

Since the Darwin approach normally is considered accurate to order \((v/c)^2\) it is sometimes stated that for consistency one must also extend the relativistic particle energy expression, \(E_m(v_a) = m_a c^2 / \sqrt{1 - v_a^2/c^2}\), to this order. There are, however, several reasons not to do this. One is that whereas such an expansion makes the particle energies qualitatively wrong at high speeds there is no reason to assume that the other terms in (4) become qualitatively wrong at high speeds [12]. In fact one can show [13] that small modifications produce potentials that are valid to arbitrary speeds. A second reason is that we are here going for the Hamiltonian formalism and in this formalism speed is not a relevant variable. The problems of finding the Hamiltonian corresponding to the Darwin Lagrangian for large systems is instead governed by a different dimensionless parameter, \(N r_c/R\), where \(N\) is the number of electrons, \(r_c\) is the classical electron radius, and \(R\) a typical length of the system [14–16].

We thus solve (10) for the velocity,  
\[ v_a = \frac{p_a - \frac{e_a}{c} A_a}{\sqrt{(m_a c)^2 + (p_a - \frac{e_a}{c} A_a)^2}}, \]  
(11)

and insert the result in (4). We note that here it is assumed that the vector potential, \(A_a\), at particle \(a\) excludes the contribution from that particle, as indicated in (8), so that self interaction is avoided. Some algebra then gives the result,  
\[ E_D = \sum_{a=1}^{N} m_a c^2 \left[ 1 + \frac{\left( p_a - \frac{e_a}{c} A_a \right) \cdot \left( p_a - \frac{e_a}{c} A_a \right)}{(m_a c)^2} \right] \]  
\[ \sqrt{1 + \frac{\left( p_a - \frac{e_a}{c} A_a \right)^2}{(m_a c)^2}} \]  
\[ + \sum_{a=1}^{N} \frac{e_a}{2} \phi_a, \]  
(12)

for the energy of our system of particles. This expression has also been derived directly by Legendre transformation from the Darwin Lagrangian \(L_D\) (Essén [10]).

The energy expression \(E_D\) should be compared to the well known energy for particles in external fields,  
\[ E_A = \sum_{a=1}^{N} m_a c^2 \left[ \frac{1 + \left( \frac{p_a - \frac{e_a}{c} A(r_a)}{m_a c} \right)^2}{(m_a c)^2} + \frac{e_a \phi(r_a)}{c} \right]. \]  
(13)

The statistical mechanics of (the non-relativistic version of) this Hamiltonian predicts zero magnetic response of a system of classical charged particles according to the, so called, Bohr-van Leeuwen theorem [17–19]. The expression (12) which is valid when the field is from the particles themselves is different and the Bohr-van Leeuwen theorem can not be invoked to make analogous predictions about its properties.

In order to find a Hamiltonian from (12) we must express the vector potentials \(A_a\) in terms of the canonical (generalized) momenta \(p_a\). In the Darwin approach the vector potential can be obtained as solution of,  
\[ \nabla^2 A = -\frac{4\pi}{c} j_t, \]  
(14)

where \(j_t\) is the transverse (divergence free) current density [20]. One can also use the ordinary current density (3) and impose zero divergence by a gauge transformation afterwards [21]. In equation (14) the second time derivative in the d’Alembert operator of the wave equation has been skipped since that term is \(\sim c^{-2}\) and use of the Coulomb gauge means that retardation is correctly handled to this order already [22]. In order to get the vector potential in terms of the \(p_a\) one might then insert,  
\[ v_a = \frac{p_a}{m_a} - \frac{e_a}{m_a c} A_a(r_a), \]  
(15)

from (11) neglecting terms of order \(c^{-2}\) and higher, into the expression (3) for the current density. For a homogeneous one component plasma (with mobile particles of mass \(m\) and charge \(e\)) this approach transforms (14) into,  
\[ \left( \nabla^2 - 4\pi \frac{e^2}{mc^2} n(r) \right) A(r) = -\frac{4\pi}{c} j_p(r), \]  
(16)

where,  
\[ n(r) = \sum_{a=1}^{N} \delta(r - r_a), \]  
(17)

and [we are assuming \(j\) instead of \(j_t\) in (14)],  
\[ j_p(r) \equiv \frac{e}{m} \sum_{a=1}^{N} p_a \delta(r - r_a), \]  
(18)
essentially without approximation. Details are given in Essén and Nordmark [21].

We now assume that we, at least for long range average purposes, can replace the discontinuous number density (17) with the smoothed volume average,

\[ \bar{n}(r) = \frac{1}{V} \sum_{a=1}^{N} \delta(r - r_a) dV. \]  

This, together with the assumption of a homogeneous, constant density plasma, \( \bar{n} = \text{constant} \), gives,

\[ \left( \nabla^2 - \frac{1}{\lambda^2} \right) A(r) = -\frac{4\pi}{c} j_p(r). \]  

Here,

\[ \lambda^2 \equiv \frac{1}{4\pi \bar{n} r_c}, \]  

\( r_c \equiv \frac{\sqrt{\bar{n}}}{4\pi} \), and \( j_p \) is given by (18). We assume, as usual, that the homogeneous one component plasma consists of particles moving against a background of smeared out charge of the opposite sign of \( e \), thus ensuring charge neutrality. The relevant solution of Eq. (20) is \[ A(r) = \frac{e}{mc} \sum_{a=1}^{N} p_a \exp(-|r - r_a|/\lambda) / |r - r_a|, \]  

a Yukawa, or exponentially screened, vector potential. If we can make this momentum space vector potential divergence free we can use it in (12) to get the Hamiltonian. We note that it has Coulomb singularities at the particle positions, \( r = r_a \), but that these vanish in the homogenous since there self interactions are assumed excluded. Only the smooth part is thus of interest.

That exponential screening should occur in a Hamiltonian describing magnetic interaction was noted by Bethe and Fröhlich [23] already in 1933. They were not aware of the work of Darwin, however. Neither were Bohm and Pines [24] who, studying collective motion of conduction electrons in 1951 also arrived at such a result, directly from a particles plus field Hamiltonian. The modern history of the exponential screening based on the Darwin approach started in 1980 with Jones and Pytte [15] who derived it in a Fourier transformed formalism, also for a homogeneous one component plasma. Their derivation was part of a debate on the usefulness of the original Darwin Hamiltonian in plasma physics.

Noting that (22) is not divergence free Essén and Nordmark [21] found the gauge transformation to the correct Coulomb gauge divergence free expression,

\[ A^c_p(r) = \frac{e}{mc} \exp(-r/\lambda) \{ g(r/\lambda) p + h(r/\lambda) (p \cdot \hat{r}) \hat{r} \}, \]  

for the vector potential from a particle with momentum \( p \) at the origin. Here,

\[ g(x) \equiv 1 - \frac{\exp(x) - (1 + x)}{x^2}, \]  

and \( h(x) \equiv 2 - 3g(x) \).

We now assume that all particles of the homogeneous one component plasma of constant number density \( \bar{n} \) have the same momenta \( p \) and calculate the vector potential at the origin by superposing contributions of the type (23) all over space. It is convenient to use spherical coordinates assuming that the momentum \( p \) is in the \( z \)-direction. The contributions from the two terms of (23) separately diverge so it is necessary to do the angular integration of the second term first. Because, \( g(x) + h(x)/3 = 2/3 \), their sum then gives us the finite result,

\[ A^c_p(r) = A(p) = \frac{2}{3} \frac{c}{e} p. \]  

Note firstly that without the exponential damping such a vector potential would diverge violently and secondly that the interplay between the screening length \( \lambda \) and the number density \( \bar{n} \) is such that both vanish from the final result of the integration.

An alternative way of finding this result is by using Eq. (20) directly. We must then first change the left hand side to the divergence free electric momentum current density \( j_{pt} \). In the present case the current density is constant, \( j_p(r) = (e/m)p \bar{n} \), so it is not clear what the divergence free version should be. Realizing that this current density is constant only because of volume averaging, one can use a result by Crisp [25] (in his Appendix A) which states that the volume integral of a vector field \( A \) is related to the volume integral of the corresponding transverse field \( A_t \) by,

\[ \int_A A(r) dV = \frac{2}{3} \int A(r) dV. \]  

When dealing with volume averages one should therefore take the transverse part to be, \( j_{pt} = (2/3)j_p = (2/3)(e/m)p \bar{n} \). Realizing that the corresponding, \( A^c_p = A_{pt} \), must then also be constant (20) gives,

\[ \left( 0 - \frac{1}{\lambda^2} \right) A^c_p = -\frac{4\pi}{c} \left( \frac{2}{3} \frac{e}{mc} p \bar{n} \right). \]  

Use of the expression (21) for \( \lambda^2 \) then again gives (25), \( A^c_p = (2/3)(e/c)p \).

Let us return to the energy (12) which, for our case of a homogeneous one component plasma of particles, all with the same momenta \( p \), gives,

\[ E_D(p) = mc^2 \frac{1 + \left( \frac{p - eA(p)}{2mc} \right)^2}{(mc)^2}, \]  

per particle, if we ignore the contribution \( e\phi_p/2 \) from the scalar potential which averages to zero (and is velocity and momentum independent in the Coulomb gauge).
Here we now insert the result (25) and find after simplification, that the Darwin energy per particle is,

\[ E_D(p) = mc^2 \frac{1 + \frac{2p^2}{mc^2}}{\sqrt{1 + \frac{p^2}{mc^2}}} \]  

(29)

One should compare this result to the usual relativistic energy of a free particle,

\[ E_0(p) = mc^2 \sqrt{1 + \frac{p^2}{mc^2}} \]  

(30)

which is obtained from (28) when \( A(p) = 0 \), i.e. for a non-magnetic plasma with uncorrelated particle motions. These two functions are plotted in Fig. 1. The connection between velocity and momentum can be found from Eq. (10). For our case, \( p(1 - 2/3) = mv/\sqrt{1 - v^2/c^2} \), so that \( p = 3mv/\sqrt{1 - v^2/c^2} \). Inductive inertia thus results in an effective mass three times the normal one.

The implications of the above result are far reaching for many aspects of astrophysical, laboratory, and metallic conduction electron plasmas. The latter, which are unable to correlate their momenta at higher temperatures, due to lattice oscillations, become able to do so at lower temperatures. Their energy as function of canonical momentum then shifts to the lower of the two curves in Fig. 1. Inductive inertia thus lowers the energy of the correlated electrons as originally suggested by Frenkel [26], see also Essén [27]. For astrophysical plasmas it is not obvious whether the nuclei or the electrons should constitute the fixed background when making the one component plasma approximation. The energy reduction is much greater if the heavy particles are considered to be mobile. Kulsrud [3] points out that one can explain the longevity of astrophysical currents as due to their large inductive inertia. Our result says that this inertia in fact reduces the energy so that it also provides a mechanism for the generation of these currents.

Recall that the approximations used to get the central result (29) is (i) the neglect of radiation, (ii) the neglect of terms of order \( c^{-2} \), and smaller in going from (11) to (15), and (iii) in the use of the smoothed density (19) instead of the discontinuous exact particle density. These approximations seem considerably less severe than those normally used in plasma physics.

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