Evolution of Mixed Dirac Particles Interacting with an External Magnetic Field

Maxim Dvornikov\textsuperscript{a,} and Jukka Maalampi\textsuperscript{a,}†

\textsuperscript{a}Department of Physics, P.O. Box 35, FIN-00014 University of Jyväskylä, Finland; IZMIRAN, 142190, Troitsk, Moscow region, Russia; Helsinki Institute of Physics, P.O. Box 64, FIN-00014 University of Helsinki, Finland

(Dated: October 7, 2018)

We study in the framework of relativistic quantum mechanics the evolution of a system of two Dirac neutrinos that mix with each other and have non-vanishing magnetic moments. The dynamics of this system in an external magnetic field is determined by solving the Pauli-Dirac equation with a given initial condition. We consider first neutrino spin-flavor oscillations in a constant magnetic field and derive an analytical expression for the transition probability of spin-flavor conversion in the limit of small magnetic interactions. We then investigate ultrarelativistic neutrinos in a transversal magnetic field and derive their wave functions and transition probabilities with no limitation for the size of transition magnetic moments. Although we consider neutrinos, our formalism is straightforwardly applicable to any spin-1/2 particles.

PACS numbers: 14.60.Pq, 14.60.St, 03.65.Pm
Keywords: particle mixing, magnetic field, neutrino spin-flavor oscillations

I. INTRODUCTION

In the neutrino oscillation phenomena observed so far oscillations occur between different neutrino flavors $\nu_\lambda$ ($\lambda = e, \mu, \tau$) so that an active left-handed neutrino goes to another active left-handed neutrino (e.g. $\nu_e^L \leftrightarrow \nu_{\mu}^L$). In some situations another type of oscillations is possible, namely spin-oscillation, where oscillation happens between an active left-handed neutrino and its inert right-handed counterpart (e.g. $\nu_e^L \leftrightarrow \nu_{\mu}^R$). Also a combination of these two oscillation types, so called spin-flavor oscillations, can happen. There oscillations take place between an active left-handed neutrino and an inert right-handed neutrino of different flavor (e.g. $\nu_e^L \leftrightarrow \nu_{\mu}^R$) \cite{1,2}.

In this paper we shall consider spin-flavor oscillations in an external magnetic field. We assume that neutrinos are Dirac particles with non-vanishing magnetic moments. Let us remind that Dirac particles can have both ordinary (diagonal) magnetic moments and transition (non-diagonal) magnetic moments, whereas for Majorana particles only transition magnetic moments are possible \cite{3}. Note that despite the recent claims of the experimental discovery of the Majorana nature of neutrinos \cite{4}, it is still an open question whether neutrinos are Dirac or Majorana particles \cite{5}. We shall use the formalism of relativistic quantum mechanics, that is, we use the Dirac equation as a starting point, which is a straightforwardly applicable to any spin-1/2 particles. One of us (MD) has previously used this formalism for studying neutrino flavor oscillations in vacuum \cite{6} and in an external axial-vector field \cite{7} (neutrino interaction with matter). Neutrino spin-flavor oscillations in electromagnetic fields of various configurations have been studied in Refs. \cite{8}. The present paper is a continuation of these previous works.

Let us consider a system of two Dirac neutrinos with non-vanishing masses and magnetic moments. In general, both the mass matrix and the matrix of magnetic moments are non-diagonal in the flavor basis of neutrino wave functions. When the mass part of the Hamiltonian is diagonalized by a unitary transformation and the original flavor basis is replaced by a new set of wave functions in the mass eigenstate basis, the matrix of magnetic moments is generally not diagonal in the new basis. This means that there will be transition magnetic moments between the mass eigenstates. We will consider magnetic moment matrices of neutrinos in various bases in Sec.\textsuperscript{III}.

In Sec.\textsuperscript{III} we will discuss the situation where the resulting magnetic moment matrix is close to diagonal, i.e. the transition magnetic moments are small compared with the diagonal magnetic moments in the mass eigenstate basis, in which case one can apply the formalism developed in Refs. \cite{6,7}. In Sec.\textsuperscript{IV} we will consider the effects of transition magnetic moments for ultrarelativistic neutrinos in a transversal magnetic field with no limitations on the size of any magnetic moments. In particular, we apply our result for studying the situation, where the transition magnetic moment is large compared with the diagonal ones. We will derive the transition probability for the process like $\nu_\beta^L \rightarrow \nu_\alpha^R$, where $\alpha, \beta$ denote two different flavors in transversal magnetic field. In Sec.\textsuperscript{V} we summarize our results.

II. ELECTRODYNAMICS OF MIXED PARTICLE STATES POSSESSING MAGNETIC MOMENTS

The magnetic moments of Dirac neutrinos are usually non-vanishing in both flavor and mass eigenstates...
Here, L (a vacuum mixing parameter) have dimension of mass.

Let us denote the magnetic moments of the two Dirac neutrinos $\nu_\alpha, \nu_\beta$ as $M_{\alpha\alpha}, M_{\beta\beta}$ and $M_{\alpha\beta}$, where the last one is called as a transition magnetic moment. The Lagrangian of the neutrinos in the presence of an external electromagnetic field $F_{\mu\nu}$ is given by

$$L(\nu_\alpha, \nu_\beta) = \sum_{\lambda=\alpha, \beta} L_0(\nu_\lambda) - (m_{\beta\lambda} \bar{\nu}_\beta \gamma_\mu \nu_\alpha + \text{h.c.})$$

Here, $L_0(\nu_\lambda) = \bar{\nu}_\lambda (i\gamma^\mu \partial_\mu - m_{\lambda\lambda}) \nu_\lambda$ and $\sigma_{\mu\nu} = (i/2)(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$, and the parameters $m_{\lambda\lambda}$ and $m_{\beta\lambda}$ (a vacuum mixing parameter) have dimension of mass.

The Lagrangian (2.1) rewritten in terms of the fields $\psi_\alpha$ through a unitary transformation

$$\nu_\alpha (r, 0) = 0, \quad \nu_\beta (r, 0) = \xi (r),$$

where $\xi (r)$ is a function to be specified. If we identify $\nu_\alpha$ as $\nu_\alpha$ or $\nu_\gamma$ and $\nu_\beta$ as $\nu_\mu$, for instance, this initial condition might correspond a situation where the source of neutrinos consists of pions and kaons which decay into muon neutrinos.

In order to eliminate the vacuum mixing term in Eq. (2.1) we introduce a new basis of the wave functions, the mass eigenstate basis $\psi_a$, $a = 1, 2$, related to the original flavor basis $\nu_\lambda$ through a unitary transformation

$$\nu_\lambda = \sum_{a=1,2} U_{\lambda a} \psi_a,$$

where the matrix $U = (U_{\lambda a})$ is parameterized in terms of a mixing angle $\theta$ in the usual manner:

$$U = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ (2.4)

The Lagrangian (2.1) rewritten in terms of the fields $\psi_a$ takes the form

$$L(\psi_1, \psi_2) = \sum_{a=1,2} L_0(\psi_a)$$

where $L_0(\psi_a) = \bar{\psi}_a (i\gamma^\mu \partial_\mu - m_a) \psi_a$ is the Lagrangian for the free neutrino $\psi_a$ with the mass $m_a$ and

$$\mu_{ab} = \sum_{\lambda\lambda'} U_{a\lambda}^{-1} M_{\lambda\lambda'} U_{b\lambda'},$$ (2.6)

is the magnetic moment matrix presented in the mass eigenstates basis. Using Eqs. (2.2)-(2.4), the initial conditions for the fermions $\psi_a$ become

$$\psi_1 (r, 0) = \sin \xi (r), \quad \psi_2 (r, 0) = \cos \xi (r).$$ (2.7)

Let us assume that the magnetic field is constant, uniform and directed along the z-axis, $B = (0, 0, B)$, and that the electric field vanishes, $E = 0$. In this case we write down the Pauli-Dirac equation for $\psi_a$, resulting from Eq. (2.6), as follows:

$$\ii \dot{\psi}_a = \mathcal{H}_a \psi_a + V \psi_a, \quad a, b = 1, 2, \quad a \neq b,$$ (2.8)

where $\mathcal{H}_a = (\alpha p) + \rho_3 m_a - \mu_a \rho_3 \Sigma_3 B$ is the Hamiltonian for the neutrino $\psi_a$ accounting for the magnetic field, $V = -\mu_3 \Sigma_3 B$ describes the interaction of the transition magnetic moment with the external magnetic field, $\mu_a = \mu_{aa}$, and $\mu = \mu_{ab} = \mu_{ba}$ are elements of the matrix $(\mu_{ab})$. Here we use the usual definitions for the Dirac matrices $\alpha = \gamma^0 \gamma^5, \beta = \gamma^0, \Sigma = \gamma^0 \gamma^5 \gamma$.

It is in order to remark that in Ref. [11] the problem of neutrino with anomalous magnetic moment was considered in the framework of Dirac-Schwinger equation in an external magnetic field (Furry representation), which leads to a wave equation that differs from our equation (2.8). The two approaches give, however, the same energy spectrum [see Eq. (2.9)].

The general solution to Eq. (2.8) can be presented as follows:

$$\psi_a (r, t) = \int \frac{d^3p}{(2\pi)^3 2 E_p} e^{\ii p r} \sum_{\xi = \pm 1} \left[ a^{(\xi)}_a (t) \psi^{(\xi)}_a (E^{(\xi)}_a) \exp (-\ii E^{(\xi)}_a t) \right. + \left. b^{(\xi)}_a (t) \psi^{(\xi)}_a (E^{(\xi)}_a) \exp (+\ii E^{(\xi)}_a t) \right],$$ (2.9)

The basis spinors $\psi^{(\xi)}_a$ and $\psi^{(\xi)}_a$, as well as the energy $E^{(\xi)}_a$, as a function of the particle momentum are given in the Appendix [12]. Our main goal is to determine the coefficients $a^{(\xi)}_a$ and $b^{(\xi)}_a$ consistent with both the initial conditions (2.7) and the evolution equation (2.8). They are in general functions of time.

III. PERTURBATIVE SOLUTION

In this section we investigate the dynamics of the mixed neutrinos system in the case where the transition magnetic moment can be considered as a small perturbation. We assume that the characteristic energy of the particle $\psi_a$ is large compared with the energy $|\mu B|$ associated with the interaction induced by the transition magnetic moment. Imposing the initial condition (2.7), we
now solve Eq. (2.8) taking the term \( V \psi \) in the Hamiltonian \( \mathcal{H}_a \) as a small correction. We express the solution of Eq. (2.8) as an expansion
\[
\psi_a(r, t) = \psi_a^{(0)}(r, t) + \psi_a^{(1)}(r, t) + \ldots ,
\]
(3.1)
where \( \psi_a^{(0)}(r, t) \) is the solution of Eq. (2.8) when \( V \psi = 0 \), that is, the eigenvalue of the Hamiltonian \( \mathcal{H}_a \). The function \( \psi_a^{(1)}(r, t) \) is linear in \( \mu B \), and the ellipses stands for terms of higher order in \( \mu B \).

Let us first search for the zeroth order solution \( \psi_a^{(0)} \). The coefficients \( a_0^{(c)} \) and \( b_0^{(c)} \) defined in Eq. (2.9) are for \( \psi_a^{(0)} \) time independent. Using the orthonormality conditions for the spinors \( u_0^{(c)} \) and \( v_0^{(c)} \) [see Eq. (A.1)] and following the method developed in Refs. [3], we find
\[
\psi_a^{(0)}(r, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i p r} S_a(p, t) \psi_a(p, 0),
\]
(3.2)
where
\[
S_a(p, t) = \sum_{\xi = \pm 1} \left[ \left( u_0^{(c)} \otimes u_0^{(c)\dagger} \right) \exp(-i E_0^{(c)} t) \\
+ \left( v_0^{(c)} \otimes v_0^{(c)\dagger} \right) \exp(+i E_0^{(c)} t) \right],
\]
(3.3)
is an analog of the Pauli-Jordan function for a spin-1/2 particle in an external magnetic field and
\[
\psi_a(p, 0) = \int d^3 r e^{-i p r} \psi_a(r, 0)
\]
is the Fourier transform of the initial wave function \( \psi_a \).

Eqs. (3.2) and (3.3) together with Eqs. (2.8) and \( \mathcal{H}_a \) [or Eq. (2.2)] allow one to describe the evolution of any system of two Dirac fermions in an external magnetic field. These expressions directly follow from the Lorentz invariant Pauli-Dirac equation (2.8), and they are valid for an arbitrary initial condition. In particular, the initial momentum of the particle can be arbitrary, though neutrinos are of course generally ultrarelativistic.

To proceed we have to specify our initial condition, that is, give the function \( \xi(r) \). Following our previous studies [4], we choose the initial condition for \( \nu_3 \) as a plane wave \( \xi(r) = e^{i r \cdot \xi_0} \), with the initial momentum being directed along the \( x \)-axis, i.e. \( \mathbf{k} = (k, 0, 0) \). Given that we have chosen the magnetic field as \( B = (0, 0, B) \), we are thus studying the evolution of neutrino wavefunction in a transverse magnetic field. We also consider neutrinos as ultrarelativistic particles, i.e. \( k \gg m, 1 \), implying \( \xi_0^1 = (1/2)(1, -1, -1, 1) \). It is easy to see that \( \xi_0 \) obeys the condition \( (1/2)(1 - \Sigma_1) \xi_0 = \xi_0 \) and is normalized to one. Hence, the spinor \( \xi(r) \) describes an ultrarelativistic particle propagating along the \( x \)-axis with its spin directed opposite to the \( x \)-axis, i.e. a left-handedly polarized neutrino.

In contrast to the mixing due to a mass matrix, which is always present, the magnetic transition only occurs in the presence of a magnetic field. Let us assume that a particle passes through a region of length \( r_0 \) where there is a nonvanishing magnetic field, i.e. \( B \neq 0 \) if \( 0 < t < t_0 \) and \( B = 0 \) if \( t < 0 \) and \( t > t_0 \) where \( t_0 \sim r_0 \). The particle emission and detection are assumed to take place while the magnetic field is switched off, at \( t_e = 0 - \delta t \) and \( t_d = t_0 + \delta t \), respectively, where \( \delta t \) is some small period of time. Hence, the operator \( \Sigma_1 \) describes the polarization state of the initial and final particle, and one can assume that the system is initially prepared to a state described in terms of the spinor \( \xi_0 \) as we have done above. Of course, the wave function must be continuous in the borderlines of the \( B = 0 \) and \( B \neq 0 \) regions.

One can now determine the field distribution of the particle \( \nu_a \), which was initially absent. With help of Eqs. (2.8), (2.2), (22) and (3.3), and by taking into account the initial condition we have chosen, we obtain the following expression for the wavefunction of the right-polarized state of \( \nu_a \):
\[
\nu_a^{(0)R}(x, t) = \frac{1}{2} \left( 1 + \Sigma_1 \right) \left[ \cos \theta \psi_1^{(0)}(x, t) - \sin \theta \psi_2^{(0)}(x, t) \right]
= i \sin \theta \cos \theta \left[ e^{-i E_0 t} \sin (\mu B t) \right.
- \left. e^{-i E_0 t} \sin (\mu B t) \right] e^{i k_x \xi_0}.
\]
(3.4)
where
\[
\mathcal{E}_{1, 2} = \sqrt{m_1^2 + k^2}, \quad \kappa_0 = (1/2)(1, 1, 1, 1).
\]
(3.5)
We have used here the identities
\[
(u^\pm \otimes u^\pm) \xi_0 = 0,
\]
(3.6)
\[
\frac{1}{2} \left( 1 + \Sigma_1 \right) \left( u^\pm \otimes u^\pm \right) \xi_0 = \pm \frac{1}{2} \kappa_0,
\]
(3.7)
which result from Eq. (A.6). Note that in the ultrarelativistic limit \( m/k \ll 1 \) one can neglect the neutrino mass dependence of the basis spinors \( u^{(c)} \) and \( v^{(c)} \) and this is why the subscripts \( a \) and \( b \) are omitted in Eq. (3.7) (see also Sec. IV and Eq. (A6) for the explicit forms of \( u^{(c)} \) and \( v^{(c)} \)). It should be mentioned that Eq. (3.6) implies that no particles with negative energies (antiparticles) are produced in neutrino interacting with an external magnetic field.

The measurable quantity is the square of the wave function \( \nu_a^{(0)R}(x, t) \), which can be interpreted as the transition probability of the process \( \nu_a \rightarrow \nu_a^R \). From Eq. (3.4) we obtain
\[
P_{\nu_a \rightarrow \nu_a^R}(t) = \left| \nu_a^{(0)R} \right|^2 = \sin^2(2\theta) \left[ \sin^2(\delta \mu B t) \cos^2(\mu B t) \\
+ \sin(\mu_1 B t) \sin(\mu_2 B t) \sin^2[\Phi(k)t] \right],
\]
(3.8)
where
\[
\Phi(k) = \frac{\delta m^2}{4k},
\]
(3.9)
is the phase of vacuum oscillations, \( \delta m^2 = m_1^2 - m_2^2 \),
\( \delta \mu = (\mu_1 - \mu_2)/2 \) and \( \mu = (\mu_1 + \mu_2)/2 \).
In the case of neutrinos having equal magnetic moments, \(\mu_1 = \mu_2 = \mu_0\), Eq. (3.1) leads to the result one would expect. Namely, Eq. (3.8) can be rewritten as \(P = P_P \cdot P_S\), where \(P_P = \sin^2(2\theta)\sin^2[\Phi(k) t]\) is the usual transition probability of flavor oscillation and \(P_S = \sin^2(\mu_0 B t)\) is the probability of the transition between different polarization states within each mass eigenstate. That is, as the magnetic moment interactions are in this case insensitive to flavor, the transitions between flavors are solely due to the mass mixing.

Let us now consider the first order correction \(\psi_{\alpha}^{(1)}(\mathbf{r}, t)\) in Eq. (3.11). Using the same method as in Ref. [7] we obtain

\[
\psi_{\alpha}^{(1)}(\mathbf{r}, t) = -i \int \frac{d^3 \mathbf{P}}{(2\pi)^3} e^{i \mathbf{P} \cdot \mathbf{r}} \sum_{\zeta = \pm 1} \left[ \left( u_{\alpha}^{(1)} \otimes v_{\alpha}^{(1)} \right)^{\dagger} \right] \\
\times \exp \left( -i E_{\alpha}^{(1)} t \right) V G_{\alpha}^{(1)} \times \exp \left( +i E_{\alpha}^{(1)} t \right) V R_{\alpha}^{(1)} \left[ \psi_b(\mathbf{P}, 0) \right],
\]

where

\[
G_{\alpha}^{(1)} = \int_0^t dt' \exp \left( +i E_{\alpha}^{(1)} t' \right) S_0(\mathbf{P}, t'),
\]

\[
R_{\alpha}^{(1)} = \int_0^t dt' \exp \left( -i E_{\alpha}^{(1)} t' \right) S_0(\mathbf{P}, t').
\]

In Eqs. (3.10) and (3.11) \(a \neq b\).

With help of Eqs. (2.3), (2.4), (3.10) and (3.11) the first-order correction to \(\nu_{\alpha}^{(0)}\) gets the form

\[
\nu_{\alpha}^{(1)R}(x, t) = i \cos 2\theta \frac{\mu B}{\Delta} \left( e^{-i \Sigma \sin \frac{\Delta t}{2}} + e^{-i \Sigma \sin \frac{\delta t}{2}} \right) e^{ikx \cdot k_0},
\]

where

\[
\Sigma = \bar{\Sigma} + \bar{\mu} B, \quad \Delta = \delta \epsilon + \delta \mu B, \\
\sigma = \bar{\epsilon} - \mu B, \quad \delta = \delta \epsilon - \delta \mu B.
\]

and

\[
\epsilon = \frac{\epsilon_1 - \epsilon_2}{2}, \quad \bar{\epsilon} = \frac{\epsilon_1 + \epsilon_2}{2}.
\]

It can be seen from Eqs. (3.12) and (3.13) that the additional constraint should be fulfilled for the perturbative approach to be valid, namely \(\delta \epsilon \neq \pm \delta \mu B\). It should be noticed that Eq. (3.12) is obtained in the ultrarelativistic limit, where small terms \(m_a/k \ll 1\) are neglected.

Let us now find out the effect of the first-order correction to the transition probability. Using Eqs. (3.4) and (3.12) we obtain

\[
P_{\alpha}^{(1)R}(t) = P_{\alpha}^{(0)R} P_{\alpha}^{(1)R} + h.c.
\]

\[
= \frac{\mu B}{4[\Phi^2(k) - (\delta \mu B)^2]} \sin 4\theta \\
\times \left\{ \Phi(k) \sin [2\Phi(k) t] \sin(2\delta \mu B t) \\
- 4\delta \mu B \left( \sin^2(\delta \mu B t) \cos^2(\mu B t) + \sin(\mu B t) \sin(\mu B t) \right) \right\}.
\]

The transition probability of spin-flavor oscillations between mass eigenstate neutrinos up to effects linear in the transition magnetic moment is given as a sum of the probabilities given in Eqs. (3.8) and (3.15).

It should be noticed that the method used above allows one to examine particles with arbitrary initial distributions \(\xi(\mathbf{r})\) since Eqs. (3.2) and (3.3), as well as Eqs. (3.10) and (3.11), are not restricted to any specific initial condition. Also, as we mentioned before, the initial momenta of particles is not restricted. This has relevance, of course, only in the case the results are applied to other particles than neutrinos. Nevertheless, our analysis is not totally general as we have assumed the transition magnetic moment small. It turns out, actually, that one can circumvent this restriction in the case of ultrarelativistic particles. That is, in the case of neutrinos one can solve, as we will do in the next Section, the Pauli-Dirac equation analytically for an arbitrary magnetic moment matrix.

**IV. EVOLUTION OF ULTRARELATIVISTIC PARTICLES WITH ARBITRARY MAGNETIC MOMENTS MATRIX**

In this section we study the influence of the transition magnetic \(\mu\) on the evolution of the mixed neutrino system without assuming \(\mu\) to be small.

Using the orthonormality conditions of the basis spinors [see Eq. (A4)] and the fact that \(\pm E_{\alpha}^{(1)}\) are the eigenvalues of the Hamiltonian \(H_a\), \(a = 1, 2\),

\[
H_a u_{\alpha}^{(1)} = +E_{\alpha}^{(1)} u_{\alpha}^{(1)}, \quad H_a v_{\alpha}^{(1)} = -E_{\alpha}^{(1)} v_{\alpha}^{(1)},
\]

one obtains from Eq. (2.9) the following ordinary differential equations for \(a_{\alpha}^{(1)}(t)\) and \(b_{\alpha}^{(1)}(t)\):

\[
\dot{a}_{\alpha}^{(1)} = \exp (+iE_{\alpha}^{(1)} t) u_{\alpha}^{(1)} V \\
\times \sum_{\zeta=\pm 1} [a_{\alpha}^{(1)}] u_{\alpha}^{(1)} \exp (-iE_{\alpha}^{(1)} t) \\
+ b_{\alpha}^{(1)} v_{\alpha}^{(1)} \exp (+iE_{\alpha}^{(1)} t)],
\]

\[
\dot{b}_{\alpha}^{(1)} = \exp (-iE_{\alpha}^{(1)} t) v_{\alpha}^{(1)} [V \\
\times \sum_{\zeta=\pm 1} [a_{\alpha}^{(1)}] u_{\alpha}^{(1)} \exp (-iE_{\alpha}^{(1)} t) \\
+ b_{\alpha}^{(1)} v_{\alpha}^{(1)} \exp (+iE_{\alpha}^{(1)} t)].
\]

Eqs. (4.2) are subject to the initial conditions
\[ a^{(c)}_a(0) = \frac{1}{(2\pi)^{3/2}} \psi_a(p,0), \]
\[ b^{(c)}_a(0) = \frac{1}{(2\pi)^{3/2}} \psi_a(p,0), \quad (4.3) \]
which result from Eq. (2.8). Note that the functions \( a^{(c)}_a \) and \( b^{(c)}_a \) are here time dependent in general, in contrast to Sec. III.

Let us choose the initial wave function of \( \nu_2 \) [see Eq. (2.2)] as \( \xi(r) = e^{i\mathbf{k} \cdot \mathbf{r}} \), with the initial momentum being aligned along the \( x \)-axis, \( \mathbf{k} = (k,0,0) \). The magnetic field is assumed to be \( \mathbf{B} = (0,0,B) \), implying that we study the propagation of neutralinos in the transversal magnetic field. Given that \( \mathbf{k} \bot \mathbf{B} \), we can rewrite the Eq. (4.2) for the function \( a^{(c)}_a \) in the form (the corresponding equation for the function \( b^{(c)}_a \) can be obtained analogously)
\[
\dot{a}^+_a = a^+_a \langle u^+_a | [V|u^+_a] \rangle \exp [i(E^+_a - E^+_0)t]
+ b^+_a \langle u^+_a | [V|v^+_a] \rangle \exp [i(E^+_a + E^+_0)t]. \quad (4.4)
\]
Here we use the explicit form of the basis spinors in the transversal magnetic field given in Eq. (A2). Note that \( \langle u^+_a | V|u^+_a \rangle = 0 \) and \( \langle u^+_a | V|v^+_a \rangle = 0 \).

A further simplification of Eqs. (4.2) is obtained when we study the ultrarelativistic initial wave function, \( k \gg m_{1,2} \), and \( \xi^F_2 = (1/2)(1, -1, -1, 1) \) (see Sec. III), in other words the system is, like in the case we studied in the previous Section, in the state \( \nu_2^F \) initially. With help of the obvious identities \( \langle u^+_a | V|u^+_a \rangle = \mp \mu B \) and \( \langle u^+_a | V|v^+_a \rangle = 0 \), which result from Eq. (A6), one can cast Eq. (4.4) into the form
\[
\dot{a}^+_a = \mp a^+_a \mu B \exp [i(E^+_a - E^+_0)t]. \quad (4.5)
\]
Let us note that the Eq. (4.5) is similar to the evolution equations for a neutrino interacting with a twisting magnetic field, which were examined in Refs. 16. On the basis of this previous study we are able to write down the solutions as
\[
a^+_1(t) = F^+ a^+_1(0) + G^+ a^+_2(0),
\]
\[
a^+_2(t) = F^{+*} a^+_2(0) - G^{+*} a^+_1(0), \quad (4.6)
\]
where
\[
F^+ = \left[ \cos \Omega_+ t - \frac{\omega_+}{2 \Omega_+} \sin \Omega_+ t \right] \exp (i\omega_+ t/2),
\]
\[
G^+ = \pm \frac{i \mu B}{\Omega_+} \sin \Omega_+ t \exp (i\omega_+ t/2), \quad (4.7)
\]
and
\[
\Omega_\pm = \sqrt{(\mu B)^2 + (\omega_\pm)^2}, \quad \omega_\pm = E^\pm_1 - E^\pm_2. \quad (4.8)
\]
The derivation of Eqs. (4.0)-(4.8) from Eqs. (4.5) is presented in Appendix B.

Using Eq. (2.9) and Eqs. (1.2)-(1.8) and the identity \( \langle \psi^{(c)} \otimes \psi^{(c)} \rangle \xi_0 = 0 \) [see Eq. (3.7)] we obtain the wave functions \( \psi_a \), \( a = 1,2 \), as,
\[
\psi_1(x,t) = \exp (-iE^+_1 t) \left( u^+ \otimes u^+ \right)
\times [F^+ \psi_1(x,0) + G^+ \psi_2(x,0)]
+ \exp (-iE^+_1 t) \left( u^- \otimes u^- \right)
\times [F^- \psi_1(x,0) + G^- \psi_2(x,0)],
\]
\[
\psi_2(x,t) = \exp (-iE^+_2 t) \left( u^+ \otimes u^+ \right)
\times [F^+ \psi_2(x,0) - G^+ \psi_1(x,0)]
+ \exp (-iE^+_2 t) \left( u^- \otimes u^- \right)
\times [F^- \psi_2(x,0) - G^- \psi_1(x,0)], \quad (4.9)
\]
which satisfy the chosen initial condition since \( G^\pm(0) = 0 \), \( F^\pm(0) = 1 \) [see Eqs. (4.7) and \( (u^+ \otimes u^+ + (u^- \otimes u^-))\psi_\alpha(x,0) = \psi_\alpha(x,0) \) [see Eq. (A6)]. Note that the subscripts \( a \) and \( b \) are again omitted in the basis spinors \( u^{(c)} \) and \( v^{(c)} \) as we assume ultrarelativistic particles.

With help of Eqs. (2.8), (2.13) and (1.6) [see also Eq. (3.3)] we receive for the right-handedly polarized component of \( \nu_\alpha \) the expression
\[
\nu^R_{\alpha}(x,t) = \frac{1}{2} \left\{ \sin \theta \cos \theta \left[ e^{-iE^+ t} (e^{i\mu B t} F^+ - e^{-i\mu B t} F^-) \right.ight.
+ e^{-iE^+ t} (e^{i\mu B t} F^+ - e^{-i\mu B t} F^-) \]
\[
- e^{-iE^+ t} (e^{i\mu B t} F^+ - e^{-i\mu B t} F^-) \]
\[
+ \cos^2 \theta e^{-iE^+ t} (e^{i\mu B t} F^+ - e^{-i\mu B t} F^-) \]
\[
+ \sin^2 \theta e^{-iE^+ t} (e^{i\mu B t} F^+ - e^{-i\mu B t} F^-) \left. \right) \right\} \exp (i\kappa_0 t). \quad (4.10)
\]
The quantities \( \xi_\alpha \) and \( \kappa_0 \) are defined in Eq. (5.5). We have used here the identity given in Eq. (3.7). Finally, taking into account Eqs. (4.7) and (4.8) it is possible to express the wave function in Eq. (4.10) in the form
\[
\nu^R_{\alpha}(x,t) = \left\{ \sin \theta \cos \theta \left[ \frac{1}{2} \left[ \frac{\omega_+}{\Omega_+} \sin(\Omega t) \exp (i\mu B t) \right.ight.ight.
\]
\[
- \frac{\omega_-}{\Omega_-} \sin(\Omega t) \exp (-i\mu B t) \left. \right] \left. \right] + i \mu B \left[ \frac{\sin(\Omega t)}{\Omega_+} \cos^2 \theta - \frac{\sin(\Omega t)}{\Omega_-} \sin^2 \theta \right] \right\} \exp (i\kappa_0 t). \quad (4.11)
\]
The transition probability for the process \( \nu_\beta^R \rightarrow \nu_\alpha^R \) can be directly obtained as the squared modulus of \( \nu_\beta^R(x,t) \) from Eq. (4.10) or Eq. (4.11), that is \( P_{\nu_\beta^R \rightarrow \nu_\alpha^R}(t) = |\langle \nu_\beta^R(x,t) | \rangle|^2 \). Notice that the probability is a function of time alone with no dependence on spatial coordinates. This is of course obvious as we have taken the initial wave function as a plane wave and the the magnetic field spatially constant.

Let us now apply the general results Eq. (4.10) or Eq. (4.11) to two special cases. We first consider the situation where \( \mu_{1,2} \gg \mu \), i.e. the case when the transition
magnetic moment is small compared with the diagonal ones. Using Eqs. (4.7) and (4.8) we find that in this case \( F^\pm \approx 1 \) and \( \Omega_\pm \approx \Omega_{\pm}/2 \), and Eq. (4.10) takes the form
\[
\nu^R_\alpha(x,t) \approx \begin{cases} 
\sin \theta \cos \theta \left[ e^{-i\xi_\mu t} \sin \mu_1 Bt - e^{-i\xi_\mu B} \sin \mu_2 Bt \right] \\
\cos 2\theta \left( e^{-i\xi_\mu B} \frac{\Delta t}{\Delta} + e^{-i\xi_\mu B} \frac{\Delta t}{\delta} \right) \end{cases} \times e^{i\kappa_0} \kappa_0.
\]
(4.12)

In the previous Section we studied this case perturbatively, and one can easily check that results obtained there [see Eqs. (3.3) and (3.12)] coincide with (4.12).

As another application of our general result we will study the situation, where the transition magnetic moments are much larger than the diagonal ones, that is \( \mu \gg \mu_{1,2} \). In this case Eqs. (4.7) gives \( F^+ \approx F^- \) and \( G^+ \approx -G^- \), and we receive from Eq. (4.10) or Eq. (4.11) for the wave function \( \nu^R_\alpha \) the expression
\[
\nu^R_\alpha(x,t) \approx i \exp(-i\xi_\mu t + ikx) \\
\times \cos(2\theta) \frac{\mu B}{\Omega} \sin(\Omega t) \kappa_0.
\]
(4.13)

where
\[
\Omega = \sqrt{(\mu B)^2 + \Phi^2(k)}.
\]
(4.14)

The transition probability for the process \( \nu^L_\beta \to \nu^R_\alpha \) is then given by
\[
P_{\nu^L_\beta \to \nu^R_\alpha}(t) = \cos^2(2\theta) \left( \frac{\mu B}{\Omega} \right)^2 \sin^2(\Omega t).
\]
(4.15)

The behavior of the system in this case is schematically illustrated in Fig. 1. It should be noticed that the analog of Eq. (4.15) was obtained in Ref. 2 where the authors studied the resonant spin-flavor precession of Dirac and Majorana neutrinos in matter under the influence of an external magnetic field in frames of the quantum mechanical approach.

Let us discuss the applicability of our results to one specific oscillation channel, \( \nu^L_\mu \to \nu^R_\tau \). According to the recent experimental data [see, e.g., Ref. 12] the mixing angle between \( \nu_\mu \) and \( \nu_\tau \) is close to its maximal value of \( \pi/4 \). In the limit of maximal mixing the magnetic moment matrix given in Eq. (2.6) takes the form
\[
(\mu_{ab}) \approx \begin{pmatrix} 2(M_{\tau\mu} + M_{\mu\tau})/2 + M_{\tau\mu} & -(M_{\tau\tau} - M_{\mu\mu})/2 \\
-(M_{\tau\tau} - M_{\mu\mu})/2 & (M_{\tau\tau} + M_{\mu\mu})/2 - M_{\tau\mu} \end{pmatrix}.
\]
(4.16)

Eqs. (3.8) and (3.15) [or Eq. (4.12)] are valid in the case this matrix is close to diagonal, i.e. when \(|(M_{\tau\tau} - M_{\mu\mu})/2| \ll |(M_{\tau\tau} + M_{\mu\mu})/2 \pm M_{\tau\mu}|\). In contrast to the mixing angles, the experimental information about the neutrino magnetic moments is very limited.

**FIG. 1:** The schematic illustration of the system evolution in the case \( \mu \gg \mu_{1,2} \). The horizontal lines of the figure correspond to various neutrino eigenstates at different moments of time \( t = 0 \) and \( t \). The expressions next to arrows correspond to the appropriate factors in the formula (4.18) of the wave function. The arrows from \( \psi_1^L(0) \) to \( \psi_1^L(0) \) and \( \psi_2^L(0) \), for example, indicate the vacuum mixing matrix transformation at \( t = 0 \) and the arrow from \( \psi_1^L(0) \) to \( \psi_2^L(0) \) the evolution of the mass eigenstates with the helicity change. The transitions \( \psi_1^L(0) \to \psi_2^R(t) \) can be described by the formula, \( \psi_{1,2}(t) = A_{l-r} \psi_{1,2}(0) \), where \( A_{l-r} = i(\mu B/\Omega) \sin \Omega t \).

Also theoretically, very little is known about their magnitude. What one knows is that the diagonal magnetic moments \( M_{\lambda\lambda} \) could be very small in the extensions of the Standard Model \( M_{\lambda\lambda} \approx 10^{-19}(m_{\lambda\lambda}/eV) \mu B \) [see, e.g., Refs. 13]. The contributions to the transition magnetic moments, \( M_{\mu\mu} \) in our case, can be much bigger close to the experimental upper limit of \( 10^{-10} \mu B \). One can see that for any conceivable values of the masses of the known neutrinos, \( M_{\mu\mu} \) and \( M_{\tau\tau} \) are orders of magnitude smaller than \( 10^{-10} \mu B \). Our results [Eq. (4.8)] are applicable in this kind of situation.

We can also consider the opposite situation of \( M_{\lambda\lambda} \gg M_{\mu\mu} \). It is implemented, e.g., in the minimally extended standard model (see Refs. 13) where the magnetic moments matrix is diagonal in the flavor basis, i.e. \( M_{\tau\mu} \) is negligible. If it is the case, one observes that \( \mu_1 = \mu_2 \) in Eq. (4.16). Thus \( \omega_+ = \omega_\omega = \omega_\tau = \Omega = \Omega_\tau \approx \Omega_\lambda = \Omega \) [see Eqs. (3.8) and (A3)]. Eq. (4.11) takes the form
\[
\nu^R_\alpha(x,t) = \frac{\omega}{2\Omega} \sin(\Omega t) \sin(\mu B t) \\
\times \exp(-i\xi_\mu t + ikx) \kappa_0,
\]
(4.17)

where we account for that \( \cos \theta = \sin \theta = 1/\sqrt{2} \). The transition probability can be calculated on the basis of Eq. (4.17),
\[
P_{\nu^L_\beta \to \nu^R_\alpha}(t) = \frac{[\Phi(k)]^2}{\Omega} \sin^2(\Omega t) \sin^2(\mu B t).
\]
(4.18)

Now the magnetic moments are \( \bar{\mu} = (M_{\tau\tau} + M_{\mu\mu})/2 \) and \( \mu = -(M_{\tau\tau} - M_{\mu\mu})/2 \). In Eq. (4.18) the phase of vacuum oscillations, \( \Phi(k) \), is determined in Eq. (3.9) and the parameter \( \Omega \) is introduced in Eq. (4.14).

At the end of this section it should be noticed that the results for the description of spin-flavor oscillations of relativistic Dirac neutrinos are consistent with the standard quantum mechanical approach based on the Schrödinger equation of evolution. The consistency is briefly discussed in Appendix C.
ACKNOWLEDGMENTS

The work has been supported by the Academy of Finland under the contracts No. 108875 and No. 104915. MD is thankful to the Russian Science Support Foun-
and
\[ u^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad u^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \]
\[ v^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad v^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (A6) \]

In Eqs. (A5) and (A6) we assume that the momentum is directed along the x-axis, \( p = (p, 0, 0) \). The subscript \( a \) is omitted in Eq. (A6) since we neglect small terms \( (m_a/p) \ll 1 \) there.

APPENDIX B: SOLUTION TO THE ORDINARY DIFFERENTIAL EQUATIONS FOR THE FUNCTIONS \( a_b^{(c)} \) AND \( b_a^{(c)} \)

Let us study the time evolution of the two-component spinor \( Z^T = (Z_1, Z_2) \) which is governed by the Schrödinger equation of the form,
\[ i\dot{Z} = \mathcal{H}Z, \quad \text{(B1)} \]
where the Hamiltonian has the following form:
\[ \mathcal{H} = \mu \mathcal{B} \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}. \quad \text{(B2)} \]

Here \( \mu, \mathcal{B} \) and \( \omega \) are the real parameters. Eq. (B1) should be supplied with the initial condition \( Z(0) \). To find the solution to Eqs. (B1) and (B2) we introduce the new spinor \( Z' \) by the relation, \( Z = \mathcal{U}Z' \), where the unitary matrix \( \mathcal{U} \) reads
\[ \mathcal{U} = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}. \quad \text{(B3)} \]

Now Eq. (B1) is rewritten in the following way:
\[ i\dot{Z}' = H'Z', \quad \text{(B4)} \]
with the new Hamiltonian \( H' \) which is obtained with help of Eqs. (B1)-(B3),
\[ H' = \mathcal{U}^\dagger \mathcal{H} \mathcal{U} - i\mathcal{U}^\dagger \dot{\mathcal{U}} = \begin{pmatrix} \omega/2 & \mu \mathcal{B} \\ \mu \mathcal{B} & -\omega/2 \end{pmatrix}. \quad \text{(B5)} \]

Note that the initial condition for the spinor \( Z'(0) \) is the same as for \( Z(0), Z'(0) = Z(0) \), due to the special form of the matrix \( \mathcal{U} \) in Eq. (B3).

Supposing that the Hamiltonian \( H' \) in Eqs. (B4) and (B5) does not depend on time we get the solution to Eq. (B4) as
\[ Z'(t) = \exp(-iH't)Z'(0) = (\cos \Omega t - i(\sigma \mathcal{B}) \sin \Omega t)Z'(0), \quad \text{(B6)} \]
where \( \mathbf{n} = (\mu \mathcal{B}, 0, \omega/2)/\Omega \) is the unit vector, \( \Omega = \sqrt{[\mathcal{B}]^2 + [\omega/2]^2} \) and \( \sigma \) are the Pauli matrices. Using Eqs. (B3) and (B6) we arrive to the expressions for the components of \( Z \) written in terms of the initial condition \( Z(0) \):
\[ Z_1(t) = (\cos \Omega t - i\frac{\omega}{2\Omega} \sin \Omega t) e^{i\omega t/2} Z_1(0) - i\frac{\mu \mathcal{B}}{\Omega} \sin(\Omega t) e^{i\omega t/2} Z_2(0), \]
\[ Z_2(t) = (\cos \Omega t + i\frac{\omega}{2\Omega} \sin \Omega t) e^{-i\omega t/2} Z_2(0) - i\frac{\mu \mathcal{B}}{\Omega} \sin(\Omega t) e^{-i\omega t/2} Z_1(0). \quad \text{(B7)} \]

Then we identify the spinor \( Z \) with different functions \( a_b^{(c)} \) and \( b_a^{(c)} \), the parameters \( \mathcal{B} \) and \( \omega \) with the strength of the magnetic field and energy differences respectively (of course, with the proper signs). With help of Eqs. (1.5) and analogous equations for the functions \( b_a^{(c)} \) one gets the list of four cases:

\begin{itemize}
  \item For \( Z^T = (a_1^+, a_2^+) \), \( \mathcal{B} = -B, \omega = \omega_+ \) and \( \Omega = \Omega_+ \);
  \item For \( Z^T = (a_1^-, a_2^-) \), \( \mathcal{B} = B, \omega = \omega_- \) and \( \Omega = \Omega_- \);
  \item For \( Z^T = (b_1^+, b_2^+) \), \( \mathcal{B} = B, \omega = -\omega_+ \) and \( \Omega = \Omega_+ \);
  \item For \( Z^T = (b_1^-, b_2^-) \), \( \mathcal{B} = -B, \omega = -\omega_- \) and \( \Omega = \Omega_- \).
\end{itemize}

Using these formulae together with Eqs. (B7) we readily arrive to Eqs. (1.6)-(1.8). Note that the dynamics of the system (B1) and (B2) is analogous to the quantum mechanical description of neutrino spin-flavor oscillation in a twisting magnetic field studied in Refs. [10].

APPENDIX C: SCHRÖDINGER DESCRIPTION OF SPIN-FLAVOR OSCILLATIONS

Let us study the Schrödinger evolution equation for the two mass eigenstates neutrinos with magnetic moments in an external transversal magnetic field,
\[ i\dot{\Psi} = \mathcal{H}\Psi, \]
\[ \mathcal{H} = \begin{pmatrix} \varepsilon_1 & 0 & -\mu_1 B & -\mu B \\ 0 & \varepsilon_2 & -\mu B & -\mu_2 B \\ -\mu_1 B & -\mu B & \varepsilon_1 & 0 \\ -\mu B & -\mu_2 B & 0 & \varepsilon_2 \end{pmatrix}, \quad \text{(C1)} \]
where mass eigenstates energies \( \varepsilon_{1,2} \) are given in Eq. (4.5). The wave function of neutrinos should be presented in the following form \[10\]: \( \Psi^T = (\psi^L_1, \psi^L_2, \psi^R_1, \psi^R_2) \), where \( \psi^L,R_1,2 \) are one-component objects. The initial condition consistent with Eq. (2.7) is
\[ \Psi^T(0) = (\sin \theta, \cos \theta, 0, 0). \quad \text{(C2)} \]

To study the quantum mechanical evolution of the system we look for a solution of Eq. (C1) of the form
\[ \Psi \sim e^{-i\lambda t}. \]

Solving the corresponding secular equation we get the eigenvalues of the Hamiltonian \( H \) in the form,

\[ \lambda_{1,2}^+ = \Sigma_+ + \Omega_+, \quad \lambda_{1,2}^- = \Sigma_- + \Omega_-, \quad (C3) \]

where \( \Sigma_{\pm} = (E_{1,2}^x + E_{1,2}^y)/2 \) and \( \Omega_{\pm} \) are given in Eq. (3.3). The energy levels in an external magnetic field \( E_{1,2}^z \) are presented in Eq. (A3). The general solution to Eq. (C1) is thus takes the form

\[
\Psi(t) = (\alpha_1 u_1 e^{-i\Omega_+ t} + \alpha_2 u_2 e^{i\Omega_+ t}) e^{-i\Sigma_+ t} + (\beta_1 v_1 e^{-i\Omega_- t} + \beta_2 v_2 e^{i\Omega_- t}) e^{-i\Sigma_- t}, \quad (C4) 
\]

where the basis spinors are given as

\[
u_{1,2} = \frac{1}{2\sqrt{\Omega_+}} \begin{pmatrix} R_+ \\ \mp \mu B/R_+ \\ R_+ \\ \mp \mu B/R_+ \end{pmatrix}, \quad (C5)
\]

the initial condition in Eq. (C2) is satisfied. We choose

\[
\lambda_{1,2}^+ = 1 - \Sigma_1 \psi_{1,2}(x, 0) = \psi_{1,2}(x, 0), \quad (C6)
\]

where the functions \( F^\pm \) and \( G^\pm \) are introduced in Eq. (1.7).

Let us compare these wave functions with the results of our approach, presented in the main text. Initial conditions for the four-component mass eigenstates wave functions are \( \psi_{1,2}(x, 0) = \sin \theta e^{i k x} \xi_0 \) and \( \psi_2(x, 0) = \cos \theta e^{i k x} \xi_0 \) [see Eq. (2.7)], where the spinor \( \xi_0 \) was introduced in Sec. III. It is easy to check that

\[
\psi_{1,2}^L(x, 0) = \frac{1}{2}(1 - \Sigma_1) \psi_{1,2}(x, 0) = \psi_{1,2}(x, 0), \quad \psi_{1,2}^R(x, 0) = \frac{1}{2}(1 + \Sigma_1) \psi_{1,2}(x, 0) = 0. \quad (C7)
\]

These expressions imply that no right-polarized neutrinos exist initially.

With help of the following identity:

\[
\frac{1}{2}(1 - \Sigma_1) (u^\pm \otimes u^\pm) \xi_0 = \frac{1}{2} \xi_0, \quad (C8)
\]

which can be derived from Eq. (A0), as well as using Eqs. (3.7) and (4.9), we obtain the helicity eigenstates of the four-component wave functions \( \psi_a, a = 1, 2, \)

\[
\psi_1^L(t) = \frac{1}{2} \left\{ e^{-iE_1^{\pm} t} F^+ + e^{-iE_2^{\pm} t} F^- \right\} \sin \theta + \left\{ e^{-iE_1^{\pm} t} G^+ + e^{-iE_2^{\pm} t} G^- \right\} \cos \theta e^{i k x} \xi_0, \quad (C9)
\]

\[
\psi_2^L(t) = \frac{1}{2} \left\{ e^{-iE_1^{\pm} t} F^+ + e^{-iE_2^{\pm} t} F^- \right\} \cos \theta - \left\{ e^{-iE_1^{\pm} t} G^+ + e^{-iE_2^{\pm} t} G^- \right\} \sin \theta e^{i k x} \xi_0, \quad (C10)
\]

\[
\psi_1^R(t) = \frac{1}{2} \left\{ e^{-iE_1^{\pm} t} F^+ - e^{-iE_2^{\pm} t} F^- \right\} \sin \theta + \left\{ e^{-iE_1^{\pm} t} G^+ - e^{-iE_2^{\pm} t} G^- \right\} \cos \theta e^{i k x} \xi_0, \quad (C11)
\]

\[
\psi_2^R(t) = \frac{1}{2} \left\{ e^{-iE_1^{\pm} t} F^+ - e^{-iE_2^{\pm} t} F^- \right\} \cos \theta - \left\{ e^{-iE_1^{\pm} t} G^+ - e^{-iE_2^{\pm} t} G^- \right\} \sin \theta e^{i k x} \xi_0, \quad (C12)
\]

where the spinor \( \xi_0 \) is given in Eq. (5.5). The comparison of Eq. (C8) with Eq. (C7) shows that Schrödinger approach gives results analogous to the results obtained in our approach based on Pauli-Dirac equation.
[1] J. Schechter and J. W. F. Valle, Phys. Rev. D 24, 1883 (1981); M. B. Voloshin, M. I. Vysotskii, and L. B. Okun', JETP 64, 446 (1986); Sov. J. Nucl. Phys. 44, 440 (1986); E. Akhmedov, Phys. Lett. B 213, 64 (1988).
[2] C.-S. Lim and W. J. Marciano, Phys. Rev. D 37, 1368 (1988).
[3] See, e.g., M. Fukugita and T. Yanagida, Physics of neutrinos and applications to astrophysics, Springer, Berlin, 2003, p. 593.
[4] H. V. Klapdor-Kleingrothaus, et al., Phys. Lett. B 586, 198 (2004), hep-ph/0404088.
[5] S. R. Elliott and J. Engel, J. Phys. G 30, R183 (2004), hep-ph/0405078.
[6] M. Dvornikov, Phys. Lett. B 610, 262 (2005), hep-ph/0411101; in Proceedings of the IPM school and conference on Lepton and Hadron Physics, Tehran, 2006, ed. by Y. Farzan, eConf C0605151 (2007), hep-ph/0609139; hep-ph/0610047.
[7] M. Dvornikov, Eur. Phys. J. C 47, 437 (2006), hep-ph/0601156.
[8] A. Egorov, A. Lobanov, and A. Studenikin, Phys. Lett. B 491, 137 (2000), hep-ph/9910476; M. S. Dvornikov and A. I. Studenikin, Phys. At. Nucl. 64, 1624 (2001); ibid. 67, 719 (2004); M. Dvornikov, ibid. 70, 342 (2007), hep-ph/0410152.
[9] K. S. Babu and V. S. Mathur, Phys. Lett. B 196, 218 (1987).
[10] W. Grimus and T. Scharnagl, Mod. Phys. Lett. A 8, 1943 (1993).
[11] A. V. Borisov, V. Ch. Zhukovsky, and A. I. Ternov, Sov. Phys. J. 31, 228 (1988).
[12] M. Maltoni, T. Schwetz, M. A. Tortola, and J. W. F. Valle, New J. Phys. 6, 122 (2004), hep-ph/0405172.
[13] K. Fujikawa and R. E. Shrock, Phys. Rev. Lett. 45, 963 (1980); M. Dvornikov and A. Studenikin, Phys. Rev. D 69, 073001 (2004), hep-ph/0305206.
[14] W.-M. Yao et al., J. Phys. G 33, 1 (2006).
[15] I. M. Ternov, V. G. Bagrov, and A. M. Khapaev, JETP 21, 613 (1965).
[16] A. Yu. Smirnov, Phys. Lett B 260, 161 (1991); E. Kh. Akhmedov, P. I. Krastev, and A. Yu. Smirnov, Z. Phys. C 48, 701 (1991).