Poisson-Lie T-Duality and non trivial monodromies

A. Cabrera *, H. Montani† & M. Zuccalli*

* Departamento de Matemáticas, Universidad de La Plata,
Calle 50 esq. 115 (1900) La Plata, Argentina
† Centro Atómico Bariloche and Instituto Balseiro
(8400) S. C. de Bariloche, Río Negro, Argentina

August 20, 2008

Abstract

We describe a general framework for studying duality among different phase spaces which share the same symmetry group H. Solutions corresponding to collective dynamics become dual in the sense that they are generated by the same curve in H. Explicit examples of phase spaces which are dual with respect to a common non trivial coadjoint orbit \( O_{c,0}(\alpha,1) \subset h^* \) are constructed on the cotangent bundles of the factors of a double Lie group \( H = N \rtimes N^* \). In the case \( H = LD \), the loop group of a Drinfeld double Lie group \( D \), a hamiltonian description of Poisson-Lie T-duality for non trivial monodromies and its relation with non trivial coadjoint orbits is obtained.

Contents

1 Introduction \hfill 2
2 Setting for duality and the diagram \hfill 4
3 Phase spaces on Lie groups, central extensions and double Lie groups. \hfill 8
   3.1 Chiral WZNW type phase spaces \hfill 8
      3.1.1 Symplectic equivalence between \( O_{c,0}(\alpha,1) \) and \( O_{c,-\alpha}(0,1) \) \hfill 11
   3.2 Double Lie groups and sigma model phase spaces \hfill 12
      3.2.1 Phase spaces on \( T^*N \) \hfill 13
      3.2.2 Phase spaces on \( T^*N^* \) \hfill 16
Poisson Lie T-Duality

4.1 Dualizable subspaces ................................................. 19
4.2 The PL T-duality scheme ............................................ 20

5 Associated hamiltonian Systems ..................................... 22
5.1 Master WZNW-type model on (T*H,ωc,θ) ........................ 24
5.2 Collective hamiltonian on the factor (N × n*,ωc) .............. 26
  5.2.1 Collective system for μ0,0 : N × n* → h0,0 .................. 26
  5.2.2 Collective system for μ0,α : N × n* → h0,−α ............... 27
5.3 Collective hamiltonian on the factor (N* × n,ωo) .............. 27

6 Loop groups and Poisson Lie T-duality for trivial monodromies 28
6.1 The chiral WZNW phase space ...................................... 30
6.2 A symplectic LD action on LG* × LG .......................... 32
6.3 T-duality diagram for non trivial monodromies ............... 33
6.4 Induced hamiltonian systems on loops groups ............... 33
  6.4.1 The induced lagrangians ................................. 35

7 Conclusions ................................................................ 37

8 Appendix 1 .................................................................. 37

9 Appendix 2 .................................................................. 38

1 Introduction .................................................................. 19

Poisson-Lie T-duality [1] refers to a non-Abelian duality between two 1 + 1 dimensional σ-models describing the motion of a string on targets which are a dual pair of Poisson-Lie groups. The lagrangians of the models are written in terms of the underlying bialgebra structure of the Lie groups, and Poisson-Lie T-duality stems from the self dual character the Drinfeld double. Indeed, classical T-duality transformation relates some dualizable subspaces of the associated phase spaces, mapping solutions reciprocally. Many different approaches have revealed the canonical character of these transformations [1] [2] [4] [5] [6]. There are also WZNW-type models with target on the associated Drinfeld double group D whose dynamics encodes in some way the equivalent sigma models on the factors [1] [7].

In reference [8], most of the known facts of T-duality were embodied in a purely hamiltonian approach, offering a unified description of its classical aspects based on the symplectic geometry of the underlying loop group phase spaces. There, PL T-duality is realized via momentum maps from the cotangent bundles of both the factor Lie groups to a pure central extension coadjoint orbit of the Drinfeld double. These moment maps are associated to the dressing symmetry inherited from the double Lie group structure. Since the hamiltonians corresponding to the σ-models are in collective form, the pre-images of coadjoint orbits through the moment maps allows us to identify the dualizable
spaces that are preserved by the dynamics. On the other hand, these orbits are symplectomorphic to the reduced phase space of one of the chiral sectors of the WZNW model with trivial monodromy, making clear the geometric content of the relation between the cotangent bundle of \( D \) and its factors. These facts are encoded in the commutative diagram

\[
\begin{array}{ccc}
(T^* LG; \omega_o) & \leftarrow & (\Omega D; \omega_{\Omega D}) \\
\downarrow \psi & & \downarrow \Phi \\
(L\mathfrak{g}_T^*; \{\},_{KK}) & \leftrightarrow & (T^* LG^*; \tilde{\omega}_o) \\
\end{array}
\]

where the left and right vertices are the phases spaces of the \( \sigma \)-models, with the canonical Poisson (symplectic) structures, \( L\mathfrak{g}_T^* \) is the dual of the centrally extended Lie algebra of \( LD \) with the Kirillov-Kostant Poisson structure, and \( \Omega D \) is the symplectic manifold of based loops. In particular, \( \mu \) and \( \tilde{\mu} \) are derived as momentum maps associated to hamiltonian actions of the centrally extended loop group \( LD_{c,0} \) on the \( \sigma \)-model phase spaces. The subsets which can related by T-duality are the pre images under \( \mu \) and \( \tilde{\mu} \) of the coadjoint orbit \( O(0,1) \simeq \Omega D \) where both momentum maps intersect and the dualizable subspaces are identified as the orbits of \( \Omega D \), their symplectic foliation. From this setting, we were able to build dual hamiltonian models by taking any suitable hamiltonian function on the loop algebra of the double and lifting it in a collective form \[16\]. For particular choices, the lagrangian formalism is reconstructed obtaining the known dual sigma and WZNW-like model models.

The present work is devoted to the extension of the framework developed in \[8\] in order to include T-duality based on non trivial coadjoint orbits of the form \( O(\alpha,1) \), leading to meaningful changes as a consequence of the non trivial monodromies.

To that end, we describe a general setting for studying duality between different hamiltonian systems with phase spaces \( P \) and \( \tilde{P} \) linked by a common hamiltonian \( H \)-action. This duality is supported on some coadjoint orbit \( O \subset \mathfrak{g}^* \) where the corresponding momentum maps intersect cleanly. Thus, diagram (1) becomes a special case of this situation.

Motivated by the previous T-duality investigations, we work out examples built on the cotangent bundles of a double Lie group \( H = N \rtimes N^* \) and on its factors \((N,N^*)\). As a corner stone of the T-duality scheme we choose a non trivial coadjoint orbit \( O(\alpha,1) \) of the centrally extended group \( H \), with \( \alpha \) in either \( \mathfrak{n} \) or \( \mathfrak{n}^* \). This orbit, in turn, can be related to a coboundary shifted trivial one. We find out some actions of \( H \) on the cotangent bundles of the factors \((N,N^*)\) whose associated equivariant momentum maps serve as the linking arrows with the coadjoint orbits mentioned before. This gives the basic structure underlying T-duality. Hence, collective dynamics completes the approach introducing the appropriate dynamics. However, we shall see that since the symmetric role...
played by the factors in the case described by diagram \([1] \) no longer holds, richer T-dual models emerge. Later, all these is applied in the case \(H = LD\) with \(D = G \ltimes G^*\), driving in a constructive way to some previously studied models.

This work is organized as follows: in Section 2 we describe a general setting for studying duality based on a common hamiltonian \(G\)-action and give some simple examples; in Section 3 we present the general geometric framework linking coadjoint orbits on \(H\) and the phase spaces on its factors \((N, N^*)\). The role played by central extensions is also analyzed, and the symmetry actions leading to the relevant momentum maps are constructed. In Section 4.2 all the kinematic aspects studied before are condensed into a new PL T-duality scheme based on collective dynamics. The construction of the resulting \(H\)-hamiltonian systems and the associated lagrangians is addressed in Section 5.

In Section 6, we illustrate the previous developments for \(H = LD\), with \(D = G \ltimes G^*\), discussing some properties of the models. Finally, some conclusions and comments are condensed in the last Section 7.

2 Setting for duality and the diagram

In this section, we generalize the geometrical framework behind \(T\)-duality which was described in \([8]\).

Let us consider a Poisson manifold \((P, \{\cdot, \cdot\}_P)\) on which a Lie group \(G\) acts by canonical transformations. Suppose further that the action is hamiltonian with \(Ad^*\)-equivariant moment map \(J : P \rightarrow g^*\), where \(g\) denotes the Lie algebra of \(G\). Recall that \(J : P \rightarrow g^*\) is a Poisson map for the (+) Kirillov-Kostant Poisson bracket \(\{\cdot, \cdot\}_{KK}\) on \(g^*\). Following \([16]\), we define

**Definition:** We say that the \(G\)-hamiltonian system \((P, \{\cdot, \cdot\}_P, G, J, H)\), with Hamilton function \(H : P \rightarrow \mathbb{R}\), has dynamics of collective motion type if \(H\) is given by the composition

\[
H = h \circ J
\]

with \(h : g^* \rightarrow \mathbb{R}\).

For this kind of systems, the dynamics is confined in a coadjoint orbit, as it is stated in the next theorem.

**Theorem:** Let \((P, \{\cdot, \cdot\}_P, G, J, h \circ J)\) be a collective \(G\)-hamiltonian system. Then for the initial value \(p_0 \in P\) s.t. \(J(p_0) = J_0 \in g^*\) the solution \(p(t)\) of the Hamilton equations of motion in \(P\) is given by

\[
p(t) = g(t) \cdot p_0
\]

with \(g(t) \in G\) such that

\[
\dot{g}g^{-1} = dh_{\xi(t)}
\]

\[
g(0) = e
\]
where $\xi(t) \in g^*$ is the solution of the Hamilton equations on $g^*$
\[ \dot{\xi}(t) = -ad_{d\xi(t)}^*\xi(t), \]
\[ \xi(0) = J_0. \]

The above result can be summarized in the following diagram
\[
(P, \mathcal{L}, h \circ J) \quad (T^*G, \omega_0, h \circ s)
\]
\[
\downarrow s \quad \downarrow J
\]
\[
(g^*, \mathcal{L}_{KK}, h)
\]
(2)

where $s(\alpha_g) = R_g^*\alpha_g$ for $\alpha_g \in T^*_gG$ is the momentum map associated to the lifting to $T^*G$ of the left action $L_g$ of $G$ on itself and $R_g$ denotes the right translation in $G$. The map $s$ is a Poisson map in relation with the Kirillov-Kostant Poisson bracket on $g^*$ defined as $\{F,H\}_{KK}(\eta) = +\langle \eta, [dF,dH] \rangle$.

**Remark:** (Groupoid actions) In fact, $T^*G \rightrightarrows g^*$ is a symplectic groupoid integrating the Poisson manifold $(g^*, \mathcal{L}_{KK})$ (9) and any such diagram as above with complete Poisson $J$, defines a $T^*G$-groupoid action on $P$. In this case, it coincides with a usual $G$-action. As in the above proposition, solutions in $P$ for collective $H$ are given by this groupoid action $\alpha_g(t) \cdot p_0$ with $\alpha_g(t)$ a solution to the corresponding collective hamiltonian eqs. on $(T^*G, \omega_0, h \circ s)$.

Notice that, if $(\tilde{P}, \mathcal{L}_{\tilde{P}}, G, \tilde{J}, h \circ \tilde{J})$ is another collective hamiltonian $G$-system, we would have the analogous diagram to (2) so we can glue both of them yielding
\[
\downarrow s \quad \downarrow \tilde{J}
\]
\[
(g^*, \mathcal{L}_{KK}, h)
\]
(3)

If both systems share some non empty set in the images of the corresponding momentum maps, then part of its dynamics can be described in a unified way, as stated in the next proposition.

**Proposition:** Let $(P, \mathcal{L}_P, G, h \circ J)$ and $(\tilde{P}, \mathcal{L}_{\tilde{P}}, G, \tilde{J}, h \circ \tilde{J})$ be collective $G$-hamiltonian systems such that $\text{Im}J \cap \text{Im}\tilde{J} \neq \emptyset$. Then, for $J_0 \in \text{Im}J \cap \text{Im}\tilde{J} \subset g^*$ and $p_0 \in J^{-1}(J_0)$, $\tilde{p}_0 \in \tilde{J}^{-1}(J_0)$, the solutions $p(t)$ and $\tilde{p}(t)$ corresponding to the initial values $p_0, \tilde{p}_0$ for the hamiltonian equations on $(P, \mathcal{L}_P, h \circ J)$ and $(\tilde{P}, \mathcal{L}_{\tilde{P}}, h \circ \tilde{J})$, respectively, are given by
\[
p(t) = g(t) \cdot p_0, \]
\[
\tilde{p}(t) = g(t) \cdot \tilde{p}_0
\]
where \( g(t) \in G \) is the curve solution to

\[
\dot{g}g^{-1} = dh_\xi(t) \\
g(0) = e
\] (4)

with \( \xi(t) \in g^* \) the solution of the hamiltonian equations on \( g^* \)

\[
\dot{\xi} = -ad_{g_\xi(t)}^* h(t) \\
\xi(0) = J_0.
\]

Thus both solutions on \( P \) and \( \tilde{P} \), with compatible initial conditions, are obtained from the same curve \( g(t) \) in \( G \). So, if we had one of them (say \( p(t) \)), we can map it through \( J \) to \( g^* \) and then, by solving eq. (4), obtain the other solution \( \tilde{p}(t) = g(t) \cdot \tilde{p}_0 \). Motivated by field theory applications we do the following definition:

**Definition:** Two collective \( G \)-hamiltonian systems \((P, \{\}, P, G, J, h \circ J)\) and \((\tilde{P}, \{\}, \tilde{P}, G, \tilde{J}, h \circ \tilde{J})\) with the same collective function \( h \) and such that there exist a subspace \( O \subset \text{Im} J \cap \text{Im} \tilde{J} \), are said to be dual to each other with respect to \( O \).

When \( \text{Im} J \cap \text{Im} \tilde{J} = \emptyset \), then duality is trivial. In general, \( \text{Im} J \cap \text{Im} \tilde{J} \) is a disjoint union of coadjoint orbits in \( g^* \).

**Remark:** (Transitivity) Note that being dual with respect to a certain fixed subspace \( O \subset g^* \) is a transitive property.

In the following sections, we shall apply this setting to systems coming from classical field theory. Meanwhile, we show some applications in simple examples.

**Example:** (Rigid body dual to a Pendulum) This example comes from [18]. The hamiltonian system \((T^*SE(2), \omega_0, h \circ J)\) describes the motion of a rigid body where \( \omega_0 \) denotes the standard symplectic structure on \( T^*SE(2) \) and \( J \) the moment map corresponding to the lifted left \( SE(2) \) action. The collective function \( h : se(2)^* \rightarrow \mathbb{R} \) is

\[
h(\beta) = \frac{1}{2} \langle \beta, N \beta \rangle
\]

for \( N : se(2)^* \simeq \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) given by the matrix

\[
N = \begin{pmatrix}
0 & 0 & 0 \\
0 & c \left( \frac{1}{I_1} - \frac{1}{I_2} \right) & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where \( c = \left( \frac{1}{I_1} - \frac{1}{I_2} \right)^{-1} \) and \( I_1 < I_2 < I_3 \) are the principal moments of inertia of the underlying rigid body. Now, consider the hamiltonian system
\((T^*S^1, \tilde{\omega}, h \circ J)\) where \(\tilde{\omega}\) denotes \((k_1 k_2)\) times the standard symplectic structure on \(T^*S^1\), with the momentum map \(\tilde{J} : T^*S^1 \to \text{se}(2)^*\) being
\[
\tilde{J}(\theta, p) = (r \sin \theta, r \cos \theta, k_1 k_2 p)
\]
Here
\[
\frac{1}{k_1^2} = \frac{1}{I_1} - \frac{1}{I_3}, \quad \frac{1}{k_2^2} = \frac{1}{I_2} - \frac{1}{I_3}, \quad \frac{1}{k_3^2} = c \left(\frac{1}{I_1} - \frac{1}{I_2}\right)
\]
and \(r = \sqrt{2K}\) denotes the constant value of the function
\[
K(\beta) = \frac{1}{2} \left(\frac{1}{I_1} - \frac{1}{I_3}\right) \beta_1^2 + \frac{1}{2} \left(\frac{1}{I_2} - \frac{1}{I_3}\right) \beta_2^2.
\]
From the dual Hamilton equations on \(T^*S^1\), one easily arrives to the following equation for \(\theta(t)\)
\[
\frac{d^2}{dt^2} \theta = -K \left(\frac{1}{I_1} - \frac{1}{I_2}\right) \sin(2\theta)
\]
which is the equation for the motion of a pendulum with angle \(\theta\). It thus follows from the above considerations that the rigid body hamiltonian system is dual to a pendulum hamiltonian system with respect to \(\mathcal{O} := \text{Im} J \subset \text{se}(2)^*\).

**Example: (Coadjoint orbits)** Generalizing the previous example, suppose that \((P, \{\}, P, G, J, h \circ J)\) is a \(G\)-hamiltonian system such that the coadjoint orbit \(\mathcal{O}_\mu \subset \text{Im} J\). Then the following diagram
\[
\begin{array}{ccc}
(P, \{\}, P, h \circ J) & \xrightarrow{\iota} & (\mathcal{O}_\mu, \omega^K, h \circ i) \\
\downarrow & & \downarrow \\
(\mathfrak{g}^*, \{\}, \mathcal{K}, k) & & \end{array}
\]
says that \((P, \{\}, P, h \circ J)\) is dual to \((\mathcal{O}_\mu, \omega^K, h \circ i)\) with respect to \(\mathcal{O}_\mu \subset \mathfrak{g}^*\), where \(\iota : \mathcal{O}_\mu \hookrightarrow \mathfrak{g}^*\) denotes the inclusion and \(\omega^K\) the Kirillov-Kostant symplectic structure on the coadjoint orbit \(\mathcal{O}_\mu\) corresponding to \(\{\} \mathcal{K}\) in \(\mathfrak{g}^*\).

**Example: (Dual groups)** Suppose that \((H, N, N^*)\) are a triple of Lie groups for which \(H\) is the corresponding perfect Drinfeld double of the Poisson-Lie groups \(N\) and \(N^*\) [13]. Moreover, suppose that there is a one cocycle
\[
C : H \longrightarrow h^*
\]
i.e. a map satisfying
\[
C(l k) = \text{Ad}_l^{H^*} C(k) + C(l), \quad l, k \in H
\]
with the additional property $C(N \subseteq H) \subseteq n^o := n$-annihilator in $h^*$ and $C(N^* \subseteq H) \subseteq n^{o*} := n^*$-annihilator in $h^*$. Then, we can consider the following diagram corresponding to (3)

\[
\begin{array}{ccc}
(T^*N, \omega_o, h \circ J) & (T^*H, \omega_C, h \circ s) & (T^*N^*, \tilde{\omega}_o, h \circ \tilde{J}) \\
\downarrow s & & \downarrow \tilde{J} \\
(h^*, \{,\}_C, h) & & \\
\end{array}
\]

The maps involved are $J(g, \alpha) = C(g) + Ad^h g \alpha$ and $\tilde{J}(\tilde{g}, X) = C(\tilde{g}) + Ad^{h^*} \tilde{g} X$. The structures are: $\omega_C$ the right invariant $C$-modified symplectic structure \[12\], $\{,\}_C$ the affine Poisson structure on $h^*$ defined by $C$. With these, $s(d, \xi) = \xi$ is the (source) Poisson map. Notice that the orbit of the $C$-affine coadjoint action on $h^*$ through $0$ is $C(H) \subset h^*$ (see below) and gives the intersection $\text{Im} J \cap \text{Im} \tilde{J}$. So, $(T^*N, \omega_o, h \circ J)$ and $(T^*N^*, \tilde{\omega}_o, h \circ \tilde{J})$ are duals to each other (and, hence, also to $(T^*H, \omega_C, h \circ s)$) with respect to $O := C(H)$. See also \[8\].

3 Phase spaces on Lie groups, central extensions and double Lie groups.

In this section, we elaborate on the structure proposed in the previous section (the last example above) for describing duality on non-trivial coadjoint orbits. Dual phase spaces are built from the factors $N$ and $N^*$ of a (perfect) double Lie group $H = N \ltimes N^*$, and the corresponding $H$-action on them is constructed by means a certain Lie algebra $h$-cocycle. In contrast with ref. \[8\], the symmetric role that the factors $N$ and $N^*$ played in the duality formulation is broken by considering solutions associated to an element $\alpha \in n^*$.

3.1 Chiral WZNW type phase spaces

Let us begin recalling some results of \[12\] with explicit considerations for coboundary modified cocycles. Let $H$ be a Lie group and $T^*H \sim H \times h^*$ its cotangent bundle trivialized by left translations. We consider on it the canonical 1-form $\theta_o$ and the symplectic form $\omega_o = -d\theta_o$, which on vectors $(v, \xi), (w, \lambda) \in T_{(l,\eta)} (H \times h^*) = T_lH \times h^*$ is

\[
\langle \omega_o, (v, \xi) \otimes (w, \lambda) \rangle_{(l,\eta)} = -\langle \xi, l^{-1}w \rangle + \langle \lambda, l^{-1}v \rangle + \langle \eta, [l^{-1}v, l^{-1}w] \rangle \quad (5)
\]

A new symplectic structure can be obtained by adding a two cocycle $c: h \otimes h \rightarrow \mathbb{R}$, derived from an $Ad^h$-cocycle $C: H \rightarrow h^*$, characterized by $c(Ad_g X, Ad_g Y) = c(X, Y) + \langle C(g^{-1}), [X, Y] \rangle$ for all $X, Y \in h$. Also recall

\[1\] The case studied in \[8\] corresponds to the $\alpha = 0$, i.e., trivial monodromy solutions case.
that $C (lk) = \text{Ad}^H_{\ell^{-1}} C (k) + C (l)$ and $\hat{c} \equiv - dC|_{c_\theta} : \mathfrak{h} \to \mathfrak{h}^*$ produces $c (\mathbf{X}, \mathbf{Y}) \equiv \langle \hat{c} (\mathbf{X}), \mathbf{Y} \rangle$. In the remaining, we fix $C$ and consider its shifting by a coboundary $B_{\theta}$ defined by $\theta \in \mathfrak{h}^*$ as

$$B_{\theta} (l) = \text{Ad}^H_{\ell^{-1}} \theta - \theta$$

defining the following shifted cocycle $C_{\theta}$ and two cocycle $c_{\theta}$,

$$C_{\theta} (l) = C (l) - \text{Ad}^H_{\ell^{-1}} \theta + \theta$$
$$c_{\theta} (\mathbf{X}, \mathbf{Y}) = c (\mathbf{X}, \mathbf{Y}) - \langle \theta, [\mathbf{X}, \mathbf{Y}] \rangle$$

The extended symplectic form $\omega_{c, \theta}$ on $T^* \mathbb{H}$, for $(v, \xi), (w, \lambda) \in T_{(l, \eta)} (\mathbb{H} \times \mathfrak{h}^*) = T_l \mathbb{H} \times \mathfrak{h}^*$ is given by

$$\langle \omega_{c, \theta}, (v, \xi) \otimes (w, \lambda) \rangle_{(l, \eta)} = \langle \omega_\alpha, (v, \xi) \otimes (w, \lambda) \rangle_{(l, \eta)} - c_{\theta} (vl^{-1}, wl^{-1})$$

which is invariant under right translations of $\mathbb{H}$. This symplectic manifold $(\mathbb{H} \times \mathfrak{h}^*, \omega_{c, \theta})$ is related to the phase space of chiral modes of the WZNW models when loops groups are considered.

Now, consider the extended coadjoint action $\hat{\text{Ad}}^H_{\theta^*}$ of the corresponding centrally extended group $\mathbb{H}_{c, \theta}$ on $\mathfrak{h}_{c, \theta}^*$, the dual of the central extended Lie algebra $\mathfrak{h}_{c, \theta}$ of $\mathfrak{h}$ by the cocycle $c_{\theta}$. It is given by

$$\hat{\text{Ad}}^H_{\theta^*} (\xi, b) = (\text{Ad}^H_{\ell^{-1}} \xi + bC_{\theta} (l), b)$$

and the linear Poisson bracket $\{\cdot, \cdot\}_{c, \theta}$ on $\mathfrak{h}_{c, \theta}^*$ by

$$\{ \langle \mathbf{X}, \cdot \rangle, \langle \mathbf{Y}, \cdot \rangle \}_{c, \theta} (\xi, 1) = \langle \xi, [\mathbf{X}, \mathbf{Y}] \rangle - c_{\theta} (\mathbf{X}, \mathbf{Y})$$

for $\mathbf{X}, \mathbf{Y} \in \mathfrak{h}$. Its symplectic leaves are the $\hat{\text{Ad}}^H_{\theta^*}$-coadjoint orbits equipped with the Kirillov-Kostant symplectic structure.

The $\hat{\text{Ad}}^H_{\theta^*}$-equivariant momentum map $\hat{J}_{c, \theta} : \mathbb{H} \times \mathfrak{h}^* \to \mathfrak{h}_{c, \theta}^*$ associated to the induced symplectic $\mathbb{H}_{c, \theta}$-action on $(\mathbb{H} \times \mathfrak{h}^*, \omega_{c, \theta})$ is

$$\hat{J}_{c, \theta} (l, \eta) = (\eta - \text{Ad}^l C_{\theta} (l), 1)$$

**Remark (Affine coadjoint action)** We observe that everything that follows can be carried out by means of the affine coadjoint action of the group $\mathbb{H}$, without extension, on $\mathfrak{h}^*$

$$\hat{\text{Ad}}^H_{\text{Aff}, \ell^{-1}} \xi = \text{Ad}^H_{\ell^{-1}} \xi + C_{\theta} (l)$$

instead of the extended one $\hat{\text{Ad}}^H_{\theta^*}$ of $\mathbb{H}_{c, \theta}$ on $\mathfrak{h}_{c, \theta}^*$. This affine action gives rise to the coadjoint affine orbits $O_{\alpha}^\text{Aff}$ and the corresponding affine Poisson bracket on $\mathfrak{h}^*$, without further reference to central extensions. However, we keep the central extension framework for simplicity.
The phase spaces \((H \times \mathfrak{h}^*, \omega_{c,0})\) play a central role in our \(T\)-duality scheme: most of its features rely on their symmetry properties and the corresponding reduced spaces are the bridge connecting \(T\)-dual systems. So let us work out a couple of related phase spaces which we shall be concerned with.

**S1**- For \(\theta = 0\), the Marsden-Weinstein reduction procedure can be applied to a regular value of the form \((\alpha, 1) \in \mathfrak{h}^*_c,0\) within the phase space \((H \times \mathfrak{h}^*, \omega_{c,0})\). We get that \(\left[\hat{j}^{R}_{c,0}\right]^{-1}(\alpha, 1) \simeq H\) is a presymplectic manifold with the restricted 2-form

\[
\tilde{\omega}_{-\alpha}(v, w) := \omega_{c,0}|_{\left[\hat{j}^{R}_{c,0}\right]^{-1}(\alpha, 1)} (v, w) = c_{-\alpha} \left(l^{-1}v, l^{-1}w\right)
\]

for \(v, w \in T_lH\). Its null distribution is spanned by the infinitesimal generators of the action of the subgroup \(H_\alpha := \ker C_{-\alpha}\), so that the reduced symplectic space is

\[
M^{(\alpha,1)}_{c,0} := \left[\hat{j}^{R}_{c,0}\right]^{-1}(\alpha, 1) / H_{\alpha} \simeq \frac{H}{H_{\alpha}}
\]

where the right action of \(H_\alpha\) on \(H\) is considered. Denoting the fiber bundle \(H^{H/H_\alpha} H/H_\alpha\), then the base \(H/H_\alpha\) is endowed with a symplectic form \(\omega_R\) defined by \(\Pi^{H/H_\alpha}_* \omega_R = \tilde{\omega}_{-\alpha}\). This form is invariant under the residual left action of \(H\) on \(H/H_\alpha\) and has associated momentum map \(\hat{\Phi}_{c,0} : H/H_\alpha \longrightarrow \mathfrak{h}^*_c,0\)

\[
\hat{\Phi}_{c,0} : H/H_\alpha \longrightarrow \mathfrak{h}^*_c,0 / \hat{\Phi}_{c,0} (l \cdot H_\alpha) = (C_{-\alpha} (l) + \alpha, 1)
\]

which is \(\hat{\text{Ad}}_{\theta=0}^{H^*_c}\)-equivariant and gives a local symplectic diffeomorphism from \((H/H_\alpha, \omega_R)\) to the coadjoint orbit \(O_{c,0}|_{(\alpha, 1)}\) equipped with the Kirillov-Kostant symplectic structure \(\omega_{c,0,K}\). Notice that the subgroup \(H_\alpha\) coincides a with the stabilizer subgroup \([H_{c,0}]_{(\alpha, 1)}\) of the point \((0, 1) \in \mathfrak{h}^*_c,-\alpha\).

**S2**- For arbitrary \(\theta = -\alpha \in \mathfrak{h}^*_c\), the reduction procedure can be applied to the regular value \((0, 1) \in \mathfrak{h}^*_c,-\alpha\) of \(\text{Im} \hat{j}^{R}_{c,-\alpha}\). The level set \(\left[\hat{j}^{R}_{c,-\alpha}\right]^{-1}(0, 1) \simeq H\) is again a presymplectic manifold with restricted 2-form

\[
\tilde{\omega}_{-\alpha}(v, w) := \omega_{c,-\alpha}|_{\left[\hat{j}^{R}_{c,-\alpha}\right]^{-1}(0, 1)} (v, w) = c_{-\alpha} \left(l^{-1}v, l^{-1}w\right)
\]

for \(v, w \in T_lH\). The null distribution of \(\tilde{\omega}_{-\alpha}\) is spanned by the infinitesimal generators of the (right) action of the subgroup \(H_\alpha\). Hence, the reduced symplectic space is

\[
M^{(0,1)}_{c,-\alpha} := \left[\hat{j}^{R}_{c,-\alpha}\right]^{-1}(0, 1) / H_{\alpha} \simeq \frac{H}{H_{\alpha}}
\]
again. The symplectic form $\omega_R$ is defined by $\Pi_{\mathcal{H}/H_{\alpha}}^* \omega_R = \tilde{\omega}_{-\alpha}$ as before. Recall that $\omega_R$ is invariant under the residual left action and that the associated $\hat{Ad}_{-\alpha}$-equivariant momentum map $\hat{\Phi}_{c,-\alpha} : \mathcal{H}/H_{\alpha} \to \frak{h}_{c,-\alpha}^*$ now reads

$$\hat{\Phi}_{c,-\alpha} (l \cdot H_{\alpha}) = (C_{-\alpha} (l), 1)$$

(10)

Moreover, $\hat{\Phi}_{c,-\alpha}$ is a local symplectic diffeomorphism onto the coadjoint orbit $\mathcal{O}_{c,-\alpha} (0,1) \subset \frak{h}_{c,-\alpha}^*$ equipped with the Kirillov-Kostant symplectic structure $\omega_{c,-\alpha}$. Notice that the subgroup $H_{\alpha}$ coincides with the stabilizer subgroup $[H_{c,-\alpha}]_{(0,1)}$ of the point $(0,1) \in \frak{h}_{c,-\alpha}^*$.

### 3.1.1 Symplectic equivalence between $\mathcal{O}_{c,0} (\alpha,1)$ and $\mathcal{O}_{c,-\alpha} (0,1)$

The above described reduced spaces can be linked through the *shifting trick* as follows. The orbits $\mathcal{O}_{c,0} (\alpha,1) \subset \frak{h}_{c,0}^*$ and $\mathcal{O}_{c,-\alpha} (0,1) \subset \frak{h}_{c,-\alpha}^*$ are both isomorphic to $\mathcal{H}/H_{\alpha}$ as symplectic manifolds and, moreover,

**Proposition:** The map $\varphi : \frak{h}_{c,\alpha}^* \to \frak{h}_{c,0}^*$

$$\varphi (\eta, 1) = (\eta + \alpha, 1)$$

(11)

is a Poisson diffeomorphism. When restricted to $\varphi : \mathcal{O}_{c,-\alpha} (0,1) \subset \frak{h}_{c,-\alpha}^* \to \mathcal{O}_{c,0} (\alpha,1) \subset \frak{h}_{c,0}^*$ it becomes a symplectic diffeomorphism.

**Proof:** The underlying vector space of $\frak{h}_{c,\alpha}^*$, $\frak{h}_{c,0}^*$ is the same $\frak{h}^* \oplus \mathbb{R}$. Introducing the Legendre transform $L_t : \frak{h}^* \oplus \mathbb{R} \to \frak{h}$ of some function $f \in C^\infty (\frak{h}_{c,0}^*)$ as

$$\langle L_t (\eta, 1), \xi \rangle = \frac{d}{dt} (\eta + t \xi, 1) \bigg|_{t=0}$$

The Poisson structures on the generic $\frak{h}_{c,\beta}^*$ is

$$\{ f, h \}_{\beta,0} (\eta, 1) = \langle \eta - \beta, [L_t (\eta, 1), L_h (\eta, 1)] \rangle + c (L_t (\eta, 1), L_h (\eta, 1))$$

Then, from this expression and having in mind that $L_t (\varphi (\eta, 1)) = L_{t \circ \varphi} (\eta, 1)$, it is immediate to see that

$$\{ f, h \}_{c,0} (\varphi (\eta, 1)) = \{ f \circ \varphi, h \circ \varphi \}_{c,-\alpha} (\eta, 1).$$

Thus the hamiltonian vector fields associated to the the functions $f \in C^\infty (\frak{h}_{c,0}^*)$ and $f \circ \varphi \in C^\infty (\frak{h}_{c,-\alpha}^*)$ are $\varphi$-related

$$\varphi^* [\hat{ad}^c_{\mathcal{L}_{\mathcal{H}} (\eta + \alpha, 1)} (\eta, 1)] = \hat{ad}^c_{\mathcal{L}_{\mathcal{H}} (\eta + \alpha, 1)} (\eta + \alpha, 1)$$

The orbits $\mathcal{O}_{c,-\alpha} (0,1) \subset \frak{h}_{c,-\alpha}^*$ and $\mathcal{O}_{c,0} (\alpha, 1) \subset \frak{h}_{c,0}^*$ are

$$\mathcal{O}_{c,-\alpha} (0,1) = \{ (C_{-\alpha} (l^{-1}), 1) / l \in \mathcal{H} \} = \frac{H}{H_{c,-\alpha} (0,1)}$$

$$\mathcal{O}_{c,0} (\alpha, 1) = \{ (C (l^{-1}) + Ad l_{\alpha}, 1) / l \in \mathcal{H} \} = \frac{H}{H_{c,0} (\alpha,1)}$$

11
where \( H^{\cdot,0}_{(0,1)} = H^c_{(0,1)} = H_{\cdot,0} = \ker C_{-\cdot} \) are the stabilizer subgroups of \((0,1) \in h^c_{-\cdot,0} \) and \((\alpha,1) \in h^c_{\cdot,0} \). The restriction of the above Poisson structures to these orbits endow them with the corresponding Kirillov-Kostant symplectic forms \( \omega^{KK,c}_{-\cdot,0}, \omega^{KK,c}_{\cdot,0} \), and the diffeomorphism \( \varphi : \mathcal{O}_{c,-\cdot,0} (0,1) \to \mathcal{O}_{c,0} (\alpha,1) \), \( \varphi (C_{-\cdot} (l^{-1})) = C_{-\cdot} (l^{-1}) + \alpha \), becomes a symplectic one. In fact, for \( \eta = C_{-\cdot} (l^{-1}) \) and after a direct computation, one recovers

\[
\begin{align*}
\left\langle \omega^{KK}_{\cdot,0}, \tilde{\alpha}^{c,0}_{\cdot,0} \right\rangle &= \left\langle \omega^{KK}_{c,0}, \tilde{\alpha}^{c,0}_{\cdot,0} \right\rangle (\eta + \alpha, 1) \\
&= \left\langle \omega^{KK}_{\cdot,0}, \tilde{\alpha}^{c,-\cdot,0}_{\cdot,0} \right\rangle (\eta, 1) \otimes \tilde{\alpha}^{c,-\cdot,0}_{\cdot,0} (\eta, 1)
\end{align*}
\]

showing that \( \varphi^{*} \omega^{KK}_{c,0} = \omega^{KK}_{\cdot,0} \).

Hence, all the above maps can be resumed in the following diagram

\[
\begin{aligned}
\begin{array}{c}
\mathcal{O}_{c,0} (\alpha,1), \omega^{KK}_{c,0} \\
\downarrow \quad \varphi \\
\mathcal{O}_{c,-\cdot,0} (0,1), \omega^{KK}_{\cdot,0}
\end{array}
\end{aligned}
\]

where all the arrows are symplectic isomorphisms. This result shall be used for describing a WZNW-type model on a double Lie group \( H \) as described in the next section.

### 3.2 Double Lie groups and sigma model phase spaces

We assume now that \( H \) is a Drinfeld double Lie group \([13, 14]\), \( H = N \ltimes N^* \) with tangent Lie bialgebra \( \mathfrak{h} = \mathfrak{n} \oplus \mathfrak{n}^* \). This bialgebra \( \mathfrak{h} \) is naturally equipped with the non-degenerate symmetric \( Ad \)-invariant bilinear form \( (\cdot, \cdot)_{\mathfrak{h}} \) provided by the pairing between \( \mathfrak{n} \) and \( \mathfrak{n}^* \) and which turns them into isotropic subspaces. Let \( \psi \) denote the identification between \( \mathfrak{h} \) and \( \mathfrak{h}^* \) induced by this bilinear form \( (\cdot, \cdot)_{\mathfrak{h}} \). For the sake of brevity, we often omit it from formulas when there is no danger of confusion.

The aim of the following subsections is to construct dual phase spaces from the factors \( N \) and \( N^* \) as described in section\([2]\) T-duality over the trivial orbit \( \mathcal{O}_{c,0} (0,1) \) was considered in ref. \([3]\) in relation to Poisson-Lie T-duality for loop groups and trivial monodromies. Now, we focus our attention into exploring hamiltonian \( H \)-actions on phase spaces \( T^*N \) and \( T^*N^* \) such that they become T-dual over a non-trivial coadjoint orbit \( \mathcal{O}_{c,0} (\alpha,1) \), with \( \alpha \) in \( \mathfrak{n} \) or \( \mathfrak{n}^* \). Notice that, once \( \alpha \) is chosen in one of the factors, the symmetric role played by \( N \) and \( N^* \) in the construction is broken. By the Poisson isomorphism \([14]\), equivalent systems will be constructed on the orbit \( \mathcal{O}_{c,-\cdot,0} (0,1) \), for \( \alpha \) in \( \mathfrak{n} \) or \( \mathfrak{n}^* \).
Hence, we can choose the momentum maps for associated $H$-actions to be valued on $(h^*_{c,0}, \{ \}, \xi_{c,0})$ or $(h^*_{c,-\alpha}, \{ \}, \xi_{c,-\alpha})$. Poisson-Lie T-duality for non-trivial monodromies in the loop group case is described in section 6.

On double Lie groups there exist reciprocal actions between the factors $N$ and $N^*$ named dressing actions [13], [14]. Since every element $l \in H$ can be written as $l = gh$, with $g \in N$ and $h \in N^*$, the product $hg$ in $H$ can be expressed as $\hat{h}g = g^h \hat{h}^\varphi$, with $g^h \in N$ and $\hat{h}^\varphi \in N^*$. The dressing action of $N^*$ on $N$ is then defined as

$$
\text{Dr} : N^* \times N \rightarrow N \quad \mid \quad \text{Dr}(\hat{h}, g) = \Pi_N(\hat{h}g) = g^\hat{h}
$$

where $\Pi_N : H \rightarrow N$ is the projector. For $\xi \in n^*$, the infinitesimal generator of this action at $g \in N$ is

$$
\xi \rightarrow \frac{d}{dt} \text{Dr}(e^{t\xi}, g) \bigg|_{t=0}
$$

such that, for $\eta \in n^*$, we have $\left[ \text{dr}(\xi)_g, \text{dr}(\eta)_g \right] = \text{dr}([\xi, \eta]_n)_g$. It satisfies the relation $\text{Ad}^{H^{-1}}_g \xi = -g^{-1} \text{dr}(\xi)_g + \text{Ad}^*_n \xi$, where $\text{Ad}^{H^{-1}} = \text{Aut}(h)$ is the adjoint action of $H$ on its Lie algebra. Then, using the projector $\Pi_n : h \rightarrow n$, we can write $\text{dr}(\xi)_g = -g \Pi_n \text{Ad}^{H^{-1}}_g \xi$.

Let us now consider the action of $H$ on itself by left translations $L_{ab} \hat{g} = ab \hat{g} h$. Its projection on the one of the factors, $N$ for instance, yields also an action of $H$ on that factor

$$
\Pi_N \left( L_{ab} \hat{g} \right) = \Pi_N \left( ab \hat{g} h \right) = ag^b
$$

The projection on the factor $N^*$ is obtained by the reversed factorization of $H$, namely $N^* \times N$, such that

$$
\Pi_{N^*} \left( L_{ba} \hat{h} \right) = \Pi_{N^*} \left( ba \hat{h} g \right) = b \hat{h} a
$$

In the next subsections, we lift these actions to $T^*N$ and $T^*N^*$ and twist them using a cocycle. The resulting ones play a central role in constructing $T$-dual phase spaces out of these cotangent bundles, turning them in hamiltonian spaces for different central extensions of the group $H$.

3.2.1 Phase spaces on $T^*N$

**Hamiltonian $H_{c,0}$-spaces** We now consider a phase space $T^*N \cong N \times n^*$, trivialized by left translations and equipped with the canonical symplectic form $\omega_\nu$. We shall realize the symmetry described above, as it was introduced in [8].

We promote this symmetry on $T^*N$ to a centrally extended one by means of an $n^*$-valued cocycle $C_{N^*} : N^* \rightarrow n^*$,

$$
\hat{d}^N_{c,0} : H_{c,0} \times (N \times n^*) \rightarrow (N \times n^*)
$$

$$
\hat{d}^N_{c,N^*} \left( ab, (g, \lambda) \right) = \left( ag^\hat{b}, \text{Ad}^{H^*}_{\left( \hat{b} \right)}^{-1} \lambda + C_{N^*} \left( \hat{b}^\varphi \right) \right)
$$

(13)
Aiming to repeat the same construction on a dual phase space, the adjoint $N$-cocycle $C^{N^*} : N^* \rightarrow n^*$ is assumed to be the restriction to the factor $N$ of an $H$-coadjoint cocycle on $C : H \rightarrow \mathfrak{h}^*$, namely $C^{N^*} := C^H|_{N^*}$, in such a way that

$$
C|_N : N \rightarrow n \\
C|_{N^*} : N^* \rightarrow n^*
$$

We shall say that, in this case, $C$ is compatible with the factor decomposition.

**Proposition:** Let $T^*N$ be identified with $N \times n^*$ via left translations and endowed with its canonical symplectic structure. The action $\tilde{d}_0^{N \times n^*}$, defined in eq.(13) from a cocycle $C$ compatible with the factor decomposition, is hamiltonian and the momentum map $\hat{\mu}_{0,0} : (N \times n^*, \omega_0) \rightarrow (\mathfrak{h}_{c,0}, \{ , \}_{c,0})$

$$
\hat{\mu}_{0,0}(g, \lambda) = \tilde{\mathrm{Ad}}_{0,\lambda}^{-1}(\psi(\lambda), 1) = \left(\psi\left(\tilde{\mathrm{Ad}}_{g}^{H} \lambda + C^{N^*}(g)\right), 1\right)
$$

is $\tilde{\mathrm{Ad}}_{g}$-equivariant.

**Proof:** The infinitesimal generator of the action (13) associated to $(X, \xi) \in \mathfrak{h}$ is the vector field

$$(X, \xi)_{N \times n^*}|_{(g, \lambda)} = \left( Xg - d\mathfrak{r}(\xi)_{g}, [Ad_{g}^{*}\xi, \lambda] - \mathfrak{c}(Ad_{g}^{*}\xi) \right)$$

By a straightforward calculation one may see that

$$
i(X, \xi)_{N \times n^*} \omega_0 \big|_{(g, \lambda)} = d(\tilde{\mathrm{Ad}}_{g, \lambda}^{H^*} + C(g), (X, \xi))$$

so that $f(X, \xi)(g, \lambda) \equiv \langle \tilde{\mathrm{Ad}}_{g, \lambda}^{H^*} + C(g), (X, \xi) \rangle$ is the hamiltonian function associated to the vector field $(X, \xi)_{N \times n^*}$. Then, $\mu_{0,0} : N \times n^* \rightarrow \mathfrak{h}^{*}_{c,0}$ defined as

$$
\mu_{0,0}(g, \lambda) = \left( \psi\left(\tilde{\mathrm{Ad}}_{g}^{H^*} \lambda + C(g)\right), 1\right)
$$

is the momentum map associated to the action (13).

Hence, since $(X, \xi)_{N \times n^*}$ is hamiltonian for all $((X, \xi), s) \in \mathfrak{h}_{c,0}$, and $\tilde{d}_0^{N \times n^*}$ leaves the canonical symplectic form invariant. Furthermore, $\mu_{0,0}$ is $\tilde{\mathrm{Ad}}$-equivariant

$$
\mu_{0,0}\left(\tilde{d}_0^{N \times n^*}(a\tilde{b}, (g, \lambda))\right) = \mu_{0,0}\left( a\tilde{b}, \tilde{\mathrm{Ad}}_{(a\tilde{b})}^{H^*-1} \lambda + C^{N^*}\left(\tilde{b}^{\mathfrak{g}}\right)\right)
$$

then

$$
\mu_{0,0}\left(\tilde{d}_0^{N \times n^*}(a\tilde{b}, (g, \lambda))\right) = \tilde{\mathrm{Ad}}_{0(a\tilde{b})}^{H^*} \mu_{0,0}(g, \lambda)
$$

as required. $\blacksquare$

As stated in the introduction, for constructing dualizable subspaces, we need to fix an $\alpha \in n^*$ and study the $\mu_{0,0}$-pre image of the orbit $\mathcal{O}_{c,0}(\alpha, 1) \subset \mathfrak{h}^{*}_{c,0}$.
**Hamiltonian $H_{c,-\alpha}$-spaces**  In view of the equivalence stated in section 3.1.1, we can think of $T^\ast N \simeq N \times n^\ast$ as phase space linked by a momentum map valued on $h_{c,-\alpha}^\ast$. This is attained by considering the coboundary shifted cocycle

$$C_{\alpha}^N (\tilde{h}) = C_{\alpha}^N (\hat{h}) + \text{Ad}_{h} \alpha - \alpha$$

for some $\alpha \in n^\ast$, thus enabling to introduce an $H_{c,-\alpha}$-action on $T^\ast N \simeq N \times n^\ast$ defined as

$$\hat{d}^N_{\alpha} : H_{c,-\alpha} \times (N \times n^\ast) \longrightarrow (N \times n^\ast)$$

$$\hat{d}^N_{\alpha} (\tilde{b}, (g, \eta)) = \left( a g^\delta, \text{Ad}_{h} \eta + C_{\alpha}^N \left( \tilde{b}^\delta \right) \right)$$

for $a \tilde{b} \in H = N \rtimes N^\ast$ and $(g, \eta) \in T^\ast N$. It is worth to remark this action is not a cotangent lift of a transformation on $N$, and that it is meaningful just for $\alpha \in n^\ast$, it does not make sense to for arbitrary $\alpha \in h^\ast$.

The shifted $H$-cyclic

$$C_{-\alpha} (l) = C (l) + \text{Ad}_{h} \alpha - \alpha$$

(16)

does not satisfy property (14), so the above action may be not hamiltonian in $(N \times n^\ast, \omega_{\alpha})$ (compare to example 2) unless some constraint is imposed on $\alpha$.

**Proposition:** Let $\alpha \in n^\ast$ then, provided the condition

$$\Pi_{n^\ast} [X, \alpha] = 0$$

(17)

is fulfilled for all $X \in n$, then the above defined shifted $H$-cocycle $C_{-\alpha}$ becomes compatible with the factor decomposition $H = N \rtimes N^\ast$. Consequently, in this case, the action $\hat{d}^N_{\alpha} : H_{c,-\alpha} \times (N \times n^\ast, \omega_{\alpha}) \longrightarrow (N \times n^\ast, \omega_{\alpha})$

defined by (15) is hamiltonian. The associated $\text{Ad}^H_{\alpha}^\ast$-equivariant momentum map $\mu_{0,\alpha} : (N \times n^\ast, \omega_{\alpha}) \longrightarrow (h_{c,-\alpha}^\ast, \{ , \}_{c,-\alpha})$ is given by

$$\mu_{0,\alpha} (g, \eta) = \text{Ad}_{h}^{H_{\alpha}^\ast} (\psi (\eta), 1) = \left( \psi \left( \text{Ad}_{\alpha}^H \eta + C_{-\alpha} (g) \right), 1 \right)$$

(18)

Alternatively, a hamiltonian $H_{c,-\alpha}$-space on $N \times n^\ast$ can be retrieved by considering a coboundary shifted symplectic form $\omega_{\alpha}$ on $T^\ast N \cong N \times n^\ast$, obtained by adding the coboundary $b_{\alpha} (X, Y) = \langle \alpha, [X, Y] \rangle_{\pi}$ to the canonical one so that, in body coordinates, it is

$$\langle \omega_{\alpha}, (v, \rho) \otimes (w, \xi) \rangle_{\omega_{\alpha}} = -\langle \rho, g^{-1} g v \rangle + \langle \xi, g^{-1} g v \rangle + \langle \eta, [g^{-1} g v, g^{-1} w] \rangle$$

for $(v, \rho), (w, \lambda) \in T_{(g, \alpha)} (N \times n^\ast) = T_{g}^\ast N \times n$.

**Proposition:** The action $\hat{d}^N_{\alpha} : H_{c,-\alpha} \times (N \times n^\ast, \omega_{\alpha}) \longrightarrow (N \times n^\ast, \omega_{\alpha})$

defined by (15) for $\alpha \in n^\ast$, is hamiltonian. It has the associated $\text{Ad}^H_{\alpha}^\ast$-equivariant momentum map $\mu_{\alpha,\alpha} : (N \times n^\ast, \omega_{\alpha}) \longrightarrow (h_{c,-\alpha}^\ast, \{ , \}_{c,-\alpha})$

$$\mu_{\alpha,\alpha} (g, \eta) = \text{Ad}_{h}^{H_{\alpha}^\ast} (\psi (\eta), 1) = \left( \psi \left( \text{Ad}_{\alpha}^H \eta + C_{-\alpha} (g) \right), 1 \right)$$

(19)
Within this formulation, for constructing dualizable subspaces, we must look at the $\mu_{0,\alpha}$-pre-image of the orbit $O_{c,-\alpha}(0,1) \subset h^*_c,-\alpha$.

3.2.2 Phase spaces on $T^*N^*$

**Hamiltonian $H_{c,0}$-spaces** In searching for some $T$-dual partners for the phase spaces on $N \times n^*$ built above, we shall consider the symplectic manifold $(N^* \times n, \tilde{\omega}_o)$ where $\tilde{\omega}_o$ is the canonical 2-form in body coordinates. However, the way is not so direct as in the pure central extension orbit case [8] and a different strategy is needed in order to complete the diagram.

Let us consider $H$ with the opposite factorization, denoted as $H \rightarrow H^\top = N^* \bowtie N$, so that every element is now written as $\tilde{h}g$ with $\tilde{h} \in N^*$ and $g \in N$. From [3,2] we get the action $b^{G^*} : H \times N^* \rightarrow N^*$ defined as $b^G(\tilde{h}a, \tilde{h}) = \tilde{b}\tilde{h}a$

with $a \in N$ and $\tilde{h}, \tilde{b} \in N^*$.

As we shall see below, the search for a Hamiltonian $H$-action on $N^* \times n^*$ will leads us to meet again the restriction (17) on $\alpha$. First, let us consider the arrow $z : (N^* \times n, \tilde{\omega}_o) \rightarrow (h, \{\cdot,\cdot\}_c, 0) \subset (h^*_c, 0, \{\cdot,\cdot\}_c, 0)$, for $\alpha \in n^*$, defined by the diagram

$$
\begin{array}{ccc}
(H \times h^*_c, \omega_c^R) & \xleftarrow{i_{\alpha}} & (N^* \times n, \tilde{\omega}_o) \\
\downarrow s & & \downarrow z \\
(b, \{\cdot,\cdot\}_c, 0) & & (h, \{\cdot,\cdot\}_c, 0)
\end{array}
$$

where

$$
i_{\alpha}(\tilde{h}, X) = (\tilde{h}, C(\tilde{h}) + Ad_{\tilde{h}}^R(X + \alpha))
$$

and the map $s$ being

$$s(l, X) = X$$

In the above diagram, $H \times h^*$ is regarded as the trivialization of $T^*H$ by right translations equipped with the symplectic structure $\omega_c^R = \omega_o^R - c \circ R$

$$
\langle \omega_c^R(v, Y) \otimes (w, Z) \rangle_{(\tilde{h}, X)} = \omega_o^R - c \left(v\tilde{h}^{-1}, w\tilde{h}^{-1}\right)
$$

and $\omega_o^R$ denotes the standard symplectic structure on $T^*H$ in space coordinates. Recall that $h \simeq h^*$ is equipped with a non degenerate symmetric bilinear form $(\cdot, \cdot)_h : h \otimes h \rightarrow h$. Finally, we recall the affine Poisson bracket $\{\cdot,\cdot\}_c^{Aff} : C^\infty(h) \otimes C^\infty(h) \rightarrow C^\infty(h)$

$$
\{\{X, -\}_h, (Y, -)_h\}_c^{Aff}(Z) = -\{[X, Y], Z\}_h - c(X, Y)
$$

with $X, Y, Z \in h$, so that $(h, \{\cdot,\cdot\}_c) \subset (h_{c,0}, \{\cdot,\cdot\}_c, 0)$ via $X \mapsto (X, 1)$. 

16
Remark (Affine coadjoint action) Via the isomorphism \( \mathfrak{h} \simeq \mathfrak{h}^* \) induced by the bilinear form on \( \mathfrak{h} \), we can work on \( (\mathfrak{h}, \{\cdot,\cdot\}_\text{Aff}) \) by considering the affine coadjoint action

\[
\text{Ad}^\mathfrak{h}_{\mathfrak{g}_{\text{Aff}}} Y = \text{Ad}^\mathfrak{h}_Y + C_\theta(l)
\]

on \( \mathfrak{h} \) instead of the full extended coadjoint action \( \hat{\text{Ad}}_{\mathfrak{g}_{\text{Aff}}} \) on \( \mathfrak{h}_{c,\theta}^* \).

It is not hard to see that the map \( i_\alpha : (N^* \times \mathfrak{n}, \tilde{\omega}_o) \longrightarrow (H^* \times \mathfrak{h}, \omega^R) \) is symplectic, i.e., \( i_\alpha^* \omega^R = \tilde{\omega}_o \) for all \( \alpha \in \mathfrak{n}^* \), and that the map \( s : (H^* \times \mathfrak{h}, \omega^R) \longrightarrow (\mathfrak{h}, \{\cdot,\cdot\}_\text{Aff}) \) is a Poisson map.

Thus, the resulting map \( z : (N^* \times \mathfrak{n}, \tilde{\omega}_o) \longrightarrow (\mathfrak{h}, \{\cdot,\cdot\}_\text{Aff}) \) is a suitable candidate to be the generator of a Hamiltonian action of \( H \) on the phase space \( N^* \times \mathfrak{n} \) and, moreover, its image \( z(N^* \times \mathfrak{n}) \) contains the orbit \( \mathcal{O}_{c,0} (\alpha, 1) \subset \mathfrak{h}^* \) as desired. Notice that, if \( \hat{\mu}_{0,\alpha} : (N^* \times \mathfrak{n}, \tilde{\omega}_o) \longrightarrow (\mathfrak{h}_{c,-\alpha}, \{\cdot,\cdot\}_{c,-\alpha}) \) denotes the dual version of the map (11), then the map \( z \) coincides with the composition \( \varphi \circ \hat{\mu}_{0,\alpha} \), where the isomorphism \( \varphi \) was given in (11). However, this candidate to momentum map fails to be a Poisson map for general \( \alpha \). This issue is addressed in the following proposition.

**Proposition:** Let us define the map \( \tilde{\mu}_{0,\alpha}^\varphi : (N^* \times \mathfrak{n}, \tilde{\omega}_o) \longrightarrow (\mathfrak{h}, \{\cdot,\cdot\}_\text{Aff}) \) as

\[
\tilde{\mu}_{0,\alpha}^\varphi (\hat{h}, X) := z = \text{Ad}^\mathfrak{h}_{\hat{h}} X + C_{-\alpha} \left( \hat{h} \right) + \alpha = \text{Ad}^\mathfrak{h}_{\hat{h}} (X + \alpha) + C \left( \hat{h} \right)
\]

(21)

Then, it is Poisson iff condition (17) is satisfied:

\[
\forall X \in \mathfrak{n}. \text{ Here } \tilde{\omega}_o \text{ is the canonical 2-form on } N^* \times \mathfrak{n} \text{ trivialized by left translations.}
\]

**Proof:** Let us sketch the guiding lines of this proof. It must be proved, for \( F, G \in C^\infty (\mathfrak{h}^*) \), that

\[
\{ F \circ \tilde{\mu}_{0,\alpha}^\varphi, G \circ \tilde{\mu}_{0,\alpha}^\varphi \}_{N^* \times \mathfrak{n}} (\hat{h}, X) = \{ F, G \}_{\text{Aff}} (\tilde{\mu}_{0,\alpha}^\varphi (\hat{h}, X))
\]

The Poisson bracket \( \{\cdot,\cdot\}_{N^* \times \mathfrak{n}} \) is the symplectic one corresponding to \( \tilde{\omega}_o \):

\[
\{ F \circ \tilde{\mu}_{0,\alpha}^\varphi, G \circ \tilde{\mu}_{0,\alpha}^\varphi \}_{N^* \times \mathfrak{n}} (\hat{h}, X) = \left( d \left( G \circ \tilde{\mu}_{0,\alpha}^\varphi \right), V_{F \circ \tilde{\mu}_{0,\alpha}^\varphi} \right)
\]

where the Hamiltonian vector field is \( V_f = (\hat{h} \delta f, \text{ad}_{\hat{h}} f X - \hat{h} df) \) and \( df = df + \delta f \in \mathfrak{n}^* \oplus \mathfrak{n} \), for \( f \in C^\infty (N^* \times \mathfrak{n}) \). The explicit expression for the differentials is

\[
\text{d} (F \circ \tilde{\mu}_{0,\alpha}^\varphi) = \hat{h}^{-1} \Pi_n \text{ad}_{\hat{h}}^X \text{Ad}^\mathfrak{h}_{\hat{h}} dF + \hat{h}^{-1} \Pi_n c_{-\alpha} (\text{Ad}^\mathfrak{h}_{\hat{h}} dF)
\]

\[
\delta (F \circ \tilde{\mu}_{0,\alpha}^\varphi) = \Pi_n \cdot \text{Ad}^\mathfrak{h}_{\hat{h}} dF
\]
that leads to
\[
\{ \mathcal{F} \circ \tilde{\mu}^c_{\alpha, \omega} \circ \tilde{\nu}^c_{\alpha, \omega} \big|_{N^* \times n} \big( \tilde{\h}, X \big) = \left\langle \Pi_n \text{ad}^{\h} X \cdot \tilde{\h}, \tilde{\h} \right\rangle + \left\langle \Pi_n \text{ad}^\h \, dG \cdot \tilde{\h}, \tilde{\h} \right\rangle + \left\langle \Pi_n \text{ad}^\h \, dF \cdot \tilde{\h}, \tilde{\h} \right\rangle.
\]
If \( \hat{c} \) is defined as in eq. [3], the restrictions \( \hat{c} \big|_n : n \rightarrow n \) and \( \hat{c} \big|_{n^*} : n^* \rightarrow n^* \) are assumed to be endomorphism of vector spaces. Then we write \( \Pi_n \hat{c} \left( \text{Ad}^\h \, dG \right) = \hat{c} \left( \Pi_n \text{Ad}^\h \, dG \right) \) and \( \Pi_n \hat{c} \left( \text{Ad}^\h \, dF \right) = \hat{c} \left( \Pi_n \text{Ad}^\h \, dF \right) \). Besides these, we use also the relations
\[
\left\langle \Pi_n \text{ad}^{\h} X \cdot \Pi_n \text{Ad}^\h \, dG, \Pi_n \text{ad}^\h \, dF \right\rangle = \left\langle \Pi_n \text{ad}^{\h} X \cdot \Pi_n \text{Ad}^\h \, dG, \Pi_n \text{Ad}^\h \, dF \right\rangle
\]
and the Lie bracket in the double Lie algebra
\[
[(X, \eta), (Z, \xi)]_\h = [(X, Z)_n - \text{ad}^\h \eta Z + \text{ad}^\h \xi X, [\eta, \xi]_n^* - \text{ad}^\h \xi \eta + \text{ad}^\h \eta \xi]
\]
for \( (X, \eta), (Z, \xi) \in \h \). After a tedious but straightforward calculations, one arrives to
\[
\{ \mathcal{F} \circ \tilde{\mu}^c_{\alpha, \omega} \circ \tilde{\nu}^c_{\alpha, \omega} \big|_{N^* \times n} \big( \tilde{\h}, X \big) - \{ \mathcal{F}, \mathcal{G} \}_\text{Aff} \left( \tilde{\mu}^c_{\alpha, \omega} \left( \tilde{\h}, X \right) \right) = \left\langle \Pi_n \text{ad}^\h \, dG, \Pi_n \text{Ad}^\h \, dF, \alpha \right\rangle + \left\langle \Pi_n \text{ad}^\h \, dF, \alpha \right\rangle.
\]
implying that \( \tilde{\mu}^c_{\alpha, \omega} \) is a Poisson map provided the right hand side vanish for arbitrary \( dG, dF \in \h \). After some manipulations, it reduces to
\[
\left\langle \Pi_n \text{ad}^\h \, dG, \Pi_n \text{Ad}^\h \, dF, \alpha \right\rangle = 0
\]
that is equivalent to require that
\[
\Pi_n \left[ \alpha, X \right] = 0
\]
for all \( X \in n \).

Example: (Lu-Weinstein doubles [13]) When \( N^* = K \) a compact simple real Lie group, e.g. \( SU(N) \), then \( H = AN \times K \) where \( N = AN \) and \( N^* = K \) are the subgroups given by the Iwasawa decomposition of \( H = K^C \). For any element \( \alpha \in t \), with \( t \subset n^* \) being the Cartan subalgebra of \( t = \text{Lie}(K) \), condition [17] is satisfied (see Appendix 1).
For $\alpha$ fulfilling condition (17), an $H$-hamiltonian action on $(N^* \times n, \tilde{\omega}_0)$ is obtained as stated below.

**Proposition:** Consider the symplectic manifold $(N^* \times n, \tilde{\omega}_0)$ where $\tilde{\omega}_0$ is the canonical 2-form in body coordinates. The map $\hat{b} : H\gamma_c, 0 \times (N^* \times n) \longrightarrow (N \times n^*)$ defined as

$$\hat{b} \left( h_a, (\tilde{h}, X) \right) = \left( \tilde{h}_a, Ad^{H}_{a_{\tilde{h}}} X + C_{-\alpha} \left( a_{\tilde{h}} \right) \right)$$

is a hamiltonian $H$-action and $\hat{\mu}_{0,\alpha}^{\gamma}$ is the associated $\tilde{Ad}^{H*}_{0}$-equivariant the momentum map.

**Remark** That $\hat{b}$ as defined above is an action on $N^* \times n$ follows from the fact that, when (17) is satisfied, then $\Pi_{n^*}C_{-\alpha} \left( a_{\tilde{h}} \right) = 0$. Thus, the expression $Ad^{H}_{a_{\tilde{h}}} X + C_{-\alpha} \left( a_{\tilde{h}} \right)$ is always $n$-valued as it should be.

Consequently, we have obtained the desired third hamiltonian $H$-space, namely $(N^* \times n, \tilde{\omega}_0) \equiv (T^*N^*, \tilde{\omega}_0)$, completing the diagram

$$\begin{array}{ccc}
T^*N & \xrightarrow{\hat{\Phi}_{c,0}} & T^*N^* \\
\downarrow \hat{\mu}_{0,0} & & \downarrow \hat{\mu}_{0,\alpha} \\
O_{c,0}(\alpha, 1) & \xleftarrow{\hat{\Phi}_{c,0}} & (h_\gamma, \{ \cdot \}_{c,0})
\end{array}$$

**Hamiltonian $H_{c,-\alpha}$-spaces** Again, the equivalence stated in section 3.1.1 enables to seek for a similar system now hanging on $O_{c,-\alpha}(0, 1) \subset h^*_{c,-\alpha}$. In doing so, we consider the map $\tilde{\mu}^\alpha : (N^* \times n, \tilde{\omega}_0) \longrightarrow (h, \{ \cdot \}_{c,-\alpha}^{Aff})$ defined as

$$\tilde{\mu}^\alpha (\tilde{h}, X) = Ad^{H}_{\tilde{h}} X + C_{-\alpha} \left( \tilde{h} \right)$$

which is Poisson for the corresponding shifted affine Poisson structure on $h \simeq h^*$

$$\{ (X, -)_b, (Y, -)_b \}_{c,-\alpha}^{Aff} (Z) = - \langle [X, Y], Z \rangle_b - c_{-\alpha}(X, Y)$$

for $X, Y, Z \in h$, hence it is a momentum map for an associated action of $H_{c,-\alpha}$ on $N^* \times n$.

## 4 Poisson Lie $T$-Duality

We now translate the approach to Poisson Lie $T$-duality developed in [8] to the hamiltonian $H$-spaces studied in previous section.

In order to connect hamiltonian vector fields on the phase spaces $T^*N$ and $T^*N^*$, we consider some coadjoint orbit lying in the intersection of the images of the corresponding equivariant momentum maps associated to the $H$-actions.
As explained in that reference, PL $T$-duality holds on some subspaces of these phases spaces, namely the dualizable subspaces. These subspaces are identified as the symplectic leaves of the presymplectic submanifolds obtained by taking the pre-images, under the corresponding momentum map, of the coadjoint orbit which we are regarding. Compatible dynamics are then implemented by collective hamiltonian functions, and Poisson Lie $T$-duality works on theses leaves mapping the solutions of these underlying collective dynamics.

In the following, we proceed to describe these dualizable subspaces and the PL $T$-duality scheme for the phase spaces described above.

## 4.1 Dualizable subspaces

Let us consider first the phase spaces on $T^*N$. In these cases, the dualizable subspaces are the symplectic leaves in the pre-images of the coadjoint orbit $O_{c,0} (\alpha, 1) \subset \mathfrak{h}_{c,0}^*$ by $\mu_{0,0}$, or in $O_{c,-\alpha} (0, 1) \subset \mathfrak{h}_{c,-\alpha}^*$ by $\mu_{0,\alpha}$, with $\alpha \in \mathfrak{n}^*$ in both cases.

In the first case, recall the corresponding momentum map $\mu_{0,0}$ is $\psi_{Ad_{0:a\hat{b}} (\alpha, 1)}$. The pre-image $\mu_{0,0}^{-1}(O_{c,0} (\alpha, 1))$ consists of those $(g, \lambda) \in N \times \mathfrak{n}^*$ such that $\mu_{0,0} (g, \lambda) = \psi_{Ad_{0:a\hat{b}} (\alpha, 1)}$ for some $a\hat{b} \in H$. Then, using definition (13), we have

$$\mu_{0,0}^{-1}(O_{c,0} (\alpha, 1)) = \{ (g, Ad_{k}^H \alpha + C (k)) \in N \times \mathfrak{n}^* / g\hat{k} \in H \} = \hat{d}_{\mathfrak{n}}^{N \times \mathfrak{n}^*} (g\hat{k}, (\alpha, 1)) / g\hat{k} \in H$$

giving rise to the following statement.

**Proposition:** $\mu_{0,0}^{-1}(O_{c,0} (\alpha, 1)) \equiv O_{N \times \mathfrak{n}^*} (e, \alpha)$, where $O_{N \times \mathfrak{n}^*} (e, \alpha)$ is the orbit through $(e, \alpha) \in N \times \mathfrak{n}^*$ under the action $\hat{d}_{\mathfrak{n}}^{N \times \mathfrak{n}^*} : H_{c,0} \times (N \times \mathfrak{n}^*) \to (N \times \mathfrak{n}^*)$, eq. (13).

Thus, every tangent vector $V \in T_{(g,\lambda)}\mu_{0,0}^{-1}(O_{c,0} (\alpha, 1))$ looks like

$$V|_{(g,\lambda)} = \hat{d}_{\mathfrak{n}}^{N \times \mathfrak{n}^*} (X)_{(g,\lambda)}$$

for some $X \in \mathfrak{h}$, showing that $T\mu_{0,0}^{-1}(O_{c,0} (\alpha, 1)) = \mu_{0,0}^{-1}T O_{c,0} (\alpha, 1)$. Hence, following a theorem by Kazhdan, Kostant and Sternberg [17], we conclude $\mu_{0,0}^{-1}(O_{c,0} (\alpha, 1))$ is a coisotropic submanifold, and the null distribution of the presymplectic form is spanned the infinitesimal generators of $[H_{c,0}(\alpha, 1)]$, the stabilizer subgroup of $(\alpha, 1) \in \mathfrak{h}_{c,0}^*$. We then have the next statement:

**Proposition:** $\mu_{0,0}^{-1}(O_{c,0} (\alpha, 1))$ is a presymplectic submanifold with the closed 2-form given by the restriction of the canonical form $\omega_{c}$,

$$\left\langle \omega_{c}, \hat{d}_{\mathfrak{n}}^{N \times \mathfrak{n}^*} (X) \otimes \hat{d}_{\mathfrak{n}}^{N \times \mathfrak{n}^*} (Y) \right\rangle_{(g,\xi)} = \left\langle \left( C_{\alpha} (a\hat{b}) + \alpha, 1 \right), \hat{ad}_{\mathfrak{k}}^N (Y) \right\rangle_{\mathfrak{h}_{c,0}}$$

20
for $X, Y \in \mathfrak{h}$, and $(g, \xi) = \tilde{d}_{\alpha}^{N \times n^*} (a\tilde{b}, (e, \alpha)) \in \mu_{0,0}^{-1} (\mathcal{O}_{c,0} (\alpha, 1))$. Its null distribution is spanned by the infinitesimal generators of the right action by the stabilizer $[H_{c,0}]_{(\alpha,1)} = \ker C_{\alpha}$ of the point $(\alpha, 1)$

$$r : [H_{c,0}]_{(\alpha,1)} \times \mu_{0,0}^{-1} (\mathcal{O}_{c,0} (\alpha, 1)) \to \mu_{0,0}^{-1} (\mathcal{O}_{c,0} (\alpha, 1))$$

$$r \left( \iota_\alpha, \tilde{d}_{\alpha}^{N \times n^*} (a\tilde{b}, (e, \alpha)) \right) = \tilde{d}_{\alpha}^{N \times n^*} (a\tilde{b}c^{-1}, (e, \alpha))$$

and the null vectors at the point $(g, \xi) \in \mu_{0,0}^{-1} (\mathcal{O}_{c,0} (\alpha, 1))$ are

$$r_{*(g,\xi)} Z_\alpha = -\tilde{d}_{\alpha}^{N \times n^*} (Ad_D^D Z_\alpha)_{(g,\xi)}$$

for all $Z_\alpha \in \text{Lie} (\ker C_{\alpha}) \subset \mathfrak{h}$.

On the other hand, in the equivalent shifted formulation explained in (3.2.1), i.e. when considering the moment map $\mu_{0,0}$ taking values in the shifted $\mathfrak{h}_{\alpha}^{c,-\alpha}$, and when condition (1.7) is satisfied, the level set $\mu_{0,0}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1))$ is

$$\mu_{0,0}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1)) = \left\{ (g, C_{-\alpha} (\tilde{b})) \in N \times n^* / g \in N, \; \tilde{b} \in N^* \right\}$$

It coincides with the $H_{c,-\alpha}$-orbit through $(e, 0)$ in $N \times n^*$

$$\mathcal{O}_{N \times n^*} (e, 0) = \left\{ \tilde{D}_{\alpha}^{N \times n^*} (g\tilde{b}, (e, 0)) \in N \times n^* / g \in N, \; \tilde{b} \in N^* \right\}$$

since $\tilde{D}_{\alpha}^{N \times n^*} (g\tilde{b}, (e, 0)) = (g, C_{-\alpha} (\tilde{b}))$. Tangent vectors $W$ to $\mu_{0,0}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1))$ at the point $(g, \xi)$ are of the form

$$W |_{(g,\xi)} = \tilde{d}_{\alpha}^{N \times n^*} (X)_{(g,\xi)}$$

for $X \in \mathfrak{h}$, and then, as above, $\mu_{0,0}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1))$ is a coisotropic submanifold yielding the analogous result:

**Proposition:** $\mu_{0,0}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1))$ is a presymplectic submanifold with the closed 2-form given by the restriction of the canonical form $\omega_\alpha$,

$$\left\langle \omega_\alpha, \tilde{d}_{\alpha}^{N \times n^*} (X) \otimes \tilde{d}_{\alpha}^{N \times n^*} (Y) \right\rangle_{(g,\xi)} = \left\langle \left( C (a\tilde{b}), 1 \right), \tilde{a}^{h_{c,-\alpha}} Y \right\rangle_{h_{c,-\alpha}}$$

for $X, Y \in \mathfrak{h}$, and $(g, \xi) = \tilde{d}_{\alpha}^{N \times n^*} (a\tilde{b}, (e, 0)) \in \mu_{0,0}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1))$. Its null distribution is spanned by the infinitesimal generators of the right action by the stabilizer $[H_{c,-\alpha}]_{(0,1)} := \ker C_{-\alpha}$ of the point $(0, 1)$

$$r : [H_{c,-\alpha}]_{(0,1)} \times \mu_{0,0}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1)) \to \mu_{0,0}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1))$$

$$r \left( \iota_\alpha, \tilde{d}_{\alpha}^{N \times n^*} (a\tilde{b}, (e, 0)) \right) = \tilde{d}_{\alpha}^{N \times n^*} (a\tilde{b}c^{-1}, (e, 0))$$
and the null vectors at the point \( \tilde{\mathcal{A}}^N \times \mathfrak{n}^* \left( \tilde{a} \hat{b}, (e, 0) \right) \in \mu_{\mathfrak{h}, \mathfrak{n}}^{-1} (\mathcal{O}(0, 1)) \) are

\[
 r_{(g, \xi)} Z_{\alpha} = - \tilde{\mathcal{A}}^N \times \mathfrak{n}^* \left( \text{Ad} C_{\alpha} Z_{\alpha} \right)_{(g, \xi)}
\]

for all \( Z_{\alpha} \in \text{Lie} (\ker C) \subset \mathfrak{h} \).

For the phase spaces on \( T^* \mathfrak{n} \), we now consider the symplectic manifold \((\mathfrak{n} \times \mathfrak{n}, \tilde{\omega}_0)\) which are involved in the current scheme for \(T^*-\text{duality} \). Hence, the dualizable subspaces are contained in the submanifold \((\tilde{\mu}_{\mathfrak{h}, \mathfrak{n}}^{-1} (\mathcal{O}_{c,0} (\alpha, 1)) \).

**Proposition:** Let \( \alpha \in \mathfrak{n}^* \) satisfying (17). Then,

\[
 (\tilde{\mu}_{\mathfrak{h}, \mathfrak{n}}^{-1} (\mathcal{O}_{c,0} (\alpha, 1))) = \left\{ \tilde{\mathcal{B}} \left( \tilde{g} h, (e, 0) \right) \in \mathfrak{n} \times \mathfrak{n}^* \mid \tilde{g} h \in \mathfrak{H} \right\}
\]

where the \( \mathfrak{H} \)-action \( \tilde{\mathcal{B}} \) is given by (22).

As in the model over \( \mathfrak{n} \), the restriction to \((\tilde{\mu}_{\mathfrak{h}, \mathfrak{n}}^{-1} (\mathcal{O}_{c,0} (\alpha, 1))) \) of the symplectic structure is degenerate.

**Proposition:** \((\tilde{\mu}_{\mathfrak{h}, \mathfrak{n}}^{-1} (\mathcal{O}_{c,0} (\alpha, 1))) \) is a presymplectic submanifold with closed two form obtained from the restriction of the canonical form \( \tilde{\omega}_0 \). The null distribution is spanned by the infinitesimal generators of the right action of \( H_{(-1,1)}^{-1} (\mathcal{O}_{c,0} (\alpha, 1)) \)

\[
 r : H_{(-1,1)}^{-1} \times (\tilde{\mu}_{\mathfrak{h}, \mathfrak{n}}^{-1} (\mathcal{O}_{c,0} (\alpha, 1))) \rightarrow (\tilde{\mu}_{\mathfrak{h}, \mathfrak{n}}^{-1} (\mathcal{O}_{c,0} (\alpha, 1)))
\]

\[
r \left( l_{\alpha} \tilde{\mathcal{A}}^N \times \mathfrak{n}^* \left( \tilde{a} \hat{b}, (e, 0) \right) \right) = \tilde{\mathcal{A}}^N \times \mathfrak{n}^* \left( \tilde{a} \hat{b} l_{\alpha}^{-1}, (e, 0) \right).
\]

Analogous characterizations hold in the corresponding shifted formulation for \( \mathfrak{n} \).

### 4.2 The PL T-duality scheme

So far, we have considered independently sigma model like models on the Lie groups \( \mathfrak{n} \), \( \mathfrak{n}^* \) and a WZNW like model on the double \( \mathfrak{H} = \mathfrak{n} \ltimes \mathfrak{n}^* \). When \( \alpha \in \mathfrak{n}^* \) fulfills (17), the hamiltonian H-actions described in the previous sections give rise to the following diagram including all the involved phase spaces
Using the geometrical or kinematical information of this diagram (25), following [8] and section 2, we know that by considering collective dynamics coming from an $h : (\mathfrak{h}_{c,0})^* \to \mathbb{R}$, the resulting hamiltonian systems:

\[
\begin{align*}
\left\{ (N \times n^*, \omega_o, H_{c,0}, \mu_{0,0}, h \circ \mu_{0,0}) , \\
(\mathcal{O}_{c,0} (\alpha, 1), \omega^{KK}, H_{c,0}, \Phi_{c,0}, h \circ \Phi_{c,0}) \\
(N^* \times n, \omega_o, H_{c,0}, \tilde{\mu}_{0,\alpha}, h \circ \tilde{\mu}_{0,\alpha}) \right\}
\end{align*}
\]

become dual to each other with respect to $\mathcal{O}_{c,0} (\alpha, 1)$.

**Remark:** (The equivalent shifted formulation) In view of the equivalence stated in section 3.1.1, the collective hamiltonian systems corresponding to the $\mathfrak{h}_{c,-\alpha}$-valued momentum maps lead to duality with respect to $\mathcal{O}_{c,-\alpha} (0, 1)$.

As stated in 2 solutions lying on the dualizable subspaces $\tilde{\mu}_{0,0}^{-1} (\mathcal{O}_{c,0} (\alpha, 1))$, $(\tilde{\mu}_{0,\alpha})^{-1} (\mathcal{O}_{c,0} (\alpha, 1))$ (equiv., in the shifted formulation, $\mu_{0,\alpha}^{-1} (\mathcal{O}_{c,-\alpha} (0, 1))$ and $(\tilde{\mu}_{\alpha})^{-1} (\mathcal{O}_{c,-\alpha} (0, 1))$) are generated by the same curve $l(t) \in H_{c,0}$ defined by

\[
\frac{dl}{dt} l^{-1} (t) = L_h (\gamma(t))
\]

\[l(0) = e\]

where $\gamma(t) \in \mathcal{O}_{c,0} (\alpha, 1) \subset \mathfrak{h}_{c,0}^*$ is the integral curve of $h$ for the WZNW like model of Section 5.1. The corresponding initial values are related by

$$O_{c,0} (\alpha, 1) \ni \gamma(0) = \mu_{0,0} (g_o, \eta_o) = \tilde{\mu}_{0,\alpha} (\tilde{h}_o, Z_o)$$
where \((g_0, \eta_0) \in N \times n^*\) and \((\tilde{h}_0, Z_0) \in N^* \times n\) stand for the initial conditions for the N and \(N^*\) sigma models of sects. 5.2 and 5.3 respectively. Thus, finally, the dual solutions can be written as:

\[
\begin{align*}
\dot{a} (l(t), (g_0, \eta_0)) & \in N \times n^* \\
[l(t)]_o & \in O_{c,0} (\alpha, 1) \simeq H/H_\alpha \\
\dot{b} (l(t), (\tilde{h}_0, Z_0)) & \in N^* \times n
\end{align*}
\]

Notice that duality transformations between \(T^*N\) and \(T^*N^*\) involve finding the curve \(l(t)\) of (26) and applying the two factorizations \(H = N \bowtie \bowtie N^* \simeq N^* \bowtie \bowtie N\). As mentioned before, this duality transformations hold on the dualizable subspaces described above.

**Remark:** (Plurality) In this context, it is clear that another decomposition \(H = M \bowtie \bowtie M^*\) of the same group shall yield plurality between the corresponding collective models on \(M, M^*, N\) and \(N^*\), as long as the moment maps associated to the \(H\)-action intersect (recall sec. 3). Also notice that, if \(P\) is another hamiltonian \(H\)-space whose moment map intersects the others, then a collective hamiltonian system on \(P\) will also be dual to the previous ones. In the first cases, being the phase spaces cotangent bundles \(T^*N\) \((T^*N^*, \text{etc.})\), it allows for a lagrangian description in terms of curves (or fields, see next sections) lying in \(N, (N^*, \text{etc.})\).

5 Associated hamiltonian Systems

In the previous section, we have been concentrated on the kinematic-geometric aspects of Poisson Lie \(T\)-duality. In the current section, we describe the hamiltonian systems’ equations associated to the dual phase spaces we have constructed endowed with the corresponding collective dynamics.

5.1 Master WZNW-type model on \((T^*H, \omega_{c,\theta})\)

The equation of motion on the phase space \((T^*H, \omega_{c,\theta})\) described in Section 3.4 for some \(\mathcal{H} \in C^\infty (H \times \mathfrak{h}^*)\), are

\[
\begin{align*}
\dot{l}^{-1}l &= \delta \mathcal{H} \\
\dot{\eta} &= ad_{\delta \mathcal{H}}^* (\eta - C_{\theta} (l^{-1})) - \tilde{e}_{\theta} (\delta \mathcal{H}) - ld\mathcal{H}
\end{align*}
\]

for \((l, \eta) \in H \times \mathfrak{h}^*\), with \(d\mathcal{H}|_{(l,\eta)} = (d\mathcal{H}, \delta \mathcal{H})_{(l,\eta)} \in T^*_H (H \times \mathfrak{h}^*)\). By setting \(\theta = 0\) one gets the equation of motion for the system 1 (S1) and, for \(\theta = -\alpha\), the equation of motion for the system 2 (S2), as described in that Section.

Collective dynamics warranties Lax type equations, so we consider some examples with quadratic Hamilton functions including some arbitrary linear self adjoint operators \(\mathbb{L}_i : \mathfrak{h} \rightarrow \mathfrak{h}\) and \(\mathbb{L}_i^* \equiv \psi \circ \mathbb{L}_i \circ \psi : \mathfrak{h}^* \rightarrow \mathfrak{h}^*, i = 1, 2, 3\), where
\[ \psi : \mathfrak{h} \rightarrow \mathfrak{b}^* \] denotes the identification induced by the symmetric Ad-invariant non degenerate bilinear form \((,)_b\), and \(\psi\) means the inverse map.

**S1** - For the symplectic manifold \((H \times \mathfrak{b}^*, \omega_{c,0})\), and their reduced space \(M_{c,0}^{(0,1)}\), we consider

\[
\mathcal{H}_{c,0}(l, \eta; \alpha) = \frac{1}{2} (Ad^*_l \eta, \mathbb{L}_2^* \eta) + (Ad^*_l \eta, \mathbb{L}_2^* (C_{-\alpha} (l + \alpha)) \eta, \\
- \frac{1}{2} \mathbb{L}_2^* (C_{-\alpha} (l + \alpha)) \eta,)
\]  

(28)

To realize the collective form of this hamiltonian function, we follow the results of item S1 in Sec. 3.1 in order to restrict the system to the Marsden-Weinstein reduced spaces \(M_{c,0}^{(\alpha,1)}\). This means to make \(\eta = Ad^*_l C(l) + \alpha\) and, because of the residual left action of \(H\) on \(H/\mathfrak{h}\), we have the associated momentum map \(\hat{\Phi}_{c,0} : H/\mathfrak{h} \rightarrow \mathfrak{b}^*\), \(\hat{\Phi}_{c,0}(l \cdot \mathbb{H}) = (C_{-\alpha} (l + \alpha)) \), which allow us retrieve the collective form

\[
\mathcal{H}_{c,0}(l; \alpha) \big|_{M_{c,0}^{(\alpha,1)}} = \frac{1}{2} \left( \hat{\Phi}_{c,0}(l \cdot \mathbb{H}), \mathbb{L}_2^* \hat{\Phi}_{c,0}(l \cdot \mathbb{H}) \right) \big|_{\mathfrak{b}^*},
\]

Hence, the Hamilton equations a restricted to

\[
\left[ J_{c,0}^R \right]^{-1} (\alpha, 1) = \{(l, Ad^*_l C(l) + \alpha) / l \in H \}
\]

are

\[
l^{-1} \dot{l} = \tilde{\psi} \left( (\mathbb{L}_2^* + L_2^*) \eta \right) \eta \\
\dot{\eta} = ad^*_l \tilde{\psi} \left( (\mathbb{L}_2^* + L_2^*) \eta \right) \eta - \hat{c} \left( \tilde{\psi} \left( (\mathbb{L}_2^* + L_2^*) \eta \right) \right)
\]

The Legendre transformation from the first equation leads to the lagrangian function \(L_{c,0} = \langle \eta, l^{-1} \dot{l} \rangle - \mathcal{H}_{c,0},\)

\[
L_{c,0} = \frac{1}{2} \left( \mathbb{L}_2^* \mathbb{L}_2^* C_{-\alpha} (l + \alpha), l^{-1} \dot{l} \right) - \frac{1}{2} \mathbb{L}_2^* (C_{-\alpha} (l + \alpha), \mathbb{L}_2^* (C_{-\alpha} (l + \alpha)) \eta,)
\]

**S2** - For the symplectic manifold \((H \times \mathfrak{b}^*, \omega_{c,-\alpha})\) and their reduced space \(M_{c,-\alpha}^{(0,1)}\), we consider the quadratic Hamilton functions

\[
\mathcal{H}_{c,-\alpha}(l, \eta; -\alpha) = \frac{1}{2} (Ad^*_l \eta, \mathbb{L}_2^* \eta) + (Ad^*_l \eta, \mathbb{L}_2^* C_{-\alpha} (l)) \eta, \\
- \frac{1}{2} \mathbb{L}_2^* (C_{-\alpha} (l), \mathbb{L}_2^* C_{-\alpha} (l)) \eta,
\]  

(29)

When restricted to \(M_{c,-\alpha}^{(0,1)}, \eta = Ad^*_l C_{-\alpha} (l),\) see item S2 in Sec. 3.1 it takes the form

\[
\mathcal{H}_{c,-\alpha}(l, \eta; -\alpha) \big|_{M_{c,-\alpha}^{(0,1)}} = \frac{1}{2} (C_{-\alpha} (l), \mathbb{L}_2^* + L_2^*) C_{-\alpha} (l) \eta,)
\]

25
where
\[ \Phi_{c,-\alpha}(l \cdot H^{\alpha}_{l}) = (C_{-\alpha}(l), 1). \]
The Hamilton equations of motion reduced to
\[ \left(j^{R}_{c,-\alpha}\right)^{-1}(0, 1) = \{(l, -C_{-\alpha}(l^{-1})) / l \in H\} \]
are
\[ l^{-1}i = \tilde{\psi} \left( (L^{\alpha}_{2} + L^{\alpha}_{1}) \eta \right) \]
\[ \eta = \text{ad}^* \left( (L^{\alpha}_{3} + L^{\alpha}_{0}) \eta \right) - \tilde{\phi}_{-\alpha} \left( (L^{\alpha}_{3} + L^{\alpha}_{2}) \eta \right) \]

We use the first equation of motion to invert the Legendre transformation in order to obtain the lagrangian function \( \mathcal{L}_{c,-\alpha} = \langle \eta, l^{-1}i \rangle - \mathcal{H}_{c,-\alpha} \).

Thus, we get
\[
\mathcal{L}_{c,-\alpha}(l, i) = \frac{1}{2} \left\langle L_{2} \psi \left( \frac{i}{l} \right), \frac{i}{l} \right\rangle - \left\langle L_{3}^{\alpha} \psi \left( \frac{C_{-\alpha}(l)}{l} \right), \frac{i}{l} \right\rangle \]
\[ + \frac{1}{2} \left( C_{-\alpha}(l), \left( L_{2}^{\alpha} + L_{1}^{\alpha} \right) C_{-\alpha}(l) \right)_h, \]

### 5.2 Collective hamiltonian on the factor \((N \times n^*, \omega_0)\)

#### 5.2.1 Collective system for \(\mu_{0,0} : N \times n^* \rightarrow \mathfrak{h}_{c,0}\)

Sigma models are now regarded as collective systems on \(N \times n^*\). T-duality scheme can be applied to the hamiltonian space \((N \times n^*, \omega_0, \mathfrak{H}_{c,0}, \mu_{0,0}, \mathfrak{h} \circ \mu_{0,0})\), for an arbitrary function \(h : \mathfrak{h}_{c,0} \rightarrow \mathbb{R}\).

For instance, a specific type of quadratic hamiltonian for \(\mu_{0,0} : N \times n^* \rightarrow \mathfrak{h}_{c,0}\) was built up in [8] for \(N = LG\), so we refer to it and give here a brief recall. Using the invariant bilinear form \((\cdot, \cdot)_h\) we propose the collective hamiltonian

\[ \mathcal{H}^{\alpha}_C(g, \eta) = \frac{1}{2} \left( \text{Ad}^*_g \eta + C(g), \text{E} \left( \text{Ad}^*_g \eta + C(g) \right) \right)_h \]  

(30)

where \(\text{E} : \mathfrak{h} \rightarrow \mathfrak{h}\) is a self adjoint linear operator. Motivated by standard Poisson-Lie T-duality [8], we further assume that \(\mathcal{E}^2 = \text{Id}\). Now, following Appendix 2, we call \(\mathcal{E}^\alpha = \text{Ad}^\alpha \cdot \text{E} \cdot \text{Ad}^\alpha\), \(\mathcal{G}^\alpha = (\Pi_n \mathcal{E} \Pi_n)^{-1} : n \rightarrow n^*\) and \(\mathcal{B}^\alpha = -\mathcal{G}^\alpha \circ \Pi_n \mathcal{E} \Pi_n : n \rightarrow n^*\). Then, Hamilton equation for \(g \in N\) is

\[ g^{-1} \dot{g} = (\mathcal{G}^\alpha)^{-1} \mathcal{B}^\alpha C(g^{-1}) + (\mathcal{G}^\alpha)^{-1} \eta \]

The lagrangian coming from the collective hamiltonian given by such symmetric operator \(\mathcal{E}\) defines a sigma model since, as a block matrix in \(\mathfrak{g} \oplus \mathfrak{g}^*\), we have

\[ \mathcal{E}_g = \left( \begin{array}{cc} -\mathcal{G}^{-1} \mathcal{B} \mathcal{G}^{-1} & \mathcal{G}^{-1} \\ \mathcal{G} - \mathcal{B} \mathcal{G}^{-1} & \mathcal{B} \end{array} \right) \]

(31)

Following ref. [8] we arrive to the sigma model lagrangian (see also Appendix 2),

\[ \mathcal{L}_{\sigma}^0 = \frac{1}{2} \left\langle (\mathcal{G} + \mathcal{B}(g^{-1} \dot{g} - C(g^{-1}))), g^{-1} \dot{g} + C(g^{-1}) \right\rangle. \]  

(32)
5.2.2 Collective system for $\mu_{0,\alpha} : N \times n^* \rightarrow \mathfrak{h}_{c,-\alpha}^*$

Let us now consider the Hamiltonian space $(N \times n^*, \omega_o, H_{c,-\alpha}, \mu_{0,\alpha}, h \circ \phi \circ \mu_{0,\alpha})$ where $\mu_{0,\alpha} : N \times n^* \rightarrow \mathfrak{h}_{c,-\alpha}^*$ was defined in (18) for $\alpha$ satisfying condition (17), $\phi : \mathfrak{h}_{c,-\alpha}^* \rightarrow \mathfrak{h}_{c,0}^*$ is the Poisson diffeomorphism introduced in 3.1.1. For $h$ being the quadratic function as in (30), we have

$$H_{\alpha} = h \circ \phi \circ \mu_{0,\alpha} = \frac{1}{2} (\phi \circ \mu_{0,\alpha}, \mathcal{E} \phi \circ \mu_{0,\alpha})_h$$

The non-degenerate symmetric bilinear form $(,)_h$ induces the identification $\psi$ between $\mathfrak{h}$ and $\mathfrak{h}^*$, so that the Hamiltonian turns out to be

$$H_{\alpha} = \frac{1}{2} \left( \phi \circ \tilde{Ad}^{H^*}_{-\alpha, g^{-1}} (\psi (\lambda), 1), \mathcal{E} \tilde{\psi} \left( \phi \circ \tilde{Ad}^{H^*}_{-\alpha, g^{-1}} (\psi (\lambda), 1) \right) \right)_h$$

Hamilton equations of motion yield

$$g^{-1} \dot{g} = \Pi_n \tilde{\psi} \left( \mathcal{E} \left( \lambda + \tilde{Ad}^H_{g^{-1}} \tilde{\psi} (C (g)) + \alpha \right) \right)$$

Taking into account the form of the operator $\mathcal{E}$ of eq. (31),

$$g^{-1} \dot{g} = G^{-1}_g B g C (g^{-1}) + G^{-1}_g (\lambda + \alpha)$$

Once again, to retrieve the Lagrangian function, we use the first Hamilton equation to invert the Legendre transformation

$$\lambda = G_g g^{-1} \dot{g} - B g C (g^{-1}) - \alpha$$

Thus, by Appendix 2,

$$\mathcal{L}^\alpha = \frac{1}{2} \left( (G_g + B_g) (g^{-1} \dot{g} - C (g^{-1})), g^{-1} \dot{g} + C (g^{-1}) \right)_h - \langle g^{-1} \dot{g}, \alpha \rangle.$$

5.3 Collective Hamiltonian on the factor $(N^* \times n, \tilde{\omega}_o)$

Let us now consider the collective dynamic system on $T^*N^* \cong N^* \times n$ defined by,

$$\tilde{H}^\alpha = h \circ \tilde{\mu}_{0,\alpha}^\alpha$$

for the momentum map $\tilde{\mu}_{0,\alpha}^\alpha : N^* \times n \rightarrow \mathfrak{h}_{c,0}^*$, defined in (21), and $h$ being the quadratic function as above. This yields

$$\tilde{H}^\alpha = \frac{1}{2} \left( \tilde{Ad}^D_h (X + \alpha) + C \left( \tilde{h} \right), \mathcal{E} \left( \tilde{Ad}^D_h (X + \alpha) + C \left( \tilde{h} \right) \right) \right)_h$$

so that the Hamilton equation of motion for $\tilde{h}$ is

$$\tilde{h}^{-1} \ddot{\tilde{h}} = \mathcal{E}_h \left( X + \alpha - C \left( \tilde{h}^{-1} \right) \right)$$
\[ E_\sim \mathbf{h} = Ad_{\mathbf{h}}^{-1} \mathcal{E} Ad_{\mathbf{h}} \]. Using the dual decomposition \( \mathbf{h} = n^* \oplus n \), we can write
\[ E_\sim \mathbf{h} = \left( \begin{array}{cc} -\tilde{G}_\mathbf{h}^{-1} & \tilde{G}_\mathbf{h}^{-1} \\ \tilde{G}_\mathbf{h} - \tilde{B}_\mathbf{h} & \tilde{B}_\mathbf{h} - \tilde{G}_\mathbf{h}^{-1} \end{array} \right) \]
with the operators \( \tilde{G}_\mathbf{h} = (\Pi_n^* \mathcal{E}_\mathbf{h} \Pi_n)^{-1} : n^* \rightarrow n \) and \( \tilde{B}_\mathbf{h} = -\tilde{G}_\mathbf{h} \circ \Pi_n^* \mathcal{E}_\mathbf{h} \Pi_n^* : n^* \rightarrow n \). The equation for \( \tilde{\mathbf{h}} \) yields
\[ X = \tilde{G}_\mathbf{h} \left( \mathbf{h}^{-1} \mathbf{h} \right) + \tilde{B}_\mathbf{h} \left( \alpha - C \left( \mathbf{h}^{-1} \right) \right) \]
We thus obtain, following Appendix 2, the following lagrangian function
\[ \tilde{\mathcal{L}}^\alpha \left( \mathbf{h}, \dot{\mathbf{h}} \right) = \frac{1}{2} \left( \mathbf{h}^{-1} \dot{\mathbf{h}} + C \left( \mathbf{h}^{-1} \right) - \alpha, \left( \tilde{G}_\mathbf{h} + \tilde{B}_\mathbf{h} \right) \left( \mathbf{h}^{-1} \dot{\mathbf{h}} - C \left( \mathbf{h}^{-1} \right) + \alpha \right) \right) \]

**Remark:** (The equivalent shifted formulation) Analogously, a model on \( T^*N^* \cong N^* \times n \) in corresponding shifted version can be obtained by proposing a Hamiltonian function
\[ \tilde{\mathcal{H}}^\alpha = h \circ \varphi \circ \tilde{\mu}^\alpha \]
for \( \tilde{\mu}^\alpha : (N^* \times n, \tilde{\omega}_o) \rightarrow (\mathfrak{h}, \{ , \}_{c,-\alpha}) \) defined in eq. (24), leading to the same Hamilton equations and lagrangian since \( \varphi \circ \tilde{\mu}^\alpha = \tilde{\mu}^\alpha_{\mathbf{h}^\alpha} \).

### 6 Loop groups and Poisson Lie T-duality for trivial monodromies

In the classical field theory context, Sigma and WZNW models can be built up on the cotangent bundle of loop groups. In particular, for the WZNW model, a pure cocycle is added to the canonical symplectic form in order to produce the decoupling of the chiral modes \[12\], leading to the fact that a general solution for the equation of motion can be written as a non linear combination of both modes. Finally, on the chiral phase space, solutions related to non-trivial extended coadjoint orbits are the ones associated to non trivial monodromies.

In this Section, we apply the results obtained in the previous sections to the case of underlying loop groups, thus addressing the case in which non trivial monodromies appear. More precisely, we consider \( H = LD \), for some Lie group \( D \) with Lie algebra \( \mathfrak{d} \) such that \( \mathfrak{h} = L\mathfrak{d} \). Moreover, \( \mathfrak{d} \) will be regarded also as a double Lie group \( D = G \ltimes G^* \) with Lie bialgebra \( \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g} \), so that \( N = LG \), \( n = Lg \), and the same for the corresponding duals.

For \( l \in LD \), \( l' \) denotes the derivative in the loop parameter \( s \in S^1 \), and we write \( vl^{-1} \) and \( l^{-1}v \) for the right and left translation of any vector field \( v \in TD \). Let \( \mathfrak{d} \) be the Lie algebra of \( D \) equipped with a non degenerate symmetric \( Ad \)-invariant bilinear form \( ( , )_\mathfrak{d} \). Frequently we will work with the subset \( L\mathfrak{d}^* \subset \mathfrak{d} \).
\((L\mathfrak{d})^\ast\) instead of \((L\mathfrak{d})^\ast\) itself, and we identify it with \(L\mathfrak{d}\) through the map \(\psi : L\mathfrak{d} \to L\mathfrak{d}^\ast\) provided by the bilinear form
\[
(\cdot, \cdot)_{L\mathfrak{d}} = \frac{1}{2\pi} \int_{S^1} (\cdot, \cdot)_\mathfrak{d}
\]
on \(L\mathfrak{d}\). In this framework, the two cocycle \(c : \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}\) of Section 3.1 is given by the bilinear form \(\Gamma_k : L\mathfrak{d} \times L\mathfrak{d} \to \mathbb{R}\)
\[
c(X, Y) \equiv \Gamma_k(X, Y) = \frac{k}{2\pi} \int_{S^1} (X(s), Y'(s))_\mathfrak{d} \, ds
\]
with \(X, Y \in L\mathfrak{d}\). It is derived from the one cocycle \(C_k : L\mathfrak{d} \to L\mathfrak{d}^\ast\),
\[
C_k(l) = k \psi (l l^{-1})
\]
\(\mathbb{R}\). We now recall some facts about the stabilizer subgroup of a point \((\eta, 1) \in L\mathfrak{d}^\ast\). If \([L\mathfrak{d}]_{(\eta, 1)}\) denotes the stabilizer, with \(\eta \in L\mathfrak{g}^\ast\), consider \(\tilde{h}_\eta(s) \in C^\infty(\mathbb{R}, G^\ast)\), \(\tilde{h}_\eta(0) = e\), and \(\tilde{M}_\eta \in G^\ast\) such that
\[
\eta(s) = \tilde{h}_\eta(s) \tilde{h}_\eta^{-1}(s)
\]
where \(\tilde{M}_\eta \in G^\ast\) is the monodromy of the \(\mathfrak{g}^\ast\)-valued map \(\eta(s)\). Then \([L\mathfrak{d}]_{(\eta, 1)}\) consists of loops of the form
\[
l(s) = \tilde{h}_\eta(s) l(0) \tilde{h}_\eta^{-1}(s)
\]
based at some point \(l(0)\) within the stabilizer \(D_{\tilde{M}_\eta}\) of \(\tilde{M}_\eta\) in \(D\)
\[
\tilde{M}_\eta = l^{-1}(0) \tilde{M}_\eta l(0)
\]
For any other point \((\beta, 1) = \tilde{A}_{m}^{L\mathfrak{d}^\ast}(\eta, 1) \in O(\eta, 1), m \in L\mathfrak{d}, \) the isotropy group is \(m^{-1}(L\mathfrak{d}_{(\eta, 1)}) m\). Writing \(\beta(s) = -l_\beta^{-1}(s) l_\beta s + 2\pi\), it is related with \(\tilde{M}_\eta\) as
\[
\tilde{M}_\eta = m^{-1}(s) \tilde{M}_\eta m (s)
\]
As a symplectic manifold, the coadjoint orbit $O_{\Gamma_k,0}(\alpha, 1) \subset L\mathfrak{g}^*$, for $\alpha \in L\mathfrak{g}^*$, is endowed with the Kirillov-Kostant symplectic form

$$\omega_{\alpha}^{\Gamma_k} = \langle [X,Y], \beta \rangle + \Gamma_k(X,Y)$$

with $(\beta, 1) = \hat{\text{Ad}}_{Ld}(\alpha, 1) \in O_{\Gamma_k,0}(\alpha, 1)$, for some $l \in LD$.

Let us define

$$\phi : LD \rightarrow O_{\Gamma_k,0}(\alpha, 1) \subset L\mathfrak{g}^*$$

and consider the pull back of $\omega_{\alpha}^{\Gamma_k}$ through $\phi$

$$\phi^*\omega_{\alpha}^{\Gamma_k} = \Gamma_k(l^{-1}d[l^{-1}dl] + \langle [l^{-1}dl, l^{-1}dl], \alpha \rangle)$$

The null distribution of this form is spanned by the Lie algebra of the stabilizer subgroup $LD(\alpha, 1)$ of $(\alpha, 1)$. Thus, if $LD/\text{LD}(\alpha, 1)$ is a smooth manifold, there is a symplectic diffeomorphism $(LD/\text{LD}(\alpha, 1), \phi^*\omega_{\alpha}^{\Gamma_k}) \rightarrow (O_{\Gamma_k,0}(\alpha, 1), \omega_{\alpha}^{\Gamma_k})$.

As a consequence we have,

**Remark:** The momentum map associated to the residual symmetry of $LD$ on $LD/\text{LD}(\alpha, 1)$ is $\phi$.

By also recalling the map $\mu_{0,0}$ from [8] (called just $\mu$ there), so far we have

$$T^*LG \xrightarrow{\mu_{0,0}} LD/\text{LD}(x) \xrightarrow{\phi} L\mathfrak{g}^*$$

We are going now to describe the other phase spaces presented in section 3 which are also related to the orbit $O_{\Gamma_k,0}(\alpha, 1)$.

### 6.1 The chiral WZNW phase space

The phase spaces described in section 5.1, for $H = LD$, correspond to a chiral sector of the WZNW model [21]. Consider the unique curve $\tilde{h}_{\alpha} \in C^\infty(\mathbb{R}, G^*)$ such that

$$\alpha(s) = \tilde{h}_{\alpha}'(s) \tilde{h}_{\alpha}^{-1}(s)$$

with $\tilde{h}_{\alpha}(0) = e$ so that $\tilde{h}_{\alpha}(2\pi) = \tilde{M}_{\alpha} = \text{Hol}(\alpha(s))$ is the holonomy of $\alpha(s) \subset \mathfrak{g}^*$. The solutions in $LD$ to the classical equation of motion of the WZNW model

$$\partial_- (\partial_+ ll^{-1}) = 0$$

are $l(s,t) = l_L(x_+ l_R^{-1}(x_-)$, which combines the chiral solutions $l_L(x_-)$ and $l_R(x_+)$, with $x_\pm = s \pm t$. In spite of the fact that $l(s,t)$ is periodic in $s$, the
chiral modes are not restricted to be closed loops. In fact, for chiral modes satisfying
\[ l^\pm(x \pm 2\pi) = l^\pm(x) \]
the solutions \( l(s,t) \) remains periodic. Let \( PD \) be the extended chiral phase space of the WZNW model (see [21])
\[ PD = \{ l : R \to D / l(x + 2\pi) = l(x)M, \text{ for some } M \in D \}. \]

Here, each \( M \in D \) is the monodromy of path \( l(x) \) satisfying
\[ l(0)Ml^{-1}(0) = Hol(l^1l^{-1}) = l(2\pi)l^{-1}(0) \]
since \( l(0)^{-1}l(2\pi) = M \).

In order to study separately the chiral degree of freedom, the following symplectic form on \( PD \) must be considered
\[ \omega^{PD}(l) = -\Gamma_k(l^{-1}dl\otimes l^{-1}dl) - \frac{1}{2} \langle (l^{-1}dl)(0) \otimes dMM^{-1} \rangle + \rho(M) \]
with \( \rho \) being a 2-form just depending on the holonomy \( M \) so that
\[ d\rho(M) = \frac{1}{6} \langle dMM^{-1} \otimes [dMM^{-1} \otimes dMM^{-1}] \rangle \]

It is worth to remark that \( \rho \) is defined just locally in some neighborhood of the identity. In ref. [21], it is shown that such \( \rho \) is given by generalized \( r \) matrices solutions of the Generalized Dynamically Modified Classical Yang-Baxter equation, defining Poisson structure chiral space \( PD \).

Let us define the map
\[ J : PD \to L\Omega^*_{\Gamma_k} \]
\[ J(l) = (l^1l^{-1}, 1) \quad (35) \]
The loop group \( LD \) acts on \( PD \) by left multiplication with associated momentum map \( J \). On the other side, the group \( D \) acts on \( PD \) by right multiplication
\[ l(x) \cdot d \to l(x)d, \]
which is a Poisson-Lie symmetry in relation to the Poisson-Lie structure on \( D \) given by the canonical \( r \)-matrix in the double Lie algebra \( \hat{\mathfrak{d}} \) and a particular election of \( \rho(M) \), as it is shown in [21]. In that case, the map \( \sigma : LD \to D^* \) defined as \( \sigma(l(x)) = l^{-1}(x)l(x + 2\pi) = M \) defines the corresponding \( D^* \)-valued momentum map.

Then, we have the following diagram
For each value $M$, we consider the pre image $\sigma^{-1}(M) \subset PD$ and define the subspace

$$E_\alpha(D) = LD \cdot \tilde{h}_\alpha = \{ l \cdot \tilde{h}_\alpha / l \in LD \}$$

which is injected into $\sigma^{-1}(M) \subset PD$ as follows

$$E_\alpha(D) \overset{\iota}{\hookrightarrow} \sigma^{-1}(M = Hol(\alpha)) \subset PD.$$ 

Because it can be identified with $LD$, the restriction of the symplectic form on $PD$ to $E_\alpha(D)$ coincides with $\mu^* \omega_{KK}\big|_l$, and

$$\mu^\alpha := J \circ \iota$$

is a presymplectic map. Hence, we have following phase spaces related by symplectic morphisms with the orbit $O_{\Gamma_k,0}(\alpha, 1)$,

$$\begin{array}{ccc}
T^*LG & E_\alpha(D) \\
\mu_0 \downarrow & \mu^\alpha \\
O_{\Gamma_k,0}(\alpha, 1) \overset{\iota}{\hookrightarrow} (L\mathfrak{d}_{\Gamma_k,0}^*\{,\}_{KK})
\end{array}$$

6.2 A symplectic $LD$ action on $LG^* \times Lg$

From Section 3.2.2 it follows that $\tilde{\mu}_{0,\alpha} : (LG^* \times Lg, \tilde{\omega}_\alpha) \longrightarrow (L\mathfrak{d}, \{ , \}_{Aff}) \subset (L\mathfrak{d}_{\Gamma_k,0}^*\{,\}_{KK})$, given by

$$\tilde{\mu}_{0,\alpha}^{\gamma}(\tilde{g}, Z) = (\tilde{g} \tilde{g}^{-1} + Ad_{\tilde{g}}^L(Z + \alpha)) , \quad (36)$$

is a Poisson map provided that condition (17) is satisfied. In the current loop group case, this means that

$$ad_{\alpha(s)}X \in \mathfrak{g} \subset \mathfrak{d}$$ 

for all $X \in \mathfrak{g}$ and all $s \in S^1$.

Remark: When $G^* = K$ is a compact simple real Lie group, e.g. $SU(N)$, any constant element $\alpha \in \mathfrak{t}$, where $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{k} = Lie(K)$, condition (37) is satisfied (see Appendix 1).

Under this condition, the loop group $LD$ acts on $LG^* \times Lg$ via

$$(\tilde{g}, Z) \overset{\iota=(h\alpha)}{\longmapsto} (\tilde{h}g^\alpha, Ad_{\tilde{g}}^L(Z + \alpha) + (a\tilde{g})' (a\tilde{g})^{-1} - \alpha)$$

32
where \( \tilde{g}^a a \tilde{g} = a \tilde{g} \). The corresponding infinitesimal generator \( X_{LG^* \times Lg}(\tilde{g}, Z) \) associated to an \( X = (\xi, X) \in LD \) at the point \( (\tilde{g}, Z) \in \) is

\[
X_{LG^* \times Lg}(\tilde{g}, Z) = \left( (\tilde{g} \Pi^*g, \Pi g \left( Ad_{LD} \tilde{g}^{-1} X' + \left[ \tilde{g}^{-1} \tilde{g}' + Z + \alpha, \Pi g (Ad_{LD} \tilde{g}^{-1} X') \right] \right) \right)
\]

where \( \Pi g^*, \Pi g \) are the projectors on the Lie subalgebras \( g^*, g \), respectively.

Moreover, this action is hamiltonian and the corresponding moment map is precisely \( \tilde{\mu}_{\phi^0, \alpha} \).

### 6.3 T-duality diagram for non trivial monodromies

Thus, considering the results of this section, we may assemble a PL T-duality diagram based on the coadjoint orbit \( O_{\Gamma_k,0}(\alpha, 1) \subset LD^*_{\Gamma_k,0} \).

This fact doesn’t affect the side corresponding to the phase space \( T^*LG \): there we have a \( \sigma \)-model describing the dynamics of a closed string valued on the target \( G \), symmetric under the hamiltonian action of the group \( LD_{\Gamma_k,0} \) given in eq. (13) and with associated momentum map \( \mu_{0,0} \) of Proposition (3.2.1), as depicted in diagram (34).

The double Lie group \( LD \) appears in T-duality diagrams through the sub-space \( E_{\alpha}(D) \) bringing the open string models into the scheme. It is injected in the extended chiral phase space of the WZNW model, \( PD \), and relates to the orbit \( (LD^*_{\Gamma_k,0}; \{, \}_{K_K}) \) by the map \( \mu^\alpha := J \circ i \).

Finally, on the dual side things become subtler: as explained just above, and in section 3.2.2 the connection of the cotangent bundle \( T^*LG^* \equiv LG^* \times Lg \) to the coadjoint orbit \( (LD^*_{\Gamma_k,0}; \{, \}_{K_K}) \) does exist provided \( \alpha \) be such that (37) is fulfilled. Under this condition, the map \( \tilde{\mu}_{0,0, \alpha} : (LG^* \times Lg, \tilde{\omega}_0) \rightarrow (LD^*_{\Gamma_k,0}; \{, \}_{K_K}) \) turns in a Poisson one, enabling to assemble the complete T-duality diagram

\[
\begin{array}{ccc}
T^*LG & \xrightarrow{\mu_{0,0}} & E_{\alpha}(D) \\
& \quad & \xleftarrow{\mu^\alpha} \\
O_{\Gamma_k,0}(\alpha, 1) & \rightarrow & (LD^*_{\Gamma_k,0}; \{, \}_{K_K}) \\
& \xrightarrow{\tilde{\mu}_{0,0, \alpha}} & \\
& \quad & \xleftarrow{\nu_{0,0, \alpha}} \\
T^*LG^* & \xrightarrow{\nu_{0,0, \alpha}} &
\end{array}
\]

### 6.4 Induced hamiltonian systems on loops groups

Following section 5 we now choose a quadratic hamiltonian function \( H : LD^*_{c,0} \sim LD_{c,0} \rightarrow \mathbb{R} \) given by

\[
H(X, 1) = \frac{1}{2} \langle X, \mathcal{E}(X) \rangle
\]
with $\mathcal{E} : \mathfrak{d} \to \mathfrak{d}$ a linear operator. The lifts to $LD/D_{\alpha(x)}$ and $PD$ are, respectively,

$$
\begin{align*}
H \circ \phi(l) &= \frac{1}{2} \langle \hat{A}d^*_l(\alpha,1), \mathcal{E}(\hat{A}d^*_l(\alpha,1)) \rangle \\
H \circ J(l) &= \frac{1}{2} \langle l^{-1}, \mathcal{E}(l^{-1}) \rangle
\end{align*}
$$

We notice that $H \circ J$ in $PD$ is invariant under the right (PL) action of $D$ and then, the corresponding momentum map shall give conserved quantities. This implies that the monodromy $\hat{M}$ is constant and that we can restrict to the subspace within $m^{-1}(\hat{M}) \subset PD$ consisting of paths with the same monodromy, for example $E_{\alpha}(D)$ defined above. The resulting dynamics is the chiral WZNW-type one corresponding to section 5.1.

In $Ld^*_x \sim Ld_x$, writing the corresponding integral curve as $\gamma(t) = \hat{A}d^*_l(t)(\alpha,1)$ for the initial value $\gamma(0) = (\alpha,1)$, we have that the corresponding equation for $l(t) \in LD$ is

$$
\frac{d}{dt} l^{-1} = \mathcal{E}(l^{-1}l + Ad_l \alpha) = \mathcal{E}((\hat{b})'(\hat{b})^{-1})
$$

(38)

with $\alpha = \hat{b} \hat{b}^{-1}$, $\hat{b}(0) = e$, $\hat{b} \in PG^* \subset PD$. Since the hamiltonian functions are in collective form (recall sec. 4.2), the integral curves in $LD/D_{\alpha(x)}$ and $PD$ will be determined by $l(t)$.

**Remark:** (dual factorizations) Writing $l = \hat{g} \hat{h}$ or $l = \hat{g} h$, we alternatively obtain

$$
\frac{d}{dt} \hat{g} \hat{g}^{-1} + Ad_{\hat{g}} \frac{d}{dt} \hat{h} \hat{h}^{-1} = \mathcal{E}(g'g^{-1} + Ad_g(Ad_{\hat{g}} \alpha + \hat{h}'\hat{h}^{-1}))
$$

$$
= \mathcal{E}((\hat{g}\hat{h})(\hat{g}\hat{h})^{-1})
$$

$$
\frac{d}{dt} \hat{g} \hat{g}^{-1} + Ad_{\hat{g}} \frac{d}{dt} h h^{-1} = \mathcal{E}(g'g^{-1} + Ad_g(Ad_{\hat{g}} \alpha + h' h^{-1}))
$$

$$
= \mathcal{E}((\hat{g}\hat{h})(\hat{g}\hat{h})^{-1}).
$$

In the first case, since $\alpha \in \mathfrak{g}^*$ and, thus, $\hat{b} \in G^*$, the terms corresponding to fields in $G$ and $G^*$ split, allowing to lift the dynamics from $\mathcal{O}_{\Gamma,\omega}(\alpha,1)$ to $LT^*G$ via $\mu$, as we already knew [8]. In turn, in the second case above, this cannot be done generally because the $G$ and $G^*$ variables are mixed up.

Now, if (37) is satisfied for a constant $\alpha \in \mathfrak{g}^*$, following Sections 5.3 and 4.2 we find that the resulting model pulls back to $(LG^* \times Lg, \hat{\omega}, h \circ \hat{\mu}^\alpha)$ and that
it becomes dual to that on $LT^*G$. The equations of motion on $\mathbb{L}G^* \times \mathbb{L}g$ are

$$
\tilde{g}^{-1}\frac{d}{dt}\tilde{g} = \Pi_G^*\left(\mathcal{E}_g(\tilde{g}^{-1}\tilde{g}' + Z + \alpha)\right)
$$

$$
\frac{d}{dt} Z = \Pi_G^*\{[Z, \tilde{g}^{-1}\frac{d}{dt}\tilde{g}] + \mathcal{E}_g a_d D^{-1}\tilde{g}'(Z + \alpha) + \mathcal{E}_g(\tilde{g}^{-1}\tilde{g}' + Z + \alpha)'
- ad_{\tilde{g}^{-1}\tilde{g}}^D \mathcal{E}_g(\tilde{g}^{-1}\tilde{g}' + Z + \alpha) - ad_{(Z + \alpha)}^D \mathcal{E}_g(\tilde{g}^{-1}\tilde{g}' + Z + \alpha)\}
$$

whose solutions are given by

$$(\tilde{g}, Z)(t) = l_t \cdot (\tilde{g}_0, Z_0)$$

when $l_t \in LD$ is a solution of equation (38) and $(\tilde{g}_0, Z_0) \in (\tilde{\mu}_{\phi_0}, \alpha)^{-1}(\alpha, 1)$.

### 6.4.1 The induced lagrangians

The action functional corresponding to the above dynamics in $LD/D\alpha(x)$, expressed in terms of $D$-valued field $l$, is

$$
S = \int \theta^a(l) - H \circ \phi(l)
$$

where $H$ is the hamiltonian given in the previous section and $\theta^a$ is a potential for the 2-form $\omega_{KK}^a$, which can be written as

$$
\theta^a(l) = \langle l' l^{-1}, dl l^{-1} \rangle + \frac{1}{6} d^{-1} \langle dl l^{-1}, [dl l^{-1}, dl l^{-1}] \rangle + 2 \langle l^{-1} dl, \alpha \rangle.
$$

This agrees with the action proposed in ref. [22] for a particular choice of $H$. By construction, this WZW-like model on $D$ is dual to the corresponding sigma models with targets $G$ and $G^*$, as described in the previous sections.

The sigma model on $G$, for a particular choice of $\mathcal{E}$, can be found in ref. [8]. The same operator $\mathcal{E}$, following section 5.3 and assuming (37) is satisfied for a constant $\alpha \in g^*$, induces a lifted hamiltonian on $(LG^* \times \mathbb{L}g, \tilde{\omega}_\alpha)$. The Legendre transformation turns out to be non singular and the resulting model is given by the lagrangian (see Appendix 2)

$$
\mathcal{L}_\alpha(\tilde{g}, \frac{d\tilde{g}}{dt}) = \left\langle \partial_- \tilde{g} \tilde{g}^{-1} - Ad_\alpha^D, ((\mathcal{B}_e + \mathcal{G}_e) + \tilde{\pi}(\tilde{g}))^{-1} (\partial_+ \tilde{g} \tilde{g}^{-1} + Ad_\alpha^D) \right\rangle
$$

(39)

where $\tilde{\pi}(\tilde{g})$ is the Poisson bivector of $G^*$. Using the fact that $\tilde{\pi}(\tilde{g}) = \tilde{\pi}(\tilde{g} e^{x\alpha})$ since (37) is satisfied, then this lagrangian can be expressed in terms of the open monodromic string variable $\tilde{m} = \tilde{g} e^{x\alpha}$, $x \in [0, 2\pi]$,

$$
\mathcal{L}_\alpha(\tilde{m}, \frac{d\tilde{m}}{dt}) = \left\langle \partial_- \tilde{m} \tilde{m}^{-1}, ((\mathcal{B}_e + \mathcal{G}_e) + \tilde{\pi}(\tilde{m}))^{-1} \partial_+ \tilde{m} \tilde{m}^{-1} \right\rangle
$$

35
Remark: This lagrangian corresponds to the one given in [22], but in our case we know by construction that the dynamics preserves the monodromy (it restricts to $(\tilde{\mu}_0^\circ)^{-1}(\mathcal{O}_k,\alpha(1)) \subset LG^* \times L\mathfrak{g}$ when the initial value lies there) and no further constraints are needed.

In the above case, for each $\alpha$ we have a dual model $L_\alpha$. One may ask if we can glue all this models together making a unique $L$ which is dual to the one in $G$ for any $\alpha$. Note that to that end, we should consider $\alpha$ as varying in a space of all possible (log of) monodromies (e.g.: in a torus $t$ inside a compact group $K$ [22]) and then consider a structure on $LG^* \times L\mathfrak{g} \times \{\alpha\}$ giving the correct dynamics.

If $\alpha \in t$ is considered as a variable, the correct dynamics is given by one which sets $\alpha = \text{const}$. To that end, we enlarge the phase space from $LG^* \times L\mathfrak{g}$ to the (also symplectic) $(LG^* \times L\mathfrak{g} \times T^*t, \tilde{\omega}(\tilde{\mathfrak{g}},Z) \oplus \omega_\alpha(\alpha,\lambda))$. There, we define

$$\tilde{\mu} : LG^* \times L\mathfrak{g} \times T^*t \longrightarrow L\mathfrak{g}$$

which is still a Poisson map. The action it generates is the same on $LG^* \times L\mathfrak{g}$, it is trivial on $t$ and is non trivial on $t^*$. Now, the image of $\tilde{\mu}$ covers all the orbits $O_{k,0}(\alpha,1)$ for $\alpha \in t$. The lagrangian associated to the (now singular since $h \circ \tilde{\mu}$ does not depend on $\lambda$) hamiltonian system $(LG^* \times L\mathfrak{g} \times T^*t, \tilde{\omega}_h \oplus \omega_\alpha, h \circ \tilde{\mu})$ is

$$L(\tilde{\mathfrak{g}}, \frac{\partial \tilde{\mathfrak{g}}}{\partial t}, \alpha, \frac{\partial \alpha}{\partial t}, \lambda)$$

where in the last added term $\lambda$ plays the role of a Lagrange multiplier, and so the dynamics for the new variables is

$$\frac{\partial \alpha}{\partial t} = 0$$
$$\frac{\partial \lambda}{\partial t} = \nabla_\alpha L_\alpha.$$  

Example: (Lu-Weinstein doubles) Let $K$ be a real simple and compact Lie group. Then $D = AN \times K$ where $G = AN$ and $G^* = K$ are the subgroups given by the Iwasawa decomposition of $K^\circ$. Now, let $T \subset K$ be a maximal torus and $t$ its Lie algebra. By choosing a constant $\alpha \in t$ and a particular hamiltonian on $L\mathfrak{g}^*_c$, we obtain a resulting model in phase space $(LG^* \times L\mathfrak{g}, \tilde{\omega})$. This model can be expressed in terms of $\tilde{\mathfrak{g}}(x)e^{t\alpha}$ exclusively, yielding a monodromic strings model similar to that of [22]. This follows from the above considerations because, in this case, $b = e^{t\alpha}$ and $\tilde{\mathfrak{g}}b = (\tilde{\mathfrak{g}}b)(h^b)$ (see Appendix 1). The resulting lagrangian is [30] (compare to the ad hoc constrained one of [22]).
7 Conclusions

We carried out an enlargement of the T-duality scheme developed in [8] in order to include coadjoint orbits with non trivial monodromy as pivotal phase space. To that end, we considered a general framework for studying duality between different phase spaces which share the same symmetry group H. Solutions corresponding to collective dynamics become dual in the sense that they are generated by the same curve in H. Explicit examples of dual phase spaces in the above sense were constructed on the cotangent bundles of the factors of a double Lie group $H = N \ltimes N^*$.

When considering duality over non trivial extended orbits $O_{c,0} (\alpha, 1)$ with $\alpha \in \mathfrak{n}^*$, some important new facts appeared, being the most significative the loss of the symmetry between the role played by the factors of the double Lie group. Also, a condition on $\alpha$, eq. (17), has to be imposed in order for the momentum maps intersecting in $O_{c,0} (\alpha, 1)$ to be compatible with the underlying hamiltonian structure (i.e. to be Poisson maps). In the loop group case, standard sigma models are now T-dually related to models with non trivial $\alpha$-monodromies, namely open string models [22], as showed in subsection 6.4.1. It is worth to remark that, in the present framework, the dynamics of open string models are monodromy preserving by construction, no further constraints need to be added.

On the other hand, since a non trivial orbit becomes related to a trivial one by changing the cocycle extension by coboundary, we introduced a second pivotal vertex in the T-duality scheme. This trivial orbit can be regarded as the phase space chiral modes of a WZNW type model, with shifted collective lagrangian, and allows for additional collective models on the cotangent bundles of the factors N and $N^*$ in the T-duality class of those hanged from the non trivial monodromy orbit.

Thus, we succeeded in to generalize the symplectic geometry approach to Poisson Lie T-duality of ref. [8], stressing the fundamental role played by coadjoint orbits of double Lie groups, central extensions and collective dynamics, also analyzing some hamiltonian an lagrangian models on the involved phase spaces.

8 Appendix 1

We are going to study the structure of the brackets in the double Lie algebra for $\mathfrak{g}$ being a simple compact real Lie algebra (e.g. $su(n)$). Recall from [24], that if $(H_i, e_i, f_i)$ with $i = 1, ..., \text{rank}(\mathfrak{g})$ denotes the elements of the Chevalley basis of $\mathfrak{g}$ for a fixed Borel subalgebra $\mathfrak{b}$, then

\begin{align*}
[H_i, H_j] &= 0 \\
[H_i, e_j] &= a_{ij} e_j \\
[H_i, f_j] &= -a_{ij} f_j
\end{align*}

(40)
where \((a_{ij})\) denotes the Cartan matrix of \(\tilde{g}\). Note that \(span\{H_i\}\) is the abelian Lie algebra of a maximal torus \(T \subset G\). The standard bialgebra structure \(\delta : \tilde{g} \rightarrow \tilde{g} \wedge \tilde{g}\) on \(\tilde{g}\) is defined by

\[
\begin{align*}
\delta(H_i) &= 0 \\
\delta(e_i) &= d_i H_i \wedge e_i \\
\delta(f_i) &= d_i H_i \wedge f_i
\end{align*}
\]

with \(d_i\) being the length of the \(i\)-th root.

Now, let \((H_i, e_i, f_i)\) denote the basis of \(g = \tilde{g}^*\) dual to \((H_i, e_i, f_i)\). We want to express the commutation relations in terms of this dual basis of the bracket on \(g\) induced by \(\delta\). Using (40), (41) and the definition

\[
\langle [X,Y], \alpha \rangle = \langle X \otimes Y, \delta(\alpha) \rangle
\]

for all \(X, Y \in g, \alpha \in \tilde{g}\), it is easy to see that in \(g\)

\[
\langle [X,Y], H_i \rangle = 0.
\]

Thus, in the double \(\mathfrak{d} = \tilde{g} \oplus g\), we have that

\[
[(H_i,0),(0,X)]_\mathfrak{d} = (0,ad_{H_i}^*X)
\]

hence, if \(\tilde{b} = exp(\Sigma_i c_i H_i) \in T \subset \tilde{G}, h \in G,\)

\[
\tilde{b} \cdot h = h \tilde{b} \cdot \tilde{b}
\]

so the dressing action of \(h\) on \(\tilde{b}\) is trivial, i.e. \(\tilde{b}^h = \tilde{b}\).

9 Appendix 2

Here we give some details on the algebra involved in passing from a 1\textsuperscript{st}-order lagrangian to a 2\textsuperscript{nd}-order sigma model lagrangian.

Let us start with a 1\textsuperscript{st}-order lagrangian, \(\dot{q}, \dot{q} \in \tilde{V}, p \in V,\)

\[
L(q,p) = \langle p, \dot{q} \rangle - \frac{1}{2} \langle p + q, \mathcal{E}(p + q) \rangle
\]

for \(\mathcal{E}\) linear operator on \(\tilde{V} \oplus V\) satisfying \(\mathcal{E}^2 = Id\) and being self adjoint with respect to the pairing \(\langle , \rangle\). From these conditions, it is easy to see that the operator \(\mathcal{G}^{-1} := \rho_{\tilde{V}} \mathcal{E} \rho_V : V \rightarrow \tilde{V}\) is invertible and that

\[
\langle \mathcal{G} \tilde{v}, \tilde{w} \rangle = \langle \tilde{v}, \mathcal{G} \tilde{w} \rangle
\]

\(\tilde{v}, \tilde{w} \in \tilde{V}\). Also, \(\mathcal{B} := (\rho_V \mathcal{E} \rho_V) \circ (\rho_V \mathcal{E} \rho_V)^{-1} : \tilde{V} \rightarrow V\) satisfies

\[
\langle \mathcal{B} \tilde{v}, \tilde{w} \rangle = - \langle \tilde{v}, \mathcal{B} \tilde{w} \rangle
\]

38
with \( \rho_V, \rho_{\tilde{V}} \) denoting the projections on \( V \) and \( \tilde{V} \), respectively. We also have the relations

\[
\rho_V \mathcal{E} \rho_{\tilde{V}} = -\mathcal{G}^{-1}B \\
\rho_V \mathcal{E} \rho_{\tilde{V}} = \mathcal{G} - \mathcal{B} \mathcal{G}^{-1}B.
\]

Euler-Lagrange equations for the variable \( p \) imply

\[
\frac{\partial L}{\partial p} = 0 \\
\dot{q} = \mathcal{E}(p + q)
\]

thus,

\[ p = \mathcal{G} \dot{q} + B \dot{q}. \]

Hence, using the above equations to rewrite the lagrangian in terms of \( \dot{q}, \dot{q}' \) we obtain

\[
L(q, \dot{q}) = \frac{1}{2} \langle \dot{q}, \mathcal{G} \dot{q} \rangle - \frac{1}{2} \langle \dot{q}, \mathcal{G} \dot{q} \rangle - \langle \dot{q}, \mathcal{B} \dot{q} \rangle \\
= \frac{1}{2} \langle \dot{q} - \dot{q}', (\mathcal{G} + \mathcal{B})(\dot{q} + \dot{q}) \rangle
\]

the sigma model like 2nd-order lagrangian.

Acknowledgments

H.M. and A.C. thanks to CONICET for financial support.

References

[1] C. Klimcik, P. Severa, Poisson-Lie T-duality and loop groups of Drinfeld doubles, Phys. Lett. B 351, 455-462 (1995), hep-th/9512040, Dual non-Abelian duality and the Drinfeld double, Phys. Lett. B 372, 65-71 (1996), hep-th/9502122, Poisson-Lie T-duality; C. Klimcik, Nucl. Phys. Proc. Suppl. 46, 116-121 (1996), hep-th/9509095.

[2] Y. Lozano, Non Abelian duality and canonical transformations, Y. Lozano, Phys. Lett. B 355, 165-170 (1995), hep-th/9503045.

[3] A. Yu. Alekseev, A. Z. Malkin, Symplectic structures associated to Lie-Poisson groups, Comm. Math. Phys. 162, 147-74 (1994), hep-th/9303038.

[4] O. Alvarez, Target space duality I: general theory, Nucl. Phys. B 584 (2003) 659-681, hep-th/0003177, Target space duality II: applications, Nucl. Phys. B 584, 682-704 (2003), hep-th/0003178.
[5] K. Sfetsos, Canonical equivalence of non-isometric sigma-models and Poisson-Lie T-duality, Nucl. Phys. B 517, 549-566 (1998), hep-th/9710163; Poisson Lie T-duality beyond the classical level and the renormalization group Phys. Lett. B 434, 365-375 (1998), hep-th/9803019.

[6] A. Stern, Hamiltonian approach to Poisson Lie T-duality, Phys. Lett. B 450, 141-148 (1999), hep-th/9811256.

[7] A. Yu. Alekseev, C. Klimcik, A.A. Tseytlin, Quantum Poisson-Lie T-duality and WZNW model, Nucl. Phys. B 458, 430-444 (1996), hep-th/9509123.

[8] A. Cabrera, H. Montani, Hamiltonian loop group actions and T-Duality for group manifolds, J. Geom. Phys. 56, 1116-1143 (2006).

[9] Coste, A., Dazord, P. and Weinstein, A., Groupoîdes symplectiques, Publications du Departement de Mathematiques de l'Universite de Lyon, I, 2/A, 1-65 (1987); Mikami, K., and Weinstein, A., Moments and reduction for symplectic groupoid actions, Publ. RIMS Kyoto Univ. 24, 121-140 (1988).

[10] J. E. Marsden, A. Weinstein, Reduction of symplectic manifolds with symmetry, Rep. Math. Phys. 5, 121-131 (1974).

[11] A. Pressley, G. Segal, Loop groups, Oxford: Clarendon Press, 1986.

[12] J. Harnad, Constrained hamiltonian systems on Lie groups, moment map reductions and central extensions, Can. J. Phys. 72, 375-388 (1994) ; John Harnad, B. A. Kupershmidt, Symplectic Geometries on $T^*\tilde{G}$, hamiltonian Group Actions and Integrable Systems, J. Geom. Phys. 16, 168-206 (1995).

[13] V. G. Drinfeld, Hamiltonian Lie groups, Lie bialgebras and the geometric meaning of the classical Yang Baxter equation, Soviet Math. Dokl. 27, 68 (1983).

[14] J.-H. Lu, A. Weinstein, Poisson Lie groups, dressing transformations and Bruhat decomposition, J. Diff. Geom. 31, 501-526 (1990).

[15] M.A. Semenov-Tian-Shansky, Dressing transformations and Poisson group actions, Publ. RIMS, Kyoto Univ. 21 (1985), 1237-1260.

[16] V. Guillemin, S. Sternberg, Symplectic technics in physics, Cambridge, Cambridge Univ. Press, 1984.

[17] D. Kazhdan, B. Kostant , S. Sternberg, Hamiltonian group actions and dynamical system of Calogero type, Commun. Pure Appl. Math. 31, 481-508 (1978).

[18] J. Marsden, T. Ratiu, Introduction to Mechanics and Symmetry, Springer Verlag 1994.
[19] C. Klimcik, *Quasitriangular WZW model*, Rev. Math. Phys. 16, 679-808, (2004), hep-th/0103118.

[20] C. Klimcik, S. Parkhomenko, *Supersymmetric gauged WZNW models as dressing cosets*, Phys. Lett. B 463, 195-200 (1999), hep-th/9906163; S.E. Parkhomenko, *Poisson-Lie T-duality and Complex Geometry in N=2 superconformal WZNW models*, Nucl. Phys. B 510, 623-639 (1998), hep-th/9706199; C. Klimcik, *Poisson-Lie T-duality and (1,1) supersymmetry*, Phys. Lett. B 414, 85-91 (1997), hep-th/9707194.

[21] J. Balog, L. Feher, L. Palla, *Chiral Extensions of the WZNW Phase Space, Poisson-Lie Symmetries and Groupoids*, Nucl. Phys. B 568 (2000) 503-542, e-Print Archive: hep-th/9910046; L. Feher, I. Marshall, *The non-Abelian momentum map for Poisson-Lie symmetries on the chiral WZNW phase space*, math.QA/0401226; J. Balog, L. Feher, L. Palla, *On the Chiral WZNW Phase Space, Exchange r-Matrices and Poisson-Lie Groupoids*, Talk given at the Superior Mathematics Seminar, Montreal, Quebec, Canada, 26 Jul - 6 Aug 1999, e-Print Archive: hep-th/9912173.

[22] C. Klimcik, S. Parkhomenko, *Monodromic strings*, hep-th/0010084; C. Klimcik, S.E. Parkhomenko, *The Poisson-Lie T-duality and zero modes*, Theor. Math. Phys. 139:834-845, (2004).

[23] L. Hlavaty, L. Snobl, *Poisson-Lie T-plurality of three-dimensional conformally invariant sigma models*, JHEP 0410:045, (2004), hep-th/0403164.

[24] V. Chari, A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press 1994.