GLOBAL-IN-TIME STRICHARTZ ESTIMATES ON NON-TRAPPING ASYMPTOTICALLY CONIC MANIFOLDS

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Abstract. We prove global-in-time Strichartz estimates without loss of derivatives for the solution of the Schrödinger equation on a class of non-trapping asymptotically conic manifolds. We obtain estimates for the full set of admissible indices, including the endpoint, in both the homogeneous and inhomogeneous cases. This result improves on the results by Tao, Wunsch and the first author in [22] and [33], which are local in time, as well as the results of the second author in [40], which are global in time but with a loss of angular derivatives. In addition, the endpoint inhomogeneous estimate is a strengthened version of the uniform Sobolev estimate recently proved by Guillarmou and the first author [15].

1. Introduction

Strichartz estimates are an essential tool for studying the behaviour of solutions to nonlinear Schrödinger equations, nonlinear wave equations, and other nonlinear dispersive equations. In particular, global-in-time Strichartz estimates are needed to show global well-posedness and scattering for these equations. The purpose of this article is to prove global-in-time Strichartz estimates for the Schrödinger equation on asymptotically conic nontrapping manifolds.

Let \((M^0, g)\) be a Riemannian manifold of dimension \(n \geq 2\), and let \(I \subset \mathbb{R}\) be a time interval. Strichartz estimates are a family of dispersive estimates on solutions \(u(t, z)\): \(I \times M^0 \to \mathbb{C}\) to the Schrödinger equation

\[
    i\partial_t u + \Delta_g u = 0, \quad u(0) = u_0(z)
\]

where \(\Delta_g\) denotes the Laplace-Beltrami operator on \((M^0, g)\). The general Strichartz estimates state that

\[
    \|u(t, z)\|_{L^q_t L^r_x(I \times M^0)} \leq C\|u_0\|_{H^s(M^0)},
\]

where \(H^s\) denotes the \(L^2\)-Sobolev space over \(M^0\), and \((q, r)\) is an admissible pair, i.e.

\[
    2 \leq q, r \leq \infty, \quad 2/q + n/r = n/2, \quad (q, r, n) \neq (2, \infty, 2).
\]

It is well known that (1.1) holds for \((M^0, g) = (\mathbb{R}^n, \delta)\) with \(s = 0\) and \(I = \mathbb{R}\).

In this paper, we continue the investigations carried out in [21] [22] concerning Strichartz inequalities on a class of non-Euclidean spaces, that is, smooth complete noncompact asymptotically conic Riemannian manifolds \((M^0, g)\) which satisfy a nontrapping condition. Here, ‘asymptotically conic’ means that \(M^0\) has an end of the form \((r_0, \infty)_r \times Y\), with metric asymptotic to \(dr^2 + r^2 h\) as \(r \to \infty\), where \((Y, h)\) is a closed Riemannian manifold of dimension \(n - 1\) (a more precise definition is given below). In [22], the first author, Tao and Wunsch established the local in time Strichartz
inequalities
\[\|e^{it\Delta_y u_0}\|_{L^2_t L^2_y([0,1] \times M^o)} \leq C \|u_0\|_{L^2(M^o)}.\]

In this paper, we establish the same inequality on the full time interval, \(t \in \mathbb{R}\). To treat an infinite time interval, the method of \cite{22} no longer works, and we take a completely new approach in this paper (see Section 1.3). Although phrased in terms of asymptotically conic manifolds we emphasize that our results apply in particular to

- Schrödinger operators \(\Delta + V\) on \(\mathbb{R}^n\), with \(V\) suitably regular and decaying at infinity;
- nontrapping metric perturbations of flat Euclidean space, with the perturbation suitably regular and decaying at infinity.

1.1. Geometric setting. Let us recall the asymptotically conic geometric setting, which is the same as in \cite{16 17 20 22}. Let \((M^o, g)\) be a complete noncompact Riemannian manifold of dimension \(n \geq 2\) with one end, diffeomorphic to \((0, \infty) \times Y\) where \(Y\) is a smooth compact connected manifold without boundary. Moreover, we assume \((M^o, g)\) is asymptotically conic which means that \(M^o\) can be compactified to a manifold \(M\) with boundary \(\partial M = Y\) such that the metric \(g\) becomes a scattering metric on \(M\). That is, in a collar neighborhood \([0, \epsilon) \times \partial M\) of \(\partial M\), \(g\) takes the form
\[g = \frac{dx^2}{x^4} + \frac{h(x)}{x^2} = \frac{dx^2}{x^4} + \frac{\sum h_{jk}(x,y)dy^j dy^k}{x^2},\]
where \(x \in C^\infty(M)\) is a boundary defining function for \(\partial M\) and \(h\) is a smooth family of metrics on \(Y\). Here we use \(y = (y_1, \cdots, y_{n-1})\) for local coordinates on \(Y = \partial M\), and the local coordinates \((x,y)\) on \(M\) near \(\partial M\). Away from \(\partial M\), we use \(z = (z_1, \cdots, z_n)\) to denote the local coordinates. Moreover if every geodesic \(z(s)\) in \(M\) reaches \(Y\) as \(s \to \pm\infty\), we say \(M\) is nontrapping. The function \(r := 1/x\) near \(x = 0\) can be thought of as a “radial” variable near infinity and \(y = (y_1, \ldots, y_{n-1})\) can be regarded as \(n - 1\) “angular” variables. Rewriting \((1.4)\) using coordinates \((r, y)\), we see that the metric is asymptotic to the exact conic metric \(dr^2 + r^2h(0)\) on \((r_0, \infty), r \times Y\) as \(r \to \infty\).

The Euclidean space \(M^o = \mathbb{R}^n\), or any compactly supported perturbation of this metric, is an example of an asymptotically conic manifold with \(Y\) equal to \(S^{n-1}\) endowed with the standard metric.

Let \((M^o, g)\) be an asymptotically conic manifold. The complex Hilbert space \(L^2(M^o)\) is given by the inner product
\[\langle f_1, f_2 \rangle_{L^2(M^o)} = \int_{M^o} f_1(z) \overline{f_2(z)} dg(z)\]
where \(dg(z) = \sqrt{g} dz\) is the measure induced by the metric \(g\). Let \(\Delta_g = \nabla^* \nabla\) be the Laplace-Beltrami operator on \(M\); our sign convention is that \(\Delta_g\) is a positive operator. Let \(V\) be a real potential function on \(M\) such that
\[V \in C^\infty(M), \quad V(x,y) = O(x^3) \text{ as } x \to 0.\]

Assume that \(n \geq 3\) and that
\[H := \Delta_g + V\]
has no zero eigenvalue or zero-resonance.

That is, there is no solution \(u\) to \(Hu = 0\) such that \(u \to 0\) at infinity. These assumptions allow us to use the results of \cite{16, 17}. 
1.2. Main results. Now we consider the Schrödinger equation
\begin{equation}
    i\partial_t u + H u = 0 \quad u(0) = u_0(z).
\end{equation}
The main purpose of this paper is to prove the following results. Notice that the endpoint estimate ($q = 2$ and $\tilde{q} = 2$) is included in both cases.

**Theorem 1.1** (Long-time homogeneous Strichartz estimate). Let $(M^o, g)$ be an asymptotically conic non-trapping manifold of dimension $n \geq 3$. Let $H = \Delta_g + V$ satisfy (1.5) and (1.6) and suppose $u$ is the solution to \( (1.7) \). Then
\begin{equation}
    \|u(t, z)\|_{L^q_t L^r_z(\mathbb{R} \times M^o)} \leq C\|u_0\|_{L^2(M^o)},
\end{equation}
where the admissible pair $(q, r) \in [2, \infty]^2$ satisfies (1.2).

**Theorem 1.2** (Long-time inhomogeneous Strichartz estimate). Let $(M^o, g)$ and $H$ be as in Theorem 1.1. Suppose that $u$ solves the inhomogeneous Schrödinger equation with zero initial data
\begin{equation}
    i\partial_t u + H u = F(t, z).
\end{equation}
Then the inhomogeneous Strichartz estimate
\begin{equation}
    \|u(t, z)\|_{L^q_t L^r_z(\mathbb{R} \times M^o)} \leq C\|F\|_{L^q_t L^r_z(\mathbb{R} \times M^o)}
\end{equation}
holds for admissible pairs $(q, r), (\tilde{q}, \tilde{r})$.

1.3. Strategy of the proof. Our argument here extends to long time and to the endpoint the Strichartz estimates in [22] where the first author, Tao and Wunsch constructed a “local” parametrix for the propagator $e^{itH}$ based on the parametrix from [20]. In that paper, Schrödinger solutions $e^{itH}u_0$ were obtained by applying the parametrix to $u_0$ and then correcting this approximate solution using Duhamel’s formula, using local smoothing estimates to control the correction term. This approach works well on a finite time interval, but cannot be expected to work on an infinite time interval as the errors accumulate over time: certainly they cannot be expected to decay to zero as $t \to \infty$, as would be required to prove $L^q$ estimates in time on an infinite interval.

The main new idea in the current paper is to express the propagator $e^{itH}$ exactly using the spectral measure $dE_{\sqrt{\|H\|}}(\lambda)$, exploiting the very precise information on the spectral measure for the Laplacian on asymptotically conic nontrapping manifolds has recently become available from the works [23], [19], [16].

After expressing the propagator in terms of an integral of the multiplier $e^{it\lambda^2}$ against the spectral measure, our strategy is to use the abstract Strichartz estimate proved in Keel-Tao [27]. Thus, with $U(t)$ denoting the (abstract) propagator, we need to show uniform $L^2 \to L^2$ estimates for $U(t)$, and $L^1 \to L^\infty$ type dispersive estimate on the $U(t)U(s)^*$ with a bound of the form $O(|t - s|^{-n/2})$. In the flat Euclidean setting, the estimates are obvious because of the explicit formula for the propagator. But in our general setting it turns out to be more complicated. It follows from [20] that the propagator $U(t)(z, z')$ fails to satisfy such a dispersive estimate at any pair of conjugate points $(z, z') \in M^o \times M^o$ (i.e. pairs $(z, z')$ where geodesics emanating from $z$ focus at $z'$). Our geometric assumptions allow conjugate points, so we need to modify the propagator such that the failure of the dispersive estimate at conjugate points is avoided.
This is possible to the $TT^*$ nature of the estimates required by the Keel-Tao formalism. Recall that the dispersive estimate required by Keel-Tao is of the form
\begin{equation}
\|U(t)U(s)^*\|_{L^1 \to L^\infty} \lesssim C|t-s|^{-n/2}.
\end{equation}
If $U(t)$ is the propagator $e^{it\mathbf{H}}$ then the operator on the left hand side is $e^{i(t-s)\mathbf{H}}$. However, nothing in the Keel-Tao formalism requires the $U(t)$ to form a group of operators. Hence we are free to break up $e^{it\mathbf{H}} = \sum_j U_j(t)$ and prove the estimate (1.11) for each $U_j$. Our choice of $U_j(t)$ (sketched directly below) means that $U_j(t)U_j(s)^*$ is essentially the kernel $e^{i(t-s)\mathbf{H}}$ localized sufficiently close to the diagonal that we avoid pairs of conjugate points, and hence can prove the dispersive estimate.

Our method of decomposing $e^{it\mathbf{H}} = \sum_j U_j(t)$ is motivated by a decomposition used in the proof in [17] of a restriction estimate for the spectral measure, that is, an estimate of the form
\begin{equation}
\|dE_{\sqrt{\mathbf{H}}}(\lambda)\|_{L^p(M^\circ) \to L^{p'}(M^\circ)} \lesssim C\lambda^{n(1/p - 1/p') - 1}, \quad 1 \leq p \leq \frac{2(n+1)}{n+3}.
\end{equation}

In [17], it was observed that to prove a restriction estimate for $dE_{\sqrt{\mathbf{H}}}(\lambda)$, it suffices (via a $TT^*$ argument) to prove the same estimate for the operators $Q_i(\lambda)dE_{\sqrt{\mathbf{H}}}(\lambda)Q_i(\lambda)^*$, where $Q_i(\lambda)$ is a partition of the identity operator in $L^2(M^\circ)$. The operators $Q_i(\lambda)$ used in [17] are pseudodifferential operators (of a certain specific type) serving to localize $dE_{\sqrt{\mathbf{H}}}(\lambda)$ in phase space close to the diagonal. The authors of [17] showed that the localized operators $Q_i(\lambda)dE_{\sqrt{\mathbf{H}}}(\lambda)Q_i(\lambda)^*$ satisfy kernel estimates analogous to those satisfied by the spectral measure for $\sqrt{\Delta}$ on flat Euclidean space:
\begin{equation}
\left|\left(Q_i(\lambda)dE_{\sqrt{\mathbf{H}}}(\lambda)Q_i(\lambda)\right)(z,z')\right| \lesssim C\lambda^{n-1-j}(1 + \lambda d(z,z'))^{-(n-1)/2+j}, \quad j \in \mathbb{N},
\end{equation}
where $dE_{\sqrt{\mathbf{H}}}(\lambda)$ is the $j$th derivative in $\lambda$ of the spectral measure, and $d$ is the Riemannian distance on $M^\circ$.

The authors of [17] hoped that (1.12) could be used as a ‘black box’ in applications of their work. Unfortunately, (1.12) seems inadequate for our present purposes. This is because, in order to obtain the dispersive estimate, we need to efficiently exploit the oscillation of the ‘spectral multiplier’ $e^{it\mathbf{H}}$, and particularly the discrepancy between the way this function oscillates relative to the oscillations (in $\lambda$) of the Schwartz kernel of the spectral measure. The second main innovation of this paper is to improve estimate (1.12) on the localized spectral measure. We show

**Proposition 1.3.** Let $(M^\circ, g)$ and $\mathbf{H}$ be in Theorem [17]. Then there exists a $\lambda$-dependent operator partition of unity on $L^2(M)$
\begin{equation}
\text{Id} = \sum_{i=1}^N Q_i(\lambda),
\end{equation}
with $N$ independent of $\lambda$, such that for each $Q_i(\lambda)$ we have either the estimate
\begin{equation}
\left(Q_i(\lambda)dE_{\sqrt{\mathbf{H}}}(\lambda)Q_i(\lambda)^*\right)(z,z') = \lambda^{n-1} \sum_{\pm} e^{\pm i\lambda d(z,z')} a_{\pm}(\lambda, z, z'),
\end{equation}
\begin{equation}
|\partial^\alpha_{\lambda} a_{\pm}(\lambda, z, z')| \lesssim C_\alpha \lambda^{-\alpha}(1 + \lambda d(z,z'))^{-\frac{n-1}{2}},
\end{equation}
where $d(\cdot, \cdot)$ is the Riemannian distance on $M^o$, or the estimate
\begin{equation}
(1.14) \quad (Q_j(\lambda)dE_{\sqrt{H}(\lambda)}Q^*_j(\lambda))(z, z') = \lambda^{n-1}b(\lambda, z, z'), \quad |\partial^\alpha b(\lambda, z, z')| \leq C_\alpha \lambda^{-\alpha}.
\end{equation}

We now define localized (in phase space) propagators $U_j(t)$ by
\begin{equation}
(1.15) \quad U_j(t) = \int_0^\infty e^{it\lambda^2} Q_j(\lambda)dE_{\sqrt{H}(\lambda)}, \quad 1 \leq j \leq N.
\end{equation}
Then the operator $U_j(t)U_j(s)^*$ is given, at least formally, by (see Lemma 5.3)
\begin{equation}
(1.16) \quad U_j(t)U_j(s)^* = \int e^{i(t-s)\lambda^2} Q_j(\lambda)dE_{\sqrt{H}(\lambda)}Q_j(\lambda)^*.
\end{equation}
However, there are subtleties involved in spectral integrals such as $(1.15)$, $(1.16)$ containing operator-valued functions. Even to show that $(1.15)$ is well-defined as a bounded operator on $L^2(M^o)$ is nontrivial. The third main innovation of this paper is to give an effective method for analyzing spectral integrals such as $(1.13)$, $(1.14)$ with operator-valued multipliers. We use a dyadic decomposition in $\lambda$ and a Cotlar-Stein almost orthogonality argument to show the well-definedness of $(1.15)$ and prove a uniform estimate on $\|U_j(t)\|_{L^2 \to L^2}$, as required by the Keel-Tao formalism.

Having made sense of $(1.16)$, we exploit the oscillations both in the multiplier $e^{i(t-s)\lambda^2}$ and in the localized spectral measure (as expressed by $(1.13)$ and $(1.14)$) to obtain the required dispersive estimate for $U_j(t)U_j(s)^*$. The homogeneous Strichartz estimate for $e^{i|H|$ then follows by applying Keel-Tao to each $U_j$ and summing over $j$.

Next we consider the inhomogeneous Strichartz estimates. As is well-known, the non-endpoint cases of the inhomogeneous estimate follow from the homogeneous estimates and the Christ-Kiselev lemma. The endpoint inhomogeneous estimate requires an additional argument, and in particular, in this case we require estimates on $U_i(t)U_j(s)^*$ for $i \neq j$. This estimate turns out to be very similar to the uniform Sobolev estimate (on asymptotically conic nontrapping manifolds) of Guillarmou and the first author [15]. We use the techniques of that paper, in particular a refined partition of the identity operator. This resemblance to the proof in [15] is more than formal: as pointed out to us by Thomas Duyckaerts and Colin Guillarmou, the inhomogeneous endpoint Strichartz estimate implies the uniform Sobolev estimate; we sketch this argument in Section 8. Thus, this part of the paper can be regarded as a time-dependent reformulation of the proof in [15], leading to a more general result.

1.4. Previous literature. Now we review some classical results about the Strichartz estimates. In the flat Euclidean space, where $M^o = \mathbb{R}^n$ and $g_{jk} = \delta_{jk}$, one can take $I = \mathbb{R}$; see Strichartz [30], Ginibre and Velo [18], Keel and Tao [27], and references therein. The now-classic paper [27] by Keel-Tao developed an abstract approach to Strichartz estimates which has become the standard approach in most subsequent literature, including this paper. Strichartz estimates for compact metric perturbations of Euclidean space were proved locally in time by Staffilani and Tataru [37], and subsequently for asymptotically Euclidean manifolds by Robbinzo-Zuily [34] and Bouclet-Tzvetkov [10], and in the asymptotically conic setting by Hassell-Tao-Wunsch [23] and Mizutani [32]. In these works, either the metric is assumed nontrapping, or the theorem holds outside a compact set. In [23] the authors proved that Strichartz estimates without loss hold on an asymptotically conic manifold with hyperbolic trapped set.
Strichartz estimates have also been studied on exact cones [13] and on asymptotically hyperbolic spaces [9].

Strichartz estimates have also been studied on compact manifolds and on manifolds with boundary. In the compact case, Strichartz estimates usually are local in time and with some loss of derivatives $s$ (i.e. the RHS of (1.8) has to be replaced by the $H^s$ norm of $u_0$). Estimates for the standard flat 2-torus were shown by Bourgain [1] to hold for any $s > 0$. For any compact manifold, Burq et al. [2] showed that the estimate holds for $s = \frac{1}{q}$ and the loss of derivatives, as well as the localization in time, is sharp on the sphere. Manifolds with boundary were studied in [5, 6], [26], [7].

Global-in-time Strichartz estimates on asymptotically Euclidean spaces have been proved in Bouclet-Tzvetkov [11] (but with a low energy cutoff), Metcalfe-Tataru [29], Marzuola-Metcalfe-Tataru [30] and Marzuola-Metcalfe-Tataru-Tohaneanu [31].

As already noted, Strichartz estimates are an essential tool for studying the behaviour of solutions to nonlinear dispersive equations. There is a vast literature on this topic, and it is beyond the scope of this introduction to review it, so we refer instead to Tao’s book [38] and the references therein.

1.5. Organization of this paper. We review the partition of the identity and properties of the microlocalized spectral measure for low energies in Section 2 and for high frequency in Section 3. In Section 4, we prove Proposition 1.3 based on the properties of the microlocalized spectral measure. Section 5 is devoted to the construction of microlocalized propagators and the proof of the $L^2$-estimates. The dispersive estimates are proved in Section 6. Finally we prove the homogeneous Strichartz estimates in Section 7 and the inhomogeneous Strichartz estimates in Section 8.

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2. Spectral measure and partition of the identity at low energies

The spectral measure for the operator $H$ for low energies was constructed in [16], on the ‘low energy space’ $M^2_{k,b}$ Here we recall the low energy space $M^2_{k,b}$ and the associated space $M^2_{k,sc}$. The latter space is needed in order to define the class of pseudodifferential operators in which our operator partition $Q_i(\lambda)$ lies.

2.1. Low energy space. The low energy space $M^2_{k,b}$ was defined in [16] (based on unpublished work of Melrose-Sá Barreto) by performing iterated blow-ups on $[0, \lambda_0] \times M^2$. This space is illustrated in Figure 1. More precisely, we define the 3-codimension corner $C_3 = \{0\} \times \partial M \times \partial M$ and the 2-codimension submanifolds

$$C_{2,L} = \{0\} \times \partial M \times M, \quad C_{2,R} = \{0\} \times M \times \partial M, \quad C_{2,C} = [0,1] \times \partial M \times \partial M.$$ 

Without loss of generalities, we assume $\lambda_0 = 1$. The space $M^2_{k,b}$ is defined by

$$M^2_{k,b} = ([0,1] \times M^2; C_3, C_{2,R}, C_{2,L}, C_{2,C})$$
with blow-down map \( \beta : M^2_{k,b} \rightarrow [0, 1] \times M^2 \). The new boundary hypersurfaces created by these blowups are labelled by

\[
rb = \text{clos}_{\beta}^{-1}([0, 1] \times M \times \partial M), \quad \text{lb} = \text{clos}_{\beta}^{-1}([0, 1] \times \partial M \times M),
\]

\( zf = \text{clos}_{\beta}^{-1}({0} \times M \times M), \)

the ‘b-face’ \( \text{bf} = \text{clos}_{\beta}^{-1}(C_{2,C} \setminus C_3) \) and

\[
\text{bf}_0 = \beta_b^{-1}(C_3), \quad \text{rb}_0 = \text{clos}_{\beta}^{-1}(C_{2,R} \setminus C_3), \quad \text{lb}_0 = \text{clos}_{\beta}^{-1}(C_{2,L} \setminus C_3).
\]

The closed lifted diagonal is given by \( \text{diag}_b = \text{clos}_{\beta}^{-1}([0, 1] \times \{(m, m); m \in M^o\}) \), and its intersection with the face \( \text{bf} \) is denoted by \( \partial_d \text{diag}_b \). We remark that \( zf \) is canonically diffeomorphic to the b-double space

\[
M^2_b = [M^2; \partial M \times \partial M].
\]

And each section \( M^2_{k,b} \cap \{\lambda = \lambda_*\} \) for fixed \( 0 < \lambda_* < 1 \) is canonically diffeomorphic to \( M^2_b \).

We define functions \( x \) and \( y \) on \( M^2_{k,b} \) by lifting from the left factor of \( M \) (near \( \partial M \)), and \( x', y' \) by lifting from the right factor of \( M \); similarly \( z, z' \) (away from \( \partial M \)). Let \( \rho = x/\lambda, \rho' = x'/\lambda \) and \( \sigma = \rho/\rho' = x/x' \). Then we can use coordinates \((y, y', \sigma, \rho', \lambda)\) near \( \text{bf} \) and away from \( \text{rb} \); while \((y, y', \sigma^{-1}, \rho, \lambda)\) near \( \text{bf} \) and away from \( \text{lb} \).

We further define the space \( M^2_{k,sc} \) to be the blowup of \( M^2_{k,b} \) at \( \partial_d \text{diag}_b \). This space is illustrated in Figure 2. In local coordinates, near \( \partial_d \text{diag}_b \), a boundary defining function for \( \text{bf} \) is given by \( x/\lambda \), and the diagonal is given by \( \sigma = 1, y = y' \). Therefore, coordinates on the interior of the new boundary hypersurface, denoted \( sc \), created by this blowup are

\[
\frac{\lambda(\sigma - 1)}{x}, \frac{\lambda(y - y')}{x}, \lambda, y'.
\]

We define \( \text{diag}_{sc} \) to the closure of the interior of \( \text{diag}_b \) lifted to \( M^2_{k,sc} \).

The following lemma will be useful in our estimates.

**Lemma 2.1.** Let \( w = (w_1, \ldots, w_n) \) denote a set of coordinates defining \( \text{diag}_b \subset M^2_{k,b} \); for example, near \( \text{bf}_0 \) or \( \text{bf} \), we can take \( w = (\sigma - 1, y_1 - y'_1, \ldots, y_{n-1} - y'_{n-1}) \). Then \( |w|/x \) is comparable to \( d(z, z') \) in a neighbourhood of \( \text{diag}_b \). Equivalently, \( |w|/\rho \) is comparable to \( \lambda d(z, z') \).

**Proof.** Away from \( \text{bf}_0 \cup \text{bf} \), \( |w|^2 \) is a quadratic defining function for \( \text{diag}_b \), and so is \( d(z, z')^2 \), hence they are comparable. Now consider what happens near \( \text{bf}_0 \) or \( \text{bf} \). Using coordinates \( w = (\sigma - 1, y_1 - y'_1, \ldots, y_{n-1} - y'_{n-1}) \), we have

\[
\frac{|w|}{x} \sim \left| \frac{\sigma - 1}{x} \right| + \left| \frac{y - y'}{x} \right|.
\]

Write \( r = 1/x \); then this is

\[
|\sigma - 1| + r |y - y'|.
\]

Given that the metric takes the form \( dr^2 + r^2 h(x, y, dy) \), where \( h \) is positive definite, we see that this is comparable to \( d(z, z') \). \( \square \)

**Remark 2.2.** In the case \( M^o = \mathbb{R}^n \), with Euclidean coordinates \( z = (z_1, \ldots, z_n) \), we can take \( w = (z_1 - z'_1, \ldots, z_n - z'_n) \).
2.2. Pseudodifferential operators on the low energy space. We use the class of pseudodifferential operators $\Psi^m_k(M; \Omega^{1/2}_{k,b})$ on $M^2_{k,sc}$ introduced by Guillarmou and the first author. By definition, these operators have Schwartz kernels which are half-densities conormal to the diagonal $\text{diag}_{sc}$, smooth on $M^2_{k,sc}$ away from the diagonal, and rapidly decreasing at all boundary hypersurfaces not meeting the diagonal, i.e. at $lb_0$, $rb_0$, $lb$ and $rb$. In addition, we will only consider those operators with kernels supported where $\rho, \rho' \leq C < \infty$. In this setting we can write the kernel in the form

$$ (2.1) \quad \chi^n \int e^{\lambda/x}((\sigma-1)\nu+(y-y')\mu) a(\lambda, \rho, y, \mu, \nu) d\mu d\nu \left| \frac{dg'd\lambda}{\lambda} \right|^{1/2} $$

where $a$ is a classical symbol of order $m$ in the $(\mu, \nu)$ variables, smooth in $(\lambda, \rho, y)$ and supported where $\rho \leq c$. We remark that the $|d\lambda/\lambda|^{1/2}$ factor is purely formal; if we write this in the form $A(z, z', \lambda)|dg'd\lambda/\lambda|^{1/2}$, then the action on a half-density $f|dg|^{1/2}$ is given by

$$ \left( \int A(z, z', \lambda) f(z')dg(z') \right)|dg(z)|^{1/2}. $$

From this representation it is easy to see the following.

Figure 1. The manifold $M^2_{k,b}$. Arrows show the direction in which the indicated function increases from 0 to $\infty$. 
Figure 2. The manifold $M^2_{k,sc}$; the dashed line is the boundary of the lifted diagonal $\Delta_{k,sc}$

Lemma 2.3. If $A \in \Psi^m_k(M; \Omega^{1/2}_{k,b})$ then $(\lambda \partial_\lambda)^N A$ is also a pseudodifferential operator of order $m$, i.e. $(\lambda \partial_\lambda)^N A \in \Psi^m_k(M; \Omega^{1/2}_{k,b})$.

Proof. It suffices to prove for $N = 1$ and use induction. If $\lambda \partial_\lambda$ hits the function $a$ in (2.1), then $a$ is still a symbol of order $m$ in the $(\mu, \nu)$ variables, smooth in $(\lambda, \rho, y)$ and supported where $\rho \leq c$. (Notice that $\rho = x/\lambda$ depends on $\lambda$ as well.) On the other hand, if $\lambda \partial_\lambda$ hits the phase, this is the same as $\nu \partial_\nu + \mu \cdot \partial_\mu$ hitting the phase, as it is homogeneous of degree 1 in both $\lambda$ and in $(\nu, \mu)$. Integrating by parts we obtain another symbol $\tilde{a}$ of order $m$. This completes the proof. \hfill $\square$

Lemma 2.4. If $A \in \Psi^m_k(M; \Omega^{1/2}_{k,b})$, and if $m < -n$, then $A$ satisfies a kernel bound

$$\left| A(z, z') \right| \leq \lambda^n \left( 1 + \lambda d(z, z') \right)^{-N}$$

for any $N \in \mathbb{N}$.

Proof. If the order $m$ is less than $-n$, then the integral (2.1) is absolutely convergent, showing that the kernel of $\lambda^{-n} A$ is uniformly bounded. Next, we note that the
differential operator
\[ 1 - \partial^2_{\nu} - \sum_{\mu} \partial^2_{\mu} \]
leaves the exponential in (2.1) invariant. By applying this \( N \) times to the exponential and then integrating by parts, we see that the integral is bounded by
\[ C_N \left( 1 + \lambda^2 \left( x^{-2} - 2 + x^{-2} |y-y'|^2 \right) \right)^{-N} \]
for any \( N \). Finally, as in the proof of Lemma 2.1, the square of the Riemannian distance on \( M \) is comparable to
\[ \frac{(\sigma - 1)^2}{x^2} + \frac{|y-y'|^2}{x^2}, \]
so the integral is bounded by \( C_N (1 + \lambda d(z,z'))^{-N} \) for any \( N \).

Corollary 2.5. If \( A \in \Psi^K(M; \Omega^{1/2}) \), and if \( m < -n \), then \( A \) is bounded \( L^2(M^o) \rightarrow L^2(M^o) \) uniformly as \( \lambda \rightarrow 0 \). The same is true for \( (\lambda \partial) N A \) for any \( N \).

Proof. This follows from the kernel bound in Lemma 2.4, the volume estimate \( cr^n \leq V(z,r) \leq Cr^n \) for the volume \( V(z,r) \) of the ball of radius \( r \) centered at \( z \in M^o \), and Schur’s test. \( \square \)

2.3. Partition of the identity. We essentially the same partition of the identity as in [17]. We briefly recall how this is defined. We choose a function \( \chi \in C^\infty(\mathbb{R}) \) of a real variable, with \( \chi = 0 \) for \( t \leq \varepsilon \) and \( \chi = 1 \) for \( t \geq 2 \varepsilon \). We define \( Q^{\text{low}}_1(\lambda) \) to be multiplication by the function \( 1 - \chi(\rho) \) (recall \( \rho = x/\lambda \)). Next, we choose \( Q'_1(\lambda) \) such that \( \text{Id} - Q'_1(\lambda) \) is microlocally equal to the identity for \( 1/2 \leq |\mu|^2 + \nu^2 \leq 3/2 \), and microsupported in \( 1/4 \leq |\mu|^2 + \nu^2 \leq 2 \). Then we define \( Q^{\text{low}}_1(\lambda) = \chi(\rho)Q'_1(\lambda) \). This means that \( \text{Id} - Q^{\text{low}}_0 - Q^{\text{low}}_1 \) is supported where \( \rho \) is small and close to the characteristic variety. We then decompose this as \( Q^{\text{low}}_2 + \cdots + Q^{\text{low}}_N \) such that each \( Q^{\text{low}}_i, i \geq 2 \) has support where \( \nu \) is contained in a small interval.

2.4. Localized spectral measure. The main result of [16] was that the spectral measure for the Laplacian on an asymptotically conic manifold is, for low energies, a Legendre distribution associated to a pair of Legendre submanifolds, the ‘propagating Legendrian’ \( L^B \) and the ‘incoming/outgoing Legendrian’ \( L^L \). Rather than explain the meaning of these statements, we discuss some consequences of this result for the localized spectral measure, by which we mean the operator \( Q(\lambda) dE_{\sqrt{H}(\lambda)} Q(\lambda)^* \) where \( Q(\lambda) \) is a member of our partition of the identity.

Proposition 2.6. Let \( Q^{\text{low}}_i(\lambda) \) be a member of the partition of the identity defined above. Let \( \eta > 0 \) be given. Then for \( i, j = 0 \) or \( 1 \), \( Q^{\text{low}}_i(\lambda) dE_{\sqrt{H}(\lambda)} Q^{\text{low}}_j(\lambda)^* \) satisfies the estimates on the RHS of (1.14), and \( Q^{\text{low}}_i(\lambda) dE_{\sqrt{H}(\lambda)} Q^{\text{low}}_i(\lambda)^*, i \geq 2 \) can be written as a finite sum of terms of the following two types:

(i) An oscillatory function of the form
\[ \lambda^{n-1} e^{\pm i \lambda d_{\text{conic}}(y,y')} a(y,y',\sigma,x,\lambda) \]
where \( a \) is supported where \( x, x' \leq \eta \) and \( d_{\partial M}(y,y') \leq \eta \) and satisfies estimate (1.14);
(ii) An oscillatory integral of the form

\[ \lambda^{n-1} \int_{\mathbb{R}^{n-1}} e^{i\Phi(y,y',\sigma,v)/\rho} b(y,y',\sigma,v,\rho,\lambda) \, dv \]

where \( b \) is smooth in all its arguments, and supported in a small neighbourhood of a point \((y_0,y_0,1,v_0,0,0)\) such that \( d_v \Phi(y_0,y_0,1,v_0) = 0 \). Moreover, writing \( w = (w_1, \ldots, w_n) \) for a set of coordinates defining \( \text{diag}_b \subset M_{k,b}^2 \), i.e. \( w = (y-y', \sigma-1) \), and \( v = (v_2, \ldots, v_n) \), one can rotate in the \( w \) variables such that the function \( \Phi = \Phi(y,w,v) \) has the properties

\[
\begin{aligned}
(a) & \quad d_v \Phi = w_j + O(w_1), \\
(b) & \quad \Phi = \sum_{j=2}^n v_j d_v \Phi + O(w_1), \\
(c) & \quad d^2_{v_j y_k} \Phi = w_1 A_{jk}, \\
(d) & \quad d_v \Phi = 0 \implies \Phi(y,y',\sigma)/x = \pm d_{\text{conic}}(y,y',\sigma/\pi,\pi)
\end{aligned}
\]

where \( A_{jk} \) is nondegenerate for all \((y,w,v)\) in the support of \( b \). Here \( d_{\text{conic}} \) is the metric on the exact cone with cross section \((\partial M,h)\).

**Remark 2.7.** The conic metric \( d_{\text{conic}} \) has an explicit expression when \( d_{\partial M}(y,y') < \pi \). Writing \( r = 1/x, r' = 1/x' = \sigma/x \), it takes the form

\[ d_{\text{conic}}(y,y',r,r') = \sqrt{r^2 + r'^2 - 2rr' \cos d_{\partial M}(y,y')} = r \sqrt{1 + \sigma^2 - 2\sigma \cos d_{\partial M}(y,y')} \]

where \( d_{\partial M} \) denotes the metric on \( \partial M \) with respect to the Riemannian metric \( h \). Notice that \( d_{\text{conic}}(y,y',r,r')/r \) is only a function of \((y,y',\sigma)\), hence (d) makes sense.

**Proof.** The first statement, about \( Q^\text{low}_i(\lambda)dE_\sqrt{H}(\lambda)Q^\text{low}_j(\lambda)^* \) for \( i,j = 0,1 \), follows from the microlocal support estimates in [17, Section 5]. In fact, \( Q^\text{low}_i(\lambda) \) has empty wavefront set, while \( Q^\text{low}_i(\lambda) \) has wavefront set disjoint from the characteristic variety of \( H - \lambda^2 \), which contains the microlocal support of \( dE_\sqrt{H}(\lambda) \). It follows that \( Q^\text{low}_i(\lambda)dE_\sqrt{H}(\lambda)Q^\text{low}_j(\lambda)^* \), for \( i, j = 0,1 \), vanishes rapidly at \( \text{bf} \), \( \text{lf} \) and \( \text{rb} \), and is polyhomogeneous conormal at the other boundary hypersurfaces of \( M_{k,b}^2 \), vanishing to order \( n-1 \) at each of these faces. The estimate \([16,14]\) follows directly.

We next discuss (ii). Everything in this statement has been proved in [17, Lemma 6.5 and Proposition 6.2] except for the statement that \( \Phi \) is given by the conic distance function when \( d_v \Phi = 0 \). This can be extracted from [16]. First observe that the phase function \( \Phi \) parametrizes the Legendre submanifold \( L^{\text{bf}} \) defined in [23, 24, 10], given in the coordinates \((y,y',\sigma,\mu,\mu',\nu,\nu')\) of [16, Section 3] by the union of the leaves \( \gamma^2 = \gamma^2(y,\eta) \),

\[
\gamma^2 = \{ (y,y',\sigma = x/x',\mu,\mu',\nu,\nu') \mid y = y(s), y' = y(s'), \mu = \eta(s) \sin s, \mu' = -\eta(s') \sin s', \nu = -\cos s, \nu' = \cos s', \sigma = \sin s/\sin s', (s,s') \in [0,\pi]^2 \}
\]

as \((y,\eta)\) ranges over the cosphere bundle of \( T^* \partial M \) and \((y(s),\eta(s))\) is the geodesic with \((y(0),\eta(0)) = (y,\eta)\). Here parametrization is meant in the sense that the set

\[ \{ d(\lambda \Phi(y,y',\sigma,v)/x) \mid d_v \Phi = 0 \} = L^{\text{bf}}. \]
In particular, $\nu$ and $\nu'$ are defined as the coefficients of $\lambda d(1/x)$ and $\lambda d(1/x')$ respectively. It is straightforward to check that the coefficients of $\lambda d(1/x)$ and $\lambda d(1/x')$ in the expression for $d(\lambda \Phi/x)$ are
\[
\nu = \Phi - \frac{\partial \Phi}{\partial \sigma}, \quad \nu' = \frac{\partial \Phi}{\partial \sigma}.
\]
It follows that $\Phi = \nu + \sigma \nu'$. Therefore, in terms of $s$ and $s'$, from (2.6) we see that
\[
d_c \Phi = 0 \implies \Phi = -\cos s + \sigma \cos s'.
\]
If we square this then we get
\[
d_c \Phi = 0 \implies \Phi^2 = \cos^2 s + \sigma^2 \cos^2 s' - 2\sigma \cos s \cos s'.
\]
We can write the RHS in the form
\[
1 - \sin^2 s + \sigma^2 (1 - \sin^2 s') - 2\sigma (\cos s - \cos s') - s \sin s'\cos s'.
\]
Noting that $\sin^2 s + \sigma^2 \sin^2 s' = 2\sigma \sin s \sin s'$, using the expression for $\sigma$ in (2.6), we see that
\[
d_c \Phi = 0 \implies \Phi^2 = 1 + \sigma^2 - 2\sigma \cos d_{\text{diff}}(y, y').
\]
We also recall from [17, Section 6] that the $Q^\text{low}(\lambda)$, $i \geq 2$, are chosen so that $\nu$ lies in some small interval on the wavefront set of $Q^\text{low}(\lambda)$. It follows that $|\nu + \nu'|$ is small on the microlocal support of $Q^\text{low}(\lambda)dE_{\text{H}}(\lambda)Q^\text{low}(\lambda)^*$, which implies using (2.6) that $s$ is close to $s'$, hence $y$ is close to $y'$. Therefore the formula (2.5) is valid, and we see that
\[
d_c \Phi = 0 \implies \Phi^2 = x^2 d_{\text{conic}}^2(y, y', \frac{1}{x}, \frac{\sigma}{x}).
\]
We now apply [17, Lemma 6.5]. This tells us that for any point in the microlocal support of $Q^\text{low}(\lambda)dE_{\text{H}}(\lambda)Q^\text{low}(\lambda)^*$, $i \geq 2$, either there is a neighbourhood in which $L$ projects diffeomorphically to the base $M^n$, or the point lies at the conormal bundle to the diagonal, i.e. $y = y'$, $\sigma = 1$, $\mu = -\mu'$, $\nu = -\nu'$. In the former case, the function $d_{\text{conic}}(y, y', \frac{1}{x}, \frac{\sigma}{x})$ can be used directly as the phase function, and we obtain the statement (i) in the Proposition. In the latter case, a phase function $\Phi$ depending on $n - 1$ variables $v_2, \ldots, v_n$ was constructed explicitly in [17, Proposition 6.2], and satisfies properties (a) – (c) of (2.4). □

Remark 2.8. It might help to give an example to show how this works. In Euclidean space, the Schwartz kernel of the spectral measure $dE_{\sqrt{\Delta}}(\lambda)$ of $\sqrt{\Delta}$ is given by
\[
dE_{\sqrt{\Delta}}(\lambda; z, z') = \frac{\lambda^{n-1}}{(2\pi)^n} \int_{\mathbb{S}^{n-1}} e^{i\lambda(z-z') \cdot \zeta} d\zeta,
\]
one can find the phase function $(z-z') \cdot \zeta$, where $\zeta \in \mathbb{S}^{n-1}$. Locally near $\zeta = (1, 0, \ldots, 0)$, we can write $\zeta = (\sqrt{1 - |v|^2}, v_2, \ldots, v_n)$. Write $x = |z|^{-1}$ and $w = (z - z')/|z|$. Then the phase function becomes
\[
\Phi = w_1 \sqrt{1 - v_2^2 - \cdots - v_n^2} + \sum_{j=2}^n w_j v_j,
\]
and we can check that properties (a) – (d) of (2.4) hold in this case.
3. Spectral measure and partition of the identity at high energies

In the previous section we recalled the partition of the identity operator and the structure of the localized spectral measure for low energy i.e. $0 < \lambda \leq \lambda_0$. We now do the same for high energies, $\lambda \in [\lambda_0, \infty)$. For the sake of convenience, we introduce the semiclassical parameter $h = \lambda^{-1}$ (which should not be confused with $h$ in the metric $g$), so that we pay our attention to the range $h \in (0, h_0]$, where $h_0 = \lambda_0^{-1}$. The spectral measure of the operator $H$ for high energy was constructed in [19] on the high energy space $X$. Our main task is to adapt each of main results in the previous section to the high energy setting.

3.1. High energy space. The high energy $X$ was defined in [19]. Now we briefly recall it. The high energy space $X$ is defined by $X = [0, h_0] \times M^2$, where $M^2 = [M^2; \partial M \times \partial M]$ is the $b$-double space defined to be the radial blowup of $M^2$ at the corner $(\partial M)^2$. Informally, $M^2$ is the blowup space so that the boundary defining functions $x/(x + x')$ and $x + x'$ are smooth near the corner $x = x' = 0$. We define the boundary hypersurfaces in $X$ by

$$rb = [0, h_0] \times M \times \partial M, \quad lb = [0, h_0] \times \partial M \times M, \quad bf = [0, h_0] \times \partial M \times \partial M,$$

and the ‘main face’ $mf = \{0\} \times M^2$. The labelling of boundary hypersurfaces is consistent with the notations defined in the low energy space, since when $\lambda \in (C^{-1}, C)$ (where $\lambda = 1/h$) the spaces both have the form $(C^{-1}, C) \times M^2_b$. Recall $\sigma = x/x'$, we can use coordinates $(y, y', \sigma, x', h)$ near $bf$ and away from $rb$, and coordinates $(y, y', \sigma^{-1}, x, h)$ near $bf$ and away from $lb$. We use coordinates $(z, z', h)$ away from $bf$, $rb$ and $lb$.

3.2. Semiclassical scattering pseudodifferential operators. We recall the space $\Psi^{m,l,k}_{sc, h}(M; s\Phi^1/2)$ of semiclassical scattering pseudodifferential operators, introduced by Wunsch and Zworski [39] based on Melrose [32], is indexed by the differential order $m$, the boundary order $l$ and the semiclassical order $k$. One can express the space of the operators in terms of the space $\Psi^{m,0,0}_{sc, h}(M; s\Phi^1/2)$ by

$$\Psi^{m,l,k}_{sc, h}(M; s\Phi^1/2) = x^l h^{-k} \Psi^{m,0,0}_{sc, h}(M; s\Phi^1/2).$$

The semiclassical pseudodifferential operator $A \in \Psi^{m,l,k}_{sc, h}$ takes the kernel

$$h^{-n} \int e^{i(z - z') \zeta/h} a(z, \zeta, h) d\zeta \left| \frac{dg d\eta dh}{h^2} \right|^{1/2}$$

in the interior, or

$$h^{-n} \int e^{i((y - y') \eta + (\sigma - 1)\nu)/(hx)} a(x, y, \eta, \nu, h) d\eta d\nu \left| \frac{dg d\eta dh}{h^2} \right|^{1/2}$$

near the boundary. Here $a$ is a symbol of order $m$ in the variable $\zeta$ or $(\eta, \nu)$ variables and satisfies $h^{k+l-1} a(0, \Phi^1/2) \in C^\infty((0, h_0) \times \ac T^* M)$, where $\sigma$ is a boundary defining function for boundary hypersurface of $\ac T^* M$ at fiber-infinity, that is $\sigma = |(\eta, \nu)|$. Since the case in the interior is same as the classical pseudos, so we only need consider the case near the boundary i.e. $a$ is supported on $x h \leq c$ for small constant $c$.

**Lemma 3.1.** If $A \in \Psi^{m,0,0}_{sc, h}(M; s\Phi^1/2)$ then $(h\partial_h)^N A$ is also a pseudodifferential operator of order $m$, i.e. $(h\partial_h)^N A \in \Psi^{m,0,0}_{sc, h}(M; s\Phi^1/2)$. 

Proof. The proof is parallel to the argument in Lemma 2.3. By induction, we only need consider \( N = 1 \). If \( h \partial_h \) hits the function \( a \) in (3.1), then \( a \) is still a symbol of order \( m \) in the \( (\eta, \nu) \) variables, smooth in \( (h, x, y) \) and supported where \( xh \leq c \). On the other hand, if \( h \partial_h \) hits the phase, this is the same as \( \nu \partial_\nu + \eta \partial_\eta \), hitting the phase, as it brings a factor which is homogeneous of degree \(-1\) in \( h \) and of degree \( 1 \) in \( (\nu, \eta) \). Integrating by parts we obtain another symbol \( \hat{a} \) of order \( m \). This completes the proof. \( \square \)

**Lemma 3.2.** If \( A \in \Psi_{sc,h}^{m,0,0}(M; \ast^g \Omega^{1/2}) \), and if \( m < -n \), then \( A \) satisfies a kernel bound
\[
\left| A(z, z') \right| \leq h^{-n} \left( 1 + h^{-1}d(z, z') \right)^{-N}
\]
for any \( N \in \mathbb{N} \).

**Proof.** This is also in the same spirit of Lemma 2.3. If the order \( m \) is less than \(-n\), then the integral (3.1) is absolutely convergent, showing that the kernel of \( h^n A \) is uniformly bounded. Next, we note that the differential operator
\[
\frac{1 - \partial_j^2 - \sum_i \partial_i^2}{1 + ((hx)^{-2}(\sigma - 1)^2 + (hx)^{-2}|y - y'|^2)}
\]
leaves the exponential in (3.1) invariant. By integrating by parts \( N \)-times, we see that the integral is bounded by
\[
C_N \left( 1 + ((hx)^{-2}(\sigma - 1)^2 + (hx)^{-2}|y - y'|^2) \right)^{-N}
\]
for any \( N \). Finally, we note that the square of the Riemannian distance on \( M \) is comparable to
\[
\frac{(\sigma - 1)^2}{x^2} + \frac{|y - y'|^2}{x^2},
\]
so the integral is bounded by \( C_N (1 + h^{-1}d(z, z'))^{-N} \) for any \( N \).

**Corollary 3.3.** If \( A \in \Psi_{sc,h}^{m,0,0}(M; \ast^g \Omega^{1/2}) \), and if \( m < -n \), then \( A \) is bounded \( L^2(M^0) \to L^2(M^0) \) uniformly as \( h \to 0 \). The same is true for \( (h \partial_h)^N A \) for any \( N \).

**Proof.** This follows from the kernel bound in Lemma 3.2 and Schur’s test, since the volume estimate \( cr^n \leq V(z, r) \leq Cr^n \) for the volume \( V(z, r) \) of the ball of radius \( r \) centred at \( z \in M^0 \). \( \square \)

### 3.3. The partition of the identity.
We follow essentially the same construction of the operator partition of the unity as in [17] which is slightly different from the low energy one. For convenience, we briefly recall how this is constructed. We choose \( Q_1^{\text{high}} \) such that \( \text{Id} - Q_1^{\text{high}} \) is microlocally equal to the identity when \( 1/2 \leq \sigma(h^2 \Delta_g) \leq 3/2 \), and microsupported in \( 1/4 \leq \sigma(h^2 \Delta_g) \leq 2 \) (here \( \sigma \) denotes the semiclassical principal symbol). Noting that \( \text{Id} - Q_1^{\text{high}} \) is microsupported close to the characteristic variety of \( h^2 \Delta_g \), we further decompose \( \text{Id} - Q_1^{\text{high}} \) as \( Q_2^{\text{high}} + \cdots + Q_{N_1}^{\text{high}} + \cdots + Q_{N_2}^{\text{high}} \) such that each \( Q_i^{\text{high}} \) has support in \( \{ x + x' < 2\epsilon \} \) where \( \nu \) is contained in a small interval; and each \( Q_i^{\text{high}} \), \( i \geq N' + 1 \) has sufficiently small microsupport in the interior of \( T^* M^0 \cap \{ x \geq \epsilon/2 \} \). Note that we may (and will) assume that \( N' = N_i \) and that \( Q_i^{\text{high}}(\lambda) = Q_i^{\text{low}}(\lambda) \) for intermediate energies, and for \( 1 \leq i \leq N_i \).
3.4. Localized spectral measure. In [19], Wunsch and the first author showed that the spectral measure for the Laplacian on this setting is, for high energy, a Legendre distribution. We give some consequences of this result for the localized spectral measure needed in this paper. As in the low energy, the microlocalized spectral measure means distribution. We give some consequences of this result for the localized spectral measure the spectral measure for the Laplacian on this setting is, for high energy, a Legendre

\[ h^{-\frac{1}{2}} e^{i\epsilon d(z',z)/h} \tilde{a}(z, z', h) \]

where \( \tilde{a} \) satisfies estimate \( \lambda \).

(iii) An oscillatory integral supported near \( x = x' = 0 \) of the form

\[ h^{-\frac{1}{2}} \int_{\mathbb{R}^{n-1}} e^{i\epsilon \Phi(z,z',v)/h} b(z, z', v, h) dv, \]

where \( b \) is smooth in all its arguments, and supported in a small neighbourhood of a point \((y_0, y_0, 1, 0, v_0, 0)\) such that \( d_v \Psi(y_0, y_0, 1, v_0) = 0 \). Moreover, writing \( w = (w_1, \ldots, w_n) \) for a set of coordinates defining \( \text{diag}_A \subset M_\beta^2 \), i.e. \( w = (y, y', \sigma - 1) \), and \( v = (v_2, \ldots, v_n) \), one can rotate in the \( w \) variables such that the function \( \Psi = \Psi(y, w, x, v) \) has the same properties \( \lambda \), as \( \Phi \) for sufficiently small \( x \).

Proof. The proof is analogous to the proof of Proposition 2.6 with the main difference being that the computation takes place over the whole of \( M_\beta^2 \) (including the interior), not just at the boundary as is the case in the low energy case. We prove (iii), i.e. we work in the interior of \( M_\beta^2 \), using coordinates \((z, z')\), with \( z \) a coordinate on the left copy of \( M^0 \), and \( z' \) on the right copy. The proof for (ii) is only notationally different.
Recall from [19] that the phase function $\Psi$ parametrizes the Legendre submanifold (3.6)
\[ L = \{(z, \zeta, z', \zeta', \tau) \mid (z, \zeta) = G_{\tau}(z', -\zeta')\} \]
where $G_{\tau}$ is the geodesic flow at time $t$ — this follows from [17] Equation 7.9 and the discussion following. To say that $\Psi$ parametrizes $L$ is to say that locally,
\[ L = \{(z, d_z\Psi, z', d\zeta, \Psi) \mid d_z\Psi = 0\}. \]
In particular, $\tau$ is equal to the value of the phase function. It follows from this description and from (3.3) that $d_v\Psi = 0$ implies that $\Psi = \pm d(z, z')$. The choice of signs here arises from the fact that any $(z, z') \in (M^0)^2$, close to but not on the diagonal, has a neighbourhood $U$ such that there are two sheets of $L$ lying above $U$. This corresponds in (3.30) to changing the signs of $\zeta, \zeta'$ and $\tau$. One sheet is parametrized by $+d(z, z')$, and the other by $-d(z, z')$. A similar statement can be made about the choice of signs in (2.2).

We now apply [17] Lemma 7.6 and (ii) of Lemma 7.7. This tells us that for any point in the microlocal support of $Q_i^{\text{high}}(\lambda) dE_i F(\lambda) Q_i^{\text{high}}(\lambda)^*$, either there is a neighbourhood in which $L$ projects diffeomorphically to the base $M^2_b$, or the point lies at the conormal bundle to the diagonal, i.e. $z = z', \zeta = -\zeta'$. In the former case, the function $\pm d(z, z')$ can be used directly as the phase function, and we obtain the statement (i) in the Proposition. In the latter case, a phase function $\Psi$ depending on $n-1$ variables $v_2, \ldots, v_n$ can be constructed following the general approach of [17] Proposition 7.5. Since this was not written down explicitly in the coordinates $(z, z')$ valid in the interior of $M^2_b$ we sketch briefly how this is done. It follows from the proof of Lemma 7.6 of [17], we can rotate coordinates so that $w_1, \zeta_2, \ldots, \zeta_n, z'$ give coordinates on $L$ locally. (The proof of Lemma 7.6 shows that one can take $(\tau, \zeta_2, \ldots, \zeta_n, z')$ but since it is also shown that $\partial z_1/\partial \tau \neq 0$, then one can substitute $z_1$ for $\tau$, and then substitute $w_1 = z_1 - z_1'$ for $z_1$.) One can therefore express the functions $w_2, \ldots, w_n$, and $\tau$ on $L$ as smooth functions $W_j(w_1, \zeta_2, \ldots, \zeta_n, z')$ and $T(w_1, \zeta_2, \ldots, \zeta_n, z')$ of these coordinates. Then the function
\[ \Psi(w, z', v) = \sum_{j=2}^n (w_j - W_j(w_1, \zeta_2, \ldots, \zeta_n, z')) v_j + T(w_1, \zeta_2, \ldots, \zeta_n, z') \]
satisfies the requirements of (3.3), and parametrizes $L$ locally. This is shown by adapting the argument of [17] Proof of Proposition 6.2 in a straightforward way (which itself is a minor variation on [25] Theorem 21.2.18), so we omit the details. This establishes part (iii) of the Proposition. When working close to $x = x' = 0$, we need to use coordinates as in [17] Proposition 7.5 and apply [17] Lemma 7.6 and (i) of Lemma 7.7, and we end up with the statement in part (ii).

Remark 3.6. The Lagrangian $L$ is smooth up to the boundary when viewed as a submanifold in the ‘scattering-fibred cotangent bundle’ described in [16]. The boundary at $bf$ is naturally isomorphic to $L^\text{bf}$ in Proposition 2.6. Correspondingly, we find that the distance function $d(z, z')$ on $M^2_b$ satisfies
\[ d(z, z') - d_{\text{conic}}(y, y', \frac{1}{x}, \frac{1}{x'}) = e(z, z') \]
is a smooth function on $M^2_\delta$, or more precisely on that part of $M^2_\delta$ where $x, x' \leq \eta$ and $d_{\partial M}(y, y') \leq \eta$ for sufficiently small $\eta$ (see [21, Lemma 9.4]). From this we see that the results of Proposition 2.6 and Proposition 3.4 are compatible, as the factor $\exp(i\lambda e(z, z'))$ which is the discrepancy between (2.2) and (3.2), and between (2.4)(d) and (3.5)(d), can be absorbed in the symbol $\tilde{a}$, respectively $b$.

Remark 3.7. The results of this paper could be extended to long range scattering metrics, as treated in [22]. However, this would require an extension of the results of [21], [19] and [16] to Lagrangian submanifolds which are only conormal, rather than smooth, at the boundary. If this were done, then the discrepancy $e(z, z')$ between the distance function and the conic distance function is no longer smooth, or even bounded, but rather is conormal at the boundary with a bound of the form $(x + x')^{-1+\epsilon}$ at the boundary of $M^2_\delta$, i.e. a bit smaller than the distance functions themselves. In this case, the correct description of the localized spectral measure is with the true distance function $d(z, z')$ as phase function, rather than (2.2), which is only true in the short range case.

4. Proof of Proposition 1.3

We now prove Proposition 1.3. We define our partition of unity $Q_i$ by combining the low energy and high energy partitions. We choose a cutoff function $\chi(\lambda)$ supported in $[0, 2]$ such that $1 - \chi$ is supported in $[1, \infty)$, and define

$$Q_1 = \chi(\lambda)(Q_{0_{\text{low}}}^{\text{low}} + Q_1^{\text{low}}), \quad Q_i = \chi(\lambda)Q_i^{\text{high}}, \quad \text{for } 2 \leq i \leq N_i;$$

$$Q_i = (1 - \chi(\lambda))Q_i^{\text{high}}, \quad \text{for } N_i + 1 \leq i \leq N := N_i + N_h.$$

We prove the Proposition initially for low energies, i.e. for $i \leq N_i$. Proposition 2.6 shows that $Q_1$ satisfies (1.14). For $Q_j$, $j \geq 2$, consider the first type of representation, (2.2), for the localized spectral measure. In this case, the result follows from Remark 3.7 which allows us to replace the conic distance function with the true distance function on the support of $a$, since we have

$$\exp(i\lambda d(z, z')) = \exp(i\lambda d_{\text{conic}}(z, z')) \exp(i\lambda e(z, z')), \quad \left| \partial_\lambda^k \exp(i\lambda e(z, z')) \right| \leq C.$$

Thus the factor $\exp(i\lambda e(z, z'))$ can be absorbed into the symbol $a$, without disturbing the validity of the estimates (1.14) (for low energies $\lambda \in \text{supp } \chi$).

So consider the second type of representation in Proposition 2.6. We need to show that

$$a(\lambda, z, z') = e^{\mp i\lambda d(z, z')} \int_{\mathbb{R}^{n-1}} e^{\pm i\lambda \Phi(y, w, v)/x} b(\lambda, \rho, y, w_1, v) dv, \quad \rho = x/\lambda,$$

satisfies (1.14), that is, in view of Lemma 2.1

$$\left| (\lambda \partial_\lambda)^\alpha a(\lambda, z, z') \right| \leq C_{\alpha}(1 + \frac{|w|}{\rho})^{-\frac{n-1}{2}}.$$

We now break the estimate into various cases. First, if $|w| \lesssim \rho$, then for $\alpha = 0$ we only need to show that $a$ is uniformly bounded, which is obvious. If we apply $\lambda \partial_\lambda$ to (1.2), then this is harmless when it hits $b$. When it hits the phase it brings down a factor $i\lambda(\mp d(z, z') \pm \Phi/x)$. Notice that $\lambda d(z, z') \sim |w|/\rho \lesssim 1$ in this case. On the other
hand, \( \lambda \Phi / x = \Phi / \rho = v \cdot d_v \Phi / \rho + O(w_1 / \rho) \), and again since \(|w| \leq \rho\) the \( O(w_1 / \rho) \) is harmless. To treat the \( v \cdot d_v \Phi / \rho \) term, we can write using (b) of (2.4)
\[
\frac{v \cdot d_v \Phi}{\rho} e^{i \Phi / \rho} = -i v \cdot d_v e^{i \Phi / \rho},
\]
and integrating by parts we see that this term is \( O(1) \) after integration.

Second, suppose that \(|w_1| \leq \rho\) but \(|w| \geq C \rho\) for some large \( C \). For large enough \( C \), this means that \( d_v \Phi \neq 0 \), for some \( j \geq 2 \), since by (a) of (2.4), we have \( d_v \Phi = w_j - O(w_1) \). So by choosing \( j \) so that \(|w_j|\) is maximal, and then \( C \) large enough, we have \(|d_v \Phi| \geq \epsilon |w|\). Then we can write
\[
e^{i \Phi / \rho} = \left( \frac{\rho d_v}{id_v \Phi} \right)^N e^{i \Phi / \rho},
\]
and integrate by parts. Each integration by parts gains us a factor of \( \rho / |w| \). Thus we can estimate (4.2) by (4.2). To proceed, we fix \((a, b, \alpha, \beta, \gamma)\) and integrate by parts. Each integration by parts gains us a factor of \( \rho / \Phi \). Let \( \tilde{\Phi} \) be the stationary phase, and in particular need property (c) of (2.4). Define
\[
(4.3)
\tilde{\Phi}(x, y, w, v) = |w_1|^{-1}(\Phi(y, w, v) \mp x d(z, z'))
\]
and let \( \omega = |w_1| / \rho \), then we need to estimate
\[
\lambda^\alpha \partial^{\alpha}_{\lambda} a(\lambda, z, z') = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} \omega^\beta \int_{\mathbb{R}^n} e^{i \omega \tilde{\Phi}(x, y, w, v)} \tilde{\Phi}^\beta (\lambda^\gamma \partial^{\gamma}_{\lambda} b)(\lambda, \rho, y, w_1, v) dv.
\]
Let \( \tilde{b} = \lambda^\gamma \partial^{\gamma}_{\lambda} b \), then \(|\partial^{\gamma}_{\lambda} \tilde{b}| \leq C_\gamma \lambda^{-\gamma} \). Thus note \( \omega \geq 1 \), it reduces to show for any \( 0 \leq \beta \leq \alpha \)
\[
(4.4)
\left| \int_{\mathbb{R}^n} e^{i \omega \tilde{\Phi}(x, y, w, v)} (\omega \tilde{\Phi})^\beta \tilde{b}(\lambda, \rho, y, w_1, v) dv \right| \leq C \omega^{-n-1}.
\]
To proceed, we fix \((x, y, w)\) with \( w \neq 0 \) (and hence \( w_1 \neq 0 \) due to our assumption that \(|w_1|\) is comparable to \(|w|\)). We use a cutoff function \( \Upsilon \) to divide the \( v \) integral into two parts: the support of \( \Upsilon \), in which \(|d_v \tilde{\Phi}| \geq \bar{\epsilon} / 2 \), and the other on the support of \( 1 - \Upsilon \), in which \(|d_v \tilde{\Phi}| \leq \bar{\epsilon} \). On the support of \( \Upsilon \), we integrate by parts in \( v \) and gain any power of \( \omega^{-1} \), proving (4.4). On the support of \( 1 - \Upsilon \), we make the variable change
\[
(v_2 \cdots, v_n) \rightarrow (\theta_2, \cdots, \theta_n), \quad \theta_i = d_v \tilde{\Phi}, \quad i = 2 \cdots, n.
\]
Note that by property (c) of (2.4),
\[
\frac{\partial \theta_j}{\partial v_k} = d^2_{v_j v_k} \tilde{\Phi} = \pm A_{jk}.
\]
The nondegeneracy of \( A_{jk} \) shows that this change of variables is locally nonsingular, provided \( \bar{\epsilon} \) is sufficiently small. Thus, for each point \( v \) in the support of \( 1 - \Upsilon \), there is a neighbourhood in which we can change variables to \( \theta \) as above. Using the compactness of the support of \( b \) in (2.3), we see that there are a finite number of neighbourhoods

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covering the intersection of the support of $\Upsilon$ and the $v$-support of $b$. For simplicity of exposition, we assume that there is only one such neighborhood $U$ below.

Denote $B_{\delta} := \{ \theta : |\theta| \leq \delta \}$, and choose a $C^\infty$ function $\chi_{B_{\delta}}(\theta)$ which equals 1 when on the set $B_{\delta}$ but equals 0 for outside $B_{2\delta}$, and with derivatives bounded by

$$|\nabla^{(j)}_{\theta} \chi_{B_{\delta}}(\theta)| \leq C\delta^{-j}.$$  

Here $\delta$ is a parameter to be chosen later (depending on $\omega$). Consider the integral (4.4) after changing variables and with the cutoff function $\chi_{B_{\delta}}(\theta)$ inserted (note that $1 - \Upsilon = 1$ on the support of $\chi_{B_{\delta}}(\theta)$, provided $\delta \leq \tilde{\epsilon}/2$):

$$\int e^{i\omega \Phi(x,y,w,\theta)}(\omega \Phi)^{\beta} b(\lambda, \rho, y, w_1, \theta) \chi_{B_{\delta}}(\theta) \frac{d\theta}{|A^{-1}(y, w, \theta)|}.$$  

Using property (d) of (2.4), we see that $\Phi = 0$ when $\theta = 0$. Also, due to our choice of $\theta$, we have $d_\theta \Phi = 0$ when $\theta = 0$. Hence $\Phi = O(|\theta|^2)$. Hence

$$|\omega^\beta \int e^{i\omega \Phi(x,y,w,\theta)}(\omega \Phi)^{\beta} b(\lambda, \rho, y, w_1, \theta) \chi_{B_{\delta}}(\theta) \frac{d\theta}{|A^{-1}(y, w, \theta)|}| \leq C(\omega \delta^2)^\beta \delta^{n-1}.$$  

It remains to treat the integral with cutoff $(1 - \chi_{B_{\delta}}(\theta))$ inserted. Notice that $|d_\theta \Phi|$ is comparable to $|\theta|$ since $d_\theta \Phi = 0$ when $\theta = 0$, and

$$d_{\theta, \theta, \theta}^2 \Phi = \sum_{k,l} (A^{-1})_{il} (A^{-1})_{jk} d_{\nu_k \nu_l} \Phi$$  

is nondegenerate when $\theta = 0$. We define the differential operator $L$ by

$$L = \frac{-id_\theta \Phi \cdot \partial_\theta}{\omega |d_\theta \Phi|^2}.$$  

Then the adjoint operator is given by

$$^*L = -L + \frac{i}{\omega} \left( \frac{\Delta_\theta \Phi}{|d_\theta \Phi|^2} - 2 \frac{d_{\theta, \theta, \theta}^2 \Phi d_{\theta, \theta} \Phi d_{\theta, \theta} \Phi}{|d_\theta \Phi|^4} \right).$$  

Since $Le^{i\omega \Phi} = e^{i\omega \Phi}$, we integrate by parts $N$ times to obtain

$$\left| \int e^{i\omega \Phi(x,y,w,\theta)}(\omega \Phi)^{\beta} b(\lambda, \rho, y, w_1, \theta)(1 - \chi_{B_{\delta}}(\theta))(1 - \Upsilon) d\theta \right| \leq C \int \left| (^*L)^N ((\omega \Phi)^{\beta} b(\lambda, \rho, y, w_1, \theta)(1 - \chi_{B_{\delta}}(\theta))(1 - \Upsilon)) \right| d\theta.$$  

Inductively we find that

$$\left| (^*L)^N ((\omega \Phi)^{\beta} b(1 - \chi_{B_{\delta}})(1 - \Upsilon)) \right| \leq C \omega^{-N+\beta} \max \{ |\theta|^{2\beta-2N}, |\theta|^{2\beta-N} \delta^{-N} \}.$$  

Choosing $N$ large enough, we get

$$\left| \int e^{i\omega \Phi(x,y,w,\theta)}(\omega \Phi)^{\beta} b(\lambda, \rho, y, w_1, \theta)(1 - \chi_{B_{\delta}})(1 - \Upsilon) d\theta \right| \leq \omega^{-N+\beta} \int_{|\theta| \geq \delta} \left( |\theta|^{2\beta-2N} + |\theta|^{2\beta-N} \delta^{-N} \right) d\theta \leq C \omega^{-N+\beta} \delta^{2\beta-2N} \delta^{n-1}.$$
Choose $\delta = \omega^{-1/2}$ to balance the two parts of the integral (with $\chi_{B_\delta}$ and with $1 - \chi_{B_\delta}$).

We finally obtain
\[
\left| \int e^{i\omega \Phi(x,y,w,\theta)}(\omega \Phi)^{\beta} \beta \omega (\lambda, \rho, y, w_1, \theta)(1 - \Upsilon) d\theta \right| \leq C\omega^{-(n-1)/2},
\]
which proves (4.4) as desired.

We next sketch how to prove (1.14) in the high energy case, $i > N_l$. In terms of Proposition 3.4, consider a term of type (ii); it suffices to show
\[
a(h, z, z') = e^{\mp id(z,z')/h} \int_{\mathbb{R}^{n-1}} e^{i\Psi(y,w,x,v)/(xh)} b(h, x, y, w_1, v) dv,
\]
satisfies
\[
\left| (h\partial_h)^\alpha a(h, z, z') \right| \leq C_n \left( 1 + \frac{|w|}{xh} \right)^{-\frac{n-1}{2}}.
\]
Notice that $\lambda = 1/h$ and $\Psi$ has the same properties (a) — (d) as $\Phi$. Therefore the low energy proof works verbatim, with the argument $x$ of $\Psi$ acting as a smooth parameter, and leads to the desired conclusion. The proof in case (iii) works in exactly the same way.

5. $L^2$ estimates

In this section, we prove $L^2 \to L^2$ estimates on microlocalized versions of the Schrödinger propagator, using the operator partition of unity $Q_i$ constructed in [17] and recalled in the previous section.

We begin by defining microlocalized propagators. First we give a formal definition. It is not immediately clear that the formal definition is well-defined, so our first task is to show this. We do so by showing that each microlocalized propagator is a bounded operator on $L^2$. This serves both to show the well-definedness of each microlocalized propagator, and to establish the $L^2 \to L^2$ estimate needed for the abstract Keel-Tao argument.

We define, as in the Introduction,
\[
U_j(t) = \int_0^\infty e^{it\lambda^2} Q_j(\lambda) dE_{\sqrt{2}n}(\lambda)
\]
where $Q_j$ is the decomposition defined in (4.1).

Our first task is to make sense of this expression. We do this by showing that each $U_j(t)$ is a bounded operator on $L^2(M^\circ)$. We have

**Proposition 5.1.** The integral (5.1) defining $U_j(t)$ are well-defined on each finite interval, and converge on $\mathbb{R}_+$ in the strong operator topology to define bounded operators on $L^2(M^\circ)$. Moreover, the operator norm of $U_j(t)$ are bounded uniformly for $t \in \mathbb{R}$. Finally, we have
\[
\sum_i U_i(t) = e^{itH}.
\]
The rest of this subsection is devoted to proving this Proposition.

Suppose that $A(\lambda)$ is a family of bounded operators on $L^2(M^\circ)$, compactly supported and $C^1$ in $\lambda \in (0, \infty)$. Integrating by parts, the integral of

$$\int_0^\infty A(\lambda) \, dE_{\sqrt{H}}(\lambda)$$

is by definition given by

$$- \int_0^\infty \left( \frac{d}{d\lambda} A(\lambda) \right) E_{\sqrt{H}}(\lambda) \, d\lambda.$$

In view of Corollaries 2.5 and 3.3, we can take $A(\lambda)$ to be a smooth function of $\lambda$ with compact support in $(0, \infty)$ times $e^{it\lambda^2} Q_i(\lambda)$. This means that the integral (5.1) is well-defined over any compact interval in $(0, \infty)$. We need to show that the integral over the whole of $\mathbb{R}_+$ converges in the strong operator topology. To do so, we introduce a dyadic partition of unity on the positive $\lambda$ axis by choosing $\phi \in C^\infty_c([1/2, 2])$, taking values in $[0, 1]$, such that

$$\sum_j \phi\left(\frac{\lambda}{2^j}\right) = 1.$$

We now define

(5.3) $$U_{i,j}(t) = - \int_0^\infty \frac{d}{d\lambda} \left( e^{it\lambda^2} \phi\left(\frac{\lambda}{2^j}\right) Q_i(\lambda) \right) E_{\sqrt{H}}(\lambda).$$

We next show that the sum over $j$ of the operators $U_{i,j}(t)$ in (5.3) is well-defined. For this we use the Cotlar-Stein lemma, which we recall here (we use the version in [14, Chapter 8]):

**Lemma 5.2 (Cotlar-Stein lemma).** Suppose that $A_j$ are a sequence of bounded linear operators on a Hilbert space $H$ such that

(5.4) $$\|A_j^* A_k\|_{H \rightarrow H} \leq (\gamma(j - k))^2, \quad \|A_j A_k^*\|_{H \rightarrow H} \leq (\gamma(j - k))^2,$$

where $\{\gamma(j)\}_{j \in \mathbb{Z}}$ is a sequence of positive constants such that $C = \sum_{j \in \mathbb{Z}} \gamma(j) < \infty$. Then for all $f \in H$, the sequence $\sum_{|j| \leq N} A_j f$ converges as $N \to \infty$ to an element $A f \in H$. The operators $A = \sum_j A_j$ and $A^* = \sum_j A_j^*$ so defined (in the strong operator topology) satisfy

(5.5) $$\|A\|_{H \rightarrow H} \leq C, \quad \|A^*\|_{H \rightarrow H} \leq C.$$

Moreover, the operator norms of $\sum_{j \in J} A_j$ and $\sum_{j \in J} A_j^*$ are bounded by $C$ for any finite subset $J$ of the integers.

We also use the following Lemma:

**Lemma 5.3.** Suppose that for $l = 1, 2$, $A_l(\lambda)$ is a family of operators compactly supported in $\lambda$ in the open interval $(0, \infty)$, and with $A_l(\lambda)$, $\partial_\lambda A_l(\lambda)$ uniformly bounded on $L^2(M^\circ)$. Define

$$B_l = \int A_l(\lambda) \, dE_{\sqrt{H}}(\lambda).$$

Then

$$B_1 B_2^* = \int A_1(\lambda) \, dE_{\sqrt{H}}(\lambda) A_2(\lambda)^*,$$
where by definition the last expression is equal to
\begin{equation}
\left( - \frac{d}{d\lambda} A_1(\lambda) \right) E_{\sqrt{\Pi}}(\lambda) A_2(\lambda) - A_1(\lambda) E_{\sqrt{\Pi}}(\lambda) \left( \frac{d}{d\lambda} A_2(\lambda) \right) .
\end{equation}

\textbf{Proof.} We compute
\begin{equation}
B_1 B_2^* = \iint \left( \frac{d}{d\lambda} A_1(\lambda) \right) E_{\sqrt{\Pi}}(\lambda) E_{\sqrt{\Pi}}(\mu) \left( \frac{d}{d\mu} A_2(\mu)^* \right) d\lambda d\mu
\end{equation}
\begin{equation}
= \iint_{\lambda < \mu} \left( \frac{d}{d\lambda} A_1(\lambda) \right) E_{\sqrt{\Pi}}(\lambda) \left( \frac{d}{d\mu} A_2(\mu)^* \right) d\lambda d\mu
\end{equation}
\begin{equation}
+ \iint_{\mu < \lambda} \left( \frac{d}{d\lambda} A_1(\lambda) \right) E_{\sqrt{\Pi}}(\mu) \left( \frac{d}{d\mu} A_2(\mu)^* \right) d\lambda d\mu
\end{equation}
\begin{equation}
= \int \left( \frac{d}{d\lambda} A_1(\lambda) \right) E_{\sqrt{\Pi}}(\lambda) \left( - A_2(\lambda)^* \right) d\lambda + \int \left( - A_1(\mu) \right) E_{\sqrt{\Pi}}(\mu) \left( \frac{d}{d\mu} A_2(\mu)^* \right) d\mu
\end{equation}
\begin{equation}
= (5.6).
\square
\end{equation}

Now we show that the sum in \([5.3]\) is well-defined. We first note a simplification: since the \(Q_i(\lambda)\) are a partition of the identity, we have
\begin{equation}
U_j := \sum_{i=1}^{N_L} U_{i,j}(t) = \int e^{it^2} \chi(\lambda) \phi \left( \frac{\lambda}{2t} \right) dE_{\sqrt{\Pi}}(\lambda),
\end{equation}
which is clearly bounded on \(L^2(M^\circ)\) with operator norm \(\leq 1\) using spectral theory. Moreover, the sum of any subset of the \(U_j\) converges strongly to an operator with norm \(\leq 1\). Due to this, we may ignore \(i = 1\) and prove the \(L^2\)-boundedness only for \(2 \leq i \leq N_L\) (in the low energy case).

We have, by Lemma \([5.3]\)
\begin{equation}
U_{i,j}(t)U_{i,k}(t)^* = \int \chi(\lambda)^2 \phi \left( \frac{\lambda}{2t} \right) \phi \left( \frac{\lambda}{2t} \right) Q_i(\lambda) dE_{\sqrt{\Pi}}(\lambda) Q_i(\lambda)^*
\end{equation}
\begin{equation}
= - \int \frac{d}{d\lambda} \chi(\lambda)^2 \phi \left( \frac{\lambda}{2t} \right) \phi \left( \frac{\lambda}{2t} \right) Q_i(\lambda) E_{\sqrt{\Pi}}(\lambda) Q_i(\lambda)^*
\end{equation}
\begin{equation}
- \int \chi(\lambda)^2 \phi \left( \frac{\lambda}{2t} \right) \phi \left( \frac{\lambda}{2t} \right) Q_i(\lambda) E_{\sqrt{\Pi}}(\lambda) \frac{d}{d\lambda} Q_i(\lambda)^*.
\end{equation}

We observe that this is independent of \(t\), and is identically zero unless \(|j - k| \leq 2\). When \(|j - k| \leq 2\), we note that the integrand is a bounded operator on \(L^2\), with an operator bound of the form \(C/\lambda\) where \(C\) is uniform, as we see from Corollary \([2.4]\) and the support property of \(\phi\). The integral is therefore uniformly bounded, as we are integrating over a dyadic interval in \(\lambda\).

We next consider the operators \(U_{i,j}^*(0)U_{i,k}(0)\), just in the case \(t = 0\). This has an expression
\begin{equation}
\iint E_{\sqrt{\Pi}}(\lambda) \frac{d}{d\lambda} \chi(\lambda)^2 \phi \left( \frac{\lambda}{2t} \right) Q_i(\lambda)^* \frac{d}{d\mu} \left( Q_i(\mu) \phi \left( \frac{\mu}{2t} \right) \chi(\mu) \right) E_{\sqrt{\Pi}}(\mu) d\lambda d\mu.
\end{equation}
Write $Q_{i,j}^*(\lambda), Q_{i,k}(\mu)$ for the operators in parentheses above. By the construction of $Q_i$ ($2 \leq i \leq N_l$), the scattering pseudodifferential operators $Q_{i,j}^*(\lambda), Q_{i,k}(\mu)$ are smooth and compactly supported in $x'/\lambda, x'/\mu$ respectively and are microlocally supported near the characteristic set. More precisely, we see the composition of the two scattering pseudodifferential operators for $2 \leq i \leq N_l$

$$Q_{i,j}^*(\lambda)Q_{i,k}(\mu)$$

$$= \int e^{-i\lambda((y-y')\cdot \eta+(\sigma-1)\nu)/x'} e^{i\mu((y'-y'')\cdot \eta'+(\sigma'-1)\nu')/x'}$$

$$\times q_{i,j}(\lambda, y', \frac{x'}{\lambda}, \eta, \nu)q_{i,k}(\mu, y', \frac{x'}{\mu}, \eta', \nu')dx'dy'd\eta d\nu d\eta' dv'$$

where $\sigma = x'/x, \sigma' = x'/x''$, and $q_{i,j}, q_{i,k}$ are smooth and polyhomogeneous in $\lambda, \mu$ and smooth compact supported in $x'/\lambda, x'/\mu, x'$. When $i \geq 2$, we have that $\nu, \eta, \nu', \eta'$ satisfy $\nu^2 + |\eta|^2 \geq 1/4$ and $\nu'^2 + |\eta'|^2 \geq 1/4$. By symmetry, we assume $\lambda > \mu$ without loss of generality. Let us introduce the operator

$$\mathcal{L} = i[\lambda(|\nu|^2 + |\eta|^2)]^{-1}(x'\eta \partial_{x'} - \nu x'^2 \partial_{x'})$$

then $\mathcal{L} e^{-i\lambda((y-y')\cdot \eta+(\sigma-1)\nu)/x'} = e^{-i\lambda((y-y')\cdot \eta+(\sigma-1)\nu)/x'}$. By using $\mathcal{L}$ to integrate by parts, we gain the factor $\lambda^{-1}$ since $|\nu|^2 + |\eta|^2$ is uniformly bounded from below; we incur a factor $\mu$ if the derivative falls on $e^{i\mu((y'-y'')\cdot \eta'+(\sigma'-1)\nu')/x'}$, or a factor of $x'$ or $x'^2/\mu$ if the derivative falls on $q_{i,j}$ or $q_{i,k}$. Since $x' \leq \mu$ on the support of $q_{i,k}$, we have an overall gain of $\mu/\lambda \sim 2^{-|j-k|}$. The $L^2$-boundedness of the spectral projection gives $\|U_{i,j}^*(0)U_{i,k}(0)\|_{L^2 \to L^2} \leq C 2^{-|j-k|}$.

It now follows from the Cotlar-Stein Lemma that $U_i(0)^*, i = 2 \ldots N_l$, is well defined as the strong limit of the sequence of operators

$$\sum_{|j| \leq l} U_{i,j}(0)^*.$$  

Consider the sequence $\sum_{|j| \leq l} U_{i,j}(t)^*$. We claim that this sequence converges strongly. We define $U_i(t)^*$ to be this limit. To prove this claim, choose an arbitrary $f \in L^2(M^0)$. We have shown that

$$\lim_{l \to \infty} \sup_{m > l} \left\| \sum_{l \leq |j| \leq m} U_{i,j}(0)^* f \right\|_2^2 = 0.$$ 

This is equivalent to

$$\lim_{l \to \infty} \sup_{m > l} \sum_{l \leq |j| \leq m} \langle U_{i,j}(0)U_{i,j'}(0)^* f, f \rangle = 0.$$ 

But we saw in [5,8] that $U_{i,j}(0)U_{i,j'}(0)^* = U_{i,j}(t)U_{i,j'}(t)^*$. Hence we have

$$\lim_{l \to \infty} \sup_{m > l} \sum_{l \leq |j|, |j'| \leq m} \langle U_{i,j}(t)U_{i,j'}(t)^* f, f \rangle = 0,$$ 

which implies that

$$\lim_{l \to \infty} \sup_{m > l} \left\| \sum_{l \leq |j| \leq m} U_{i,j}(t)^* f \right\|_2^2 = 0.$$
Hence the sequence \( \sum_{|j|\leq l} U_{i,j}(t)^* f \) converges for every \( f \in L^2(M^\circ) \), i.e. the sequence \( \sum_{j \geq -l} U_{i,j}(t)^* \) converges strongly. We see from this that the integral
\[
\int e^{-it\lambda^2}dE(\lambda)Q_i(\lambda)^*
\]
converges in the strong topology, hence defines \( U_i(t)^* \). Finally we show that the operator norm of \( U_i(t)^* \) is bounded uniformly in \( t \). Since \( \sum_{|j|\leq l} U_{i,j}(t)^* \) converges in the strong operator topology, we have
\[
\|U_i(t)^*\| \leq \sup_{l \to \infty} \| \sum_{|j|\leq l} U_{i,j}(t)^* \|
\]
But we have
\[
\| \sum_{|j|\leq l} U_{i,j}(t)^* \|^2 = \| \sum_{|j|\leq l} U_{i,j}(t)U_{i,j'}(t)^* \| = \| \sum_{|j|,|j'|\leq l} U_{i,j}(0)U_{i,j'}(0)^* \| = \| \sum_{|j|\leq l} U_{i,j}(0)^* \|^2
\]
and the operator norm of \( \sum_{|j|\leq l} U_{i,j}(0)^* \) is bounded uniformly in \( l \) by the estimates proved above using the Cotlar-Stein Lemma.

Next, we treat the high energy part of \( U_i(t) \). As in the low energy case, we may ignore \( i = N + 1 \) and prove the \( L^2 \)-boundedness only for \( N_i + 2 \leq i \leq N \).

The argument is almost exactly as before: with \( U_{i,j}(t) \) defined as in [15,3], we bound the operator norms of \( U_{i,j}(t)U_{i,k}(t)^* \) and \( U_{i,j}(t)^*U_{i,k}(t) \). The argument for \( U_{i,j}(t)U_{i,k}(t)^* \) works exactly as above; we find that this operator is zero unless \( |j - k| \leq 2 \), and is uniformly bounded in that case. The argument for \( U_{i,j}(0)^*U_{i,k}(0) \) is also almost exactly the same as above. It is clear that each of these operators is bounded uniformly in \( i, k \), so it remains to prove a decaying estimate when \( |j - k| \) is large. Recall that the \( Q_i^{\text{high}}(\lambda) \) either have kernels supported where \( x + x' < \epsilon \) or are microsupported away from the boundary. In the former case, the low energy proof works verbatim, while in the latter case, the argument is only slightly different. For completeness we give the argument in the latter case. Suppose that
\[
\frac{d}{d\lambda} \left((1 - \chi)(\lambda)\phi\left(\frac{\lambda}{2}\right)Q_i^{\text{high}}(\lambda)^*\right)
\]
takes the form, where \( z \) are local coordinates in a coordinate patch in the interior of \( M^\circ \) covering the projection of the microsupport of \( Q_i^{\text{high}}(\lambda) \),
\[
\lambda^n \int e^{i\lambda(z-z')}\zeta q_{i,j}(z',\zeta,\lambda) d\zeta,
\]
where \( q_{i,j} \) is supported where \( \lambda \sim 2^j \), \( |\zeta|^2 \sim 1 \) and is such that \( D_z^\alpha D_{\zeta}^\beta q_{i,j} \) is bounded by \( C\lambda^{-1} \). Then we can write the composition
\[
\frac{d}{d\lambda} \left((1 - \chi)(\lambda)\phi\left(\frac{\lambda}{2}\right)Q_i^{\text{high}}(\lambda)^*\right) \frac{d}{d\mu} \left(Q_i^{\text{high}}(\mu)\phi\left(\frac{\mu}{2}\right)(1 - \chi)(\mu)\right)
\]
in the form
\[
\lambda^n \mu^n \int \int e^{i\lambda(z-z'')}\zeta q_{i,j}(z'',\zeta,\lambda) e^{i\mu(z''-z'')}\zeta q_{i,k}(z'',\zeta',\mu) d\zeta d\zeta' dz''.
\]
(5.9)
Assume without loss of generality that \( j > k \), i.e. \( \lambda > \mu \) on the support of the integrand. We note that the differential operator

\[
L = i \zeta \cdot \partial z'' \lambda |\zeta|^2
\]

leaves \( e^{i\lambda(z-z')} \zeta \) invariant, so we can apply it to this phase factor in the integral \([5.9]\). Integrating by parts, the \( \partial z'' \) derivative either hits the other phase factor \( e^{i\mu(z''-z')} \zeta' \), in which case we incur a factor of \( \mu \), or it hits one of the symbols \( q_{i,j} \) or \( q_{i,k} \), in which case we incur no factor. So we gain a factor of either \( \mu / \lambda \sim 2^{-|j-k|} \), or \( 1 / \lambda \) which is even better since \( \mu > 1 \) or else the integrand vanishes due to the \( 1 - \chi \) factor. This completes the Cotlar-Stein estimates for \( U_i(0) \). We deduce the uniform estimates for \( U_i(t) \) exactly as in the low energy case.

This completes the proof of Proposition 5.1.

Remark 5.4. We would like to point out that this argument allows us to avoid using a Littlewood-Paley type decomposition in this setting. Littlewood-Paley type estimates were established in [8] for the asymptotically conic manifold in the form of

\[
\| f \|_{L^p} \lesssim \left( \sum_{k \geq 0} \| \phi(2^{-2k} \Delta_g) f \|_{L^p}^2 \right)^{1/2} + \sum_{k \leq 0} \| \phi(2^{-2k} \Delta_g) f \|_{L^p}.
\]

6. Dispersive estimates

In this section, we use stationary phase and Proposition 1.3 to establish the microlocalized dispersive estimates.

**Proposition 6.1** (Microlocalized dispersive estimates). Let \( Q_i(\lambda) \) be defined in \([3.1]\). Then for all integers \( i \geq 1 \), the kernel estimate

\[
\left| \int_0^\infty e^{it\lambda^2} (Q_i(\lambda) dE, \sqrt{H}(\lambda) Q_i^*(\lambda)) (z, z') d\lambda \right| \leq C |t|^{-\frac{n}{2}}
\]

holds for a constant \( C \) independent of points \( z, z' \in M^\circ \).

**Proof.** The key to the proof is to use the estimates in Proposition 1.3. We first consider \( i = 1 \) and \( i = N_l + 1 \), i.e. taking the case that either \( Q_{i,0}^{\text{low}}(\lambda) + Q_i^{\text{low}}(\lambda) \) or \( Q_i^{\text{high}}(\lambda) \). Then we can use the estimate \([1.14]\) to obtain for \( i = 1 \) or \( N_l + 1 \)

\[
\left| \left( \frac{d}{d\lambda} \right)^N (Q_i(\lambda) dE, \sqrt{H}(\lambda) Q_i^*(\lambda)) \right| \leq C_N \lambda^{n-1-N} \quad \forall N \in \mathbb{N}.
\]

Let \( \delta \) be a small constant to be chosen later. Recall that we chose \( \phi \in C_c^\infty([\frac{1}{2}, 2]) \) such that \( \sum_j \phi(2^{-j} \lambda) = 1 \); we denote \( \phi_0(\lambda) = \sum_{j \leq 0} \phi(2^{-j} \lambda) \). Then

\[
\left| \int_0^\infty e^{it\lambda^2} (Q_i(\lambda) dE, \sqrt{H}(\lambda) Q_i^*(\lambda)) (z, z') \phi_0(\frac{\lambda}{\delta}) d\lambda \right| \leq C \int_0^\delta \lambda^{n-1} d\lambda \leq C \delta^n.
\]
We use integration by parts $N$ times to obtain, using (6.2),
\[
\left| \int_{0}^{\infty} e^{it\lambda^2} \sum_{j \geq 0} \phi(\frac{\lambda}{2^j}) (Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)}Q_i^*(\lambda))(z, z')d\lambda \right|
\leq \sum_{j \geq 0} \left| \int_{0}^{\infty} \left( \frac{1}{2\lambda \partial \lambda} \right)^N (e^{it\lambda^2}) \phi(\frac{\lambda}{2^j}) (Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)}Q_i^*(\lambda))(z, z')d\lambda \right|
\leq C_N |t|^{-N} \sum_{j \geq 0} \int_{2j-1}^{2j+1} \lambda^{n-1-2N}d\lambda \leq C_N |t|^{-N/2} \delta^{n-2N}.
\]
Choosing $\delta = |t|^{-\frac{1}{2}}$, we have thus proved
\[
(6.3) \quad \left| \int_{0}^{\infty} e^{it\lambda^2} (Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)}Q_i^*(\lambda))(z, z')d\lambda \right| \leq C_N |t|^{-\frac{n}{2}}.
\]
Now we consider the case $i \geq 2$, $i \neq N_1 + 1$. Let $r = d(z, z')$ and $\bar{r} = rt^{-\frac{1}{2}}$. In this case, we write the kernel of by Proposition 1.3
\[
(6.4) \quad \int_{0}^{\infty} e^{it\lambda^2} (Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)}Q_i^*(\lambda))(z, z')d\lambda
= \int_{0}^{\infty} e^{it\lambda^2} e^{-ir\lambda \lambda^{-1}a(\lambda, z, z')}d\lambda = t^{-\frac{1}{2}} \int_{0}^{\infty} e^{it\lambda^2} e^{-ir\lambda \lambda^{-1}a(\lambda^{-1/2}, z, z')}d\lambda,
\]
where $a$ satisfies estimates
\[
|\partial^\alpha a(\lambda, z, z')| \leq C_\alpha \lambda^{-\alpha}(1 + \lambda d(z, z'))^{-\frac{n-1}{2}}.
\]
and therefore
\[
(6.5) \quad \left| \partial^\alpha (a(t^{-1/2}, z, z')) \right| \leq C_\alpha \lambda^{-\alpha}(1 + \lambda \bar{r})^{-\frac{n-1}{2}}.
\]
We divide the RHS of (6.4) into two pieces using the partition of unity above. It suffices to prove that there exists a constant $C$ independent of $\bar{r}$ such that
\[
I := \left| \int_{0}^{\infty} e^{it\lambda^2} e^{-ir\lambda \lambda^{-1}a(t^{-1/2}, z, z')}\phi(\lambda)d\lambda \right| \leq C,
\]
\[
II := \sum_{j \geq 0} \int_{0}^{\infty} e^{it\lambda^2} e^{-ir\lambda \lambda^{-1}a(t^{-1/2}, z, z')}\phi(\frac{\lambda}{2^j})d\lambda \leq C.
\]
The first one is obvious, since $\lambda \leq 1$. Define $A_1 = \{ \lambda : \bar{r} \lambda \leq 1/4 \}$ and $A_2 = \{ \lambda : \bar{r} \lambda \geq 1/4 \}$. Now we write $II = II_1 + II_2$, where
\[
II_1 = \sum_{j \geq 0} \int_{\lambda \in A_1} e^{it\lambda^2} e^{-ir\lambda \lambda^{-1}a(t^{-1/2}, z, z')}\phi(\frac{\lambda}{2^j})d\lambda.
\]
\[
II_2 = \int_{\lambda \in A_2} e^{it\lambda^2} e^{-ir\lambda \lambda^{-1}a(t^{-1/2}, z, z')}\phi(\lambda)d\lambda.
\]
Let $\Phi(\lambda, \bar{r}) = \lambda^2 - \bar{r}\lambda$. We first consider $II_1$. Since $\lambda \in A_1$, and $\lambda \geq 1/2$ on the support of the $\phi$ factor, the integrand is only nonzero when $\bar{r} \leq 1/2$. Therefore $|\partial^\alpha \Phi| = 2\lambda - \bar{r} \geq
 Define the operator \( L = L(\lambda, \bar{r}) = (2\lambda - \bar{r})^{-1}\partial_\lambda \). By (6.5) and using integration by parts, we obtain for \( N > n/2 \)
\[
II_1 \leq \sum_{j \geq 0} \left| \int_{\lambda \in A_1} e^{i\lambda^2} e^{-iF\lambda} \lambda^{-1} a(t^{-1/2}\lambda, z, z') \varphi(\frac{\lambda}{2^j}) d\lambda \right|
\]
\[
= \sum_{j \geq 0} \left| \int_{\lambda \in A_1} L^N e^{i(\lambda^2 - F\lambda)} \left[ \lambda^{-1} a(t^{-1/2}\lambda, z, z') \varphi(\frac{\lambda}{2^j}) \right] d\lambda \right|
\]
\[
\leq C_N \sum_{j \geq 0} \int_{|\lambda| \sim 2^j} \lambda^{-1-2N} d\lambda \leq C_N.
\]
Finally we consider \( II_2 \). Here we need a further decomposition, based on the size of \( \partial_\lambda \Phi \).
\[
II_2 \leq \left| \int_{\lambda \in A_2} e^{i\lambda^2} e^{-iF\lambda} \lambda^{-1} a(t^{-1/2}\lambda, z, z') (1 - \varphi(0)) \varphi(2\lambda - \bar{r}) d\lambda \right|
\]
\[
+ \sum_{j \geq 0} \left| \int_{\lambda \in A_2} e^{i\lambda^2} e^{-iF\lambda} \lambda^{-1} a(t^{-1/2}\lambda, z, z') (1 - \varphi(0)) \varphi(\frac{2\lambda - \bar{r}}{2^j}) d\lambda \right|
\]
\[
:= II_2^1 + II_2^2.
\]
If \( \bar{r} \leq 10 \), then for the integrand of \( II_2^1 \) to be nonzero we must have \( \lambda \leq 10 \), due to the \( \varphi_0 \) factor. Then it is easy to see that \( II_2^1 \) is bounded. If \( \bar{r} \geq 10 \), we have \( \bar{r} \sim \lambda \) since \( |2\lambda - \bar{r}| \leq 1 \). Hence
\[
II_2^1 \leq \int_{|2\lambda - \bar{r}| \leq 1} \lambda^{-1}(1 + \bar{r})^{-\frac{M+1}{2}} d\lambda \leq C.
\]
Now we consider the second term. Integrating by parts, we show by (6.5)
\[
II_2^2 \leq \sum_{j \geq 0} \left| \int_{\lambda \in A_2} e^{i\lambda^2} e^{-iF\lambda} \lambda^{-1} a(t^{-1/2}\lambda, z, z') (1 - \varphi(0)) \varphi(\frac{2\lambda - \bar{r}}{2^j}) d\lambda \right|
\]
\[
= \sum_{j \geq 0} \left| \int_{\lambda \in A_2} L^N e^{i(\lambda^2 - F\lambda)} \left[ \lambda^{-1} a(t^{-1/2}\lambda, z, z') (1 - \varphi(0)) \varphi(\frac{2\lambda - \bar{r}}{2^j}) \right] d\lambda \right|
\]
\[
\leq C_N \sum_{j \geq 0} 2^{-jN} \int_{|2\lambda - \bar{r}| \sim 2^j} \lambda^{-1}(1 + \bar{r})^{-\frac{M+1}{2}} d\lambda.
\]
If \( \bar{r} \leq 2^{j+1} \), then \( \lambda \leq 2^{j+2} \) on the support of the integrand. The \( j \)th term can then be estimated by \( C_N 2^{-jN} 2^{(j+2)n/2} \), which is summable for \( N > n \). Otherwise, we have \( \lambda \sim \bar{r} \), which means the integrand is bounded and we estimate the \( j \)th term by \( C_N 2^{-jN} 2^j \), which is summable for \( N > 1 \). Therefore we have completed the proof of Proposition 6.1.

7. Homogeneous Strichartz estimates

We use the \( L^2 \)-estimates and the microlocalized dispersive estimates to conclude the proof of Theorem 1.1. By Proposition 6.1, we have for all \( t \in \mathbb{R} \) and all \( u_0 \in L^2 \)
\[
\|U(t)u_0\|_{L^2(M^\circ)} \lesssim \|u_0\|_{L^2(M^\circ)};
\]
By Lemma 5.3
\[ U_i(s)U_i^* (t)f = \int_0^\infty e^{i(s-t)\lambda^2} Q_i(\lambda)dE_\sqrt{\Pi}(\lambda)Q_i^*(\lambda)f. \]

Hence we have the following decay estimates by Proposition 6.1
\[ \| U_i(s)U_i^* (t)f \|_{L^\infty} \lesssim |t-s|^{-n/2}\| f \|_{L^1}. \]

As a consequence of the Keel-Tao abstract Strichartz estimate in [27], we have
\[ \| U_i(t)u_0 \|_{L^q(R;L^r(M^c))} \lesssim \| u_0 \|_{L^2(M^c)}, \]
where \((q, r)\) is sharp \( \frac{n}{2}\)-admissible, that is, \( q, r \geq 2, (q, r, n) \neq (2, \infty, 2) \) and \( 2/q + n/r = n/2 \). By the definition of \( U_i(t) \) based on the construction of \( Q_i \), we see that
\[ e^{itH} = \sum_{i=1}^N U_i(t). \]

Therefore we have proved the long-time homogeneous Strichartz estimate.

8. INHOMOGENEOUS STRICHTZART ESTIMATES

In this section, we prove Theorem 1.2 including at the endpoint \((q, r) = (\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})\) for \( n \geq 3 \). Let \( U(t) = e^{itH} : L^2 \to L^2 \). We have already proved that
\[ \| U(t)u_0 \|_{L^q_tL^r_x} \lesssim \| u_0 \|_{L^2} \]
holds for all \((q, r)\) satisfying (1.2). By duality, the estimate is equivalent to
\[ \left\| \int_\mathbb{R} U(t)U^*(s)F(s)ds \right\|_{L^q_tL^r_x} \lesssim \| F \|_{L^{q'}_tL^{r'}_x}, \]
where both \((q, r)\) and \((\tilde{q}, \tilde{r})\) satisfy (1.2). By the Christ-Kiselev lemma [12], we obtain for \( q > \tilde{q}' \)
(8.1)
\[ \left\| \int_{s<t} U(t)U^*(s)F(s)ds \right\|_{L^q_tL^r_x} \lesssim \| F \|_{L^{q'}_tL^{r'}_x}. \]

Notice that \( \tilde{q}' \leq 2 \leq q \), therefore we have proved all inhomogeneous Strichartz estimates except the endpoint \((q, r) = (\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})\). To treat the endpoint, we need show the bilinear form estimate
(8.2)
\[ |T(F, G)| \lesssim \| F \|_{L^2_tL^2_x} \| G \|_{L^2_tL^2_x}, \]
where \( r = 2n/(n-2) \) and \( T(F, G) \) is the bilinear form
(8.3)
\[ T(F, G) = \int_{s<t} \langle U(t)U^*(s)F(s), G(t) \rangle_{L^2} dsdt. \]

Theorem 1.2 follows from

**Proposition 8.1.** There exists a partition of the identity \( Q_i(\lambda) \) on \( L^2(M^c) \) such that, with \( U_i(t) \) defined as in (3.1), there exists a constant \( C \) such that for each pair \((i, j)\), either
(8.4)
\[ \int_{s<t} \langle U_i(t)U_j^*(s)F(s), G(t) \rangle_{L^2} dsdt \lesssim C \| F \|_{L^2_tL^2_x} \| G \|_{L^2_tL^2_x}. \]
or
\[(8.5) \quad \int_{s > t} \langle U_i(t)U_i^*(s)F(s), G(t) \rangle _{L^2} \, ds dt \leq C \|F\|_{L^2_t L^\infty_x} \|G\|_{L^2_t L^\infty'_x}.
\]

Proof of Theorem 1.2 assuming Proposition 8.1. We have proved that for all \(1 \leq i \leq N\),
\[\|U_i(t)u_0\|_{L^2_t L^\infty_x} \lesssim \|u_0\|_{L^2},\]

hence it follows by duality that for all \(1 \leq i, j \leq N\),
\[(8.6) \quad \int_{\mathbb{R}^2} \langle U_i(t)U_i^*(s)F(s), G(t) \rangle _{L^2} \, ds dt \leq C \|F\|_{L^2_t L^\infty_x} \|G\|_{L^2_t L^\infty'_x}.
\]

Therefore, if (8.5) is proved, then by subtracting from (8.6) we obtain (8.4). Then, by summing over all \(i\) and \(j\), we obtain \(8.2\).

To prove Proposition 8.1 we use the following lemma that follows essentially from [15, Section 5].

**Lemma 8.2.** The partition of the identity \(Q_i(\lambda)\) can be chosen so that the pairs of indices \((i, j)\), \(1 \leq i, j \leq N\), can be divided into three classes,
\[
\{1, \ldots, N\}^2 = J_{\text{near}} \cup J_{\text{not--out}} \cup J_{\text{not--inc}},
\]

so that

- if \((i, j) \in J_{\text{near}}\), then \(Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)}Q_j(\lambda)^*\) satisfies the conclusions of Proposition 8.3, 
- if \((i, j) \in J_{\text{non--inc}}\), then \(Q_i(\lambda)\) is not incoming-related to \(Q_j(\lambda)\) in the sense that no point in the operator wavefront set (microlocal support) of \(Q_i(\lambda)\) is related to a point in the operator wavefront set of \(Q_j(\lambda)\) by backward bicharacteristic flow;
- if \((i, j) \in J_{\text{non--out}}\), then \(Q_i(\lambda)\) is not outgoing-related to \(Q_j(\lambda)\) in the sense that no point in the operator wavefront set of \(Q_i(\lambda)\) is related to a point in the operator wavefront set of \(Q_j(\lambda)\) by forward bicharacteristic flow.

**Proof.** We modify our high-energy partition of the identity as in done in [15, Section 5], and then define the overall partition of unity as in (4.1).

First suppose that both \(Q_i\) and \(Q_j\) are low energy operators (the case \(i, j \leq N_1\)). In that case, it is shown in [15, Lemma 5.4] that either \(Q_i\) is not incoming-related or not outgoing-related to \(Q_j\), or that the microsupports of \(Q_i\) and \(Q_j\) satisfy the following condition: \((y, \mu, \nu) \in WF(Q_i)\) and \((y', \mu', \nu') \in WF(Q_j)\) implies that \(\nu - \nu'\) is small. In the latter case, it then follows that the microlocal support of \(Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)}Q_j(\lambda)^*\) satisfies the condition of \([17, \text{Lemma 6.5}]\). This shows that \(Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)}Q_j(\lambda)^*\) can be written in terms of phase functions as in Proposition 2.6. The proof of Proposition 1.3 then applies and shows that the first alternative in the Lemma holds.

Similarly, if \(Q_i\) and \(Q_j\) are high energy operators (the case \(i, j \text{ both } > N_1\)) then it is shown in [15, Lemma 5.3] that either (1) \(Q_i\) is not incoming-related or (2) not outgoing-related to \(Q_j\), or (3) that \(q \in WF'(Q_i)\) and \(q' \in WF'(Q_j)\) implies that \(x(q)\) and \(x(q')\) are both small, and that \(\nu(q) - \nu(q')\) is small, or (4) that \(x(q)\) and \(x(q')\) are both at least \(\epsilon > 0\), and that \(q\) and \(q'\) are close in \(T^*M^3\). In cases (3) and (4), it follows that the microlocal support of \(Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)}Q_j(\lambda)^*\) satisfies the condition
of [17] Lemma 7.7]. This shows that \( Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)Q_j(\lambda)^*} \) can be written in terms of phase functions as in Proposition [18]. The proof of Proposition [18] then applies and shows that the first alternative in the Lemma holds.

On the other hand, if one of the operators is low-energy and the other is high-energy, then notice that \( Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)Q_j(\lambda)^*} \) is only nonzero for intermediate energies \( \lambda \in [1, 2] \). Assuming without loss of generality that the high-energy operator is \( Q_j = (1 - \chi(\lambda))Q_{j - N}^{\text{high}} \), either \( j \leq 2N \) (in which case \( Q_j(\lambda) \) is equal to a multiple of the low energy operator \( Q_{j - N}(\lambda) \) in this range) or \( j \geq 2N + 1 \) (in which case \( Q_j \) is microsupported in the interior, i.e. for \( x \geq \epsilon/2 \) for some small \( \epsilon > 0 \)). In the former case, the Lemma follows exactly as for pairs of low energy operators. In the latter case, \( Q_j(\lambda) \) has no operator wavefront set at all for \( \lambda \in [1, 2] \), since the wavefront set for finite values of \( \lambda \) is at the boundary, i.e. at \( x = 0 \); in this case, \((i, j)\) is in both \( J_{\text{not-inc}} \) and \( J_{\text{not-out}} \). This completes the proof.

We exploit the not-incoming or not-outgoing property of \( Q_i(\lambda) \) with respect to \( Q_j(\lambda) \) in the following two lemmas.

**Lemma 8.3.** Let \( Q_i(\lambda), Q_j(\lambda) \) be such that \( Q_i \) is not outgoing-related to \( Q_j \). Then, for \( \lambda \leq 2 \), and as a multiple of \( |dgd\gamma|^{1/2}|d\lambda| \), the Schwartz kernel of \( Q_i(\lambda)dE_{\sqrt{\Pi}(\lambda)Q_j(\lambda)^*} \) can be expressed as the sum of a finite number of terms of the form

\[
\lambda^{n-1} \int_{\mathbb{R}^k} e^{i\lambda \Phi(y,y',\sigma,v)/x} \left( \frac{2'}{2} \right)^{(n-1)/2-k/2} a(\lambda, y, y', \sigma, \frac{x'}{\lambda}, v) dv \quad \text{or}
\]

\[
\lambda^{n-1} \int_{\mathbb{R}^k} \int_{0}^{\infty} e^{i\lambda \Phi(y,y',\sigma,v,s)/x} \left( \frac{2'}{2} \right)^{(n-1)/2-k/2} s^{n-2} a(\lambda, y, y', \sigma, \frac{x'}{\lambda}, v, s) ds dv
\]

in the region \( \sigma = x/x' \leq 2 \), \( x'/\lambda \leq 2 \), or

\[
\lambda^{n-1} a(\lambda, y, y', \sigma, x'/\lambda)
\]

in the region \( \sigma = x/x' \leq 2 \), \( x'/\lambda \geq 1 \), where in each case, \( \Phi < -\epsilon < 0 \) and \( a \) is a smooth function compactly supported in the \( v \) and \( s \) variables (where present), such that \( (|\lambda\partial\lambda|)^N a \leq C_N \) for all \( N \in \mathbb{N} \). In each case, we may assume that \( k \leq n - 1 \); if \( k = 0 \) in (8.7) or \( k = 1 \) in (8.8), then there is no variable \( v \), and no \( v \)-integral. The key point is that in each expression, the phase function is strictly negative.

If, instead, \( Q_i \) is not incoming-related to \( Q_j \), then the same conclusion holds with the reversed sign: the Schwartz kernel can be written as a finite sum of terms with a strictly positive phase function.

**Remark 8.4.** For \( \sigma \geq 1/2 \), the Schwartz kernel has a similar description, as follows immediately from the symmetry of the kernel under interchanging the left and right variables.

**Proof.** The statement that the Schwartz kernel has the indicated forms above follows immediately from the description of the spectral measure in [16] Theorem 3.10] as a Legendre distribution in the class \( I^{m,p,r_{lb},r_{rb}}(M_{k,b}^2, (L^p, L^q); \Omega_{k,b}^{1/2/2}) \), where \( m = -1/2, p = (n - 2)/2, r_{lb} = r_{rb} = (n - 1)/2 \). The bound on \( k \) follows from the fact that \( k \) can be taken as the drop in rank of the projection from \( L^p \) to the base \( (\partial M)^2 \times (0, \infty) \), which is the front face (that is, the face created by blowup) of \( M_{k,b}^2 \). We claim that
the drop in rank is at most \( n - 1 \), which proves that we may assume that \( k \leq n - 1 \). To prove this claim, we show that the differentials \( dy_1, \ldots, dy_{n-1} \) and at least one of \( d\sigma, dy_1, \ldots, dy_{n-1} \) are linearly independent on \( L \). This can be seen from the description of \( L \) as the flow from the set
\[
\{(y, y, 1, \mu, -\mu, \nu, -\mu) \mid \nu^2 + h(\mu) = 1\},
\]
using the coordinates of \((2.6)\), by the flow of the vector field \( V \), which is the vector field given by \( x^{-1} \) times the Hamilton vector field of the principal symbol of \( \Delta \) acting in the right variables on \( M_{*}^{2} \). In fact \( V_{r} = \sin s' \partial_{s'} \) in the coordinates \((s, s')\) on the leaves \( \gamma^{2} \) of \((2.6)\), and takes the form (see [24, Eq. (2.26)] or [16, Eq. (3.5)])
\[
2\nu' \frac{\partial}{\partial \sigma} - 2\nu' \mu' \cdot \frac{\partial}{\partial \mu'} + h' \frac{\partial}{\partial y'} + \left( \frac{\partial h'}{\partial \mu'} \frac{\partial}{\partial y'} - \frac{\partial h'}{\partial y'} \frac{\partial}{\partial \mu'} \right), \quad h' = h(y', \mu') = \sum_{i,j} h_{ij}(y') \mu_{i} \mu_{j}.
\]

It is clear that \( dy_{1}, \ldots, dy_{n-1} \) are linearly independent at the initial set \((8.10)\). Moreover, their Lie derivative with respect to \( V \) vanishes, so they are linearly dependent on all of \( L^{bf} \). Also, since \( h' + \nu^2 = 1 \) on \( L^{bf} \), either the \( \partial_{\sigma} \) or the \( \partial_{y'} \) component of the vector field \( V \) does not vanish, unless \( \sigma = 0 \), showing that either \( d\sigma \) or one of the \( dy' \) do not vanish at each point of \( L^{bf} \) for \( \sigma \neq 0 \). But it was shown in [24] that \( L^{bf} \) is transversal to the boundary at \( \sigma = 0 \), which means that \( d\sigma \neq 0 \) on \( L^{bf} \) when \( \sigma \) is small. This proves the claim.

We next show that \( \Phi \) can be taken to be strictly negative. We use the microlocal support estimates from [17]. Applying [17, Corollary 5.3], we find that the microlocal support of \( Q_{i}(\lambda) d\sigma \pi_{\lambda}(\lambda) Q_{j}(\lambda)^* \) is contained in that part of \( L^{bf} \) where \((in the notation of (2.6)) s < s' \) (since the initial set \((8.10)\) corresponds to \( s = s' \), \( \partial_{s'} \), respectively \( \partial_{s} \) moves in the outgoing, resp. incoming, direction along the flow). Repeating the calculation following (2.9) we see that the value of \( \Phi \) ‘on the Legendrian’ is \( \Phi = -\cos s + \sigma \cos s' = (\sin s')^{-1} \sin(s - s') \), which is strictly negative. By restricting the support of the symbol \( a \) in \((8.7) \rightarrow (8.9)\), we can assume that \( \Phi \) is negative everywhere on the support of the integrand.

**Lemma 8.5.** Let \( Q_{i}(\lambda), Q_{j}(\lambda) \) be such that \( Q_{i} \) is not outgoing-related to \( Q_{j} \). Then, for \( \lambda \geq 1 \), and as a multiple of \( |d\sigma dy'|^{1/2} |d\lambda| \), the Schwartz kernel of \( Q_{i}(\lambda) d\sigma \pi_{\lambda}(\lambda) Q_{j}(\lambda)^* \) can be written in terms of a finite number of oscillatory integrals of the form
\[
\begin{align*}
(8.11) & \int_{\mathbb{R}^{k}} e^{i\lambda \Phi(y, y', \sigma, x, v)/x} \lambda^{n-1+k/2} x^{(n-1)/2-k/2} a(\lambda, y, y', \sigma, x, v) dv \\
(8.12) & \int_{\mathbb{R}^{k-1}} \int_{0}^{\infty} e^{i\lambda \Phi(y, y', \sigma, x, v, s)/x} \lambda^{n-1+k/2} x^{(n-1)/2-k/2} s^{n-2} a(\lambda, y, y', \sigma, x, v, s) ds dv
\end{align*}
\]
in the region \( \sigma = x/x' \leq 2, x \leq \delta, \) or
\[
(8.13) \int_{\mathbb{R}^{k}} e^{i\lambda \Phi(z, z', \nu)} \lambda^{n-1+k/2} a(\lambda, z, z', v) dv
\]
in the region \( x \geq \delta, x' \geq \delta \), where in each case, \( \Phi < -\epsilon < 0 \) and \( a \) is a smooth function compactly supported in the \( v \) and \( s \) variables (where present), such that \( |(\partial \lambda)^{N} a| \leq C_{N} \).

In each case, we may assume that \( k \leq n - 1 \); if \( k = 0 \) in \((8.11) \) or \((8.13)\), or \( k = 1 \)
in (8.12) then there is no variable \( v \), and no \( v \)-integral. Again, the key point is that in each expression, the phase function is strictly negative.

If, instead, \( Q_i \) is not incoming-related to \( Q_j \), then the same conclusion holds with the reversed sign: the Schwartz kernel can be written as a finite sum of terms with a strictly positive phase function.

Proof. The proof is essentially identical to that of Lemma 8.3 and is omitted. \( \square \)

Next we establish dispersive estimates for \( U_i(t)U_j(s)^* \):

Lemma 8.6. We have the following estimates on \( U_i(t)U_j(s)^* \):

1. If \( (i, j) \in J_{\text{near}} \), then for all \( t \neq s \) we have

\[
\| U_i(t)U_j^*(s) \|_{L^1 \rightarrow L^\infty} \leq C|t - s|^{-\frac{n}{2}}, \tag{8.14}
\]

2. If \( (i, j) \) such that \( Q_i \) is not outgoing related to \( Q_j \), and \( t < s \), then

\[
\| U_i(t)U_j^*(s) \|_{L^1 \rightarrow L^\infty} \leq C|t - s|^{-\frac{n}{2}}, \tag{8.15}
\]

3. Similarly, if \( (i, j) \) such that \( Q_i \) is not incoming related to \( Q_j \), and \( s < t \), then

\[
\| U_i(t)U_j^*(s) \|_{L^1 \rightarrow L^\infty} \leq C|t - s|^{-\frac{n}{2}}. \tag{8.16}
\]

Proof. The estimate (8.14) is essentially proved in Proposition 6.1 since we can use Proposition 1.3. Assume that \( Q_i \) is not incoming-related to \( Q_j \), and consider (8.10). By Lemma 8.3, \( U_i(t)U_j(s)^* \) is given by

\[
\int_0^\infty e^{i(t-s)\lambda^2} (Q_i(\lambda)dE_{\sqrt{\overline{H}}}(\lambda)Q_j^*(\lambda))(z, z')d\lambda. \tag{8.17}
\]

Then we need to show that for \( s < t \)

\[
\left| \int_0^\infty e^{i(t-s)\lambda^2} (Q_i(\lambda)dE_{\sqrt{\overline{H}}}(\lambda)Q_j^*(\lambda))(z, z')d\lambda \right| \leq C|t - s|^{-\frac{n}{2}}. \tag{8.18}
\]

**Case 1**, \( t - s \geq 1 \). We introduce a dyadic partition of unity in \( \lambda \). Let \( \phi \in C^\infty_c([\frac{1}{2}, 2]) \) be as in Section 3 such that \( \sum_j \phi(2^{-j}\sqrt{t-s}\lambda) = 1 \), define

\[
\phi_0(\sqrt{t-s}\lambda) = \sum_{j \leq 0} \phi(2^{-j}\sqrt{t-s}\lambda), \quad \phi_j(\lambda) := \phi(2^{-j}\lambda)
\]

and insert

\[
1 = \phi_0(\sqrt{t-s}\lambda) + \sum_{j \geq 1} \phi_j(\sqrt{t-s}\lambda), \quad \phi_j(\lambda) := \phi(2^{-j}\lambda)
\]

into the integral (8.17). In addition, we substitute for \( Q_i(\lambda)dE_{\sqrt{\overline{H}}}(\lambda)Q_j^*(\lambda) \) one of the expressions in Lemmas 8.3 and 8.5. Since \( t - s \geq 1 \), for the \( \phi_0 \) term, only the low energy expressions are relevant. The estimate follows immediately from noticing that these expressions are pointwise bounded by \( C\lambda^{n-1} \), using the fact that \( k \leq n - 1 \) in these expressions.

To treat the \( \phi_j \) terms for \( j \geq 1 \), we substitute again one of the expressions in Lemmas 8.3 and 8.5. For notational simplicity we consider the expression (8.14), but
the argument is similar in the other cases. We scale the $\lambda$ variable and obtain the expression

\begin{equation}
\int_0^\infty \int_{\mathbb{R}^k} e^{i(t-s)\lambda^2 + i\lambda \Phi(z,z',v)} \lambda^{n-1+k/2} a(\lambda, z, z', v) \phi_j(\sqrt{t-s} \lambda) \, dv \, d\lambda = (t-s)^{-\frac{k}{2} - \frac{1}{4}} \int_0^\infty \int_{\mathbb{R}^k} e^{i\left(\lambda^2 + \frac{\sqrt{s}(z,z',v)}{\sqrt{t-s}}\right)} \lambda^{n-1+k/2} a\left(\frac{\lambda}{\sqrt{t-s}}, y, y', \sigma, v\right) \phi_j(\lambda) \, dv \, d\lambda
\end{equation}

where $\overline{\lambda} = \sqrt{t-s} \lambda$. We observe that the overall exponential factor is invariant under the differential operator

$$L = -i \frac{4 \overline{\lambda}^2 + 4 \lambda \Phi / \sqrt{t-s}}{(2 \lambda^2 + \lambda \Phi / \sqrt{t-s})^2}.$$ 

The adjoint of this is

$$L^t = -L + \frac{i}{2 \lambda^2 + \lambda \Phi / \sqrt{t-s}} - i \frac{4 \lambda^2 + \lambda \Phi / \sqrt{t-s}}{(2 \lambda^2 + \lambda \Phi / \sqrt{t-s})^2}.$$ 

We apply $L^N$ to the exponential factors, and integrate by parts $N$ times. Since $\Phi \geq 0$ according to Lemma 8.3 and since we have an estimate $|(-\lambda \partial_\lambda)^N a| \leq C_N$, each time we integrate by parts we gain a factor $\overline{\lambda}^{-2} \sim 2^{-2j}$. It follows that the integral with $\phi(2^{-j} \overline{\lambda})$ inserted is bounded by $(t-s)^{-n/2} \sim (2N-n-2j/2)$ uniformly for $t-s \leq 1$. Hence we prove (8.16) by summing over $j \geq 0$. The argument to prove (8.15) is analogous.

**Case 2.** $t-s \leq 1$. In this case, we use a dyadic decomposition in terms of the original variable $\lambda$. We consider the integral (8.17), insert the dyadic decomposition

$$1 = \sum_{j=0}^{\infty} \phi_j(\lambda),$$

and substitute for $Q_j(\lambda) dE_{\sqrt{\lambda}}(\lambda) Q_j^*(\lambda)$ one of the expressions in Lemmas 8.3 and 8.5.

For the case $j = 0$, the estimate follows immediately from the uniform boundedness of (8.7) \, (8.9). For the cases $j \geq 1$, we use the expressions in Lemma 8.3 and observe that the overall exponential factor is invariant under the differential operator

$$L = \frac{-i}{2(t-s)\lambda^2 + \lambda \Phi} \lambda \frac{\partial}{\partial \lambda}.$$ 

The adjoint of this is

$$L^t = -L + \frac{i}{2(t-s)\lambda^2 + \lambda \Phi} - i \frac{4(t-s)\lambda^2 + \lambda \Phi}{(2(t-s)\lambda^2 + \lambda \Phi)^2}.$$ 

We apply $L$ $N$-times to the exponential factors, and integrate by parts. Since $\Phi \geq \varepsilon > 0$ according to Lemma 8.3 and since we have an estimate $|(-\lambda \partial_\lambda)^N a| \leq C_N$, each time we integrate by parts we gain a factor $\lambda^{-1} \sim 2^{-j}$. It follows that the integral with $\phi(2^{-j} \lambda)$ inserted is bounded by $2^{-j(N-n-2j/2)}$ uniformly for $t-s \leq 1$. Hence we prove (8.16) by summing over $j \geq 0$. The argument to prove (8.15) is analogous. □

**Remark 8.7.** Notice that, in the cases (8.15) and (8.16), there is a lot of ‘slack’ in the estimates. This is because the sign of $t-s$ has the favourable sign relative to the sign of the phase function, so that the overall phase in integrals such as (8.19) are never stationary. Then integration by parts give us more decay than needed to prove the
estimates. This is important because it overcomes the growth of the spectral measure as $\lambda \to \infty$ at conjugate points: at pairs of conjugate points we have $k > 0$ and we see from, say, [13] that the spectral measure will not obey the localized (near the diagonal) estimates of Proposition 8.4 by a factor $\lambda^{k/2}$. The geometric meaning of $k$ is the drop in rank of the projection from $L$ down to $M^2$, hence is positive precisely at pairs of conjugate points.

We now complete the proof of Theorem 1.2 by proving Proposition 8.1.

Proof of Proposition 8.1. We use a partition of the identity as in Lemma 8.2. In the case that $(i, j) \in J_{\text{near}}$, we have the dispersive estimate (8.14). This allows us to apply the argument of [27] Sections 4–7 to obtain (8.4). In the case that $(i, j) \in J_{\text{non-out}}$, we obtain (8.5) since we have the dispersive estimate (8.10) for $s < t$. Finally, in the case that $(i, j) \in J_{\text{non-inc}}$, we obtain (8.5) since we have the dispersive estimate (8.15) for $s > t$. □

Remark 8.8. The endpoint inhomogeneous Strichartz estimate is closely related to the uniform Sobolev estimate

$$
(8.20) \quad \|(H - \alpha)^{-1}\|_{L^r \to L^{r'}} \leq C, \quad r = \frac{2n}{n + 2}.
$$

where $C$ is independent of $\alpha \in \mathbb{C}$. This estimate was proved by [28] for the flat Laplacian, and by [13] for the Laplacian on nontrapping asymptotically conic manifolds (it was also shown in [15] that (8.20) holds for $r \in [2n/(n + 2), 2(n + 1)/(n + 3)]$ with a power of $\alpha$ on the RHS). In fact, it was pointed out to the authors by Thomas Duyckaerts and Colin Guillarmou that the endpoint inhomogeneous Strichartz estimate implies the uniform Sobolev estimate (8.20). To see this, we choose $w \in C^\infty_\text{c}(M^0)$ and $\chi(t)$ equal to 1 on $[-T, T]$ and zero for $|t| \geq T + 1$, and let $u(t, z) = \chi(t)e^{i\alpha t}w(z)$. Then

$$(i\partial_t + H)u = F(t, z), \quad F(t, z) := \chi(t)e^{i\alpha t}(H - \alpha)w(z) + i\chi'(t)e^{i\alpha t}w(z).$$

Applying the endpoint inhomogeneous Strichartz estimate, we obtain

$$
\|u\|_{L^2_t L^{r'}_z} \leq C\|F\|_{L^2_t L^r_z}.
$$

From the specific form of $u$ and $F$ we have

$$
\|u\|_{L^2_t L^{r'}_z} = \sqrt{2T}\|w\|_{L^{r'}} + O(1), \quad \|F\|_{L^2_t L^r_z} = \sqrt{2T}\|(H - \alpha)w\|_{L^r} + O(1).
$$

Taking the limit $T \to \infty$ we find that

$$
\|w\|_{L^{r'}} \leq C\|(H - \alpha)w\|_{L^r},
$$

which implies the uniform Sobolev estimate.

In the other direction, suppose that the uniform Sobolev estimate holds. If $u$ and $F$ satisfy (1.3), then taking the Fourier transform in $t$ we find that

$$(8.21) \quad (H - \alpha)\hat{u}(\alpha, z) = \hat{F}(\alpha, z).$$

Suppose for a moment that the following statement were true: “Fourier transformation in $t$ is a bounded linear map from $L^2(\mathbb{R}_t; L^p(M^0))$ to $L^2(\mathbb{R}_t; L^p(M^0))$ for $p = r', r$”. Using this and the uniform Sobolev inequality, applied to (8.21), we would obtain the inhomogeneous Strichartz estimate. Unfortunately, the statement in quotation marks is known to be false, so this argument is purely heuristic. Nevertheless, it illustrates
the close relation between the two estimates. It would be interesting to know if there are general conditions under which the two estimates are equivalent.

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