Recursive calculation of connection formulas for systems of differential equations of Okubo normal form

Toshiaki Yokoyama
Department of Mathematics, Chiba Institute of Technology, Narashino 275-0023, Japan

Abstract

We study the structure of analytic continuation of solutions of an even rank system of linear ordinary differential equations of Okubo normal form (ONF). We develop an adjustment of the method by using the Euler integral for evaluating the connection formulas of the Gauss hypergeometric function \( \genfrac{[}{]}{0pt}{}{2}{1}(\alpha, \beta, \gamma; x) \) to the system of ONF. We obtain recursive relations between connection coefficients for the system of ONF and ones for the underlying system of half rank.

1 Introduction

The Gauss hypergeometric differential equation

\[
x(1-x)\frac{d^2u}{dx^2} + \{c - (1 + a + b)x\} \frac{du}{dx} - abu = 0
\]  

(1.1)

is transformed into a system of first-order differential equations of the form

\[
\left(xI_2 - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \frac{d}{dx} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 - c \\ -(1 - c + a)(1 - c + b) \\ c - 1 - a - b \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},
\]

where

\[
u_1 = u, \quad u_2 = \frac{du}{dx} - (1 - c)u.
\]

(1.2)

As a generalization of this system, Okubo ([10], [11]) studies a rank \( n \) system of linear differential equations of the form

\[
xI_n - T \frac{du}{dx} = Au,
\]

(1.3)

where \( u \) is an \( n \)-vector of unknown functions, \( A \) is an \( n \times n \) constant matrix, \( T \) is an \( n \times n \) diagonal constant matrix, and \( I_n \) denotes the \( n \times n \) identity matrix. We call \((1.3)\) a system of linear differential equations of \textit{Okubo normal form} (ONF).

In this paper we shall study the structure of analytic continuation of solutions of a rank \( 2n \) system of ONF of the form

\[
xI_{2n} - T \frac{dU}{dx} = A_P U,
\]

(1.4)

where \( U = U(x) \) is a \( 2n \)-vector of unknown functions, \( T, A \in \text{M}(n; \mathbb{C}), \; P \in \text{GL}(n; \mathbb{C}) \) are matrices such that \( T \) and \( A' = P^{-1}AP \) are diagonal, \( t \) is a complex number satisfying \( \det(tI_n - T) \neq 0 \), and \( \rho_j \; (j = 1, 2) \) are complex constants.
The Gauss hypergeometric equation is intimately related to the Euler-Darboux equation. Our system \([1.4]\) is obtained through the process of making the relation of the two equations clear. Set

\[
E(\alpha, \beta, \gamma) = (\xi - \eta)^2 \frac{\partial^2}{\partial \xi \partial \eta} - \alpha(\xi - \eta) \frac{\partial}{\partial \xi} - \beta(\eta - \xi) \frac{\partial}{\partial \eta} - \gamma.
\]

In the case that \(\gamma = 0\), which is done by using the relation

\[
E(\alpha, \beta, \gamma)(\xi - \eta)^\delta = (\xi - \eta)^\delta E(\alpha + \delta, \beta + \delta, \gamma + \delta(\delta + \alpha + \beta - 1)) \tag{1.5}
\]

with a suitable choice of the parameter \(\delta\), the partial differential equation

\[
E(\alpha, \beta, 0)f = 0 \tag{1.6}
\]

is called the Euler-Darboux equation. Darboux (\[3, \S 347\]) shows that by setting \(f = \xi^\lambda \varphi(t), \ t = \eta/\xi\), with a fixed constant \(\lambda\), the equation \((1.6)\) is reduced to the ordinary differential equation

\[
t(1 - t) \frac{d^2 \varphi}{dt^2} + \{1 - \lambda - \beta - (1 - \lambda + \alpha)t\} \frac{d\varphi}{dt} - \lambda \alpha \varphi = 0,
\]

which is the hypergeometric equation with the parameters \(a, b, c\) replaced by \(\alpha, \lambda - \beta, 1 - \lambda - \beta\), respectively.

Set

\[
L_2(\lambda) = \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} - \lambda.
\]

Miller (\[9\]) notices that the space of particular solutions of \((1.6)\) of the form \(f = \xi^\lambda \varphi(\eta/\xi)\) coincides with the space of solutions of the system of partial differential equations

\[
\begin{cases}
E(\alpha, \beta, 0)f = 0, \\
L_2(\lambda)f = 0,
\end{cases} \tag{1.7}
\]

since the second equation implies \(f = \xi^\lambda \varphi(\eta/\xi)\). Let us consider a 1-dimensional section of this system obtained by taking \(\eta\) for a constant, and show that the section is also reduced to the hypergeometric equation. We first make the substitution \(f = (\xi - \eta)^{-\beta}g\). Then, using the relation \((1.5)\) and the relation

\[
L_2(\lambda)(\xi - \eta)^\delta = (\xi - \eta)^\delta L_2(\lambda - \delta),
\]

we obtain the system of partial differential equations

\[
\begin{cases}
E(\alpha - \beta, 0, (1 - \alpha)\beta)g = 0, \\
L_2(\lambda + \beta)g = 0.
\end{cases} \tag{1.8}
\]

Eliminating the term of \(\partial^2 g/\partial \xi \partial \eta\) from this system by

\[
(\xi - \eta)^2 \frac{\partial}{\partial \xi} L_2(\lambda + \beta)g - \eta E(\alpha - \beta, 0, (1 - \alpha)\beta)g, \tag{1.9}
\]

we obtain

\[
\xi(\xi - \eta)^2 \frac{\partial^2 g}{\partial \xi^2} + \{(1 - \lambda - \beta)\xi - (1 - \lambda - \alpha)\eta\}(\xi - \eta) \frac{\partial g}{\partial \xi} + (1 - \alpha)\beta \eta g = 0. \tag{1.10}
\]

As a differential equation in one variable \(\xi\), this equation is Fuchsian and has the Riemann scheme (the list of exponents)

\[
\begin{pmatrix}
\xi = 0 & \xi = \infty & \xi = \eta \\
0 & 0 & 1 - \alpha \\
\lambda + \alpha & -\lambda - \beta & \beta
\end{pmatrix}.
\]
So, if we write
\[ \alpha = 1 - a, \ \beta = b, \ \lambda = a - c \quad \text{or} \quad \alpha = 1 - b, \ \beta = a, \ \lambda = b - c \] (1.11)
and change the variable \( \xi \) to \( x \) by
\[ x = 1 + \frac{\eta}{\xi - \eta} \] (1.12)
which carries the points \( \xi = 0, \infty, \eta \) into the points \( x = 0, 1, \infty \), respectively, then we can transform the equation (1.10) into the equation (1.1). Note that by virtue of the substitution (1.11), we can write the system (1.8) in the form
\[ \begin{align*}
E(1 - a - b, 0, ab)g &= 0, \\
L_2(a + b - c)g &= 0,
\end{align*} \] (1.13)
which is not altered if \( a \) and \( b \) are interchanged.

Darboux ([3, §354]) furthermore shows that an integral of the form
\[ f(\xi, \eta) = \int_{\eta}^{\xi} (\xi - \zeta)^{-\beta}(\eta - \zeta)^{-\alpha} h(\zeta) \, d\zeta \] (1.14)
becomes a solution of the equation (1.6) under the condition \( \Re \alpha < 0, \Re \beta < 0 \). Set
\[ L_1(\lambda) = \xi \frac{d}{d\xi} - \lambda. \]
If \( h(\zeta) \) is a solution of the ordinary differential equation
\[ L_1(\lambda + \alpha + \beta - 1)h = 0, \] (1.15)
then the integral (1.14) satisfies \( L_2(\lambda)f = 0 \) and hence becomes a solution of the system (1.7). Indeed, since
\[ f_{\xi}(\xi, \eta) = \int_{\eta}^{\xi} (-\beta)(\xi - \zeta)^{-\beta - 1}(\eta - \zeta)^{-\alpha} h(\zeta) \, d\zeta, \]
\[ f_{\eta}(\xi, \eta) = \int_{\eta}^{\xi} (-\alpha)(\xi - \zeta)^{-\beta}(\eta - \zeta)^{-\alpha - 1} h(\zeta) \, d\zeta, \]
we have
\[ \begin{align*}
\xi f_{\xi}(\xi, \eta) + \eta f_{\eta}(\xi, \eta) + (\alpha + \beta)f(\xi, \eta) \\
&= \int_{\eta}^{\xi} \{\beta(\xi - \zeta)^{-\beta - 1}(\eta - \zeta)^{-\alpha} + \alpha(-\zeta)(\xi - \zeta)^{-\beta}(\eta - \zeta)^{-\alpha - 1}\} h(\zeta) \, d\zeta \\
&= -\int_{\eta}^{\xi} \frac{\partial}{\partial \zeta}\{(\xi - \zeta)^{-\beta}(\eta - \zeta)^{-\alpha}\} \zeta h(\zeta) \, d\zeta \\
&= \int_{\eta}^{\xi} (\xi - \zeta)^{-\beta}(\eta - \zeta)^{-\alpha} \frac{d}{d\zeta}\{\zeta h(\zeta)\} \, d\zeta \\
&= (\lambda + \alpha + \beta)f(\xi, \eta),
\end{align*} \] (1.16)
which leads to \( L_2(\lambda)f(\xi, \eta) = 0 \). Here we have used \( \frac{d}{d\zeta}\{\zeta h(\zeta)\} = h(\zeta) + \zeta \frac{d}{d\zeta} h(\zeta) = (\lambda + \alpha + \beta)h(\zeta) \) in the last equality.

It should be noted that the equation (1.15) is a rank 1 system of ONF and the integral (1.14) is considered as an extension in two variables of the Euler transformation
\[ f(\zeta) = \int_{\tau}^{\zeta} (\zeta - \tau)^{-\alpha - \beta} h(\tau) \, d\tau \]
which carries a solution of the equation (1.15) into a solution of the equation \( L_1(\lambda)f = 0 \). In view of affinity of the system of ONF for the Euler transformation we may extend \( L_1(\lambda) \), \( L_2(\lambda) \) to rank \( n \) differential operators of ONF as

\[
L_1(T', A; \rho) = (\zeta I_n - T')\frac{d}{d\zeta} - \rho I_n - A,
\]

\[
L_2(T', A; \rho) = (\xi I_n - T')\frac{\partial}{\partial \xi} + (\eta I_n - T')\frac{\partial}{\partial \eta} - \rho I_n - A,
\]

where \( A \) is an \( n \times n \) constant matrix and \( T' \) is an \( n \times n \) diagonal constant matrix. For a solution \( w(\zeta) \) of the system of ONF

\[
L_1(T', A; \rho + \alpha + \beta - 1)w = 0
\]

we define

\[
u(\xi, \eta) = \int_{\eta}^{\xi} (\xi - \zeta)^{-\beta}(\eta - \zeta)^{-\alpha}w(\zeta)\,d\zeta
\]

provided that \( \Re \alpha < 0 \) and \( \Re \beta < 0 \). Then, similarly to the integral (1.14), \( u(\xi, \eta) \) becomes a solution of the system of partial differential equations

\[
\begin{aligned}
E(\alpha, \beta, 0)u &= 0, \\
L_2(T', A; \rho)u &= 0.
\end{aligned}
\]  

(1.19)

The operator \( L_2(T', A; \rho) \) also satisfies the relation

\[
L_2(T', A; \rho)(\xi - \eta)^{\delta} = (\xi - \eta)^{\delta}L_2(T', A; \rho - \delta).
\]

So, similarly to (1.8) and (1.13), setting

\[
\alpha = 1 + \rho_j', \quad \beta = -\rho_j', \quad \rho = -1 - \rho_j', \quad \text{and} \quad u = (\xi - \eta)^{\rho_j}v,
\]

where \( j \in \{1, 2\} \) and \( j' \) denotes the complement of \( j \) in \( \{1, 2\} \), namely, \( (j, j') = (1, 2) \) or \( (2, 1) \), we can transform the system (1.19) into the system

\[
\begin{aligned}
E(1 + \rho_1 + \rho_2, 0, \rho_1 \rho_2)v &= 0, \\
L_2(T', A; -1 - \rho_1 - \rho_2)v &= 0,
\end{aligned}
\]

(1.20)

which is not altered if \( \rho_1 \) and \( \rho_2 \) are interchanged. The quantities \( \rho_1, \rho_2 \) and \( A \) in (1.20) correspond to \( -a, -b \) and \( 1 - c \) in (1.13), respectively.

Our system (1.4) is obtained as a 1-dimensional section of the system (1.20). The author introduced the system (1.4) firstly in [15], and then used it to establish an algorithm for constructing all irreducible semisimple systems of ONF having rigid monodromy in [16]. Haraoka ([4]) constructs integral representations of solutions of the system (1.4) using the integral equivalent to (1.18), and shows by following the author’s algorithm that solutions of all irreducible semisimple systems of ONF can be represented by the integral of Euler type.

In this paper we study connection problems between local solutions of the system (1.4) by using the integral (1.18). We are especially concerned with the relation between connection coefficients of the system (1.4) and ones of the system (1.17).

2 Preliminary

In this section we give transformations for obtaining the system (1.4) from the system (1.20), and recall Haraoka’s results on integral representations of solutions of the system (1.4).
2.1 Transformation of equations

Set

\[ M(T', A) = -\frac{\partial}{\partial \xi}((\xi - \eta)(\xi I_n - T') - A(\eta I_n - T')), \]
\[ N(T', A; \rho) = (\xi - \eta)L_2(T', A; \rho) + M(T', A) \]

\[ = (\xi - \eta)(\eta I_n - T') \frac{\partial}{\partial \eta} - \rho(\xi - \eta)I_n - A(\xi I_n - T'), \]

and write

\[ E = E(1 + \rho_1 + \rho_2, 0, \rho_1 \rho_2), \]
\[ L_2 = L_2(T', A; -1 - \rho_1 - \rho_2), \]
\[ M = M(T', A), \]
\[ N = N(T', A; -1 - \rho_1 - \rho_2) \]

for short.

Remark 2.1. Applying the change of variable \((1.12)\) to the differential operator defining \(u_2\) in \((1.2)\), we obtain

\[ x \frac{d}{dx} = (1 - c) \frac{1}{\eta} \left( -\frac{\partial}{\partial \xi}((\xi - \eta)\xi \frac{d}{d\xi} - (1 - c)\eta) \right). \]

The differential operator \(M(T', A)\) corresponds to this operator.

Similarly to \((1.3)\), we have the following proposition.

**Proposition 2.1.** We have

\[ (\xi - \eta)^2 \frac{\partial}{\partial \xi}L_2 - (\eta I_n - T')E = - \left( (\xi - \eta) \frac{\partial}{\partial \xi} + (\rho_1 + \rho_2)I_n - A \right) \]
\[ = (A - \rho_1 I_n)(A - \rho_2 I_n)(\eta I_n - T'), \]

\[ (\xi - \eta)^2 \frac{\partial}{\partial \eta}L_2 - (\xi I_n - T')E - (\rho_1 + \rho_2 + 1)(\xi - \eta)L_2 \]
\[ = \left( \frac{\partial}{\partial \eta} - (\rho_1 + \rho_2)I_n + A \right) \]
\[ N + (A - \rho_1 I_n)(A - \rho_2 I_n)(\xi I_n - T'). \]

**Proof.** By direct calculation.

**Proposition 2.2.** Set \(v_1 = (\eta I_n - T')v\) and \(v_2 = Mv\). Then the system of partial differential equations \((1.3)\) is transformed into a system of first-order partial differential equations for a 2n-vector \(\left( \begin{array}{c} v_1 \\ v_2 \end{array} \right)\) of the form

\[
\begin{cases}
(\xi - \eta) \frac{\partial v_1}{\partial \xi} = -((\xi I_n - T')^{-1}(\eta I_n - T'))A v_1 - ((\xi I_n - T')^{-1}(\eta I_n - T'))v_2, \\
(\xi - \eta) \frac{\partial v_2}{\partial \xi} = (A - \rho_1 I_n)(A - \rho_2 I_n) v_1 + (A - (\rho_1 + \rho_2)I_n)v_2, \\
(\xi - \eta) \frac{\partial v_1}{\partial \eta} = \{A(\xi I_n - T') - (\rho_1 + \rho_2)(\xi - \eta)I_n\}(\eta I_n - T')^{-1}v_1 + v_2, \\
(\xi - \eta) \frac{\partial v_2}{\partial \eta} = -((A - \rho_1 I_n)(A - \rho_2 I_n)(\xi I_n - T')^{-1}(\eta I_n - T'))v_1 - (A - (\rho_1 + \rho_2)I_n)v_2.
\end{cases}
\]

**Proof.** The system of partial differential equations

\[
\begin{cases}
(\xi - \eta)^2 \frac{\partial}{\partial \xi}L_2 - (\eta I_n - T')E \end{cases} v = 0,
\]
\[
(\xi - \eta)^2 \frac{\partial}{\partial \eta}L_2 - (\xi I_n - T')E - (\rho_1 + \rho_2 + 1)(\xi - \eta)L_2 \end{cases} v = 0,
\]
\[ L_2v = 0 \]
is equivalent to the system (1.20). By virtue of Proposition 2.1 and the definition of the operators \( M \) and \( N \) we can write this system in the form
\[
\begin{cases}
(\xi - \eta) \frac{\partial}{\partial \xi} + (\rho_1 + \rho_2)I_n - A \\
M - (A - \rho_1 I_n)(A - \rho_2 I_n)(\eta I_n - T')
\end{cases} v = 0,
\begin{cases}
(\xi - \eta) \frac{\partial}{\partial \eta} - (\rho_1 + \rho_2)I_n + A \\
N + (A - \rho_1 I_n)(A - \rho_2 I_n)(\xi I_n - T')
\end{cases} v = 0,
\]
\((M - N)v = 0.
\]

The first equation of this system and \( Mv - v_2 = 0 \) are equivalent to the first two equations of the system (2.1). Besides, the second equation and \( Nv - v_2 = 0 \) are equivalent to the last two equations of the system (2.1). \( \square \)

Using the relations
\[
(\xi I_n - T')^{-1}(\eta I_n - T') = (\xi - \eta) \left\{ \frac{1}{\xi - \eta} I_n - (\xi I_n - T')^{-1} \right\},
\]
\[
(\xi I_n - T')(\eta I_n - T')^{-1} = (\xi - \eta) \left\{ \frac{1}{\xi - \eta} I_n + (\eta I_n - T')^{-1} \right\},
\]
we can write the system (2.1) in a Pfaffian system of the form
\[
dV = \left\{ \begin{pmatrix} (\xi I_n - T')^{-1} \\ O \end{pmatrix} A d\xi + \left( A - (\rho_1 + \rho_2)I_{2n} \right) \left( (\eta I_n - T')^{-1} \\ O \right) d\eta - A \frac{d(\xi - \eta)}{\xi - \eta} \right\} V,
\]
where
\[
V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad A = \begin{pmatrix} A \\ -(A - \rho_1 I_n)(A - \rho_2 I_n) \quad (\rho_1 + \rho_2)I_n - A \end{pmatrix}.
\]
For a fixed constant \( \eta_0 \) the \( 2n \)-vector function \( V(\xi) = V(\xi; \eta_0) \) satisfies
\[
\frac{dV}{d\xi} = \left( \begin{pmatrix} (\xi I_n - T')^{-1} \\ O \end{pmatrix} - \frac{1}{\xi - \eta_0} I_{2n} \right) AV.
\]

**Proposition 2.3.** Assume that \( \eta_0 \) satisfies \( \det(T' - \eta_0 I_n) \neq 0 \). The change of variables
\[
\xi = \eta_0 + \frac{1}{x - t} \quad \text{and} \quad V = \begin{pmatrix} I_n \\ P \end{pmatrix} U,
\]
where \( t \) is a constant, transforms the system (2.3) into the system of ONF
\[
\begin{pmatrix} xI_{2n} - \begin{pmatrix} T \\ tI_n \end{pmatrix} \end{pmatrix} \frac{dU}{dx} = AP U,
\]
where
\[
T = tI_n + (T' - \eta_0 I_n)^{-1}.
\]

**Proof.** The change of variable (2.3) leads to
\[
\begin{pmatrix} (\xi I_n - T')^{-1} \\ O \end{pmatrix} - \frac{1}{\xi - \eta_0} I_{2n} = -(x - t)^2 \begin{pmatrix} xI_{2n} - \begin{pmatrix} T \\ tI_n \end{pmatrix} \end{pmatrix}^{-1}.
\]

Substituting this formula into the right hand side of
\[
\frac{dV}{dx} = \frac{d\xi}{dx} \frac{dV}{d\xi} = \frac{1}{(x - t)^2} \left( \begin{pmatrix} (\xi I_n - T')^{-1} \\ O \end{pmatrix} - \frac{1}{\xi - \eta_0} I_{2n} \right) AV,
\]
and then substituting \( V = \begin{pmatrix} I_n \\ P \end{pmatrix} U \), we obtain the system (2.5). \( \square \)
2.2 Integral representation of solutions

We denote by $w(\rho, \zeta)$ a solution of the system of ONF

$$L_1(T', A; \rho)w = 0. \quad (2.6)$$

Recall the notation $j'$ for $j = 1, 2$:

$$j' = \begin{cases} 
2 & \text{for } j = 1, \\
1 & \text{for } j = 2.
\end{cases}$$

**Proposition 2.4.** For $j = 1, 2$, let $w(-\rho_j - 1; \zeta)$ be a solution of the system \[2.6\] with $\rho = -\rho_j - 1$. Then the integral

$$v(\xi, \eta) = (\xi - \eta)^{-\rho_j} \int_C (\xi - \zeta)^{\rho_j} (\eta - \zeta)^{-\rho_{j'} - 1} w(-\rho_j - 1; \zeta) \, d\zeta$$

is a solution of the system \[2.6\], provided that

$$\left[ (\xi - \zeta)^{\rho_j} (\eta - \zeta)^{-\rho_{j'} - 1} (\zeta I_n - T') w(-\rho_j - 1; \zeta) \right]_C = 0. \quad (2.7)$$

**Proof.** The change of variables $v = (\xi - \eta)^{-\rho_j} u$ leads to a system

\[
\begin{align*}
E(\rho_{j'} + 1, -\rho_j, 0)u &= 0, \\
L_2(T', A; -\rho_{j'} - 1)u &= 0.
\end{align*}
\]

It is trivial that the integral

$$u(\xi, \eta) = \int_C (\xi - \zeta)^{\rho_j} (\eta - \zeta)^{-\rho_{j'} - 1} w(-\rho_j - 1; \zeta) \, d\zeta$$

satisfies the first equation of this system. Similarly to \[1.16\], we have

$$\begin{align*}
&((\xi I_n - T')u_{\xi}(\xi, \eta) + (\eta I_n - T')u_{\eta}(\xi, \eta) + (-\rho_j + \rho_{j'} + 1)u(\xi, \eta) \\
&= -\int_C \frac{\partial}{\partial \zeta} \{(\xi - \zeta)^{\rho_j} (\eta - \zeta)^{-\rho_{j'} - 1}\} (\zeta I_n - T') w(-\rho_j - 1; \zeta) \, d\zeta \\
&= \int_C (\xi - \zeta)^{\rho_j} (\eta - \zeta)^{-\rho_{j'} - 1} \frac{d}{d\zeta} \{(\zeta I_n - T') w(-\rho_j - 1; \zeta)\} \, d\zeta \\
&\quad - \left[ (\xi - \zeta)^{\rho_j} (\eta - \zeta)^{-\rho_{j'} - 1} (\zeta I_n - T') w(-\rho_j - 1; \zeta) \right]_C \\
&= (A - \rho_j I_n)u(\xi, \eta),
\end{align*}$$

which leads to $L_2(T', A; -\rho_{j'} - 1)u(\xi, \eta) = 0$. Here in the last equality we have used \[2.7\] and $\frac{d}{d\zeta} \{(\zeta I_n - T') w(-\rho_j - 1; \zeta)\} = (A - \rho_j I_n)w(-\rho_j - 1; \zeta)$.

Combining Propositions 2.4 and 2.2 we see that for $j = 1, 2$, the integral

$$V(\xi, \eta) = \left( \frac{\eta I_n - T'}{M(T', A)} \right) \int_C \left( \frac{\xi - \zeta}{\eta - \zeta} \right)^{\rho_j} (\eta - \zeta)^{-\rho_{j'} - 1} w(-\rho_j - 1; \zeta) \, d\zeta$$

becomes a solution of the Pfaffian system \[2.2\] under the condition \[2.7\]. Substituting $\eta = \eta_0$ and then $\xi = \eta_0 + \frac{1}{x - t}$, we obtain the following proposition.
Proposition 2.5. For \( j = 1, 2 \), let \( w(-\rho_j - 1; \zeta) \) be a solution of the system (2.6) with \( \rho = -\rho_j - 1 \). Then the integral
\[
V(\xi) = V(\xi, \eta_0) \equiv \left( \eta_0 I_n - T' \right) M_{\eta_0}(T', A) \int_C \left( \frac{\xi - \zeta}{\zeta - \eta_0} \right) (\eta_0 - \zeta)^{-\rho_j - 1} w(-\rho_j - 1; \zeta) d\zeta,
\]
(2.8)
where \( M_{\eta_0}(T', A) \) denotes the operator \( M(T', A) \) with \( \eta \) replaced by \( \eta_0 \), becomes a solution of the system (2.3). Moreover, the integral
\[
U(x) = \left( I_n \atop P^{-1} \right) V(\eta_0 + \frac{1}{x - t}, \eta_0) = \left( \eta_0 I_n - T' \right) M_{\eta_0}(T', A) \int_C \left\{ 1 + (x - t)(\eta_0 - \zeta) \right\} (\eta_0 - \zeta)^{-\rho_j - 1} w(-\rho_j - 1; \zeta) d\zeta,
\]
(2.9)
where
\[ M_{\xi, \eta_0}(T', A) = \left\{ I_n + (x - t)(\eta_0 I_n - T') \right\} \frac{\partial}{\partial x} - A(\eta_0 I_n - T'), \]
becomes a solution of the system of ONF (2.5).

Even if the integrals (2.8) and (2.9) are divergent, they make sense in the sense of the finite part of a divergent integral (see [7. 2.3.3]). For example, in the case that \( C \) is a segment from \( \eta_0 \) to \( \xi \) in (2.8), the integral (2.8) is divergent if \( \Re \rho_j \leq -1 \) or \( \Re \rho_j' \geq 0 \); however, the integral makes sense in the sense of the finite part if \( \rho_j \notin \mathbb{Z}_{<0} \) and \( \rho_j' \notin \mathbb{Z}_{\geq 0} \). We do not assume the conditions for convergence of the integrals stated below, and treat them as the finite part of a divergent integral when they are divergent.

3 Solutions characterized by local behavior

We reverse our way of investigation. Namely, we start with the system (1.4) and then determine the system (2.6).

3.1 Assumptions

Consider the system (1.4). We write \( T \) in the form
\[
T = \begin{pmatrix}
t_1 I_{n_1} & \cdots \\
\vdots & \ddots \\
t_p I_{n_p}
\end{pmatrix}
\]
and set
\[ t = t_{p+1}, \]
where \( t_i \neq t_j \) \( (1 \leq i \neq j \leq p + 1) \) and \( n_1 + n_2 + \cdots + n_p = n \). We assume that
\[ \text{No three } t_i \text{ } (1 \leq i \leq p + 1) \text{ lie on a straight line.} \]

Renumbering the \( t_i \)'s if necessary, we fix an assignment of the arguments \( \theta_i = \arg(t_i - t_{p+1}) \) \( (1 \leq i \leq p) \) such that
\[ \theta_1 < \theta_2 < \cdots < \theta_p < \theta_1 + 2\pi. \]

Besides, we fix a real number \( \theta_{p+1} \) such that
\[ \theta_p < \theta_{p+1} < \theta_1 + 2\pi. \]
When we write
\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1p} \\
A_{21} & A_{22} & \cdots & A_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
A_{p1} & A_{p2} & \cdots & A_{pp}
\end{pmatrix}
\]
in the same partition as \(T\), namely, \(A_{ij} \in M(n_i, n_j; \mathbb{C})\), we may assume that the diagonal blocks \(A_{ii}\) (1 \(\leq i \leq p\)) are of Jordan canonical form, since a transformation of the form \(U = \begin{pmatrix} Q & 
\end{pmatrix}\), where \(Q = \begin{pmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_p
\end{pmatrix}, \ Q_i \in GL(n_i; \mathbb{C})
\]
changes the system (1.4) into the same system with \(A_{ij}\) and \(P\) replaced by \(Q_i^{-1}A_{ij}Q_j\) and \(Q^{-1}P\), respectively. We assume that \(A_{11}, A_{22}, \ldots, A_{pp}\) are diagonal, and set for 1 \(\leq i \leq p\)
\[
A_{ii} = \begin{pmatrix}
\lambda_{i,1}I_{\ell_{i,1}} \\
\lambda_{i,2}I_{\ell_{i,2}} \\
\vdots \\
\lambda_{i,r}I_{\ell_{i,r}}
\end{pmatrix},
\]
where \(\lambda_{i,k} \neq \lambda_{i,h} (k \neq h)\) and \(\ell_{i,1} + \ell_{i,2} + \cdots + \ell_{i,r} = n_i\). We write \(A' = P^{-1}AP\) in the form
\[
A' = \begin{pmatrix}
\mu_1I_{m_1} \\
\mu_2I_{m_2} \\
\vdots \\
\mu_qI_{m_q}
\end{pmatrix},
\]
where \(\mu_k \neq \mu_h (k \neq h)\) and \(m_1 + m_2 + \cdots + m_q = n\). Throughout this paper we assume that
\[
\begin{align*}
\lambda_{i,k}, \lambda_{i,k} - \lambda_{i,h} & \not\in \mathbb{Z} \quad \text{for } 1 \leq i \leq p, 1 \leq k \neq h \leq r_i, \\
\mu_k - \mu_h & \not\in \mathbb{Z} \quad \text{for } 1 \leq k \neq h \leq q, \\
\rho_j, \rho_1 - \rho_2 & \not\in \mathbb{Z} \quad \text{for } 1 \leq j \leq 2, \\
\rho_j - \lambda_{i,k} & \not\in \mathbb{Z} \quad \text{for } 1 \leq j \leq 2, 1 \leq i \leq p, 1 \leq k \leq r_i.
\end{align*}
\]

3.2 Nonholomorphic solutions near singular points: generic case

First, we treat the case that none of the \(\rho_j\)'s is an eigenvalue of the matrix \(A\). In this case we assume that
\[
\rho_j - \mu_k \not\in \mathbb{Z} \quad \text{for } 1 \leq j \leq 2, 1 \leq k \leq q
\]
in addition to (3.1)–(3.4). The system (1.4) is a Fuchsian system with singularities \(x = t_1, \ldots, t_{p+1}\) and \(\infty\). Note that
\[
A_P \sim \begin{pmatrix}
\rho_1I_n \\
\rho_2I_n
\end{pmatrix}.
\]
The Riemann scheme of the system (1.4) is
\[
\begin{aligned}
  x &= \ell_1 \quad \cdots \quad x = \ell_p & x &= \ell_{p+1} & x &= \infty \\
  0(n_1) & \quad \cdots & 0(n_p) & 0(n) & - \rho_1(n) \\
  \lambda_1(\ell_1,1) & \quad \cdots & \lambda_p(\ell_p,1) & \rho_1 + \rho_2 - \mu_1(m_1) & - \rho_2(n) \\
  & \vdots & & \vdots & \vdots \\
  \lambda_1(\ell_1,r_1) & \quad \cdots & \lambda_p(\ell_p,r_p) & \rho_1 + \rho_2 - \mu_q(m_q) \\
\end{aligned}
\]

where \( \lambda(\ell) \) denotes an exponent \( \lambda \) with its multiplicity \( \ell \), and \( \bar{n}_i = 2n - n_i \) (1 \( \leq i \leq p \)).

Applying the general theory of local solutions near a regular singular point (e.g. [7, Chapter 1]), we have the following theorems. We use the notation \( \varepsilon_m(k) = \text{the k-th unit m-vector} \).

**Theorem 3.1.** For \( 1 \leq i \leq p, 1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k} \) there exists a unique solution of the system (1.4) of the form
\[
U_{i,k,h}(x) = (x - t_i)^{\lambda_i,h} \sum_{m=0}^{\infty} G_{i,k,h}(m)(x - t_i)^m
\]
with
\[
G_{i,k,h}(0) = \varepsilon_{2n}(n_1 + \cdots + n_{i-1} + \ell_{i,1} + \cdots + \ell_{i,k-1} + h)
\]
which is convergent for \( |x - t_i| < \min_{1 \leq k \leq p+1, k \neq i} |t_k - t_i| \). Besides, for \( 1 \leq k \leq q, 1 \leq h \leq m_k \) there exists a unique solution of the system (1.4) of the form
\[
U_{p+1,k,h}(x) = (x - t_{p+1})^{\rho_1 + \rho_2 - \mu_k} \sum_{m=0}^{\infty} G_{p+1,k,h}(m)(x - t_{p+1})^m
\]
with
\[
G_{p+1,k,h}(0) = \varepsilon_{2n}(n + m_1 + \cdots + m_{k-1} + h)
\]
which is convergent for \( |x - t_{p+1}| < \min_{1 \leq k \leq p} |t_k - t_{p+1}| \).

**Theorem 3.2.** For \( 1 \leq j \leq 2, 1 \leq h \leq n \) there exists a unique solution of the system (1.4) of the form
\[
U_{\infty,j,h}(x) = \left( \frac{1}{x - t_{p+1}} \right)^{-\rho_j} \sum_{m=0}^{\infty} G_{\infty,j,h}(m) \left( \frac{1}{x - t_{p+1}} \right)^m
\]
with
\[
G_{\infty,j,h}(0) = (A_P - \rho_j I_{2n})\varepsilon_{2n}(n + h)
\]
\[
= \left( \rho_j I_n - A^T \right)\varepsilon_n(h)
\]
which is convergent for \( |x - t_{p+1}| > \max_{1 \leq k \leq p} |t_k - t_{p+1}| \).

### 3.3 Reducible case (i)

Next, we treat the case that one of the \( \rho_j \)'s is an eigenvalue of the matrix \( A \). Put
\[
\rho_2 = \mu_q
\]
and assume that
\[
\rho_1 - \mu_k \notin \mathbb{Z} \quad \text{for} \quad 1 \leq k \leq q \quad \text{(3.6)}
\]
in addition to (3.1)–(3.4). In this case the coefficient $A_P$ has the form

$$A_P = \left( \begin{array}{c|c} A'_P & * \\ \hline 0 & \rho_1 I_{m_q} \end{array} \right), \quad A'_P \in \mathbb{M}(2n - m_q; \mathbb{C}),$$

and the system has a solution of the form $U = \begin{pmatrix} U' \\ 0_{m_q} \end{pmatrix}$. Here $U'$ satisfies the system of ONF of rank $2n - m_q$

$$(xI_{2n-m_q} - T') \frac{dU'}{dx} = A'_P U', \quad T' = \begin{pmatrix} T \\ tI_{n-m_q} \end{pmatrix}.$$

Note that

$$\bar{\lambda} = 2 \leq n - q.$$
with
\[ G'_{\infty, \mu_q, h}(0) = \begin{cases} (A_P - \rho_1 I_{2n-m_q}) \varepsilon_{2n-m_q} (n+h) & \text{for } 1 \leq h \leq n-m_q, \\ (P \varepsilon_n(h)) (0_{n-m_q}) & \text{for } n-m_q+1 \leq h \leq n. \end{cases} \]

The series are convergent for \(|x-t_{p+1}| > \max_{1 \leq k \leq P} |t_k - t_{p+1}|.

### 3.4 Reducible case (ii)

Lastly, we treat the case that both of the \(\rho_j\)'s are an eigenvalue of the matrix \(A\). Put \(\rho_1 = \mu_{q-1}\) and \(\rho_2 = \mu_q\) with the conditions (3.1–3.4). In this case the coefficient \(A_P\) has the form
\[ A_P = \begin{pmatrix} A''_{p} \\ O \end{pmatrix} \begin{pmatrix} \mu_q I_{q-1} \\ \mu_q I_{m_q} \end{pmatrix}, \quad A''_{p} \in \mathbb{M}(2n - m_{q-1} - m_q; \mathbb{C}), \]

and the system (3.1) has a solution of the form \(U = \begin{pmatrix} U'' \\ 0_{m_q-1+m_q} \end{pmatrix}\). Here \(U''\) satisfies the system of ONF of rank \(2n - m_{q-1} - m_q\)
\[ (xI_{2n-m_q-1-m_q} - T'') \frac{dU''}{dx} = A''_{p} U'', \quad T'' = \begin{pmatrix} T \\ tI_{n-m_q-1-m_q} \end{pmatrix}. \]  
(3.8)

Note that
\[ A''_{p} \sim \begin{pmatrix} \mu_{q-1} I_{n-m_q} \\ \mu_q I_{m_q} \end{pmatrix}. \]

The Riemann scheme of the system (3.8) is
\[
\begin{align*}
\begin{array}{cccc}
x = t_1 & \cdots & x = t_p & x = t_{p+1} \\
0 (\tilde{n}'_p) & \cdots & 0 (\tilde{n}'_p) & 0 (n) \quad - \mu_{q-1} (n-m_q) \\
\lambda_{1,1} (\ell_{1,1}) & \cdots & \lambda_{p,1} (\ell_{p,1}) & \mu_{q-1} + \mu_q - 1 (m_1) \quad - \mu_q (n-m_q-1) \\
\vdots & \vdots & \vdots & \vdots \\
\lambda_{1,r_1} (\ell_{1,r_1}) & \cdots & \lambda_{p,r_p} (\ell_{p,r_p}) & \mu_{q-1} + \mu_q - 2 (m_{p-2}) \\
\end{array}
\end{align*}
\]

where \(\tilde{n}'_i = 2n - m_{q-1} - m_q - n_i (1 \leq i \leq p)\).

**Theorem 3.5.** For \(1 \leq i \leq p, \ 1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i,k}\) there exists a unique solution of the system (3.8) of the form
\[ U_{t_{i,k},h}(x) = (x-t_i)^{\lambda_{i,k}} \sum_{m=0}^{\infty} G'_{t_{i,k},h}(m)(x-t_i)^m \]
with
\[ G'_{t_{i,k},h}(0) = \varepsilon_{2n-m_{q-1}-m_q} (n_1 + \cdots + n_{i-1} + \ell_{i,1} + \cdots + \ell_{i,k-1} + h) \]

which is convergent for \(|x-t_i| < \min_{1 \leq k \leq P, \ell_{i,k} \neq 1} |t_k - t_i|\). Besides, for \(1 \leq k \leq q-2, \ 1 \leq h \leq m_k\) there exists a unique solution of the system (3.8) of the form
\[ U_{t_{p+1,k},h}(x) = (x-t_{p+1})^{\mu_{q-1} + \mu_q - m_k} \sum_{m=0}^{\infty} G''_{t_{p+1,k},h}(m)(x-t_{p+1})^m \]
with
\[ G''_{t_{p+1,k},h}(0) = \varepsilon_{2n-m_{q-1}-m_q} (n + m_1 + \cdots + m_{k-1} + h) \]

which is convergent for \(|x-t_{p+1}| < \min_{1 \leq k \leq P} |t_k - t_{p+1}|\).
Theorem 3.6. For $1 \leq h \leq n - m_q$ there exists a unique solution of the system (3.5) of the form

$$U_{\infty, \mu_q-1, h}^n(x) = \left(\frac{1}{x - t_{p+1}}\right)^{-\mu_q-1} \sum_{m=0}^{\infty} G_{\infty, \mu_q-1, h}^n(m) \left(\frac{1}{x - t_{p+1}}\right)^m$$

with

$$G_{\infty, \mu_q-1, h}^n(0) = \left\{ \begin{array}{ll}
(A_p^\mu - \mu_q I_{2n-m_q-1-m_q}) \xi_{2n-m_q-1-m_q}(n+h) & \text{for } 1 \leq h \leq n - m_q - 1 - m_q, \\
(P \xi_n(h)) & \text{for } n - m_q - 1 - m_q + 1 \leq h \leq n - m_q.
\end{array} \right.$$

Besides, for $1 \leq h \leq n - m_q - 1 - m_q$ and $n - m_q + 1 \leq h \leq n$ there exists a unique solution of the system (3.5) of the form

$$U_{\infty, \mu_q, h}^n(x) = \left(\frac{1}{x - t_{p+1}}\right)^{-\mu_q} \sum_{m=0}^{\infty} G_{\infty, \mu_q, h}^n(m) \left(\frac{1}{x - t_{p+1}}\right)^m$$

with

$$G_{\infty, \mu_q, h}^n(0) = \left\{ \begin{array}{ll}
(A_p^\mu - \mu_q-1 I_{2n-m_q-1-m_q}) \xi_{2n-m_q-1-m_q}(n+h) & \text{for } 1 \leq h \leq n - m_q - 1 - m_q, \\
(P \xi_n(h)) & \text{for } n - m_q + 1 \leq h \leq n.
\end{array} \right.$$

The series are convergent for $|x - t_{p+1}| > \max_{1 \leq k \leq p} |t_k - t_{p+1}|$.

4 Solutions of the underlying system

Let us fix a complex number $\eta_0$. We define $t'_i$ $(1 \leq i \leq p)$ by

$$t'_i = \eta_0 + \frac{1}{t_i - t_{p+1}},$$

and define

$$T' = \begin{pmatrix}
t'_1 I_{n_1} \\
t'_2 I_{n_2} \\
\vdots \\
t'_p I_{n_p}
\end{pmatrix}.$$ (4.1)

Using this $T'$ and the matrix $A$ in the system (1.4), we set up the system of ONF

$$(\zeta I_n - T') \frac{dw}{d\zeta} = (\rho I_n + A)w,$$ (4.2)

which we call the underlying system associated with the system (1.4).

We define the assignment of the arguments $\theta'_i = \arg(t'_i - \eta_0)$ $(1 \leq i \leq p)$ by

$$\theta'_i = -\arg(t_i - t_{p+1}) = -\theta_i.$$

Moreover, we set

$$\theta'_\infty = -\theta_{p+1}.$$

Note that the $\theta'_i$'s satisfy

$$\theta'_\infty < \theta'_p < \theta'_{p-1} < \cdots < \theta'_2 < \theta'_1 < \theta'_{p+1} + 2\pi.$$

In addition to the assumptions (3.1)–(3.4) we assume that

$$\mu_j, \lambda_{i,k} - \mu_j \notin \mathbb{Z} \quad \text{for } 1 \leq j \leq q, \ 1 \leq i \leq p, \ 1 \leq k \leq r_i.$$ (4.3)

Moreover, we assume that the parameter $\rho$ satisfies

$$\rho + \lambda_{i,k} \notin \mathbb{Z} \quad \text{for } 1 \leq i \leq p, \ 1 \leq k \leq r_i.$$ (4.4)
4.1 Nonholomorphic solutions near singular points

The Riemann scheme of the system (4.2) is

\[
\begin{align*}
\zeta &= t'_1 \\
0 (n-n_1) &= 0 (n-n_p) \\
\rho + \lambda_{1,1} (\ell_{1,1}) &= \rho + \lambda_{p,1} (\ell_{p,1}) \\
\vdots &= \vdots \\
\rho + \lambda_{1,r_1} (\ell_{1,r_1}) &= \rho + \lambda_{p,r_p} (\ell_{p,r_p})
\end{align*}
\]

Theorem 4.1. For \(1 \leq i \leq p, 1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k}\), there exists a unique solution of the system (4.2) of the form

\[
w_{t'_i,k,h}(\rho; \zeta) = (\zeta - t'_i)^{\rho + \lambda_{i,k}} \sum_{n=0}^{\infty} \frac{\Gamma(\rho + \lambda_{i,k} + 1)}{\Gamma(\rho + \lambda_{i,k} + 1 + m)} g_{t'_i,k,h}(m)(\zeta - t'_i)^m
\]

with

\[
g_{t'_i,k,h}(0) = \varepsilon_n (n_1 + \cdots + n_{i-1} + \ell_{i,1} + \cdots + \ell_{i,k} + 1),
\]

which is convergent for \(|\zeta - t'_i| < R'_{\ell_{i,k}}|, R'_{t'_i}\) denoting \(\min_{1 \leq k \leq p, k \neq i} |t'_k - t'_i|\).

Theorem 4.2. For \(1 \leq k \leq q, 1 \leq h \leq \ell_{k,h}\), there exists a unique solution of the system (4.2) of the form

\[
w_{\infty,k,h}(\rho; \zeta) = \left(\frac{1}{\zeta - \eta_0}\right)^{-\rho - \mu_k} \sum_{m=0}^{\infty} \frac{\Gamma(-\rho - \mu_k + m)}{\Gamma(-\rho - \mu_k)} g_{\infty,k,h}(m) \left(\frac{1}{\zeta - \eta_0}\right)^m
\]

with

\[
g_{\infty,k,h}(0) = P\varepsilon_n (m_1 + \cdots + m_{k-1} + 1),
\]

which is convergent for \(|\zeta - \eta_0| > R'_\ell|, R'_\eta\) denoting \(\max_{1 \leq k \leq p} |t'_k - \eta_0|\).

Remark 4.1. The coefficients \(g_{t'_i,k,h}(m)\) and \(g_{\infty,k,h}(m)\) for \(m = 0, 1, 2, \ldots\) do not depend on \(\rho\).

Set

\[
P' = \mathbb{C} \setminus \bigcup_{i=1}^{p} \{\eta_0 + (t'_i - \eta_0)s | 1 \leq s < \infty\}.
\]

In \(P'\) we specify the branch of \(w_{t'_i,k,h}(\rho; \zeta)\) by the assignment of argument

\[
\arg(\zeta - t'_i) \in (\theta'_i - 2\pi, \theta'_i)
\]

for \(1 \leq i \leq p\).

Theorem 4.3. Assume that \(\rho' - \rho \notin \mathbb{Z}_{\geq 0}\) in addition to (4.4) for \(\rho\) and \(\rho'\). For \(\zeta \in P'\) we have

\[
w_{t'_i,k,h}(\rho; \zeta) = \frac{\Gamma(\rho + \lambda_{i,k} + 1)}{\Gamma(\rho - \rho') \Gamma(\rho' + \lambda_{i,k} + 1)} \int_{t'_i}^{\zeta} (\zeta - \tau)^{\rho - \rho' - 1} w_{t'_i,k,h}(\rho'; \tau) d\tau,
\]

where the path of integration is a segment or a curve in \(P'\) with initial point \(t'_i\) and terminal point \(\zeta\), and the branch of the integrand is determined by the following assignment of the arguments

\[
\arg(\zeta - \tau) = \arg(\tau - t'_i) = \arg(\zeta - t'_i)
\]

for \(\zeta\) sufficiently close to \(t'_i\).
Proof. It suffices to prove (4.8) for $\zeta$ near $t'_i$. Set

$$u(\zeta) = \int_{t'_i}^{\zeta} (\zeta - \tau)^{\rho - \rho' - 1} w_{t'_i, k, h}(\rho'; \tau) \, d\tau. \quad (4.9)$$

By the way similar to (1.16) we can easily see that $u(\zeta)$ becomes a solution of (4.2). Substituting the expansion (4.4) with $\rho$ replaced by $\rho'$, we have

$$u(\zeta) = \sum_{m=0}^{\infty} \frac{\Gamma(\rho' + \lambda_{i,k} + 1)}{\Gamma(\rho' + \lambda_{i,k} + 1 + m)} g_{t'_i, k, h}(m) \int_{t'_i}^{\zeta} (\zeta - \tau)^{\rho - \rho' - 1} (\tau - t'_i)^{\rho' + \lambda_{i,k} + m} \, d\tau. \quad \text{Changing the variable of integration } \tau \text{ to } s \text{ by } \tau = t'_i + (\zeta - t'_i)s, \text{ we obtain}$$

$$\int_{t'_i}^{\zeta} (\zeta - \tau)^{\rho - \rho' - 1} (\tau - t'_i)^{\rho' + \lambda_{i,k} + m} \, d\tau = (\zeta - t'_i)^{\rho + \lambda_{i,k} + m} \int_{0}^{1} (1 - s)^{\rho - \rho' - 1} s^{\rho' + \lambda_{i,k} + m} \, ds$$

$$= (\zeta - t'_i)^{\rho + \lambda_{i,k} + m} \frac{\Gamma(\rho - \rho')\Gamma(\rho' + \lambda_{i,k} + m + 1)}{\Gamma(\rho + \lambda_{i,k} + m + 1)}$$

and hence

$$u(\zeta) = \frac{\Gamma(\rho - \rho')\Gamma(\rho' + \lambda_{i,k} + 1)}{\Gamma(\rho + \lambda_{i,k} + 1)} \sum_{m=0}^{\infty} \frac{\Gamma(\rho + \lambda_{i,k} + 1)}{\Gamma(\rho + \lambda_{i,k} + 1 + m)} g_{t'_i, k, h}(m)(\zeta - t'_i)^{\rho + \lambda_{i,k} + m}. \quad \text{This implies (4.5).} \quad \Box$$

4.2 Holomorphic solutions in the plane cut from one singular point

Set

$$\mathcal{P}'_i = \mathbb{C} \setminus \{\eta_0 + (t'_i - \eta_0)s \mid 1 \leq s < \infty\}$$

for $1 \leq i \leq p$.

**Theorem 4.4.** For $1 \leq i \leq p$, $1 \leq k \leq r_i$, $1 \leq h \leq \ell_{i,k}$ there exists a unique solution $\tilde{w}_{t'_i, k, h}(\rho; \zeta)$ of the system (4.3) such that

- $\tilde{w}_{t'_i, k, h}(\rho; \zeta)$ is holomorphic in $\mathcal{P}'_i$,
- $\tilde{w}_{t'_i, k, h}(\rho; \zeta) = w_{t'_i, k, h}(\rho; \zeta) + \text{hol}(\zeta - t'_i)$ near $\zeta = t'_i$.

**Theorem 4.5.** Assume that $\rho + \mu_k \notin \mathbb{Z}_{\geq 0}$ for $1 \leq k \leq q$. Under the specification (4.4) we have

$$\tilde{w}_{t'_i, k, h}(\rho; \zeta) = -\frac{e^{-\pi \sqrt{-1} \Gamma(\rho' + \lambda_{i,k})}}{\Gamma(-\rho - \lambda_{i,k})\Gamma(\rho' + \lambda_{i,k} + 1)} \int_{t'_i}^{\infty} (\zeta - \tau)^{\rho - \rho' - 1} \tilde{w}_{t'_i, k, h}(\rho'; \tau) \, d\tau \quad (4.10)$$

for $\zeta \in \mathcal{P}'_i$, where the path of integration is the ray from $t'_i$ to $\infty$ along the right-hand side of the cut $\arg(\zeta - t'_i) = \theta'_i$, and the branch of the integrand is determined by the following assignment of the arguments

$$\left\{ \begin{array}{l}
\arg(\tau - t'_i) = \theta'_i, \\
\arg(\zeta - \tau) \in [\arg(\zeta - t'_i), \theta'_i - \pi]^*,
\end{array} \right. \quad [\alpha, \beta]^* \text{ denoting the interval } [\min(\alpha, \beta), \max(\alpha, \beta)].$$
Proof. Set
\[ \hat{u}(\zeta) = \int_{t'_i}^{\infty} (\zeta - \tau)^{\rho - \rho' - 1} w_{t'_i, k, h} (\rho' ; \tau) d\tau. \]

Similarly to the integral (4.9) in the proof of Theorem 4.3 we see that \( \hat{u}(\zeta) \) becomes a solution of (4.2). It is trivial that \( \hat{u}(\zeta) \) is holomorphic in \( P'_i \). In order to find the behavior of \( \hat{u}(\zeta) \) near \( \zeta = t'_i \) we shall analytically continue it along a circle of center \( t'_i \) with sufficiently small radius in the positive direction. The analytic continuation is found by deforming the path of integration as \( \zeta \) travels along the circle. Figure 4.1 illustrates the deformation of the path of integration. From the last picture of Figure 4.1 we see that the analytic continuation of \( \hat{u}(\zeta) \) leads to

\[ \hat{u}(\zeta) + \left( e^{2\pi\sqrt{-1}(\rho - \rho')} - 1 \right) e^{2\pi\sqrt{-1}(\rho' + \lambda_{i,k})} u(\zeta), \]

where \( u(\zeta) \) denotes the integral (4.9) in the proof of Theorem 4.3.

On the other hand, when we analytically continue \( \hat{w}_{t'_i, k, h}(\rho; \zeta) \) along the circle, we obtain

\[ \hat{w}_{t'_i, k, h}(\rho; \zeta) + \left( e^{2\pi\sqrt{-1}(\rho + \lambda_{i,k})} - 1 \right) w_{t'_i, k, h}(\rho; \zeta). \]

So the solution defined by

\[ \hat{w}_{t'_i, k, h}(\rho; \zeta) - \frac{e^{2\pi\sqrt{-1}(\rho + \lambda_{i,k})} - 1}{(e^{2\pi\sqrt{-1}(\rho - \rho')} - 1)e^{2\pi\sqrt{-1}(\rho' + \lambda_{i,k})} \Gamma(\rho - \rho') \Gamma(\rho' + \lambda_{i,k} + 1)} \hat{u}(\zeta) \]

is not only holomorphic in \( P'_i \) but also single-valued near \( \zeta = t'_i \), and then must be 0 under the assumption (4.4). Lastly, using the formula of \( \Gamma \)-function \( \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z} \), we obtain (4.10). \( \square \)
4.3 Connection coefficients for the underlying system

In this subsection we explain the dependence on $\rho$ of connection coefficients for the system \([4.2]\). The following results are due to R. Schäfke. See his papers \([12]\) and \([13]\) in detail.

First, we consider the connection coefficients between solutions near finite singular points. In $\mathcal{P}'$, under the specification \([4.7]\), let us express the following results are due to R. Schäfke. See his papers \([12]\) and \([13]\) in detail.

\[ w_{\nu',k,h}(\rho;\zeta) = \sum_{i=1}^{r_{\nu,k}} \sum_{k=1}^{r_{\nu,k}} c_{\nu',k,h;\nu',k,h}(\rho) w_{\nu',k,h}(\rho;\zeta) + \text{hol}(\zeta - t_{\nu}'). \]  

\[ (4.11) \]

**Theorem 4.6** \([13] \text{ (3.6) Satz}\). We have

\[ c_{\nu',k,h;\nu',k,h}(\rho) = \frac{e^{\pi \sqrt{-\imath} \rho} \Gamma(\rho + \lambda_{i,k} + 1) \Gamma(-\rho - \lambda_{\nu,k})}{\Gamma(\lambda_{i,k} + 1) \Gamma(-\lambda_{\nu,k})} c_{\nu',k,h;\nu',k,h} \]  

\[ (4.12) \]

for $1 \leq i \neq \nu \leq q$, $1 \leq k \leq r_{i,k}$, $1 \leq h \leq \ell_{i,k}$, $1 \leq \tilde{k} \leq r_{\nu,k}$, $1 \leq \tilde{h} \leq \ell_{\nu,k}$, where $c_{\nu',k,h;\nu',k,h} = c_{\nu',k,h;\nu',k,h}(0)$.

**Proof.** Let $L_{\nu'}^{+}$ (resp. $L_{\nu'}^{-}$) be a curve in $\mathcal{P}'$ starting from $t_{\nu}'$ and going to $\infty$ along the left-hand (resp. right-hand) side of the cut $\arg(\zeta - t_{\nu}') = \theta_{\nu}'$ and $H_{\nu}$ the Hankel loop in $\mathcal{P}'$ surrounding the cut $\arg(\zeta - t_{\nu}') = \theta_{\nu}'$ with sufficiently small radius (see Figure 4.2 (a)). For $|\arg z + \theta_{\nu}'| < \pi/2$ we define

\[ y_{\nu',k,h}(L_{\nu'}^{+};z) = \frac{z^{\rho+1}}{\Gamma(\rho + \lambda_{i,k} + 1)} \int_{L_{\nu'}^{+}} e^{-\zeta z} w_{\nu',k,h}(\rho;\zeta) d\zeta, \]

\[ y_{\nu',k,h}(H_{\nu};z) = \frac{z^{\rho+1}}{(1 - e^{-2\pi \sqrt{-\imath} (\rho + \lambda_{i,k})}) \Gamma(\rho + \lambda_{v,k} + 1)} \int_{H_{\nu}} e^{-\zeta z} w_{\nu',k,h}(\rho;\zeta) d\zeta. \]

It is well known \([2, 6, 11]\) that these satisfy the system of Poincaré rank one

\[ z \frac{dy}{dz} = -(A + zT') y, \]  

\[ (4.13) \]

which does not depend on $\rho$. Moreover, $y_{\nu',k,h}(L_{\nu'}^{+};z)$ and $y_{\nu',k,h}(H_{\nu};z)$ themselves do not depend on $\rho$. Indeed, substituting \([4.8]\) and then reversing the order of integration, we have

\[ y_{\nu',k,h}(L_{\nu'}^{+};z) = \frac{z^{\rho+1}}{\Gamma(\rho - \rho') \Gamma(\rho' + \lambda_{i,k} + 1)} \int_{L_{\nu'}^{+}} \left\{ \int_{L_{\nu'}^{+}(\tau)} e^{-\zeta z} (\zeta - \tau)^{\rho' - 1} d\zeta \right\} w_{\nu',k,h}(\rho';\tau) d\tau, \]

\[ y_{\nu',k,h}(H_{\nu};z) = \frac{z^{\rho+1}}{\Gamma(\rho - \rho') \Gamma(\rho' + \lambda_{i,k} + 1)} \int_{H_{\nu}} \left\{ \int_{H_{\nu}(\tau)} e^{-\zeta z} (\zeta - \tau)^{\rho' - 1} d\zeta \right\} w_{\nu',k,h}(\rho';\tau) d\tau, \]

\[ (4.14) \]
we have
\[
y_{t',k,h}(L_{i,v}^+(\tau);z) = \frac{z^{\rho+1}}{\Gamma(\rho + \lambda_{i,k} + 1)} \int_{L_{i,v}^+} e^{-z^\tau} w_{t',k,h}(\rho';\tau) d\tau,
\]
we have
\[
y_{t',k,h}(L_{i,v}^+(\tau);z) = \frac{z^{\rho+1}}{\Gamma(\rho + \lambda_{i,k} + 1)} \int_{L_{i,v}^+} e^{-z^\tau} w_{t',k,h}(\rho';\tau) d\tau,
\]
which means that \( y_{t',k,h}(L_{i,v}^+(\tau);z) \) does not depend on \( \rho \). As for \( y_{t',k,h}(H_{\nu};z) \), using the representation
\[
y_{t',k,h}(H_{\nu};z) = \frac{z^{\rho+1}}{\Gamma(\rho + \lambda_{i,k} + 1)} \int_{H_{\nu}} e^{-z^\tau} w_{t',k,h}(\rho';\tau) d\tau,
\]
we can prove its independence from \( \rho \) similarly to \( y_{t',k,h}(L_{i,v}^+(\tau);z) \).

Next, we consider the difference of \( y_{t',k,h}(L_{i,v}^+(\tau);z) \) and \( y_{t',k,h}(L_{i,v}^-(\tau);z) \). Substituting (4.11) into
\[
y_{t',k,h}(L_{i,v}^+(\tau);z) - y_{t',k,h}(L_{i,v}^-(\tau);z) = \frac{z^{\rho+1}}{\Gamma(\rho + \lambda_{i,k} + 1)} \int_{H_{\nu}} e^{-z^\tau} w_{t',k,h}(\rho';\tau) d\tau,
\]
we have
\[
y_{t',k,h}(L_{i,v}^+(\tau);z) - y_{t',k,h}(L_{i,v}^-(\tau);z) = \sum_{k=1}^{\rho} \sum_{h=1}^{\ell} \left( 1 - e^{-2\pi \sqrt{-1}(\rho + \lambda_{i,k})} \right) \Gamma(\rho + \lambda_{i,k} + 1) \frac{c_{t',k,h}(H_{\nu};z)}{\Gamma(\rho + \lambda_{i,k} + 1)} \bar{w}_{t',k,h}(\rho';\tau) d\tau.
\]

Since the \( y_{t',k,h}(H_{\nu};z) \)'s are linearly independent, the coefficients must be uniquely determined and hence not depend on \( \rho \). The equality of the coefficients and those for \( \rho = 0 \) leads to (4.12).

Next, we consider the connection coefficients between solutions near a finite singular point and ones near infinity. Set
\[
\mathcal{P} = \mathbb{C} \setminus \left( \bigcup_{i=1}^{p} \{ \eta_0 + (t'_i - \eta_0)s \mid 0 \leq s \leq 1 \} \bigcup \{ \eta_0 + se^{\sqrt{-1}t'_0} \mid 0 \leq s < \infty \} \right).
\]
For \( \zeta \in \mathcal{P} \), we determine the assignment of argument of \( \zeta - \eta_0 \) and \( \zeta - t'_i \) (1 \( \leq i \leq p \)) as
\[
\begin{align*}
\arg(\zeta - \eta_0) &\in (\theta'_0, \theta'_0 + 2\pi), \\
\arg(\zeta - t'_i) &\in (\theta'_i - \pi, \theta'_i + \pi) \text{ near } \zeta = t'_i.
\end{align*}
\]
In \( \mathcal{P} \) let us express \( w_{t',k,h}(\rho;\zeta) \) by a linear combination of solutions near \( \zeta = \infty \) as
\[
w_{t',k,h}(\rho;\zeta) = \sum_{k=1}^{m} \sum_{h=1}^{\ell} c_{\infty,k,h}(t'_i,k,h)(\rho) w_{\infty,k,h}(\rho;\zeta).
\]

**Theorem 4.7.** We have
\[
c_{\infty,k,h}(t'_i,k,h)(-\mu_i - 1) = 0
\]
for \( 1 \leq i \leq q, \ 1 \leq h \leq m_i, \ 1 \leq i \leq p, \ 1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i,k}. \)

To prove this theorem we prepare a lemma. We denote by \( [ \ ]_m \) the \( m \)-th row of a matrix or the \( m \)-th component of a column vector.
Lemma 4.8 (cf. [4] Proposition 3.1). Suppose that all of the residue matrices $A_i$ ($1 \leq i \leq p$) of a rank $n$ Fuchsian system

$$\frac{du}{d\zeta} = \left( \sum_{i=1}^{p} \frac{1}{\zeta - t_i} A_i \right) u$$

satisfy

$$[A_i]_m = 0 \quad \text{for} \quad n' + 1 \leq m \leq n' + n'' ,$$

where $n' + n'' \leq n$. Then for $n' + 1 \leq m \leq n' + n''$ the $m$-th component of any solution of this system is a constant. Moreover, if the analytic continuation of a solution $u(\zeta)$ along a closed curve encircling a singular point coincides with itself multiplied by a constant different from 1, then the solution $u(\zeta)$ satisfies

$$[u(\zeta)]_m = 0 \quad \text{for} \quad n' + 1 \leq m \leq n' + n'' .$$

Proof. Trivial.

Proof of Theorem 4.9. By direct calculation we see that $u(\zeta) = P^{-1}(\zeta I_n - T')w_{t',k,h}(-\mu_l - 1; \zeta)$ satisfies the system

$$\frac{du}{d\zeta} = \left( \sum_{i=1}^{p} \frac{1}{\zeta - t'_i} (A' - \mu_l I_n) B_i \right) u,$$

where the matrices $B_i$ are defined by $P^{-1}(\zeta I_n - T')^{-1} P = \sum_{i=1}^{p} \frac{1}{\zeta - t'_i} B_i$. Hence by Lemma 4.8 we have

$$\left[ P^{-1}(\zeta I_n - T')w_{t'_i,k,h}(-\mu_l - 1; \zeta) \right]_m = 0$$

for $m_1 + \cdots + m_{l-1} + 1 \leq m \leq m_1 + \cdots + m_l$. Similarly, for $k \neq l$ we have

$$\left[ P^{-1}(\zeta I_n - T')w_{\infty,k,h}(-\mu_l - 1; \zeta) \right]_m = 0$$

for $m_1 + \cdots + m_{l-1} + 1 \leq m \leq m_1 + \cdots + m_l$. On the other hand, we have

$$\left[ P^{-1}(\zeta I_n - T')w_{\infty,\tilde{t},h}(-\mu_l - 1; \zeta) \right]_m = 1$$

for $m = m_1 + \cdots + m_{l-1} + \tilde{h}$. These facts and the $(m_1 + \cdots + m_{l-1} + \tilde{h})$-th component of the relation (4.14) with $\rho = -\mu_l - 1$ multiplied by $P^{-1}(\zeta I_n - T')$ lead to (4.15).

Theorem 4.9. Assume that $\rho + \mu_k \notin \mathbb{Z}_{\geq 0}$ for $1 \leq k \leq q$. We have

$$c_{\infty,\tilde{h},t_i',k,h}(\rho) = \frac{\Gamma(\rho + \lambda_i,k + 1) \Gamma(\mu_k + 1)}{\Gamma(\rho + \mu_k + 1) \Gamma(\lambda_i,k + 1)} c_{\infty,\tilde{h},t_i',k,h}$$

(4.16)

for $1 \leq \tilde{h} \leq q$, $1 \leq \tilde{h} \leq m_k$, $1 \leq i \leq p$, $1 \leq k \leq r_i$, $1 \leq \lambda_i,k$, where $c_{\infty,\tilde{h},t_i',k,h} = c_{\infty,\tilde{h},t_i',k,h}(0)$.

Proof. Let $M_i^{-}$ (resp. $M_i^{+}$) be a curve in $\bar{P}'$ starting from $t'_i$ and going to $\infty$ along the left-hand (resp. right-hand) side of the cut arg$(\zeta - \eta_0) = \theta'_{\infty}$, and $H_{\infty}$ the Hankel loop in $\bar{P}'$ surrounding the cut arg$(\zeta - \eta_0) = \theta'_{\infty}$ with sufficiently large radius (see Figure 4.2 (b)). For $| \arg z + \theta'_{\infty} | < \pi/2$ we define

$$y_{t'_i,k,h}(M_i^{-}; z) = \frac{z^{\rho+1}}{\Gamma(\rho + \lambda_i,k + 1)} \int_{M_i^{-}} e^{-z\zeta} w_{t'_i,k,h}(\rho; \zeta) d\zeta,$$

$$y_{t'_i,k,h}(M_i^{+}; z) = \frac{z^{\rho+1}}{\Gamma(\rho + \lambda_i,k + 1)} \int_{M_i^{+}} e^{-z\zeta} w_{t'_i,k,h}(\rho; \zeta) d\zeta,$$

$$y_{\infty,k,h}(H_{\infty}; z) = \frac{z^{\rho+1}}{2\pi \sqrt{-1} e^{\pi i \rho + \pi \rho + 1}} \int_{H_{\infty}} e^{-z\zeta} w_{\infty,k,h}(\rho; \zeta) d\zeta.$$
These are not only solutions of the system \((4.13)\) but also independent from \(\rho\). The independence from \(\rho\) of \(y_{t',k,h}(M_i^+;z)\) can be proved by the same way as that of \(y_{t',k,h}(L_{1,0};z)\) thanks to

\[
\int_{M_i^+(\tau)} e^{-z\zeta} (\zeta - \tau)^{\rho - \rho'} - 1 \, d\zeta = \Gamma(\rho - \rho') z^\rho - \rho e^{-z\tau},
\]

\[
\int_{M_i^+(\tau)} e^{-z\zeta} (\zeta - \tau)^{\rho - \rho'} - 1 \, d\zeta = e^{2\pi\sqrt{-1}(\rho - \rho')} \Gamma(\rho - \rho') z^\rho - \rho e^{-z\tau},
\]

where \(M_i^+(\tau)\) denotes the subpath of \(M_i^+\) from \(\tau\) to \(\infty\). As for \(y_{\infty,k,h}(H_{\infty};z)\), substituting \((4.5)\) and then integrating term by term, we have

\[
y_{\infty,k,h}(H_{\infty};z) = z^{\rho + 1} \sum_{m=0}^{\infty} \frac{\Gamma(-\rho - \mu_k + m)}{2\pi \sqrt{-1} e^{\pi \sqrt{-1}(\rho + \mu_k + 1)}} g_{\infty,k,h}(m) \int_{H_{\infty}} e^{-z\zeta} (\zeta - \eta_0)^{\rho + \mu_k - m} \, d\zeta.
\]

Since

\[
\int_{H_{\infty}} e^{-z\zeta} (\zeta - \eta_0)^{\rho + \mu_k - m} \, d\zeta = \left(2\pi \sqrt{-1} e^{\pi \sqrt{-1}(\rho + \mu_k + 1)} \right) \frac{\Gamma(\rho + \mu_k - m + 1)}{\Gamma(-\rho - \mu_k + m)} e^{-\eta_0 z} z^{-\rho - \mu_k + m - 1},
\]

we obtain

\[
y_{\infty,k,h}(H_{\infty};z) = e^{-\eta_0 z} z^{-\mu_k} \sum_{m=0}^{\infty} (-1)^m g_{\infty,k,h}(m) z^m,
\]

which does not depend on \(\rho\).

Now let us consider the difference of \(e^{2\pi\sqrt{-1}\rho y_{t',k,h}(M_i^+;z)} \) and \(y_{t',k,h}(M_i^-;z)\). Substituting \((4.14)\) into

\[
e^{2\pi\sqrt{-1}\rho y_{t',k,h}(M_i^+;z)} - y_{t',k,h}(M_i^-;z) = \frac{z^{\rho + 1}}{\Gamma(\rho + \lambda_{i,k} + 1)} \int_{H_{\infty}} e^{-z\zeta} w_{t',k,h}(\rho; \zeta) \, d\zeta,
\]

we have

\[
e^{2\pi\sqrt{-1}\rho y_{t',k,h}(M_i^+;z)} - y_{t',k,h}(M_i^-;z) = \sum_{k=1}^{q} \sum_{l=1}^{m_k} \frac{2\pi \sqrt{-1} e^{\pi \sqrt{-1}(\rho + \mu_{k,l} + 1)}}{\Gamma(-\rho - \mu_l + \lambda_{l,k} + 1)} c_{\infty,k,h,t',k,h}(\rho) y_{\infty,k,h}(H_{\infty};z).
\]

For any \(l, 1 \leq l \leq q\), replacing \(\rho\) by \(-\mu_l - 1\) in this formula and subtracting the resulting formula from this formula, we find

\[
\left(e^{2\pi\sqrt{-1}\rho} - e^{-2\pi\sqrt{-1}\mu_l}\right) y_{t',k,h}(M_i^+;z) = \sum_{l=1}^{m_k} \frac{2\pi \sqrt{-1} e^{\pi \sqrt{-1}(\rho + \mu_{k,l} + 1)}}{\Gamma(-\rho - \mu_l + \lambda_{l,k} + 1)} c_{\infty,l,h,t',k,h}(\rho) y_{\infty,l,h}(H_{\infty};z) + \text{a linear combination of } y_{\infty,k,h}(H_{\infty};z) \text{ for } \tilde{k} \neq l
\]

by virtue of \((4.15)\) in Theorem \((4.4)\). Since the \(y_{\infty,k,h}(H_{\infty};z)\)'s are linearly independent, the quantity

\[
\frac{2\pi \sqrt{-1} e^{\pi \sqrt{-1}(\rho + \mu_{k,l} + 1)}}{\left(e^{2\pi\sqrt{-1}\rho} - e^{-2\pi\sqrt{-1}\mu_l}\right) \Gamma(-\rho - \mu_l + \lambda_{l,k} + 1)} c_{\infty,l,h,t',k,h}(\rho) = \frac{e^{2\pi\sqrt{-1}\mu_l} \Gamma(\rho + \mu_l + 1)}{\Gamma(\rho + \lambda_{l,k} + 1)} c_{\infty,l,h,t',k,h}(\rho)
\]

must be uniquely determined and hence not depend on \(\rho\), which leads to \((4.16)\).
5 Integrals associated with the solutions of the underlying system

For $1 \leq i \leq p$ we set

$$
\phi_i^- = \max(\theta_i'_{i+1} + \delta, \theta_i' - \pi),
$$

$$
\phi_i^+ = \min(\theta_i'_{i-1} - \delta, \theta_i' + \pi),
$$

where $\theta_0' = \theta_0' + 2\pi$, $\theta_p' + 1 = \theta_p'$, and $\delta$ is a sufficiently small positive number, and set

$$
S_i^{-} = \{ \zeta \mid \arg(\zeta - \eta_0) \in (\phi_i^-, \theta_i') \},
$$

$$
S_i^{+} = \{ \zeta \mid \arg(\zeta - \eta_0) \in (\theta_i', \phi_i^+) \}.
$$

For $\xi \in S_i^{\pm}$ we consider the integrals

$$
W_{\text{sing}, k, h}^{(S_i^{\pm}; \nu_1, \nu_2; \xi)} = \left( \frac{\eta_0 I_{n - T'} + T'}{P^{-1} M_{\nu_0} (T', A)} \right) \int_{a}^{b} \frac{(\zeta - \eta_0)^{-\nu_1} - \nu_2^{-1}}{(\zeta - \eta_0)^{1 - \nu_1}} w_{\text{sing}, k, h}(\zeta)^{\nu_1} d\zeta, \tag{5.1}
$$

where $\nu_1, \nu_2$ are complex parameters, $a, b \in \{ \xi, \eta_0, t_i', \infty \}$, $a \neq b$, $\text{sing} = t_i'$ or $\infty$. The path of integration for $W_{\text{sing}, k, h}^{(S_i^{\pm}; \nu_1, \nu_2; \xi)}$ is the segment or the ray $\overline{ab}$ from $a$ to $b$ indicated in Figure 5.1. Among the twelve possible choice of combinations of $\text{sing}$ and $\overline{ab}$ we investigate the following seven cases:

$$
W_{\xi, k, h}^{(S_i^{\pm}; \eta_0; \nu_1, \nu_2; \xi)} = W_{\xi, k, h}^{(S_i^{\pm}; t_i' + \nu_1, \nu_2; \xi)} = W_{\xi, k, h}^{(S_i^{\pm}; t_i' + \nu_1, \nu_2; \xi)} = W_{\xi, k, h}^{(S_i^{\pm}; t_i' + \nu_1, \nu_2; \xi)}.
$$

For $\xi \in S_i^{\pm}$ we determine the assignment of argument of $\xi - t_i'$ as

$$
\arg(\xi - t_i') \in (\theta_i' - \pi, \theta_i') \quad \text{for} \, \xi \in S_i^{-},
$$

$$
\arg(\xi - t_i') \in (\theta_i' - 2\pi, \theta_i' - \pi) \quad \text{for} \, \xi \in S_i^{+}. \tag{5.2}
$$

The branches of $W_{\text{sing}, k, h}^{(S_i^{\pm}; \nu_1, \nu_2; \xi)}$ along $\overline{ab}$ are determined by the assignment of $\arg(\xi - \zeta), \arg(\zeta - \eta_0)$ and $\arg(\zeta - t_i')$ tabulated in Table 5.1 or 5.2. Similarly to the integral (2.8) in Proposition 2.5 the integrals (5.1) satisfy the system

$$
\frac{dW}{d\xi} = \left( \left( (\xi I_n - T')^{-1} O \right) - \frac{1}{\xi - \eta_0} I_{2n} \right) \left( -A'(\nu_1 I_n) (A' - \nu_2 I_n) P^{-1} \right) \left( \nu_1 + \nu_2 \right) I_n - A' \right) W. \tag{5.3}
$$
Table 5.1: Assignment of branches in case $\xi \in S_i^-$

| $\eta_0 \xi$ | $\theta_i' - \pi, \chi_i$ |
| $t_i' \xi$ | $\theta_i' - \pi$ |
| $\eta_0 t_i'$ | $\theta_i' - \pi$ |
| $\infty \xi$ | $\phi_i - \pi$ |

$\psi = \arg(\xi - \eta_0) \in (\phi_i - \pi, \theta_i')$,  
$\chi_i = \arg(\xi - t_i') \in (\theta_i' - \pi, \theta_i')$.

Table 5.2: Assignment of branches in case $\xi \in S_i^+$

| $\eta_0 \xi$ | $\chi_i - \theta_i' - \pi$ |
| $t_i' \xi$ | $\chi_i - \theta_i' - \pi$ |
| $\eta_0 t_i'$ | $\chi_i - \theta_i' - \pi$ |
| $\infty \xi$ | $\phi_i + \pi$ |

$\psi = \arg(\xi - \eta_0) \in (\theta_i', \phi_i^+)$,  
$\chi_i = \arg(\xi - t_i') \in (\theta_i' - 2\pi, \theta_i' - \pi)$.
5.1 Integrals associated with the solutions near finite singular points

We use the notation 
\[ \Sigma'_{i} = \{ \eta_0 + (t_i' - \eta_0)s \mid s \in J \} \]
for \( J = [0, 1], [1, \infty], \) and so on.

**Theorem 5.1.** Assume that \( \nu_1 \notin \mathbb{Z}_{<0} \) and \( \nu_2 \notin \mathbb{Z}_{\geq 0} \). The analytic continuation of the integral \( W_{\nu',k,h}^{(S_{-}^{i} ; \eta_0 \xi)}(\nu_1, \nu_2; \xi) \) across the open segment \( \Sigma'_{i}^{(0, 1)} \) into \( S_i^+ \) coincides with the integral \( W_{\nu',k,h}^{(S_{+}^{i} ; \eta_0 \xi)}(\nu_1, \nu_2; \xi) \). Moreover, as \( \xi \rightarrow \eta_0, \xi \in S_i^{-} \cup \Sigma_i^{(0, 1)} \cup S_i^+, \) we have
\[
W_{\nu',k,h}^{(S_{+}^{i} ; \eta_0 \xi)}(\nu_1, \nu_2; \xi)
= (\xi - \eta_0)^{-\nu_2} \left\{ \frac{\Gamma(\nu_1 + 1)\Gamma(-\nu_2)}{\Gamma(-\nu_2 + 1)} \left( \frac{P}{\nu_2 I_0 - A'} \right) P^{-1}(\eta_0 I_0 - T') w_{\nu',k,h}(-\nu_1 - 1; 0) + O(\xi - \eta_0) \right\}.
\]

**Proof.** The equality of the analytic continuation of \( W_{\nu',k,h}^{(S_{-}^{i} ; \eta_0 \xi)} \) and the integral \( W_{\nu',k,h}^{(S_{+}^{i} ; \eta_0 \xi)} \) follows from the continuity of the arguments \( \arg(\xi - \zeta), \arg(\zeta - \eta_0) \) and \( \arg(\zeta - \eta_0) \) in \( S_i^+ \cup \Sigma_i^{(0, 1)} \cup S_i^+ \), where we define \( \arg(\xi - \zeta) = \theta'_i, \arg(\zeta - \eta_0) = \theta'_i \) and \( \arg(\zeta - \eta_0) = \theta'_i - \pi \) for \( \xi \in \Sigma_i^{(0, 1)} \).

Note that \( w_{\nu',k,h}(-\nu_1 - 1; \xi) \) is holomorphic for \( \xi \in \mathbb{C} \setminus \bigcup_{i=1}^{\infty} \Sigma_i^{(1, \infty)} \), where we specify \( \theta'_i - 2\pi < \arg(\xi - \eta_0) < \theta'_i \). Provided that \( |\zeta - \eta_0| < \min_{1 \leq k \leq p} |t_k' - \eta_0| \), we expand \( w_{\nu',k,h}(-\nu_1 - 1; \xi) \) in powers of \( \zeta - \eta_0 \):
\[
w_{\nu',k,h}(-\nu_1 - 1; \xi) = \sum_{m=0}^{\infty} g_{\nu',k,h}^m(-\nu_1 - 1; m)(\zeta - \eta_0)^m.
\]

Substituting this expansion into \( W_{\nu',k,h}^{(S_{+}^{i} ; \eta_0 \xi)}(\nu_1, \nu_2; \xi) \), we obtain
\[
W_{\nu',k,h}^{(S_{+}^{i} ; \eta_0 \xi)}(\nu_1, \nu_2; \xi) = \left( \frac{\eta_0 I_0 - T'}{P^{-1}M_{\eta_0}(T', A)} \right) \sum_{m=0}^{\infty} \frac{\Gamma(\nu_1 + 1)\Gamma(m - \nu_2)}{\Gamma(m - \nu_2 + 1)} g_{\nu',k,h}^m(-\nu_1 - 1; m) \int_{\eta_0}^{\xi} \left( \frac{\xi - \zeta}{\zeta - \eta_0} \right)^{\nu_1} (\zeta - \eta_0)^{m-\nu_2-1} d\zeta.
\]

Changing the variable of integration \( \zeta \) to \( s \) by \( \zeta = \eta_0 + (\xi - \eta_0)s \), we obtain
\[
\int_{\eta_0}^{\xi} \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} (\zeta - \eta_0)^{m-\nu_2-1} d\zeta = (\xi - \eta_0)^{m-\nu_2} \int_{0}^{1} (1 - s)^{\nu_1}s^{m-\nu_2-1} ds = \frac{\Gamma(\nu_1 + 1)\Gamma(m - \nu_2)}{\Gamma(m + \nu_1 - \nu_2 + 1)} (\xi - \eta_0)^{m-\nu_2}.
\]

Then we have
\[
W_{\nu',k,h}^{(S_{+}^{i} ; \eta_0 \xi)}(\nu_1, \nu_2; \xi)
= \left( \frac{\eta_0 I_0 - T'}{P^{-1}M_{\eta_0}(T', A)} \right) \sum_{m=0}^{\infty} \frac{\Gamma(\nu_1 + 1)\Gamma(m - \nu_2)}{\Gamma(m + \nu_1 - \nu_2 + 1)} g_{\nu',k,h}^m(-\nu_1 - 1; m)(\xi - \eta_0)^{m-\nu_2}
= (\xi - \eta_0)^{-\nu_2} \left\{ \frac{\Gamma(\nu_1 + 1)\Gamma(-\nu_2)}{\Gamma(-\nu_2 + 1)} \left( \frac{\eta_0 I_0 - T'}{P^{-1}(\nu_1 - \nu_2 + 1)} \right) g_{\nu',k,h}^m(-\nu_1 - 1; 0) + O(\xi - \eta_0) \right\},
\]
which leads to (5.4). □
**Theorem 5.2.** Assume that \( \nu_1 \not\in \mathbb{Z}_{<0} \) and \( \nu_1 - \lambda_{i,k} \not\in \mathbb{Z}_{\geq 0} \). The analytic continuation of the integral \( W^{(S_i^+)_{(0) \times (0)}}_{t_i,k}(\nu_1,\nu_2;\xi) \) across the open segment \( \Sigma_i^{(0,1)} \) into \( S_i^+ \) coincides with the integral \( W^{(S_i^+)_{(0) \times (0)}}_{t_i,k}(\nu_1,\nu_2;\xi) \).

Moreover, as \( \xi \to t'_i, \xi \in S_i^+ \cup \Sigma_i^{(0,1)} \cup S_i^+ \), we have

\[
W^{(S_i^+)_{(0) \times (0)}}_{t_i,k}(\nu_1,\nu_2;\xi) = -(t'_i - \eta_0)^{-\nu_1 - \nu_2} \Gamma(\nu_1 + 1)\Gamma(\lambda_{i,k} - \nu_1) \frac{\Gamma(\lambda_{i,k} + 1)}{\Gamma(\lambda_{i,k} + 1)} 
\times (\xi - t'_i)^{\lambda_{i,k}} \{\varepsilon_{2n}(n_1 + \cdots + n_{i-1} + \ell_{i,1} + \cdots + \ell_{i,k-1} + h) + O(\xi - t'_i)\}. \tag{5.5}
\]

To prove this theorem we prepare a lemma.

**Lemma 5.3 ([4] Proposition 2.5).** The integral

\[
v(\xi) = \int_a^b \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} (\zeta - \eta_0)^{-\nu_2 - 1} w(-\nu_1 - 1; \zeta) d\zeta,
\]

where \( w(-\nu_1 - 1; \zeta) \) is a solution of \( [4.2] \), satisfies

\[
M_{\nu_0}(T',A)v(\xi) = (\nu_2 I_n - A) \int_a^b \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} (\zeta - \eta_0)^{-\nu_2 - 1} (\zeta I_n - T') w(-\nu_1 - 1; \zeta) d\zeta, \tag{5.6}
\]

provided that

\[
\left[ (\xi - \zeta)^{\nu_1} (\zeta - \eta_0)^{-\nu_2} (\zeta I_n - T') w(-\nu_1 - 1; \zeta) \right]_a^b = 0. \tag{5.7}
\]

**Proof.** Since

\[
\frac{d}{d\xi} v(\xi) = \int_a^b \frac{\nu_1}{(\xi - \eta_0)^2} \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1 - 1} (\zeta - \eta_0)^{-\nu_2 - 1} w(-\nu_1 - 1; \zeta) d\zeta,
\]

we have

\[
- (\xi - \eta_0)(\xi I_n - T') \frac{d}{d\xi} v(\xi)
\]

\[
= - \int_a^b \nu_1 \left\{ (\xi - \zeta) I_n + (\zeta I_n - T') \right\} \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1 - 1} (\zeta - \eta_0)^{-\nu_2} w(-\nu_1 - 1; \zeta) d\zeta
\]

\[
= - \int_a^b \nu_1 \left[ (\xi - \zeta)^{\nu_1} (\zeta I_n - T') \frac{d}{d\zeta} \left( \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} \right) \right] (\zeta - \eta_0)^{-\nu_2 - 1} w(-\nu_1 - 1; \zeta) d\zeta
\]

\[
= - \int_a^b \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} \left[ (\xi - \zeta)^{\nu_1} (\zeta I_n - T') w(-\nu_1 - 1; \zeta) + \frac{d}{d\zeta} \left( (\zeta - \eta_0)^{-\nu_2} w(-\nu_1 - 1; \zeta) \right) \right] d\zeta
\]

\[
+ (\xi - \eta_0)^{-\nu_1} \left[ (\xi - \zeta)^{\nu_1} (\zeta - \eta_0)^{-\nu_2} w(-\nu_1 - 1; \zeta) \right]_a^b
\]

\[
= A(\eta_0 I_n - T') v(\xi) + (\nu_2 I_n - A) \int_a^b \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} (\zeta - \eta_0)^{-\nu_2 - 1} (\zeta I_n - T') w(-\nu_1 - 1; \zeta) d\zeta,
\]

which leads to (5.6). Here we have used (5.7) and

\[
\frac{d}{d\xi} \left\{ (\zeta - \eta_0)^{-\nu_2} (\zeta I_n - T') w(-\nu_1 - 1; \zeta) \right\}
\]

\[
= -\nu_2 (\zeta - \eta_0)^{-\nu_2 - 1} (\zeta I_n - T') w(-\nu_1 - 1; \zeta) + (\zeta - \eta_0)^{-\nu_2} (A - \nu_1 I_n) w(-\nu_1 - 1; \zeta)
\]

\[
= - (\zeta - \eta_0)^{-\nu_2 - 1} \left\{ A(\eta_0 I_n - T') + (\nu_2 I_n - A)(\zeta I_n - T') \right\} w(-\nu_1 - 1; \zeta) - \nu_1 (\zeta - \eta_0)^{-\nu_2} w(-\nu_1 - 1; \zeta)
\]

in the last equality. ∎
Proof of Theorem 5.3. The equality of the analytic continuation of $W_{t_i', k, h}^{(S_i^+; t_i')}(\xi, \nu_1, \nu_2; \xi)$ and the integral $W_{t_i', k, h}^{(S_i^+; t_i')}(\xi, \nu_1, \nu_2; \xi)$ follows from the continuity of the arguments $\arg(\xi - \zeta)$, $\arg(\zeta - \eta_0)$ and $\arg(\zeta - t_i')$ in $S_i^{(1)} \cup \Sigma_i^{(0,1)} \cup S_i^{(1)}$, where we define $\arg(\xi - \zeta) = \theta_i' - \pi$, $\arg(\zeta - \eta_0) = \theta_i' + \pi$ and $\arg(\zeta - t_i') = \theta_i' - \pi$ for $\xi \in \Sigma_i^{(0,1)}$.

Applying Lemma 5.3 we have

$$W_{t_i', k, h}^{(S_i^+; t_i')}(\nu_1, \nu_2; \xi) = \sum_{m=0}^{\infty} F_{t_i', k, h}(m) \int_{t_i'}^{\xi} \frac{\xi - \zeta}{\xi - \eta_0}^{\nu_1} (\zeta - \eta_0)^{-\nu_2-1} (P^{-1}(\nu_2 I_n - A)(\zeta I_n - T')) w_{t_i', k, h}(-\nu_1 - 1; \xi) d\zeta.$$  

Provided that $|\xi - t_i'| < \min_{k \neq i} |t_k' - t_i'|$, substituting (4.5) with $\rho = -\nu_1 - 1$ and

$$\left( P^{-1}(\nu_2 I_n - A)(\zeta I_n - T') \right) = \left( P^{-1}(\nu_2 I_n - A)(t_i' I_n - T') \right) + (\zeta - t_i') \left( P^{-1}(\nu_2 I_n - A) \right),$$

we obtain

$$W_{t_i', k, h}^{(S_i^+; t_i')}(\nu_1, \nu_2; \xi) = \sum_{m=0}^{\infty} F_{t_i', k, h}(m) \int_{t_i'}^{\xi} \frac{\xi - \zeta}{\xi - \eta_0}^{\nu_1} (\zeta - \eta_0)^{-\nu_2-1} (\zeta - t_i')^{m+\lambda_i,k - \nu_1 - 1} d\zeta,$$

where the initial coefficient is given by

$$F_{t_i', k, h}(0) = \left( P^{-1}(\nu_2 I_n - A)(t_i' I_n - T') \right) g_{t_i', k, h}(0) = -(t_i' - \eta_0) \varepsilon_2 n_i + \cdots + \ell_i + \cdots + \ell_i - 1 + h).$$

Changing the variable of integration $\zeta$ to $s$ by $\zeta = t_i' + (\xi - t_i') (s)$, we obtain

$$\int_{t_i'}^{\xi} \frac{\xi - \zeta}{\xi - \eta_0}^{\nu_1} (\zeta - \eta_0)^{-\nu_2-1} (\zeta - t_i')^{m+\lambda_i,k - \nu_1 - 1} d\zeta$$

$$= (\xi - \eta_0)^{-\nu_1} (\xi - t_i')^{m+\lambda_i,k} (t_i' - \eta_0)^{-\nu_2-1} \int_{0}^{1} (1-s)^{\nu_1} \left( 1 - \frac{\xi - t_i'}{\eta_0 - t_i'} s \right)^{-\nu_2-1} s^{m+\lambda_i,k - \nu_1 - 1} d s,$$

where $\arg(t_i' - \eta_0) = \theta_i'$ and hence arg $(1 - \frac{\xi - t_i'}{\eta_0 - t_i'} s) \in [0, \psi - \theta_i'] \subset (-\pi, \pi)$. So the integral on the right-hand side is expressed by the Gauss hypergeometric function as

$$\frac{\Gamma(\nu_1 + 1) \Gamma(m + \lambda_i,k - \nu_1)}{\Gamma(m + \lambda_i,k + 1)} \frac{1}{2} F_1 \left( m + \lambda_i,k - \nu_1, \nu_2 + 1, \frac{\xi - t_i'}{\eta_0 - t_i'} \right).$$

Then we have

$$W_{t_i', k, h}^{(S_i^+; t_i')}(\nu_1, \nu_2; \xi) = \Gamma(\nu_1 + 1)(\xi - \eta_0)^{-\nu_1} (t_i' - \eta_0)^{-\nu_2-1} (\xi - t_i')^{\lambda_i,k} \times \sum_{m=0}^{\infty} \frac{\Gamma(m + \lambda_i,k - \nu_1)}{\Gamma(m + \lambda_i,k + 1)} \frac{1}{2} F_1 \left( m + \lambda_i,k - \nu_1, \nu_2 + 1, \frac{\xi - t_i'}{\eta_0 - t_i'} \right) (\xi - t_i')^m.$$

Combining this with $(\xi - \eta_0)^{-\nu_1} = (t_i' - \eta_0)^{-\nu_1} (1 - \frac{\xi - t_i'}{\eta_0 - t_i'} s)^{-\nu_1}$, we have [53].

We next consider the integrals $W_{t_i', k, h}^{(S_i^+; t_i')}(\nu_1, \nu_2; \xi)$, which make sense if $\xi$ is in $C \setminus \Sigma_i^{(0,1)}$.

**Theorem 5.4.** Assume that $\nu_2 \notin \mathbb{Z}_{\geq 0}$ and $\nu_1 - \lambda_i,k \notin \mathbb{Z}_{\geq 0}$. The integrals $W_{t_i', k, h}^{(S_i^+; t_i')}(\nu_1, \nu_2; \xi)$ are analytically continued to a 2n-vector function that is holomorphic and single-valued in $C \setminus \Sigma_i^{(0,1)}$. Moreover, the integral
Proof. It is trivial that \( W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)}(\nu_1, \nu_2; \xi) \) is holomorphic in \( \xi \in \mathbb{C} \setminus [0, 1] \). Since the factor
\[
\left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1}
\]
is single-valued in \( \xi \in \mathbb{C} \setminus [0, 1] \) for \( \zeta \in \Sigma_i^{[0, 1]} \), \( W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)}(\nu_1, \nu_2; \xi) \) is also single-valued in \( \xi \in \mathbb{C} \setminus [0, 1] \).

If we extend the assignment of the arguments \( \arg(\xi - t'_i) \) and \( \arg(\xi - \zeta) \) for \( \xi \in S_i^+ \) into \( S_i^- \cup (1, \infty) \cup S_i^+ \) continuously, then for \( \xi \in S_i^+ \) we have
\[
\begin{cases}
\arg(\xi - t'_i) \in (\theta'_1, \theta'_2 + \pi), \\
\arg(\xi - \zeta) \in [\arg(\xi - \eta_0), \arg(\xi - t'_i)] \subset (\theta'_1, \theta'_2 + \pi).
\end{cases}
\]

On the other hand, the assignment of \( \arg(\xi - \zeta) \) for \( W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)} \) prescribed in Table 5.2 satisfies
\[
\arg(\xi - \zeta) \in [\psi - 2\pi, \chi_1] \subset (\theta'_1 - 2\pi, \theta'_2 - \pi).
\]
This means that the analytic continuation of \( W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)} \) across the ray \( \Sigma_i^{(1, \infty)} \) into \( S_i^+ \) differs from \( W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)} \) in the multiplier factor \( e^{2\pi \sqrt{-1} \nu_1} \).

To find the behavior of \( W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)}(\nu_1, \nu_2; \xi) \) near \( \zeta = t'_i \) we analytically continue it along a circle of center \( t'_i \) with sufficiently small radius in the positive direction (see Figure 5.2 (a) for \( W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)} \) or (b) for \( W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)} \)). Carefully tracing how the arguments change along the path, we obtain
\[
W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)}(\nu_1, \nu_2; \xi) + e^{\pm} \left( 1 - e^{2\pi \sqrt{-1} \nu_1} \right) W_{t'_i, k, h}^{(S^+_{i}; \nu_0 t'_i)}(\nu_1, \nu_2; \xi),
\]

Figure 5.2: Deformation of the path \( \eta_0 t'_i \)
where $\epsilon^- = 1$ for $W^{(S_i^+; \nu_0, t_i')}_t$, while $\epsilon^+ = e^{2\pi \sqrt{T} \lambda_i, k - \mu_1}$ for $W^{(S_i^+; \nu_0, t_i')}_t$. This means that

$$W^{(S_i^+; \nu_0, t_i')}_t(\nu_1, \nu_2; \xi) = -\frac{e^{\pm\left(2\pi \sqrt{-1}\nu_1 - 1\right)}}{e^{2\pi \sqrt{-1}\lambda_i, k - 1}} W^{(S_i^+; \nu_0, t_i')}_t(\nu_1, \nu_2; \xi) + \text{hol}(\xi - t_i')$$

near $\xi = t_i'$, which leads to (5.8).

Now we shall investigate the integral $W^{(S_i^+; t_i', \infty)}(\nu_1, \nu_2; \xi)$. Define the integral

$$\tilde{W}^{(S_i^+; t_i', \infty)}(\nu_1, \nu_2; \xi) = \left(\frac{\eta_1 I_{\nu_0} - T'}{P^{-1}M_{\theta_0}(T', A)}\right)^{\nu_1} \int_{\eta_0}^{\xi} \left(\frac{\xi - \zeta}{\xi - \eta_0}\right)^{-\nu_2} (\zeta - \eta_0)^{-\nu_2 - 1} \tilde{w}_{t_i', k, h}(-\nu_1; -\xi) d\zeta,$$

where $\tilde{w}_{t_i', k, h}(-\nu_1; -\xi)$ is the solution of the underlying system (4.2) with $\rho = -\nu_1 - 1$ stated in Theorem 4.4.

**Theorem 5.5.** Assume that $\nu_1 \notin \mathbb{Z}_{\geq 0}$, $\nu_1 - \lambda_i, k \notin \mathbb{Z}_{\geq 0}$, $\nu_2 \notin \mathbb{Z}_{< 0}$, and $\nu_2 - \mu_1 \notin \mathbb{Z}_{< 0}$ for $1 \leq l \leq q$. The analytic continuation of the integral $W^{(S_i^+; t_i', \infty)}(\nu_1, \nu_2; \xi)$ across the open segment $\Sigma_i^{(0,1)}$ into $S_i^{+}$ coincides with the integral $W^{(S_i^+; t_i', \infty)}(\nu_1, \nu_2; \xi)$. Moreover, the integral $W^{(S_i^+; t_i', \infty)}(\nu_1, \nu_2; \xi)$ multiplied by $(\xi - \eta_0)^{\nu_2}$ is holomorphic for $\xi \in \mathbb{P}_i$, where $\mathbb{P}_i = \mathbb{C} \setminus \Sigma_i^{[1, \infty]}$. Besides, we have

$$W^{(S_i^+; t_i', \infty)}(\nu_1, \nu_2; \xi) = -\frac{e^{\pi \sqrt{-1}(\lambda_i, k - \nu_1 - \nu_2)\Gamma(\nu_2 - \lambda_i, k) + 1)\Gamma(\lambda_i, k - \nu_1)\Gamma(\nu_2 + 1)\Gamma(-\nu_1)}{\tilde{W}^{(S_i^+; t_i', \infty)}(\nu_1, \nu_2; \xi)}.$$

**Proof.** The equality of the analytic continuation of $W^{(S_i^+; t_i', \infty)}$ and the integral $W^{(S_i^+; t_i', \infty)}$ follows from the continuity of the arguments $\arg(\xi - \zeta), \arg(\xi - \eta_0)$ and $\arg(\xi - t_i')$ in $S_i^+ \cup \Sigma_i^{(0,1)} \cup S_i^{+}$, where we define $\arg(\xi - \zeta) = \beta_i' - \pi, \arg(\xi - \eta_0) = \beta_i'$ and $\arg(\xi - t_i') = \beta_i'$ for $\xi \in \Sigma_i^{(0,1)}$.

It is trivial that $W^{(S_i^+; t_i', \infty)}(\nu_1, \nu_2; \xi)$ multiplied by $(\xi - \eta_0)^{\nu_2}$ is holomorphic in $\mathbb{C} \setminus \Sigma_i^{[1, \infty]}$.

Set

$$v(\xi) = \int_{\xi'}^{\xi} \left(\frac{\xi - \zeta}{\xi - \eta_0}\right)^{\nu_2} (\zeta - \eta_0)^{-\nu_2 - 1} \tilde{w}_{t_i', k, h}(-\nu_1; -\xi) d\zeta,$$

where $\arg(\zeta - \tau) \in [\chi, \beta_i' - \pi] \subset [\beta_i' - 2\pi, \beta_i']$. Setting $\zeta - \tau = e^{\pi \sqrt{-1}(\tau - \zeta)}$ and $\zeta = \eta_0 + (\xi - \eta_0)s$, we find

$$\int_{\eta_0}^{\xi} \left(\frac{\xi - \zeta}{\xi - \eta_0}\right)^{\nu_2} (\zeta - \eta_0)^{-\nu_2 - 1} (\zeta - \tau)^{\nu_1 - \nu_2 - 1} d\zeta$$

$$= (\xi - \eta_0)^{\nu_1} e^{-\pi \sqrt{-1}(\tau - \eta_0)} \left(1 - \frac{\xi - \eta_0}{\tau - \eta_0}\right)^{\nu_1 - \nu_2 - 1} d\zeta,$$
where \( \arg(\tau - \eta_0) = \theta'_i \) and \( \arg \left( 1 - \frac{\xi - \eta_0}{\tau - \eta_0} \right) \in (-\pi, \pi) \). So the integral on the right-hand side is expressed by the Gauss hypergeometric function as

\[
\frac{\Gamma(\nu_2 + 1)\Gamma(-\nu_1)}{\Gamma(\nu_2 - \nu_1 + 1)} {}_2F_1 \left( \frac{\nu_2 - \nu_1 + 1, -\nu_1}{\nu_2 - \nu_1 + 1} ; \frac{\xi - \eta_0}{\tau - \eta_0} \right),
\]

which is reduced to

\[
\frac{\Gamma(\nu_2 + 1)\Gamma(-\nu_1)}{\Gamma(\nu_2 - \nu_1 + 1)} \left( \frac{e^{\pi \sqrt{-1}(\xi - \tau)}}{\tau - \eta_0} \right)^{\nu_1},
\]

where \( \arg(\xi - \tau) \in [\chi_i, \theta'_i - \pi] \subset (\theta'_i - 2\pi, \theta'_i) \). Thus we have

\[
\int_{\eta_0}^{\xi} \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_2} (\eta_0 - \zeta)^{-\nu_1-1} (\xi - \zeta)^{-\nu_1} d\zeta = e^{\pi \sqrt{-1}(\nu_1+\nu_2-\lambda_i,k)} \frac{\Gamma(\nu_2 + 1)\Gamma(-\nu_1)}{\Gamma(\nu_2 - \nu_1 + 1)} \left( \frac{\xi - \tau}{\eta_0} \right)^{\nu_1} (\tau - \eta_0)^{-\nu_2-1}.
\]

Substituting this formula into (5.10), we have

\[
\vec{v}(\xi) = -e^{\pi \sqrt{-1}(\nu_1+\nu_2-\lambda_i,k)} \frac{\Gamma(\nu_2 + 1)\Gamma(-\nu_1)}{\Gamma(\nu_2 - \lambda_i,k + 1)\Gamma(\lambda_i,k - \nu_1)} \vec{v}(\xi).
\]

This implies (5.9). \( \Box \)

5.2 Integrals associated with the solutions near infinity

Similarly to Theorem 5.1, we can prove the following theorem.

**Theorem 5.6.** Assume that \( \nu_1 \notin \mathbb{Z}_{<0} \) and \( \nu_2 \notin \mathbb{Z}_{\geq 0} \). As \( \xi \to \eta_0, \xi \in S_i^\pm \), we have

\[
W_{\infty,k,h}^{(S_i^\pm; \eta_0)}(\nu_1, \nu_2; \xi) = \langle S_i^0 \rangle \left( \frac{\Gamma(\nu_1 + 1)\Gamma(-\nu_2)}{\Gamma(\nu_1 - \nu_2 + 1)} \left( \frac{P}{\nu_2 I_n - A'} \right) P^{-1}(\eta_0 I_n - T') w_{\infty,k,h} \right| \xi \rangle + O(\xi - \eta_0),
\]

where \( w_{\infty,k,h} \) denotes the analytic continuation of \( w_{\infty,k,h}(\nu_1 - 1; \xi) \) into a neighborhood of \( \zeta = \eta_0 \) through the sector \( S_i^\pm \) (the double-signs correspond).

**Proof.** Omitted. \( \Box \)

**Theorem 5.7.** Assume that \( \nu_1 \notin \mathbb{Z}_{<0} \) and \( \nu_2 - \mu_k \notin \mathbb{Z}_{<0} \). The analytic continuation of the integral \( W_{\infty,k,h}^{(S_i^\pm; \eta_0)}(\nu_1, \nu_2; \xi) \) across the open ray \( \Sigma_i^{(1,\infty)} \) into \( S_i^{k(+)} \) is equal to the integral \( W_{\infty,k,h}^{(S_i^{k(+)}; \eta_0)}(\nu_1, \nu_2; \xi) \) multiplied by \( e^{2\pi \sqrt{-1}\nu_1} \). Moreover, as \( \xi \to \infty, \xi \in S_i^0 \cup \Sigma_i^{(1,\infty)} \cup S_i^{k(+)} \), we have

\[
W_{\infty,k,h}^{(S_i^{k(+)}; \eta_0)}(\nu_1, \nu_2; \xi) = e^{\pi \sqrt{-1}\nu_1} \frac{\Gamma(\nu_1 + 1)\Gamma(\nu_2 - \mu_k + 1)}{\Gamma(\nu_1 + \nu_2 - \mu_k + 1)} \left( \xi - \eta_0 \right)^{\nu_2} \left( \frac{1}{\xi - \eta_0} \right)^{\nu_1} \left\{ \varepsilon_{2n}(n + m_1 + \cdots + m_{k-1} + h) + O \left( \frac{1}{\xi - \eta_0} \right) \right\},
\]

where the double-signs correspond.

**Proof.** If we extend the assignment of \( \arg(\xi - \zeta) \) for \( \xi \in S_i^0 \) into \( S_i^0 \cup \Sigma_i^{(1,\infty)} \cup S_i^{k(+)} \) continuously, then for \( \xi \in S_i^{k(+)} \) we have

\[
\arg(\xi - \zeta) = \psi + \pi.
\]
On the other hand, the assignment of \( \arg(\xi - \zeta) \) for \( W_{\infty,k,h}^{(S_i^1,\infty)} \) prescribed in Table 5.2 is

\[
\arg(\xi - \zeta) = \psi - \pi.
\]

This means that the analytic continuation of \( W_{\infty,k,h}^{(S_i^1,\infty)} \) across the ray \( \Sigma_i^{(1,\infty)} \) into \( S_i^\pm \) differs from \( W_{\infty,k,h}^{(S_i^1,\infty)} \) in the multiplier factor \( e^{2\pi \sqrt{-1} \nu_1} \).

Assume that \( |\xi - \eta_0| > \max_{1 \leq k \leq p} |t_k' - \eta_0| \). Substituting (4.6) with \( \rho = -\nu_1 - 1 \) into \( W_{\infty,k,h}^{(S_i^1,\infty)} \), we obtain

\[
W_{\infty,k,h}^{(S_i^1,\infty)}(\nu_1, \nu_2; \xi) = \left( \eta_0 I_n - T' \right) \sum_{m=0}^\infty \frac{\Gamma(\nu_1 - \mu_k + 1 + m)}{\Gamma(\nu_1 - \mu_k + m)} \int_0^\infty d\zeta \left( \frac{\xi - \zeta}{\zeta - \eta_0} \right)^{\nu_1} W_{\infty,k,h}(m) \int_0^\infty \left( \frac{\xi - \xi'}{\xi - \eta_0} \right)^{\nu_2 - \nu_1 + \mu_k - m - 2} d\zeta.
\]

Allowing for the assignment of the arguments of \( \xi - \zeta \), we have

\[
\int_0^\infty \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} W_{\infty,k,h}(m) \int_0^\infty \left( \frac{\xi - \xi'}{\xi - \eta_0} \right)^{\nu_2 - \nu_1 + \mu_k - m - 2} d\zeta = e^{\pm \pi \sqrt{-1} \nu_1} (\xi - \eta_0)^{-\nu_1} \int_0^\infty (\xi - \xi')^{-\nu_2} (\xi - \eta_0)^{-\nu_2 - \nu_1 + \mu_k - m - 2} d\zeta,
\]

where \( \arg(\xi - \zeta) = \arg(\xi - \eta_0) = \psi \) and the double-signs correspond. Changing the variable of integration \( \zeta \) to \( s \) by \( \zeta = \eta_0 + \frac{s - \eta_0}{\xi - \eta_0} \), we can easily see that the integral on the right-hand side is equal to

\[
-(\xi - \eta_0)^{-\nu_2 + \mu_k - m - 1} \frac{\Gamma(\nu_1 + 1)\Gamma(\nu_2 - \mu_k + m + 2)}{\Gamma(\nu_1 + \nu_2 - \mu_k + m + 2)}.
\]

Hence we have

\[
W_{\infty,k,h}^{(S_i^1,\infty)}(\nu_1, \nu_2; \xi) = -e^{\pm \pi \sqrt{-1} \nu_1} \Gamma(\nu_1 + 1) \times \left( \eta_0 I_n - T' \right) \sum_{m=0}^\infty \frac{\Gamma(\nu_1 - \mu_k + 1 + m)}{\Gamma(\nu_1 - \mu_k + m)} \int_0^\infty \left( \frac{\xi - \xi'}{\xi - \eta_0} \right)^{\nu_2 - \nu_1 + \mu_k - m - 1} \frac{\Gamma(\nu_2 - \mu_k + 1)\Gamma(\nu_2 - \mu_k + m + 2)}{\Gamma(\nu_1 + \nu_2 - \mu_k + m + 2)} d\zeta
\]

\[
= e^{\pm \pi \sqrt{-1} \nu_1} \Gamma(\nu_1 + 1) \times \left( \xi - \eta_0 \right)^{-\nu_2 + \mu_k - m} \left\{ \frac{\Gamma(\nu_2 - \mu_k + 1)}{\Gamma(\nu_1 + \nu_2 - \mu_k + m + 2)} \left( -\nu_1 - \nu_2 - \mu_k + 1 \right) \Gamma(\nu_1 + \nu_2 - \mu_k + m + 2) \right\},
\]

which implies (5.11).

**Theorem 5.8.** Assume that \( \nu_2 \notin \mathbb{Z}_{\geq 0}, \nu_2 - \mu_k \notin \mathbb{Z}_{< 0} \) and \( \nu_1 + \nu_2 - \mu_k \notin \mathbb{Z} \). The integral \( W_{\infty,k,h}^{(S_i^1,\eta_0,\infty)}(\nu_1, \nu_2; \xi) \) is holomorphic for \( \xi \in \mathbb{C} \setminus \{ \eta_0 + s e^{\sqrt{-1} \phi_i} | s \geq 0 \} \). Moreover, near \( \xi = \infty, \xi \in S_i^\pm \), we have

\[
W_{\infty,k,h}^{(S_i^1,\eta_0,\infty)}(\nu_1, \nu_2; \xi) = e^{\pm \pi \sqrt{-1} (\mu_k - \nu_2)} \sin \pi \nu_1 \sin \pi (\mu_k - \nu_1 - \nu_2) \frac{\Gamma(\nu_1 + \nu_2 - \mu_k + m + 2)}{\Gamma(\nu_1 + \nu_2 - \mu_k + m + 2)} W_{\infty,k,h}^{(S_i^1,\infty)}(\nu_1, \nu_2; \xi) + \text{hol} \left( \frac{1}{\xi - \eta_0} \right).
\]

Here the double-signs correspond.

**Proof.** It is trivial that \( W_{\infty,k,h}^{(S_i^1,\eta_0,\infty)}(\nu_1, \nu_2; \xi) \) is holomorphic in \( \mathbb{C} \setminus \{ \eta_0 + s e^{\sqrt{-1} \phi_i} | s \geq 0 \} \).

To find the behavior of \( W_{\infty,k,h}^{(S_i^1,\eta_0,\infty)}(\nu_1, \nu_2; \xi) \) near \( \xi = \infty \) we analytically continue \( W_{\infty,k,h}^{(S_i^1,\eta_0,\infty)}(\nu_1, \nu_2; \xi) \) along a circle of center \( \eta_0 \) with sufficiently large radius in the positive direction. Then from Figure 5.3 for
Figure 5.3: Deformation of the path $\eta_0^\infty$ for $\xi \in S_i^-$

Figure 5.4: Deformation of the path $\eta_0^\infty$ for $\xi \in S_i^+$

\[ W(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty}) \text{ or from Figure } 5.4 \text{ for } W(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty}) \text{ we see that the analytic continuation of } W(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty}) \text{ is} \]

\[ W(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty})(\nu_1,\nu_2;\xi) + \epsilon \left( 1 - e^{-2\pi \sqrt{-1} \nu_1} \right) W(S_{\infty,k,h}^{\pm,\eta_0,\infty})(\nu_1,\nu_2;\xi), \]

where $\epsilon^- = 1$ for $W(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty})$ while $\epsilon^+ = e^{2\pi \sqrt{-1} (\mu_k - \nu_2)}$ for $W(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty})$. This means that

\[ W(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty})(\nu_1,\nu_2;\xi) = \frac{\epsilon^- \left( 1 - e^{-2\pi \sqrt{-1} \nu_1} \right)}{e^{2\pi \sqrt{-1} (\mu_k - \nu_1 - \nu_2)} - 1} W(S_{\infty,k,h}^{\pm,\eta_0,\infty})(\nu_1,\nu_2;\xi) + \text{hol} \left( \frac{1}{\xi - \eta_0} \right) \]

near $\xi = \infty$, which leads to (5.12).

5.3 Reducible cases

The following proposition sophisticates Haraoka’s result [4, Proposition 3.3 and Corollary 3.2]. Recall the notation $[\phantom{\nu_1}]_m$ that denotes the $m$-th component of a column vector.

**Proposition 5.9.** The integrals $W_{\text{sing},k,h}^{(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty})}$ treated in Theorems 5.1, 5.2, 5.4, 5.6, 5.7 and 5.8 satisfy

\[ \left[ W_{\text{sing},k,h}^{(S_{\nu_1,\nu_2}^{\pm,\eta_0,\infty})} \right]_m = 0 \quad \text{for } n + m_1 + \cdots + m_{l-1} + 1 \leq m \leq n + m_1 + \cdots + m_{l-1} \quad (5.13) \]

except for $W_{\infty,k,h}^{(S_{\nu_1,\mu_k}^{\pm,\eta_0,\infty})}$ and $W_{\infty,k,h}^{(S_{\nu_1,\mu_k}^{\pm,\eta_0,\infty})}$, and

\[ \left[ W_{\text{sing},k,h}^{(S_{\nu_1,\mu_k}^{\pm,\eta_0,\infty})} \right]_m = 0 \quad \text{for } n + m_1 + \cdots + m_{l-1} + 1 \leq m \leq n + m_1 + \cdots + m_{l-1} \quad (5.14) \]
except for \( W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\mu_k, \nu_2; \xi) \), \( a, b \in \{\xi, \eta_0, \infty\} \).

**Proof.** We can apply Lemma 5.3 to the integrals \( W^{(S^{\pm}_{\text{sing},k,h})}_{\text{sing},k,h} (\nu_1, \nu_2; \xi) \) except for \( W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_k; \xi) \) and \( W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_k; \xi) \), and obtain

\[
W^{(S^{\pm}_{\text{sing},k,h})}_{\text{sing},k,h} (\nu_1, \nu_2; \xi) = \int_a^b \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} (\zeta - \eta_0)^{-\nu_2 - 1} \left( (\eta_0 I_n - T') w^{(S^{\pm}_{\text{sing},k,h})}_{\text{sing},k,h} (\nu_1 - 1; \zeta) \right) \, d\zeta.
\]

Combining this expression with the definition of \( A' \) and with the facts stated in the proof of Theorem 4.7 we obtain (5.13) and (5.14), respectively. \( \square \)

## 6 Relations of the integrals

In this section we investigate relations of the integrals defined in the preceding section, and then translate the relations to those for solutions of the system (1.4) in the next section.

We assume that

\[
\begin{align*}
\nu_j & \notin \mathbb{Z} \quad \text{for } 1 \leq j \leq 2, \\
\nu_j - \lambda_i & \notin \mathbb{Z}_{\geq 0} \quad \text{for } 1 \leq j \leq 2, \ 1 \leq i \leq p, \ 1 \leq k \leq r_i, \\
\nu_j - \mu_k & \notin \mathbb{Z}_{\leq 0} \quad \text{for } 1 \leq j \leq 2, \ 1 \leq k \leq q.
\end{align*}
\]

**Theorem 6.1.** For \( \xi \in S^+_i \) we have

\[
W^{(S^{\pm}_{\text{sing},k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) - W^{(S^{\pm}_{\text{sing},k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) - W^{(S^{\pm}_{\text{sing},k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) = 0,
\]

\[
W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) - W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) - W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) = 0,
\]

and for \( \xi \in S^+_i \)

\[
e^{-2\pi \sqrt{-1} \nu_2} W^{(S^{\pm}_{\text{sing},k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) - W^{(S^{\pm}_{\text{sing},k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) - W^{(S^{\pm}_{\text{sing},k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) = 0,
\]

\[
e^{-2\pi \sqrt{-1} \nu_2} W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) - W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) - W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) = 0.
\]

**Proof.** Applying the Cauchy theorem to the triangles \( \triangle(\eta_0, \eta_0, t'_i) \) and \( \triangle(\eta_0, \xi, \infty) \), we have

\[
\int_{\partial \triangle(\eta_0, \eta_0, t'_i)} \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} (\zeta - \eta_0)^{-\nu_2 - 1} w^{(S^{\pm}_{t'_i,k,h})}_{t'_i,k,h} (\eta_0 - 1; \zeta) \, d\zeta = 0
\]

and

\[
\int_{\partial \triangle(\eta_0, \xi, \infty)} \left( \frac{\xi - \zeta}{\xi - \eta_0} \right)^{\nu_1} (\zeta - \eta_0)^{-\nu_2 - 1} w^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\eta_0 - 1; \zeta) \, d\zeta = 0,
\]

respectively. Comparing the branches of these integrals and those of the integrals \( W^{(S^{\pm}_{\text{sing},k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) \) and \( W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) \), we can easily see that the relations above hold. \( \square \)

Set

\[
\begin{align*}
V^{(S^{\pm}_{t'_i,k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) &= -\frac{\Gamma(-\lambda_i + 1)}{\Gamma(\nu_1 + 1)\Gamma(-\lambda_i + \nu_1)} W^{(S^{\pm}_{t'_i,k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi), \\
V^{(S^{\pm}_{t'_i,k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi) &= \frac{\Gamma(-\lambda_i + 1\nu)}{\Gamma(-\lambda_i + \nu_1)} W^{(S^{\pm}_{t'_i,k,h})}_{t'_i,k,h} (\nu_1, \nu_2; \xi), \\
V^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) &= -\frac{\Gamma(-\lambda_i + 1\nu)}{\Gamma(\nu_1 + 1)\Gamma(\nu_2 - \mu_k + 1)} W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi), \\
V^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi) &= \frac{\Gamma(-\lambda_i + 1\nu)}{\Gamma(\nu_1 + 1)\Gamma(\nu_2 - \mu_k + 1)} W^{(S^{\pm}_{\infty,k,h})}_{\infty,k,h} (\nu_1, \nu_2; \xi),
\end{align*}
\]
where the double-signs correspond. As a corollary to Theorems 5.1, 5.2, 5.7 and 5.8, we have the following theorem.

**Theorem 6.2** (cf. [5 §7.4]). Each of the \( V^{(S^+_{-})}_{\text{sing},k,h}(\nu_1,\nu_2;\xi) \)'s is not altered if \( \nu_1 \) and \( \nu_2 \) are interchanged.

**Proof.** The integrals \( V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) \) and \( V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) \) not only satisfy the system (5.3) but also have the same asymptotic behavior as \( \xi \to t'_i \). This implies that the integrals coincide with each other. As for other integrals, the coincidence when \( \nu_1 \) and \( \nu_2 \) are interchanged follows from their asymptotic behavior also.

**Remark 6.1.** From Theorems 5.2, 5.4 and 5.7 we see that \( V^{(S^+_{-})}_{\text{sing},k,h}(\nu_1,\nu_2;\xi) \) and its analytic continuation satisfy

\[
V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) = V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) \quad \text{for } \xi \in S^+_i \cup S^+_{i} \cup S^+_i,
\]

\[
V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) = V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) \quad \text{for } \xi \in \mathbb{C} \setminus \Sigma_i^{(0,1)}.
\]

Combining (5.8) with (5.12), we obtain the following relations.

Rewriting the relations stated in Theorem 6.1 in terms of \( V^{(S^+_{-})}_{\text{sing},k,h}(\nu_1,\nu_2;\xi) \), we obtain

\[
e^{-\pi\sqrt{-1}\nu_2} W^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) + \frac{e^{\pi\sqrt{-1}\nu_2} \Gamma(\nu_1 + 1) \Gamma(-\lambda_i,k - \nu_1)}{\Gamma(-\lambda_i,k + 1)} V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) - \frac{\Gamma(\lambda_i,k - \nu_1) \Gamma(-\lambda_i,k)}{\Gamma(-\nu_1)} V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) = 0
\]

and

\[
e^{-\pi\sqrt{-1}\nu_2} W^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) + \frac{e^{\pi\sqrt{-1}\nu_2} \Gamma(\nu_1 + 1) \Gamma(-\lambda_i,k - \nu_1)}{\Gamma(-\lambda_i,k + 1)} V^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) - \frac{\Gamma(\lambda_i,k - \nu_1) \Gamma(-\lambda_i,k)}{\Gamma(-\nu_1)} V^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) = 0,
\]

where the double-signs correspond.

Combining (6.4) with (5.8) and (6.5) with (5.12), we obtain the following relations.

**Theorem 6.3.** We have

\[
W^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) = \frac{e^{\pi\sqrt{-1}\nu_2} \Gamma(\nu_1 + 1) \Gamma(-\lambda_i,k - \nu_1)}{\Gamma(-\lambda_i,k + 1)} V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) + \text{hol}(\xi - t'_i)
\]

for \( \xi \in S^+_i \) near \( t'_i \), and

\[
W^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) = -\frac{\Gamma(\nu_1 + 1) \Gamma(-\lambda_i,k - \nu_1)}{\Gamma(-\lambda_i,k + 1)} V^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) + \text{hol}(\frac{1}{\xi - \eta_0})
\]

for \( \xi \in S^+_i \) near \( \infty \).

**Proof.** Representing \( \xi \) by \( V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) \) and \( V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) \), we have

\[
V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) = e^{\pi\sqrt{-1}\lambda_i,k} V^{(S^+_{-})}_{t_i,k,h}(\nu_1,\nu_2;\xi) + \text{hol}(\xi - t'_i)
\]

Substituting this into (6.4), we obtain (6.6). Similarly, representing \( \xi \) by \( V^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) \) and \( V^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) \), we have

\[
V^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) = e^{\pi\sqrt{-1}\lambda_i,k} V^{(S^+_{-})}_{\infty,k,h}(\nu_1,\nu_2;\xi) + \text{hol}(\frac{1}{\xi - \eta_0})
\]

Substituting this into (6.5), we obtain (6.7).
In the relations (6.4) and (6.5) interchanging $\nu_1$ and $\nu_2$ and then using Theorem 6.2 we obtain
\begin{equation}
e^{-\pi \sqrt{-1} t_2} W_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_2, \nu_1; \xi) + \frac{e^{\pm \pi \sqrt{-1} t_2} \Gamma(\nu_2 + 1) \Gamma(\lambda_i,k - \nu_2)}{\Gamma(\lambda_i,k + 1)} V_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_1, \nu_2; \xi) \end{equation}
(6.8)

and
\begin{equation}W_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_2, \nu_1; \xi) + \frac{e^{\pm \pi \sqrt{-1} t_2} \Gamma(\nu_2 + 1) \Gamma(\lambda_i,k - \nu_2)}{\Gamma(\lambda_i,k + 1)} V_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_1, \nu_2; \xi) \end{equation}
(6.9)

where the double-signs correspond. Solving the simultaneous equations (6.4) and (6.8) with respect to $V_{\ell',k,h}^{(S_i^{\pm}\xi)}$ and $V_{\ell',k,h}^{(S_i^{\pm}\eta_0 t'_i)}$, and (6.5) and (6.9) with respect to $V_{\ell',k,h}^{(S_i^{\pm}\xi)}$ and $V_{\ell',k,h}^{(S_i^{\pm}\eta_0 \infty)}$, we have the following relations.

**Theorem 6.4.** For $\xi \in S_i^{\pm}$ we have
\begin{equation}V_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_1, \nu_2; \xi) = -\sum_{j=1}^{2} \frac{e^{-\pi \sqrt{-1} t_j} \sin \pi \nu_j \Gamma(\lambda_i,k + 1)}{\sin \pi (\nu_j - \nu_j') \Gamma(\nu_j + 1) \Gamma(\lambda_i,k - \nu_j)} W_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_j, \nu_j'; \xi), \end{equation}
(6.10)

and
\begin{equation}V_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_1, \nu_2; \xi) = -\sum_{j=1}^{2} \frac{\sin \pi \nu_j \Gamma(\nu_1 + \nu_2 - \mu_k + 1)}{\sin \pi (\nu_j - \nu_j') \Gamma(\nu_j + 1) \Gamma(\nu_j - \mu_k + 1)} W_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_j, \nu_j'; \xi), \end{equation}
(6.11)

where the double-signs correspond and $j'$ denotes the complement of $j$ in $\{1, 2\}$, namely, $(j, j') = (1, 2), (2, 1)$.

Now, let us recall the connection formulas (4.11) in $\mathcal{P}'$ and (4.14) in $\hat{\mathcal{P}}'$. Note that we have determined the assignment of argument of $\zeta - t'_i$ as

$$\arg(\zeta - t'_i) \in \begin{cases} (\theta'_i - 2\pi, \theta'_i) & \text{for } \xi \in \mathcal{P}', \\ (\theta'_i - \pi, \theta'_i + \pi) & \text{for } \xi \in \hat{\mathcal{P}}' \end{cases}$$

while

$$\arg(\zeta - t'_i) \in \begin{cases} (\theta'_i - \pi, \theta'_i) & \text{for } \xi \in S_i^-, \\ (\theta'_i - 2\pi, \theta'_i - \pi) & \text{for } \xi \in S_i^+ \end{cases}$$

**Theorem 6.5.** For $\xi \in S_i^+ \cap S_{i+1}^+$ we have
\begin{equation}V_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_1, \nu_2; \xi) = \sum_{k=1}^{r_i+1} \sum_{h=1}^{t_i+1,k} c_{t_i+1,\tilde{k},h} V_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_1, \nu_2; \xi) + \text{hol}(\xi - t'_{i+1}), \end{equation}
(6.12)

and for $\xi \in S_i^+ \cap S_{i-1}^-$
\begin{equation}V_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_1, \nu_2; \xi) = \sum_{k=1}^{r_{i-1}} \sum_{h=1}^{t_{i-1},k} c_{t_{i-1},\tilde{k},h} V_{\ell',k,h}^{(S_i^{\pm}\xi)}(\nu_1, \nu_2; \xi) + \text{hol}(\xi - t'_{i-1}), \end{equation}
(6.13)

where $c_{t_{i\pm},\tilde{k},h} = c_{t_{i\pm},\tilde{k},h}(0)$. 

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Proof. We give the proof only for (6.12). Substituting (4.11) with \( \nu = i + 1 \) and \( \rho = -\nu_j - 1 \) into the integral \( W^{(S_{i+1}^-; \eta_0 \xi)}_{\ell_{i+1}^+} (\nu_j, \nu_j'; \xi) \), we have

\[
W^{(S_{i+1}^-; \eta_0 \xi)}_{\ell_{i+1}^+} (\nu_j, \nu_j'; \xi) = \sum_{k=1}^{r+1} \sum_{h=1}^{r+1} c_{\ell_{i+1}^+, k, h, t_{i+1}'; k, h, t_{i+1}'} (\nu_j - 1) W^{(S_{i+1}^-; \eta_0 \xi)}_{\ell_{i+1}^+} (\nu_j, \nu_j'; \xi) + \text{hol}(\xi - t_{i+1}') .
\]

Here the holomorphy of the integral of the term \( \text{hol}(\zeta - t_{i+1}') \) can be proven by representing the term as a linear combination of \( \bar{w}_{\ell_{i+1}^+, k, h} (\nu_j - 1; \zeta) \) \((\nu \neq i + 1)\) and then applying Theorem 6.6. Substituting this relation into (6.10) and then substituting (4.12) with \( \nu = i + 1 \) and \( \rho = -\nu_j - 1 \) into the result, we have

\[
V_{\ell_{i+1}^+, k, h}^{(S_{i+1}^-; \eta_0 \xi)} (\nu_j, \nu_j') = \sum_{k=1}^{r+1} \sum_{h=1}^{r+1} \sum_{j=1}^{2} e^{-2\pi \sqrt{-1} \nu_j} \sin \pi \nu_j \Gamma(\nu_j - \lambda_{i+1, k, h} + 1) \sin \pi (\nu_j' - \nu_j) \Gamma(\nu_j + 1) \Gamma(-\lambda_{i+1, k, h}) c_{\ell_{i+1}^+, k, h, t_{i+1}'; k, h, t_{i+1}'} W^{(S_{i+1}^-; \eta_0 \xi)}_{\ell_{i+1}^+} (\nu_j, \nu_j'; \xi) + \text{hol}(\xi - t_{i+1}').
\]

Lastly, substituting (6.6) with \( \nu \) replaced by \( i + 1 \), we have

\[
V_{\ell_{i+1}^+, k, h}^{(S_{i+1}^-; \eta_0 \xi)} (\nu_1, \nu_2; \xi) = \sum_{k=1}^{r+1} \sum_{h=1}^{r+1} \sum_{j=1}^{2} e^{-\pi \sqrt{-1} \nu_j} \sin \pi \nu_j \sin(\pi (\nu_j' - \nu_j)) c_{\ell_{i+1}^+, k, h, t_{i+1}'; k, h, t_{i+1}'} V^{(S_{i+1}^-; \eta_0 \xi)}_{\ell_{i+1}^+} (\nu_1, \nu_2; \xi) + \text{hol}(\xi - t_{i+1}'),
\]

which implies (6.12) since the identity

\[
2 \sum_{j=1}^{2} \frac{e^{-\pi \sqrt{-1} \nu_j} \sin \pi \nu_j}{\sin \pi (\nu_j' - \nu_j)} = 1
\]

holds. \( \square \)

**Theorem 6.6.** For \( \xi \in S_{i}^{\ell^-} \) we have

\[
V_{\ell_{i}^-, k, h}^{(S_{i}^{\ell^-}; \eta_0 \xi)} (\nu_1, \nu_2; \xi) = \sum_{k=1}^{q} \sum_{h=1}^{m_k} \frac{\Gamma(\mu_k + 1) \Gamma(\mu_k - \nu_1 - \nu_2)}{\Gamma(\mu_k - \nu_1) \Gamma(\mu_k - \nu_2)} c_{\infty, k, h, t_{i}'; k, h, t_{i}'} V_{\infty, k, h}^{(S_{i}^{\ell^-}; \eta_0 \xi)} (\nu_1, \nu_2; \xi) + \text{hol} \left( \frac{1}{\xi - \eta_0} \right),
\]

and for \( \xi \in S_{i}^{\ell^+} \)

\[
V_{\ell_{i}^+, k, h}^{(S_{i}^{\ell^+}; \eta_0 \xi)} (\nu_1, \nu_2; \xi) = e^{-2\pi \sqrt{-1} \lambda_{i, k}} \sum_{k=1}^{q} \sum_{h=1}^{m_k} \frac{\Gamma(\mu_k + 1) \Gamma(\mu_k - \nu_1 - \nu_2)}{\Gamma(\mu_k - \nu_1) \Gamma(\mu_k - \nu_2)} c_{\infty, k, h, t_{i}'; k, h, t_{i}'} V_{\infty, k, h}^{(S_{i}^{\ell^+}; \eta_0 \xi)} (\nu_1, \nu_2; \xi)
\]

\[
+ \text{hol} \left( \frac{1}{\xi - \eta_0} \right)
\]

where \( c_{\infty, k, h, t_{i}'; k, h, t_{i}'} = c_{\infty, k, h, t_{i}'; k, h, t_{i}'} (0) \).

**Proof.** First, we prove (6.15). Substituting (4.11) with \( \rho = -\nu_j - 1 \) into the integral \( W^{(S_{i}^{\ell^-}; \eta_0 \xi)}_{\ell_{i}^+, k, h} (\nu_j, \nu_j'; \xi) \), we have

\[
W^{(S_{i}^{\ell^-}; \eta_0 \xi)}_{\ell_{i}^+, k, h} (\nu_j, \nu_j'; \xi) = \sum_{k=1}^{q} \sum_{h=1}^{m_k} c_{\infty, k, h, t_{i}'; k, h, t_{i}'} (\nu_j - 1) W^{(S_{i}^{\ell^-}; \eta_0 \xi)}_{\infty, k, h} (\nu_j, \nu_j'; \xi).
\]

Substituting this relation into (6.10) and then substituting (4.12) with \( \rho = -\nu_j - 1 \) into the result, we obtain

\[
V_{\ell_{i}^+, k, h}^{(S_{i}^{\ell^-}; \eta_0 \xi)} (\nu_1, \nu_2; \xi) = -\sum_{k=1}^{q} \sum_{h=1}^{m_k} \sum_{j=1}^{2} e^{-\pi \sqrt{-1} \nu_j} \sin \pi \nu_j \Gamma(\mu_k + 1) \sin(\pi (\nu_j' - \nu_j)) c_{\infty, k, h, t_{i}'; k, h, t_{i}'} V_{\infty, k, h}^{(S_{i}^{\ell^-}; \eta_0 \xi)} (\nu_j, \nu_j'; \xi).
\]
Lastly, substituting (6.7), we have

\[ W_{\nu_1, \nu_2; \xi}^{(S_i^+; t', \xi)} (\nu_1, \nu_2; \xi) = \sum_{k=1}^{q} \sum_{h=1}^{m_k} \sum_{j=1}^{2} e^{\frac{\pi}{\sqrt{-1}} \nu_j \sin \pi \nu_j} \frac{\Gamma(\mu_k + 1) \Gamma(\mu_k - \nu_1 - \nu_2)}{\Gamma(\mu_k + 1) \Gamma(\mu_k - \nu_2)} c_{\infty, k, h, t'; k, h, t} W_{\nu_1, \nu_2; \xi}^{(S_i^+; \xi)} (\nu_1, \nu_2; \xi) + \text{hol} \left( \frac{1}{\xi - \eta_0} \right), \]

which leads to (6.15) by virtue of the identity (6.14).

Next, we prove (6.16). Note that the assignment of argument of \( \zeta - t'_i \) in \( S_i^+ \) differs from that in \( P' \). We analytically continue \( W_{\nu_1, \nu_2; \xi}^{(S_i^+; t, \xi)} \) and \( W_{\nu_1, \nu_2; \xi}^{(S_i^+; \xi)} \) across the ray \( \sum_{\nu_1}^{(1, \infty)} \) into \( S_i^+ \). The analytic continuation is found by deforming the path of integration \( \eta_0 \xi \) to the path \( \gamma(\eta_0 \xi) \) indicated in Figure 6.1 (b), and the relation

\[ W_{\nu_1, \nu_2; \xi}^{(S_i^+; \gamma(\eta_0 \xi))} (\nu_1, \nu_2; \xi) = \sum_{k=1}^{q} \sum_{h=1}^{m_k} e^{\frac{\pi}{\sqrt{-1}} \nu_j \sin \pi \nu_j} W_{\nu_1, \nu_2; \xi}^{(S_i^+; \gamma(\eta_0 \xi))} (\nu_1, \nu_2; \xi) \]

holds. Here, from Figure 6.1 (b) we see that the relation

\[ W_{\nu_1, \nu_2; \xi}^{(S_i^+; \gamma(\eta_0 \xi))} (\nu_1, \nu_2; \xi) \equiv e^{2\sqrt{-1} \nu_j \sin \pi \nu_j} W_{\nu_1, \nu_2; \xi}^{(S_i^+; t, \xi)} (\nu_1, \nu_2; \xi) \]

holds. Besides, from Theorem 5.7 we see that the relation

\[ W_{\nu_1, \nu_2; \xi}^{(S_i^+; \gamma(\eta_0 \xi))} (\nu_1, \nu_2; \xi) \equiv e^{2\sqrt{-1} \nu_j \sin \pi \nu_j} W_{\nu_1, \nu_2; \xi}^{(S_i^+; \xi)} (\nu_1, \nu_2; \xi) \]

holds. Using these relations instead of the first and the second relations of Theorem 6.1 we can establish the relation (6.16) in a similar way to (6.15). \( \square \)

7 Connection formulas

Taking account of the transformation (2.4), we see that the integrals

\[ W_{\nu_1, \nu_2; \xi}^{(S_i^+; \xi)} (\nu_1, \nu_2; \xi), \quad V_{\nu_1, \nu_2; \xi}^{(S_i^+; \xi)} (\nu_1, \nu_2; \xi) \]

become solutions of the system (1.3). We shall establish relations between these integrals and the local solutions \( U_{\nu_1, \nu_2; \xi} \) defined in Section 3 and then rewrite the connection formulas among the integrals \( W_{\nu_1, \nu_2; \xi}^{(S_i^+; \xi)} (\nu_1, \nu_2; \xi) \) and \( V_{\nu_1, \nu_2; \xi}^{(S_i^+; \xi)} (\nu_1, \nu_2; \xi) \) by means of \( U_{\nu_1, \nu_2; \xi} \).
From now on in the case that the endpoint \( b \) of the path \( ab \) is equal to \( \xi \) we omit \( \xi \) in the superscript, namely, we write
\[
W_{t_i',k,h}^{(S_{i-}^{\pm};\eta_0)} = W_{t_i',k,h}^{(S_{i-}^{\pm};\eta_0;\xi)}, \quad V_{t_i',k,h}^{(S_{i-}^{\pm};\xi)} = V_{t_i',k,h}^{(S_{i-}^{\pm};\xi)},
\]
\[
W_{\infty,k,h}^{(S_{i-}^{\pm};\eta_0)} = W_{\infty,k,h}^{(S_{i-}^{\pm};\eta_0;\xi)}, \quad V_{\infty,k,h}^{(S_{i-}^{\pm};\eta_0)} = V_{\infty,k,h}^{(S_{i-}^{\pm};\eta_0;\xi)}.
\]
For \( 1 \leq i \leq p \) we set
\[
S_i^+ = \{ x \mid \theta_i < \arg(x - t_{p+1}) < \min(\theta_{i+1} - \delta, \theta_i + \pi) \},
\]
\[
S_i^- = \{ x \mid \max(\theta_{i-1} + \delta, \theta_i - \pi) < \arg(x - t_{p+1}) < \theta_i \},
\]
where \( \theta_0 = \theta_{p+1} - 2\pi \). Note that \( \xi \in S_i^- \) (resp. \( S_i^+ \)) corresponds to \( x \in S_i^+ \) (resp. \( S_i^- \)) through the transformation \( \mathcal{L} \).

7.1 Generic case

First, we consider the connection formulas for the system \( (1.4) \) in the case that none of the \( \rho_j \)'s is an eigenvalue of the matrix \( A \). We assume the conditions \( (3.1)-(3.4), (3.5) \) and \( (4.3) \).

**Proposition 7.1.** For \( 1 \leq i \leq p \) we have
\[
W_{t_i',k,h}^{(S_{i-}^{\pm};\eta_0)}(\rho_j, \rho_j'; \eta_0 + \frac{1}{x - t_{p+1}}) = \frac{\Gamma(\rho_j + 1)\Gamma(-\rho_j')}{\Gamma(\rho_j - \rho_j' + 1)} \sum_{\hat{h}=1}^{n} \gamma_{\hat{h};t_i',k,h}(\rho; \eta_0)(1 \leq \hat{h} \leq n),
\]
for \( x \in S_{i}^{\mp}, \) where
\[
\gamma_{\hat{h};t_i',k,h}(\rho) = [P^{-1}]_{\hat{h}}(\eta_0 I_n - T')w_{t_i',k,h}(\rho; \eta_0)(1 \leq \hat{h} \leq n),
\]
and
\[
W_{\infty,k,h}^{(S_{i-}^{\pm};\eta_0)}(\rho_j, \rho_j'; \eta_0 + \frac{1}{x - t_{p+1}}) = \frac{\Gamma(\rho_j + 1)\Gamma(-\rho_j')}{\Gamma(\rho_j - \rho_j' + 1)} \sum_{\hat{h}=1}^{n} \gamma_{\hat{h};\infty,k,h}(\rho; \eta_0)(1 \leq \hat{h} \leq n),
\]
for \( x \in S_{i}^{\mp}, \) where
\[
\gamma_{\hat{h};\infty,k,h}(\rho) = [P^{-1}]_{\hat{h}}(\eta_0 I_n - T')w_{\infty,k,h}|_{S_{i}^{\mp}}(\rho; \eta_0)(1 \leq \hat{h} \leq n).
\]

**Proof.** Under the transformation \( \mathcal{L} \), \( \xi \rightarrow \eta_0 \) is equivalent to \( x \rightarrow \infty \). From Theorem 5.1 we find
\[
\frac{\Gamma(\rho_j - \rho_j' + 1)}{\Gamma(\rho_j + 1)\Gamma(-\rho_j')} W_{t_i',k,h}^{(S_{i-}^{\pm};\eta_0)}(\rho_j, \rho_j'; \eta_0 + \frac{1}{x - t_{p+1}}) = (x - t_{p+1})^{\rho_j' - \rho_j} \left( \left( \rho_j I_n - A' \right)^P(\eta_0 I_n - T')w_{t_i',k,h}(\rho_j - 1; \eta_0) + O \left( \frac{1}{x - t_{p+1}} \right) \right)
\]
as \( x \rightarrow \infty, x \in S_{i}^{\mp} \). Here the initial term satisfies
\[
\left( \rho_j I_n - A' \right)^P(\eta_0 I_n - T')w_{t_i',k,h}(\rho_j - 1; \eta_0) = \sum_{\hat{h}=1}^{n} \gamma_{\hat{h};t_i',k,h}(\rho_j - 1; \eta_0),
\]
which implies \( (7.1) \). Similarly, from Theorem 5.6 we obtain \( (7.2) \).

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For $x \in S_i^+$ we determine the assignment of argument of $x - t_i$ as

$$\arg(x - t_i) \in (\theta_i, \theta_i + \pi) \quad \text{for } x \in S_i^+,$$

$$\arg(x - t_i) \in (\theta_i - \pi, \theta_i) \quad \text{for } x \in S_i^-.$$  

(7.3)

Under the assignment (5.2) and (7.3) we have

$$\xi - t'_i = \frac{e^{-\pi \sqrt{-1}}}{t_i - t_{p+1}} \cdot \frac{x - t_i}{x - t_{p+1}}.$$

(7.4)

**Proposition 7.2.** For $1 \leq i \leq p$ we have

$$V_{t_i,k,h}^{(S_i^\pm,t'_i)}(\rho_1, \rho_2; \eta_0 + \frac{1}{x - t_{p+1}}) = e^{-\pi \sqrt{-1}\lambda_{i,k}}(t_i - t_{p+1})^{\rho_1 + \rho_2 - 2\lambda_{i,k}}U_{t_i,k,h}(x) \quad (1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k})$$

for $x \in S_i^\pm$.  

**Proof.** Note that

$$\xi - t'_i = \frac{1}{(t_i - t_{p+1})^2} \cdot (x - t_i) \cdot \{1 + O(x - t_i)\}$$

as $x \to t_i$, where

$$\arg\{1 + O(x - t_i)\} = \arg\left(\frac{t_i - t_{p+1}}{x - t_{p+1}}\right) \in (-\pi, \pi).$$

Substituting (7.3) into (5.3), we find

$$V_{t_i,k,h}^{(S_i^\pm,t'_i)}(\rho_1, \rho_2; \eta_0 + \frac{1}{x - t_{p+1}}) = \left(1 - \frac{1}{t_i - t_{p+1}}\right)^{-\rho_1 - \rho_2} \left(\frac{e^{-\pi \sqrt{-1}}}{(t_i - t_{p+1})^2}\right)^{\lambda_{i,k}}$$

$$\times (x - t_i)^{\lambda_{i,k}} \{\varepsilon_2(n_1 + \cdots + n_{i-1} + \ell_{i,1} + \cdots + \ell_{i,k-1} + h) + O(x - t_i)\}$$

as $x \to t_i$, $x \in S_i^\pm$.  This implies (7.4). \hfill \Box

**Proposition 7.3.** We have

$$V_{t_{p+1},k,h}^{(S_i^\pm,\infty)}(\rho_1, \rho_2; \eta_0 + \frac{1}{x - t_{p+1}}) = U_{t_{p+1},k,h}(x) \quad (1 \leq k \leq q, 1 \leq h \leq m_k)$$

for $x \in S_i^\pm$.  

**Proof.** Note that $\xi \to \infty$ is equivalent to $x \to t_{p+1}$.  From Theorem 5.7 we obtain

$$V_{t_{p+1},k,h}^{(S_i^\pm,\infty)}(\rho_1, \rho_2; \eta_0 + \frac{1}{x - t_{p+1}}) = (x - t_{p+1})^{\rho_1 + \rho_2 - \mu_k} \{\varepsilon_2(n + m_1 + \cdots + m_{k-1} + h) + O(x - t_{p+1})\}$$

as $x \to t_{p+1}, x \in S_i^\pm$.  This implies (7.6). \hfill \Box

Combining Theorems 6.3, 6.3 and 6.3 with Propositions 7.1, 7.3 and 7.2 we obtain the following conclusions.

**Theorem 7.4.** For $1 \leq i \leq p$ the coefficients $C_{\infty,k,h}^{(\xi,i)}$ in the connection formula

$$U_{i,k,h}(x) = \sum_{j=1}^{2} \sum_{k=1}^{n} C_{\infty,j,i,k,h} U_{\infty,j,k,h}(x) \quad (1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k})$$

(7.7)
for \( x \in \mathcal{S}_i^\pm \) are given by

\[
C_{\infty,j,h,t+1,k,h} = e^{\pi \sqrt{-1} (\lambda_{i,k} - \rho_j)} (t_i - t_{p+1})^{2\lambda_{i,k} - \rho_1 - \rho_2} \frac{\Gamma(\rho_j - \rho_j') \Gamma(\lambda_{i,k} + 1)}{\Gamma(\rho_j - \rho_j') \Gamma(\lambda_{i,k} + 1)} \gamma_{\tilde{h},t',k,h} (-\rho_j - 1) \tag{7.8}
\]

(1 \leq j \leq 2, 1 \leq \tilde{h} \leq n, 1 \leq k \leq \ell_i, 1 \leq \tilde{h} \leq \ell_i,k).

Besides, the coefficients \( C^{(\mathcal{S}_i^\pm)}_{\infty,j,h,t+1,k,h} \) in the connection formula

\[
U_{t+1,k,h}(x) = \sum_{j=1}^{2} \sum_{h=1}^{n} C^{(\mathcal{S}_i^\pm)}_{\infty,j,h,t+1,k,h} U_{\infty,j,h}(x) \quad (1 \leq k \leq q, 1 \leq h \leq m_k) \tag{7.9}
\]

for \( x \in \mathcal{S}_i^\pm \) are given by

\[
C^{(\mathcal{S}_i^\pm)}_{\infty,j,h,t+1,k,h} = \frac{\Gamma(\rho_j - \rho_j') \Gamma(\rho_1 + \rho_2 - \mu_k + 1)}{\Gamma(\rho_j - \rho_j') \Gamma(\rho_1 + \rho_2 - \mu_k + 1)} \gamma_{\tilde{h},\infty,k,h} (-\rho_j - 1) \tag{7.10}
\]

(1 \leq j \leq 2, 1 \leq \tilde{h} \leq n, 1 \leq k \leq q, 1 \leq h \leq m_k).

**Proof.** Substituting (7.4) and (7.3) into (6.10) \( \nu_1 = \nu_2 = \rho_2 \) and then interchanging \( j \) and \( j' \), we obtain (7.7) with (7.8). Similarly, substituting (7.2) and (7.3) into (6.11) \( \nu_1 = \nu_2 = \rho_2 \) and then interchanging \( j \) and \( j' \), we obtain (7.9) with (7.10).

**Theorem 7.5.** For \( 1 \leq i \leq p - 1 \) the coefficients \( C_{t+i,\tilde{k},h,t+i,k,h} \) in the connection formula

\[
U_{t+i,k,h}(x) = \sum_{k=1}^{r_{i+1}} \sum_{\tilde{h}=1}^{\ell_{i+1,k}} C_{t+i,\tilde{k},h,t+i,k,h} U_{t+i,\tilde{k},h}(x) + \text{hol}(x - t_{i+1}) \quad (1 \leq k \leq \ell_i, 1 \leq \tilde{h} \leq \ell_{i+1,k}) \tag{7.11}
\]

for \( x \in \mathcal{S}_i^+ \cup \mathcal{S}_{i+1}^- \) are given by

\[
C_{t+i+1,\tilde{k},h,t+i,k,h} = \frac{e^{\pi \sqrt{-1} \lambda_{i,k}} (t_i - t_{p+1})^{2\lambda_{i,k} - \rho_1 - \rho_2}}{e^{\pi \sqrt{-1} \lambda_{i+1,k}} (t_{i+1} - t_{p+1})^{2\lambda_{i+1,k} - \rho_1 - \rho_2}} C_{t+i,\tilde{k},h,t+i,k,h} \tag{7.12}
\]

(1 \leq k \leq \ell_{i+1}, 1 \leq \tilde{h} \leq \ell_{i+1,k}, 1 \leq k \leq \ell_i, 1 \leq \tilde{h} \leq \ell_{i,k}).

Besides, for \( 2 \leq i \leq p \) the coefficients \( C_{t+i,\tilde{k},h,t+i,k,h} \) in the connection formula

\[
U_{t+i,k,h}(x) = \sum_{k=1}^{r_{i-1}} \sum_{\tilde{h}=1}^{\ell_{i-1,k}} C_{t+i,\tilde{k},h,t+i,k,h} U_{t+i,\tilde{k},h}(x) + \text{hol}(x - t_{i-1}) \quad (1 \leq k \leq \ell_i, 1 \leq \tilde{h} \leq \ell_{i-1,k}) \tag{7.13}
\]

for \( x \in \mathcal{S}_i^- \cup \mathcal{S}_{i-1}^+ \) are given by

\[
C_{t+i-1,\tilde{k},h,t+i,k,h} = \frac{e^{\pi \sqrt{-1} \lambda_{i,k}} (t_i - t_{p+1})^{2\lambda_{i,k} - \rho_1 - \rho_2}}{e^{\pi \sqrt{-1} \lambda_{i-1,k}} (t_{i-1} - t_{p+1})^{2\lambda_{i-1,k} - \rho_1 - \rho_2}} C_{t+i,\tilde{k},h,t+i,k,h} \tag{7.14}
\]

(1 \leq k \leq \ell_{i-1}, 1 \leq \tilde{h} \leq \ell_{i-1,k}, 1 \leq k \leq \ell_i, 1 \leq \tilde{h} \leq \ell_{i,k}).

**Proof.** Substituting (7.3) and the same expression with \( i \) replaced by \( i + 1 \) into (6.12) \( \nu_1 = \nu_2 = \rho_2 \), we obtain (7.11) with (7.12). Similarly, substituting (7.3) and the same expression with \( i \) replaced by \( i - 1 \) into (6.13) \( \nu_1 = \nu_2 = \rho_2 \), we obtain (7.13) with (7.14).
Theorem 7.6. For $1 \leq i \leq p$ the coefficients $C_{t_{p+1}, \tilde{k}, h, t_i, k, h}^\pm$ in the connection formula

$$U_{t_i, k, h}(x) = \sum_{k=1}^{m_k} C_{t_{p+1}, \tilde{k}, h, t_i, k, h}^\pm U_{t_{p+1}, \tilde{k}, h}(x) + \text{hol}(x - t_{p+1}) \quad (1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i, k}) \tag{7.15}$$

for $x \in S_i^\pm$ are given by

$$C_{t_{p+1}, \tilde{k}, h, t_i, k, h}^\pm = e^{\pm \sqrt{-1} \lambda_i, k (t_i - t_{p+1})} \lambda_i, k - \rho_1 - \rho_2 \Gamma(\mu_\tilde{k} + 1) \Gamma(\mu_\tilde{k} - \rho_1 - \rho_2) \frac{\Gamma(\mu_i - \rho_1) \Gamma(\mu_i - \rho_2)}{\Gamma(\mu_\tilde{k} - \rho_1) \Gamma(\mu_\tilde{k} - \rho_2)} C_{\infty, \tilde{k}, h, t_i, k, h} \tag{7.16}$$

for $1 \leq \tilde{k} \leq q_i, \ 1 \leq \tilde{h} \leq m_{\tilde{k}}, \ 1 \leq k \leq r_i$, $1 \leq h \leq \ell_{i, k}$, where the double-signs correspond.

Proof. Substituting (7.3) and (7.6) into (6.15) $\nu_1 = \nu_2 = \rho_2$ for $x \in S_i^+$ and (6.16) $\nu_1 = \nu_2 = \rho_2$ for $x \in S_i^-$, we obtain (7.15) with (7.16).

Remark 7.1. The quantities $\gamma_{\tilde{h}, t_i', k, h}(-\rho_{\tilde{h}'}) - 1)$ and $\gamma_{\tilde{h}, \infty, k, h}(-\rho_{\tilde{h}'}) - 1)$ do not depend on $\eta_0$, since $u(\sigma) = ((\sigma + \eta_0)I_n - T') w(\rho; \sigma + \eta_0)$ satisfies the system

$$\frac{du}{d\sigma} = ((\rho + 1)I_n + A)(\sigma I_n - (T - t_{p+1}I_n)^{-1}) u,$$

which does not depend on $\rho$, and $(\eta_0 I_n - T') w(\rho; \eta_0) = u(0)$. The coefficients $c_{t_i', k, h, t_i', k, h}$ and $c_{\infty, \tilde{k}, h, t_i', k, h}$ do not depend on $\eta_0$ also.

7.2 Reducible case (i)

Next, we consider the connection formulas for the system (3.7). We assume the conditions (3.4), (3.5), and (4.3). Thanks to Proposition 5.9 we obtain the following results.

Proposition 7.7. For $1 \leq i \leq p$ we have

$$W_{t_i', k, h}^{(S_i^\pm, \eta_0)}(\rho_1, \mu_\eta; \eta_0 + \frac{1}{x - t_{p+1}}) = \frac{\Gamma(\rho_1 + 1) \Gamma(-\mu_\eta)}{\Gamma(\rho_1 - \mu_\eta + 1)} \sum_{h=1}^{m_{\eta}} \gamma_{\tilde{h}, t_i', k, h}(-\rho_1 - 1) \left( U'_{\infty, \mu_\eta, h}(x) \right)_{0_{m_\eta}} \tag{7.17}$$

for $1 \leq k \leq r_i$, $1 \leq h \leq \ell_{i, k}$.

$$W_{t_i', k, h}^{(S_i^\pm, \eta_0)}(\mu_\eta, \rho_1; \eta_0 + \frac{1}{x - t_{p+1}}) = \frac{\Gamma(\mu_\eta + 1) \Gamma(-\rho_1)}{\Gamma(\mu_\eta - \rho_1 + 1)} \sum_{h=1}^{m_{\eta}} \gamma_{\tilde{h}, t_i', k, h}(-\mu_\eta - 1) \left( U'_{\infty, 1, h}(x) \right)_{0_{m_\eta}} \tag{7.18}$$

for $1 \leq k \leq r_i$, $1 \leq h \leq \ell_{i, k}$.

for $x \in S_i^\pm$. For $x \in S_i^\pm$. For $x \in S_i^\pm$. For $x \in S_i^\pm$. For $x \in S_i^\pm$. 39
Proposition 7.8. For $1 \leq i \leq p$ we have
\[
V_{t',k,h}^{(S_i^\pm;i')} (\rho_1, \mu_q; \eta_0 + \frac{1}{x - t_{p+1}}) = e^{-\pi \sqrt{-1} \lambda_i, k} (t_i - t_{p+1})^{\rho_1 + \mu_q - 2 \lambda_i, k} \left( \begin{pmatrix} U'_{t_i,k,h}(x) \\ 0_{m_q} \end{pmatrix} \right) \quad (1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i,k})
\]
for $x \in S_i^\pm$.

Proposition 7.9. We have
\[
V_{\infty,k,h}^{(S_i^\pm;\infty)} (\rho_1, \mu_q; \eta_0 + \frac{1}{x - t_{p+1}}) = \left( \begin{pmatrix} U'_{t_{p+1},k,h}(x) \\ 0_{m_q} \end{pmatrix} \right) \quad (1 \leq k \leq q - 1, \ 1 \leq h \leq m_k)
\]
for $x \in S_i^\mp$.

Theorem 7.10. For $1 \leq i \leq p$ the coefficients $C'_{\infty,1,h;t_i,k,h}$ and $C'_{\infty,\mu_q,h;t_i,k,h}$ in the connection formula
\[
U'_{t_i,k,h}(x) = \sum_{h=1}^{n-m_q} C'_{\infty,1,h;t_i,k,h} U'_{\infty,1,h}(x) + \sum_{h=1}^{n} C'_{\infty,\mu_q,h;t_i,k,h} U'_{\infty,\mu_q,h}(x) \quad (1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i,k})
\]
for $x \in S_i^\pm$ are given by
\[
C'_{\infty,1,h;t_i,k,h} = e^{\pi \sqrt{-1} \lambda_{i,k} - \mu_q} (t_i - t_{p+1})^{2 \lambda_{i,k} - \rho_1 - \mu_q \Gamma(\rho_1 - \mu_q)\Gamma(\lambda_{i,k} + 1)} \frac{\Gamma(\rho_1 - \mu_q)\Gamma(\lambda_{i,k} + 1)}{\Gamma(\rho_1 + 1)\Gamma(\lambda_{i,k} - \mu_q)} \gamma_{h;t_i,k,h} (-\mu_q - 1)
\]
\[
(1 \leq h \leq n - m_q, \ 1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i,k}),
\]
\[
C'_{\infty,\mu_q,h;t_i,k,h} = e^{\pi \sqrt{-1} \lambda_{i,k} - \mu_q} (t_i - t_{p+1})^{2 \lambda_{i,k} - \rho_1 - \mu_q \Gamma(\rho_1 - \mu_q)\Gamma(\lambda_{i,k} + 1)} \frac{\Gamma(\rho_1 - \mu_q)\Gamma(\lambda_{i,k} + 1)}{\Gamma(\mu_q + 1)\Gamma(\lambda_{i,k} - \mu_q)} \gamma_{h;t_i,k,h} (-\rho_1 - 1)
\]
\[
(1 \leq h \leq n, \ 1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i,k}).
\]
Besides, the coefficients $C'_{\infty,1,h;1,t_p+1,k,h}$ and $C'_{\infty,\mu_q,h;1,t_p+1,k,h}$ in the connection formula
\[
U'_{t_{p+1},k,h}(x) = \sum_{h=1}^{n-m_q} C'_{\infty,1,h;1,t_p+1,k,h} U'_{\infty,1,h}(x) + \sum_{h=1}^{n} C'_{\infty,\mu_q,h;1,t_p+1,k,h} U'_{\infty,\mu_q,h}(x) \quad (1 \leq k \leq q - 1, \ 1 \leq h \leq m_k)
\]
for $x \in S_i^\pm$ are given by
\[
C'_{\infty,1,h;1,t_p+1,k,h} = \frac{\Gamma(\rho_1 - \mu_q)\Gamma(\rho_1 + \mu_q - \mu_k + 1)}{\Gamma(\rho_1 + 1)\Gamma(\rho_1 - \mu_k)} \gamma_{h;\infty,k,h} (-\mu_q - 1)
\]
\[
(1 \leq h \leq n - m_q, \ 1 \leq k \leq q - 1, \ 1 \leq h \leq m_k),
\]
\[
C'_{\infty,\mu_q,h;1,t_p+1,k,h} = \frac{\Gamma(\rho_1 - \mu_q)\Gamma(\rho_1 + \mu_q - \mu_k + 1)}{\Gamma(\mu_q + 1)\Gamma(\mu_q - \mu_k + 1)} \gamma_{h;\infty,k,h} (-\rho_1 - 1)
\]
\[
(1 \leq h \leq n, \ 1 \leq k \leq q - 1, \ 1 \leq h \leq m_k).
\]

Theorem 7.11. For $1 \leq i \leq p - 1$ the coefficients $C'_{t_{i+1},k,h;1,k,h}$ in the connection formula
\[
U'_{t_i,k,h}(x) = \sum_{k=1}^{r_i} \sum_{h=1}^{\ell_{i,k}} C_{t_{i+1},k,h;t_i,k,h} U'_{t_{i+1},k,h}(x) + \text{hol}(x - t_{i+1}) \quad (1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i,k})
\]
for $x \in S_i^+ \cup S_{i+1}$ are given by
\[
C'_{t_{i+1},k,h;t_i,k,h} = \frac{e^{\pi \sqrt{-1} \lambda_{i,k}} (t_i - t_{p+1})^{2 \lambda_{i,k} - \rho_1 - \mu_q}}{e^{\pi \sqrt{-1} \lambda_{i,k}} (t_{i+1} - t_{p+1})^{2 \lambda_{i,k} - \rho_1 - \mu_q}} C'_{t_{i+1},k,h;t_{i+1},k,h}
\]
\[
(1 \leq k \leq r_{i+1}, \ 1 \leq h \leq \ell_{i+1,k}, \ 1 \leq k \leq r_i, \ 1 \leq h \leq \ell_{i,k}).
\]
Besides, for $2 \leq i \leq p$ the coefficients $C'_{\ell_{i-1},k,h;\ell_{i},k,h}$ in the connection formula

$$U'_{t_{i-1},k,h}(x) = \sum_{k=1}^{r_{i-1}} \sum_{h=1}^{\ell_{i,k}} C'_{t_{i-1},k,h;\ell_{i},k,h} U'_{t_{i},k,h}(x) + \text{hol}(x - t_{i-1}) \quad (1 \leq k \leq r_{i}, \ 1 \leq h \leq \ell_{i,k})$$

for $x \in S^+_i \cup S^+_{i-1}$ are given by

$$C'_{t_{i-1},k,h;\ell_{i},k,h} = \frac{e^{\pi \sqrt{-1} t_{i-1} \mu} (t_{i-1} - t_{p+1})^{2\lambda_{i-1,k} - \mu}}{e^{\pi \sqrt{-1} t_{i-1} \mu} (t_{i-1} - t_{p+1})^{2\lambda_{i-1,k} - \mu} c_{t_{i-1},k,h'};h} \left(1 \leq k \leq r_{i-1}, \ 1 \leq h \leq \ell_{i-1,k}, \ 1 \leq k \leq r_{i}, \ 1 \leq h \leq \ell_{i,k}\right).$$

**Theorem 7.12.** For $1 \leq i \leq p$ the coefficients $C'^{\pm}_{t_{p+1},k,h;\ell_{i},k,h}$ in the connection formula

$$U'_{t_{i-1},k,h}(x) = \sum_{k=1}^{q-1} \sum_{h=1}^{m_{i,k}} C'^{\pm}_{t_{p+1},k,h;\ell_{i},k,h} U'_{t_{p+1},k,h}(x) + \text{hol}(x - t_{p+1}) \quad (1 \leq k \leq r_{i}, \ 1 \leq h \leq \ell_{i,k})$$

for $x \in S^+_i$ are given by

$$C'^{\pm}_{t_{p+1},k,h;\ell_{i},k,h} = e^{\pm \pi \sqrt{-1} t_{p+1} \mu} (t_{p+1} - t_{p+1})^{2\lambda_{i-1,k} - \mu} \frac{\Gamma(\mu_{i,k} + 1)\Gamma(\mu_{i,k} - \mu_{i,k} - q)\Gamma(\mu_{i,k} - \mu_{i,k} + 1)\Gamma(\mu_{i,k} - \mu_{i,k} - q)}{\Gamma(\mu_{i,k} - \mu_{i,k})\Gamma(\mu_{i,k} - \mu_{i,k} - q)} c_{\infty,k,h';h} \left(1 \leq k \leq q - 1, \ 1 \leq h \leq m_{i,k}, \ 1 \leq k \leq r_{i}, \ 1 \leq h \leq \ell_{i,k}\right),$$

where the double-signs correspond.

### 7.3 Reducible case (ii)

Finally, we consider the connection formulas for the system (8). We assume the conditions (3.11–3.14) and (4.3). We use the notation of an index set

$$\Lambda = \{1, \ldots, n\} \setminus \{m_{1} + \cdots + m_{1-1} + 1, \ldots, m_{1} + \cdots + m_{l}\}$$

for $l = q - 1, q$.

**Proposition 7.13.** For $1 \leq i \leq p$ we have

$$W_{\xi_{i},k,h}^{(S^+_i)}(\mu_{q-1}, \mu_{q}; \eta_{0} + \frac{1}{x - t_{p+1}}) \frac{\Gamma(\mu_{q-1} + 1)\Gamma(-\mu_{q}}{\Gamma(\mu_{q-1} - \mu_{q} + 1)} \sum_{\eta_{0} \in \Lambda} \gamma_{h;i',k,h}(-\mu_{q-1} - 1) \left(U''_{t_{i},k,h}(x) \right)_{0_{m_{q-1} + m_{q}}} \left(1 \leq k \leq r_{i}, \ 1 \leq h \leq \ell_{i,k}\right),$$

$$W_{\xi_{i},k,h}^{(S^+_i)}(\mu_{q-1}, \mu_{q}; \eta_{0} + \frac{1}{x - t_{p+1}}) \frac{\Gamma(\mu_{q-1} + 1)\Gamma(-\mu_{q}}{\Gamma(\mu_{q-1} - \mu_{q} + 1)} \sum_{\eta_{0} \in \Lambda} \gamma_{h;i',k,h}(-\mu_{q-1} - 1) \left(U''_{t_{i},k,h}(x) \right)_{0_{m_{q-1} + m_{q}}} \left(1 \leq k \leq r_{i}, \ 1 \leq h \leq \ell_{i,k}\right),$$

for $x \in S^+_i$, and

$$W_{\infty,k,h}^{(S^+_i)}(\mu_{q-1}, \mu_{q}; \eta_{0} + \frac{1}{x - t_{p+1}}) \frac{\Gamma(\mu_{q-1} + 1)\Gamma(-\mu_{q}}{\Gamma(\mu_{q-1} - \mu_{q} + 1)} \sum_{\eta_{0} \in \Lambda} \gamma_{h;i',k,h}(-\mu_{q-1} - 1) \left(U''_{t_{i},k,h}(x) \right)_{0_{m_{q-1} + m_{q}}} \left(1 \leq k \neq q - 1 \leq q, \ 1 \leq h \leq m_{k}\right),$$

$$W_{\infty,k,h}^{(S^+_i)}(\mu_{q-1}, \mu_{q}; \eta_{0} + \frac{1}{x - t_{p+1}}) \frac{\Gamma(\mu_{q-1} + 1)\Gamma(-\mu_{q}}{\Gamma(\mu_{q-1} - \mu_{q} + 1)} \sum_{\eta_{0} \in \Lambda} \gamma_{h;i',k,h}(-\mu_{q-1} - 1) \left(U''_{t_{i},k,h}(x) \right)_{0_{m_{q-1} + m_{q}}} \left(1 \leq k \leq q - 1, \ 1 \leq h \leq m_{k}\right)$$

for $l = q - 1, q$. 
for \( x \in \mathcal{S}^i \).

**Proposition 7.14.** We have
\[
V_{t_i,k,h}^{(\mathcal{S}^\pm,t_i)}(\mu_q-1,\mu_q;\eta_0 + \frac{1}{x-t_{p+1}}) = e^{-\pi \sqrt{-1} \lambda_{i,k}(t_i - t_{p+1}) (\mu_q-1 + \mu_q - 2\lambda_{i,k})} \left( \frac{U''_{t_i,k,h}(x)}{0_{m_{q-1}+m_q}} \right) (1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k})
\]
for \( x \in \mathcal{S}^\pm \).

**Proposition 7.15.** We have
\[
V_{\infty,k,h}^{(\mathcal{S}^\pm,\infty)}(\mu_q-1,\mu_q;\eta_0 + \frac{1}{x-t_{p+1}}) = \left( \frac{U''_{t_{p+1},k,h}(x)}{0_{m_{q-1}+m_q}} \right) (1 \leq k \leq q-2, 1 \leq h \leq m_k)
\]
for \( x \in \mathcal{S}^\pm \).

**Theorem 7.16.** For \( 1 \leq i \leq p \) the coefficients \( C''_{\infty,\mu_q-1,\mu_q,h},k,h \) and \( C''_{\infty,\mu_q,\mu_q,h},k,h \) in the connection formula
\[
U''_{t_i,k,h}(x) = \sum_{h \in \Lambda_q} C''_{\infty,\mu_q-1,\mu_q,h},k,h U''_{\infty,\mu_q-1,h}(x) + \sum_{h \in \Lambda_{q-1}} C''_{\infty,\mu_q,\mu_q,h},k,h U''_{\infty,\mu_q,h}(x) (1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k})
\]
for \( x \in \mathcal{S}^i \) are given by
\[
C''_{\infty,\mu_q-1,\mu_q,h},k,h = e^{\pi \sqrt{-1} (\lambda_{i,k} - \mu_q)(t_i - t_{p+1})} (\mu_q-1 - \mu_q) \Gamma(\lambda_{i,k} + 1) \Gamma(\mu_q-1 + 1) \gamma_{h,t_i,k,h} (-\mu_q - 1)
\]
\((h \in \Lambda_q, 1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k})\),
\[
C''_{\infty,\mu_q,\mu_q,h},k,h = e^{\pi \sqrt{-1} (\lambda_{i,k} - \mu_q-1)(t_i - t_{p+1})} (\mu_q - \mu_q-1) \Gamma(\lambda_{i,k} + 1) \Gamma(\mu_q + 1) \gamma_{h,t_i,k,h} (-\mu_q - 1)
\]
\((h \in \Lambda_{q-1}, 1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k})\).

Besides, the coefficients \( C''_{\infty,\mu_q-1,\mu_q,h},k,h \) and \( C''_{\infty,\mu_q,\mu_q,h},k,h \) in the connection formula
\[
U''_{t_{p+1},k,h}(x) = \sum_{h \in \Lambda_q} C''_{\infty,\mu_q-1,\mu_q,h},k,h U''_{\infty,\mu_q-1,h}(x) + \sum_{h \in \Lambda_{q-1}} C''_{\infty,\mu_q,\mu_q,h},k,h U''_{\infty,\mu_q,h}(x) (1 \leq k \leq q-2, 1 \leq h \leq m_k)
\]
for \( x \in \mathcal{S}^i \) are given by
\[
C''_{\infty,\mu_q-1,\mu_q,h},k,h = \frac{\Gamma(\mu_q-1 - \mu_q) \Gamma(\mu_q-1 + \mu_q - \mu_k + 1)}{\Gamma(\mu_q-1 + 1) \Gamma(\mu_q-1 - \mu_k + 1)} \gamma_{h,\infty,k,h} (-\mu_q - 1)
\]
\((h \in \Lambda_q, 1 \leq k \leq q-2, 1 \leq h \leq m_k)\),
\[
C''_{\infty,\mu_q,\mu_q,h},k,h = \frac{\Gamma(\mu_q - \mu_q-1) \Gamma(\mu_q-1 + \mu_q - \mu_k + 1)}{\Gamma(\mu_q + 1) \Gamma(\mu_q - \mu_k + 1)} \gamma_{h,\infty,k,h} (-\mu_q - 1)
\]
\((h \in \Lambda_{q-1}, 1 \leq k \leq q-2, 1 \leq h \leq m_k)\).

**Theorem 7.17.** For \( 1 \leq i \leq p-1 \) the coefficients \( C''_{t_{i+1},k,h},t_{i+1},k,h \) in the connection formula
\[
U''_{t_i,k,h}(x) = \sum_{k=1}^{r_{i+1}} \sum_{h=1}^{\ell_{i+1,k}} C''_{t_{i+1},k,h},t_{i+1},k,h U''_{t_{i+1},k,h}(x) + \text{hol}(x - t_{i+1}) (1 \leq k \leq r_i, 1 \leq h \leq \ell_{i,k})
\]
for \( x \in S_i^+ \cup S_{i+1} \) are given by

\[
C_{t_{i+1}, \tilde{k}, \tilde{h}; t, k, h}'' = \frac{e^{\sqrt{-1} \pi \lambda_{i, k}} (t_i - t_{p+1})^{2 \lambda_{i, k} - \mu_{q-1} - \mu_q}}{e^{\sqrt{-1} \pi \lambda_{i+1, \tilde{k}} (t_{i+1} - t_{p+1})^{2 \lambda_{i+1, \tilde{k}} - \mu_{q-1} - \mu_q}} \tilde{c}_{t_{i+1}, \tilde{k}, \tilde{h}; t, k, h}
\]

\[
(1 \leq \tilde{k} \leq t_{i+1}, 1 \leq \tilde{h} \leq t_{i+1, \tilde{k}}, 1 \leq k \leq r_i, 1 \leq h \leq \ell_{i, k}).
\]

Besides, for \( 2 \leq i \leq p \) the coefficients \( C_{t_{i-1}, \tilde{k}, \tilde{h}; t, k, h}'' \) in the connection formula

\[
U_{t_{i-1}, k, h}(x) = \sum_{k=1}^{r_{i-1, k}} \sum_{h=1}^{\ell_{i-1, k}} C_{t_{i-1}, \tilde{k}, \tilde{h}; t, k, h}'' U_{t_{i-1}, \tilde{k}, \tilde{h}; t, k, h}'(x) + \text{hol}(x - t_{i-1}) \quad (1 \leq k \leq r_i, 1 \leq h \leq \ell_{i, k})
\]

for \( x \in S_i^- \cup S_{i-1}^+ \) are given by

\[
C_{t_{i-1}, \tilde{k}, \tilde{h}; t, k, h}'' = \frac{e^{\sqrt{-1} \pi \lambda_{i, k}} (t_{i-1} - t_{p+1})^{2 \lambda_{i, k} - \mu_{q-1} - \mu_q}}{e^{\sqrt{-1} \pi \lambda_{i-1, \tilde{k}} (t_{i-1} - t_{p+1})^{2 \lambda_{i-1, \tilde{k}} - \mu_{q-1} - \mu_q}} \tilde{c}_{t_{i-1}, \tilde{k}, \tilde{h}; t, k, h}
\]

\[
(1 \leq \tilde{k} \leq r_{i-1}, 1 \leq \tilde{h} \leq t_{i-1, \tilde{k}}, 1 \leq k \leq r_i, 1 \leq h \leq \ell_{i, k}).
\]

**Theorem 7.18.** For \( 1 \leq i \leq p \) the coefficients \( C_{t_{p+1}, \tilde{k}, \tilde{h}; t, k, h}'' \) in the connection formula

\[
U_{t_{p+1}, k, h}(x) = \sum_{k=1}^{q-2} \sum_{h=1}^{m_k} C_{t_{p+1}, \tilde{k}, \tilde{h}; t, k, h}'' U_{t_{p+1}, \tilde{k}, \tilde{h}; t, k, h}'(x) + \text{hol}(x - t_{p+1}) \quad (1 \leq k \leq r_i, 1 \leq h \leq \ell_{i, k})
\]

for \( x \in S_i^+ \) are given by

\[
C_{t_{p+1}, \tilde{k}, \tilde{h}; t, k, h}'' = e^{\pi \sqrt{-1} \lambda_{i, k}} (t_i - t_{p+1})^{2 \lambda_{i, k} - \mu_{q-1} - \mu_q} C_{t_{p+1}, \tilde{k}, \tilde{h}; t, k, h}
\]

\[
(1 \leq \tilde{k} \leq q - 2, 1 \leq \tilde{h} \leq m_{\tilde{k}}, 1 \leq k \leq r_i, 1 \leq h \leq \ell_{i, k}),
\]

where the double-signs correspond.

**References**

[1] W. Balser, W. B. Jurkat and D. A. Lutz, On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities, I, *SIAM J. Math. Anal.*, 12 (1981), 691–721.

[2] G. D. Birkhoff, Singular points of ordinary linear differential equations, *Trans. Amer. Math. Soc.*, 10 (1909), 436–470.

[3] G. Darboux, “Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal”, tome II, Gauthier-Villars, Paris, 1915.

[4] Y. Haraoka, Integral representations of solutions of differential equations free from accessory parameters, *Adv. Math.*, 169 (2002), 187–240.

[5] E. Hille, “Ordinary Differential Equations in the Complex Domain”, John Wiley & Sons, Inc., New York, 1976.

[6] E. L. Ince, “Ordinary Differential Equations”, Dover, New York, 1956.
[7] K. Iwasaki, H. Kimura, S. Shimomura and M. Yoshida, “From Gauss to Painlevé: A Modern Theory of Special Functions”, Vieweg, Wiesbaden, 1991.

[8] M. Kohno, “Global Analysis in Linear Differential Equations”, Kluwer Academic Publishers, Dordrecht, 1999.

[9] W. Miller, Jr., Symmetries of differential equations. The hypergeometric and Euler-Darboux equations, *SIAM J. Math. Anal.*, 4 (1973), 314–328.

[10] K. Okubo, Connection problems for systems of linear differential equations, in “Japan-United States Seminar on Ordinary Differential Equations (Kyoto, 1971)”, 238–248. Lecture Notes in Math., 243, Springer, Berlin, 1971.

[11] K. Okubo, “On the group of Fuchsian equations”, Seminar Reports of Tokyo Metropolitan University, Tokyo, 1987.

[12] R. Schäfke, Über das globale analytische Verhalten der Lösungen der über die Laplacetransformation zusammenhängenden Differentialgleichungen \( tx' = (A + tB)x \) und \( (s - B)v' = (\rho - A)v \), Doctoral Dissertation, University of Essen, Germany, 1979.

[13] R. Schäfke, Über das globale Verhalten der Normallösungen von \( x'(t) = (B + t^{-1}A)x(t) \) und zweier Arten von assoziierten Funktionen, *Math. Nachr.*, 121 (1985), 123–145.

[14] N. Takayama, Propagation of singularities of solutions of the Euler-Darboux equation and a global structure of the space of holonomic solutions I, *Funkcial. Ekvac.*, 35 (1992), 343–403; II, *ibid.*, 36 (1993), 187–234.

[15] T. Yokoyama, A system of total differential equations of two variables and its monodromy group, *Funk. Ekvac.*, 35 (1992), 65–93.

[16] T. Yokoyama, Construction of systems of differential equations of Okubo normal form with rigid monodromy, *Math. Nachr.*, 279 (2006), 327–348.