IDEMPOTENTS OF LARGE NORM AND
HOMOMORPHISMS OF FOURIER ALGEBRAS

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Abstract. We provide necessary and sufficient conditions for the existence of idempotents of arbitrarily large norms in the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of a locally compact group $G$. We prove that the existence of idempotents of arbitrarily large norm in $B(G)$ implies the existence of homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$ for every locally compact group $H$. A partial converse is also obtained: the existence of homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$ for some amenable locally compact group $H$ implies the existence of idempotents of arbitrarily large norm in $B(G)$.

1. Introduction

Let $G$ be a locally compact group. The Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ of the group $G$ were introduced by Eymard [7]. The Fourier-Stieltjes algebra of $G$ consists of the matrix coefficients $(\pi(\cdot)\xi, \eta)$ of all continuous unitary representations $\pi$ of $G$, while the Fourier algebra of $G$ consists of the matrix coefficients of the left regular representation of $G$. If $G$ is abelian, $A(G)$ and $B(G)$ can be identified, via the Fourier transform, with $L^1(\hat{G})$ and the measure algebra $M(\hat{G})$ of the dual group $\hat{G}$, respectively. In [2] Cohen characterized the homomorphisms from $A(H)$ into $B(G)$ in terms of piecewise affine maps when $H, G$ are locally compact abelian groups. To obtain his result he proved a characterization of idempotents in $B(G)$ [3]. Host in [10] extended the characterization of idempotents in $B(G)$ to general locally compact groups.

Homomorphisms of Fourier algebras for locally compact groups were studied by Ilie [11] and Ilie and Spronk [12]. They characterized completely bounded homomorphisms from $A(H)$ into $B(G)$ for locally compact groups $H, G$ with $H$ amenable in terms of piecewise affine maps [12] (see also [5]).

Let $G, H$ be locally compact groups. Let $K$ be a subgroup of $G$ and $C$ a left coset of $K$ in $G$. A map $\alpha : C \rightarrow H$ is called affine if there exists a continuous homomorphism $\theta : \hat{K} \rightarrow H$ and elements $s_0 \in H, t_0 \in G$ such that $C = t_0^{-1}K$ and

$$\alpha(t) = s_0\theta(t_0 t).$$

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for all \( t \in C \).

A map \( \alpha : Y \to H \) is called **piecewise affine** if \( Y \) can be written as a disjoint union \( Y = \bigcup_{i=1}^{m} Y_i \), where each \( Y_i \) belongs to the open coset ring \( \Omega_0(G) \), such that each restriction \( \alpha|_{Y_i} \) extends to an affine map \( \alpha_i : C_i \to H \) defined on an open coset \( C_i \supseteq Y_i \).

Recall that the open coset ring \( \Omega_0(G) \) is the ring generated by the open cosets of the group \( G \).

Let \( Y \) be an open and closed subset of \( G \) and \( \alpha \) a piecewise affine map \( Y \to H \). Define \( \rho : A(H) \to B(G) \) by

\[
(1.1) \quad \rho(u)(t) = \begin{cases} u \circ \alpha(t), & t \in Y \\ 0, & t \in G \setminus Y \end{cases}
\]

Ilie and Spronk proved in [12, Proposition 3.1 and Theorem 3.7] that \( \rho \) is a completely bounded homomorphism and that, if the group \( H \) is amenable, every completely bounded homomorphism \( \rho : A(H) \to B(G) \) is of this form.

**Notation** The symbol \( \chi_F \) denotes the characteristic function of a set \( F \).

To motivate our work, consider the following simple example of completely bounded homomorphisms from \( A(\mathbb{Z}) \) to \( A(\mathbb{Z}) \) of arbitrarily large norm:

For \( F = \{-k, \ldots, k\} \subseteq \mathbb{Z} \) we define the map \( \rho_F : A(\mathbb{Z}) \to A(\mathbb{Z}) \) by

\[
\rho_F(u)(j) = \begin{cases} u(0), & j \in F \\ 0, & j \notin F \end{cases}
\]

Then it follows from [11] that \( \rho_F \) is a completely bounded homomorphism.

Consider the function \( u_0 : \mathbb{Z} \to \mathbb{Z} \) given by \( u_0(i) = \delta_{i,0} \). Clearly \( u_0 \in A(\mathbb{Z}) \) and \( \|u_0\|_{A(\mathbb{Z})} \leq 1 \).

Since \( \rho_F(u_0) = \chi_F \), its Fourier transform is \( \hat{\rho_F(u_0)}(z) = \hat{\chi}_F(z) = \sum_{i \in F} z^{-i} \) and so

\[
\|\rho_F\| \geq \|\rho_F(u_0)\|_{A(\mathbb{Z})} = \|\chi_F\|_{A(\mathbb{Z})} = \|\hat{\chi}_F\|_{L^1(\mathbb{T})}
\]

\[
= \int_{\mathbb{T}} \left| \sum_{i \in F} z^{-i} \right| dz = \int_{0}^{2\pi} \left| \frac{D_k(x)}{2\pi} \right| dx
\]

where \( D_k \) is the Dirichlet kernel, and it is known that the \( L^1 \) norm of \( D_k \) grows like \( \log k \).

In the above example we used the existence of idempotents in \( A(\mathbb{Z}) \) of large norm to construct homomorphisms of \( A(\mathbb{Z}) \) with large norm.

In this work we study the following questions: a) for which locally compact groups \( G \) there exist idempotents of arbitrarily large norms in the Fourier algebra \( A(G) \) (resp. in the Fourier-Stieltjes algebra \( B(G) \)) and b) how is the existence of idempotents of arbitrarily large norm related to the existence of homomorphisms of arbitrarily large norm between Fourier algebras.

We provide necessary and sufficient conditions for the existence of idempotents of arbitrarily large norms in the Fourier algebra \( A(G) \) (resp. in the Fourier-Stieltjes algebra \( B(G) \)) of a locally compact group \( G \). To prove our
results we reduce the problem to the case where \( G \) is totally disconnected. Then we first consider the case where \( G \) is a discrete group in Proposition 2.2, and for the general case we use a result of Leiderman, Morris and Tkachenko for totally disconnected groups \([14]\). We also prove that the existence of idempotents of arbitrarily large norm in \( B(G) \) implies the existence of homomorphisms of arbitrarily large norm from \( A(H) \) into \( B(G) \) for every locally compact group \( H \). Finally, we obtain a partial converse: the existence of homomorphisms of arbitrarily large norm from \( A(H) \) into \( B(G) \) for some amenable locally compact group \( H \) implies the existence of idempotents of arbitrarily large norm in \( B(G) \).

2. Norms of Idempotents

Let \( G \) be a locally compact group. A function \( u : G \to \mathbb{C} \) is called a multiplier of \( A(G) \) if \( uA(G) \subseteq A(G) \). In this case the map \( m_u : A(G) \to A(G) : v \mapsto uv \) is bounded. In case it is completely bounded we call \( u \) a completely bounded multiplier. We denote by \( M_{cb}A(G) \) the algebra of completely bounded multipliers of \( A(G) \).

The space \( M_{cb}A(G) \) inherits the operator space structure from the space \( CB(A(G)) \) of completely bounded maps \( A(G) \to A(G) \). We write \( \|u\|_{CB(A(G))} \), and simply \( \|u\|_{cb} \) when there is no risk of confusion, for the completely bounded norm \( \|m_u\|_{CB(A(G))} \) of an element \( u \in M_{cb}A(G) \). Note that \( B(G) \) (and also \( A(G) \)) inherits the operator space structure from \( M_{cb}A(G) \).

It is shown in \([6]\) that the space \( M_{cb}A(G) \) is the dual of the normed space \((L^1(G), \|\cdot\|_{Q(G)})\) where the norm \( \|\cdot\|_{Q(G)} \) is given by

\[
\|f\|_{Q(G)} = \sup \left\{ \left\| \int_G f(s)\phi(s)ds \right\| : \phi \in M_{cb}A(G), \|\phi\|_{cb} \leq 1 \right\} , \quad f \in L^1(G) .
\]

We shall use the following Theorem, combining \([18, \text{Corollary 6.3 (iv)}]\) and \([7, \text{2.26, Corollaire 3 and 3.25, Proposition}]\):

**Theorem 2.1** (Eymard, Spronk). Let \( G \) be a locally compact group and \( H \) a closed, normal subgroup of \( G \). Let \( \pi : G \to G/H \) be the quotient map. The map

\[
j_\pi : M_{cb}(A(G/H)) \to M_{cb}(A(G)) : u \mapsto u \circ \pi
\]

is a complete isometry. Moreover, \( j_\pi(B(G/H)) \subseteq B(G) \); if \( H \) is compact, then \( j_\pi(A(G/H)) \subseteq A(G) \).

**Proposition 2.2.** Let \( G \) be a discrete infinite group. Then

\[
\sup\{\|\chi_F\|_{cb} : F \subseteq G \text{ finite}\} = +\infty.
\]

**Proof.** Assuming that

\[
\sup\{\|\chi_F\|_{cb} : F \subseteq G \text{ finite}\} = M < +\infty ,
\]
we shall prove that $\ell^\infty(G) \subseteq M_{cb}(A(G))$. This means that $G$ is a strong
Leinert set, which, by a result of Pisier [16, Theorem 3.3], implies that $G$
must be finite.

If $v$ is an extreme point of the positive part $\Omega$ of the unit ball of $\ell^\infty(G)$,
then $v(t) \in \{0, 1\}$ for all $t \in G$. Thus for any finite $F \subseteq G$, the function $\chi_F v$
takes values in $\{0, 1\}$ and so $\chi_F v = \chi_{F'}$ for some finite subset $F'$ of $G$. Thus
$$\|\chi_F v\|_{cb} \leq M$$
by our assumption.

Now fix an arbitrary $u \in \Omega$ and a finite subset $F \subseteq G$. By the Krein-
Milman theorem, $u$ is a weak-* limit of a net $(u_i)$ of convex combinations
of extreme points of $\Omega$. By the previous paragraph, each $u_i$ will satisfy
$$\|\chi_F u_i\|_{cb} \leq M.$$

Since $M_{cb}A(G)$ is the dual of $(\ell^1(G), \| \cdot \|_{Q(G)})$, given $\varepsilon > 0$ there exists
$f \in \ell^1(G)$ with $\|f\|_{Q(G)} \leq 1$ such that
$$\|\chi_F u\|_{cb} - \varepsilon < \sum_{t \in G} (\chi_F u f)(t).$$

Now
$$\sum_{t \in G} (\chi_F u f)(t) = \lim_i \sum_{t} (\chi_F u_i f)(t),$$

since $u$ is the weak-* limit of the net $(u_i)$ and so
$$\sum_{t \in G} (\chi_F u f)(t) = \lim_i \sum_{t} (\chi_F u_i f)(t) = \lim_i \langle f, \chi_F u_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $\ell^1(G)$ and $M_{cb}(G)$. But
$$|\langle f, \chi_F u_i \rangle| \leq \|f\|_{Q(G)} \|\chi_F u_i\|_{cb} \leq M$$
for each $i$, and therefore
$$\|\chi_F u\|_{cb} - \varepsilon < \sum_{t \in G} (\chi_F u f)(t) = \lim_i \langle f, \chi_F u_i \rangle \leq M.$$

Thus, for all nonnegative $u$ in the unit ball of $\ell^\infty(G)$ we have
$$\sup_{u \in \Omega} \|\chi_F u\|_{cb} \leq M$$
for every finite subset $F \subseteq G$. In particular, if $u \in c_{oo}(G)$ is nonnegative
then $\frac{u}{\|u\|_\infty} \in \Omega$ and thus
$$\|u\|_{cb} \leq M \|u\|_\infty.$$
We conclude that the norms \( \| \cdot \|_{cb} \) and \( \| \cdot \|_{\infty} \) are equivalent on \( c_{00}(G) \). Thus the identity map \( id : (c_{00}(G), \| \cdot \|_{\infty}) \to (M_{cb}(A(G)), \| \cdot \|_{cb}) \) is continuous. Since \( M_{cb}(A(G)) \) is a dual Banach space we can consider the unique weak* continuous extension of \( id \) to its double dual \( \ell^\infty(G) \) (see e.g. [1, Lemma A.2.2.]) which we denote by \( T \):

\[
T : (\ell^\infty(G), \| \cdot \|_{\infty}) = (c_{00}(G), \| \cdot \|_{\infty})^{**} \to (M_{cb}(A(G)), \| \cdot \|_{cb}).
\]

We claim that \( T \) is the identity. If \( u \in \ell^\infty(G) \); we will show that \( u = Tu \). Indeed, if \( (u_i) \) is a net in \( c_{00}(G) \) such that \( u = \lim_i u_i \) in the weak* topology \( \sigma(\ell^\infty(G), \ell^1(G)) \), then \( (Tu_i) \) converges to \( Tu \) is the weak* topology of \( M_{cb}(A(G)) \), and hence pointwise (since \( (Tu_i(t) = \langle Tu, \delta_t \rangle \) for \( t \in G \)). Thus, for all \( t \in G \),

\[
(Tu)(t) = \lim_i (Tu_i)(t) = \lim_i u_i(t) = u(t)
\]

since \( u = \lim_i u_i \) in the weak* topology \( \sigma(\ell^\infty(G), \ell^1(G)) \) and hence point-wise. This proves our claim.

We have shown that \( \ell^\infty(G) \subseteq M_{cb}(A(G)) \) and thus \( G \) must be finite, as observed above. \( \square \)

Since \( \|u\|_{cb} \leq \|u\|_{B(G)} \) when \( u \in B(G) \) [6, Corollary 1.8], we obtain the following

**Corollary 2.3.** If \( G \) is a discrete infinite group then

\[
\sup_F \{ \| \chi_F \|_{A(G)} : F \subseteq G \text{ finite} \} = +\infty.
\]

**Note** We thank the referee for providing the following alternative argument for Corollary 2.3:

Using the analogous arguments as in the proof of Proposition 2.2, we can show that if \( \sup_F \{ \| \chi_F \|_{A(G)} : F \subseteq G \text{ finite} \} < +\infty \) then \( \ell^\infty(G) = B(G) \).

It follows from this equality that \( G \) must be finite. Indeed if \( \ell^\infty(G) = B(G) \), then \( \ell^1(G) = C^\ast(G) \) with equivalent norms. However, this would imply that \( \ell^1(G) \) is Arens regular, which by [19] shows that \( G \) is finite.

**Theorem 2.4.** Let \( G \) be an infinite totally disconnected group. Then

\[
\sup_F \{ \| \chi_F \|_{cb} : F \subseteq G, \chi_F \in A(G) \} = +\infty.
\]

**Proof.** From the theorem of van Dantzig [4], [9, Theorem II.7.7] there exists a compact open subgroup \( H \subseteq G \).

If \( H \) is finite then \( \{e\} \) is an open subgroup of \( G \) and thus \( G \) is discrete. In this case the conclusion follows from Proposition 2.2.

If \( H \) is infinite, by Theorem 2.6 in [14] there exists a closed normal subgroup \( N \) of \( H \) such that the quotient \( H/N \) is homeomorphic to a countably infinite product of finite groups. We write \( K = H/N \). Clearly \( K \) is compact and separable. We also denote by \( K_d \) the group \( K \) with the discrete topology. The inclusion

\[
l : K_d \to K
\]
is a continuous homomorphism; thus it induces a contractive homomorphism
\[ \rho : A(K) \to B(K_d) : u \to u \circ \iota. \]
Let \( \epsilon > 0 \). By Proposition 2.2 there exists a finite \( F \subseteq K \) such that \( \chi_F \in A(K_d) \) and
\[ \| \chi_F \|_{A(K_d)} > \epsilon. \]
Since \( K \) is a totally disconnected and separable group, there exists a decreasing sequence of compact-open subgroups such that
\[ \bigcap_{n=1}^{\infty} K_n = \{ e \} \quad \text{and hence} \quad \bigcap_{n=1}^{\infty} FK_n = F. \]
Now \( FK_n \) is a finite disjoint union of sets of the form \( x_iK_n \) where \( x_i \in F \) and since each \( K_n \) is a compact open subgroup, \( \chi_{x_iK_n} \) is in \( A(K) \) and has norm 1. Indeed, the constant function 1 on the compact group \( K_n \) belongs to \( A(K_n) \) and has norm 1. It follows from [7, Proposition 3.21 (1)] that \( \chi_{x_iK_n} \) belongs to \( A(K_n) \) and has norm 1 and hence the same holds for its translate \( \chi_{x_iK_n} \). Thus \( \chi_{FK_n} \) is in \( A(K) \) and the sequence \( (\| \chi_{FK_n} \|_{A(K)})_n \) is bounded by the cardinality of \( F \). Since \( \rho \) is bounded, the sequence \( (\| \chi_{FK_n} \|_{B(K_d)})_n \) is also bounded. For all \( f \in \ell^1(K_d) \), since \( (\chi_{FK_n})_n \) converges pointwise to \( \chi_F \), by dominated convergence we have
\[ \lim_n \sum_{t \in K_d} f(t)\chi_{FK_n}(t) = \sum_{t \in K_d} f(t)\chi_F(t). \]
Since \( \ell^1(K_d) \) is dense in the predual \( C^*(K_d) \) of \( B(K_d) \), we obtain
\[ \text{w*-}\lim_n \chi_{FK_n} = \chi_F \]
in the weak* topology of \( B(K_d) \). Therefore \( \sup_n \| \chi_{FK_n} \|_{B(K_d)} > \epsilon \) and hence
\[ \sup_n \| \chi_{FK_n} \|_{A(K)} > \epsilon \]
which implies that there exists \( \chi_{FK_n} \in A(K) \) such that
\[ \| \chi_{FK_n} \|_{A(K)} > \epsilon. \]
This shows that
\[ \sup\{\| \chi_V \|_{A(H/N)} : V \subseteq H/N, \chi_V \in A(H/N)\} = +\infty \]
and since \( H/N \) is compact, it follows that
\[ \sup\{\| \chi_V \|_{cb} : V \subseteq H/N, \chi_V \in A(H/N)\} = +\infty. \]
Let \( \pi : H \to H/N \) be the quotient map. It follows from Theorem 2.1 that if \( F \subseteq H/N \) is such that \( \chi_F \in A(H/N) \), then \( \chi_F \circ \pi = \chi_{\pi^{-1}(F)} \in A(H) \) and
\[ \| \chi_F \|_{CB(A(H/N))} = \| \chi_F \circ \pi \|_{CB(A(H))} = \| \chi_{\pi^{-1}(F)} \|_{CB(A(H))}. \]
We conclude that
\[ \sup\{\| \chi_F \|_{cb} : F \subseteq H, \chi_F \in A(H)\} = +\infty. \]
Let $F \subseteq H$ be such that $\chi_F \in A(H)$. There exists $v \in A(G)$ such that $v|_H = \chi_F$ [7, 3.21(1)]. By [7, 3.21(2)], $\chi_H \in A(G)$ and hence $\chi_F = v\chi_H \in A(G)$.

Since by [18, Corollary 6.3(iii)] the map

$$M_c b A(G) \to M_c b A(H) : u \to u|_H$$

is completely contractive, we obtain

$$\|\chi_F\|_{CB(A(G))} \geq \|\chi_F|_H\|_{CB(A(H))} = \|\chi_F\|_{CB(A(H))}.$$

We conclude that

$$\sup\{\|\chi_F\|_{cb} : F \subseteq G, \chi_F \in A(G)\} = +\infty.$$

□

Note the crucial use of [14] in obtaining a countable family $(K_n)$ of compact open subgroups with $\cap_{n=1}^{\infty} K_n = \{e\}$.

The proof of the above theorem is not constructive. Below we provide a different proof for the case where $G$ is an infinite direct product of finite groups. Ilie and Spronk [12, Theorem 2.1] proved that if $\chi_F$ is an idempotent in $B(G)$, then $\|\chi_F\|_{B(G)} = 1$ if and only if $F$ is a coset of an open subgroup of $G$. Forrest and Runde in [8] and Stan in [17] proved that if the cb-norm of an idempotent $\chi_F \in B(G)$ satisfies $\|\chi_F\|_{cb} < \frac{2}{\sqrt{3}}$ then $F$ is a coset of an open subgroup of $G$. The ‘gap’ $\left[1, \frac{2}{\sqrt{3}}\right)$ was improved by Mudge and Pham in [15] to $\left[1, \frac{1+\sqrt{2}}{2}\right)$.

Proposition 2.5. Let $G$ be an infinite direct product of finite groups. Then

$$\sup\{\|\chi_F\|_{cb} : F \subseteq G, \chi_F \in A(G)\} = +\infty.$$

Proof. Since $G$ is compact, $A(G) = B(G)$ and $\|u\|_{cb} = \|u\|_{A(G)}$ for all $u \in A(G)$ [13, Corollary 5.4.11] and hence it is sufficient to prove the proposition for $\|\cdot\|_{A(G)}$.

Let $G_0$ be a finite group with $|G_0| \geq 3$. Then there exists $A \subseteq G_0$ such that $A$ is not a coset of a subgroup of $G_0$ (for example, take $A$ such that $|A|$ does not divide $|G_0|$). Then, since $A \in \Omega_0(G_0)$, it follows from [15] that

$$\|\chi_A\|_{A(G_0)} \geq \frac{1+\sqrt{2}}{2}.$$

Now consider finite groups $G_1, G_2, \ldots, G_n$ with $|G_i| \geq 3$ for all $i = 1, 2, \ldots, n$ and set $G = \prod_{i=1}^{n} G_i$. Choose $A_i \in G_i$ with $\|\chi_{A_i}\|_{A(G_i)} \geq \frac{1+\sqrt{2}}{2}$ and set $A = A_1 \times A_2 \times \cdots \times A_n$. Since $A(G)$ is isometrically isomorphic to the operator space projective tensor product $A(G_1) \hat{\otimes} A(G_2) \hat{\otimes} \cdots A(G_n)$ [13, Lemma 4.1.2] we obtain

$$\|\chi_A\|_{A(G)} = \|\chi_{A_1} \otimes \chi_{A_2} \otimes \cdots \chi_{A_n}\|_{A(G_1) \hat{\otimes} A(G_2) \hat{\otimes} \cdots \hat{\otimes} A(G_n)} \geq \left(\frac{1+\sqrt{2}}{2}\right)^n.$$

Now let $G = \prod_{i=1}^{\infty} G_i$ be an infinite product of finite groups $G_i$. Without loss of generality we may assume that $|G_i| \geq 3$ for all $i \in \mathbb{N}$ (lumping
and hence equal to \(G\) if and only if it is of the form \(\chi_{A}\).

Together some of the \(G_i\)’s if necessary. Set \(H_n = \prod_{i=n+1}^{\infty} G_i\) and let \(\pi_n\) be the quotient map \(G \to G/H_n\). It follows from Theorem 2.1 that the map \(A(G/H_n) \to A(G) : u \mapsto u \circ \pi_n\) is isometric. Since \(G/H_n \cong \prod_{i=1}^{n} G_i\), we can choose \(A \subseteq G/H_n\) such that \(\|\chi_A\|_{A(G/H_n)} \geq (\frac{1+\sqrt{2}}{2})^n\). Setting \(F = \pi_n^{-1}(A)\), we obtain \(\|\chi_F\|_{cb} = \|\chi_F\|_{A(G)} \geq (\frac{1+\sqrt{2}}{2})^n\) and the conclusion follows.

**Corollary 2.6.** Let \(G\) be a locally compact group and \(G_0\) be the connected component of \(e \in G\). If the quotient \(G/G_0\) is infinite then
\[
\sup\{\|\chi_F\|_{cb} : F \subseteq G, \chi_F \in B(G)\} = +\infty.
\]

**Proof.** Since \(G/G_0\) is infinite and totally disconnected, it follows from Theorem 2.4 that
\[
\sup\{\|\chi_F\|_{cb} : F \subseteq G/G_0, \chi_F \in A(G/G_0)\} = +\infty.
\]

Let \(\pi : H \to G/G_0\) be the quotient map. Since \(\chi_{\pi^{-1}(F)} = \chi_F \circ \pi\), it follows from Theorem 2.1 that
\[
\sup\{\|\chi_F\|_{cb} : F \subseteq G, \chi_F \in B(G)\} = +\infty.
\]

**Remark 2.7.** Let \(G\) be a locally compact group and \(N\) a closed normal subgroup of \(G\). It follows from Theorem 2.1 that if
\[
\sup\{\|\chi_F\|_{cb} : F \subseteq G/N, \chi_F \in B(G/N)\} = +\infty
\]
then
\[
\sup\{\|\chi_F\|_{cb} : F \subseteq G, \chi_F \in B(G)\} = +\infty.
\]

### 3. Groups with idempotents with large norms

Let \(G\) be a locally compact group and \(G_0\) the connected component of the identity of \(G\). In this section we show that \(B(G)\) contains idempotents of arbitrarily large norms if and only if \(G/G_0\) is infinite. We also prove a related result for \(A(G)\). It follows from [10] that an idempotent is in \(B(G)\) if and only if it is of the form \(\chi_F\) with \(F\) in the open coset ring of \(G\).

Let \(H\) be an open subgroup of \(G\). Then \(H \cap G_0\) is open and closed in \(G_0\), and hence equal to \(G_0\); thus \(H \supseteq G_0\). Since \(G_0\) is contained in every open subgroup of \(G\), we have that that if \(E\) is an left coset of an open subgroup in \(G\), then \(E = EG_0\). It is easy to see that if \(E\) is a left coset of an open subgroup \(G\), we also have \(E^c = E^c G_0\) (here \(E^c\) is the complement of \(E\)).

**Lemma 3.1.** Let \(X\) be in the open coset ring \(\Omega_0(G)\). Then \(X = X G_0\).

**Proof.** We show that if \(X = X G_0\) and \(Y = Y G_0\), then \(X \cap Y = (X \cap Y) G_0\).

Indeed, let \(z \in X \cap Y\) and \(g \in G_0\). Then \(z = x g' = y g''\) for some \(x \in X\), \(y \in Y\) and \(g', g'' \in G_0\). Then \(z g = x g' g = y g'' g\) and since \(x g' g \in X\) and \(y g'' g \in Y\) we obtain \(z g \in X \cap Y\). In view of the above remark on complements, the assertion follows.
Corollary 3.2. Let $\phi$ be the map defined on $\Omega_0(G)$ by $X \mapsto q(X)$ (where $q : G \to G/G_0$ is the quotient map). Then $\phi$ is a ring isomorphism from $\Omega_0(G)$ onto $\Omega_0(G/G_0)$.

Proof. Clearly $\phi(X \cap Y) = \phi(X) \cap \phi(Y)$ and $\phi(X^c) = \phi(X)^c$. Let $aK$ be a coset of an open subgroup in $G/G_0$. Then $\phi^{-1}(aK)$ is a coset of an open subgroup in $G$ and $\phi(\phi^{-1}(aK)) = aK$. Finally, $XG_0 = YG_0$ for $X, Y \in \Omega_0(G)$ implies $X = Y$, hence $\phi$ is injective. □

Theorem 3.3. Let $G$ be a locally compact group and $G_0$ be the connected component of $e \in G$. The following are equivalent

1. The quotient $G/G_0$ is infinite and $G_0$ is compact
2. $\sup\{\|\chi_F\|_{cb} : \chi_F \in A(G)\} = +\infty$.
3. $\sup\{\|\chi_F\|_{A(G)} : \chi_F \in A(G)\} = +\infty$.

Proof. That (1) implies (2) follows from Theorems 2.1 and 2.4.

That (2) implies (3) follows since $\|\chi_F\|_{cb} \leq \|\chi_F\|_{A(G)}$.

We show that (3) implies (1): If $G_0$ is not compact, it follows from Lemma 3.1 that there are no idempotents in $A(G)$. If $G/G_0$ is finite, the open coset ring is finite from Corollary 3.2, and hence the set of idempotents is finite. □

The proof of the following theorem is similar.

Theorem 3.4. Let $G$ be a locally compact group and $G_0$ be the connected component of $e \in G$. The following are equivalent

1. The quotient $G/G_0$ is infinite.
2. $\sup\{\|\chi_F\|_{cb} : \chi_F \in B(G)\} = +\infty$.
3. $\sup\{\|\chi_F\|_{B(G)} : \chi_F \in B(G)\} = +\infty$.

4. NORMS OF HOMOMORPHISMS

In this section we show that if $G$ is a locally compact group with connected component of the identity $G_0$ such that $G/G_0$ is infinite, and $H$ is a locally compact group, then there exist homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$. We also prove that if there exists an amenable group $H$ such that homomorphisms of arbitrarily large norm from $A(H)$ into $B(G)$ exist, then $G/G_0$ is infinite.

Proposition 4.1. Let $G, H$ be locally compact groups and $F \in \Omega_0(G)$. For $u \in A(H)$ we define

$$\rho_F(u)(t) = \begin{cases} u(e), & t \in F \\ 0, & t \notin F \end{cases}$$

Then the map $\rho_F$ is a completely bounded homomorphism $A(H) \to B(G)$ and

$$\|\rho_F\|_{cb} = \|\rho_F\| = \|\chi_F\|_{B(G)}.$$
Proof. It follows from [12, Theorem 3.1] that the map $\rho_F$ is a completely bounded homomorphism. Choose $u \in A(H)$ such that $u(e) = 1$ and $\|u\|_{A(H)} \leq 1$. Then
\[ \|\rho_F\| \geq \|\rho_F(u)\|_{B(G)} = \|u(e)\chi_F\|_{B(G)} = \|\chi_F\|_{B(G)}. \]

We also have, for $u \in A(H)$,
\[ \|\rho_F(u)\|_{B(G)} = \|u(e)\chi_F\|_{B(G)} = |u(e)|\|\chi_F\|_{B(G)} \leq \|u\|_{A(H)}\|\chi_F\|_{B(G)}, \]
and hence $\|\rho_F\| = \|\chi_F\|_{B(G)}$.

Since the image of $\rho_F$ is one-dimensional, it follows that $\|\rho_F\|_{cb} = \|\rho_F\| = \|\chi_F\|_{B(G)}$.

□

Applying Theorem 3.4 to Proposition 4.1, we obtain the following Corollary 4.2.

**Corollary 4.2.** Let $G, H$ be locally compact groups and assume that
\[ \sup\{\|\chi_F\|_{cb} : F \in \Omega_0(G)\} = +\infty. \]

Then
\[ \sup\{\|\rho : A(H) \to B(G)\| : \rho \text{ is a cb homomorphism}\} = +\infty. \]

**Theorem 4.3.** Let $G$ be a locally compact group and $G_0$ be the connected component of $e \in G$. The following are equivalent:

(i) $\sup\{\|\rho : A(H) \to B(G)\| : \rho \text{ is a cb homomorphism}\} = +\infty$.

(ii) There exists an amenable locally compact group $H$ such that
\[ \sup\{\|\rho : A(H) \to B(G)\| : \rho \text{ is a cb homomorphism}\} = +\infty. \]

(iii) The group $G/G_0$ is infinite.

(iv) $\sup\{\|\chi_F\|_{B(G)} : F \in \Omega_0(G)\} = +\infty$.

**Proof.** Clearly (i) implies (ii). The equivalence (iii) ⇔ (iv) follows from Theorem 3.4. Also the implication (iv) ⇒ (i) follows from Corollary 4.2.

It remains to show that (ii) implies (iii). Suppose not, i.e. that $|G/G_0| < +\infty$. By Corollary 3.2, there exists $m \in \mathbb{N}$ such that $|\Omega_0(G)| \leq m$. Let $\rho : A(H) \to B(G)$ be a completely bounded homomorphism. By [12, Theorem 3.7], $\rho$ is of the form 1.1 for some $Y \in \Omega_0(G)$ and a piecewise affine map $\alpha : Y \to H$. By Proposition 3.1 in [12] we have that
\[ \|\rho\|_{cb} \leq m \cdot \sum_{F \in \Omega_0(G)} \|\chi_F\|_{B(G)} \leq m^2 \max\{\|\chi_F\|_{B(G)} : F \in \Omega_0(G)\} \]
which is a contradiction. □

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