On Holomorphic Factorization in Asymptotically AdS 3D Gravity

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Abstract

This paper studies aspects of “holography” for Euclidean signature pure gravity on asymptotically AdS 3-manifolds. This theory can be described as SL(2, C) CS theory. However, not all configurations of CS theory correspond to asymptotically AdS 3-manifolds. We show that configurations that do have the metric interpretation are parameterized by the so-called projective structures on the boundary. The corresponding asymptotic phase space is shown to be the cotangent bundle over the Schottky space of the boundary. This singles out a “gravitational” sector of the SL(2, C) CS theory. It is over this sector that the path integral has to be taken to obtain the gravity partition function. We sketch an argument for holomorphic factorization of this partition function.

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1 Introduction

In this paper we study certain aspects of holography for negative cosmological constant gravity in 2+1 dimensions. The theory we consider is that of pure gravity; the only field is the metric. This should be contrasted to the by now standard setup of AdS$_3$/CFT$_2$ correspondence, in which the 3-dimensional theory contains, in addition to the metric (an infinite number of) other fields. Our main aim is to shed some light on “holography” in the pure gravity context. Namely, as was shown more than ten years ago by Brown and Henneaux [1], the algebra of asymptotic symmetries of negative cosmological constant 2+1 gravity is the Virasoro algebra of certain central charge. Thus, the corresponding quantum theory, if exists, must contain the same algebra among its symmetries and is therefore a conformal field theory. This argument of course applies not only to pure gravity, but also to any 3-dimensional theory containing it, in particular to the system arising in the AdS$_3$/CFT$_2$ correspondence of string theory. In that case the CFT is known: using the Maldacena limit argument [2] one conjectures the CFT to be the sigma model describing the low-energy dynamics of the D1/D5 system, see, e.g., the review [3] for more detail. There is no analogous D-brane argument for the pure gravity case, so the question which CFT, if any, gives a “holographic” description of pure gravity cannot be answered this way. However, pure gravity is a topological field theory. It has been known since the work of Witten [4] that 3d TQFT’s are intimately related to 2d CFT’s. One thus might suspect that some “holographic” description arises this way. This paper is aimed at studying aspects of this “holography”.

Some readers may object our usage of term “holography” to describe a TQFT/CFT relation. Indeed, the bulk theory here has no propagating degrees of freedom. Holographic relations which are encountered in string theory are, on the other hand, between a local theory with propagating degrees of freedom in bulk and a local theory on the boundary. This is much more non-trivial than a TQFT/CFT relation. Still, in the TQFT/CFT context certain quantities of interest from the CFT point of view can be very effectively calculated using the bulk theory, and vice versa. It is in this limited sense that a TQFT/CFT relation is an example of holography.

Some may view this holography as trivial. However, as we shall attempt to demonstrate in this paper, this is not so. First of all, although gravity in 2+1 dimensions can be rewritten as a CS theory, the relevant gauge group is non-compact. We are thus entering a realm of non-compact TQFT’s, which is much less studied than the compact case. The problem here is that, by analogy with the compact gauge group case, one expects quantum groups to be relevant, but now these are non-compact. Even though non-compact quantum groups are studied to some extent, one does not seem to understand them well enough to define the corresponding non-compact TQFT’s. The second point is that in the usual well-understood compact CS TQFT/CFT correspondence one has a relation only to a holomorphic sector of the CFT. More precisely, the statement is that the Hilbert space $\mathcal{H}$ of the holomorphic conformal blocks of the group $G$ WZW CFT on a Riemann surface $X$ essentially coincides with the Hilbert space of CS theory for group $G$ on a 3D manifold whose boundary is $X$. In particular, the WZW holomorphic conformal blocks are just certain states of the quantum CS theory. On the other hand, the partition function of any CFT is constructed from both holomorphic and anti-holomorphic conformal blocks; one says that it holomorphically
factorizes. CS TQFT only gives one chiral sector. To obtain the structure relevant for the full CFT, one needs two CS theories. The arising holographic correspondence is rather non-trivial already in the compact gauge group case, see [5]. As it was shown in this work, given a “chiral” TQFT, e.g., CS theory with gauge group \( G \), there exists certain other TQFT, essentially given by two copies of CS, such that the full CFT partition function receives the interpretation of a state of this TQFT. For the compact gauge group case this TQFT is given by the so-called Turaev-Viro model, see [5] for more detail. The present paper is a step toward a non-compact version of that story.

Thus, in the present paper we study the relation between a non-compact TQFT and the corresponding holographic (full) CFT. The TQFT in question is Euclidean 3d gravity with negative cosmological constant. This Euclidean theory is interesting in its own right. Indeed, classically this is essentially the theory of hyperbolic 3-manifolds – an extremely rich subject that has been under active study for the last few decades. It is a very interesting problem to construct the corresponding quantum theory. This theory is expected to define new knot invariants, and may become of importance in 3D topology. The Euclidean quantum theory also plays an important role in the construction [6] of the Lorentzian signature theory.

A relation between 3D gravity and a (full) CFT was anticipated already in [7], and we were influenced by this work when writing the present paper. The author notices that the gravity action can be written as a difference of two CS actions. At the level of the partition function this suggests holomorphic factorization, a feature characteristic of a CFT. The author suggested that the relevant full CFT is the quantum Liouville theory. Another work relevant in this regard is [8]. This paper showed how the Liouville theory on the boundary arises in asymptotically AdS gravity.

In the present paper we shall argue that the partition function of 3D gravity on an asymptotically AdS manifold reduces to a full CFT partition function on the boundary. Our argument is in the spirit of [7] and is to show that the partition function holomorphically factorizes. However, we are not claiming that the CFT in question is Liouville theory. In fact, the quantum Liouville theory is known to be related to \( SL(2,\mathbb{R}) \) CS theory, not \( SL(2,\mathbb{C}) \). Thus, the CFT that arises from 3-d gravity in the way described in this paper is most probably not related to the Liouville theory. It is some other CFT, whose exact nature is still to be determined. We partially characterize this CFT by describing the relevant phase space.

The organization of this paper is as follows. We start by describing our main results and conclusions. Section 3 gives the action formulation of the theory we are dealing with. We describe the asymptotic phase space in section 4. The partition function is studied in section 5.

2 Overview and main results

In this paper we are interested in the quantum theory of negative cosmological constant gravity in three dimensions, for the case of Euclidean signature. The action for this theory can be rewritten as the difference of two \( SL(2,\mathbb{C}) \) CS actions, see below. The corresponding CS quantum theory was studied in [9]. However, the quantization procedure described there is not directly relevant in the asymptotically AdS case, because it does not in any way incorporate the important asymptotic
structure. Work [9] uses a parameterization of the SL(2, C) CS phase space as the cotangent bundle over the moduli space of flat SU(2) connections. As we shall see below, in the context of asymptotically AdS gravity certain other parameterization is relevant.

The SL(2, C) WZW theory, or, more precisely, certain associated coset theories were also actively studied in contexts other than 3D gravity. Thus, the gauged SL(2, C)/SU(2) WZW theory appears prominently in the usual SU(2) WZW or CS theories. Here a problem of finding the scalar product of CS states reduces to a problem of evaluating the SL(2, C)/SU(2) coset theory path integral, see [10] and references therein. The coset theory is rather well understood, in particular one knows explicitly the spectrum and the structure constants of the three point function, see [11]. Recently, some progress has also been made in Liouville theory, which is a close relative of the SL(2, C)/SU(2) coset model. Thus the work of Ponsot and Teschner [12] has proved the Liouville bootstrap by reducing the problem to a question about representations of a certain non-compact quantum group. Related is the development of quantum Teichmüller spaces, see [13] and [14], whose theory builds, essentially, upon representation theory of the same quantum group. All these results are potentially relevant for AdS 3D gravity.

On the physics side, AdS 3D gravity has been studied extensively, both classical aspects and the quantization. Probably the most popular approach to the quantum theory is that based on the algebra of asymptotic symmetries, see, e.g., [15] and references therein. It originated in the paper by Brown and Henneaux [1]. Studying the algebra of asymptotic symmetries of the Lorentzian theory, they noticed that this algebra essentially coincides with the Virasoro algebra of certain central charge. The central charge depends on the cosmological constant, and was found to be equal to $c = 3l/2G$, where $l = 1/\sqrt{-\Lambda}$ and $G$ is Newton’s constant. This means that a theory describing asymptotically AdS gravity must be a conformal field theory of this central charge. Coussaert, Henneaux and van Driel [8] then showed that the Einstein-Hilbert action reduces on shell, as a consequence of asymptotically AdS boundary conditions, to the action of Liouville theory. This promoted Liouville theory into a good candidate for the quantum theory of AdS 3D gravity. However, as we shall argue in this paper, the actual holographic theory is not the Liouville, but certain other, possibly related to it theory. The CS formulation was also used in tackling the quantization problem. Thus, works [16, 17] and more recently [18] showed that the BTZ BH entropy can be obtained from the SU(2) CS partition function by an analytic continuation.

Having completed this brief review, let us outline the main constructions of the present paper. The paper consists of two main parts. In the first part we analyze the structure of the space of classical solutions of the theory. The main aim here will be to understand the asymptotic structure of the CS connections. The second part is devoted to an analysis of the gravity partition function. Here we will need the results from the first part to state precisely over what class of fields the path integrals are taken.

The structure of the space of classical solutions is conveniently summarized in a notion of asymptotic phase space. It is introduced in section 4. This phase space is just the space of classical solutions of equations of motion, equipped with the symplectic structure that is evaluated at the asymptotic infinity. For our purposes of analyzing the partition function we only need to understand
the structure of the space of solutions, more precisely a certain parameterization of this space. The symplectic structure on this space does not play any role, at least directly. Indeed, what we consider in this paper is the partition function, which is given by the path integral of $e^{-I_{gr}}$. The symplectic structure on the space of solutions would be relevant if we considered the path integral with an imaginary unit in the exponential, that is a quantum state. Such quantum states would be essentially given by the quantization of our asymptotic phase space. As an aside remark let us note that these states play important role in the Lorentzian signature quantum theory, see [6]. However, in the present paper we shall not use them. Thus, the asymptotic phase space we get is not to be quantized, at least not in the present paper. The main rational for introducing it in the present paper is to point out that a natural symplectic structure on the space of solutions is also the one induced by the gravity action.

As we explain in detail in section 4, the spaces appearing as solutions of the theory are handlebodies. The simplest example to keep in mind is the AdS space itself, which, for Euclidean signature that we are considering, is just the unit ball, its boundary being a sphere. For a more general space the conformal boundary at infinity is some Riemann surface, and the space itself is a handlebody. To understand the structure of the space of solutions, let us recall that, since there are no local DOF in 3D gravity, all constant negative curvature spaces look locally like AdS. Thus, they can all be obtained from AdS by discrete identifications. It is a standard result that the moduli space of such manifolds with a Riemann surface $X$ as the boundary is parametrized by homomorphisms from $\pi_1(X)$ into the isometry group, which in this case is $\text{SL}(2, \mathbb{C})$, modulo the overall action of $\text{SL}(2, \mathbb{C})$. Our first result is that homomorphisms that arise in asymptotically AdS context are of a special type. Namely, they are those associated with the so-called projective structures on $X$. Thus, the space of solutions of our theory is parametrized by moduli of the boundary $X$ and a projective structure on $X$. It is known that this space is naturally a bundle over the moduli space, namely the cotangent bundle $T^*T_g$, where $T_g$ is the Teichmüller space at genus $g$.

Kawai [19] has shown that the symplectic structure on this space arising from its embedding into $\text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$ coincides with the usual cotangent bundle symplectic structure on $T^*T_g$. The first of this symplectic structures is essentially that of CS theory, and thus also the gravitational one. Thus, the gravitational symplectic structure evaluated at the asymptotic infinity coincides with the one on $T^*T_g$. Actually, as we shall see below, the phase space that appears is the cotangent bundle over the so-called Schottky space. This is related to the fact that the boundary in our approach is uniformized by the Schottky, not Fuchsian groups. The Schottky space is a certain quotient of the Teichmüller space. Summarizing, we get:

**Result 1** The asymptotic phase space of Euclidean AdS 3D gravity is the cotangent bundle over the Schottky space of the boundary.

It is interesting to note that the same phase space is known to appear in 3D gravity in a different setting. For zero cosmological constant, in the usual Lorentzian signature case, it is a well-known result [20] that the reduced phase space, for spacetimes of topology $X \times \mathbb{R}$, where $X$ is some Riemann surface, is the cotangent bundle over the Teichmüller space of $X$. To arrive to this result one uses, see [20], the usual geometrodynamics description, and a special time-slicing
by hypersurfaces of constant York time $T = \text{Tr}K$, where $K$ is the hypersurface extrinsic curvature. Witten [21] arrived at the same result using the Chern-Simons formulation. Let us note that the holomorphic factorization of the partition function of a 3D theory is related to the fact that its phase space is $T^*T_g$. Indeed, the cotangent bundle $T^*T_g$ can be naturally identified with the space $T_g \times T_g$. Therefore, quantum states of such a theory, which are square integrable functions on the Teichmüller space, can be also realized as $|\Psi|^2$ of states $\Psi$ obtained by quantization of the Teichmüller space. Thus, interestingly, in spite of the fact that the action of zero cosmological constant 3D gravity is not two CS actions, and the Verlinde [7] argument for holomorphic factorization does not apply, the theory can still be expected to exhibit some analog of this property.

In section 5 we turn to an analysis of the partition function. It is given by the path integral of $e^{-I_{gr}[g]}$. Our point of departure is a representation of the gravitational partition function as a path integral over the CS connections. As we show in section 3, the boundary terms of the gravity action are exactly such that in the CS formulation one gets the following action:

$$
I[A, \bar{A}] := -iI_{CS}[A] + iI_{CS}^+[\bar{A}] + 2 \int d^2z \text{Tr}A_z\bar{A}_z. \tag{2.1}
$$

Here $I_{CS}[A], I_{CS}^+[\bar{A}]$ are CS actions suitable for fixing $A_z, \bar{A}_z$ correspondingly, see (A.6), (A.5). The key point is that the real gravity action $I_{gr}$ gets represented as $iI_{CS}$ plus its complex conjugate. Thus, we consider the partition function, which is the path integral of the exponential of $(-I_{gr})$. This path integral is not a quantum state of our system precisely because there is no $i$ in the exponential. However, it gets represented in the CS formulation as a product of two CS path integrals with the imaginary unit in the exponential, or in other words, two CS quantum states. This is clearly reminiscent of the holomorphic factorization.

To further analyze the structure of the partition function we need to specify over which class of connections the path integral is taken. We show in section 4 that CS connections appearing as classical solutions of our theory have the following asymptotic structure. They are pure gauge:

$$
A \sim \left(m_{T^\mu}\, F_\mu \, h_\varphi \, r\right)^{-1}d(m_{T^\mu}\, F_\mu \, h_\varphi \, r), \quad \bar{A} \sim (\bar{r} \, \bar{h}_\varphi \, \bar{F}_\bar{\mu} \, \bar{m}_{T^\nu})d(\bar{r} \, \bar{h}_\varphi \, \bar{F}_\bar{\mu} \, \bar{m}_{T^\nu})^{-1} \tag{2.2}
$$

Here $m_{T^\mu}, F_\mu, h_\varphi, r$ and correspondingly the other connection are certain (multi-valued) matrix-valued functions on $X$, to be given below. The matrices $F_\mu$ and $h_\varphi$ depend in a certain way on the Beltrami differential $\mu$ and the Liouville field $\varphi$ correspondingly. Beltrami differential $\mu$ parameterizes the conformal structure on the boundary $X$. This is achieved by fixing a reference conformal structure. Let $z$ be a complex coordinate on the reference surface such that $|dz|^2$ gives the corresponding conformal structure. Then $|dz + \mu d\bar{z}|^2$ is a metric in a different conformal class. Because of the conformal anomaly to be discussed below, everything depends not just on the conformal class of the metric, but also on a representative in each class. The Liouville field $\varphi$ parameterizes different representatives in the same conformal class. A representative is given by the metric $e^{\varphi}|dz + \mu d\bar{z}|^2$. The matrix $m_{T^\mu}$ depends in a special way on a quadratic differential $T^\mu$ on $X^\mu$ that is related to a projective structure. The matrix $r$ is constant on $X$ and only depends on the radial coordinate.

The dependence of the connections on the radial coordinate is such that the action (2.1) contains only the logarithmic divergence. There are no terms in (2.1) containing the area-type divergence.
One can take care of the logarithmic divergence simply by introducing new connections $a, \bar{a}$, such that the original connections are gauge transforms of the new ones:

$$A = a^r, \quad \bar{A} = \bar{r}a.$$  \hfill (2.3)

The new connections can be restricted to the boundary. The action (2.1) considered as a functional of the connections $a, \bar{a}$ is explicitly finite. It however contains a conformal anomaly coming from the last term in (2.1). We will define the CS path integral as an integral over $a, \bar{a}$.

We first analyze the genus zero case and then make comments as to the general situation. The path integral can be taken in two steps. One first integrates over the bulk, keeping the connections on the boundary fixed. Both $Da$ and $D\bar{a}$ are the usual CS path integrals. For both connections the result is the exponential of the WZW action:

$$\int Da \, e^{-iI_{CS}[a]} = e^{-I_{WZW}[g]}, \quad a|_{\partial M} = g^{-1}dg, \quad g = m T^\mu F_\mu h_\varphi,$$  \hfill (2.4)

$$\int D\bar{a} \, e^{iI_{CS}[\bar{a}]} = e^{-I_{WZW}[\bar{g}]}, \quad \bar{a}|_{\partial M} = \bar{g} \bar{d}g^{-1}, \quad \bar{g} = \bar{h}_\varphi \bar{F}_\mu \bar{m}_{\bar{F}_\mu}.  \hfill (2.5)$$

The result of the bulk integration is thus exponential of a new action. An important step is to realize that the WZW action $I_{WZW}[m T^\mu F_\mu]$ is essentially the Polyakov light-cone gauge action [22]. In other words, we have:

$$I_{WZW}[m T^\mu F_\mu] = \int d^2 z T^\mu - W[\mu], \quad I_{WZW}[\bar{F}_\mu \bar{m}_{\bar{F}_\mu}] = \int d^2 z T\bar{\mu} - W[\bar{\mu}].$$ \hfill (2.6)

Here $T$ is a certain quadratic differential obtained from $T^\mu$. When $a$ is a solutions of classical equations of motion, that is flat, the quadratic differential $T$ satisfies an equation involving $\mu$. For example, when $\mu = 0$ (no deformation of the reference surface $X$) $T$ must be holomorphic. The quantity $W[\mu]$ above is the Polyakov action. It is a known function of $\mu$, which satisfies $\partial W[\mu]/\partial \mu = S(f^\mu, z)$, where $f^\mu$ is the quasi-conformal mapping for $\mu$ and $S$ stands for the Schwartzian derivative. Using all these facts one gets an explicit expression for the result of the bulk path integral.

The next step is to integrate over the boundary data. The partition function we are interested in is a functional of a conformal structure on the surface, and also of a representative in this conformal class. Thus, it is a function of the Beltrami differential $\mu, \bar{\mu}$, and of the Liouville field $\varphi$. To get this function one has to integrate over the quadratic differential $T$ on $X$. Since one should integrate over all field configurations, not just classical solutions, there are no additional constraints (like holomorphicity) that $T$ has to satisfy. Thus, one finds that the partition function has the following simple structure:

$$Z_{gr}[\varphi, \mu, \bar{\mu}] = \int DTDT\bar{T} \, e^{-\int d^2 z T^\mu \mu - \int d^2 z T\bar{\mu} + W[\mu] + W[\bar{\mu}] + \bar{K}[\varphi, T, \bar{T}, \mu, \bar{\mu}].$$  \hfill (2.7)

Here $K[\varphi, T, \bar{T}, \mu, \bar{\mu}]$ is a certain functional, given in section 5. An important fact is that it is a quadratic polynomial in $T, \bar{T}$. The integral over $T, \bar{T}$ can thus be easily taken. Shifting the integration variables, and absorbing the result of a Gaussian integral into a definition of the measure, one gets:
Result 2  The partition function at genus zero holomorphically factorizes according to:

\[ Z_{gr}[\varphi, \mu, \bar{\mu}] = e^{S_L[\varphi, \mu, \bar{\mu}] + K[\mu, \bar{\mu}]} \left( e^{-W[\mu]} e^{-W[\bar{\mu}]} \right). \tag{2.8} \]

Here \( S_L[\varphi, \mu, \bar{\mu}] \) is the Liouville action in the background \(|dz + \mu d\bar{z}|^2\). The quantity \( K[\mu, \bar{\mu}] \) is a certain functional of the Beltrami differential. The above result is exactly what one expects as a holomorphically factorized partition function at genus zero. We comment on a higher genus case in section 5. We sketch an argument for holomorphic factorization similar to that of Witten [23]. The argument is to interpret the partition function as a certain inner product.

3 Actions

We start by defining the action for the theory, both in the geometrodynamics and the CS formulations.

Geometrodynamics

On a manifold with boundary one usually uses the following action

\[ -\frac{1}{2} \int d^3 x \sqrt{g} (R + 2) - \int d^2 x \sqrt{q} K. \tag{3.1} \]

We have put \( 8\pi G = l = 1 \). For asymptotically AdS spaces this action diverges. One of the two types of divergences can be canceled [24] by adding an “area” term. The action becomes:

\[ -\frac{1}{2} \int d^3 x \sqrt{g} (R + 2) - \int d^2 x \sqrt{q} (K - 1). \tag{3.2} \]

The boundary condition for which (3.3) gives a well-defined variational principle is that the induced boundary metric is held fixed. However, the boundary in our case is not a true boundary; rather it is only the conformal boundary of the space. Thus, what is really being kept fixed in the variational principle is the conformal class of the metric at the asymptotic infinity. The Euclidean path integral with these boundary conditions gives the canonical ensemble partition function, in which the intensive thermodynamical parameters (temperature etc.) are kept fixed.

It turns out, however, that from the point of view of the Chern-Simons formulation that we shall review shortly, a certain other action is more natural. Namely, instead of fixing the induced boundary metric, it is more convenient to fix the spin connection. In this case no trace of extrinsic curvature term needs to be added. However, one still needs the area term to cancel the divergence. Thus, the action that we are going to use is:

\[ I_{gr} = -\frac{1}{2} \int d^3 x \sqrt{g} (R + 2) - \int d^2 x \sqrt{q}. \tag{3.3} \]

This action can be viewed as suitable for computing the micro-canonical ensemble partition function, in which the energy, etc. is kept fixed at the boundary.

The Chern-Simons formulation
The CS formulation of AdS 3D gravity has been extensively discussed in the literature, see, e.g., [15]. In this formulation Euclidean AdS 3D gravity becomes the SL(2, C) CS theory. This group is not semi-simple and thus there are two possible choices of the trace to be used when writing the action. As was explained in [21], the trace to be used to get gravity is as follows. Let $J^i$ be generators of rotations: $[J^i, J^j] = \epsilon^{ijk} J^k$, and $P^i$ be generators of boosts: $[P^i, P^j] = -\epsilon^{ijk} J^k$, $[P^i, J^j] = \epsilon^{ijk} P^k$. The trace to be used is such that $\text{Tr}(J_i P_j) \sim \delta^{ij}$ and the trace of $J_i$ with $J_i$ and $P_i$ with $P_i$ is zero. It is customary to choose $J^i = -i\sigma^i$, $P^i = \sigma^i$, where $\sigma^i$ are the usual Pauli matrices. Then the trace can be written as $\text{Tr} = -\frac{1}{2} \text{Im} \text{Tr}$, where $\text{Tr}$ is the usual matrix trace. On the other hand, the imaginary part can be represented as the difference of the quantity and its complex conjugate. Thus, the action can be written using the ordinary matrix trace at the expense of having to subtract the complex conjugate action. The complex conjugate action can be thought of as the CS action of the complex conjugate connection. Thus, one has to work with both the original and the complex conjugate connections simultaneously.

Let us describe this in more detail. With our choice of conventions (spelled out in Appendix B) the two matrix valued CS connections are given by:

$$A = w + \frac{i}{2} e, \quad \bar{A} = w - \frac{i}{2} e. \quad (3.4)$$

They are complex and with our conventions $\bar{A} = -(A)^\dagger$, where $\dagger$ denotes Hermitian conjugation. The quantities $w, e$ are the matrix valued spin connection and the frame field correspondingly, see Appendix B for more details. The “bulk” CS action for $A$ is:

$$\tilde{I}_{\text{CS}}[A] = \frac{1}{2} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (3.5)$$

The CS coupling constant, which is usually present in front of the action in the combination $k/4\pi$ was set to $k = 2\pi$. This is for consistency with our choice $8\pi G = l = 1$. Using the decomposition (3.4) of $A, \bar{A}$ into $w, e$ one gets:

$$-i\tilde{I}_{\text{CS}}[A] + i\tilde{I}_{\text{CS}}[\bar{A}] = \int_M \text{Tr} \left( e \wedge f(w) - \frac{1}{12} e \wedge e \wedge e \right) + \frac{1}{2} \int_{\partial M} \text{Tr}(e \wedge w). \quad (3.6)$$

The bulk term here is the usual Palatini action. When connection $w$ satisfies its equation of motion, requiring that it is the spin connection compatible with $e$, the action reduces to:

$$-i\tilde{I}_{\text{CS}}[A] + i\tilde{I}_{\text{CS}}[\bar{A}] \to -\frac{1}{2} \int d^3x \sqrt{g}(R + 2) - \frac{1}{2} \int d^2x \sqrt{q}K. \quad (3.7)$$

We note that the boundary term here, although different from the one in (3.3), also regularizes the action in the sense that the action is at most logarithmically divergent.

Since we want the CS formulation action to reduce on shell to the action (3.3), we need some extra boundary terms. As is clear from (3.7), the following quantity must be added:

$$+\frac{1}{2} \int d^2x \sqrt{q}K - \int d^2x \sqrt{q}$$

The first term here is

$$-\frac{1}{2} \int \text{Tr} e \wedge w = -\frac{1}{2i} \int \text{Tr} A \wedge \bar{A} = -\int d^2z \text{Tr}(A_z \bar{A}_z - \bar{A}_z A_z).$$
Here we have introduced $d^2z = dz \wedge d\bar{z}/2i$. The area term can also be expressed in terms of the CS connections. We have:

$$\int d^2x \sqrt{q} = -\int d^2z \text{Tr}(e_ze_\bar{z}) = \int d^2z \text{Tr}(A - \bar{A})_z(A - \bar{A})_{\bar{z}}.$$  

The two terms combine into:

$$-\int d^2z \text{Tr}(A_zA_{\bar{z}} + \bar{A}_z\bar{A}_{\bar{z}} - 2A_{\bar{z}}\bar{A}_z).$$

Adding this expression to the bulk CS actions one gets:

$$I[A, \bar{A}] = -iI_{CS}[A] + iI_{CS}[\bar{A}] + 2\int d^2z \text{Tr} A_z\bar{A}_{\bar{z}}.$$  

(3.8)

Here $I_{CS}[A]$ are the CS actions suitable for fixing $A_z, A_{\bar{z}}$ on the boundary correspondingly, see Appendix A. We find it non-trivial that the boundary terms of the geometrodynamics action combine in the CS formulation into two “holomorphic” CS actions, plus a term that mixes them. This is certainly suggestive of the holomorphic factorization.

### 4 The Asymptotic Phase Space

The purpose of this section is to understand in detail the structure of the space of classical solutions of our theory. In particular, we will analyze the asymptotic structure of the CS connections. Facts derived in this section will be used in an essential way in section 5, when we discuss the gravity path integral.

We summarize all the facts we obtain in this section in a notion of the asymptotic phase space. As we have briefly explained in the introduction, this is just the space of solutions of equations of motion equipped with a natural symplectic structure that is induced by the gravity action. The symplectic structure is evaluated at the conformal boundary. The phase space we introduce is a Euclidean AdS$_3$ analog of the asymptotic phase space of 4D gravity, see [25]. The motivation in 4D comes from the idea of asymptotic quantization put forward by Ashtekar. He proposed to isolate radiative degrees of freedom in exact general relativity and then use the usual symplectic methods to quantize them. This was achieved by introducing an “initial value” formulation with certain free data at future and past null infinities, instead of the usual extrinsic curvature and the metric on a spatial hypersurface. The asymptotic free data are parametrized by a certain connection field. The phase space is then the space of certain equivalence classes of connections at future and past null infinity, with a rather natural symplectic structure, see [25]. This phase space can be quantized using the usual methods. Our phase space is similar, except for the fact that we are working with Euclidean metrics. The phase space will be similarly parametrized by certain data at infinity, and the gravitational action induces a certain symplectic structure. One could quantize this phase space, the resulting states turn out to be the analytic continuations of the states of Lorentzian signature theory, see [6]. However, the main object of the present paper is not a quantum state, but the partition function. The difference is that while the first can be realized as the path integral of $e^{iI_{gr}}$, the later is the path integral with no imaginary unit in the exponential.
The spaces that appear as classical solutions of our theory are Euclidean constant negative curvature manifolds that are asymptotically AdS. A precise definition of asymptotically AdS spaces was given, for the case of 3D, in [1]. A nice coordinate-free description valid in any dimension can be found in [27]. Both works treat Lorentzian signature spacetimes, but with appropriate modifications the definition can be used also in our Euclidean context. We restrict our attention only to spaces that have the asymptotic boundary consisting of a single component. The boundary is then a Riemann surface. In this paper, for simplicity, we shall consider only the case of compact Riemann surfaces, that is, no punctures or conical singularities. Our analysis can be generalized to include punctures (and conical singularities) but we will not consider this in the present paper. Let us note in passing the physical interpretation of 3D hyperbolic spaces having a compact Riemann surface as the asymptotic boundary. As was argued in [28], these spaces should be interpreted as Euclidean continuations of multiple black hole solutions of [29]. A particular case of the boundary being a torus is the usual Euclidean version of the BTZ black hole.

Let us see now what is the structure of the space of such 3D hyperbolic manifolds. Since there are no local DOF, different geometries are obtained as quotients of AdS$^3$, or the hyperbolic space $H^3$, by a discrete subgroup of its isometry group, which is SL(2, C). Such spaces $M$ can be parametrized by homomorphisms $\phi : \pi_1(M) \to \text{SL}(2, \mathbb{C})$, modulo the overall action of SL(2, C). The image of $\pi_1(M)$ under $\phi$ is just the discrete group that is used to obtain the space: $M = H^3/\phi(\pi_1(M))$. Since our spaces $M$ have the topology of a handlebody, so that some of cycles on the boundary are contractible inside $M$, the fundamental group of $M$ is smaller than that of $\partial M$. However, as we shall see, it is natural to allow singularities inside $M$. Then the fundamental group of $M$ coincides with that of $X$: $\pi_1(M) = \pi_1(X)$. Thus, solutions of equations of motion are parametrized by homomorphisms $\phi \in \text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))$ modulo conjugation. The space of such homomorphisms has a natural symplectic structure, discussed in [30] and thus becomes a phase space. For $X$ being a compact genus $g$ Riemann surface the (complex) dimension of this space is $6g-6$. The described phase space, namely the space of homomorphisms $\phi \in \text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$, is also the reduced phase space in the CS description. Indeed, as is discussed in, e.g., [21], the reduced phase space of CS theory on $X \times \mathbb{R}$ is parametrized by homomorphisms of $\pi_1(X)$ into the gauge group in question, in our case SL(2, C), modulo conjugation. Since gravity is SL(2, C) CS theory, the natural symplectic structure on $\phi \in \text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$ is also the one induced by the gravity action.

So far we have in no way used the asymptotic structure. As we shall see, the asymptotic boundary conditions restrict the type of homomorphisms $\phi \in \text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))$ that can arise at infinity. The allowed homomorphisms turn out to be those associated with projective structures on $X$. This restricts one to a special smaller space, which still has the complex dimension $6g-6$. This space is parametrized by sections of the cotangent bundle over the Schottky space space $\mathcal{S}_g$. We remind the reader that the Schottky space is a quotient of the Teichmuller space $T_g$ with respect to some of the modular transformations, see more on this below. Unlike the Teichmuller space, the Schottky space is not simply connected. On the cotangent bundle to the Schottky space the CS symplectic structure is known to reduce to the canonical cotangent bundle symplectic structure, see

$\ast$There is also a very rich class of hyperbolic manifolds arising as complements of links in $S^3$. These play a major role in 3D topology, see, e.g., [26] for a review. We do not consider these spaces here.
[19]. Thus, using the asymptotically AdS boundary conditions one obtains the cotangent bundle over $\mathcal{G}_g$ as the phase space.

The phase space $\text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$ of $\text{SL}(2, \mathbb{C})$ CS theory is known to contain rather nasty singularities, see, e.g., [30]. However, it is non-singular near the homomorphisms that come from projective structures, see [19] and references therein. Thus, asymptotically AdS boundary conditions serve as a regulator, throwing away singular parts of the CS phase space. Thus, asymptotically AdS gravity is different from the $\text{SL}(2, \mathbb{C})$ CS theory in that the phase space of the theory, even though of the same dimension, is smaller. Only certain CS configurations have the metric interpretation. To obtain a theory related to gravity one should only quantize this sector of CS. This can be compared with other examples of CS-gravity correspondence. Thus, a well studied example is that of positive cosmological constant Euclidean signature theory. The relevant CS gauge group in that case is $\text{SO}(4) \sim \text{SU}(2) \times \text{SU}(2)$. However, in this example it is impossible to make a restriction to those CS configurations that have a metric interpretation. More precisely, what is considered as gauge in CS theory should not be treated as gauge in gravity. Thus, the reduced phase spaces of two theories are different, see, e.g., [31] for a good discussion of this. Quantization of CS theory gives a theory that in no obvious way is related to quantum gravity. On the other hand, in our case there is a clear cut sector of $\text{SL}(2, \mathbb{C})$ CS that has a gravitational interpretation. It consists of those points $\phi \in \text{Hom}(\pi_1(X), \text{SL}(2, \mathbb{C}))$ in the CS phase space that come from projective structures on the boundary. By quantizing this sector of CS one should get a theory that is quantum gravity, unlike the case of positive cosmological sector, in which it is not known how to select a gravitational sector. We consider the description of the gravitational sector of $\text{SL}(2, \mathbb{C})$ CS theory as one of the most important results of this paper.

There is another natural phase space that is associated with asymptotically AdS 3D gravity. As we discuss in the next subsection, there is a large class of 3D spaces arising from the Schottky uniformization of Riemann surfaces. These spaces are not the most general ones appearing in asymptotically AdS 3D gravity. More precisely, the most general solution is allowed to have singularities inside the space. The restriction to non-singular solutions gives exactly the spaces obtained via Schottky uniformization. One can consider the restriction of the phase space $T^*\mathcal{G}_g$ to this smaller space of solutions. This smaller space is $\mathcal{G}_g$ itself, and it is a Lagrangian sub-manifold in $T^*\mathcal{G}_g$.

To understand why homomorphisms that arise in asymptotically AdS gravity are restricted to those coming from projective structures we need to describe in more detail how the spaces in question can be obtained by identifications of points.

### 4.1 Asymptotically AdS spaces via Schottky groups

In this subsection we describe how a large class of asymptotically AdS 3D manifolds can be obtained using Schottky groups. The spaces described in this subsection are not the most general asymptotically AdS manifolds, as we shall see. However, once they are understood, the structure of the whole space of solutions will become clear. The material presented here is widely known, see, e.g., [32].
We will mostly use the Poincare upper half-space model for \( \mathbb{H}^3 \). In this model the metric on \( \mathbb{H}^3 \) is given by:

\[
 ds^2 = \frac{1}{\xi^2} (d\xi^2 + |dy|^2). 
\]

We have put the radius of curvature \( l = 1 \). The boundary of \( \mathbb{H}^3 \) is at \( \xi = 0 \), and is just the (extended) complex plane \( \mathbb{C} \), \( y \) is a complex coordinate on the boundary.

The isometry group of \( \mathbb{H}^3 \) is denoted by \( \text{M"ob}(\mathbb{H}^3) \). It can be identified with the group of linear fractional transformations \( \text{M"ob} = \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm I\} \). This is done by considering the action of \( \text{M"ob} \) on the boundary \( \mathbb{C} \) of \( \mathbb{H}^3 \). It acts by fractional linear transformations on \( y \), or, the same, by conformal transformations. Any fractional linear transformation can be expressed as a composition of an even number of inversions with respect to circles or lines in \( \mathbb{C} \). To construct an isometry of \( \mathbb{H}^3 \) that corresponds to a particular fractional linear transformation one has to extend the corresponding circles or lines to half-spheres or half-planes in \( \mathbb{H}^3 \). The isometry of \( \mathbb{H}^3 \) is then given by the same composition of inversions in these half-spheres and half-planes. This is called a Poincare extension of an element of \( \text{M"ob}(\mathbb{H}^3) \). The Poincare extension can be realized explicitly using quaternions, see, e.g., [32].

A large class of asymptotically AdS spaces whose boundary is a compact genus \( g \) Riemann surface can be obtained using the so-called Schottky groups. A Schottky group \( \Sigma \) is a group that is freely (that is, no relations) generated by a finite number of strictly loxodromic (that is, no elliptic, no parabolic) elements of \( \text{M"ob} \). A Schottky group is called \( \text{marked} \) if one chooses in it a set of generators \( L_1, \ldots, L_g \) satisfying no relations. It is easiest to understand the structure of \( \mathbb{H}^3/\Sigma \) by considering the action of \( \Sigma \) on the boundary of \( \mathbb{H}^3 \). Let us denote the completion of the set of fixed points of this action by \( \Delta \), and the complement of \( \Delta \) in \( \mathbb{C} \) by \( \Omega \). \( \Sigma \) acts on \( \Omega \) properly discontinuously, and \( \Omega/\Sigma \sim X \), where \( X \) is a genus \( g \) Riemann surface. This is easiest to understand by introducing a fundamental region for the action of \( \Sigma \). Recall that a fundamental region \( D \in \Omega \) is a region no two points inside of which are related by a transformation from \( \Sigma \) and such that any point in \( \Omega \) is obtainable as an image of a point in \( D \). A fundamental region for \( \Sigma \) can be obtained by picking up a set of \( g \) pairs of non-intersecting circles (or, more generally, Jordan curves) \( C_1, \ldots, C_g \) and \( C'_1, \ldots, C'_g \), such that \( C'_i = -L_i(C_i) \) (minus denotes the change of orientation), which all lie outside of each other. The region outside of these circles is a fundamental region \( D \). The surface \( X \) can be obtained from \( D \) by identifying the circles forming its boundary pairwise. As it is not hard to see, the surface one gets by identifying the circles –boundaries of the fundamental region \( D \)– is indeed a \( g \)-handled sphere.

The classical \textit{retro-section} theorem due to Koebe, see, e.g., [33], states that all Riemann surfaces can be obtained this way. Note, however, that the 3D space one obtains depends not only on the conformal structure of surface \( X \) (its boundary), but also on the Schottky data: a set of \( g \) non-intersecting curves on \( X \). One obtains the Schottky uniformization of \( X \) cutting it precisely along this set of curves. Thus, there is not a single Schottky uniformization of a given Riemann surface \( X \), but infinitely many of them. Similarly, there is not a single 3D manifold corresponding to a given conformal structure on the boundary, but infinitely many such 3D manifolds. These 3-manifolds can be thought of as different fillings of the surface \( X \). One specifies a 3-manifolds by saying which
set of boundary cycles is contractible inside $M$. Moreover, as we shall see below, even when the conformal structure of $X$ is fixed and the Schottky data are chosen, there is still a whole family of asymptotically AdS 3D manifolds approaching at infinity this surface $X$. As we explain below, this family is parametrized by a projective structure on $X$. However, these general manifolds are singular inside, as we shall see.

4.2 The Fefferman-Graham asymptotic expansion

To understand why the 3D spaces obtained via Schottky uniformization are not most general asymptotically AdS spaces we need few more facts. It can be shown that asymptotically AdS boundary conditions imply that asymptotically the metric has the following simple form:

$$ds^2 = \frac{d\rho^2}{\rho^2} + q_{ab} dx^a dx^b,$$

(4.2)

where

$$q_{ab} = \frac{1}{\rho^2}q_{ab}^{(0)} + q_{ab}^{(1)} + \rho^2 q_{ab}^{(2)} + \ldots$$

(4.3)

One can use Einstein equations to show that the trace part of $q_{ab}^{(1)}$ is completely determined by $q_{ab}^{(0)}$. The traceless part, however, is free. Once this trace-free part is specified, all other terms in the expansion are determined, see [34, 35]. In 3D, the freedom in the traceless part of $q_{ab}$, as was noticed by, e.g., Banados [15] is exactly that of choosing a quadratic differential on the boundary. Holomorphic quadratic differentials are in one-to-one correspondence with equivalence classes of projective structures, see Appendix C. This is in agreement with the anticipated result that a general solution is parametrized by both a conformal structure and an equivalence class of projective structures on $X$.

For our purpose of analyzing the partition function, and also to prove that the asymptotic phase space is the cotangent bundle over the Teichmuller space, we need an explicit parameterization of the space of solutions. To obtain it, we shall find an explicit expression for the metric on a 3D space obtained via Schottky uniformization. It will then become quite clear how to modify this metric to obtain the most general asymptotically AdS 3D space.

4.3 The Banados, Skenderis-Solodukhin and Rooman-Spindel metrics

The result of Banados [15] is that in 3D, for the case of flat boundary metrics, the Fefferman-Graham expansion (4.2) stops at order $\rho^2$. This result was initially obtained for the case of flat boundary (genus one), but was later proved in full generality by Skenderis and Solodukhin [36]. What these authors obtained is exactly the most general solution of asymptotically AdS gravity. It was later shown by Rooman and Spindel [37] how this most general solution can be obtained by a coordinate transformation from the BTZ metric.

Instead of simply borrowing the most general asymptotically AdS metric from [36, 37] we sketch another derivation of it, which makes clear the relation between the spaces obtained via the Schottky groups and the metric [36, 37]. We use essentially the same idea as in [37], however, our derivation
is much simpler, for we apply a coordinate transformation to AdS space, and not to the BTZ BH space as in [37].

The idea of our derivation is to find a coordinate system in AdS that is compatible with identifications from the Schottky group $\Sigma$. The condition of compatibility is that the AdS metric when written in this new coordinates is invariant under the transformations from the Schottky group $\Sigma$. This metric then descends to a metric in the quotient space. Using the same method as in [37], one finds that such a coordinate system is given by:

$$
\begin{align*}
\xi &= \frac{\rho e^{-\varphi/2}}{1 + \frac{1}{4}\rho^2 e^{-\varphi} |\varphi_w|^2}, \\
y &= w + \frac{\varphi_w}{2} \frac{\rho^2 e^{-\varphi}}{1 + \frac{1}{4}\rho^2 e^{-\varphi} |\varphi_w|^2}.
\end{align*}
$$

(4.4)

The key quantity in this expressions is the canonical Liouville field $\varphi$. It is a (real) function of the complex coordinate $w \in \Omega$:

$$
\varphi(Lw) = \varphi(w) - \ln |L'|^2.
$$

(4.5)

The Liouville field can be constructed from the map between the Schottky and Fuchsian uniformization domains, see [38] for more details. The field $\varphi$ has a property that its stress-energy tensor $T^\varphi$ is equal to the Schwartzian derivative of the map $J^{-1} : \Omega \to H$, where $\Omega$ is the domain of discontinuity of $\Sigma$ and $H$ is the hyperbolic plane uniformizing the Riemann surface $X$, see [38].

The coordinates (4.4) are compatible with the identifications. This follows from the fact that AdS$_3$ metric (4.1), when written in coordinates $\rho, w, \bar{w}$ is invariant under transformations from $\Sigma$. Indeed, metric (4.1) expressed in terms of the new coordinates becomes:

$$
\begin{align*}
\eta^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2} e^{\varphi} dwd\bar{w} + \frac{1}{2} T^\varphi dw^2 + \frac{1}{2} \bar{T}^\varphi d\bar{w}^2 + R dwd\bar{w} \\
&+ \frac{1}{4} \rho^2 e^{-\varphi}(T^\varphi dw + R d\bar{w})(\bar{T}^\varphi d\bar{w} + R dw).
\end{align*}
$$

(4.6)

Here we have introduced:

$$
T^\varphi = \varphi_{ww} - \frac{1}{2} \varphi_w^2, \quad R = \varphi_{w\bar{w}}.
$$

(4.7)

The first quantity is just the stress-energy tensor of the Liouville field $\varphi$, the second is related to the curvature scalar of the 2D metric $e^{\varphi} |dw|^2$. Using the transformation property (4.5) of $\varphi$ one can show that:

$$
(T^\varphi \circ L)(L')^2 = T^\varphi, \quad (R \circ L)\overline{LL'} = R.
$$

(4.8)

This immediately implies that (4.6) is invariant under the transformations $w \to L \circ w$ for all generators $L_i$ of $\Sigma_g$.

The metric we just wrote is of the same form as the most general one obtained by Skenderis and Solodukhin [36]. Indeed, it can be written as:

$$
\begin{align*}
\eta^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2} \left(1 + \frac{\rho^2}{2} g_{(2)} g_{(0)}^{-1}\right) g_{(0)} \left(1 + \frac{\rho^2}{2} g_{(2)} g_{(0)}^{-1}\right),
\end{align*}
$$

(4.9)
\[ g_{(2)ij} = \frac{1}{2} \left( R(0)g_{(2)ij} + T_{ij} \right). \]

The notations here are self-explanatory. The metric (4.6) is also the same as the one obtained by Rooman and Spindel [37]. Note, however, that this metric was obtained in [37] by applying a similar coordinate transformation to the BTZ black hole metric, not to the AdS space.\(^\dagger\) Thus, the coordinate transformation (4.4) which relates the AdS metric and the one given by (4.6) is new. It can be used, for example, to determine the range of coordinates \( \rho, w \) in (4.6), a problem discussed in [36].

It is now not hard to see how to modify the metric (4.6) to obtain the most general asymptotically AdS manifold. Indeed, the quantity \( T^\varphi \), which was obtained to be equal to the Liouville stress-energy tensor, does not have to be related to \( \varphi \). The metric (4.6) is an exact solution of Einstein equations for any meromorphic function \( T(w) \) added to \( T^\varphi \). Note, however, that the metric is non-singular inside the 3-manifold only when one uses \( T^\varphi \). Indeed, non-singular manifolds are the ones coming from the Schottky uniformization. A non-singular inside manifold must have exactly \( g \) cycles on the boundary that are contractible in it. This is exactly the property of Schottky manifolds. When \( T \) is arbitrary, the metric (4.6) is still a solution of Einstein equations near the boundary, but it does not glue globally into a non-singular metric inside. The typical singularity one obtains is a line of conical singularities inside.

It is thus clear how to obtain the most general metric from (4.6). One has to consider a general holomorphic quadratic differential added to \( T^\varphi \). We choose to parameterize this as \( T^\varphi - T \), where \( T^\varphi \) is the stress-energy tensor of the Liouville field \( \varphi \) and \( T \) is some holomorphic quadratic differential for our Schottky group \( \Sigma \): \( T(Lw)(L')^2 = T(w), L \in \Sigma \). The most general asymptotically AdS metric then becomes:

\[
ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2} e^{\varphi} dw d\bar{w} + \frac{1}{2}(T^\varphi - T)dw^2 + \frac{1}{2}(T^\varphi - T)d\bar{w}^2 + R dw d\bar{w} + \frac{1}{4}\rho^2 e^{-\varphi}(T^\varphi - T)dw + Rdw)((T^\varphi - T)d\bar{w} + Rd\bar{w}).
\]

Here \( R = \varphi_w\bar{w} \), and \( \varphi \) is the (unique) solution of Liouville equation that has transformation properties (4.5). The above metric is of the same form as the most general asymptotically AdS metric obtained in [36]. Having this explicit expression for the metric one can calculate the corresponding CS connections. This is done in Appendix B. Their asymptotic behavior is analyzed in the next subsection.

### 4.4 The asymptotic structure of the CS connections

The asymptotic form of the CS connections corresponding to the metric (4.6) is obtained in Appendix B, formulas (B.16), (B.17). As is explained in the previous subsections, to get the connections corresponding to (4.10) we have to replace \( T^\varphi \) by \( T^\varphi - T \) in all the formulas.

\(^\dagger\)The fact that we have obtained the same metric starting from AdS is not surprising. Indeed, BTZ metric itself can be obtained from AdS by a coordinate transformation.
It is not hard to notice that the dependence on $\rho$ is correctly reproduced by introducing certain new connections independent of $\rho$. Namely, one can see that

$$
A = (a)^{r} = r^{-1}ar + r^{-1}dr, \quad \bar{A} = \bar{r}(\bar{a}) = \bar{r}\bar{a}r^{-1} + \bar{r}d\bar{r}^{-1},
$$

(4.11)

where

$$
a_w = \left( -\frac{1}{4} \varphi_w, \frac{1}{2} e^{-\varphi/2}(T \varphi - T) \right), \quad a_{\bar{w}} = \left( \frac{1}{4} \varphi_{\bar{w}}, \frac{1}{2} e^{-\varphi/2} R \right)
$$

(4.12)

$$
\bar{a}_w = \left( -\frac{1}{4} \varphi_w, 0 \right), \quad \bar{a}_{\bar{w}} = \left( \frac{1}{2} e^{-\varphi/2} (\bar{T} \varphi - \bar{T}) - \frac{1}{4} \varphi_{\bar{w}} \right)
$$

(4.13)

and no $a_\rho, \bar{a}_\rho$ components. The complex matrix $r$ is given by:

$$
r = \left( \frac{1}{\sqrt{\rho}} 0 \right),
$$

(4.14)

and $\bar{r}$ is the complex conjugate of $r$. It does not matter which branch of the square root is chosen in (4.14). Note that the new connections $a, \bar{a}$ no longer satisfy the relation $\bar{a}$ is minus the hermitian conjugate of $a$. This happened because we used a $\sqrt{\rho}$ in the matrix defining the gauge transformation. The new relation is that $\bar{a}$ is the hermitian conjugate of $a$, and the diagonal components change sign. We did not have to introduce the factor of $\sqrt{\rho}$ in the gauge transformation parameter (4.14). This would result in having some cumbersome factors of $i$ in the connections $a, \bar{a}$. We find our choice for $a, \bar{a}$ more convenient. The price one pays is a little more complicated relation between the two connections.

It is not hard to see that the $a_w, \bar{a}_w$ components of the connections are pure gauge:

$$
a_{\tilde{w}} = h_{\varphi}^{-1} \partial_w h_{\varphi}, \quad \bar{a}_w = (\bar{h}_{\varphi}) \partial_w (\bar{h}_{\varphi})^{-1},
$$

(4.15)

where

$$
h_{\varphi} = \left( e^{\varphi/4} \frac{1}{2} \varphi_w e^{-\varphi/4} \right), \quad \bar{h}_{\varphi} = \left( e^{\varphi/4} 0 \right)
$$

(4.16)

We are thus led to new connections, which we will denote by $\alpha, \bar{\alpha}$. The connections $a, \bar{a}$ are then the gauge transforms of $\alpha, \bar{\alpha}$:

$$
a = (\alpha)^{h_{\varphi}}, \quad \bar{a} = \bar{h}_{\varphi}(\bar{\alpha}),
$$

(4.17)

with

$$
\alpha = \left( \begin{array}{cc} 0 & -\frac{1}{2} T \\ 1 & 0 \end{array} \right) dw, \quad \bar{\alpha} = \left( \begin{array}{cc} 0 & 1 \\ -\frac{1}{2} T & 0 \end{array} \right) d\bar{w}.
$$

(4.18)

What we have found (4.18) as the asymptotic form of the connections are exactly the canonical connections $\alpha, \bar{\alpha}$ in holomorphic and anti-holomorphic vector bundles $E, \bar{E}$ of rank 2 over $X = \Omega/\Sigma$, see Appendix C.
4.5 General parameterization of the connections

What we have found in the previous subsection, is that the CS connections $\alpha, \bar{\alpha}$ that arise from the most general classical solution given by (4.10) are of the form (4.17) with the connections $\alpha, \bar{\alpha}$ being the canonical connections in the holomorphic and anti-holomorphic vector bundles $E, \bar{E}$. These connections can in turn be represented as gauge transforms, see (C.10). Thus, we get:

$$a = (m^\alpha h_\alpha)^{-1}d(m^\alpha h_\alpha), \quad \bar{a} = (\bar{h}_\alpha \bar{m}^\alpha)^{-1}d(\bar{h}_\alpha \bar{m}^\alpha).$$

In this subsection we introduce a somewhat more general parameterization of $a, \bar{a}$.

Let us recall that the CS connections $a, \bar{a}$ depend on the conformal structure of $X$, and this dependence comes through $\varphi$ and $T$, because $\varphi$ is the Liouville field for $\Sigma$, see (4.5), and $T$ is a holomorphic quadratic differential for $\Sigma$. They also depend on a projective structure $f$, on which $T$ depends. It is somewhat inconvenient to have the dependence on the moduli enter through $\varphi$. Instead, as is usual in CFT, let us introduce a reference Riemann surface $X$, and then consider quasi-conformal deformations of it, parametrized by a Beltrami differential. One then gets more general connections $a, \bar{a}$ that depend on the conformal structure of the reference surface $X$, on a Beltrami differential $\mu$, and on a projective structure on the surface $X^\mu$. These new connections are essentially the ones considered, e.g., by Kawai [19]. One obtains them considering the holomorphic vector bundle $E^\mu$ over the deformed Riemann surface $X^\mu$. There is the usual connection (C.9) in $E^\mu$, with $T = T^\mu$ being a holomorphic quadratic differential for the deformed Schottky group $\Sigma^\mu$. As is explained in [19], this connection can be pulled back to the bundle $E$ over the reference surface $X$. In fact, $E^\mu$ is isomorphic to $E$. The isomorphism is described by a matrix-valued function $F_\mu(w)$:

$$F_\mu = \begin{pmatrix} \frac{1}{2} f_\mu^{-1/2} & \frac{d}{dw}(f_\mu^{-1/2}) \\ 0 & (f_\mu^{1/2}) \end{pmatrix}. \tag{4.20}$$

Here $f_\mu$ is a solution of the Beltrami equation $f_\mu' = \mu f_\mu$ on the Schottky domain. The matrix $F_\mu$ satisfies the intertwining property:

$$\eta_{L^\mu}(f_\mu(w)) = F_\mu(Lw) \eta_{L_\mu}(w) F_\mu^{-1}(w). \tag{4.21}$$

Pulling back the canonical connection $\alpha$ from $E^\mu$ to $E$ we get a new flat connection:

$$\alpha^\mu = F_\mu^{-1} f_\mu^\ast \alpha F_\mu + F_\mu^{-1}dF_\mu = \begin{pmatrix} 0 & -\frac{1}{2} T \\ \frac{1}{2} & 0 \end{pmatrix} \frac{d}{dw} + \begin{pmatrix} -\frac{1}{2} \mu_w & -\frac{1}{2} (T^\mu + \mu_{ww}) \\ \frac{1}{2} \mu_w & 0 \end{pmatrix} d\bar{w}. \tag{4.22}$$

Here

$$T = T^\mu(f_\mu(w))(f_\mu^\ast)^2 + S(f_\mu, w) \tag{4.23}$$

is a quadratic differentials for $\Gamma$, which is however no longer holomorphic in $H$. In fact, when $T^\mu$ is holomorphic, that is comes from a projective structure on $X^\mu$, one has:

$$(\partial_{\bar{w}} - \mu \partial_w - 2 \mu_w) T = \mu_{www}. \tag{4.24}$$

Similarly, one gets the other connection by pulling back $\bar{\alpha}$ from $\bar{E}^\mu$ with the help of a matrix-valued function $\bar{F}_\mu$.
Thus, we get our desired final parameterization of the CS connections:

\[ \alpha^\mu = \bar{F}_\mu f^{\mu*} \alpha \bar{F}_\mu^{-1} + \bar{F}_\mu d\bar{F}_\mu^{-1} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} T & 0 \end{pmatrix} d\bar{w} + \begin{pmatrix} \frac{1}{2} \bar{\mu} \phi & -\bar{\mu} \\ -\frac{1}{2} (T \bar{\mu} + \bar{\mu} \bar{\omega}) & -\frac{1}{2} \bar{\mu} \bar{\omega} \end{pmatrix} dw. \tag{4.26} \]

Thus, instead of having all the dependence on the moduli in the Liouville field \( \varphi \) and the quadratic differential \( T \), we introduce the new connections \( \alpha^\mu, \bar{\alpha}^\mu \) that explicitly depend on the Beltrami differential \( \mu \). These are both flat connections, but they are connections on different bundles, namely on \( E, \bar{E} \). Thus, one cannot simply use them in the action (2.1), in particular because the cross-term would not be invariantly defined. However, there exists yet another bundle, to which both of these connections can be mapped by a gauge transformation. The transformed connections become connections on the same bundle, and the cross-term in the action is defined. These pulled back connections are just the gauge transforms of \( \alpha^\mu, \bar{\alpha}^\mu \) with matrices \( h_\varphi, \bar{h}_\varphi \) \((4.16)\). Thus, we get our desired final parameterization of the CS connections:

\[ \alpha^\mu = (m_{T^\mu} F_\mu)^{-1} d(m_{T^\mu} F_\mu), \quad \bar{\alpha}^\mu = (\bar{F}_\mu \tilde{m}_{T^\mu}) d(\bar{F}_\mu \tilde{m}_{T^\mu})^{-1}, \tag{4.27} \]

and

\[ \mathbf{a} = (\alpha^\mu) h_\varphi, \quad \bar{\mathbf{a}} = \bar{h}_\varphi (\bar{\alpha}^\mu). \tag{4.28} \]

As we have said, the connections \( \mathbf{a}, \bar{\mathbf{a}} \) are connections on the same bundle over \( X \), which we will denote by \( P \). The corresponding factor of automorphy is:

\[ M = \begin{pmatrix} \frac{1}{\gamma^\varphi} & 0 \\ 0 & \left( \frac{1}{\gamma^\varphi} \right)^{-1} \end{pmatrix}. \tag{4.29} \]

Thus, the connections transform as:

\[ \gamma^\varphi \mathbf{a} = M^{-1} \mathbf{a} M + M dM, \quad \gamma^\varphi \bar{\mathbf{a}} = M^{-1} \bar{\mathbf{a}} M + M dM. \tag{4.30} \]

Let us also note the transformation properties of the matrices \( m_{T^\mu}, F_\mu, h_\varphi \). We have:

\[ m_{T^\mu}(Lw) = \chi^\varphi_{L^\mu} m_{T^\mu}(w) \eta^{-1}_L(w), \quad \tilde{m}_{T^\mu}(Lw) = \bar{\eta}_L^{-1}(w) \tilde{m}_{T^\mu}(w) \bar{\chi}_L^\mu, \]

\[ (m_{T^\mu} F_\mu)(Lw) = \chi^\varphi_{L^\mu} (m_{T^\mu} F_\mu)(w) \eta^{-1}_L(w), \quad (\bar{F}_\mu \tilde{m}_{T^\mu})(Lw) = \bar{\eta}_L^{-1}(w) (\bar{F}_\mu \tilde{m}_{T^\mu})(w) \bar{\chi}_L^\mu, \]

and

\[ g(Lw) = \chi^\varphi_{L^\mu} g(w) M, \quad \bar{g}(Lw) = M^{-1} \bar{g}(w) \bar{\chi}_L^\mu, \tag{4.31} \]

where we have introduced

\[ g = m_{T^\mu} F_\mu h_\varphi, \quad \bar{g} = \bar{h}_\varphi \bar{F}_\mu \tilde{m}_{T^\mu}. \tag{4.32} \]

### 5 The partition function

Having the parameterization (4.27), (4.28) at our disposal, we are ready to study the partition function. It is given by the path integral of \( e^{-I_{D^\varphi}} \) over metrics. As we have shown in section 3,
the geometrodynamics action, when written in the CS formulation, becomes a simple sum of two CS actions, plus a cross-term that mixes them, see (3.8). Thus, the partition function can be represented as the CS path integral. The boundary data that are kept fixed are the conformal structure parametrized by the Beltrami differential $\mu$ and the Liouville field $\varphi$. As is not hard to show, the action diverges logarithmically for the connection field with the asymptotic behavior (B.16), (B.17). There is no other divergence. The area type divergence was already taken care of. To make the integrals well-defined, we should subtract the logarithmically divergent term. A nice way to do this is to replace the connections $A, \bar{A}$ by the gauge transformed connections $a, \bar{a}$, see (4.11). This takes care of the divergence but introduces a conformal anomaly, as expected of a CFT partition function. Thus, we define the partition function as:

$$Z_{\text{gr}}[\varphi, \mu, \bar{\mu}] = \int \mathcal{D}a \mathcal{D}\bar{a} \ e^{-iI_{CS}^-[a]+iI_{CS}^+[a]-2\int d^2w \ Tr a_{\phi} \bar{a}_\psi}. \quad (5.1)$$

Let us first analyze the genus zero case. The path integral can be taken in two steps. One first integrates over the bulk, keeping the connections on the boundary fixed. Both $\mathcal{D}a$ and $\mathcal{D}\bar{a}$ are the usual CS path integrals. For both connections the result is the exponential of the WZW action:

$$\int \mathcal{D}a \ e^{-iI_{CS}^-[a]} = e^{-I_{\text{WZW}}[g]}, \quad a_{\partial M} = \bar{g}^{-1}dg, \quad (5.2)$$

$$\int \mathcal{D}\bar{a} \ e^{iI_{CS}^+[a]} = e^{-I_{\text{WZW}}[\bar{g}]}, \quad \bar{a}_{\partial M} = \bar{g}dg^{-1}. \quad (5.3)$$

Here $g, \bar{g}$ are the matrix-valued functions introduced in (4.32). The result of the bulk integration is thus exponential of a new action, which we shall denote by $-I[\varphi, \mu, \bar{\mu}, T, \bar{T}]$. Using the Wiegman-Polyakov identity (A.13), one finds that this action is given by:

$$I[\varphi, \mu, \bar{\mu}, T, \bar{T}] = I_{\text{WZW}}[g, \bar{g}]. \quad (5.4)$$

To analyze the structure of the partition function another representation turns out to be more convenient. As is not hard to show,

$$I[\varphi, \mu, \bar{\mu}, T, \bar{T}] = I_{\text{WZW}}[m_{\nu} F_{\mu}] + I_{\text{WZW}}[\bar{F}_{\bar{\bar{\mu}}} \bar{m}_{\bar{\nu}}] + I_{\text{WZW}}[g_{\varphi}, \bar{g}_{\bar{\varphi}}, \alpha_\mu, \bar{\alpha}_\bar{\varphi}]. \quad (5.5)$$

Here $g_{\varphi} = h_{\varphi} \bar{h}_{\varphi}$ and

$$I_{\text{WZW}}[g, \alpha_\mu, \alpha_{\bar{\varphi}}] = I_{\text{WZW}}[g] + 2\int d^2w \ Tr (g\alpha_\mu g^{-1}\alpha_{\bar{\varphi}} + \alpha_\mu g^{-1}\partial_{\bar{\varphi}} g + \alpha_{\bar{\varphi}} g \partial_{\mu} g^{-1}) \quad (5.6)$$

is the usual gauged WZW action. An important step is to realize that the action $I_{\text{WZW}}[m_{\nu} F_{\mu}]$ is the Polyakov light-cone gauge action [22]. In other words, we have:

$$I_{\text{WZW}}[m_{\nu} F_{\mu}] = \int d^2w T_{\mu} - W[\mu], \quad I_{\text{WZW}}[\bar{F}_{\bar{\bar{\mu}}} \bar{m}_{\bar{\nu}}] = \int d^2w \bar{T}_{\bar{\bar{\mu}}} - W[\bar{\mu}]. \quad (5.7)$$

Here $T$ and $S$ are given by (4.23), and $W[\mu]$ is the Polyakov action:

$$W[\mu] = -\frac{1}{2} \int d^2w \frac{f_{\mu}}{f_{\bar{\mu}}} \mu_w. \quad (5.8)$$

The functional $W[\mu]$ satisfies $\partial W[\mu]/\partial \mu = S(f^\mu, w)$. 
We can use the representation (5.5) together with (5.7) to integrate over the boundary data. Since we would like the resulting partition function to depend on the Beltrami differential $\mu, \bar{\mu}$ and $\varphi$, one only has the quadratic differential $T$ to integrate over. Thus, we have

$$Z_{\text{gr}}[\varphi, \mu, \bar{\mu}] = \int DT \bar{D} e^{-I[\varphi, \mu, \bar{\mu}; T, \bar{T}]}.$$  \hspace{1cm}(5.9)$$

Here the integral is taken over all quadratic differentials $T$ for $\Gamma$. One does not impose any equations like (4.24) on $T$. Let us introduce

$$K[\varphi, T, \bar{T}, \mu, \bar{\mu}] = I_{\text{WZW}}[g_\varphi, \bar{\alpha}_w, \bar{\alpha}_{\bar{w}}].$$  \hspace{1cm}(5.10)$$

This function can be explicitly calculated. It is of the form:

$$K[\varphi, T, \bar{T}, \mu, \bar{\mu}] = \int d^2 w \frac{1}{2} e^{-\varphi} |\mu|^2 T \bar{T} + T(...)+\bar{T}(...) + \ldots ,$$  \hspace{1cm}(5.11)$$

where (...) denote certain terms depending on $\varphi$ and $\mu$. Using (5.7), one gets:

$$Z_{\text{gr}}[\varphi, \mu, \bar{\mu}] = \int DT \bar{D} e^{\int d^2 w T \mu - \int d^2 w \bar{T} \bar{\mu} - K[\varphi, T, \bar{T}, \mu, \bar{\mu}]} \left[ e^{-W[\mu]} e^{-W[\bar{\mu}]} \right].$$  \hspace{1cm}(5.12)$$

Thus, what one gets is exactly the structure expected at genus zero. The expression in the square brackets is just the product of holomorphic and anti-holomorphic conformal blocks. Indeed, in the genus zero case the Hilbert space is one-dimensional, and the holomorphic conformal block is given by $\Psi[\mu] = e^{-W[\mu]}$. It satisfies the conformal Ward identity:

$$\left( \partial_{\bar{w}} - \mu \partial_w - 2\mu_w \right) \frac{\delta W}{\delta \mu} = \mu_{www}.$$  \hspace{1cm}(5.13)$$

The prefactor is also the expected one having to do with the conformal anomaly. Indeed, it is not hard to show that, for $\mu = 0$, the prefactor is exactly the Liouville action for $\varphi$. When $\mu = 0$ the expression in the exponential becomes simply $-I_{\text{WZW}}[g_\varphi]$. This can be shown to be:

$$I_{\text{WZW}}[g_\varphi] = -\frac{1}{2} \int d^2 w \left( |\varphi_w|^2 - e^{-\varphi} R^2 \right).$$  \hspace{1cm}(5.14)$$

Although this is not the usual Liouville action, which has $e^\varphi$ as the second term, (5.14) does lead to the Liouville equation as its equation of motion. It is thus a version of the Liouville action. Thus, the prefactor in (5.12), at least for $\mu = 0$ is exactly the expected $e^{\text{SL}[\varphi]}$. The prefactor that comes from the integration over $T, \bar{T}$ is absorbed in the definition of the measure. For general $\mu$ one has to carry the integration over $T$. It can be done by completing the square. One then obtains a certain function of $\varphi$ and $\mu$ as the prefactor. We know that the dependence on $\varphi$ must be that of the Liouville action $S_{\text{L}}[\varphi, \mu, \bar{\mu}]$ in the background metric $|dw + \mu d\bar{w}|^2$. There is also an additional term, which we shall denote as $K[\mu, \bar{\mu}]$, that we will not attempt to calculate. Summarizing, in the genus zero case one gets the holomorphically factorized partition function:

$$Z_{\text{gr}}[\varphi, \mu, \bar{\mu}] = e^{S_{\text{L}}[\varphi, \mu, \bar{\mu}]+K[\mu, \bar{\mu}]} \left[ e^{-W[\mu]} e^{-W[\bar{\mu}]} \right].$$  \hspace{1cm}(5.15)$$

Let us now turn to a more complicated higher genus case. The first complication that arises in this case is that the WZW actions arising as the result (5.2) of the CS path integral are not
well-defined. Indeed, the functions $g, \bar{g}$ on $H$ do not descend to matrix-valued functions on $X$ because of their complicated transformation property (4.3.1). Thus, the WZW action functional is not well-defined on such $g, \bar{g}$. However, the full action, that is the WZW action (5.4) of the product $g\bar{g}$ is well-defined, at least in some cases, because

$$(g\bar{g})(Lw) = \chi_{L}^*(g\bar{g})(w)\bar{\chi}_{L}^*.$$  \hspace{1cm} (5.16)

Here $\chi_{L}, \bar{\chi}_{L}$ are constant matrices, which are the monodromy representations of the holomorphic and anti-holomorphic projective structures on $X$, see Appendix C. One can convince oneself that, in the case $\chi_{L} \in SU(1,1), \forall \gamma$ so that $\chi_{L}^*\bar{\chi}_{L}^* = 1$, the WZW action (5.4) is well-defined. Indeed, one only has to worry about the non-local WZ term. To define it, one extends $G$ to a function $G(t, w, \bar{w})$ such that $G(1, w, \bar{w}) = g\bar{g}$ and $G(0, w, \bar{w}) = 1$. Then the boundary terms $\partial w(\ldots), \partial \bar{w}(\ldots)$ that arise in the WZ term cancel pairwise from the components of the boundary related by $\gamma$. However, the above condition on the monodromy matrices is too restrictive and we don’t want to impose it. Instead, we have to assume that the WZW actions $I_{WZW}[g], I_{WZW}[\bar{g}]$ can be separately defined. It must be possible to define them using a procedure similar to that used in [39] to define a higher genus analog of the Polyakov action. As is shown in [39], one can define it as an integral of the same integrand as in (5.8) over a fundamental region on $H$ plus a certain set of boundary terms, which make the action independent of a choice of the fundamental region, and make the variational principle well-defined. We shall assume that a similar procedure can be used to define the WZW actions (5.7). The dependence on $T$ in this action may not be that simple as in (5.7). In fact, one does expect $T$ to enter the boundary terms that are necessary to make the action well-defined. We shall also assume that one can make sense of the WZW action $I_{WZW}[g, \varphi]$ in (5.5). Indeed, as we saw, this is essentially the Liouville action for $\varphi$, which also does make sense on higher genus surfaces, see [38].

To show holomorphic factorization in the higher genus case we, following Witten [23], interpret (2.7) as a certain inner product. Witten noticed that the partition function $Z_{WZW}[A]$ of the WZW model coupled to the gauge field $A$ can be represented as an inner product:

$$Z_{WZW}[A] = \int DB \left| \chi[A, B] \right|^2 e^{\int d^2w B_w B_{\bar{w}}}. \hspace{1cm} (5.17)$$

The key is that $\chi(A, B)$ depends on the holomorphic component of $A$ and anti-holomorphic component of $B$, and thus can be interpreted as a special state in $\mathcal{H}_{CS} \otimes \overline{\mathcal{H}}_{CS}$. It can be decomposed over a basis of states in $\mathcal{H}_{CS}$:

$$\chi(A, B) = \sum_{I, J} \chi^{IJ} \Psi_I[A] \Psi_J[B]. \hspace{1cm} (5.18)$$

Integrating over $B$ in (5.17) one gets a holomorphically factorized partition function:

$$Z_{WZW}[A] = \sum_{I, J} N^{IJ} \Psi_I[A] \Psi_J[A]. \hspace{1cm} (5.19)$$

Actually, for the gauged WZW model $\chi^2 = \chi$ and $\chi^{IJ}$ is just the identity matrix. This means that $N^{IJ}$ is the inverse of the matrix giving the inner product of CS states:

$$N^{IJ} = (N_{IJ})^{-1}, \hspace{1cm} N_{IJ} = \int DA \Psi_I[A] \Psi_J[A]. \hspace{1cm} (5.20)$$
Integrating over $A$ one gets the partition function of the gauged WZW model, which, as can be seen from the above argument, is the dimension of the CS Hilbert space. One does not expect a similar “projective” property of $\chi$ in 3D gravity.

Witten’s argument can be applied in our case if one notices that the partition function similarly be represented as an inner product:

$$Z_{gr}[^{\varphi, \mu, \bar{\mu}]_T} = e^{K[^{\varphi, \mu, \bar{\mu}]_T]} \int D\mathcal{T} D\bar{\mathcal{T}} \left| \chi[^{\mu, T}]^{\varphi, \mu, \bar{\mu}} \right|^2 e^{-\frac{1}{2} \int d^2w \epsilon^{-\varphi} \bar{T} T},$$

(5.21)

where $\chi[^{\mu, T}]^{\varphi, \mu, \bar{\mu}}$ can be thought of as a state in $H_{\Sigma_g} \otimes \overline{H_{\Sigma_g}}$. Here $H_{\Sigma_g}$ is the Hilbert space obtained by quantizing the Schottky space $\Sigma_g$. This needs some explanation. The first argument of $\chi[^{\mu, T}]^{\varphi, \mu, \bar{\mu}}$ is the Beltrami differential for $\Sigma$, which is a holomorphic coordinate on $\Sigma_g$. Thus, as a function of $\mu$, $\chi[^{\mu, T}]^{\varphi, \mu, \bar{\mu}}$ can be thought of as a state in $H_{\Sigma_g}$. The second argument is a quadratic differential $T$. To understand why it can also be thought of as a holomorphic coordinate on $\Sigma_g$, let us recall some elements of Teichmuller theory. As is well-known, there are two different realizations of the Teichmuller space $T_g = T(\Gamma)$, where $\Gamma$ is the Fuchsian group for the reference surface $X$. In one realization $T(\Gamma)$ is the space of all Fuchsian groups $\Gamma^\mu$ obtained by quasi-conformal deformations $f^\mu : \mathbb{H} \rightarrow \mathbb{H}$, $f^\mu = \mu f^\mu$. In order for $\Gamma^\mu = f^\mu \circ \Gamma \circ f^\mu^{-1}$ to be a Fuchsian group, the Beltrami differential must satisfy certain “reality condition”, namely $\mu(z, \bar{z}) = \bar{\mu}$. In the second realization $T(\Gamma)$ is the space of all quasi-Fuchsian groups. In this case $\mu(z, \bar{z}) = 0$ in the lower half-plane $\mathbb{H}$. In this second case there is the so-called Bers embedding of $T(\Gamma)$ into the space of quadratic differentials for $\Gamma$ holomorphic in the lower half-plane $\mathbb{H}$. Such quadratic differentials thus provide holomorphic coordinates on $\Sigma_g$. In this case one can decompose:

$$\chi[^{\mu, T}]^{\varphi, \mu, \bar{\mu}} = \sum_{IJ} \chi^{IJ}[\mu] \Psi_I[^{\mu}] \Psi_J[^{T}].$$

(5.22)

Using this decomposition, one gets a holomorphically factorized partition function in the form:

$$Z_{gr}[^{\varphi, \mu, \bar{\mu}]_T} = e^{K[^{\varphi, \mu, \bar{\mu}]_T]} \sum_{IJ} N^{IJ}[\mu] \Psi_I[^{\mu}] \Psi_J[^{T}].$$

(5.23)

The quantities $N^{IJ}$ can in principle depend on the moduli of the reference surface $X$. In the case of CS theory it is known that a basis of the conformal blocks $\Psi_I[A]$ can be chosen in such a way that $N^{IJ}$ are moduli independent. It would be important to establish an analogous property in our case. Similarly to the CS case, it would require constructing a vector bundle over $\Sigma_g$, whose fibers are isomorphic to $H_{\Sigma_g}$, and constructing a projectively flat connection in this bundle. We leave this to future research.

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A CS and WZW actions

The bulk CS action is given by:

$$\tilde{I}_{CS}[A] = \frac{1}{2} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$  \hspace{1cm} (A.1)

Variation of the bulk CS action gives a term proportional to the equation of motion plus a boundary term:

$$\delta \left( \tilde{I}_{CS}[A] \right) = \int_M \text{Tr} \left( \delta A \wedge F(A) \right) - \frac{1}{2} \int_{\partial M} \text{Tr} A \wedge \delta A. \hspace{1cm} (A.2)$$

To have a well-defined action principle on a manifold with boundary one must specify boundary conditions. The standard procedure is to fix a complex structure on the boundary and fix either the holomorphic or anti-holomorphic part of the connection. We shall use the following rule to switch between differential form and complex notation:

$$\int_{\partial M} A \wedge B = \int_{\partial M} dw \wedge d\bar{w} \left( A_w B_{\bar{w}} - A_{\bar{w}} B_w \right) = 2i \int_{\partial M} d^2 w \left( A_w B_{\bar{w}} - A_{\bar{w}} B_w \right), \hspace{1cm} (A.3)$$

where we have introduced the area element

$$d^2 w = \frac{dw \wedge d\bar{w}}{2i}. \hspace{1cm} (A.4)$$

With this convention, the action suitable for boundary condition $A_w$ kept fixed is:

$$I_{CS}^+[A] = \tilde{I}_{CS}[A] + i \int d^2 w \text{Tr} A_w A_{\bar{w}}. \hspace{1cm} (A.5)$$

The action for $A_{\bar{w}}$ fixed is:

$$I_{CS}^-[A] = \tilde{I}_{CS}[A] - i \int d^2 w \text{Tr} A_w A_{\bar{w}}. \hspace{1cm} (A.6)$$

Under a gauge transformation

$$A \rightarrow A^g = g^{-1} A g + g^{-1} d g \hspace{1cm} (A.7)$$

the bulk action transforms as

$$\tilde{I}_{CS}[A^g] = \tilde{I}_{CS}[A] + C(g, A), \hspace{1cm} (A.8)$$
with the co-cycle
\[
C(g, A) = -\frac{1}{6} \int_M \text{Tr} (g^{-1} dg)^3 + \frac{1}{2} \int_{\partial M} \text{Tr} (A \wedge dg g^{-1}).
\] (A.9)

Behavior of the chiral actions under gauge transformations is as follows:
\[
I^+_{\text{CS}}[A^\text{g}] = I^+_{\text{CS}}[A] - iI^+_{\text{WZW}}[g] + 2i \int d^2w \text{Tr} A_w \partial_w gg^{-1},
\] (A.10)
\[
I_{\text{CS}}[A^\text{g}] = I_{\text{CS}}[A] + iI_{\text{WZW}}[g] - 2i \int d^2w \text{Tr} A_w \partial_w gg^{-1}.
\] (A.11)

Here the two WZW actions are given by:
\[
I^+_{\text{WZW}}[g] = -\frac{1}{4} \int d^2x \text{Tr} (g^{-1} \partial^\mu gg^{-1} \partial^\nu g) + \frac{i}{6} \int \text{Tr} (g^{-1} dg)^3 =
= -\int d^2w \text{Tr} (g^{-1} \partial^w gg^{-1} \partial_w g) + \frac{i}{6} \int \text{Tr} (g^{-1} dg)^3.
\] (A.12)

The minus sign in front of the first term is standard. It makes the non-linear sigma-model action (first term) positive for \(g\) unitary. The action \(I^+_{\text{WZW}}\) gives as its equation of motion that the current \(J_\omega = g^{-1} \partial^\omega g\) is conserved \(\partial_w J_\omega = 0\), while \(I_{\text{WZW}}\) implies that \(J_w = g^{-1} \partial_w g\) is conserved. Note that \(I^-_{\text{WZW}}[g] = I^+_{\text{WZW}}[g^{-1}]\).

We shall also need the Polyakov-Wiegman identity:
\[
I^+_{\text{WZW}}[gh] = I^+_{\text{WZW}}[g] + I^+_{\text{WZW}}[h] - 2 \int d^2w \text{Tr} g^{-1} \partial_w g \partial_\omega hh^{-1},
\] (A.13)
\[
I^-_{\text{WZW}}[gh] = I^-_{\text{WZW}}[g] + I^-_{\text{WZW}}[h] - 2 \int d^2w \text{Tr} g^{-1} \partial_w g \partial_\omega hh^{-1}.
\] (A.14)

**B Chern-Simons connections**

In this appendix we obtain explicit expressions for the complex CS connections corresponding to the metric (4.6). Let us rewrite the metric (4.6) in the complex frame form as:
\[
ds^2 = \left(\frac{d\rho}{\rho}\right)^2 + \theta_w \theta_\bar{w},
\] (B.1)
where
\[
\theta_w = \frac{1}{2} \rho e^{-\varphi/2} T^\varphi dw + \frac{1}{\rho} e^{\varphi/2} (1 + \frac{1}{2} \rho^2 e^{-\varphi} R) d\bar{w},
\] (B.2)
and \(\theta_\bar{w} = \bar{\theta}_w\). Let us also introduce the real frame:
\[
e_1 = d\rho/\rho, \quad e_2 = \frac{1}{2} (\theta_w + \theta_\bar{w}), \quad e_3 = \frac{1}{2i} (\theta_w - \theta_\bar{w}).
\] (B.3)

It is not hard to find the spin connection coefficients. One gets:
\[
w_{12} = \frac{1}{2\rho} e^{\varphi/2} \left[\left(1 - \frac{1}{2} \rho^2 e^{-\varphi} (R + T^\varphi)\right) dw + \left(1 - \frac{1}{2} \rho^2 e^{-\varphi} (R + \bar{T}^\varphi)\right) d\bar{w}\right],
\] (B.4)
\[
w_{31} = \frac{i}{2\rho} e^{\varphi/2} \left[\left(1 - \frac{1}{2} \rho^2 e^{-\varphi} (R - \bar{T}^\varphi)\right) d\bar{w} - \left(1 - \frac{1}{2} \rho^2 e^{-\varphi} (R - T^\varphi)\right) dw\right],
\]
\[
w_{23} = \frac{1}{2i} (C dw - \bar{C} d\bar{w}).
\]
Here the quantity $C$ is given by a rather complicated expression:

$$
C = \frac{1}{(1 + \frac{1}{2}\rho^2 e^{-\varphi} R)^2 - \rho^2 e^{-2\varphi} T^\varphi T^\varphi} \left[ \varphi_w + \rho^2 e^{-\varphi} (R_w - T^\varphi_w) 
+ \frac{1}{4} \rho^4 e^{-2\varphi} (2T^\varphi T^\varphi_w + 2RT^\varphi \varphi_w + 2RR_w - T^\varphi T^\varphi \varphi_w - R^2 \varphi_w - 2RT^\varphi_w - 2T^\varphi R_w) \right].
$$

(B.5)

Here we gave the full expression without using the fact that $T^\varphi, T^\varphi$ are conserved. Note that in the limit $\rho \to 0$

$$
C = \varphi_w.
$$

(B.6)

We also note that in the flat case $R = 0$, and using the conservation laws for $T^\varphi, T^\varphi$ one gets $C = \varphi_w$ exactly and not just in the $\rho \to 0$ limit.

Let us now find the CS connections. It is customary to represent these in the matrix form. We introduce the anti-hermitian matrices $J^1 = i\sigma^3, J^2 = i\sigma^2, J^3 = i\sigma^1$, where $\sigma^i$ are the standard Pauli matrices. Then define:

$$
e = e_i J^i = \begin{pmatrix} id\rho/\rho & \theta_w \\
-\theta_w & -id\rho/\rho \end{pmatrix}.
$$

(B.7)

The metric is then given by:

$$
ds^2 = -\frac{1}{2} \text{Tr}(ee).
$$

(B.8)

The CS connections are defined as:

$$
A_i = w_i + i e_i, \quad \bar{A} = w_i - i e_i,
$$

(B.9)

where

$$
w_i = \frac{1}{2} \epsilon_{ijk} w_{jk}.
$$

(B.10)

Let us also introduce the matrix valued connections $A, \bar{A}$:

$$
A = \frac{1}{2} A_i J^i, \quad \bar{A} = \frac{1}{2} \bar{A}_i J^i,
$$

(B.11)

or, equivalently

$$
A = w + \frac{i}{2} e, \quad \bar{A} = w - \frac{i}{2} e,
$$

(B.12)

where $w = (1/2)w_i J^i$. The factor of $1/2$ in the decomposition of the connections into $J^i$ is adjusted so that the curvatures of $A, \bar{A}$ are given by the usual expressions $F(A) = dA + A \wedge A$ and similarly for $\bar{A}$. Using the expressions (B.4) for the spin connection we get:

$$
A = \begin{pmatrix}
-\frac{d\varphi}{2\rho} - \frac{1}{4} C dw + \frac{1}{4} \bar{C} dw \\
\frac{i}{2} \rho e^{-\varphi/2} (T^\varphi dw + Rd\bar{w}) \\
-\frac{1}{4} \rho e^{\varphi/2} dw + \frac{d\varphi}{2\rho} + \frac{1}{4} C dw - \frac{1}{4} \bar{C} dw
\end{pmatrix}
$$

(B.13)

$$
\bar{A} = \begin{pmatrix}
+\frac{d\varphi}{2\rho} - \frac{1}{4} C dw + \frac{1}{4} \bar{C} dw \\
\frac{i}{2} \rho e^{-\varphi/2} (R dw + T^\varphi dw) \\
\frac{1}{2} \rho e^{-\varphi/2} (R dw + T^\varphi dw) - \frac{d\varphi}{2\rho} + \frac{1}{4} C dw - \frac{1}{4} \bar{C} dw
\end{pmatrix}
$$

(B.14)

Note that

$$
\bar{A} = -(A)^\dagger.
$$

(B.15)
The above expressions for the CS connections are also found in [41]. In the genus one case the boundary geometry is flat $R = 0$ and these expressions essentially coincide with the ones found, e.g., by Banados [15]. To see this, we note that in the flat case the Liouville field is given by the sum of holomorphic and anti-holomorphic pieces $\varphi = A + \bar{A}$, and $e^{A/2}$ can be absorbed into the complex coordinate $w$. This removes the exponentials and the factors proportional to $C$ on the diagonal and the resulting connections are exactly those of [15] (up to some sign differences which stem from a difference in conventions).

We will also need the expressions for the connections $A, \bar{A}$ in the limit $\rho \rightarrow 0$. These are obtained by replacing $C$ in (B.13), (B.14) with $\varphi_w$. One gets, in components:

$$A_\rho \sim \begin{pmatrix} -\frac{1}{2\rho} & 0 \\ 0 & \frac{1}{2\rho} \end{pmatrix}, \quad A_w \sim \begin{pmatrix} \frac{i}{4} \varphi_w e^{-\varphi/2} T^\varphi \\ -\frac{i}{4} \varphi_w e^{\varphi/2} \end{pmatrix}, \quad A_\bar{w} \sim \begin{pmatrix} \frac{i}{4} \varphi_\bar{w} \bar{T} \varphi \\ -\frac{i}{4} \varphi_\bar{w} \end{pmatrix}$$

(B.16)

and

$$\bar{A}_\rho \sim \begin{pmatrix} \frac{1}{2\rho} & 0 \\ 0 & -\frac{1}{2\rho} \end{pmatrix}, \quad \bar{A}_w \sim \begin{pmatrix} \frac{i}{4} \varphi_w e^{-\varphi/2} R \\ \frac{i}{4} \varphi_w \end{pmatrix}, \quad \bar{A}_\bar{w} \sim \begin{pmatrix} \frac{i}{4} \varphi_\bar{w} \bar{T} \varphi \\ -\frac{i}{4} \varphi_\bar{w} \end{pmatrix}$$

(B.17)

C Projective structures

There is a one-to-one correspondence between holomorphic quadratic differentials on a Riemann surface and equivalence classes of projective structures. Thus, the DOF described by $T, \bar{T}$ are those of a projective structure. Let us review this correspondence. We use [42] as the main source. We give a description in terms of Fuchsian groups and Fuchsian uniformization. Analogous facts hold in the Schottky picture, except that one uses the whole complex plane instead of the hyperbolic plane $\mathbf{H}$.

A projective structure on a Riemann surface $X = \mathbf{H}/\Gamma$ is a complex analytic function $f(z)$ on the covering space $H$ that satisfies:

$$f(\gamma \circ z) = \chi_\gamma \circ f(z), \quad \forall \gamma \in \Gamma.$$  

(C.1)

Here $\chi_\gamma$ is a representation of $\gamma \in \Gamma$ in the group M"ob that acts on $f(z)$ by a fractional linear transformation. Thus, projective structures on $X$ are in one-to-one correspondence with representations $\chi \in \text{Hom}(\Gamma, \text{M"ob})$. Inequivalent projective structures are defined as corresponding to inequivalent representations, where equivalent representations are those related by a conjugation in M"ob.

Here are some examples of projective structures. The simplest example is that of the Fuchsian projective structure. In this case the function $f$ is the identity: $f(z) = z$, and the representation $\chi$ of $\Gamma$ is the Fuchsian group itself. The second example is the Schottky projective structure. Having a Schottky uniformization of a given Riemann surface $X$ (it depends on a choice of the maximal set of non-intersecting, homotopy non-trivial and non-equivalent set of curves on $X$), there is a map
$J(z)$ from the unit disc $H$ to the complex plane of the Schottky uniformization:

\[ \begin{array}{ccc}
H & \xrightarrow{J} & \Omega \\
\pi & \downarrow & \pi \\
X & \swarrow & \pi
\end{array} \quad \text{(C.2)} \]

Here $\pi$ is the quotient map $\pi : \Omega \to \Omega/\Sigma = X$, where $\Sigma$ is the corresponding Schottky group.

The map $J(z)$ gives the Schottky projective structure:

\[ J(A \circ z) = J(z), \quad J(B \circ z) = L \circ J(z). \quad \text{(C.3)} \]

Here $A \in \Gamma$ correspond to the elements of $\pi_1(X)$ that go around the handles along which one cuts to obtain the Schottky uniformization, and $L \in \text{SL}(2, \mathbb{C})$ are the generators of the Schottky group.

Let us now discuss a one-to-one correspondence between equivalence classes of projective structures and holomorphic quadratic differentials for $\Gamma$. The relation is that the Schwartzian derivative $S(f)$ of the function $f(z)$ defining the projective structure gives a holomorphic quadratic differential: $S(f) \circ \gamma (\gamma')^2 = S(f)$. The opposite is also true. Namely, given a holomorphic quadratic differential $T$, there exists a solution $f$ of the Schwartz equation

\[ S(f; z) = T. \quad \text{(C.4)} \]

The solution is unique modulo the natural action of M"{o}b on the left. Thus, a quadratic differential defines a projective structure up to equivalence. All in all, we have:

\[ f(z) - \text{proj. structure} \iff S(f; z) \, dz^2 - \text{holom. quadratic differential for } \Gamma \quad \text{(C.5)} \]

There is a canonical lift of the representation $\xi \in \text{Hom}(\Gamma, \text{M"{o}b})$ to a representation in $\text{SL}(2, \mathbb{C})$. It is given by the so-called *monodromy representation* $\chi^*$ of a projective structure. Namely, consider the so-called Fuchs equation:

\[ u'' + \frac{1}{2} T u = 0. \quad \text{(C.6)} \]

Here $T$ is a holomorphic quadratic differential on $X = \mathbb{H}/\Gamma$. The monodromy group of this equation gives a representation $\chi^*$ of $\Gamma$ in $\text{SL}(2, \mathbb{C})$. To obtain the monodromy representation one uses the following simple lemma [43]:

**Lemma 1** If $u, v$ are two linearly independent solutions of the Fuchs equation (C.6), then

\[ \frac{1}{\sqrt{\gamma}} \begin{pmatrix} u \\ v \end{pmatrix} (\gamma z) \]

also satisfies the same equation with respect to $z$.

This immediately means that

\[ \frac{1}{\sqrt{\gamma}} \begin{pmatrix} u \\ v \end{pmatrix} (\gamma z) = \chi^*_{\gamma} \begin{pmatrix} u \\ v \end{pmatrix}(z), \quad \text{(C.7)} \]
where $\chi^*_\gamma$ is some matrix independent of $z$. This is the monodromy representation. The ratio of two linearly independent solutions $f = u/v$ satisfies $S(f; z) = T$ and gives a projective structure.

The last fact we need is a relation between projective structures and equivalence classes of rank 2 holomorphic complex vector bundles over $X$. Given the monodromy representation $\chi^* \in \text{Hom}(\Gamma, \text{SL}(2, \mathbb{C}))$ of a projective structure, one gets a holomorphic vector bundle over $X$ as a quotient of the trivial bundle $\mathbb{C}^2 \times H$ over $H$. The equivalence relation used to get the quotient is: $\mathbb{C}^2 \times H \ni \{F, z\} \sim \{\chi^*(\gamma) F, \gamma \circ z\}$. Here $F$ is a vector $F = (u, v)$.

In practice it is more convenient to work with a somewhat different, but related bundle over $X$. The following definitions and facts are from [43, 19]. Let us introduce the following holomorphic rank 2 vector bundle $E$ over $X$: $E = (z, F)/\sim$, where the equivalence relation is: $(z, F) \sim (\gamma z, \eta_\gamma(z) F)$. Here $\eta_\gamma(z)$ is a factor of automorphy, given by:

$$\eta_\gamma(z) = \begin{pmatrix} (\gamma')^{-1/2} & \frac{d}{dz} (\gamma')^{-1/2} \\ 0 & (\gamma')^{1/2} \end{pmatrix}. \quad (C.8)$$

Consider the following holomorphic connection in $E$:

$$\alpha = \begin{pmatrix} 0 & -\frac{1}{2} T \\ 1 & 0 \end{pmatrix} dz. \quad (C.9)$$

Here $T$ is some holomorphic quadratic differential for $\Gamma$. As is not hard to check, this connection can also be represented as:

$$\alpha = m^{-1} dm, \quad (C.10)$$

where $m$ is given by:

$$m = \begin{pmatrix} u & u_z \\ v & v_z \end{pmatrix}, \quad (C.11)$$

and $u, v$ are two linearly independent solutions of the Fuchs equation (C.6) for $T$. The holomorphic connection $\alpha$ is a connection in the bundle $E$ for it satisfies the following transformation property:

$$\eta_{\gamma}^{-1} \gamma^* \alpha \eta_\gamma + \eta_{\gamma}^{-1} d\eta_\gamma = \alpha. \quad (C.12)$$

Here $\gamma^* \alpha$ is the pullback of the connection $\alpha$ under the mapping $z \rightarrow \gamma z$. Let us finally note one more lemma showing a relation between the holonomy of the connection $\alpha$ and the monodromy matrix $\chi^*_\gamma$.

**Lemma 2** Let matrix $m(z) \in \text{SL}(2, \mathbb{C})$ be given by (C.11) with $u, v$ being two linearly independent solutions of the Fuchs equation. Then

$$m(\gamma z) \eta_\gamma(z) = \chi^*_\gamma m(z). \quad (C.13)$$

This shows that the holonomy of the connection $\alpha$ along some cycle on $X$ is essentially (up to the factor of automorphy) the corresponding monodromy matrix $\chi^*$.

As is discussed in [42], not all classes of representations $\chi^* \in \text{Hom}(\Gamma, \text{SL}(2, \mathbb{C}))$ correspond to projective structures. The representations that are excluded are the unitary and reducible ones, or the ones that become unitary or reducible when restricted to any subgroup of finite index in $\Gamma$. 

To summarize, there is a set of relations between projective structures $f$, holomorphic quadratic differentials $T = S(f)$, solutions $u, v : u/v = f$ of the Fuchs equation (C.6) for $T$, and holomorphic rank 2 vector bundles with the canonical holomorphic connection given by (C.9).

Let us now quickly state analogous facts for anti-holomorphic projective structures. Equivalence classes of such are in one-to-one correspondence with anti-holomorphic quadratic differentials $\overline{T}$. One can similarly introduce the anti-holomorphic Fuchs equation

\[
\ddot{\bar{u}} + \frac{1}{2} \overline{T} \bar{u} = 0. \quad (C.14)
\]

**Lemma 1'** If $\bar{u}, \bar{v}$ are two linearly independent solutions of the Fuchs equation (C.14), then

\[
\frac{1}{\sqrt{\gamma}} \begin{pmatrix} \bar{u} & -\bar{v} \end{pmatrix} (\gamma z)
\]

also satisfies the same equation with respect to $\bar{z}$.

This means that

\[
\frac{1}{\sqrt{\gamma}} \begin{pmatrix} \bar{u} & -\bar{v} \end{pmatrix} (\gamma z) = \begin{pmatrix} \bar{u} & -\bar{v} \end{pmatrix} (z) \bar{\chi}_\gamma^*.
\]  

(C.15)

Here $\bar{\chi}_\gamma^*$ is related to $\chi_\gamma^*$ as follows:

\[
\chi_\gamma^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \bar{\chi}_\gamma^* = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}.
\]

One can similarly introduce an anti-holomorphic vector bundle $\overline{E}$. It is defined using the following factor of automorphy:

\[
\overline{\eta}_\gamma(\bar{z}) = \begin{pmatrix} (\gamma')^{-1/2} & 0 \\ -\frac{d}{dz} (\gamma')^{-1/2} & (\gamma')^{1/2} \end{pmatrix}.
\]

(C.17)

The connection

\[
\bar{\alpha} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} \overline{T} & 0 \end{pmatrix} d\bar{z}
\]

is a connection on $\overline{E}$:

\[
\overline{\eta}_\gamma \gamma^* \bar{\alpha} \overline{\eta}_\gamma^{-1} + \bar{\eta}_\gamma d\overline{\eta}_\gamma^{-1} = \bar{\alpha}.
\]

(C.19)

The connection (C.18) can be represented as

\[
\bar{\alpha} = \overline{\mathbf{m}} d\overline{\mathbf{m}}^{-1},
\]

(C.20)

where

\[
\overline{\mathbf{m}} = \begin{pmatrix} \bar{u} & -\bar{v} \\ -\bar{u}_z & \bar{v}_z \end{pmatrix}.
\]

(C.21)

**Lemma 2'** The matrix $\overline{\mathbf{m}}$ satisfies

\[
\overline{\eta}_\gamma(z) \overline{\mathbf{m}}(\gamma z) = \overline{\mathbf{m}}(z) \bar{\chi}_\gamma^*.
\]

(C.22)
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