Obtaining mass parameters of compact objects from red-blue shifts emitted by geodesic particles around them.

Ricardo Becerril\textsuperscript{1}, Susana Valdez-Alvarado\textsuperscript{2} and Ulises Nucamendi\textsuperscript{1},
\textsuperscript{1}Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo. Edif. C-3, 58040 Morelia, Michoacán, México,
\textsuperscript{2}Facultad de Ciencias de la Universidad Autónoma del Estado de México, Instituto Literario No. 100, C.P. 50000, Toluca, Estado de México, México.

(Dated: October 7, 2016)

The mass parameters of compact objects such as Boson Stars, Schwarzschild, Reissner Nordstrom and Kerr black holes are computed in terms of the measurable redshift-blueshift \((z_{\text{red}}, z_{\text{blue}})\) of photons emitted by particles moving along circular geodesics around these objects and the radius of their orbits. We found bounds for the values of \((z_{\text{red}}, z_{\text{blue}})\) that may be observed. For the case of Kerr black hole, recent observational estimates of SgrA
\textsuperscript{*} mass and rotation parameter are employed to determine the corresponding values of these red-blue shifts.

I. INTRODUCTION

The increasing amount of evidence that many galaxies contain a supermassive black hole at their center \cite{Herrera-2001}, motivated Herrera and Nucamendi (hereafter referred as H-N) to develop a theoretical approach to obtain the mass and rotation parameter of a Kerr black hole in terms of the redshift and rotation parameter of a Kerr black hole in terms of that data set. In this paper, we provide a brief summary of H-N theoretical scheme in the section II. In order to have a self contained paper, we provide a brief description of how this can be accomplished. Particularly, the mass of the black hole for SgrA
\textsuperscript{*} and its corresponding angular momentum that have been recently estimated \cite{Narayan-2007}: \(M \sim 2.7 \times 10^{6} M_{\odot}\) and \(a \sim 0.9939 M\) are employed in our analysis. In addition, the mass parameter of axialsymmetric non-rotating compact objects such as Schwarzschild and Reissner-Nordstrom black holes as well as Boson-Stars is found in terms of the red-blue shift of light and the orbit radius of emitting particles. In order to have a self contained paper, we provide a brief summary of H-N theoretical scheme in the section II. In sections III and IV we deal with the non-rotating examples above mentioned and the rotating Kerr black hole respectively.

II. THEORETICAL APPROACH

H-N considered a rotating axialsymmetric space-time in spherical coordinates \((x^{\mu}) = (t, r, \theta, \phi)\). The geodesic trajectory followed by a massive particle in this space-time can be obtained by solving the Euler-Lagrange equations

\[
\frac{\partial \mathcal{L}}{\partial x^{\mu}} - \frac{d}{d\tau} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}} \right) = 0,
\]

with the Lagrangian \(\mathcal{L}\) given by

\[
\mathcal{L} = \frac{1}{2} \left[ g_{tt} \dot{t}^2 + 2 g_{t\phi} \dot{t} \dot{\phi} + g_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 + g_{\phi\phi} \dot{\phi}^2 \right],
\]

being \(\dot{x}^{\mu} = \frac{dx^{\mu}}{d\tau}\) and \(\tau\) the proper time. It is assumed that the metric depends solely on \(r\) and \(\theta\); thus, the space time is endowed with two commuting Killing vectors \([\xi, \psi] = 0\) which read: \(\xi = (1, 0, 0, 0), \psi = (0, 0, 0, 1)\).

Since \(g_{\mu\nu} = g_{\mu\nu}(r, \theta)\), there are two quantities that are conserved along the geodesics

\[
p_{t} = \frac{\partial \mathcal{L}}{\partial \dot{t}} = g_{tt} \dot{t} + g_{t\phi} \dot{\phi} = g_{tt} U^{t} + g_{t\phi} U^{\phi} = -E,
\]

\[
p_{\phi} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = g_{\phi\phi} \dot{\phi} = g_{\phi\phi} U^{\phi} = L,
\]

where \(U^{\mu} = (U^{t}, U^{r}, U^{\theta}, U^{\phi})\) is the 4-velocity which is normalized to unity rendering

\[
-1 = g_{tt}(U^{t})^2 + g_{rr}(U^{r})^2 + g_{\theta\theta}(U^{\theta})^2 + g_{\phi\phi}(U^{\phi})^2 + g_{t\phi} U^{t} U^{\phi}.
\]

Two of these 4-velocity components can be found by inverting \cite{Herrera-2001}:

\[
U^{t} = \frac{g_{\phi\phi} E + g_{t\phi} L}{g_{\phi\phi} - g_{t\phi} g_{\phi\phi}}, \quad U^{\phi} = \frac{g_{t\phi} E + g_{tt} L}{g_{\phi\phi} - g_{t\phi} g_{\phi\phi}}.
\]

Inserting (5) in (4) one obtains

\[
g_{rr} (U^{r})^2 + V_{eff} = 0,
\]
where $V_{eff}$ is an effective potential given by

$$V_{eff} = 1 + g_{t\phi} (U^\phi)^2 - \frac{E^2_g g_{\phi\phi} + L^2 g_{tt} + 2ELg_{t\phi}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}.$$  \hfill (7)

The goal is to write the parameters of an axially symmetric space-time in terms of the observational red and blue shifts $z_{red}$ and $z_{blue}$ of light emitted by massive particles moving around a compact object. These photons have 4-momentum $k^\mu = (k^t, k^r, k^\phi, k^\theta)$ that move along null geodesics $k_{\mu}k^\mu = 0$. Using the same Lagrangian \cite{12} one gets two conserved quantities

$$- E\gamma = g_{tt}k^t + g_{\phi\phi}k^\phi, \quad L\gamma = g_{\phi\phi}k^t + g_{t\phi}k^\phi.$$  \hfill (8)

The frequency shift $z$ associated to the emission and detection of photons is defined as

$$1 + z = \frac{\omega_e}{\omega_d},$$  \hfill (9)

where $\omega_e$ is the frequency emitted by an observer moving with the massive particle at the emission point $e$ and $\omega_d$ the frequency detected by an observer far away from the source of emission. These frequencies are given by

$$\omega_e = -k_{\mu}U^\mu|_e, \quad \omega_d = -k_{\mu}U^\mu|_d.$$  \hfill (10)

$U^\mu_e$ and $U^\mu_d$ are the 4-velocity of the emitter and detector respectively. If the detector is located far away from the source ($r \to \infty$) then $U^\mu_d = (1, 0, 0, 0)$ since $U^\mu_d U^\mu_d \to 0$, whereas $U^\mu = E = 1$. The frequency $\omega_e = -k_{\mu}U^\mu|_e$ is explicitly given by

$$\omega_e = \left( E\gamma U^t - L\gamma U^\phi - g_{rr}U^r k^r - g_{t\phi}U^\phi k^\phi \right) |_e,$$

with a similar expression for $\omega_d$. As a result \cite{12} becomes

$$1 + z = \frac{(E\gamma U^t - L\gamma U^\phi - g_{rr}U^r k^r - g_{t\phi}U^\phi k^\phi) |_e}{(E\gamma U^t - L\gamma U^\phi - g_{rr}U^r k^r - g_{t\phi}U^\phi k^\phi) |_d}.$$  \hfill (11)

This is an expression for the red and/or blue shifts of light emitted by massive particles that are orbiting around a compact object measured by a distant observer. The apparent impact parameter $b \equiv \frac{L}{\omega_e}$ of photons, that is to say, the minimum distance to the origin $r = 0$ was introduced for convenience. Due to the fact that $E\gamma$ and $L\gamma$ are preserved along null geodesics all the way from emission to detection one has that $b_e = b_d$. On the other hand, a set of massive particles (that could be a set of stars) that may be orbiting around a compact object (that could be a black hole) is expanding as a whole and it has a redshift $z_c$. Yet all those particles are individually moving having therefore, an individual redshift.

Astronomers define a kinematic redshift as $z_{kin} = z - z_c$, and some report their data in terms of $z_{kin}$. $z_c$ corresponds to a frequency shift of a photon emitted by a static particle located at $b = 0$ thus

$$1 + z_e = \frac{(E\gamma U^t)|_e}{(E\gamma U^t)|_d} = \frac{U^t_e}{U^t_d}.$$  \hfill (12)

The kinematic redshift $z_{kin} = (1 + z) - (1 + z_e)$ can be written as

$$z_{kin} = \frac{(U^t - bU^\phi - \frac{1}{E\gamma}g_{rr}U^r k^r - \frac{1}{E\gamma}g_{t\phi}U^\phi k^\phi)|_e}{(U^t - bU^\phi - \frac{1}{E\gamma}g_{rr}U^r k^r - \frac{1}{E\gamma}g_{t\phi}U^\phi k^\phi)|_d} - \frac{U^t_e}{U^t_d}.$$  \hfill (13)

The analysis can be performed with either $z_{kin}$ using \cite{13} or $z$ using \cite{11}. We work with $z_{kin}$ in this paper. The general expression \cite{13} is simplified for circular orbits ($U^r = 0$) in the equatorial plane ($U^\theta = 0$)

$$z_{kin} = \frac{U^t_e U^t_d b_d - U^t_e U^t_d U^t_d}{U_d^2(U_d^t - b_d U_d^t)}.$$  \hfill (14)

In \cite{13} what is still needed is to take into account light bending due to gravitational field, that is to say, to find $b = b(r)$. The criteria employed in \cite{2} to construct this mapping is to choose the maximum value of $z$ at a fixed distance from the observed center of the source at a fixed $b$. Inverting \cite{3} to obtain $k^\mu = k^\mu(g_{\alpha\beta}, E, L)$ and inserting this expression into $k_{\mu}k^\mu = 0$ with $k^r = 0$ and $k^\theta = 0$ one arrives at

$$b_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{tt}}.$$  \hfill (15)

$b_{\pm}$ can be evaluated at the emitter or detector position. Since in general there are two different values of $b_{\pm}$, there will be two different values of $z$ of photons emitted by a receding ($z_1$) or an approaching object ($z_2$) with respect to a distant observer. These kinematic shifts of photons emitted either side of the central value $b = 0$ read

$$z_1 = \frac{U^t_e U^t_d b_{d_+} - U^t_d U^t_e b_{e_-}}{U^t_d(U^t_d - U^t_d U^t_d)},$$  \hfill (16)

$$z_2 = \frac{U^t_e U^t_d b_{d_-} - U^t_d U^t_e b_{e_+}}{U^t_d(U^t_d - U^t_d U^t_d)}.$$  \hfill (17)

In the next section we shall apply this formalism to non-rotating compact objects.
III. NON-ROTATING SPACE-TIMES

In order to apply H-N approach, it is necessary to have a Killing tensor $K_{\mu\nu}$ of the space-time to be analyzed, this implies the existence of an additional constant of motion $C = K_{\mu\nu}U^\mu U^\nu$. $C$ is not needed in the case of non-rotating space-times, that is to say, for $g_{t\phi} = 0$ or when particles are orbiting just on the equatorial plane. In the present section, we study the relationship between the observed redshift (blueshift) of photons emitted by particles traveling along circular and equatorial paths around non-rotating compact objects and the mass parameter of these objects. Since $g_{t\phi}$ vanishes, the apparent impact parameter becomes $b_\pm = \pm \sqrt{-g_{t\phi}/g_{tt}}$ and the effective potential \( V_{eff} \) acquires a rather simple form

\[
V_{eff} = 1 + \frac{E^2}{g_{tt}} + \frac{L^2}{g_{t\phi}}.
\]

For circular orbits $V_{eff}$ and its derivative $\frac{dV_{eff}}{dr}$ vanish. From these two conditions one finds two general expressions for the constants of motion $E^2$ and $L^2$ for any non-rotating axisymmetric space-time

\[
E^2 = -\frac{g_{t\phi}^2}{g_{tt}g_{t\phi} - g_{t\phi}g_{t\phi}}, \tag{19}
\]

\[
L^2 = \frac{g_{t\phi}^2}{g_{tt}g_{t\phi} - g_{t\phi}g_{t\phi}}, \tag{20}
\]

where primes denote derivative with respect to $r$. In order to guarantee stability of these circular orbits, $V''_{eff} > 0$ must hold. The general expression for $V''_{eff}$ is

\[
V''_{eff} = -E^2 \left[ g_{tt}g_{tt} - 2g_{t\phi}g_{t\phi} \right] - L^2 \left[ g_{t\phi}g_{t\phi} - 2g_{t\phi}g_{t\phi} \right]
\]

\[
= g_{t\phi}^2 g_{tt} - g_{tt}g_{t\phi} - 2g_{t\phi}g_{t\phi} + 2g_{t\phi}g_{t\phi} + g_{t\phi}g_{t\phi}, \tag{21}
\]

where \( 19 \) and \( 20 \) were employed in the last step. Using the explicit form of $E$ and $L$, \( 19 \) and \( 20 \), in \( 21 \) one obtains expression for the 4-velocities in terms of solely the metric components

\[
U^\phi = \sqrt{\frac{g_{tt}}{g_{tt}g_{t\phi} - g_{t\phi}g_{t\phi}}}, \quad U^t = -\sqrt{\frac{-g_{t\phi}}{g_{tt}g_{t\phi} - g_{t\phi}g_{t\phi}}}.
\]

From which the angular velocity of particles in these circular paths becomes

\[
\Omega = \frac{-g_{tt}}{g_{t\phi}}. \tag{23}
\]

Since $b_+ = -b_-$, the redshift $z_1 = z_{red}$ and blueshift $z_2 = z_{blue}$ are equal but with opposite sign: $z_1 = -z_2$, the explicit expression is

\[
12M \left( r_c - 2M \right) \left( r_c - 3M \right)
\]

Furthermore, if the detector is located far away from the compact object $r_d \rightarrow \infty$, and as we mentioned before, $U^\mu \rightarrow (1, 0, 0, 0)$. Thus \( 24 \) becomes

\[
z_1 = U^\phi b_{c\pm} = \sqrt{\frac{-g_{t\phi}g_{tt}}{g_{tt}(g_{tt}g_{t\phi} - g_{t\phi}g_{t\phi})}}. \tag{25}
\]

A. Schwarzschild Black Hole

As our first working example of a non-rotating space-time, we consider the Schwarzschild black hole, for which the relevant metric components are $g_{tt} = -(1 - 2M/r)$ and $g_{t\phi} = r^2 \sin^2 \theta$. Inserting these components in \( 25 \) with $\theta = \pi/2$ one finds

\[
z^2 = \frac{r_c M}{(r_c - 2M)(r_c - 3M)}, \tag{26}
\]

which is a relationship between the measured red-shift $z$, the mass parameter of a Schwarzschild black hole $M$ and the radius $r_c$ of a massive particle’s circular orbit that emits light and of course, $r_c > 3M$. The relationship \( 26 \) is equivalent to

\[
M = r_c F(z) \quad \text{where} \quad F(z) = 1 + 5z^2 + 10z^4 + z^4, \tag{27}
\]

On the other hand, circular orbits are stable as long as $V''_{eff} > 0$, from \( 21 \) $V''_{eff}$ reads

\[
V''_{eff} = \frac{2M(r_c - 6M)}{r_c^2(r_c - 2M)(r_c - 3M)}, \tag{28}
\]

which is positive provided that $r_c > 6M$; therefore, $\frac{M}{r_c} = F^{-1} > 6$ which is fulfilled if and only if $|z| < 1/\sqrt{2}$ and solely for the minus sign $F_-(z)$. Hence, a measurement of the redshift $z$ of light emitted by a particle that follows a circular orbit of radius $r_e$ in the equatorial plane around a Schwarzschild black hole will have a mass parameter determined by $M = r_c F_-(z)$, and $z$ must be $|z| < 1/\sqrt{2}$. The energy, angular momentum, velocities $U^t$, $U^\phi$ and the angular velocity of the emitter, can be computed from \( 19 \), \( 20 \), \( 22 \) and \( 23 \) and written as function of the measurable redshift $z$ and radius $r_c$ of the circular photons source’s orbit by using \( 27 \).
any multiple of the solar mass, this is to say, by the black hole mass parameter. Reissner-Nordström space-time which represents a electrically charged black hole, whose relevant metric components are \( g_{tt} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \) where \( Q \) is the electric charge parameter and \( g_{\phi\phi} = r^2 \sin^2 \theta \). For circular equatorial orbits of the photon source, the redshift reads

\[
\begin{align*}
E^2 &= \frac{(r_c - 2M)^2}{r_c(r_c - 3M)} = \frac{(1 - 2F_-(z))^2}{r_c(1 - 3F_-(z))}, \\
L^2 &= \frac{Mr_c^2}{r_c - 3M} = \frac{r_c^2 F_-(z)}{1 - 3F_-(z)}, \\
U^t &= \sqrt{\frac{r_c}{r_c - 3M}} = \frac{1}{\sqrt{1 - 3F_-(z)}}, \\
U^\phi &= \frac{1}{r_c} \sqrt{\frac{M}{r_c - 3M}} = \frac{1}{r_c} \sqrt{\frac{F_-(z)}{1 - 3F_-(z)}}, \\
\Omega &= \sqrt{\frac{M}{r_c^3}} = \sqrt{\frac{F_-(z)}{r_c^2}}.
\end{align*}
\]

(29)

(30)

(31)

The function \( M = M(r, z) = rF_-(z) \) is in geometrized units (\( G = c = 1 \)). In order to plot it, we scale \( M \) and \( r \) by any multiple of the solar mass, this is to say, by \( pM_\odot \), for \( Sr = 2p = 2.72 \times 10^6 \). Figure 1 shows this scaled relation \( M = M(r, z) \) which is symmetric with respect to the shift \( z (z_{red} > 0, z_{blue} < 0) \).

![Figure 1](image1.png)

FIG. 1. It is shown the mass parameter \( M \) as a function of redshift \( (z > 0) \) or blueshift \( (z < 0) \) and the radius \( r \) of an eventual circular orbit of a photon emitter. \( M \) and \( r \) are in geometrized units and scaled by \( pM_\odot \) where \( p \) is an arbitrary factor of proportionality.

Given a set of \( N \) pairs \( \{r, z\}_i \) of observed redshifts \( z \) (blueshifts) of emitters traveling around a Schwarzschild black hole along circular orbits of radii \( r \), a Bayesian statistical analysis might be carried out in order to estimate the black hole mass parameter.

**B. The Reissner-Nordström Black Hole**

Our next non-rotating working example is the Reissner-Nordström space-time which represents an electrically charged black hole, whose relevant metric components are \( g_{tt} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \) where \( Q \) is the electric charge parameter and \( g_{\phi\phi} = r^2 \sin^2 \theta \). For circular equatorial orbits of the photon source, the redshift reads

\[
z^2 = \frac{r_c^2(Mr_c - Q^2)}{(r_c^2 - 3Mr_c + 2Q^2)(r_c^2 - 2Mr_c + Q^2)}.
\]

(32)

This relationship is equivalent to

\[
M = r_cG_{\pm}(r_c, z^2, Q^2),
\]

(33)

where

\[
G_{\pm} = \frac{1}{12z^2} \left[ (5z^2 + 1) + \frac{7Q^2z^2}{r_c^2} \right.
\]

\[
\left. + \left( z^4 + 10z^2 + 1 + \frac{z^2Q^2}{r_c^2} \left[ \frac{z^2Q^2}{r_c^2} - 2(z^2 + 5) \right] \right)^{1/2} \right]
\]

(34)

In this case, the conserved quantities \( E^2 \) and \( L^2 \) are

\[
E^2 = \frac{(Q^2 + r_c(r_c - 2M))^2}{r_c^2(2Q^2 + r_c(r_c - 3M))},
\]

(35)

\[
L^2 = \frac{r_c^2(Mr_c - Q^2)}{2Q^2 + r_c(r_c - 3M)}.
\]

(36)

\( E^2 \) and \( L^2 \) are real only if \( r_c^2 - 3Mr_c + 2Q^2 > 0 \) and \( Mr_c - Q^2 > 0 \). Therefore, \( z^2 \) is positive provided that \( r_c^2 - 2Mr_c + Q^2 > 0 \). As it is known, in this metric, one distinguishes three regions: \( 0 < r < r_- \), \( r_- < r < r_c \) and \( r_+ < r \), where \( r_\pm = M \pm \sqrt{M^2 - Q^2} \) are the roots of \( r^2 - 2Mr + Q^2 = 0 \), which are real and distinct only if \( M^2 > Q^2 \) stands. The surface \( r = r_+ \) is an event horizon similar to that \( r = 2M \) for the Schwarzschild’s metric \( [10] \). Since \( r > r_+ \) implies \( r^2 - 2Mr + Q^2 > 0 \), our analysis is performed for \( r > r_+ \), that is, outside the event horizon.

The stability of circular equatorial orbits requirement

\[
V_{eff}'' = \frac{Mr_c(18Q^2 + 2r_c^2 - 12Mr_c - 8Q^4)}{r_c^2(2Q^2 + r_c(r_c - 3M))(Q^2 + r_c(r_c - 2M))} > 0,
\]

(37)

tells us that \( Mr_c(9Q^2 + r_c^2 - 6Mr_c - 4Q^4) > 0 \). Inserting \( M = r_cG_{\pm} \) into this last condition would yield, in principle, an inequality that may bound the values of the redshift \( z \), as it was the case for Schwarzschild. This inequality turns out to be cumbersome to be analyzed analytically; hence, the analysis was performed numerically in the following manner: given values of \( Q^2 \) and \( r_c \), we vary \( z^2 \) and compute \( M = r_cG_{\pm}(z^2, Q^2, r_c) \).
for each value of $z^2$. With this value $M$ at hand, we check whether the four conditions are all satisfied: (i) $M^2 > Q^2$, (ii) $r^2 - 3Mr + 2Q^2 > 0$, (iii) $Mr - Q^2 > 0$ and (iv) $Mr(9Q^2 + r^2 - 6Mr) - 4Q^4 > 0$. The second and third inequalities guarantee that, one indeed, has circular and equatorial orbits, the fourth stems from $V'_{c_{eff}} > 0$. We look for the minimum and maximum value of $z$ for which these four conditions are fulfilled. This process is repeated for several values of $Q^2$ and $r_c$. For $Q = 0$, the result for Schwarzschild ($|z| < 1/\sqrt{2}$) is recovered. Figure 2 shows the surfaces $z_{\min} = z_{\min}(r_c, Q^2)$ and $z_{\max} = z_{\max}(r_c, Q^2)$. Only for frequency shifts $z$ such that $|z| \in (z_{\min}, z_{\max})$, the corresponding values $M = M(z^2, Q^2, r_c) = r_c G_- \rho_m$ are acceptable.

The velocities $U^\phi$ and $U^t$ of photons emitters orbiting in circular and equatorial paths are

$$U^\phi = \frac{1}{r_c} \sqrt{\frac{Mr_c - Q^2}{r_c^2 (2Q^2 + r_c(r_c - 3M))}}$$

$$U^t = \sqrt{\frac{r_c^2}{2Q^2 + r_c(r_c - 3M)}}.$$

and their angular velocity is given by

$$\Omega = \sqrt{\frac{Mr_c - Q^2}{r_c^2}}.$$

Since $M = r_c G_-(z^2, r_c, Q^2)$, these 4-velocity components and $\Omega$ are actually functions of the redshift $z$, the radius of the circular orbit $r_c$ and the parameter $Q^2$. Unlike the Schwarzschild black hole, there is not an analytic relationship of the mass parameter $M$ in terms only of the measurable variables $z$ and $r$, it depends also on $Q^2$. At any rate, given a set of the observables $(z, r)$, Bayesian statistical analysis would provide an estimate for both parameters $M$ and $Q$.

### C. Boson Stars

Colpi et al performed a study of self-interacting Boson stars which were modeled by a complex scalar field endowed with a quartic potential $V = \frac{\lambda}{2} |\phi|^4 + \frac{\Lambda}{4} |\phi|^2$. The stability analysis yielded equilibrium configurations along either an stable and unstable branch [5, 6]. We will be concerned with stable equilibrium configurations of Boson stars for which the metric reads

$$ds^2 = -a^2(r)dt^2 + a^2(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

The components $g_{rr} = a^2(r)$ and $g_{\theta\theta} = -a^{-2}(r)$ are found by solving

$$\begin{align*}
\frac{da}{dx} &= \frac{\alpha}{2} \left[ \frac{1 - a^2 x}{a^2} + a^2 x \left( \frac{\Omega^2}{a^2} + 1 + \frac{\Lambda}{\Delta} \right) \phi^2 + \frac{\phi'}{a} \right], \\
\frac{d\phi}{dx} &= \frac{\alpha}{2} \left[ \frac{a^2 - 1 - x}{a^2} + a^2 x \left( \frac{\Omega^2}{a^2} - 1 - \frac{\Lambda}{\Delta} \right) \phi^2 + \frac{\phi'}{a} \right],
\end{align*}$$

where, for numerical purposes, we have introduced the following dimensionless variables: $x = mr$, $\phi = \sqrt{4\pi G\phi}$, $\Lambda = \lambda/4\pi G m^2$ and $\Omega = \omega/m$, where $m$ is the mass of complex scalar field $\phi$, $\omega$ its frequency and $\lambda$ the dimensionless self-coupling of the scalar. Here $'$ represents the derivative with respect to $x$.

For the complex scalar field, we consider a harmonic form $\Phi(t, r) = \phi(r)e^{-i \omega t}$ and solve the Klein-Gordon equation, that in terms of the dimensionless variables, takes the form

$$\ddot{\phi} + \left(1 - \frac{\Omega^2}{\alpha^2} + \frac{\Lambda}{\Delta} \right) \phi^2 \phi - \left(\frac{\alpha}{\alpha} - \frac{\alpha'}{a} + \frac{2}{a^2} \right) \dot{\phi} = 0.$$

The boundary conditions for the metric functions and the scalar field, in order to guarantee regularity at the origin and asymptotic flatness at infinity, are: $a(0) = 1$, $\alpha(0) = 1$, $\phi(0) = \phi_0$, $\phi'(0) = 0$, $\lim a(r) = \lim 1/a(x)$ and $\lim \phi(x) \approx 0$.

The system is basically an eigenvalue problem for the frequency of the boson star $\omega$ as a function of a parameter, the so called, central value of the scalar field $\phi_0$ which determines the mass $M$ of a boson star. This system can be solved by using the shooting method [9].
The table below, shows the values of the masses corresponding to stable and unstable boson stars for both $\Lambda = 0$ and $\Lambda = 100$.

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\Lambda & 0 & 100 \\
\hline
M_{\text{stable}} & M_{\text{crit}} & M_{\text{crit}} & M_{\text{stable}} & M_{\text{crit}} \\
\hline
\phi_0 & M_T & \phi_0 & M_T & \phi_0 & M_T \\
\hline
0.05 & 0.416 & 0.29 & 0.620 & 0.04 & 1.371 & 0.10 & 2.249 \\
0.25 & 0.020 & 0.80 & 0.431 & 0.08 & 2.227 & 0.16 & 1.892 \\
\hline
\end{array}
$$

One can also note that for configurations with the same value of mass, but different self-interacting parameter, the maximum redshift increases as $\Lambda$ decreases. For large values of $x$, the redshift for all configurations converge to the same values (see fig. 5).
IV. KERR BLACK HOLE

Explicit expressions for the shifts $z_1$ and $z_2$ computed at either side of $b = 0$ were found by H-N

$$z_1 = \pm \sqrt{M} \left( 2aM + r_c \sqrt{r_c^2 - 2Mr_c + a^2} \right) / r_c^{3/4}(r_c - 2M) \sqrt{r_c^2 - 3Mr_c^{1/2} + 2aM^{1/2}},$$

$$z_2 = \pm \sqrt{M} \left( 2aM - r_c \sqrt{r_c^2 - 2Mr_c + a^2} \right) / r_c^{3/4}(r_c - 2M) \sqrt{r_c^2 - 3Mr_c^{1/2} + 2aM^{1/2}} \tag{45}$$

Upper signs corresponds to co-rotating orbits and lower signs to counter-rotating orbits. From (45) the rotating parameter $a$ as a function of the mass parameter $M$, the radius of circular equatorial orbits $r_c$ of particles around the Kerr black hole emitting light and the corresponding $z_1$ and $z_2$ turns out to be

$$a^2(\alpha, \beta, r_c, M) = r_c^3(r_c - 2M)\alpha / 4M^2\beta - r_c^2\alpha, \tag{46}$$

where $\alpha \equiv (z_1 + z_2)^2$ and $\beta \equiv (z_1 - z_2)^2$. Nonetheless, there is not an explicit expression to find the mass parameter $M$, instead, there is an eight order polynomial for it derived also from (45). In this section, we carry out a numerical analysis to study how $M$ varies with $r_c$, and the shifts $z_1$ and $z_2$ detected by a far away observer. The metric components of the Kerr black hole in the Boyer-Lindquist coordinates are given by

$$g_{tt} = \left( 1 - \frac{2Mr}{\Sigma} \right), \quad g_{\phi\phi} = -\frac{2Mar \sin^2 \theta}{\Sigma},$$

$$g_{rr} = \frac{r^2 + a^2 + 2Ma^2r\sin^2 \theta}{\Sigma} \sin^2 \theta, \quad g_{\theta\theta} = \Sigma, \tag{47}$$

where

$$\Delta \equiv r^2 + a^2 - 2Mr, \quad \Sigma \equiv r^2 + a^2\cos^2 \theta,$$

with the restriction $M^2 \geq a^2$. For circular and equatorial orbits, the two conserved quantities are

$${E} = \frac{r^3/2 - 2M \sqrt{r} \pm a\sqrt{M}}{r^{3/4} \sqrt{r^3/2 - 3M \sqrt{r} \pm 2a\sqrt{M}}},$$

$${L} = \frac{\pm \sqrt{M}(r^2 + 2aM \sqrt{r} + a^2)}{r^{3/4} \sqrt{r^3/2 - 3M \sqrt{r} \pm 2a\sqrt{M}}} \tag{48}$$

Co-rotating orbits (upper signs) have $L > 0$ whereas counter-rotating (lower signs) orbits have $L < 0$. In order to have real values for $E$ and $L$, and thereby circular orbits, it is necessary that

$$r^{3/2} - 3M \sqrt{r} \pm 2a\sqrt{M} \geq 0. \tag{49}$$

Circular-equatorial orbits can be either bound or unbound. The later type are those for which, given a small outward perturbation, the particle will go to infinity, one has bound orbits otherwise. There are bound orbits provided that

$$r > r_{mb} = 2M \mp 2\sqrt{M}\sqrt{M \mp a} \tag{50}$$

is satisfied. Not all bound orbits are stable, only those whose radius satisfies $V_{eff}(r) \geq 0$ are stable [4]. This condition is akin to

$$r \geq r_{ms} = M \left[ 3 + Z_2 + \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)} \right],$$

$$Z_1 = 1 + \left( 1 - \frac{a^2}{M^2} \right)^{1/3} \left[ \left( 1 + \frac{a}{M} \right)^{1/3} + \left( 1 - \frac{a}{M} \right)^{1/3} \right],$$

$$Z_2 = \sqrt{3} a^2 / M^2 + Z_1^2. \tag{51}$$

$M$ can not be written as an explicit function of $r_c$, $\alpha$ and $\beta$, or equivalently as a function of $r_c$, $z_1$ and $z_2$. In order to find the mass parameter $M$, one has to numerically find the roots of the eight order polynomial derived from (45)

$$F(M) = \left[ 16r_cM^3 - (4\beta M^2 - \alpha r_c^2)(r_c - 2M)(r_c - 3M) \right]^2 - 4\alpha r_c^2M(r_c - 2M)^2(4\beta M^2 - \alpha r_c^2). \tag{52}$$

It is convenient to normalized $M$ by an arbitrary $M_{max}$ as $\tilde{M} = M / M_{max}$ thereby $0 < \tilde{M} \leq 1$. $r_c$ is also scaled with $M_{max}$ as $\tilde{r}_c = r_c / M_{max}$. $M_{max}$ may be chosen again as $p M_{50}$. We will work with the $M$ and $r_c$ variables henceforth but we will drop the tildes.
For a given value of the radius of the emitter’s circular path \( r_c \), one sets the size of the parameter domain \( D = (z_{1\text{min}}, z_{1\text{max}}) \times (z_{2\text{min}}, z_{2\text{max}}) \) where a search of these polynomial’s roots is carried out. The polynomial \( F \) has the following properties: \( F(M; r_c, z_1, z_2) = F(M; r_c, z_2, z_1) = F(M; r_c, -z_1, -z_2) \) which is useful for choosing \( D \). Recalling that the two different values of \( z \) correspond to photons emitted by a receding \( (z_1) \) or an approaching object \( (z_2) \) with respect to a distant observer, an opposite domain would be \( D = (0, z_{1\text{max}}) \times (-z_{2\text{min}}, 0) \). At each point \( q = (z_1, z_2) \in D \), \( F \) is numerically solved to attain \( M = M(q; r_c) \). One starts with a given fixed value of \( r_c \), and search in our domain \( D \) for the subset \( D_{r_c} \) where roots of \( F(M; r_c, q) = 0 \) exist. In principle, there may be up to eight real roots \( M \) (or none) at \( q \in D \). If there is at least one root, the corresponding \( a^2 \) is computed using (46) and we test whether \( M^2 \geq a^2 \) actually holds. If this is the case, \( r^{3/2} - 3M \sqrt{\pm 2aM} \geq 0 \) should be tested to determine for which roots of \( P(M; r_c, q) \) there is indeed, a circular orbit. Moreover, this inequality tells us what type of orbit we are dealing with at \( q \), either a co or counter-rotating one. We discard those roots of the polynomial \( F \) at a point \( q \in D \) that do not fulfill the conditions for circular, bound \( (r > r_{\text{mb}}) \) and stable \( (r > r_{\text{rms}}) \) orbits. What we have found is that, not in every single point \( q \in D \), there is a root of \( F(M) = 0 \) that leads us to a circular stable orbit of radius \( r_c \) followed by a photon emitter particle, only in a subset \( D_{r_c} \subset D \) such a mass parameter exists.

Furthermore, in all the surveys we have done on domains with different sizes and different values of \( r_c \), in almost every point \( q \in D_{r_c} \), the mass \( M \) obtained is unique, so is the rotation parameter \( a \). There is a tiny region \( D_{\text{double}} \subset D_{r_c} \) where two roots at \( q \in D_{r_c} \) exist, these two roots are very close to each other, the difference between each pair, is typically of order \( 10^{-2} \) or smaller. Figure 6 shows the bounds of the frequency shifts where there is a mass parameter corresponding to circular stable corotating orbits of photon emitters. In the subset \( D_{r_c} \) of the parameter space \( (z_1, z_2) \) there is a single (red region) and a double (black region) root \( (M) \) of the polynomial \( F(M; r_c, z_1, z_2) = 0 \) for \( r_c = 3 \). There is a rather small region in \( D \) where retrograde orbits are allowed. That region is not shown in Figure 6. At any rate, in spiral galaxies, most of the stars have direct rather than retrograde orbits. Figure 7 presents the mass parameter \( M = M(r_c, z_1, z_2) \) for \( r_c = 1 \) and \( r_c = 3 \).

For some values of the mass parameter \( M \), figure 8 shows the set of points \( \{ (z_1, z_2, r_c) \} \) corresponding to those values of \( M \). If a set of observations \( \{ (z_{\text{red}}, z_{\text{blue}}, r_c) \} \) of redshifts-blueshifts coming from emitters in circular orbits of radii \( r_c \) laid along and around a curve corresponding to a value \( M \), that specific value would be an estimate of the Kerr black hole mass \( M \).

If we select the estimate of the putative black hole mass at the center of our galaxy \( M = 2.72 \times 10^6 M_\odot \) to define \( \tilde{r} = r/M \) and \( a = pM = 0.9939 M \), the expressions of the frequency shifts become

\[
z_1 = \pm \left( 2p + \tilde{r} \sqrt{\tilde{r}^2 - 2\tilde{r} + 1} \right) \frac{1}{\tilde{r}^3/4 (\tilde{r} - 2) \sqrt{3/2 - 3\tilde{r}^1/2} \pm 2p}
\]

\[
z_2 = \pm \left( 2p - \tilde{r} \sqrt{\tilde{r}^2 - 2\tilde{r} + 1} \right) \frac{1}{\tilde{r}^3/4 (\tilde{r} - 2) \sqrt{3/2 - 3\tilde{r}^1/2} \pm 2p},
\]
whose plots are shown in figure 9 for the corotating case. As $r/M \to 2$, $z_1 \to \infty$. Negative values of $z_1$ are found for $r_c < 2$, that might be due to the very strong dragging of the black hole over the emitter. As $r/M$ increases, $z_{red} \to -z_{blue}$ as is the case for the Schwarzschild black hole, whose plot is also shown (dashed curves) and starts at $r = 6$ as it should be.

V. FINAL REMARKS

In this paper we have applied the theoretical approach developed by H-N to determine the mass parameter of compact objects in terms of the frequency shifts $z$ of light emitted by particles traveling along circular geodesics of radii $r_c$ around those objects. For the Schwarzschild and Reissner Nordstr"om black holes, we found an explicit formula $M = M(z, r_c)$ and $M = M(z, r_c, Q^2)$ respectively, and bounds for $z$. Not all values of $z$ would be detected from a far away observer. For Boson Stars, $z$ increases as the radius of the orbits increases and reaches a maximum shown in figure 4. For different equilibrium configurations this $z_{max}$ increases as the central value $\phi_0$ increases regardless that configuration lays on the stable or unstable branch. The curve $z(\phi_{crit})$ seems to be the limit of all $z(\phi)$ for large radii. For configurations with a fixed $M$ but different $\Lambda$, $z_{max}$ decreases as $\Lambda$ increases. It would be interesting to perform a similar analysis for rotating boson stars, this work is progress.

For the Kerr black hole, the mass parameter obtained as a root of the polynomial $F(N; r_c, z_1, z_2)$ is nearly unique. There is a small region in the space $D$ where there are double roots. The plot of the redshift and blueshift as a function of $r_c$ for the putative black hole at the center of our galaxy was also presented. Recently, a black hole with scalar hair was constructed by Carlos Herdeiro and Eugen Radu [11]. It would be interesting to construct the curve $z = z(r_c)$ for a given $M$ for such space-time and compare it with the one presented here for the Kerr black hole to determine the effect of hair.

Acknowledgments

R. B. is grateful to professor Harry Swinney for his warm hospitality at the Center for Nonlinear Dynamics of the University of Texas at Austin where part of this work was carried out. U. N. and R. B. acknowledge partial support by CIC-UMSNH. S. V. acknowledges support by CONACyT, under retention grant. The authors thank SNI and PRODEP-SEP for support.

[1] M. B. Begelman, Evidence for black holes, Science 300, 1898 (2003). Z. Q. Shen, K. Y. Lo, M.-C. Liang, P. T. P. Ho, and J.-H. Zhao, A size of $\approx 1$ au for the radio source Sgr A* at the center of the Milky Way, Nature (London) 438, 62 (2005). A. M. Ghez, S. Salim, N. N. Weinberg, J. R. Lu, T. Do, J. K. Dunn, K. Matthews, M. R. Morris, S. Yelda, E. E. Becklin, T. Kremenek, M. Milosavljevic, and J. Naiman, Measuring distance and propereties of the Milky Way’s central supermassive black hole with stellar orbits, Astrophys. J. 689, 1044 (2008). M. R. Morris, L. Meyer, and A. M. Ghez, Galactic center research: Manifestations of the central black hole, Res. Astron. Astrophys. 12, 995 (2012)

[2] Alfredo Herrera and Ulises Nucamendi, Kerr black hole
parameters in terms of the redshift/blueshift of photons emitted by geodesic particles, Phys. Rev. D 92, 045024 (2015).

[3] B. Aschenbach, N. Grosso, D. Porquet and P. Predehl, X-ray flares reveal mass and angular momentum of the Galactic Center black hole, A & A 417, 7178 (2004).

[4] James M. Bardeen, William H. Press and Saul A. Teukolsky. Rotating black holes: locally nonrotating frames, energy extractio, and scalar synchrotron radiation. Astrophy. J. 178 347 (1972).

[5] Monica Colpi, Stuart L. Shapiro and Ira Wasserman. Boson stars: Gravitational equilibria of self-interacting scalar fields. Phys. Rev. Letts. 57 (1986).

[6] Remo Ruffini, Silvano Bonazzola. System of selfgravitating particles in general reality and the concept of an equation of state. Phys. Rev. 187 1767-1783 (1969).

[7] Susana Valdez, Carlos Palenzuela, Daniela Alic and Luis Urena. Dynamical evolution of fermion-boson stars. Phy. Rev D 87, 084040 (2013).

[8] S. Shapiro and S. Teukolsky. In Black Holes, White Dwarfs and Neutron Stars: The Physics of Compact Objets (Wiley-VCH, New York, 1983).

[9] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, in Numerical Recipes in C: The Art of Scientific Computing (Cambridge University Press, Cambridge, 1992).

[10] S. Chandrasekhar. The Mathematical Theory of Black Holes. Clarendon Press, Oxford (1992).

[11] Carlos A.R. Herdeiro and Eugen Radu. Kerr Black Holes with Scalar Hair Phys. Rev. Lett. 112, 221101 (2014).