2-D GRAVITY AS GAUGE THEORIES WITH EXTENDED GROUPS

DANIEL CANGEMI
Department of Physics
University of California at Los Angeles
405 Hilgard Avenue
Los Angeles, CA 90024 (U.S.A.)

Abstract

The interaction of matter with gravity in two dimensional spacetimes can be supplemented with a geometrical force analogous to a Lorentz force produced on a surface by a constant perpendicular magnetic field. In the special case of constant curvature, the relevant symmetry does not lead to the de Sitter or the Poincaré algebra but to an extension of them by a central element. This richer structure suggests to construct a gauge theory of 2-D gravity that reproduces the Jackiw-Teitelboim model and the string inspired model. Moreover matter can be coupled in a gauge invariant fashion. Classical and quantized results are discussed.

Introduction

The beautiful success of General Relativity and the key role played by gauge theories in the description of fundamental interactions are two main reasons leading physicists to be interested in differential geometry. On the one hand, particles follow geodesics of spacetime, on the other hand, gauge potentials are identified with connections on some principal bundle. Moreover, it is tempting to exploit the local symmetries of General Relativity to write it as a gauge theory. Attempts in this direction turn out to be rather successful in lower dimensional gravities. In 2+1 dimensions, it is recognized that planar gravity is described by a Chern-Simons model. In this note, I will consider the even simpler case of 1+1 dimensions, where a gauge theoretical formulation of lineal gravity has a natural setting using an extended Poincaré group or, more generally, an extended de Sitter group.

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group; the extension is related to a geometrical force, which exists only in that particular dimension.

Gravity in 1+1 dimensions

The reduction of General Relativity to 1+1 dimensions is not straightforward because of the vanishing of the Einstein tensor. There are two main proposals for lineal gravities.

One is obtained with a dimensional reduction of the Einstein-Hilbert action in 2+1 dimensions [18].

\[ I_{JT} = \frac{1}{2\pi k} \int d^2 x \sqrt{-g} \eta (R - \Lambda) \]  

(1)

The Lagrange multiplier, \( \eta \), enforces constant curvature, \( R = \Lambda \).

The other proposal [2] is inspired by string theory on a two dimensional target space (it can alternatively be viewed as an s-wave approximation of 3+1 gravity [10]).

\[ I_{SI} = \frac{1}{2\pi k} \int d^2 x \sqrt{-\bar{g}} e^{-2\phi} (\bar{R} + 4\bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \lambda) \]  

(2)

Its classical solutions are \( \bar{g}_{\mu\nu} = h_{\mu\nu}/(M - \lambda(x - \bar{x})^2) \), where \( h_{\mu\nu} = \text{diag}(1, -1) \) is the flat spacetime metric. The value \( M = 0 \) corresponds to a flat metric (vacuum solution), whereas the cases \( M \neq 0 \) have the characteristics of a black hole. The action (2) takes a simpler form with a change of variables [15], \( g_{\mu\nu} = \exp(-2\phi)\bar{g}_{\mu\nu}, \eta = \exp(-2\phi) \).

\[ I_{SI} = \frac{1}{2\pi k} \int d^2 x \sqrt{-g} (\eta R - \lambda) \]  

(3)

The Lagrange multiplier, \( \eta \), now enforces zero curvature, \( R = 0 \). Proposals (1) and (2) suggest the more general action [15, 13, 1].

\[ I_g = \frac{1}{2\pi k} \int d^2 x \sqrt{-g} \left( \eta (R - \Lambda) - \lambda \right) \]  

(4)

In view of the string inspired model (2), the “stringy” metric \( \bar{g}_{\mu\nu} \) is conformally related to \( g_{\mu\nu}, \bar{g}_{\mu\nu} = g_{\mu\nu}/\eta \). However, there is no definite reason to prefer one or the other as the physical metric [8].

Let us end this section by recalling an equivalent formulation of geometry where \((g_{\mu\nu}, R)\) is substituted with \((e^a_\mu, \omega_\mu)\). The Zweibein, \( e^a_\mu \), is related to the metric, \( g_{\mu\nu} = e^a_\mu h_{ab} e^b_\nu \), and the spin-connection, \( \omega_\mu \), to the curvature, \( d\omega = R \text{vol}/2 \) (vol is the volume two-form). Moreover, a space without torsion implies a relation between the Zweibein and the spin-connection, \( de^a + e^a_\nu \omega e^b_\nu = 0 \) (\( \epsilon_{ab} \) is the antisymmetric two-tensor with value \( \epsilon^{01} = 1 \)).
Point particle motion on the line

The gauge symmetry hidden in the action (4) becomes obvious if one studies
the motion of a particle on the line. The interaction of a point particle
in a background geometry is usually described by the geodesic equation.
However, in two dimensions (and only in this dimension), the right side
of that equation may be supplemented by a force term of a geometrical
nature [6].

\[ \frac{d}{d\tau} \frac{m \dot{x}^\mu}{\sqrt{\dot{x}^\alpha g_{\alpha\beta} \dot{x}^\beta}} + \frac{1}{\sqrt{\dot{x}^\alpha g_{\alpha\beta} \dot{x}^\beta}} \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = \mathcal{F}(R) g^{\mu\nu} \sqrt{-g} \epsilon_{\nu\rho} \dot{x}^\rho \] (5)

This equation is still general covariant and invariant under reparametriza-
tion provided \( \mathcal{F}(R) \) is a scalar function. We will restrict ourself to linear
examples, \( \mathcal{F}(R) = -B - A R/2 \). Due to its similarity with electromagnetism
(which is not included here), the generalized geodesic equation (5) is ob-
tained from the variation of the action,

\[ I_m = - \int d\tau \left[ m \sqrt{\dot{x}^\mu(g_{\mu\nu}(x(\tau))\dot{x}^\nu(\tau))} \right. \\
\left. + \dot{x}^\mu(\tau) \left( A \omega^\mu(x(\tau)) + Ba_\mu(x(\tau)) \right) \right] \] (6)

where \( \omega \) is the spin-connection and \( a \) a one-form satisfying the exactness
condition \( da = \text{vol} \).

It is easy to check that for constant curvature this action is invariant
under a change of coordinates defined by a Killing vector field. Constant
curvature spacetimes (with trivial topology, which we assume here) are max-
imally symmetric and thus possess three independent Killing vectors fields,
\( \xi_{(J)}, \xi_{(0)}, \xi_{(1)} \). By Noether’s theorem, they generate three conserved currents.

\[ \xi^\mu_{(J)} = \epsilon^\mu_{\nu} x^\nu \rightarrow J \]  
\[ \xi^\mu_{(a)} = \delta^\mu_{a}(1 - \frac{A}{8} x^2) + \frac{A}{4} h_{\mu\nu} x^\nu x^\mu \rightarrow P_a \quad (a = 0, 1) \] (7)

With the canonical symplectic structure \( \left[ \frac{\partial L}{\partial x^\nu}, x^\mu \right] = \delta^\mu_{\nu} \), these currents fulfill
the algebra,

\[ [P_a, J] = \epsilon^b_a P_b \]  
\[ [P_a, P_b] = \epsilon_{ab} \left( \frac{1}{2} J + \mathcal{B}_\Lambda I \right) \] (8)

where \( I \) is a central element acting by 1 in the representation [F].

Due to the presence of a geometrical force, we do not get the de Sitter
algebra in its expected form; more specifically, in the flat case, \( \Lambda = 0 \), we
do not recover the Poincaré algebra but a central extension of it. For \( \mathcal{B} \neq 0 \), this algebra possesses a non-degenerate, invariant inner product,

\[
\text{h}_{AB} = \langle Q_A, Q_B \rangle = \begin{pmatrix}
    h_{ab} & 0 & 0 \\
    0 & \frac{(m/B\Lambda)^2}{1-\frac{1}{2}(m/B\Lambda)^2} & \frac{1}{B\Lambda} \\
    0 & \frac{1}{B\Lambda} & \frac{\Lambda/2\Lambda^2}{1-\frac{1}{2}(m/B\Lambda)^2}
\end{pmatrix}
\] (9)

(\( A, B = 0, 1, 2, 3; Q_a = P_a; Q_2 = J; Q_3 = I \)), which depends on a real parameter \( m \). The Casimir \( Q_A h^{AB} Q_B \) in the representation (7) coincides with the Hamiltonian for a particle of mass \( m \). It can be shown that the freedom in the parameter \( m \) corresponds in the case \( \Lambda = 0 \) to a global symmetry [14] also found in the dilaton model [17] where its anomaly plays a crucial role in the existence of Hawking radiation [8].

**Gauge formulation of the gravity sector**

We suggest to use this enhanced group structure for a gauge description of gravity. A connection will be thus a one-form of the type

\[
A = e^a P_a + \omega J + B\Lambda a I
\] (10)

with curvature two-form

\[
F = dA + A^2
\] (11)

\[
= (de^a + e^d \omega e^b) P_a + (d\omega + \frac{A}{2} e^a \epsilon_{ab} e^b) J + B\Lambda (da + \frac{1}{2} e^a \epsilon_{ab} e^b) I
\]

The components of \( F \) reproduce geometrical quantities if we interpret \( e^a \) as a Zweibein and \( \omega \) as a spin-connection: The two first components are the torsion relating the Zweibein to the spin-connection, the third one equals \((R - \Lambda)\text{vol}/2\) and the last one \((da - \text{vol})\). Using a scalar function with value in the adjoint representation of the gauge group, \( \eta = \eta^a P_a + \eta^2 J + \eta^3 I \), and the non-degenerate inner product (9), we build a gauge invariant action,

\[
I'_{g} = \frac{1}{2\pi k} \int \langle \eta, F \rangle
\]

\[
= \frac{1}{2\pi k} \int \left[ \eta_a (de^a + e^d \omega e^b) - \frac{1}{1-\frac{1}{2}(m/B\Lambda)^2} \right]
\]

\[
\left( (m/B\Lambda)^2 \eta^2 - \frac{1}{B\Lambda^2} \eta^3 \right) (d\omega + \frac{A}{2} e^a \epsilon_{ab} e^b)
\]

\[
+ \frac{1}{1-\frac{1}{2}(m/B\Lambda)^2} \left( - (\eta^2 + \frac{\Lambda}{2B\Lambda^2} \eta^3) (da + \frac{1}{2} e^a \epsilon_{ab} e^b) \right)
\] (12)

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which not only reproduces the action (13) with
\[
\eta = \frac{1}{1 - \frac{1}{2}(m/B\Lambda)^2} \left( \frac{1}{2}(m/B\Lambda)^2 \eta^2 - \frac{1}{25} \eta^3 \right)
\]
\[
\lambda = \frac{1}{1 - \frac{1}{2}(m/B\Lambda)^2} \left( -\eta^2 + \frac{\Lambda}{25} \eta^3 \right)
\]  
but also provides a one-form, whose classical value, \( da = \text{vol} \), is the one needed to construct the matter action (10).

Besides the zero curvature condition, \( F = 0 \), we also get an equation for the scalar function, \( D_\mu \eta = 0 \). This set of equations is easily solved by the general solution
\[
A = U^{-1} dU
\]
\[
\eta = U^{-1} \eta(0) U
\]  
for any group element \( U \) and constant gauge algebra element \( \eta(0) \). Of course, \( U \) has to be chosen carefully in order to reproduce a geometric solution associated to a non-degenerate metric \( g_{\mu\nu} \) [4, 14]. The “stringy” metric \( \tilde{g}_{\mu\nu} = g_{\mu\nu}/\eta \) then takes the form of a static black hole, for \( \Lambda = 0 \), \( g_{\mu\nu} = h_{\mu\nu}/(M - \lambda(x - \bar{x})^2) \). Nevertheless the physical content of the model will not depend on this choice and \( U = 1 \) i.e., \( e^a = \omega = a = 0 \), is perfectly admissible. This is sometimes referred as the unbroken phase. The physics should be contained in the gauge invariant part of \( \eta \).

\[
\langle \eta, \eta \rangle = \langle \eta(0), \eta(0) \rangle = M
\]
\[
\langle \eta, I \rangle = \langle \eta(0), I \rangle = \lambda/B\Lambda
\]  

The gauge theoretical approach relates the number of free parameters in the classical solutions \( (M, \lambda, \bar{x}^0, \bar{x}^1) \) to the dimension (four) of the gauge group. It introduces also the cosmological constant \( \lambda \) as a dynamical variable. The parameters \( M \) and \( \lambda \) are gauge invariant quantities and describe the physical content of the theory, as we will see in the next section.

**Quantization of the gravity sector**

A gauge theoretical setting allows a more tractable way to deal with quantization. We present here the canonical quantum structure of gravity without matter; it is simple and interesting, even if, in the absence of matter, there are no propagating degrees of freedom. We write the action (13) in its Hamiltonian form.

\[
I_g' = \frac{1}{2\pi k} \int d^2 x e^{\mu\nu} \langle \eta, F_{\mu\nu} \rangle
\]
\[= \frac{1}{2\pi k} \int dt dx \left( \langle \eta, \partial_0 A_1 \rangle + \langle A_0, D_1 \eta \rangle \right) - \frac{1}{2\pi k} \int dt dx \partial_1 \langle \eta, A_0 \rangle
\]
The Hamiltonian is a sum of constraints

\[ G^A = -\partial_1 \eta^A + f_{BC}^A A_1^B \eta^C \] (17)

\( A, B, C = 0, 1, 2, 3 \) are the gauge group indices, which are raised and lowered with the inner product \( h_{AB} \), and \( f_{BC}^A \) are the structure constants of the gauge group. The spatial component of the gauge connection is canonically conjugate to \( \eta \) and we postulate the usual commutation relations

\[ [\eta_A(x), A_B^B(y)] = i 2\pi k \delta^B_A \delta(x - y) \] (18)

With these commutation relations, the algebra of constraints coincides, as usual in gauge theories, with the original gauge algebra.

\[ [G_A(x), G_B(y)] = if_{ABC} G_C(x) \delta(x - y) \] (19)

In a Schrödinger picture, we consider states as functionals of \( \eta_A(x) \), \( \Psi[\eta] \), on which \( A_1^A(x) \) acts by functional derivation, \( (2\pi k/i)(\delta/\delta\eta_A(x)) \).

Physical states are those annihilated by the constraints \( G_A \) and they satisfy the differential equations

\[
\begin{align*}
(\partial_1 \eta_a - i 2\pi k \epsilon_a^b \eta_b \frac{\delta}{\delta \eta^c} + i 2\pi k \eta^2 \epsilon_{ab} \frac{\delta}{\delta \eta_b}) \Psi &= 0 \\
(\partial_1 \eta^2 + i 2\pi k \Lambda \epsilon^a_{\bar{b}} \eta_{\bar{b}} \frac{\delta}{\delta \eta_a}) \Psi &= 0 \\
(\partial_1 \eta^3 + i 2\pi k \Lambda \epsilon^a_{\bar{b}} \eta_{\bar{b}} \frac{\delta}{\delta \eta_a}) \Psi &= 0
\end{align*}
\] (20)

These equations are solved by the functionals

\[ \Psi[\eta_A] = \exp \left( \frac{i}{2\pi k} \int dx \eta^2 \epsilon_{ab} \eta_b \frac{\delta}{\delta \eta_a} \right) \psi(M, \lambda) \bigg|_{\langle \eta, I \rangle = \Lambda/B_\Lambda} \] (21)

with support on the constant gauge invariant combinations \( \langle \eta, I \rangle = M \) and \( \langle I, \eta \rangle = \lambda/B_\Lambda \); \( \psi \) is a function of the variables \( M \) and \( \lambda \). The physical states depend on the two values \( M \) and \( \lambda \), which coincide for classical solutions with the two parameters of the black hole configuration. Let us now couple matter to this gravity.

**Coupling to matter**

The coupling to matter follows the one discussed before, see Eq. (6). It is possible to find a gauge invariant formulation of it either for point particle or for fields, cf. Ref. [6]. The gauge invariant actions are of the form

\[ I_m[A, p(\tau), \xi^a], \quad I_m[A, \bar{\psi}, \bar{\psi}, \psi, \xi^a], \ldots \] (22)
where the additional field $\xi^a$ acts like a Higgs field that insures the gauge invariance of the action. The essential feature of this coupling is that it does not involve $\eta$. In this gauge formulation, the matter is coupled to the metric $g_{\mu\nu}$, whereas in the geometrical point of view people use mainly a coupling to $\bar{g}_{\mu\nu}$. But, since their coupling is conformal, it is not really different at the classical level. Nevertheless, this difference could have its importance once we proceed to the quantization [8]. Notice that our coupling breaks conformal invariance at the classical level even in the massless case. Namely, the trace of the energy-momentum is proportional to the additional force strength, $\mathcal{B}$ and at the quantum level its vacuum expectation picks up an additional term, $R/24\pi$ [6].

The equations of motion are modified in the following way

$$F = 0$$
$$D_\mu \eta = 2\pi k J^5_\mu \tag{23}$$

where $(J^5_\mu)_A = -\epsilon^{\mu\nu}(\delta I'_m/\delta A^A_\nu)$ is the axial current. Let us consider the point particle. Outside the particle trajectory, $J^5_\mu$ is zero and the equations are those of pure gravity. We have two sets of four constant parameters on each side of the trajectory, whose differences are fixed by the particle characteristics. The shift in $M$ and $\lambda$ implies a transition from a pure gravity state to another when crossing the particle line; this is usually interpreted [4] as a black hole created by an in-falling particle. The shift in $\bar{x}$ is a basic ingredient in deriving a Hawking radiation [2] for the “stringy” metric, $\bar{g}_{\mu\nu}$.

Our formulation reproduces interesting features of linear gravity. But being a gauge theory, we are able to discuss in a straightforward manner issues concerning gauge charges or quantization.

**A gauge definition of mass**

The definition of mass and angular-momentum is an ill-defined concept in General Relativity. Different methods lead to different results [3]. However, when one has a gauge invariance, Noether’s procedure uniquely define conserved currents and charges. In our model, $I'_g + I'_m$, an infinitesimal gauge transformation $\theta$ generates an explicit conserved current

$$j^\mu_\theta = \frac{1}{\pi k} \epsilon^{\mu\nu} \partial_\nu \langle \eta, \theta \rangle \tag{24}$$

and a conserved charge.

$$Q_\theta = \int dx^1 j^0_\theta = \frac{1}{\pi k} \langle \eta, \theta \rangle |_{x^1=+\infty} |_{x^1=-\infty} \tag{25}$$
The question is which $\theta$ define energy. Obviously, energy should be related with infinitesimal diffeomorphisms in a time-like Killing direction.

But, in topological field theory ($F = 0$), infinitesimal diffeomorphisms are equivalent to infinitesimal gauge transformations \[13\].

\[
L_f A_\mu = f^\alpha \partial_\alpha A_\mu + \partial_\mu f^\alpha A_\alpha = D_\mu (f^\alpha A_\alpha) + f^\alpha F_{\alpha\mu} \tag{26}
\]

An infinitesimal diffeomorphism, $f^\alpha$, is identified with an infinitesimal gauge transformation, $f^\alpha A_\alpha$. It is thus associated to the conserved charge

\[
Q_f = \frac{1}{\pi k} \langle \eta, f^\alpha A_\alpha \rangle \big|_{x^1=+\infty}^{x^1=-\infty} \tag{27}
\]

and energy $E$ is defined for a time-like Killing vector $f^\alpha$.

In the absence of matter, the contributions at $x^1 = +\infty$ and $x^1 = -\infty$ are identical, which implies $E = 0$. When matter is included, due to the jump of the value of $\eta$ across the particle trajectory, the contributions are different and gives a non zero energy, $E = \langle \eta, \eta \rangle = M$, in full agreement with the ADM definition.

**Conclusions**

In this brief note, I have shown how General Relativity and gauge theory can be combined in 1+1 dimensional spacetime. Once the gauge group is recognized, we are able to produce a gauge theory, which encompasses the Jackiw-Teitelboim and the string inspired models. The inclusion of matter in a gauge invariant way is possible and provides a model, which not only reproduces previous results but also provides a natural way to define gauge invariant and conserved quantities, as energy, and to deal with quantization. Another interesting feature of the model is the introduction of the cosmological constant as a dynamical variable \[12\]. Supersymmetric extensions have been studied in relation to a positive energy theorem \[16\] and for a topological description of supergravity \[7\]. The quantization of pure gravity has shown how the physical states depend on gauge invariants. The quantization of the full model deserves further study. It would also be interesting to consider topological effects occurring in the definition of the one-form $a$ and in the resolution of $F = 0$ \[11\].

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