A COMPARISON PRINCIPLE FOR THE POROUS MEDIUM EQUATION AND ITS CONSEQUENCES

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Abstract. We prove a comparison principle for the porous medium equation in more general open sets in $\mathbb{R}^{n+1}$ than space-time cylinders. We apply this result in two related contexts: we establish a connection between a potential theoretic notion of the obstacle problem and a notion based on a variational inequality. We also prove the basic properties of the PME capacity, in particular that there exists a capacitary extremal which gives the capacity for compact sets.

1. Introduction

We study the porous medium equation (PME for short)

$$\frac{\partial u}{\partial t} - \Delta u^m = 0,$$

(1.1)

where $m > 1$. This equation is an important prototype of a nonlinear parabolic equation. The equation is degenerate, meaning that the modulus of ellipticity vanishes when the solution is zero. The name stems from modeling the flow of a gas in a porous medium: the continuity equation, Darcy’s law, and an equation of state for the gas lead to (1.1) for the density of the gas, after scaling out various physical constants. For more information about this equation, including numerous further references, we refer to the monographs [8] and [17].

The comparison principle is a fundamental tool in the theory of elliptic and parabolic equations. In particular, it can be used to define a class of supersolutions which is the counterpart for superharmonic functions in classical potential theory: we call a function a semicontinuous supersolution, if it satisfies the comparison principle with respect to continuous solutions. The definition is due to F. Riesz [15], and it makes the development of a nonlinear potential theory feasible.

The comparison principle for parabolic equations is usually formulated for space-time cylinders, meaning sets of the form $\Omega_T = \Omega \times (0, T)$. The boundary values are then taken over the parabolic boundary, where only the initial and lateral boundaries are taken into account. However, one often encounters situations where one would like to apply the comparison principle in sets which are not space-time cylinders. Thus our main objective is to establish a comparison principle for the PME in more general open sets in $\mathbb{R}^{n+1}$. Such a result is occasionally called the elliptic comparison principle, in reference to the fact that the time variable no longer has a special role. Moreover, the elliptic comparison principle can be used to develop the Perron method in general space-time domains, see [3, 12]. We also present two applications where such a comparison principle is indispensable.

2010 Mathematics Subject Classification. Primary 35K55; Secondary: 31C15, 35K86.

The first author was partially supported by Academy of Finland, project #259224, and by the Swedish Research Council, #637-2014-6822.

A part of the research reported in this work was done during the authors’ stay at the Institute Mittag-Leffler (Djursholm, Sweden).
For the heat equation, when \( m = 1 \), one may add constants to solutions. A comparison principle for general open sets then follows from the space-time cylinder case by a straightforward exhaustion argument. For the PME, there is a comparison principle over cylindrical domains, but adding constants is no longer possible. Our idea for circumventing this difficulty is to multiply one of the functions being compared by a constant close to one. The modified function is no longer a solution, but it still satisfies the PME with an error-term on the right hand side. The error-term vanishes as the multiplicative constant tends to one. The comparison principle for the original functions then follows by the usual duality proof, modified to account for the error-term. Our argument yields a comparison principle for open sets of the form \( \Omega_T \setminus K \), where \( K \) is a compact set.

As the first application, we consider the obstacle problem. Roughly speaking, this amounts to finding a solution to a PDE subject to the constraint that the solution stays above a given function, the obstacle. Here we use a potential theoretic method for solving the problem: we define the solution to the obstacle problem to be the infimum of all supersolutions lying above the obstacle (réduit). For smooth enough obstacles the réduit is the smallest supersolution above the obstacle. The concept of réduit is standard in classical potential theory, and it has been utilized in a nonlinear parabolic context in [14]. Existence and uniqueness follow in a straightforward manner, at least for continuous obstacles. However, the relation between the smallest supersolution and the variational solutions to obstacle problems constructed in [4] is not obvious. In this direction, we prove that the smallest supersolution is also a variational solution for sufficiently smooth obstacles. This follows from two facts. First, we prove that the smallest supersolution can always be approximated by variational solutions. Second, the notion of variational solution is stable with respect to the convergence of the obstacles in certain norms, see [4]. The converse of this, i.e. whether a variational solution agrees with the smallest supersolution, remains a very interesting open problem.

The second application is a notion of parabolic capacity for the PME. This concept is defined via a measure data problem, as in [11] for the parabolic \( p \)-Laplacian. See also [18, 19] and the references therein for the capacity for the heat equation. We prove the basic properties of the capacity related to the PME, such as countable subadditivity and the existence of the capacitary extremal of a compact set. Our comparison principle plays a key role in the latter argument.

The paper is organized as follows. In Section 2, we recall the necessary background material, in particular various notions of supersolutions. Section 3 contains the proof of the comparison principle, and Section 4 is concerned with the obstacle problem. Finally, the basic properties of capacity are proved in Section 5.

2. Weak supersolutions and semicontinuous supersolutions

Let \( \Omega \) be an open and bounded subset of \( \mathbb{R}^n \), and let \( 0 < t_1 < t_2 < T \). We use the notation \( \Omega_T = \Omega \times (0, T) \) and \( U_{t_1, t_2} = U \times (t_1, t_2) \), where \( U \subset \Omega \) is open. The parabolic boundary \( \partial_p U_{t_1, t_2} \) of a space-time cylinder \( U_{t_1, t_2} \) consists of the initial and lateral boundaries, i.e.

\[
\partial_p U_{t_1, t_2} = (\partial \Omega \times \{t_1\}) \cup (\partial U \times \{t_1, t_2\}).
\]

The notation \( U_{t_1, t_2} \subset \Omega_T \) means that the closure \( \overline{U_{t_1, t_2}} \) is compact and \( \overline{U_{t_1, t_2}} \subset \Omega_T \).

We use \( H^1(\Omega) \) to denote the usual Sobolev space, the space of functions \( u \) in \( L^2(\Omega) \) such that the weak gradient exists and also belongs to \( L^2(\Omega) \). The norm of \( H^1(\Omega) \) is defined by

\[
\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|
abla u\|_{L^2(\Omega)}^2.
\]
The Sobolev space with zero boundary values, denoted by $H^2_0(\Omega)$, is the completion of $C_0^\infty(\Omega)$ with respect to the norm of $H^2(\Omega)$.

The parabolic Sobolev space $L^2(0,T;H^1(\Omega))$ consists of measurable functions $u: \Omega_T \to [-\infty, \infty]$ such that $x \mapsto u(x,t)$ belongs to $H^1(\Omega)$ for almost all $t \in (0,T)$, and
\[
\int_{\Omega_T} |u|^2 + |\nabla u|^2 \, dx \, dt < \infty. \quad \text{(2.1)}
\]

The definition of $L^2(0,T;H^1_0(\Omega))$ is identical, apart from the requirement that $x \mapsto u(x,t)$ belongs to $H^1_0(\Omega)$. We say that $u$ belongs to $L^2_{\text{loc}}(0,T;H^1_0(\Omega))$ if $u \in L^2(t_1,t_2;H^1(U))$ for all $U_{t_1,t_2} \Subset \Omega_T$.

Supersolutions to the porous medium equation are defined in the weak sense in the parabolic Sobolev space.

**Definition 2.1.** A nonnegative function $u: \Omega_T \to \mathbb{R}$ is a weak supersolution of the equation
\[
\frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in } \Omega_T, \quad \text{if } u^m \in L^2_{\text{loc}}(0,T;H^1_0(\Omega)) \quad \text{and}
\int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \, dx \, dt \geq 0,
\]
for all positive, smooth test functions $\varphi$ compactly supported in $\Omega_T$. The definition of weak subsolutions is similar; the inequality is simply reversed. Weak solutions are defined as functions that are both super- and subsolutions.

Weak solutions are locally Hölder continuous, after a possible redefinition on a set of measure zero. See [7], [8], [9], [17], or [20].

We have also the following class of supersolutions.

**Definition 2.2.** A function $u: \Omega_T \to [0,\infty]$ is a semicontinuous supersolution, if

1. $u$ is lower semicontinuous,
2. $u$ is finite in a dense subset of $\Omega_T$, and
3. the following parabolic comparison principle holds: Let $U_{t_1,t_2} \Subset \Omega$, and let $h$ be a solution to (2.1) which is continuous in $U_{t_1,t_2}$. Then, if $h \leq u$ on $\partial_h U_{t_1,t_2}$, $h \leq u$ also in $U_{t_1,t_2}$.

Note that a semicontinuous supersolution is defined in every point. Every weak supersolution is a semicontinuous supersolution provided that a proper pointwise representative is chosen. This is a consequence of the following lemma.

**Lemma 2.3** ([2]). Let $u$ be a nonnegative weak supersolution to the porous medium equation in $\Omega \times (t_1,t_2)$. Then $u$ has a lower semicontinuous representative.

In the other direction, a bounded semicontinuous supersolution is also a weak supersolution, as shown in [13]. If unbounded functions are allowed, then the class of semicontinuous supersolutions is strictly larger, since the Barenblatt solution is a semicontinuous supersolution, but is not a weak supersolution, see [13].

**Lemma 2.4** ([13]). Let $u$ be a weak supersolution such that $|u| \leq M < \infty$. Then
\[
\iint_{\Omega_T} \eta^2 |\nabla u|^2 \, dx \, dt \leq 16M^{2m-1} \int_{\Omega} |\nabla \eta|^2 \, dx + 6M^{m+1} \int_{\Omega} \eta^2 \, dx,
\]
for all nonnegative functions $\eta \in C_0^\infty(\Omega)$.

An application of the Riesz representation theorem shows that for each weak supersolution $u$, there exists a positive Radon measure $\mu_u$ such that
\[
\iint_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_T} \varphi \, d\mu_u,
\]
for all smooth compactly supported functions \( \varphi \). This is the Riesz measure of \( u \). The integrals on the left hand side do not depend on the particular pointwise representative of a supersolution. Thus a weak supersolution \( u \) and its lower semicontinuous regularization \( \overline{u} \) have the same Riesz measures.

**Lemma 2.5.** If \( u \) and \( v \) are weak supersolutions in \( \Omega_\infty \), \( u,v = 0 \) on \( \partial_p \Omega_\infty \), \( u^m,v^m \in L^2(0,\infty;H^1_0(\Omega)) \), and \( \mu_v \leq \mu_u \), then \( v \leq u \) a.e. in \( \Omega_\infty \).

**Proof.** Let \( \varphi \in C^\infty_0(\Omega_\infty) \) be nonnegative. By subtracting the equations satisfied by \( u \) and \( v \) and using the assumption about the measures, we have

\[
\int_{\Omega_\infty} -(u - v)\varphi_t + \nabla(u^m - v^m) \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_\infty} \varphi \, d\mu_u - \int_{\Omega_\infty} \varphi \, d\mu_v \geq 0.
\]

By a standard approximation argument using the fact that \( u^m,v^m \in L^2(0,\infty;H^1_0(\Omega)) \), we may also take the test functions \( \varphi \in C^\infty_0(\Omega_\infty) \) so that \( \varphi = 0 \) on the lateral boundary \( \partial \Omega \times (0,\infty) \). We apply the Green’s formula to get

\[
\int_{\Omega_\infty} -(u - v)\varphi_t - (u^m - v^m)\Delta \varphi \, dx \, dt \geq 0.
\]

The fact that \( v \leq u \) follows from this inequality by repeating the standard duality proof for the comparison principle for the PME, see e.g. \([7, \text{Lemma 5}], [8, \text{Theorem 1.1.1}]\), or \([17, \text{Theorem 6.5}]\). \( \square \)

**Lemma 2.6.** Let \( u_i, i = 1,2,\ldots \) is a uniformly bounded sequence of weak supersolutions in \( \Omega_\infty \) such that \( u_i \rightharpoonup u \) a.e. in \( \Omega_\infty \). Then \( u \) is a weak supersolution in \( \Omega_\infty \) and

\[
\lim_{i \to \infty} \int_{\Omega_\infty} \phi \, d\mu_{u_i} = \int_{\Omega_\infty} \phi \, d\mu_u,
\]

for every \( \phi \in C^\infty_0(\Omega_\infty) \).

**Proof.** Due to the uniform bound on the functions \( u_i \), it easily follows that

\[
\int_{\Omega_\infty} -\phi \frac{\partial \phi}{\partial t} - u^m \Delta \phi \, dx \, dt \geq 0.
\]

An application of Lemma 2.4 on each \( u_i \) implies that \( \nabla u^m \in L^2_{\text{loc}}(\Omega_\infty) \). This, together with (2.2) yields that \( u \) is a weak supersolution. The claim about the measures follows from the computation

\[
\lim_{i \to \infty} \int_{\Omega_\infty} \phi \, d\mu_{u_i} = \lim_{i \to \infty} \int_{\Omega_\infty} -\phi \frac{\partial \phi}{\partial t} + \nabla u_i^m \cdot \nabla \phi \, dx \, dt = \int_{\Omega_\infty} -\phi \frac{\partial \phi}{\partial t} + \nabla u^m \cdot \nabla \phi \, dx \, dt = \int_{\Omega_\infty} \phi \, d\mu_u.
\]

We will frequently use the following characterization of the weak convergence of measures. See \([10, \text{Theorem 1, p. 54}]\) for the proof.

**Theorem 2.7.** Let \( \mu \) and \( \mu_k, k = 1,2,3,\ldots \), be Radon measures on \( \mathbb{R}^n \). Then the following statements are equivalent.

1. For all compactly supported smooth functions \( \phi \), one has

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \phi \, d\mu_k = \int_{\mathbb{R}^n} \phi \, d\mu.
\]

2. For all compact sets \( K \), one has

\[
\limsup_{k \to \infty} \mu_k(K) \leq \mu(K).
\]
3. A Comparison Principle

The core of our arguments is a suitable form of the comparison principle, which we will prove in this section. We will work extensively with finite unions of space-time cylinders, so we begin by introducing some notation for such sets. For space-time cylinders \( U_{t_1,t_2} = U \times (t_1,t_2) \), we denote the lateral boundary by

\[
S(U_{t_1,t_2}) = \partial U \times (t_1,t_2).
\]

For a cylinder the definition of the parabolic boundary is standard, but for finite unions of space-time cylinders we will recall the definitions. The lateral boundary of a finite union of space-time cylinders \( U_{t_1,t_2} \) is then given by

\[
S(\cup U_{t_1,t_2}) := (\cup S(U_{t_1,t_2})) \setminus (\cup U_{t_1,t_2}).
\]

We also denote the tops of \( \cup U_{t_1,t_2} \) by

\[
T(\cup U_{t_1,t_2}) := (\cup U_{t_1,t_2} \times \{t_1\}) \setminus (\cup U_{t_1,t_2}),
\]

and the bottoms similarly as

\[
B(\cup U_{t_1,t_2}) := (\cup U_{t_1,t_2} \times \{t_2\}) \setminus (\cup U_{t_1,t_2}).
\]

Thus the parabolic boundary of \( Q = \cup U_{t_1,t_2} \) is \( S(Q) \cup B(Q) \) and the parabolic boundary of backwards in time equations becomes \( S(Q) \cup T(Q) \).

We want to use the very weak (i.e., distributional) formulation of the porous medium equation, so we consider smooth test functions \( \phi \in C^\infty(Q) \) where \( Q = \cup U_{t_1,t_2} \) such that \( \phi = 0 \) on \( S(Q) \). Note that the gradient of \( \phi \) does not necessarily vanish on \( S(Q) \). In the following we will always work with \( \Omega \) a smooth domain. Let us now write the PME in terms of the above class of test functions. Assume at first that \( \phi \) has compact support in space. Then a standard approximation argument shows that we may write the definition of weak solutions as

\[
\int_Q \left[ -u \phi_t + \nabla u^m \cdot \nabla \phi \right] dx \, dt + \int_{T(Q)} u \phi \, dx + \int_{B(Q)} u \phi \, dx = 0.
\]

After this, we may pass from compactly supported test functions to test functions vanishing on the sides \( S(Q) \), since \( u \) and \( \nabla u^m \) are in \( L^2(Q) \). Now, apply Green’s formula, which is justified by the usual trace theorem, to get

\[
\int_{T(Q)} u \phi \, dx - \int_{B(Q)} u \phi \, dx + \int_Q \left[ -u \phi_t - u^m \Delta \phi \right] dx \, dt
\]

\[
+ \int_{S(Q)} u^m \partial_n \phi \, d\sigma \, dt = 0. \tag{3.1}
\]

A similar argument can be carried out for weak supersolutions and subsolutions. In these cases, we get the appropriate inequalities in the final form. This formulation will be our starting point in the proof of the comparison principle.

**Theorem 3.1.** Let \( K \) be a compact set in \( \Omega_T \) where \( \Omega \) is a smooth domain, let \( u \) be a nonnegative upper semicontinuous function which is a continuous weak supersolution in \( \Omega_T \setminus K \) and satisfies \( u^m \in L^2(0,T;H^1(\Omega)) \). Let \( v \) be a non-negative lower semicontinuous function which is a weak supersolution in \( \Omega_T \setminus K \) and satisfies \( v^m \in L^2(0,T;H^1(\Omega)) \), \( v > 0 \) on \( K \) and \( u \leq v \) on \( K \cup \partial K \Omega_T \). Then \( u \leq v \) in \( \Omega_T \).
Proof. We let \( \epsilon > 0 \), and denote
\[
D_{\epsilon} = \left\{ (x,t) \in \Omega_T : \frac{u}{1 + \epsilon} \geq v \right\}.
\]
The function \( \frac{u}{1 + \epsilon} - v \) is upper semicontinuous, so that the set \( D_{\epsilon} \) is closed in \( \Omega_T \). Moreover, the set \( D_{\epsilon} \) does not intersect the set \( K \), since \( u \leq v \) on \( K \) and \( \inf_K v > 0 \). Since \( K \) is compact, there is a positive distance between \( D_{\epsilon} \) and \( K \). Thus we can cover \( D_{\epsilon} \) with a finite collection of space time cylinders not intersecting \( K \). Denote the covering set by \( \tilde{D}_{\epsilon}^F \), and note that since \( D_{\epsilon_1} \subset D_{\epsilon_2} \) for \( \epsilon_1 > \epsilon_2 \) we may choose the coverings for different values of \( \epsilon \) so that \( \tilde{D}_{\epsilon_1}^F \subset \tilde{D}_{\epsilon_2}^F \). Let then \( D_{\epsilon}^F = \Omega_T \cap \tilde{D}_{\epsilon}^F \). The set \( D_{\epsilon}^F \) is still a finite union of space-time cylinders, and the function \( u \) is a weak solution in \( D_{\epsilon}^F \).

Let \( u_{\epsilon} = \frac{u}{1 + \epsilon} \). We want to compare \( u_{\epsilon} \) with \( v \) in \( D_{\epsilon}^F \). To this end, note first that \( u_{\epsilon} < v \) on \( \partial D_{\epsilon}^F \). Further, the function \( u_{\epsilon} \) is a solution to
\[
(u_{\epsilon})_t - \Delta(u_{\epsilon})^m = f := \frac{(1 + \epsilon)^{m-1} - 1}{(1 + \epsilon)^m} \Delta u^m,
\]
interpreted in the sense of distributions. To see this, we compute
\[
[u_{\epsilon}]_t - \Delta[u_{\epsilon}]^m = \left[ u_{\epsilon} - u \right]_t - \Delta\left[ u_{\epsilon}^m - u^m \right] = \left( \frac{1}{1 + \epsilon} - 1 \right) u_t - \left( \frac{1}{(1 + \epsilon)^m} - 1 \right) \Delta u^m = \left( \frac{1}{1 + \epsilon} - \frac{1}{(1 + \epsilon)^m} \right) \Delta u^m = \frac{(1 + \epsilon)^{m-1} - 1}{(1 + \epsilon)^m} \Delta u^m.
\]

We aim at adapting the proof of the comparison principle for the PME, see e.g. [7]. To proceed let
\[
D_{\epsilon,s}^F = \left\{ (x,t) \in D_{\epsilon}^F : t \leq s \right\},
\]
and take positive functions \( \phi \in C^\infty_0(D_{\epsilon,s}^F) \), and \( \psi \in C^\infty(D_{\epsilon,s}^F) \) which vanishes on \( S(D_{\epsilon,s}^F) \), \( \partial_{\epsilon} \psi \leq 0 \) on \( S(D_{\epsilon,s}^F) \) and so that \( \psi \) equals \( \phi \) on \( T(D_{\epsilon,s}^F) \). Denote also \( b = u_{\epsilon} - v \) for brevity. Since \( b \) is negative and consequently also \( u_{\epsilon}^m - v^m \) on \( \partial D_{\epsilon}^F \), we have from (3.1) and (3.2), since \( \partial_{\epsilon} \psi \leq 0 \) on \( S(D_{\epsilon}^F) \)

\[
\int_{D_{\epsilon,s}^F} f \psi \, dx \, dt = \int_{T(D_{\epsilon,s}^F)} b \psi \, dx \, dt - \int_{S(D_{\epsilon,s}^F)} b \psi \, dx \, dt + \int_{D_{\epsilon,s}^F} \psi_t \, dx \, dt - \int_{D_{\epsilon,s}^F} [u_{\epsilon}^m - v^m] \Delta \psi \, dx \, dt + \int_{T(D_{\epsilon,s}^F)} [u_{\epsilon}^m - v^m] \partial_{\epsilon} \psi \, d\sigma \, dt
\]

\[
\geq \int_{T(D_{\epsilon,s}^F)} b \psi_t \, dx \, dt - \int_{D_{\epsilon,s}^F} \psi_t \, dx \, dt - \int_{D_{\epsilon,s}^F} [u_{\epsilon}^m - v^m] \Delta \psi \, dx \, dt.
\]

We can rewrite this as
\[
\int_{T(D_{\epsilon,s}^F)} b \psi_t \, dx \, dt \leq \int_{D_{\epsilon,s}^F} b(\psi_t + a \Delta \psi) \, dx \, dt + \int_{D_{\epsilon,s}^F} f \psi \, dx \, dt,
\]
where
\[
a = \begin{cases} \frac{u_{\epsilon}^m - v^m}{u_{\epsilon} - v}, & \text{if } u_{\epsilon} \neq v, \\ 0, & \text{if } u_{\epsilon} = v. \end{cases}
\]

Next we use a regularization to make the term \( \psi_t + a \Delta \psi \) small in the above inequality. To do this let \( a_k, k = 1, 2, \ldots, \) be smooth functions in \( D_{\epsilon,s}^F \) such that
\[
\frac{1}{k} \leq a_k \leq k,
\]
and
\[
\int_{D_{\epsilon,s}^F} \frac{(a_k - a)^2}{a_k} \, dx \, dt \to 0 \quad \text{as} \quad k \to \infty.
\]
We replace the function $\psi$ in (3.3) by the solution $\psi_k$ to the following boundary value problem
\[
\begin{aligned}
  u_t + a_k \Delta u &= 0, & & \text{in } D_{c,s}^F, \\
  u(x,s) &= \phi(x,s), & & \text{on } T(D_{c,s}^F), \\
  u &= 0, & & \text{on } S(D_{c,s}^F),
\end{aligned}
\]
and get by Hölder’s inequality
\[
\int_{T(D_{c,s}^F)} (u_t - v) \phi \, dx \leq \int_{D_{c,s}^F} b(a-a_k) \Delta \psi_k \, dx \, dt - \int_{D_{c,s}^F} f \psi_k \, dx \, dt \\
\leq \left[ \int_{D_{c,s}^F} b\frac{(a-a_k)^2}{a_k} \, dx \, dt \right]^{1/2} \left[ \int_{D_{c,s}^F} a_k (\Delta \psi_k)^2 \, dx \, dt \right]^{1/2} + \int_{D_{c,s}^F} f \psi_k \, dx \, dt. 
\]

(3.6)

To continue, we need to estimate the term on the right hand side of (3.6) containing the quantity $a_k (\Delta \psi_k)^2$ independently of $k$. To do this we follow the calculations of [7]. We use the equation (3.5) for $\psi_k$ and integrate by parts, first in time and then in space, which gives
\[
\int_{D_{c,s}^F} a_k \Delta \psi_k \Delta \psi_k \, dx \, dt = - \int_{D_{c,s}^F} \psi_k \Delta \psi_k \, dx \, dt \\
- \int_{T(D_{c,s}^F)} \phi \Delta \psi_k \, dx \, dt + \int_{B(D_{c,s}^F)} \psi_k \Delta \psi_k \, dx \\
= \int_{D_{c,s}^F} \psi_k \Delta [\psi_k] \, dx \, dt + \int_{T(D_{c,s}^F)} \phi \Delta \psi_k \, dx \, dt + \int_{B(D_{c,s}^F)} \Delta \psi_k \, dx \\
\leq - \int_{S(D_{c,s}^F)} \partial_n \psi_k \psi_k \, dx \, dt + \int_{D_{c,s}^F} \Delta \psi_k \psi_k \, dx \, dt + \int_{T(D_{c,s}^F)} |\nabla \phi|^2 \, dx. 
\]

(3.7)

Note now that the first term on the right hand side in (3.7) vanishes, since for almost every $t \leq s$ we have $[\psi_k]_t = 0$ on $S[D_{c,s}^F]$ due to the fact that $\psi_k$ vanishes smoothly on the boundary. For the second term on the right hand side in (3.7), we use the first line in (3.7). This implies that
\[
\int_{D_{c,s}^F} a_k (\Delta \psi_k)^2 \, dx \, dt \leq \frac{1}{2} \int_{T(D_{c,s}^F)} |\nabla \phi|^2 \, dx. 
\]

(3.8)

With the estimate (3.8) in hand, we see from (3.4) and the fact that $b$ is bounded that
\[
\left( \int_{D_{c,s}^F} b\frac{(a-a_k)^2}{a_k} \, dx \, dt \right)^{1/2} \left( \int_{D_{c,s}^F} a_k (\Delta \psi_k)^2 \, dx \, dt \right)^{1/2} \to 0 \quad \text{as} \quad k \to 0. 
\]

(3.9)

To proceed we need to take care of the term involving $f$ on the right hand side in (3.6). Recall that, as a distribution
\[
f = \frac{(1+\epsilon)^{m-1} - 1}{(1+\epsilon)^m} \Delta u^m.
\]

Since the function $\psi_k$ vanishes on the lateral boundary $S(D_{c,s}^F)$ of $D_{c,s}^F$, we have
\[
\int_{D_{c,s}^F} f \psi_k \, dx \, dt = \frac{(1+\epsilon)^{m-1} - 1}{(1+\epsilon)^m} \int_{D_{c,s}^F} \nabla u^m \cdot \nabla \psi_k \, dx \, dt \\
\leq \frac{(1+\epsilon)^{m-1} - 1}{(1+\epsilon)^m} \left( \int_{D_{c,s}^F} |\nabla u^m|^2 \, dx \, dt \right)^{1/2} \left( \int_{D_{c,s}^F} |\nabla \psi_k|^2 \, dx \, dt \right)^{1/2}. 
\]

(3.10)

By the assumption $u^m \in L^2(0,T;H_0^1(\Omega))$, we see that the first integral is bounded independent of $k$ and $\epsilon$. 
Next we need estimate the $L^2$-norm of $|\nabla \psi_k|$ independently of $k$, which we do as in \cite[p. 133]{17}. Multiply the equation (3.5) for $\psi_k$ by the test function $\theta = \Delta \psi_k \chi(t)$, where $\chi(0) = 1/2$ and $\chi(s) = 1$; thus $\chi_t \approx \frac{1}{t^2}$. Next we integrate by parts, first in space and then in time, and get

$$
0 = \int_{D_{r,s}^c} [\psi_k]_t\Delta \psi_k \chi \, dx \, dt + \int_{D_{r,s}^c} a_k(\Delta \psi_k)^2 \chi \, dx \, dt
$$

$$
= -\int_{D_{r,s}^c} \nabla (\psi_k)_t \cdot \nabla \psi_k \chi \, dx \, dt + \int_{D_{r,s}^c} a_k(\Delta \psi_k)^2 \chi \, dx \, dt
$$

$$
= -\frac{1}{2} \int_{D_{r,s}^c} [|\nabla \psi_k|^2] \chi \, dx \, dt + \int_{D_{r,s}^c} a_k(\Delta \psi_k)^2 \chi \, dx \, dt
$$

$$
= \frac{1}{2} \int_{D_{r,s}^c} (|\nabla \psi_k|^2) \chi_t \, dx \, dt - \frac{1}{2} \int_{T(D_{r,s}^c)} |\nabla \psi_k|^2 \chi
$$

$$
+ \frac{1}{2} \int_{B(D_{r,s}^c)} |\nabla \psi_k|^2 \chi + \int_{D_{r,s}^c} a_k(\Delta \psi_k)^2 \chi \, dx \, dt,
$$

using (3.5) and (3.11) we get

$$
\frac{1}{8} \int_{D_{r,s}^c} (|\nabla \psi_k|^2) \, dx \, dt + \int_{D_{r,s}^c} a_k(\Delta \psi_k)^2 \chi \, dx \, dt
$$

$$
\leq C \left( \int_{T(D_{r,s}^c)} |\nabla \psi_k|^2 \chi \, dx - \int_{B(D_{r,s}^c)} |\nabla \psi_k|^2 \chi \, dx \right) \leq C \int_{T(D_{r,s}^c)} |\nabla \phi|^2 \, dx. \quad (3.12)
$$

Combining (3.6), (3.9), (3.10) and (3.12), we have so far established

$$
\int_{T(D_{r,s}^c)} (u - v) \phi \, dx \leq C \frac{(1 + \epsilon)^{m-1} - 1}{(1 + \epsilon)^m} \int_{T(D_{r,s}^c)} |\nabla \phi|^2 \, dx. \quad (3.13)
$$

Before letting $\epsilon \to 0$, we still need to check that

$$
\int_{T(D_{r,s}^c)} |\nabla \phi|^2 \, dx \leq C,
$$

for some constant $C$ not depending on $\epsilon > 0$. We are free to assume that $\phi \in C_0^\infty(D_{0,s})$ for some $\epsilon_0$. Then, since $D_{0,s} \subset D_{r,s}$ for $\epsilon \leq \epsilon_0$, we have $T(D_{0,s}) \cap D_{0,s} \subset T(D_{0,s})$ which proves the desired bound. Thus, letting $\epsilon \to 0$ in (3.13), we get that

$$
\int_{D_{0,s} \cap \{(x, t) : u \geq v \}} (u - v) \phi \, dx \leq 0.
$$

Since this holds for any positive $\phi$, we obtain that $u \leq v$ a.e. in $\Omega_T \cap [\mathbb{R}^n \times \{s\}]$ for any $s$, and then also in $\Omega_T$.

The crucial point in the proof above is that we can approximate the set

$$
D_{r,s} = \left\{ (x, t) \in \Omega_T : \frac{u}{1 + \epsilon} \geq v \right\},
$$

by finite unions of space time boxes while staying inside the set where $u$ is a weak solution. Thus we can also deduce the following Theorem:

**Theorem 3.2.** Let $E$ be an open set in $\mathbb{R}^{n+1}$, let $u$ be a non-negative continuous weak solution in $E$ such that

$$
\int_E \left[ |u|^m + |\nabla u|^2 \right] \, dx \, dt < \infty.
$$

Let $v$ be a non-negative lower semicontinuous weak supersolution such that

$$
\int_E \left[ |v|^m + |\nabla v|^2 \right] \, dx \, dt < \infty,
$$

by finite unions of space time boxes while staying inside the set where $u$ is a weak solution. Thus we can also deduce the following Theorem:
v > 0 on ∂E and \( u \leq v \) on ∂E. Then \( u \leq v \) in E. Furthermore if a connected component of ∂E is the boundary of a finite union of space-time cylinders then we can remove the assumption \( v > 0 \) on that component.

4. The obstacle problem

In this section, we construct solutions to the obstacle problem by a potential theoretic method. More specifically, we call a function \( u \) a solution to the obstacle problem if it is the smallest supersolution above the given obstacle function \( \psi \).

Existence and uniqueness are fairly easily established for this notion of solution to the obstacle problem. However, the relationship between the variational solutions studied in [4] and the smallest supersolution is not immediately clear. In this direction, we apply the comparison principle established earlier to prove that the smallest supersolution is also a variational solution, provided that the obstacle is sufficiently regular. This is a consequence of two facts: first, we prove that the smallest supersolution is a point-wise limit of variational solutions. Second, variational solutions are stable with respect to convergence of the obstacles in a suitable norm.

We expect that the converse is also true, i.e. that a variational solution is the smallest supersolution. However, our version of the comparison principle in general domains is not strong enough to prove this.

First we describe the notion of smallest supersolution in more detail.

**Definition 4.1.** Let \( \psi \) be a positive, bounded measurable function in \( \Omega_\infty \), and denote
\[
U_\psi = \{ v \text{ is a semicontinuous supersolution in } \Omega_\infty : v \geq \psi \text{ in } \Omega_\infty \}.
\]
We define the réduite (or reduced function) of \( \psi \) as
\[
R_\psi = \inf \{ v : v \in U_\psi \}.
\]
For a measurable set \( E \), we abbreviate \( R_E = R_{\chi_E} \). We denote by \( \bar{R}_\psi \) (lower semi-continuous) ess liminf-regularization of \( R_\psi \). The function \( \bar{R}_\psi \) is usually called the balayage of \( \psi \).

The terms réduite and balayage come from classical potential theory. The notion is due to Poincaré. We will need the following basic theorem, for which the proof is standard, but we reproduce it here for the reader’s convenience.

**Theorem 4.2.** The balayage \( \bar{R}_\psi \) is a semicontinuous supersolution in \( \Omega_T \).

**Proof.** Pick a space-time cylinder \( U_{t_1,t_2} @ \Omega_T \) and a weak solution \( u \) which is continuous in \( \overline{U_{t_1,t_2}} \) with \( u \leq \bar{R}_\psi \) on \( \partial_\mu U_{t_1,t_2} \). Then also \( u \leq v \) on \( \partial_\mu U_{t_1,t_2} \) for \( v \in U_\psi \), and by comparison the same holds in \( U_{t_1,t_2} \). We take the infimum over \( v \) to get that \( u \leq R_\psi \). Since \( \bar{R} = u \) by the continuity of \( u \), we conclude that \( u \leq \bar{R}_\psi \). \( \square \)

Note that in general, \( R_\psi \) might not be lower semicontinuous, and \( \bar{R}_\psi \) might not be above \( \psi \) in every point. However, for continuous \( \psi \) it holds that \( \bar{R}_\psi \geq \psi \) everywhere. This together with Theorem 4.2 implies that \( \bar{R}_\psi \) is the unique smallest semicontinuous supersolution above the obstacle \( \psi \). By the smallest supersolution, we mean a function \( u \in U_\psi \) with the property that
\[
u u \leq v \text{ for all } v \in U_\psi.
\]
A semicontinuous supersolution with the property (4.1) is unique, if it exists; indeed, if there are two functions \( u_1, u_2 \in U_\psi \) satisfying (4.1), then two applications of (4.1) give the inequalities \( u_1 \leq u_2 \) and \( u_2 \leq u_1 \), so that \( u_1 = u_2 \).
The next aim is to relate the smallest supersolution to the variational solutions to the obstacle problem constructed in [4]. We first recall some facts from [4]. We consider nonnegative obstacle functions \( \psi \) defined on \( \Omega_T \), with compact support and satisfying

\[
\psi^m \in L^2(0, T; H^1_0(\Omega)), \quad \partial_t(\psi^m) \in L^\infty(\Omega_T). \tag{4.2}
\]

The class of admissible functions for the obstacle problem is defined by

\[
K_\psi(\Omega_T) := \{ v : \Omega_T \to [0, \infty] : v^m \in L^2(0, T; H^1_0(\Omega)), v \geq \psi \text{ a.e. on } \Omega_T \}.
\]

Note that \( \psi \in K_\psi \), and therefore \( K_\psi \neq \emptyset \).

With the above classes, we can state the definition of a strong solution to the obstacle problem.

**Definition 4.3.** A nonnegative function \( u \in K_\psi(\Omega_T) \) is a strong solution to the porous medium equation if \( \partial_t u \in L^2(0, T; H^{-1}(\Omega)) \) and

\[
\int_0^T (\partial_t u, \alpha(v^m - u^m)) \, dt + \int_{\Omega_T} \alpha \nabla u^m \cdot \nabla (v^m - u^m) \, dz \geq 0,
\]

holds for all comparison maps \( v \in K_\psi(\Omega_T) \) and every Lipschitz continuous cut-off function \( \alpha : [0, T] \to [0, \infty] \) with \( \alpha(T) = 0 \).

The cutoff function \( \alpha \) is needed for making this definition consistent with the definition of weak solutions to the obstacle problem, which we will recall later.

For the existence of strong solutions, we still need the assumption

\[
\psi := \partial_t \psi - \Delta \psi^m \in L^\infty(\Omega_T). \tag{4.3}
\]

The following result can be extracted from [4, Theorem 2.6].

**Theorem 4.4.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) with a smooth boundary. Assume that the obstacle \( \psi \) satisfies the regularity conditions (4.2) and (4.3). Then there exists a strong solution \( u \) to the obstacle problem for the PME in the sense of Definition 4.3 satisfying \( u^m \in L^2(0, T; H^1_0(\Omega)) \) and \( u(\cdot, 0) = 0 \).

The function \( u \) is also locally H"older continuous, and satisfies \( u \geq \psi \) everywhere in \( \Omega_T \). Further, \( u \) is a weak supersolution to the porous medium equation in \( \Omega_T \), and a weak solution in the open set \( \{ z \in \Omega_T : u(z) > \psi(z) \} \).

We now wish to show that \( u \) in Theorem 4.4 is a weak solution in the larger set

\[
[\Omega_T \setminus \text{supp}(\psi)] \cup \{ z \in \Omega_T : u(z) > \psi(z) \}.
\]

With this in mind we recall the following form of a partition of unity.

**Lemma 4.5** (Partition of Unity). Let \( U_1, U_2, \ldots, U_n \) be open sets, and let \( K \) be a compact set such that \( K \subset U_1 \cup U_2 \cup \cdots \cup U_n \). Then there exist functions \( \eta_i \in C^\infty_0(U_i) \) such that

\[
\sum_{i=1}^n \eta_i = 1 \quad \text{on} \quad K.
\]

**Proof.** For a version where the functions \( \eta_i \) are continuous, see [16, Theorem 2.13, p. 40]. The fact that one may also choose smooth functions follows easily from the continuous version by applying a suitable mollification. \( \square \)

**Lemma 4.6.** The strong solution to the obstacle problem given by Theorem 4.4 is also a weak solution to the PME in the set \( \Omega_T \setminus \text{supp}(\psi) \).

**Proof.** Let \( \delta > 0 \) be a number, and let \( \eta_\delta : \mathbb{R} \to [0, 1] \) be a Lipschitz function with \( \eta(s) = 0 \) for \( s \leq -\delta \), \( \eta(s) = 1 \) for \( s \geq 0 \), and \( |\eta'(s)| \leq 1/\delta \). The solution \( u \) is constructed in [4] as the uniform limit as \( \delta \to 0 \) of solutions to

\[
\partial_t u^\delta - \Delta u^\delta = \eta_\delta(\psi^m - u^m_\delta)(\partial_t \psi - \Delta \psi^m)^+.
\]
The claim now follows from the fact that \((\partial_t \psi - \Delta \psi^m)_+ = 0\) in \(\Omega_T \setminus \text{supp}(\psi)\).

**Theorem 4.7.** Let \(\psi\) be a nonnegative, compactly supported function satisfying the regularity assumptions (4.2) and (4.3), let \(u\) be the strong solution to the obstacle problem given by Theorem 4.4 with obstacle \(\psi\), and denote \(K = \text{supp}(\psi) \cap \{u = \psi\}\). Then \(u\) is a weak solution in \(\Omega_\infty \setminus K\).

**Proof.** We first show that the function \(u\) is a weak solution in \(\Omega_\infty \setminus K\). Denote \(U_1 = \Omega_\infty \setminus \text{supp}(\psi)\) and \(U_2 = \{u > \psi\}\). These sets are open, and

\[
\Omega_\infty \setminus K = U_1 \cup U_2.
\]

Further, \(u\) is a weak solution in \(U_1\) and in \(U_2\). The claim concerning the set \(U_1\) is Lemma 4.6, and the claim about \(U_2\) is a part of Theorem 4.4. To show that \(u\) is a solution also in \(U_1 \cup U_2\), let \(\varphi \in C^\infty_0(U_1 \cup U_2)\). An application of Lemma 4.5 shows that there are functions \(\eta_i \in C^\infty_0(U_1), i = 1, 2\), such that \(\eta_1 + \eta_2 = 1\) on the support of \(\varphi\). By applying the fact that \(u\) is a weak solution in \(U_1\) and in \(U_2\), we get

\[
\int_{\Omega_\infty} -u \partial_t \varphi + \nabla u^m \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_\infty} -u \partial_t (\varphi \eta_i) + \nabla u^m \cdot \nabla (\varphi \eta_i) \, dx \, dt = 0.
\]

Since this holds for any test function \(\varphi\), \(u\) is a weak solution in \(U_1 \cup U_2\).

We are now ready to proceed with the approximation result.

**Theorem 4.8.** Let \(\psi\) be continuous and compactly supported in \(\Omega_T\). Then the smallest supersolution \(\bar{R}_\psi\) is an increasing limit of strong solutions \(w_j\) to the obstacle problem with smooth compactly supported obstacles \(\phi_j\) increasing to \(\psi\).

**Proof.** Let \(U_j = \{\psi > 1/j\}\) for \(j = 1, 2, \ldots\), \(K_j = \overline{U}_j\). First note that \(K_j\) and \(K_{j+1}\) have a positive distance between them. Thus if we let \(h_j = (\sqrt{\psi} - 1/\sqrt{j})_+\), we see that \(h_{j+1} - h_j\) is strictly positive in \(K_j\). By a mollification argument it can easily be seen that for each \(j = 1, \ldots\) there exists a function \(f_j \in C^\infty_0(K_{j+1})\) such that

\[
h_j \leq f_j \leq h_{j+1}.
\]

Taking \(\phi_j = f_j^2\), \(j = 1, \ldots\), we immediately see that \(\phi_j^m \in C^2_0(K_{j+1})\), thus it satisfies (4.2) and (4.3). Moreover by construction we get

\[
\phi_1 < \phi_2 < \ldots < \psi, \quad \phi_j \to \psi \text{ as } j \to \infty.
\]

Let \(w_j\) be the strong solutions to the \(\phi_j\)-obstacle problems. Since \(\bar{R}_\psi \geq \psi\) by the continuity of \(\psi\), we have \(w_j < \bar{R}_\psi\) on \(K = \partial(\{w_j = \phi_j\} \cap \text{supp} \phi_j)\). Also note that \(K \subset U_{j+1}\), whence \(\bar{R}_\psi > \frac{1}{j+1} > 0\) on \(K\). This allows us to use the comparison principle of Theorem 3.1 together with Theorem 4.7 to get that \(w_j \leq \bar{R}_\psi\). A similar argument shows that \(w_j \leq w_{j+1}\). Thus \(w = \lim_{j \to \infty} w_j\) is a semicontinuous supersolution as an increasing limit of continuous supersolutions, and \(w \leq \bar{R}_\psi\). To finish the proof, we have that \(w \geq \psi\) everywhere in \(\Omega_T\), whence \(R_\psi \leq w\). Thus

\[
w \leq \bar{R}_\psi \leq R_\psi \leq w,
\]

and the proof is complete.

The final step is to combine the approximation result with a stability result for variational solutions to conclude that the smallest supersolution is also a variational solution. We recall some more facts from [4], in particular the notion of a weak variational solution, for which stability with respect to the obstacles can be established.

For the notion of weak solutions, we use the class of admissible comparison functions

\[
K^m_\psi(\Omega_T) = \{v \in K_\psi(\Omega_T) : \partial_t (v^m) \in L^{\frac{m+1}{m}}(\Omega_T)\}.
\]
We need to make sense of the time term in the variational inequality when we do not know that $\partial_t u$ belongs to the dual of the parabolic Sobolev space. We do this as in [1] and [4]. We recall the notation
\[
\langle \partial_t u, \alpha \eta(v^m - u^m) \rangle_{\omega_0} = \int_{\Omega_T} \eta \left[ \alpha \left( \frac{1}{m+1} u^{m+1} - u^m \right) - \alpha u \partial_t v^m \right] \, dx \, dt
+ \alpha(0) \int_{\Omega} \eta \left( \frac{1}{m+1} u_0^{m+1} - u_0 v^m(\cdot, 0) \right) \, dx
\]
where $u_0 \in L^{m+1}(\Omega)$ is a function giving the initial values of the solution, and $\alpha$ is a nonnegative Lipschitz continuous cutoff function depending only on the time variable with $\alpha(T) = 0$. The role of the function $\alpha$ is to eliminate the final time term, as we do not know in general whether $u$ is continuous in time. Observe that if $\partial_t u \in L^2(0, T; H^{-1}(\Omega))$, we have
\[
\int_0^T \langle \partial_t u, \alpha \eta(v^m - u^m) \rangle \, dt = \langle \partial_t u, \alpha \eta(v^m - u^m) \rangle_{\omega_0}.
\]
This follows formally from integration by parts, and the rigorous justification is given in [4, Lemma 3.2]. This makes the following definition consistent with the definition of strong solutions in the previous section, i.e. strong solutions are also weak solutions.

**Definition 4.9.** A nonnegative function $u \in K_\psi(\Omega_T)$ is a weak solution to the obstacle problem for the porous medium equation if the inequality
\[
\langle \partial_t u, \alpha \eta(v^m - u^m) \rangle_{\omega_0} + \int_{\Omega_T} \alpha \nabla u^m \cdot \nabla \left( \eta(v^m - u^m) \right) \, dz \geq 0
\]
holds true for all comparison maps $v \in K'_\psi(\Omega_T)$ and every nonnegative, Lipschitz continuous cut-off function depending only on the time variable with $\alpha(T) = 0$.

**Theorem 4.10.** Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary. Assume that the obstacle $\psi$ satisfies the regularity condition (4.2). Then there exists a weak solution $u$ to the obstacle problem for the porous medium equation in the sense of Definition 4.9 satisfying $v^m \in L^2(0, T; H^1_0(\Omega))$. Again, $u$ is a weak supersolution to the porous medium equation in $\Omega_T$.

The following theorem may be extracted from the proof of Theorem 2.7 in [4].

**Theorem 4.11.** Let $\psi_i$ be a sequence of obstacles satisfying (4.2) with compact support in $\Omega_T$ such that
\[
\psi_i^m \rightarrow v^m \text{ in } L^2(0, T; H^1_0(\Omega)), \quad \text{and} \quad \partial_t(\psi_i^m) \rightarrow \partial_t(v^m) \text{ in } L^{\frac{2n}{n+2}}(\Omega_T),
\]
furthermore let $u_i$ be the respective variational weak solutions to the obstacle problem with obstacle $\psi_i$, see Theorem 4.10.

Then there is a function $u \in L^\infty(0, T; L^{m+1}(\Omega))$ with $u^m \in L^2(0, T; H_0^1(\Omega))$ and $u(\cdot, 0) = 0$ such that, up to subsequences,
\[
\psi_i \rightarrow u \text{ a.e.,} \quad u_i^m \rightarrow u^m \text{ in } L^2(\Omega_T), \quad \text{and} \quad \nabla u_i^m \rightarrow \nabla u^m \text{ weakly in } L^2(\Omega_T).
\]
Furthermore, $u$ is a variational weak solution to the obstacle problem with obstacle $\psi$ and initial values zero.

Since strong variational solutions are also weak variational solutions, we get the following theorem as an immediate consequence of Theorems 4.8 and 4.11.

**Theorem 4.12.** Let $\psi$ be a continuous function with compact support in $\Omega_T$ satisfying the regularity assumptions (4.2). Then the smallest supersolution $\mathcal{R}_\psi$ is also a variational weak solution.
A general converse for Theorem 4.12 remains open. We record the following partial result for use in Section 5.

**Theorem 4.13.** Let \( \psi \) be a smooth obstacle with \( \psi > 0 \) in \( \Omega_T \). Then any variational strong solution \( u \) to the obstacle problem coming from Theorem 4.4 satisfies \( u = \bar{R}_\psi \).

**Proof.** Since \( \psi > 0 \), any strong solution is strictly positive inside \( \Omega_T \). Thus, given a semicontinuous supersolution \( v \in \mathcal{U}_\psi \), we may apply Theorem 3.1 on the set \( \{ u > \psi \} \) to conclude that \( u \leq v \). Since \( u \in \mathcal{U}_\psi \), we get \( u = \bar{R}_\psi \). \( \square \)

## 5. Parabolic capacity for the porous medium equation

In this section, we define the parabolic capacity for the porous medium equation and establish its basic properties.

**Definition 5.1.** The PME capacity of an arbitrary subset \( E \) of \( \Omega_\infty \) is

\[
\text{cap}(E) = \sup \{ \mu(\Omega_\infty) : 0 \leq u_\mu \leq 1, \text{supp}(\mu) \subset E \},
\]

where \( \mu \) is a positive Radon measure, and \( u_\mu \) is a weak supersolution with \( u_\mu = 0 \) on \( \partial_0 \Omega_\infty \), and a weak solution to the measure data problem

\[
(u_\mu)_t - \Delta u_\mu^n = \mu.
\]

Our next result is that there exists a capacitary extremal for the PME capacity of a compact set \( K \), i.e. a semicontinuous supersolution \( u \) such that \( \text{cap}(K) = \mu_u(K) \). We need the following two lemmas.

**Lemma 5.2.** Let \( \psi \) be a smooth, positive compactly supported function, and set

\[
\psi_\varepsilon = (\psi^m + \varepsilon^m)^{1/m} \quad \text{and} \quad v_\varepsilon = \bar{R}_{\psi_\varepsilon}.
\]

Then the limit function

\[
v = \lim_{\varepsilon \to 0} v_\varepsilon
\]

is a continuous weak supersolution and a weak variational solution to the obstacle problem with obstacle \( \psi \) in \( \Omega_T \). Further, \( v \) is a weak solution in the open set \( \{ v > \psi \} \).

**Proof.** The existence of the point-wise limit as \( \varepsilon \to 0 \) follows from the fact that \( \bar{R}_{\psi_\varepsilon_1} \leq \bar{R}_{\psi_\varepsilon_2} \) if \( \varepsilon_1 \leq \varepsilon_2 \). The limit \( v \) is an upper semicontinuous weak supersolution as a decreasing limit of continuous weak supersolutions, and \( v \geq \psi \) since \( v_\varepsilon \geq \psi_\varepsilon \).

By Theorem 4.13, we may take \( v_\varepsilon \) to be a strong variational solution to the obstacle problem. Hence \( v \) is a weak variational solution to the obstacle problem by [4]. The continuity follows from [5].

Since \( v_\varepsilon \) is a variational strong solution to the obstacle problem, it is weak solution in the set \( \{ v_\varepsilon > \psi_\varepsilon \} \). If \( K \) is now a compact set contained in \( \{ v > \psi \} \), we have that \( K \) is also contained in \( \{ v_\varepsilon > \psi_\varepsilon \} \) for all sufficiently small \( \varepsilon \), since

\[
v_\varepsilon - \psi_\varepsilon \geq \inf_K (v - \psi) - \varepsilon
\]

by the inequalities \( v_\varepsilon \geq v, -\psi_\varepsilon = -((\psi^m + \varepsilon^m)^{1/m}) \geq -\psi - \varepsilon \), and the fact that \( \inf_K (v - \psi) > 0 \). Thus

\[
\int_{\Omega_T} -v_\varepsilon \partial_t \varphi + \nabla v_\varepsilon^n \cdot \nabla \varphi \, dx \, dt = \lim_{\varepsilon \to 0} \int_{\Omega_T} -v_\varepsilon \partial_t \varphi + \nabla v_\varepsilon^n \cdot \nabla \varphi \, dx \, dt = 0
\]

for all smooth test functions \( \varphi \) with support in \( K \). Since \( K \) was arbitrary, \( v \) is a weak solution. \( \square \)
The next lemma is the key step in constructing the capacitary extremal. For the proof, we record the following estimate. Let \( u \) be a positive weak supersolution in \( \Omega_\infty \), \( u \) vanishing on the lateral boundary of \( \Omega_\infty \). Suppose in addition that there exists a time \( t_0 \geq 0 \) so that \( u \) is a weak solution in \( \Omega_{t_0, \infty} \). Then

\[
    u(x, t) \leq c(t - t_0)^{-\frac{5}{2}}
\]

(5.1)

for all \( t > t_0 \) with a constant depending only on \( n, m \), and the diameter of \( \Omega \). This is the so-called universal estimate. See Proposition 5.17 in [17] for the proof.

**Lemma 5.3.** Let \( K \) be a compact subset of \( \Omega_\infty \). Assume that \( u \) and \( v \) are lower semicontinuous weak supersolutions in \( \Omega_\infty \) and that \( u \) is continuous in \( \overline{\Omega}_\infty \), and a weak solution after a time \( T \) such that \( K \subset \Omega_T \). Moreover, assume that \( u > 1 \) in \( K \), \( u = 0 \) on \( \partial_p \Omega_\infty \), \( 0 \leq v \leq 1 \) in \( \Omega_\infty \), and \( v = 0 \) on \( \partial_p \Omega_\infty \). Then

\[
    \mu_v(K) \leq \mu_u(\Omega_\infty).
\]

**Proof.** Let \( 0 \leq \psi_i \in C^\infty_0(\Omega_\infty) \) be an increasing sequence of smooth obstacles converging to \( v \) such that \( \psi_i < v \) and \( \psi_i < \psi_{i+1} \). Denote the perturbed obstacles \( \psi_i' = (\psi_i^m + \epsilon)^{1/m} \), and the corresponding solutions to the obstacle problem by \( v_i' \). Further, let \( v_i = \lim_{\epsilon \to 0} v_i' \) be the weak supersolutions in \( \Omega_\infty \) constructed in Lemma 5.2. We argue as in [13, proof of Theorem 3.2, p. 148–149] to see that \( v_i \leq v_{i+1} \leq v \) and \( v_i \to v \) as \( i \to \infty \). Finally, we have that

\[
    \sup_{\Omega_\infty} v_i' = \sup_{\Omega_\infty} \psi_i' \leq 1 + \epsilon
\]

and

\[
    \sup_{\Omega_\infty} v_i = \sup_{\Omega_\infty} \psi_i \leq 1.
\]

We define the supersolution

\[
    w_i' = \min(v_i', u).
\]

By lower semicontinuity, the set \( \{ u > 1 \} \) is open, and \( K \) is compactly contained in it. This allows us to construct a compact set \( K' \) and an open set \( U \) such that \( K \subset U \subset K' \subset \{ u > 1 \} \). If \( \epsilon \) is small enough we know that \( 1 + \epsilon < u \) in \( K' \), so that \( w_i'' = v_i' \) in \( K' \). Hence for such \( \epsilon \) we have for \( \phi' = 1 \) on \( U \) and \( \phi'' = 0 \) outside \( K' \) that

\[
    \mu_{w_i''}(U) \leq \int_U \phi' d\mu_{w_i'} = \int_U \phi' d\mu_{w_i'} \leq \mu_{w_i''}(K').
\]

(5.2)

Next note that since \( u = 0 \) on \( \partial_p \Omega \) and \( u \leq \epsilon \) for sufficiently large times by (5.1), there is a compact set \( K'' \supset K' \) in \( \Omega_\infty \) such that in \( \Omega_\infty \setminus K'' \) we have \( w_i'' = u \) since \( v_i' \geq \epsilon \). Hence we obtain for \( \phi'' \in C^\infty_0(\Omega_\infty) \) such that \( \phi'' = 1 \) on \( K'' \) that

\[
    \int_{\Omega_\infty} \phi'' d\mu_{w_i'} = \int_{\Omega_\infty} -w_i'' \frac{\partial \phi''}{\partial t} + \nabla (w_i'')^m \cdot \nabla \phi'' \, dx \, dt
    = \int_{\Omega_\infty} -w_i'' \frac{\partial \phi''}{\partial t} + \nabla u^m \cdot \nabla \phi'' \, dx \, dt
    = \int_{\Omega_\infty} \phi'' d\mu_u.
\]

Thus we obtain the estimate

\[
    \mu_{w_i''}(K') \leq \int_{\Omega_\infty} \phi'' d\mu_{w_i'} = \int_{\Omega_\infty} \phi'' d\mu_u \leq \mu_u(\Omega_\infty).
\]

(5.3)

We combine (5.2) and (5.3) to get the inequality

\[
    \mu_{v_i'}(U) \leq \mu_u(\Omega_\infty).
\]
By construction $v_i^c \to v_1$ point-wise, thus from Lemma 2.6 we get that $\mu_{v_i^c} \to \mu_{v_1}$ weakly. By the standard properties of weak convergence of measures, see Theorem 2.7, we get that

$$\mu_{v_1}(U) \leq \liminf_{c \to 0} \mu_{v_i^c}(U) \leq \mu_u(\Omega_\infty).$$

The sequence $(v_i)$ is increasing, and converges point-wise to the original supersolution $v$. Again from Lemma 2.6 we get the weak convergence of the corresponding measures. Another application of Theorem 2.7 now shows that

$$\mu_u(K) \leq \mu_{v_i}(U) \leq \liminf_{i \to \infty} \mu_{v_i}(U) \leq \mu_u(\Omega_\infty),$$

and the proof is complete. \hfill \Box

A consequence of Theorem 3.1, is that in the special case that we have a decreasing sequence of smooth obstacles converging to a characteristic function of a compact set, the obstacle problem is stable. If we had a full elliptic comparison principle, this lemma would hold for a decreasing sequence of smooth obstacles converging to an upper semi-continuous obstacle.

**Lemma 5.4.** Let $K \subset \Omega_\infty$ be a compact set. Let $E_i \in \Omega_\infty$, $i = 1, \ldots$ be a shrinking sequence of open sets such that $E_{i+1} \Subset E_i$

$$\bigcap_{i=1}^{\infty} E_i = K.$$ 

Assume that the non-negative functions $\psi_i : \Omega_\infty \to \mathbb{R}$ are supported in $E_i$, satisfy (4.2) and (4.3), and $\psi_i \geq \chi_K$, $i = 1, 2, \ldots$, is a decreasing sequence such that $\psi_i \to \chi_K$ point-wise in $\Omega_\infty$ as $i \to \infty$. Then $R_{\psi_i} \to R_K$ point-wise in $\Omega_\infty$ and $\mu_{R_{\psi_i}} \to \mu_{R_K}$ weakly as $i \to \infty$.

**Proof.** By Theorem 4.7, the functions $R_{\psi_i}$ are continuous. Thus an application of Lemma 2.6 shows that $u = \lim_{i \to \infty} R_{\psi_i}$ is an upper semicontinuous weak supersolution, and the respective measures also converge weakly. Further,

$$u \geq R_K,$$

since $R_{\psi_i} \geq R_K$ for each $i$.

The lemma now follows if we prove the opposite inequality. To this end, note first and that from Theorem 4.7, $R_{\psi_i}$ is a weak solution in $\{ R_{\psi_i} > \psi_i \} \cup (\Omega_\infty \setminus \text{supp}(\psi_i))$, so that the support of the measure $\mu_{R_{\psi_i}}$ is contained in $\text{supp}(\psi_i) \subset E_i$. These sets shrink to $K$, and the measures $\mu_{R_{\psi_i}}$ converge weakly to $\mu_u$. Thus $\text{supp}(\mu_u) \subset K$, which implies that $u$ is a weak solution in $\Omega_\infty \setminus K$.

If now $v \geq \chi_K$ is an arbitrary semicontinuous supersolution with $v = 0$ on $\partial_p \Omega_\infty$, it follows from Theorem 3.1 that $u \leq v$. We take the infimum over $v$ to get that

$$u \leq R_K,$$

and the proof is complete. \hfill \Box

A consequence of the stability Lemma 5.4 is that we have stability of the balayage with respect to decreasing sequences of compact sets.

**Lemma 5.5.** Let $K_i \subset \Omega_\infty$, $i = 1, 2, \ldots$, be a decreasing sequence of compact sets and denote $K = \bigcap_{i=1}^{\infty} K_i$. Then $RK_i$ is a decreasing sequence converging to $RK$, moreover $\mu_{RK_i}$ converges to $\mu_{RK}$, weakly as $i \to \infty$.

**Proof.** Let us construct $E_i = \{ d((x, t); K_i) < c/i \}$, $i = 1, \ldots$, for a small constant $c < 1$ such that $E_i \subset \Omega_\infty$, then the sequence $E_i$ satisfies the requirements of Lemma 5.4.
Let us now construct smooth functions $\hat{\psi}_i \in C_0^\infty(E_i)$ such that $\hat{\psi}_i = 1$ on $K_i$, then let $\psi_i = [\hat{\psi}_i]^2$, and we have that $R_{\psi_i} \geq R_{K_i}$ by construction. As in the proof of Theorem 4.8, the sequence $\psi_i$ will satisfy (4.2) and (4.3). It is now clear that the sequence $\psi_i$ satisfies all requirements of Lemma 5.4 and thus we get that $R_{\psi_i} \to R_{K_i}$ and consequently also $R_{K_i} \to R_{K}$, furthermore using Lemma 2.6 we see that the measures $\mu_{R_{K_i}}$ converge weakly to $\mu_{R_{K_i}}$ as $i \to \infty$. □

Theorem 5.6. Let $K$ be a compact subset of $\Omega_\infty$. Then

$$\text{cap}(K) = \mu_{R_{K}}(K).$$

Proof. Since $R_{K}$ is a semicontinuous supersolution such that $0 \leq R_{K} \leq 1$, it follows immediately from the definition of the PME capacity that

$$\mu_{R_{K}}(K) \leq \text{cap}(K),$$

since $R_{K}$ is a solution outside $K$.

To prove the opposite inequality, let first $K' \equiv \Omega_\infty$ be a compact set such that $K \subset K'$. To be able to use Lemma 5.4 we will let $E_i \subset K'$, $i = 1, \ldots$ be a shrinking sequence of open sets such that

$$\bigcap_{i=1}^\infty E_i = K.$$

Let $\hat{\psi}_i \in C_0^\infty(E_i)$, $i = 1, \ldots$, be a decreasing sequence of smooth functions converging to $\chi_K$ point-wise in $\Omega_\infty$ as $i \to \infty$, and such that

$$\hat{\psi}_i = \sqrt{1 + \frac{1}{2^i}} \text{ on } K.$$

Consider now the functions $\psi_i = [\hat{\psi}_i]^2$, then $\psi_i^m \in C_0^\infty(E_i)$, and it is a decreasing sequence of functions converging to $\chi_K$ point-wise in $\Omega_\infty$ as $i \to \infty$, such that

$$\psi_i = 1 + \frac{1}{2^i} \text{ on } K,$$

moreover $\psi_i$ satisfies (4.2) and (4.3) for all $m > 1$. Denote by $u_i$ the corresponding solutions to the obstacle problems with obstacle $\psi_i$. Let now $v$ be a weak supersolution in $\Omega_\infty$ such that $0 \leq v \leq 1$ and $v = 0$ on $\partial_p \Omega_\infty$. Then it follows from Lemma 5.3 that

$$\mu_v(K) \leq \mu_{u_i}(\Omega_\infty) = \mu_{u_i}(K').$$

We use Lemma 5.4 to see that $\mu_{u_i} \to \mu_{R_{K}}$ weakly. The claim now follows from the above estimate, since

$$\limsup_{i \to \infty} \mu_{u_i}(K') \leq \mu_{R_{K}}(K') = \mu_{R_{K}}(K)$$

by Theorem 2.7. □

We have now developed all the technical tools needed to establish the basic properties of the PME capacity, including that it is a regular, subadditive capacity.

Theorem 5.7. The PME capacity has the following properties.

(1) Countable subadditivity: In other words if $E_i$, $i = 1, 2, \ldots$, be arbitrary subsets of $\Omega_\infty$ and $E = \bigcup_{i=1}^\infty E_i$, one has

$$\text{cap}(E) \leq \sum_{i=1}^\infty \text{cap}(E_i).$$
(2) Stability with respect to increasing sequences of sets: Let $E_i, i = 1, 2, \ldots$, be arbitrary subsets of $\Omega_\infty$ with the property $E_1 \subset E_2 \subset \cdots$ and denote $E = \bigcup_{i=1}^{\infty} E_i$. Then
\[
\lim_{i \to \infty} \text{cap}(E_i) = \text{cap}(E).
\]

(3) Stability with respect to decreasing sequences of compact sets: Let $K_i \subset \Omega_\infty, i = 1, 2, \ldots$, be a decreasing sequence of compact sets and denote $K = \bigcap_{i=1}^{\infty} K_i$. Then
\[
\lim_{i \to \infty} \text{cap}(K_i) = \text{cap}(K).
\]

(4) Let $U \Subset \Omega_\infty$ be an open set. Then
\[
\text{cap}(U) = \mu_{R_\epsilon}(\Omega_\infty).
\]

Proof. From the methods developed in [11] we see that (1) and (2) follow from Lemma 2.5. Property (3) is a consequence of Theorem 5.6 and Lemma 5.5. Property (4) follows from (2), Theorem 5.6, and Lemma 2.6 as in [11, Lemma 5.9]. □

In conclusion we have established more than enough to say that Borel sets are Choquet capacitable:

**Theorem 5.8.** The PME capacity is Choquet capacitable (inner regular). This means that for all Borel sets $E \subset \Omega_\infty$ it holds that
\[
\text{cap}(E) = \sup \{\text{cap}(K) : K \subset E, K \text{ compact}\}.
\]

Proof. Since the capacity is monotone, stable with respect to increasing sequences of sets (Theorem 5.7, (2)) and stable with respect to decreasing sequences of compact sets (Theorem 5.7, (3)), it is a regular capacity and hence the claim follows from Choquet’s capacitability theorem [6, Theorem 9.3, p. 155]. □

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