INTERNAL ZONOTOPAL ALGEBRAS AND THE MONOMIAL REFLECTION GROUPS $G(m, 1, n)$

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Abstract. The group $G(m, 1, n)$ consists of $n$-by-$n$ monomial matrices whose entries are $m$th roots of unity. It is generated by $n$ complex reflections acting on $\mathbb{C}^n$. The reflecting hyperplanes give rise to a (hyperplane) arrangement $G \subset \mathbb{C}^n$. The internal zonotopal algebra of an arrangement is a finite dimensional algebra first studied by Holtz and Ron. Its dimension is the number of bases of the associated matroid with zero internal activity. In this paper we study the structure of the internal zonotopal algebra of the Gale dual of the reflection arrangement of $G(m, 1, n)$, as a representation of this group. Our main result is a formula for the top degree component as an induced representation. We also provide results on representation stability, a connection to the Whitehouse representation in type A, and an analog of decreasing trees in type B.

1. Introduction

A meta-problem in the theory of hyperplane arrangements starts with an algebraic object $M(A)$ derived from an arrangement $A$ and determines the extent to which the intersection lattice $L(A)$ determines $M(A)$ up to isomorphism. The prototypical example of this associates to an arrangement $A \subset V$ the cohomology ring of the complement $H^*(V - A; \mathbb{Z})$. The structure of this ring was determined by Orlik and Solomon [19] — it is determined by the combinatorics of no-broken-circuit subsets of the associated matroid.

There is an equivariant version of the meta-problem where the arrangement $A$ is fixed by the action of a group $W$ acting linearly on $V$. When this happens one wants to determine the structure of $M(A)$ as a representation of $W$. The prototypical example here is that of a real reflection group $W$ and its reflection arrangement $A \subset V$. One wants to understand the structure of $H^*(V - A; \mathbb{Z})$ as a representation of $W$. For complex reflection groups this problem was first studied by Orlik and Solomon [20] and later by many others.

Of recent interest are the so-called zonotopal algebras of a hyperplane arrangement [1, 2, 12, 14, 15]. The zonotopal ideals of an arrangement $A \subset V$ are ideals in $Sym(V)$ generated by powers of elements of $v$. Specifically, the $k$th zonotopal ideal of $A$ is

$$I_{A,k} = \langle h^{\max(\rho_A(h)+k+1,0)} : h \in V \rangle$$

where $\rho_A(h)$ is the number of hyperplanes in $A$ that do not contain $h$. The quotient $S_{A,k} = Sym(V)/I_{A,k}$ is the $k$th zonotopal algebra of $A$. Holtz and Ron [12] single out the cases $k = -2, -1, 0$ as being of particular interest, and call these the internal, central and external zonotopal algebras of $A$. The internal case $k = -2$ exhibits dramatic subtleties in comparison to all other cases $k > -1$ [2, 14,
The initial motivation for studying the internal zonotopal ideal was that its Macaulay inverse system consists of those partial differential operators that leave continuous the box spline associated to \( \mathcal{A} \) (assuming \( \mathcal{A} \) is unimodular) [15, Corollary 9].

In this paper we study the internal zonotopal algebra of certain arrangements coming from complex reflection groups \( W \). For numerological reasons we do not study the naturally occurring reflection arrangement of \( W \), but instead its Gale dual. We are specifically interested in the structure of this algebra as representations of \( W \).

**Theorem.** Let \( \mathcal{G} \) be the reflection arrangement of one of the monomial groups \( G(m, 1, n) \subset GL_n(\mathbb{C}) \), consisting of \( n \)-by-\( n \) permutation matrices whose non-zero entries are \( m \)th roots of unity (\( m > 1 \) and \( n \geq 3 \)). Let \( \mathcal{G}^\perp \) denote the Gale dual of \( \mathcal{G} \). Then the degree \( n-1 \) component of \( S_{\mathcal{G}^\perp}^{-2} \) is isomorphic to

\[
\text{Ind}_{C}^{W}(\chi)
\]

as a representation of \( W \). Here \( C \subset W \) is the group generated by an \( n \)-cycle and the scalar matrix which multiplies by \( e^{2\pi i/m} \), and \( \chi \) is the character taking values \( e^{2\pi i/n} \) and \( e^{2\pi i(n-1)/m} \) at the respective generators of \( C \).

When \( n \) and \( m \) are coprime, \( \text{Ind}_{C}^{W}(\chi) \approx \text{Ind}_{C'}^{W}(e^{2\pi i/n} \cdot e^{2\pi i(n-1)/m}) \), where \( C' \) is the cyclic subgroup of \( W \) generated by a Coxeter element.

We complement this result with several others on the structure of the internal zonotopal algebra \( S_{\mathcal{G}^\perp}^{-2} \). We study the type A case of the above theorem, when \( m = 1 \). Based on our theorem above, it is perhaps unsurprising that the Lie representation appears in type A, however our perspective makes obvious the “hidden” action of a larger symmetric group than is necessary on the Lie representation, as observed by Mathieu [18] and Robinson and Whitehouse [22]. For all \( G(m, 1, n) \), we give explicit generators of the internal zonotopal ideal and then use this description to prove finite generation in the sense of Sam and Snowden [25] and representation stability as described by Gan and Li [10]. We generalize a recurrence relation for the Whitehouse representation which factorizes the regular representation of \( G(m, 1, n) \), \( m > 1 \). In type B, when \( m = 2 \), we use a Gröbner basis for the internal zonotopal ideal to compute an analog of decreasing trees, which are classic combinatorial objects.

When \( m = 2 \) the main result bears some similarity to results of N. Bergeron [3, Theorem 5.1] and, separately, Douglas [6]. Both authors were interested in the structure of the cohomology ring of the complement of reflection arrangement of \( G(2, 1, n) \). The character of this representation is also as induced from the same subgroup described in our main theorem, albeit induced from a different character.

This \textit{a fortiori} similarity cannot be explained by a general relationship between the internal zonotopal algebra and the cohomology of the hyperplane complement.

2. A worked example

Before we begin in earnest, we work a complete example of the main theorem for the group \( W = G(2, 1, 3) \). This is the hyperoctahedral group of signed 3-by-3 permutation matrices, which has \( 2^3 \cdot 3! = 48 \) elements. \( W \) acts on \( \mathbb{C}^3 \) in the
natural way. The corresponding reflection arrangement $\mathcal{G} \subset \mathbb{C}^3$ is defined by the 9 hyperplanes,

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_1 \pm x_2 = 0, \quad x_1 \pm x_3 = 0, \quad x_2 \pm x_3 = 0.$$ 

Here, $x_1, x_2, x_3$ are the standard basis vectors of $\mathbb{C}^3$.

The Gale dual $\mathcal{G}^\perp$ of $\mathcal{G}$ is an arrangement of 9 hyperplanes in a 6 dimensional vector space $K$, which has basis

$$y_{12}^0, y_{13}^0, y_{23}^0, y_{12}^1, y_{13}^1, y_{23}^1.$$ 

The superscripts are read modulo 2, the subscripts of the $y^0$’s are anticommutative ($y_{11}^0 = -y_{11}^0$) and the subscripts of the $y^1$’s are commutative ($y_{13}^1 = y_{13}^1$). The reflection group $W$ acts on this vector space, where an honest permutation matrix (i.e., $(0/1)$-matrix) permutes the subscripts of the $y$’s and the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

acts by fixing $y_{23}^0$ and $y_{12}^1$, while it sends $y_{12}^0$ to $-y_{12}^1$ and $y_{13}^0$ to $-y_{13}^1$. This material is discussed in Section 6 where it is shown that these rules really do determine a representation of $W$.

The internal zonotopal ideal $I_{\mathcal{G}^\perp -2}$ is quadratically generated in $Sym(K)$ by

$$(y_{12}^0)^2, (y_{12}^1)^2, \quad (y_{13}^0)^2, (y_{13}^1)^2, \quad (y_{23}^0)^2, (y_{23}^1)^2, \quad y_{12}^0 y_{12}^1, \quad y_{13}^0 y_{13}^1, \quad y_{23}^0 y_{23}^1,$$

along with

$$(y_{12} + y_{23} + y_{31})^2, (z_{12} + y_{23} + z_{31})^2, (z_{21} + y_{13} + z_{32})^2, (z_{31} + y_{12} + z_{23})^2.$$ 

This ideal is $W$-stable; see Theorem 7.1. One can readily compute with a computer algebra system like Macaulay2 [17] (which we used extensively to conjecture our results) that the Hilbert series of the quotient $Sym(K)/I_{\mathcal{G}^\perp -2}$ is $1 + 6q + 8q^2 = (1 + 2q)(1 + 4q)$. This follows from Corollary 6.4. A Gröbner basis for this ideal is studied in Section 10.

From the Hilbert series one readily obtains that the largest degree component of $Sym(K)/I_{\mathcal{G}^\perp -2}$ that is non-zero is an 8-dimensional representation of $W$. The main theorem of this paper, Theorem 9.1, identifies this character as an induced character, namely,

$$\text{Ind}^W_C (\chi)$$

where $C$ is the abelian subgroup of $W$ with generators

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and $\chi$ takes the values $e^{2\pi i/3}$ and $(-1)^{3-1} = 1$, respectively, at the generators of $C$. Since $C$ is an abelian group of order 6 it must be that $C$ is cyclic, and the cyclic generator must be Coxeter element $c \in W$. It follows that the 8 dimensional top of $Sym(K)/I_{\mathcal{G}^\perp -2}$ is isomorphic to $\text{Ind}^W_C (e^{2\pi i/3})$ as a representation of $W$. This is Corollary 9.2.
The group $G(2, 1, 2)$ is a subgroup of $G(2, 1, 3)$ by viewing the former as signed 3-by-3 permutation matrices of the form
\[
\begin{bmatrix}
* & * & 0 \\
* & * & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
There are 8 such matrices. We may thus view the 8 dimensional top of $Sym(K)/I_{G,2,-2}$ as a representation of $G(2, 1, 2)$. Corollary 7.2 identifies this as the regular representation of $G(2, 1, 2)$.

3. Representation theory background

In this section we briefly recall some notions from representation theory. All the material needed in this paper can be found in, e.g., [8, Chapters 1–3].

A representation of a finite group $G$ is a group homomorphism $G \to GL(V)$, where $V$ is a finite dimensional vector space (say, over the complex numbers). The simplest example of a representation of $G$ is the group algebra $\mathbb{C}G$, whose basis elements are the elements of $G$ and where $G$ acts on the left by multiplication in the group.

The group algebra $\mathbb{C}G$ is a finite dimensional algebra and a representation of $G$ is equivalent to a (left) module over $\mathbb{C}G$. We conflate the two notions freely.

Let $G$ be a group and $H \subset G$ a subgroup. Given a representation $U$ of $H$ one may extend scalars to obtain a representation of $G$:

\[
U \rightsquigarrow \text{Ind}^G_H(U) := \mathbb{C}G \otimes_{\mathbb{C}H} U.
\]

This process is called induction, and $\text{Ind}^G_H(U)$ is called an induced representation. The dimension of $\text{Ind}^G_H(U)$ is $\text{dim}(U) \cdot |G|/|H|$. The associativity of tensor products proves that there is a natural isomorphism of representation of $F$,

\[
\text{Ind}^F_G(\text{Ind}^G_H(U)) \approx \text{Ind}^F_H(U).
\]

This is referred to as the transitivity of induction.

Say that $U$ is a representation of $H$ and is contained in a representation $V$ of $G$. By definition, there is a map $\mathbb{C}G \otimes_{\mathbb{C}H} U \to V$. If the $G$ orbit of $U$ is all of $V$ then the above map is surjective. If, in addition, the dimension of $V$ is exactly $\text{dim}(U) \cdot |G|/|H|$, the above map is an isomorphism and hence $V \approx \text{Ind}^G_H(U)$. In the sequel we will identify induced representation by demonstrating these properties.

The character of a representation $V$ of $G$ is the function $\chi_V : G \to \mathbb{C}$, $g \mapsto \text{trace}(g : V \to V)$. This function determines $V$ up to isomorphism, and is constant on conjugacy classes of $G$. When $G$ is cyclic it suffices to describe a one dimensional representation by its character value at a generator, as we will often do. The character of $\text{Ind}^G_H(U)$ is easy to determine from the character of $U$. The induced character at $g$ is zero unless $g$ is conjugate to an element of $H$. On elements $h \in H$, the induced character is equal to $|G|/|H||h^G| \sum_{x \in H \cap h^G} \chi_U(x)$. Here $h^G$ is the conjugacy class of $h$.

Example 3.1 (Stanley [23, Lemma 7.2]). Let $C$ be the cyclic subgroup generated by an $n$-cycle $c$ in $\mathfrak{S}_n$. Abusing notation, let $e^{2\pi i/n}$ denote the one dimensional
representation of \( C \) where \( c \) acts by \( e^{2\pi i/n} \). Then, the character of \( \text{Ind}_C^{\mathbb{C}^n}(e^{2\pi i/n}) \) at \( c^d \), for \( d \) a divisor of \( n \), is
\[
\frac{n!}{n \cdot (n!/(n/d)^d)!} \sum_{1 \leq k \leq n \atop k \text{ prime to } n/d} e^{2\pi i dk/n} = (d-1)!/(n/d)^{d-1} \mu(n/d).
\]
Here \( \mu \) is the usual number-theoretic Möbius function.

4. Zonotopal Algebras

Let \( \mathcal{A} \) be a central arrangement of hyperplanes in a complex finite dimensional vector space \( V \). We let \( M = M(\mathcal{A}) \) denote the matroid of \( \mathcal{A} \). Recall that this is the simplicial complex on \( \mathcal{A} \) whose faces (alias independent sets) are collections of hyperplanes whose defining linear forms are linearly independent. Maximal faces in \( M \) are referred to as its bases.

Define a function \( \rho_{\mathcal{A}} : V \to \mathbb{N} \) by the rule
\[
\rho_{\mathcal{A}}(h) = \text{the number of hyperplanes in } \mathcal{A} \text{ not containing } h,
\]
and use this to define the ideal
\[
I_{\mathcal{A},k} = \langle h^\max\{\rho_{\mathcal{A}}(h)+k+1,0\} : h \in V \rangle \subset \text{Sym}(V).
\]
The quotient \( S_{\mathcal{A},k} := \text{Sym}(V)/I_{\mathcal{A},k} \) has Krull dimension zero, and so it is a finite dimensional complex vector space. The ring \( \text{Sym}(V) \) is graded in the usual way, and we denote the \( m \)th graded piece of a graded module over this ring by \((-)^m\).
The ideal \( I_{\mathcal{A},k} \) is graded, since it is generated by powers of linear forms, which are homogeneous. The Hilbert series of \( S_{\mathcal{A},k} \) is the generating function for the dimensions of the graded pieces of this graded \( \text{Sym}(V) \)-module. Since \((S_{\mathcal{A},k})_m = 0 \) for \( m \) sufficiently large there is a biggest \( m \) for which \((S_{\mathcal{A},k})_m \neq 0 \) and we call this the top of quotient and denote it by \( S_{\mathcal{A},k}(\text{top}) \) to ease the proliferation of superscripts and subscripts.

Example 4.1. Let \( \mathcal{A} \) be the arrangement in \( \mathbb{C}^2 \) whose hyperplanes are defined by \( x = 0 \), \( y = 0 \) and \( x \pm y = 0 \). The real picture is shown below by the black lines.

Then, \( h_2^3 \in I_{\mathcal{A},-2} \) and \( h_2^2 \in I_{\mathcal{A},-2} \). One computes that \( \langle x^2, y^2, (x \pm y)^2 \rangle \subset I_{\mathcal{A},-2} \).

4.1. Hilbert series. We will describe the Hilbert series of \( S_{\mathcal{A},k} \) in terms of the matroid of \( \mathcal{A} \). To do so, we need the Tutte polynomial of a matroid. We take the most expedient route. Given a matroid \( M \) with ground set \( E \), the rank of a subset \( S \) of \( E \), is size of a largest independent set of \( M \) contained in \( S \). The Tutte polynomial of a matroid \( M \) with ground set \( E \) and rank function \( \text{rk} : 2^E \to \mathbb{N} \) is the bivariate polynomial
\[
T_M(p,q) = \sum_{A \subseteq E} (p-1)^{\text{rk}(E)-\text{rk}(A)}(q-1)^{|A|-\text{rk}(A)}.
\]
The Tutte polynomial of a matroid is universal in the sense that any matroid invariant taking values in an abelian group satisfying a generalized deletion-contraction identity must be an evaluation of the Tutte polynomial.

For \( k \geq -2 \) there is a short exact sequence relating \( S_{A,k} \) to \( S_{A \setminus H,k} \) and \( S_{A/H,k} \), where \( A \setminus H \) is \( A \) with one of its defining hyperplanes \( H \) removed, and \( A/H \) is the arrangement obtained by intersecting \( A \) with \( H \). When interpreted at the level of Hilbert series this becomes a deletion-contraction relation and we have the following result.

**Theorem 4.2.** Let \( A \) be a central, essential arrangement of hyperplanes in \( V \). Let \( M = M(A) \) denote the matroid of the arrangement \( A \), which consists of \( m \) hyperplanes. The Hilbert series of \( S_{A,k} \) is equal to

1. \( q^{m-rk(M)}T_M(1+q,1/q) \) if \( k = 0 \);
2. \( q^{m-rk(M)}T_M(1,1/q) \) if \( k = -1 \);
3. \( q^{m-rk(M)}T_M(0,1/q) \) if \( k = -2 \).

The algebras \( S_{A,k} \) occurring in the theorem are, respectively, referred to as the external, central and internal zonotopal algebras of \( A \) by Holtz and Ron. Parts (1) and (2) were discovered by many authors in many contexts (we mention only \([1, 2, 12]\)), but the interested reader should look to these for more extensive references. The internal zonotopal algebra is more subtle than its central and external counterparts, and was not discovered until the work of Holtz and Ron \([12]\) where they proved (3).

In the external and central cases, bases of the Macaulay inverse system of the associated ideals are given by certain products of linear forms defining \( A \). Analogous results were conjectured to hold in the internal case and an incorrect proof was given in \([1]\). This was corrected in \([2]\), and a combinatorial basis was later given by Lenz \([15]\) when \( A \) is unimodular. For non-unimodular \( A \), there is no known canonical basis of the Macaulay inverse system of \( I_{A,-2} \) described by the matroid of \( A \). This is particularly striking since the dimension of \( S_{A,-2} \) is \( T_M(0,1) \), a very well known number: it enumerates the number of bases of \( M \) with internal activity zero and is the reduced Euler characteristic of the independent set complex of \( M \). Finding a combinatorial basis for \( S_{A,-2} \) for general \( A \) is a tantalizing open problem.

A smaller generating set for \( I_{A,k} \) suffices than the one given when \( k \in \{-2, -1, 0\} \). To describe the smaller generating set, we recall that a line of the arrangement \( A \) is a one-dimensional intersection of hyperplanes in \( A \).

**Theorem 4.3** (Ardila–Postnikov \([1]\)). Let \( A \) be a central, essential arrangement of hyperplanes in \( V \). The ideal

\[ I'_{A,k} = \langle h^{\varepsilon(h)+k+1} : h \text{ spans a line of } A \rangle \]

is equal to \( I_{A,k} \) for \( k \in \{-2, -1, 0\} \).

**Example 4.4.** Continuing Example 4.1, the containment \( \langle x^2, y^2, (x \pm y)^2 \rangle \subset I_{A,-2} \) is an equality by the previous result. The Tutte polynomial of \( A \) is \( p^2 + q^2 + 2p + 2q \), by definition. The Hilbert series of \( S_{A,-2} \) is \( 1 + 2q \), as can be found by the computation of the ideal, or using Theorem 4.2.
A comprehensive treatment of reflection groups in unitary spaces is Lehrer and Taylor’s book [16]. Let $V$ be a complex vector space. A (generalized) reflection is an element of $GL(V)$ that fixes a hyperplane point-wise and has finite order.

A complex reflection group is a finite subgroup of $GL(V)$ generated by reflections. Such groups have been classified by Shephard and Todd, and Serre and Chevallay. There appears a single infinite family of groups $G(me,e,n)$, $m,e,n \geq 1$, called the monomial groups as well as 34 exceptional groups. The monomial group $G(me,e,n)$ consists of $n$-by-$n$ permutation matrices whose entries are $me$-th roots of unit, the product of which is a $e$-th root of unity.

Associated to a complex reflection group $W$ is its reflection arrangement $A \subset V$. This is the arrangement of hyperplanes that are fixed by all the reflections $W$.

The reflection arrangement $G = G_{m,1,n}$ associated to the group $G(m,1,n)$ ($m > 1$) is defined in $V = \mathbb{C}^n$ by the coordinate hyperplanes $x_i = 0$ ($1 \leq i \leq n$) and the hyperplanes

$$x_i - e^{2\pi ik/m}x_j \quad (1 \leq i < j \leq n, \ 1 \leq k \leq m).$$

**Example 5.1.** The arrangement in Example 4.1 is the reflection arrangement of $G(2,1,2)$. The group $G(2,1,2)$ is generated by the two matrices displayed below.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \in G(2,1,2).$$

The group $G(1,1,n)$ is the symmetric group, which we will usually write as $\mathfrak{S}_n$. There is an obvious injection $\mathfrak{S}_n \to G(m,1,n)$, a fact we will make often use of. The group $G(m,1,n)$ can also be thought of as the wreath product $\mathbb{Z}_m \wr \mathfrak{S}_n$.

### 5.1. Invariants and degrees

The Chevalley–Shephard–Todd theorem characterizes complex reflection groups as those subgroups $W \subset GL(V)$ for which the ring of polynomial invariants $\text{Sym}(V^*)^W$ is itself a polynomial ring. Algebraically independent generators of the invariant ring will not be unique, but their degrees $d_1 \leq d_2 \leq \cdots \leq d_{\ell}$ will be and are referred to as the **degrees of** $W$.

The Coxeter number of $W$ is the largest degree of $W$ and will be denoted by $h$. For reflection groups $W \subset GL(V)$ generated by $\dim(V)$ reflections (which are called well-generated) the integers $d'_i$ satisfying $d'_i + d'_{n-i+1} = h$ are called the **codegrees** of $W$.

The following result can be found in [16, Appendix D.2].

**Proposition 5.2.** The degrees of $G(m,1,n)$ are $m,2m,3m,\ldots, mn$. The codegrees of $G(m,1,n)$ are $0,m,2m,\ldots,(n-1)m$.

The Tutte polynomial evaluation $\chi_A(q) = (-1)^{\ell(A)}T_A(1-q,0)$ is called the characteristic polynomial of the arrangement. The polynomial $(-q)^{\ell(A)}\chi_A(-1/q)$ is Hilbert series of the cohomology ring of the complement of the arrangement $V - A$. When $A$ is a reflection arrangement the relationship to the codegrees of $W$ is this.

**Theorem 5.3** (Orlik–Solomon [20, Theorem 5.5]). Let $A$ be the reflection arrangement of a well-generated complex reflection group $W$ with codegrees $d'_1, \ldots, d'_n$. 

\[ \chi_A(q) = (-1)^{\ell(A)}T_A(1-q,0). \]
Then,
\[ q^{r_k(A)} T_A(1 + 1/q, 0) = \prod_{i=1}^{n} (1 + (1 + d'_i)q) \]
is the Hilbert series of \( H^*(V - A; \mathbb{Z}) \).

**Example 5.4.** In our running example, the Tutte polynomial of \( A \) is \( p^2 + q^2 + 2p + 2q \). We see that
\[ q^{r_k(A)} T_A(1 + 1/q, 0) = q^2((1 + 1/q)^2 + 2(1 + 1/q)) = (1 + q)(1 + 3q). \]
The codegrees of \( G(2, 1, 2) \) are 0 and 2.

6. Gale Duality

**6.1. Definition and properties.** Any arrangement of hyperplanes has a dual arrangement, determined as follows. If \( A \subset V \) is an arrangement then we can construct \( C^A \), a vector space with a basis in bijection with the hyperplanes defining \( A \).

Any choice of linear functionals defining \( A \) determines a linear map \( C^A \to V^* \). The kernel of this map is denoted \( K \), so we have an exact sequence
\[ 0 \to K \to C^A \to V^*. \]

Dualizing gives an exact sequence
\[ 0 \leftarrow K^* \leftarrow (C^A)^* \leftarrow V. \]

Since there is a natural basis of \( (C^A)^* \) indexed by the hyperplanes in \( A \), the map \( (C^A)^* \to K^* \) determines an arrangement in \( K \). This is the Gale dual of \( A \), which we denote by \( A^\perp \). Assuming that the map to \( V^* \) above is surjective, Gale duality is an honest duality in the sense that \( (A^\perp)^\perp = A \).

Gale duality can be interpreted at the level of matroids. Let \( M \) be a matroid whose ground set we denote by \( E \) and whose bases are denoted \( B(M) \). The dual matroid \( M^\perp \) of \( M \) has ground set \( E \) and
\[ B(M^\perp) = \{ E - B : B \in B(M) \}. \]
If \( M \) is the matroid associated to an arrangement \( A \) then \( M^\perp \) is the matroid associated to \( A^\perp \).

**Proposition 6.1 ([5, Proposition 6.2.4]).** Let \( M \) be a matroid and \( M^\perp \) its dual matroid. Then the Tutte polynomials of \( M \) and \( M^\perp \) are related by
\[ T_{M^\perp}(x, y) = T_M(y, x). \]

**Example 6.2.** In our running example, \( A \) is self-dual. In general, to find the coordinates of the Gale dual of an arrangement whose normal vectors are the columns of the block matrix \([I | A] \), where \( I \) is an appropriately sized identity matrix, one computes \([ -A^\dagger | I] \). In the present example we have
\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & -1
\end{bmatrix}
\sim
\begin{bmatrix}
-1 & -1 & 1 & 0 \\
-1 & 1 & 0 & 1
\end{bmatrix},
\]
and after relabeling and scaling the columns, we get the same arrangement. The Tutte polynomial of \( A \) is \( p^2 + q^2 + 2p + 2q \), which is visibly symmetric in \( p \) and \( q \).
We employ the previous result to compute Hilbert series as such.

**Proposition 6.3.** Let $A \subset V$ be the reflection arrangement of well-generated complex reflection group $W$. Say that the codegrees of $W$ are $d'_1, \ldots, d'_n$ Then the Hilbert series of $S_{A^+, -2}$ is

$$\prod_{i=1}^{n} (1 + d'_i q)$$

In particular, the dimension of the degree $n - 1$ piece of $S_{A^+, -2}$ is the product of the non-zero codegrees of $W$.

**Proof.** It follows from Theorem 4.2 that the Hilbert series in question is the Tutte polynomial evaluation $q^n T_A(0, 1/q)$. By Proposition 6.1 this is $q^n T_A(1/q, 0)$ and by the Theorem 5.3 we obtain

$$q^n T_A(1/q, 0) = \prod_{i=1}^{n} (1 + d'_i q).$$

□

**Corollary 6.4.** Let $G$ be the reflection arrangement of the group $G(m, 1, n)$ and let $G^\perp$ be its dual. Then the Hilbert series of $S_{G^+, -2}$ is

$$(1 + mq)(1 + 2mq) \ldots (1 + (n-1)mq).$$

**Proof.** This follows from the previous result since the codegrees of $G(m, 1, n)$ are $0, m, 2m, \ldots , (n-1)m$. □

6.2. **Group actions and duality.** Assume that $V$ is a representation of a group $W$ and that $W$ stabilizes $A$ in the sense that for any $w \in W$, $wA = A$. The arrangement $A^\perp$ can be made to carry an action of $W$ in a natural way, as we now explain. Using the natural action of $W$ on $V^*$, the elements of $W$ act by permuting and rescaling of the linear functionals defining $A$. This affords an action of $W$ on $C^A$, whereby $W$ acts by generalized permutation matrices. This action is cooked up so that the map $C^A \to V^*$ is equivariant.

Since $K \subset C^A$ is the kernel of a map of $W$-modules it is $W$ invariant. Since the map $(C^A)^* \to K^*$ is a map of $W$-modules we see that the arrangement $A^\perp$ is fixed in $K$ by the natural action of $W$.

6.3. **The Gale dual of the reflection arrangement of $G(m, 1, n)$.** Let $G \subset C^n$ denote the reflection arrangement associated to the group $G(m, 1, n)$. We assume that $m > 1$ until Section 8. In this section we give an explicit coordinatization of the Gale dual $G^\perp$ and describe how $W = G(m, 1, n)$ acts on the dual space $K$ in which $G^\perp$ lives.

We begin by labeling the hyperplanes that define $G$. To ease the notation (only slightly) we will write $\omega$ for $e^{2\pi i/m}$. Let $h_i$ denote the hyperplane defined by the vanishing of the $i$-th coordinate function $e_i = 0$ in $C^n$. Let $h_{ij}^k$ denote the hyperplane defined by $e_i - \omega^k e_j = 0$, $i < j$. We will use the convention that the superscript in this notation is read modulo $m$ and that

$$h_{ij}^k = -\omega^k h_{ij}^{-k}.$$

The $h$’s form the basis for the vector space $C^G$ and the linear map $C^G \to C^n$ sends $h_i \mapsto e_i$ and $h_{ij}^k \mapsto e_i - \omega^k e_j$, where we are identifying $C^n$ with its dual space.
The group $W$ is generated by the $n$-by-$n$ permutation matrices, which we identify with the symmetric group $\mathfrak{S}_n$, and the diagonal matrices $g_i$ that have all diagonal entries 1 except the $i$-th which is $\omega$. The action of $W$ on $\mathbb{C}^\mathfrak{S}_n$ is described in the following proposition.

**Proposition 6.5.** Given $\pi \in \mathfrak{S}_n \subset W$,

$$
\pi h_i = h_{\pi(i)}, \quad \pi h_{ij}^k = h_{\pi(i)\pi(j)}^k.
$$

For all $i, j$,

$$
g_i h_i = \omega h_i, \quad g_i h_{ji}^k = \omega h_{ji}^{k-1}, \quad g_j h_{ij}^k = h_{ij}^{k+1}.
$$

If $\ell \notin \{i, j\}$ then $g_\ell h_{ij}^k = h_{ij}^k$.

The kernel $K$ of the map $\mathbb{C}^\mathfrak{S}_n \to \mathbb{C}^n$ is $m\binom{n}{2}$ dimensional and has basis given by

$$
y_{ij}^k = (h_i - \omega^k h_j) - h_{ij}^k, \quad (1 \leq i < j, 0 \leq k < m).
$$

As before, we define $y_{ij}^k = -\omega y_{ji}^{-k}$ and read the superscript $k$ modulo $m$. The arrangement $\mathcal{G}^\perp \subset K$ comes from restricting the linear functionals dual to the $h$’s to $K$. The action of $W$ on $K$ is immediate from the previous proposition.

**Proposition 6.6.** Given $\pi \in \mathfrak{S}_n \subset W$,

$$
\pi y_{ij}^k = y_{\pi(i)\pi(j)}^k.
$$

For all $i, j$,

$$
g_i y_{ij}^k = \omega y_{ij}^{k-1}, \quad g_j y_{ij}^k = y_{ij}^{k+1}.
$$

If $\ell \notin \{i, j\}$ then $g_\ell y_{ij}^k = y_{ij}^k$.

An elementary computation shows the following result.

**Proposition 6.7.** The product $g_1 g_2 \cdots g_n \in W$ is central in $W$. It acts by multiplication by $\omega$ on $K$, and hence multiplication by $\omega^{n-1}$ on $\text{Sym}^{n-1}(K)$ and on $S_{\mathcal{G}^\perp, -2}$.

### 7. The internal zonotopal ideal

In this section we describe in coordinates the defining ideal of $S_{\mathcal{G}^\perp, -2}$. We will use this to prove representation stability in the sense of Church, Ellenberg and Farb and developed for wreath products with symmetric groups by Gan and Li [10] and Sam and Snowden [25].

#### 7.1. Generators.

Recall that $\mathcal{G}^\perp \subset K$, where $K$ has coordinates $y_{ij}^k$ with $1 \leq i < j \leq n$ and $0 \leq k \leq m - 1$.

**Theorem 7.1.** Let $\mathcal{G}$ be the reflection arrangement of $W = G(m, 1, n)$ ($m \geq 1$). Then the ideal defining $S_{\mathcal{G}^\perp, -2}$ in $\text{Sym}(K)$ is the sum of

$$
J_1 = \langle y_{ij}^k y_{ij}^{k'} : 1 \leq i < j \leq n, 0 \leq k, k' < m \rangle
$$

with the smallest $W$-stable ideal $J_2 \subset \text{Sym}(K)$ containing the single element $y_{ij}^0 y_{jk}^0 + y_{ik}^0 y_{kj}^0 + y_{ij}^0 y_{jk}^0$. (Note that $J_1$ is the smallest $W$ stable ideal containing the elements $y_{12}^k y_{12}^{k'}$, $0 \leq k, k' \leq m - 1$.)
Proof. We let \( I \) denote the ideal of \( S_{G''} \). We first show that \( J_1, J_2 \subset I \). Consider the element \( y^k_{ij} = h_i - \omega^k h_j - h^k_{ij} \in K \). The hyperplanes in \( G' \) that do not contain this vector are \( h_i, h_j \) and \( h^k_{ij} \), hence \( (y^k_{ij})^2 \in I \). Similarly, the number of hyperplanes in \( G' \) that do not contain \( y^k_{ij} = (\omega^k - \omega^k)h_j - h^k_{ij} + h^0_{ij} \) is 3, hence \( (y^k_{ij} - y^k_{ij})^2 \in I \). This proves \( J_1 \subset I \). Next we consider \( y^0_{ij} + y^0_{jk} + y^0_{ki} = -(h^0_{ij} + h^0_{jk} + h^0_{ki}) \), which is contained in all but 3 hyperplanes in \( G'' \), so its square is in \( I \). Subtracting off the squared variables from \( (y^0_{ij} + y^0_{jk} + y^0_{ki})^2 \) shows that \( J_2 \subset I \).

We claim that \( \text{Sym}(K)/(J_1 + J_2) \) has the same dimension as \( \text{Sym}(K)/I \) as a complex vector space. The non-zero monomials in \( \text{Sym}(K)/J_1 \) are in obvious bijection with graphs on vertex set \([n]\) whose edges are labeled with the numbers \( 0, 1, \ldots, m - 1 \). We will show that the monomials of graphs with cycles are zero in the quotient.

Suppose that we have a monomial in \( \text{Sym}(K)/(J_1 + J_2) \) whose corresponding graph has a 3-cycle. We can use the \( W \) invariance of this quotient to assume the three cycle has two edges labeled 0. For example, if we had the monomial \( y^1_{12}y^2_{23}y^3_{31} \) then we can apply \( g^2_1g^2_2 \) and work with a multiple of \( y^0_{12}y^0_{23}y^0_{31} \). Rewriting the transformed monomial using the generators of \( J_2 \) given above, we will obtain a sum of two monomials both of which reduce to zero modulo \( J_1 \).

Suppose now that we have a monomial whose corresponding graph has a cycle of length longer than 3. Similar to the argument above we can use the elements in \( J_2 \) to rewrite this monomial as a sum of monomials of graphs with shorter cycles. See Fig. 1. The resulting sum then reduces to zero in the quotient by induction on the length of a cycle. We conclude that \( \text{Sym}(K)/(J_1 + J_2) \) is spanned by monomials in bijection with forests on vertex set \([n]\) whose edges are labeled with the numbers \( 0, 1, \ldots, m - 1 \).

Using the relations in \( J_2 \), we see that out of the forests that span \( \text{Sym}(K)/(J_1 + J_2) \), we can use only those where each connected component of the underlying forest has its highest label appearing as a leaf. This will follow by induction on the degree of the highest labeled vertex in each component. Say that the degree of the vertex with highest label \( v_0 \) in a component is at least two, with neighbors \( v_1 \) and \( v_2 \). Using a relation that swaps out an edge \( v_0v_1 \) or \( v_0v_2 \) with \( v_1v_2 \) makes the degree of \( v_0 \) go down by one. Suppose we have a tree that is not a path, where \( v_0 \) is a leaf. Say that \( v_1 \) is the unique neighbor of \( v_0 \). If \( v_1 \) has degree larger than two then it has neighbors \( v_2 \) and \( v_3 \). The relation that swaps an edge \( v_1v_2 \) or \( v_1v_3 \) with

![Figure 1. Rewriting a monomial corresponding to a graph with a cycle of length larger than 3 in terms of two monomials whose graphs have cycles of smaller length.](image)
v_2v_3 decreases the degree of v_1. By induction, a spanning set of monomials for the internal zonotopal algebra is indexed by forests where each component is a path, and the highest labeled vertex in each path is degree one.

The number such forests with exactly one connected component is at once seen to be m^{n-1}(n-1)!, which is the dimension of the degree n-1 piece of Sym(K)/I = S_{G^+,-2}. A calculation with the exponential formula shows that the number of such forests on n vertices is 1 \cdot (m+1) \cdot (2m+1) \cdots ((n-1)m+1), which is the dimension of S_{G^+,-2}.

We conclude that dim Sym(K)/I = dim Sym(K)/(J_1 + J_2), and since they are both finite dimensional algebras I = J_1 + J_2, which is what we wanted to show. 

**Corollary 7.2.** As a representation of W = G(m, 1, n), S_{G^+,-2}(top) is cyclic with a generator y_{12}^0 y_{23}^0 \cdots y_{(n-1)}^0. As a representation of G(m, 1, n-1), S_{G^+,-2}(top) is isomorphic to the regular representation with generator y_1^0 y_2^0 \cdots y_{n(n-1)}^0.

**Remark 7.3.** There is another way to see that the monomials of graphs containing cycles are zero in the quotient. Indeed, if i_1, i_2, \ldots, i_\ell forms a simple cycle then, by definition, (y_{i_1 i_2}^0 + y_{i_2 i_3}^0 + \cdots + y_{i_\ell i_1}^0)^\ell is zero in the quotient. Expanding, the only square free monomial appearing is y_{i_1 i_2}^0 y_{i_2 i_3}^0 \cdots y_{i_\ell i_1}^0, which proves this monomial is in the internal zonotopal ideal. For \ell \leq n-1 we can act by an appropriate product of the g_i's to get an arbitrary edge labeled cycle.

**Example 7.4.** Consider the case n = 3. The monomials in Sym^2(K) that are non-zero in S_{G^+,-2}(top) are the 2^m+1 monomials of the form

\[ y_{12}^k y_{23}^{k'}, \quad y_{21}^k y_{13}^{k'}, \]

where 0 \leq k, k' < m are arbitrary. These monomials form a basis for S_{G^+,-2}(top).

### 7.2. Representation Stability

In this section we write W_n := G(m, 1, n). We show that for fixed degrees k the sequence of representations \((S_{G_n,-2}^{-1})_{n \geq 3}\) of W_n behave in an organized way explained by the phenomenon of representation stability.

The category \(FI_{\mathbb{Z}/m}\) was introduced by Sam and Snowden in [25]. Its objects are finite sets and a map \(R \to S\) between two sets is a pair \((f, \rho)\) where \(f : R \to S\) is an injection and \(\rho : R \to \mathbb{Z}/m\). The composition of \((f, \rho) : R \to S\) with \((g, \sigma) : S \to T\) is defined by \((g \circ f, \tau) \text{ where } \tau(x) = \sigma(f(x))\rho(x)\).

A (complex) representation of \(FI_{\mathbb{Z}/m}\) is a sequence \(M = (M_n)_{n \geq 0}\) of finite dimensional complex representations \(M_n\) of \(W_n\), together with a sequence of compatible transition maps \(M_n \to M_{n+1}\) that are \(W_n\)-equivariant. Here compatible means that if \(w = w_1 \oplus w_2 \in W_{n+1}\) where \(w_1 \in W_n\), then the action of \(w_1\) on \(M_n\) agrees with the action of \(w\) on the image of \(M_n\) in \(M_{n+1}\). A simple example to keep in mind here is the sequence where \(M_n\) is the sign representation of \(S_n\) for all \(n\). The identity maps \(M_n \to M_{n+1}\) do not form a compatible sequence (indeed the sign of \((12)(34)\) is not the same as the sign of \((12)\)). On the other hand, the trivial representation of \(S_n\) along with the identity maps do form a compatible sequence.

Denote the reflection arrangement associated to the group \(W_n\) by \(G_n\), instead of the usual \(G\).
Proposition 7.5. For each degree $k \geq 0$, the sequences of representations $((S_{\mathcal{G}^+_{n,-2}})_n)_{n \geq 0}$ and $((I_{\mathcal{G}^+_{n,-2}})_n)_{n \geq 0}$ are representations of the category $\text{FI}_{\mathbb{Z}/m}$.

Proof. Let $K_n$ denote the ambient space of the arrangement $\mathcal{G}^+_n$. Then $(K_n)_n$ is a $\text{FI}_{\mathbb{Z}/m}$ module since given $w \in G(m,1,n+\ell)$ which is an extension of an element $w_1 \in G(m,1,n)$, we immediately see that the action of $w_1$ on $K_n$ agrees with the action of $w$ on the image of $K_n$ in $K_{n+\ell}$.

It follows that the graded pieces of $\text{Sym}(K_n)$ give $\text{FI}_{\mathbb{Z}/m}$ modules. Let $I_n \subset \text{Sym}(K_n)$ denote the internal zonotopal ideal of $\mathcal{G}^+_n$. Restricting the inclusions $\text{Sym}(K_n) \to \text{Sym}(K_{n+1})$ to $I_n$ and apply Theorem 7.1 to see that the graded pieces of $I_n$ form $\text{FI}_{\mathbb{Z}/m}$ modules. Finally, taking quotients shows that the graded pieces of $S_{\mathcal{G}^+_{n,-2}}$ form $\text{FI}_{\mathbb{Z}/m}$-modules. $\square$

A representation $M$ of $\text{FI}_{\mathbb{Z}/m}$ is said to be finitely generated if there are finitely many elements in the spaces $M_n$ such that the smallest $\text{FI}_{\mathbb{Z}/m}$ submodule of $M$ containing these elements is exactly $M$ itself.

Proposition 7.6. For each degree $k \geq 0$, the sequences of representations $((S_{\mathcal{G}^+_{n,-2}})_n)_{n \geq 0}$ and $((I_{\mathcal{G}^+_{n,-2}})_n)_{n \geq 0}$ are finitely generated.

Proof. Theorem 7.1 shows finite generation of the graded pieces of the sequence of ideals directly. This also follows since the sequence $(\text{Sym}(K_n)_n)$ is finitely generated and the main theorem of [25] proves that subrepresentations of finitely generated objects are again finitely generated. The graded pieces of the sequence $(S_{\mathcal{G}^+_{n,-2}})_n$ are finitely generated since they are quotients of a finitely generated object. $\square$

The importance of being finitely generated here is that it implies representation stability, as described in [10, Definition 1.10]. Recall that the irreducible (complex) representations of the wreath product $W_n \approx \mathbb{Z}/m \wr \mathfrak{S}_n$ are indexed by $m$-tuples of partitions $\Lambda = (\lambda^i)$ satisfying $|\Lambda| = \sum_{i=1}^m |\lambda^i| = n$. We denote this irreducible by $L(\Lambda)$; we will not need the precise construction here. The entries of this $m$-tuple $\Lambda$ are in bijection with the irreducible characters of $\mathbb{Z}/m$ and we assume that $\lambda^1$ corresponds to the trivial character. For $n$ larger than $|\Lambda| + \lambda^1$ we let $\Delta[n]$ denote the partition obtained by adding a single part of length $n - |\Lambda|$ to $\lambda^1$. We say that a sequence $(M_n)_{n \geq 0}$ of $W_n$ representations is representation stable if for all $n \gg 0$ the following conditions hold:

1. The map $M_n \to M_{n+1}$ is injective.
2. The image of $M_n$ in $M_{n+1}$ generates $M_{n+1}$ as a $W_{n+1}$-module.
3. The irreducible decomposition is given by
   \[
   M_n \approx \bigoplus_{\Lambda} L(\Delta[n])^\oplus m_{\Lambda}
   \]
   for some integers $m_{\Lambda} \geq 0$ that do not depend on $n$.

Gan and Li prove [10, Theorem 1.12] that $(M_n)_{n \geq 0}$ is finitely generated if and only if it is representation stable. Applying their result we have the following.

Theorem 7.7. For each degree $k \geq 0$, the sequences of representations $((S_{\mathcal{G}^+_{n,-2}})_n)_{n \geq 0}$ and $((I_{\mathcal{G}^+_{n,-2}})_n)_{n \geq 0}$ are representation stable.
Example 7.8. One consequence of representation stability of a sequence \((M_n)_{n \geq 0}\) is that \(\dim(M_n)\) is eventually polynomial in \(n\). In our case the dimension \(d_{n,k}\) of \((S_{G_n}^{\top}, -2)\) satisfies the recurrence
\[
d_{n,k} = d_{n-1,k} + (n-1)md_{n-1,k-1}.
\]
This follows by computing the coefficient of \(q^k\) in the Hilbert series,
\[
(1 + mq)(1 + 2mq) \cdots (1 + (n-1)mq),
\]
in two ways. For fixed \(k\), \(d_{n,k}\) is a polynomial in \(n\) for every \(n \geq 0\).

8. Type A

In this section we investigate the case when \(m = 1\) in \(G(m, 1, n)\), so that our reflection group is the symmetric group of \(n\)-by-\(n\) permutation matrices. To emphasize the dependence on \(n\) here we will write \(S_n\) instead of \(W\). The main theorem in this section, Theorem 8.1, will be crucial in our proof of the main theorem of this paper, Theorem 9.1.

8.1. Statement of the type A result. The reflection arrangement of \(S_n\) is the well-studied braid arrangement \(\mathcal{A} \subset \mathbb{C}^n\) whose defining hyperplanes are
\[
x_j - x_i, \quad (1 \leq i < j \leq n).
\]
The Macaulay inverse system of the central zonotopal ideal \(I_{\mathcal{A}, -1}\) has dimension \(n^{n-2}\), and was studied by the author and Rhoades in [4]. There it was shown to be a representation of the symmetric group \(S_n\) that restricted to the well-studied parking representation of \(S_{n-1}\). This is the representation with basis given by sequences \(p = (p_1, \ldots, p_{n-1})\) whose non-decreasing rearrangement \(q\) satisfies \(q_j \leq j\) for all \(1 \leq j \leq n-1\).

To identify the Gale dual of \(\mathcal{A}\) we label the hyperplane \(x_i - x_j = 0\) by \(h_{ij}\). Those \(h_{ij}\) with \(1 \leq i < j \leq n\) form a basis for \(\mathbb{C}^\mathcal{A}\), and we stipulate that \(h_{ij} = -h_{ji}\). The kernel \(K\) of the natural map \(\mathbb{C}^\mathcal{A} \rightarrow \mathbb{C}^n\) is \(\binom{n}{2} - n = \binom{n-1}{2}\) dimensional, and a basis is given by
\[
y_{ij} = h_{ij} - h_{i\in} - h_{j\in} \quad (1 \leq i < j \leq n-1).
\]
One can check that \(y_{ji} = -y_{ij}\). This kernel can be naturally identified with the cycle space of the complete graph on \(n\) vertices, which is spanned by the characteristic vectors of length 3 oriented cycles. A basis is then given by those 3 cycles that visit the vertex labeled \(n\).

The transposition \((kn) \in S_n\) acts on the variables \(y_{ij}\) by the rule,
\[
(kn)_{ij} = \begin{cases} 
y_{ij} + y_{jk} + y_{ki}, & \text{if } k \notin \{i, j\}, \\
-y_{ij} & \text{if } k \in \{i, j\}. 
\end{cases}
\]
Notice that the action of \(S_{n-1} \subset S_n\) on \(K\) agrees with action set forth in Section 6.3 with \(m = 1\) and \(n\) replaced by \(n-1\). We have the following analog of Theorem 9.1.

Theorem 8.1. Let \(\mathcal{A}\) denote the reflection arrangement of \(S_n\). Let \(c \in S_{n-1} \subset S_n\) denote a any \((n-1)\)-cycle. As a representation of \(S_{n-1}\), \(S_{A^{\top}, -2}(\text{top})\) is isomorphic to \(\text{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n-1}}(e^{2\pi ij/(n-1)})\). As a representation of \(S_{n-2} \subset S_{n-1}\), \(S_{A^{\top}, -2}(\text{top})\) is isomorphic to the regular representation.
We will prove this result in the next subsection.

The induced representation here \( \text{Ind}_{S_{n-1}}^{\mathfrak{S}_n}(e^{2\pi i/(n-1)}) \) is called the Lie representation, which we write as \( \text{Lie}_{n-1} \). It is the representation of \( \mathfrak{S}_{n-1} \) afforded by the multilinear component of the free Lie algebra on \( n-1 \) letters, as is shown by the computation of the latter character by Klyachko [13, Corollary 1]. It was shown by Stanley [23, Theorem 7.3] that the tensor product of the Lie representation with the sign character is isomorphic to the top cohomology of the partition lattice, and this is isomorphic to the top cohomology of the complement of the braid arrangement \( C^n - A \).

8.2. The ideal of \( S_{A^+, -2} \) and the Whitehouse representation. We now compute a generating set for the defining ideal of \( S_{A^+, -2} \). It is possible to do this in the same manner as \( G(m, 1, n), m > 1 \). However, some novel features appear in the approach we take in this section.

This ideal of \( S_{A^+, -2} \) is an ideal in the polynomial ring \( \text{Sym}(K) = \mathbb{C}[y_{ij} : 1 \leq i < j \leq n-1] \).

**Lemma 8.2.** The ideal \( I_{A^+, -2} \) is equal to

\[
\langle y_{ij}^2 : 1 \leq i < j \leq n \rangle + \langle y_{ij}y_{ik} + y_{jk}y_{kj} + y_{ki}y_{jk} : 1 \leq i < j < k \leq n \rangle
\]

We start by proving one containment.

**Proposition 8.3.** The ideal \( I = I_{A^+, -2} \) contains \( J \).

**Proof.** Consider the linear functional \( h^*_{k\ell} \) dual to \( h_{k\ell} \), restricted to \( K \). To show that \( y_{ij}^2 \in I \) we must compute the number of \( h^*_{k\ell} \), \( k < \ell \), that do not vanish on \( y_{ij} \). This number is 3 and hence \( y_{ij}^{3-2+1} = y_{ij}^2 \in I \).

The ideal \( I \) is stable under the action of \( \mathfrak{S}_n \), and hence \( (kn)y_{ij}^2 = (y_{ij} + y_{jk} + y_{kn})^2 \in I \). Subtracting off the quadratic terms proves that \( I \) contains \( y_{ij}y_{ki} + y_{ji}y_{kj} + y_{ki}y_{jk} \). \( \Box \)

**Proposition 8.4.** The set of polynomials,

\[
y_{ij}^2, \quad (1 \leq i < j \leq n-1), \quad -y_{ij}y_{ik} + y_{ij}y_{jk} - y_{ik}y_{jk}, \quad (1 \leq i < j < k \leq n-1).
\]

forms a Gröbner basis under any term order where the leading terms are underlined above.

**Proof.** We use the fact that syzygies of polynomials with relatively prime leading terms need not be computed [7, Exercise 15.20]. By symmetry, the computation reduces to the case when \( n = 4 \). This case is easily checked with a computer (e.g., using [17]). \( \Box \)

**Proof of Lemma 8.2.** The ideal of leading terms of \( J \) is \( \langle y_{ij}^2 : 1 \leq i < j \leq n-1 \rangle + \langle y_{ij}y_{jk} : 1 \leq i < j < k \leq n-1 \rangle \). It follows that a basis for \( \text{Sym}(K)/J \) consists of square-free monomials in the \( y_{ij} \) that avoid \( y_{ij}y_{ik} \), for \( 1 \leq i < j < k \leq n \). These monomials are in obvious bijection with the set of decreasing forests on \( n-1 \) vertices. These are forests with vertex set \( [n-1] \) where, in every component, each path directed away from the largest vertex in that component decreases. Adding a vertex labeled \( n \) and connecting the largest vertex in each component to \( n \), we
obtain a bijection between decreasing forests on \( n - 1 \) vertices, and decreasing trees on \( n \) vertices. By \([24, \text{Proposition 1.5.5}]\), there are exactly \((n - 1)!\) such trees.

It follows that the dimension of \( \text{Sym}(K)/J \) as a vector space is \((n - 1)!\). Since this is the dimension of \( S_{A^+_{n-2}} \) (recall, its Hilbert series is \((1 + q)(1 + 2q) \cdots (1 + (n - 2)q)\) and there is a containment between \( J \) and \( I_{A^+_{n-2}} \), the two ideals are equal. 

\[\square\]

**Proof of Theorem 8.1.** By Lemma 8.2, we have identified \( S_{A^+_{n-2}} \) with the graded \( \mathfrak{S}_{n-1} \) module Mathieu studies in \([18, \text{Theorem 3.1}]\). The top degree component of this module is, by his definition, the \( \mathfrak{S}_{n-1} \) module \( \text{Lie}_{n-1} \). By the result of Klyachko \([13, \text{Corollary 1}]\) this is the stated induced character. Since the induced character is zero at every element not conjugate to a power of an \( n \) cycle, it is zero at every element of \( \mathfrak{S}_{n-2} \) except the identity. It follows that the restriction of \( S_{A^+_{n-2}} \) (top) to \( \mathfrak{S}_{n-2} \) is isomorphic to the regular representation. 

\[\square\]

The structure of \( S_{A^+_{n-2}} \) (top) as a representation of \( \mathfrak{S}_n \) is more subtle, but also turns out to have been studied in work of Mathieu \([18]\) and Gaiffi \([9]\). To state this result we need to define the Whitehouse representation of the symmetric group \( \mathfrak{S}_n \). This is the (\textit{a priori} virtual) representation,

\[\text{Wh}_n := \text{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} (\text{Lie}_{n-1}) - \text{Lie}_n,\]

which was first studied by Kontsevich in the context of free Lie algebras and later by Robinson and Whitehouse \([22]\).

**Theorem 8.5.** There is an isomorphism of representations of \( \mathfrak{S}_n \), \( S_{A^+_{n-2}} \) (top) \( \approx \) \( \text{Wh}_n \).

To prove Theorem 8.5 we will need a result of Sundaram.

**Proposition 8.6** (Sundaram \([26, \text{Lemma 3.1}]\)). Let \( W_n \) and \( V_n \) be (possibly virtual) representations of \( \mathfrak{S}_n \), and let \( 1^\perp \) denote the orthogonal complement of the trivial representation in \( \mathbb{C}^n \). Then the following are equivalent:

1. \( W_n \otimes 1^\perp \approx V_n \).
2. \( W_n \oplus V_n \approx \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} W_n \).

**Proof of Theorem 8.5.** We have identified \( S_{A^+_{n-2}} \) with the graded \( \mathfrak{S}_n \) module Mathieu studies in \([18, \text{Theorem 3.3}]\). We denote the top degree part of this \( \mathfrak{S}_n \) module by \( Q_n \). In \([18, \text{Theorem 4.4}]\) it is shown that there is an isomorphism of \( \mathfrak{S}_n \)-modules,

\[Q_n \otimes 1^\perp \approx \left( \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} (Q_{n+1}) \right).\]

Using Theorem 8.1 and Proposition 8.6,

\[Q_n \oplus \left( \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} (Q_{n+1}) \right) \approx Q_n \oplus \text{Lie}_n \]

\[\approx \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \left( \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} Q_n \right) \]

\[\approx \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \left( \text{Lie}_{n-1} \right).\]

This proves that \( Q_n \approx \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} (\text{Lie}_{n-1}) - \text{Lie}_n \). 

\[\square\]
Our goal in this section is to prove the main result of the paper.

**Theorem 9.1.** Let \( G \) be the reflection arrangement associated to the group \( W = G(m, 1, n) \). Let \( C \subset W \) be generated by an \( n \) cycle \( c \in S_n \subset W \) and \( g_1 g_2 \ldots g_n \in W \). There is an isomorphism of representations,

\[
S_{G^+,-2}(\text{top}) \cong \text{Ind}^W_C(\chi),
\]

where \( \chi(c) = e^{2\pi i/n} \) and \( \chi(g_1 g_2 \ldots g_n) = \omega^{n-1} \).

Recall that \( \omega \) is a primitive \( n \)th root of unity.

**Proof.** Consider the \( S_n \) submodule \( V_n \) generated by \( y_1^0 y_2^0 \ldots y_{(n-1)}^0 \) in \( S_{G^+,-2}(\text{top}) \). This module is dimension \( (n-1)! \) by Corollary 7.2 since the monomials of paths where \( n \) is an endpoint form a basis. The quotient,

\[
S_{G^+,-2}/\langle y_{ij}^k : 1 \leq i \neq j \leq n, k \neq 0 \mod m \rangle
\]

is isomorphic as an \( S_n \) module to \( S_{A^+,-2} \), where \( A \) is the reflection arrangement of \( G(1, 1, n+1) \) (i.e., the braid arrangement). This follows from the description of the ideals in Theorem 7.1 and Lemma 8.2. The resulting composite

\[
V_n \to S_{A^+,-2}(\text{top})
\]

is an isomorphism of \( S_n \) modules and hence \( V_n \approx \text{Lie}_n \).

By Proposition 6.7 and Theorem 8.1, it follows that as a representation of \( \langle S_n, g_1 \cdot \cdot \cdot g_n \rangle \),

\[
V_n \cong \text{Ind}_{\langle S_n, g_1 \cdot \cdot \cdot g_n \rangle}^{\langle S_n, g_1 \cdot \cdot \cdot g_n \rangle}(\chi),
\]

where \( c \) is any \( n \)-cycle and \( \chi \) is the character of defined by \( \chi(c) = e^{2\pi i/n} \) and \( \chi(g_1 \cdot \cdot \cdot g_n) = \omega^{n-1} \).

By Corollary 7.2 the \( W_n \) orbit of \( V_n \) is all of \( S_{G^+,-2}(\text{top}) \) and so there is a surjective map of \( W_n \) modules

\[
\text{Ind}^W_{\langle S_n, g_1 \cdot \cdot \cdot g_n \rangle}(V_n) \to S_{G^+,-2}(\text{top}).
\]

The dimension of the module on the left is \( n!m^n/(n!m) \cdot \dim V_n = (n-1)!m^{n-1} \).

Since this is the dimension of the module on the right, the map above is an isomorphism. By the transitivity of induction Eq. (1), we have

\[
\text{Ind}^W_{\langle S_n, g_1 \cdot \cdot \cdot g_n \rangle}(\chi) \approx \text{Ind}^W_{\langle S_n, g_1 \cdot \cdot \cdot g_n \rangle} \left( \text{Ind}_{\langle S_n, g_1 \cdot \cdot \cdot g_n \rangle}^{\langle S_n, g_1 \cdot \cdot \cdot g_n \rangle}(\chi) \right)
\]

\[
\approx \text{Ind}^W_{\langle S_n, g_1 \cdot \cdot \cdot g_n \rangle}(V_n)
\]

\[
\approx S_{G^+,-2}(\text{top}). \quad \square
\]

**Corollary 9.2.** If \( n \) and \( m \) are coprime then

\[
S_{G^+,-2}(\text{top}) \approx \text{Ind}_C^W(e^{2\pi i/n} \cdot \omega^{n-1}),
\]

where \( c \in W \) is a Coxeter element.

A Coxeter element in a well generated reflection group is a product, in any order, of the generators of the group. Equivalently [21], a Coxeter element is an element of \( W = G(m, 1, n) \) that has an eigenvalue of multiplicative order \( mn \). The multiplicative order in \( W \) of any such element is \( mn \).
Proof. In the defining representation of \( W \), an \( n \)-cycle \( c \) will have \( e^{2\pi i/n} \) as an eigenvalue and \( g_1 \cdots g_n \) has \( \omega = e^{2\pi i/m} \) as its eigenvalues. Because \( (g_1 \cdots g_n) \) is a scalar matrix, it follows that \( c \cdot (g_1 \cdots g_n) \) has \( e^{2\pi i(m+n)/mn} \) as an eigenvalue. Since this is a primitive \( mn \)th root of unity, this element is a Coxeter element. Since Coxeter elements in \( W \) have order \( mn \), it follows that the cyclic group appearing in Theorem 9.1 is generated by \( c \cdot (g_1 \cdots g_n) \) and the character \( \chi \) evaluated at \( c \cdot (g_1 \cdots g_n) \) is \( e^{2\pi i/n} \cdot \omega^{n-1} \). \( \square \)

9.1. Factorization of the regular representation. One might hope, based on what happens in type A, that \( S_{G_{n+1}}^{+} \) could be made to carry an action of \( G(m, 1, n+1) \). However, there does not appear to be an action generalizing Eq. (2). The goal of this section is to prove an analog of the result of Mathieu [18, Theorem 4.4] occurring in the proof of Theorem 8.5, namely,

\[
\text{Res}^{S_{n+1}}(Q_{n+1}) \approx Q_n \otimes 1^\perp.
\]

**Proposition 9.3.** Let \( E = E_0 \oplus E_1 \) be the graded representation of \( G(m, 1, n) \) that is trivial in degree 0 and equal to \( \text{Ind}^{G(m,1,n)}_{G(m,1,n-1)} 1 \) in degree 1. Let \( G_n \) be the reflection arrangement of \( G(m, 1, n) \). Then there is an isomorphism of graded \( G(m, 1, n) \)-modules,

\[
\text{Res}^{G(m,1,n+1)}_{G(m,1,n)}(S_{G_{n+1}}^+) \approx S_{G_{n+1}}^- \otimes E.
\]

Taking the top degree piece of both sides and applying Corollary 7.2 we obtain the following result.

**Corollary 9.4.** Maintaining the notation of Proposition 9.3, there is an isomorphism of \( G(m, 1, n) \)-modules,

\[
\mathbb{C}[G(m, 1, n)] \approx S_{G_{n+1}}^+ \otimes E_1.
\]

We can unravel this using the proof of Sundaram’s Proposition 8.6, which brought us to the Whitehouse representation in Theorem 8.5. We obtain a much more underwhelming result, since we arrive at the simple statement

\[
\text{Ind}^{G(m,1,n+1)}_{G(m,1,n)} \mathbb{C}[G(m, 1, n)] \approx \mathbb{C}[G(m, 1, n+1)].
\]

**Proof of Proposition 9.3.** From the proof of Theorem 7.1, a linear basis for \( S_{G_{n+1}}^+ \) consists of monomials indexed by certain forests on vertex set \([n+1]\). Specifically, the edges are labeled with \( 0, 1, \ldots, m-1 \), each components is a path, and the largest labeled vertex in each path has degree one. This means that there is a basis indexed by monomials that are either (a) not divisible by any variable of the form \( y_{(n+1)i}^k \) or (b) divisible by exactly one variable of the form \( y_{(n+1)i}^k \).

Let \( F = F_0 \oplus F_1 \) be trivial in degree 0 and in degree 1 have basis \( y_{(n+1)i}^k \), \( 1 \leq i \leq n, 0 \leq k \leq m-1 \). The multiplication map \( S_{G_{n+1}}^- \otimes F \to S_{G_{n+1}}^+ \) is surjective and is \( G(m, 1, n) \)-equivariant. Since the Hilbert series of both sides match, we get an isomorphism.

The group \( G(m, 1, n) \) acts by permuting the variables \( y_{(n+1)i}^k \). The action is transitive on the set of variables. The stabilizer of \( y_{(n+1)i}^0 \) is precisely \( G(m, 1, n-1) \) and this proves that \( E_1 \approx F_1 \) as representations of \( G(m, 1, n) \). \( \square \)
10. **Type B decreasing trees**

In this section we compute a Gröbner basis of the ideal \( I_{G_{-2}} \) when \( m = 2 \), where \( W \) is the hyperoctohedral group. We use this to give a non-trivial generalization of the notion of decreasing trees, as discussed in Section 8.

We start with a definition. A \( \pm \)tree on \( n \) vertices is a rooted tree with vertex set \([n]\) and root vertex \( n \), together with a \( \{+,−\}\)-coloring of its edges. A decreasing \( \pm \)tree on \( n \) vertices is a \( \pm \)tree on \( n \) vertices such that on any path directed away from the root,

1. along every edge labeled \(-\) the vertex labels decrease,
2. for \( i < j \) and arbitrary \( k \), there is no path of the form \( j \rightarrow i \rightarrow k \),
3. for all \( i_1 < i_2 < i_3 < i_4 \), there is no subpath of any of the three forms, or their reverse, \( i_4 \rightarrow i_1 \rightarrow i_2 \rightarrow i_3 \), \( i_4 \rightarrow i_1 \rightarrow i_3 \rightarrow i_2 \), \( i_4 \rightarrow i_2 \rightarrow i_1 \rightarrow i_3 \).

If we ignore the possibility of edges labeled \( + \) this definition reduces to the usual definition of decreasing trees.

**Example 10.1.** Here are the 8 decreasing \( \pm \)trees on 3 vertices.

\[
\begin{array}{cccc}
3 & 3 & 3 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 2 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 2 \\
\end{array}
\]

**Example 10.2.** There are 48 decreasing \( \pm \)trees on 4 vertices. Exactly 8 such trees are isomorphic to a path rooted at an endpoint, and these are displayed below.

\[
\begin{array}{cccccccc}
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
3 & 3 & 3 & 3 & 3 & 2 & 2 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
2 & 2 & 2 & 2 & 1 & 3 & 3 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 \\
\end{array}
\]

**Theorem 10.3.** There are \( 2^{n-1}(n-1)! \) decreasing \( \pm \)trees on \( n \) vertices.
Proof. We know from the proof of Theorem 7.1 that the graphs corresponding to monomials not in $I_{G+,-2}$ are forests on $[n]$ whose edges are labeled 0 and 1. We change every edge labeled 0 to an edge labeled $-$ likewise with 1 and $+$. We now show that the monomials of decreasing $\pm$-trees form a basis for the quotient $S_{G+,-2}(\top)$.

For this, we claim that a Gröbner basis of $I_{G+,-2}$ is furnished by the following monomials:

$$(y_{ij}^0)^2, \quad (y_{ij}^0)^2, \quad y_{ij}^0 y_{ij}^1, \quad y_{ij}^0 y_{jk}^1 y_{ki}^1.$$  

where $i, j$ and $k$ range of distinct triples of integers, as well as the polynomials

$$\begin{align*}
y_{ij}^0 y_{ik}^0 + y_{ij}^0 y_{jk}^0 + y_{ik}^0 y_{kj}^0, & \quad y_{ji}^0 y_{ik}^0 + y_{ij}^1 y_{ik}^1 - y_{ik}^0 y_{kj}^1, \\
y_{ij}^1 y_{ik}^1 y_{ij}^1 y_{ik}^1, & \quad y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1 - y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1 + y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1, \\
y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1, & \quad y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1 - y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1 + y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1, \\
y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1, & \quad y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1 - y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1 + y_{ij}^1 y_{ij}^1 y_{ij}^1 y_{ij}^1.
\end{align*}$$  

where the indices range over distinct tuples of integers, and finally given $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ the polynomials,

$$\begin{align*}
y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 - y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 - y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 + y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1, \\
y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 - y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 - y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 + y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1, \\
y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 - y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 - y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1 + y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_1 i_4}^1.
\end{align*}$$  

We use any term order where the underlined monomials are leading terms (such as graded-reverse lexicographic order with the $y^0$’s coming before the $y^1$’s).

To see this one applies the Buchberger algorithm to the generators of $I_{G+,-2}$ given in Theorem 7.1. By [7, Exercise 15.20] we can reduce the computation to the case $n = 4$, since we need not resolve the syzygies of pair of polynomials with relatively prime leading terms. This case is, once again, handled at once by a computer to produce the described Gröbner basis.

A bijective proof of this result generalizing the one for ordinary decreasing trees is not known.

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