Exit time of turbulent signals: a way to detect the intermediate dissipative range.

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The exit time statistics of experimental turbulent data is analyzed. By looking at the exit-time moments (Inverse Structure Functions) it is possible to have a direct measurement of scaling properties of the laminar statistics. It turns out that the Inverse Structure Functions show a much more extended Intermediate Dissipative Range than the Structure Functions, leading to the first clear evidence of the universal properties of such a range of scales.

In stationary isotropic turbulent flows, a net flux of energy establishes in the inertial range, i.e. from forced scales, $L_0$, down to the dissipative scale, $\tau_d$. Energy is transferred through a statistically scaling-invariant process, characterized by a strongly non-gaussian (intermittent) activity. Understanding the statistical properties of intermittency is one of the most challenging open problems in three dimensional fully developed turbulence. In isotropic turbulence, the most studied statistical indicators are the longitudinal structure functions, i.e. moments of the velocity increments at distance $R$ in the direction of $\hat{R}$:

$$S_p(R) = \langle (v(x + R) - v(x)) \cdot \hat{R} \rangle^p.$$  \hspace{1cm} (1)

Typically, one is forced to analyze one-dimensional string of data: the output of hot-wire anemometer. In these cases Taylor Frozen-Turbulence Hypothesis is used in order to bridge measurements in space with measurements in time. Within the Taylor Hypothesis, one has the large-scale typical time, $T_0 = L_0/U_0$, and the dissipative time, $t_d = r_d/U_0$, where $U_0$ is the large scale velocity field, $L_0$ is the scale of the energy injection and $r_d$ is the Kolmogorov dissipative scale. As a function of time increment, $\tau$, structure functions assume the form: $S_p(\tau) = \langle [v(t + \tau) - v(t)]^p \rangle$. It is well known that for time increment corresponding to the inertial range, $\tau_d \ll \tau \ll T_0$, structure functions develop an anomalous scaling behavior: $S_p(\tau) \sim \tau^{\zeta_p}$, where $\zeta_p$ is a non linear function, while far inside the dissipative range, $\tau \ll \tau_d$, they must show the laminar scaling: $S_p(\tau) \sim \tau^p$.

Beside the huge amount of theoretical, experimental and numerical studies devoted to the understanding of velocity fluctuations in the inertial range (see \cite{cencini2017} for a recent overview), only few -mainly theoretical- attempts have focused on the Intermediate Dissipation Range (IDR) \cite{vulpiani2019, cencini2019}. By IDR we mean the range of scales, $\tau \sim \tau_d$, in between the two power law ranges: the inertial and the dissipative range.

The very existence of the IDR is relevant for the understanding of many theoretical and practical issues. Among them we cite: the modelizations of small scales for optimizing Large Eddy Simulations; the possible influence of small scales statistics on macroscopic global quantities, e.g. drag-reduction due to the presence of very diluted polymers in the fluid \cite{biferale2019}; the validity of the Refined Kolmogorov Hypothesis, i.e. the bridge between inertial and dissipative statistics.

A non-trivial IDR is connected to the presence of intermittent fluctuations in the inertial range. Namely, anomalous scaling law characterized by the exponents $\zeta(p)\,\text{,}$ can be explained by assuming that velocity fluctuations in the inertial range are characterized by a spectrum of different local scaling exponents: $\delta_v = v(t + \tau) - v(t) \sim \tau^h$ with the probability to observe at scale $\tau$ a value $h$ given by $P_r(h) \sim \tau^{3-D(h)}$. This is the celebrated multifractal picture of the energy cascade which has been confirmed by many independent experiments \cite{cencini2019}. The non trivial dissipative statistics can be explained by defining the dissipative cut-off as the scale where the local Reynolds number is order of unity:

$$Re(\tau_d) = \frac{\nu r_d}{U_0} \sim O(1).$$  \hspace{1cm} (2)

By inverting (2) we immediately obtain a prediction of a fluctuating $\tau_d$:

$$\tau_d(h) \sim \nu^{1/(1+h)},$$

where for sake of simplicity we have assumed the large scale velocity, $U_0$, and the outer scale, $L_0$, both fixed to one.

In this letter we propose, and measure in experimental and synthetic data, a set of new observable which are able to highlight the IDR properties. The main idea is to take a one-dimensional string of turbulent data, $v(t)$, and to analyze the statistical properties of the exit times from a set of defined velocity-thresholds. Roughly speaking a kind of Inverse Structure Functions.

This analysis allow us to give the first clear evidence of non-trivial intermittent fluctuations of the dissipative cut-off in turbulent signals. A similar approach has already been exploited for studying the particle separation statistics \cite{cencini2019}. Recently, exit-time moments have also been studied in the time evolution of a shell model \cite{vulpiani2019}. The letter is organized as follows. First we define the exit-time probability density function and we motivate why this PDF is dominated by the IDR. Then, we present the data analysis performed in high-Reynolds number turbulent flows and in synthetic multi-affine signals \cite{vulpiani2019}. Finally, we summarize the evidences supporting a non trivial IDR and discuss possible further investigations. Fluctuations of viscous cut-off are particularly important for all those regions in the fluid where the velocity field is

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locally smooth, i.e. the local fluctuating Reynolds number is small. In this case, the matching between non-linear and viscous terms happens at scales much larger than the Kolmogorov scale, $\tau_d \sim \nu^{-3/4}$. It is natural, therefore, to look for observable which feel mainly laminar events. A possible choice is to measure the exit-time moments through a set of velocity thresholds. More precisely, given a reference initial time $t_0$ with velocity $v(t_0)$, we define $\tau(\delta v)$ as the first time necessary to have an absolute variation equal to $\delta v$ in the velocity data, i.e. $|v(t_0) - v(t_0 + \tau(\delta v))| = \delta v$. By scanning the whole time series we recover the probability density functions of $\tau(\delta v)$ at varying $\delta v$ from the typical large scale values down to the smallest dissipative values. Positive moments of $\tau(\delta v)$ are dominated by events with a smooth velocity field, i.e. laminar bursts in the turbulent cascade. Let us define the Inverse Structure Functions (Inverse-SF) as:

$$\Sigma_p(\delta v) \equiv \langle \tau^p(\delta v) \rangle \ .$$

(3)

According to the multifractal description we suppose that, for velocity thresholds corresponding to inertial range values of the velocity differences, $\delta v \ll v_M \equiv \delta T_0 v$, the following dimensional relation is valid:

$$\delta v \sim \tau^h \rightarrow \tau(\delta v) \sim \delta v^{1/h} ,$$

(4)

with a probability to observe a value $\tau$ for the exit time given by inverting the multifractal probability, i.e.

$$P(\tau \sim \delta v^{1/h}) \sim \delta v^{[3-D(h)]/h}$$

(5)

Made this ansatz, one can write down a prediction for the Inverse-SF, $\Sigma_p(\delta v)$ evaluated for velocity thresholds within the inertial range:

$$\Sigma_p(\delta v) \sim \int_{h_{\min}}^{h_{\max}} dh \, \delta v^{[p+3-D(h)]/h} \sim \delta v^{\chi(p)}$$

(6)

where the RHS has been obtained by a saddle point estimate:

$$\chi(p) = \min_h \{ [p+3-D(h)]/h \} .$$

(7)

Let us now consider the IDR properties. For each $p$ the saddle point evaluation selects a particular $h = h_s(p)$ where the minimum is reached. Let us also remark that from (4) we have an estimate for the minimum value assumed by the velocity in the inertial range given a certain singularity $h$: $v_m(h) = \delta v / (1+h)$. Therefore, the smallest value at which the scaling $\Sigma_p(\delta v) \sim \delta v^{\chi(p)}$ still holds depends on both $\nu$ and $h$. Namely, $\delta v_m(p) \sim \nu^{h_s(p)/(1+h_s(p))}$. The most important consequence is that for $\delta v \ll \delta v_m(p)$ the integral is not any more dominated by the saddle point value but by the maximum value still dynamically alive at that velocity difference, $1/h(\delta v) = -1 - \log(\nu)/\log(\delta v)$. This leads for $\delta v \ll \delta v_m(p)$ to a pseudo-algebraic law:

$$\Sigma_p(\delta v) \sim \delta v^{[p+3-D(h(\delta v))] / h(\delta v)} .$$

(8)

The presence of this $p$-dependent velocity range, intermediate between the inertial range, $\Sigma_p(\delta v) \sim \delta v^{\chi(p)}$, and the far dissipative scaling, $\Sigma_p(\delta v) \sim \delta v^p$, is the IDR signature. Then, it is easy to show that Inverse-SF should display an enlarged IDR. Indeed, for the usual direct structure functions the saddle point $h_s(p)$ value is reached for $h < 1/3$. This pushes the IDR to a range of scales very difficult to observe experimentally. On the other hand, as regards the Inverse-SF, the saddle point estimate of positive moments is always reached for $h_s(p) > 1/3$. This is an indication that we are probing the laminar part of the velocity statistics. Therefore, the presence of the IDR must be felt much earlier in the range of available velocity fluctuations. Indeed, if $h_s(p) > 1/3$, the typical velocity field at which the IDR shows up is given by $\delta v_m(p) \sim \nu^{h_s(p)/(1+h_s(p))}$, that is much larger than the Kolmogorov value $\delta v_{d_d} \sim \nu^{1/4}$.

In Fig. 1 we plot $\Sigma_1(\delta v)$. The straight lines shows the dissipative range behavior (dashed) $\Sigma_1(\delta v) \sim \delta v$, and the inertial range non intermittent behavior (dotted) $\Sigma_1(\delta v) \sim (\delta v)^3$. The inset shows the direct structure function $S_1(\tau)$ with superimposed the intermittent slope $\chi(1) = .39$.

Let us first make a technical remark. If one wants to compare the predictions with the experimental data, it is necessary to perform the average over the time-statistics in a weighted way. This is due to the fact that by looking at the exit-time statistics we are not sampling the time-series uniformly, i.e. the higher the value of $\tau(\delta v)$, the longer it is detectable in the time series. Let us call $\tau_1(\delta v), \tau_2(\delta v), \ldots, \tau_N(\delta v)$ the string of exit time values obtained by analyzing the velocity string data consecutively for a given $\delta v$. $N$ is the number of times for which $\tau(\delta v)$ reaches a given threshold. It is easy to realize that the sequential time average of any observable based on exit-time statistics, $\langle \tau^p(\delta v) \rangle \equiv \left< \tau^p(\delta v) \right>_i \equiv (1/N) \sum_{i=1}^N \tau_i^p$, is connected to the
uniformly-in-time multifractal average, \(<\cdot\equiv\int dh(\cdot)\), by the relation:

\[
<\tau^p(\delta v)> = \sum_{i=1}^{N} \tau_i^p / \sum_{i=1}^{N} \tau_i = <\tau^{p+1}>_t / <\tau>_t,
\]

where \(\tau_i / \sum_{i=1}^{N} \tau_i\) takes into account the non-uniformity in time. Let us now go back to Fig. 1. One can see that the scaling is very poor. Indeed, it is not possible to extract any quantitative prediction about the inertial range slope. For this reason, we have only drawn the dimensional non-intermittent slope and the dissipative slope as a possible qualitative references. On the other hand, (inset of Fig. 1), the scaling behavior of the direct structure functions \(<|\delta v(\tau)|>\sim \tau^{\zeta(1)}(\cdot)\) is quite clear in a wide range of scales. This is a clear evidence of IDR’s contamination into the whole range of available velocity values for the Inverse-SF cases. Similar results (not shown) are found for higher orders \(\Sigma_p\) structure functions.

In order to better understand the scaling properties of \(\Sigma_p(\delta v)\) we investigate a synthetic multi-affine field obtained by combining successive multiplications of Langevin dynamics \([10]\). The advantage of using a synthetic field is that one can control analytically the scaling properties of direct structure functions in order to have the same scaling laws observed in experimental data. The signal we used is sequential in time. Therefore, it does not present a superposed hierarchical structure as other multi-affine field proposed in the past \([12, 13]\). An IDR can be introduced in the synthetic signals by smoothing the original dynamics on a moving time-window of size \(\delta T\). Imposing a smoothing time-window is equivalent to fixing the Reynolds numbers, \(Re \sim \delta T^{-4/3}\). The purpose to introduce this stochastic multi-affine field is twofold. First we want to reach high Reynolds numbers enough to test the inverse-multifractal formula \([6]\). Second, we want to test that the very extended IDR observed in the experimental data, see Fig. 1, is also observed in this stochastic field. This would support the claim that the experimental result is the evidence of an extended IDR. In Fig. 2 we show the Inverse-SF, \(\Sigma_1(\delta v)\), measured in the multifractal synthetic signal at high-Reynolds numbers. The observed scaling exponent, \(\chi(1)\), is in agreement with the prediction \([6]\). The same agreement also holds for higher moments. In Table 1, we compare the best fit to the \(\Sigma_p(\delta v)\) measured on the synthetic field with the inversion formula \([6]\). As for the comparison between the theoretical expectation \([6]\) and the synthetic data let us note the following points. First, in \([10]\) it was proved that the signal possesses the given direct-structure functions exponents for positive moments, i.e. the \(\zeta(p)\) exponents are in a one-to-one correspondence with the \(D(h)\) curve for \(h < 1/3\). Nothing was possible to be proved for observables feeling the \(h > 1/3\) interval and therefore the agreement between the inversion formula \([6]\) and the numerical results cannot be proved analytically. Second, because the synthetic signal is defined by using Langevin processes, the less singular \(h\)-exponents expected to contribute to the saddle-point \([6]\) is \(h = 0.5\). Therefore, the theoretical prediction, \(\chi(0.5)\), in Table 1 has been obtained by imposing \(h_{max} = 0.5\).

Let us now go back to the most interesting question about the statistical properties of the IDR. In order to study this question we have smoothed the stochastic field, \(v(t)\), by performing a running-time average over a time-window, \(\delta T\). Then we compare Inverse-SF scaling properties at varying Reynolds numbers, i.e. for different dissipative cut-off : \(Re \sim \delta T^{-4/3}\). The expression \([10]\) predicts the possibility to obtain a data collapse of all curves with different Reynolds numbers by rescaling the Inverse-SF as follows \([10]\):

\[
-\frac{\ln(\Sigma_p(\delta v))}{\ln(\delta T/\delta T_0)} \text{ vs. } -\frac{\ln(\delta v/U)}{\ln(\delta T/\delta T_0)},
\]

where \(U\) and \(\delta T_0\) are adjustable dimensional parameters. Within the same experimental (or synthetic) set up they are Reynolds number independent (i.e. \(\delta T\) independent).
\[ \ln(\delta v) / \ln(\nu) \] only. Therefore, identifying \( Re \propto \nu^{-1} \), the relation (11) directly follows from (8). This rescaling was originally proposed as a possible test of IDR for direct structure functions in (2) but, as already discussed above, for the latter observable it is very difficult to detect any IDR due to the extremely small scales involved (1).

Fig. 3 shows the rescaling (10) of the Inverse-SF, \( \Sigma_1(\delta v) \), for the synthetic field at different Reynolds numbers and for the experimental signals. As it is possible to see, the data-collapse is very good for both the synthetic and experimental signal. This is a clear evidence that the poor scaling range observed in Fig. 1 for the experimental signal can be explained as the signature of the IDR. The same behavior holds for higher moments (not shown).

It is interesting to remark that for a self-affine signal \( (D(h) = \delta(h - 1/3)) \), the IDR is highly reduced and the Inverse-SF, scaling trivially as \( \Sigma_p(\delta v) \sim (\delta v)^{3p} \), do not bring any new information.

Let us summarize the results obtained and the open problems. First, by defining the exit-time moments, \( \Sigma_p(\delta v) \), we argued that they must be dominated by the laminar part of the energy cascade. This implies that they depend only on the part of \( D(h) \) which falls to the right of its maximum, i.e. \( h > 1/3 \). These \( h \)'s values are not testable by the direct structure functions (12). Inverse-SF are the natural tool to test any model concerning velocity fluctuations less singular than the Kolmogorov value \( \delta v \sim \nu^{1/3} \).

Second, by analyzing high-Reynolds data and synthetic fields, we have proved that the extension of the IDR for \( \Sigma_p(\delta v) \) is magnified. Moreover, the rescaling (11) based on the assumption (2) gives a good data collapse of all curves for different Reynolds numbers. This is a clear evidence of the IDR.

Many questions are still open. First, the analysis of a wider set of experimental data could make it possible to quantify the agreement of the data-collapse with the prediction based on (2) and (8). Indeed, it is easy to realize that, by using different parameterization for the onset of the viscous range, one would have predicted the existence of an extended IDR for \( \Sigma_p(\delta v) \) but with a slightly different rescaling procedure (2). The quality of experimental data available to us is not high enough to distinguish between the two different predictions. Analyzing different experimental data-sets, at different Reynolds numbers, could also make it possible to better explore \( D(h) \) for \( h > 1/3 \). This is an important question because doubts about the universality of these \( D(h) \) values may be raised on the basis of the usual energy cascade picture. For example, as discussed above, in the Langevin synthetic data a good agreement between the multifractal prediction and the numerical data is obtained by imposing \( h_{\text{max}} = 0.5 \), similarly in true turbulent data other \( h_{\text{max}} \) values could appear depending on the physical mechanism driving the energy transfer at large scales.

Once the attention is focused on the exit-time statistics, different questions connected to the entropic properties of the exit-times can also be asked. For these kind of questions there are no \textit{a priori} reasons to believe that the information coded in the direct-statistics is similar to the information coded in the inverse-statistics. Work is in progress in this direction.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( p \) & 1 & 2 & 3 & 4 & 5 \\
\hline
\( \chi_{\text{syn}}(p) \) & 2.32(4) & 4.40(8) & 6.38(8) & 8.35(1) & 10.1(2) \\
\( \chi_{\text{th}}(p) \) & 2.32 & 4.34 & 6.34 & 8.35 & 10.35 \\
\hline
\end{tabular}
\caption{Comparison between the Inverse-SF scaling exponents \( \chi_{\text{syn}}(p) \) measured in the synthetic signal and the inversion of the theoretical multifractal prediction (11), \( \chi_{\text{th}}(p) \). The synthetic signal has been defined such that the \( D(h) \) function leads to the same set of experimental \( \zeta(p) \) exponents for the direct structure functions.}
\end{table}

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