SU(5)+Adjoint Higgs Model at Finite Temperature

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Abstract

A three-dimensional effective theory of the finite-temperature SU(5)+adjoint Higgs model is constructed using the method of dimensional reduction. The resulting theory can be used for computer simulations of the GUT phase transition of the early universe. In this paper the transition is analyzed at two-loop level in perturbation theory. The structure of the effective theory does not essentially depend on the matter contents of the original four-dimensional one. Therefore the analysis of this paper applies also to more realistic theories.

1 Introduction

The simplest candidate for a theory unifying all the known gauge interactions is the SU(5) model proposed by Georgi and Glashow in 1974 [1]. Even though this model is known to conflict with experiments, it might still give a qualitatively correct picture of the unification. For example the supersymmetric version of this model is compatible with the experiments.

Irrespective of the exact form of the unified theory, it is generally believed that in the early stages of the evolution of our universe, there was a phase transition in which the unified symmetry broke down to the residual SU(3)×SU(2)×U(1) symmetry. The details of this transition may have had important cosmological consequences. The change in vacuum energy density may have given rise to an exponential expansion of the universe [2]. The recently in [3] proposed scenario of thermal inflation is even more closely connected to the GUT transition. Due to the non-trivial first homotopy of the residual symmetry group a large amount of magnetic monopoles may have been created in the transition,

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having dramatic effects on the evolution of the universe [4,5]. Recently it has
also been proposed that the monopole problem could be solved by symmetry
non-restoration of the SU(5) theory at high temperature [6–8].

Various properties of the high-temperature SU(5) model have been analysed by
many authors [9–12]. Since the model as such is too complicated for efficient
computer simulations, the studies have been mostly based on perturbation
theory. It is well known that many of the properties of the phase transitions
cannot be obtained by perturbative calculations, since the massless particles
cause infrared divergences [13]. However, the only existing lattice simulations
are more than a decade old and are therefore very inaccurate according to
present standards [14]. The method of dimensional reduction [15–17] gives a
way to split the problem into two parts, one of which can be handled perturba-
tively and the other with computer simulations. The idea is to first integrate
out the Matsubara modes which at high temperatures decouple from the zero
mode and then integrate out the heavy temporal component of the gauge field.
The resulting 3d theory can then be simulated on a lattice using the results
of [18]. This procedure has been successfully applied to the electroweak phase
transition [19] as well as to other models with relatively simple gauge symme-
tries [20–24]. The SU(5) gauge group has a much more complicated structure
and has some qualitatively new features since in the Lie algebra of SU(N)
with \( N > 3 \) there are two different symmetric tensors of order four while for
\( N < 4 \) there is only one. This makes the study of the model interesting also
independently of all the cosmological applications.

The paper is organized as follows. In Section 2 we define the model consid-
ered. In Sections 3 and 4 we apply the method of dimensional reduction to
the model. In Sections 5 and 6 we examine the relations between 4d and 3d
parameters and discuss the properties of the resulting effective theory. The
two appendices discuss the group theoretical factors and the evaluation of the
two-loop effective potential.

## 2 SU(5)+adjoint Higgs

The original model suggested in [1] consists of two scalar fields, one in adjoint
and one in fundamental representation of the gauge group SU(5), and two
fermion fields, one of which is in \( 5^* \) and the other in \( 10 \) representation. To get
a more realistic theory, one could even add a third scalar field, which would
be in \( 45 \) representation [25]. Also a model with an additional SU(5) singlet
scalar field has been suggested as a scenario for inflation of the universe [26].
All these models contain many free parameters, whose values cannot be fixed
by experiments, since their effects are only visible near the GUT energy scale.
Some constraints can be derived for example from cosmological consequences.
However, from the running of the Standard Model gauge couplings, the value of the gauge coupling constant at the GUT scale can be seen to be $g \approx 0.39$.

For simplicity, we shall consider a model with an adjoint scalar field only. Therefore the theory is not really physical and we will not fix the parameter values to any phenomenologically favourable ones, but we will instead explore the whole space of the parameters. Nevertheless, the calculations can be straightforwardly extended to include the remaining fields, all of which would be integrated out during dimensional reduction. Therefore the resulting effective theory would be of exactly the same form, only the relations between the 3d and 4d parameters would differ. This also applies to supersymmetric extensions of SU(5), although in that case a more complicated effective theory may be needed [27].

Our Minkowskian Lagrangian is

$$\mathcal{L}_M = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr}(D_\mu \Phi D^\mu \Phi) - m^2 \text{Tr} \Phi^2 - \lambda_1 (\text{Tr} \Phi^2)^2 - \lambda_2 \text{Tr} \Phi^4, \quad (1)$$

where $A_\mu = A_\mu^A T^A$ is the gauge field and $\Phi = \Phi^A T^A$ is the Higgs field. The generators $T^A$ of the symmetry group are Hermitian and normalized such that $\text{Tr} T^A T^B = \frac{1}{2} \delta^{AB}$. Their properties are discussed in more detail in Appendix A. The covariant derivative is $D_\mu \Phi = \partial_\mu \Phi + ig[A_\mu, \Phi]$. One should note that many authors have chosen a different convention. For example Langacker [28] uses $g_{\text{Langacker}} = -\sqrt{2} g$.

Perturbatively we have in our system 24 massless vector fields and 24 scalar fields in the symmetric phase. In the physical broken phase, there are 12 massless and 12 massive vectors and 12 scalar fields. Eight of the scalars form an octet in SU(3), three form an triplet in SU(2) and one is charged with respect to U(1). The massless vectors are the gauge bosons of the residual symmetry. On tree-level the observable parameters in the broken phase are the masses of the fields, given in Eqs. (17), (18). When one takes the loop corrections into account, they are replaced by the pole masses. These differ from the renormalized tree-level masses, when $\overline{\text{MS}}$ scheme is used.

Because of confinement, the real spectrum of the theory differs completely from the perturbative one. All the observable fields are SU(5) singlets and are therefore composite operators of the perturbative fields. The simplest gauge invariant operators are the following:

$$\text{Tr} \Phi^n, \quad \text{Tr} F_{\mu\nu} \Phi^{n-1}, \quad 2 \leq n \leq 5. \quad (2)$$

In the symmetric phase these operators represent bound states but in the broken phase they can be identified with the perturbative fields. All the ob-
servables of the system are correlators of these, or more complicated operators, which can be interpreted as bound states also in the broken phase.

At a finite temperature $T$ the system can be described with the same Lagrangian with an Euclidian time dimension in which the bosonic fields are periodic with period $\frac{1}{T}$. Let us therefore replace the time variable $t$ with $-i\tau$. For the calculations we will adopt the Landau gauge and hence we write the Euclidian Lagrangian as

$$\mathcal{L}_E = \frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr}(D_\mu \Phi)^2 + m^2 \text{Tr} \Phi^2 + \lambda_1 (\text{Tr} \Phi^2)^2 + \lambda_2 \text{Tr} \Phi^4$$

$$+ \frac{1}{2\xi} \left( \partial_\mu A^A_\mu \right)^2 + \partial_\mu \overline{c}_A \partial_\mu c^A - g f^{ABC} \partial_\mu c^A \partial_\mu c^B,$$

where $c$ is a ghost field and $\xi \to 0$.

3 Superheavy scale

The first task is to integrate over the superheavy, non-static Matsubara modes. At this stage all the fermions of the theory would disappear, since they do not have a static mode owing to their antiperiodicity. The time component $A_0$ of the gauge field will become an adjoint scalar field. We shall start by writing as an Ansatz the most general Lagrangian with at most quartic interactions which respects the desired symmetries and contains two adjoint scalar fields and a gauge field:

$$\mathcal{L}_3 = \frac{1}{4} F^A_{ij} F^{A}_{ij} + \text{Tr}(D_\mu \Phi)^2 + m_3^2 \text{Tr} \Phi^2 + \lambda_1' (\text{Tr} \Phi^2)^2 + \lambda_2' \text{Tr} \Phi^4$$

$$+ \text{Tr}(D_\mu A_0)^2 + m_3^2 \text{Tr} A_0^2 + \kappa_1 (\text{Tr} A_0^2)^2 + \kappa_2 \text{Tr} A_0^4$$

$$+ \alpha_1 \text{Tr} \Phi^2 \text{Tr} A_0^2 + \alpha_2 (\text{Tr} \Phi A_0)^2 + \alpha_3 \text{Tr} \Phi^2 A_0^2 + \alpha_4 \text{Tr} \Phi A_0 \Phi A_0.$$ (4)

The fields and the coupling constants have dimensions

$$[A_\mu] = [\Phi] = [g_3] = \text{GeV}, \quad [\lambda_1'] = [\kappa_i] = [\alpha_i] = [m_3] = \text{GeV}. \quad (5)$$

A renormalizable theory could have interaction vertices of as many as six fields but we neglect these higher-order corrections [17]. We shall now extract the parameters of (4) and the correct normalization of the fields by comparing the two- and four-point Green’s functions of the finite temperature 4d theory and of the 3d theory. This has been explained in great detail in [17].

For the normalization of the fields the parts proportional to the momentum squared $k^2$ are needed in two-point correlators $\langle \Phi \Phi \rangle$, $\langle A_0 A_0 \rangle$ and $\langle A_i A_j \rangle$. The
corresponding diagrams are shown in Fig. 1. We will write down only the part coming from the non-static modes of the diagrams. The high temperature approximation of the non-zero modes gives

\[(1.1) = \frac{k^2}{16\pi^2}g^2\delta^{AB}\frac{15}{\epsilon_b}, \quad (6)\]
\[(1.2) = \frac{k^2}{16\pi^2}g^2\delta^{AB}\left(\frac{10}{\epsilon_b} - \frac{25}{3}\right), \quad (7)\]
\[(1.3) = \frac{k^2}{16\pi^2}(\delta_{ij} - \frac{k_i k_j}{k^2})g^2\delta^{AB}\left(\frac{10}{\epsilon_b} + \frac{5}{3}\right). \quad (8)\]

Here

\[\frac{1}{\epsilon_b} = \frac{1}{\bar{\epsilon}} + L_b = \frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{(4\pi T)^2}\right) + 2\gamma_E, \quad (9)\]

where \(\gamma_E \approx 0.577216\) is the Euler-Mascheroni constant.

We will also need the Debye mass of the \(A_0\)-field. This is given by the constant
part of $\langle A_0 A_0 \rangle$. The corresponding diagrams are also given in Fig. 1. We get

\begin{equation}
(1.4) = -g^2 \delta^{AB} \left( \frac{5}{2} T^2 + \frac{m^2}{16\pi^2} 10 \right). \tag{10}
\end{equation}

The effective mass of the Higgs field $\Phi$ could also be evaluated in a similar way, but we need it to two-loop order and it is easier to calculate it using the effective potential. We shall postpone this, since in the calculation the counterterms from the four-point correlators are needed. The gauge field $A_i$ does not acquire a mass in dimensional reduction due to gauge invariance.

The three-dimensional gauge coupling constant $g_3$ is most easily obtained from the $\langle \Phi \Phi A_i A_j \rangle$-correlator. This consists of the diagrams shown in Fig. 2. The result is

\begin{equation}
(2.1) = \frac{g^4}{16\pi^2} \delta_{ij} (f^{ACE} f^{BDE} + f^{ADE} f^{BCE}) \frac{15}{2} \frac{1}{\epsilon_b}. \tag{11}
\end{equation}

Exactly the same diagrams give also the $\langle \Phi \Phi A_0 A_0 \rangle$-correlator:

\begin{align*}
(2.2) &= \frac{g^2}{16\pi^2} \left[ -(12g^2 + 20\lambda_1) \delta^{AB} \delta^{CD} - (6g^2 + 2\lambda_2)(\delta^{AP} \delta^{BC} + \delta^{AC} \delta^{BD}) \\
&\quad - (15g^2 - 10\lambda_2) d^{ABE} d^{CDE} \\
&\quad + \left( \frac{15}{2} \frac{1}{\epsilon_b} g^2 - 5g^2 - 4\lambda_1 - \frac{8}{5} \lambda_2 \right) (f^{ACE} f^{BDE} + f^{ADE} f^{BCE}) \right]. \tag{12}
\end{align*}

We need also the self-coupling of $A_0$, for which one gets

\begin{equation}
(2.3) = \frac{g^4}{16\pi^2} \left[ (\delta^{AB} \delta^{CD} + \text{perm.}) + \frac{5}{8} (d^{ABE} d^{CDE} + \text{perm.}) \right] (-16). \tag{13}
\end{equation}

The notation “perm.” denotes all the possible different permutations of indices, that is

\begin{align*}
\delta^{AB} \delta^{CD} + \text{perm.} &= \delta^{AB} \delta^{CD} + \delta^{AC} \delta^{BD} + \delta^{AD} \delta^{BC}, \\
d^{ABE} d^{CDE} + \text{perm.} &= d^{ABE} d^{CDE} + d^{ACE} d^{BDE} + d^{ADE} d^{BDE}. \tag{14}
\end{align*}

The last four-point function to be considered is $\langle \Phi \Phi \Phi \Phi \rangle$, which gives

\begin{equation}
(2.4) = \frac{1}{16\pi^2} \left[ \left( 64\lambda_1^2 + \frac{212}{5} \lambda_1 \lambda_2 + \frac{232}{20} \lambda_2^2 + 12g^4 \right) (\delta^{AB} \delta^{CD} + \text{perm.}) \right].
\end{equation}
Figure 2. The four-point diagrams needed in the super-heavy integration

\[ + \left( 12\lambda_1\lambda_2 + \frac{32}{5}\lambda_2^2 + \frac{15}{2} g^4 \right) \left( d^{ABE} d^{CDE} + \text{perm.} \right) \frac{1}{\epsilon_6} \]

\[ - g^4 \left[ 8(\delta^{AB}\delta^{CD} + \text{perm.}) + 5(d^{ABE} d^{CDE} + \text{perm.}) \right]. \]  

(15)
Let us now calculate the effective mass of the Higgs field using the effective potential. For this we must shift the Higgs field. At this point the direction of the shift is irrelevant, but we choose the physical one

$$\Phi \rightarrow \Phi + \frac{v}{\sqrt{15}} \text{Diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2}) = \Phi + v\tau_1. \quad (16)$$

After the shift we have 12 massive vector bosons with mass

$$M^2 = \frac{5}{12}g^2 v^2. \quad (17)$$

There are also four different kinds of Higgs fields, one for each factor group of the residual group and one Goldstone. They have the masses

$$m_1^2 = m^2 + \left(\lambda_1 + \frac{2}{5}\lambda_2\right) v^2, \quad m_2^2 = m^2 + \left(\lambda_1 + \frac{7}{30}\lambda_2\right) v^2,$$

$$m_3^2 = m^2 + \left(\lambda_1 + \frac{9}{10}\lambda_2\right) v^2, \quad m_4^2 = m^2 + \left(3\lambda_1 + \frac{7}{10}\lambda_2\right) v^2. \quad (18)$$

We will also need explicit expressions for the renormalization counterterms in the broken phase to one-loop order. The mass counterterms in the \(\overline{\text{MS}}\)-scheme are

$$\delta m_1^2 = \frac{1}{16\pi^2} \frac{1}{\varepsilon} \left[ m^2 \left(26\lambda_1 + \frac{47}{5}\lambda_2\right) + v^2 \left(32\lambda_1^2 + \frac{118}{5}\lambda_1\lambda_2 + \frac{148}{25}\lambda_2^2 + \frac{15}{2}g^4\right) \right], \quad (19)$$

$$\delta m_2^2 = \frac{1}{16\pi^2} \frac{1}{\varepsilon} \left[ m^2 \left(26\lambda_1 + \frac{47}{5}\lambda_2\right) + v^2 \left(32\lambda_1^2 + \frac{108}{5}\lambda_1\lambda_2 + \frac{364}{75}\lambda_2^2 + \frac{25}{4}g^4\right) \right], \quad (20)$$

$$\delta m_3^2 = \frac{1}{16\pi^2} \frac{1}{\varepsilon} \left[ m^2 \left(26\lambda_1 + \frac{47}{5}\lambda_2\right) + v^2 \left(32\lambda_1^2 + \frac{148}{5}\lambda_1\lambda_2 + \frac{228}{25}\lambda_2^2 + \frac{45}{4}g^4\right) \right], \quad (21)$$

$$\delta m_4^2 = \frac{1}{16\pi^2} \frac{1}{\varepsilon} \left[ m^2 \left(26\lambda_1 + \frac{47}{5}\lambda_2\right) + v^2 \left(96\lambda_1^2 + \frac{324}{5}\lambda_1\lambda_2 + \frac{364}{25}\lambda_2^2 + \frac{75}{4}g^4\right) \right], \quad (22)$$

$$\delta M^2 = \frac{1}{16\pi^2} \frac{1}{\varepsilon} \frac{125}{8}g^4 v^2, \quad (23)$$

and the wave function counterterms are
Figure 3. The diagrams for the effective potential of the Higgs field

\[
\delta Z_\Phi = \frac{g^2}{16\pi^2} \frac{1}{15}, \quad (24)
\]

\[
\delta Z_A = \frac{g^2}{16\pi^2} \frac{1}{10}. \quad (25)
\]

The one-particle irreducible vacuum diagrams needed are shown in Fig. 3. We need only the part of the result which is quadratic in \(v\). We have written down also the terms quartic in \(v\) to one-loop order just as a check of the result (15). Using the color factors given in Appendix B we could make the full two-loop calculation, but it is not necessary here. In the high temperature approximation the result is

\[
V(v) = V^{3d}(v) + \text{constant}
\]

\[
+ \frac{1}{2} v^2 \left\{ \left( \frac{5}{4} g^2 + \frac{13}{6} \lambda_1 + \frac{47}{60} \lambda_2 \right) T^2 - \frac{m^2}{16\pi^2} (26\lambda_1 + \frac{47}{5} \lambda_2) L_b 
+ \frac{T^2}{16\pi^2} \left[ \left( \frac{1}{\varepsilon} - 4 \log \frac{3T}{\mu} - 4c \right) \left( 13\lambda_1^2 + \frac{47}{5} \lambda_1 \lambda_2 \right) 
+ \frac{493}{100} \lambda_2^2 - 65 g^2 \lambda_1 - \frac{47}{2 g^2} \lambda_2 \right] 
+ L_b \left( -\frac{25}{2} g^4 + \frac{65}{2} g^2 \lambda_1 + \frac{47}{4} g^2 \lambda_2 - \frac{208}{3} \lambda_1^2 - \frac{752}{15} \lambda_1 \lambda_2 - \frac{922}{75} \lambda_2^2 \right) 
+ \frac{275}{12} g^4 + \frac{65}{3} g^2 \lambda_1 + \frac{47}{6} g^2 \lambda_2 \right\} \}
\]
+ \frac{1}{24} v^4 \left( \frac{1}{16\pi^2} \left[ \left( \frac{25}{2} \left( -\frac{75}{2} L_b \right) g^4 \right) \right. \\
- \left( 192\lambda_1^2 + \frac{648}{5}\lambda_1 \lambda_2 + \frac{728}{25}\lambda_2^2 \right) L_b \right], \tag{26}
\end{equation}

where \( c = \frac{1}{2} \left( \log \frac{8\pi}{9} + \frac{\zeta(2)}{\zeta(2)} - 2\gamma_E \right) \).

Now we have all the necessary Green’s functions to fix the three-dimensional parameters. This is done by matching the corresponding results in both theories taking into account that the normalization of the fields is also different. At one-loop level one only has to add corrections to the coupling constants and the field renormalization. The relations between the three-dimensional and four-dimensional fields are

\[
\Phi^{3d}_2 = \frac{1}{T} \left[ 1 - \frac{g^2}{16\pi^2} 15 L_b \right] \Phi^2, \tag{27}
\]

\[
(A^{3d}_0)^2 = \frac{1}{T} \left[ 1 - \frac{g^2}{16\pi^2} \left( 10 L_b - \frac{25}{3} \right) \right] A_0^2, \tag{28}
\]

\[
(A^{3d}_i)^2 = \frac{1}{T} \left[ 1 - \frac{g^2}{16\pi^2} \left( 10 L_b + \frac{5}{3} \right) \right] A_i^2. \tag{29}
\]

The parameters of the three-dimensional theory written as functions of those of the original 4d theory are

\[
g^2_3 = g^2 T \left[ 1 + \frac{g^2}{16\pi^2} \left( \frac{35}{2} L_b + \frac{5}{3} \right) \right], \tag{30}
\]

\[
\kappa_1 = \frac{g^4 T}{16\pi^2} 6, \tag{31}
\]

\[
\kappa_2 = \frac{g^4 T}{16\pi^2} 10, \tag{32}
\]

\[
\lambda'_1 = \lambda_1 T - \frac{T}{16\pi^2} \left[ \left( 32\lambda_1^2 + \frac{94}{5}\lambda_1 \lambda_2 + \frac{84}{25}\lambda_2^2 + \frac{9}{2}g^4 - 30g^2 \lambda_1 \right) L_b \right. \\
- \left. 3g^4 \right], \tag{33}
\]

\[
\lambda'_2 = \lambda_2 T - \frac{T}{16\pi^2} \left[ \left( 12\lambda_1 \lambda_2 + \frac{32}{5}\lambda_2^2 + \frac{15}{2} g^4 - 30g^2 \lambda_2 \right) L_b - 5g^4 \right], \tag{34}
\]

\[
\alpha_1 = \frac{g^2 T}{16\pi^2} \left( 6g^2 + 24\lambda_1 \right), \tag{35}
\]

\[
\alpha_2 = \frac{g^2 T}{16\pi^2} \left( 12g^2 + 4\lambda_2 \right), \tag{36}
\]

\[
\alpha_3 = 2g^2 T \left[ 1 + \frac{1}{16\pi^2} \left( \frac{35}{2} L_b g^2 + \frac{35}{3} g^2 + 4\lambda_1 - \frac{42}{5} \lambda_2 \right) \right], \tag{37}
\]
\[ \alpha_4 = -2g^2T \left[ 1 + \frac{1}{16\pi^2} \left( \frac{35}{2}Lb g^2 - \frac{10}{3}g^2 + 4\lambda_1 + \frac{8}{5}\lambda_2 \right) \right], \quad (38) \]

\[ m_D^2 = \frac{5}{2}g^2T^2, \quad (39) \]

\[ m_3^2 = \tilde{m}^2 + \frac{T}{12} \left( 15g_3^2 + 26\lambda_1 + \frac{47}{5}\lambda_2' \right) \]

\[ + \frac{1}{16\pi^2} \left( \frac{25}{2}g_3^2 + 26\lambda_1 + \frac{47}{5}\lambda_2' \right) \]

\[ + \frac{1}{16\pi^2} \left[ 10g_3^2 \left( 26\lambda_1' + \frac{47}{5}\lambda_2' \right) - 52\lambda_2'^2 \right] \]

\[ - \frac{188}{5}\lambda_1'\lambda_2' - \frac{493}{25}\lambda_2'^2 \left( \log \frac{3T}{\mu} + c \right), \quad (40) \]

\[ \tilde{m}^2 = m^2 \left[ 1 + \frac{1}{16\pi^2} \left( 15g^2 - 26\lambda_1 - \frac{47}{5}\lambda_2 \right) L_b \right]. \quad (41) \]

In every equation the \( \mu \)-dependence of \( L_b \) cancels the \( \mu \)-dependence of the coupling constants. Therefore none of the coupling constants of the three-dimensional theory runs. From Eq. (40) one can read the running of the three-dimensional mass as a function of the scale \( \mu \),

\[ \mu \frac{\partial m_3^2(\mu)}{\partial \mu} = -\frac{1}{16\pi^2} f_{2m} \]

\[ = -\frac{1}{16\pi^2} \left[ 10g_3^2 \left( 26\lambda_1' + \frac{47}{5}\lambda_2' \right) - 52\lambda_2'^2 - \frac{188}{5}\lambda_1'\lambda_2' - \frac{493}{25}\lambda_2'^2 \right] \quad (42) \]

As we can see later in Eq. (64), the running due to the coupling of the two adjoint scalar fields \( \Phi \) and \( A_0 \) cancels when the coupling constants \( \alpha \) have the values given in Eqs. (35)–(38). Therefore the \( A_0 \) field does not contribute to \( f_{2m} \) and the result (42) is the same also for a theory with only one adjoint scalar field. This result can be generalized to the case of SU(\( N \)) symmetry, and is

\[ f_{2m} = 2Ng_3^2 \left( (N^2 + 1)\lambda_1' + (2N^2 - 3)\lambda_2' \right) / N \]

\[ - 2(N^2 + 1)\lambda_1'^2 - 4(2N^2 - 3)\lambda_1'\lambda_2' / N - (N^4 - 6N^2 + 18)\lambda_2'^2 / N^2. \quad (43) \]

Since the three-dimensional theory is superrenormalizable, this result is exact for a theory with only one adjoint Higgs.

Thus we have now constructed a three-dimensional theory with a gauge field and two adjoint Higgses, which describes the same physics as the original theory in a sense that the static correlators of the three-dimensional theory coincide with those of the four-dimensional one at order \( \mathcal{O}(g^4) \). However, the
Figure 4. The diagrams needed for the heavy integration. The solid line represents the $A_0$ scalar field.

theory is still unnecessarily complicated. At high temperature the $A_0$ field is namely heavy, since its mass is proportional to the temperature. Thus one can integrate it out as well. We will do that in the next section.

4 Heavy scale

Integrating out the heavy $A_0$ field is also described in [17]. For that task we need all the two- and four-point diagrams containing the $A_0$ field. They are shown in Fig. 4.

On one-loop level there are no momentum dependent two-point $\langle \Phi \Phi \rangle$-diagrams with the $A_0$ field. Therefore the normalization of the Higgs field does not change. For the gauge field $A_i$ we do instead have one diagram:

$$ (4.1) = - \frac{g_3^2}{4\pi m_D} \frac{5}{12} \delta^{AB} (k^2 \delta_{ij} - k_i k_j). $$

(44)

The correction to the gauge coupling can be most easily evaluated from the $\langle \Phi \Phi A_i A_i \rangle$-correlator. There are two such diagrams with the $A_0$ field, but they cancel each other:
Thus the only effect to the gauge coupling will be from the new normalization of the gauge field.

The correction for the Higgs self-interaction can be obtained from the only $\langle \Phi \Phi \Phi \Phi \rangle$-diagram with $A_0$-lines. The result is

$$
\begin{align*}
(4.3) &= \frac{1}{16\pi m_D} \left[ \left( 24\alpha_1^2 + 2\alpha_1\alpha_2 + \frac{48}{5}\alpha_1\alpha_3 + \frac{2}{5}\alpha_2\alpha_3 \right.ight. \\
&\quad + \left. 24\alpha_3^2 - \frac{2}{5}\alpha_1\alpha_4 + \frac{2}{5}\alpha_2\alpha_3 - \frac{2}{25}\alpha_3\alpha_4 + \frac{24}{25}\alpha_4^2 \right) (\delta^{AB}\delta^{CD} + \text{perm}) \\
&\quad + \left( \alpha_2\alpha_3 + \frac{21}{20}\alpha_3^2 + \alpha_2\alpha_4 - \frac{2}{5}\alpha_3\alpha_4 - \frac{1}{5}\alpha_4^2 \right) (d^{ABE}\delta^{CDE} + \text{perm}) \right].
\end{align*}
$$

To get the mass correction of the Higgs field we shall evaluate the $A_0$-dependent part of the two-loop effective potential. Breaking the symmetry gives for the gauge field $A_i$ and the Higgs field $\Phi$ the same masses as before in Eqs. (17), (18). The $A_0$ field will acquire the masses

$$
\begin{align*}
M_1^2 &= m_D^2 + \left( \frac{1}{2}\alpha_1 + \frac{1}{15}\alpha_3 + \frac{1}{15}\alpha_4 \right) v^2, \\
M_2^2 &= m_D^2 + \left( \frac{1}{2}\alpha_1 + \frac{13}{120}\alpha_3 - \frac{1}{10}\alpha_4 \right) v^2, \\
M_3^2 &= m_D^2 + \left( \frac{1}{2}\alpha_1 + \frac{3}{20}\alpha_3 + \frac{3}{20}\alpha_4 \right) v^2, \\
M_4^2 &= m_D^2 + \left( \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{7}{60}\alpha_3 + \frac{7}{60}\alpha_4 \right) v^2.
\end{align*}
$$

Now the desired part of the effective potential can be obtained from the six vacuum diagrams shown in Fig. 4. They give

$$
\begin{align*}
(4.4.1) &= 8C_8(M_1) + 12C_8(M_2) + 3C_8(M_3) + C_8(M_4), \\
(4.4.2) &= -(20\kappa_1 + 10\kappa_2) D_{SS}(M_1, M_1) - (42\kappa_1 + 15\kappa_2) D_{SS}(M_2, M_2) \\
&\quad - \left( \frac{15}{4}\kappa_1 + \frac{15}{8}\kappa_2 \right) D_{SS}(M_3, M_3) - \left( \frac{3}{4}\kappa_1 + \frac{7}{40}\kappa_2 \right) D_{SS}(M_4, M_4) \\
&\quad - (48\kappa_1 + 16\kappa_2) D_{SS}(M_1, M_2) - 12\kappa_1 D_{SS}(M_1, M_3)
\end{align*}
$$
Using these results we can construct an effective three-dimensional theory with
\[ C \]
where the fields are related to the original fields as follows:
\[ \mathcal{L} = \frac{1}{4} F_{ij} F^{ij} + \operatorname{Tr}(D_i \Phi)^2 + m^2 \operatorname{Tr}\Phi^2 + \bar{\lambda}_1 (\operatorname{Tr}\Phi^3)^2 + \bar{\lambda}_2 \operatorname{Tr}\Phi^4, \]
(58)

where the fields are related to the original fields as follows:
\[ \Phi^2 = \Phi_{3d}^2 \]
\[ = \frac{1}{T} \left[ 1 - \frac{g^2}{16\pi^2} 15L_b \right] \Phi^2, \]  
(59)
\[ \mathcal{A}_i^2 = \left[ 1 + \frac{5}{48\pi m_D} \right] (A_i^{3d})^2 \]
\[ = \frac{1}{T} \left[ 1 + \frac{g}{24\pi} \sqrt{\frac{5}{2}} - \frac{g^2}{16\pi^2} \left( 10L_b + \frac{5}{3} \right) \right] A_i^2. \]  
(60)

The relation of the parameters of the theory to those of the original theory is

\[ \mathcal{g}^2 = g_3^2 \left[ 1 - \frac{5}{48\pi m_D} g_2^2 \right] \]
\[ = g^2 T \left[ 1 - \frac{g}{24\pi} \sqrt{\frac{5}{2}} + \frac{g^2}{16\pi^2} \left( \frac{35}{2} L_b + \frac{5}{3} \right) \right], \]  
(61)
\[ \bar{\lambda}_1 = \lambda'_1 - \frac{1}{16\pi m_D} \left( 12\alpha_1^2 + \alpha_1\alpha_2 + \frac{1}{2} \alpha_2^2 + \frac{24}{5} \alpha_1 \alpha_3 \right. \]
\[ + \left. \frac{27}{100} \alpha_3^2 - \frac{1}{5} \alpha_1 \alpha_4 + \frac{1}{25} \alpha_3 \alpha_4 + \frac{13}{25} \alpha_4^2 \right) \]
\[ = T \left\{ \lambda_1 - \frac{3g^3}{8\sqrt{10\pi}} \right. \]
\[ - \left. \frac{1}{16\pi^2} \left[ \left( 32\lambda_1^2 + \frac{94}{5} \lambda_1 \lambda_2 + \frac{84}{25} \lambda_2^2 + \frac{9}{2} g^2 - 30g^2 \lambda_1 \right) L_b - 3g^4 \right] \right\}, \]  
(62)
\[ \bar{\lambda}_2 = \lambda'_2 - \frac{1}{16\pi m_D} \left( \alpha_2 \alpha_3 + \frac{21}{20} \alpha_3^2 + \alpha_2 \alpha_4 - \frac{2}{5} \alpha_3 \alpha_4 - \frac{1}{5} \alpha_4^2 \right) \]
\[ = T \left\{ \lambda_2 - \frac{g^3}{8\pi} \sqrt{\frac{5}{2}} \right. \]
\[ - \left. \frac{1}{16\pi^2} \left[ \left( 12\lambda_1 \lambda_2 + \frac{32}{5} \lambda_2^2 + \frac{15}{2} g^4 - 30g^2 \lambda_2 \right) L_b - 5g^4 \right] \right\}, \]  
(63)
\[ \bar{m}^2 = \frac{m_3^2}{4\pi} \left( 12\alpha_1 + \frac{1}{2} \alpha_2 + \frac{12}{5} \alpha_3 - \frac{1}{10} \alpha_4 \right) \]
\[ + \frac{1}{16\pi^2} \left[ \left( \frac{1}{\varepsilon} + 1 + 4 \log \frac{\mu}{2m_D} \right) g_3^2 \left( 30\alpha_1 + \frac{5}{4} \alpha_2 + 6\alpha_3 - \frac{1}{4} \alpha_4 - \frac{25}{8} g_3 \right) \right. \]
\[ - \left. \left( \frac{1}{\varepsilon} + 1 + 4 \log \frac{\mu}{m_3 + 2m_D} \right) \left( 3\alpha_1^2 + \frac{1}{4} \alpha_1 \alpha_2 + \frac{25}{16} \alpha_2^2 + \frac{6}{5} \alpha_1 \alpha_3 \right. \right. \]
\[ + \frac{23}{40} \alpha_2 \alpha_3 + \frac{537}{800} \alpha_3^2 - \frac{1}{20} \alpha_1 \alpha_4 + \frac{6}{5} \alpha_2 \alpha_4 - \frac{11}{50} \alpha_3 \alpha_4 + \frac{581}{400} \alpha_4^2 \right] \]
\[ + \frac{25}{8} g_3^4 + \left( 13\kappa_1 + \frac{47}{10} \kappa_2 \right) \left( 12\alpha_1 + \frac{1}{2} \alpha_2 + \frac{12}{5} \alpha_3 - \frac{1}{10} \alpha_4 \right) \]
\[ m_2 = m_3 - \frac{5g^3T^2}{4\pi} \sqrt{\frac{5}{2}} + \frac{g^4T^2}{16\pi^2} \left( \frac{75}{2} \log \frac{m_3 + 2m_D}{2m_D} - \frac{25}{4} \right). \] (64)

One should note that for the correct values of the parameters \( \alpha \) (35)–(38) the expression (64) is not divergent since the divergences cancel each other. Therefore the new effective mass \( \overline{m}^2 \) runs in the same way as \( m_3^2 \) as a function of the scale \( \mu \). In particular, Eq. (42) is correct also in this case.

5 Parameters of the effective theory

Let us now study concretely the values of the parameters of the effective theory (58) and their relation to the parameters of the original 4d theory (1). As discussed in Sect. 2, it is reasonable to fix only the gauge coupling constant, and keep the other parameters free. Thus we will only assume that the parameters of the theory are such that the construction of the effective theory is legitimate.

The effective theory has four parameters: \( g^2 \), \( \lambda_1 \), \( \lambda_2 \) and \( \overline{m}^2 \). However, these are all dimensionful quantities and we can choose one to fix the scale and express the theory in terms of three dimensionless parameters

\[ y = \frac{\overline{m}^2(g^2)}{g^4}, \quad x_1 = \frac{\lambda_1}{g^2}, \quad x_2 = \frac{\lambda_2}{g^2}. \] (65)

Here \( y \) has been defined in the \( \overline{\text{MS}} \)-scheme with the renormalization scale \( \mu = g^4 \). When the 4d parameters have been renormalized at the scale \( \mu = 4\pi T e^{-\gamma_E} \) and \( g = 0.39 \), we can use Eqs. (17), (18) to replace the 4d parameters with tree-level masses in Eqs. (61)–(64) and we obtain

\[
\begin{align*}
    x_1 &= 0.21 \frac{m_1^2}{M^2} - 0.59 \frac{m_1^4}{M^4} - 0.012, \quad x_2 = 2.52 \frac{m_1^2}{M^2} - 0.020, \\
    y &= 6.80 + 6.08 \frac{m_1^2}{M^2} - 1.45 \frac{m_1^4}{M^4} + 3.95 \frac{m_1^2}{M^4} - 0.038 \frac{m_4^2}{M^4} \\
    &\quad - 0.12 \frac{m_1^2 m_4^2}{M^4} - 21.9 \frac{m_4^2}{T^2},
\end{align*}
\] (66)

where we have neglected the logarithmic dependence on the temperature. Here \( M \), \( m_1 \) and \( m_4 \) are the tree-level broken phase masses of the massive vectors, the SU(3) octet scalars and the U(1) charged scalar, respectively. As discussed in Section 2, the physical pole masses differ from these due to loop corrections. From Eq. (66) we see that as the temperature decreases, \( x_1 \) and \( x_2 \) stay constant and \( y \) decreases. Since the transition takes place near \( y = 0 \), we can get
a simple approximation for the critical temperature by setting \( y(T = T_c) = 0 \) in Eq. (66). In the next section we derive more accurate estimates.

6 Properties of the effective theory

In a system with an SU(\(N\)) gauge field coupled to a scalar field in fundamental representation, it has been shown that the “symmetric” and the “broken” phase are analytically connected to each other [31]. This has been confirmed by numerical simulations in the case \( N = 2 \) [32]. In [31] the authors expect that for SU(\(N\))+adjoint Higgs there could be a symmetry breaking related to the \( Z_N \) symmetry in which the Higgs field transforms trivially. Nevertheless, lattice results have given evidence that for SU(2)+adjoint Higgs there is no distinction between the phases and for some values of the parameters the phase transition does not exist [24]. It is therefore uncertain, to what extent we can consider the phase transition to be a symmetry breakdown as suggested by the perturbative interpretation.

Even though it is well known that only non-perturbative, numerical computations can give reliable results in the vicinity of a phase transition, we will inspect the perturbative effective potential in order to obtain information about the phase diagram. For small values of \( x_1 \) and \( x_2 \) this gives the correct result.

The essential shape of the potential is already found in the one-loop approximation, taking into account only the gauge field loop. This gives the effective potential

\[
V(\Phi) = y \text{Tr} \Phi^2 + x_1 (\text{Tr} \Phi^2)^2 + x_2 \text{Tr} \Phi^4 - \frac{1}{6\pi} \sum_{i,j=1}^{5} |\Phi_i - \Phi_j|^3,
\]

where \( \Phi = \text{Diag}(\Phi_1, \ldots, \Phi_5) \) and \( \sum_i \Phi_i = 0 \) and we have scaled \( \Phi \) and \( V(\Phi) \) to dimensionless quantities. The Higgs vev can always be transformed to this diagonal form. Here \( y \) denotes actually the value at the scale \( \mu \),

\[
y(\mu) = y - \frac{f_{2m}}{16\pi^2} \log \frac{\mu}{g^2}.
\]

In this approximation the scale \( \mu \) cannot be fixed.

In perturbation theory the transition is of first order, due to the cubic term in (67). We can evaluate the critical surface \( y_c = y_c(x_1, x_2) \), where the minima of \( V(\Phi) \) are degenerate. In our present approximation this can be calculated
analytically. Let us write $\Phi = v\tau$, where $v$ is a real number and $\text{Tr}\tau^2 = \frac{1}{2}$. Then we obtain
\[
y_c = \frac{1}{18\pi^2} \frac{(\sum_{i,j} |\tau_i - \tau_j|^3)^2}{x_1 + 4x_2 \text{Tr}\tau^4}.
\] (69)

This result depends on the direction $\tau$ of the symmetry breakdown. We shall discuss mainly the physically most interesting directions
\[
\tau_1 = \frac{1}{\sqrt{15}} \text{Diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2}), \quad \tau_2 = \frac{1}{2\sqrt{10}} \text{Diag}(1, 1, 1, 1, -4),
\] (70)

which lead to the residual symmetries of the Standard Model and SU(4) $\times$ U(1), respectively. It has been shown [29] that on tree-level these are the only possible minima. Quantum corrections may change the situation and for some values of parameters there may be other minima, even a global one. The corresponding critical surfaces are
\[
y_c(\tau_1) = \frac{625}{36\pi^2} \frac{1}{30x_1 + 7x_2}, \quad y_c(\tau_2) = \frac{625}{36\pi^2} \frac{1}{20x_1 + 13x_2}.
\] (71)

From this we see that the critical surfaces cross at the line $x_1 = \frac{7}{5}x_2$. This ratio is connected to the $Z_5$ symmetry of the SU(5) gauge theory at finite temperature [30]. We can namely proceed with the dimensional reduction for a pure SU(5) theory using essentially the diagrams already calculated and end up with the ratio $x_1 = \frac{3}{5}x_2$ for the effective self-coupling of the $A_0$ field. The origin and the minima corresponding to the directions $\tau_1$ and $\tau_2$ are then identified by the $Z_5$ symmetry, which shows up in our calculation just as the degeneracy in this approximation. For the other possible minima SU(3) $\times$ [U(1)]$^2$ and [SU(2)]$^2$ $\times$ [U(1)]$^2$, the critical surfaces lie lower than the ones shown above for all the parameter values.

In addition to the critical surface between the symmetric phase and the broken phases we are also interested in phase transitions or critical surfaces between different broken phases. It turns out that $\tau_1$ and $\tau_2$ are the only shifts that can give an absolute minimum for any values of the parameters. Therefore only the critical surface between those two phases is of interest. It consists of the points in the parameter space in which the minima in the directions $\tau_1$ and $\tau_2$ are degenerate. Unfortunately we cannot write a simple expression for that surface as in Eq. (71).

One can easily improve the accuracy of this one-loop approximation. The effective potential can be calculated to two-loop order using essentially the color factors calculated previously. However, for some diagrams this must be done by explicitly fixing the direction of the shift. Therefore the result cannot
be written in a general form as in (67). We have completed this calculation for the shift directions $\tau_1$ and $\tau_2$ and the expressions for the effective potential are shown in Appendix B. In principle one could write down the complete one-loop effective potential for a shift in an arbitrary direction, but the resulting expression is much too complicated and we have omitted it.

The phase diagram of the system in two-loop approximation is shown in Fig. 5. We have only evaluated the potential in two broken minima and therefore the possible presence of any other phases is not shown. The phase diagram consists of three domains in which the true minimum is in the corresponding phase. The boundaries between the domains correspond to phase transitions. This result is correct only for small values of $x_1$ and $x_2$, but the theory can be simulated on a lattice to obtain the correct critical surface. It is interesting to speculate whether the phase transition becomes a crossover as $x_1$ and $x_2$ grow and become about 0.1. This effect has been found for an SU(2) gauge field coupled to a Higgs field in the fundamental [32] or adjoint [24] representation. If the same is true for SU(5), the GUT phase transition might never had happened.

We cannot get any direct information about the kinetics of the phase transition from the effective 3d theory, since it only describes the equilibrium behaviour.
As the parameters cross a critical surface the two corresponding minima are degenerate, which gives rise to phase transition. Owing to the supercooling effects the transition does not take place instantaneously, but the system may stay in a metastable state for some time. A reliable analysis of the kinetics of the phase transition is possible only after the correct values of $T_c$, latent heat and interface tension are determined from lattice simulations.

7 Conclusions

We have constructed a three-dimensional theory that describes the equilibrium behaviour of a high-temperature SU(5)+adjoint Higgs system correctly up to order $\mathcal{O}(g^3)$. This is done by determining the parameter values of the effective theory by comparing the Green’s functions of the two theories. The procedure is free from infrared divergences, but solves some of the problems concerning numerical simulations of the original 4d theory. The resulting theory contains no fermions and has parametrically only one energy scale. Also the smaller number of dimensions allows one to use much larger lattice sizes. Therefore one can expect to have much more accurate results about the phase transition than previously.

We have proceeded with the task of dimensional reduction only for a non-realistic theory with no fermions. The fermion content of the theory would only change the values of the parameters of the effective theory, not the structure of the theory itself, since due to their antiperiodicity the fermions have only super-heavy Matsubara modes of mass $\sim \pi T$, which are integrated out. Near the phase transition the other scalar fields than the adjoint one are also heavy and therefore do not affect the structure of the effective theory. Thus we get already now reliable qualitative results from the effective theory. The inclusion of the realistic particle spectrum is only a mechanical task and involves no new technical difficulties. One might also extend the analysis to a supersymmetric version of the theory. Even then an unambiguous picture of the phase transition cannot be obtained because of the large freedom in the choice of the parameters of the four-dimensional theory.

There are a few problems with the construction presented in this paper. First of all it is questionable how well one can apply the standard high-temperature formalism in the era before the GUT transition. The system may actually not be in thermodynamical equilibrium. This is the case in most of the models of inflation. The effects of the high curvature of the universe should perhaps be taken into account [33]. Of course the formalism can describe only systems in an equilibrium and therefore it can be used only to study some of the features of the phase transition. However, there are many important properties which can be extracted from the equilibrium behavior near the transition. The effective three-dimensional theory gives a correct picture of the equilibrium.
behaviour only at high temperatures and when the four-dimensional coupling constants are small. These demands are well satisfied in SU(5).

The previous numerical studies of the electroweak phase transition have shown that for large values of the Higgs self-coupling constant the transition ends. We expect the same phenomenon to occur also in the case of SU(5) theory, but we have to wait for the numerical simulations to confirm this conjecture. The absence of the transition may have some important cosmological consequences. It might affect the problem with the overabundance of magnetic monopoles. It has been suggested [6–8] that in the absence of a transition the monopole problem could be solved without inflation. The reliable non-perturbative results of the SU(5) phase transition can also be used to make the analysis of the possible inflationary phase of the evolution of the universe much more accurate.

Acknowledgements

I wish to thank K. Kajantie, M. Laine and M. Shaposhnikov for useful discussions. This work was supported by the Academy of Finland.

A Properties of the SU(5) generators

In the case of SU(5) symmetry, the structure of the vertices is much richer than for example in SU(2). Here is a somewhat complete list of the relations needed in the calculations of this paper. Some of the group-theoretical factors necessary for the calculation of the effective potential must be evaluated explicitly by using a specific choice of generators and we shall omit them here. The relations of this appendix can be obtained using the methods of Kaplan and Resnikoff [34].

First we give some relations and conventions for the generators of the fundamental representation of SU($N$), which are defined to be Hermitian:

\[
[T^A, T^B] = if^{ABC}T^C, \tag{A.1}
\]

\[
\text{Tr}T^AT^B = \frac{1}{2}\delta^{AB}, \tag{A.2}
\]

\[
T^AT^B = \frac{1}{2N}\delta^{AB} + \frac{1}{2}d^{ABC}T^C + \frac{i}{2}f^{ABC}T^C, \tag{A.3}
\]

\[
(T^AT^A)_{ab} = \frac{N^2 - 1}{2N}\delta_{ab}, \tag{A.4}
\]
In order to discuss the properties of the generators of the adjoint representation, let us define two matrices

\[(F^A)_{BC} = -i f^{ABC}, \quad (D^A)_{BC} = d^{ABC}.\]  

For these we have the usual Jacobi identities

\[f^{ABE} f^{CDE} + f^{CBE} f^{DAE} + f^{DBE} f^{ACE} = 0,\]

\[f^{ABE} d^{CDE} + f^{CBE} d^{DAE} + f^{DBE} d^{ACE} = 0.\]  

Traces of two matrices:

\[\text{Tr} F^A F^B = f^{ACD} f^{BCD} = N \delta^{AB},\]

\[\text{Tr} D^A D^B = d^{ACD} d^{BCD} = \frac{N^2 - 4}{N} \delta^{AB}.\]  

Traces of three matrices:

\[\text{Tr} F^A F^B F^C = \frac{N}{2} i f^{ABC},\]

\[\text{Tr} F^A D^B D^C = \frac{N^2 - 4}{2N} i f^{ABC},\]

\[\text{Tr} D^A F^B F^C = \frac{N}{2} d^{ABC},\]

\[\text{Tr} D^A D^B D^C = \frac{N^2 - 12}{2N} d^{ABC}.\]  

Traces of four matrices:

\[\text{Tr} F^A F^B F^C F^D = \delta^{AB} \delta^{CD} + \delta^{AD} \delta^{BC}\]

\[+ \frac{N}{4} (d^{ABE} d^{CDE} - d^{ACE} d^{BDE} + d^{ADE} d^{BCE}),\]

\[\text{Tr} F^A F^B D^C D^D = \delta^{AB} \delta^{CD} - \delta^{AD} \delta^{BC}\]

\[+ \frac{N}{4} (d^{ABE} d^{CDE} + d^{ACE} d^{BDE} - d^{ADE} d^{BCE}).\]
When the symmetry is broken to the symmetry group $SU(3) \times \tau$
interesting directions

$$\text{Tr} D^A D^B D^C D^D = \frac{N^2}{N^2} \delta^{AB} \delta^{CD} + \frac{N^2}{N^2} \delta^{AD} \delta^{BC} + \frac{24}{N} d^{AB} d^{CD}$$

$$- \frac{N}{4} d^{ACE} d^{BDE} + \frac{N^2 - 8}{4N} d^{ADE} d^{BCE}$$

$$+ \frac{2}{N} (f^{ABE} f^{CDE} + f^{ADE} f^{BCE}).$$

\section*{B Two-loop effective potential of the 3d theory}

In this appendix we give the expressions for the effective potential of the
Higgs field in the three-dimensional theory broken to the two physically most
interesting directions $\tau_1$ and $\tau_2$.

When the symmetry is broken to the symmetry group $SU(3) \times SU(2) \times U(1)$,
$\Phi \mapsto \Phi + v \tau_1$, the effective potential is

$$V(v) = \frac{1}{2} y v^2 + \left( \frac{1}{4} x_1 + \frac{7}{120} x_2 \right) v^4$$

$$+ 8 C_S(m_1) + 12 C_S(m_2) + 3 C_S(m_3) + C_S(m_4) + 12 C_V(M)$$

$$- (20 x_1 + 10 x_2) D_{SS}(m_1, m_1) - (42 x_1 + 15 x_2) D_{SS}(m_2, m_2)$$

$$- \left( \frac{15}{4} x_1 + \frac{15}{8} x_2 \right) D_{SS}(m_3, m_3) - \left( \frac{3}{4} x_1 + \frac{7}{40} x_2 \right) D_{SS}(m_4, m_4)$$

$$- (48 x_1 + 16 x_2) D_{SS}(m_1, m_2) - 12 x_2 D_{SS}(m_1, m_3)$$

$$- \left( \frac{8}{5} x_2 \right) D_{SS}(m_1, m_4) - (18 x_1 + 9 x_2) D_{SS}(m_2, m_3)$$

$$- \left( \frac{4 x_1 + \frac{7}{2} x_2}{5} \right) D_{SS}(m_2, m_4) - \left( \frac{3}{2} x_1 + \frac{27}{20} x_2 \right) D_{SS}(m_3, m_4)$$

$$- 6 D_{SV}(m_1, 0) - \frac{15}{2} D_{SV}(m_2, 0) - \frac{3}{2} D_{SV}(m_3, 0) - 4 D_{SV}(m_1, M)$$

$$- \frac{15}{2} D_{SV}(m_2, M) - \frac{9}{4} D_{SV}(m_3, M) - \frac{5}{4} D_{SV}(m_4, M)$$

$$- \frac{15}{4} D_{VV}(0, 0) - \frac{15}{2} D_{VV}(M, 0) - \frac{15}{4} D_{VV}(M, M)$$

$$- 6 D_{SSV}(m_1, m_1, 0) - \frac{15}{2} D_{SSV}(m_2, m_2, 0) - \frac{3}{2} D_{SSV}(m_3, m_3, 0)$$

$$- 8 D_{SSV}(m_1, m_2, M) - \frac{9}{2} D_{SSV}(m_2, m_3, M) - \frac{5}{2} D_{SSV}(m_2, m_4, M)$$

$$- \frac{5}{2} D_{VVV}(0, 0, 0) - \frac{15}{2} D_{VVV}(0, M, M)$$

$$- 30 D_{\eta \eta \eta}(0) - 30 D_{\eta \eta \eta}(M)$$

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\[-v^2 \left( \frac{8}{3} x_2^2 D_{SSS}(m_1, m_1, m_1) + 8 \left( x_1 + \frac{2}{3} x_2 \right)^2 D_{SSS}(m_1, m_1, m_4) \right) + \frac{4}{15} x_2^2 D_{SSS}(m_1, m_2, m_2) + \frac{12}{5} x_2^2 D_{SSS}(m_2, m_2, m_3) \]
\[+12 \left( x_1 + \frac{7}{30} x_2 \right)^2 D_{SSS}(m_2, m_2, m_4) \]
\[+3 \left( x_1 + \frac{9}{10} x_2 \right)^2 D_{SSS}(m_3, m_3, m_4) \]
\[+3 \left( x_1 + \frac{7}{30} x_2 \right)^2 D_{SSS}(m_4, m_4, m_4) \]
\[+ \frac{5}{3} D_{SVV}(m_1, M, M) + \frac{15}{16} D_{SVV}(m_3, M, M) \]
\[+ \frac{25}{28} D_{SVV}(m_4, M, M) + \frac{25}{16} D_{SVV}(m_2, 0, M) \right), \quad (B.1)\]

where the masses are

\[M^2 = \frac{5}{12} v^2,\]
\[m_1^2 = y + \left( x_1 + \frac{2}{5} x_2 \right) v^2, \quad m_2^2 = y + \left( x_1 + \frac{7}{30} x_2 \right) v^2,\]
\[m_3^2 = y + \left( x_1 + \frac{9}{10} x_2 \right) v^2, \quad m_4^2 = y + \left( 3x_1 + \frac{7}{10} x_2 \right) v^2. \quad (B.2)\]

When the shift is made to the direction $\Phi \mapsto \Phi + v \tau_2$, the symmetry is broken down to SU(4)×U(1) and the effective potential becomes

\[V(v) = \frac{1}{2} y v^2 + \left( \frac{1}{4} x_1 + \frac{13}{80} x_2 \right) v^4 \]
\[+ 15 C_S(m_1) + 8 C_S(m_2) + C_S(m_3) + 8 C_V(M) \]
\[- \left( \frac{255}{4} x_1 + \frac{435}{16} x_2 \right) D_{SS}(m_1, m_1) - (20 x_1 + 10 x_2) D_{SS}(m_2, m_2) \]
\[- \left( \frac{3}{4} x_1 + \frac{39}{80} x_2 \right) D_{SS}(m_3, m_3) - (60 x_1 + 15 x_2) D_{SS}(m_4, m_2) \]
\[- \left( \frac{15}{2} x_1 + \frac{9}{8} x_2 \right) D_{SS}(m_1, m_3) - \left( 4x_1 + \frac{13}{5} x_2 \right) D_{SS}(m_2, m_3) \]
\[-15 D_{SV}(m_1, 0) - 5 D_{SV}(m_2, 0) - \frac{15}{4} D_{SV}(m_1, M) \]
\[-5 D_{SV}(m_2, M) - \frac{5}{4} D_{SV}(m_3, M) \]
\[- \frac{15}{2} D_{VV}(0, 0) - 5 D_{VV}(M, 0) - \frac{5}{2} D_{VV}(M, M) \]
\[-15 D_{SSV}(m_1, m_1, 0) - 5 D_{SSV}(m_2, m_2, 0) \]

\[24\]
\[- \frac{15}{2} D_{SSV}(m_1, m_2, M) - \frac{5}{2} D_{SSV}(m_2, m_3, M) - 5D_{VVV}(0, 0, 0) - 5D_{VVV}(0, M, M) - 40D_{\eta\eta V}(0) - 20D_{\eta\eta V}(M) - v^2 \left( \frac{27}{8} x_2^2 D_{SSS}(m_1, m_1, m_1) + 15 \left( x_1 + \frac{3}{20} x_2 \right)^2 D_{SSS}(m_1, m_1, m_3) \right. \\
\left. + \frac{3}{2} x_2^2 D_{SSS}(m_1, m_2, m_2) + 8 \left( x_1 + \frac{13}{20} x_2 \right)^2 D_{SSS}(m_2, m_2, m_3) \right. \\
\left. + 3 \left( x_1 + \frac{13}{20} x_2 \right)^2 D_{SSS}(m_3, m_3, m_3) \right. \\
\left. + \frac{75}{32} D_{SVV}(m_1, M, M) + \frac{25}{16} D_{SVV}(m_2, 0, M) \right. \\
\left. + \frac{25}{32} D_{SVV}(m_3, M, M) \right),
\tag{B.3}
\end{align*}

where the masses are
\[ M^2 = \frac{5}{8} v^2, \quad m_1^2 = y + \left( x_1 + \frac{3}{20} x_2 \right) v^2, \]
\[ m_2^2 = y + \left( x_1 + \frac{13}{20} x_2 \right) v^2, \quad m_3^2 = y + \left( 3 x_1 + \frac{39}{20} x_2 \right) v^2. \tag{B.4} \]

The explicit expressions for the \( C \) and \( D \) functions are
\[ C_S(m) = -\frac{m^3}{12\pi}, \]
\[ C_V(m) = -\frac{m^3}{6\pi}, \]
\[ D_{SSS}(m_1, m_2, m_3) = \frac{1}{16\pi^2} \left( \frac{1}{2} + \log \frac{\mu}{m_1 + m_2 + m_3} \right), \]
\[ D_{SS}(m_1, m_2) = -\frac{m_1 m_2}{16\pi^2}, \]
\[ D_{SV}(m_1, m_2) = 4D_{SS}(m_1, m_2), \]
\[ D_{VV}(m_1, m_2) = \frac{16}{3} D_{SS}(m_1, m_2), \]
\[ D_{SSV}(m_1, m_2, M) = (M^2 - 2m_1^2 - 2m_2^2)D_{SSS}(m_1, m_2, M) \]
\[ + \frac{(m_1^2 - m_2^2)^2}{M^2} (D_{SSS}(m_1, m_2, M) - D_{SSS}(m_1, m_2, 0)) \]
\[ + \frac{1}{16\pi^2 M} ((m_1 + m_2)(M^2 + (m_1 - m_2)^2) \]
\[ - M m_1 m_2), \]
\[ D_{\eta\eta V}(M) = -\frac{M^2}{4} D_{SSS}(M, 0, 0) \]

\[ D_{VVV}(0, M, M) = \frac{1}{16\pi^2 M^2} \left( \frac{29}{12} + 8 \log 2 - 10 \log \frac{\mu}{M} \right) \]

\[ D_{SVV}(m, 0, M) = \frac{1}{16\pi^2} \left( \frac{3}{2} - \frac{2m}{M} + \frac{2m^2}{M^2} \log \frac{m + M}{m} + 6 \log \frac{\mu}{m + M} \right), \]

\[ D_{SVV}(m, M, M) = \frac{1}{16\pi^2} \left( 3 - \frac{2m}{M} - \frac{m^2}{M^2} - \frac{m^4}{M^4} \log \frac{m}{m + 2M} + 2 \frac{m^2 - M^2}{M^4} \log \frac{m + M}{m + 2M} + 6 \log \frac{\mu}{m + 2M} \right). \]

(B.5)

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