On potentials of Itô’s Processes with Drift in $L_{d+1}$

N. V. Krylov

Received: 17 March 2021 / Accepted: 3 November 2021 / Published online: 11 January 2022
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract
This paper is a natural continuation of Krylov (2020), where strong Markov processes are constructed in the time inhomogeneous setting with Borel measurable uniformly bounded and uniformly nondegenerate diffusion and drift in $L_{d+1}(\mathbb{R}^{d+1})$. Here we study some properties of these processes such as the probability to pass through narrow tubes, higher summability of Green’s functions, and so on. The results seem to be new even if the diffusion is constant.

Keywords Itô’s equations with singular drift · Potentials of diffusion processes

Mathematics Subject Classification (2010) 60H10 · 60J60

1 Introduction

Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, \ldots, x^d), \ d \geq 2$. Fix some $p_0, q_0 \in [1, \infty)$ such that
\[
\frac{d}{p_0} + \frac{1}{q_0} = 1. \tag{1.1}
\]

It is proved in [12] that Itô’s stochastic equations of the form
\[
x_s = x + \int_0^s \sigma(t + r, x_r) \, dw_r + \int_0^s b(t + r, x_r) \, dr \tag{1.2}
\]
admit weak solutions, where $w_s$ is a $d$-dimensional Wiener process, $\sigma$ is a uniformly non-degenerate, bounded, Borel function with values in the set of symmetric $d \times d$ matrices, $b$ is a Borel measurable $\mathbb{R}^d$-valued function given on $\mathbb{R}^{d+1} = (-\infty, \infty) \times \mathbb{R}^{d}$ such that
\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |b(t, x)|^{p_0} \, dx \right)^{q_0/p_0} \, dt < \infty \tag{1.3}
\]
if $p_0 \geq q_0$ or
\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} |b(t, x)|^{q_0} \, dt \right)^{p_0/q_0} \, dx < \infty \tag{1.4}
\]

$\# \quad \# \quad \#$

N. V. Krylov
nkrylov@umn.edu

1 University of Minnesota, 127 Vincent Hall, Minneapolis, MN 55455, USA
if $p_0 \leq q_0$. Observe that the case $p_0 = q_0 = d + 1$ is not excluded and in this case the condition becomes $b \in L_{d+1}([\mathbb{R}^{d+1}])$.

The goal of this article is to study some properties of such solutions or Markov processes whose trajectories are solutions of Eq. 1.2. In particular, in Section 2 for more or less general processes of diffusion type we derive several estimates of Aleksandrov type by using Lebesgue spaces with mixed norms like (in case $t = 0, x = 0$ in Eq. 1.2)

$$E \int_0^\infty e^{-t} f(t, x_t) \, dt \leq N \|f\|_{L_{p,q}},$$

provided that $d/p + 1/q \leq 1$.

We also show that expected time when the process $(t, x_t)$, starting at $(0, 0)$, exits from $[0, R^2] \times \{x : |x| < R\}$ is comparable to $R^2$. This and sharpening of the estimates of Section 2 achieved in Section 3 play a crucial role in Section 4 where we show a significant improvement of the Aleksandrov estimates in the direction of lowering the powers of summability of $f$ in Eq. 1.5 to $d_0/p + 1/q \leq 1$ with $d_0 < d$. Time homogeneous versions of these estimates are also given.

In the same Section 2 we give some estimates of the distribution of the exit times from cylinders, which are heavily used in the sequel. We also prove that, for any $0 \leq s \leq t < \infty$,

$$E \sup_{r \in [s, t]} |x_r - x_s|^n \leq N(|t - s|^{n/2} + |t - s|^n).$$

It is to be said that instead of Eqs. 1.3 or 1.4, which are not invariant under self-similar transformations, we impose a slightly stronger assumption on $b$, that is invariant.

As we mentioned above, in Section 4 we improve the results of Section 2 in what concerns the Aleksandrov estimates, which allows us to prove Itô’s formula for $W^{1,2}_{p,q}(Q)$-functions if $d_0/p + 1/q \leq 1$.

In Section 5 we discuss some applications of our results to the theory of parabolic equations. Itô’s formula is the main instrument here. We prove the qualitative form of the parabolic Aleksandrov maximum principle for $u \in W^{1,2}_{p,q}$ with $d_0/p + 1/q \leq 1$ (Theorem 5.1). In the case of bounded $b$ and $p = q = d + 1$ in the parabolic case and $p = d$ in the elliptic case the result of Theorem 5.1 is “classical” (about 50 year old). It was generalized by Cabrè [2], Escauriaza [4], and Fok [6] in the elliptic case when $p < d$ (close to $d$) again when $b$ is bounded. In [3] a parabolic version of these results, extending some earlier results by Wang, are given for $L_p$-viscosity solutions with $p < d + 1$ (close to $d + 1$) when $b$ is bounded. However, it is worth noting that in the elliptic case it may happen that $b \notin L_d$ and the equation is still solvable (see, for instance, [8]). In our situation we have some freedom in choosing $p, q$ and $b \in L_{p_0,q_0}$, but we only treat true solutions. Theorem 5.1 covers Theorem 2.4 of [3] on the account of having mixed norms and $b \in L_{p_0,q_0}$.

One more result in this section is aimed at applications to the theory of fully nonlinear parabolic equations with lower order coefficients in $L_{p,q}$. We prove a theorem allowing one to pass to the limit under the sign of fully nonlinear operator when the arguments (functions) converge only weakly and give its application to linear equations.

It is worth mentioning that there is a vast literature about stochastic equations when Eq. 1.1 is replaced with $d/p + 2/q \leq 1$. This condition is much stronger than ours. Still we refer the reader to the recent articles [1, 14, 18] and the references therein for the discussion of many powerful and exciting results obtained under this stronger condition. There are also many papers when this condition is considerably relaxed on the account of imposing various regularity conditions on $\sigma$ and $b$ and/or considering random initial conditions with bounded density, see, for instance, [19, 20] and the references therein. Restricting the situation to
the one when $\sigma$ and $b$ are independent of time allows one to relax the above conditions significantly further, see, for instance, [8] and the references therein.

As we have mentioned we apply our results to elliptic and parabolic equations. Observe that they are in nondivergence form. These have almost nothing to do with similar estimates for divergence form equations.

Introduce

\[ BR = \{ x \in \mathbb{R}^d : |x| < R \}, \quad B_R(x) = x + B_R, \quad C_{T,R} = [0, T) \times B_R, \]
\[ C_{T,R}(t, x) = (t, x) + C_{T,R}, \quad C_R(t, x) = C_{R^2,R}(t, x), \quad C_R = C_R(0, 0), \]
\[ D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_iD_j, \quad \partial_t = \frac{\partial}{\partial t}. \]

For $p, q \in [1, \infty]$ and domains $Q \subset \mathbb{R}^{d+1}$ we introduce the space $L_{p,q}(Q)$ as the space of Borel functions on $Q$ such that

\[ \| f \|_{L_{p,q}(Q)}^q := \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} I_Q(t, x)|f(t, x)|^p \, dx \right)^{q/p} \, dt < \infty \]
if $p \geq q$ or

\[ \| f \|_{L_{p,q}(Q)}^p := \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} I_Q(t, x)|f(t, x)|^q \, dt \right)^{p/q} \, dx < \infty \]
if $p \leq q$ with natural interpretation of these definitions if $p = \infty$ or $q = \infty$. If $Q = \mathbb{R}^{d+1}$, we drop $Q$ in the above notation. Observe that $p$ is associated with $x$ and $q$ with $t$ and the interior integral is always elevated to the power $\leq 1$. In case $p = q = d + 1$ we abbreviate $L_{d+1,d+1} = L_{d+1}$. For the set of functions on $\mathbb{R}^d$ summable to the $p$th power we use the notation $L_p(\mathbb{R}^d)$. Of course, if we write, say $f \in L_{p,q,\text{loc}}$ we mean that $f \in L_{p,q}(Q)$ for any bounded $Q$.

If $\Gamma$ is a measurable subset of $\mathbb{R}^{d+1}$ we denote by $|\Gamma|$ its Lebesgue measure. The same notation is used for measurable subsets of $\mathbb{R}^d$ with $d$-dimensional Lebesgue measure. We hope that it will be clear from the context which Lebesgue measure is used. If $\Gamma$ is a measurable subset of $\mathbb{R}^{d+1}$ and $f$ is a function on $\Gamma$ we denote

\[ \int_{\Gamma} f \, dx \, dt = \frac{1}{|\Gamma|} \int \Gamma f \, dx \, dt. \]

In case $f$ is a function on a measurable subset $\Gamma$ of $\mathbb{R}^d$ we set

\[ \int_{\Gamma} f \, dx = \frac{1}{|\Gamma|} \int \Gamma f \, dx. \]

Throughout the article $\tilde{\tilde{\epsilon}}$ is a fixed constant, $\tilde{\tilde{\epsilon}} \in (0, \infty)$. Various constants are denoted by the same symbol $N$ even though the constants may be different. Usually, $N$ comes without arguments, but if we write, say $N = N(a, b, ...)$, we mean that $N$ depends only on the contents of the parentheses. These constants even denoted the same way as $N(a, b, ...)$ may be all different. There is, however, one exception, $\tilde{\tilde{N}} = \tilde{\tilde{N}}(d, p_0, \delta)$ always stands for a constant from Theorem 2.3.

2 The Case of General Diffusion Type Processes with Drift in $L_{p_0,q_0}$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\mathcal{F}_t, t \geq 0$, be an increasing family of complete $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$, let $w_t$ be an $\mathbb{R}^d$-valued Wiener process relative to $\mathcal{F}_t$. Fix $\delta \in (0, 1)$ and denote by $\mathbb{S}_\delta$ the set of $d \times d$ symmetric matrices whose eigenvalues are
between $\delta$ and $\delta^{-1}$. Assume that on $\Omega \times [0, \infty)$ we are given a measurable $\mathbb{S}_\delta$-valued $\mathcal{F}_t$-adapted function $\sigma_t = \sigma_t(\omega)$ and a measurable $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted function $b_t$, such that
\[ \int_0^T |b_t| \, dt < \infty \]
for any $T \in (0, \infty)$ and $\omega$. Define
\[ x_t = \int_0^t \sigma_s \, dw_s + \int_0^t b_s \, ds. \tag{2.1} \]

**Assumption 2.1** We are given a function $h \in L_{p_0,q_0}, \text{loc}$ such that
\[ |b_t| \leq h(t,x_t). \]
Furthermore, there exists a bounded nondecreasing function $\tilde{b}_R$, $R \in (0, \infty)$, such that for any $(t,x) \in \mathbb{R}^{d+1}$ and $R \in (0, \infty)$ we have
\[ \|h\|_{L_{p_0,q_0}(C_R(t,x))}^{q_0} \leq \tilde{b}_R R. \tag{2.2} \]

Observe that if $p_0 = q_0 = d + 1$ and $h \in L_{d+2}$ (a typical case in the theory of parabolic equations), then Eq. 2.2 is satisfied with $\tilde{b}_R = \tilde{b} = \|h\|_{L_{d+2}}^{d+1}$ because by Hölder’s inequality
\[ \|h\|_{L_{d+1}(C_R(t,x))}^{d+1} \leq \tilde{b}_R R. \]
On the other hand, it may happen that Eq. 2.2 is satisfied with $p_0 = q_0 = d + 1$ but $h \notin L_{d+2,\text{loc}}$.

**Example 2.1** Take $\alpha \in (0,d)$, $\beta \in (0,1)$ such that $\alpha + 2\beta = d + 1$ and consider the function $g(t,x) = |t|^{-\beta}|x|^{-\alpha}$. Observe that
\[ \int_{C_R(t,x)} g(s,y) \, dy \, ds = R \int_{C_{1}(t',x')} g(s,y) \, dy \, ds, \]
where $t' = t/R^2$, $x' = x/R$. Obviously, the last integral is a bounded function of $(t',x')$. Hence, the function $h = g^{1/(d+1)}$ satisfies (2.2) with $p_0 = q_0 = d + 1$. As is easy to see for any $p > d + 1$ one can find $\alpha$ and $\beta$ above such that $h \notin L_{p,\text{loc}}$.

Note that if $h$ is bounded and has compact support, Eq. 2.2 is certainly satisfied. A condition very similar to Eq. 2.2 appeared before in [15].

The following is a particular case of Theorem 4.5 of [12]. In Theorem 2.1 $p = \infty$ is allowed (and then $q = 1$).

**Theorem 2.1** Suppose that Assumption 2.1 is satisfied and
\[ p, q \in [1, \infty], \quad \frac{d}{p} + \frac{1}{q} = 1. \tag{2.3} \]
Then for any Borel $f \geq 0$ and stopping time $\gamma$
\[ E \int_0^\gamma f(t,x_t) \, dt \leq N(d,p_0,\delta) \left( A + \|h\|_{L_{p_0,q_0}}^{2q_0} \right)^{d/(2p)} \|f\|_{L_{p,q}}, \tag{2.4} \]
where $A = E_\gamma$.

Our first goal is to estimate $A$ and eliminate it from Eq. 2.4. For $T, R \in (0, \infty)$ introduce
\[ \tau_{T,R}(x) = \inf \{ t \geq 0 : (t, x + x_t) \notin C_{T,R} \}, \quad \tau_R(x) = \tau_{R^2,R}(x), \quad \tau_R = \tau_{R}(0). \]
Lemma 2.2. We have
\[ A := E \tau_R(x) \leq R^2, \]  
and consequently, assuming (2.3), for any Borel nonnegative \( f \)
\[ E \int_0^{\tau_R(x)} f(t, x_t) \, dt \leq N(d, p_0, \delta)(1 + \tilde{b}_R)^{d/p} R^{d/p} \| f \|_{L_{p,q}}, \]  
(2.6)

Proof. Obviously, \( \tau_R \leq R^2 \) and Eq. 2.5 follows. After that Eq. 2.6 follows from Eqs. 2.2 and 2.4. The lemma is proved.

Estimate (2.5) says that in the typical case of nondegenerate diffusion \( \tau_R \) is of order not more than \( R^2 \). A very important fact, which is implied by Corollary 2.7 and will allow us to obtain higher order summability of Green’s functions, is that \( \tau_R \) is of order not less than \( R^2 \). To show this we need an additional assumption appearing after the following result, in which
\[ \tau'_R(x) = \inf\{t \geq 0 : x + x_t \notin B_R\}, \quad \gamma_R(x) = \inf\{t \geq 0 : x + x_t \in \bar{B}_R\}. \]  
(2.7)

Theorem 2.3. There are constants \( \bar{\xi} = \bar{\xi}(d, \delta) \in (0, 1) \) and \( \bar{N} = \bar{N}(d, p_0, \delta) \) continuously depending on \( \delta \) such that if, for an \( R \in (0, \infty) \), we have
\[ \bar{N} \bar{b}_R \leq 1, \]  
(2.8)

then for \( |x| \leq R \)
\[ P(\tau_R(x) = R^2) \leq 1 - \bar{\xi}, \quad P(\tau_R = R^2) \geq \bar{\xi}. \]  
(2.9)

Moreover for \( n = 1, 2, ... \) and \( |x| \leq R \)
\[ P(\tau'_R(x) \geq nR^2) = P(\tau_{nR^2, R}(x) = nR^2) \leq (1 - \bar{\xi})^n, \]  
(2.10)

so that \( E \tau'_R(x) \leq N(d, \delta)R^2 \) and
\[ I := E \int_0^{\tau'_R(x)} h(t, x_t) \, dt \leq N(d, p_0, \delta)(1 + \tilde{b}_R)^{d/p} R^{1/q_0} \]  
(2.11)

Furthermore, the probability starting from a point in the closed ball of radius \( R/16 \) with center in \( \bar{B}_{R/2} \) to reach the ball \( \bar{B}_{R/16} \) before exiting from \( B_R \) is bigger than \( \bar{\xi} \): for any \( x, y \) with \( |y| \leq R/2 \) and \( |x - y| \leq R/16 \)
\[ P(\tau'_R(x) > \gamma R/16(x)) \geq \bar{\xi}. \]  
(2.12)

Remark 2.1. The last statement of the theorem might look awkward because it just says that for any \( x \in \bar{B}_{R/16} \) estimate (2.12) holds. Mentioning \( y \) seems superfluous. The goal of introducing \( y \) is that Eq. 2.12 shows that starting from any point in \( B_{R/16}(y) \) the process reaches \( \bar{B}_{R/16} \) with positive probability without exiting from \( B_R \), thus “moves in the direction” of \(-y\), no matter where in \( B_{R/16}(y) \) the starting point is.

We first prove an auxiliary result, in which
\[ m_t = \int_0^t \sigma_x \, dw, \quad a_t = (1/2) \sigma_t \sigma_t^*. \]

Lemma 2.4. (i) There exists \( \kappa = \kappa(d) > 0 \) such that for
\[ \psi(x, t) = R^{-4} \left( R^2 - 4|x|^2 \right)^2 \phi_t, \quad \phi_t = \exp \int_0^t \kappa R^{-2} \, tr a_s \, ds \]
the process \( \psi(m_t, t) \) is a local submartingale.
Take a \( \xi \in C_0^\infty(\mathbb{R}) \) such that it is even, nonnegative, and decreasing on \((0, \infty)\). For \( T \in (0, \infty) \) and \( x \in \mathbb{R} \) and \( t \leq T \) define \( u(t, x) = E\xi(x + w_{t-}^1) \). Also take \( x \in \mathbb{R}_d \) and set
\[
 r_t = \frac{(x + m_t, a_t(x + m_t))}{|x + m_t|^2} \quad (0/0 := 1), \quad \eta_t = 2 \int_0^t r_s ds.
\]
Then the process \( u(\eta_t, |x + m_t|) \) is a supermartingale before \( \eta_t \) reaches \( T \), in particular, on \([0, \delta^2 T)\).

(iii) There exists \( \alpha = \alpha(d, \delta) > 1 \) such that for \( u(x) = |x|^{-\alpha} \) and any nonzero \( x \in \mathbb{R}_d \) the process \( u(|x + m_t|) \) is a submartingale before \( x + m_t \) hits the origin.

Proof. (i) It is easy to see that for a \( \kappa = \kappa(d) > 0 \) we have \( \kappa \mu^2 - 16\mu + 32d^{-1}(1 - \mu) \geq 0 \) for all \( \mu \), which implies that for all \( \lambda \)
\[
\kappa(1 - 4\lambda^2)^2 - 16(1 - 4\lambda^2) + 128d^{-1}\lambda^2 \geq 0. \tag{2.13}
\]
It follows that
\[
R^4 \phi_{t-}^{-1} d\psi(m_t, t) = \kappa(1 - 4|m_t|^2)^2 R^{-2} \tau a_t dt - 8(R^2 - 4|m_t|^2)(2m_t dm_t + 2\tau a_t dt) + 128(m_t, a_t m_t) dt \geq dM_t,
\]
where \( M_t \) is a local martingale. This proves (i).

(ii) Observe that \( u \) is smooth, even in \( x \), and satisfies \( \partial_t u + (1/2)\Delta u = 0 \). Furthermore, as is easy to see \( u'(t, x) \leq 0 \) for \( x \geq 0 \). It follows by Itô’s formula that before \( \eta_t \) reaches \( T \) we have (dropping obvious values of some arguments)
\[
du(\eta_t, |x + m_t|) = r_t(2\partial_t u + u'') dt + \frac{u'}{|x + m_t|}(\tau a_t - r_t) dt + dM_t,
\]
where \( M_t \) is a stochastic integral. Here the second term with \( dt \) is negative since \( u' \leq 0 \), and this proves that \( u(\eta_t, |x + m_t|) \) is a local supermartingale. Since it is nonnegative, it is a supermartingale.

Assertion (iii) is proved by simple application of Itô’s formula (see, for instance, the proof of Lemma 2.2 in [17]). The lemma is proved.

Proof of Theorem 2.3 . Notice that by Eq. 2.6
\[
E \int_0^\tau_R |h(t, x_t)| dt \leq N(1 + \bar{b}_R) d/p_0 \bar{r}^{1/q_0} \cdot R. \tag{2.14}
\]
Furthermore, observe that for \( \gamma \) defined as the minimum of \( R^2 \) and the first exit time of \( m_t \) from \( B_{R/2} \) it holds that \( \phi_{\gamma} \leq e^{x d/\delta} \). Hence, by Lemma 2.4 (i)
\[
1 = \psi(0, 0) \leq E\psi(m_\gamma, \gamma) \leq e^{x d/\delta^2} P(\sup_{t \leq R^2} |m_t| < R/2)
\]
and, since \( \tau_R \leq R^2 \),
\[
P(\sup_{t \leq \tau_R} |m_t| < R/2) \geq 2\tilde{\xi}(d, \delta) > 0.
\]
Also note that
\[
P(\tau_R < R^2) \leq P(\int_0^\tau_R |h(t, x_t)| dt \geq R/2) + P(\sup_{t \leq \tau_R} |m_t| \geq R/2).
\]
Therefore, we get the right estimate in Eq. 2.9 for \( 2N(1 + \bar{b}_R) d/p_0 \bar{r}^{1/q_0} \leq \tilde{\xi} \).

On the other hand, take \( \xi \) such that \( \xi(x) = \eta(x/R) \), where \( \eta(x) = 1 \) for \( |x| \leq 2 \) and take \( T = \delta^2 R^2 \), in which case \( u(0, x) \leq u(0, 0) < 1 \) and \( u(0, 0) \) depends only on \( \delta \) (and
Among the previous definitions, 

$$P(\sup_{t \leq R^2} |x + m_t| < 2R) \leq P(|x + m_{\mu}| < 2R) \leq Eu(\eta_{\mu}, |x + m_{\mu}|) \leq u(0, x) \leq u(0, 0).$$

Hence,

$$P(\tau_R(x) < R^2) \geq P(\int_{\tau_R}^{\tau'_{R}(x)} |h(t, x_t)| dt \leq R/2, \sup_{t \leq R^2} |x + m_t| \leq 2R) \geq 1 - P(\int_{\tau_R}^{\tau'_{R}(x)} |h(t, x_t)| dt \geq R/2) - P(\sup_{t \leq R^2} |x + m_t| \leq 2R)$$

and it is clear how to adjust (2.8) to get both inequalities in Eq. 2.9 with perhaps different from the above one. Estimate (2.10) is obtained by iterations.

To prove (2.11) come back to Eq. 2.14 and denote by \(J\) its right-hand side. Then use the conditional version of Eq. 2.14 to see that

$$I = \infty \sum_{n=1}^{\infty} EI_{\tau(n-1)R^2} \tau'_{R}(x) R^2, R(x) > \tau'_{R}(x) = \sum_{n=1}^{\infty} (1 - \xi)^{n-1}.$$}

This yields (2.11).

To prove (2.12) use assertion (iii) of Lemma 2.4 to conclude that

$$du(|x + x_t|) \geq b'_t \nu_t u(|x + x_t|) dt + dM_t,$$

where \(M_t\) is a local martingale. For our \(x\), on the time interval, which we denote \((0, v)\), when \(x + x_t \in B_{R} \setminus \bar{B}_{R/16}\) we have \(|Du(|x + x_t|) t \leq N(d, \alpha) R^{-\alpha - 1}\). Furthermore, at starting point \(u(x) \geq (9R/16) - ^{\alpha}\). Consequently and by Eq. 2.11

$$(9R/16)^{-\alpha} \leq NR^{-\alpha - 1} E \int_{0}^{\tau'_{R}(x)} h(t, x_t) dt + P(v = \tau'_{R}(x)) R^{-\alpha} + P(v = \gamma_{R/16}(x)) (R/16)^{-\alpha},$$

$$(16/9)^{\alpha} \leq N_1(1 + \bar{b}_R)^d/p_{0} \tilde{b}_{R}^{1/q_0} + 1 - P(\tau'_{R}(x) > \gamma_{R/16}(x)) + 16^\alpha P(\tau'_{R}(x) > \gamma_{R/16}(x)).$$

It follows easily that Eq. 2.12 holds with \(\xi\) perhaps different from the above ones, once a relation like (2.8) holds. The continuity of \(\hat{N}\) in Eq. 2.8 and of \(\xi(d, \delta)\) with respect to \(\delta\) is established by inspecting the above proof. The theorem is proved.

**Assumption 2.2** There exists \(R \in (0, \infty]\) such that

$$\hat{N}(d, p_0, \delta) \tilde{b}_R < 1. \quad (2.15)$$

This assumption as well as Assumption 2.1 is supposed to hold throughout the article. Set

$$\lambda = R^{-2}.$$
Corollary 2.5 For \( \mu \in [0, 1] \) and \( R \leq R \) we have
\[
Ee^{-\mu R^{-2} \tau_R} \leq e^{-\mu \bar{\xi}/2}.
\] (2.16)

Indeed, the derivative with respect to \( \mu \) of the left-hand side of Eq. 2.16 is
\[
-R^{-2} E \tau_R e^{-\mu R^{-2} \tau_R} \leq -e^{-\mu} R^{-2} P(\tau_R = R^2) \leq -e^{-\mu \bar{\xi}},
\]
where the last inequality follows from Eq. 2.9. By integrating we find
\[
Ee^{-\mu R^{-2} \tau_R} - 1 \leq (e^{-\mu} - 1) \bar{\xi},
\]
which after using
\[
e^{-\mu} - 1 \leq -\mu/2, \quad 1 - \mu \bar{\xi} / 2 \leq e^{-\mu \bar{\xi} / 2}
\]
leads to Eq. 2.16.

Theorem 2.6 For any \( \lambda, R > 0 \) we have
\[
E e^{-\lambda R^{-2} \tau_R} \leq e^{\bar{\xi}/2} e^{-\sqrt{\lambda} R \bar{\xi}/2} = \begin{cases} e^{\bar{\xi}/2} e^{-\sqrt{\lambda} R \bar{\xi}/2} & \text{if } \lambda \geq \hat{\lambda} \\ e^{\bar{\xi}/2} e^{-\lambda R \bar{\xi}/2} & \text{if } \lambda \leq \hat{\lambda}, \end{cases}
\] (2.17)

where \( \hat{\lambda} = \lambda \min(1, \lambda / \bar{\lambda}). \)

In particular, for any \( R > 0 \) and \( t \leq RR \bar{\xi}/4 \) we have
\[
P(\tau_R \leq t) \leq e^{\bar{\xi}/2} \exp \left( -\frac{\bar{\xi}^2 R^2}{16t} \right).
\] (2.18)

Proof. Take an integer \( n \geq 1 \), introduce \( \tau^k, k = 1, \ldots, n \), as the first exit time of \((t, x_t)\) from \( CR/n(\tau^k-1, x_{\tau^k-1}) \) after \( \tau^k-1 \) \((\tau^0 := 0)\). If
\[
\lambda \leq n^2 / R^2, \quad R/n \leq R, \quad \text{that is } \quad n \geq R \sqrt{\lambda} \max(1, / \sqrt{\lambda} R),
\]
then by Eq. 2.16 with \( \mu = (R/n)^2 \lambda \) we have
\[
E \left( e^{-\lambda (\tau^k-1)} | \mathcal{F}_{\tau^k} \right) \leq e^{-(R/n)^2 \lambda \bar{\xi}/2}.
\]

Hence,
\[
E e^{-\lambda \tau_R} \leq E \prod_{k=1}^n e^{-\lambda (\tau^k-1)} \leq e^{-R^2n^{-1} \lambda \bar{\xi}/2}.
\] (2.19)

If \( \lambda R^2 \geq 1 \), we take \( n = \lceil R \sqrt{\lambda} \rceil \) and use that \( R^2n^{-1} \lambda \geq R \sqrt{\lambda} \lambda - 1 \). If \( \lambda R^2 \leq 1 \), we take \( n = \lceil R / R \rceil \) and use that \( R^2n^{-1} \lambda \geq RRR \lambda - 1 \). This proves (2.17).

To prove (2.18) observe that if \( \lambda \geq \hat{\lambda} \)
\[
P(\tau_R \leq t) = P(\exp(-\lambda \tau_R) \geq \exp(-\lambda t)) \leq \exp(\bar{\xi}/2 + \lambda t - \sqrt{\lambda} R \bar{\xi} / 2).
\]

For \( \sqrt{\lambda} = R \bar{\xi}/(4t) \) we get (2.18) provided \( R \bar{\xi}/(4t) \geq R^{-1} \). The theorem is proved.

Recall that \( R \) is fixed throughout the article.

Corollary 2.7 Let \( \Lambda \in (0, \infty) \). Then there is a constant \( N = N(R, \bar{R}, \Lambda, \bar{\xi}) \) such that for any \( R \in (0, \bar{R}), \lambda \in [0, \Lambda] \)
\[
NE \tau_R \geq R^2, \quad NE \int_0^{\tau_R} e^{-\lambda t} dt \geq R^2.
\] (2.20)
Indeed, for any $v \leq \frac{R\bar{\xi}}{4R}$ and $R \in (0, \bar{R})$ we have $vR^2 \leq R\frac{R\bar{\xi}}{4}$ so that

$$E\tau_R \geq vR^2 P(\tau_R > vR^2) \geq vR^2 \left(1 - e^{\xi/2} \exp\left(-\frac{\xi^2}{16v}\right)\right).$$

$$E \int_0^{\tau_R} e^{-\lambda t} dt = \lambda^{-1} E(1 - e^{-\lambda\tau_R}) \geq \lambda^{-1} E I_{\tau_R > vR^2}(1 - e^{-\lambda\nu R^2})$$

$$= \lambda^{-1} P(\tau_R > vR^2)(1 - e^{-\lambda\nu R^2})$$

$$\geq \lambda^{-1} \left(1 - e^{\xi/2} \exp\left(-\frac{\xi^2}{16v}\right)\right)(1 - e^{-\lambda\nu R^2}),$$

which yields (2.20) for an appropriate small $v = v(R, \bar{R}, \lambda, \bar{\xi}) > 0$.

**Corollary 2.8** For any $n > 0$ and $0 \leq s \leq t$ we have

$$E \sup_{r \in [s, t]} |x_r - x_s|^n \leq N(|t - s|^{n/2} + |t - s|^n),$$

(2.21)

where $N = N(n, R, \bar{\xi})$.

Indeed, clearly we may assume that $s = 0$. Then for $v_0 = 4(R\bar{\xi})^{-1}$ and $\mu \geq t v_0$ we have $t \leq \mu \frac{R\bar{\xi}}{4}$ and

$$P(\sup_{r \leq t} |x_r| \geq \mu) \leq P(\tau_\mu \leq t) \leq e^{\xi/2} \exp\left(-\frac{\mu^2\xi^2}{16t}\right).$$

Consequently,

$$E \sup_{r \leq t} |x_r|^n = n \int_0^\infty \mu^{n-1} P(\sup_{r \leq t} |x_r| \geq \mu) d\mu \leq n \int_0^{t v_0} \mu^{n-1} d\mu$$

$$+ n e^{\xi/2} \int_0^\infty \mu^{n-1} \exp\left(-\frac{\mu^2\xi^2}{16t}\right) d\mu,$$

and the result follows.

A few more general results are related to going through a long “sausage”.

**Theorem 2.9** Let $R \in (0, R]$, $x, y \in \mathbb{R}^d$ and $16|x - y| \geq 3R$. For $r > 0$ denote by $S_r(x, y)$ the open convex hull of $B_r(x) \cup B_r(y)$. Then there exist $T_0, T_1$, depending only on $\bar{\xi}$, such that $0 < T_0 < T_1 < \infty$ and the probability $\pi$ that $x + x_t$ will reach $\bar{B}_{R/16}(y)$ before exiting from $S_R(x, y)$ and this will happen on the time interval $[nT_0R^2, nT_1R^2]$ is greater than $\pi^0_n$, where

$$n = \left\lceil \frac{16|x - y| + R}{4R} \right\rceil$$

and $\pi_0 = \bar{\xi}/3$.

**Proof** We may assume that $y = 0$. Introduce $\tau(x)$ as the first time $x + x_t$ reaches $\bar{B}_{R/16}$ and $\gamma(x)$ as the first time it exits from $S_R(x, 0)$. Owing to $16|x| \geq 3R$, we have $n \geq 1$ and we are going to use the induction on $n$ with the induction hypothesis that

$$\left\lceil \frac{16|x| + R}{4R} \right\rceil = n \implies P(\gamma(x) > \tau(x) \in [nT_0R^2, nT_1R^2]) \geq \pi^0_n.$$

If $n = 1$, $3R/16 \leq |x| < 7R/16$ and by Theorem 2.3 we have $P(\tau^R_n(x) > \tau(x)) \geq \bar{\xi}$. Furthermore, in light of Theorem 2.3, there is $T_1 = T_1(\bar{\xi})$ such that $P(\tau^R_{T_1}(x) > T_1R^2) \leq \bar{\xi},$
Using Eq. 2.18 we also see that there is $T_0 = T_0(\xi) < T_1$ such that $P(\tau(x) \leq T_0 R^2) \leq \xi/3$. Hence, $P(\gamma(x) > \tau(x) \in [T_0 R^2, T_1 R^2]) \geq \xi/3 = \pi_0$. This justifies the start of the induction.

Assuming that our hypothesis is true for some $n \geq 1$ suppose that $(n+2)R/4 > |x| + R/16 \geq (n+1)R/4$. In that case, let $z = nRx/(4|x|)$, $\tau_z$ be the first time $x_\tau$ reaches $\bar{B}_{R/16}(z)$, and let $\gamma_z$ be the first time it exits from $S_{R}(x,z)$. As is easy to see

$$P(\gamma(x) > \tau(x) \in [(n+1)T_0 R^2, (n+1)T_1 R^2]) \geq P(\gamma_z > \tau_z \in [T_0 R^2, T_1 R^2], \gamma(x_\tau) > \tau(x_\tau) \in [nT_0 R^2, nT_1 R^2]) = El_{\gamma_z > \tau_z \in [T_0 R^2, T_1 R^2]} P(\gamma(x_\tau) > \tau(x_\tau) \in [nT_0 R^2, nT_1 R^2] | F_{\tau_z}).$$

Observe that on the set $\tau_z < \infty$ we have $nR/4 \leq |x_\tau| + R/16 < (n+1)R/4$, so that, by the conditional version of our induction hypothesis, the conditional probability above is greater than $\pi_0^n$. Then just by shifting the origin to $z$ and using the first part of the proof we obtain our result for $n+1$ in place of $n$. The theorem is proved. \hfill \Box

**Remark 2.2** Observe that, for any fixed $x$, $y$, the interval $[nT_0 R^2, nT_1 R^2]$ is as close to zero as we wish if we choose $R$ small enough. Then, of course, the corresponding probability will be quite small but $> 0$.

**Corollary 2.10** Let $R \leq \bar{R}$, $\kappa \in [0, 1)$, and $|x| \leq \kappa R$. Then for any $T > 0$

$$NP(\tau'_R(x) > T) \geq e^{-vT/(1-\kappa)R^2}, \quad (2.22)$$

where $N$ and $v > 0$ depend only on $\xi$.

Indeed, passing from $B_R$ to $B_{(1-\kappa)R}(x)$ shows that we may assume that $x = 0$ and $\kappa = 0$. In that case, consider meandering of $x_\tau$ between $\bar{B}_{R/16}$ and $\partial B_{R/16}(y)$ where $|y| = R/4$ without exiting from $B_R$. As is easy to deduce from Theorem 2.9, given that the $n$th loop happened, with probability $\pi_0^n$ the next loop will occur and take at least $4R^2T_0$ of time. Thus the $n$th loop will happen and will take at least $4nR^2T_0$ of time with probability at least $\pi_0^n$. It follows that, for any $n$,

$$P(\tau'_R \geq 4nR^2T_0) \geq \pi_0^n,$$

and this yields (2.22) for $x = 0$ and $\kappa = 0$.

The following complements Corollary 2.10.

**Corollary 2.11** Let $R \in (0, \bar{R})$. Then there exists a constant $N$, depending only on $\xi$, $\bar{R}$, $R$, such that, for any $T > 0$,

$$P(\tau'_R > T) \leq Ne^{-T/(NR^2)}.$$ 

Indeed, if $R \leq \bar{R}$, the result follows from Theorem 2.3. For $R \geq \bar{R}$, take a point $y$ such that $|y| = \bar{R} + \bar{R}$, for any $x$ define $\gamma(x)$ as the first time $x + x_\tau$ hits $\bar{B}_{R/16}(y)$, and set

$$n_0 = \left\lceil \frac{16(R + 2\bar{R}) + R}{4\bar{R}} \right\rceil.$$ 

It follows from Theorem 2.9 that for any $x \in B_R$

$$P(\tau'_R(x) \leq n_0T_1 R^2) \geq P(\gamma(x) \leq n_0T_1 R^2) \geq \pi_0^{n_0}.$$ 

Hence

$$P(\tau'_R(x) > n_0T_1 R^2) \leq 1 - \pi_0^{n_0}$$ 

and the result follows from Khasminski’s lemma.
3 Mixed Norm Estimates of Potentials of Stochastic Processes

Here we are moving toward estimating the resolvents of Markov diffusion processes in $L_{p,q}$. Recall that the process $x_t$ is introduced by Eq. 2.1.

**Lemma 3.1** Assume (2.3). Then there is a constant $N$, depending only on $\delta$, $d$, $p_0$, and $\tilde{b}_\infty$, such that for any $t_0 \geq 0$, $x_0 \in \mathbb{R}^d$, $\lambda > 0$, and Borel nonnegative $f$ vanishing outside $C_{\hat{\lambda}^{-1/2}}(t_0, x_0)$ we have

$$E \int_0^\infty e^{-\lambda t} f(t, x_t) dt \leq N\hat{\lambda}^{-d/(2p)} \Phi_{\lambda}(t_0, x_0) \|f\|_{L_{p,q}},$$  \tag{3.1}

where $\Phi_{\lambda}(t, x) = e^{-\sqrt{\lambda}(\sqrt{\lambda} + |x|)\xi/4}$, $\hat{\lambda}$ is defined in Theorem 2.6 and $\tilde{\xi}$ is introduced in Theorem 2.3.

Proof. Fix $\rho = N(\tilde{\xi})\hat{\lambda}^{-1/2} > 0$ such that the right-hand side of Eq. 2.17 equals $1/2$ when $R = \rho$. Then introduce $\tau^0$ as the first time $(t, x_t)$ hits $C_{\hat{\lambda}^{-1/2}}(t_0, x_0)$ and set $\gamma^0$ as the first time after $\tau^0$ the process $(t, x_t)$ exits from $C_{\hat{\lambda}^{-1/2} + \rho}(t_0, x_0)$. We define recursively $\tau^k$, $k = 1, 2, \ldots$, as the first time after $\gamma^{k-1}$ the process $(t, x_t)$ hits $C_{\hat{\lambda}^{-1/2}}(t_0, x_0)$ and $\gamma^k$ as the first time after $\tau^k$ the process $(t, x_t)$ exits from $C_{\hat{\lambda}^{-1/2} + \rho}(t_0, x_0)$.

These stopping times are either infinite or lie between $t_0$ and $t_0 + \hat{\lambda}^{-1}$. Therefore, the left-hand side of Eq. 3.1 equals

$$E \sum_{k=0}^\infty e^{-\lambda \tau^k} I_k,$$

where

$$I_k = I_{\tau^k > t_0} E \left( \int_{\tau^k(t_0 + \hat{\lambda}^{-1})}^{\gamma^k(t_0 + \hat{\lambda}^{-1})} e^{-\lambda(t - t')} f(t, x_t) dt \mid \mathcal{F}_{\tau^k} \right).$$

Here on the set, where $\tau^k > t_0$,

$$\int_{\tau^k(t_0 + \hat{\lambda}^{-1})}^{\gamma^k(t_0 + \hat{\lambda}^{-1})} dt = \gamma^k - \tau^k \leq \hat{\lambda}^{-1}.$$

Using this after estimating the norm of $\hat{h}$ in $C_{\hat{\lambda}^{-1/2} + \rho}(t_0, x_0)$ we infer from Eq. 2.4 that $I_k \leq N\hat{\lambda}^{-d/(2p)} \|f\|_{L_{p,q}}$, where $N = N(d, p_0, \delta, \tilde{b}_\infty)$.

Next, observe that, if $\sqrt{t_0} > |x_0|$, then $\tau^0$ is bigger than the first exit time of $(t, x_t)$ from $C_{\sqrt{t_0}}$, and by Theorem 2.6

$$Ee^{-\lambda \tau^0} \leq Ne^{-\sqrt{\lambda}\sqrt{t_0} \tilde{\xi}/2}.$$  

In case $\sqrt{t_0} \leq |x_0|$ and $|x_0| > \hat{\lambda}^{-1/2}$ our $\tau^0$ is bigger than the first exit time of $(t, x_t)$ from $C_{|x_0| - \hat{\lambda}^{-1/2}}$, and

$$Ee^{-\lambda \tau^0} \leq Ne^{-\sqrt{\lambda}(|x_0| - \hat{\lambda}^{-1/2}) \tilde{\xi}/2}.$$  

The last estimate (with $N = 1$) also holds if $|x_0| \leq \hat{\lambda}^{-1/2}$, so that in case $\sqrt{t_0} \leq |x_0|$

$$Ee^{-\lambda \tau^0} \leq Ne^{-\sqrt{\lambda}|x_0| \tilde{\xi}/2}$$

and we conclude that in all cases

$$Ee^{-\lambda \tau^0} \leq Ne^{-\sqrt{\lambda}(|x_0| + \sqrt{t_0}) \tilde{\xi}/4}.$$
Furthermore, by the choice of $\rho$ and Theorem 2.6

$$E\left(e^{-\lambda(y^k - \tau^k)} \mid F_{\tau^k}\right) \leq \frac{1}{2},$$

$$Ee^{-\lambda\tau^k} = Ee^{-\lambda(y^k - 1)} E\left(e^{-\lambda(y^k - 1)} \mid F_{y^k}\right) \leq \frac{1}{2}Ee^{-\lambda y^k - 1},$$

so that

$$Ee^{-\lambda\tau^k} \leq \frac{1}{4}Ee^{-\lambda y^k - 1}, \quad E e^{-\lambda\tau^k} \leq 4^{-k}Ee^{-\lambda \tau^0}.$$

Recalling (3.2) we see that the left-hand side of Eq. 3.1 is dominated by $N\Phi_\lambda(t_0, x_0)\|f\|_{L_{p,q}}$ and the lemma is proved.

The following theorem shows that the time spent by $(t, x_t)$ in cylinders $C_1(0, x)$ decays very fast as $|x| \to \infty$.

**Theorem 3.2** Suppose that

$$p, q \in [1, \infty], \quad \nu := 1 - \frac{d}{p} - \frac{1}{q} \geq 0. \quad (3.3)$$

Then there is a constant $N = N(\delta, d, p, q, \nu)$ such that for any $\lambda > 0$ and Borel nonnegative $f$ we have

$$I := E\int_0^\infty e^{-\lambda t} f(t, x_t) \, dt \leq N\lambda^{-\nu} \lambda^{d/(2p)}\|\Psi_\lambda^{1-v} f\|_{L_{p,q}}, \quad (3.4)$$

where $\Psi_\lambda(t, x) = \exp(-\sqrt{\lambda}(|x| + \sqrt{t})\xi/16)$.

**Proof.** First assume that $\nu = 0$. Take a nonnegative $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ of the type $\dot{\lambda}(d + 2)/2 \eta(\dot{\lambda}t, \sqrt{\lambda}x)$ with support in $C_{\dot{\lambda}^{-1/2}}$ and unit integral and for $(t, x), (s, y) \in \mathbb{R}^{d+1}$ set

$$f_{s,y}(t, x) = f(t, x) \zeta(t - s, x - y).$$

Clearly, due to Lemma 3.1,

$$I = \int_0^\infty \int_{\mathbb{R}^d} E\int_0^\infty e^{-\lambda t} f_{s,y}(t, x_t) \, dt \, dy \, ds \leq N\dot{\lambda}^{-d/(2p)} \int_0^\infty \int_{\mathbb{R}^d} \Phi_\lambda(s, y)\|f_{s,y}\|_{L_{p,q}} \, dy \, ds.$$ 

**Case** $p \geq q$. Then $q < \infty$ and we introduce

$$M_1^{q/(q-1)} = \int_0^\infty \int_{\mathbb{R}^d} \Phi_\lambda^{q/(2q-2)}(s, y) \, dy \, ds,$$

$$M_2^{p/(p-q)} = \int_0^\infty \int_{\mathbb{R}^d} \Phi_\lambda^{pq/(4p-4q)}(s, y) \, dy \, ds, \quad p \neq q, \quad M_2 = 1, \quad p = q.$$

It follows by Hölder’s inequality that

$$\dot{\lambda}^{d/(2p)} I \leq NM_1\left(\int_0^\infty \int_{\mathbb{R}^d} \Phi_\lambda^{q/(2q-2)}(s, y)\int_0^\infty \left(\int_{\mathbb{R}^d} f_{s,y}^p(t, x) \, dx\right)^{q/p} \, dt \, dy \, ds\right)^{1/q}$$

$$= NM_1\left(\int_0^\infty dt \left(\int_0^\infty \int_{\mathbb{R}^d} \Phi_\lambda^{q/(2q-2)}(s, y)\left(\int_{\mathbb{R}^d} f_{s,y}^p(t, x) \, dx\right)^{q/p} \, dy \, ds\right)^{1/q}\right)$$

$$\leq NM_1M_2^{1/q}\left(\int_0^\infty dt \left(\int_0^\infty \int_{\mathbb{R}^d} \Phi_\lambda^{q/(4q-4)}(s, y)f_{s,y}^p(t, x) \, dy \, ds \, dx\right)^{q/p}\right)^{1/q}.$$
We replace \( \Phi^{p/4}_\lambda(x, y) \) by \( \Phi^{p/4}_\lambda(t, x) \) taking into account that these values are comparable as long as \( \zeta(t - s, x - y) \neq 0 \). After that integrating over \( dyds \) and computing \( M_1, M_2 \) lead immediately to Eq. 3.4.

**Case \( p < q \)** It follows by Hölder’s inequality that
\[
\dot{\lambda}^{d/(2p)} I \leq N M_3 \left( \int_0^\infty \int_{\mathbb{R}^d} \Phi^{p/2}_\lambda(s, y) \int_{\mathbb{R}^d} \left( \int_0^\infty f(s,y,t,x) dt \right)^{pq/d} dx dy ds \right)^{1/p}
\]
\[
\leq N M_3 M_4 \left( \int_{\mathbb{R}^d} dx \left( \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty \Phi^{q/4}_\lambda(s, y) f^{q/4}(s,y,t,x) dt dy ds \right)^{p/q} \right)^{1/p},
\]
where
\[
M_3^{p/(p-1)} = \int_0^\infty \int_{\mathbb{R}^d} \Phi^{p/(2p-2)}_\lambda(s, y) dy ds,
\]
\[
M_4^{q/(q-p)} = \int_0^\infty \int_{\mathbb{R}^d} \Phi^{pq/(4q-4p)}_\lambda(s, y) dy ds.
\]
This leads to Eq. 3.4 as above. The theorem is proved if \( \nu = 0 \).

If \( \nu \in (0, 1) \), by Hölder’s inequality the left-hand side of Eq. 3.4 is dominated by \( I_1 I_2 \), where
\[
I_1^{1/\nu} = E \int_0^\infty e^{-\lambda t} f(t,x_t) dt = 1/\lambda,
\]
\[
I_2^{1/(1-\nu)} = E \int_0^\infty e^{-\lambda t} f^{1/(1-\nu)}(t,x_t) dt.
\]
Here
\[
\frac{d}{p - p\nu} + \frac{1}{q - q\nu} = 1
\]
so that by the case that \( \nu = 0 \)
\[
I_2^{1/(1-\nu)} \leq N \lambda^{d/(2p-2p\nu)} \| \Psi f^{1/(1-\nu)} \|_{L_{p-\\nu,q-\\nu}}
\]
\[
= N \lambda^{d/(2p-2p\nu)} \| \Psi f \|_{L_{p,q}}^{1/(1-\nu)}.
\]
This leads to (3.4) again. Finally, if \( \nu = 1 \) so that \( p = q = \infty \), the left-hand side of Eq. 3.4 is obviously dominated by \( \lambda^{-1} \sup f \), so that Eq. 3.4 holds with \( N = 1 \). The theorem is proved.

By taking \( q = \infty \) and \( f(t,x) = f(x) \) we come to the following, which extends Corollary 2.5 of [11] to the case of time dependent drift \( b \in L_{p_0,q_0,loc} \). It is further generalized by relaxing the restriction on \( p \) in Theorem 4.8.

**Corollary 3.3** Let \( p \in [d, \infty] \). Then for any \( \lambda > 0 \) and Borel nonnegative \( f(x) \) we have
\[
E \int_0^\infty e^{-\lambda t} f(x_t) dt \leq N \lambda^{-1+d/p} \| \Psi f \|_{L_p(\mathbb{R}^d)},
\]  (3.5)
where \( \Psi(x) = \exp(-\sqrt{\lambda} |x|^{2}/16) \) and \( N = N(\delta, d, p, p_0, \tilde{b}_\infty) \).

Next results are dealing with the exit times of the process \( x_t \) rather than \( (t, x_t) \). We will need them while showing an improved integrability of Green’s functions.

Estimate (3.6) below in case \( b \) is bounded was the starting point for the theory of time homogeneous controlled diffusion processes about fifty years ago.
Lemma 3.4 Let $p \in [d, \infty]$. Then for any Borel nonnegative $f(x)$, $R \leq \bar{R}$, and $x \in \mathbb{R}^d$

$$E \int_0^{\tau_{k}(x)} f(x_t) \, dt \leq N(\delta, d, \bar{b}, \bar{R}, R, p_0)R^{2-d/p}\|f\|_{L_p(\mathbb{R}^d)}.$$ (3.6)

Proof. Define $q$ from Eq. 2.3 and observe that for $k = 1, 2, \ldots$, $\Delta_k = [(k-1)R^2, kR^2)$ and

$$f_k(t, x) := I_{\Delta_k}(t) f(x),$$

according to Eq. 2.4, on the set where $\tau'_R(x) \geq (k-1)R^2$ we have

$$E \left( \int_{(k-1)R^2}^{(kR^2) \wedge \tau'_R(x)} f(x_t) \, dt \mid \mathcal{F}_{(k-1)R^2} \right)$$

$$\leq \int_{(k-1)R^2}^{(kR^2) \wedge \tau'_R(x)} f_k(t, x_t) \, dt \mid \mathcal{F}_{(k-1)R^2}$$

$$\leq N\left( R^2 + \|h\|_{L^q_0(\Delta_k \times B_R)}^{d/(2p)} \right) R^{2-q} \|f\|_{L_p(\mathbb{R}^d)}$$

$$\leq N(1 + \bar{b}^2 \bar{R}^d) R^{2-d/p} \|f\|_{L_p(\mathbb{R}^d)}.$$ (4.1)

It follows that

$$E \int_0^{\tau_{k}(x)} f(x_t) \, dt = \sum_{k=1}^{\infty} EI\tau_{k}(x) \geq (k-1)R^2 \int_{(k-1)R^2}^{(kR^2) \wedge \tau'_R(x)} f(x_t) \, dt$$

$$\leq NR^{2-d/p} \|f\|_{L_p(\mathbb{R}^d)} \sum_{k=1}^{\infty} P(\tau'_R(x) \geq (k-1)R^2).$$

By Corollary 2.11 each of the probabilities in the last sum is less than $Ne^{-k/N}$ and this proves the lemma.

4 Green’s Functions

Here is a straightforward (see, for instance, the comment after (3.2) in [11]) consequence of Eq. 3.4. Recall that the process $x_t$ is introduced by Eq. 2.1.

Theorem 4.1 Assume (3.3) and take $\lambda > 0$. Then there exists a constant $N = N(\delta, d, p, q, p_0, \bar{b}^{\infty})$ and a nonnegative Borel function $G_{\lambda}(t, x)$ (Green’s function of $(\cdot, x)$) on $\mathbb{R}^{d+1}$ such that $G_{\lambda}(t, x) = 0$ for $t \leq 0$ and for any Borel nonnegative $f$ given on $\mathbb{R}^{d+1}$ we have

$$E \int_0^{\infty} e^{-\lambda t} f(t, x_t) \, dt = \int_{\mathbb{R}^{d+1}} f(t, x) G_{\lambda}(t, x) \, dxdt,$$

$$\|\Psi^{-1}_{\lambda} G_{\lambda}\|_{L^p_{p,q}} \leq N\lambda^{-\nu} \lambda^{-d/(2p)},$$ (4.1)

where we use the notation

$$\|u\|_{L^p_{p,q}} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |u(t, x)|^p \, dx \right)^{q'/p'} \, dt \right)^{1/q'} \text{ if } p \geq q,$$

$$\|u\|_{L^p_{p,q}} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |u(t, x)|^{q'} \, dx \right)^{p'/q'} \, dt \right)^{1/p'} \text{ if } p < q,$$

and $p' = p/(p-1)$, $q' = q/(q-1).$
The highest power of pure \((p = q = d + 1)\) summability of \(G_\lambda\) guaranteed by this theorem is \(1 + 1/d\). It turns out that, actually, \(G_\lambda\) is summable to a higher power. The proof of this is based on the parabolic version of Gehring’s lemma from [7].

Introduce \(Q\) as the set of cylinders \(C_R(t, x)\), \(R > 0, t \geq 0, x \in \mathbb{R}^d\). For \(Q = C_R(t, x) \in Q\) let \(2Q = C_{2R}(t, x)\). If \(Q \in Q\) and \(Q = C_R(t, x)\), we call \(R\) the radius of \(Q\).

**Theorem 4.2** Let \(\lambda \in (0, \infty)\). Then there exist \(d_0 \in (1, d)\) and a constant \(N\), depending only on \(\delta, d, R_0, p_0, \lambda\), such that for any \(Q \in Q\) of radius \(R \leq R/2\) and \(p \geq d_0 + 1\), we have \[\|G_\lambda\|_{L_p/(p-1)(Q)} \leq NR^{-(d+2)/p}\|G_\lambda\|_{L_1(2Q)},\] (4.2) which is equivalently rewritten as
\[
\left(\int_Q G_\lambda^{p/(p-1)}(t, x) dx dt\right)^{(p-1)/p} \leq N \int_{2Q} G_\lambda dx dt.
\]

Proof. We basically follow the idea in [5]. Take \(Q \in Q\) of radius \(R \leq R/2\) and define recursively
\[
\gamma^1 = \inf\{t \geq 0 : (t, x_t) \in \bar{Q}\}, \quad \tau^1 = \inf\{t \geq \gamma^1 : (t, x_t) \notin 2Q\},
\]
\[
\gamma^{n+1} = \inf\{t \geq \tau^n : (t, x_t) \in \bar{Q}\}, \quad \tau^{n+1} = \inf\{t \geq \gamma^{n+1} : (t, x_t) \notin 2Q\}.
\]
Then for any nonnegative Borel \(f\) vanishing outside \(Q\) with \(\|f\|_{L_{d+1}(Q)} = 1\) we have
\[
\int_Q f G_\lambda(t, x) dx dt = E \int_0^\infty e^{-\lambda t} f(t, x_t) dt
\]
\[
= \sum_{n=1}^\infty E e^{-\lambda \gamma^n} E\left(\int_{\gamma^n}^{\tau^n} e^{-\lambda(t-\gamma^n)} f(t, x_t) dt \mid \mathcal{F}_{\gamma^n}\right).
\]
Next we use the conditional version of Eq. 2.6 to see that the conditional expectations above are less than \(NR^{d/(d+1)}\). After that we use the conditional version of Corollary 2.7 to get that
\[
R^2 \leq NE\left(\int_{\gamma^n}^{\tau^n} e^{-\lambda(t-\gamma^n)} dt \mid \mathcal{F}_{\gamma^n}\right).
\]
Then we obtain
\[
\int_Q f G_\lambda(t, x) dx dt \leq NR^{-(d+2)/(d+1)} \sum_{n=1}^\infty E e^{-\lambda \gamma^n} E\left(\int_{\gamma^n}^{\tau^n} e^{-\lambda(t-\gamma^n)} dt \mid \mathcal{F}_{\gamma^n}\right)
\]
\[
= NR^{-(d+2)/(d+1)} \sum_{n=1}^\infty E \int_{\gamma^n}^{\tau^n} e^{-\lambda t} dt
\]
\[
\leq NR^{-(d+2)/(d+1)} \int_0^\infty e^{-\lambda t} I_{2Q}(t, x_t) dt
\]
\[
= NR^{-(d+2)/(d+1)} \int_{2Q} G_\lambda(t, x) dx dt.
\]
The arbitrariness of \(f\) implies that
\[
\left(\int_Q G_\lambda^{(d+1)/d}(t, x) dx dt\right)^{d/(d+1)} \leq N \int_{2Q} G_\lambda(t, x) dx dt.
\]
Now the assertion of the theorem for \(p = d_0\) follows directly from the parabolic version of the famous Gehring’s lemma stated as Proposition 1.3 in [7]. For larger \(p\) it suffices to use Hölder’s inequality. The theorem is proved.
The parameters $d_0$ and $N$ in Theorem 4.2 may depend on $\delta, d, R, p_0, \lambda$. What is important for the future is that $d_0$ below is independent of $\lambda$.

**Theorem 4.3** There exists $d_0 \in (1, d)$, depending only on $\delta, d, R, p_0$, such that for any $p \geq d_0 + 1$ and $\lambda > 0$

\[
\left( \int_0^\infty \int_\mathbb{R}^d G_\lambda^{p/(p-1)}(t, x) \, dx \, dt \right)^{(p-1)/p} \leq N(\delta, d, R, p_0, \lambda, p).
\]

Furthermore, the above constant $N(\delta, d, R, p_0, \lambda, p)$ can be taken in the form

\[
N(\delta, d, R, p_0, \lambda, p) \approx (d+2)/(2p-1),
\]

where

\[
\tilde{\lambda}_p = \lambda(\lambda \wedge 1)^{d/(2p-d-2)}.
\]

Proof. Represent $\mathbb{R}^{d+1} = \{0, \infty\} \times \mathbb{R}^d$ as the union of countably many $Q_1, Q_2, \ldots \subset \mathbb{Q}$ of radius $R/2$ so that each point in $\mathbb{R}^{d+1}$ belongs to no more than $m(d)$ of the $2^d Q_i$'s and let $d_0$ be taken from Theorem 4.2 with $\lambda = 1$. Then for $\lambda \geq 1$

\[
\|G_\lambda\|_{L_p/(p-1)(\mathbb{R}^{d+1}_+)} \leq \|G_1\|_{L_p/(p-1)(\mathbb{R}^{d+1}_+)} \leq \sum_i \|G_1\|_{L_p(2Q_i)} \leq N_1 \|G_1\|_{L_1(\mathbb{R}^{d+1}_+)} = N_1.
\]

If $\lambda \in (0, 1)$ we take nonnegative $f \in L_p(\mathbb{R}^{d+1}_+)$ and observe that

\[
J := E \int_0^\infty e^{-\lambda t} f(t, x_t) \, dt = \sum_{n=0}^\infty e^{-\lambda n} \int_n^{n+1} e^{-\lambda(t-n)} f(t, x_t) \, dt
\]

By the first case each expectation in the sum is dominated by $N \|f I_{[n,n+1]}\|_{L_p}$. Therefore

\[
J \leq N \sum_{n=0}^\infty e^{-\lambda n} \|f I_{[n,n+1]}\|_{L_p} \leq N \|1-e^{-\lambda}\|_{L_p(\mathbb{R}^{d+1}_+)} \leq N \|f\|_{L_p(\mathbb{R}^{d+1}_+)},
\]

where the second inequality follows from Hölder’s inequality. This takes care of the case that $\lambda \in (0, 1)$ in both statements of the theorem.

To prove the second statement in case $\lambda \geq 1$ consider the process $(t, y_t)$, where $y_t = \sqrt{\lambda} x_t/\lambda$. We have

\[
y_t = \int_0^t \sigma_s/\lambda \, d(\sqrt{\lambda} w_{s/\lambda}) + \int_0^t \lambda^{-1/2} b_s/\lambda \, ds,
\]

where $\sqrt{\lambda} w_{s/\lambda}$ is a Wiener process and

\[
|\lambda^{-1/2} b_s/\lambda| \leq \lambda^{-1/2} h(s/\lambda, x_t/\lambda) =: \tilde{h}(s, y_s).
\]

Observe that

\[
\|\tilde{h}\|_{L_{p_0,q_0}(C_R(t,x))} = \sqrt{\lambda}\|h\|_{L_{p_0,q_0}(C_R/\sqrt{\lambda}x/\sqrt{\lambda})} \leq \tilde{b}_R/\sqrt{\lambda} R \leq \tilde{b}_R R.
\]
It follows that the above theory is applicable to \((t, y_t)\) and provides estimates with the same constants as for \((t, x_t)\). In particular, for any \(p \geq d_0 + 1\) (with \(d_0\) found above) and Borel nonnegative \(f(t, x)\)

\[
I := E \int_0^\infty e^{-t} f(t, y_t) \, dt \leq NN(\delta, d, R, p_0, p) \| f \|_{L_p(\mathbb{R}_+^{d+1})}.
\]

After that it only remains to note that

\[
I = \lambda E \int_0^\infty e^{-\lambda t} g(t, x_t) \, dt,
\]

where \(g(t, x) = f(\lambda t, \sqrt{\lambda} x)\) and

\[
\| f \|_{L_p(\mathbb{R}_+^{d+1})} = \lambda^{(d+2)/(2p)} \| g \|_{L_p(\mathbb{R}_+^{d+1})}.
\]

The theorem is proved.

Similar improvement of integrability occurs for the Green’s function of \(x_t\) rather than \((t, x_t)\). Here is a straightforward consequence of Eq. 3.5.

**Theorem 4.4** Let \(p \in [d, \infty)\). Then for any \(\lambda > 0\) there exists a nonnegative Borel function \(g_\lambda(x)\) (Green’s function of \(x\) on \(\mathbb{R}^d\)) such that for any Borel nonnegative \(f\) given on \(\mathbb{R}^d\) we have

\[
E \int_0^\infty e^{-\lambda t} f(x_t) \, dt = \int_{\mathbb{R}^d} f(x) g_\lambda(x) \, dx,
\]

\[
\| \Psi_\lambda^{-d/p} g_\lambda \|_{L_p(\mathbb{R}^d)} \leq N\lambda^{-1+d/p} \lambda^{-d/(2p)},
\]

(4.3)

where \(\Psi_\lambda(x) = \exp(-\sqrt{\lambda}|x|\theta/16)\), \(p' = p/(p - 1)\), and \(N\) depends only on \(\delta, d, R, p, p_0, b_\infty\).

According to this theorem this Green’s function is summable to the power \(d/(d - 1)\). Again it turns out that this power can be increased. If \(B\) is an open ball in \(\mathbb{R}^d\) by \(2B\) we denote the concentric open ball of twice the radius of \(B\).

**Theorem 4.5** Let \(\lambda \in (0, \infty)\). Then there exist \(d_0 \in (1, d)\) and a constant \(N\), depending only on \(d, R, \lambda, p\), such that for any ball \(B\) of radius \(R \leq R/2\) and \(p \geq d_0\), we have

\[
\| g_\lambda \|_{L_p((p-1)(B))} \leq N R^{-d/p} \| g_\lambda \|_{L_1(2B)},
\]

(4.4)

which is equivalently rewritten as

\[
\left( \int_B g_\lambda^{p/(p-1)} \, dx \right)^{(p-1)/p} \leq N \int_{2B} g_\lambda \, dx.
\]

Proof. We again follow the idea in [5]. Take a ball \(B\) of radius \(R \leq R/2\) and define recursively

\[
\gamma^1 = \inf\{t \geq 0 : x_t \in \bar{B}\}, \quad \tau^1 = \inf\{t \geq \gamma^1 : x_t \notin 2B\},
\]

\[
\gamma^{n+1} = \inf\{t \geq \tau^n : x_t \in \bar{B}\}, \quad \tau^{n+1} = \inf\{t \geq \gamma^{n+1} : x_t \notin 2B\}.
\]
Then for any nonnegative Borel $f$ vanishing outside $B$ with $\|f\|_{L_d(B)} = 1$ we have

$$\int_B fg_\lambda(x) \, dx = E \int_0^\infty e^{-\lambda t} f(x_t) \, dt$$

$$= \sum_{n=1}^\infty E e^{-\lambda \gamma_n} E \left( \int_{\gamma_n}^{\tau_n} e^{-\lambda (t-\gamma_n)} f(x_t) \, dt \mid \mathcal{F}_{\gamma_n} \right).$$

Next we use the conditional version of Eq. 3.6 to see that the conditional expectations above are less than $NR$. After that we use the conditional version of Corollary 2.7 to get that

$$R^2 \leq NE \left( \int_{\gamma_n}^{\tau_n} e^{-\lambda (t-\gamma_n)} \, dt \mid \mathcal{F}_{\gamma_n} \right).$$

Then we obtain

$$\int_B fg_\lambda(x) \, dx \leq NR^{-1} \sum_{n=1}^\infty E e^{-\lambda \gamma_n} E \left( \int_{\gamma_n}^{\tau_n} e^{-\lambda (t-\gamma_n)} \, dt \mid \mathcal{F}_{\gamma_n} \right)$$

$$= NR^{-1} \sum_{n=1}^\infty E \int_{\gamma_n}^{\tau_n} e^{-\lambda t} \, dt$$

$$\leq NR^{-1} E \int_0^\infty e^{-\lambda t} I_{2B}(x_t) \, dt = NR^{-1} \int_{2B} g_\lambda(x) \, dx.$$

The arbitrariness of $f$ implies that

$$\left( \int_B g_\lambda^{d/(d-1)}(x) \, dx \right)^{(d-1)/d} \leq N \int_{2B} g_\lambda(x) \, dx,$$

and again it only remains to use Gehring’s lemma in case $p = d$. For larger $p$ it suffices to use Hölder’s inequality. The theorem is proved.

By mimicking the proof of Theorem 4.3 one gets its “elliptic” counterpart.

**Theorem 4.6** There exists $d_0 \in (1, d)$, depending only on $\delta$, $d$, $R$, $p_0$, such that for any $p \geq d_0$ and $\lambda > 0$

$$\left( \int_{\mathbb{R}^d} g_\lambda^{p/(p-1)}(x) \, dx \right)^{(p-1)/p} \leq N(\delta, d, R, p_0, p)^\lambda_{e,p}^{d/(2p)-1},$$

where

$$\lambda_{e,p} = \lambda(1 \wedge \lambda)^{d/(2p-d)}.$$ 

**Remark 4.1** Below by $d_0$ we denote the largest of the $d_0$’s from Theorems 4.3 and 4.6 and observe that, as the simple example of $d^{ij} = \delta^{ij}$ and $b \equiv 0$ shows, $d_0 > d/2$.

Next, we present an improved mixed-norm parabolic Aleksandrov estimates by following the interpolation arguments in Nazarov [16]. The improvement consists of $d_0$ in place of $d$ in Eq. 4.5.

**Lemma 4.7** Suppose that

$$p, q \in [1, \infty], \quad \frac{d_0}{p} + \frac{1}{q} = 1. \quad (4.5)$$
Then for any Borel \( f(t,x) \geq 0 \)
\[
I := E \int_0^\infty e^{-\lambda t} f(t, x_t) \, dt \leq N_{d_0+1}^{-\frac{(2d_0-d)}{(2p)}} \| f \|_{L_{p,q}},
\]  
(4.6)
where \( N = N(\delta, d, R, p, p_0) \).

**Proof** If \( p = d_0 + 1 \), then \( q = d_0 + 1 \) and Eq. 4.6 follows from Theorem 4.3. In other terms, for any Borel \( f(t,x) \geq 0 \)
\[
E \int_0^\infty e^{-\lambda t} f(t, x_t) \, dt = \int_Q G_\lambda(t, x) f(t, x) \, dx \, dt
\]
\[
\leq N_{d_0+1}^{-\frac{(2d_0-d)}{(2d_0+2)}} \| f \|_{L_{d_0+1}}.
\]

If \( p = d_0 \) and \( q = \infty \) estimate (4.6) follows from Theorem 4.6 since
\[
I \leq E \int_0^\infty e^{-\lambda t} \sup_{s \geq 0} f(s, x_s) \, dt = \int_{\mathbb{R}^d} g_\lambda(x) \sup_{s \geq 0} f(s, x) \, dx
\]
\[
\leq N_{e,d_0}^{-\frac{(2d_0-d)}{(2d_0)}} \left( \int_{\mathbb{R}^d} \sup_{s \geq 0} f(d_0, s, x) \, dx \right)^{1/d_0} = N_{e,d_0}^{-\frac{(2d_0-d)}{(2d_0)}} \| f \|_{L_{d_0,\infty}}
\]
and as is easy to check \( \tilde{\lambda}_{e,d_0} = \tilde{\lambda}_{d_0+1} \).

If \( p = \infty \) and \( q = 1 \)
\[
I \leq \int_0^\infty \sup_x f(t, x) \, dt = \| f \|_{L_{\infty,1}}.
\]

We will use these facts in an interpolation argument. In case \( \infty > p > d_0 + 1 \) we have \( p > q \) and set \( \beta = p/(d_0 + 1) \) and \( \alpha = \beta/(\beta - 1) \). Take a nonnegative \( g(t) \) such that \((f(t,x)g(t))/g(t) = f(t,x) (0/0 = 0)\) and use Hölder’s inequality to conclude that
\[
I \leq I_1 I_2 ,
\]
where
\[
I_1 = \left( \int_0^\infty g^{-\alpha}(t) \, dt \right)^{1/\alpha},
\]
\[
I_2 = \left( E \int_0^\infty e^{-\lambda t} g^\beta(t) f^\beta(t, x_t) \, dt \right)^{1/\beta}
\]
\[
\leq N_{d_0+1}^{-\frac{(2d_0-d)}{(2p)}} \left( \int_0^\infty g(d_0+1)^{\beta}(t) \left( \int_{\mathbb{R}^d} f(d_0+1)^{\beta}(t, x) \, dx \right) \, dt \right)^{1/(d_0\beta + \beta)}.
\]

For \( g \) found from
\[
g^{-\alpha}(t) = g(d_0+1)^{\beta}(t) \int_{\mathbb{R}^d} f(d_0+1)^{\beta}(t, x) \, dx
\]
we get (4.6) and this takes care of the case that \( \infty > p > d_0 + 1 \).

If \( \infty > q > d_0 + 1 \) we have \( p < q \) and set \( \beta = q/(q - d_0 - 1) \) and \( \alpha = \beta/(\beta - 1) \). Take a nonnegative \( g(x) \) such that \((f(t,x)g(x))/g(x) = f(t,x) (0/0 = 0)\) and use Hölder’s inequality to conclude that
\[
I \leq I_1 I_2 ,
\]
where
\[
I_1 = \left( E \int_0^\infty e^{-\lambda t} g^{-\alpha}(x_t) \, dt \right)^{1/\alpha}
\]
\[
\leq N_{d_0+1}^{-(d-2d_0)/(2d_0\alpha)} \left( \int_{\mathbb{R}^d} g^{-d_0\alpha}(x) \left( \int_0^\infty f(d_0+1)^{\alpha}(t, x) \, dt \right) \, dx \right)^{1/(d_0\alpha + \alpha)}.
\]
For $g$ found from
\[ g^{-d_0 \beta}(x) = g^{(d_0+1)\alpha}(x) \int_0^\infty f^{(d_0+1)\alpha}(t, x) \, dt \]
we get (4.6) after simple manipulations and this proves the lemma.

Using this lemma instead of Eq. 2.6 and just repeating the proof of Lemma 3.1 we come
to a natural counterpart of the latter and then by literally repeating the proof of Theorem 3.2
we come to the following result, that is a version of Theorem 4.1 of Nazarov [16] in which
d/p + 1/q ≤ 1 that is stronger than ours, but in which the assumption on $h$ is weaker. A
proper probabilistic version of Theorem 4.1 of Nazarov [16] is found in [12]. Recall that
\[ \mathbb{R}^{d+1}_+ = \{ t \geq 0 \} \cap \mathbb{R}^{d+1}. \]

**Theorem 4.8** Suppose
\[ p, q \in [1, \infty], \quad \nu := 1 - \frac{d_0}{p} - \frac{1}{q} \geq 0. \tag{4.7} \]
Then there is $N = N(\delta, d, p, q, p_0, \tilde{b}_\infty)$ such that for any $\lambda > 0$ and Borel nonnegative $f$ we have
\[ E \int_0^\infty e^{-\lambda t} f(t, x_t) \, dt \leq N e^{-\nu + \nu^2/(2d_0) / p_0 \nu} f \| L_{p,q}(\mathbb{R}^{d+1}_+) \|
\tag{4.8} \]
where $\Psi_\lambda(t, x) = \exp(-\sqrt{\lambda \nu x^2 + \sqrt{t} \xi^2})$. In particular, if $f$ is independent of $t, p \geq d_0,$
and $q = \infty$
\[ E \int_0^\infty e^{-\lambda t} f(x_t) \, dt \leq N e^{-\nu - d/(2p) \nu} \| \tilde{b}_{\nu} \| L_p(\mathbb{R}^d), \]
where $\tilde{b}_{\nu}(x) = \exp(-\sqrt{\lambda x^2 \xi^2})$.

**Theorem 4.9** Assume that Eq. 4.7 holds. Then
(i) for any $n = 1, 2, \ldots, \text{nonnegative Borel } f$ on $\mathbb{R}^{d+1}_+$ and $T \leq 1$ we have
\[ E \left[ \int_0^T f(t, x_t) \, dt \right]^n \leq n! N^n T^n \| \Psi^{\nu/(1-v)/n} f \|^n_{L_{p,q}(\mathbb{R}^{d+1}_+)} \] \tag{4.9}
where $N = N(\delta, d, p, q, p_0, \tilde{b}_\infty)$ and $\chi = \nu + (2d_0 - d)/(2p)$;
(ii) for any nonnegative Borel $f$ on $\mathbb{R}^{d+1}_+$ and $T \geq 1$ we have
\[ I := E \int_0^T f(t, x_t) \, dt \leq N T^{1-1/q} \| \Psi^{\nu/(1-v)} f \|^n_{L_{p,q}(\mathbb{R}^{d+1}_+)} \] \tag{4.10}
where $N = N(\delta, d, p, q, p_0, \tilde{b}_\infty)$.

Proof. The proof of (i) proceeds by induction on $n$ and is achieved by almost literally
repeating the proof of Theorem 2.7 of [11]. The induction hypothesis is that for all $(t, x) \in \mathbb{R}^{d+1}_+$ and $\kappa \in [0, 1/n]$
\[ E \left[ \int_0^T f(t + s, x + x_s) \, ds \right]^n \leq n! N^n T^n \Psi^{\nu/(1-v)/\kappa}(t, x) \| \Psi^{\nu/(1-v)\kappa} f \|^n_{L_{p,q}(\mathbb{R}^{d+1}_+)} \] \tag{4.11}
We will discuss in detail only the case of $n = 1$. In that case observe that $\lambda := 1/T \geq 1$ which allows us to use Theorem 4.8 in the following computations when $\kappa \in [0, 1]$:

$$e^{-\lambda T} E \int_0^T f(t + s, x + x_s) ds \leq E \int_0^\infty e^{-\lambda s} f(t + s, x + x_s) ds$$

$$\leq N \lambda^{-\kappa} \|f(t + \cdot, x + \cdot)\|_{L_{p,q}(\mathbb{R}^{d+1})}^{(1-v)\kappa}$$

$$\leq N \lambda^{-\kappa} \Psi_\kappa(x) \|f\|_{L_{p,q}(\mathbb{R}^{d+1})}^{(1-v)\kappa},$$

where the last inequality is due to the fact that, for $x, y \in \mathbb{R}^d$, $\Psi_\kappa(s, y) \leq \Psi_\kappa(t + s, x + y) \Psi_\kappa^{-1}(t, x)$. Since $\lambda T = 1$, we get (4.11) with $n = 1$.

While proving (4.10) we may assume that $T = k$, where $k \geq 1$ is an integer. Then note that owing to Eq. 4.9 for any integer $n \geq 0$

$$E(\int_n^{n+1} f(t, x_t) dt \mid \mathcal{F}_n) \leq N \|\Psi_1^{1-v} f I_{[n,n+1]}\|_{L_{p,q}}.$$

Hence,

$$I \leq \sum_{n=0}^{k-1} \|\Psi_1^{1-v} f I_{[n,n+1]}\|_{L_{p,q}} =: NJ.$$

If $p \geq q$, we have by Hölder’s inequality

$$J \leq k^{(q-1)/q} \|\Psi_1^{1-v} f\|_{L_{p,q}}.$$

If $p < q$

$$J \leq k^{(p-1)/p} \left( \int_{\mathbb{R}^d} \sum_{n=0}^{k-1} \left( \int_n^{n+1} \Psi_1^{1-v} f^q(t, x) dt \right)^{p/q} dx \right)^{1/p}$$

$$\leq k^{(p-1)/p + (q-p)/(qp)} \|\Psi_1^{1-v} f\|_{L_{p,q}},$$

which yields (4.10) since $(p - 1)/p + (q - p)/(qp) = 1 - 1/q$. The theorem is proved.

Next theorem improves estimate (2.6) in what concerns the restrictions on $p, q$.

**Theorem 4.10** Assume that Eq. 4.7 holds with $v = 0$. Then for any $R \in (0, \bar{R})$, $x$, and Borel nonnegative $f$ given on $C_R$, we have

$$E \int_0^{\tau_{R}(x)} f(t, x + x_t) dt \leq N R^{(2d_0-d)/p} \|f\|_{L_{p,q}(C_R)},$$

where $N = N(\delta, d, R, p, p_0, \bar{b}, \bar{R})$.

Proof. Since $\tau_{R}(x) \leq R^2$, the left-hand side of Eq. 4.12 is smaller than

$$e^{\lambda R^2} E \int_0^\infty e^{-\lambda t} I_{C_R} f(t, x + x_t) dt$$

for any $\lambda > 0$. We estimate the last expectation by using Theorem 4.8, observe that, for $\lambda = R^{-2}$ we have $N_{\lambda d_0+1} \geq \lambda$ owing to $R \leq \bar{R}$, and then immediately come to Eq. 4.12, however, with $N$ depending on $\bar{b}_\infty$ in place of $\bar{b}_\bar{R}$. To see that this replacement is not needed, it suffices to note that the left-hand side of Eq. 4.12 will not change if we replace $b_t$ with $b_t I_{t < \tau_{\bar{R}}(x)}$ which admits the estimate

$$|b_t I_{t < \tau_{\bar{R}}(x)}| \leq h I_{C_{\bar{R}}(0, -x)}(t, x_t).$$

The theorem is proved.
Theorem 4.9 allows us to prove Itô’s formula for functions \( u \in W_{p,q}^{1,2}(Q) \), where \( Q \) is a domain in \( \mathbb{R}^{d+1} \) and

\[
W_{p,q}^{1,2}(Q) = \{ v : v, \partial_t v, Dv, D^2v \in L_{p,q}(Q) \}
\]

with norm introduced in a natural way. Before, the formula was known only for (smooth, Itô, and) \( W_{d+1}^{1,2} \)-functions and processes with bounded drifts or for \( W_2^{d,0} \)-functions in case the drift of the process is dominated by \( h(x_t) \) with \( h \in L_d \) (see [11]).

The following extends Theorem 2.10.1 of [9].

**Theorem 4.11** Assume that Eq. 4.7 holds with \( \nu = 0 \) and \( p < \infty, q < \infty \). Let \( Q \) be a bounded domain in \( \mathbb{R}^{d+1} \), \( 0 \in Q \), \( b \) be bounded, and \( u \in W_{p,q}^{1,2}(Q) \cap C(\overline{Q}) \). Then, for \( \tau \) defined as the first exit time of \( (t, x_t) \) from \( Q \) and for all \( t \geq 0 \),

\[
u(t \wedge \tau, x_t \wedge \tau) = u(0,0) + \int_0^{t \wedge \tau} D_tu(s, x_s) dm^i_s + \int_0^{t \wedge \tau} \left[ \partial_t u(s, x_s) + a^{ij}_s D_{ij} u(s, x_s) + b^i_s D_i u(s, x_s) \right] ds
\]

(4.13)

and the stochastic integral above is a square-integrable martingale.

**Proof.** First assume that \( u \) is smooth and its derivatives are bounded. Then Eq. 4.13 holds by Itô’s formula and, moreover, by denoting \( \tau^n = n \wedge \tau \) for any \( n \geq 0 \) we have

\[
E \int_{\tau^n}^{\tau^{n+1}} |Du(s, x_s)|^2 ds \leq NE \left( \int_{\tau^n}^{\tau^{n+1}} D_tu(s, x_s) dm^i_s \right)^2
\]

\[
= NE \left( u(\tau^{n+1}, x_{\tau^{n+1}}) - u(\tau^n, x_{\tau^n}) \right.
\]

\[
- \int_{\tau^n}^{\tau^{n+1}} \left[ \partial_t u(s, x_s) + a^{ij}_s D_{ij} u(s, x_s) + b^i_s D_i u(s, x_s) \right] ds \bigg)^2
\]

\[
\leq N \sup_{\overline{Q}} |u| + NE \left( \int_{\tau^n}^{\tau^{n+1}} I_Q(|\partial_t u| + |Du| + |D^2u|)(s, x_s) ds \right)^2.
\]

(4.14)

Since \( Q \) is bounded, \( \tau \) is bounded as well and in light of Theorem 4.9 we conclude that

\[
E \int_0^{\tau} |Du(s, x_s)|^2 ds \leq N \sup_{\overline{Q}} |u| + N \| \partial_t u, Du, D^2u \|_{L_{p,q}(Q)},
\]

(4.14)

where \( N \) are independent of \( u \) and \( Q \) as long as the size of \( Q \) in the \( t \)-direction is under control. Owing to Fatou’s theorem, this estimate is also true for those \( u \in W_{p,q}^{1,2}(Q) \cap C(\overline{Q}) \) that can be approximated uniformly and in the \( W_{p,q}^{1,2}(Q) \)-norm by smooth functions with bounded derivatives (recall that \( p < \infty, q < \infty \)). For our \( u \) there is no guarantee that such approximation is possible. However, mollifiers do such approximations in any subdomain \( Q' \subset \overline{Q}' \subset Q \). Hence, Eq. 4.14 holds for our \( u \) if we replace \( Q \) by \( Q' \) (containing \( (0,0) \)). Setting \( Q' \uparrow Q \) proves (4.14) in the generals case and proves the last assertion of the theorem.

After that Eq. 4.13 with \( Q' \) in place of \( Q \) is proved by routine approximation of \( u \) by smooth functions. Setting \( Q' \uparrow Q \) finally proves (4.13). The theorem is proved.
5 Application to Parabolic Equations

Fix a constant $\delta \in (0, 1)$ and recall that by $\mathbb{S}_\delta$ we denote the set of $d \times d$-symmetric matrices whose eigenvalues are between $\delta$ and $\delta^{-1}$. In this section we impose the following.

**Assumption 5.1**  
(i) On $\mathbb{R}^{d+1}$ we are given Borel measurable $\sigma(t,x)$ and $b(t,x)$ with values in $\mathbb{S}_\delta$ and in $\mathbb{R}^d$ respectively.
(ii) We are given $p_0, q_0 \in [1, \infty)$ satisfying (1.1) and a function $h(t, x)$ satisfying (2.2) and such that $|b| \leq h$.
(iii) Assumption 2.2 is satisfied.

Introduce $a = \sigma^2$ and set

$$Lu(t,x) = (1/2)a^{ij}(t,x)D_{ij}u(t,x) + b^i(t,x)D_iu(t,x).$$

For a domain $Q \subset \mathbb{R}^{d+1}$ one denotes by $\partial'Q$ its parabolic boundary defined as the set of all points on $\partial Q$ which are endpoints of continuous curves of type $(t,x_t)$, $t \in [a,b]$, which start in $Q$ and belong to $\bar{Q}$ for all $t < b$.

The following has the same flavor as Nazarov’s Theorem 4.1 of [16] or Theorem 4.3 of [12]. We get a wider range of $p, q$ on the account of restricting $b$. Here is a qualitative form of the maximum principle.

**Theorem 5.1** Let $0 < R \leq \bar{R}$, domain $Q \subset C_R$, and assume that Eq. 4.7 holds with $v = 0$, $p < \infty, q < \infty$, and that we are given a function $u \in W^{1,2}_{p,q,loc}(\bar{Q}) \cap C(Q)$. Take a function $c \geq 0$ on $Q$. Then on $Q$

$$u \leq NR^{(2d_0-d)/p}\|I_{Q,u>0}(\partial_t u + Lu - cu)\|_{L_{p,q}} + \sup_{\partial'Q}u_+,$$  

(5.1)

where $N = N(\delta, d, \bar{R}, p_0, \bar{b}_R)$. In particular (the maximum principle), if $\partial_t u + Lu - cu \geq 0$ in $Q$ and $u \leq 0$ on $\partial'Q$, then $u \leq 0$ in $Q$.

Proof. Obviously the right-hand side of Eq. 5.1 decreases if we replace $c$ with zero. Hence we may assume that $c = 0$. Also, we need to prove (5.1) only in $Q \cap \{u > 0\}$ on the parabolic boundary of which either $u = 0$ or $u \leq \sup_{\partial'Q}u_+$. Therefore, we may assume that $u > 0$ in $Q$.

Then for $\varepsilon > 0$ define $Q^\varepsilon$ as the collection of $(t, x) \in Q$ such that the closed ball in $\mathbb{R}^{d+1}$ centered at $(t, x)$ with radius $\varepsilon$ lies in $Q$. Obviously $Q^\varepsilon$ is open. It is not hard to prove (see, for instance, the proof of Lemma 3.1.13 in [10]) that $\text{dist}(\partial'Q, \partial'Q^\varepsilon) = \varepsilon$. It follows, owing to the continuity of $u$ and the monotone convergence theorem, that it suffices to prove Eq. 5.1 with $Q^\varepsilon$ in place of $Q$. As a consequence of that we may assume that $u \in W^{1,2}_{p,q}(\bar{Q})$.

This gives us the opportunity to replace $L$ in Eq. 5.1 with $L_n := I_{|b| \geq n\Delta} + I_{|b| < n\Delta}L$ (is the Laplacian) and then pass to the limit by the dominated convergence and monotone convergence theorems. Hence, we may assume that $b$ is bounded. In this situation for fixed $(t, x) \in Q$ by Itô’s formula we have

$$u(t,x) = Eu(t, t, x_\tau) + E\int_0^\tau f(t+s, x_\tau) \, ds,$$  

(5.2)

where $f = - (\partial_t u + Lu)$, $x_\tau$ is a solution of Eq. 1.2 on a probability space, and $\tau$ is the first exit time of $(t + s, x_\tau)$ from $Q$. After that, to prove (5.1), it only remains to use Theorem 4.10 and the fact that $(t + \tau, x_\tau) \in \partial'Q$. The theorem is proved.
Needless to say that applying Theorem 5.1 to $-u$ in place of $u$ one also gets the lower estimate of $u$.

**Remark 5.1** The statement of Theorem 5.1 may look futile because there is no guarantee that even for an $u \in W^{1,2}_{p,q}(Q)$ the norm in Eq. 5.1 is finite. However, there is an important case (see [13]) when $a = (\delta^{ij})$, $b \in L^{d+1}_{d+1}$, and the equation $\partial_t u + \Delta u + b^j D_j u = f$ has solutions in $W^{1,2}_p$ with $Du \in L_r$, $p \in (1, d+1)$, $r = p(d+1)/(d+1-p)$, on the account of $f \in L_r$. In that case $b^j D_j u \in L^p$.

The last result we present is needed for constructing the theory of fully nonlinear parabolic equations with drift terms in $L_{p,q}$ following the path in [10].

**Theorem 5.2** Assume that Eq. 4.7 holds with $p < \infty$, $q < \infty$ and take $R \in (0, \infty]$. Then there exists constants $N, \kappa > 0$, depending only on $d, \delta, R, p, q, p_0$, $\bar{b}_\infty$, such that for any $\lambda \geq 1$ and $u \in W^{1,2}_{p,q}$, $\text{loc} (CR) \cap C(\bar{CR}) (C_\infty = \mathbb{R}^{d+1}_+)$ we have

$$u(t, x) = E e^{\lambda R} u(t + \tau_R, x_{\tau_R}) - E \int_0^{\tau_R} e^{-\lambda t} f(t + s, x_s) ds =: I(t, x) + J(t, x),$$

where $f = \lambda u - Lu - \partial_t u$, $\tau_R$ is the first exit time of $(t + s, x_s)$ from $CR$ and $x_s$ is a solution of Eq. 1.2.

Here, thanks to Eq. 2.17

$$I(t, x) \leq N e^{-\kappa R^{\sqrt{\lambda}}} \sup_{\partial' CR} u, \quad \|IC_{R/2} I\|_{L_{p,q}} \leq N R^{d/p+2/q} e^{-\kappa R^{\sqrt{\lambda}}} \sup_{\partial' CR} u, \quad (5.4)$$

where $N, \kappa > 0$ depend only on $d, \delta, R, p, q, p_0$.

To estimate $J$ we define $f$ as zero outside $CR$ and observe that

$$J(t, x) \leq N e^{-\eta} \int_0^{\infty} F^{q/p}(t, s, x_s) ds,$$

where $\eta = \nu + (2d_0 - d)/(2p)$.

If $p \geq q$, Eq. 5.5 implies that

$$\int_{\mathbb{R}^d} |\lambda^{\eta} J(t, x)|^p dx \leq N \int_{\mathbb{R}^d} \left( \int_0^{\infty} F^{q/p}(t, s, x_s) ds \right)^{p/q} dx, \quad (5.6)$$
where
\[ F(t, s, x) = \int_{\mathbb{R}^d} \Psi_\lambda^{(1-v)p}(s, y) f^p(t+s, x+y) \, dy. \]

By Minkowski’s inequality the integral on the right in Eq. 5.6 is dominated by
\[ \left( \int_0^\infty \left( \int_{\mathbb{R}^d} F(t, s, x) \, dx \right)^{q/p} \, ds \right)^{p/q}, \]
where
\[ \int_{\mathbb{R}^d} F(t, s, x) \, dx = \int_{\mathbb{R}^d} f^p(t+s, y) \, dy \int_{\mathbb{R}^d} \Psi_\lambda^{(1-v)p}(s, y) \, dy \leq N\lambda^{-d/2} e^{-\mu \sqrt{kx}} \int_{\mathbb{R}^d} f^p(t+s, y) \, dy, \]
with \( \mu = \mu(\delta, p, q, R) > 0. \) Below by \( \mu \) we denote all such constants. It follows that
\[ \int_{\mathbb{R}^d} |\lambda \eta \bar{J}(t, x)|^q \, dt \leq N\lambda^{-d/2} e^{-\mu \sqrt{k|y|}} \int_{\mathbb{R}^d} f^q(t+s, x+y) \, ds \]
which along with Eq. 5.4 yield (5.3).

If \( q \geq p, \)
\[ \int_0^\infty |\lambda \eta \bar{J}(t, x)|^q \, dt \leq N \int_0^\infty \left( \int_{\mathbb{R}^d} F^{p/q}(t, x, y) \, dy \right)^{q/p} \, dt \]
where
\[ F(t, x, y) = \int_{\mathbb{R}^d} \Psi_\lambda^{(1-v)q}(s, y) f^q(t+s, x+y) \, ds. \]

By Minkowski’s inequality
\[ \left( \int_0^\infty |\lambda \eta \bar{J}(t, x)|^q \, dt \right)^{p/q} \leq N \int_{\mathbb{R}^d} \left( \int_0^\infty F(t, x, y) \, dt \right)^{p/q} \, dy, \]
where
\[ \int_0^\infty F(t, x, y) \, dt \leq \int_0^\infty f^q(s, x+y) \, ds \int_0^\infty \Psi_\lambda^{(1-v)q}(s, y) \, ds \leq N\lambda^{-1} e^{-\mu \sqrt{k|y|}} \int_0^\infty f^q(s, x+y) \, ds. \]
Hence,
\[ \left( \int_0^\infty |\lambda \eta \bar{J}(t, x)|^q \, dt \right)^{p/q} \leq N\lambda^{-p/q} \int_{\mathbb{R}^d} e^{-\mu \sqrt{k|y|}} \left( \int_0^\infty f^q(s, x+y) \, ds \right)^{p/q} \, dy, \]
\[ \|\lambda \eta \bar{J}\|_{L_{p,q}(\mathbb{R}^{d+1})}^p \leq N \lambda^{-p/q-d/2} \|f\|_{L_{p,q}(\mathbb{R}^{d+1})}^p, \]
and we again come to Eq. 5.3. The theorem is proved.

The full strength of Theorem 5.2 is seen in the theory of fully nonlinear equations. But even for linear ones one gets a nontrivial information as, for instance, in the following theorem which, in particular, implies that the operator \( L + \partial_t \) with the domain
\[ \{u \in W^{1,2}_{p,q,\text{loc}}(C_R) \cap C(\overline{C_R}) : Lu + \partial_t u \in L_{p,q}(C_R), u|_{\partial' C_R} = 0\} \]
is a closed operator in \( L_{p,q}(C_R). \)
Theorem 5.3 Assume that Eq. 4.7 holds with \( p < \infty, q < \infty \) and take \( R \in (0, \infty) \).
Suppose we are given \( u_0, u_1, ... \in W^{1,2}_{p,q,\text{loc}}(CR) \cap C(\bar{CR}) \) and \( f \in L_{p,q}(CR) \) such that \( f_n := Lu_n + \partial_t u_n \in L_{p,q}(CR) \) for \( n \geq 0 \),
\[
\sup_{n \geq 1} \sup_{\partial CR} |u_n| < \infty, \quad \|f_n - f\|_{L_{p,q}(CR)} + \|u_n - u_0\|_{L_{p,q}(CR)} \to 0
\]
as \( n \to \infty \). Then \( Lu_0 + \partial_t u_0 = f \) in \( CR \).

Proof. Take a smooth \( \psi \) on \( CR \) and apply (5.3) to \( u_n - u + \psi/\lambda \) in place of \( u \). Then pass to the limit as \( n \to \infty \) to find
\[
\|\psi_+\|_{L_{p,q}(CR/2)} \leq N_1 \lambda R^{d/p+2/q} e^{-\kappa R \sqrt{\lambda}} + N_2 \|\psi - f + Lu_0 + \partial_t u_0 - (L\psi + \partial_t \psi)/\lambda\|_{L_{p,q}(CR)},
\]
where \( N_2 \) is independent of \( \lambda \) and \( \psi \) and \( N_1 \) is independent of \( \lambda \). By setting \( \lambda \to \infty \) we get
\[
\|\psi_+\|_{L_{p,q}(CR/2)} \leq N_2 \|\psi - f + Lu_0 + \partial_t u_0\|_{L_{p,q}(CR)}.
\]
This is true if \( \psi \) is smooth enough and by approximation is true for any \( \psi \in L_{p,q}(CR) \). For \( \psi = f - Lu_0 - \partial_t u_0 \) we get that \( f - Lu_0 - \partial_t u_0 \leq 0 \) in \( CR/2 \). The reader understands that here as well as in Eq. 5.3 one can take any number \( < R \) in place of \( R/2 \). Hence, \( f - Lu_0 - \partial_t u_0 \leq 0 \) in \( CR \). Passing to \( -u_n, -f \) yields \( f - Lu_0 - \partial_t u_0 \geq 0 \) and proves the theorem.

Acknowledgements The author is sincerely grateful to the referees for their comments which helped improve the presentation.

Declarations

Conflict of Interests The manuscript contains no data and there are no conflict of interest.

References

1. Beck, L., Flandoli, F., Gubinelli, M., Maurelli, M.: Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness. Electron. J. Probab. 24(136), 1–72 (2019)
2. Cabrè, X.: On the Alexandroff-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations. Comm. Pure Appl. Math. 48, 539–570 (1995)
3. Crandall, M.G., Kocan, M., Święch, A.: \( L^p \)-theory for fully nonlinear uniformly parabolic equations. Comm. Partial Differ. Equ 25, 1997–2053 (2000)
4. Escauriaza, L.: \( W^{2,n} \) a priori estimates for solutions to fully non-linear equations. India. Univ. Math. J. 42, 413–423 (1993)
5. Fabes, E.B., Stroock, D.W.: The \( L^p \)-integrability of Green’s functions and fundamental solutions for elliptic and parabolic equations. Duke Math. J. 51(4), 997–1016 (1984)
6. Fok, P.: Some maximum principles and continuity estimates for Fully Nonlinear Elliptic Equations of Second Order. Ph.D Thesis, Santa Barbara (1996)
7. Giaquinta, M., Struwe, M.: On the partial regularity of weak solutions of nonlinear parabolic systems. Math. Z. 179, 437–451 (1982)
8. Kinzebulatov, D., Semenov, Y.A.: Stochastic differential equations with singular (form-bounded) drift, arXiv:1904.01268
9. Krylov, N.V.: Controlled diffusion processes, Nauka, Moscow, 1977 in Russian; English transl. Springer (1980)
10. Krylov, N.V.: Sobolev and Viscosity Solutions for Fully Nonlinear Elliptic and Parabolic Equations, Mathematical Surveys and Monographs, vol. 233. Amer. Math. Soc., Providence, RI (2018)
11. Krylov, N.V.: On stochastic equations with drift in \( L_d \). Ann. Prob. 49(5), 2371–2398 (2021)
12. Krylov, N.V.: On time inhomogeneous stochastic Itô equations with drift in \( L_{d+1} \), arXiv: 2005.08831 (2020)
13. Krylov, N.V.: On the heat equation with drift in \( L_{d+1} \), arXiv:2101.00119
14. Nam, Kyeongsik.: Stochastic differential equations with critical drifts, arXiv:1802.00074 (2018)
15. Nazarov, A.I.: A centennial of the Zaremba-Hopf-Oleinik lemma. SIAM J. Math. Anal. 44(1), 437–453 (2012)
16. Nazarov, A.I.: Interpolation of linear spaces and estimates for the maximum of a solution for parabolic equations, Partial differential equations, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat., Novosibirsk, 50–72 in Russian; translated into English as On the maximum principle for parabolic equations with unbounded coefficients, arXiv:1507.05232 (1987)
17. Safonov, M.V.: Non-Divergence Elliptic Equations of Second Order with Unbounded Drift. Nonlinear Partial Differ. Equ. Related Top. 211–232. Amer. Math. Soc. Transl. Ser. 2, 229, Adv. Math. Sci., 64, Amer. Math. Soc., Providence (2010)
18. Xie, Longjie., Zhang, Xicheng.: Ergodicity of stochastic differential equations with jumps and singular coefficients. Ann. l’Inst. Henri Poincaré - Probab. Stat. 56(1), 175–229 (2020)
19. Zhao, Guohuan.: Stochastic Lagrangian flows for SDEs with rough coefficients, arXiv:1911.05562v2
20. Zhang, Xicheng., Zhao, Guohuan.: Stochastic Lagrangian path for Leray solutions of 3D Navier-Stokes equations, arXiv:1904.04387

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.