Discrete curves in $\mathbb{CP}^1$ and the Toda lattice.

Tim Hoffmann* and Nadja Kutz†

October 31, 2018

Abstract

In this paper we investigate flows on discrete curves in $\mathbb{C}^2$, $\mathbb{CP}^1$, and $\mathbb{C}$. A novel interpretation of the one dimensional Toda lattice hierarchy and reductions thereof as flows on discrete curves will be given.

1 Introduction

In this paper we investigate flows on discrete curves in $\mathbb{C}^2$, $\mathbb{CP}^1$, and $\mathbb{C}$. By a discrete curve $\gamma$ in $K$ we mean a map $\gamma : \mathbb{Z} \rightarrow K$. A flow on $\gamma$ is a smooth variation of $\gamma$. The description of these curves is mainly motivated by the picture of discrete curves in $\mathbb{CP}^1$, in other words we imagine curves in $\mathbb{C}^2$ as curves in $\mathbb{CP}^1$, which are lifted to homogeneous coordinates; and curves in $\mathbb{C}$, as curves in $\mathbb{CP}^1$, which do not hit infinity. In particular this view allows us to find a novel interpretation of the wellknown one dimensional Toda lattice hierarchy in terms of flows on discrete curves. The Toda lattice hierarchy is a set of equations, including the Toda equation:

$$\ddot{q}_k = e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}}.$$

The Toda equation is sometimes also called first flow equation of the Toda lattice hierarchy, it was discovered by Toda in 1967 ([Tod89]). A good overview about the vast literature about the Toda lattice can be found in ([FT86, Sur02]). The paper is organized as follows.

In section 2 we define discrete curves in $\mathbb{C}^2$ and $\mathbb{CP}^1$. A discrete analog of the Schwartzian derivative in terms of cross-ratios of four neighboring points

*Supported by the DFG sfb288. email: hoffmann@math.tu-berlin.de
†Supported by the Servicezentrum Humboldt University of Berlin. email: nadja@math.tu-berlin.de
will be defined. A zero curvature or Lax representation for flows on that discrete curves will be given.

In section 3 we will restrict ourselves to the case of so-called discrete (conformal) arc length parametrized curves. In 3.1 we define certain flows on arc length parametrized curves which in the turn induce flows for the cross-ratios of that curve. It will be shown that for these flows the cross-ratios give solutions to equations of the Volterra hierarchy.

In 3.2 the arc length parametrized curves in \( \mathbb{C}^2 \) viewed as curves in \( \mathbb{C} \mathbb{P}^1 \) in homogeneous coordinates will be reduced to curves in \( \mathbb{C} \), by assuming that the second coordinate does not vanish. We call this reduction Euclidean reduction. It will be shown that if one starts with a flow on a discrete arc length parametrized curve in \( \mathbb{C}^2 \), such that the corresponding cross-ratios are solutions to the second flow of the Volterra hierarchy, which is also called the discrete KdV flow, then the corresponding curve obtained by Euclidean reduction satisfies a discrete version of the mKdV flow. This gives a discrete version of a geometrical interpretation of the Miura transformation.

In 3.3 a nice geometrical interpretation of Bäcklund transformations of arc length parametrized discrete curves in \( \mathbb{C}^2 \) will be given. In the turn a time discrete version of the Volterra equation, which is the first flow equation of the Volterra hierarchy, is derived.

In section 4 determinants of two and three neighboring points of a discrete curve in \( \mathbb{C}^2 \) will be identified with the Flaschka-Manakov variables of the Toda lattice. The Toda flows define flow directions for curves in \( \mathbb{C}^2 \).

In 5 the flows on the discrete curves in \( \mathbb{C}^2 \) given by Toda lattice hierarchy will be further investigated. In particular in 5.1 the flows corresponding to the "first" three flows will be given explicitly.

In 5.2 we derive several geometrical features of these flows. In particular in 5.2.2 we will look at flows on curves which are compatible with the known reduction \([\text{Sur}02]\) of the Toda lattice hierarchy to the Volterra lattice hierarchy by setting the Flaschka-Manakov variable \( p_k = 0 \). This reduction is different from from the reduction in 3.1, where among others \( g_k = \exp((q_k - q_{k+1})/2) = \text{const} \). Nevertheless the cross ratios of such curves evolve again with equations of the Volterra hierarchy. Hence there exist two geometrically motivated reductions of the Toda lattice hierarchy to the Volterra hierarchy. The case \( p_k = 0 \) and \( g_k = \text{const} \) lies in the intersection of both reductions, it belongs to the trivial solutions of the Toda lattice hierarchy equations. The corresponding class of curves, which are invariant under this trivial flow are quadrics, as will be shown in that subsection.

In the subsection 5.2.2 we will again look for a class of curves, which is invariant (up to euclidian motion and tangential flow) under a certain Toda flow. This time we find that in the Euclidian reduction socalled discrete
planar elastic curves are invariant under the discrete mKdV flow.

2 Flows on discrete curves

2.1 Flows on discrete curves in $\mathbb{C}^2$

Let $c : \mathbb{Z} \rightarrow \mathbb{C} \mathbb{P}^1$, $k \mapsto c_k$ be a discrete curve in the complex projective space. We assume $c$ is immersed, i.e. $c_{k-1}, c_k$ and $c_{k+1}$ are pairwise disjoined. By introducing homogeneous coordinates, we can lift $c$ to a map

$$\gamma : \mathbb{Z} \rightarrow \mathbb{C}^2$$
$$k \mapsto \gamma_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix} \quad (2.1)$$

with $c_k = x_ky_k^{-1}$. Obviously $\gamma$ is not uniquely defined: For $\lambda : \mathbb{Z} \rightarrow \mathbb{C}^*$, $\lambda_k \gamma_k$ is also a valid lift.

Define:

$$g_k = \det(\gamma_k, \gamma_{k+1}) = x_ky_{k+1} - y_kx_{k+1} \quad (2.2)$$
$$u_k = \det(\gamma_{k-1}, \gamma_{k+1}) \quad (2.3)$$

The following lemma can be straightforwardly obtained by using the above definitions (2.3):

**Lemma 1**

$$\gamma_{k+1} = \frac{1}{g_{k-1}}(u_k\gamma_k - g_k\gamma_{k-1}).$$

If the variables $u_k$ and $g_k$ and initial points $\gamma_0$ and $\gamma_1$ are given, then lemma (4) is a recursive definition of a discrete curve. This may look as an odd way to define a discrete curve; nevertheless later on the power of this definition will become more apparent. In particular the variables $u_k$ and $g_k$ will be related to the Flaschka-Manakov [Fla74a, Fla74b, Man74] variables of the one dimensional Toda lattice.

Note that after choosing an initial $\gamma_0$ it is always possible ($c$ is immersed) to find $\gamma_k$ such that:

$$g_k = 1 \quad (2.4)$$

for all $k$. We will call discrete curves $\gamma$ with property (2.4) conformal arc length parametrized. Hence the variables $g_k$ measure the deviation from arc length parameterization.
Definition 1 The cross-ratio of four points \(a, b, c, d \in \mathbb{C}^2\) is defined by

\[
\text{cr}(a, b, c, d) = \frac{\det(a, b) \det(c, d)}{\det(b, c) \det(d, a)}.
\]

Let us denote the cross-ratio of four neighboring points of \(\gamma\) by \(Q\):

\[
Q_k := \text{cr}(\gamma_{k-1}, \gamma_k, \gamma_{k+2}, \gamma_{k+1}) = \frac{g_{k-1}g_{k+1}}{u_ku_{k+1}}.
\]  

(2.5)

Now \(\det(\gamma_k, \frac{\gamma_{k+1} - \gamma_{k-1}}{g_k + g_{k-1}}) = 1\). This means that \(\gamma_k\) and \(\frac{\gamma_{k+1} - \gamma_{k-1}}{g_k + g_{k-1}}\) are linearly independent. Hence an arbitrary flow on (or variation of) \(\gamma\) can be written in the following way:

\[
\frac{d}{dt}\gamma_k = \dot{\gamma}_k = \alpha_k\gamma_k + \frac{\beta_k}{u_k}(\gamma_{k+1} - \gamma_{k-1}), \quad \alpha_k, \beta_k \in \mathbb{C}. \tag{2.6}
\]

The variables \(\alpha_k, \beta_k \in \mathbb{C}\) are arbitrary, for convenience we sometimes use the normalized variable:

\[
\hat{\beta}_k = \beta_k(g_{k+1} + g_{k-1}). \tag{2.7}
\]

Lemma 2 A flow on the discrete curve \(\gamma\) given by (2.4) generates the following flow on the variables \(g_k\):

\[
\dot{g}_k = g_k(\alpha_{k+1} + \alpha_k) + \beta_{k+1} - \beta_k. \tag{2.8}
\]

Proof Differentiate \(g_k\) in (2.3):

\[
\dot{g}_k = \det(\dot{\gamma}_k, \gamma_{k+1}) + \det(\gamma_k, \dot{\gamma}_{k+1})
= \det(\alpha_k\gamma_k + \frac{\beta_k}{u_k}(\gamma_{k+1} - \gamma_{k-1}), \gamma_{k+1})
+ \det(\gamma_k, \alpha_{k+1}\gamma_{k+1} + \frac{\beta_{k+1}}{u_{k+1}}(\gamma_{k+2} - \gamma_k))
= g_k\alpha - \beta_k + g_k\alpha_{k+1} + \beta_{k+1}. \tag{2.9}
\]

If the flow is arc length preserving, i.e. in particular \(\dot{g}_k = 0\) f.a. \(k \in \mathbb{Z}\) then there exists an obvious trivial solution for (2.8): Choosing \(\beta \equiv 0\) induces \(\alpha_{k+1} = -\alpha_k\). This flow corresponds to the freedom of the initial choice of \(\gamma_0\) and has no effect on the corresponding curve \(c\) in \(\mathbb{C}P^1\). Note also that in this case (2.8) is a linear equation. So one can always add any two flows solving it.
Lemma 3 A flow on the discrete curve \( \gamma \) given by (2.6) generates the following flow on the variables \( u_k \):\[
\dot{u}_k = u_k(\alpha_{k-1} + \alpha_{k+1}) + \beta_{k-1}\frac{g_k}{u_{k-1}g_{k-1}}(g_{k-2} + g_{k-1}) - \beta_{k+1}\frac{g_{k-1}}{u_{k+1}g_k}(g_k + g_{k+1})
\]
(2.10)
or equivalently\[
\dot{u}_k = \frac{\alpha_{k-1} + \alpha_{k+1}}{u_k} + \beta_{k-1}\frac{Q_{k-1}}{g_k-2g_{k-1}}(g_{k-2} + g_{k-1}) - \beta_{k+1}\frac{Q_k}{g_kg_{k+1}}(g_k + g_{k+1})
\]
(2.11)
\[
Proof\quad \dot{u}_k = \text{det}(\dot{\gamma}_{k-1}, \gamma_{k+1}) + \text{det}(\gamma_{k-1}, \dot{\gamma}_{k+1}) = \alpha_{k-1}u_k + g_k\frac{\beta_{k-1}}{u_{k-1}} - \frac{\beta_{k-1}}{u_{k-1}}\text{det}(\gamma_{k-2}, \gamma_{k+1}) + \alpha_{k+1}u_k - g_{k+1}\frac{\beta_{k+1}}{u_{k+1}} - \frac{\beta_{k+1}}{u_{k+1}}\text{det}(\gamma_{k-1}, \gamma_{k+2})
\]
(2.12)
Using lemma 2 one gets \[
\text{det}(\gamma_{k-1}, \gamma_{k+2}) = \frac{1}{g_k}u_ku_{k+1} - g_{k+1}g_k^{-1}
\]
which gives the result. \(\square\)

Lemma 4 A flow on the discrete curve \( \gamma \) given by (2.6) generates the following flow on the cross-ratios \( Q_k \):\[
\frac{\dot{Q}_k}{Q_k} = (Q_k - 1)(\beta_{k+1}(\frac{1}{g_{k+1}} + \frac{1}{g_k}) - \beta_k(\frac{1}{g_k} + \frac{1}{g_{k-1}})) + Q_{k+1}\beta_{k+2}(\frac{1}{g_{k+2}} + \frac{1}{g_{k+1}}) - Q_{k-1}\beta_{k-1}(\frac{1}{g_{k-1}} + \frac{1}{g_{k-2}})
\]
(2.13)
\[
Proof\quad \text{Using the definition of the cross-ratio (2.5) and lemmas 3 and 2 the assertion follows immediately.}\ \(\square\)

2.2 Flows on discrete curves in \( \mathbb{CP}^1 \)

Lemma 5 A flow on the discrete curve \( \gamma \) given by (2.6) generates the following flow on the non-lifted curve \( c \) in \( \mathbb{CP}^1 \) whenever \( c \) does not hit \( \infty \):\[
\dot{c}_k = \beta_k\frac{g_k + g_{k-1}}{2g_kg_{k-1}}\frac{2(c_{k+1} - c_k)(c_k - c_{k-1})}{c_{k+1} - c_{k-1}}
\]
(2.14)
Equation (2.14) can be written in the form

$$ \dot{c}_k = \frac{\beta_k}{M^h(g_k, g_{k+1})} M^h(c_{k+1} - c_k, c_k - c_{k-1}) $$

where $M^h$ denotes the harmonic mean. In general the flow of the non-lifted curve will depend on the chosen lift, since the $g_k$ depend on the choice. Note however, that the $\alpha_k$ do not contribute to the evolution of the non-lifted curve $c$.

**Proof** For a given lift $\gamma = (x_k, y_k)$ the non-lifted curve may be reconstructed by $c_k = x_k / y_k$ whenever $y_k \neq 0$ which means that the curve does not hit $\infty$. Now insert this and the definitions of $g$ and $u$ in both sides of equation (2.14).

### 2.3 Zero Curvature representation

Define:

$$ F_k = \begin{pmatrix} \gamma^T_k \\ \gamma^T_{k-1} \end{pmatrix} = \begin{pmatrix} x_k & y_k \\ x_{k-1} & y_{k-1} \end{pmatrix}. $$

Note that if the curve $\gamma$ is conformal arc length parametrized (2.4) then $F_k \in SL(2, \mathbb{C})$ for all $k \in \mathbb{Z}$.

**Proposition 2.1** Let $\alpha_k, \beta_k \in \mathbb{C}$ be arbitrary and $g_k, u_k$ be as defined in (2.3). Then

$$ F_{k+1} = L_k F_k \quad \dot{F}_k = V_k F_k $$

with

$$ L_k = \begin{pmatrix} 1 & -g_k \\ 0 & 1 \end{pmatrix}, \quad V_k = \begin{pmatrix} \alpha_k + \frac{1}{g_{k-1}} \beta_k & -(1 + \frac{g_k}{g_{k-1}}) \beta_k \\ (1 + \frac{g_k}{g_{k-1}}) \beta_k & \alpha_{k-1} - \frac{1}{g_{k-1}} \beta_{k-1} \end{pmatrix}. $$

The compatibility equation

$$ \dot{L}_k = V_{k+1} L_k - L_k V_k $$

is satisfied for all $\alpha_k, \beta_k \in \mathbb{C}$.

The compatibility equation (2.17) is also called zero curvature equation.

**Proof** The construction of $L_k$ is obvious with lemma 1. The construction of $V_k$ follows also quite straightforwardly from lemma 1 and the definition of the flow on $\gamma$ in (2.6). The compatibility equation (2.17) holds by construction. The flows on $g_k$ and $u_k$ were constructed by using a well defined flow on $\gamma$ for which in particular $\frac{d}{dt} (L_k \gamma_k) = V_{k+1} \gamma_{k+1}$ (which gives (2.17)). Nevertheless (2.17) can also easily be checked directly.
3 Conformal arc length parametrized curves

3.1 Conformal arc length parametrized curves in $\mathbb{C}^2$

Let $c$ be a discrete curve in $\mathbb{C}P^1$. Up to Möbius transformations (or up to the choice of $c_0$, $c_1$, and $c_2$) $c$ is completely determined by the cross-ratios $Q_k$. If one scales all $Q_k$ with a non-vanishing factor $\lambda$ one gets—again up to Möbius transformations—a new discrete curve $c(\lambda)$. We call the family of all such curves the associated family of $c = c(1)$.

As mentioned in the beginning, one has a choice when lifting a discrete curve from $\mathbb{C}P^1$ to $\mathbb{C}^2$. This choice can be fixed by prescribing the determinants of successive points. A natural choice is here to set $g \equiv 1$ which we called conformal arc length parametrization and we will discuss this choice here. However, in section 4 we will see, that other normalizations are likewise meaningful.

We will now investigate flows on $\gamma$ that preserve the conformal arc length. The condition for this is, that $\dot{g}_k = 0$, which implies by equation 2.8

$$\alpha_{k+1} + \alpha_k = \beta_{k+1} - \beta_k$$

(3.18)

for $\alpha$ and $\beta$ from equation (2.6).

Recall that the conformal arc length condition leaves us with an initial choice of $\gamma_k$ and that this freedom corresponds to a trivial flow with $\beta \equiv 0$ and $\alpha_{k+1} = -\alpha_k$. This flow is a first example of a (conformal) arc length preserving flow. It does not change the curve $c$ in $CP^1$ though, since only the $\beta_k$ contribute to the evolution of the non-lifted curve.

If we choose $\beta \equiv 1/2$ and $\alpha \equiv 0$. We get for the curve

$$\dot{\gamma}_k = \frac{1}{2u_k}(\gamma_{k+1} - \gamma_{k-1}).$$

(3.19)

This is what we will call the conformal tangential flow. Then $\dot{u}_k = \frac{1}{u_{k-1}} - \frac{1}{u_{k+1}}$ and $Q$ will solve the famous Volterra model [FT86, Sur02]:

$$\dot{Q}_k = Q_k(Q_{k+1} - Q_{k-1}).$$

(3.20)

If we want this equation for the whole associated family of $\gamma$ we must scale time by $\lambda$:

$$\lambda\dot{Q}_k(\lambda) = Q_k(\lambda)(Q_{k+1}(\lambda) - Q_{k-1}(\lambda))$$

One obtains the next higher flow of the Volterra hierarchy [Sur02] when one chooses $\beta_k = 1/2(Q_{k-1} + Q_k + 1)$. This implies

$$\dot{Q}_k = Q_k(Q_{k+1}(Q_{k+2} + Q_{k+1} + Q_k) - Q_{k-1}(Q_k + Q_{k-1} + Q_{k-2})).$$

(3.21)
Conjecture

Let

\[ \beta_{n+1}^{new} - \beta_k^{new} = \frac{\dot{Q}_k}{Q_k}. \]  \hspace{1cm} (3.22)

Given flows \( \frac{\dot{Q}_k}{Q_k} \) this defines the variables \( \beta_k^{new} \) up to a constant. Now observe that starting with the flow \( \frac{\dot{Q}_k}{Q_k} = 0 \) gives \( \beta_k = a_1 \), where \( a_1 \in \mathbb{C} \) is an arbitrary constant. Inserting these \( \beta_k \) into the flow equation (2.13) in the reduced case \( g_k = 1 \) gives in the turn a new flow equation

\[ \frac{\dot{Q}_k}{Q_k} = a_1(Q_{k+1} - Q_{k-1}), \]

which is (up to the constant \( a_1 \)) the Volterra equation. Now inserting this Volterra equation into equation (3.22) gives new \( \beta_k \) as

\[ \beta_k := a_1(Q_k + Q_{k-1}) + a_2 \]

which in the turn give the following flows on the cross-ratios:

\[ \frac{\dot{Q}_k}{Q_k} = a_1 ((Q_k - 1)(Q_{k+1} - Q_{k-1})) + a_1(Q_{k+1}(Q_{k+2} + Q_{k+1}) - Q_{k-1}(Q_{k-1} + Q_{k-2})) + a_2(Q_{k+1} - Q_{k-1}). \]

This is (up the constant \( a_1 \)) the next higher flow in the Volterra hierarchy plus a Volterra term if \( a_2 \neq 0 \).

We conjecture that all higher flows of the Volterra hierarchy can be obtained in this way. There are strong indications that this holds also in the continuous case [Pin]. A proof of this conjecture would be beyond the scope of this article, we postpone this to a later publication.

To make contact with the classical results we will now derive the 2 × 2-Lax representation of the Volterra model for our tangential flow:

Define the gauge matrix

\[ E_k := \prod_{i=0}^{k-1} u_i \begin{pmatrix} 1 & 0 \\ 0 & u_k \end{pmatrix} \]  \hspace{1cm} (3.23)

and set

\[ \tilde{L}_k := E_{k+1}^{-1} L_k E_k = \begin{pmatrix} 1 & -1 \\ Q_k & 0 \end{pmatrix} \]  \hspace{1cm} (3.24)
This gauge of $L_k$ implies the following change for $V_k$:

$$
\tilde{V}_k := E_k^{-1}V_k E_k + E_k^{-1}\dot{E}_k
= \begin{pmatrix}
1 + Q_{k-1} & -1 \\
Q_{k-1} & Q_k
\end{pmatrix} + \begin{pmatrix}
Q_{-1} - \frac{1}{2} & 0 \\
0 & Q_{-1} - \frac{1}{2}
\end{pmatrix}.
$$

(3.25)

The second matrix summand may be omitted since it is constant. If we now transpose the system, reverse the direction of the $k$-labeling and introduce the spectral parameter $\lambda$ as mentioned above we end with the two matrices:

$$
\begin{align*}
L^\gamma(\lambda) &= \begin{pmatrix}
1 & \lambda Q_k \\
-1 & 0
\end{pmatrix}
V^\gamma(\lambda) &= \begin{pmatrix}
1 + \lambda Q_{k+1} & \lambda Q_{k+1} \\
-1 & \lambda Q_k
\end{pmatrix}
\end{align*}
$$

(3.26)

with the compatibility condition $\lambda \dot{L}_k^\gamma(\lambda) = V_k^\gamma(\lambda)L_k^\gamma(\lambda)-L_k^\gamma(\lambda)V_{k-1}^\gamma(\lambda)$. This is up to the change $\lambda \to \lambda^{-2}$ and a gauge transformation with

$$
\tilde{E} = \begin{pmatrix}
\lambda^{1/2} & 0 \\
0 & \lambda^{-1/2}
\end{pmatrix}
$$

the known form of the Volterra Lax-pair [Sur02].

### 3.2 Euclidean reduction

Let $c$ be a discrete curve in $\mathbb{C}$ and set $S_k := c_{k+1} - c_k$. If $|S_k| = 1$ holds we call $c$ arc length parametrized.

Remember, that a flow on $c$ can be described via lemma [1]. For an arc length parameterized curve $c$ the curvature $\kappa$ is defined as follows:

$$
\kappa_k = 2\tan\frac{\angle(S_{k-1}, S_k)}{2}.
$$

(3.27)

$\kappa$ can be computed in the following way:

$$
\kappa_k = 2i \frac{1 - \frac{S_k}{S_{k-1}}}{1 + \frac{S_k}{S_{k-1}}}
$$

(3.28)

In the arc length parametrized case we can write

$$
M^h(c_{k+1} - c_k, c_k - c_{k-1}) = 2\frac{S_{k-1}S_k}{S_{k-1} + S_k} = \frac{S_{k-1} + S_k}{1 + \langle S_{k-1}, S_k \rangle}
$$
since
\[ \left\langle 2 \frac{S_{k-1}S_k}{S_{k-1} + S_k}, S_{k-1} \right\rangle = \text{Re}\left(2 \frac{S_{k-1}S_k}{S_{k-1} + S_k} \overline{S_{k-1}}\right) = 1 = \left\langle \frac{S_{k-1} + S_k}{1 + \langle S_{k-1}, S_k \rangle}, S_{k-1} \right\rangle \]
and the same for the scalar product with \( S_k \).

Let us compute how the discrete curvature evolves: Write
\[ \dot{S}_k = i \mu_k S_k \quad (3.29) \]
and since then
\[ \frac{d}{dt} S_k = (\mu_k - \mu_{k-1}) i \frac{S_k}{S_{k-1}} \]
we get
\[ \dot{\kappa} = 2i \left( \frac{d}{dt} S_k \right) \frac{-2}{(1 + \frac{S_k}{S_{k-1}})^2} = (\mu_k - \mu_{k-1})(1 + \frac{\kappa_k^2}{4}) \quad (3.30) \]

On the other hand using equations (3.29) and (3.28) one can calculate \( \mu_k \) to be
\[ \mu_k = \frac{1}{2} (\beta_{k+1} \kappa_{k+1} - \beta_k \kappa_k) - i (\beta_{k+1} + \beta_k) \quad (3.31) \]
In the case \( \beta = 2 \) this implies for the evolution of the discrete curvature \( \kappa \)
\[ \frac{\dot{\kappa}}{1 + \frac{\kappa_k^2}{4}} = \kappa_{k+1} - \kappa_{k-1} \quad (3.32) \]
and as the flow on the discrete curve we get the well known tangential flow \[\text{DS99, BS99}]:\]
\[ \dot{c} = \frac{S_- + S_+}{1 + \langle S_-, S \rangle}. \]

Now let us rewrite \( Q \) to get an interpretation for the choice of \( \beta \) that gives the second Volterra flow \( \beta_k = (Q_{k-1} + Q_k + 1)/2 \):
\[ Q_k = \frac{S_{k-1}S_{k+1}}{(S_{k-1} + S_k)(S_k + S_{k+1})} = ((1 + \frac{S_k}{S_{k-1}})(\frac{S_k}{S_{k+1}} + 1))^{-1} \]
\[ = ((1 + \frac{2i - \kappa_k}{2i + \kappa_k})(1 + \frac{2i + \kappa_{k+1}}{2i - \kappa_{k+1}}))^{-1} = -\frac{1}{16}(2i + \kappa_k)(2i - \kappa_{k+1}) \]
\[ = \frac{1}{16}(2i(\kappa_{k+1} - \kappa_k) + (\kappa_k \kappa_{k+1}) + 4). \]

With this on hand we can calculate
\[ \mu_k = \frac{11}{32} (\kappa_{k+1} + \kappa_k) + \frac{1}{64} \left( \left( \frac{\kappa_{k+1}^2}{4} + 1 \right)(\kappa_{k+2} + \kappa_k) + \left( \frac{\kappa_k^2}{4} + 1 \right)(\kappa_{k+1} + \kappa_{k-1}) \right) \]
which leaves us with
\[ \frac{\dot{\kappa}}{1 + \frac{\kappa_k^2}{4}} = \frac{11}{32} (\kappa_{k+1} - \kappa_{k-1}) + \frac{1}{64} \left( \left( \frac{\kappa_{k+1}^2}{4} + 1 \right)(\kappa_{k+2} + \kappa_k) - \left( \frac{\kappa_{k-1}^2}{4} + 1 \right)(\kappa_k + \kappa_{k-2}) \right) \quad (3.33) \]
for the evolution of the discrete curvature \( \kappa \). This is—up to a tangential flow part which can be removed by adjusting the constant term in the choice of \( \beta \)—a discretization of the mKdV equation:

\[
\dot{\kappa} = \kappa''' + \frac{3}{2} \kappa^2 \kappa'.
\] (3.34)

Therefore we will call the flow that comes from the second Volterra flow discrete mKdV flow:

\[
\dot{c} = \frac{1}{32} \left( (\kappa_k \kappa_{k-1} + \kappa_{k+1} \kappa_k) + 2i(\kappa_{k+1} - \kappa_{k-1}) + 1 \right) M^h(S_{k-1}, S_k) \] (3.35)

**Lemma 6** The discrete tangential flow and the discrete mKdV flow both preserve the discrete arc length parameterization.

**Proof** We calculate \( \langle S_k, \dot{S}_k \rangle \) for a general flow:

\[
\langle S_k, \dot{S}_k \rangle = \text{Re}(\overline{S_k}(\beta_{k+1} M^h(S_k, S_{k+1}) - \beta_k M^h(S_{k-1}, S_k)))
\]

\[
= 2 \text{Re} \left( \frac{\beta_{k+1}}{1 + s_{k+1}} - \frac{\beta_k}{1 + s_{k-1}} \right) = 2 \text{Re}(\beta_{k+1}(1 + i\kappa_{k+1}) - \beta_k(1 - i\kappa_k)).
\]

So the condition for a flow of the form \( \dot{c} = \beta M^h(S_{k-1}, S_k) \) to preserve the discrete arc length is

\[
\text{Re}(\beta_{k+1} - \beta_k) = \text{Im}(\kappa_{k+1} \beta_{k+1} + \kappa_k \beta_k).
\] (3.36)

for the tangential flow this clearly holds. In the case of the mKdV flow it is an easy exercise to show equation (3.36).

In section 5.2.2 we will see that the discrete mKdV flow is connected to so called discrete elastic curves.

### 3.3 Discrete flows

As in the previous section let \( \gamma \) be the lift of a immersed discrete curve in \( \mathbb{CP}^1 \) into \( \mathbb{C}^2 \) satisfying the normalization (3.18).

**Lemma 7** Given an initial \( \tilde{\gamma}_0 \) and a complex parameter \( \mu \) there is an unique map \( \tilde{\gamma} : \mathbb{Z} \rightarrow \mathbb{C}^2 \) satisfying normalization (3.18) and

\[
\mu = \text{cr}(\gamma_k, \gamma_{k+1}, \tilde{\gamma}_{k+1}, \tilde{\gamma}_k).
\] (3.37)

We will call \( \tilde{\gamma} \) a Bäcklund transform of \( \gamma \).
Proof Solving equation (3.37) for \( \tilde{\gamma}_{k+1} \) gives that \( \tilde{\gamma}_{k+1} \) is a Möbius transform of \( \tilde{\gamma}_k \).

Lemma 8 If \( \tilde{\gamma} \) is a Bäcklund transform of \( \gamma \) with parameter \( \mu \) then

\[
\tilde{Q}_k = Q_k \frac{s_k}{s_{k+1}}, \\
(1 - \mu)Q_k = \frac{(1 - s_k)(s_{k+1} - 1)}{(1 - s_k)(s_{k+1} - 1)}
\]

with \( s_k = \text{cr}(\gamma_{k-1}, \tilde{\gamma}_k, \gamma_{k+1}, \gamma_k) \).

Proof Due to the properties of the cross-ratio (a useful table of the identities can be found in [HJHP99]) we have

\[
1 - \mu = \text{cr}(\gamma_k, \tilde{\gamma}_{k+1}, \gamma_{k+1}, \tilde{\gamma}_k) = \frac{\det(\gamma_k, \tilde{\gamma}_{k+1}) \det(\gamma_{k+1}, \tilde{\gamma}_k)}{\det(\tilde{\gamma}_k, \gamma_k) \det(\gamma_{k+1}, \tilde{\gamma}_k)}
\]

\[
Q_k = \frac{\det(\gamma_{k-1}, \gamma_k) \det(\gamma_{k+2}, \gamma_{k+1})}{\det(\gamma_k, \gamma_{k+1}) \det(\gamma_{k+1}, \gamma_k)}
\]

\[
\frac{1}{1 - s_k} = \text{cr}(\gamma_{k-1}, \gamma_k, \tilde{\gamma}_{k+1}, \gamma_{k+1}) = \frac{\det(\gamma_{k-1}, \gamma_k) \det(\tilde{\gamma}_{k+1}, \gamma_{k+1})}{\det(\gamma_k, \gamma_{k+1}) \det(\gamma_{k+1}, \gamma_{k-1})}
\]

\[
\frac{s_{k+1}}{s_{k+1} - 1} = \text{cr}(\gamma_k, \tilde{\gamma}_{k+1}, \gamma_{k+1}, \gamma_{k+2}) = \frac{\det(\gamma_k, \tilde{\gamma}_{k+1}) \det(\gamma_{k+1}, \gamma_{k+2})}{\det(\gamma_{k+1}, \gamma_{k+1}) \det(\gamma_{k+2}, \gamma_{k})}.
\]

Multiplying the first two and the second two equations proves the second statement. If we set \( \tilde{s}_k = \text{cr}(\gamma_{k-1}, \gamma_k, \tilde{\gamma}_{k+1}, \gamma_{k+1}) \) we see that \( \frac{s_k}{s_{k+1}} = 1 \) and therefore

\[
(1 - \mu)\tilde{Q}_k = \frac{\tilde{s}_{k+1}}{(1 - \tilde{s}_k)(s_{k+1} - 1)} = \frac{1}{(1 - s_k)(s_{k+1} - 1)} = \frac{s_k}{(1 - s_k)(s_{k+1} - 1)}
\]

which proves the first statement. \( \square \)

If \( c \) is a periodic curve with period \( N \), we can ask for \( \tilde{c} \) to be periodic too. Since the map sending \( c_0 \) to \( c_N \) is a Möbius transformation it has at least one but in general two fix-points. These special choices of initial points give two Bäcklund transforms that can be viewed as past and future in a discrete time evolution.

We will now show, that this Bäcklund transformation can serve as a discretization of the tangential flow since the evolution on the \( Q \)'s are a discrete version of the Volterra model.

The discretization of the Volterra model first appeared in Tsujimoto, e. al. 1993. We will refer to the version stated in [Sur02]. There it is given in
the form

$$\tilde{\alpha}_k = \frac{\alpha_{k+1}}{\beta_k} \tag{3.39}$$

$$\beta_k - h\alpha_k = \frac{\beta_{k-1}}{\beta_{k-1} - h\alpha_{k-1}} \tag{3.40}$$

with $h$ being the discretization constant.

**Theorem 9** Let $\tilde{Q}$ be a Bäcklund transform of $Q$ with parameter $\mu$. The map sending $Q_k$ to $\tilde{Q}_{k+1}$ is the discrete time Volterra model (3.39) with $\alpha_k = Q_k$, $\tilde{\alpha}_k = Q_{k+1}$, $\frac{\beta_k}{h} = \frac{Q_k}{s_{k+1}}$ and $h = \mu - 1$.

**Proof** With the settings from the theorem we have

$$\tilde{\alpha}_k = \tilde{Q}_{k+1} = Q_{k+1} \frac{s_{k+1}}{s_{k+2}} = Q_k \frac{Q_{k+1} s_{k+1}}{s_{k+2} Q_k} = \alpha_k \frac{\beta_{k+1}}{\beta_k}$$

and on the other hand

$$\beta_k - hQ_k = (\mu - 1)Q_k \left( \frac{1}{s_{k+1}} - 1 \right) = \frac{1}{1 - s_k}$$

and

$$\frac{\beta_{k-1}}{\beta_{k-1} - hQ_{k-1}} = \frac{hQ_k}{s_k} \left( \frac{1}{s_k} - 1 \right) = \frac{1}{1 - s_k}.$$

This proves the theorem.

The continued Bäcklund transformations give rise to maps $\gamma : \mathbb{Z}^2 \to \mathbb{C}P^1$ that can be viewed as discrete conformal maps—especially in the case when $\mu$ is real negative (which is quite far from the tangential flow, that is approximated with $\mu \approx 1$) [BP96, BP99, HJMNP01].

On the other hand in case or real $\mu$ the transformation is not restricted to the plane: Four points with real cross-ratio always lie on a circle. Thus the map that sends $\tilde{\gamma}_k$ to $\tilde{\gamma}_{k+1}$ is well defined in any dimension. Maps from $\mathbb{Z}^2$ to $\mathbb{R}^3$ with cross-ratio -1 for all elementary quadrilaterals serve as discretization of isothermic surfaces and have been investigated in [BP96].

\footnote{More general one can demand $cr = \frac{\alpha_m}{\beta_m}$—see Chapter [BP96].}


4 Flows on discrete curves and the Toda lattice

Definition 4.1 Let $\lambda \in \mathbb{C}$ be arbitrary. Define

\[
p_k := \frac{1}{g_k g_{k-1}} u_k - \lambda
\]

\[
e^{-\frac{q_{k+1} + g_k}{g_k}} := g_k
\]

Clearly the above definitions are not unique. $p_k$ and $g_k$ will be identified with the Flaschka-Manakov variables of the Toda lattice hierarchy [Fla74a, Fla74b, Man74]. $\lambda$ will be the corresponding spectral parameter.

With the above definitions at hand we are now able to state the following correspondence with the Toda lattice hierarchy.

Theorem 4.2 Denote

\[
V_k := \begin{pmatrix}
  v_{11}^k & v_{12}^k \\
  v_{21}^k & v_{22}^k
\end{pmatrix}
\]

Define

\[
\alpha_k := v_{11}^k + \frac{v_{12}^k u_k}{g_{k-1} + g_k}, \quad \beta_k := -\frac{v_{12}^k g_{k-1} u_k}{g_{k-1} + g_k}.
\]

By (2.6) and with definition (4.43), $\alpha_k$ and $\beta_k$ define a certain flow on discrete curves in $\mathbb{C}^2$. Let $V_k$ be a Lax representation matrix corresponding to the n-th Toda flow in the notations as in [FT86]. Then the Lax matrices $V_k$ of the discrete curve flow in (2.17) together with the definitions (4.43) are identical to the above $V_k$. Hence the compatibility equation (2.17) for a flow on discrete curves in $\mathbb{C}^2$ corresponding to the definitions (4.43), is the compatibility equation of the n-th Toda flow.

Proof Setting

\[
-g_{k-1}^{-1} u_k^{-1} (g_{k-1} + g_k) \beta_k \overset{!}{=} v_{12}^k
\]

\[
g_{k-1}^{-1} u_{k-1}^{-1} (g_{k-2} + g_{k-1}) \beta_{k-1} \overset{!}{=} v_{21}^k
\]

results in the constraint

\[
v_{12}^k \overset{!}{=} -v_{21}^k \frac{g_k}{g_{k-1}}
\]

But this constraint is just the 22-component of the compatibility equation (2.17) for general $V_k$ and the Toda $L_k$, hence it is satisfied by all $V_k$ of the
Toda hierarchy. Hence the variables $\beta_k$ are well defined. Likewise the second constraint obtained by setting

$$\alpha_k + \frac{1}{g_{k-1}} \beta_k \overset{!}{=} v_k^{11}$$  \hspace{1cm} (4.46)
$$\alpha_{k-1} - \frac{1}{g_{k-1}} \beta_{k-1} \overset{!}{=} v_k^{22}$$  \hspace{1cm} (4.47)

together with the 22-component gives the 21 component of (2.17). Hence the variables $\alpha_k$ are well defined. The 11- and 12-component are giving the Toda field equations.

**Remark 4.3** The above matrices can be regauged into the matrices $\hat{L}_k = \Omega_{k+1}^{-1} L_k \Omega_k^{-1}$ and $\hat{V}_k = \Omega_k V_k \Omega_k^{-1} + \dot{\Omega}_k \Omega_k^{-1}$ with

$$\Omega_k = \begin{pmatrix} e^{\frac{q_k}{2}} & 0 \\ 0 & -e^{\frac{-q_{k-1}}{2}} \end{pmatrix}.$$  

$\hat{L}_k$ and $\hat{V}_k$ are then the 2 by 2 matrix representation of the usual Flaschka-Manakov matrices [Sur02]. One has $tr V_k = tr \hat{V}_k = \frac{1}{2}(\dot{q}_k - \dot{q}_{k-1})$. On the other hand by the definition of the matrices $V_k$ in proposition 2.3 one has

$$\frac{\dot{q}_k}{g_k} = \frac{1}{2}(\dot{q}_k - \dot{q}_{k+1}) = tr \hat{V}_{k+1}.$$  \hspace{1cm} (4.48)

therefore

$$\dot{q}_k = tr \hat{V}_{k+1}.$$  \hspace{1cm} (4.49)

5  **The first three Toda lattice hierarchy flows and reductions of them**

5.1  **The first three Toda flows**

The flow directions for a discrete curve $\gamma$ is given by a specific choice of the variables $\alpha_k$, $\beta_k$ (compare with (2.6)). In the following we will choose $\alpha_k$, $\beta_k$ in such a way that the corresponding evolution for the determinants $g_k$, and $u_k$ (2.3) is the evolution of the canonical variables of the toda lattice hierarchy. In this section we will look at the flow directions given by the “first” three flows of the Toda lattice hierarchy.
Proposition 5.1 (First flow) Using (2.7) define the following flow directions for $\gamma$:

$$
\alpha^{TL1}_k = -\frac{1}{2}(p_k + \lambda) \frac{g_{k-1} - g_k}{g_{k-1} + g_k} = - \frac{u_k}{2} \frac{g_{k-1} - g_k}{g_{k-1} + g_k}
$$  \hspace{1cm} (5.50)

$$
\beta^{TL1}_k = -\frac{g_{k-1} g_k (p_k + \lambda)}{g_{k-1} + g_k} = - \frac{u_k}{g_{k-1} + g_k}
$$  \hspace{1cm} (5.51)

The induced flow on the determinants $g_k$ and $u_k$ (2.3) is with definitions (4.41), (4.42) given by the first Toda lattice flow.

Proof In accordance with theorem 4.2 we obtain the following compatibility matrices:

$$
L_k = \begin{pmatrix}
  g_k(p_k + \lambda) & -\frac{g_k}{g_{k-1}} \\
  1 & 0
\end{pmatrix}
$$  \hspace{1cm} (5.52)

$$
V_k = \begin{pmatrix}
  -\frac{1}{2}(p_k + \lambda) & g_{k-1}^{-1} \\
  -g_{k-1}^{-1} & \frac{1}{2}(p_{k-1} + \lambda)
\end{pmatrix}
$$  \hspace{1cm} (5.53)

These matrices are wellknown [FT86] and give the Toda lattice equations, which are called the first flow of the Toda lattice hierarchy:

$$
\dot{g}_k = tr V_{k+1} = \frac{1}{2}(p_k - p_{k+1})
$$  \hspace{1cm} (5.54)

$$
\dot{p}_k = g_k^{-2} - g_{k-1}^{-2}
$$  \hspace{1cm} (5.55)

Remark 5.2 Due to remark 4.3 the variables $\dot{q}_k$ evolve with $\dot{q}_k = p_k + \lambda$. Hence

$$
\ddot{q}_k = g_k^{-2} - g_{k-1}^{-2}
$$

which is the wellknown Toda lattice equation.

Analogously the flows for the next two higher flows [Sur02] can be determined:

Proposition 5.3 (second flow) Define

$$
\alpha^{TL2}_k = -\frac{1}{2}(p_k^2 - \lambda^2) \frac{g_{k-1} - g_k}{g_{k-1} + g_k} - \frac{1}{2}(g_k^{-2} - 2g_{k-1}^{-2} + g_{k-2}^{-2})
$$  \hspace{1cm} (5.56)

$$
\beta^{TL2}_k = -\frac{g_{k-1} g_k (p_k^2 - \lambda^2)}{g_{k-1} + g_k}
$$  \hspace{1cm} (5.57)
The induced flow on $g_k$ and $p_k$ (2.3) is with definitions (4.41), (4.42) given by the second flow of the Toda lattice hierarchy [Sur02]:

$$\frac{\dot{g}_k}{g_k} = \frac{1}{2}(p_{k+1}^2 - p_k^2 + g_{k+1}^{-2} - g_{k-1}^2) \quad (5.58)$$

$$\dot{p}_k = g_k^{-2}(p_{k+1}^2 + p_k^2 - g_{k-1}^{-2}(p_k + p_{k-1})) \quad (5.59)$$

Remark 5.4 $\dot{q}_k = g_k^{-2} + g_{k-1}^{-2} + p_k^2 - \lambda^2$.

Proposition 5.5 (third flow) Define

$$\alpha_{TL}^{T3} = \frac{1}{2}g_{k-1}^{-1}g_k(p_{k}^3 + \lambda^3 - 2(p_k + \lambda)g_k^{-1}g_{k-1}^{-1}) \quad (5.60)$$

$$-\frac{1}{2}g_k^{-2}(2p_k + p_{k+1}) + \frac{1}{2}g_{k-1}^{-2}(p_k + 2p_k) \quad (5.61)$$

$$\beta_{TL}^{T3} = \frac{g_{k-1}g_k(p_k + \lambda)}{g_k^{-1} + g_k}(g_{k-1}^{-2} + g_k^{-2} + p_k^2 - \lambda p_k + \lambda^2). \quad (5.62)$$

The induced flow on $g_k$ and $p_k$ (2.3) is with definitions (4.41), (4.42) given by the third flow of the Toda lattice hierarchy [Sur02]:

$$\frac{\dot{g}_k}{g_k} = \frac{1}{2}[(p_{k+1}^3 + p_k + p_{k+2}g_{k+1}^{-2} + 2p_{k+1}g_{k+1}^{-2} + p_k + p_{k+1}g_k^{-2})$$

$$- (p_{k-1}^3 + p_k - p_{k-1}g_k^{-2} + p_k g_{k-1}^{-2} + 2p_k g_{k-1}^{-2})]$$

$$\dot{p}_k = g_k^{-2}(p_{k+1}^2 + p_k^2 + p_{k+1}p_k + g_{k+1}^{-2} + g_k^{-2}) - g_{k-1}^{-2}(p_k^2 + p_{k-1}^2 + p_k p_{k-1} + g_{k-1}^{-2} + g_k^{-2}) \quad (5.64)$$

Remark 5.6 $\dot{q}_k = g_k^{-2}(2p_k + p_{k+1}) + g_{k-1}^{-2}(p_{k-1} + 2p_k) + p_k^3 + \lambda^3$.

5.2 Towards geometrical interpretations of the Toda lattice

In the following we look at the reduction $p_k = 0$ and determine the geometric shape of curves, which are invariant under the trivial Toda and the discrete KdV flow.
5.2.1 Reduction $p_k = 0$ and invariant curves under the trivial Toda flow

In the previous section one reduced the discrete curves in $\mathbb{C}^2$ to so-called arclength parametrized curves, by setting $g_k = 1$ for all $k$. It was conjectured that for that reduction there exist certain flow directions (3.22) which let the cross-ratios evolve according to flows in the Volterra hierarchy.

On the other hand it is a fact [Sur02] that the Toda lattice hierarchy reduces to the Volterra hierarchy for flows with an even enumeration number if $p_k = 0$. Indeed, looking at the above equations one sees easily that if $p_k = 0$ then the evolution of the $g_k^{-2}$ in equation 5.58 is given by the Volterra equation. In the same manner the third flow admits no reduction but the fourth flow would again give an equation for the $g_k^{-2}$ which can be identified with a discrete version of the KdV equation [Sur02]. We observe that for the case $p_k = 0$ the cross-ratio is given by

$$Q_k = \lambda g_k^{-2}.$$

Hence as a direct consequence

**Proposition 5.7** If $\lambda = 1$, $p_k = 0$ then the flow of the cross-ratios $Q_k$, as given by the flow of the 2n’th Toda lattice hierarchy via theorem 4.2 satisfies the equations of the n’th Volterra hierarchy.

As can be seen by looking at the above Toda lattice hierarchy equations, the reduction $g_k = \text{const}$ is generally not compatible with the equations. It is compatible if $p_k = \text{const}$ (and especially $p_k = 0$, which can always be achieved by a change of $\lambda$), which gives the trivial evolution $\dot{g}_k = 0, \dot{p}_k = 0$ for all flows of the Toda hierarchy. Hence the two reductions $\dot{g}_k = 0$ and $p_k = 0$ with their corresponding Volterra hierarchy flows, seem to be in some kind of duality, where only the trivial reduction $g_k = \text{const} \land p_k = \text{const}$ $\leftrightarrow u_k = \text{const} \land g_k = \text{const}$ seems to lie in the intersection of the two pictures. We were interested in what kind of curves belong to this most trivial solution of the Toda flows. The following proposition shows that if the variables $u_k = u$ and $g_k = g$ are constants, then $\gamma$ lies on a quadric, or in other words in this case $\gamma$ defines a **discrete quadric** in $\mathbb{C}^2$.

**Proposition 5.8**

a) Let $\gamma_0$, $\gamma_1$ be fixed initial conditions for a discrete curve $\gamma : \mathbb{Z} \rightarrow \mathbb{C}^2$ with

$$\langle M \gamma_0, \gamma_0 \rangle = 1 \quad (5.65)$$
$$\langle M \gamma_1, \gamma_1 \rangle = 1 \quad (5.66)$$
$$\langle M \gamma_0, \gamma_1 \rangle = \frac{u}{2g} \quad (5.67)$$
$$\langle M \gamma_1, \gamma_0 \rangle = \frac{u}{2g} \quad (5.68)$$
where $\langle \gamma, \gamma \rangle = x^2 + y^2$ is the naive complexification of the real scalar product to $\mathbb{C}^2$ and $M$ is a symmetric 2 by 2 matrix; $g = \det(\gamma_0, \gamma_1) \neq 0$ and $u$ is a free parameter. Then $M$ is uniquely defined by conditions 5.68 and $\det M = \frac{1}{g^2}(1 - u^2)$.

b) Define a discrete curve with the above initial conditions recursively by:

$$\gamma_{k+2} := \frac{u}{g} \gamma_{k+1} - \gamma_k$$

then

$$\langle M\gamma_k, \gamma_{k+1} \rangle = \frac{u}{2g}$$

$$\langle M\gamma_k, \gamma_k \rangle = 1 \quad \text{for all} \quad k \in \mathbb{Z}. \quad (5.70)$$

**Proof**

a) After a lengthy calculation $M$ can be derived from the conditions (5.68) as

$$M = \frac{1}{g^2} \left( \begin{array}{cc} y_0^2 + y_1^2 - 2y_0y_1 \frac{u}{2g} & (x_0y_1 + y_0x_1) \frac{u}{2g} - (x_0y_0 + x_1y_1) \\ (x_0y_1 + y_0x_1) \frac{u}{2g} - (x_0y_0 + x_1y_1) & x_0^2 + x_1^2 - 2x_0x_1 \frac{u}{2g} \end{array} \right)$$

b) $\langle M\gamma_2, \gamma_1 \rangle = \frac{u}{g} \langle M\gamma_1, \gamma_1 \rangle - \langle M\gamma_0, \gamma_1 \rangle = \frac{u}{g} \cdot 1 - \frac{u}{2g}$, hence by induction $\langle M\gamma_k, \gamma_{k+1} \rangle = \frac{u}{2g}$ for all $k \in \mathbb{Z}$. From this it follows that

$$\langle M\gamma_2, \gamma_2 \rangle = \frac{u^2}{g^2} \langle M\gamma_1, \gamma_1 \rangle + \langle M\gamma_0, \gamma_0 \rangle - 2 \frac{u}{g} \langle M\gamma_1, \gamma_0 \rangle = 1$$

and hence by induction $\langle M\gamma_k, \gamma_k \rangle = 1$ for all $k \in \mathbb{Z}$.

Looking at the variables $\alpha_k, \beta_k$ corresponding to the first three Toda flows (e.g. (5.51)) the case $p_k = \text{const} \wedge g_k = \text{const}$ gives the following evolution on the curves:

$$\dot{\gamma}_k = \text{const} (\gamma_{k+1} - \gamma_{k-1}) \quad (5.72)$$

where the constant const varies corresponding to the considered Toda flow.

**Proposition 5.9** A flow direction as in (5.72) along a discrete quadric $\gamma$ (as defined in proposition 5.8) is tangent to the (smooth) quadric on which the points of $\gamma$ lie.

**Proof** We have to show that if $\dot{\gamma}_k = \frac{d}{dt} \gamma_k = \rho(\gamma_{k+1} - \gamma_{k-1})$ then

$$\frac{d}{dt}\big|_0 < M\gamma_k(t), \gamma_k(t) >= 0 \quad \text{and} \quad \frac{d}{dt}\big|_0 \langle M\gamma_k(t), \gamma_{k+1}(t) \rangle = 0.$$
Now
\[
\frac{d}{dt} \langle M \gamma_k(t), \gamma_k(t) \rangle = \langle M \dot{\gamma}_k, \gamma_k \rangle + \langle M \gamma_k, \dot{\gamma}_k \rangle = 2\rho (\langle (M \gamma_{k+1}, \gamma_k) - (M \gamma_{k-1}, \gamma_k) \rangle
\]
Using lemma 1 the proof for the second assertion works analogously.

By the above, a class of curves (namely quadrics) was sorted out by looking at trivial solutions to the Toda lattice equations. The movement of these curves under the corresponding Toda flow (tangent to the quadric) was very geometrical and simple.

In the next section a similar construction will be done. Here it will be shown in the euclidian reduction \(3.2\), that discrete curves which evolve under the discrete mKdV flow simply by a translation, define so-called discrete elastic curves.

### 5.2.2 Invariant curves under the discrete mKdV flow: Discrete generalized elastic curves

Let \( c \) be a discrete arc length parametrized curve in \( \mathbb{C} \) as discussed in section \( 5.2 \).

**Definition 2** A discrete regular arc length parametrized curve \( c: \{0, \ldots, N\} \rightarrow \mathbb{C} \) is called planar elastic curve if it is an critical point to the functional
\[
\sum_{j=1}^{N-1} \log(1 + \kappa_j^4) \quad (5.73)
\]
The admissible variations preserve the arc length, \( c_N - c_0 \) and the tangents at the end points.

It can be shown \([BHS]\) that the curvature of a discrete elastic curve obeys the following equation:
\[
\kappa_{k+1} = 2a \frac{\kappa_k}{1 + \kappa_k^4} - \kappa_{k-1}. \quad (5.74)
\]
for some real constant \( a \).

We now want to know, which class of curves is invariant under the discrete mKdV flow (up to euclidean motion and some tangential flow). It will turn out that that discrete elastic curves are a special case in that class.
Since discrete arc length parametrized curves are determined by their curvature (3.27) up to euclidean motion, it is sufficient to impose the constraint that the curvature must not change up to the changes made by the tangential flow (3.32). In other words: we ask for solutions \( \kappa \) for which (3.32) is a multiple of (3.33):

\[
(a - \frac{11}{32})(\kappa_{k+1} - \kappa_{k-1}) = \frac{1}{64} \left( \frac{\kappa_{k+1}^2}{4} + 1 \right) (\kappa_{k+2} + \kappa_k) - \left( \frac{\kappa_{k-1}^2}{4} + 1 \right) (\kappa_k + \kappa_{k-2})
\]

(5.75)

One can “integrate” this equation twice and get the following lemma:

**Lemma 10** The curvature of a discrete curve, that evolves up to some tangential flow by Euclidean motion under the mKdV flow satisfies

\[
\kappa_{k+1} = \frac{2a\kappa_k}{1 + \kappa_k^2} - \kappa_{k-1} + b + c_k
\]

(5.76)

for some constants \( a \) and \( b \) and a function \( c \) with \( c_{k+1} = -c_k \).

**Proof** Nothing left to show.

In the case \( b = 0 \) and \( c \equiv 0 \) this gives the equation for planar elastic curves (5.74). Figure 1 shows three closed discrete generalized elastic curves.

### 6 Conclusion

In this paper we introduced a novel view onto the one dimensional Toda lattice hierarchy in terms of flows on discrete curves. In particular this view allowed us to give a geometric meaning to the trivial Toda flow and the mKdV flow which are special flows within the Toda lattice hierarchy. This was achieved by classifying that class of curves, which is invariant under the corresponding flow. It would be interesting to investigate discrete curves that belong to other Toda flows in this sense. In fact we view our exposition rather as a starting point for a more thorough investigation of this subject.

If one finds a meaningful symplectic structure on the space of curves \( \gamma \), then by the novel view onto the Toda lattice hierarchy this may lead to a symplectic structure for the “vertex operators” \( F_n \). We are currently working on that issue.

The theory of discrete curves is in close connection to the theory of discrete surfaces. In addition it was already known to Darboux [Dol97] that certain invariants on surfaces admitting a conjugate net parametrization satisfy the two dimensional Toda lattice equation. It is an interesting question
to see whether our approach can be transferred to discrete surfaces and the two dimensional Toda lattice hierarchy [UT84].

It would be interesting to see, whether our approach could also be applied to generalized Toda Systems [Kos79]. We haven’t thought about that yet.

Acknowledgements

We like to thank Ulrich Pinkall for deep and interesting discussions and many inspiring ideas. We like to thank Yuri Suris and Leon Takhtajan for helpful hints.

References

[BHS] A. I. Bobenko, T. Hoffmann, and W. Schief. Discrete differential geometry. unpublished.
[BP96] A. Bobenko and U. Pinkall. Discrete isothermic surfaces. *J. reine angew. Math.*, 475:178–208, 1996.

[BP99] A. Bobenko and U. Pinkall. Discretization of surfaces and integrable systems. In Bobenko A. and Seiler R., editors, *Discrete Integrable Geometry and Physics*. Oxford University Press, 1999.

[BS99] A. Bobenko and Y. Suris. Discrete time Lagrangian mechanics on Lie groups, with an application to the Lagrange top. *Comm. Math. Phys.*, 204:147–188, 1999.

[Dol97] A. Doliwa. Geometric discretization of the Toda system. *Phys. Lett. A*, 234:187–192, 1997.

[DS99] A. Doliva and Santini. Geometry of discrete curves and lattices and integrable difference equations. In A. Bobenko and R. Seiler, editors, *Discrete integrable geometry and physics*, chapter Part I 6. Oxford University Press, 1999.

[Fla74a] H. Flaschka. The Toda lattice I. existence of integrals. *Phys. Rev. B*, 9:1924–1925, 1974.

[Fla74b] H. Flaschka. The Toda lattice II. inverse scattering solution. *Progr. Theor. Phys.*, 51:703–716, 1974.

[FT86] L. D. Faddeev and L. A. Takhtajan. *Hamiltonian methods in the theory of solitons*. Springer, 1986.

[HJHP99] U. Hertrich-Jeromin, T. Hoffmann, and U. Pinkall. A discrete version of the Darboux transformation for isothermic surfaces. In A. Bobenko and R. Seiler, editors, *Discrete Integrable Geometry and Physics*, pages 59–81. Oxford University Press, 1999.

[HJMP01] U. Hertrich-Jeromin, I. McIntosh, P. Norman, and F. Pedit. Periodic discrete conformal maps. *J. reine angew. Math.*, 534:129–153, 2001.

[Kos79] B. Kostant. The solution to a generalized toda lattice and representation theory. *Adv. Math.*, 34:195–338, 1979.

[Man74] S. V. Manakov. On the complete integrability ond stochasization in discrete dynamical systems. *Sov. Phys. JEPT*, 40:269–274, 1974.
[Pin] U. Pinkall. private communication.

[Sur02] Yu. Suris. The problem of integrable discretization: Hamiltonian approach. to be published, 2002.

[Tod89] M. Toda. Theory of nonlinear Lattices. Springer-Verlag, 1989.

[UT84] K. Ueno and K. Takasaki. Toda lattice hierachy. Adv. Stud. Pure Math., 4(1–95), 1984.