GENERALIZED SCHRÖDINGER REPRESENTATION 
IN BRST–QUANTIZATION

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Abstract

An analysis of the state space in the BRST–quantization in the Schrödinger representation is performed on the basis of the results obtained earlier in the framework of the Fock space representation. It is shown that to get satisfactory results it is necessary to have from the very beginning a meaningful definition of the total state space.
1 Introduction

Now the BRST–quantization method is the most popular method for the covariant quantization of gauge–invariant systems. This method is based on the concept of the BRST–symmetry, being a special type of symmetry generated by a nilpotent operator $\hat{\Omega}$ \[^{1}\]. The procedure of the BRST–quantization in its simplest form looks as follows \[^{2}\]. Starting with a given gauge–invariant system, characterized by a gauge–invariant Lagrangian, we extend the configuration space by adding ghost and antighost variables, and construct a BRST–invariant effective Lagrangian. This effective Lagrangian, unlike the original one, is nondegenerate. The usual methods of quantization applied to the effective Lagrangian leads us to the total state space of the BRST–quantization, where the BRST–symmetry generator $\hat{\Omega}$ acts. The physical subspace is specified then by the condition

$$\hat{\Omega}\Psi = 0. \tag{1.1}$$

Since we have

$$\hat{\Omega}^2 = 0, \tag{1.2}$$

then the vectors of the form

$$\Psi = \hat{\Omega}\Phi \tag{1.3}$$

evidently belong to the physical subspace. Moreover, such vectors are orthogonal to any physical vector. Factorizing the physical subspace by the subspace, formed by the vectors of form (1.3), we come to the physical state space of the system.

The BRST–charge $\hat{\Omega}$ is by definition a hermitian operator. It is quite clear that a hermitian nilpotent operator in a space with a positive definite scalar product is trivial. Hence, the total state space must have an indefinite scalar product, and the question on positivity of the scalar product in the physical state space arises. In fact, the complete answer to this question is yet unknown. For a special class of systems with a quadratic BRST–charge a strict proof of the positivity of the scalar product in the physical state space was given in Ref. \[^{3}\]. The proof was based on the representation of the total state space as a $\mathbb{Z}_2$–graded Krein space. The relevant operators were there creation and annihilation operators. There were a few attempts to analyze the structure of the total state space starting with the operators of generalized coordinates and momenta in the Schrödinger representation, but these attempts have not led, from our point of view, to satisfactory results. In the present paper we translate the results of Ref. \[^{3}\] to the language of the Schrödinger representation and discuss the drawbacks of the previous considerations.

The main difference between the consideration given here and the attempts made earlier is indefiniteness of the scalar product in the sector of the bosonic unphysical operators. This obstacle leads to the necessity of using the so–called generalized Schrödinger representation for the operators of generalized coordinates and momenta \[^{4}\]. The next very important point, allowing to get satisfactory results, is the usage of an auxiliary positive definite scalar product in the total state space. This scalar product defines the norm which allows to get rid of the superfluous physical states arising in a naive treatment of the problem.
2 Formulation of the problem

There are two different approaches to constructing a BRST–invariant effective theory starting from a given gauge–invariant system. In the first approach, discussed in the introduction, the initial object is a gauge–invariant Lagrangian. Following the second approach, one starts with the Hamiltonian description of the initial system. The constraints, arising in this description, determine the form of the corresponding BRST–charge (for a review see Refs. [5, 6]). In fact, both approaches are essentially equivalent (see Ref. [7] and references therein). In the present consideration we are interested only in the form of the BRST–charge and follow the Hamiltonian approach.

Let us consider the hamiltonian system with the phase space described by generalized coordinates $q_i, Q_\alpha, i = 1, \ldots, n, \alpha = 1, \ldots, m$, and generalized momenta $p_i, P_\alpha$, having the usual nonzero Poisson brackets:

$$\{p_i, q_j\} = -\delta_{ij}, \quad \{P_\alpha, Q_\beta\} = -\delta_{\alpha\beta}. \quad (2.1)$$

Suppose that there are $n$ first class constraints [8] of the form

$$p_i = 0. \quad (2.2)$$

To construct the BRST–charge [5, 6], associated with the system under consideration, we enlarge the phase space of the system adding to the initial (even) coordinates odd coordinates $\theta_i, \pi_i, i = 1, \ldots, n$, with the nonzero Poisson brackets

$$\{\pi_i, \theta_j\} = -\delta_{ij}. \quad (2.3)$$

According to the general scheme [5, 6], the BRST–charge in our case has the form

$$\Omega = \theta_i p_i. \quad (2.4)$$

We shall treat the commuting and anticommuting variables as even and odd elements of a Grassmann algebra $\mathbb{G}$ with countably infinite number of generators over the field of complex numbers $\mathbb{C}$ [9, 10, 11]. Suppose that the Grassmann algebra $\mathbb{G}$ is supplied with an involution, so that we can define the concept of a real variable. It is natural to consider the even variables $q_i, Q_\alpha, p_i$ and $P_\alpha$ as real variables. We suppose also that the odd generalized coordinates $\theta_i$ are real, while the odd generalized momenta $\pi_i$ are imaginary [11]. Denoting the involution in $\mathbb{G}$ by an asterisk, we can write

$$q_i^* = q_i, \quad p_i^* = p_i, \quad Q_\alpha^* = Q_\alpha, \quad P_\alpha^* = P_\alpha, \quad (2.5)$$

$$\theta_i^* = \theta_i, \quad \pi_i^* = -\pi_i. \quad (2.6)$$

In quantum theory we have the set of the operators $\hat{q}_i, \hat{Q}_\alpha, \hat{\theta}_i$, corresponding to the generalized coordinates, and the operators $\hat{p}_i, \hat{P}_\alpha, \hat{\pi}_i$, corresponding to the generalized momenta. These operators irreducibly act in a Hilbert space $\mathcal{H}$ with an indefinite scalar product $(~,~)$. According to Eqs. (2.5), (2.6) we suppose that

$$\hat{q}_i^\dagger = \hat{q}_i, \quad \hat{p}_i^\dagger = \hat{p}_i, \quad \hat{Q}_\alpha^\dagger = \hat{Q}_\alpha, \quad \hat{P}_\alpha^\dagger = \hat{P}_\alpha, \quad (2.7)$$

$$\hat{\theta}_i^\dagger = \hat{\theta}_i, \quad \hat{\pi}_i^\dagger = -\hat{\pi}_i. \quad (2.8)$$
where $\dagger$ means the hermitian conjugation with respect to the scalar product $(, )$. The operators, we have introduced, satisfy the following commutation relations
\begin{align*}
[\hat{p}_i, \hat{q}_j] &= -i\delta_{ij}, \quad [\hat{P}_\alpha, \hat{Q}_\beta] = -i\delta_{\alpha\beta}, \quad [\hat{\pi}_i, \hat{\theta}_j] = -i\delta_{ij}, \tag{2.9}
\end{align*}
where only the nontrivial relations are written. Note that the symbol $[,]$ means here the generalized commutator $[3]$. According to the classical expression, we suppose that the BRST–charge in quantum theory has the form
\begin{equation}
\hat{\Omega} = \hat{\theta}_i \hat{\pi}_i. \tag{2.10}
\end{equation}
It is clear that the operator $\hat{\Omega}$ is hermitian and nilpotent.

It is convenient to consider a concrete representation of the operators we are dealing with. The most popular here is the Schrödinger representation. In this representation the state space $\mathcal{H}$ of the system under consideration is formed by the functions of commuting variables $q_a, Q_\alpha$ and anticommuting variables $\theta_a$, having the form
\begin{equation}
\Psi(q, Q, \theta) = \sum_{k=0}^{n} \frac{1}{k!} \Psi_{i_1...i_k}(q, Q) \theta_{i_1} \ldots \theta_{i_k}, \tag{2.11}
\end{equation}
where the functions $\Psi_{i_1...i_k}(q, Q)$ take values in $G$. Actually we restrict ourselves to the functions $\Psi_{i_1...i_k}(q, Q)$ being continuations of ordinary functions of $n + m$ real variables $[10]$.

We define the integral of a function $\Psi(q, Q, \theta)$ over commuting and anticommuting variables in accordance with Ref. $[10]$, with the only difference that the integration over the odd variables $\theta_i$ is normalized by
\begin{equation}
\int \Psi(q, Q, \theta) d^n \theta = \Psi_{1...n}(q, Q). \tag{2.12}
\end{equation}
The scalar product in $\mathcal{H}$ is defined as
\begin{equation}
(\Psi, \Phi) = (-i)^{n(n-1)/2} \int \Psi^*(q, Q, \theta) \Phi(q, Q, \theta) d^n q d^m Q d^n \theta. \tag{2.13}
\end{equation}
The operators, corresponding to the generalized coordinates, are realized in the Schrödinger representation as multiplication operators:
\begin{align*}
\hat{q}_i \Psi &= q_i \Psi, \quad \hat{Q}_\alpha \Psi = Q_\alpha \Psi, \quad \hat{\theta}_i \Psi = \theta_i \Psi, \tag{2.14}
\end{align*}
while the operators, corresponding to the generalized momenta, are proportional to differentiation operators:
\begin{align*}
\hat{p}_i \Psi &= -i \frac{\partial \Psi}{\partial q_i}, \quad \hat{P}_\alpha \Psi = -i \frac{\partial \Psi}{\partial Q_\alpha}, \quad \hat{\pi}_i \Psi = -i \frac{\partial \Psi}{\partial \theta_i}. \tag{2.15}
\end{align*}
It can be shown that such definition of the operators and the scalar product leads us to relations $(2.7)-(2.9)$. 


For the action of the BRST–charge on an arbitrary state vector \( \Psi(q, Q, \theta) \) we get the expression
\[
(\hat{\Omega}\Psi)(q, Q, \theta) = -i \sum_{k=0}^{n-1} \frac{1}{k!} \left( \hat{p}_{[i_1}^{(k)} \Psi_{i_2...i_{k+1}]} \right) (q, Q) \theta_{i_1} \cdots \theta_{i_{k+1}},
\] (2.16)
where the square brackets means the antisymmetrization. From the above expression it follows that the BRST–invariance condition (1.1) is equivalent in our case to the set of relations
\[
\hat{p}_{[i_1}^{(k)} \Psi_{i_2...i_{k+1}]} = 0, \quad k = 0, \ldots, n-1.
\] (2.17)
In particular, for \( k = 0, 1 \) we have
\[
\hat{p}_i^{(0)} \Psi = 0,
\] (2.18)
\[
\hat{p}_i^{(1)} \Psi_j - \hat{p}_j^{(1)} \Psi_i = 0.
\] (2.19)
From Eq. (2.19) it follows that there exists a function \( \Phi(q, Q) \) such that
\[
\Psi^{(1)}_i = \hat{p}_i \Phi.
\] (2.20)
In fact, it can be shown that there exists a set of functions \( \Phi^{(k)}_{i_1...i_k}(q, Q) \), such that
\[
\Psi^{(k+1)}_{i_1...i_{k+1}} = (k + 1)\hat{p}_{[i_1}^{(k)} \Phi^{(k)}_{i_2...i_{k+1}]}], \quad k = 0, \ldots, n-1.
\] (2.21)
Thus, any BRST–invariant state vector can be represented in the form
\[
\Psi = \Psi^{(0)} + \hat{\Omega}\Phi,
\] (2.22)
where the function \( \Psi^{(0)}(q, Q) \) satisfies relations (2.18). To prove the validity of this statement, it is convenient to establish an analogy of the action of the operator \( \hat{\Omega} \) with the action of the exterior derivative operator. To this end, let us treat the variables \( Q_\alpha \) as parameters, and associate with a state vector of form (2.11) the differential form
\[
\tilde{\Psi} = \sum_{k=0}^{n} \frac{1}{k!} \Psi^{(k)}_{i_1...i_k}(q, Q) dq_{i_1} \wedge \cdots \wedge dq_{i_k}.
\] (2.23)
It is not difficult to get convinced that
\[
\tilde{\Omega}\Psi = -id\tilde{\Psi}.
\] (2.24)
Using now the Poincaré lemma [12], we easily come to representation (2.22).

Factorizing out the state vectors of form (1.3), we conclude that the physical state space is formed by functions which do not depend on \( q_i \) and \( \theta_i \). This result is, at a first sight, very attractive, because it establishes a direct correspondence between the physical state space in the BRST–quantization and the state space, arising in the Dirac quantization of
the system \cite{8}. From the other side, we see that the scalar product for the physical state vectors is undefined. We can get for it either zero or infinite value depending on the order of the integration over the commuting and anticommuting variables we shall choose. This discouraging result forces us to consider the above reasonings more carefully.

The first observation, we can make, is that we have not actually given a strict definition of the total state space. What functions of commuting and anticommuting variables do really belong to it? Recall that in a usual situation, which we encounter in quantum mechanics, we consider as the vectors, describing the states of a system in the Schrödinger representation, the square integrable functions. For such functions the scalar product takes finite values. In our case the situation is more complicated. If we define the state space as the set of vectors having finite scalar square, we shall discover that not any pair of such vectors has finite scalar product.

In any case a strict definition of the state space may destroy the consideration given above, because it may happen that the vector $\Phi$, entering representation \cite{222}, does not belong to the state space, and the physical state space is different from that we have gotten in the above consideration.

Recall also that the functions, which do not depend on $q_i$ and $\theta_i$, correspond to the state vectors which trivially satisfy the BRST–invariance condition, and the scalar product for such states is undefined. Hence, any reasonable definition of the state space should allow to get rid of such vectors.

Note here that in Ref. \cite{3} there was given a strict consideration of the problem we are discussing here. The main difference from the present paper was the usage in Ref. \cite{3} of creation and annihilation operators in the Fock space representation instead of operators of generalized coordinates and momenta in the Schrödinger representation which we use here. It is interesting to translate the results of Ref. \cite{3} to the language of the Schrödinger representation and compare them with the results obtained here.

3 State space of model

Let us begin with the discussion of the state space of the model considered in Ref. \cite{3}. First recall that the classical formulation of the BRST–quantization involves both commuting and anticommuting variables, furthermore it is supposed that an involution for these variables is introduced. Accordingly, in quantum theory the corresponding operators act in a $\mathbb{Z}_2$–graded Hilbert space, and the involution corresponds to the hermitian conjugation in this space.

For the model considered in Ref. \cite{3} there are the operators $A_\alpha$ and $A_\alpha^\dagger$ corresponding to physical particles, and the operators $a_i$, $\bar{a}_i$, $c_i$, $\bar{c}_i$ and $a_i^\dagger$, $\bar{a}_i^\dagger$, $c_i^\dagger$, $\bar{c}_i^\dagger$ corresponding to unphysical particles. It is supposed that the operators of physical and unphysical particles act in a $\mathbb{Z}_2$–graded Hilbert space $\mathcal{H}$. The scalar product in $\mathcal{H}$ is denoted by $\langle \ , \ \rangle$, and the hermitian conjugate operator of an operator $O$ is denoted by $O^\dagger$. It is assumed that

$$|A_\alpha| = \bar{0}, \quad |a_i| = |\bar{a}_i| = \bar{0}, \quad |c_i| = |\bar{c}_i| = \bar{1}. \quad (3.1)$$

Here and below we follow the notations of Ref. \cite{3}. The nontrivial commutation relations for the operators of physical and unphysical particles have the form

$$[A_\alpha, A_\beta^\dagger] = \delta_{\alpha\beta}, \quad (3.2)$$
In these relations, \([ , ]\) denotes the generalized commutator \([3]\). The form of the commutation relations allows us to call the operators with a dagger and the operators without it, creation and annihilation operators, respectively.

Suppose that in \(\mathcal{H}\) there exists a unique vacuum vector \(\Psi_0\), such that
\[
A_\alpha \Psi_0 = 0,
\]
\[
a_i \Psi_0 = \bar{a}_i \Psi_0 = 0,
\]
\[
c_i \Psi_0 = \bar{c}_i \Psi_0 = 0,
\]
\[
\langle \Psi_0, \Psi_0 \rangle = 1.
\]

Due to a non–canonical form of the commutation relations (3.3) and (3.4) the state space of the system is an indefinite metric space. It is worth to note here that the indefiniteness of the metric is connected not only with the odd ghost operators \(c_i\) and \(\bar{c}_i\), but also with the even operators \(a_i\) and \(\bar{a}_i\) corresponding to the gauge degrees of freedom. To demonstrate the indefiniteness of the metrics it is convenient to introduce a new set of even unphysical operators defined by
\[
a_{0i} = \frac{1}{\sqrt{2}} (\bar{a}_i - a_i), \quad a_{0i}^\dagger = \frac{1}{\sqrt{2}} (\bar{a}_i^\dagger - a_i^\dagger),
\]
\[
a_{1i} = \frac{1}{\sqrt{2}} (\bar{a}_i + a_i), \quad a_{1i}^\dagger = \frac{1}{\sqrt{2}} (\bar{a}_i^\dagger + a_i^\dagger).
\]

As it follows from Eq. (3.6) the operators \(a_{0i}\) and \(a_{1i}\) annhilate the vacuum vector:
\[
a_{0i} \Psi_0 = a_{1i} \Psi_0 = 0.
\]

The nontrivial commutation relations for the new operators have the form
\[
[a_{0i}, a_{0j}^\dagger] = -\delta_{ij}, \quad [a_{1i}, a_{1j}^\dagger] = \delta_{ij}.
\]

Consider the vector \(a_{0i}^\dagger \Psi_0\); from relations (3.11) and (3.12) we see that the scalar square of this vector is equal to \(-1\). Thus, we actually deal with the state space with indefinite metric.

As it was noted in \([3]\), the consideration of the state space in the BRST–quantization becomes rigorous if we treat it as a \(\mathbb{Z}_2\)–graded Krein space \([13, 14, 3]\). Let us describe the corresponding construction. Consider the Fock space \(\mathcal{H}\) where the operators \(A_\alpha\), \(a_i\), \(\bar{a}_i\), \(c_i\), \(\bar{c}_i\) act as ordinary annihilation operators. Here \(A_\alpha\), \(a_i\), \(\bar{a}_i\) are boson annihilation operators, while \(c_i\), \(\bar{c}_i\) are fermion annihilation operators. As it was noted above the space \(\mathcal{H}\) should be a \(\mathbb{Z}_2\)–graded linear space, with \(A_\alpha\), \(a_i\), \(\bar{a}_i\) being even operators, and \(c_i\), \(\bar{c}_i\) being odd operators. We introduce in \(\mathcal{H}\) the corresponding structure of a \(\mathbb{Z}_2\)–graded linear space as follows.

Denote the positive definite scalar product in \(\mathcal{H}\) by \((\ , \ )\). The hermitian conjugation with respect to this scalar product will be denoted by a star. Hence, \(A_\alpha^\star\), \(a_i^\star\), \(\bar{a}_i^\star\), \(c_i^\star\), \(\bar{c}_i^\star\) are the creation operators, corresponding to the annihilation operators \(A_\alpha\), \(a_i\), \(\bar{a}_i\), \(c_i\), \(\bar{c}_i\). The annihilation and creation operators are supposed to satisfy the ordinary commutation relations
or anticommutation relations. The vectors of $H$, generated by the action of the creation operators on the vacuum vector $\Psi_0$ form a basis in $H$. Let us assume that a vector generated by an even (odd) number of fermion creation operators and any number of boson creation operators is even (odd). In this the vacuum vector $\Psi_0$ is considered to be even. An arbitrary vector of $H$ is said to be even (odd) if it can be represented as a linear combination of even (odd) basis vectors. It is clear that we have actually introduced in $H$ the required structure of $\mathbb{Z}_2$–graded Hilbert space [3]. Taking into account the fact that in a $\mathbb{Z}_2$–graded Hilbert space the parity of the hermitian conjugate operator coincides with the parity of the original one, we can write the commutation relation of the creation and annihilation operators with the help of the generalized commutator operation, as

$$[A_{i}, A_{j}] = \delta_{ij},$$

$$[a_{i}, a_{j}^\dagger] = \delta_{ij},$$

$$[c_{i}, c_{j}^\dagger] = \delta_{ij},$$

Introduce now in $H$ the structure of a $\mathbb{Z}_2$–graded Krein space. To this end, define the operator $J$ with the help of the relations

$$JA_{i}J^{-1} = A_{i},$$

$$Ja_{i}J^{-1} = a_{i},$$

$$Jc_{i}J^{-1} = c_{i},$$

$$J\Psi_0 = \Psi_0.$$

It can be easily shown that the operator $J$ is a hermitian operator, satisfying the relation $J^2 = I$. Note also that $J$ is an even operator. Thus, $J$ defines in $H$ the structure of a $\mathbb{Z}_2$–graded Krein space [3]. The corresponding indefinite scalar product $\langle \ , \ \rangle$ is related to the positive definite scalar product $(\ , \ )$ by the equality

$$\langle \phi, \psi \rangle \equiv (\phi, J\psi).$$

Note also that the hermitian conjugation with respect to the indefinite scalar product is related to the hermitian conjugation with respect to the positive definite scalar product by the relation

$$O^\dagger = JO^\dagger J$$

for any operator $O$. From this relation and from Eqs (3.16)–(3.18) it follows that

$$A_{i}^\dagger = A_{i}^\dagger,$$

$$a_{i}^\dagger = a_{i}^\dagger,$$

$$c_{i}^\dagger = c_{i}^\dagger.$$
can be used to prove the positive definiteness of the scalar product in the physical state space for the case of the quadratic BRST–charge [3].

Without any loss of generality, we can suppose that the greek and latin indices take just one value. We assume that this is the case and denote the corresponding operators by the same letters without indices.

4 Generalized Schrödinger representation

From the consideration of the previous section it follows that we can consider the state space $\mathcal{H}$ as the tensor product of three spaces $\mathcal{H}_a$, $\mathcal{H}_A$ and $\mathcal{H}_c$:

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_A \otimes \mathcal{H}_c. \quad (4.1)$$

The space $\mathcal{H}_a$ is the representation space for the even unphysical creation and annihilation operators, the space $\mathcal{H}_c$ is the representation space for the odd unphysical creation and annihilation operators, while $\mathcal{H}_A$ is the corresponding space for the physical operators. Here the operator $J$ and the vacuum vector $\Psi_0$ factorize as

$$J = J_a \otimes J_A \otimes J_c, \quad (4.2)$$

$$\Psi_0 = \Psi_{0a} \otimes \Psi_{0A} \otimes \Psi_{0c}. \quad (4.3)$$

4.1 Even unphysical operators

First consider the space $\mathcal{H}_a$. Introduce the operators $\hat{v}_r$ and $\hat{u}_r$, $r = 0, 1$ defined by

$$\hat{v}_r = \frac{1}{\sqrt{2}} (a_r + a_r^*), \quad \hat{u}_r = \frac{1}{i\sqrt{2}} (a_r - a_r^*). \quad (4.4)$$

The operators $\hat{v}_r$, $\hat{u}_r$ are hermitian with respect to the positive definite scalar product $(,)$:

$$\hat{v}_r^* = \hat{v}_r, \quad \hat{u}_r^* = \hat{u}_r \quad (4.5)$$

and satisfy the commutation relations

$$[\hat{v}_r, \hat{v}_s] = 0, \quad [\hat{u}_r, \hat{u}_s] = 0, \quad [\hat{v}_r, \hat{u}_s] = i\delta_{rs}. \quad (4.6)$$

Thus, we can consider $\mathcal{H}_a$ as the space formed by square integrable function of two real variables $v_0$ and $v_1$ with the scalar product

$$(\Psi, \Phi) = \int \Psi^*(v)\Phi(v)d^2v. \quad (4.8)$$

The Schrödinger representation for the operators $\hat{v}_r$ and $\hat{u}_r$ is

$$\hat{v}_r \Psi = v_r \psi, \quad \hat{u}_r \Psi = -i \frac{\partial \Psi}{\partial v_r}. \quad (4.9)$$
Hence, for the annihilation and creation operators we have
\[
\hat{a}_r \Psi = \frac{1}{\sqrt{2}} \left( v_r + \frac{\partial}{\partial v_r} \right) \Psi, \quad \hat{a}^\dagger_r \Psi = \frac{1}{\sqrt{2}} \left( v_r - \frac{\partial}{\partial v_r} \right) \Psi.
\] (4.10)

The vacuum vector \( \Psi_{0a} \) in the space \( \mathcal{H}_a \) has the form
\[
\Psi_{0a}(v) = \frac{1}{\sqrt{\pi e^{-\frac{1}{2} (v_0^2 + v_1^2)}}}.
\] (4.11)

Let us consider the action of the operator \( J_a \) on \( \mathcal{H}_a \). From the definition of the operators \( a_r \) we get
\[
J_a a_0 J_a = -a_0, \quad J_a a_1 J_a = a_1.
\] (4.12)

From these relations it follows that
\[
J_a \hat{v}_0 J_a = -\hat{v}_0, \quad J_a \hat{u}_0 J_a = -\hat{u}_0,
\] (4.13)

\[
J_a \hat{v}_1 J_a = \hat{v}_1, \quad J_a \hat{u}_1 J_a = \hat{u}_1.
\] (4.14)

These equalities, together with the condition of the invariance of the vacuum vector \( \Psi_{0a} \) give
\[
(J \Psi)(v_0, v_1) = \Psi(-v_0, v_1).
\] (4.15)

We can write this result shortly as
\[
(J \Psi)(v) = \Psi(\sigma(v)),
\] (4.16)

where \( \sigma(v_0, v_1) = (-v_0, v_1) \). For the indefinite scalar product we get the expression
\[
\langle \Psi, \Phi \rangle = \int \Psi^*(\sigma(v)) \Phi(v) d^2v.
\] (4.17)

The operators \( \hat{v}_0 \) and \( \hat{u}_0 \) are not hermitian with respect to the indefinite scalar product \( \langle , \rangle \); actually we have
\[
\hat{v}_0^\dagger = -\hat{v}_0, \quad \hat{u}_0^\dagger = -\hat{u}_0.
\] (4.18)

Since the indefinite scalar product is more fundamental for our problem, it is desirable to introduce new operators, which are hermitian with respect to it \( [L] \). To this end, let us consider the operators \( \hat{q}_r \) and \( \hat{p}_r \), defined by
\[
\hat{q}_0 = -i \hat{v}_0, \quad \hat{p}_0 = i \hat{u}_0,
\] (4.19)

\[
\hat{q}_1 = \hat{v}_1, \quad \hat{p}_1 = \hat{u}_1.
\] (4.20)

For these operators we have
\[
\hat{q}_r = \hat{q}_r, \quad \hat{p}_r = \hat{p}_r.
\] (4.21)

The commutation relations are again of the canonical form
\[
[\hat{q}_r, \hat{q}_s] = 0, \quad [\hat{p}_r, \hat{p}_s] = 0,
\] (4.22)

\[
[\hat{q}_r, \hat{p}_s] = i \delta_{rs}.
\] (4.23)
Introducing the variables
\[ q_0 = -iv_0, \quad q_1 = v_1, \quad (4.24) \]
we can consider \( H_a \) as the space formed by functions of these variables. For the operators \( \hat{q}_r \) and \( \hat{p}_r \) we have
\[ \hat{q}_r \Psi = q_r \Psi, \quad \hat{p}_r \Psi = -i \partial \Psi / \partial q_r, \quad (4.25) \]
while the expressions for the scalar products takes the form
\[ (\Psi, \Phi) = i \int \Psi^*(q) \Phi(q) d^2q, \quad (4.26) \]
\[ \langle \Psi, \Phi \rangle = i \int \Psi^*(\theta) \Phi(\theta) d^2\theta, \quad (4.27) \]
where \( d^2q = dq_0 dq_1 = -idv_0 dv_1. \)

### 4.2 Odd unphysical operators

Introduce now the operators of the odd generalized coordinates \( \hat{\theta}_r \) and generalized momenta \( \hat{\pi}_r \):
\[ \hat{\theta}_0 = \frac{1}{\sqrt{2}} (c^\dagger + c), \quad \hat{\theta}_1 = \frac{1}{i \sqrt{2}} (c - c^\dagger), \quad (4.28) \]
\[ \hat{\pi}_0 = \frac{1}{i \sqrt{2}} (\bar{c} + \bar{c}^\dagger), \quad \hat{\pi}_1 = \frac{1}{\sqrt{2}} (-\bar{c} + \bar{c}^\dagger). \quad (4.29) \]
The operators \( \hat{\theta}_r \) and \( \hat{\pi}_r \) satisfy the canonical commutation relations
\[ [\hat{\theta}_r, \hat{\theta}_s] = 0, \quad [\hat{\pi}_r, \hat{\pi}_s] = 0, \quad (4.30) \]
\[ [\hat{\pi}_r, \hat{\theta}_s] = -i \delta_{rs}. \quad (4.31) \]
Hence we can take as the space \( H_c \) the space of functions of two real anticommuting variables \( \theta_r, r = 0, 1 \), and define the operators \( \hat{\theta}_r \) and \( \hat{\pi}_r \) as
\[ \hat{\theta}_r \Psi = \theta_r \Psi, \quad \hat{\pi}_r \Psi = -i \partial \Psi / \partial \theta_r, \quad (4.32) \]
The indefinite scalar product in this case have the form
\[ \langle \Psi, \Phi \rangle = -i \int \Psi^*(\theta) \Phi(\theta) d^2\theta. \quad (4.33) \]
It is not difficult to get convinced that the operators \( \hat{\theta}_r \) are hermitian, while the operators \( \hat{\pi}_r \) are antihermitian with respect to the scalar product \( \langle \ , \ \rangle \).

The annihilation operators \( c \) and \( \bar{c} \) act on a state vector as
\[ c \Psi = \frac{1}{\sqrt{2}} (\theta_0 + i \theta_1) \Psi, \quad \bar{c} \Psi = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \theta_0} + i \frac{\partial}{\partial \theta_1} \right) \Psi. \quad (4.34) \]
Hence the vacuum vector $\Psi_0c$ has the form
\[
\Psi_{0c}(\theta) = \frac{1}{\sqrt{2}}(\theta_0 + i\theta_1)
\] (4.35)

From the definition of the operators $\hat{\theta}_r$ and $\hat{\pi}_r$ it follows that
\[
J_c\hat{\theta}_rJ_c = i\hat{\pi}_r, \quad J_c\hat{\pi}_rJ_c = -i\hat{\theta}_r.
\] (4.36)

Taking into account the invariance of the vacuum vector under the action of the operator $J_c$, we can get the following expression for the action of $J_c$ on an arbitrary state vector:
\[
J_c\Psi(\theta) = -i \int e^{(\theta_0\eta_0 + \theta_1\eta_1)}\Psi(\eta)d^2\eta
\] (4.37)

From this relation we conclude that the positive definite scalar product in $\mathcal{H}_c$ has the form
\[
(\Psi, \Phi) = -\int \Psi^*(\theta)e^{(\theta_0\eta_0 + \theta_1\eta_1)}\Phi(\eta)d^2\theta d^2\eta.
\] (4.38)

### 4.3 Physical operators and total state space

There are no problems with the construction of the Schrödinger representation for the operators of physical particles. Introducing the operators of the generalized coordinate $\hat{Q}$ and generalized momentum $\hat{P}$:
\[
\hat{Q} = \frac{1}{\sqrt{2}}(A + A^*), \quad \hat{P} = \frac{1}{i\sqrt{2}}(A - A^*),
\] (4.39)
we consider $\mathcal{H}_A$ as the space of square integrable function of a real variable $Q$, and represent $\hat{Q}$ and $\hat{P}$ as
\[
\hat{Q}\Psi = Q\Psi, \quad \hat{P}\Psi = -i\frac{\partial \Psi}{\partial Q}.
\] (4.40)

The operator $J_A$ acts in $\mathcal{H}_A$ as the unit operator, and the scalar products $(\ , \ )$ and $\langle \ , \ \rangle$ coincide:
\[
(\Psi, \Phi) = \langle \Psi, \Phi \rangle = \int \Psi^*(Q)\Phi(Q)dQ.
\] (4.41)

Summarizing the above consideration, we can say that the total state space is the space formed by functions of the variables $q_r, Q$ and $\theta_r$. All these variables are real, except the variable $q_0$, which is imaginary. The positive definite scalar product in the total state space has the form
\[
(\Psi, \Phi) = -i \int \Psi^*(q, Q, \theta)e^{(\theta_0\eta_0 + \theta_1\eta_1)}\Phi(q, Q, \eta)d^2qd^2\theta d^2\eta,
\] (4.42)
while for the indefinite scalar product we have the expression
\[
\langle \Psi, \Phi \rangle = \int \Psi^*(q^*, Q, \theta)\Phi(q, Q, \theta)d^2qd^2\theta.
\] (4.43)

The operator $J$, connecting the indefinite and positive definite scalar products, act on a state vector $\Psi$ as
\[
(J\Psi)(q, Q, \theta) = -i \int e^{(\theta_0\eta_0 + \theta_1\eta_1)}\Psi(q^*, Q, \eta)d^2\eta.
\] (4.44)
5 BRST–charge and physical state space

It can be shown that a general quadratic BRST–charge can be written as

$$\hat{\Omega} = \sqrt{2} (a^\dagger c + c^\dagger a).$$  \hspace{1cm} (5.1)

Proceeding to the operators of the generalized coordinates and momenta we get for \(\hat{\Omega}\) the following expression

$$\hat{\Omega} = (\hat{p}_0 + \hat{q}_1)\hat{\theta}_0 + (\hat{p}_1 + \hat{q}_0)\hat{\theta}_1.$$  \hspace{1cm} (5.2)

In Ref. [3] it was shown that in the case under consideration any BRST–invariant state vector can be represented in the form

$$\Psi = \Psi' + \hat{\Omega}\Phi,$$  \hspace{1cm} (5.3)

where the vector \(\Psi'\) satisfy the relations

$$a\Psi' = \tilde{a}\Psi' = 0, \quad c\Psi' = \tilde{c}\Psi' = 0.$$  \hspace{1cm} (5.4)

In other words, the vector \(\Psi'\) does not contain unphysical particles. Note that the vector \(\Psi'\) is defined by the vector \(\Psi\) uniquely. Moreover, different physical vectors, satisfying relations (5.4), correspond to different physical states. Using the Schrödinger representation, we get for \(\Psi'\) the expression

$$\Psi'(q, Q, \theta) = \frac{1}{\sqrt{2\pi}} (\theta_0 + i\theta_1)e^{\frac{1}{2}(\bar{q}_0^2 - q_1^2)}\psi(Q).$$  \hspace{1cm} (5.5)

Thus, the physical state space can be parameterized by the square integrable functions \(\psi(Q)\).

The BRST–charge, given by Eq. (5.2), is different from the BRST–charge, we considered in section 2. Note that after the transformation

$$\hat{p}_0 + \hat{q}_1 \rightarrow \hat{p}_0, \quad \hat{p}_1 + \hat{q}_0 \rightarrow \hat{p}_1$$  \hspace{1cm} (5.6)

we come to the BRST–charge of form (2.10). This transformation does not change commutators, and after it we get the following Schrödinger representation for the operators \(\hat{q}_r^*\) and \(\hat{p}_r^*\):

$$\hat{q}_0 \Psi = q_0 \Psi, \quad \hat{q}_1 \Psi = q_1 \Psi,$$  \hspace{1cm} (5.7)

$$\hat{p}_0 \Psi = \left(-i\frac{\partial}{\partial q_0} + q_1\right)\Psi, \quad \hat{p}_1 \Psi = \left(-i\frac{\partial}{\partial q_1} + q_0\right)\Psi.$$  \hspace{1cm} (5.8)

To get the usual representation we should multiply the state vector by the factor \(\exp(-iq_0q_1)\). After this we obtain for the vector \(\Psi'\) the following expression

$$\Psi'(q, Q, \theta) = \frac{1}{\sqrt{2\pi}} (\theta_0 + i\theta_1)e^{\frac{1}{2}((\bar{q}_0^2 - q_1^2)^2)}\psi(Q).$$  \hspace{1cm} (5.9)

Note here that the multiplication of the state vectors by the factor \(\exp(-iq_0q_1)\) does not change the expression for the indefinite scalar product \(\langle , \rangle\), while for the positive definite scalar product we get

$$(\Psi, \Phi) = -i \int \Psi^*(q, Q, \theta)e^{(\theta_0\eta_0 + \theta_1\eta_1)}e^{2i(q_0\eta_0 + q_1\eta_1)}\Phi(q, Q, \eta)d^2q d^2Qd^2\theta d^2\eta.$$  \hspace{1cm} (5.10)
Let us now compare the results we have obtained with the discussion given in section 2. Formally we have the same Schrödinger representation as in section 2. The difference is in the fact that the variable $q_0$ is now imaginary and the scalar product have form (4.43). Besides, we also have a positive definite scalar product defined on the state space. Actually the total state space $\mathcal{H}$ in our approach is defined as the space of the functions $\Psi$ of the variables $q_r, Q$ and $\theta_r$, such that $(\Psi, \Psi) < \infty$. The arguments based on the Poincaré lemma are also applicable in our case, but taking into account the results of Ref. [3], we conclude that we cannot now transform a general physical state vector to the state vector having no dependence from odd variables. The physical state vectors having no dependence from odd variables, should be excluded from the consideration, because the positive definite scalar product is not defined for them.

6 Conclusion

In the present paper we have compared the consideration of the state space in the BRST–quantization made in Ref. [3] with the help of creation and annihilation operators in the Fock representation, with one based on the operators of generalized coordinates and momenta in the Schrödinger representation. It appeared that the naive treatment of the problem leads to unsatisfactory results due to the absence of a meaningful definition of the total phase space.

Stress also that in accordance with our consideration some variables describing the initial gauge–invariant system should be quantized with indefinite metric, this conclusion is in accordance with a general consideration of the properties of the BRST–quantization performed by R. Marnelius with collaborators (see Ref. [19] and references therein).

Unfortunately, we do not see now a way to generalize the results obtained in the present paper to the case of constraints forming a general nonabelian algebra. Its seems very likely that such a generalization should be based on the consideration of the structure of the corresponding gauge group manifold, but it is not clear for us how to introduce an indefinite metrics in the space of functions on the gauge group. This question is quite nontrivial, and a short remark made in this respect in Ref. [19] is, from our point of view, not enough.

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