The geometry and phase structure of non-relativistic branes

Nabamita Banerjee\textsuperscript{1}, Suvankar Dutta\textsuperscript{2} and Dileep P Jatkar\textsuperscript{3}

\textsuperscript{1} ITF, Utrecht University, Utrecht, the Netherlands
\textsuperscript{2} Department of Physics, Swansea University, Swansea, UK
\textsuperscript{3} Harish-Chandra Research Institute, Chhatnag Road, Allahabad, India

E-mail: N.Banerjee@uu.nl, pysd@swan.ac.uk and dileep@hri.res.in

Received 24 March 2011, in final form 1 June 2011
Published 11 July 2011
Online at stacks.iop.org/CQG/28/165002

Abstract
We use the solution-generating technique, the $TsT$ transformation, to obtain new solutions in type II string theory as well as in M-theory. We explicitly work out examples starting with rotating D3 and M2 branes as well as the D1–D5-p system. Among a variety of solutions, we find many of them having asymptotic Schrödinger symmetry. We also devise a new method of deriving the free energy of black brane systems, which is more efficient than the Euclidean action procedure. We test our method on known examples before applying it to new asymptotically Schrödinger backgrounds. We study the phase structure of these backgrounds by analyzing the free energy thus derived.

PACS numbers: 11.25.-w, 11.25.Yb

1. Introduction and summary

AdS/CFT correspondence relates conformal field theories in $d$ spacetime dimensions to superstring/M-theory compactified down to anti-de Sitter (AdS) spacetime in $d+1$ dimensions [1, 2]. This correspondence has been extended to non-conformal field theories by relating it to asymptotically AdS spacetimes in the bulk. It has also been generalized to non-conformal branes [3].

Another direction in which this correspondence is extended is in the exploration of holographic duals of strongly coupled non-relativistic conformal field theories [4–11]. These non-relativistic conformal field theories have Schrödinger group symmetry [12–20]. This symmetry consists of the usual Galilean invariance, the scaling symmetry as well as the particle number symmetry\textsuperscript{4}. There has been some activity along this direction where solution-generating techniques have been used to obtain bulk geometries with asymptotic Schrödinger

\textsuperscript{4} The relation of Schrödinger symmetry and the Newton–Cartan theory dates back to the work of Duval \textit{et al} [21]. For other related works, see [22–24].
symmetry \cite{10, 25–28}. A large class of asymptotic Schrödinger geometries have been obtained by applying these techniques to various brane configurations \cite{25, 26}.

Interplay between the $\text{TsT}$ transformation and the near-horizon extremal limit was studied earlier in the case of spinning D3 brane problem \cite{26}. It was shown that the procedure of near-horizon extremal limit does not commute with the $\text{TsT}$ transformation in the case of both null $\text{TsT}$ and space-like $\text{TsT}$ transformation of spinning D3 brane solutions. In this paper, we study $\text{TsT}$ transformations of D brane systems and their near-horizon extremal limit. Section 2 mostly contains the review of known results, where we write down general properties of $\text{TsT}$ transformations and list relevant formulae which will be useful in subsequent sections. In section 3, we re-examine the spinning D3 brane case both for null and space-like $\text{TsT}$.\footnote{We define null and space-like $\text{TsT}$ transformations in the next section.} We find that our results are in agreement with earlier results in the case of null $\text{TsT}$ transformations but contrary to earlier claims, we find that the space-like $\text{TsT}$ transformation does commute with the near-horizon extremal limit. Later in the same section, we look at the transformation of the D1–D5-p system with respect to null and space-like $\text{TsT}$. We find that the near-horizon extremal limit of this system for null $\text{TsT}$ is independent of the parameter $\gamma$, which appears in the $\text{TsT}$ transformation. As a result, the near-horizon extremal limit of the D1–D5-p system before and after the null $\text{TsT}$ transformation gives rise to an identical background.

We then proceed to apply this method to the rotating M2 brane solution\footnote{Similar Schrödinger backgrounds from M-theory were considered earlier in \cite{29, 30}.} in section 4. Since T-duality is not a symmetry of M-theory, we can implement the $\text{TsT}$ transformation by first reducing the M2 brane solution to type IIA theory and then carrying out the $\text{TsT}$ transformation on it. We then oxidize the resulting solution back to M-theory. There are a couple of ways of reducing the M2 brane to the type IIA string theory solution. One of them gives rise to the fundamental string solution, whereas the other gives the D2 branes solution. The action of the $\text{TsT}$ transformation on these two solutions gives different backgrounds. In the case of the fundamental string solution, the null $\text{TsT}$ transformation gives rise to a background which is not asymptotically Schrödinger-type geometry. The scalar $M$, which parametrizes the scale factor in front of the light-cone part of the metric as well as the behavior of the dilaton, does not become constant asymptotically. In order to obtain a better behaved solution we work in a suitable type IIB frame, which can be obtained by performing a T-duality on the type IIA background followed by an S-duality. In this type IIB frame, we carry out the $\text{TsT}$ transformation. We then revert to the M-theory metric in two different ways. We can either follow the same steps in reverse order, i.e. performing S-duality, followed by T-duality to type IIA theory and then lift to M-theory or we can directly T-dualize the background into the type IIA background and lift to M-theory. These two procedures give rise to two different M-theory solutions. In the former case, we end up with a geometry which has a decoupled metric piece corresponding to $\mathbb{C}P^3 \times S^1$ and the full metric asymptotically approaches $\text{AdS}_3 \times \mathbb{C}P^3 \times S^1$. In the latter case, we have a decoupled $\mathbb{C}P^3$ metric and the circle is fibered over the three-dimensional non-compact space. This space asymptotically becomes a circle fibration over $\text{AdS}_3$ with a decoupled $\mathbb{C}P^3$ metric.

In section 5, we discuss the phase structure of the $\text{TsT}$ transformed black holes. The phase structure of the black hole spacetime can be obtained from its free energy. Therefore, one needs to compute the free energy from the Euclidean path integral of the black hole spacetime, $F = -T \ln Z_\Sigma$, where $Z_\Sigma$ is the partition function. For pure gravity, the partition function $Z_\Sigma$ is given by

\begin{equation}
Z_\Sigma(X) = \int [Dg] e^{-I_3(X)},
\end{equation}

\footnote{\textsuperscript{2} We define null and space-like $\text{TsT}$ transformations in the next section.}
where $I_S(X)$ is the Euclidean effective action on $X$. In the large-$N$ limit (or $G_N \to 0$, classical supergravity), it is given by

$$I_S \sim \frac{1}{G_N} \int d^{d+1}x \sqrt{g} R = N^2 K(X). \quad (1.2)$$

In general, there may be several possible $X$s given a fixed boundary geometry $M$. In that case, there is no particular way to pick one specific spacetime. Therefore, one has to consider all possible $X$s and replace (1.1) by the sum over all such integrals. In a semi-classical limit, i.e. $N \to \infty$, one can write the partition function in the following way:

$$Z_S(X) = \sum_i e^{-N^2 K_{cl}(X^0_i)}, \quad (1.3)$$

where $X^0_i$s are classical solutions of the Einstein equation with some fixed boundary geometry.

One can note that there is a natural mechanism for a singularity or phase transition that would arise only in the large-$N$ limit. At large $N$, the sum will be dominated by those $X^0_i$ for which $F(X^0_i)$ is smallest. Therefore, at a point at which $F(X^0_i) = F(X^0_j)$ for some $i \neq j$, one may jump from one branch to the other. This signifies a phase transition in the large-$N$ theory.

To study the phase transition, we need to compute the on-shell action. In this paper, we develop a new method to compute the Euclidean on-shell action. In the usual way of computing the on-shell action, one encounters large volume divergence and needs to regularize the action by either subtracting the contribution from the background spacetime or adding counterterms to the original action. In practice, these methods are cumbersome especially when one has several matter fields. We use Wald’s formula to compute the entropy of the black hole which depends only on the near-horizon geometry. We derive the temperature of the Euclidean black hole by computing the periodicity of the Euclidean time circle. We then use these two variables to compute the on-shell action of the black hole using the following relations:

$$S - \beta \left( \frac{\partial I}{\partial \beta} \right)_\mu I = 0 \quad \text{for a fixed chemical potential } \mu, \quad (1.4)$$

$$S - \beta \left( \frac{\partial I}{\partial \beta} \right)_q I = 0 \quad \text{for a fixed charge } q.$$

When we integrate these first-order equations, we encounter an integration constant. We fix this constant by demanding the free energy of the background spacetime to be zero. We describe this procedure in detail in section 5 and work out an illustrative example before applying it to the non-relativistic case. The Euclidean on-shell action derived in this manner for black hole spacetimes is then used to study the phase structure. We conclude with the discussion of our results.

2. The TsT transformation

In this section, as mentioned in the introduction, we review known results about the TsT transformation. For more details, the reader is referred to [25, 26, 28, 31] and references therein, where this method was used in a variety of contexts. We discuss general properties of TsT transformations, which is a solution-generating technique in string theory. The basic idea of the solution-generating technique is to exploit the fact that low-energy supergravity theory has more symmetry than full string theory. A solution to supergravity equations of motion can be shown to be a solution to string theory. A symmetry transformation of this solution with respect to a supergravity symmetry does not give rise to a new solution in

We are considering only gravity for simplicity.
supergravity. However, if the transformation employed is not a symmetry of full string theory, then the transformed solution can be interpreted as a new solution of string theory. In the case of the $TsT$ transformations, T-dualities are performed along the isometry direction, whereas the $s$-transformation mixes the dualized coordinate with another isometry direction. This transformation generically changes the asymptotics of the resulting metric but is guaranteed to be a solution to the supergravity equations of motion. We spell out the procedure involved in the $TsT$ transformation shortly but before that we briefly mention two metrics, which give rise to non-relativistic isometries in the boundary conformal field theories and appear in the $TsT$ transformed solutions.

The holographic dual of a $d$ spatial dimensional Galilean CFT is given by [4, 5, 7]

$$dx^2 = r^2 \left( -2 \frac{du}{dv} - r^{2(z-1)} \frac{dr}{r^2} + \sum_{i=1}^{d} (dx^i)^2 \right) + \frac{dr^2}{r^2}, \quad (2.1)$$

For $z = 1$, the spacetime becomes pure AdS$_{d+1}$ whose dual theory corresponds to a conformal field theory in $(d + 1, 1)$ spacetime dimensions.

For $z > 1$, the bulk geometry corresponds to a dual to non-relativistic field theory with the Galilean transformations along with scaling as a global symmetry. While the coordinate $u$ is interpreted as a boundary time coordinate, the null direction $v$ is taken to be compact and the quantized momentum along $v$ is identified with the particle number of Galilean CFT. We call this spacetime Sch$_{d+3}$, where $z$ is the dynamical exponent of the Schrödinger spacetime.

Let us also note that the bulk geometry corresponding to the Lifshitz spacetime is given by

$$dx^2 = r^2 \left( -2^{2(z-1)} \frac{dr}{r^2} + \sum_{i=1}^{d} (dx^i)^2 \right) + \frac{dr^2}{r^2}, \quad (2.2)$$

where $t$ is the boundary time coordinate of the boundary theory.

We now discuss the generic form of the $TsT$ transformed geometry. We use the unhatted variables to denote the metric, dilaton and the second-rank anti-symmetric tensor before the $TsT$ transformation and the hatted variables to denote them after $TsT$. In the case of Ramond–Ramond (RR) field strengths, our convention is to use $F_q$ to denote $q$-form before $TsT$ and $\hat{F}_q$ to denote that after $TsT$. In the next section, we apply this technique to specific examples in string theory. The $TsT$ transformations can be applied to those metrics which have at least two isometry directions. In the absence of NS–NS two-form field $B$, it is convenient to cast the metric in the following form:

$$dx^2 = (A_1 d\chi + K_1)^2 + (A_2 d\psi + A_3 d\chi + K_2)^2 + ds_5^2, \quad (2.3)$$

where $ds_5^2$ and the one-forms $K_1$ and $K_2$ do not depend on the isometry directions $\chi$ and $\psi$. The $TsT$ transformations then correspond to performing T-duality along the $\psi$ direction, followed by a shift along $\chi$, i.e. $\chi \rightarrow \chi - \gamma \psi$, and then performing T-duality back along the $\psi$ direction. Note that under the Buscher duality transformations [32], the pure metric background can give rise to a non-trivial dilaton and an anti-symmetric tensor field background. For example, if the original metric is of the form (2.3), then the end result of the $TsT$ transformation is

$$d\tilde{s}^2 = \mathcal{M}(A_1 d\chi + K_1)^2 + \mathcal{M}(A_2 d\psi + A_3 d\chi + K_2)^2 + ds_5^2,$$

$$e^{2\Phi} = \mathcal{M} e^{2\Phi}, \quad \mathcal{M} = (1 + \gamma^2 A_1^2 A_2^2)^{-1},$$

$$\tilde{B} = -\gamma \mathcal{M} A_1 A_2 (A_1 d\chi + K_1) \wedge (A_2 d\psi + A_3 d\chi + K_2).$$

In this simplified situation, where $B = 0$ and only $p$-form RR field strength is non-vanishing, we can easily find the effect of $TsT$ on the RR field. Suppose that this $p$-form field
strength has components along both the $\psi$ and $\chi$ directions, then after the $\text{TsT}$ transformation, we obtain the $(p-2)$-form field strength

$$\mathcal{F}_{p-2} = \gamma \iota_{x} \iota_{\psi} F_p,$$

in addition to the original $p$-form field strength. As a result, the effective $p$-form field strength is then given by

$$F_p = \mathcal{F}_p + \mathcal{F}_{p-2} \wedge \hat{B}.$$  

In general, the original background can have a non-trivial dilaton as well as the antisymmetric field strength. To find the $\text{TsT}$ transformed form of the metric and the NS–NS field strength, it is useful to define $\hat{E}_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$. Under the $\text{TsT}$ transformation, $E_{\mu\nu}$ transforms as

$$\hat{E}_{\mu\nu} = \mathcal{M} \left[ E_{\mu\nu} + \gamma \left[ \det \begin{pmatrix} E_{\psi X} & E_{\psi Y} \\ E_{\mu X} & E_{\mu Y} \end{pmatrix} - \det \begin{pmatrix} E_{X \psi} & E_{X Y} \\ E_{\mu \psi} & E_{\mu Y} \end{pmatrix} \right] + \gamma^2 \det \begin{pmatrix} E_{\psi} & E_{\phi} \\ E_{X} & E_{XX} \end{pmatrix} \right],$$

where

$$\mathcal{M} = \left[ 1 + \gamma (E_{\psi X} - E_{X \psi}) + \gamma^2 \det \begin{pmatrix} E_{\psi X} & E_{\phi} \\ E_{X} & E_{XX} \end{pmatrix} \right]^{-1}. $$

We can now read out the transformed metric and the NS–NS antisymmetric field from equation (2.7) by isolating the symmetric and antisymmetric parts of $\hat{E}$, namely

$$\hat{g}_{\mu\nu} = \text{Sym}[\hat{E}_{\mu\nu}], \quad \hat{B}_{\mu\nu} = \text{AntiSym}[\hat{E}_{\mu\nu}].$$

The dilaton transforms as $e^{2\Phi} = \mathcal{M} e^{2\phi}$. The $\text{TsT}$ transformation also acts on the RR field strengths. For general backgrounds, the RR fields and the field strengths after $\text{TsT}$ are related to those before $\text{TsT}$, and

$$\sum_q \hat{C}_q \wedge \exp(\hat{B}) = \sum_q C_q \wedge \exp(B) + \gamma \iota_{x} \iota_{\psi} \sum_q C_q \wedge \exp(B),$$

$$\sum_q \hat{F}_q \wedge \exp(\hat{B}) = \sum_q F_q \wedge \exp(B) + \gamma \iota_{x} \iota_{\psi} \sum_q F_q \wedge \exp(B),$$

where $B$ is the Neveu–Schwarz sector antisymmetric tensor before the $\text{TsT}$ transformation and $\hat{B}$ is that after $\text{TsT}$. In the next two sections, we apply these general $\text{TsT}$ transformation rules to different string theory and M-theory geometries to generate deformed solutions.

3. The $\text{TsT}$ transformation of string geometry

In this section, we discuss the effect of the $\text{TsT}$ transformation of Dp brane geometry and study how their asymptotic and near-horizon geometries change under this transformation. In particular, we consider two different D brane configurations, rotating D3 brane geometry and the D1–D5-p system. Both arise as solutions of type IIB string theory. We perform the null and space-like $\text{TsT}$ transformations in both the cases and discuss the behavior of geometries asymptotically as well as in the near-horizon limit. We also study the commutativity of the extremal near-horizon limit and $\text{TsT}$ transformation in both the cases.
3.1. Rotating D3 brane

We start with a review on the non-relativistic extension of the rotating non-extremal D3 brane solution. Let us consider D3 branes rotating along three isometry directions of transverse $S^5$ space with equal angular momenta in all three planes. From a five-dimensional point of view (compactifying over $S^3$), this system can be viewed as a charged AdS$_5$ black brane with $U(1)^3$ symmetry. The boundary theory corresponds to a conformal field theory with three equal global $U(1)$ charges. This system has been studied in details in [26]. In most of the cases, our results are in agreement with their; however, there are some differences in the case of space-like $TsT$.

We work in a ten-dimensional set-up and analyze the effect of both the null $TsT$ and space-like $TsT$ transformations. Since the $TsT$ transformation is a symmetry of supergravity, the non-relativistic geometries resulting out of these transformations are solutions of type IIB theory. At the end, we take the extremal near-horizon limit of both $TsT$ transformed solutions. As mentioned in the introduction, our results agree with earlier results in the literature about the non-commutativity of null $TsT$ transformation and the near-horizon extremal limit. In the case of space-like $TsT$, however, we find that it commutes with the near-horizon extremal limit. The non-extremal metric\(^8\) in the decoupling limit [1] of the rotating D3 brane geometry is given by

\[
d s^2 = \frac{r^2}{f} \left(-f \, dt^2 + dx^2 + dy^2 + dz^2 + \sum_{i=1}^{3} \frac{t^2 \, dr^2}{f} + t^2 (d \alpha^2 + \sin^2 \alpha \, d \beta^2 + \mu_i^2 (d \xi_i + A)^2 + \mu_2^2 (d \xi_2 + A)^2 + \mu_3^2 (d \xi_3 + A)^2) \right),
\]

(3.1)

\[
F_5 = (1 + \ast) \left[ -\frac{4 r^2}{f^2} dt \wedge dr + \frac{Q}{f^2} d \left( \sum_{i=1}^{3} \mu_i^2 d \xi_i \right) \wedge dx \wedge dy \wedge dz \right],
\]

(3.2)

where the angular functions $\mu_i$ are parameterized as $\mu_1 = \cos \alpha$, $\mu_2 = \sin \alpha \cos \beta$, $\mu_3 = \sin \alpha \sin \beta$, and the metric functions are as follows:

\[
f(r) = \left( 1 - \frac{r_0^2}{r^2} \right) \left( 1 + \frac{r_0^2}{r_0^2 - r^2} \right), \quad A = A_t \, dt = \frac{Q}{f^2} \left( \frac{1}{r_0^2} - \frac{1}{r^2} \right) \, dt.
\]

(3.3)

In what follows we consider three angular momenta to be equal and denote them by $Q$. As mentioned in the previous section, we need two isometry directions to carry out the $TsT$ transformation. We use one of the D3 brane worldvolume directions as an isometry direction. The second isometry direction that we use is from the $CP^2$ direction. To extract that it is suitable to write the $S^5$ metric as a $U(1)$ fibration over $CP^2$, we have

\[
d s^2_{S^5} = (d \psi + \mathcal{P} + A)^2 + d s^2_{CP^2},
\]

(3.4)

where the one-form $A$ is defined in (3.3) and

\[
\mathcal{P} = \frac{1}{3} \left( \chi_1 + d \chi_2 - \sin^2 \alpha \, d \chi_2 \sin^2 \beta - d \chi_1 \cos^2 \beta \right),
\]

\[
\psi = \frac{1}{3} (\phi_1 + \phi_2 + \phi_3), \quad \chi_1 = \phi_1 - \phi_2, \quad \chi_2 = \phi_1 + \phi_2.
\]

(3.5)

Note that the curvature of the one-form $\mathcal{P}$ is proportional to the K"ahler form $\omega_{CP^2}$ on $CP^2$, namely $-\frac{\alpha}{2} d \mathcal{P} = \omega_{CP^2}$, and the metric of $CP^2$ can be expressed as

\[
d s^2_{CP^2} = d \alpha^2 + \sin^2 \alpha \, d \beta^2 + \sin^2 \alpha \, \cos^2 \alpha \, d \chi_1 + \sin^2 \alpha \, \cos^2 \beta \, d \chi_2 + \sin^2 \alpha \, \sin^2 \beta \, \cos^2 \beta \, (d \chi_1 - d \chi_2)^2.
\]

(3.7)

Thus, the rotating D3 brane solution that we are looking at has all three angular momenta equal. In the following subsections, we study the non-relativistic extension of this geometry.

---

\(^8\) We follow the notation of [26].
3.1.1. The null TsT transformation. We start with the null TsT transformation. Before performing the transformation, we first need to define light-cone coordinates, $x^\pm = (t \pm y)/2$. The vielbeins $e^i$, which are derived from the original metric, are given by

\[
e^0 = \frac{2r}{\sqrt{1-f}} \mathrm{d}x^+, \quad e^1 = \frac{r[(1-f) \mathrm{d}x^- - (1+f) \mathrm{d}x^+]}{l\sqrt{1-f}}, \quad e^2 = \frac{r}{l} \mathrm{d}x, \quad e^3 = \frac{r}{l} \mathrm{d}z,
\]

\[
e^4 = \frac{dr}{r\sqrt{1-f}}, \quad e^5 = l \mathrm{d}x, \quad e^6 = l \sin \alpha \, d\beta,
\]

\[
e^7 = \frac{l}{2} \sin 2\alpha (\cos^2 \beta \, d\chi_1 + \sin^2 \beta \, d\chi_2),
\]

\[
e^8 = l \sin \alpha \sin \beta \cos (d\xi_1 - d\xi_2), \quad e^9 = l(d\psi + \mathcal{P} + A).
\]

To connect up to the notation of section 2, we use $x^-$ as the $x$ direction in the TsT transformation described there. In the light-cone coordinates, different functions appearing in metric (2.3) take the following form:

\[A_1 = \frac{r}{l} \sqrt{1-f}, \quad A_2 = l, \quad A_3 = A_l, \quad K_1 = -\frac{r}{l} \left(1 + f\right) \mathrm{d}x^+,
\]

\[K_2 = \frac{1}{3} \left(d\chi_1 + d\chi_2\right) - \sin^2 \alpha (d\chi_1 \cos^2 \beta + d\chi_2 \sin^2 \beta) + A_l \mathrm{d}x^+.
\]

Note that the NS–NS two-form $B$ is zero for the D3 brane solution (3.1). Following the general TsT transformation procedure outlined in section 2, we write down the new solution obtained from equation (3.1). The TsT procedure is implemented after expressing (3.1) in terms of the light-cone coordinates, $x^\pm$. The new background, which also solves type IIB equations of motion, is

\[\mathrm{d}s^2 = \mathcal{M}(A_1 \mathrm{d}x^- + K_1)^2 + \mathcal{M}(A_2 \mathrm{d}\psi + A_3 \mathrm{d}x^- + K_2)^2 + \mathrm{d}z^2,
\]

\[e^{2\Phi} = \mathcal{M} = \left(1 + \gamma r^2 (1 - f)\right)^{-1}, \quad \tilde{C}_2 = -\gamma l^2 A_l \omega_{\mathbb{CP}^2},
\]

\[\tilde{B} = -\gamma r \mathcal{M} \sqrt{1-f} (A_1 \mathrm{d}x^- + K_1) \wedge (A_2 \mathrm{d}\psi + A_3 \mathrm{d}x^- + K_2),
\]

\[\mathcal{F}_5 = F_5 + \mathcal{H}_1 \wedge \mathcal{H}_2 = (1 + *)G_5
\]

\[= (1 + *) \left[ -\frac{4}{l^2} e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 + \frac{2Q}{l^3 \sqrt{1-f}} (\sqrt{f} e^0 + e^1) \wedge e^2 \wedge e^3 \wedge \omega_{\mathbb{CP}^2} \right].
\]

The Poincaré dual is defined using * with respect to this TsT transformed metric. The asymptotic geometry is found by taking the $r \to \infty$ limit and keeping leading terms

\[
\mathrm{d}s^2 \to \frac{r^2}{l^2} \left( -4 \gamma r^2 (\mathrm{d}x^+)^2 - 4 \mathrm{d}x^- \mathrm{d}x^+ + \mathrm{d}x^2 + \mathrm{d}y^2 \right)
\]

\[+ \frac{l^2}{r^2} \mathrm{d}r^2 + l^2 \left( (\mathrm{d}\psi + \mathcal{P} + A_{\infty})^2 + \mathrm{d}z^2 \right),
\]

\[\exp(2\Phi) \to 1, \quad \tilde{B} \to \gamma r^2 \mathrm{d}x^+ \wedge (\mathrm{d}\psi + \mathcal{P} + A_{\infty}), \quad \tilde{C}_2 \to -\gamma l^2 A_{\infty} \omega_{\mathbb{CP}^2}.
\]

Defining rescaled light-cone coordinates as $u = 2\gamma l x^+$ and $v = \frac{1}{l^2} x^-$, it is easy to see that the above metric takes the form (2.1) at the boundary $r \to \infty$ with the dynamical exponent $z = 2$.

**Extremal near-horizon limit.** We are interested in studying the near-horizon extremal limit of this deformed geometry. As is well known, the undeformed geometry (3.1) has an AdS2
factor in its extremal near-horizon limit. The extremal limit is arrived at by taking \( Q = \sqrt{2} r_0^3 \).

Taking the near-horizon limit of the extremal geometry is little subtle and usually one needs to do an appropriate scaling of coordinates to obtain the near-horizon geometry. We consider the following scaling limit:

\[
\rho = \frac{r - r_0}{\beta}, \quad \tau = \beta t, \quad \text{with} \quad \beta \to 0. \tag{3.12}
\]

This gives the near-horizon extremal geometry as

\[
ds^2 = \frac{r_0^2}{l^2} \left( -\frac{12\rho^2}{r_0^2} \, dr^2 + \mathcal{M}_0 \, dy^2 \right) + \frac{r_0^2}{l^2} (dx^2 + dz^2) + \frac{l^2}{12\rho^2} \, d\rho^2
+ l^2 \left( \mathcal{M}_0 \left( d\psi + P + \frac{2\sqrt{2}\rho}{l^2} \, dr \right)^2 + ds_{\mathbb{CP}^2}^2 \right), \tag{3.13}
\]

\[e^{2\phi} = \mathcal{M}_0, \quad \hat{B} = \gamma r_0^2 \mathcal{M}_0 \, dy \wedge (d\psi + P + \frac{2\sqrt{2}\rho}{l^2} \, dr), \quad \hat{C}_2 = 0,
\]

\[\mathcal{F}_5 = (1 + \gamma) \left[ -\frac{4r_0^3}{l^4} \, dt \wedge d\rho - \frac{2\sqrt{2}r_0^3}{l^4} \omega_{\mathbb{CP}^2} \right] \wedge dx \wedge dy \wedge dz, \tag{3.14}
\]

where \( \mathcal{M}_0 = \left( 1 + \gamma r_0^2 \right)^{-1} \).

Thus we see that, while the asymptotic geometry has Schrödinger isometry, we do obtain the AdS\(_2\) factor in the near-horizon extremal limit. The same geometry (3.13) can also be obtained if we start from the extremal near-horizon solution (which has an AdS\(_2\) factor in it), perform null \( TsT \) and then take the above scaling limit (3.12). It is worth mentioning that in this case before taking the scaling limit, the near-horizon geometry does not have any AdS\(_2\) isometry, as pointed out in [26].

### 3.1.2. The space-like \( TsT \) transformation.

For the space-like \( TsT \) transformation, we do not use the light-cone coordinates; we instead choose \( y \) as the \( \chi \) direction and perform the twist in the \( y \) direction. The deformed geometry now looks like

\[
ds^2 = \frac{r_0^2}{l^2} (-f \, dr^2) + \mathcal{M} \frac{r_0^2}{l^2} \, dy^2 + \frac{r_0^2}{l^2} (dx^2 + dz^2) + \frac{l^2}{r_0^2} \, dr^2 + l^2 \left( \mathcal{M}(d\psi + P + A)^2 + ds_{\mathbb{CP}^2}^2 \right),
\]

\[e^{2\phi} = \mathcal{M}, \quad \hat{B} = -\gamma r_0^2 \mathcal{M} \, dy \wedge (d\psi + P + A), \quad \hat{C}_2 = -\gamma l^2 A_0 \omega_{\mathbb{CP}^2},
\]

\[\mathcal{F}_5 = (1 + \gamma) G_5 = (1 + \gamma) \left[ \left( -\frac{4r_0^3}{l^4} \, dt \wedge dr - \frac{2Q}{l^4} \omega_{\mathbb{CP}^2} \right) \wedge dx \wedge dy \wedge dz \right], \tag{3.14}
\]

where \( \mathcal{M} = (1 + \gamma r_0^2)^{-1} \).

The near-horizon extremal limit gives a similar result as above, only with an opposite sign for the two-form field \( \hat{B} \). It also solves the equations of motion as expected. For both cases, we see that the near-horizon extremal geometry does depend on the shift parameter \( \gamma \). We see that the situation is different in the D1–D5-p system that we study in the following section. Thus, we see that the extremal near-horizon limit of the \( TsT \) transformed geometry and the \( TsT \) transform of the near-horizon extremal geometry are not the same. We summarize our observations in the (non-)commutative diagram (see figure 1).

### 3.2. D1–D5-p system

We now look at another solution of type IIB theory. This is given by the D1–D5 brane system with momentum \( p \), where \( Q_1 \) number of D1 branes wrap the compact direction \( x_5 \), \( Q_5 \) D5 brane
wraps $x_5$ and a compact four-manifold $M_4$, and the momentum $n$ is along the circle $x_5$. We show that in the D1–D5-p system, we do not encounter a commutative or pentagonal diagram (see figure 1) we obtained in the rotating D3 brane case. In this case, as we demonstrate below, we start with the near-horizon geometry of D1–D5-p, transform it by $TsT$, followed by a near-horizon limit which gives us back the original near-horizon geometry. Thus, in a sense the diagram in this case is just triangular, signifying that the near-horizon geometry of the $TsT$ transformed solution has no new information compared to the near-horizon geometry of the D1–D5-p system. This fact has an interesting implication on the entropy of non-relativistic black holes. In the string frame, the metric, the dilaton and the RR three-form field strength $F_3$ corresponding to this solution are given by [33]

$$\begin{align*}
\text{d}s^2_{str} &= \frac{1}{\sqrt{H_1 H_2}} \left[ -\text{d}t^2 + \text{d}x_5^2 + \frac{c_3}{r^2} (\coth \sigma \text{d}r + \text{d}x_5)^2 + H_1 \text{d}M_4^2 \right] + \sqrt{H_1 H_2} \left[ f^{-1} \text{d}r^2 + r^2 \text{d}\Omega_3^2 \right], \\
e^{-2\phi} &= \left( 1 + \frac{r_0^2 \sinh^2 \gamma}{r^2} \right) \left( 1 + \frac{r_0^2 \sinh^2 \alpha}{r^2} \right)^{-1}, \\
F_3 &= 2 \sqrt{c_1 (c_1 + r_0^2)} \frac{r^3 H_1^2}{r_0^3} \text{d}r \wedge \text{d}x_5 \wedge \text{d}r + 2 \sqrt{c_2 (c_2 + r_0^2)} \frac{r^3}{r^3} \epsilon_3,
\end{align*}$$

(3.15)

where $\epsilon_3$ is the volume element of 3-sphere. The harmonic functions appearing in the metric and the corresponding parameters are

$$\begin{align*}
H_1 &= 1 + \frac{c_1}{r^2}, \quad H_2 = 1 + \frac{c_2}{r^2}, \quad f = 1 - \frac{r_0^2}{r^2}, \\
c_1 &= r_0^2 \sinh^2 \alpha, \quad c_2 = r_0^2 \sinh^2 \gamma, \quad c_3 = r_0^2 \sinh^2 \sigma.
\end{align*}$$

(3.16)
The corresponding brane charges and the momentum are expressed in terms of $c_i$, the volume of $M_4$, $V$, the radius of $x_5$ circle, $R$, and the string coupling $g$:

\[ Q_i = \frac{V}{4\pi^2 g} \int * F_3 = \frac{V}{2g} 2 \sqrt{c_1 (c_1 + r_0^2)}, \]

\[ Q_5 = \frac{1}{4\pi^2 g} \int F_3 = \frac{1}{2g} 2 \sqrt{c_2 (c_2 + r_0^2)}, \quad n = \frac{R^2 V r_0^2}{2g^2} \sinh 2\alpha. \]  

(3.18)

The extremal limit of this background corresponds to taking $\alpha$, $\gamma$, $\sigma$ to $\infty$ and $r_0 \to 0$ keeping $c_1$, $c_2$, $c_3$ fixed, i.e.

\[ \lim_{r_0 \to 0} \sinh^2 \alpha \to c_1, \quad \lim_{r_0 \to 0} \sinh^2 \gamma \to c_2, \quad \lim_{r_0 \to 0} \sinh^2 \sigma \to c_3. \]  

(3.19)

We now determine the near-horizon geometry of the extremal solution. The near-horizon extremal geometry, as expected, has an AdS$_2$ factor. To obtain this, we first carry out the following coordinate transformation:

\[ r = \sqrt{c_3} x, \quad x_5 = y - t, \quad \tau = \lambda t, \]  

(3.20)

and then we take $\lambda \to 0$ which corresponds to the near-horizon limit. The near-horizon extremal metric becomes

\[ ds^2_{\text{str}} = \frac{\sqrt{c_1 c_2}}{4} \left( -4c_3 \frac{\rho^2}{c_1 c_2} dt^2 + \frac{d\rho^2}{\rho^2} + \frac{c_3}{\sqrt{c_1 c_2}} dy d\tau + \frac{c_1}{c_2} dM_4^2 + \sqrt{c_1 c_2} d\Omega_5^2. \]  

(3.21)

Changing the variable $\rho = 2\hat{\rho} \frac{1}{\sqrt{c_3 c_5}}$, we obtain

\[ ds^2_{\text{str}} = \frac{\sqrt{c_1 c_2}}{4} \left( -\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} + \frac{c_3}{\sqrt{c_1 c_2}} \left( dy - \frac{1}{2} \frac{c_1 c_2}{c_3} \rho \ d\tau \right)^2 + \sqrt{\frac{c_1}{c_2}} dM_4^2 + \sqrt{c_1 c_2} d\Omega_5^2. \]  

(3.22)

Thus, we obtain the AdS$_2$ factor with momentum along the $y$ direction. The three-form field strength in this limit is given by

\[ F_3 = \frac{1}{2} \frac{\sqrt{c_1 c_2}}{c_1} d\tau \wedge dy \wedge d\rho + \frac{2c_2}{\rho^3} \epsilon_3. \]  

(3.23)

Next, we apply both null and space-like $TsT$ transformations on this geometry and find the deformed solutions.

**3.2.1. The null $TsT$ transformation.** We work with full ten-dimensional geometry (3.15). We start with the null $TsT$ transformation; it is then convenient to define the null coordinates as

\[ x^\pm = \frac{1}{2} (t \pm x_5). \]  

(3.24)

Now, in terms of these new coordinates, we can rewrite the D1–D5 metric as

\[ ds^2 = (A_1 \, dx^- + K_1)^2 + (A_2 \, d\psi + A_3 \, dx^- + K_2)^2 + ds_8^2, \]  

(3.25)

where

\[ A_1 = \sqrt{c_3} \left( \coth \sigma - 1 \right), \quad A_2 = r (H_1 H_2)^{i}, \quad A_3 = 0, \]

\[ K_1 = -\frac{2r^2 - c_3 (\coth^2 \sigma - 1)}{r (H_1 H_2)^{i} \sqrt{c_3} (\coth \sigma - 1)} \, dx^*, \quad K_2 = r (H_1 H_2)^{i} \frac{1}{2} \cos \theta \, d\phi. \]  

(3.26)
and all other components of the metric are written in $dx_s^2$. $dx_3^2$ does not play any role in what follows. The $S^3$ metric is written as a Hopf fibration over $S^2$. While the fibered coordinate is written explicitly, the $S^2$ metric is part of $dx_3^2$. More concretely,

$$r^2 \sqrt{H_1 H_2} \, d \Omega_3^2 = (A_2 \, d \psi + A_3 \, dx^- + K_2)^2 + \frac{A_2}{4} (d \theta^2 + \sin^2 \theta \, d \phi^2). \quad (3.27)$$

As far as the NS–NS two-form field $B$ is concerned, the situation is similar to the D3 brane case, namely the $B$ field is vanishing in this background. Thus, we can readily write down the $TsT$ transformed geometry as (2.4), where

$$M = (1 + \gamma^2 A_2^2 A_3^2)^{-1} = (1 + \gamma^2 c_3 (\coth \sigma - 1))^{-1}. \quad (3.28)$$

Again the three-form field strength $F_3$ does not change under this transformation.

We can rewrite the above solution in terms of the original coordinates as

$$d s^2_{\text{eff}} = \frac{M}{\sqrt{H_1 H_2}} \left[ -dr^2 + dx_3^2 + \frac{c_3}{r^2} (\coth \sigma \, dr + dx_3) dx_3 + \gamma^2 (c_3 (\coth^2 \sigma - 1) - r^2) (dr + dx_3)^2 \right]$$

$$+ \frac{H_1}{H_2} \, d M_4^2 + \sqrt{H_1 H_2} \frac{d r^2}{f} + M(A_2 \, d \psi + A_3 \, dx^- + K_2)^2 + \frac{B}{4} (d \theta^2 + \sin^2 \theta \, d \phi^2)$$

$$e^{\psi} = M$$

$$\hat{B} = -\gamma M A_2 (c_3 (\coth \sigma - 1) - r^2) (\coth \sigma \, dr + dx_3) \wedge (A_2 \, d \psi + A_3 \, dx^- + K_2), \quad (3.29)$$

and $F_3$ does not change its form under this transformation.

**Extremal near-horizon limit.** We want to study the extremal near-horizon geometry of this non-relativistic black hole. The limit is defined in (3.19) and (3.20). In this limit, $\tanh \sigma \to 1$ and we can easily check that the geometry is the same as the one given in (3.22); in particular, the NS–NS two-form field $B$ vanishes. This fact has an interesting implication on the entropy of the non-relativistic black holes. Since extremal near-horizon geometry is identical for both original and $TsT$ transformed cases, it implies that the entropy of the non-relativistic black hole is identical to that of the relativistic one even after taking higher derivative corrections into account.

**3.2.2. The space-like $TsT$ transformation.** Let us now look at the effect of space-like $TsT$ transformations on the D1–D5-p system. In this case, we choose the two symmetry directions as $x_5$ and $\psi$. We again rewrite the metric as

$$d s^2 = (A_1 \, dx_5 + K_1)^2 + (A_2 \, d \psi + A_3 \, dx^- + K_2)^2 + d \bar{s}_3^2, \quad (3.30)$$

where

$$A_1 = \left( 1 + \frac{c_3}{r^2} \right) \frac{1}{(H_1 H_2)^{\frac{1}{2}}}, \quad A_2 = r (H_1 H_2)^{\frac{1}{2}}, \quad A_3 = 0, \quad K_1 = \frac{c_3 \coth \sigma}{r^2 (H_1 H_2)^{\frac{1}{2}} (1 + \frac{c_3}{r^2})^{\frac{1}{2}}} \, dr, \quad K_2 = r (H_1 H_2)^{\frac{1}{2}} \frac{1}{2} \cos \theta \, d \phi. \quad (3.31)$$

Now following the similar procedure as described above we can write the space-like $TsT$ transformed geometry, and it looks like (2.4), with $\chi$ as $x_5$ and metric functions as given in (3.31). $M$ takes the following form:

$$M = (1 + \gamma^2 r^2 \left( 1 + \frac{c_3}{r^2} \right))^{-1}, \quad (3.32)$$

9 In [25], the authors found the geometry for the $n = 0$ case.
and the explicit form of the background is
\[
d^2 s_{\text{str}} = \frac{\mathcal{M}}{\sqrt{H_1 H_2}} \left[ -\frac{\ell^2 \gamma}{r^2} (\coth \gamma dr + dx_5)^2 + \gamma^2 (c_3 \coth^2 \gamma - 1 - r^2) dr^2 \right]
+ \frac{H_1}{H_2} \frac{\ell^2}{f} dM_4^2 + \frac{H_1 H_2}{f} d^2 \mathcal{A} + \mathcal{M}(A_2 d\psi + A_3 dx^- + K_2)^2 + \frac{A_2}{4} (d\phi^2 + \sin^2 \theta d\phi^2),
\]
e^{2\Phi} = \mathcal{M},
\]
\[
\hat{B} = -\gamma \mathcal{M} A_2 dx_5 \wedge (A_2 d\psi + A_3 dx + K_2).
\]

We can again study the extremal near-horizon limit of this geometry and we find that it is not independent of the shift parameter \( \gamma \), quite unlike the null TsT transformed geometry. In the extremal limit, \( \mathcal{M} \rightarrow \mathcal{M}_0 = (1 + \gamma^2 c_3)^{-1} \), and hence the overall \( \gamma \) dependence does not drop out. The two-form NS–NS field is also non-trivial in this limit.

3.2.3. TsT of the non-extremal D1–D5 system: Bañados–Teitelboim–Zanelli (BTZ) black hole and CFT1. In this section, we consider the decoupling limit of the non-extremal D1–D5 system which is BTZ \( \times S^3 \times T^4 \). The boundary of this AdS black hole is two dimensional.

We start with metric (3.15) and consider the following decoupling limit:\(^{10}\)

\[
\alpha' \rightarrow 0, \quad r \rightarrow 0, \quad u_0 \rightarrow 0,
\]

\[
\alpha, \gamma, \sigma \rightarrow \infty,
\]

with
\[
U = \frac{r}{\alpha'}, \quad U_0 = \frac{r_0}{\alpha'} \text{ fixed}.
\]

In this limit, metric (3.15) becomes
\[
d s^2 = \alpha' \left[ \frac{U^2}{l^2} (dr^2 + dx_5^2) + \frac{U_0^2}{l^2} (\cosh \sigma dr + \sinh \sigma dx_5)^2 + \frac{dU^2}{U^2 - U_0^2} \right]
+ l^2 d\Omega_3^2 + \sqrt{\frac{Q_1}{v Q_5}} (dx_4^2 + \ldots + dx_5^2),
\]

After performing the coordinate redefinition
\[
u = \frac{(U^2 + U_0^2 \sinh^2 \sigma)}{l^2}, \quad \tau = lt, \quad r_+ = \frac{U_0 \cosh \sigma}{l}, \quad r_- = \frac{U_0 \sinh \sigma}{l},
\]

the BTZ black hole metric becomes
\[
d s^2 = \alpha' \left[ -\frac{\nu^2}{l^2} f d\tau^2 + \nu^2 \left( dx_5 + \frac{r_+ r_-}{u^2 l} dr \right)^2 + \frac{l^2}{u^2 f} d\tau^2 + l^2 d\Omega_3^2 + \sqrt{\frac{Q_1}{v Q_5}} (dx_4^2 + \ldots + dx_5^2) \right],
\]

where different quantities appearing in the metric are
\[
v = \frac{V_4}{16\pi \alpha'^2} \text{ fixed}, \quad l = (g_6 Q_1 Q_3)^{1/4},
\]
\[
g_6 = \frac{g_s}{\sqrt{v}}, \quad f = \left( 1 - \frac{r_+}{u^2} \right) \left( 1 - \frac{r_-}{u^2} \right),
\]

where \( V_4 \) is the volume of \( T^4 \) and \( g_s \) is the string coupling. We carry out only the null TsT transformation on the BTZ black hole solution. As usual, we again define light-cone

\(^{10}\) See [34] for a detailed discussion.
coordinates as \( x^\pm = (t \pm l x_5)/2 \) and follow the procedure outlined earlier. The final form of the \( TsT \) transformed geometry is

\[
d^s_2 = M(A_1 \, dx^- + K_1 \, dx^+)^2 + M \left( A_2 \, d\psi + \frac{1}{2} \cos \theta \, d\phi \right)^2 - A_4 (dx^+)^2 \\
+ \frac{l^2}{4} (d\theta^2 + \sin^2 \theta \, d\phi^2) + \frac{Q_1}{\sqrt{Q_5}} (dx_6^2 + \cdots + dx_9^2),
\]

where

\[
A_1 = \frac{r_r - r_\pm}{l}, \quad A_2 = l, \quad K_1 = \frac{r_r + r_\pm - 2r^2}{l(r_+ - r_-)},
\]

\[
A_4 = \frac{4(r^2 - r_\pm^2)(r_r^2 - r_\pm^2)}{l^2(r_r - r_\pm)^2}, \quad M = (1 + \gamma^2(r_r - r_\pm^2))^{-1}.
\]

This metric has asymptotic Schrödinger symmetry. This can be seen by carrying out the large-\( r \) asymptotic expansion of the above geometry and in particular, the asymptotic metric in light-cone coordinates has the form

\[
d^s_2 = M \left[ - \frac{4\gamma^2 r^4}{l^2} (dx^+)^2 - \frac{2r^2}{l^2} \, dx^+ \, dx^- + \left( d\psi + \frac{1}{2} \cos \theta \, d\phi \right)^2 \right] \\
+ \frac{l^2}{4} \left( d\theta^2 + \sin^2 \theta \, d\phi^2 + \frac{4 \, dr^2}{r^2} \right).
\]

It is evident that the asymptotic Schrödinger background has the dynamical exponent \( z = 2 \).

It is worth pointing out at this point that the boundary field theory in this case is one dimensional. In other words, this geometry is dual to the quantum mechanical system. Given the fact that the bulk geometry asymptotes to the Schrödinger spacetime with dynamical exponent \( z = 2 \) implies that the boundary theory is conformal quantum mechanics. This is because for \( z = 2 \), Schrödinger spacetimes have additional symmetry corresponding to special conformal transformations. Recall that conformal quantum mechanics plays a pivotal role in Sen’s quantum entropy function [35]. It would be interesting to see if a similar technique could be extended to non-relativistic black holes.

### 4. The \( TsT \) transformation of rotating M2 brane

In this section, we look at the rotating M2 brane solutions in the 11-dimensional supergravity (low-energy limit of M-theory). These M2 branes can have four angular momenta in the transverse plane and they source the three-form potential in the 11 dimensions. As in the case of the D3 brane, we start with a rotating M2 brane solution with all angular momenta equal. This background in the decoupling limit takes the form

\[
ds^2 = -\frac{f(r)}{H^2} \, dt^2 + H^2 \left( f(r) \, d\rho^2 + \rho^2 \, d\Omega^2_{2,k} \right) + g^{-2} \sum_{i=1}^{4} \left( d\mu_i^2 + \mu_i^2 (d\phi_i + A_i)^2 \right),
\]

\[
C^M_3 = -8g^3 \rho^3 H^3 \, dt \wedge d\Omega^2_{2,k} + 2\sqrt{q}(\mu + kq) \sum_i \mu_i^2 \, d\phi_i \wedge d\Omega^2_{2,k},
\]

where

\[
f = k - \frac{\mu}{r} + 4g^2 r^2 H^4, \quad H = 1 + \frac{q}{r}.
\]

We will now carry out the following coordinate transformation:

\[
r = \rho + q, \quad m = \mu + 2qk, \quad Q = \sqrt{q}(\mu + kq).
\]
From now on we will work in the $k = 0$ limit, which implies $d\Omega^2_{\mathbb{C}P^3} = dy_1^2 + dy_2^2$. We also redefine the $M2$ brane worldvolume coordinates, $\tilde{y} = 2g\tilde{x}$, and also redefine the coupling constant as $g = \frac{1}{4}$ so that we can bring the boundary metric in the form $r^2(-dr^2 + d\tilde{x}^2) + M_7$.

The metric and the three-form field strength expressed in terms of these variables become

$$
\begin{aligned}
&\text{ds}^2 = -r^2 f(r) \, dr^2 + \frac{dr^2}{r^2 f(r)} + \frac{r^2}{g^2} (dx_1^2 + dx_2^2) + 4g^2 \sum_{i=1}^4 (d\mu_i^2 + \mu_i^2 (d\phi_i + A)^2), \\
&C_3^M = -\tilde{g} r^3 \, dt \wedge dx_1 \wedge dx_2 + \frac{2Q}{g^2} \sum_{i=1}^4 \mu_i^2 d\phi_i \wedge dx_1 \wedge dx_2,
\end{aligned}
$$

(4.4)

and

$$
\begin{aligned}
&A = \frac{Q}{r\tilde{g}} \, dt, \\
&f = \tilde{g}^2 - \frac{m}{r^3} + \frac{Q^2}{r^4}.
\end{aligned}
$$

(4.5)

The $S^7$ metric can be written in the Hopf fibration form

$$
\sum_{i=1}^4 (d\mu_i^2 + \mu_i^2 (d\phi_i + A)^2) = (d\psi + {\mathcal P} + A)^2 + d\xi^2_{\mathbb{C}P^3},
$$

(4.6)

where

$$
\begin{aligned}
&\phi_1 = \psi + \frac{1}{4}(\xi_1 + \xi_2 + \xi_3), \\
&\phi_2 = \psi + \frac{1}{4}(\xi_2 + \xi_3 - 3\xi_1), \\
&\phi_3 = \psi + \frac{1}{4}(\xi_3 + \xi_1 - 3\xi_2), \\
&\phi_4 = \psi + \frac{1}{4}(\xi_1 + \xi_2 - 3\xi_3).
\end{aligned}
$$

(4.7)

The advantage of this form of the metric is in the convenience of expressing equal angular momenta in four transverse planes. They essentially translate into momentum along the $\psi$ direction. The one-form $P$ is

$$
\begin{aligned}
&P = \frac{1}{4} (d\xi_1 + d\xi_2 + d\xi_3) - \mu_3^2 d\xi_1 - \mu_1^2 d\xi_2 - \mu_2^2 d\xi_3,
\end{aligned}
$$

(4.8)

and the explicit form of the $\mathbb{C}P^3$ metric is

$$
\begin{aligned}
&d\xi_{\mathbb{C}P^3}^2 = d\alpha^2 + \sin^2 \alpha \, d\beta^2 + \sin^2 \alpha \, \sin^2 \beta \, dy^2 \\
&\quad + \cos^2 \alpha \, \sin^2 \alpha (d\xi_1 + (d\xi_1 - d\xi_3) \cos^2 \beta + (d\xi_2 - d\xi_3) \cos^2 \gamma \sin^2 \beta)^2 \\
&\quad + \sin^2 \alpha \, \sin^2 \beta \cos^2 \beta (d\xi_1 - d\xi_3) + (-d\xi_2 + d\xi_3) \cos^2 \gamma)^2 \\
&\quad + \cos^2 \gamma \, \sin^2 \gamma \sin^2 \beta \sin^2 \alpha (d\xi_2 - d\xi_3)^2.
\end{aligned}
$$

(4.9)

In terms of the Hopf fibration, the metric and three-form fields for the rotating $M2$ brane solution are given by

$$
\begin{aligned}
&\text{ds}^2 = -r^2 f(r) \, dr^2 + \frac{dr^2}{r^2 f(r)} + r^2 (dx_1^2 + dx_2^2) + 4(d\psi + {\mathcal P} + A)^2 + 4 d\xi_{\mathbb{C}P^3}^2, \\
&C_3^M = -r^3 \, dt \wedge dx_1 \wedge dx_2 + 2Q (d\psi + {\mathcal P}) \wedge dx_1 \wedge dx_2.
\end{aligned}
$$

(4.10)

Here, for simplicity we have set $\tilde{g} = 1$.

Since T-duality is not a symmetry of M-theory, we can carry out the $TsT$ transformation only after re-expressing this solution in terms of string theory variables. For this purpose, it is useful to write down the metric and three-form field in terms of vielbeins, which for this background are
4.1. Reduction along the worldvolume $x_2$ direction

Let us consider that the worldvolume direction $x_2$ is compact. We can then dimensionally reduce the M2 brane solution to a ten-dimensional solution of type IIA theory. This ten-dimensional solution will only have the NS–NS sector fields turned on. Now following (4.14),
we can read out the metric, dilaton and NS–NS two-form field

\[
d s^2_{\text{IIA}} = r \left[ -\frac{r^2 f(r)}{r^2 f(r)} \, dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \, dx_1^2 + 4(d\psi + \mathcal{P} + A)^2 + 4 \, ds_{\text{CP}^3}^2 \right],
\]

\[
B_{\mu
u} = C^{\mu
u}_{\psi x_1} = -r^3 \, dt \wedge dx_1 + 2Q(d\psi + \mathcal{P}) \wedge dx_1,
\]

\[
\exp \left[ \frac{4}{3} \Phi \right] = \exp[2\sigma] = r^2.
\]

The metric and the two-form field are given by

\[
d s^2_{10} = r\left( -(e^0)^2 + (e^1)^2 + (e^2)^2 + 4((e^3)^2 + \cdots + (e^8)^2) + (e^9)^2 \right),
\]

\[
B = 16r \left( 1 + \frac{Q}{4r^4} \right) \sqrt{\frac{e^0 \wedge e^1}{4f}} + \frac{2Q}{r\sqrt{1-f}} (\sqrt{f} e^0 \wedge e^1) \wedge e^9
\]

\[
H = 24r \left( \frac{1}{f} e^0 \wedge e^1 \wedge e^2 + \frac{4Q}{r\sqrt{1-f}} e_{\text{CP}^3} \wedge (\sqrt{f} e^0 \wedge e^1) \right).
\]

We follow the TsT transformation rules given in section 2 ((2.7), (2.10)) and obtain

\[
d s^2 = \mathcal{M}[(A_1 dx^- + K_1 dx^+)^2 + (A_2 d\psi + 2\sqrt{r} \mathcal{P} + A_3 dx^- + K_2 dx^+)^2]
\]

\[
- A_4 (dx^+)^2 + 4r \, ds_{\text{CP}^3} + \frac{dr^2}{rf},
\]

\[
\hat{B} = \mathcal{M}[B(1 - 2Q\gamma) - \gamma A_1 A_2 (A_1 dx^- + K_1 dx^+) \wedge (A_2 d\psi + 2\sqrt{r} \mathcal{P} + A_3 dx^- + K_2 dx^+)],
\]

where

\[
A_1 = r \sqrt{r(1 - f)}, \quad A_2 = 2\sqrt{r}, \quad K_1 = -\frac{r^{3/2}(1 + f)}{\sqrt{1 - f}}, \quad A_3 = K_2 = 2A_4 \sqrt{r},
\]

\[
A_4 = \frac{2\sqrt{r^3}}{\sqrt{1 - f}}, \quad \hat{K}_1 = K_1 + \gamma \frac{4f Q r^{3/2}}{\sqrt{1 - f}}, \quad \hat{K}_2 = K_2 - \gamma \frac{4(2Q^2 + r^4)}{\sqrt{r}}
\]

\[
\mathcal{M} = (1 - 4\gamma Q + 4\gamma^2 (Q^2 + (1 - f)r^4))^{-1}.
\]

By carrying out the expansion of the metric for large \( r \), it is easy to see that the above solution does not have asymptotic Schrödinger geometry. It is particularly easy to see that \( \mathcal{M} \) does not become constant as we approach the boundary; it instead vanishes. Thus, although the above field configuration is a solution of type IIA theory and therefore its uplift along \( x_1 \) is a solution in M-theory, it does not possess either asymptotic AdS or Schrödinger symmetry. We will therefore not pursue the detailed study of this background any further.

We can nevertheless obtain interesting M-theory solutions with asymptotic AdS symmetry if we follow a strategy of first performing a chain of duality transformations and then carrying out the TsT transformation. In the process, we have multiple ways of lifting the solution back to M-theory. Following will be our strategy of obtaining a new solution from the type IIA background that we obtained by reducing the rotating M2 brane solution along the \( x^2 \) direction.

1. We first T-dualize the type IIA solution (4.17) along the \( \psi \) direction. This takes us to type IIB theory. We continue to refer to this T-dual direction as \( \psi \).
2. Next we perform an S-duality transformation on this solution.
3. We then carry out the usual TsT transformation on this S-dual solution. The T-duality is performed on \( \psi \) and the shift is performed along one of the light-cone directions defined after equation (4.11).
(4) We can then go back to the type IIA solution, either by
(a) performing another S-duality transformation on the $TsT$ transformed geometry and
then T-dualize it back to type IIA or
(b) by directly T-dualizing the deformed geometry to type-IIA.

(5) Finally, we uplift the resulting type IIA solution obtained by either method to M-theory.

We provide details of intermediate steps in appendix B and present only the final $TsT$
transformed solution uplifted to M-theory here. If we follow step 4(a), then the M-theory
metric becomes

$$d\gamma_{M}^{2} = r^{2}M[(1 - f - 4f\gamma^{2})(dx^{+})^{2} - 2(1 + f)dx^{+}dx^{-} + (1 - f)(dx^{-})^{2}]$$

$$+ \frac{dr^{2}}{r^{2}f} + 4(d\psi^{2} + ds_{\mathbb{C}P^{3}}^{2}) + e^\frac{1}{2}(dx_{2} + C_{T2}^{\mu}dx^{\mu})^{2},$$

$$C_{3}^{M} = \frac{1}{3!}C_{5\mu\nu\rho}^{T2}dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} + \frac{1}{2}B_{\mu\nu}^{T2}dx^{\mu} \wedge dx^{\nu} \wedge dx_{2},$$

with $M = (1 + \gamma^{2}(1 - f))^{-1}$. The expressions for $C_{T2}^{\mu}$ and $C_{3\mu\nu\rho}^{T2}$ are given in appendix B.

In the case when we directly T-dualize the $TsT$ transformed solution using step 4(b), the following M-theory geometry is obtained:

$$d\gamma_{M}^{2} = (1 - f)r^{2}((dx^{-})^{2} + (dx^{+})^{2} - \frac{1 + f}{1 - f}dx^{+}dx^{-}) + \frac{1}{f}r^{2}dr^{2} + Mr^{2}d\psi^{2}$$

$$+ r^{2}\gamma((1 + f)dx^{+} - (1 - f)dx^{-})d\psi + 4(ds_{\mathbb{C}P^{3}}^{2} + (dx_{2} + C_{T2}^{\mu}dx^{\mu})^{2}),$$

$$C_{3}^{M} = \frac{1}{3!}C_{5\mu\nu\rho}^{T2}dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} + \frac{1}{2}B_{\mu\nu}^{T2}dx^{\mu} \wedge dx^{\nu} \wedge dx_{2}.$$  \hspace{1cm} (4.23)

The asymptotic geometry is given by

$$d\gamma^{2} = \frac{dr^{2}}{r^{2}} + r^{2}(-4dx^{+}dx^{-} + 4\gamma dx^{+}d\psi + d\psi^{2})$$

$$+ 4\left(dx_{2} + \sum\left(-\frac{1}{4} + \mu_{i}^{2}\right)dx_{i}\right)^{2} + 4ds_{\mathbb{C}P^{3}}^{2},$$ \hspace{1cm} (4.24)

Although the deformed M-theory solution has a non-trivial $\gamma$ dependence and is well
defined at the boundary, it is easy to check that this metric is asymptotically AdS. It does not have
a Schrödinger-like behavior at the boundary.

4.2. Reduction along the transverse $\psi$ direction

We can also reduce the M-theory solution along the $\psi$ direction, the corresponding type IIA
solution has both NS–NS and R–R sector turned on. Again following the general rules given
in (4.14), the type IIA solution that we obtain is as follows:

$$d\gamma_{IIA}^{2} = 2\left(-r^{2}fdr^{2} + \frac{dr^{2}}{r^{2}f} + r^{2}(dx_{2}^{2} + dx_{2}^{2}) + 4ds_{\mathbb{C}P^{3}}^{2}\right),$$ \hspace{1cm} (4.25)

$$e^{2\Phi} = 8,$$

$$B_{2} = 2Qdx_{1} \wedge dx_{2}, \quad C_{\mu} = P \wedge A, \quad C_{3} = (-r^{3}dr + 2Qp) \wedge dx_{1} \wedge dx_{2}.$$  \hspace{1cm} (4.26)

We perform the $TsT$ transformation on this solution. For this purpose, we again define
the light-cone directions as given after equation (4.11). In terms of the vielbeins, the metric
looks as

$$d\gamma^{2} = 2(-(e^{0})^{2} + (e^{1})^{2} + (e^{2})^{2} + 4((e^{3})^{2} + \cdots + (e^{8})^{2} + (e^{10})^{2})).$$ \hspace{1cm} (4.27)

We also redefine $\xi_{1}, \xi_{2}$ and $\xi_{3}$ in terms of $\chi_{1} = \xi_{1}, \chi_{2} = \xi_{1} - \xi_{2}$ and $\chi_{3} = \xi_{1} - \xi_{3}$. The $\mathbb{C}P^{3}$ metric can now be written using the new variables as

$$4ds_{\mathbb{C}P^{3}}^{2} = \sin^{2}2\alpha(d\chi_{1} + N)^{2} + dx_{2}^{2},$$ \hspace{1cm} (4.28)
where
\[ N = d\chi_2 \cos^2 \beta + d\chi_3 \cos^2 \delta \sin^2 \beta, \]
and \(ds^2\) is the remaining part of the \(\mathbb{CP}^3\) metric consisting of \(\alpha, \beta, \delta, \chi_2\) and \(\chi_3\).

Next we perform the T-duality transformation along the \(\chi_1\) and the shift along the \(x^-\) direction. The final \(T\) \(T\) transformed solution takes the following form:
\[
d\hat{s}^2 = 2M \left( (e^1)^2 + \left( e^6 + 2\gamma \frac{Q}{r} \sin 2\alpha e^{10} \right)^2 \right) + 2\left( -\left( e^0 \right)^2 + (e^2)^2 + 4\left((e^3)^2 + (e^5)^2 + (e^7)^2 + (e^{10})^2 \right) \right). \tag{4.30} \]

In coordinate basis, the full solution looks like
\[
d\hat{s}^2 = \mathcal{M}\left((A_1 dx^- + K_1 dx^+)^2 + (A_2(d\chi_1 + N) + 2\gamma Q \sin 2\alpha dx_2)^2 - D(dx^+)^2 \right) + 2\left( \frac{dr^2}{r^2} + r^2 dx^2_1 + dx^2_2 \right),
\]
\[\hat{B} = \mathcal{M}(B - \gamma A_3A_2(A_1 dx^- + K_1 dx^+) \wedge A_2(d\chi_1 + N) + 16\gamma^2 f Q r^2 \sin^2 2\alpha dx^+ \wedge dx_2),\]
\[e^{\delta\phi} = \frac{8}{\mathcal{M}}, \quad \hat{C}_1 = \mathcal{P} + A + 2\gamma Q \cos \alpha dx_2, \quad \hat{C}_3 = C_3 + (\mathcal{P} + A) \wedge B - \hat{C}_1 \wedge \hat{B}. \tag{4.31} \]

Various functions appearing in the metric and other background fields are
\[A_1 = \sqrt{2(1 - f)r}, \quad A_2 = \sqrt{2} \sin 2\alpha,\]
\[K_1 = -\frac{\sqrt{2}(1 + f)r}{\sqrt{1 - f}}, \quad A_4 = \frac{2\sqrt{2} f r}{\sqrt{1 - f}}. \tag{4.32} \]

The asymptotic metric (after uplifting to 11 dimensions) is given by
\[
ds^2 = \left( -16\gamma^2 \sin^2 2\alpha r^4 (dx^+)^2 - 4r^2 dx^+ dx^- + r^2 dx^2_1 + \frac{dr^2}{r^2} \right)
+ \frac{1}{2}(A_2(d\chi_1 + N) + 2\gamma Q \sin 2\alpha dx_2)^2 + dx^2_2 + 4(dx^+ \wedge \hat{C}_1)^2. \tag{4.33} \]

This metric has an interesting feature that its \(g_{++}\) component depends on \(\alpha\), which is one of the \(\mathbb{CP}^3\) coordinates. As a result, the four-dimensional space is non-trivially fibered over the periodic coordinate \(\alpha\). Interestingly, the \(\alpha\) dependence of \(g_{++}\) is such that for \(\alpha = 0, \pi/2\) and \(\pi\), the asymptotic metric reduces to pure \(AdS_4\) spacetime, whereas for all other values of \(\alpha\), we obtain the asymptotically Schrödinger background. We get back to these special values of \(\alpha\) in the concluding section. Defining the rescaled light-cone coordinates as
\[u = 2\nu x^+, \quad v = \frac{1}{\nu} x^-, \quad \nu = 2\gamma \sin 2\alpha, \tag{4.34} \]
it is easy to see that the above metric takes the form (2.1) at the boundary \(r \to \infty\) with the dynamical exponent \(z = 2\). The fact that the asymptotic geometry is nontrivially fibered over a compact coordinate of \(\mathbb{CP}^3\) has interesting implications for thermodynamics. We find it convenient to isolate the coordinate \(\alpha\) from the rest of the \(\mathbb{CP}^3\) coordinates and club it with the noncompact coordinates in order to give unambiguous definitions of thermodynamic quantities. In particular, we leave out the \(\alpha\) direction and integrate out the rest of the \(\mathbb{CP}^3\) coordinates along with the Hopf fiber coordinate \(\psi\). This leaves us with a five-dimensional spacetime, which, as mentioned above, looks asymptotically like a four-dimensional Schrödinger space fibered over the \(\alpha\) circle. All the thermodynamic quantities associated with the black hole in the interior of this spacetime will depend on where we are located in the \(\alpha\) direction.
5. Thermodynamics and phase structure

We study the thermodynamics and the possible phase transition of these non-relativistic systems to other non-relativistic systems. To understand the phase structure of a black hole spacetime, one needs to compute its free energy [36] which is given by $W = I/\beta$, where $I$ is the (Euclidean) on-shell action and $\beta$ is the inverse temperature. In asymptotically AdS space, the on-shell action has large volume divergences\(^\text{11}\) and we need to regularize this action by either using boundary counterterms [37] or subtracting the contribution of the background spacetime [38]. A detailed discussion of the background subtraction method can be found in [39].

In the Hawking–Page formalism (or the background subtraction method), one has to match the geometry of the black hole spacetime and background spacetime at some constant $r = \tilde{R}$ hyper-surface. This matching fixes the temperature of the background spacetime in terms of that of the black hole spacetime, and at the end one takes the cutoff $\tilde{R} \to \infty$. In [12, 13], it has been argued that for black holes in asymptotically Schrödinger space, the subtraction method is subtle as the null circle becomes degenerate for the background spacetime on the constant $\tilde{R}$ hyper-surface, whereas for the black hole spacetime, it is not. Therefore, they introduced an ‘ad hoc’ prescription to compute a renormalized on-shell action.

It is worth emphasizing that the Euclidean method, by itself, is sufficient to compute all thermodynamic quantities, which satisfy the first law, starting from the renormalized on-shell action. The on-shell action (or more precisely the free energy) captures the phase structure of the black hole spacetime. Therefore, our goal here is to compute the free energy so that we can analyze the phase structure of these solutions. While the Euclidean method is sufficient, it can become quite cumbersome for complicated field configurations. Our new solutions with Schrödinger asymptotics do indeed have several fields turned on. While it is still possible to evaluate the Euclidean action, we will momentarily define an alternate method for computing the on-shell Euclidean action. First of all, note that the Hawking temperature can be determined either by computing surface gravity on the horizon or by computing the periodicity of the Euclidean time circle. In either case, it suffices to have the knowledge of the near-horizon geometry. Secondly, recall that the entropy of the black hole spacetime can be determined in two different ways. One way is to evaluate the Euclidean on-shell action in the black hole spacetime and then using thermodynamic relations to derive expression for entropy. Another method is to use Wald’s formula to directly derive black hole entropy. In all known relativistic examples, both these methods give rise to the same formula for entropy [39]. The strategy we employ rests on the equality of these two ways of deriving the entropy. We first determine the entropy using Wald’s method and turn the Euclidean action method on its head to derive the on-shell action by integrating the Wald entropy. This requires solving a first-order inhomogeneous differential equation. The on-shell action derived in this manner contains an ambiguity corresponding to the constant of integration. This ambiguity can be fixed by demanding that the free energy vanishes for a fixed background spacetime. We subsequently write down the free energy for the black hole spacetime using this on-shell action and the Hawking temperature derived from either of the methods mentioned above. We illustrate this method for the AdS Reissner–Nordström black hole in $d+1$ dimensions; however, we eventually apply it to the asymptotically Schrödinger backgrounds. This method becomes particularly efficient in the case of the $TsT$ transformed geometries where a variety of fields are turned on due to the solution-generating technique. While the Euclidean action

\(^{11}\) For asymptotically flat black holes, the divergence comes from the Gibbons–Hawking boundary term.
can nevertheless be determined by conventional methods, our method turns out to be much more efficient in deriving desired results.

Let us start with the AdS–Reissner–Nordström black hole in $d+1$ dimensions. The solution is given by

$$\begin{align*}
    ds^2 &= -V(r)\, dt^2 + \frac{dr^2}{V(r)} + r^2 \, d\Omega_{k,d-1}^2, \\
    A(r) &= \left( -\frac{q}{c} \frac{r^{d-2}}{d-2} + \mu \right) \, dt,
\end{align*}$$

where

$$
    V(r) = k - \frac{m}{r^{d-2}} + \frac{q^2}{r^{2d-4}} + \frac{r^2}{L^2}, \quad c = \frac{2(d-2)}{d-1}.
$$

Here $k$ determines the horizon topology. For a flat horizon (black brane), $k = 0$, whereas $k = 1$ ($-1$) for spherical (hyperboloid) horizons. The asymptotic value of the gauge field $A_t$ is defined to be the chemical potential,

$$
    \mu = \frac{1}{c} \frac{q}{r^{d-2}}.
$$

The temperature of the black hole can be computed from its surface gravity at the horizon defined as [8]

$$
    \kappa^2 = -\frac{1}{2} (\nabla^a \zeta^b)(\nabla_a \zeta_b)|_H, \tag{5.4}
$$

where $\zeta^a$ is the null generator of the horizon. The Hawking temperature is then $T = \frac{1}{2\pi} \kappa$. We can also find the temperature as the inverse of the period of the Euclidean time circle. Thus, the classical solution has a finite temperature

$$
    T = \frac{1}{\beta} = \left[ \frac{4\pi L^2 r_i^{2d-3}}{\left( dr_i^{d-2} + k(d-2) L^2 r_i^{2d-4} - (d-2) q^2 L^2 \right)^{1}} \right], \tag{5.5}
$$

where $r_i$ is the position of the outer horizon.

We compute entropy using Wald’s formula. For two-derivative gravity theory, the formula gives entropy as the horizon area $A_{\text{Horizon}}$. To employ Wald’s formula, we need the metric to be written in Einstein’s frame. We find the horizon area [40] by writing the metric in the Boyer–Lindquist coordinate, so that there is an $r$-coordinate which does not mix with others\(^\text{12}\). We can then write the metric as

$$
    g_{\mu\nu} = \begin{pmatrix} g_{rr} & 0 \\ 0 & P_{ij} \end{pmatrix}. \tag{5.6}
$$

Note that the time coordinate $t$ appears in the metric $P_{ij}$. The local definition of the horizon is a fixed $(t, r)$ surface where $g_{rr}$ diverges and the determinant of $P_{ij}$ vanishes. The cofactor of $P_{ij}$ is

$$
    A^{\mu\nu} = g^{\mu\nu} \det P. \tag{5.7}
$$

The area of the horizon can then be written as $A_{\text{Horizon}} = \sqrt{A^{\mu\nu}}$. Using this formulation, we obtain the entropy

$$
    S = \frac{A_{\text{Horizon}}}{4G} = \frac{V_{k,d-1} r_i^{d-1}}{4G}, \tag{5.8}
$$

where $V_{k,d-1}$ is the volume of the transverse space. In the following subsection, we will use thermodynamic relations to derive the on-shell action using the expression for entropy given in (5.8).

\(^{12}\) All the metrics that we have considered in this paper are in the Boyer–Lindquist coordinates.
5.1. Computation of the on-shell action

We now use the expressions derived above for the entropy and the temperature and substitute them into thermodynamics relations to compute the on-shell action or equivalently the free energy. The expression for free energy depends on the ensemble we are working on. In the present situation, we have two ensembles: fixed potential ensemble and fixed charge ensemble [41, 42].

**Fixed potential case.** In this ensemble, the relation between entropy and free energy is given by

\[ S - \beta \left( \frac{\partial I}{\partial \beta} \right)_\mu + I = 0. \quad (5.9) \]

Once we determine the on-shell action \( I \), we can compute other thermodynamic quantities such as the energy and the physical charge using

\[ E = \left( \frac{\partial I}{\partial \beta} \right)_\mu - \mu \left( \frac{\partial I}{\partial \mu} \right)_\beta, \quad Q = -\frac{1}{\beta} \left( \frac{\partial I}{\partial \mu} \right)_\beta. \quad (5.10) \]

Now let us solve equation (5.9) using (5.5) and (5.8). The solution is given by

\[ I = -\frac{V_{k,d-1}}{16\pi GL^2} \left( L^2 r_{*}^{d-2} (c^2 \mu^2 - k) + (r_{*}^{n} - 4c_1) \right). \quad (5.11) \]

However as we have pointed earlier, we have an undetermined constant \( c_1 \), which has to be fixed. This is achieved by employing the following strategy: the free energy of the black hole spacetime is measured with respect to ‘some’ background spacetime. We, therefore, choose the scale such that the free energy of the background is zero. With this choice, we can fix the constant \( c_1 \). If we want to measure the free energy of the black hole spacetime with respect to global AdS, then we set the free energy of the global AdS spacetime to zero. The global AdS spacetime corresponds to \( r_* = 0 \) keeping \( \mu \) fixed in this ensemble\(^{13}\). This implies

\[ W \equiv \left. \frac{I}{\beta} \right|_{r_* \to 0} = 0. \quad (5.12) \]

This uniquely fixes \( c_1 = 0 \). Therefore, the on-shell action is given by

\[ I = -\frac{V_{k,d-1}}{16\pi GL^2} \left( L^2 r_{*}^{d-2} (c^2 \mu^2 - k) + r_{*}^{n} \right). \quad (5.13) \]

This result was obtained in [41] and the complete phase structure of the AdS–Reissner–Nordström black hole was analyzed there\(^{14}\).

**Fixed charge case.** One can also consider thermodynamic ensemble where the charge parameter \( q \) is fixed. In this case, the thermodynamic relations are given by

\[ S - \beta \left( \frac{\partial I}{\partial \beta} \right)_q + I = 0 \quad (5.14) \]

and

\[ E = \left( \frac{\partial I}{\partial \beta} \right)_q, \quad \mu = \frac{1}{\beta} \left( \frac{\partial I}{\partial \mu} \right)_\beta. \quad (5.15) \]

\(^{13}\) For black branes, we chose the background to be an AdS soliton and set the free energy of the soliton to zero.

\(^{14}\) Also see [43] for the black brane phase structure.
We use (5.14) to compute $I$ and it is given by

$$I = \frac{V_d}{16\pi GL^2} \left( kL^2 r_e^{-2} - r_e^d + (2d - 3)q^2 L^2 r_e^{2-d} + 4c_1 \right). \tag{5.16}$$

For this ensemble, we cannot measure the free energy of the black hole spacetime with respect to global AdS as the latter has zero charge. We instead measure the free energy with respect to the extremal solution. The extremal solution has the same charge $q$ as the black hole, whose free energy we wish to derive and its horizon is located at $r_e$, where

$$d r_e^{2d-2} + L^2 (d-2)(q^2 r_e^4 - k r_e^2 d) = 0. \tag{5.17}$$

In this case we are comparing the free energy of the black hole with that of the extremal solution. The constant of integration, $c_1$, is thus determined by setting the free energy for the extremal black hole to zero:

$$W = \left. I \right|_{r_e \to r_e, q \text{fix}} = 0 \Rightarrow c_1 = \frac{1}{4r_e^{d+2}} \left( (2d-3)q^2 r_e^4 + kr_e^{2d} \right). \tag{5.18}$$

The Euclidean action for the black hole is therefore given by

$$I = \frac{V_d}{16\pi GL^2} \left( kL^2 r_e^{-2} - r_e^d + (2d - 3)q^2 L^2 r_e^{2-d} \right)$$

$$- \frac{2k(d-1)}{d} L^2 r_e^{d-2} - \frac{2(d-1)^2}{d} \frac{q^2 L^2}{r_e^{d-2}}. \tag{5.19}$$

This result was also found in [41]. For the $k = 0$ (black brane) case, we see that the black brane geometry is always dominant over the background geometry. Thus, we see that our strategy gives the desired result in the case of the AdS–Reissner–Nordström black hole in $d+1$ dimensions. In the following subsection, we use this strategy to study the thermodynamics of non-relativistic black objects.

### 5.2. Thermodynamics of non-relativistic black objects

We use our strategy to perform similar computations but this time for $(d+1)$-dimensional non-relativistic black objects (black branes and black holes) and study their phase structure. Such black objects can be obtained by the dimensional reduction of various non-relativistic branes (D3, M2) along the transverse space. As we have seen in the last subsection, for studying the phase structure of a black hole spacetime, we need to know its free energy. Using thermodynamic relations, we can obtain the free energy from the knowledge of the temperature and the entropy of the black hole.

To find the temperature of the non-relativistic black objects, we can use the surface gravity defined in (5.4). The null generator $\zeta$ is proportional to $\frac{\partial}{\partial t}$. In analogy with asymptotically flat space, where the null generator has unit norm asymptotically, we can fix the normalization of $\zeta$. We demand that the component of $\zeta$ along the boundary (non-relativistic CFT) time translation has unit coefficient. Since the generator of time translation in the boundary theory is $\frac{\partial}{\partial \tau}$, we obtain

$$\zeta = \frac{1}{\nu} \frac{\partial}{\partial t} = \frac{\partial}{\partial u} + \frac{1}{2v^2} \frac{\partial}{\partial v}, \tag{5.20}$$

where $\nu$ is the scaling parameter of the light-cone directions, which is required for obtaining the boundary $r \to \infty$ metric as in (2.1). In particular, $\nu = \gamma f$ for non-relativistic rotating D3 brane geometry (3.11) and $\nu = 2\gamma \sin 2\alpha$ for non-relativistic rotating M2 brane geometry.
We can also find the temperature as the inverse of the period of the Euclidean time circle. Either of these procedures gives us

$$T = \frac{1}{\beta} = \left[ \frac{4\pi r_r^{2d-3} \nu}{\text{d} r_+^{d-2} + k (d-2) r_+^{2d-4} - (d-2) q^2} \right]^{-1}. \hspace{1cm} (5.21)$$

To compute the entropy, we need to compute the horizon area $A_{\text{horizon}}$. We can use (5.6) and (5.7), provided we keep in mind that the correct time direction for the non-relativistic black objects is $u$. This gives us the entropy as

$$S = \frac{A_{\text{horizon}}}{4G} = \frac{V_{k,d-1} r_u^{d-1} \nu}{4G}. \hspace{1cm} (5.22)$$

5.2.1. On-shell action for the fixed potential case. To study the phase structure, we have to first decide on the choice of ensemble. Now since the non-relativistic theories can be realized as a deformed version of relativistic quantum field theories, it is natural to allow the particle number to fluctuate. Thus, grand canonical ensemble is the right choice of ensemble in this case. Hence, we study the phase structure of non-relativistic systems only for the fixed potential ensemble. The strategy described above can be readily applied to the fixed charge case; however, we do not pursue that line here. We hope to report on that elsewhere.

To compute the on-shell action, we use the temperature and entropy as input and use relation (5.9). In this case, we have two chemical potentials corresponding to two conserved charges. As the Killing generator $\zeta$ of the event horizon has a component along the light-like direction $\nu$ (5.20), the corresponding momentum $\frac{\partial}{\partial \nu}$ is a conserved charge and we have a chemical potential for the $\nu$-translation given by $\mu_1 = 1/(2\nu^2)$. The other chemical potential is associated with the boundary value of the gauge field (5.3). With $u$ being the correct boundary time direction, we decompose $A = A_t \text{d}t = A_u \text{d}u + A_v \text{d}v$. Thus, the second chemical potential is defined as

$$\mu_2 = \lim_{r \to \infty} A_u = \frac{1}{2c^2} q \frac{v}{r_u^{d-2}} = \frac{\sqrt{\mu_1}}{\sqrt{2c} r_u^{d-2}}. \hspace{1cm} (5.23)$$

Replacing the charges in terms of the corresponding chemical potentials, the temperature in grand canonical ensemble is given by

$$\frac{1}{T} = \beta = \frac{2\sqrt{2\pi r_+} \sqrt{\mu_1}}{-2c^2 \text{d} \mu_2^2 + 4c^2 \mu_2^2 + (d-2) k \mu_1 + \text{d} r_+^2 \mu_1}. \hspace{1cm} (5.24)$$

We can then solve equation (5.9) to write the on-shell action as

$$I = \frac{-2\sqrt{2c^2 \mu_2^2 r_u^2} + 8c_1 r_u^2 \sqrt{\mu_1} + \sqrt{2} k \mu_1 r_u^d - 2k \mu_1 r_u^{d+2}}{8r_+ \sqrt{\mu_1} (-2c^2 \mu_2^2 + 4c^2 \mu_2^2 + (d-2) k \mu_1 + \text{d} r_+^2 \mu_1)}. \hspace{1cm} (5.25)$$

The corresponding free energy is given by

$$W = \frac{r_u^d (k \mu_1 - 2c^2 \mu_2^2)}{16\pi r_+^2 \mu_1} + 4\sqrt{2c_1 r_u^2 \sqrt{\mu_1} - \mu_1 r_u^{d+2}}. \hspace{1cm} (5.26)$$

As in the relativistic case, we have an undetermined constant $c_1$ that needs to be fixed. Choosing the non-relativistic extremal black hole/brane geometry\(^{15}\) as the background and setting its free energy to zero, we obtain $c_1 \to 0$ and the free energy becomes

$$W = \frac{r_u^d (k \mu_1 - 2c^2 \mu_2^2) - \mu_1 r_u^{d+2}}{16\pi r_+^2 \mu_1}. \hspace{1cm} (5.27)$$

\(^{15}\) This geometry is obtained by setting $r_+ \to 0$ and $Q \to 0$ of the higher dimensional brane solution.
Thus, from (5.27) we see that black branes \((k = 0)\) are always dominant\(^{16}\), whereas there exists a phase transition between the black hole \((k = 1)\) phase and the background spacetime when
\[
\frac{\mu_1}{\mu_2} = \frac{2c^2}{1 - r_c^2}. \quad (5.28)
\]
When the ratio \(\frac{\mu_1}{\mu_2}\) is less than its critical value (5.28), the black hole geometry will be dominant over the background. This is the usual Hawking–Page transition. Also, if we assume that both the chemical potentials are positive, then \(r_c < 1\).

It is clear from the equation of \(\beta\) (5.24) that there are two distinct behaviors of \(\beta\) as a function of the ratio of two chemical potentials, namely \(\mu_1/\mu_2^2\). For \(\frac{\mu_1}{\mu_2^2} < 2c^2\), \(\beta\) diverges when\(^{17}\)
\[
\beta = \frac{\sqrt{(d-2)(\mu_1 - 2c^2\mu_2^2)}}{d\mu_1 - c^2\mu_2^2},
\]
whereas it smoothly goes to zero as \(r_c \to 0\) for \(\frac{\mu_1}{\mu_2^2} > 2c^2\). In the first case, there exists a unique black hole associated with each temperature and this branch dominates the thermodynamics (free energy is always negative). In the second case, for a fixed chemical potential, there exists a nucleation temperature
\[
\frac{1}{T_n} = \beta_n = \frac{2\pi^2}{d(d - 2)(\mu_1 - 2c^2\mu_2^2)}, \quad (5.29)
\]
at which two black holes with the same horizon radii are formed with \(r_n = \sqrt{\frac{(d-2)(\mu_1 - 2c^2\mu_2^2)}}{d\mu_1 - c^2\mu_2^2}\). As we increase the temperature, one of them becomes smaller (small black hole) and the other one becomes larger (big black hole). For temperatures greater than \(T_n\), these two black holes have the horizon radius \(r_n = \lambda r_n\), where \(\lambda > 1\) for the big black hole and \(\lambda < 1\) for the small black hole. We compute free energies for these two black holes and it turns out that the free energy for small black hole is always positive and therefore this phase is unstable, whereas the big black hole phase is dominant over the background spacetime for \(\lambda > \sqrt{\frac{d}{d-2}}\) which is compatible with (5.28). The Hawking–Page transition temperature is
\[
T_{\text{HP}} = (d - 1)\sqrt{(\mu_1 - 2c^2\mu_2^2)/(2\pi^2)}.
\]
We can also find conserved charges corresponding to the chemical potentials. They are given by\(^{18}\)
\[
P_1 = \left. \frac{\delta W}{\delta \mu_1} \right|_{T, \mu_2} = -\frac{\nu^2 r_c^{d-2}}{16\pi} \left( d - 2 \right) \left( k r_c^2 + q^2 r_c^2 + 2k r_c^2 \right),
\]
\[
P_2 = \left. \frac{\delta W}{\delta \mu_2} \right|_{T, \mu_1} = \frac{\sqrt{2(d - 2)(d - 1)} q \nu}{4\pi}.
\] (5.31)

These are the physical charges for generic \((d + 1)\)-dimensional non-relativistic black objects. In particular, for five-dimensional charged non-relativistic black branes, the above results match exactly with those of [26]\(^{19}\). The charge \(P_2\) obtained for the non-relativistic case is \(2\nu\) times the relativistic charge of [41]. This is easily understood from the fact that the gauge field here is \(A_{\mu}\).

\(^{16}\)The black brane can undergo a transition to the AdS soliton geometry [26, 44].

\(^{17}\)We have considered \(d > 2\).

\(^{18}\)We rewrite \(r_n\) in terms of temperature as
\[
r_n = \frac{\sqrt{2\pi T} \sqrt{\mu_1}}{\sqrt{d \mu_1}} = \frac{\sqrt{2\pi T} \sqrt{\mu_1}}{\sqrt{(2c^2 T^2 - (d - 2)d (k \mu_1 - 2c^2 \mu_2^2))}}.
\] (5.30)

\(^{19}\)Note that the chemical potential \(\mu_2\) is \(\sqrt{d/2}\) times their \(\mu_2\).
6. Discussion

We analyzed $TsT$ transformed geometries associated with brane configurations in type II string theory as well as in M-theory. One of these configurations was studied earlier in the literature. We find that the space-like $TsT$ transformation in the rotating D3 brane case commutes with the extremal near-horizon limit. The geometry obtained by $TsT$ transforming the D1–D5-p solution has the feature that in the extremal near-horizon limit, it reduces to the undeformed extremal near-horizon geometry. The reason is that in this limit, the $\gamma$ dependence drops out completely, which essentially erases the memory of the shift transformation. In the case of the BTZ black holes, the dual field theory is non-relativistic conformal quantum mechanics.

The $TsT$ transformation of M2 brane geometry is carried out by using a set of U-duality transformations. We carry out these transformations by first reducing the system to the type II set-up and then carrying out $TsT$ and lift it back to the M-theory solution. Among various ways of generating a new M-theory solution, we find that the null $TsT$ transformation on the type IIA background obtained by reduction along the $\psi$ direction has the feature that the metric is singular if one of the compact coordinates $\alpha = 0, \pi/2, \pi$. Asymptotically this metric approaches the Schrödinger metric for generic $\alpha$ but for these special values of $\alpha = 0, \pi/2, \pi$, the space becomes AdS. However, for precisely these values of $\alpha$, the metric has curvature singularity. One way to get around this is to consider blow-ups of $\mathbb{CP}^3$ at these points so that the singularity is removed. These spaces are $\mathbb{CP}^3$ analogs of del Pezzo spaces and the M-theory solution is well behaved on such blown-up spaces.

We also analyzed thermodynamics of these geometries. We have taken a novel approach to derive the free energy of the system and our results match with known results in the literature both for fixed charge and fixed chemical potential ensemble. Using the expression for free energy thus derived, we have analyzed the phase structure of generic non-relativistic $(d+1)$-dimensional black objects (holes and branes). Since these non-relativistic systems are derived from the relativistic ones, it is natural to choose the ensemble which allows changing particle number; thus, the grand canonical ensemble is a natural choice in this case. We have shown that for a specific range of chemical potentials (5.28), there exists a Hawking–Page transition from the non-relativistic black hole to non-relativistic extremal brane geometries. This thermodynamic analysis can be applied to $(d+1)$-dimensional systems for $d > 2$ only. In particular, it cannot be used for the BTZ black hole case. The entropy and temperature computed for the BTZ background are real quantities; however, if we compute them from the action, both of them turn out to be imaginary. While it can be shown that the temperature is analytic, the entropy is not. As a result, the knowledge of real temperature and entropy is not sufficient to deduce the action and hence the free energy uniquely by inverting the thermodynamic relation. This is related to the fact that the Euclidean time circle has imaginary periodicity.

It would be interesting to understand the relation of the transformed BTZ solution to non-relativistic conformal quantum mechanics better. In particular, the extension of the quantum entropy function to these cases would be quite illuminating. The $(1+1)$-dimensional non-relativistic conformal field theory obtained from M2 branes may be relevant to the physics of the Burgers equation which is a non-relativistic Navier–Stokes equation in one spatial dimension. We hope to address these issues in future.

Acknowledgments

We would like to thank J David, R Gopakumar, P Kumar, S Paul Chowdhury, M Rangamani, S Raju, A Sen and S Vandoren for useful discussions. DPJ thanks ITF, Utrecht University, for
their warm hospitality during the course of this work. NB and SD thank ISM2011 and HRI for their hospitality during the completion of the work. The work of NB was supported by NWO Veni grant, the Netherlands.

Appendix A. T-duality transformation rules for NS–NS and RR fields

For the self-consistency of this paper, in this appendix, we jot down the T-duality transformation rules for both NS–NS and RR fields. For details, the reader can refer to the original articles [32, 45] and [46]. We however closely follow the notation of [31].

**NS–NS fields.** Let us denote the T-dualized direction by \( \psi \) and other directions are denoted by \( a, b, \ldots \). The transformation of the NS–NS fields, i.e. the metric, the NS–NS two-forms and the dilaton under T-duality is as follows:

\[
\begin{align*}
g'_{\psi \psi} &= \frac{1}{g_{\psi \psi}}, \\
g'_{a \psi} &= \frac{g_{a \psi}}{g_{\psi \psi}}, \\
g'_{ab} &= g_{ab} = \frac{g_{a \psi} g_{b \psi} + B_{a \psi} B_{b \psi}}{g_{\psi \psi}}, \\
B'_{a \psi} &= \frac{g_{a \psi}}{g_{\psi \psi}}, \\
B'_{ab} &= B_{ab} = \frac{g_{a \psi} B_{b \psi} + B_{a \psi} g_{b \psi}}{g_{\psi \psi}}, \\
\Phi'_1 &= \Phi' = \Phi - \frac{1}{2} \ln g_{\psi \psi}.
\end{align*}
\]

(A.1)

**RR fields.** For stating the transformations of a different RR field, we first define some notation. First, we decompose a \( p \)-form \( N_p \) as

\[
N_p = \overline{N}_p + N_p[\psi] \wedge dx^1 + \ldots + dx^p,
\]

where \( N_p \) does not contain any \( \psi \) component and \( N_p[\psi] \) is a \((p - 1)\)-form defined as

\[
(N_p[\psi])_{a_1 \ldots a_{p-1}} = (N_p)_{a_1 \ldots a_{p-1}}[\psi] = (N_p)_{a_1 \ldots a_{p-1}} \wedge dx^1 + \ldots + dx^{p-1}.
\]

(A.3)

Let us also define one-forms

\[
\begin{align*}
J &= \frac{g_{a \psi}}{g_{\psi \psi}} dx^a, \\
b &= B_{[\psi]} + dx_1 + 1 \sum_{i=1}^{4} \left( \frac{1}{4} - \mu_i \right) dx_i \wedge dx_1.
\end{align*}
\]

Let these definitions, the T-duality transformation rule for RR fields is

\[
C'_p = C_{p+1[i]} + \tilde{C}_{p-1} \wedge b + C_{p-1[i]} \wedge b \wedge J,
\]

(A.4)

and similar transformations for the RR field strengths, \( \mathcal{F}_p \).

Appendix B. TsT transformed metrics

We have laid down the strategy that we follow in applying the TsT transformation to the M-theory solution in subsection 4.1. Here we give intermediate steps of those set of duality transformations on rotating the M2 brane solution. The solution after step 1 is

\[
\begin{align*}
\mathcal{d}s^2_T &= r \left[ -r^2 f(r) \mathcal{d}r^2 + \mathcal{d}x_1^2 + \mathcal{d}x_2^2 + \mathcal{d}x_3^2 \right] + r^3 \left( 1 + \frac{Q^2}{r^4} \right) \mathcal{d}x_4^2 + \mathcal{d}\psi^2, \\
B^T &= -(r^3 + 2A_t Q) \mathcal{d}x_1 \wedge \mathcal{d}x_2 + \mathcal{d}A_t \wedge \mathcal{d}x_1 + \mathcal{d}A_t \wedge \mathcal{d}\psi + \sum_{i=1}^{4} \left( \frac{1}{4} - \mu_i \right) \mathcal{d}x_i \wedge \mathcal{d}\psi, \\
\Phi^T &= \Phi - \frac{1}{2} \ln 4r = \ln \frac{r}{2}, \\
C^T_0 &= C^T_2 = C^T_4 = 0.
\end{align*}
\]

(B.1)

This is a solution in type IIB theory. The string coupling is still divergent in the large-\( r \) limit. We use the S-duality symmetry of type IIB theory to transform this solution to a dual frame. The S-duality transformation leaves the metric in Einstein frame and the self-dual four-form
field invariant and transforms the dilaton–axion as well as the NS–NS and RR two-form fields. Note that the string frame metric does transform under S-duality,

$\text{d}^2 \hat{S}_1 = e^{-\phi T_1} \text{d}^2 S_1$, \hspace{1cm} $e^{\phi a T_1} = \frac{2}{r}$, \hspace{1cm} $B^S = C^{T_1} = 0,$

$C^S_2 = -B^{T_1}$, \hspace{1cm} $C^S_4 = C^{T_1} = 0$.

In the S-dual frame, the string coupling becomes weak in the large-$r$ limit. We then apply the $T s T$ transformation on the above solution after writing it in terms of the light-cone coordinates defined after equation (4.11). Using (2.4), we obtain the result

$\text{d} \hat{S}^2 = M(A_1 \text{d} x^- + K_1 \text{d} x^+)^2 + M(A_2 \text{d} \psi + A_5 \text{d} x^- + A_6 \text{d} x^+)^2$

$- A_4 (\text{d} x^+)^2 + 2 \left( 4 \text{d} x^2_{\Sigma \Omega} + \frac{\text{d} x^2}{r^2} \right)$

$\hat{B} = -\gamma A_1 A_2 M(A_1 \text{d} x^- + K_1 \text{d} x^+)^\wedge (A_2 \text{d} \psi + A_5 \text{d} x^- + A_6 \text{d} x^+)$

$e^{2 \phi} = e^{2 \phi T_1} M, \hspace{1cm} \hat{C}_2 = C^S_2, \hspace{1cm} \hat{C}_0 = \gamma A_1, \hspace{1cm} M = (1 + \gamma^2 (1 - f))^{-1},$ \hspace{1cm} (B.2)

where the remaining RR sector fields vanish, and

$A_1 = r \sqrt{2(1 - f)}, \hspace{1cm} K_1 = -\sqrt{2r(1 + f)} / \sqrt{1 - f},$

$A_2 = \frac{1}{\sqrt{2r}}, \hspace{1cm} A_5 = -A_6 = \frac{\sqrt{2} Q}{r}, \hspace{1cm} A_4 = \frac{8 f r^2}{1 - f}.$ \hspace{1cm} (B.3)

We then follow steps 4 and 5 to obtain the $T s T$ transformed solution in type IIA theory and finally to M-theory. As mentioned in our strategy, there are two ways to arrive at the solution in the type IIA frame. The solution after step 4(a) is

$\text{d} s_{T_2}^2 = e^{\frac{1}{2} \phi} \left[ \text{d} x^2 M((1 - f - 4 f \gamma^2)(\text{d} x^+)^2 - 2(1 + f) \text{d} x^- \text{d} x^+(1 - f)(\text{d} x^-)^2)$

$+ \frac{\text{d} x^2}{r^2 f} + 4(\text{d} \psi^2 + \text{d} x^2_{\Sigma \Omega}) \right],$ \hspace{1cm} $B_{T_2}^2 = -2 Q(\text{d} x^+ \text{d} x^-) \wedge \text{d} \psi,$

$C^T_1 = C^S_2 + C_0 \wedge \hat{b} = \gamma \left[ M A_1 A_2 (A_1 \text{d} x^- + K_1 \text{d} x^+)^\wedge A_6 \hat{b} \right]$ \hspace{1cm} (B.4)

$C^T_2 = -2(r^2 + 2 A_1 Q) \text{d} x^+ \text{d} x^- \wedge \hat{b} + \hat{C}_2 \wedge b \wedge \hat{J}, \hspace{1cm} C^T_3 = 0,$

where

$\hat{b} = -A_4 (\text{d} x^+ \wedge \text{d} x^-) - \sum_{i=1}^{3} \left( \frac{1}{4} - \mu_{i+1}^2 \right) \text{d} \xi_i \wedge \text{d} \psi, \hspace{1cm} \hat{J} = \frac{A_5 \text{d} x^- + A_6 \text{d} x^+}{A_2}.$ \hspace{1cm} (B.5)

On the other hand, if we directly T-dualize the $T s T$ transformed metric, as stated in step 4(b), it gives the following geometry:

$\text{d} s_{T_2}^2 = 2(1 - f) r^2 \left( (\text{d} x^-)^2 + (\text{d} x^+)^2 - 2 \frac{1 + f}{1 - f} \text{d} x^- \text{d} x^+ \right) + 2 M^{-1} r^2 \text{d} \psi^2$

$+ 2 r^2 \gamma((1 + f) \text{d} x^- - (1 - f) \text{d} x^+) \wedge \text{d} \psi + \frac{2}{f r^2} \text{d} x^2_{\Sigma \Omega}$,

$\Phi_{T_2} = \frac{3}{2} \ln 2, \hspace{1cm} B_{T_2} = 2 Q(\text{d} x^- - \text{d} x^+) \wedge \text{d} \psi,$

$C^T_1 = \hat{C}_2 \wedge b = -A_4 (\text{d} x^+ \wedge \text{d} x^-) - \sum_{i=1}^{3} \left( \frac{1}{4} - \mu_{i+1}^2 \right) \text{d} \xi_i \wedge \gamma A_1 b,$

$C^T_2 = -2(r^2 + 2 A_1 Q) \text{d} x^+ \wedge \text{d} x^- \wedge b + \hat{C}_2 \wedge b \wedge \hat{J},$ \hspace{1cm} (B.6)
where

\[ b = -\gamma A_1 A_2^2 \mathcal{M} (A_1 \, dx^- + K_1 \, dx^+) + d\psi, \quad J = \frac{A_2 \, dx^- + A_6 \, dx^+}{A_2}. \]  

Finally, we uplift these solutions to M-theory. The corresponding M-theory geometry is given in section 4.

References

[1] Maldacena J M 1998 The large N limit of superconformal field theories and supergravity Adv. Theor. Math. Phys. 2 231 (arXiv:hep-th/9711200)

[2] Aharony O, Gubser S S, Maldacena J M, Ooguri H and Oz Y 2000 Large N field theories, string theory and gravity Phys. Rep. 323 183 (arXiv:hep-th/9905111)

[3] Kantscheider I, Skenderis K and Taylor M 2008 Precision holography for non-conformal branes J. High Energy Phys. JHEP09(2008)094 (arXiv:0807.3324 [hep-th])

[4] Son D T 2008 Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry Phys. Rev. D 78 046003 (arXiv:0804.3972 [hep-th])

[5] Balasubramanian K and McGreevy J 2008 Gravity duals for non-relativistic CFTs Phys. Rev. Lett. 101 061601 (arXiv:0804.4053 [hep-th])

[6] Goldberger W D 2009 AdS/CFT duality for non-relativistic field theory J. High Energy Phys. JHEP03(2009)069 (arXiv:0806.2867 [hep-th])

[7] Adams A, Balasubramanian K and McGreevy J 2008 Hot spacetimes for cold atoms J. High Energy Phys. JHEP11(2008)059 (arXiv:0807.1011 [hep-th])

[8] Herzog C P, Rangamani M and Ross S F 2008 Heating up Galilean holography J. High Energy Phys. JHEP11(2008)080 (arXiv:0807.1099 [hep-th])

[9] Bhattacharyya S, Minwalla S and Wadia S R 2009 The incompressible non-relativistic Navier–Stokes equation from gravity J. High Energy Phys. JHEP08(2009)059 (arXiv:0810.1545 [hep-th])

[10] Maldacena J, Martelli D and Tachikawa Y 2008 Comments on string theory backgrounds with non-relativistic conformal symmetry J. High Energy Phys. JHEP10(2008)072 (arXiv:0807.1100 [hep-th])

[11] Bobev N and Kundu A 2009 Deformations of holographic duals to non-relativistic CFTs J. High Energy Phys. JHEP07(2009)098 (arXiv:0904.2873 [hep-th])

[12] Bobev N, Kundu A and Pilch K 2009 Supersymmetric IIB solutions with Schrödinger symmetry J. High Energy Phys. JHEP07(2009)107 (arXiv:0905.0673 [hep-th])

[13] Yamada D 2009 Thermodynamics of black holes in Schrödinger space Class. Quantum Grav. 26 075006 (arXiv:0809.4928 [hep-th])

[14] Kim B S and Yamada D 2010 Properties of Schrödinger black holes from AdS space arXiv:1008.3286 [hep-th]

[15] Adams A, Brown C M, DeWolfe O and Rosen C 2009 Charged Schrödinger black holes Phys. Rev. D 80 125018 (arXiv:0907.1920 [hep-th])

[16] Rangamani M, Ross S F, Son D T and Thompson E G 2009 Conformal non-relativistic hydrodynamics from gravity J. High Energy Phys. JHEP01(2009)075 (arXiv:0811.2049 [hep-th])

[17] Singh H 2010 Galilean type IIA backgrounds and a map arXiv:1007.0866 [hep-th]

[18] Singh H 2009 Galilean anti-de Sitter spacetime in Romans theory Phys. Lett. B 682 225 (arXiv:0909.1692 [hep-th])

[19] Brattan D K 2010 Charged, conformal non-relativistic hydrodynamics J. High Energy Phys. JHEP10(2010)015 (arXiv:1003.0797 [hep-th])

[20] Panigrahi K L and Roy S 2010 Drag force in a hot non-relativistic, non-commutative Yang–Mills plasma J. High Energy Phys. JHEP04(2010)003 (arXiv:1001.2904 [hep-th])

[21] Mukhopadhyay A 2010 A covariant form of the Navier–Stokes equation for the Galilean conformal algebra J. High Energy Phys. JHEP01(2010)100 (arXiv:0908.0797 [hep-th])

[22] Duval C, Burdet G, Künze H P and Perrin M 1985 Bargmann structures and Newton–Cartan theory Phys. Rev. D 31 1841

[23] Duval C, Gibbons G W and Horvathy P 1991 Celestial mechanics, conformal structures and gravitational waves Phys. Rev. D 43 3907–22 (arXiv:hep-th/0512188)

[24] Duval C, Hassaine M and Horvathy P A 2009 The geometry of Schrödinger symmetry in gravity background/non-relativistic CFT Ann. Phys. 324 1158–67 (arXiv:0809.3128 [hep-th])

[25] Duval C and Horvathy P A 2009 Non-relativistic conformal symmetries and Newton–Cartan structures J. Phys. A: Math. Theor. 42 465206 (arXiv:0904.0531 [math-ph])
[25] Mazzucato L, Oz Y and Theisen S 2009 Non-relativistic branes J. High Energy Phys. JHEP04(2009)073 (arXiv:0810.3673 [hep-th])
[26] Imeroni E and Sinha A 2009 Non-relativistic metrics with extremal limits J. High Energy Phys. JHEP09(2009)096 (arXiv:0907.1892 [hep-th])
[27] Pal S S 2009 Non-relativistic supersymmetric Dp branes Class. Quantum Grav. 26 245014 (arXiv:0904.3620 [hep-th])
[28] Imeroni E and Sinha A 2009 Non-relativistic metrics with extremal limits J. High Energy Phys. JHEP09(2009)096 (arXiv:0907.1892 [hep-th])
[29] Colgain E O, Varela O and Yavartanoo H 2009 Non-relativistic M-theory solutions based on Kaehler–Einstein spaces J. High Energy Phys. JHEP07(2009)081 (arXiv:0906.0261 [hep-th])
[30] Ooguri H and Park C S 2010 Supersymmetric non-relativistic geometries in M-theory Nucl. Phys. B 824 136 (arXiv:0905.1954 [hep-th])
[31] Pal S S 2009 Null Melvin twist to Sakai–Sugimoto model arXiv:0808.3042 [hep-th]
[32] Colgain E O, Varela O and Yavartanoo H 2009 Non-relativistic M-theory solutions based on Kaehler–Einstein spaces J. High Energy Phys. JHEP09(2009)096 (arXiv:0907.1892 [hep-th])
[33] Imeroni E 2008 On deformed gauge theories and their string/M-theory duals J. High Energy Phys. JHEP10(2008)026 (arXiv:0808.1271 [hep-th])
[34] Buscher T H 1987 A symmetry of the string background field equations Phys. Lett. B 194 59
[35] Horowitz G T, Maldacena J M and Strominger A 1996 Phys. Lett. B 383 151 (arXiv:hep-th/9603109)
[36] David J R, Mandal G and Wadia S R 2002 Phys. Rep. 369 549 (arXiv:hep-th/0203048)
[37] Sen A 2009 Quantum entropy function from AdS(2)/CFT(1) correspondence Int. J. Mod. Phys. A 24 4225–44 (arXiv:0809.3304 [hep-th])
[38] Cvetic M and Gubser S S 1999 Phases of R-charged black holes, spinning branes and strongly coupled gauge theories J. High Energy Phys. JHEP04(1999)024 (arXiv:hep-th/9902195)
[39] Balasubramanian V and Kraus P 1999 Spacetime and the holographic renormalization group Phys. Rev. Lett. 83 3605 (arXiv:hep-th/9903190)
Balasubramanian V and Kraus P 1999 A stress tensor for anti-de Sitter gravity Commun. Math. Phys. 208 413 (arXiv:hep-th/9902121)
[40] Hawking S W and Page D N 1983 Thermodynamics of black holes in anti-de Sitter space Commun. Math. Phys. 87 577
[41] Dutta S and Gopakumar R 2006 On Euclidean and noetherian entropies in AdS space Phys. Rev. D 74 044007 (arXiv:hep-th/0604070)
[42] Gimon E G, Hashimoto A, Hubeny V E, Lumin O and Rangamani M 2003 Black strings in asymptotically plane wave geometries J. High Energy Phys. JHEP08(2003)035 (arXiv:hep-th/0306131)
[43] Chamblin A, Emparan R, Johnson C V and Myers R C 1999 Charged AdS black holes and catastrophic holography Phys. Rev. D 60 064018 (arXiv:hep-th/9902170)
[44] Cvetic M et al 1999 Embedding AdS black holes in ten and eleven dimensions Nucl. Phys. B 558 96 (arXiv:hep-th/9903214)
[45] Lu J X, Roy S and Xiao Z 2011 Phase transitions and critical behavior of black branes in canonical ensemble J. High Energy Phys. JHEP01(2011)133 (arXiv:1011.5198 [hep-th])
Lu J X, Roy S and Xiao Z 2011 Phase structure of black branes in grand canonical ensemble arXiv:1011.5198 [hep-th]
[46] Banerjee N and Dutta S 2007 Phase transition of electrically charged Ricci-flat black holes J. High Energy Phys. JHEP07(2007)047 (arXiv:0705.2682 [hep-th])
[47] Maharana J and Schwarz J H 1993 Noncompact symmetries in string theory Nucl. Phys. B 390 3 (arXiv:hep-th/9207016)
[48] Hassan S F 2000 T duality, space-time spinors and RR fields in curved backgrounds Nucl. Phys. B 568 145–61 (arXiv:hep-th/9907152)