Abstract. In this paper, we show that dyadic paraproducts $\pi_b$ with $b$ in dyadic BMO are bounded on matrix weighted $L^p(W)$ if $W$ is a matrix $A_p$ weight.

1. Introduction

Let $D$ be the collection of dyadic subintervals of $\mathbb{R}$. Given some $I \in D$, let $h_I$ be the associated Haar basis function defined by

$$h_I = |I|^{-\frac{1}{2}}(\chi_{I_+} - \chi_{I_-})$$

where $I_-$ is the left half of $I$ and $I_+$ is the right half of $I$. Given a locally integrable function $b$, the densely defined dyadic paraproduct $\pi_b$ on $L^p(\mathbb{R})$ is defined as

$$\pi_b f = \sum_{I \in D} b_I m_I fh_I$$

where $b_I := \langle b, h_I \rangle_{L^2(\mathbb{R})}$ and $m_I f$ is the average of $f$ over $I$. Note that dyadic paraproducts are often considered as dyadic “toy” models of non convolution singular integral operators. Moreover, it is classical and easy to show that any singular integral operator can be written as a sum of dyadic paraproducts and a singular integral operator that is “close to a being a convolution operator.” For these reasons, dyadic paraproducts have generated considerable interest in recent years (see [11] for a history and further discussion of dyadic paraproducts.)
Now, let dyadic BMO (for short, BMO) be the space of functions $b$ on $\mathbb{R}$ (modulo constants) that satisfies
\[
\sup_{I \in \mathcal{D}} \frac{1}{|I|} \int_I |b - m_I b|^2 \, dx < \infty. \tag{1.1}
\]
It is classical and well known that a dyadic paraproduct $\pi_b$ is bounded on $L^p(\mathbb{R})$ for any $1 < p < \infty$ if and only if $b \in \text{BMO}$.

It is often useful to have weighted norm inequalities for operators that are related to singular integral operators. Note that if $w$ is a positive a.e. function on $\mathbb{R}$ (i.e. a “weight”) and $L^p(w)$ is the $L^p$ space on $\mathbb{R}$ with respect to the measure $w(x) \, dx$, then a deep and celebrated result from the 1970’s is that a singular integral operator $T$ is bounded on weighted $L^p(w)$ if $w$ satisfies the so called “$A_p$ condition”
\[
\sup_{I \subset \mathbb{R}} \left( \frac{1}{|I|} \int_I w \, dx \right) \left( \frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty \tag{1.1}
\]
where the supremum is taken over all intervals $I$ (see [1, 4]). Note also that this condition is necessary when $T$ is the Hilbert transform [4]. Since dyadic paraproducts are “toy” models of certain singular integral operators, one might guess that dyadic paraproducts $\pi_b$ with symbols $b \in \text{BMO}$ are bounded on weighted $L^p(w)$ if $w \in A_p$, and in fact this was proven true in [5].

Now suppose that $n \in \mathbb{N}$ and suppose that $B = (B_{ij})_{i,j=1}^n$ is an $n \times n$ matrix valued function on $\mathbb{R}$ where each entry is locally integrable. One can then define the densely defined paraproduct $\pi_B$ on vector $L^p(\mathbb{R}; \mathbb{C}^n)$ by
\[
\pi_B \vec{f} = \sum_{I \in \mathcal{D}} \left( B_I m_I \vec{f} \right) h_I.
\]
where $B_I$ is the $n \times n$ matrix valued function with entries $((B_{ij})_{I})_{i,j=1}^n$ and $m_I \vec{f}$ is the $\mathbb{C}^n$ valued function with entries $(m_I \vec{f})_i = (\vec{f}_I)_i$. Such matrix symbolled dyadic paraproducts have generated considerable interest in recent years (for example, see [5, 7, 8, 10]) and their boundedness properties on unweighted $L^p(\mathbb{R}; \mathbb{C}^n)$ for $1 < p < \infty$ are still not fully understood. However, it is known that $\pi_B$ is bounded on each $L^p(\mathbb{R}; \mathbb{C}^n)$ for $1 < p < \infty$ if $B$ satisfies
\[
\sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} \| B_I^* B_J \| < \infty \quad \tag{1.2}
\]
where here (and throughout this paper) \( \| \cdot \| \) is the operator norm and \( D(I) \) is the collection of dyadic subintervals of \( I \) including \( I \) itself. Note that if \( b \) is a scalar function, then it is elementary to check that the seminorms induced by (1.1) and (1.2) are equal.

Naturally, one can also consider weighted norm inequalities on matrix weighted \( L^p \) where \( W \) is an \( n \times n \) matrix valued function on \( \mathbb{R} \) that is positive definite a.e. (i.e. a “matrix weight”) and \( L^p(W) \) is the weighted Banach space of measurable \( \mathbb{C}^n \) valued functions with norm

\[
\| \vec{f} \|_{L^p(W)} := \left( \int_{\mathbb{R}} |W(x)^\frac{1}{p} \vec{f}(x)|^p \, dx \right)^{\frac{1}{p}}
\]

(to avoid possible confusion, we will always use a capital \( W \) for a matrix weight and a lower case \( w \) for a scalar weight and we will adopt a similar convention for symbols of dyadic paraproducts.)

In fact, suppose we are given an operator \( T \) on scalar \( L^p(\mathbb{R}) \) and extend \( T \) to \( L^p(\mathbb{R}; \mathbb{C}^n) \) naturally by \( (T\vec{f})_i = T(f_i) \) where \( \vec{f} \) has components \( f_i \). Then necessary and sufficient conditions on a matrix weight \( W \) where the Hilbert transform \( H \) is bounded on \( L^p(W) \) for \( 1 < p < \infty \) were proven in \([9,14]\). Moreover, this condition was shown in \([3]\) (where these operators acted on functions defined on \( \mathbb{R}^d \) for any \( d \in \mathbb{N} \)) to be sufficient for singular integral operators to be bounded on \( L^p(W) \). Unfortunately, this condition (which will be called the matrix \( A_p \) condition) can not be written in a nice and simple form like (1.1). Moreover, since it requires some preliminary results and definitions, we will leave the definition of matrix \( A_p \) weights for the next section.

An obvious question is then whether matrix symbolled dyadic paraproducts \( \pi_B \) where \( B \) satisfies (1.2) are bounded on \( L^p(W) \) for \( 1 < p < \infty \) if \( W \) is a matrix \( A_p \) weight. The main result of this paper will be an affirmative answer to this question in the special case where \( B \) is of the form \( B = b(Id_{n \times n}) \) where \( b \in BMO^d \) and \( Id_{n \times n} \) is the \( n \times n \) identity matrix. Note that in this case, we have \( \pi_B = \pi_b \) where \( \pi_b \) is naturally extended to \( L^p(\mathbb{R}; \mathbb{C}^n) \). Thus, we will prove the following:

**Theorem 1.1.** If \( 1 < p < \infty \), \( b \in BMO^d \), and \( W \) is a matrix \( A_p \) weight, then the dyadic paraproduct \( \pi_b \) is bounded on \( L^p(W) \).

In the last section, we will briefly comment about the general situation where \( B \) is a matrix valued function satisfying (1.2). Note that throughout the paper, \( C \) will denote a constant that may change from
equation to equation, or even from line to line, and that depends only on (possibly) $n$, $p$, and the matrix weight $W$.

## 2. Reduction to Haar multipliers

Now we will describe the matrix $A_p$ condition that was briefly discussed in the introduction. We will solely base our discussion and definition of matrix $A_p$ weights on [3], since we will only need results that are contained there. It was shown in [3] that for any matrix weight $W$ with locally integrable entries and any interval $I$, there exists a positive definite matrix $V_I$ where

$$|I|^{-\frac{1}{p'}}\|\chi_I W^{\frac{1}{p'}}\vec{v}\|_{L^p} \leq |V_I \vec{v}| \leq n^{\frac{1}{2}}|I|^{-\frac{1}{p'}}\|\chi_I W^{\frac{1}{p'}}\vec{v}\|_{L^p}$$

for any $\vec{v} \in \mathbb{C}^n$, where $\| \cdot \|_{L^p}$ is the canonical $L^p(\mathbb{R}; \mathbb{C}^n)$ norm. Intuitively, $V_I$ should be thought of as the “$L^p$ average of $W^{\frac{1}{p'}}$ over $I$” and the reader should keep this in mind throughout the paper. Similarly, it was shown in [3] that for any matrix weight $W$ where $W^{-\frac{1}{p'}}$ has locally integrable entries and any interval $I$, there exists a positive definite $V'_I$ such that

$$|I|^{-\frac{1}{p'}}\|\chi_I W^{-\frac{1}{p'}}\vec{v}\|_{L^{p'}} \leq |V'_I \vec{v}| \leq n^{\frac{1}{2}}|I|^{-\frac{1}{p'}}\|\chi_I W^{-\frac{1}{p'}}\vec{v}\|_{L^{p'}}$$

where $p'$ is the conjugate exponent of $p$. Note that it is easy to see that

$$|V_I V'_I \vec{e}| \geq |\vec{e}| \quad (2.1)$$

for all intervals $I$ and $\vec{e} \in \mathbb{C}^n$. Following [3], we say that $W$ is a matrix $A_p$ weight if the product $V_I V'_I$ of the operators $V_I$ and $V'_I$ are uniformly bounded with respect to all intervals $I$ (in the operator norm). It is not difficult to show that this definition is equivalent to the definitions of matrix $A_p$ weights in [9,14]. For yet another equivalent definition of matrix $A_p$ weights, see [13].

The proof of Theorem 1.1 will be an adaption of the scalar techniques in [6] to the matrix case, and in particular, the proof of Theorem 1.1 will be reduced to analyzing the constant Haar multiplier $M_{W}^{\frac{1}{p'}}$, which is defined by

$$M_{W}^{\frac{1}{p'}} \vec{f} := \sum_{I \in \mathcal{D}} V_I \vec{f}_I h_I.$$ 

Here, $\vec{f}_I$ is the vector with components $(\vec{f}_I)_i := (f_i)_I$. 
Trivially we will have that $\pi_b$ is bounded on $L^p(W)$ if $W^{1\over p}\pi_bW^{-1\over p}$ is bounded on $L^p(\mathbb{R}; \mathbb{C}^n)$, and note that

$$W^{1\over p}\pi_bW^{-1\over p} = W^{1\over p}(M_W^{1\over p})^{-1}\left(M_W^{1\over p}\pi_bW^{-1\over p}\right).$$

We will end this section by showing that $M_W^{1\over p}\pi_bW^{-1\over p}$ is bounded on $L^p(\mathbb{R}; \mathbb{C}^n)$ and in the last section show that $W^{1\over p}(M_W^{1\over p})^{-1}$ is bounded on $L^p(\mathbb{R}; \mathbb{C}^n)$.

Also, note that in the scalar case one only needs to prove Theorem 1.1 when $p = 2$, since then an extrapolation argument (see [2]) will prove the theorem for $1 < p < \infty$. Since extrapolation is not available in the matrix weighted case, we will use a Littlewood-Paley theory argument instead. In particular, we will use the following result, which follows easily from [11], p. 14.

**Lemma 2.1.** Let $S(\vec{f})$ denote the (dyadic) square function given by

$$S(\vec{f})(x) = \left(\sum_{I \in D} \frac{|f_I|^2}{|I|} \chi_I(x)\right)^{1\over 2}.$$

Then $\vec{f} \in L^p(\mathbb{R}; \mathbb{C}^n)$ if and only if $S(\vec{f}) \in L^p(\mathbb{R})$ (and the unweighted $L^p$ norms of the two are equivalent with constant only depending on $p$ and $n$.)

Similar to [6][12], a “reverse Hölder inequality” will play a crucial role in our subsequent arguments. In particular, we will need the following result from [3]

**Lemma 2.2.** If $W$ is a matrix $A_p$ weight, then there exists $\delta > 0$ and constants $C = C_{W;\delta,q}$ such that for all intervals $I$, we have

$$\frac{1}{|I|} \int_I \|V_I W^{-\frac{1}{p}}(y)\|^q \, dy \leq C \text{ for all } q < p' + \delta$$

and

$$\frac{1}{|I|} \int_I \|W^{\frac{1}{p}}(y)V_I'\|^q \, dy \leq C \text{ for all } q < p + \delta$$

**Lemma 2.3.** If $W$ is a matrix $A_p$ weight, then $M_W^{1\over p}\pi_bW^{-1\over p}$ is bounded on $L^p(\mathbb{R}; \mathbb{C}^n)$. 
Proof. Given \( \tilde{f} \in L^p(\mathbb{R}; \mathbb{C}^n) \), let \( \tilde{\psi} = M_{W^-\tilde{f}}^p \tilde{f} \) so by definition we have that

\[
\|S(\tilde{\psi})\|_{L^p}^p = \int_{\mathbb{R}} \left( \sum_{I \in D} \frac{|b_I|^2}{|I|} |V_I m_I (W^{-\frac{1}{p}} \tilde{f})|^2 \chi_I(t) \right)^{\frac{p}{2}} \, dt.
\]

But if \( \epsilon > 0 \) is small enough, then Lemma 2.2 and Hölder’s inequality gives us that

\[
|V_I m_I (W^{-\frac{1}{p}} \tilde{f})| = \frac{1}{|I|} \left| \int_I V_I W^{-\frac{1}{p}} (y) \tilde{f}(y) \, dy \right|
\leq \left( \frac{1}{|I|} \int_I \|V_I W^{-\frac{1}{p}} (y)\|^{p' + p'\epsilon} \, dy \right)^{\frac{1}{p' + p'\epsilon}} \left( \frac{1}{|I|} \int_I |\tilde{f}(y)|^{\frac{p + p\epsilon}{1 + p\epsilon}} \, dy \right)^{\frac{1 + p\epsilon}{p + p\epsilon}}
\leq C \left( \frac{1}{|I|} \int_I |\tilde{f}(y)|^{\frac{p + p\epsilon}{1 + p\epsilon}} \, dy \right)^{\frac{1 + p\epsilon}{p + p\epsilon}}.
\]

Thus, we are reduced to estimating

\[
\int_{\mathbb{R}} \left( \sum_{I \in D} \frac{|b_I|^2}{|I|} \left( m_I \left| \tilde{f} \right|^{\frac{p + p\epsilon}{1 + p\epsilon}} \right)^{\frac{2 + 2p\epsilon}{p + p\epsilon}} \chi_I(t) \right)^{\frac{p}{2}} \, dt.
\]

However, an application of Carleson’s Lemma (Lemma 5.3 in [11]) gives us that

\[
\int_{\mathbb{R}} \left( \sum_{I \in D} \frac{|b_I|^2}{|I|} \left( m_I \left| \tilde{f} \right|^{\frac{p + p\epsilon}{1 + p\epsilon}} \right)^{\frac{2 + 2p\epsilon}{p + p\epsilon}} \chi_I(t) \right)^{\frac{p}{2}} \, dt
\leq \|b\|^p_{BMO_d} \left( \int_{\mathbb{R}} \left( \sup_{I \in D, x \in I} \left( m_I \left| \tilde{f} \right|^{\frac{p + p\epsilon}{1 + p\epsilon}} \right)^{\frac{2 + 2p\epsilon}{p + p\epsilon}} \chi_I(t) \frac{|I|}{|I|} \right) \, dx \right)^{\frac{p}{2}} \, dt.
\]

\[
\leq \|b\|^p_{BMO_d} \left( \int_{\mathbb{R}} \left( \sup_{I \in D, x \in I} \left( m_I \left| \tilde{f} \right|^{\frac{p + p\epsilon}{1 + p\epsilon}} \right)^{\frac{2 + 2p\epsilon}{p + p\epsilon}} \chi_I(t) \frac{|I|}{|I|} \right) \, dx \right)^{\frac{p}{2}} \, dt.
\]

\[
\leq \|b\|^p_{BMO_d} \left( \int_{\mathbb{R}} \left( M^d(\left| \tilde{f} \right|^{\frac{p + p\epsilon}{1 + p\epsilon}})(t) \right)^{\frac{p + p\epsilon}{p + p\epsilon}} \, dt \right)
\leq C \|b\|^p_{BMO_d} \int_{\mathbb{R}} |\tilde{f}(t)|^p \, dt
\]

where \( M^d \) is the ordinary dyadic maximal function, which is well known to be a bounded sublinear operator on \( L^p(\mathbb{R}) \) for any \( p > 1 \). \( \square \)
3. Proof of Theorem 1.1

In this section we will complete the proof of Theorem 1.1. As discussed in the previous section, this will be done by showing that \( W^{\frac{1}{p}}(M_{W}^{\frac{1}{p}})^{-1} \) is bounded on \( L^p(\mathbb{R}; \mathbb{C}^n) \) whenever \( W \) is a matrix \( A_p \) weight. We will do this by adapting the stopping time arguments in [6,12]. To that end, let \( W \) be a matrix \( A_p \) weight. For any interval \( I \in \mathcal{D} \) and some fixed \( \lambda \) (that will be specified later,) let \( \mathcal{J}(I) \) be the collection of maximal \( J \in \mathcal{D}(I) \) such that

\[
\frac{1}{|J|} \int_J \|W^{\frac{1}{p}}(x)V_I^p\|^p \, dx > \lambda \quad \text{or} \quad \frac{1}{|J|} \int_J \|W^{-\frac{1}{p}}(x)\|^p \, dx > \lambda. \tag{3.1}
\]

Also, let \( \mathcal{F}(I) \) be the collection of dyadic subintervals of \( I \) not contained in any interval \( J \in \mathcal{J}(I) \). Since \( W \) is a matrix \( A_p \) weight, note that we immediately have \( J \in \mathcal{F}(J) \) for any \( J \in \mathcal{D}(I) \).

Let \( \mathcal{J}^0(I) := \{I\} \) and inductively define \( \mathcal{J}^j(I) \) and \( \mathcal{F}^j(I) \) for \( j \geq 1 \) by \( \mathcal{J}^j(I) := \bigcup_{J \in \mathcal{J}^{j-1}(I)} J \) and \( \mathcal{F}^j(I) := \bigcup_{J \in \mathcal{J}^{j-1}(I)} \mathcal{F}(J) \) for any positive integer \( j \). Clearly the intervals in \( \mathcal{J}^j(I) \) for \( j > 0 \) are pairwise disjoint. Furthermore, since \( J \in \mathcal{F}(J) \) for any \( J \in \mathcal{D}(I) \), we have that \( \mathcal{D}(I) = \bigcup_{j=0}^{\infty} \mathcal{F}^j(I) \). We will slightly abuse notation and write \( \bigcup \mathcal{J}(I) \) for the set \( \bigcup_{J \in \mathcal{J}(I)} J \) and write \( |\bigcup \mathcal{J}(I)| \) for \( |\bigcup_{J \in \mathcal{J}(I)} J| \). The next lemma will show that \( \mathcal{J} \) is a decaying stopping time in the sense of [6].

**Lemma 3.1.** For \( \lambda > 0 \) large enough, there exists \( 0 < c < 1 \) such that

\[
|\bigcup \mathcal{J}^j(I)| \leq c^j |I| \quad \text{for every} \quad I \in \mathcal{D} \quad \text{(where} \quad c \text{is independent of} \quad I). \]

**Proof.** For \( I \in \mathcal{D} \), let \( \mathcal{G}(I) \) denote the collection of maximal \( J \in \mathcal{D}(I) \) such that the first inequality (but not necessarily the second inequality) in (3.1) holds. For fixed \( I \in \mathcal{D} \), enumerate the intervals in \( \mathcal{G}^j \) as \( \{I_j\} \).

We will first show that there exists \( 0 < c < 1 \) where \( |\bigcup \mathcal{G}^j(I)| \leq c^j |I| \) for every \( I \in \mathcal{D} \). Clearly by iteration we can assume that \( j = 1 \). Note that

\[
\|W^{\frac{1}{p}}(x)V_I^p\|^p \leq \lambda
\]
a.e. on \( G^I := I \setminus \bigcup \mathcal{G}(I) \) so that

\[
\int_{G^I} \|W^{\frac{1}{p}}(x)V_I^p\|^p \, dx \leq \lambda |G^I|.
\]

Clearly it is enough to show that there exists \( \alpha > 0 \) (independent of \( I \)) such that \( |G^I| \geq \alpha |I| \). Assume that this is false, so that there
exists $I \in D$ such that $|G_I| \leq \frac{|I|}{M}$ for some large $M$ (to be specified momentarily), which means that
\[
\int_{G_I} \|W^\frac{1}{p}(x)V'_I\|^p \, dx \leq \frac{|I|}{M}, \tag{3.2}
\]
However, using (2.1) and the definition of $V_I$, we have that
\[
\int_I \|W^\frac{1}{p}(x)V'_I\|^p \, dx \geq C|I|
\]
which combined with (3.2) gives us that
\[
\sum_j \int_{I_j} \|W^\frac{1}{p}(x)V'_I\|^p \, dx = \int_I \|W^\frac{1}{p}(x)V'_I\|^p \, dx - \int_{G_I} \|W^\frac{1}{p}(x)V'_I\|^p \, dx
\]
\[
\geq \left(C - \frac{1}{M}\right)|I| \geq \frac{C}{2}|I| \tag{3.3}
\]
as long as $M$ is set large enough. Combining (3.3) with the maximality of each $I_j$, we have that
\[
C|I| \leq \lambda \sum_j |I_j|. \tag{3.4}
\]
However, an application of Lemma 2.2 and Hölder’s inequality tells us that there exists $q > p$ such that
\[
C|I| \geq \sum_j |I_j|
\]
\[
\geq C \int_{I_j} \|W^\frac{1}{p}(x)V'_I\|^q \, dx
\]
\[
\geq \frac{C}{|I_j|^\frac{q-1}{p}} \left( \int_{I_j} \|W^\frac{1}{p}(x)V'_I\|^p \, dx \right)^{\frac{q}{q-1}}
\]
\[
\geq C\lambda^q \sum_j |I_j|.
\]
This, combined with (3.4) tells us that
\[
C \geq \lambda^{\frac{q}{p}-1}
\]
which is an obvious contradiction for large enough $\lambda$ since $C$ is independent of $\lambda$.

To avoid confusion, we will now write $G_\lambda(J)$ to indicate which $\lambda$ we are using in the definition of our stopping time. For $\lambda > 0$ large enough, we now show that there exists $C' > 1$ depending on $n$ and $p$
where \( K \in \mathcal{G}_{\mathcal{C}'\lambda^2}(J) \) implies that \( K \subset \tilde{K} \) for some \( \tilde{K} \in \mathcal{G}_{\lambda}^2(J) \). To do this, let \( L \subset J \) be the unique dyadic interval in \( \mathcal{G}_{\lambda}(J) \) such that \( K \subset L \). Thus, we have that

\[
C'\lambda^2 \leq \frac{1}{|K|} \int_K \|W_{\tilde{\rho}}(x)V_j'\|^p \, dx \leq \|(V_L')^{-1}V_j'\|^p \left( \frac{1}{|K|} \int_K \|W_{\tilde{\rho}}(x)V_L'\|^p \, dx \right).
\]

(3.5)

However, it follows easily from (2.1) and the definition of \( V_L \) that

\[
\|(V_L')^{-1}V_j'\|^p \leq \|V_L V_j'\|^p \leq \frac{C}{|L|} \int_L \|W_{\tilde{\rho}}(x)V_j'\|^p \, dx \leq C\lambda
\]

(3.6)

since \( L \in \mathcal{G}_{\lambda}(J) \). Plugging this into (3.5) and setting \( C' = C \) where \( C \) comes from (3.6) then tells us that there exists \( \tilde{K} \in \mathcal{G}_{\lambda}(L) \subset \mathcal{G}_{\lambda}^2(J) \) where \( K \subset \tilde{K} \). Thus, we have that

\[
|\mathcal{G}_{\mathcal{C}'\lambda^2}(J)| \leq |\mathcal{G}_{\lambda}^2(J)| \leq c^2 |J|.
\]

By iterating this last inequality and letting \( \lambda \) be large enough, we can assume that \( |\mathcal{G}_{\lambda}(J)| \leq \frac{1}{4} |J| \).

To finish the proof, let \( \tilde{\mathcal{G}}_{\lambda}(I) \) denote the collection of maximal \( J \in \mathcal{D}(I) \) such that the second inequality (but not necessarily the first inequality) in (3.1) holds. By Lemma 2.2 and the same arguments as above, we can choose \( \lambda \) large enough so that \( |\tilde{\mathcal{G}}_{\lambda}(I)| \leq \frac{1}{4}, \) which means that

\[
|\mathcal{F}_\lambda(I)| \leq |\mathcal{G}_{\lambda}(I)| + |\tilde{\mathcal{G}}_{\lambda}(I)| \leq \frac{1}{2} |I|.
\]

(3.7)

The proof is now completed by iterating (3.7).

The next main result will be an \("L^p\) Cotlar-Stein lemma\) (Lemma 3.2) that is a vector version of Lemma 8 in [6]. We will need a few preliminary definitions to state this result. Let \( \mathcal{J}^j := \mathcal{J}^j([0,1]) \) and \( \mathcal{F}^j := \mathcal{F}^j([0,1]) \). Now for each \( j \in \mathbb{N} \) let \( \Delta_j \) be defined by

\[
\Delta_j \vec{f} := \sum_{I \in \mathcal{F}^j} \vec{f}_I h_I,
\]

and write \( \vec{f}_j := \Delta_j \vec{f} \).

**Lemma 3.2.** Let the \( \mathcal{F}^j \)'s be as above and write \( T_j := T \Delta_j \) for any linear operator \( T \) on \( \mathbb{C}^n \) valued functions defined on \( \mathbb{R} \). Suppose that
\[ T = \sum_{j=1}^{\infty} T_j \quad \text{and suppose that there exists } C > 0 \quad \text{and } 0 < c < 1 \quad \text{such that for every } j, k \in \mathbb{Z}, \quad \text{one has} \]

\[ \int_{\mathbb{R}} |T_j \tilde{f}|^p |T_k \tilde{f}|^p \, dx \leq C c^{j-k} \|\tilde{f}_j\|_{L^p}^p \|\tilde{f}_k\|_{L^p}^p, \]

then \( T \) is bounded on \( L^p(\mathbb{R}; \mathbb{C}^n) \).

**Proof.** It follows directly from Lemma 7 in [6] and elementary linear algebra that

\[ \sum_{j=1}^{\infty} \|\tilde{f}_j\|_{L^p}^p \leq C \|\tilde{f}\|_{L^p}^p \]

whenever \( f \in L^p(\mathbb{R}; \mathbb{C}^n) \). The proof of Lemma 3.2 is now identical to the proof of Lemma 8 in [6]. \( \square \)

**Theorem 3.3.** If \( W \) is a matrix \( A_p \) weight for \( 1 < p < \infty \) then \( W^{1/2} (M_{W}^{1/2})^{-1} \) is bounded on \( L^p(\mathbb{R}; \mathbb{C}^n) \).

**Proof.** As in [6], it is enough to prove that the operator \( T \) defined by

\[ Tf(x) := \sum_{I \in D([0,1])} W^{1/2}_I(x)V_I^{-1} \tilde{f}_I h_I(x) \]

is bounded on \( L^p(\mathbb{R}; \mathbb{C}^n) \). Note that we clearly have \( T = \sum_{j=1}^{\infty} T_j \).

For each \( I \in D \), let

\[ M_I \tilde{f} := \sum_{J \in F(I)} V_J^{-1} \tilde{f}_J h_J \]

so that

\[ T_j \tilde{f} = \sum_{I \in J^{j-1}} W^{1/2}_I M_I \tilde{f}. \]

We first claim that the operators \((V_I')^{-1} M_I \) for \( I \in J^{j-1} \) are uniformly bounded on \( L^p(\mathbb{R}; \mathbb{C}^n) \), and in particular, we claim that

\[ \|(V_I')^{-1} M_I \tilde{f}\|_{L^p}^p \leq C \lambda^p \|\tilde{f}\|_{L^p}^p \]

whenever \( I \in J^{j-1} \). It is well known (and can be easily proved by Littlewood-Paley theory) that a constant Haar multiplier \( T_a f := \sum_{I \in D} a_I f_I h_I \) is bounded on \( L^p(\mathbb{R}) \) if and only if \( a := \{a_I\}_{I \in D} \in \ell^\infty \), and in particular, there exists \( C > 0 \) such that

\[ \|T_a f\|_{L^p}^p \leq C \|a\|_{\ell^\infty}^p \|f\|_{L^p}^p. \] (3.8)
For any $I \in J^{j-1}$ and $J \in \mathcal{F}(I)$, it follows from (2.1) and the definition of $V'_j$ that for any orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{C}^n$, we have

\[
\|(V'_I)^{-1}V'_j\|_p \leq \|V_I V'_j\|_p \|(V'_I)^{-1}V'_j\|_p \\
\leq \|V_I V'_j\|_p \|V'_j\|_p \\
\leq C \sum_{i=1}^n |V'_j V_I e_i|^p \\
\leq C \sum_{i=1}^n \left( \frac{1}{|J|} \int_J |W^{-\frac{1}{p'}}(x) V_I e_i|^{p'} \, dx \right)^{\frac{p}{p'}} \\
\leq C \left( \frac{1}{|J|} \int_J \|V_I W^{-\frac{1}{p'}}(x)\|^{p'} \, dx \right)^{\frac{p}{p'}} \\
\leq C \lambda^{\frac{p}{p'}} \tag{3.9}
\]

where the last inequality follows from the fact that $J \in \mathcal{F}(I)$. It follows easily from elementary linear algebra, (3.8), and (3.9) that

\[
\|(V'_I)^{-1}M_I \tilde{f}\|_p = \|(V'_I)^{-1}M_I \tilde{f}\|_p \\
\leq C \lambda^{\frac{p}{p'}} \|\tilde{f}\|_{L^p} \\
\leq C \lambda^{\frac{p}{p'}} \|f\|_{L^p}.
\]

Now we will show that each $T_j$ is bounded. To that end, we estimate:

\[
\int_{\mathbb{R}} |T_j f|^p \, dx = \int_{\bigcup_{\xi < j} \setminus \bigcup_{\xi \leq j}} |T_j f|^p \, dx + \int_{\bigcup_{\xi \geq j} \setminus \bigcup_{\xi < j}} |T_j f|^p \, dx \\
= (A) + (B).
\]
We estimate (A) first as follows:

\[(A) = \sum_{J \in \mathcal{J}^{-1}} \int_{J \setminus \bigcup \mathcal{J}(J)} |T_j \vec{f}|^p \, dx \]

\[= \sum_{J \in \mathcal{J}^{-1}} \int_{J \setminus \bigcup \mathcal{J}(J)} |W^J \beta(x) M_J \vec{f}(x)|^p \, dx \]

\[\leq \sum_{J \in \mathcal{J}^{-1}} \int_{J \setminus \bigcup \mathcal{J}(J)} \|W^J \beta(x) V'_J\|_p |(V'_J)^{-1} M_J \vec{f}(x)|^p \, dx \]

\[\leq \lambda \sum_{J \in \mathcal{J}^{-1}} \int J |(V'_J)^{-1} M_J \vec{f}|^p \, dx \]

\[\leq C \lambda \beta^{p+1} \|f_j\|^p_{L^p}, \]

\[\leq C \lambda \beta^{p+1} \|\vec{f}\|^p_{L^p}, \]

As for (B), note that \(M_J \vec{f}\) is constant on \(I \in \mathcal{J}(J)\), and so we will refer to this constant by \(M_J \vec{f}(I)\). We then estimate (B) as follows:

\[(B) = \int_{\bigcup \mathcal{J}} |T_j \vec{f}|^p \, dx \]

\[\leq \sum_{J \in \mathcal{J}^{-1}} \sum_{I \in \mathcal{J}(J)} \int I |W^J \beta(x) M_J \vec{f}|^p \, dx \]

\[\leq \sum_{J \in \mathcal{J}^{-1}} \sum_{I \in \mathcal{J}(J)} |I|| (V'_J)^{-1} M_J \vec{f}(I) | \left( \frac{1}{|I|} \int I \|W^J \beta(x) V'_J\|^p \, dx \right) \]

\[\leq 2\lambda \sum_{J \in \mathcal{J}^{-1}} \sum_{I \in \mathcal{J}(J)} |I|| (V'_J)^{-1} M_J \vec{f}(I) | \]

\[= 2\lambda \sum_{J \in \mathcal{J}^{-1}} \sum_{I \in \mathcal{J}(J)} \int I |(V'_J)^{-1} M_J \vec{f}(x)|^p \, dx \]

\[\leq C \lambda \beta^{p+1} \|f_j\|^p_{L^p}, \]

\[\leq C \lambda \beta^{p+1} \|\vec{f}\|^p_{L^p}. \] (3.10)

To finish the proof, we claim that there exists \(0 < c < 1\) such that

\[\int_{\bigcup \mathcal{J}^{k-1}} |T_j \vec{f}|^p \, dx \leq Cc^{k-j} \|\vec{f}\|^p_{L^p}\]
whenever $k > j$. If we define $M_j \tilde{f}$ as

$$M_j \tilde{f} := \sum_{I \in J^{k-1}} M_I \tilde{f},$$

then $M_j \tilde{f}$ is constant on $J \in J^j$. Thus, we have that

$$\int_{\bigcup J^{k-1}} |T_j \tilde{f}|^p dx = \sum_{J \in J^j} \sum_{I \in J^{k-j-1}(J)} \int_I |W_{\tilde{\nu}}^{1/2}(x) M_j \tilde{f}(J)|^p dx \leq \sum_{J \in J^j} \sum_{I \in J^{k-j-1}(J)} |J| |(V'_j)^{-1} M_j \tilde{f}(J)|^p \left( \frac{1}{|J|} \int_I \|W_{\tilde{\nu}}^{1/2}(x)V'_j\|^p dx \right)$$

However,

$$|J| |(V'_j)^{-1} M_j \tilde{f}(J)|^p \leq |J| |V'_j M_j \tilde{f}(J)|^p \leq C \int_J |W_{\tilde{\nu}}^{1/2}(x) M_j \tilde{f}(J)|^p dx = \int_J |T_j \tilde{f}(x)|^p dx$$ (3.11)

On the other hand, picking some $q > p$ in Lemma 2.2 and combining Hölder’s inequality with Lemma 3.1, we have that

$$\frac{1}{|J|} \sum_{I \in J^{k-j-1}(J)} \int_I |W_{\tilde{\nu}}^{1/2}(x)V'_j|^p dx \leq \frac{1}{|J|} \int_{\bigcup J^{k-j-1}(J)} |W_{\tilde{\nu}}^{1/2}(x)V'_j|^p dx \leq \frac{1}{|J|} \left( \int_{\bigcup J^{k-j-1}(J)} |W_{\tilde{\nu}}^{1/2}(x)V'_j|^q dx \right)^{\frac{p}{q}} (c^{k-j-1}|J|)^{\frac{1-p}{q}} \leq c^{(k-j-1)(1-\frac{p}{q})} \left( \frac{1}{|J|} \int_J |W_{\tilde{\nu}}^{1/2}(x)V'_j|^q dx \right)^{\frac{p}{q}} \leq Cc^{(k-j-1)(1-\frac{p}{q})}$$ (3.12)

Combining (3.11) with (3.12), we get that

$$\int_{\bigcup J^{k-1}} |T_j \tilde{f}|^p dx \leq Cc^{(k-j-1)(1-\frac{p}{q})} \int_{\bigcup J^{j}} |T_j \tilde{f}|^p dx \leq C\lambda^{p+1} c^{(k-j-1)(1-\frac{p}{q})} \|f\|_{L^p}^p.$$ 

Note that this estimate combined with the Cauchy Schwarz inequality and Lemma 3.2 completes the proof since $T_k \tilde{f}$ is supported on $\bigcup J^{k-1}$. 
4. Open problems.

Finally in this paper we will discuss some interesting open questions. Note that Theorem 1 was proved in the scalar case in [6] using the constant Haar multiplier

\[ \tilde{M}^{\frac{1}{p}}_W f = \sum_{I \in D} (m_{Iw})^{\frac{1}{p}} f_I h_I. \]

Even though \( V_I \) is intuitively the “\( L^p \) average of \( W^{\frac{1}{p}} \) over \( I \),” it should be clear to the reader that it is crucial that we define our constant Haar multiplier in terms of \( V_I \) for our arguments to work. However, it would be very interesting to know if \( W^{\frac{1}{p}}(\tilde{M}^{\frac{1}{p}}_W)^{-1} \) is bounded on \( L^p(\mathbb{R}; \mathbb{C}^n) \) if \( W \) is a matrix \( A_p \) weight and \( 1 < p < \infty \).

Now suppose that \( B \) is an \( n \times n \) matrix valued function satisfying (1.2). By an argument that is very similar to the proof of Lemma 2.3, we would have that \( M^{\frac{1}{p}}_W \pi_B W^{-\frac{1}{p}} \) is bounded on \( L^p(\mathbb{R}; \mathbb{C}^n) \) if the matrix weight \( W \) satisfied the condition

\[ \sup_{I \subseteq \mathbb{R}} \| V_I \| \| V'_I \| < \infty. \] (3.1)

Obviously (3.1) is much stronger than the matrix \( A_p \) condition, and note that the appearance of (3.1) comes from combining the arguments in the proof of Lemma 2.3 with trivial operator norm inequalities, which essentially “throws away” the extra noncommutativity that does not appear when \( b \) is a scalar function. For this reason, we will conjecture that Lemma 2.3 still holds for \( \pi_B \) when \( B \) is a matrix valued function satisfying (1.2), which would imply that Theorem 1.1 is true in this situation.

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