Maximizing the number of $x$-colorings of 4-chromatic graphs

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Abstract

Let $\mathcal{C}_4(n)$ be the family of all connected 4-chromatic graphs of order $n$. Given an integer $x \geq 4$, we consider the problem of finding the maximum number of $x$-colorings of a graph in $\mathcal{C}_4(n)$. It was conjectured that the maximum number of $x$-colorings is equal to $(x)_4(x-1)^{n-4}$ and the extremal graphs are those which have clique number 4 and size $n + 2$.

In this article, we reduce this problem to a finite family of graphs. We show that there exist a finite family $\mathcal{F}$ of connected 4-chromatic graphs such that if the number of $x$-colorings of every graph $G$ in $\mathcal{F}$ is less than $(x)_4(x-1)^{|V(G)|-4}$ then the conjecture holds to be true.

Keywords: $x$-colouring, chromatic number, $k$-chromatic, chromatic polynomial, $k$-connected, subdivision, theta graph

1 Introduction

In recent years problems of maximizing the number of colorings over various families of graphs have received a considerable amount of attention in the literature, see, for example, [1, 5, 6, 7, 8, 10, 14, 16]. A natural graph family to look at is the family of connected graphs with fixed chromatic number and fixed order. Let $\mathcal{C}_k(n)$ be the family of all connected $k$-chromatic graphs of order $n$. What is the maximum number of $k$-colorings among all graphs in $\mathcal{C}_k(n)$? Or more generally, for an integer $x \geq k$, what is the maximum number of $x$-colorings of a graph in $\mathcal{C}_k(n)$ and what are the extremal graphs? The answer to this question depends on the chromatic number $k$. When $k \leq 3$, the answer to this question is known and when $k \geq 4$ the problem is wide open. It is well known that (see, for example, [4]) for $k = 2$ and $x \geq 2$, the maximum number of $x$-colorings of a graph in $\mathcal{C}_2(n)$ is equal to $x(x-1)^{n-1}$, and extremal graphs are trees when $x \geq 3$. For $k = 3$, Tomescu [13] settled the problem by showing the following:

Theorem 1.1. [13] If $G$ is a graph in $\mathcal{C}_3(n)$ then

$$\pi(G, x) \leq (x - 1)^n - (x - 1) \quad \text{for odd } n$$

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and

\[ \pi(G, x) \leq (x - 1)^n - (x - 1)^2 \]

for every integer \( x \geq 3 \). Furthermore, the extremal graph is the odd cycle \( C_n \) when \( n \) is odd and odd cycle with a vertex of degree 1 attached to the cycle (denoted \( C_{n-1}^1 \)) when \( n \) is even.

Let \( \mathcal{C}_k^*(n) \) be the set of all graphs in \( \mathcal{C}_k(n) \) which have clique number \( k \) and size \( \binom{k}{2} + n - k \) (see Figure 1). It is easy to see that if \( G \in \mathcal{C}_k^*(n) \) then \( \pi(G, x) = (x)_{\downarrow k} (x - 1)^{n-k} \) where \( (x)_{\downarrow k} \) is the \( k \)th falling factorial \( x(x-1)(x-2) \cdots (x-k+1) \). Tomescu \([12]\) conjectured that when \( k \geq 4 \), the maximum number of \( k \)-colorings of a graph in \( \mathcal{C}_k(n) \) is equal to \( k!(k-1)^{n-k} \) and extremal graphs belong to \( \mathcal{C}_k^*(n) \).

**Conjecture 1.2.** \([12]\) If \( G \in \mathcal{C}_k(n) \) where \( k \geq 4 \) then

\[ \pi(G, k) \leq k!(k-1)^{n-k} \]

and extremal graphs belong to \( \mathcal{C}_k^*(n) \).

The conjecture above was later extended to all \( x \)-colorings with \( x \geq 4 \).

**Conjecture 1.3.** \([4, \text{pg. 315}]\) Let \( G \) be a graph in \( \mathcal{C}_k(n) \) where \( k \geq 4 \). Then for every \( x \in \mathbb{N} \) with \( x \geq k \)

\[ \pi(G, x) \leq (x)_{\downarrow k} (x - 1)^{n-k}. \]

Moreover, the equality holds if and only if \( G \) belongs to \( \mathcal{C}_k^*(n) \).

Several authors have studied Conjecture 1.3. In \([13]\), Conjecture 1.3 was proven for \( k = 4 \) under the additional condition that graphs are planar:

**Theorem 1.4.** \([13]\) If \( G \) is a planar graph in \( \mathcal{C}_4(n) \) then

\[ \pi(G, x) \leq (x)_{\downarrow 4} (x - 1)^{n-4} \]

for every integer \( x \geq 4 \) and furthermore equality holds if and only if \( G \) belongs to \( \mathcal{C}_4^*(n) \).

Also, in \([1]\) Conjecture 1.3 was proven for every \( k \geq 4 \), provided that \( x \geq n - 2 + \left( \binom{n}{2} - \binom{k}{2} - n + k \right)^2 \), and in \([7]\) it was proven for every \( k \geq 4 \) under the additional condition that independence number of the graphs is at most 2. In this article, our main result is Theorem 3.5 which reduces this conjecture (for \( k = 4 \)) to a finite family of 4-chromatic graphs.
2 Terminology and background

Let \( V(G) \) and \( E(G) \) be the vertex set and edge set of a (finite, undirected) graph \( G \), respectively. The order of \( G \) is \( |V(G)| \) and the size of \( G \) is \( |E(G)| \). For a nonnegative integer \( x \), a (proper) \( x \)-coloring of \( G \) is a function \( f : V(G) \to \{1, \ldots, x\} \) such that \( f(u) \neq f(v) \) for every \( uv \in E(G) \). The chromatic number \( \chi(G) \) is smallest \( x \) for which \( G \) has a \( x \)-coloring and \( G \) is called \( k \)-chromatic if \( \chi(G) = k \). Let \( \pi(G, x) \) denote the chromatic polynomial of \( G \). For nonnegative integers \( x \), the polynomial \( \pi(G, x) \) counts the number of \( x \)-colorings of \( G \).

Let \( G + e \) be the graph obtained from \( G \) by adding an edge \( e \) and \( G/e \) be the graph formed from \( G \) by contracting edge \( e \). For \( e \notin E(G) \), observe that

\[
\chi(G) = \min\{\chi(G + e), \chi(G/e)\}
\]

and

\[
|\chi(G + e) - \chi(G/e)| \leq 1. \tag{1}
\]

The well known Addition-Contraction Formula says that

\[
\pi(G, x) = \pi(G + e, x) + \pi(G/e, x).
\]

A graph \( G \) is called the \( r \)-clique sum of \( G_1, G_2, \ldots, G_n \) if \( G = G_1 \cup G_2 \cup \cdots G_n \) and \( G_1 \cap G_2 \cap \cdots G_n \) induces a complete graph \( K_r \) in \( G \) (see Figure 2). In this case the Complete Cutset Theorem says that

\[
\pi(G, x) = \prod_{i=1}^{n} \pi(G_i, x) \quad \left(\begin{array}{c} x \end{array}\right)_r^{n-1}.
\]

A subset \( S \) of the vertices of a graph \( G \) is called a cutset of \( G \) if \( G - S \) has more than one component. A connected graph is called \( k \)-connected if there does not exist a set of \( k - 1 \) vertices whose removal disconnects the graph. A block of a graph \( G \) is a maximal 2-connected subgraph of \( G \). A connected graph \( G \) is called a cactus graph if every block of
Figure 2: The 2-clique sum of $C_3$, $C_4$, $K_4$.

$G$ is either an edge or a cycle. If $B_1, \ldots, B_n$ be the blocks of a connected graph $G$ then by the Complete Cutset Theorem,

$$\pi(G, x) = \frac{1}{x^{n-1}} \prod_{i=1}^{n} \pi(B_i, x) \quad (2)$$

The chromatic polynomial of a cycle graph $C_n$ is given by

$$\pi(C_n, x) = (x - 1)^n + (-1)^n(x - 1).$$

A graph $G'$ is called a subdivision of $G$ if $G'$ is obtained from $G$ by replacing edges of $G$ with paths whose endpoints are the vertices of the edges. Let $K_{p,q}$ denote the complete bipartite graph with partitions of size $p$ and $q$. The $t$-spoke wheel, denoted by $W_t$, has vertices $v_0, v_1, \ldots, v_t$ where $v_1, v_2, \ldots, v_t$ form a cycle, and $v_0$ is adjacent to all of $v_1, v_2, \ldots, v_t$. Let $V_t$ denote the graph whose vertex set is $\{u_1, u_2, \ldots, u_t, v_2, \ldots, v_{t-1}\}$ and edge set is

$$\{u_i u_{i+1}\}_{i=1}^{t-1} \cup \{v_i v_{i+1}\}_{i=2}^{t-2} \cup \{u_i v_1\}_{i=2}^{t-1} \cup \{u_1 v_2, u_t v_{t-1}, u_1 u_t\}$$

see Figure 3.

**Proposition 2.1.** If $H$ is a connected subgraph of a connected graph $G$, then for all $x \in \mathbb{N}$,

$$\pi(G, x) \leq \pi(H, x)(x - 1)^{|V(G)| - |V(H)|}.$$

**Proof.** Let $G'$ be a minimal connected spanning subgraph of $G$ which contains $H$. Then, by the Complete Cutset Theorem, $\pi(G', x) = \pi(H, x)(x - 1)^{|V(G)| - |V(H)|}$. Every $x$-coloring of $G$ is an $x$-coloring of $G'$. Hence, $\pi(G', x) \geq \pi(G, x)$. Thus the result follows. \qed

**Proposition 2.2.** \cite{7} Let $G \in \mathcal{C}_k(n)$ and $\omega(G) = k$. Then for all $x \in \mathbb{N}$ with $x \geq k$,

$$\pi(G, x) \leq (x)_k (x - 1)^{n - k}$$

with equality if and only if $G \in \mathcal{C}_k^*(n)$. 

\[4\]
3 Proof of the main result

To prove our main result, we need the following three lemmas whose proofs are provided in Section 4.

**Lemma 3.1.** Let \( x \in \mathbb{N} \) be such that \( x \geq 4 \). Suppose that for every noncomplete 3-connected 4-chromatic graph \( H \), the inequality \( \pi(H, x) < (x)_{\downarrow 4} (x - 1)^{|V(H)| - 4} \) holds. Then, for every connected 4-chromatic graph \( G \) the inequality \( \pi(G, x) \leq (x)_{\downarrow 4} (x - 1)^{|V(G)| - 4} \) holds with equality if and only if \( G \in \mathcal{C}_4(|V(G)|) \).

**Lemma 3.2.** Let \( G \) be a subdivision of \( K_{3,10} \) and \( |V(G)| = n \). Then,
\[
\pi(G, x) < (x)_{\downarrow 4} (x - 1)^{n - 4}
\]
for every real number \( x \geq 3.95 \).

**Lemma 3.3.** Let \( G \) be a cactus graph of order \( n \) which has 6 cycles. Then,
\[
\pi(G, x) < (x)_{\downarrow 4} (x - 1)^{n - 4}
\]
for every real number \( x \geq 3.998 \).

We also make use of the following result.

**Theorem 3.4.** \([11] \) For every integer \( t \geq 3 \), there is an integer \( N = f(t) \) such that every 3-connected graph with at least \( N \) vertices contains a subgraph isomorphic to a subdivision of one of \( W_t, V_t, \) and \( K_{3,t} \).

![Figure 3: The graph \( V_t \) in Theorem 3.4](image)

**Theorem 3.5.** There exists a finite family \( \mathcal{F} \) of 3-connected nonplanar 4-chromatic graphs such that if every graph \( G \) in \( \mathcal{F} \) satisfies \( \pi(G, x) < (x)_{\downarrow 4} (x - 1)^{|V(G)| - 4} \) for all \( x \in \mathbb{N} \) with \( x \geq 4 \), then Conjecture [1.3] holds to be true.
Proof. Take \( t = 12 \) in Theorem \[\text{3.4}\] and let \( N = f(12) \). Let \( F \) be the family of all 3-connected nonplanar 4-chromatic graphs of order less than \( N \). Assume that for every graph \( G \) in \( F \), the inequality \( \pi(G, x) < (x)_{4,4}(x - 1)^{|V(G)| - 4} \) holds for every integer \( x \geq 4 \). Now we shall show that Conjecture \[\text{1.3}\] holds to be true. Let \( x \in \mathbb{N} \) with \( x \geq 4 \). By Lemma \[\text{3.1}\] it suffices to show that every noncomplete 3-connected 4-chromatic graph \( H \) satisfies \( \pi(H, x) < (x)_{4,4}(x - 1)^{|V(H)| - 4} \). Let \( H \) be a 3-connected 4-chromatic graph. By Theorem \[\text{1.4}\] we may assume that \( H \) is nonplanar. If \(|V(H)| < N\) then the result holds by the assumption. So we may assume that \(|V(H)| \geq N\). By Theorem \[\text{3.4}\] \( H \) contains a subgraph isomorphic to a subdivision of \( W_{12}, V_{12} \) and \( K_{3,12} \). If \( H \) contains a subgraph isomorphic to a subdivision of \( K_{3,12} \) then the result follows by Proposition \[\text{2.1}\] and Lemma \[\text{3.2}\]. If \( H \) contains a subgraph isomorphic to a subdivision of \( W_{12} \) or \( V_{12} \) then \( H \) contains a subgraph isomorphic to cactus graph having 6 cycles. Therefore the result follows from Proposition \[\text{2.1}\] and Lemma \[\text{3.3}\].

4 Proofs of lemmas used in the proof of the main result

4.1 Reduction to 3-connected graphs

Let \( S \) be a set of vertices in a graph \( G \). An \( S \)-lobe of \( G \) is an induced subgraph of \( G \) whose vertex set consists of \( S \) and the vertices of a component of \( G - S \). A \( k \)-chromatic graph \( G \) is called \( k \)-critical if \( \chi(H) < \chi(G) \) for every proper subgraph \( H \) of \( G \).

**Proposition 4.1.** \[\text{[15] pg. 218}\] Let \( G \) be a \( k \)-critical graph with a cutset \( S = \{x, y\} \). Then

(i) \( xy \notin E(G) \), and

(ii) \( G \) has exactly two \( S \)-lobes and they can be named \( G_1, G_2 \) such that \( G_1 + xy \) is \( k \)-critical and \( G_2 / xy \) is \( k \)-critical.

**Proof of Lemma \[\text{3.1}\]** We proceed by induction on the number of edges. If \( G \in \mathcal{E}_4^*(|V(G)|) \), then the equality \( \pi(G, x) = (x)_{4,4}(x - 1)^{|V(G)| - 4} \) holds and the result is clear. The minimum number of edges of a connected 4-chromatic graph \( G \) which does not belong to \( \mathcal{E}_4^*(|V(G)|) \) is 8 and the extremal graph is the union of a \( K_4 \) and \( K_3 \) which intersect in an edge. So \( \pi(G, x) = (x)_{4,4}(x - 1)^8 = (x)_{4,4}(x - 2) \) and the strict inequality \( (x)_{4,4}(x - 2) < (x)_{4,4}(x - 1) \) holds.

Now suppose that \( G \) is a connected 4-chromatic graph with \(|E(G)| > 8 \) and \( G \notin \mathcal{E}_4^*(|V(G)|) \).

If \( G \) is not 2-connected, then \( G \) has a block \( B \) such that \(|E(B)| < |E(G)| \) and \( \chi(B) = 4 \) as \( \chi(G) = \max\{\chi(B) : B \text{ is a block of } G\} \). If \( B \cong K_4 \) then the result follows by Proposition \[\text{2.2}\]. Suppose \( B \ncong K_4 \), then \( B \notin \mathcal{E}_4^*(|V(B)|) \) as \( B \) is 2-connected and the only 2-connected graph in \( \mathcal{E}_4^*(|V(B)|) \) is the complete graph. By the induction hypothesis we have \( \pi(B, x) < (x)_{4,4}(x - 1)^{|V(B)| - 4} \). By Proposition \[\text{2.1}\] we have \( \pi(G, x) \leq \pi(B, x)(x - 1)^{|V(G)| - |V(B)|} \). Hence we get \( \pi(G, x) < (x)_{4,4}(x - 1)^{|V(G)| - 4} \).

Now we may assume that \( G \) is 2-connected. If \( G \) is not 4-critical then there is an edge \( e \in E(G) \) such that \( \chi(G - e) = 4 \). Also \( G - e \) is connected as \( G \) is 2-connected. If \( G - e \) is not 2-connected then we can repeat the same argument as in the previous case to show that \( \pi(G - e, x) < (x)_{4,4}(x - 1)^{|V(G - e)| - 4} \) with equality if and only if \( G - e \in \mathcal{E}_4^*(|V(G - e)|) \). Note
that $V(G) = V(G-e)$. If $G-e \in \mathcal{C}_4^*(|V(G)|)$ then $\chi(G/e) \geq 4$ and hence $\pi(G/e, x) > 0$. If $G-e \notin \mathcal{C}_4^*(|V(G)|)$ then $\pi(G-e, x) < (x)_{\downarrow 4}(x-1)^{|V(G)|-4}$ by the induction hypothesis. In each case we get

$$\pi(G, x) = \pi(G-e, x) - \pi(G/e, x) < (x)_{\downarrow 4}(x-1)^{|V(G)|-4}. $$

For the rest of the proof we may assume that $G$ is a 4-critical graph and $G$ is not 3-connected. Let $S = \{u, v\}$ be a cutset of $G$. By Proposition 4.1, $uv \notin E(G)$ and $G$ has exactly two $S$-lobes and they can be named as $G_1$, $G_2$ such that $G_1 + uv$ is 4-critical and $G_2/uv$ is 4-critical. So by the induction hypothesis, we have

$$\pi(G_1 + uv, x) \leq (x)_{\downarrow 4}(x-1)^{|V(G_1+uv)|-4}$$

and

$$\pi(G_2/uv, x) \leq (x)_{\downarrow 4}(x-1)^{|V(G_2/uv)|-4}. $$

By the observation in 4.1, the inequalities $3 \leq \chi(G_2 + uv) \leq 5$ and $3 \leq \chi(G_1/uv) \leq 5$ hold. If $\chi(G_2 + uv) = 3$ then by Theorem 1.1

$$\pi(G_2 + uv, x) \leq (x-1)^{|V(G_2+uv)|} - (x-1).$$

If $\chi(G_2 + uv) \geq 4$ then let $G'$ be a 4-chromatic connected spanning subgraph of $G_2 + uv$. By the induction hypothesis,

$$\pi(G', x) \leq (x)_{\downarrow 4}(x-1)^{|V(G')|-4} = (x)_{\downarrow 4}(x-1)^{|V(G_2+uv)|-4}. $$

Since $\pi(G_2 + uv, x) \leq \pi(G', x)$, we get $\pi(G_2 + uv, x) \leq (x)_{\downarrow 4}(x-1)^{|V(G_2+uv)|-4}$. Now it is easy to check that

$$(x)_{\downarrow 4}(x-1)^{|V(G_2+uv)|-4} \leq (x-1)^{|V(G_2+uv)|} - (x-1).$$

Hence, in each case we have

$$\pi(G_2 + uv, x) \leq (x-1)^{|V(G_2+uv)|} - (x-1).$$

Similarly, we also have

$$\pi(G_1/uv, x) \leq (x-1)^{|V(G_1/uv)|} - (x-1).$$

By the Complete Cutset Theorem,

$$\pi(G + uv, x) = \frac{\pi(G_1 + uv, x) \pi(G_2 + uv, x)}{x(x-1)}$$

$$\leq \frac{(x)_{\downarrow 4}(x-1)^{|V(G_1+uv)|-4} ((x-1)^{|V(G_2+uv)|} - (x-1))}{x(x-1)}$$

$$= \frac{(x)_{\downarrow 4}((x-1)^{|V(G)|-3} - (x-1)^{|V(G_1)|-4})}{x}.$$
Thus the result follows.

(see, for example, [2] for details).

we shall first analyze theta graphs and a subdivision of polynomialsof the graph and a certainsubdivision of $K_4$.

$\pi(G/uv, x) = \frac{\pi(G_1/uv, x) \pi(G_2/uv, x)}{x}$

$\leq \frac{(x-1)^{V(G_1/uv)} - (x-1)}{x} \frac{\pi(G_2/uv)|^{4}}{x}$

$= \frac{(x-1)^{V(G_1/uv)} - (x-1)}{x} \frac{\pi(G_2/uv)|^{4}}{x}$

as $\pi(G_1/uv) = |V(G_1)| - 1, |V(G_2/uv)| = |V(G_2)| - 1$. Now, let $|V(G)| = n, |V(G_1)| = n_1$ and $|V(G_2)| = n_2$. Then,

$\pi(G, x) = \pi(G + uw, x) + \pi(G/uw, x)$

$\leq \frac{(x-1)^{n-3} - (x-1)}{x} \frac{\pi(G_1)|^{n-4} - (x-1)^{n_1-4}}{x}$

$= \frac{(x-1)^{n-3} + (x-1)^{n-4} - (x-1)^{n_1-4} - (x-1)^{n_2-4}}{x}$

$= \frac{(x-1)^{n_1-4} - (x-1)^{n_2-4}}{x} - \frac{(x-1)^{n_2-4}}{x}$

$< (x-1)^{n_4 - 4}$

Thus the result follows. $\square$

4.2 Proof of Lemma 3.2

The chromatic polynomial of a subdivision of $K_{3,t}$ can be calculated using the chromatic polynomials of theta graphs and a certain subdivision of $K_4$. So, in order to prove Lemma 3.2, we shall first analyze theta graphs and a subdivision of $K_4$.

4.2.1 Theta graphs

A theta graph $\theta_{s_1, s_2, s_3}$ is formed by taking a pair of vertices $u, v$ and joining them by three internally disjoint paths of sizes $s_1, s_2, s_3$ (see Figure 1). By the Addition-Contraction Formula, it is easy to see that

$$\pi(\theta_{s_1, s_2, s_3}, x) = \frac{3 \prod_{i=1}^{3} ((x-1)^{s_i+1} + (-1)^{s_i+1}(x-1))}{(x(x-1))^2} + \frac{3 \prod_{i=1}^{3} ((x-1)^{s_i} + (-1)^{s_i}(x-1))}{x^2}$$

(see, for example, [2] for details).
Lemma 4.2. $\pi(\theta_{s_1,s_2,s_3}, x + 1)$ is equal to

$$\frac{x}{x + 1} \left( x \left( \sum_{i=1}^{3} s_i \right)^{-1} + (-1)^{s_1 + s_2} x^{s_3} + (-1)^{s_1 + s_3} x^{s_2} + (-1)^{s_2 + s_3} x^{s_1} + (-1)^{\sum_{i=1}^{3} s_i} (x - 1) \right).$$

Proof. Using the formula given in [3],

$$\pi(\theta_{s_1,s_2,s_3}, x + 1) = \prod_{i=1}^{3} \left( x^{s_i+1} + (-1)^{s_i+1} x \right) \frac{x^2(x+1)^2}{x^2(x+1)} + \prod_{i=1}^{3} \left( x^{s_i} + (-1)^s x \right) \frac{(x+1)^2}{x^2}$$

Calculations show that the latter is equal to

$$\frac{x}{(x+1)^2} \left( \prod_{i=1}^{3} \left( x^{s_i+1} + (-1)^{s_i+1} \right) + x^2 \prod_{i=1}^{3} \left( x^{s_i-1} + (-1)^{s_i} \right) \right).$$

Now we rewrite the latter as

$$\frac{x}{(x+1)^2} (x^{s_1+s_2+s_3-1}(x+1) + (-1)^{s_2+s_3} x^{s_1} (x+1) + (-1)^{s_1+s_3} x^{s_2} (x+1)$$

$$+ (-1)^{s_1+s_2} x^{s_3} (x+1) + (-1)^{s_1+s_2+s_3} x^2 - (-1)^{s_1+s_2+s_3} x)$$

which simplifies to

$$\frac{x}{x+1} \left( x^{s_1+s_2+s_3-1} + (-1)^{s_1+s_2} x^{s_3} + (-1)^{s_1+s_3} x^{s_2} + (-1)^{s_2+s_3} x^{s_1} + (-1)^{s_1+s_2+s_3} (x - 1) \right).$$

Definition 4.3. Given $a, b, c \in \mathbb{Z}^+$, we define a function $G_{a,b,c}$ by

$$G_{a,b,c}(x) = \begin{cases} 
1 + \frac{3}{x^2} + \frac{1}{x^2} & \text{if none of } a, b, c \text{ is equal to } 1 \\
1 + \frac{2}{x^2} + \frac{1}{x^2} & \text{if exactly one of } a, b, c \text{ is equal to } 1 \\
1 + \frac{1}{x^2} + \frac{1}{x^2} & \text{if exactly two of } a, b, c \text{ are equal to } 1 \\
1 + \frac{2}{x} + \frac{1}{x^2} & \text{if all of } a, b, c \text{ are equal to } 1 
\end{cases}$$
Lemma 4.4. Let \( a, b, c \in \mathbb{Z}^+ \). Then for every real number \( x \geq 1 \),

\[
\pi(\theta_{a,b,c} x + 1) \leq \frac{x^{a+b+c}}{x+1} G_{a,b,c}(x)
\]

Proof. By Lemma 4.2, \( \pi(\theta_{a,b,c} x + 1) \) is equal to

\[
\frac{x}{x+1} \left( x^{a+b+c-1} + (-1)^{a+b} x^c + (-1)^{a+c} x^b + (-1)^{b+c} x^a + (-1)^{a+b+c} (x-1) \right).
\]

So, it suffices to show that

\[
x^{a+b+c-1} + (-1)^{a+b} x^c + (-1)^{a+c} x^b + (-1)^{b+c} x^a + (-1)^{a+b+c} (x-1) \leq G_{a,b,c}(x) x^{a+b+c-1}. \tag{4}
\]

To prove the inequality in \((4)\), we consider several cases.

Case 1: \( a, b, c \geq 2 \).

By the definition of \( G_{a,b,c} \),

\[
G_{a,b,c}(x) x^{a+b+c-1} = x^{a+b+c-1} + 3x^{a+b+c-4} + x^{a+b+c-5}.
\]

Each of \((-1)^{a+b} x^c\), \((-1)^{a+c} x^b\) and \((-1)^{b+c} x^a\) is at most \( x^{a+b+c-4} \). So,

\[
(-1)^{a+b} x^c + (-1)^{a+c} x^b + (-1)^{b+c} x^a \leq 3x^{a+b+c-4}.
\]

Also, it is clear that \((-1)^{a+b+c} (x-1) \leq x^{a+b+c-5} \). Now the inequality in \((4)\) follows.

Case 2: exactly one of \( a, b \) and \( c \) is equal to 1.

Without loss, we may assume that \( a = 1 \) and \( b, c \geq 2 \). By the definition of \( G_{a,b,c} \),

\[
G_{a,b,c}(x) x^{a+b+c-1} = x^{b+c} + 2x^{b+c-3} + x^{b+c-6}.
\]

Also, the left side of \((4)\) is equal to

\[
x^{b+c} + (-1)^{1+b} x^c + (-1)^{1+c} x^b + (-1)^{b+c}.
\]

If \( b = c = 2 \) then \( G_{a,b,c}(x) x^{a+b+c-1} \) is equal to \( x^4 + 2x + x^{-2} \) and the left side of \((4)\) is equal to \( x^4 - 2x^2 + 1 \leq x^4 + 2x + x^{-2} \). And it is clear that \( x^4 - 2x^2 + 1 \leq x^4 + 2x + x^{-2} \).

If exactly one of \( b \) and \( c \) is equal to 2, say, \( b = 2 \) and \( c \geq 3 \), then \( G_{a,b,c}(x) x^{a+b+c-1} \) is equal to \( x^{c+2} + 2x^{c-1} + x^{c-4} \) and the left side of \((4)\) is equal to \( x^{c+2} - x^c + (-1)^{c+1} x^2 + (-1)^c \). Now it is easy to see that \( x^{c+2} - x^c + (-1)^{c+1} x^2 + (-1)^c \leq x^{c+2} + 2x^{c-1} + x^{c-4} \) since \( c \geq 3 \).

If \( b, c \geq 3 \) then each of \((-1)^{1+b} x^c\) and \((-1)^{1+c} x^b\) is at most \( x^{b+c-3} \). So, \((-1)^{1+b} x^c + (-1)^{1+c} x^b \leq 2x^{b+c-3} \). Also, \((-1)^{b+c} \leq x^{b+c-6} \). Therefore,

\[
x^{b+c} + (-1)^{1+b} x^c + (-1)^{1+c} x^b + (-1)^{b+c} \leq x^{b+c} + 2x^{b+c-3} + x^{b+c-6}.
\]

Case 3: exactly two of \( a, b \) and \( c \) is equal to 1.
Without loss, we may assume that \( a = b = 1 \) and \( c \geq 2 \). By the definition of \( G_{a,b,c} \),
\[
G_{a,b,c}(x) x^{a+b+c-1} = x^{1+c} + x^c + x^{c-2} + x^{c-3}.
\]
Also, the left side of (4) is equal to
\[
x^{1+c} + x^c + (-1)^{1+c}(x + 1).
\]
It is easy to see that \((-1)^{1+c}(x + 1) \leq x^{c-2} + x^{c-3}\) since \( c \geq 2 \). So,
\[
x^{1+c} + x^c + (-1)^{1+c}(x + 1) \leq x^{1+c} + x^c + x^{c-2} + x^{c-3}.
\]

Case 4: \( a = b = c = 1 \).

By the definition of \( G_{a,b,c} \),
\[
G_{a,b,c}(x) x^{a+b+c-1} = x^2 + 2x + 1.
\]
The left side of (4) is also equal to \( x^2 + 2x + 1 \). Therefore the result follows. \( \square \)

**Lemma 4.5.** Let \( a, b, c \in \mathbb{Z}^+ \) be such that at least one of \( a, b, c \) is at least 2. Then,
\[
\pi(\theta_{a,b,c}, x + 1) \leq \frac{x^{a+b+c}}{x + 1} \left( 1 + \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^4} \right)
\]
for every real number \( x \geq \sqrt{2} \).

**Proof.** It is straightforward to check that
\[
\frac{3}{x^3} + \frac{1}{x^4} \leq \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^4}
\]
and
\[
\frac{2}{x^3} + \frac{1}{x^6} \leq \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^4}
\]
for all real \( x \geq \sqrt{2} \). Thus the result follows by Lemma 4.4. \( \square \)

### 4.2.2 A subdivision of \( K_4 \)

Let \( SK_{4}^{s_1,s_2,s_3} \) denote a subdivision of \( K_4 \) such that three edges of \( K_4 \) are replaced with paths of sizes \( s_1, s_2 \) and \( s_3 \), and all the other edges of \( K_4 \) are left undivided (see Figure 5). If \( uv \) is an undivided edge of \( K_4 \), then
\[
SK_{4}^{s_1,s_2,s_3} - uv \cong \theta_{s_1+1,s_2,s_3+1}
\]
and
\[
SK_{4}^{s_1,s_2,s_3} / uv \cong \theta_{s_1,s_2+1,s_3}.
\]
Therefore,
\[
\pi(SK_{4}^{s_1,s_2,s_3}, x) = \pi(\theta_{s_1+1,s_2,s_3+1}, x) - \pi(\theta_{s_1,s_2+1,s_3}, x)
\] (5)
Lemma 4.6. Let \( s_1, s_2, s_3 \in \mathbb{Z}^+ \) and \( x \) be a real number with \( x \geq 2 \). Then,

\[
\pi(SK_4^{s_1,s_2,s_3}, x + 1) \leq \frac{x - 1}{x + 1} x^{s_1+s_2+s_3+1} \left( 1 + \frac{2}{x^2} \right).
\]

Proof. Using (5) and Lemma 4.2, calculations show that \( \pi((SK_4)^{s_1,s_2,s_3}, x + 1) \) is equal to

\[
\frac{x(x-1)}{x+1} \left( \sum_{i=1}^{3} s_i + (-1)^{s_1+s_2+1} x^{s_3} + (-1)^{s_1+s_3+1} x^{s_2} + (-1)^{s_2+s_3+1} x^{s_1} + 2(-1)^{\sum_{i=1}^{3} s_i} \right).
\]

Now, all of \( s_1 + s_2, s_1 + s_3, s_2 + s_3 \) cannot be odd at the same time. So at least one of \( s_1 + s_2, s_1 + s_3, s_2 + s_3 \) is even. So this means that at least one of the terms \( (-1)^{s_1+s_2+1} x^{s_3}, (-1)^{s_1+s_3+1} x^{s_2}, (-1)^{s_2+s_3+1} x^{s_1} \) is negative. Therefore it is easy to see that \( \sum_{i=1}^{3} s_i \) is at most \( \frac{3}{x+1} \left( 1 + \frac{2}{x^2} \right) \) for every real \( x \geq 2 \). Thus the result follows. \( \square \)

4.2.3 A subdivision of \( K_{3,t} \)

Lemma 4.7. Let \( \{a,b,c\} \) and \( \{v_1,v_2,\ldots,v_t\} \) be the bipartition of the graph \( K_{3,t} \). Let \( G \) be a subdivision of \( K_{3,t} \) such that the edge \( av_i \) (resp. \( bv_i \) and \( cv_i \)) of \( K_{3,t} \) is replaced with a path of size \( a_i \) (resp. \( b_i \) and \( c_i \)) for \( i = 1,\ldots,t \). Then \( \pi(G,x) \) is equal to

\[
\frac{\prod_{i=1}^{t} \pi(\theta_{a_i,b_i,c_i}, x)}{(x(x-1))^{t-1}} + \frac{\prod_{i=1}^{t} \pi(\theta_{a_i,b_i+1,c_i}, x)}{(x(x-1))^{t-1}} + \frac{\prod_{i=1}^{t} \pi(\theta_{a_i,b_i,c_i+1}, x)}{(x(x-1))^{t-1}} + \frac{\prod_{i=1}^{t} \pi(SK_4^{a_i,b_i,c_i}, x)}{(x(x-1)(x-2))^{t-1}} + \frac{\prod_{i=1}^{t} \pi(\theta_{a_i,b_i,c_i}, x)}{x^{t-1}}.
\]
Proof. We apply the addition contraction formula successively. Let $A = G + ab$ and $B = G/ab$. So,

$$\pi(G, x) = \pi(A, x) + \pi(B, x).$$

Let $u$ be the vertex of $B$ which is obtained by contracting $a$ and $b$. $B_1 = B + uc$ and $B_2 = B/uc$. So,

$$\pi(B, x) = \pi(B_1, x) + \pi(B_2, x).$$

Let $A_1 = A + bc$ and $A_2 = A/bc$. So,

$$\pi(A, x) = \pi(A_1, x) + \pi(A_2, x).$$

Let $A_1^1 = A_1 + ac$ and $A_2^1 = A_1/ac$. So,

$$\pi(A, x) = \pi(A_1^1, x) + \pi(A_2^1, x).$$

Hence, we obtain that

$$\pi(G, x) = \pi(A_1^1, x) + \pi(A_2^1, x) + \pi(A_2, x) + \pi(B_1, x) + \pi(B_2, x).$$

Now we use the Complete Cutset Theorem to find the chromatic polynomials of the graphs $A_1^1$, $A_2^1$, $A_2$, $B_1$, $B_2$. Observe that $A_1^1$ is the 3-clique sum of $(SK_4)^{a_1,b_1,c_1}, \ldots, (SK_4)^{a_t,b_t,c_t}$. Hence,

$$\pi(A_1^1, x) = \prod_{i=1}^t \pi(SK_4^{a_i,b_i,c_i}, x) \frac{x(x-1)(x-2)}{(x-1)^{t-1}}$$

The graph $A_2^1$ is the 2-clique sum of $\theta_{a_1,b_1+1,c_1}, \ldots, \theta_{a_t,b_t+1,c_t}$, so

$$\pi(A_2^1, x) = \prod_{i=1}^t \pi(\theta_{a_i,b_i+1,c_i}, x) \frac{x(x-1)}{(x-1)^{t-1}}$$

Similarly, $A_2$ is the 2-clique sum of $\theta_{a_1+1,b_1,c_1}, \ldots, \theta_{a_t+1,b_t,c_t}$; $B_1$ is the 2-clique sum of $\theta_{a_1,b_1,c_1+1}, \ldots, \theta_{a_t,b_t,c_t+1}$; $B_2$ is the 1-clique sum of $\theta_{a_1,b_1,c_1}, \ldots, \theta_{a_t,b_t,c_t}$. Therefore,

$$\pi(A_2, x) = \prod_{i=1}^t \pi(\theta_{a_i+1,b_i,c_i}, x) \frac{x(x-1)}{(x-1)^{t-1}}$$

$$\pi(B_1, x) = \prod_{i=1}^t \pi(\theta_{a_i,b_i,c_i+1}, x) \frac{x(x-1)}{(x-1)^{t-1}}$$

$$\pi(B_2, x) = \prod_{i=1}^t \pi(\theta_{a_i,b_i,c_i}, x) x^{t-1}$$

Thus, the result follows. \qed
Lemma 4.8. Let \( \{a, b, c\} \) and \( \{v_1, v_2, \ldots, v_t\} \) be the bipartition of the graph \( K_{3,t} \). Let \( G \) be a subdivision of \( K_{3,t} \) such that the edge \( av_i \) (resp. \( bv_i \) and \( cv_i \)) of \( K_{3,t} \) is replaced with a path of size \( a_i \) (resp. \( b_i \) and \( c_i \)) for \( i = 1, \ldots, t \). Define

\[
F(x, t) = 3 \left( 1 + \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^4} \right)^t + \frac{1}{x} \left( 1 + \frac{2}{x} + \frac{1}{x^2} \right)^t + (x - 1) \left( 1 + \frac{2}{x^2} \right)^t.
\]

Then for every real \( x \geq 2 \),

\[
\pi(G, x + 1) \leq \frac{x^{n+2t-2}}{(x + 1)^{2t-1}} F(x, t).
\]

Proof. By Lemma 4.5, each of \( \pi(\theta_{a_i+1, b_i, c_i}, x + 1), \pi(\theta_{a_i, b_i+1, c_i}, x + 1), \pi(\theta_{a_i, b_i, c_i+1}, x + 1) \) is at most

\[
x^{a_i+b_i+c_i+1} \left( 1 + \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^4} \right)
\]

for every real \( x \geq \sqrt{2} \). Also, \( 1 + \frac{2}{x} + \frac{1}{x^2} \geq 1 + \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^4} \) holds for all \( x \geq 1 \). Hence Lemmas 4.4 and 4.5 yield

\[
\pi(\theta_{a_i, b_i, c_i}, x + 1) \leq x^{a_i+b_i+c_i} \left( 1 + \frac{2}{x} + \frac{1}{x^2} \right).
\]

By Lemma 4.6 we also have

\[
\pi(SK_{4}^{a_i,b_i,c_i}, x + 1) \leq \frac{x - 1}{x + 1} x^{a_i+b_i+c_i+1} \left( 1 + \frac{2}{x^2} \right)
\]

for every real \( x \geq 2 \). Observe that

\[
n + 2t - 3 = \sum_{i=1}^{t} (a_i + b_i + c_i).
\]

Hence,

\[
x^{n+2t-3} = \prod_{i=1}^{t} x^{a_i+b_i+c_i}.
\]

Now by Lemma 4.7 for every real \( x \geq 2 \), we get

\[
\pi(G, x + 1) \leq x^{n+2t-2} \left( \frac{3}{(x + 1)^{2t-1}} \left( 1 + \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^4} \right)^t \right) + x^{n+2t-2} \left( \frac{1}{x(x + 1)^{2t-1}} \left( 1 + \frac{2}{x} + \frac{1}{x^2} \right)^t \right) + x^{n+2t-2} \left( \frac{(x - 1)}{(x + 1)^{2t-1}} \left( 1 + \frac{2}{x^2} \right)^t \right) = \frac{x^{n+2t-2}}{(x + 1)^{2t-1}} F(x, t).
\]

\[\square\]
Proof of Lemma 3.2  We shall show that
\[ \pi(G, x + 1) < (x + 1)_{14} x^{n-4} \]
holds for every real number \( x \geq 2.95 \). Take \( t = 10 \) for the rest of the proof. Recall that
\[ F(x, t) = 3 \left( 1 + \frac{1}{x} + \frac{1}{x^3} + \frac{1}{x^4} \right)^t + \frac{1}{x} \left( 1 + \frac{2}{x} + \frac{1}{x^2} \right)^t + (x - 1) \left( 1 + \frac{2}{x^2} \right)^t. \]
By Lemma 4.8, it suffices to show that
\[ \frac{x^{n+2t-2}}{(x + 1)^{2t-1}} F(x, t) < (x + 1)_{14} x^{n-4} \]
which is equivalent to showing that
\[ x^{2t+2} F(x, t) < (x + 1)^{2t-1} (x + 1)_{14}. \]
Calculations show that \( x^{2t+2} F(x, t) \) is equal to
\[ x^{-2t+1} \left( 3x(x^4 + x^3 + x + 1)^t + (x^4 + 2x^3 + x^2)^t + x(x - 1)(x^4 + 2x^2)^t \right). \]
So we shall show that
\[ q(x) := 3x(x^4 + x^3 + x + 1)^t + (x^4 + 2x^3 + x^2)^t + x(x - 1)(x^4 + 2x^2)^t \]
is less than \( r(x) := x^{2t-1}(x + 1)^{2t-1} (x + 1)_{14} \) for all \( x \geq 2.95 \). Let
\[ p(x) = r(x) - q(x). \]
Calculations show that for \( t = 10 \), the polynomial \( p(x) \) has positive leading coefficient and the largest real root of \( p(x) \) is 2.9408 . . . . Thus the result follows.

4.3 Proof of Lemma 3.3

Lemma 4.9. Let \( G \) be a cactus graph with \( t \) edges and \( p \) cycles \( C_1, \ldots, C_p \) where \( |V(C_i)| = n_i \) for \( i = 1, \ldots, p \). Then
\[ \pi(G, x) = \frac{(x - 1)^t_{x+1}}{x^{p-1}} \prod_{i=1}^{p} ((x - 1)^{n_{i-1}} + (-1)^{n_{i}}) \]
Proof. By the formula given in equation (2),
\[ \pi(G, x) = \frac{1}{x^{t_p-1}} (x(x - 1))^t \prod_{i=1}^{p} \pi(C_i, x) = \frac{(x - 1)^t}{x^{p-1}} \prod_{i=1}^{p} \pi(C_i, x). \]
Since \( \pi(C_i, x) = (x - 1)^{n_i} + (-1)^{n_i}(x - 1) \), the latter simplifies to
\[
\frac{(x - 1)^{t+p}}{x^{p-1}} \prod_{i=1}^{p} ((x - 1)^{n_i-1} + (-1)^{n_i}).
\]
Thus the result follows. \( \square \)

**Lemma 4.10.** Let \( p, N_1, \ldots, N_p \in \mathbb{Z}^+ \) be such that \( N = \sum_{i=1}^{p} N_i \) and \( N_1, \ldots, N_p \geq 3 \). Then,
\[
\prod_{i=1}^{p} (x^{N_i} + 1) \leq x^{N-3p} \left( x + \frac{1}{3x^2} \right)^{3p}
\]
for every real \( x \geq 1 \).

**Proof.**
\[
\prod_{i=1}^{p} (x^{N_i} + 1) \leq \sum_{i=0}^{p} \binom{p}{i} x^{N_i-3i} = x^{N-3p} \sum_{i=0}^{p} \binom{p}{i} x^{3p-3i} \leq x^{N-3p} \sum_{i=0}^{3p} \binom{3p}{i} x^{3p-i} \left( \frac{1}{3x^2} \right)^i = x^{N-3p} \left( x + \frac{1}{3x^2} \right)^{3p}
\]
where the last inequality holds since
\[
\binom{p}{i} \leq \frac{1}{3!} \binom{3p}{i}
\]
for all \( i = 0, \ldots, p \). \( \square \)

**Lemma 4.11.** Let \( G \) be a cactus graph of order \( n \) with \( t \) edges and \( p \) cycles \( C_1, \ldots, C_p \) where \( |V(C_i)| = n_i \) for \( i = 1, \ldots, p \). Then,
\[
\pi(G, x + 1) \leq \frac{x^{n-8p-1} (3x^3 + 1)^{3p}}{3^{3p} (x + 1)^{p-1}}
\]
for every real \( x \geq 1 \).

**Proof.** Assume that exactly \( l \) of the cycles \( C_1, \ldots, C_p \) are equal to \( C_3 \) where \( 0 \leq l \leq p \). Without loss we may assume \( n_1, \ldots, n_l = 3 \) and \( n_{l+1}, \ldots, n_p \geq 4 \). Also observe that
\[
n = t - p + 1 + \sum_{i=1}^{p} n_i
\]
holds. Now,

\[
\pi(G, x + 1) = \frac{x^{l+p}}{(x + 1)^{p-1}} \prod_{i=1}^{p} (x^{a_i - 1} + (-1)^{n_i}) 
\]

(6)

\[
\leq \frac{x^{l+p}}{(x + 1)^{p-1}} \prod_{i=1}^{l} (x^{a_i - 1} - 1) \prod_{i=l+1}^{p} (x^{a_i - 1} + (-1)^{n_i}) 
\]

(7)

\[
\leq \frac{x^{l+p}}{(x + 1)^{p-1}} \prod_{i=1}^{l} x^{a_i - 1} \prod_{i=l+1}^{p} (x^{a_i - 1} + 1) 
\]

(8)

\[
\leq \frac{x^{l+p}}{(x + 1)^{p-1}} \sum_{i=1}^{l} (n_i - 1) \left( \sum_{i=l+1}^{p} (n_i - 1) \right)^{3(p-l)} \left( x + \frac{1}{3x^2} \right)^{3(p-l)} 
\]

(9)

\[
\leq \frac{x^{l+p}}{(x + 1)^{p-1}} \sum_{i=1}^{l} (n_i - 1) \left( \sum_{i=l+1}^{p} (n_i - 1) \right)^{3p} \left( x + \frac{1}{3x^2} \right)^{3p} 
\]

(10)

\[
= \frac{x^{n-2p-1}}{(x + 1)^{p-1}} \left( x + \frac{1}{3x^2} \right)^{3p} 
\]

(11)

\[
= \frac{x^{n-8p-1} (3x^3 + 1)^{3p}}{3^p (x + 1)^{p-1}} 
\]

(12)

(13)

where (6) follows by Lemma 4.9, (7) holds as \((-1)^{n_i} = -1\) for \(i = 1, \ldots, l\); (8) follows because \(x^{a_i - 1} - 1 \leq x^{a_i - 1}\) and \(x^{a_i - 1} + (-1)^{n_i} \leq x^{a_i - 1} + 1\); (9) holds by Lemma 4.10 (note that if \(l = 0\) then \(\prod_{i=1}^{l} x^{a_i - 1} = 1\) and if \(l = p\) then \(\prod_{i=l+1}^{p} (x^{a_i - 1} + 1) = 1\); (10) holds since \(x^{3l} (x + \frac{1}{3x^2})^{-3l} \leq 1\); (11) is clear; (12) holds because \(\left( \sum_{i=1}^{p} (n_i - 1) \right) - 3p = n - t - 1 - 3p\); (13) follows by a routine simplification. Therefore we obtain the desired result.

**Proof of Lemma 3.3** We shall show that \(\pi(G, x + 1) < (x + 1)_{\mathbb{Q}} x^{n-4}\) holds for every real number \(x \geq 2.998\). Let \(p = 6\). By Lemma 1.11, it suffices to show that

\[
\frac{x^{n-8p-1} (3x^3 + 1)^{3p}}{3^p (x + 1)^{p-1}} \leq (x + 1)_{\mathbb{Q}} x^{n-4} 
\]

which is equivalent to showing that

\[
(3x^3 + 1)^{3p} \leq 3^p x^{8p-3} (x + 1)_{\mathbb{Q}} (x + 1)^{p-1}. 
\]

Let

\[
q(x) = 3^p x^{8p-3} (x + 1)_{\mathbb{Q}} (x + 1)^{p-1} - (3x^3 + 1)^{3p}. 
\]

Calculations show that the polynomial \(q(x)\) has positive leading coefficient and the largest real root of \(q(x)\) is equal to 2.99791\ldots. Hence the result follows.
5 Concluding Remarks

To prove our main result we reduced the problem to 3-connected graphs and made use of typical subgraphs of 3-connected graphs which are large enough. Existence of such typical subgraphs guarantee that the number of x-colorings cannot exceed the desired upper bound. Consequently a natural question to ask is what typical subgraphs do 4-chromatic graphs have and can we make use of such subgraphs to settle the problem? A well known result due to Dirac [3] says that every 4-chromatic graph has a subgraph that is a subdivision of $K_4$. But unfortunately existence of a subdivision of $K_4$ is not helpful. For example, consider $G = SK_3^4, 4$, which is depicted in Figure 5. Then $G$ has 12 vertices,

$$\pi(G, x) = x^{12} - 14x^{11} + 90x^{10} - 352x^9 + 935x^8 - \cdots$$

and

$$(x)_{\downarrow 4}(x - 1)^8 = x^{12} - 14x^{11} + 87x^{10} - 318x^9 + 762x^8 - \cdots.$$  

Calculations show that for every real $x > 2$,

$$\pi(G, x) \not\approx (x)_{\downarrow 4}(x - 1)^8.$$  

Also, Conjecture [13] (for $k = 4$) was proven in [13] for planar graphs. Therefore, by Lemma [3.1] it suffices to restrict our attention to 3-connected nonplanar graphs. It is known that every 3-connected nonplanar graph distinct from $K_5$ contains a subdivision of $K_{3,3}$ (see, for example, [9]). Note that if $G$ is a subdivision of $K_{3,3}$ then the inequality $\pi(G, x) < (x)_{\downarrow 4}(x - 1)^{\text{V(G)} - 4}$ does not hold for every $x \geq 4$, however we believe that it holds for $x \geq 7.405$.

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