A Classical-Logic View of a Paraconsistent Logic

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Abstract. This paper is concerned with the first-order paraconsistent logic LPQ\textsuperscript{\neg,f}. A sequent-style natural deduction proof system for this logic is presented and, for this proof system, both a model-theoretic justification and a logical justification by means of an embedding into first-order classical logic is given. For no logic that is essentially the same as LPQ\textsuperscript{\neg,f}, a natural deduction proof system is currently available in the literature. The given embedding provides both a classical-logic explanation of this logic and a logical justification of its proof system. The major properties of LPQ\textsuperscript{\neg,f} are also treated.

Keywords: paraconsistent logic, classical logic, natural deduction, logical consequence, logical equivalence, embedding

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1 Introduction

A set of formulas is contradictory if there exists a formula such that both that formula and the negation of that formula can be deduced from it. In classical logic, every formula can be deduced from every contradictory set of formulas. A paraconsistent logic is a logic in which not every formula can be deduced from every contradictory set of formulas.

In [11], Priest proposed the paraconsistent propositional logic LP (Logic of Paradox) and its first-order extension LPQ. The paraconsistent logic considered in this paper, called LPQ\textsuperscript{\neg,f}, is LPQ enriched with a falsity constant and an implication connective for which the standard deduction theorem holds. A sequent-style natural deduction proof system for LPQ\textsuperscript{\neg,f} is presented. In addition to the usual model-theoretic justification of the proof system, a logical justification by means of an embedding into first-order classical logic is given. Classical logic is used meta-logically here: the embedding provides a classical-logic explanation of LPQ\textsuperscript{\neg,f}.

LPQ\textsuperscript{\neg,f} is essentially the same as CLuNs, J\textsuperscript{\neg}_3, J\textsuperscript{\neg}_3*, and LP\textsuperscript{\neg} [10]. The proof systems for these logics available in the literature are Hilbert systems for the first two logics and a Gentzen-style sequent system for the last one. To fill the gap, a natural deduction proof system for LPQ\textsuperscript{\neg,f} is given in this paper. An important
reason to present a justification of this proof system by means of an embedding into classical logic is to draw attention to the viewpoint that, although it may be convenient to use a paraconsistent logic like $\text{LPQ}^{\supset, F}$ if contradictory formulas have to be dealt with, classical logic is the ultima ratio of formal reasoning.

The only difference between CLuNs and $\text{LPQ}^{\supset, F}$ is that the former has a bi-implication connective and the latter does not have that connective. However, the bi-implication connective of CLuNs is definable in $\text{LPQ}^{\supset, F}$. $J_3^*$ and $\text{LP}^0$ do not have the falsity constant and the implication connective of $\text{LPQ}^{\supset, F}$. Instead, each of $J_3^*$ and $\text{LP}^0$ has a connective that is foreign to classical logic. However, the constants and connectives of $\text{LPQ}^{\supset, F}$ are definable in terms of those of each of these logics and vice versa. That is why it is said that $\text{LPQ}^{\supset, F}$ is essentially the same as these logics. A plus of $\text{LPQ}^{\supset, F}$ is that it does not have a connective that is foreign to classical logic.

The major properties of $\text{LPQ}^{\supset, F}$ concerning its logical consequence relation and its logical equivalence relation are also treated in this paper. The properties in question that concern its logical consequence relation are generally considered desirable properties of a reasonable first-order paraconsistent logic. It turns out that 13 classical laws of logical equivalence that also hold for the logical equivalence relation of $\text{LPQ}^{\supset, F}$ are sufficient to distinguish $\text{LPQ}^{\supset, F}$ completely from the infinitely many three-valued first-order paraconsistent logics with the desirable properties referred to above.

The structure of this paper is as follows. First, the language of the paraconsistent logic $\text{LPQ}^{\supset, F}$ is described (Section 2). Next, a sequent-style natural deduction proof system for $\text{LPQ}^{\supset, F}$ is presented (Section 3). After that, a model-theoretic justification of this proof system is given (Section 4). Then, a justification of this proof system by means of an embedding into classical logic is given (Section 5). Following this, the major properties of $\text{LPQ}^{\supset, F}$ are treated (Sections 6). Finally, some concluding remarks are made (Section 7).

2 The Language of $\text{LPQ}^{\supset, F}$

In this section the language of the paraconsistent logic $\text{LPQ}^{\supset, F}$ is described. First, the assumptions which are made about function and predicate symbols are given and the notion of a signature is introduced. Next, the terms and formulas of $\text{LPQ}^{\supset, F}$ are defined for a fixed but arbitrary signature. Thereafter, notational conventions and abbreviations are presented and some remarks about free variables and substitution are made. In coming sections, the proof system of $\text{LPQ}^{\supset, F}$ and the interpretation of the terms and formulas of $\text{LPQ}^{\supset, F}$ are defined for a fixed but arbitrary signature.

2.1 Signatures

It is assumed that the following has been given: (a) a countably infinite set $\mathcal{V}$ of variable symbols, (b) for each $n \in \mathbb{N}$, a countably infinite set $\mathcal{F}_n$ of function symbols of arity $n$, and, (c) for each $n \in \mathbb{N}$, a countably infinite set $\mathcal{P}_n$ of predicate
symbols of arity $n$. It is also assumed that all these sets and the set $\{=\}$ are mutually disjoint. We write $SYM$ for the set $\mathcal{V} \cup \bigcup \{ \mathcal{F}_n \mid n \in \mathbb{N} \} \cup \bigcup \{ \mathcal{P}_n \mid n \in \mathbb{N} \}$.

Function symbols of arity 0 are also known as constant symbols and predicate symbols of arity 0 are also known as proposition symbols.

A signature $\Sigma$ is a subset of $SYM \setminus \mathcal{V}$. We write $F_n(\Sigma)$ and $P_n(\Sigma)$, where $\Sigma$ is a signature and $n \in \mathbb{N}$, for the sets $\Sigma \cap \mathcal{F}_n$ and $\Sigma \cap \mathcal{P}_n$, respectively.

The language of LPQ$^{\Sigma}$ will be defined for a fixed but arbitrary signature $\Sigma$. This language will be called the language of LPQ$^{\Sigma}$ over $\Sigma$ or shortly the language of LPQ$^{\Sigma}(\Sigma)$. The corresponding proof system and interpretation will be called the proof system of LPQ$^{\Sigma}(\Sigma)$ and the interpretation of LPQ$^{\Sigma}(\Sigma)$.

### 2.2 Terms and formulas

The language of LPQ$^{\Sigma}(\Sigma)$ contains terms and formulas. They are constructed according to the formation rules given below.

The set of all terms of LPQ$^{\Sigma}(\Sigma)$, written $T_{\text{LPQ}^{\Sigma}(\Sigma)}$, is inductively defined by the following formation rules:

1. if $x \in \mathcal{V}$, then $x \in T_{\text{LPQ}^{\Sigma}(\Sigma)}$;
2. if $c \in F_0(\Sigma)$, then $c \in T_{\text{LPQ}^{\Sigma}(\Sigma)}$;
3. if $f \in F_{n+1}(\Sigma)$ and $t_1, \ldots, t_{n+1} \in T_{\text{LPQ}^{\Sigma}(\Sigma)}$, then $f(t_1, \ldots, t_{n+1}) \in T_{\text{LPQ}^{\Sigma}(\Sigma)}$.

The set of all formulas of LPQ$^{\Sigma}(\Sigma)$, written $F_{\text{LPQ}^{\Sigma}(\Sigma)}$, is inductively defined by the following formation rules:

1. if $F \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$;
2. if $p \in P_0(\Sigma)$, then $p \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$;
3. if $P \in P_{n+1}(\Sigma)$ and $t_1, \ldots, t_{n+1} \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$, then $P(t_1, \ldots, t_{n+1}) \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$;
4. if $t_1, t_2 \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$, then $t_1 = t_2 \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$;
5. if $A \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$, then $\neg A \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$;
6. if $A_1, A_2 \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$, then $A_1 \land A_2, A_1 \lor A_2, A_1 \supset A_2, A_1 \equiv A_2 \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$;
7. if $x \in \mathcal{V}$ and $A \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$, then $\forall x \cdot A, \exists x \cdot A \in F_{\text{LPQ}^{\Sigma}(\Sigma)}$.

The propositional fragment of $F_{\text{LPQ}^{\Sigma}(\Sigma)}$, written $P_{\text{LPQ}^{\Sigma}(\Sigma)}$, is the subset of $F_{\text{LPQ}^{\Sigma}(\Sigma)}$ inductively defined by the formation rules 1, 2, 5, and 6.

For the connectives $\neg$, $\land$, $\lor$, and $\supset$ and the quantifiers $\forall$ and $\exists$, the classical truth-conditions and falsehood-conditions are retained. Except for implications, a formula is classified as both-true-and-false exactly when it cannot be classified as true or false by these conditions.

### 2.3 Notational conventions and abbreviations

In the sequel, some notational conventions and abbreviations will be used.

The following will sometimes be used without mentioning (with or without subscripts): $x$ as a syntactic variable ranging over all variable symbols from $\mathcal{V}$;
as a syntactic variable ranging over all terms from \( T_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \), \( \mathcal{F} \) as a syntactic variable ranging over all formulas from \( T_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \), and \( \Gamma \) as a syntactic variable ranging over all finite sets of formulas from \( T_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \).

The string representation of terms and formulas suggested by the formation rules given above can lead to syntactic ambiguities. Parentheses are used to avoid such ambiguities. The need to use parentheses is reduced by ranking the precedence of the logical connectives \( \neg, \land, \lor, \supset \). The enumeration presents this order from the highest precedence to the lowest precedence. Moreover, the scope of the quantifiers extends as far as possible to the right and \( \forall x_1 \cdots \forall x_n \cdot A \) is usually written as \( \forall x_1, \ldots, x_n \cdot A \).

Non-equality, truth, and bi-implication are defined as abbreviations: \( t_1 \neq t_2 \) stands for \( \neg(t_1 = t_2) \), \( \top \) stands for \( \neg \bot \), \( A_1 \equiv A_2 \) stands for \( (A_1 \supset A_2) \land (A_2 \supset A_1) \).

## 2.4 Free variables and substitution

Free variables of a term or formula and substitution for variables in a term or formula are defined in the usual way. We write \( \text{free}(e) \), where \( e \) is a term from \( T_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \) or a formula from \( F_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \), for the set of free variables of \( e \). We write \( \text{free}(\Gamma) \), where \( \Gamma \) is a finite set of formulas from \( F_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \), for \( \bigcup \{ \text{free}(A) \mid A \in \Gamma \} \).

Let \( x \) be a variable symbol from \( V \), \( t \) be a term from \( T_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \), and \( e \) be a term from \( T_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \) or a formula from \( F_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \). Then \( [x := t]e \) is the result of replacing the free occurrences of the variable symbol \( x \) in \( e \) by the term \( t \), avoiding—by means of renaming of bound variables—free variables becoming bound in \( t \).

## 3 Proof System of \( \mathcal{LPQ}^{\supset \mathcal{F}}(\Sigma) \)

The proof system of \( \mathcal{LPQ}^{\supset \mathcal{F}}(\Sigma) \) is formulated as a sequent-style natural deduction proof system. This means that the inference rules have sequents as premises and conclusions. First, the notion of a sequent is introduced. Next, the inference rules of the proof system of \( \mathcal{LPQ}^{\supset \mathcal{F}}(\Sigma) \) are presented. Then, the notion of a derivation of a sequent from a set of sequents and the notion of a proof of a sequent are introduced. An extension of the proof system of \( \mathcal{LPQ}^{\supset \mathcal{F}}(\Sigma) \) which can serve as a proof system for first-order classical logic is also described.

### 3.1 Sequents

In \( \mathcal{LPQ}^{\supset \mathcal{F}}(\Sigma) \), a sequent is an expression of the form \( \Gamma \vdash A \), where \( \Gamma \) is a finite set of formulas from \( T_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \) and \( A \) is a formula from \( F_{\mathcal{LPQ} \supset \mathcal{F}}(\Sigma) \). We write \( \vdash A \) instead of \( \emptyset \vdash A \). Moreover, we write \( \Gamma, \Gamma' \) for \( \Gamma \cup \Gamma' \) and \( A \) for \( \{ A \} \) on the left-hand side of a sequent.

The intended meaning of the sequent \( \Gamma \vdash A \) is that the formula \( A \) is a logical consequence of the formulas \( \Gamma \). There are several sensible notions of logical consequence in the case where formulas can be classified as both-true-and-false.
The notion underlying LPQ^{>\cdot} is precisely defined in Section 4. It corresponds to the intuitive idea that one can draw conclusions that are not false from premises that are not false. Sequents are proved by (natural deduction) proofs obtained by using the rules of inference given below.

### 3.2 Rules of inference

The sequent-style natural deduction proof system of LPQ^{>\cdot}(\Sigma) consists of the inference rules given in Table 1. In this table, \(x\) is a syntactic variable ranging over all variable symbols from \(V\), \(t_1\), \(t_2\), and \(t\) are syntactic variables ranging over all terms from \(\mathcal{T}\), and \(A\) is a syntactic variable ranging over all formulas from \(\mathcal{F}\).
over all terms from $\mathcal{T}_{\text{LPQ}}(\Sigma)$, and $A_1$, $A_2$, $A_3$, and $A$ are syntactic variables ranging over all formulas from $\mathcal{F}_{\text{LPQ}}(\Sigma)$. Double lines indicate a two-way inference rule.

### 3.3 Derivations and proofs

In LPQ($\Sigma$), a **derivation** of a sequent $\Gamma \vdash A$ from a finite set of sequents $\mathcal{H}$ is a finite sequence $\langle s_1, \ldots, s_n \rangle$ of sequents such that $s_n$ equals $\Gamma \vdash A$ and, for each $i \in \{1, \ldots, n\}$, one of the following conditions holds:

- $s_i \in \mathcal{H}$;
- $s_i$ is the conclusion of an instance of some inference rule from the proof system of LPQ($\Sigma$) whose premises are among $s_1, \ldots, s_{i-1}$.

A **proof** of a sequent $\Gamma \vdash A$ is a derivation of $\Gamma \vdash A$ from the empty set of sequents. A sequent $\Gamma \vdash A$ is said to be **provable** if there exists a proof of $\Gamma \vdash A$.

An inference rule that does not belong to the inference rules of some proof system is called a **derived inference rule** if there exists a derivation of the conclusion from the premises, using the inference rules of that proof system, for each instance of the rule.

The difference between CLuNs and LPQ$^{\supseteq}$ is that bi-implication is a logical connective in CLuNs and must be defined as an abbreviation in LPQ$^{\supseteq}$. In [8], a proof system of CLuNs is presented which is formulated as a Hilbert system. Removing the axiom schemas $A \equiv 1$, $A \equiv 2$, and $A \equiv 3$ from this proof system and taking formulas of the form $A_1 \equiv A_2$ in this proof system as abbreviations yields a proof system of LPQ$^{\supseteq}$ formulated as a Hilbert system. Henceforth, this proof system will be referred to as the H proof system of LPQ$^{\supseteq}$ and the proof system presented in Section 3.2 will be referred to as the ND proof system of LPQ$^{\supseteq}$.

### 3.4 A proof system of CL($\Sigma$)

The name CL is used to denote a version of classical logic that has the same logical constants, connectives, and quantifiers as LPQ$^{\supseteq}$. In CL, the same assumptions about symbols are made as in LPQ$^{\supseteq}$ and the notion of a signature is defined as in LPQ$^{\supseteq}$. The languages of CL($\Sigma$) and LPQ$^{\supseteq}$($\Sigma$) are the same. A natural deduction proof system of CL($\Sigma$) can be obtained by adding the following inference rule to the ND proof system of LPQ$^{\supseteq}$($\Sigma$):

\[
\begin{array}{c}
\Gamma \vdash A_1 \\
\Gamma \vdash \neg A_1 \\
\hline
\Gamma \vdash A_2
\end{array}
\]

This proof system is known to be sound and complete.\footnote{If we replace the inference rule EM by the inference rule C in the ND proof system of LPQ$^{\supseteq}$($\Sigma$), then we obtain a sound and complete proof system of the paraconsistent analogue of LPQ$^{\supseteq}$. The propositional part of that logic ($\mathcal{L}^{\supseteq}$) is studied in [9].} There exist better known
alternatives to it, but this proof system is arguably the most appropriate one in this paper.

In Section 5, the sequents of $\text{LPQ}^{\geq F}(\Sigma)$ will be translated to sequents of $\text{CL}(\Sigma')$ ($\Sigma'$ is a particular signature related to $\Sigma$). The translation concerned has the property that what can be derived remains the same after translation. This implies that the inference rules of the proof system of $\text{LPQ}^{\geq F}(\Sigma)$ become derived inference rules of the above-mentioned proof system of $\text{CL}(\Sigma')$ after translation. Thus, the translation provides a logical justification for the inference rules of $\text{LPQ}^{\geq F}(\Sigma)$. A model-theoretic justification is afforded by the interpretation given in Section 4.

4 Interpretation of Terms and Formulas of $\text{LPQ}^{\geq F}(\Sigma)$

The proof system of $\text{LPQ}^{\geq F}$ is based on the interpretation of the terms and formulas of $\text{LPQ}^{\geq F}(\Sigma)$ presented below: the inference rules preserve validity under this interpretation. The interpretation is given relative to a structure and an assignment. First, the notion of a structure and the notion of an assignment are introduced. Next, the interpretation of the terms and formulas of $\text{LPQ}^{\geq F}(\Sigma)$ is presented.

4.1 Structures

The terms from $\mathcal{T}_{\text{LPQ}^{\geq F}(\Sigma)}$ and the formulas from $\mathcal{F}_{\text{LPQ}^{\geq F}(\Sigma)}$ are interpreted in structures which consist of a non-empty domain of individuals and an interpretation of every symbol in the signature $\Sigma$ and the equality symbol. The domain of truth values consists of three values: $t$ (true), $f$ (false), and $b$ (both true and false).

A structure $A$ of $\text{LPQ}^{\geq F}(\Sigma)$ consists of:
- a set $\mathcal{U}^A$, the domain of $A$, such that $\mathcal{U}^A \neq \emptyset$ and $\mathcal{U}^A \cap \{t, f, b\} = \emptyset$;
- for each $c \in F_0(\Sigma)$,
  an element $c^A \in \mathcal{U}^A$;
- for each $n \in \mathbb{N}$, for each $f \in F_{n+1}(\Sigma)$,
  a function $f^A : \underbrace{\mathcal{U}^A \times \cdots \times \mathcal{U}^A}_{n+1 \text{ times}} \to \mathcal{U}^A$;
- for each $p \in P_0(\Sigma)$,
  an element $p^A \in \{t, f, b\}$;
- for each $n \in \mathbb{N}$, for each $P \in P_{n+1}(\Sigma)$,
  a function $P^A : \underbrace{\mathcal{U}^A \times \cdots \times \mathcal{U}^A}_{n+1 \text{ times}} \to \{t, f, b\}$;
- a function $=^A : \mathcal{U}^A \times \mathcal{U}^A \to \{t, f, b\}$ such that, for each $d \in \mathcal{U}^A$,
  $=^A(d, d) = t$ or $=^A(d, d) = b$.

Instead of $w^A$ we write $w$ when it is clear from the context that the interpretation of symbol $w$ in structure $A$ is meant.
4.2 Assignments

An assignment in a structure $A$ of LPQ$_{\Sigma}^{\geq}(\Sigma)$ assigns elements from $U^A$ to the variable symbols from $V$. The interpretation of the terms from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$ and the formulas from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$ in $A$ is given with respect to an assignment $\alpha$ in $A$.

Let $A$ be a structure of LPQ$_{\Sigma}^{\geq}(\Sigma)$. Then an assignment in $A$ is a function $\alpha : V \rightarrow U^A$. For every assignment $\alpha$ in $A$, variable symbol $x \in V$, and element $d \in U^A$, we write $\alpha(x \rightarrow d)$ for the assignment $\alpha'$ in $A$ such that $\alpha'(x) = d$ and $\alpha'(y) = \alpha(y)$ if $y \neq x$.

4.3 Interpretation

The interpretation of the terms from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$ is given by a function mapping term $t$ to the value of $t$ in $A$ under assignment $\alpha$. Similarly, the interpretation of the formulas from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$ is given by a function mapping formula $A$, structure $A$ and assignment $\alpha$ in $A$ to the element of $\{t,f,b\}$ that is the truth value of $A$ in $A$ under assignment $\alpha$. We write $[t]_\alpha^A$ and $[A]_\alpha^A$ for these interpretations.

The interpretation functions for the terms from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$ and the formulas from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$ are inductively defined in Table 2. In this table, $x$ is a syntactic variable ranging over all variable symbols from $V$, $c$ is a syntactic variable ranging over all function symbols from $F_0(\Sigma)$, $f$ is a syntactic variable ranging over all function symbols from $F_{n+1}(\Sigma)$ (where $n$ is understood from the context), $p$ is a syntactic variable ranging over all predicate symbols from $P_{n}(\Sigma)$, $P$ is a syntactic variable ranging over all predicate symbols from $P_{n+1}(\Sigma)$ (where $n$ is understood from the context), $t_1, \ldots, t_{n+1}$ are syntactic variables ranging over all terms from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$, and $A_1, A_2, \ldots, A_k$ are syntactic variables ranging over all formulas from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$.

The logical consequence relation of LPQ$_{\Sigma}^{\geq}(\Sigma)$ is based on the idea that a formula $A$ holds in a structure $A$ under an assignment $\alpha$ in $A$ if $[A]_\alpha^A \in \{t,b\}$.

Let $\Gamma$ be a finite set of formulas from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$ and $A$ be a formula from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$. Then $A$ is a logical consequence of $\Gamma$, written $\Gamma \models A$, if for all structures $A$ of LPQ$_{\Sigma}^{\geq}(\Sigma)$, for all assignments $\alpha$ in $A$, $[A]_\alpha^A = f$ for some $A' \in \Gamma$ or $[A]_\alpha^A = b$.

As mentioned before, the difference between CLuNs and LPQ$_{\Sigma}^{\geq}$ is that bi-implication is a logical connective in CLuNs and must be defined as an abbreviation in LPQ$_{\Sigma}^{\geq}$. In [3], an interpretation of the formulas of CLuNs is presented whose restriction to formulas without occurrences of the bi-implication connective is essentially the same as the interpretation of the formulas of LPQ$_{\Sigma}^{\geq}$ given above. The soundness and completeness properties for the Hilbert proof system of CLuNs proved in [3] directly carry over to LPQ$_{\Sigma}^{\geq}$.

Theorem 1. The ND proof system of LPQ$_{\Sigma}^{\geq}(\Sigma)$ presented in Section 4.2 is sound and complete, i.e., for each finite set $\Gamma$ of formulas from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$ and each formula $A$ from $T_{LPQ_{\geq}^{\geq}(\Sigma)}$, $\Gamma \models A$ is provable in the ND proof system of LPQ$_{\Sigma}^{\geq}(\Sigma)$ iff $\Gamma \vdash A$. 

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$\vdash$ of found.
the H system, a corresponding derived inference rule of the ND
system can be
found for the H system, (b) for each inference rule of the ND
system different from I,
Because it is known from [3] that these properties hold for the H
proof of $\vdash$ A in the ND system, using that (a) the standard deduction
theorem holds for the H system of LPQ $\Sigma$

| $f(t_1, \ldots, t_{n+1})_\alpha^A$ | $f^A([t_1]_\alpha^A, \ldots, [t_{n+1}]_\alpha^A)$ |
|---------------------------------|------------------------------------------|
| $[P(t_1, \ldots, t_{n+1})]_\alpha^A$ | $p^A([t_1]_\alpha^A, \ldots, [t_{n+1}]_\alpha^A)$ |
| $[t_1 = t_2]_\alpha^A$ | $=^A ([t_1]_\alpha^A, [t_2]_\alpha^A)$ |
| $[\neg A]_\alpha^A$ | (t if $[A]_\alpha^A = f$
| | otherwise, f if $[A]_\alpha^A = t$ |
| $[A_1 \land A_2]_\alpha^A$ | (t if $[A_1]_\alpha^A = t$ and $[A_2]_\alpha^A = t$
| | otherwise, f if $[A_1]_\alpha^A = f$ or $[A_2]_\alpha^A = f$ |
| $[A_1 \lor A_2]_\alpha^A$ | (t if $[A_1]_\alpha^A = t$ or $[A_2]_\alpha^A = t$
| | otherwise, f if $[A_1]_\alpha^A = f$ and $[A_2]_\alpha^A = f$ |
| $[A_1 \supset A_2]_\alpha^A$ | (t if $[A_1]_\alpha^A = f$ or $[A_2]_\alpha^A = t$
| | otherwise, f if $[A_1]_\alpha^A \neq f$ and $[A_2]_\alpha^A = f$ |
| $[\forall x.A]_\alpha^A$ | (t if, for all $d \in U^A$, $[A]_{\alpha(x \rightarrow d)}^A = t$
| | otherwise, f if, for some $d \in U^A$, $[A]_{\alpha(x \rightarrow d)}^A = f$ |
| $[\exists x.A]_\alpha^A$ | (t if, for some $d \in U^A$, $[A]_{\alpha(x \rightarrow d)}^A = t$
| | otherwise, f if, for all $d \in U^A$, $[A]_{\alpha(x \rightarrow d)}^A = f$ |

**Proof.** Because it is known from [3] that these properties hold for the H proof system of LPQ $\Sigma$, it is sufficient to prove that, for each finite set $\Gamma$ of formulas from $\mathcal{F}_{LPQ}(\Sigma)$ and each formula $A$ from $\mathcal{F}_{LPQ}(\Sigma)$, $\vdash A$ is provable in the H system of LPQ $\Sigma$ if and only if $\vdash A$ is provable in the ND system of LPQ $\Sigma$.

The only if part is straightforwardly proved by induction on the length of the proof of $\vdash A$ in the H system, using that (a) for each axiom $A'$ of the H system, $\vdash A'$ can be proved in the ND system and (b) for each inference rule of the H system, a corresponding derived inference rule of the ND system can be found.

The if part is straightforwardly proved by induction on the length of the proof of $\vdash A$ in the ND system, using that (a) the standard deduction theorem holds for the H system, (b) for each inference rule of the ND system different from I,
\(\vdash\), \(\forall\)-I, and \(\exists\)-E, there exists a corresponding axiom of the H system, (c) for each of the inference rules \(\vdash\)-E, \(\forall\)-I, and \(\exists\)-E, a corresponding derived inference rule of the H system can be found, and (d) \(\vdash A \supset A\) can be proved in the H system.

In addition to the notion of logical consequence, the notions of logical equivalence and consistency are semantic notions that are relevant for a paraconsistent logic. The logical equivalence relation is semantically defined as it is semantically defined in classical logic. Let \(A_1\) and \(A_2\) be formulas from \(F_{\text{LPQ} \supset} \Sigma\). Then \(A_1\) is logically equivalent to \(A_2\), written \(A_1 \iff A_2\), iff for all structures \(A\) of \(\text{LPQ} \supset \Sigma\), for all assignments \(\alpha\) in \(A\), \([A_1]^A_\alpha = [A_2]^A_\alpha\). The consistency property is not semantically definable in classical logic. Let \(A_1\) and \(A_2\) be formulas from \(F_{\text{LPQ} \supset} \Sigma\). Then \(A\) is consistent iff for all structures \(A\) of \(\text{LPQ} \supset \Sigma\), for all assignments \(\alpha\) in \(A\), \([A_1]^A_\alpha \neq b\).

It should be mentioned that, unlike in classical logic, it does not hold in three-valued paraconsistent logics that logical equivalence is the same as logical consequence and its inverse.

5 Embedding of \(\text{LPQ}^{\supset \Sigma}\) into \(\text{CL}(\Sigma)\)

To give a classical-logic view of \(\text{LPQ}^{\supset \Sigma}\), the terms, formulas and sequents of \(\text{LPQ}^{\supset \Sigma}\) are translated in this section to terms, formulas and sequents, respectively, of \(\text{CL}(\Sigma)\), where \(\Sigma\) is a signature obtained from the signature \(\Sigma\) as defined below. The mappings concerned provide a uniform embedding of \(\text{LPQ}^{\supset \Sigma}\) into \(\text{CL}(\Sigma)\). What can be proved remains the same after translation. Thus, the mappings provide both a classical-logic explanation of \(\text{LPQ}^{\supset \Sigma}\) and a logical justification of its proof system.

5.1 Translation

In the translation, a canonical mapping from \(\Sigma\) to \(\text{SYM} \setminus \mathcal{V}\) is assumed. For each \(w \in \Sigma\), we write \(\overline{w}\) for the symbol to which \(w\) is mapped. The mapping concerned is further assumed to be injective and such that:

- if \(w \in F_n(\Sigma)\), then \(\overline{w} = w\);
- if \(w \in P_n(\Sigma)\), then \(\overline{w} \in F_n\).

The signature \(\overline{\Sigma}\) is defined by \(\overline{\Sigma} = \{\overline{w} \mid w \in \Sigma\}\).

Moreover, it is assumed that \(\text{true}, \text{false}, \text{both} \in F_0\), \(B, U \in P_1\), and \(\text{eq} \in P_2\). It is also assumed that \(X_1, X_2, \ldots \in \mathcal{V}\).

For the translation of terms from \(\text{LPQ}^{\supset \Sigma}\), one translation function is used:

\(\langle \_ \rangle : \text{LPQ}^{\supset \Sigma} \rightarrow \text{CL}(\overline{\Sigma})\)
and for the translation of formulas from $\mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma)$, three translation functions are used:

\[
\begin{align*}
\llbracket - \rrbracket^t &: \mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma) \to \mathcal{T}_{CL}(\Sigma), \\
\llbracket - \rrbracket^f &: \mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma) \to \mathcal{T}_{CL}(\Sigma), \\
\llbracket - \rrbracket^b &: \mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma) \to \mathcal{T}_{CL}(\Sigma).
\end{align*}
\]

For a formula $A$ from $\mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma)$, there are three translations of $A$. $\llbracket A \rrbracket^t$ is a formula of $CL(\Sigma)$ stating that the formula $A$ of $LPQ^{\rightarrow \Sigma}(\Sigma)$ is true in $LPQ^{\rightarrow \Sigma}(\Sigma)$. Likewise, $\llbracket A \rrbracket^f$ is a formula of $CL(\Sigma)$ stating that the formula $A$ of $LPQ^{\rightarrow \Sigma}(\Sigma)$ is false in $LPQ^{\rightarrow \Sigma}(\Sigma)$ and $\llbracket A \rrbracket^b$ is a formula of $CL(\Sigma)$ stating that the formula $A$ of $LPQ^{\rightarrow \Sigma}(\Sigma)$ is both true and false in $LPQ^{\rightarrow \Sigma}(\Sigma)$.

The translation functions for the terms from $\mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma)$ and the formulas from $\mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma)$ are inductively defined in Table 3. In this table, $x$ is a syntactic variable ranging over all variable symbols from $V$, $c$ is a syntactic variable ranging over all function symbols from $F_0(\Sigma)$, $f$ is a syntactic variable ranging over all function symbols from $F_n(\Sigma)$ (where $n$ is understood from the context), $p$ is a syntactic variable ranging over all predicate symbols from $P_0(\Sigma)$, $P$ is a syntactic variable ranging over all predicate symbols from $P_n(\Sigma)$ (where $n$ is understood from the context), $t_1, \ldots, t_{n+1}$ are syntactic variables ranging over all terms from $\mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma)$, and $A_1, A_2, \ldots, A_n$ are syntactic variables ranging over all formulas from $\mathcal{T}_{LPQ^{\rightarrow \Sigma}}(\Sigma)$.

The translation rules strongly resemble the interpretation rules of $LPQ^{\rightarrow \Sigma}(\Sigma)$ that are given in Section 3. In the rules for the mapping $\llbracket - \rrbracket^t$ correspond to the truth-conditions and the rules for the mapping $\llbracket - \rrbracket^b$ correspond to the falsehood-conditions.

A translation for sequents of $LPQ^{\rightarrow \Sigma}(\Sigma)$ can also be devised:

\[
\llbracket \Gamma \vdash A \rrbracket = \text{Ax}(\Sigma, \Gamma \cup \{A\}) \cup \{\llbracket A' \rrbracket^t \lor \llbracket A' \rrbracket^b \mid A' \in \Gamma \} \vdash \llbracket A \rrbracket^t \lor \llbracket A \rrbracket^b,
\]

where $\text{Ax}(\Sigma, \Gamma \cup \{A\})$ consists of the following formulas:

- true $\neq$ false $\land$ true $\neq$ both $\land$ false $\neq$ both;
- $\forall X_1 \cdot B(X_1) \equiv (X_1 = \text{true} \lor X_1 = \text{false} \lor X_1 = \text{both})$;
- $\exists X_1 \cdot U(X_1)$;
- $\forall X_1 \cdot \lnot(U(X_1) \equiv B(X_1))$;
- $U(\overline{\sigma})$ for each $c \in F_0(\Sigma)$;
- $\forall X_1, \ldots, X_{n+1} \cdot U(X_1) \land \ldots \land U(X_{n+1}) \supset U(\overline{\sigma}(X_1, \ldots, X_{n+1}))$ for each $f \in F_n(\Sigma)$, for each $n \in \mathbb{N}$;
- $B(\overline{\sigma})$ for each $p \in P_0(\Sigma)$;
- $\forall X_1, \ldots, X_{n+1} \cdot U(X_1) \land \ldots \land U(X_{n+1}) \supset B(\overline{\sigma}(X_1, \ldots, X_{n+1}))$ for each $P \in P_n(\Sigma)$, for each $n \in \mathbb{N}$;
- $\forall X_1, X_2 \cdot U(X_1) \land U(X_2) \supset B(\text{eq}(X_1, X_2))$;
- $\forall X_1 \cdot \text{eq}(X_1, X_1) = \text{true} \lor \text{eq}(X_1, X_1) = \text{both}$;
- $U(x)$ for each $x \in \text{free}(\Gamma \cup \{A\})$. 

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Ax(Σ, Γ ∪ {A}) contains formulas asserting that the domain of truth values contains exactly three elements, the domain of individuals contains at least one element and is disjoint from the domain of truth values, application of a function yields an element from the domain of individuals, application of a predicate, including the equality predicate, yields an element from the domain of truth values, application of the equality predicate does not yield false if the arguments are identical, and free variables range over the domain of individuals.

5.2 Embedding

An important property of the translation of sequents of LPQ_{f>(Σ)} to sequents of CL(Σ ∪ {true, false, both, B, U, eq}) presented above is that what can be proved
remains the same after translation. This means that the translation provides a uniform embedding of LPQ\(\vdash_f\)(\(\Sigma\)) into CL(\(\Sigma\) \(\cup\) \{true, false, both, B, U, eq\}).

**Theorem 2.** For each finite set \(\Gamma\) of formulas from \(\mathcal{F}_{LPQ\vdash_f}(\Sigma)\) and each formula \(A\) from \(\mathcal{F}_{LPQ\vdash_f}(\Sigma)\), \(\Gamma \vdash A\) is provable in LPQ\(\vdash_f\)(\(\Sigma\)) iff \(\langle \Gamma \vdash A \rangle\) is provable in CL(\(\Sigma\) \(\cup\) \{true, false, both, B, U, eq\}).

**Proof.** The only if part is easily proved by induction on the length of a proof of \(\Gamma \vdash A\) and case distinction on the last inference rule applied, using that the ND proof system for CL(\(\Sigma\) \(\cup\) \{true, false, both, B, U, eq\}) described in Section 3.4 contains all inference rules of LPQ\(\vdash_f\)(\(\Sigma\)).

The if part is proved making use of Theorem 1. Let \(A\) be a structure of LPQ\(\vdash_f\)(\(\Sigma\)). Then \(A\) can be transformed in a natural way into a structure \(A^*\) of CL(\(\Sigma\) \(\cup\) \{true, false, both, B, U, eq\}) with the following properties: \([A]^A_\alpha = t\) iff \([\langle A \rangle]^A_\alpha = t\), \([A]^A_\alpha = f\) iff \([\langle A \rangle]^A_\alpha = f\), and \([A]^A_\alpha = b\) iff \([\langle A \rangle]^A_\alpha = t\) (for all assignments \(\alpha\) in \(A\)). Now assume that \(A\) is a counter-model for \(\Gamma \vdash A\). Then, for its above-mentioned properties, \(A^*\) is a counter-model for \(\langle \Gamma \vdash A \rangle\).

From this, by Theorem 1 the if part follows immediately. \(\square\)

The translation of sequents extends to inference rules in the obvious way.

**Corollary 1.** The translation of the inference rules of the presented proof system of LPQ\(\vdash_f\)(\(\Sigma\)) are derived inference rules of the proof system of CL(\(\Sigma\) \(\cup\) \{true, false, both, B, U, eq\}) described in Section 3.4.

### 6 Major Properties of LPQ\(\vdash_f\)

In this section, the major properties of LPQ\(\vdash_f\) concerning its logical consequence relation and its logical equivalence relation are presented.

#### 6.1 The logical consequence relation of LPQ\(\vdash_f\)

Below, the properties of LPQ\(\vdash_f\) concerning its logical consequence relation are presented that are generally considered to be desirable properties of a reasonable first-order paraconsistent logic. The symbol \(\vdash_{CL}\) is used to denote the logical consequence relation of CL.

The following are properties of LPQ\(\vdash_f\) concerning its logical consequence relation:

(a) containment in classical logic: \(\vdash \subseteq \vdash_{CL}\);

(b) proper basic connectives and quantifiers: for all \(\Gamma \subseteq \mathcal{F}_{LPQ\vdash_f}(\Sigma)\), \(A_1, A_2, A_3 \in \mathcal{F}_{LPQ\vdash_f}(\Sigma)\), and \(x \in \mathcal{V}\):

   (b1) \(\Gamma, A_1 \vdash A_2\) iff \(\Gamma \vdash A_1 \supset A_2\);

   (b2) \(\Gamma \vdash A_1 \land A_2\) iff \(\Gamma \vdash A_1\) and \(\Gamma \vdash A_2\);

   (b3) \(\Gamma, A_1 \lor A_2 \vdash A_3\) iff \(\Gamma, A_1 \vdash A_3\) and \(\Gamma, A_2 \vdash A_3\);

   (b4) \(\Gamma \vdash \forall x. A\) iff \(\Gamma \vdash A\), provided \(x \notin \text{free}(\Gamma)\);

   (b5) \(\Gamma, \exists x. A \vdash A_2\) iff \(\Gamma, A_1 \vdash A_2\), provided \(x \notin \text{free}(\Gamma \cup \{A_2\})\);
(c) weak maximal paraconsistency relative to classical logic with respect to the propositional fragment: for all $A \in \mathcal{P}_{\text{LPQ}^{3,f}}(\Sigma)$ with $\not\models A$ and $\models_{\text{CL}} A$, for the minimal consequence relation $\vdash'$ with $\vdash \subseteq \vdash'$ and $\vdash A$, for all formulas $A' \in \mathcal{P}_{\text{LPQ}^{3,f}}(\Sigma), \vdash A'$ iff $\models_{\text{CL}} A'$;

(d) strong maximal absolute paraconsistency with respect to the propositional fragment: for all first-order logics $\mathcal{L}$ with the same logical constants, connectives, and quantifiers as LPQ$^{3,f}$ and a consequence relation $\vdash'$ such that 
\[
\{ \Gamma \vdash A \mid \Gamma \cup \{ A \} \subseteq \mathcal{P}_{\text{LPQ}^{3,f}}(\Sigma) \} \subset \{ \Gamma \vdash A' \mid \Gamma \cup \{ A \} \subseteq \mathcal{P}_{\text{LPQ}^{3,f}}(\Sigma) \},
\]
$\mathcal{L}$ is not paraconsistent;

(e) internalization consistency: $A$ is consistent iff $\vdash (A \supset F) \lor (\neg A \supset F)$;

(f) internalization logical equivalence: $A_1 \Leftrightarrow A_2$ iff $\vdash (A_1 \equiv A_2) \land (\neg A_1 \equiv \neg A_2)$.

With the exception of properties (b$_4$) and (b$_5$), these properties are inherited from the propositional part of LPQ$^{3,f}$ (cf. [9]). Properties (a)–(c) indicate that LPQ$^{3,f}$ retains much of first-order classical logic. Properties (a), (b$_1$), (c), and (d) make the propositional part of LPQ$^{3,f}$ an ideal paraconsistent logic according to Definition 21 in [2]. By property (e), the propositional part of LPQ$^{3,f}$ is also a logic of formal inconsistency according to Definition 23 in [6].

From Theorem 4.42 in [1], it is known that there are exactly 8192 different three-valued paraconsistent propositional logics with properties (a), (b$_1$), (b$_2$), and (b$_3$). From Corollary 4.74 in [1], it is known that the propositional part of LPQ$^{3,f}$ is the strongest three-valued paraconsistent propositional logic with property (a) in the sense that for each three-valued paraconsistent propositional logic with property (a) there exists a logical consequence preserving translation of its formulas into formulas of the propositional part of LPQ$^{3,f}$.

6.2 The logical equivalence relation of LPQ$^{3,f}$

There are infinitely many different three-valued first-order paraconsistent logics with properties (a) and (b). This means that these properties, which concern the logical consequence relation of a logic, have no discriminating power. The same holds for properties (c)–(f) because each three-valued first-order paraconsistent logics with properties (a) and (b) has these properties as well (cf. [9]).

Below, properties concerning the logical equivalence relation of a logic are used for discrimination. It turns out that 13 classical laws of logical equivalence that also hold for the logical equivalence relation of LPQ$^{3,f}$ are sufficient to distinguish LPQ$^{3,f}$ completely from all other three-valued first-order paraconsistent logics with properties (a) and (b).

The laws in question are the identity, annihilation, idempotent, and commutative laws for conjunction and disjunction, the double negation law, two laws that uniquely characterize implication, and two laws that concern universal and existential quantification.

**Theorem 3.** The logical equivalence relation of LPQ$^{3,f}$ satisfies laws (1)–(13) from Table 4.
Table 4. Distinguishing laws of logical equivalence for LPQ\(^{3,f}\)

| (1) \(A \land F \Leftrightarrow F\) | (2) \(A \lor T \Leftrightarrow T\) |
| (3) \(A \land T \Leftrightarrow A\) | (4) \(A \lor F \Leftrightarrow A\) |
| (5) \(A \land A \Leftrightarrow A\) | (6) \(A \lor A \Leftrightarrow A\) |
| (7) \(A_1 \land A_2 \Leftrightarrow A_2 \land A_1\) | (8) \(A_1 \lor A_2 \Leftrightarrow A_2 \lor A_1\) |
| (9) \(\neg \neg A \Leftrightarrow A\) | (10) \(F \supset A \Leftrightarrow T\) |
| (12) \(\forall x \bullet (A_1 \land A_2) \Leftrightarrow (\forall x \bullet A_1) \land A_2\) | (13) \(\exists x \bullet (A_1 \lor A_2) \Leftrightarrow (\exists x \bullet A_1) \lor A_2\) |

Proof. For each of the laws (1)–(13), with the exception of law (11), satisfaction follows directly from the definition of the interpretation function for formulas given in Table 2. For law (11), we first have to establish that \(\llbracket A \lor \neg A \rrbracket^A \in \{t, b\}\). □

Moreover, among the infinitely many three-valued first-order paraconsistent logics with properties (a) and (b), LPQ\(^{3,f}\) is the only one whose logical equivalence relation satisfies all laws given in Table 4.

Theorem 4. There is exactly one three-valued first-order paraconsistent logic with properties (a) and (b) of which the logical equivalence relation satisfies laws (1)–(13) from Table 4.

Proof. We know from Theorem 4.2 in [9] that for each of the logical connectives there are laws among laws (1)–(11) that exclude all but one of its possible interpretations. Moreover, given the remaining interpretations of \(\land\) and \(\lor\), it is not hard to see that laws (12) and (13) cannot hold if the interpretations \(\forall\) and \(\exists\) differ from their interpretations in LPQ\(^{3,f}\). □

It follows immediately from property (b\(_2\)) that the logical equivalence relation of every three-valued first-order paraconsistent logics with properties (a) and (b) satisfies law (1) from Table 4. It follows immediately from the proof of Theorem 4 that all proper subsets of laws (2)–(13) from Table 4 are insufficient to distinguish LPQ\(^{3,f}\) completely from the other three-valued first-order paraconsistent logics with properties (a) and (b). The next corollary also follow immediately from the proof of Theorem 4.

Corollary 2. There are exactly 16 three-valued first-order paraconsistent logics with properties (a) and (b) of which the logical equivalence relation satisfies laws (1)–(9), (12), and (13) from Table 4.

\(^2\) The paracomplete analogue of LPQ\(^{3,f}\) is the only three-valued first-order paracomplete logic with properties (a) and (b) whose logical equivalence relation satisfies the laws from Table 4, with laws (10) and (11) replaced by (10\(^\prime\)) \(T \supset A \Leftrightarrow A\), and (11\(^\prime\)) \((A_1 \land \neg A_1) \supset A_2 \Leftrightarrow T\) (cf. [9]).
It should be mentioned that the logical equivalence relation of \( \text{LPQ} \supset \text{F} \) does not only satisfy the identity, annihilation, idempotent and commutative laws for conjunction and disjunction but also other basic classical laws for conjunction and disjunction, including the absorption, associative, distributive and de Morgan’s laws (cf. [9]).

6.3 On the closeness of \( \text{LPQ} \supset \text{F} \) to CL

Below, the different properties \( \text{LPQ} \supset \text{F} \) related to closeness to CL are briefly discussed.

\( \text{LPQ} \supset \text{F} \) is a paraconsistent logic whose properties concerning its logical consequence relation include virtually all properties that have been proposed as desirable properties of such a logic. The properties concerned are related to closeness to CL.

If closeness to CL is considered important, the above-mentioned properties concerning the logical equivalence relation concerning conjunction, disjunction, negation, universal quantification and existential should arguably also be taken as desirable properties of a paraconsistent logic.

Moreover, \( \text{LPQ} \supset \text{F} \) has no connective or quantifier that is foreign to CL and the inference rules of its natural deduction proof system are all known from CL:

- except for the inference rules concerning the negation connective, the inference rules are the ones found in all natural deduction proof systems for CL;
- the inference rules concerning the negation connective are a rule that corresponds to the law of the excluded middle and rules that correspond to the de Morgan’s laws for all connectives and quantifiers;
- the rule corresponding to the law of the excluded middle is also found in natural deduction proof systems for CL and the rules corresponding to the de Morgan’s laws are well-known derived rules of natural deduction proof systems for CL.

This means that natural deduction reasoning in the setting of \( \text{LPQ} \supset \text{F} \) differs from classical natural deduction reasoning only by slightly different, but classically justifiable, reasoning about negations.

7 Concluding Remarks

The paraconsistent logic \( \text{LPQ} \supset \text{F} \) has been presented. A sequent-style natural deduction proof system has been given for this logic. A natural deduction proof system has not been given before in the literature for one of the logics that are essentially the same as \( \text{LPQ} \supset \text{F} \). In addition to the model-theoretic justification of the proof system, a logical justification by means of an embedding into classical logic has been given. Thus, a classical-logic view of \( \text{LPQ} \supset \text{F} \) has been provided. It appears that a classical-logic view of a paraconsistent logic has not been given before.
A classical-logic view of a paracomplete logic has been given before in [8], following the same approach as in the current paper. It is likely that this approach works for all truth-functional finitely-valued logics. The approach concerned is reminiscent of the method, described in [5], to reduce the many-valued interpretation of the formulas of a truth-functional finitely-valued logic to a two-valued interpretation.

In [4], an application of the propositional part of \( \text{LPQ}^\succ \text{F} \) can be found. The properties of the logical equivalence relation that are essential for that application reduce the number of three-valued paraconsistent propositional logics with properties (a) and (b) that are applicable to one, namely the propositional part of \( \text{LPQ}^\succ \text{F} \). This strengthens the impression that \( \text{LPQ}^\succ \text{F} \) is among the paraconsistent logics that deserve most attention. However, the question arises whether a paraconsistent logic is really needed to deal with contradictory sets of formulas. The embedding of \( \text{LPQ}^\succ \text{F} \) into CL given in this paper shows that it can be dealt with in classical logic but in a much less convenient way.

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