FINITE IRREDUCIBLE CONFORMAL MODULES OF RANK TWO LIE CONFORMAL ALGEBRAS

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Abstract. In the present paper, we prove that any finite non-trivial irreducible module over a rank two Lie conformal algebra $H$ is of rank one. We also describe the actions of $H$ on its finite irreducible modules explicitly. Moreover, we show that all finite non-trivial irreducible modules of finite Lie conformal algebras whose semisimple quotient is the Virasoro Lie conformal algebra are of rank one.

1. INTRODUCTION

The notion of a Lie conformal algebra was introduced in \cite{K} providing an axiomatic description of properties of the operator product expansion in conformal field theory. There are many other fields closely related to Lie conformal algebras such as vertex algebras, linearly compact Lie algebras and integrable systems. In the view of \cite{BDK}, the notion of a Lie conformal algebra is a generalization of that of classical Lie algebra, i.e. it is just the Lie algebra over a pseudo-tensor category. Thus, Lie conformal algebras have many “conformal analogue” notions and properties as ordinary Lie algebras such as semisimple Lie conformal algebras, cohomology groups of Lie conformal algebras, conformal modules. It was proved in \cite{DK} that, up to isomorphism, every finite semisimple Lie conformal algebra is a direct sum of Lie conformal algebras of the form: $\text{Vir}, \text{Vir} \ltimes \text{Cur}(g), \text{Cur}(g)$ where $g$ is a finite dimensional semisimple Lie algebra. The cohomology theory of Lie conformal algebras was also established in \cite{BKV}. As ordinary Lie algebras, the second cohomology group of a Lie conformal algebra calculates all abelian extensions of this Lie conformal algebra by its modules (coefficients of the cohomology group). The rank one and two Lie conformal algebras were classified in \cite{DK} and \cite{K, HF, BCH, W1} respectively. The rank three Lie conformal algebras were partially classified in \cite{W}. The problem of classifying all finite irreducible conformal modules of a given Lie conformal algebra was also studied by many authors. Finite irreducible conformal modules of a semisimple Lie conformal algebra were classified in \cite{CK}. Finite irreducible conformal modules of other important Lie conformal algebras were investigated in \cite{WY, LW, SXY} and so on. In the present paper, we prove that any finite non-trivial irreducible module over a rank two Lie conformal algebra $H$ is of rank one. We also describe the actions of $H$ on its finite irreducible modules explicitly. The abelian extensions of some free rank two Lie
conformal algebras by their rank one modules are also computed. Moreover, we show that all finite non-trivial modules of finite Lie conformal algebras whose quotient is the Virasoro Lie conformal algebra are of rank one.

The paper is organized as follows. In Section 2, we recall some basic notions and properties of Lie conformal algebras. In Section 3, we study finite non-trivial irreducible modules of free rank two Lie conformal algebras explicitly. As an application, we prove that any finite non-trivial irreducible modules of finite Lie conformal algebras whose semisimple quotient is the Virasoro Lie conformal algebra must be free of rank one. In addition, the abelian extensions of some free rank two Lie conformal algebras by their rank one modules are also computed.

Through this paper, denote $\mathbb{C}$ and $\mathbb{Z}_+$ the sets of all complex numbers and all nonnegative integers respectively. In addition, all vector spaces and tensor products are over $\mathbb{C}$. For any vector space $V$, we use $V[\lambda]$ to denote the set of polynomials of $\lambda$ with coefficients in $V$.

2. Preliminary

In this section, we recall some basic definitions and results of Lie conformal algebras and their modules.

**Definition 2.1.** A Lie conformal algebra $A$ is a $\mathbb{C}[\partial]$-module together with a $\mathbb{C}$-linear map (call $\lambda$-bracket) $A \otimes A \to A[\lambda]$, $a \otimes b \mapsto [a_\lambda b]$, satisfying the following axioms

\[
[\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b], \quad \text{(conformal sequilinearity)},
\]

\[
[a_\lambda b] = [b, \lambda - a] \quad \text{(skew - symmetry)},
\]

\[
[a_\lambda [b_\mu c]] = [[a_\lambda b][\lambda_+ c] + [b_\mu [a_\lambda c]] \quad \text{(Jacobi identity)},
\]

for all $a, b, c \in A$.

By means of conformal sequilinearity, we can define a Lie conformal algebra by giving the $\lambda$-brackets on its generators over $\mathbb{C}[\partial]$. In addition, the rank of a Lie conformal algebra is its rank as a $\mathbb{C}[\partial]$-module. We say a Lie conformal algebra finite if it is finitely generated as a $\mathbb{C}[\partial]$-module.

**Example 2.2.** (DK) The Virasoro Lie conformal algebra $Vir = \mathbb{C}[\partial] L$ is a free $\mathbb{C}[\partial]$-module of rank one, whose $\lambda$-brackets are determined by $[L L] = (\partial + 2\lambda)L$. Further, it is well known that $Vir$ is a simple Lie conformal algebra.

For a Lie algebra $g$, the current Lie conformal algebra $Cur g$ is a free $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial] \otimes g$ equipped with the $\lambda$-brackets:

\[
[a_\lambda b] = [a, b], \quad \text{for} \quad a, b \in g.
\]

The Lie conformal algebra $Vir \ltimes Cur g$ is defined by the following $\lambda$-brackets

\[
[L_\lambda a] = (\partial + \lambda)a, \quad \text{for all} \quad a \in g.
\]

**Example 2.3.** For $a, b \in \mathbb{C}$, the Lie conformal algebra $W(a, b) = \mathbb{C}[\partial] L \oplus \mathbb{C}[\partial] Y$ is a free rank two $\mathbb{C}[\partial]$-module with:

\[
[L_\lambda Y] = (\partial + a \lambda + b)Y, \quad [L_\lambda L] = (\partial + 2\lambda)L, \quad [Y_\lambda Y] = 0.
\]

Suppose $A$ is a Lie conformal algebra. For any $a, b \in A$, we write

\[
[a_\lambda b] = \sum_{j \in \mathbb{Z}_+} (a_{(j)} b) \lambda^j.
\]
For every $j \in \mathbb{Z}_+$, we have the $\mathbb{C}$-linear map: $A \otimes A \rightarrow A$, $a \otimes b \mapsto a^{(j)}b$, which is called $j$-th product. A $\mathbb{C}[[\partial]]$-submodule $B$ of $A$ is called a subalgebra of $A$ if $B$ is closed under all $j$-th products, that is, $a^{(j)}b \in B$ for any $a, b \in B$ and $j \in \mathbb{Z}_+$. A subalgebra $B$ is called an ideal of $A$ if $a^{(j)}b \in B$ for any $a, b \in B$, $j \in \mathbb{Z}_+$. For example, $\text{Cur}(\mathfrak{g})$ is an ideal of $\text{Vir} \ltimes \text{Cur}(\mathfrak{g})$, and $\text{Vir}$ is a subalgebra of $\text{Vir} \ltimes \text{Cur}(\mathfrak{g})$.

The derived algebra of a Lie conformal algebra $A$ is the vector space $A' = \text{span}_\mathbb{C}\{a(n)b \mid a, b \in A, n \in \mathbb{Z}_+,\}$. It is easy to check that $A'$ is an ideal of $A$. Define $A^{(1)} = A'$ and $A^{(n+1)} = [A^{(n)}]'$ for any $n \geq 1$. Then we get the derived series

$$A = A^{(0)} \supset A^{(1)} \supset \cdots A^{(n)} \supset A^{(n+1)} \supset \cdots$$

of $A$. If there exists $N \in \mathbb{Z}_+$ such that $A^{(n)} = 0$ for any $n \geq N$, then $A$ is said to be solvable. Note that the sum of two solvable ideals of a Lie conformal algebra $A$ is still a solvable ideal. Thus if $M$ is a finite Lie conformal algebra, the maximal solvable ideal exists and is unique which we denote by $\text{Rad}(M)$. A Lie conformal algebra is semisimple if its radical is zero. Thus $M/\text{Rad}(M)$ is semisimple for any finite Lie conformal algebra $M$.

It was proved in [DK] that any finite semisimple Lie conformal algebra is the direct sum of the following Lie conformal algebras:

$$\text{Vir}, \text{Cur}(\mathfrak{g}), \text{Vir} \ltimes \text{Cur}(\mathfrak{g}),$$

where $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra.

Next, let us recall the definition of conformal module of a Lie conformal algebra $A$.

**Definition 2.4.** For a Lie conformal algebra $A$, a conformal $A$-module $M$ is a $\mathbb{C}[[\partial]]$-module with a $\mathbb{C}$-linear map $A \otimes M \rightarrow M[\lambda]$, $a \otimes m \mapsto a_\lambda m$, satisfying the following axioms:

$$\partial a_\lambda m = -\lambda(a_\lambda m), \quad a_\lambda \partial m = (\partial + \lambda) a_\lambda m,$$

$$a_\lambda(b_\mu m) = [a_\lambda b]_{\lambda + \mu} m + b_\mu(a_\lambda m),$$

for all $a, b \in A$ and $m \in M$.

Since we only consider conformal modules in this paper, the conformal $A$-modules are simply called $A$-modules. If an $A$-module $M$ is finitely generated as a $\mathbb{C}[[\partial]]$-module, then we call it a finite $A$-module. The rank of $M$ is its rank as a $\mathbb{C}[[\partial]]$-module. The notions of a submodule and an irreducible submodule are defined by using $j$-th products in the obvious way.

An $A$-module $M$ is said to be trivial if $a_\lambda m = 0$ for any $a \in A$ and $m \in M$. Suppose that $M$ is a finite $A$-module. Let $\text{Tor}(M) := \{m \in M \mid f(\partial)m = 0 \text{ for some nonzero } f(\partial) \in \mathbb{C}[[\partial]]\}$. Then $\text{Tor}(M)$ is a trivial $A$-submodule of $M$ by [DK, Lemma 8.2]. Thus any finite dimensional $A$-submodule of $M$ is trivial. Let $\mathbb{C}_u = \mathbb{C}$ as a vector space for some $u \in \mathbb{C}$. Then $\mathbb{C}_u$ becomes a one-dimensional $A$-module with $\partial c = uc$ and $a_\lambda c = 0$ for any $a \in A$ and any $c \in \mathbb{C}_u$. Since any finite dimensional $A$-submodule of $M$ is trivial, any irreducible finite dimensional $A$-module is in such form. Hence we mainly focus on studying finite non-trivial $A$-modules in the sequel.

**Example 2.5.** Regular module. For any Lie conformal algebra $A$, we can regard $A$ as an $A$-module with respect to the $\lambda$-brackets of $A$. Thus $\text{Tor}(A)$ is in the center of $A$. 

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Suppose $M$ is a finite $A$-module. Consider the natural semiprodut Lie conformal algebra $A \times M$. Then $\text{Tor}(A)$ is in the center of $A \times M$. Thus $\text{Tor}(A)$ acts trivially on $M$. Hence we also focus on finite Lie conformal algebras which are free as $\mathbb{C}[[\partial]]$-modules in the sequel.

The following proposition is well known.

**Proposition 2.1.** ([DK, Theorem 8.2]) Any non-trivial free rank one $\text{Vir}$-module has the form $M_{a,b} = \mathbb{C}[\partial]v$, such that

$$L_{\lambda}v = (\partial + a\lambda + b)v,$$

for $a, b \in \mathbb{C}$. Moreover, if $a \neq 0$, then $M_{a,b}$ is irreducible and any non-trivial finite irreducible $\text{Vir}$-module is in such form. If $a = 0$, then $M_{0,b}$ has the unique finite irreducible proper $\text{Vir}$-submodule $(\partial + b)M_{0,b}$, which is isomorphic to $M_{1,b}$.

Finally, let us recall the (extended) annihilation Lie algebra of a Lie conformal algebra. The annihilation Lie algebra $\text{Lie}(A)^+$ of a Lie conformal algebra $A$ is the vector space spanned by $\{a(n) \mid a \in A, \ n \in \mathbb{Z}_+\}$ with relations

$$(\partial a)(n) = -na(n-1), \ (a + b)(n) = a(n) + b(n), \ (ka)(n) = ka(n),$$

for $a, b \in A$ and $k \in \mathbb{C}$, and the Lie brackets of $\text{Lie}(A)^+$ are given by

$$[a(m), b(n)] = \sum_{i \in \mathbb{Z}_+} \binom{m}{i} (a(i)b)(m+n-i).$$

The extended annihilation Lie algebra $\text{Lie}(A)^c$ is the Lie algebra $\mathbb{C}\partial \times \text{Lie}(A)^+$ with the brackets $[\partial, a(n)] = -na(n-1)$.

**Example 2.6.** Suppose $A = \mathbb{C}[\partial]L$ is the Virasoro Lie conformal algebra. Define $L_n = L(n+1)$ for $n \geq -1$. Then $\text{Lie}(A)^+$ is a Lie algebra with basis $\{L_n, n \geq -1\}$ and brackets as following:

$$[L_m, L_n] = (m-n)L_{m+n},$$

for $m, n \in \mathbb{Z}_+$. One can see that $\text{Lie}(A)^+ \cong \text{Vect}(\mathbb{C})$ where $\text{Vect}(\mathbb{C})$ is the regular vector fields over $\mathbb{C}$.

A $\text{Lie}(A)^c$-module $V$ is said to be conformal if for any $v \in V$ and $a \in A$, there exists $N(a,v) \in \mathbb{Z}_+$ such that $a(n)v = 0$ when $n > N(a,v)$. In fact, any $A$-module determines a unique conformal $\text{Lie}(A)^c$-module and vice versa.

**Proposition 2.2.** (see [CK]) Suppose that $V$ is a finite $A$-module. For any $a \in A$, $v \in V$, we write

$$(2.2) \quad a_{\lambda}v = \sum_{j \in \mathbb{Z}_+} (a(j)v) \frac{\lambda^j}{j!}.$$

Then $V$ is a conformal $\text{Lie}(A)^c$-module with the action:

$$a(n) \cdot v = a(n)v, \quad \partial \cdot v = \partial v.$$

Conversely, if $V$ is a conformal $\text{Lie}(A)^+$-module, then $V$ is also an $A$-module with the action:

$$(2.3) \quad a_{\lambda}v = \sum_{j \in \mathbb{Z}_+} (a(j) \cdot v) \frac{\lambda^j}{j!}.$$

In particular, $V$ is a finite irreducible $A$-module if and only if $V$ is an irreducible conformal $\text{Lie}(A)^c$-module.
3. Irreducible Modules of Free Rank Two Lie Conformal Algebras

In this section, we will study finite non-trivial irreducible modules of free rank two Lie conformal algebras. As an application, we prove that any finite non-trivial irreducible modules of a finite Lie conformal algebra $M$ whose semisimple quotient is the Virasoro Lie conformal algebra, that is, $M/\text{Rad}(M) = \text{Vir}$, must be free rank of one as a $\mathbb{C}[\partial]$-module. Finally, the abelian extensions of some free rank two Lie conformal algebras by their rank one modules are also computed.

We use $\mathcal{H}$ to denote a free rank two Lie conformal algebra in the sequel. If $\mathcal{H}$ is semisimple, then $\mathcal{H}$ is the direct sum of two Virasoro Lie conformal algebras, where these two algebras are ideals of $\mathcal{H}$. As for the non-semisimple case, we have the following proposition.

**Proposition 3.1.** ([BCH Theorem 2.21]) If $\mathcal{H}$ is solvable, then there is a basis $\{A, B\}$ such that

\[
[B_A B] = 0, \quad [A_B B] = P_1(\partial, \lambda) B, \quad [A_A A] = Q_1(\partial, \lambda) B, \quad \text{where } P_1(\partial, \lambda), Q_1(\partial, \lambda) \in \mathbb{C}[\partial, \lambda].
\]

Suppose that $\mathcal{H}$ is neither solvable nor semisimple. Then there is a basis $\{A, B\}$ such that

\[
[B_A B] = 0, \quad [A_B B] = \delta(\partial + a\lambda + b) B, \\
[A_A A] = (\partial + 2\lambda) A + Q(\partial, \lambda) B, \quad \delta \in \{0, 1\},
\]

where $a, b \in \mathbb{C}$ and $Q(\partial, \lambda)$ is some polynomial depending on $a, b, \delta$. More explicitly, $Q(\partial, \lambda) = 0$ if $b \neq 0$ or $\delta = 0$. Otherwise, we have

| $a \in \mathbb{C}$ | $Q(\partial, \lambda), \beta, \gamma \in \mathbb{C}$ |
|------------------|--------------------------------------------------|
| 1                | $\beta(2\lambda + \partial)$                     |
| 0                | $\beta(2\lambda + \partial)(\lambda^2 + \lambda \partial) + \gamma(2\lambda + \partial) \partial$ |
| -1               | $\beta(2\lambda + \partial) \partial^2 + \gamma(2\lambda + \partial)(\lambda^2 + \lambda \partial) \partial$ |
| -4               | $\beta(2\lambda + \partial)(\lambda^2 + \lambda \partial)^2$ |
| -6               | $\beta(2\lambda + \partial)[11(\lambda^2 + \lambda \partial)^3 + 2(\lambda^2 + \lambda \partial)^3 \partial^2]$ |

**Remark 3.1.** Let $\mathcal{H}$ be a free rank two Lie conformal algebra. Then *Ado Theorem* holds for $\mathcal{H}$, that is, $\mathcal{H}$ has a finite faithful module. [PK Theorem 3] states that any Lie conformal algebra in the form $\mathcal{A} = S \ltimes \mathcal{R}$, where $S$ is a finite semisimple Lie conformal algebra or $S = 0$ and $\mathcal{R}$ is the radical ideal of $\mathcal{A}$, has a finite faithful module. Thus it is clear when $\mathcal{H}$ is semisimple or solvable. Suppose $\mathcal{H} = \mathbb{C}[\partial] A \oplus \mathbb{C}[\partial] B$ is a Lie conformal algebra defined in [3,2]. If $Q(\partial, \lambda)$ is trivial, then by [PK Theorem 3], $\mathcal{H}$ has a finite faithful module. If $Q(\partial, \lambda)$ is non-trivial, we claim that $\mathcal{H}$ is centerless. Suppose that $f(\partial) A + g(\partial) B \in \mathcal{H}$ is in the center of $\mathcal{H}$. Then

\[
[(f(\partial) A + g(\partial) B) \lambda A] = f(-\lambda)(\partial + 2\lambda) A - g(-\lambda)(\partial \partial - a\partial - a\lambda + b) B = 0.
\]

Thus, $f(\partial) = g(\partial) = 0$. Hence the regular module is a finite faithful module of $\mathcal{H}$ in this case.

**Proposition 3.2.** Set $\mathcal{H} = \mathbb{C}[\partial] A \oplus \mathbb{C}[\partial] B$ as direct sum of ideals, that is $[A_A B] = 0$, where $\mathbb{C}[\partial] A$ is the Virasoro Lie conformal algebra. Let $V$ be a finite $\mathcal{H}$-module such that $A$ acts non-trivially on $V$. Then there exists $u \in V$ such that $A_u = (\partial + a\lambda + \beta)u$ for some $a, \beta \in \mathbb{C}$ and $B_u = 0$. 

Proof. Consider $V$ as a $\mathbb{C}[\partial]A$-module. Thus by Proposition 2.2, there exists $u \in V$ such that $A_\lambda u = (\partial + \alpha \lambda + \beta) u$ for some $\alpha, \beta \in \mathbb{C}$. Assume that $B_\lambda u = \sum_{i \in \mathbb{Z}_+} u_i \lambda^i$, where $u_i \in V$. Then

$$\begin{align*}
A(0)(B_\lambda u) = \sum_{i \in \mathbb{Z}_+} (A(0) \cdot u_i) \lambda^i = B_\lambda (A(0) \cdot u) = (\partial + \lambda + \beta) \sum_{i \in \mathbb{Z}_+} u_i \lambda^i.
\end{align*}$$

By comparing the coefficients of $\lambda$ of Equation (3.3), we have $B_\lambda u = 0$ which implies that $\mathbb{C}[\partial]u$ is an $\mathcal{H}$-module.

Proposition 3.3. Suppose that $\mathcal{H}$ is the Lie conformal algebra defined in (3.2) with $\delta = 1$ and $V$ is a non-trivial finite $\mathcal{H}$-module. Then there exists a nonzero element $u \in V$ such that $\mathbb{C}[\partial]u$ is an $\mathcal{H}$-submodule of rank one as a $\mathbb{C}[\partial]$-module. In particular, if $V$ is irreducible, then $V = \mathbb{C}[\partial]u$.

Proof. If $Q(\partial, \lambda)$ is not a constant, let us use $d$ to denote the total degree of $Q(\partial, \lambda)$. Otherwise, we let $d = 1$.

Then the corresponding extended annihilation Lie algebra $\text{Lie}(\mathcal{H})^e = \mathbb{C}\partial \oplus \sum_{n \geq 0} \mathbb{C}A(n) \oplus \sum_{n \geq 0} \mathbb{C}B(n)$, whose brackets are determined by

$$\begin{align*}
[A(m), A(n)] &= (m-n)A(m+n-1) + \sum_{i=0}^d a_i B(m+n-i), \\
[A(m), B(n)] &= ((a-1)m-n)B(n+m-1) + bB(n+m), \\
[\partial, A(n)] &= -nA(n-1), \quad [\partial, B(n)] = -nB(n-1), \quad [B(m), B(n)] = 0,
\end{align*}$$

where $a_i \in \mathbb{C}$ for $0 \leq i \leq d$ depending on $Q(\partial, \lambda)$.

Define $\mathcal{L}_p = \text{span}_\mathbb{C}\{A(s+d), B(s)|s \geq p\}$ for $p \geq 0$. Since $V$ is a non-trivial conformal $\mathcal{L}_0$-module, by Proposition 2.2, there exists a minimal integer $N \geq 0$ such that $U := \{v \in V|\mathcal{L}_N \cdot v = 0\} \neq \{0\}$. In addition, $U$ is finite dimensional by Lemma 14.4 of [BDK]. Let $\mathcal{N} = \mathbb{C}(\partial - A(0)) \oplus \mathcal{L}_0 \oplus \sum_{i=1}^{d-1} \mathbb{C}A(i)$. Then $\mathcal{N}$ is a subalgebra of $\text{Lie}(\mathcal{H})^e$. Moreover, we have $[\mathcal{N}, \mathcal{L}_p] \subset \mathcal{L}_p$ for any $p \geq 0$. Thus $U \subset V$ is a nonzero non-trivial finite dimensional $\mathcal{N}/\mathcal{L}_N$-module. Note that

$$\mathcal{N}^{(1)} := [\mathcal{N}, \mathcal{N}] \subset \mathcal{L}_0 \oplus \sum_{i=1}^{d-1} \mathbb{C}A(i).$$

Define inductively $\mathcal{N}^{(i+1)} := [\mathcal{N}^{(i)}, \mathcal{N}^{(i)}]$. Then $\mathcal{N}/\mathcal{L}_N$ is a solvable Lie algebra. Therefore, by Lie’s Theorem, there exists a nonzero element $u \in U$ and a linear function $\xi$ on $\mathcal{N}$ such that $A(0) \cdot u = (\partial + \beta) u$ for some $\beta \in \mathbb{C}$, $A(i) \cdot u = \xi(A(i)) u$ for $i > 0$ and $B(j) \cdot u = \xi(B(j)) u$ for $j \geq 0$. Thus $A_\lambda u = (\partial + p(\lambda)) u$ and $B_\lambda u = f(\lambda) u$ for some $p(\lambda), f(\lambda) \in \mathbb{C}[\lambda]$. Hence, $\mathbb{C}[\partial]u$ is a free rank one $\mathcal{H}$-submodule of $V$. 

By Propositions 2.2 and Proposition 3.3, for any free rank two Lie conformal algebra $\mathcal{H}$, it is easy to see that any finite non-trivial irreducible module of $\mathcal{H}$ must be free of rank one.

More explicitly, we have the following theorem as the main results of this paper.

Theorem 3.2. Suppose that $\mathcal{H} = \mathbb{C}[\partial]A \oplus \mathbb{C}[\partial]B$ is a Lie conformal algebra of rank two. Then any non-trivial finite irreducible $\mathcal{H}$-module is free of rank one. Moreover,
if \( V = \mathbb{C}[\partial]v \) is a non-trivial irreducible \( \mathcal{H} \)-module, then the action of \( \mathcal{H} \) on \( V \) has to be one of the following cases:

(i) If \( \mathcal{H} \) is solvable with the relations (3.1), then we have \( A_{\lambda}v = \phi_A(\lambda)v \), \( B_{\lambda}v = \phi_B(\lambda)v \), where \( \phi_A(\lambda), \phi_B(\lambda) \) are not zero simultaneously. Moreover, \( \phi_B(\lambda) \neq 0 \) only if \( p(\partial, \lambda) = Q(\partial, \lambda) = 0 \).

(ii) Suppose that \( \mathcal{H} \) is the Lie conformal algebra defined in (3.3) with \( \delta = 1 \). Then

\[
A_{\lambda}v = (\partial + \alpha\lambda + \beta)v, \quad B_{\lambda}v = \gamma v,
\]
where \( \alpha, \beta, \gamma \in \mathbb{C} \) such that \( \gamma \neq 0 \) only if \( a = 1, b = 0 \) and \( Q(\partial, \lambda) = 0 \). Further, if \( \gamma = 0 \), then \( \alpha \neq 0 \).

(iii) Suppose that \( \mathcal{H} \) is the Lie conformal algebra defined in (3.4) with \( \delta = 0 \), then either

\[
A_{\lambda}v = (\partial + \alpha\lambda + \beta)v, \quad B_{\lambda}v = 0, \quad \text{for some } \beta, \ 0 \neq \alpha \in \mathbb{C},
\]
or

\[
A_{\lambda}v = 0, \quad B_{\lambda}v = \phi(\lambda)v, \quad \text{for some nonzero } \phi(\lambda) \in \mathbb{C}[\lambda].
\]

(iv) If \( \mathcal{H} = \mathbb{C}[\partial]A \oplus \mathbb{C}[\partial]B \) is a direct sum of two Virasoro Lie conformal algebras with \( [A_{\lambda}B] = 0 \), then either

\[
A_{\lambda}v = (\partial + \alpha_1\lambda + \beta_1)v, \quad B_{\lambda}v = 0, \quad \text{for some } \beta_1, \ 0 \neq \alpha_1 \in \mathbb{C},
\]
or

\[
A_{\lambda}v = 0, \quad B_{\lambda}v = (\partial + \alpha_2\lambda + \beta_2)v, \quad \text{for some } \beta_2, \ 0 \neq \alpha_2 \in \mathbb{C}.
\]

Proof. (i) It is clear from the definition and Theorem 8.4 of [DK].

(ii) If \( \mathcal{H} \) is neither solvable nor semisimple, then by Proposition 3.3 \( A_{\lambda}v = (\partial + p(\lambda))v \) and \( B_{\lambda}v = f(\lambda)v \) for some \( p(\lambda), f(\lambda) \in \mathbb{C}[\lambda] \). Plugging these into the equations:

\[
[A_{\lambda}B]_{\lambda+\mu}v = A_{\lambda}B_{\mu}v - B_{\mu}A_{\lambda}v,
\]

\[
[A_{\lambda}A]_{\lambda+\mu}v = A_{\lambda}A_{\mu}v - A_{\mu}A_{\lambda}v,
\]
we have

\[
f(\lambda + \mu)(-\lambda - \mu + a\lambda + b) = -\mu f(\mu),
\]

\[
(\lambda - \mu)p(\lambda + \mu) + Q(-\lambda - \mu, \lambda)f(\lambda + \mu) = \lambda p(\lambda) - \mu p(\mu).
\]

If \( f(\lambda) \neq 0 \), Equation (3.6) forces that \( a = 1, b = 0 \) and \( f(\lambda) = \gamma \) for some nonzero \( \gamma \in \mathbb{C} \). In this case, \( Q(\lambda, \mu) = 0 \). Otherwise, Equation (3.7) implies that \( Q(\lambda, \mu) \) is a 2-coboundary in \( C^2(Vir, M_{1,0}) \), which contradicts to the construction of \( Q(\lambda, \mu) \).

Thus, in any case, \( Q(-\lambda - \mu, \lambda)f(\lambda + \mu) = 0 \). The remainder of (ii) is obvious from Proposition 3.2

(iii) and (iv) are immediate results by Proposition 3.2

As an application, we have the following theorem.

**Theorem 3.3.** Let \( M \) be a free finite Lie conformal algebra such that \( M/\text{Rad}(M) = Vir \). Then any non-trivial finite \( M \)-module \( V \) must contain a free rank one \( M \)-submodule.
Proof. Note that \( \text{Rad}(M) \) is the radical of \( M \). Then by the conformal Lie’s Theorem, a nonzero eigenspace \( V_\phi \) exists, where \( V_\phi = \{ v \in V | m_\lambda v = (\phi(m))_{\lambda = -\lambda} v, \forall m \in \text{Rad}(M) \} \) for some \( \phi \in \text{Hom}_{C[\partial]}(\text{Rad}(M), C[\partial]) \). By Proposition 14.1 of [BDK], \( V_\phi \) spans an \( M \)-module, which is free by Lemma 4.1 of [BDK]. Hence, we may assume that \( V = C[\partial] \otimes C V_\phi \). Note that \( V_\phi \) is a finite dimensional vector space according to that \( V \) is a finite \( M \)-module.

Thus we may assume that \( V = C[\partial] \otimes C V_\phi \) for some \( \phi \in \text{Hom}_{C[\partial]}(\text{Rad}(M), C[\partial]) \). Since \( \text{Vir} \) is free as a \( C[\partial] \)-module, \( M = C[\partial]L \oplus \text{Rad}(M) \) where

\[
[L_\lambda L] = (\partial + 2\lambda)L \mod \text{Rad}(M).
\]

Suppose \( \{ M_i | 1 \leq i \leq k \} \) is a \( C[\partial] \)-basis of \( \text{Rad}(M) \). If \( L \) or \( \text{Rad}(M) \) acts trivially on \( V \), then the theorem is obvious. Hence let us assume that both \( L \) and \( M_1 \) act non-trivially on \( V \). Therefore, \( \phi(M_1) \neq 0 \).

Define \( \ker(V) = \{ a \in M | a_1 u = 0, \forall u \in V \} \). Then \( \ker(V) \) is an ideal of \( M \). We are going to prove that \( (C[\partial]L \oplus C[\partial]M_1) \cap \ker V = \{ 0 \} \). Assume that there exist \( f(\partial) \) and \( g(\partial) \in C[\partial] \) such that \( (f(\partial)L + g(\partial)M_1)v = 0 \) for any \( v \in V_\phi \). On one hand, \( f(\partial) = 0 \) implies that \( g(\partial) = 0 \). On the other hand, if \( f(\partial) \neq 0 \), \( L_\lambda v \in V_\phi \) which implies that \( V_\phi \) is a finite dimensional \( M \)-module. In this case, \( M \) acts trivially on \( V_\phi \) which contradicts to our assumption. As a consequence, \( \text{rank}(M/\ker(V)) \geq 2 \) and \( \text{rank}(\ker(V)) \leq k - 1 \).

Define \( T_i = \phi(M_i)M_1 - \phi(M_1)M_i \) for \( 2 \geq i \geq k \). Let \( T \) be the \( C[\partial] \)-module generated by \( \{ T_2, T_3, \ldots, T_k \} \). Then \( T \subseteq \ker(V) \) and \( \text{rank}(T) = k - 1 \). Thus \( k - 1 \leq \text{rank}(T) \leq \text{rank}(\ker(V)) \leq k - 1 \). Hence \( \text{rank}(M/\ker(V)) = 2 \). Then, this reduces to the study of finite irreducible modules of rank two Lie conformal algebras. Now by Theorem 3.2 we complete the proof. \( \square \)

Remark 3.4. Since the torsion part of a finite Lie conformal algebra acts trivially on any module of this Lie conformal algebra, by Theorem 3.3 any finite non-trivial irreducible modules of a finite Lie conformal algebra \( M \) satisfying \( M/\text{Rad}(M) = \text{Vir} \) is free of rank one. Therefore, any finite non-trivial irreducible modules of Lie conformal algebras such as \( \mathcal{W}(a, b) \), Schrödinger-Virasoro type Lie conformal algebras in [LHW] is free of rank one. This result provides a unified method to determine finite irreducible modules of such Lie conformal algebras.

We finish this paper by computing the abelian extensions of some free rank two Lie conformal algebras by their non-trivial free rank one modules. As in Theorem 3.1 of [BK], these abelian extensions can be described in terms of the second cohomology groups.

**Proposition 3.4.** Let \( \mathcal{H} = C[\partial]A \oplus C[\partial]B \) be a rank two Lie conformal algebra with \( \lambda \)-brackets:

\[
[A_\lambda A] = (\partial + 2\lambda)A, \quad [A_\lambda B] = 0, \quad [B_\lambda B] = \delta(\partial + 2\lambda)B, \quad \delta \in \{0, 1\}.
\]

Suppose \( M_{a, b} = C[\partial]v \) is a rank one \( \mathcal{H} \)-module defined as following:

\[
A_\lambda v = (\partial + a\lambda + b)v, \quad B_\lambda v = 0,
\]

for some \( a, b \in C \). Then \( H^2(\mathcal{H}, M_{a, b}) \cong H^2(C[\partial]A, M_{a, b}) \).

**Proof.** Suppose \( \mathcal{E} \) is an abelian extension of \( \mathcal{H} \) by \( M_{a, b} \). Then the nontrivial \( \lambda \)-brackets of \( \mathcal{E} \) are given by

\[
[A_\lambda A] = (\partial + 2\lambda)A + Q_1(\partial, \lambda)v, \quad [A_\lambda v] = (\partial + a\lambda + b)v,
\]

\[
[B_\lambda A] = \delta(\partial + 2\lambda)A, \quad [B_\lambda B] = \delta(\partial + 2\lambda)B, \quad \delta \in \{0, 1\}.
\]
\[ [A_\lambda B] = Q_2(\partial, \lambda)v, \quad [B_\lambda B] = \delta(\partial + 2\lambda)B + Q_3(\partial, \lambda)v, \delta \in \{0, 1\}, \]

for some \( Q_1(\partial, \lambda), Q_2(\partial, \lambda), Q_3(\partial, \lambda) \in \mathbb{C}[\partial, \lambda] \). The Jacobi-identites

\[
[\lambda[A_\mu B]] = [[A_\lambda A]_{\lambda+\mu} B] + [A_\mu [A_\lambda B]],
\]

\[
[\lambda[B_\mu B]] = [[A_\lambda B]_{\lambda+\mu} B] + [B_\mu [A_\lambda B]],
\]

provide

\[(\lambda - \mu)Q_2(\partial, \lambda + \mu) = Q_2(\partial + \lambda, \mu)(\partial + a\lambda + b) - Q_2(\partial + \mu, \lambda)(\partial + a\mu + b) \quad (3.8)\]

and

\[
\delta(\partial + \lambda + 2\mu)Q_2(\partial, \lambda) + Q_3(\partial + \lambda, \mu)(\partial + a\lambda + b) = 0. \quad (3.9)
\]

Applying \( \lambda = 0 \) in \( (3.9) \),

\[ Q_3(\partial, \mu) = -\delta(\partial + 2\mu)f(\partial) \]

where \( f(\partial) = \frac{Q_2(\partial, 0)}{\partial + b} \). Similarly, applying \( \mu = 0 \) in \( (3.9) \), we get

\[ Q_2(\partial, \lambda) = (\partial + a\lambda + b)f(\partial + \lambda). \quad (3.10) \]

Define \( \tilde{B} = B - f(\partial)v \). Thus \( \mathcal{E} = \mathbb{C}[\partial]A \oplus \mathbb{C}[\partial]\tilde{B} \oplus M_{a,b} \) as direct sum of \( \mathbb{C}[\partial] \)-modules. Moreover, it is straightforward to check that \([A_\lambda \tilde{B}] = 0 \) and \([B_\lambda \tilde{B}] = \delta(\partial + 2\lambda)\tilde{B} \). Thus, by a suitable choice of basis of \( \mathcal{E} \) as a free \( \mathbb{C}[\partial] \)-module, we can take \( Q_2(\partial, \lambda) = Q_3(\partial, \lambda) = 0 \). This completes the proof. \( \square \)

**Proposition 3.5.** Let \( \mathcal{H} = \mathbb{C}[\partial]A \oplus \mathbb{C}[\partial]B \) be a rank two Lie conformal algebra with \( \lambda \)-brackets:

\[ [A_\lambda A] = (\partial + 2\lambda)A, \quad [A_\lambda B] = 0, \quad [B_\lambda B] = 0. \]

Suppose \( V = \mathbb{C}[\partial]v \) is a rank one \( \mathcal{H} \)-module defined as following:

\[ A_\lambda v = 0, \quad B_\lambda v = \phi(\lambda)v, \quad \phi(\lambda) \in \mathbb{C}[\partial]. \]

Then \( H^2(\mathcal{H}, V) \cong H^2(\mathbb{C}[\partial]B, V) \).

**Proof.** Suppose \( \mathcal{E} \) is an abelian extension of \( \mathcal{H} \) by \( V \). Then the nontrivial \( \lambda \)-brackets of \( \mathcal{E} \) are given by

\[ [A_\lambda A] = (\partial + 2\lambda)A + Q_1(\partial, \lambda)v, \quad [B_\lambda v] = \phi(\lambda)v, \]

\[ [B_\lambda A] = Q_2(\partial, \lambda)v, \quad [B_\lambda B] = Q_3(\partial, \lambda)v, \]

for some \( Q_1(\partial, \lambda), Q_2(\partial, \lambda), Q_3(\partial, \lambda) \in \mathbb{C}[\partial, \lambda] \). The Jacobi-identites

\[ [\lambda[A_\mu A]] = [[A_\lambda A]_{\lambda+\mu} A] + [A_\mu [A_\lambda A]], \]

\[ [B_\lambda [A_\mu A]] = [[B_\lambda A]_{\lambda+\mu} A] + [B_\mu [A_\lambda A]], \]

provide

\[(\lambda - \mu)Q_1(\partial, \lambda + \mu) = (\partial + \lambda + 2\mu)Q_1(\partial, \lambda) - (\partial + 2\lambda + \mu)Q_1(\partial, \mu) \quad (3.11)\]

and

\[(\partial + \lambda + 2\mu)Q_2(\partial, \lambda) + \phi(\lambda)Q_1(\partial + \lambda, \mu) = 0. \quad (3.12)\]

Setting \( \mu = 0 \) in both \( (3.11) \) and \( (3.12) \), we get

\[ Q_1(\partial, \lambda) = (\partial + 2\lambda)f(\partial), \]

where \( f(\partial) = \frac{Q_2(\partial, 0)}{\partial} \), and

\[ Q_2(\partial, \lambda) = -\phi(\lambda)f(\partial + \lambda). \]
Define \( \tilde{A} = A + f(\partial)v \). Thus \( \mathcal{E} = \mathbb{C}[\partial]\tilde{A} \oplus \mathbb{C}[\partial]B \oplus V \) as direct sum of \( \mathbb{C}[\partial] \)-modules. Moreover, it is straightforward to check that \( [B, \tilde{A}] = 0 \) and \( [\tilde{A}, \tilde{A}] = (\partial + 2\lambda)\tilde{A} \). Thus, by a suitable choice of basis of \( \mathcal{E} \) as a free \( \mathbb{C}[\partial] \)-module, we can take \( Q_1(\partial, \lambda) = Q_2(\partial, \lambda) = 0 \). This completes the proof. \( \square \)

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