ISOGONIC AND ISODYNAMIC POINTS OF A SIMPLEX
IN A REAL AFFINE SPACE

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ABSTRACT. A non-equilateral triangle in a Euclidean plane has exactly two isogonic and two
isodynamic points. There are a number of different but equivalent characterizations
of these triangle centers. The aim of this paper is to work out characteristic properties
of isogonic and isodynamic centers of simplices that can be transferred to higher dimensions.
In addition, a geometric description of the Weiszfeld algorithm for calculating the Fermat
point of a simplex is given.

1. Introduction and terminology.

Let \( A_1, \ldots, A_{n+1} \) be \( n+1 \) affinely independent points in a real Euclidean-affine space
\( \mathcal{A} \) of dimension \( n > 1 \). We work with barycentric coordinates with respect to the tuple
\((A_1, \ldots, A_{n+1})\). Given a point \( P \in \mathcal{A} \), we write \( P = [p_1, \ldots, p_{n+1}] \) (resp. \( P = [p_1 : \ldots : p_{n+1}] \)), if \( p_1, \ldots, p_{n+1} \) are the absolute (resp. homogeneous) coordinates of \( P \) with
respect to \((A_1, \ldots, A_{n+1})\). The set \( \Sigma = \{ P = [p_1, \ldots, p_{n+1}] \mid p_i \geq 0 \text{ for } i = 1, \ldots, n+1 \} \) is
an \( n \)-simplex. The affine subspace \( A_1 \cup \cdots \cup A_{k+1} \) spanned by \( k+1 \) points \( A_{i_1}, \ldots, A_{i_{k+1}} \in \{ A_1, \ldots, A_{n+1} \} \) is called \( k \)-sideplane of \( \Sigma \), whilst the \( k \)-simplex \( \Sigma \cap (A_{i_1} \cup \cdots \cup A_{i_{k+1}}) =: \Sigma_{i_1, \ldots, i_{k+1}} \) is called a \( k \)-face of \( \Sigma \). Instead of \( 1 \)-sideplanes we usually speak of \( 1 \)-faces, and the \( 1 \)-faces are also called \( 1 \)-planes, the \((n-1)\)-faces \( 1 \)-facets of \( \Sigma \).

Let \( d_{ij} \) be the distance between two vertices \( A_i, A_j \) of \( \Sigma \), then the squared distance between
two points \( P = [p_1, \ldots, p_{n+1}] \) and \( Q = [q_1, \ldots, q_{n+1}] \) is given by

\[
\sum_{1 \leq i < j \leq n+1} (d_{ij})^2 (p_i - q_i)(p_j - q_j)
\]

\text{Remarks.}

- If \( \mathcal{A} \) is the real vector space \( \mathbb{R}^n \) with dot product \( \cdot \) and Euclidean distance
function \( d(P, Q) = \sqrt{(P - Q) \cdot (P - Q)} \), then the equation \((\star)\) applies, as it was shown
by Coxeter [2].
- As we know from Galilean and from Lorentz-Minkowski geometry, squared distances
between points can take values that are non-positive real numbers (cf. [8]). But in Euclidean-
affine spaces squared distances have to be positive, and, moreover, volumes of simplices of
dimension \( \leq n \) take only positive values.

2. Properties of the isogonic and of the isodynamic points in the planar case \( n=2 \).

In this paper we want to find properties that characterize isogonic and isodynamic points
of an \( n \)-simplex. At first, we examine the 2-dimensional case. Instead of giving a definition
for these centers, we list several well known properties which characterize these points in
a unique way, cf. [3,10,13,17,19,23].

A point \( P \) is an isodynamic point of a triangle \( A_1A_2A_3 \) if any of the following (equivalent)
statements is true:

\text{Date: April 13, 2021.}
The mirror image of triangle $A_1A_2A_3$ with respect to a circle with center $P$ is an equilateral triangle.

Let $C_i$ denote the intersection of the tripolar of the symmedian $K$ (defined as the isogonal conjugate of the centroid $G$) with the sideline opposite the vertex $A_i$ and let $\mathcal{C}_i$ denote the circle with center $C_i$ passing through the vertex $A_i$, $i = 1, 2, 3$; then $P$ is a common point of the three circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$.

Let $\mathcal{C}$ denote the circumcircle of triangle $A_1A_2A_3$ and $T_A \mathcal{C}$ denote the tangent of $\mathcal{C}$ at the point $A_i$, let $C_i^*$ denote the intersection of $T_A \mathcal{C}$ with the sideline opposite $A_i$ and $\mathcal{C}_i^*$ denote the circle with center $C_i^*$ passing through the vertex $A_i$, $i = 1, 2, 3$; then $P$ is a common point of the three circles $\mathcal{C}_1^*$, $\mathcal{C}_2^*$, $\mathcal{C}_3^*$.

The pedal triangle of $P$ is equilateral.

Let us assume that the triangle $A_1A_2A_3$ is not equilateral. Then there exist precisely two isodynamic points $J_1$, $J_2$, and the following statements hold:

$J_1$, $J_2$ both lie on the Brocard axis, a line through the symmedian $O$ of triangle $A_1A_2A_3$.

One of the two isodynamic points is a point inside the triangle, and the other is the mirror image of this point with respect to the circumcircle.

The points $O$, $J_1$, $K$, $J_2$ form a harmonic range.

The circles $\mathcal{C}_1^*$, $\mathcal{C}_2^*$, $\mathcal{C}_3^*$ are called the Apollonian circles of the triangle $A_1A_2A_3$. A point $Q$ is a point on $\mathcal{C}_i^*$ precisely when

$$\frac{d(Q, A_j)}{d(Q, A_k)} = \frac{d_{ij}}{d_{ik}}, \quad \{i, j, k\} = \{1, 2, 3\} \quad (**) .$$

Let $T_{J_k} \mathcal{C}_i^*$ be the tangent of $\mathcal{C}_i^*$ at the point $J_k$, $k = 1, 2$ and $i = 1, 2, 3$. Then:

The three singular conics $T_{J_1} \mathcal{C}_1^* \cup T_{J_2} \mathcal{C}_2^* \cup T_{J_3} \mathcal{C}_3^*$ are congruent in pairs, $k = 1, 2$. In other words, these are unions of two lines with the same angle of intersection.

A point $P$ is an isogonic point (often also called isogonic center) of triangle $A_1A_2A_3$ if any of the three following (equivalent) statements is true:

1. The three singular conics $(P \cup A_1) \cup (P \cup A_2)$, $(P \cup A_2) \cup (P \cup A_3)$, $(P \cup A_3) \cup (P \cup A_1)$ are congruent in pairs.
2. The inversion of triangle $A_1A_2A_3$ in a circle with center $P$ leads to an equilateral triangle.
3. The antipedal triangle of $P$ is equilateral.

There exist exactly two isogonic points, $F_1$ and $F_2$. These are the isogonal conjugates of the points $J_1$ and $J_2$, respectively.

If $F_1$ is inside the triangle $A_1A_2A_3$, then it is the Fermat-Torricelli point of the triangle, it minimizes the function $P \mapsto d(P, A_1) + d(P, A_2) + d(P, A_3)$.

Remarks. Let us look at the situation in elliptic and in hyperbolic planes. In these planes there are two points - let us call them $J_1$ and $J_2$ - which satisfy (2). Both are points on the line $K \cup O$. One more point on this line is the Lemoine point $K$, a point whose tripolar is orthogonal to $K \cup O$ and meets the sidelines of triangle $A_1A_2A_3$ in $C_1^*, C_2^*, C_3^*$. In
In this section we generalize results and ideas published by P. Yiu [24] to higher dimensions.

3. Generalized Apollonian spheres.

In this section we generalize results and ideas published by P. Yiu [24] to higher dimensions.

We come back to the general case of an $n$-simplex $\Sigma$ with vertices $A_1, \ldots, A_{n+1}$. Let $P = \{p_1 : \cdots : p_{n+1}\}$ be a point not on any of the $(n-1)$-sideplanes of $\Sigma$, thus $p_1p_2 \cdots p_{n+1} \neq 0$. The $(n-1)$-plane $P \cup A_3 \cup \cdots \cup A_{n+1}$ meets the line $A_1 \cup A_2$ at the point $P_{12} := \{p_1 : p_2 : 0 : \cdots : 0\}$, while the $\Sigma$-polar plane of $P$, is the $(n-1)$-plane $P^2 := \{[x_1 : \cdots : x_{n+1} \mid \frac{x_1}{p_1} + \cdots + \frac{x_{n+1}}{p_{n+1}} = 0\}$ meets the line $A_1 \cup A_2$ at the point $P_{12}^* := [-p_1 : p_2 : 0 : \cdots : 0]$. The midpoint of these two points, $Q_{12} := \frac{1}{2}P_{12} + \frac{1}{2}P_{12}^* = [-p_1^2 : p_2^2 : 0 : \cdots : 0]$, is a point on the $\Sigma$-polar plane of the barycentric square $P^2$ of $P$, as can be easily checked.

The $(n-1)$-sphere with diameter $\mathbb{P}_{12}, P_{12}^*$ is $Q_{12} := \{Q = tP_{12} + (1-t)P_{12}^* \mid 0 \leq t \leq 1\}$ and center $Q_{12}$ is denoted by $S_{12}$. This sphere $S_{12}$ meets the circumsphere of $\Sigma$ orthogonally.

Proof. The points $A_1, P_{12}, A_2, P_{12}^*$ form a harmonic range. Therefore the points $A_1, A_2$ are inversive with respect to $S_{12}$, and every sphere through $A_1$ and $A_2$, especially the circumsphere of $\Sigma$, is orthogonal to $S_{12}$. □

The points $P_{ij}, P_{ij}^*, Q_{ij}$ and $(n-1)$-spheres $S_{ij}, 1 \leq i \leq j \leq n+1$ are defined likewise. Following Yiu, we will call these spheres generalized Apollonian spheres of $\Sigma$ with respect to $P$. The sphere $S_{ij}$ is the locus of points $R$ satisfying $d(A_i, R) : d(A_j, R) = 1/|p_i| : 1/|p_j|$. From this follows that $R$ is a common point of two spheres $S_{ij}, S_{jk}, 1 \leq i < j < k \leq n+1$, precisely when $d(A_i, R) : d(A_j, R) : d(A_k, R) = 1/|p_i| : 1/|p_j| : 1/|p_k|$. The two points $J_1$ and $J_2$ also satisfy (9), and the four points $O, J_1, K, J_2$ form a harmonic range. But statements (1), (3), (4), (7) do not apply, in general. The equation given in (5) is responsible for the name of the two centers; it has to be replaced in the hyperbolic case by the equation

$$\sinh(\frac{1}{2}d(P, A_1)) \sinh(\frac{1}{2}d_{23}) = \sinh(\frac{1}{2}d(P, A_2)) \sinh(\frac{1}{2}d_{13}) = \sinh(\frac{1}{2}d(P, A_3)) \sinh(\frac{1}{2}d_{12})$$

and in the elliptic case by

$$\sin(\frac{1}{2}d(P, A_1)) = \sin(\frac{1}{2}d_{23}) = \sin(\frac{1}{2}d(P, A_2)) \sin(\frac{1}{2}d_{13}) = \sin(\frac{1}{2}d(P, A_3)) \sin(\frac{1}{2}d_{12}).$$

The name Apollonian circles for the circles $C_1^*, C_2^*, C_3^*$ is justified also in planes with nonzero Gaussian curvature, because their equations are very similar to equation (8), cf. [9].

In elliptic and in hyperbolic planes, there are two points, $F_1$ and $F_2$, say, for which statement (10) is true (cf. [9]), and if one of these points $F_1, F_2$ lies inside this triangle, this point is the Fermat-Torricell point of the triangle (cf. [11]). There are strong indications (based on experiments with GeoGebra) that $F_1$ and $F_2$ are exactly the points that satisfy condition (11), see [9]. But (12) does not necessarily apply to them, and $F_1$ and $F_2$ are, in general, not isogonal conjugates of $J_1$ and $J_2$.

What is the situation like in a Lorentz-Minkowski plane or in a Galilean plane? Apollonian circles exist in a Lorentz-Minkowski plane, however their common points are points on the line at infinity. In a Galilean plane, each Apollonian circle consists of two parallel lines and all these six lines meet at the absolute pole. (A nonsingular circle touches the line at infinity at the absolute pole, and this point is the center of the circle.) So there are no isodynamic points in Galilean and Lorentz-Minkowski planes, nor are there isogonic points.
Figure 1. Three generalized Appollonian circles. All figures were created with the software program GeoGebra [25].

But then $R$ must be a point on $S_{ik}$, as well. The radical $(n-1)$-plane of the three spheres is perpendicular to the line through their centers and contains the circumcenter $O$. Therefore, the intersection of all these radical $(n-1)$-planes is the line through $O$ perpendicular to the $\Sigma$-polar plane of $P^2$. As a consequence, all $(n-1)$-spheres $S_{ij}$ pass through a point of this line if any of these spheres does. If such a point exists, we call it a generalized isodynamic point of $\Sigma$ with respect to $P$. If there is exactly one generalized isodynamic point, then it is a point on the circumcircle. If there are two such points, they are inversives with respect to the circumcircle.

The "classical" case: $P = $ incenter $I$. In this case, the spheres $S_{ij}$ are the (proper) Apollonian spheres. Sruba’s paper [21] suggests that the line through $O$ perpendicular to the $\Sigma$-polar plane of $K = I^2$ always meets an Apollonian sphere at two points, which are then called isodynamic points. However, this is not the case, not even for $n = 3$. We present a counter example. There exists a tetrahedron with sidelengths $(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}) = (13, 11, 9, 12, 5, 11)$. We will show that the Apollonian spheres $S_{12}, S_{13}, S_{23}$ of this tetrahedron do not meet. Let $a_i$ denote the area of the facet opposite vertex $A_i$; then $a_1 = 6\sqrt{21}, a_2 = \frac{9}{4}\sqrt{403}, a_3 = \frac{9}{4}\sqrt{51}, a_4 = 6\sqrt{105}$. The line $A_4 \cup K$ meets the sideplane $A_1 \cup A_2 \cup A_3$ at the point $R = [a_1^2 : a_2^2 : a_3^2 : 0]$. The intersection of the spheres $S_{12}, S_{13}, S_{23}$ with the plane $A_1 \cup A_2 \cup A_3$ are the generalized Apollonian circles of the point $R$ and triangle $\Delta = A_1 A_2 A_3$. P. Yiu [23] gives a criterion to decide whether or not these circles have points in common: They have no common points precisely when the point

$$Q = \left[ d_{23}^2 \frac{d_{23}^2}{a_4^2} - \frac{d_{13}^2}{a_2^2} - \frac{d_{12}^2}{a_3^2}, \quad d_{13}^2 \left( \frac{d_{13}^2}{a_2^2} - \frac{d_{12}^2}{a_3^2} - \frac{d_{23}^2}{a_4^2} \right), \quad d_{12}^2 \left( \frac{d_{12}^2}{a_3^2} - \frac{d_{23}^2}{a_4^2} \right), \quad d_{23}^2 \left( \frac{d_{23}^2}{a_4^2} - \frac{d_{13}^2}{a_2^2} \right) \right] : 0$$

lies outside the circumcircle of $\Delta$. 
Since the distance between $Q = \left[ \frac{-23}{210}, \frac{121}{315}, \frac{169}{680} \right]$, and the circumcenter of
$\Delta$, $O_{\Delta} = \left[ \frac{\sqrt{23}}{210}, \frac{121}{315}, \frac{169}{680} \right]$, is bigger than the circumradius of $\Delta$, the Apollonian spheres
$S_{12}, S_{13}, S_{23}$ of this tetrahedron do not meet, and isodynamic points, when defined as common
points of the Apollonian spheres, do not exist for this tetrahedron.

Remark. C. Pohoata and V. Zajic [20] presented a generalization of the Apollonian circles
different from that introduced by P. Yu.

4. Isogonic points of a simplex.

From [4] we adopt the following terminology: A simplex is regular if all its edges are the
same length, a simplex is equiareal if all its facets have the same $(n-1)$-volume, and it is
equifacetal if all its facets are congruent.

In dimension $3$, every equiareal simplex is equifacetal; but in higher dimensions this is not the
case.

Let us define:

A point $P$ is an isogonic point of an $n$-simplex $\Sigma$ if its antipedal simplex is equiareal.

It follows immediately, that a point $P$ is an isogonic point of an $n$-simplex $\Sigma$ if and only if
its mirror image $\Sigma^*$ in an $(n-1)$-sphere with center $P$ is equiareal. The simplex $\Sigma^*$ and the
antipedal simplex are similar simplices.

Before moving to higher dimensions $n$, we shall see that our definition is adequate for di-

mension $n = 3$: If the antipedal simplex is equiarial, it is also equifacetal, and the four triads
of lines $(P \cup A_i) \cup (P \cup A_j) \cup (P \cup A_k)$, $1 \leq i < j < k \leq 4$, are congruent in pairs. On the
other hand, this congruence also implies that the antipedal simplex is equifacetal. But it is
not the congruence, it is the equiangularity that justifies the name isogonic point. Moreover,
if $P$ is an isogonic point inside the tetrahedron $\Sigma$, then this point is the Fermat-Torricelli
point of $\Sigma$, see [1]. Let us call the Fermat-Torricelli point $F$.

In 1937 A. Weiszfeld published a method to calculate the barycentric coordinates of $F$.
Starting from a point $F_0 = [f_{0,1} : \cdots : f_{0,n+1}]$ with $f_{0,1} \cdots f_{0,n+1} \neq 0$, $F$ is the limit
of an iteration process with a step: $F_{i+1} = \{f_{i+1,1} : \cdots : f_{i+1,n+1}\}$, $f_{i+1,k} = f_{i,k}/d(F_i, A_k)$,
see [22, 4, 14, 15]. A geometric description of this process is given in the next section.

5. $Z^*$-correspondence.

Let $P = [p_1 : \cdots : p_{n+1}]$ be a point not on any $(n-1)$-sideplane of $\Sigma$, $S$ an $(n-1)$-sphere
with center $P$, and let $\Sigma^* = A_1^* A_2^* \cdots A_n^*$ be the polar simplex of $\Sigma$ with respect to $S$. The
simplices $\Sigma$ and $\Sigma^*$ are orthologic, and both orthologic centers coincide at $P$.
Therefore the barycentric coordinates of $P$ with respect to $\Sigma^*$ agree with the barycentric
coordinates of $P$ with respect to $\Sigma$.

Proof of the last statement. We can assume that $p_1 + \cdots + p_{n+1} = 1$ and that $S$ has radius $1$.
It suffices to show that $\sum_i p_i A_i^* = P$. First observe that the vector $n_i := \frac{\text{sgn}(p_i)}{d(P, A_i^*)}(A_i^* - P)$
is an outward unit normal vector of the $(n-1)$-dimensional surface of $\Sigma$. When we denote
the $(n-1)$-volume of the facet opposite vertex $A_i$ by $a_i$, then $\sum_i a_i n_i$ is the zero vector. It
The $I^*$-transversal and the $I^*$-correspondent of a point $P$. Triangle $A_1^{[P]}A_2^{[P]}A_3^{[P]}$ is the antipedal triangle of $P$.

follows:

$$\sum_i p_i A_i^* = \sum_i p_i P + \sum_i p_i (A_i^* - P)$$

$$= P + \sum_i \left( \frac{a_i}{n d(P, A_i^*) \sum_j a_j} \right) \left( d(P, A_i^*) n_i \right)$$

$$= P + \frac{1}{n \sum_i a_i} \sum_i a_i n_i = P.$$

Now we take a point $Z^*$ having homogeneous coordinates $z_1^* : \cdots : z_{n+1}^*$ with respect to $\Sigma^*$. If $Z^*$ is different from $P$, its polar $(n-1)$-plane with respect to $S$ is a hyperplane in $A$ and is called the $Z^*$-transversal of $P$. The $\Sigma$-pole of this hyperplane is called the $Z^*$-correspondent of $P$, cf. [6]. It makes sense to choose the centroid $G$ of $\Sigma$ as the $P^*$-correspondent of $P$.

Let us denote the $Z^*$-correspondent of $P$ by $P#Z^*$, then

$$P#Z^* = \left[ \frac{p_1}{z_1^*} : \cdots : \frac{p_{n+1}}{z_{n+1}^*} \right] \quad (\ast\ast\ast).$$

**Proof.** Here only an outline of a proof of this equation is presented; a more detailed proof is given in [6].

We assign to each point $R \in A$ and each hyperplane $h$ not passing through $P$ a real number
\[ \lambda_p = \lambda_p(R, h) \] by
\[ \lambda_p(R, h) = 0, \text{ if } R = P \text{ or if the line } P \lor R \text{ does not meet } h \]
and
\[ \lambda_p(R, h) = \frac{1}{t}, \text{ if the point } P + t(R - P) \text{ is the intersection of } P \lor R \text{ and } h. \]

Some properties of \( \lambda_p \) are listed below:
- For any points \( R_1, R_2 \), for any real numbers \( t_1, t_2 \) and for a hyperplane \( h \) of \( (A) \),
  \[ \lambda_p(t_1 R_1 + t_2 R_2, h) = t_1 \lambda_p(R_1, h) + t_2 \lambda_p(R_1, h). \]
- A point \( R \) lies on a hyperplane \( h \) if and only if \( \lambda_p(R, h) = 1 \).
- Let us denote the \((n-1)\)-sideplane of \( \Sigma \) opposite of the vertex \( A_i \) by \( h_i, i = 1, \ldots, n+1 \). If \( P = [p_1, \ldots, p_{n+1}] \), then \( \lambda_p(A_i, h_i) = 1 - 1/p_i \). It can be easily checked that the \( i \)-th barycentric coordinate of the point \( P + \frac{p_i}{p_{i+1}}(A_i - P) \) is zero.

If \( \psi \) is the mapping that assigns to each point \( R \) its \( S \)-polar hyperplane, then
\[ \psi(A^*_i) = h_i = \{ R \mid \lambda(R, h_i) = 1 \}, \]
\[ \psi(z^*_i A^*_i) = \{ R \mid z^*_i \lambda_p(R, h_i) = 1 \} \]
and \( \psi(\sum_i z^*_i A^*_i) = \{ R \mid \sum_i z^*_i \lambda_p(R, h_i) = 1 \} \).

We calculate the intersection of a sideline \( A_i \lor A_j \) of \( \Sigma \) with the hyperplane \( \psi(Z^*) \). For simplicity, we take \((i, j) = (1, 2)\) and determine the real number \( x \) such that
\[ \lambda_p(xA_1 + (1-x)A_2, \psi(Z^*)) = 1. \]

\[ 1 = \lambda_p(xA_1 + (1-x)A_2, \psi(Z^*)) = \frac{p_1}{z^*_1} + (1-x)\frac{p_2}{z^*_2} \]
and \( 1-x = -\frac{p_2/z^*_2}{p_1/z^*_1 - p_2/z^*_2}. \)

In the following we are particularly interested in the \( Z^* \)-correspondents for \( Z^* = G^* \)
(centroid of \( \Sigma^* \)), \( Z^* = I^* \) (incenter of \( \Sigma^* \)) and \( Q^* = K^* \) (symmedian of \( \Sigma^* \)):
- \( P\#G^* = P \)
- \( P\#I^* = [\frac{p_1}{a_i^*} : \cdots : \frac{p_{n+1}}{a_i^*}] = [\frac{\text{sgn}(p_i)}{d(P, A_i)} : \cdots : \frac{\text{sgn}(p_{n+1})}{d(P, A_{n+1})}], a_i^* = (n-1)\text{-volume of the facet of } \Sigma^* \text{ opposite } A_i^*. \)

Explanation: \( p_j = p_j^* = \text{sgn}(p_j)\frac{a_j^*}{d(P, A_j)} \)
- \( P\#K^* = [\frac{p_1}{(a_i^*)^2} : \cdots : \frac{p_{n+1}}{(a_i^*)^2}] = \frac{1}{p_1(d(P, A_1)^2)} : \cdots : \frac{1}{p_{n+1}(d(P, A_{n+1})^2)}. \)

For a point \( P = [p_1, \ldots, p_{n+1}] \) not on any sideline of \( \Sigma \) consider the sequences \((Q_i)_{i \in \mathbb{N}}\)
and \((R_i)_{i \in \mathbb{N}}\) given by \( Q_0 = R_0 = P \) and \( Q_{i+1} = \frac{1}{d(Q_i; A_i)} : \cdots : \frac{1}{d(Q_i; A_{n+1})} \)
\[ R_{i+1} = \frac{|p_i|d(R_i; A_1)^2}{|p_{n+1}|d(R_i; A_{n+1})^2}. \]
All points \( Q_i, R_i, i > 0 \), are points inside \( \Sigma \), and \( Q_{i+1} = Q_i P#I^*, R_{i+1} = R_i P#K^*. \) For
Figure 3. The first points of the sequences \((Q_i)\subseteq\mathbb{N}\) and \((R_i)\subseteq\mathbb{N}\) with \(Q_0 = R_0 = P\) and \(Q_{i+1} = Q^* \# I^*, R_{i+1} = R^* \# K^*\) are shown. \(F_1\) and \(F_2\) are the two isogonic points.

\(Q_1\) and \(R_1\) we have \(Q_1 = P^*Z^*\) with \(Z^* = [\text{sgn}(p_1) a_1^* : \ldots : \text{sgn}(p_{n+1}) a_{n+1}^*]\) and \(R_1 = P^*Z^*\) with \(Z^* = [\text{sgn}(p_1)(a_1^*)^2 : \ldots : \text{sgn}(p_{n+1})(a_{n+1}^*)^2]\). Both sequences converge to the Fermat-Torricelli point \(F\), even if this is a vertex of \(\Sigma\). It seems that the second sequence in comparison to the first converges twice as fast (see Figure 3). But more importantly, the second sequence avoids calculating square roots. If the Fermat-Torricelli point of a simplex lies inside the simplex, then it is an isogonic point.

6. A 3-simplex with five isogonic points.

In dimension \(n > 2\), simplices, in general, have more than two isogonic points. We present a 3-simplex with five isogonic points. Consider in the Euclidean space \((\mathbb{R}^4, \cdot)\) the simplex \(\Sigma\) with the four vertices

\[
A_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 \\ 8 \\ 0 \\ 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 6 \end{pmatrix}.
\]

In order to calculate the isogonic points, we first calculate their isogonally conjugated points. These are the points with an equiareal pedal simplex. For a point \(P\) which is not a vertex of \(\Sigma\) let \(\Sigma_P\) denote the pedal simplex of \(P\) and let \(G_P\) and \(I_P\) denote the centroid and the incenter of \(\Sigma_P\), respectively. We define a sequence of points \((P_i)\subseteq\mathbb{N}\) by \(P_0 := P\) and \(P_{i+1} := P_n + G_P - I_P\). Obviously, this sequence is a constant sequence if and only if \(P\) is the isogonal conjugate of an isogonic point. Experiments with the interactive geometry tool GeoGebra [25] suggest that each of these sequences converges to one of these limit

Finally, we present the two isodynamic points of $\Sigma$ points (rounded to 12 decimal places):

\[
\begin{align*}
L_0 &= [0.266996565955, 0.275481800939, 0.217355830792, 0.240165802314], \\
L_1 &= [-4.180629474014, 2.569387212447, 1.60211303329, 1.009129223238], \\
L_2 &= [1.193250865914, -1.252645952150, 0.354761022780, 0.704634063455], \\
L_3 &= [0.713260932730, 0.358215195120, -0.616627271982, 0.545151144132], \\
L_4 &= [0.657546390333, 0.802131717931, 0.639088262811, -1.098766371077].
\end{align*}
\]

The pedal triangles of these points have facets with an area

\[
\begin{align*}
a_0 &= 2.404772767371, \\
a_1 &= 122.12536031480, \\
a_2 &= 19.392997370805, \\
a_3 &= 9.848601171111, \\
a_4 &= 18.965046082427.
\end{align*}
\]

The isogonic points of $\Sigma$ are

\[
\begin{align*}
F_0 &= [0.369979160947, 0.229493298326, 0.163611619856, 0.236915925371], \\
F_1 &= [-0.297000489955, 0.309278164652, 0.279002561033, 0.708719744707], \\
F_2 &= [0.388102931405, -0.236608485604, 0.469943106828, 0.378562447371], \\
F_3 &= [0.382915343108, 0.487963317698, -0.159452369671, 0.288573708865], \\
F_4 &= [0.645021938255, 0.338403751068, 0.238914519123, -0.222340208446].
\end{align*}
\]

The antipedal triangles of these points have facets with an area

\[
\begin{align*}
\tilde{a}_0 &= 241.637142362610, \\
\tilde{a}_1 &= 60.087819904352, \\
\tilde{a}_2 &= 31.387257487815, \\
\tilde{a}_3 &= 5.647726265255, \\
\tilde{a}_4 &= 31.003305976553.
\end{align*}
\]

Finally, we present the two isodynamic points of $\Sigma$:

\[
\begin{align*}
J_1 &= [0.206439675828, 0.327649375007, 0.263085414624, 0.202825534542], \\
J_2 &= [2.954833710960, -0.575606610593, -1.403778427224, 0.024551326857].
\end{align*}
\]

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