We identify multiresolution subspaces giving rise via Hankel transforms to Bessel functions. They emerge as orthogonal systems derived from geometric Hilbert-space considerations, the same way the wavelet functions from a multiresolution scaling wavelet construction arise from a scale of Hilbert spaces. We study the theory of representations of the $C^*$-algebra $O_{\nu+1}$ arising from this multiresolution analysis.

I. INTRODUCTION

The starting point for the multiresolution analysis from wavelet theory is a system $U, \{T_j\}_{j \in \mathbb{Z}}$, of unitary operators with the property that the underlying Hilbert space $\mathcal{H}$ contains a vector $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, satisfying

$$U \varphi = \sum_j a_j T_j \varphi$$

(1)

for some sequence $\{a_j\}$ of scalars. In addition, the operator system $\{U, T_j\}$ must satisfy a non-trivial commutation relation. In the case of wavelets, it is

$$UT_j U^{-1} = T_{NJ}, \quad j \in \mathbb{Z},$$

(2)

where $N$ is the scaling number, or equivalently the number of subbands in the corresponding multiresolution. When this structure is present, there is a way to recover the spectral theory of the problem at hand from representations of an associated $C^*$-algebra. In the case of orthogonal wavelets, we may take this $C^*$-algebra to be the Cuntz algebra. In that case, the operators $T_j$ may be represented on $L^2(\mathbb{R})$ as translations,

$$(T_j \xi)(x) = \xi(x-j), \quad \xi \in L^2(\mathbb{R}),$$

and $U$ may be taken as scaling $U \xi = N^{-1/2} \xi(x/N)$. This system clearly satisfies (3). (For a variety of other examples of this, the reader is referred to Ref. 1. The setup there applies to dynamical systems of $N$-to-1 self-maps: for example, those of complex dynamics and Julia sets.) In the wavelet case, a multiresolution is built from a solution $\varphi \in L^2(\mathbb{R})$ to the scaling identity (1). The numbers $\{a_j\}_{j \in \mathbb{Z}}$ from (1) must then satisfy the orthogonality relations

$$\sum_k a_k = \sqrt{N}, \quad \sum_k \bar{a}_k a_{k+2m} = \delta_{0,m}. $$

(3)

In this case, the analysis is based on the Fourier transform: Take

$$m_0(e^{it}) = \sum_k a_k e^{ikt}. $$

(4)

Then (in the wavelet case, following Ref. 2) a solution to (1) will have a product formula

$$\hat{\varphi}(t) = \prod_{j=1}^{\infty} m_0(t/N^j), $$

(5)

up to a constant multiple. The Cuntz algebra $O_N$ enters the picture as follows: Formula (3) is not practical for computations, and the analysis of orthogonality relations is done better by reference to the Cuntz relations, see (11)–(12) below. Setting
\[ W (\{ \xi_j \}) := \sum_j \xi_j \varphi (x - j), \]  

and using (3), we get an isometry \( W \) of \( \ell^2 \) into a subspace of \( L^2 (\mathbb{R}) \), the resolution subspace. Setting

\[
(S_0 f) (z) := \sqrt{N} m_0 (z) f (z^N), \quad f \in L^2 (\mathbb{T}),
\]

and using \( L^2 (\mathbb{T}) \cong \ell^2 \) by the Fourier series, we establish the following crucial intertwining identity:

\[ WS_0 = UW, \]

so that \( U \) is a unitary extension of the isometry \( S_0 \). We showed in Refs. 3 and 8 that functions \( m_1, \ldots, m_{N-1} \in L^\infty (\mathbb{T}) \) may then be chosen such that the corresponding matrix

\[
\left( m_j \left( e^{i(t+k\pi/N)} \right) \right)_{j,k=0}^{N-1}
\]

is in \( U_N (\mathbb{C}) \) for a.a. \( t \). Then it follows that the operators

\[ S_j f (z) := \sqrt{N} m_j (z) f (z^N), \quad f \in L^2 (\mathbb{T}), \]

will yield a representation of the Cuntz relations; see (11)–(12) below. Conversely, if (10) is given to satisfy the Cuntz relations, then the matrix in (8) takes values in \( U_N (\mathbb{C}) \).

The present paper aims at an analogous construction, but based instead on the Bessel functions, i.e., we use the Bessel functions in (4) in place of the usual Fourier basis \( \{ e^{ikt} \}_{k \in \mathbb{Z}} \); see (20) below. If \( \nu \) is the parameter of the Bessel function \( J_\nu \), then we show that \( N = \nu + 1 \) is an admissible scaling for a multiresolution construction.

The motivation for doing a multiresolution construction based on a wider variety of special functions, other than the Fourier basis, derives in part from the rather restrictive axiom system dictated by the traditional setting. It is known that many applications require a more general mathematical setup. Moreover, our present approach also throws light on special-function theory, and may be of independent interest for that reason.

We will apply multiresolutions to the Hankel transform and the Bessel functions of parameter \( \nu \). Our analysis is especially well suited for the introduction of a quantum variable \( q \) in such a way that variations in \( q \) lead to a better understanding of an associated family of deformations. Our use of the Cuntz algebra is motivated by Refs. 3 and 5.

The Cuntz algebras have been used independently in operator algebra theory and in the study of multiresolution wavelets, and our present paper aims to both make this connection explicit, and as well make use of it in the analysis of special functions. The \( q \)-deformations of the special functions may be of independent interest. This deformation is related to, but different from, those which have appeared in Refs. 12–16.

II. THE CUNTZ ALGEBRA AND ITERATED FUNCTION SYSTEMS

We now consider representations \( \pi \) of the Cuntz algebra \( O_{\nu+1} \) coming from multiresolution analysis based on Hankel transforms. In Section II we give some preliminaries on Hankel transforms on \( L^2 (\mathbb{R}) \). We then construct wavelets arising from multiresolutions with scaling \( \nu + 1 \) using Hankel transforms on \( L^2 (\mathbb{C}) \), relative to an appropriate measure on \( \mathbb{C} \). The map from wavelets into representations is described. We establish connections between certain representations of \( O_{\nu+1} \) and Hankel wavelets arising from that multiresolution analysis.

Recall that \( O_{\nu+1} \) is the \( C^* \)-algebra generated by \( \nu + 1, \nu \in \mathbb{N} \), isometries \( S_0, \ldots, S_\nu \) satisfying

\[
S_i^* S_j = \delta_{ij} 1
\]

and

\[
\sum_{i=0}^\nu S_i S_i^* = 1.
\]

The representations we will consider are realized on Hilbert spaces \( H = L^2 (\Omega, d\mu) \) where \( \Omega \) is a measure space (to be specified below) and \( \mu \) is a probability measure on \( \Omega \).

We define the representations in terms of certain maps...
We will apply this in Section V to the Riemann surface of \( \sqrt{z} \).

In Section IV we develop a \( q \)-parametric multiresolution wavelet analysis in \( L^2(\mathbb{C}, d\mu_q) \) where \( d\mu_q \) is a \( q \)-measure, as in Ref. [16] by using \( q \)-Hankel transforms.

A class of \( q \)-parametric representations of the \( C^* \)-algebra \( O_{\nu+1} \) is found. We further identify a class of representations of the Cuntz algebra which has the structure of compact quantum groups of type A [17].

### III. Hankel Transforms and a Multiresolution Analysis

In this section we construct a multiresolution using Hankel transforms. We start by giving some basic definitions on Hankel transforms.

Let us recall that the Hankel transform of order \( \alpha \) of a function \( f \), denoted by \( \hat{f} \), is defined by

\[
\hat{f}(t) = \int_0^\infty J_\alpha(xt) f(x) \, dx \tag{14}
\]

where

\[
J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\alpha+k+1)} \left(\frac{x}{2}\right)^{2k}
\]

is the Bessel function of order \( \alpha \) and 

\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt, \quad \text{Re}(z) > 0.
\]

If we multiply both sides of (14) by \( J_\alpha(yt) \, t \) and integrate from \( t = 0 \) to \( \infty \) we obtain

\[
\int_0^\infty J_\alpha(yt) \hat{f}(t) \, dt = f(y) = \int_0^\infty J_\alpha(yt) t \int_0^\infty J_\alpha(xt) f(x) \, dx \, dt. \tag{15}
\]

The integral transform on the left-hand side of (15) is equal to \( f(y) \) for suitable functions \( f \), by the Hankel inversion theorem [18]. The resulting double integral is called the Hankel Fourier-Bessel integral

\[
f(y) = \int_0^\infty J_\alpha(yt) \left(\int_0^\infty J_\alpha(xt) f(x) \, dx\right) t \, dt. \tag{16}
\]

It can be written as the following transform pair

\[
g(t) = \int_0^\infty J_\alpha(yt) f(y) y \, dy, \tag{17}
\]

\[
f(y) = \int_0^\infty J_\alpha(yt) g(t) \, dt.
\]

A Plancherel type result can be easily derived for this transform: if \( F(\rho) \) and \( G(\rho) \) are Hankel transforms of \( f(x) \) and \( g(x) \) respectively, then we have

\[
\int_0^\infty \rho F(\rho) G(\rho) \, d\rho = \int_0^\infty \rho F(\rho) \int_0^\infty x g(x) J_\nu(\rho x) \, dx \, d\rho
\]

\[
= \int_0^\infty x g(x) \left(\int_0^\infty \rho F(\rho) J_\nu(\rho x) \, d\rho\right) \, dx = \int_0^\infty x f(x) g(x) \, dx.
\]

Let us give some preliminaries on the standard multiresolution wavelet analysis of scale \( \nu, \nu \in \mathbb{Z} \). Following Refs. [2,19] we define scaling by \( \nu \) on \( L^2(\mathbb{R}) \) by

\[
(U \xi)(x) = (\nu + 1)^{-\frac{\nu}{2}} \xi \left(\frac{x}{\nu + 1}\right)
\]
and translation by 1 on $L^2(\mathbb{R})$ by

$$(T\xi)(x) = \xi(x - 1).$$

As mentioned in the Introduction, it is our aim here to adapt the theory of multiresolutions from wavelet theory to the analysis of the Bessel functions via the Hankel transform. The classical theory is based on recurrence algorithms which we show adapt very naturally to the multiresolutions. But our analysis will still be based on the “classical” identities for the special functions (see, e.g., Refs. 22–27).

A scaling function is a function $\varphi \in L^2(\mathbb{R})$ such that if $V_0$ is the closed linear span of all translates $T^k\varphi$, $k \in \mathbb{Z}$, then $\varphi$ has the following four properties

i) $\{T^k\varphi : k \in \mathbb{Z}\}$ is an orthonormal set in $L^2(\mathbb{R})$;

ii) $U\varphi \in V_0$;

iii) $\bigwedge_{n \in \mathbb{Z}} U^n V_0 = \{0\}$;

iv) $\bigvee_{n \in \mathbb{Z}} U^n V_0 = L^2(\mathbb{R})$.

The simplest example of a scaling function is the characteristic function of the interval $[0,1]$, i.e., the Haar function. By i) we may define an isometry

$$F_\varphi : V_0 \longrightarrow L^2(\mathbb{R}), \quad \xi \longmapsto m,$$

as follows. The scaling by $\nu$ on $L^2(\mathbb{R})$ is defined by the unitary operator $U$ given by $(U\xi)(x) = (\nu + 1)^{-\frac{1}{2}} \xi\left((\nu + 1)^{-1} x\right)$ for $\xi \in L^2(\mathbb{R})$, $x \in \mathbb{R}$, and the translation as the following operator $(T\xi)(x) = \xi(x - 1)$.

We consider the scaling Haar function $\varphi$ given as the sum $\varphi(x) = h(1 - x) - h(-x)$ of Heaviside functions $h$.

Let $V_0$ be the linear span of $\{\varphi^{(k)}(x) \equiv x^k \varphi(x - k)\}_{k \in \mathbb{Z}}$. Then $V_0$ is a closed subspace of $L^2(\mathbb{R})$ with respect to the following scalar product: $\langle f \mid g \rangle = \int f(x)g(x) \, dx$. Then $V_0$ is spanned by $\{x^\nu \varphi(x - k)\}_{k \in \mathbb{Z}}$. Also we have $\bigcap_{n \in \mathbb{Z}} U^n V_0 = \{0\}$ and $\bigvee U^n V_0 = L^2(\mathbb{R})$.

Let $\xi \in L^2(\mathbb{R})$, and assume that $\xi(x) = \sum b_k \varphi(x - k) x^\nu$.

By applying the Hankel transform $H_\nu(\cdot,t)$ to both sides of the above equality we get:

$$H_\nu(\xi(x),t) = \sum_{k} b_k H_\nu (\varphi(x - k) x^\nu, t)$$

$$= \sum_{k} b_k H_\nu (h(k+1-x) x^\nu, t) - \sum_{k} b_k H_\nu (h(k-x) x^\nu, t)$$

$$= \left[ \sum_{k} b_k (k+1)^{\nu+1} J_{\nu+1} (t(k+1)) \right] - \sum_{k} b_k k^{\nu+1} J_{\nu+1} (kt) \cdot H_0 \left( \frac{1}{x}, t \right).$$

To write the above expression in a more compact form we use the addition formula for Bessel functions

$$J_n(x + y) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}(y).$$

Then we get

$$H_\nu(\xi(x),t) = \sum_{k} b_k \left[ (k+1)^{\nu+1} J_{\nu+1} (kt + t) - k^{\nu} J_{\nu+1} (kt) \right] H_0 \left( \frac{1}{x}, t \right)$$

$$= \sum_{k} b_k \left[ (k+1)^{\nu+1} \sum_{h} J_h(tk) J_{\nu+1-h}(t) - k^{\nu} J_{\nu+1}(kt) \right] H_0 \left( \frac{1}{x}, t \right).$$

Define

$$m_0(t) = \sum_{k} b_k \left[ (k+1)^{\nu+1} \sum_{h} J_h(tk) J_{\nu+1-h}(t) - k^{\nu} J_{\nu+1}(kt) \right].$$

$$m_0(t) = \sum_{k} b_k \left[ (k+1)^{\nu+1} \sum_{h} J_h(tk) J_{\nu+1-h}(t) - k^{\nu} J_{\nu+1}(kt) \right].$$
Here we have considered \( L^2(\mathbb{R}, \mu) \) with \( d\mu(x) = x \, dx \). By using the Plancherel Theorem, and the orthogonality of the Haar functions, we get

\[
\frac{\delta_{k,0}}{2(\nu + 1)} = \int_0^\infty \varphi^{(k)}(x) \varphi^{(0)}(x) \, x \, dx
\]

\[
= \int_0^\infty \left[ H_\nu(x^{\nu} [h(1 + x) - h(-x)]), t) \right] H_\nu(x^{\nu} [h(1 - x) - h(-x)], t) \, t \, dt
\]

\[
= \sum_{j \in \mathbb{Z}} \int_{j}^{j+1} \left[ H_\nu(x^{\nu}[h(k + 1 - x) - h(k - x)], t) \right] H_\nu(x^{\nu}[h(1 - x) - h(-x)], t) \, t \, dt.
\]

Thus the latter, upon a change of variables, can be rewritten as

\[
\int_0^1 \sum_{j \in \mathbb{Z}} [H_\nu(x^{\nu}[h(k + 1 - x) - h(k - x)], t) \right] H_\nu(x^{\nu}[h(1 - x) - h(-x)], t) \, (t + j) \, dt.
\]

We used the following obvious fact:

\[
\frac{1}{2(\nu + 1)} = \int_0^1 \frac{dt}{2(\nu + 1)}.
\]

On comparing (23) and (22) for \( k = 0 \), we get

\[
\sum_{j \in \mathbb{Z}} H^2_\nu(x^{\nu}[h(1 - x) - h(-x)], t) - \frac{1}{2(\nu + 1)} = 0, \quad \text{a.e.}
\]

On the other hand, in view of (21), (20), (19) and (18) the left-hand side of this equality can be rewritten in terms of \( m_0 \) as follows

\[
\sum_{j \in \mathbb{Z}} \left|m_0(t + j)\right|^2 \left|H_0(1/z, t + j)\right|^2 = \frac{1}{2(\nu + 1)}, \quad \text{a.e.}
\]

To get a direct connection with representations of \( O_{\nu+1} \), we need to consider our new multiresolutions on the complex plane \( \mathbb{C} \). Assume \( \varphi \) to be a step function on \( \mathbb{C} \), defined for \( |z| \leq 1 \) by

\[
\varphi \left( |z| e^{i \text{Arg}(z)} \right) = \begin{cases} 1 & \text{if } 0 \leq \text{Arg}(z) \leq \alpha, \\ 0 & \text{otherwise,} \end{cases}
\]

where \( \alpha = \frac{2\pi}{m} \). Take then \( V_0 \) to be the span of \( \{ \varphi \left( |z| + k \right) \exp(i \text{Arg}(z) + N\alpha) \} \), with \( k, m \in \mathbb{Z}, 1 \leq N \leq m \). Let

\[
U \xi(z) = (\nu + 1)^{-1/2} \xi \left( \frac{z}{\nu + 1} \right)
\]

be the scaling operator. Let

\[
V_j = \text{closed span} \left\{ \varphi \left[ \left( \frac{|z|}{(\nu + 1)^j} + k \right) e^{i \text{Arg}(z) + N\alpha} \right] \right\}_{k \in \mathbb{Z}, 1 \leq N \leq m}.
\]

Consider \( L^2(\mathbb{C}, d\mu) \) where the measure \( d\mu(z) = z^\nu \, dz \), and \( dz \) denotes the planar measure on \( \mathbb{C} \). Assume \( U \varphi \in V_0 \), i.e.,

\[
(U \varphi)(z) = \sum_k a_k \varphi \left( |z| + k \right) \exp(i \text{Arg}(z) + N\alpha). \]

**Proposition 1** With the assumptions above, we conclude that the properties i)–iv) of a multiresolution are satisfied.
Proof. i) follows from the fact that the \( \varphi \)'s have disjoint support on \( L^2(\mathbb{C},d\mu) \). ii) holds for Haar functions and iii) follows from i). By the density of step functions on \( L^2(\mathbb{C},d\mu) \) also iv) follows.

If \( \xi \in V_{-j} \) then \( U^j \xi \in V_0 \). Since

\[
\varphi \in V_0 \subset V_{-1} \quad \text{and} \quad \left\{ \varphi \left[ \frac{|z|}{\nu+1} + k \right] \exp \left( i (\text{Arg}(z) + N\alpha) \right) \right\}
\]

are orthonormal in \( V_{-1} \), we have

\[
\varphi(z) = \sum_k a_k \varphi \left[ \frac{|z|}{\nu+1} + k \right] \exp \left( i (\text{Arg}(z) + N\alpha) \right),
\]

so by applying the Hankel transform of order \( \nu \) we get

\[
H_\nu \left( \varphi \left[ \frac{|z|}{\nu+1} + k \right] \exp \left( i (\text{Arg}(z) + N\alpha) \right) ; t \right) = m_0(t) H_0 \left( \frac{1}{z} ; t \right).
\]

Using the orthogonality of \( \varphi [(|z|+k) \exp \left( i (\text{Arg}(z) + N\alpha) \right)]_{k \in \mathbb{Z}} \) we have

\[
\langle \varphi^{(k,N)} , \varphi^{(0,0)} \rangle = \int_{\mathbb{C}} \varphi^{(k,N)}(z) \overline{\varphi^{(0,0)}(z)} \, d\mu(z)
\]

\[
= \int_0^{2\pi} \int_0^{\infty} \varphi [(|z|+k) \exp \left( i (\text{Arg}(z) + N\alpha) \right)] \overline{\varphi [\, |z| \exp \left( i \text{Arg}(z) \right)]} \times |z|^{\nu+1} \exp \left( i \text{Arg}(z) (\nu + 1) \right) d|z| \, d\text{Arg}(z)
\]

\[
= \int_0^1 |z|^{\nu+1} \int_0^{\pi} \exp \left( i \text{Arg}(z) (\nu + 1) \right) \delta_{N,0} \, d|z| \int_0^{\pi} \delta_{N,0} \, d\text{Arg}(z)
\]

\[
= \frac{1}{\nu + 2} \frac{\delta_{k,0} e^{i\alpha(\nu+1)} - 1}{i(\nu+1)} \delta_{N,0}.
\]

By the Plancherel theorem, we then have

\[
\frac{1}{\nu + 2} \frac{\delta_{k,0} e^{i\alpha(\nu+1)} - 1}{i(\nu+1)} \delta_{N,0} = \int_{\mathbb{C}} H_\nu \left( \varphi^{(k,N)}(z) ; t \right) \overline{H_\nu \left( \varphi^{(0,0)}(z) ; t \right)} \, d\mu(t).
\]

The left-hand side can then be rewritten as

\[
\int_0^1 \int_0^{\alpha} H_\nu \left( \varphi^{(k,N)}(z) ; t \right) \overline{H_\nu \left( \varphi^{(0,0)}(z) ; t \right)} \, dt \, d\text{Arg}(t).
\]

Upon a change of variable letting \( \theta = \text{Arg}(t+2\pi j) \), the latter equals

\[
\int_0^1 |t|^{\nu+1} \, dt \int_0^{\alpha} e^{i\theta(\nu+1)} \sum_j H_\nu \left( \varphi^{(k,N)}(z) ; |t| e^{i\theta} \right) \overline{H_\nu \left( \varphi^{(0,0)}(z) ; |t| e^{i\theta} \right)} \, d\theta.
\]

Comparing the previous two formulae for \( k = N = 0 \) we get

\[
\sum_j \left| H_\nu \left( \varphi^{(0,0)} ; |t| e^{i\theta} \right) \right|^2 = \frac{1}{\nu + 2} \frac{e^{i\alpha(\nu+1)} - 1}{i(\nu+1)} = 0 \quad \text{a.e.}
\]

Rewriting the above in terms of \( m_0 \) we have

\[
\sum_j \left| m_0 \left( te^{2\pi ij} \right) \right|^2 \left| H_0 \left( \frac{1}{z} ; |t| e^{i\theta} \right) \right|^2 = \frac{1}{\nu + 2} \frac{e^{i\alpha(\nu+1)} - 1}{i(\nu+1)},
\]

since
From (25) we get

\[
\int_0^1 |t|^{\nu+1} \, d|t| \int_0^\alpha e^{i\theta(t^{\nu+1})} \sum_j m_0 \left( t e^{2\pi ij} \right) |^2 \left| H_0 \left( \frac{1}{z}; t e^{i\theta} \right) \right|^2 \, d\theta = \int_0^1 |t|^{\nu+1} \, d|t| \int_0^\alpha e^{i\theta(t^{\nu+1})} \sum_j m_0 \left( t e^{2\pi ij} \right) \frac{1}{|t|} \, d\theta
\]

\[
= \int_0^1 |t|^{-\nu-1} \, d|t| \int_0^\alpha e^{i\theta(t^{\nu+1})} \sum_j m_0 \left( t e^{2\pi ij} \right) |^2 \, d\theta.
\]

From (25) we get

\[
\int_0^1 |t|^{-\nu-1} \, d|t| \int_0^\alpha e^{i\theta(t^{\nu+1})} \sum_j m_0 \left( t e^{2\pi ij} \right) |^2 \, d\theta = \frac{1}{\nu + 2} \delta_{k,0} e^{i\alpha(t^{\nu+1})} - \frac{1}{i (\nu + 1)} \delta_{N,0}
\]

\[
= \frac{\nu}{\nu + 2} \int_0^1 |t|^{-\nu-1} \, d|t| \int_0^\alpha e^{i\theta(t^{\nu+1})} \, d\theta.
\]

Thus

\[
\left( \frac{1}{\nu + 1} \right) \sum_j m_0 \left( t e^{2\pi ij/(\nu+1)} \right) |^2 = \frac{\nu}{\nu + 2}.
\]

Set \( c = \frac{\nu}{\nu + 2} \); thus we have \( \left( \frac{1}{c(\nu+1)} \right) \sum_j m_0 \left( t e^{2\pi ij/(\nu+1)} \right) |^2 = 1 \).

Then in fact, as in Ref. 3 Thm. 5.1.1, we have proved the following result.

**Theorem 2** If the ladder of the closed subspaces \( \{V_j\}_{j \in \mathbb{Z}} \) in \( L^2(\mathbb{C}) \) satisfy properties i)–iv), then there exists an associated orthonormal wavelet basis \( \{\psi_{jk} : j, k \in \mathbb{Z}\} \) for \( L^2(\mathbb{C}) \) such that

\[
(U \varphi)(z) = \sum_k a_k \varphi([(z) + k]) \exp\{i(\text{Arg}(z) + N\alpha)\}
\]

holds. One possibility for the wavelet is \( \psi \) corresponding to \( \varphi \) such that

\[
H_\nu \left( \varphi \left( [(z) + k]) \exp\{i(\text{Arg}(z) + N\alpha)\} \right); (\nu + 1) t \right) = m_0 (t) H_0 \left( \frac{1}{z}; t \right)
\]

is satisfied.

**Completion of proof.** Observe that the Bessel functions have a “multiplicative periodicity” on the unit circle in the following sense:

\[
J_\nu \left( z e^{\pi ik} \right) = e^{\pi ik} J_\nu (z)
\]

From the above (25), this implies that

\[
c^{-1} \sum_{j=0}^\nu m_0 \left( z e^{2\pi ij/(\nu+1)} \right) |^2 = (\nu + 1).
\]

Given \( m_0 \) satisfying (26) there exists \( \{m_i, i = 1, \ldots, \nu\} \) from Corollary 4.2 of Ref. 4 such that

\[
\sum_{j=0}^\nu c^{-1} m_k \left( (\nu + 1) \right) m_{k'} \left( (\nu + 1) \right) = \delta_{kk'} (\nu + 1).
\]

Thus, reformulating the orthogonality conditions, we get that the following matrix,

\[
M(z) = \frac{1}{\sqrt{c (\nu + 1)}} \begin{pmatrix}
    m_0 (\sigma_0 (z)) & m_0 (\sigma_1 (z)) & \ldots & m_0 (\sigma_\nu (z)) \\
    m_1 (\sigma_0 (z)) & m_1 (\sigma_1 (z)) & \ldots & m_1 (\sigma_\nu (z)) \\
    \vdots & \vdots & & \vdots \\
    m_\nu (\sigma_0 (z)) & m_\nu (\sigma_1 (z)) & \ldots & m_\nu (\sigma_\nu (z))
\end{pmatrix},
\]
is unitary for almost all $z \in \mathbb{C}$.

Let $O_{\nu+1}$ be the $C^*$-algebra generated by $\nu + 1$ isometries $S_0, S_1, \ldots, S_\nu$ satisfying:

$$S_i^* S_j = \delta_{i,j} 1, \quad \sum_{i=0}^{\nu} S_i S_i^* = 1.$$ 

The representations we consider are now realized on the Hilbert space $H = L^2(\mathbb{C}, d\mu)$ where the measure $\mu$ is given by $d\mu(z) = z^{\nu} dz$.

As in Ref. 19 the representation of the Cuntz algebra is defined in terms of certain maps $\sigma_i: \Omega \rightarrow \Omega$, as in (13), such that $\mu(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0$ for $i \neq j$, and of measurable functions $m_0, \ldots, m_{\nu}: \mathbb{C} \rightarrow \mathbb{C}$. Also

$$\int f(z) \, d\mu(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} \rho_r \int f(\sigma_r(z)) \, d\mu(z), \quad (27)$$

where $\{\rho_r\}$ is a (finite) probability distribution on the cyclic group $\mathbb{Z}_{\nu+1}$.

The representations take the following form on $L^2(\mathbb{C}, d\mu)$:

$$(S_k \xi)(z) = m_k(z) \xi(z^{\nu+1}),$$

where the functions $m_k$ are obtained from the above multiresolution construction. It is easy to verify that it is a representation of $O_{\nu+1}$, and that

$$(S_k^* S_k \xi)(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} c^{-1} \rho_r m_k(\sigma_r(z)) \xi(\sigma_r(z)).$$

Thus:

$$(S_k^* S_k \xi)(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} c^{-1} \rho_r m_k(\sigma_r(z)) m_k'(\sigma_r(z)) \xi(\sigma_r(z))$$

$$= \delta_{k,k'} \xi(z),$$

by the unitarity of the matrix $M(z)$. Similarly we may verify that

$$\sum_{k \in \mathbb{Z}_{\nu+1}} (S_k S_k^* \xi)(z) = \xi(z).$$

As a result, we then have a representation of $O_{\nu+1}$. \hfill \Box

**IV. A q-PARAMETRIC CONSTRUCTION OF $m_0$**

Let us now turn to a $q$-parametric construction of $m_0$. We start by giving a $q$-extension of the Hankel Fourier-Bessel integral. We use the orthogonality relations from the following result (Theorem 3.10, p. 35 of Ref. 14).

**Theorem 3** For $|x| < q^{-1/2}$, $n, m \in \mathbb{Z}$, we have

$$\delta_{m,n} = \sum_{k=-\infty}^{\infty} x^{k+n} q^{\frac{k+n}{2}} (x q^2; q^2)^{\infty} \Phi_{1,1} \left( \begin{array}{c} 0 \\ x^2 q \end{array} \bigg| q, q^{n+k+1} \right)$$

$$\times x^{k+m} q^{\frac{k+m}{2}} (x q^2; q^2)^{\infty} \Phi_{1,1} \left( \begin{array}{c} 0 \\ x^2 q \end{array} \bigg| q, q^{m+k+1} \right)$$

where the sum is absolutely convergent, uniformly on compact subsets of the open disk $|x| < q^{-1/2}$. 

8
We prove that the orthogonality relation of the above theorem is a \( q \)-analogue of the Hankel Fourier-Bessel integral \( [16] \). To simplify notations, replace \( q \) by \( q^2 \) and \( x \) by \( q^a \). For \( \Re(\alpha) > -1 \) this gives

\[
\delta_{m,n} = \sum_{k=-\infty}^{\infty} q^{(\alpha+1)(k+n)} \frac{(q^{2a+2}; q^2)_\infty}{(q^2; q^2)_\infty} \Phi_{1,1} \left( \begin{array}{c} 0 \\ q^{2a+2} \end{array} \right| q^2, q^{2a+2k+2} \right) \times q^{(\alpha+1)(k+n)} \frac{(q^{2a+2}; q^2)_\infty}{(q^2; q^2)_\infty} \Phi_{1,1} \left( \begin{array}{c} 0 \\ q^{2a+2} \end{array} \right| q^2, q^{2a+2k+2} \right).
\]

(28)

Now rewrite (28) as the transform pair

\[
g(q^n) = \sum_{k=-\infty}^{\infty} q^{(\alpha+1)(k+n)} \frac{(q^{2a+2}; q^2)_\infty}{(q^2; q^2)_\infty} \Phi_{1,1} \left( \begin{array}{c} 0 \\ q^{2a+2} \end{array} \right| q^2, q^{2a+2k+2} \right) f(q^k),
\]

\[
f(q^k) = \sum_{k=-\infty}^{\infty} q^{(\alpha+1)(k+n)} \frac{(q^{2a+2}; q^2)_\infty}{(q^2; q^2)_\infty} \Phi_{1,1} \left( \begin{array}{c} 0 \\ q^{2a+2} \end{array} \right| q^2, q^{2a+2k+2} \right) g(q^n),
\]

where \( f, g \) are \( L^2 \)-functions on the set \( \{ q^k : k \in \mathbb{Z} \} \) with respect to the counting measure. Insert now \( J_\alpha(x; q) \), i.e., the Hahn-Exton \( q \)-Bessel function given by

\[
J_\alpha(x; q) = \frac{(q^{a+1}; q)_\infty}{(q; q)_\infty} x^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)/2} x^{2k}}{(q^{a+1}; q)_k (q; q)_k} = \frac{(q^{a+1}; q)_\infty}{(q; q)_\infty} x^\alpha \Phi_{1,1} \left( \begin{array}{c} 0 \\ q^{a+1} \end{array} \right| q, x^2 q \right)
\]

and replace \( f(q^k) \) and \( g(q^n) \) by \( q^k f(q^k) \) and \( q^a g(q^n) \) respectively. This implies that \( xf(x) \) and \( xg(x) \) have to be \( L^2 \)-functions on the set \( \{ q^k : k \in \mathbb{Z} \} \). Hence we have

\[
g(q^n) = \sum_{k=-\infty}^{\infty} q^{2k} J_\alpha(q^{k+n}; q^2) f(q^k),
\]

(29)

\[
f(q^k) = \sum_{k=-\infty}^{\infty} q^{2n} J_\alpha(q^{k+n}; q^2) g(q^n),
\]

and the result follows.

**Remark 4** When \( q \to 1 \) with the condition

\[
\frac{\log(1-q)}{\log q} \in 2\mathbb{Z},
\]

we can replace \( q^k \) and \( q^n \) in (29) with \( (1-q)^{\frac{1}{2}} q^k \) and \( (1-q)^{\frac{1}{2}} q^n \) respectively. By using the following \( q \)-integral notation,\( [16] \)

\[
\int_0^\infty f(t) \, dt = (1-q) \sum_{k=-\infty}^{\infty} f(q^k) q^k,
\]

(30)

then (29) takes the form

\[
g(\lambda) = \int_0^\infty f(x) J_\alpha((1-q) \lambda x; q^2) \, dx\, q^x(x),
\]

\[
f(x) = \int_0^\infty g(\lambda) J_\alpha((1-q) \lambda x; q^2) \, \lambda \, d_\lambda(\lambda),
\]

where \( \lambda \) in the first identity, and \( x \) in the second identity, take the values \( q^n \), \( n \in \mathbb{Z} \). For \( q \to 1 \) we therefore obtain, at least formally, the Hankel transform pair
We construct a \( q \)-analogue of a multiresolution via \( q \)-Hankel transforms. To achieve that, let us proceed as we did in the previous section; but now we replace the Hankel transform by the deformed one using a \( q \)-measure. Let us consider as before the space \( L^2(\mathbb{C}) \), but with the measure \( d\mu(z) \) replaced by the \( q \)-measure \( d\mu_q(z) \), i.e., \( d\mu_q(z) = z^\nu \, d\mu(z) \). Assume \( \varphi \) to be the function on \( \mathbb{C} \) defined for \( |z| \leq 1 \) by

\[
\varphi(|z| \, e^{i \text{Arg}(z)}) = \begin{cases} 1 & \text{if } 0 \leq \text{Arg}(z) \leq \alpha, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( \alpha = \frac{2\pi}{m} \). Take then \( V_0 \) to be the closed span of \( \{ \varphi([|z| + k) \exp(i \text{Arg}(z) + N\alpha))] \} \), with \( k, m \in \mathbb{Z}, 1 \leq N \leq m \). Let \( U \) be the scaling operator \( [\mathfrak{F}] \). Let

\[
V_j = \text{span} \left\{ \varphi \left( \left( \frac{|z|}{\nu + 1} \right)^j + k \right) e^{i(\text{Arg}(z) + N\alpha)} \right\}_{j \in \mathbb{Z}, 1 \leq N \leq m}.
\]

Let \( \xi \) be a function on \( L^2(\mathbb{C}, d\mu_q(z)) \) given by

\[
\xi(z) = \sum_k a_k \{ \varphi([|z| + k) \exp(i \text{Arg}(z) + N\alpha))] \}.
\]

Assume \( U\varphi \in V_0 \), i.e.,

\[
(U\varphi)(z) = \sum_k a_k \varphi([|z| + k) \exp(i \text{Arg}(z) + N\alpha))].
\]

**Proposition 5** With the assumptions above, we conclude that the properties i)--iv) of a multiresolution are satisfied.

**Proof.** i) follows from the fact that the \( \varphi \)'s have disjoint support on \( L^2(\mathbb{C}, d\mu_q) \). ii) holds as before and iii) follows from i). By the density of step functions on \( L^2(\mathbb{C}, d\mu_q) \) also iv) follows.

If \( \xi \in V_{-j} \) then \( U^j\xi \in V_0 \). By applying the \( q \)-Hankel transform of order \( \nu \) we get

\[
H_\nu^q(\varphi([|z| + k) \exp(i \text{Arg}(z) + N\alpha)]; t) = m_0(t) H_\nu^q \left( \frac{1}{z}; t \right).
\]

where we denote by \( H_\nu^q(z; t) \) the \( q \)-Hankel transform to avoid confusion. The Plancherel Theorem for Hankel transforms extends in a natural way to the case of \( q \)-Hankel transforms where it takes the following form:

\[
\int \int_C tF(t) G(t) \, d\mu_q(t) = \int \int_C zF(z) \, g(z) \, d\mu_q(z),
\]

where \( d\mu_q(z) \) is the \( q \)-measure [39].

Then by using the orthogonality of \( \varphi([|z| + k) \exp(i \text{Arg}(z) + N\alpha))]_{k \in \mathbb{Z}} \) and the following fact:

\[
\int_0^1 |z|^{\nu+1} \, d\mu_q(z) = \frac{1 - q}{1 - q^\nu+2},
\]

we have

\[
\langle \varphi^{(k,N)} | \varphi^{(0,0)} \rangle \equiv \int_C \varphi^{(k,N)}(z) \overline{\varphi^{(0,0)}(z)} \, d\mu_q(z) = \int_0^{2\pi} \varphi([|z| + k) \exp(i \text{Arg}(z) + N\alpha)] \overline{\varphi([|z| \exp(i \text{Arg}(z))]} \\
\times |z|^{\nu+1} \exp(i \text{Arg}(z) (\nu + 1)) \, d|z| \, d\text{Arg}(z) = \int_0^1 |z|^{\nu+1} \delta_{k,0} \, d|z| \int_0^\alpha \exp(i \text{Arg}(z) (\nu + 1)) \, \delta_N,0 \, d\text{Arg}(z) = \frac{1 - q}{1 - q^{\nu+2}} \delta_{k,0} e^{i\alpha(\nu+1)} - \frac{1}{i (\nu + 1)} \delta_N,0.
\]

(31)
By the Plancherel theorem, we then have
\[
\frac{1 - q}{1 - q^{\nu + 2}} e^{i\alpha (\nu + 1)} - 1 \frac{\delta_{k,0}}{i (\nu + 1)} \delta_{N,0} = \int_{\mathbb{C}} H^q_{\nu} \left( \varphi^{(k,N)} (z); \right) t \frac{H^q_{\nu} \left( \varphi^{(0,0)} (z); \right) t}{d \mu_q (t)}.
\]

The left-hand side can then be rewritten as
\[
\int_0^1 \int_0^\alpha H^q_{\nu} \left( \varphi^{(k,N)} (z); \right) t \frac{H^q_{\nu} \left( \varphi^{(0,0)} (z); \right) t}{d \mu_q (t)} d \theta.
\]

Upon a change of variable letting \( \theta = \text{Arg} (t + 2\pi j) \), the latter equals
\[
\int_0^1 |t|^{\nu + 1} d_q |t| \int_0^\alpha e^{i\theta (\nu + 1)} \sum_j H^q_{\nu} \left( \varphi^{(k,N)} (z); \right) t e^{i\theta} \frac{H^q_{\nu} \left( \varphi^{(0,0)} (z); \right) t}{d \theta}.
\]

Comparing the previous two formulae for \( k = N = 0 \) we get
\[
\sum_j \left| H^q_{\nu} \left( \varphi^{(0,0)}; \right) t e^{i\theta} \right|^2 - \frac{1 - q}{1 - q^{\nu + 2}} e^{i\alpha (\nu + 1)} - 1 \frac{\delta_{k,0}}{i (\nu + 1)} = 0 \text{ a.e.}
\]

Rewriting the above in terms of \( m_0 \) we have
\[
\sum_j \left| m_0 \left( t e^{2\pi i j} \right) \frac{t}{e^{i\theta}} \right|^2 H^q_{\nu} \left( \frac{1}{z}; \right) t e^{i\theta} \left| H^q_{\nu} \left( \varphi^{(0,0)} (z); \right) t \right| e^{i\theta} = \frac{1 - q}{1 - q^{\nu + 2}} \frac{e^{i\alpha (\nu + 1)} - 1}{i (\nu + 1)}.
\]

Since from (31)
\[
\int_0^1 |t|^{\nu - 1} d_q |t| \int_0^\alpha |m_0 \left( t e^{2\pi i j} \right)|^2 d \theta = \frac{1 - q}{1 - q^{\nu + 2}} \frac{e^{i\alpha (\nu + 1)} - 1}{i (\nu + 1)} \delta_{k,0} \delta_{N,0}
\]
\[
\quad = \frac{1 - q}{1 - q^{\nu + 2}} \int_0^1 |t|^{\nu - 1} d_q |t| \int_0^\alpha e^{i\theta (\nu + 1)} d \theta,
\]

thus
\[
\left( \frac{1}{\nu + 1} \right) \sum_j \left| m_0 \left( t e^{2\pi i j / (\nu + 1)} \right) \right|^2 = \frac{1 - q}{1 - q^{\nu + 2}}.
\]

Set \( c_q = \frac{1 - q}{1 - q^{\nu + 2}} \); thus we have \( \frac{1}{c_q (\nu + 1)} \sum_j \left| m_0 \left( t e^{2\pi i j / (\nu + 1)} \right) \right|^2 = 1 \).

Observe that the Bessel functions have a “multiplicative periodicity” on the unit circle in the following sense:
\[
J_{\nu} \left( z e^{\pi i k} \right) = e^{\pi i k \nu} J_{\nu} \left( z \right).
\]

This implies that
\[
c_q^{-1} \sum_{j=0}^{\nu} \left| m_0 \left( t e^{2\pi i j / (\nu + 1)} \right) \right|^2 = (\nu + 1).
\]

Thus a \( q \)-analogue of Theorem 3 holds. As in the previous section we construct representations of the Cuntz algebra in terms of the functions \( m_i \); whose existence is guaranteed from Corollary 4.2 of Ref. 3.

As before we construct representations of the algebra \( O_{\nu + 1} \) associated to the above multiresolution for the deformed case. The representations are realized on a Hilbert space \( H = L^2 (\mathbb{C}, d \mu_q) \) where the measure is given by \( d \mu_q (z) = z^{\nu} d_q z \).

We now turn to the representation of the Cuntz algebra. It is given in terms of certain maps
\[
\sigma_k \colon \Omega \rightarrow \Omega, \quad \sigma_k \left( z \right) = \sigma_0 \left( z \right) e^{i2\pi / (\nu + 1)},
\]

where
\[
\sigma_k \left( z \right) = \sigma_0 \left( z \right) e^{i2\pi / (\nu + 1)},
\]
\[ \sigma_0(z)^{\nu+1} = z, \]
such that \( \mu_q(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0 \) for \( i \neq j \). Also

\[
\int_{\mathbb{C}} f(z) \, d\mu_q(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} \rho_r \int_{\mathbb{C}} f(\sigma_r(z)) \, d\mu_q(z). \tag{33}
\]

In fact, we have \( \mu_q(\sigma_r(E)) = \rho_r \mu_q(E) \), for Borel subsets \( E \subset \mathbb{C} \).

The representation takes the following form on \( L^2(\mathbb{C}, d\mu_q) \):

\[
(S_k \xi)(z) = m_k(z) \xi(z^{\nu+1}),
\]
where the functions \( m_k \) are obtained from the above multiresolution construction. Then we have

\[
(S_k^* \xi)(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} c^{-1}_q \rho_r m_k(\sigma_r(z)) \xi(\sigma_r(z)).
\]

Thus:

\[
(S_k^* S_k \xi)(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} c^{-1}_q \rho_r m_k(\sigma_r(z)) m_{k'}(\sigma_{r'}(z)) \xi(\sigma_{r'}(z))
= \delta_{k,k'} \xi(z),
\]
by the unitarity of the matrix \( M(z) \). We have used the convention \( \sigma(z) = z^{\nu+1} \) and the fact that \( \sigma \circ \sigma_r = \text{id} \) for all \( r \). It is easy similarly to verify that

\[
\sum_{k \in \mathbb{Z}_{\nu+1}} (S_k S_k^* \xi)(z) = \xi(z).
\]

As a result, we then have a representation of \( O_{\nu+1} \).

V. MULTiresolution ANALYSIS

We study now a particular case of a construction of a multiresolution. We then see how to construct a representation
of the Cuntz algebra. It is interesting to see that for the corresponding representation so constructed we get a \( q \)-number related to the modulus of a Markov trace \( \tau_{\nu+1} \) for compact quantum groups of type \( A_{\nu+1} \).

Let us now consider

\[ V_0 = \text{closed span} \left\{ [h(q^k - z) - h(q^{k+1} - z)] \right\}_{k \in \mathbb{Z}} \]
and the step function is given by

\[
\varphi^{(k)}_{\nu}(z, q) = [h(q^k - z) - h(q^{k+1} - z)]. \tag{34}
\]

Set

\[
[q+1]_q^2 = \frac{1-q^{2(\nu+1)}}{1-q^2}
\]
and define the scaling

\[
U f(z) = (\nu + 1)^{-\frac{1}{2}} f\left((\nu + 1)^{-1} z\right).
\]

Assume \( U \varphi^{(k)}_{\nu} \in V_0 \), then

\[
U \varphi^{(k)}_{\nu}(z, q) = \sum_k a_k [h(q^k - z) - h(q^{k+1} - z)].
\]

It follows
\[ U^j \varphi^k_{\nu} (z, q) = \sum_k a_k \left[ h \left( q^k - \frac{z}{(\nu + 1)^j} \right) - h \left( q^{k+1} - \frac{z}{(\nu + 1)^j} \right) \right]. \]

Let \( V_j = U^j V_0 \), so that if \( f \in V_j \), \( U^{-j} f \in V_0 \). The set \( \{ \varphi^k_{\nu} \}_{k \in \mathbb{Z}} \) is an orthonormal set in \( L^2 (\mathbb{C}) \). In fact

\[ h \left( q^k - z \right) - h \left( q^{k+1} - z \right) = \begin{cases} 1 & \text{if } q^{k+1} < |z| < q^k, \\ 0 & \text{otherwise}, \end{cases} \]

are defined for \( q^{k+1} < |z| < q^k \) in the annulus of \( r = q^{k+1}, R = q^k \). It follows that the set \( \{ \varphi^k_{\nu} (z, q) \}_{k \in \mathbb{Z}} \) is orthogonal in \( L^2 (\mathbb{C}) \) since the functions \( \varphi^k_{\nu} \) have disjoint support. Actually the set is orthogonal in \( L^2 (\mathbb{T}) \) since for \( k \to \infty, q^k \to 0 \), and for \( k \to 0, q^k \to 1 \). Let

\[ \xi(z) = \sum_k a_k \left[ h \left( q^k - z \right) - h \left( q^{k+1} - z \right) \right]. \quad (35) \]

By applying the \( q \)-Hankel transform on both sides of (35) we get then

\[ \hat{\xi} (t) = \sum_k a_k H^q_0 \left[ (h(q^k - z) - h(q^{k+1} - z)); t \right], \]

which implies

\[ \hat{\xi} (t) = \sum_k a_k \left[ q^{k(\nu+1)} J_{\nu+1} ((1-q)t q^k; q) - q^{(k+1)(\nu+1)} J_{\nu+1} ((1-q)t q^{k+1}; q) \right] \times H^q_0 \left( \frac{1}{z}; t \right) = \sum_k a_k q^{k(\nu+1)} \left[ J_{\nu+1} ((1-q)t q^k; q) - q^{-\nu+1} J_{\nu+1} ((1-q)t q^{k+1}; q) \right] \times H^q_0 \left( \frac{1}{z}; t \right). \]

Using the Plancherel theorem for \( q \)-Hankel transforms and orthogonality of \( \{ \varphi^k_{\nu} \}_{k \in \mathbb{Z}} \) as before, since we have

\[ \delta_{k,0} = 1 = \frac{1}{1-q} \int_q^1 t \, dq, \]

the left-hand side becomes then

\[ 0 = \int_q^1 \left[ \sum_{j \in \mathbb{Z}} q^{2j} \overline{H^q_0 ((h(q^k-z) - h(q^{k+1} - z)); q^j s)}) H^q_0 ((h((1-z) - h(q^1-z)); q^j s)) \right] \, dq, \]

so that almost everywhere,

\[ \sum_{j \in \mathbb{Z}} q^{2j} \overline{H^q_0 ((h(q^k-z) - h(q^{k+1} - z)); q^j s)}) H^q_0 ((h((1-z) - h(q^1-z)); q^j s)) \]

\[ = \frac{1}{1-q^{2(\nu+1)}}, \]

Now we have by using the above:
\[ \sum_{j \in \mathbb{Z}} q^{2j} H_0^q (h (q^k - z) - h (q^{k+1} - z); q^j t)) \cdot H_0^q ((h (1-z) - h (q^1 - z); q^j t)) \\
= \sum_{j \in \mathbb{Z}} |m_0 (tq^j)|^2 \left| H_0^q \left( \frac{1}{z}; tq^j \right) \right|^2 . \]

Hence we have:

\[ \sum_{j \in \mathbb{Z}} |m_0 (tq^j)|^2 \left| H_0^q \left( \frac{1}{z}; tq^j \right) \right|^2 = \frac{1}{1-q^{2(\nu+1)}} . \]

By a similar argument as above we get the special property for the function \( m_0 \):

\[ \sum_{j \in \mathbb{Z}} |m_0 (tq^j)|^2 \left| H_0^q \left( \frac{1}{z}; tq^j \right) \right|^2 = \frac{1}{1-q^{2(\nu+1)}} . \] (36)

Observe that in this case since \( q \leq |t| \leq 1 \) and then from \( q \leq q^{1-j} \leq |t| \) \( q^{-j} \leq 1 \) we have \( |t| \leq q^j \leq 1 \) and then

\[ j \leq \frac{\log |t|}{\log q} . \]

For \( |t| = q \), \( j = 1 \) and for \( |t| = 1 \), \( j = 0 \) Hence the sum in (36) reduces to a finite sum, by using a similar argument as for the Haar wavelet multiresolution. For a scale \( \nu + 1 \) we thus have

\[ \sum_{j=0}^{\nu} |m_0 (tq^j)|^2 \left| H_0^q \left( \frac{1}{z}; tq^j \right) \right|^2 = \frac{1}{1-q^{2(\nu+1)}} . \]

In this case we should note that \( |H_0^q \left( \frac{1}{z}; tq^j \right)|^2 = q^{-2j} \). Thus it follows:

\[ \sum_{j=0}^{\nu} q^{-2j} |m_0 (tq^j)|^2 = \frac{1}{1-q^{2(\nu+1)}} . \]

Set \( d_q = \frac{1}{1-q} \): then \( d_q^{-1} \sum_{j=0}^{\nu} q^{-2j} |m_0 (tq^j)|^2 = 1 \).

With the function \( m_0 \) given choose \( m_1, \ldots, m_\nu \) in \( L^2 (T, d\mu) \) such that

\[ \sum_{j=0}^{\nu} q^{-2j} m_r (tq^j) \overline{m_r (tq^j)} = \delta_{r,r'} \frac{1}{1-q^{2(\nu+1)}} . \] (37)

Define the functions \( \psi_1, \psi_2, \ldots, \psi_\nu \) by the formula:

\[ H_\nu^q \left( \psi^{(j,m)}_r (z); t (\nu + 1) \right) = m_r (t) H_0^q \left( \frac{1}{z}; t \right) . \] (38)

Concretely the functions in (38) are \( \psi^{(j,m)}_r (z) = \psi_r \left( (\nu + 1)^{-m} z - q^j \right) \). Then using (35) and (38) it follows that

\[ \left\{ (\nu + 1)^{-m} \psi^{(j,m)}_r (z) \right\}_{j,m} \]

is an orthogonal basis for the space \( V_{-1} \cap V_0^+ \) and then by iii) and iv) they form an orthogonal basis for \( L^2 (C, d\mu_q) \).

Now reformulating (37), the orthonormality of \( \left\{ (\nu + 1)^{-m} \psi^{(j,m)}_r (z) \right\}_{j,m} \) is equivalent to the following matrix,

\[ M (t) = \frac{1}{\sqrt{d_q (\nu + 1)}} \begin{pmatrix} \sqrt{\rho_0 m_0 (\sigma_0 (t))} & \sqrt{\rho_1 m_0 (\sigma_1 (t))} & \cdots & \sqrt{\rho_\nu m_0 (\sigma_\nu (t))} \\ \sqrt{\rho_0 m_1 (\sigma_0 (t))} & \sqrt{\rho_1 m_1 (\sigma_1 (t))} & \cdots & \sqrt{\rho_\nu m_1 (\sigma_\nu (t))} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\rho_0 m_\nu (\sigma_0 (t))} & \sqrt{\rho_1 m_\nu (\sigma_1 (t))} & \cdots & \sqrt{\rho_\nu m_\nu (\sigma_\nu (t))} \end{pmatrix} , \]

14
being unitary, where $\rho_j = q^{-2j}$.

The class of representations of the algebra $O_{\nu+1}$ associated to the above multiresolution construction is given as in the previous cases in terms of the functions $m_i$ and of the maps $\sigma_i$. The representations are realized on the Hilbert space $H = L^2(\mathbb{C}, d\mu_q)$ where the measure is $d\mu_q(z) = z^{\nu} d_q z$. A similar construction works for the case $q = 1$ where we use classical Bessel functions and the usual Hankel transform.

Define the representation of the Cuntz algebra in terms of certain maps (analogous to (32)): 

$$\sigma_i: \Omega \rightarrow \Omega, \quad \sigma_i(z) = \sigma_0(z) q^i,$$

where 

$$\sigma_0(z)^{\nu+1} = z,$$

such that $\mu_q(\sigma_i(\Omega) \cap \sigma_j(\Omega)) = 0$ for $i \neq j$. Hence, the system (13) here will be the $N$-sheeted Riemann surface of $\sqrt[\nu]{z}$. Also 

$$\int_{\mathbb{C}} f(z) d\mu_q(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} \rho_r \int_{\mathbb{C}} f(\sigma_r(z)) d\mu_q(z),$$

(39)

which is the analogue of (27). In fact, $\mu_q(\sigma_r(E)) = \rho_r \mu_q(E)$ with $\rho_i = q^{-2i}$, for Borel sets $E \subset \mathbb{C}$.

The representation takes the following form on $L^2(\mathbb{C}, d\mu_q)$:

$$(S_k \xi)(z) = m_k(z) \xi(z^{\nu+1}),$$

where the functions $m_k$ are obtained from the above multiresolution construction. By using (39) we have 

$$(S_k^* \xi)(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} d_q^{-1} \rho_r m_k(\sigma_r(z)) \xi(\sigma_r(z)).$$

Thus:

$$(S_k^* S_k \xi)(z) = \sum_{r \in \mathbb{Z}_{\nu+1}} d_q^{-1} \rho_r m_k(\sigma_r(z)) m_k(\sigma_r(z)) \xi(\sigma_r(z))$$

$$= \delta_{k,k^*} \xi(z),$$

by the unitarity of the matrix $M(z)$. It is easy to verify that 

$$\sum_{k \in \mathbb{Z}_{\nu+1}} (S_k S_k^* \xi)(z) = \xi(z).$$

We then have a representation of $O_{\nu+1}$. The interesting feature in this case is the fact that the $q$-number

$$\frac{1}{1 - q^{2\nu+2}} = \frac{1}{1 - q^2 [\nu + 1]_q^{-2}}$$

appearing in the orthogonality relations is exactly a multiple of the modulus of the Markov trace associated to the compact quantum group of type A.

Then we can perform a Fourier-type analysis over the cyclic group $\mathbb{Z}_{\nu+1}$ introducing 

$$A_{i,j}(z) = \left(\frac{1}{1 - q^{2(\nu+1)}}\right)^{-1} \sum_{\omega^{\nu+1} = z} \omega^{-j} m_i(\omega)$$

and the inverse transform

$$m_i(z) = \sum_{j=0}^{\nu} z^j A_{i,j}(z^{\nu+1}).$$
VI. TIGHT FRAMES, DEFORMED TIGHT FRAMES AND REPRESENTATIONS OF $O_{\nu+1}$

In this section we construct tight frames giving rise to certain representations of the Cuntz algebra.

The representations we will consider are realized on a Hilbert space $H = L^2(\Omega, \mu)$ where $\Omega$ is a measure space and $\mu$ is a probability measure on $\Omega$.

A frame is a set of non-independent vectors which can be used to construct an explicit and complete expansion for every vector in the space. Thus we have the following definition:

**Definition 6** A family of functions $\{ \varphi_j \}_{j \in J}$ in a Hilbert space $H$ is called a frame if there exist $A > 0$, $B < \infty$ so that for all $f$ in $H$ we have:

$$A \|f\| \leq \sum_{j \in J} |\langle f | \varphi_j \rangle|^2 \leq B \|f\|^2.$$ 

We call $A$ and $B$ the frame bounds. If the two frame bounds are equal then as in Ref. 2 the frame will be called a tight frame. Thus in a tight frame we have, for all $f \in H$,

$$\sum_{j \in J} |\langle f | \varphi_j \rangle|^2 = A \|f\|^2,$$

where $\langle f | \varphi_j \rangle$ are the Fourier coefficients.

We construct tight frames but instead of a Fourier transform we use the Hankel transform as defined in the previous sections. We will see that the construction will then extend to a $q$-deformed tight frame.

Let us start with functions $m_0, m_1, \ldots, m_\nu : T \to C$ such that the following $\nu + 1 \times \nu + 1$ matrix

$$M(t) = \frac{1}{\sqrt{\nu + 1}} \begin{pmatrix} m_0(\sigma_0(t)) & m_0(\sigma_1(t)) & \cdots & m_0(\sigma_\nu(t)) \\ m_1(\sigma_0(t)) & m_1(\sigma_1(t)) & \cdots & m_1(\sigma_\nu(t)) \\ \vdots & \vdots & \ddots & \vdots \\ m_\nu(\sigma_0(t)) & m_\nu(\sigma_1(t)) & \cdots & m_\nu(\sigma_\nu(t)) \end{pmatrix}$$

is unitary for almost all $z \in T$. Assume that $m_0(0) = 1$ and that the following infinite product:

$$H_k(\varphi(z); t) = \prod_{l=1}^{\infty} m_0((\nu + 1)^{-l} t)$$

converges pointwise almost everywhere. By Ref. 3 it follows from the condition

$$\sum_{j=0}^{\nu} |m_0(te^{\pi i j})|^2 = 1$$

that $H_k(\varphi(z); t) \in L^2(T)$, and that $\|\varphi\|_2 \leq 1$. Let us now define $\psi_1, \psi_2, \ldots, \psi_\nu$ by the formula:

$$H_{k+j}(\psi_r(j,m)(z); t(\nu + 1)) = m_r(t) H_0\left(\frac{1}{z}; t \right).$$

Then we have that the system

$$\left\{ \psi_r(j,m)(z) \right\}_{j,m}$$

is not an orthogonal set, so $\left\{ \psi_r(j,m)(z) \right\}_{j,m}$ is not an orthogonal basis for $L^2(R)$, but only a tight frame in the sense that

$$\sum_{m,j,r} \left| \left\langle f | \psi_r(j,m)(z) \right\rangle \right|^2 = \|f\|^2$$

for all $f \in L^2(R)$. 

16
Let us specialize to the following case on the space \( L^2(T, d\mu(z) = z^{-1} dz) \):

\[
m_0(z) = \sum_{k \in \mathbb{Z}} b_k J_k(z) \quad \text{and} \quad m_r(\sigma_j(z)) = \sum_{k \in \mathbb{Z}} b_k J_{k+r}(ze^{\pi i j})
\]

where \( \sigma_j(z) = \sigma_0(z)e^{\pi i j} \).

The unitarity of the matrix \( M(z) \) implies the following conditions:

1. For the diagonal entries we have:

\[
\left| \sum_{r=0}^{\nu} m_r(z e^{\pi i r}) \right|^2 = \sum_{r=0}^{\nu} \sum_{k,l} b_k b_l J_k(z) J_l(z) e^{\pi i (k-l)}.
\]

Since we have the following:

\[
\int_{|z|=1} \sum_{r=0}^{\nu} |m_0(\sigma_r(z))|^2 d\mu(z) = 1 = 1 \frac{1}{2\pi i} \int_{|z|=1} d\mu(z),
\]

for \( k = l \) the Residue Theorem gives the following:

\[
1 = (\nu + 1) \sum_k |b_k|^2 \frac{1}{(k!2^k)^2}.
\]

2. For the off-diagonal entries, i.e., for \( k' \neq l' \), we have:

\[
\sum_{r=0}^{\nu} m_{k'}(te^{\pi i r}) m_{l'}(te^{\pi i r}) = 0;
\]

we get \( \sum_{k,l} b_k b_l c_{\nu+k} = 0 \) and then \( \sum_{k,l} b_k c_{\nu+k+l'} = 0, l' = n + k' \), by using the “multiplicative periodicity” of the Bessel functions with respect to the argument. Define now \( \psi_1, \psi_2, \ldots, \psi_\nu \) such that

\[
(\nu + 1)^{\frac{1}{2}} H_{k+j} \psi_r((z);(\nu + 1) t)) = m_j(t) H_0 \left( \frac{1}{z}; t \right).
\]

Then the \( \left\{ \psi_r^{(j,m)}(z) \right\}_{j,m} \) are not orthogonal but they satisfy:

\[
\sum_{m,j,r} \left| \left\langle f \left| \psi_r^{(j,m)}(z) \right\rangle \right| \right|^2 = \| f \|^2
\]

for all \( f \in L^2(\mathbb{R}) \). This follows as in Ref. 2, Prop. 6.2.3, from the unitarity of the matrix of the \( (m_{i,j})_{i,j} = (m_i(\sigma_j(z)))_{i,j} \) and from the formula (40). It then follows that the set \( \left\{ \psi_r^{(j,m)}(z) \right\}_{j,m} \) is a tight frame.

Let us look at the case of the deformed representations of the algebra \( O_{\nu+1} \). See Ref. 29 for a class of deformed representations of the Cuntz algebra related to the Hahn-Exton \( q \)-Bessel functions. We consider the space \( L^2(T, d\mu) \) as before, where we take the measure given by

\[
d\mu(z) = z^{-1} dz,
\]

where we use Hahn-Exton \( q \)-Bessel functions instead of classical Bessel functions.

Define the operators \( S_k \) on \( L^2(T, d\mu) \) by:

\[
(S_k \xi)(z) = m_k(z) \xi(z^{\nu+1}),
\]

where

\[
m_0(z) = \sum_{k \in \mathbb{Z}} b_k J_k(z; q)
\]

and

17
m_r (σ_j (z)) = \sum_{k \in \mathbb{Z}} b_k J_{k+r} (z q^j; q) .

Hence we have:

(S^*_k \xi) (z) = \sum_{r \in \mathbb{Z}^{\nu+1}} \rho_r m_k (σ_r (z)) \xi (σ_r (z)) .

Thus:

(S^*_k S_k \xi) (z) = \sum_{r \in \mathbb{Z}^{\nu+1}} \rho_r m_k (σ_r (z)) m_k (σ_r (z)) \xi (σ_r (z))

= \delta_{k,k'} \xi (z),

by using the unitarity of the matrix $M (z)$.

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