Quantum Nonlocality Enhanced by Homogenization

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Homogenization proposed in [Y.-C Wu and M. Žukowski, Phys. Rev. A 85, 022119 (2012)] is a procedure to transform a tight Bell inequality with partial correlations into a full-correlation form that is also tight. In this paper, we check the homogenizations of two families of n-partite Bell inequalities: the Hardy inequality and the tight Bell inequality without quantum violation. For Hardy’s inequalities, their homogenizations bear stronger quantum violation for the maximally entangled state; the tight Bell inequalities without quantum violation give the boundary of quantum and supra-quantum, but their homogenizations do not have the similar properties. We find their homogenization are violated by the maximally entangled state. Numerically computation shows the the domains of quantum violation of homogenized Hardy’s inequalities for the generalized GHZ states are smaller than those of Hardy’s inequalities.

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I. INTRODUCTION

The problem of the possibility of a local realistic interpretation of quantum mechanics was first addressed in the discussion between Einstein, Podolsky, Rosen (EPR) [1] and Bohr [2]. In order to settle down the philosophical debate, Bell proposed an experimental scheme in 1964 [3]. Bell’s seminal paper contains an inequality, which holds for local realistic correlations but can be violated by quantum mechanical correlations. In the work of Clauser, Horne, Shimony, and Holt (CHSH) [4] a new inequality was derived, which, comparing with the original Bell’s expression, can be more applicable to real experimental setups. An inequality with lower-order (partial) correlations is usually called a CH-type Bell inequality, whereas an inequality involving only highest-order (full) correlations is usually referred to as a CHSH-type Bell inequalities.

To identify quantum nonlocality (QN), one needs a set of complementary observables for each party. It is anticipated that the ability to detect QN becomes stronger as the number of observables (settings) increases. For instance, the 2-setting CHSH inequality determines the visibility for the Werner state as 1/√2, while a 465-setting inequality [5] decreases the visibility to 0.7056, which is slightly smaller than 1/√2.

On the other hand, although the CHSH inequality can detect the QN for all two-qubit pure entangled states, for more parties it is not desirable to invoke the full-correlation inequality, like the CHSH inequality. In fact, the first Bell inequality to identify QN for the whole domain of the generalized GHZ state \( \cos \theta |000\rangle + \sin \theta |111\rangle \) is comprised of partial correlations.

In general, in constructing Bell inequalities there is a trade-off between full correlations and the ability to detect QN for more generalized states. Wu and Žukowski [6] show how to transform a CH-type inequality into a CHSH-type inequality, the tightness being preserved. In this paper, we use the procedure of homogenization to compare the quantum violation for various states. Through the homogenization, the setting for each party is increased by one. This strengthens the quantum violation for the maximally entangled state, while the violation regimes for the nonmaximally entangled state could be narrowed.

The organization of this paper is as follows. In Sec. II, we briefly review the procedure of homogenization. Then, in Sec. III, we focus on the generalized GHZ state and compare its quantum violation of the Hardy inequality before and after the homogenization. Likewise, in Sec. IV we study the influence of the homogenization on a family of tight Bell inequalities without quantum violation. We discuss the results in Sec. V and propose a possible reason for the enhanced QN by the homogenization.

II. BRIEF REVIEW ON HOMOGENIZATION

In general, the CHSH- and CH-type inequalities read

\[ \sum_{ij} \omega_{ij} a_i b_j \leq 1, \quad (1) \]

\[ 0 \leq I(\hat{a}, \hat{b}) = c + \sum_i \alpha_i a_i + \sum_j \beta_j b_j + \sum_{ij} \gamma_{ij} a_i b_j \leq M, \quad (2) \]
where \( \omega_{ij}, \alpha_i, \beta_j \) and \( \gamma_{ij} \) are real coefficients, \( c \) is a real constant, \( M \) is the classical upperbound of \( \| \mathbf{r} \| \), and \( a_i, b_j \) are dichotomic observables taking values \( \pm 1 \). Wu and Żukowski [6] showed that (2) can be transformed into (1) by homogenization, and that if (2) is tight then (1) is also tight. Specifically, the homogeneous expression is obtained as

\[
H(I) = c' a_0 b_0 + \sum_i \alpha_i a_i b_0 + \sum_j \beta_j b_j a_0 + \sum_{ij} \gamma_{ij} a_i b_j, (3)
\]

with

\[
-\frac{M}{2} \leq \frac{1}{a_0 b_0} H(I) \leq \frac{M}{2}, \quad c' = c - M/2. \quad (4)
\]

III. HARDY’S NONLOCALITY INEQUALITIES

Cereceda [7] extend Hardy’s nonlocality proof for two spin-1/2 particles [8] to the case of \( n \) spin-1/2 particles configured in the generalized GHZ state. We now show that the maximal quantum violation of the homogenized Hardy’s inequality is stronger than Hardy’s original inequality. The \( n \)-qubit CH-type Hardy’s inequality reads [7]

\[
p(0|0_20_3 \cdots 0_n|0_10_20_3 \cdots 0_n) \\
\leq p(1|1_21_3 \cdots 1_n|1_12_3 \cdots 1_n) \\
+ p(0|0_20_3 \cdots 0_n|1_10_20_3 \cdots 0_n) \\
+ \cdots + p(0|0_10_20_3 \cdots 0_n|1_10_20_3 \cdots 1_n), \quad (5)
\]

where \( p(i_1i_2i_3 \cdots i_n|j_1j_2j_3 \cdots j_n) \) denotes the joint probability of obtaining result \( i_k \) under setting \( j_k \) for the \( k \)-th qubit, with \( i_k, j_k = 0, 1 \) and \( k \) running from 1 to \( n \). In the following context the subscript for each party could be omitted with no confusion. Let us rewrite (5) in the form of correlations. For \( n = 3 \),

\[
|5 - A_1 - B_1 - C_1 - A_2 B_1 - A_2 C_1 - A_1 B_2 - B_2 C_1 - A_1 B_2 C_2 - A_1 B_2 C_1 - A_1 B_2 C_1 - A_1 B_1 C_2|/8 \leq 1, \quad (6)
\]

where \( A_1 = p(0|0|1) - p(1|0|0) \) and \( A_2 = p(0|0|1) - p(1|1|0) \) are observables for the first party, and likewise for \( B_1, C_1 \). It is violated maximally by the three-qubit GHZ state by a factor of 1.17546. The homogenized inequality can be written as

\[
|20 - 2(A_1 + B_1 + C_1 + D_1) - A_1 B_1 - A_1 C_1 - A_1 D_1 - B_1 C_1 - B_1 D_1 - C_1 D_1 - A_2 B_1 - A_2 C_1 - A_2 D_1 - A_2 C_1 - A_2 B_1 - A_2 C_1 - B_1 C_2 - C_1 D_2 - A_2 B_2 - A_2 C_2 - A_2 D_2 - B_2 C_2 - B_2 D_2 - C_2 D_2 - A_2 B_2 C_2 - A_2 B_2 C_1 - A_2 B_2 C_1 - A_2 B_2 C_1 - A_2 B_2 C_2 - A_2 B_2 D_2 - A_2 B_2 D_2 - A_2 B_2 C_2 - D_1 - A_2 B_2 D_1 - A_2 B_2 D_1 - A_2 B_2 D_1|/24 \leq 1, \quad (8)
\]

For \( n = 4 \),

\[
|20 - 2(A_1 + B_1 + C_1 + D_1) - A_1 B_1 - A_1 C_1 - A_1 D_1 - B_1 C_1 - B_1 D_1 - C_1 D_1 - A_2 B_1 - A_2 C_1 - A_2 D_1 - A_2 C_1 - A_2 B_1 - A_2 C_1 - B_1 C_2 - C_1 D_2 - A_2 B_2 - A_2 C_2 - A_2 D_2 - B_2 C_2 - B_2 D_2 - C_2 D_2 - A_2 B_2 C_2 - A_2 B_2 C_1 - A_2 B_2 C_1 - A_2 B_2 C_1 - A_2 B_2 C_2 - A_2 B_2 D_2 - A_2 B_2 D_2 - A_2 B_2 C_2 - D_1 - A_2 B_2 D_1 - A_2 B_2 D_1 - A_2 B_2 D_1|/24 \leq 1, \quad (8)
\]

It is violated maximally by the three-qubit GHZ state by a factor of 1.06904. The homogenized inequality can be written as

\[
|20 - 2(A_1 + B_1 + C_1 + D_1) - A_1 B_1 - A_1 C_1 - A_1 D_1 - B_1 C_1 - B_1 D_1 - C_1 D_1 - A_2 B_1 - A_2 C_1 - A_2 D_1 - A_2 C_1 - A_2 B_1 - A_2 C_1 - B_1 C_2 - C_1 D_2 - A_2 B_2 - A_2 C_2 - A_2 D_2 - B_2 C_2 - B_2 D_2 - C_2 D_2 - A_2 B_2 C_2 - A_2 B_2 C_1 - A_2 B_2 C_1 - A_2 B_2 C_1 - A_2 B_2 C_2 - A_2 B_2 D_2 - A_2 B_2 D_2 - A_2 B_2 C_2 - D_1 - A_2 B_2 D_1 - A_2 B_2 D_1 - A_2 B_2 D_1|/24 \leq 1, \quad (8)
\]

It is violated maximally by the three-qubit GHZ state by a factor of 1.9524 (see Fig. 1).

IV. BELL INEQUALITIES WITH NO QUANTUM VIOLATION

The quantum correlations (QC) are in general more stronger than classical correlations (CC). Augustaki et al. [9] showed how unextendable product bases (UPBs) that satisfy a given requirement give rise to a family of tight Bell inequalities without quantum violation. In these situations, QC and CC perform equally well in information tasks, while the supraquantum nonsignaling correlations do provide an advantage over CC. Thus
such inequalities pinpoint the facet of polytope that separate quantum and supraquantum correlations, and provide better understandings of different sets of correlations which serve as valuable information resources.

Bell inequality with non-negative weights \( q_j \), which always can be assumed to obey \( 0 \leq q_j \leq 1 \).

\[
\sum_j q_j p(a_j | x_j) \leq \max \{q_j\},
\]

(10)

From the initial set of orthogonal product vectors, Augusiak et al. proved that all these inequalities are not violated by QC [3] [10]. By local unitaries and permutations of particles, all UPBs can be brought to \( S_0^i = S_0 = \{|0\rangle, |1\rangle\} \) and \( S_1^i = S_1 = \{|e\rangle, |\overline{e}\rangle\} \) \( (i = 1, 2, 3) \). We assign conditional probabilities in the following way: \( |000\rangle \rightarrow p(000|000), |\overline{1}\rangle \rightarrow p(110|011), |e1\rangle \rightarrow p(011|101), |\overline{e}1\rangle \rightarrow p(101|110) \). Then, we can get the tight Bell inequality with no quantum violation found in [11] and [12]. For odd \( n \), the inequality can be written as

\[
\sum_{k=0}^{(n-1)/2} \sum_{i_1 < \ldots < i_{k}=1} \sum_{n} T_{1\ldots i_{k}} p(0|0) \leq 1,
\]

(11)

and for even \( n \)

\[
\sum_{k=0}^{(n-2)/2} \sum_{i_1 < \ldots < i_{k}=2} \sum_{n} T_{1\ldots i_{k}} [p(0|0) + p(0 \ldots 01|10 \ldots 0)] \leq 1.
\]

(12)

Here \( 0 = (0, \ldots, 0) \), and \( T_{1\ldots i_{k}} \) denotes a filp \( (0 \leftrightarrow 1) \) of input bits and output bits at positions \( i_1, \ldots, i_{k} \) and \( i_1 - 1, \ldots, i_{k} - 1 \) (if \( i_j = 1 \), then \( i_{j-1} = n \)), respectively.

For \( n = 3 \), we obtain the inequality

\[
p(000|000) + p(101|110) + p(011|101) + p(110|011) \leq 1.
\]

(13)

For \( n = 4 \),

\[
p(0000|0000) + p(0011|1000) + p(0110|0011) + p(1011|0110) + p(1100|1110) \leq 1.
\]

(14)

Rewriting (13), the symbols in inequalities of probability \( 0 \to 1 \) and \( 1 \to 2 \) into the correlation function, we obtain

\[
|A_1 B_1 - A_2 B_1 + A_1 B_2 - A_2 B_2 + A_1 C_1 + A_2 C_1 + A_1 C_2 - A_2 C_2 + A_1 C_1 + A_2 C_2 - A_1 B_2 C_2 + A_1 B_2 C_2|/4 \leq 1.
\]

(15)

This inequality cannot be violated in quantum mechanics. However, let us see its homogenized inequality:

\[
|A_1 B_1 C_0 - A_2 B_1 C_0 + A_1 B_2 C_0 - A_2 B_2 C_0 + A_1 B_0 C_1 + A_2 B_0 C_1 - A_1 B_0 C_1 - A_2 B_0 C_1 - A_0 B_0 C_0 - A_0 B_0 C_0 + A_0 B_0 C_1 + A_1 B_0 C_1 + A_2 B_0 C_1 + A_1 B_2 C_2 + A_2 B_2 C_2|/4 \leq 1
\]

(16)

which is equivalent to the inequality in Ref. [13]. It is violated maximally by the three-qubit GHZ state by a factor of 2 (see Fig. 1).

For \( n = 4 \)

\[
|2A_1 C_1 + 2B_1 C_1 - 2A_1 C_2 - 2A_2 C_2 + 2B_1 C_2 - 2B_2 C_2 + A_1 B_1 C_1 + A_2 B_1 C_1 + A_1 B_2 C_1 + A_2 B_2 C_1 + A_1 B_1 C_2 - A_2 B_1 C_2 + A_1 B_2 C_2 - A_2 B_2 C_2 + A_1 C_1 C_1 - A_2 C_1 C_1 - A_1 C_2 C_1 - A_2 C_2 C_1 + A_1 B_1 C_1 + A_2 B_1 C_1 |2 + A_1 C_1 C_1 + A_2 C_1 C_1 + A_1 C_2 C_1 + A_2 C_2 C_1 + A_1 B_1 C_2 + A_2 B_1 C_2 + A_1 B_2 C_2 + A_2 B_2 C_2 + A_1 C_1 C_2 + A_2 C_1 C_2 + A_1 C_2 C_2 + A_2 C_2 C_2|/8 \leq 1,
\]

(17)

whose homogenized inequality can be written as

\[
|2A_1 B_0 C_1 D_0 + 2A_2 B_0 C_1 D_0 + 2A_0 B_1 C_1 D_0 - 2A_0 B_2 C_1 D_0 - 2A_1 B_0 C_2 D_0 + 2A_0 B_2 C_2 D_0 - 2A_2 B_0 C_2 D_0 + 2A_1 B_1 C_1 D_0 + 2A_1 B_2 C_1 D_0 + 2A_2 B_1 C_1 D_0 + 2A_2 B_2 C_1 D_0 + 2A_1 B_0 C_1 D_1 + 2A_1 B_2 C_1 D_1 - 2A_2 B_0 C_1 D_1 - 2A_2 B_2 C_1 D_1 + 2A_1 B_1 C_1 D_2 + 2A_1 B_2 C_1 D_2 - 2A_2 B_1 C_1 D_2 - 2A_2 B_2 C_1 D_2 + 2A_1 B_0 C_2 D_2 + 2A_1 B_2 C_2 D_2 - 2A_2 B_0 C_2 D_2 - 2A_2 B_2 C_2 D_2 + 2A_1 B_1 C_2 D_2 + 2A_1 B_2 C_2 D_2 |/8 \leq 1
\]

(18)

It is violated maximally by the three-qubit GHZ state by a factor of 2.44134 (see Fig. 1).
FIG. 2: The exclusivity graph for \( p(110|011) \). The sum of probabilities of pairwise exclusive events cannot exceed 1.

V. DISCUSSION AND CONCLUSIONS

To summarize, we have studied the influence of homogenization on enhancing the quantum nonlocality of two families of Bell inequalities. Through the homogenization, the quantum violation of the Hardy inequality has become stronger for the GHZ state, while the parameter domain for violation regime of the generalized GHZ state has been narrowed. On the other hand, quantum violation has appeared again for the family of tight Bell inequalities without quantum violation. The reason for this is that a tight Bell inequality without quantum violation can be represented by a complete graph \( [14] \) (see Fig. 2), i.e., event probabilities are pairwise exclusive, so that, according to graph theory, the independence number is equal to the Lovász number \( [15] \), indicating the coincidence of the classical upperbound with the quantum maximum. However, by homogenization, it is transformed into a full-correlation form, whose corresponding graph is no longer complete. In this regard, the independence number is in general less than than the Lovász number, rendering a quantum violation possible.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] N. Bohr, Phys. Rev. 48, 696 (1935).
[3] J. S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964).
[4] J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett. 23, 880 (1969).
[5] T. Vértesi, Phys. Rev. A 78, 032112 (2008).
[6] Y.-C. Wu and M. ˙Zukowski, Phys. Rev. A 85, 022119 (2012).
[7] J. L. Cereceda, Phys. Lett. A 327, 433 (2004).
[8] L. Hardy, Phys. Rev. Lett. 71, 1665 (1993).
[9] R. Augusiak, J. Stasinska, C. Hadley, J.K. Korbicz, M. Lewenstein, and A. Acín, Phys. Rev. Lett. 107, 070401 (2011).
[10] R. Augusiak et al., Phys. Rev. A 85, 042113 (2012).
[11] C. Śliwa, Phys. Lett. A 317, 165-168 (2003).
[12] M. L. Almeida, J. D. Bancal, N. Brunner, A. Acín, N. Gisin, and S. Pironio, Phys. Rev. Lett. 104, 230404 (2010).
[13] M. Wieśniak, P. Badziag, and M. ˙Zukowski, Phys. Rev. A 76, 012110 (2007).
[14] A. Cabello, S. Severini, and A. Winter, Phys. Rev. Lett. 112, 040401 (2014).
[15] L. Lovász, IEEE Trans. Inf. Theory 25, 1 (1979).