INSTABILITY OF COUPLED SYSTEMS WITH DELAY

REINHARD RACKE
Department of Mathematics and Statistics
University of Konstanz, 78457 Konstanz, Germany

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ABSTRACT. We consider linear initial-boundary value problems that are a coupling like second-order thermoelasticity, or the thermoelastic plate equation or its generalization (the $\alpha$-$\beta$-system introduced in [1, 26]). Now, there is a delay term given in part of the coupled system, and we demonstrate that the expected inherent damping will not prevent the system from not being stable; indeed, the systems will shown to be ill-posed: a sequence of bounded initial data may lead to exploding solutions (at any fixed time).

1. Introduction. It is well-known that delay problems like the simplest one of parabolic type,

$$\theta_t(t,x) = \Delta \theta(t - \tau),$$

with a delay parameter $\tau > 0$, or of hyperbolic type,

$$u_{tt}(t,x) = \Delta u(t - \tau),$$

with usual boundary and initial conditions are not well-posed. Their instability is given in the sense that there is a sequence of initial data remaining bounded, while the corresponding solutions, at a fixed time, go to infinity in an exponential manner, see Jordan, Dai & Mickens [12] and Dreher, Quintanilla & Racke [9], or Prüß [29], in particular for connections to Volterra equations. Indeed, it was shown in [9] that the same phenomenon of instability is given for a general class of problems of the type

$$\frac{d^n}{dt^n} u(t) = A u(t - \tau),$$

$n \in \mathbb{N}$ fixed, whenever $(-A)$ is linear operator in a Banach space having a sequence of real eigenvalues $(\lambda_k)_k$ such that $0 < \lambda_k \to \infty$ as $k \to \infty$.

Delay equations are well motivated from the applications, cf. [29], Chandrasekharaih [5], Bátikai & Piazzera [4]. For example, to have an alternative to the classical heat equation, which corresponds to $\tau = 0$ and shows the physically not justified phenomenon of infinite propagation speed of signals, introducing a delay in the constitutive law can be done as follows: Heat conduction is usually described by means of the energy equation

$$\theta_t + \gamma \text{ div } q = 0$$

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for the temperature $\theta$ and the heat flux vector $q$. With the constitutive law

$$q(t + \tau, \cdot) = -\kappa \nabla \theta(t, \cdot),$$

where $\gamma, \kappa > 0$, which expresses that a change of the temperature gradient at time $t$ is effective in the heat flux only with a delay $\tau > 0$, we obtain the delay equation

$$\theta_t(t, \cdot) = \kappa \gamma \Delta \theta(t - \tau, \cdot).$$

Adding certain non-delay terms, e.g. $\Delta \theta(t, x)$ on the right-hand side of (1), is already sufficient to obtain a well-posed problem, cf. [29, 4]. In the case of constitutive equations involving two or three phase-lags (rather than one in (5)), it is shown by Quintanilla in [30, 31] that a combination of the constitutive equations with a two temperature heat conduction theory may lead to well-posed problems, but also, depending on the relations between the phase-lag parameters, may lead to unstable problems, see [32].

Here, we consider coupled systems where there is an expected damping effect of a second differential equation in comparison to the first one, and we ask if there is still instability (ill-posedness) or whether we can have well-posedness. We shall answer this with several ill-posedness results.

For well-posedness results for wave equations with delay terms (in the interior), there are a number of papers by Nicaise and co-authors Ammari, Fridman, Pignotti, and Valein [27, 28, 10, 2], in [27] also with instability results. For works with delay terms in the boundary conditions, see the references in the paper [14].

For the system of coupled wave equations of Timoshenko type with delay terms of the type

$$\rho_1 \phi_{tt}(t, x) - K(\phi_x + \psi)_x(t, x) = 0, \quad (7)$$
$$\rho_2 \psi_{tt}(t, x) - b \psi_{xx}(t, x) + K(\phi_x + \psi) + \mu_1 \psi_t(t, x) + \mu_2 \psi_t(t - \tau, x) = 0, \quad (8)$$

the well-posedness (under certain conditions on $\mu_1, \mu_2$) was investigated by Said-Houari & Laskri [37] and extended to a time-varying delay term — replacing $\psi_t(t - \tau, x)$ by $\psi_t(t - \tau(t), x)$ — in the work of Kirane, Said-Houari & Anwar [14].

Here we consider coupled systems of different types. A typical first example is the coupling arising in thermoelasticity. In one dimension, we have the hyperbolic-parabolic system

$$u_{tt}(t, x) - au_{xx}(t - \tau, x) + b \theta_x(t, x) = 0, \quad (9)$$
$$\theta_t(t, x) - d \theta_{xx}(t, x) + bu_{tx}(t, x) = 0, \quad (10)$$

where $u$ describes the displacement, and $\theta$ is the temperature difference, and where $t \geq 0$ and $x \in (0, L) \subset \mathbb{R}$ with $L > 0$. To complete the initial-boundary value problem, we consider the boundary conditions

$$u(t, x) = \theta_x(t, x) = 0,$$

for $t \geq 0$ and $x \in \{0, L\}$, and initial conditions for $u_t(0, \cdot), u(s, \cdot)$, for $-\tau \leq s \leq 0$, and for $\theta(0, \cdot)$. The damping through heat conduction essentially given in (10) is for classical thermoelasticity — corresponding to $\tau = 0$ — strong enough to compose an exponentially stable system (modulo constant functions $\theta$ due to the boundary condition), thus strongly impacting the oscillating part of the (pure) wave equation ($u_{tt} - au_{xx} = 0$), see Racke [35] or Jiang & Racke [11] for extensive surveys, or, more specific, Racke [33, 34].
Here, we shall prove that the system with delay (9), (10) is not well-posed and
instable, that is, the damping through heat conduction turns out to be not strong
enough now; the instable system part \( (u_{tt}(t, \cdot) - au_{xx}(t-\tau, \cdot) = 0) \) will predominate.

Then even more expected, the same will happen for the system, where one delay is
given in the equation for the temperature, i.e. for
\[
\begin{align*}
  u_{tt}(t, x) - au_{xx}(t, x) + b\theta_x(t, x) &= 0, \\
  \theta_t(t, x) - d\theta_{xx}(t-\tau, x) + bu_{tt}(t, x) &= 0.
\end{align*}
\]  

The classical thermoelastic plate equation, a coupling of the plate equation (with
the Schrödinger equation behind) with heat conduction will then be investigated in
the same way. Here we have the system
\[
\begin{align*}
  u_{tt}(t, x) + a\Delta^2 u(t-\tau, x) + b\Delta \theta(t, x) &= 0, \\
  \theta_t(t, x) - d\Delta \theta(t, x) - b\Delta u_t(t, x) &= 0,
\end{align*}
\]
where \( x \in G \subset \mathbb{R}^n, n \in \mathbb{N} \) now, \( G \) bounded, and the corresponding one with the
delay term as in (12), (13). Initial conditions are given as usual, and we consider
the boundary conditions
\[
u(t, x) = \Delta u(t, x) = \theta(t, x) = 0
\]
for \( t \geq 0, x \in \partial G \). These thermoelastic plate equations (14), (15) — in comparison
to the thermoelastic system above being, for \( \tau = 0 \), exponentially stable also in
space dimension \( n \geq 2 \) and generating an analytic semigroup — will also turn out
to be ill-posed now.

The thermoelastic plate system (14), (15) has been widely discussed in particular
for bounded reference configurations \( G \ni x \), see the work of Kim [13], Muñoz
Rivera & Racke [25], Liu & Zheng [23], Avalos & Lasiecka [3], Lasiecka & Triggiani
[16, 17, 18, 19] for the question of exponential stability of the associated semigroup
(for various boundary conditions), and Russell [36], Liu & Renardy [20], Liu & Liu
[21], Liu & Yong [22] for proving its analyticity, see also the book of Liu & Zheng
[23] for a survey. For results in exterior domains see, for example, Muñoz Rivera &
Racke [26], Denk, Racke & Shibata [7, 8].

The thermoelastic plate system (14), (15) is a special case of the so-called \( \alpha, \beta \-
system, now with delay,
\[
\begin{align*}
  u_{tt}(t) + aAu(t-\tau) - bA^2 \theta(t) &= 0, \\
  \theta_t(t) + dA^2 \theta(t) + bA^2 u_t(t) &= 0,
\end{align*}
\]
for functions \( u, \theta : [0, \infty) \to \mathcal{H}, \) with \( A \) being a self-adjoint operator in the Hilbert
space \( H \), having a countable complete orthonormal system of eigenfunctions \( (\phi_j)_j \)
with corresponding eigenvalues \( 0 < \lambda_j \to \infty \) as \( j \to \infty \). The thermoelastic plate e-
quations appear with \( \alpha = \beta = \frac{1}{2} \) and \( A = (-\Delta_D)^2 \), where \( -\Delta_D \) denotes the Laplace
operator realized in \( L^2(G) \) on some bounded domain \( G \) in \( \mathbb{R}^n \) with Dirichlet bound-
ary conditions. This original \( \alpha, \beta \)-system without delay \( (\tau = 0) \) was introduced
by Muñoz Rivera & Racke [26] and, independently, by Anmar Khodja & Benab-
dallah [1], and investigated with respect to exponential stability and analyticity of
the associated semigroup, the latter also for the Cauchy problem, where \( \Omega = \mathbb{R}^n \, \)
and, more general, \( A = (-\Delta)^n \) for \( n > 0 \) arbitrary, and in arbitrary \( L^p \)-spaces for
\( 1 < p < \infty \), see Denk & Racke [6]. It was shown that we have a strong smoothing
property for parameters \( (\beta, \alpha) \) in the region \( \mathcal{A}_{sm} \) (see Figure 1.1), where
\[
\mathcal{A}_{sm} := \{ (\beta, \alpha) \mid 1 - 2\beta < \alpha < 2\beta, \alpha > 2\beta - 1 \}, \quad (\tau = 0),
\]
and that the analyticity (in $L^p(\mathbb{R}^n)$) is given in the region $\mathcal{A}_{an}$ (see Figure 1.2),

$$\mathcal{A}_{an} := \{ (\beta, \alpha) \mid \alpha \geq \beta, \alpha \leq 2\beta - 1/2 \}, \quad (\tau = 0). \quad (20)$$

Here, we shall show that the $\alpha$-$\beta$-system with delay (17), (18) is not well-posed in the region $\mathcal{A}_{1 in}$ (see Figure 1.3), where

$$\mathcal{A}_{1 in} := \{ (\beta, \alpha) \mid 0 \leq \beta \leq \alpha \leq 1, \alpha \geq \frac{1}{2}, (\beta, \alpha) \neq (1,1) \}. \quad (21)$$

A similar result will hold for the related system

$$u_{tt}(t) + aAu(t) - bA^{\beta}\theta(t) = 0, \quad (22)$$

$$\theta_t(t) + dA^\alpha\theta(t - \tau) + bA^{\beta}u_t(t) = 0, \quad (23)$$

in the region $\mathcal{A}_{2 in}$ (see Figure 1.4), where

$$\mathcal{A}_{2 in} := \{ (\beta, \alpha) \mid 0 < \beta \leq \alpha \leq 1, \ (\beta, \alpha) \neq (1,1) \}. \quad (24)$$

It is interesting to notice that there are differences comparing the regions $\mathcal{A}_{1 in}$ and $\mathcal{A}_{2 in}$. For the former, the damping through the main equation for $\theta$ has to be weak enough (“$\alpha \geq \frac{1}{2}$”) to still guarantee the ill-posedness suggested by the main equation with delay for $u$.

The behavior in the regions outside $\mathcal{A}_{1 in}$, $j = 1, 2$, is an open question.

We remark that the $\alpha$-$\beta$-system (without delay) has been recognized to possibly describe also viscoelastic systems, and, with respect to smoothing properties, even the second-order thermoelastic system from above, although in the latter case it is (first) formally not of this type; but after deriving a single differential equation of third order in time for $u$ (or $\theta$) only, the $\alpha$-$\beta$-formalism applies, see [26].
The methods to prove the results mentioned up to now will be to construct exponentially growing solutions with the help of an ansatz through eigenfunctions, and then modifying and extending ideas from [9] to the situation of coupled systems given here.
We remark that it would be possible to study the delay term at different places, actually to discuss systems like
\begin{align*}
  u_{tt}(t) + aAu(t - g_1\tau) + bA^\beta\theta(t - g_2\tau) &= 0, \\
  \theta_t(t) + dA^\alpha\theta(t - g_3\tau) + bA^\beta u_t(t - g_4\tau) &= 0,
\end{align*}
where \(g_j \in \{0, 1\}, j = 1, 2, 3, 4\). Here we studied the cases \(g_1 - 1 = g_2 = g_3 = g_4 = 0\) and \(g_1 = g_2 = g_3 - 1 = g_4 = 0\) only, for simplicity of the presentation.

The paper is organized as follows: In Section 2 we shall discuss the second-order thermoelastic systems and prove that the systems with delay are not well-posed, and in Section 3 the thermoelastic plate equations are discussed in a similar manner. In Section 4 the \(\alpha\)-\(\beta\)-system with delay will be studied proving the ill-posedness in a certain parameter region. In the appendix, we recall some arguments from [9].

\(L^p\) denotes the usual \(L^p\)-space of Lebesgue-integrable functions, \(\|\cdot\|\) and \(\|\cdot\|_H\) denote the norm in \(L^2\) and in a Hilbert space \(H\), respectively, and \(\frac{d}{dt}\) or subscripts \(t\) or \(x\) denote (partial) derivatives.

2. Second-order thermoelasticity with delay terms. We consider the thermoelastic system with delay given in (9), (10) and the related system (12), (13). The parameters \(a, b, d\) appearing are positive constants, \(\tau > 0\) is the — in applications often relatively small — relaxation parameter.

Both systems are completed with the boundary condition (11) and with initial conditions
\begin{align*}
  u(s, \cdot) = u^0(s), \ (-\tau \leq s \leq 0), \ u_t(0, \cdot) = u^1, \ \theta(0, \cdot) = \theta^0,
\end{align*}
and
\begin{align*}
  u(0, \cdot) = u^0, \ u_t(0, \cdot) = u^1, \ \theta(s, 0) = \theta^0(s), \ (-\tau \leq s \leq 0),
\end{align*}
respectively.
These systems are shown to be not well-posed, i.e. we prove

**Theorem 2.1.** (i) The initial-boundary value problem with delay (9), (10), (11), (27) is not well-posed. There exists a sequence \((u_j, \theta_j)\) of solutions with \(L^2\)-norm \(\|u_j(t, \cdot)\|\) tending to infinity \((as \ j \to \infty)\) for any fixed \(t > 0\), while for the initial data the norms \(\sup_{-\tau \leq s \leq 0} \|(u_j^0(s), u_j^1(s), \theta_j^0)\|\) remain bounded. Actually, \(h_j\) remains bounded. Additionally we have initial conditions for \(u_j\) and \(\theta_j\) such that \(h_j(t) \to \infty\) as \(j \to \infty\), while the initial data remain bounded. Actually, \(h_j\) will be of the form \(h_j(t) = e^{i\omega_j t}\) with \(\Re \omega_j \to \infty\) as \(j \to \infty\), see below.

Let \(\lambda_j := \frac{\omega_j}{L}\). Plugging the ansatz (29), (30) into the differential equations (9), (10) we conclude that \((h_j, g_j)\) should satisfy (as necessary and sufficient condition)

\[
h_j''(t) + a\lambda_j^2 h_j(t - \tau) - b\lambda_j g_j(t) = 0,
\]

(31)

\[
g_j'(t) + d\lambda_j^2 g_j(t) + b\alpha_j h_j'(t) = 0,
\]

(32)

where a prime "\(t\)" denotes a one-dimensional derivative. Additionally we have initial conditions for \(h_j\) and for \(g_j\) that will be specified below.

(31) implies

\[
b\lambda_j g_j(t) = h_j''(t) + a\lambda_j^2 h_j(t - \tau)
\]

(33)

and

\[
b\lambda_j g_j'(t) = h_j'''(t) + a\lambda_j^2 h_j'(t - \tau).
\]

(34)

(32) implies

\[
b\lambda_j g_j'(t) + d\lambda_j^2 b\lambda_j g_j(t) + b^2\lambda_j^2 h_j'(t) = 0.
\]

(35)

Combining (33) - (35) we obtain

\[
h_j'''(t) + d\lambda_j^2 h_j'(t) + b^2\lambda_j^2 h_j'(t - \tau) + a\lambda_j^2 h_j(t - \tau) = 0.
\]

(36)

Conversely, if \(h_j\) satisfies (36) and \(g_j\) is defined by (32) with

\[
g_j(0) = \frac{1}{b\lambda_j} (h_j''(0) + a\lambda_j^2 h_j(-\tau)),
\]

(37)

then \((h_j, g_j)\) also satisfies (31). This can be seen as follows: Let

\[
w_j(t) := h_j''(t) + a\lambda_j^2 h_j(t - \tau) - b\lambda_j g_j(t),
\]

(38)
then we have, using (36), (32),
\[
\frac{d}{dt} w_\tau(t) + d\lambda_j^2 w_\tau(t) = h_j''(t) + a\lambda_j^2 h_j'(t-\tau) - b\lambda_j g_j(t)
\]
\[
+ d\lambda_j^2 h_j''(t) + a\lambda_j^2 h_j'(t-\tau) - b\lambda_j d^2 g_j(t)
\]
\[
= h_j''(t) + d\lambda_j^2 h_j''(t) + a\lambda_j^2 h_j'(t-\tau)
\]
\[
+ a\lambda_j^2 h_j(t-\tau) - b\lambda_j (g_j'(t) + d\lambda_j^2 g_j(t))
\]
\[
= 0.
\]
Moreover,
\[
w_\tau(0) = h_j''(0) + a\lambda_j^2 h_j(-\tau) - b\lambda_j g_j(0) = 0,
\]
by (37). Thus, by (39), (40), we conclude
\[
w_\tau(t) = 0, \quad t \geq 0,
\]
hence (31) is satisfied (which was to be proved). Now we can make the following ansatz for \(h_j\):
\[
h_j(t) = \frac{1}{\omega_j^2} e^{\omega_j t},
\]
where \((\omega_j)_j\) will be determined such that \(\Re \omega_j \to \infty\) as \(j \to \infty\).
The initial data for \(h_j\) will remain bounded as \(j \to \infty\),
\[
h_j(s) = \frac{1}{\omega_j^2} e^{\omega_j s}, \quad -\tau \leq s \leq 0, \quad h_j'(0) = \frac{1}{\omega_j}.
\]
Then \(g_j\) will be determined by (32) with initial value (37), i.e.
\[
g_j(0) := \frac{1}{b\lambda_j} (1 + \frac{a\lambda_j^2}{\omega_j^2} e^{-\omega_j \tau}),
\]
and \((g_j(0))_j\) will also be shown to be a bounded sequence. In order to satisfy the equation (36) with the ansatz (41) it is sufficient and necessary that \(\omega_j\) satisfies
\[
\omega_j^3 + d\lambda_j^2 \omega_j^2 + (b^2 \lambda_j^2 + a\lambda_j^2 e^{-\omega_j \tau}) \omega_j = -ad\lambda_j^2 e^{-\omega_j \tau}.
\]
If we can find \((\omega_j)_j\) such that the following three conditions (45) - (47) are satisfied, then part (i) of Theorem 2.1 will be proved.
For a subsequence \((\omega_{j_k})_k\), \(j_k \to \infty\) as \(k \to \infty\),
\[
\Re \omega_{j_k} \to \infty \quad \text{as } k \to \infty,
\]
\[
\sup_k \left| \frac{\lambda_{j_k}^2}{\omega_{j_k}^2} e^{-\omega_{j_k} \tau} \right| < \infty,
\]
(to assure the boundedness of \((g_j(0))_j\)),
\[
\left| \frac{e^{\omega_{j_k} t}}{\omega_{j_k}^2} \right| \to \infty \quad \text{as } k \to \infty
\]
(to assure that \(h_{j_k}(t) \to \infty\)). We shall now prove the solvability of (44) and the properties (45) - (47). For simplicity we (first for a while) drop the index \(j\), i.e. we write \(\omega = \omega_j, \lambda = \lambda_j\), and so on. Then (44) is equivalent to
\[
\omega^2 (1 + \frac{\omega}{d\lambda^2} + \frac{b^2/d + a/de^{-\omega \tau}}{\omega}) = -a\lambda^2 e^{-\omega \tau}.
\]
To solve this we make the ansatz (as in [9])

$$\omega = \mu(1 + \zeta),$$

(49)

where $|\zeta| < \frac{1}{2}$, and where $\mu$ solves

$$\mu^2 = -a \lambda^2 e^{-\tau \mu}.$$  

(50)

This problem (50) has solutions $\mu_{jk}$, for a subsequence $j_k \to \infty$, with $\Re \mu_{jk} \to \infty$ as $k \to \infty$, according to the proof of Theorem 2.1 in [9], which we repeat in the appendix for the reader’s convenience. Then (48) is equivalent to solving

$$(1 + \zeta)^2(1 + g(\zeta)) = e^{-\tau \mu \zeta},$$

(51)

where

$$q(\zeta) := \frac{\mu(1 + \zeta)}{d \lambda^2} + \frac{b^2/d + a/de^{-\tau \mu(1 + \zeta)}}{\mu(1 + \zeta)}.$$  

(52)

(51) is equivalent to

$$1 - e^{-\tau \mu \zeta} + \left( g(\zeta) + (2\zeta + \zeta^2)(1 + g(\zeta)) \right) = 0.$$  

(53)

$f$ and $g$ are holomorphic in $\Omega := \{ \zeta \mid |\zeta| < \frac{1}{10\tau |\mu|} \} \equiv B(0, \frac{1}{10\tau |\mu|})$. $f$ has in $\Omega$ exactly one zero ($\zeta = 0$) since w.l.o.g. $\frac{1}{10\tau |\mu|} < \frac{1}{2}$. $f$ satisfies on $\partial \Omega:

$$|f(\zeta)| \geq \inf_{10|\zeta| = 1} |e^{-\tau \mu \zeta} - 1| = \inf_{10|\zeta| = 1} |e^{-z} - 1| =: \bar{f} > 0,$$

$\bar{f}$ being independent of $\mu$. Moreover, we have on $\partial \Omega$

$$|\zeta| \leq \frac{C}{|\mu|},$$

(55)

where $C = \frac{1}{10\tau}$ here, but $C$ will denote different constants being independent of $j$ in the sequel. We notice that, w.l.o.g., $10\tau |\mu| < 1/2$, and that

$$|q(\zeta)| \leq C \left( |\mu| \lambda^2 + \frac{1}{|\mu|} \right) \leq \frac{C}{|\mu|},$$

(56)

where we used (50) and $|e^{-\tau \mu \zeta}| \leq e^{1/10}$ as well as

$$|\mu| \lambda^2 = \frac{1}{|\mu|} a e^{-\tau \Re \mu} \leq \frac{C}{|\mu|}.$$  

(57)

We conclude from (56) and (55)

$$|g(\zeta)| \leq \frac{C}{|\mu|}.$$  

(58)

For $\mu$ large enough we have $|f| > |g|$, by (54), (58), hence we get with the classical theorem of Rouché that there is exactly one solution $\zeta = \zeta_{jk}$ in $\Omega = \Omega_j = B(0, \frac{1}{10\tau |\mu_{jk}|})$, and $\omega_{jk} = \mu_{jk}(1 + \zeta_{jk})$ solves (44). To prove (45) we observe that

$$|\arg(\omega_{jk})| \leq |\arg(\mu_{jk})| + |\arg(1 + \zeta_{jk})| \leq \frac{\pi}{8} + \frac{\pi}{4} < \frac{\pi}{2},$$

(59)

where we used that $|\zeta_{jk}| < \frac{1}{2}$ and the fact that $0 < \arg(\mu_{jk}) \leq \frac{\pi}{2}$ which arises from the construction of $\mu_{jk}$ in the proof of Theorem 1.1 in [9], see (161) in the appendix. As a consequence we conclude the validity of (45). Using (50) we get

$$\frac{\lambda^2}{\omega^2_{jk}} e^{-\tau \omega_{jk}} \leq C |e^{-\tau \mu_{jk} \zeta_{jk}}| \leq C,$$

where $\omega_{jk}$ is defined in (44).
which assures (46).

Finally, we prove (47) as follows. For $0 < \varepsilon < 1$ we have
\[
|\frac{e^{\omega_j k t}}{\omega_j^2}| \geq C \frac{e^{\mu_j k t}}{\mu_j^2} |e^{\mu_j \zeta_j k t}| = C \frac{e^{\varepsilon \mu_j k t}}{\mu_j^2} |e^{\mu_j (1-\varepsilon + \zeta_j) k t}|.
\] (60)

Using (161) again and observing
\[
|\frac{e^{\varepsilon \mu_j k t}}{\mu_j^2}| \geq \frac{e^{\varepsilon \cos(\pi) \mu_j k t}}{\mu_j^2} \to \infty \text{ as } k \to \infty,
\] (61)
as well as
\[
|\arg(\mu_j (1 - \varepsilon + \zeta_j))| \leq \frac{\pi}{8} + |\arg(1 - \varepsilon + \zeta_j)| < \frac{\pi}{2}
\]for $\varepsilon$ small enough, implying
\[
|e^{\mu_j k (1-\varepsilon + \zeta_j) t}| \geq 1,
\] (62)
we get, combining (60) - (62) the desired relation (47), and the proof of part (i) is finished.

We remark that the behavior of $g_j$ is open.

Now we prove (ii): Making the same ansatz for $(u_j, \theta_j)$ as in (29), (30) we derive the equations
\[
h_j''(t) + a \lambda_j^2 h_j(t) = b \lambda_j g_j(t),
\] (63)
\[
g_j'(t) + d \lambda_j^2 g_j(t - \tau) + b \lambda_j h_j'(t) = 0,
\] (64)
from which the differential equation
\[
g_j''(t) + d \lambda_j^2 g_j''(t - \tau) + b^2 \lambda_j^2 g_j(t) + a \lambda_j^2 g_j'(t) + ad \lambda_j^4 g_j(t - \tau) = 0
\] (65)
follows (cp. with (36)).

If $g_j$ solves (65) (with given initial conditions) and if $h_j$ solves (63) with initial conditions satisfying
\[
h_j(0) = \frac{1}{b a \lambda_j^4} (b^2 \lambda_j^2 g_j(0) + g_j''(0) + d \lambda_j^2 g_j'(-\tau)),
\] (66)
\[
h_j'(0) = -\frac{1}{b \lambda_j} (g_j'(0) + d \lambda_j^2 g_j(-\tau)),
\] (67)
then $(h_j, g_j)$ solves (63), (64), which can be seen looking at
\[
w_j(t) := g_j'(t) + d \lambda_j^2 g_j(t - \tau) + b \lambda_j h_j'(t)
\] (68)
and deriving the relation
\[
\frac{d^2}{dt^2} w_j(t) + a \lambda_j^2 w_j(t) = 0, \quad w_j(0) = 0, \quad \frac{d}{dt} w_j(0) = 0,
\] (69)
the latter given by (66), (67). This implies $w_j(t) = 0$ for $t \geq 0$, which is equivalent to (64).

For $g_j$ we make the ansatz
\[
g_j(t) = \frac{1}{\omega_j^2} e^{\omega_j t},
\] (70)
implying the boundedness of the initial data $g_j(s), -\tau \leq s \leq 0$, $g'(0), g''(0)$, as $j \to \infty$, since $\Re \omega_j \to \infty$ will be shown ($k \to \infty, j_k \to \infty$). $h_j$ will then be determined by (63), (66), (67), and the data prescribed in (66), (67) will be shown to be bounded too.
In order to satisfy the equation (65) with the ansatz (70) it is sufficient (and necessary) that \( \omega_j \) satisfies
\[
\omega_j^2 + d \lambda_j^2 \omega_j e^{-\tau \omega_j} + b^2 \lambda_j^2 \omega_j + a \lambda_j^2 \omega_j = -ad \lambda_j^2 e^{-\tau \omega_j}.
\] (71)

If we can find \((\omega_j)_j\) such that the conditions (45) - (47) are satisfied, then part (ii) will be proved.

Dropping the index \( j \) again, (71) is equivalent to
\[
\omega(1 + \frac{\omega^2}{(a + b^2)\lambda_j^2}) + \frac{d}{a + b^2} \omega e^{-\tau \omega} = -\frac{ad}{a + b^2} \lambda_j^2 e^{-\tau \omega}.
\] (72)

**Remark:** Whether we choose \( \omega^2 \) (as in part (i), cp. (48)) or \( \omega \) (in (72)) as a factor depends on the powers of \( \lambda_j \) and the position of \( e^{-\tau \omega} \) on the left-hand side. The aim is to get the right estimate (76) for \( q \) below (cp. (56)).

Let \( \mu = \mu_{j_k} \) be the solution(s) to
\[
\mu = -\frac{ad}{a + b^2} \lambda_j^2 e^{-\tau \mu}
\] (73)

(for the existence see again the appendix), satisfying \( \Re \mu_{j_k} \to \infty \) for a subsequence as \( j_k \to \infty \) for \( k \to \infty \). Then, with the ansatz (49), (72) is equivalent to solving
\[
(1 - e^{-\tau \mu}) + (q(\zeta) + \zeta + \zeta q(\zeta)) = 0,
\] (74)

where
\[
q(\zeta) := \frac{\mu^2 (1 + \zeta)^2}{(a + b^2)\lambda_j^2} + \frac{d}{a + b^2} \mu (1 + \zeta) e^{-\tau \mu (1 + \zeta)}.
\] (75)

\( f \) and \( g \) are holomorphic in \( \Omega := B(0, \frac{1}{107\| \mu \|}) \). \( f \) has exactly one zero \((\zeta = 0) \) in \( \Omega \) since w.l.o.g. \( \frac{1}{107\| \mu \|} < \frac{1}{2} \). On \( \partial \Omega \) we have again (54), (55) and
\[
|q(\zeta)| \leq \frac{C}{|\mu|} \frac{|\mu^2 e^{-\tau \mu}|}{|\mu|} \leq \frac{C}{|\mu|},
\] (76)

since \( |\mu|^2 e^{-\tau \Re \mu} = |\mu|^2 e^{-\tau |\mu| \cos(\arg(\mu))} \to 0 \). This follows because \( \cos(\arg(\mu)) \geq \cos(\frac{\pi}{8}) \), where we used (161) again. Hence
\[
|g(\zeta)| \leq \frac{C}{|\mu|},
\] (77)

and (54), (77) combined with Rouche’s theorem gives exactly one solution \( \zeta = \zeta_{j_k} \) in \( \Omega \) to (74), and \( \omega_{j_k} = \mu_{j_k} (1 + \zeta_{j_k}) \) solves (71). The relation (45) is proved as in (59) (replacing \( \pi/8 \) by \( \pi/4 \)).

Using (49) we get
\[
|\frac{\lambda_{j_k} e^{-\tau \omega_{j_k}}}{\omega_{j_k}^2} \frac{1}{a d} | \frac{1}{|\lambda_{j_k} \omega_{j_k}|} |1 + \zeta_{j_k}| e^{-\tau \mu_{j_k} \zeta_{j_k}}| \leq C,
\]
proving (47). Finally, (46) follows as in the proof of part (i), see (60) - (62) (replacing \( \pi/8 \) by \( \pi/4 \)). Thus, Theorem 2.1 is proved. \( \Box \)

**Remark:** We have proved the exploding of the solution in \( L^2 \) for bounded data in \( L^2 \). Coming from semigroup theory in the case \( \tau = 0 \) (without delay), one might argue that usually \( V := (u_x, u_t, \theta) \) in \( L^2 \) or \( V := (u, u_t, \theta) \) with gradient norm for \( u \) is considered, and, hence, one should prove the exploding of \( u_x \) in \( L^2 \) for data
with bounded norm $u_0^0$. But this can also be achieved replacing in part (i) — for example — the ansatz (41) by
\[
h_j(t) = \frac{1}{\lambda_j \omega_j^2} e^{\omega_j t}.
\] (78)

Then everything (but one thing) carries over literally; only the arguments to prove now
\[
\left| \frac{e^{\omega_j t}}{\lambda_j \omega_j^2} \right| \to 0, \quad k \to \infty,
\]
given before in (60) - (62) have to be slightly modified in (61) to
\[
\left| \frac{e^{\mu_j t}}{\lambda_j \omega_j^2} \right|^2 = \left| \frac{e^{2\mu_j t}}{\lambda_j^2 \omega_j^4} \right| = d \frac{e^{2(2t-\tau)\mu_j \cos(\arg(\mu_j))}}{|\omega_j|^6},
\] (79)
where we used (73) again, hence
\[
\left| \frac{e^{\mu_j t}}{\lambda_j \omega_j^2} \right| \to \infty \text{ as } k \to \infty
\] (80)
if
\[
t > \frac{\tau}{2\varepsilon}.
\] (81)
That is, we obtain the instability for $t > 0$ satisfying (81).

3. Thermoelastic plates with delay terms. With the same methods as for the second-order thermoelastic systems in Section 2, we can deal with the system for thermoelastic plates with delay (14), (15) and with the related system
\[
u_{tt}(t,x) + a \Delta^2 u(t,x) + b \Delta \theta(t,x) = 0,
\] (82)
\[
\theta_t(t,x) - d \Delta \theta(t-\tau,x) - b \Delta u_t(t,x) = 0,
\] (83)
where $u, \theta : [0, \infty) \times G \to \mathbb{R}$, and $a, b, d, \tau > 0$ as before, and $G$ is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$.

Additionally, one has the boundary conditions (16) and initial conditions,
\[
u(s, \cdot) = u_0^0(s), (-\tau \leq s \leq 0), \quad u_i(0, \cdot) = u^1, \quad \theta(0, \cdot) = \theta^0,
\] (84)
and
\[
u(0, \cdot) = u^0, \quad u_t(0, \cdot) = u^1, \quad \theta(s, \cdot) = \theta^0(s), (-\tau \leq s \leq 0),
\] (85)
respectively.

Replacing $\lambda_j = \frac{j\pi}{L}$ and $\varphi_j$ from (29) by the eigenvalues $(\lambda_j)_j$ of the Laplace operator $(-\Delta_D)$ in $L^2(G)$ with Dirichlet boundary conditions, we can make the ansatz (cp. (29), (30)).
\[
u = u_j(t,x) = \varphi_j(x) h_j(t),
\] (86)
\[
\theta = \theta_j(t,x) = \varphi_j(x) g_j(t).
\] (87)

Then the methods of the proof of Theorem (2.1) carry over, and we have

**Theorem 3.1.** (i) The initial-boundary value problem with delay (14), (15), (16), (84) is not well-posed. There exists a sequence $(u_j, \theta_j)_j$ of solutions with $L^2$-norm $\|u_j(t, \cdot)\|$ tending to infinity (as $j \to \infty$) for any fixed $t > 0$, while for the initial data the norms $\sup_{-\tau \leq s \leq 0} \|u_0^0(s), u^1, \theta^0\|$ remain bounded.

(ii) The corresponding statement on ill-posedness also holds for the initial-boundary value problem with delay (82), (83), (16), (85).
The remarks at the end of Section 2 including (here) the $L^2$-norm of $\Delta u_j$ carry over mutatis mutandis.

We do not give details of the proof since the thermoelastic plate systems with delay (14), (15), resp. (82), (83) are a special case of the general $\alpha$-$\beta$-system with delay to be studied in the next section.

4. The $\alpha$-$\beta$-system with delay terms. As a generalization of the thermoelastic plate system — well discussed for $\tau = 0$ — we study the $\alpha$-$\beta$-system with delay (17), (18) and the related system (22), (23) for functions $u, \theta : [0, \infty) \to \mathcal{H}$, $\mathcal{H}$ a separable Hilbert space, $A$ being a linear self-adjoint operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$, having a complete orthonormal system of eigenfunctions $(\varphi_j)_j$ with corresponding eigenvalues $0 < \lambda_j \to \infty$ as $j \to \infty$. $a, b, d, \tau > 0$ are as before, and $0 \leq \beta \leq \alpha \leq 1$ (88) are parameters. The thermoelastic plate system from Section 3 is given by $\alpha = \beta = \frac{1}{2}$ and $A = (-\Delta_D)^2$. As usual, we have the conditions

$$u(t) \in D(A), \theta(t) \in D(A^\alpha), t \geq 0,$$

and initial conditions,

$$u(s) = u^0(s), (-\tau \leq s \leq 0), \quad u_t(0) = u^1, \quad \theta(0) = \theta^0,$$

and

$$u(0) = u^0, \quad u_t(0) = u^1, \quad \theta(s) = \theta^0(s), (-\tau \leq s \leq 0),$$

respectively.

For $(\beta, \alpha)$ in the region

$$A^1_{in} = \{(\beta, \alpha)| 0 \leq \beta \leq \alpha \leq 1, \beta \geq \frac{1}{2}, (\beta, \alpha) \neq (1, 1)\},$$

see Figure 1.3, we get the following ill-posedness result for the delay problem (17), (18):

**Theorem 4.1.** Let $(\beta, \alpha) \in A^1_{in}$. Then the delay problem (17), (18), (89), (90) is not well-posed. There exists a sequence $(u_j(t), \theta_j(t))_j$ of solutions with norm $\|u_j(t)\|_\mathcal{H}$ tending to infinity (as $j \to \infty$) for any fixed $t$, while for the initial data the norms $\sup_{-\tau \leq s \leq 0} \|u^0_j(s), u^1_j, \theta^0_j\|_\mathcal{H}$ remain bounded.

For $(\beta, \alpha)$ in the region

$$A^2_{in} := \{(\beta, \alpha)| 0 < \beta \leq \alpha \leq 1, (\beta, \alpha) \neq (1, 1)\},$$

see Figure 1.4, we get the following ill-posedness result for the delay problem (22), (23):

**Theorem 4.2.** Let $(\beta, \alpha) \in A^2_{in}$. Then the delay problem (22), (23), (89), (91) is not well-posed. There exists a sequence $(u_j(t), \theta_j(t))_j$ of solutions with norm $\|\theta_j(t)\|_\mathcal{H}$ tending to infinity (as $j \to \infty$) for any fixed $t$, while for the initial data the norms $\sup_{-\tau \leq s \leq 0} \|u^0_j, u^1_j, \theta^0_j(s)\|_\mathcal{H}$ remain bounded.

**Proof Theorem 4.1.** : We make the ansatz

$$u = u_j(t) = h_j(t) \varphi_j, \quad (94)$$

$$\theta = \theta_j(t) = g_j(t) \varphi_j. \quad (95)$$
As in the specific examples in Sections 2 and 3, we look for a solution \( h_j \) to the third-order equation
\[
h_j'''(t) + d\lambda_j^2 h_j'(t) + b^2 \lambda_j^{2\beta} h_j'(t) + a\lambda_j h_j'(t - \tau) + ad\lambda_j^{1+\alpha} h_j(t - \tau) = 0,
\]
(96)
derived from the ansatz (94), (95), and, then, for \( g_j \) satisfying
\[
g_j'(t) + d\lambda_j^2 g_j(t) = -b\lambda_j^2 h_j'(t),
\]
(97)
with
\[
g_j(0) := \frac{1}{b\lambda_j^2} (h_j''(0) + a\lambda_j h_j(-\tau)).
\]
(98)
Then \((u_j, \theta_j)\) satisfy (17), (18), cp. the arguments in (38) - (40).

Making the ansatz
\[
h_j(t) = \frac{1}{\omega_j^2} e^{\omega_j t},
\]
(99)
we shall obtain \((\omega_j)_j\) such that \(\Re \omega_j \to \infty\) as \(j \to \infty\), at least for a subsequence \((\omega_{jk})_k\). In order to satisfy the equations (96) with the ansatz (99) it is sufficient (and necessary) that \(\omega_j\) satisfies
\[
\omega_j^3 + d\lambda_j^2 \omega_j^2 + (b^2 \lambda_j^{2\beta} + a\lambda_j e^{-\tau \omega_j}) \omega_j = -ad\lambda_j^{1+\alpha} e^{-\tau \omega_j}.
\]
(100)
If we find \((\omega_j)_j\) (or a subsequence \((\omega_{jk})_k\)) such that the conditions
\[
\Re \omega_{jk} \to \infty \text{ as } k \to \infty,
\]
(101)
\[
\sup_k \left| \frac{\lambda_j^{2-\beta} e^{-\tau \omega_{jk}}}{\omega_j^2} \right| < \infty
\]
(102)
(to assure the boundedness of the data), and
\[
\text{for } t > 0: \left| \frac{e^{\omega_{jk} t}}{\omega_j^2} \right| \to \infty \text{ as } k \to \infty,
\]
(103)
are satisfied, then Theorem 4.1 will be proved. To get (101) - (103) we have to distinguish two cases (in order to guarantee the boundedness of the corresponding functions \(q\) below).

Case 1:
\[
\alpha < 2\beta.
\]
(104)
Now, (100) is equivalent to
\[
\omega_j \left(1 + \frac{\omega_j^2}{b^2 \lambda_j^{2\beta}} + \frac{d\omega_j}{b^2 \lambda_j^{2\beta-\alpha}} + \frac{a}{b^2} \lambda_j^{1-2\beta} e^{-\tau \omega_j} \right) = -\frac{ad}{b^2} \lambda_j^{1+\alpha-2\beta} e^{-\tau \omega_j}.
\]
(105)
We make the ansatz (49), where \(\mu\) is the solution to
\[
\mu = -\frac{ad}{b^2} \lambda_j^{1+\alpha-2\beta} e^{-\tau \mu},
\]
(106)
see the appendix, since \(1 + \alpha - 2\beta > 0\) due to \((\beta, \alpha) \neq (1,1)\).

Then, solving (105) is equivalent to solving
\[
(1 - e^{-\tau \mu \zeta}) + (q(\zeta) + \zeta + \zeta q(\zeta)) = 0,
\]
(107)
where
\[
q(\zeta) := \frac{\mu^2 (1 + \zeta)^2}{b^2 \lambda_j^{2\beta}} + \frac{d\mu (1 + \zeta)}{b^2 \lambda_j^{2\beta-\alpha}} + \frac{a}{b^2} \lambda_j^{1-2\beta} e^{-\tau \mu (1+\zeta)}.
\]
(108)
\( f \) and \( g \) are holomorphic in \( \Omega := B(0, \frac{1}{10|\mu|}) \), where \( 10\tau|\mu| > 2 \) w.l.o.g. \( f \) has exactly one zero (\( \zeta = 0 \)) in \( \Omega \). On \( \partial \Omega \) we have again (54) and (55), and

\[
|q(\zeta)| \leq \frac{C}{|\mu|} \left( \left| \frac{\mu^3}{\lambda^{2\beta}} \right| + \left| \frac{\mu^2}{\lambda^{2\beta-\alpha}} \right| + \left| \frac{\mu^2}{\lambda^{2\alpha+\beta}} \right| \right),
\]

(109)

where we used (106) which also yields, for \( q \in \{2\beta, 2\beta - \alpha, 2\alpha + \beta\} \), and for \( \delta_\mu := g/(1 + 2\alpha - \beta) \),

\[
\frac{1}{\lambda^e} = \left( \frac{ad}{b^2} \right) \delta_\mu e^{-\tau \delta_\mu \Re \mu},
\]

(110)

which implies

\[
|p_1(\mu)| \leq C \sum_{m=1}^{3} |\mu|^{3-\delta_{\mu, m}} e^{-\tau \delta_{\mu, m} \Re \mu},
\]

(111)

where

\[
\delta_0 := 2\beta, \delta_1 := 2\beta - \alpha, \delta_2 := 2\alpha + \beta
\]

such that \( \delta_{\mu, m} > 0, m = 1, 2, 3 \), by our assumption (104).

**Remark:** \( 3 - \delta_{\mu, m} > 0, m = 1, 2, 3, \)

We conclude for \( m = 1, 2, 3 \)

\[
|\mu_{jk}|^{3-\delta_{\mu, m}} e^{-\tau \delta_{\mu, m} \Re \mu_{jk}} \to 0 \text{ as } k \to \infty
\]

(112)

(since \( \cos \arg(\mu_{jk}) \geq c_0 > 0 \) as before). Combining (109) - (112) we conclude

\[
|g(\zeta)| \leq \frac{C}{|\mu|},
\]

(113)

and, hence, using again Rouché’s theorem, that there is exactly one solution \( \zeta_{jk} \) in \( \Omega \) to (107), and \( \omega_{jk} = \mu_{jk}(1 + \zeta_{jk}) \) solves (100). The relation (101) is proved as in (59). Regarding (102) we estimate, using (106),

\[
\left| \frac{\lambda_{jk}^{2-\beta} e^{-\tau \omega_{jk}}}{\omega_{jk}^{2}} \right| \leq \frac{C}{\lambda_{jk}^{2\alpha-1}} \leq C,
\]

since \( \alpha \geq 1/2. \)

Finally, (103) follows as in the proof of part (i) of Theorem (2.1), see (60) - (62). This gives the assertion in case 1.

Case 2:

\[
\alpha \geq 2\beta.
\]

(114)

Proceeding in the same spirit as before, we recognize (100) to be equivalent to (dropping the index \( j \) again)

\[
\omega^2 \left( 1 + \frac{\omega}{d\lambda^\alpha} + \frac{b^2}{\lambda^{\alpha-2\beta} \omega} + \frac{a\lambda^{1-\alpha}}{d\omega} e^{-\tau \omega} \right) = -a\lambda e^{-\tau \omega},
\]

(115)

for which we make the ansatz (49), \( \mu \) solving

\[
\mu^2 = -a\lambda e^{-\tau \mu}
\]

(116)

according to the proof of Theorem 1.1 from [9], see the appendix. Then (115) is equivalent to

\[
(1 - e^{-\tau \mu \zeta}) + (q(\zeta) + 2\zeta + \zeta^2 + (2\zeta + \zeta^2) q(\zeta)) = 0,
\]

(117)
where
\[ q(\zeta) := \frac{\mu(1 + \zeta)}{d\lambda^{\alpha}} + \frac{b^2}{\lambda^{\alpha-2}\mu(1 + \zeta)} + \frac{a\lambda^{1-\alpha}}{d\mu(1 + \zeta)} e^{-\tau\mu(1+\zeta)}. \] (118)

If is the same as in case 1. We estimate
\[ |q(\zeta)| \leq \frac{C}{|\mu|} \bigg( |\frac{\mu^2}{\lambda^{\alpha}}| + 1 + |\lambda^{1-\alpha} e^{-\tau\mu} e^{-\tau\mu\zeta}| \bigg) \leq \frac{C}{|\mu|}, \] (119)
since, by (116),
\[ \left| \frac{\mu^2}{\lambda^{\alpha}} \right| = a^\alpha |\mu|^{2(1-\alpha)} e^{-\alpha\tau |\mu| \cos(\arg(\mu))} \to 0, \]
because \( \alpha > 0 \), and, analogously,
\[ |\lambda^{1-\alpha} e^{-\tau\mu} e^{-\tau\mu\zeta}| \leq C |\mu|^{2(1-\alpha)} e^{-\alpha\tau |\mu| \cos(\arg(\mu))} \to 0. \]
This implies
\[ |g(\zeta)| \leq \frac{C}{|\mu|}, \] (120)
and, with Rouché’s theorem, we get a unique solution \( \zeta_{jk} \) in \( \Omega \) to (117), and \( \omega_{jk} = \mu_{jk}(1 + \zeta_{jk}) \) solves (105). The remaining relations (101) - (103) follow as in case 1.

Proof of Theorem 4.2.: Making the same ansatz (94), (95) as in the proof of Theorem 4.1 we look for a solution \( g_j \) to
\[ g_j'''(t) + d\lambda_j^{\alpha} g_j''(t - \tau) + a\lambda_j^{\alpha^2} g_j'(t) + b^2 \lambda_j^{2\alpha} g_j(t) + ad\lambda_j^{1+\alpha} g_j(t - \tau) = 0, \] (121)
and, then, for \( h_j \) satisfying
\[ h_j''(t) + a\lambda_j h_j(t) = b\lambda_j^{\beta} g_j(t), \] (122)
with
\[ h_j(0) = \frac{1}{ab\lambda_j^{1+\beta}} \left( b^2 \lambda_j^{2\alpha} g_j(0) + g_j''(0) + d\lambda_j^{\alpha} g_j'(-\tau) \right), \] (123)
\[ h_j'(0) = -\frac{1}{b\lambda_j^{\beta}} \left( g_j'(0) + a\lambda_j^{\alpha} g(-\tau) \right). \] (124)
Then \( (u_j, \theta_j) \) satisfy (22), (23), cp. the arguments in (68), (69).

For \( g_j \) we make the ansatz
\[ g_j(t) = \frac{1}{\omega_j^\alpha} e^{\omega_j t}. \] (125)

We shall obtain \( (\omega_{jk})_k \) such that \( \Re \omega_{jk} \to \infty \) as \( j_k \to \infty \) as \( k \to \infty \). To satisfy (121) by (125) it is sufficient (and necessary) that \( \omega_j \) satisfies
\[ \omega_j^3 + d\lambda_j^{\alpha} e^{-\tau\omega_j} \omega_j^2 + (a\lambda_j + b^2 \lambda_j^{2\alpha}) \omega_j = -ad\lambda_j^{1+\alpha} e^{-\tau\omega_j}. \] (126)
We have to find \( (\omega_j)_j \) resp. a subsequence \( (\omega_{jk})_k \) such that the conditions (101)
\[ \sup_k |\lambda_j^{\alpha-\beta} e^{-\tau\omega_{jk}}| < \infty, \] (127)
(to assure the boundedness of the data), and (103) are satisfied; then the theorem will be proved.

Case 1:
\[ 0 < \beta \leq 1/2. \] (128)
Then (126) is equivalent to
\[
\omega_j \left(1 - \frac{b^2}{a \lambda_j^{1-2\beta}} + \frac{\omega_j^2}{a \lambda_j} + \frac{d}{a} \lambda_j^{-1} e^{-\tau \omega_j} \right) = -d \lambda_j^\alpha e^{-\tau \omega_j}. \tag{129}
\]

The ansatz (49), where \( \mu \) is the solution to
\[
\mu = -d \lambda \, e^{-\tau \mu}, \tag{130}
\]
according to Theorem 1.1 in [9], observe \( \alpha > 0 \), yields that solving (129) is equivalent to solving
\[
\left(1 - e^{-\tau \mu} \right) (q(\zeta) + \zeta + \zeta q(\zeta)) = 0, \tag{131}
\]
where
\[
q(\zeta) := -\frac{b^2}{a \lambda^{1-2\beta}} + \frac{\mu^2 (1 + \zeta)^2}{a \lambda} + \frac{d}{a} \lambda^{-1} \mu (1 + \zeta) e^{-\tau \mu (1 + \zeta)}. \tag{132}
\]

With (54), (55) analogously, we additionally get on \( \partial \Omega \) (with \( \Omega = B(0, \frac{1}{10 \tau |\mu|}) \) again)
\[
|q(\zeta)| \leq C \left| \frac{\mu}{\lambda^{1-2\beta}} \right| + \left| \frac{\mu^3 \lambda}{\lambda} \right| + |\lambda^{-1} \mu^2 e^{-\tau \mu}| \tag{133}
\]
W.l.o.g. we assume \( \beta < 1/2 \) since for \( \beta = 1/2 \) we may divide (126) by \( (a + b^2) \lambda \) and replace (130) by
\[
\mu = -\frac{ad}{a + b^2} \lambda \, e^{-\tau \mu}, \tag{134}
\]
and (132) by
\[
q(\zeta) = -\frac{b^2}{a \lambda^{1-2\beta}} + \frac{\mu^2 (1 + \zeta)^2}{(a + b^2) \lambda} + \frac{d}{a} \lambda^{-1} \mu (1 + \zeta) e^{-\tau \mu (1 + \zeta)}, \tag{135}
\]
and then (133) by
\[
|q(\zeta)| \leq C \left| \frac{\mu}{\lambda^{1-2\beta}} \right| + |\lambda^{-1} \mu^2 e^{-\tau \mu}| \tag{136}
\]
and finally continue as follows (now for \( \beta < 1/2 \) again).

By (130) we have for \( \varrho > 0 \)
\[
\frac{1}{\lambda} = d \varrho \, e^{-\tau \delta_\varrho \Re \mu} \tag{137}
\]
with
\[
\delta_\varrho := \frac{\varrho}{\alpha}. \tag{138}
\]
as well as
\[
e^{-\tau \mu} = -\frac{1}{d} \frac{\mu}{\lambda^{\alpha}}. \tag{139}
\]
Hence, using (130) again,
\[
|p_2(\mu)| \leq C \left( \left| \frac{\mu}{\lambda^{1-2\beta}} \right| + \left| \frac{\mu^3}{\lambda} \right| + \left| \frac{\mu^3}{\lambda^{2\beta}} \right| \right) \leq C \sum_{m=1}^3 |\mu|^{3-\varrho_m} e^{-\tau \delta_\varrho_m \Re \mu}, \tag{140}
\]
where
\[
\varrho_1 := 1 - 2\beta, \varrho_2 := 1, \varrho_3 := 1, \tag{141}
\]
assuring \( \varrho_m > 0, m = 1, 2, 3 \), since \( \beta < 1/2 \) by our assumptions.
As in the proofs before, we successively conclude

$$|p_2(\mu)| \leq C, \quad |q(\zeta)| \leq \frac{C}{|\mu|},$$

hence

$$|q(\zeta)| \leq \frac{C}{|\mu|}. \quad (142)$$

This implies again the existence of exactly one solution $\zeta_{jk}$ in $\Omega$ to (131), and $\omega_{jk} = \mu_{jk}(1 + \zeta_{jk})$ solves (126).

The relation (101) is proved as in (59). Regarding (127) we estimate, using $\beta > 0$ and (130),

$$|\lambda^{\alpha - \beta} e^{-\tau \omega_{jk}}| \leq C|\mu_{jk}|^{1 - \frac{\beta}{2}} e^{-\frac{\beta}{2} \|\mu_{jk}\|} \leq C. \quad (143)$$

Finally, (103) follows as in the proof of part (i) of Theorem 2.1, see (60) – (62). This proves the assertion in case 1.

Case 2: $1/2 < \beta < 1.$

Now, (126) is equivalent to

$$\omega_j \left(1 + \frac{a}{b^2 \lambda_{2j}^2 - 1} + \frac{\omega_j^2}{b^2 \lambda_{2j}^2} + \frac{d\omega_j}{b^2 \lambda_{2j}^2} e^{-\tau \omega_j} \right) = -\frac{ad}{b^2} \lambda^{1 + \alpha - 2\beta} e^{-\tau \mu}. \quad (145)$$

We observe that for $\beta < 1$ we have $1 + \alpha - 2\beta > 0$. The ansatz (49) where $\mu$ solves

$$\mu = -\frac{ad}{b^2} \lambda^{1 + \alpha - 2\beta} e^{-\tau \mu}, \quad (146)$$

plugged into (145) yields to solve

$$\left(1 - e^{-\tau \mu \zeta}\right) (q(\zeta) + \zeta + \zeta q(\zeta)) = 0 \quad (147)$$

where

$$q(\zeta) := \frac{a}{b^2 \lambda_{2j}^2 - 1} + \frac{\mu^2(1 + \zeta)^2}{b^2 \lambda_{2j}^2} + \frac{d\mu(1 + \zeta)}{b^2 \lambda_{2j}^2} e^{-\tau \mu(1 + \zeta)}.$$ Estimating

$$|q(\zeta)| \leq \frac{C}{|\mu|} \left|\frac{\mu}{\lambda^{2j} - 1} + \frac{\mu^3}{\lambda_{2j}^3} + |\lambda^{\alpha - 2\beta} \mu^2 e^{-\tau \mu}| \right|, \quad (148)$$

using — by (146) — for $\varrho > 0$,

$$\frac{1}{\lambda^e} = \left(\frac{ad}{b^2}\right) \frac{\varrho e^{-\tau \delta e \|\mu\|}}{|\mu|^\delta_e}, \quad (149)$$

where

$$\delta_e := \frac{\varrho}{1 + \alpha - 2\beta}, \quad (150)$$

and

$$e^{-\tau \mu} = -\frac{b^2}{ad} \frac{\mu}{\lambda^{1 + \alpha - 2\beta}}, \quad (151)$$

we obtain

$$|p_3(\mu)| \leq C \left|\frac{\mu}{\lambda^{2j} - 1} + \frac{\mu^3}{\lambda_{2j}^3} + \frac{\mu^3}{\lambda^e}\right| \leq C \sum_{m=1}^{3} |\mu|^{3 - \vartheta_m} e^{-\tau \delta_{\vartheta_m} \|\mu\|}, \quad (152)$$
where
\[ g_1 := 2\beta - 1, \quad g_2 := 2\beta, \quad g_3 := 1 \]
(153)
satisfy \( g_m > 0, \quad m = 1, 2, 3. \)

As before we conclude
\[ |p_3(\mu)| \leq C, \quad |q(\zeta)| \leq \frac{C}{|\mu|}, \]
implying the existence of exactly one solution \( \zeta_{jk} \) in \( \Omega \) to (131), and \( \omega_{jk} = \mu_{jk}(1 + \zeta_{jk}) \) solves (126). The relation (101) is proved as in (59). Regarding (127) we estimate, using (130),
\[ |\lambda_j^{\alpha - \beta} e^{-\tau \omega_{jk}}| \leq C \frac{\mu_{jk}}{\lambda_j^{\alpha - \beta}} \leq C |\mu_{jk}|^{-\frac{b}{1+\alpha-\beta}} e^{-\tau \frac{b}{1+\alpha-\beta}} \leq C. \]
(154)

Finally, (103) follows as in case 1. This finishes the proof of Theorem 4.2.

We notice the interesting difference \( A_1 \neq A_2 \). For (17), (18) the damping through \( \theta \) has to be weak enough ("\( \alpha \geq \frac{1}{2} \)") to still guarantee the ill-posedness suggested by the equation (17) for \( u \).

One should also notice that the conditions \( (\beta, \alpha) \in A_j \) in \( j = 1, 2 \), are sufficient for the instability, the behavior in the region outside \( A_j \) is an open question.

It would be possible to study with the methods above even more general systems like
\[ u_{tt}(t) + Au(t - g_1\tau) + bA^\beta \theta(t - g_2\tau) = 0, \]
\[ \theta_t(t) + dA^\alpha \theta(t - g_3\tau) + bA^\beta u_t(t - g_4\tau) = 0, \]
(155)
where \( g_j \in \{0, 1\}, \quad j = 1, 2, 3, 4. \) Here the cases \( g_1 = g_2 = g_3 = g_4 = 0 \) and \( g_1 = g_2 = g_3 = 1 = g_4 = 0 \) were treated, for simplicity.

5. Appendix. For the reader’s convenience, we recall the essential parts of the proof of Theorem 1.1 in [9] which we used in the previous sections. We look for solutions \( \omega_t \) to the equation
\[ \omega^n_t = e^{-\omega\tau} \xi_t, \]
(157)
where \( \xi_t \to -\infty \) as \( l \to \infty \), and \( n \in \mathbb{N}. \) Dropping the index \( l \) for simplicity and writing \( \omega = re^{i\varphi} \) with \( 0 \leq r < \infty \) and \( 0 \leq \varphi < 2\pi \), we get from (157)
\[ \xi = r^n e^{in\varphi} e^{i\varphi} = r^n e^{r\tau \cos \varphi} e^{i(n\varphi + r\tau \sin \varphi)}. \]
(158)
Since \( \xi < 0 \), we wish to solve
\[ r^n e^{r\tau \cos \varphi} = |\xi|, \quad n\varphi + r\tau \sin \varphi = \pi, \]
(159)
which implies
\[ r = \frac{\pi - n\varphi}{\tau \sin \varphi}. \]
(160)
It is advantageous to place an additional restriction on \( \varphi \):
\[ 0 < \varphi \leq \frac{\pi}{4n}. \]
(161)
Substituting (160) into (158) we obtain
\[ \psi(\varphi) := (\pi - n\varphi)^n e^{(\pi - n\varphi) \cot \varphi} - \frac{|\xi|^n \sin^n \varphi}{\tau^n \sin \varphi} = 0. \]
(162)
Our aim is to show that (162) always has a zero in \((0, \pi/(4n))\) whenever \(|\xi|\) is large enough.
We have \(\lim_{\varphi_{l,10}} \psi(\varphi) = \infty\) and
\[
\psi\left(\frac{\pi}{4n}\right) = \left(\frac{3\pi}{4}\right)^n e^{\frac{3\pi}{4} \cos\left(\frac{\pi}{4n}\right)} - |\xi|^n \sin^n\left(\frac{\pi}{4n}\right) \to -\infty \text{ as } |\xi| \to \infty,
\]
which gives us a number \(\varphi \in (0, \frac{\pi}{4n})\) solving (162). Hence, using
\[
\cos \varphi_{l} \geq \cos \frac{\pi}{4n} > 0,
\]
due to (161), there is an \(m \in \mathbb{N}\) such that for all \(l \geq m\) there is a solution \(\omega_{l} = r_{l} e^{i\varphi_{l}}\) to (157) such that
\[
\Re \omega_{l} = r_{l} \cos \varphi_{l} \to \infty \text{ as } l \to \infty, \quad r_{l} \to \infty \text{ as } l \to \infty,
\]
where we used \(\infty \leftarrow |\xi_{l}| = r_{l}^{n} e^{\tau_{l} \cos \varphi_{l}}\).

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E-mail address: reinhard.racke@uni-konstanz.de