Krein–Adler transformations for shape-invariant potentials and pseudo virtual states

Satoru Odake\(^1\) and Ryu Sasaki\(^2\)

\(^1\) Department of Physics, Shinshu University, Matsumoto 390-8621, Japan
\(^2\) Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

E-mail: ryu@yukawa.kyoto-u.ac.jp

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Abstract

For 11 examples of one-dimensional quantum mechanics with shape-invariant potentials, the Darboux–Crum transformations in terms of multiple pseudo virtual state wavefunctions are shown to be equivalent to Krein–Adler transformations, deleting multiple eigenstates with shifted parameters. These are based upon infinitely many polynomial Wronskian identities of classical orthogonal polynomials, i.e. the Hermite, Laguerre and Jacobi polynomials, which constitute the main part of the eigenfunctions of various quantum mechanical systems with shape-invariant potentials.

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1. Introduction

Virtual state wavefunctions are essential for the construction of the multi-indexed Laguerre and Jacobi polynomials \([1, 2]\). They are polynomial-type solutions of one-dimensional Schrödinger equations for shape-invariant potentials \([3–5]\). They are characterized as having negative energies (the groundstate has zero energy) and no zeros in the physical domain, and they and their reciprocals are square non-integrable. By dropping the condition of no zeros and requiring the reciprocals to be square integrable at both boundaries \((2, 32)\), pseudo virtual state wavefunctions are obtained. In most cases, the virtual and pseudo virtual state wavefunctions are obtained from the eigenfunctions by twisting the parameter(s) based on the discrete symmetries of the Hamiltonian \([1]\). Starting from a shape-invariant potential, a Darboux transformation \([6, 7]\) in terms of a nodeless pseudo virtual state wavefunction \(\tilde{\phi}(x)\) with energy \(\tilde{E}\) produces a solvable system with an extra eigenstate below the original groundstate with energy \(\tilde{E}\) and eigenfunction \(\tilde{\phi}(x)^{-1}\). This method of generating a solvable system by ‘adding an eigenstate’ below the groundstate has been known for many years, starting from the simplest harmonic oscillator potential examples \([8]\) and followed by the work of many other authors \([9–14]\). As remarked by Adler \([15]\) for the harmonic oscillator case and generalized
by the present authors [16] for other potentials, such a system can be derived by special
types of Krein–Adler transformations—that is, the Krein–Adler transformation for a system
with negatively shifted parameters in which the created state will be the groundstate. The
transformation uses all the eigenstates between the new and the original groundstates.
In this paper we present a straightforward generalization of the above result for various
shape-invariant potentials listed in section 3: the Coulomb potential with the centrifugal barrier
(C), the Kepler problem in spherical space (K), the Morse potential (M), the soliton potential
(s), the Rosen–Morse (RM) potential, the hyperbolic symmetric top II (hst), the Kepler problem
in hyperbolic space (Kh), the hyperbolic Darboux–Pöschl–Teller (hDPT) potential, on top of
the well-known harmonic oscillator (H), the radial oscillator (L) and the Darboux–Pöschl–
Teller (DPT) potential (J). They are divided into two groups according to the eigenfunction
patterns in section 3.1. We mainly follow Infeld–Hull [4] for the naming of potentials. A
Darboux–Crum transformation in terms of multiple pseudo virtual state wavefunctions is
equivalent to a certain Krein–Adler transformation deleting multiple eigenstates with shifted
parameters. In contrast to the use of genuine virtual state wavefunctions [1], not all choices
of the multiple pseudo virtual states would generate singularity free systems. The singularity
free conditions of the obtained system are supplied by the known ones for the Krein–Adler
transformations [15].

Underlying the above equivalence are infinitely many polynomial Wronskian identities
relating Wronskians of polynomials with twisted parameters to those of shifted parameters.
These identities imply the equality of the deformed potentials with the twisted and shifted
parameters. This in turn guarantees the equivalence of all the other eigenstate wavefunctions.
We present the polynomial Wronskian identities for Group A: the harmonic oscillator (H),
the radial oscillator (L), the DPT potential (J) and some others. For Group B, the identities
take slightly different forms—determinants of various polynomials with twisted and shifted
parameters. The infinitely many polynomial Wronskian identities are the consequences of the
fundamental Wronskian (determinant) identity (2.12) as demonstrated in section 4.

This paper is organized as follows. The essence of Darboux–Crum transformations for
the Schrödinger equation in one dimension is recapitulated in section 2.1. The definitions
of virtual states and pseudo virtual states are given in section 2.2. In section 3 two groups
of eigenfunction patterns are introduced in section 3.1 and related Wronskian expressions
are explored in section 3.2. Details of the eleven examples of shape-invariant systems are
provided in sections 3.3–3.13. Section 4 is the main part of this paper. We demonstrate the
equivalence of the Darboux–Crum transformations in terms of multiple pseudo virtual states
to Krein–Adler transformations in terms of multiple eigenstates with shifted parameters.
The underlying polynomial Wronskian identities are proven with their more general determinant
identities. The final section consists of a summary and comments.

2. Darboux–Crum and Krein–Adler transformations

Darboux transformations in general [6] apply to generic second order differential equations of
Schrödinger form
\[ \mathcal{H} = -\frac{d^2}{dx^2} + U(x), \quad \mathcal{H}\psi(x) = \mathcal{E}\psi(x) \quad (\mathcal{E}, U(x) \in \mathbb{C}), \]  
without further structures of quantum mechanics, e.g. boundary conditions, self-adjointness of
\( \mathcal{H} \), Hilbert space, etc. In the next subsection, we summarize the formulas of multiple Darboux
transformations, which are purely algebraic.
2.1. General structure

Let \( \{ \psi_j(x), \tilde{E}_j \} \ (j = 1, 2, \ldots, M) \) be distinct solutions of the original Schrödinger equation (2.1):

\[
\mathcal{H}_E \psi_j(x) = \tilde{E}_j \psi_j(x) \quad (\tilde{E}_j \in \mathbb{C} : j = 1, 2, \ldots, M),
\]

to be called seed solutions. By picking up one of the above seed solutions, say \( \psi_1(x) \), we form new functions with the above solution \( \psi(x) \) and the rest of \( \{ \psi_k(x), \tilde{E}_k \} \ (k \neq 1) \):

\[
\Psi^{[1]}(x) \overset{\text{def}}{=} \frac{W[\psi_1, \psi](x)}{\psi_1(x)} = \frac{\psi_1(x) \partial_x \psi(x) - \partial_x \psi_1(x) \psi(x)}{\psi_1(x)} , \quad \Psi^{[1]}_{1,k}(x) \overset{\text{def}}{=} \frac{W[\psi_1, \psi_{1,k}](x)}{\psi_1(x)} .
\]

(2.3)

It is elementary to show that \( \Psi^{[1]}(x), \psi^{-1}_1(x) \overset{\text{def}}{=} \psi_1(x)^{-1} \) and \( \Psi^{[1]}_{1,k}(x) \) are solutions of a new Schrödinger equation of a deformed Hamiltonian \( \mathcal{H}^{[1]} \)

\[
\mathcal{H}^{[1]} = -\frac{d^2}{dx^2} + U^{[1]}(x), \quad U^{[1]}(x) \overset{\text{def}}{=} U(x) - 2\theta_x^2 \log |\psi_1(x)|,
\]

with the same energies \( E \), \( \tilde{E}_1 \) and \( \tilde{E}_k \):

\[
\mathcal{H}^{[1]} \psi^{[1]}(x) = E \psi^{[1]}(x), \quad \mathcal{H}^{[1]} \psi^{-1}_1(x) = \tilde{E}_1 \psi^{-1}_1(x),
\]

\[
\mathcal{H}^{[1]} \psi^{[1]}_{1,k}(x) = \tilde{E}_k \psi^{[1]}_{1,k}(x) \quad (k \neq 1).
\]

(2.5)

By repeating the above Darboux transformation \( M \) times, we obtain new functions

\[
\Psi^{[M]}(x) \overset{\text{def}}{=} \frac{W[\psi_1, \psi_2, \ldots, \psi_M, \psi](x)}{W[\psi_1, \psi_2, \ldots, \psi_M](x)},
\]

\[
\Psi^{[M]}_{j}(x) \overset{\text{def}}{=} \frac{W[\psi_1, \psi_2, \ldots, \psi_j, \ldots, \psi_M](x)}{W[\psi_1, \psi_2, \ldots, \psi_M](x)} \quad (j = 1, 2, \ldots, M),
\]

(2.7)

which satisfy an \( M \)th deformed Schrödinger equation with the energies \( E \) and \( \tilde{E}_j \) [7, 15]:

\[
\mathcal{H}^{[M]} = -\frac{d^2}{dx^2} + U^{[M]}(x), \quad U^{[M]}(x) \overset{\text{def}}{=} U(x) - 2\theta_x^2 \log |W[\psi_1, \psi_2, \ldots, \psi_M](x)|,
\]

\[
\mathcal{H}^{[M]} \Psi^{[M]}(x) = E \Psi^{[M]}(x), \quad \mathcal{H}^{[M]} \Psi^{[M]}_{j}(x) = \tilde{E}_j \Psi^{[M]}_{j}(x) \quad (j = 1, 2, \ldots, M).
\]

(2.10)

Here \( W[f_1, f_2, \ldots, f_n](x) \) is a Wronskian

\[
W[f_1, f_2, \ldots, f_n](x) \overset{\text{def}}{=} \det \left( \frac{d^{j-1} f_k(x)}{dx^{j-1}} \right)_{1 \leq j, k \leq n}.
\]

For \( n = 0 \), we set \( W[\cdot](x) = 1 \) and \( W[f_1, f_2, \ldots, f_j, \ldots, f_n](x) \) means that \( f_j(x) \) is excluded from the Wronskian. In deriving the determinant formulas (2.7)–(2.8) and (2.9) use is made of the properties of the Wronskian

\[
W[gf_1, gf_2, \ldots, gf_n](x) = g(x)^n W[f_1, f_2, \ldots, f_n](x),
\]

(2.11)

\[
W[W[f_1, f_2, \ldots, f_n, g], W[f_1, f_2, \ldots, f_n, h]](x) = W[f_1, f_2, \ldots, f_n](x) \cdot W[f_1, f_2, \ldots, f_n, g, h](x) \quad (n > 0).
\]

(2.12)

Another useful property of the Wronskian is that it is invariant when the derivative \( \frac{d}{dx} \) is replaced by an arbitrary ‘covariant derivative’ \( D \), with an arbitrary smooth function \( q:(x) \):

\[
D \overset{\text{def}}{=} \frac{d}{dx} - q(x), \quad W[f_1, f_2, \ldots, f_n](x) = \det(D_{j-1} \cdots D_1 f_k(x))_{1 \leq j, k \leq n},
\]

(2.13)
with \( D_{j-1} \cdots D_2 D_1 \big|_{j=1} = 1 \). Under the change of variable \( x \to \eta(x) \), the Wronskian behaves as

\[
f_j(x) = F_j(\eta(x)), \quad W[f_1, f_2, \ldots, f_n](x) = \left( \frac{d\eta(x)}{dx} \right)^{-n(n-1)} W[F_1, F_2, \ldots, F_n](\eta(x)).
\]

(2.14)

Obviously the potential \( U^{[M]}(x) \) (2.9) is independent of the order of the seed solutions. The zeros of the seed solution \( \psi_i(x) \) (the Wronskian \( W[\psi_1, \ldots, \psi_M](x) \)) induce singularities of the potential \( U^{[1]}(x) \) in (2.4) \( (U^{[M]}(x) \) in (2.9)).

We apply Darboux transformations for various extensions of exactly solvable potentials \( U(x) \in \mathbb{R} \) in one-dimensional quantum mechanics defined in an interval \( x_1 < x < x_2 \). We assume that the potential is smooth in the interval and the system has a finite (or an infinite) number of discrete eigenstates with a vanishing groundstate energy:

\[
\mathcal{H}\phi_n(x) = E_n\phi_n(x) \quad (0 \leq n \leq n_{\text{max}} \text{ or } n \in \mathbb{Z}_{\geq 0}),
\]

(2.15)

\[
0 = E_0 < E_1 < E_2 < \cdots,
\]

\[
(\phi_{n+1}, \phi_n) \equiv \int_{x_1}^{x_2} \! dx \phi_{n+1}(x)\phi_n(x) = h_n\delta_{nn}, \quad h_n > 0.
\]

(2.16)

Hereafter we consider real solutions only and use the term eigenstates in their strict sense, i.e. they only apply to those with square integrable wavefunctions and correspondingly the eigenvalues. The zero groundstate energy condition can always be achieved by adjusting the constant part of the potential \( U(x) \). Then the Hamiltonian is positive semi-definite and it has a simple factorized form expressed in terms of the groundstate wavefunction \( \phi_0(x) \) which has no node in \((x_1, x_2)\):

\[
\mathcal{H} = -\frac{d^2}{dx^2} + U(x) = A^\dagger A, \quad U(x) = \frac{\partial^2 \phi_0(x)}{\phi_0(x)} = (\partial_x w(x))^2 + \partial^2_x w(x),
\]

(2.17)

\[
A \equiv \frac{d}{dx} - \partial_x w(x), \quad A^\dagger = -\frac{d}{dx} - \partial_x w(x), \quad w(x) \in \mathbb{R}, \quad \phi_0(x) = e^{w(x)}.
\]

(2.18)

For all the examples of solvable potentials to be discussed in this paper, the main part of the discrete eigenfunctions \( [\phi_n(x)] \) are polynomials \( P_n(\eta(x)) \), in a certain function \( \eta(x) \), which is called the sinusoidal coordinate [18]. The original systems to be extended usually contain some parameter(s), \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and the parameter dependence is denoted by \( \mathcal{H}(\lambda) \), \( A(\lambda), E_n(\lambda), \phi_n(\lambda), P_n(\eta(x); \lambda) \) etc. We consider the shape-invariant [5] original systems only, which are characterized by the condition:

\[
A(\lambda)A(\lambda)^\dagger = A(\lambda + \delta)^\dagger A(\lambda + \delta) + E_I(\lambda),
\]

(2.19)

or equivalently

\[
(\partial_x w(x; \lambda))^2 - \partial^2_x w(x; \lambda) = (\partial_x w(x; \lambda + \delta))^2 + \partial^2_x w(x; \lambda + \delta) + E_I(\lambda).
\]

(2.20)

Here \( \delta \) is the shift of the parameters.

It should be remarked that the pair of Hamiltonians, \( A^\dagger A \) and \( AA^\dagger \) together with the corresponding Darboux–Crum transformations constitute the basic ingredients of the supersymmetric quantum mechanics [5]. The series of multiple Darboux–Crum transformations (2.5)–(2.16) can also be formulated in the language of supersymmetric quantum mechanics. We believe the formulation in section 2.1 is simpler and more direct than that of susy quantum mechanics.

The explicit extensions depend on the choices of the seed solutions \( \{\psi_j(x), \tilde{E}_j\} \) \((j = 1, \ldots, M)\). The obvious choices are a subset of the discrete eigenfunctions \( \{\phi_j(x), \tilde{E}_j\} \) \((j \in D)\),
in which \( \mathcal{D} \) is a subset of the index set of the total discrete eigenfunctions. By using those from the groundstate on \( \phi_0, \phi_1, \ldots, \phi_{M-1} (\mathcal{D} = \{0, 1, \ldots, M - 1\}) \) successively, Crum [7] has derived an essentially iso-spectral extension

\[
\mathcal{H}^{[M]} \phi^{[M]}_n(x) = \mathcal{E}^{[M]}_n \phi^{[M]}_n(x) \quad (n = M, M + 1, \ldots),
\]

(2.21)

\[
\phi^{[M]}_n(x) \equiv \frac{W[\phi_0, \phi_1, \ldots, \phi_{M-1}](x)}{W[\phi_0, \phi_1, \ldots, \phi_{M-1}](x)}, \quad \left( \mathcal{E}^{[M]}_n, \phi^{[M]}_n \right) = \prod_{j=0}^{M-1} (\mathcal{E}_n - \mathcal{E}_j) \cdot \hbar_n \delta_{nm},
\]

(2.22)

\[
U^{[M]}(x) \equiv U(x) - 2 \hbar^2 \log |W[\phi_0, \phi_1, \ldots, \phi_{M-1}](x)|.
\]

(2.23)

in which the potential \( U^{[s]}(x) (s = 1, 2, \ldots, M) \) at each step is non-singular. Shape-invariance means simply

\[
U^{[s]}(x; \lambda) = U(x; \lambda + s \delta) + E_s(\lambda) \quad (s = 1, 2, \ldots, M),
\]

(2.24)

\[
\phi^{[s]}_n(x; \lambda) \propto \phi_n(x; \lambda + s \delta) \quad (s = 1, 2, \ldots, M; \ n = 0, 1, \ldots). \tag{2.25}
\]

By allowing gaps in the final spectrum, Krein and Adler [15] have generalized Crum’s results for \( \mathcal{D} \equiv \{d_1, d_2, \ldots, d_M\} \) \((d_j \in \mathbb{Z}_{\geq 0} : \text{mutually distinct})\):

\[
\mathcal{H}^{[M]} \phi^{[M]}_n(x) = \mathcal{E}^{[M]}_n \phi^{[M]}_n(x) \quad (n \notin \mathcal{D}),
\]

(2.26)

\[
\phi^{[M]}_n(x) \equiv \frac{W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}, \phi_0](x)}{W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x)}, \quad \left( \mathcal{E}^{[M]}_n, \phi^{[M]}_n \right) = \prod_{j=1}^{M} (\mathcal{E}_n - \mathcal{E}_{d_j}) \cdot \hbar_n \delta_{nm},
\]

(2.27)

\[
U^{[M]}(x) \equiv U(x) - 2 \hbar^2 \log |W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x)|.
\]

(2.28)

The potential \( U^{[M]}(x) \) is non-singular when the set \( \mathcal{D} \), which specifies the gaps, \( \phi^{[M]}_{d_j}(x) \equiv 0 \) \((d_j \in \mathcal{D})\), satisfies the conditions [15, 16]:

\[
\text{Krein–Adler conditions : } \prod_{j=1}^{M} (m - d_j) \geq 0 \quad (\forall m \in \mathbb{Z}_{\geq 0}). \tag{2.29}
\]

These simply mean that the gaps are even numbers of consecutive levels. These extensions are well-known. In the next subsection, we consider seed functions which are not eigenfunctions.

2.2. Virtual and pseudo virtual states

Now let us consider extensions in terms of non-eigen seed functions. For the sake of simplicity, we first consider the cases of exactly iso-spectral extensions (deformations). The seed functions \( \{\varphi_j(x), \tilde{E}_j\} (j = 1, 2, \ldots, M) \) satisfying the following conditions are called virtual state wavefunctions:

1. No zeros in \(x_1 < x < x_2\), i.e. \(\varphi_j(x) > 0\) or \(\varphi_j(x) < 0\) in \(x_1 < x < x_2\).
2. Negative energy \(\tilde{E}_j < 0\).
3. \(\varphi_j(x)\) is also a polynomial type solution, like the original eigenfunctions, see (3.7).
4. Square non-integrability, \(\langle \varphi_j, \varphi_j \rangle = \infty\).
5. Reciprocal square non-integrability, \(\langle \varphi_j^{-1}, \varphi_j^{-1} \rangle = \infty\).

\[\text{They are different from the ‘virtual energy levels’ in quantum scattering theory [17].}\]
Of course these conditions are not totally independent. The negative energy condition is necessary for the positivity of the norm as seen from the norm formula (2.27), since a similar formula is valid for the virtual state wavefunction cases when $E_{d_i}$ is replaced by $\tilde{E}_j$.

When the first condition is dropped and the reciprocal is required to be square integrable at both boundaries, $(x_1, x_1 + \epsilon)$, $(x_2 - \epsilon, x_2)$, $\epsilon > 0$ (see (2.32)), such seed functions are called pseudo virtual state wavefunctions. When the system is extended in terms of a pseudo virtual state wavefunction $\psi_j(x)$, the new Hamiltonian $\mathcal{H}^{[1]}$ has an extra eigenstate $\psi_j^{-1}(x)$ with the eigenvalue $\tilde{E}_j$, if the potential is non-singular. The extra state is below the original groundstate and $\mathcal{H}^{[1]}$ is no longer iso-spectral with $\mathcal{H}$. This is a consequence of (2.5). Its non-singularity is not guaranteed, either.

When extended in terms of $M$ pseudo virtual state wavefunctions $\{\psi_j(x), \tilde{E}_j\}$ ($j = 1, 2, \ldots, M$), the resulting Hamiltonian $\mathcal{H}^{[M]}$ has $M$ additional eigenstates $\tilde{\psi}_j^{[M]}(x)$ (2.8), if the potential $U^{[M]}(x)$ is non-singular. They are all below the original groundstate.

Since $\psi_j(x)$ is finite in $x_1 < x < x_2$, the non-square integrability can only be caused by the virtual state wavefunctions belong to either of the following type I and II and the pseudo virtual state wavefunctions belong to type III:

- **Type I**: $\int_{x_1}^{x_1 + \epsilon} dx \psi_j(x)^2 < \infty, \int_{x_2 - \epsilon}^{x_2} dx \psi_j(x)^2 = \infty,$

  \[ \& \int_{x_1}^{x_1 + \epsilon} dx \psi_j(x)^{-2} = \infty, \int_{x_2 - \epsilon}^{x_2} dx \psi_j(x)^{-2} < \infty, \]  

- **Type II**: $\int_{x_1}^{x_1 + \epsilon} dx \psi_j(x)^2 = \infty, \int_{x_2 - \epsilon}^{x_2} dx \psi_j(x)^2 < \infty,$

  \[ \& \int_{x_1}^{x_1 + \epsilon} dx \psi_j(x)^{-2} < \infty, \int_{x_2 - \epsilon}^{x_2} dx \psi_j(x)^{-2} = \infty, \]  

- **Type III**: $\int_{x_1}^{x_1 + \epsilon} dx \psi_j(x)^2 = \infty$ or $\int_{x_2 - \epsilon}^{x_2} dx \psi_j(x)^2 = \infty,$

  \[ \& \int_{x_1}^{x_1 + \epsilon} dx \psi_j(x)^{-2} < \infty, \int_{x_2 - \epsilon}^{x_2} dx \psi_j(x)^{-2} < \infty. \]  

An appropriate modification is needed when $x_2 = +\infty$ and/or $x_1 = -\infty$. Hereafter we denote the pseudo state wavefunctions by $\{\phi_n\}$, which are the main ingredients of this paper.

The Darboux–Crum transformations in terms of type I and II virtual state wavefunctions have been applied to achieve exactly iso-spectral deformations of the radial oscillator potential and the DPT potentials [1], which generate the multi-indexed Laguerre and Jacobi polynomials.

In this paper we show that the Darboux–Crum transformations in terms of a set of $\mathcal{D} \equiv \{d_1, d_2, \ldots, d_M\}$ pseudo virtual state wavefunctions

\[ \mathcal{H}^{[M]} \phi_n^{[M]}(x) = E_n \phi_n^{[M]}(x), \]  

\[ \phi_n^{[M]}(x) \equiv \frac{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}, \phi_n]}{W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}]}(x), \quad (\phi_m^{[M]}, \phi_n^{[M]}) = \prod_{j=1}^{M} \left( E_n - \tilde{E}_{d_j} \right) \cdot h_n \delta_{mn}, \]  

\[ U^{[M]}(x) \equiv U(x) - 2\bar{a}^2 \log W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}](x), \]  

are equivalent to a system generated by Krein–Adler transformations (2.26)–(2.28) specified by a certain set $\mathcal{D}$ (4.2) of eigenfunctions with shifted parameters for various systems with shape-invariant potentials, which are listed in the subsequent section. In contrast to the extensions in
terms of the virtual states, these extensions in terms of pseudo virtual states are not iso-spectral, and the obtained systems are not shape-invariant.

As seen in each example, the virtual and pseudo virtual wavefunctions are generated from the eigenstate wavefunctions through twisting of parameters based on the discrete symmetries of the original Hamiltonian.

3. Examples of shape-invariant quantum mechanical systems

Here we provide the essence of shape-invariant and thus exactly solvable one-dimensional quantum mechanical systems. The first five examples sections 3.3–3.7 have infinitely many discrete eigenstates, whereas the rest sections 3.8–3.13 have finitely many eigenstates. They are divided into two groups of eigenfunction patterns. The eigenfunctions \( \{ \phi_n \} \) and the pseudo virtual state wavefunctions \( \{ \tilde{\phi}_v \} \)

\[
\mathcal{H}(\lambda)\phi_n(x; \lambda) = \mathcal{E}_n(\lambda)\phi_n(x; \lambda) \quad (0 \leq n \leq n_{\text{max}}(\lambda) \quad \text{or} \quad n \in \mathbb{Z}_{\geq 0}),
\]

\[
\mathcal{H}(\lambda)\tilde{\phi}_v(x; \lambda) = \tilde{\mathcal{E}}_v(\lambda)\tilde{\phi}_v(x; \lambda) \quad (v \in \mathcal{V}(\lambda)),
\]

are listed in detail for reference purposes. Here \( \mathcal{V}(\lambda) \) is the index set of the pseudo virtual state wavefunctions, which is specified for each example. The type I and II virtual state wavefunctions for systems with finitely many eigenstates will be discussed in a separate paper [19].

The basic tools of shape-invariant systems (2.19)–(2.20) are the forward shift and backward shift relations:

\[
\mathcal{A}(\lambda)\phi_n(x; \lambda) = f_n(\lambda)\phi_{n-1}(x; \lambda + \delta),
\]

\[
\mathcal{A}(\lambda)^{-1}\phi_{n-1}(x; \lambda + \delta) = b_{n-1}(\lambda)\phi_n(x; \lambda),
\]

\[
\mathcal{E}_n(\lambda) = f_n(\lambda)b_{n-1}(\lambda).
\]

For the parameters of shape-invariant transformation \( \lambda \rightarrow \lambda + \delta \), we use the symbol \( g \) to denote an increasing (\( \delta = 1 \)) parameter, \( h \) for a decreasing (\( \delta = -1 \)) parameter and \( \mu \) for an unchanging (\( \delta = 0 \)) parameter, except for the DPT potentials (J), in which \( g \) and \( h \) are both increasing parameters. Throughout this paper we assume that the parameters \( g \) and/or \( h \) take generic values, that is, not integers or half odd integers.

In the next subsection, we introduce two groups of the eigenfunction patterns of the eleven examples of shape-invariant systems, sections 3.3–3.13. Then the basics of the Wronskians for the two groups are discussed in section 3.2. The explicit formulas of the eigenvalues, eigenfunctions, pseudo virtual state wavefunctions and the discrete symmetries of the eleven shape-invariant systems are provided in sections 3.3–3.13 for reference purposes. Among them, we also report type II virtual state wavefunctions for two potentials, (C) section 3.6 and (Kh) section 3.12. These have been reported in [11] and [14], respectively, in connection with the \( M = 1 \) rational extensions. We stress that multi-indexed orthogonal polynomials can be constructed for these two potentials in exactly the same way as in [1]. For another type of virtual state wavefunctions for the potentials with finitely many discrete eigenstates sections 3.8–3.13, see a subsequent publication [19]. Sections 3.3–3.13 could be skipped in the first reading.
3.1. Two groups of eigenfunctions patterns

In all the eleven Hamiltonian systems sections 3.3–3.13 the energy formula of the pseudo virtual state is related to that of the ‘eigenstate’ with a negative ‘degree’:

\[ \tilde{E}_v (\lambda) = E_{-v-1} (\lambda). \]  

(3.6)

These Hamiltonian systems are divided into two groups of eigenfunction patterns:

\[ \phi_{0v}(x; \lambda) = \phi_{0v}(x; \lambda) \eta_n(\eta(x); \lambda), \quad \phi_{0v}(x; \lambda) = \begin{cases} \phi_0(x; \lambda) & : \text{Group A} \\ \phi_0(x; \lambda + n\delta) & : \text{Group B} \end{cases}. \]  

(3.7)

Seven potentials (H), (L), (J), (M), (s), (hst) and (hDPT) belong to group A and four potentials (C), (K), (RM) and (Kh) belong to group B. Group A has a simple structure. The ‘n’ dependence is carried only by the degree of the polynomials (except for (M)). They all satisfy the closure relations [18] and \( \eta(x) \) is called a sinusoidal coordinate. The groundstate wavefunctions of this group satisfy

\[ \frac{\phi_0(x; \lambda + \delta)}{\phi_0(x; \lambda)} = e^{\frac{2i}{\pi} \frac{d\eta(x)}{dx}}, \quad c_x = \begin{cases} 1 & : H, s, \text{hst} \\ 2 & : L \\ -4 & : J \\ -1 & : M \\ 4 & : \text{hDPT} \end{cases}. \]  

(3.8)

Group B has a more complicated structure. The ‘n’ dependence appears also in the other factor and in the parameter of the polynomials \( \alpha_n, \beta_n \).

The pseudo virtual state wavefunctions are obtained from the eigenstate wavefunctions by twisting \( \lambda \rightarrow \tau(\lambda) \) (and \( x \rightarrow ix \) for (H) and (L)):

\[ \tilde{\phi}_v(x; \lambda) = \phi_v(x; \tau(\lambda)) = \tilde{\phi}_{0v}(x; \lambda) \xi_v(\eta(x); \lambda), \quad \tilde{\phi}_{0v}(x; \lambda) = \begin{cases} \phi_0(x; \tau(\lambda)) & : \text{Group A} \\ \phi_0(x; \tau(\lambda) + v\delta) & : \text{Group B} \end{cases}, \quad \xi_v(\eta; \lambda) = P_v(\eta; \tau(\lambda)), \]  

(3.9)

with a slight modification for (H) and (L):

\[ \begin{align*} 
\text{(H)} : & \quad \tilde{\phi}_v(x; \lambda) = i^{v+1} \phi_v(ix) = \tilde{\phi}_{0v}(x) \xi_v(\eta(x)), \\
& \quad \tilde{\phi}_{0v}(x) = \phi_0(x), \quad \xi_v(\eta) = i^{v} P_v(\eta), \quad \xi_v(\eta; \lambda) = P_v(-\eta; \tau(\lambda)). 
\end{align*} \]  

(3.10)

\[ \begin{align*} 
\text{(L)} : & \quad \tilde{\phi}_v(x; \lambda) = i^{v-1} \phi_v(ix; \tau(\lambda)) = \tilde{\phi}_{0v}(x; \lambda) \xi_v(\eta(x); \lambda), \\
& \quad \tilde{\phi}_{0v}(x; \lambda) = i^{v-1} \phi_0(ix; \tau(\lambda)), \quad \xi_v(\eta; \lambda) = P_v(-\eta; \tau(\lambda)). 
\end{align*} \]  

(3.11)

The twist operation \( \tau \) satisfies \( \tau(\lambda + \alpha \delta) = (\lambda - \alpha \delta) \) (\( \alpha \in \mathbb{C} \)).

Corresponding to the forward and backward shift relations of the eigenfunctions (3.3)–(3.4) and the energy factorization formula (3.5), those for the pseudo virtual state wavefunction read:

\[ A(\lambda) \tilde{\phi}_v(x; \lambda) = -\epsilon b_v(\lambda) \tilde{\phi}_{v+1}(x; \lambda + \delta), \quad \epsilon = \begin{cases} -1 & : H \\ 1 & : \text{others} \end{cases}, \]  

(3.12)

\[ A(\lambda) \tilde{\phi}_{v+1}(x; \lambda + \delta) = -\epsilon' f_{v+1}(\lambda) \tilde{\phi}_v(x; \lambda), \quad \epsilon' = \begin{cases} -1 & : L \\ 1 & : \text{others} \end{cases}, \]  

(3.13)

\[ \tilde{E}_v(\lambda) = \epsilon \epsilon' f_{v+1}(\lambda) b_v(\lambda) = f_{v-1}(\lambda) b_{v-2}(\lambda) = E_{-v-1}(\lambda). \]  

(3.14)
3.2. Wronskian formulas

Based on the eigenfunction patterns, the Wronskians of the eigenfunctions and the pseudo virtual state wavefunctions for the set $\mathcal{D} = \{d_1, d_2, \ldots, d_M\}$

$$W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x; \lambda), \quad W[\Phi_{d_1}, \Phi_{d_2}, \ldots, \Phi_{d_M}](x; \lambda)$$

are reduced to determinant formulas of the polynomials. Since the latter is obtained from the former by twisting, we present derivations of the former.

For group A, the reduction to the Wronskian of the polynomials is achieved by the Wronskian formulas (2.11) and (2.14) ($\tilde{P}_n(x; \lambda) = P_n(\eta(x); \lambda)$):

$$W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x; \lambda) = \phi_0(x; \lambda)^M W[\tilde{P}_{d_1}, \tilde{P}_{d_2}, \ldots, \tilde{P}_{d_M}](x; \lambda)$$

$$= \phi_0(x; \lambda)^M \left( \frac{d\eta(x)}{dx} \right)^{\frac{1}{2}M(M-1)} W[P_{d_1}, P_{d_2}, \ldots, P_{d_M}](\eta(x); \lambda). \quad (3.15)$$

For group B, the derivative operator $\partial_{\lambda}^{j-1}$ in the Wronskian is replaced by ‘covariant derivatives’ $A(\lambda + (j-2)\delta) \cdots A(\lambda + \delta)A(\lambda)$ (2.13) and use is made of the forward shift relation (3.3) to obtain

$$A(\lambda + (j-2)\delta) \cdots A(\lambda + \delta)A(\lambda)\phi_0(x; \lambda) = \prod_{i=0}^{j-2} f_{n-i}(\lambda + i\delta) \cdot \phi_{n-j+1}(x; \lambda + (j-1)\delta)$$

$$= \prod_{i=0}^{j-2} f_{n-i}(\lambda + i\delta) \cdot \phi_0(x; \lambda + n\delta)P_{n-j+1}(\eta(x); \lambda + (j-1)\delta). \quad (3.16)$$

We have

$$W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x; \lambda) = \det \left( \prod_{i=0}^{j-2} f_{d_{n-i}}(\lambda + i\delta) \cdot \phi_{d_{n-j+1}}(x; \lambda + (j-1)\delta) \right)_{1 \leq i, j \leq M}$$

$$= \prod_{k=1}^{M} \phi_0(x; \lambda + d_k \delta) \cdot \det \left( \prod_{i=0}^{j-2} f_{d_{n-i}}(\lambda + i\delta) \cdot P_{d_{n-j+1}}(\eta(x); \lambda + (j-1)\delta) \right)_{1 \leq i, j \leq M}. \quad (3.17)$$

This method is also applicable to group A, in which $\prod_{k=1}^{M} \phi_0(x; \lambda + d_k \delta)$ is replaced by

$$\prod_{k=1}^{M} \phi_0(x; \lambda + (j-1)\lambda) = \phi_0(x; \lambda)^M \left( \frac{d\eta(x)}{dx} \right)^{\frac{1}{2}M(M-1)}.$$

Let us summarize the results:

$$W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x; \lambda) = \tilde{A}_D(x; \lambda) \tilde{\Xi}_D(\eta(x); \lambda), \quad (3.18)$$

$$\tilde{A}_D(x; \lambda) = \begin{cases} \phi_0(x; \lambda)^M \left( \frac{d\eta(x)}{dx} \right)^{\frac{1}{2}M(M-1)} & \text{: Group A} \\ \prod_{k=1}^{M} \phi_0(x; \lambda + d_k \delta) & \text{: Group B} \end{cases} \quad (3.19)$$

$$\tilde{\Xi}_D(\eta; \lambda) = \begin{cases} \tilde{\Xi}_D(\eta; \lambda) = \frac{1}{2}M(M-1)W[P_{d_1}, P_{d_2}, \ldots, P_{d_M}](\eta; \lambda) & \text{: Group A} \\ \det(\tilde{X}_{j,k}(\eta; \lambda))_{1 \leq i, k \leq M} & \text{: Group B} \end{cases} \quad (3.20)$$

$$\tilde{X}_{j,k}(\eta; \lambda) = \prod_{i=0}^{j-2} f_{d_{n-i}}(\lambda + i\delta) \cdot P_{d_{n-j+1}}(\eta; \lambda + (j-1)\delta) \quad (3.21)$$
The pseudo virtual state wavefunction Wronskian is simply obtained by the twisting:
\[
W[\tilde{\phi}_d, \tilde{\phi}_{d_1}, \ldots, \tilde{\phi}_{d_M}](x; \lambda) = A_D(x; \lambda) \Xi_D(\eta(x); \lambda),
\] (3.22)

\[
A_D(x; \lambda) = \left\{ \begin{array}{ll}
\tilde{\phi}_{00}(x; \lambda) \lambda^M \left( e^{\frac{i}{2} \int_{w(x; \lambda)}^{x} \frac{d\xi}{\xi}} \right)^{\frac{1}{2}M(M-1)} & : \text{Group A} \\
\prod_{k=1}^{M} \phi_0(x; \lambda) + d_k \delta & : \text{Group B}
\end{array} \right.
\] (3.23)

\[
\Xi_D(\eta; \lambda) = \left\{ \begin{array}{ll}
\tilde{\Xi}_D(\eta; \lambda) & : \text{Group A} \\
\det \left( X_{j,k}^D(\eta; \lambda) \right)_{1 \leq j,k \leq M} & : \text{Group B}
\end{array} \right.
\] (3.24)

\[
X_{j,k}^D(\eta; \lambda) = \prod_{i=0}^{j-2} f_{d_{i-1}}(t(\lambda) + i\delta) \cdot \tilde{\xi}_{d_{i-j+1}}(\eta; \lambda - (j-1)\delta).
\] (3.25)

(Note that the expressions of \( \tilde{\Xi}_D \) and \( \Xi_D \) for group B are valid for group A as well.) Their relations are
\[
A_D(x; \lambda) = \tilde{A}_D(x; \lambda), \quad \Xi_D(\eta; \lambda) = \tilde{\Xi}_D(\eta; \lambda),
\] (3.26)

with a slight modification for (H) and (L):
\[
H : A_D(x; \lambda) = \tilde{A}_D(\eta(x)), \quad \Xi_D(\eta; \lambda) = e^{-\delta \eta} \tilde{\Xi}_D(\eta),
\]
\[
L : A_D(x; \lambda) = e^{i(1-\frac{2M}{M})} A_D(\eta(x); \lambda), \quad \Xi_D(\eta; \lambda) = e^{iM(M-1)} \tilde{\Xi}_D(\eta; \lambda).
\]

Both \( \tilde{\Xi}_D(\eta; \lambda) \) and \( \Xi_D(\eta; \lambda) \) are polynomials in \( \eta \) and their degrees are generically \( \ell_D \):
\[
\ell_D = \sum_{j=1}^{M} d_j - \frac{1}{2}M(M-1).
\] (3.27)

**Remark.** Strictly speaking, the notation \( D \) of the polynomials \( \tilde{\Xi}_D \) and \( \Xi_D \) represents an ordered set. By changing the order of the \( d_j \), \( \tilde{\Xi}_D \) and \( \Xi_D \) may change sign. On the other hand the functions \( \tilde{A}_D \) and \( A_D \) are invariant under the permutations of the \( d_j \). In order to avoid excessive appearance of \( \pm \) signs in the general formulas involving \( \tilde{\Xi}_D \) and \( \Xi_D \), etc, we adopt the following convention. The formulas in section 4 involving the Wronskians and determinants of various polynomials depending on \( D \), \( \tilde{D} \) and other sets are understood to be true up to a multiplicative constant \( \pm 1 \) coming from the multi-linearity of the determinants. This does not affect the main propositions, e.g. (4.19)–(4.20) in proposition 4.2.

### 3.3. Harmonic oscillator (H)

The well-known system of the harmonic oscillator has infinitely many eigenstates:
\[
\lambda : \text{none}, \quad \delta : \text{none}, \quad -\infty < x < \infty,
\]
\[
u(x; \lambda) = -\frac{1}{2}x^2, \quad U(x; \lambda) = x^2 - 1,
\]
\[
\mathcal{E}_n(\lambda) = 2n, \quad \mathcal{E}(x; \lambda) = x, \quad f_n(\lambda) = 2n, \quad b_{n-1}(\lambda) = 1,
\]
\[
\tilde{\phi}_n(x; \lambda) = \phi_0(x; \lambda) P_n(\eta(x); \lambda), \quad \phi_0(x; \lambda) = e^{-\lambda x^2}, \quad P_n(\eta; \lambda) = H_n(\eta),
\]
\[
h_n(\lambda) = 2^n n! \sqrt{\pi}.
\]

Here \( H_n(x) \) is the Hermite polynomial. The system satisfies the **closure relation** introduced in [18]. The pseudo virtual state wavefunctions are obtained by the following discrete symmetry:
\[
\mathcal{H}(\lambda) = -\mathcal{H}(\lambda)|_{x \to ix} + \mathcal{E}_{-1}(\lambda),
\]
\[
\tilde{\phi}_n(x; \lambda) = i^{\gamma} \phi_{n}(ix; \lambda) = e^{i\gamma} i^{\gamma} H_n(ix) \quad (\nu \in \mathbb{Z}_{\geq 0}),
\]
\[
\tilde{\mathcal{E}}(\lambda) = -\mathcal{E}_1(\lambda) + \mathcal{E}_{-1}(\lambda) = \mathcal{E}_{-\nu-1}(\lambda).
\]
3.4. Radial oscillator (L)

The radial oscillator potential has also infinitely many discrete eigenstates in the specified parameter range:

\[ \lambda = g, \quad \delta = 1, \quad 0 < x < \infty, \quad g > \frac{1}{\pi}, \]

\[ w(x; \lambda) = -\frac{1}{2}x^2 + g \log x, \quad U(x; \lambda) = x^2 + \frac{g(g - 1)}{x^2} - (1 + 2g), \]

\[ \mathcal{E}_n(\lambda) = 4n, \quad \eta(x) = x^2, \quad f_\eta(\lambda) = -2, \quad b_{n-1}(\lambda) = -2n, \]

\[ \phi_n(x; \lambda) = \phi_0(x; \lambda) P_n(\eta(x); \lambda), \quad \phi_0(x; \lambda) = e^{-\frac{1}{2}x^2}, \quad P_n(\eta; \lambda) = L_{n-g}^{(g-\frac{1}{2})}(\eta), \]

\[ \hbar_n(\lambda) = \frac{1}{2n!} \Gamma(n + g + \frac{1}{2}). \]

Here \( L_{n-g}^{(g-\frac{1}{2})}(\eta) \) is the Laguerre polynomial. The system satisfies the closure relation [18].

There are three types of discrete symmetries:

- **Type I**: \( \mathcal{H}(\lambda) = -\mathcal{H}(\lambda)|_{x \rightarrow -x} - 2(1 + 2g), \)
- **Type II**: \( \mathcal{H}(\lambda) = \mathcal{H}(\lambda)|_{x \rightarrow -x} + E_{-1}(\lambda), \)
- **Type III**: \( \mathcal{H}(\lambda) = \mathcal{H}(\lambda)|_{x \rightarrow -x} + E_{-1}(\lambda), \)

and the pseudo virtual states are generated by type III:

\[ \phi_0(x; \lambda) = i^{n-1}\phi_0(ix; \lambda) = e^{\frac{i}{2}x^2}P_n(\eta(x); \lambda) \quad (n \in \mathbb{Z}_{\geq 0}), \]

\[ \mathcal{E}_n(\lambda) = -\mathcal{E}_n(t(\lambda)) + E_{-1}(\lambda) = E_{-v-1}(\lambda). \]

The type I and II virtual states are obtained by using type I and II discrete symmetries [1].

3.5. The DPT potential (J)

The DPT potential has also infinitely many discrete eigenstates in the specified parameter range:

\[ \lambda = (g, h), \quad \delta = (1, 1), \quad 0 < x < \frac{\pi}{2}, \quad g, h > \frac{3}{2}, \]

\[ w(x; \lambda) = g \log \sin x + h \log \cos x, \quad U(x; \lambda) = \frac{g(g - 1)}{2 \sin^2 x} + \frac{h(h - 1)}{2 \cos^2 x} - (g + h)^2, \]

\[ \mathcal{E}_n(\lambda) = 4n(n + g + h), \quad \eta(x) = \cos 2x, \quad f_\eta(\lambda) = -2(n + g + h), \quad b_{n-1}(\lambda) = -2n, \]

\[ \phi_n(x; \lambda) = \phi_0(x; \lambda) P_n(\eta(x); \lambda), \quad \phi_0(x; \lambda) = (\sin x)^g(\cos x)^h, \quad P_n(\eta; \lambda) = P_n^{(g-\frac{1}{2}, h-\frac{1}{2})}(\eta), \]

\[ \hbar_n(\lambda) = \frac{1}{2n!} \Gamma(n + g + \frac{1}{2}) \Gamma(n + h + \frac{1}{2}) \Gamma(n + g + h). \]

Here \( P_n^{(g, h)}(\eta) \) is the Jacobi polynomial. The system satisfies the closure relation [18].

There are three types of discrete symmetries:

- **Type I**: \( \mathcal{H}(\lambda) = \mathcal{H}(t^i(\lambda)) + (1 + 2g)(1 - 2h), \quad t^i(\lambda) = (g, 1 - h), \)
- **Type II**: \( \mathcal{H}(\lambda) = \mathcal{H}(t^0(\lambda)) + (1 - 2g)(1 + 2h), \quad t^0(\lambda) = (1 - g, h), \)
- **Type III**: \( \mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + E_{-1}(\lambda), \quad t = t^0 \circ t^1, \quad t(\lambda) = (1 - g, 1 - h), \)

and the pseudo virtual states are generated by type III:

\[ \phi_0(x; \lambda) = \phi_0(x; t(\lambda)) = (\sin x)^{1-g}(\cos x)^{1-h}P_n(\eta(x); t(\lambda)) \quad (0 \leq v < g + h - 1), \]

\[ \mathcal{E}_n(\lambda) = \mathcal{E}_n(t(\lambda)) + E_{-1}(\lambda) = E_{-v-1}(\lambda). \]

The type I and II virtual states are obtained by using type I and II discrete symmetries [1]. The Hamiltonian \( \mathcal{H} \) has also the ‘left–right’ mirror symmetry \( x \rightarrow \frac{\pi}{2} - x \), \( g \leftrightarrow h \).
3.6. The Coulomb potential plus the centrifugal barrier (C)

The system has also infinitely many discrete eigenstates in the specified parameter range:
\( \lambda = g, \quad \delta = 1, \quad 0 < x < \infty, \quad g > \frac{1}{2}, \)
\( w(x; \lambda) = g \log x - \frac{x}{g}, \quad U(x; \lambda) = \frac{g(g-1)}{x^2} - \frac{2}{x} + \frac{1}{g^2}, \)
\( \mathcal{E}_n(\lambda) = \frac{1}{g^2} - \frac{1}{(g+n)^2}, \quad \eta(x) = x^{-1}, \quad f_n(\lambda) = -\frac{2}{g(g+n)^2}, \quad b_{n-1}(\lambda) = \frac{-n(2g+n)}{g}, \)
\( \phi_n(x; \lambda) = e^{-\frac{g}{n} x} P_n(\eta(x); \lambda), \quad \phi_0(x; \lambda) = e^{-\frac{g}{2} x}, \quad P_n(\eta; \lambda) = \eta^n P_n^{(2g-1)} \left( \frac{2}{g+n} \eta^{-1} \right) \).
\( h_n(\lambda) = \left( \frac{g+n}{2} \right)^{2g+2} \frac{4}{n!} \Gamma(2g+n). \)

The discrete symmetry and the pseudo virtual state wavefunctions are:
\( \mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + \mathcal{E}_{-1}(\lambda), \quad t(\lambda) = 1 - g, \)
\( \tilde{\phi}_n(x; \lambda) = \phi_n(x; t(\lambda)) = e^{\frac{g}{n} x} x^{1-\frac{1}{g}} P_n(\eta(x); t(\lambda)) \quad (0 \leq v < g - 1), \)
\( \tilde{\mathcal{E}}_v(\lambda) = \mathcal{E}_v(t(\lambda)) + \mathcal{E}_{-1}(\lambda) = \mathcal{E}_{-v-1}(\lambda). \)

For \( g - 1 < v < 2g - 1 \left( g > \frac{1}{2} \right) \), the discrete symmetry generates the type II virtual states \( \tilde{\phi}_n(x; \lambda) \) [11].

This system can be obtained from the Kepler problem in spherical space section 3.7 in a certain limiting procedure.

3.7. The Kepler problem in spherical space (K)

The system has also infinitely many discrete eigenstates in the specified parameter range:
\( \lambda = (g, \mu), \quad \delta = (1, 0), \quad 0 < x < \pi, \quad g > \frac{1}{2}, \quad \mu > 0, \)
\( w(x; \lambda) = g \log x - \frac{\mu}{g}, \quad U(x; \lambda) = g \frac{(g-1)}{\sin^2 x} - 2 \mu \cot x + \frac{\mu^2}{g^2} - g^2, \)
\( \mathcal{E}_n(\lambda) = (g+n)^2 - g^2 - \frac{\mu^2}{g^2} - \frac{g^2}{(g+n)^2}, \quad \eta(x) = \cot x, \)
\( f_n(\lambda) = \frac{g^2(g+n)^2 - \mu^2}{g(g+n)^2}, \quad b_{n-1}(\lambda) = \frac{n(2g+n)}{g}, \)
\( \phi_n(x; \lambda) = e^{\frac{\mu}{n} \sin^2 x} P_n(\eta(x); \lambda), \quad \phi_0(x; \lambda) = e^{-\frac{\mu}{2} \sin^2 x}, \quad P_n(\eta; \lambda) = \eta^n P_n^{(2g-1)}(\eta), \quad \alpha_n = -g - n + \frac{\mu}{g+n} i, \quad \beta_n = -g - n - \frac{\mu}{g+n} i, \)
\( h_n(\lambda) = \frac{e^{\frac{\mu}{2} \sin^2 x} 2^{1-2(g+n)} \pi (g+n) \Gamma(2g+n) \Gamma(g+n)}{n! (g+n)^2 + \frac{\mu}{g+n} i} \Gamma(g+n) \Gamma(g-n). \)

The discrete symmetry and the pseudo virtual state wavefunctions are:
\( \mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + \mathcal{E}_{-1}(\lambda), \quad t(\lambda) = (1-g, \mu), \)
\( \tilde{\phi}_n(x; \lambda) = \phi_n(x; t(\lambda)) = e^{\frac{\mu}{n} \sin^2 x} P_n(\eta(x); t(\lambda)) \quad (0 \leq v < 2g - 1), \)
\( \tilde{\mathcal{E}}_v(\lambda) = \mathcal{E}_v(t(\lambda)) + \mathcal{E}_{-1}(\lambda) = \mathcal{E}_{-v-1}(\lambda). \)
The system has finitely many discrete eigenstates $0 \leq n \leq n_{\text{max}}(\lambda) = [h\lambda]$ in the specified parameter range ($[a]$ denotes the greatest integer not exceeding and not equal to $a$):

$\lambda = (h, \mu), \quad \delta = (-1, 0), \quad -\infty < x < \infty, \quad h, \mu > 0$,

$w(x; \lambda) = hx - \mu e^x, \quad U(x; \lambda) = \mu^2 e^{2x} - \mu(2h + 1)e^x + h^2$,

$\mathcal{E}_n(\lambda) = h^2 - (h - n)^2, \quad \eta(x) = e^{-x}, \quad f_n(\lambda) = \frac{n - 2h}{2\mu}, \quad b_{n-1}(\lambda) = -2n\mu$,

$\phi_n(x; \lambda) = \phi_0(x; \lambda)P_n(\eta(x); \lambda), \quad \phi_0(x; \lambda) = \mu x e^{-\mu x}$,

$P_n(\eta; \lambda) = (2\mu \eta^{-1})^{-n}L^\infty_{n_{\text{max}}}((2h - n + 1)/(2\mu))$,

The system satisfies the closure relation [18].

The discrete symmetry and the pseudo virtual state wavefunctions are:

$\mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + \mathcal{E}_{-1}(\lambda), \quad t(\lambda) = (-1 - h, -\mu)$,

$\tilde{\phi}_n(x; \lambda) = \phi_n(x; \lambda)P_n(\eta(x); \lambda), \quad \phi_0(x; \lambda) = (\cosh x)^{-h}$,

$P_n(\eta(x); \lambda) = (\cosh x)^{-h}P_n^{(h-n,h-n)}(\tanh x), \quad h_0(\lambda) = \frac{\Gamma(2h - n + 1)}{n!(-n + 1)!\Gamma(2h - n + 1)}$.

The system satisfies the closure relation [18]. One can rewrite $P_n(\eta; \lambda)$ as

$P_n(\eta; \lambda) = \left(\frac{h - \left[\frac{n + 1}{2}\right]}{h - n + 1}\right)\mu^\frac{1}{2}P_n^{(-h-\frac{1}{2},-h-\frac{1}{2})}(i\eta)$,

where $[a]$ denotes the greatest integer not exceeding $a$.

The discrete symmetry and the pseudo virtual state wavefunctions are:

$\mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + \mathcal{E}_{-1}(\lambda), \quad t(\lambda) = (-1 - h, -\mu)$,

$\tilde{\phi}_n(x; \lambda) = \phi_n(x; \lambda)P_n(\eta(x); \lambda), \quad \phi_0(x; \lambda) = (\cosh x)^{-h}$,

$\tilde{E}_n(\lambda) = \tilde{E}_n(t(\lambda)) + \mathcal{E}_{-1}(\lambda) = \mathcal{E}_{-v-1}(\lambda)$.

This system can be obtained from the RM potential section 3.10 by taking the $\mu \to 0$ limit.
3.10. The RM potential

This potential is also called the RM II potential. The system has finitely many discrete eigenstates $0 \leq n \leq n_{\text{max}}(\lambda) = [h - \sqrt{\mu}]$ in the specified parameter range:

$\lambda = (h, \mu), \; \delta = (-1, 0), \; -\infty < x < \infty, \; h > \sqrt{\mu} > 0,$

$w(x; \lambda) = -h \log \cosh x - \frac{\mu}{h} x, \; U(x; \lambda) = -\frac{h(h+1)}{\cosh^2 x} + 2\mu \tanh x + h^2 + \frac{\mu^2}{h^2},$

$E_n(\lambda) = h^2 - (h - n)^2 + \frac{\mu^2}{h^2}, \; \eta(x) = \tanh x,$

$f_n(\lambda) = \frac{h^2(h - n)^2 - \mu^2}{h(h - n)^2}, \; b_{n-1}(\lambda) = \frac{n(2h - n)}{h},$

$\phi_0(x; \lambda) = e^{-\frac{\mu}{2h} x}(\cosh x)^{h+n} P_n(\eta(x); \lambda), \; \alpha_n = h - n + \frac{\mu}{h - n}, \; \beta_n = h - n - \frac{\mu}{h - n},$

$h_n(\lambda) = \frac{2^{2h-2n}(h - n)\Gamma(h + \frac{\mu}{h - n} + 1)\Gamma(h - \frac{\mu}{h - n} + 1)}{n!(h - n)^2 \left(\frac{\mu}{h - n}\right)^n \Gamma(2h - n + 1)},$

By taking the limit $\mu \to 0$, the soliton potential section 3.9 is obtained. Based on the symmetry $x \to -x$, $\mu \to -\mu$, positive $\mu$ is selected.

The discrete symmetry and the pseudo virtual state wavefunctions are:

$\mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + E_{-1}(\lambda), \; t(\lambda) = (-1, -h, \mu),$

$\tilde{\phi}_v(x; \lambda) = \phi_v(x; t(\lambda)) = e^{\frac{\mu}{\pi h} x}(\cosh x)^h P_{v}(\eta(x); t(\lambda)) \; (v \in \mathbb{Z}_{\geq 0}),$

$E_v(\lambda) = E_v(t(\lambda)) + E_{-1}(\lambda) = E_{-v-1}(\lambda).$

3.11. The hyperbolic symmetric top II (hst)

The system has finitely many discrete eigenstates $0 \leq n \leq n_{\text{max}}(\lambda) = [h]'$ in the specified parameter range:

$\lambda = (h, \mu), \; \delta = (-1, 0), \; -\infty < x < \infty, \; h, \mu > 0,$

$w(x; \lambda) = -h \log \cosh x - \mu \tan^{-1} \sinh x,$

$U(x; \lambda) = -h(h + 1) + \mu^2(2h + 1) \sinh x + h^2,$

$E_n(\lambda) = h^2 - (h - n)^2, \; \eta(x) = \sinh x, \; f_n(\lambda) = \frac{n - 2h}{2}, \; b_{n-1}(\lambda) = -2n,$

$\phi_0(x; \lambda) = \phi_0(x; \lambda) P_n(\eta(x); \lambda), \; \phi_0(x; \lambda) = e^{-\mu \tan^{-1} \sinh x}(\cosh x)^{-h},$

$P_{n}(\eta; \lambda) = i^{-n} g_{\alpha, \beta}^{(n)}(\eta), \; \alpha = -h - \frac{1}{2} - i\mu, \; \beta = -h - \frac{1}{2} + i\mu,$

$h_n(\lambda) = \frac{\pi n! \Gamma(2h - n + 1)}{\Gamma(2h - n + 1 + i\mu) \Gamma(2h - n + 1 - i\mu)}.$

The system satisfies the closure relation [18].

The discrete symmetry and the pseudo virtual state wavefunctions are:

$\mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + E_{-1}(\lambda), \; t(\lambda) = (-1, -h, -\mu),$

$\tilde{\phi}_v(x; \lambda) = \phi_v(x; t(\lambda)) = e^{\mu \tan^{-1} \sinh x}(\cosh x)^{h+1} P_{v}(\eta(x); t(\lambda)) \; (v \in \mathbb{Z}_{\geq 0}),$

$E_v(\lambda) = E_v(t(\lambda)) + E_{-1}(\lambda) = E_{-v-1}(\lambda).$
3.12. The Kepler problem in hyperbolic space (Kh)

This potential is also called the Eckart potential. It has finitely many discrete eigenstates \(0 \leq n \leq n_{\text{max}}(\lambda) = \lfloor \sqrt{\mu} - g \rfloor^\prime\) in the specified parameter range:

\[
\lambda = (g, \mu), \quad \delta = (1, 0), \quad 0 < x < \infty, \quad \sqrt{\mu} > g > \frac{1}{2},
\]

\[
w(x; \lambda) = g \log \sinh x - \frac{\mu}{g} x, \quad U(x; \lambda) = \frac{g(g - 1)}{\sinh^2 x} - 2\mu \coth x + g^2 + \frac{\mu^2}{g^2},
\]

\[
\mathcal{E}_n(\lambda) = g^2 - (g + n)^2 + \frac{\mu^2}{g^2} - \frac{\mu^2}{(g + n)^2}, \quad \eta(x) = \coth x,
\]

\[
f_n(\lambda) = \frac{\mu^2 - g^2(g + n)^2}{g(g + n)^2}, \quad b_{n-1}(\lambda) = \frac{n(2g + n)}{g},
\]

\[
\phi_n(x; \lambda) = e^{-\frac{\pi}{2} i} \Gamma(\frac{1}{2} + \frac{\mu}{g}) \Gamma(g + n), \quad \phi_0(x; \lambda) = e^{-\frac{\pi}{2} i} \Gamma(\frac{1}{2} + \frac{\mu}{g}),
\]

\[
P_n(\eta; \lambda) = P_n^{\alpha_n, \beta_n}(\eta), \quad \alpha_n = -g - n + \frac{\mu}{g} n, \quad \beta_n = -g - n - \frac{\mu}{g} n,
\]

\[
h_n(\lambda) = \frac{(g + n) \Gamma(1 - g + \frac{\mu}{g} n) \Gamma(2g + n)}{2^{2g + 2n} \Gamma(\frac{1}{2} - g - n)^2 \Gamma(g + \frac{\mu}{g} n)^2}.
\]

The discrete symmetry and the pseudo virtual state wavefunctions are:

\[
\mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + \mathcal{E}_{-1}(\lambda), \quad t(\lambda) = (1 - g, \mu),
\]

\[
\phi_0(x; \lambda) = \phi_0(x; t(\lambda)) = e^{-\frac{\pi}{2} i} \Gamma(\frac{1}{2} + \frac{\mu}{g}) \Gamma(\frac{1}{2} + \frac{\mu}{g}),
\]

\[
\mathcal{E}_n(\lambda) = \mathcal{E}_n(t(\lambda)) + \mathcal{E}_{-1}(\lambda) = \mathcal{E}_{-n-1}(\lambda) \quad (0 \leq n < g - 1, \quad v > \frac{\mu}{g} + g - 1).
\]

For \(g - 1 < v < 2g - 1 \left( g > \frac{1}{2} \right)\), the discrete symmetry generates the type II virtual states \(\phi_n(x; \lambda)\) [14].

3.13. The hDPT potential

It has finitely many discrete eigenstates \(0 \leq n \leq n_{\text{max}}(\lambda) = \lfloor \frac{h - g}{2} \rfloor^\prime\) in the specified parameter range:

\[
\lambda = (g, h), \quad \delta = (1, -1), \quad 0 < x < \infty, \quad h > g > \frac{1}{2},
\]

\[
w(x; \lambda) = g \log \sinh x - h \log \cosh x, \quad U(x; \lambda) = \frac{g(g - 1)}{\sinh^2 x} - \frac{h(h + 1)}{\cosh^2 x} + (h - g)^2,
\]

\[
\mathcal{E}_n(\lambda) = 4n(h - g - n), \quad \eta(x) = \cosh 2x, \quad f_n(\lambda) = 2(n + g - h), \quad b_{n-1}(\lambda) = -2n,
\]

\[
\phi_n(x; \lambda) = \phi_0(x; \lambda) P_n(\eta(x; \lambda)), \quad \phi_0(x; \lambda) = (\sinh x)^g (\cosh x)^{-h},
\]

\[
P_n(\eta; \lambda) = P_n^{(-\frac{1}{2}, -\frac{1}{2})}(\eta), \quad h_n(\lambda) = \frac{\Gamma(n + g + \frac{1}{2}) \Gamma(h - g - n + 1)}{2n!(h - g - 2n) \Gamma(h - n + 1 + \frac{1}{2})}.
\]

The eigenvalues can be also expressed as \(\mathcal{E}_n(\lambda) = 4 \left( (\frac{h - g}{2})^2 - (\frac{h - g}{2} - n)^2 \right)^\prime\). The system satisfies the closure relation [18].

Three types of discrete symmetries are:

Type I : \(\mathcal{H}(\lambda) = \mathcal{H}(t^1(\lambda)) - (1 + 2g)(1 + 2h), \quad t^1(\lambda) = (g, -1 - h),\)

Type II : \(\mathcal{H}(\lambda) = \mathcal{H}(t^1(\lambda)) - (1 - 2g)(1 - 2h), \quad t^1(\lambda) = (1 - g, h),\)

Type III : \(\mathcal{H}(\lambda) = \mathcal{H}(t(\lambda)) + \mathcal{E}_{-1}(\lambda), \quad t = t^1 \circ t^1, \quad t(\lambda) = (1 - g, -1 - h).\)

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The pseudo virtual state wavefunctions are generated by type III:
\[
\hat{\phi}_n(x; \lambda) = \phi_n(x; t(\lambda)) = (\sinh x)^{\frac{1}{2}}(\cosh x)^{b+1} P_b(\eta(x); t(\lambda)) \quad (v \in \mathbb{Z}_{\geq 0}),
\]
\[
\tilde{E}_n(\lambda) = E_n(t(\lambda)) + E_{-1}(\lambda) = E_{-v-1}(\lambda).
\]

The type I and II virtual states are obtained by using type I and II discrete symmetries.

The Morse potential section 3.8 is obtained by the following limit:
\[
x = \frac{1}{2}(x^M - a), \quad g = \frac{1}{2} \mu e^\alpha, \quad \hbar = \frac{1}{2} \mu e^\alpha + 2\hbar^M + \alpha, \quad \lim_{a \to \infty} \mathcal{H}(\lambda) = 4\hbar^M(\lambda^M),
\]
together with the eigenfunctions.

4. Main results

Let us introduce appropriate symbols and notation for stating the main results. Let \( D \) be a set of distinct non-negative integers. We introduce an integer \( N \) and fix it to be not less than the maximum of \( D \):
\[
N \geq \max(D).
\]
Let us define a set of distinct non-negative integers \( \tilde{D} = [0, 1, \ldots, N]\setminus\{d_1, d_2, \ldots, d_M\} \) together with the shifted parameters \( \tilde{\lambda} \):
\[
\tilde{D} \overset{\text{def}}{=} \{0, 1, \ldots, \tilde{d}_1, \ldots, \tilde{d}_2, \ldots, \tilde{d}_M, \ldots, N\} = \{e_1, e_2, \ldots, e_{N+1-M}\},
\]
\[
d_j \overset{\text{def}}{=} N - d_j, \quad \tilde{\lambda} \overset{\text{def}}{=} \lambda - (N+1)\delta.
\]

Starting from the well-defined original system \((3.1)\), the system after Darboux–Crum transformations in terms of a set of pseudo virtual state wavefunctions \( D \) is described by the Hamiltonian \( \mathcal{H}^{DC} \):
\[
\mathcal{H}^{DC} = -\frac{d^2}{dx^2} + U^{DC}(x),
\]
\[
U^{DC}(x) = U(x; \lambda) - 2\alpha^2 \log |W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x; \lambda)|.
\]

General theory presented in section 2 states that, if the Hamiltonian \( \mathcal{H}^{DC} \) is non-singular, the eigenstates are given by \( \Phi_n^{DC} \) and \( \Phi_j^{DC} \):
\[
\Phi_n^{DC}(x) = \frac{W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}; \phi_n](x; \lambda)}{W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x; \lambda)} \quad (n = 0, 1, \ldots, n_{\max}(\lambda) \text{ or } \infty),
\]
\[
\Phi_j^{DC}(x) = \frac{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \ldots, \tilde{\phi}_{d_M}; \phi_j](x; \lambda)}{W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}](x; \lambda)} \quad (j = 1, 2, \ldots, M),
\]
\[
\mathcal{H}^{DC} \Phi_n^{DC}(x) = \mathcal{E}_n(\lambda) \Phi_n^{DC}(x), \quad \mathcal{H}^{DC} \Phi_j^{DC}(x) = \mathcal{E}_{-j-1}(\lambda) \Phi_j^{DC}(x).
\]

The system after Krein–Adler transformations in terms of \( \tilde{D} \) with shifted parameters \( \tilde{\lambda} \) is described by the Hamiltonian \( \mathcal{H}^{KA} \):
\[
\mathcal{H}^{KA} = -\frac{d^2}{dx^2} + U^{KA}(x),
\]
\[
U^{KA}(x) = U(x; \tilde{\lambda}) - 2\alpha^2 \log |W[\phi_0, \phi_1, \ldots, \tilde{\phi}_{d_1}, \ldots, \tilde{\phi}_{d_M}, \ldots, \phi_N](x; \tilde{\lambda})|.
\]
The differential equations (4.4), Hamiltonians (4.5), and the illustration (figure 1) would be helpful to understand the proposition. The form of the singularity free conditions of the potential are algebraic and they hold irrespective of the non-singularity of $\mathcal{H}^{DC}$ and $\mathcal{H}^{KA}$.

Our main results hold for any one of the shape-invariant quantum mechanical systems in sections 3.3–3.13.

**Proposition 4.1.** The two systems with $\mathcal{H}^{DC}$ and $\mathcal{H}^{KA}$ are equivalent. To be more specific, the equality of the potentials and the eigenfunctions read:

$$\Phi_n^{KA}(x) = \frac{W[\Phi_0, \Phi_1, \ldots, \Phi_{d_1}, \ldots, \Phi_{d_2}, \ldots, \Phi_N](x; \lambda)}{W[\Phi_0, \Phi_1, \ldots, \tilde{\Phi}_d, \ldots, \tilde{\Phi}_{d_2}, \ldots, \Phi_N](x; \bar{\lambda})} \quad (n = 0, 1, \ldots, n_{\max}(\lambda) \text{ or } \infty),$$

$$\Phi_j^{KA}(x) = \frac{W[\Phi_0, \Phi_1, \ldots, \tilde{\Phi}_d, \ldots, \tilde{\Phi}_{d_2}, \ldots, \Phi_N](x; \lambda)}{W[\Phi_0, \Phi_1, \ldots, \tilde{\Phi}_d, \ldots, \tilde{\Phi}_{d_2}, \ldots, \Phi_N](x; \lambda)} \quad (j = 1, 2, \ldots, M).$$

$$\mathcal{H}^{KA}\Phi_n^{KA}(x) = \mathcal{E}_{N+1+n}(\lambda)\Phi_n^{KA}(x), \quad \mathcal{H}^{KA}\Phi_j^{KA}(x) = \mathcal{E}_{d_j}(\lambda)\Phi_j^{KA}(x). \quad (4.6)$$

Note that $n_{\max}(\lambda)$ in sections 3.8–3.13 satisfies

$$n_{\max}(\lambda) - (N + 1) = n_{\max}(\lambda). \quad (4.7)$$

The differential equations (4.4) and (4.6) (with $n \in \mathbb{Z}_{\geq 0}$) hold irrespective of the non-singularity of $\mathcal{H}^{DC}$ and $\mathcal{H}^{KA}$.

The singularity free conditions of the potential are

$$\prod_{j=1}^{N+1-M} (m - e_j) \geq 0 \quad (\forall m \in \mathbb{Z}_{\geq 0}). \quad (4.11)$$

For $M = 1$, $D = \{d_1\}$, $\tilde{D} = \{0, 1, \ldots, \tilde{d}_1, \ldots, N\}$, the above conditions are satisfied by even $d_1$, $d_1 \in 2\mathbb{Z}_{\geq 0}$. In other words, the pseudo virtual state wavefunctions $|\tilde{\Phi}_v\rangle$ for even $v$ are nodeless. The above equalities (up to multiplicative factors) (4.8)–(4.10) (4.9) with $n \in \mathbb{Z}_{\geq 0}$ are algebraic and they hold irrespective of the non-singularity conditions (4.11).

The following energy formulas satisfied by the eleven systems

$$\mathcal{E}_n(\lambda) - \mathcal{E}_{N-1}(\lambda) = \mathcal{E}_{N+1+n}(\bar{\lambda}),$$

$$\mathcal{E}_{v-1}(\lambda) - \mathcal{E}_{N-1}(\lambda) = \mathcal{E}_{N+1+n}(\bar{\lambda}), \quad (4.12)$$

and the illustration (figure 1) would be helpful to understand the proposition. The form of the energy curve is that of the DPT potential but the situation is similar in all other ten potentials.

If the equality of the potentials (4.8) is shown, then, under the condition (4.11), the Hamiltonians $\mathcal{H}^{DC} - \mathcal{E}_{N-1}(\lambda) = \mathcal{H}^{KA}$ are non-singular, which implies (4.9) and (4.10). Therefore the remaining task is to show (4.8).

The basic ingredients of proposition 4.1 are the two Wronskians

$$W[\Phi_0, \Phi_1, \ldots, \tilde{\Phi}_d, \ldots, \tilde{\Phi}_{d_2}, \ldots, \Phi_N](x; \lambda), \quad W[\Phi_0, \Phi_1, \ldots, \tilde{\Phi}_d, \ldots, \tilde{\Phi}_{d_2}, \ldots, \Phi_N](x; \bar{\lambda}),$$

which determine potentials (4.8) and eventually all the eigenfunctions (4.9)–(4.10). By using the Wronskian formulas (3.22) and (3.18) they are expressed by the determinants of...
By this identity, the equality of the potentials (4.8) reduces to

Since both polynomials are of the same degree

important relation between

These two polynomials

be expressed as

F(x, N, \lambda)

From this independence and the obvious relation

F can be expressed as

It is easy to show the following identity for any of the 11 systems:

By this identity, the equality of the potentials (4.8) reduces to

Since both polynomials are of the same degree \ell_D, this means that the two polynomials are proportional. We state this as

**Proposition 4.2.** Two polynomials characterizing \( {\mathcal{H}}^{DC} \) and \( {\mathcal{H}}^{KA} \), \( {\Xi}_D(\eta; \lambda) \) and \( {\tilde{\Xi}}_D(\eta; \tilde{\lambda}) \), are proportional:

\[
{\Xi}_D(\eta; \lambda) \propto {\tilde{\Xi}}_D(\eta; \tilde{\lambda}).
\] (4.19)
In particular, this means polynomial Wronskian identities

\[ W[\xi_{d_1}, \ldots, \xi_{d_v}](\eta; \lambda) \propto W[P_0, \ldots, \bar{P}_{d_1}, \ldots, \bar{P}_{d_v}, \ldots, P_N](\eta; \lambda), \]

(4.20)

for all the systems in group A. Recall that \( \xi_1(\eta; \lambda) = P_1(\eta; \ell(\lambda)) \) (3.9) (with a slight modification for (H) and (L), (3.10)-(3.11)). For the simplest case of \( M = 1, N = d = \ell \Rightarrow D = \ell \), the Wronskian identities are reported as (A.22) in [16] for the three types of the classical orthogonal polynomials, Hermite (H), Laguerre (L) and Jacobi (J). To the best of our knowledge, the general polynomial Wronskian identities (4.20) have not been reported before.

The proof of proposition 4.2 is done by induction in \( M \).

**First step.** In the first step we prove (4.19) for \( M = 1, N \geq d = \ell \equiv v \):

\[ \xi_v(\eta; \lambda) \propto \tilde{\Xi}_{0,1,\ldots,v}(\eta; \lambda). \]

(4.21)

Recall that the differential equations (4.4) and (4.6) hold, and

\[ \tilde{\phi}_v(x; \lambda) = A_{[v]}(x; \lambda)\xi_v(\eta(x); \lambda), \]

(4.22)

\[ W[\phi_0, \phi_1, \ldots, \phi_N](x; \lambda) = \tilde{A}_{[0,1,\ldots,N]}(x; \lambda)\tilde{\Xi}_{0,1,\ldots,N}(\eta; \lambda), \]

\[ W[\phi_0, \phi_1, \ldots, \phi_N](x; \lambda) = \tilde{A}_{[0,1,\ldots,N]}(x; \lambda)\tilde{\Xi}_{0,1,\ldots,N}(\eta; \lambda). \]

(4.23)

By substituting (4.22) into the differential equation \( H^{DC}_{[v]}(\tilde{\phi}^{DC}_1(x) = E_{v-1}(\lambda)\tilde{\phi}^{DC}_1(x) \) (4.4), it becomes

\[ \bar{a}_x^2 \bar{f} + 2\bar{a}_x \bar{a}_x \log \frac{G_u}{G} \bar{a}_x \bar{f} + \left( \bar{a}_x^2 A_{[v]}(x; \lambda) - U(x; \lambda) + 2\bar{a}_x \log F(x, N, \lambda) \right) \bar{f} = 0, \]

(4.24)

where \( f = \xi_v(\eta(x); \lambda) \). By substituting (4.23) into the differential equation \( H^{KA}_{[v]}(\tilde{\phi}^{KA}_1(x) = E_v(\lambda)\tilde{\phi}^{KA}_1(x) \) (4.6) and using \( \tilde{\Xi}_{0,1,\ldots,N}(\eta(x); \lambda) = \text{constant} \) (see (4.38)), we obtain

\[ \bar{a}_x^2 \tilde{f} + 2\bar{a}_x \log \frac{G_v}{G} \bar{a}_x \tilde{f} + \left( \bar{a}_x^2 G_v + \bar{a}_x^2 G_v - 2\bar{a}_x G_v \right) \tilde{f} = 0, \]

(4.25)

where \( \tilde{f} = \tilde{\Xi}_{0,1,\ldots,v}(\eta(x); \lambda) \), \( G_v = \tilde{A}_{[0,1,\ldots,v]}(x; \lambda) \) and \( G = \tilde{A}_{[0,1,\ldots,N]}(x; \lambda) \). By using (4.15) and (4.16), we find \( A_{[v]}(x; \lambda)/G_v = F(x, N, \lambda) = 1/G \). The above equation (4.25) is rewritten as

\[ \bar{a}_x^2 \tilde{f} + 2\bar{a}_x A_{[v]}(x; \lambda) \bar{a}_x \tilde{f} + \left( \bar{a}_x^2 A_{[v]}(x; \lambda) - U(x; \lambda) + 2\bar{a}_x^2 \log F(x, N, \lambda) \right) \tilde{f} = 0. \]

(4.26)

From (4.12) and (4.17), we have

\[ U(x; \lambda) - E_{v-1}(\lambda) = U(x; \lambda) - E_v(\lambda) + 2\bar{a}_x^2 \log F(x, N, \lambda). \]

Thus \( f = \xi_v(\eta(x); \lambda) \) and \( \tilde{f} = \tilde{\Xi}_{0,1,\ldots,v}(\eta(x); \lambda) \) satisfy the same differential equation. Since both of them are polynomials of degree \( v \) in \( \eta \), they should be proportional for generic parameters [20].

**Second step.** Assume that (4.19) holds until \( M (M \geq 1) \); we will show that it also holds for \( M + 1 \).

Before presenting a general proof, we illustrate the outline by taking the simple case of group A (4.20). We have shown the \( M = 1 \) case:

\[ \xi_v(\eta; \lambda) \propto W[P_0, \ldots, \bar{P}_{d_1}, \ldots, P_N](\eta; \lambda). \]
By using the algebraic Wronskian identity (2.12), we have
\[
W[ξ_{d1}, ξ_{d2}, ξ_{d3}, . . . , ξ_{dN}] = W[W[P_0, . . . , P_N], W[P_0, . . . , P_N]](η; 0)
\]
\[
= W[P_0, . . . , P_N](η; 0) - W[P_0, . . . , P_N](η; 0)
\]
where we have used \( W[P_0, P_1, . . . , P_N](η; 0) = \text{constant} \) (see (4.38)). This is the \( M = 2 \) result. For \( M = 3 \), we use the \( M = 2 \) results to obtain
\[
ξ_{d1}(η; λ) \cdot W[ξ_{d1}, ξ_{d2}, ξ_{d3}, . . . , ξ_{dN}] = W[ξ_{d1}, ξ_{d2}, ξ_{d3}, . . . , ξ_{dN}](η; λ)
\]
\[
\propto W[ξ_{d1}, ξ_{d2}, ξ_{d3}, . . . , ξ_{dN}](η; λ)
\]
This establishes the identities for \( M = 3 \). Higher \( M \) identities follow in a similar way.

Let us present a general proof. Equation (4.19) is equivalent to
\[
\frac{W[Φ_{d1}, Φ_{d2}, . . . , Φ_{dN}] (x; λ)}{A_D (x; λ)} \propto \frac{W[φ_0, φ_1, . . . , φ_d_N] (x; λ)}{A_D (x; λ)}
\]
which implies
\[
W[Φ_{d1}, Φ_{d2}, . . . , Φ_{dN}] (x; λ) \propto F(x, N, λ)W[φ_0, φ_1, . . . , φ_d_N] (x; λ).
\]
Assume that (4.19) holds till \( M (M \geq 1) \). By using the Wronskian identity (2.12), we obtain
\[
W[Φ_{d1}, . . . , Φ_{dM+1}] (x; λ) \propto W[Φ_{d1}, . . . , Φ_{dM}, Φ_{dM+1}] (x; λ)
\]
\[
\propto W[F(x, N, λ)W[φ_0, . . . , φ_{dM}, φ_{dM+1}], W[φ_0, . . . , φ_{dM}, φ_{dM+1}]] (x; λ)
\]
\[
= F(x, N, λ)^2W[φ_0, . . . , φ_{d1}, . . . , φ_{dM}, φ_{dM+1}, . . . , φ_N] (x; λ)
\]
\[
\propto F(x, N, λ)^2W[φ_0, . . . , φ_{d1}, . . . , φ_{dM}, φ_{dM+1}, . . . , φ_N] (x; λ)
\]
This leads to
\[
\frac{W[Φ_{d1}, . . . , Φ_{dM+1}] (x; λ)}{A_{D(d, . . . , dM+1)} (x; λ)} \propto \frac{F(x, N, λ)A_{D(d, . . . , dM+1)} (x; λ)}{A_{D(d, . . . , dM+1)} (x; λ)} \frac{W[φ_0, . . . , φ_{d1}, . . . , φ_{dM+1}, . . . , φ_N] (x; λ)}{A_{D(d, . . . , dM+1, . . . , N)} (x; λ)}
\]
\[
= \frac{W[φ_0, . . . , φ_{d1}, . . . , φ_{dM+1}, . . . , φ_N] (x; λ)}{A_{D(d, . . . , dM+1, . . . , N)} (x; λ)},
\]
which means
\[
Ξ_{D(d, . . . , dM+1)} (η; λ) \propto \Xi_{D(d, . . . , dM+1, . . . , N)} (η; 0).
\]
This concludes the induction proof of (4.19) and the proof of propositions 4.1 and 4.2 is completed.

In order to obtain the rational forms (the ratios of polynomials) of the eigenfunctions (4.9)–(4.10) in proposition 4.1, we need the Wronskian expressions of the numerators:

\[
W[\phi_{d_1}, \phi_{d_2}, \ldots, \phi_{d_M}, \phi_{0}] (x; \lambda) = A_{D,n} (x; \lambda) P_{D,n} (\eta(x); \lambda).
\]

\[
A_{D,n} (x; \lambda) = \begin{cases} 
\overline{\phi}_{(0j)} (x; \lambda)^{M} \phi_{0} (x; \lambda) \left( e^{-\frac{dM(u)}{dx}} \right)^{\frac{1}{2} M(M-1)-M} & \text{: Group A} \vspace{1em} \\
\prod_{k=1}^{M} \phi_{0} (x; (\ell(\lambda) + d_1 \delta)) \cdot \phi_{0} (x; \lambda + n \delta) & \text{: Group B}
\end{cases}
\]

Here \( P_{D,n} (\eta; \lambda) \) is a polynomial in \( \eta \) and its degree is generically \( \ell_D + M + n \). This leads to

\[
\frac{W[\phi_{d_1}, \ldots, \phi_{d_M}, \phi_{0}] (x; \lambda)}{W[\phi_{d_1}, \ldots, \phi_{d_M}] (x; \lambda)} = \frac{P_{D,n} (\eta(x); \lambda)}{E_D (\eta(x); \lambda)} \times \begin{cases} 
\phi_0 (x; \lambda - M \delta) & \text{: Group A} \vspace{1em} \\
\phi_0 (x; \lambda + n \delta) & \text{: Group B}
\end{cases}
\]

Equation (3.18) leads to

\[
\frac{W[\phi_{d_1}, \ldots, \phi_{d_M}, \phi_{0}] (x; \lambda)}{W[\phi_{d_1}, \ldots, \phi_{d_M}] (x; \lambda)} = \frac{\tilde{P}_{D,n} (\eta(x); \lambda)}{E_D (\eta(x); \lambda)} \times \begin{cases} 
\phi_0 (x; \lambda - M \delta) & \text{: Group A} \vspace{1em} \\
\phi_0 (x; \lambda + n \delta) & \text{: Group B}
\end{cases}
\]

\[
\tilde{P}_{D,n} (\eta; \lambda) \defeq \tilde{E}_{(d_1, \ldots, d_M)} (\eta; \lambda) : \text{degree} = \ell_D - (N + 1 - M) + (N + 1 + n) = \ell_D + M - n.
\]

In the present case we have

\[
\frac{W[\phi_{\ell_1}, \ldots, \phi_{\ell_{2M+1+n}}, \phi_{0}] (\lambda; \rho)}{W[\phi_{\ell_1}, \ldots, \phi_{\ell_{2M+1+n}}] (x; \lambda)} = \frac{\tilde{P}_{D, N+1+n} (\eta(x); \lambda)}{E_D (\eta(x); \lambda)} \times \begin{cases} 
\phi_0 (x; \lambda - M \delta) & \text{: Group A} \vspace{1em} \\
\phi_0 (x; \lambda + n \delta) & \text{: Group B}
\end{cases}
\]

where the degree of \( \tilde{P}_{D, N+1+n} (\eta; \lambda) \) is \( \ell_D - (N + 1 - M) + (N + 1 + n) = \ell_D + M + n \). Therefore (4.9) implies

\[
P_{D,n} (\eta; \lambda) \propto \tilde{P}_{D, N+1+n} (\eta; \lambda) \quad (n \in \mathbb{Z}_{\geq 0}).
\]

Similarly (4.10) implies

\[
\tilde{E}_{(d_1, \ldots, d_M)} (\eta; \lambda) \propto \tilde{P}_{D, j} (\eta; \lambda) \quad (j = 1, 2, \ldots, M).
\]

Let us introduce \( D_\downarrow \) based on \( D = \{d_1, d_2, \ldots, d_M\} \):

\[
D_\downarrow = \{d_1 \pm 1, d_2 \pm 1, \ldots, d_M \pm 1\}.
\]

The newly introduced polynomials at \( n = 0 \) are related to the old ones:

\[
\tilde{P}_{D,0} (\eta; \lambda) \propto \tilde{E}_{D_\downarrow} (\eta; \lambda + \delta),
\]

\[
P_{D,0} (\eta; \lambda) \propto \tilde{E}_{D_\downarrow} (\eta; \lambda + \delta).
\]

Based on these formulas, the following relations follow:

\[
f_D (x; \lambda) \defeq \log \left| \frac{W[\phi_{d_1}, \ldots, \phi_{d_M}, \phi_{0}] (x; \lambda)}{W[\phi_{d_1}, \ldots, \phi_{d_M}] (x; \lambda)} \right| \quad (\text{min} d_j \geq 2),
\]

\[
(\partial_\lambda f_D (x; \lambda))^2 - \partial^2_\lambda f_D (x; \lambda) = (\partial_\lambda f_D (x; \lambda + \delta))^2 + \partial^2_\lambda f_D (x; \lambda + \delta) + \xi_1 (\lambda),
\]

\[
f_D (x; \lambda) \defeq \log \left| \frac{W[\phi_{d_1}, \ldots, \phi_{d_M}, \phi_{0}] (x; \lambda)}{W[\phi_{d_1}, \ldots, \phi_{d_M}] (x; \lambda)} \right|,
\]

\[
(\partial_\lambda f_D (x; \lambda))^2 - \partial^2_\lambda f_D (x; \lambda) = (\partial_\lambda f_D (x; \lambda + \delta))^2 + \partial^2_\lambda f_D (x; \lambda + \delta) + \xi_1 (\lambda),
\]

which have the same forms as the shape-invariance relation (2.20) but they do not mean the shape-invariance. The \( M = 1 \) case for (M), (RM) and (Kh) was presented as 'enlarged' shape-invariance in [13, 14].
In the rest of this section we provide the proofs of (4.33)–(4.34). For (4.33), we note that the forward shift relation (3.3) can be written as

$$\frac{d}{dx} \phi_n(x; \lambda) = f_0(x) \phi_{n-1}(x; \lambda + \delta).$$

(4.37)

By using this, we obtain

$$W[\phi_{d_1}, \ldots, \phi_{d_M}, \phi_0](x; \lambda)$$

$$\phi_0(x; \lambda) M+1 W \left[ \phi_{d_1}, \ldots, \phi_{d_M}, \phi_0 \right] (x; \lambda)
= \phi_0(x; \lambda) M+1 W \left[ \phi_{d_1}, \ldots, \phi_{d_M}, \phi_0 \right] (x; \lambda)
= \phi_0(x; \lambda) M+1 (-1)^M W \left[ \frac{d}{dx} \phi_{d_1}, \ldots, \frac{d}{dx} \phi_{d_M} \right] (x; \lambda)
= \phi_0(x; \lambda) M+1 (-1)^M W \left[ f_0(x), \phi_{d_1-1}(x; \lambda + \delta), \ldots, \phi_{d_M}(x) \right.\phi_{d_M-1}(x; \lambda + \delta) \phi_0(x; \lambda) \left] (x)
= (-1)^M \prod_{j=1}^M f_{d_j}(x) \cdot \phi_0(x; \lambda) W[\phi_{d_1-1}, \ldots, \phi_{d_M-1}](x; \lambda + \delta).

This can be rewritten as

$$\frac{\tilde{Z}_{[d_1, \ldots, d_M, 0]}(\eta(x); \lambda)}{\hat{Z}_{[d_1, \ldots, d_M, 0]}(\eta(x); \lambda + \delta)} = (-1)^M \prod_{j=1}^M f_{d_j}(x) \cdot \phi_0(x; \lambda) \tilde{A}_{[d_1, \ldots, d_M, 0]}(\eta(x); \lambda + \delta),$$

and the fact that its right-hand side is a constant

$$\phi_0(x; \lambda) \tilde{A}_{[d_1, \ldots, d_M, 0]}(\eta(x); \lambda + \delta) = 1,$$

can be verified for any of the systems listed in sections 3.3–3.13. This concludes the proof of (4.33). We remark that (4.33) is

$$\frac{\tilde{Z}_{[0,1, \ldots, N]}(\eta; \lambda)}{\tilde{Z}_{[0,1, \ldots, N-1]}(\eta; \lambda + \delta)} \propto \frac{\tilde{A}_{[d_1, \ldots, d_M, 0]}(\eta; \lambda + N\delta)}{\tilde{A}_{[d_1, \ldots, d_M, 0]}(\eta; \lambda)} \propto \cdots \propto \tilde{Z}_{[0]}(\eta; \lambda + N\delta) = \text{constant.}$$

(4.38)

For (4.34), we note that the forward shift relation for \( \tilde{\phi}_\delta(x; \lambda) \) (3.12) can be rewritten as

$$\frac{d}{dx} \tilde{\phi}_\delta(x; \lambda) = -\epsilon b_\delta(\lambda) \tilde{\phi}_{\delta+1}(x; \lambda + \delta) \phi_0(x; \lambda).$$

(4.39)

By using this, we obtain

$$W[\tilde{\phi}_{d_1}, \ldots, \tilde{\phi}_{d_M}, \phi_0](x; \lambda)$$

$$\phi_0(x; \lambda) M+1 W \left[ \tilde{\phi}_{d_1}, \ldots, \tilde{\phi}_{d_M}, \phi_0 \right] (x; \lambda)
= \phi_0(x; \lambda) M+1 W \left[ \tilde{\phi}_{d_1}, \ldots, \tilde{\phi}_{d_M}, \phi_0 \right] (x; \lambda)
= \phi_0(x; \lambda) M+1 (-1)^M W \left[ \frac{d}{dx} \tilde{\phi}_{d_1}, \ldots, \frac{d}{dx} \tilde{\phi}_{d_M} \right] (x; \lambda)
= \phi_0(x; \lambda) M+1 (-1)^M W \left[ -\epsilon b_{d_1}(\lambda) \tilde{\phi}_{d_1+1}(x; \lambda + \delta) \phi_0(x; \lambda), \ldots, -\epsilon b_{d_M}(\lambda) \tilde{\phi}_{d_M+1}(x; \lambda + \delta) \phi_0(x; \lambda) \right.\phi_0(x; \lambda) \left] (x)
= \prod_{j=1}^M \epsilon b_{d_j}(\lambda) \cdot \phi_0(x; \lambda) W[\tilde{\phi}_{d_1+1}, \ldots, \tilde{\phi}_{d_M+1}](x; \lambda + \delta).

This can be rewritten as

$$\frac{P_{[d_1, \ldots, d_M, 0]}(\eta(x); \lambda)}{\tilde{Z}_{[d_1+1, \ldots, d_M+1]}(\eta(x); \lambda + \delta)} = \prod_{j=1}^M \epsilon b_{d_j}(\lambda) \cdot \phi_0(x; \lambda) \tilde{A}_{[d_1+1, \ldots, d_M+1]}(\eta(x); \lambda + \delta) \tilde{A}_{[d_1, \ldots, d_M, 0]}(\eta(x); \lambda).$$
and the fact that its right-hand side is a constant
\[ \frac{\phi_0(x; \lambda)A_{d_1+1,...,d_M+1}(x; \lambda + \delta)}{A_{d_1,...,d_M},0(x; \lambda)} = 1, \]
can be verified for any of the systems listed in sections 3.3–3.13. This concludes the proof of (4.34).

5. Summary and comments

In the context of rational extensions of solvable potentials, the concept of the pseudo virtual state wavefunctions is introduced. They are obtained by relaxing two conditions of the virtual state wavefunctions; the reciprocals are square integrable at both boundaries and they need not be nodeless. A Darboux–Crum transformation in terms of a pseudo virtual state wavefunction will produce a new eigenstate below the original ground state. The same system can be derived by a special type of Krein–Adler transformation with negatively shifted parameters. The main results of this paper is the equivalence of the Darboux–Crum transformation in terms of multiple pseudo virtual states and Krein–Adler transformations in terms of multiple eigenstates with negatively shifted parameters. This is based on polynomial Wronskian identities, which are generalizations of those reported by the present authors [16] a few years ago. The equivalence holds for most of the known shape-invariant potentials consisting of eleven explicit examples having finite as well as infinite discrete eigenstates.

The type II virtual state wavefunctions have been obtained by the discrete symmetries for two potentials, (C) section 3.6 and (Kh) section 3.12. They have been used in the context of \( M = 1 \) rational extensions of these potentials in [11] and [14]. Multi-indexed extensions of these two potentials can be constructed in exactly the same way as in [1]. In a separate publication [19], we will discuss rational extensions in terms of genuine virtual state wavefunctions for shape-invariant potentials having finitely many discrete eigenstates. They have different features from those with infinitely many eigenstates, which have been explored in connection with the multi-indexed orthogonal polynomials [1].

The multi-indexed Laguerre and Jacobi orthogonal polynomials are labeled by the multi-index \( D \), but different multi-index sets may give the same multi-indexed polynomials, e.g. equations (50)–(51) in [1]. The proposition 4.2 gives its generalization. By applying the twist based on the type II discrete symmetry to (4.19), the lhs becomes the denominator polynomial with multiple type I virtual state deletion and the rhs becomes that of type II.

After completing this paper, we became aware of a recent work [21], which discusses some rational extensions of the harmonic oscillator. They correspond to some special examples of the equivalence for the harmonic oscillator (H) and \( M = 2 \).

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References

[1] Odake S and Sasaki R 2011 Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials Phys. Lett. B 702 164–70 (arXiv:1105.0508 [math-ph])
[2] Gómez-Ullate D, Kamran N and Milson R 2012 Two-step Darboux transformations and exceptional Laguerre polynomials J. Math. Anal. Appl. 387 410–8 (arXiv:1103.5724 [math-ph])
[3] Gendenshtein L E 1983 Derivation of exact spectra of the Schroedinger equation by means of supersymmetry JETP Lett. 38 356–9
[4] Infeld L and Hull T E 1951 The factorization method Rev. Mod. Phys. 23 21–68
[5] Cooper F, Khare A and Sukhatme U 1995 Supersymmetry and quantum mechanics Phys. Rep. 251 267–385
[6] Darboux G 1888 Théorie Générale Des Surfaces vol 2 (Paris: Gauthier-Villars)
[7] Crum M M 1955 Associated Sturm–Liouville systems Q. J. Math. 6 121–7 (arXiv:physics/9908019)
[8] Dubov S Yu, Eleonskii V M and Kulagin N E 1992 Equidistant spectra of anharmonic oscillators Sov. Phys.—JETP 75 466–51
[9] Gómez-Ullate D, Kamran N and Wilson R 2004 The Darboux transformation and algebraic deformations of shape-invariant potentials J. Phys. A: Math. Gen. 37 1789–804 (arXiv:quant-ph/0308062)
[10] Gómez-Ullate D, Kamran N and Wilson R 2004 Supersymmetry and algebraic Darboux transformations J. Phys. A: Math. Gen. 37 10065–78 (arXiv:nlin.SI/0402052)
[11] Grandati Y 2011 Solvable rational extensions of the isotonic oscillator Ann. Phys. 326 2074–90 (arXiv:1101.0055 [math-ph])
[12] Ho C-L 2011 Prepotential approach to solvable rational extensions of harmonic oscillator and Morse potentials J. Math. Phys. 52 122107 (arXiv:1105.3670 [math-ph])
[13] Quesne C 2012 Revisiting (quasi-)exactly solvable rational extensions of the Morse potential Int. J. Mod. Phys. A 27 1250073 (arXiv:1203.1812 [math-ph])
[14] Quesne C 2012 Novel enlarged shape invariance property and exactly solvable rational extensions of the Rosen–Morse II and Eckart potentials SIGMA 8 080 (arXiv:1208.6165 [math-ph])
[15] Krein M G 1957 On continuous analogue of a formula of Christoffel from the theory of orthogonal polynomials Dokl. Akad. Nauk CCCP 113 970–3 (in Russian)
Adler V È 1994 A modification of Crum’s method Theor. Math. Phys. 101 1381–6
[16] Garcia-Gutiérrez L, Odake S and Sasaki R 2010 Modification of Crum’s theorem for ‘discrete’ quantum mechanics Prog. Theor. Phys. 124 1–26 (arXiv:1004.0289 [math-ph])
[17] Schiff L J 1968 Quantum Mechanics 3rd edn (New York: McGraw-Hill)
[18] Odake S and Sasaki R 2006 Unified theory of annihilation-creation operators for solvable (‘discrete’) quantum mechanics J. Math. Phys. 47 102102 (arXiv:quant-ph/0605215)
Odake S and Sasaki R 2006 Exact solution in the Heisenberg picture and annihilation-creation operators Phys. Lett. B 641 112–7 (arXiv:quant-ph/0605221)
[19] Odake S and Sasaki R 2013 Extensions of solvable potentials with finitely many discrete eigenstates J. Phys. A: Math. Theor. 46 235203 (arXiv:1301.3980 [math-ph])
[20] Calogero F and Ge Y 2012 Can the general solution of the second-order ODE characterizing Jacobi polynomials be polynomial? J. Phys. A: Math. Theor. 45 095206
[21] Marquette I and Quesne C 2013 Two-step rational extensions of the harmonic oscillator: exceptional orthogonal polynomials and ladder operators J. Phys. A: Math. Theor. 46 155201 (arXiv:1212.3474 [math-ph])