Inertial and Hodge–Tate weights of crystalline representations

Robin Bartlett

Abstract. Let $K$ be an unramified extension of $\mathbb{Q}_p$ and $\rho: G_K \to \text{GL}_n(\mathbb{Z}_p)$ a crystalline representation. If the Hodge–Tate weights of $\rho$ differ by at most $p$ then we show that these weights are contained in a natural collection of weights depending only on the restriction to inertia of $\overline{\rho} = \rho \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. Our methods involve the study of a full subcategory of $p$-torsion Breuil–Kisin modules which we view as extending Fontaine–Laffaille theory to filtrations of length $p$.

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1. Introduction

Let $K/\mathbb{Q}_p$ be a finite unramified extension with residue field $k$. In this paper we show that if the Hodge–Tate weights of a crystalline representation $\rho$ of $G_K$ are sufficiently small then these weights are encoded in an explicit way by the reduction of $\rho$ modulo $p$. Using Fontaine–Laffaille theory this is known for Hodge–Tate weights differing by at most $p - 1$; we will treat weights differing by at most $p$. Our techniques are local and involve the study of a full subcategory of $p$-torsion Breuil–Kisin modules, which we view as extending (p-torsion) Fontaine–Laffaille theory to filtrations of length $p$.

To state our result let $\mathbb{Z}_p^n$ denote the set of $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \leq \ldots \leq \lambda_n$. In Section 2 we show how to attach to any continuous $\overline{\rho}: G_K \to \text{GL}_n(\mathbb{F}_p)$ a subset

$$\text{Inert}(\overline{\rho}) \subset (\mathbb{Z}_p^n)^{\text{Hom}_{\mathbb{F}_p}(k, \mathbb{F}_p)}$$

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This subset depends only on the restriction to inertia of the semi-simplification of \( \overline{\rho} \), and does so in an explicit fashion. We typically write an element of Inert(\( \overline{\rho} \)) as 
\((\lambda_\tau)_{\tau \in \text{Hom}_p(k, \overline{\rho})}\) with \(\lambda_\tau = (\lambda_1, \tau \leq \ldots \leq \lambda_n, \tau)\).

Throughout Hodge–Tate weights are normalised so that the cyclotomic character has weight \(-1\).

**Theorem A.** Let \( \rho: G_K \to \text{GL}_n(\mathbb{Z}_p) \) be a crystalline representation. For each \( \tau \in \text{Hom}_p(k, \mathbb{Z}_p) \) let \( \lambda_\tau \in \mathbb{Z}_+^n \) denote the \( \tau \)-Hodge–Tate weights of \( \rho \). If \( \lambda_n, \tau - \lambda_1, \tau \leq p \) for all \( \tau \) then
\[(\lambda_\tau)_{\tau \in \text{Inert}(\overline{\rho})}\]

When \( n = 2 \) and \( p > 2 \) the result is a theorem of Gee–Liu–Savitt [11]. When \( n = 2 \) and \( p = 2 \) the result is due to Wang [17]. In this paper we extend their methods to higher dimensions.

As already mentioned, when \( \lambda_n, \tau - \lambda_1, \tau \leq p - 1 \) the Theorem A is a straightforward consequence of Fontaine–Laffaille theory, so the main content of our result is that it applies to Hodge–Tate weights differing by \( p \). On the other hand the Theorem A does not hold if the condition \( \lambda_n, \tau - \lambda_1, \tau \leq p \) is relaxed. For example, there exist irreducible two dimensional crystalline representations \( \rho \) of \( G_{\mathbb{Q}_p} \) with Hodge–Tate weights \((-p - 1, 0)\), whose reduction modulo \( p \) have the form \( \overline{\rho} = (\chi_{\text{cyc}} \circ \chi_{\text{cyc}}, 0) \), see [3, Théorème 3.2.1]. Here \( \chi_{\text{cyc}} \) denotes the cyclotomic character. It is easy to check that \((-p - 1, 0)\) is not an element of Inert(\( \overline{\rho} \)).

Our motivation comes from the weight part of (generalisations of) Serre’s modularity conjecture. As a corollary of our result we can prove some new cases of weight elimination for mod \( p \) representations associated to automorphic representations on unitary groups of rank \( n \). To be more precise let \( F \) be an imaginary CM field in which \( p \) is unramified and fix an isomorphism \( \nu: \mathbb{Q}_p \cong \mathbb{C} \). Attached to any RACSDC (regular, algebraic, conjugate self-dual, and cuspidal) automorphic representation \( \Pi \) of \( \text{GL}_n(A_F) \) there is a continuous irreducible \( \tau_{v,p}(\Pi): G_F \to \text{GL}_n(\mathbb{Q}_p) \), cf. the main result of [6]. If \( \Pi \) is unramified above \( p \) then \( \tau_{v,p}(\Pi) \) is crystalline above \( p \), and if \( \lambda = (\lambda_\kappa)_{\kappa} \in (Z_{+}^n)^{\text{Hom}(F, \mathbb{C})} \) is the weight of \( \Pi \) then the \( \kappa \)-Hodge–Tate weights\(^1\) of \( \tau_{v,p}(\Pi) \) equal
\[
\lambda_\kappa + (0, 1, \ldots, n - 1)
\]

Therefore, if \( W(\overline{\tau})_{\text{inert}} \subset (Z_{+}^n)^{\text{Hom}(F, \mathbb{C})} \) consists of \( (\lambda_\kappa)_{\kappa} \) such that \( \lambda_\kappa + (0, 1, \ldots, n - 1) \in \text{Inert}(\overline{\tau})_{v} \), Theorem A implies

**Corollary B.** Let \( \overline{\tau}: G_F \to \text{GL}_n(\mathbb{F}_p) \) be irreducible and continuous. Let \( W(\overline{\tau})_{\text{aut}} \) denote the set of weights \( \lambda \in (Z_{+}^n)^{\text{Hom}(F, \mathbb{C})} \) such that there exists an RACSDC automorphic representation \( \Pi \) of \( \text{GL}_n(A_F) \) which is unramified at \( p \), has weight \( \lambda \), and is such that \( \tau_{\kappa, p}(\Pi) \cong \overline{\tau} \). Then
\[
W(\overline{\tau})_{\leq p - n + 1} \subset W(\overline{\tau})_{\text{inert}}^{\text{aut}}
\]
where for \( * \in \{\text{aut, inert}\} \), \( W(\overline{\tau})_{\leq p - n + 1}^* \) is the subset containing \( (\lambda_\kappa)_{\kappa} \in W(\overline{\tau})^* \) with \( \lambda_n, \kappa - \lambda_1, \kappa \leq p - n + 1 \).

\(^1\)Using \( \kappa \) we can identify \( \kappa \in \text{Hom}(F, \mathbb{C}) \) with pairs \( (v, \overline{\tau}) \) where \( v \) is a place of \( F \) above \( p \) and \( \overline{\tau} \in \text{Hom}(F_v, \mathbb{F}_p) \). Since \( p \) is unramified in \( F \), \( \overline{\tau} \) can be identified with \( \tau \in \text{Hom}_p(k_v, \mathbb{F}_p) \) where \( k_v \) denotes the residue field of \( F_v \). The \( \kappa \)-th Hodge–Tate weights of \( \tau_{\kappa, \Pi} \) are then the \( \tau \)-th Hodge–Tate weights of \( \tau_{v,p}(\Pi) \) at \( v \).
We point out that while the Corollary B involves only distinct Hodge–Tate weights, due to the regularity assumptions on our automorphic representations, Theorem A does not require such distinctness.

If $\mathcal{R}$ is assumed to arise from some potentially diagonalisable RACSDC automorphic representation (a notion introduced in [2]) and if we assume $r_v$ is semi-simple for each $v | p$ then, under a Taylor–Wiles hypothesis, the inclusion in the Corollary B is an equality. This follows from e.g. [1, Theorem 3.1.3].

To conclude this introduction we briefly explain our proof of the theorem; let us do this by sketching the content of the various sections in this paper. In the first two sections we recall some basic notions; in Section 2 we define the set $\text{Inert}(\mathcal{R})$ and in Section 3 we give some elementary results on filtered modules. In Section 4 we recall the notion of a Breuil–Kisin module, and recall how to associate to them Galois representations. Breuil–Kisin modules killed by $p$ admit a natural set of weights and in Section 5 we define what it means for a $p$-torsion Breuil–Kisin module to be strongly divisible; its weights must be contained in $[0, p]$ and a certain explicit condition on its $\varphi$ must be satisfied. We view the category of strongly divisible Breuil–Kisin modules $\text{Mod}_{SD}^k$ as an extension of $p$-torsion Fontaine–Laffaille theory to filtrations of length $p$. We establish two important properties of $\text{Mod}_{SD}^k$. The first main property (Proposition 5.4.7) is shown in Section 5 and states that $\text{Mod}_{SD}^k$ is stable under subquotients, and that weights behave well along short exact sequences. The second main property (Proposition 6.4.1) is proved in Section 6 and concerns the structure of simple objects in $M \in \text{Mod}_{SD}^k$. We show that for such $M$ the weights of $M$ coincide with the inertial weights of the associated Galois representation. These two properties mirror the situation for Fontaine–Laffaille theory. However, unlike in Fontaine–Laffaille theory, it is not the case that simple $M \in \text{Mod}_{SD}^k$ are determined by their weights together with their associated Galois representation. This complicates the proofs considerably. Thus, while there are similarities between $\text{Mod}_{SD}^k$ and Fontaine–Laffaille theory in some respects, the former category is more complicated, reflecting the fact that the reduction of crystalline representations with Hodge–Tate weights in $[0, p]$ is genuinely more subtle than for weights in the Fontaine–Laffaille range. In the final section we recall a theorem of Gee–Liu–Savitt [11] which relates $\text{Mod}_{SD}^k$ with the reduction modulo $p$ of those crystalline representations with Hodge–Tate weights contained in $[0, p]$. Using this, and the two properties of $\text{Mod}_{SD}^k$ described above, it is straightforward to deduce Theorem A.

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1.1. Notation. Throughout we let $k$ denote a finite field of characteristic $p > 0$ and write $K_0 = W(k)[1/p]$. In the introduction we took $K = K_0$; however some of our constructions are valid for arbitrary finite extensions so now allow $K$ to denote a totally ramified extension of $K_0$ of degree $e$, with ring of integers $\mathcal{O}_K$. At certain points it will be necessary to assume $K = K_0$. 
Let $C$ denote the completion of an algebraic closure $\overline{K}$ of $K$ and let $\mathcal{O}_C$ be its ring of integers, with residue field $\overline{k}$. We write $G_K = \text{Gal}(\overline{K}/K)$ and $v_p$ for the valuation on $C$ normalised so that $v_p(p) = 1$.

We fix a uniformiser $\pi \in K$ and a compatible system $\pi^{1/p^n} \in \overline{K}$ of $p^n$-th roots of $\pi$. Many constructions in this paper depend upon these choices. Set $K_\infty = K(\pi^{1/p^n})$ and $G_{K_\infty} = \text{Gal}(\overline{K}/K_\infty)$.

Let $\mu_{p^n}(\overline{K})$ denote the group of $p^n$-th roots of unity in $\overline{K}$ and write $\mathbb{Z}_p(1)$ for the free rank one $\mathbb{Z}_p$-module

$$\lim_{\leftarrow} \mu_{p^n}(\overline{K})$$

Let $\chi_{\text{cycl}}: G_K \to \mathbb{Z}_p^\times$ denote the character though which $G_K$ acts on $\mathbb{Z}_p(1)$.

Let $E/\mathbb{Q}_p$ denote a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. We assume throughout that $K_0 \subset E$. This will be our coefficient field in which the representations we consider will be valued.

If $A$ is any ring of characteristic $p$ we let $\varphi: A \to A$ denote the homomorphism $x \mapsto x^p$. If $A$ is perfect (i.e. $\varphi$ is an automorphism) we let $W(A)$ denote the ring of Witt vectors of $A$ and write $\varphi: W(A) \to W(A)$ for the automorphism lifting $\varphi$ on $A$.

## 2. Inertial Weights

In this section we recall the structure of irreducible torsion representations of $G_K$ and $G_{K_\infty}$. We then define the set $W(\mathfrak{p})^{\text{inert}}$ from the introduction.

### 2.1. Tame ramification

Let $K^\text{ur}$ and $K^t$ be the maximal unramified and maximal tamely ramified extension of $K$ respectively. Set $I^t = \text{Gal}(K^t/K^\text{ur})$. As in [15, Proposition 2] there is an isomorphism

$$s: I^t \to \lim_{\leftarrow} l^\times$$

where in the limit $l$ runs over finite extensions of $k$ with transition maps given by norm maps. This isomorphism sends $\sigma \mapsto (s(\sigma))_l$ where $s(\sigma)_l$ is the image in the residue field of $K^t$ of the Card($l^\times$)-th root of unity

$$\sigma(\pi^{1/\text{Card}(l^\times)})/\pi^{1/\text{Card}(l^\times)} \in K^t$$

Here $\pi^{1/\text{Card}(l^\times)}$ is any Card($l^\times$)-th root of $\pi$; $s(\sigma)_l$ does not depend upon any of these choices. Via $s$ we define the fundamental character

$$\omega_I: I^t \to l^\times$$

For $\theta \in \text{Hom}_{\mathbb{F}_p}(l, \overline{\mathbb{F}}_p)$ then define $\omega_\theta = \theta \circ \omega_I$. Note this is a power of $\omega_I$ and $\omega_{\theta \circ \varphi} = \omega_\theta^p$.

**Lemma 2.1.1.** Any continuous $\chi: I^t \to \overline{\mathbb{F}}_p^\times$ extends to a continuous character of $\text{Gal}(K^t/K)$ if and only if there exist integers $(r_\tau)_{\tau \in \text{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)}$ such that $\chi = \prod_{\tau} \omega_\tau^{r_\tau}$.

**Proof.** Since $1 \to I^t \to \text{Gal}(K^t/K) \to G_k \to 1$ is split, $\chi$ extends to $\text{Gal}(K^t/K)$ if and only if $\chi$ is stable under the conjugation action of $G_k$ on $I^t$. Via $s$ this action is given by the natural action of $G_k$ on $\lim_{\leftarrow} l^\times$, and so $\chi$ extends if and only if $\chi^{l^{\text{inert}}_{\mathbb{F}_p}} = \chi$. After [15, Proposition 5] this is equivalent to asking that $\chi$ be a power of $\omega_k$, thus a product as in the lemma. \qed
In particular we see each $\omega_l$ extends to a character of $G_L$ where $L/K$ is the unramified extension with residue field $l$. Such an extension is well defined only up to twisting by an unramified character. Our fixed choice of uniformiser $\pi \in K$ allows us to define a canonical choice of extension by sending $\sigma \in G_L$ onto the image in the residue field of the element $\sigma(\pi^{1/{\text{Card}(l^\times)})/\pi^{1/{\text{Card}(l^\times)}) \in K^\times$ where $\pi^{1/{\text{Card}(l^\times)}}$ is an $\text{Card}(l^\times)$-th root of $\pi$. We shall denote this character again by $\omega_l: G_L \to \overline{\mathbb{F}}_p$. Also, for $\theta \in \text{Hom}_{\mathbb{F}_p}(l, \overline{\mathbb{F}}_p)$ we write $\omega_\theta = \theta \circ \omega_l$, as characters of $G_L$.

For an extension $L/K$ write $\text{Ind}^K_L V$ in place of $\text{Ind}_{\text{Gal}(\overline{K}/K)}^K \overline{V}$.

**Lemma 2.1.2.** If $V$ is a continuous irreducible representation of $G_K$ on a finite dimensional $\overline{\mathbb{F}}_p$-vector space then $V \cong \text{Ind}^K_L \chi$, where $L/K$ is an unramified extension of degree $\dim_{\overline{\mathbb{F}}_p} V$ and $\chi: G_L \to \overline{\mathbb{F}}_p$ is a continuous irreducible representation of $G_L$.

**Proof.** As $V$ is irreducible the $G_K$-action factors through $G = \text{Gal}(K^\times/K)$ by [15, Proposition 4]. Since $l^\times$ is abelian of order prime to $p$, $V|_l$ is a sum of $\overline{\mathbb{F}}_p$-valued characters. If $r \in G_K$ and $\chi: l^\times \to \overline{\mathbb{F}}_p$ is a character define a new character by $\chi^{(1)}(r) = \chi(r^{-1} \sigma)$. If $l^\times$ acts on $V|_l$ by $\chi$ then $l^\times$ acts on $\chi(V)$ by $\chi^{(1)}$; thus $G_K$ acts on the set of $\chi$ appearing in $V|_l$. Fix $\chi$ appearing in $V|_l$ and let $H \subset G$ be the normal subgroup containing $l^\times$, corresponding to the stabiliser of $\chi$ in $G_K$. By the orbit-stabiliser theorem $[G : H] \leq \dim_{\overline{\mathbb{F}}_p} V$.

Frobenius reciprocity gives a non-zero map $V|_H \to \text{Ind}^H_H \chi$. If $L/K$ is the unramified extension corresponding to $H$ then since the image of $H$ in $G_K$ stabilises $\chi$, this character can be extended to $H$ as in Lemma 2.1.1. Thus $\text{Ind}^H_H \chi = \chi \otimes \text{Ind}^H_H 1$. Since $\text{Ind}^H_H \chi$ is a discrete $H$-module we can find a finite dimensional subrepresentation $R \subset \text{Ind}^H_H \chi$ so that $V|_H$ is mapped into $\chi \otimes R$. As Gal($L^w/L$) is abelian $R$ admits a composition series $0 = R_0 \subset \ldots \subset R_0 = R$ such that each $R_i/R_{i+1}$ is one-dimensional. If $i$ is the largest integer such that $V|_H \to \text{Ind}^H_H V$ factors through $\chi \otimes R_i$ then $V|_H \to \chi \otimes R_i/R_{i+1}$ is non-zero. Frobenius reciprocity gives a non-zero map $V \to \text{Ind}^K_K(\chi \otimes R_i/R_{i+1})$ which, $V$ being irreducible, is injective. Thus $[G : H] = \dim_{\overline{\mathbb{F}}_p} \text{Ind}^K_K(\chi \otimes R_i/R_{i+1}) \geq \dim_{\overline{\mathbb{F}}_p} V$. The inequality of the first paragraph implies $[G : H] = \dim_{\overline{\mathbb{F}}_p} V$ and so this map is an isomorphism.

**Definition 2.1.3.** Let $\mathbf{p}$ be a continuous representation of $G_K$ on an $n$-dimensional $\overline{\mathbb{F}}_p$-vector space. After Lemma 2.1.2 there exist continuous characters $\zeta: G_{L_\zeta} \to \overline{\mathbb{F}}_p$ with $L_{\zeta}/K$ finite unramified, such that

$$
\mathbf{p}^{ss} \cong \bigoplus_{\zeta} \text{Ind}^K_{L_\zeta} \zeta
$$

(2.1.4)

with each summand irreducible. Let $l_\zeta/k$ denote the residue field of $L_\zeta$. After Lemma 2.1.1 there are integers $(r_\theta, \zeta)_{\theta \in \text{Hom}_{\mathbb{F}_p}(l_\zeta, \overline{\mathbb{F}}_p)}$ such that

$$
\zeta|_{l_\tau} = \prod \omega_{l_\zeta}^{-r_{\theta, \zeta}}
$$

Any such collection of $r_{\theta, \zeta}$ defines a weight $\lambda = (\lambda_\tau)_{\tau \in \text{Hom}_{\mathbb{F}_p}(k, \overline{\mathbb{F}}_p)}$ via $\lambda_\tau = \{ r_{\theta, \zeta} \mid \theta|_k = \tau \}$. Define $\text{Inert}(\mathbf{p})$ to be the set of $\lambda$ obtained in this way.

It is easy to check that $W(\mathbf{p})|_{\text{inert}}$ depends only on $\mathbf{p}^{ss}|_{\text{inert}}$. 
2.2. $G_{K_{\infty}}$-representations.

Lemma 2.2.1. Let $K_{\infty}^i = K_{\infty}K^i$. Then restriction defines an isomorphism $\text{Gal}(K_{\infty}^i/K_{\infty}) \to \text{Gal}(K^i/K)$. If $L/K$ is a tamely ramified extension this isomorphism identifies $\text{Gal}(L_{\infty}/K_{\infty})$ with $\text{Gal}(L/K)$ where $L_{\infty} = LK_{\infty}$.

Proof. Since $K_{\infty}/K$ is totally wildly ramified we have $K_{\infty} \cap K^i = K$. The lemma then follows from Galois theory.

Corollary 2.2.2. Restriction describes an equivalence between the category of semi-simple continuous $G_K$-representations on finite dimensional $\overline{\mathbb{F}_p}$-vector spaces, and the analogous category of $G_{K_{\infty}}$-representations.

Proof. The action of $G_K$ on such a semi-simple representation factors through $\text{Gal}(K^i/K)$ as $\text{Gal}(\overline{K}/K^i)$ is pro-$p$. Likewise the action of $G_{K_{\infty}}$ factors through $\text{Gal}(K_{\infty}^i/K_{\infty})$, and so the result follows from Lemma 2.2.1.

3. Filtrations

This section contains some elementary results on filtered modules; they will be useful later. Consider a commutative ring $A$ and a collection of ideals $(F^iA)_{i \in \mathbb{Z}}$ satisfying

$$F^{i+1}A \subset F^iA, \quad (F^iA)(F^jA) \subset F^{i+j}A, \quad F^iA = A \text{ for } i << 0$$

Then the category $\text{Fil}(A)$ of filtered $A$-modules consists of $A$-modules $M$ equipped with a collection of $A$-sub-modules $(F^iM)_{i \in \mathbb{Z}}$ satisfying

$$F^{i+1}M \subset F^iM, \quad (F^iA)(F^jM) \subset F^{i+j}M, \quad F^iM = M \text{ for } i << 0$$

Morphisms are maps $f: M \to N$ of $A$-modules such that $f(F^iM) \subset F^iN$ for all $i$. If $M$ is an object of $\text{Fil}(A)$ we set $\text{gr}(M) = \bigoplus_i \text{gr}^i(M)$ where $\text{gr}^i(M) = F^iM/F^{i+1}M$. The module $\text{gr}(A)$ admits an obvious structure of a ring and each $\text{gr}(M)$ admits the structure of a module over $\text{gr}(A)$.

3.1. Strict maps. If $M$ is an object of $\text{Fil}(A)$ and $N \subset M$ is an $A$-sub-module the induced filtration on $N$ is that given by $F^iN = N \cap F^iM$. If $f: M \to N$ is a surjective $A$-module homomorphism the quotient filtration on $N$ is that given by $F^iN = f(F^iM)$.

Remark 3.1.1. For any morphism $f: M \to N$ in $\text{Fil}(A)$ there is a sequence

$$\text{ker}(f) \to M \to \text{coim}(f) \to \text{im}(f) \to N \to \text{coker}(f)$$

in $\text{Fil}(A)$. The modules $\text{ker}(f) \subset M$ and $\text{im}(f) \subset N$ are each equipped with the induced filtration. The modules $\text{coker}(f)$ and $\text{coim}(f)$ are equipped with the quotient filtration, coming from $N$ and $M$ respectively.

Definition 3.1.2. A morphism $f: M \to N$ in $\text{Fil}(A)$ is strict if $F^iN \cap f(M) = f(F^iM)$ for all $i \in \mathbb{Z}$. Equivalently $f$ is strict if $\text{coim}(f) \to \text{im} f$ is an isomorphism in $\text{Fil}(A)$.

Notation 3.1.3. The filtration on $A$ induces the structure of a topological ring on $A$; the $F^iA$ form a basis of open neighbourhoods of zero. Similarly the filtration on an object $M$ of $\text{Fil}(A)$ gives $M$ the structure of a topological $A$-module. Then

- $M$ is discrete if and only if $F^iM = 0$ for $i >> 0$;
- $M$ is Hausdorff if and only if $\cap F^iM = 0$;
\textbullet M \text{ is complete if and only if the natural map } M \to \varprojlim M/F^i M \text{ is an isomorphism.}

**Lemma 3.1.4.** Let \( f : M \to N \) be a morphism in \( \text{Fil}(A) \) which is an isomorphism of \( A \)-modules.

1. Then \( f \) is an isomorphism in \( \text{Fil}(A) \) if and only if \( \text{gr}^i(M) \to \text{gr}^i(N) \) is injective for all \( i \).
2. If \( M \) is complete and \( N \) Hausdorff then \( f \) is an isomorphism in \( \text{Fil}(A) \) if and only if \( \text{gr}^i(M) \to \text{gr}^i(N) \) is surjective for all \( i \).

**Proof.** The following diagram commutes and has exact rows.

\[
\begin{array}{ccc}
0 & \to & F^{i+1}M \\
& \downarrow & \downarrow \\
0 & \to & F^{i+1}N
\end{array}
\]

Since \( M \to N \) is an isomorphism of \( A \)-modules the leftmost and central vertical arrows are injective. For (1) use the snake lemma to obtain an exact sequence \( 0 \to \ker c \to \coker(a) \to \coker(b) \to \coker(c) \). One proves \( F^i M \to F^i N \) is surjective by increasing induction on \( i \); using as the base case the fact that \( F^i M \to F^i N \) is surjective for \( i < < 0 \), since \( F^i M = M \) for \( i < < 0 \). For (2) argue as in [16, Proposition 6]. \( \square \)

**Lemma 3.1.5.** Let \( f : M \to N \) be a morphism in \( \text{Fil}(A) \). Then the following are equivalent.

1. \( f \) is strict;
2. \( \text{gr}(\ker(f)) \to \text{gr}(M) \to \text{gr}(N) \) is exact;
3. \( 0 \to \text{gr}(\ker(f)) \to \text{gr}(M) \to \text{gr}(N) \to \text{gr}(\coker(f)) \to 0 \) is exact.

If \( M \) is complete and \( N \) is Hausdorff then the same is true with (2) replaced by

(2') \( \text{gr}(M) \to \text{gr}(N) \to \text{gr}(\coker(f)) \) is exact for all \( i \);

**Proof.** It is straightforward to check (without any conditions on \( M \) and \( N \)) that (2) is equivalent to \( \text{gr}^i \text{coim}(f) \to \text{gr}^i \text{im}(f) \) being injective for all \( i \), that (2') is equivalent to this map being surjective for all \( i \), and that (3) is equivalent to this map being an isomorphism for all \( i \). Thus (1) \( \iff \) (2) \( \iff \) (3) follows from Lemma 3.1.4(1) applied to the morphism \( \text{coim}(f) \to \text{im}(f) \). Similarly (1) \( \iff \) (2') \( \iff \) (3) follows from Lemma 3.1.4(2), noting that \( M \) being complete implies \( \text{coim}(f) \) is complete and \( N \) being Hausdorff implies \( \text{im}(f) \) is Hausdorff. \( \square \)

**Corollary 3.1.6.** Let \( M \) be a Hausdorff object of \( \text{Fil}(A) \) with \( A \) complete. Suppose \( (m_j) \) is a finite collection of elements of \( M \) and suppose that there are integers \( r_j \) such that \( m_j \in F^{r_j} M \). Let \( \overline{m_j} \) denote the image of \( m_j \) in \( \text{gr}^{r_j}(M) \). If the \( \overline{m_j} \) generate \( \text{gr}(M) \) over \( \text{gr}(A) \) then \( M \) is complete and the \( m_j \) generate \( M \). Further

\[ F^i M = \sum_j (F^{i-r_j} A)m_j \]

If the \( \overline{m_j} \) form a \( \text{gr}(A) \)-basis of \( \text{gr}(M) \) then the \( m_j \) are an \( A \)-basis of \( M \).

**Proof.** Argue as in [16, Corollary] using the second part of Lemma 3.1.5. \( \square \)
3.2. Adapted Bases. We now put ourselves in the following situation. Let $a \in A$ be a nonzerodivisor and equip $A$ with the $a$-adic filtration (so $F^nA = a^nA$). Let $M$ be a finite free $A$-module and let $N \subset M[\frac{1}{a}]$ be a finitely generated $A$-submodule with $N[\frac{1}{a}] = M[\frac{1}{a}]$. Make $N$ into an object of $\text{Fil}(A)$ by setting $F^iN = a^iM \cap N$.

**Lemma 3.2.1.** Suppose that $A$ is complete. Give $N/a$ the quotient filtration and suppose that a finite collection $(g_i)$ of elements of $N$ is given along with integers $(r_i)$ such that $g_i \in F^{r_i}N$. If the images of $g_i$ in $\text{gr}^i(N/a)$ form a $\text{gr}(A/a) = A/a$-basis of $N$ then the $(g_i)$ form a basis of $N$ and the $(a^{-r_i}g_i)$ form a basis of $M$.

**Proof.** The induced filtration on the kernel $aN$ of $N \to N/a$ is given by $F^i(aN) = aN \cap F^iN = aF^{i-1}N$ (because $a$ is not a zerodivisor). Lemma 3.1.5 implies there is an exact sequence

$$0 \to \text{gr}^{i-1}(N) \xrightarrow{a} \text{gr}^i(N) \to \text{gr}^i(N/a) \to 0$$

Thus $\text{gr}(N)/a = \text{gr}(N/a)$ where $a \in \text{gr}(A)$ denotes the homogeneous element of degree 1 represented by $a \in A$. It is then easy to see (e.g. using the graded version of Nakayama’s lemma) that the images of the $g_i$ in $\text{gr}(N)$ generate this module over $\text{gr}(A)$. Since $\cap_i a^ig(A) = 0$ they are also $\text{gr}(A)$-linearly independent. As $N$ is finitely generated $N$ is Hausdorff and so we may apply Corollary 3.1.6 to deduce that the $(g_i)$ form an $A$-basis of $N$ and that

$$F^nN = \sum (F^{n-r_i}A)g_i$$

As the $g_i$ are $A$-linearly independent the $(a^{-r_i}g_i)$ are $A$-linearly independent. To show they generate $M$ take $m \in M$ and $n$ large enough that $a^n m \in N$. Then $a^nm \in F^nN$ and so $a^nm = \sum a_i g_i$ with $a_i \in F^{n-r_i}A$. It follows that $m = \sum (a^{n-r_i}a_i)(a^{-r_i}g_i)$ and so, since $(a^{n-r})F^{n-r}A \subset A$, we are done. □

3.3. Filtered Vector Spaces. Finally we give criteria to determine when two filtrations on a vector space are the same.

**Lemma 3.3.1.** Suppose $A = k$ is a field and let $V$ be an $k$-vector space equipped with two discrete filtrations $G^iV \subset F^iV$. Then

$$\sum i \dim_k \text{gr}^i_G(V) \leq \sum i \dim_k \text{gr}^i_F(V)$$

with equality if and only if $G = F$.

**Proof.** Since $\dim_k \text{gr}^i_F(V) = \dim_k F^iV - \dim_k F^{i+1}V$ we have

$$\sum i \dim_k \text{gr}^i_F(V) = \sum \dim_k F^iV$$

Likewise when $F$ is replaced by the filtration $G$. As $G^iV \subset F^iV$, $\dim_k G^iV \leq \dim_k F^iV$; the desired inequality follows. This inequality is an equality if and only if $\dim_k G^iV = \dim_k F^iV$ for all $i$, i.e. if and only if $G = F$. □

**Notation 3.3.2.** Say that a sequence of morphisms $M \to N \to P$ in $\text{Fil}(A)$ is exact if it is exact as a sequence of $A$-modules and if $M \to N$ is strict. Lemma 3.1.5 implies that a sequence $0 \to M \to N \to P \to 0$ in $\text{Fil}(A)$ which is exact in the category of $A$-modules is exact in $\text{Fil}(A)$ if and only if $0 \to \text{gr}(M) \to \text{gr}(N) \to \text{gr}(P) \to 0$ is an exact sequence of $A$-modules.
Corollary 3.3.3. Suppose $A = k$ is a field and let $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ be a sequence of finite dimensional discrete objects in $\Fil(k)$ which is exact in the category of $k$-vector spaces. If $f$ (respectively $g$) is strict then
\[
\sum i \dim_k \gr^i(N) \leq \sum i \dim_k \gr^i(M) + \sum i \dim_k \gr^i(P) \quad (\text{respectively} \geq)
\]
Conversely if one of $f$ or $g$ is strict then equality implies the sequence is exact in $\Fil(k)$.

Proof. As $P$ is discrete we can apply Lemma 3.3.1 to deduce that
\[
\sum i \dim_k \gr^i(N/M) \leq \sum i \dim_k \gr^i(P)
\]
with equality if and only if $g$ is strict. If $f$ is strict Lemma 3.1.5 tells us that $0 \to \gr(M) \to \gr(N) \to \gr(N/M) \to 0$ is exact, and so
\[
\sum i \dim_k \gr^i(N) = \sum i \dim_k \gr^i(M) + \sum i \dim_k \gr^i(N/M)
\]
The lemma follows when we assume $f$ is strict. If $g$ is strict one argues similarly, applying Lemma 3.3.1 to the map $M \to \ker(g)$. □

4. Breuil–Kisin Modules

4.1. Etale $\phi$-modules. First we recall the description of $G_{K_\infty}$-representations given by etale $\phi$-modules.

Definition 4.1.1. Let $\mathcal{O}_{C^p}$ be the inverse limit of the system
\[
\mathcal{O}_C/p \leftarrow \mathcal{O}_C/p \leftarrow \mathcal{O}_C/p \leftarrow \ldots
\]
with transition maps $x \mapsto x^p$. This is a perfect integrally closed ring of characteristic $p$. There is a multiplicative identification $\mathcal{O}_{C^p} = \varprojlim \mathcal{O}_C$ (the limit again taken with respect to the transition maps $x \mapsto x^p$) given by
\[
(\mathfrak{m}_n) \mapsto \left( \lim_{m \to \infty} x_m^{p_m+n} \right)_n
\]
where $x_m \in \mathcal{O}_C$ is any lift of $\mathfrak{m}_n$. We write $x \mapsto x^\ell$ for the projection onto the first coordinate $\mathcal{O}_{C^p} \to \mathcal{O}_C$. Let $C^p$ denote the field of fractions of $\mathcal{O}_{C^p}$. The formula $v^\ell(x) = v_p(x^\ell)$ defines a valuation on $C^p$ for which it is complete. The field $C^p$ is also algebraically closed. Further, the action of $G_K$ on $\mathcal{O}_C$ induces a continuous action of $G_K$ on $\mathcal{O}_{C^p}$ and $C^p$.

Notation 4.1.2. Let $\mathfrak{S} = W(k)[[u]]$ and $A_{\inf} = W(\mathcal{O}_{C^p})$. Both rings are equipped with a $\mathbb{Z}_p$-linear endomorphism $\phi$; on $A_{\inf}$ this is the usual Witt vector Frobenius and on $\mathfrak{S}$ it is given by $\sum a_i u^i \mapsto \sum \phi(a_i) u^{ip}$. The system $\pi^\ell / \pi^{\ell'}$ defines an element $\pi^\ell / \pi^{\ell'} \in \mathcal{O}_{C^p}$ and we embed $\mathfrak{S} \to A_{\inf}$ by mapping $u \mapsto [\pi^\ell]$ (where $[\cdot]$ denotes the Teichmuller lifting). This embedding is compatible with $\phi$. Let $\mathcal{O}_\mathfrak{E}$ denote the $p$-adic completion of $\mathfrak{S}[1/p]$. Then $\phi$ on $\mathfrak{S}$ extends to $\mathcal{O}_\mathfrak{E}$ and the embedding $\mathfrak{S} \to A_{\inf}$ extends to a $\phi$-equivariant embedding $\mathcal{O}_\mathfrak{E} \to W(C^p)$.

By functoriality there are $\phi$-equivariant $G_K$-actions on $A_{\inf} = W(\mathcal{O}_{C^p})$ and $W(C^p)$ lifting those modulo $p$. 
Definition 4.1.3. An etale $\varphi$-module is a finitely generated $\mathcal{O}_E$-module $M^\text{et}$ equipped with an isomorphism

$$\varphi_{M^\text{et}} : M^\text{et} \otimes_{\mathcal{O}_E,\varphi} \mathcal{O}_E \xrightarrow{\sim} M^\text{et}$$

We may interpret $\varphi_{M^\text{et}}$ as a $\varphi$-semilinear map $M^\text{et} \to M^\text{et}$ via $m \mapsto \varphi_{M^\text{et}}(m \otimes 1)$. When there is no risk of confusion we shall write $\varphi$ in place of $\varphi_{M^\text{et}}$. Let $\text{Mod}^\text{et}_K$ denote the abelian category of etale $\varphi$-modules.

Construction 4.1.4. Since the action of $G_{K^\infty}$ on $\mathbb{C}^p$ fixes $\pi^p$ the $\mathbb{Z}_p$-module

$$T(M^\text{et}) = (M^\text{et} \otimes_{\mathcal{O}_E} W(\mathbb{C}^p))_{\varphi=1}$$

admits a $\mathbb{Z}_p$-linear action of $G_{K^\infty}$ (given by the trivial action on $M^\text{et}$ and natural $G_{K^\infty}$-action on $W(\mathbb{C}^p)$). This describes a functor from $\text{Mod}^\text{et}_K$ to the category of finitely generated $\mathbb{Z}_p$-modules equipped with a continuous $\mathbb{Z}_p$-linear $G_{K^\infty}$-action.

Proposition 4.1.5 (Fontaine). The functor $M^\text{et} \mapsto T(M^\text{et})$ is an exact equivalence of categories. The representation $T(M^\text{et})$ is determined up to isomorphism by the existence of a $\varphi, G_{K^\infty}$-equivariant identification

$$M^\text{et} \otimes_{\mathcal{O}_E} W(\mathbb{C}^p) = T(M^\text{et}) \otimes_{\mathbb{Z}_p} W(\mathbb{C}^p)$$

Proof. The embedding $\mathcal{O}_E \to W(\mathbb{C}^p)$ reduces modulo $p$ to an inclusion of $k((u))$ in $\mathbb{C}^p$. The completion of $K^\infty$ is a perfectoid field in the sense of [14], whose tilt is the completed perfection of $k((u)) \subset \mathbb{C}^p$. It follows from [14, Theorem 3.7] that the action of $G_{K^\infty}$ on $\mathbb{C}^p$ identifies $G_K = G_{k((u))}$. Let $\mathcal{O}_{\mathbb{C}^p}$ be the $p$-adic completion of the Cohen ring (i.e. the discrete valuation ring of characteristic zero with uniformizer $p$) with residue field $k((u))^{\text{sep}}$. Then $\mathcal{O}_{\mathbb{C}^p}$ may be identified as a subring of $W(\mathbb{C}^p)$ stable under the action of $G_{K^\infty}$ and $\varphi$. The proposition with $T(M^\text{et})$ replaced by $T'(M^\text{et}) := (M^\text{et} \otimes_{\mathcal{O}_E} \mathcal{O}_{\mathbb{C}^p})_{\varphi=1}$ follows from [10, Proposition 1.2.6] applied with $E = k((u))$. It therefore suffices to show the inclusion $T'(M^\text{et}) \subset T(M^\text{et})$ is an equality. Since we know there are $\varphi$-equivariant identifications

$$M^\text{et} \otimes_{\mathcal{O}_E} W(\mathbb{C}^p) = T'(M^\text{et}) \otimes_{\mathbb{Z}_p} W(\mathbb{C}^p)$$

the equality follows by taking $\varphi$-invariants. \qed

4.2. Breuil–Kisin modules. Breuil–Kisin modules appear as special $\mathcal{S}$-lattices inside etale $\varphi$-modules.

Definition 4.2.1. A Breuil–Kisin module is a finitely generated $\mathcal{S}$-module $M$ equipped with an isomorphism

$$\varphi_M : M \otimes_{\mathcal{S},\varphi} \mathbb{S}[\frac{1}{\varphi}] \xrightarrow{\sim} M[\frac{1}{\varphi}]$$

Here $E(u) \in \mathcal{S}$ denotes the minimal polynomial of $\pi$ over $K_0$. We may interpret $\varphi_M$ as a $\varphi$-semilinear map $M \to M[\frac{1}{\varphi}]$ via $m \mapsto \varphi_M(m \otimes 1)$. When there is no risk of confusion we write $\varphi$ in place of $\varphi_M$. Let $\text{Mod}^\text{BK}_K$ denote the abelian category of Breuil–Kisin modules.

Notation 4.2.2. If $M \in \text{Mod}^\text{BK}_K$ we write $M^\varphi \subset M[\frac{1}{\varphi}]$ for the image of

$$M \to M \otimes_{\mathcal{S},\varphi} \mathbb{S}[\frac{1}{\varphi}] \xrightarrow{\varphi_M} M[\frac{1}{\varphi}]$$

More generally we use this notation whenever $A$ is any ring equipped with a Frobenius $\varphi$ and $M$ is an $A$-module equipped with a map $\varphi_M : M \otimes_{\varphi,A} A[\frac{1}{a}] \to M[\frac{1}{a}]$ for some $a \in A$. Then $M^\varphi := \varphi_M(M \otimes 1) \subset M[\frac{1}{a}]$. 

Construction 4.2.3. Note $E(u)$ is a unit in $O_E$. Thus if $M \in \text{Mod}_K^{BK}$ then $M \otimes_{O_E} O_E$ is an etale $\varphi$-module and

$$T(M) := T(M \otimes_{O_E} O_E) = (M \otimes_{O_E} W(C^o))^\varphi = 1$$

defines a functor from $\text{Mod}_K^{BK}$ to the category of continuous $G_{K_\infty}$-representations on finitely generated $\mathbb{Z}_p$-modules. Since $\mathcal{S} \to O_E$ is flat Proposition 4.1.5 implies $M \mapsto T(M)$ is exact on $\text{Mod}_K^{BK}$.

Remark 4.2.4. Kisin [12, Proposition 2.1.12] has shown $M \mapsto T(M)$ is fully faithful when restricted to Breuil–Kisin modules which are free over $\mathcal{S}$. However if one does not restrict to Breuil–Kisin modules which are free over $\mathcal{S}$ then this is not true.

Construction 4.2.5. For $M, N \in \text{Mod}_K^{BK}$ the $\mathcal{S}$-module

$$\text{Hom}(M, N) := \text{Hom}_{\mathcal{S}}(M, N)$$

of $\mathcal{S}$-linear homomorphisms $M \to N$ is made into an object of $\text{Mod}_K^{BK}(O)$ as follows. Since $\varphi: \mathcal{S} \to \mathcal{S}$ is flat the natural map $\text{Hom}_{\mathcal{S}}(M, N)^O \otimes_{\varphi} \mathcal{S}[\frac{1}{E}] \to \text{Hom}_{O_E}[\varphi](M \otimes_{\varphi} \mathcal{S}[\frac{1}{E}], N \otimes_{\varphi} \mathcal{S}[\frac{1}{E}])$ is an isomorphism. Similarly the natural map $\text{Hom}_{\mathcal{S}}(\mathcal{O}[[M]], \mathcal{O}[[N]]) \to \text{Hom}_{O_E}[\varphi](\mathcal{O}[[M]], \mathcal{O}[[N]])$ is an isomorphism. As such the isomorphism

$$\text{Hom}_{\mathcal{S}}[\varphi](M \otimes_{\varphi} \mathcal{S}[\frac{1}{E}], N \otimes_{\varphi} \mathcal{S}[\frac{1}{E}]) \to \text{Hom}_{O_E}[\varphi](\mathcal{O}[[M]], \mathcal{O}[[N]])$$
given by $f \mapsto \varphi_N \circ f \circ \varphi_M^{-1}$ makes $\text{Hom}(M, N)$ into a Breuil–Kisin module. Note that

$$T(\text{Hom}(M, N)) = \text{Hom}_{\mathcal{S}}(T(M), T(N))$$
as $G_{K_\infty}$-representations, where the $G_{K_\infty}$-action on the right is via $\sigma(f) = \sigma \circ f \circ \sigma^{-1}$.

4.3. Coefficients. In practice we are interested in representations valued in extensions of $\mathbb{Z}_p$. For this reason we introduce a variant of $\text{Mod}_K^{BK}$.

Definition 4.3.1. Recall the $\mathbb{Z}_p$-algebra $O$ defined in Subsection 1.1. A Breuil–Kisin module with $O$-action is a pair $(M, \iota)$ where $M \in \text{Mod}_K^{BK}$ and $\iota$ is a $\mathbb{Z}_p$-algebra homomorphism $\iota: O \to \text{End}_{BK}(M)$. Equivalently a Breuil–Kisin module with $O$-action is an $\mathcal{S}_O = \mathcal{S} \otimes_{\mathbb{Z}_p} O$-module $M$ equipped with an isomorphism

$$M \otimes_{\varphi, \mathcal{S}_O} \mathcal{S}_O[\frac{1}{E}] \xrightarrow{\sim} M[\frac{1}{E}]$$

Here $\varphi$ on $\mathcal{S}_O$ denotes the $O$-linear extension of $\varphi$ on $\mathcal{S}$. Let $\text{Mod}_K^{BK}(O)$ denote the category of Breuil–Kisin modules with $O$-action.

Remark 4.3.2. By functoriality $M \mapsto T(M)$ induces an exact functor from $\text{Mod}_K^{BK}(O)$ into the category of continuous representations of $G_{K_\infty}$ on finitely generated $O$-modules.

Construction 4.3.3. Let $M, N \in \text{Mod}_K^{BK}(O)$. Then

$$\text{Hom}(M, N)^O := \text{Hom}_{\mathcal{S}_O \otimes_{\mathbb{Z}_p} O}(M, N)$$
is made into an object of $\text{Mod}_K^{BK}(O)$ as in Construction 4.2.5. Again we have

$$T(\text{Hom}(M, N)^O) = \text{Hom}_{O}(T(M), T(N))$$
as $G_{K_\infty}$-representations.
Construction 4.3.4. View $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}$ as an $\mathcal{O}[[u]]$-algebra via $\sum a_i u^i \mapsto \sum u^i \otimes a_i$. Recall that $K_0 \subset E$ by assumption. Thus $k \subset \mathbb{F}$ and so the map

$$(\sum a_i u^i) \otimes b \mapsto (\sum \tau(a_i) b u^i)_\tau$$

describes an isomorphism of $\mathcal{O}[[u]]$-algebras $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O} \to \prod_\tau \mathcal{O}[[u]]$, the product running over $\tau \in \operatorname{Hom}_{\mathbb{F}_p}(k, \mathbb{F})$ (we abusively write $\tau$ also for its extension to an embedding $\tau: W(k) \to \mathcal{O}$). Let $\bar{\tau} \in \mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}$ be the idempotent corresponding to $\tau$. As $\bar{\tau}$ is determined by the property $(a \otimes 1)\bar{\tau} = (1 \otimes \tau(a))\bar{\tau}$ for $a \in W(k)$, the map $\varphi \otimes 1$ sends

$$\bar{\tau} \otimes \varphi \mapsto \bar{\tau}$$

If $M \in \operatorname{Mod}_{BK}^K(\mathcal{O})$ we set $M_\tau = \bar{\tau} M$ which we view as an $\mathcal{O}[[u]]$-algebra. By the above $\varphi_M$ restricts to a map

$$M_{\tau \otimes \varphi} \otimes_{\varphi, \mathcal{O}[[u]]} \mathcal{O}[[u]] \to M_{\tau}[1/\varphi(E)]$$

which becomes an isomorphism after inverting $\tau(E)$. Here $\varphi$ on $\mathcal{O}[[u]]$ is that induced by $\varphi \otimes 1$ on $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}$, i.e. is given by $\sum a_i u^i \mapsto \sum a_i u^i p$.

Corollary 4.3.6. If $M \in \operatorname{Mod}_{BK}^K(\mathcal{O})$ is free as an $\mathcal{G}$-module then it is free as an $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}$-module.

Proof. If $M$ is free over $\mathcal{G}$ then each $M_\tau$ is free over $\mathcal{O}[[u]]$. By (4.3.5) the rank of $M_\tau$ over $\mathcal{O}[[u]]$ does not depend on $\tau$ so $M = \prod_\tau M_\tau$ is free over $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}$. □

Similarly

Corollary 4.3.7. Let $\varpi \in \mathcal{O}$ be a uniformiser and suppose $M \in \operatorname{Mod}_{BK}^K(\mathcal{O})$ is $\varpi$-torsion. If $M$ is free as an $\mathcal{G}/p = k[[u]]$-module then it is free as a module over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$.

5. Strongly Divisibility

5.1. Torsion Breuil–Kisin modules.

Definition 5.1.1. Denote by $\operatorname{Mod}_{BK}^K \subset \operatorname{Mod}_{BK}^k$ the full subcategory whose objects are modules which are free over $\mathcal{G}/p = k[[u]]$.

Remark 5.1.2. An $M \in \operatorname{Mod}_{BK}^K$ is the same thing as a $k[[u]]$-lattice inside an etale $\varphi$-module over $\mathcal{O}_E/p = k((u))$ because $E(u) \equiv u^p$ modulo $p$.

Lemma 5.1.3. The functor $M \mapsto T(M)$ restricts to an essentially surjective functor from $\operatorname{Mod}_{BK}^K$ to the category of continuous representations of $G_{K_\infty}$ on finite dimensional $\mathbb{F}_p$-vector spaces. If $M \in \operatorname{Mod}_{BK}^K$ and

$$0 \to T_1 \to T(M) \to T_2 \to 0$$

is an exact sequence of $G_{K_\infty}$-representations then there exists a unique exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

in $\operatorname{Mod}_{BK}^K$ such that $T(M_i) = T_i$.

In particular there are many $p$-torsion Breuil–Kisin modules giving rise to the same etale $\varphi$-module. This is in contrast to the integral situation, see Remark 4.2.4.
Proof. If $T$ is an $F_p$-representation of $G_{K_\infty}$ then there exists a $p$-torsion $M^\text{et} \in \text{Mod}_K^{\text{et}}$ such that $T(M^\text{et}) = T$. Remark 5.1.2 shows that any $k[[u]]$-lattice $M \subset M^\text{et}$ is an object of $\text{Mod}_k^{\text{BK}}$ with $T(M) = T$.

For the second part, there exists an exact sequence $0 \to M_1^\text{et} \to M^\text{et} \to M_2^\text{et} \to 0$ such that $T(M_1^\text{et}) = T$, and such that $M^\text{et} = M[\frac{1}{u}]$. Since $M_2$ is torsion-free we must have $M_1 = M \cap M_1^\text{et}$ and $M_2 = \text{Im}(M) \cap M_2^\text{et}$. \hfill $\square$

Construction 5.1.4. Let $M \in \text{Mod}_k^{\text{BK}}$. A composition series of $M$ is a filtration

$$0 = M_n \subset \ldots \subset M_0 = M$$

by sub-Breuil–Kisin modules such that each $M_i/M_{i+1}$ is an irreducible object (i.e. admits no non-zero proper sub-objects $N \in \text{Mod}_k^{\text{BK}}$ such that the cokernel of $N \to M_i/M_{i+1}$ is $k[[u]]$-torsion-free) of $\text{Mod}_k^{\text{BK}}$. Lemma 5.1.3 implies being irreducible is equivalent to asking that $T(M_i/M_{i+1})$ is an irreducible $G_{K_\infty}$-representation. Lemma 5.1.3 also implies that composition series for $M$ are in bijection with composition series for $T(M)$.

Warning 5.1.5. It is not the case that the set of irreducible factors of a composition series is independent of the choice of composition series.

5.2. Strong Divisibility. In this subsection we define a full-subcategory $\text{Mod}_k^{\text{SD}} \subset \text{Mod}_k^{\text{BK}}$ which we view as an extension of $p$-torsion Fontaine–Laffaille theory to filtrations of length $p$.

Construction 5.2.1. Let $M$ be an object of $\text{Mod}_k^{\text{BK}}$. Recall $M^\varphi$ is the $k[[u]]$-sub-module of $M[\frac{1}{u}]$ generated by $\varphi(M)$. Equip $M^\varphi$ with a filtration given by $F^iM^\varphi = M^\varphi \cap u^iM$. Let $M_k^\varphi = M^\varphi / u$. We equip this $k$-vector space with the quotient filtration.

Definition 5.2.2. If $M \in \text{Mod}_k^{\text{BK}}$ let $\text{Weight}(M)$ be the multiset of integers containing $i$ with multiplicity

$$\dim_k \text{gr}^i(M_k^\varphi)$$

Construction 5.2.3. Similarly to Construction 5.2.1 we equip $M$ with a filtration by setting $F^iM = \{ m \in M \mid \varphi(m) \in u^iM \}$. The semilinear injection

$$\varphi: M \hookrightarrow M^\varphi$$

is then a morphism of filtered modules. Let $M_k = M / u$. We equip this $k$-vector space with the quotient filtration.

Lemma 5.2.4. The injection $\varphi: M \hookrightarrow M^\varphi$ induces a functorial $k$-semilinear automorphism of filtered vector spaces

$$M_k \to M_k^\varphi$$

Proof. All that needs to be checked is that $\varphi: M \to M^\varphi$ induces a $k$-semilinear isomorphism $M_k \to M_k^\varphi$. As $M_k$ and $M_k^\varphi$ have the same dimension over $k$ we only need to check surjectivity. As $M^\varphi$ is the $k[[u]]$-module generated by $\varphi(M) \subset M[\frac{1}{u}]$ surjectivity follows because $\varphi$ is an automorphism on $k = k[[u]] / u$. \hfill $\square$

Lemma 5.2.5. Let $M$ be an object of $\text{Mod}_k^{\text{BK}}$. The following are equivalent:

1. The map $M_k \to M_k^\varphi$ is an isomorphism of filtered modules.
(2) There exists a \( k[[u]] \)-basis \( (f_i) \) of \( M \) and integers \( (r_i) \) such that \( (u^{r_i} f_i) \) is a \( k[[u^p]] \)-basis of \( \varphi(M) \).

**Proof.** Suppose \( M_k \rightarrow M_k^\varphi \) is an isomorphism of filtered modules. We can find integers \( r_i \) and elements \( g_i \in F^j M \) whose images in \( \varphi(M) \) form a \( k \)-basis. As the induced map \( \varphi(M_k) \rightarrow \varphi(M_k^\varphi) \) is an isomorphism it follows that the images of \( \varphi(g_i) \in \varphi(M) \) in \( \varphi(M_k^\varphi) \) form a \( k \)-basis. Applying Lemma 3.2.1 with \( M = M, N = M^\varphi \) and \( a \in A \) equal to \( u \in k[[u]] \) proves that (1) implies (2) with \( f_i = u^{-r_i} \varphi(g_i) \).

To prove (2) implies (1) we use the \( f_i \) to give explicit descriptions of the filtration on \( M_k^\varphi \). Since \( \varphi(M) \) generates \( M^\varphi \) over \( k[[u]] \) every \( m \in M^\varphi \) can be written as \( m = \sum \alpha_i(u^{r_i} f_i) \) with \( \alpha_i \in k[[u]] \). If \( m \in F^j M^\varphi \) then \( \alpha_i \in u^{\text{max}(j \cdot r_i,0)} k[[u]] = F^{j-r_i} k[[u]] \) since the \( f_i \) form a basis of \( M \). Hence

\[
F^j M^\varphi = \sum (F^{j-r_i} k[[u]])(u^{r_i} f_i)
\]

and so \( F^j M_k^\varphi = \sum_{r_i \geq j} k \overline{f}_i \) where \( \overline{f}_i \) denotes the image of \( u^{r_i} f_i \) in \( M_k^\varphi \). If \( g_i \in M \) is such that \( \varphi(g_i) = u^{r_i} f_i \) we have \( g_i \in F^j M \) if \( r_i \geq j \). If \( \overline{g}_i \) denotes the image of \( g_i \) in \( M_k \) then since the map \( M_k \rightarrow M_k^\varphi \) sends \( \overline{g}_i \rightarrow \overline{f}_i \), it induces surjections \( F^j M_k \rightarrow F^j M_k^\varphi \). Thus \( M_k \rightarrow M_k^\varphi \) is an isomorphism in \( \text{Fil}(k) \). \( \square \)

**Remark 5.2.6.** Note that if we have a basis as in (2) of Lemma 5.2.5 then the above proof shows that \( \varphi(M_k^\varphi) = \sum_{r_i = j} k \overline{f}_i \). Thus the multiset \( \{ r_i \} \) is equal to \( \text{Weight}(M) \).

**Remark 5.2.7.** Isomorphism classes of objects in \( \text{Mod}_k^{BK} \) can be described explicitly. Choosing a basis and considering the matrix of \( \varphi: M \rightarrow M[1] \) with respect to that basis describes a bijection

\[
\text{(5.2.8)} \quad \left\{ \text{isomorphism classes of rank } n \text{ objects of } \text{Mod}_k^{BK} \right\} \leftrightarrow \text{GL}_n(k[[u]])/\sim
\]

Here \( A \sim B \) if there exists \( C \in \text{GL}_n(k[[u]]) \) such that \( A = C^{-1} B \varphi(C) \). Recall that any invertible matrix over \( k((u)) \) can be written as \( C_1 \Lambda C_2 \) where \( \Lambda = \text{diag}(u^{r_i}) \) and \( C_i \in \text{GL}_n(k[[u]]) \).

- If \( M \) is an object of \( \text{Mod}_k^{BK} \) corresponding under (5.2.8) to a \( \varphi \)-conjugacy class represented by \( C_1 \Lambda C_2 \) then the \( (r_i) = \text{Weight}(M) \).
- The isomorphism classes of Breuil–Kisin modules satisfying the equivalent conditions of Lemma 5.2.5 identify, via (5.2.8), with \( \varphi \)-conjugacy classes represented by matrices \( C_1 \Lambda \) with \( C_1 \in \text{GL}_n(k[[u]]) \) and \( \Lambda = \text{diag}(u^{r_i}) \).

**Definition 5.2.9.** Let \( \text{Mod}_k^{BK, SD} \subset \text{Mod}_k^{BK} \) denote the full subcategory whose objects satisfy the equivalent conditions of Lemma 5.2.5 and have \( \text{Weight}(M) \subset [0,p] \). We say such \( M \) are strongly divisible.

**5.3. Strong Divisibility with Coefficients.** We reproduce the previous subsection allowing \( O \)-coefficients.

**Definition 5.3.1.** Let \( \text{Mod}_k^{BK}(O) \) denote the full subcategory of \( \text{Mod}_k^{BK}(O) \) whose objects are finite free over \( k[[u]] \otimes_{\mathcal{F}} \mathcal{F} \). This is equivalent to being free over \( k[[u]] \) and killed by \( \varpi \) after Corollary 4.3.7.

**Remark 5.3.2.** As in Construction 4.3.4 each \( M \in \text{Mod}_k^{BK}(O) \) decomposes as

\[
M = \prod_{\tau \in \text{Hom}_p(k,\mathbb{F})} M_{\tau}
\]
with each $M_{\tau}$ a finite free module over $F[[u]]$. Since the filtration on $M$ is by $k[[u]] \otimes_k F$-sub-modules this is a decomposition of filtered modules. Thus $M_k = \prod_{\tau} M_{k,\tau}$ as filtered modules (each $M_{k,\tau}$ being a filtered $F$-vector space). Analogous statements hold for $M^{\varphi}$ and $M_k^{\varphi}$.

**Definition 5.3.3.** For $\tau \in \text{Hom}_{\mathbb{Q}}(k, F)$ let $\text{Weight}_{\tau}(M)$ be the multiset of integers which contains $i$ with multiplicity equal to

$$\dim_F \text{gr}^i(M_{k,\tau}^{\varphi})$$

Since $M_k^{\varphi} = \prod_{\tau} M_{k,\tau}^{\varphi}$ we have that $\text{Weight}(M)$ equals the union over all $\tau$ of $[F : k]$ copies of $\text{Weight}_{\tau}(M)$.

The following is a version of Lemma 5.2.5 for objects of $\text{Mod}_k^{\text{BK}}(\mathcal{O})$ and is proved in exactly the same fashion.

**Lemma 5.3.4.** Let $M$ be an object of $\text{Mod}_k^{\text{BK}}(\mathcal{O})$. Then the following are equivalent:

1. The semilinear map $M_k \rightarrow M_k^{\varphi}$ is an isomorphism of filtered modules.
2. For $\tau \in \text{Hom}_{\mathbb{Q}}(k, F)$ there exists an $F[[u]]$-basis $(f_i)$ of $M_{\tau}$ and integers $(r_i)$ such that $(u^{r_i} f_i)$ is an $F[[u^p]]$-basis of $\varphi(M_{\tau})$.

**Remark 5.3.5.** As in Remark 5.2.6 if bases as in (2) of Lemma 5.3.4 exist then the multiset $\{r_{i,\tau}\}$ equals $\text{Weight}_{\tau}(M)$.

**Remark 5.3.6.** There is the following analogue of Remark 5.2.7 for $\text{Mod}_k^{\text{BK}}(\mathcal{O})$. Choosing $F[[u]]$-bases for each $M_{\tau}$ and taking the matrices representing $\varphi$ with respect to these bases describes a bijection

$$\left\{ \text{isomorphism classes of rank } n \text{ objects of } \text{Mod}_k^{\text{BK}}(\mathcal{O}) \right\} \leftrightarrow \text{GL}_n(F[[u]])/\sim$$

where $f = [K : \mathbb{Q}_p]$ and where two $f$-tuples of matrices satisfy $(A_{\tau}) \sim (B_{\tau})$ if there exist $C_{\tau} \in \text{GL}_n(F[[u]])$ such that $A_{\tau} = C_{\tau}^{-1} B_{\tau} \varphi(C_{\tau} \circ \varphi)$ for all $\tau$. Each $A_{\tau}$ can be written as $C_{\tau} \Lambda_{\tau} C_{\tau}^t$ with $C_{\tau}, C_{\tau}^t \in \text{GL}_n(F[[u]])$ and $\Lambda_{\tau} = \text{diag}(u^{r_{i,\tau}})$.

- The multiset $\{r_{i,\tau}\}$ is the multiset $\text{Weight}_{\tau}(M)$.
- The $M$ which satisfy Lemma 5.3.4 correspond to classes represented by an $f$-tuple of matrices $(A_{\tau})$ such that each $A_{\tau} = C_{\tau} \Lambda_{\tau}$.

**Definition 5.3.7.** Let $\text{Mod}_k^{\text{SD}}(\mathcal{O}) \subset \text{Mod}_k^{\text{BK}}(\mathcal{O})$ denote the full subcategory whose objects are strongly divisible when viewed as objects of $\text{Mod}_k^{\text{BK}}$.

**5.4. Subquotients.** We now show $\text{Mod}_k^{\text{SD}}$ and $\text{Mod}_k^{\text{SD}}(\mathcal{O})$ are closed under subquotients.

**Remark 5.4.1.** If $M \in \text{Mod}_k^{\text{BK}}$ then there are exact sequences

$$0 \rightarrow \text{gr}^{i-1}(M^{\varphi}) \xrightarrow{u} \text{gr}^i(M^{\varphi}) \rightarrow \text{gr}^i(M_k^{\varphi}) \rightarrow 0$$

$$0 \rightarrow \text{gr}^{i-p}(M) \xrightarrow{u} \text{gr}^i(M) \rightarrow \text{gr}^i(M_k) \rightarrow 0$$

The first is just the exact sequence (3.2.2) in the case $M = M$ and $N = M^{\varphi}$ with $A = k[[u]]$ and $a = u$. The second exact sequence is obtained similarly (using that $F^i(uM) = u(F^{i-p}M)$).

**Lemma 5.4.2.** Let $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ be an exact sequence in $\text{Mod}_k^{\text{BK}}$. 
(1) The map $N \to P$ is strict when viewed as a map of filtered modules if and only if $0 \to M_k \to N_k \to P_k \to 0$ is an exact sequence in $\text{Fil}(k)$ in the sense of Notation 3.3.2.

(2) The map $N^\varphi \to P^\varphi$ is strict if and only if $0 \to M^\varphi_k \to N^\varphi_k \to P^\varphi_k \to 0$ is exact in $\text{Fil}(k)$

(3) Statement (2) is equivalent to $M^\varphi_k \to N^\varphi_k$ being strict, which is equivalent to $N^\varphi_k \to P^\varphi_k$ being strict.

**Proof.** Note that $M \to N$ is strict as a map of filtered modules. To see this suppose $m \in M \cap F^i N$, then $\varphi(m) \in \varphi(M) \cap u_i^i N \subset M[\frac{1}{u_i}] \cap u_i^i N$. Since $M \to N$ has $u$-torsion-free cokernel $M[\frac{1}{u}] \cap u_i^i N = u_i M$. Thus $m \in F^i M$. Similarly $M^\varphi \to N^\varphi$ is strict. Hence $N \to P$ is strict if and only if $0 \to \text{gr}^i(M) \to \text{gr}^i(N) \to \text{gr}^i(P) \to 0$ is exact for each $i$ and likewise $N^\varphi \to P^\varphi$ is strict if and only if $0 \to \text{gr}^i(M^\varphi) \to \text{gr}^i(N^\varphi) \to \text{gr}^i(P^\varphi) \to 0$ is exact (Lemma 3.1.5).

Using the second exact sequence of Remark 5.4.1 we obtain the following commutative diagram with exact rows.

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \text{gr}^{i-p}(M) & \text{gr}^i(M) & \text{gr}^i(M_k) & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \text{gr}^{i-p}(N) & \text{gr}^i(N) & \text{gr}^i(N_k) & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \text{gr}^{i-p}(P) & \text{gr}^i(P) & \text{gr}^i(P_k) & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

The previous paragraph shows that if $N \to P$ is strict then the left and middle columns are exact, and so the right column is exact also. Conversely if the right column is exact then one proves the middle column is exact by increasing induction on $i$ (for small enough $i$ the left column will be zero). This proves (1). The same argument but with the diagram replaced with the diagram obtained by considering the first exact sequence of Remark 5.4.1 proves (2) also.

It remains to show that if $M^\varphi_k \to N^\varphi_k$ or $N^\varphi_k \to P^\varphi_k$ is strict then $0 \to M^\varphi_k \to N^\varphi_k \to P^\varphi_k \to 0$ is exact. It suffices to show that $\sum_{i \in \text{Weight}(M)} i + \sum_{i \in \text{Weight}(P)} i = \sum_{i \in \text{Weight}(N)} i$ after Corollary 3.3.3. Remark 5.2.7 says that $\sum_{i \in \text{Weight}(M)} i$ equals the $u$-adic valuation of the determinant of $\varphi: M \to M[\frac{1}{u}]$ (in any choice of basis). Since this is clearly additive on exact sequences the lemma follows. $\square$

**Lemma 5.4.3.** Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\text{Mod}_k^{\text{BK}}$. Suppose $M$ and $P$ satisfy the equivalent conditions of Lemma 5.2.5. If $N \to P$ is strict then $N$ satisfies the equivalent conditions of Lemma 5.2.5 also.

**Proof.** Consider the following commutative diagram.

\[
\begin{array}{cccc}
0 & \to & \text{gr}^i(M^\varphi_k) & \to & \text{gr}^i(N^\varphi_k) & \to & \text{gr}^i(P^\varphi_k) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \text{gr}^i(M_k) & \to & \text{gr}^i(N_k) & \to & \text{gr}^i(P_k) & \to & 0
\end{array}
\]

The left and right vertical arrows are isomorphisms by assumption. Since $N \to P$ is strict, part (1) of Lemma 5.4.2 implies the bottom row is exact. Thus $\text{gr}^i(N^\varphi_k) \to
Lemma 5.4.2 then implies the top row is exact. We conclude that $N_k \to N_k^\varphi$ is an isomorphism in $\Fil(k)$.

\textbf{Lemma 5.4.4.} Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\Mod^\text{BK}_k$. Suppose that $N$ satisfies the equivalent conditions of Lemma 5.2.5 and that $M_k \to N_k$ is strict. Then $N \to P$ is strict and $M$ and $P$ also satisfy the equivalent conditions of Lemma 5.2.5.

\textbf{Proof.} The following diagram of objects in $\Fil(k)$ commutes.

\[
\begin{array}{ccc}
M_k^\varphi & \to & N_k^\varphi \\
\uparrow & & \uparrow \\
M_k & \to & N_k
\end{array}
\]

As maps of $k$-vector spaces the horizontal arrows are injective and the vertical arrows are isomorphisms. By assumption the maps $M_k \to N_k$ and $N_k \to N_k^\varphi$ are strict. It follows that $M_k^\varphi \to N_k^\varphi$ and $M_k \to M_k^\varphi$ are strict also.

The following is also a commutative diagram in $\Fil(k)$.

\[
\begin{array}{ccc}
N_k^\varphi & \to & P_k^\varphi \\
\uparrow & & \uparrow \\
N_k & \to & P_k
\end{array}
\]

As maps of $k$-vector spaces the vertical maps are isomorphisms and the horizontal arrows are surjections. By assumption the maps $M_k \to N_k^\varphi$ and $N_k \to N_k$ are strict. Thus $M$ and $P$ are as in Lemma 5.2.5 and after (1) of Lemma 5.4.2 we know $N \to P$ is strict.

\textbf{Lemma 5.4.5.} Suppose $N$ is strongly divisible. If $0 \to M \to N \to P \to 0$ is an exact sequence in $\Mod^\text{BK}_k$ then $M_k \to N_k$ is strict.

\textbf{Proof.} We have a commutative diagram with exact rows (Remark 5.4.1)

\[
\begin{array}{ccc}
0 & \to & \gr^{i-p}(M) \to \gr^i(M) \to \gr^i(M_k) \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \gr^{i-p}(N) \to \gr^i(N) \to \gr^i(N_k) \to 0
\end{array}
\]

One knows that $M \to N$ is strict (as was shown in the first paragraph of the proof of Lemma 5.4.2) so the left and middle vertical arrows are injective by Lemma 3.1.5. We have to show $\alpha$ is injective for every $i$.

For injectivity of $\alpha$ when $i < p$ we argue as follows. As $\Weight(N) \subset [0, p]$, and because $N_k \cong N_k^\varphi$, we have $\gr^i(N_k) = 0$ for $i < 0$. Hence $\gr^i(N) = \gr^{i-p}(N)$ for $i < 0$. This implies $\gr^i(N) = 0$ for $i < 0$ because for small enough $i$, $F^i N = N$. Using the diagram we deduce that $\gr^i(M) = 0$ for $i < 0$ also, and that for $i < p$ we have $\gr^i(M) = \gr^i(M_k)$ and $\gr^i(N) = \gr^i(N_k)$. This proves $\alpha$ is injective when $i < p$.

For injectivity of $\alpha$ when $i \geq p$ it suffices to show $F^i N_k = 0$ for $i > p$ (because then $F^i M_k = 0$ for $i > p$ so $\alpha$ is just the zero map when $i > p$ and when $i = p$, $\alpha$ is the inclusion $F^i M_k \to F^i N_k$). Let us prove this is the case. Since $\Weight(N) \subset [0, p]$ we have $\gr^i(N_k) = 0$ for $i > p$; it suffices to show $F^i N_k = 0$ for $i > p$.
Proposition 5.4.6. Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\text{Mod}^B_k$. Then $M$ and $P$ are strongly divisible and the sequence

$$0 \to M^\circ_k \to N^\circ_k \to P^\circ_k \to 0$$

is exact in $\text{Fil}(k)$. Thus $\text{Weight}(N) = \text{Weight}(M) \cup \text{Weight}(P)$.

Proof. (1) follows from Lemma 5.4.2, Lemma 5.4.4 and Lemma 5.4.5. For (2) use Lemma 5.4.3.

Proposition 5.4.7. Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\text{Mod}^B_k(\mathcal{O})$. Then $M$ and $P$ are both strongly divisible and for each $\tau \in \text{Hom}_{\mathbb{Z}_p}(k, \mathbb{F})$ we have $\text{Weight}_\tau(N) = \text{Weight}_\tau(M) \cup \text{Weight}_\tau(P)$.

Proof. This is immediate from Proposition 5.4.6. In particular we point out that the exact sequence in (1) of Proposition 5.4.6 is functorial and so is an exact sequence of $k \otimes_{\mathbb{F}_p} \mathbb{F}$-modules. Thus it decomposes into exact sequences

$$0 \to M^\circ_{k, \tau} \to N^\circ_{k, \tau} \to P^\circ_{k, \tau} \to 0$$

which shows $\text{Weight}_\tau(N) = \text{Weight}_\tau(M) \cup \text{Weight}_\tau(P)$.

6. Irreducible Objects

Provided $\mathbb{F}$ is sufficiently large, irreducible $\mathbb{F}$-representations of $G_K$ and $G_{K, \infty}$ are induced from characters (Lemma 2.1.2). In this section we investigate the extent with which this is true for objects of $\text{Mod}^S_k(\mathcal{O})$. Throughout assume $k \subset \mathbb{F}$.

6.1. Rank ones. Recall from Construction 4.3.4 how $\mathcal{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ is made into an $\mathcal{O}[[u]]$-algebra. Then $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ becomes an $\mathbb{F}[[u]]$-algebra. Also let $e_\tau \in k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ denote the image of the idempotent $\overline{e}_\tau \in \mathcal{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$ defined in Construction 4.3.4.

The next lemma is proven by an easy change of basis argument (see [11, Lemma 6.2])

Lemma 6.1.1. Fix $\tau_0 \in \text{Hom}_{\mathbb{Z}_p}(k, \mathbb{F})$. Let $M \in \text{Mod}^B_k(\mathcal{O})$ be of rank one over $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$. Then $M$ is isomorphic to a Breuil–Kisin module

$$N = k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}, \quad \varphi_N(1) = (x) \sum u^{r_\tau} e_\tau$$

where $r_\tau \in \mathbb{Z}$ and where $(x) = xe_{\tau_0} + \sum_{\tau \neq \tau_0} e_\tau$ for some $x \in \mathbb{F}^\times$.

Remark 6.1.2. If $N$ is as in Lemma 6.1.1 then $\text{Weight}_r(N) = \{r\}$. Note also that $N$ satisfies the equivalent conditions of Lemma 5.3.4. Thus $N \in \text{Mod}^S_k(\mathcal{O})$ if and only if $r_\tau \in [0, p]$. 

p. But $N_k$ is both Hausdorff (being a quotient of $N$, which is Hausdorff) and a finite dimensional $k$-vector space, this forces $F^i N_k$ to vanish for large $i$. So we are done.
Proposition 6.1.3. If $N$ is as in Lemma 6.1.1 then the $G_{K_\infty}$-action on $T(N)$ is through the restriction to $G_{K_\infty}$ of the character 

$$\psi_x \prod_{\tau} \omega_{\tau}^{-\tau}$$

Here $\psi_x$ denotes the unramified character sending the geometric Frobenius to $x$, and the $\omega_{\tau}$ are the characters defined in the paragraph after the proof of Lemma 2.1.1.

Proof. This is [11, Proposition 6.7]. However note that in loc. cit. they contravariantly associate a $G_{K_\infty}$-representation to Breuil–Kisin module; this is why the character appearing here is the inverse of that in loc. cit. \hfill $\Box$

### 6.2. Crystalline $G_K$-actions.

For this subsection assume that $K = K_0$. This does not change the category $\text{Mod}_{\text{SD}}^\infty(\mathcal{O})$ but it does change the embedding $k[[u]] \to \mathcal{O}_{C^0}$. In particular it means $\nu^\flat(u) = 1$. The motivation for the following definition comes from a result of Gee–Liu–Savitt (see Theorem 7.1.4).

**Definition 6.2.1.** A crystalline $G_K$-action on $M \in \text{Mod}_{k}^{\text{BK}}(\mathcal{O})$ is a continuous $\varphi$-equivariant $\mathcal{O}$-linear $\mathcal{O}_{C^0}$-semilinear action of $G_K$ on $M \otimes_{k[[u]]} \mathcal{O}_{C^0}$ such that $(\sigma - 1)(m) = 0$ for all $m \in M$ and $\sigma \in G_{K_\infty}$, and such that

$$(\sigma - 1)(m) \in M \otimes_{k[[u]]} u^{p/p-1}\mathcal{O}_{C^0}$$

for all $m \in M$ and $\sigma \in G_K$.

**Remark 6.2.2.** One can prove that $M \in \text{Mod}_{k}^{\text{BK}}(\mathcal{O})$ lies in $\text{Mod}_{k}^{\text{SD}}(\mathcal{O})$ if and only if $\text{Weight}(M) \subset [0,p]$ and $M$ admits a crystalline $G_K$-action. We hope to report on this in a future work.

**Remark 6.2.3.** Crystalline $G_K$-actions are $\varphi$-equivariant and so induce an extension to $G_K$ of the $G_{K_\infty}$-action on $T(M)$. This extension makes the identification

$$M \otimes_{k[[u]]} C^\flat = T(M) \otimes_{\mathbb{F}_p} C^\flat$$

g$K$-equivariant. Also this extension is continuous since the subspace topology $\mathbb{F}_p \subset C^\flat$ is the discrete topology.

**Lemma 6.2.4.** Suppose $N \in \text{Mod}_{k}^{\text{BK}}(\mathcal{O})$ admits a crystalline $G_K$-action. If $0 \to T_1 \to T(N) \to T_2 \to 0$ is an exact sequence of $G_K$-representations (with the $G_K$-action on $T(N)$ as in Remark 6.2.3) and

$$0 \to M \to N \to P \to 0$$

is the corresponding exact sequence in $\text{Mod}_{k}^{\text{BK}}(\mathcal{O})$ (Lemma 5.1.3), then the $G_K$-action on $N \otimes_{k[[u]]} \mathcal{O}_{C^0}$ induces crystalline $G_K$-actions on $M$ and $P$.

Proof. The exact sequence

$$0 \to T_1 \otimes_{\mathbb{F}_p} C^\flat \to T(N) \otimes_{\mathbb{F}_p} C^\flat \to T_2 \otimes_{\mathbb{F}_p} C^\flat \to 0$$

is $G_K$-equivariant. Since $M \otimes_{k[[u]]} C^\flat = T_1 \otimes_{\mathbb{F}_p} C^\flat$ it follows that if $m \in M$ and $\sigma \in G_K$ then $\sigma(m) \in M \otimes_{k[[u]]} C^\flat$. Thus

$$\sigma(m) - m \in (N \otimes_{k[[u]]} u^{p/p-1}\mathcal{O}_{C^0}) \cap (M \otimes_{k[[u]]} C^\flat)$$

Since the quotient $N/M = P$ is free over $k[[u]]$ this intersection equals $M \otimes_{k[[u]]} u^{p/p-1}\mathcal{O}_{C^0}$. Thus the crystalline $G_K$-action on $N$ restricts to a crystalline $G_K$-action on $M$. This implies the crystalline $G_K$-action on $N \otimes_{k[[u]]} \mathcal{O}_{C^0}$ descends
through the surjection $N \otimes_{k[[u]]} \mathcal{O}_C \to P \otimes_{k[[u]]} \mathcal{O}_C$ to a $G_K$-action satisfying $\sigma(p) - p \in P \otimes_{k[[u]]} u^{p/(p-1)}\mathcal{O}_C$. \hfill \qed

**Corollary 6.2.5.** If $M$ and $N \in \text{Mod}_{BK}^{{\mathcal{O}}} (\mathcal{O})$ admit crystalline $G_K$-actions and $M \to N$ is a morphism inducing a $G_K$-equivariant map $T(M) \to T(N)$ then $M \otimes_{k[[u]]} \mathcal{O}_C \to N \otimes_{k[[u]]} \mathcal{O}_C$ is $G_K$-equivariant.

**Example 6.2.6.** If $N$ is as in Lemma 6.1.1 then an easy calculation shows that the following describes a continuous $\varphi$-equivariant $\mathcal{O}$-linear $\mathcal{O}_C$,-semilinear $G_K$-action on $M \otimes_{k[[u]]} \mathcal{O}_C$.

$$\sigma(e_\tau) = \eta(\sigma)^{\Theta_\tau} e_\tau$$

Here $\eta(\sigma) \in \mathcal{O}_C$, is the unique $p^{|k: \mathbb{F}_p|}/\mathbb{F}_p$ - 1-th root of $\sigma(u)/u$ whose image in $\mathcal{O}_C/m_{C} = \mathbb{F}_p$ is 1, and

$$\Theta_\tau := \sum_{0}^{[k: \mathbb{F}_p]-1} r_{\tau \varphi \varphi}p^j.$$

For this to be a crystalline $G_K$-action we need to check that $\eta(\sigma)^{\Theta_\tau} - 1 \in u^{p/(p-1)}\mathcal{O}_C$.

This follows from Lemma 6.2.7 below. As there is at most one way to extend a continuous character $\chi: G_K \to \mathbb{F}^\times$ to $G_K$ this is the unique crystalline $G_K$-action on $N$.

**Lemma 6.2.7.** Let $n \in \mathbb{Z}$. Let $\sigma \in G_K$ and suppose that $\sigma(u)/u$ is a $\mathbb{Z}_p$-generator of $p^n \mathbb{Z}_p(1)$. Then $v^h(\eta(\sigma)^n - 1)$ is $\frac{p^{1+\nu_p(n)+m}}{p-1}$.

**Proof.** This is follows from the calculation that if $\epsilon = (\epsilon_1, \epsilon_2, \ldots)$ is a $\mathbb{Z}_p$-generator of $\mathbb{Z}_p(1)$ then $v^h(\epsilon) - 1 = \frac{p^m}{p-1}$ which is proven in e.g. [8, §5.1.2]. \hfill \qed

### 6.3. Induction and Restriction.

**Notation 6.3.1.** Let $L/K$ be the unramified extension corresponding to a finite extension $l/k$, and let $L_{\infty} = K_{\infty} L$. Set $\mathfrak{S}_L = W(l)[[u]]$. Extension of scalars along the inclusion $f: \mathfrak{S} \to \mathfrak{S}_L$ describes a functor

$$f^* : \text{Mod}_{BK}^L \to \text{Mod}_{BK}^K$$

For $M \in \text{Mod}_{BK}^L$ the module $f^* M = M \otimes_{\mathfrak{S}} \mathfrak{S}_L$ is made into a Breuil–Kisin module via the semilinear map $m \otimes s \mapsto \varphi_M (m) \otimes \varphi(s)$; this map induces the isomorphism

$$(\varphi f^* M)[\frac{1}{E}] = (f^* \varphi M)[\frac{1}{E}] = f^*(\varphi M[\frac{1}{E}]) \xrightarrow{f \varphi_M} f^*(M)[\frac{1}{E}] = (f^* M)[\frac{1}{E}]$$

where the first = comes from the fact that $\varphi \circ f = f \circ \varphi$. The natural isomorphism $f^* M \otimes_{\mathfrak{S}_L} W(C^0) \cong M \otimes_{\mathfrak{S}} W(C^0)$ is clearly $\varphi, G_{L_{\infty}}$-equivariant so $T(f^* M) = T(M)|_{C_{L_{\infty}}}$.

**Notation 6.3.2.** With notation as in Notation 6.3.1, restriction of scalars along $f$ induces a functor

$$f_* : \text{Mod}_{BK}^K \to \text{Mod}_{BK}^L$$

If $M \in \text{Mod}_{BK}^L$ we equip $f_* M$ with the obvious semilinear map $m \mapsto \varphi_M (m)$. Let us verify that this makes $f_* M$ into a Breuil–Kisin module. The semilinear map induces the composite:

$$(\varphi f_* M)[\frac{1}{E}] \to (f_* \varphi M)[\frac{1}{E}] = f_* (\varphi M[\frac{1}{E}]) \xrightarrow{f \varphi_M} f_* (M[\frac{1}{E}]) = (f_* M)[\frac{1}{E}]$$
which we claim is an isomorphism. It suffices to check the natural map \( \varphi^* f_*, M \to f_* \varphi^* M \) is an isomorphism, and this follows because the commutative diagram

\[
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{f} & \mathcal{S}_{L} \\
\varphi \uparrow & & \uparrow \varphi \\
\mathcal{S} & \xrightarrow{f} & \mathcal{S}_{L}
\end{array}
\]

is a pushout.

**Lemma 6.3.3.** For all \( M \in \text{Mod}_{K}^{\text{BK}} \) and \( N \in \text{Mod}_{L}^{\text{BK}} \) there are functorial isomorphisms

\[
\text{Hom}(M, f_*, N) \cong f_* \text{Hom}(\varphi^* M, N)
\]

in \( \text{Mod}_{K}^{\text{BK}} \).

**Proof.** The standard adjunction between \( f^* \) and \( f_* \) provides functorial \( \mathcal{S} \)-linear isomorphisms \( \text{Hom}_{\mathcal{S}}(M, f_*, N) \to \text{Hom}_{\mathcal{S}}(\varphi^* M, N) \). Explicitly this map sends \( \alpha \) onto the homomorphism \( m \otimes s \mapsto \varphi(m) s \). As this is \( \varphi \)-equivariant we get isomorphisms as claimed.

**Lemma 6.3.4.** Let \( N \in \text{Mod}_{L}^{\text{BK}} \). Then there are functorial identifications \( \iota_N : T(f_*, N) \to \text{Ind}^{K_{\infty}}_{L_{\infty}} T(N) \) such that the diagram

\[
\begin{array}{ccc}
\text{Hom}_{BK}(M, f_*, N) & \xrightarrow{6.3.3} & \text{Hom}_{BK}(\varphi^* M, N) \\
\downarrow_{g \mapsto \iota_N \circ T(g)} & & \downarrow T \\
\text{Hom}_{G_{K_{\infty}}}(T(M), \text{Ind}^{K_{\infty}}_{L_{\infty}} T(N)) & \xrightarrow{(\text{Frob})} & \text{Hom}_{G_{L_{\infty}}}(T(M), \text{Ind}^{K_{\infty}}_{L_{\infty}} T(N))
\end{array}
\]

commutes for all \( M \in \text{Mod}_{K}^{\text{BK}} \). The top horizontal arrow is obtained from the identification in Lemma 6.3.3 by taking \( \varphi \)-invariants, and the lower horizontal arrow is given by Frobenius reciprocity.

**Proof.** Let \( \mathcal{O}_{E, L} \) be the \( p \)-adic completion of \( \mathcal{S}_L[e^{1/p}] \). The map \( f : \mathcal{S} \to \mathcal{S}_L \) extends to a map \( f : \mathcal{O}_E \to \mathcal{O}_{E, L} \) and so we can make sense of the operations \( f^* \) and \( f_* \) on etale \( \varphi \)-modules. Write \( M^\text{et} = M \otimes_{\mathcal{S}} \mathcal{O}_E \) and \( N^\text{et} = N \otimes_{\mathcal{S}_L} \mathcal{O}_{E, L} \). Then clearly \( f^*(M^\text{et}) = (f^* M)^\text{et} \), and because \( \mathcal{O}_{E, L} = \mathcal{O}_E \otimes_{\mathcal{S}} \mathcal{S}_L \mathcal{S}_L \) we also have that \( f_*(N^\text{et}) = (f_* N)^\text{et} \). We obtain maps

\[
\begin{align*}
\text{Hom}_{BK}(M, f_*, N) & \to \text{Hom}_{\mathcal{S}}(M^\text{et}, f_*, N^\text{et}), \\
\text{Hom}_{BK}(f^* M, N) & \to \text{Hom}_{\mathcal{S}}(f^* M^\text{et}, N^\text{et})
\end{align*}
\]

which commute with \( T \). The analogue of Lemma 6.3.3 in the setting of etale \( \varphi \)-modules is proved in exactly the same way, and the obtained identification is compatible with the maps above. Thus to prove the lemma we may replace \( \text{Hom}_{BK} \) with \( \text{Hom}_{\mathcal{S}} \) (homsets in the category of etale \( \varphi \)-modules) and \( M \) and \( N \) with \( M^\text{et} \) and \( N^\text{et} \) in the diagram of the lemma.

Since \( M^\text{et} \to T(M^\text{et}) \) is an equivalence of categories, the map \( (\text{Frob}) \circ T \circ (6.3.3) \circ T^{-1} \) describes an identification

\[
(6.3.5) \quad \text{Hom}_{\mathcal{S}_{K_{\infty}}}(V, T(f_*, N)) \to \text{Hom}_{\mathcal{S}_{K_{\infty}}}(V, \text{Ind}^{K_{\infty}}_{L_{\infty}} T(N))
\]

for any continuous \( G_{K_{\infty}} \) representation \( V \) on a finitely generated \( \mathbb{Z}_p \)-module. As (6.3.5) is functorial in \( V \) Yoneda’s lemma provides the isomorphism \( \iota_N \). As (6.3.5) is functorial in \( N \) we see that \( \iota_N \) is functorial.

**Lemma 6.3.6.** Assume \( k \subset l \subset F \).
(1) If $M \in \text{Mod}^{\text{SD}}_k(\mathcal{O})$ then $f^* M \in \text{Mod}^{\text{SD}}_l(\mathcal{O})$ and for each $\theta \in \text{Hom}_F(l, \mathbb{F})$ we have

$$\text{Weight}_\theta(f^* M) = \text{Weight}_{\theta|_k}(M)$$

(2) If $N \in \text{Mod}^{\text{SD}}_l(\mathcal{O})$ then $f_* N \in \text{Mod}^{\text{SD}}_l(\mathcal{O})$ and

$$\text{Weight}_\tau(f_* N) = \bigcup_{\theta|_k = \tau} \text{Weight}_\theta(N)$$

**Proof.** By functoriality both $f^*$ and $f_*$ preserve $\mathcal{O}$-actions. Note that the inclusion $k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F} \to l[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ sends the idempotents $e_\tau \mapsto \sum_{\theta|_k = \tau} e_\theta$. Thus $(f^* M)_\theta = M_{\theta|_k}$ and $(f_* N)_\tau = \prod_{\theta|_k = \tau} N_\theta$. Both (1) and (2) then follow by verifying the second condition of Lemma 6.3.4.

**Example 6.3.7.** Let $L/K$ be as in Notation 6.3.1 and $N \in \text{Mod}^{\text{BK}}_l(\mathcal{O})$ be as in Lemma 6.1.1, so that

$$N = l[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}, \quad \varphi_N(1) = (x) \sum_{\theta \in \text{Hom}_F(l, \mathbb{F})} u^{r_\theta} e_\theta$$

for some $x \in \mathbb{F}^\times$ and $r_\theta \in \mathbb{Z}$. In Example 6.2.6 it was shown $N$ admits a crystalline $GL$-action. An identical calculation shows that the same formula

$$\sigma(e_\theta) = \eta(\sigma)_{\Theta_\theta} e_\theta, \quad \Theta_\theta = \sum_{i=0}^{[l:F_p]-1} p^i r_{\Theta_\theta, \varphi^i}$$

but with $\sigma \in G_K$, defines a crystalline $G_K$-action on $f_* N$.

**6.4. Main Result.** For the rest of this section fix $M \in \text{Mod}^{\text{SD}}_l(\mathcal{O})$ and let $l/k$ be the extension of degree $\dim_\mathbb{F} T(M)$. Let $L/K$ be the unramified extension corresponding to $l/k$. Assume $l \subset \mathbb{F}$. As in Notation 6.3.1 there is a map $f : \mathfrak{S} \to \mathfrak{S}_L$. Our aim is to prove

**Proposition 6.4.1.** Suppose $K = K_0$ and that $M$ admits a crystalline $G_K$-action as in Subsection 6.2. Suppose $T(M)$ is irreducible as a $G_{K_\infty}$-representation and that $\mathbb{F}$ is sufficiently large so that Lemma 2.1.2 applies to $T(M)$.

Then there exists $M' \in \text{Mod}^{\text{SD}}_l(\mathcal{O})$ of rank one over $l[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}$ such that $T(f_* M') = T(M)$ and such that $\text{Weight}_\tau(f_* M') = \text{Weight}_\tau(M)$.

We give the proof over the next three subsections. Also, in Subsection 6.8 we give an example showing it is not necessarily true that $M = f_* M'$. Before this we record a corollary of the proposition.

**Corollary 6.4.2.** If $M$ as in Proposition 6.4.1 then there are integers $r_\theta$ for $\theta \in \text{Hom}_F(l, \mathbb{F})$ such that $T(M) = \psi \otimes \text{Ind}_{l_k}^{L_k}(\prod_{\theta} \omega_\theta^{-r_\theta})$ for some unramified character $\psi$, and such that

$$\text{Weight}_\tau(M) = \{ r_\theta \mid \theta|_k = \tau \}$$

**Proof.** Apply Proposition 6.1.3 to $M'$ and then use Lemma 6.3.6(2). \[\square\]
6.5. Proof of Proposition 6.4.1: a first approximation. We begin by providing a first approximation of \( M \) by a rank one object \( N \in \text{Mod}^i_{BK}(\mathcal{O}) \). For this subsection we need only that \( M \in \text{Mod}^i_{SD}(\mathcal{O}) \) has \( T(M) \cong \text{Ind}^K_{L\infty} \chi \) for some character \( \chi : G_{L\infty} \to \mathbb{F}^\times \) (if \( p = 2 \) we will also need that \( \chi \) is not an unramified twist of the trivial character). In particular we shall not use that \( T(M) \) is irreducible, or that \( M \) admits a crystalline \( G_K \)-action.

There is a non-zero map \( T(M)|_{G_{L\infty}} \to \chi \) corresponding under Frobenius reciprocity to the isomorphism \( T(M) \cong \text{Ind}^K_{L\infty} \chi \). Thus there is a surjection \( f^* M \to N \) where \( N \in \text{Mod}^i_{BK}(\mathcal{O}) \) is of rank one with \( T(N) = \chi \) (Lemma 5.1.3). Applying Lemma 6.3.4 to \( f^* M \to N \) we obtain a map

\[
(6.5.1) \quad M \to f_* N
\]

which after applying \( T \) induces an isomorphism. Thus (6.5.1) becomes an isomorphism after inverting \( u \); in particular it is injective.

As in Lemma 6.1.1 we may suppose

\[
(6.5.2) \quad N = k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}, \quad \varphi_N(1) = (x) \sum_{\theta \in \text{Hom}_{\mathbb{F}_p}(1, \mathbb{F})} u^{r\theta} e_\theta
\]

where \( x \in \mathbb{F}^\times \) and \( r_\theta \in \mathbb{Z} \). Note \( f^* M \in \text{Mod}^i_{SD}(\mathcal{O}) \) since \( M \in \text{Mod}^i_{SD}(\mathcal{O}) \) by Lemma 6.3.6. Therefore \( N \in \text{Mod}^i_{SD}(\mathcal{O}) \) and

\[
r_\theta \in \text{Weight}_{\theta|_{BK}}(M)
\]

by Proposition 5.4.7. In particular \( r_\theta \in [0, p] \).

Simplification. To prove the proposition we may assume \( x = 1 \) in (6.5.2) as we now explain. Let \( ur_x \in \text{Mod}^i_{BK}(\mathcal{O}) \) be the rank one object given by

\[
ur_x = k[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}, \quad \varphi_{ur_x}(1) = xe_{\tau_0} + \sum_{\tau \neq \tau_0} e_\tau
\]

and set \( \wtilde{M} = \text{Hom}(ur_x, M)^\mathcal{O} \) (recall Construction 4.3.3). One checks that \( \wtilde{M} \in \text{Mod}^i_{SD}(\mathcal{O}) \) and that \( \text{Weight}_{ur_x}(\wtilde{M}) = \text{Weight}_{ur_x}(M) \) for each \( \tau \) by verifying that condition (2) of Lemma 5.3.4 holds. With notation for \( \psi_x \) as in Proposition 6.1.3

\[
T(\wtilde{M}) = \text{Hom}(\psi_x, \text{Ind}^K_{L\infty} T(N)) = \text{Ind}^K_{L\infty} (\chi_{au}^{r\theta})
\]

cf. the last sentence of Construction 4.3.3. This shows that if the proposition is proved for \( \wtilde{M} \) it is proved for \( M \). It also shows that if one repeats the construction of the first two paragraphs of the proof with \( \wtilde{M} \) one obtains (6.5.2) with \( x = 1 \).

Assume from now on that \( x = 1 \) in (6.5.2). Since \( M \to f_* N \) is an isomorphism after inverting \( u \) we can define integers \( \delta_\theta \in \mathbb{Z}_{\geq 0} \) minimal amongst those satisfying \( u^{\delta_\theta} e_\theta \in M \)

Let \( P \subset N \) be the rank one object of \( \text{Mod}^i_{BK}(\mathcal{O}) \) generated by the \( u^{\delta_\theta} e_\theta \). We claim \( P \in \text{Mod}^i_{SD}(\mathcal{O}) \) and \( \text{Weight}_{ur_x}(P) \subset \text{Weight}_{ur_x}(M) \). After Proposition 5.4.7 it suffices to give an injection \( P \to f^* M \) with torsion-free cokernel. The map\(^3\)

\[
u^{\delta_\theta} e_\theta \mapsto e_\theta(u^{\delta_\theta} e_\theta \otimes 1)
\]

\(^3\)Here \( u^{\delta_\theta} e_\theta \) is an element of \( M \) and \( u^{\delta_\theta} e_\theta \otimes 1 \) is an element of \( f^* M = M \otimes_{\mathcal{O}} \mathcal{S}_L \).
is such an injection. This is a morphism of Breuil–Kisin modules and has torsion-free cokernel by the definition of $\delta_\theta$. As $\text{Weight}_\theta(P) = \{r_\theta + p\delta_\theta, -\delta_\theta\}$ we conclude

$$r_\theta + p\delta_\theta, -\delta_\theta \in \text{Weight}_\theta(M).$$

In particular $r_\theta + p\delta_\theta, -\delta_\theta \in [0, p]$. As $r_\theta \in [0, p]$, $p\delta_\theta, -\delta_\theta \leq p$ and so

$$(p^{[\ell:F_p]} - 1)\delta_\theta = \sum_{i=0}^{[\ell:F_p]-1} p^i(p\delta_\theta, -\delta_\theta) \leq p(p^{[\ell:F_p]} - 1)/(p - 1)$$

which implies $\delta_\theta \in [0, 1]$ if $p > 2$, and $\delta_\theta \in [0, 2]$ if $p = 2$. If $p = 2$ and $\delta_\theta = 2$ then, as $r_\theta + p\delta_\theta, -\delta_\theta \in [0, p]$, we must have $\delta_\theta = 2$ and $r_\theta = 0$. Thus $r_\theta = 0$ for all $\theta \in \text{Hom}_{\mathbb{F}_p}(L, F)$ and so $\chi$ is the trivial character. At the start of this subsection we assumed when $p = 2$ this was not the case, and so $\delta_\theta \in [0, 1]$ when $p = 2$ also.

At this point we have proved enough to deduce the following.

**Corollary 6.5.3.** Suppose that $M \in \text{Mod}_{\mathbb{F}_p}(\mathcal{O})$ with $T(M)$ irreducible and $\text{Weight}_\theta(M) \subset [0, p - 1]$. Then $M = f_*N$.

**Proof.** In fact our argument shows the result holds when $T(M) = \text{Ind}_{L_{\mathbb{F}_p}}^{K_{\mathbb{F}_p}} \chi$ with $\chi$ not (an unramified twist of) the trivial character. In this case the arguments of this subsection are valid. If $\delta_\theta = 1$ for all $\theta \in \text{Hom}_{\mathbb{F}_p}(L, F)$ then $r_\theta + p - 1 \in [0, p - 1]$ so all the $r_\theta = 0$, but then $\chi$ is the trivial character. If $\delta_\theta = 2$ and $\delta_\theta = 0$ then $r_\theta + p \in [0, p - 1]$ which is impossible. Thus $\delta_\theta = 0$ for all $\theta$ and $M = f_*N$. 

**6.6. Proof of Proposition 6.4.1 under an assumption.** Now we give a construction which produces $M'$ as in Proposition 6.4.1 as a sub-module of $N$, provided a certain assumption holds.

Unlike the previous subsection we will require that $T(M)$ be irreducible and we will use that $M$ admits a crystalline $G_K$-action. This allows us to invoke the following lemma.

**Lemma 6.6.1.** Fix $i \in \mathbb{Z}$. If $\sum \alpha_\theta e_\theta \in M$ with $\alpha_\theta \in \mathbb{F}$ then

$$\sum_{\alpha_\theta e_\theta \in M} \alpha_\theta e_\theta \in M$$

**Proof.** Note $f_*N$ admits a crystalline $G_K$-action by Example 6.3.7. The map $T(M) \to T(f_*N)$ induced by $M \to f_*N$ is an isomorphism of $G_K$-representations by construction. It must therefore be an isomorphism of $G_K$-representations by Corollary 2.2.2 (where both $T(M)$ and $T(f_*N)$ are made into $G_K$-representations via the respective crystalline $G_K$-actions). The inclusion $M \otimes_{k[u]} \mathcal{O}_{C^0} \to (f_*N) \otimes_{k[u]} \mathcal{O}_{C^0}$ is then $G_K$-equivariant by Corollary 6.2.5. Therefore, if $\sum \alpha_\theta e_\theta \in M$ and $\sigma \in G_K$

$$(\sigma - 1)(\sum \alpha_\theta e_\theta) = \sum \alpha_\theta(\eta(\sigma)^{\Theta_\theta} - 1) e_\theta \in M \otimes_{k[u]} \mathcal{O}_{C^0}$$

Choose $\sigma$ such that $\sigma(u)/u$ is a $\mathbb{Z}_p$-generator of $\mathbb{Z}_p(1)$. Then $\varphi(\eta(\sigma)^i - 1) = p^{\varphi(i)+1}/(p - 1)$ by Lemma 6.2.7. It follows that

$$\sum \alpha_\theta \left( \frac{\eta(\sigma)^{\Theta_\theta} - 1}{\eta(\sigma) - 1} \right) e_\theta \in M \otimes_{k[u]} \mathcal{O}_{C^0}.$$
Note that \( \frac{n(\sigma)\theta_{a-1}}{\theta_{i-1}} = 1 + \eta(\sigma) + \ldots + \eta(\sigma)^{a-1} \equiv \Theta_i \) modulo \( u^{p-1} \mathcal{O}_C \) and that \( \Theta_i \equiv r_\theta \) in \( \mathcal{O}_C \) (as \( p = 0 \) in this ring). Recall in the previous subsection we show \( \delta_\theta \equiv [0,1] \), therefore \( w_\theta \in M \) for every \( \theta \). It follows that

\[
(6.6.2) \quad \sum \alpha_\theta e_\theta \in M \Rightarrow \sum \alpha_\theta r_\theta e_\theta \in (M \otimes k[[u]] \mathcal{O}_C) \cap N = M
\]

whenever \( \alpha_\theta \in \mathbb{F} \). The equality \( (M \otimes k[[u]] \mathcal{O}_C) \cap N = M \) follows from \([5, \S3.5 Proposition 10]\) because \( k[[u]] \to \mathcal{O}_C \) is faithfully flat. It is straightforward to check that (6.6.2) is equivalent to asking that for each \( i \in \mathbb{Z} \)

\[
\sum \alpha_{\theta^i} e_\theta \in M \Rightarrow \sum \alpha_\theta e_\theta \in M
\]

whenever \( \alpha_\theta \in \mathbb{F} \).

**Remark 6.6.3.** As well as the previous lemma we make repeated use of the observations:

- If \( m \in M \) then \( \varphi(m) \in M \).
- If \( m \in M \) is such that \( \varphi(m) \in u^{p+1}M \) then \( m \in uM \).

The first follows because \( \text{Weight}(M) \) consists of positive integers. The second follows because \( \text{gr}^i(M_k) = 0 \) for \( i > p \) and so \( F^{p+1}M_k = F^{p+2}M_k = \ldots = 0 \). Therefore the image of \( F^{p+1}M \) in \( M_k \) is zero, i.e. \( F^{p+1}M \subset uM \). Thus, if \( \varphi(m) \in u^{p+1}M \) then \( m \in uM \).

**Construction 6.6.4.** Fix \( \theta_0 \in \text{Hom}_p(l, \mathbb{F}) \) and set \( \theta_m = \theta_0 \circ \varphi^m \) for \( 0 \leq m \leq [l : F_p] - 1 \). Define \( X \subset \text{Hom}_p(l, \mathbb{F}) \) by asserting

\[
(6.6.5) \quad \theta_i \not\in X \iff \text{there exists } \alpha_{\theta_j} \in \mathbb{F} \text{ such that } e_{\theta_i} + \sum_{0 \leq j < i} \alpha_{\theta_j} e_{\theta_j} \in M
\]

Note the construction of \( X \) depends upon the choice of \( \theta_0 \).

We shall use that \( X \) satisfies the following two properties.

**Lemma 6.6.6.**

1. If for \( \theta \in X \) there exist \( \alpha_\theta \in \mathbb{F} \) such that

\[
\sum_{\theta \in X} \alpha_\theta e_\theta \in M
\]

then all \( \alpha_\theta = 0 \).

2. If \( \theta \not\in X \) then there exists a unique

\[
e_{\theta} + \sum \alpha_\kappa e_\kappa \in M, \quad \alpha_\kappa \in \mathbb{F}
\]

with the sum running over \( \kappa \in X \) such that \( r_\theta \equiv r_\kappa \) modulo \( p \) and such that \( \theta|k = \kappa|k \). In particular the sum lies in \( M_{\theta|k} \).

**Proof.** If in (1) not all the \( \alpha_\theta = 0 \) there will be a largest \( i \) such that \( \alpha_{\theta_i} \neq 0 \) and \( \theta_i \in X \); then \( e_{\theta_i} + \sum_{\theta \neq \theta_i} \frac{\alpha_\theta}{\alpha_{\theta_i}} e_\theta \in M \) which contradicts the fact that \( \theta_i \in X \).

Uniqueness in (2) follows from (1) so we only need to show existence. We claim there exists \( \alpha_\kappa \in \mathbb{F} \) such that \( e_{\theta} + \sum_{\kappa \in X} \alpha_\kappa e_\kappa \in M \); note if this is the case then by Lemma 6.6.1 and the fact that \( M = \prod M_p \) there will exist a sum as in (2). Write \( \theta = \theta_i \). We prove the claim by induction on \( i \). If \( i \) is minimal amongst \( i \) with \( \theta_i \not\in X \) then in (6.6.5) each \( \theta_j \in X \) so the claim holds. For general \( i \) and any sum as in (6.6.5) some \( \theta_j \) may not be contained in \( X \), however our inductive hypothesis then
implies there exist \( \beta_\kappa \in \mathbb{F} \) such that \( \alpha_{\theta_j} e_{\theta_j} + \sum_{\kappa \in X} \beta_\kappa e_\kappa \in M \), and so the claim holds in this case also. \( \square \)

**Lemma 6.6.7.** Suppose \( \theta_0 \) is chosen so that

1. if \( \theta \in X \) and \( \theta \circ \varphi \notin X \) then \( r_\theta > 0 \),
2. if \( \theta \notin X \) and \( \theta \circ \varphi \in X \) then \( r_\theta = 0 \).

Then the conclusion of Proposition 6.4.1 holds.

**Proof.** We claim the rank one object \( M' \in \text{Mod}_I^{\text{BK}}(O) \) with \( (M')_\theta \) generated by

\[
 f_\theta = \begin{cases} 
 e_\theta & \text{if } \theta \notin X \\
 u e_\theta & \text{if } \theta \in X
\end{cases}
\]

will satisfy the conditions of Proposition 6.4.1. Clearly \( T(M') = T(N) \) so \( T(M) = T(f, M') \). Thus we only need to check that \( \text{Weight}_r(M) = \text{Weight}_r(f, M') \). If \( g_{\theta \circ \varphi} = f_{\theta \circ \varphi} \) (the reason for this notation should become clear after reading the below) then there are relations

(C1) \[ \varphi(g_{\theta \circ \varphi}) = u^{p+r_\theta} f_\theta \quad \text{if } \theta \notin X, \theta \circ \varphi \in X \]
(C2) \[ \varphi(g_{\theta \circ \varphi}) = u^{r_\theta} f_\theta \quad \text{if } \theta, \theta \circ \varphi \notin X \]
(C3) \[ \varphi(g_{\theta \circ \varphi}) = u^{r_\theta - 1} f_\theta \quad \text{if } \theta \in X, \theta \circ \varphi \not\in X \]
(C4) \[ \varphi(g_{\theta \circ \varphi}) = u^{r_\theta + p - 1} f_\theta \quad \text{if } \theta, \theta \circ \varphi \in X \]

Therefore, using Lemma 5.3.4 and Remark 5.3.5, in order to show \( \text{Weight}_r(M) = \text{Weight}_r(f, M') \) for each \( \tau \in \text{Hom}_F(l, F) \) it suffices to produce \( \mathbb{F}[u] \)-bases \( (g_{\theta \circ \varphi}) f_{\theta \circ \varphi} \) \( (\theta \circ \varphi) \in \tau \) of \( M_{\tau \circ \varphi} \) and \( f_{\theta \circ \varphi} \) \( (\theta \circ \varphi) \in \tau \) of \( M_\tau \) such that the relations (C1)-(C4) hold. We construct such \( g_{\theta \circ \varphi} \) and \( f_\theta \) case-by-case, as follows.

- Consider (C1), i.e. \( \theta \notin X \) and \( \theta \circ \varphi \in X \). By hypothesis \( r_\theta = 0 \). By Lemma 6.6.6(2) there are \( \alpha_\kappa \in \mathbb{F} \) such that

\[
 f_\theta := e_\theta + \sum_{\kappa \in X} \alpha_\kappa e_\kappa \in M_\tau
\]

with the sum running over \( \kappa \in X \) with \( r_\kappa \equiv 0 \) modulo \( p \). Note that if \( e_{\kappa \circ \varphi} \notin M \) then \( r_\kappa \in [0, 1] \) (because \( p \delta_{\kappa \circ \varphi} - \delta_\kappa + r_\kappa = p - \delta_\kappa + r_\kappa \in [0, p] \)). Therefore if \( \kappa \circ \varphi \in X \) (so that \( e_{\kappa \circ \varphi} \notin M \)) then \( r_\kappa = 0 \). On the other hand if \( \kappa \circ \varphi \notin X \) then \( r_\kappa > 0 \) by hypothesis. Therefore \( r_\kappa = p \) and \( e_{\kappa \circ \varphi} \in M \). As a consequence

\[
 g_{\theta \circ \varphi} := u e_{\theta \circ \varphi} + \sum_{\kappa \circ \varphi \in X} u \alpha_\kappa e_{\kappa \circ \varphi} + \sum_{\kappa \circ \varphi \in \kappa} \alpha_\kappa e_{\kappa \circ \varphi} \]

satisfies \( \varphi(g_{\theta \circ \varphi}) = u^p f_\theta \). Therefore \( \varphi(u g_{\theta \circ \varphi}) \in u^{2p} M_\tau \) and so by Remark 6.6.3 we deduce \( g_{\theta \circ \varphi} \in M_{\tau \circ \varphi} \).

- Consider (C2), i.e. \( \theta \notin X \) and \( \theta \circ \varphi \notin X \), and suppose \( r_\theta = 0 \). Using Lemma 6.6.6(2) there are \( \alpha_{\kappa \circ \varphi} \in \mathbb{F} \) such that

\[
 g_{\theta \circ \varphi} := e_{\theta \circ \varphi} + \sum_{\kappa \circ \varphi \in X} \alpha_{\kappa \circ \varphi} e_{\kappa \circ \varphi} \in M_{\tau \circ \varphi}
\]

Define \( f_\theta := \varphi(g_{\theta \circ \varphi}) \). Then \( f_\theta \in M_\tau \) by Remark 6.6.3 and

\[
 f_\theta = e_\theta + \sum_{\kappa \circ \varphi \in X} \alpha_{\kappa \circ \varphi} u^{r_\kappa} e_\kappa
\]
The next two lemmas finish the proof.

- Consider (C2), i.e. \( \theta \not\in X \) and \( \theta \circ \varphi \not\in X \), and suppose \( r_\theta > 0 \). By Lemma 6.6.6(2) there are \( \alpha_\kappa \in \mathbb{F} \) such that

\[
    f_\theta := e_\theta + \sum \alpha_\kappa e_\kappa \in M_f
\]

with the sum running over \( \kappa \in X \) satisfying \( r_\kappa \equiv r_\theta \mod p \). Define

\[
    g_{\theta \circ \varphi} := e_{\theta \circ \varphi} + \sum_{r_\kappa > 0} \alpha_\kappa e_{\kappa \circ \varphi} + u \sum_{r_\kappa = 0} \alpha_\kappa e_{\kappa \circ \varphi}
\]

Note the second sum can appear only if \( r_\theta = p \). We have \( \varphi(g_{\theta \circ \varphi}) = u^{r_\theta} f_\theta \) and \( \varphi(u_{\theta \circ \varphi}) = u^{p + r_\theta} f_\theta \). As \( r_\theta > 0 \) Remark 6.6.3 implies \( g_{\theta \circ \varphi} \in M_{r_\theta \circ \varphi} \).

- Consider (C4), i.e. \( \theta \in X \) and \( \theta \circ \varphi \not\in X \). By hypothesis \( r_\theta > 0 \). Define

\[
    g_{\theta \circ \varphi} := e_{\theta \circ \varphi} + \sum_{\kappa \circ \varphi \in X} \alpha_{\kappa \circ \varphi} e_{\kappa \circ \varphi} \in M_{r_\theta \circ \varphi}
\]

as in Lemma 6.6.6(2) and define

\[
    f_{\theta} := u e_\theta + \sum_{\kappa \circ \varphi \in X} \alpha_{\kappa \circ \varphi} u^{r_\kappa} e_\kappa
\]

Now if \( e_{\theta \circ \varphi} \in M \) then by uniqueness of Lemma 6.6.6(2), \( g_{\theta \circ \varphi} = e_{\theta \circ \varphi} \) and \( f_\theta = u e_\theta \) so \( \varphi(g_{\theta \circ \varphi}) = u^{r_\theta} f_\theta \). If \( e_{\theta \circ \varphi} \not\in M \) then \( r_\theta = 1 \) (because \( p \delta_{\theta \circ \varphi} - \delta_\theta + r_\theta = p - 1 + r_\theta \in [0, p] \)) and so \( \varphi(g_{\theta \circ \varphi}) = f_\theta = u^{r_\theta} f_\theta \). In particular we see that \( f_\theta \in M_f \).

- Consider (C4), i.e. \( \theta \in X \) and \( \theta \circ \varphi \in X \). Put \( g_{\theta \circ \varphi} := u e_{\theta \circ \varphi} \) and \( f_\theta := u e_\theta \).

Then \( \varphi(g_{\theta \circ \varphi}) = u^{r_\theta + p - 1} f_\theta \).

The next two lemmas finish the proof. \( \square \)

**Lemma 6.6.8.** The \( (f_\theta)_{|\kappa = r} \) just defined form a \( \mathbb{F}[u] \)-basis of \( M_f \).

**Proof.** Let \( W \) be the \( \mathbb{F}[u] \)-span of all the \( f_\theta \), so that \( W \subseteq M \). Since the \( \mathbb{F}[u] \)-rank of \( M \) equals \( |l : \mathbb{F}_q| \), the number of the \( f_\theta \), it suffices to show \( W = M \).

The first step is to show \( u e_\theta \in W \) for each \( \theta \). If \( \theta \) is as in (C4) this then is obvious. It is also obvious if \( \theta \) is as in (C3) and \( e_{\theta \circ \varphi} \in M \) for then \( f_\theta = u e_\theta \). If \( \theta \) is as in (C1) then \( u f_\theta = u e_\theta + \sum u e_\kappa e_\kappa \) where the sum runs over \( \kappa \in X \) such that if \( \kappa \circ \varphi \not\in X \) then \( e_{\kappa \circ \varphi} \in M \); by the previous two sentences we deduce \( u e_\theta \in W \). At this point we’ve shown \( u e_\kappa \in W \) if \( \kappa \circ \varphi \not\in X \). If \( \theta \) is as in (C3) but with \( e_{\theta \circ \varphi} \not\in M \) then

\[
    f_\theta = u e_\theta + \sum_{\kappa \circ \varphi \in X} \alpha_{\kappa \circ \varphi} u^{r_\kappa} e_\kappa
\]

We know the \( u^{r_\kappa} e_\kappa \in W \) when \( r_\kappa > 0 \) so we have that

\[
    u e_\theta + \sum_{\kappa \circ \varphi \in X} \alpha_{\kappa \circ \varphi} e_\kappa \in W
\]

Split this sum up as

\[
    u e_\theta + \sum_{\kappa \circ \varphi \in X: \kappa \in X} \alpha_{\kappa \circ \varphi} e_\kappa + \sum_{\kappa \circ \varphi \in X: \kappa \not\in X} \alpha_{\kappa \circ \varphi} e_\kappa \in W
\]
If $\kappa \circ \varphi \in X$ and $\kappa \not\in X$ then $\kappa$ is as in (C1) and so $f_\kappa = e_\kappa + \sum_{i \in X} \alpha_i e_i \in W$. It follows there are $\beta_i \in \mathbb{F}$ such that

$$ue_\theta + \sum_{i \in X} \beta_i e_i \in W$$

However then $\sum_{i \in X} \beta_i e_i \in M$ since $ue_\theta \in M$, which implies by Lemma 6.6.6(1) that all $\beta_i = 0$. Thus $ue_\theta \in W$. At this point we know $ue_\theta \in W$ except if $\theta$ is as in (C2), i.e., $\theta \not\in X$ and $\theta \circ \varphi \not\in X$. In this case

$$uf_\theta = \begin{cases} 
ue_\theta + \sum_{i \in X} \alpha_i e_i & \text{if } r_\theta > 0 \\
ue_\theta + \sum_{i \in \varphi \neq X} \alpha_i e_i & \text{if } r_\theta = 0
\end{cases}$$

and so that $ue_\theta \in W$ follows from all the cases we have previously worked out.

To finish the proof note that if $Q \subset N$ is the $\mathbb{F}$-vector space spanned by the $e_\kappa$ with $\kappa \in X$ then $Q \cap M = 0$ by Lemma 6.6.6(1). If $\theta$ is as in (C1) then $e_\theta - f_\theta \in Q$ by definition. Using this and the fact that $ue_\theta \in W$ for all $\theta$ we see additionally that if $\theta$ is as in (C2) with $r_\theta = 0$, then there exists $w \in W$ such that

$$e_\theta - w \in Q$$

This is also true if $\theta$ is as in (C2) with $r_\theta > 0$ since then $e_\theta - f_\theta \in Q$. Thus if $\theta \not\in X$ there exists $w \in W$ such that $e_\theta - w \in Q$. Now take an arbitrary element $z = \sum \alpha_\theta e_\theta \in M$. We need to show it lies in $W$. We can assume $\alpha_\theta \in \mathbb{F}$ because we know $ue_\theta \in W$ for all $\theta$. By the above we can find $w \in W$ such that $z - w \in Q$; however since $z - w \in M$ we conclude that $z = w$. Thus the $(f_\theta)$ generate $M$ which proves the lemma.

**Lemma 6.6.9.** The $(g_{\theta \circ \varphi})$ just defined form a $\mathbb{F}[[u]]$-basis of $M_{\tau \circ \varphi}$.

**Proof.** The idea is the same as the previous lemma, but the details are slightly different. Again let $W \subset M$ be the sub-$\mathbb{F}[[u]]$-module spanned by the $g_{\theta \circ \varphi}$. Again we show $W = M$.

First we show $ue_{\theta \circ \varphi} \in W$ for all $\theta$. If $\theta$ is as in (C4) then this is clear. It is also clear if $\theta$ is as in (C3) and $e_{\theta \circ \varphi} \in M$ because in this case $g_{\theta \circ \varphi} = e_{\theta \circ \varphi}$. If $\theta$ is as in (C1) then

$$g_{\theta \circ \varphi} = ue_{\theta \circ \varphi} + \sum_{r_\kappa = 0}^{p} \alpha_{\kappa} e_{\kappa \circ \varphi} + \sum_{r_\kappa = p}^{M} \alpha_{\kappa} e_{\kappa \circ \varphi} \in M_\tau$$

with each $\kappa \in X$, so using the two previous cases we deduce $ue_{\theta \circ \varphi} \in W$. In particular we’ve checked $ue_{\theta \circ \varphi} \in W$ whenever $\theta \circ \varphi \in X$. If $\theta$ is as in (C3) with $e_{\theta \circ \varphi} \not\in M$, or as in (C2) with $r_\theta = 0$ then $ue_{\theta \circ \varphi} \in W$ since $ug_{\theta \circ \varphi} = ue_{\theta \circ \varphi} + \sum_{\kappa \in X} \alpha_\kappa e_\kappa$, and each $ue_\kappa \in W$ by the above. At this point the only remaining case is when $\theta$ is as in (C2) with $r_\theta > 0$. If $\theta$ is as in (C2) with $r_\theta > 0$ then

$$ug_{\theta \circ \varphi} = ue_{\theta \circ \varphi} + \sum_{r_\kappa > 0} \alpha_{\kappa} e_{\kappa \circ \varphi} + u^2 \sum_{r_\kappa = 0} \alpha_{\kappa} e_{\kappa \circ \varphi}$$

with each $\kappa \in X$. As we’ve shown above that if $\kappa \in X$ then $ue_{\kappa \circ \varphi} \in W$ we deduce that $ue_{\theta \circ \varphi} \in W$. This completes the proof that $ue_\theta \in W$ for all $\theta$.

We finish the proof just as in the previous lemma. Let $Q \subset N$ be the $\mathbb{F}$-span of the $e_\kappa$ with $\kappa \in X$, so that $Q \cap M = 0$. If $\theta$ is as in (C3) then $e_{\theta \circ \varphi} - g_{\theta \circ \varphi} \in Q$ by construction. Using this and the fact that $ue_\theta \in W$ for all $\theta$ we also see that if $\theta$ is as in (C2) with $r_\theta > 0$ then there exists $w \in W$ such that $e_{\theta \circ \varphi} - w \in Q$. This is
also true if \( \theta \) is as in (C2) with \( r_\theta = 0 \), for then \( g_{\theta \circ \varphi} - e_{\theta \circ \varphi} \in Q \). This shows that if \( \theta \circ \varphi \not\in X \) then there exists a \( w \in W \) such that
\[
e_{\theta \circ \varphi} - w \in Q
\]
Thus for any general element \( Z = \sum \alpha_\theta e_\theta \in M \) there exists \( w \in W \) such that \( Z - w \in Q \cap M \). We conclude \( Z = w \in W \) which finishes the proof. \( \square \)

### 6.7. Verifying the hypothesis in Lemma 6.6.7

Continue with the notation of the previous subsection.

**Lemma 6.7.1.** Suppose \( \theta_0 \) is such that \( r_{\theta_0|p|^{-1}} = 0 \). Then condition (2) of Lemma 6.6.7 holds.

**Proof.** We must show \( \theta \not\in X \) and \( \theta \circ \varphi \in X \) implies \( r_\theta = 0 \). If \( \theta = \theta_{[l:F]}^{-1} \) then \( r_{\theta_{[l:F]}^{-1}} = 0 \) by assumption there is nothing to prove. Now suppose \( \theta = \theta_i \) with \( i \not\in \lceil l : F \rceil - 1 \) so that \( \theta_{i+1} \) is defined. As \( \theta_{i+1} \in X \), \( \delta_{i+1} = 1 \) and so \( r_{\theta_i} + p - \delta_{\theta_i} \in [0, p] \). Hence \( r_{\theta_i} \in (0, 1] \). Suppose for a contradiction that \( r_{\theta_i} = 1 \). As \( \theta_i \not\in X \) there exists \( \alpha_{\theta_i} \in F \) such that
\[
z := e_{\theta_i} + \sum_{0 \leq j < i} \alpha_{\theta_j} e_{\theta_j} \in M
\]
As in Lemma 6.6.6(2) we can assume \( \alpha_{\theta_j} = 0 \) unless \( r_{\theta_j} \equiv r_{\theta_i} \equiv 1 \) modulo \( p \). Thus \( r_{\theta_j} = 1 \) when \( \alpha_{\theta_j} \neq 0 \). If \( z' = e_{\theta_{i+1}} + \sum \alpha_{\theta_j} e_{\theta_{j+1}} \) then \( \varphi(uz') = u^{p+1}z \) and so \( z' \in M \) by Remark 6.6.3. Therefore \( \theta_{i+1} \not\in X \) which is a contradiction. \( \square \)

**Lemma 6.7.2.** If the two conditions of Lemma 6.6.7 do not hold for any choice of \( \theta_0 \) then \( M \) is reducible.

**Proof.** For varying \( I < \lceil l : F \rceil \) and \( \theta \) consider the collection of sums
\[
e_\theta + \sum_{0 < j \leq I} \alpha_{\theta \circ \varphi^j} e_{\theta \circ \varphi^j} \in M
\]
with \( \alpha_{\theta \circ \varphi^j} \in F \). If this set is empty then \( X = \text{Hom}_F(l, F) \) and so the conditions of Lemma 6.6.7 would be trivially verified. Thus we may consider the smallest \( I \) such that a sum as in (6.7.3) exists. For a given \( \theta \), there can be at most one such sum; if there were two their difference would have length \( < I \) and so must be zero.

We now show that for any sum as in (6.7.3), \( r_\theta = r_{\theta_0 \circ \varphi^j} \) whenever \( \alpha_{\theta \circ \varphi^j} \neq 0 \). Lemma 6.6.1 implies \( r_{\theta} \equiv r_{\theta_0 \circ \varphi^j} \) modulo \( p \) whenever \( \alpha_{\theta_0 \circ \varphi^j} \neq 0 \). As \( r_{\theta_0}, r_{\theta_0 \circ \varphi^j} \in [0, p] \) this congruence will be an equality except possibly if \( r_{\theta_0} = 0 \) or \( p \). In this case set
\[
z = u^{\gamma \gamma} e_{\theta_0 \circ \varphi^j} + \sum_{0 < j \leq I} \alpha_{\theta \circ \varphi^j} u^{\gamma_j} e_{\theta \circ \varphi^j+i-1}
\]
where \( \gamma_j = 0 \) if \( r_{\theta_0 \circ \varphi^j} = p \) and \( \gamma_j = 1 \) if \( r_{\theta_0 \circ \varphi^j} = 0 \), and likewise \( \gamma = 0 \) if \( r_{\theta} = p \) and \( \gamma = 1 \) if \( r_{\theta} = 0 \). Then \( \varphi(uz) \) equals \( u^{2p} \) multiplied by (6.7.3), and so by Remark 6.6.3, \( z \in M \). If all the \( \gamma_j, \gamma \) are equal then all the \( r_{\theta_j} \) are equal and our claim follows. If \( \gamma = 1 \) and not all \( \gamma_j = 1 \) then \( z - u e_{\theta_0 \circ \varphi^j} - \sum \gamma_{j=1} u \alpha_{\theta_0 \circ \varphi^j} e_{\theta_0 \circ \varphi^j+i} \in M \) is non-zero, contradicting the minimality of \( I \). Thus, not all the \( \gamma, \gamma_j \) being equal implies \( r_{\theta} = p \). Since \( r_{\theta} + p \delta_{\theta_0 \circ \varphi^j} - \delta_{\theta} \in [0, p] \) we must have \( \delta_{\theta_0 \circ \varphi^j} = 0 \), and so \( e_{\theta_0 \circ \varphi^j} \in M \). Therefore \( I = 0 \) and (6.7.3) reads \( e_{\theta} \in M \) and the claim is immediate.


Now fix a sum as in (6.7.3). We shall show, under the hypotheses of the lemma, that there exist $\beta_{\theta \circ \varphi} \in \mathbb{F}$ such that

$$e_{\theta \circ \varphi} \left( I \right) + \sum_{0 < j \leq I} \beta_{\theta \circ \varphi} e_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid \in M$$

and such that $\beta_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid = \alpha_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid$ if $\alpha_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid \neq 0$. Let us explain why this implies $T(M)$ is reducible. Note that possibly $\beta_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid = 0$ while $\alpha_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid = 0$. Actually this cannot happen because if it did then applying this construction to $\theta \circ \varphi$, and then $\theta \circ \varphi^2$, and so on till we get to $\theta \circ \varphi^{|e_{\theta \circ \varphi}|-1}$, would yield another sum as in (6.7.3) with more non-zero terms, which contradicts uniqueness. Thus (6.7.4) must be equal to $r_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid = r_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid + 1$; the second paragraph of the proof then implies that there exist $\beta_{\theta \circ \varphi} \mid e_{\theta \circ \varphi} \mid \in \mathbb{F}$ such that condition (2) of Lemma 6.6.7 is trivially verified. Therefore condition (1) of Lemma 6.6.7 cannot hold, i.e. there exists $\kappa \in X$ such that $\kappa \circ \varphi \in X$ and $r_{\kappa} = 0$. Write $\kappa = \theta_t$. If $t = \lfloor l : \mathbb{F}_p \rfloor - 1$ then $\kappa \circ \varphi \in X$ which implies $e_{\theta_t} \in M$. Then $I = 0$, and (6.7.3) reads $e_{\theta} \in M$, so we can take (6.7.4) equal to $e_{\theta_0}$. If $i \neq \lfloor l : \mathbb{F}_p \rfloor - 1$ then $\kappa \circ \varphi = \theta_{i+1}$. As $\theta_{i+1} \notin X$ there must exist $\beta_{\theta_i} \in \mathbb{F}$ such that

$$e_{\theta_{i+1}} \left( I \right) + \sum_{0 \leq j < i+1} \beta_{\theta_j} e_{\theta_j} \in M$$

Since $r_{\kappa} = 0$ and $r_{\theta} = 0$, applying $\varphi$ gives that $e_{\theta_t} + \beta_{\theta_t} e_{\theta_0} + \sum_{0 \leq j < i+1} \beta_{\theta_j} e_{\theta_{j+1}} \in M$. Using that $u e_{\theta_{j+1}} \in M$ and that $\theta_{j+1} = \theta \circ \varphi$ it follows that

$$e_{\theta_t} + \beta_{\theta_t} e_{\theta_0} + \sum_{0 \leq j < i+1} \beta_{\theta_j} e_{\theta_{j+1}} = \beta_{\theta_0} e_{\theta_0} + \sum_{0 \leq j < i+1} \beta_{\theta_j} e_{\theta_{j+1}} \in M$$

where $\beta_{\theta_t} = \beta_{\theta_i}$ unless $\theta_{i+1} > 0$ in which case $\beta'_{\theta_j} = 0$. On the right hand side of (6.7.6) we’ve set $\beta'_{\theta_i+1} = 1$. Note $\beta_{\theta_0} \neq 0$ because otherwise the left hand side of (6.7.6) would contradict the fact that $\kappa = \theta_t \in X$. We can rewrite $\beta_{\theta_0}$ times (6.7.3) as $\beta_{\theta_0} e_{\theta_0} + \sum_{0 \leq j < I} \beta_{\theta_0} \alpha_{\theta \circ \varphi} e_{\theta_{j+1}}$, and so if $i+1 > I$ then the difference between this element and (6.7.6) is a sum which contradicts the fact that $\theta_t \in X$. Also, we cannot have $i+1 < I$ as then (6.7.6) contradicts the minimality of $I$. Therefore $i+1 = I$ and so $\beta_{\theta_0} \cdot (6.7.6)$ equals (6.7.3) by uniqueness of (6.7.3). Hence $\beta_{\theta_0} \cdot (6.7.5)$ is the sum asked for in (6.7.4).

6.8. An example. We conclude this section by giving an example of $M$ as in Proposition 6.4.1 with $M \neq f_* N$. Take $K = \mathbb{Q}_p$ and let $L/K$ be of degree 5. We shall exhibit $M$ as a sub-module of $f_* N$ where $N$ is the rank one object of $\text{Mod}^{\text{gp}}(O)$ given by

$$N = l[[u]] \otimes_{\mathbb{F}_p} \mathbb{F}, \quad \varphi_N \mid e_{\theta \circ \varphi} \mid = u^r e_{\theta \circ \varphi} + u^n e_{\theta \circ \varphi^2} + e_{\theta \circ \varphi} + u^ne_{\theta \circ \varphi} + e_{\theta}$$
Here we have fixed a \( \theta \in \text{Hom}_F(l,F) \) and \( n, x \) are integers satisfying \( 1 \leq n \leq p, 0 \leq x < p \).

Let \( M \subset f_* N \) be the sub-module generated over \( \mathbb{F}[u] \) by \( \epsilon_{\theta \circ \varphi}, \epsilon_{\theta \circ \varphi^2} + \epsilon_{\theta \circ \varphi^3}, u \epsilon_{\theta \circ \varphi}, \epsilon_\theta \). One computes that

\[
\varphi(\epsilon_{\theta \circ \varphi}, \epsilon_{\theta \circ \varphi^3}, \epsilon_{\theta \circ \varphi^2}, u \epsilon_{\theta \circ \varphi}, \epsilon_\theta) = (\epsilon_{\theta \circ \varphi}, \epsilon_{\theta \circ \varphi^3} + \epsilon_{\theta \circ \varphi^2}, u \epsilon_{\theta \circ \varphi}, \epsilon_\theta) X
\]

where

\[
X = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
u^n & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & u^{n-1} & 0 & 0 \\
0 & 0 & 0 & u^p & 0 \\
0 & 0 & 0 & 0 & u^x \\
1 & 0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

which shows that \( M \) is strongly divisible. One checks that the induced map \( f^* M \to N \) is surjective which shows that \( M \neq f_*(N') \) for any rank one \( N' \).

7. Crystalline Representations

In this section we state the key results which relate \( \text{Mod}^{\text{BD}}_K(O) \) with crystalline representations. We then give a proof of the theorem from the introduction.

7.1. Crystalline Representations and Breuil–Kisin modules. As in \[8\] let \( B_{dR} \) denote Fontaine’s ring of \( p \)-adic periods, and \( B_{\text{crys}} \subset B_{dR} \) the ring of crystalline periods. As in \[9\] a \( p \)-adic representation \( V \) of \( G_K \) is crystalline if

\[
D_{\text{crys}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K}
\]

has \( K_0 \)-dimension equal to \( \dim_{\mathbb{Q}_p} V \). The inclusion \( B_{\text{crys}} \otimes_{K_0} K \subset B_{dR} \) induces an equality \( D_{\text{crys}}(V)_K := D_{\text{crys}}(V) \otimes_{K_0} K = (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} \) which allows us to equip \( D_{\text{crys}}(V)_K \) with the filtration

\[
F^i D_{\text{crys}}(V)_K := (V \otimes_{\mathbb{Q}_p} t^i B_{dR}^+)^{G_K}
\]

Here \( B_{dR}^+ \subset B_{dR} \) is the discrete valuation ring with field of fractions \( B_{dR}^+ \), and \( t \) is any choice of uniformiser.

Theorem 7.1.1 (Kisin). There is a fully faithful functor \( T \mapsto M(T) \) which sends a crystalline \( \mathbb{Z}_p \)-lattice onto an object of \( \text{Mod}^{\text{BK}}_K \) which is free over \( \mathfrak{O} \). The Breuil–Kisin module \( M(T) \) is uniquely determined by the fact that \( T(M(T)) = T \mid_{G_{K_0}} \).

Proof. This is the main result of \[12\]. The formulation we give here is taken from \[4, \text{Theorem 4.4}\].

Notation 7.1.2. A crystalline \( \mathfrak{O} \)-lattice is a \( G_K \)-stable \( \mathfrak{O} \)-lattice inside a continuous representation of \( G_K \) on a finite dimensional \( E \)-vector space which is crystalline when viewed as a \( \mathbb{Q}_p \)-representation. By functoriality \( M \mapsto T(M) \) restricts to a functor from the category of crystalline \( \mathfrak{O} \)-lattices into \( \text{Mod}^{\text{BK}}_K(\mathfrak{O}) \).

Definition 7.1.3. If \( V \) is a crystalline representation on an \( E \)-vector space then \( D_{\text{crys}}(V) \) is a free module over \( K_0 \otimes_{\mathbb{Q}_p} E \) of rank \( \dim_E V \) and so \( D_{\text{crys}}(V)_K \) is a free \( K_0 \otimes_{\mathbb{Q}_p} E \)-module of rank \( \epsilon \dim_E V \). If \( K_0 \subset E \) then as in Construction 4.3.4 there is a decomposition

\[
D_{\text{crys}}(V)_K = \prod_{\tau \in \text{Hom}_p(k,F)} D_{\text{crys}}(V)_{K,\tau}
\]
with each $D_{\text{crys}}(V)_{K,\tau}$ a filtered $E$-vector space of dimension $e \dim_E V$. Define the $	au$-th Hodge–Tate weights of $V$ to be the multiset $\text{HT}_\tau(V)$ which contains $i$ with multiplicity
\[
\dim_E g^i(D_{\text{crys}}(V)_{K,\tau})
\]
With these normalisations the cyclotomic character has $	au$-th Hodge–Tate weights $\{-1, \ldots, -1\} (e \text{ copies of } -1)$.

We need the following two results of Gee–Liu–Savitt. Fix a $\mathbb{Z}_p$-generator $\epsilon = (\epsilon_1, \epsilon_2, \ldots) \in \mathbb{Z}_p(1)$ which we view as an element of $\mathcal{O}_{C^0}$ and let $\mu = [\epsilon] - 1 \in A_{\text{inf}}$.

**Theorem 7.1.4** (Gee–Liu–Savitt). If $T$ is a crystalline $\mathcal{O}$-lattice then there exists a continuous $\varphi$-equivariant $\mathcal{O}$-linear and $A_{\text{inf}}$-semilinear action of $G_K$ on $M(T) \otimes_{E_A} A_{\text{inf}}$ such that for $m \in M(T)$, $(\sigma - 1)(m) = 0$ when $\sigma \in G_{K_\infty}$, and
\[
(\sigma - 1)(m) \in M(T) \otimes_{E_A} [\pi^\varphi]^{-1}(\mu)A_{\text{inf}}
\]
when $\sigma \in G_K$.

**Proof.** A proof of given in the appendix. Note however that this statement is essentially given by [11, Corollary 4.10] when $p > 2$ and the argument in the appendix is an immediate extension of the arguments of [11].

**Lemma 6.2.7** shows the $v^p$-valuation of the image of $[\pi^\varphi]^{-1}(\mu)$ in $\mathcal{O}_{C^0}$ equals $1 + \frac{1}{p-1}$. If $K = K_0$ then $v^p(u) = 1$ and so Theorem 7.1.4 implies if $T$ is a crystalline $\mathcal{O}$-lattice then $M(T) \otimes_{E_A} F$ admits a crystalline $G_K$-action.

**Theorem 7.1.5** (Gee–Liu–Savitt, Wang). Suppose $K = K_0$. If $p = 2$ choose $\pi$ so that $K_\infty \cap K(\mu_{p^\infty}) = K$. If $T$ is a crystalline $\mathcal{O}$-lattice such that $\text{HT}_\tau(V) \subset [0, p]$ where $V = T \otimes_{E_A} E$, then $\overline{M} := M(T) \otimes_{E_A} F \in \text{Mod}^{\text{SD}}_k(O)$ and $\text{Weight}_\tau(\overline{M}) = \text{HT}_\tau(V)$.

**Proof.** When $p > 2$ this follows by reducing the description of $M(T)$ given in [11, Theorem 4.22] modulo any uniformiser of $\mathcal{O}$. The case $p = 2$ follows similarly using [17, Theorem 4.2] (note that the existence of a $\pi$ as stated is proven in [17, Lemma 2.1]).

**7.2. Proof of main theorem.** In this subsection we assume $K = K_0$. We can now give the proof of the theorem in the introduction. Recall that if $\overline{\rho}: G_K \rightarrow \text{GL}_n(\overline{\mathbb{F}}_p)$ is a continuous representation then in Definition 2.1.3 we defined the set $\text{Inert}(\overline{\rho})$.

**Theorem 7.2.1.** Let $K = K_0$. Let $\rho : G_K \rightarrow \text{GL}_n(\mathbb{Z}_p)$ be crystalline and suppose that $\text{HT}_\tau(\rho) = (\lambda_{1,\tau} \leq \ldots \leq \lambda_{n,\tau})$ with $\lambda_{n,\tau} - \lambda_{1,\tau} \leq p$. Then
\[
(\lambda_{\tau}) \in \text{Inert}(\overline{\rho})
\]

**Proof.** Choose a coefficient field $E$ so that $\rho$ is defined over $\mathcal{O}$. Via a straightforward twisting argument we may suppose $\text{HT}_\tau(\rho) \in [0, p]$. Let $M(\rho) \in \text{Mod}^{\text{SD}}_k(O)$ be the associated Breuil–Kisin module. By Theorem 7.1.5, $\overline{M} = M(\rho) \otimes_{E} F \in \text{Mod}^{\text{SD}}_k(O)$ and $\text{HT}_\tau(\rho) = \text{Weight}_\tau(\overline{M})$. Theorem 7.1.4 implies $\overline{M}$ admits a crystalline $G_K$-action.

\[\text{It is important when referencing both [11] and [17] to keep track of differences in normalisa-}
\[\text{tion. In both these references } G_{K_\infty}\text{-representations are attached contravariantly to Breuil–Kisin}
\[\text{modules and their Hodge–Tate weights are normalised to be the negative of ours.}\]
Choose a $G_K$-composition series of $\rho \otimes_\mathcal{O} \mathbb{F}$. Enlarging $E$ if necessary we can suppose that Lemma 2.1.2 holds for each Jordan–Hölder factor. Let $0 = \overline{M}_n \subset \ldots \subset \overline{M}_0 = \overline{M}$ be the corresponding composition series of $\overline{M}$. By Lemma 6.2.4 each of $\overline{M}_i/\overline{M}_{i+1}$ admit crystalline $G_K$-actions compatible with the $G_K$-action on $\overline{M}$. By Proposition 5.4.7 each of $\overline{M}_i/\overline{M}_{i+1} \in \text{Mod}^{\text{SD}}(\mathcal{O})$ and $\text{Weight}_r(\overline{M}) = \bigcup_i \text{Weight}_r(M_i/M_{i+1})$. Therefore each $\overline{M}_i/\overline{M}_{i+1}$ satisfy the conditions of Corollary 6.4.2; it is then clear that $(\lambda_r) \in \text{Inert}(\overline{p})$.

Appendix A.

In this appendix we give a proof of Theorem 7.1.4. The argument is only a slight variation of one appearing in [11, Subsection 4.1]. By functoriality it will suffice to prove the theorem when $T$ is a crystalline $\mathbb{Z}_p$-lattice (i.e. when $\mathcal{O} = \mathbb{Z}_p$).

A.1. Galois actions. As usual $K/K_0$ is a totally ramified extension of degree $e$. Let $T$ by a crystalline representation of $G_K$ on a finite free $\mathbb{Z}_p$-module and let $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Additionally set $D = D_{\text{crys}}(V)$ and $M = M(T)$ from Theorem 7.1.1.

Notation A.1.1. Let $\mathcal{O}^{\text{rig}}$ be the subring of $K_0[[u]]$ consisting of power series which converge on the open unit disk. The embedding $\mathcal{O} \to A_{\text{inf}}$ from Notation 4.1.2 extends to a $\varphi$-equivariant embedding $\mathcal{O}^{\text{rig}} \to B_{\text{crys}}^+$. Inside $\mathcal{O}^{\text{rig}}$ the product

$$\lambda = \prod_{n=0}^{\infty} \varphi^n(\frac{E(u)}{E(0)})$$

converges and the inclusion $\mathcal{O}^{\text{rig}} \to B_{\text{crys}}^+$ sends $\varphi(\lambda)$ onto a unit so we obtain an inclusion $\mathcal{O}^{\text{rig}}[\frac{1}{\varphi(\lambda)}] \to B_{\text{crys}}^+$.

Examining the construction of $M$ given in [12] it follows that there are $\varphi$-equivariant identifications

(A.1.2) $$M^\varphi \otimes_{\mathcal{O}} \mathcal{O}^{\text{rig}}[\frac{1}{\varphi(\lambda)}] \cong D \otimes_{K_0} \mathcal{O}^{\text{rig}}[\frac{1}{\varphi(\lambda)}]$$

(see [12, Lemma 1.2.6]). This identification is also $G_{K_{\text{max}}}$-equivariant (for the trivial $G_{K_{\text{max}}}$-action on both sides). Via this identification we obtain functorial $\varphi, G_{K_{\text{max}}}$-equivariant identifications

(A.1.3) $$M \otimes_{\mathcal{O}} B_{\text{crys}} \cong D \otimes_{K_0} B_{\text{crys}} \cong T \otimes_{\mathbb{Z}_p} B_{\text{crys}}$$

for which the right $\cong$ is in fact $G_K$-equivariant. Via these identifications $M \otimes_{\mathcal{O}} B_{\text{crys}}$ is equipped with a natural $G_K$-action which semi-linearly extends the trivial action of $G_{K_{\text{max}}}$ on $M$, and commutes with $\varphi$. To prove Theorem 7.1.4 we show this $G_K$-action stabilizes $M \otimes_{\mathcal{O}} A_{\text{inf}}$, acts continuously, and is such that

$$(\sigma - 1)(m) \in M \otimes_{\mathcal{O}} [\tau^\varphi] \varphi^{-1}(\mu) A_{\text{inf}}$$

for $m \in M$ and $\sigma \in G_{K_{\text{max}}}$. Since the $G_K$-action commutes with $\varphi$ this last condition is equivalent to asking that $(\sigma - 1)(m) \in M^\varphi \otimes_{\mathcal{O}} [\tau^\varphi] \varphi^{-1}(\mu) A_{\text{inf}}$ when $m \in \varphi(M)$.

A.2. Period rings. Recall there is a homomorphism $\mathbb{Z}_p(1) \to B_{\text{dR}}$ sending $\zeta \mapsto \log(|\zeta|) := \sum_{n \geq 1} (-1)^{n+1} \frac{|\zeta|}{n} \in B_{\text{dR}}^+$. Let $\epsilon$ be the $\mathbb{Z}_p$-generator of $\mathbb{Z}_p(1)$ fixed in the sentence before Theorem 7.1.4, and set $t = \log(|\epsilon|)$.

For topological reasons it is convenient to use the ring $B_{\text{max}}^+$ and $B_{\text{max}}$ introduced in [7], in place of $B_{\text{crys}}^+$ and $B_{\text{crys}}$ (in particular Lemma A.2.2 below does not hold with $B_{\text{max}}^+$ replaced by $B_{\text{crys}}^+$).
We must show that $x_n u^n \in B_{dr}$ with $x_n \in A_{inf}$ a sequence converging $p$-adically to zero. This is a $G_K$-stable subring of $B_{dr}$ containing $A_{inf}$. The $\varphi$ on $A_{inf}$ extends uniquely to a $\varphi$ on $A_{max}$. Also $A_{max}$ is $p$-adically complete. Set $B_{max}^+ = A_{max}[\frac{1}{p}]$ and, noting that $t \in A_{max}$, set $B_{max} = B_{max}^+[\frac{1}{t}]$.

Recall $A_{cr} \subset B_{dr}$ can be defined as the subring of elements $\sum_{n \geq 0} x_n u^n$ with $x_n \in A_{inf}$ a sequence converging $p$-adically to zero. As $v_p(n!) \leq n$ and $n \leq v_p((pn)!)$ we see $\varphi(A_{max}) \subset A_{cr} \subset A_{max}$, and so $\varphi(B_{max}^+) \subset B_{cr}^* \subset B_{max}^+$ and $\varphi(B_{max}) \subset B_{cr} \subset B_{max}$.

**Lemma A.2.2.** Equip $B_{max}^+$ with the topology described by asserting that $(p^n A_{max})_{n \geq 0}$ forms a basis of open neighbourhoods of $0$. Then $B_{max}^+$ is complete and any principal ideal $a B_{max}^+ \subset B_{max}$ is closed.

**Proof.** Completeness is immediate since $A_{max}$ is $p$-adically complete. To check $a B_{max}^+$ is closed consider a sequence $b_i \in a B_{max}^+$ converging to $b \in B_{max}^+$. We must show $b \in a B_{max}^+$. Since $B_{max}^+$ is a complete domain it suffices to show $\frac{b}{a}$ converges in $B_{max}^+$. This follows from [7, Proposition III.2.1] which asserts that if $||x|| = \inf_{n \geq 0} \sum_{n \geq 0} x_n u^n \in A_{max}$) $p^n$ then $p^{-1}||x|| ||y|| \leq ||xy||$. Hence $||\frac{b}{a} - \frac{b_i}{a}|| \leq \frac{1}{||a||} ||b_i - b_j||$, and so $\frac{b}{a}$ converges in $B_{max}^+$. □

**A.3. Monodromy and Galois.** We can tensor (A.1.2) with $B_{max}^+$ to obtain identifications

(A.3.1) \[ M^\varphi \otimes E B_{max}^+ \cong D \otimes K_0 B_{max}^+ \]

which are $\varphi, G_K$-equivariant. With $B_{max}^+$ topologised as in Lemma A.2.2 this $G_K$-action is continuous, since the $G_K$-action on $B_{max}^+$ is continuous.

**Construction A.3.2.** Inverting $\lambda$ in (A.1.2) allows us to equip $M \otimes E \mathcal{O}_{rig}[\frac{1}{\lambda}]$ with a differential operator $N$ over $\partial = -u \frac{\partial}{\partial u}$ by asserting that $N(d) = 0$ for $d \in D$. Note that $N \varphi = \varphi N$ since the same relation holds for $\partial$. Also, because $\partial(\mathcal{O}_{rig}[\frac{1}{\lambda}]) \subset u \mathcal{O}_{rig}[\frac{1}{\lambda}]$, we have that $N(M \otimes E \mathcal{O}_{rig}[\frac{1}{\lambda}]) \subset M \otimes E u \mathcal{O}_{rig}[\frac{1}{\lambda}]$ for every $m \in M \otimes E \mathcal{O}_{rig}[\frac{1}{\lambda}]$.

For $\sigma \in G_K$ let $\epsilon(\sigma) \in \mathbb{Z}_p(1)$ be the cocycle defined by $\epsilon(\sigma)_n = \sigma(\pi^{1/p^n})/\pi^{1/p^n}$. Then we have

**Lemma A.3.3.** The action of $G_K$ on $M^\varphi \otimes E B_{max}^+$ is given by

\[
\sigma(m \otimes a) = \sum_{n \geq 0} N^n(m) \otimes \sigma(a)(-\log(\epsilon(\sigma)))^n/n!
\]

where $m \in M^\varphi \otimes E \mathcal{O}_{rig}[\frac{1}{\varphi(\lambda)}]$ and $a \in B_{max}^+$. \[\sum_{n \geq 0} (-\log(\epsilon(\sigma)))^n/n! \partial^n(f)\]
converges in $B^{\pm}_{\text{max}}$ to $\sigma(f)$. It suffices to consider $f = u^{i}$. Then $\sigma(f) = [\epsilon(\sigma)]^{i}u^{i}$. On the other hand, since $\vartheta^{\nu}(u^{i}) = (-i)^{\nu}u^{i}$, (A.3.4) equals $\exp\{\log([\epsilon(\sigma)]^{i})\}f$. If this sum converges it will do so to $[\epsilon(\sigma)]^{i}u^{i}$ which proves the lemma.

To show convergence it suffices to show $\frac{\log([\epsilon(\sigma)])}{n!}$ lies in $A_{\text{max}}$ and in this ring converges $p$-adically to zero. Note that $\log([\epsilon(\sigma)]) = \alpha t$ for some $\alpha \in \mathbb{Z}_{p}$. The proof of [7, Lemme III.3.9] shows that $t \in pA_{\text{max}}$ if $p > 2$ and $t \in p^{2}A_{\text{max}}$ if $p = 2$. Thus convergence of $\frac{\log([\epsilon(\sigma)])}{n!}$ follows because $\frac{\alpha t}{n!} \in \mathbb{Z}_{p}$ converges $p$-adically to zero when $p > 2$, and $\frac{\alpha t}{n!}$ converges to zero when $p = 2$.

The following lemma completes our proof of Theorem 7.1.4.

**Lemma A.3.5.** The module $M^{\varphi} \otimes A_{\text{inf}}$ is stable under the action of $G_{K}$. Moreover, if $m \in \varphi(M)$ and $\sigma \in G_{K}$ then

$$(\sigma - 1)(m) \in M^{\varphi} \otimes A_{\text{inf}}[\frac{1}{n}]$$

**Proof.** Since $\varphi(M) \otimes A_{\text{inf}} = M^{\varphi} \otimes A_{\text{inf}}$ it suffices to prove the second statement. Using [4, Lemma 4.26] applied to the Breuil–Kisin–Fargues module $M^{\varphi} \otimes A_{\text{inf}}$ (note the embedding $\varphi \rightarrow A_{\text{inf}}$ in loc. cit.) is obtained from our embedding by composing with $\varphi$ which is why $M^{\varphi} \otimes A_{\text{inf}}$, rather than $M \otimes A_{\text{inf}}$ is a Breuil–Kisin–Fargues module) shows that as sub-modules of (A.1.3) we have

$$M^{\varphi} \otimes A_{\text{inf}}[\frac{1}{n}] \cong T \otimes_{\mathbb{Z}_{p}} A_{\text{inf}}[\frac{1}{n}]$$

Therefore $(\sigma - 1)(m) \in M^{\varphi} \otimes A_{\text{inf}}[\frac{1}{n}]$ for $m \in \varphi(M)$.

On the other hand, we claim that as a consequence of Lemma A.3.3 we have $(\sigma - 1)(m) \in M^{\varphi} \otimes [\pi^{n}]^{p}B^{+}_{\text{max}}$. To see this recall from the previous lemma that

$$(\sigma - 1)(m) = \sum_{n \geq 1} N^{n}(m) \otimes \frac{(-\log([\epsilon(\sigma)]))}{n!}$$

Since $m \in \varphi(M)$ and $N\varphi = \varphi N$ we have $N^{n}(m) \in \varphi(N(M \otimes O^{\text{rig}}[\frac{1}{n}]))) \subset \varphi(uM \otimes O^{\text{rig}}[\frac{1}{n}]) \subset M^{\varphi} \otimes [\pi^{n}]^{p}B^{+}_{\text{max}}$ for $n \geq 1$. By [7, Lemme III.3.9] we also have $\log([\epsilon(\sigma)]) \in A_{\text{max}} = \mu A_{\text{max}}$. Therefore each term of (A.3.6) lies in $M^{\varphi} \otimes [\pi^{n}]^{p}B^{+}_{\text{max}}$. Lemma A.2.2 then implies (A.3.6) converges in $M^{\varphi} \otimes [\pi^{n}]^{p}B^{+}_{\text{max}}$, which verifies the claim.

To complete the proof it suffices to show that $A_{\text{inf}}[\frac{1}{n}] \cap [\pi^{n}]^{p}B^{+}_{\text{max}} = [\pi^{n}]^{p}\mu A_{\text{inf}}$. This follows from the next two facts. The first fact is that if $a \in A_{\text{inf}} \cap [\pi^{n}]^{p}B^{+}_{\text{max}}$ then $a \in [\pi^{n}]A_{\text{inf}}$. This is proven with $B^{+}_{\text{max}}$ replaced by $B^{+}_{\text{crys}}$ in [13, Lemma 3.2.2]. Using that $\varphi(B^{+}_{\text{max}}) \subset B^{+}_{\text{crys}}$ we deduce the same applies for $B^{+}_{\text{max}}$. The second fact is that if $a \in A_{\text{inf}} \cap \mu^{n}B^{+}_{\text{max}}$ then $a \in \mu^{n}A_{\text{inf}}$. It suffices to prove this when $n = 1$. The homomorphism $\theta : A_{\text{inf}} \rightarrow \mathcal{O}_{C}$ given by $\sum x_{i}p^{i} \mapsto \sum x_{i}^{p}p^{i}$ extends to $\theta : B^{+}_{\text{max}} \rightarrow C$, and since $\theta(\varphi^{n}(\mu)) = 0$ we must have $\theta(\varphi^{n}(a)) = 0$ for all $n \geq 1$. By [8, Proposition 5.1.3] the ideal $\{a \in A_{\text{inf}} | \varphi^{n}(a) \in \ker \theta \text{ for all } n \geq 1\}$ equals $\mu A_{\text{inf}}$. This proves the second fact, and so the lemma.

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E-mail address: robinbartlett18@mpim-bonn.mpg.de