ON BLOW-UP “TWISTORS”
FOR THE NAVIER–STOKES EQUATIONS IN $\mathbb{R}^3$:
A VIEW FROM REACTION-DIFFUSION THEORY

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ABSTRACT. Formation of blow-up singularities for the Navier–Stokes equations (NSEs)
\[ u_t + (u \cdot \nabla)u = -\nabla p + \Delta u, \quad \text{div } u = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+, \]
with bounded data $u_0$ is discussed. Using natural links with blow-up theory for nonlinear reaction-diffusion PDEs, some possibilities to construct special self-similar and other related solutions that are characterized by blow-up swirl with the angular speed near the blow-up time (this represents simplest $\omega$-limits of rescaled orbits as periodic ones)
\[ \varphi(t) \sim -\sigma \ln(T - t) \quad \Rightarrow \quad \dot{\varphi}(t) \sim \frac{\sigma}{T - t} \rightarrow \infty \quad \text{as } t \rightarrow T^- \quad (\sigma \neq 0). \]
This is done in cylindrical polar coordinates $\{r, \varphi, z\}$ in $\mathbb{R}^3$, using the restriction of the NSEs to the linear subspace $W_2 = \text{Span}\{1, z\}$. Similarly, blow-up twistors with axis precessions in the spherical geometry $\{r, \theta, \varphi\}$ are introduced.

It is shown that other blow-up patterns (a “screwing in tornado”) may correspond to a slow “centre-stable manifold-like drift” about Slezkin–Landau singular or other equilibria of the NSEs. Some approaches to blow-up singularities can be applied to 3D Euler’s equations and to well-posed Burnett equations in 7D (i.e., the NSEs with $\Delta \mapsto -\Delta^2$). Though most of blow-up scenarios were not justified even at a qualitative level, the author hopes that the proposed approaches to families of blow-up and other patterns, including those with blow-up swirl, will give some extra insight into the micro-scale “turbulent” structure of the NSEs.

The discussion of possible types of blow-up patterns for the NSEs is going in conjunction with some other classic nonlinear PDEs of mathematical physics.

1. Introduction: Navier–Stokes equations and blow-up
1.1. Two faces of the open problem via blow-up formulation: first discussion around self-similar blow-up. It is well-understood (see e.g., [148, Ch. 5]) that the fundamental open problem of fluid mechanics\footnote{The Millennium Prize Problem for the Clay Institute; see Fefferman [58].} and PDE theory on global existence or
nonexistence of bounded smooth solutions of the Navier–Stokes equations (the NSEs),

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\nabla p + \Delta u, \\
\text{div } u &= 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \quad (u = (u, v, w)),
\end{align*}
\]

with arbitrary bounded divergence-free \(L^2\)-data \(u_0\), is two-fold:

From a standard evolution PDE and blow-up point of view, there exist two possibility: the positive answer, i.e., existence of a global bounded solution (then necessarily it is unique by the existing local semigroup of bounded smooth solutions) is equivalent to

\[
\exists \text{ global bounded solution } \neq \exists \text{ finite-time blow-up at any } T > 0.
\]

On the other hand, the negative answer, i.e., nonexistence in general of a global bounded solution, is equivalent to the following:

\[
\not\exists \text{ global bounded solution } = \exists \text{ a finite-time blow-up pattern},
\]

which corresponds to bounded \(L^2\)-initial data. In both cases, we mean blow-up in \(L^\infty(\mathbb{R}^N)\) of bounded smooth solutions at the first blow-up time \(t = T\).

It seems obvious that both scenarios (1.2) and (1.3) assume a detailed study of possible blow-up behaviour of solutions as \(t \to T^-\), and this cannot be avoided and represents a clear alternative face of this fundamental open problem of PDE theory. Of course, (1.2) would be achieved without any blow-up study provided that a new technique (e.g., a new conservation law and/or a monotonicity formulae) could be invented for the NSEs. However, the long history of this Millennium Open Problem suggests that this is expected to be extremely difficult. Anyway, this can happen since the Navier–Stokes equations can indeed inherit from their universal physical nature (a system comprising Newton’s Second Law, the continuity, and a basic viscosity) some extra still hidden and unknown continuous, discrete, or other symmetries/monotonicity/symplectic/dissipative, etc. features.

It is definite that, during a few last years, the direction of the attacking this open problem was clearly partially changed and a seriously increasing number of papers along the evolution blow-up scenarios (1.2) and (1.3) were published. In particular, a rather complete negative answer was achieved supporting somehow (1.2) in the following way:

\[
\text{for } (1.1), \text{ a standard self-similar blow-up as } t \to T^- \text{ is impossible.}
\]

In a most general evolution setting, this is due to the work by Hou and Li [103], based on the crucial nonexistence result in Nečas–Růžička–Šverák [162] proved via the Maximum Principle (the MP). The ban (1.4) was proven in [103] by solid semigroup theory that is adequate to classic evolution approaches to nonlinear PDEs with blow-up (we present a spectral discussion of (1.4) in Section 2). In fact, it was proved that any convergence in \(L^p(\mathbb{R}^3)\), with \(p > 3\), as \(t \to T^-\) to a rescaled self-similar profile means that the solution remains bounded at \(t = T\) (so it is a removable singularity). This approach does not

\[\footnotesize^2\text{Claude Louis Marie Henri Navier, 1785-1836, and George Gabriel Stokes, 1819-1903.}\]
exhaust all the possibilities of, say, almost self-similar or other (Type II) blow-up, but, nevertheless, is a new convincing fact on possible singularities in the fluid model (1.1).

The negative result (1.4) completes a long remarkable history of the study of similarity blow-up singularities for the NSEs that was initiated by J. Leray in 1933–34 [136, 137], who actually posed a deeper problem on both backward and forward phenomena:

\[
\text{Leray's blow-up scenario:} \quad \text{self-similar blow-up as } t \to T^- (t < T) \quad \text{and similarity collapse of singularities as } t \to T^+ (t > T);
\]

see his precise statements and a discussion on these principal issues in Section 2.

1.2. Similar open regularity problems for \( N = 4 \) and for the well-posed Burnett equations. It is important to recognize that, unlike its great sounding and well-established reputation even among non-experts, the NSEs Millennium problem is just a “remnant” of a general fundamental difficulty of PDE theory in the twenty first century, which seems cannot principally be understood by classic modern techniques, to say nothing about a rigorous proof. Therefore, most general new concepts covering a wider PDE area are in great demand.

To justify that, one does not need to address essentially other classes of nonlinear PDEs and/or systems, and can just slightly generalize the classic NSEs. First, let us mention that a similar and not less difficult (but indeed more exotic) open problem on existence/nonexistence of global classical solutions persists for the Navier–Stokes equations in dimension \( N = 4 \); see Scheffer (1978) [188] and recent developments in [48].

Second, as a next neighbouring example of a similar nature associated with applications, consider the well-posed Burnett equations

\[
\begin{align*}
\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p - \Delta^2 \mathbf{u}, & \text{div } \mathbf{u} &= 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+,
\end{align*}
\]

where, in comparison with the standard model (1.1), the Laplacian squared with the correct sign \(-\Delta^2\) on the right-hand side is posed. Note that here one deals with the linear solenoidal bi-harmonic flow induced by the operator \( D_t + \Delta^2 \), which is parabolic but assumes no any order-preserving and other properties of the semigroup \( e^{-\Delta^2 t} \) (which are somehow naturally partially inherited from the positive Gaussian for \( D_t - \Delta \) in the NSEs (1.1)). Surely, both problems, the NSEs (1.1) in \( \mathbb{R}^3 \) and the Burnett equations (1.6) in \( \mathbb{R}^7 \) look rather ridiculous and seem cannot have a real application. But for PDE theory these can be considered as some key and canonical representatives exhibiting the necessary principal difficulties (or course, there are other examples like that, e.g., supercritical nonlinear Schrödinger or Ginzburg–Landau equations to be used and discussed as well).

The more complicated model (1.6) appears on the basis of Grad’s method in Chapman–Enskog expansions for hydrodynamics. In particular, equations (1.6) were studied in [74] as a particular case of Kuramoto–Sivashinsky-type PDEs (see on a derivation therein also), where it was shown that, for the smooth orbits, the following embedding holds:

\[
L^{\frac{4}{3}}(\mathbb{R}^N) \quad \implies \quad L^{\infty}(\mathbb{R}^N) \quad (N > 6).
\]
This is an analogy for (1.1), where the idea of a similar transition $L^3(\mathbb{R}^3) \Rightarrow L^\infty(\mathbb{R}^3)$ goes back to Leray (1934); see a survey in the next section. Fixing

(1.8) 

$N = 7$ (or any $N \geq 7$),

we arrive at a similar open problem on existence/nonexistence of global smooth solutions. For $N \leq 6$ the proof is easier. In fact, this is the analogy of $N = 2$ for (1.1), where existence-uniqueness in the Cauchy problem is due to Leray (1933) [135] (extended by Hopf in 1951 [101] and Ladyzhenskaya (1958) for the IBVPs [128] [129]. Obviously, with such a huge growth of the order of PDEs involved and the dimension $N$ of the Euclidean space for the Laplacian and the convective term (to say nothing about the non order-preserving properties of the higher-order semigroup $e^{-\Delta^l}$), a detailed and convincing analysis of (1.2) and (1.3), together with a complete description of blow-up patterns, seems entirely illusive and non-achievable.

1.3. On universality of the open $L^p \Rightarrow L^\infty$ problem in PDE theory. As we have just mentioned above, roughly speaking, the Millennium Prize Problem, posed specially for the NSEs, is, in a loose sense, “non-unique”, since similar open regularity problems (or not that lighter significance) occur for many evolution PDEs of various types. We list a few of them, where the difficult open mathematical aspects of global existence and/or blow-up are associated with the following factors:

(i) supercritical Sobolev parameter range of the principal operator (hence, standard or very enhanced embedding-interpolation techniques fails), and, in fact, as a corollary,

(ii) multi-dimensional space $x \in \mathbb{R}^N$, with $N \geq 3$, at least (this leaves a lot of room for constructing various $L^\infty$ blow-up patterns via self-similarity, angular swirl, axis precessions, linearization, matching, etc.).

We now list those PDEs, where we give a few recent basic references to feel the subject.

(I) Supercritical defocusing nonlinear Schrödinger equation (NLSE) (see [150] [215])

(1.9) 

$- i u_t = \Delta u - |u|^{p-1}u, \quad \text{with} \quad p > p_S(2) = \frac{N+2}{N-2} \quad (N \geq 3)$;

(II) $2m$th-order supercritical semilinear heat equation with absorption ($m = 1$ is covered by the MP; see [80] and [33], where the result in § 4 for $p > p_S(2m)$ applies to small solutions only):

(1.10) 

$u_t = -(-\Delta)^m u - |u|^{p-1}u, \quad \text{with} \quad p > p_S(2m) = \frac{N+2m}{N-2m} \quad (N > 2m, \ m \geq 2)$;

(III) The semilinear supercritical wave equations (see [106] [221], as most recent guides)

(1.11) 

$u_{tt} = \Delta u - |u|^{p-1}u, \quad \text{with} \quad p > p_S(2) = \frac{N+2}{N-2} \quad (N \geq 3)$.

One can add to those “supercritical” PDEs some others of a different structure such as the Kuramoto–Sivashinsky equations for $l = 1, 2, \ldots$ [74]

(1.12) 

$u_t = -(-\Delta)^l u + (-\Delta)^l u + \frac{1}{p} \sum_{k} d_k D_k(|u|^p), \quad |d| = 1, \quad p > p_0 = 1 + \frac{2(l+1)}{N}$.

Here, $p_0$ is not the Sobolev critical exponent, though precisely for $p > p_0$, $L^2 \neq L^\infty$ by blow-up scaling, [74] § 5. On the other hand, a more exotic applied models exhibit similar
fundamental difficulties such as the following nonlinear dispersion equation (see [69, 75] for references and some details)

\[(1.13)\quad u_t = -D_x[(\Delta)^m u] - D_x(|u|^{p-1}u), \quad \text{with} \quad p > ps(2m) = \frac{N+2m}{N-2m}.
\]

In view of the conservation properties for the models \((1.12)\) and \((1.13)\), these, though being local, can be more adequate to the nonlinear NSEs \((1.1)\), than the others above.

In most of the cases, the operator on the right-hand sides satisfying for \(u \in C_0^\infty(\mathbb{R}^N)\)

\[(1.14)\quad A(u) = - (\Delta)^m u - |u|^{p-1}u \implies \langle A(u), u \rangle = - \int |D^mu|^2 - \int |u|^{p+1} \leq 0,
\]

is indeed coercive and monotone in the metric of \(L^2(\mathbb{R}^N)\), which always helps for global existence-uniqueness of sufficiently smooth solutions of these evolution PDEs. For the NLS \((1.9)\), this gives a stronger conservation laws than for the focusing equation with the “source-like” term \(|u|^{p-1}u\). Evidently, replacing \(\Delta\) in \((1.9)\) and \((1.11)\) by \((-\Delta)^m, m \geq 2\) moves the supercritical range to that in \((1.10)\). On the other hand, introducing quasilinear differential operators \(- (\Delta)^m |u|^{\sigma} u\) with \(\sigma > 0\) moves the critical exponent to \(ps(2m, \sigma) = (\sigma+1)\frac{N+2m}{N-2m}\). Similar supercritical PDEs can contain \(m\)th-order \(p\)-Laplacian operators, such as the one for \(m = 2, \sigma > 0\),

\[(1.15)\quad A(u) = - \Delta(|\Delta u|^{\sigma} u) - |u|^{p-1}u, \quad \langle A(u), u \rangle = - \int |\Delta u|^{\sigma+2} - \int |u|^{p+1} \leq 0.
\]

However, the lack of embedding-interpolation techniques to get \(L^\infty\)-bounds, which can be expressed as the lack of compact Sobolev embedding of the corresponding spaces for bounded domains \(\Omega \subset \mathbb{R}^N\) (this analogy is not straightforward and is used as a consistent illustration only)

\[(1.16)\quad H^m(\Omega) \not\subset L^{p+1}(\Omega) \quad \text{for} \quad p > ps(2m),
\]

actually presents the core of the problem: it is not clear how and when bounded solutions can attain in a finite blow-up time a “singular blow-up component” in \(L^\infty\). For the operator in \((1.15)\), a similar supercritical demand reads

\[(1.17)\quad W_{\sigma+2}^{2}(\Omega) \not\subset L^{p+1}(\Omega) \quad \text{for} \quad p > ps(4, \sigma) = \frac{(\sigma+1)N+2(\sigma+2)}{N-2(\sigma+2)} , \quad N > 2(\sigma+2).
\]

In the given supercritical Sobolev ranges, finite mass/energy blow-up patterns for \((1.9)\)–\((1.13)\) are unknown, as well as global existence of arbitrary (non-small) solutions.

It is curious that for the NSEs with the same absorption mechanism as above,

\[(1.18)\quad u_t + (u \cdot \nabla)u = - \nabla p + \Delta u - |u|^{p-1}u, \quad \text{div} u = 0 \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+,
\]

by the same reasons and similar to \((1.10)\), the global existence of smooth solutions is guaranteed [27] in the subcritical Sobolev range only: for

\[(1.19)\quad p \leq 5 = \frac{N+2}{N-2}|_{N=3} \quad \text{(and} \quad p \geq \frac{7}{2} \text{by another natural reason)}.
\]

We thus claim that, even for the PDEs with local nonlinearities \((1.9)\)–\((1.11)\) (and similar higher-order others), the study of the admissible types of possible blow-up patterns can represent an important and constructive problem, with the results that can be key also for the non-local parabolic flows such as \((1.1)\), \((1.6)\), \((1.18)\), etc. Moreover, it seems reasonable first to clarify the blow-up origins in some of looking similar and simpler (hopefully, yes,
since (1.1) is both nonlocal and vector-valued unlike the others) local supercritical PDEs, and next to extend the approaches to the non-local NSEs (1.1); though, obviously, the former ones are not that attractive and, unfortunately, are not related to “millennium” issues (however, many PDE experts very well recognize how important these are for general PDE theory).

1.4. Main synthetic goal of the paper and on the proposed style of research. For the author, who for almost thirty years dealt with blow-up singularities in various nonlinear PDEs and mainly in reaction-diffusion systems of different orders, the appearance of the paper [103] was a crucial sign. Actually, this announced that, if blow-up singularities in the model (1.1) are possible, these must be of interesting, complicated enough, non-similarity, and non-symmetric nature, though anyway a long story of various unsuccessful attempts to reconstruct those blow-up singularities (if any) suggested that this is expected to be a difficult affair. This convinced us to look now carefully at such a complicated model at the Navier–Stokes equations (a system of four PDEs with \((x,t) \in \mathbb{R}^3 \times \mathbb{R}_+\)) from the point of view of standard blow-up theory. Though the author, who was dealing with uncomparably simpler PDEs, which however generated a number of still open problems (further comments on this matter will be given later), naturally expected that a definite, to say nothing about a rigorous, singularity construction might not be convincingly done just in view of a general complexity of the model.

Let us specify the actual main goal of this essay. Of course, clearly, some part of the author general motivation is associated with the Millenium (1.2)–(1.3) Problem. On the other hand, there exists the second half of the motivation, which the author honestly regards as not as less important (and even more valuable for general PDE theory).

More clearly, in his opinion, performing a partial or most complete classification of singularities for (1.1) and other related nonlinear models of practical interest along the lines of blow-up theory becomes nowadays a fundamental mathematical direction. Overall, this is about a description of possible complicated micro configurations that an evolution system can create, or,

\[(1.20) \quad \text{Goal: } \begin{array}{c}
\text{for NSEs: describe all “turbulent” incompressible fluid} \\
\text{configurations on possibly minimal micro-}(x,t)\text{-scales,}
\end{array} \]

regardless whether these are developed in finite-time blow-up or non blow-up manners. Actually, (1.20) does not assume that the solutions must be of a finite kinetic energy; it is just necessary that the behaviour of solutions at infinity (as \(x \to \infty\)) does not play any essential role for formation of patterns. Otherwise, this would mean posing various “boundary conditions” at infinity that can immensely increase the variety of admissible patterns. Describing some approaches to (1.20) is our actual MAIN SYNTHETIC GOAL.

\[3\]The first preprint of this paper, which was available to the author, dated 24th August 2007.
Concerning the style of a structural characterization of our concepts and ideas, we are oriented to perform our research in a such unified manner that, at least, formally,

(1.21) **Style**: approaches cover both NSEs (1.1) and the Burnett equations (1.6).

This gives a room for possible future author’s apology in the sense that

(1.22) if (1.2) were proved, the goal would be the 7D Burnett equations (1.6)

(or other complicated nonlinear evolution PDEs with similar principal difficulties of $L^\infty$-bounds). Of course, (1.6) will be studied less, but we will indeed seriously present basics of necessary related linear and nonlinear operator theory covering the case of the bi-harmonic operator $-\Delta^2 u$ or the $2m$th-order one $-(-\Delta)^m u$ for any $m \geq 2$. It is clear that any rigorous justifying the conclusions for (1.6) would be incredibly difficult or even will be never achieved, but anyway this cannot prevent us from performing the research: *the necessity of a deep mathematical study of various nonlinear PDEs without any hope of strict formulations of many results is already clearly an inevitable feature of modern PDE theory associated with several types of higher-order singular or degenerate equations*.

1.5. **Layout of the paper.** The main steps of our analysis are as follows:

- **Blow-up survey (Section 2).** We begin with a necessary short survey devoted to classic and recent blow-up results for the Navier–Stokes and some for Euler equations.

- **First application of blow-up scaling: Hermitian structure of zero sets (Section 3).** This is devoted to a first application of blow-up scaling to the NSEs. Namely, we show that the local structure of multiple zeros of regular solutions can be governed by special vector solenoidal *Hermite polynomials* as eigenfunctions of the adjoint rescaled linear operator. We claim that this study is a natural and unavoidable step for further deeper discussion and classification of all types of micro-scale, single-point configurations that can be generated by the evolution system in finite time. These sets of singular patterns are assumed to include also possible $L^\infty$-blow-up patterns, which are essentially nonlinear and, for revealing of those, demands complicated matching procedures, where the Hermitian polynomial space-time structures will be key.

- **Third blow-up scaling: singularities in NSEs and EEs (Section 4).** This is another version of blow-up scaling showing that any blow-up in the NSEs must be supported by a bounded “NS-entropy” solution of the EEs, which is defined on larger space-time subsets.

- **Blow-up twistor mechanism (Section 5).** According to typical blow-up results, which are well-known for a wide audience of mathematicians working with nonlinear parabolic, hyperbolic, dispersion, Boussinesq, and other evolution PDEs, it is clear that, regardless a pretty strong negative result in (1.4), the story of the open problem is far away not only from being solved but even reasonable understood. We intend to show some extra ways how the Navier–Stokes equations can create complicated blow-up patterns. The main difficulty is indeed to detect a suitable and adequate for fluid vortex models mechanism

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4"The main goal of a mathematician is not proving a theorem, but an effective investigation of the problem...,” A.N. Kolmogorov, 1980s.
of blow-up swirl (rotation) leading to an essential reduction of the system (1.1), which can be at least discussed to clarify the way of construction of such blow-up patterns that have been called in Section 5 blow-up twistors. Namely, we show that, in cylindrical polar coordinates \( \{r, \varphi, z\} \), NSEs (1.1) allow a consistent restriction to the 2D linear subspace

\[
W_2 = \text{Span}\{1, z\}
\]

and moreover this subspace is partially invariant under the nonlinear operators involved (this is understood in the sense of mappings, [76 Ch. 3, 7]). Such blow-up patterns belong to \( W_2 \) and lead to involved nonlinear systems, which are very difficult to study, but, anyway, this admits the natural similarity logarithmic travelling wave (TW) mechanism of blow-up vortex swirl about the \( z \)-axis, with the angular dependence

\[
(1.24) \quad \varphi(t) = -\sigma \ln(T - t) \implies \dot{\varphi}(\tau) = \frac{\sigma}{T - t} \to \infty \text{ as } t \to T^- \quad \text{where } \sigma \neq 0.
\]

In other words, we pose an extra logTW angular dependence in cylindrical coordinates

\[
(1.25) \quad \varphi = \mu - \sigma \ln(T - t), \quad \text{where } \mu \in (0, 2\pi) \text{ is the rescaled angle,
}\]

which inserts into the system a new “nonlinear eigenvalue” \( \sigma \in \mathbb{R} \). As usual, assuming such an extra evolution freedom in this swirling dynamical system extends the overall possibility to get suitable blow-up patterns, possibly even in the self-similar form.\(^5\) Inserting the blow-up swirl (1.25) into (1.5) revives this Leray scenario, since now, in the rescaled variables, we observe not a stabilization to a point (already prohibited by the MP), but convergence to a periodic orbit. Roughly speaking, this falls into the scope of a much more difficult dynamics corresponding to the “Poincare–Bendixson Theorem” (existence of blow-up), or to “Dulac’s Negative Criterion” (nonexistence), as in classic ODE theory [176], but the current PDEs are infinite-dimensional and nonlocal. But this is not the end of the story even if a nonexistence Dulac-like result would have been proved: the rescaled orbits may converge to various quasi-periodic orbits with arbitrarily large number of fundamental frequencies (some details to be also discussed).

Meantime, we state this our observation as: the NSEs

\[
(1.26) \quad \text{on } W_2 \text{ admit a (self-similar) mechanism of blow-up swirl at a point.}
\]

We are not able to study somehow rigorously the evolution dynamical systems that occur and propose a few ideas how various blow-up twistors can occur via evolution close to certain invariant manifolds associated with the similarity flow. Though the fluid model (1.1) naturally supports formation and evolution of vortices (“von Kármán’s streets”), it should be noted that, mathematically speaking, taking into account the rotational torsion-like

\(^5\) Recently, the blow-up \( \ln(T - t) \) factors are more boldly appear for the NSEs; see e.g., [36 § 1.2].

\(^6\) Introducing “twistors”, we mean a specific self-similar angular blow-up mechanism (1.25) and around with further perturbations and approximate similarity features to be described shortly. The fact that blow-up in the NSEs could be connected with a “tornado-type” structures is well-recognized. Q.v. Sinai [193 p. 730]: “… A negative answer [to existence of strong solutions of the 3D NSEs] could be connected with solutions which develop singularities in finite time like a tornado-type solution where infinite vorticity appears at some particular points in time and space.” (Underlying is author’s.)
“spiral wave” mechanism (1.26) and other related axis and vertex precession phenomena includes into the nonstationary rescaled system extra velocity and other parameters, being nonlinear eigenvalues, that, as usual, improves the overall probability to get a necessary pattern by matching of various local blow-up and non-singular flows.

We introduce and discuss these examples as a warning showing that a reasonable simple treatment of the scenario (1.3) cannot be expected. We postpone until Section 8 construction and analysis of a most involved blow-up pattern with the swirl for (1.1) in the spherical coordinates \( \{r, \theta, \varphi\} \), where this gets much more complicated and seems does not admit any lower-dimensional reductions. Overall, these lead to (1.26).

Actually, our blow-up concepts are rather general and are not bounded by the framework of the classic model (1.1). In particular, as a key ingredient, we show possible generating mechanisms of countable families of other blow-up patterns, which can exhibit different properties. In other words, we propose the following statement, which is well-understood in reaction-diffusion theory (see comments below):

\[
(1.27) \quad \text{in general, “self-similarity ban” (1.4) does not prevent blow-up in NSEs (1.1).}
\]

For higher-order systems of PDEs with (1.4), the blow-up story is about to begin, and the present paper pretends to be just a first step along a “blow-up R–D direction”.

Of course, (1.1) is a dynamical system that admits the strong \textit{a priori} control of the \( L^2 \)-norm of the solutions at the blow-up time \( t = T \), to say nothing about the evident presence of the MP [162] in the stationary rescaled form of the equations, but, possibly, the system is complicated enough to get over such an obstacle in an evolution way. Therefore, we will present in Section 7.7 a first discussion of various ways how to overcame the ban (1.4) keeping blow-up similarity rescaled variables. We show that a thorough study of blow-up evolution on the quasi-stationary manifold of Slezin–Landau’s singular solutions for a submerged jet [196, 132] (see also [200]) could provide us with other types of patterns.

Thus, the introduced blow-up twistor mechanism shows that the Navier–Stokes equations in \( \mathbb{R}^3 \), which naturally support vorticity-type evolution (as a necessary feature of this basic fluid model), can develop blow-up rotational angular phenomena in finite time at a fixed stagnation point of the flow (Section 3), which naturally leads to periodic \( \omega \)-limits. It is not still clear if this twistor construction may lead to a truly localized blow-up swirl-like singularity. By the partial invariance of the twistor (it belongs to the 2D subspace (1.23)), it has a velocity field that is unbounded in the \( z \)-direction. In our discussion, we show that the branching phenomena that could lead to \( z \)-localization of such a perturbed twistor, imply a very difficult evolution matching-like problem, which seems cannot be tackled rigorously still. In Section 8 in the spherical geometry, we involve extra precession axis and vertex mechanisms that make the problem more complicated but built a bridge to more realistic generalized quasi-periodic (or periodic) blow-up twistor behaviour.

We do not rule out existence of other types of rotational singularities for (1.1), especially in view of the “multiplicity curse” of blow-up scenarios; see (1.29) below. The mathematics

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7See bibliographical comments and references in Section 7 and on further extensions in Appendix C.
of such singularity structures promises to be unbelievably difficult since assumes sorting out an infinite (at least a countable) number of various possibilities. This “countability curse” for blow-up asymptotics will be explained later. In view of that, possibly, it is not an exaggeration to say that it would be much easier and much more pleasant to settle (1.2), and hence would forget about this “awkward” multiple blow-up stuff. This is a warning again: in reality, blow-up theory is well-known to be very difficult even for looking very simple PDE models! However, for the NSEs, in author’s opinion,

(1.28) both claims (1.2) and (1.3) assume equally difficult proofs by blow-up scaling.

In other words, to find the right answer, one needs to pass through a sequence of very difficult steps of analysis that in most issues are overlapping in both approaches.

The following possible important feature of the blow-up twistors deserves mentioning: in view of their extreme rotational nature, they can create vorticity and sometimes velocity fields that tend to zero as \( t \to T^− \) in the standard weak (integral) sense in \( L^p_{loc} \). Therefore, though we observe that the vorticity gets infinite as \( t \to T^− \), the local total mass (the integral over a small shrinking neighbourhood of the stagnation point) becomes negligible and just disappears at the blow-up time.

**Remark:** On infinite family of patterns with regional and global blow-up. It turns out [70] that there exists an infinite countable family of blow-up patterns on \( W_2 \) in the cylindrical coordinates. As a compensation for a lack of proper mathematics here, in [70], we managed to justify much simpler and rigorously confirmed construction of blow-up space jets that exhibit effective regional or global blow-up in the radial variable. This example underlines another important feature of blow-up for (1.1) (though these always are of infinite energy):

(1.29) there can exist an infinite countable family of blow-up patterns.

In particular, there exists effective regional blow-up of the \( z \)-component of the vector field in the radial \( r \)-direction, i.e., the blow-up wave does not propagate and is of a standing type. This is a good sign, but of course the velocity field is unbounded in the \( z \)-direction, so that the patterns have infinite energy. A proper “bending” of the \( z \)-axis to create a kind of “blow-up ring” is rather suspicious, since also will exhibit infinite kinetic energy as \( t \to T^− \). A more complicated geometry is necessary for the next refined blow-up construction. Another important feature is

(1.30) these blow-up patterns converge to similarity solutions of Euler’s equations.

In the rescaled sense, the convergence turns out to be uniform in \( \mathbb{R}^3 \)! We do not know whether the phenomenon (1.30) is expected to be generic for other hypothetical blow-up patterns of finite energy.

Similarly, in [70], for the 3D *Euler’s equation* (EEs),

(1.31) \( \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p, \quad \text{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+, \)

\[8\] Leonhard Paul Euler, 1707-1783.
the restriction to $W_2$ is shown to admit patterns with single point blow-up in the $r$-variable.

- **Blow-up patterns with swirl and convergence to EEs** (1.31) (Section 4). This is about possible extensions of the results in [70] to the non-radial geometry with the blow-up swirl included, which is a training for more complicated blow-up structures to come.

- **Blow-up about Slezkin–Landau and other equilibria** (Section 7). This can lead to blow-up patterns by a kind of “centre-stable manifold” analysis by using various singular or regular stationary solutions of the NSEs. Since some results are derived on the basis of the famous Slezkin–Landau singular steady states, we include some history of these solutions and put translations of Slezkin’s rare notes of 1934 and 1954 in Appendices A and B at the end of the paper to be followed by a further discussion in Appendix C.

As a rather general conclusion, we state the following “steady-TW exercise” for the rescaled NSEs, which should be solved before even talking about (1.2) or (1.3):

\[
\sigma \nu + \frac{1}{2} y \cdot \nabla \nu + \frac{1}{2} v + (v \cdot \nabla) v = -\nabla p + \Delta v, \quad \text{div } v = 0,
\]

where $\sigma \in \mathbb{R}$ is the nonlinear eigenvalue introduced as in (1.25) being the blow-up angular velocity. For $\sigma = 0$, (1.32) is Leray’s classic rescaled stationary problem with a number of nonexistence results, [162, 208, 152] (though not everything is still known, [204]).

For $\sigma \neq 0$, suitable profiles in (1.32) represent the simplest case of non self-similar blow-up with the omega-limit set consisting of a periodic orbit. Nevertheless, plugging into the system a single new real eigenvalue $\sigma$ can be non-sufficient for settling the question on possible non-trivial $\omega$-limits of rescaled blow-up orbits, so we will need to introduce further mechanisms of axis precessions leading to quasi-periodic and other motions.

In general, even for $\sigma = 0$, the stationary (nonlocal elliptic) problem:

\[
(U \cdot \nabla) U = -\nabla P + \Delta U, \quad \text{div } U = 0,
\]

remains still open (though some key steps have been already made, [208, 204, 117, 118], which inspire some optimism that the situation is under a proper control; see below). As a clue exercise to such a complexity, in Section 7, we demonstrate that complicated oscillatory sign changing singular equilibria occur for the simplest elliptic problem from R–D theory in the supercritical Sobolev range:

\[
\Delta u + |u|^{p-1} u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad p \geq p_8 = \frac{N+2}{N-2}, \quad N \geq 3.
\]

Non-radial singularity structures for (1.34) were not addressed in the literature and are unknown. These can generate extremely complicated blow-up patterns in the corresponding parabolic R–D flow; see (1.63) below.

In fact, the proposed swirl behaviour (1.25) naturally corresponds to the rotational invariance of some symmetric singular equilibria in (1.33). In general, for possible more complicated singular stationary profiles $U(x)$, new types of “rotations” are necessary to
introduce and understand. One of a possible scenario of blow-up in the NSEs is as follows:

\[ \text{the orbit } \{u(t)\} \text{ evolves as } t \to T^- \text{ “close” to some singular stationary } \]
\[ \text{manifolds (1.33) being “trapped” in their complicated structure.} \]

Then “swirling type” of the orbit behaviour close to \( x = 0 \) as \( t \to T^- \) on shrinking compact subsets will naturally and entirely depend on the singular steady manifold involved and can be extremely complicated leading to any of multi-dimensional quasi-periodic or even chaotic attractors.

- **Twistors in spherical geometry (Section 8).** We next perform a formal study of a blow-up twistor in the spherical coordinates that are necessary for creating a truly spatially localized pattern. We postpone our final conclusions inherited from the previous analysis until Section 9. As a by-product, we again naturally arrive at the problem (1.33), which is difficult and open, regardless a good progress made recently on understanding of the scaling nature and uniqueness of famous Slezkin–Landau singular solutions of a submerged jet, [[208, 204, 155]]; see Section 5 for details. Thus, we again discuss the possibility that finite kinetic energy blow-up twistors may be generated in a small shrinking as \( t \to T^- \) vicinity of EVERY “proper” singular steady states from (1.33), which possibly is not precisely of homogeneity \(-1\) in \( r = |x| \) and possessing torsion-precession mechanism of its swirl axis. Other, more regular “steady states” are also of importance.

Thus, on the basis of the given fundamental model (1.1), we will show some mathematical tools that are necessary to tackle general difficult problems of blow-up and non-blow-up (then a local smooth solution becomes global and hence unique) for complicated higher-order PDEs.

1.6. **On some reaction-diffusion and parabolic analogies to be applied: first exercise on Type I and Type II blow-up patterns.** Nowadays, blow-up PDE theory, as a self-contained subject, embraces a wide range of various nonlinear evolution models; we list a few monographs from the 1980s and later periods up to 2007, [[1, 11, 66, 76, 79, 154, 174, 183, 187, 199]], where further extensions and references can be found. Most of them are mainly and specially devoted to blow-up behaviour in nonlinear partial differential equations of parabolic and hyperbolic types (those two PDE areas are well-established for blow-up since the 1950s) and contain key literature and various blow-up results achieved in the last fifty years. During this long time, a huge amount of beautiful and difficult conclusions were obtained resulting in deep and sometimes exhausting understanding and complete classification of blow-up patterns for some nonlinear parabolic and other models.

Therefore, for those who are constantly working in these PDE areas, it is not surprising that the crucial two the so-called Millennium Problems such as

(I) **Global existence or nonexistence of smooth solutions of the NSEs in \( \mathbb{R}^3 \),** and

(II) **The Poincaré Conjecture** (see further comments below),

both, in their already existing (for (II)) and possible (for (I)) ways of the solution, heavily rely on the study of blow-up solutions of the nonlinear evolution equations involved.
Frank–Kamenetskii equation: self-similar (Type I) and Type II fast blow-up patterns. In discussing further consequences related to both problems (1.2) and (1.3), we apply some general results of blow-up theory developed for various reaction-diffusion equations and systems, which turn out to be fruitful in application to blow-up for the more complicated PDEs such as (1.1). In particular, it is well-known that even for simple looking semilinear parabolic reaction-diffusion PDEs such as the classic Frank-Kamenetskii equation (1938) [60] developed in combustion theory of solid fuels (also called the solid fuel model; first blow-up results in related ODE models are due to Todes, 1933)

\[ u_t = \Delta u + e^u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \]

the property (1.29) holds. Namely, the first family of “linearized” blow-up patterns is constructed by linearization techniques and further matching of centre and stable manifolds orbits, where the latter one is infinite-dimensional causing the eventual countability of the family; see explanations below. Moreover, it is crucial that this family of blow-up asymptotics can exhaust all possible types of blow-up behaviour that is available in the model. Such delicate results for \( N = 1 \) or 2 and for other R–D PDEs (1.63) are due to Velázquez [213]. Hence, the family of blow-up patterns for (1.36) from [213, 214] is evolutionary complete (a notion introduced in [65], where further references can be found).

For \( N \geq 3 \), (1.36) possesses non-trivial self-similar blow-up (Type I) patterns, which are called nonlinear eigenfunctions, and, in addition, the set of linearized patterns is more involved and can include other countable families. The total family of blow-up patterns gets more complicated, so its evolution completeness is unknown representing a difficult open problem for (1.36).

It is curious that that this remains open even for the radial equation (1.36) in \( \mathbb{R}^3 \),

\[ u_t = u_{rr} + \frac{2}{r} u_r + e^u \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}_+ \quad (u = u(r, t), \ r = |x| > 0, \ u_r(0, t) \equiv 0). \]

To illustrate this fact and our future arguments in Section 5.12 we briefly explain how a countable family of non self-similar patterns of Type II can occur.

**Similarity blow-up patterns.** Performing in (1.37) the standard self-similar blow-up scaling yields the rescaled equation, which are known to exist for \( N \geq 3 \) [11],

\[ u(r, t) = -\ln(T - t) + v(y, \tau), \quad y = \frac{r}{\sqrt{T - t}}, \quad \tau = -\ln(T - t) \]

\[ \implies v_r = H(v) \equiv \Delta_N v - \frac{1}{2} y v_y + e^v - 1. \]

The self-similar rescaled profiles \( f(y) \neq 0 \) are its good equilibria (we omit details since will be talking a lot about similarity blow-up later on):

\[ H(f) = 0 \quad \text{in} \quad \mathbb{R}^N, \quad f(y) \text{ has at most logarithmic growth as } y \to \infty. \]

According to (1.38), each \( f(y) \) defines a Type I self-similar blow-up (see below).

**Non-similarity blow-up patterns.** Equation (1.39) in dimension \( N \geq 3 \) admits a singular equilibrium of the form

\[ V(y) = \ln \frac{2(N-2)}{|y|^2} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \]

The terms “Type I, II” were borrowed from Hamilton [94], where Type II is also called slow blow-up. 

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9The terms “Type I, II” were borrowed from Hamilton [94], where Type II is also called slow blow-up.
Then some blow-up patterns can evolve as \( \tau \to +\infty \) (\( t \to T^- \)) close to this “stationary manifold”.

**Inner Region I expansion: linearization.** To see the applicability of this idea, we perform the linearization in (1.38),

\[
(1.41) \quad v(\tau) = V + Y(\tau) \implies Y_\tau = \Delta_N Y - \frac{1}{2} y Y_y + \frac{2(N-2)}{|y|^2} Y + \mathbf{D}(Y),
\]

where \( \mathbf{D} \) is a quadratic perturbation as \( Y \to 0 \) in a suitable metric. By classic theory of self-adjoint operators \[15\], the linear operator with the inverse square potential in (1.41)

\[
(1.42) \quad \mathbf{H}'(V) = \Delta_N - \frac{1}{2} y \cdot \nabla + \frac{2(N-2)}{|y|^2} I
\]
is well posed in \( H^2_\rho'(\mathbb{R}^N) \) (with the weight as in (2.24)), i.e., has a compact resolvent and a real discrete spectrum, provided that the inverse square potential is not too much singular at the origin \( y = 0 \). Namely, one needs that \( 2(N-2) \) is less than the constant \( c_H \) of embedding \( H^1_0(B_1) \subset L^2_{|x|^{-1}}(B_1) \) by the Hardy classic inequality\[10\]

\[
(1.43) \quad \int_{B_1} \frac{|u|^2}{|x|^2} \, dx \leq c_H \int_{B_1} |\nabla u|^2 \, dx \implies 2(N-2) \leq c_H = \left( \frac{N-2}{2} \right)^2, \quad \text{i.e.}, \quad N \geq 10.
\]

Thus, in integer dimensions \( N \geq 11 \) (\( N = 10 \) has own peculiarities; see \[73\]), the operator (1.42) has a discrete spectrum and radial eigenfunctions satisfying

\[
(1.44) \quad \sigma(\mathbf{H}'(V)) = \{\lambda_k, k = 0, 2, 4, \ldots\}; \quad \psi^*_k(y) \sim b_k y^k + \ldots, \quad y \to +\infty,
\]

and \( b_k \) and \( c_k \) are some normalization constants. The orthonormal set of eigenfunctions \( \Phi^* = \{\psi^*_k\} \) is then complete and closed in \( L^2_{|x|}'(\mathbb{R}^N) \) in the radial setting.

Hence, for \( N \geq 11 \), equation (1.37) admits very special asymptotic patterns; see \[17, 72\] (global solutions and a survey) and \[98, 59\] (blow-up solutions and a survey). In particular, concerning fast blow-up patterns of Type I\[1\], these are obtained by matching of the linearized behaviour on the steady singular manifold for any \( k = 2, 4, \ldots \) such that \( \lambda_k < 0 \):

\[
(1.45) \quad v_k(y, t) = V(y) - C_k e^{\lambda_k \tau} \psi^*_k(y) + \ldots \sim F_k(y, \tau) \equiv -2 \ln y - e^{\lambda_k \tau} y^{-\delta} \quad \text{as} \quad y \to 0,
\]

with a bounded flow at the origin \( r = 0 \); see \[98, 99, 47\] for first results in this direction, and further references above. The correct \( L^\infty \)-rate of blow-up behaviour is obtained by calculating the absolute maximum of \( F_k(y, \tau) \): for \( \tau \gg 1 \),

\[
(1.46) \quad (F_k)'_y = 0 \quad \text{at} \quad y_k \sim e^{\frac{\lambda_k \tau}{2}} \quad \implies \quad \sup_y F_k(y, \tau) = F_k(y_k, \tau) \sim \alpha_k \tau, \quad \alpha_k = \frac{2^{\lambda_k}}{\delta} > 1.
\]

**Inner Region II: quasi-stationary regular flow.** Thus, in Region II close enough to \( r = 0 \), one needs to solve the original equation (1.37) with the condition as in (1.46), which

\[\text{It’s idea goes back to the 1920s, } \text{[95], and in this form was already used by Leray in 1934 [137]; see Section [73] for extra details.}\]

\[\text{In blow-up R–D theory, Type II assumes faster non-self-similar growth.}\]
suggests the following scaling therein:

\[(1.47)\]
\[u(0, t) = -\alpha_k \ln(T - t) \implies u(r, t) = -\alpha_k \ln(T - t) + w(\xi, s),\]

where \(\xi = \frac{r}{(T-t)^{\alpha_k/2}}\) and \(s = \frac{1}{\alpha_k-1} (T - t)^{1-\alpha_k} \to +\infty.\)

Substituting this into the F–K equation, after elementary manipulations yields the following perturbed problem for \(w\):

\[(1.48)\]
\[w_{ss} = \Delta_\xi w + e^w - \frac{\alpha_k}{\alpha_k - 1} \frac{1}{s} \left(1 + \frac{1}{2} w\xi\right) \quad \text{for} \quad s \gg 1.\]

Recalling that according to \((1.46)\), we have to have that \(w(0, t) = 0\), general stability theory for such blow-up singularity problems \([79]\) suggests, that since \((1.48)\) is a perturbed gradient flow, there is the stabilization to the unique bounded stationary solution: uniformly on compact subsets in \(\xi\), as \(s \to +\infty\),

\[(1.49)\]
\[w(\xi, s) \to W(\xi), \quad \text{where} \quad \Delta_\xi W + e^W = 0, \quad W(0) = 0 \quad (W(\xi) \sim -2\ln\xi, \quad \xi \gg 1).\]

Eventually, in this Region II, the current blow-up pattern behaves as:

\[(1.50)\]
\[u(r, t) \sim \alpha_k |\ln(T - t)| + W\left(\frac{r}{(T-t)^{\alpha_k/2}}\right) \quad \text{as} \quad t \to T^-.\]

Thus, \((1.38)\) gives the asymptotic structure of such blow-up patterns \(\{u_k(r, t), \ k \geq 2\}\) with the following fast blow-up rate:

\[(1.51)\]
\[u_k(0, t) \sim \left(1 + \frac{2|\lambda_k|}{\delta}\right)|\ln(T - t)| \quad \text{as} \quad t \to T^- \quad (\lambda_k \sim -\frac{k}{2}, \quad k \gg 1).\]

These Type II non self-similar blow-up patterns can have arbitrarily fast growth for \(k \gg 1\) than the standard Type I similarity divergence associated with the ODE

\[(1.52)\]
\[u' = e^u \implies u(t) = |\ln(T - t)| \quad \text{as} \quad t \to T^-.

For \(N = 10\), the origin \(y = 0\) is in the limit-point case for the operator \((1.42)\), so that its deficiency indices are \((2, 2)\) \([160]\), and there still exists its proper self-adjoint extension with a discrete spectrum and compact resolvent, so that \((1.44)\) holds; see details of an application in \([73]\). This also allows to construct blow-up patterns with a slightly different blow-up rates.

On the other hand, for \(N \in [3, 9]\), instead of \((1.44)\), the origin 0 is oscillatory: as \(y \to 0\),

\[(1.53)\]
\[\psi^*(y) \sim y^{-\frac{N-2}{2}}[C_1 \cos(b \ln y) + C_2 \sin(b \ln y)], \quad \text{where} \quad b = \sqrt{(N-2)(10-N)}.\]

This yields that any eigenfunction of any self-adjoint extension of \((1.42)\) (existing by classic theory \([160]\)) will make the function \((1.41)\) to be sign-changing and oscillatory near the matching origin \(y = 0\). Indeed, for nonnegative solutions, this prohibits any matching at all, so that Type II blow-up patterns are then nonexistent, which is rigorously proved in Matano–Merle \([146]\). Therefore, it is plausible that for radial nonnegative solutions in dimension \(3 \leq N \leq 9\), the only blow-up patterns are exhausted by the family of linearized ones (as in Section \(5.12\)) and the nonlinear self-similar solutions \((1.39)\), though there is no proof of such a completeness. However, in the nonradial geometry for solutions of changing sign such a classification blow-up conclusion is far away from being even formally justified.
In view of (1.4), one of our tricks is to check whether Type II non-similarity blow-up patterns can be constructed for the NSEs, though this should be much harder of course.

**Quasilinear reaction-diffusion extension: countable sets of blow-up patterns.** For a quasilinear extension of (1.36) in 1D, with the so-called \( p \)-Laplacian operator (here \( p \mapsto \sigma \)),

\[
(1.54) \quad u_t = (|u_x|^\sigma u_x)_x + e^u \quad (\sigma \geq 0),
\]
a countable set of blow-up patterns was constructed in [21]. As an intrinsic feature of essentially quasilinear problems, it was shown that, depending on \( \sigma > 0 \), first patterns represent nonlinear eigenfunctions, i.e., are self-similar, while the rest are constructed by the linearization techniques and matching as for (1.36). For all \( \sigma > \sigma_\infty = 0.60... \), all blow-up patterns are nonlinear, i.e., then (1.54) admits countable families of self-similar solutions. Moreover, it was also shown that, in view of the strong degeneracy of the \( p \)-Laplacian operator in (1.54), there exist other blow-up similarity solutions, which can compose uncountable families. It is still an open problem to prove evolution completeness of these blow-up families. We reflect this discussion as follows: in the family (1.29),

\[
(1.55) \quad \text{there may exist infinitely many nonlinear or “linearized” blow-up patterns.}
\]

**Higher-order parabolic extensions.** Some types of blow-up singularities are already known for higher-order reaction-diffusion equations such as

\[
(1.56) \quad u_t = -(-\Delta)^m u + |u|^{p-1} u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad (m \geq 2, \ p > 1)
\]

\[
(1.57) \quad \text{and} \quad u_t = -\Delta^2 u - \Delta(|u|^{p-1} u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad (p > 1);
\]

see [22, 56, 55, 64] and references therein concerning other models with blow-up. The latter limit unstable Cahn–Hilliard equation (1.57) has the divergence form and hence the flow preserves the total mass of \( L^1 \)-solutions (in this sense this mimics a similar feature of (1.1)). Equation (1.57) is shown to obey Leray’s blow-up scenario (1.5) [70]. Note that the questions on a complete description of nonlinear and linearized blow-up patterns for (1.56) or (1.57) and evolution completeness of the whole countable family remain open even in the one-dimensional case \( N = 1 \) and \( m = 2 \) (the fourth-order diffusion only).

**1.7. Second discussion around the NSEs.** Thus, the negative conclusion (1.4) rules out a standard self-similar way of blow-up in the Navier–Stokes equations. Anyway, regardless known deep ideas on interaction of “vortices tubes” and others (mainly coming from fluid dynamics and less mathematically developed), it seems we cannot still imagine how complicated other non self-similar individual “linearized” blow-up patterns can be (for instance, these can be partially invariant, i.e., “partially nonlinear”, or non-invariant at all), to say nothing about possible interaction of a number of different related blow-up “vortex tubes”. This is our aim to give, on the basis of a reaction-diffusion blow-up experience, a possible new insight into the world of imaginary singularities for (1.1).

In view of a more complicated nature of the Navier–Stokes equations that are a system of four equations in the four-dimensional space-time continuum\(^{12}\) \( \mathbb{R}^3 \times \mathbb{R}_+ \), the slogan

\(^{12}\)I.e., a “dynamical system \( 4 \times 4 = (4 \text{ dependent} \times 4 \text{ independent variables}) \)”.
should contain more surprises. At least, one can expect a formally infinite number of possible types of blow-up to be checked out, and, to achieve the positive answer (1.2), all of them should be ruled out by the assumption of finite energy and/or others. It is not an exaggeration to assume that (1.1) may admit a countable set of different countable (or even uncountable) scenarios of formation of single point blow-up patterns that admit local construction (on shrinking subset to the blow-up point) by some kind of combination of linearized and nonlinear eigenvalue techniques. Then the crucial fact on the finiteness of the energy of the globally extended patterns can be checked only after matching (or non-matching) of these local singularity structures with surrounding bundles of less blowing up or even bounded “tails”. Such matching procedures (which are responsible for existence or nonexistence of the patterns) are also expected to be extremely difficult.

Vice versa, to achieve the negative claim (1.3), in order to construct a suitable blow-up pattern, which prohibits global evolution of bounded smooth finite kinetic energy solutions, one then needs to sort out, possibly again, a huge (infinite and even uncountable) number of a priori unknown blow-up solution structures.

Note that, for many other models including various nonlinear higher-order PDEs, for which there is no any hope to control existence and/or nonexistence of blow-up singularities by some conservation/monotonicity/order-preserving/etc., mechanisms (that are nonexistent for sure), the careful study of (1.2) and (1.3) is unavoidable, is very difficult indeed, and, in many cases, in a full generality, is not doable at all. The latter means that this problem is not analytically solvable, i.e., constructive conditions that guarantee (1.2) (or (1.3)) cannot be derived. Nevertheless, this is not a manifestation of any kind of a “PDE agnosticism”, since the most stable and generic asymptotic structures can and must be studied and understood by any, rigorous or not, mathematical means. The illusive is a full description of all the possible singularity patterns and their evolution completeness (see a proper setting for this below) to eventually guarantee (1.2) or (1.3).

1.8. On a related blow-up parabolic area: Ricci flows and the Poincaré Conjecture. Another amazing geometry and PDE area, which is not less famous nowadays, is the study of blow-up structure of Ricci flows of metrics in $\mathbb{R}^3$. Following the logic of evolution blow-up PDE analysis, it can be characterized that Perel’mann’s new monotonicity formulae and principals of his blow-up surgery at finite-time singularities of the Ricci flow made it possible to guarantee to get a proper global extension beyond all blow-ups and with a kind of necessary “symmetrization” at the finite extinction time. In particular, this allowed to prove the Poincaré Conjecture, see [31] for details, history, references, and recent development. Proposed for this kind of analysis by Richard Hamilton in 1982 the Ricci flow for a family of Riemannian metrics $g(t) = \{g_{ij}(t)\}$:

$$ g(x, t) : \quad g_t = -2\text{Ric} \quad g, \quad \text{or, for components,} \quad (g_{ij})_t = -2R_{ij}, $$

where $\text{Ric} = \{R_{ij}\}$ is the Ricci curvature of $g$, represents a system of parabolic PDEs for the components, which overall obeys the Maximum Principle (the MP) and other

\[^{13}\text{A closed, smooth, and simply connected 3-manifold is homeomorphic to } S^3. \text{ This still remains the only solved Millennium Problem among other seven.}\]
related classic properties of parabolic flows. The scalar curvature equation for the scalar curvature $R = (g^{ij}R_{ij})(x,t)$ is a semilinear parabolic PDE with a quadratic nonlinearity,

$$R_t = \Delta R + 2|Ric g|^2,$$

where $\Delta \leq 0$ is the corresponding Laplacian. Moreover, in dimension 2, the scalar curvature $R(x,t)$ then takes the form of a standard semilinear quadratic R–D equation

$$R_t = \Delta R + R^2 \quad (N = 2).$$

This shows a (obviously, well-known) link between Ricci flows singularities with standard and well-developed blow-up R–D theory. Similar to (1.36), blow-up for this equation has a long history and embraces hundreds of publications in the 1980-90s in almost all leading world journals on nonlinear PDEs and a few monographs mentioned above. For future use, let us note that a key idea of its generic non-trivial blow-up behaviour with logarithmic (the appearance of $\ln(T - t)$-term is due to a centre subspace behaviour as explained in Section 5.12 so we avoid further comments),

$$\sim \sqrt{\ln(T - t)} \quad \text{as} \quad t \to T^-,\quad (1.61)$$

deformation of similarity blow-up structures dates back to Hocking, Stuartson, and Stuart in 1972, [100]. It was proved rigorously twenty years later that the actual structurally stable blow-up behaviour for (1.60) as $t \to T^-$ close to the blow-up point $x = 0$ is given by

$$R(x,t) = \frac{1}{T-t} f_*\left(\frac{x}{\sqrt{(T-t)\ln(T-t)}}\right)(1 + o(1)), \quad \text{where} \quad f_*(y) = \frac{1}{1+c_*|y|^2}, \quad (1.62)$$

and $c_* > 0$ is a constant depending on the dimension $N$ only ($c_* = \frac{1}{4}$ for $N = 1$); see [213] and [187, p. 312] for references and results. A full account of results on blow-up for semilinear equations such as (1.60) can be found in the most recent monograph by Quittner and Suplet [183]. Formula (1.62) is universal for dimensions $N < 6$, i.e., in the subcritical Sobolev range for the elliptic operator in (1.60), with the general $|u|^{p-1}u$ term,

$$u_t = \Delta u + |u|^{p-1}u, \quad \text{with} \quad p = 2 < p_{\text{Sobolev}} = \frac{N+2}{N-2} \quad (N > 2). \quad (1.63)$$

As an accompanying key feature for further use here, the structurally stable (generic) blow-up pattern (1.62) perfectly serves as a powerful confirmation of the slogan (1.27). Indeed, the similarity scalings for the equation (1.60) are standard and simple,

$$R(x,t) = \frac{1}{T-t} \hat{R}, \quad y = \frac{x}{\sqrt{T-t}'}, \quad (1.64)$$

so do not contain any of logarithmic factors as in (1.61) and (1.62), which are created in the blow-up evolution as $t \to T^-$ for almost arbitrary suitable solutions with $L^\infty \cap L^1$-data.

It is indeed also worth mentioning that since the function $f_*(y)$ in (1.62) is radial, this expresses the phenomenon of strong symmetrization of blow-up structures near blow-up time that is true for almost all admissible non-constant solutions of (1.60) (classification of singularities is also a key feature of Perelman’s proof, which the crucial blow-up surgery
is based upon, \[31\ Ch. 7\]. We will constantly use the generic blow-up behaviour (1.62) as a source for further speculations concerning the model (1.1).

It is remarkable that one of the key ideas of Perel’mann’s proof was surgery followed by a necessary classification of possible blow-up singularities, which can occur for the Ricci flow (1.58). Recall our main target (1.20) for the NSEs.

1.9. Reaction-diffusion: a new Type II blow-up patterns for \( p = p_S \). The parabolic equation (1.63) in the critical case \( p = p_S \) for \( N = 3 \) reveals another type of construction of Type II non self-similar blow-up patterns. Namely, in the critical Sobolev case, (1.63) admits the Loewner–Nirenberg stationary solution (1974) \[142\]

\[
u \equiv \frac{N(N-2)}{N(N-2)+|x|^2} \equiv \sqrt{\frac{3}{3+|x|^2}} \quad \text{for} \quad N = 3.
\]

This is indeed a very special and remarkable explicit solution, so that, similar to the scenario in Section 1.6, blow-up can occur about the 1D manifold of such rescaled equilibria. Namely, following [59], we explain how this can happen. First, as in (1.38), we perform the standard self-similar scaling in (1.63), with \( p = 5 \) for \( N = 3 \),

\[
u(x, t) = (T - t)^{-\frac{1}{p-1}} v(y, \tau), \quad y = \frac{x}{\sqrt{T - t}}, \quad \tau = -\ln(T - t), \quad \text{so}
\]

\[
u_\tau = H(v) \equiv \Delta v - \frac{1}{2} y \cdot \nabla v - \frac{1}{p-1} v + |v|^{p-1} v.
\]

Second, let us assume that \( v(y, \tau) \) behaves for \( \tau \gg 1 \) being close to the stationary manifold composed of equilibria (1.65), i.e., for some unknown function \( \varphi(\tau) \rightarrow +\infty \) as \( \tau \rightarrow +\infty \),

\[
u(y, \tau) \approx \nu_\tau \nu(y, \tau) \approx \varphi(\tau)(\rho_\tau(y) y)
\]

on the corresponding shrinking compact subsets in the new variable \( \zeta = \varphi^{\frac{p-1}{2}}(\tau) y \). It then follows that, in the case \( N = 3 \), on the solutions (1.65) in terms of the original rescaled variable \( y \) (see computations in [59], p. 2963); our notations have been slightly changed)

\[
|\nu(y, \tau)|^{p-1} v(y, \tau) \rightarrow \frac{4\pi^2}{\varphi(\tau)} \delta(y) \quad \text{as} \quad \tau \rightarrow +\infty
\]

in the sense of distributions. Therefore, equation (1.66) takes asymptotically the form

\[
u_\tau = H(v) \equiv \Delta v - \frac{1}{2} y \cdot \nabla v - \frac{1}{4} v + \frac{4\pi^2}{\varphi(\tau)} \delta(y) + \ldots \quad \text{for} \quad \tau \gg 1.
\]

Thus, we get an eigenvalue problem for Hermite’s classic operator [14, p. 48]:

\[
B^* = \Delta v - \frac{1}{2} y \cdot \nabla v, \quad \text{where} \quad \sigma(B^*) = \{ \lambda_\beta = -\frac{\beta}{2}, \quad |\beta| = 0, 1, 2, \ldots \},
\]

defined in \( L_{\rho^*}^2(\mathbb{R}^3) \), \( \rho^*(y) = e^{-|y|^2/4} \), with the domain \( H_{\rho^*}^2(\mathbb{R}^3) \), to be used later. It follows from (1.69) that one can try the following regular parts of such Type II blow-up patterns: balancing two terms in (1.69) yields

\[
u_\beta(y, \tau) = e^{(\lambda_\beta - \frac{1}{\beta})\tau} H_\beta(y) + \ldots \quad \Rightarrow \quad \varphi_\beta(\tau) \sim e^{\frac{2\beta}{\beta+1} \tau} \quad \text{for} \quad \tau \gg 1,
\]

\[14\] The same operator occurs for the rescaled equation (1.60), where the factor \( \sqrt{|\ln(T - t)|} = \sqrt{\tau} \) is due to the eigenspace behaviour with \( \lambda_2 = -1 \) and the second Hermite polynomial \( H_2(y) = \frac{1}{\sqrt{2}} (1 - y^2) \).
where \( H_\beta(y) \) are Hermite polynomials as the eigenfunctions of (1.70). Together with the scaling in (1.66), this yields a countable family of complicated blow-up structures. To reveal the actual space-time and changing sign structures of such Type II patterns, special matching procedures apply. In [59], this analysis has been performed in the radial geometry, though (and this is also key for us to recognize) still no rigorous justification of the existence of such blow-up scenarios is available.

1.10. Further comments on the “RD-sense” of this essay. As was already announced, the present paper is a general view from blow-up reaction-diffusion theory to possible “singularities” (including, e.g., multiple zeros) of the 3D NSEs. In other words, the main intention of the author is to involve some RD-experience into the NSEs study. As is easily detectable by experts in the area, this special issue dictated, in particular, a quite special kind of survey and references involved in the list, which by no means reflects the actual history and modern trends of NSEs and Euler’s equations theory (to say nothing about R–D theory as well). Nevertheless, the author hopes that a “detached observer view” can deliver some new useful accents to the area.

The subjects of the R–D and the NSEs do not coincide at all and seems their overlapping is suspicious in many places. Since the NSEs subject is huge and extremely difficult, the author apologizes for any inconvenience caused by his approximate constructions and/or speculations that, from classic fluid dynamic PDE views, can be classified as rather non-consistent or well known somehow. It seems understandable that, in view of not that optimistic final conclusions (cf. (9.4)), which were predicted in advance, the author, with a completely different background, did not have a very strong motivation to make more clear some of the matching constructions when this looked being possible (not often, unfortunately). The “semi-mathematical” (and sometimes logically incomplete) language of presenting the speculations reflects the obvious truth: each elementary step in this long story of singularities/blow-up for the NSEs could take years to fix (and never known how many). So, the author took the risk to show quite a discontinuous route (a “jumping frog”-style) to the end of this story, of course, with a full understanding how risky this way could be in mathematics, when many steps and concepts of his speculations can be attacked by justified incinerating critics from some experts from the field. Anyway, the author hopes that the attentive Reader will find some ideas and concepts of blow-up from the R–D systems useful, even if these were not presented on a sufficiently rigorous basis and costs; and even will find satisfactory a rigorous (“almost”, i.e., if fixed in a reasonable finite time) construction of a countable family of blow-up space jets for (1.1).

Indeed, the combination of all the possible tools of singularity analysis, leading to a success for the NSEs (1.1), will serve as a solid and reliable basis for further development of modern theory of higher-order PDEs. In this context, blow-up scaling methods become more and more penetrating into the core of PDE theory being natural and unavoidable tools of evolution analysis of higher-order nonlinear equations in all its three “hipostases”:

(i) existence, (ii) uniqueness, and (iii) asymptotic behaviour.
2. SOME FACTS OF SINGULARITY HISTORY FOR THE NAVIER–STOKES EQUATIONS:

LERAY’S BLOW-UP SOLUTIONS, SINGULAR POINTS, SPECTRA, AND PATTERNS

We present here a short survey on global solvability, singularity formation, and other classic facts concerning the Navier–Stokes equations (1.1), and stress a special attention to some blow-up issues, which will be used in what follows. A perfect and detailed overview of main mathematical results concerning the NSEs is available in Taylor [206, Ch. 17], which includes several aspects to be quoted below without proper referencing.

2.1. Leray–Hopf (“turbulent”) solutions of finite kinetic energy. It is a classic matter that the energy $L^2$-norm is natural for (1.1). After multiplication by $u$ in the metric $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^N)}$ and integration by parts, the convective and pressure terms vanish on smooth enough functions $u(x, t)$ with sufficiently fast (say, exponential) decay at infinity,

\begin{equation}
\langle (u \cdot \nabla)u, u \rangle = 0 \quad \text{and} \quad -\langle \nabla p, u \rangle = \langle p, \nabla \cdot u \rangle = 0.
\end{equation}

Therefore, on such smooth solutions, we have the instantaneous rate of dissipation of kinetic energy given by

\begin{equation}
\frac{1}{2} \frac{d}{dt} \| u(t) \|^2_2 = -\| Du(t) \|^2_2 \quad \Rightarrow \quad \| u(t) \|^2_2 + 2 \int_0^t \| Du(s) \|^2_2 ds = \| u_0 \|^2_2, \quad t \geq 0.
\end{equation}

Actually, the estimate in (2.2) with the inequality sign “$\leq$” is the energy inequality for Leray–Hopf weak solutions of (1.1) (in 1933, Leray also called such weak solutions “turbulent” [137, p. 231, 241] and compared these with regular ones; see also Lions [140, Ch. 1, §6] for a discussion); q.v. e.g., [182] and references therein. (2.2) is also a crucial identity for general turbulence theory. E.g., the famous Kolmogorov–Obukhov power “K-41” law (1941) [119, 165] for the energy spectrum of turbulent fluctuations for wave numbers $k$ from the so-called inertial range,

\begin{equation}
E(k) = C \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}, \quad \text{where} \quad \varepsilon = \frac{1}{Re} \langle |Du|^2 \rangle,
\end{equation}

uses this rate of dissipation of kinetic energy for high Reynolds numbers $Re$ ($\langle \cdot \rangle$ is an invariant measure of calculating expected values).

Note another, weaker “conservation laws”, since the convective terms in the NSEs are in divergent form ($\otimes$ is the tensor product of vectors in $\mathbb{R}^3$):

\begin{equation}
(u \cdot \nabla)u = \text{div} (u \otimes u) \equiv (u u)_{x_1} + (v u)_{x_2} + (w u)_{x_3}.
\end{equation}

Hence, integrating over $\mathbb{R}^3$ with the necessary decay at infinity, one can get that the total “masses” of the velocity components are preserved:

\begin{equation}
\frac{d}{dt} \int u(x, t) \, dx, \quad \frac{d}{dt} \int v(x, t) \, dx, \quad \frac{d}{dt} \int w(x, t) \, dx = 0, \quad \text{i.e.,}
\end{equation}

\begin{equation}
\int u(t) \equiv \int u_0, \quad \int v(t) \equiv \int v_0, \quad \int w(t) \equiv \int w_0 \quad \text{for} \quad t \geq 0.
\end{equation}

Of course, as estimates, (2.5) are weaker than (2.2), and the mass “semi-norms” by taking the absolute values of the integrals there are difficult to apply for solutions of changing sign. Though, as sharp conservation properties, these can be valuable in construction (or prohibiting) special sensitive blow-up patterns. Moreover, for some parabolic problems
with the mass conservation (and no \( L^2 \)-control), estimates such as (2.5) can play a key role for global extension of blow-up solutions beyond blow-up, for \( t > T \). A discussion on the application of Leray’s blow-up scenario (1.5) to the limit Cahn–Hilliard equation (1.57) is presented in [70].

Nevertheless, the \( L^2 \)-bounds as in (2.2) are not sufficient to control the \( L^\infty \)-non-blowing up property of solutions, and this is the origin of extensive mathematical research in the last seventy years or so. Recall that global weak solutions of (1.1) satisfying

\[
  u \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; H^1(\mathbb{R}^3))
\]

were already constructed by Leray [136, 137] (1933), and Hopf [101] (1951). We recommend recent papers [6, 16, 29, 37, 43] as a guide to various results of local and global (for small and other data) theory of (1.1), including analyticity results in both spatial and temporal variables; see [49, 224] for a modern overview.

2.2. Blow-up self-similar singularities with finite energy are nonexistent: on Leray’s scenario of backward and forward blow-up self-similar singularities. It seems that the original idea that the classic fundamental problem of the unique solvability of (1.1) in \( \mathbb{R}^3 \), i.e., existence of a global smooth bounded \( L^2 \)-solution, is associated with existence or nonexistence of certain blow-up singularities as \( t \to T^- \), goes back to Th. von Kármán; see [109, 110, 111] and survey [13].

As usual, similarity solutions, as a next manner to further specify the behaviour, can be attributed to an invariant group of scaling transformations: if \( \{u(x, t), p(x, t)\} \) is a solution of the NSEs (1.1), then

\[
  \{u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)\}
\]

is also a solution for any \( \lambda \in \mathbb{R} \).

Setting here \( \lambda = \frac{1}{\sqrt{T-t}} \) formally yields two types of self-similar solutions of (1.1), the blow-up and the global ones. Let us now focus on key historical aspects of such a study.

Namely, in 1933, J. Leray [136, 137] proposed a mathematical question to look for blow-up in (1.1) driven by the self-similar solutions

\[
  u(x, t) = \frac{1}{\sqrt{T-t}} U(y), \quad p(x, t) = \frac{1}{T-t} P(y), \quad \text{where} \quad y = \frac{x}{\sqrt{T-t}}.
\]

This Leray’s statement is well known and was stressed to in many papers [137, p. 225]:

\[
  \ldots \text{la solution des équations de Navier dont il s’agit est:}
  \]

\[
  (3.12) \quad u_i(x, t) = [2\alpha(T-t)]^{-\frac{1}{2}} U_i[(2\alpha(T-t))^{-\frac{1}{2}} x] \quad \text{(} t < T \text{)}
  \]

(\( \lambda x \) désigne le point de coordonnées \( \lambda x_1, \lambda x_2, \lambda x_3 \).)

\[
  \]

15 The author apologizes for not being able to trace out von Kármán’s original work (or a lecture?), where the idea of singularity of the velocity field \( \sim (T-t)^{-\alpha} \) (\( \alpha = \frac{3}{5}, \frac{5}{3}, \text{ or } \frac{2}{5} \)) appeared first.

16 Such a blow-up backward continuation variable \( y \) for 1D linear parabolic equations was already systematically used by Strum in 1836 [203]; on his backward-forward continuation analysis, see [66, p. 4].

17 Here and later, boxing and underlying are author’s.
However, at the end of the same paper in *Acta mathematica*, Leray returned once more to this similarity blow-up problem and now his question is also truly remarkable [137, p. 245] (here (3.11) is the system (2.9) below for the similarity profiles in (2.7)):

"**Remarque:** Si le système (3.11) possède une solution non nulle $U_i(x)$ cette solution permet de construire un exemple très simple de solution turbulente c’est le vecteur $U_i(x,t)$ égal à

$$
(2.8) \quad [2\alpha(T - t)]^{-\frac{1}{2}} U_i\left[2\alpha(T - t)\right]^{-\frac{1}{2}} x
$$

pour $t < T$ et à o pour $t > T$; il existe une seule époque d’irrégularité: $T$.

"**Therefore, Leray in (2.8) posed both principal questions on existence of self-similar solutions of blow-up backward type for $t < T$ and of the standard forward type for $t > T$, which are naturally supposed to “coincide” (in which sense? – in general, a difficult question of extended semigroup theory) at the unique singularity point $t = T$. Thus, this is a principal setting not only for a similarity way of formation of a blow-up singularity as $t \to T^-$, but also for self-similar continuation of the solution for $t > T$, i.e., beyond blow-up time, when it becomes again regular and bounded. Even for simple parabolic reaction-diffusion equations such as (1.36) and (1.60), though such “peaking” blow-up self-similar solutions are known, a theory of such an incomplete blow-up is far away from being well-understood and contains a number of open problems; see references and results in [77] and in survey [78] (as usual, further recent progress already achieved in this direction can be then traced out via MathSciNet). For the Cahn-Hilliard equation (1.57), this is done in [70] in the lines of Leray’s scenario (1.5).

Thus, substituting (2.7) into (1.1) yields for $U$ and $P$ the following “stationary” system (3.11) in [137, p. 225]:

$$
(2.9) \quad \frac{1}{2} U + \frac{1}{2} (y \cdot \nabla) U + (U \cdot \nabla) U = -\nabla P + \Delta U, \quad \text{div} U = 0 \quad \text{in} \quad \mathbb{R}^3.
$$

During last twelve years, a number of enhanced negative answers concerning existence of such non-trivial similarity patterns (2.7), (2.9) were obtained. The key ingredient [162] of such nonexistence proofs is the Maximum Principle [18]:

$$
(2.10) \quad II = \frac{1}{2} |U|^2 + \frac{1}{2} y \cdot U + P \quad \text{satisfies} \quad -\Delta II + (U + \frac{1}{2} y) \cdot \nabla II = -|\text{curl} U|^2 \leq 0;
$$

see further details in [34, 152, 208], and the advanced and negative nonstationary PDE answer in [103]. Let us note an existence result in [48] for $N = 4$.

Regardless the nonexistence of the similarity blow-up (2.7) and especially in connection with, the following Leray conclusion deserves the attention [137, p. 224]. For the function

$$
(2.11) \quad V(t) = \sup_x |u(x,t)| \quad \text{for} \quad t < T:
$$
Thus, Leray was the first who derived this estimate from below for a solution to have blow-up at $t = T$. This estimate led Leray to pose the question on blow-up similarity solutions (2.7) (not just the fact that this would be a simple dimensional way for the NSEs to develop a finite-time singularity as $t \to T^-$ with an analogous extension for $t > T$). Indeed, otherwise, it was shown that the integral in the estimate on p. 223 for the successive approximations $\{u^{(n)}_i(x, t)\}_i$ will never exhibit a necessary divergence (more precisely, here a Dini–Osgood-type integral condition is supposed to occur). In other words, Leray’s estimate (3.9) above then suggests that a fast-type blow-up, which is not self-similar (i.e., of Type II), is the only possible.

We devote a notable part of the present paper to discussing such type of fast blow-up that is unbounded in the rescaled variables introduced in (2.7) and (2.18) below.

Thus, nonexistence (1.4) of Leray’s similarity solutions (2.7) and other related local types of self-similar blow-up is a definite step towards better understanding of the singularity nature for the Navier–Stokes equations. This does not settle the problem of singularity formation, since there might be other ways for (1.1) to create singularities as $t \to T^-$ rather than the purely self-similar scenario (2.7); see the monograph [148] for details. Other concepts of such a multiplicity are discussed below.

Concerning blow-up of infinite energy solutions, consider the strain field (q.v. [161]):

$$u = (-\zeta(t)x + u(x, y, t), -\zeta(t)y + v(x, y, t), 2\zeta(t)z),$$

where $\zeta(t) = \|\omega(\cdot, t)\|_\infty$, and the vorticity is structurally associated with famous Oseen’s vortex (1910) [170].

$$\omega(r, t) = \frac{5}{2(T-t)} e^{-\frac{r^2}{T-t}} (r^2 = x^2 + y^2), \text{ so that } \text{curl } u = (0, 0, w(r, t)).$$

The additional velocity field associated with an analogous vorticity in the cylindrical coordinates $\{r, \varphi, z\}$ is [159] (here $\Gamma > 0$ is a constant)

$$u = (0, v(r, t), 0), \text{ where } v(r, t) = \frac{\Gamma}{2\pi r} \left(1 - e^{-\frac{r^2}{4(T-t)}}\right) (\omega = (0, 0, \omega(r, t))).$$

19 This Leray’s integral approach can be characterized as a forerunner type for a number of later famous estimates via “control of vorticity” including that by Beale, Kato, and Majda [10] for Euler’s equations (on recent extensions, see [223]).
2.3. **On blow-up in Euler’s equations.** A similar nonexistence of self-similar solutions and some other types of blow-up has been obtained for the *Euler equations* \((1.31)\) in \(\mathbb{R}^3\) (derived by L. Euler in the middle of the eighteenth century [54], see the most recent survey by Constantin [42] for a modern mathematical activity exposition around); see [34, 45, 102], and references therein. Local existence of smooth solutions for \((1.31)\) has been known from 1920s; see Lichtenstein [139] (and [8] for a full history); see also later results of the 1970s by Kato [113]. Existence of global classical solutions is still open.

Pelz [175] and Gibbon [85] contain interesting surveys on various mathematical, symmetry, and more physical ideas concerning possible blow-up scenarios including numerical aspects (for the latter, see Kerr [116] for completeness concerning recent discussions on numerical blow-up issues).

Infinite energy solutions of \((1.31)\) do blow-up, for instance, according to the following separable solution in the cylindrical coordinates [87]:

\[
(2.15) \quad u = (u(r, t), 0, z\gamma(r, t)), \quad \text{where} \quad \gamma(r, t) = -\frac{e^{-r^2}}{T-t}, \quad u(r, t) = \frac{1}{2r}\left(1-e^{-r^2}\right).
\]

Blow-up of more general solutions of this form \(u(x, y, z) = (u(x, y, t), v(x, y, t), z\gamma(x, y, t))\) was studied in detail in [41], where, in particular, non-power blow-up rate was observed,

\[
\gamma \sim \frac{1}{(T-t)\ln(T-t)} \quad \text{as} \quad t \to T^-.
\]

Notice also that, in a bounded (interior or exterior) domain in rescaled variables, Euler’s equations \((1.31)\) admit non-trivial similarity solutions [96, 97] (cf. \((2.7)\) for \(\alpha = \beta = \frac{1}{2}\))

\[
(2.16) \quad u(x, t) = \frac{1}{(T-t)^{\alpha}} U(y), \quad y = \frac{x}{(T-t)^{\beta}} \quad \text{in the range} \quad \beta \in \left[\frac{2}{3}, 1\right] \quad \text{and} \quad \alpha + \beta = 1.
\]

In the original spatial \(x\)-variables, this self-similar blow-up is supported by boundary conditions for a domain that shrinks into a point as \(t \to 1^-\). On the other hand, conditions (looking still rather non-constructive) of pointwise (in \(L^\infty\)) blow-up for \((1.31)\) were introduced in [35].

We believe that some ideas of our blow-up swirl analysis can be also applied to Euler’s equations \((1.31)\), and this is reflected in [70]. Moreover, in [70], we perform the asymptotic construction of blow-up patterns of the NSEs \((1.1)\) converging as \(t \to T^-\) to similarity solutions of the EEs \((1.31)\). However, in general, \((1.31)\) is a special subject that, in several circumstances, demands different approaches for constructing families of blow-up patterns.

---

20 It seems that first strongly physically motivated arguments in favor of blow-up phenomena for Euler’s equations were due to Onsager [169] (1949).

21 In published form the incompressible equations appeared only in 1761, while a preliminary version was presented to Berlin Academy in 1752; see the full history in [40].

22 "The blow-up problem for the Euler equations is a major open problem of PDE theory, of far greater physical importance than the blow-up problem for the Navier–Stokes equation, which of course is known to nonspecialists because it is a Clay Millennium Problem," [42, p. 607].
2.4. **Singular blow-up set has zero measure.** There exists another classic direction of the singularity theory for the Navier–Stokes equations that was originated by Leray himself [137] (see also details in [53]) and in Caffarelli, Kohn, and Nirenberg [26]. It was shown that the one-dimensional Hausdorff measure of the singular (blow-up) points in a time-space cylinder is equal to zero and these are contained in a space-time set of the Hausdorff dimension \( \leq \frac{1}{2} \). We refer to [93, 164, 192] for further development and references. In particular, at a given moment \( t_0 \), the admitted number of singular points can be finite; see [163, 191] and presented references.

Incidentally, among other results including Leray’s one in [137], a refined criterion is obtained in [192], saying that, if \( T = 1 \) is the first singular (blow-up) moment for a solution \( u(x,t) \) of (1.1), then

\[
\lim_{t \to 1^-} \frac{1}{1-t} \int_0^1 \int_{\mathbb{R}^3} |u(x,t)|^3 \, dx \, dt = +\infty.
\]

The condition (2.17) is consistent with Leray–Prodi–Serrin–Ladyzhenskaya regularity \( L^{p,q} \) criteria and other more recent researches; see key references, history, details, and results concerning this huge existence-regularity-blow-up business around the NSEs in [53, 63, 134, 148, 182, 192, 202, 219]; [57] represents a modern panorama of such studies, which also commented that Ohyama’s result in 1960 [166] was obtained before Serrin’s one in 1962 [194]. (2.17) is also associated with Kato’s class of unique mild solutions (in \( \mathbb{R}^N \)), [115]; see details and key references in [29, 63, 216].

2.5. **Leray rescaled variables.** We perform the nonstationary scaling as in (2.7), \( T = 1 \),

\[
u(x,t) = \frac{1}{\sqrt{1-t}} \hat{u}(y,\tau), \quad y = \frac{x}{\sqrt{1-t}}, \quad \tau = -\ln(1-t) \to +\infty \text{ as } t \to 1^-.
\]

This yields the rescaled equations for \( \hat{u} = (\hat{u}^1, \hat{u}^2, \hat{u}^3)^T \) and \( P \),

\[
\hat{u}_\tau = \Delta \hat{u} - \frac{1}{2} (\hat{u} \cdot \nabla) \hat{u} - \frac{1}{2} \hat{u} - (\hat{u} \cdot \nabla) \hat{u} - \nabla P, \quad \text{div } \hat{u} = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+.
\]

In particular, after scaling, (2.17) takes the form

\[
\lim_{\tau \to +\infty} e^\tau \int_\tau^{+\infty} e^{-s} \left( \int_{\mathbb{R}^3} |\hat{u}(y,s)|^3 \, dy \right) \, ds = +\infty,
\]

so that, if \( t = 1 \) is singular, then the solution of the rescaled equations (2.19) must diverge (blow-up) as \( \tau \to +\infty \) in \( L^3(\mathbb{R}^3) \).

As a standard next step, we exclude the pressure from the equations (2.19),

\[
\hat{u}_\tau = H(\hat{u}) \equiv (B^* - \frac{1}{2} I) \hat{u} - \mathbb{P} (\hat{u} \cdot \nabla) \hat{u} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+,
\]

where \( \mathbb{P}v = v - \nabla \Delta^{-1}(\nabla \cdot v) \) (\( \|\mathbb{P}\| = 1 \))
is the Leray–Hopf projector of \( (L^2(\mathbb{R}^3))^3 \) onto the subspace \( \{ w \in (L^2)^3 : \text{div } w = 0 \} \) of solenoidal vector fields.\(^{23}\) Another representation is \( \mathbb{P}v = (v_1 - R_1 \sigma, v_2 - R_2 \sigma, v_3 - R_3 \sigma)^T \).

\(^{23}\)This emphasizes the unpleasant fact that the NSEs are a *nonlocal* parabolic problem, so that a somehow full use of order-preserving properties of the semigroup is illusive; though some “remnants” of the Maximum Principle (cf. (2.10)) for such flows may remain and actually appear from time to time.
where $R_j$ are the Riesz transforms, with symbols $\xi_j/|\xi|$, and $\sigma = R_1v_1 + R_2v_2 + R_3v_3$. We then first apply $\mathbb{P}$ to the original velocity equation in (1.1) and next use the blow-up rescaling (2.18). Using the fundamental solution of $\Delta$ in $\mathbb{R}^N$, $N \geq 3$ ($\sigma_N$ is the surface area of the unit ball $B_1 \subset \mathbb{R}^N$)

\begin{equation}
 b_N(y) = -\frac{1}{(N-2)\sigma_N \ |y|^{N-2}}, \quad \text{where} \quad \sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)},
\end{equation} 

the operator in (2.21) is written in the form of Leray’s formulation \cite[p. 32]{148}

\begin{equation}
 \mathbf{H}(\dot{u}) \equiv (\mathbf{B}^* - \frac{1}{2} I) \dot{u} - (\dot{u} \cdot \nabla) \dot{u} + C_3 \int_{\mathbb{R}^3} \frac{y - z}{|y - z|^3} \text{tr}(\nabla \dot{u}(z, \tau))^2 \, dz,
\end{equation} 

where $\text{tr}(\nabla \dot{u}(z, \tau))^2 = \sum_{(i,j)} \dot{u}_i^j \dot{u}_j^i$, and $C_N = \frac{1}{\sigma_N} > 0$.

2.6. Hermitian spectral theory of the blowing up rescaled operator: point spectrum and solenoidal polynomial eigenfunctions. In fact, the rescaled equations (2.21) (or the original ones (2.19)) are truly remarkable. Writing first linear operators on the right-hand side of (2.21) in a divergent form,

\begin{equation}
 \tilde{\mathbf{B}}^* \dot{u} \equiv (\mathbf{B}^* - \frac{1}{2} I) \dot{u} = \Delta \dot{u} - \frac{1}{2} (y \cdot \nabla) \dot{u} - \frac{1}{2} \dot{u} \equiv \frac{1}{\rho^*} \Delta (\rho^* \nabla \dot{u}) - \frac{1}{2} \dot{u},
\end{equation} 

where the weight is $\rho^*(y) = e^{-|y|^2}$, we observe that the actual rescaled evolution is now restricted to the weighted $L^2$-space $L^2_{\rho^*}(\mathbb{R}^3)$, with the exponential weight $\rho^*(y)$. Here, $\tilde{\mathbf{B}}^* = \mathbf{B}^* - \frac{1}{2} I$ is a shifted adjoint Hermite operator with the point spectrum \cite[p. 48]{15}.

\begin{equation}
 \sigma(\tilde{\mathbf{B}}^*) = \{ \lambda_k = -k - \frac{1}{2}, \quad k = |\beta| = 0, 1, 2, \ldots \} \quad (\beta \text{ is a multiindex}),
\end{equation} 

where each $\lambda_k$ has the multiplicity $\frac{(k+1)(k+2)}{2}$ for $N = 3$, or the binomial number $C_{N+k-1}^k$. The corresponding complete and closed set of eigenfunctions $\Phi^* = \{ \psi^*_\beta(y) \}$ is composed from separable Hermite polynomials. Similar spectral and eigenfunction properties can be also attributed to blow-up problems for 2mth-order parabolic PDEs such as (1.56); see e.g., \cite{51}, where a correspondence between the “blow-up” space $L^2_{\rho^*}$ and $L^2_{\rho}$ for the global evolution, with $\rho^* = \frac{1}{\rho}$, is more clearly explained. The bi-orthonormality holds:

\begin{equation}
 \langle \psi^*_\beta, \psi^*_\gamma \rangle = \delta_{\beta\gamma} \quad \text{for any } \beta, \gamma.
\end{equation} 

Note another important for us property of Hermite polynomials:

\begin{equation}
 \forall \psi^*_\beta, \quad \text{any derivative } D^k \psi^*_\beta \text{ is also an eigenfunction with } k = |\beta| - |\gamma| \geq 0.
\end{equation} 

Recall that \cite{15}:

\begin{equation}
 \text{polynomial set } \Phi^* \text{ is complete and closed in } L^2_{\rho^*}(\mathbb{R}^3).
\end{equation} 

We need to consider eigenfunction expansions in the solenoidal restriction

\begin{equation}
 \bar{L}^2_{\rho^*}(\mathbb{R}^3) = L^2_{\rho^*}(\mathbb{R}^3)^3 \cap \{ \text{div } \mathbf{v} = 0 \}.
\end{equation} 

Indeed, among the polynomials $\Phi^* = \{ \psi^*_\beta \}$ there are many that well-suit the solenoidal fields. Namely, introducing the eigenspaces

\begin{equation}
 \Phi_k^* = \text{Span } \{ \psi^*_\beta, \ |\beta| = k \}, \quad k \geq 1,
\end{equation} 

in view of (2.27) \( \text{div} \) plays a role of a “shift operator” in the sense that

\[
(2.30) \quad \text{div} : \Phi^*_k \rightarrow \Phi^*_{k-1}.
\]

We next define the corresponding solenoidal eigenspaces as follows (see also Section 2.8):

\[
(2.31) \quad S^*_k = \{ \mathbf{v}^* = (v^*_1, v^*_2, v^*_3) : \text{div} \mathbf{v}^* = 0, \; v^*_i \in \Phi^*_k \}, \quad \text{where dim } S^*_k = k(k+2);
\]

see [81] [82] and further references therein. Actually, [81] deals with global asymptotics as \( t \rightarrow +\infty \), where the adjoint operator \( \mathsf{B} \) in (2.49) occurs. Since \( \mathsf{B} \) is self-adjoint in \( L^p_\rho(\mathbb{R}^3) \), several results from [82] Append. A are applied to \( \mathsf{B}^* \) (cf. Section 2.8). For a full collection, see [20] for further asymptotic expansions and extensions of these ideas.

In particular, those solenoidal Hermite polynomial eigenfunctions of \( \mathsf{B}^* \) can be chosen as follows [82, p. 2166-69] (the choice is obviously not unique; normalization constants are omitted):

\[
\begin{align*}
\lambda_1 = -\frac{1}{7} & : \quad \mathbf{v}^*_{11} = \begin{bmatrix} 0 \\ -y_3 \\ y_2 \end{bmatrix}, \quad \mathbf{v}^*_{12} = \begin{bmatrix} y_3 \\ 0 \\ -y_1 \end{bmatrix}, \quad \mathbf{v}^*_{13} = \begin{bmatrix} -y_2 \\ y_1 \\ 0 \end{bmatrix} \quad (\text{dim } S^*_1 = 3); \\
\lambda_2 = -1 & : \quad \mathbf{v}^*_{21} = \begin{bmatrix} 4 - y_2^2 - y_3^2 \\ y_1 y_2 \\ y_1 y_3 \end{bmatrix}, \quad \mathbf{v}^*_{22} = \begin{bmatrix} y_1 y_2 \\ 4 - y_1^2 - y_3^2 \\ y_2 y_3 \end{bmatrix}, \quad \mathbf{v}^*_{23} = \begin{bmatrix} y_1 y_3 \\ y_2 y_3 \\ 4 - y_1^2 - y_2^2 \end{bmatrix}, \\
\mathbf{v}^*_{24} = - \begin{bmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{bmatrix}, \quad \mathbf{v}^*_{25} = - \begin{bmatrix} y_2 y_3 \\ 0 \\ -y_2 y_1 \end{bmatrix}, \\
\mathbf{v}^*_{26} = \begin{bmatrix} -y_2 y_3 \\ y_2 y_3 \\ y_1 y_3 \end{bmatrix}, \quad \mathbf{v}^*_{27} = \begin{bmatrix} y_1 y_2 \\ y_3^2 - y_1^2 \\ -y_2 y_3 \end{bmatrix}, \quad \mathbf{v}^*_{28} = \begin{bmatrix} y_2^2 - y_3^2 \\ -y_1 y_2 \\ y_1 y_3 \end{bmatrix} \quad (\text{dim } S^*_2 = 8), \quad \text{etc.}
\end{align*}
\]

We need the following final conclusion. By (2.28), the set of vectors \( \Phi^{*3} \) is complete and closed in \( L^p_\rho(\mathbb{R}^3)^3 \), so that

\[
(2.33) \quad \forall \mathbf{v} \in L^2_\rho(\mathbb{R}^3)^3 \quad \Rightarrow \quad \mathbf{v} = \sum_{(\beta)} c_\beta \mathbf{v}^*_\beta, \quad \mathbf{v}^*_\beta \in \Phi^{*3}_k, \quad k = |\beta| \geq 1.
\]

It then follows from (2.26), (2.30) that

\[
(2.34) \quad \text{polynomial set } \hat{\Phi}^* = \Phi^{*3} \cap \{ \text{div} \mathbf{v} = 0 \} \text{ is complete and closed in } \hat{L}^2_\rho(\mathbb{R}^3).
\]

In what follows, we always assume that we deal with “solenoidal” asymptotics involving eigenfunctions as in (2.31).

For Burnett equations (1.6), as we have promised to go with in parallel, the blow-up rescaling and elements of linear solenoidal spectral theory are found in Section 3.2.

\textsuperscript{24}Note a standard result of functional analysis: polynomials are complete in any weighted \( L^p \)-space with an exponentially decaying weight; see the analyticity argument in Kolmogorov–Fomin [120, p. 431].
2.7. On a countable set of quasi-periodic singularities: first formalities. Thus, according to the criterion (2.20), the moment $T = 1$ is not a singular (and hence regular) point, if the corresponding locally smooth solution of (2.21) does not blow-up as $\tau \to +\infty$ in a suitable functional setting. Thus, the problem of global existence and uniqueness of a smooth solutions of the NSEs in $\mathbb{R}^3$ reduces to nonexistence of blow-up in infinite time for the rescaled system (2.19) or (2.21). In such a framework, this problem falls into the scope of blow-up/non-blow-up theory for nonlinear evolution PDEs.

Let us first discuss a simple corollary that follows from the above spectral properties of (2.24). Since by assumption $T = 1$ is the first blow-up point of $u(x,t)$, we study solutions of (2.21) that are globally defined in $\mathbb{R}^3 \times \mathbb{R}_+$, i.e., do not blow-up in finite $\tau$. Moreover, the scaling (2.18) implies that we are looking for orbits with very sharp $L^2$-divergence,

$$\|\hat{u}(\tau)\|_2^2 = c_1 e^{\frac{1}{2} \tau}(1 + o(1)) \quad \text{as} \quad \tau \to +\infty,$$

where $c_1 = \|u(\cdot,1)\|_2^2 > 0$. Consider the energy identity for smooth rescaled solutions

$$\frac{1}{2} \frac{d}{d\tau}\|\hat{u}(\tau)\|_2^2 d\tau = -\|D\hat{u}(\tau)\|_2^2 + \frac{1}{4} \|\hat{u}(\tau)\|_2^2.$$

Solving it together with (2.35) yields the following control of the gradient $D\hat{u}(\tau)$:

$$\|u(\tau)\|_2^2 = (c_1 + 2 \int_0^\infty e^{-\frac{\tau}{2}} \|D\hat{u}(s)\|_2^2 ds) e^{\frac{\tau}{2}} \quad \Longrightarrow \quad \int_0^\infty e^{-\frac{\tau}{2}} \|D\hat{u}(s)\|_2^2 ds < \infty.$$

In particular, for any $\varepsilon > 0$, the following measure is always finite and satisfies:

$$\text{meas}\{s \gg 1 : \|D\hat{u}(s)\|_2^2 \geq \varepsilon e^{\frac{\tau}{2}}\} = o\left(\frac{1}{\varepsilon}\right) \quad \text{as} \quad \varepsilon \to 0^+.$$

Indeed, in comparison with (2.35), this shows certain “degeneracy” for $\tau \gg 1$ of the $L^2$-norm of the gradient $\Delta\hat{u}(\tau)$ relative to that of $\hat{u}(\tau)$. At least, this shows that the gradient cannot have a uniform exponential growth as in (2.35) and should be much slower. In general, this does not imply essential consequences, since, according to (2.24),

the actual blow-up evolution of $\hat{u}(\tau)$ as $\tau \to +\infty$

is restricted to the space $L^2_{\rho^*}(\mathbb{R}^N)$, rather than $L^2(\mathbb{R}^N)$.

Therefore, in particular, the exponential $L^2$-divergence of the orbit (2.35) on expanding spatial subsets as $y \to \infty$ can be “invisible” in the metric $L^2_{\rho^*}(\mathbb{R}^N)$ with the exponentially decaying weight $\rho^*(y) = e^{-|y|^2/4}$.

Thus, this is the class of global orbits of interest, and we are looking for the structure of these $\omega$-limit sets, which, for such smooth orbits, are defined via local uniform convergence.

As usual in dynamical system theory, one first discusses the case when the orbit $\{\hat{u}(\tau)\}$ approaches the simplest invariant manifold being a point. Thus, assume that

$$\hat{u}(\tau) \to \bar{u} \quad \text{as} \quad \tau \to +\infty,$$

where, for future use, we suppose convergence in $L^q(\mathbb{R}^3)$, with $q > 3$ (note that a standard topology is expected to be that of $L^2_{\rho^*}(\mathbb{R}^3)$ or the corresponding Sobolev one $H^2_{\rho^*}(\mathbb{R}^3)$).
Then \( \bar{u} \) is necessarily a self-similar profile, so that by global and local nonexistence results \[34, 152, 162, 208 \]

\[ \bar{u} = 0. \]

Note that the spectrum (2.25) of the linearized operator implies that, with a proper control of the quadratically small convection term (this is easy in \( L^2_{\rho^*} \) with the exponentially decaying weight), the non-stationary small solutions \( \hat{u}(\tau) \) must satisfy

\[ |\hat{u}(\tau)| \sim O\left(e^{-\frac{\tau}{2}}\right) \quad \text{as} \quad \tau \to +\infty, \]

where \( \lambda_0 = -\frac{1}{2} \) is precisely the spectral gap in (2.25). More precisely, as a new application of spectral theory, we have the following:

**Proposition 2.1.** Assume that, along a subsequence \( \{\tau_k\} \to +\infty \),

\[ \hat{u}(y, \tau_k) \to 0 \quad \text{uniformly in} \quad L^\infty \cap L^2_{\rho^*}. \]

Then \( t = T \) is not a blow-up time for \( u(x, t) \) (in other words, the singularity is removable).

**Proof.** Consider the sequence of solutions \( \{\hat{u}_k(y, s) = \hat{u}(y, \tau_k + s)\} \) with vanishing initial data in \( L^\infty \) according to (2.42). Using and well-developed spectral properties of the linearized operator \( B^* \) in (2.24) defined in \( L^2_{\rho^*}(\mathbb{R}^N) \), with generalized Hermite polynomials as a complete and closed set of eigenfunctions. Therefore, according to classic asymptotic parabolic theory (see e.g., [145]), we conclude that for any sufficiently large \( k \),

\[ \hat{u}_k(y, s) \sim O\left(e^{-\frac{s}{2}}\right), \quad s \gg 1 \quad \Rightarrow \quad \hat{u}(y, \tau) \sim O\left(e^{-\frac{\tau}{2}}\right), \quad \tau \gg 1. \]

Overall, taking into account Leray’s scaling (2.18), this yields \( (T = 1): \)

\[ u(x, t) \sim (T - t)^{-\frac{1}{2}}O\left(e^{-\frac{\tau}{2}}\right) = O(1) \quad \text{as} \quad t \to T^-, \]

so that \( u(x, t) \) is uniformly bounded at \( t = T \).

Thus, equation (2.21) does not allow stabilization to an equilibrium, since this corresponds to the no-blow-up case. Further, as usual in textbooks on dynamical systems, the next candidate for being the corresponding invariant manifold is a periodic orbit of finite period \( T^* > 25 \). In [178, p. 1218], this conjecture was connected with the study of the complex Ginzburg–Landau equation

\[ i u_t + (1 - i \varepsilon) \Delta u + (1 + i \delta)|u|^{2\sigma}u = f \quad \text{in} \quad \mathbb{R}^N \times (0, T), \]

where \( \varepsilon > 0, \delta \geq 0 \) and \( \sigma > \frac{2}{N} \). It was pointed out in [178] that the model (2.45) exhibits the same scaling, similar energy control, and local semigroup theory; see also [149, § 4] for some related estimates of such a periodic behaviour of an unknown structure. It was pointed out in [220] (see also [141]) that the CGLE such as (2.45) can be a good PDE

\[ \text{□} \]

\[ 25 \] The role of \( \tau \)-periodic motion for rescaled blow-up solutions of the Euler equations (1.31) was pointed out in [180] (however, the nonexistence conclusion on p. 218 therein looks rather suspicious).

\[ 26 \] However, unlike (1.11), it was shown in [178] and in other papers cited therein that (2.45) admits a lot of (countable families of?) blow-up similarity profiles, and this makes it more analogous to the Cahn–Hilliard model (1.57); see [50], where a countable blow-up family was detected even for \( N = 1 \), and [70], where Leray’s scenario was shown to apply.
system modelling regularity and other questions regarding the NSEs (1.1). In particular, global existence of weak solutions can be obtained analogously to Leray’s proof. Incidentally, concerning the singular set of blow-up points for (2.46) of zero measure [220], it is known that this set is restricted to a bounded domain, [141].

Actually, this would mean proving for (2.21) the Poincaré–Bendixson theorem saying, essentially, that if a rescaled orbit of (2.21) satisfying (2.36), (2.37) does not stabilize to an equilibrium (and actually (2.44) makes this impossible), then it convergence to a simple closed curve,

\[(2.46) \quad \omega(\hat{u}_0) \text{ consists of a } T_1 \text{-periodic orbit } \Gamma_1.\]

According to (2.39), the natural metric of convergence as \(\tau \to +\infty\) in (2.46) is then assumed to be that of \(L^2\), while the \(L^2\)-divergence (2.35) occurs on subsets with \(|y| \gg 1\), which does not affect the convergence. A more clear discussion of such blow-up periodic orbits will be postponed until Section 7.4, where a proper spectral theory is under scrutiny.

Of course, (2.46) would be the best and very pleasant case. Indeed, for general DSs of such complexity, (2.46) is a very difficult open problem. Hence, the first main principal difficulty is how to predict a possible structure of such blow-up “periodic orbits”. Actually, this is one of our main goals, and the blow-up angular swirl mechanism (1.25) in Section 5 is a natural argument in support of the periodic motion (2.46).

Continuing using the logic of standard dynamical system theory, we next would have to assume that a periodic \(\omega\)-limit set would have been ruled out (i.e., being nonexistent) for a given rescaled orbit \(\{\hat{u}(\tau)\}\). Then we should, e.g., conjecture a countable set of other possibilities of evolution geometrically related to tori in \(\mathbb{R}^{n+1}\) for \(n = 2, 3, 4, \ldots\),

\[(2.47) \quad \omega(\hat{u}_0) \text{ consists of a quasi-periodic orbit } \Gamma_n \text{ driven by } n \text{ fund. frequencies.}\]

where \(n = 1\) leads to (2.46). To get a quasi-periodic motion for \(n = 2\) on the invariant \(\omega\)-limit set, we introduce in Section 8 the idea of precessions of the swirl axis, which leads to extremely difficult and open mathematics. For \(n \geq 3\), such a clear visual geometric interpretation of the scenarios (2.47) is not that easy or straightforward. A spectral background for bifurcation of such patterns is difficult and obscure (cf. Section 7.4).

Eventually, under the assumption of nonexistence of all of those types of \(\omega\)-limits in (2.47), passing to the limit \(n \to \infty\) would then have led to a kind of a strange attractor for the dynamical system (2.21), which can be an extremely complicated invariant manifold to be proved to exist. Recall that, for proving nonexistence of \(L^\infty\)-singularities

\[27\text{Is there any hope that a nonstationary version of the elliptic differential inequality (2.10) applied to the nonlocal parabolic PDE (2.21) can rule out at least some of special quasi-periodic oscillations about the unique equilibrium 0? (seems, not sufficient, and no hope).}\]

\[28\text{The proof of existence of a robust strange attractor for the E. Lorenz dynamical system in } \mathbb{R}^3 \text{[144] ("3 \times 1", i.e., uncomparably easier), describing thermal fluid convection with some relations to the NSEs, proposed in 1963 was declared by Smale as one of the several challenging problems for the twenty-first century (1998), which eventually took nearly four decades to complete; see [209, 210] and also [153].}\]
for (1.1), all these infinite number of possibilities have to be ruled out (via a new energy/monotonicity/spectral, etc. control).

2.8. Global similarity solutions defined for all \( t > 0 \) do exist. Solenoidal linearized patterns. In contrast to blow-up, as usual for typical parabolic reaction-diffusion-absorption problems, global similarity solutions of (1.1) without blow-up, i.e., (2.7) with \( T - t \rightarrow t > 0 \) occur more frequently and correspond to the following scaling and rescaled equations (the invariant of the scaling group involved, \( y = x/\sqrt{t} \), is sometimes called the Bolzman substitution\(^{29}\)):

\[
\mathbf{u}(x,t) = \frac{1}{\sqrt{t}} \hat{\mathbf{u}}(y,\tau), \quad y = \frac{x}{\sqrt{t}}, \quad \tau = \ln t \rightarrow +\infty \quad \text{as} \quad t \rightarrow +\infty,
\]

where \( \hat{\mathbf{u}} = \Delta \hat{\mathbf{u}} + \frac{1}{2} (y \cdot \nabla) \hat{\mathbf{u}} + \frac{1}{2} \hat{\mathbf{u}} - P (\hat{\mathbf{u}} \cdot \nabla) \hat{\mathbf{u}} \) in \( \mathbb{R}^3 \times \mathbb{R}_+ \).

Here we obtain the linear operator

\[
\tilde{\mathbf{B}} = \mathbf{B} - I,
\]

where \( \mathbf{B} \) is adjoint to \( \mathbf{B}^* \) in the metric of the dual space \( L^2(\mathbb{R}^3) \). The spectrum of \( \tilde{\mathbf{B}} \), which has a self-adjoint (Friedrichs) extension in \( L^2_\rho \), where \( \rho = \frac{1}{\rho} \) \([15]\), is

\[
\sigma(\tilde{\mathbf{B}}) = \{ \lambda_k = -\frac{k^2}{2} - 1, \quad k = |\beta| = 0, 1, 2, ... \}.
\]

Non-trivial self-similar profiles, i.e., stationary solutions of (2.48), do exist and describe asymptotics as \( t \rightarrow +\infty \) of various sufficiently small solutions; see \([30, 29, 19, 81, 82, 89, 92, 177]\). In \([177]\), a simple criterion of asymptotic similarity form for solutions was obtained. Note that Slezkin–Landau singular stationary solutions (7.1) are self-similar.

In addition, by (2.50), 0 is asymptotically stable, and this makes it possible to construct fast decaying solutions on each 1D stable manifolds with the asymptotic behaviour\(^{30}\)

\[
\mathbf{u}_\beta(x,t) \sim t^{k} \mathbf{v}_\beta \left( \frac{x}{\sqrt{t}} \right) + ... \quad \text{as} \quad t \rightarrow \infty, \quad \mathbf{v}_\beta = \mathbf{v}_\beta^* F \in S_k
\]

are solenoidal eigenfunctions of \( \mathbf{B} \) defined in Section 2.6. Namely, taking

\[
\mathbf{v} = (v_1, v_2, v_3)^T \in S_k, \quad v_i \in \Phi_k = \{ \psi_\beta = \left( \frac{-1}{\sqrt{\beta}} \right)^{|\beta|} D^\beta F(y), \quad |\beta| = k \},
\]

where \( F \) stands for the rescaled Gaussian (see (5.60) with \( N = 3 \)), we have that

\[
\text{div} \mathbf{v} = (v_1)_{y_1} + (v_2)_{y_2} + (v_3)_{y_3} = \text{div}(\mathbf{v}^* F) \equiv (\text{div} \mathbf{v}^*) F - \frac{1}{2} y \cdot \mathbf{v}^* F.
\]

This establishes a one-to-one correspondence between solenoidal eigenfunction classes \( S_k^* \) in (2.31) for \( \mathbf{B}^* \) and \( S_k \) in (2.51) for \( \mathbf{B} \); see (2.32) for the first eigenfunctions \( \mathbf{v}_\beta = \mathbf{v}_\beta^* F \). Therefore, \( \dim S_k = k(k+2) \), etc.; see details and rather involved proofs of the asymptotics (2.51) for \( k = 1 \) and 2 in \([81]\). We will deal with patterns such as (2.51) later on.

\(^{29}\)Similarity solutions were used by Weierstrass around 1870, and by Bolzmann around 1890; this rescaled variable \( y \) in parabolic PDEs was widely used by Sturm in 1836 \([203]\) (and possibly even before?)

\(^{30}\)We present here only the first term of expansion; as usual in dynamical system theory, other terms in the case of “resonance” can contain \( \ln t \)-factors (\( \text{q.v.} \) \([4]\) for a typical PDE application); this phenomenon was shown to exist for the NSEs in \( \mathbb{R}^2 \) \([82, \text{p. 236}] \).
It follows from \((2.32)\) that there exists the corresponding eigenfunctions of \(\tilde{B}\) with \(\lambda_1 = -\frac{3}{2}\) given by \((2.50)\). Hence, by scaling \((2.48)\), for instance, there exists the asymptotic pattern for \(t \gg 1\) (the rate \(O(t^{-2})\) in \(L^\infty\) is thus sharp for \(u_0 \in \hat{L}_2^2(\mathbb{R}^3)\))

\[
(2.54) \quad u_1(x, t) \sim \frac{1}{t^2} v_1 \left( \frac{x}{\sqrt{t}} \right), \quad \text{where} \quad v_1(y) = \frac{1}{4\sqrt{2}} (y_3, y_3, 0)^T e^{-\frac{|y|^2}{4}}.
\]

In the half space \(x \in \mathbb{R}^3 \cap \{x_3 > 0\}\) with no slip boundary condition \(u|_{x_3=0} = 0\), the decay rate of solutions is also of interest (clearly, some of the polynomials in \((2.32)\) are good for that); see [39] for recent developments.

Note again that the calculus of solenoidal eigenfunction classes look like being specially designed to suit the divergence-free flows not only for both types of scalings, blow-up \((2.18)\) and the global one \((2.48)\), but also for the Burnett equations \((1.6)\), where classes \(S_k\) can be defined for any necessary \(2m\)th-order linear operators \([51]\). The adjoint ones \(S_k^*\) then are also composed from solenoidal generalized Hermite polynomials only.

3. First application of Hermitian spectral theory: Sturmian local structure of zero sets of bounded solutions and unique continuation

3.1. Nodal sets for the Stokes problem and NSEs. Here we perform a first step towards the classification problem \((1.20)\). Namely, we assume that at the point \((x, t) = (0, 1)\) the solution \(\hat{u}(y, t)\) is uniformly bounded and is such that the eigenfunction expansion of the corresponding rescaled function satisfying \((2.40)\),

\[
(3.1) \quad \hat{u}(y, \tau) = \sum_{\beta} c_\beta(\tau) v_\beta^*(y), \quad \text{where} \quad c_\beta v_\beta^* = (c_{1\beta}, c_{2\beta}, c_{3\beta})^T \in \hat{L}_2^2(\mathbb{R}^3),
\]

converges in \(\hat{L}_2^2(\mathbb{R}^3)\), and moreover, uniformly on compact subsets. These convergence questions of polynomial series are standard; see \([51, 67]\), where further references and details are given. Then the expansion coefficients satisfy the following dynamical system:

\[
(3.2) \quad \begin{cases} 
\dot{c}_\beta = (\lambda_\beta - \frac{1}{2}) c_\beta + \sum_{\alpha, \gamma} d_{\alpha\gamma\beta} c_\alpha c_\gamma & \text{for any } |\beta| \geq 0, \\
\text{where} & d_{\alpha\gamma\beta} = -\langle \mathbb{P} (\hat{v}_\alpha^* \cdot \nabla) \hat{v}_\gamma^*, v_\beta \rangle \text{ for all } \alpha, \gamma.
\end{cases}
\]

It is natural to assume that the quadratic sum on the right-hand side converges for the given smooth rescaled solution \(\hat{u}(y, \tau)\). Recall that, according to scaling \((2.18)\), we deal with bounded and uniformly exponentially small functions satisfying

\[
(3.3) \quad |\hat{u}(y, \tau)| \leq C e^{-\frac{\tau}{2}} \text{ in } \mathbb{R}^3 \times \mathbb{R}_+.
\]

The system \((3.2)\) is difficult for a general study, and, of course, it contains the answer to the existence/nonexistence problem, provided that \((0, 1)\) is a singular point of the solution \(u(x, t)\). For regular points, it can provide us with a typical classification of nodal sets of solutions. This kind of study was first performed by Sturm in 1836 for linear 1D parabolic equations \([203]\); see historical and other details in \([67, \text{Ch. 1}]\).

Thus, following these lines, we clarify local zero sets of solutions of the NSEs at regular points. Assume that

\[
(3.4) \quad u(0, 1) = 0,
\]
which can be always achieved by constant shifting \( u(x,t) \mapsto u(x,t) - u(0,1) \). In this connection, recall that the first eigenfunctions with \( \lambda_\beta = 0 \)
\[
\begin{align*}
\nu_\beta^0(y) &\sim (1,1,1)^T, (1,1,0)^T, (1,0,1)^T, \ldots,
\end{align*}
\]
are the only ones that have empty nodal sets of some of its components. Then, bearing in mind the blow-up scaling term \((1 - t)^{-\frac{3}{2}} \equiv e^{\frac{3}{2} \tau}\) in (2.18), we have to assume that
\[
\begin{align*}
c_0(\tau) &= 0 \quad \text{or} \quad c_0(\tau) \to 0 \quad \text{as} \quad \tau \to +\infty \quad \text{exponentially faster than} \quad e^{-\frac{3}{2} \tau}.
\end{align*}
\]

Polynomial nodal sets for the Stokes problem. A first clue to a correct understanding of the DS (3.2) is given by the Stokes problem, i.e., without the nonlinear convection term,
\[
\begin{align*}
\nu_t &= -\nabla p + \Delta \nu, \\
\text{div} \; \nu &= 0.
\end{align*}
\]
Then (3.2) becomes linear diagonal and is easily solved:
\[
\begin{align*}
\dot{c}_\beta &= (\lambda_\beta - \frac{1}{2})c_\beta \quad \Rightarrow \quad c_\beta(\tau) = c_\beta(0)e^{-\frac{(1+|\beta|)\tau}{2}} \quad \text{for any} \quad |\beta| \geq 0.
\end{align*}
\]
Therefore, according to (3.1), all possible multiple zero asymptotics for the Stokes problem (its local “micro-scale turbulence”) is described by finite solenoidal Hermite polynomials, and the zero sets of rescaled velocity components also asymptotically, as \( \tau \to +\infty \) (i.e., \( t \to 1^- \)) obey the nodal Hermite structures.

NSEs. Consider the full nonlinear dynamical system (3.2), which on integration is
\[
\begin{align*}
\dot{c}_\beta &= (\lambda_\beta - \frac{1}{2})c_\beta \quad \Rightarrow \quad c_\beta(\tau) = c_\beta(0)e^{-\frac{(1+|\beta|)\tau}{2}} \quad \text{for any} \quad |\beta| \geq 0.
\end{align*}
\]
It follows that the nonlinear quadratic terms in (3.9), under certain assumptions, can affect the rate of decay of solutions near the multiple zero. As usual in calculus, this indeterminacy can be tackled by L’Hospital rule.

Since we are mainly interested in the study of nodal structures of solutions by using the eigenfunction expansion (3.1), we naturally need to assume that it is possible to choose the leading decaying term (or a linear combination of terms) in this sum as \( \tau \to +\infty \). Then obviously these leading terms will asymptotically describe the Hermitian polynomial structure of nodal sets as \( t \to 1^- \). For PDEs with local nonlinearities, this is done in a standard manner as in [67, § 4]; in the nonlocal case, this seems can cause technical difficulties. However, the DS (3.2) looks (but illusionary) as being obtained from a problem with local nonlinearities. In other words, the nonlocal nature of the NSEs is hidden in (3.2) in the structure of the quadratic sum coefficients \( \{d_{\alpha,\gamma,\beta}\} \), and this do not affect the nodal set behaviour for some class of multiple zeros. We will check this as follows:

We consider a “resonance class” of multiple zeros. Namely, let us assume there exist a multiindex subset \( B \) and a function \( h(\tau) \to 0 \) such that
\[
\begin{align*}
c_\beta(\tau) &\sim h(\tau) \quad \text{as} \quad \tau \to +\infty \quad \text{for any} \quad \beta \in B, \\
|c_\beta(\tau)| &\ll |h(\tau)| \quad \text{as} \quad \tau \to +\infty \quad \text{for any} \quad \beta \notin B.
\end{align*}
\]
In other words, only the coefficients \( \{c_\beta(\tau), \beta \in B\} \) are assumed to define the nodal set via (3.1), and other terms are negligible as \( \tau \to +\infty \). Under the natural assumption of
a strong enough convergence of the quadratic sums in (3.2) (this can be expected not to be the case for singular blow-up points only), taking the ODEs from (3.2) for each \( \beta \in B \) yields, for \( \tau \gg 1 \),
\[
(3.11) \quad \dot{c}_\beta = (\lambda_\beta - \frac{1}{2})c_\beta + o(c_\beta), \quad \text{where} \quad c_\beta(\tau) \sim h(\tau).
\]
Hence, the asymptotic balancing of these equations must assume that as \( \tau \to +\infty \)
\[
(3.12) \quad \dot{h} \sim (\lambda_\beta - \frac{1}{2})h \quad \Rightarrow \quad c_\beta(\tau) \sim h(\tau) \sim e^{(-\frac{1}{2} - \frac{1}{2})\tau} \quad \text{and} \quad |\beta| = k,
\]
where we may omit lower-order multipliers. Thus, there exists a \( k \geq 1 \) such that \( |\beta| = k \) for any \( \beta \in B \). One can see that for such “resonance” multiple zeros, the nonlocal quadratic term in (3.2) is not important. Thus, in the resonance zero class prescribed by (3.10), as \( \tau \to +\infty \), on compact subsets in \( y \), similar to Stokes’ problem,
\[
(3.13) \quad \text{the nodal set of } \hat{u}(y, \tau) \text{ is governed by some solenoidal Hermitian polynomials.}
\]
Note that the conclusion that, locally, for any zero of finite order at \((0,1)\),
\[
(3.14) \quad \text{nodal sets of } u(x,t) \text{ are governed by finite-degree polynomials}
\]
is trivially true for any sufficiently smooth solution. Indeed, this follows from the Taylor expansion of such solutions
\[
(3.15) \quad u(x,t) = \sum_{(\mu,\nu)} \frac{(-1)^\nu}{\mu!\nu!} \left( D_{tt}^{\mu\nu}u \right)(0,1) x^\mu (1-t)^\nu + R(x,t),
\]
where \( R \) stands for a higher-order remainder. Translating (3.15) via (2.18) into the expansion for \( \hat{u}(y, \tau) \) yields some polynomial structure, so (3.14) is obviously true. Thus, the principal feature of (3.13) is that the Hermite polynomials count only therein.

Obviously, for the nonlocal problem (2.21), there exist other non-resonance zeros. Indeed, let \((0,1)\) be a zero of \( u(x,t) \) of a finite order \( m \geq 1 \), i.e., as \( x \to 0 \),
\[
(3.16) \quad u(x,1) \sim x^\sigma, \quad \text{with} \quad |\sigma| = m.
\]
We now use the following expansion:
\[
(3.17) \quad u(x,t) = u(x,1) - u_t(x,1)(1-t) + \frac{1}{2!} u_{tt}(x,1)(1-t)^2 + \ldots,
\]
where, by (2.21), all the time-derivatives \( D_t^\mu u(x,0) \) can be calculated:
\[
(3.18) \quad u_t(x,1) = \Delta u(x,1) + (\mathbb{P}(u \cdot \nabla)u)(x,1) \sim x^{\sigma-2} + (\mathbb{P}(u \cdot \nabla)u)(x,1),
\]
with a natural meaning of \( \Delta x^\sigma \sim x^{\sigma-2} \). If the nonlocal term is negligible here and for other time-derivatives, i.e.,
\[
u \in \mathbb{N}, \quad \nu \leq \alpha(n) = \frac{d}{d+1} \quad \text{and} \quad \alpha(n) \leq \alpha(n+1),
(3.19) \quad u_t(x,1) \sim x^{\sigma-2}, \quad u_{tt}(x,1) \sim x^{\sigma-4}, \ldots,
\]
then according to (3.17) this leads to a Hermitian structure of nodal sets. In fact, this corresponds to the pioneering zero-set calculus performed by Sturm in 1836; see his original computations in [66, p. 3].
In general, the nonlocal term in (3.18) is not specified by a local structure of the zero under consideration, so, obviously, can essentially affect the zero evolution. For instance, as a hint, we can have the following zero:

\[ u(0, 1) = C \neq 0 \implies u(x, t) \sim x^\sigma + (1 - t) \sim e^{-\tau}(1 + z^\sigma), \quad z = \frac{x}{(1-t)^{1/m}}, \]

so this nodal set is governed by the rescaled variable \( z \), which is different from the standard similarity one \( y \) in (2.18). Of course, due to the nonlocality of the equation, many other types of zeros can be described. Actually, such non-resonance zeros can be governed by sufficiently arbitrary polynomials as the general expansion (3.15) suggests.

Finally, the proof that zeros of infinite order are not possible (and, as usual in such Carleman and Agmon-type uniqueness results, this occurs for \( u \equiv 0 \) only) is a difficult technical problem; see an example in [67, § 6.2]. For analytic in \( y \) solutions of the NSEs (see references and results in [49, 224]), this problem is nonexistent, and then in (3.13) the degree of the solenoidal vector Hermite polynomials is always finite, though can be arbitrarily large.

Note another straightforward consequence of this analysis that this gives the following conventional unique continuations result: let (3.11) hold, \( (0, 1) \) be a resonance zero\(^{31}\), and at least one component of the nodal set of \( \hat{u}(y, \tau) \) does not obey (3.13). Then

\[ u \equiv 0 \quad \text{everywhere}. \]

Of course, this is not that surprising since the result is just included in the existing and properly converging eigenfunction expansion (3.1) under the assumption (3.10).

For elliptic equations \( P(x, D)u = 0 \), this has the natural counterpart on strong unique continuation property saying that nontrivial solutions cannot have zeros of infinite order; a result first proved by Carleman in 1939 for \( P = -\Delta + V, \quad V \in \mathcal{L}_\infty^{\text{loc}}, \quad \mathbb{R}^2 \) [32]; see [50, 205] for further references and modern extensions.

Thus, this is the first application of solenoidal Hermitian polynomial vector fields for regular solutions of the NSEs. We expect that, due to the DS (3.2), some “traces” of such an analysis and Hermite polynomials should be seen in the fully nonlinear study of \( \hat{u}(y, \tau) \) at the singular blow-up point \( (0, 1) \), where, instead of (3.4), we have to assume that, in the sense of \( \limsup_{x, t} \),

\[ |u(0, 1)| = +\infty. \]

3.2. Burnett equations. For (1.6), the blow-up scaling (2.18) is replaced by

\[ u(x, t) = (1 - t)^{-\frac{3}{4}} \hat{u}(y, \tau), \quad y = \frac{x}{(1-t)^{3/4}}, \]

so that the rescaled system (2.21) takes a similar form

\[ \hat{u}_\tau = H(\hat{u}) \equiv (B^* - \frac{3}{4} I)\hat{u} - \mathbb{P}(\hat{u} \cdot \nabla)\hat{u} \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+. \]

\(^{31}\)Indeed, this is hard to check; for PDEs with local nonlinearities, this assumption is not needed, so that such a unique continuation theorem makes full sense, [67].
The spectral theory of the given here adjoint operator
\[ B^* = -\Delta^2 - \frac{1}{4} y \cdot \nabla, \quad \text{where} \quad \sigma(B^*) = \left\{ \lambda_\beta = -\frac{|\beta|^4}{4}, \quad |\beta| = 0, 1, 2, \ldots \right\} \]
with eigenfunctions being generalized Hermite polynomials is available in [51]; a solenoidal extension in the same lines is needed. Therefore, under the same assumptions, the polynomial structure of nodal sets is guaranteed for the corresponding Stokes-like and Burnett equations (and for an arbitrary \( 2m \)-th-order of the viscosity \(-\Delta^m u \) therein).

4. Second application of blow-up scaling: on convergence to the EEs

We now present the results of a general application of another related blow-up scaling for establishing a connection between blow-up in the NSEs and singularities in the EEs. We use the blow-up scaling in the form of [80, §2], which received other applications and extensions in [68, 33, 74], etc. As usual, such a rescaling near blow-up time leads to ancient solutions in Hamilton’s notation [94], which has been a typical technique of R–D theory; see various form of its application in [187, 79] and others. In [74], this scaling technique applies to the NSEs and Burnett equations in \( \mathbb{R}^N \) to present a simple treatment of the corresponding Leray–Prodi–Serrin–Ladyzhenskaya regularity \( L^{p,q} \) criteria and other estimates. Ancient solutions of the 3D NSEs allowed recently to get new non-blow-up results for axi-symmetric flows; see [118] and also Section 5.12.

We begin with a definition, which settles an “evolution” concept of entropy solutions for the EEs. In blow-up R–D theory, such concepts of “entropy-viscosity” come from extended semigroup theory, where proper (blow-up or singular) solutions are only those, which can be obtained by regular approximations. For problems with the MP, such extended semigroup theory leads to the unique continuation (partially or completely unbounded) of any blow-up solutions beyond blow-up time \( t > T \), [60, §6.2].

The questions of the vanishing viscosity limits in the NSEs to get the EEs are classical in fluid mechanics and lead in general to a number of fundamental open problems; see a clear statement of such questions in [8, p. 422], where necessary topologies of convergence are prescribed. For general (not necessarily bounded) solutions, the estimates (2.2) suggest the weak-* topology of \( L^\infty(\mathbb{R}_,, L^2(\mathbb{R}^3)) \), while for bounded solutions this can be improved (the reason for the uniform boundedness is associated with the blow-up scaling to be applied). However, we are not obliged to deal with a specific convergence, especially since the necessary minimal topology for in what follows is still unknown.

**Definition 4.1.** A function (“distribution”) \( u(\cdot, t) \) is said to be a bounded NS-entropy (i.e., Navier–Stokes-entropy) solution of the EEs (1.31) if it is obtained as the limit
\[ u_k \to u \quad \text{as} \quad k \to \infty \quad \text{weak-}* \quad \text{in} \quad L^\infty(\mathbb{R}^3 \times \mathbb{R}_-) \]
of a sequence of bounded classical solutions \( \{u_k(x, t)\} \),
\[ |u_k(x, t)| \leq 1 \quad \text{and} \quad \|u_k(\cdot, t)\|_2 \leq C \quad \text{for} \quad k \geq 0, \]
of the NSEs with vanishing viscosity coefficients,
\[ u_k : \quad u_t + P(u \cdot \nabla)u = \delta_k \Delta u, \quad \text{where} \quad \delta_k \to 0^+ \quad \text{as} \quad k \to \infty. \]
The given in (4.1) weak-* topology of convergence follows from the fact that, by (4.2), the sequence \( \{u_k\} \) is bounded in \( L^\infty(\mathbb{R}^3 \times \mathbb{R}^-) \). Note that we do not specify in which sense then \( u \) satisfies the EEs,

\[
(4.4) \quad u_t + P(u \cdot \nabla)u = 0,
\]

and we even do not demand \( u \) to be any kind of weak solution of (4.4). Note that this is not always the case even for nonlinear dispersion equations (NDEs) such as [69, 75]

\[
(4.5) \quad u_t = (uu_x)_x, \quad u_t = (uu_x)_x, \quad u_t = uu_{xxx}, \quad \text{and} \quad u_t = -(uu_x)_{xxx}, \quad \text{etc.}
\]

On the other hand, the regularity concepts are not always an option in singularity theory: it is known that sometimes even analytic extensions of solutions are not a proper one, i.e., proper (extremal) solutions have finite regularity; see an example in [66, p. 140].

Thus, according to (4.1), the concept of the NS-entropy for the EEs includes the (natural, indeed) way of regular approximations of its solutions. In this sense, this is quite similar to conservation laws theory, e.g., for the 1D Euler equation

\[
(4.6) \quad u_t + uu_x = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+,
\]

with \( L^1 \)-data \( u_0 \). The parabolic approximation is as follows:

\[
(4.7) \quad u = \lim_{\varepsilon \to 0} u_\varepsilon, \quad \text{where} \quad u_\varepsilon : \quad u_t + uu_x = \varepsilon u_{xx}, \quad u(x,0) = u_0(x),
\]

with the convergence in \( L^1 \). Of course, classic entropy concepts apply directly to (4.6) (say, in the sense of Oleinik and Kruzhkov), but the entropy regularization description (4.7) is completely self-consistent; see [198] for details. Note that conservation laws such as (4.6) are natural zero-level models for the 3D EEs, where both features of the divergence and the \( L^2 \)-control are available. Then Tartar–Murat’s compensation compactness approach could be also natural, but, unfortunately, seem not applicable to singularly perturbed EEs [4.3].

Further “distributional” properties of NS-entropy solutions of the EEs are unknown, and these compose the core of the problem; see below.

4.1. The NSEs. Thus, we assume that there exist sequences \( \{t_k\} \to T^- \leq \infty, \{x_k\} \subset \mathbb{R}^N, \) and \( \{C_k\} \to +\infty \) such that the solution \( u(x,t) \) of (1.1) becomes unbounded:

\[
(4.8) \quad \sup_{\mathbb{R}^N \times [0,t_k]} |u(x,t)| = |u(x_k,t_k)| = C_k \to +\infty \quad \text{as} \quad k \to \infty.
\]

As in [80, § 2], we then perform the change

\[
(4.9) \quad u_k(x,t) \equiv u(x_k + x, t_k + t) = C_k w_k(y,s), \quad \text{where} \quad x = a_k y, \quad t = a_k^2 s,
\]

where the sequence \( \{a_k\} \) is such that the \( L^2 \)-norm is preserved after rescaling, i.e.,

\[
(4.10) \quad \|u_k(t)\|_2 = \|w_k(s)\|_2 \quad \implies \quad a_k = C_k^{-\frac{2}{3}} \to 0.
\]

Taking the NSEs in the nonlocal from as in (2.21), we then obtain the following rescaled equations for \( w = w_k(y,s) \):

\[
(4.11) \quad w_s + \delta_k P(w \cdot \nabla)w = \Delta w, \quad \text{where} \quad \delta_k = C_k^{\frac{3}{4}}.
\]
Next, after time shifting, $s \mapsto s-s_0$, with a fixed arbitrarily large $s_0 > 0$, the solutions and data satisfy the uniform bounds: for all $k \gg 1$

\begin{equation}
|w_k(s)| \leq 1 \quad \text{and} \quad \|w_k(s)\|_2 \leq C \quad \text{for all} \quad s \in [-s_0, 0].
\end{equation}

The principle (and obvious, otherwise global existence would trivially follow) fact is that

\begin{equation}
\delta_k = C_k^{1/3} \to +\infty \quad \text{as} \quad k \to \infty \quad (\text{actually meaning that} \quad \mathcal{L}^2 \not\Rightarrow \mathcal{L}^\infty),
\end{equation}

so that (4.11) is a singularly perturbed problem to be analyzed as follows:

We divide the equation (4.11) by $\delta_k$ and introduce the new time $\bar{s} = \delta_k s$ to get for $w_k = w_k(\bar{s})$ the equation

\begin{equation}
\frac{1}{\delta_k} w_{\bar{s}} + P(w \cdot \nabla)w = \frac{1}{\delta_k} \Delta w.
\end{equation}

In view of (4.12), the sequence is converging weak-$*$ in $L^\infty(\mathbb{R}^3 \times \mathbb{R}^-)$:

\begin{equation}
w_k(\bar{s}) \rightharpoonup \bar{w}(\bar{s}) \quad \text{as} \quad k \to \infty
\end{equation}

(the convergence also takes place in better topologies). We then need to pass to the limit in the rescaled NSEs (4.14), where the right-hand side vanishes in the weak sense in view of the sufficient regularity (4.12). Concerning the quadratic convection term, in view of its divergence (2.4), one needs extra assumptions to get in the limit a weak formulation of the resulting EEs (locally, for sufficiently smooth solutions, this is well-known due to pioneering results by Kato (1983), Temam and Wang, etc., see survey [8, § 4], [217], and references therein). Using Definition 4.1 as a standard conclusion from (4.14), we obtain:

**Proposition 4.1.** Under the above hypotheses, the following holds:

(i) $\bar{w}$ in (4.15) is a bounded ancient NS-entropy solution of the EEs

\begin{equation}
\frac{4}{ds} \int |\bar{w}|^2 = 0 \quad (\leq 0),
\end{equation}

which is monotone on evolution orbits. Therefore, the omega-limit set in the topology of $C_{\text{loc}}(\mathbb{R}^3)$ of smooth orbits consists of equilibria, on which $V(\bar{w}) = \int |\bar{w}|^2$ vanishes locally on compact subsets, so $\bar{w} = 0$. In other words, we use the fact that smooth $L^2$-solutions of EEs decays to zero in time uniformly on compact subsets. Therefore, such solutions cannot satisfy the last normalization condition in (4.16). Actually, we do not need a deeper discussion here about these rather obscure aspects of EEs theory, since, moreover, writing (4.14) without an extra time-rescaling, i.e., keeping the variable $s$,

\begin{equation}
\frac{1}{\delta_k} w_{s} + P(w \cdot \nabla)w = \frac{1}{\delta_k} \Delta w.
\end{equation}

yields as $k \to \infty$ that $\bar{w}$ is a weak stationary bounded NS-entropy solution of the EEs. Finally, one can replace (1.2) by the following formal:
**Corollary 4.1.** If the only bounded stationary NS-entropy solution of the EEs with the additional to (4.1) uniform convergence on compact subsets is trivial, then (1.2) holds.

The convergence (4.15) embraces larger compact subsets in x than the self-similar one according to (2.7):

\[(4.19) \quad |x - x_k| = O(C_k^{-\frac{4}{3}}), \quad \text{while} \quad |x - x_k|_{(2.7)} = O(C_k^{-1}).\]

Admissible solutions of the problem (4.16) are generally unknown. Thus, (4.15) shows a general relation between blow-up in the NSEs and NS-entropy solutions of the EEs: special bounded singular solutions of the EEs must create and support blow-up for the NSEs, which eventually must evolve as \(t \to T^-\) on compact subsets that are smaller than those in (4.19) for the EEs. It general, the “almost trivial” case (cf. Corollary 4.1) (4.20)

\[\bar{w}(y) = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \{0\}\]

cannot be excluded. Obviously, for (4.20) to be treated seriously, one needs an extra stronger “N-topology” of convergence in (4.1), which should be adequate to the both NSEs and EEs and is unknown. On the other hand, in [70], we present almost “explicit” blow-up infinite energy patterns, where nontrivial limits like (4.15) take place. Some principles of formation of Type II blow-up patterns are discussed in Section 7.6.

**4.2. Burnett equations.** This is similar, consult [74] for final estimates.

**5. Construction of blow-up twistors on a 2D linear subspace in \(\mathbb{R}^3\)**

In this section, we demonstrate the fact coming from reaction-diffusion theory that the Navier–Stokes equations can admit blow-up behaviour generating a blow-up swirl (1.24) at a stagnation point with accelerating and eventually infinite angular speed as \(t \to T^-\). Such a blow-up behaviour is associated with the so-called logarithmic travelling waves in the tangential angular direction, which are group-invariant solutions that occur for some nonlinear diffusion-combustion PDEs. We explain such a behaviour in the cylindrical geometry, where the axis of rotation is fixed to be extended to more realistic spherical geometry in Section 8. To this end, we will perform a partially invariant construction (understood not in a standard way of invariance under a group of transformations).

Even on the 2D subspace \(W_2\) given in (1.23), the analysis of existence of such blow-up twistors is very difficult and leads to involved nonlinear systems that we are not able to study rigorously.\(^{32}\) In view of this, we will show another hypothetical way of constructing other related non-similarity patterns of unknown rotational-like singularity nature in conjunction with the multiplicity claim (1.29).

We begin by noting that the Navier–Stokes equations (1.1) inheriting for given data a strong spatial symmetry (for instance, cylindrically axisymmetric irrotational flows), actually reduce their “effective” dimension by one (a very formal and rough issue) and then get into the global existence and uniqueness case of classical solutions for \(N = 2\).

\(^{32}\)This blow-up swirl behaviour is naturally attached to the Poincaré–Bendixson-type conclusion (2.46).
This idea goes back to Ukhovskii–Yudovich [211] and Ladyzhenskaya [130] concerning axisymmetric geometry; see [193, 222] for proper detailed definitions of such global smooth flows and recent results and [147] for helical symmetries. Formally speaking, the present patterns with some features of axisymmetry could be not complicated enough to generate singularities with finite kinetic energy, so we will need further mental efforts to remove such a restriction to blow-up; see Section 8. On the other hand, global regularity of general axisymmetric flows is also not known, so that blow-up patterns may be possible even in this simplified geometry of the linear dependence on $z$, but, probably, after extra precession-type or other necessary manipulations with this axis of rotation; see below.

5.1. Navier–Stokes equations in cylindrical coordinates. We introduce the standard cylindrical coordinates $(r, \varphi, z)$ in $\mathbb{R}^3$, where

$$r^2 = x^2 + y^2 \quad \text{and} \quad e_r = (\sin \varphi, \cos \varphi, 0), \quad e_\varphi = (-\cos \varphi, \sin \varphi, 0), \quad e_z = (0, 0, 1).$$

Denote the corresponding velocity field and the pressure as follows:

$$u = (u_r, v_\varphi, w_z) = (U, V, W) \equiv U e_r + V e_\varphi + W e_z \quad \text{and} \quad P,$$

where $V$ stands for the swirl component of velocity. Then equations (1.1) take the form

$$\begin{align*}
&\frac{d}{dt} U - \frac{1}{r} V^2 = -P_r + \Delta U - \frac{2}{r^2} V_\varphi - \frac{1}{r} U, \\
&\frac{d}{dt} V + \frac{1}{r} UV = -\frac{1}{r} P_\varphi + \Delta V + \frac{2}{r^2} U_\varphi - \frac{1}{r} V, \\
&\frac{d}{dt} W = -P_z + \Delta W, \\
&U_r + \frac{1}{r} U + \frac{1}{r} V_\varphi + W_z = 0.
\end{align*}$$

Here, the full time-derivative $\frac{d}{dt}$ and the Laplacian $\Delta$ are given by

$$\begin{align*}
\frac{d}{dt} &= D_t + U D_r + \frac{1}{r} V D_\varphi + W D_z, \\
\Delta &= \Delta_2 + D_{zz} \equiv D_{rr}^2 + \frac{1}{r} D_r + \frac{1}{r^2} D_\varphi^2 + D_{zz},
\end{align*}$$

$D(\cdot)$ being partial derivatives, and $\Delta_2$ the Laplacian in the polar variables $(r, \varphi)$.

5.2. On axi-symmetric flows: towards global existence. A flow (or a vector field) is called axi-symmetric if it is invariant under rotations in $\varphi$. The principal advantage of such flows is that the $V$-equation in (5.3) takes the form of a standard parabolic one,

$$V_t + UV_r + WV_z = \Delta_r V - \left(\frac{1}{r} U + \frac{1}{r^2} V_\varphi\right) V,$$

so that some of the MP, Harnack’s inequalities, Nash–Moser iterations, De Giorgi–Nash estimates, and other tools of classic parabolic theory can be expected to take a part. For instance, the standard change for the 3D Laplacian gives the divergent equation:

$$r V = \hat{V} \quad \implies \quad \hat{V}_t + UV_r + W\hat{V}_z + \frac{2}{r} \hat{V}_r = \Delta_r \hat{V},$$

and hence by comparison, via the control on the parabolic boundary,

$$|\hat{V}(r, z, t)| \leq C \quad \implies \quad |V(r, z, t)| \leq \frac{C}{r}.$$
A more involved use of the MP yields some important results; e.g., to prohibit Type I blow-up [118, 38]:

\[ |u(x,t)| \leq \frac{C}{\sqrt{T-t}} \quad \implies \text{no blow-up at } t = T^- . \]

Note again that axi-symmetric flows with no swirl, i.e., with \( V \equiv 0 \), are regular; see Ukhovskii–Yudovich [211], Ladyzhenskaya [130], and more recent results in [193, 222, 118, 38] (the regularity remains under the presence of helical symmetries [147]).

Thus, for axi-symmetric settings, the \( V \)-equation is simple, (5.5), while the third \( W \)-equation in (5.3) gives the pressure,

\[ P = \int_{z}^{\infty} \left( \frac{d}{dz} W - \Delta W \right) \quad (W|_{z=\infty} = 0). \]

Finally, we calculate \( W \) from the last div-equation,

\[ W = \int_{z}^{\infty} \frac{1}{r} (rU)_r. \]

Plugging all this into the \( U \)-equation yields

\[ U_t + UU_r + WU_z = \Delta U + \frac{1}{r} V^2 - \frac{1}{r^2} U \]

\[ + \int \int \left\{ (\Delta \frac{1}{r} (rU)_r)_r - \left( \frac{1}{r} (rU_t)_r \right)_r \right. \]

\[ - \left[ U \left( \frac{1}{r} (rU)_r \right)_r \right] - \left[ \frac{1}{r} (rU)_r \frac{1}{r} (rU)_r \right]_r \}. \]

As we have seen in (2.21), the first two integral linear terms on the right-hand side in (5.10) can be incorporated into the main derivatives \( \Delta U \) and \( U_t \), since these have the necessary good signs, so that the equation reads

\[ (I + L_1)U_t + UU_r + WU_z = (\Delta + L_2)U + \frac{1}{r} V^2 - \frac{1}{r^2} U \]

\[ - \int \int \left\{ U \left( \frac{1}{r} (rU)_r \right)_r + \left[ \frac{1}{r} (rU)_r \frac{1}{r} (rU)_r \right]_r \right. \}, \]

where \( L_1 > 0 \) and \( L_2 < 0 \) (in the metric of \( L^2 \)) are pseudo-differential operators with easily computed symbols (these are directly related to the projector \( \mathbb{P} \))

\[ L_1 = -D_r \left( \frac{1}{r} D_r (rI) \right) \equiv -\Delta_r + \frac{1}{r^2} I, \quad L_2 = \int_{r}^{\infty} \int_{r}^{\infty} D_r \left( \Delta \frac{1}{r} D_r (rI) \right). \]

Let us present some typical RD-like speculations concerning blow-up. Assume that, in the \( (U,V) \)-system (5.11), (5.5), a blow-up occurs at the origin \( r = 0, z = 0 \) as \( t \to T^- \). Then, in view of the \( L^2 \)-boundedness, this must be single-point blow-up. Then, the linear terms in the system are not essential as \( t \to T^- \) in comparison with a number of quadratic ones. This also related to the linear nonlocal terms, where integration over the necessarily rescaled spatial variables diminishes their rates of divergence. Therefore, the quadratic pointwise and integral terms only in the above \( (U,V) \)-system can be responsible for blow-up. We then need to consider two cases:

**Case I: nonlocal quadratic terms are negligible in (5.11).** Then, in the limit \( t \to T^- \), the \( U \)-equation becomes asymptotically pointwise in main terms, and the system reads:

\[
\begin{cases}
U_t + UU_r + WU_z = \Delta U + \frac{1}{r} V^2, \\
V_t + UV_r + WV_z = \Delta V - \frac{1}{r} UV.
\end{cases}
\]
It then follows by a standard comparison (barrier) arguments that the growth of all the localized solutions are controlled by the ODE system for supersolutions,

\[
\begin{align*}
U_t &= \frac{1}{r} V^2; \\
V_t &= -\frac{1}{r} UV \\
\frac{dU}{dV} &= -V \\
U^2 + V^2 &= C,
\end{align*}
\]

which thus assumes no blow-up at all.

**Case II: V is not essential for blow-up.** Then, performing a preliminary passage to the limit as \(\tau = \tau_k + s \to \infty\) \((\tau = -\ln(T - t))\), we arrive at the system with \(V \equiv 0\), which is known to admit no blow-up.

**Case III: quadratic nonlocal terms reinforce \(\frac{1}{r} V^2\).** Thus, this is the only possible case.

However, a careful analysis of those two integral terms in (5.11) shows that both have the wrong sign to do that. Without pretending to any rigorous conclusions, we speculate about the sign of the integrals in (5.15) using standard clues, which look naive, but very often give correct answers in many RD-type problems.

Indeed, integrating over a small neighbourhood of the maximum point in \(U\) (say, \(x = 0\)), where single-point blow-up occurs, these terms have the signs of the following integrals (others have a similar nature and can be checked out analogously):

\[
\begin{align*}
\sim -\int_{z}^{\infty} \int_{z}^{\infty} U_r \Delta_r U \quad \text{and} \quad \sim -\int_{z}^{\infty} \int_{z}^{\infty} \Delta_r U \frac{1}{r} U_z.
\end{align*}
\]

Thus, we assume that the main sign-dominant part in these integrals is delivered by integration over sufficiently small neighbourhood of the maximum point. Then, since there \(U_r \leq 0\) (we think that the internal area, where \(U_r\) can be positive, is not dominated in the integral), \(\Delta_r U \leq 0\), and \(U_z \leq 0\), these integrals cannot be essentially positive on shrinking compact subsets to create an additional new type of “nonlocal” blow-up. In other words, the quadratic integral operators have the tendencies to be non-positive on such typical axi-symmetric flows and actually assure their extra stabilization.

A difficult scrutinized analysis is necessary to fix such a non-blow-up conclusion. However, this is a good sign for us, since we see that, for blow-up, a special \(\varphi\)-acceleration as \(t \to T^-\) is crucially needed to create extra positive quadratic integral terms to generate blow-up, where the TW dependence (1.25) to be introduced is the simplest opportunity.

### 5.3. Positive integral quadratic \(\varphi\)-dependent operators: a rout to blow-up.

We briefly review such an opportunity. Thus, (5.9) reads (we box \(D_{\varphi}\)-dependent operators)

\[
W = \int_{z}^{\infty} \left[ \frac{1}{r} (rU)_r + \frac{1}{r} V_\varphi \right].
\]

This adds into the right-hand side of the \(U\)-equation (5.11) the following extra operators:

\[
\begin{align*}
U_t + UU_r + \frac{1}{r} VU_\varphi + WU_z &= \Delta U + \frac{1}{r} V^2 - \cdots - \int_{z}^{\infty} \int_{z}^{\infty} \left[ U \left( \frac{1}{r} V_\varphi \right)_r \right] \\
&\quad + \frac{1}{r} V \left( \frac{1}{r} V_\varphi \right)_r + \left[ \frac{1}{r} (rU)_r + \frac{1}{r} V_\varphi \right] \left[ \frac{1}{r} (rU)_r + \frac{1}{r} V_\varphi \right] \left[ \frac{1}{r} (rU)_r + \frac{1}{r} V_\varphi \right]_r.
\end{align*}
\]

The \(V\)-equation, due to the pressure term \(-\frac{1}{r} P_\varphi\), also gets a new complicated operator,

\[
\begin{align*}
V_t + UV_r + \frac{1}{r} VV_\varphi + WV_z &= \Delta V - \frac{1}{r} UV \left[ \int_{z}^{\infty} \left( \frac{d}{dt} W - \Delta W \right)_\varphi \right]
\end{align*}
\]
Therefore, plugging (5.16) yields several $\varphi$-dependent quadratic operators listed below:

\begin{equation}
V_t + UV_r + \frac{1}{r}VV_{\varphi} + WV_z = \Delta V - \frac{1}{r}UV - \ldots
\end{equation}

\begin{equation}
- \frac{1}{r} \int_{r}^{\infty} \int_{r}^{z} \{ U \left( \frac{1}{r} (rU)_r + \frac{1}{r} V_r \right) + \frac{1}{r} V \left[ \frac{1}{r} (rU)_r + \frac{1}{r} V_{\varphi} \right] \}_{\varphi} + \left[ \frac{1}{r} (rU)_r + \frac{1}{r} V_{\varphi} \right] \frac{1}{r} \left( rU \right)_r + \frac{1}{r} V_{\varphi} \} \, \varphi.
\end{equation}

Thus, the control of those new swirl-type operators in (5.17) and (5.19) becomes much more difficult and this settles the core difficulty of the blow-up/non-blow-up problem for the NSEs. Now the $\varphi$-derivatives are essentially involved into evolution, including even the third-order one in the integral term containing $\frac{1}{r}^{2} VV_{\varphi\varphi\varphi}$. In particular, the simplest way to introduce the $\varphi$-dependence is the proposed blow-up swirl mechanism in (1.25), which actually has a self-similar form and corresponds to periodic omega-limits. Thus, it seems plausible that some of the integrals in (5.19) can be divergent enough to generate blow-up, unless all of them have the opposite negative sign, or their mutual interaction prevents singularities. Recall that each integration in $z$, according to the similarity variable $y$ in (2.18) yields the multiplier $\sqrt{T-t}$, which is not powerful enough to compensate other singular scalings in $(U, V, r)$ there. Note again that, since $U \to +\infty$, the regularizing anti-blow-up term $-\frac{1}{r} UV$ in the $V$-equation must be defeated by some of the divergent nonlocal terms with a $\varphi$-swirl. As in R–D blow-up theory, all such similarity of approximate self-similar balances eventually lead to complicated open problems for nonlocal systems, which later on we will attack and explain by using other, more local, blow-up approaches.

On the other hand, for purposes of proving global solvability, the above $(U, V)$-system (5.17), (5.19) (it is a simplified one) looks most reasonable. Then a careful step by step analysis of nonlinear mutual interaction of all the operators in (5.19) is unavoidable.

5.4. Generalized von Kármán solutions on a partially invariant linear subspace: first term of general expansion. We look for velocity field and pressure of the form

\begin{equation}
U = U(r, \varphi, t), \quad V = V(r, \varphi, t), \quad W = z\tilde{W}(r, \varphi, t), \quad \text{and} \quad P = P(r, \varphi, t).
\end{equation}

This dependence models a class of dynamical stretched 3D vortex flows including Burgers’ vortices (1948) [24]. The structure of such solutions also corresponds to the earlier (1921) classic von Kármán swirling flow solutions of (1.1) exhibiting typical linear dependencies on two independent spatial variables $x$ and $y$ [110, 112],

\begin{equation}
\begin{cases}
u = f'(z)x - g(z)y, \\
v = f'(z)x + g(z)y, \\
w = -2f(z), \\
p = -2[f'(z) + f^2(z)],
\end{cases}
\end{equation}

where functions $f$ and $g$ satisfy a system of two nonlinear ODEs,

\begin{equation}
\begin{cases}
f'''' + 2f f'' - (f')^2 + g^2 = 0, \\
g'' + 2fg' - 2f'g = 0.
\end{cases}
\end{equation}
These solutions have been applied to various problems of fluid dynamics; see Berker [13]. Note that, unlike von Kármán similarity solutions (5.21) that induce ODEs (5.22), the reduction (5.20) leads to a more complicated system of nonlinear PDEs, which anyway is simpler than the original model. Stationary solutions of the Euler equations (1.31) with a linear $z$-dependence of $w$ as in (5.20) were also used by Oseen (1927) [171] (cf. the structure (2.15) for Oseen’s vortex); see more details in [87].

Actually, (5.20) means that the vector field $\mathbf{u}$ and $p$ belong to a 2D linear subspace,

$$ (5.23) \quad \mathbf{u}, \ p \in W_2 = \text{Span}\{1, z\}. $$

Then some of the equations (5.25) below (projections of (5.3) onto $W_2$) express the partial invariance of $W_2$ with respect to the nonlinear operators in (5.3); see [76, Ch. 7] for further details and examples. Here the invariance of an $l$-dimensional linear subspace $W_l$ under a given nonlinear operator $\mathbf{A}$ is understood in the usual mapping sense,

$$ \mathbf{A}(W_l) \subseteq W_l. $$

Partial invariance means that only a subset of $W_l$ satisfies this inclusion. Similar 2D restrictions on $W_2 = \text{Span}\{1, y\}$ exist for the Navier–Stokes equations in dimension $N = 2$; see [76, p. 34] and references therein. The corresponding blow-up solutions are described in [79, Ch. 8].

It is key to note that the third velocity component $W$ in (5.20) becomes the first term in the asymptotic expansion of general solutions, for which

$$ (5.24) \quad W = z\bar{W} + \sum_{k \geq 2} z^k \bar{W}_k. $$

Then, as usual in asymptotic expansion theory [107], the first term is governed by the most nonlinear system (to be studied), while the rest of the coefficients solve “linearized” systems (that, in view of equations for other components, can be also rather difficult, at least first ones). In general, with a great luck, analytic expansions such as (5.24) might lead to solutions that are spatially localized in $z$; see further discussions below.

Thus, substituting (5.20) into (5.3) yields

$$ (5.25) \quad \begin{cases} U_t + UU_r + \frac{1}{r} VU_\varphi - \frac{1}{r} V^2 = -P_r + \Delta_2 U - \frac{2}{r} V_\varphi - \frac{1}{r} U, \\ V_t + UV_r + \frac{1}{r} VV_\varphi + \frac{1}{r} UV = -\frac{1}{r} P_\varphi + \Delta_2 V + \frac{2}{r} U_\varphi - \frac{1}{r} V, \\ \bar{W}_t + U\bar{W}_r + \frac{1}{r} \bar{W}V_\varphi + \bar{W}^2 = \Delta_2 \bar{W}, \\ U_r + \frac{1}{r} U + \frac{1}{r} V_\varphi + \bar{W} = 0. \end{cases} $$

The last equation in (5.25) easily determines $U$ in terms of $W$ and $V$ (cf. (5.16)),

$$ (5.26) \quad U = -\frac{1}{r} \int_0^r z\bar{W}(z, \varphi, t) \, dz - \frac{1}{r} \int_0^r V_\varphi(z, \varphi, t) \, dz. $$
As customary, the first equation in (5.25) is a pressure one that \textit{a posteriori} is going to define the pressure. Excluding $P$ from first two equations yields the following PDE:

$$
(5.27) \quad [U_t + UU_r + \frac{1}{r} VU_\varphi - \frac{1}{r} V^2 - \Delta_2 U + \frac{2}{r^2} V_\varphi + \frac{1}{r^2} U]_\varphi = \left[ r \left(V_t + UV_r + \frac{1}{r} VV_\varphi + \frac{1}{r} UV - \Delta_2 V - \frac{2}{r^2} U_\varphi + \frac{1}{r^2} V\right) \right]_r.
$$

It is not difficult to see that this awkward equation is just a pseudo-parabolic PDE that also can be reduced to a more standard semilinear nonlocal form. Therefore, (5.27) emphasizes the fact that the NSEs (1.1) admit a unique local semigroup of smooth solutions.

Thus, taking the third equation,

$$
(5.28) \quad \bar{W}_t + U \bar{W}_r + \frac{1}{r} V \bar{W}_\varphi + \bar{W}^2 = \Delta_2 \bar{W} \quad \text{(with the constraint (5.26))},
$$

we arrive at the system (5.27), (5.28) for two unknowns $\bar{W}$ and $V$. For convenience, we now return to the original system (5.25) for performing further blow-up scaling.

5.5. Blow-up twistor variables and rescaled equations. We introduce next the following blow-up rescaled independent variables, where, for convenience, the blow-up time is reduced to $T = 0$:

$$
(5.29) \quad U = \frac{1}{\sqrt{-t}} u, \quad V = \frac{1}{\sqrt{-t}} v, \quad \bar{W} = \frac{1}{(-t)} w, \quad \text{and} \quad P = \frac{1}{(-t)} p.
$$

The rescaled dependent variables are given by:

$$
(5.30) \quad y = \frac{(-t)}{\sqrt{-t}}, \quad \varphi = \mu - \sigma \ln(-t), \quad \text{and} \quad \tau = -\ln(-t),
$$

where $\sigma \neq 0$ is an unknown parameter, which cannot be scaled out and plays a role of a \textit{nonlinear eigenvalue} for the future “stationary” problem. As usual in blow-up problems, the new time variable gets infinite,

$$
(5.31) \quad \tau = -\ln(-t) \to +\infty \quad \text{as} \quad t \to 0^-.
$$

It is crucial that, according to (5.30), on compact intervals in the rescaled angle $\mu$,

$$
(5.32) \quad \varphi = \mu - \sigma \ln(-t) \equiv \mu + \sigma \tau \to \infty \quad \text{as} \quad \tau \to +\infty \quad \text{(i.e.,} \quad t \to 0^-).
$$

Therefore, in the rescaled angle variable, the dependence $\varphi = \mu + \sigma \tau$ reflects a \textit{travelling wave} angular behaviour, where $\sigma$ is the \textit{wave angular speed}. The angular dependence in (5.32) corresponds to infinite acceleration of rotation of all the velocity components about the $z$-axis, and therefore we call special pattern solutions (5.29), associated with scaling (5.30), \textit{blow-up twistors}.

Remark 1: on non blow-up flows with spiral symmetry. Such solutions of the NSEs (1.1) in the cylindrical coordinates with the usual TW-type angular dependence

$$
(5.33) \quad \varphi = \mu + \sigma t
$$

are well known and were studied by Bytev \cite{25} in 1972; see also the results of the group classification of the NSEs in \cite{2}. Unlike (5.29), the standard invariant of the group of translations associated with (5.33) does not allow blow-up. Indeed, logTWs and simply TWs are different group-invariant solutions, and the former ones assume extra invariance relative to a group of scalings.
Remark 2: on tornado-type blow-up in complex NSEs. Tornado-type blow-up behaviour of the complex version of the NSEs, for which blow-up was established in Li–Sinai [138], was confirmed in [5] by numerical methods (though the achieved numerical evidence concerning the structure of such singularities still looks rather unsufficient).

In view of (1.4), let us now check the nature of the scalings in (5.29), (5.30). According to full similarity rescaling in (2.7), the \( z \)-variable is

\[
\zeta = \frac{z}{\sqrt{-t}} \implies zW = \frac{1}{\sqrt{-t}} \zeta \hat{w},
\]

and hence all these scaling factors are self-similar. However, (1.4) is not a warning for us, since logarithmic angular TWs do not belong to the framework of standard similarity patterns (in fact, these are related to periodic orbits). In addition, we are going to use nonstationary evolution governed by the PDEs (5.35) including patterns without a similarity stabilization. On the other hand, for axisymmetric flows, it is known that the blow-up must be of Type II, i.e., with the blow-up rate faster than self-similar [118, 38].

Substituting (5.29), (5.30) into the general system (5.25) yields the following nonstationary rescaled equations:

\[
\begin{cases}
    u_r + \frac{1}{2} y u_y + \frac{1}{2} u + \sigma u_\mu + uu_y + \frac{1}{y} vu_\mu - \frac{1}{y^2} v^2 = -p_y + \Delta_2 u - \frac{2}{y} v_\mu - \frac{1}{y^2} u, \\
    v_r + \frac{1}{2} y v_y + \frac{1}{2} v + \sigma v_\mu + vv_y + \frac{1}{y} vv_\mu + \frac{1}{y} uv = -\frac{1}{y} p_\mu + \Delta_2 v + \frac{2}{y^2} u_\mu - \frac{1}{y^2} v, \\
    w_r + \frac{1}{2} y w_y + w + \sigma w_\mu + uw_\mu + \frac{1}{y} vw_\mu + w^2 = \Delta_2 w, \\
    u_y + \frac{1}{y} u + \frac{1}{y} v_\mu + w = 0.
\end{cases}
\]

The pressure and the constraint (5.26) interpretation remain the same as for (5.25), so (5.35) is a nonstationary nonlocal system of two equations for unknowns \( w \) and \( v \). Recall that the 2-Laplacian \( \Delta_2 \) now takes the full form

\[
\Delta_2 = D_y^2 + \frac{1}{y} D_y + \frac{1}{y^2} D_\mu^2.
\]

5.6. Remark: log-TWs in reaction-diffusion systems. Using blow-up logarithmic travelling waves comes as a fruitful idea from reaction-diffusion theory; see examples in [187, pp. 105, 308, 411]. For instance, the classic quadratic porous medium equations with source and convection in 1D (a canonical combustion problem with regional blow-up, [187, Ch. 4])

\[
u_t = (uu_x)_x + uu_x + u^2
\]

admits the following blow-up logarithmic travelling waves:

\[
u(x, t) = \frac{1}{(-t)} f(y), \quad y = x - \sigma \ln(-t) \implies f + \sigma f' = (ff')' + ff' + f^2.
\]

The scaling group-invariant nature of such logTWs seems was first obtained by Ovsiannikov in 1959 [172], who performed a full group classification of the nonlinear heat equation

\[
u_t = (k(u)u_x)_x,
\]
for arbitrary functions $k(u)$. In particular, such invariant solutions appear for the porous medium and fast diffusion equations for $k(u) = u^n$, $n \neq 0$:

$$u_t = (u^nu_x)_x \implies \exists u(x, t) = t^{-\frac{1}{n}} f(x + \sigma \ln t), \quad \text{where} \quad -\frac{1}{n} f + \sigma f' = (f^n f')'. $$

Blow-up angular dependence such as in (5.32) was studied later on in [7], where the corresponding similarity solutions for the reaction-diffusion equation with source

$$u_t = \nabla \cdot (u^\sigma \nabla u) + u^\beta \quad \text{in} \quad \mathbb{R}^2 \times (0, T) \quad (\beta > 1, \sigma > 0),$$

were indicated by reducing the PDE to a quasilinear elliptic problem (it seems, there is no still a rigorous proof of existence of such patterns). For parabolic models such as (5.37), that are order-preserving via the MP and do not have a natural vorticity mechanism, such “spiral waves” as $t \to T^-$ must be generated by large enough initial data specially “rotationally” distributed in $\mathbb{R}^2$. For the fluid model (1.1) with typical vorticity features, this can be different.

5.7. **On some auxiliary properties of twistor structures.** As usual, these are stationary solutions of (5.35), which are independent of the time-variable $\tau$, i.e., in the original variables,

$$\begin{aligned}
\hat{U}(r, \varphi, t) &= \frac{1}{\sqrt{-t}} \hat{u}(y, \mu), \\
\hat{V}(r, \varphi, t) &= \frac{1}{\sqrt{-t}} \hat{v}(y, \mu), \\
\hat{W}(r, \varphi, t) &= \frac{1}{(-t)} \hat{w}(y, \mu), \\
\hat{P}(r, \varphi, t) &= \frac{1}{(-t)} \hat{p}(y, \mu).
\end{aligned}$$

The rescaled profiles $\hat{u}$, $\hat{v}$, $\hat{w}$, and $\hat{p}$ solve the corresponding stationary system,

$$\begin{aligned}
\frac{1}{y} y \hat{u}_y + \frac{1}{2} \hat{u} + \sigma \hat{u}_\mu + \hat{u}_y + \frac{1}{y} \hat{v} \hat{u}_\mu - \frac{1}{y} \hat{v}^2 &= -\hat{p}_y + \Delta_2 \hat{u} - \frac{2}{y^2} \hat{v}_\mu - \frac{1}{y^2} \hat{u}, \\
\frac{1}{y} y \hat{v}_y + \frac{1}{2} \hat{v} + \sigma \hat{v}_\mu + \hat{v}_y + \frac{1}{y} \hat{v} \hat{v}_\mu + \frac{1}{y} \hat{w} \hat{v}_\mu &= -\frac{1}{y} \hat{p}_\mu + \Delta_2 \hat{v} + \frac{2}{y^2} \hat{v}_\mu - \frac{1}{y^2} \hat{v}, \\
\frac{1}{y} y \hat{w}_y + \hat{w} + \sigma \hat{w}_\mu + \hat{w}_y + \frac{1}{y} \hat{v} \hat{w}_\mu + \frac{1}{y} \hat{w} \hat{w}_\mu + \frac{1}{y} \hat{w}^2 &= \Delta_2 \hat{w}, \\
\hat{u}_y + \frac{1}{y} \hat{u} + \frac{1}{y} \hat{v}_\mu + \hat{w} &= 0.
\end{aligned}$$

We now present the full system for $\hat{w}$, $\hat{v}$. First, we perform the reflection,

$$w \mapsto -w.$$ 

Second, we have from the last equation in (5.39)

$$\hat{u} = \frac{1}{y} \int_0^y z \hat{w} \, dz - \frac{1}{y} \int_0^y \hat{v}_\mu \, dz.
$$

Then the $\hat{w}$-equation reads

$$\begin{aligned}
\Delta_2 \hat{w} - \frac{1}{y} y \hat{w}_y - \hat{w} + \hat{w}^2 - \sigma \hat{w}_\mu \\
- \frac{1}{y} \left( \int_0^y z \hat{w} \, dz \right) \hat{w}_y + \frac{1}{y} \left( \int_0^y \hat{v}_\mu \, dz \right) \hat{w}_y - \frac{1}{y} \hat{v} \hat{w}_\mu &= 0.
\end{aligned}$$

(5.42)
Thirdly, the \((\hat{u}, \hat{v})\)-equation (with (5.41)) takes the form

\[
\left[\frac{1}{2} y \hat{u}_y + \frac{1}{2} \hat{u} + \sigma \hat{u}_\mu + \hat{u} \hat{u}_y + \frac{1}{y} \hat{v} \hat{u}_\mu - \frac{1}{y} \hat{v}^2 - \Delta_2 \hat{u} + \frac{2}{y^2} \hat{v}_\mu + \frac{1}{y} \frac{1}{y} \hat{u}\right]_y = \left[y \left(\frac{1}{2} y \hat{v}_y + \frac{1}{2} \hat{v} + \sigma \hat{v}_\mu + \hat{u} \hat{v}_y + \frac{1}{y} \hat{v} \hat{v}_\mu + \frac{1}{y} \hat{v} - \Delta_2 \hat{v} - \frac{2}{y} \hat{u}_\mu + \frac{1}{y} \frac{1}{y} \hat{v}\right)\right]_y.
\]

(5.43)

Thus, we arrive at the system of two PDEs (5.42), (5.43), with the nonlocal constraint (5.41), which, as we have seen, actually means the presence of an extra first-order PDE.

This system with the real parameter \(\sigma \neq 0\) is indeed complicated. We note that the Laplacian \(\Delta_2\) in (5.36) includes second-order derivatives in both radial \(y\) and the angular \(\mu\) variables. The latter dependence is crucial for such similarity twistors. We do not intend and do not plan to study this system somehow rigorously, though some results, in particular, about asymptotic distributions as \(y \to +\infty\) can be obtained and will be of importance in what follows.

Obviously, in view of (5.20), these similarity twistors as special kind of blow-up vortices are not spatially localized in the \(z\)-direction, so they have infinite energy always, and, for their existence, a whole \(\mathbb{R}^3\)-space should be taken into account (we skip at this moment their expansion meaning (5.24)). But as typical in reaction-diffusion equations such as (1.36), (1.54), (1.56), and many others, such similarity or approximate similarity blow-up structures can appear from local finite and well-spatially-localized data. Here there appear hard questions of their evolution stability (in which intermediate sense?) or behaviour close to centre and/or stable manifolds associated with their partial similarity space-time geometry. These questions are addressed to the full non-stationary system (5.35).

As we have mentioned, in the case of nonexistence of such “stationary” swirl patterns, some time-dependent perturbations of these structures can lead to various non self-similar blow-up twistors that evolve close to invariant manifolds associated with the scaling (5.38). Actually, precisely this always happens to the semilinear scalar curvature equation (1.60) for \(N < 6\) and for the Frank–Kamenetskii equation (1.36) for \(N = 1\) or 2. Unfortunately, this will lead to much more difficult mathematics than for a simpler purely stationary similarity one (that does not look easy at all, of course).

5.8. **On extension beyond blow-up for \(t > T\).** This is connected with Leray’s blow-up scenario (1.5). We do not discuss such difficult issues here in any detail and just mention that, as typical, the extension after blow-up is a secondary question relative to the blow-up ones on the behaviour as \(t \to T^-\), which still remains mysterious.

Let us also note that, concerning blow-up twistors with the angular variable (5.32), a natural way of extensions of possible blow-up solutions is to use the forward variable

\[
\varphi = \mu + \hat{\sigma} \ln t \quad \text{for} \quad t > 0
\]

(5.44)

and \(-t \mapsto t\) in (5.29), where \(\hat{\sigma} \neq \sigma\), in general (but sign \(\sigma = \text{sign} \hat{\sigma}\) to keep the direction of rotation), is an extra parameter of matching at \(t = T = 0\). Principles of proper matching of a blow-up flow for \(t < 0\) with a more regular one for \(t > 0\) for (1.1) are not clear. For simpler nonlinear PDEs such similarity matching is possible; see [70] for (1.57).
5.9. **Discussion: on auxiliary properties of blow-up twistors.** Let us look for some auxiliary properties of the “stationary” profiles \( \hat{u}, \hat{v}, \hat{w} \) as solutions of (5.41)–(5.43). To reveal a first key feature, assume for a moment that these profiles are of changing sign, so that rescaling (5.29) would mean *non-uniform blow-up*, where components diverge as \( t \to 0^- \) to \( \pm \infty \) on some subsets.

We next need to point out another curious possible feature of such solutions that is of importance for general understanding of the nature of such singularities. Let us demand that the similarity profiles have *angular periodic* behaviour with zero mean as \( y \to \infty \).

For instance, assume for the component \( \hat{w}(y,\mu) \) that

\[
\hat{w}(y,\mu) = \frac{C(\mu)}{y^2} + \ldots \quad \text{as} \quad y \to +\infty, \quad \text{where} \quad \int_0^{2\pi} C(\mu) \, d\mu = 0.
\]

Then, in view of the behaviour of the rescaled variables in (5.30) near blow-up time, under some extra hypotheses, this can imply, by the blowing up rotational behaviour, that

\[
W(r,\varphi,t) \to 0 \quad \text{as} \quad t \to 0^-,
\]

in the weak sense on the corresponding subsets in the rescaled variables. Indeed, in view of the zero mean in (5.45), the equality (5.46) is then a manifestation of Riemann’s Lemma from Fourier transform theory. As more usual and typical, under weaker assumptions, such solutions locally represent an oscillating “building block” (see [17]) for \( x \approx 0, \, t \to 0^- \) in terms of the corresponding vorticity \( \omega = \nabla \times u \), where \( u \) is then defined by the *Biot–Savart law*,

\[
u(x,t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y,t) \, dy \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+.
\]

The property (5.46) (and also relative to the vorticity, which is easier) then represents special type of singularities that are almost “invisible” (in fact, efficiently “nonexistent”) close to the blow-up time in the natural integral (weak) sense. Recall that this could be a key feature since all the differential equations of fluid mechanics are derived from kinetic equations with *integral* operators in collision-like terms by approximating typical integral kernels involved by kernels with pointwise supports (Grad’s method in Chapman–Enskog expansions). Therefore, the integral, average meaning of coherent macro-structures and micro-singularities are of practical importance only.

Another principal question is as follows: can, in the non-stationary setting, the “zero mean oscillation property” such as (5.46) in a neighbourhood of the origin affect and diminish the asymptotic behaviour of the corresponding solutions as \( y \to \infty \) to get them into finite energy class? It seems that this is not that essential for the present cylindrical geometry, but can be key for the spherical one with the unique single point at the origin; see Section 8.

5.10. **Discussion: on asymptotic stability of twistors.** We begin with the following first observation concerning evolutionary significance of similarity scaling:

Assume, not taken into account possible energy characteristics, that a suitable non-trivial solution of the stationary system (5.42), (5.43) exists. Then the next and a more
difficult step is to study its asymptotic stability in the framework of the full non-stationary system (5.35). This stabilization phenomenon then becomes a difficult open problem. Observe that there is no chance to prove that (5.35) is a gradient dynamical system in any admissible metric. Recall that the negative result in [103, Th. 1, §2] establishing the self-similar ban is proved under the a priori assumption on convergence to a fixed rescaled profile (then necessarily this profile must be a stationary solution and hence it is zero by [162]); and finally, the exponential decay via spectral and semigroup characteristics of the rescaled infinitesimal generator $B^*_{\rho^\ast}$ in $L^2_{\rho^\ast}(\mathbb{R}^3)$ implies that the solution $u(x, t)$ is bounded at $t = 0^−$.

In this connection, it is worth mentioning that, as known from reaction-diffusion theory, regardless bad energy and other global properties of twistor profiles, good rescaled solutions can stabilize to them in a local topology, e.g., in view of interior regularity,

$$\text{(5.48)}$$

uniformly on compact subsets

in the rescale variables. Of course, uniform stabilization in $\mathbb{R}^3$ is then impossible in view of the energy discrepancy.

5.11. Discussion: on linearized blow-up patterns about a constant equilibrium, (I). Assume next that either (i) any suitable non-trivial steady profiles satisfying (5.42), (5.43) are nonexistent, or (ii) there exists a non-trivial solution (both possibilities are suitable for us). What kind of other behaviour do we then expect of the nonstationary system (5.35)? In other words, is there any hope to get a kind of entirely non-self-similar blow-up twistor behaviour?

The idea of such “linearized construction” is as follows. We first need to fix a family of simpler local solutions or “almost” (say, slightly perturbed) solutions of the dynamical system under consideration. As the next step, we linearize the flow about this family and use the orbits on the corresponding stable or centre manifolds to match them with the surrounding orbits of the necessary behaviour and regularity. These matching procedures, though being rigorously justifies for some simpler parabolic PDEs such as (1.36) or (1.60) (we have presented the references), remain open for many other quasilinear and higher-order parabolic reaction-diffusion models, and, surely, this will be the case for (1.1).

We show how then a countable family of the so-called linearized blow-up patterns (unlike the above similarity patterns that are nonlinear eigenfunctions) can be constructed. Consider the nonstationary $\psi$-equation from (5.35) bearing in mind the reflection (5.40),

$$\text{(5.49)}$$

$$w_\tau = \Delta_2 w - \frac{1}{2} y w_y - w + w^2 - u w_y - \sigma w_\mu - \frac{1}{y} v w_\mu.$$ 

We see that (5.49) admits the constant equilibrium

$$\text{(5.50)}$$

$$w_\ast = 1, \quad \text{i.e.,} \quad \bar{W}(r, \varphi, t) \equiv \left(\frac{1}{r}\right)^\mu,$$

which corresponds to uniform global blow-up as $t \to 0^-$ in the whole space. This is the easiest exact solution, which we are going to linearize the flow about. It is known from reaction-diffusion theory (the proof is obvious for systems with the MP) that in order to blow-up at the fixed time $t = 0$, the rescaled solution $w(y, \tau)$ should always
be “sufficiently” close (in a certain metric) to the constant blow-up profile (5.50), since otherwise, essentially smaller solutions must exhibit larger blow-up times, so these become exponentially small as \( \tau \to +\infty \) after rescaling (5.29); see a spectral justification later on.

Thus, we perform the standard linearization about (5.50) by setting \( w = 1 + Y \)

to get the “linearized” equations (we have taken into account the \( u \)-representation (5.41)):

\[
Y_\tau = \Delta_2 Y - y Y_y + Y - \sigma Y_\mu - \frac{1}{y} v Y_\mu + \frac{1}{y} \left( \int_0^y v_\nu \right) Y_y + D(Y),
\]

where \( D(Y) = Y^2 - \frac{1}{y} \left( \int_0^y z Y \right) Y_y \),

is a quadratic perturbation as \( Y \to 0 \). Consider the radial part of the linear operator \( B^* \) in (5.52) excluding at this moment its tangential angular first- and second-order operators including that in \( \Delta_2 \),

\[
\tilde{B}^* = \Delta_2 - y D_y + I.
\]

Changing the radial variable yields

\[
y \mapsto \frac{y}{\sqrt{2}} \quad \Rightarrow \quad \tilde{B}^* = 2 \left( \Delta_2 - \frac{1}{2} y D_y + \frac{1}{2} I \right),
\]

where we observe the adjoint Hermite operator (cf. (1.70))

\[
B^* = \Delta_2 - \frac{1}{2} y D_y, \quad \text{with} \quad B = \Delta_2 + \frac{1}{2} y D_y + I \quad \text{in} \quad L^2(\mathbb{R}^2).
\]

On the other hand, \( B \) is self-adjoint in the weighted space \( L^2_{\rho}(\mathbb{R}^2) \), with the weight

\[
\rho(y) = e^{\frac{y^2}{4}},
\]

so that this falls into the scope of classic theory of linear self-adjoint operators, \[15\].

In particular, the adjoint operator \( B^* \) in the adjoint space \( L^2_{\rho^*}(\mathbb{R}^2) \), where

\[
\rho^*(y) = \frac{1}{\rho(y)} = e^{-\frac{y^2}{4}},
\]

has the discrete spectrum (here we take into account the restricted radial symmetry)

\[
\sigma(B^*) = \{ \lambda_{2k} = -k, \quad k = 0, 1, 2, \ldots \},
\]

and the eigenfunctions are normalized Hermite polynomials \[15\] p. 48

\[
\psi_{2k}^*(y) = c_{2k} H_{2k}(y), \quad c_{2k} = \frac{2^{2k}}{\sqrt{(2k)!}}, \quad k = 0, 1, 2, \ldots.
\]

As was already pointed out, in the general non-radial setting in \( \mathbb{R}^N \), all these polynomials are obtained by differentiating the rescaled Gaussian \( F(y) \) of the fundamental solution of the heat operator \( D_t - \Delta_N \),

\[
F(y) = \frac{1}{(4\pi)^{N/2}} e^{-\frac{|y|^2}{4}} \quad (N = 2),
\]

so that the following generating formula holds:

\[
\psi_{\beta}^*(y) = \frac{1}{F(\beta)} c_\beta D^\beta F(y) \equiv c_\beta H_{\beta}(y), \quad \text{where} \quad c_\beta = \frac{2^{[\beta]}}{\sqrt{\beta!}}.
\]

\[34\] The non-radial eigenvalue problem with \( \sigma \neq 0 \) is studied in Section 6.
and \( \beta = (\beta_1, ..., \beta_N) \), with \(|\beta| = \beta_1 + ... + \beta_N\), is a multiindex. This set of polynomials include all the angular-dependent eigenfunctions, so these are complete and closed in the whole weighted space \( L^2_{\mu_\ast}(\mathbb{R}^2) \). Note that, unlike Section 2.6, we do not need solenoidal test for eigenfunctions involved.

It then follows from (5.54) that

\[
\sigma(\tilde{B}^\ast) = \left\{ \lambda_{2k} = 1 - 2k, \quad k = 0, 1, 2, ... \right\}.
\]

The first unstable mode with \( k = 0 \) and \( \lambda_0 = 1 \) corresponds to the unavoidable unstability of the scaling (5.38) with respect to small perturbation of the blow-up time, when we replace \( 0 \mapsto T \approx 0. \) On the manifold of solutions with the same blow-up time, this unstable mode plays no role and is excluded.

Thus, the first actual mode takes place for \( k = 1 \), with

\[
\lambda_2 = -1 \quad \text{and} \quad \psi_2^\ast(y) = \hat{c}_2 \left( 1 - \frac{1}{4} y^2 \right).
\]

This corresponds to patterns with the following behaviour for large \( \tau \gg 1 \) in the inner region that is characterized by arbitrarily large compact subsets in \( y \):

\[
\psi_2^\ast 2(y) \approx 1 + e^{-\tau} C \psi_2^\ast(y) + ... \quad (C > 0),
\]

where the tangential operator in (5.52) should be now also taken into account. This makes the spectral theory more involved (other equations are also included) and can lead to complicated computations. For the purely radial case, analogous computations will be performed in the next section.

Meantime, we deal with the conventional expansion (5.64), which should be matched with the outer behaviour for \( y \gg 1 \). The corresponding rescaled variable in this outer region is also seen from (5.64) by using the quadratic behaviour of \( H_2(y) \) as \( y \to \infty \). Indeed, this gives the first term of the stationary expansion in the new rescaled variable:

\[
\psi_2^\ast 2(y) \approx 1 - \hat{c}_2 C e^{-\tau} y^2 + ... \quad \text{for} \quad y \gg 1, \quad \text{where} \quad \zeta = e^{-\frac{1}{2} \tau} y.
\]

In a similar manner, we find other patterns with the inner behaviour governed by other stable 1D eigenspaces of \( \tilde{B}^\ast \), where

\[
\psi_2^\ast 2k(y) \approx 1 + e^{\lambda_{2k} \tau} C \psi_2^\ast 2k(y) + ... , \quad \text{where} \quad k = 1, 2, 3, ..., \]

and \( \psi_2^\ast 2k(y) \) are other higher-degree non-monotone Hermite polynomials. In general, this can lead to a countable set of different blow-up patterns; see details in [70].

For the non-radial case, where all the tangential operators should be taken into account, the matching procedure with the outer region can be very complicated and cannot be justified rigorously for the full model. This is done in [70] for simpler blow-up axisymmetric jets that are admitted by the Navier–Stokes equations in \( \mathbb{R}^3 \).

In general, on matching, we arrive at a special countable family of blow-up patterns that are characterized by a stronger and wider propagation than the (existing or not) self-similarity scaling (5.29) suggests. For instance, for the scalar curvature equation (1.60), the first generic blow-up pattern (1.62), which is characterized by a slow drift on
a centre manifold corresponding to the similarity rescaled variables (1.64), shows a faster propagation in the $x$-direction with the extra logarithmic factor (here $T = 0$)

$$\sim \sqrt{\tau} = \sqrt{|\ln(-t)|} \to +\infty \text{ as } t \to 0^-.$$  

Such an expanding blow-up twistor can be not that adequate to the nature of single-point blow-up for the Navier–Stokes equations (at least, what we could expect).

Therefore, we begin discussing the second, probably, more realistic situation.

5.12. Discussion: on linearized patterns about singular equilibria, (II). We now will use other equilibria of the equations (1.1), which are not constant and hence more and better localized about the stagnation point $x = 0$.

**Singular radial equilibria.** The first candidate is singular stationary solutions that exhibit strong singularities at $x = 0$, and are possibly not any weak, very weak, or mild solutions at all. This is not that important for us, since we are going to use them just for linearizing and next remove the singularity by matching with a regular bundle at the singularity point.

As an easy illustration of the type of calculus to be performed later on, let us look for simple homogeneity $-1$ separable stationary solutions of the original system (5.25),

\begin{equation}
(5.67)
\begin{align*}
U(r, \varphi) &= \frac{A(\varphi)}{r}, & V(r, \varphi) &= \frac{B(\varphi)}{r}, & W(r, \varphi) &= \frac{C(\varphi)}{r^2}, & P(r, \varphi) &= \frac{D(\varphi)}{r^2}.
\end{align*}
\end{equation}

These separable structures are invariant under the scaling group in (5.29), so that the same singular patterns in the variables (5.30) can be used for the study of twistor behaviour in the rescaled system (5.35).

Substituting (5.67) into (5.25) yields the following system on the unit circle $S^1$ in $\mathbb{R}^2$:

\begin{equation}
(5.68)
\begin{align*}
A'' - BA' - 2B' + A^2 + B^2 + 2D &= 0, \\
B'' - BB' + 2A' - D' &= 0, \\
C'' - BC' + 4C + 2AC - C^2 &= 0, \\
B' + C &= 0.
\end{align*}
\end{equation}

The second equation is integrated once that, on using the last one, gives $D$ as a quadratic function of other unknowns,

\begin{equation}
(5.69)
\begin{align*}
D &= 2A - \frac{1}{2} B^2 - C + M_0 \quad (M_0 = \text{const.})
\end{align*}
\end{equation}

On substitution into (5.68), we obtain an easier system

\begin{equation}
(5.70)
\begin{align*}
A'' - BA' + A^2 + 4A + 2M_0 &= 0, \\
C'' - BC' + 4C + 2AC - C^2 &= 0, \\
B' + C &= 0.
\end{align*}
\end{equation}

Substituting $C = -B'$ from the last equation yields two equations for functions $A$ and $B$,

\begin{equation}
(5.71)
\begin{align*}
A'' - BA' + A^2 + 4A + 2M_0 &= 0, \\
B'' - BB'' + (B')^2 + 2(A + 2)B' &= 0,
\end{align*}
\end{equation}
which still remain a difficult fifth-order ODE on $S^1$ with an arbitrary parameter $M_0 \in \mathbb{R}$, which can play a role of a nonlinear eigenvalue. In particular, it is easy to check that there are no explicit trigonometric solutions, where

$$A, B \in W_3 = \text{Span} \{1, \cos l\varphi, \cos l\varphi \} \quad (l \in \mathbb{N}).$$

Finally, we note that there exists simpler “irrotational” (no swirl) solutions (5.67):

$$C = 0 \quad \text{and} \quad A^2 + B^2 + 2D = 0,$$

about which first linearized analysis can be naturally began with.

It is worth mentioning here that Šverák [204] proved that all singular stationary homogeneity $-1$ equilibria of the NSEs in $\mathbb{R}^3 \setminus \{0\}$ are equivalent, up to orthogonal transformations, to Slezkin–Landau’s solutions; a stronger result was obtained in [155], see Section [7]. Since singularities of (5.67) are concentrated at the $z$-axis $\{r = 0\}$, we cannot use the result of [204], though it does give a hope to avoid to scrutinize the general system (5.70), which looks not that easy at all. In addition, there is a strong nonexistence result in [118, § 5] for axi-symmetric ancient solutions of the NSEs:

$$|\mathbf{u}(x, t)| \leq \frac{C}{\sqrt{x_1^2 + x_2^2}} \quad \text{in} \quad \mathbb{R}^3 \times (-\infty, 0) \implies \mathbf{u} \equiv 0.$$

It seems this does not directly apply to equilibria (5.67) and similar others, since these are singular at $r = 0$ (i.e., are not bounded ancient solutions). Anyway, it seems that some singular equilibria can be eventually ruled out but not all of them.

Thus, by $\mathbf{U}$ we denote a certain singular equilibrium of the NSEs, not necessarily given by the homogeneous formulae (5.67). In general, the description of all the possible singular stationary orbits at $x = 0$ (or at $r = 0$) leads to very difficult study of ill-posed elliptic evolution equations as explained in Section [8.1].

**Construction of blow-up patterns: in need of non self-adjoint theory.** Thus, we are not going to study the system (5.70) in detail here, and will concentrate on the principles of construction of blow-up twistors using linearization about the manifold corresponding to such singularities. Denoting by $\mathbf{u}$ the 3-vector of the rescaled variables $\mathbf{u} = (u, v, w)$, we write (5.35) as a dynamical system for the rescaled variable (cf. (2.21))

$$\mathbf{u}' = \mathbf{H}(\mathbf{u}) \quad \text{for} \quad \tau > 0,$$

where, as in (2.21) or (2.23), the pressure variable is excluded by projecting the solution space onto the kernel of the gradient operator. Thus, we have fixed $\mathbf{U}(y, \mu) \neq 0$ as a solenoidal stationary singular solutions of the rescaled equation (5.74).

We next perform the linearization by setting

$$\mathbf{u}(\tau) = \mathbf{U} + \mathbf{Y}(\tau)$$

I.e., defined for all $t < 0$, [94]. By scaling (2.6), any $L^\infty$ blow-up solution $\mathbf{u}(x, t)$ generates a non-trivial ancient uniformly bounded one $\mathbf{v}(x, t) = \frac{1}{C_k} \mathbf{u}(x_k + \frac{t}{C_k}, t_k + \frac{t}{C_k})$, where $C_k = \sup_x |\mathbf{u}(x, t_k)| = |\mathbf{u}(x_k, t_k)| \to +\infty$ (on the applications, see [118]). This reflects typical scaling tools of R–D theory [79]; cf. Section [4] and applications to solvability and bounds for nonlinear parabolic equations in [80, § 2].
to get the linear equation for $Y$,

$$\dot{Y} = H'(U)Y + D(Y) \quad \text{for } \tau > 0,$$

where $D$ denotes a nonlinear perturbation, which is quadratic as $Y \to 0$.

It follows from the general steady structure such as (5.67) that the first step of such a construction is to find the point spectrum of the linear integral (pseudo-differential) matrix operator $B^* = H'(U)$ for vector-valued functions with periodic in $\mu$ (and possibly singular at the origin) coefficients. Thus, we define the Inner Region I as a family of special space-time subsets, where the linearized equation holds asymptotically:

$$\text{Inner Region I: } \dot{Y} = H'(U)Y + \ldots \quad \text{for } \tau \gg 1.$$ 

Recall that, by construction, $H'(U)$ is assumed to act in a solenoidal vector field.

A number of accompanying questions arise. E.g., one can pose the following:

**Question (i):** $B^* = H'(U)$ is not self-adjoint in no weighted $L^2$-spaces;

**Question (ii):** what is the domain of $B^*$ (hopefully a kind of Sobolev space $H^2_\rho(\mathbb{R}^3)$)?

**Question (iii):** what is the space and domain of the adjoint operator $B$?

**Question (iv):** it is not clear why $B^*$ and $B$ could have enough or at least some real (or complex) eigenvalues, compact resolvent (by compact embedding $H^2_\rho \subset L^2_\rho$?), which conditions at the singular origin?), and bi-orthogonality property of bases (if any);

**Question (v):** why the eigenfunctions of $B^*$ can be at least approximately associated with some structures of finite polynomials (possibly separable in the angular direction)?

**Question (vi) on solenoidal eigenfunctions and classes.** Though this is a natural part of the eigenvalue problem for the pair $\{B^*, B\}$, let us recall the key: assuming the $-1$ homogeneity (and if not?) of the equilibrium such as (5.67) employed, we are obliged to perform a “solenoidization” to get a sufficient amount of (possibly even some polynomial where again the angle separation is assumed?) eigenfunctions $\psi_{\beta}$, which will generated eigenspaces of solenoidal fields as in (2.31); ..., etc.

For the singular S–L solutions (7.1), this will be continued in Section 7, where we also discuss in greater detail the questions of matching of various asymptotic inner and outer regions.

Nevertheless, the parabolic experience of doing linearized theory for $2m$th-order PDEs such as (1.56) shows that the corresponding linear operators $B$ and $B^*$ are also non self-adjoint (the only self-adjoint case is for $m = 1$; cf. (2.24)), but admit real point spectrum only, [51]. However, the matrix spectrum problem for the pseudo-differential operator $B^*$ in (5.76) and for the corresponding formally (in the topology of $L^2$) adjoint operator $B$ are much more difficult and remain open; see Section 7 for further details and related important comments in [204].

Thus, as for an illustration, assume that $B^*$ has a real point spectrum

$$\sigma(B^*) = \{\lambda_k, \quad k = |\beta| = 0, 1, 2, \ldots\} \quad (\beta \text{ is a multiindex}),$$

$56$
where the first positive eigenvalue,
\[
\lambda_0 > 0 \quad (= 1, \text{ it seems}),
\]
reflects the natural unstability with respect to the change of the blow-up time. We also assume that, as a standard fact from linear operator theory (that also needs a difficult
approving), the eigenfunctions \( \{ v_\beta \} \) are assumed to be bi-orthonormal to the corresponding adjoint basis \( \{ v_\beta^* \} \), and both bases are complete, closed, and form Riesz-type bases in the corresponding solenoidal spaces.

We are interested in eigenvalues \( \lambda_k \leq 0 \). The most interesting case occurs when \( \lambda_k = 0 \), since it corresponds to a behaviour close to the centre subspace of \( B^* \). Note that proving existence of a centre or stable invariant manifolds for the full nonlinear problem (5.74) is a very difficult problem. As a formal hint, assuming that the kernel of \( B^* \) is one-dimensional spanned by an eigenfunctions \( v_\beta^* \) (for any dimension, the analysis is similar with a system to occur), we are looking for solutions in the form
\[
(5.80) \quad  \hat{Y}_0(y, \tau) \sim c_\beta(\tau) \hat{v}_\beta^* + \hat{w}(\tau) \quad \text{for} \quad \tau \gg 1 \quad (v_\beta^* \in S_k, \quad cv^* = (c_1v_1^*, c_2v_2^*, c_3v_3^*)),
\]
where the remainder \( \hat{w}(\tau) \) is supposed to be orthogonal to \( \ker B^* \) in the dual metric between the spaces for \( B \) and \( B^* \) denoted by \( \langle \cdot, \cdot \rangle \). Substituting into (5.76) and multiplying by \( v_\beta \) yields the following asymptotic ODE system:
\[
(5.81) \quad  \dot{c}_\beta = \langle D(c_\beta v_\beta^*), v_\beta \rangle + \ldots \equiv Q_\beta(c_\beta) + \ldots,
\]
where \( Q_\beta(c_\beta) \) is a vector quadratic form in \( \mathbb{R}^3 \). Looking for a standard solution with a power decay yields a quadratic algebraic system on the coefficients,
\[
(5.82) \quad c_\beta(\tau) = \frac{A_\beta}{\tau} + \ldots \quad \implies \quad Q_\beta(A_\beta) = -A_\beta.
\]
Assuming that the matrix equation in (5.82) has a solution \( A_\beta \neq 0 \), we obtain the following asymptotic pattern:
\[
(5.83) \quad  \hat{Y}_0(y, \tau) \sim \frac{A_\beta}{\tau} v_\beta^*(y) + \ldots \quad \text{as} \quad \tau \to +\infty \quad (\lambda_k = 0).
\]
In this way, we observe a slow logarithmic drift, with \( \tau = -\ln(-t) \), relative to the centre subspace of \( H(U) \).

For \( \lambda_k < 0 \), with \( k = |\beta| \), we arrive at more standard stable subspace patterns
\[
(5.84) \quad  \hat{Y}_k(\tau) \sim e^{\lambda_k \tau} C_\beta v_\beta^* + \ldots \quad (\lambda_k < 0).
\]
Both (5.81) and (5.84) describe for \( \tau \gg 1 \) asymptotically small perturbations in (5.75) of a steady singular solenoidal field \( U \), so we have to choose solenoidal eigenfunctions.

**On outer matching:** *Outer Region.* The main idea of matching of these patterns with the outer region, which is most remote is explained in [70], where we are assuming that \( v_\beta^*(y) \) have a polynomial decay, for some coefficients \( \delta_k > 0 \), \( k = |\beta| \),
\[
(5.85) \quad |v_\beta^*(y)| \sim b_k y^{-\delta_k} + \ldots \quad \text{as} \quad y \to +\infty.
\]

\[\text{[36]Recall (2.31) for such eigenfunction expansions.}\]
Here we arrive at a simpler first-order matrix operator, since in the new rescaled variables of the outer region, all the Laplacians form asymptotically small singular perturbations simply meaning that Euler’s equations are dominant here. Of course, the resulting singularly perturbed dynamical system is difficult to tackle to pass to the limit $\tau \to +\infty$.

Note the following crucial property of such an extension into the next outer region II, where solution is much less: we are not now obliged to keep there the self-similar power-like decay at infinity. Moreover, we now can match this blow-up behaviour with the exponentially decay associated with the heat semigroup $e^{\Delta_2 \tau}$ for the equations such as (5.49) and others. This matching is not a key difficulty of the blow-up twistor construction and deals with not that singular orbits. Again, we take into account patterns (2.31).

On principles of singular inner matching: Inner Region II. The last crucial step is to match the centre (5.83) and stable (5.84) subspace patterns with the regular and bounded solenoidal flow in the singular inner region close to the origin $y = 0$. This is the most involved matter, where the divergence-free asymptotics satisfying something like (2.31) become key. The singular inner matching in a countable number of various cases is responsible for existence or nonexistence of solenoidal blow-up patterns and is the main open problem.

Assuming for definiteness that such matching exists, we predict some other evolution features of the resulting blow-up patterns. Here we formally suppose that the presented asymptotic expansions go along the solenoidal classes like in (2.31), so that such spatial-temporal structures can be used for some preliminary estimates of blow-up evolution.

Assume first that in the stable subspace representation (5.84), the eigenfunction is singular at $y = 0$ with the following behaviour ($\gamma_k > 1$):

$$|v_{\beta}^*(y)| \sim d_k y^{-\gamma_k} + ... \quad \text{as} \quad y \to 0.$$  

Therefore, the pattern (5.84) has the asymptotic behaviour in the singular inner region of the type (we omit here non-essential constants and neglect other minor multipliers)

$$\hat{u}_k(y, \tau) \sim \frac{\Delta}{y} - e^{\lambda_k \tau} C_{\beta} y^{-\gamma_k} + .....$$  

Calculating as in (1.46) (see [47]) the maximum point in $y$ of the function in (5.87) yields

$$\sup_{y > 0} |\hat{u}_k(y, \tau)| \sim e^{-\lambda_k^{-1} \tau} \to +\infty \quad \text{as} \quad \tau \to +\infty \quad (\lambda_k < 0).$$

Under the above assumptions, this is about a right bound on regular hypothetical continuation of the pattern $\hat{u}_k$ from an exponentially small neighbourhood of the singular state $U$ to smooth bounded solutions for $y \approx 0$. Note that the behaviour of the pattern as $y \to 0$ can be very complicated including blow-up swirl-like features. This gives a preliminary estimate of the rate of this non self-similar blow-up in the $u$-variable

$$\|u_k(\cdot, t)\|_{\infty} \sim (-t)^{-\frac{1}{2} + \frac{\lambda_k}{\gamma_k - 1}} \quad \text{as} \quad t \to 0^- \quad (k = |\beta| \gg 1).$$

Similarly, for the centre subspace pattern (5.83) (if any), this very rough estimate imply the following perturbation of the self-similar rate:

$$\|u_0(\cdot, t)\|_{\infty} \sim \frac{1}{\sqrt{-t}} |\ln(-t)|^{-\frac{1}{\gamma_k - 1}} \quad \text{as} \quad t \to 0^-.$$
See Section 7.6 for further necessary properties of such a matching.

**Remark: back to reaction-diffusion.** For single parabolic equations such as (1.36), (1.60) (for \( N > 6 \)), or a more general combustion model

\[ u_t = \Delta u + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \quad (p > 1), \]

such examples of creating non-similarity asymptotics by matched asymptotic expansion techniques are known and have been rigorously justified; see [17, 59, 72, 77, 98], and references therein. Even for these reasonably simple parabolic PDEs, in the supercritical Sobolev range as in (1.34), the set of singular stationary solutions can be rather complicated that affect the evolution properties of blow-up and other asymptotics; see [146, 156, 157] and Section 7.10. For solutions of (5.91) of changing sign (that are always taking place in our case), the blow-up patterns for \( p = p_S \) become more involved and their actual existence is still not fully justified rigorously, [59].

The reaction-diffusion equation (5.91) does not assume any conservation of energy, but, however, there are examples of the so called incomplete blow-up, when the solution \( u(x,t) \) becomes unbounded in \( L^\infty(\mathbb{R}^N) \) at a single moment \( t = T \) only; see [77] and earlier references therein concerning equation (1.36), as well as [157, 158] for more recent extensions and achievements. For the model (1.1), blow-up moments are incomplete, in view of the \( a \text{ priori} \) bound (2.2).

It seems this quite a bounded mathematical experience cannot be directly translated to the Navier–Stokes rescaled equations (5.74).

We have discussed the second principle to generate another countable set of blow-up twistors. No doubt that there are other ways to get various countable families of such patterns, which we are still not aware of and cannot even imagine.

5.13. **Discussion: on a very formal way to create a vertex and branching to localized smooth blow-up twistors; a twistor ring.** Here we discuss another important issue concerning the twistors on the 2D subspace \( \text{Span} \{1, z\} \).

(i) A vertex is by truncation of the \( W \)-component in (5.20) via the positive part,

\[ W = (z)_+ \tilde{W}, \]

so that \( W = 0 \) for \( z < 0 \). This leads to a weak solution of (1.1), so a certain effort is necessary to check whether this gives a Leray–Hopf solution in the sense of the inequality in (2.2) for all \( t < 0 \), i.e., before the blow-up occurs. On the other hand, in view of the existence of a local semigroup of smooth bounded solutions, existence of such solutions with a vertex assumes performing a certain smoothing at the stagnation point at \( x = 0 \). This cannot be done in the above self-similar manner, so the proof of such a “smooth branching” (if any) from the given non-smooth blow-up twistor represents a typical difficult open problem of modern theory of nonlinear PDEs.

(ii) Similarly, there occurs another, not simpler, open problem of “smooth branching” from this unbounded twistor of other, partially and asymptotically self-similar blow-up twistors that might be sufficiently spatially localized as \( t \to 0^- \). As we have mentioned,
a possible way to check a possibility of such branching consists of using asymptotic expansion theory to create solutions via series like \((5.24)\). Obviously, this leads to difficult nonlinear (first ones) and further linearized systems for the expansion coefficients and represents another face of this open branching problem.

(iii) There is another formal geometric way to create blow-up patterns by deforming the axis of their symmetry. For instance, let us assume, in view of our pretty local analysis of such blow-up behaviour for small \(z \approx 0\), that the \(z\)-axis can be bend into a finite ring such that at some points similar blow-up structures oriented in the tangential \(z\)-direction occur with a certain periodic behaviour along the ring generatrix\(^{37}\). This leads to “twistor rings” with a very difficult open accompanying mathematics.

5.14. On twistors in Euler equations. Similar blow-up twistors, with analogous concepts of their swirl-like and other extensions, can be formally detected for the Euler equations \((1.31)\), where the calculations are supposed to be simpler but actually they are not. Without the “curse of local smoothness” for the Navier–Stokes equations \((1.1)\), for the Euler ones \((1.31)\) containing first-order differential operators only without a typical interior regularity, the necessary branching phenomena are easier to justify, so there is a hope of doing these more rigorously. However, one can see that, neglecting Laplacians \(\Delta_2\) in the systems \((5.25)\) and \((5.39)\), indeed simplifies the analysis but the remaining PDEs are still very difficult to understand rigorously that demands new concepts of solutions and entropy regularity. The stationary system \((5.41)–(5.43)\) with no Laplacian operators \(\Delta_2\)’s takes the form

\[
\begin{align*}
\frac{-1}{2} y \hat{w} y - \hat{w} - \sigma \hat{w} \mu + \hat{w}^2 & \\
\frac{-1}{y} \left( \int_0^y z \hat{w} \, dz \right) \hat{w} y + \frac{1}{y} \left( \int_0^y \hat{v} \mu \, dz \right) \hat{w} y - \frac{1}{y} \hat{v} \hat{w} \mu = 0. \\
\left[ \frac{1}{2} y \hat{u} y + \frac{1}{2} \hat{u} + \sigma \hat{u} \mu + \hat{u} \hat{u} y + \frac{1}{y} \hat{v} \hat{u} \mu - \frac{1}{2} \hat{v}^2 + \frac{2}{y} \hat{v} \mu + \frac{1}{y} \hat{u} \mu \right] \mu & \\
= \left[ y \left( \frac{1}{2} y \hat{v} y + \frac{1}{2} \hat{v} + \sigma \hat{v} \mu + \hat{v} \hat{v} y + \frac{1}{y} \hat{u} \hat{v} \mu + \frac{1}{y} \hat{u} \hat{v} - \frac{2}{y} \hat{u} \mu + \frac{1}{y} \hat{v} \hat{v} \right) \right] y,
\end{align*}
\]

with the same nonlocal constraint \((5.41)\). It is still a difficult system of a first and a second-order nonlocal evolution PDEs, i.e., actually, is a system of higher-order equations.

As we know, regardless its solvability, the linearization of this system about constant, singular, or other equilibria defines the linear operators that may generate other countable families of non-self-similar blow-up patterns with the evolution on centre and/or stable eigenspaces, as in Sections 5.11 and 5.12.

It is important to mention that, following the lines of this construction, we necessarily arrive at the non-stationary system corresponding to \((5.93)\). Since odd-order operators are dominated here, a proper “entropy” setting for solutions with weak and strong discontinuities will be necessary to check evolution consistency of the blow-up patterns constructed\(^{38}\).

\(^{37}\)For the Euler equations \((1.31)\), see \([180]\).

\(^{38}\)Recall that there is still no a successful notion of weak solutions of Euler equations in 3D.
This is expected to be a hard problem that, with a clear inevitability, accompanies this pointwise blow-up analysis of the Euler equations.

Finally, we believe that general concepts of swirling rotations, axis precessions, and vertex motion of periodic or quasi-periodic nature developed in Section 8 can be applied to the Euler equations [131], but we will not develop these here and return to the Navier–Stokes ones [1.1]; see [180] for further refreshing and rather exotic ideas. Recall that typical “rolling-up mechanisms” for appeared bubble caps and other swirling features of formation of blow-up singularities were observed numerically even in the axisymmetric setting; see [91, 181], etc., though these scenarios of blow-up seem remain still under scrutiny, while some of them have been ruled out in other works.

6. On non-radial blow-up patterns on \( W_2 \): eigenfunctions with swirl

In this section, we show how to extend some ideas coming from the simpler model proposed in Ohkitani [168] with a countable set of blow-up solutions constructed in [70]. We return to the NSEs restricted to the subspace \( W_2 = \text{Span}\{1, z\} \) in Section 5.4.

6.1. The system on the subspace \( W_2 \) with swirl and angular dependence. For convenience, we replace

\[
\tilde{W} \mapsto W,
\]

so that the system of three equations from Section 5.4 takes the form

\[
\begin{align*}
W_t &= \Delta_2 W - U W_r - \frac{1}{r} V W_\varphi + W^2, \\
U &= \frac{1}{r} \int_0^r z W(z, \varphi, t) \, dz - \frac{1}{r} \int_0^r V_\varphi(z, \varphi, t) \, dz, \\
\left[U_t + U U_r + \frac{1}{r} V U_\varphi - \frac{1}{r} V^2 - \Delta_2 U + \frac{2}{r} V_\varphi - \frac{1}{r} U \right]_\varphi \\
&= \left[r \left(V_t + UV_r + \frac{1}{r} VV_\varphi - \frac{1}{r} UV - \Delta_2 V - \frac{2}{r} V_\varphi + \frac{1}{r} V \right) \right]_r.
\end{align*}
\]

Indeed, this is a very difficult system. As we have mentioned, its axi-symmetric version admits further, more rigorous study, [168, 70]. Here we have demonstrated other aspects of such solutions.

Thus, we apply the same scaling as in (5.29), (5.30) to the general system (6.2) to get

\[
\begin{align*}
w_r &= \Delta_2 w - \frac{1}{2} y w_y - w - \sigma w_\mu - u w_\mu - \frac{1}{y} w v_\mu + w^2, \\
u &= \frac{1}{y} \int_0^y z w(z, \mu, \tau) \, dz - \frac{1}{y} \int_0^y v_\mu(z, \mu, \tau) \, dz, \\
\left[u_r + \sigma u_\mu + u w_\mu + \frac{1}{y} v u_\mu - \frac{1}{y} v^2 - \Delta_2 u + \frac{1}{2} y u_\mu + \frac{1}{2} u + \frac{2}{y^2} v_\mu + \frac{1}{y} u \right]_\mu \\
&= \left[y \left(v_r + \sigma v_\mu + u v_\mu + \frac{1}{y} v v_\mu + \frac{1}{y} u v - \Delta_2 v + \frac{1}{2} y v_\mu + \frac{1}{2} v - \frac{2}{y^2} u_\mu + \frac{1}{y^2} v \right) \right]_\mu.
\end{align*}
\]

The linearization \( w = 1 + Y \) as in (5.51) yields the linearized equation (cf. (5.52))

\[
Y_\tau = B^* Y + Y - \sigma Y_\mu - u Y_y - \frac{1}{y} v Y_\mu + D(Y),
\]

where \( D \) is as in (5.52) and \( B^* \) is the adjoint Hermite operator (5.55). We next use the second equation for \( u \) in (6.3) to get

\[
u = \frac{1}{2} + \frac{1}{y} \int_0^y z Y \, dz - \frac{1}{y} \int_0^y v_\mu \, dz.
\]
Assume now that
\[ Y(y, \mu, \tau) \] and \[ v(y, \mu, \tau) \] are exponentially small for \( \tau \gg 1 \).

Then by (6.5) \( u = \frac{y}{2} + \ldots \) up to exponentially small perturbations, so that small solutions of (6.4) are still governed by the linear operator (5.53)
\[ (6.6) \]
\[ \mathbf{B}^* Y + Y - \sigma Y_\mu \equiv \tilde{\mathbf{B}}^* Y - \sigma Y_\mu + \ldots \quad \text{as} \quad \tau \to +\infty. \]

Consider first the full operator with the rotational part with the adjoint one in \( L^2(\mathbb{R}^2) \)
\[ (6.7) \]
\[ \mathbf{B}^* = \mathbf{B} = \Delta_2 + yD_y + 3I \]
\[ \mathbf{L}_\sigma = \tilde{\mathbf{B}}^* - \sigma D_\mu \quad \text{and} \quad \mathbf{L}_\sigma = \tilde{\mathbf{B}} + \sigma D_\mu \]
\[ \sigma \equiv \frac{\mu}{\tau} \]
\[ \tilde{\mathbf{B}}^* = \Delta_2 + yD_y + 3I \]
\[ \sigma \equiv \frac{\mu}{\tau} \]
\[ \text{we see that the essential use of the angular operators makes it rather difficult. Obviously, any radial Hermite polynomial remains an eigenfunction for any } \sigma \text{. Finding a suitable point spectrum and non-radial eigenfunctions with essential } \mu \text{-dependence for (6.9) can be a difficult problem. One can see that a standard separation of variables is non-applicable for (6.9).} \]

Thus, a principal question is whether the operator (6.8) has enough real eigenvalues for construction of linearized blow-up patterns. Let us show that in this sense (6.8) is quite suitable. As an illustration of a local approach to (6.9), we apply the classic techniques \[ \text{to trace out branching of eigenfunctions from } \psi_\beta \text{ of } \tilde{\mathbf{B}}^* \text{ at } \sigma = 0. \]

Let the kernel of \[ \mathbf{B}^* - \lambda_\beta I \] has the dimension \( M \geq 2 \). Then looking for the expansion
\[ (6.10) \]
\[ \psi^*(\sigma) = \sum_{k=1}^{M} c_k(\sigma) \psi_{\beta,k}^* + \sigma \varphi + \ldots (\varphi \perp \psi_{\beta}^*) \quad \text{and} \quad \lambda(\sigma) = \lambda_\beta + \sigma s + \ldots, \]
\[ (6.11) \]
we obtain the following problem (\( \lambda_\beta \) are as given in (6.13)):
\[ (\mathbf{B}^* - \lambda_\beta I) \varphi = \sum_{(k)} c_k(0) [s \psi_{\beta,k}^* + (\psi_{\beta,k}^*)'_\mu]. \]
This yields \( M \) orthogonality conditions of solvability
\[ (6.12) \]
\[ \langle \sum_{(k)} c_k(0) [s \psi_{\beta,k}^* + (\psi_{\beta,k}^*)'_\mu], \psi_{\beta,j} \rangle = 0 \quad \text{for} \quad j = 1, 2, \ldots, M, \]
that define the eigenfunctions of \( \tilde{\mathbf{B}}^* \), from which branching is available. Then (6.11) yields the unique solution \( \varphi \) that shows the branching evolution in (6.10) (as usual, the dimension of the kernel then can play a part). In other words, (6.8) in the inverse integral form can be treated as a compact perturbation of a self-adjoint operator, and classic perturbation theory applies \[ \text{to calculate the deformation of the real point spectrum.} \]

This shows that there exists branching for small angular speeds \( |\sigma| \) of eigenfunctions with swirl from standard Hermite polynomials for \( \sigma = 0 \). For compact inverse integral operators involved, these are continuous \( \sigma \)-curves are indefinitely extensible and can end up at other bifurcation point only or can be unbounded; see [23]. In other words, the
swirl-dependent operators (6.11) have enough real eigenfunctions for using in necessary construction of linearized blow-up patterns. Some of the related questions of linear operator theory for (6.11) remain obscure; e.g., eigenfunction closure for such non-symmetric cases (but indeed this looks doable though can be rather technical).

It is curious that even in the case \( \sigma = 0 \), we can find essentially non-radial patterns, which can give an insight into a swirl structure of blow-up patterns that have a clear \( \varphi \)-dependence. Namely, we now take into account all the non-radial Hermite polynomial eigenfunctions (5.61) of \( B^* \) with the spectrum

\[
\sigma(B^*) = \{ \lambda_\beta = 1 - |\beta|, \ |\beta| = 0, 1, 2, ... \}.
\]

Therefore, there exist solutions of (6.4) with the asymptotic behaviour as \( \tau \to +\infty \),

\[
w = 1 - e^{\lambda_\beta \tau} \psi_\beta(y, \mu) + ... \quad \text{for any} \quad |\beta| > 1,
\]

provided that (6.6) holds on compact subsets in \( y \). Obviously, the third equation in (6.3) admits exponentially small solutions \( v \) for \( u \) being an exponentially small perturbation of \( y^2 \) according to (6.5). Then patterns (6.14) in the inner region make sense.

The extensions of such patterns into the outer region is similar. We have:

\[
\psi_\beta(y, \mu) \sim y^{|\beta|} f_j(\mu) \quad \text{as} \quad y \to +\infty,
\]

where the first entry \( y \) means \((y_1, y_2)^T\) and \( f_j \) are homogeneous harmonic polynomials being the eigenfunctions of the Laplace–Beltrami operator \( \Delta_\mu = D_\mu^2 \) on the circle \( S^1 \subset \mathbb{R}^2 \),

\[
\Delta_\mu f_j = -j^2 f_j \quad \text{on} \quad S^1.
\]

Therefore, (6.14) yields the outer variable

\[
w \sim 1 - |\zeta|^{|\beta|} f_j(\mu), \quad \text{where} \quad \zeta = y e^{\rho_\beta \tau}, \quad \rho_\beta = \frac{\lambda_\beta}{|\beta|} < 0.
\]

Finally, rewriting the first equation in (6.3) in terms of the new variable \( \zeta > 0 \) by the change corresponding to the region governed by the new spatial rescaled variable \( \zeta \), we recast the original equation by setting (this change is explained in detail in [70]) \( \theta(\zeta, \tau) = w(y, \tau) \). This yields the following perturbed Hamilton–Jacobi (Euler) equation:

\[
\theta_\tau = A_\beta(\theta) + e^{2\rho_\beta \tau} \Delta_2 \theta + e^{\rho_\beta \tau} \left[ \frac{1}{\zeta} \left( \int_0^\zeta v_\mu \right) - \frac{1}{\zeta} \left( \int_0^\zeta z_\theta \right) \theta_\zeta - \frac{1}{\zeta} \left( \int_0^\zeta z_\theta \right) \theta_\zeta - \theta + \theta^2 \right],
\]

where \( A_\beta(\theta) = -(\rho_\beta + \frac{1}{2}) \zeta \theta_\zeta - \frac{1}{\zeta} \left( \int_0^\zeta z_\theta \right) \theta_\zeta - \theta + \theta^2 \).

Thus, eventually, on passage to the limit \( \tau \to \infty \), we obtain the steady problem

\[
-(\rho_\beta + \frac{1}{2}) \zeta h_\zeta - \frac{1}{\zeta} \left( \int_0^\zeta z h \right) h_\zeta - h + h^2 = 0, \quad h(\zeta) \sim 1 - \zeta^{|\beta|} f_j(\mu), \quad \zeta \to 0.
\]

For each fixed \( \mu \in [0, 2\pi) \), this is an ODE, which was the object of a detailed study in [70]. Therefore, for any \( f_j(\mu) > 0 \), such a profile always exists and is compactly supported with non-radial support. Unfortunately, for \( f_j(\mu) < 0 \), the ODE (6.19) gives
a monotone increasing solution which moreover blow-up in finite ζ. Therefore, such non-radial patterns can be constructed in any connected sector \( S_j = \{ \mu : f_j(\mu) > 0 \} \) with zero Dirichlet conditions at the boundary rays. There holds:

\[
\zeta = x(-t)^{\frac{|\beta|-2}{2|\beta|}} = \text{separable standing-wave blow-up for } |\beta| = 2.
\]

This case includes the radial pattern already studied and others, which are no symmetric.

We hope that using non-trivial rotation \( \sigma \neq 0 \) will supply us, via the eigenvalue problem (6.9), some extra new eigenfunctions that allow us to get a non-radial blow-up pattern with swirl in the whole space.

7. On blow-up patterns concentrated about Slezkin–Landau singular solutions of a “submerged jet”: first example of linearization

We now discuss some ways to construct necessary blow-up patterns to be applied in greater detail in Section 8 by using the spherical geometry. Namely, we introduce implications of using the following classic singular solutions of the NSEs:

7.1. Singular homogeneous equilibria and their properties. In 1934, Slezkin [196] (see also comments in [197]) showed that (1.1) admit special stationary solutions with the singularity and spatial decay of the velocity field \( \sim \frac{1}{r} \) by reducing the problem to a linear hypergeometric-type ODE; see more details in Appendix C. In 1944 [39], Landau [132] found a family of explicit solutions of that type describing steady flows induced by a point source, which leads to the setting of a submerge jet that is oriented along the positive part \( Oz \) of the \( z \)-axis. This gives the following one-parameter family of the explicit Slezkin–Landau singular stationary solutions \( \{u_{SL}(x), p_{SL}(x)\} \), of (1.1):

\[
\begin{align*}
  u(x) &= \frac{2(cz-r)x}{(cz-z)^2r}, \\
  v(x) &= \frac{2(cz-r)y}{(cz-z)^2r}, \\
  w(x) &= \frac{2(cz^2 - 2cz + cx^2)}{(cz-z)^2r}, \\
  p(x) &= \frac{4(cz-r)}{(cz-z)^2r},
\end{align*}
\]

where \( r^2 = x^2 + y^2 + z^2 = |x|^2 \), and \( c \in \mathbb{R} \) is a constant such that \( |c| > 1 \). In view of a strong singularity at \( x = 0 \), the existence of such a solution demands an extra force at the origin, so actually (7.1) is a fundamental solution of the stationary operators of the NSEs satisfying, in the sense of distributions (this computation was already done in Landau’s original work [132, p. 300]; see also [21 pp. 2-9], [133, p. 182], or [29, p. 250]),

\[
(u \cdot \nabla)u + \nabla p - \Delta u = b(c)\delta(x)j, \quad \nabla \cdot u = 0, \quad \text{where } j = (0, 0, 1)^T \text{ and } b(c) = \frac{8\pi c}{3(c^2-1)} [2 + 6c - 3c(c^2 - 1) \ln \left( \frac{c+1}{c-1} \right)] = \frac{16\pi}{3c^2} - \frac{32\pi}{3c^4} + \ldots \text{ as } c \to +\infty.
\]
Thus, physically speaking, this means that (7.3) the steady S–L solution demands a permanent fluid injection at the origin \( x = 0 \).

Thus, \( u_{\text{SL}} \) satisfies (this will be used in the rescaled blow-up variables \( y \)):

\[
u_{\text{SL}}(x) \sim \frac{1}{c|x|} \quad \text{as} \quad x \to 0, \quad \text{and, more precisely,}
\]

\[
u_{\text{SL}}(x) = \frac{1}{c} u_0 + O\left(\frac{1}{c^2}\right) \quad \text{as} \quad c \to +\infty, \quad \text{where} \quad u_0 = \left(\frac{2z}{r^3}, \frac{2y}{r^3}, \frac{2}{r}\right)^T,
\]

\[|u_{\text{SL}}(x)| \to \infty \quad \text{as} \quad c \to 1^+ \quad \text{on the semiaxis } O_2^+ = \{ x = y = 0, z \geq 0 \}.
\]

To illustrate the last property of unbounded steady profiles, consider the first component in (7.1) for a fixed \( z > 0 \) (for \( z = 0 \), the estimate is similar with \( 4 \mapsto 2 \) at the end):

\[
u(x)|_{c=1} = \frac{2x}{r(\sqrt{x^2+y^2+z^2})} = \frac{4x(1+o(1))}{r^2(x^2+y^2)} \to \infty \quad \text{as} \quad x^2 + y^2 \to 0.
\]

This is rather impressive (and promising for blow-up around): it turns out that arbitrarily large vector fields (of special structure) can be locked in a singular steady pattern.

In the spherical coordinates, the flow (7.1), being symmetric about the polar (\( O_z \)-) axis, reads (this is the actual Slezkin [196] and Landau form [132]; see also [133, p. 82])

\[
\begin{align*}
u(r, \theta) &= \frac{1+\cos^2 \theta - 2c \cos \theta}{r(c-\cos \theta)^2}, \\
v(r, \theta) &= \frac{2 \sin \theta}{r(c-\cos \theta)}, \quad w = 0, \\
p(r, \theta) &= \frac{4(c \cos \theta - 1)}{r^2(c-\cos \theta)^2}.
\end{align*}
\]

Then, in accordance with the divergence (7.5), we have

\[
v(r, \theta)|_{c=1} = \frac{2 \cos \theta}{r \sin \frac{\theta}{2}} \to \infty \quad \text{as} \quad \theta \to 0.
\]

The singular S–L solutions are \( L^2 \) locally, but not globally:

\[
u_{\text{SL}} \in L^2_{\text{loc}}(\mathbb{R}^3), \quad \text{but} \quad u_{\text{SL}} \not\in L^2(\mathbb{R}^3) \quad \text{for all} \quad |c| > 1.
\]

It is also worth noting for further blow-up use of (7.9) that the total mass flux through any closed surface around the origin is equal to zero [133, p. 83]; cf. the “vanishing oscillatory property” (5.46). It turns out that (7.1) are the only possible stationary homogeneous of degree \(-1\), regular except the origin \((0,0,0)\) solutions of (1.1) [204]; see also earlier result [208] proved under the \( z \)-axis symmetry assumption. One can see that (7.1) are invariant under the scaling group in (2.18). Note that the whole set of possible steady singularities of the NSEs is still not fully known; even in the class

\[
u(x)| \leq C_\ast \frac{r}{|x|}, \quad \text{where} \quad C_\ast > 0 \quad \text{is a constant.}
\]

However, an essential first step in this direction was done in Miura–Tsai [155], who proved: if \( u \) is any very weak steady solution of the NSE in \( \mathbb{R}^3 \setminus \{0\} \), then

\[
u(x) \text{ satisfies (7.9) for some small } C_\ast > 0 \quad \implies \quad u = u_{\text{SL}} \text{ for some } |c| > 1,
\]

where, by (7.5), (7.7), \( c \) is assumed to be large enough. In any case, the S–L solutions (7.1) should play a crucial role, since these are expected to be isolated from other singular
ones (if any); see further details in [204], [155]. In addition, there are no smaller singular equilibria (see the result and earlier references in [117]): if \( \mathbf{u}, p \in C^\infty(B_R \setminus \{0\}) \), then
\[
|\mathbf{u}(x)| = o\left(\frac{1}{|x|}\right) \quad \text{as} \quad x \to 0 \quad \implies \quad 0 \text{ is removable and } \mathbf{u} \in C^\infty \text{ there.}
\]

A similar removable singularity theorem for the nonstationary NSEs [125], where (7.11) is assumed to be valid for a weak solution \( \mathbf{u}(x, t) \) in some set \( B_r \times (0, t_0) \), includes a smallness condition on \( \mathbf{u} \) in \( L^\infty(0, t_0; L^3) \).

Note that in addition to the symmetry of these solutions about the \( z \)-axis, (7.1) also implies that such a flow does not exhibit any swirl, i.e., the angular component \( w = u_\varphi \equiv 0 \) in (7.6). This, in view of global existence results for cylindrically axisymmetric (irrotational) flows [211], [130] (see [222] for details), indeed, reduces the chances to get a reasonable blow-up pattern moving along the corresponding quasi-stationary manifolds (a finite-dimensional centre, if any, or a stable one of infinite dimension). Nevertheless, the situation with the steady manifold induced by (7.1) is not that hopeless. For instance, instead of the “standard” similarity swirl given by the log-law (5.32), one can use a slower rotational mechanism by setting
\[
\varphi = \mu + \sigma \kappa(t), \quad \text{where } (-t)\kappa'(t) \to 0 \quad \text{as} \quad t \to 0^-.
\]

Then, the stationary term \( \sigma \mathbf{u}_\mu \) in (5.35) is replaced by the asymptotically vanishing one,
\[
\sigma \mathbf{u}_\mu \mapsto \sigma(-t)\kappa'(t)\mathbf{u}_\mu \to 0 \quad \text{(e.g., } \kappa(t) = -\frac{\ln(-t)}{\ln|\ln(-t)|} \text{ for } t \approx 0^- \text{, with } \delta > 0),
\]
on bounded smooth orbits. Therefore, for a suitable class of solutions, passing to the limit along a sequence \( \{\tau_k\} \to +\infty \) by setting \( \tau \mapsto \tau_k + \tau \) will fix us the previous limit irrotational equations admitting the “stationary” singularity (7.1). Of course, this leads to a delicate matching procedure of connecting the slow whirling flow characterized by (7.12) with Slezkin–Landau’s “quasi-steady” solutions (7.1), which actually should determine the function \( \kappa(\tau) \). We return to other extensions and use of (7.1) in Section 8.

7.2. On some hypothetical extensions. On the other hand, it is principal for blow-up patterns to know whether the rescaled NSEs admit other solutions with swirl. In general, classification of singular states for the NSEs is a difficult open problem (see [204]), though it is a necessary step for understanding of existence/nonexistence of finite energy blow-up patterns for such dynamical systems.

For instance, it is crucial to check existence of singular solutions with the TW dependence in the angular direction, with
\[
\varphi \mapsto \varphi + \sigma t.
\]
This gives the nonlinear eigenvalue problem (we assume that the singularity has not been essentially changed, so we may keep the same its \( \delta \)-interpretation, for simplicity)
\[
\sigma \mathbf{u}_\varphi + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \Delta \mathbf{u} = b(\sigma)\delta(x)\mathbf{j}, \quad \nabla \cdot \mathbf{u} = 0.
\]
For \( \sigma_0 = 0 \) and \( b(0) = b(c) \), this gives the S–L solutions satisfying (7.2). Do other eigenvalues \( \{\sigma_k \in \mathbb{R}\} \) exist? A partial negative answer is available (see [155] as a guide),
but it does not cover the whole range. The corresponding singular states \( \{ u^k \} \) then can be used for constructing various blow-up patterns.

In the rescaled NSEs (2.21), (2.23), one needs to know existence of regular states with the blow-up angular dependence (5.30) (for \( \sigma = 0 \), no solutions [162])

\[
(7.16) \quad \sigma \bar{u} + H(\bar{u}) = 0 \quad \text{in} \quad \mathbb{R}^3, \quad \bar{u} \in L^2_{loc}(\mathbb{R}^3) \quad (\bar{u} \not\in L^2(\mathbb{R}^3) \text{ in general.})
\]

Here \( \bar{u}(y) \) may be assumed to be bounded at \( y = 0 \), so that it is natural to suppose that, along some subsequences or continuous branches (see Section 7.8 for Type II using),

\[
(7.17) \quad \bar{u}(0) = \bar{C} \quad \text{and} \quad \bar{u} \to \bar{u}_{SL}(c) \quad \text{as} \quad |\bar{C}| \to +\infty \quad (|c| > 1)
\]

uniformly on compact subsets in \( \mathbb{R}^3 \setminus \{0\} \). In other words, the singular S–L solutions could serve as “envelopes” of the regular ones (this is obscure and questionable). Then, it seems, at least a two-parameter family of such regular profiles might be expected. In any case, the extra nonlinear eigenvalues \( \{ \sigma_k \} \) (or other perturbation-like mechanisms) can essentially affect the structure of those hypothetical regular swirling states \( \{ \bar{u}_k \} \) solving (7.16). We again state that these speculations are made under the clear absence of any clue on existence of those singular quasi-steady patterns with various blow-up swirls. Both clear mathematically existence or nonexistence conclusions are desperately needed here.

7.3. On possible blow-up patterns: towards “swirling tornado” about S–L singular equilibria. Inner Region I. We now perform first steps of the blow-up strategy according to the blow-up scenario developed in Section 1.6 now applied to the S–L singular stationary solution. So we are going to check whether a “blow-up swirling tornado” (not a twistor, since we do not always apply directly the logTW mechanism, though it is not excluded) can appear by approaching the \( L^\infty \)-singularity at \( x = 0 \) by drifting as \( t \to T^- \) by “screwing in” about this singular equilibrium structure. There is a clear suspicion that this can hardly happen without, according to (7.3), a permanent fluid injection, which is not available for bounded \( L^2 \)-solutions. Note that the hypothetical blow-up tornado is going to occur very fast, during a miserable time scale, when the total injection is negligible. Anyway, this physical negative motivation should find some clear mathematical issues in support or not.

Recall that the behaviour in Inner Region I is assumed to be characterized by the linearized problem (5.77). Thus, following the same standard principles of matching, linearization of the general equation (2.21) about \( u_{SL} \) via (5.75) yields, in Inner Region I, the equation (5.76) with quite a tricky linear pseudo-differential (integro-differential) operator obtained from (2.23), which in \( C^\infty_0(\mathbb{R}^3 \setminus \{0\}) \) reads

\[
H'(\bar{u}_{SL})Y = (B^* - \frac{1}{2} I) Y - \mathbb{P}CY, \quad \text{where} \quad CY = \text{div} \left( \bar{u}_{SL} \otimes Y + Y \otimes \bar{u}_{SL} \right)
\]

\[
+ C_3 \int_{\mathbb{R}^3} \frac{y \cdot \bar{u}_{SL} \otimes Y - \bar{u}_{SL} \otimes Y \cdot \bar{u}_{SL}}{|y|^3} \sum_{(i,j)} \left( \bar{u}_{SL,i}^j Y_{i,j}^j + \bar{u}_{SL,j}^i Y_{i,j}^i \right) \equiv J_1 Y + J_2 Y.
\]

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Here, $J_1$ includes all the local differential terms, while $J_2$ the integral one. Note that, in general, according to (7.16), the above linearized operator must include the operator

$\mathbf{(7.19)} \quad \ldots - \sigma D_\mu Y + \ldots$, so (7.18) contains two free parameters: $c$ and $\sigma$. 

with a nonlinear eigenvalue $\sigma$ coming from the stationary problem already containing the parameter $c > 1$. In a most general setting, the linearization is performed relative to any unknown S–L-type singular steady states delivered by (7.16). For simplicity, we will continue to deal with the standard S–L solutions, actually meaning that, in addition to other branching ideas, one needs also (cf. Section 6.1):

$\mathbf{(7.20)}$ 

to develop branching of eigenfunctions from the logTW speed $\sigma = 0$, 

though possibly this is not an issue since demanding too much from the operator (7.18) ($\lambda = 0$ must be in the spectrum). On the other hand, even if (7.20) makes sense, it is then also plausible that such a branching is available (or not) for $|\sigma|$ small only, and proper eigenfunctions may occur via “saddle-node” bifurcations for large $|\sigma|$, which are very difficult to detect. In any case, further detailing of these arguments seems excessive, since, even without (7.20) and/or others, the study gets very complicated with already too many open problems to appear.

In all the cases, we observe in (7.18) singular unbounded coefficients as $y \to \infty$ as well as $y \to 0$. A proper treatment of $y = \infty$ is settled via the space $L^2_{\rho^*}(\mathbb{R}^3)$, as usual. The singular point $y = 0$ is much worse.

**Local operator: discrete spectrum by improved Hardy–Leray inequality.** Namely, using (7.4) yields that the singularity of the potentials in the local terms,

$\mathbf{(7.21)} \quad CY \equiv (\hat{u}_{SL} \cdot \nabla)Y + (Y \cdot \nabla)\hat{u}_{SL} \sim \left(\frac{y}{c|y|^2} \cdot \nabla \right)Y + (Y \cdot \nabla)\frac{1}{c|y|},$

is well covered by Hardy’s classic inequality at least for all large $c > 0$, since both terms “act” like the inverse square potential. Thus, according to (1.43), we need

$\mathbf{(7.22)} \quad J_1 Y \sim \Delta Y + \frac{Y}{c|y|^2} \quad \text{as} \quad y \to 0 \quad \Longrightarrow \quad \sim \frac{1}{c} \leq \frac{(N-2)^2}{4} \big|_{N=3} = \frac{1}{4}.$

Note that, for axi-symmetric solenoidal flows, the Hardy optimal constant $c_H = \frac{(N-2)^2}{4}$ becomes better, and, as shown in Costin–Maz’ya [44], can be replaced by

$\mathbf{(7.23)} \quad c_{H, \text{axi–sol.}} = \frac{(N-2)^2}{4} \left(\frac{N^2+2N+4}{N^2+2N-4}\right)_{N=3} = \frac{25}{68} = 0.3676... > \frac{1}{4}.$

In other words, the operator $\Delta Y + \frac{25}{68} \frac{Y}{|x|^2}$ in the axi-symmetric solenoidal field admits a proper setting in $L^2$ for $x \approx 0$. Note that the inequality

$\mathbf{(7.24)} \quad \frac{1}{4} \int_{\mathbb{R}^3} \frac{|Y|^2}{|x|^2} \leq \int_{\mathbb{R}^3} |\nabla Y|^2 \quad (N = 3)$

already appeared in the same Leray’s pioneering paper in 1934 [137]. Thus, $\frac{1}{4}$ in (7.24) can be replaced by $\frac{25}{68}$ that improves the admissible range of $c$’s according to (7.22). Hence, we obtain the first important and even principal conclusion:

$\mathbf{(7.25)} \quad \text{for any } c \gg 1, \text{ the local linearized operator } J_1 \text{ is well-posed.}$

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Therefore, this local part of the linearized operator has a compact resolvent, a discrete spectrum in the same weighted $L^2$-space, etc., though several questions including completeness and closure of the eigenfunction set and others remain open and difficult in this non self-adjoint case. Moreover, it follows from (7.18) and (7.21) that for $c \gg 1$, with a proper functional setting, $J_1$ has eigenvalues that are close to those in (2.23) for $B^* - \frac{1}{2} I$.

In addition, by classic perturbation theory (see Kato [114]), it can then be shown that

$$\text{eigenfunctions } \{\hat{v}_\beta^*\} \text{ of } J_1 \text{ can be obtained from generalized }$$

(7.26)

Hermite polynomials $\{v^*_\beta\}$ via branching at $c = +\infty$.

This justifies existence of infinitely many real eigenvalues of $J_1$, at least, for $c \gg 1$ (global extension of bifurcation $c$-branches to finite values of $c > 1$ is a difficult open problem).

Since in (2.21) $P$ is a projector in $(L^2)^3$, $\|P\| = 1$, it seems reasonable to expect that some part of discrete spectrum may be governed by the local differential operator $J_1$. Consider the full linearized equation

$$Y_\tau = (B^* - \frac{1}{2} I)Y - P CY + D(Y) \equiv H'(\hat{u}_{SL})Y + D(Y),$$

(7.27)

where $D$ is a quadratic perturbation. Assuming that there exists a proper eigenvalue $\lambda_\beta \in \sigma_p(H'(\hat{u}_{SL})) \neq \emptyset$, with $\text{Re} \lambda_\beta \leq 0$, we expect the behaviour $Y(\tau) \sim c e^{\lambda_\beta \tau} \hat{v}_\beta^*$ for $\tau \gg 1$, if $\text{Re} \lambda_\beta < 0$, or (as in Section 5.12) $Y(\tau) \sim c_\beta(\tau)\hat{v}_\beta^*$ close to a centre subspace, if $\text{Re} \lambda_\beta = 0$. [Both to be matched with more regular flows around.] Substituting this into (7.27) yields the eigenvalue problem, where the nonlocal term implies a tricky setting,

$$\text{(7.28) } (B^* - \frac{1}{2} I)\hat{v}_\beta^* - P \hat{C} \hat{v}_\beta^* = \lambda_\beta \hat{v}_\beta^*, \text{ for } \hat{v}_\beta^* \in \tilde{H}^2_\beta(\mathbb{R}^3), \text{ at least.}$$

Again, it should be noted that the angular operator (7.19) must enter this eigenvalue problem. We claim that this extra technicality can be tackled by separation in the angular variable, though this leads to more difficult calculus and a proper adaptation of further conclusions are not easy but doable. However, this shows a natural opportunity to be retained within local spectral theory for $J_1$. Since, as we have seen, $PC \to 0$ as $c \to +\infty$, this demands solving the following problem for generalized Hermite polynomials $\{v^*_\beta \in \Phi^k_\beta, k = |\beta|\}$ as in Section 2.6

(7.29)

$\text{for which } v^*_\beta, \text{ the linearized term } C v^*_\beta \text{ is or } \text{“almost” solenoidal?}$

Obviously, (7.28) shows that if $C v^*_\beta$ is fully divergence-free, then $P = I$ on it, so the local operator $J_1$ takes the full power, and this settles the spectral problem for $c \gg 1$. As we expect, this will allow us to start a series-like expansion in terms of the small parameter $\frac{1}{c}$ of the necessary div-free eigenfunction of the full problem (7.28).

We did not check if such eigenfunctions are actually existent, but, in a whole, the problem (7.29) looks doable. Indeed, we do not expect to find such polynomials $v^*_\beta$ satisfying (7.29) for sufficiently small $|\beta| = 0, 1$ or 2, etc. The first such $l = |\beta|$, for which (7.29) holds with necessary accuracy, would show a complicated geometry (with remnants of swirl, axis precession, and so on), which would eventually affect the global structure of a possible blow-up pattern. Once we have found such solenoidal generalized Hermite polynomials satisfying (7.29), this starts the not less simpler procedure of constructing
those “twistors” by matching with more regular flows close to the singular point \( y = 0 \) and outside the pattern blow-up region (the latter is supposed to be simpler). Of course, it is too early discussing any seriously this nonlinear matching matter (but we will, noting also that we have already mentioned some formal principles of such procedures that are well established in blow-up R–D theory). But indeed the linearized spectral analysis about the rescaled S–L solution is assumed to show some basics of formation of such mysterious blow-up patterns including their strong angular motion (the eigenfunctions are supposed to be essentially non-radial), “precession” features of their axis of rotation (to be discussed in greater detail), and even their possible merging into a closed path of several individual twistors (hopefully, not infinite).

**Nonlocal operator: an obscure functional setting.** Anyway, despite speculations concerning the more or less standard local operator \( J_1 \) (with basics related to Hardy–Leray inequalities), one cannot avoid analyzing the nonlocal integral operator \( J_2 \).

The spectral questions for the nonlocal operator in (7.18) become more involved. Consider \( J_2 \) in \( C^\infty_0 (\mathbb{R}^3 \setminus \{0\}) \), where, on integration by parts,

\[
J_2 Y \equiv C_3 \int_{\mathbb{R}^3} \frac{y - z}{|y - z|^3} \sum_{(i,j)} \left( \dot{u}_{SL}^i Y_j^i z_i + \ddot{u}_{SL}^j Y_i^i z_j \right)
\]

(7.30)

\[
= -C_3 \int_{\mathbb{R}^3} \sum_{(i,j)} \left[ \left( \frac{y - z}{|y - z|^3} \dot{u}_{SL}^i Y_j^i \right) z_i Y_j^j + \left( \frac{y - z}{|y - z|^3} \ddot{u}_{SL}^j Y_i^i \right) z_j Y_i^j \right].
\]

We obtain singular integral operators with the kernels, formally exhibiting the behaviour

(7.31) \( K(y, z) = K_1(y, z) + K_2(y, z) \), where \( |K_1| \sim \frac{1}{c |y - z|^2 |z|^3}, |K_2| \sim \frac{1}{c |y - z|^3 |z|^2} \).

**First view of \( K_1 \): too singular, a condition at 0 needed.** The kernel \( K_1 \) is standard polar relative to the first multiplier having \( \sim \frac{1}{|y - z|^2 |z|^3} \), with \( \mu = 2 < N = 3 \). However, the second multiplier, at first sight, exhibits a non-integrable singularity \( \frac{1}{|z|^3} \) at \( z = 0 \) in \( \mathbb{R}^3 \) demands special conditions at the origin e.g., of the type (otherwise, the integrals in (7.30) ought to be understood in a generalized \( v.p. \) sense, see below)

(7.32) \( |Y(z)| = O(\rho(|z|)) \rightarrow 0 \) as \( z \rightarrow 0 \), where \( \int_0 \rho(r) \frac{dr}{r} < \infty \).

On the one hand, it seems clear that any such condition (7.32) on the behaviour at the origin can reduce a hope to find a proper eigenfunction of (7.18). On the other hand, it is well known that eigenvalues of pseudo-differential operators can be of infinite multiplicity (e.g., for those with constant symbol), i.e., the admitted behaviour at singularity points can be extremely various (unlike the differential operators, for which such a behaviour is restricted by their orders and asymptotics of the coefficients). However, even for the local differential operators, imposing such conditions is not that hopeless, as the following example shows:

**Example: differential operator with a locally non-integrable potential.** Consider for simplicity a scalar differential operator associated with \( J_1 \) in (7.18), with an extra inverse cubic potential \( \not\in L^2_{\text{loc}}(\mathbb{R}^3) \) as in (7.31):

(7.33) \( E = B^* - \frac{1}{2} I - C - \frac{A}{|z|^3} I \) (\( A \neq 0 \)).
Obviously, looking for possible eigenfunctions, the behaviour of solutions close to the singular origin 0 is defined by the principal part, so that, as \( z \to 0 \),
\[
\mathbf{E}\psi^* = \lambda \psi^* \implies \Delta \psi^* - \frac{A}{|z|^3} \psi^* + \ldots = 0 \implies \psi^*(z) \sim \omega \left( \frac{1}{|z|} \right) |y|^{-\delta} e^{\alpha |z|^\alpha},
\]
where \( \omega \) on the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) makes the angular separation. Substituting the last WKBJ-type expansion term into the operator \((\psi^*)'' + \frac{2}{|z|} (\psi^*)' - \frac{A}{|z|^3} \psi^* = 0 \) yields
\[
\alpha = -\frac{1}{2}, \quad \delta = \frac{3}{4}, \quad \text{and} \quad a^2 = 4A \implies a_\pm = \pm 2\sqrt{A} \quad \text{for} \quad A > 0
\]
\[(A > 0 \text{ corresponds to “stable” potential, i.e., monotone principal operator). Thus, close to 0, there exist two asymptotic bundles of solutions with essentially different behaviours:}
\[
\psi^*_-(z) \sim |z|^{-\frac{3}{4}} e^{-2\sqrt{A} |z|} \to 0 \quad \text{and} \quad \psi^*_+(z) \sim |z|^{-\frac{3}{4}} e^{2\sqrt{A} |z|} \to \infty.
\]
Therefore, posing the condition (cf. (7.32))
\[
Y(0) = 0
\]
does not spoil at all the eigenvalue problem for \( \mathbf{E} \) and just eliminates those singular behaviours at 0 (half of all of them) making the spectrum discrete. In linear operator theory, the same is usually and naturally done by assuming \( Y \in L^2(B_1) \), which nevertheless has come true after certain asymptotic analysis presented above. For the principal radial ordinary differential operator, this means that the singular point \( |z| = 0 \) is in the limit point case (an index characterization is also available for the elliptic setting).

On the contrary (and this case seems more correctly describes the nature of such an “unstable” potential in (7.31), though not that obviously), if \( A < 0 \), then (7.35) yields \( a_\pm = \pm 2i \sqrt{|A|} \), so that both bundles are singular and oscillatory: as \( z \to 0 \),
\[
\psi^*_-(z) \sim |z|^{-\frac{3}{4}} \cos \left( 2\sqrt{\frac{A}{|z|}} \right), \quad \psi^*_+(z) \sim |z|^{-\frac{3}{4}} \sin \left( 2\sqrt{\frac{A}{|z|}} \right), \quad \psi^*_\pm \in L^2_{\text{loc}};
\]
the limit circle case for the operator \( \Delta - \frac{A}{|z|^3} I \). Then the discrete spectrum of \( \mathbf{E} \) is also achieved by posing special conditions at the singularity, \[160\], and then any eigenfunctions exhibit the singular behaviour (7.38). It is another hard problem to check whether (7.38) allows to match such an oscillatory bundle with smoother \( L^\infty \)-flow near the origin \( z = 0 \) for \( \tau \gg 1 \); see below. Therefore, in the case of total oscillatory bundle (7.38), the condition such as (7.37) makes no sense and, evidently, destroys the eigenvalue problem giving \( \sigma(\mathbf{E}) = \emptyset \).

Note finally, that the oscillatory-type behaviour similar to (7.38), in general, is not an absolute obstacle for getting from such “eigenfunctions” a proper blow-up pattern. We discuss this for some R–D equations in Section 7.10. Then, for the behaviour like (7.38), the integral operator as in (7.30) should be understood in a (canonical) regularized sense, see below.

**Singular kernel \( K_2 \) is of Calderón–Zygmund type.** The integral operator with the kernel \( K_2 \) in (7.31) is singular. However, in view of the divergence form of the operator \( \mathbf{C} \) in (7.18), there is a hope (not yet fully justified) that this kernel falls into the scope of Calderón–Zygmund’s classic result (1952) \[28\] saying that such an operator can be
bounded in $L^p(\mathbb{R}^3)$ (in the local sense, meaning that, as usual, we cut-off the infinity by the appropriate weight $\rho$) for some $p > 1$; see [206, Ch. 5, § 5] for modern overview and references. Most of the text-books on pseudo-differential operator theory quote such fundamental results as being its origin; see [151, p. 278]. Note that the conditions of the fundamental Calderón–Zygmund result in the standard form [206, p. 16] do not directly cover the singularities of $K_2$, so that an extra hard work is essential. In addition, one surely needs extensions of such boundedness results to $L^p$-spaces of vector-valued functions $Y$ (such a study was already initiated by Calderón himself in 1962); these questions being well understood; see references and a survey on further operator-valued issues in [105].

**Back to $K_1$: the operator at 0 in a c.r. sense.** Indeed, such a possibility to “overrun” (if possible) the very restrictive condition [7.32] is to take into account the specific divergence part of the $|z|^3$-terms in (7.30). To explain this, let us fix $i = j = 1$ in the first term, where for $z \approx 0$ and $y \neq 0$, for $Y^1 \in C^\infty_0$, we have that the singular part in the regularized value sense vanishes:

$$ (7.39) \quad \sim C_3 \frac{y}{|y|^3} Y^1(0) \int_{B_\varepsilon} (\hat{u}_{SL})_{z_1 z_1} \, dz = 0, $$

since the second-order $z_1$-derivative, according to (7.1), is an odd function in $z_1$. Namely, the corresponding part of the integral operator in (7.30) acts like the standard distribution $P_{\frac{1}{z_1}}$ with the regularization

$$ (7.40) \quad \langle P_{\frac{1}{z_1}}, \varphi \rangle = \int_{0}^{\infty} \frac{\varphi(s) - \varphi(-s) - 2s \varphi'(0)}{s^3} \, ds \quad \text{for} \quad \varphi \in C^\infty_0(\mathbb{R}), $$

which is canonical, c.r. (i.e., this regularization keeps the linear properties of the functionals, as well as the differentiation). A similar c.r.-property is observed for the whole $Y^1$-term appeared in (7.30) that consists of three members: $2[(\hat{u}_{SL})_{z_1 z_1} + (\hat{u}_{SL})_{z_1 z_2} + (\hat{u}_{SL})_{z_1 z_3}]$. Note that all of them are odd relative $z_1$, so the indefinite integral is even. It seems that direct $L^p$-theory is not applicable to such c.r.-integral operators that thus deserve further study. Elsewhere, in other, less singular terms, the standard v.p.-sense c.r.

$$ \langle P_{\frac{1}{z_1}}, \varphi \rangle = \int_{-\infty}^{\infty} \frac{\varphi(s) - \varphi(-s)}{s} \, ds = \int_{0}^{\infty} \frac{\varphi(s) - \varphi(-s)}{s} \, ds $$

in the Cauchy sense can occurs, leading to more standard integral operators with typical singularities of the Hilbert transform in $z_1$ (a Calderón–Zygmund operator in $\mathbb{R}$), etc.

Continuing the fruitful idea of eigenfunction branching at $c = +\infty$ as in the local case above, we obtain an open problem for the whole pseudo-differential operator (7.18):

$$ (7.41) \quad \text{In which operator topology, does} \quad H'(\hat{u}_{SL}) \to B^* - \frac{1}{2} I \text{ as } c \to +\infty? $$

Or the limit always contains a certain “singular” part? The answer is principal.

Note that any extra condition such as (7.32) can be inconsistent with admitted “weakly singular” behaviour corresponding to the local operator $J_1$ composed from $B^*$ and $C$ in (7.21) provided that the latter ones are “dominant”. Then this case becomes rather standard and, as in blow-up R–D theory (see Section 1.6), is governed by improved Hardy’s inequalities discussed above.
However, if the nonlocal operator $J_2$ is “dominant” close to the origin\textsuperscript{40}, which seems
unavoidable and represents the main difficulty of blow-up in the NSEs (cf. the uncertain polynomial micro-structure of multiple zeros detected in Section 3 by the same reason),
then the conditions such as \eqref{7.32} can be natural, giving rise to the necessity of different
spectral theory, which is entirely unknown and represents an open problem. We do
not know whether the linearized pseudo-differential operator \eqref{7.18} with such a strongly
singular kernel (and hence with a nonregular full symbol; see an overview in [206, p. 45])
has a nontrivial discrete spectrum in a proper functional setting involving conditions at
the singularity and Lorentz–Marcinkiewicz or Zygmund-type spaces (see [206, p. 37] with
applications to nonlinear PDEs in this Taylor’s volume).

Note that, plausibly, by some mysterious reason, such a pseudo-differential operator
may admit just a single proper eigenfunction (for further use in blow-up matching, with a
positive or negative conclusion, it does not matter), which cannot be detected in principle
by any general advanced spectral theory. This is a difficult problem, where numerics for
such involved nonlocal operators, can be key, though a definite justified answer: “yes”
or “no” to existence of proper eigenfunctions and hence centre-stable eigenspaces, can be
very questionable.

Overall, spectral properties of the pseudo-differential operator \eqref{7.30} are of great de-
mand (its symbol is not still well understood) and can be key for existence/nonexistence
of proper blow-up patterns on centre-stable manifolds created by the S–L exact steady
singular solutions. Actually, even the negative result on nonexistence would play a role
for concentrating on other more involved scenarios of blow-up to be focused on. Of course,
as usual, since the S–L solutions \eqref{7.0} are axi-symmetric with no swirl, our analysis is
assumed to include blow-up swirling mechanism as in \eqref{5.32}, so that the resulting oper-
ators contain typical angular terms like $-\sigma D_\mu$ as in \eqref{7.19}, with, possibly, sufficiently
large angular speeds $\sigma$ (this assumes complicated separation angular techniques). Thus,
Questions (i)–(vi) from Section 5.12 do deserve further study in this case, and, with a
certain luck, will give a first insight into blow-up singularities for the NSEs.

Let us note that, after matching of the linearized patterns on the stable (centre) man-
ifold for $J_{1,2}$ with more regular flow close to the origin (this is even more difficult open
problem, which makes no sense if spectral theory is still unavailable; Sections 1.6 and
future 7.6 may be consulted for an idea), a typical shape of the resulting patterns (ac-
ccording to \eqref{7.32} or other restrictions) will have a form of swirling “tornado” about a S–L
singular steady solution, which is self-focused onto the origin $x = 0$ as $t \to T^- = 0^-$. F

\textbf{Final conclusion: spectral results are expected by branching at $c = +\infty$.} Despite various
difficulties and open problems already detected, our final conclusion is not fully negative.
Though the extra condition such as \eqref{7.32} looks rather restrictive and even frightening,
the branching idea at $c = +\infty$ correlated with this well, since:

\begin{equation}
\text{(7.42) there exist infinitely many Hermite polynomials $v_\beta^*$ satisfying \eqref{7.32}.}
\end{equation}

\textsuperscript{40}Actually, as seen from $P$ in \eqref{2.21}, both local and nonlocal operator parts are of a “similar power”.}

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For instance, (2.32) implies that those are $v^*_{11}$, $v^*_{12}$, $v^*_{13}$, $v^*_{21}$, $v^*_{25}$, $v^*_{26}$, $v^*_{27}$, $v^*_{28}$, etc. However, obviously conditions such as (7.32) can mean that $\hat{u}(y, \tau)$ is always singular at $y = 0$, so (7.29) is indeed more preferable.

Thus, within the positive reflection of the hard demand (7.41) and in view of the desired feature (7.42), we expect that

$$H'(\hat{u}_{\text{SL}})$$

may have several real eigenvalues close to $\lambda_\beta = -|\beta|^2$,

and the total number of those gets infinite as $c \to +\infty$. Indeed, in this limit, we observe a “convergence” to the Hermite operator $B^* - \frac{1}{2} I$. In other words, the above analysis makes it possible to start a real procedure of checking whether at least one from this countable set of linearized structures admits a proper matching to get a finite energy blow-up pattern for the NSEs (or all of them are hopeless, which nevertheless would not prove nonexistence of blow-up since there are other blow-up scenarios). In other words,

$$S\text{--L solutions with } c \gg 1 \text{ are not still forbidden for blow-up evolution around,}$$

while extensions of the branches to finite $c > 1$ is even more promising, but indeed extremely difficult. Then the still mysterious matching procedure gets principal, which itself can cross out all the previous “spectral” speculations and illusive achievements.

### 7.4. First application of spectral theory: towards periodic blow-up patterns.

As a first elementary application of the above spectral discussion, returning to simpler periodic blow-up orbits inducing (2.46), we pose a straightforward question on the Andronov–Hopf (A–H) classic scenario of bifurcation of periodic orbits from the singular equilibrium $\hat{u}_{\text{SL}}$. Namely, we state:

$$\text{to check if an A–H bifurcation can occur at some } \sigma = \sigma_{\text{AH}}, \ c = c_{\text{AH}} \text{ for the operator (7.18), (7.19), i.e., } \exists \imath \omega \in \sigma(H'(\hat{u}_{\text{SL}}) + \sigma_{\text{AH}} D_\mu), \ \omega \neq 0.$$  

The asymptotic behaviour of the corresponding eigenfunction $v^*_{\beta}(y)$ as $y \to 0$ is also of crucial importance to get a periodic pattern by matching with the regular bounded flow for $y \approx 0$. In case of both positive answers, this would lead to a periodic twistor blow-up pattern for $\sigma \approx \sigma_{\text{AH}}$ that gives rise to the $\omega$-limit set (2.46). Respectively, existence of multiple eigenvalues $\imath \omega_1, \ldots, \imath \omega_n$ may lead to quasi-periodic orbits obeying (2.47). These are hard open hypothetical questions.

Let us finish this brief discussion with a negative result concerning bifurcation from 0:

**Proposition 7.1.** For the rescaled equation (2.21) with the swirl operator (7.19), an A–H bifurcation from 0 is impossible.

**Proof.** Consider the eigenvalue problem for the linearized operator about 0:

$$\left( B^* - \frac{1}{2} I \right) v^* - \sigma D_\mu v^* = \lambda v^* \quad \text{in} \quad L^2_{\rho^*}.$$  

Using the symmetry (2.24) and divergence of $D_\mu$ on $S^1 = (0, 2\pi)$, we get

$$\frac{\lambda + \lambda}{2} \| v^* \|_{L^2_{\rho^*}} = -\| \nabla v^* \|_{L^2_{\rho^*}} - \frac{1}{2} \| v^* \|_{L^2_{\rho^*}} \quad \implies \quad \text{Re} \lambda < 0 \text{ for any } \sigma \in \mathbb{R}. \quad \Box$$
7.5. Inner Region II: matching with smoother solenoidal flow near the origin.

Assume that the previously posed spectral problem has been solved successfully, so we have found the actual Inner Region I, where (5.77) holds. For simplicity, we assume a stable subspace behaviour:

\begin{equation}
\text{Inner Region I : } \hat{u}(y, \tau) \sim \hat{u}_{SL}(y) + e^{\lambda \gamma^*} \hat{v}(y) \quad \text{for} \quad \tau \gg 1,
\end{equation}
on a certain “maximal” set \( y \in \Upsilon(\tau) \), where such an expansion is applicable. Roughly speaking, it is given by

\begin{equation}
\Upsilon(\tau) \sim \{ r(\tau) < |y| < R(\tau) \},
\end{equation}
where \( R(\tau) \) characterizes the outer matching with smooth and almost regular flow, while much smaller \( r(\tau) \to 0 \) is the crucial sphere, on which the inner matching with the flow bounded at the origin is supposed to occur. Thus, in particular,

\begin{equation}
\text{set } \Upsilon(\tau) \text{ essentially depends on the eigenfunction } \hat{v}^*_\gamma(y) \text{ behaviour as } y \to 0.
\end{equation}

Indeed, as a proper illustration, \( \Upsilon(\tau) \) is different for the first monotone patterns as in (7.36) and for the singular oscillatory ones in (7.38). For other types of possible eigenfunctions, the actual matching as in (5.80)–(5.88) defines the inner boundary of \( \Upsilon(\tau) \).

Concerning the outer boundary of the radius \( R(\tau) \), which is not that essential, it is estimated as follows: since \( \hat{v}^*_\gamma(y) \) is assumed to be close to a Hermite polynomial for \( y \gg 1 \) of order \( k \) large and hence \( \lambda_\gamma(c) = \frac{k+1}{2} \), with \(|\gamma| = k\), this radius is characterized by matching with zero: for \( \tau \gg 1 \),

\begin{equation}
\hat{u}_{SL}(y) + e^{\lambda \gamma^*} \hat{v}^*_\gamma(y) \sim 0 \implies \frac{1}{|y|} - e^{-\frac{k+1}{2} |y|^k} \sim 0 \implies R(\tau) \sim e^{\frac{1}{2}},
\end{equation}
as should be via the basic kinetic energy estimate (2.2) in the rescaled variables (2.18). However, as we will see, this outer region cannot be important for the inner matching.

In Inner Region II, we have to return to the full original rescaled equation (2.21), where for convenience we set \( \mathbb{P} = I - Q \),

\begin{equation}
\hat{u}_\tau = (B^* - \frac{1}{2} I) \hat{u} + (\hat{u} \cdot \nabla) \hat{u} + Q(\hat{u} \cdot \nabla) \hat{u},
\end{equation}
where the convection term \((\hat{u} \cdot \nabla) \hat{u}\) is negligible in comparison with the linear \( B^* \)-term: this is the actual definition of Region II. The non-local term keeps be influential and even leading therein (we have seen a similar nonlocal phenomenon in the study of polynomial micro-structure of multiple zeros in Section 3, where the integral term was shown to essentially deform the Hermitian solenoidal fields).

Thus, in Region II, the following asymptotic equation occurs:

\begin{equation}
\hat{u}_\tau = (B^* - \frac{1}{2} I) \hat{u} + f^*(\hat{u}) + ..., \quad \text{where } f^*(\hat{u}) = Q(\hat{u} \cdot \nabla) \hat{u} \quad \text{as } \tau \to +\infty.
\end{equation}

It follows (or this can be a key assumption to be checked) that, in view of the expansion (7.38), the forcing term in (7.53) can be estimated as follows:

\begin{equation}
f^*(\hat{u}) \sim f^*(\hat{u}_{SL}) \sim Q_{T(\tau)}(\hat{u}_{SL} \cdot \nabla) \hat{u}_{SL},
\end{equation}
where in the last term we assume integration in (2.23) over \( \Upsilon(\tau) \) only, which represents the leading-order expansion term of the nonlocal term. If the latter is not true, one needs
to include also the nonlocal portion that depends on the still unknown solution expansion in Region I, that makes the analysis more complicated (but formally doable). The Outer Region, where the solution is much smaller, is assumed to produce no essential influence on the integral.

Thus, the possibility of matching of Regions I and II depends on existence of a smooth bounded solution of the following limit problem:

\[ \dot{u}_r = (B^* - \frac{i}{2} I) \dot{u} + f_r^*(\tau), \]

where \( f_r^*(\tau) = Q_{\tau}(\dot{u}_{SL}) \) for \( \tau \gg 1 \).

It is convenient to use the generalized Hermite polynomials to describe the solution:

\[ \dot{u}(\tau) = \sum c_\beta v_\beta^*, \]

where \( c_\beta = (\lambda_\beta - \frac{1}{2}) c_\beta + \langle f_\tau^*(\tau), v_\beta \rangle \).

The first condition (by no means, a necessary and/or sufficient) of proper matching reads then as follows:

\[ \text{the auxiliary problem (7.56) has an } L^\infty\text{-solution for all } \tau \gg 1 \]

(otherwise, \( u_0 \not\in L^\infty \)). As a second one, for a rough checking of matching, consider the equation for the leading Fourier coefficient for \( \beta = 0 \), with \( \lambda_0 = 0 \):

\[ \hat{c}_0 = -\frac{1}{2} c_0 + \langle f_r^*(\tau), v_0 \rangle, \]

where \( v_0 \) is a constant vector (a polynomial of degree zero). Integrating this and assuming a slow growth divergence, one can expect a “quasi-stationary” behaviour given by \( 7.58 \):

\[ c_0 \sim 2 \langle f_r^*(\tau), v_0 \rangle \text{ for } \tau \gg 1. \]

Assuming that \( r(\tau) \to 0 \) (otherwise, no blow-up as \( \tau \to +\infty \)), matching the stable manifold behaviour about the S-L profile \( \dot{u}_{SL} \) assumes, at least, that, for \( \tau \gg 1 \),

\[ \hat{u}(0, \tau) \sim c_0(\tau) \sim \dot{u}_{SL}(\infty)|_{|x| \sim r(\tau)} \implies |c_0(\tau)| \sim \frac{1}{r(\tau)}. \]

Involving other expansion coefficients \( \{c_\beta(\tau)\} \) will lead to a similar but more complicated relation with \( r(\tau) \). Altogether, \( 7.59 \) and \( 7.60 \) define a complicated nonlinear integral equation for the expansion coefficient \( c_0(\tau) \) and the matching radius \( r(\tau) \). On integration in the non-local term given via \( 2.23 \), i.e.,

\[ f_r^*(\tau) \sim \int Y(\tau) \frac{y-z}{|y-z|^3} \sum_{i,j} \hat{u}_{SL,z}^i \hat{u}_{SL,z}^j, \]

where \( |\hat{u}_{SL,z}^i| \sim \frac{1}{|z|^2} \),

it follows that the RHS in \( 7.59 \) behaves as \( \sim \frac{1}{r} \) as \( r \to 0 \), which is satisfactory for a possible matching purpose (at least, an obvious contradiction for matching is not an immediate option). Surely, these are just rough matching estimates, and further more difficult matching study along the lines of \( 7.56 \), \( 7.59 \) is necessary.

Thus, due to \( 7.56 \), the “quality” of the eigenfunction \( \hat{v}_r^*(y) \) as \( y \to 0 \) is essentially involved into \( 7.57 \) via the set of integration \( \Upsilon(\tau) \) for \( \tau \gg 1 \). This somehow involves verifying a certain “integrability” (sufficient integral convergence) over \( \Upsilon(\tau) \), where the asymptotics of the eigenfunction \( \hat{v}_r^*(y) \) as \( y \to 0 \) is key. In other words, this eigenfunction must belong to a “proper functional class” to make the above computations meaningful. In general, similar to the construction in Section \( 1.6 \) the actual blow-up structure in Region...
II will indeed depend on the behaviour of $\hat{v}_\gamma^*(y)$ as $y \to 0$ (unknown), and hopefully will look like a smoother “quasi-stationary” (driven by $\Delta$ only) evolution. On the other hand, even for very bad, singular and/or highly oscillatory eigenfunctions $\hat{v}_\gamma^*$, there is still some plausible hope that the integrals in (7.56) still properly converge in such a manner that (7.57) holds (this can be checked), and the matching with a bounded flow for $y \approx 0$ can be purely “accidental”. This means that (7.57) can be valid for some special and possibly very singular eigenfunctions $\hat{v}_\gamma^*$, so that the functional setting for the eigenvalue problem for the occurred pseudo-differential operator can be treated in a wide sense.

Further matching conditions, involving also the already obtained Region I expansions (and the Outer Region if necessary), may be important, which eventually give the matched solutions in terms of a converging bounded functional series. Finally, in view of the precaution (7.3), another fruitful idea is to diminish the fluid injection by assuming that (7.62)

$$c = c(\tau) \to +\infty \quad \text{as} \quad \tau \to +\infty,$$

which is an individual subject of Section 7.7.

7.6. A scaling view to formation of Type II blow-up solutions: heteroclinic orbits are necessary. We then need to consider the full rescaled equation taking into account the angular operator (7.19):

$$(7.63) \hat{u}_r + \sigma \hat{u}_\mu + P(\hat{u} \cdot \nabla)\hat{u} = \Delta \hat{u} - \frac{1}{2} y \cdot \nabla \hat{u} - \frac{1}{2} \hat{u}.$$

Assuming that the blow-up is faster than the self-similar one, i.e., is of Type II and

$$(7.64) \sup_y |\hat{u}(y, \tau_k)| = C_k \to \infty \quad \text{as} \quad \{\tau_k\} \to \infty,$$

we apply the $C_k$-scaling technique as in Section 4 by setting

$$(7.65) \hat{u} = C_k w, \quad y = y_k + a_k z, \quad \tau = \tau_k + a_k^2 s, \quad \mu \mapsto a_k^2 \mu, \quad \text{where} \quad a_k = \frac{1}{C_k} \to 0.$$

The resulting equation for $w_k$ takes the following perturbed form:

$$(7.66) w_s + \sigma w_\mu + P(w \cdot \nabla)w = \Delta w - \frac{1}{C_k} \left(\frac{1}{2} z \cdot \nabla w + \frac{1}{2} w\right).$$

In view of uniform boundedness and further regularity of the sequence $\{w_k\}$ on compact subsets in $\mathbb{R}^3 \times \mathbb{R}$, we can pass to the limit $k \to \infty$ in the weak sense in (7.66) to conclude that $w_k(s)$ must approach a regular solution $W$ satisfying the NSEs

$$(7.67) W_s + \sigma W_\mu + P(W \cdot \nabla)W = \Delta W \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}, \quad \|W(0)\|_\infty = 1.$$

Note that $W(z, s) \not\equiv 0$ is an ancient solution, which is defined for all $s \leq 0$. At the same time, by construction, it is also a future solution, which must defined for all $s > 0$. Indeed, one can see that if $W(s)$ blows up at some finite $s = S^- > 0$, this would contradict the Type II solution $w(y, \tau)$ is globally defined for all $\tau > 0$. Thus, by scaling of the Type II blow-up orbit (7.65), we arrive at the problem (7.67), which defines:

$$(7.68) \{\text{heteroclinic solution} \ W(z, s) \not\equiv 0\} = \{\text{ancient for} \ s < 0\} \cup \{\text{future for} \ s > 0\}.$$

Note that constant stationary solutions $W_0$ of (7.67) are the simplest possibilities, which being rather suspicious (too elementary) have not still been ruled out. We do not know whether or not non-constant or non-steady global bounded regular solutions of
exist for some $\sigma \neq 0$. Note that in the stabilization problem to a constant solution $W_0$, which appear by linearization:

$$W(s) = W_0 + v(s) \implies v_s + \sigma v_\mu + \mathbb{P}(W_0 \cdot \nabla)v = \Delta v,$$

the linear operator does not have a clear discrete spectrum, unlike the one appeared in Region I. Therefore, the actual exponential stabilization rate in Region II is inherited from the spectral problem in Region I, assumed to be properly solved beforehand. In any case, this analysis shows a quasi-stationary nature of formation of Type II blow-up patterns on shrinking spatial subsets around blow-up points. Of course, the actual details of such a behaviour can be much more complicated involving various extra singularity mechanisms.

Regardless a clear lack of rigorous arguments here on existence of Type II blow-up patterns, a negative conclusion occurs (surely, not surprisingly):

(7.70) if the hypothetical heteroclinic patterns $W(z, s)$ from (7.67) do not match for $|z| \gg 1$ typical spatial structures in (7.1) or those in (7.16), tornado-type blow-up around the S–L singularities is not possible.

7.7. Blow-up drift on the c-manifold of S–L solutions. As we know from R–D theory, the above blow-up scenario does not exhaust all the types of possible singular patterns. As a new evolution possibility, we again consider the S–L solutions, but currently we assume a centre-subspace-like evolution on their whole manifold.

We begin with the first linearized procedure to calculate a first approximation of such a behaviour. Namely, following [72], we assume that slow blow-up evolution occurs “along” an unknown functional dependence $c = c(\tau)$ in (7.18), i.e., via (7.62),

$$\hat{u}(\tau) = \hat{u}_{\text{SL}}(c(\tau)) + Y(\tau) \in L^2_p(\mathbb{R}^3), \text{ where } c(\tau) \to +\infty \text{ as } \tau \to +\infty.$$

Then, instead of (5.76), using the second asymptotics in (7.4), we substitute (7.71) into the NSEs with the corresponding extra force on the right-hand side as in (7.2) of the form $= \frac{16}{c(\tau)} \delta(y) j + \ldots$. Omitting higher-order terms, we obtain the following perturbed inhomogeneous PDE:

$$Y_\tau - \frac{\hat{c}(\tau)}{\hat{c}^2(\tau)} \hat{u}_0 = H'(\hat{u}_{\text{SL}}(c(\tau)))Y + D(Y) + \ldots$$

$$(7.72) \equiv (B^* - \frac{1}{2} I)Y - \frac{1}{c} [((\hat{u}_0 \cdot \nabla)Y + (Y \cdot \nabla)\hat{u}_0) - \frac{1}{c} C_3 \int_{\mathbb{R}^3} (\cdot)(\hat{u}_0, Y) + \ldots].$$

In the last term, we mean the nonlocal operator in (7.18) defines (still formally) at the constant vector field $\hat{u}_0$, with a special truncation at the origin.

We next looking for a solution governed by the leading perturbation in (7.72),

$$Y(\tau) = \varphi(\tau) \Psi^*(y) + w(\tau), \text{ where } \varphi(\tau) = -\frac{\hat{c}(\tau)}{\hat{c}^2(\tau)} \to 0 \text{ as } \tau \to +\infty,$$

and we assume that $w(\tau) \perp \Psi^*$ in the dual metric, i.e., with a proper definition of the adjoint element (a linear functional),

$$\langle w(\tau), \Psi \rangle \equiv 0 \quad (\Psi \in L^2_p(\mathbb{R}^3)).$$
Substituting (7.73) into (7.72) yields

\[(7.75) \quad \dot{\varphi} \psi^* + \varphi \dot{u}_0 = \varphi \left( B^* - \frac{1}{2} I \right) \psi^* - \frac{c}{\tau} \langle \psi^* \rangle (\hat{u}_0, \psi^*) + \ldots, \]

where \([\cdot](\hat{u}_0, \psi^*)\) denotes the linear operators in (7.72) at \(\hat{u}_0\) (with possibly a necessary truncation at 0).

**Countable set of exponential patterns.** Formally, then it follows that (7.75) admits a countable set of linear solenoidal blow-up patterns (here \(\lambda_k = -\frac{k}{2}\))

\[(7.76) \quad \mathcal{Y}_k(y, \tau) = a_k(\tau) v_k^*(y) + \ldots, \quad \text{where} \quad \dot{a}_k - \frac{c}{\tau} a_k = (\lambda_k - \frac{1}{2}) a_k.\]

In particular, we have the following special family of functions \(\{c_k(\tau)\}\):

\[(7.77) \quad c_k(\tau) = e^{(\frac{k}{2} - \lambda_k) \tau} \implies a_k(\tau) = (\frac{1}{2} - \lambda_k) \tau e^{(\lambda_k - \frac{k}{2}) \tau} \quad \text{for} \quad \tau \gg 1.\]

**Power decay pattern.** Assuming now that in (7.73),

\[(7.78) \quad |\dot{\varphi}(\tau)| \ll |\varphi(\tau)| \quad \text{for} \quad \tau \gg 1,\]

we mean that \(c(\tau)\) is a slow growing function not of an exponential form. Then the only possible way to balance the terms in (7.75) is the vector \(\psi^*\) to satisfy:

\[(7.79) \quad \left( B^* - \frac{1}{2} I \right) \psi^* = \hat{u}_0.\]

Since the operator is invertible, there exists a unique solution \(\psi^*\) constructed by an eigenfunction expansion via solenoidal Hermite polynomials, so \(\psi^*\) is also solenoidal. It follows from (7.4) that this linear procedure gives a similar singularity at the origin (cf. below with a nonlinear one),

\[(7.80) \quad |\psi^*(y)| \sim \frac{1}{|y|} \quad \text{as} \quad y \to 0.\]

The further balance in (7.75) is then obtained, as usual, by multiplication by the adjoint orthonormal element \(\Psi\) that yields an asymptotic ODE for \(\varphi(\tau)\) for \(\tau \gg 1\):

\[(7.81) \quad \dot{\varphi} = \gamma_0 \frac{c}{\tau} + \ldots, \quad \text{where} \quad \gamma_0 = \langle [\cdot](\hat{u}_0, \psi^*), \psi \rangle.\]

We must admit that actually, \(\gamma_0\) is not supposed to be a constant, since in this dual product, a further matching of the pattern with a bounded flow for \(y \approx 0\) should be assumed. This matching (a cut-off for \(|y| \ll 1\) procedure makes the integral in (7.81) finite for all \(\tau \gg 1\), but then we conclude that

\[(7.82) \quad \gamma_0 = \gamma_0(\tau) \to \infty \quad \text{as} \quad \tau \to +\infty \quad \text{(e.g.,} \quad \gamma_0 \sim \ln \tau),\]

though the divergence turns out to be slower than that for \(\varphi(\tau)\), so this does not essentially affect the leading term in (7.81). Thus, we arrive at the equation for \(\varphi(\tau)\):

\[(7.83) \quad \left( \frac{\varphi'}{\tau} \right)' = \gamma_0 \frac{c}{\tau^2} + \ldots \quad \implies \quad c(\tau) \sim -\frac{\gamma_0}{2} \tau + \ldots \quad \text{as} \quad \tau \to +\infty\]

(recall that we do not exclude the case, e.g., \(c(\tau) \sim \tau \ln \tau \text{ for} \quad \tau \gg 1\)).

Hence, up to lower-order multipliers, we get the following expansion of such solenoidal patterns, which is convenient to write in terms of a series of the form (again, lower-order, logarithmic-like factors are not taken into account):

\[(7.84) \quad \hat{u}(y, \tau) \sim \frac{1}{\tau} \hat{u}_0 + \frac{1}{\tau^2} \hat{u}_1 + \frac{1}{\tau^3} \hat{u}_2 + \ldots,\]
where for convenience we put \( \hat{u}_1 \sim \Psi^* \). To get the coefficients \( \{ \hat{u}_k \} \) of this nonlinear expansion, one needs to use the full rescaled equation (2.21) with the operator (2.23). It then follows that the equation for the third coefficient \( \hat{u}_2 \) takes the form

\[
(7.85) \quad -\hat{u}_1 = (\mathbf{B}^* - \frac{1}{2} I) \hat{u}_2 - (\hat{u}_0, \nabla) \hat{u}_1 - (\hat{u}_1, \nabla) \hat{u}_0 - C_3 \int_{\mathbb{R}^3} \cdot (\hat{u}_0, \hat{u}_1),
\]

where in the last term we again assume a certain evolution cut-off procedure near the origin. Then we obtain the singularity for \( \hat{u}_2(y) \):

\[
(7.86) \quad - (\hat{u}_0, \nabla) \hat{u}_1 \sim \frac{1}{|y|^4} \quad \Rightarrow \quad \hat{u}_2(y) \sim \frac{\ln|y|}{|y|} \quad \text{as} \quad y \to 0,
\]

i.e., we have found that the third term is more singular at the origin than two previous ones. Therefore, similar to the procedure described in our first Type II blow-up structure in (1.45) for the Frank–Kamenetskii equation, we observe that on small compact subsets in \( y \), a model one-sided (say from above) behaviour of the rescaled vector field can be roughly estimated as

\[
(7.87) \quad \left| \hat{u}(y, \tau) \right| \sim U(y, \tau) \sim \frac{1}{\tau} \frac{1}{|y|} + \frac{1}{\tau^2} \frac{1}{|y|} + \frac{1}{\tau^3} \frac{\ln|y|}{|y|} + \ldots.
\]

Taking into account the first and the third terms, the absolute positive maximum of the scalar function \( U(y, \tau) \) is attained at

\[
(7.88) \quad |y| \sim e^{-\tau^2} \to 0, \quad \text{and hence} \quad \sup_y U(y, \tau) \sim \frac{\alpha^2}{\tau} \quad \text{for} \quad \tau \gg 1.
\]

Once we have known the radius (7.88) of truncation of the singularity structure of \( U \) at the origin, all computations can be redone to see its actual influence on the final asymptotics. In view of scaling (2.18), overall, this yields a formal expansion (7.84) of the solenoidal vector field for a quadratic NSEs system, which is singular at the origin, but can admit some kind of critical surface inflection-like behaviour at the level set \((T = 1)\)

\[
(7.89) \quad \left| \mathbf{u}(x, t) \right| \sim \frac{e^{\ln(T-t)^2}}{\sqrt{T-t} \ln(T-t)} \quad \text{as} \quad t \to T^-.
\]

In other words, values (7.89) for \( t \approx T^- \), the singular vector field \( \mathbf{u}(x, t) \) is expected to exhibit certain special transitional behaviour, which possibly can be used for further necessary matching and branching to create more reasonable blow-up patterns.

The pattern with special level sets as in (7.88) is obtained in the simplest situation, where no “resonance” between different terms of the expansion (7.87), which itself can generate logarithmic factors, is assumed. In general, for quadratic dynamical systems as (2.21), there can be several types of asymptotic centre-manifold-type expansions, which are very sensitive and depend on resonance conditions of various terms involved; see invariant manifold theory in [145]. The validity of such conditions are very difficult to check, especially in the presence of extra outer matchings involved. However, in our opinion, (7.88) correctly characterizes some features of the blow-up behaviour of such patterns up to some extra exponents and/or slower growing factors. A justification of such a behaviour requires a quite involved matching analysis of fully using eigenfunction expansions on the vector solenoidal Hermitian polynomials and related delicate and technical matching procedures involved.
7.8. On a possibility of blow-up on regular equilibria. We now follow the discussion in Section 1.9. These structures are more exotic but also reasonably well-known in reaction-diffusion theory; cf. [59] and [72]. For simplicity, we take \( \sigma = 0 \) in (7.16) or (7.15), then a new perturbation occurs) and study a possibility of blow-up behaviour

(7.90) \( \hat{u}(y, \tau) = \varphi(\tau)\hat{u}(\varphi(\tau)y) + Y \), where \( \varphi(\tau) \to \infty \) as \( \tau \to +\infty \).

To this end, we first introduce the blow-up variables

(7.91) \( \hat{u}(y, \tau) = \varphi(\tau)\hat{v}(z, s), \quad z = \varphi(\tau)y, \quad \varphi^2(\tau) \, d\tau = ds \)

where \( \hat{v} \) solves the following rescaled non-autonomous PDE:

(7.92) \( \hat{v}_s = H(\hat{v}) - \rho(s)[(z \cdot \nabla)\hat{v} + \hat{v}] \), where \( \rho(s) = (\frac{\dot{\varphi}}{\varphi^2} + \frac{1}{2} \frac{1}{\varphi^2})(\tau) \).

If \( \sigma \neq 0 \), we introduce the angular TW dependence \( \mu = \hat{\mu} + s \), so that the extra term \( -\sigma \hat{v}_\mu \) should be put into the right-hand side (cf. (1.32)), which does not change the concepts of matching. It then follows that we may estimate \( \hat{v}(s) \) as follows:

(7.93) \( \hat{v}(s) = \hat{u} + \rho(s)W \), provided that \( \frac{d}{\rho} \to 0, \ s \to +\infty \)

(i.e., \( \rho(s) \) is not exponentially decaying and is algebraic), where \( W \) is a proper solutions of the linearized inhomogeneous equation

(7.94) \( H'(\hat{u})W - [(z \cdot \nabla)\hat{u} + \hat{u}] = 0. \)

In order to get possible acceptable families of functions \( \{\varphi_k(\tau)\} \), we return to the original rescaled variable \( y \) and use (7.90) in (2.21), (2.23) to get for \( Y \) the equation

(7.95) \( Y_\tau = (B^* - \frac{1}{2} I)Y + H(\varphi(\tau)\hat{u}(\varphi(\tau)y)) + \ldots, \)

where we omit higher-order terms. By the assumption (7.17), we have from (7.2) that the main inhomogeneous term in (7.95) can be estimates as follows:

(7.96) \( H(\varphi(\tau)\hat{u}(\varphi(\tau)y)) = -\frac{16\pi}{c(\tau)} \delta(y) + \ldots \implies Y_\tau = (B^* - \frac{1}{2} I)Y - \frac{16\pi}{c(\tau)} \delta(y) + \ldots. \)

We then balance the terms similar to (7.77):

(7.97) \( Y_k(\tau) = e^{(\lambda_k - \frac{1}{2})\tau}v_k^* + \ldots \) and \( c_k(\tau) = e^{(\frac{1}{2} - \lambda_k)\tau} \) for \( \tau \gg 1. \)

Comparing (7.90), (7.97) and (7.91), (7.93) yields the matching condition, which we take in the simplest form to catch the exponential factors only:

(7.98) \( \rho(s) \equiv \frac{\dot{\varphi}}{\varphi^2} + \frac{1}{2} \frac{1}{\varphi^2} \sim \frac{1}{\varphi} e^{(\lambda_k - \frac{1}{2})\tau} \implies \varphi_k(\tau) \sim \left( \frac{1}{2} - \lambda_k \right) e^{(\frac{1}{2} - \lambda_k)\tau} \to \infty. \)

Let us first check the consistency of the expansion (7.91) for (7.91):

(7.99) \( s \sim e^{(k+1)\tau} \implies \rho_k(s) \sim \frac{1}{s} \) for \( s \gg 1, \)

so that each \( \rho_k(s) \) has the desired algebraic (and non-integrable) decay at infinity.

Recall that these are blow-up patterns constructed via a hypothetical evolution on the manifolds of “steady” (or “TW-swirl” for \( \sigma \neq 0 \)) solutions, which includes both families of the singular S-L profiles and the regular ones as in (7.16). Including the angular eigenvalue \( \sigma \neq 0 \) will make the formal analysis more complicated to say nothing about a rigorous justification (a finite energy interpretation of such blow-up patterns is also hard).
Towards Burnett equations. Finally, we note that, for the Burnett equations (1.6), a similar formal analysis can be performed relative to the singular solutions with a different behaviour near the origin,

\begin{equation}
    u_{SS}(x) \sim \frac{1}{|x|^5} \quad \text{as} \quad x \to 0,
\end{equation}

though proving existence of such steady structures is a difficult problem, as well as a rigorous mathematical justification of the expansion and matching procedures for both families of singular and bounded states. Concerning the linearization approach as in Section 7.8 it can be performed in similar lines by using spectral theory and the generalized Hermite polynomials for the operator (3.21), 51. The possible ways of “interaction” with the spectral characteristics of the nonlocal operator $J_2$ are out of question here.

On the other hand, it is not excluded that (1.6) can admit a purely self-similar blow-up with various multiple patterns (a finite or countable set? – seems should be finite as higher-order parabolic flows suggest [22]; however, the solenoidal restriction can indeed easily spoil Leray’s-type “similarity party” even here). Of course, the nonexistence proof based on the MP ideas from [162] is not applicable here. Thus, (1.6) can admit much more complicated families of blow-up patterns (it seems that numerics can help here being however very difficult), though the classification problem (1.20) remains a principle difficult issue, which, as expected, will never be completely solved (too difficult, and not that essential?).

Complicated oscillatory singular equilibria for an R–D equation. Here, as a key illustration to some of our speculations, we briefly consider (1.34). In radial geometry, such Emden (1907)–Fowler (1914) equations have been most carefully studied in ODE theory. First detailed classification of solutions were obtained in Gel’fand [84] (this ODE section is known be written by Barenblatt) and Joseph–Lundgren [108]; see further references and applications for blow-up in [66, § 6.5]. We briefly comment on still difficult and seems not completely well-understood oscillatory properties of singular solutions.

First of all, (1.34) admits the homogeneous singular stationary solutions (SSS):

\begin{equation}
    U(r) = \pm C_* r^{-\mu}, \quad \text{where} \quad \mu = \frac{2}{p-1}, \quad C_* = [\mu(N-2-\mu)]^{1-p} \quad (p > \frac{N}{N-2}).
\end{equation}

To describe others of changing sign, we introduce the oscillatory component $\varphi$ by

\begin{equation}
    u(r) = r^{-\mu} \varphi(-\ln r) : \quad \varphi'' + (2\mu + 2 - N) \varphi' + \mu(\mu + 2 - N) \varphi + |\varphi|^{p-1} \varphi = 0.
\end{equation}

Figure 1 shows the non-oscillatory character of singular solutions in the subcritical Sobolev range $p < p_S$ by shooting from $s = 0 \ (r = 1)$. Both Figures 1(a) and (b) explain that as $r \to 0$, i.e., $s \to +\infty$, the oscillatory component stabilizes to constants $\pm C_*$ as in (7.101), excluding countable set of regular patterns \{u_k, k \geq 0\}, which are bounded at $r = 0$ and have a fixed number of sign changes. Here (a) shows shooting the first profile $u_0(r) > 0$ with no zeros, while (b) corresponds to shooting of $u_1(r)$ with a single zero on $r \in (0, 1) \ (s \in (0, +\infty))$. Surrounding those regular $u_k(r)$ are continuous families of singular solutions exhibiting the behaviour (7.101) as $r \to 0$. 

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Figure 1. Shooting the first (a) and the second (b) regular equilibria via ODE in (7.102) for $p = 4 < p_S = 5, N = 3$.

Next Figure 2 shows the crucial appearance of infinitely many oscillatory periodic orbits $\varphi(s)$ at the critical value $p = p_S$, where (7.102) takes the autonomous variational form

$$\varphi'' - \frac{(N-2)^2}{8}\varphi + |\varphi|^{p-1}\varphi = 0.$$  

Finally, in Figure 3 we show how the oscillatory character of singular solutions dramatically changes for $p > p_S$, where nonlinear spiral out behaviour replaces the bounded periodic one. In (a), we take $p = 6 > p_S = 5$ for $N = 3$. In (b), the highly oscillatory behaviour is shown for $p = 2$ and $N = 17$, i.e., above the uniqueness $[77]$ critical exponent $p^* = 1 + \frac{4}{N-4-2\sqrt{N-1}}$ for $N \geq 11$ ($p^* = \frac{9}{5} < 2$ for $N = 17$), which plays a role in a number of blow-up problems for this R–D equation; see $[77]$ and $[146]$ for further details. Thus, the spiral-type oscillations for large $p > p_S$ get arbitrarily large, as $r \to 0$, so singular equilibria stay arbitrarily far from the homogeneous one (7.101) (is this possible for the steady NSEs (1.33) relative to the homogeneous S–L solutions (7.1)?)?

Thus, the singular equilibrium manifold for (1.34) is rather complicated even in the radial case. Of course, in the radial geometry, the blow-up analysis of the corresponding parabolic equation (1.63) is essentially simplified by using the Sturmian argument of intersection comparison (the number of intersections of different solutions cannot increase with time; C. Sturm, 1836), and this can prohibit some scenarios of blow-up; see various applications in $[66, 77, 146]$. In the non-radial geometry, even for (1.63), this advantage is almost nonexistent, the singular equilibria can get much more complicated and are unknown. On the other hand, the MP remains then in place, but its application to control, simplify, or prohibit possible blow-up structures is unclear.

Thus, the non-radial geometry in such parabolic problems can result in existence of new blow-up patterns with evolution “close” to such singular manifolds, where specific “swirl-torsion” blow-up phenomena may occur. Of course, the matching conditions such
Remark: singular equilibria in the bi-harmonic case and open problems. The analogous elliptic problem for the Burnett equations (1.6) is the bi-harmonic one,

\[ -\Delta^2 u + |u|^{p-1} u = 0, \]
which admits a similar to (7.102) substitution with \( \mu = \frac{4}{p-1} \) and the ODE:

\[
\begin{align*}
-\varphi^{(4)} - A \varphi'' - B \varphi' - C \varphi - D \varphi + |\varphi|^{p-1} \varphi &= 0, \\
A &= 2(2\mu + 4 - N), \\
B &= 6\mu^2 + 18\mu + 11 + (N - 1)(N - 9 - 6\mu), \\
C &= 2[2\mu^3 + 9\mu^2 + 11\mu + 3 + (N - 1)(N - 3)(\mu + 1) - 3\mu^2 - 6\mu - 2]], \\
D &= \mu(\mu + 2)[(\mu + 1)(\mu + 3) + (N - 1)(N - 5 - 2\mu)].
\end{align*}
\]

This is a harder equation than (7.102), and its complexity shows how difficult proofs of global or blow-up bounds on solutions of the corresponding parabolic PDE (1.56), \( m = 2 \), can be for \( p \geq p_S = \frac{N+4}{N-4} \), with \( N > 4 \). The homogeneous SSS of (7.105) is

\[
U(r) = \pm C_* r^{-\mu}, \quad \text{where} \quad \mu = \frac{4}{p-1}, \quad C_* = D^{\frac{1}{p-1}} > 0,
\]

existing for \( p > \frac{N}{N-4} \), \( N > 4 \), or \( p < \frac{N+2}{N-2} \), \( N > 2 \).

Global radial solutions of (7.105) for \( p > p_S \) are obtained in [83], where further references can be found.

Again, blow-up patterns can be created by the manifold of singular equilibria with completely unknown non-radial structure (with no traces of the MP); the analogy of the matching (7.57) is supposed to be taken into account; see [71] for a discussion. Any general estimates of Type I or II blow-up for (1.56) for \( p \geq p_S \) are absent. Moreover, we claim that a full and complete description of all the (non-radial) blow-up patterns for (1.56) will never be achieved. Similar difficulties concerning existence and nonexistence of various blow-up patterns occur for supercritical nonlinear Schrödinger equations such as (1.9) (see key references on this subject in Merle–Raphael [150] and Visan [215]) and for many other important higher-order PDEs (see the list around (1.9)) and systems of the twenty-first century PDE mathematics/applications.

8. On complicated blow-up patterns with swirl and precessions

Using the previous ideas of possible concepts of blow-up in the equations (1.1), we continue the construction of more refined structures of a full complexity. We now use spherical coordinates that allow us to fix more complicated singular stationary solenoidal fields, in a neighbourhood of which some non-steady blow-up phenomena may occur.

8.1. Basic solutions with swirl in spherical coordinates. Consider the equations (1.1) in the spherical polar coordinate system, with \( u = (u_r, u_\theta, u_\varphi) \equiv (u, v, w) \):

\[
\begin{align*}
\begin{cases}
\left. \begin{array}{l}
\frac{1}{r} u_t + uu_r + \frac{1}{r} v u_\theta + \frac{1}{r \sin \theta} w u_\varphi - \frac{1}{r^2} v^2 - \frac{1}{r} u^2 \\
\end{array} \right) = -p_r + \Delta_3 u - \frac{2}{r^2} u - \frac{2 \cot \theta}{r^2} v - \frac{2 \cot \theta}{r^2} u_\varphi, \\
\left. \begin{array}{l}
\frac{1}{r} v_t + uv_r + \frac{1}{r} u v_\theta + \frac{1}{r \sin \theta} w v_\varphi - \frac{1}{r^2} u^2 - \frac{1}{r} v^2 \\
\end{array} \right) = -p_\theta + \Delta_3 v - \frac{1}{r \sin \theta} \cot \theta v + \frac{1}{r} u_\theta - \frac{2 \cot \theta}{r \sin \theta} w_\varphi, \\
\left. \begin{array}{l}
\frac{1}{r} w_t + uw_r + \frac{1}{r} u w_\theta + \frac{1}{r \sin \theta} w u_\varphi - \frac{1}{r^2} v^2 - \frac{1}{r} w^2 \\
\end{array} \right) = -p_\varphi + \Delta_3 w - \frac{1}{r \sin \theta} \cot \theta w + \frac{2 \cot \theta}{r \sin \theta} u_\varphi + \frac{2 \cot \theta}{r \sin \theta} v_\varphi, \\
u_r + \frac{2}{r} u + \frac{1}{r} v_\theta + \frac{\cot \theta}{r} v + \frac{1}{r \sin \theta} w_\varphi = 0.
\end{cases}
\end{align*}
\]

(8.1)
Here $\Delta_3$ denotes the spherical $\mathbb{R}^3$-Laplacian:

\begin{equation}
\Delta_3 h = \frac{1}{r} (r^2 h_r)_r + \frac{1}{r \sin \theta} (\sin \theta h_\theta)_\theta + \frac{1}{r^2 \sin^2 \theta} h_{\varphi \varphi}.
\end{equation}

**Homogeneous singular stationary states.** Similar to (5.67), we begin with simpler singular stationary solutions of (8.1) of the homogeneity $-1$,

\begin{equation}
(u, v, w) = \frac{1}{r} (\hat{u}, \hat{v}, \hat{w})(\varphi, \theta), \quad p = \frac{1}{r^2} \hat{p}(\varphi, \theta).
\end{equation}

This solution representation means that the operators of the stationary equations (8.1), in view of their homogeneity, perform the following mappings of linear subspaces:

\begin{equation}
W_1 = \text{Span} \{ \frac{1}{r} \} \to \hat{W}_2 = \text{Span} \{ \frac{1}{r^2}, \frac{1}{r} \}.
\end{equation}

Substituting yields the following PDE system for the four unknowns $(\hat{u}, \hat{v}, \hat{w}, \hat{p})$:

\begin{equation}
\begin{cases}
-\hat{u}^2 + \hat{v}_\theta + \frac{1}{\sin \theta} \hat{w}_\varphi - \hat{v}^2 - \hat{w}^2 \\
= 2\hat{p} + \hat{u}_\theta + \cot \theta \hat{u}_\varphi + \frac{1}{\sin^2 \theta} \hat{u}_\varphi - 2\hat{u}_\theta - 2\hat{v}_\theta - 2\cot \theta \hat{v}_\varphi - \frac{2}{\sin \theta} \hat{w}_\varphi, \\
\hat{v}_\theta + \frac{1}{\sin \theta} \hat{v}_\varphi - \cot \theta \hat{v}^2 \\
= -\hat{p}_\theta + \hat{v}_\theta + \cot \theta \hat{v}_\varphi + \frac{1}{\sin^2 \theta} \hat{v}_\varphi - \frac{1}{\sin^2 \theta} \hat{v} + 2\hat{u}_\theta - \frac{2\cot \theta}{\sin \theta} \hat{w}_\varphi, \\
\hat{v}_\varphi + \cot \theta \hat{v}_\theta + \frac{1}{\sin \theta} \hat{w}_\varphi \\
= -\frac{1}{\sin \theta} \hat{p}_\varphi + \hat{w}_\theta + \cot \theta \hat{w}_\varphi + \frac{1}{\sin^2 \theta} \hat{w}_\varphi - \frac{1}{\sin^2 \theta} \hat{w} + \frac{2}{\sin \theta} \hat{u}_\varphi + \frac{2\cot \theta}{\sin \theta} \hat{v}_\varphi, \\
\hat{u} + \hat{v}_\theta + \cot \theta \hat{v} + \frac{1}{\sin \theta} \hat{w}_\varphi = 0.
\end{cases}
\end{equation}

Indeed, the system looks rather frightening for studying in general. As we have mentioned, fortunately, it was proved in [204] that, up to an isometry, the only non-trivial $(-1)$-homogeneity stationary solutions in $\mathbb{R}^3 \setminus \{0\}$ are the Slezkin–Landau ones (7.1) (it was also conjectured there that branching from $u_{\text{SL}}$ is impossible). Further extensions, showing the exceptional role of the S-L solutions, are obtained in [155]. In any case, (8.5) shows the range of typical difficulties concerning systems that inevitably occur while searching for other types of stationary or non-stationary singularity manifolds for linearization.

Nevertheless, according to our “linearized strategy” of further construction of blow-up patterns (and for answering questions (1.2) or (1.3)),

\begin{equation}
\text{all properties of all solutions of (8.5) should be known in detail.}
\end{equation}

Indeed, singular stationary solutions (8.3), (8.5) can be key for a possible successful matching to create a blow-up pattern. The second step is then to answer:

\begin{equation}
\text{Questions (i)–(vi), § 5.12 for operators linearized about all solutions of (8.5).}
\end{equation}

To underline the complexity of the problems (8.6) and (8.7), we will stress below the attention to some known examples of particular solutions of that type.

**On general singular stationary states: elliptic evolution.** Of course, the system (8.5) does not contain all the necessary singular stationary states for (8.1), which might be important for blow-up constructions. The general representation of such solutions takes the form

\begin{equation}
(u, v, w) = \frac{1}{r} (\hat{u}, \hat{v}, \hat{w})(\varphi, \theta, s), \quad p = \frac{1}{r^2} \hat{p}(\varphi, \theta, s),
\end{equation}

\text{for blow-up constructions. The general representation of such solutions takes the form}
where \( s = -\ln r \to +\infty \) as \( r \to 0^+ \) plays the role of the time-variable. Then

\[
(8.9) \quad u_r = -\frac{1}{r^2} (\ddot{u}_s + \dot{u}), \quad u_{rr} = \frac{1}{r^2} (\dddot{u}_s + 3\ddot{u}_s + 2\dot{u}), \quad \text{etc.,}
\]

so that the stationary system \((8.1)\), according to the variable \( s \), takes the form of “elliptic evolution” equations for \( s \gg 1 \),

\[
(8.10) \quad \{ u_{ss} = \ldots, \quad v_{ss} = \ldots, \quad w_{ss} = \ldots, \quad u_s = \ldots \}
\]

In general regularity linear PDEs theory, such “blow-up” scalings lead to complicated spectral theory of pencils of linear operators, whose spectrum and root functions classify all possible types of singularities occurred. We refer to seminal Kondrat’ev’s papers in the 1960s [121, 122] and monographs by Maz’ya with collaborators [123, 124] (further extensions via MathSciNet).

As is well-known (since Hadamard’s classic example), such an elliptic-like evolution is ill-posed and almost all of the orbits are destroyed before reaching the singularity point \( s = +\infty \) \((r = 0)\), but anyway the rest of the orbits that are defined for all \( s \gg 1 \) describe all possible singular steady states for \((8.1)\). Such an analysis has been effectively implemented for a number of single semilinear elliptic problems, even in the case of non-Lipschitz nonlinearities; see e.g., a machinery and a full list of references in [14]. Indeed, for the stationary system of four PDEs as in \((8.1)\), the problem of identifying all possible global evolution trajectories reaching \( s = +\infty \) is extremely difficult. Recall again that each such a non-trivial singular stationary state with possible inclinations and precessions of the swirl axis (see further developments below) can be key for construction of a blow-up pattern by some kind of a linearization and matching procedures. Of course, eventually we cannot escape the problem \((8.7)\) for each of those singular stationary solutions \((8.8)\).

In other words, the singular stationary problem \((1.33)\) plays a first key role for understanding of the blow-up mechanism of the NSEs, to say nothing about a similar problem with rotations to be represented later on.

8.2. Scaling and introducing blowing up angular singularity with precessions.

We return to the general system \((8.1)\) in the standard rescaled blow-up variables

\[
(8.11) \quad u = \frac{1}{\sqrt{-t}} \dot{u}, \quad p = \frac{1}{(-t)^{\frac{3}{2}}} \dot{p}, \quad y = \frac{r}{\sqrt{-t}}, \quad \tau = -\ln(-t),
\]

where, as one of the possibilities, we first assume the similarity swirling mechanism of accelerating rotation in the angle \( \varphi \),

\[
(8.12) \quad \varphi = \mu - \sigma \ln(-t) \equiv \mu + \sigma \tau \quad (\sigma \neq 0).
\]

As we have seen, using the variable \((8.12)\) introduces into the system an extra parameter \( \sigma \in \mathbb{R} \) being a nonlinear eigenvalue that increases the probability of successful matching of flows in various regions.
Substituting (8.11), (8.12) into (8.1) yields the following system:

\[
\begin{cases}
\ddot{u} + \frac{1}{y} y \dot{u} + \frac{1}{2} \dot{u} + \sigma \ddot{u}_\mu + \dot{u} \ddot{y} + \frac{1}{y} \ddot{u}_\theta + \frac{1}{y} \dot{u} \ddot{u}_\mu + \frac{1}{y} \ddot{u}_\mu - \frac{1}{y} \dot{u}^2 - \frac{1}{y} \ddot{u}^2 \\
= -\hat{p} + \Delta_3 \ddot{u} - \frac{2}{y^2} \ddot{u} - \frac{2}{y^2} \dot{u} \ddot{u}_\mu - \frac{2}{y^2} \dot{u} \ddot{u}_\mu - \frac{2}{y^2} \ddot{u}_\mu,
\end{cases}
\]

(8.13)

\[
\dot{v} + \frac{1}{2} \frac{1}{y} \ddot{v} + \frac{1}{2} \ddot{v} + \sigma \dot{v}_\mu + \dot{v} \ddot{u} + \frac{1}{y} \ddot{v}_\theta + \frac{1}{y} \dot{v} \ddot{v}_\mu + \frac{1}{y} \ddot{v}_\mu - \frac{1}{y} \dot{v}^2 - \frac{1}{y} \ddot{v}^2 \\
= -\frac{1}{y} \dot{p}_\theta + \Delta_3 \ddot{v} - \frac{1}{y^2} \sin^2 \theta \dot{v} + \frac{2}{y^2} \ddot{u}_\mu - \frac{2}{y^2} \dot{u} \ddot{u}_\mu - \frac{2}{y^2} \dot{u} \ddot{u}_\mu - \frac{2}{y^2} \ddot{u}_\mu,
\]

\[
\dot{w} + \frac{1}{2} \frac{1}{y} \ddot{w} + \frac{1}{2} \ddot{w} + \sigma \dot{w}_\mu + \dot{w} \ddot{u} + \frac{1}{y} \ddot{w}_\theta + \frac{1}{y} \dot{w} \ddot{w}_\mu + \frac{1}{y} \ddot{w}_\mu - \frac{1}{y} \dot{w}^2 - \frac{1}{y} \ddot{w}^2 \\
= -\frac{1}{y^2} \dot{p}_\mu + \Delta_3 \ddot{w} - \frac{1}{y^2} \sin^2 \theta \ddot{w} + \frac{1}{y^2} \ddot{u}_\mu + \frac{1}{y^2} \dot{u} \ddot{u}_\mu + \frac{1}{y^2} \dot{u} \ddot{u}_\mu + \frac{1}{y^2} \ddot{u}_\mu,
\]

\[
\dot{u} + \frac{2}{y} \ddot{u} + \frac{1}{y} \dot{u} \ddot{u}_\mu + \frac{1}{y} \ddot{u}_\mu = 0.
\]

The rescaled Laplacian is now

\[
(8.14) \quad \Delta_3 h = \frac{1}{y^2} (y^2 h_y) + \frac{1}{y^2 \sin \theta} (\sin \theta h_\theta) + \frac{1}{y^2 \sin \theta^2} h_{\mu\mu}.
\]

Recall that, in view of the ban (1.4), for \( \sigma = 0 \) non-trivial self-similar, i.e., stationary, solutions of (8.13) are in fact non-existent. More generally, we are supposed to perform a matching asymptotic expansion construction of blow-up patterns for (8.13) using some already known quasi-stationary manifolds. It should be mentioned the possibility of taking \( \sigma = 0 \) for the \( \varphi \)-independence solutions

\[
(8.15) \quad \hat{u} = u(r, \theta, \tau), \quad \hat{p} = p(r, \theta, \tau).
\]

The long history of difficulties in proving global existence of solutions (8.15) [193] suggests that a blow-up pattern can be revealed even in this restricted geometry.

However, it seems that a most reliable approach to blow-up patterns should comprise:

(i) either the blow-up swirl mechanism (8.12) with \( \sigma \neq 0 \), or

(ii) the asymptotically slowing down mechanism such as in (7.12), (7.13), formally corresponding to the case

\[
(8.16) \quad \sigma = \sigma(\tau) \to 0 \quad \text{as} \quad \tau \to +\infty \quad \text{in} \quad (8.12).
\]

Both cases seem can suit the linearized construction about the Slezkin–Landau solutions (7.6) or others more regular, which assumes a difficult spectral analysis of the linear operator as in (7.18), where the linearization is now performed relative to the nonlinear operators in (8.13). Note that for the asymptotically \( \sigma = 0 \) in any similarity annulus \( \left\{ \frac{1}{c} \sqrt{-t} \leq r \leq C \sqrt{-t} \right\} \), with arbitrary \( C > 1 \), the construction remains the same and includes general eigenfunctions of (7.18) having a \( \varphi \)-dependence. Again, Questions (i)–(v) from Section 5.12 appear. Assume that we can answer these questions for some particular setting of singular quasi-stationary manifold. For instance, then as a by-product, this would actually mean that we may look for a point spectrum of \( B^* \) with the eigenfunction behaviour of the third \( w \)-component of the linearization (5.75) about the singular equilibrium, say, \( U = u_{SL} \).

\[
(8.17) \quad Y_3(\tau) \sim e^{\lambda \tau} \psi^*_\beta(r, \theta, \varphi) + ... \to 0 \quad \text{as} \quad \tau \to +\infty,
\]

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where \( \lambda_k \in \sigma_p(B^*) \), \( k = |\beta| \), is such that \( \Re \lambda_k < 0 \) (or simply \( \lambda_k < 0 \) for a real eigenvalue that also can be expected). In other words, the swirling blow-up behaviour then will occur on smaller shrinking subsets \( \{ r = o(\sqrt{-t}) \} \) as \( t \to 0^- \), where one can expect both scenarios associated with (8.12) or (7.12), (7.13), or with a constant, independent of \( \varphi \) rotation as for solutions (8.15).

Let us mention the following aspect from Section 5.9: can the possible zero mean oscillation property similar to (5.46) (in the original \((x,t)\)-variables) affect the asymptotic behaviour of solutions as \( x \to \infty \) in such a way that the patterns will attain a finite energy? Actually, this is related to the hard problem of the asymptotic behaviour for the NSEs in a compliment of a bounded smooth domain, on the boundary of which blow-up swirl rotations with the zero mean as in (5.46) is prescribed. Possibly, this could affect the asymptotics of the solutions which become better localized in the \( L^2 \)-sense. The difficulty of this question dramatically increases if a possible axis precession (see below) is taken into account.

Thus, it seems that, according to the above scenarios, there is no chance to study possible admissible solutions of the resulting systems such as (8.13) in a reasonable and reliable generality and mathematical strictness. So we will continue to develop some necessary formal arguments in an attempt to give further hints for understanding such potential complicated blow-up patterns. As in Sections 5.12 and 7 we will assume that in some intermediate region the blow-up behaviour goes along the quasi-stationary manifolds of singular steady solutions of (8.13), which, hopefully, have asymptotically the form (8.3), with \( r \to y \), i.e., do not exhibit somehow essential dependence on the \( \varphi \)-torsion (which may get crucial for smaller \( y \)). Then, as we have seen, a complicated linearized operator such as in (5.76) and (7.18) occurs. The main difficulty is not studying its spectral properties in suitable weighted topology, but the matching procedures with bounded orbits for \( y \approx 0 \).

For large \( y \gg 1 \), we always assume that there exists a possibility of matching the inner region with a properly deformed rescaled kernel of the fundamental solution of the operator \( D_r - \Delta_3 \), since all the evolution equations in (8.13) contain the necessary counterparts. Observe that the resulting patterns are not supposed to be of a simple self-similar form, so they do not exhibit any uniform homogeneity as in (8.3) for steady profiles and/or \( L^\infty \)-boundedness as non-stationary similarity solutions after scaling (8.11).

An axis precession mechanism. In order to further increase the probability of such a matching of various manifolds for systems like (8.13), in order to suit (2.47), it is necessary and natural for such swirling/vortex flows to introduce an additional axis precession mechanism for these blow-up twistors. This cannot be described explicitly or by a system on a lower-dimensional subspace. However, there is a standard asymptotic approach.

Thus, we first need to plug into system another parameter in order to change the axisymmetric (with swirl) geometry of the solutions involved. As it has been mentioned, any strong symmetry (or quasi-symmetry) constraint in the Navier–Stokes equations reduces the dimension of the space, so approach them closer to the regular \( \mathbb{R}^2 \)-case. Therefore, thinking about creating even more complicated geometry and not hesitating to perform rather weird transformations with such unknown solutions, we will show a way how to
perform a necessary perturbation of those solution structures. As a first simple formal illustration, let us take into account an extra slow motion in the $\theta$-angle by setting

\begin{equation}
\theta = \rho + \varepsilon \tau, \quad \text{where} \quad |\varepsilon| \ll 1.
\end{equation}

This will mean introducing into the system (8.13) extra $O(\varepsilon)$-operators, i.e.,

\begin{equation}
\ldots + \varepsilon \hat{u}_\rho + \ldots, \quad \ldots + \varepsilon \hat{v}_\rho + \ldots, \quad \ldots + \varepsilon \hat{w}_\rho + \ldots \quad \text{into the LHSs of equations.}
\end{equation}

Using in all the equations the corresponding $\varepsilon$-expansions such as

\begin{equation}
\sin \theta = \sin(\rho + \varepsilon \tau) = \sin \rho + \varepsilon \tau \cos \rho + \ldots, \quad \frac{1}{\sin \theta} = \frac{1}{\sin \rho} - \varepsilon \tau \cot \rho \frac{\cos \rho}{\sin \rho} + \ldots,
\end{equation}

e tc., gives extra entries of $\varepsilon$ into the system. According to asymptotic expansion theory (see Il’in [107] for typical difficult methods and further references) and not taking into account at this moment singularities introduced by (8.20) into the PDE system, this makes it possible to look for solutions in the form of formal expansions

\begin{equation}
(\hat{u}, \hat{v}, \hat{w}, \hat{p}) = (\hat{u}_0, \hat{v}_0, \hat{w}_0, \hat{p}_0) + \varepsilon (\hat{u}_1, \hat{v}_1, \hat{w}_1, \hat{p}_1) + \ldots.
\end{equation}

By a standard procedure, being fully developed, the series (8.21) gives a unique formal representation of a certain formal solution. The main step is the first one, for $\varepsilon = 0$ that leads to the above nonlinear systems (which were assumed can be solved). The next terms ($\hat{u}_k, \hat{v}_k, \hat{w}_k, \hat{p}_k$) are obtained by iterating some linear systems, that is easier, and existence of a solution can be checked by standard (but not always simple) arguments. The convergence of such series, as usual, is not required, and is often extremely difficult to guarantee even for lower-order PDEs, where the rate of convergence or asymptotics are also hardly understandable. Typically, such series serve as asymptotic ones, i.e., each next term correctly describes the behaviour of the actual solution as $\varepsilon \to 0$.

Thus, (8.21) is now assumed to describe branching of suitable solutions from the unperturbed one at $\varepsilon = 0$ with no fast swirl. Here, we refer to classic bifurcation-branching theory for equations with compact nonlinear integral operators, [12, 46, 126, 212], etc. Proving the actual branching is a deadly difficult problem, especially, since we are interested in quite special solutions only. Therefore, we do not concentrate on branching phenomena and continue to reveal possible features of such blow-up twistors.

Note that (8.18) indicates a tendency of the desired precession with potentially unbounded deviation of the swirl axis. To avoid such an unpleasant (and seems non-realistic?) pattern, we assume that the actual precession is governed by (cf. (7.12))

\begin{equation}
\theta = \rho + \varepsilon \kappa(\tau, \varepsilon), \quad \text{where} \quad \kappa(\tau, \varepsilon) \text{ is bounded,}
\end{equation}

so that (8.18) contains the first term of the $\tau$-expansion, and extra difficult asymptotic matching theory occurs.

**Periodic and quasi-periodic axis precession.** However, even the axis precession according to the simplified dependence (8.18) can give insight into the actual behaviour of such

\footnote{We discuss this simplified version of an $\varepsilon$-expansion for convenience. In general, depending on the kernel of the linearized operator and other factors, the expansion can depend on the small parameter $\varepsilon^{1/l}$, where integer $l \geq 1$ is defined from the solvability of the corresponding nonlinear algebraic systems on the coefficients; see general bifurcation-branching theory, e.g., [212].}
matching blow-up patterns. In particular, as a formal illustration, ignoring at the moment a periodic-like behaviour in \( \tau \), consider both angular dependencies of the swirl axis on the unit sphere \( S^2 \subset \mathbb{R}^3 \).

\[
\begin{cases}
\varphi = \sigma \tau, \\
\theta = \varepsilon \tau.
\end{cases}
\]

(8.23)

Since, as in the classic representation of periodic and quasi-periodic motion on a torus in \( \mathbb{R}^3 \), the angle behaviour is understood modulo \( 2\pi \), so we can have both scenarios on the sphere \( S^2 \). Namely, assume for a moment that a matching procedure has turned out to be successful for a given pair of eigenvalues \( (\sigma, \varepsilon) \). Then (8.23) indicates that the \textit{periodic scenario} of the axis evolution associated with (2.46) (but not entirely; see below) takes place provided that

\[
\frac{\sigma}{\varepsilon} \in \mathbb{Q}
\]

is rational, and, \textit{vice versa}, we have a quasi-periodic precession if

\[
\frac{\sigma}{\varepsilon} \text{ is an irrational number.}
\]

(8.25)

The periodic scenario (8.24) looks being the simplest one to create a blow-up singularity in the dynamical system (2.41) provided stabilization to a point is forbidden by (1.4) (if to follow the principle of Occam’s Raso\textsuperscript{42}). But of course, this does not rule out other patterns, which actually, do not look more complicated and admit entirely similar characterization (8.25) (though we do not exclude the non-precision case \( \varepsilon = 0 \)). For (8.22), we can observe a more realistic scenario of very small precession exhibiting periodic or quasi-periodic motion of the axis.

\textit{Briefly on precession of the vertex.} It is quite natural that including both swirl and axis precession mechanisms will also require an extra “precession of the vertex” (as an acceptable analogy with a rotating top on a sufficiently smooth surface suggests); otherwise such a complicated non-symmetric vortex evolution with swirl and precession having a fixed stagnation point would not be possible. This assumes the slow-variable change of the original rescaled coordinate system,

\[
\hat{x} \mapsto \hat{x} + a(\tau),
\]

(8.26)

where hopefully \( a(\tau) \) can be a bounded function with also periodic or quasi-periodic behaviour as \( \tau \to \infty \).\textsuperscript{43} This will involve into the systems such as (8.11) extra perturbed operators according to the change

\[
u_{\tau} \mapsto \nu_{\tau} + (a' \cdot \nabla)u,
\]

(8.27)

\textsuperscript{42}W. Ockam’s LEX PARSIMONIAE: “entia non sunt multiplicanda praeter necessitatem”.

\textsuperscript{43}For reaction-diffusion equations such as (1.36) or (1.60), the \textit{blow-up set} is introduced \( B[u_0] = \{ x_0 \in \mathbb{R}^N : \exists \{ x_n \} \to x_0 \text{ and } \{ t_n \} \to T^- \text{ such as } |u(x_n, t_n)| \to \infty \} \). It is then proved that \( B \) is closed (e.g., is a point; see Friedman–McLeod \textsuperscript{62} for a pioneering approach), and next blow-up scaling is performed relative to an \( x_0 \in B[u_0] \) by setting \( y = \frac{x - x_0}{\sqrt[4]{T - t_0}} \), etc.
so that these extra terms can be responsible for additional evolution blow-up phenomena (even in the asymptotically vanishing case $a'(\tau) \to 0$, since this also might support a quasi-periodic-like motion; the integrable case $\int^\infty a'(\tau) \, d\tau < \infty$ makes (8.26) non-essential) similar to those induced by swirling (7.12) and precession (8.22). All three mechanisms taking altogether lead to a possibility to attempt to construct a blow-up twistor pattern originated in the outer region by the Slezkin–Landau singular steady solutions (7.6) or others, which possibly are still unknown.

In conclusion, let us mention that we do not stress attention to a possible more physical-mechanical interpretation of the blow-up patterns under speculations. Namely, we do not know and cannot imagine how many actual twistors should be involved in such a “hypothetical fluid configuration” to produce such a pattern under the fixed blow-up frame and zoom. We just recall that, as a consistent part of our “swirling-like philosophy”, we mean that complicated blow-up patterns can be locally trapped as $t \to T^-$ by the quasi-stationary singularity manifold described by the problems (1.33), (1.32), and others. Such a blow-up drift along those singular manifolds will itself define the type of generalized swirling that is necessary to support the evolution.

Mechanical interpretations of various solutions and patterns have always been very effective, difficult, and involved techniques of modern applied and mathematical fluid dynamics, which can be also efficient after a blow-up scaling, but possibly could fail in view of the fact that the blow-up micro-structure of the NSEs might have nothing to do with their classic macro-coherent structures studied during almost two centuries. A possibly acceptable example is as follows: the local atomic (micro-scaled) structure of a desk, where this paper is about to be finished, has nothing to do with its global (macro) properties. This is up to the obvious fact that here the micro- and macro-mechanics are different: the quantum and Newton’s ones. A similar phenomenon occurs for the NSEs: the operators of the original PDEs (1.1) and of the blow-up rescaled ones (2.21) live in completely different spaces, $L^2(\mathbb{R}^3)$ and $L^2_{\rho^*}(\mathbb{R}^3)$.

Again on $L^\infty$-bounded or unbounded rescaled orbits $\{\hat{u}(\tau)\}$. Finally, we add to this formal description of possible blow-up patterns of twistor type the following necessary observation based on our previous analysis. Namely, according to matching concepts revealed in Sections 5.11, 5.12, and 7 there exist two cases (sub-scenarios relative to the above):

(i) **Type I**: the rescaled orbit $\{\hat{u}(\tau)\}$ is bounded in $L^\infty$ as $\tau \to +\infty$. Then this corresponds to scenarios (2.40) and (2.47), as before, and

(ii) **Type II**: $\|\hat{u}(\tau)\|_\infty \to \infty$ as $\tau \to +\infty$. This can happen according to the matching analysis and can be driven by an exponential (5.88) or power-like as in (5.90) divergence (or others) depending on the manifold that was used for matching purposes. Of course, this does not ruled out the quasi-periodic $L^\infty$ behaviour (5.12) of bounded orbits in the necessary new rescaled variables.

In both cases, the blow-up patterns remain exhibiting similar properties of swirl and precession but, possibly, on different spatio-temporal subsets.
9. Final remarks

9.1. Micro-structure of turbulence. As the attentive Reader has noticed, the main goal of the present essay is not about achieving or even essentially approaching a definite answer to the fundamental open problem: (1.2) or (1.3) for the NSEs (1.1). Actually, this is more about approaching better understanding the Goal (1.20). In other words, this is about presenting some ideas on a description of

\[(9.1) \text{ existing micro-scaled fluid configurations appearing from smooth data.}\]

We recall that even for bounded smooth solutions, revealing such micro-structure of multiple zeros at regular points (Section 3) led us to some difficult problems, though their Hermite polynomial solenoidal structure was partially justified.

Concerning singular points, in (9.1), the term “micro-scaled” is key meaning to look for fluid configurations that are seen via microscopic “blow-up rescaling zoom” as in (2.7).\[44\] This means a description and a classification of the admitted “local turbulent micro-structures” of the Navier–Stokes equations (1.1). The problem of “micro-structure” can be posed for any linear or nonlinear evolution PDE of parabolic, hyperbolic, nonlinear dispersion, etc., types, and gets very complicated even for simple models (cf. [67, § 9] as a short introduction to this involved subject).

As a by-product, solving (9.1) would also mean the negative answer (1.3) provided that the family of those configurations would include some blow-up patterns. Not pretending at all to giving any comprehensive insight into the problem (9.1), we just have shown that including a standard similarity “log-torsion” mechanism produces a number of very complicated singular stationary (cf. (8.6)) or evolutionary dynamical systems as some lower-dimensional reduction of the NSEs. This also implies that a detailed study of such reduced, but still very complicated, dynamical systems and corresponding equilibria are necessary and unavoidable steps that should have been passed before even thinking about attacking the Millennium Problem (along the proposed lines).

9.2. A final pessimistic expectation in general PDE theory. As a consequence of all the above blow-up speculations, we first state the simplest claim: for the NSEs,

\[(9.2) \text{ from the side of (1.3), checking all blow-up configurations is impossible.}\]

More precisely, involving the positive part (1.2), it can be emphasized that, for a sufficiently wide class of complicated dynamical systems “\(M \times (N + 1)\)” such as (2.21) with \(M = 4, N = 3\) (the numbers of dependent and independent variables involved), with a

\[44\]In general, this reflects a small part of fluid dynamic problems of fundamental importance; e.g., cf. A.M. Lyapunov Master’s Thesis “On Stability of Spheroidal Equilibrium Forms of a Rotating Fluid” (S.-Petersburg University, 27th January, 1885; Supervisor: P.L. Chebyshov), which was a forerunner of Lyapunov’s stability theory and a starting point for his correspondence since 1885 with H. Poincaré, who also in 1885 obtained a linearized system of Euler equations about a rotating fluid pattern as a rigid body around the \(z\)-axis; see [179] for an extra account.
similar mathematics including a global energy control of solutions (not enough to guarantee \(L^\infty\)-bounds by embedding and/or interpolation), divergence of operators, scaling laws, and other related and necessary properties.\(^{45}\)

\[(9.3)\] a definite answer to claims like (1.3) and (1.2), in general, is impossible.

As we have seen, for the NSEs (1.1), some ideas of blow-up focusing can be understood via standard asymptotic language, though a full justification could also take years or do not admit such at all. For the fourth-order bi-harmonic operator as in (1.6), being also parabolic but with no order-preserving and nonlocal properties, a similar proof often can be characterized as being completely illusive. Moreover, here a self-similar blow-up is then rather plausible as for (1.56), \(m = 2\), [22], though will be extremely difficult to prove. In other words, there is a huge probability that

\[(9.4)\] Problem: (1.2) or (1.3), can be analytically non-solvable.

Rephrasing the above, problems of such complexity from the side (1.2) can be “accidentally solvable”\(^{46}\) when special tricks associated with these equations and operators ONLY (not robust techniques admitting perturbations, i.e., “structurally stable”) can rule out some essential part of the core difficulties; all modulo (1.22). But this “accidental” feature is rather unlikely: both (1.2) and (1.3) are too much related to each other, and proofs should pass through a lot of similar extremely difficult stages. We should accept that, in modern nonlinear higher-order PDE theory, many standard results, which had been perfectly solved for lower-order counterparts by inventing great mathematical methods by great mathematicians, do not admit rigorous setting in principle. In this case, looking and searching for results admitting rigorous proofs would be a wrong idea, also contradicting Kolmogorov’s thoughts; see a footnote in Section 1.4.

To this end, we recall that a definite, positive or negative, answer in particular assumes a rigorous checking whether (2.47) is true or not for any \(n \geq 2\). Let us also remind that, unlike (8.23), in general, we are talking about a quasi-periodic motion in an infinite-dimensional functional space for \(\hat{u}(\tau)\), so that possible localization of that is a very difficult problem. Recall that such quasiperiodic motions can be trapped in a vicinity of singular “equilibria” (1.32) or (1.33) of very complicated unknown structure. In its turn, via a matching approach, this means checking whether the corresponding asymptotic bundles do or do not overlap for any \(n = 1, 2, 3, \ldots\). This creates a restriction that cannot be checked analytically in general because it depends on unknown and unpredictable conditions of matching of various pair of infinite-dimensional local vector bundles. In view of the energy control, those matching look like checking if, for a suitable 2D restriction, there

\(^{45}\)Just in case, modulo (1.22); recent almost purely “parabolic” and MP-type results in [118, 38] (I am sure that further, even stronger papers in this directions will follow soon) for nonexistence of Type I blow-up for axi-symmetric flows inspired some optimism, though, as we have tried to show in Sections 5.2, 5.3, and 7, this can be just the beginning of a very long road to success.

\(^{46}\)Or, at least, it then could be not an exaggeration to comment that, posing this Millennium Problem, it would be also useful to specify, which millennium it is supposed to be.
exists a heteroclinic path connecting two saddles for a given dynamical system on a 2D manifold. For general DSs, existence/nonexistence of such a path cannot be predicted, and moreover, DSs with such heteroclinic orbits are known to be structurally unstable (Andronov–Pontriagin–Peixoto’s theorem, 1937–57). It then seems that such a matching is not possible for almost all DSs like that. This is true, but since, according to (2.47), at least a countable (or more than that) number of such possibilities is supposed to be checked, this changes the dimension of the parameter space and, eventually, makes the problem to be analytically non-solvable.

If, in reality, the justification of the negative claim (1.3) would have been proved of being of a geometric “configuration” as a matching of the type saddle–saddle on some manifold for establishing existence of a blow-up pattern with finite energy, it would clearly suggest that any enhancements of functional space and corresponding facilities for proving (1.2) by more and more refined interpolation-embedding techniques or similar would be entirely hopeless. Unless there is a parameter “gap” prohibiting such matchings uniformly in $n \geq 1$ in (2.47) and for other types of patterns. Hence, the existence problem takes a principally other background and becomes the question of blow-up scaling.

In other words, then the NSEs regularity problem should be classified as being of “pointwise sense”, i.e., its global solvability is not controlled by any possible a priori bounds in $L^p$ or related Sobolev spaces, so it is hopeless to try to derive those. Nowadays, more and more nonlinear evolution PDEs (for instance, supercritical nonlinear Schrödinger-type or sufficiently multi-dimensional Burnett equations) penetrate into this class, which demands principally new mathematics of the truly twenty first century. One should be ready to recognize that, in this PDE class, most of typical mathematical problems will be analytically non-solvable. This by no means diminishes the role of the pure mathematics. On the contrary, this implies that the whole mathematical culture will be needed to build a necessary well-organized understanding of the problem, under the pressure that no even a hope for any definite rigorous answers exists.

Eventually, the author quite bravely expresses his personal opinion, which in view of the possible feature (9.4) might have some sense:

$$\text{(9.5)}$$

for (1.1), global existence is more plausible than blow-up.

Actually, this is a pure probability based upon above long speculations and discussions. However, global existence issues, though expected to be somehow related to blowing up ones, were not the subject of the paper.

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Appendix A. English Translation of N.A. Slezkin’s paper

ON AN INTEGRABILITY CASE OF FULL DIFFERENTIAL EQUATIONS OF THE MOTION OF A VISCOUS FLUID

N. A. Slezkin

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If the motion of viscous fluid is stationary and the axisymmetry of the flow takes place, then, as is known, the stream function satisfies the following differential equation in the cylindrical coordinates:

\[
(A.1) \quad -\frac{1}{r} \frac{\partial \psi}{\partial z} \frac{\partial D \psi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial D \psi}{\partial z} + \frac{2}{r^2} \frac{\partial ^2 \psi}{\partial z^2} D \psi = \nu D \frac{\partial ^2 \psi}{\partial r^2},
\]

where

\[
D = \frac{\partial ^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial ^2}{\partial z^2},
\]

or, on introduction of the conical coordinate \( \rho \) and \( \cos \theta = \tau \), the equation (A.1) takes the form:

\[
(A.2) \quad \frac{1}{\rho^2} \left[ \frac{\partial \psi}{\partial \rho} \frac{\partial D \psi}{\partial \tau} - \frac{\partial \psi}{\partial \tau} \frac{\partial D \psi}{\partial \rho} + 2 \left( \frac{\tau}{1-\tau^2} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho} \frac{\partial \psi}{\partial \tau} \right) D \psi \right] = \nu D \frac{\partial ^2 \psi}{\partial \rho^2},
\]

where

\[
D = \frac{\partial ^2}{\partial \rho^2} + \frac{1-\tau^2}{\rho^2} \frac{\partial ^2}{\partial \tau^2}.
\]

Under a certain assumption on the form of the function \( \psi \), equation (A.2) can be reduced to an ordinary differential equation of fourth order, and the latter to a Riccati equation. Indeed, set

\[
(A.3) \quad \psi = \rho f(\tau).
\]

Then we have:

\[
D \psi = \frac{1-\tau^2}{\rho} f'',
\]

and the equation (A.2) takes the form:

\[
(A.4) \quad ff''' + 3f'f'' = \nu [(1-\tau^2)f^{IV} - 4\tau f'''].
\]

The left-hand side of this equation can be represented as

\[
f f''' + 3f'f'' = \frac{1}{2} (f^2)'',
\]

while the right-hand one as

\[
\nu [(1-\tau^2)f' + 2\tau f]''.
\]

and then (A.4) is rewritten as follows:

\[
(A.5) \quad \left( \frac{1}{2} f^2 \right)''' = \nu [(1-\tau^2)f' + 2\tau f]''.
\]

Integrating it three times, we obtain:

\[
(A.6) \quad \frac{1}{2} f^2 - \nu [(1-\tau^2)f' + 2\tau f] = C_0 + C_1 \tau + C_2 \tau^2,
\]

where \( C_0, C_1, C_2 \) are constants of integration.

Equation (A.6) can be rewritten as:

\[
(A.7) \quad f' = \frac{1}{2\nu(1-\tau^2)} f^2 - \frac{2\tau}{1-\tau^2} f + \frac{C_0 + C_1 \tau + C_2 \tau^2}{1-\tau^2}.
\]

Thus, we have obtained a Riccati differential equation of the form that is not explicitly integrated. On substitution

\[
f = -2\nu(1-\tau^2) \frac{d\ln y}{d\tau}
\]

this reduces to the linear differential equation of the 2nd order:

\[
(A.8) \quad \frac{d^2 y}{d\tau^2} + \frac{C_0 + C_1 \tau + C_2 \tau^2}{1-\tau^2} y = 0.
\]
The solution of the latter one can be studied by using methods of analytic theory of differential equations.

[Uchenije Zapiski MGU, No. II, 1934.]

ÜBER EINEN INTEGRIERBAREN FALL DER VOLLSTÄNDIGEN BEWEGUNGSGLEICHUNGEN EINER ZÄHEN FLÜSSIGKEIT

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(Zusammenfassung)

Es wird der Fall untersucht, in dem die Gleichung für die Stromfunktion sich auf eine gewöhnliche Gleichung, und diese letztere auf eine Riccatische Gleichung zurückführen lässt.

[Wissenschaftliche Berichte der Moskauer Staatsuniversität, H. II, 1934.]

Appendix B. English Translation of N.A. Slezkin’s paper [197]

Remark on the notes of Yu. V. Rumer, “The problem of a submerged jet”[1] and of L.G. Loitianskii, “Propagation of a whirling jet into an infinite spaces filled with the same fluid”[2]

N. A. Slezkin (Moscow)

In both notes, it was pointed out that the problem on a laminar submerged jet was first considered by L.D. Landau (Mechanics of Continuum Media[3]) and that the solution of this problem represents a new case of explicit integration of equations of motion of a viscous fluid. The note of such kind in the second part is not correct.

In the note by N. A. Slezkin “On an integrability case of full differential equations of the motion of a viscous fluid”[4] published in Uchenije Zapiski MGU, No. II, 1934, it was shown that the full differential equation for the stream function of a steady axisymmetric motion in the spherical coordinates $r$ and $\tau = \cos \theta$

$$
\frac{1}{r^2} \left[ \frac{\partial \psi}{\partial r} \frac{\partial D\psi}{\partial r} - \frac{\partial \psi}{\partial \tau} \frac{\partial D\psi}{\partial \tau} + 2(\frac{\tau}{1-\tau^2} + \frac{1}{r} \frac{\partial \psi}{\partial r})D\psi \right] = \nu D^2\psi
$$

$$
D = \frac{\partial^2}{\partial r^2} + \frac{1-\tau^2}{r^2} \frac{\partial^2}{\partial \tau^2}
$$

under the assumption $\psi = rf(\tau)$ reduces to the equation

$$
f' = \frac{1}{2\nu(1-\tau^2)} f^2 - \frac{2\tau}{1-\tau^2} f + \frac{C_0 + C_1 \tau + C_2 \tau^2}{1-\tau^2}
$$
which by the substitution \[ f = -2\nu(1 - \tau)^2 \frac{d\ln y}{d\tau} \] was reduced to the linear differential equation

\[
\frac{d^2 y}{d\tau^2} + \frac{C_0 + C_1\tau + C_2\tau^2}{1 - \tau} y = 0
\]

Equating the constants \( C_0, C_1, C_2 \) to zero, we arrive at the solution

\[ f = -2\nu(1 - \tau^2) \frac{1}{A + \tau} \]

which was obtained by L. D. Landau within the study of the problem of a submerged jet. This case of integration of full differential equations of the motion of a viscous fluid was referred to in L. I. Sedov’s book\(^5\) (p. 104) and in Rosenblatt’s paper\(^6\).

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APPENDIX C. ON SOME SIMPLER EXACT SINGULAR STEADY SOLUTIONS: MORE HISTORY

For further application to blow-up patterns, we will need to revise some stationary solutions of (1.1) that are singular (unbounded) at the origin \( x = 0 \). As above, such singular equilibria can be used for attempting to construct different blow-up patterns that are not bounded in the rescaled similarity variables as in (2.7). Constructing various exact or explicit particular solutions of the NSEs is a classic, effective, diverse, and, at the same time, difficult way of understanding fluid flows. Some basic ideas go back to Oseen and von Kármán. There are various other approaches to constructing other, exact or explicit, less and non-singular solutions of the Navier–Stokes equations; see monographs \[133, 173\] and surveys in \[86, 167\], with a number of references therein.

Singular solutions (8.3) of the homogeneity \(-1\) have a long history. Actually, Slezin–Landau’s explicit solutions (7.1) belong to the same class. As we have mentioned, the first ODE reduction was due to Slezin \[196\] in 1934. For convenience of the Reader, in Appendix A, we present the English translation of Slezin’s short note \[196\], while Appendix B contains the translation of his further note \[197\] (1954), so both rather convincingly show that Slezin’s ODE analysis

\[47\text{Clearly, a misprint: should be } (1 - \tau^2). - VAG.\]
where \( \chi \) performed by Strakchowitsch (1931) \cite{201} by looking for stream functions in the forms

\[
(C.1) \quad \psi = \frac{\chi(r)}{r} + \alpha z \quad \text{and} \quad \psi = \chi(r) + \frac{\alpha z^2}{2}.
\]

Based on these Strakchowitsch’s and Slezkin’s \cite{196} representations, Rosenblatt (1936) \cite{185} constructed solutions with

\[
(C.2) \quad \psi = f(r)z + f_1(r) \quad \Rightarrow \quad \left\{ \begin{array}{l}
\left( \frac{\beta}{r} \right)'' + \left[ \frac{1}{r} \left( \frac{\beta}{r} \right)' \right]' + f \left[ \frac{1}{r} \left( \frac{\beta}{r} \right)' \right]' - \frac{\beta}{r} \left( \frac{\beta}{r} \right)' = 0, \\
\left( \frac{\alpha}{r} \right)'' + \left[ \frac{1}{r} \left( \frac{\alpha}{r} \right)' \right]' + f \left[ \frac{1}{r} \left( \frac{\alpha}{r} \right)' \right]' - \frac{\alpha}{r} \left( \frac{\alpha}{r} \right)' = 0.
\end{array} \right.
\]

Both equations admit straightforward integration ones. It was shown that the equations admit non-singular solutions given by infinite power series. Therefore, it seems that Slezkin \cite{196} was the first who looked for exact stationary fluid flows that are \textit{singular} at the origin \( x = 0 \).

In 1950, Yaceev \cite{218} performed a detailed construction (in fact, quite similar to Slezkin’s one; cf. Appendix A) of such exact solutions without torsion and \( w = 0 \), where

\[
(C.3) \quad u = \frac{\dot{u}(\theta)}{r}, \quad v = \frac{\dot{v}(\theta)}{r}, \quad w = 0, \quad p = \frac{\dot{p}(\theta)}{r}.
\]

These assumptions essentially simplify the system \cite{8,5} that now takes the form

\[
(C.4) \quad \dot{u} = -\dot{v}' - \cot \theta \dot{v},
\]

and substituting into the second one, on integration once, yields the pressure,

\[
(C.5) \quad \dot{p} = -\dot{v}' - \cot \theta \dot{v} - \frac{\dot{u}}{\sin^2 \theta} \dot{v} + 2\dot{u}, \quad \dot{u} + \dot{v}' + \cot \theta \dot{v} = 0.
\]

The last equation gives the first velocity component

\[
(C.6) \quad \dot{u} = -\dot{v}' - \cot \theta \dot{v}, \quad \dot{p} = -2\dot{v}' + \frac{2(b(\cos \theta - \alpha))}{\sin^2 \theta}, \quad \dot{v} = -\frac{\theta}{\chi(\theta)}.
\]

where \( \chi(\theta) \) is given by the hyper-geometric functions

\[
(C.7) \quad \chi(\theta) = (\cos \theta)^\gamma (\sin \theta)^{1 + \alpha + \beta - \gamma} \times \left[ c_2 F(\alpha, \beta, \gamma, \cos^2 \frac{\theta}{2}) + c_2 F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, \cos^2 \frac{\theta}{2}) \right],
\]

where \( c_{1,2} \in \mathbb{R} \). The relations between constants \( a, b, c \) and \( \alpha, \beta, \gamma \) are as follows:

\[
(C.8) \quad \left\{ \begin{array}{l}
a = \gamma^2 - (1 + \alpha + \beta)\gamma + \frac{1}{2} (\alpha + \beta)^2 - \frac{1}{2}, \\
b = (\alpha + \beta - 1)\gamma - \frac{1}{2} (\alpha + \beta) + \frac{1}{2}, \quad c = \frac{1}{2} [(\alpha - \beta)^2 - 1].
\end{array} \right.
\]
The hyper-geometric function $F = F(z)$ in (C.7) satisfying the ODE
\[(C.9) \quad z(1 - z)F'' + [c - (a + b + 1)z]F' - abF = 0,\]
is regular at the origin and is given by the Kummer power series,
\[(C.10) \quad F(a, b, c, z) = 1 + \frac{ab}{c} z + \frac{1}{2!} \frac{a(a+1)b(b+1)}{c(c+1)} z^2 + \ldots,\]
where $c \neq 0$ and $c \neq -l$, with $l \in \mathbb{N}$. This converges uniformly in $\{|z| < 1\}$ and also on the unit circle $\{|z| = 1\}$ if $a + b - c < 0$. Note that the reduction to a linear ODE related to (C.9) was already available in Slezkin [196]. It follows from the expression for $\hat{p}$ in (C.6) that sufficiently regular solutions demand $a = b = 0$. The solution (7.1) in these coordinates yields (7.6) and is a particular case of (C.6); see [132] and [133, p. 82].

Similar exact singular solutions were obtained by Squire (1951) [200] (see Pai [173, p. 72] for extra comments), where (cf. (C.2))
\[(C.11) \quad u = \frac{f'(\theta)}{r \sin \theta}, \quad v = -\frac{f(\theta)}{r \sin \theta}, \quad w = 0.\]
Then, similarly to (C.4), we have from the second equation in (8.1) that
\[(C.12) \quad p_\theta = -vv_\theta + \frac{1}{r} u_\theta \implies p = -\frac{1}{2} v^2 + \frac{1}{r} u + \frac{c_1}{r} \quad (c_1 \in \mathbb{R}),\]
where the constant of integration is taken in the form $\hat{p}$ for further convenience. The first equation in (8.5) then yields for $f = f(\xi)$, with $\xi = \cos \theta$, the following equation:
\[(C.13) \quad (f')^2 + ff'' = 2f' + [(1 - \xi^2)f'''] - 2c_1.\]
Integrating leads to the first-order quadratic Bernoulli-type equation $f^2 = 4\xi f + 2(1 - \xi^2)f' - 2(c_1\xi^2 + c_2\xi + c_3)$, so that $(\alpha, \beta, b$ depend on $c_{1,2,3})$
\[(C.14) \quad f(\xi) = \alpha(1 + \xi) + \beta(1 + \xi) + \frac{2(1-\xi^2)(1+\xi)^2}{(1-\xi)^2} \left[ b - \int_1^{\xi} \left( \frac{1+\eta}{1-\eta} \right)^{\beta} d\eta \right]^{-1}.\]
The regular case $\alpha = \beta = 0$ can be interpreted as a jet issuing from a nozzle.

A further asymptotic extension of the exact solutions (C.2) was performed two years later by Rumer [186], who looked for solutions on the subspace $W_3 = \text{Span}\{1, \frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3}\}$
\[(C.15) \quad u = \frac{u(\theta)}{r} + \frac{u(\theta)}{r^2}, \quad v = \frac{v(\theta)}{r} + \frac{v(\theta)}{r^2}, \quad w = 0, \quad p = \frac{\hat{p}(\theta)}{r^2} + \frac{\hat{p}(\theta)}{r^3}.\]
It turns out that a certain extension of solutions (C.6), (C.7) exists in the case $a = b = c = 0$ in (C.8), but these are not exact solutions so that (C.15) gives the next asymptotic term of expansion in terms of $\frac{1}{r}$. Further, Loicyanskii [143] in 1953 proposed to look for solutions with torsion in the cylindrical coordinates $\{r, \varphi, z\}$ in terms of power series for $u, w \neq 0$ small,
\[(C.16) \quad (u, v, w) : \quad \sum_{n=1}^{\infty} a_n(\eta)\frac{1}{r^n}, \quad \text{where} \quad \eta = \frac{r}{z},\]
which are not converging and are singular at the origin. The formula (C.16) may supply us with some other singular solutions, that, though not admitting explicit representations, can be used in blow-up analysis. In general, it seems that the method of formal asymptotic expansions is an appropriate way to treat our more general and difficult system (8.5) and further related ones.
to be introduced. We refer to Kurdyumov [127] for more recent developments associated with such exact solutions of (1.1).

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