Quantum chaos for the unitary Fermi gas from the generalized Boltzmann equations

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Abstract
In this paper, we study the chaotic behavior of the unitary Fermi gas in both high- and low-temperature limits by calculating the quantum Lyapunov exponent defined in terms of the out-of-time-order correlator. We take the method of generalized Boltzmann equations, derived from the augmented Keldysh approach (Aleiner et al 2016 Ann. Phys. 375, 378). At high temperatures, the system is described by weakly interacting fermions with two spin components and the Lyapunov exponent is found to be \( \lambda_L = 21 \frac{n}{\pi^2} \). Here \( n \) is the density of fermions for a single-spin component. In the low-temperature limit, the system is a superfluid and can be described by phonon modes. Using the effective action derived in (Son and Wingate 2006 Ann. Phys. 321, 197), we find \( \lambda_L = 9 \times 10^2 \bigg( \frac{L}{T} \bigg)^4 T \) where \( T_F \) is the Fermi energy. By comparing these to existing results of heat conductivity, we find that \( D_E \ll v^2/\lambda_L \) where \( D_E \) is the energy diffusion constant and \( v \) is some typical velocity.

We argue that this is related to the conservation law for such systems with quasi-particles.

Keywords: quantum chaos, unitary fermi gas, boltzmann equation

(Some figures may appear in colour only in the online journal)

1. Introduction
In recent years, the out-of-time-order correlator (OTOC), which is proposed to diagnose the quantum chaos, has drawn a lot of attention in the gravity, condensed matter and quantum information communities. An OTOC \( F_{WV}(t) \) for operator \( W \) and \( V \) with proper regularization is defined as [1–3]

\[
F_{WV}(t) = \text{tr}[\sqrt{\rho} W(t) V(t) \sqrt{\rho} W(t) V(t)] / Z, \tag{1}
\]

Here \( \rho = \exp(-\beta H) \) and \( Z \) is the thermal partition function. Let us consider systems with some small parameters which, for example, could be \( 1/N \) for a model with \( N \) local degree of freedoms. For such systems, at an intermediate time scale, \( F_{WV}(t) \) is believed to have an exponential deviation behavior \( F_{WV}(t) \sim c_0 \exp(\lambda_V t) \). Here \( c_0 \) is some constant and \( \epsilon \) is a small parameter. \( \lambda_V \) is defined as the quantum Lyapunov exponent and can be related to the classical Lyapunov exponent under semi-classical approximation [3]. A time scale, Lyapunov time \( \tau_L \), can be defined as \( 1/\lambda_L \). Remarkably, the quantum Lyapunov exponent has been proved to be upper-bounded by \( 2\pi/\beta \) for any quantum mechanical systems [4] and is saturated by models with gravity duals [2, 5, 6], including celebrated SYK models [7–9].

In condensed matter physics, an important related question is the exact relation between the information scrambling and the thermalization of a closed system. Although intuitively the information scrambling describes the loss of memories for a closed system, which implies local thermal equilibrium, there are also examples where the thermalization time \( \tau_{eq} \gg \tau_L \) [10]. Lyapunov exponents are also found to be closely related to transport behaviors where some bounds are proposed for general diffusion constants [10–13]. Moreover, it is found that the relation \( \frac{v^2}{\beta} \tau_L \sim D_E \) holds for holographic models [14] and SYK chains [15] where \( v_\beta \) is the speed of information spreading and \( D_E \) is the energy diffusion constant. Whether similar relations hold for realistic models is an interesting question.

To get some understanding of these problems, it is helpful to study the chaotic behavior of some realistic models, especially those with possible holographic description, and
Table 1. A summary of results for the quai-particle (QP), Lyapunov exponent $\lambda_f = 1/\tau_f$, typical velocity $v$, combination $v^2 \tau_f$ and energy diffusion constant $D_E$ of unitary Fermi gas in high and low temperature.

| QP           | $\lambda_f$ | $v$      | $v^2 \tau_f$ | $D_E$          |
|--------------|-------------|----------|--------------|----------------|
| High-T fermion | $21n/T^{1/2}$ | $\sqrt{3}T^{1/2}/2$ | $14T^{1/2}/n$ | $0.33T^{1/2}/n$ [34] |
| Low-T phonon  | $9 \times 10^3 T^3/T_F^2$ | $\sqrt{T_F^3}/2$ | $3 \times 10^{-3} T_F^6/T^5$ | $\sim T_F/T$ [34, 35] |

It is difficult to study the unitary Fermi gas for an arbitrary temperature due to the absence of a small parameter. One possible choice is to introduce large-$N$ factors to suppress the quantum fluctuation. However, this will not lead to a controlled calculation if we set $N$ finite finally. In this work we will focus on the high-temperature limit and the low-temperature limit where controlled analysis exists. The system can then be described by either dilute interacting fermions at high temperature or phonons in the low-temperature limit [22]. These two limits are separated by a superfluid transition. The main results are summarized in Table 1. We find that $D_E \ll v^2/\lambda_f$, here $D_E$ is the energy diffusion constant and $v$ is some typical velocity. Here, by typical velocity, we mean the typical group velocity for quasi-particles at a certain temperature, which is equivalent to the butterfly velocity in such systems with quasi-particles [14, 23].

The method of generalized Boltzmann equations derived from augmented Keldysh approach [24] to study OTOCs. It is an analogy of the traditional Boltzmann equation for the evolution of distribution functions [25], which predicts the behavior of the normal-ordered correlators. This method has been shown [26] to directly relate to the Bethe–Salpeter equation method [9, 26–33] for models with well-defined quasi-particles. As explained latter, the advantage of this method is the existence of a shortcut to directly writing out the generalized Boltzmann equations without field-theory derivations. Since the Boltzmann equations exist even for classical systems, it would be interesting to study the reduction of quantum chaos to classical chaos by this method in the future.

The plan of this paper is the following. In section 2 we firstly give a brief review of the path integral in an augmented Keldysh contour using the example of a microscopic model for the unitary Fermi gas. Then, using this path integral formula, we derive the generalized Boltzmann equations, which are the counterpart of the traditional Boltzmann equations in the traditional Keldysh approach with single forward and backward evolution. In section 3 we give a shortcut to the generalized Boltzmann equations without a field-theory calculation in the augmented Keldysh approach. We discuss some properties of the generalized Boltzmann equations and present the results for the Lyapunov exponent for high temperature in section 4. In section 5, we study the quantum chaos at low temperature using the effective phonon description. Some remarks and outlooks can be found in section 6.

2. The generalized Boltzmann equations for the unitary Fermi gas

In this section we give a brief review of the augmented Keldysh approach [24] and derive the generalized Boltzmann equations for contact interacting fermions. The relation between OTOC and the generalized Boltzmann equations is firstly proposed in [24], and studied by adding a source term in [26]. We also give some intuitive arguments in the appendix based on the evolution of thermal field doubled states (TFD) [36].

To study OTOC (1), which contains two-forward/backward evolution and a split-thermal-density matrix, one should double the time contour in the traditional Keldysh approach [25] and insert a matrix element $\sqrt{T}$ between two copies (different from the arrangement in [25]). A schematic of the time contour in shown in figure 1, where we have labeled two copies by $u/d$ and forward/backward evolution by $+/\rightarrow$. We then have four different fermion fields $\psi_{\pm,a/d}$ for each spin species, $\uparrow$ or $\downarrow$. For the two-component contact interacting fermions, the partition function reads as follows:

$$Z = \int \mathcal{D}\psi \mathcal{D}\tilde{\psi} \exp(i \int dt \mathcal{L}[\psi_{\pm,a/d}(x, t)]),$$

(2)
\[ \mathcal{L} = \int dx \left( i \bar{\psi} (\hat{G}^0)^{-1} \gamma \psi - g \sum_{m-u/d} \sum_{\ell = \pm} \xi \bar{\psi}(x_m,1) \gamma \psi(x_m,1) \right) \]  

Here we have omitted all indexes for fermion field in the \( G^0 \) term. \( g \) is the bare interaction strength and \( \xi = \pm \) labels different contours. The interaction term is diagonal in different contours with a different sign for the forward and the backward evolution. The quadratic term leads to mixing between different contours because of the non-zero matrix element at \( t = \pm \). This is similar to the traditional Keldysh approach. Similarly, as in thermal equilibrium by going back to the operator representation. We write out the Schwinger–Dyson equation in real time: 

\[ \left((\hat{G}^0)^{-1} - \Sigma \right) \circ G = I. \]  

Here we should keep in mind that the Green’s function and self-energy are all matrices of space, time, spin, 1/2 and \( u/d \) indices. Since the unitarity still holds, the Green’s function and the self-energy still have the specific causal structure in 1/2 index: 

\[ G = \begin{pmatrix} G_R & G_K \\ 0 & G_A \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_R & \Sigma_K \\ 0 & \Sigma_A \end{pmatrix}. \]  

This is the same form as the results in the traditional Keldysh approach [25], although now each operator is a matrix with \( u/d \) indexes. We call \( F \) the distribution matrix. Up to now the derivation is exact. To proceed we need to take a semi-classical approximation by assuming a slow variation in both space and time. Mathematically, this calls for the Wigner transformation defined as 

\[ \mathcal{W} \left[ A(x_1, x_2) \right] = \int \frac{dx_f}{(2\pi)^3} \left( x + \frac{x_r}{2}, \frac{x_i}{2} \right) \exp(i p \cdot x_r). \]  

which separates the center-of-mass coordinate \( x \) and semi-classical momentum \( p \) and simply reduces to Fourier transformation for systems with translational symmetry. A similar definition works for time and frequency space. To the leading order in fluctuation \( \nabla_i \cdot \nabla_f \), we have 

\[ \mathcal{W}[A \circ B] \approx A B + \frac{i}{2} (\nabla_i A \cdot \nabla_f B - \nabla_i B \cdot \nabla_f A) + ... \]  

This expansion gives the generalized Boltzmann equation for the distribution matrix \( F \) to the leading order: 

\[ (\partial_t + \mathbf{p} \cdot \nabla_f) F(\mathbf{p}) = i \Sigma_k - i (\Sigma_R F - F \Sigma_A) \equiv \text{St}(\mathbf{p}, F). \]  

Here we have defined \( \text{St}[F] \) whose diagonal element in \( u/d \) space is proportional to the collision integral [37] in the traditional Boltzmann equation. We have also set the frequency \( F \) on-shell, which works to the leading order because \( F \) always appears together with the spectral function. 

We now want to work out the explicit form of \( \text{St}[F] \) for the unitary Fermi gas in the high-temperature limit. We firstly perform a Hubbard–Stratonovich transformation, which introduces pair fields. The interaction part of the action is then
Figure 2. Diagrams for the self-energy of fermions and pairs that dominate in the high-temperature limit. Solid lines are propagators for fermions and wavy lines represent pairs.

The self-energy diagrams shown in figure 2, which are the dominant contribution in the high-temperature limit for dilute gases [38]. Alternatively, one could introduce large-N indexes to suppress the fluctuation [39]. For the self-energy, we also take the leading order of the gradient expansion for the Wigner transformation. Omitted the time dependence in Green’s function for simplicity. The self-energy matrix $\Sigma_{\alpha\sigma}$ in with implicit 1/2 indexes is given by

$$\Sigma_{\alpha\sigma}(p) = -\frac{1}{\beta} \int [T_{\alpha\sigma}^{\text{pair}}(k - p)^T + (k - p)^T T_{\alpha\sigma}^{\text{pair}}],$$

Here we define the self-energy for pairs by $g^{-1} = (g_{\sigma}^{-1} + \Pi)$, one could show that

$$g_K = -g_k \Pi_k g_A, \quad g_{K/A} = g_k (g_{\sigma}^{-1} + \Pi_{K/A}) g_A,$$

where $\Pi_{\text{cl}} = 0, \Pi_{\text{clq}} = \Pi_{\text{p}}, \Pi_{\text{ql}} = \Pi_{\text{p}}$, and $\Pi_{\text{qq}} = \Pi_{\text{p}}$.

Expanding in terms of $z = \exp(\mu/T)$, which is valid at high temperature, we can approximate $g_{\text{R,mm}}(k_0, k) = \frac{1}{1/4\pi a_h - i\hbar k - k^2/4}/(2\pi)$, which is the two-body scattering matrix in a vacuum by using the renormalization relation

$$\frac{1}{g_{\sigma}} = \frac{1}{4\pi a_h} + \int \frac{d^2k}{(2\pi)^3} \frac{1}{k^2},$$

which relates $g_{\sigma}$ to the physical scattering length $a_h$. Straight forward derivations based on equation (17) and (19) lead to the final answer for $\Sigma[F]$. For $m = m'$, we have the scattering term in the traditional Boltzmann equation

$$\Sigma_{\text{mm'}}(p) = \frac{1}{4} \int \frac{d^2p_1 d^2p_2 d^3p_3}{(2\pi)^9} T(p, p_1) \times \left(-L_{-1}(p_1) F_{\text{mm'}}(p) + F_{\text{mm'}}(p_3)ight) + F_{\text{mm'}}(p_2) - F_{\text{mm'}}(p_1) - F_{\text{mm'}}(p_1) F_{\text{mm'}}(p_3),$$

with

$$T = \frac{(4\pi a_h)^2}{1 + |p - p'_f|^2 a_h^2/4} (2\pi)^4 \delta^{(4)}(p + p_1 - p_2 - p_3),$$

and

$$L = F_{\text{mm'}}(p_2) F_{\text{mm'}}(p_3) - F_{\text{mm'}}(p_1) F_{\text{mm'}}(p_3) - F_{\text{mm'}}(p_1) F_{\text{mm'}}(p_2) + 1.$$
term proportional to \( F(p) \) comes from \((\Sigma G F - F\Sigma A)\), which is more or less the same for either \( uu/dd \) terms or \( ud/du \) components since the contribution is always from the diagonal components of self-energy. Else terms are from \( \Sigma A \) which contain contributions from \( G_{0}^{u} \) or \( G_{0}^{d} \) for \( m = m' \) while only the contribution from \( G_{0}^{d} \) exists for \( m \neq m' \). As a result, for \( m \neq m' \) there is only one term \(~F\)\(^{3}\). The label of \( m \) and \( m' \) in this term can be written out directly by considering the leading-order scattering diagram shown in figure 3 where we use the vertex before the Hubbard–Stratonovich transformation with a four-fermion interaction.

Based on this analysis, the shortcut to the generalized Boltzmann equations can be summarized as follows:

1. Write out the traditional Boltzmann equation in terms of distribution function \( f(p, t) \) based on the transition rate, which counts the number of particles in a certain momentum.
2. Define the function \( F(p, t) = 1 \mp 2f(p, t) \) for fermions/bosons, translate the Boltzmann equation of \( f(p, t) \) to the equation of \( F(p, t) \). (This gives equation (20), where the factor of \( 1/4 \) is because of the factor of \( 2 \) in the definition of \( F \) above.)
3. Separate out the term proportional to \( F(p) \) in the traditional Boltzmann equation as \( LF(p) \). Write a term \((L[F_{u}]+L[F_{d}])F_{mm}(p)/2 \) in the generalized Boltzmann equation.
4. For the remaining terms in the traditional Boltzmann equation, only keep the term with largest number of \( F \) and add labels of \( mm' \) or \( m'm \) to them as discussed previously by considering the leading-order diagram.

One could straightforwardly show these rules lead to exactly the same equation as the one in equation (23). Also one could verify that it is true for other models studied in [25]. As an example, we will use them when analyzing the low-temperature chaotic behavior of the unitary Fermi gas.

**4. Quantum chaos in the high-temperature limit**

Before proceeding to solve the generalized Boltzmann equation for the unitary Fermi gas, we analyze some of the properties of generalized Boltzmann equations using the example of interacting fermions. The evolution of diagonal terms in the distribution matrix \( F_{uu} \) and \( F_{dd} \) only depends on themselves. The existence of the \( H \)–theorem [37] guarantees their relaxation to thermal equilibrium. For a small deviation, one could linearize the Boltzmann equations. We could defined the deviation from thermal equilibrium by

\[
F_{mm}(p) = F_{mm}^{0}(p) + \delta F_{mm}(p).
\]

One class of modes has both non-vanishing \( \delta F_{uu} \) and \( \delta F_{dd} \). The relaxation of them mean \( \delta F_{uu,\alpha}(t, p) = \delta F_{dd,\alpha}(t, p) = -2\delta F_{uu}(t, p) = -2\delta F_{dd}(t, p) \) with \( \lambda_{\alpha} > 0 \). At the same time, one could verify that the solution for \( \delta F_{uu} \) and \( \delta F_{dd} \) is given by

\[
\begin{align*}
\delta F_{uu,\alpha}(t, p) &= -2\delta F_{uu}(t, p)\exp(-\lambda_{\alpha} t), \\
\delta F_{dd,\alpha}(t, p) &= -2\delta F_{dd}(t, p)\exp(-\lambda_{\alpha} t).
\end{align*}
\]

Here \( \alpha \) labels different solutions in this class. As a result, all components of distribution matrices relax to the equilibrium value. Another possible class of solutions satisfies \( \delta F_{uu,\alpha}(t, p) = \delta F_{dd,\alpha}(t, p) = 0 \), which means the diagonal part is always in equilibrium. However, the off-diagonal part is non trivial and given by

\[
\begin{align*}
\delta F_{ud,\alpha}(t, p) &= -2\delta F_{ud}(t, p)\exp(-\lambda_{\alpha} t), \\
\delta F_{du,\alpha}(t, p) &= -2\delta F_{du}(t, p)\exp(-\lambda_{\alpha} t).
\end{align*}
\]

However, now \( \lambda_{\alpha} \) may become negative. For a general initial condition, the solution should be a superposition of all these eigenmodes. If the perturbation in \( F_{uu} \) and \( F_{dd} \) is larger than that of \( F_{uu} \) and \( F_{dd} \), such that the coefficients of terms in equation (25) are positive\(^{1}\), we expect an exponential deviation from thermal equilibrium for the off-diagonal components of the distribution matrix and the exponent, which is given by the negative \( \lambda_{\alpha} \) with the largest absolute value, and gives the Lyapunov exponent \( \lambda_{L} = \max_{\alpha} \lambda_{\alpha} \).

With this understanding, we proceed to solve the generalized Boltzmann equations. Since we are interested in the Lyapunov exponent, we just set the diagonal components to be at thermal equilibrium \( F_{uu}(p) = F_{dd}(p) = (1 - 2n_{F}(p^{2}/2, \mu))F_{0}(p) \) [24, 26]. We assume a spin-independent and spatial-homogeneous initial condition. Linearizing the equation, we get

\[
\begin{align*}
S_{mm}(p) &= \frac{1}{4} \int \frac{dp_{1}dp_{2}dp_{3}dp_{4}}{(2\pi)^{9}} \langle f(p_{1}, p_{2}) \rangle (-L^{0}(p_{1})\delta F_{mm'}(p_{1}), \\
& \quad - F_{m'm}(p_{1})F_{mm}(p_{2})\delta F_{mm'}(p_{3}), \\
& \quad - F_{m'm}(p_{1})\delta F_{mm}(p_{2}), \\
& \quad F_{mm}(p_{1}) - \delta F_{m'm}(p_{1})F_{mm}(p_{2})F_{mm'}(p_{3}) \rangle
\end{align*}
\]

(26)

with

\[
L^{0} = F^{0}(p_{1})F^{0}(p_{2}) - F^{0}(p_{1})F^{0}(p_{3}) - F^{0}(p_{1})F^{0}(p_{2}) + 1,
\]

(27)

and \( F_{0}(p) = -F_{0}(p) = \frac{1}{\cosh((p^{2}/2 - \mu)/2)} \). In an equivalent Bethe–Salpeter calculation, the first term in equation (26)

\(^{1}\) This depends on the short-time behavior of the perturbation. In a special set-up one could straight forwardly show this is true for free-fermion systems.
corresponds to the self-energy of Green’s functions, while other terms correspond to a convolution with kernels with one or two rungs.

Since the calculation is controlled in the high-temperature limit, we keep all terms to the leading order of \( r_t \).

terms correspond to a convolution with kernels with one or two rungs.

To this leading order, the Lyapunov exponent should be proportional to \( z \). In the unitary limit with \( a_z = \infty \), one expects \( \lambda_L \propto T^2 \). We show simplified expressions directly used in numerics in the appendix. In the weakly interacting limit, where \( a_z \to 0^+ \), \( \lambda_L \propto \frac{1}{a_z} T^2 \), our results are symmetric for \( a_z \to -a_z \), because we do not consider the distribution of bosons, which may be interpreted as considering physics in the upper branch. Numerical results for Lyapunov exponents as a function of scattering length are shown in figure 4, where we have determined \( \lambda_L \approx \frac{1}{a_z} T^2 \approx 21 \frac{a_z}{27} \) in the unitary limit, in which \( n \) is the density for a single-spin component. The result is parametrically smaller than the chaos bound. When we lower the temperature, \( z \) becomes larger. The approximation used here becomes unreliable when \( z \) is \( O(1) \), which is close to the superfluid transition.

We are interested in the combination \( v_T T_L \), which can be compared to the diffusion constant. In present case, the typical velocity is thermal velocity and we find \( v_T T_L \sim 14T^3/2T \) with \( v_T \approx 3T/2 \) to be the typical velocity of the system. It is interesting to compare this result with the energy diffusion constant. The heat conductivity \( \kappa \) is calculated in [34] by a variational method of Boltzmann equations in the high-temperature limit, and the result is found to be

\[ \kappa = \frac{225}{128 \pi^4} T^3/2. \]

By using Einstein’s relation \( D_E = \kappa/c_V \) with heat capacity \( c_V \), we find \( D_E \sim 0.33T^3/2 \). We will give arguments to this after studying the low-temperature case.

5. Quantum chaos in the low-temperature limit: effective field theory

The calculation with a microscopic-fermionic model is not controlled at low temperature. As a result, we choose to use the effective description in terms of phonons for the unitary Fermi gas in the low-temperature limit. This is reasonable since when the system is deep in the superfluid phase, the only low-energy excitation is phonon. Due to the (non-relativistic) conformal symmetry of the system, the effective theory for phonons can be determined up to several coefficients [22], which are then determined using certain approximations [40–43].

To the leading order in gradient expansion, the effective action for phonon field \( \phi \) is

\[ \mathcal{L}_\phi = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\nu^2}{2} (\nabla \phi)^2 - g_1 [(\partial_\mu \phi)^3 - 9v_T^2 \partial_\mu \phi (\nabla \phi)^2], \]

with \( v_T^2 = \frac{2\mu}{T} \) and \( g_1 = \frac{4\mu^2 \xi T^2}{378 \hbar \mu^2} \), where \( \mu = \xi_n T_F \) and \( T_F \) is determined by the density of the fermions in the non-interacting limit. We take the Bertsch parameter \( \xi_n \) to be \( \sim 0.4 \) here [44]. We will firstly set \( v_T = 1 \) and finally add them back by dimensional analysis.

In this action, the dispersion of the phonon is linear with \( \epsilon_k = v_T k \). However, whether a real process \( \phi(p_1) \rightarrow \phi(p_2) \) and \( \phi(p_3) \) (and its inverse) can occur will depend on the next-to-leading-order correction to linear dispersion. In [40] it is found that the correction is \( \epsilon_{k'} = v_T k + u k^2 \) with a positive \( u \). As a result, the splitting of a single phonon into two is allowed by conservation laws and at low-temperature physics should be dominated by such process. Here we keep to \( u^0 \) and all particles move in the same direction.

Now we would like to derive the generalized Boltzmann equation using our shortcut. Diagrams for the decay or formation of phonons are shown in figure 5. Summing up these contributions, for homogeneous perturbation the evolution of distribution function \( f(p, t) \) is described by [25, 34]

\[ \frac{\partial f(p, t)}{\partial t} = I^{(a)}[f, f] + I^{(b)}[f, f], \]

where the collision integrals are given by

\[ I^{(a)} = \int \frac{dp_1 dp_2}{8p_1 p_2 (2\pi)^3} \left| \mathcal{M}(p, p_1; p_2) \right|^2 \delta^{(4)}(p_2 - p_1 - p) \]

\[ \left[ -f(p_1) (1 + f(p_2)) f(p) \right] + (1 + f(p_1)) f(p_2) (1 + f(p)), \]

(31)
have the scattering amplitude $F(p)$. Each row cancels out in thermal equilibrium.

$$F^{(1)} = \frac{1}{2} \int \frac{d^4p_1 d^4p_2}{8pp_1p_2(2\pi)^2} |M(p_1, p_2; p)|^2 \delta^4(p_2 + p_1 - p)$$
\[= (1 + f(p_1))(1 + f(p_2))f(p) + f(p_1)f(p_2)(1 + f(p)), \]

(32)

where the difference of factor is from the symmetry factor. By using the fact that the momentum of all particles is parallel, we have the scattering amplitude $|M(p_1, p_2; p)|^2 = (488\delta_{pp_1p_2})^2$. Now we proceed to perform steps 2-4 of the shortcut, and the result is as follows:

$$\frac{\partial F(p, t)_{\mu\nu}}{\partial t} = St_{\mu\nu}(p),$$

(33)

with

$$St_{\mu\nu}(p) = St^{(1)}_{\mu\nu}(p) + St^{(2)}_{\mu\nu}(p)$$

(34)

$$St^{(1)}_{\mu\nu}(p) = \frac{1}{2} \int \frac{d^4p_1 d^4p_2}{8pp_1p_2(2\pi)^2} |M(p_1, p_2; p_2)|^2 \delta^4(p_2 - p_1 - p)$$
$$\times [(F(p_2) - F(p_1))F_{\mu\nu}(p) + F_{\mu\nu}(p_1)F_{\mu\nu}(p_2)].$$

(35)

$$St^{(2)}_{\mu\nu}(p) = \frac{1}{4} \int \frac{d^4p_1 d^4p_2}{8pp_1p_2(2\pi)^2} |M(p_1, p_2; p_2)|^2 \delta^4(p_2 - p_1 - p)$$
$$\times [(F(p_2) - F(p_1))F_{\mu\nu}(p) + F_{\mu\nu}(p_1)F_{\mu\nu}(p_2)].$$

(36)

The label of $m$ and $m'$ is read out from the one-loop self-energy diagram. For a real boson, in thermal equilibrium we have

$$F(0)(p) = 1 + 2n_B(p), \quad F_{\mu\nu}^0(p) = F_{\mu\nu}^0(\infty) = \frac{1}{\sinh(p/2T)}.$$  

(37)

One could check equations (35) and (36) vanish for such solutions. The $\delta$ function is easily integrated out and we solve the eigenvalue of linearized generalized Boltzmann equations numerically. Now we assume $F_{\mu\nu} = F_{\mu\nu}$ for all time. By power-counting and putting back $v_s$, the result should be $\lambda_L = C T^3/T^3$. The Lyapunov exponent is found to be $C \approx 9 \times 10^7$. The Lyapunov exponent in low temperature then decreases much quicker ($\sim T^2$) than the chaotic bound ($\sim T$).

Because the typical velocity scale $v_s$ does not depend on temperature, we have $v_s^2 \tau_L \sim T^3/T^2$. However, $D_L = \kappa/c_v \sim T^2/T^2$ because, as found in [34, 35], $\kappa \propto T^2$ and for phonon gas $c_v \propto T^3$. Again, we find that $D_L \ll v_s^2 \tau_L$ in the low-energy limit.

We attribute such results to momentum conservation, which reduces the efficiency of energy transport significantly. For example, in the low-energy limit, to the leading order the energy current by phonons is given by $J_E = \int v_s p |f(p)|$. However, if the dispersion is strictly linear $v_s = v_s$ then this vanishes due to momentum conservation, if we start from an initial state with vanishing total momentum. Previous studies of heat capacity due to phonons in the low-temperature limit of the unitary Fermi gas indeed consider the correction of dispersion beyond linear [34, 35]. Similarly, in the high-temperature case $J_E = \int v_s^2 p |f(p)|$. If we approximate the energy $p^2/2$ as $T^3/T^2$, which is the expectation from thermal distribution, it again vanishes, indicating a large part of the thermal energy can not lead to thermal transport. Nevertheless, this suppression is much larger for the phonon case, where we see a parametric difference between $D_L$ and $v_s^2 \tau_L$. We expect this to be a general mechanism for systems with quasi-particles.

6. Summary and outlooks

In this work we have studied the chaotic behavior of the unitary Fermi gas in the high- and low-temperature limit by using the generalized Boltzmann equations. In the high-temperature limit, we use the microscopic model and find $\lambda_L = 21 \frac{3}{T^2}$ in the unitary limit. In the low-temperature limit, we utilize the effective field theory of phonons where $\lambda_L = 9 \times 10^7 \frac{T}{T^3}$. By comparing with previous results, we find $D_L \ll v_s^2 \tau_L$ with a typical velocity scale $v_s$ and we explain this as a result of momentum conservation. We also propose a shortcut to the generalized Boltzmann equation by using the traditional Boltzmann equation.

One interesting question is whether we relate the calculation for the quantum Lyapunov exponent from the generalized Boltzmann equation to classical chaos, since in a classical system we can also write down the traditional Boltzmann equations and then modify them to write out the equation for chaos. This may provide greater understanding about the relation between classical and quantum chaos. Another problem relates to exploring the full-temperature regime in certain large-$N$ generalizations. It would be interesting if one could write our same matrix model, which is related to the unitary Fermi gas, where the Lyapunov exponent is not suppressed by $1/N$ factor.

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Appendix A. Relation between an OTOC and generalized Boltzmann equation

We could get some intuition of using some kind of Boltzmann equations to study the behavior of OTOC. To avoid possible singularity, one could assume some explicit cut-off or take some lattice model. To begin with, we consider a doubled system, prepared in a thermal field doubled state at inverse temperature $\beta$ [36]:

$$|\psi\rangle = \sum_n \exp(-\beta E_n/2) |n_u\rangle |n_d\rangle / Z_\beta. \quad (A1)$$

Here we use $u/d$ to label the different Hilbert spaces. Now we perturb the system by applying an operator $W_d(0)$ to the $u$ system, go give

$$|\tilde{\psi}(t = 0)\rangle = W_d(0)|\psi\rangle. \quad (A2)$$

Now we begin to evolve the system. Instead of using $H = H_u + H_d$, here we choose to use $H = H_u - H_d$, which makes $|\psi\rangle$ an eigenstate of the Hamiltonian. Then after time $t$ we do some measurements. We choose to measure the operator $V^\dagger_u V_u$ or the correlation function $V^\dagger_u V_u$. For $V^\dagger_u V_u$, we have

$$\langle \tilde{\psi}(t = 0)|V^\dagger_u V_u(t)|\tilde{\psi}(t = 0)\rangle = \langle W^\dagger(0)V^\dagger(t)V(t)W(0)\rangle_\beta. \quad (A3)$$

In the last equation, the measurement is done in a single system with temperature $\beta$. If we take the $V$ and $W$ to be the annihilation operator of particles, this is just an experiment of kicking out one atom and then study the evolution of particle density. In a standard semi-classical approximation [25], the evolution of this density distribution can be described by the traditional Boltzmann equation, with an initial value determined by its value at $t = 0$. This is the $uu$ part discussed in equation (20).

For the $u/d$ correlation function $V^\dagger_u V_d$, we have

$$\langle \tilde{\psi}(t = 0)|V^\dagger_u V_d(t)|\tilde{\psi}(t = 0)\rangle = \text{tr}[\hat{\rho} W^\dagger(0)V^\dagger(t)V(t)W(0)] / Z. \quad (A4)$$

This turns out to be an OTOC. Nevertheless, we expect both definitions lead to similar behavior to the same Lyapunov exponents in the long-time limit. One way to see this is by realizing the homogeneous part of the self-consistent equation for them in long-time limit should be the same. The similarity between equations (A3) and (A4) suggest an unified semi-classical equation may exist, which can describe the evolution of an OTOC from its initial condition at small $t$. This is the idea of the generalized Boltzmann equation. The scrambling of information, which is described by the vanishing of an OTOC, implies the TFD, after the thermalization of this doubled system, and will behave like a tensor product of two thermal ensembles locally.

Appendix B. Simplified expressions for generalized Boltzmann equations in the high-temperature limit

Here we give the simplified expressions, used directly in numerics, for generalized Boltzmann equations in the high-temperature limit. We assume that for mode with maximal Lyapunov exponent $F_{\text{max}}(p) = F_{\text{max}}(\rho)$ is rotational invariant and satisfies $F_{\text{ud}} = -F_{\text{du}}$, motivated by the unperturbed solution. In $St_{\text{ad}}[p, F]$, the term proportional to $\delta F_{\text{ad}}(p)$ is given by

$$\int_0^\infty \frac{2p^2 dp_1}{\pi} f(p, p_1)(-4e^{-\frac{p^2}{2}}) e^{-\frac{p_1^2}{2}} \delta F_{\text{ad}}(p), \quad (B1)$$

with

$$f(p, p_1) = \frac{2}{a^4 pp_1} [a^2(2|p - p_1| + a^2pp_1 - 2|p + p_1|]$$

$$- 8 \log \frac{4 + a^2|p - p_1|}{4 + a^2|p + p_1|}. \quad (B2)$$

The term proportional to $\delta F_{\text{ad}}(p_1)$ is given by

$$\int_0^\infty \frac{2p^2 dp_1}{\pi} f(p, p_1)(4e^{-\frac{p^2}{2}}) e^{-\frac{p_1^2}{2}} \delta F_{\text{ad}}(p_1), \quad (B3)$$

where we have used $F_{\text{ad}} = -F_{\text{du}}$. Finally, for the last term proportional to $F_{\text{d}d}(p_2)$, we have

$$\int \frac{p^2 dp_1}{\pi} f(p, p_1)(2e^{-\frac{p^2}{2}}) e^{-\frac{p_1^2}{2}} \delta F_{\text{ad}}(p_2)$$

$$\times \delta((p_2^2 - 2p^2 p_1 \cos \theta_1) - (p_1^2 + p_2^2 - 2pp_1 \cos \theta_2)). \quad (B4)$$

where we could further integrate over $t$ by solving the constraint imposed by the $\delta$ function. In numerics, we only find a single-positive eigenvalue and the Lyapunov exponent is well defined.

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