A fractional model of the diffusion equation and its analytical solution using Laplace transform

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Abstract In this study, the homotopy perturbation transform method (HPTM) is performed to give analytical solutions of the time fractional diffusion equation. The HPTM is a combined form of the Laplace transform and homotopy perturbation methods. The numerical solutions obtained by the proposed method indicate that the approach is easy to implement and accurate. These results reveal that the proposed method is very effective and simple in performing a solution to the fractional partial differential equation. A solution has been plotted for different values of \( \alpha \), and some numerical illustrations are given.

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1. Introduction

In the past century, notable contributions have been made to both the theory and application of fractional differential equations. These equations are increasingly used to model problems in research areas as diverse as dynamical systems, mechanical systems, control, chaos, chaos synchronization, continuous-time random walks, anomalous diffusive and subdiffusive systems, unification of diffusion and wave propagation phenomena and others. The advantage of the fractional order system is that it allows greater degrees of freedom in the model. An integer order differential operator is a local operator, whereas the fractional order differential operator is non-local, in the sense that it takes into account the fact that a future state not only depends upon the present state, but also upon the history of all its previous states. For this realistic property, fractional order systems have become popular. Another reason behind using fractional order derivatives is that these are naturally related to the systems with memory, which prevails for most physical and scientific system models. The book by Oldham and Spanier [1] has played a key role in the development of the subject. Some fundamental results related to solving fractional differential equations may be found in books of Miller and Ross [2], Podlubny [3] and Kilbas et al. [4].

Our concern in this work is to consider the numerical solution of the time-fractional diffusion equation. The free motion of the particle is modeled by the classical diffusion equation. The time fractional diffusion equation [5,6] under external force is represented as follows:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial}{\partial x} (F(x)u(x,t)),
\]

\[0 < \alpha \leq 1, \ D > 0,\]

(1.1)

where \( u(x,t) \) represent the probability density finding a particle at the point \( x \) in the time instant \( t \), the positive constant \( D \) depends on the temperature, the friction coefficient, the universal gas constant and finally on the Avogordo number, \( F(x) \) is the external force. In the present paper, we have considered two examples. In the first example, we consider Eq. (1.1) for \( D = 1 \) and \( F(x) = -x \). In the second example, we consider two dimensional parabolic equations with nonlocal boundary
conditions, which arise in many important problems of mathematical physics, such as thermo-elasticity, inverse problems, theory of thermal stresses, medical engineering, control theory, chemical diffusion, heat conduction processes, population dynamics, vibration problems, nuclear reactor dynamics, medical science, biochemistry and certain biological processes.

The homotopy perturbation method was first proposed by the Chinese mathematician, He [7–11]. Considerable research work has been conducted recently in applying the homotopy perturbation method to a class of linear and non-linear equations including Yildirim and Kocak [12], Yildirim and Gulkanat [13], Yildirim [14], Kumar et al. [15], Khan et al. [16–18], Golbabai and Sayevand [19] and Khan et al. [20]. The proposed method is a coupling of the Laplace transformation, the homotopy perturbation method and He's polynomials, and is mainly due to Ghorbani [21,22]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear partial differential equations using various methods combined with the Laplace transform. Among these are the Laplace decomposition methods by Khan and Hussain [23], Jafari et al. [24] and the Laplace homotopy perturbation method by Khan et al. [25,26] and Madani et al. [27].

In this article, a sincere attempt has been made to explore the analytical expressions of probability density functions of time fractional diffusion equations for different fractional Brownian motions, and also for standard motion. The numerical results of the problems are presented graphically.

2. Basic definitions of fractional calculus

In this section, we give some basic definitions and properties of the fractional calculus theory which shall be used in this paper:

Definition 2.1. A real function, \( f(t), t > 0, \) is said to be in space \( C_{\mu} \), \( \mu \in \mathbb{R} \) if there exists a real number, \( p > \mu \), such that \( f(t) = t^p f_1(t) \) where \( f_1(t) \in C(0, \infty) \), and if it is said to be in space \( C_n \) if, and only if, \( f^{(n)} \in C_{\mu}, n \in \mathbb{N} \).

Definition 2.2. The left sided Riemann–Liouville fractional integral operator of order \( \mu \geq 0 \), of function \( f \in C_{\mu} \), \( \alpha > -1 \) is defined [28,29] as:

\[
L^{\mu} f(t) = \begin{cases} 
\frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, & \mu > 0, t > 0, \\
0, & \mu = 0
\end{cases}
\]

where \( \Gamma(\cdot) \) is the well-known Gamma function.

Definition 2.3. The left side Caputo fractional derivative of \( \dot{f}, f \in C^m_{\alpha}, m \in N \cup \{0\} \) is defined by Podlubny [3] and Samko et al. [30] as:

\[
D^{\mu}_{\alpha} f(t) = \frac{\partial^{m} f(t)}{\partial t^{m}} = \begin{cases} 
\frac{1}{\Gamma(m-\mu)} \int_0^t (t-\tau)^{m-\mu-1} f(\tau) d\tau, & \mu < m, m \in N, \\
\frac{\partial^{m} f(t)}{\partial t^{m}}, & \mu = m.
\end{cases}
\]

Note that by [3,30],

(i) \( L^{\mu} f(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(x,t)}{t^{\mu+1}} d\tau, \mu > 0, t > 0, \)

(ii) \( D^{\mu}_{\alpha} f(x,t) = L^{\mu}_{1-\mu} f(x,t), \quad 0 < \mu < 1.\)

Definition 2.4. The Mittag-Leffler function, \( E_{\alpha}(z) \), with \( \alpha > 0 \), is defined by the following series representation, valid in the whole complex plane by Mainardi [31]:

\[
E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}.
\]

Definition 2.5. The Laplace transform of continuous (or almost piecewise continuous) function \( f(t) \) in \( [0, \infty) \) is defined as

\[
F(s) = L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt.
\]

where \( s \) is real or complex number.

Definition 2.6. The Laplace transform, \( L[f(t)] \), of the Riemann–Liouville fractional integral is defined by Miller and Ross [2] as:

\[
L^{\mu} f(t) = s^{-\mu} F(s).
\]

Definition 2.7. The Laplace transform, \( L[f(t)] \), of the Caputo fractional derivative is defined by Miller and Ross [2] as:

\[
L[D^{\mu}_{\alpha} f(t)] = s^{\mu} F(s) - \sum_{k=0}^{\mu-1} s^{\mu-k-1} f^{(k)}(0),
\]

\[\quad n - 1 < \alpha \leq n.\]

3. Fractional homotopy perturbation transform method

In order to elucidate the solution procedure of the fractional Laplace homotopy perturbation method, we consider the following nonlinear fractional differential equation:

\[
D^{\mu}_{\alpha} u(x,t) + R[x]u(x,t) + N[x]u(x,t) = q(x,t), \quad t > 0, \; x \in \mathbb{R}, \; 0 < \alpha \leq 1,
\]

\[
u(0) = h(x),
\]

where \( D^{\mu}_{\alpha} = \frac{\partial^{m}}{\partial t^{m}} \) is the linear operator in \( x \), \( N[x] \) is the general nonlinear operator in \( x \), and \( q(x,t) \) is continuous functions. Now, the methodology consists of applying the Laplace transform first on both sides of Eq. (3.1). Thus, we get:

\[
L[D^{\mu}_{\alpha} u(x,t)] + L[R[x]u(x,t) + N[x]u(x,t)] = L[q(x,t)],
\]

Now, using the development property of the Laplace transform, we have:

\[
L[u(x,t)] = s^{-1} h(x) - s^{-\alpha} L[q(x,t)]
\]

\[
+ s^{-\alpha} L[R[x]u(x,t) + N[x]u(x,t)].
\]

Operating the inverse Laplace transform on both sides in Eq. (3.3), we get:

\[
u(x,t) = G(x,t) - L^{-1}(s^{-\alpha} L[R[x]u(x,t) + N[x]u(x,t)]),
\]

where \( G(x,t) \) represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in \( p \), as given below:

\[
u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u),
\]

where the homotopy parameter, \( p \), is considered as a small parameter \((p \in [0,1])\). The nonlinear term can be decomposed as:

\[
\nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u),
\]
where $H_n$ are He’s polynomials of $u_0, u_1, u_2, \ldots, u_n$, which can be calculated by the following formula:

$$H_n(u_0, u_1, u_2, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N \left( \sum_{i=0}^{\infty} p^i u_i \right) \right],$$

where $n = 0, 1, 2, 3, \ldots$.

Substituting Eqs. (3.5) and (3.6) in Eq. (3.4) and using HPM by He [7–11], we get:

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) - p \times \left( L^{-1} \left[ s^{-\alpha} L \left[ R \sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right). \quad (3.7)$$

This is a coupling of the Laplace transform and homotopy perturbation methods using He’s polynomials. Now, equating the coefficient of corresponding power of $p$ on both sides, the following approximations are obtained as:

$$p^0 : u_0(x, t) = G(x, t),$$

$$p^n : u_n(x, t) = L^{-1} \left( s^{-\alpha} L \left[ R \sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right), \quad (n > 0, n \in \mathbb{N}).$$

Proceeding in this same manner, the rest of the components, $u_n(x, t)$, can be completely obtained, and the series solution is thus entirely determined.

Finally, we approximate the analytical solution, $u(x, t)$, by truncated series:

$$u(x, t) = \lim_{N \to \infty} \sum_{n=1}^{N} u_n(x, t). \quad (3.9)$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [32].

4. Numerical examples

In this section, two examples of time fractional diffusion equations are solved to demonstrate the performance and efficiency of the HPM with the coupling of the Laplace transform.

Example 4.1. In this example, we consider the following time-fractional diffusion equation as follows:

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = \frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial}{\partial x} (x u(x, t)), \quad (4.1)$$

$u(x, 0) = f(x).$

The methodology consists of applying the Laplace transform first on both sides of Eq. (4.1). Thus, we get:

$$L[D^\alpha_t u(x, t)] = L[u_{xx} + (x u)_x]. \quad (4.2)$$

Using the differentiation property of the Laplace transform in Eq. (4.2), we get:

$$L[u(x, t)] = s^{-\alpha} f(x) + s^{-\alpha} L[u_{xx} + (x u)_x]. \quad (4.3)$$

Applying the inverse Laplace transform on both sides, we get:

$$u(x, t) = f(x) + L^{-1} \left( s^{-\alpha} L[u_{xx} + (x u)_x] \right). \quad (4.4)$$

By applying the aforesaid homotopy perturbation method, we have:

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = f(x) + p \left( L^{-1} \left( s^{-\alpha} L[u_{xx} + (x u)_x] \right) \right). \quad (4.5)$$

Equating the coefficient of the like power of $p$ on both sides in Eq. (4.5), we get:

$$p^0 : u_0(x, t) = f(x),$$

$$p^n : u_n(x, t) = L^{-1} \left( s^{-\alpha} L[u_{xx} + (x u)_x] \right), \quad (n > 0, n \in \mathbb{N}) \quad (4.6)$$

Case 4.1.1. Let us consider $f(x) = 1$. By applying the aforesaid method in the first iteration, we have:

$$u_0(x, t) = 1.$$

The subsequent terms are:

$$u_1(x, t) = \frac{t^{\alpha}}{\Gamma(\alpha + 1)}, \quad u_2(x, t) = \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)},$$

$$u_3(x, t) = \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}, \ldots, \quad u_n(x, t) = \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}.$$

Using the above terms, the solution, $u(x, t)$, is:

$$u(x, t) = \lim_{p \to 0} \sum_{n=0}^{\infty} p^n u_n(x, t) = 1 + \frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} = E_{\alpha}(t^\alpha). \quad (4.7)$$

If $\alpha = 1/2$, then we get:

$$u(x, t) = E_{1/2}(\sqrt{t}).$$

The above result is in complete agreement with Das [6].

Case 4.1.2. Now, by considering $u_0(x, t) = f(x) = x$, we get:

$$u_1(x, t) = \frac{x(2t^\alpha)}{\Gamma(\alpha + 1)}, \quad u_2(x, t) = \frac{x(2t^\alpha)^2}{\Gamma(2\alpha + 1)},$$

$$u_3(x, t) = \frac{x(2t^\alpha)^3}{\Gamma(3\alpha + 1)}, \ldots, \quad u_n(x, t) = \frac{x(2t^\alpha)^n}{\Gamma(n\alpha + 1)}.$$

Therefore, the exact solution is given as:

$$u(x, t) = x \left( 1 + \frac{(2t^\alpha)}{\Gamma(\alpha + 1)} + \frac{(2t^\alpha)^2}{\Gamma(2\alpha + 1)} + \frac{(2t^\alpha)^3}{\Gamma(3\alpha + 1)} + \cdots + \frac{(2t^\alpha)^n}{\Gamma(n\alpha + 1)} + \cdots \right) = x \sum_{k=0}^{\infty} \frac{(2t^\alpha)^k}{\Gamma(k\alpha + 1)} = x E_{\alpha}(2t^\alpha).$$

For fractional derivatives, $\alpha = 1/2$, we get:

$$u(x, t) = x E_{1/2}(2\sqrt{t}).$$

Again, we note that the result obtained by LHPM is the same as that of Ray and Bera [5], and Das [6].
Case 4.1.3. When \( f(x) = x^2 \), then we get:

\[
    u_0(x, t) = x^2,
    u_1(x, t) = (2 + 3x^2) \frac{t^\alpha}{\Gamma(\alpha + 1)},
    u_2(x, t) = (8 + 9x^2) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)},
    u_3(x, t) = (26 + 27x^2) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)}, \ldots
    u_n(x, t) = [x^2 + (1 + x^2)(3^n - 1)] \frac{p_n^\alpha}{\Gamma(n\alpha + 1)}.
\]

Therefore, the exact solution for a given case is:

\[
    u(x, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x, t) = x^2 + (2 + 3x^2) \frac{t^\alpha}{\Gamma(\alpha + 1)} + (8 + 9x^2) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots + (x^2 + (1 + x^2)(3^n - 1)) \frac{p_n^\alpha}{\Gamma(n\alpha + 1)} + \cdots
\]

\[
    = \sum_{k=0}^{\infty} \frac{R_k(t)}{\Gamma(\alpha k + 1)} = E_0 (Rt^\alpha),
\]

where \( R_k = x^2 + (1 + x^2)(3^k - 1) \).

The above result is in complete agreement with Das [6] for the value of \( \alpha = 1/2 \). The LHPP solution is almost valid for a large wide range of times, which shows that the present method can solve a fractional diffusion equation with a high degree of accuracy.

Example 4.2. In this example, we consider the following two dimensional diffusion equation [33]:

\[
    \frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} \equiv \frac{\partial^\alpha u(x, y, t)}{\partial x^\alpha} + \frac{\partial^\alpha u(x, y, t)}{\partial y^\alpha},
\]

\[
    0 < \alpha \leq 1, \quad a \leq x \leq b, \quad c \leq y \leq d,
\]

with initial condition \( u(x, y, 0) = \sin x \sin y \), which is easily seen to have the exact solution, \( u(x, y, t) = e^{-2t} \sin x \sin y \).

Taking the Laplace transform on both sides in Eq. (4.8), we get:

\[
    L[u(x, y, t)] = \sin x \sin y + s \cdot L[(D_x^\alpha + D_y^\alpha) u].
\]

(4.9)

Applying the inverse Laplace transform on both sides, we get:

\[
    u(x, y, t) = \sin x \sin y + L^{-1} \left[ s^{-\alpha}L[(D_x^\alpha + D_y^\alpha) u] \right].
\]

By applying the aforesaid homotopy perturbation method, we have:

\[
    \sum_{n=0}^{\infty} p^n u_n(x, y, t) = \sin x \sin y + p \left( \sum_{n=0}^{\infty} p^n L^{-1} \left[ s^{-\alpha}L[(D_x^\alpha + D_y^\alpha) u] \right] \right).
\]

(4.10)

Equating the corresponding power of \( p \) on both sides, we get:

\[
    p^0 : u_0(x, y, t) = \sin x \sin y,
    p^1 : u_1(x, y, t) = L^{-1} \left[ s^{-\alpha}L[(D_x^\alpha + D_y^\alpha) u_0(x, y, t)] \right] = -2 \sin x \sin y \frac{t^\alpha}{\Gamma(\alpha + 1)},
    p^2 : u_2(x, y, t) = L^{-1} \left[ s^{-\alpha}L[(D_x^\alpha + D_y^\alpha) u_1(x, y, t)] \right] = 4 \sin x \sin y \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)},
    p^3 : u_3(x, y, t) = L^{-1} \left[ s^{-\alpha}L[(D_x^\alpha + D_y^\alpha) u_2(x, y, t)] \right] = -8 \sin x \sin y \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)},
\]

\[
    \vdots
    p^n : u_n(x, y, t) = L^{-1} \left[ s^{-\alpha}L[(D_x^\alpha + D_y^\alpha) u_{n-1}(x, y, t)] \right] = (-2)^n \sin x \sin y \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)}.
\]

Using the above terms, the solution, \( u(x, y, t) \), is:

\[
    u(x, y, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x, y, t) = \sin x \sin y \left( 1 - \frac{2}{\Gamma(\alpha + 1)} + 4 \frac{t^\alpha}{\Gamma(2\alpha + 1)} + \cdots + \frac{(2\alpha)!}{\Gamma(3\alpha + 1)} + \cdots \right),
\]

\[
    = \sin x \sin y E_0 \left( -2t^\alpha \right).
\]

Now, for the standard case, i.e. for \( \alpha = 1 \), this series has the closed form of the solution, \( u(x, y, t) = e^{-2t} \sin x \sin y \), which is an exact solution of the given two dimensional diffusion equation (4.8) for \( \alpha = 1 \).

5. Numerical result and discussion

In this section, the evaluation results of the solution for Eq. (4.1) are depicted through Figures 1–7 for various values of \( x \) and \( t \) at different values of \( \alpha \). Figure 1 shows the behavior of the approximate solution obtained by HPTM for Case 4.1.1 of Example 4.1. It is seen from Figure 1 that displacement \( u(x, t) \) increases with increases in \( t \) for different values of \( \alpha \). Figures 2–4 shows the evolution result for Case 4.1.2 when we are taking \( f(x) = x \). Figure 2 shows the three dimensional probability density function for the Case 4.1.2 at \( \alpha = 1/2 \). Figure 3 represents the two dimensional displacement, \( u(x, t) \), at the constant value of \( x = 1 \). It is seen from both Figures 2 and 3 that displacement \( u(x, t) \) increases with increases in \( x \) and \( t \). Both Figures 2 and 3 have complete agreement with the result of Das [6]. Figure 4 shows the behavior of the approximate solution obtained by HPTM for Case 4.1.2 of Example 4.1. It is seen from Figure 4 that displacement \( u(x, t) \) increases with increases in \( t \) for different values of \( \alpha \).

Figures 5–7 show the behavior of the approximate solution obtained by HPTM for Case 4.1.3 of Example 4.1. We can see a similar behavior in the approximate solution, \( u(x, t) \), for Case 3 of Example 4.1. as in Case 4.1.1. Figure 5 represents the three dimensional displacement for the value of \( f(x) = x^2 \) at the constant value of \( \alpha = 1/2 \). Figure 6 represent the two dimensional displacement, \( u(x, t) \), at the constant value of \( x = 1 \). It is seen from both Figures 5 and 6 that displacement \( u(x, t) \) increases with increases in \( x \) and \( t \). The numerical results in all cases have complete agreement with Ray and Bera [5] and Das [6]. It is seen from Figure 7 that displacement \( u(x, t) \) increases with increases in \( t \) for different values of \( \alpha \).

Figures 8–10 show the evaluation results of the approximate solution obtained by the HPTM for Example 4.2. Figures 8 and 9 show the comparison between the exact solution.
Figure 1: Plot of the solution $u(x, t)$ with respect to $x$ and time $t$ at $\alpha = 1/2$ for Case 1 in Example 4.1.

Figure 2: Plot of the solution $u(x, t)$ vs. time $t$ and at $x = 1$ for Case 1 in Example 4.1.

Figure 3: Plot of $u(x, t)$ vs. time $t$ at $x = 1$ for different value of $\alpha$ for Case 1 in Example 4.1.

Figure 4: Plot of the solution $u(x, t)$ with respect to $x$ and time $t$ at $\alpha = 1/2$ for Case 2 in Example 4.1.

Figure 5: Plot of the solution $u(x, t)$ vs. time $t$ at $x = 1$ for Case 2 in Example 4.1.

Figure 6: Plot of $u(x, t)$ vs. time $t$ at $x = 1$ for different value of $\alpha$ for Case 2 in Example 4.1.

Figure 7: Plot of $u(x, t)$ vs. time $t$ at $x = 1$ for different value of $\alpha$ for Case 3 in Example 4.1.

and approximate solution obtained by the present method. Figure 10 shows the behavior of the approximate solution, $u(x, t)$, for different values of $\alpha = 0.7, 0.8, 0.9$, and for the standard two dimensional time fractional diffusion equation, i.e. at the value $\alpha = 1$ of Eq. (4.8). It is seen from Figure 10 that the solution obtained by the present method decreases very rapidly with the increases in $t$ at the value of $x = y = 1$.

6. Conclusion

This paper develops an effective modification of the homotopy perturbation method, which is coupled with the Laplace transform and He’s polynomials, and its validity is studied in a wide range with two examples of time fractional diffusion equations. From the results, it is clear that the Laplace homotopy perturbation method yields very accurate approximate solutions using only a few iterates. Thus, it is
concluded that the homotopy perturbation transform method becomes more powerful and efficient than before in finding analytical, as well as numerical, solutions for a wide class of linear and nonlinear fractional differential equations. It provides more realistic series solutions that converge very rapidly in real physical problems.

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