Supporting Information

for

Nonequilibrium Kondo effect in a graphene-coupled quantum dot in the presence of a magnetic field

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Derivation of the QD retarded Green’s function, of self-energies and verification of the developed analytical method
Appendix A: Green’s function of the dot

The retarded Green’s function is defined as \( \langle\langle A(t)|B(0)\rangle\rangle_r = -i\theta(t)\langle\{A(t),B(0)\}\rangle \), where \( A \) and \( B \) are fermionic operators and \( \theta(t) \) is the Heaviside function [1-3]. Its Fourier transform reads \( \langle\langle A|B\rangle\rangle_{\omega}^r \). The equation of motion of the retarded Green’s function in energy space is \( \omega^+ \langle\langle A|B\rangle\rangle_{\omega}^r + \langle\langle [H,A]|B\rangle\rangle_{\omega}^r = \langle\{A,B\}\rangle \), where \( \omega^+ = \omega + i\delta \), with \( \delta \) being a positive infinitesimal [1,3]. We can define the dot retarded Green’s function as \( G_{d\sigma}^r(\omega) = \langle\langle d_\sigma|d_\sigma^\dagger\rangle\rangle_{\omega}^r \) when replacing \( A(B) \) by \( d_\sigma(d_\sigma^\dagger) \) in the above notations. The equation of motion for \( G_{d\sigma}^r(\omega) \) is:

\[
(\omega^+ - \epsilon_{d\sigma})G_{d\sigma}^r(\omega) = 1 + U\langle\langle d_\sigma n_\sigma^\dagger\rangle\rangle_{\omega}^r + \sum_{\alpha,s,k_{\text{c}}} \int_{k_{\text{c}}}^{+k_{\text{c}}} dk V(k)\langle\langle c_{\alpha\sigma s^\dagger}d_\sigma^\dagger\rangle\rangle_{\omega}^r.
\]  

(S1)

The equation for the term \( \langle\langle c_{\alpha\sigma s^\dagger}d_\sigma^\dagger\rangle\rangle_{\omega}^r \) reads:

\[
\langle\langle c_{\alpha\sigma s^\dagger}d_\sigma^\dagger\rangle\rangle_{\omega}^r = \frac{V(k)}{\omega^+ - \epsilon_k} \langle\langle d_\sigma|d_\sigma^\dagger\rangle\rangle_{\omega}^r.
\]  

(S2)

We define the \( \Sigma_0^r(\omega) \) self-energy as:

\[
\Sigma_0^r(\omega) = \sum_{\alpha,s,k_{\text{c}}}^{+k_{\text{c}}} \int_{k_{\text{c}}}^{k_{\text{c}}} dk V(k)\omega^+ - \epsilon_k = -2\eta \left( \omega \ln \left| \frac{D^2 - \omega^2}{\omega^2} \right| + i\pi |\omega| \theta(D - |\omega|) \right),
\]  

(S3)

where we used the Eq. (S31) and introduced a dimensionless parameter as \( \eta = 2(\bar{V}/\hbar v_F)^2 \). By substituting Eq. (S2) into Eq. (S1) with Eq. (S3) we have:

\[
(\omega^+ - \epsilon_{d\sigma} - \Sigma_0^r(\omega))G_{d\sigma}^r(\omega) = 1 + U\langle\langle d_\sigma n_\sigma^\dagger\rangle\rangle_{\omega}^r.
\]  

(S4)
The equation of motion for \( \langle \langle \langle d_\sigma n_\bar{\sigma} | d_\bar{\sigma}^\dagger \rangle \rangle \rangle \) is:

\[
(\omega^+ - \varepsilon_{d\sigma} - U) \langle \langle \langle d_\sigma n_\bar{\sigma} | d_\bar{\sigma}^\dagger \rangle \rangle \rangle_{\omega} = \langle n_\bar{\sigma} \rangle \\
+ \sum_{\alpha, \bar{\sigma}} \int_{-k_c}^{+k_c} dkV(k) \left[ \langle \langle c_{\alpha \bar{\sigma}} d_\alpha | d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} + \langle \langle d_\alpha^\dagger c_{\alpha \bar{\sigma}} d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} - \langle \langle c_{\alpha \bar{\sigma}}^\dagger d_\alpha^\dagger | d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} \right].
\] (S5)

To determine the Green’s function of the quantum dot, we need to calculate the new higher-order correlation functions that appear on the right-hand side of Eq. (S5). The equations of motion for these terms are expressed as:

\[
\Omega^{(0)}_{k\bar{\sigma}}(\omega) \langle \langle \langle c_{\alpha \bar{\sigma}} n_\bar{\sigma} | d_\bar{\sigma}^\dagger \rangle \rangle \rangle_{\omega} = V(k) \langle \langle \langle d_\sigma n_\bar{\sigma} | d_\bar{\sigma}^\dagger \rangle \rangle \rangle_{\omega} \\
- \sum_{\alpha', \bar{\sigma}'} \int_{-k_c}^{+k_c} dk'V(k') \left[ \langle \langle c_{\alpha \bar{\sigma}} c_{\alpha' \bar{\sigma}'} d_\alpha | d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} + \langle \langle c_{\alpha \bar{\sigma}}^\dagger c_{\alpha' \bar{\sigma}'}^\dagger d_\bar{\sigma}^\dagger | d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} \right],
\] (S6)

\[
\Omega^{(1)}_{k\bar{\sigma}}(\omega) \langle \langle \langle d_\sigma^\dagger c_{\alpha \bar{\sigma}} d_\alpha | d_\bar{\sigma}^\dagger \rangle \rangle \rangle_{\omega} = \langle \langle d_\sigma^\dagger c_{\alpha \bar{\sigma}} \rangle \rangle_{\omega} + V(k) \langle \langle \langle d_\sigma n_\bar{\sigma} | d_\bar{\sigma}^\dagger \rangle \rangle \rangle_{\omega} \\
+ \sum_{\alpha', \bar{\sigma}'} \int_{-k_c}^{+k_c} dk'V(k') \left[ \langle \langle d_\sigma^\dagger c_{\alpha \bar{\sigma}} c_{\alpha' \bar{\sigma}'} d_\alpha | d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} - \langle \langle c_{\alpha' \bar{\sigma}'}^\dagger c_{\alpha \bar{\sigma}}^\dagger d_\bar{\sigma}^\dagger | d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} \right],
\] (S7)

\[
\Omega^{(2)}_{k\bar{\sigma}}(\omega) \langle \langle \langle c_{\alpha \bar{\sigma}}^\dagger d_\alpha | d_\bar{\sigma}^\dagger \rangle \rangle \rangle_{\omega} = \langle \langle c_{\alpha \bar{\sigma}}^\dagger d_\alpha \rangle \rangle_{\omega} - V(k) \langle \langle \langle d_\sigma n_\bar{\sigma} | d_\bar{\sigma}^\dagger \rangle \rangle \rangle_{\omega} \\
+ \sum_{\alpha', \bar{\sigma}'} \int_{-k_c}^{+k_c} dk'V(k') \left[ \langle \langle c_{\alpha \bar{\sigma}}^\dagger c_{\alpha' \bar{\sigma}'} d_\alpha | d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} - \langle \langle c_{\alpha \bar{\sigma}}^\dagger c_{\alpha' \bar{\sigma}'}^\dagger d_\bar{\sigma}^\dagger | d_\bar{\sigma}^\dagger \rangle \rangle_{\omega} \right],
\] (S8)

where the following notations are used: \( \Omega^{(0)}_{k\bar{\sigma}}(\omega) \equiv \omega^+ - \varepsilon_k, \Omega^{(1)}_{k\bar{\sigma}}(\omega) \equiv \omega^+ - \varepsilon_k + (\varepsilon_{d\bar{\sigma}} - \varepsilon_{d\sigma}) \) and \( \Omega^{(2)}_{k\bar{\sigma}}(\omega) \equiv \omega^+ - \varepsilon_k - (\varepsilon_{d\bar{\sigma}} + \varepsilon_{d\sigma}) - U \). To obtain an analytical formula for the Green’s function of the quantum dot, we have to truncate the higher-order correlation functions that appear in Eqs. (S6)-(S8) by using an approximation method. In order to do this, we apply the broadly used Lacroix decoupling scheme [2] that leads to close the infinite number of higher-order correlation functions. By performing the approximations and substituting the resulting equations into the Eq. (S5) for the
Green’s function of the quantum dot, we obtain:

\[
G_{d\sigma}^{\Omega}(\omega) = \frac{\Pi_{\sigma}^{(1)}(\omega) + U \left[ \langle n_{\bar{\sigma}} \rangle + \Pi_{\sigma}^{(2)}(\omega) \right]}{\Pi_{\sigma}^{(1)}(\omega) \left[ \omega - \epsilon_{d\sigma} - \Sigma_{0}^{\tau}(\omega) \right] - U \Pi_{\sigma}^{(3)}(\omega)},
\]

where we introduced the notations:

\[
\Pi_{\sigma}^{(1)}(\omega) = \omega - \epsilon_{d\sigma} - \Sigma_{0}^{\tau}(\omega) - \sum_{\alpha,s}^{+k_{c}} \int d\omega(k)^{2} \left[ \frac{1}{\Omega_{k\sigma}^{(1)}} + \frac{1}{\Omega_{k\sigma}^{(2)}} \right],
\]

\[
\Pi_{\sigma}^{(2)}(\omega) = \sum_{\alpha,s}^{+k_{c}} \int d\omega(k)^{2} \left[ \frac{\langle d_{\sigma}^{\dagger} c_{\alpha \sigma}^{\dagger} \rangle}{\Omega_{k\sigma}^{(1)}} - \frac{\langle c_{\alpha \sigma}^{\dagger} d_{\sigma} \rangle}{\Omega_{k\sigma}^{(2)}} \right],
\]

\[
\Pi_{\sigma}^{(3)}(\omega) = \Sigma_{0}^{\tau}(\omega) \Pi_{\sigma}^{(2)}(\omega)
\]

The average values of the mixing operators in the above relations are treated non-self-consistently, following the Meir approximation [4], i.e. \( \langle d_{\sigma}^{\dagger} c_{\alpha \sigma}^{\dagger} \rangle = \langle c_{\alpha \sigma}^{\dagger} d_{\sigma} \rangle \approx 0 \) and \( \langle c_{\alpha' \sigma'}^{\dagger} c_{\alpha \sigma}^{\dagger} c_{\alpha \sigma} c_{\alpha' \sigma'} \rangle \approx f_{\alpha}(\epsilon_{k}) \delta_{\alpha \alpha'} \delta_{\sigma \sigma'} \delta(k - k'). \) In this case \( \Pi_{\sigma}^{(2)}(\omega) \approx 0, \) the retarded Green’s function for the quantum dot reduces:

\[
G_{d\sigma}^{\Omega}(\omega) = \frac{1 - \langle n_{\bar{\sigma}} \rangle}{\omega - \epsilon_{d\sigma} - \Sigma_{0}^{\tau}(\omega) + U \left[ \sum_{\omega} \left( \omega_{\Sigma_{1\sigma}(\omega) + \Sigma_{2\sigma}(\omega)}^{\Sigma_{1\sigma}(\omega) + \Sigma_{2\sigma}(\omega)} \right) \right] + \langle n_{\sigma} \rangle} + \frac{\langle n_{\sigma} \rangle}{\omega - \epsilon_{d\sigma} - \Sigma_{0}^{\tau}(\omega) - U \sum_{\omega} \left( \omega_{\Sigma_{1\sigma}(\omega) + \Sigma_{2\sigma}(\omega)}^{\Sigma_{1\sigma}(\omega) + \Sigma_{2\sigma}(\omega)} \right) + \langle n_{\sigma} \rangle}.
\]

where we defined the following self-energies by the relations:

\[
\Sigma_{i\sigma}(\omega) = \sum_{\alpha,s}^{+k_{c}} \int d\omega(k)^{2} \frac{V(k)^{2}}{\Omega_{k\sigma}^{(i)}} = -2 \eta \left[ \omega_{\sigma} \ln \left| \frac{D^{2} - \omega_{\sigma}^{2}}{\omega_{\sigma}^{2}} \right| + i \pi \omega_{\sigma} |D| \right], \quad i = 1, 2,
\]
with shorthand notations: \( \omega_{1\sigma} = \omega - \sigma \Delta \varepsilon_d \) and \( \omega_{2\sigma} = \omega - 2\varepsilon_d - U \). Therefore, we have \( \Sigma_{3\sigma}(\omega) = \sum_{\alpha,s} \int_{k_c} \frac{d^3 k}{(2\pi)^3} \delta \Omega_{3\sigma}^{(1)}(\omega) f_{\alpha}(\varepsilon_k) = \sum_{\alpha} \Sigma_{3\sigma}(\omega) \) and \( \Sigma_{4\sigma}(\omega) = \sum_{\alpha,s} \int_{k_c} \frac{d^3 k}{(2\pi)^3} \delta \Omega_{4\sigma}^{(2)}(\omega) f_{\alpha}(\varepsilon_k) = \sum_{\alpha} \Sigma_{4\sigma}(\omega) \) with solutions:

\[
\Sigma_{3\sigma}^{(-)}(\omega) = \eta D + \omega_{1\sigma} \ln \left| \frac{\mu_{\alpha} - \omega_{1\sigma} - 2T}{D + \omega_{1\sigma}} \right| - R_{3\sigma}^\alpha(\omega) - iJ_{3\sigma}^\alpha(\omega),
\]

\[
\Sigma_{3\sigma}^{(0)}(\omega) = \eta \left\{ D - T + \omega_{1\sigma} \ln \left| \frac{\omega_{1\sigma} + 2T}{D + \omega_{1\sigma}} \right| \right.
\]
\[
\left. - \frac{\omega_{1\sigma}}{2} \left( 1 - \frac{\omega_{1\sigma}}{2T} \right) \ln \left| \frac{(\omega_{1\sigma} + 2T)(\omega_{1\sigma} - 2T)}{\omega_{1\sigma}^2} \right| \right) - iJ_{3\sigma}^\alpha(\omega),
\]

\[
\Sigma_{3\sigma}^{(+)}(\omega) = \eta \left\{ D - \omega_{1\sigma} \ln \left| \frac{(D + \omega_{1\sigma})(\mu_{\alpha} - \omega_{1\sigma} - 2T)}{\omega_{1\sigma}^2} \right| + R_{3\sigma}^\alpha(\omega) - iJ_{3\sigma}^\alpha(\omega),
\]

where we introduced the following relations:

\[
R_{3\sigma}^\alpha(\omega) = \omega_{1\sigma} - \mu_{\alpha} + \frac{\omega_{1\sigma}}{2} \left( 1 + \frac{\mu_{\alpha} - \omega_{1\sigma} - 2T}{2T} \right) \ln \left| \frac{\mu_{\alpha} - \omega_{1\sigma} - 2T}{\mu_{\alpha} - \omega_{1\sigma} + 2T} \right|,
\]

\[
J_{3\sigma}^\alpha(\omega) = \frac{\pi}{2} \omega_{1\sigma} \left[ 1 + \tanh \left( \frac{\mu_{\alpha} - \omega_{1\sigma}}{2T} \right) \right] \theta(D + \omega_{1\sigma})
\]

and:

\[
\Sigma_{4\sigma}^{(-)}(\omega) = -\eta \left\{ D + \omega_{2\sigma} \ln \left| \frac{D - \omega_{2\sigma}}{\mu_{\alpha} + \omega_{2\sigma} - 2T} \right| + R_{4\sigma}^\alpha(\omega) + iJ_{4\sigma}^\alpha(\omega),
\]

\[
\Sigma_{4\sigma}^{(0)}(\omega) = -\eta \left\{ D - T + \omega_{2\sigma} \ln \left| \frac{D - \omega_{2\sigma}}{\omega_{2\sigma} - 2T} \right| \right.
\]
\[
\left. + \frac{\omega_{2\sigma}}{2} \left( 1 + \frac{\omega_{2\sigma}}{2T} \right) \ln \left| \frac{(\omega_{2\sigma} - 2T)(\omega_{2\sigma} + 2T)}{\omega_{2\sigma}^2} \right| \right) + iJ_{4\sigma}^\alpha(\omega),
\]

\[
\Sigma_{4\sigma}^{(+)}(\omega) = -\eta \left\{ D + \omega_{2\sigma} \ln \left| \frac{(D - \omega_{2\sigma})(\mu_{\alpha} + \omega_{2\sigma} - 2T)}{\omega_{2\sigma}^2} \right| - R_{4\sigma}^\alpha(\omega) + iJ_{4\sigma}^\alpha(\omega),
\]

where we have:

\[
R_{4\sigma}^\alpha(\omega) = \omega_{2\sigma} + \mu_{\alpha} + \frac{\omega_{2\sigma}}{2} \left( 1 + \frac{\mu_{\alpha} + \omega_{2\sigma} - 2T}{2T} \right) \ln \left| \frac{\mu_{\alpha} + \omega_{2\sigma} - 2T}{\mu_{\alpha} + \omega_{2\sigma} + 2T} \right|,
\]

S4
\[ J_{4\sigma}^\alpha(\omega) = \frac{\pi}{2}|\omega_{2\sigma}| \left[ 1 + \tanh \left( \frac{\mu_\alpha + \omega_{2\sigma}}{2T} \right) \right] \theta(D - \omega_{2\sigma}), \]  

(S24)

where \( \gamma = -, 0 \) and \( + \) correspond to the cases: \(-D < \mu_\alpha \lesssim 0, \mu_\alpha = 0 \) and \( 0 \lesssim \mu_\alpha < D \).

**Appendix B: Derivation of the self-energies at finite temperature**

In this section, we show a simple method to deduce self-energies \( \Sigma_{3\sigma}(\omega) \) and \( \Sigma_{4\sigma}(\omega) \) for finite temperatures. We introduce the self-energies by:

\[
\Sigma_{3\sigma}(\omega) = \sum_{\alpha,s}^{+k_c} \int d\epsilon \frac{V(k)^2}{\Omega(k\sigma)} f(\epsilon - \mu_\alpha) = \sum_{\alpha} \Sigma_{3\sigma}^{(\gamma)}(\omega) \tag{S25}
\]

and:

\[
\Sigma_{4\sigma}(\omega) = \sum_{\alpha,s}^{+k_c} \int d\epsilon \frac{V(k)^2}{\Omega(k\sigma)} f(\epsilon - \mu_\alpha) = \sum_{\alpha} \Sigma_{4\sigma}^{(\gamma)}(\omega), \tag{S26}
\]

with:

\[
\Sigma_{3\sigma}^{(\gamma)}(\omega) = \eta \int_{-D}^{+D} d\epsilon \frac{|\epsilon| f(\epsilon - \mu_\alpha)}{\epsilon + \omega_{1\sigma} + i\delta} = \eta \Sigma_{3\sigma}^{(\gamma)}(\omega) \tag{S27}
\]

and:

\[
\Sigma_{4\sigma}^{(\gamma)}(\omega) = \eta \int_{-D}^{+D} d\epsilon \frac{|\epsilon| f(\epsilon - \mu_\alpha)}{\epsilon + \omega_{2\sigma} + i\delta} = \eta \Sigma_{4\sigma}^{(\gamma)}(\omega), \tag{S28}
\]

where we introduced the notation \( \epsilon = \hbar v_F k \). Note that \( \Sigma_{3\sigma}^{(\gamma)}(\omega) \) and \( \Sigma_{4\sigma}^{(\gamma)}(\omega) \) explicitly depend on \( \omega \) through \( \omega_{1\sigma} \) and \( \omega_{2\sigma} \), respectively. Furthermore, by changing the variable \( \beta(\epsilon - \mu_\alpha) = x \) where \( \beta = 1/T \), then the Fermi function \( f(\epsilon - \mu_\alpha) \) can be expressed as:

\[
f(x) = \frac{1}{2} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right], \tag{S29}
\]
where \( \tanh(x/2) \) has the properties [5]:

\[
\tanh\left(\frac{x}{2}\right) \approx \begin{cases} 
-1 & \text{if } x < -2 \\
\frac{x}{2} & \text{if } -2 < x < 2 \\
+1 & \text{if } x > 2.
\end{cases}
\]  

(S30)

The following calculations will be based on the properties of function \( \tanh(x/2) \) outlined in Eq. (S30). We also use the Dirac identity [6]:

\[
\frac{1}{x \pm i \eta} = \mathcal{P} \frac{1}{x} \mp i \pi \delta(x),
\]  

(S31)

where \( \eta \) is a positive infinitesimal, \( \mathcal{P} \) is the Cauchy principal value and \( \delta(x) \) being the Dirac delta function. We implicitly applied this relation to deduce \( \Sigma_r^r(\omega) \) and \( \Sigma_i^i(\omega) \) in Eqs. (S3) and (S14).

From Eq. (S27) we write for \( I_{3\sigma}^{\alpha(\gamma)}(\omega) = I_{3\sigma}^{\alpha(\gamma)1}(\omega) - I_{3\sigma}^{\alpha(\gamma)2}(\omega) \) with:

\[
I_{3\sigma}^{\alpha(\gamma)1}(\omega) = \int_{-D}^{0} d\varepsilon \frac{\varepsilon \cdot f(\varepsilon - \mu_{\alpha})}{\varepsilon - \omega_{1\sigma} - i\delta}.
\]  

(S32)

\[
I_{3\sigma}^{\alpha(\gamma)2}(\omega) = \int_{0}^{D} d\varepsilon \frac{\varepsilon \cdot f(\varepsilon - \mu_{\alpha})}{\varepsilon - \omega_{1\sigma} - i\delta}.
\]  

(S33)

Firstly, we assume that \( 0 \lesssim \mu_{\alpha} < D \) and \( 0 \lesssim \omega_{1\sigma} < D' \) where \( D' > D \) is arbitrarily introduced. Using Eqs. (S29)-(S31) then \( I_{3\sigma}^{\alpha(+)1}(\omega) \) and \( I_{3\sigma}^{\alpha(+)2}(\omega) \) can be calculated as:

\[
I_{3\sigma}^{\alpha(+)1}(\omega) \approx \frac{1}{2\beta} \int_{-\beta(D+\mu_{\alpha})}^{-\beta\mu_{\alpha}} \frac{dx(x + \beta\mu_{\alpha})}{x + \beta(\mu_{\alpha} - \omega_{1\sigma})} \left[1 - \tanh\left(\frac{x}{2}\right)\right]
\]  

\[
\approx D + \omega_{1\sigma} \ln \left|\frac{D + \omega_{1\sigma}}{-\omega_{1\sigma}}\right|.
\]  

(S34)
where the imaginary part in the integral has been neglected due to the limits of integration, and:

\[ I_{3\sigma}^{(\alpha^+)^2}(\omega) = \frac{1}{2\beta} \int_{-\beta\mu_\alpha}^{\beta(D-\mu_\alpha)} \frac{dx(x+\beta\mu_\alpha)}{x+\beta(\mu_\alpha-\omega_1\sigma)-i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \]

\[ \approx -\omega_1\sigma \ln \left| \frac{\omega_1\sigma}{\mu_\alpha-\omega_1\sigma-2T} - \frac{\omega_1\sigma}{2} \left( 1 + \frac{\mu_\alpha-\omega_1\sigma}{2T} \right) \ln \left| \frac{\mu_\alpha-\omega_1\sigma-2T}{\mu_\alpha-\omega_1\sigma+2T} \right| + \mu_\alpha-\omega_1\sigma + i\pi \omega_1\sigma \left[ 1 + \tanh \left( \frac{\mu_\alpha-\omega_1\sigma}{2T} \right) \right], \]

(S35)

where \( \eta = \delta \beta \to 0^+ \). We consider the case: \( 0 \leq \mu_\alpha < D \) and \( -D' < \omega_1\sigma \leq 0 \). Introducing \( \omega_{1\sigma}^+ = -\omega_1\sigma \) in the same way we find:

\[ I_{3\sigma}^{(\alpha^+)^1}(\omega) = \frac{1}{2\beta} \int_{-\beta(D+\mu_\alpha)}^{-\beta\mu_\alpha} \frac{dx(x+\beta\mu_\alpha)}{x+\beta(\mu_\alpha+\omega_{1\sigma}^+)-i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \]

\[ \approx D - \omega_{1\sigma}^+ \ln \left| \frac{\omega_{1\sigma}^+}{D-\omega_{1\sigma}^+} - \frac{i\pi}{2} \omega_{1\sigma}^+ \left[ 1 + \tanh \left( \frac{\mu_\alpha+\omega_{1\sigma}^+}{2T} \right) \right] \theta(D-\omega_{1\sigma}^+). \]

Furthermore, we have:

\[ I_{3\sigma}^{(\alpha^+)^2}(\omega) \approx \frac{1}{2\beta} \int_{-\beta\mu_\alpha}^{\beta(D-\mu_\alpha)} \frac{dx(x+\beta\mu_\alpha)}{x+\beta(\mu_\alpha+\omega_{1\sigma}^+)-i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx \mu_\alpha + \omega_{1\sigma}^+ \]

\[ + \omega_{1\sigma}^+ \ln \left| \frac{\omega_{1\sigma}^+}{\mu_\alpha+\omega_{1\sigma}^+-2T} - \frac{\omega_{1\sigma}^+}{2} \left( 1 + \frac{\mu_\alpha+\omega_{1\sigma}^+}{2T} \right) \ln \left| \frac{\mu_\alpha+\omega_{1\sigma}^++2T}{\mu_\alpha+\omega_{1\sigma}^+-2T} \right| \right. \]

(S36)

Combining Eqs. (S34)-(S37) we simply obtain \( I_{3\sigma}^{(\alpha^+)} \) defined for the full range of the energy, \(-D' < \omega_1\sigma < D'\):

\[ I_{3\sigma}^{(\alpha^+)}(\omega) \approx D - \mu_\alpha + \omega_1\sigma - \omega_1\sigma \ln \left| \frac{(D+\omega_1\sigma)(\mu_\alpha-\omega_1\sigma-2T)}{\omega_1^2} \right| \]

\[ + \frac{\omega_1\sigma}{2} \left( 1 + \frac{\mu_\alpha-\omega_1\sigma}{2T} \right) \ln \left| \frac{\mu_\alpha-\omega_1\sigma-2T}{\mu_\alpha-\omega_1\sigma+2T} \right| \]

\[ - \frac{i\pi}{2} |\omega_1\sigma| \left[ 1 + \tanh \left( \frac{\mu_\alpha-\omega_1\sigma}{2T} \right) \right] \theta(D+\omega_1\sigma). \]

(S38)
Assuming that $-D < \mu_\alpha \lesssim 0$ and $0 < \omega_1 \sigma < D'$ and introducing $\mu^+_{\alpha} = -\mu_\alpha$, thus, one finds:

$$I^\alpha_{3\sigma}^{-1}(\omega) \approx \frac{1}{2\beta} \int_{-\beta(D-\mu^+_{\alpha})}^{\beta\mu^+_{\alpha}} \frac{dx(x-\beta\mu^+_{\alpha})}{x-\beta(\mu^+_{\alpha} + \omega_1 \sigma)} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx D - \mu^+_{\alpha} - \omega_1 \sigma$$  \hspace{1cm} (S39)

$$+ \omega_1 \sigma \ln \left| \frac{\mu^+_{\alpha} + \omega_1 \sigma + 2T}{D + \omega_1 \sigma} \right| + \omega_1 \sigma (1 - \frac{\mu^+_{\alpha} + \omega_1 \sigma}{2T}) \ln \left| \frac{\mu^+_{\alpha} + \omega_1 \sigma - 2T}{\mu^+_{\alpha} + \omega_1 \sigma + 2T} \right|.$$  \hspace{1cm} (S40)

In the case of $-D < \mu_\alpha \lesssim 0$, $-D' < \omega_1 \sigma \lesssim 0$ and introducing $\mu^+_{\alpha} = -\mu_\alpha$ and $\omega^+_{1\sigma} = -\omega_1 \sigma$ we have:

$$I^\alpha_{3\sigma}^{-1}(\omega) \approx \frac{1}{2\beta} \int_{-\beta(D+\mu^+_{\alpha})}^{\beta\mu^+_{\alpha}} \frac{dx(x-\beta\mu^+_{\alpha})}{x-\beta(\mu^+_{\alpha} - \omega^+_{1\sigma}) - i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx D - \mu^+_{\alpha} + \omega^+_{1\sigma}$$  \hspace{1cm} (S41)

$$- \omega^+_{1\sigma} \ln \left| \frac{\mu^+_{\alpha} - \omega^+_{1\sigma} + 2T}{D - \omega^+_{1\sigma}} \right| - \omega^+_{1\sigma} \left( 1 - \frac{\mu^+_{\alpha} - \omega^+_{1\sigma}}{2T} \right) \ln \left| \frac{\mu^+_{\alpha} - \omega^+_{1\sigma} - 2T}{\mu^+_{\alpha} - \omega^+_{1\sigma} + 2T} \right|.$$

$$- i \frac{\pi}{2} \omega^+_{1\sigma} \left[ 1 - \tanh \left( \frac{\mu^+_{\alpha} - \omega^+_{1\sigma}}{2T} \right) \right] \theta(D - \omega^+_{1\sigma}),$$

$$I^\alpha_{3\sigma}^{-2}(\omega) \approx \frac{1}{2\beta} \int_{\beta\mu^+_{\alpha}}^{\beta(D+\mu^+_{\alpha})} \frac{dx(x-\beta\mu^+_{\alpha})}{x-\beta(\mu^+_{\alpha} - \omega^+_{1\sigma})} \left[ 1 - \tan \left( \frac{x}{2} \right) \right] \approx 0.$$  \hspace{1cm} (S42)

Comparing Eqs. (S39)-(S40) with Eqs. (S41)-(S42) one finds the $I^\alpha_{3\sigma}^{-}(\omega)$ for the entire energy domain, $-D' < \omega_1 \sigma < D'$:

$$I^\alpha_{3\sigma}^{-}(\omega) \approx D + \mu_\alpha - \omega_1 \sigma + \omega_1 \sigma \ln \left| \frac{\mu_\alpha - \omega_1 \sigma - 2T}{D + \omega_1 \sigma} \right|$$

$$+ \frac{\omega_1 \sigma}{2} \left( 1 + \frac{\mu_\alpha - \omega_1 \sigma}{2T} \right) \ln \left| \frac{\mu_\alpha - \omega_1 \sigma + 2T}{\mu_\alpha - \omega_1 \sigma - 2T} \right|$$

$$- i \frac{\pi}{2} \omega_1 \sigma \left[ 1 + \tanh \left( \frac{\mu_\alpha - \omega_1 \sigma}{2T} \right) \right] \theta(D + \omega_1 \sigma).$$  \hspace{1cm} (S43)
Now, we consider the case \( \mu_\alpha = 0 \) and \( 0 \lesssim \omega_1 \lessgtr D' \) and find:

\[
I_{3\sigma}^{\alpha(0)1}(\omega) \approx \frac{1}{2\beta} \int_{-\beta D}^{0} \frac{xdx}{x - \beta \omega_1 \sigma - i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx D + \omega_1 \sigma \ln \left| \frac{\omega_1 \sigma + 2T}{D + \omega_1 \sigma} \right| + \frac{\omega_1 \sigma}{2} \left( 1 - \frac{\omega_1 \sigma}{2T} \right) \ln \left| \frac{\omega_1 \sigma}{\omega_1 \sigma + 2T} \right| - \frac{T}{2} - \frac{\omega_1 \sigma}{2}. \tag{S44}
\]

\[
I_{3\sigma}^{\alpha(0)2}(\omega) = \frac{1}{2\beta} \int_{0}^{\beta D} \frac{xdx}{x - \beta \omega_1 \sigma - i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx \frac{T}{2} - \frac{\omega_1 \sigma}{2} + \frac{\omega_1 \sigma}{2} \left( 1 - \frac{\omega_1 \sigma - 2T}{\omega_1 \sigma} \right) \ln \left| \frac{\omega_1 \sigma - 2T}{\omega_1 \sigma} \right| + \frac{\pi}{2} \omega_1 \sigma \left[ 1 - \tanh \left( \frac{\omega_1 \sigma - 2T}{2T} \right) \right]. \tag{S45}
\]

For \( \mu_\alpha = 0 \) and \(-D' < \omega_1 \sigma \lesssim 0 \) with \( \omega_1^+ = -\omega_1 \sigma \) we have:

\[
I_{3\sigma}^{\alpha(0)1}(\omega) = \frac{1}{2\beta} \int_{-\beta D}^{0} \frac{xdx}{x + \beta \omega_1 \sigma - i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx D - T + \frac{\omega_1^+ \sigma}{2} \approx \frac{T}{2} + \frac{\omega_1^+ \sigma}{2} \left( 1 + \frac{\omega_1^+ \sigma}{2T} \right) \ln \left| \frac{\omega_1^+ \sigma + 2T}{\omega_1^+ \sigma} \right| \tag{S46}
\]

\[
I_{3\sigma}^{\alpha(0)2}(\omega) \approx \frac{1}{2\beta} \int_{0}^{\beta D} \frac{xdx}{x + \beta \omega_1 \sigma - i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx \frac{T}{2} + \frac{\omega_1^+ \sigma}{2} - \frac{\omega_1^+ \sigma}{2} \left( 1 + \frac{\omega_1^+ \sigma}{2T} \right) \ln \left| \frac{\omega_1^+ \sigma + 2T}{\omega_1^+ \sigma} \right| \tag{S47}
\]

Using Eqs. (S44)-(S47) for the full energy domain \( I_{3\sigma}^{\alpha(0)} \) can be expressed as:

\[
I_{3\sigma}^{\alpha(0)}(\omega) \approx D - T + \omega_1 \sigma \ln \left| \frac{\omega_1 \sigma + 2T}{D + \omega_1 \sigma} \right| - \frac{\omega_1 \sigma}{2} \left( 1 - \frac{\omega_1 \sigma}{2T} \right) \ln \left| \frac{\omega_1 \sigma + 2T}{\omega_1 \sigma} \right| \left( \frac{\omega_1 \sigma - 2T}{\omega_1 \sigma} \right) \tag{S48}
\]

\[
- \frac{\pi}{2} \omega_1 \sigma \left[ 1 - \tanh \left( \frac{\omega_1 \sigma}{2T} \right) \right] \theta(D + \omega_1 \sigma). \]
Comparing Eqs. (S38), (S43) and (S48), Eqs. (S15)-(S19) can be introduced. In similar way, we can write $I_{4\sigma}^{(\gamma)}(\omega) = -I_{4\sigma}^{(\gamma)1}(\omega) + I_{4\sigma}^{(\gamma)2}(\omega)$ with:

$$I_{4\sigma}^{(\gamma)1}(\omega) = \int_{-D}^{0} d\varepsilon \frac{\varepsilon \cdot f(\varepsilon - \mu_\alpha)}{\varepsilon + \omega_2\sigma + i\delta},$$  \hspace{1cm} (S49)$$

$$I_{4\sigma}^{(\gamma)2}(\omega) = \int_{0}^{D} d\varepsilon \frac{\varepsilon \cdot f(\varepsilon - \mu_\alpha)}{\varepsilon + \omega_2\sigma + i\delta}.$$  \hspace{1cm} (S50)

We assume that $0 \lesssim \mu_\alpha < D$ and $0 \lesssim \omega_2\sigma < D'$, one finds:

$$I_{4\sigma}^{(+)}1(\omega) \approx \frac{1}{2\beta} \frac{-\beta \mu_\alpha}{-\beta(D+\mu_\alpha)} \left[ \int \frac{dx(x+\beta \mu_\alpha)}{x+\beta(\mu_\alpha+\omega_2\sigma)} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \right] \approx D - \omega_2 \sigma \ln \left| \frac{\omega_2 \sigma}{D - \omega_2 \sigma} \right| + i \frac{\pi}{2} \omega_2 \sigma \left[ 1 + \tanh \left( \frac{\mu_\alpha + \omega_2 \sigma}{2T} \right) \right] \Theta(D - \omega_2 \sigma),$$  \hspace{1cm} (S51)$$

$$I_{4\sigma}^{(+)}2(\omega) \approx \frac{1}{2\beta} \frac{\beta(D+\mu_\alpha)}{-\beta \mu_\alpha} \left[ \int \frac{dx(x+\beta \mu_\alpha)}{x+\beta(\mu_\alpha+\omega_2\sigma)} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \right] \approx \mu_\alpha + \omega_2 \sigma$$

$$+ \omega_2 \sigma \ln \left| \frac{\omega_2 \sigma}{\mu_\alpha + \omega_2 \sigma - 2T} \right| + \frac{\omega_2 \sigma}{\mu_\alpha + \omega_2 \sigma + 2T} \left( 1 + \frac{\mu_\alpha + \omega_2 \sigma}{2T} \right) \ln \left| \frac{\mu_\alpha + \omega_2 \sigma - 2T}{\mu_\alpha + \omega_2 \sigma + 2T} \right|. $$  \hspace{1cm} (S52)

We consider the case $0 \lesssim \mu_\alpha < D$ and $-D' \lesssim \omega_2 \sigma \lesssim 0$, introducing $\omega_{2\sigma}^+ = -\omega_2 \sigma$, in the same way we have:

$$I_{4\sigma}^{(+)}1(\omega) \approx \frac{1}{2\beta} \frac{-\beta \mu_\alpha}{-\beta(D+\mu_\alpha)} \left[ \int \frac{dx(x+\beta \mu_\alpha)}{x+\beta(\mu_\alpha - \omega_{2\sigma}^+)} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \right] \approx D + \omega_{2\sigma}^+ \ln \left| \frac{\omega_{2\sigma}^+}{D + \omega_{2\sigma}^+} \right|,$$  \hspace{1cm} (S53)
\( \chi_{4\sigma}^{\alpha(\cdot)}(\omega) = \frac{1}{2\beta} \int_{\frac{\mu\sigma}{d-\mu\sigma}}^{\beta} \frac{dx(x+\beta\mu\sigma)}{x+\beta(\mu\sigma-\omega\sigma) + i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx \mu\sigma - \omega\sigma \)

\[ \approx D + \mu\sigma + \omega\sigma + \omega\sigma \ln \left| \frac{\omega\sigma}{(D - \omega\sigma)(\mu\sigma + \omega\sigma - 2T)} \right| \]

\[ + \frac{\omega\sigma}{2} \left( 1 + \frac{\mu\sigma + \omega\sigma}{2T} \right) \ln \left| \frac{\mu\sigma + \omega\sigma - 2T}{\mu\sigma + \omega\sigma + 2T} \right| \]

\[ - i \frac{\pi}{2} \omega\sigma \left[ 1 + \tanh \left( \frac{\mu\sigma + \omega\sigma}{2T} \right) \right] \theta(D - \omega\sigma). \]

For \(-D < \mu\sigma \leq 0\) and \(0 < \omega\sigma < D\) with \(\mu\sigma = -\mu\sigma\) we find:

\( \chi_{4\sigma}^{\alpha(-)}^{1}(\omega) = \frac{1}{2\beta} \int_{-\beta(D-\mu\sigma)}^{\beta} \frac{dx(x-\beta\mu\sigma)}{x-\beta(\mu\sigma + \omega\sigma) + i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx \mu\sigma - \omega\sigma \)

\[ \approx D - \mu\sigma + \omega\sigma \ln \left| \frac{\mu\sigma + \omega\sigma + 2T}{D - \omega\sigma} \right| - \omega\sigma \frac{2}{2T} \left( 1 - \frac{\mu\sigma - \omega\sigma}{\mu\sigma - \omega\sigma + 2T} \right) \ln \left| \frac{\mu\sigma + \omega\sigma - 2T}{\mu\sigma + \omega\sigma + 2T} \right| \]

\[ + i \frac{\pi}{2} \omega\sigma \left[ 1 - \tanh \left( \frac{\mu\sigma + \omega\sigma}{2T} \right) \right] \theta(D - \omega\sigma). \]

\( \chi_{4\sigma}^{\alpha(-)}^{2}(\omega) \approx 0. \)

In the case of \(-D < \mu\sigma \leq 0\) and \(-D' < \omega\sigma \leq 0\) we have:

\[ \chi_{4\sigma}^{\alpha(-)}^{1}(\omega) \approx \frac{1}{2\beta} \int_{-\beta(D-\mu\sigma)}^{\beta} \frac{dx(x-\beta\mu\sigma)}{x-\beta(\mu\sigma + \omega\sigma) + i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx D - \mu\sigma + \omega\sigma \]

\[ + \omega\sigma \frac{2}{2T} \left( 1 - \frac{\mu\sigma + \omega\sigma}{\mu\sigma + \omega\sigma + 2T} \right) \ln \left| \frac{\mu\sigma + \omega\sigma - 2T}{\mu\sigma + \omega\sigma + 2T} \right| \]

S11
We now take the case $\mu = 0$ with $0 < \omega_2 \sigma < D'$ and find:

\[
I_{4\sigma}^{\alpha(0)1}(\omega) = \frac{1}{2\beta} \int_{-\beta D}^{\beta D} \frac{xdx}{x + \beta \omega_2 \sigma + i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx D - \frac{T}{2} + \frac{\omega_2 \sigma}{2}
\]

(S61)

\[
I_{4\sigma}^{\alpha(0)2}(\omega) \approx \frac{1}{2\beta} \int_{-\beta D}^{\beta D} \frac{xdx}{x + \beta \omega_2 \sigma} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx \frac{T}{2} + \frac{\omega_2 \sigma}{2} - \frac{\omega_2 \sigma}{2} \left( 1 + \frac{\omega_2 \sigma}{2T} \right) \ln \left| \frac{\omega_2 T + 2T}{\omega_2 \sigma} \right|
\]

(S62)

For $\mu = 0$ and $-D' < \omega_2 \sigma < 0$ with $\omega_2^+ = -\omega_2$, we obtain:

\[
I_{4\sigma}^{\alpha(0)1}(\omega) \approx \frac{1}{2\beta} \int_{-\beta D}^{\beta D} \frac{xdx}{x + \beta \omega_2^+} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx D - \frac{T}{2} - \frac{\omega_2^+}{2}
\]

(S63)
\[ f^{(0)}_{4\sigma}(\omega) = \frac{1}{2\beta} \int_{0}^{\beta D} \frac{x \, dx}{x - B \omega \sigma + i\eta} \left[ 1 - \tanh \left( \frac{x}{2} \right) \right] \approx \frac{T}{2} - \frac{\omega \sigma}{2} (S64) \]

\[ + \frac{\omega \sigma}{2} \left( 1 - \frac{\omega \sigma}{2T} \right) \ln \left| \frac{\omega \sigma - 2T}{\omega \sigma + 2T} \right| - i \frac{\pi}{2} \omega \sigma \left[ 1 - \tanh \left( \frac{\omega \sigma}{2T} \right) \right]. \]

Comparing Eqs. (S61)-(S62) with Eqs. (S63)-(S64), one finds the \( f^{(0)}_{4\sigma} \) for the full energy domain:

\[ f^{(0)}_{4\sigma}(\omega) \approx -D + T + \omega \sigma \ln \left| \frac{\omega \sigma - 2T}{D - \omega \sigma} \right| \]

\[ + \frac{\omega \sigma}{2} \left( 1 + \frac{\omega \sigma}{2T} \right) \ln \left| \frac{\omega \sigma - 2T}{\omega \sigma + 2T} \right| \]

\[ - i \frac{\pi}{2} \omega \sigma \left[ 1 + \tanh \left( \frac{\omega \sigma}{2T} \right) \right] \theta(D - \omega \sigma). \]

(S65)

In the same way, comparing Eqs. (S55), (S60) and (S65), Eqs. (S20)-(S24) can be introduced.

Note that these results are valid as well at low temperatures. For absolute zero temperature we can substitute \( f_\alpha(\varepsilon_k) \) with the Heaviside function, i.e., \( f_\alpha(\varepsilon_k) = \theta(\mu_\alpha - \varepsilon_k) \), and using the method presented above the integrals can be calculated.

**Appendix C: The verification of an analytical solution**

In this section, we compare our analytical results for \( \Sigma_3(\omega) \) presented in Appendix B with those of Z.-G. Zhu and J. Berakdar in Ref. [7]. In order to do this, we introduce the following integral:

\[ I(\omega) = \int_{-D}^{D} d\varepsilon \frac{D}{-\varepsilon + \omega + i\delta}. \]

(S66)

Z.-G. Zhu and J. Berakdar applied a contour integral method in complex plane and found that:

\[ I(\omega) = \frac{|D|}{2} \ln \left( \frac{|D^2 - \omega^2|}{(2\pi T)^2} \right) - |\mu| \psi(z) - \frac{1}{2} \left[ \omega \ln \left( \frac{|D^2 - \omega^2|}{\omega^2} \right) + i\pi |\omega| \theta(D - |\omega|) \right]. \]

(S67)

where \( z = \frac{1}{2} + \frac{\omega - \mu}{2\pi i T} \) and \( \psi(z) \) is the digamma function. It can be shown that the integrating function in Eq. (S66) is not a holomorphic function, and thus the contour integral method can not be applied
for $I(\omega)$. As we shall see, their results differ from those obtained by numerical calculations (see Figure S1). Our analytical results, given by $I^{(+)}_{\alpha,\sigma}(\omega)$ in Appendix B, are in better agreement with the numerical calculations. Therefore, it can be verified that the relation (S67) does not accurately reproduce the case of absolute zero temperature.

![Figure S1:](image)

**Figure S1:** (a) Real part of $I(\omega)$ as a function of energy. (b) Imaginary part of $I(\omega)$ as a function of energy. The chemical potential is fixed at $\mu/D = 0.05$ and the temperature is set to be $T/D = 5 \cdot 10^{-5}$. It can be observed that our analytical results show a good agreement with the numerical calculations. The results of Z.-G. Zhu and J. Berakdar present a departure from the numerical calculations due to the mathematical method used by them.

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