REGULARITY OF THE MODULI SPACE
OF INSTANTON BUNDLES $MI_{\mathbb{P}^3}(5)$.

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§0. Introduction.

Instanton vector bundles were defined by Atiyah, Drinfeld, Hitchin and Manin [ADHM] in order to construct all the self-dual solutions of the Yang-Mills equation over $S^4$. A mathematical instanton bundle $E$ on $\mathbb{P}^3 := \mathbb{P}^3(\mathbb{C})$ can be defined as the cohomology bundle of a monad

$$O(-1)^k \rightarrow O^{2k+2} \rightarrow O(1)^k$$

on $\mathbb{P}^3$, where $c_2(E) = k$. This is equivalent to the condition that $E$ is a stable bundle of rank 2 on $\mathbb{P}^3$ such that $c_1(E) = 0$, $c_2(E) = k$ and $H^1(E(-2)) = 0$. If $E$ is a mathematical instanton bundle, then it is easy to check by using Hirzebruch-Riemann-Roch Theorem that $h^1(S^2E) - h^2(S^2E) = 8k - 3$ and so $8k - 3$ is the expected dimension of the moduli space of mathematical instanton bundles $MI_{\mathbb{P}^3}(k)$. It is not known if the moduli space $MI_{\mathbb{P}^3}(k)$ is a regular variety of pure dimension $8k - 3$. The answer is affirmative in the cases $1 \leq k \leq 4$ and this was proved in [H], [ES] and [LeP]. In this article we extend these results to the case $k = 5$. More precisely, we prove the following

**Theorem 0.1.** For $2 \leq k \leq 5$ the moduli space $MI_{\mathbb{P}^3}(k)$ of mathematical instantons is a regular variety of pure dimension $8k - 3$.

Our result should be compared with [AO2], where it is proved that the closure of $MI_{\mathbb{P}^3}(5)$ in the Maruyama scheme of vector bundles of rank 2 with $c_1 = 0$, $c_2 = 5$ contains singular points. Our proof requires tools both from invariant theory and algebraic geometry.

In §1 we prove some algebraic lemmas. In §2 we give an invariant theoretical description of $MI_{\mathbb{P}^3}(k)$ and we prove some result about unstable planes. Finally, in §3 we prove Theorem 0.1.

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§1. An invariant theoretical description of \( M_{I_{23}}(k) \).

Our first goal is to describe the moduli space \( M_{I_{23}}(k) \) in terms of invariant theory and prove that for any \( A \in M_{I_{23}}(k) \) and any plane \( H \) in \( \mathbb{P}^3 \), \( h^0(E_H) \leq 1 \). The group \( SL_{2k+2} \) acts canonically on the space \( \mathbb{C}^{2k+2} \). Let \( h_1, \ldots, h_{2k+2} \) be the standard basis of the space \( \mathbb{C}^{2k+2} \) and \( h_1^*, \ldots, h_{2k+2}^* \) be the dual basis of the dual space \( \mathbb{C}^{(2k+2)*} \). Consider the 2-form

\[
\omega = \sum_{1 \leq i \leq k+1} b_i^* \wedge h_{k+1+i}^* = \sum_{1 \leq i \leq k+1} (h_i^* \otimes h_{k+1+i}^* - h_{k+1+i}^* \otimes h_i^*) \in \wedge^2 \mathbb{C}^{(2k+2)*}.
\]

Let \( Sp_{2k+2} \) be the stabilizer of the 2-form \( \omega \) in the group \( SL_{2k+2} \). The 2-form \( \omega \) defines canonically the \( Sp_{2k+2} \)-bundle \( \mathbb{C}^{(2k+2)*} \) on \( \mathbb{C}^{2k+2} \). We have the canonical actions of the group \( SL_4 \times SL_k \times Sp_{2k+2} \) on the spaces \( \mathbb{C}^4, \mathbb{C}^{4*}, \mathbb{C}^{2k+2}, \mathbb{C}^k, \mathbb{C}^{k*}, \mathbb{C}^4 \otimes \mathbb{C}^k, \ldots \).

We have the quadratic \((SL_4 \times SL_k \times Sp_{2k+2})\)-morphism

\[
\gamma : \mathbb{C}^{4*} \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2} \rightarrow S^2 \mathbb{C}^{4*} \otimes \wedge^2 \mathbb{C}^k,
\]

\[
(f_i^* \otimes b_j^* \otimes h_l) \rightarrow \frac{1}{2} \omega(h_l, h_{ij}) (f_i^* f_j^*) \otimes (b_i^* \wedge b_j^*)
\]

and the bilinear \((SL_4 \times SL_k \times Sp_{2k+2})\)-morphisms

\[
\beta : \mathbb{C}^{4*} \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2} \times \mathbb{C}^{2k+2} \rightarrow \mathbb{C}^{4*} \otimes \mathbb{C}^k,
\]

\[
\beta(f_i^* \otimes b_j^* \otimes h_l, h_{ij}) = \omega(h_l, h_{ij}) f_i^* \otimes b_j^*
\]

and

\[
\varepsilon : \mathbb{C}^{4*} \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2} \times \mathbb{C}^4 \otimes \mathbb{C}^k \rightarrow \mathbb{C}^{2k+2},
\]

\[
(f_i^* \otimes b_j^* \otimes h_l, f_i \otimes b_j) \rightarrow \delta_{ii} \delta_{jj} h_l
\]

Consider the following conditions for an element \( A \in \mathbb{C}^{4*} \otimes \mathbb{C}^* \otimes \mathbb{C}^{2k+2} \)

\((E_1)\) \( \varepsilon(A, f \otimes b) \neq 0 \) for all \( 0 \neq f \in \mathbb{C}^4, 0 \neq b \in \mathbb{C}^k \),

\((E_2)\) \( \gamma(A) = 0 \),

\((E_3)\) \( \beta(A, h) \neq 0 \) for all \( 0 \neq h \in \mathbb{C}^{2k+2} \).

An element \( A \in \mathbb{C}^{4*} \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2} \) defines the sheaf morphism \( \mathcal{O}^{2k+2} \xrightarrow{A} \mathcal{O}(1)^{k} \).

This morphism and the symplectic structure over \( \mathcal{O}^{2k+2} \) define the sequence

\[
\mathcal{O}(-1)^k \xrightarrow{f_A^*} \mathcal{O}^{2k+2} \xrightarrow{f_A} \mathcal{O}(1)^k.
\]

The condition \((E_1)\) means that \( f_A \) is surjective or that \( \text{Ker } f_A \) is locally free. The condition \((E_2)\) means that the above sequence is a complex. Therefore, \((E_1)\) and \((E_2)\) together mean that it is a monad according to [BH]. The condition \((E_3)\) means moreover that the cohomology bundle \( E \) of the monad is a stable vector bundle. It is well known (see e.g. [AO1], Th. 2.8) that the conditions \((E_1)\) and \((E_2)\) imply \((E_3)\).

Set

\[
I_i = \{ A \in \mathbb{C}^{4*} \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2} \mid \text{the condition } (E_i) \text{ holds for } A \},
\]

and the sequence

\[
\mathcal{O}(-1)^k \xrightarrow{f_A^*} \mathcal{O}^{2k+2} \xrightarrow{f_A} \mathcal{O}(1)^k
\]
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$I : = I_1 \cap I_2 \cap I_3 = I_1 \cap I_2$.

Recall that the quotient $I/G$, where $G = \text{SL}_k \times \text{Sp}_{2k+2} \times \mathbb{C}^*$ is isomorphic to $\text{MI}_{P^3}(k)$ (see [BH]). Moreover, the stabilizer of any $A \in I$ in the group $G$ is equal to the kernel $H$ of the action of $G$ on $I$ and $\dim(H) = 0$. Therefore, any $G$-orbit $G \cdot A$, where $A \in I$, is isomorphic to $G/H$ and $\dim(G \cdot A) = \dim(G) = 3k^2 + 5k + 3$. Consider the canonical morphism

$$\pi : I \to I/G = \text{MI}_{P^3}(k).$$

The results above imply the following fact:

**Lemma 1.1.** For any $A \in I$ we have

$$\dim(T_{\pi(A)} \text{MI}_{P^3}(k)) = \dim(T_A I) - 3k^2 - 5k - 3.$$

**Theorem 1.2.** Let $E$ be an instanton bundle on $\mathbb{P}^3$ and let $H$ be a plane. Then $h^0(E|_H) \leq 1$.

**Proof.** The bundle $E$ is the cohomology bundle of a monad

$$\mathcal{O}(-1)^k \longrightarrow \mathcal{O}^{2k+2} \overset{b}{\longrightarrow} \mathcal{O}(1)^k,$$

where $c_2(E) = k$. Notice that it is enough to show that $h^0(K|_H) \leq 1$, where $K := \text{Ker} b$. A section $s \in H^0(K|_H)$ induces the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\mathcal{O}_H & \overset{s}{\longrightarrow} & K_H & \overset{i}{\longrightarrow} & \mathcal{O}_H^{2k+2} & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & T & \longrightarrow & \mathcal{O}_H^{2k+1} & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & \\
\end{array}
$$

Since $i \circ s$ is injective, we see that $s$ splits $K|_H$ as $K|_H \simeq \mathcal{O}_H \oplus T$. Therefore it is enough to show that $H^0(T) = 0$. Now $T$ is a bundle of rank $k + 1$ with $\det T = \mathcal{O}(-k)$ and thus

$$T \simeq \wedge^k T^*(-k).$$

We will prove a little more, indeed we show that

\[(1.1) \quad H^0(\wedge^i T^*(-i)) = 0 \quad \text{for} \quad 1 \leq i \leq k.\]

We prove (1.1) by induction. For $i = 1$ (1.1) holds by the above diagram. Now, let us consider

$$0 \longrightarrow E|_H \longrightarrow \mathcal{O}_H \oplus T^* \longrightarrow \mathcal{O}_H^k(1) \longrightarrow 0.$$
Let us remember that, since $E$ is a rank 2 bundle with $c_1(E) = 0$, we have $\wedge^2 E \cong \mathcal{O}$. Therefore, the second wedge power of $T^*$ twisted by $\mathcal{O}_H(-2)$ gives

$$0 \rightarrow \mathcal{O}_H(-2) \rightarrow \wedge^2 T^*(-2) \oplus T^*(-2) \rightarrow T^*(-1)^k \oplus \mathcal{O}_H(-1)^k$$

which proves (1.1) for $i = 2$. Moreover, the $i$-th wedge power of $T^*$ twisted by $\mathcal{O}(-i)$ gives

$$0 \rightarrow \wedge^i T^*(-i) \oplus \wedge^{i-1} T^*(-i) \rightarrow [\wedge^{i-1} T^*(-i+1)]^k \oplus [\wedge^{i-2} T^*(-i+1)]^k$$

and this sequence provides the inductive step.

□

**Definition.** $W(E) = \{H \in \mathbb{P}^{3*} \mid h^0(E|_H) \neq 0\}$ is called the variety (scheme) of unstable planes of $E$. Its scheme structure is defined as the degeneracy locus of the mapping

$$H^1(E(-1)) \otimes \mathcal{O} \rightarrow H^1(E) \otimes \mathcal{O}(1)$$

over $\mathbb{P}^{3*}$ (Theorem 1.2 shows that this map drops rank at most by one).

For an element $A \in \mathbb{C}^{4*} \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2}$ define the subvariety

$$X_A = \{(\overrightarrow{\tau}, \overrightarrow{\nu}) \in \mathbb{P}^{3*} \times \mathbb{P}^{k-1*} \mid f^* \otimes b^* \in \text{Im}(\beta(A, \cdot))\}.$$

**Lemma 1.3.** Let $q_1$ be the projection of $\mathbb{P}^{3*} \times \mathbb{P}^{k-1*}$ on $\mathbb{P}^{3*}$. We have $W(E) = q_1(X_A)$ and the fiber of the projection $X_A \rightarrow q_1(X_A)$ over $H$ is isomorphic to $\mathbb{P}(H^0(E|_H))$.

**Proof.** With the notations of the proof of Theorem 1.2 we have that $H \in W(E)$ iff $h^0(K|_H) \neq 0$. We have $H^0(K|_H) = \text{Ker}(\mathbb{C}^{2k+2} \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^4)$. Now suppose that $\overrightarrow{\tau}$ corresponds to $H$, then the existence of a nonzero $\alpha \in H^0(K|_H)$ is equivalent to $\beta(A, \alpha) = f^* \otimes b^*$, where $(\overrightarrow{\tau}, \overrightarrow{\nu}) \in \mathbb{P}^{3*} \times \mathbb{P}^{k-1*}$.

□

**Corollary 1.4.** The morphism $X_A \rightarrow q_1(X_A)$ is an isomorphism of the underlying varieties, in particular $\dim X_A = \dim q_1(X_A)$.

Recall that special 't Hooft bundles are the instanton bundles such that $h^0(E(1)) = 2$. They can be defined through the Serre correspondence by $k + 1$ skew lines lying on a smooth quadric surface [H]. We need the following special case of a theorem of J. Coanda [Co].

**Theorem 1.5.** If $E$ is an instanton bundle such that $\dim W(E) \geq 2$, then $E$ is a special 't Hooft bundle and $W(E)$ is a quadric surface.

It is known ([H]) that special 't Hooft bundles are smooth points in the moduli space of instanton bundles.

**Corollary 1.6.** If $\dim X_A \geq 2$, then $A$ corresponds to a smooth point in the moduli space of mathematical instanton bundles.

A reformulation of this Corollary into the invariant theoretical language is as follows.
Corollary 1.7. For any $A^0 \in I$ such that $\dim(X_{A^0}) \geq 2$, we have

$$\dim(T_{\pi(A^0)}MI_{P3}(k)) = 8k - 3.$$  

Lemma 1.8. Suppose $A^0 \in I$; then $\dim(T_{A^0}I) > 3k^2 + 13k$ if and only if there exists $0 \neq S^0 \in S^2\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^k$ such that $\xi(A^0, S^0) = 0$, where $\xi$ is the bilinear $SL_4 \times SL_k \times Sp_{2k+2}$-morphism defined by

$$\xi : \mathbb{C}^{4*} \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2} \times S^2\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^k \to \mathbb{C}^4 \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2},$$

$$(f^*_i \otimes b^*_j \otimes h_i, f_i f_{i'} \otimes b_{j'} \wedge b_{j''}) \mapsto (\delta_{i'i'}f_{i'} + \delta_{j'j''}b_{j''} - \delta_{j'j'}b_{j'}) \otimes h_i.$$

Proof. We have $\dim(T_{A^0}I) > 3k^2 + 13k$ iff the differential $d\gamma|_{A^0}$ is nonsurjective. The differential $d\gamma|_{A^0}$ is nonsurjective iff $(d\gamma|_{A^0})^*(S^0) = 0$ for some $0 \neq S^0 \in S^2\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^k$. It can be easily checked that

$$(d\gamma|_{A})^*(S^0) = \xi(A, S).$$

Hence, $\dim(T_{A^0}I) > 3k^2 + 13k$ iff $\xi(A^0, S^0) = 0$ for some $0 \neq S^0 \in S^2\mathbb{C}^4 \otimes \wedge^2\mathbb{C}^k$. \hfill \Box

For the convenience of the reader we give a cohomological interpretation of Lemma 1.8. Let $E^0$ be the instanton bundle defined by $A^0 \in I$ as the cohomology bundle of monad (1.1). By Lemma 1.1 and deformation theory the assumption $\dim(T_{A^0}I) > 3k^2 + 13k$ is equivalent to $h^1(S^2E^0) = \dim(T_{\pi(A^0)}MI_{P3}(k)) > 8k - 3$. Therefore, the assumption of Lemma 1.8 is equivalent to $H^2(S^2E^0) \neq 0$. The second symmetric power of the left hand side of (1.1) gives $H^2(S^2E^0) \simeq H^2(S^2(Ker f_{A^0})).$ The second symmetric power of the right hand side of (1.1) gives

$$H^2(S^2(Ker f_{A^0})) \simeq Coker \left[ H^0(\mathcal{O}(1)) \otimes \mathbb{C}^{4*} \otimes \mathbb{C}^{2k+2*} \xrightarrow{\Phi} H^0(\mathcal{O}(2)) \otimes \wedge^2(\mathbb{C}^{k*}) \right].$$

Lemma 1.8 follows because the dual of $\Phi$ can be identified with $\xi(A^0, \cdot)$. \hfill \Box

Section 2. Algebraic Lemmas.

Among all this section we prove some algebraic lemmas that we will use in order to prove our main result.

Lemma 2.1. Suppose $R$ is a nonzero block-matrix:

$$R = \begin{pmatrix} R^1 \\ R^2 \end{pmatrix},$$

where $R^i$ is a skew-symmetric matrix of size $k \times k$; then there exists a column $v_0$ of height $k$ such that

$$Rv_0 = \begin{pmatrix} \lambda_1 u_0 \\ \lambda_2 u_0 \end{pmatrix} \neq 0$$

for some column $u_0$ of height $k$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Proof. Suppose that $det(R^1) \neq 0$. In this case set $v_0 \in Ker(R^2 - \mu_0 R^1)$, where $\mu_0$ is a root of the equation $det(R^2 - \mu R^1) = 0$. \hfill \Box
Suppose that \( \det(R^1) = 0 \). One can assume that
\[
R^1 = \begin{pmatrix} R_{11}^1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R^2 = \begin{pmatrix} R_{11}^2 & R_{12}^2 \\ R_{21}^2 & R_{22}^2 \end{pmatrix},
\]
where \( R_{11}^1 \) is a skew-symmetric matrix of size \( k' \times k' \), \( k' < k \), \( \det(R_{11}^1) \neq 0 \) and \( R_{11}^2 \) is a skew-symmetric matrix of size \( k' \times k' \). If \( R_{12}^2 \neq 0 \) or \( R_{22}^2 \neq 0 \), then we set \( v_0 = \begin{pmatrix} 0 \\ v'_0 \end{pmatrix} \) for some \( v'_0 \) such that \( R_{12}^2 v'_0 \neq 0 \) or \( R_{22}^2 v'_0 \neq 0 \). If \( R_{12}^2 = 0 \) and \( R_{22}^2 = 0 \), then \( R_{21}^2 = 0 \) and we set \( v_0 = \begin{pmatrix} v'_0 \\ 0 \end{pmatrix} \), where
\[
\begin{pmatrix} R_{11}^1 & v'_0 \end{pmatrix} = \begin{pmatrix} \lambda_1 u'_0 \\ \lambda_2 u'_0 \end{pmatrix} \neq 0.
\]
\[\square\]

Consider the linear spaces \( \mathbb{C}^4 \) and \( \mathbb{C}^k \). Let \( f_1, \ldots, f_4 \) be the standard basis of \( \mathbb{C}^4 \) and let \( f_1', \ldots, f_4' \) be the dual basis of the dual space \( \mathbb{C}^{4*} \). Let \( b_1, \ldots, b_k \) be the standard basis of \( \mathbb{C}^k \) and let \( b_1', \ldots, b_k' \) be the dual basis of the dual space \( \mathbb{C}^{k*} \). The group \( \text{SL}_4 \) acts canonically on the space \( \mathbb{C}^4 \) and the group \( \text{SL}_k \) acts canonically on the space \( \mathbb{C}^k \). So the actions of the group \( \text{SL}_4 \times \text{SL}_k \) are defined on the spaces \( \mathbb{C}^4, \mathbb{C}^{4*}, \mathbb{C}^k, \mathbb{C}^{k*}, \mathbb{C}^4 \otimes \mathbb{C}^k, \ldots \)

Consider the linear space \( S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \). For an element \( S \in S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \) define
\[
\text{rk}(S) = \dim(\text{Im}(\rho(S, \cdot))),
\]
where
\[
\rho: S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \otimes \mathbb{C}^{4*} \otimes \mathbb{C}^{k*} \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^k,
\]
\[
(f_i f_i^* \otimes b_j \wedge b_j^*, f_i' f_i'^* \otimes b_j') \mapsto (\delta_{ij} f_i + \delta_{ij'} f_i') \otimes (\delta_{jj'} b_j - \delta_{jj'} b_j')
\]
is the bilinear \( (\text{SL}_4 \times \text{SL}_k) \)-morphism. Note that \( \text{rk}(S) \) is an even number. The following lemma is the only place in the paper where we need the assumption \( k \leq 5 \).

**Lemma 2.2.** Suppose \( 2 \leq k \leq 5 \) and consider \( S \in S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \), such that \( 2 \leq \text{rk}(S) \leq 2k - 2 \). Then one of the following conditions holds:

1. \( \rho(S, B^{\oplus}) = f_0 \otimes b_0 \neq 0 \) for some \( B^{\oplus} \in \mathbb{C}^{4*} \otimes \mathbb{C}^{k*} \), \( f_0 \in \mathbb{C}^4 \), \( b_0 \in \mathbb{C}^k \).
2. \( \text{rk}(S) = 6 \) and there exists \( 0 \neq f^{\oplus} \in \mathbb{C}^{4*} \) such that \( \rho(S, f^{\oplus} \otimes b^*) = 0 \) for all \( b^* \in \mathbb{C}^{k*} \).
3. \( \text{rk}(S) = 8 \) and \( \dim(Z_S) \geq 2 \), where
\[
Z_S = \{(f, b) \in \mathbb{P}^{3*} \times \mathbb{P}^{k-1*} \mid \rho(S, f^* \otimes b^*) = 0\},
\]
\[
\mathbb{P}^{3*} = \mathbb{P}\mathbb{C}^{4*}, \quad \mathbb{P}^{k-1*} = \mathbb{P}\mathbb{C}^{k*}.
\]

**Proof.** Consider the coordinate expression of \( S \) in the bases \( \{f_i\} \) and \( \{b_i\} \):
\[
S = \sigma_{ij} f_i f_j' \otimes b_i \wedge b_j.
\]
We get a block matrix $\sigma$ defined by

$$
\sigma = (\sigma_{ij})_{1 \leq i,j \leq k} = \begin{pmatrix}
0 & \sigma_{12} & \ldots & \sigma_{1k} \\
\sigma_{21} & 0 & \ldots & \sigma_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{k1} & \sigma_{k2} & \ldots & 0
\end{pmatrix},
$$

where $\sigma_{ij} = (\sigma_{ij}^{lp})_{1 \leq l,p \leq 4}$ is a symmetric matrix of size $4 \times 4$, $\sigma_{ij} = -\sigma_{ji}$. There is a second coordinate expression

$$
S = \hat{\sigma}_{ij}^{lp} f_i f_j \otimes b_l \wedge b_p,
$$

and we get a second block matrix $\hat{\sigma}$ defined by

$$
\hat{\sigma} = (\hat{\sigma}_{ij})_{1 \leq i,j \leq 4} = \begin{pmatrix}
\hat{\sigma}_{11} & \hat{\sigma}_{12} & \hat{\sigma}_{13} & \hat{\sigma}_{14} \\
\hat{\sigma}_{21} & \hat{\sigma}_{22} & \hat{\sigma}_{23} & \hat{\sigma}_{24} \\
\hat{\sigma}_{31} & \hat{\sigma}_{32} & \hat{\sigma}_{33} & \hat{\sigma}_{34} \\
\hat{\sigma}_{41} & \hat{\sigma}_{42} & \hat{\sigma}_{43} & \hat{\sigma}_{44}
\end{pmatrix},
$$

where $\hat{\sigma}_{ij} = (\hat{\sigma}_{ij}^{lp})_{1 \leq l,p \leq k}$ is a skew-symmetric matrix of size $k \times k$, $\hat{\sigma}_{ij} = \hat{\sigma}_{ji}$.

Transform the basis $\{b_i\}$ and obtain

$$
r \overset{\text{def}}{=} \text{rk}(\sigma^{12}) = \max_{\{(c_{ij})_{1 \leq i,j \leq k}\}} \{\text{rk}(c_{1i} \sigma_{ij} c_{2j})\}.
$$

We have

$$
2k - 2 \geq \text{rk}(S) = \text{rk}(\sigma) = \text{rk}(\hat{\sigma}) \geq 2 \text{rk}(\sigma^{12}) = 2r.
$$

Therefore, one of the following cases holds:

(a) $r = 1$ or $2$,

(b) $r = 3$, $\text{rk}(\sigma) = 6$, and $k \geq 4$,

(c) $r = 4$, $\text{rk}(\sigma) = 8$, and $k = 5$,

(d) $r = 3$, $\text{rk}(\sigma) = 8$, and $k = 5$.

Transform the basis $\{f_i\}$ and obtain

$$
\sigma_{lp}^{12} = \begin{cases}
1 & \text{if } 1 \leq l = p \leq r, \\
0 & \text{if } l \neq p \text{ or } l = p > r.
\end{cases}
$$

From (2.1) it follows that $\sigma_{lp}^{ij} = 0$ for $l,p > r$ whence

$$
\hat{\sigma}_{ij} = 0 \quad \text{for } i,j > r.
$$

(a). Consider the case (a).

In this case we prove that the condition (1) holds, i.e., we prove that there exists a column $f^0$ of height 4 and columns $b^0, B^{*01}, \ldots, B^{*04}$ of height $k$ such that

$$
\hat{\sigma} \begin{pmatrix}
B^{*01} \\
\vdots \\
B^{*04}
\end{pmatrix} = \begin{pmatrix}
f^{*01}_1 \\
\vdots \\
f^{*04}_1
\end{pmatrix} \neq 0.
$$
But this easily follows from Lemma 2.1 and (2.3).

(b). Consider the case (b).

In this case we prove that the condition (2) holds, i.e. we prove that there exists a column $f^{*0}$ of height 4 such that

\begin{equation}
\sigma \begin{pmatrix} b_1^* f^{*0} \\ \vdots \\ b_k^* f^{*0} \end{pmatrix} = 0
\end{equation}

for any column $b^*$ of height $k$. From the condition $\text{rk}(\sigma) = 6$ and (2.1) it follows that

$$
\sigma^{ij} = \begin{pmatrix}
\sigma_{11}^{ij} & \sigma_{12}^{ij} & \sigma_{13}^{ij} & 0 \\
\sigma_{21}^{ij} & \sigma_{22}^{ij} & \sigma_{23}^{ij} & 0 \\
\sigma_{31}^{ij} & \sigma_{32}^{ij} & \sigma_{33}^{ij} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

From this for

$$
f^{*0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

it easily follows (2.4)

(c). Consider the case (c).

In this case we prove that the condition (3) holds. We have:

$$
Z_S = \{ (f^*, b^*) : \left( \begin{array}{c} f_1^* \\ \vdots \\ f_4^* \\ f_5^* \end{array} \right), \left( \begin{array}{c} b_1^* \\ \vdots \\ b_4^* \\ b_5^* \end{array} \right) | \sigma \begin{pmatrix} b_i^* f^* \\ \vdots \\ b_5^* f^* \end{pmatrix} = 0 \}.
$$

Consider the matrix

$$
\tilde{\sigma} = \begin{pmatrix}
0 & E_4 & \sigma_{13} & \sigma_{14} & \sigma_{15} \\
-E_4 & 0 & \sigma_{23} & \sigma_{24} & \sigma_{25}
\end{pmatrix},
$$

where $E_4$ is the identity matrix of size $4 \times 4$. The 8 rows of the matrix $\tilde{\sigma}$ are the first 8 rows of the matrix $\sigma$. Since $\text{rk}(\sigma) = 8 = \text{rk}(\tilde{\sigma})$, for a matrix $P$ of size $20 \times p$ we have:

\begin{equation}
\sigma P = 0 \iff \tilde{\sigma} P = 0.
\end{equation}

For $3 \leq i \leq 5$ consider the following matrix $P_i$ of size $20 \times 4$:

$$
P_i = \begin{pmatrix}
-a_2^{2i} \\
-\sigma_{11}^{ij} \\
P_3 \\
P_4 \\
P_5
\end{pmatrix},
$$

where $P_i = -E_4$ and $P_{ij} = 0$ for $j \neq i$. We see that $\tilde{\sigma} \cdot P_i = 0$. From (2.5) it follows that $\sigma \cdot P_i = 0$ or

$$
\sigma^{ij} = \sigma_{1j} \sigma_{2i} - \sigma_{2j} \sigma_{1i}, \quad 3 \leq j \leq 5.
$$
From this we obtain
\[
0 = \sigma_{ji} + (\sigma_{ij})^\top = \sigma_{1j} \sigma_{2i} - \sigma_{2j} \sigma_{1i} + (\sigma_{1i} \sigma_{2j} - \sigma_{2i} \sigma_{1j})^\top = [\sigma_{1j}, \sigma_{2i}] + [\sigma_{2j}, \sigma_{1i}], \quad 3 \leq i, j \leq 5.
\]

One can rewrite these equations into the following compact form:
\[
(\sigma_{ji})_{3 \leq i, j \leq 5} = \begin{pmatrix}
\sigma_{j1} & \sigma_{j2} & \cdots & \sigma_{j5}
\end{pmatrix}
\begin{pmatrix}
\sigma_{1}^\top & \sigma_{2}^\top & \cdots & \sigma_{5}^\top
\end{pmatrix} = 0
\]
for all \(t_1, t_2, t_3 \in \mathbb{C}\).

**Claim 1.** For every \((b_3^*, b_4^*, b_5^*) \neq (0, 0, 0)\) there exists \((b_1^*, b_2^*)\) and a nonzero column \(f^*\) of height 4 such that
\[
\sigma \begin{pmatrix}
\sigma_{1}^\top f^* \\
\vdots \\
\sigma_{5}^\top f^*
\end{pmatrix} = 0.
\]

**Proof of Claim 1.** From (2.6) it follows that the symmetric matrices
\[
b_3^* \sigma_{13} + b_4^* \sigma_{14} + b_5^* \sigma_{15}, \quad b_3^* \sigma_{23} + b_4^* \sigma_{24} + b_5^* \sigma_{25}
\]
commute therefore they have a common eigenvector \(f^*\) with the eigenvalues \(b_2^*, -b_1^*\) respectively. We have
\[
\tilde{\sigma} \begin{pmatrix}
\sigma_{1}^\top f^* \\
\vdots \\
\sigma_{5}^\top f^*
\end{pmatrix} = 0
\]
and from this and (2.5) Claim 1 follows.

From Claim 1 it follows that \(\dim(Z_S) \geq 2\).

(d). Consider the case (d).

In this case we prove that the condition (3) holds, i.e., we prove that \(\dim(Z_S) \geq 2\).

**Claim 2.** Suppose \(N \subset \mathbb{P} \mathbb{C}^{5*}\) is a line in general position; then there exists \(0 \neq f^* \in \mathbb{C}^*\) with \(b^* \in N\) such that \(\rho(S, f^* \otimes b^*) = 0\).

**Proof of Claim 2.** One can assume that \(N = \overline{(b_1, b_2)}\). We have to prove that there exists a column \(f^*\) of height 4 and \(b_1^*, b_2^* \in \mathbb{C}\), \((b_1^*, b_2^*) \neq (0, 0)\) such that
\[
(\sigma_{j}^\top f^*_{j=4} = 0, \quad (b_1^*, b_2^*) \neq (0, 0))
\]

Consider the 4th and 8th rows of the matrix \(\sigma\):
\[
\text{row}_4(\sigma) = (0, \ldots, 0, \sigma_{41}^{13}, \sigma_{42}^{13}, \ldots, \sigma_{45}^{15}, \sigma_{44}^{15}),
\]
\[
\text{row}_8(\sigma) = (0, \ldots, 0, \sigma_{41}^{23}, \sigma_{42}^{23}, \ldots, \sigma_{45}^{25}, \sigma_{44}^{25}).
\]
We want to show that row 4(\sigma) and rows 8(\sigma) are linearly dependent. Suppose that row 4(\sigma) and rows 8(\sigma) are linearly independent, then the first 8 rows of the matrix \sigma are linearly independent. Since rk(\sigma) = 8, we see that every row of \sigma is a linear combination of the first 8 rows. From row 4(\sigma) \neq 0 it follows that \sigma_i^{j_1} \neq 0 for some 3 \leq i \leq 5, 1 \leq j \leq 4. Since \sigma_i^{j_1} = -\sigma_i^{j_2} \neq 0, we see that (4(i - 1) + j)th row of the matrix \sigma is not a linear combination of the first 8 rows. This contradiction proves that row 4(\sigma) and rows 8(\sigma) are linearly dependent.

Finally, to obtain (2.7) we take

\[ f^{*0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \]

and \( b_1^*, b_2^* \) such that \((b_1^*, b_2^*) \neq (0, 0)\) and \( b_1^* \text{row}_4(\sigma) + b_2 \text{row}_8(\sigma) = 0 \).

From Claim 2 it follows that \( \dim(\mathbb{Z}_S) \geq 3 > 2 \).

§3. The proof of Theorem 0.1.

**Lemma 3.1.** Consider elements \( A^0 \in \mathbb{C}^4 \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2} \) and \( S^0 \in S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \) such that \( \xi(A^0, S^0) = 0 \).

1. Suppose

\[ \tau : \mathbb{C}^4 \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2} \times S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \times \mathbb{C}^{2k+2} \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^k \]

is an arbitrary trilinear \((\text{SL}_4 \times \text{SL}_k \times \text{Sp}_{2k+2})\)-morphism; then

\[ \tau(A^0, S^0, \mathbb{C}^{2k+2}) = 0 \]

2. Suppose

\[ \alpha : \mathbb{C}^4 \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2} \times S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \times \mathbb{C}^{2k+2} \rightarrow \mathbb{C}^{2k+2} \]

is an arbitrary trilinear \((\text{SL}_4 \times \text{SL}_k \times \text{Sp}_{2k+2})\)-morphism; then

\[ \alpha(A^0, S^0, \mathbb{C}^4 \otimes \mathbb{C}^{k*}) = 0 \]

**Proof.**

1. Consider the following nontrivial trilinear \( \text{SL}_4 \times \text{SL}_k \times \text{Sp}_{2k+2} \)-morphism:

\[ \tau_0 : \mathbb{C}^4 \otimes \mathbb{C}^{k*} \otimes \mathbb{C}^{2k+2} \times S^2 \mathbb{C}^4 \otimes \wedge^2 \mathbb{C}^k \times \mathbb{C}^{2k+2} \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^k, \]

\( (A, S, h) \mapsto \varphi(\xi(A, S), h), \)
where
\[ \kappa : \mathbb{C}^4 \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2} \times \mathbb{C}^{2k+2} \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^k, \]
\[ (f_i \otimes b_j \otimes h_{i'j'}) \mapsto \omega(h_{i'j'})f_i \otimes b_j \]
is the bilinear \( SL_4 \times SL_k \times Sp_{2k+2} \)-morphism.

On the other hand the \((SL_4 \times SL_k \times Sp_{2k+2})\)-module
\[ (\mathbb{C}^4 \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2}) \otimes (S^2 \mathbb{C}^4 \otimes \Lambda^2 \mathbb{C}^k) \otimes \mathbb{C}^{2k+2} \]
contains the irreducible \( SL_4 \times SL_k \times Sp_{2k+2} \)-module \( \mathbb{C}^4 \otimes \mathbb{C}^k \) with multiplicity 1. Therefore, there exists a unique, up to a scalar factor, nontrivial trilinear
\( SL_4 \times SL_k \times Sp_{2k+2} \)-morphism
\[ \tau : \mathbb{C}^4 \otimes \mathbb{C}^k \otimes \mathbb{C}^{2k+2} \times (S^2 \mathbb{C}^4 \otimes \Lambda^2 \mathbb{C}^k) \times \mathbb{C}^{2k+2} \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^k. \]
Thus, \( \tau = c\tau_0 \) for some \( c \in \mathbb{C} \) and we get
\[ \tau(A^0, S^0, C^{2k+2}) = c\tau_0(A^0, S^0, C^{2k+2}) = c\kappa(\xi(A^0, S^0), C^{2k+2}) = 0. \]

(2) Consider the linear mapping \( \alpha(A^0, S^0, \cdot)^* \) dual to \( \alpha(A^0, S^0, \cdot) \). From (1) it follows that \( \alpha(A^0, S^0, \cdot)^* = 0 \). Thus, \( \alpha(A^0, S^0, \mathbb{C}^4 \otimes \mathbb{C}^k) = 0. \)

\[ \square \]

**Proof of Theorem 0.1.**

We suppose that there exists \( A^0 \in I \) such that \( \dim(T_{\pi(A^0)}MI_{\mathbb{C}}^3(k)) > 8k - 3 \), \( 2 \leq k \leq 5 \) and we obtain a contradiction.

From Corollary 1.7 it follows that
\[ (3.1) \quad \dim(X_{A^0}) \leq 2 \]
and by Lemma 1.1 we have \( \dim(T_{A^0}I) > 3k^2 + 18k \). Hence, by Lemma 1.8 there exists \( 0 \neq S^0 \in S^2 \mathbb{C}^4 \otimes \Lambda^2 \mathbb{C}^k \) such that
\[ (3.2) \quad \xi(A^0, S^0) = 0. \]
Consider the following composition of linear mappings
\[ \rho(S^0, \cdot) \circ \beta(A^0, \cdot) : \mathbb{C}^{2k+2} \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^k, \quad h \mapsto \rho(S^0, \beta(A^0, h)), \]
where \( \beta \) is defined in \( \S1 \) and \( \rho \) is defined in \( \S2 \). From (3.2) and Lemma 1.8 (1) it follows that \( \rho(S^0, \cdot) \circ \beta(A^0, \cdot) = 0 \) or
\[ (3.3) \quad \Im(\beta(A^0, \cdot)) \subset \ker(\rho(S^0, \cdot)). \]
On the other hand, by (E3) we have \( \text{rk}(\beta(A^0, \cdot)) = 2k + 2 \) and with (3.3) this gives us
\[ (3.4) \quad \text{rk}(\rho(S^0, \cdot)) \leq 2k - 2. \]

Therefore, from (3.4) it follows that one of the conditions (1) - (3) of Lemma 2.2 holds for \( S = S^0 \).
I. Consider the case when the condition (1) of Lemma 2.2 holds for $S = S^0$. In this case, consider the following composition of linear mappings

$$
\varepsilon(A^0, \cdot) \circ \rho(S^0, \cdot) : \mathbb{C}^{4*} \otimes \mathbb{C}^{k*} \rightarrow \mathbb{C}^{2k^2 + 2}, \quad B^* \mapsto \varepsilon(A^0, \rho(S^0, B^*)).
$$

From Lemma 1.8 (2) it follows that $$\varepsilon(A^0, \cdot) \circ \rho(S^0, \cdot) = 0.$$ By the condition (1) of Lemma 2.2 there exists $B^* \notin \mathbb{C}^{4*} \otimes \mathbb{C}^{k*}$ such that $$\rho(S^0, B^*) = f^0 \otimes b^0 \neq 0.$$ Thus, we have $$\varepsilon(A^0, f^0 \otimes b^0) = \varepsilon(A^0, \rho(S^0, B^*)) = 0$$ and therefore $A^0 \notin I_1$. But this contradicts the fact that $A^0 \in I$.

II. Consider the case when the condition (2) of Lemma 2.2 holds for $S = S^0$. From (3.4) it follows that $k = 4$ or $k = 5$. By the condition (2) of Lemma 2.2 we have \{f^0\} \times \mathbb{C}^{k*} \subset \ker(\rho(S^0, \cdot)).$ On the other hand, we have (3.3) and

$$\dim(\ker(\rho(S^0, \cdot))) - \dim(\operatorname{Im}(\beta(A^0, \cdot))) = \begin{cases} 
0 & \text{if } k = 4, \\
2 & \text{if } k = 5.
\end{cases}$$

Therefore $\operatorname{Im}(\beta(A^0, \cdot)) \supset \{f^0\} \times M$ for some linear subspace $M \subset \mathbb{C}^{k*}$ of dimension $\geq 3$. But this contradicts (3.1).

III. Consider the case when the condition (3) of Lemma 2.2 holds for $S = S^0$. From (3.4) it follows that $k = 5$. Thus

$$\dim(\operatorname{Im}(\beta(A^0, \cdot))) = 12 = \dim(\ker(\rho(S^0, \cdot)))$$

and from this, together with (3.3), it follows that $\operatorname{Im}(\beta(A^0, \cdot)) = \ker(\rho(S^0, \cdot))$. Therefore $X_{A^0} = Z_{S^0}$. From this and the condition (3) of Lemma 2.2 we obtain $\dim(X_{A^0}) = \dim(Z_{S^0}) \geq 2$. But this again contradicts (3.1).

□

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