Cobordism maps on periodic Floer homology induced by elementary Lefschetz fibrations

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Abstract

Periodic Floer homology (PFH) is a Gromov–Floer type invariant for fibered three–
manifolds with Hamiltonian structures. The cobordism maps on periodic Floer homology
induced by symplectic cobordisms are currently only defined indirectly by using Seiberg–
Witten theory. In this paper, we investigate the cobordism maps induced by a class of
symplectic cobordisms constructed by P. Seidel, called elementary Lefschetz fibrations. We
define the PFH cobordism maps induced by elementary Lefschetz fibrations in terms of
holomorphic curves. Moreover, we compute these maps for some cases.

1 Introduction and Main results

Let $\pi : Y \to S^1$ be a surface fibration over the circle together with a closed
fiberwise non–degenerate 2–form over $Y$. Such a 2–form is called an admissible
2–form or a Hamiltonian structure. Assume that the fiber of $\pi$ is a connected
oriented surface, possibly with boundary. Under the above setup, M. Hutchings
introduces a Gromov–Floer type invariant called periodic Floer homology (abbre-
viated as PFH) $PFH_*(Y, \omega)$ [7]. The definition will be reviewed later. Y–J. Lee
and C. H. Taubes show that PFH is isomorphic to a version of Seiberg–Witten
cohomology [14].

Given a Lefschetz fibration $\pi_X : W \to B$ together with a closed fiberwise non–
degenerate 2–form $\omega_X$ such that $\partial W = Y$ and $\omega_W|_Y = \omega$, it is expected that the
Lefschetz fibration induces a homomorphism $PFH(W, \omega_W) : PFH(Y, \omega) \to \Lambda$, where $\Lambda$ is a local coefficient system. Such a homomorphism is called a cobor-
dism map. We call the triple $(W, \pi_W, \omega_W)$ a fiberwise symplectic cobordism. The
2–form $\omega_W$ is called a fiberwise symplectic form or an admissible 2–form. Note
that $\omega_W$ is not necessary symplectic. But we can construct a natural symplectic
form $\Omega_W$ by taking $\Omega_W = \omega_W + \pi_W^* \omega_B$, where $\omega_B$ is a large volume form of $B$. 
The purpose of this paper is to understand the cobordism maps induced by a special class of fiberwise symplectic cobordisms, called elementary Lefschetz fibrations, in terms of holomorphic curves. Such Lefschetz fibrations are constructed by Seidel [16], [17]. We use $(X, \pi_X, \omega_X)$ to denote the elementary Lefschetz fibration throughout.

**Elementary Lefschetz fibrations** Roughly speaking, an elementary Lefschetz fibration $(X, \pi_X, \omega_X)$ consists of a Lefschetz fibration $(X, \omega_X)$ over a disk with exactly one critical point together with a fiberwise symplectic form $\omega_X$. Let us briefly review its construction. More details are also given in Section 2.2. Start with the local model

$$\pi : \mathbb{C}^2_{x=(x_1,x_2)} \to \mathbb{C} \quad x \to x_1^2 + x_2^2,$$

we first construct an exact Lefschetz fibration $(E, \pi_E)$ by a suitable cutting of (1). More precisely, for $\lambda, \delta > 0$, define $E = \{x \in \mathbb{C}^2||x|^4 - |\pi(x)|^2 \leq 4\lambda^2, |\pi(x)| \leq \delta\}$ and $\pi_E = \pi|_E$. Then $E$ is a Lefschetz fibration over a disk $D_\delta$ with radius less than or equal to $\delta$. $(E, \pi_E)$ has a unique critical point at the origin. A regular fiber of $\pi_E$ is a cylinder of length $2\lambda$. Note that $E$ is a manifold with corners. The boundary of $E$ can be divided into the vertical boundary $\partial_v E = \pi^{-1}(\partial D_\delta)$ and the horizontal boundary $\partial_h E = \sqcup_{z \in D_\delta} \partial \pi^{-1}(z)$. Topologically, $(X, \pi_X)$ is obtained by gluing $(E, \pi_E)$ with a trivial fibration $S \times D_\delta$ over a disk along their horizontal boundaries, i.e., $X = E \cup_{\partial \pi^{-1}(z) \sim \{z\} \times \partial S} (S \times D_\delta)$, where $S$ is an oriented compact surface with two boundary components.

A candidate of the fiberwise symplectic form over $E$ is the restriction of the standard symplectic form $\omega_{\mathbb{C}^2}|_E$. For the purpose of gluing $\omega_{\mathbb{C}^2}|_E$ with an area form $\omega_S$ of $S$, we need to modify $\omega_{\mathbb{C}^2}|_E$ such that it is “trivial near $\partial_h E$”. $\omega_X$ is the result of this gluing. We will explain more details about this point in Section 2.2.

Note that the construction of $(X, \pi_X, \omega_X)$ depends on many choices such as $\lambda, \delta, \omega_S$, etc. But we still prefer to call it ‘the elementary Lefschetz fibration’ because they are isotropic to each other in the sense: Given two elementary Lefschetz fibrations $(X, \pi_X, \omega_X)$ and $(X', \pi_X', \omega'_X)$ constructed by different data, then there is a diffeomorphism $F : X \to X'$ preserving the Lefschetz fibration structure such that $[F^*\omega'_X - \omega_X] = 0 \in H^2(X, \mathbb{R})$. Also, the Floer homology and its cobordism maps under consideration should not depend on this deformation.

**Results** The main result of this paper is to construct the cobordism maps induced by the elementary Lefschetz fibrations by using holomorphic curves. More-
over, we compute them for some cases. We assume that the fiber $F$ of $X$ is a closed surface with genus $g(F) \geq 2$ and the vanishing cycle is non–separating; unless otherwise stated.

**Theorem 1.** Let $(X, \pi_X, \omega_X)$ be an elementary Lefschetz fibration. Fix an integer $Q$. Assume that $\int_F \omega_X \geq Q + 1$ and $Q \neq g(F) - 1$.

**A.** Then for a generic $\Omega_{X}$–tame almost complex structure $J$ which is sufficiently close to $J_{h}(X, \omega_X)$ (see Definition 2.3), we have a well–defined homomorphism

$$PFH(X, \omega_X)_{J} : A(X) \otimes PFH_{*}(Y, \omega, Q) \rightarrow \mathbb{Z},$$

where $A(X) = [U] \otimes \Lambda^{*}H_{1}(X, \mathbb{Z})$ and $U$ is the $U$–map (see [2]). The homomorphism is defined by counting $J$–holomorphic curves.

**B.** Moreover, we have:

- If $Q > g(F) - 1$, then
  $$PFH(X, \omega_X)_{J}((\Pi_{a} e_{a}) e_{0}^{m_{0}} e_{1}^{m_{1}}) = 1$$
  and $PFH(X, \omega_X)_{J}$ maps the other ECH generators to zero. Here $a$ are critical points of a Morse function $f_{S}$ with $\nabla^{2} f_{S}(a) > 0$. Each $e_{a}$ is the periodic orbit corresponding to $a$, and $\{e_{i}\}_{i=0,1}$ are the only two degree 1 elliptic periodic orbits of the Dehn twist. For the precise definition, please refer to Section 2.3.

- If $g(F) - 1 \geq 2Q$, then
  $$PFH(X, \omega_X)_{J}(e^{m}) = 1$$
  and maps the other generators of $PFH_{*}(Y, \omega, Q)$ to zero. In this case, $e_{0}$ and $e_{1}$ are homologous and $e$ is their homology class.

It is not difficult to see that the cobordism map $PFH(X, \omega_X)_{J}$ can split into

$$PFH(X, \omega_X)_{J} = \sum_{\Gamma_X \in H_{2}(X, \partial X, \mathbb{Z})} PFH(X, \omega_X, \Gamma_X)_{J}.$$  

Some of these components are invariant under the blow–up (see Lemma 5.2 of [3]). Thus we have the following corollary.

**Corollary 1.1.** Let $\{x_{i}\}_{i=1}^{k} \subset X$ be a finite set of points such that $\pi_X(x_{i}) \neq \pi_X(x_{j})$ for $i \neq j$ and $\pi_X(x_{i}) \neq 0$ for all $i$. Let $(\pi_X' : X' \rightarrow D, \omega_X')$ be the blow–up of $X$ at $\{x_{i}\}_{i=1}^{k}$. For any $\Gamma_X \in H_{2}(X, \partial X, \mathbb{Z})$ regarded as a homology class in $X'$, then $PFH(X', \omega_X', \Gamma_X)_{J'}$ satisfies the same conclusions in Theorem 1.
Remark 1.1. In general, neither PFH nor its cobordism maps can be defined with $\mathbb{Z}_2$– or $\mathbb{Z}$–coefficients. One needs certain monotonicity assumptions or introducing the local coefficient system (see [7], [14]). But all the objects we are considering satisfy the monotonicity properties if $Q \neq g(F) - 1$. For details please see Lemma 5.1 of [7] and Remark 4.2. We use $\mathbb{Z}$–coefficient throughout.

Remark 1.2. The assumption $Q \neq g(F) - 1$ is only used to guarantee that the monotonicity properties are true and it plays no role elsewhere. Therefore, the statement of part $A.$ in Theorem 1 is still true if we replace the $\mathbb{Z}$–coefficient by a local coefficient system.

The following two remarks concern the relations between $PFH(X, \omega_X)_J$ and the cobordism maps $HP_{sw}(X, \Omega_X)$ defined in [3]. The construction of $HP_{sw}(X, \Omega_X)$ is parallel to Hutchings and Taubes’s results for embedded contact homology [13]. It relies heavily on the isomorphism “PFH=SWF” [14] and the Seiberg–Witten theory.

Remark 1.3. Suppose that $Q > g(F) - 1$ and the cobordism maps $PFH(X, \omega_X)_J$ are defined by using almost complex structures compatible with the symplectic form. ([3] calls them cobordism–admissible.) Then we can show that $PFH(X, \omega_X)_J$ agrees with the one $HP_{sw}(X, \Omega_X)$ defined in [3]. In particular, $PFH(X, \omega_X)_J$ is independent of the choice of such a class of almost complex structures.

The idea of the proof is to establish a 1–1 correspondence between the holomorphic curves and solutions to Seiberg–Witten equations perturbed by the symplectic form. It requires many aspects of the Seiberg–Witten equations and argument from Taubes’s series of papers [18], [19], so it has beyond the scope of this paper. We refer the reader to Section 8 of [3] where proves a parallel result for other Lefschetz fibrations. [3] summarizes the main argument of Taubes and explains why the argument can be adapted to our setting. The only difference here with [3] is that there are extra covers of holomorphic planes, called special holomorphic planes, contributed to the cobordism maps. To deal with such curves, the argument has already been carried out by C.Gerig [5]. For a class of symplectic cobordisms, Gerig proves that the cobordism maps on embedded contact homology defined by holomorphic curves agree with the cobordism maps on Seiberg–Witten Floer cohomology via the isomorphism “ECH=SWF” in [19].

Note that Gerig’s setup differs from ours in the following two aspects: Firstly, the triple $(Y, \pi, \omega)$ is replaced by a contact 3–manifold in [5]. Also, the periodic orbits are changed to be Reeb orbits. Secondly, the Seiberg–Witten equations are perturbed by $\omega$ and $\Omega_X$ in [3] while they are changed to be the contact form and
the symplectic form of a symplectic cap respectively. These changes only influence the proof of the “$\text{SW} \Rightarrow \text{Gr}$” degeneration because the other arguments mainly take place locally near the holomorphic curves. The “$\text{SW} \Rightarrow \text{Gr}$” degeneration in our setting has been proved in Proposition 5.12 of [3]. Therefore, the proof in [5] can be applied to our setup with only notational changes.

**Remark 1.4.** For the case $Q \leq g(F) - 1$, the author conjectures that the conclusion in the above remark should be still true. The only difference in the current case is that we need to use certain tame almost complex structures to define PFH (see 1.1.4 of [14]) and its cobordism maps; otherwise, the fibers of $Y$ violate the compactness of the moduli space. However, the techniques in Taubes’s papers require a compatible almost complex structure rather than a tame one. To overcome this issue, a solution is provided by Lee and Taubes [14]. For each tame almost complex structure $J$ in the symplectization of $Y$, they modify the symplectic form such that $J$ is compatible with this new symplectic form. If their construction can be generalized to the cobordism case, then we may run the same argument as in the case $Q > g(F) - 1$ to define $HP_{sw}(X, \Omega_X)$ and show that $HP_{sw}(X, \Omega_X) = PFH(X, \omega_X)_J$.

We believe that $PFH(X, \omega_X)_J$ is independent of the almost complex structures defined it. If $2Q \leq g(F) - 1$, then we already see this from the computation in Theorem 1.

**Motivations** There are several motivations for this paper. First, while PFH is defined in terms of holomorphic curves, the cobordism maps currently are only defined by using the Seiberg–Witten theory. Even we can use the Seiberg–Witten theory as a ‘black box’ to construct the cobordism maps on PFH, it is still meaningful to understand the cobordism maps in terms of holomorphic curves. This could help with finding higher–dimensional analogues of this invariant. Also, in many applications, the holomorphic curves definition seems more suited for computation. Secondly, in the previous result [3], the author defines the cobordism maps by using the holomorphic curve methods for fiberwise symplectic cobordisms satisfying certain technical assumptions (♠). The elementary Lefschetz fibrations are the simplest non–trivial examples that do not satisfy the assumptions (♠). Therefore, the results here help with generalizing the results in [3] to more situations. Thirdly, the computation here should help with computing the cobordism maps $PFH(W, \omega_W)_J$, where $(W, \pi_W, \omega_W)$ is the fiberwise symplectic cobordism constructed in [17]. It is a Lefschetz fibration over an annulus with only one critical point.
Idea of the proof  We end up the introduction by summarizing the idea of the proof. First, we establish a combinatorial formula for ECH index in Section 3. From this index formula, we can see there are only a few generators that can be mapped to non–zero. Secondly, there are special sections of \((X, \pi_X)\), called horizontal sections, whose positive ends are asymptotic to these potential generators. Thanks to Seidel’s observation, these sections are holomorphic curves for a suitable choice of almost complex structures. Finally, the energy constraints ensure that the horizontal sections are the only holomorphic curves that contributed to the cobordism maps. The above three ingredients lead to Theorem 1.

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2 Preliminaries

2.1 Review of ECH index and periodic Floer homology

In this section, we review those aspects of periodic Floer homology that we need in the proof of Theorem 1. For more details, please refer to [9].

Orbit set  Let \((Y, \pi)\) be a surface fibration over the circle together with a Hamiltonian structure \(\omega\). The 2–form \(\omega\) gives a splitting of \(TY = TY^\text{vert} \oplus TY^\text{hor}\), where \(TY^\text{vert} = \ker \pi_*\) and \(TY^\text{hor}\) is the \(\omega\)–orthogonal complement of \(TY^\text{vert}\). The Reeb vector field \(R\) of \((Y, \pi, \omega)\) is characterized by conditions:

\[
R \in \Gamma(TY^\text{hor}) \quad \text{and} \quad \pi_*(R) = \partial_t,
\]

where \(\partial_t\) is the coordinate vector field of \(S^1\). A periodic orbit is a smooth map \(\gamma : \mathbb{R}/q\mathbb{Z} \to Y\) satisfying the ODE \(\partial_t \gamma = R \circ \gamma\) for some \(q > 0\). The number \(q\) is called the period or degree of the periodic orbit. We say that a periodic orbit \(\gamma\) is elliptic if the eigenvalues of the linearized return map \(P_\gamma\) are on the unit circle, positive hyperbolic if the eigenvalues of \(P_\gamma\) are real positive numbers, and negative hyperbolic if the eigenvalues of \(P_\gamma\) are real negative numbers.
An orbit set $\alpha = \sum_i m_i \alpha_i$ is a finite formal sum of periodic orbits, where $\alpha_i$ are distinct, non-degenerate, irreducible embedded periodic orbits and $m_i$ are positive integers. An orbit set $\alpha$ is called an ECH generator if $m_i = 1$ whenever $\alpha_i$ is a hyperbolic orbit. In the rest of the paper, we write an orbit set using multiplicative notation $\alpha = \Pi_i \alpha_i^{m_i}$ instead of summation notation. The following definition is useful when we define the cobordism maps on PFH.

**Definition 2.1.** (see [10] Definition 4.1) Fix $Q > 0$. Let $\gamma$ be an embedded elliptic orbit with degree $q \leq Q$.

- $\gamma$ is called $Q$-positive elliptic if the rotation number $\theta$ is in $(0, \frac{q}{Q}) \mod 1$.
- $\gamma$ is called $Q$-negative elliptic if the rotation number $\theta$ is in $(-\frac{q}{Q}, 0) \mod 1$.

The ECH index Let $(W, \pi_W, \omega_W)$ be a fiberwise symplectic cobordism from $(Y, \pi, \omega)$ to $(Y', \pi', \omega')$, where $(W, \pi_W)$ is a Lefschetz fibration such that $\pi_W^{-1}(\partial B) = Y \cup (-Y')$, and $\omega_W$ is a fiberwise symplectic form which agrees with $\omega$ and $\omega'$ along $Y$ and $Y'$ respectively.

Given orbit sets $\alpha = \Pi_i \alpha_i^{m_i}$ and $\beta = \Pi_j \beta_j^{n_j}$ on $Y$ and $Y'$ respectively, define the space of relative homology classes $H_2(W, \alpha, \beta)$. A typical element in $H_2(W, \alpha, \beta)$ is a 2–chain $Z$ in $W$ such that $\partial Z = \sum_i m_i \alpha_i - \sum_j n_j \beta_j$, modulo the boundary of 3–chains. Note that $H_2(W, \alpha, \beta)$ is an affine space over $H_2(W, \mathbb{Z})$. Given $Z \in H_2(W, \alpha, \beta)$ and trivializations $\tau$ of $\ker \pi_*|_\alpha$ and $\ker \pi'_*|_\beta$, the ECH index is defined by

$$I(\alpha, \beta, Z) = c_\tau(Z) + Q_\tau(Z) + \sum_i \sum_{p=1}^{m_i} CZ_\tau(\alpha_i^p) - \sum_j \sum_{q=1}^{n_j} CZ_\tau(\beta_j^q),$$

where $c_\tau(Z)$ and $Q_\tau(Z)$ are respectively the relative Chern number and the relative self-intersection number (see [6] and [8]), and $CZ_\tau$ is the Conley–Zehnder index. The ECH index $I$ depends only on orbit sets $\alpha$, $\beta$, and the relative homology class $Z$.

**Almost complex structure** Recall that the symplectic form on $W$ is defined by $\Omega_W = \omega_W + \pi^* \omega_B$, where $\omega_B$ is a large volume form of $B$. We can define the symplectic completion $(\overline{W}, \omega_W)$ by adding cylindrical ends. Moreover, the fibration structure can be extended to the completion. More details can be found in Section 2.3 of [3]. To introduce holomorphic curves in $\overline{W}$, we need the following definition of almost complex structures.
Definition 2.2. An almost complex structure $J$ on $\overline{W}$ is called $\Omega_W$–tame if $J$ satisfies the following properties:

1. On the cylindrical ends, $J$ is $\mathbb{R}_s$–invariant and $J(\partial_s) = R$, where $R$ is the Reeb vector field. Also, we require that $J$ sends $\ker \pi_*$ to itself along periodic orbits with degree less than or equal to $Q$.

2. $J$ is $\Omega_W$–tame.

3. Identify a neighbourhood of a critical point of $\pi_W$ with the local model \([\mathbf{II}]\). Then $J$ agrees with the standard complex structure $J_0$ in that neighbourhood.

The space of $\Omega_W$–tame almost complex structures is denoted by $J_{tame}(W,\omega_W)$. It carries a natural $C^\infty$ topology.

For the purpose of computing the cobordism maps, we require a special class of almost complex structures that preserve the vertical and horizontal bundles. Recall that the admissible 2–form $\omega_W$ splits $TW$ into $TW_{hor}$ and $TW_{vert}$, where $TW_{vert} = \ker \pi_{W*}$ and $TW_{hor} = \{v \in TW | v \perp_{\omega_W} TW_{vert}\}$. Let $\overline{B}$ be the completion of $B$ by adding cylindrical ends. Let $(s,t)$ be the coordinates of the cylindrical ends. We fix a $\omega_B$–compatible complex structure $j_B$ such that $j_B(\partial_s) = \partial_t$ on the cylindrical ends.

Definition 2.3. Let $J_h(W,\omega_W) \subset J_{tame}(W,\omega_W)$ be the subspace of $\Omega_W$–tame almost complex structures on $\overline{W}$ with the following properties:

1. Away from critical points of $\pi_W$, $J(TW_{hor}) \subset TW_{hor}$ and $J(TW_{vert}) \subset TW_{vert}$.

2. Away from critical points of $\pi_W$, $J|_{\ker \pi_{W*}}$ is compatible with $\omega_W$.

3. $(\pi_W)_*$ is complex linear with respect to $J$ and $j_B$, i.e., $j_B \circ (\pi_W)_* = (\pi_W)_* \circ J$.

Remark 2.1. For a generic $J \in J_h(W,\omega_W)$, all simple holomorphic curves can achieve transversality except for the fibers and horizontal sections. See Proposition 21.1 of [17]. For the definition of horizontal sections, please refer to Definition 4.6.

Notation. The definition of $J_h(W,\omega_W)$ depends obviously on the choice of $j_B$, but it plays no role in our argument. To simplify the notation, we suppress $j_B$ from the notation.
Holomorphic currents  Let \((\overline{W}, \pi_W, \omega_W)\) be the symplectic completion of \((W, \pi_W, \omega_W)\). Fix an \(\Omega_W\)-tame almost complex structure \(J\). A \(J\)-holomorphic current \(C = \sum d_a C_a\) from \(\alpha\) to \(\beta\) is a finite formal sum of \(J\)-holomorphic curves such that \(C\) is asymptotic to \(\alpha\) and \(\beta\) respectively in the current sense, where \(C_a\) are distinct, irreducible, somewhere injective \(J\)-holomorphic curves with finite energy \(\int_{C_a} \omega_W < \infty\) and \(d_a\) are positive integers. The number 
\[
E(C) = \int_C \omega_W = \sum_a d_a \int_{C_a} \omega_W
\]
is called \(\omega_W\)-energy. Let \(\mathcal{M}_i^j(\alpha, \beta, Z)\) denote the space of holomorphic currents from \(\alpha\) to \(\beta\) with ECH index \(I = i\) and relative homology class \(Z\).

**Definition 2.4.** A holomorphic current \(C = \sum d_a C_a\) is called embedded if \(d_a = 1\) for any \(a\) and \(C_a\) are pairwise disjoint embedded holomorphic curves.

**Definition of periodic Floer homology**  Now let us return to the definition of PFH. Fix an integer \(Q\). The chain complex \(\text{PFC}(Y, \omega, Q)\) of PFH is a free module generated by ECH generators with degree \(Q\). Consider the special case that \(\overline{W} = \mathbb{R} \times Y\) and \(J\) is a generic \(\mathbb{R}\)-invariant \(\Omega_W\)-tame almost complex structure. The differential of PFH is defined by 
\[
\langle \partial \alpha, \beta \rangle = \sum_{Z \in H_2(Y, \alpha, \beta)} \left( \# \mathcal{M}_i^j(\alpha, \beta, Z) / \mathbb{R} \right)
\]
(2)
The obstruction gluing argument in [11] and [12] show that \(\partial^2 = 0\). The homology of \((\text{PFC}(Y, \omega, Q), \partial)\) is called periodic Floer homology, denoted by \(PFH_*(Y, \omega, Q)\).

**Cobordism maps on PFH**  Let \((W, \pi_W, \omega_W)\) be a fiberwise symplectic cobordism from \((Y, \pi, \omega)\) to \((Y', \pi', \omega')\). It is expected to define the cobordism maps in chain level by 
\[
\langle PFC(W, \omega_W), \alpha, \beta \rangle = \sum_{Z \in H_2(W, \alpha, \beta)} \left( \# \mathcal{M}_i^j(\alpha, \beta, Z) \right).
\]
(3)
However, the above formula doesn’t make sense in general due to the appearance of holomorphic currents with negative ECH index. The reasons are explained in Section 5.5 of [9]. But in our special case, we show that it actually works.

### 2.2 Elementary Lefschetz fibration

In this subsection, we give more details about the elementary Lefschetz fibration \((X, \pi_X, \omega_X)\). To this end, let us first return to the exact Lefschetz fibration \((E, \pi_E, \omega_{C^2}|_E)\). What follows here paraphrase parts of the accounts in [17], [16].
Let \( T = T^*S^1 \) be the cotangent bundle of the circle. The standard coordinates on \( T \) are
\[
T = \{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 : <u, v> = 0 \mid |v| = 1 \}.
\]
In terms of the coordinates, the geodesic flow \( \sigma_t \) on \( T \) is
\[
\sigma_t(u, v) = \left( \cos(2\pi t)u -\sin(2\pi t)|u|v, \cos(2\pi t)v + \sin(2\pi t)\frac{u}{|u|} \right).
\]
Sometimes it is more convenient to identify \( T \) with a cylinder \( \mathbb{R}_x \times (\mathbb{R}_y/\mathbb{Z}) \) via changing of coordinates
\[
u = ix e^{i2\pi y}, \quad v = e^{i2\pi y}.
\]
Let \( T_\lambda = \{(u, v) \in T \mid |u| \leq \lambda \} \). Let \( V = \cup_{z \in \partial S} V_z \cup (0, 0) \) be the union of all the vanishing cycles (include \((0, 0)\)) of \( E \), where \( V_z = \sqrt{z} \{ (0, v) \in T \} \subset \pi_E^{-1}(z) \). Away from \( V, E \) can be identified with a trivial bundle over a disk via the following diffeomorphism:
\[
\Phi : E - V \rightarrow D_\delta \times (T_\lambda - T_0)
\]
\[
\Phi(x) = (\pi(x), \sigma_4(-Im(\hat{x})|\text{Re}(\hat{x})|, \text{Re}(\hat{x})|\text{Re}(\hat{x})|^{-1})) = (\pi(x), \sigma_4(\Phi_2(\hat{x}))),
\]
where \( \Phi_2(x) = (-Im(\hat{x})|\text{Re}(\hat{x})|, \text{Re}(\hat{x})|\text{Re}(\hat{x})|^{-1}) \) and \( \hat{x} = e^{-i\pi t}x \) provided that \( \pi(x) = re^{2\pi it} \). \( \Phi \) satisfies the following properties:

1. \( \Phi \) is a diffeomorphism on each fiber.

2. \( (\Phi^{-1})^*\theta_{C^2} = \theta_T - \tilde{R}_r(|u|)dt \), where \( \theta_{C^2} = \frac{i}{4} \sum_{k=1}^2 (z_k d\bar{z}_k - \bar{z}_k dz_k), \tilde{R}_r(t) = \frac{t}{2} - \frac{i}{\delta} \sqrt{t^2 + 4t^2} \) and \( \theta_T = xdy \) in terms of coordinates \( (4) \).

As mentioned in the Introduction, we need to modify \( \omega_{C^2} \) so that it is “trivial” near the horizontal boundary in the sense that it agrees with \( d\theta_T \). Fix \( 0 < \delta_0 << 1 \).
Take a cutoff function \( g \) such that \( g(t) = 0 \) near \( t = 0 \) and \( g(t) = 1 \) where \( t \geq \lambda - \delta_0 \). Define a 1-form by
\[
\theta_E = \Phi^*(\theta_T + (g(|x|) - 1)\tilde{R}_r(|x|)dt).
\]
Then the 2-form \( \omega_E = d\theta_E \) satisfies the requirement.

Let \( N(\partial h)E = D_\delta \times ([-\lambda, -\lambda + \delta_0] \cup [\lambda - \delta_0, \lambda]) \times S^1 \) be a neighborhood of the horizontal boundary. Let \((S, \omega_S)\) be a connected symplectic surface with boundary \( \partial S = S^1 \cup (-S^1) \). A collar neighborhood of \( \partial S \) is identified with \( N(\partial S) = ([-\lambda, -\lambda + \delta_0] \cup [\lambda - \delta_0, \lambda]) \times S^1 \). Choose \( \omega_S \) such that \( \omega_S|_{N(\partial S)} = d\theta_T \). Take a trivial fiberwise symplectic fibration over a disk \((D_\delta \times S, \omega_S)\). Then \( \pi_X : X \rightarrow D_\delta \) is obtained by gluing \( E \) with \( D_\delta \times S \) via identifying \( N(\partial h)E \) and
$D_\delta \times N(\partial S)$. The admissible 2–form $\omega_X$ is obtained by gluing $\omega_E$ with $\omega_S$ in the obvious way. The fiber of $X$ is $F = S \cup T_\lambda$ and $Y = \partial_\delta E \cup (S \times S^1)$.

Lemma 18.4 of [17] shows that the symplectic monodromy $\phi : \pi^{-1}(\delta) \to \pi^{-1}(\delta)$ of $(E, \omega_E)$ is given by

$$\phi(u, v) = \begin{cases} \sigma R'_r(|u|)(u, v) & \text{if } u \neq 0 \\ (0, -v) & \text{if } u = 0 \end{cases}.$$ 

Here $R_r(t) = (1 - g(t)) \tilde{R}_r(t)$. The above monodromy is called the Dehn twist. We extend $\phi$ to be $Id$ over $S$.

Although $\phi$ depends on the choices of $\delta, \lambda$, and the cutoff function $g$, it is unique up to a Hamiltonian isotopy. (See Lemma 2.1 of [17].) Since the periodic Floer homology is invariant under a Hamiltonian isotopy of the monodromy (see Corollary 1.1 of [14]), we don’t emphasize these choices unless otherwise stated.

In the rest of the paper, we always assume that $\delta = 1$ and denote $R(t) = R_1(t)$. Also, we choose a suitable cutoff function $g$ such that $0 \leq R'(|t|) \leq \frac{1}{2}$ and $R''(|t|) \leq 0$.

The following lemma concerns the condition $\int_F \omega_X \geq Q + 1$ in Theorem 1. We can always construct an admissible 2–form satisfying this condition.

**Lemma 2.5.** Given $L > 0$, we can find an admissible 2–form $\omega_X$ such that $\int_F \omega_X > L$.

**Proof.** Let $\omega_X$ be the admissible 2–form constructed before. We modify $\omega_X$ such that $\int_F \omega_X > L$ as follows.

Let $U = ([-\lambda - 2\delta_0, -\lambda + \delta_0] \cup [\lambda - \delta_0, \lambda + 2\delta_0]) \times S^1_y$ be a collar neighborhood of $\partial S$. Take a function $f : S \to \mathbb{R}$ supported in $U$ such that $f = c_1$ on $[\lambda - \delta_0, \lambda + \delta_0] \times S^1$ and $f = c_2$ on $([-\lambda - \delta_0, -\lambda + \delta_0] \times S^1)$, where $c_1$ and $c_2$ are positive constants. Write $\omega_S = d\theta_S$ for some 1–form $\theta_S$. Define $\theta'_S = \theta_S + f dy$. Then we still have $\omega'_S = d\theta'_S = \omega_E$ on $N(\partial S)$ by definition. Thus we can glue $\omega'_S$ with $\omega_E$ together as before. The result is called $\omega'_X$. By Stokes’ theorem, we have

$$\int_F \omega'_X = \int_F \omega_X + c_1 - c_2.$$ 

Take $c_1 - c_2 > L$. Then we get the conclusion.

**Lemma 2.6.** We have $H_2(X, \mathbb{Z}) = \mathbb{Z}$. Moreover, the generator is represented by the fiber.

**Proof.** Recall that $X$ is obtained by gluing $E$ and $D \times S$ along their horizontal boundaries. The conclusion follows from the Mayer–Vietoris theorem for the singular homology.
2.3 Periodic orbits

In this section, we describe the periodic orbits on \((Y, \pi, \omega)\). Under the diffeomorphism \([5]\) and the coordinates \([4]\), the Dehn twist is

\[
\phi(x, y) = \begin{cases} 
(x, y + R'(x)) & \text{if } x > 0 \\
(x, y - R'(-x)) & \text{if } x < 0 \\
(x, y \pm \frac{1}{2}) & \text{if } x = 0.
\end{cases}
\]

At each \(x_0\) such that \(R'(x_0) = \frac{p}{q}\), the torus \(T_{x_0} = S_1 \times \{x_0\} \times S_y\) is foliated by embedded periodic orbits. Each periodic orbit is of the form \(\gamma_{\xi}(\tau) = (\tau, x_0, y_0 + R'(x_0)\tau)\) where \(x_0 > 0\) and \(\gamma_{\xi}(\tau) = (\tau, x_0, y_0 - R'(-x_0)\tau)\) if \(x_0 < 0\) (When \(x_0 < 0\), we write \(R'(-x_0) = 1 - \frac{\xi}{q}\)), where \(\tau \in \mathbb{R}/(q\mathbb{Z})\). The integers \(p, q\) here are relatively prime. The torus \(T_{x_0}\) is called a Morse–Bott torus.

To ensure that the periodic orbits are non–degenerate, we need to perturb the admissible 2–form \(\omega\). At each Morse–Bott torus \(T_{x_0}\), we can perform a standard perturbation such that there are only two periodic orbits that survive (see \([2]\)). One is elliptic and the other one is positive hyperbolic, denoted by \(e_{\xi q}\) and \(h_{\xi q}\) respectively. These two periodic orbits are corresponding to the minimum and maximum of a perfect Morse function \(f_{T_{x_0}}\) on the circle. For any fixed integer \(Q\), we can arrange that all the periodic orbits on \(\partial_v E\) with degree less than or equal to \(Q\) are either \(e_{\xi q}\) or \(h_{\xi q}\).

To describe the periodic orbits in the trivial part \(S^1 \times S\) of \(Y\), let \(f_S : S \to \mathbb{R}\) be a small Morse function such that \(\nabla f_S\) is transversal to \(\partial S\) and there are critical points \(\{p_i\}_{i=0,1}\) and \(\{q_i\}_{i=0,1}\) on \(N(\partial S)\) satisfying \(\nabla^2 f_S(p_i) > 0\) and \(tr(\nabla^2 f_S(q_i)) = 0\). In addition, there are no other critical points on \(N(\partial S)\).

For sufficiently small \(f_S\), the periodic orbits with degree less than or equal to \(Q\) over \((S^1 \times S, \omega_S + df_S \wedge dt)\) only consists of constant orbits at critical points of \(f_S\). We always use \(a \in \text{Crit}(f_S)\) to denote the critical point on \(S - N(\partial S)\). If \(tr(\nabla^2 f_S(a)) \neq 0\), then the corresponding periodic orbit is elliptic, denoted by \(e_a\). Moreover, \(e_a\) is either \(Q\)–positive or \(Q\)–negative depending on the sign of \(\nabla^2 f_S(a)\) accordingly. If \(tr(\nabla^2 f_S(a)) = 0\), then the corresponding periodic orbit is positive hyperbolic, denoted by \(h_a\). \(\{e_i\}_{i=0,1}\) and \(\{h_i\}_{i=0,1}\) are respectively elliptic orbits and hyperbolic orbits corresponding to \(\{p_i\}_{i=0,1}\) and \(\{q_i\}_{i=0,1}\).

In conclusion, we can arrange that all the periodic orbits with degree less than or equal to \(Q\) are either \(e_{\xi q}\) or \(h_{\xi q}\) or \(e_a\) or \(h_a\). Keep in mind that the periodic orbits here are either \(Q\)–positive elliptic or \(Q\)–negative elliptic or positive hyperbolic.
Remark 2.2. In order to ensure that the elementary Lefschetz fibration is non-negative in the sense of Definition 4.6, the functions \( f_{\tau_0} \) and \( f_S : S \to \mathbb{R} \) are chosen to be non-negative and their minimums are zero.

Remark 2.3. The description here in fact is the same as [7]. The model of the Dehn twist using by Hutchings and Sullivan is slightly different from ours here. But we can transfer our model to theirs via changing of coordinates and a Hamiltonian isotropy.

Define \( f : [-\lambda, \lambda] \times S^1_y \to [0, 1] \times S^1_t \) by sending \((x, y)\) to \((\frac{x}{2\lambda} + \frac{1}{2}, y)\). Let \( H(x) = R(x) + \frac{x^2}{4\lambda} - \frac{x}{2} \) be a Hamiltonian function. Let \( \phi_H \) be the time-1 flow of the Hamiltonian vector field of \( H \). Then

\[
f \circ \phi_H \circ f^{-1}(s, t) = \begin{cases} 
(s, t - s + 1) & \text{if } \frac{1}{2} \leq s \leq 1 \\
(s, t - s) & \text{if } 0 \leq s < \frac{1}{2}
\end{cases}
= (s, t - s).
\]

coincides with the model in [7].

For the purpose of computing the ECH index, we want to express the periodic orbits in terms of coordinates \( x = (x_1, x_2) \). The result is summarized in the following lemma.

Lemma 2.7. Let \( x_0 \in [-\lambda, \lambda] \) such that \( R'(x_0) = \frac{\ell}{q} \) and \( \gamma_{x_0} \) is the periodic orbit at \( x_0 \). Let \( x(\tau) = (x_1(\tau), x_2(\tau)) = \Phi^{-1}(\gamma_{x_0}), \tau \in \mathbb{R}/(q\mathbb{Z}) \). We have the following two cases: If \( x_0 \geq 0 \), then

\[
\begin{align*}
\begin{cases} 
 x_1 &= \frac{1}{2}e^{h+i\phi_0}e^{2\pi i \frac{\ell}{q}} + \frac{1}{2}e^{-h-i\phi_0}e^{2\pi i \frac{\ell}{q}} \\
x_2 &= -\frac{i}{2}e^{h+i\phi_0}e^{2\pi i \frac{\ell}{q}} + \frac{i}{2}e^{-h-i\phi_0}e^{2\pi i \frac{\ell}{q}} 
\end{cases}
\end{align*}
\tag{6}
\]

for some constant \( h = h(x_0) \geq 0 \).

If \( x_0 \leq 0 \), then

\[
\begin{align*}
\begin{cases} 
 x_1 &= \frac{1}{2}e^{-h-i\phi_0}e^{2\pi i \frac{\ell}{q}} + \frac{1}{2}e^{h+i\phi_0}e^{2\pi i \frac{\ell}{q}} \\
x_2 &= \frac{i}{2}e^{-h-i\phi_0}e^{2\pi i \frac{\ell}{q}} - \frac{i}{2}e^{h+i\phi_0}e^{2\pi i \frac{\ell}{q}} 
\end{cases}
\end{align*}
\tag{7}
\]

for some constant \( h = h(x_0) \geq 0 \). In both cases, \( h = 0 \) if and only if \( x_0 = 0 \).

Proof. Firstly, one can check that the geodesic flow \( \sigma_t \) can be written as

\[
\sigma_t(x, y) = \begin{cases} 
(x, y \pm t) & \text{if } \pm x > 0 \\
(0, y \pm \frac{1}{2}) & \text{if } x = 0.
\end{cases}
\]

under the identification [4]. Let \( x \in E \) and \( \dot{x} = e^{-i\theta}x = \dot{p} + i\dot{q} \). Write \( \Phi_2(\dot{x}) = (u, v) \in T \) as \( v = \frac{\dot{p}}{||\dot{p}||} = e^{i2\pi \theta} \) and \( u = -\dot{q}||\dot{p}|| = \pm||\dot{p}||i\dot{q}e^{i2\pi \theta} \).
If \( \gamma_\mathbb{R}(\tau) = (\tau, x_0, y_0 + \frac{q}{q}\tau) \) and \( x_0 > 0 \), then

\[
\sigma_\mathbb{R}^t(\Phi_2(\mathbf{x})) = \sigma_\mathbb{R}^t(-\hat{q}|\hat{p}|, \frac{\hat{p}}{|\hat{p}|}) = (|\hat{p}| |\hat{q}|, \theta + \frac{t}{2}).
\]

under the identification \([4]\). Therefore, \( t = \tau, \theta = y_0 + \frac{q}{q}\tau - \frac{1}{2}t \) and \( x_0 = |\hat{p}| |\hat{q}| \).

By relations \(|\hat{p}|^2 - |\hat{q}|^2 = 1\), \( \hat{p}, \hat{q} \geq 0 \) and \( x_0^2 = |\hat{p}|^2 |\hat{q}|^2 \), we get

\[
|\hat{p}|^2 = \sqrt{x_0^2 + \frac{1}{4} + \frac{1}{2}} \tag{8}
\]

\[
|\hat{q}|^2 = \sqrt{x_0^2 + \frac{1}{4} - \frac{1}{2}}.
\]

Write \( |\hat{p}| = \frac{e^h + e^{-h}}{2} \) and \( |\hat{q}| = \frac{e^h - e^{-h}}{2} \). Follows from the definition, we have

\[
\begin{align*}
p &= \cos(\pi t)\hat{p} - \sin(\pi t)\hat{q} \\
q &= \sin(\pi t)\hat{p} + \cos(\pi t)\hat{q}.
\end{align*}
\]

Using the relations \( \frac{\hat{p}}{|\hat{p}|} = e^{i2\pi \theta} \) and \( -\hat{q}|\hat{p}| = i|\hat{q}| |\hat{p}| e^{i2\pi \theta} \), we have

\[
\begin{align*}
p &= \cos(\pi t)\hat{p} - \sin(\pi t)\hat{q} = \frac{1}{2} e^h e^{i2\pi(\theta + \frac{t}{2})} + \frac{1}{2} e^{-h} e^{i2\pi(\theta - \frac{t}{2})} \\
q &= \sin(\pi t)\hat{p} + \cos(\pi t)\hat{q} = i e^{-h} e^{i2\pi(\theta - \frac{t}{2})} - i e^h e^{i2\pi(\theta + \frac{t}{2})}.
\end{align*}
\]

Then

\[
\begin{align*}
x_1 &= \frac{1}{2} e^h e^{i2\pi(\theta + \frac{t}{2})} + \frac{1}{2} e^{-h} e^{-i2\pi(\theta - \frac{t}{2})} \\
x_2 &= i e^{-h} e^{-i2\pi(\theta - \frac{t}{2})} - i e^h e^{i2\pi(\theta + \frac{t}{2})}.
\end{align*}
\]

Replace \( \theta \) by \( y_0 + \frac{q}{q}\tau - \frac{1}{2}t \) and \( t = \tau \), then we get the result.

For the case that \( \gamma_\mathbb{R}(\tau) = (\tau, x_0, y_0 + (\frac{q}{q} - 1)\tau) \) and \( x_0 < 0 \), the argument is similar. We left the details to the reader. We can regard the periodic orbit at \( x_0 = 0 \) as a limit of the periodic orbits at \( x \neq 0 \) and \( x \to 0 \). The periodic orbit at \( x_0 = 0 \) is

\[
\begin{align*}
x_1 &= \frac{1}{2} e^{i(-y_0 + \frac{1}{2}\tau)} + \frac{1}{2} e^{i(y_0 + \frac{1}{2}\tau)} \\
x_2 &= i \frac{1}{2} e^{i(-y_0 + \frac{1}{2}\tau)} - i \frac{1}{2} e^{i(y_0 + \frac{1}{2}\tau)}.
\end{align*} \tag{9}
\]

\[\square\]

3 A formula for the ECH index

In this section, we deduce a combinatorial formula for the ECH index as in \([7]\), \([10]\). There may be other smarter ways to compute the ECH index, but here
we construct several surfaces in $X$ explicitly and then compute the ECH index directly from the definition.

First, let us consider the ECH index of relative homology classes in $E$. Note that $H_2(E, \mathbb{Z}) = 0$, there is a unique element $Z_\alpha$ in $H_2(E, \alpha)$ for each orbit set $\alpha$. Therefore, we denote the ECH index, the relative Chern number, and the relative self–intersection number by $I(\alpha)$, $c_\tau(\alpha)$ and $Q_\tau(\alpha)$ respectively. The result on the ECH index is as follows.

**Theorem 3.1.** Let $\pi : E \to D$ be the exact Lefschetz fibration defined in Introduction and orbit set $\alpha = \Pi_{\gamma_{i\alpha}}$ satisfying $\frac{p_i}{q_i} \geq \frac{p_j}{q_j}$ for $i \leq j$. Then the ECH index is

$$I(\alpha) = Q + P(Q - P) - \sum_{i<j}(p_iq_j - p_jq_i) - e(\alpha),$$

where $e(\alpha)$ is the total multilpity of elliptic orbits in $\alpha$, $P = \sum_i p_i$ and $Q = \sum_i q_i$.

**Proof.** We compute the quantities $c_\tau$, $Q_\tau$ and $CZ_\tau$ in Lemmas 3.4, 3.7 and (14) respectively. Their proof will appear in the upcoming subsection.

We can follow [7] to rewrite the above formula for ECH index in the following way. Let $w_j = \sum_{i=0}^j(p_i, q_i)$. Let $P(\alpha)$ be the convex path in the plane consisting of straight line segments between the points $w_{j-1}$ and $w_j$, oriented so that the origin is the initial endpoint. Let $\Lambda_\alpha$ be the region in the plane which is enclosed by $P(\alpha)$ and the line segment from $(0,0)$ to $(P,0)$ and the line segment from $(P,0)$ to $(P,Q)$. The area of $\Lambda_\alpha$ is

$$2\text{Area}(\Lambda_\alpha) = PQ - \sum_{i<j}(p_iq_j - p_jq_i).$$

Therefore, we can rewrite the ECH index as

$$I(\alpha) = Q + 2\text{Area}(\Lambda_\alpha) - P^2 - e(\alpha).$$

**Remark 3.1.** The additive property of the ECH index implies that the ECH index of a relative class in $\mathbb{R} \times \partial E$ is $I(\alpha) - I(\beta)$. It agrees with the index computation (Proposition 3.2) in [7].

**Corollary 3.2.** $I(\alpha) \geq 0$ and equality holds if and only if $\alpha = e_1^m e_0^n$ for some $m, n \geq 0$.

**Proof.** Since $0 \leq \frac{p_i}{q_i} \leq 1$, the triangle determined by $(0,0)$, $(0,P)$ and $(P,P)$ is inside the region $\Lambda_\alpha$. Therefore, $P^2 \leq 2\text{Area}(\Lambda_\alpha)$. By definition, $e(\alpha) \leq Q$. Therefore, $I(\alpha) \geq 0$. The equality holds if and only if $Q = e(\alpha)$ and $P^2 = 2\text{Area}(\Lambda_\alpha)$. The only possibility is that $\alpha = e_1^m e_0^n$. 

15
Now let us consider the ECH index of relative homology classes in the “trivial” part \( D \times S \). The result is as follows.

**Lemma 3.3.** Let \( a \) be a critical point of \( f_S \). Define \( C_a = \{ a \} \times D \). Let \( m_a \) be a non–negative integer. Then we have the following three cases:

- If \( \nabla^2 f_S(a) > 0 \), then the corresponding periodic orbit \( e_a \) is elliptic with Conley–Zehnder index \( CZ_\tau(e_a) = -1 \). In addition, \( I(m_aC_a) = 0 \).

- If \( \nabla^2 f_S(a) < 0 \), then the corresponding periodic orbit \( e_a \) is elliptic with Conley–Zehnder index \( CZ_\tau(e_a) = 1 \). In addition, \( I(m_aC_a) = 2m_a \).

- Finally, if \( tr(\nabla^2 f_S(a)) = 0 \), then the corresponding periodic orbit \( h_a \) is positive hyperbolic with Conley–Zehnder index \( CZ_\tau(e_a) = 0 \). In addition, \( I(m_aC_a) = m_a \).

Here \( \tau \) is the trivialization from a fixed trivialization of \( TS \) and the standard trivialization of \( TD \).

**Proof.** Let \((x, y)\) be local coordinates at \( a \). It is easy to check that the linearized Reeb flow is

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
f_{Sxy}(a) & f_{Syy}(a) \\
-f_{Sxx}(a) & f_{Sxy}(a)
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}.
\]

(12)

By (12), we know that if \( tr(\nabla^2 f_S(a)) \neq 0 \), then it is an elliptic orbit. Otherwise, it is a positive hyperbolic orbit. Also, when \( tr(\nabla^2 f_S(a)) \neq 0 \), the sign of rotation number of \( e_a \) is equal to \(-\text{sign}(\nabla^2 f_S(a))\). In conclusion, \( e_a \) is \( Q \)-negative if \( \nabla^2 f_S(a) > 0 \) and \( e_a \) is \( Q \)-positive if \( \nabla^2 f_S(a) < 0 \).

Finally, it is easy to check that \( c_\tau(C_a) = 1 \) and \( Q_\tau(C_a) = 0 \) from the definition. These two ingredients lead to the statements of the lemma. \( \square \)

Given an orbit set \( \alpha = \Pi_i \alpha_i^{m_i} \), we define a reference class \( Z_\alpha = \sum_i m_i[S_i] \in H_2(X, \alpha) \), where \( S_i \) are surfaces defined as follows. If \( \alpha_i = \gamma_{a_i}^{m_i} \), then we define a surface \( S_i \) to be the image of \( u \) in (15) or (16) accordingly. If \( \alpha_i \) is a critical point \( a \) of \( f_S \), then we define \( S_i \) by \( S_i = \{ a \} \times D \). It is worth noting that \( u \) doesn’t intersect \( \{ a \} \times D \). Write \( \alpha = (\Pi_a \gamma_a^{m_a})\Pi_i \gamma_{a_i}^{m_i} \), where \( \gamma_a \) are the constant periodic orbits at critical points \( a \). Then we have

\[
I(Z_\alpha) = \sum_a I(m_a(\{ a \} \times D)) + I(\Pi_i \gamma_{a_i}^{m_i}).
\]

Recall that \( H_2(X, \alpha) \) is an affine space over \( H_2(X, Z) \). By lemma 2.6 any \( Z \in H_2(X, \alpha) \) is of the form \( Z = Z_\alpha + m[F] \). Let us denote \( I_m(\alpha) = I(Z_\alpha + m[F]) \). Therefore, in general we have

\[
I_m(\alpha) = I_0(\alpha) + 2m(Q + 1 - g(F)).
\]

(13)
3.1 Proof of Theorem 3.1

In this subsection, we compute the relative Chern number \( c_\tau \), the relative self intersection number \( Q_\tau \) and the Conley–Zehnder index \( CZ_\tau \). Theorem 3.1 follows directly from these computations.

**Trivialization**  Let us first clarify the trivialization of what we are using. Restrict the map \( \Phi \) on the boundary \( \partial_v E \), we have a diffeomorphism \( \Phi : \partial_v E - V \to S^1_t \times (T\lambda - T\nu) = S^1_t \times ([-\lambda, 0] \cup (0, \lambda]) \times S^1_y \). The tangent bundle of \( T\lambda = [-\lambda, \lambda] \times S^1_y \) has a canonical trivialization. Using \( \Phi \) to pull back this canonical trivialization, we get a trivialization \( \tau \) along each periodic orbit at \( x \neq 0 \).

Let us write \( \partial_v E - V = \partial_v E^+ \cup \partial_v E^- \) and \( \Phi_{\pm} = \Phi|_{\partial_v E^\pm} \). \( \Phi_+ \) and \( \Phi_- \) do not match at \( V \) and thus \( \Phi \) cannot be extended to the whole \( \partial_v E \). But \( \Phi_+ \) and \( \Phi_- \) can extend to \( \partial_v E^\pm \) and give trivializations \( \Phi_+ : \partial_v E^+ \to S^1_t \times [0, \lambda] \times S^1_y \) and \( \Phi_- : \partial_v E^- \to S^1_t \times [-\lambda, 0] \times S^1_y \) respectively. Using \( \Phi_{\pm} \) to pull back the canonical trivialization, hence there are two trivializations \( \tau_{\pm} \) along the periodic orbits at \( x_0 = 0 \). But there is no difference between using \( \tau_+ \) and \( \tau_- \) when we compute \( c_\tau, Q_\tau, CZ_\tau \). So we just use the same notation \( \tau \) to denote one of them.

**Conley–Zehnder index**  Since \( R''(x) < 0 \), the elliptic orbit \( e^{k}_{\lambda} \) has small negative rotation number with respect to the trivialization \( \tau \). As long as the perturbation \( f_{T\lambda} \) is small enough, we have

\[
CZ_\tau(e^{k}_{\lambda}) = -1, \quad CZ_\tau(h^{k}_{\lambda}) = 0, \quad (14)
\]

for any \( k \leq Q \).

**Relative intersection number**  The result about the relative self–intersection is as follows:

**Lemma 3.4.** For \( 0 \leq \frac{p}{q}, \frac{p'}{q}' \leq 1 \), then \( Q_\tau(\gamma_{\frac{p}{q}}, \gamma_{\frac{p'}{q}'}) = \min\{p(q' - p'), p'(q - p)\} \). Assume that \( \alpha = \Pi_i \gamma_{\frac{p_i}{q_i}} \) and \( \frac{p_i}{q_i} \geq \frac{p_j}{q_j} \) for \( i \leq j \). Then

\[
Q_\tau(\alpha) = P(Q - P) - \sum_{i<j}(p_i q_j - p_j q_i),
\]

where \( P = \sum_i p_i \) and \( Q = \sum_i q_i \).

It is worth noting that \( Q_\tau \) is quadratic in the sense that

\[
Q_\tau(\alpha) = \sum_i Q_\tau(\gamma_{\frac{p_i}{q_i}}) + 2 \sum_{i<j} Q_\tau(\gamma_{\frac{p_i}{q_i}}, \gamma_{\frac{p_j}{q_j}}).
\]
Turn out it suffices to compute $Q_{\tau}(\gamma_{\frac{p}{q}})$ and $Q_{\tau}(\gamma_{\frac{p}{q}}, \gamma_{\frac{p}{q}})$. The idea of computing $Q_{\tau}(\gamma_{\frac{p}{q}})$ and $Q_{\tau}(\gamma_{\frac{p}{q}}, \gamma_{\frac{p}{q}})$ is to express them as intersection numbers of two surfaces. To this ends, let us construct a surface $u : D_z \to E$ which is asymptotic to $\gamma_{\frac{p}{q}}$ as follows.

Let $\epsilon_i(r) : [0, \infty) \to [0, \infty)$ be cutoff functions with the following properties:

- $\epsilon_i(r)$ is nonincreasing and it is supported in $r \leq 2\delta$.
- When $r \leq \delta$, $\epsilon_i(r)$ is constant. The constant is still denoted by $\epsilon_i$ and $\epsilon_i \ll \delta$.

If $0 \leq \frac{p}{q} \leq \frac{1}{2}$, define $u(z)$ by

$$
(1 + \epsilon_1)\frac{1}{2}e^{h+iy_0}z^p + (1 + \epsilon_2)\frac{1}{2}e^{-h-iy_0}z^q - \epsilon_3, -(1 + \epsilon_1)\frac{1}{2}e^{h-iy_0}z^p + (1 + \epsilon_2)\frac{1}{2}e^{-h-iy_0}z^q - \epsilon_3.
$$

If $\frac{1}{2} \leq \frac{p}{q} \leq 1$, define $u(z)$ by

$$
(1 + \epsilon_1)\frac{1}{2}e^{-h+iy_0}z^q - \epsilon_3, -(1 + \epsilon_1)\frac{1}{2}e^{-h+iy_0}z^p + (1 + \epsilon_2)\frac{1}{2}e^{h+iy_0}z^q - \epsilon_3.
$$

Note that these two definitions coincide when $\frac{p}{q} = \frac{1}{2}$. By construction, $u$ is asymptotic to $\gamma_{\frac{p}{q}}(t)$ as $z$ tends to boundary of $D$.

**Lemma 3.5.** The map $u$ satisfies the following properties:

1. For sufficiently small $\epsilon_i$ and $\delta$, then $u$ is embedded except at $z = 0$.

2. Let $\frac{p'}{q'} < \frac{p}{q}$ and $v$ be the $\frac{p'}{q'}$–version of (15) or (16) accordingly. If $v$ doesn’t involve the zero–order term, i.e., $\epsilon_3 = 0$, then for sufficiently small $\epsilon_i$ and $\delta$, the intersection points of $u$ and $v$ lie in the region $\{0 < |z| \leq \delta\} \times \{0 < |w| \leq \delta\}$.

3. The intersection of $u$ and $v$ are transversal and the sign of the intersection points are positive.

**Proof.** 1. It is straightforward to check that $u$ is immersed except at $z = 0$. Moreover, $u$ is 1–1 onto its image for sufficiently small $\epsilon_i$. To see this, note that the unperturbed version of $u$ ($\epsilon_i = 0$) is 1–1 onto its image because $p$ and $q$ are relatively prime. By using the limit argument and the fact that $u$ is immersion, we can deduce the same conclusion for sufficiently small $\epsilon_i$.

2. There are three possibilities of the order of $\frac{p}{q}, \frac{p'}{q'} : 0 \leq \frac{p'}{q'} < \frac{p}{q} \leq \frac{1}{2}, \frac{1}{2} \leq \frac{p'}{q'} \leq \frac{p}{q} \leq 1$ and $\frac{p'}{q'} < \frac{1}{2} < \frac{p}{q} \leq 1$. We prove the statement case by case. Without loss of generality, we assume that $y_0 = 0$ in (15), (16).
Let us consider the case that $0 \leq \frac{p'}{q'} < \frac{p}{q} \leq \frac{1}{2}$. Let $(z, w)$ be an intersection point of $u$ and $v$. It satisfies the equations

\[
\begin{cases}
(1 + \epsilon_1)\frac{1}{2}e^{h}z^{p} + (1 + \epsilon_2)\frac{1}{2}e^{-h}z^{q-p} + \epsilon_3 = (1 + \epsilon_4)\frac{1}{2}e^{k}w^{p'} + (1 + \epsilon_5)\frac{1}{2}e^{-k}w^{q'-p'} \\
-(1 + \epsilon_1)\frac{i}{2}e^{h}z^{p} + (1 + \epsilon_2)\frac{i}{2}e^{-h}z^{q-p} = -(1 + \epsilon_4)\frac{i}{2}e^{k}w^{p'} + (1 + \epsilon_5)\frac{i}{2}e^{-k}w^{q'-p'}.
\end{cases}
\]

These are equivalent to

\[
\begin{cases}
(1 + \epsilon_1)e^{h}z^{p} + \epsilon_3 = (1 + \epsilon_4)e^{k}w^{p'} \\
(1 + \epsilon_2)e^{-h}z^{q-p} + \epsilon_3 = (1 + \epsilon_5)e^{-k}w^{q'-p'}.
\end{cases}
\]

(17)

Assume that $0 < \epsilon_i \ll \delta \ll 1$. Then the solutions to (17) lie inside either $\{|z|, |w| \leq \delta\}$ or $\{|z|, |w| \geq c_0^{-1}\}$ for some constant $c_0 \geq 1$ and $c_0^{-1} \gg \delta$. To see this, if $|w| > \delta$, then (17) implies

\[
|z|^p \geq \frac{1}{1 + \epsilon_1} \left((1 + \epsilon_4)e^{k-h}|w|^{p'} - \epsilon_3 e^{-h}\right) \geq c_0^{-1} |w|^{p'};
\]

\[
|z|^{q-p} \leq \frac{1}{1 + \epsilon_2} \left((1 + \epsilon_5)e^{h-k}|w|^{q'-p'} + \epsilon_3 e^h\right) \leq c_0 |w|^{q'-p'}.
\]

Therefore, $|w|^{p'q'-p'q} \geq c_0^{-q}$. Similarly, if $|z| > \delta$, then we can deduce that $|z| \geq c_0^{-1}$. Note that the cases $\{|z| \leq \delta, |w| \geq c_0^{-1}\}$ and $\{|w| \leq \delta, |z| \geq c_0^{-1}\}$ cannot happen by (17).

In the case that $|w|, |z| \geq c_0^{-1} > 2\delta$, then $\epsilon_i = 0$ and (17) becomes

\[
\begin{cases}
e^{h}z^{p} = e^{k}w^{p'} \\
e^{-h}z^{q-p} = -e^{-k}w^{q'-p'}.
\end{cases}
\]

By the above equations, we have

\[
\log |z| = \frac{k - h}{pq' - p'q}q', \quad \log |w| = \frac{k - h}{pq' - p'q}.
\]

Assume that $R'(x_0) = \frac{p}{q}$ and $R'(x_1) = \frac{p'}{q'}$. Then $0 < x_0 < x_1$ because $R'' < 0$. By equations (8), it is easy to check that $k > h$. Then the norms $|z|, |w| > 1$ which are not in our domain. Finally, $(0, 0), (z, 0)$ and $(0, w)$ cannot be intersection points for suitable choices of $\epsilon_i$. This can be checked directly.

For the case that $\frac{1}{2} \leq \frac{p'}{q'} < \frac{p}{q} \leq 1$, we get

\[
\begin{cases}
(1 + \epsilon_1)e^{h}z^{q-p} + \epsilon_3 = (1 + \epsilon_4)e^{k}w^{q'-p'} \\
(1 + \epsilon_2)e^{-h}z^{p} + \epsilon_3 = (1 + \epsilon_5)e^{-k}w^{p'}.
\end{cases}
\]

(18)
at the intersection point. The same argument can show that the solutions to (18) lie inside either \(|z|, |w| \leq \delta\) or \(|z|, |w| \geq c_0^{-1}\) for some constant \(c_0 \geq 1\) and \(c_0^{-1} \gg \delta\). Note that \(\epsilon_i = 0\) where \(|w|, |z| \geq c_0^{-1} > 2\delta\). If \(|w|, |z| \geq c_0^{-1}\), then (18) implies that

\[
\log |z| = \frac{h - k}{pq' - p'q}(q'), \quad \log |w| = \frac{h - k}{pq' - p'q}(q).
\]

Assume that \(R'(x_0) = \frac{p}{q}\) and \(R'(x_1) = \frac{p'}{q'}\). Then \(x_0 < x_1 < 0\) because of \(R'' < 0\). By the definition of \(h\) and \(k\), it is easy to check that \(h > k\). Then the norms \(|z|, |w| > 1\) which are not in our domain.

Again, \((0, 0), (z, 0)\) and \((0, w)\) cannot be intersection points for suitable choices of \(\epsilon_i\).

For the case that \(\frac{p'}{q'} < \frac{1}{2} < \frac{p}{q} \leq 1\), we have

\[
\begin{align*}
(1 + \epsilon_1)e^{h'zq' - p} + \epsilon_3 &= (1 + \epsilon_3)e^{-k'w'q' - p'} \\
(1 + \epsilon_2)e^{-h'zq' + p'} + \epsilon_3 &= (1 + \epsilon_4)e^{k'w'q' - p'}
\end{align*}
\]

(19)

The same argument can show that the solutions to (19) lie inside either \(|z|, |w| \leq \delta\) or \(|z|, |w| \geq c_0^{-1}\) for some constant \(c_0 \geq 1\) and \(c_0^{-1} \gg \delta\). As before, if \(|w|, |z| \geq c_0^{-1} > 2\delta\), then they satisfy

\[
\log |z| = \frac{h + k}{pq' - p'q}(q'), \quad \log |w| = \frac{h + k}{pq' - p'q}(q).
\]

Again they are not in our domain.

The intersection points of the forms \((0, 0), (z, 0)\) and \((0, w)\) can be ruled out as the other cases.

3. In the region \(\{0 < |z| \leq \delta\} \times \{0 < |w| \leq \delta\}\), \(\epsilon_i\) are constants. The statement follows from that the coordinate functions \(x_1\) and \(x_2\) are holomorphic with respect to \(z\) and \(w\).

\(\square\)

Let us consider the case that \(\frac{p}{q} = \frac{p'}{q'}\). Without loss of generality, we only consider the case that \(x_0 > 0\) and \(R'(x_0) = \frac{p}{q} < \frac{1}{2}\). The argument for the other cases are the same, we left the details to the reader. Let

\[
u(z) = \left((1 + \epsilon_1)\frac{1}{2}e^{h'z^p} + \frac{1}{2}e^{-h'z^q - p} + \epsilon_2, -(1 + \epsilon_1)\frac{i}{2}e^{h'z^p} + \frac{i}{2}e^{-h'z^q - p}\right)
\]

and

\[
v(z) = \left(\frac{1}{2}e^{h + i\gamma_0 w^p} + \frac{1}{2}e^{-h - i\gamma_0 w^q - p}, -\frac{i}{2}e^{h + i\gamma_0 z^p} + \frac{i}{2}e^{-h - i\gamma_0 z^q - p}\right).
\]
Lemma 3.6. The maps $u$ and $v$ satisfy the following properties:

1. For sufficiently small $\epsilon_i$ and $\delta$, then $u$ and $v$ are embedded except at $z = 0$.

2. Assume that $\epsilon_1 \gg \epsilon_2$ and $y_0$ is generic. Then all the intersection points of $u$ and $v$ lie in the region $\{0 < |z| \leq \delta\} \times \{0 < |w| \leq 2\delta\}$.

3. The intersection of $u$ and $v$ are transversal and the sign of the intersection points are positive.

Proof. We only prove the second statement and the proof of the other two is the same as in Lemma 3.5. Let $(z, w)$ be an intersection point. Then it satisfies equations

$$
\begin{align*}
(1 + \epsilon_1) & \frac{1}{2} e^{h} z^{p} + \frac{1}{2} e^{-h} z^{q-p} + \epsilon_2 = \frac{1}{2} e^{h} w^{p} + \frac{1}{2} e^{-h} w^{q-p} \\
- (1 + \epsilon_1) & \frac{i}{2} e^{h} z^{p} + \frac{i}{2} e^{-h} z^{q-p} = -\frac{i}{2} e^{h} w^{p} + \frac{i}{2} e^{-h} w^{q-p}.
\end{align*}
$$

The above equations are equivalent to

$$
\begin{align*}
(1 + \epsilon_1) e^{h} z^{p} + \epsilon_2 &= e^{h+i\imath y_0} w^{p} \\
e^{-h} z^{q-p} + \epsilon_2 &= e^{-h-i\imath y_0} w^{q-p}.
\end{align*}
$$

If $|z| \geq 2\delta$, then we have $\epsilon_i = 0$, $z^p = e^{i\imath y_0} w^p$ and $z^q = w^q$. Using the polar coordinates, it is easy to check that $\frac{2}{q} = \frac{w + 2\pi k}{2\pi l}$ for some $k, l \in \mathbb{Z}$. This is impossible for a generic $y_0$. Thus there are no intersection points in the region $|z| \geq 2\delta$.

If $\delta \leq |z| < 2\delta$, then (20) implies that

$$
(1 + \epsilon_1) |z|^p - \frac{1}{2} e^{-h} \epsilon_2 \leq |w|^p;
$$

$$
|w|^{q-p} \leq |z|^{q-p} + e^h \epsilon_2.
$$

Hence,

$$
(1 + \epsilon_1) |z|^p \leq |z|^p \left(1 + e^h \frac{\epsilon_2}{|z|^{q-p}}\right) \frac{e^{q-p}}{q-p} + e^{-h} \epsilon_2 \leq |z|^p + c_0 \frac{e^h \epsilon_2}{|z|^{q-2p}} + e^{-h} \epsilon_2.
$$

Therefore,

$$
\delta^p \epsilon_1 \leq c_0 \delta^{q-p} \epsilon_2.
$$

This is impossible provided that we choose $\epsilon_1 = 100 c_0 \delta^{q-p} \epsilon_2$. In conclusion, the only possibility is that $0 \leq |z| < \delta$. From (20), it is easy to deduce that $0 \leq |w| \leq 2\delta$.

Again, it is straightforward to check that $(0, 0)$, $(z, 0)$ and $(0, w)$ cannot be intersection points. \qed
Proof of Lemma 3.4. For \( \frac{p}{q} \neq \frac{p'}{q'} \), then \( Q_\tau(\gamma_{\frac{p}{q}}, \gamma_{\frac{p'}{q}'}) \) equals to the intersection number of \( u \) and \( v \). The intersection number equals to the number of solutions to equations (17) or (18) or (19) which lie inside the region \( \{0 < |z| \leq \delta\} \times \{0 < |w| \leq \delta\} \).

Keep in mind that our \( \epsilon_i \) are constants in this region rather than functions. Replace the functions \( \epsilon_i \) in (17), (18) and (19) by constants. According to the theorem of Bernshtein [1] (also see Chapter 3 of [15]), equations (17) or (18) or (19) have total \( \max \{p(q' - p'), p'(q - p)\} \) solutions in \( \mathbb{C}^* \times \mathbb{C}^* \). However, there are \( |pq' - qp'| \) solutions outside the region \( \{0 < |z|, |w| < \delta\} \). In fact, the estimates in the proof of Lemma 3.5 implies that these solutions lie in the region \( \{1 \leq |z|, |w|\} \). To see this, note that the unperturbed equations (17) or (18) or (19) (\( \epsilon_i = 0 \)) have exactly \( |pq' - qp'| \) solutions in the region \( \{1 < |z|, |w|\} \). By the limit argument, the same conclusion holds provided that \( \epsilon_i \) are small enough. In conclusion, we have

\[
Q_\tau(\gamma_{\frac{p}{q}}, \gamma_{\frac{p'}{q}'}) = \max \{p(q' - p'), p'(q - p)\} - |pq' - qp'| = \min \{p(q' - p'), p'(q - p)\}.
\]

Note that \( Q_\tau(e_{\frac{p}{q}}) = Q_\tau(h_{\frac{p}{q}}) = Q_\tau(e_{\frac{p}{q}}, h_{\frac{p}{q}}) \) with respect to our trivializations. So the relative self intersection number equals to the number of solutions of (20) in the region \( \{0 < |z|, |w| < \delta\} \). By Bernshtein’s theorem [1] and the same argument as before, we have

\[
Q_\tau(e_{\frac{p}{q}}, h_{\frac{p}{q}}) = p(q - p).
\]

Relative Chern number

Lemma 3.7. Let \( \alpha = \Pi_i \gamma_{\frac{p_i}{q_i}} \) be an orbit set with \( P = \sum_i p_i \) and \( Q = \sum_i q_i \). Then we have \( c_\tau(\alpha) = Q \).

Proof. Note that the relative Chern number satisfies the additive property

\[
c_\tau(\alpha) = \sum_i c_\tau(\gamma_{\frac{p_i}{q_i}}).
\]

Thus it suffices to compute \( c_\tau(\gamma_{\frac{p}{q}}) \).

By definition, we have

\[
4x^2 = 4|\rho|^2 |q|^2 = (|x_1|^2 + |x_2|^2)^2 - |\pi(x_1, x_2)|^2 = 2|x_1|^2|x_2|^2 - x_1^2x_2^2 - x_1^2\bar{x}_2^2.
\]

By differentiating both side of the above identity, we obtain

\[
4xdx = (x_1\bar{x}_2 - \bar{x}_1x_2)(x_2d\bar{x}_1 - \bar{x}_2dx_1 + \bar{x}_1dx_2 - x_1d\bar{x}_2).
\]
Note that $4x^2 = -(x_1 \bar{x}_2 - \bar{x}_1 x_2)^2 = 4(\text{Im}(x_1 \bar{x}_2))^2$. Along the periodic orbits at $x \neq 0$, we can check that $x = \text{Im}(x_1 \bar{x}_2)$. Therefore,

$$dx = \frac{i}{2}(x_2 d\bar{x}_1 - \bar{x}_2 dx_1 + \bar{x}_1 dx_2 - x_1 d\bar{x}_2)$$

along the periodic orbit. In addition, it can be defined along the periodic orbit at $x = 0$.

Let $J \in J_b(E, \omega_E)$ be an almost complex structure such that $J(\partial_x) = \partial_y$ over $1 - \delta \leq r \leq 1$ and $J = J_0$ near the critical point. Therefore, $dx + idy = dx + iJdx \pm iR_x'(\pm x)dt$ over the region where $x \neq 0$ and $1 - \delta \leq r \leq 1$. $T^{1,0}_jE$ is generated by

$$ds + idt, \quad dx + iJdx \pm \frac{R_x'(\pm x)}{2}(ds + idt).$$

Define a section $\psi = (ds + idt) \wedge (dx + iJdx \pm \frac{R_x'(\pm x)}{2}(ds + idt)) = \frac{z}{|z|^2} \wedge (dx + iJdx)$, where $z = \pi(x)$. Near the critical point of $\pi_E$, $dx + iJdx = dx + iJ_0dx = -i\bar{x}_2dx_1 + i\bar{x}_1dx_2$. Thus

$$\psi = \frac{2z}{|z|^2} (x_1dx_1 + x_2dx_2) \wedge (-i\bar{x}_2dx_1 + i\bar{x}_1dx_2) = 2i\bar{z}dx_1 \wedge dx_2.$$

Therefore, we can extend $\psi$ to a section of $T^{2,0}_jE$ over the whole $E$.

Obviously, $\psi$ is nowhere vanishing except at $z = 0$. Let $u : D \rightarrow E$ be the unperturbed version of (15) or (16). Then $\psi|_u$ vanishes at the critical point and the vanishing order is $-q$. Therefore, $c_r(\gamma_{\frac{q}{q}}) = -\#\psi^{-1}(0) = q$.

\[\square\]

4 Proof of the main results

4.1 Energy constraint

Before we prove Theorem\[1\] let us write down a formula for the $\omega_x$–energy and deduce a constraint on the relative homology class.

Let $Z_\alpha \in H_2(E, \alpha)$. Its $\omega_E$–energy is denoted by $E(\alpha)$. By Stokes’ theorem, $E(\alpha) = \int_\alpha \theta_E$. By a direct computation, we have

$$E(\gamma_{\frac{q}{q}}) = \begin{cases} 
  x_0p - R(x_0)q & \text{at } x \geq 0 \text{ such that } R'(x_0) = \frac{p}{q} \\
  x_0(p - q) - R(-x_0)q & \text{at } x \leq 0 \text{ such that } R'(x_0) = \frac{p}{q}
\end{cases}.$$

Therefore,

$$E(\alpha) = \sum_i (|x_i| R'(|x_i|) - R(|x_i|)) q_i.$$
By the definition of $R(|x|)$, it is easy to show that $E(\alpha) \leq Q$. Note that we have $E(S_a) = 0$ for $S_a = \{a\} \times D$. Thus for $Z_\alpha \in H_2(X, \alpha)$, we still have $E(Z_\alpha) \leq Q$. After being perturbed by $f_S$ and $\{f_{T_{a_0}}\}$, the energy $E(Z_\alpha) \leq (1 + \varepsilon)Q$ for a small $\varepsilon > 0$.

**Remark 4.1.** Take $f_{T_{a_0}}$ such that the minimum is 0. Then after the Morse–Bott perturbations, we have $E(e_\varphi) = E(\gamma_\varphi)$.

For a general class $Z_\alpha + m[F] \in H_2(X, \alpha)$, the $\omega_X$–energy is

$$E(Z_\alpha + m[F]) = E(Z_\alpha) + m \int_F \omega_X.$$  \hfill (21)

**Remark 4.2.** By combining (21) and (13), we know that the ECH index and the energy satisfy a relation

$$I(Z_1) - I(Z_2) = c(E(Z_1) - E(Z_2))$$

for some constant $c$. Note that $c \neq 0$ if $Q \neq g(F) - 1$. This property ensures that the right hand side of (3) is a finite sum. Thus the cobordism maps can be defined with $Z_2$– or $\mathbb{Z}$–coefficients.

**Definition 4.1** (See [16], [17].) Fix $J \in J_h(W, \omega_W)$. A tuple $(\pi_W : W \to B, \omega_W, J)$ is called non–negative if at any point $w$ away from the critical points of $\pi_W$, then $\omega_W|_{TW_{\text{hor}}} = \rho(w)(\pi_W^*\omega_B)|_{TW_{\text{hor}}}$, where $\omega_B \in \Omega^2(B)$ is a positive volume form and $\rho$ is a non–negative function.

**Remark 4.3.** An important consequence of this definition is that if $(\pi_W : W \to B, \omega_W, J)$ is non–negative, then any $J$–holomorphic current $C$ has non–negative $\omega_W$–energy. The reason is that

$$E(C) = \int_C |TC^{\text{vert}}|^2 + \int_{\pi_W(C)} \rho|C\omega_B$$

for every simple holomorphic curve $C$ away from the critical points of $\pi_W$, where $TC^{\text{vert}}$ is the vertical component of $TC$. Moreover, the holomorphic curve with “minimal” energy is very rigid in the following sense: If $E(C) = \int_{\pi_W(C)} \rho|C\omega_B$, then $C$ is horizontal, i.e., $TC \subset TW_{\text{hor}}$.

The elementary Lefschetz fibration $(X, \pi_X, \omega_X)$ together with $J \in J_h(X, \omega_X)$ is non–negative. (See Lemmas 18.3, Proposition 19.10 of [17].) Also, the tuple $(X, \pi_X, \omega_X, J)$ is still non–negative after being perturbed by $f_S, \{f_{T_{a_0}}\}$ as long as these functions are non–negative.
Lemma 4.2. Given a positive integer $Q$, if $\int_F \omega_X \geq Q + 1$, then we have the following property: Fix $J \in \mathcal{H}(X, \omega_X)$. Then for any orbit set $\alpha$ with degree less than or equal to $Q$ and $m < 0$, there is no holomorphic current $\mathcal{C}$ with relative homology class $Z_\alpha + m[F]$.

Proof. According to the above discussion, we know that $E(\alpha + m[F]) \leq (1 + \varepsilon)Q + m \int_F \omega_X$. If $m < 0$, then $E(\alpha + m[F]) < 0$. However, our $(X, \pi_X, \omega_X, J)$ is non-negative, thus we have $E(\mathcal{C}) \geq 0$ for any $J$-holomorphic current. Therefore, the class $Z_\alpha + m[F]$ has no holomorphic representative. $\square$

Lemma 4.3. Given a positive integer $Q$, if $\int_F \omega_X \geq Q + 1$, then the conclusion in Lemma 4.2 still holds for any $\Omega_X$-tame almost complex structure $J$ which is sufficiently close to $\mathcal{H}(X, \omega_X)$.

Proof. Given $\varepsilon_0 > 0$, then for any $\Omega_X$-tame almost complex structure $J$ which is sufficiently close to $\mathcal{H}(X, \omega_X)$, we have $E(\mathcal{C}) \geq -\varepsilon_0$ for any $J$-holomorphic current. Otherwise, we can find a sequence of $\Omega_X$-tame almost complex structures $\{J_n\}_{n=1}^\infty$ and $J_n$-holomorphic currents $\mathcal{C}_n$ such that $\{J_n\}_{n=1}^\infty$ converges to $J_\infty \in \mathcal{H}(X, \omega_X)$ and $E(\mathcal{C}_n) < -\varepsilon_0$. According to Taubes’s Gromov compactness (see Lemma 6.8 of [13]), $\{\mathcal{C}_n\}_{n=1}^\infty$ converges to a $J_\infty$-holomorphic current (possibly broken) $\mathcal{C}_\infty$ in current sense. Therefore, $E(\mathcal{C}_\infty) \leq -\varepsilon_0 < 0$ which contradicts $J_\infty \in \mathcal{H}(X, \omega_X)$.

The rest of the argument is the same as in Lemma 4.2. $\square$

4.2 Cobordism maps

In this subsection, we show that definition (3) makes sense. The proof is essentially the same as [4]. The argument there can use to show that if $(Y, \pi, \omega)$ only consists of $Q$-negative elliptic orbits and positive hyperbolic orbits, then the ECH index is non-negative for every holomorphic curve. However, the conclusion is not true in general if $(Y, \pi, \omega)$ also contains $Q$-positive elliptic orbits. But in our case, the conclusion still holds due to the explicit computation of the ECH index and Fredholm index.

Lemma 4.4. Let $C \in \mathcal{M}(\alpha, Z_\alpha + m[F])$ be an irreducible holomorphic curve. Let $q = [\alpha] \cdot [F] \leq Q$. Then the Fredholm index of the holomorphic curve $C$ is

$$\text{ind}C = 2g(C) - 2 + h(C) + 2q + 4m(1 - g(F)) + 2e_+(C),$$

where $e_+(C)$ is the number of ends at $Q$-positive elliptic orbits and $h(C)$ is the number of ends at hyperbolic orbits.
Proof. By Lemma 3.7 and the Adjunction formula, we have
\[ c_\tau(Z_\alpha + m[F]) = q + mc_1([F]) = q + 2m(1 - g(F)). \]

The conclusion follows from the definition of the Fredholm index, (14) and Lemma 3.3. \qed

Given an irreducible simple holomorphic curve \( C \in \mathcal{M}^J(\alpha) \), define \( e_Q(C) \) to be the total multiplicity of all \( Q \)–negative elliptic orbits in \( \alpha \). \( C \) is a special holomorphic plane if it is a holomorphic plane with \( I(C) = \text{ind}C = 0 \) and has exactly one positive end at a simple \( Q \)–negative elliptic orbit. (See Definition 3.15 of [4].)

In [10], Hutchings defines a self–intersection number \( C \star C \) for a simple holomorphic curve \( C \). For its definition, please see the following lemma. Here \( C \star C \) agrees with the usual self–intersection number when \( C \) is closed. To ensure that the ECH index is non–negative, we need to show that this self–intersection is non–negative.

**Lemma 4.5.** Let \( C \in \mathcal{M}^J(Z_\alpha + m[F]) \) be a simple irreducible holomorphic curve and \( q = [\alpha] \cdot [F] \leq Q \). Suppose that \( g(F) - 1 \geq Q \) and \( J \) is a generic almost complex structure which is sufficiently close to \( J_h(X, \omega_X) \). Then
\[ 2C \star C = 2g(C) - 2 + \text{ind}C + h(C) + 2e_Q(C) + 4\delta(C) \geq 0. \]

In particular, \( I(C) \geq 0 \). In addition, if \( I(C) = \text{ind}C = 0 \), then \( C \star C = 0 \) if and only if \( C \) is a special holomorphic plane which satisfies \( q = e_Q(C) = 1 \) and \( h(C) = e_+(C) = 0 \).

**Proof.** We argue by contradiction. Assume that \( C \star C < 0 \). Then we must have \( g(C) = e_Q(C) = \delta(C) = 0 \).

If \( h(C) = 0 \), then \( \text{ind}C \) is even and hence \( \text{ind}C = 0 \). According to Lemma 4.4, we have
\[ -1 + e_+(C) + q + 2m(1 - g(F)) = 0. \]

On the other hand, \( e_+(C) + q + 2m(1 - g(F)) \leq 2q - 2mQ \). Keep in mind that \( m \geq 0 \) because of Lemma 4.3. As a result, \( m = 0 \). Then \( \text{ind}C = 2q - 2 + 2e_+(C) = 0 \) implies that \( e_+(C) = 0 \) and \( q = 1 \). Therefore, \( C \) is closed with degree 1. This is impossible.

If \( h(C) = 1 \), then \( \text{ind}C = 0 \); otherwise \( C \star C \geq 0 \). However, by Lemma 4.4, we know that \( \text{ind}C \) is odd. We get a contradiction.

In conclusion, \( C \star C \geq 0 \). Follows from Proposition 4.8 of [10] and Theorem 4.15 of [8], we have \( I(C) \geq 0 \).
Now let us prove the last statement. Assume that \( \text{ind}C = 0 \) and \( C \ast C = 0 \). Then we have \( \delta(C) = 0 \) and
\[
2g(C) - 2 + h(C) + 2e_Q(C) = 0.
\]
If \( g(C) = 0 \), then there are two possibilities:
1. \( h(C) = 0 \) and \( e_Q(C) = 1 \);
2. \( h(C) = 2 \) and \( e_Q(C) = 0 \).

On the other hand, \( \text{ind}C = 0 \) and Lemma 4.4 implies that
\[
h(C) + 2q + 2e_+(C) + 4m(1 - g(F)) = 2. \tag{22}
\]
It is worth noting that \( h(C) + 2q + 2e_+(C) + 4m(1 - g(F)) \leq 4q - 4mQ \) due to our assumption. In either cases, we have \( m = 0 \); otherwise, the left hand side of (22) is strictly less then 2.

In the first case that \( h(C) = 0 \) and \( e_Q(C) = 1 \), then \( 2q + 2e_+(C) = 2 \) implies that \( q = 1 \) and \( e_+(C) = 0 \). Consequently, \( C \) is a special holomorphic plane with one positive end at a simple \( Q \)-negative periodic orbit and has no other ends. In the second case, \( q + e_+(C) = 0 \) implies that \( q = 0 \). This is impossible.

If \( g(C) = 1 \), then \( C \ast C = 0 \) implies that \( h(C) = e_Q(C) = 0 \). By \( \text{ind}C = 0 \) and Lemma 4.4 we have
\[
2q + 2e_+(C) + 4m(1 - g(F)) = 0. \tag{23}
\]
Suppose that \( Q < g(F) - 1 \). If \( m \geq 1 \), then the left hand side of (23) is strictly less than zero. If \( m = 0 \), then \( g = e_+(C) = 0 \). \( C \) is a closed curve and \( [C] = n[F] \) because of Lemma 2.6. The Adjunction formula and our assumption \( g(F) \geq 2 \) implies that \( C \) cannot have genus 1.

If \( Q = g(F) - 1 \), then \( C \) can not be closed as before. Also, \( h(C) = e_Q(C) = 0 \) implies that all the ends of \( C \) are asymptotic to \( Q \)-positive elliptic orbits. By Lemma 3.3, \( I(C) \geq 2 \) contradicts with \( I(C) = 0 \).

**Definition 4.6.** Let \( u : \overline{D} \to \overline{X} \) be a section which is asymptotic to a periodic orbit with degree 1. The section \( u \) is called horizontal if \( \text{Im}(du) \subset T\overline{X}^{\text{hor}} \).

Note that for any \( J \in \mathcal{J}_h(X, \omega_X) \), the horizontal section is \( J \)-holomorphic.

**Proof of the part A. of Theorem 1**
Proof. To define the cobordism map, let us consider the moduli space $\mathcal{M}_0^q(\alpha + m[F])$ and $\mathcal{C} = \sum_k d_k C_k \in \mathcal{M}_0^q(\alpha + m[F])$. Due to Lemmas 4.2 and 4.3, we always assume that $m \geq 0$.

Let us first consider the case that $Q > g(F) - 1$. Choose a generic almost complex structure $J \in \mathcal{J}_h(X, \omega_X)$. Since $Q > g(F) - 1$, the element $\mathcal{C} \in \mathcal{M}_0^q(\alpha, Z_\alpha + m[F])$ cannot contain a closed component; otherwise $I \geq 2$. (See Lemma 5.7 of [3].)

By Lemma 3.3 and Theorem 3.1 and (13), $I(\mathcal{C}) = I_m(\alpha) \geq 0$ for any orbit set and $m \geq 0$. Moreover, $I = 0$ if and only if $m = 0$ and the orbits set is of the form $\alpha = (\Pi_a e_a^0 e_0^m e_1^m)$, where $a$ is a critical point of $f_S$ with $\nabla^2 f_S(a) > 0$. Thus let us assume $\alpha = (\Pi_a e_a^0 e_0^m e_1^m)$ and $m = 0$. By the energy formula and Remark 4.1, we have $E(\alpha) = \sum_a f_S(a)$.

We claim that the only element in $\mathcal{M}_0^q(\alpha)$ is of the form $\mathcal{C} = \sum_a m_a C_{e_a} + m_0 C_{e_0} + m_1 C_{e_1}$, where $C_{e_*}$ is a special holomorphic plane. To see this, let $\mathcal{C} = \sum_k d_k C_k$. First note that all periodic orbits in $\alpha$ are $Q$–negative elliptic, therefore $C_k \ast C_k \geq 0$ by definition. Proposition 4.8 of [10] and Theorem 4.15 of [8] imply that $\text{ind} C_k = I(C_k) = 0$ for any $k$. By the fact that $\pi_X : C_k \rightarrow \overline{\mathcal{D}}$ is holomorphic and the Riemann–Hurwitz formula, we have

$$2g(C_k) - 2 = b - e_-(C_k) - q_k,$$

where $q_k$ is the degree of $\pi_X|_{C_k}$, $e_-(C_k)$ is the number of ends of $C_k$ and $b \geq 0$ is a certain counting the branched points. According to Lemma 4.4, we have

$$0 = \text{ind} C_k = 2g(C_k) - 2 + 2q_k = b + q_k - e_-(C_k).$$

Consequently, $b = g(C_k) = 0$ and $q_k = e_-(C_k) = 1$. Thus $C_k$ is a special holomorphic plane. By Remark 4.3, $C_k$ actually is a horizontal section because $E(C_k) = 0$ or $f_S(a)$.

Let $\tilde{C}$ be an unbranched cover of $C_{e_*}$ (disjoint copies of $C_{e_*}$) with $\text{ind} \tilde{C} = 0$. It satisfies $\text{ind} \tilde{C} > 2g(\tilde{C}) - 2 + h_+(\tilde{C})$, where $h_+(\tilde{C})$ is the number of ends at even periodic orbits ($h_+(\tilde{C}) = 0$ in our case). By the automatic transversality theorem [20], $\tilde{C}$ is Fredholm regular for any $J \in \mathcal{J}_h(X, \omega_X)$. Therefore, $\mathcal{M}_0^q(\alpha)$ only consists of a single element and hence it is compact automatically. Then we define $\text{PFC}(X, \omega_X)_J \alpha = \mathcal{M}_0^q(\alpha)$; otherwise, $\text{PFC}(X, \omega_X)_J = 0$.

To show that it is a chain map, i.e., $\text{PFC}(X, \omega_X)_J \circ \partial = 0$. Let $\beta$ be an ECH generator and $\mathcal{C}' = \sum_k d_k' C_{k}' \in \mathcal{M}_0^q(\beta + m[F])$ be a holomorphic current with ECH index 1. According to Lemma 3.3 and Theorem 3.1 and (13), $I_m(\beta) = 1$ implies that $m = 0$ and $\beta$ doesn’t involve $Q$–positive elliptic orbits. By definition,
\[ C_k' \ast C_k' \geq 0. \] In this case, one can show that \( C' \) consists of embedded holomorphic curves and special holomorphic planes by using the ECH inequalities in [8] and [10]. Thus the moduli space \( \mathcal{M}_1'(\beta) \) is a 1-dimensional manifold. Note that the ECH index of holomorphic curves are non-negative by Lemma 3.3 and Theorem 3.1 and (13). Therefore, a broken holomorphic curve arising as a limit of curves in \( \mathcal{M}_1'(\beta) \) consists of a holomorphic curve with \( I = 1 \) in the top level, a curve with \( I = 0 \) in the cobordism level, and connectors (branched covers of trivial cylinders) in between. Make use of the gluing analysis in [11] and [12], we have \( PFC(X, \omega_X)_J \circ \partial \beta = 0 \).

Now we turn to consider the case that \( Q \leq g(F) - 1 \). Fix a generic \( \Omega_X \)-tame almost complex \( J \) structure which is sufficiently close to \( J_h(X, \omega_X) \). By Lemma \( 2.6 \), we know that a closed holomorphic curve has homology class \( m[F] \) for \( m \geq 0 \). However, \( I(m[F]) < 0 \), this is impossible for a generic \( J \). Hence, there are no closed components in \( C \). According to Proposition 4.8 of [10] and Corollary 4.5, \( I(C) = 0 \) implies that

1. \( I(C_k) = \text{ind}C_k = 0 \) and \( C_k \) is embedded for each \( k \).
2. \( C_k \ast C_l = 0 \) for \( k \neq l \).
3. (a) If \( C_k \ast C_k > 0 \), then \( d_k = 1 \).
   (b) If \( C_k \ast C_k = 0 \), then \( C_k \) is a special holomorphic plane which is asymptotic to a simple \( Q \)-negative periodic orbit with degree 1.

Therefore, \( C = C_0 + \sum_a d_a C_{e_a} + d_1 C_{e_1} + d_0 C_{e_0} \), where \( C_0 \) is an embedded holomorphic current with \( I = 0 \) and \( C_{e_a} \) are special holomorphic planes. One can use the same argument in [4] (Proposition 3.16) to show that \( \mathcal{M}_0(\alpha) \) is a finite set. As a consequence, the definition (3) makes sense in this case. To show that \( PFC(X, \omega_X)_J \) is a chain map, the argument is the same as the case \( Q > g(F) - 1 \). We left the details to readers.

Finally, in either cases, it is not difficult to extend the cobordism maps over the ring \( A(X) \). For more details, please see [4].

\[ \square \]

**Proof of the part B of Theorem 1**

**Proof.** Assume that \( Q > g(F) - 1 \). According to the proof of part A, we know that \( PFC(X, \omega_X)_J \alpha = 0 \) unless that \( \alpha = (\Pi_a c_a^{m_a}) c_0^{m_0} c_1^{m_1} \), where \( a \) are critical points of \( f_S \) with \( \nabla^2 f_S(a) > 0 \). Moreover, there is only one element
in $\mathcal{M}_0^J(\alpha)$ which consists of copies of horizontal sections. According to [4], the sign of such an element is positive. In sum, $\# \mathcal{M}_0^J(\alpha) = 1$.

Now let us consider the second case that $2Q + 1 \leq g(F)$. Assume that the Morse function $f_S : S \to \mathbb{R}$ is chosen as in the proof of Theorem 5.3 of [7]. Then $(Y, \pi, \omega)$ has no $Q$–negative elliptic orbits with degree 1 except $e_0, e_1$. By Theorem 5.3 of [7] and our assumption, the generators of $PFH_*(Y, \omega, Q)$ are represented by a linear combination of orbits sets $\alpha = (\Pi_a h_a)\alpha_0$, where $a$ are saddle points of $f_S$. Let us denote the degree of $\Pi_a h_a$ by $q$. Then $\alpha_0$ is an orbit set of index $0 \leq I_0(\alpha_0) \leq 2Q' - 1$ and $Q' = Q - q$.

By Lemma 3.3 and Theorem 3.1 and (13),

$$I_m(\alpha) = q + I_0(\alpha_0) + 2m(Q + 1 - g(F)).$$

Since $2Q + 1 \leq g(F)$, $I_m(\alpha) \leq q + I_0(\alpha_0) - 2mQ < 2Q - 2mQ$. According to Lemma 3.3 we may assume that $m \geq 0$. Therefore, $I_m(\alpha) < 0$ unless $m = 0$. When $m = 0$, $I_0(\alpha) = 0$ if and only if $q_+ = q = I_0(\alpha_0) = 0$. In particular, $\alpha = e_0^{m_0} e_1^{m_1} = e^{m_0+m_1}$.

Recall that for all $J \in \mathcal{J}_h(X, \omega_X)$, transversality of $\mathcal{M}_0^J(e_0^{m_0} e_1^{m_1})$ can be achieved. If $J \in \mathcal{J}_{\text{tame}}(X, \omega_X)$ is sufficiently close to $\mathcal{J}_h(X, \omega_X)$, then $\mathcal{M}_0^J(e_0^{m_0} e_1^{m_1})$ also consists of a single element with a positive sign.

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