MAXIMAL DIGRAPHS WHOSE HERMITIAN SPECTRAL RADIUS IS AT MOST 2

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Abstract. We classify maximal digraphs whose Hermitian spectral radius is at most 2.

1. Introduction

Smith [13] and Lemmens and Seidel [9] showed that a connected simple graph whose (0,1)-adjacency matrix has spectral radius at most 2 is a subgraph of one of the following graphs:

As is well known, these are extended Dynkin diagrams of the irreducible root lattices of types A, D, and E. Indeed, if \( A \) is the adjacency matrix of a graph with spectral radius at most 2, then \( -A + 2I \) is a positive semidefinite matrix, which thus can be seen as the Gram matrix of a set \( \Sigma \) of vectors in \( \mathbb{R}^n \) such that \( (x, x) = 2 \) and \( (x, y) \in \{0, -1\} \) for all \( x \neq y \in \Sigma \). Therefore, \( \Sigma \) is contained in a fundamental root system of a root lattice [1].

The aim of the present work is to generalize this result to the class of digraphs. A digraph (a directed or mixed graph) \( \Delta \) consists of a finite set \( V \) of vertices together with a subset \( E \subseteq V \times V \) of ordered pairs of distinct elements of \( V \). If \( (x, y) \in E \) and \( (y, x) \notin E \), then we call \( (x, y) \) an arc or directed edge. If both \( (x, y) \in E \) and \( (y, x) \in E \), then the pair \( \{x, y\} \) is said to form a digon of \( \Delta \).

The Hermitian adjacency matrix \( H = H(\Delta) \) of \( \Delta \) was independently defined by Liu and Li [10], and Guo and Mohar [5] as a Hermitian matrix \( H \in \mathbb{C}^{V \times V} \) with

\[ (x, y) = (Hx, y) = (\Sigma_{x \neq y} \Sigma_{x \neq z} (x, z)) \]

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entries given by

\[
(H)_{xy} = \begin{cases} 
1 & \text{if } (x, y) \in E, (y, x) \in E, \\
i & \text{if } (x, y) \in E, (y, x) \notin E, \\
-i & \text{if } (x, y) \notin E, (y, x) \in E, \\
0 & \text{otherwise.}
\end{cases}
\]

In what follows, by the Hermitian spectral radius, or simply, spectral radius \(\rho(\Delta)\) of a digraph \(\Delta\), we mean the spectral radius of its Hermitian adjacency matrix \(H(\Delta)\). Guo and Mohar [6] studied digraphs whose spectral radius is less than 2. In this paper, we give a classification of digraphs whose spectral radius is at most 2.

In order to state our theorem, we need to define a number of graphs. The graphs \(\Delta_{2k}^{(1)}, \Delta_{2k}^{(i)}, \Delta_8^\dagger, \Delta_{14}^\dagger\) or \(\Delta_{16}^\dagger\) are depicted in Fig. 2–6. The labels attached to the vertices in Fig. 2–5 will be explained in Section 3. We denote by \(P_n\) the path graph on \(n\) vertices, and by \(C_n\) the (undirected) cycle graph on \(n\) vertices. We denote by \(D_n\) the
Let $\tilde{C}_n$ be the digraph obtained from $D_n$ by reversing the direction of one of the arcs. Let $\tilde{C}'_n$ be the digraph obtained from $D_n$ by replacing one of the arcs by a digon. Let $\tilde{C}''_n$ be the digraph obtained from $D_n$ by taking two consecutive arcs and then replacing the first one by a digon and reversing the direction of the second. See Figure 7 for illustrations of these four graphs. For positive integers $a, b, c$, let $Y_{a,b,c}$ be the tree obtained by taking paths $P_{a+1}, P_{b+1}, P_{c+1}$ and identifying an end vertex of each into a single vertex. Note that $Y_{a,b,c}$ has $a + b + c + 1$ vertices. For nonnegative integers $a_1, a_2, a_3, a_4$, let $\Box_{a_1,a_2,a_3,a_4}$ be a digraph obtained from $\tilde{C}_4$ with (consecutive) vertices $v_1, v_2, v_3, v_4$ by adding directed paths on $a_i + 1$ vertices that are attached to $v_i$ for each $i = 1, 2, 3, 4$. The digraphs $\tilde{U}_1$ and $\tilde{U}_6$ can be found in Fig. 11. The bipartite signed graphs $U_1, \ldots, U_{11}$ are depicted in Fig. 10.

By a subdigraph of a digraph $\Delta = (V, E)$, we mean a digraph of the form $(W, E \cap (W \times W))$, where $W$ is a subset of $V$. It is clear that the Hermitian adjacency matrix of a subdigraph of $\Delta$ is a principal submatrix of $H(\Delta)$. See Definition 7 for
a definition of canonical digraphs of a bipartite signed graph. Our main result is the following theorem. Since switching equivalence (see Definition 2) preserves Hermitian spectra of digraphs, we give our classification of connected digraphs whose Hermitian spectral radius is at most 2 up to switching equivalence.

**Theorem 1.** Let $\Delta$ be a connected digraph. If $\rho(\Delta) \leq 2$, then $\Delta$ is switching equivalent to a subdigraph of one of the following:

(i) $\Delta_{2k}^{(i)}$,
(ii) $\Delta_{2k}^{(i)}$,
(iii) $\Delta_8^\dagger, \Delta_{14}$ or $\Delta_{16}$.

Moreover, if $\rho(\Delta) < 2$, then $\Delta$ is switching equivalent to a subdigraph of one of the following:

(iv) $D_n$, where $n \not\equiv 0 \pmod{4}$,
(v) $\tilde{C}_n$, where $n \not\equiv 2 \pmod{4}$,
(vi) $\tilde{C}_n^\prime$, where $n \not\equiv 1 \pmod{4}$,
(vii) $\tilde{C}_n^\prime$, where $n \not\equiv 3 \pmod{4}$,
(viii) a path $P_n$,
(ix) $\square_{a,0,c,0}$ for some $a,c \geq 0$,
(x) $Y_{a,1,1}$ for some $a \geq 1$,
(xi) $\tilde{U}_1$,
(xii) $\tilde{U}_6$,
(xiii) canonical digraphs of the bipartite signed graphs $U_1, \ldots, U_{11}$.

Conversely, every digraph in (i)–(iii) has spectral radius 2, and every digraph in (iv)–(xiii) has spectral radius less than 2.

There are a number of proper subdigraphs of $\Delta_{2k}^{(i)}$ and $\Delta_{2k}^{(i)}$ having spectral radius 2. A partial classification of maximal digraphs with spectral radius at most 2 has been obtained in [14] under the assumption that the underlying graph is $C_4$-free.

Note that the spectral radius is monotone with respect to taking subdigraphs: if $\Delta'$ is a subdigraph of $\Delta$, then $\rho(\Delta') \leq \rho(\Delta)$. This reduces the problem to consideration of a number of forbidden subdigraphs, and this led to the classification in [6].

We shall exploit another approach in this paper. Note that symmetric (Hermitian) matrices over a ring of algebraic integers having all their eigenvalues in the interval $[-2,2]$ are called cyclotomic, and are of independent interest in algebraic number theory. Integer cyclotomic matrices were described by McKee and Smyth [14], whose proof also involved the classical root systems. Their result was further extended by Greaves [4] to Hermitian cyclotomic matrices over the Eisenstein and Gaussian integers.

Since a Hermitian adjacency matrix is a Hermitian matrix over the Gaussian integers, the results of Greaves [14] contain, however, do not immediately imply those of Guo and Mohar [6] due to the following obstacles. First of all, the classification in [4] is given up to equivalence, which is weaker than switching equivalence for digraphs (see Section 2). Secondly, the classification in [4] does not explicitly list matrices with all their eigenvalues in the open interval $(-2,2)$. 
We proceed as follows. We first show that each of \( Z[i] \)-graphs, corresponding to maximal indecomposable Hermitian cyclotomic matrices determined in \([4]\), gives rise to a unique switching equivalence class of digraphs. This implies that every digraph with spectral radius at most 2 is switching equivalent to a subdigraph of one of the digraphs in Theorem \([4]\)(i)-(iii) This classification contains digraphs with spectral radius less than 2 classified by Guo and Mohar \([6]\), so in principle, one can try to derive their result by looking at subdigraphs of digraphs with spectral radius 2. Instead, we show that such a classification follows from the results of McKee and Smyth \([11]\) via the notion of the associated signed graph of a digraph, which we introduce in Section \(2\). In doing so, we found a counterexample to the statement of \([6]\, Lemma \, 4.8(b)]\), leading to an omission in \([6]\, Theorem \, 4.15\]. We thus complete the statement of \([6]\, Theorem \, 4.15\) by supplying the missing digraph which is the canonical digraph of \( U_7 \) (see Fig. 8) in our Theorem \([1]\)(xiii).

After giving preliminaries in Section \(2\), we give a proof of the first part of Theorem \([1]\) in Section \(3\). The second part of Theorem \([1]\) is proved in Section \(4\). Finally, in Section \(5\), we establish a correspondence between the digraphs with spectral radius at most 2 and the Gaussian root lattices. We also characterize digraphs with smallest eigenvalue greater than \(-\sqrt{2}\), which strengthens \([5]\, Proposition \, 8.6]\).

2. Preliminaries

2.1. Equivalence relations on matrices. In this subsection, we discuss equivalence relations defined in \([4]\,[12]\). Let \( n \) be a positive integer, and let \( H_n \) denote the set of all Hermitian matrices of order \( n \) with entries in \( \{0, \pm 1, \pm i\} \), where \( i = \sqrt{-1} \). Let \( U_n(Z[i]) \) be the subgroup of the complex unitary group consisting of those matrices which have entries in \( Z[i] \). Then \( U_n(Z[i]) \) is generated by the permutation matrices together with the diagonal matrices with diagonal entries in \( \{\pm 1, \pm i\} \). For two matrices \( A, B \in H_n \), we say that \( A \) is strongly equivalent to \( B \), written \( A \approx B \), if \( A = QBQ^* \) or \( A = Q^*BQ \) for some \( Q \in U_n(Z[i]) \). The matrices \( A \) and \( B \) are called equivalent, written \( A \sim B \), if \( A \) is strongly equivalent to \( B \) or \(-B\).

Definition 2. We say that a digraph \( \Delta' \) is obtained from \( \Delta \) by four-way switching if \( H(\Delta') = QH(\Delta)Q^* \) for some \( Q \in U_n(Z[i]) \). The digraph whose Hermitian adjacency matrix is \( H(\Delta) \) is called the converse of a digraph \( \Delta \). We say that two digraphs are switching equivalent (see \([12]\)) if one can be obtained from the other by a sequence of four-way switchings and operations of taking the converse.

We remark that the four-way switching was defined in \([5]\) by modifying the set of arcs with respect to a certain partition of the vertex set. Our definition is equivalent to the one in \([5]\).

Lemma 3. Let \( \Delta_1 \) and \( \Delta_2 \) be digraphs with respective Hermitian adjacency matrices \( H_1 \) and \( H_2 \). Then the following statements are equivalent:

(i) \( \Delta_1 \) and \( \Delta_2 \) are switching equivalent,

(ii) \( H_1 \) and \( H_2 \) are strongly equivalent.

Proof. Immediate from the definitions. \( \square \)
2.2. Associated signed graphs. A signed graph $S$ is a triple $(V, E^+, E^-)$ of a set $V$ of vertices, a set $E^+$ of 2-subsets of $V$ (called positive edges), and a set $E^-$ of 2-subsets of $V$ (called negative edges) such that $E^+ \cap E^- = \emptyset$. We depict positive (resp. negative) edges by solid (resp. dashed) lines (see Fig. 9, 10). The adjacency matrix of a signed graph $S = (V, E^+, E^-)$ is the matrix $A(S)$ whose rows and columns are indexed by $V$ such that its $(x, y)$-entry is 1, $-1$, 0 according to $\{x, y\} \in E^+, E^-$, otherwise, respectively. We say two signed graphs $S, S'$ are strongly equivalent (resp. equivalent) if $A(S) \approx A(S')$ (resp. $A(S) \sim A(S')$). By a subgraph of a signed graph $S$, we mean a signed graph of the form $(W, E_W^+, E_W^-)$, where $W$ is a subset of $V$, and $E_W^\pm$ is the subset of $E^\pm$ consisting of those 2-subsets that are contained in $W$. It is clear that the adjacency matrix of a subgraph of $S$ is a principal submatrix of $A(S)$.

Given a connected digraph $\Delta$ with Hermitian adjacency matrix $H = A + iB$, where $A$ is a symmetric $(0, 1)$-matrix and $B$ is a skew symmetric $(0, \pm 1)$-matrix, the associated signed graph $S(\Delta)$ of $\Delta$ is the signed graph with adjacency matrix

$$C = \begin{bmatrix} A & B \\ B^\top & A \end{bmatrix}.$$  

Lemma 4. For a digraph $\Delta$, the Hermitian spectrum of $\Delta$ is the same as that of the $S(\Delta)$ in which multiplicities are doubled.

Proof. Since $H$ is Hermitian, we have

$$\begin{bmatrix} I & 0 \\ -iI & I \end{bmatrix} \begin{bmatrix} A & B \\ B^\top & A \end{bmatrix} \begin{bmatrix} I & 0 \\ iI & I \end{bmatrix} = \begin{bmatrix} A + iB & B \\ 0 & A - iB \end{bmatrix}.$$  

This implies

$$\det(xI - C) = (\det(xI - H))^2,$$

and hence the eigenvalues of $C$ are the same as those of $H$, with multiplicities doubled. $\square$

We can characterize $S(\Delta) = (V_1 \cup V_2, E^+ \cup E^-)$ as follows. The vertex set $V_1 \cup V_2$ is the disjoint union of two copies $V_i = \{x_i \mid x \in V(\Delta)\}$ of the vertex set $V(\Delta)$ of the digraph $\Delta$. The edges of $S(\Delta)$ are the following:

(i) If $\{x, y\}$ is a digon in $\Delta$, then $\{x_1, y_1\}, \{x_2, y_2\} \in E^+$;
(ii) If $(x, y)$ is an arc in $\Delta$, then $\{x_1, y_2\} \in E^+$ and $\{x_2, y_1\} \in E^-.$

For a digraph $\Delta$ (or a signed graph $S$), let $G(\Delta)$ ($G(S)$, respectively) denote the underlying graph of $\Delta$ (or $S$, respectively), i.e., a graph obtained from $\Delta$ (from $S$) by replacing all of its arcs (signed edges, respectively) with undirected edges. We say a digraph (or a signed graph) is connected if its underlying graph is connected. For a vertex $x$ of a graph $G$, we denote by $\deg_G(x)$ the degree of $x$.

Lemma 5. With reference to the above description of $S(\Delta)$, the following statements hold for all $x \in V(\Delta)$:

(i) the vertices $x_1$ and $x_2$ are not adjacent in $G(S(\Delta))$,
(ii) the vertices $x_1$ and $x_2$ have no common neighbors in $G(S(\Delta))$,
(iii) $\deg_{G(S(\Delta))}(x_1) = \deg_{G(S(\Delta))}(x_2) = \deg_G(x)$.
Proof. Immediate. □

Lemma 6. The associated signed graph of a connected digraph $\Delta$ is either connected, or has two connected components, say $S_1$ and $S_2$. The former case occurs precisely when there is a cycle in $\Delta$ containing an odd number of arcs. In the latter case, $H(\Delta)$, $A(S_1)$ and $A(S_2)$ are strongly equivalent.

Proof. Note that $S(\Delta)$ is connected if and only if there exists a vertex $x$ of $\Delta$ such that $x_1$ and $x_2$ are connected by a path in $S(\Delta)$. This condition is equivalent to the existence of a cycle containing an odd number of arcs. Suppose that $S(\Delta)$ is disconnected, and let $S_1$ be a connected component of $S(\Delta)$. Let $A_1$, $A_2$ and $B_1$ denote the submatrix of (1) corresponding to $(V_1 \cap S_1)^2$, $(V_1 \setminus S_1)^2$ and $(V_1 \cap S_1) \times (V_2 \cap S_1)$, respectively. Since $B = -B^\top$ in (1), the matrix (1) has the form

$$
\begin{bmatrix}
A_1 & 0 & 0 & B_1 \\
0 & A_2 & -B_1^\top & 0 \\
0 & -B_1 & A_1 & 0 \\
B_1^\top & 0 & 0 & A_2
\end{bmatrix}.
$$

This implies that $S_1$ and the other connected component have adjacency matrix

$$
\begin{bmatrix}
A_1 & \pm B_1 \\
\pm B_1^\top & A_2
\end{bmatrix}
$$

which are strongly equivalent to the Hermitian adjacency matrix

$$
\begin{bmatrix}
A_1 & iB_1 \\
-iB_1^\top & A_2
\end{bmatrix}
$$

of $\Delta$. □

For example, the associated signed graph of $\Delta_8^\dag$ (see Figure 6) is connected, since $\Delta_8^\dag$ contains a directed triangle. The associated signed graphs of $\Delta_{14}$ and $\Delta_{16}$ (see Figure 6) have two connected components since, after removing digons, the underlying graphs are bipartite.

Definition 7. Let $S$ be a connected bipartite signed graph with bipartition $V(S) = V_1 \cup V_2$. Let $D$ be the diagonal matrix whose rows and columns are indexed by $V(S)$, and whose $(x, x)$-entry is 1 or $i$ according to $x \in V_1$ or $V_2$. Then $D^* A(S) D$ is the Hermitian adjacency matrix of a digraph $\Delta$. We call $\Delta$ a canonical digraph of the bipartite signed graph $S$.

As an example, a canonical digraph of the bipartite signed graph $U_7$ in Fig. 10 is shown in Fig. 8. A canonical digraph is not uniquely determined, but it is unique up to switching equivalence.

Since the underlying graph of a canonical digraph $\Delta$ of a bipartite signed graph is bipartite, Lemma 7 shows that the associated signed graph of $\Delta$ is disconnected.

Lemma 8. Let $S$ be a connected bipartite signed graph. For a digraph $\Delta$, $H(\Delta)$ is strongly equivalent to $A(S)$ if and only if $\Delta$ is switching equivalent to a canonical digraph of $S$. 
Proof. Let $\Delta_0$ be a canonical digraph of $S$. If $\Delta$ is switching equivalent to $\Delta_0$, then $H(\Delta) \approx H(\Delta_0) \approx A(S)$ by Lemma 3 and construction.

Conversely, suppose $H(\Delta) \approx A(S)$. Since $S$ is bipartite, every cycle in a canonical digraph $\Delta_0$ of $S$ has an even number of arcs. By Lemma 6, $S(\Delta_0)$ has two connected components, say $S_1$ and $S_2$. It is easy to see that $A(S_i) \approx A(S)$ for $i = 1, 2$, and hence by Lemma 6 we have $H(\Delta_0) \approx A(S_1) \approx A(S) \approx H(\Delta)$. Thus, $\Delta$ is switching equivalent to $\Delta_0$ by Lemma 3. □

Lemma 9. Let $U$ be a bipartite signed graph, and let $\Delta_0$ be a canonical digraph of $U$. Let $\Delta$ be a digraph such that $S(\Delta)$ is disconnected, and $S(\Delta)$ has a connected component which is equivalent to some subgraph of $U$. Then $\Delta$ is switching equivalent to some subdigraph of $\Delta_0$.

Proof. By the assumption, $S(\Delta)$ has a connected component $S_1$ which is equivalent to some subgraph of $U$. Since $U$ is bipartite, so is $U'$. Thus $A(U')$ is strongly equivalent to $-A(U')$. This implies that $A(S_1) \approx A(U')$. Since $A(U) \approx H(\Delta_0)$ by Lemma 8 we have $A(U') \approx H(\Delta_0')$ for some subdigraph $\Delta_0'$ of $\Delta_0$. Since $H(\Delta) \approx A(S_1)$ by Lemma 6 we conclude $H(\Delta) \approx H(\Delta_0')$. The result follows from Lemma 3. □

3. Spectral radius at most 2

Hermitian adjacency matrices with spectral radius at most 2 are considered in a broader context in [4]. Greaves [4 Theorem 3.1] classified, among others, cyclotomic matrices over Gaussian integers with unit entries and zero diagonals, under the assumption of maximality and indecomposability. It is also shown in [4 Theorem 3.4] that every indecomposable cyclotomic matrix over Gaussian integers is a principal submatrix of a maximal one. These matrices are described in terms of weighted digraphs in [4], which are not digraphs in our sense. Alternatively, we can describe each of these matrices $H$ by giving a set of vectors whose Gram matrix is $H + 2I$. For convenience, if $H + 2I$ is the Gram matrix of a set of vectors each of which has squared norm 2, then we call $H$ the displaced Gram matrix of this set. We denote by $e_1, e_2, \ldots, e_n$, the standard orthonormal basis of the vector space $\mathbb{C}^n$.

For a positive integer $k$, we define a matrix $T_{2k}^{(1)}$ as the displaced Gram matrix of the set of vectors

$$\{e_p \pm e_{p+1} \mid 1 \leq p \leq k\},$$
where indices are read modulo $k$. We also define a matrix $T_{2k}^{(i)}$ as the displaced Gram matrix of the set of vectors
\[ \{e_p \pm e_{p+1} \mid 1 \leq p < k\} \cup \{ie_k \pm e_1\}.\]

We define three more matrices $S_8^\dagger$, $S_{14}$, and $S_{16}$, as follows:

\[
S_8^\dagger = \begin{bmatrix} 0 & -1 & -1 & i & 1 & 0 & 0 & 0 \\ -1 & 0 & i & -1 & 0 & 1 & 0 & 0 \\ -1 & -i & 0 & 1 & 0 & 0 & 1 & 0 \\ -i & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & -i & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -i & 1 \\ 0 & 0 & 1 & 0 & 1 & i & 0 & -1 \\ 0 & 0 & 0 & 1 & i & 1 & -1 & 0 \end{bmatrix},
\]

\[
S_{14} = \begin{bmatrix} 0 & M \\ M^\top & 0 \end{bmatrix},
\]

\[
S_{16} = \begin{bmatrix} C + C^\top & -C + C^\top \\ C - C^\top & C^3 + C^5 \end{bmatrix},
\]

where $M$ and $C$ are the circulant matrices of order 7 and 8, with first row $[1, 1, 0, 1, 0, 0, -1]$ and $[0, 1, 0, \ldots, 0]$, respectively.

**Theorem 10** ([4 Theorems 3.1 and 3.4]). Let $H$ be an indecomposable Hermitian matrix with spectral radius at most 2 that has only zeros on the diagonal and whose nonzero entries are from $\{\pm 1, \pm i\}$. Then $H$ is equivalent to a principal submatrix of one of the matrices $T_{2k}^{(1)}$, $T_{2k}^{(i)}$, $S_8^\dagger$, $S_{14}$, and $S_{16}$, which are maximal subject to these conditions.

The reader might think that Theorem 10 immediately implies the classification of maximal digraphs with spectral radius at most 2 up to switching equivalence. In view of Lemma 3, however, switching equivalence of digraphs amounts to strong equivalence of their Hermitian adjacency matrices. Since Theorem 10 classifies possible matrices up to equivalence, but not strong equivalence, $H$ may not be strongly equivalent to $-H$ for those $H$ appearing in Theorem 10. We shall show however, that for each of the matrices $H$ appearing in Theorem 10 $H$ turns out to be strongly equivalent to $-H$. Moreover, we shall show that $H$ is strongly equivalent to the Hermitian adjacency matrix of a digraph.

Recall that the digraphs $\Delta_{2k}^{(1)}$ and $\Delta_{2k}^{(i)}$ are defined in Fig. 2–5.

**Proposition 11.** Let $x = 1$ or $i$. Every Hermitian matrix which is equivalent to $T_{2k}^{(x)}$ is strongly equivalent to $H(\Delta_{2k}^{(x)})$.

**Proof.** Observe that the matrix $T_{2k}^{(x)}$ is of the form
\[
\begin{bmatrix} A & B \\ B & -A \end{bmatrix}.
\]
Then \( QT_{2k}^{(x)} Q^* = -T_{2k}^{(x)} \) for

\[
Q = \begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}.
\]

Thus, it suffices to show that \( T_{2k}^{(x)} \) is strongly equivalent to \( H(\Delta_{2k}^{(x)}) \).

Suppose first \( x = 1 \). If \( k \) is even, then \( T_{2k}^{(1)} \) is strongly equivalent to the displaced Gram matrix of the set of vectors

\[
\{ e_p \pm e_{p+1} \mid 1 \leq p < k, \text{ } p \text{ even} \} \cup \{i(e_p \pm e_{p+1}) \mid 1 \leq p \leq k, \text{ } p \text{ odd} \}
\]

and this is the matrix \( H(\Delta_{2k}^{(1)}) \) (see Fig. 2). Next suppose \( k \) is odd. Then \( T_{2k}^{(1)} \) is strongly equivalent to the displaced Gram matrix of the set of vectors

\[
\{ e_p \pm e_{p+1} \mid 1 \leq p < k, \text{ } p \text{ even} \} \cup \{i(e_p \pm e_{p+1}) \mid 1 \leq p < k, \text{ } p \text{ odd} \}
\]

\[
\cup \{i(e_k + e_1)\} \cup \{-i(e_k - e_1)\},
\]

and this is the matrix \( H(\Delta_{2k}^{(1)}) \) (see Fig. 3).

Next suppose \( x = i \). If \( k \) is odd, then \( T_{2k}^{(i)} \) is strongly equivalent to the displaced Gram matrix of the set of vectors

\[
\{ e_p \pm e_{p+1} \mid 1 \leq p \leq k, \text{ } p \text{ even} \} \cup \{i(e_p \pm e_{p+1}) \mid 1 \leq p \leq k, \text{ } p \text{ odd} \} \cup \{ie_k \pm e_1\},
\]

and this is the matrix \( H(\Delta_{2k}^{(i)}) \) (see Fig. 4).

\[\square\]

**Proposition 12.** For \( S = S_8^1, S_{14} \) or \( S_{16} \), every Hermitian matrix which is equivalent to \( S \) is strongly equivalent to \( H(\Delta_8^1), H(\Delta_{14}) \) or \( H(\Delta_{16}) \), respectively.

**Proof.** For each \( S = S_8^1, S_{14} \) or \( S_{16} \), it suffices to show the Hermitian adjacency matrix of the corresponding digraph is strongly equivalent to \( S \) and \(-S\).

Let \( D_1 \) and \( D_2 \) be the diagonal matrices with diagonal entries

\[-i, i, i, 1, 1, 1, 1, -i\] \( \text{and} \) \( [1, 1, 1, -i, i, -i, -i, -1] \),

respectively. Then \( D_1 S_8^1 D_1^* = -D_2 S_8^1 D_2 \) is the Hermitian adjacency matrix of \( \Delta_8^1 \).

Let \( D_1 \) be the diagonal matrix with diagonal entries \([1, 1, 1, 1, \ldots, i] \), where \( 1 \) and \( i \) are repeated 7 times each. Then \( D_1 S_{14} D_1^* = -D_1 S_{14} D_1 \) is the Hermitian adjacency matrix of \( \Delta_{14} \).

Let \( D_1 \) and \( D_2 \) be the diagonal matrices with diagonal entries

\([i, \ldots, i, 1, 1, \ldots, 1] \), \( \text{and} \) \( [i, -i, i, -i, i, -i, i, -i, 1, -1, 1, -1, 1, -1, -1, -1] \),

respectively. Then \( D_1 S_{16} D_1^* = -D_2 S_{16} D_2 \) is the Hermitian adjacency matrix of \( \Delta_{16} \). \( \square \)
Let $\Delta$ be a connected digraph with $\rho(\Delta) \leq 2$. Let $H$ be the Hermitian adjacency matrix of $\Delta$. Then by Theorem $\text{(10)}$, $H$ is equivalent to a principal submatrix of one of the matrices

$$
T^{(1)}_{2k}, \ T^{(i)}_{2k}, \ S_8^+, \ S_{14}, \text{ or } S_{16}. 
$$

This implies that $H$ is a principal submatrix of a Hermitian matrix $T$ which is equivalent to one of the matrices in $(2)$. By Propositions $\text{(11)}$ and $\text{(12)}$, $T$ is strongly equivalent to one of the matrices

$$
H(\Delta^{(1)}_{2k}), \ H(\Delta^{(i)}_{2k}), \ H(\Delta_8^+), \ H(\Delta_{14}) \text{ or } H(\Delta_{16}).
$$

This implies that $H$ is strongly equivalent to a principal submatrix of one of the matrices in $(3)$. By Lemma $\text{3}$, $\Delta$ is switching equivalent to a subdigraph of a digraph in Theorem $\text{(1)(i)-(iii)}$.

Conversely, the digraphs in Theorem $\text{(1)(i)-(iii)}$ have spectral radius at most 2 by Theorem $\text{(10)}$. The digraphs in Theorem $\text{(1)(i)-(ii)}$ contain a subdigraph which is switching equivalent to $C_4$, so they have spectral radius exactly 2. It can be checked directly that the digraphs in Theorem $\text{(1)(iii)}$ have spectral radius exactly 2. This completes the proof of the first part of Theorem $\text{(1)}$.

4. Spectral radius less than 2

Throughout this section, we let $\Delta$ be a connected digraph with spectral radius less than 2, and complete the proof of the second part of Theorem $\text{(1)}$. Lemma $\text{4}$ shows that $\mathcal{S}(\Delta)$ has spectral radius less than 2. Connected signed graphs with spectral radius less than 2 are essentially classified by the following theorem due to McKee and Smyth.

**Theorem 13** (\cite{11, Theorem 4}). Up to equivalence, the connected signed graphs maximal with respect to having all their eigenvalues in $(-2, 2)$ are the eleven 8-vertex sporadic examples $U_1, \ldots, U_{11}$ shown in Fig. 10 and the infinite family $O_{2k}$ of $2k$-cycles with one negative edge for $2k \geq 8$, shown in Fig. 7. Further, every connected signed graph having all its eigenvalues in $(-2, 2)$ is either contained in a maximal one, or is a subgraph of one of the signed graphs $Q_{hk}$ of Fig. 9 for $h + k \geq 4$.

In the notation of \cite{6}, the digraphs $C_3$ and $D_3$ have isomorphic associated signed graph, which is $O_6$ in the notation of \cite{11}. This means that there are digraphs which are not switching equivalent, but their associated signed graphs are switching equivalent. Thus, the results of \cite{11} does not immediately imply those of \cite{6}. In this section, we show how to derive the results of \cite{6} from \cite{11}.

**Lemma 14.** If $\mathcal{S}(\Delta)$ has a connected component which is equivalent to a subgraph of $O_{2k}$, then $\Delta$ is switching equivalent to one of the digraphs in Theorem $\text{(1)(iv)-(viii)}$.

**Proof.** Suppose first $\mathcal{S}(\Delta)$ has a component which is equivalent to $O_{2k}$. Then $G(\mathcal{S}(\Delta))$ is $C_{2k}$ or $2C_{2k}$ by Lemma $\text{5}$. In the former case, Lemma $\text{5}$ implies that $G(\Delta)$ is $C_k$, while in the latter case, Lemma $\text{5}$ implies that $G(\Delta)$ is $C_{2k}$. Then the classification follows from \cite{6, Theorem 3.9} (see also \cite{5, Proposition 8.8}).
If $S(\Delta)$ has a component which is equivalent to a proper subgraph of $O_{2k}$, then such a component is a path. If $S(\Delta)$ is connected, then by By Lemma 5 (iii), the sum of the degrees of vertices of $G(\Delta)$ is odd, which is impossible. Thus $S(\Delta)$ consists of two paths. Then by Lemma 6, $H(\Delta)$ is strongly equivalent to the adjacency matrix of a path. By Lemma 3, $\Delta$ is switching equivalent to a path. □

**Lemma 15.** If $S(\Delta)$ has a connected component which is equivalent to a subgraph of $Q_{hk}$, then $\Delta$ is switching equivalent to one of the digraphs in Theorem 1 (viii) – (x).

**Proof.** Lemma 5 shows that $S(\Delta)$ is disconnected. Indeed, for example, for $Q_{hk}$ with $h, k > 0$, there are exactly two vertices of degree 3, and they have common neighbors. Thus, by Lemma 6, $G(\Delta)$ is a subgraph of $G(Q_{hk})$. Suppose $\Delta$ contains a quadrangle $Q$. Then by Lemma 14, $Q$ is switching equivalent to $\tilde{C}_4$, $\tilde{C}_4'$ or $\tilde{C}_4''$. In the latter two cases, by Lemma 5, $S(Q)$ is connected, and hence an octagon, a contradiction. Thus $Q$ is switching equivalent to $\tilde{C}_4$. This means that $\Delta$ is switching equivalent to $\Box_{a,0,c,0}$ for some $a,c \geq 0$.

Now we can assume that $G(\Delta)$ is a proper subgraph of $G(Q_{hk})$ not containing a quadrangle. Then $G(\Delta)$ is $Y_{a,1,1}$ or a path, and hence $\Delta$ is switching equivalent to $Y_{a,1,1}$ or a path. □

**Lemma 16.** If $S(\Delta)$ is connected and equivalent to a subgraph of $U_i$ for some $i$, then $i = 1$ or 6, and $\Delta$ is switching equivalent to a path of length at most 3, $D_3$, $\tilde{C}_3$, $\tilde{U}_1$ or $\tilde{U}_6$.

**Proof.** First, if $S(\Delta)$ is connected and equivalent to a proper subgraph of $U_i$, then $\Delta$ has at most three vertices. Then it is routine to check that $\Delta$ is switching equivalent to a path of length at most 3, $D_3$ or $\tilde{C}_3$.

Now assume that $S(\Delta)$ is equivalent to $U_i$. Then the underlying graph of $U_i$ must satisfy the conditions of Lemma 5. The only graphs satisfying the conditions are $U_1, U_6$.

If $S(\Delta)$ is equivalent to $U_1$, then the underlying graph of $\Delta$ is $K_4$. Then it is routine to check that $\Delta$ is switching equivalent to $\tilde{U}_1$.

If $S(\Delta)$ is equivalent to $U_6$, then the underlying graph of $\Delta$ is the triangle with a pendant edge attached. Then it is routine to check that $\Delta$ is switching equivalent to $\tilde{U}_6$. □
Lemma 17. If $S(\Delta)$ is disconnected and a connected component of $S(\Delta)$ is equivalent to a subgraph of $U_i$ for some $i$, then $\Delta$ is switching equivalent to a subdigraph of a canonical digraph of $U_i$.

Proof. Observe that all of the signed graphs $U_1, \ldots, U_{11}$ are bipartite. The result follows from Lemmas 3 and 9. \hfill \Box

Now we are ready to complete the proof of the second part of Theorem 1. By Lemma 4, the associated signed graph $S(\Delta)$ has spectral radius less than 2. By Theorem 13, up to equivalence, each connected component of $S(\Delta)$ is either contained in a maximal one $O_{2k}, U_1, \ldots, U_{11}$, or is a subgraph of $Q_{hk}$ for some $h, k$ with $h + k \geq 4$. If $S(\Delta)$ has a connected component which is equivalent to a subgraph of $O_{2k}$ (resp.
Q_{hk}), then Lemma 14 (resp. Lemma 15) shows that ∆ is switching equivalent to one of the digraphs in Theorem 1 (iv)–(viii) (resp. (viii)–(x)). If \( S(\Delta) \) is connected and equivalent to a subgraph of \( U_i \), then Lemma 14 shows that \( \Delta \) is switching equivalent to one of the digraphs in Theorem 1 (iv)–(viii) (resp. (viii)–(x)). If \( S(\Delta) \) is disconnected and has a connected component which is equivalent to a subgraph of \( U_i \), then Lemma 17 shows that \( \Delta \) is switching equivalent to a subdigraph of one of the digraphs in Theorem 1 (xiii).

Conversely, every digraph in (iv)–(xiii) has spectral radius less than 2 by Lemma 4 and Theorem 13. This completes the proof of Theorem 1.

Theorem 1 and [6, Theorem 4.15] give the same infinite families of digraphs \( D_n, \tilde{C}_n, \tilde{C}_n', \tilde{C}_n'', P_n, \Box_{a,0,c,0} \) and \( Y_{a,1,1} \). Since Theorem 1 only claims that every digraph with spectral radius less than 2 is a subdigraph of a digraph listed there, those which are listed in [6, Theorem 4.15] but are not maximal do not appear in Theorem 1.

Table 1 gives the list of graphs apart from the infinite families in Theorem 1 and [6, Theorem 4.15]. The first column gives the item labels in [6, Theorem 4.15], and the second column gives the name of the digraphs in each item, except item (g) for which the digraph has no name. The third column gives the names of digraphs in our notation, where \( \Delta(U) \) denotes a canonical digraph of a bipartite signed graph \( U \). The last column will be explained in Section 5. The digraph \( \Delta(U_7) \) is missing in [6], giving a counterexample to the statement of [6, Lemma 4.8(b)].

| Notation in [6] | Theorem 1 | lattice |
|-----------------|-----------|---------|
| (f) Y_{4,2,1}   | \Delta(U_5) | \( E_8 \otimes \mathbb{Z}[i] \) |
| Y_{3,2,1}       | \subset Y_{4,2,1} | \( E_7 \otimes \mathbb{Z}[i] \) |
| (g) —           | U_6       | \( E_8^c \) |
| (h) Y_1         | U_1       | \( E_8^c \) |
| (j) \Box_{3,1,0,0} | \Delta(U_3) | \( E_8 \otimes \mathbb{Z}[i] \) |
| \Box_{2,1,1,0}  | \Delta(U_2) | \( E_8 \otimes \mathbb{Z}[i] \) |
| \Box_{1,1,1,1}  | \Delta(U_9) | \( E_8 \otimes \mathbb{Z}[i] \) |
| (k) X_1         | \subset X_2 | \( E_6 \otimes \mathbb{Z}[i] \) |
| X_2             | \subset X_3 | \( E_7 \otimes \mathbb{Z}[i] \) |
| X_3             | \Delta(U_{11}) | \( E_8 \otimes \mathbb{Z}[i] \) |
| X_4             | \Delta(U_4) | \( E_8 \otimes \mathbb{Z}[i] \) |
| X_5             | \subset X_6 | \( E_7 \otimes \mathbb{Z}[i] \) |
| X_6             | \Delta(U_8) | \( E_8 \otimes \mathbb{Z}[i] \) |
| X_7             | \Delta(U_{10}) | \( E_8 \otimes \mathbb{Z}[i] \) |
| X_8             | \Delta(U_1) | \( E_8 \otimes \mathbb{Z}[i] \) |
| (l) X_9         | \subset X_{10} | \( E_7 \otimes \mathbb{Z}[i] \) |
| X_{10}          | \Delta(U_6) | \( E_8 \otimes \mathbb{Z}[i] \) |
| —               | \Delta(U_7) | \( E_8 \otimes \mathbb{Z}[i] \) |

Table 1. Comparison of Theorem 1 and [6, Theorem 4.15]
5. Concluding remarks

By a $\mathbb{Z}$-lattice (or simply, a lattice), we mean a free $\mathbb{Z}$-module equipped with a positive definite symmetric bilinear form. A $\mathbb{Z}[i]$-lattice, which we call a Gaussian lattice, can be defined in an analogous manner using a Hermitian form instead of a symmetric bilinear form. Since the real part of a positive definite Hermitian form is a positive definite symmetric bilinear form, a Gaussian lattice $\Lambda$ is also a $\mathbb{Z}$-lattice by regarding $\Lambda$ as a $\mathbb{Z}$-module. Conversely, if $L$ is a lattice, then one can equip a positive definite Hermitian form on $L \otimes \mathbb{Z}[i]$, making it a Gaussian lattice. Not every Gaussian lattice is obtained in this way.

A (Gaussian) lattice is called a (Gaussian) root lattice if it is generated by its set of vectors of squared norm 2. A (Gaussian) lattice is said to be irreducible if it is not an orthogonal direct sum of proper sublattices. Note that a (Gaussian) root lattice is irreducible if and only if the non-orthogonality graph on the set of its roots is connected. Using the classification of irreducible root lattices [2], every irreducible Gaussian root lattice $\Lambda$ is either $L \otimes \mathbb{Z}[i]$ for some irreducible root lattice $L$, or $\Lambda$ is irreducible as a $\mathbb{Z}$-lattice. The latter possibilities are classified by [8, Lemma 3.1]:

(i) The root lattice of type $D_{2n}$ may be regarded as a Gaussian lattice of rank $n$

$$D_{2n}^{\mathbb{Z}} = \{ \sum_{j=1}^{n} a_j e_j \mid a_j \in \mathbb{Z}[i], \sum_{j=1}^{n} a_j \in (1 + i)\mathbb{Z}[i] \},$$

where $e_1, \ldots, e_n$ are the standard orthonormal basis.

(ii) The root lattice of type $E_8$ may be regarded as a Gaussian lattice $E_8^{\mathbb{Z}}$ of rank 4 (see [7, p. 373]).

Every connected digraph with spectral radius at most 2 gives rise to an irreducible Gaussian root lattice. In fact, if $H$ is the Hermitian adjacency matrix of a digraph and $H$ has spectral radius at most 2, then $2I - H$ is positive semidefinite. This implies that there is a Gaussian lattice $\Lambda$ generated by a set $X$ of vectors of squared norm 2 in $\Lambda$ such that $-H$ is the displaced Gram matrix of $X$. Thus, the lattice $\Lambda$ is a Gaussian root lattice. In particular, every maximal digraph with spectral radius 2 in Theorem 1 generates one of the Gaussian lattices (i) or (ii) or $L \otimes \mathbb{Z}[i]$ for some irreducible root lattice $L$. Also, every digraph with spectral radius less than 2 gives a basis of a Gaussian root lattice. Table 2 (resp. the last column of Table 1) shows the correspondences between the maximal digraphs with spectral radius exactly 2 (resp. less than 2) and the Gaussian root lattices. It is worth mentioning that the associated signed graph $S(\Delta)$ of a digraph $\Delta$ with spectral radius at most 2 is connected if and only if the corresponding Gaussian root lattice $\Lambda$ is irreducible as a $\mathbb{Z}$-lattice. Indeed, $X$ is a subset of $\Lambda$ whose displaced Gram matrix is $-H(\Delta)$, then the real part of the displaced Gram matrix of $X \cup iX$ is the adjacency matrix of $S(\Delta)$. Thus, $\Lambda$ is irreducible as a $\mathbb{Z}$-lattice if and only if $S(\Delta)$ is connected.

As is well known, a connected simple graph with smallest eigenvalue at least $-1$ is a complete graph. In fact, there are no graphs with smallest eigenvalue in $(-\sqrt{2}, -1)$. Guo and Mohar [5, Proposition 8.6] determined digraphs with the same spectrum as a complete graph. It turns out that digraphs with the same spectrum as a complete
graph are switching equivalent to a complete graph. We strengthen [5, Proposition 8.6] by showing that a digraph with smallest eigenvalue greater than $-\sqrt{2}$ is switching equivalent to a complete graph.

For a Hermitian matrix $H$, we denote by $\lambda_{\min}(H)$ the smallest eigenvalue of $H$, and for a digraph $\Delta$, we write $\lambda_{\min}(\Delta) = \lambda_{\min}(H(\Delta))$.

**Proposition 18.** Let $\Delta$ be a connected digraph with $\lambda_{\min}(\Delta) > -\sqrt{2}$. Then $\Delta$ is switching equivalent to a complete graph.

**Proof.** By Lemma 4, $\lambda_{\min}(A(S(\Delta))) > -\sqrt{2}$. Then each connected component of $S(\Delta)$ is switching equivalent to a complete graph by [3, Prop. 4.7]. Since $S(\Delta)$ cannot be complete by Lemma 5 (i), $S(\Delta)$ is disconnected. Now Lemma 6 implies that $H(\Delta)$ is strongly equivalent to the adjacency matrix of a complete graph. The result then follows from Lemma 3.

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