MAGNETIC PROPERTIES OF UNCONVENTIONAL SUPERCONDUCTORS

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Abstract

The article reviews recent developments on magnetic properties of superconductors with anisotropic Cooper pairing. In particular, we show how the concept of broken symmetries is applied to the investigation of the mixed state in superconductors with a multicomponent order parameter. Starting from the phenomenological description in the framework of the generalized Ginzburg-Landau theory, we discuss different types of quantized vortices appearing at $H_{c1}$ in states with and without time-reversal breaking. General classification of superconducting phase transitions in a uniform magnetic field at $H_{c2}$ is constructed. Vortex lattices of different forms are found in the vicinity of the upper critical field. Symmetry arguments are used to classify phase transitions inside the mixed state. Special attention is given to results which can be obtained analytically. Also special emphasis is put on the open questions of the theory.
1 Introduction

The BCS microscopic theory of superconductivity [7] was preceded by the phenomenological approach to the superconducting phase transition developed in the works by Ginzburg, Landau [20] and Abrikosov [1]. Considering the superconducting order parameter as a scalar complex wave function which breaks gauge symmetry, one can explain many remarkable properties of superconductors: the Meisner effect, critical behavior of bulk samples and thin films, two types of superconductivity, and so on. The microscopic approach developed later clarified the physical meaning of the complex order parameter and established the relation between phenomenological constants of the Ginzburg-Landau (GL) functional and microscopic characteristics of superconducting metal.

One of the basic points of the BCS theory is the assumption that electrons are paired in a fully isotropic s-wave state with both total spin $S$ and orbital momentum $L$ of the Cooper pairs equal to zero. The possibility of the unconventional superconductivity with anisotropic, non s-wave pairing has been discussed since early 60th. In particular, it was shown [10,57] that anisotropic Cooper pairs are formed if the interaction between fermion quasiparticles is attractive for at least one value of the relative orbital momentum.

The feature of the phenomenological theory of unconventional superconductors is that they are described by the multicomponent order parameter with a more complicated structure of the GL functional. This leads to qualitatively new properties of unconventional superconductivity. The behavior of superconductors with a multicomponent order parameter in magnetic field is the subject of this review.

Until the beginning of the 80th, superfluid $^3\text{He}$ was the only known example of an unconventional pairing. However, because of the absence of electric charge, the interaction of superfluid component in $^3\text{He}$ with magnetic field is restricted to the weak spin paramagnetism of Cooper pairs, and lies therefore beyond our attention. (The complete review on superfluidity of $^3\text{He}$ is given in [73].)

Since the discovery of superconductivity in so called heavy fermion compounds [17,23,51], they have been extensively discussed as new possible candidates for anisotropic pairing. Some of them, such as CeCu$_2$Si$_2$, UBe$_{13}$, URu$_2$Si$_2$, UPt$_3$, UPd$_2$Al$_3$, UNi$_2$Al$_3$, reveal various unusual properties, which could be attributed to unconventional superconducting states [36,66,68].

The most convincing evidence of the nontrivial superconductivity in heavy fermions is the complicated $H$–$T$ phase diagram of superconducting UPt$_3$ (Fig. 1). In numerous experiments the splitting of the superconducting phase transition in zero magnetic field, the discontinuity in slope (kink) of $H_{c2}(T)$, two different superconducting states at $H = 0$ and three different vortex phases have been observed (see [36,68] and reference there in). Such multiphase diagram could not be interpreted in terms of the usual GL theory. It seems natural to explain it assuming that superconductivity in UPt$_3$ is described by a multicomponent order parameter with different minimums of corresponding GL functional, and phase transitions take place between them.

A number of questions about magnetic properties of unconventional superconductors are inspired by these experiments: what are the common and different features of the unconventional and s-
wave superconductors, what are the phase transitions in the mixed state of superconductors with a multicomponent order parameter? In this paper we report on the recent progress in this field achieved since the publication of previous reviews $^{[22,65]}$. We concentrate mostly on theoretical aspects of the description of the vortex state in unconventional superconductors (the list of the discussed questions is given at the end of Sec. 1.3). We do not suppose to present here details of the concrete models designed to explain peculiar properties of UPt$_3$, this was done recently in a number of works $^{[27,43,61]}$. However, we use these models as an illustration of the general approach and try to stress, when it is possible, which particular properties of the mixed state of multicomponent superconductors can be used for reliable identification of an unconventional superconductivity in heavy fermions.

Three following Subsections have an introductorily character and give the outline of the GL theory of unconventional superconductors, the main points of the Abrikosov theory of the mixed state of conventional $s$-wave superconductors and the properties of multicomponent superconductors at $H = 0$.

### 1.1 Ginzburg-Landau Theory

In this Subsection we consider GL theory of unconventional superconductivity developed in $^{[8,72,75]}$. We define the order parameter and the symmetry group of the space uniform state of unconventional superconductors (at $H = 0$) which will be used to describe the mixed state in the next Sections. The two-component model, which is commonly used for the description of unconventional superconductivity in UPt$_3$, is considered as a particular example of unconventional superconductivity.

The superconducting second order phase transition in the ordinary case results in the breaking of normal state gauge symmetry. To give the phenomenological description of the phase transition in an unconventional superconductor, we first define the symmetry group of a normal metal, which can be broken when the condensate of Cooper pairs appears. This group includes:

- The gauge group $U(1)$ of multiplication of electron wave functions $\Psi$ by an arbitrary phase factor: $\Psi \rightarrow e^{i\alpha} \Psi$.

- The space group of the crystal lattice.

- The group of transformations of the spin space.

- The time reversal operation $R$.

Two remarks are relevant at this point.

(i) The space group, generally speaking, includes different combinations of translations, rotations and reflections. However, because of the large size of Cooper pairs (which at $T = 0$ is of the order of coherence length $\xi_0$) in comparison to lattice constant $a$ (in BCS theory $a/\xi_0 \sim T_c/E_F$), the discrete crystal structure has no effect on the classification of superconducting states. Therefore,
a superconductor can be considered as uniform but anisotropic medium with the space symmetry
$T \times G$, where $T$ is the abelian group of continuous three dimensional translations, and $G$ is the
point group of the crystal. Classification of superconducting states were done taking into account
discrete crystal structure in $^{[52,47]}$.

(ii) Transformations of the spin space are separate from real space transformations only if the
spin-orbital interaction is negligible (as in the case of superfluid $^3$He). In the opposite case of
strong spin-orbital coupling, one should assume that point group $G$ acts in both coordinate and
spin spaces. Since the strong spin-orbital coupling seems to be relevant for properties of heavy
fermion compounds, we will concentrate on the latter case. Note, however, that the spin-orbital
interaction does not lift the Kramers degeneracy, and superconducting electrons are characterized
by the two-component wave function $\Psi_{\alpha}$. This pseudospin degeneracy is important for the triplet-
singlet classification of the superconducting state $^{[4,72,75]}$ (see below).

Finally, the symmetry group of the normal state $\mathcal{G}$ can be written as follows:

$$\mathcal{G} = T \times G \times R \times U(1) \ .$$  \hspace{1cm} (1)

The superconducting order parameter is the condensate wave function

$$\Delta_{\alpha\beta}(r_1, r_2) = \langle \hat{\Psi}_\alpha(r_1) \hat{\Psi}_\beta(r_2) \rangle \ .$$  \hspace{1cm} (2)

It is more convenient to rewrite (2) in Wigner variables: Cooper pair center of mass coordinate $r = (r_1 + r_2)/2$ and position on the Fermi surface $k$, which is obtained by Fourier transformation: $(r_1 - r_2) \rightarrow k$. The electron energy spectrum is reconstructed in a thin layer near the Fermi surface. Hence, the dependence of $\Delta_{\alpha\beta}(k,r)$ on the direction of $k$ is more important than on its modulus. In this Subsection we assume that the superconducting state is homogeneous in space what means that $\Delta_{\alpha\beta}$ does not depend on $r$. For this reason we will not discuss here translational properties of the order parameter.

The possible superconducting states are classified according to the symmetry of $\Delta_{\alpha\beta}(k)$. We
have already mentioned that the superconducting phase transition breaks the gauge invariance $U(1)$. Hence possible residual symmetries of $\Delta_{\alpha\beta}(k)$ may be found by listing all subgroups of group $\mathcal{G}$ which do not contain the group $U(1)$ as a separate subgroup $^{[8,75]}$. For a conventional superconductor this subgroup is equal to $G \times R$. If it is smaller than $G \times R$, the order parameter $\Delta_{\alpha\beta}(k)$ is defined to be unconventional.

Not all subgroups of $\mathcal{G}$ correspond to energetically stable superconducting phases. The application of the Landau symmetry approach restricts the list of possible superconducting phases arising just below $T_c$.

First of all, the superconducting phase transition corresponds to a particular irreducible rep-
resentation of the point group $G$. More exactly, the order parameter $\Delta_{\alpha\beta}(k)$ should be expanded
over the basis functions $\Phi_{\alpha\beta}^i(k)$ of the given irreducible representation in the vicinity of the critical
temperature:

$$\Delta_{\alpha\beta}(k, r) = \sum_i \eta_i(r) \Phi_{\alpha\beta}^i(k) \ .$$  \hspace{1cm} (3)
Another important property of $\Delta_{\alpha\beta}(k)$ follows from the anticommutation relation on fermion operators in (2): $\Delta_{\alpha\beta}(k) = -\Delta_{\beta\alpha}(-k)$. Therefore, for crystals with space inversion symmetry, the superconducting pairing occurs either in the singlet, space-even states with $\hat{\Delta}(k) = i\hat{\sigma}_y\psi(k)$ (where $\psi(k) = \psi(-k)$) or in the triplet, space-odd states with $\hat{\Delta}(k) = i\hat{\sigma}_y(\hat{\sigma}\cdot\mathbf{d}(k))$ (where $\mathbf{d}(k) = -\mathbf{d}(-k)$).

As the gap in the quasiparticles energy spectrum is proportional to the superconducting order parameter, the zeroes of the latter play important role for low temperature thermodynamics of a superconductor \cite{22,65}. The symmetry analysis \cite{8,75} shows that in the triplet case the types of zeros in the energy gap always correspond to points on the Fermi surface, whereas whole lines of zeros are possible for the singlet pairing.

When this phenomenological approach is applied to heavy fermion superconductors, the following one- (1D) and multidimensional (2D or 3D) irreducible representations for different point groups $G$ are usually considered (we use notations of \cite{33}):

1D: $A_1, A_2, B_1, B_2$ and 2D: $E_{1g}, E_{1u}, E_{2u}$ for the hexagonal group $D_{6h}$ of UPt$_3$, UPd$_2$Al$_3$, UNi$_2$Al$_3$.

1D: $A_1, A_2$, and 2D: $E$ for the tetragonal group $D_{4h}$ of CeCu$_2$Si$_2$, URu$_2$Si$_2$.

1D: $A_1, A_2$, 2D: $E$ and 3D: $F_1, F_2$ for the cubic group $O_h$ of UBe$_{13}$.

The explicit form of basis functions of these irreducible representations is given in \cite{22,65}.

Near $T_c$ free energy can be obtained as an expansion in powers of the order parameter which is invariant under the action of group $G$.

Consider first the case of 1D irreducible representation of $G$. The symmetry of the order parameter is determined by the basis function $\hat{\Phi}(k)$. When this representation is identical, $\hat{\Phi}(k)$ has the full symmetry of the group $G \times R$ and corresponds to the conventional Cooper pairing. For nonidentical representations corresponding to the unconventional pairing some symmetry elements of $G$ change phase of $\hat{\Phi}(k)$ by $\pi$, and, consequently, the symmetry group of superconducting state is expressed in the form $G'(G') \times R$, where $G'$ is an invariant subgroup of $G$ \cite{75}. From this follows that the modulus $|\hat{\Phi}(k)|$ still has the full crystal symmetry and the superconducting state is nonmagnetic. In both cases the order parameters are one-component and their space variations are described by one complex amplitude $\Psi(r)$. For many purposes it is sufficient to write the free energy density up to the forth order in powers of $\Psi(r)$. As a result one obtains the usual Ginzburg-Landau functional:

$$ F = -\alpha(T)|\Psi|^2 + \beta|\Psi|^4 + K|D_k\Psi|^2 + \frac{\hbar^2}{8\pi} - \frac{\mathbf{B} \mathbf{H}}{4\pi} , $$

\begin{equation}
\alpha(T) = \alpha' \left( 1 - \frac{T}{T_c} \right) , \quad D_k = \partial_k - i\frac{2e}{\hbar c} A_k , \quad k = x, y, z ,
\end{equation}

where $\alpha, \beta, K$ are positive phenomenological constants. Magnetic induction $\mathbf{B}$ is a spatial average of the microscopic magnetic field: $\mathbf{B} = \langle \mathbf{h}(r) \rangle$ ($\mathbf{h} = \text{rot}\mathbf{A}$). Note that for uniaxial (biaxial) crystals, the coefficient $K$ should be replaced by a uniaxial (biaxial) second rank tensor.

Thus the magnetic properties of one-component unconventional superconductors and conventional superconductors are similar in the framework of GL theory. For this reason we will be interested only in multicomponent unconventional superconductors which correspond to the multidimensional irreducible representations of $G$.  

In our review we will consider magnetic properties of unconventional superconductors mostly on the example of two-dimensional irreducible representations \( E_1 \) of group \( D_{6h} \) with basis functions \( \hat{\Phi}_x(k) = k_x k_z \), \( \hat{\Phi}_y(k) = k_y k_z \) in the case of even order parameter \( E_1^g \), or \( \hat{\Phi}_x(k) = \hat{z} k_x \), \( \hat{\Phi}_y(k) = \hat{z} k_y \) for odd representation \( E_1^u \). Vector \( \eta_i \) in (3) has two components \( \eta_x \) and \( \eta_y \), and the GL functional can be written as follows:

\[
F = -\alpha(T)(\eta^\ast \cdot \eta) + \beta_1(\eta^\ast \cdot \eta)^2 + \beta_2|\eta - \eta^\ast|^2 + K_1 D_i^\ast \eta_j^i D_i^\ast \eta_j + K_2 D_i^\ast \eta_i \eta_j + \frac{\hbar^2}{8\pi} - \frac{\mathbf{BH}}{4\pi}, \quad i, j = x, y ,
\]

(5)

\[ \hat{\Delta}(k, r) = \eta_x(r) \hat{\Phi}_x(k) + \eta_y(r) \hat{\Phi}_y(k) . \]

Our choice is explained by two reasons.

(i) The symmetry group \( D_{6h} \) of functional (3) is the same as of the normal state of the heavy fermion superconductor \( \text{UPt}_3 \). It is the \( E_1 \) irreducible representation that is involved in many models of interpretation of the complicated phase diagram of \( \text{UPt}_3 \) \([24,28,45,74]\).

(ii) The 2D model (3) is, in a sense, the simplest example of multicomponent order parameter which possesses the nontrivial properties of the mixed state. Many properties of the GL functional (3) may be straightforwardly transferred on another more complicated models of superconducting pairing in \( \text{UPt}_3 \) \([19,53,83]\).

The symmetry of the multicomponent order parameter is determined by the coefficients \( \eta_i \) in expansion (3). Therefore, in order to find the residual symmetry of the superconducting state, the corresponding GL functional should be minimized. The complete list of the possible residual symmetries for the unconventional superconductivity in heavy fermions is presented in \([22,65]\). The residual symmetries of the uniform superconducting state in \( E_1 \) model will be discussed in Sec. 1.3.

### 1.2 Magnetic Properties of Usual Superconductors

One of fundamental properties of superconductors is the expulsion of magnetic field from a sample (the Meisner effect). Depending on the material, the Meisner effect can be either complete, as in type-I superconductors, or partial, as in type-II superconductors. In the latter case magnetic field is dispersed through the material in the form of flux lines or vortices, each of them containing a unit of magnetic flux \( \phi_0 = \frac{hc}{2|e|} = 2.07 \times 10^{-7} \text{ G-cm}^2 \). Note that heavy fermion superconductors demonstrate properties of the type-II superconductors.

Having in mind the discussion of the mixed state in unconventional superconductors, we survey the main magnetic properties of usual type-II superconductors with \( s \)-pairing, which can be understood on the basis of the GL functional (4). Varying energy (4) with respect to \( \delta \Psi^\ast \) and \( \delta \mathbf{A} \), one gets the GL equations:

\[
-\alpha \Psi + 2\beta |\Psi|^2 \Psi - KD_i D_i \Psi = 0 ,
\]

(6)

\[
(\nabla \times (\nabla \times \mathbf{A})) = \frac{4\pi}{c} \mathbf{j} , \quad \mathbf{j} = \frac{2e}{i\hbar} K (\Psi^\ast \nabla \Psi - \Psi \nabla \Psi^\ast) - \frac{8e^2 K}{\hbar^2 c} |\Psi|^2 \mathbf{A} .
\]
According to the classical results of Abrikosov \cite{1}, magnetic properties of type-II superconductors are quite different in different parts of the \( H-T \) phase diagram.

For magnetic fields smaller than the lower critical field \( H_{c1} \), circular surface currents screen the external field and magnetic flux does not penetrate inside the sample. The amplitude of superconducting order parameter is fixed: \(|\Psi_0|^2 = \alpha/2\beta\), whereas the phase of \( \Psi_0 \) is arbitrary. The gain of the energy in superconducting state defines the thermodynamical critical field \( H_c \): \( F_n - F_s = H_c^2 / 8\pi \).

For \( H > H_{c1} \) the flux begins to penetrate inside the material in the form of vortices. A single vortex line has an axially symmetric structure:

\[
\Psi(r) = \Psi(r) \exp(-i\varphi), \quad h(r) = h(r) \hat{z},
\]

(\( r \) and \( \varphi \) are the polar coordinates) and carries one quanta of magnetic flux \cite{16,59}. The amplitude \( \Psi(r) \) is suppressed to zero at the vortex center. It restores to the equilibrium value \( \Psi_0 \) at a radius of the vortex core, which is of the order of the coherence length

\[
\xi = \left( \frac{K}{\alpha} \right)^{1/2}.
\]

The magnetic field is maximal at the vortex center and approximately equal to \( 2H_{c1} \). The supercurrents (which are beyond the vortex core) screen the magnetic field and \( h(r) \) tends to zero outside the screening region which is of the order of the London penetration depth

\[
\lambda = \left( \frac{\hbar^2 e^2 \beta}{16\pi^2 \alpha K} \right)^{1/2}.
\]

The ratio of the penetration depth to the coherence length defines a dimensionless temperature independent (near \( T_c \)) GL parameter \( \kappa \), whose value determines the type of superconductor. For type-II superconductors \( \kappa > 1/\sqrt{2} \). It is the screening supercurrents \( j_s = c \phi_0 / (8\pi^2 \lambda^2 r) \), \( \xi \ll r \ll \lambda \) which bring for large \( \kappa \) the main logarithmic contribution to the vortex energy per unit length or “flux line tension:”

\[
\varepsilon_L = \frac{\phi_0^2}{(4\pi \lambda)^2} \left( \ln \frac{\lambda}{\xi} + \epsilon \right),
\]

where the small numerical constant \( \epsilon \approx 0.08 \) is due to the vortex core energy. The lower critical field is defined by the expression:

\[
H_{c1} = \frac{4\pi}{\phi_0} \cdot \varepsilon_L = \frac{\phi_0}{4\pi \lambda^2} \left( \ln \frac{\lambda}{\xi} + \epsilon \right).
\]

When the intervortex distance \( r_L \) is larger than \( \lambda \), it is sufficient to take into account interaction only between nearest neighbors through pair potential \( U = \phi_0 / 4\pi \cdot h(r_L) \). In the intermediate region \( \xi \ll r_L \leq \lambda \) \( (H > H_c) \) vortex lines form a dense lattice. Single vortex interacts with all neighbors in the region of the area \( \lambda^2 \).

When increasing field reduces the distance between vortices to the order of \( \xi \), the superconductivity is destroyed. To calculate this the upper critical field \( H_{c2} \), one should consider the GL equation
neglecting the diamagnetic field generated by superconducting currents and the nonlinear term, which are small near $H_{c2}$. Thus we derive the linearized GL equation:

$$\alpha \Psi = -K \left( \nabla - i \frac{2e}{\hbar c} A \right)^2 \Psi,$$  \hspace{1cm} (12)

which formally coincides with the Schrödinger equation for an electron in a magnetic field. This equation is known to have a discrete set of Landau levels. Using the analogy with the Landau level quantization, we obtain a set of eigen (critical) fields:

$$H_n = \frac{\hbar c \alpha}{2|e|K(2n+1)} = \frac{\phi_0}{2\pi \xi^2} \cdot \frac{1}{2n+1}.$$  \hspace{1cm} (13)

The upper critical field corresponds to the highest $H_n$ and, therefore, to the lowest Landau level with $n = 0$:

$$H_{c2} = \frac{\phi_0}{2\pi \xi^2}.$$  \hspace{1cm} (14)

Vortices form a regular lattice of equilateral triangles at the entire field region between $H_{c1}$ and $H_{c2}$.

1.3 New Features of Multi-Component Superconductors

As we have seen in previous Subsection, the energy functional \((\mathcal{H})\) leads to the particular magnetic properties. These properties will be the same (at least near $T_c$) for all one-component superconductors, since they are also described by functional \((\mathcal{H})\) or its anisotropic modifications. Therefore we are interested mainly in superconductors described by a multicomponent order parameter.

The dimensionality of the order parameter is that of the irreducible representation to which it belongs. Additional degrees of freedom always correspond to the states of the Cooper pairs degenerated with respect to the direction of the internal angular momentum, it does not matter whether of spin or orbital origin. For example, in the case of the 2D order parameter, the degeneracy exists between the states with “up” or “down” direction of the angular momentum $i(\hat{\eta}^* \times \hat{\eta})$.

In the case of GL energy \((\mathcal{H})\), the equilibrium order parameter satisfies the following nonlinear equations:

$$\alpha \eta_i - 2\beta_1 \eta_i (\eta_j^* \eta_j) - 2\beta_2 \eta_i^* (\eta_j \eta_j) + K_1 D_j D_j \eta_i + K_2 D_i D_j \eta_j + K_3 D_j D_i \eta_j + K_4 D_z D_z \eta_i = 0,$$  \hspace{1cm} (15)

$$j_i = -\frac{4e}{\hbar} \text{Im} \left[ K_1 \eta_j^* D_i \eta_j + K_2 \eta_i^* D_j \eta_j + K_3 \eta_j^* D_j \eta_i + K_4 \delta_{iz} \eta_i^* D_z \eta_j \right].$$

New properties in comparison with ordinary superconductors appear already without magnetic field. Depending on the sign of $\beta_2$, the resulting superconducting phases are:

$$\tilde{\Delta}(\mathbf{k}) = \left( \frac{\alpha}{4\beta_1} \right)^{1/2} (\hat{\Phi}_1(\mathbf{k}) + i\hat{\Phi}_2(\mathbf{k})),$$  \hspace{1cm} (|\hat{\Phi}_1| = |\hat{\Phi}_2|, \hat{\Phi}_1 \perp \hat{\Phi}_2), \hspace{1cm} \beta_2 > 0,$$  \hspace{1cm} (\hat{\Phi} = \hat{\Phi}^*), \hspace{1cm} \beta_2 < 0.$$  \hspace{1cm} (16)

$$\tilde{\Delta}(\mathbf{k}) = \left( \frac{\alpha}{2\beta_{12}} \right)^{1/2} \hat{\Phi}(\mathbf{k}),$$  \hspace{1cm} (17)
To be consistent, the energy should be bounded from below for each phase (16) and (17) what leads to the positive definiteness of the forth order terms:

$$\beta_1 > 0, \quad \beta_1 + \beta_2 = \beta_{12} > 0.$$  \hspace{1cm} (18)

For $\beta_2 > 0$ the ground state of the GL functional (3) has two-fold degeneracy with respect to the appearance of either $\hat{\Phi}_+ = \hat{\Phi}_1 + i\hat{\Phi}_2$ or $\hat{\Phi}_- = \hat{\Phi}_1 - i\hat{\Phi}_2$ phases. Both phases are conjugated with respect to the time inversion. Physically, these phases correspond to the superconducting states with definite values of the projection of internal angular momentum of Cooper pairs like in $^3$He-A. Symmetry group of $\hat{\Phi}_\pm(\mathbf{k})$ phase includes arbitrary rotations about $\hat{z}$ combined with multiplication by the phase factor $\exp(\mp i\varphi)\hat{L}_\varphi$, and rotations by an angle of $\pi$ about $\hat{x}$ combined with the time reversal $RU_{2x}$. Discrete degeneracy of the energy minimum leads to the two types of superconducting domains.

For $\beta_2 < 0$ the order parameter is real up to the complex factor. The symmetry group of the superconducting state in this case is $D_2(C_2) \times R$. The direction of the vector $\hat{\Phi}(\mathbf{k})$ is fixed by the sixth order term $\eta_+^3\eta_-^3 + \eta_-^3\eta_+^3$ in the GL functional ($\eta_\pm = \eta_x \pm i\eta_y$). We have not included this term in (3) since it is small near $T_c$ in comparison with the other terms. In this case the ground state possesses continuous degeneracy with respect to the direction of $\hat{\Phi}$. Therefore smooth texture of the order parameter $\hat{\Delta}(\mathbf{k}, \mathbf{r}) = \mathbf{n}(\mathbf{r}) \cdot \hat{\Phi}(\mathbf{k})$ should be expected in real samples rather than domains.

Derivation of the free energy functional (3) in the framework of the Landau theory requires as a necessary condition positiveness of the gradient terms without any magnetic field (i.e., if $\mathbf{A}(\mathbf{r}) \equiv 0$), which leads to:

$$K_1 > 0, \quad K_4 > 0, \quad K_{123} = K_1 + K_2 + K_3 > 0.$$  \hspace{1cm} (19)

However within this range of parameters it is possible the magnetic instability of the uniform superconducting state $^{54}$, which results in a spatially varying order parameter in zero applied field $^{81}$. In fact exact restrictions connected to the existence of the lower energy boundary for such laminar phases do not differ strongly from the above necessary conditions $^{81}$. Therefore in the following consideration we impose only conditions (19). It will be useful to introduce dimensionless parameters $C = (K_2 + K_3)/2K_1$, $D = (K_2 - K_3)/2K_1$. Then, according to (19), we have $1 + 2C > 0$.

Neglecting by influence of the order parameter $\hat{\Delta}$ on the pairing potential, one can obtain in the weak-coupling approximation the following relations between phenomenological parameters for $E_{1g}$ representation $^{22}$: $\beta_2 = 0.5\beta_1$, $K_1 = K_2 = K_3$. Note that the approximate particle-hole symmetry of Bogoljubov equations near the Fermi surface results in a relatively small value of $D \sim (T_c/E_F)^2$ $^{22,14,55}$.

Returning to the magnetic properties of unconventional superconductors, we could expect the same qualitative features in their behavior, namely, penetration of separate vortices inside the material at fields higher than $H_{c1}$, formation of the vortex lattice, which becomes denser with increasing external field, and destruction of superconductivity at $H = H_{c2}$ when the distance between vortices becomes of the order of $\xi$. 


However, the GL equations (15) for the 2D order parameter have a more complicated structure than those for the $s$-wave superconductor (6). This is why new properties of the mixed state of unconventional superconductor are expected, and the well established results for usual superconductors should be reexamined in the multicomponent case.

The structure of the mixed state of a multicomponent superconductor is considered in the following Sections of this review. Here we give a short outline of the problems to be discussed.

1. The multicomponent structure of the order parameter and two different types of ground state lead to the possibility of the existence of different types of vortices carrying an integer flux, which are discussed in Sec. 2. Unlike the case of usual superconductors, the vortices can have a nonsingular structure of the core and be nonaxisymmetric. Since in reality only the most energetically favorable vortices exist, one should calculate the energy for each type of vortices and choose the vortex with the lowest energy. Note that the favorability of single-quantum vortices over multiple-quantum vortices is also questionable.

2. In order to calculate the upper critical field $H_{c2}(T)$ for unconventional superconductors, the multicomponent linearized GL equations have to be solved. The puzzling feature of solutions of such equations described in Sec. 3 is that all components of the order parameter participate in the formation of the Abrikosov vortex lattice, and, hence, the internal structure of the mixed state near $H_{c2}$ is rather complicated. The main question to be considered in this context in Sec. 4 is: whether the lattice has the regular triangular form as for $s$-wave superconductors or it is distorted. Another surprising fact is the existence of several superconducting phases from different Landau levels with close values of critical fields. It will be shown in Sec. 5 that the admixture of such phases inevitably leads to structural phase transitions in the Abrikosov lattice below $H_{c2}$.

In addition to the review of already published works we have included new original results in Secs. 3.5 and 4.3.

In fact, depending on the parameters in multicomponent GL equations, both nontrivial vortices near $H_{c1}$ and structural phase transitions in the Abrikosov lattice near $H_{c2}$ may occur. The crucial open question of the mixed state theory of the multicomponent superconductors is relation of these two limiting cases:

— How are nonaxisymmetric vortices placed in the lattice near $H_{c1}$?
— What is the behavior of such vortex lattice with the increase of magnetic field?
— What kinds of phase transitions can occur in the intermediate region between $H_{c1}$ and $H_{c2}$ in unconventional superconductors?

### 2 Vortices in Unconventional Superconductors

The vortices in a superfluid liquid with the multicomponent order parameter were intensively studied during the last two decades in connection with rotating superfluid $^3$He (for review see [60]). It was shown that the response of such a liquid to rotation is formation of topologically stable vortices of
the order parameter. At large distances from the vortex core the slow space variation of the order parameter \( \Delta(k, r) \) can be described as a long-range texture of the uniform order parameter \( \Delta(k) \) which is fixed by the forth order terms in the GL functional. (For \(^3\)He it is the space variation of A or B phase.) However, in the vicinity of the vortex core the bulk phase is destroyed and all components of the order parameter have the same order of magnitude. We define the characteristic size of this region as \( \tilde{\xi} \).

The analog of the superfluid liquid rotation is the response of the superconductor to the external magnetic field. Before translating the results of superfluid \(^3\)He theory on the case of the multicomponent superconductors, we should stress that there is a feature which makes the difference between superconducting and superfluid vortices: the first ones have the finite size \( \lambda \) due to the screening effect of the currents, whereas the size of superfluid vortices is infinite.

Therefore, three length scales enter the problem of vortices in multicomponent superconductors: the vortex size (the London penetration depth) \( \lambda \), the radius of the vortex “hard” core (the coherence length) \( \xi \), and the radius of the region around the vortex core \( \tilde{\xi} \) where all \( k \)-components of the order parameter are mixed. Note, however, that such a stratification of the vortex on the characteristic scales has physical sense only under the condition that \( \lambda \gg \tilde{\xi} \gg \xi \) where the values \( \lambda, \tilde{\xi}, \xi \) are defined by the parameters of multicomponent GL functional. Having in mind this case, we will discuss magnetic vortices in the two-component model (5), which have been studied in most details. We will assume also that the magnetic field is parallel to the principal crystal axis.

The properties of vortices and the scales of \( \lambda, \tilde{\xi}, \xi \) strongly depend on the bulk superconducting state. As we have already mentioned in Sec. 1.3 there are two stable minimums of the GL functional (5) describing homogeneous state for different signs of \( \beta_2 \). There is either a complex order parameter (broken time-reversal phase) \( \hat{\Delta}(k) \sim \hat{\Phi}_1 + i \hat{\Phi}_2, (\beta_2 > 0) \) or a real one (vector phase) \( \hat{\Delta}(k) \sim \hat{\Phi}, (\beta_2 < 0) \). We will consider both cases separately in Secs. 2.1 and 2.2.

### 2.1 Vortices in the Broken Time-Reversal Phase

To begin with the discussion of vortices in the broken time-reversal phase, note that the ground state is doubly degenerate with respect to the direction of the angular momentum of Cooper pairs. Due to this discrete degeneracy, the domain structure (with phases \( \hat{\Phi}_+ (k), \hat{\Phi}_- (k) \)) in the ground state of the two-component superconductor is expected \(^{[75]}\). Thus, the vortices originating from both phases can exist inside the superconductor. Note also that the domain structure itself contributes to the magnetic properties of the unconventional superconductors because of two important effects: (i) the internal (“up” and “down”) magnetization of Cooper pairs owning to their orbital momentum (this bulk magnetization is assumed to be small due the microscopic electron-hole symmetry \(^{[76]}\)); (ii) the magnetization owning to persistent surface currents which circulate around domains and originate due to the inhomogeneity of the order parameter (on the scale \( \tilde{\xi} \) near the domain boundary (these currents are due to boundary effects of the bulk magnetization \(^{[76]}\)). The low field magnetic response of unconventional superconductors due to the domain structure was considered in detail in \(^{[63,64,65]}\).
Note that an external magnetic field lifts the \( \Phi_{\pm}(k) \) degeneracy and makes one sort of domains more favorable. Therefore, the ground (vortex) state of unconventional superconductor in an external field is rather complicated and history (field or zero-field cooling experiments) dependent. In this review we will neglect for simplicity the effects of domain- and internal-magnetization and will assume that the vortices exist in the bulk of domains (the latter means that the domain sizes are larger than the penetration length). We will discuss the bulk vortices in both phases \( \Phi_{+}(k) \) and \( \Phi_{-}(k) \) simultaneously using “±” subscript.

2.1.1 Global vortex structure

The long range structure of \( n \)-quantum vortex is given by:

\[
\hat{\Delta}_{\pm}(k, r) = \Psi_{\pm}(r)(\Phi_{1}(k) \pm i\Phi_{2}(k)) ,
\]

where phase of \( \Psi_{\pm}(r) \) varies on \( 2\pi n \) about the vortex center. Substituting (20) into (15), we obtain the effective GL equation for the space variation of \( \Psi_{\pm}(r) \):

\[
\left( -\alpha \mp \frac{2e}{\hbar c} h(r)DK_{1} \right) \Psi_{\pm} + 4\beta_{1}|\Psi_{\pm}|^{2}\Psi_{\pm} - K_{1}(1 + C)D_{i}D_{i}\Psi_{\pm} = 0 .
\]

The term proportional to \( D \) is responsible for the above mentioned interaction of the angular momentum of Cooper pairs with local magnetic field. If we neglect by this term (see Sec. 1.3), Eq. (21) becomes the same as the usual GL equation (6) for \( s \)-wave superconductor. Then by analogy we find the size of the vortex (penetration depth) in the time-reversal breaking phase:

\[
\lambda_{t} = \left( \frac{\hbar^{2}c^{2}\beta_{1}}{16\pi e^{2}\alpha K_{1}(1 + C)} \right)^{1/2} .
\]

We can also define the coherence length which corresponds to the size of the vortex core:

\[
\xi_{t} = \left( \frac{K_{1}(1 + C)}{\alpha} \right)^{1/2} .
\]

However, this length has a limited physical sense for small \( \beta_{2} \) because in the region \( r \leq \xi_{t} \) other components of the order parameter are admixed, and Eq. (21) does not hold. In this region the gradient energy due to the texture of \( \Phi_{\pm} \)

\[
F_{\text{grad}} \sim \frac{K_{1}(1 + C)}{\xi_{t}^{2}} F_{\text{bulk}}
\]

becomes comparable to the contribution to the uniform part of the GL energy due to other admixed components of the order parameter:

\[
\delta F_{\text{bulk}} \sim \frac{\beta_{2}}{\beta_{1}} F_{\text{bulk}} .
\]
Equating $\delta F_{\text{bulk}}$ to $F_{\text{grad}}$, we obtain the characteristic scale $\tilde{\xi}_t$:

$$
\tilde{\xi}_t = \xi_t \left( \frac{\beta_1}{\beta_2} \right)^{1/2}.
$$

(26)

The same estimation is valid also for the width of the domain wall between $\Phi_+$ and $\Phi_-$ phases [25, 64].

By the analogy with usual $s$-wave superconductor, we can conclude that the one-quantum vortex has the smallest energy and $H_{c1}$ is defined by formulas (7) and (11) with a formal substitution $\xi \to \tilde{\xi}_t$, $\epsilon \to \tilde{\epsilon}_t$, where $\tilde{\epsilon}_t$ is the energy of the vortex core in the region $r < \tilde{\xi}_t$. In the region $r \gg \tilde{\xi}_t$ the vortex (20) is described by the same function (7): $\Psi_\pm(r) = \Psi(r)e^{-i\phi}$ (with substitution $\xi_t$, $\lambda_t$) like a usual Abrikosov vortex.

These statements are valid provided $\lambda_t \gg \tilde{\xi}_t$, which is the case for not very small positive $\beta_2$. In particular, we have assumed that the main vortex energy is concentrated in the screening currents region $\lambda_t > r > \tilde{\xi}_t$, what means that $\ln(\lambda_t/\tilde{\xi}_t) \gg \tilde{\epsilon}_t$.

2.1.2 Vortex core structure

The investigation of the vortex core structure at $\beta_2 > 0$ was done in [70, 71, 6]. Our description of the vortex structure is based on the results of these papers. We will use the analytical arguments of [6] to illustrate the appearance of vortices with nontrivial core structure and then present the results of the numerical calculations of [70] which were done in a broad range of parameters.

In the region $r \leq \tilde{\xi}_t$ all components of the order parameter are mixed:

$$
\hat{\Delta}_\pm (k, r) = (\hat{\Phi}_1(k) \pm i\hat{\Phi}_2(k))\Psi_\pm(r) + (\hat{\Phi}_1(k) \mp i\hat{\Phi}_2(k))X_\mp(r).
$$

(27)

Functions $\Psi_\pm(r)$, $X_\mp(r)$ can be determined from solution of the GL equations (15). Since at $r \gg \xi_t$ vortex (27) should gradually turn into (20), we have at the large distances $\Psi_\pm(r) \to \Psi(r)e^{-i\phi}$ and $X_\mp(r) \to 0$.

Axisymmetric vortices

In general, there exist different solutions of Eqs. (15) obeying the boundary conditions $\Psi_\pm|_{r \to \infty} = \sqrt{\alpha/4\beta_1} e^{-i\phi}$, $X_\mp|_{r \to \infty} = 0$. One of the extremums of GL functional always corresponds to the “most symmetric vortex,” which possesses all symmetries possible for line defects in condensed matter [60]. According to [60] the maximal symmetry group of a line defect includes continuous rotations about its axis with multiplications by a phase factor, the reflection $\hat{\sigma}_h$ in the perpendicular plane, and the combined symmetry $RU_2$. The last two discrete symmetries have no effect on vortex classification in the model of 2D superconducting order parameter.

The generator of continuous subgroup and corresponding condition of invariance have form:

$$
\dot{Q} \hat{\Delta}_\pm^a (k, r) = 0, \quad \dot{Q} = \dot{L}_z - q\hat{I}.
$$

(28)
Here \( \hat{I} \) is the generator of \( U(1) \) group, and \( q \) is an integer which has the meaning of the total angular momentum eigenvalue of the vortex. The symmetry group of the axisymmetric vortex is \( \{ e^{-iq\phi} \hat{L}_\phi, RU_2, \hat{\sigma}_h \} \). The solutions of Eq. (28) for the case of one-quantum vortices in the phases \( \Phi_\pm \) are

\[
\begin{align*}
\Psi_\pm^a(r) &= \Psi_\pm(r) e^{-i\phi}, \quad X_\pm^a(r) = X_\pm(r) e^{i\phi}, \quad q = 0, \\
\Psi_\pm^a(r) &= \Psi_\pm(r) e^{-i\phi}, \quad X_\pm^a(r) = X_\pm(r) e^{-3i\phi}, \quad q = -2.
\end{align*}
\]

Functions \( \Psi_\pm(r) \) and \( X_\pm(r) \) vanish at \( r = 0 \) making a “hard” core of the axisymmetric one-quantum vortex. At large distances \(|\Psi_\pm|^2 \approx \alpha/4\beta_1\), and amplitude \( X_\pm \) is small. Linearizing Eq. (31) in \( X_\pm \) in this region, one comes to the following equation:

\[
\left( -\alpha + 4(\beta_1 + 2\beta_2)|\Psi_\pm|^2 - K_1(1 + C)(D_x^2 + D_y^2) \right) X_\pm = K_1 C(D_x \pm iD_y)^2\Psi_\pm.
\]

Substituting \( \Psi_\pm = \sqrt{\alpha/4\beta_1} e^{-i\phi} \) in (30), one obtains:

\[
\begin{align*}
X_\pm^a(r) &\approx -\frac{C}{2(1+C)} \left( \frac{\xi_t}{r} \right)^2 \left( \frac{\alpha}{4\beta_1} \right)^{1/2} e^{i\phi}, \\
X_\pm^a(r) &\approx \frac{3C}{2(1+C)} \left( \frac{\xi_t}{r} \right)^2 \left( \frac{\alpha}{4\beta_1} \right)^{1/2} e^{-3i\phi}.
\end{align*}
\]

From (31) one can easily estimate the region where \( X_\pm^a \) becomes comparable to \( \Psi_\pm \). For \( C \geq 1 \) its sizes coincide with \( \xi_t \). In order to find behavior of \( \Psi_\pm(r) \) and \( X_\pm(r) \) at \( r \to 0 \), one should solve both GL equations (13) simultaneously. The admixture of the other components changes the usual asymptotic \( \Psi_\pm(r) \sim r \) of superconducting amplitude at the vortex core.

**Nonaxisymmetric vortices**

Although the axially symmetric vortex solution is always extremum, it is not necessarily the absolute minimum of the GL functional. Obviously, it is the minimum as \( \xi_t \to 0 \) (\( \beta_2 \gg \beta_1 \)), when \( \Psi_\pm(r) \) is defined by Eq. (30) almost throughout the entire space. Functions \( \Psi_\pm^a(r) \) are solutions of (30) in all space also for \( C = 0 \) when the additional amplitude in axisymmetric vortex \( X_\pm(r) \equiv 0 \). Due to this feature the case of \( C = 0 \) is a convenient starting point for consideration of nonaxisymmetric vortices.

In this case Eq. (30) becomes uniform and takes form of the eigenvalue problem for the ordinary Schrödinger equation on \( X_\pm \) in a potential well with \( U(r) = -\alpha + 4(\beta_1 + 2\beta_2)\Psi_\pm^2(r) \). A two-dimensional well always possesses a bound state, which is placed on some finite energy below the value of potential at infinity \( U(r)|_{r \to \infty} = \alpha\beta_2/2\beta_1 > 0 \). Therefore at \( C = 0 \) the lowest energy level of (30) should pass through 0 as \( \beta_2 \) decreases. This means that the vortex \( \hat{\Delta}_\pm^a(k, r) = \Psi_\pm^a(r) \hat{\Phi}_\pm(k) \) becomes unstable towards perturbation of the form \( X_\pm^0(r) \hat{\Phi}_\pm(k) \). Here \( X_\pm^0(r) \) is a real axisymmetric function exponentially decreasing at \( r \gg \xi_t \) and with maximum at \( r = 0 \), which corresponds to the lowest level in the well.

The critical value of \( \beta = \beta_2/\beta_1 \) corresponding to such an instability in the vortex core is \( \beta_c = 0.24 \) according to [70]. Using trial functions Barash and Mel’nikov [6] have found a slightly different value:
0.37. However, subsequent numerical investigations of one-component odd-parity model \cite{46}, which reduces in a magnetic field to model \cite{5} with $C = 0$, have suggested the same value for $\beta_c$ as in \cite{70}. The order parameter amplitude at the center of such \textit{nonsingular} vortex grows as $(\beta_c - \beta)^{1/2}$.

When $C$ is exactly equal to zero, sum $|\Psi_\pm(r)|^2 + |X^0(r)|^2$ depends only on $r$. This leads to axial symmetry of the whole vortex. Such a symmetry of the nonsingular vortex, however, is “accidental” and disappears for any small value $C$.

For small $C$ solution of Eq. (30) is the superposition $X_\pm(r) = X_\pm^0(r) + X^0(r)$. Consequently, $X_\pm(r)$ is not already invariant under arbitrary rotations and axisymmetry is spontaneously broken. Moreover, using (29) we come to the following conclusions about symmetry of the nonsingular vortices:

(i) Vortices originating in $\Phi_-$ state are one-quantum, axisymmetric and singular for $\beta_2/\beta_1 > \beta_-^c(C)$. For $\beta_2/\beta_1 < \beta_-^c(C)$ they have \textit{nonaxisymmetric} triangular structure (Fig. 2a). The residual symmetry group is: \{$e^{i\pi/3}\tilde{L}_{2\pi/3}, \hat{\sigma}_h, R\hat{\sigma}_v$\}, where $\hat{\sigma}_v$ means the reflections with respect to the triangular medians.

(ii) One-quantum vortices in $\Phi_+$ state have axisymmetric singular structure for $\beta_2/\beta_1 > \beta_+^c(C)$. For $\beta_2/\beta_1 < \beta_+^c(C)$ vortex cores are of \textit{nonaxisymmetric} “crescent” shape (see Fig. 2b). The residual vortex symmetry is of the vector type and contains only two elements: \{$\hat{\sigma}_h, R\hat{\sigma}_v$\}, where $\hat{\sigma}_v$ is the reflection in the symmetry plane of “crescent.”

The boundary of the axisymmetric vortex instability $\beta_+^\pm(C)$ was found by Tokuyasu, Hess and Sauls \cite{70} numerically (see Fig. 3). The results of \cite{6} at small $C$ qualitatively coincide with \cite{70}.

For topological reasons, only the vortices with an integer number of flux quanta can exist in the bulk of time-reversal breaking phase. However, vortices carrying fractional number of magnetic flux quanta are known on a domain boundary between phases $\Phi_+$ and $\Phi_-$ \cite{65}. Moreover, two fractional vortices could have a lower energy than a single-quantum vortex. Starting on this picture, Izumov and Laptev \cite{25} suggested a scenario in which a bulk vortex decays forming a small domain of the time-conjugated phase and two fractional vortices which exist on the closed boundary of the new domain. The loss in the gradient energy due to the domain boundary is compensated by the gain of energy due to the vortex decay. Note that the vortex core instability in the phase $\Phi_+$ with the formation of the nonaxisymmetric crescent structure and phase $\Phi_-$ inside the vortex core is an extreme case of the Izumov and Laptev scenario when the domain size is sufficiently small.

Developing the analogy with vortices in $s$-wave superconductor established by Eq. (21) we can estimate vortex core energy $\tilde{\epsilon}_l$ as $(\ln \xi_t/\xi_t + \epsilon)$. Then for those $\beta_2$, which satisfy condition $\xi_t/\xi_t > \lambda_t/\xi_t$ or $\beta_2 < \beta_1/\sqrt{\kappa}$, the main part of the vortex line tension is the energy of the vortex core $\tilde{\epsilon}_l$, rather than the hydrodynamics part proportional to $\ln(\lambda_t/\xi_t)$. In this case the energetic favorability of the one-quantum vortices in comparison with multi-quantum ones is broken. For example, the \textit{two-quantum axisymmetric vortex} in the phase $\Phi_+$ has according to (28) nonvanishing amplitude $X_\pm^a(r)$ at the center and therefore smaller core energy $\tilde{\epsilon}_l$ than one-quantum vortex. Its existence in the particular parameter region was found numerically in \cite{71}.
Spontaneous breaking of vortex axisymmetry was known before in superfluid $^3$He-B. In particular, in the parameter region the core of so-called $v$ vortex is subjected to the "director type" instability [60]. As it has been shown above the axial symmetry breaking in the vortex cores is quite general phenomenon in the case of the multicomponent order parameter. Ordering of such nonaxisymmetric vortices in the lattice at $H > H_{c1}$ is still an open question.

2.2 Vortices in the Vector Phase

2.2.1 Global vortex structure

For $\beta_2 < 0$, the equilibrium order parameter is $\hat{\Delta}(k) = (n \cdot \hat{\Phi}(k))\Psi$, where $n$ is a unit vector. Far from the vortex core the long range texture of the order parameter preserves this form:

$$\hat{\Delta}(k, r) = (n(r) \cdot \hat{\Phi}(k))\Psi(r).$$

Substituting (32) into (3) and keeping the gradient terms responsible for Goldstone modes, we have:

$$F = -\alpha|\Psi|^2 + \beta_{12}|\Psi|^4 + K_1 D_i^* \Psi^* D_i \Psi + K_{23}(n_i D_i^* \Psi^*)(n_j D_j \Psi) +$$

$$+ |\Psi|^2(K_1(\text{rot } n)^2 + K_{123}(\text{div } n)^2),$$

(33)

There are two topological reasons for the formation of stable line defects in the vector state (32): (i) the phase of $\Psi(r)$ changes on $2\pi n$ in making a complete circuit; (ii) the vector $n(r)$ sweeps the unit circle $m$ times when moving around the vortex. Therefore, possible defects are classified by two quantum numbers $(n, m)$, both which are integer or half-integer (for general aspects of topological classification see [49]).

Note, however, that the phase variation about the vortex leads to supercurrents which screen the magnetic field outside the region of the size $\lambda$. For this reason the vortex energy has the upper cutoff parameter $\lambda$ and is therefore finite. The variation of the vector $n(r)$ does not produce screening currents. Therefore, for $m \neq 0$, the contribution to the defect energy $\sim \Delta^2 \int (K_1(\text{rot } n)^2 + K_{123}(\text{div } n)^2)dV$ is logarithmically divergent beyond the core. It makes the defects with $m \neq 0$ less favorable than the pure phase vortices with $m = 0$ but $n \neq 0$.

Since we are interested in the vortices with minimal energy, we will consider only those with uniform distribution of $n(r)$ at large distances. Assuming that $n(r)$ is uniform, we rewrite the effective GL equation for $\Delta(k, r) = \Psi(r)(n \cdot \hat{\Phi}(k))$ as

$$-\alpha \Psi + 2\beta_{12}|\Psi|^2\Psi - K_1 D_i D_i \Psi - 2K_1 C(n_i D_i)^2\Psi = 0.\tag{34}$$

This equation has a form of the one-component GL equation for a crystal with anisotropy axis $n$ and magnetic field $H$ directed perpendicular to $n$. Using scaling transformation of $[31]$:

$$r_{||} = r_{||}' \sqrt{1 + 2C}, \quad r_{\perp} = r_{\perp}' \sqrt{1 + 2C},$$

(35)
we reduce (34) to the isotropic GL equation

\[- \alpha \Psi + 2 \beta_{12} |\Psi|^2 \Psi - K_1 \sqrt{1 + 2C} D_i D^i \Psi = 0.\]  

From (36) we conclude the following:

(i) the one-quantum vortex \((n = 1)\) is the most favorable one;

(ii) the sizes of the vortex corresponding to the anisotropic London penetration length are:

\[\lambda_{\parallel} = \lambda_v \sqrt{1 + 2C}, \quad \lambda_{\perp} = \frac{\lambda_v}{\sqrt{1 + 2C}},\]  

where

\[\lambda_v = \left( \frac{\hbar^2 e^2 \beta_{12}}{16 \pi e^2 \alpha K_1 \sqrt{1 + 2C}} \right)^{1/2};\]  

(iii) the coherence length corresponding to Eq. (36) is defined as

\[\xi_v = \left( \frac{K_1 \sqrt{1 + 2C}}{\alpha} \right)^{1/2};\]  

(iv) at distances larger than \(\tilde{\xi}_v\) the vortex has the form:

\[\hat{\Delta}(k, r) = \Psi(r) (n \cdot \hat{\Phi}(k)) = \Psi(r') e^{-i \phi'} (n \cdot \hat{\Phi}(k)),\]  

where function \(\Psi(r')\) has the same dependence as the usual Abrikosov vortex \(\Psi\). As in the case of the vortices in the broken time-reversal phase, the characteristic scale of \(\xi_v\) is calculated by comparing the contribution to the free energy due to admixing of the \(\hat{\Phi}_\pm\) component and the characteristic gradient energy on the scale \(\tilde{\xi}_v\). Since the density of the gradient energy is anisotropic, the region where all components of the order parameter are mixed is defined by the anisotropic scales

\[\tilde{\xi}_{\parallel} = \tilde{\xi}_v \sqrt{1 + 2C}, \quad \tilde{\xi}_{\perp} = \frac{\tilde{\xi}_v}{\sqrt{1 + 2C}},\]  

where

\[\tilde{\xi}_v = \xi_v \left( \frac{-\beta_1}{\beta_2} \right)^{1/2}.\]  

The lower critical field for the vortices in the vector phase is calculated from (11) by substitution \(\lambda \rightarrow \lambda_v, \xi \rightarrow \tilde{\xi}_v, \epsilon \rightarrow \tilde{\epsilon}_v\). Note again that the above description of the long range vortex structure is valid only if \(\lambda_v \gg \tilde{\xi}_v\).

2.2.2 Vortex core structure

The calculation of the vortex core structure for \(\beta_2 < 0\) has not been done yet. Therefore we will outline only expected qualitative features of them.

Similarly to the case of \(\beta_2 > 0\), we start from the description of the “most symmetric vortex.” In order to find its “full symmetry,” we note that the rotational symmetry of the vortex is broken
due to the vector \( \mathbf{n} \): \( \{ \hat{L}_\pi, R U_{2\pi}, P \} \). The general structure of the “most symmetric vortex” inside the \( \tilde{\xi}_v \) region is given by

\[
\hat{\Delta}(k, r) = (n \cdot \hat{\Phi}(k))\psi(r) + i(m \cdot \hat{\Phi}(k))\chi(r),
\]

where \( m \) denotes the unit vector which is perpendicular to \( n \) and \( H \). Functions \( \psi(r) \), \( \chi(r) \) have \( D_2 \) symmetry, and their phases vary by \( 2\pi \) about \( r = 0 \). Outside the core region \( \psi(r) \to \Psi(r')e^{-i\phi'} \), \( \chi(r) \to 0 \). Both amplitudes \( \psi(r) \) and \( \chi(r) \) vanish at the vortex center.

If \( \tilde{\xi}_\parallel, \tilde{\xi}_\perp \) are of the order of \( \xi_\parallel, \xi_\perp \), the “most symmetric vortex” is the most favorable. When \( \tilde{\xi}_v \) becomes sufficiently large in comparison with \( \xi_\nu \), one can expect the appearance of vortices with a lower symmetry as is the case with the vortices in the broken time-reversal phase. However, this question requires further investigations.

### 2.3 Discussion

Summarizing this Section, let us discuss what conclusions we can draw from the outlined picture of vortices in unconventional superconductors and what problems they pose for further studies.

The appearance of the unconventional vortices was mostly studied in the framework of the two-component model when \( H \parallel \mathbf{c} \). It was shown that in some range of parameters the vortices originating from the broken time-reversal phase \( (\beta_2 > 0) \) have the nonaxially symmetric structure of the core, unlike the vortices in usual superconductors which are always axisymmetric. It was also shown that the domain structure of the ground state of a superconductor plays an important role in its magnetic properties.

The instability of the vortex core with axisymmetry breaking, the domain formation and the vortex interaction with domain boundaries are certainly the features connected to each other. However, the most favorable ground state of such a system in a finite magnetic field near \( H_{c1} \) and, in particular, the exact symmetry of the lattice of nonaxial vortices are still unknown.

The vortices in a vector phase \( (\beta_2 < 0) \) were studied much less. It is known that the vortices have an anisotropic shape in the direction of \( \hat{\Phi}(k) \). However, the structure of the vortex core (depending on the parameter \( \beta_2 \)) needs further investigations.

In the foregoing the vortices have been treated in the limit when a vortex can be clearly divided into two principal regions: the region of the screening currents, in which the main vortex energy is concentrated, and the vortex core region, in which the axisymmetry instabilities occur. This limit corresponds to the restriction \( \lambda \gg \tilde{\xi} \) which is the case whenever \( \beta_2 \) is not too close to zero. Note, however, that this case is in a sense the most trivial one because all the results were obtained by the straightforward generalization of two famous theories: Abrikosov theory of the vortex structure (for the region \( r > \tilde{\xi} \)) and the theory of the vortex core instability in the superfluid \( ^3\text{He} \) \(^{[60]} \) (adopted for the two-component order parameter \( \hat{\Phi}_1 \pm i\hat{\Phi}_2 \) or \( \hat{\Phi} \)).

A very interesting problem comes out from the above discussion: what is the structure of the vortices at small \( \beta_2 \) when the condition \( \lambda > \tilde{\xi} \) is not satisfied? In this case not only the currents
which originate from the gradient of phase of the order parameter but also the currents due to the variation of all other admixed components of the order parameter contribute to the screening of magnetic field. Another related question is how the vortices in the broken time-reversal phase transform to those in the real (vector) phase when the parameter $\beta_2$ changes from $\beta_2 > 0$ to $\beta_2 < 0$. Note that these problems require a careful consideration because even the ground state of a two-component superconductor is nonuniform at small $\beta_2$ ($H = 0$) and has a space-modulated laminar structure $^{[54,81]}$. Generating of vortices in this state (and therefore the problem of $H_{c1}$) is also not clear enough $^{[29,56]}$.

As we have seen, magnetic vortices in a two-component superconductor have a lot of unusual features. However, the two-component model is the simplest one which describes the unconventional superconductivity. Therefore, one can expect even more peculiar new features in other multicomponent superconductors. In this connection, it is worthwhile to mention the mixed state in the triplet superconductor with a weak spin-orbital coupling which was studied by Burlachkov and Kopnin $^{[12]}$. They showed that in magnetic field spin-vectors of such a superconductor form a distinct texture which allows the applied field to penetrate inside a superconductor without forming singularities. It is remarkable that the lower critical field for such a texture can be much smaller than corresponding field for the usual singular vortices.

### 3 The Upper Critical Field in Unconventional Superconductors

The calculation of the upper critical field for the superconductors with multicomponent order parameter is analogous to the case of $s$-wave superconductors (see Sec. 1.2). Starting from the multicomponent GL energy functional, we derive the corresponding linearized GL equations under assumption that magnetic field inside a superconductor is uniform and equals to the external applied field. Therefore, the calculation of the upper critical field reduces to the search for eigenvalues of the system of the linearized GL equations and then to the choice of the critical (eigen) field with the maximal value.

In Sec. 3.1 we present characteristic examples for which linearized GL equations for multicomponent order parameter admit analytical solutions. We will see that, despite the complicated structure, their eigensolutions can be classified by definite quantum numbers. The nature of this classification is a consequence of the symmetry of the normal metal in magnetic field (Sec. 3.2). Hence, properties of the superconducting phases can be discussed without solving the linearized GL equations and general conditions of a kink on the $H_{c2}(T)$ curve (the problem related to the phase diagram of UPt$_3$) can be formulated (Sec. 3.3). In Sec. 3.4 we will enumerate the specific features of the $H_{c2}$ anisotropy which make it possible to distinguish the unconventional superconductivity from the usual one. Finally, in Sec. 3.5 we consider the possibility of the appearance of superconducting phases modulated along the field direction.
3.1 Upper Critical Field in the Two-Component Model

In this Subsection we consider two cases with $H \parallel c$ and $H \perp c$ when linearized GL equations for the two-component order parameter can be solved analytically \[11,67,74,79\]. The question of $H_{c2}$ anisotropy for arbitrary directed field will be considered in Sec. 3.4.

To obtain the linearized GL equation, we neglect in (15) the forth order terms. For $H \parallel \hat{z}$ these equations are

\[
\alpha \left( \frac{\eta_+}{\eta_-} \right) = -K_1 \left( \frac{(1 + C)(D_x^2 + D_y^2) - \frac{2e\hbar D}{h} HD}{C(D_x - iD_y)^2} - \frac{C(D_x + iD_y)^2}{(1 + C)(D_x^2 + D_y^2) + \frac{2e\hbar D}{h} HD} \right) \left( \frac{\eta_+}{\eta_-} \right),
\]

where $\eta_{\pm} = \eta_x \pm i\eta_y$. In (44) we set $D_z \eta = 0$, since the variation of the order parameter along $H$ only decreases $H_{c2}$. Defining the creation and annihilation operators

\[
a = \left( \frac{\hbar c}{4|e|H} \right)^{1/2} (D_x - iD_y), \quad [a, a^+] = 1,
\]

one obtains the eigenfunctions of (44) as combination of the $r$-dependent Landau levels wave functions $f_n(r)$ (for which $a^+ a f_n = (n + \frac{1}{2}) f_n$) and $k$-dependent basis functions $\hat{\Phi}_i(k)$:

\[
\hat{\Delta}_n(k, r) = f_n(r)(\hat{\Phi}_x(k) - i\hat{\Phi}_y(k)) + f_{n-2}(r)(\hat{\Phi}_x(k) + i\hat{\Phi}_y(k)),
\]

\[
\lambda_n = (2n - 1)(1 + C) \pm \sqrt{(2 + 2C - D)^2 + 4C^2n(n - 1)}.
\]

Their respective eigenfields $H_n$ are related to the eigenvalues $\lambda_n$ of the linear problem by the expression

\[
H_n = \frac{\hbar c \alpha}{2|e|K_1\lambda_n},
\]

and $H_{c2} = \max\{H_n\}$. The solution with the lowest eigenvalue is either

\[
\hat{\Delta}_a(k, r) = f_0(r)(\hat{\Phi}_x(k) - i\hat{\Phi}_y(k)),
\]

\[
\lambda_a = \lambda_0 = 1 + C - D, \quad \text{for } D > \frac{C^2}{1 + C},
\]

or

\[
\hat{\Delta}_{SK}(k, r) = f_0(r)(\hat{\Phi}_x(k) + i\hat{\Phi}_y(k)) + \omega f_2(r)(\hat{\Phi}_x(k) - i\hat{\Phi}_y(k)),
\]

\[
\lambda_{SK} = \lambda_2 = 3(1 + C) - \sqrt{8C^2 + (2 + 2C - D)^2},
\]

\[
\omega = \frac{\lambda_2 - 1 - C - D}{2\sqrt{2C}}, \quad \text{for } D < \frac{C^2}{1 + C}.
\]

The new feature of the anisotropic pairing which can be seeing from (46) is the nonfactorized dependence of $\hat{\Delta}$ on $k$ and $r$. Scharnberg and Klemm (SK) were the first who discovered this fact solving Gor’kov equations for $p$-wave pairing in the weak-coupling limit \[62\]. These solutions were obtained within the phenomenological Ginzburg-Landau approach in \[67,79\]. Note that the solutions
contain those products of $k$- and $r$-dependent functions, for which the sum of the projection of the Cooper pair angular momentum $m$ and Landau level number $n$ is the same. The corresponding quantum number $N = n + m$ was introduced in \[37\]. We shall call $N$ as “generalized Landau level number.” The symmetry reasons for the appearance of such quantum number will be given in the next Subsection.

For different values of phenomenological constants $C$, $D$, the smallest eigenvalue is either $\lambda_a$ or $\lambda_{SK}$. If the approximate particle-hole symmetry exists near the Fermi surface (e.g. in the weak-coupling regime) then $D \approx 0$. Therefore it is the SK-phase \[13\] which should appear at $H_{c2}$ in the “real” two-component superconductor (see Fig. 4).

Due to the cylindrical symmetry of the GL functional \[5\], $H_{c2}$ is isotropic in the basal plane. Taking $H \parallel \hat{x}$, we obtain from \[15\] the following system of equations $(K = K_4/K_1)$:

$$\alpha \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} = -K_1 \begin{pmatrix} (1 + 2C)D_x^2 + D_y^2 + KD_z^2 & 2CD_xD_y \\ 2CD_xD_y & D_x^2 + (1 + 2C)D_y^2 + KD_z^2 \end{pmatrix} \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix}. \quad (50)$$

This system can be solved analytically only under the condition $D_x\eta = 0$ (see Sec. 3.5) when two equations in \[50\] become decoupled and coincide with the one-component GL equations for anisotropic $s$-wave superconductors \[31\]. Using the scaling transformation, one obtains in this case the eigenfunctions of \[50\] with the lowest eigenvalues \[11\]:

$$\Delta_1 (k, r) = f_0 (s_1y, s_1^{-1}z) \hat{\Phi}_x \hat{\Phi}_z (k), \quad \lambda_1 = \sqrt{K}, \quad s_1 = \sqrt[4]{K}, \quad (51)$$

$$\Delta_2 (k, r) = f_0 (s_2y, s_2^{-1}z) \hat{\Phi}_y \hat{\Phi}_z (k), \quad \lambda_2 = \sqrt{K(1 + 2C)}, \quad s_2 = \sqrt[4]{K/(1 + 2C)}. \quad (52)$$

Therefore, for $C > 0$, the phase \[51\] corresponds to the upper critical field, whereas for $C < 0$ the phase \[52\] appears in the vicinity of $H_{c2}$.

Scaled zero Landau level functions \[51\] and \[52\] are superpositions of all even Landau levels. Therefore the above classification by quantum number $N$ reduces to $N$ over mod 2. In addition, solutions \[46\], \[51\] and \[52\] are characterized by the parity with respect to the reflection in the plane perpendicular to the filed direction.

### 3.2 Symmetry Classification of Superconducting Phases near $H_{c2}$

The appearance of the Abrikosov vortex lattice in type-II superconductors at $H_{c2}(T)$ is a second order phase transition. Hence, it can be considered in the framework of the Landau phase transition theory \[34\] irrespectively of particular model assumptions, such as the GL approximation. The corresponding symmetry approach was developed recently in the works \[37, 40, 82\] (see also \[18\]).

The appearance of the vortex lattice results in spontaneous breaking of the symmetry group of the normal state in a magnetic field. Let us first consider the Cooper pairing in an isotropic
metal, when \( G = SO_3 \times P \). In the presence of the magnetic field \( H \) directed along \( \mathbf{z} \) axis the total symmetry group \( \mathcal{G} \) is reduced to

\[
\mathcal{G} = T \times D_{\infty h}(C_{\infty h}) \times U(1) .
\]

The group \( D_{\infty h}(C_{\infty h}) \) contains the subgroup \( C_{\infty h} \) of arbitrary rotations about \( H \) with the reflection in the perpendicular plane \( \hat{\sigma}_h \) and the combined symmetry elements \( RU_2 \) and \( R\hat{\sigma}_v \). Here \( U_2 \) are rotations through angle \( \pi \) about axes perpendicular to \( H \), and \( \hat{\sigma}_v \) are reflections in planes containing \( H \).

The phase transition of a continuous type implies that the order parameter obeys the linear equation on the transition line:

\[
\hat{\mathcal{L}} \{ H, T \} \Delta(k, r) = 0 .
\]

This equation can be considered as a symbolic form of either differential GL equations, e.g. \([44]\) and \([50]\), or integral Gor’kov equations, see \([38, 62]\). The linear operator \( \hat{\mathcal{L}} \) is invariant under the action of the symmetry group \( \mathcal{G} \). In general, the analytic solution of \((54)\) is a complicated problem. But it is known that different solutions of linear equations belong to different irreducible representations of the corresponding symmetry group. This classification is model independent and valid on the whole line \( H_{c2}(T) \).

Note that the group \((53)\) has the same structure as group \((1)\) except for the breaking of time reversal symmetry. For instance, subgroup \( T \) is an abelian group of three-dimensional translations. However, in magnetic field symmetry operators acting on the order parameter include gauge transformations (along with the changes of space variables), which return the chosen vector potential \( A(r) \) to its original form:

\[
\hat{T}_a = \exp \left\{ -i \frac{2e}{\hbar c} \int_0^r \left[ A_k(r' + a) - A_k(r') \right] \, dr_k' \right\} \cdot \exp(a\nabla) ,
\]

\[
\hat{L}_\varphi = \exp \left\{ -i \frac{2e}{\hbar c} \int_0^r \left[ s_{kp} A_p(s_{qj} r_q') - A_k(r_j') \right] \, dr_k' \right\} \cdot \exp(i\varphi(\hat{l}_z + \hat{l}_k)) ,
\]

where

\[
\hat{l}_z = -i(x\partial_y - y\partial_x) , \quad \hat{l}_k = i \left( \frac{\partial}{\partial k} \times k \right) , \quad s_{kp} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} .
\]

We wrote the gauge transformation for the Cooper pair as that for a single particle with a total charge of \( 2e \) only for the sake of brevity. The exact transformation operators acting on the order parameter \((4)\) include exponents of the sum of two integrals, both in the above form, with upper limits \( r_1 \) and \( r_2 \) and multiplied by charge \( e \). Our simplification does not change commutation rules for \( \hat{T}_a \). Therefore, symmetry classification given below is valid not only in the GL regime, in which \( r \) varies over distances \( \xi \gg \xi_0 \), but also at \( T \to 0 \) when the space variations are of the order the Cooper pair size.

For operators \((53)\) the following relations hold:

\[
\hat{T}_a \hat{T}_b = \hat{T}_{a+b} \exp \left\{ -i \frac{2e}{\hbar c} \int_0^a \left[ A_k(r' + b) - A_k(r') \right] \, dr_k' \right\} ,
\]

\[ (56) \]
\[ \hat{T}_a \hat{T}_b = \hat{T}_b \hat{T}_a \exp \left\{ \frac{2e}{\hbar c} \mathbf{H} [\mathbf{a} \times \mathbf{b}] \right\}, \quad (57) \]
\[ \hat{L}_{\varphi_1} \hat{L}_{\varphi_2} = \hat{L}_{\varphi_1 + \varphi_2}. \quad (58) \]

Although the two-dimensional translations in \( x - y \) plane commute with each other, this is not true for the corresponding magnetic operators \((57)\). As follows from \((56)\), the operators \( \hat{T}_a \) form not an ordinary vector representation of the abelian group \( T \), but a ray one (e.g., see \([13]\)). Different choices of the gauge for the same distribution of the magnetic field correspond to the equivalent ray representations. Near \( H_{c2} \) the superconducting order parameter is characterized by its transformation properties under the action of magnetic operators or, equivalently, by the corresponding irreducible ray representation.

In order to classify the different irreducible ray representations, we introduce the generators of infinitesimal ray transformations:
\[ \hat{\imath}_x = -i\partial_x - \frac{2e}{\hbar c} (A_x + Hy), \]
\[ \hat{\imath}_y = -i\partial_y - \frac{2e}{\hbar c} (A_y - Hx), \]
\[ \hat{L}_z = \hat{l}_z + \hat{l}_k - \frac{2e}{\hbar c} \left( xA_y - yA_x - \frac{H}{2} (x^2 + y^2) \right), \quad (59) \]
which satisfy the following commutation relations:
\[ [\hat{\imath}_x, \hat{\imath}_y] = -\frac{2e}{\hbar c} Hi, \quad [\hat{\imath}_x, \hat{L}_z] = -i\hat{\imath}_y, \quad [\hat{\imath}_y, \hat{L}_z] = i\hat{\imath}_x. \quad (60) \]

In the case of axially symmetric gauge \( \mathbf{A} = (-\frac{1}{2}Hy, \frac{1}{2}Hx, 0) \) magnetic translations are reduced to the exponent functions from usual magnetic generators \([9, 78]\):
\[ \hat{\imath}_x = -i\partial_x - \frac{eH}{\hbar c} y, \quad \hat{\imath}_y = -i\partial_y + \frac{eH}{\hbar c} x, \quad (61) \]
while \( \hat{L}_\varphi \) is the ordinary rotation.

By virtue of \((60)\), we introduce the Casimir operator \( \hat{J} \), which commutes with operators \((55)\) and operator \( \hat{L} \). Its eigenvalues enumerate irreducible representations for a given ray representation:
\[ \hat{J} = \frac{\hbar c}{4|e|H} (\hat{\imath}_x^2 + \hat{\imath}_y^2) + \hat{L}_z - \frac{1}{2}. \quad (62) \]

One can find the proof of the commutation relations of \( \hat{J} \) and derivation of \((59)\) in \([82]\). Using lowering and raising operators \((13)\), we can write \( \hat{J} \) in the another form:
\[ \hat{J} = \hat{a}^+ \hat{a} + \hat{l}_k. \quad (63) \]

It is useful to construct the general type of eigenfunctions of operator \( \hat{J} \) (and therefore \( \hat{L} \)):
\[ \hat{\Delta}_N^\pm (k, r) = \sum_{n=m} A_n(H, T) f_n(r) \hat{\Psi}^\pm_m(k), \quad (64) \]
where \( \hat{\Psi}^\pm_m(\mathbf{k}) \) is the wave function corresponding to the definite projection \( m \) of the internal angular momentum on \( \hat{z} \). From the symmetry point of view, each product \( f_n(r) \hat{\Psi}^\pm_m(\mathbf{k}) \) with \( n + m = N \) can be considered as one from the equivalent choices for the basis functions of the \( N \)th irreducible ray representation. Then constants \( A_n(H,T) \) depend on the construction of the operator \( \hat{\mathcal{L}} \) in the particular model. In addition to \( N \), we define another quantum number \( \sigma = \pm \), which is the parity under reflection \( \hat{\sigma}_h \) in the plane perpendicular to \( \mathbf{H} \).

It can be easily seen that phases \( (46) \) present a specific case of \( (64) \). The quantum numbers \((N,\sigma)\), which were introduced in the previous Subsection, are conditioned by the operators \( \hat{\mathcal{J}}, \hat{\sigma}_h \). Note that the number \( N \) generalizes the Landau level number \( n \): when the wave function does not depend on the second variable \( \mathbf{k} \), the \( N \)-level classification reduces to usual Landau levels.

In the case of an anisotropic crystal, the rotational symmetry is broken. If, however, magnetic field is directed along a \( p \)-fold axis, one can see that \( N \)-level classification reduces to classification in \( N \) over \( \text{mod} \) \( p \). If magnetic field is directed in the basal plane of hexagonal crystal (along \( C_2 \) axis) the above classification is reduced to the parity of \( N \), as we have seen for the eigenfunctions \( (51) \) and \( (52) \).

In the absence of spin-orbital coupling spin projection on the field direction provides another quantum number, which distinguishes irreducible representations. If the spin-orbital interaction is strong, the mixing of singlet and triplet superconductivity is possible in a magnetic field. From the symmetry point of view, the space inversion \( P \) transforms basis functions of a particular Landau level through each other. Therefore, functions \( (64) \) do not possess any parity under \( P \). Hence at \( H_{c2} \) the triplet states on which the same irreducible ray representation of the group \( \mathcal{G} \) is realized should arise simultaneously with the singlet states, although the pairing type occurred at \( H = 0 \) has near \( H_{c2} \) the dominant amplitude.

In conclusion, one can apply the above classification scheme for all systems described by the symmetry group \( (53) \). Two-dimensional electron systems under a magnetic field are the most interesting among others. They were considered, e.g., in the works \(^{[32,77]} \), where particular level operators analogous to Casimir operator \( (32) \) were introduced. It is general symmetry properties of the system which make it possible such procedure in all cases: for a single electron, interacting electrons, electrons with spin-orbital coupling, or electron-phonon interaction.

### 3.3 Kink in the Upper Critical Field

The peculiar \( H-T \) phase diagram of superconducting UPt\(_3\) with two jumps in specific heat at temperatures \( T_{c1} \) and \( T_{c2} \) and with the intersection of two critical fields \( H_1(T) \) and \( H_2(T) \) at the kink point (see Fig. 1) is the main argument in favor of the anisotropic pairing in heavy fermions and of the multicomponent superconducting order parameter in UPt\(_3\) in particular \(^{[68]} \). The splitting of phase transition is believed to be explained by the closeness of critical temperatures of two superconducting states. The critical fields \( H_1(T) \) and \( H_2(T) \) are interpreted as upper critical fields of these phases.
The explanation of the phase diagram of UPt$_3$ has two different aspects: (i) what is the nature of the splitting of the phase transition, and (ii) under what conditions the double superconducting phase transition leads to the phase diagram similar to that in Fig. 1.

A comprehensive discussion of the first point can be found in [27,43,61]. We only enumerate the most popular models of $T_c$ splitting in UPt$_3$.

(1) Antiferromagnetic (AFM) symmetry breaking field models. (a) In some of these models [24,28,45,61] it is assumed that the unconventional superconductivity in UPt$_3$ is described by the two-component order parameter $\eta = (\eta_1, \eta_2)$ in accordance with (3) which corresponds to a 2D irreducible representation of the point crystal group $D_6$, either $E_1$ or $E_2$. A weak AFM order with staggered moments in the basal plane lowers the symmetry group of the system from $D_6$ to $D_2$ and therefore splits $T_c$. (b) A more complicated variant [53] assumes a one-dimensional irreducible representation of $D_6$ for superconducting order parameter but with additional degrees of freedom due to the triplet pairing and weak spin-orbit coupling. The splitting is due to the interaction of AFM moments with the total spin of the Cooper pairs.

(2) The model of nearly isotropic $d$-wave pairing. In this picture [83] UPt$_3$ is a $d$-wave isotropic superconductor with small splitting of critical temperature according to the irreducible representations of $D_{6h}$ group. Two of these representations $A_{1g}$ and $E_{1g}$ are responsible for the phase transitions in UPt$_3$.

(3) Accidental degeneracy models [19,28] unlike the other models do not explain the physical reasons for $T_c$ splitting. They only suggest that the closeness of critical temperatures of two irreducible representations of $D_{6h}$ group is accidental.

The answer on the second question can be given using the quantum numbers technique of Sec. 3.2. The upper critical field corresponds to the maximal eigenfield of the linearized equations. The crossing of two eigenfields $H_1(T)$ and $H_2(T)$ leads to the kink in $H_{c2}(T) = \max\{H_1(T), H_2(T)\}$. Using the general quantum mechanical rule that only symmetry different eigenlevels of the linear problem can cross, we came to the following conclusion:

The kink is possible only if the eigenfunctions related to $H_1(T)$ and $H_2(T)$ have different quantum numbers $N$ or $\sigma$. Otherwise the kink should be absent or at least smeared.

Application of this rule to the cited above models of $T_c$ splitting allows to verify easily their relevance to the phase diagram of UPt$_3$. In fact several models [19,53,61,83] satisfactory explain stability of the kink in $H_{c2}(T)$ for different orientations of $\mathbf{H}$, and a new experimental information is needed to select one of them.

### 3.4 Anisotropy of the Upper Critical Field

In conventional anisotropic superconductors the upper critical field can be calculated by taking into account the anisotropy of gradient terms in the GL functional. One should replace the gradient term
in \( [\text{3}] \) by \( K_{ij}D_i^*\Psi^*D_j\Psi \) (in the BCS theory, \( K_{ij} \) is proportional to the inverse tensor of the effective masses of conduction electrons). The second rank tensor \( K_{ij} \) has three independent components \( (K_{xx}, K_{yy}, K_{zz}) \) for biaxial crystals, two components \( (K_\parallel = K_{zz}, K_\perp = K_{xx} = K_{yy}) \) for uniaxial crystals and only one component \( (K_{xx} = K_{yy} = K_{zz}) \) for cubic crystals. Therefore, in uniaxial crystals (e.g., with hexagonal or tetragonal structure) the angular dependence of \( H_{c2}(\vartheta) \) has the form of an axisymmetric ellipsoid with the main axes: \( H_{c2}^\parallel = \hbar c\alpha/2|e|K_\perp \), \( H_{c2}^\perp = \hbar c\alpha/2|e|/(K_\perp K_\parallel)^{1/2} \) (see Fig. 5). For a cubic crystal the upper critical field in the GL region near \( T_c \) is isotropic.

Beyond the GL region the higher gradient terms slightly break the ellipsoidal form of the angular dependence of \( H_{c2} \) and restore the anisotropy of \( H_{c2} \) which coincides with the full point symmetry group of the crystal. However, this contribution to \( H_{c2} \) is of the order of \( (1 - T/T_c)^2 \), and it does not change the slope of the upper critical field \( H_{c2}' = (dH_{c2}/dT)|_{T=T_c} \). Recently different types of upper critical field anisotropy in the higher orders of \( (1 - T/T_c) \) have been discussed for conventional and unconventional superconductors in \([50]\). 

Turning back to the unconventional superconductivity, one should expect that anisotropy properties of \( H_{c2}' \) are different from those for the \( s \)-wave superconductor due to a more complicated structure of the GL equations. The problem of \( H_{c2}' \) anisotropy for different multicomponent order parameters for the crystal structures relevant to the heavy fermions was investigated in \([21,11,44]\). The following unusual for \( s \)-wave superconductors features were found:

- **For the cubic crystals** like UBe\(_{13} \) (point group \( O_h \)), \( H_{c2}' \) should always have the cubic anisotropy for the different field orientations.

- **For the tetragonal crystals** like CeCu\(_2\)Si\(_2\), URu\(_2\)Si\(_2\) (\( D_{4h} \)), the square-like anisotropy of \( H_{c2}' \) is expected when the magnetic field lies in the basal plane (see Fig. 6)

- **For the hexagonal crystals** like UPt\(_3\), UPd\(_2\)Al\(_3\), UNi\(_2\)Al\(_3\) (\( D_{6h} \)), there are no additional features of \( H_{c2}' \) anisotropy with respect to \( s \)-wave superconductors. \( H_{c2}' \) is isotropic in the basal plane and has an uniaxial outplane anisotropy.

Note that these types of “unusual” anisotropy are connected to the symmetry of corresponding GL functionals. However, such anisotropy was found for neither UBe\(_{13} \) \([3]\), nor for CeCu\(_2\)Si\(_2\) \([66]\). Moreover, the upper critical field in CeCu\(_2\)Si\(_2\) shows unexpected isotropy for all field directions in the linear regime near \( T_c \) \([66]\). This discrepancy can be explained by the weak sensitivity of the electron-electron pairing interaction to the crystal anisotropy (e.g., it is the case for the \( p \)-wave pairing \([38]\)).

Another characteristic feature of the anisotropy of the upper critical field in the multicomponent superconductors is the nonelliptical, nonmonotonous behavior of \( H_{c2}(\vartheta) \) for uniaxial crystals when the applied field changes its direction from the principal axis to the basal plane. This behavior is connected with the existence of several solutions with close critical fields for arbitrary directions of \( \mathbf{H} \) rather than with the symmetry of GL equations \([80]\). The investigation of the GL equations for two-component model \([15]\) in \([5,80]\) shows that the angular dependence of \( H_{c2}' \) can have pronounced
minimum in a particular region of the phenomenological parameters $C$ and $K$ (see Fig. 7). Such behavior of $H_{c2}(\vartheta)$ was also found beyond GL approximation in [58], what confirms the universality of this feature.

Unfortunately, the available experimental data on the anisotropy of the upper critical field in UPt$_3$ [69] are not clear enough to verify such a possibility. The observation of the nonmonotonous behavior of $H_{c2}(\vartheta)$ at uniaxial crystals CeCu$_2$Si$_2$, UPt$_3$, URu$_2$Si$_2$, UPd$_2$Al$_3$, or UNi$_2$Al$_3$ would be a good argument in favor of unusual superconductivity in heavy fermions.

The striking feature of the superconductivity in UPt$_3$ is the crossover in the anisotropy ratio $H_{c2}^{\perp}/H_{c2}^{\parallel}$ with temperature, which is less than one near $T_c$ and greater than one at $T = 0$. Possible explanation of this experimental fact in terms of unconventional superconductivity is suggested in [15].

At the end of this Subsection, we shall consider the quantum number classification for solution of the linearized anisotropic GL equations. When magnetic field is directed at an arbitrary angle with respect to the principal axis of a uniaxial system, the point symmetry group $D_{\infty h}(C_{\infty h})$ reduces to $G' = \{E, P, RU_{2z}\}$, where $\hat{x}$ axis lies in the basal plane perpendicularly to $H$. At the first sight, there are no quantum numbers for solutions of linearized equations in this case because rotational symmetry about $H$ is completely broken.

However, the GL functional constructed for a single irreducible representation with a particular parity possesses an additional hidden symmetry. The action of space inversion $P$ on a solution transitionally invariant along $H$ reduces to the rotation by angle $\pi$ in the plane perpendicular to the magnetic field. Therefore, these solutions can be described by the parity of the Landau level number $N$, as in the case of $H$ directed along the two-fold axis. For this reason, the vortex lattice in conventional superconductors has the form of the distorted triangular lattice with a two-fold symmetry at all intermediate directions of $H$.

3.5 Modulation of Order Parameter Along Magnetic Field

The appearance of superconducting phases modulated along field direction near $H_{c2}$ was discussed from the beginning of investigations of upper critical field in nontrivial superconductors [11]. The discussion has been renewed last time in connection with the phase diagram of UPt$_3$ [18,39]. Here we formulate analytical approach to this problem on the example of two-component model.

If the order parameter is allowed to vary along $H$, an additional quantum number, momentum $p_H$, appears. The reflection $\hat{\sigma}_h$ transforms functions with $p_H$ and $-p_H$ into each other, hence, they have the same eigenvalue. In Sec. 3.1 we have seen that for $H \perp c$ Eq. (50) could not be solved analytically for arbitrary $p_x \neq 0$. Therefore, in this case we must verify the possibility that solutions with nonzero $p_x$ have smaller eigenvalues than (51) or (52).

It is obvious that solutions with $p_x \gg 1/l_H$ have larger eigenvalues because magnetic field may be considered as a perturbation to positively definite gradient terms (19) in this case. We can check analytically also an opposite case of $p_x \ll 1/l_H$ when instead nonzero $p_x$ can be treated as a
perturbation to eigenvalues (61) and (62).

Let us consider Hamiltonian (50) as a sum of three parts \( \hat{H}_0 + \hat{H}_1 + \hat{H}_2 \), where \( \hat{H}_0 \) does not depend on \( p_x \), \( \hat{H}_1 \) depends linearly and \( \hat{H}_2 \) depends quadratically on \( p_x \). Note that \( \hat{H}_1 \) is an off-diagonal operator, therefore its contribution is of the order of \( p_x^2 \) and negative for the lowest eigenvalue. Eigenvalues and eigenfunctions of \( \hat{H}_0 \) are known:

\[
L_n = \frac{2|e|HK_1}{\hbar c} \sqrt{K} (2n + 1) , \quad \eta = (f_n, 0) , \quad (65)
\]
\[
M_n = \frac{2|e|HK_1}{\hbar c} \sqrt{K(1 + 2C)} (2n + 1) , \quad \eta = (0, g_n) , \quad (66)
\]

where \( f_n \) and \( g_n \) are introduced in (51) and (52) wave functions of scaled Landau levels. They are characterized by the parity under reflection \( \hat{\sigma}_x \) and by the parity under rotation by angle \( \pi \) about \( \hat{x} \). The variation of solutions along \( \hat{x} \) breaks the classification according to the parity under reflection, but the parity under rotation by angle \( \pi \) is still conserved. Then if we take \( \eta_x \) as a superposition of even Landau levels, \( \eta_y \) must be a sum of wave functions from odd Landau levels and vice versa.

In accordance with this symmetry classification, the second order perturbation theory gives the following corrections to the eigenvalues \( L_0 \) and \( M_0 \):

\[
L_0(p_x) - L_0 = L_0 + K_{123} p_x^2 - K_{23}^2 p_x^2 \sum_{n=2k+1} |\langle f_0 | D_y | g_n \rangle|^2 \left( \frac{1}{M_n - L_0} \right) , \quad (67)
\]
\[
M_0(p_x) = M_0 + K_1 p_x^2 - K_{23}^2 p_x^2 \sum_{n=2k+1} |\langle g_0 | D_y | f_n \rangle|^2 \left( \frac{1}{L_n - M_0} \right) . \quad (68)
\]

Substituting \( L_n \) and \( M_n \) to (51) and (52), we find:

\[
L_0(p_x) - L_0 = K_1 (1 + C) p_x^2 \left( 1 + \varepsilon - 2 \varepsilon^2 \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} (1 + q)^3 \sum_{k=0}^{\infty} q^{2k} \frac{(2k)!}{2^{2k} (k!)^2} \right) \times \left[ \frac{1}{2(1 + \varepsilon)} - \frac{\varepsilon}{((1 + \varepsilon)(4k + 3) - \sqrt{1 - \varepsilon^2})(1 + \varepsilon + \sqrt{1 - \varepsilon^2})} \right] ,
\]

\[
\varepsilon = C , \quad q = \frac{\varepsilon}{1 + \sqrt{1 - \varepsilon^2}} , \quad (69)
\]

and the same expression for \( M_0(p_x) - M_0 \) with the only change \( \varepsilon \to -\varepsilon \). If \( C > 0 \), then \( L_0 \) is the smallest eigenvalue for solutions uniform in the field direction. A lower estimate on \( L_0(p_x) \) is obtained neglecting the second term in square brackets. Then, using \( \sum_{k=0}^{\infty} q^{2k} \frac{(2k)!}{2^{2k} (k!)^2} = \frac{1}{\sqrt{1-q^2}} \), we come to:

\[
L_0(p_x) - L_0 > K_1 (1 + C) p_x^2 \left( 1 + \varepsilon - \varepsilon^2 \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} (1 + q)^3 \frac{(1 + q)^3}{\sqrt{1 - \varepsilon^2}} \right) . \quad (70)
\]

It may be easily checked that the right hand side of Eq. (70) is always positive for any \( C > 0 \) \((0 < \varepsilon < 1)\). The same is true for function \( M(p_x) \) in the opposite case when \(-0.5 < C < 0 \) \((-1 < \varepsilon < 0)\). Thus, we may conclude that modulation of superconducting phase near \( H_{c2} \) along \( \mathbf{H} \) are unfavorable in both cases \( p_x \ll 1/l_H \) and \( p_x \gg 1/l_H \). It is very improbable that such modulations will appear for intermediate values of \( p_x \).
While a solution with the lowest eigenvalue, e.g. $L_0$ for $C > 0$, is stable against small nonhomogeneous perturbations with $p_x \neq 0$, this is not always true for the next level $M_0(p_x)$. It follows from (68) that when $M_0 \to L_1$ (i.e., $C \to 4$) the corresponding denominator tends to zero, giving a large negative contribution to the coefficient at $p_x^2$. Thus, at $C = 4$ the level $M_0$ is certainly unstable against modulations with particular wavelength. The numerical investigation of $M_0(p_x)$ shows that coefficient at $p_x^2$ changes sign from $+$ to $-$ at $C = 0.946$.

A well developed modulated structure at $C > 0.946$ can not be analyzed on the basis of the second order perturbation theory. An infinite system of secular equations must be solved in this case. Variational calculations show that for $C = 4$ ($\varepsilon = 4/5$) when $M_0 = 3$ (in of units $2|e|HK_1\sqrt{K}/\hbar c$) the next level after $L_0 = 1$ has the eigenvalue $M_0(p_x^*) = 2.24$ with the modulation wavelength $\lambda = 2\pi/p_x^* \sim 4.7l_H$. The discussed instability of the second solution of the linearized GL equations is important for consideration of phase transitions inside mixed state (see Sec. 5).

4 Vortex Lattices Near $H_{c2}$

The next necessary step in the investigation of the superconducting phase transition at the upper critical field is determination of the vortex lattice form near $H_{c2}$. We describe in this Section nontrivial features of this procedure applied to the multi-component solutions of linear equations like SK-phase (49).

The symmetry approach to second order phase transitions [34] allows us to make a few general conclusions before fulfilling calculations. First of all, the free energy near $H_{c2}$ has the form of the Landau expansion in powers of $\hat{\Delta}$. The structure of the order parameter $\Delta(k, r)$ is determined by the solution of the linear equations (54) and corresponds to the particular irreducible ray representation $(N, \sigma)$. The superconducting amplitude $\hat{\Delta}$ is proportional to $(H_{c2} - H)^{1/2}$. The degeneracy of the order parameter is lifted by the forth order terms of the Landau functional. While the second order term is unique and can be always presented as $|\hat{\Delta}_N^\sigma|^2$, the forth order terms, generally, become nonlocal at low temperatures. For this reason, we consider only a restricted problem — investigation of nonlinear equations in GL approximation.

We derive below the energy parameter for multicomponent GL functionals which generalies corresponding Abrikosov formula (Sec. 4.1). In Sec. 4.2 we discuss general procedure of finding the most stable lattice form and consider symmetry properties of the energy parameter under different choices of the lattice parametrisation. Sec. 4.3 is devoted to the search of the energetically favorable forms of the Abrikosov lattice for the SK-phase (11), which should appear near $H_{c2}$ in the two-component superconductor. Symmetry group of the vortex lattice is constructed in Sec. 4.4.

4.1 Average Energy Density

In this Subsection we construct the energy parameter for an unconventional superconductor, which determines the form of the vortex lattice. For this purpose we must take the order parameter in
an unchanged form as a solution of the linearized GL equations with the largest critical field. The relative accuracy of this approximation is \((1 - H/H_{c2})\). The vector potential \(A_0\) (rot \(A_0 = H_{c2}\)), which enters the linearized GL equations, must be changed in order to include the diamagnetic field of superconducting currents: \(h = \text{rot} \, A = H + h_s\), \(\text{rot} \, h_s = 4\pi/c \cdot j_s\). Neglecting by the boundary effects, we come to the following functional:

\[
F = \frac{H - H_{c2}}{4\pi} \langle |h_s| \rangle + \langle |\hat{\Delta}|^4 \rangle - \frac{H^2}{8\pi} - \frac{\langle h_s^2 \rangle}{8\pi},
\]

where \(\langle \ldots \rangle\) denotes spatial average. The forth order terms in (71) are written in a symbolical form, which implies the sum over all the possible invariants constructed from the components \(\eta_i\).

Thus the calculation of \(F\) reduces to the derivation of the diamagnetic field \(h_s\) with the help of the Maxwell equations [16]. For one-component superconductors, in which spatial dependence of the order parameter is restricted to zero Landau level functions, this field may be found analytically:

\[
h_s = \frac{8\pi eK}{\bar{h}c} |\Psi(r)|^2.
\]

Since the calculation of \(h_s\) for the multicomponent SK-phase becomes more difficult, we restrict ourselves to the case of \(\kappa \gg 1\). This allows us to neglect by the last term in (71), which is always of the order of \(\langle |\hat{\Delta}|^4 \rangle /\kappa^2\). Thus we should find only \(\langle |h_s| \rangle\). In the chosen approximation \(h_s\) is the quadratic form of \(\Delta_N(k,r)\). The magnetic moment of the unit volume is a quantity invariant under action of the symmetry group (53). Therefore, the average \(\langle |h_s| \rangle\) is the second order invariant constructed from \(\hat{\Delta}(k,r)\), and, consequently, \(\langle |h_s| \rangle \propto \langle |\hat{\Delta}_N(k,r)|^2 \rangle\). The numerical factor in this relation does not depend on the particular structure of solution, because formally at \(H = 0\) the first term in (71) must be equal to \(-\alpha \langle |\hat{\Delta}|^2 \rangle\). Thus for large \(\kappa\) the multicomponent GL energy is:

\[
F = -\alpha \left(1 - \frac{H}{H_{c2}}\right) \langle |\hat{\Delta}|^2 \rangle + \langle |\hat{\Delta}|^4 \rangle - \frac{H^2}{8\pi}.
\]

Minimizing (72) by the amplitude of \(\hat{\Delta}\), we get

\[
F = -\frac{\alpha}{4\delta} \left(1 - \frac{H}{H_{c2}}\right)^2 - \frac{H^2}{8\pi},
\]

where

\[
\delta = \frac{\langle |\hat{\Delta}|^4 \rangle}{\langle |\hat{\Delta}|^2 \rangle^2}
\]

is the energetic parameter of GL model which generalies usual Abrikosov parameter for ordinary superconductors. The absolute minimum of \(\delta\) determines the geometrical form of the vortex lattice near \(H_{c2}\). For conventional superconductors it is

\[
\delta_0 = \beta \frac{\langle |f_0|^4 \rangle}{\langle |f_0|^2 \rangle^2},
\]

in accordance with the Abrikosov formula [1]. Vortex lattice form for axial phase (48) of the two-component model is determined by the same parameter (75) with \(\beta\) changed on \(\beta_1\). Substituting (15) in (74) we obtain that for the SK-phase the energy parameter has the following form:

\[
\delta_{SK} = \frac{\beta_1\langle |f_0|^4 + \omega^4 |f_2|^4 \rangle + 2\omega^2(\beta_1 + 2\beta_2)|f_0|^2|f_2|^2}{(1 + \omega^2)^2\langle |f_0|^2 \rangle^2}.
\]
We discuss energy minimums of these parameters in the Sec. 4.3.

4.2 Properties of Lattice Solutions

Before search for the favorable lattice form we construct in this Section general periodic solutions on different Landau levels which then should be substituted in (74)–(76). We will choose particular parametrisation for two-dimensional vortex lattices and discuss its symmetry properties which are helpful in further numerical calculations.

The vortex lattice is defined by the discrete group of translations through \( a \) and \( b \):

\[
\hat{T}_a \Delta(k, r) = \exp(i\varphi_1) \Delta(k, r), \quad \hat{T}_b \Delta(k, r) = \exp(i\varphi_2) \Delta(k, r),
\]

with arbitrary phase factors \( \varphi_1 \) and \( \varphi_2 \). It follows from (77) that magnetic translations for this subgroup commute. Thus, due to (57) any periodical solution possesses an integer number of flux quanta per unit cell \([9,78]\).

We consider only lattices with one quantum of flux per unit cell. All distances are measured in the units of the magnetic length \( l_\text{m}^2 = \hbar c/|e|H \), and the vector potential is chosen in the Landau gauge \( A = (-Hy, 0, 0) \). Using the explicit form of the magnetic operator in this gauge \( \hat{T}_a = \exp(-ia_y x) \exp(a \nabla) \), it is possible to construct the lattice solution on the \( n \)th Landau level

\[
f_n(r \mid \tau) = \sum_m \exp\left\{-\pi im^2 + \pi im(\rho + 1) + \frac{2\pi i}{a} \left( m - \frac{1}{2} \right) x - \frac{1}{2} \left[ y - \left( m - \frac{1}{2} \right) b \sin \alpha \right]^2 \right\} \times H_n \left[ y - \left( m - \frac{1}{2} \right) b \sin \alpha \right],
\]

which satisfies (74) with \( a = (a, 0) \), \( b = (b \cos \alpha, b \sin \alpha) \), \( \varphi_1 = \pi \), \( \varphi_2 = \pi(\rho + 1) \). The area of the unit cell is fixed by the condition of flux quantization: \( ab \sin \alpha = 2\pi \). The lattice form depends on the complex parameter \( \tau = \rho + i\sigma = b/a \exp(i\alpha) \). The parameters \( \tau_h = \exp(\pi i/3) \) and \( \tau_s = i \) correspond to lattices with hexagonal and square symmetries. The periodic function \( f_0(r \mid \tau) \) has one zero per unit cell located at \((0, 0)\).

The functions (78) can be written as:

\[
f_0(r \mid \tau) = \vartheta_1(\frac{z}{a} \mid \tau) \exp\left(-\frac{y^2}{2}\right), \quad f_n(r \mid \tau) \propto (a^+)^n f_0(r \mid \tau),
\]

where \( z = x + iy \), \( \vartheta_1(z \mid \tau) \) is the Jacobi theta-function, and the bar denotes complex conjugation. The change of the vector potential affects only the exponential factor in the expression for \( f_0(r \mid \tau) \), leaving the \( \tau \)-dependence through \( \vartheta_1(z/a \mid \tau) \) for all gauges. The theta-functions are the only quasiperiodical analytical functions. This leads to the uniqueness of \( f_n(r \mid \tau) \) with given periods \( a, b \) and phases \( \varphi_1, \varphi_2 \). Substituting (78) into (74), we obtain all possible one-quantum vortex lattices near \( H_\text{c2} \).

In order to find which type of lattice is realized, one should substitute \( \hat{\Delta}_N(k, r \mid \tau) \) into the nonlinear energy functional, calculate the free energy \( F(\tau) \) for each \( \tau \), and select \( \tau \) which corresponds
to the absolute minimum of $F(\tau)$. Before calculations we establish some symmetry properties of function $F(\tau)$ according to \(^{82}\), which generalizes ideas of \(^{59}\).

The choice of the basis for two-dimensional Bravé lattice is not unique. The same Bravé lattice may be described by different $\tau$. Instead of $a$ and $b$ any pair of their integer combinations $ka + lb$ and $ma + nb$, such that $kn - lm = \pm 1$, can be taken as basis vectors. This transformation changes parameter $\tau$ to $\tau' = (m + n\tau)/(k + l\tau)$. The matrices $\begin{pmatrix} k & l \\ m & n \end{pmatrix}$, which do not change the orientation of the basis form the modular group $SL(2, \mathbb{Z})$. Together with substitution $a \rightarrow -a$, $b \rightarrow b$ they represent a complete set of possible parameterizations of the original Bravé lattice.

In a physical problem, the symmetry of the periodical system can be lower than the symmetry of the corresponding Bravé lattice. The state is specified not only by the lattice, but also by additional parameters describing the structure and orientation of each lattice site. For such systems transformation in the space of its parameters that consists only of a different choice of basis vectors changes the initial state. Thus we are dealing with the presence or absence of the symmetry under the action of the group $SL(2, \mathbb{Z}) \times \mathbb{Z}_2$ in the complex plane of parameter $\tau$.

Functions $f_n(\tau \mid \rho \mid r)$ (or $\hat{\Delta}_N(k, \tau \mid r)$) are uniquely determined by their lattice, and, consequently, they should be symmetrical under $SL(2, \mathbb{Z}) \times \mathbb{Z}_2$ group. This statement can be verified by establishing the following properties of $\hat{\Delta}_N(k, \tau \mid r)$ as a function of $\tau$ (see proof in \(^{83}\)):

\begin{equation}
\hat{\Delta}_N(k, r \mid \tau + 1) = \hat{\Delta}_N(k, r \mid \tau),
\end{equation}

\begin{equation}
\hat{\Delta}_N(k, r \mid -1/\tau) = \sqrt{\tau^*} e^{-iN\varphi} \exp \left[ \frac{\pi i}{4} \left( \rho - 3 - \frac{\rho}{\rho^2 + \sigma^2} \right) \right] \hat{L}_\varphi \hat{\Delta}_N(k, r \mid \tau),
\end{equation}

where $\exp(i\varphi) = \sqrt{\tau/\tau^*}$, and

\begin{equation}
\hat{\Delta}_N(k, r \mid -\tau^*) = RU_{2y} \hat{\Delta}_N(k, r \mid \tau).
\end{equation}

Substituting $\hat{\Delta}_N(k, r \mid \tau)$ to an arbitrary functional $F$ invariant under group $\mathcal{G}$, we obtain energy $F(\tau)$, which is not changed under $SL(2, \mathbb{Z}) \times \mathbb{Z}_2$ transformations:

\begin{equation}
F(\tau) = F \left( \frac{m + n\tau}{k + l\tau} \right) = F(-\tau^*).
\end{equation}

The fundamental region of $SL(2, \mathbb{Z}) \times \mathbb{Z}_2$ group in complex plane contains all different one-quantum lattices. A specific example of the fundamental region is shown in Fig. 8. Their vertices $\tau_h$ and $\tau_s$ are always stationary points of $F(\tau)$. While at the point $\tau_s$ functional can achieve all types of extremums, in $\tau_h$ only a local maximum or a local minimum are possible. The absolute minimum of $F(\tau)$ can be located, of course, not only at these symmetrical points, but somewhere in the fundamental region. Moreover, such exotic behavior is characteristic for unconventional pairing (see Sec. 4.3).

It is interesting that similar theoretical constructions have appeared independently in the theory of Weinberg-Salam model for the electroweak interactions \(^{41,42}\). Vacuum reconstruction in high

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magnetic fields in this theory is described by energy functional on complex wave function $\Psi$ with nonlocal forth order term arising from coupling with additional bosonic field. This nonlocality in the forth order terms could results in its turn in exotic forms of lattice solutions.

### 4.3 Vortex Lattice in Two-Component Model

The explicit dependence of the energy parameter for one-component superconductors from $\tau$ can be obtained after substitution of $f_0(\mathbf{r} | \tau)$ into (75). This calculation was done in [59]:

$$\delta_0(\rho, \sigma) = \beta \sqrt{\sigma} \sum_{m,n} \cos(2\pi \rho mn) e^{-\pi \sigma (m^2 + n^2)} \, .$$

(84)

Here summation is fulfilled over all integer $m$ and $n$. The absolute minimum of the function $\delta_0(\rho, \sigma)$ is achieved at the point $\tau_h$ of the fundamental region which corresponds to the regular triangular lattice.

The energy parameter of the SK-phase (76) can be presented as:

$$\delta_{SK}(\rho, \sigma) = \frac{\beta_1}{(1 + \omega^2)^2} \left( \delta_{00}(\rho, \sigma) + 2\omega^2(1 + 2\beta)\delta_{02}(\rho, \sigma) + \omega^4\delta_{22}(\rho, \sigma) \right) ,$$

(85)

where parameter $\omega$ is given by (49) and $\beta = \beta_2/\beta_1$. Explicit expressions for $\delta_{pq}(\rho, \sigma)$ can be found in Appendix. Positions of the absolute minimums are different for each function $\delta_{pq}(\rho, \sigma)$. Therefore, variation of phenomenological parameters $\beta$ and $\omega$ leads to the rich phase diagram for the vortex lattice form near $H_{c2}$. Note that, according to (49), the following restriction $0 < \omega < 1/\sqrt{2}$ is fulfilled in the region of the existence of SK-phase near $H_{c2}$.

The numerical investigation of $\delta_{SK}(\rho, \sigma)$ was done first in [48]. But the phase diagram found there contradicts in fact general requirements of the phenomenological theory, because it predicts two coinciding phase transitions lines between triangular and square lattices and between square and rectangular lattices. The correct phase diagram for the position of the absolute minimum of function $\delta_{SK}(\rho, \sigma)$ versus phenomenological parameters $\beta$ and $\omega^2$ is presented in Fig. 9. There are different regions where the energetically favorable solution corresponds to the regular triangular, square, rectangular, and distorted triangular lattices. Phase transitions lines in Fig. 9 between square and the rectangular lattices and between square and distorted triangular lattice IV are of the second order, all other transitions are of the first order. For negative $\beta$ the only stable form of the vortex lattice is a triangular one.

The characteristic scale of the difference between various extremums of $\delta_{SK}(\rho, \sigma)$ is of the order of $10^{-4}$–$10^{-6}$ compared to $10^{-2}$ for $\delta_0(\rho, \sigma)$. Note that the point $(\beta, \omega^2) = (0.5, 0.1)$ predicted by BCS theory for $E_{1g}$ representation is situated in the region of stability of the hexagonal lattice. It is interesting that for $\delta_{SK}(\rho, \sigma)$ the absolute minimum always lies on the boundary of the fundamental region.

In Sec. 4.2 we have argued that the one-quantum vortex lattice with hexagonal symmetry always corresponds to the extremum of the energy functional for arbitrary superconducting order parameter
\[ \hat{\Delta}_N(\mathbf{k}, \mathbf{r}). \] Now we can see from Fig. 9 that corresponding stability region of the hexagonal lattice for the SK-phase is rather large.

### 4.4 Symmetry Group of the Vortex Lattice

To complete the discussion of vortex lattices near \( H_{c2} \) we should write explicitly corresponding symmetry groups, which include magnetic translations and rotations \((55)\) in combinations with nontrivial phase factors.

The transformation rules of \( \hat{\Delta}_N(\mathbf{k}, \mathbf{r}) \) near \( H_{c2} \) under magnetic translations by the lattice periods are defined by \((77)\) and \((78)\). Vortex lattices corresponding to different phases \( \varphi_1 \) and \( \varphi_2 \) go over into each other under magnetic translations which do not coincide with any integer combination of the basis vectors. Displaced solutions \( \hat{T}_\rho \hat{\Delta}_N(\mathbf{k}, \mathbf{r}) \) are transformed like Bloch wave functions with wave vectors lying in the Brillouin zone of the background lattice \( \hat{\Delta}_N(\mathbf{k}, \mathbf{r}) \) \([49]\).

We are interested mainly in the transformation properties of the triangular lattice, since it has a particular region of stability for arbitrary \( N \). Using identity \(-1/\tau_h = \tau_h\) one can obtain from \((80)-(82)\) following properties under rotation by angle \( \pi/3 \) about arbitrary vortex axis \( \Gamma \):

\[
\hat{L}_{\pi/3} \hat{\Delta}_{N,\Gamma}(\mathbf{k}, \mathbf{r} | \tau_h) = e^{-i\pi(1-N)/3} \hat{\Delta}_{N,\Gamma}(\mathbf{k}, \mathbf{r} | \tau_h),
\]

and under time-reversal together with rotation by angle \( \pi \) about an axis perpendicular to the direction of magnetic field:

\[
RU_{2y} \hat{\Delta}_{N,\Gamma}(\mathbf{k}, \mathbf{r} | \tau_h) = \hat{\Delta}_{N,\Gamma}(\mathbf{k}, \mathbf{r} | \tau_h).
\]

Combining the elements listed above, we can write the symmetry group of the vortex lattice as

\[
\mathcal{G}_\Gamma = \left\{ e^{i\pi/3} \hat{T}_a, \ e^{i\pi/2} \hat{T}_b, \ e^{i\pi(1-N)/3} \hat{L}_{\pi/3}, \ \sigma \hat{\sigma}_h, \ RU_{2y} \right\}. \tag{88}
\]

All other elements are formed as proper combinations of this basis. For example, the superconducting order parameter \( \hat{\Delta}_{N,\Gamma}(\mathbf{k}, \mathbf{r} | \tau_h) \) also possesses symmetries under rotations by angle \( 2\pi/3 \) about an axis passing through the center of the triangle \( K \) and on angle \( \pi \) about the middle of the triangle side \( M \) (see Fig. 10). Considering two other displaced vortex lattices \( \hat{\Delta}_{N,K}(\mathbf{k}, \mathbf{r} | \tau_h) \) and \( \hat{\Delta}_{N,M}(\mathbf{k}, \mathbf{r} | \tau_h) \), whose points \( K \) and \( M \) coincide with the center of coordinate \( \Gamma \), we can establish their symmetries with respect to \( \Gamma \):

\[
\mathcal{G}_K = \left\{ e^{-i\pi/3} \hat{T}_a, \ e^{-i\pi/6} \hat{T}_b, \ e^{-2i\pi N/3} \hat{L}_{2\pi/3}, \ \sigma \hat{\sigma}_h, \ e^{i\pi} RU_{2y} \right\}, \tag{89}
\]

\[
\mathcal{G}_M = \left\{ e^{i\pi} \hat{T}_a, \ e^{-i\pi/2} \hat{T}_b, \ e^{i\pi} \hat{L}_{\pi}, \ \sigma \hat{\sigma}_h, \ e^{i\pi} RU_{2y} \right\}. \tag{90}
\]

The symmetry properties of square lattice \( \hat{\Delta}_{N,\Gamma}(\mathbf{k}, \mathbf{r} | \tau_s) \) and of all others from Sec. 4.3 can be investigated by the similar way.

We have derived transformation properties \((88)-(91)\) using explicit form of \( \hat{\Delta}_N(\mathbf{k}, \mathbf{r} | \tau) \) which is valid only for a constant magnetic field inside a sample, i.e., when \( H \rightarrow H_{c2} \). Below \( H_{c2} \) the magnetic
field is modulated by superconducting currents. However, Eqs. (55) allows introduction of discrete magnetic translations and rotations in this case too. Therefore, the mixed state is invariant under symmetry group (88) in the finite region $H^* < H < H_{c2}$ until a new phase transition occurs.

5 Phase Transitions in the Mixed State

Perhaps the most unusual feature of the mixed state of an unconventional superconductor is the possibility of the phase transitions between different vortex lattices.

As we have mentioned in Sec. 1.2, in usual superconductors the character of vortex interaction strongly depends on the distance between vortices. In the vicinity of the lower critical field $H_{c1}$ vortex lattice is formed through the interactions between nearest neighbors. At fields $H > H_c$ vortex interacts with all neighbors within circle of the radius $\lambda$. If, however, $H < H_{c2}$ the average distance between vortices $r_L$ is much greater than coherence length $\xi$ and the intersection of cores of different vortices is negligible. In this case only long range hydrodynamics repulsion participates in the lattice formation.

As vortices in unconventional superconductors have the same structure at distances $r > \tilde{\xi}$, all above arguments are valid for them too. The cores intersection becomes important for intervortex distance $r_L \sim \max\{\xi, \tilde{\xi}\}$. Therefore all deviations from the usual Abrikosov theory of vortex lattice formation can occur only in the field region near the upper critical field. The convenient tool for investigation of vortex lattices in that region is symmetry approach developed in Secs. 3 and 4.

In Sec. 5.1 we formulate analytical approach to the investigation of phase transitions near $H_{c2}$ and show that the closeness of two critical (eigen) fields in the linear problem always leads to additional phase transitions inside mixed state. Sec. 5.2 is devoted to the symmetry consideration of structural phase transitions in the vortex lattices. Finally, in Sec. 5.3 we briefly discuss possible second order phase transitions below upper critical field on the examples of two-component GL functional $\mathcal{F}$ and accidental degeneracy models.

5.1 General Approach

The space distribution of the order parameter $\hat{\Delta}(k, r)$ at a given temperature $T$ and an external magnetic field $H$ ($H_{c1} < H < H_{c2}$) is defined by the minimum of the energy functional $\mathcal{F}\{\hat{\Delta}(k, r), H, T\}$. To minimize $\mathcal{F}$, one needs to solve the nonlinear (GL) equations: $\delta\mathcal{F}/\delta\hat{\Delta}^* = 0, \delta\mathcal{F}/\delta A = 0$ which can be symbolically written as:

$$\hat{L}\{\hat{\Delta}(k, r), H, T\} = 0.$$  \hspace{1cm} (91)

Their solution near $H_{c2}(T)$ is the Abrikosov lattice with a particular symmetry group $\mathcal{G}_T$.

Suppose that at some critical field $H^*(T)$ the second order structural phase transition in the vortex lattice does take place. Speaking in terms of the phase transition theory, $H^*(T)$ is the bifurcation line below which Eq. (91) has two different solutions: the old one $\hat{\Delta}_1(k, r)$ invariant
under the group $G_T$ and the new one $\hat{\Delta}_1(k, r) + \hat{\Delta}_2(k, r)$ with a lower symmetry $\tilde{G}$. At $H < H^*(T)$
the energy functional changes the character of the extremum for solution $\hat{\Delta}_1(k, r)$ from the absolute
minimum to the local maximum, whereas $\hat{\Delta}_1(k, r) + \hat{\Delta}_2(k, r)$ corresponds to its minimum. The
small perturbation $\hat{\Delta}_2(k, r)$ belongs to some nonunit irreducible representation of the group $G_T$.
Since the phase transition is of the second order, the amplitude of $\hat{\Delta}_2(k, r)$ at $H < H^*$ is small as
$(H^* - H)^{1/2}$. Linearizing (91) in the small value $\hat{\Delta}_2$, one obtains the equation for the $H^*(T)$ as a
condition on the magnetic field at which the linear eigenvalue problem

$$\hat{\mathcal{L}}\{\hat{\Delta}_1(k, r), H^*, T\} \hat{\Delta}_2(k, r) = 0$$

(92)

has a solution. The linear operator $\hat{\mathcal{L}}$ depends on the space distribution of the order parameter
$\hat{\Delta}_1(k, r)$ which in its turn is the solution of (91) taken at the phase transition line $H^*(T)$.

To specify the influence of the background solution $\hat{\Delta}_1(k, r)$ on the appearance of $\hat{\Delta}_2(k, r)$, we
separate the part $\hat{\mathcal{L}}_0$ from the operator $\hat{\mathcal{L}}$ which does not depend on $\hat{\Delta}$ and present (92) in the
following form:

$$\hat{\mathcal{L}}_0\{H^*(T)\} \hat{\Delta}_2(k, r) + \hat{\mathcal{L}}_1\{\hat{\Delta}_1(k, r), H^*(T)\} \hat{\Delta}_2(k, r) = 0.$$  \hspace{0.5cm} (93)

In the vicinity of $H_{c2}$ the second term in (93) is small and can be considered as a perturbation to
the equation

$$\hat{\mathcal{L}}_0\{H, T\} \hat{\Delta}(k, r) = 0,$$ \hspace{0.5cm} (94)

which in fact coincides with (74). The last equation has different critical fields $H_i(T)$, and the
maximal of them $H_1(T)$ corresponds to $H_{c2}(T)$. Let us assume that there exists another eigenfield
$H_2(T)$ of Eq. (74) which is sufficiently close to $H_1(T)$. Then one can derive similarly to (72) the
following two-order parameter functional:

$$F = -\alpha_1 \left(1 - \frac{H}{H_1}\right) \langle |\hat{\Delta}_1|^2 \rangle + \langle |\hat{\Delta}_1|^4 \rangle - \alpha_2 \left(1 - \frac{H}{H_2}\right) \langle |\hat{\Delta}_2|^2 \rangle + \langle |\hat{\Delta}_2|^4 \rangle + \langle \hat{\Delta}_1^2 \hat{\Delta}_2^2 \rangle,$$ \hspace{0.5cm} (95)

which represents free energy for fields close to $H_{c2}$.

Due to the infinite degeneracy of both order parameters, finding all extremums of the energy
functional (92) is a rather complicated problem even in the case when spatial dependencies of both
$\hat{\Delta}_1$ and $\hat{\Delta}_2$ are described by zero Landau level functions. However, without interaction between $\hat{\Delta}_1$ and $\hat{\Delta}_2$ the problem of the phase transitions described by functional (95) becomes trivial. There
will be two successive second order phase transitions at $H = H_1$ and $H = H_2$. Turning on of the
interaction term $\langle \hat{\Delta}_1^2 \hat{\Delta}_2^2 \rangle$ affects only the lower transition. At least in the finite range of parameters
where interaction term is small, the type of the lower transition is not changed and the critical field
$H_2(T)$ is slightly renormalized to $H^*(T)$, which corresponds to the structural phase transition in the vortex lattice. This case is considered in the two following Subsections.

If other eigenfields of (94) are far from $H_{c2}(T)$, this analytic approach is not applicable, because
Eq. (93) can not be reduced to (94) due to the significant admixture of all other Landau levels.
However, the continuous change of parameters of the system, which move $H_2$ from $H_1$ to lower
fields, raises a hope that in some range of parameters the phase transition in the mixed state still
exists.
5.2 Structural Phase Transitions

According to the Landau theory, if the second order phase transition in the vortex lattice occurs, the initial symmetry cannot be broken to an arbitrary subgroup of $G_\Gamma$. The residual symmetry group $\tilde{G} \subset G_\Gamma$ is defined as the symmetry of the new lattice $\hat{\Delta}_1(\mathbf{k}, \mathbf{r}) + \hat{\Delta}_2(\mathbf{k}, \mathbf{r})$. To classify the possible ways of symmetry breaking in the vortex lattice $\hat{\Delta}_1(\mathbf{k}, \mathbf{r})$ one should (i) find all irreducible representations of the group $G_\Gamma$, (ii) decompose the new order parameter $\hat{\Delta}_2(\mathbf{k}, \mathbf{r})$ over the basis functions of a particular representation, (iii) write the corresponding Landau functional for chosen representation, and (iv) determine all its minimums with their residual symmetry groups $\tilde{G}$.

We do not intend to develop here this general approach. Staying on the point of view of the above Subsection, we only show how this symmetry analysis must be applied in the particular case of the energy functional (95).

Previously we have shown that the appearance of the phase transition in the vortex lattice at $H = H^*(T)$ should be connected to some eigenlevel of (94). The quantum numbers $(N_2, \sigma_2)$ of $\hat{\Delta}_2(\mathbf{k}, \mathbf{r})$ are different from quantum numbers $(N_1, \sigma_1)$ of $\hat{\Delta}_1(\mathbf{k}, \mathbf{r})$. Therefore, these solutions are orthogonal: $\langle \hat{\Delta}_1^* \hat{\Delta}_2 \rangle = 0$. Moreover, the existence of a structural phase transition at $H = H^*(T)$ on a nonunit irreducible representation of the group $G_\Gamma$ also means that $\langle |\hat{\Delta}_1|^2 \hat{\Delta}_1^* \hat{\Delta}_2 \rangle = 0$. Otherwise nuclei of $\hat{\Delta}_2$ would arise at $H = H_{c2}(T)$ simultaneously with the vortex lattice $\hat{\Delta}_1$, as is the case for all Landau level functions with $n \pmod{6} = 0$ in conventional superconductors $^{35}$.

Irreducible representations of two-dimensional space group $G_\Gamma$ are characterized by the wave vector $\mathbf{q}$ lying in the Brillouin zone of the vortex lattice. To obtain a set of corresponding Bloch functions from those belonging to the Landau level $N_2$, we should construct one-quantum lattices $\hat{\Delta}_2$ of the same form as $\hat{\Delta}_1$. Then, as we have discussed in Sec. 4.4, there exists one-to-one correspondence between the displacement of periodical solutions $\hat{\Delta}_2$ in the unit cell of $\hat{\Delta}_1$ and the wave vector $\mathbf{q}$ in the Brillouin zone.

According to the Lifshitz criteria, only symmetrical points of the Brillouin zone are important for the structural phase transitions in a crystal lattice $^{34}$. For the regular triangular lattice these are the center of the hexagon, its vertex, and the center of the hexagon side. These points of the Brillouin zone correspond to the following displacements: $\rho_\Gamma = 0$; $\rho_{K,K'} = \pm (a + b)/3$; $\rho_{M,M',M''} = a/2, b/2, (a - b)/2$. The existence of equivalent displacements is a consequence of rotational symmetry around $\Gamma$. In the case of a distorted triangular lattice, e.g. when $\mathbf{H} \perp \mathbf{c}$, this degeneracy is lifted.

The shape of the stable lattice structure $\hat{\Delta}_1 + \hat{\Delta}_2$ on the transition line $H^*(T)$ is determined by interaction in the quartic terms: $\langle |\hat{\Delta}_1|^2 |\hat{\Delta}_2|^2 \rangle$ and, if it exists, $\langle \hat{\Delta}_1^2 \hat{\Delta}_2^2 + \text{c.c.} \rangle$. Depending on the sign of these interactions, all relative locations between $\hat{\Delta}_1$ and $\hat{\Delta}_2$: $\Gamma-\Gamma$, $\Gamma-K$, $\Gamma-M$ are possible.

The residual symmetry group of the mixed state at $H < H^*$ is determined by

$$\tilde{G} = G_\Gamma^{N_1} \cap G_\Phi^{N_2}$$

with appropriate $Q = \Gamma, K$ or $M$. 36
If the vortices of the new lattice coincide with those of $\hat{\Delta}_1(k, r)$, only rotational symmetry is broken below $H^\ast$. This implies the distortion of the initial hexagonal lattice.

If new vortices appear between the old ones in $M$ position, the operator $e^{-i\varphi_2}\hat{T}_b$ transforms $\hat{\Delta}_1^{(M)} + \hat{\Delta}_2^{(M)}$ into $\hat{\Delta}_1^{(I)} - \hat{\Delta}_2^{(M)}$ (see (88) and (90)). In this case the unit cell of the whole lattice is two times larger than separate cells of lattices $\hat{\Delta}_1$ and $\hat{\Delta}_2$. It follows from the same arguments that $\hat{\Delta}_1^{(I)} + \hat{\Delta}_2^{(K)}$ carries three quanta of the magnetic flux per unit cell. Such period multiplications can be either complete or partial when elementary translations $\hat{T}_a$ or $\hat{T}_b$ belong to $\hat{G}$ in combination with proper rotations or reflections.

### 5.3 Examples

#### 5.3.1 Closeness of two eigenfields

For $s$-wave superconductors the upper critical field always corresponds to the zero Landau level. The first Landau level has a critical field three times as small. For multicomponent superconductor it is possible situation when eigenfields of different phases lie sufficiently close to each other. This leads to structural phase transitions inside the mixed state.

To illustrate this case, we use the example of the two-component superconducting order parameter. As we have seen in Sec. 3.1, depending on the constants $C$ and $D$ two different phases can arise near $H_{c2}$ for $H \parallel c$. For example, if $\lambda_{SK} < \lambda_a$, the SK-phase with the generalized Landau level number $N_1 = +1$ appears at $H_{c2} = H_{SK}$. If, however, $\lambda_a$ is slightly larger than $\lambda_{SK}$, the critical field $H_a$ for the axial phase with $N_2 = -1$ is close to $H_{c2}$. When the regular triangular lattice is favorable for SK-phase, the nuclei of axial phase do not appear simultaneously with SK-phase at $H_{c2}$ because triangular lattices for phases with different $N$ are invariant under different hexagonal groups $G^N_1$ (88).

To investigate possible phase transitions in the vicinity of $H_{c2}$ one should use energy functional (D5) with $\alpha_1 = \alpha_2$, nonlinear terms (77) and (78), and the interaction term of the form ($\kappa \gg 1$):

$$\langle \hat{\Delta}_2^2 \hat{\Delta}_{SK}^2 \rangle = 2\beta_1\omega^2(|f_0|^2|f_{SK}^2|^2) + (\beta_1 + 2\beta_2)(|f_0|^2|f_{SK}^2|^2).$$

(97)

According to the results of [82], the instability at $H = H^\ast$ (minimum of (77)) corresponds to the appearance of $\Gamma-K$ structure if $\beta_2/\beta_1 > -0.5 - 0.12 \omega^2$ with the residual symmetry group

$$\tilde{G} = \{e^{i\pi/2}\hat{T}_{a+b}, \ e^{-i\pi/2}\hat{T}_{a-b}, \ e^{-2i\pi/3}\hat{L}_{2\pi/3}^{(K)}, \ \sigma\hat{\sigma}_h, \ RU_{2y}\}.$$ (98)

If $-0.5 - 0.12 \omega^2 > \beta_2/\beta_1 > -0.5 - 0.35 \omega^2$ new vortices are located at $M$ positions, and

$$\tilde{G} = \{e^{i\pi}\hat{T}_a, \ e^{i\pi}\hat{T}_b, \ \sigma\hat{\sigma}_h, \ RU_{2y}\}.$$ (99)

If $-0.5 - 0.41 \omega^2 > \beta_2/\beta_1 > -1$ two lattices $\Delta_{SK}$ and $\Delta_a$ are not displaced. Their symmetry group is

$$\tilde{G} = \{e^{i\pi}\hat{T}_a, \ e^{i\pi/2}\hat{T}_b, \ \hat{L}_\pi, \ \sigma\hat{\sigma}_h, \ RU_{2y}\}.$$ (100)
For parameters $-0.35 \omega^2 > \beta_2/\beta_1 + 0.5 > -0.41 \omega^2$ the minimum is achieved for nonsymmetric displacements between $M$ and $\Gamma$ points.

One can see that the period multiplication in the first two cases is complete. That is, e.g., the diamagnetic field of superconducting currents $h_s$ is periodical only under translations from (98) and (99).

The phase transition at $H = H^*$ leads to the phase diagram of the two-component GL functional (1) presented schematically in Fig. 4.

For $H \perp c$ there are also two phases (51) and (52) with close values of critical fields whenever $C$ is small. Since these phases differ in the quantum number $\sigma$ and in the parity of $N$, their mixing is prevented at $H_{c2}$. Possible structural phase transitions in this case were partially considered in [26,82]. The complete analysis over the entire region of phenomenological parameters must take into account the possibility of the appearance of phases modulated along $H$ (see Sec. 3.5), and has not been done yet. Note that the available experimental data on the vortex lattice restructuring at $H = H^*(T)$ in UPt$_3$ correspond to this orientation of magnetic field relative to the crystal axes [30].

The example in this Subsection shows that for any pairing type (even in an anisotropic crystal with the magnetic field directed along the rotational axis) in some range of coefficients of energy functional on the $H$–$T$ phase diagram there exist a phase transition line at $H_{c1}(T) < H^*(T) < H_{c2}(T)$. How large is this region is the question of numerical calculations.

5.3.2 Intersection of critical fields belonging to the different pairing types

Let us assume that at $H = 0$ two different superconducting states with close critical temperatures $T_a$ and $T_b$ ($T_a > T_b$) do occur. Phases $a$ and $b$ may belong to the different pairing types ($s + p$ or $s + d$) in an isotropic metal or to the two irreducible representations of the point group $G$ in anisotropic crystal. Here we will not discuss the physical reasons for this accidental closeness of $T_a$ and $T_b$ but restrict ourselves to the question: what is the $H$–$T$ phase diagram of the system in this case?

The coexistence of two superconducting phases in magnetic field is described by some GL energy functional:

$$ F = F_a\{\hat{\Delta}_a\} + F_b\{\hat{\Delta}_b\} + F_{ab}\{\hat{\Delta}_a, \hat{\Delta}_b\}, \quad (101) $$

which takes the form of (13) near the upper critical field. The solution of the problem of $H$–$T$ phase diagram is trivial if there is no interaction $F_{ab}\{\hat{\Delta}_a, \hat{\Delta}_b\}$: it will be the sum of two independent diagrams for the phases $\hat{\Delta}_a$, $\hat{\Delta}_b$ with the upper critical fields $H_a$, $H_b$, and under the condition $dH_a/dT > dH_b/dT$ it will resemble the phase diagram of UPt$_3$ (Fig. 11).

Being described by different irreducible representations of the group $G$ at $H = 0$, the phases $\hat{\Delta}_a$, $\hat{\Delta}_b$ can also belong to the eigensolutions of linearized equations of a different symmetry. Following Sec. 3.2, we should attribute the different quantum numbers $(N_a, \sigma_a)$, $(N_b, \sigma_b)$ to the critical fields $H_a$, $H_b$. The interaction term $F_{ab}\{\hat{\Delta}_a, \hat{\Delta}_b\}$ does not change the symmetry classification of the eigensolutions of the linear problem. It leads to the renormalization of $H_a(T)$ and $H_b(T)$ curves.
inside the mixed state but does not smear the kink in $H_{c2}(T) = \max\{H_a(T), H_b(T)\}$. Phase diagrams for different combinations of irreducible representations for $\hat{\Delta}_a$ and $\hat{\Delta}_b$ based on Landau expansion near $H_{c2}$ have been partially considered in $^{28,19,83}$. Note, that without mixing of $\Delta_a$ and $\Delta_b$ in the higher order gradient terms the model $^{101}$ exhibit only partial breaking of translational symmetry below $H^*(T)$.

The field intersection is also possible in the above example of 2D order parameter. The small uniaxial anisotropy in the basal plane $(|\eta_x|^2 - |\eta_y|^2)$ splits $T_c$ into: $T_c^+$ for $\Phi_y(k)$ phase and $T_c^-$ for $\Phi_x(k)$ phase. If the magnetic field is directed along $\hat{x}$, there exist an intersection of two critical fields: $H^y(T) \sim (T_c^+ - T)/(K_{123}K_4)^{1/2}$ and $H^x(T) \sim (T_c^- - T)/(K_1K_4)^{1/2}$. In fact it is the first model proposed for the phase diagram of UPt$_3$ $^{24,45}$.

Concluding this Section, we want to emphasize that the quantum number technique is very important for establishing the nature of the phase transition at $H^*(T)$ as well as for investigation of the stability of the kink. The exact sequence of the phase transitions for arbitrary strength of the interaction term in the energy functional $^{13}$ is still unknown. By numerical calculations in $^{48}$ it was shown the possibility of a phase transition of the first order for large strength of interaction, which occurs before the discussed second order structural transitions.

To summarize, we have reviewed the equilibrium properties of the mixed state of superconductors with the multicomponent order parameter. The main feature of unconventional pairing is the reduced symmetry of the superconducting state in comparison to the normal one. This results in a large variety of unusual properties such as power law temperature dependencies of thermodynamic characteristics, domain structure due to the presence of degenerate states, the angular dependence of the Josephson current, and others, which were considered in details in previous reviews $^{22,65}$. But perhaps the most pronounced breakdown of various symmetries occurs when vortices penetrates inside the volume of a multicomponent superconductor. We have considered nonaxisymmetric vortices and flux lattices of different forms. Structural phase transitions in the Abrikosov lattice are the main qualitatively new feature of unconventional superconductivity which, in our opinion, should be primarily used for establishing unusual superconductivity experimentally.

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APPENDIX

The energy parameter (74) for unconventional superconductors is constructed in general case from the repeated blocks which are integrals of periodic functions from different Landau levels. In the case of the SK-phase they are [80]:

\[
\delta_{00}(\rho, \sigma) = \frac{\langle |f_0|^4 \rangle}{\langle |f_0|^2 \rangle^2} = \sqrt{\sigma} \sum_{m,n} \cos(2\pi \rho mn) e^{-\pi \sigma (m^2 + n^2)} ,
\]

(102)

\[
\delta_{02}(\rho, \sigma) = \frac{\langle |f_0|^2 |f_2|^2 \rangle}{\langle |f_0|^2 \rangle \langle |f_2|^2 \rangle} = \frac{\sqrt{\sigma}}{2} \sum_{m,n} \cos(2\pi \rho mn) e^{-\pi \sigma (m^2 + n^2)} \left( \frac{3}{4} - \pi \sigma (m^2 + n^2) + \pi^2 \sigma^2 (m^2 - n^2)^2 \right) ,
\]

(103)

\[
\delta_{22}(\rho, \sigma) = \frac{\langle |f_2|^4 \rangle}{\langle |f_2|^2 \rangle^2} = \frac{\sqrt{\sigma}}{4} \sum_{m,n} \cos(2\pi \rho mn) e^{-\pi \sigma (m^2 + n^2)} \left( \frac{41}{16} - \frac{13}{2} \pi \sigma (m^2 + n^2) + 3\pi^2 \sigma^2 (m^2 + n^2)^2 + \frac{19}{2} \pi^2 \sigma^2 (m^2 - n^2)^2 - 6\pi^3 \sigma^3 (n^2 + m^2)(n^2 - m^2)^2 + \pi^4 \sigma^4 (n^2 - m^2)^4 \right) .
\]

(104)

Behavior of these functions on different lines in the fundamental region is shown in Fig. 12. The corresponding absolute minimums are located at the following points: for \( \delta_{00}(\rho, \sigma) \) at \( \rho = 0.5 \), \( \sigma = \sqrt{3}/2 \), for \( \delta_{02}(\rho, \sigma) \) at \( \rho = 0 \), \( \sigma = 2.6 \), for \( \delta_{22}(\rho, \sigma) \) at \( \rho = 0.5 \), \( \sigma = \sqrt{3}/2 \).

References

[1] A.A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1442 (1957) [Sov. Phys. JETP 5, 1174 (1957)].

[2] S. Adenwalla, S.W. Lin, Z. Zhao et al., Phys. Rev. Lett. 65, 2298 (1990).

[3] N.E. Alekseevsky, A.V. Mitin, V.I. Nizhankovsky et al., Pis’ma Zh. Eksp. Teor. Fiz. 41, 335 (1985) [Sov. Phys. JETP Lett. 41, 410 (1985)].

[4] P.W. Anderson, Phys. Rev. B 30, 4000 (1984).

[5] Yu.S. Barash, A.V. Galaktionov, Zh. Eksp. Teor. Fiz. 101, 1689 (1992) [JETP 74, 904 (1992)].

[6] Yu.S. Barash, A.S. Mel’nikov, Zh. Eksp. Teor. Fiz. 100, 307 (1991) [Sov. Phys. JETP 73, 170 (1991)].

[7] J. Bardeen, L.N. Cooper, and J.R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[8] E.I. Blount, Phys. Rev. B 32, 2935 (1985).
[9] E. Brown, Phys. Rev. 133, A1038 (1964).

[10] K.A. Brueckner, T. Soda, P.W. Anderson, P. Morel, Phys. Rev. 118, 1442 (1960).

[11] L.I. Burlachkov, Zh. Eksp. Teor. Fiz. 89, 1382 (1985) [Sov. Phys. JETP 62, 800 (1985)].

[12] L.I. Burlachkov, N.B. Kopnin, Zh. Eksp. Teor. Fiz. 92, 1110 (1987) [Sov. Phys. JETP 65, 630 (1987)].

[13] A.O. Burut and R. Rączka, Theory of Group Representations and Applications, PWN, Warszawa 1977.

[14] C. Choi and P. Muzikar, Phys. Rev. B 40, 5144 (1989).

[15] C. Choi and J.A. Sauls, Phys. Rev. Lett. 66, 484 (1991), Phys. Rev. B 48, 13684 (1993).

[16] P.G. De Gennes, Superconductivity of Metals and Alloys, Benjamin, New York 1966.

[17] Z. Fisk, D. Hess, C. Pethick et al., Science 239, 33 (1988).

[18] A. Garg, Phys. Rev. Lett. 69, 676 (1992).

[19] A. Garg and D.-C. Chen, Phys. Rev. B 49, 479, (1994).

[20] V.L. Ginzburg and L.D. Landau, Zh. Eksp. Teor. Fiz. 20 1064 (1950).

[21] L.P. Gor’kov, Pis’ma Zh. Eksp. Teor. Fiz. 40, 351 (1984) [Sov. Phys. JETP Lett. 40, 1155 (1984)].

[22] L.P. Gor’kov, Sov. Sci. Rev. A 9, 1 (1987).

[23] N. Grewe and F. Steglich, Heavy Fermions, in “Handbook on Physics and Chemistry of Rare Earths,” vol. 14, Elsevier, Amsterdam 1991.

[24] D.W. Hess, T.A. Tokuyasu, and J.A. Sauls, J. Phys.: Cond. Matt. 1, 8135 (1989).

[25] Yu.A. Izumov, V.M. Laptev, Phase Transit. 20, 95 (1990).

[26] R. Joynt, Europhys. Lett. 16, 289 (1991).

[27] R. Joynt, J. Magn. & Magn. Mater. 108, 31 (1992).

[28] R. Joynt, V.P. Mineev, G.E. Volovik, and M.E. Zhitomirsky, Phys. Rev. B 42, 2014 (1990).

[29] T. Kita, Phys. Rev. B 43, 5343 (1991).

[30] R.N. Kleiman, C. Broholm, G. Aeppli et al., Phys. Rev. Lett. 69, 3120 (1992).

[31] R.A. Klemm and J.R. Clem, Phys. Rev. B 21, 1868 (1980).
[32] W. Kohn, Phys. Rev. 123, 1242 (1961).
[33] L.D. Landau, E.M. Lifshitz, Quantum Mechanics, Pergamon, Oxford 1980.
[34] L.D. Landau, E.M. Lifshitz, Statistical Physics, Part I, Pergamon, Oxford 1980.
[35] G. Lasher, Phys. Rev. 140, A523 (1965).
[36] H. v. Löhneysen, Physica B 197, 551 (1994).
[37] I.A. Luk’yanchuk, Jour. de Phys. I 1, 1155 (1991).
[38] I.A. Luk’yanchuk, V.P. Mineev, Zh. Eksp. Teor. Fiz. 93, 2045 (1987) [Sov. Phys. JETP 66, 1168 (1987)].
[39] I. Luk’yanchuk, M. Sigrist, M. Zhitomirsky, Phys. Rev. Lett. 71, 1957(C) (1993).
[40] I.A. Luk’yanchuk, M.E. Zhitomirsky, Physica C 206, 373 (1993).
[41] S.W. MacDowell and O. Trönkwist, Phys. Rev. D 45, 3833 (1992).
[42] S.W. MacDowell, Nucl. Phys. B 398, 516 (1993).
[43] K. Machida, Prog. Theor. Phys. 108, 229 (1992).
[44] K. Machida, M. Ozaki, and T. Ohmi, J. Phys. Soc. Jpn. 54, 1552 (1985).
[45] K. Machida, M. Ozaki, and T. Ohmi, J. Phys. Soc. Jpn. 58, 4116 (1989).
[46] K. Machida, T. Fujita, and T. Ohmi, J. Phys. Soc. Jpn. 62, 680 (1993).
[47] V.I. Marchenko, Zh. Eksp. Teor. Fiz. 93, 583 (1987) [Sov. Phys. JETP 66, 331 (1987)].
[48] A.S. Mel’nikov, Zh. Eksp. Teor. Fiz. 101, 1978 (1992) [JETP 74, 1059 (1992)].
[49] V.P. Mineev, Sov. Sci. Rev. A 2, 173 (1980).
[50] V.P. Mineev, Ann. de Phys., to be published (1994).
[51] H.R. Ott, in “Progress in Low Temperature Physics” XI, p. 215, North Holland, Amsterdam 1987.
[52] M. Ozaki, K. Machida, and T. Ohmi, Prog. Theor. Phys. 75, 442 (1986).
[53] M. Ozaki and K. Machida, J. Phys. Soc. Jpn. 61, 1277 (1992).
[54] M. Palumbo, P. Muzikar, and J.A. Sauls, Phys. Rev. B 42, 2681 (1990).
[55] M. Palumbo, C. Choi, and P. Muzikar, Physica B 165–166, 1095 (1990).
[56] M. Palumbo and P. Muzikar, Phys. Rev. B 45, 12620 (1992).
[57] L.P. Pitaevsky, Zh. Eksp. Teor. Fiz. 37, 1794 (1959) [Sov. Phys. JETP 10, 1267 (1960)].
[58] C.T. Rieck, Ph. D. Thesis, unpublished, Hamburg, 1991.
[59] D. Saint-James, G. Sarma, E.J. Tomas, Type II Superconductivity, Pergamon, Oxford 1969.
[60] M. Salomaa and G.E. Volovik, Rev. Mod. Phys. 59, 533 (1987).
[61] J.A. Sauls, Adv. Phys., to be published (1994).
[62] K. Scharnberg, R.A. Klemm, Phys. Rev. B 22, 5233 (1980).
[63] M. Sigrist, N. Ogawa, and K. Ueda, J. Phys. Soc. Jpn. 60, 2341 (1991).
[64] M. Sigrist, T.M. Rice, and K. Ueda, Phys. Rev. Lett. 63, 1727 (1989).
[65] M. Sigrist and K. Ueda, Rev. Mod. Phys. 63, 239 (1991).
[66] G.R. Stewart, Rev. Mod. Phys. 56, 755 (1984).
[67] S.K. Sundaram and R. Joynt, Phys. Rev. B 40, 8780 (1989).
[68] L. Taillefer, J. Flouquet, G.G. Lonzarich, Physica B 169, 257 (1991).
[69] P. Thalmeier, B. Wolf, D. Weber et al., Physica C 175, 61 (1991).
[70] T.A. Tokuyasu, D.W. Hess, and J.A. Sauls, Phys. Rev. B 41, 8891 (1990).
[71] T.A. Tokuyasu and J.A. Sauls, Physica B 165–166, 347 (1990).
[72] K. Ueda and T.M. Rice, Phys. Rev. B 31, 7144 (1985).
[73] D. Vollhardt and P. Wölfle, The Superfluid Phases of $^3$He, Taylor & Francis, New York, 1990.
[74] G.E. Volovik, J. Phys. C 21, L221 (1987).
[75] G.E. Volovik, L.P. Gor’kov, Zh. Eksp. Teor. Fiz. 88, 1412 (1985) [Sov. Phys. JETP 61, 843 (1985)].
[76] G.E. Volovik, V.P. Mineev, Zh. Eksp. Teor. Fiz. 81, 989 (1981) [Sov. Phys. JETP 54, 524 (1981)].
[77] S.K. Yip, Phys. Rev. B 40, 3682 (1989).
[78] J. Zak, Phys. Rev. 134, A1602, A1607 (1964).
[79] M.E. Zhitomirsky, Pis’ma Zh. Eksp. Teor. Fiz. 49, 333 (1989) [Sov. Phys. JETP Lett. 49, 379 (1989)].
[80] M.E. Zhitomirsky, Zh. Eksp. Teor. Fiz. 97, 1346 (1990) [Sov. Phys. JETP 70, 760 (1990)].

[81] M.E. Zhitomirsky, Pis’ma Zh. Eksp. Teor. Fiz. 55, 460 (1992) [JETP Lett. 55, 472 (1992)].

[82] M.E. Zhitomirsky, I.A. Luk’yanchuk, Zh. Eksp. Teor. Fiz. 101, 1954 (1992) [JETP 74, 1046 (1992)].

[83] M.E. Zhitomirsky, I.A. Luk’yanchuk, Pis’ma Zh. Eksp. Teor. Fiz. 58, 127 (1993) [JETP Lett. 58, 131 (1993)].
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Fig. 1. Phase diagram of superconducting states in UPt₃ for (a) $\mathbf{H} \parallel \mathbf{c}$ and (b) $\mathbf{H} \perp \mathbf{c}$ (after [2]).

Fig. 2. Contour plot of the order parameter modulus for nonaxisymmetric vortices in the two-component model with the GL parameters $\beta_2 = 0.1\beta_1$ and $K_1 = K_2 = K_3$; (a) triangular vortex for $\tilde{\Delta} \sim (\tilde{\Phi}_1 - i\tilde{\Phi}_2)e^{-i\varphi}$, (b) “crescent” vortex for $\tilde{\Delta} \sim (\tilde{\Phi}_1 + i\tilde{\Phi}_2)e^{-i\varphi}$ (after [70]).

Fig. 3. Phase diagram of axisymmetry instability at the vortex core in the two-component model (after [70]).

Fig. 4. The eigen critical fields and schematic $H$–$T$ phase diagram for $\mathbf{H} \parallel \mathbf{c}$ in the two-component model when $D < C^2/(1 + C)$. For $D > C^2/(1 + C)$ the order of eigenfields is reversed.

Fig. 5. Outplane anisotropy of $H_{c2}(\vartheta)$ in uniaxial $s$-wave superconductors; $K_{\parallel} : K_{\perp} = (a) 1 : 2$, (b) $2 : 1$, (c) $4 : 1$.

Fig. 6. Anisotropy of $H_{c2}(\varphi)$ in the basal plane for $E_1$ irreducible representation of $D_4$ group, $K_5$ is the coefficient at the “tetragonal” invariant in gradient terms $[11]$ and $K_2 = K_3 = K_1$; $K_1 : K_5 = (a) 2 : 1$, (b) $1 : 1$, (c) $1 : 2$.

Fig. 7. Angular dependence of $H_{c2}(\vartheta)$ between $\mathbf{c}$-axis and the basal plane in the two-component model (5), $\varepsilon = C/(1 + C)$, $K = K_4/K_1$; (a) $\varepsilon = 0.3$, $K = 0.8$, (b) $\varepsilon = 0.3$, $K = 1$, (c) $\varepsilon = 0.3$, $K = 1.2$, (d) $\varepsilon = 0.2$, $K = 1$.

Fig. 8. The fundamental region of the $SL(2,\mathbb{Z}) \times \mathbb{Z}_2$ group in the complex plane of the parameter $\tau = \rho + i\sigma$.

Fig. 9. Phase diagram of the vortex lattice form near $H_{c2}$ for SK-phase. Region I corresponds to the regular triangular lattice, region II — to the square lattice, region III — to the rectangular lattice, regions IV and V — to the distorted triangular lattices with $45^\circ < \alpha < 60^\circ$ and $\alpha > 60^\circ$ ($\alpha < 30^\circ$) respectively. Point corresponds to the weak-coupling parameters. Details of the phase boundaries intersections are shown on (b).

Fig. 10. Symmetry elements of the triangular vortex lattice.

Fig. 11. Schematic $H$–$T$ phase diagram of the accidental degeneracy model.

Fig. 12. Behavior of functions (a) $\delta_{00}(\rho, \sigma)$, (b) $\delta_{02}(\rho, \sigma)$, (c) $\delta_{22}(\rho, \sigma)$ on lines $\rho = 0$ and $\rho = 0.5$ in the fundamental region.