Undecidable properties of self-affine sets and multi-tape automata

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Joint work with Jarkko Kari

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- **Self-affine sets**: “fractals” (object of study)
- **Multi-tape automata**: computational point of view (tool)
Example 1

- $f_1 : x \mapsto \frac{1}{2}x$
- $f_2 : x \mapsto \frac{1}{2}x + \left(\frac{1}{2}\right)$
- $f_3 : x \mapsto \frac{1}{2}x + \left(\frac{0}{1/2}\right)$
Example 1

- \( f_1 : x \mapsto \frac{1}{2}x \)
- \( f_2 : x \mapsto \frac{1}{2}x + \left( \frac{1}{0/2} \right) \)
- \( f_3 : x \mapsto \frac{1}{2}x + \left( \frac{0}{1/2} \right) \)

\[
X = f_1(X) \cup f_2(X) \cup f_3(X) \subseteq \mathbb{R}^2
\]
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1. \( f_1 : x \mapsto \frac{1}{2} x \)
2. \( f_2 : x \mapsto \frac{1}{2} x + \left( \frac{1}{0} \right) \)
3. \( f_3 : x \mapsto \frac{1}{2} x + \left( \frac{0}{1/2} \right) \)

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- $f_1: x \mapsto \frac{1}{2} x$
- $f_2: x \mapsto \frac{1}{2} x + \left(\frac{1}{0}\right)\frac{1}{2}$
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\[X = f_1(X) \cup f_2(X) \cup f_3(X) \subseteq \mathbb{R}^2\]
Example 1

- $f_1: x \mapsto \frac{1}{2} x$
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- $f_3: x \mapsto \frac{1}{2} x + \left( \begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix} \right)$

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\[ X = f_1(X) \cup f_2(X) \cup f_3(X) \subseteq \mathbb{R}^2 \]

**Theorem** (Hutchinson 1981)

Let $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ be contracting maps.
There is a unique nonempty compact set $X \subseteq \mathbb{R}^d$ such that

\[ X = f_1(X) \cup \cdots \cup f_n(X). \]

**Definition:** $f_1, \ldots, f_n$ is an iterated function system (IFS)
Affine iterated function systems

- Restrict to affine maps: \( f_i : x \mapsto M_i x + v_i \)
Affine iterated function systems

- Restrict to affine maps: \( f_i : x \mapsto M_i x + v_i \)

Interesting topological objects:
- Connectedness, homeomorphism to a disk, ...
- Nonempty interior
- Tiling properties
- Lebesgue measure, fractal dimensions

Very few criteria on the \( f_i \) are known for the above properties.
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- Interesting topological objects:
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We give “negative” answers to the nonempty interior question.
Example 2

- $f_1 : x \mapsto \frac{1}{2} x$
- $f_2 : x \mapsto \frac{1}{2} x + \left( \begin{array}{c} 1/2 \\ 0 \end{array} \right)$
- $f_3 : x \mapsto \frac{1}{2} x + \left( \begin{array}{c} 0 \\ 1/2 \end{array} \right)$
- $f_4 : x \mapsto \frac{1}{2} x + \left( \begin{array}{c} -1/2 \\ -1/2 \end{array} \right)$
Example 2

- $f_1 : x \mapsto \frac{1}{2} x$
- $f_2 : x \mapsto \frac{1}{2} x + \left( \frac{1}{2} \right)$
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- $f_4 : x \mapsto \frac{1}{2} x + \left( -\frac{1}{2} \right)$
Example 2

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- $f_2 : x \mapsto \frac{1}{2}x + \left( \frac{1}{0^2} \right)$
- $f_3 : x \mapsto \frac{1}{2}x + \left( \frac{0}{1/2} \right)$
- $f_4 : x \mapsto \frac{1}{2}x + \left( \frac{-1}{-1/2} \right)$
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Example 2

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Empty vs nonempty interior

Example 1:
Empty interior because \(4\mu(X) = \mu(X) + \mu(X) + \mu(X) = 3\mu(X)\)
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Example 2:
Nonempty interior (tiling properties)
Empty vs nonempty interior

Example 1: Empty interior because $4\mu(X) = \mu(X) + \mu(X) + \mu(X) = 3\mu(X)$

Example 2: Nonempty interior (tiling properties)

Decision problem: given rational affine maps $f_i$, does the fractal have empty interior?
Empty vs nonempty interior

Example 1: Empty interior because $4\mu(X) = \mu(X) + \mu(X) + \mu(X) = 3\mu(X)$

Example 2: Nonempty interior (tiling properties)

Decision problem: given rational affine maps $f_i$, does the fractal have empty interior?

- There is an algorithm [Bondarenko-Kravchenko 2011] if the affine $f_i$ all have the same contraction matrix (and verify a few other conditions)
Empty vs nonempty interior

Example 1: Empty interior because $4\mu(X) = \mu(X) + \mu(X) + \mu(X) = 3\mu(X)$

Example 2: Nonempty interior (tiling properties)

Decision problem: given rational affine maps $f_i$, does the fractal have empty interior?

- There is an algorithm [Bondarenko-Kravchenko 2011] if the affine $f_i$ all have the same contraction matrix (and verify a few other conditions)
- Almost nothing is known otherwise (even particular instances are difficult)
- Let’s now go towards undecidability
Graph-IFS (GIFS)

\[
\begin{align*}
    f_1 : x & \mapsto \frac{1}{2} x \\
    f_2 : x & \mapsto \frac{1}{2} x + \left( \begin{array}{c} 1/2 \\ 0 \end{array} \right) \\
    f_3 : x & \mapsto \frac{1}{2} x + \left( \begin{array}{c} 0 \\ 1/2 \end{array} \right)
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\[ f_4 : x \mapsto \frac{1}{2} x + \left( \frac{1}{2} \right) \]
\[ f_5 : x \mapsto \frac{1}{2} x + \left( \frac{1/2}{3/4} \right) \]
\[ f_6 : x \mapsto \frac{1}{2} x + \left( \frac{0}{3/4} \right) \]

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  f_2 : x & \mapsto \frac{1}{2} x + (\frac{1}{2} \ 0) \\
  f_3 : x & \mapsto \frac{1}{2} x + (0 \ 1/2) \\
  f_4 : x & \mapsto \frac{1}{2} x + (\frac{1}{2} \ 1/2) \\
  f_5 : x & \mapsto \frac{1}{2} x + (\frac{1}{2} \ 3/4) \\
  f_6 : x & \mapsto \frac{1}{2} x + (\frac{3}{4} \ 3/4)
\end{align*}
\]

\[
\begin{aligned}
  X &= f_1(X) \cup f_2(X) \cup f_3(X) \cup f_4(Y) \\
  Y &= f_5(Y) \cup f_6(X)
\end{aligned}
\]
Graph-IFS (GIFS)

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\begin{align*}
    f_1 : x & \mapsto \frac{1}{2} x \\
    f_2 : x & \mapsto \frac{1}{2} x + \left( \frac{1}{2} \right) \\
    f_3 : x & \mapsto \frac{1}{2} x + \left( 0 \right) \\
    f_4 : x & \mapsto \frac{1}{2} x + \left( \frac{1}{2} \right) \\
    f_5 : x & \mapsto \frac{1}{2} x + \left( \frac{1}{3} \right) \\
    f_6 : x & \mapsto \frac{1}{2} x + \left( \frac{3}{4} \right)
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Graph-IFS (GIFS)

\[ f_1 : x \mapsto \frac{1}{2}x \quad f_4 : x \mapsto \frac{1}{2}x + \left( \frac{1}{2} \right) \]
\[ f_2 : x \mapsto \frac{1}{2}x + \left( \frac{1}{2} \right) \quad f_5 : x \mapsto \frac{1}{2}x + \left( \frac{1}{3} \right) \]
\[ f_3 : x \mapsto \frac{1}{2}x + \left( \frac{1}{2} \right) \quad f_6 : x \mapsto \frac{1}{2}x + \left( \frac{3}{4} \right) \]

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\begin{cases}
X = f_1(X) \cup f_2(X) \cup f_3(X) \cup f_4(Y) \\
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\end{cases}
\]
Graph-IFS (GIFS)

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\begin{align*}
&f_1: x \mapsto \frac{1}{2} x & f_4: x \mapsto \frac{1}{2} x + \left(\frac{1}{2}\right) \\
&f_2: x \mapsto \frac{1}{2} x + \left(\frac{1}{2}\right) & f_5: x \mapsto \frac{1}{2} x + \left(\frac{1}{2}\right) \\
&f_3: x \mapsto \frac{1}{2} x + \left(\frac{0}{1/2}\right) & f_6: x \mapsto \frac{1}{2} x + \left(\frac{3/4}{3/4}\right)
\end{align*}
\]

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\begin{cases}
X = f_1(X) \cup f_2(X) \cup f_3(X) \cup f_4(Y) \\
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    f_4 : x &\mapsto \frac{1}{2} x + \left( \frac{1/2}{1/2} \right) \\
    f_5 : x &\mapsto \frac{1}{2} x + \left( \frac{1/2}{3/4} \right) \\
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\end{align*}
\]
Computational tools: multi-tape automata
Multi-tape automata

\(d\)-tape automaton:
- alphabet \( \mathcal{A} = A_1 \times \cdots \times A_d \)
- states \( Q \)
- transitions \( Q \times (A_1^+ \times \cdots \times A_d^+) \rightarrow Q \)

\[
\begin{align*}
000 & | 11 \\
01 & | 00 \\
01 & | 001 \\
1 & | 0 \\
1 & | 001
\end{align*}
\]

\( \mathcal{A} = \{0, 1\} \times \{0, 1\} \)
\( Q = \{X, Y\} \)
Multi-tape automata

\(d\)-tape automaton:

- alphabet \( A = A_1 \times \cdots \times A_d \)
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\[ A = \{0, 1\} \times \{0, 1\} \]
\[ Q = \{X, Y\} \]

Accepted infinite word starting from \( X \):
Multi-tape automata

\textit{d}-tape automaton:

\begin{itemize}
  \item alphabet \( A = A_1 \times \cdots \times A_d \)
  \item states \( Q \)
  \item transitions \( Q \times (A_1^+ \times \cdots \times A_d^+) \rightarrow Q \)
\end{itemize}

\[ A = \{0, 1\} \times \{0, 1\} \]

\[ Q = \{X, Y\} \]

Accepted infinite word starting from \( X \):

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\begin{align*}
000 & \mid 11 \\
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Multi-tape automata

\[ d \]-tape automaton:

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- states \( \mathcal{Q} \)
- transitions \( \mathcal{Q} \times (A_1^+ \times \cdots \times A_d^+) \rightarrow \mathcal{Q} \)

\[ \mathcal{A} = \{0, 1\} \times \{0, 1\} \]

\[ \mathcal{Q} = \{X, Y\} \]

Accepted infinite word starting from \( X \):

\[
\begin{align*}
0001 \\
110
\end{align*}
\]
Multi-tape automata

\(d\)-tape automaton:

- alphabet \(A = A_1 \times \cdots \times A_d\)
- states \(Q\)
- transitions \(Q \times (A_1^+ \times \cdots \times A_d^+) \rightarrow Q\)

\(X\)

\(Y\)

\(\mathcal{A} = \{0, 1\} \times \{0, 1\}\)

\(Q = \{X, Y\}\)

Accepted infinite word starting from \(X\):

000101
11000
Multi-tape automata

d-tape automaton:
- alphabet \( \mathcal{A} = A_1 \times \cdots \times A_d \)
- states \( Q \)
- transitions \( Q \times (A_1^+ \times \cdots \times A_d^+) \to Q \)

Accepted infinite word starting from \( X \):

000101000
1100011
Multi-tape automata

\(d\)-tape automaton:

- alphabet \(A = A_1 \times \cdots \times A_d\)
- states \(Q\)
- transitions \(Q \times (A_1^+ \times \cdots \times A_d^+) \rightarrow Q\)

\[A = \{0, 1\} \times \{0, 1\}\]
\[Q = \{X, Y\}\]

Accepted infinite word starting from \(X\):

0001010001
11000110
Multi-tape automata

\textbf{\textit{d}-tape automaton:}

\begin{itemize}
  \item alphabet \( \mathcal{A} = A_1 \times \cdots \times A_d \)
  \item states \( Q \)
  \item transitions \( Q \times (A_1^+ \times \cdots \times A_d^+) \rightarrow Q \)
\end{itemize}

\[ \mathcal{A} = \{0, 1\} \times \{0, 1\} \]
\[ Q = \{X, Y\} \]

Accepted infinite word starting from \( X \):

\begin{align*}
  &0001010001000 \\
  &1100011011
\end{align*}
Multi-tape automata

d-tape automaton:

- alphabet $\mathcal{A} = A_1 \times \cdots \times A_d$
- states $Q$
- transitions $Q \times (A_1^+ \times \cdots \times A_d^+) \rightarrow Q$

Accepted infinite word starting from $X$:

$$0001010001000 \ldots \in \mathcal{A}^\infty = (\{0, 1\} \times \{0, 1\})^\infty$$
Multi-tape automaton $\mapsto$ GIFS

Transition $u \mid v$ (tape alphabets $A_1, A_2$)
Transition $u|v$ (tape alphabets $A_1, A_2$)

Mapping $f(x, y) = (|A_1|^{|u|} 0 \ 0 |A_2|^{|v|} 0 \ 0.01 \ldots u|u|)(x, y) + (0.01 \ldots v|v|)$
Multi-tape automaton \(\mapsto\) GIFS

Transition \(u|v\)

Mapping \(f(x) = \begin{pmatrix} |A_1| & -|u| \\ 0 & |A_2| & -|v| \end{pmatrix} (x) + \begin{pmatrix} 0.u_1 \ldots u_{|u|} \\ 0.v_1 \ldots v_{|v|} \end{pmatrix}\)

Automaton:

\[
\begin{array}{c}
01 | 00 \\
\rightarrow X \\
000 | 11 \\
\rightarrow Y \\
1 | 001 \\
\rightarrow 1 | 0 \\
\end{array}
\]
Multi-tape automaton $\mapsto$ GIFS

Transition $u \mid v$ (tape alphabets $A_1$, $A_2$)

Mapping $f(x) = \begin{pmatrix} |A_1| - |u| & 0 \\ 0 & |A_2| - |v| \end{pmatrix} (x, y) + \begin{pmatrix} 0.u_1 \ldots u_{|u|} \\ 0.v_1 \ldots v_{|v|} \end{pmatrix}$

Automaton:

Associated GIFS:

$\left( \frac{1}{8} 0 \right) (x, y) + \left( 0.0000 \right)$

$\left( \frac{1}{4} 0 \right) (x, y) + \left( 0.01 \right)$

$\left( \frac{1}{2} 0 \right) (x, y) + \left( 0.1 \right)$
### Multi-tape automaton $\mapsto$ GIFS

**Dictionary:**

| Automaton | GIFS fractal |
|-----------|--------------|
| states    | GIFS fractals|
| edges     | GIFS mappings|
| #tapes    | dimension of fractals |
| alphabet $A_i$ | base-|$|A_i$| representation of $i$th coordinate |
Multi-tape automaton \[\rightarrow\] GIFS

\[
\begin{align*}
1011 | 11
\end{align*}
\]

\[\begin{align*}
f(x, y) &= \begin{pmatrix} 1/16 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0.x_1x_2 \ldots \\ 0.y_1y_2 \ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}
\end{align*}\]
Multi-tape automaton $\implies$ GIFS

$1011 \mid 11$

$f(x, y) = \begin{pmatrix} 1/16 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0.x_1 x_2 \ldots \\ 0.y_1 y_2 \ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$

$= \begin{pmatrix} 0.0000x_1 x_2 \ldots \\ 0.00y_1 y_2 \ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix}$
Multi-tape automaton  $\xrightarrow{\text{GIFS}}$

- $\begin{array}{c}
1011 | 11 \\
↓
\end{array}$

- $f(x, y) = \left( \begin{array}{cc}
\frac{1}{16} & 0 \\
0 & \frac{1}{4}
\end{array} \right) \left( \begin{array}{c}
0.x_1x_2\ldots \\
0.y_1y_2\ldots
\end{array} \right) + \left( \begin{array}{c}
0.1011 \\
0.11
\end{array} \right)$

- $= \left( \begin{array}{c}
0.0000x_1x_2\ldots \\
0.00y_1y_2\ldots
\end{array} \right) + \left( \begin{array}{c}
0.1011 \\
0.11
\end{array} \right)$

- $= \left( \begin{array}{c}
0.1011x_1x_2\ldots \\
0.11y_1y_2\ldots
\end{array} \right)$
Multi-tape automaton $\mapsto$ GIFS

1011 | 11

$\bullet \quad \rightarrow \quad \bullet$

$f(x, y) = (\frac{1}{16} \ 0)(0.x_1x_2\ldots) + (0.1011)
= (0.0000x_1x_2\ldots) + (0.1011)
= (0.1011x_1x_2\ldots)
= (0.11y_1y_2\ldots)$

Key correspondence

GIFS fractal associated with automaton $\mathcal{M}$

$$\{ (0.x_1x_2\ldots) \in \mathbb{R}^2 : (x_1x_2\ldots) \text{ accepted by } \mathcal{M} \}$$
Multi-tape automaton $\mapsto$ GIFS

Example:

\[
\begin{align*}
&0|0 \\
&1|0 \quad 0|1 \\
\end{align*}
\]

Automaton

\[
\begin{align*}
&\frac{1}{2}(x) + (0) \\
&\frac{1}{2}(y) + (0_{1/2}) \quad \frac{1}{2}(y) + (1/2)
\end{align*}
\]

GIFS

\[
= \left\{ \left( \begin{array}{c} 0.x_1x_2 \cdots \\ 0.y_1y_2 \cdots \end{array} \right) : (x_n, y_n) \neq (1, 1), \ \forall n \geq 1 \right\}
\]
Multi-tape automaton ➞ GIFS

Example:

\[0|0 \quad 10|10
0|1 \quad X \quad Y
1|0 \quad 11|11
10|11\]
Main idea

Language theoretical properties of accepted language

Topological properties of fractal set
Main idea

Language theoretical properties of accepted language $\iff$ Topological properties of fractal set

Example (in 2D) [Dube 1993, original idea]

Automaton accepts one word of the form $\left(\begin{array}{c} 0.x_1x_2\ldots \\ 0.x_1x_2\ldots \end{array}\right)$ $\iff$ $X$ intersects the diagonal $\{(x, x) : x \in [0, 1]\}$
Language universality $\iff$ nonempty interior

**Fact 1:** $\mathcal{M}$ is universal $\iff X = [0, 1]^d$
Language universality \iff nonempty interior

Fact 1: $\mathcal{M}$ is universal \iff $X = [0, 1]^d$

- Example (universal): one state, transitions $0|0$, $0|1$, $1|0$, $1|1$

- Example (universal with prefix $1$ but not universal): one state, transitions $1$, $10$, $00$ (one-dimensional)

$f_1(x) = \frac{x}{2} + \frac{1}{2}$

$f_2(x) = \frac{x}{4}$

$f_3(x) = \frac{x}{4} + \frac{1}{2}$
Language universality \iff nonempty interior

Fact 1: \( \mathcal{M} \) is universal \iff \( X = [0, 1]^d \)

- Example (universal): one state, transitions 0\( |0 \), 0\( |1 \), 1\( |0 \), 1\( |1 \)
- Example (not universal): one state, transitions 0\( |0 \), 0\( |1 \), 1\( |0 \)
Fact 1: $\mathcal{M}$ is universal $\iff X = [0, 1]^d$

- Example (universal): one state, transitions 0|0, 0|1, 1|0, 1|1
- Example (not universal): one state, transitions 0|0, 0|1, 1|0

Fact 2: $\mathcal{M}$ is prefix-universal $\iff X$ has nonempty interior
Language universality $\iff$ nonempty interior

**Fact 1:** $\mathcal{M}$ is universal $\iff X = [0, 1]^d$

- **Example (universal):** one state, transitions $0|0$, $0|1$, $1|0$, $1|1$
- **Example (not universal):** one state, transitions $0|0$, $0|1$, $1|0$

**Fact 2:** $\mathcal{M}$ is prefix-universal $\iff X$ has nonempty interior

- **Example (universal with prefix 1 but not universal):**
  one state, transitions $1, 10, 00$ (one-dimensional)

\[ f_1(x) = \frac{x}{2} + \frac{1}{2} \]
\[ f_2(x) = \frac{x}{4} \]
\[ f_3(x) = \frac{x}{4} + \frac{1}{2} \]
Main result

**Theorem [J-Kari 2013]**

For 3-state, 2-tape automata:

- universality is undecidable
- prefix-universality is undecidable
Main result

**Theorem** [J-Kari 2013]

For 3-state, 2-tape automata:
- universality is undecidable
- prefix-universality is undecidable

**Corollary**

For 2D affine graph-IFS with 3 states:
- \([0, 1]^2\) is undecidable
- empty interior is undecidable
Proof idea

**Post correspondence problem** (undecidable)

Given $n$ pairs of words $(u_1, v_1), \ldots, (u_n, v_n)$, does there exist $i_1, \ldots, i_k$ such that $u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$?

Examples:
- There is a solution: $(u_1, v_1) = (aa, aab), (u_2, v_2) = (bb, ba), (u_3, v_3) = (abb, b)$. $(i_1, i_2, i_3, i_4) = (1, 2, 1, 3)$ is a solution because $u_1 u_2 u_1 u_3 = aa bb aa abb = aab ba aab b = v_1 v_2 v_1 v_3$.
- No solution exists: $(u_1, v_1) = (aa, bab), (u_2, v_2) = (ab, ba)$.
- No solution exists: $(u_1, v_1) = (a, ab), (u_2, v_2) = (ba, ab)$.

**Variant:**
- Infinite-PCP
Proof idea

**Post correspondence problem** (undecidable)

Given $n$ pairs of words $(u_1, v_1), \ldots, (u_n, v_n)$, does there exists $i_1, \ldots, i_k$ such that $u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$?

**Examples:**

- There is a solution:
  
  $(u_1, v_1) = (a a, a a b), \quad (u_2, v_2) = (b b, b a), \quad (u_3, v_3) = (a b b, b)$

  $(i_1, i_2, i_3, i_4) = (1, 2, 1, 3)$ is a solution because

  $u_1 u_2 u_1 u_3 = a a \ b b \ a a \ a b b = a a b \ b a \ a a b \ b = v_1 v_2 v_1 v_3$

- No solution exists:
  
  $(u_1, v_1) = (a a, b a b), \quad (u_2, v_2) = (a b, b a)$

- No solution exists:
  
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Proof idea

Post correspondence problem (undecidable)

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Examples:

- There is a solution:
  \[(u_1, v_1) = (aa, aab), \quad (u_2, v_2) = (bb, ba), \quad (u_3, v_3) = (abb, b)\]
  \[(i_1, i_2, i_3, i_4) = (1, 2, 1, 3)\] is a solution because
  \[u_1 u_2 u_1 u_3 = aa \, bb \, aa \, abb = aab \, ba \, aab \, b = v_1 v_2 v_1 v_3\]

- No solution exists:
  \[(u_1, v_1) = (aa, bab), \quad (u_2, v_2) = (ab, ba)\]
  \[(u_1, v_1) = (a, ab), \quad (u_2, v_2) = (ba, ab)\]

Variant: infinite-PCP
Proof idea

**Theorem [Dube 1993]**

It is undecidable if the attractor of 2-dimensional IFS intersects the diagonal $\{(x, x) : x \in [0, 1]\}$
Proof idea

**Theorem [Dube 1993]**

It is undecidable if the attractor of 2-dimensional IFS intersects the diagonal \( \{(x, x) : x \in [0, 1]\} \)

- PCP instance \((u_1, v_1), \ldots, (u_n, v_n)\)
  \[
  \mapsto \text{1-state automaton } M \text{ with transitions } u_i|v_i
  \]
Proof idea

**Theorem [Dube 1993]**

It is undecidable if the attractor of 2-dimensional IFS intersects the diagonal \( \{(x, x) : x \in [0, 1]\} \)

- PCP instance \((u_1, v_1), \ldots, (u_n, v_n)\)
  \(\mapsto\) 1-state automaton \(M\) with transitions \(u_i \mid v_i\)
- \(\exists\) infinite-PCP solution
  \(\iff\) \(M\) accepts a configuration \((x_1x_2 \ldots)\)
Proof idea

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It is undecidable if the attractor of 2-dimensional IFS intersects the diagonal \( \{(x, x) : x \in [0, 1]\} \)

- PCP instance \((u_1, v_1), \ldots, (u_n, v_n)\)
  \[\mapsto\] 1-state automaton \(M\) with transitions \(u_i|v_i\)

- \(\exists \) infinite-PCP solution
  \[\iff\] \(M\) accepts a configuration \((x_1x_2\ldots)\) \((x_1x_2\ldots)\)

  \[\iff\] attractor contains a point \((0.x_1x_2\ldots)\) \((0.x_1x_2\ldots)\)
Proof idea

**Theorem [Dube 1993]**

It is undecidable if the attractor of 2-dimensional IFS intersects the diagonal \( \{(x, x) : x \in [0, 1]\} \)

- PCP instance \((u_1, v_1), \ldots, (u_n, v_n)\)  
  \(\mapsto\) 1-state automaton \(M\) with transitions \(u_i | v_i\)

- \(\exists\) infinite-PCP solution

  \(\iff M\) accepts a configuration \((x_1 x_2 \ldots)\)

  \(\iff\) attractor contains a point \((0.x_1 x_2 \ldots)\)

  \(\iff\) attractor \(\cap\) diagonal \(\neq\) \(\emptyset\)

- **So:** “attractor \(\cap\) diagonal \(\neq\) \(\emptyset\)” is undecidable
Proof idea

Theorem [J-Kari 2013]

For 3-state, 2-tape automata:

▶ universality is undecidable
▶ prefix-universality is undecidable

▶ Universality: reduce PCP
▶ Prefix-universality: reduce a variant of PCP ("prefix-PCP")
Conclusion & perspectives

- Some very particular families: **decidable**
- Affine 2D GIFS with 3 states: **undecidable**

Köszönöm a figyelmet
Conclusion & perspectives

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- What happens in **intermediate** cases?
  - 1D?
  - One-state GIFS? (*i.e.*, IFS)

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Conclusion & perspectives

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- Links automata ↔ topology:

| Property of the $d$-tape automaton | Topological property                        |
|-----------------------------------|---------------------------------------------|
| ∃ configurations with $=$ tapes   | Intersects the diagonal [Dube]              |
| Is universal                      | Is equal to $[0, 1]^d$                      |
| Has universal prefixes            | Has nonempty interior                      |
| ?                                 | Is connected                               |
| ?                                 | Is totally disconnected                    |
| Compute language entropy          | Compute fractal dimension                  |
Conclusion & perspectives

- Some very particular families: **decidable**
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