Global dynamics of the Buckingham’s two-body problem

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Abstract The equations of motion of the Buckingham system are the ones of a two-body problem defined by the Hamiltonian

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + A e^{-B \sqrt{x^2 + y^2}} - \frac{M}{(x^2 + y^2)^3}, \]

where \( A, B \) and \( M \) are positive constants. The angular momentum \( p_\theta = \dot{x}p_y - \dot{y}p_x \) and this Hamiltonian are two independent first integrals in involution. We denote by \( I_h \) (respectively, \( I_c \)) the set of points of the phase space where \( H \) (respectively, \( p_\theta \)) takes the value \( h \) (respectively, \( c \)). Due to the fact that \( H \) and \( p_\theta \) are first integrals, the sets \( I_h \) and \( I_{hc} = I_h \cap I_c \) are invariant under the flow of the Buckingham systems. We describe the global dynamics of the Buckingham system describing the foliation of its phase space by the invariant sets \( I_h \), the foliation of the invariant set \( I_h \) by its invariant subsets \( I_{hc} \), and the foliation of invariant set \( I_{hc} \) by the orbits of the system.

Keywords Buckingham equations · Hill regions · Global dynamics

1 Introduction and statement of the main results

A simplification of the classical Lennard-Jones potential (see Lennard-Jones 1931; Mioc et al. 2008) for studying the motion under the attractive or repulsive gravitational and intermolecular forces is the Buckingham potential

\[ U(x, y) = A e^{-B \sqrt{x^2 + y^2}} - \frac{M}{(x^2 + y^2)^3}, \]

introduced in 1928 for studying the state equation for gaseous helium, neon or argon, see Buckingham (1938). Here \( A, B \) and \( M \) are positive parameters. More precisely, this potential describes the van der Waals energy in the interaction of two atoms at the distance \( \sqrt{x^2 + y^2} \) (one of the atoms is at the origin), and the Pauli repulsion energy.

Recently the Buckingham potential has been studied in Popescu (2015), Popescu and Pricopi (2017), Pricopi and Popescu (2016). In these papers the authors studied their equilibria and their orbits using the McGehee coordinates and taking into account when the energy \( H \) is negative, zero or positive.

In this paper we describe the global dynamics of the Buckingham’s two-body problem, i.e. of the Hamiltonian

\[ H(x, y, p_x, p_y) = \frac{1}{2} (p_x^2 + p_y^2) + A e^{-B \sqrt{x^2 + y^2}} - \frac{M}{(x^2 + y^2)^3}. \]

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We change to canonical polar coordinates (see page 154 of Meyer et al. 2009) by setting
\[ x = r \cos \theta, \quad y = r \sin \theta, \quad p_x = p_r \cos \theta - \frac{p_\theta}{r} \sin \theta, \]
\[ p_y = p_r \sin \theta + \frac{p_\theta}{r} \cos \theta. \]

In these new variables the Hamiltonian system is
\[ \dot{r} = \frac{\partial H}{\partial p_r} = pr, \]
\[ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2}, \]
\[ \dot{p}_r = -\frac{\partial H}{\partial r} = AB e^{-Br} - \frac{6M}{r^7} + \frac{p_\theta^2}{r^5}, \]
\[ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0, \]
where
\[ H = \frac{1}{2} \left( \frac{r^2}{\theta} + \frac{p_\theta^2}{r^5} \right) + Ae^{-Br} - \frac{M}{r^6}. \]

The angular momentum \( p_\theta \) and the Hamiltonian \( H \) are two independent first integrals in involution. So the Hamiltonian system (1) is completely integrable in the sense of Liouville–Arnold, for more information on the completely integrable Hamiltonian systems see for instance Abraham and Marsden (1978), Arnold (1978), Meyer et al. (2009).

If \( \mathbb{R}^+ = (0, \infty) \) then the phase space of the Buckingham system is \( E = \mathbb{R}^+ \times S^1 \times \mathbb{R}^2 \) where \( r \in \mathbb{R}^+, \theta \in S^1 \) and \( (p_r, p_\theta) \in \mathbb{R}^2 \). The sets
\[ I_h = \{(r, \theta, p_r, p_\theta) \in E : H(r, \theta, p_r, p_\theta) = h \}, \]
\[ I_{hc} = \{(r, \theta, p_r, p_\theta) \in E : H(r, \theta, p_r, p_\theta) = h, p_\theta = c \}, \]
are invariant by the Hamiltonian flow of system (1), because \( H \) and \( p_\theta \) are first integrals. In other words, if a solution curve of the Hamiltonian system (1) has a point in \( I_h \) or in \( I_{hc} \), all this solution curve is contained into \( I_h \) or \( I_{hc} \), respectively.

The main results of this paper will the descriptions of the foliation of the phase space \( E = \mathbb{R}^+ \times S^1 \times \mathbb{R}^2 \) by the invariant sets \( I_h \), the foliation of the invariant set \( I_h \) by its invariant subsets \( I_{hc} \), and the foliation of invariant set \( I_{hc} \) by the orbits of the Buckingham system.

These foliations provide a good description of the global dynamics of the Hamiltonian systems (1) when \( K \) varies, where
\[ K = \left( \frac{7}{eB} \right)^7 - \frac{6M}{AB}. \]

Here \( e \) is the number \( e = 2.718282 \ldots \)
sets, or how the invariant sets \( I_{hc} \) foliate the energy sets \( I_h \), or how the energy levels \( I_h \) foliate the phase space \( E \). In this paper we solve all these questions for the Buckingham system. For a generic study of the invariant sets \( I_{hc} \) for Hamiltonian systems of two degrees of freedom having a central potential, see Llibre and Nunes (1994).

### 3 The topology of \( I_h \) and \( I_{hc} \)

The map \( H : E \rightarrow \mathbb{R} \) has a critical at the point \((r, \theta, p_r, p_\theta) \in E\) if it is an equilibrium of system (1). If a point of \( E \) is not critical, then it is regular. If there is a critical point belonging to \( H^{-1}(h) = I_h \), then \( h \in \mathbb{R} \) is a critical value of the map \( H : E \rightarrow \mathbb{R} \). If the value \( h \in \mathbb{R} \) is not critical, then it is a regular value. The set \( I_h \) is a 3-dimensional manifold if \( h \) is a regular value of the map \( H : E \rightarrow \mathbb{R} \); for more details see for instance Hirsch (1976).

Since \( r > 0 \) the critical points of \( H \) must satisfy
\[
g(r) = r^7 e^{-Br} - \frac{6M}{AB} = 0, \quad p_r = p_\theta = 0.
\]

So the set of critical points of \( H \) is
\[
C = \{(r, \theta, 0, 0) \in E : g(r) = 0 \text{ and } \theta \in \mathbb{S}^1\}
\]
\[
\approx \{(r, 0, 0) : g(r) = 0\} \times \mathbb{S}^1.
\]

We need to study the zeroes of the function \( g(r) \). Note that
\[
\frac{dg}{dr} = -e^{-Br}r^6(Br - 7)
\]
and so \( g \) has a maximum at \( r = 7/B \) and
\[
g\left(\frac{7}{B}\right) = \left(\frac{7}{eB}\right)^7.
\]

Note that the set of critical points \( C \) of \( H \) is equal to
\[
\emptyset \quad \text{if } K < 0,
\]
\[
\{(7/B, \theta, 0, 0) \in E : \theta \in \mathbb{S}^1\} \quad \text{if } K = 0,
\]
\[
\{(r_1, \theta, 0, 0) \in E : \theta \in \mathbb{S}^1\} \cup \{(r_2, \theta, 0, 0) \in E : \theta \in \mathbb{S}^1\}
\]
\[
\text{if } K > 0,
\]

where \( r_1, r_2 \) are such that \( g(r_1) = g(r_2) = 0 \) with \( r_1 < 7/B < r_2 \). Hence, the critical value is \(-Ae^{-7}/6\) if \( K = 0 \), and the values \(-Mr_1^6 + Ae^{Br_1}\) or \(-Mr_2^6 + Ae^{Br_2}\) if \( K > 0 \).

Let \( \pi : E \rightarrow \mathbb{R}^+ \times \mathbb{S}^1 \) be the natural projection from the phase space \( E \) to the configuration space \( \mathbb{R}^+ \times \mathbb{S}^1 \). Then for each \( h \in \mathbb{R} \) the Hill region \( R_h \) of \( I_h \) is defined by \( R_h = \pi(I_h) \). So, if
\[
f_h(r) = h + \frac{M}{r^6} - Ae^{-Br}
\]
we have
\[
R_h = \{(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^1 : f_h(r) \geq 0\}
\]
\[
\approx \{r \in \mathbb{R}^+ : f_h(r) \geq 0\} \times \mathbb{S}^1,
\]

where here \( \approx \) means diffeomorphic to.

Note that
\[
\lim_{r \rightarrow +\infty} f_h(r) = h \quad \text{and} \quad \lim_{r \rightarrow 0} f_h(r) = +\infty.
\]

Moreover, when \( K > 0 \) we have that \( f_h(r) \) has a minimum and a maximum which we denote by \( f_1 \) and \( f_2 \), respectively.

We compute the energy levels \( I_h \) in two different ways. The first way is more direct, and the second way allows additionally to deduce the foliation of \( I_h \) by the invariant sets \( I_{hc} \).

From the definition of \( I_h \) we have that
\[
I_h = \bigcup_{(r, \theta) \in R_h} E(r, \theta),
\]

where
\[
E(r, \theta) = \left\{(r, \theta, p_r, p_\theta) \in E : p^2_r + p^2_\theta = 2f_h(r)\right\}.
\]

Clearly for each \((r, \theta)\) given the set \( E(r, \theta) \) is an ellipse, a point, or the empty set if the point \((r, \theta)\) belongs to the interior of \( R_h \), to the boundary of \( R_h \), or does not belong to \( R_h \), respectively. Therefore, from (2), it follows easily the topology of \( I_h \) according with the different values of \( h \) and \( K \).

Another way of computing again the invariant energy levels \( I_h \) is using
\[
I_h = \{(r, \theta, p_r, p_\theta) \in E : H(r, \theta, p_r, p_\theta) = h\}
\]
\[
\approx g^{-1}(h) \times \mathbb{S}^1,
\]

where \( g(r, p_r, p_\theta) = H(r, \theta, p_r, p_\theta) \).

If \( h \in \mathbb{R} \) is a regular value of the map \( g : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( g^{-1}(h) \neq \emptyset \), then \( g^{-1}(h) \) is a surface of \( \mathbb{R}^+ \times \mathbb{R}^2 \).

It is easy to verify that the intersection of \( g^{-1}(h) \) with \( \{r = r_0 \text{ constant}\} \), is an ellipse, a point, or the empty set according to \( f_h(r_0) \) is positive, zero, or negative, respectively. So by studying the union of the ellipses or points of the form \( g^{-1}(h) \cap \{r = r_0\} \) moving \( r_0 > 0 \), we obtain the sets \( g^{-1}(h) \). Therefore, from (3), we calculate in a different way the topology of the energy levels \( I_h \).

We note that knowing the sets \( g^{-1}(h) \), from
\[
I_{hc} = I_h \cap \{p_\theta = c\}
\]
\[
= \left\{(r, p_r) : p_r = \pm \sqrt{2f_h(r) - \frac{c^2}{r^2}}\right\} \times \mathbb{S}^1
\]
\[
\approx g^{-1}(h) \times \mathbb{S}^1,
\]

(4)
we can compute the invariant sets $I_{hc}$. Consequently, we can describe the foliation of $I_h$ by $I_{hc}$ when $c$ varies.

We note that a point $(r^*, \theta^*, p^*_r, p^*_\theta)$ is \textit{regular} for the map $(H, p_\theta)$ if the rank of the matrix

$$
\left( \frac{\partial H}{\partial r} \frac{\partial H}{\partial \theta} \frac{\partial H}{\partial p_r} \frac{\partial H}{\partial p_\theta} \right)_{(r, \theta, p_r, p_\theta) = (r^*, \theta^*, p^*_r, p^*_\theta)}
$$

is 2, and a point which is not regular is \textit{critical}. A value $(h, c) \in \mathbb{R}^2$ is \textit{regular} if the set $(H, p_\theta)^{-1}((h, c))$ does not contain any critical point, otherwise it is a \textit{critical value}.

We consider different cases.

\textbf{Case 1: $K \leq 0$ and $h > 0$} In Fig. 1(a) if $K < 0$ the graphic of $f_h(r)$ decreases monotonically from $+\infty$ to $h$, while for $K = 0$ the graphic decreases having an inflexion point at $r = 7/B$. It follows from Fig. 1(a) that under these assumptions the zero velocity curve is empty and so the Hill region is $R_h = \mathbb{R}^2 \setminus \{(0, 0)\}$. Moreover $I_h$ is diffeomorphic to $(\mathbb{R}^2 \setminus \{(0, 0)\}) \times S^1$ because for each point of $R_h$ we have a topological circle of velocities. The boundary of $I_h$ with $r = 0$ corresponds to the collision manifold and the boundary with $r = +\infty$ to the infinity manifold.

Note that $g^{-1}(h)$ is the topological cylinder of Fig. 2.

When we consider the curves $y_{hc} = g^{-1}(h) \cap \{p_\theta = c\}$ for each $c \in \mathbb{R}$, we need to distinguish three subcases. There exist two values of $c$, $\pm c_1$, for which the curve $I_{hc}$ is not a manifold, and for these two values the curve $y_{hc}$ is homeomorphic to “the shape of the letter X”. Since $I_{hc, \pm c_1}$ is not a manifold, then the value $(h, \pm c_1)$ is not regular for the function $(H, p_\theta)$.

If $|c| > c_1$, the curve $y_{hc}$ has two components homeomorphic to $\mathbb{R}$, one defined in $0 < r \leq r_1$ and the other defined in $r_2 \leq r < +\infty$ with $r_1 < r_2$, see Fig. 3(a).

If $|c| = c_1$, the curve $y_{hc}$ has only one component topologically homeomorphic to the shape of the letter $X$, which can be obtained as a limiting case of Fig. 2 when $|c| \to c_1$ and $r_1$ and $r_2$ tend to the same value, see Fig. 3(b).

If $|c| < c_1$, the curve $y_{hc}$ has two components, each one defined for all $r > 0$ and homeomorphic to $\mathbb{R}$, see Fig. 3.

Clearly the manifold $I_h$ is diffeomorphic to a solid torus of dimension three without the boundary and without the central circular axis. Using topology techniques we have that a solid torus of dimension three without the boundary and without the central circular axis is homeomorphic to $\mathbb{S}^3 \setminus \{S^1 \cup S^1\}$. We claim that the foliation of $I_h$ by the subsets $I_{hc}$ varying $c$ can be obtained by rotating Fig. 4 around the $u$ axis. Now we prove the claim.

Assume that $|c| > c_1$. From Fig. 3(a) the manifold $I_{hc}$ is formed by two cylinders. In the cylinder for which $r \in (0, r_1]$ the orbits start in ejection and end in collision. On the cylinder for which $r \in [r_2, +\infty)$ the orbits start and end at infinity hyperbolically, i.e. with radial velocity $|\dot{r}| \neq 0$.

Assume that $c = c_1$ (respectively, $c = -c_1$). From Fig. 3(a) the invariant set $I_{hc}$ is formed by a periodic orbit $\alpha$ (respectively, $\beta$) and four cylinders $C_i, i = 1, 2, 3, 4$, having a common boundary formed by the periodic orbit $\alpha$ (respectively, $\beta$). The cylinders $C_1$ and $C_4$ are the stable manifold of $\alpha$ (respectively, $\beta$), and the cylinders $C_2$ and $C_3$ are the unstable manifold of $\alpha$ (respectively, $\beta$). On the cylinder $C_1$ the orbits start at ejection and end in $\alpha$ (respectively, $\beta$), on the cylinders $C_2$ the orbits start in $\alpha$ (respectively, $\beta$) and end in collision, on the cylinder $C_3$ the orbits start in $\alpha$ (respectively, $\beta$) and end at infinity hyperbolically, and finally the orbits on the cylinder $C_4$ start at infinity hyperbolically and end in $\alpha$ (respectively, $\beta$).

Assume that $|c| < c_1$. From Fig. 3(c) the manifold $I_{hc}$ is formed by two cylinders. On one of the cylinders the orbits start in ejection and end at infinity hyperbolically and on the other cylinder the orbits start at infinity hyperbolically and end in collision.

\textbf{Case 2: $K \leq 0$ and $h = 0$} It follows from Fig. 1(b) that the Hill region is again $R_h = \mathbb{R}^2 \setminus \{(0, 0)\}$, and so $I_h$ is homeo-
Fig. 3 Graphs of the curves $\gamma_{hc}$ when $K \leq 0$ and $h \geq 0$: (a) For $|c| > c_1$, (b) For $|c| = c_1$, (c) For $|c| < c_1$

Fig. 4 Manifold $I_h/S^1$ for $K \leq 0$ and $h > 0$

Fig. 5 The surface $g^{-1}(h)$ for $K \leq 0$ and $h = 0$

Fig. 6 Graphs of the curves $\gamma_{hc}$ when $K \leq 0$ and $h = 0$: (a) For $c \in \mathbb{R} \setminus \{0\}$, (b) For $c = 0$

Fig. 7 Manifold $I_h/S^1$ for $K \leq 0$ and $h = 0$

We claim that the foliation of $I_h$ by the subsets $I_{hc}$ varying $c$ can be obtained by rotating Fig. 7 around the $u$ axis. Now we prove the claim.

Assume that $c \in \mathbb{R} \setminus \{0\}$. From Fig. 8 we have that the manifold $I_{hc}$ is formed by one cylinder. In this cylinder the orbits start in ejection and end in collision.

Assume that $c = 0$. From Fig. 6(b) the invariant set $I_{hc}$ is formed by two cylinders. The orbits in the cylinder $p_r > 0$ start in ejection and end at infinity parabolically, i.e., they reach the infinity with zero radial velocity. In the cylinder
$p_r < 0$ the orbits start at infinity parabolically and end at collision.

**Case 3: $K \leq 0$ and $h < 0$** It follows from Fig. 1(b) that the zero velocity curve is a circle centered at the origin of coordinates. Then the Hill region $\pi(I_h)$ is a punctured closed disc centered at the origin whose boundary is the zero velocity curve, that is, $R_h$ is diffeomorphic to $(0, a] \times S^1$ where $a > 0$. So $I_h$ is diffeomorphic to an open solid torus whose boundary corresponds to the collision manifold, the central axis of this solid torus is the zero velocity curve. Indeed, $R_h \approx \{(r, \theta) \in (0, a] \times S^1\}$. The circle $r = a$ is the zero velocity curve. So for each ray $(0, a] \times \{\theta = \theta_0\}$ we have a circle of velocities for each point $(r, \theta_0)$ with $r \in (0, a)$. The radius of this circle varies continuously with $r$ and becomes zero at $r = a$. Taking into account all these circles together and varying $\theta_0 \in S^1$ we obtain the solid torus without the boundary with a “central axis” formed by the zero velocity circle, and the boundary of this solid torus corresponds to the collision manifold. Applying topology arguments we have that a solid torus without the boundary is homeomorphic to $S^3 \setminus S^1$, so $I_h \approx S^3 \setminus S^1$.

On the other hand the surface $g^{-1}(h)$ is the topological plane of Fig. 8.

The curves $\gamma_{hc} = g^{-1}(h) \cap \{p_\theta = c\}$ for each $c \in \mathbb{R}$ are defined for all $r$ in $0 < r < r(c)$ and are homeomorphic to $\mathbb{R}$. The manifold $I_h$ can be obtained by rotating Fig. 9 around the $u$ axis. In this picture we can see one cylinder $I_{hc}$ for every $c \in \mathbb{R}$ foliating $I_h$. The orbits on this cylinder start in ejection and end in collision. We note that only the orbits on the cylinder $I_{h0}$ pass through the zero velocity curve.

**Case 4: $K > 0$ and $2h < f_1 < f_2$** We consider different subcases, see Fig. 10(a).

Subcase 4.1: $f_1 > 0$ The dynamics of this case with $h > 0$ is the same as the one of Case 1 and with $h = 0$ is the same as the one of Case 2, while the dynamics in this case with $h < 0 < f_1$ is dynamically the same as the one of Case 3.

Subcase 4.2: $f_1 = 0$ From Fig. 10(a) we see that the zero velocity curve is formed by two circles centered at the origin of coordinates. The Hill region is formed by a closed disc of radius $d$ centered at the origin without the origin, the external boundary of this disc is the biggest circle $r = d$ of the zero velocity curve and additionally there is a circle $r = a < d$ of the zero velocity curve contained in the interior of this disc.

The part of the Hill region $\{(r, \theta) : r \in (0, a), \theta \in S^1\}$ corresponds to the projection of an open solid torus contained in $I_h$ whose boundary is the collision manifold. This open solid torus is similar to the one of Case 3. We claim that the part of the Hill region $\{(r, \theta) : r \in [a, d], \theta \in S^1\}$ corresponds to the projection of a set homeomorphic to $S^2 \times S^1$ contained in $I_h$. Indeed, for each $\theta_0 \in S^1$ and for each point of the segment $\{(r, \theta_0) : r \in [a, d]\}$ we have a circle of velocity which reduces to a point at $r = a$ and $r = d$. So a sphere $S^2$ of velocities of $I_h$ projects onto the segment $\{(r, \theta_0) : r \in [a, d]\}$ for all $\theta_0 \in S^1$. Hence the claim is proved.

In short, $I_h$ is formed by the union of an open solid torus and $S^2 \times S^1$ but the central axis of the solid torus formed by the circle of zero velocity $r = a$ is identified with a circle of $S^2 \times S^1$.

The set $I_h$ also can be obtained rotating Fig. 11 with respect to the $u$-axis. In Fig. 11 we have drawn the surface $g^{-1}(h)$ for $K > 0$ and $h = 0$.

The invariant sets $I_{hc}$ in this case are not manifolds of dimension two at three values of $c$, namely $-c_1$, 0 and $c_1$. Consequently the values $(h, \pm c_1)$ and $(h, 0)$ are not regular for the function $(H, p_\theta)$.

If $|c| > c_1$ from Fig. 12(a) we have that $I_{hc}$ is a cylinder. The orbits on this cylinder are of ejection-collision.
Fig. 10 Graph of the function $f_h(r)$ when $K > 0$: (a) For $2h < f_1$, (b) For $2h = f_1$, (c) For $2h > f_1$

If $|c| = c_1$ from Fig. 12(b) we obtain that $I_{hc}$ is the union of a cylinder with a periodic orbit. Again the orbits of the cylinder are of ejection-collision.

Fig. 11 The surface $g^{-1}(h)$ for $K > 0$ and $0 = f_1 < 2h$

Fig. 12 Graph of the curves $\gamma_{hc}$ when $K > 0$ and $2h < f_1 = 0$: (a) For $|c| > c_1$, (b) For $|c| = c_1$, (c) For $0 < |c| < c_1$, (d) For $c = 0$

If $0 < |c| < c_1$ from Fig. 12(c) we get that $I_{hc}$ is the union of a cylinder with a torus. Again the orbits on the cylinder are of ejection-collision. The orbits on the torus can be either quasiperiodic and consequently dense in the torus, or periodic.

If $c = 0$ from Fig. 12(d) we have that $I_{hc}$ has only one component formed by three cylinders which share the circle of singular points $r = a$. The cylinder $C_1$ is formed by orbits of ejection which end in the circle $r = a$. The orbits on the cylinder $C_3$ start at the circle $r = a$ and end at collision, and the orbits on the cylinder $C_2$ start and end at $r = a$.

In summary the foliation of $I_h$ but the sets $I_{hc}$ varying $c$ can be obtained by rotating Fig. 13(a) around the $\nu$ axis, with the union of $S^2 \times S^1$ of Fig. 13(b), and identifying the zero velocity circle $r = a$ contained in both figures. In Fig. 13(b) appears the foliation of $S^2 \times S^1$ by the invariant sets $I_{hc}$. This foliation corresponds to the Hopf foliation of $S^3 \approx S^2 \times S^1$.

We must identify the points (of the two surfaces of the cones glued by their bases) which are symmetric with respect to the plane containing the common bases.

Subcase 4.3: $f_1 < 0 < f_2$. It follows from Fig. 10(a) that the zero velocity curve is formed by three circles centered at the origin of coordinates. The Hill region $\pi(I_h)$ has two connected components, one is formed by a closed disc centered at the origin without the origin, the external boundary of this disc is the smallest circle of the zero velocity curve, and the other component is a closed crown whose two boundaries are the other two circles of the zero velocity curve. The manifold $I_h$ has two connected components, one is homeomorphic to a solid torus having its central axis
formed by the smallest circle of the zero velocity curve and its boundary corresponds to the collision manifold (see for more details Case 3), and the other component is homeomorphic to $S^2 \times S^1$ containing the two biggest circles of the zero velocity curve (see for more details Subcase 4.2). The set $I_h$ which is homeomorphic to the union of an open solid torus with $S^2 \times S^1$ can also be obtained rotating Fig. 14(a) with respect to the $u$-axis and adding to it the Fig. 14(b). In this last figure it is presented the foliation of the sphere $S^3$ identifying the points (of the surfaces of the cones glued by their bases) which are symmetric with respect to the plane containing the common bases. In Fig. 14(b) we also can see how the flow moves on the tori $I_{hc}$. In Fig. 14 we have drawn the surface $g^{-1}(h)$ for $K > 0$ and $f_1 < 0 < f_2$.

There exist two values of $c$, $\pm c_1 \neq 0$ for which the set $I_{hc}$ is not a manifold of dimension two, so the values $(h, \pm c_1)$ are not regular for the function $(H, p_\theta)$.

If $|c| > c_1$, from Fig. 15(a) we have that $I_{hc}$ is a cylinder whose orbits are of ejection-collision.

If $|c| = c_1$ from Fig. 12(b) we obtain that $I_{hc}$ is the union of a cylinder with a periodic orbit and the orbits in the cylinder are of ejection-collision.

Finally, if $|c| < c_1$, from Fig. 12(c) we have that $I_{hc}$ is the union of a cylinder with a torus. Again the orbits on the cylinder are of ejection-collision, and the orbits on the torus can be either quasiperiodic, or periodic.

In short, the foliation of $I_h$ by the sets $I_{hc}$ varying $c$ can be obtained by rotating Fig. 13(a) around the $u$-axis, with the union of $S^2 \times S^1$ of Fig. 13(b).

**Subcase 4.4: $f_2 = 0$**

It follows from Fig. 10(a) that the zero velocity curve is formed by two circles centered at the origin of coordinates. The Hill region $\pi(I_h)$ has two connected components, one is formed by a closed disc centered at the origin without the origin, the external boundary of this disc is the smallest circle of the zero velocity curve, and the other component is the biggest circle of the zero velocity curve. So $I_h$ has two connected components, one homeomorphic to a solid torus having as central axis the smallest circle of the zero velocity curve, whose boundary corresponds to the collision manifold (see Case 3), and the other is the biggest circle of the zero velocity curve, i.e. a circle of equilibria. Here the surface $g^{-1}(h)$ is the one given in Fig. 8 with an additional isolated point. The foliation of $I_h$ by the invariant sets $I_{hc}$ is a cylinder if $c \in \mathbb{R}\setminus\{0\}$, and the union of cylinder with a circle of equilibria if $c = 0$.

**Subcase 4.5: $f_2 < 0$**

The dynamics of this case is the same as the one of Case 3.

**Case 5: if $K > 0$ and $f_1 = 2h < f_2$**

We consider different subcases, see Fig. 10(b).

**Subcase 5.1: $0 < f_1$, or $f_1 < 0 < f_2$, or $f_2 = 0$, or $f_2 < 0$**

The dynamics of case $0 < f_1$ is the same as the one of Case 1. The dynamics of case $f_1 < 0 < f_2$ is the same as
the one of Subcase 4.3. The dynamics of case $f_2 = 0$ is the same as the one of Subcase 4.4, and the dynamics of case $f_2 < 0$ is the same as the one of Case 3.

**Subcase 5.2: $f_1 = 2h = 0$** It follows from Fig. 10(b) that the zero velocity curve is formed by a circle centered at the origin of coordinates, the Hill region $\pi(I_0) \equiv \mathbb{R}^2 \setminus \{(0,0)\}$, but it contains in its interior the zero velocity curve, and $I_h$ is homeomorphic to two solid tori without the boundary which have identified their “central axis” with the zero velocity curve, the boundary of one of the solid tori is the collision manifold and the boundary of the other solid tori is the infinity manifold.

From Fig. 15 the surface $g^{-1}(0)$ topologically is formed by two planes with a common point identified. The curves $g_\alpha = g^{-1}(0) \cap \{p_\theta = c\}$ for $c \in \mathbb{R} \setminus \{0\}$ are topologically the ones of Fig. 3(a), and for $c = 0$ are topologically the ones of Fig. 3(b). The foliation of $I_h$ by the invariant sets $I_{hc}$ varying

### Table 1

| $I_h$ | $g^{-1}(h)$ | $I_{hc}$ | $I_h/\mathbb{S}^1$ |
|-------|-------------|----------|---------------------|
| $h > 0$ | $\mathbb{S}^3 \setminus \{\mathbb{S}^1 \cup \mathbb{S}^1\}$ | 2 | $\bigcup_1 \mathbb{S}^1 \times \mathbb{R}$ if $|c| > c_1$ | 4 |
| | | | $\bigcup_2 \mathbb{S}^1 \times \mathbb{R}$ if $|c| = c_1$ | |
| | | | $\bigcup_3 \mathbb{S}^1 \times \mathbb{R}$ if $|c| < c_1$ | |
| $h = 0$ | $\mathbb{S}^3 \setminus \{\mathbb{S}^1 \cup \mathbb{S}^1\}$ | 5 | $\mathbb{S}^1 \times \mathbb{R}$ if $c \in \mathbb{R} \setminus \{0\}$ | 7 |
| | | | $\bigcup_2 \mathbb{S}^1 \times \mathbb{R}$ if $c = 0$ | |
| $h < 0$ | $\mathbb{S}^3 \setminus \mathbb{S}^1$ | 8 | $\mathbb{S}^1 \times \mathbb{R}$ if $c \in \mathbb{R}$ | 9 |

### Table 2

| $I_h$ | $g^{-1}(h)$ | $I_{hc}$ | $I_h/\mathbb{S}^1$ |
|-------|-------------|----------|---------------------|
| $f_1 > 0, h > 0$ | $\mathbb{S}^3 \setminus \{\mathbb{S}^1 \cup \mathbb{S}^1\}$ | 2 | $\bigcup_1 \mathbb{S}^1 \times \mathbb{R}$ if $|c| > c_1$ | 4 |
| | | | $\bigcup_2 \mathbb{S}^1 \times \mathbb{R}$ if $|c| = c_1$ | |
| | | | $\bigcup_3 \mathbb{S}^1 \times \mathbb{R}$ if $|c| < c_1$ | |
| $h = 0$ | $\mathbb{S}^3 \setminus \{\mathbb{S}^1 \cup \mathbb{S}^1\}$ | 5 | $\mathbb{S}^1 \times \mathbb{R}$ if $c \in \mathbb{R} \setminus \{0\}$ | 7 |
| | | | $\bigcup_2 \mathbb{S}^1 \times \mathbb{R}$ if $c = 0$ | |
| $h < 0 < f_1$ | $\mathbb{S}^3 \setminus \mathbb{S}^1$ | 8 | $\mathbb{S}^1 \times \mathbb{R}$ if $c \in \mathbb{R}$ | 9 |
| $f_1 = 0$ | $(\mathbb{S}^3 \setminus \mathbb{S}^1) \cup \mathbb{S}^1 \times \mathbb{S}^1$ | 11 | $\mathbb{S}^1 \times \mathbb{R}$ if $|c| > c_1$ | 13 |
| | | | $\bigcup_1 \mathbb{S}^1 \times \mathbb{R}$ if $|c| = c_1$ | |
| | | | $\bigcup_2 \mathbb{S}^1 \times \mathbb{R}$ if $|c| < c_1$ | |
| $f_1 < 0 < f_2$ | $(\mathbb{S}^3 \setminus \mathbb{S}^1) \cup \mathbb{S}^1 \times \mathbb{S}^1$ | 14 | $\mathbb{S}^1 \times \mathbb{R}$ if $|c| > c_1$ | 13 |
| | | | $\bigcup_1 \mathbb{S}^1 \times \mathbb{R}$ if $|c| = c_1$ | |
| | | | $\bigcup_2 \mathbb{S}^1 \times \mathbb{R}$ if $|c| < c_1$ | |
| $f_2 = 0$ | $(\mathbb{S}^3 \setminus \mathbb{S}^1) \cup \mathbb{S}^1$ | 8* | $\mathbb{S}^1 \times \mathbb{R}$ if $c \in \mathbb{R} \setminus \{0\}$ | 9* |
| | | | $\bigcup_1 \mathbb{S}^1 \times \mathbb{R}$ if $c = 0$ | |
| $f_2 < 0$ | $\mathbb{S}^3 \setminus \mathbb{S}^1$ | 8 | $\mathbb{S}^1 \times \mathbb{R}$ if $c \in \mathbb{R}$ | 9 |
$c$ can be obtained by rotating Fig. 9 with an additional circle $S^1$ of equilibria.

If $c \in \mathbb{R}\setminus\{0\}$ then $I_{0c}$ is formed by two cylinders. In one cylinder the orbits are of ejection-collision, and in the other cylinder the orbits start and end at infinity parabolically.

If $c = 0$ the topology of $I_{00}$ and the dynamics on it is the same as for the sets $I_{hc}$ of Case 1, with the difference that the periodic orbit $\alpha$ in $I_{hc}$ now becomes a circle of equilibria.

The foliation of $I_0$ by the sets $I_{0c}$ varying $c$ can be obtained by rotating Fig. 16 around the $u$-axis. After the rotation with respect to the $u$-axis the two central axes of both open solid tori in the zero velocity curve must be identified.

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**Table 3** The invariant set $I_h$ and its foliation by $I_{hc}$ for $K > 0$ and $f_1 = 2h < f_2$

| $f_1$ | $I_h$ | $g^{-1}(h)$ | $I_{hc}$ | $I_h/S^1$ |
|-------|-------|-------------|----------|-----------|
| $f_1 > 0$ | $S^1 \cup \{S^1 \cup S^1\}$ | 2 | $U_2 S^1 \times \mathbb{R}$ | if $|c| > c_1$ | 4 |
| | | | $U_4 S^1 \times \mathbb{R}$ | if $|c| = c_1$ | |
| | | | $U_2 S^1 \times \mathbb{R}$ | if $|c| < c_1$ | |
| $f_1 = 2h = 0$ | $U_2^g (S^3 \setminus S^1)$ | 15 | $U_2 S^1 \times \mathbb{R}$ | if $c \in \mathbb{R}\setminus\{0\}$ | 16 |
| | | | $U_4^g S^1 \times \mathbb{R}$ | if $c = 0$ | |
| $f_1 < 0 < f_2$ | $(S^3 \setminus S^1) \cup S^2 \times S^1$ | 14 | $S^1 \times \mathbb{R}$ | if $|c| > c_1$ | 13 |
| | | | $S^1 \cup S^1 \times \mathbb{R}$ | if $|c| = c_1$ | |
| | | | $S^1 \times \mathbb{R} \cup S^1 \times S^1$ | if $|c| < c_1$ | |
| $f_2 = 0$ | $(S^3 \setminus S^1) \cup S^1$ | 8* | $S^1 \times \mathbb{R}$ | if $c \in \mathbb{R}\setminus\{0\}$ | 9* |
| | | | $S^1 \times \mathbb{R} \cup S^1$ | if $c = 0$ | |
| $f_2 < 0$ | $S^3 \setminus S^1$ | 8 | $S^1 \times \mathbb{R}$ | if $c \in \mathbb{R}$ | 9 |

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**Table 4** The invariant set $I_h$ and its foliation by $I_{hc}$ for $K > 0$ and $f_1 < 2h < f_2$. The $15^*$ in the column $g^{-1}(h)$ indicates that the surface $g^{-1}(h)$ is the one of Fig. 15 but now the two surfaces of Fig. 15 do not share a common point. The $16^*$ in the column $I_h/S^1$ indicates that the foliation of $I_h$ by the invariant sets $I_{hc}$ is the same as the foliation in Fig. 16 without identifying the two central axes of both solid tori.

| $f_1$ | $I_h$ | $g^{-1}(h)$ | $I_{hc}$ | $I_h/S^1$ |
|-------|-------|-------------|----------|-----------|
| $f_1 > 0$ | $S^1 \setminus \{S^1 \cup S^1\}$ | 2 | $U_2 S^1 \times \mathbb{R}$ | if $|c| > c_1$ | 4 |
| | | | $U_4 S^1 \times \mathbb{R}$ | if $|c| = c_1$ | |
| | | | $U_2 S^1 \times \mathbb{R}$ | if $|c| < c_1$ | |
| $f_1 = 0$ | $U_2^g (S^3 \setminus S^1)$ | 15 | $U_2 S^1 \times \mathbb{R}$ | if $c \in \mathbb{R}\setminus\{0\}$ | 16 |
| | | | $U_4^g S^1 \times \mathbb{R}$ | if $c = 0$ | |
| $f_1 < 0 \leq 2h$ | $U_4(S^3 \setminus S^1)$ | 15* | $U_2 S^1 \times \mathbb{R}$ | if $c \in \mathbb{R}\setminus\{0\}$ | 16* |
| | | | $U_4 S^1 \times \mathbb{R}$ | if $c = 0$ | |
| $2h < 0 < f_2$ | $(S^3 \setminus S^1) \cup S^2 \times S^1$ | 14 | $S^1 \times \mathbb{R}$ | if $|c| > c_1$ | 13 |
| | | | $S^1 \cup S^1 \times \mathbb{R}$ | if $|c| = c_1$ | |
| | | | $S^1 \times \mathbb{R} \cup S^1 \times S^1$ | if $|c| < c_1$ | |
| $f_2 = 0$ | $(S^3 \setminus S^1) \cup S^1$ | 8* | $S^1 \times \mathbb{R}$ | if $c \in \mathbb{R}\setminus\{0\}$ | 9* |
| | | | $S^1 \times \mathbb{R} \cup S^1$ | if $c = 0$ | |
| $f_2 < 0$ | $S^3 \setminus S^1$ | 8 | $S^1 \times \mathbb{R}$ | if $c \in \mathbb{R}$ | 9 |
Case 6: if $K > 0$ and $f_1 < 2h < f_2$  We consider different subcases, see Fig. 10(c).

Subcase 6.1: $0 < f_1$, or $f_1 = 0$, or $2h < f_2$, or $f_2 = 0$, or $f_2 < 0$  The dynamics in the case $0 < f_1$ is the same as the one of Case 1. The dynamics in the case $f_1 = 0$ is the same as the one of Subcase 5.2 but the orbits starting and ending at infinity are now hyperbolic instead of parabolic. The dynamics in the case $2h < 0 < f_2$ is the same as the one of Subcase 4.3. Moreover, the dynamics in the case $f_2 = 0$ is the same as the one of Subcase 4.4, and finally the dynamics of the last case $f_2 < 0$ is the same as the one of Case 3.

Subcase 6.2: $f_1 < 0 \leq 2h$  It follows from Fig. 13(c) that the zero velocity curve is formed by two circles centered at the origin. The Hill region $\pi(I_h)$ is formed by two topological discs, one is a closed disc centered at the origin without the origin whose external boundary is the smallest circle of the zero velocity curve, the other topological disc is of the form $[r_1, \infty) \times S^1$ and the circle $r = r_1$ is the biggest circle of the zero velocity curve.

The manifold $I_h$ is homeomorphic to the union of two open solid tori, each solid torus has its central axis formed by one of the circles of the zero velocity curve, and the boundary of the solid torus having as central axis the smallest circle of the zero velocity is the collision manifold and the boundary of the other solid torus is the infinity manifold. Hence, $I_h$ is diffeomorphic to $(S^3 \setminus S^1) \cup (S^3 \setminus S^1)$. The surface $g^{-1}(h)$ is the one of Fig. 15 but now the two surfaces of that figure do not share a common point.

The foliation of $I_h$ by the sets $I_{hc}$ and the dynamics on each $I_{hc}$ is described in Fig. 16 without identifying the two central axes of both tori, because now the zero velocity curve is formed by two distinct circles.

In Tables 1, 2, 3, and 4, we summarize the foliation of $I_h$ by $I_{hc}$ for all the values of $K$ and $h$.

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