RATIONALITY OF MOTIVIC ZETA FUNCTION AND CUT-AND-PASTE PROBLEM

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ABSTRACT. Assuming the positive solution to the Cut-and-paste problem we prove that the motivic zeta function remains irrational after inverting $L$.

1. Introduction

Fix a field $F$ and let $K_0[\mathcal{V}_F]$ denote the Grothendieck ring of varieties over $F$. That is $K_0[\mathcal{V}_F]$ is the abelian group which is generated by isomorphism classes of $F$-varieties with relations

$$[X] = [Y] + [X \setminus Y]$$

if $Y \subset X$ is a closed subvariety. The product in $K_0[\mathcal{V}_F]$ is defined as

$$[X] \cdot [Y] = [X \times_F Y]$$

In [LaLu1] we have asked the following question:

Cut-and-paste problem. Let $Z_1, \ldots, Z_k; W_1, \ldots W_l$ be $F$-varieties and consider the disjoint unions $X = \coprod Z_i$ and $Y = \coprod W_j$. Suppose that $[X] = [Y]$. Is it possible to decompose $X$ and $Y$ into locally closed subvarieties

$$X = \coprod_{i=1}^k X_i, \quad Y = \coprod_{i=1}^k Y_i$$

such that for each $i$ the varieties $X_i$ and $Y_i$ are isomorphic?

Some positive results for this problem are obtained in the paper [LiSeb]. They prove that the solution to the problem is positive (in characteristic zero) if 1) $\dim X \leq 1$, 2) $X$ is a smooth connected projective surface, 3) $X$ contains only finite many rational curves.

In this note we want to relate the Cut-and-paste problem to the question of rationality of the motivic zeta function

$$\zeta_X(t) = \sum_{n=0}^{\infty} [\text{Sym}^n X] t^n \in K_0[\mathcal{V}_F][[t]]$$

This motivic zeta function was introduced by Kapranov in [Ka], where he proves that $\zeta_X(t)$ is rational if $\dim X \leq 1$. He also says that it is natural to expect rationality of $\zeta_X(t)$ for any variety $X$. 

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This conjecture of Kapranov was disproved in [LaLu1] and [LaLu2], where we show that the motivic zeta function of a surface \( X \) is rational if and only if \( X \) has Kodaira dimension \(-\infty\) (for \( F = \mathbb{C} \)). The proof of this uses a ring homomorphism \( K_0(V_C) \to H \) which factors through the quotient \( K_0(V_C)/L \), where \( L = [\mathbb{A}^1] \). Hence the question of rationality of the motivic zeta function in the localized ring \( K_0[V_F][L^{-1}] \) is still open.

In the paper [DeLoe] the authors conjecture (Conjecture 7.5.1) that \( \zeta_X(t) \) is rational in \( K_0[V_F][L^{-1}] \).

In this article we prove that the positive solution to the Cut-and-paste problem implies that \( \zeta_X(t) \) is not rational in \( K_0[V_F][L^{-1}] \). This follows easily from our results in [LaLu1].

We thank Ravi Vakil, whose beautiful recent lecture in Indiana University on motivic Grothendieck ring prompted us to think again about the subject.

2. Rationality of power series in the coefficients of a ring

Let \( A \) be a commutative ring with 1. We recall and compare various notions of rationality of power series with coefficients in \( A \).

**Definition 2.1.** A power series \( f(t) \in A[[t]] \) is **globally rational** if and only if there exist polynomials \( g(t), h(t) \in A[t] \) such that \( f(t) \) is the unique solution of \( g(t)x = h(t) \).

**Definition 2.2.** A power series \( f(t) = \sum_{i=0}^{\infty} a_i t^i \in A[[t]] \) is **determinantally rational** if and only if there exist integers \( m \) and \( n \) such that

\[
\det \begin{pmatrix}
  a_i & a_{i+1} & \ldots & a_{i+m} \\
  a_{i+1} & a_{i+2} & \ldots & a_{i+m+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{i+m} & a_{i+m+1} & \ldots & a_{i+2m}
\end{pmatrix} = 0
\]

for all \( i > n \).

It is classical that the Definition 2.1 is equivalent to Definition 2.2 if \( A \) is a field.

**Definition 2.3.** A power series \( f(t) \in A[[t]] \) is **pointwise rational** if and only if for all homomorphisms \( \Phi \) from \( A \) to a field, \( \Phi(f) \) is rational by either of the two previous definitions.

These definitions are related by the following proposition [LaLu2], Prop. 2.4:

**Proposition 2.4.** Any globally rational power series is determinantally rational, and any determinantally rational power series is pointwise rational. Neither converse holds for a general coefficient ring \( A \). All three conditions are equivalent when \( A \) is an integral domain.

It is known that the ring \( K_0[V_F] \) has zero divisors [Po].
3. Cut-and-Paste Problem and Rationality of $\zeta_X(t)$

The following theorem was proved in [LaLu2], Thm. 7.6 and Cor. 3.8:

**Theorem 3.1.** Let $X$ be a complex surface of Kodaira dimension $\geq 0$. Then the zeta function $\zeta_X(t) \in K_0[V_C][[t]]$ is not pointwise rational.

On the positive side it is relatively easy to prove the following theorem [LaLu2], Thm. 3.9:

**Theorem 3.2.** If $X$ is a surface with the Kodaira dimension $-\infty$, then the zeta function $\zeta_X(t) \in K_0[V_C][[t]]$ is globally rational.

Let $Y$ be a smooth projective variety of dimension $d$. Recall that the polynomial

$$h_Y(s) := 1 + h^{1,0}(Y)s + h^{2,0}(Y)t^2 + \ldots + h^{d,0}(Y)s^d$$

is a birational invariant of $Y$ [Hart], Ch. II, Exercise 8.8. Here $h^{i,0}(Y) = \dim H^0(Y, \Omega^i_Y)$. Therefore we may (in characteristic zero) define $h_Z(t)$ for any variety $Z$, not necessarily smooth and projective, as

$$h_Z(s) = h_Y(s)$$

where $Y$ is any smooth projective model of $Z$. The Künneth formula for the Hodge structure on the cohomology of the constant sheaf $\mathbb{C}$ implies that $h_Y(s)$ is even a stable birational invariant of $Y$, i.e.

$$h_Y(s) = h_{Y \times \mathbb{P}^n}(s)$$

The integer $P_g(Y) := h^{d,0}(Y)$ is the geometric genus of $Y$.

Here we prove the following theorem:

**Theorem 3.3.** Let $X$ be a complex surface with $P_g(X) \geq 2$. Assume that the Cut-and-Paste problem has a positive solution. Then the zeta function $\zeta_X(t) \in K_0[V_C][L^{-1}][[t]]$ is not determinantly rational.

**Proof.** Put $X^{(n)} := \text{Sym}^n X$. If the zeta function $\zeta_X(t) \in K_0[V_C][L^{-1}][[t]]$ is determinantly rational then there exist integers $n > 0$ and $n_0 > 0$ such that for each $m > n_0$ the determinant

$$\text{det} \begin{pmatrix} X^{(m)} & X^{(m+1)} & \ldots & X^{(m+n)} \\ X^{(m+1)} & X^{(m+2)} & \ldots & X^{(m+n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(m+n)} & X^{(m+n+1)} & \ldots & X^{(m+2n)} \end{pmatrix}$$

is not determinantally rational.
equals zero in the ring $K_0[\mathcal{V}_C][L^{-1}]$. This determinant is the sum

$$(3.2) \quad \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) X^{(m-1+\sigma(1))} \times X^{(m+\sigma(2))} \times \ldots \times X^{(m+n-1+\sigma(n+1))}$$

The assumption that the determinant is zero in $K_0[\mathcal{V}_C][L^{-1}]$ means that the quantity $3.2$ when multiplied by some power $L^N$ is zero in $K_0[\mathcal{V}_C]$. Then the positive solution to the Cut-and-paste problem implies that the various products in the alternating sum $3.2$ when multiplied by $L^N$ become pairwise birational (since all of them have the same dimension).

Note that the product

$$X^{(m)} \times X^{(m+2)} \times \ldots \times X^{(m+2n)}$$

appears exactly once in $3.2$. Now we get a contradiction with the following claim, which is proved on p. 11 in [LaLu1]:

Claim. For infinitely many $m > 0$ the equality

$$P_g(X^{(m)} \times \ldots \times X^{(m+2n)}) = P_g(X^{(m-1+\sigma(1))} \times \ldots \times X^{(m+n-1+\sigma(n+1))})$$

implies that $\sigma = 1$. \hfill \Box

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