Entire solutions with exponential growth for an elliptic system modelling phase separation

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Abstract

We prove the existence of entire solutions with exponential growth for the semilinear elliptic system

\[
\begin{aligned}
-\Delta u &= -uv^2 & \text{in } \mathbb{R}^N, \\
-\Delta v &= -u^2v & \text{in } \mathbb{R}^N, \\
u, v &> 0,
\end{aligned}
\]

for every \( N \geq 2 \). Our construction is based on an approximation procedure, whose convergence is ensured by suitable Almgren-type monotonicity formulae. The construction of the resulting solutions is extended to systems with \( k \) components, for every \( k > 2 \); in this case, the proof is much more involved and is achieved by approximation of solutions with exponential growth by means of solutions with algebraic growth of increasing degree, translating the limit

\[
\lim_{d \to +\infty} 3 \left( 1 + \frac{z}{d} \right)^d = e^z \sin y
\]

in the present setting.

Keywords: elliptic system, phase separation, Almgren monotonicity formulae, entire solutions, exponential growth

Mathematics Subject Classification: 35B08, 35B40, 35A01, 35J46, 35Q40

(Some figures may appear in colour only in the online journal)
1. Introduction and main results

In this paper we investigate the existence of entire solutions with exponential growth for the semilinear elliptic system

\[
\begin{cases}
-\Delta u = -uv^2, \\
-\Delta v = -u^2v, \\
u, v > 0,
\end{cases}
\]

in \(\mathbb{R}^2\) (thus in \(\mathbb{R}^N\) for every \(N \geq 2\)). System (1), which appears in the study of phase-separation phenomena for Bose–Einstein condensates with multiple states, has been intensively studied in the last few years; we refer in particular to [1, 3–5, 9, 10], where physical motivations are discussed and a precise description of the phase separation is derived, and to [1, 2] where existence and qualitative properties of entire solutions are central topics. In [9], it is proved that if \((u, v)\) is an entire solution to (1) and is globally \(\alpha\)-Hölder continuous for some \(\alpha \in (0, 1)\), then one between \(u\) and \(v\) is constant while the other is identically 0. On the other hand, in [1] the authors show that there exists a nontrivial solution for the system of ordinary differential equations

\[
\begin{cases}
-u'' = -uv^2 \text{ in } \mathbb{R}, \\
v'' = -u^2v \text{ in } \mathbb{R}, \\
u, v > 0,
\end{cases}
\]

which is reflectionally symmetric with respect to a point of \(\mathbb{R}^2\), in the sense that there exists \(t_0 \in \mathbb{R}\) such that \(u(t_0 + t) = v(t_0 - t)\) for every \(t \in \mathbb{R}\), and has linear growth: there exists \(C > 0\) such that \(u(t) + v(t) \leqslant C(1 + |t|)\) for every \(t \in \mathbb{R}\). Reference [2] completes the study of the one-dimensional problem with the proof of the uniqueness of the positive one-dimensional profile, up to translations and scaling. Always in [2], the authors construct entire solutions to (1) with algebraic growth for any integer rate of growth greater than 1; here and in the rest of the paper we say that \((u, v)\) has algebraic growth if there exist \(p \geq 1\) and \(C > 0\) such that \(u(x) + v(x) \leqslant C(1 + |x|^p)\) for every \(x \in \mathbb{R}^N\). The solutions constructed in [2] are not one-dimensional, and are modelled on (we will be more precise later, see remark 1.2) the homogeneous harmonic polynomials \(\mathbb{R}(z^d)\), for every \(d \geq 2\). There is a deep relationship between entire solutions to (1) and harmonic functions; this relationship has been established in [5, 9]. For instance, in case \((u, v)\) has algebraic growth, it is possible to show that up to a subsequence, the blow-down family, defined by

\[
(u_R(x), v_R(x)) = \frac{R^{N-1}}{\int_{\partial B_R(0)} u^2 + v^2} (u(Rx), v(Rx)),
\]

is uniformly convergent in every compact subset of \(\mathbb{R}^N\), as \(R \to +\infty\), to a limiting profile \((\Psi^+, \Psi^-)\), where \(\Psi\) is a homogeneous harmonic polynomial (see theorem 1.4 in [2]).

To conclude this bibliographic introduction, we have to mention that major efforts have been made recently in order to prove classification results and in particular the one-dimensional symmetry of solutions to (1). This is motivated by the relationship between (1) and the Allen–Cahn equation, which has been established in [1], and led the authors to formulate a De Giorgi-type and a Gibbons-type conjecture for solutions to (1); for results in this direction, we refer to [1, 2, 6, 7, 11].

Motivated by the quoted achievements, we wonder if system (1) has solutions with super-algebraic growth. We can give a positive answer to this question proving the existence of solutions with exponential growth. In our construction we adapt the same line of reasoning introduced in the proof of theorem 1.3 of [2]. Therein, the authors proved the existence of
solutions to (1) with the same symmetry of the function $\Re(z^d)$ in any bounded ball $B_R(0) \subset \mathbb{R}^2$, with boundary conditions $u = (\Re(z^d))^+$, $v = (\Re(z^d))^-$ on $\partial B_R(0)$. By means of suitable monotonicity formulae, they could pass to the limit for $R \to +\infty$ obtaining convergence (up to a subsequence) for the previous family to a nontrivial entire solution. In this sense, their solutions are modelled on the harmonic functions $\Re(z^d)$.

Here, having in mind the construction of solutions with exponential growth, and recalling the relationship between the entire solution of our system and harmonic functions, we start by considering

$$\Phi(x, y) := \cosh x \sin y.$$ 

The first of our main results is the following.

**Theorem 1.1.** There exists an entire solution $(u, v) \in (C^\infty(\mathbb{R}^2))^2$ to system (1) such that

1. $u(x, y + 2\pi) = u(x, y)$ and $v(x, y + 2\pi) = v(x, y)$;
2. $u(-x, y) = u(x, y)$ and $v(-x, y) = v(x, y)$;
3. The symmetries
   $$v(x, y) = u(x, y - \pi), \quad u(x, \pi - y) = v(x, \pi + y),$$
   $$u\left(x, \frac{\pi}{2} + y\right) = u\left(x, \frac{\pi}{2} - y\right), \quad v\left(x, \frac{3}{2}\pi + y\right) = v\left(x, \frac{3}{2}\pi - y\right)$$

   hold;
4. $u - v > 0$ in $\{\Phi > 0\}$ and $v - u > 0$ in $\{\Phi < 0\}$;
5. $u > \Phi^+$ and $v > \Phi^-$ in $\mathbb{R}^2$;
6. the function (Almgren quotient)
   $$r \mapsto \frac{\int_{\{r\} \times [0, 2\pi]} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\} \times [0, 2\pi]} u^2 + v^2}$$

   is nondecreasing, and
   $$\lim_{r \to +\infty} \frac{\int_{\{r\} \times [0, 2\pi]} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\} \times [0, 2\pi]} u^2 + v^2} = 1;$$
7. there exists the limit
   $$\lim_{r \to +\infty} \frac{1}{e^{2r}} \int_{\{r\} \times [0, 2\pi]} u^2 + v^2 =: \alpha \in (0, +\infty).$$

**Remark 1.2.** This solution is modelled on the harmonic function $\Phi$, in the sense that it inherits the symmetries of $(\Phi^+, \Phi^-)$ and has the same rate of growth of $\Phi$.

**Remark 1.3.** Point (7) of the theorem gives a lower and a upper bound to the rate of growth of the quadratic mean of $(u, v)$ on $\{r\} \times [0, 2\pi]$ when $r$ varies:

$$\left(\int_{\{r\} \times [0, 2\pi]} u^2 + v^2\right)^{\frac{1}{2}} = O(e^r) \quad \text{as} \ r \to +\infty.$$

The domain of integration takes into account the periodicity of $(u, v)$. The quadratic mean of $(u, v)$ on $\{r\} \times [0, 2\pi]$ has exponential growth, and the rate of growth is the same of the function $e^r$, which in turns has the same rate of growth of $\Phi$. Note that the coefficient 1 in the exponent of $e^r$ coincides with the limit as $r \to +\infty$ of the Almgren quotient defined in point (6).
Remark 1.4. With a scaling argument, it is not difficult to prove the existence of entire solutions with exponential growth of order $\lambda$ for every $\lambda > 0$ (in the previous sense). To see this, let

$$(u_\lambda(x, y), v_\lambda(x, y)) = (\lambda u(\lambda x, \lambda y), \lambda v(\lambda x, \lambda y)).$$

It is straightforward to check that $(u_\lambda, v_\lambda)$ is still a solution to (1) in the plane, is $2\pi/\lambda$-periodic in $y$, and has exponential growth of order $\lambda$ in $x$, that is,

$$\left(\int_{[0, 2\pi/\lambda]} u^2 + v^2 \right)^{1/2} = O(e^{r}) \quad \text{as} \quad r \to +\infty.$$ 

This marks a relevant difference between entire solutions with polynomial growth and entire solutions with exponential growth: while in the former case the admissible rates of growth are quantized (theorem 1.4 of [2]), in the latter one we can prescribe any positive real value as the rate of growth. One can consider the solution $(u_\lambda, v_\lambda)$ as related to the harmonic function $\cosh(\lambda x) \sin(\lambda y)$. This reveals that there exists a correspondence $\{(u_\lambda, v_\lambda) : \lambda > 0\} \leftrightarrow \{\sin(\lambda x) \cosh(\lambda y) : \lambda > 0\}$. Due to the invariance under translations and rotations of problem (1), the family $\{(u_\lambda, v_\lambda) : \lambda > 0\}$ can equivalently be related to the families of harmonic functions $[\cosh(\lambda x)[C_1 \cos(\lambda y) + C_2 \sin(\lambda y)]]$ or $[C_3 \cos(\lambda x) + C_4 \sin(\lambda x)] \cosh(\lambda y) : \lambda > 0\}$, where $C_1, C_2, C_3, C_4 \in \mathbb{R}$.

Remark 1.4 reveals that, starting from the solution found in theorem 1.1, we can build infinitely many entire solutions with different exponential growth. However, noting that system (1) is invariant under rotations, translations and scaling, intuitively speaking they are all the same solution. We wonder if there exists an entire solution of (1) having exponential growth which cannot be obtained by the one found in theorem 1.1 through a rotation, a translation or a scaling; the answer is affirmative. We denote

$$\Gamma(x, y) := e^x \sin y.$$ 

Theorem 1.5. There exists an entire solution $(u, v) \in (C^\infty(\mathbb{R}^2))^2$ to system (1) which satisfies points (1) and (3) of theorem 1.1; moreover,

(2) for every $r \in \mathbb{R}$

$$\int_{(-\infty, r) \times (0, 2\pi)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2 < +\infty;$$

(4) $u - v > 0$ in $[\Gamma > 0]$ and $v - u > 0$ in $[\Gamma < 0]$;

(5) $u > \Gamma^+$ and $v > \Gamma^-$ in $\mathbb{R}^2$;

(6) the function (Almgren quotient)

$$r \mapsto \frac{\int_{(-\infty, r) \times (0, 2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2 v^2}{\int_{(r) \times (0, 2\pi)} u^2 + v^2}$$

is nondecreasing, and

$$\lim_{r \to +\infty} \frac{\int_{(-\infty, r) \times (0, 2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2 v^2}{\int_{(r) \times (0, 2\pi)} u^2 + v^2} = 1;$$

(7) there exist the limits

$$\lim_{r \to +\infty} \frac{1}{2r^2} \int_{(r) \times (0, 2\pi)} u^2 + v^2 =: \beta \in (0, +\infty) \quad \text{and} \quad \lim_{r \to -\infty} \int_{(r) \times (0, 2\pi)} u^2 + v^2 = 0.$$

Remark 1.6. This solution is modelled on the harmonic function $\Gamma$. As explained in remark 1.3, it is possible to obtain a family of entire solutions, which is in correspondence with a family of harmonic functions.
Remark 1.7. Note that the Almgren quotients that we defined in theorems 1.1 and 1.5 are different. They are both different from the Almgren quotient which has been defined in [2].

We can generalize our existence result to the case of systems with many components. To be precise, given an integer \( k \), we will construct a solution \((u_1, \ldots, u_k)\) of
\[
\begin{cases}
-\Delta u_i = -u_i \sum_{j \neq i} u_j^2, \\
u_i > 0,
\end{cases}
\]
in the whole plane \( \mathbb{R}^2 \) having the same growth and the same symmetries of \( \Gamma \). Here and in the paper we consider the indices \( \text{mod } k \).

Theorem 1.8. There exists an entire solution \((u_1, \ldots, u_k) \in (C^\infty(\mathbb{R}^2))^k\) to system (2) such that, for every \( i = 1, \ldots, k \),

(1) \( u_i(x, y + k\pi) = u_i(x, y) \);

(2) the symmetries
\[
u_i+1(x, y) = u_i(x, y - \pi), \quad u_1(x, \frac{\pi}{2} + y) = u_1(x, \frac{\pi}{2} - y)
\]
hold;

(3) for every \( r \in \mathbb{R} \):
\[
\int_{(\mathbb{R} \times (0,2\pi))} |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 < +\infty;
\]

(4) the function (Almgren quotient)
\[
r \mapsto \int_{(\mathbb{R} \times (0,2\pi))} \sum_{i=1}^{k} |\nabla u_i|^2 + 2 \sum_{1 \leq i < j \leq k} u_i^2 u_j^2
\]
is nondecreasing, and
\[
\lim_{r \to +\infty} \int_{(\mathbb{R} \times (0,2\pi))} \sum_{i=1}^{k} |\nabla u_i|^2 + 2 \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 = 1;
\]

(5) there exist the limits
\[
\lim_{r \to +\infty} \frac{1}{2^r} \int_{(\mathbb{R} \times (0,2\pi))} \sum_{i=1}^{k} u_i^2 =: \gamma \in (0, +\infty) \quad \text{and} \quad \lim_{r \to -\infty} \int_{(\mathbb{R} \times (0,2\pi))} \sum_{i=1}^{k} u_i^2 = 0.
\]

This solution is modelled on \( \Gamma \). It is also possible to construct a solution of (2) modelled on \( \Phi \); we give only the sketch of the proof of such a result in the forthcoming remark 5.15.

Our last main result is the counterpart of theorem 1.4 of [2] in our setting. This can be quite surprising because, as we already observed, we cannot expect a quantization of the admissible rates of growth dealing with solutions with exponential growth, see remark 1.4. Nevertheless, if we consider solutions which are periodic in one direction, prescribing a period such a quantization can be recovered.

Theorem 1.9. Let \((u, v)\) be a nontrivial solution of (1) in \( \mathbb{R}^2 \) which is 2\pi-periodic in \( y \), and such that one of the following situations occurs:

(i) it holds that
\[
\lim_{r \to -\infty} \int_{[r] \times (0,2\pi]} u^2 + v^2 = 0,
\]
and
\[
d := \lim_{r \to +\infty} \frac{\int_{(\mathbb{R} \times (0,2\pi])} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\int_{[r] \times (0,2\pi]} u^2 + v^2} < +\infty.
\]
(ii) $\partial_t u = 0 = \partial_x v$ on $[a] \times [0, 2\pi]$ for some $a \in \mathbb{R}$, and 

$$d := \lim_{r \to +\infty} \frac{\int_{[-r, r] \times [0, 2\pi]} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\int_{[r, r] \times [0, 2\pi]} u^2 + v^2} < +\infty.$$ 

Then $d$ is a positive integer, 

$$\left( \int_{[r] \times [0, 2\pi]} u^2 + v^2 \right)^{\frac{1}{2}} = O(e^{dr}) \quad \text{as} \quad r \to +\infty,$$

and, up to a subsequence, the family $\{(u_R, v_R)\}$ defined by 

$$(u_R(x, y), v_R(x, y)) := \frac{1}{\sqrt{\int_{[r] \times [0, 2\pi]} u^2 + v^2}} (u(x + R, y), v(x + R, y))$$

converges in $C^0_{lo}(\mathbb{R}^2)$ and in $H^1_{lo}(\mathbb{R}^2)$, as $R \to +\infty$, to $(\Psi^+, \Psi^-)$, where the function $\Psi$ is harmonic with exponential growth of order $d$: $\Psi(x, y) = e^{dz}(C_1 \cos(dy) + C_2 \sin(dy))$ for some $C_1, C_2 \in \mathbb{R}$.

**Notation.** We will deal with functions defined in domains of type $(a, b) \times \mathbb{R}$, where $a < b$ are extended real numbers ($a = -\infty$ and $b = +\infty$ are admissible). We will often assume that $(u_1, \ldots, u_k)$ is $k\pi$-periodic in $y$; therefore, we can think to $(u_1, \ldots, u_k)$ as defined on the cylinder 

$$C_{(a, b)} := (a, b) \times S_k \quad \text{where} \quad S_k = \mathbb{R}/(k\pi \mathbb{Z}).$$

We will also denote $\Sigma_r := [r] \times S_k$. In case $b > 0$, $a = -b$, we will simply write $C_b$ instead of $C_{(-b, b)}$ to simplify the notation.

**Plan of the paper.** In section 2 we will prove some monotonicity formulae which will be used in the rest of the paper. We can deal with two types of solutions: solutions satisfying a homogeneous Neumann condition defined in a cylinder $C_{(a, b)}$ with $a > -\infty$, or solutions defined in a semi-infinite cylinder of type $C_{(-\infty, b)}$ and decaying at $x \to -\infty$. For the sake of completeness and having in mind to use some monotonicity formulae in the proof of theorem 1.8, we will always consider the case of systems with $k$ components.

The proof of theorem 1.1 will be the object of section 3. It follows the same sketch of the proof as theorem 1.3 in [2]: we start by showing that for any $R > 0$ there exists a solution $(u_R, v_R)$ to (1) in the cylinder $C_R$, with Dirichlet boundary condition 

$$u_R = \Phi^+ \quad \text{and} \quad v_R = \Phi^- \quad \text{on} \quad [-R, R] \times [0, 2\pi],$$

and exhibiting the same symmetries of $(\Phi^+, \Phi^-)$. In order to obtain a solution defined in the whole $C_\infty$, we wish to prove the $C^2_{lo}(C_\infty)$ convergence of the family $\{(u_R, v_R) : R > 1\}$, as $R \to +\infty$. To show that this convergence occurs, we will exploit the monotonicity formulae proved in section 2.1. With respect to theorem 1.3 of [2], major difficulties arise in the precise characterization of the growth of $(u, v)$, points (6) and (7) of theorem 1.1.

In section 4 we will prove theorem 1.5. One could be tempted to try to adapt the proof of theorem 1.1 replacing $\Phi$ with $\Gamma$. Unfortunately, in such a situation we could not exploit the results of section 2.1; this is related to the lack of the even symmetry in the $x$ variable of the function $\Gamma$. A possible way to overcome this problem is to work in semi-infinite cylinders $C_{(-\infty, R)}$ and use the monotonicity formulae proved in section 2.2. But working in an unbounded set introduces further complications: for instance, the compactness of the Sobolev embedding and of some trace operators, a property that we will use many times in section 3, does not hold in $C_{(-\infty, R)}$. Although we believe that this kind of obstacle can be overcome, we
propose a different approach for the construction of solutions modelled on \( \Gamma \), which is based on the elementary limit
\[
\lim_{R \to +\infty} \Phi_R(x, y) = \Gamma(x, y) \quad \forall (x, y) \in \mathbb{R}^2,
\]
where \( \Phi_R(x, y) = 2e^{-R \cosh(x + R)} \sin y \). We will prove the existence of a solution \((u_R, v_R)\) of (1) in \( C_{(-3R, R)}\) with Dirichlet boundary condition
\[
u_R = \Phi_R^+ \quad \text{and} \quad v_R = \Phi_R^- \quad \text{on} \quad \{-3R, R\} \times [0, 2\pi],
\]
and exhibiting the same symmetries of \((\Phi_R^+, \Phi_R^-)\). Then, using again the results of section 2, we will pass to the limit as \( R \to +\infty \) proving the compactness of \( \{(u_R, v_R)\} \).

Section 5 is devoted to the study of systems with many components. As in [2] the authors could prove in one shot an existence theorem for 2 or \( k \) components (there are no substantial changes in the proofs), it is natural to wonder whether here we can simply adapt step by step the construction carried out in sections 3 and 4, or not. Unfortunately, the answer is negative: following the sketch of the proof of theorem 1.1, we can adapt most of the results of sections 3 and 4 with minor changes, but with respect to the results of sections 3.2 and 4.2, we cannot show that the limit of the sequence \((u_1, u_2, \ldots, u_k, R)\) does not vanish (this fact follows from a subtle technical point). This is why we have to use a completely different argument which is not based on the existence of solutions for the system of \( k \) components in bounded cylinders (or in semi-infinite cylinders), but rests on theorem 1.6 of [2]. Roughly speaking, we will obtain the existence of a solution of (2) with exponential growth as a limit of solutions of the same system having algebraic growth. Roughly speaking, we translate the limit
\[
\lim_{d \to +\infty} \Im \left( \left( 1 + \frac{z}{d} \right)^d \right) = e^x \sin y,
\]
where \( \Im \) denotes the imaginary part of a complex function, in terms of solutions to (1): we consider a sequence of solutions to (1) with polynomial growth of order \( d \), and, after suitable scaling, we show that it converges to a solution of (1) having exponential growth, that is,
\[
\int_{[r] \times [0, k\pi]} \sum_{i=1}^{k} u_i^2 = O(e^{2r}) \quad \text{as} \quad r \to +\infty.
\]
The proof of theorem 1.9 will be the object of section 6.

2. Almgren-type monotonicity formulae

Let \( k \geq 2 \) be a fixed integer. In this section we are going to prove some monotonicity formulae for solutions of
\[
\begin{cases}
-\Delta u_i = -u_i \sum_{j \neq i} u_j^2, \\
u_i > 0
\end{cases}
\]
defined in a cylinder \( C_{(a, b)} \) (this means that we assume from the beginning that \((u_1, \ldots, u_k)\) is \( k\pi \)-periodic in \( y \)). We will use many times the following general result.

**Lemma 2.1.** Let \((u_1, \ldots, u_k)\) be a solution of (2) in \( C_{(a, b)} \). Then the function
\[
r \mapsto \int_{\Sigma_p} \sum_{i=1}^{k} \left| \nabla u_i \right|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 - 2 \int_{\Sigma_p} \sum_{i=1}^{k} (\partial_y u_i)^2
\]
is constant in \((a, b)\).
Proof. Let $a < r_1 < r_2 < b$. We test equation (3) with \((\partial_i u_1, \ldots, \partial_i u_k)\) in \(C_{(r_1, r_2)}\): for every \(i\) it results that
\[
\int_{C_{(r_1, r_2)}} \frac{1}{2} \partial_i \left( |\nabla u_i| ^2 \right) + \left( \sum_{j \neq i} u_j^2 \right) u_i \partial_i u_i = \int_{\Sigma_2} (\partial_i u_i)^2 - \int_{\Sigma_1} (\partial_i u_i)^2.
\]
Summing for \(i = 1, \ldots, k\) we obtain
\[
\int_{C_{(r_1, r_2)}} \partial_i \left( \sum_i |\nabla u_i|^2 + \sum_{i < j} u_i^2 u_j^2 \right) = 2 \int_{\Sigma_2} \sum_i (\partial_i u_i)^2 - 2 \int_{\Sigma_1} \sum_i (\partial_i u_i)^2,
\]
which gives the thesis. \(\square\)

2.1. Solutions with Neumann boundary conditions

In this section we are interested in solutions to (3) defined in \(C_{(a, b)}\) (thus \(k\pi\)-periodic in \(y\)), with \(a > -\infty\) and \(b \in (a, +\infty]\), and satisfying a homogeneous Neumann boundary condition on \(\Sigma_a\), that is,
\[
\partial x u_i = 0 \text{ on } \Sigma_a, \quad \text{for every } i = 1, \ldots, k. \tag{4}
\]
First, we observed that under this assumption lemma 2.1 implies the following:

**Lemma 2.2.** Let \((u_1, \ldots, u_k)\) be a solution of (3) in \(C_{(a, b)}\), such that (4) holds true. For every \(r \in (a, b)\) the following identity holds:
\[
\int_{E_r} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 = 2 \int_{\Sigma_r} \sum_{i=1}^k (\partial_i u_i)^2 + \int_{\Sigma_r} \sum_{1 \leq i < j \leq k} u_i^2 u_j^2.
\]

For a solution \((u_1, \ldots, u_k)\) of (3) in \(C_{(a, b)}\) satisfying (4), we define
\[
E_{\text{sym}}(r) := \int_{C_{(a, r)}} \sum_{i=1}^k |\nabla u_i|^2 + 2 \sum_{1 \leq i < j \leq k} u_i^2 u_j^2,
\]
\[
E_{\overline{\text{sym}}}(r) := \int_{C_{(a, r)}} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2,
\]
\[
H(r) := \int_{\Sigma_r} \sum_{i=1}^k u_i^2.
\]

**Remark 2.3.** The index sym denotes the fact that, as we will see, the quantities \(E_{\text{sym}}\) and \(E_{\overline{\text{sym}}}\) are well suited to describe the growth of the solution \((u_1, \ldots, u_k)\) only if \((u_1, \ldots, u_k)\) satisfies condition (4), which can be considered as a symmetry condition. In fact, under (4) one can extend \((u_1, \ldots, u_k)\) on \(C_{(2\pi-a, b)}\) by even symmetry in the \(x\) variable.

By regularity, \(E\), \(E\) and \(H\) are smooth. A direct computation shows that they are nondecreasing functions: in particular,
\[
H'(r) = 2 \int_{E_r} \sum_{i} u_i \partial x u_i = 2E(r), \tag{5}
\]
where the last identity follows from the divergence theorem and the boundary conditions of \((u_1, \ldots, u_k)\). Our next result consist in showing that also the ratio between \(E\) (or \(E\)) and \(H\) is nondecreasing.
Proposition 2.4. Let \((u_1, \ldots, u_k)\) be a solution of (3) in \(C(a,b)\) such that (4) holds true. The Almgren quotient \(N^{\text{sym}}(r) := E^{\text{sym}}(r)/H(r)\) is nondecreasing in \((a, b)\). Moreover

\[
\int_a^r \frac{\sum_{i<j} u_i^2 u_j^2}{H(s)} \, ds \leq N(r).
\]

Analogously, the function (which we will call Almgren quotient, too) \(\mathfrak{N}^{\text{sym}}(r) := E^{\text{sym}}(r)/H(r)\) is nondecreasing in \((a, b)\).

In the rest of this subsection we will briefly write \(E, E, N\) and \(N\) instead of \(E^{\text{sym}}, E^{\text{sym}}, N^{\text{sym}}\) and \(N^{\text{sym}}\) to ease the notation.

Proof. Since \((u, v) \in H^1_{\text{loc}}(C(a,b))\) is nontrivial, \(E\) and \(H\) are positive in \((a, b)\) and bounded for \(r\) bounded. We compute, by means of lemma 2.2,

\[
E'(r) = \int_{\Sigma} \sum_i |\nabla u_i|^2 + 2 \sum_{i<j} u_i^2 u_j^2
\]

\[
= \int_{\Sigma} 2 \sum_i (\partial_x u_i)^2 + \sum_{i<j} u_i^2 u_j^2 + \int_{\Sigma} (\partial_y u_i)^2 + \sum_{i<j} u_i^2 u_j^2.
\]

Using the previous identity and (5) we are in position to compute the logarithmic derivative of \(N\):

\[
N'(r) = \frac{E'(r)}{E(r)} - \frac{H'(r)}{H(r)}
\]

\[
\geq 2 \left( \frac{\int_{\Sigma} \sum_i (\partial_x u_i)^2}{\int_{\Sigma} \sum_i u_i^2} - \frac{\int_{\Sigma} \sum_i u_i \partial_x u_i}{\int_{\Sigma} \sum_i u_i^2} \right) + \frac{\int_{\Sigma} \sum_{i<j} u_i^2 u_j^2}{E(r)} \geq 0,
\]

where we used the Cauchy–Schwarz and the Young inequalities. As a consequence, \(N\) is nondecreasing in \((a, b)\). Note also that

\[
N'(r) \geq \frac{\int_{\Sigma} \sum_{i<j} u_i^2 u_j^2}{H(r)} \Rightarrow N(r) \geq \int_a^r \frac{\sum_{i<j} u_i^2 u_j^2}{H(s)} \, ds
\]

for every \(r > a\). The same argument can be adapted with minor changes to prove the monotonicity of \(\mathfrak{N}\). □

As a first consequence, we have the following:

Corollary 2.5. Let \((u_1, \ldots, u_k)\) be a solution of (3) in \(C(a,b)\) such that (4) holds.

(i) If \(N(r) \geq d\) for \(r \geq s > a\), then

\[
\frac{H(r_1)}{e^{d r_1}} \leq \frac{H(r_2)}{e^{d r_2}} \quad \forall s \leq r_1 < r_2 < b,
\]

(ii) If \(N(r) \leq d\) for \(r \leq t < b\), then

\[
\frac{H(r_1)}{e^{d r_1}} \geq \frac{H(r_2)}{e^{d r_2}} \quad \forall a < r_1 < r_2 \leq t.
\]

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Theorem 2.6. Let \( k \) be a fixed integer and let \( \Lambda > 1 \). Let 
\[
\mathcal{L}(k, \Lambda) := \min \left\{ \sum_{i=1}^{\infty} \sum_{1 \leq i < j \leq k} f_i^2 f_j^2 \left| f_1, \ldots, f_k \in H^1([0, 2\pi]), \int_0^{2\pi} \sum_{i=1}^k f_i^2 = 1 \right| \right\},
\]
where the indices are counted mod \( k \). There exists \( C > 0 \) such that 
\[
\left( \frac{k}{2} \right)^2 - C \Lambda^{-1/4} \leq \mathcal{L}(k, \Lambda) \leq \left( \frac{k}{2} \right)^2.
\]

Remark 2.7. Having in mind to apply theorem 2.6 on \( 2\pi \)-periodic functions, note that the condition \( f_1(\pi + t) = f_1(\pi - t) \) can be replaced by \( f_1(t + \tau) = f_1(t - \tau) \) for any \( \tau \in [0, 2\pi) \).

For a fixed \( r_0 \in (a, b) \), let us introduce
\[
\psi(r; r_0) := \int_{r_0}^{r} \frac{ds}{H(s)^{1/4}}.
\]
The function \( \psi \) is positive and increasing in \( \mathbb{R}^+ \); thanks to point (i) of corollary 2.5 and to the monotonicity of \( N \), whenever \((u, v)\) is nontrivial \( \psi \) is bounded by a quantity depending only on \( H(r_0) \) and \( N(r_0) \). To be precise:
\[
\psi(r; r_0) \leq 2 \frac{e^{-\frac{1}{2}N(r_0)\rho}}{H(r_0)^{1/4}} \left[ e^{-\frac{1}{2}N(r_0)\rho} - e^{-\frac{1}{2}N(r_0)\rho} \right].
\]
This, together with the monotonicity of \( \psi(\cdot; r_0) \), implies that if \( b = +\infty \) then there exists the limit
\[
\lim_{r \to +\infty} \psi(r; r_0) < +\infty.
\]

Lemma 2.8. Let \((u_1, \ldots, u_k)\) be a solution of (1) in \( C_{(a, b)} \) such that (4) holds. Let \( r_0 \in (a, b) \), and assume that 
\[
u_{i+1}(x, y) = u_i(x, y - \pi) \quad \text{and} \quad u_1(x, \tau + y) = u_1(x, \tau - y),
\]
where \( \tau \in [0, k\pi) \). There exists \( C > 0 \) such that the function \( r \mapsto \frac{E(r)}{e^{2r}}e^{C\psi(r; r_0)} \) is nondecreasing in \( r \) for \( r > r_0 \).

Proof. Recalling identity (5), we compute the logarithmic derivative 
\[
\frac{d}{dr} \log \left( \frac{E(r)}{e^{2r}} \right) = -2 + \frac{1}{2} \sum_{i \neq j} (\partial_i u_i)^2 + \frac{1}{2} \sum_{i \neq j} (\partial_i u_i)^2 + 2 \sum_{i \neq j} u_i u_j^2 u_j^2.
\]
To apply theorem 2.6, we observe that \( \Sigma_r = [r] \times [0, k\pi] \), so that
\[
\int_{\Sigma_r} (\partial_i u_i)^2 + 2 \sum_{i \neq j} u_i^2 u_j^2 = \int_0^{2\pi} \sum_i (\partial_i u_i(r, y))^2 + 2 \sum_{i \neq j} u_i(r, y)^2 u_j(r, y)^2 \, dy
\]
\[
\quad = \frac{2}{k} \int_0^{2\pi} \left[ \sum_i (\partial_i \tilde{u}_i(r, y))^2 + 2 \left( \frac{k}{2} \right)^2 \sum_{i \neq j} \tilde{u}_i(r, y)^2 \tilde{u}_j(r, y)^2 \right] \, dy,
\]
\[
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\]
where $\tilde{u}_i(r, y) = u_i(r, ky/2)$. By a scaling argument, thanks to assumption (9) (see also remark 2.7), we can say that for every $\Lambda > 1/2$ the following holds:
\[
\int_0^{2\pi} \sum_i (\partial_y \tilde{u}_i(r, y))^2 + \left(\frac{k}{2}\right)^2 \int_0^{2\pi} \sum_{i < j} \tilde{u}_i(r, y)^2 \tilde{u}_j(r, y)^2 \, dy \geq \mathcal{L} \left( k, 2\Lambda \left(\frac{k}{2}\right)^2 \right) \int_0^{2\pi} \sum_i \tilde{u}_i(r, y)^2 \, dy = \frac{2}{k} \mathcal{L} \left( k, 2\Lambda \left(\frac{k}{2}\right)^2 \right) \int_0^{2\pi} \sum_i u_i^2.
\]

The choice $\Lambda = 2H(r)/k$ yields
\[
\int_0^{2\pi} \sum_i (\partial_y \tilde{u}_i(r, y))^2 + 2 \left(\frac{k}{2}\right)^2 \int_0^{2\pi} \sum_{i < j} \tilde{u}_i(r, y)^2 \tilde{u}_j(r, y)^2 \, dy \geq \frac{2}{k} \mathcal{L} \left( k, kH(r) \right) \int_0^{2\pi} \sum_i u_i^2,
\]
and coming back to (11) we obtain
\[
\int_0^{2\pi} \sum_i (\partial_y u_i)^2 + 2 \sum_{i < j} u_i^2 u_j^2 \geq \left(\frac{2}{k}\right)^2 \mathcal{L} \left( k, kH(r) \right) \int_0^{2\pi} \sum_i u_i^2.
\]

Plugging this estimate into equation (10), we see that
\[
\frac{d}{dr} \log \left( \frac{E(r)}{e^{2r}} \right) \geq -2 + \frac{\int_0^{2\pi} \sum_i (\partial_y u_i)^2 + \left(\frac{k}{2}\right)^2 \mathcal{L} \left( k, kH(r) \right) \int_0^{2\pi} \sum_i u_i^2}{\int_0^{2\pi} \sum_i u_i \partial_y u_i} \geq -2 + \frac{2}{k} \sqrt{\mathcal{L} \left( k, kH(r) \right)} \geq -C_H(r)^{1/2},
\]
where we used theorem 2.6. An integration gives the thesis. \(\square\)

**Lemma 2.9.** Let $(u_1, \ldots, u_k)$ be a nontrivial solution of (3) in $C(a, +\infty)$, and assume that (4) and (9) hold. If $d := \lim_{r \to +\infty} N(r) < +\infty$, then $d \geq 1$ and
\[
\lim_{r \to +\infty} \frac{E(r)}{e^{2r}} = 0.
\]

**Proof.** Let us fix $r_0 > a$. First, from the previous lemma and the estimate (8), we deduce that there exists the limit
\[
l := \lim_{r \to +\infty} \frac{E(r)}{e^{2r}} = 0.
\]
Recalling that $\varphi(r; r_0)$ is bounded, it results that
\[
\frac{E(r)}{e^{2r}} \geq e^{-c_{\varphi(r; r_0)} E(r_0)} \frac{E(r_0)}{e^{2r_0}} \geq C > 0 \quad \forall r > r_0,
\]
so that the value $l$ is strictly greater than 0. Now, assume by contradiction that $d = \lim_{r \to +\infty} N(r) < 1$. The monotonicity of $N$ implies $N(r) \leq d$ for every $r > 0$. Hence, from corollary 2.5 we deduce
\[
\frac{H(r)}{e^{2dr}} \leq \frac{H(r_0)}{e^{2dr_0}} \quad \forall r > r_0 \Rightarrow \lim_{r \to +\infty} \sup_{r \to +\infty} \frac{H(r)}{e^{2dr}} < +\infty \Rightarrow \lim_{r \to +\infty} \frac{H(r)}{e^{2r}} = 0,
\]
which in turn gives
\[
0 < l = \lim_{r \to +\infty} \frac{E(r)}{e^{2r}} = \lim_{r \to +\infty} N(r) \lim_{r \to +\infty} \frac{H(r)}{e^{2r}} = 0,
\]
a contradiction. \(\square\)
2.2. Solutions with finite energy in unbounded cylinders

In what follows we consider a solution $(u_1, \ldots, u_k)$ of (3) defined in an unbounded cylinder $C_{(-\infty, b)}$, with $b \in \mathbb{R}$ (the choice $b = +\infty$ is admissible). In this setting we assume that $(u_1, \ldots, u_k)$ has a sufficiently fast decay as $x \to -\infty$, in the sense that

$$H(r) := \int_{\Sigma_r} \sum_{i=1}^k u_i^2 \to 0 \quad \text{as} \quad r \to -\infty. \quad (12)$$

First of all, we can show that under assumption (12) $(u_1, \ldots, u_k)$ has finite energy in $C_{(-\infty, b)}$.

**Lemma 2.10.** Let $(u_1, \ldots, u_k)$ be a solution of (2) in $C_{(-\infty, b)}$, such that (12) holds. Then

$$E_{unb}(r) := \int_{C_{(-\infty, r)}} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 < +\infty \quad \forall r < b.$$

The index unb stands for the fact that the energy is evaluated in an unbounded cylinder, and will be omitted in the rest of the subsection.

**Proof.** Firstly, being a classical solution, $(u_1, \ldots, u_k) \in H^1_{loc}(C_{(-\infty, b)})$. Thus, under assumption (12), there exists $C > 0$ such that $H(r) \leq C$ for every $r < b$.

Let $r_0 < b$. Let us introduce, for $r > 0$, the functional

$$e(r) := \int_{C_{(-\infty, r)}} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2.$$

For the sake of simplicity, in the rest of the proof we assume $r_0 = 0$ (thus $b > 0$). By direct computation and an application of lemma 2.1, we find

$$e'(r) = \int_{\Sigma_r} \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 = 2 \int_{\Sigma_r} \sum_i (\partial_x u_i)^2 + \int_{\Sigma_0} \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 - 2 \int_{\Sigma_0} \sum_i (\partial_x u_i)^2,$$

that is

$$\int_{\Sigma_r} \sum_i (\partial_x u_i)^2 = \frac{1}{2} e'(r) + C_0.$$ 

On the other hand, testing equation (2) in $C_{(-r, 0)}$ by $(u_1, \ldots, u_k)$ and summing for $i = 1, \ldots, k$, we find

$$e(r) \leq \int_{C_{(-r, 0)}} \sum_i |\nabla u_i|^2 + 2 \sum_{i<j} u_i^2 u_j^2 = \int_{\Sigma_r} \sum_i u_i \partial_x u_i - \int_{\Sigma_r} \sum_i u_i \partial_x u_i$$

$$\leq \int_{\Sigma_0} \sum_i u_i \partial_x u_i + \left( \int_{\Sigma_r} (\partial_x u_i)^2 \right)^\frac{1}{2} \left( \int_{\Sigma_r} \sum_i u_i^2 \right)^\frac{1}{2}.$$

Let us assume by contradiction that $e(r) \to +\infty$ as $r \to +\infty$. Taking the square of the previous inequality, using the boundedness of $H$ and assumption (12), we have

$$\begin{cases} 
\frac{1}{C_0^2} (e(r) + C_1)^2 - 2C_0 \leq e'(r) & \text{for } r > \bar{r}, \\
e(\bar{r}) > 0.
\end{cases}$$

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for some $C_0$, $C_1 > 0$ and $\tilde{r}$ sufficiently large. Any solution to the previous differential inequality blows up in finite time, in contradiction to the fact that $(u_1, \ldots, u_k) \in H_1^{loc}(C_{(-\infty,b)})$. As a consequence $e$ is bounded and, by regularity,

$$\int_{C_{(-\infty,r)}} |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 < +\infty \quad \forall r < b. \quad \square$$

**Remark 2.11.** As a by-product of the previous lemma, if $(u_1, \ldots, u_k)$ solves (2) in $C_{(-\infty,b)}$ and (12) holds, then

$$\lim_{r \to -\infty} E(r) = 0.$$ 

Having in mind to recover the monotonicity formulae of the previous subsection in the present situation, we cannot adapt the proof of lemma 2.2, where assumption (4) played an important role. However, we can obtain a similar result with a different proof.

**Lemma 2.12.** Let $(u_1, \ldots, u_k)$ be a solution to (1) in $C_{(-\infty,b)}$, such that (12) holds. Then

$$\int_{\Sigma_1 r} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 = 2 \int_{\Sigma_1 r} \sum_{i=1}^k (\partial x u_i)^2$$

for every $r < b$. 

**Proof.** We use the method of the variations of the domains: for $\psi \in C_c^1(-\infty,r)$, we consider $u_{i,\varepsilon}(r, y) = u_i(r + \varepsilon \psi(r), y)$. It is possible to see $(u_{1,\varepsilon}, \ldots, u_{k,\varepsilon})$ as a smooth variations of $(u_1, \ldots, u_k)$ with compact support in $C_{(-\infty,r)}$. To proceed, we explicitly remark that any solution to (2) is critical for the energy functional

$$J(v_1, \ldots, v_k) := \int_{C_{(-\infty,b)}} \sum_{i=1}^k |\nabla v_i|^2 + \sum_{1 \leq i < j \leq k} v_i^2 v_j^2$$

with respect to variations with compact support in $C_c^\infty(C_{(-\infty,b)})$. Note that $J(u_1, \ldots, u_k) = E(b)$. As $(u_1, \ldots, u_k)$ is a smooth solution of (2) with finite energy $E(r)$, it follows that

$$0 = \lim_{\varepsilon \to 0} \int_{C_{(-\infty,r)}} \frac{\partial}{\partial \varepsilon} \left( \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 - E(r) \right)$$

$$= \int_{C_{(-\infty,r)}} \frac{\partial}{\partial \varepsilon} \left( \sum_{i=1}^k |\nabla u_i(x + \varepsilon \psi(x), y)|^2 + \sum_{1 \leq i < j \leq k} (u_i^2(x + \varepsilon \psi(x), y) u_j^2(x + \varepsilon \psi(x), y)) \right) \left| dx \, dy \right|_{\varepsilon=0}$$

$$+ 2 \lim_{\varepsilon \to 0} \int_{C_{(-\infty,r)}} \psi'(x) \sum_{i=1}^k (\partial x u_i)^2(x + \varepsilon \psi(x)) \left| dx \, dy \right|_{\varepsilon=0}$$

$$= \int_{C_{(-\infty,r)}} \left( 2 \sum_{i=1}^k (\partial x u_i)^2 - \left( \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 \right) \right) \psi'$$

for every $\psi \in C_c^1(-\infty,r)$. Since $E(r) < +\infty$, for every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset C_{(-\infty,r)}$ such that

$$\int_{C_{(-\infty,r)} \setminus K_\varepsilon} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 < \varepsilon.$$
Let \( \psi \in C^1(-\infty, r) \) be such that \( \|\psi\|_{C^1(-\infty, r)} < +\infty \) and \( \psi = 0 \) in a neighbourhood of \( r \).

It is possible to write \( \psi = \psi_1 + \psi_2 \) where \( \psi_1 \in C^1_c(-\infty, r) \) and \( \text{supp} \psi_2 \times (\mathbb{R}/k\pi \mathbb{Z}) \subset (C(-\infty, r) \setminus K_\epsilon) \). Therefore, from (13) it follows that

\[
\int_{C(-\infty, r)} \left( 2 \sum_i (\partial_x u_i)^2 - \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi' \leq 3 \|\psi\|_{C^1(-\infty, r)} \int_{C(-\infty, r) \setminus K_\epsilon} \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) < C \epsilon.
\]

Since \( \epsilon \) has been arbitrarily chosen, we obtain

\[
\int_{C(-\infty, r)} \left( 2 \sum_i (\partial_x u_i)^2 - \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi' = 0 \quad (14)
\]

for every \( \psi \in C^1(-\infty, r) \) such that \( \|\psi\|_{C^1(-\infty, r)} < +\infty \) and \( \psi = 0 \) in a neighbourhood of \( r \).

Now, let \( \psi \in C^1((-\infty, r]) \) be such that \( \|\psi\|_{C^1((-\infty, r])} < +\infty \). For a given \( \epsilon > 0 \), we introduce a cut-off function \( \eta \in C^\infty(\mathbb{R}) \) such that

\[
\eta(s) = \begin{cases} 1 & \text{if } s \leq r - \epsilon, \\ 0 & \text{if } s \geq r - \epsilon/2. \end{cases}
\]

Since \( \eta \psi \in C^1(-\infty, r), \|\eta \psi\|_{C^1(-\infty, r)} < +\infty \) and \( \eta \psi = 0 \) in a neighbourhood of \( r \), from (14) we deduce

\[
\int_{C(-\infty, r)} \left( 2 \sum_i (\partial_x u_i)^2 - \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \eta \psi' = \int_{C(-\infty, r)} \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 - 2 \sum_i (\partial_x u_i)^2 \right) \eta \psi. \quad (15)
\]

Denoting by

\[
\gamma = \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 - 2 \sum_i (\partial_x u_i)^2 \right) \psi,
\]

the right-hand side is

\[
\int_0^{k\pi} \left( \int_{r-\epsilon}^r \eta'(x) \gamma(x, y) \, dx \right) \, dy = - \int_0^{k\pi} \gamma(r - \epsilon, y) \, dy - \int_0^{k\pi} \left( \int_{r-\epsilon}^r \eta(x) \partial_x \gamma(x, y) \, dx \right) \, dy \]

\[
= \int_{\Sigma_\epsilon} \left( 2 \sum_i (\partial_x u_i)^2 - \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi + o(1) \]

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as $\varepsilon \to 0$, where the last identity follows from the regularity of $(u_1, \ldots, u_k)$ and from the $C^1$-boundedness of $\psi$ and $\eta$. Passing to the limit as $\varepsilon \to 0$ in equation (15), we deduce that for every $\psi \in C^1((-\infty, r])$ such that $\|\psi\|_{C^1((-\infty, r])} < +\infty$ it results that
\[
\int_{C(-\infty, r)} \left( 2 \sum_i (\partial_x u_i)^2 - \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi' = \int_{\Sigma_1 r} \left( 2 \sum_i (\partial_x u_i)^2 - \left( \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi.
\]
Choosing $\psi = 1$ we obtain the thesis. □

This result permits us to prove an Almgren monotonicity formula for a solution $(u_1, \ldots, u_k)$ of (2) in $C_{(-\infty, b)}$ such that (12) holds. For such a solution, let us set
\[
E_{\text{unb}}(r) := \int_{C(-\infty, r)} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2.
\]
We will briefly write $E$ in the rest of the subsection. Clearly, lemma 2.10 and the fact that $E(r) \to 0$ as $r \to -\infty$ (see remark 2.11) imply that
\[
E(r) < +\infty \quad \forall r < b \quad \text{and} \quad \lim_{r \to -\infty} E(r) = 0. \tag{16}
\]
By regularity, $E$, $E$ and $H$ are smooth. A direct computation shows that $E$ and $E$ are increasing in $r$. As far as $H$ is concerned, with respect to the previous subsection we cannot deduce identity (5) by means of a simple integration by parts, since we are working in an unbounded domain; but in light of lemma 2.12 the same result holds.

**Lemma 2.13.** Let $(u_1, \ldots, u_k)$ be a solution to (2) in $C_{(-\infty, b)}$ such that (12) holds. Then
\[
H'(r) = 2 \int_{C_{(-\infty, r)}} \sum_{i=1}^k u_i \partial_x u_i = 2E(r)
\]
for every $r < b$. In particular, $H$ is nondecreasing.

**Proof.** For every $s < r < b$, the divergence theorem and the periodicity of $(u_1, \ldots, u_k)$ imply that
\[
E(r) = E(s) + \int_{C_{(s, r)}} \sum_i |\nabla u_i|^2 + 2 \sum_{1 \leq i < j} u_i^2 u_j^2
\]
\[
= E(s) - \int_{\Sigma_s} \sum_i u_i \partial_x u_i + \int_{\Sigma_r} \sum_i u_i \partial_x u_i. \tag{17}
\]
We consider the second term on the right-hand side. Let $\eta \in C_\infty(-1, 1)$ be a nonnegative cut-off function, even with respect to $r = 0$, such that $\eta(0) = 1$ and $\eta \leq 1$ in $(-1, 1)$. Let $\eta_s(x) = \eta(x-s)$; testing equation (3) with $u_i \eta_s$ in $C_{(s-1, s)}$, we find
\[
\int_{C_{(s-1, s)}} \nabla u_i \cdot \nabla (u_i \eta_s) + u_i^2 \sum_{i \neq j} \eta_s = \int_{\Sigma_s} u_i \partial_x u_i.
\]
Summing up for $i = 1, \ldots, k$, we obtain
\[
\int_{\Sigma_s} \sum_i u_i \partial_x u_i = \int_{C_{(s-1, s)}} \sum_i (u_i \partial_x u_i, \eta_s' + |\nabla u_i|^2 \eta_s) + 2 \sum_{1 < j} u_i^2 u_j^2 \eta_s
\]
\[
\leq C(\eta') \sum_i \|u_i\|^2_{H^1(C_{(s-1, s)})} + E(s), \tag{18}
\]
where the last estimate follows from the Hölder inequality. We claim that

\[ \sum_i \|u_i\|_{H^1(C(s\rightarrow s-1),s)} \to 0 \quad \text{as } s \to -\infty. \]

This is a consequence of the Poincaré inequality

\[ \int_{C(s \rightarrow s-1),s} u^2 \leq C \left( \int_{\Sigma s} u^2 + \int_{C(s \rightarrow s-1),s} |\nabla u|^2 \right) \quad \forall u \in H^1(C(s \rightarrow s-1),s) \]

together with assumption (12) and the fact that \( E(s) \to 0 \) as \( s \to -\infty \) (see (16)). Thus, from equation (18) we deduce that

\[ \lim_{s \to -\infty} \int_{\Sigma s} \sum_i u_i \partial_x u_i = 0, \]

which in turn can be used in (17) to obtain the thesis:

\[ E(r) = \lim_{s \to -\infty} \left( E(s) - \int_{\Sigma s} \sum_i u_i \partial_x u_i + \int_{\Sigma s} \sum_i u_i \partial_y u_i \right) = \int_{\Sigma s} \sum_i u_i \partial_x u_i. \]

□

In light of the previous results, the proof of the following statements is a straightforward modification of the proofs of proposition 2.4, corollary 2.5 and lemmas 2.8 and 2.9.

**Proposition 2.14.** Let \((u_1, \ldots, u_k)\) be a solution of (3) in \( C(-\infty, b) \) such that (12) holds. The Almgren quotient \( N^\text{unb}(r) := E^\text{unb}(r)/H(r) \) is nondecreasing. Moreover,

\[ \int_{-\infty}^{r} \frac{\int_{\Sigma s} \sum_{i<j} u_i^2 u_j^2}{H(s)} \, ds \leq N(r). \]

Analogously, the function \( N^{\text{unb}}(r) := E^{\text{unb}}(r)/H(r) \) is nondecreasing.

We will briefly write \( N \) and \( \mathcal{N} \) instead of \( N^\text{unb} \) and \( N^{\text{unb}} \) in the rest of this subsection.

**Corollary 2.15.** Let \((u_1, \ldots, u_k)\) be a solution of (3) in \( C(-\infty, b) \) such that (12) holds.

(i) If \( N(r) \geq d \) for \( r \geq s \), then

\[ \frac{H(r_1)}{e^{2\delta r_1}} \leq \frac{H(r_2)}{e^{2\delta r_2}} \quad \forall \, s \leq r_1 < r_2 < b, \]

(ii) If \( N(r) \leq d \) for \( r \leq t < b \), then

\[ \frac{H(r_1)}{e^{2\delta r_1}} \geq \frac{H(r_2)}{e^{2\delta r_2}} \quad \forall \, r_1 < r_2 \leq t. \]

Assuming that

\[ u_{i+1}(x, y) = u_i(x, y - \pi) \quad \text{and} \quad u_1(x, \tau + y) = u_1(x, \tau - y) \quad (19) \]

for some \( \tau \in [0, k\pi) \), it is possible to prove that the function \( r \mapsto \frac{E(r)}{e^{2\delta r}} \) is nondecreasing, where the definition of \( \psi \) has been given in (6). A direct consequence is the following result.

**Lemma 2.16.** Let \((u_1, \ldots, u_k)\) be a nontrivial solution of (3) in \( C_{\infty} \), and assume that (12) and (19) hold. If \( d := \lim_{r \to +\infty} N(r) < +\infty \), then \( d \geq 1 \) and

\[ \lim_{r \to +\infty} \frac{E(r)}{e^{2\delta r}} > 0. \]

**Remark 2.17.** The achievements of this section hold true for positive solutions to

\[ -\Delta u_i = -\beta u_i \sum_{j \neq i} u_j^2, \]

with the energy density \( \sum_i |\nabla u_i|^2 + 2 \sum_{i<j} u_i^2 u_j^2 \) replaced by \( \sum_i |\nabla u_i|^2 + 2 \beta \sum_{i<j} u_i^2 u_j^2 \).
2.3. Monotonicity formulae for harmonic functions

Here we prove some monotonicity formulae for harmonic functions of the plane which are $2\pi$ periodic in one variable. In what follows, in the definition of $C_{(a,b)}$ and $\Sigma_r$ we mean $k = 2$.

The following results, which will be useful in section 6, can be proved by slightly modifying the arguments of the previous subsection, so we omit most of the proofs.

In what follows we consider a harmonic function $\Psi$ defined in an unbounded cylinder $C(-\infty, b)$, with $b \in \mathbb{R}$ or $b = +\infty$. We assume that

$$H(r; \Psi) := \int_{\Sigma_r} |\nabla \Psi|^2 \to 0 \quad \text{as } r \to -\infty.$$  \hfill (20)

**Lemma 2.18.** Let $\Psi$ be a harmonic function in $C(-\infty, b)$ such that (20) holds true. Then

(i) for every $r \in \mathbb{R}$ it results that $E^{\text{unb}}(r; \Psi) := \int_{C(-\infty,r)} |\nabla \Psi|^2 < +\infty$;

(ii) it results that

$$\int_{\Sigma_r} |\nabla \Psi|^2 = 2 \int_{\Sigma_r} (\partial_x \Psi)^2.$$  \hfill (21)

**Proposition 2.19.** Let $\Psi$ be a nontrivial harmonic function in $C(-\infty, b)$, such that (20) holds true. The Almgren quotient

$$N^{\text{unb}}(r; \Psi) := \frac{\int_{C(-\infty,r)} |\nabla \Psi|^2}{\int_{\Sigma_r} \Psi^2}$$

is nondecreasing in $r$. If $N(\cdot; \Psi)$ is constant for $r$ in some nonempty open interval $(r_1, r_2)$, then $N(r; \Psi)$ is constant for all $r \in \mathbb{R}$ and there exists a positive integer $d \in \mathbb{N}$ such that $N(r; \Psi) = d$; furthermore,

$$\Psi(x, y) = [C_1 \cos(dy) + C_2 \sin(dy)] e^{\lambda x}$$

for some $C_1, C_2 \in \mathbb{R}$.

**Proof.** The Almgren quotient is well defined, thanks to lemma 2.18. To prove its monotonicity, we compute the logarithmic derivative and proceed as in proposition 2.4: this reveals that $N^{\text{unb}}(r; \Psi)$ is nondecreasing, and it turns out that it can be constant in $(r_1, r_2)$ only if

$$\int_{\Sigma_r} |\partial_x \Psi|^2 \int_{\Sigma_r} \Psi^2 = \left( \int_{\Sigma_r} \Psi \partial_x \Psi \right)^2$$

for every $r \in (r_1, r_2)$. From the Cauchy–Schwarz inequality, we evince that it must be $\partial_y \Psi = \lambda \Psi$ on $\Sigma_r$ for some constant $\lambda \in \mathbb{R}$ and for every $r \in (r_1, r_2)$. Solving the differential equation, we find that $\Psi$ is of the form $\Psi(x, y) = \psi(y)e^{\lambda x}$. This, together with the equation $\Delta \Psi = 0$, yields

$$\psi'' + \lambda^2 \psi = 0 \Rightarrow \Psi(x, y) = [C_1 \cos(\lambda y) + C_2 \sin(\lambda y)] e^{\lambda x} \quad \forall (x, y) \in (r_1, r_2) \times \mathbb{R},$$

and $\Psi$ can be uniquely extended to $\mathbb{R}^2$ by the unique continuation principle for harmonic functions. Since $\Psi$ satisfies condition (20) and is nontrivial, it follows that $\lambda > 0$. The proof is complete, recalling the periodicity in $y$ of the function $\Psi$ and computing its Almgren quotient. \hfill $\Box$

3. Proof of theorem 1.1

In this section we construct a solution to (1) modelled on the harmonic function $\Phi(x, y) = \cosh x \sin y$. 

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3.1. Existence in bounded cylinders

For every \( R > 0 \) we construct a solution \((u_R, v_R)\) to

\[
\begin{align*}
-\Delta u &= -uv^2 \quad \text{in } C_R, \\
-\Delta v &= -u^2v \quad \text{in } C_R, \\
u, v &> 0
\end{align*}
\]  
(22a)

(equivalently, we can consider the problem in \((-R, R) \times (0, 2\pi)\) with periodic boundary condition on the sides \([-R, R] \times \{0, 2\pi\}\) with Dirichlet boundary condition \(u = \Phi^+, \quad v = \Phi^-\) on \(\Sigma_R \cup \Sigma_{-R}\),

and exhibiting the same symmetries of \((\Phi^+, \Phi^-)\). To be precise:

**Proposition 3.1.** There exists a solution \((u_R, v_R)\) to problem (22a) with the prescribed boundary conditions (22b), such that

1. \(u_R(-x, y) = u_R(x, y)\) and \(v_R(-x, y) = v_R(x, y)\),
2. the symmetries
   \(v_R(x, y) = u_R(x, y - \pi)\), \(u_R(x, \pi - y) = v_R(x, \pi + y)\),
   \(u_R\left(x, \frac{\pi}{2} + y\right) = u_R\left(x, \frac{\pi}{2} - y\right)\), \(v_R\left(x, \frac{3}{2}\pi + y\right) = v_R\left(x, \frac{3}{2}\pi - y\right)\)

hold,
3. \(u_R - v_R > 0\) in \(\{\Phi > 0\}\) and \(v_R - u_R > 0\) in \(\{\Phi < 0\}\),
4. \(u_R > \Phi^+\) and \(v_R > \Phi^-\).

**Remark 3.2.** In light of the evenness of \((u_R, v_R)\) in \(x\), it results that \(\partial_x u = \partial_x v\) on \(\Sigma_0\). As a consequence, the monotonicity formulae proved in section 2.1 hold true for \((u_R, v_R)\) in the semi-cylinder \(C(0, R)\).

In order to keep the notation as simple as possible, in what follows we will refer to a solution of (22a)–(22b) as to a solution of (22).

**Proof.**

Let

\[
\mathcal{U}_R := \left\{ (u, v) \in \left( H^1(C_R) \right)^2 \mid \begin{align*}
&u = \Phi^+, \ v = \Phi^- \text{ on } \Sigma_R \cup \Sigma_{-R}, \ u \geq 0, \\
&u - v \geq 0 \text{ in } \{\Phi \geq 0\}, \\
v(x, y) = u(x, y - \pi), \ u(-x, y) = u(x, y), \\
u(x, \pi - y) = v(x, \pi + y), \ u\left(x, \frac{\pi}{2} + y\right) = u\left(x, \frac{\pi}{2} - y\right) \end{align*} \right\}.
\]

Note that if \((u, v) \in \mathcal{U}_R\) then \(v\) is nonnegative, even in \(x\) and symmetric in \(y\) with respect to \(3\pi/2\); moreover, \(u - v \leq 0\) in \(\{\Phi < 0\}\). It is immediate to check that \(\mathcal{U}_R\) is weakly closed with respect to the \(H^1\) topology. We seek solutions of (22) as minimizers of the energy functional

\[
J(u, v) := \int_{C_R} |\nabla u|^2 + |\nabla v|^2 + u^2v^2
\]

in \(\mathcal{U}_R\). The existence of at least one minimizer is given by the direct method of the calculus of variations; for the coercivity of the functional \(J\), we use the following Poincaré inequality:

\[
\int_{C_R} u^2 \leq C \left( \int_{\Sigma_{-R}} u^2 + \int_{C_R} |\nabla u|^2 \right) \quad \forall u \in H^1(C_R),
\]

(23)
where \( C \) depends only on \( R \). To show that a minimizer satisfies equation (22), we consider the parabolic problem

\[
\begin{align*}
U_t - \Delta U &= -UV^2 & \text{in } (0, +\infty) \times C_R, \\
V_t - \Delta V &= -U^2V & \text{in } (0, +\infty) \times C_R, \\
U &= \Phi^+, \ V = \Phi^- & \text{on } (0, +\infty) \times (\Sigma_R \cup \Sigma_{-R}),
\end{align*}
\]

(24)

with initial condition in \( \mathcal{U}_R \). There exists a unique local solution \((U, V)\); by the parabolic maximum and minimum principle, if follows that

\[
0 \leq U \leq \sup_{C_R} \Phi^+ \quad \text{and} \quad 0 \leq V \leq \sup_{C_R} \Phi^-.
\]

This control reveals that \((U, V)\) can be uniquely extended in the whole \((0, +\infty)\). Since

\[
\frac{d}{dt} \int_{C_R} (U(t, \cdot, \cdot), V(t, \cdot, \cdot)) = -2 \int_{C_R} (U^2 + V^2) \leq 0,
\]

(25)

that is, the energy is a Lyapunov functional, from the parabolic theory it follows that for every sequence \( t_i \to +\infty \) there exists a subsequence \((t_j)\) such that \((U(t_j), V(t_j, \cdot, \cdot))\) converges to a solution \((u, v)\) of (22). Therefore, in order to prove that \((u_R, v_R)\) solves (22), it is sufficient to show that there exists an initial condition in \( \mathcal{U}_R \) such that the limiting profile \((u, v)\) coincides with \((u_R, v_R)\). We use the fact that

\[
\mathcal{U}_R \text{ is positively invariant under the parabolic flow.}
\]

(26)

To prove this claim, we first note that by the symmetry of initial and boundary conditions and by the uniqueness of the solution to problem (24), we have

\[
\begin{align*}
V(t, x, y) &= U(t, x, y - \pi), & U(t, -x, y) &= U(t, x, y), \\
V(t, x, \pi + y) &= U(t, x, \pi - y), & U \left(t, x, \frac{\pi}{2} + y\right) &= U \left(t, x, \frac{\pi}{2} - y\right).
\end{align*}
\]

(27)

This implies

\[
U(t, x, \pi) - V(t, x, \pi) = 0 \quad \forall (t, x) \in (0, +\infty) \times [-R, R].
\]

Furthermore, using (27) and the periodicity of \((U, V)\)

\[
\begin{align*}
U(t, x, 0) - V(t, x, 0) &= U(t, x, 0) - V(t, x, 2\pi) = 0 & \forall (t, x) \in (0, +\infty) \times [-R, R], \\
U(t, x, 2\pi) - V(t, x, 2\pi) &= U(t, x, 2\pi) - V(t, x, 0) = 0 & \forall (t, x) \in (0, +\infty) \times [-R, R].
\end{align*}
\]

This means that \( U - V = 0 \) on \( \Phi = 0 \). Let us introduce \( D_R := [\Phi > 0] \cap C_R \). For each initial datum in \( \mathcal{U}_R \), we have

\[
\begin{align*}
(U - V)_t - \Delta(U - V) &= UV(U - V) & \text{in } (0, +\infty) \times D_R, \\
U - V &\geq 0 & \text{on } [0] \times D_R, \\
U - V &\geq 0 & \text{on } [0, +\infty) \times \partial D_R.
\end{align*}
\]

(28)

The parabolic minimum principle implies that \( U - V \geq 0 \) in \((0, +\infty) \times D_R \). This completes the proof of the claim.

Let us consider equation (24) with the initial conditions \( U(0, x, y) = u_R(x, y), \ V(0, x, y) = v_R(x, y) \); let us denote \((U^R, V^R)\) the corresponding solution. On one side, by minimality,

\[
J(u_R, v_R) \leq J(U^R(t, \cdot, \cdot), V^R(t, \cdot, \cdot)) \quad \forall t \in (0, +\infty);
\]

we point out that this comparison is possible because of (26). On the other side, by (25),

\[
J(U^R(t, \cdot, \cdot), V^R(t, \cdot, \cdot)) \leq J(u_R, v_R) \quad \forall t \in (0, +\infty).
\]
We deduce that \( J(U_R, V_R) \) is constant, which in turn implies (we can use again (25)),
\[
U_R(t, x, y) = V_R(t, x, y) \equiv 0 \quad \Rightarrow \quad U_R(t, x, y) = u_R(x, y), \quad V_R(t, x, y) = v_R(x, y).
\]
By the above argument, as \((u_R, v_R)\) coincides with the asymptotic profile of a solution of the parabolic problem (24), it solves (22). Points (1)–(3) of the thesis are satisfied due to the positive invariance of \( U_R \). The strong maximum principle yields \( u_R > 0 \) and \( v_R > 0 \).

Moreover,
\[
\begin{align*}
-\Delta (u_R - v_R - \Phi) &= u_R v_R (u_R - v_R) \geq 0 \quad \text{in} \ D_R \\
(u_R - v_R - \Phi) &= 0 \quad \text{on} \ \partial D_R
\end{align*}
\]
so that by the strong maximum principle and the fact that \( u_R, v_R > 0 \) we deduce \( u_R > \Phi^+ \) and \( v_R > \Phi^- \).

**Remark 3.3.** The existence of a positive solution of (22) satisfying the conditions (1)–(2) of the proposition can be proved by means of the celebrated Palais’ principle of symmetric criticality. To do this, it is sufficient to minimize the functional \( J \) in the weakly closed set
\[
\{(u, v) \in (H^1(C_R))^2 \mid u = \Phi^+ \text{ on } \Sigma_R \cup \Sigma_- R, \quad \begin{cases} u(x, y) = u(x, y - \pi), u(-x, y) = u(x, y), \\ u(x, \pi - y) = v(x, \pi + y), u\left(x, \frac{\pi}{2} + y\right) = u\left(x, \frac{\pi}{2} - y\right). \end{cases} \}
\]
and apply the maximum principle. We chose a more complicated proof since we will strongly use the pointwise estimates given by point (4).

### 3.2. Compactness of the family \( \{(u_R, v_R)\} \)

In this section, we aim at proving that, up to a subsequence, the family \( \{(u_R, v_R) : R > 1\} \) obtained in proposition 3.1 converges, as \( R \to +\infty \), to a solution \((u, v)\) of (1) defined in the whole \( C_\infty \). Then, by looking at \((u, v)\) as defined in \( \mathbb{R}^2 \) (this is possible thanks to the periodicity), we obtain a solution of (1) satisfying the conditions (1)–(5) of theorem 1.1. At a later stage, we will also obtain the estimates of points (6) and (7).

We denote \( E_R, E_R, H_R, N_R \) and \( \Omega_R \) the functions \( E^{\text{sym}}, H, E^{\text{sym}}, N^{\text{sym}} \) and \( \Omega^{\text{sym}} \) (which have been defined in section 2.1) when referred to \((u_R, v_R)\). As observed in remark 3.2, for these quantities the results of section 2.1 apply.

We will obtain compactness of the sequence \((u_R, v_R)\) using some uniform-in-\( R \) control on \( N_R \) and \( H_R \). We start with a uniform (in both \( r \) and \( R \)) upper bound for the Almgren quotients \( N_R(r) \).

**Lemma 3.4.** \( N_R(r) \leq 2 \) holds, for every \( R > 0 \) and \( r \in (0, R) \).

**Proof.** It is an easy consequence of the monotonicity of \( N_R \) and of the minimality of \((u_R, v_R)\) for the functional \( J \) in \( U_R \); noting that \( J(u_R, v_R) = E_R(R) \), we compute
\[
N_R(r) \leq N_R(R) \leq \frac{2E_R(R)}{H_R(R)} \leq \frac{2}{\int_{C_{0, R}} \Phi^2} \int_{C_{0, R}} |\nabla \Phi|^2 = 2 \tanh R.
\]
We used the fact that the restriction of \( (\Phi^+, \Phi^-) \) in \( C_R \) is an element of \( U_R \) for every \( R \), and the boundary condition of \((u_R, v_R)\) on \( \Sigma_R \).

In the proof of the following lemma we will exploit the compactness of the local trace operator \( T_E \) : \( u \in H^1(C_{0, 1}) \mapsto u|_{\Sigma} \in L^2(\Sigma_1) \).

**Lemma 3.5.** There exists \( C > 0 \) such that \( H_R(1) \leq C \) for every \( R > 1 \).
Proof. By contradiction, assume that $H_{R_n}(1) \to +\infty$ for a sequence $R_n \to +\infty$. Let us introduce the sequence of scaled functions

$$(\hat{u}_n(x, y), \hat{v}_n(x, y)) := \frac{1}{\sqrt{H_{R_n}(1)}} (u_{R_n}(x, y), v_{R_n}(x, y)).$$

We wish to prove a convergence result for such a sequence, in order to obtain a uniform lower bound for $N_{R_n}(1)$. In a natural way, the scaling leads us to consider, for $r \in (0, 1)$, the quantities

$$\hat{E}_n(r) := \int_{C_{(r,)}^1} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 + 2H_{R_n}(1) \hat{u}_n \hat{v}_n^2,$$
$$\hat{H}_n(r) := \int_{\Sigma_1} \hat{u}_n^2 + \hat{v}_n^2,$$
$$\hat{N}_n(r) := \frac{\hat{E}_n(r)}{\hat{H}_n(r)}.$$ 

By construction and thanks to lemma 3.4, it holds that $\hat{H}_n(1) = 1$ and $\hat{N}_n(r) = N_{R_n}(r) \leq 2$. Hence

$$\int_{C_{(r,)}^1} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 \leq \hat{E}_n(1) = \hat{N}_n(1) \hat{H}_n(1) \leq 2,$$  

(29)

which gives a uniform bound in the $H^1(C_{(0,1)})$ norm of the sequence $(\hat{u}_n, \hat{v}_n)$ (we can use a Poincaré inequality of type (23)). Then, we can extract a subsequence which converges weakly in $H^1(C_{(0,1)})$ to some limiting profile $(\hat{u}, \hat{v})$, which is nontrivial in light of the compactness of the local trace operator $T_{\Sigma_1}$ and of the fact that $\hat{H}_n(1) = 1$. Since

$$\mathcal{V} := \left\{(u, v) \in (H^1(C_{(0,1)}))^2 \middle| \begin{array}{l}
u - v \geq 0 \text{ in } \Phi \geq 0, \nu(x, y) = u(x, y - \pi), \\
u(x, \pi - y) = v(x, \pi + y), \nu \left( x, \frac{\pi}{2} + y \right) = u \left( x, \frac{\pi}{2} - y \right) \end{array} \right\}$$

is closed in the weak $H^1(C_{(0,1)})$ topology and $(\hat{u}_n|_{C_{(0,1)}}, \hat{v}_n|_{C_{(0,1)}}) \in \mathcal{V}$ for every $n$, $\hat{u}$ and $\hat{v}$ are nonnegative functions with the same symmetries of $(u_R, v_R)$; moreover, we can show that $(\hat{u}, \hat{v})$ satisfies the segregation condition $\hat{u} \hat{v} = 0$ a.e. in $C_{(0,1)}$. In fact, by the compactness of the Sobolev embedding $H^1(C_{(0,1)}) \hookrightarrow L^4(C_{(0,1)})$ we deduce that the interaction term

$$I(u, v) := \int_{C_{(0,1)}^1} u^2 v^2$$

is continuous in the weak topology of $(H^1(C_{(0,1)}))^2$. From the estimate (29), we infer

$$2H_{R_n}(1) I(\hat{u}_n, \hat{v}_n) \leq \hat{E}_n(1) \leq 2;$$

passing to the limit as $n \to +\infty$, we conclude

$$I(\hat{u}, \hat{v}) = \lim_{n \to +\infty} I(\hat{u}_n, \hat{v}_n) = 0 \Rightarrow \hat{u} \hat{v} = 0 \quad \text{a.e. in } C_{(0,1)}.$$ 

Moreover, from the compactness of the local trace operator $T_{\Sigma_1}$, we also deduce $\int_{\Sigma_1} \hat{u}^2 + \hat{v}^2 = 1$. Let us consider the functional

$$J^\infty(u, v) := \int_{C_{(0,1)}^1} |\nabla u|^2 + |\nabla v|^2,$$

defined in the set

$$\mathcal{M} := \left\{(u, v) \in (H^1(C_{(0,1)}))^2 \middle| \begin{array}{l}J_{\Sigma_1} u^2 + v^2 = 1, \\
v(x, y) = u(x, y - \pi), \nu v = 0 \text{ a.e. in } C_1. \end{array} \right\}$$

Due to the compactness of the trace operator, one can check that $\mathcal{M}$ is closed in the weak $(H^1(C_{(0,1)}))^2$ topology. It is clear that $(\hat{u}, \hat{v}) \in \mathcal{M}$. We claim that

$$\inf_{(u, v) \in \mathcal{M}} J^\infty(u, v) =: m > 0.$$
In fact, let us assume by contradiction that the infimum is 0: since the set $\mathcal{M}$ is weakly closed and $J^\infty$ is weakly lower semi-continuous and coercive, there exists $(\tilde{u}, \tilde{v})$ such that $J^\infty(\tilde{u}, \tilde{v}) = 0$. It follows that $(\tilde{u}, \tilde{v})$ is a vector of constant functions; the symmetry and the segregation condition imply that $(\tilde{u}, \tilde{v}) \equiv (0, 0)$, but this is in contrast with the fact that $(\tilde{u}, \tilde{v}) \in \mathcal{M}$. Thus, the weak convergence of the sequence $(\hat{u}_n, \hat{v}_n)$ entails

$$\liminf_{n \to \infty} \hat{N}_n(1) \geq \liminf_{n \to \infty} \int_{C_{1,1}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 \geq m > 0,$$

so that whenever $n$ is sufficiently large

$$\hat{N}_n(1) = \hat{N}_n(1) \geq \frac{1}{2} m.$$  \hfill (30)

Thanks to lemma 3.4 we know that $\frac{1}{2} m \leq N_R(1) \leq 2$, and from the assumption $H_{R_n}(1) \to +\infty$ we deduce that (recall (7))

$$\varphi_{R_n}(r; 1) \equiv \int_1^r \frac{ds}{H_{R_n}(s)^{1/4}} \leq 2 \frac{\psi_{R_n}(1)}{H_{R_n}(1)^{1/4} N_{R_n}(1)} \left[ e^{-\frac{1}{2} N_{R_n}(1)} - e^{-\frac{1}{2} N_{R_n}(1)r} \right] \to 0$$

as $n \to \infty$, for every $r > 1$. In particular, there exists $C > 0$ such that

$$\varphi_{R_n}(r; 1) \leq C \quad \forall 1 \leq r \leq R_n, \forall n.$$ \hfill (31)

This implies that the sequence $(E_{R_n}(1))_n$ is bounded. To see this, we first note that $(u_{R_n}, v_{R_n})$ satisfies the symmetry condition (9), which is necessary to apply lemma 2.8; consequently, the variational characterization of $(u_{R_n}, v_{R_n})$ (see also the proof of lemma 3.4 and (31)) implies that

$$\frac{E_{R_n}(1)}{e^2} \leq e^{C_{R_n}(1)} \frac{E_{R_n}(R_n)}{e^{2R_n}} \leq 2 C \frac{E_{R_n}(R_n)}{e^{2R_n}} \leq C \left[ \frac{\int_{C_{R_n}} |\nabla \Phi|^2}{e^{2R_n}} \right] = C \frac{\sinh R_n \cosh R_n}{e^{2R_n}} \leq C,$$

where $C$ does not depend on $n$. Since $(E_{R_n}(1))_n$ is bounded and $(H_{R_n}(1))_n$ tends to infinity, we obtain

$$\lim_{n \to \infty} N_{R_n}(1) = \lim_{n \to \infty} \frac{E_{R_n}(1)}{H_{R_n}(1)} = 0,$$

in contradiction to (30).

**Proposition 3.6.** There exists a subsequence of $(u_R, v_R)$ which converges in $C^2_{\text{loc}}(C_{\infty})$, as $R \to +\infty$, to a solution $(u, v)$ of (1) in the whole $C_{\infty}$. This solution satisfies point (2)–(5) of theorem 1.1, and its Almgren quotient $N$ is such that

$$N(r) \leq 2 \quad \forall r > 0 \quad \text{and} \quad \lim_{r \to +\infty} N(r) \geq 1.$$

**Proof.** As $H_R(1)$ is bounded in $R$ and $N_R(1) \leq 2$, $E_R(1)$ is bounded in $R$. By means of a Poincaré inequality of type (23), this induces a uniform-in-$R$ bound for the $H^1(C_{0,1})$ norm of $(u_R, v_R)$, which in turn, by the compactness of the trace operator, gives a uniform-in-$R$ bound for the $L^2(\partial C_{0,1})$ norm. Due to the subharmonicity of $(u_R, v_R)$, the $L^2(\partial C_{0,1})$ bound provides a uniform-in-$R$ bound for the $L^\infty$ norm of $(u_R, v_R)$ in every compact subset of $C_{0,1}$; the regularity theory for elliptic equations (see [8]) ensures that, up to a subsequence, $(u_R, v_R)$ converges in $C^2_{\text{loc}}(C_{0,1})$, as $R \to +\infty$, to a solution $(u^1, v^1)$ of (1) in $C_{0,1}$. As each $(u_R, v_R)$
is even in $x$, this solution can be extended by even symmetry in $x$ to $C_1$, and here satisfies conditions (1)–(4) of proposition 3.1 (hence both $u^1$ and $v^1$ are nontrivial). The previous argument can be iterated: indeed, by corollary 2.5 and lemma 3.4, we deduce

$$H_R(r) \leq \frac{H_R(1)}{e^4} e^{-4r} \leq Ce^{4r} \quad \forall r > 1;$$

that is, a uniform-in-$R$ bound for $H_R(1)$ induces a uniform-in-$R$ bound for $H_R(r)$ for every $r > 1$. As a consequence, we obtain, for every $r > 1$, a solution $(u', v')$ to equation (1) in $C_r$. A diagonal selection gives the existence of a solution $(u_1, v_1)$ satisfying conditions (1)–(4) of proposition 3.1 (hence both $u_1$ and $v_1$ are nontrivial). The previous argument can be iterated: indeed, by corollary 2.5 and lemma 3.4, we deduce

$$HR(r) \leq HR(1) e^{-4r} \leq C e^{4r} \quad \forall r > 1;$$

that is, a uniform-in-$R$ bound for $HR(1)$ induces a uniform-in-$R$ bound for $HR(r)$ for every $r > 1$. As a consequence, we obtain, for every $r > 1$, a solution $(u, v)$ to equation (1) in $C_r$. A diagonal selection gives the existence of a solution $(u, v)$ to (1) in the whole $C^\infty$. This solution inherits by $(u', v')$ conditions (1)–(4) of proposition 3.1, and thanks to the $C^2_{loc}(C^\infty)$ convergence and lemma 3.4 the following holds:

$$N(r) = \frac{\int_{C(0, r)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\Sigma} u^2 + v^2} \leq 2 \quad \forall r > 0.$$

From lemma 2.9, which we can apply in light of the symmetries of $(u, v)$, we conclude

$$\lim_{r \to +\infty} N(r) \geq 1. \quad \square$$

The following lemma completes the proof of point (6) of theorem 1.1. After that, by means of the pointwise estimates $u > \Phi^+$ and $v > \Phi^-$ and corollary 2.5, it is straightforward to obtain also point (7).

**Lemma 3.7.** $d := \lim_{r \to +\infty} N(r) = 1$ holds.

**Proof.** In light of the fact that $d \geq 1$, it is sufficient to show that $d \leq 1$. Let $(u_{R_n}, v_{R_n})$ be the convergent subsequence found in proposition 3.6, which we will simply denote as $\{(u_n, v_n)\}$. For $r > 0$ we let

$$f_n(r) := \frac{\int_{C(0, r)} u_n^2v_n^2}{HR_n(r)}, \quad g_n(r) := \frac{\int_{\Sigma} u_n^2v_n^2}{HR_n(r)}.$$

With $f$ and $g$ we identify the same quantities computed for the limiting profile $(u, v)$. Observe that $f_n, g_n, f$ and $g$ are continuous and nonnegative. By definition,

$$f_n(r) \leq \frac{1}{2} N_{R_n}(r) \leq 1 \quad \forall r > 0,$$

where we used lemma 3.4. The uniform convergence of $(u_n, v_n)$ implies that $f_n \to f$ and $g_n \to g$ uniformly on compact intervals, while by proposition 2.4 we have

$$\int_0^r g_n(s) \, ds \leq N_{R_n}(r) \quad \text{and} \quad \int_0^r g(s) \, ds \leq N(r),$$

so that in particular $g_n \in L^1(0, R)$ and $g \in L^1(\mathbb{R}^+).$ By means of the monotonicity formula for the Almgren quotient $\mathcal{R}$, proposition 2.4, it is possible to refine the computation in lemma 3.4:

$$N_{R_n}(r) = \mathcal{R}_{R_n}(r) + f_n(r) \leq \mathcal{R}_{R_n}(R_n) + f_n(r) \leq 1 + f_n(r).$$

In light of the strong $H^1_{loc}(C^\infty)$ convergence of $(u_n, v_n)$ to $(u, v)$, we deduce

$$N(r) \leq 1 + \lim_{n \to +\infty} f_n(r) = 1 + f(r).$$

We have to show that $f(r) \to 0$ as $r \to +\infty$. To prove this, we begin by computing the logarithmic derivative of $f_n$:

$$\frac{f_n'(r)}{f_n(r)} = \frac{\int_{C(0, r)} u_n^2v_n^2}{\int_{C(0, r)} u_n^2v_n^2} - 2 \frac{E_{R_n}(r)}{HR_n(r)} \frac{g_n(r)}{f_n(r)} = 2N_{R_n}(r).$$
where we used the fact that $H'_R(r) = 2E_R(r)$, see equation (5). Exploiting the strong $H^1$ convergence of the sequence $\{(u_n, v_n)\}$ and the fact that $\lim_{n \to +\infty} N(r_n) \geq 1$, we deduce that there exist $r_0, \delta > 0$ such that $N(r_0) > \delta$ for every $n$ sufficiently large. Consequently, $f_n$ satisfies the inequality
\[
 f'_n(r) + 2\delta f_n(r) \leq g_n(r) \quad \text{for } r \in (r_0, R).
\]

Multiplying by $e^{2\delta r}$ and integrating in $(r_1, r_2)$ for $r_0 < r_1 < r_2 < R$, we obtain
\[
 f_n(r_2) \leq e^{2\delta(r_1-r_2)} f_n(r_1) + \int_{r_1}^{r_2} g_n(s) e^{2\delta(s-r_2)} ds \leq e^{2\delta(r_1-r_2)} + \int_{r_1}^{r_2} g_n(s) ds,
\]
where we used the estimate (32). This implies
\[
 f(r_2) \leq e^{2\delta(r_1-r_2)} + \int_{r_1}^{r_2} g(s) ds \quad \text{for } r_0 < r_1 < r_2.
\]

Since $g \in L^1(\mathbb{R}^+)$ and $f \geq 0$, choosing $r_1 = r_2/2$ we find
\[
 \limsup_{r \to +\infty} f(r) = 0 = \lim_{r \to +\infty} f(r).
\]

4. Proof of theorem 1.5

In this section we construct a solution to (1) modelled on the harmonic function $\Gamma(x, y) = e^x \sin y$. Our construction is based on the trivial observation that $\Phi_1 R(x, y) = 2 \cosh(x + R) e^{-R} \sin y \to \Gamma(x, y)$ as $R \to +\infty$.

4.1. Existence in bounded cylinders

As a first step, using the same line of reasoning developed in proposition 3.1, it is possible to show the existence of solution to the system
\[
 \begin{cases}
 -\Delta u = -uv^2 & \text{in } C(-3R, R), \\
 -\Delta v = -u^2v & \text{in } C(-3R, R), \\
 u, v > 0
 \end{cases}
 \quad (33a)
\]
(equivalently, we can consider the problem in the rectangle $(-3R, R) \times (0, 2\pi)$ with periodic boundary condition on the sides $[-3R, R] \times [0, 2\pi]$) and such that
\[
 u_R = \Phi^+_R, \quad v_R = \Phi^-_R \quad \text{on } \Sigma_R \cup \Sigma_{-3R}.
\]

More precisely:

**Proposition 4.1.** There exists a solution $(u_R, v_R)$ to problem (33a) with the prescribed boundary conditions (33b), such that

1. $u_R(-R - x, y) = u_R(-R + x, y)$ and $v_R(-R - x, y) = v_R(-R + x, y)$,
2. the symmetries
\[
 u_R(x, \pi - y) = u_R(x, \pi + y), \quad v_R(x, \pi - y) = v_R(x, \pi + y),
\]
\[
 u_R \left( x, \frac{\pi}{2} + y \right) = u_R \left( x, \frac{\pi}{2} - y \right), \quad v_R \left( x, \frac{3}{2} \pi + y \right) = v_R \left( x, \frac{3}{2} \pi - y \right)
\]
hold,
3. $u_R - v_R > 0$ in $[\Phi_R > 0]$ and $v_R - u_R > 0$ in $[\Phi_R < 0]$,
4. $u_R > (\Phi_R)^+$ and $v_R > (\Phi_R)^-$.
Sketch of proof. One can recast the proof of proposition 3.1 in this setting.

Remark 4.2. In light of point (1) of the proposition, it results that
\[ \partial_t u_R = 0 = \partial_t v_R \quad \text{on } \Sigma_{-R}. \]
Therefore, the monotonicity formulae proved in section 2.1 hold true for \((u_R, v_R)\) in the semi-cylinder \(C_R\).

4.2. Compactness of the family \(\{u_R, v_R\}\)

As in the previous section, we denote by \(E_R, E_{sym}, N_R, N_{sym}\) the functions defined in section 2.1 when referred to \((u_R, v_R)\). We follow here the same line of reasoning adopted in section 3.2. First, it is not difficult to modify the proof of lemmas 3.4 and 3.5 obtaining the following estimates:

Lemma 4.3. There holds \(N_R(r) \leq 2\) for every \(R > 0\) and \(r \in (-R, R)\).

Lemma 4.4. There exists \(C > 0\) such that \(H_R(1) \leq C\) for every \(R > 1\).

We are in position to show that the family \(\{u_R, v_R\}\) is compact, in the following sense.

Proposition 4.5. There exists a subsequence of \(\{u_R, v_R\}\) which converges in \(C^2_{loc}(C_\infty)\), as \(R \to +\infty\), to a solution \((u, v)\) of (1) in the whole \(C_\infty\). This solution has the properties (2)–(4) of proposition 4.1.

Proof. As \(H_R(1)\) is bounded in \(R\) and \(N_R(1) \leq 2\), the quantity \(E_R(1)\) is bounded in \(R\) as well, and a fortiori
\[ \int_{C_1} |\nabla u_R|^2 + |\nabla v_R|^2 \leq C \quad \forall R > 1. \]

This estimate, the boundedness of \(H_R(1)\) and a Poincaré inequality of type (23) imply that \(\{u_R, v_R\}\) is bounded in \(H^1(C_1)\). Consequently, it is possible to argue as in the proof of proposition 3.6 and obtain the existence of a subsequence of \(\{u_R, v_R\}\) which converges in \(C^2_{loc}(C_1)\) to a solution \((u^1, v^1)\) of (1) in \(C_1\), which inherits by \(\{u_R, v_R\}\) the properties (2)–(4) of proposition 4.1. In light of corollary 2.5 and lemma 4.3, this procedure can be iterated: in fact
\[ H_R(r) \leq \frac{H_R(1)}{e^{4r}} e^{4r} \leq C e^{4r} \quad \forall r > 1, \]
so that applying the previous argument we obtain a subsequence of \(\{u_R, v_R\}\), which converges in \(C^2_{loc}(C_r)\) to a solution \((u', v')\) of (1) in \(C_r\), and inherits by \(\{u_R, v_R\}\) properties (2)–(4) of proposition 4.1. A diagonal selection gives the existence of a solution \((u, v)\) of (1) in the whole \(C_\infty\), and this solution enjoys properties (2)–(4) of proposition 4.1.

Remark 4.6. The monotonicity formulae proved in section 2.1 do not apply on \((u, v)\), because passing to the limit we lose the Neumann condition \(\partial_x u_R = 0 = \partial_x v_R\) on \(\Sigma_{-R}\).

In the next lemma, we show that \((u, v)\) is a solution with finite energy, so that the achievements proved in section 2.2 applies.
Lemma 4.7. Let \((u, v)\) be the solution found in proposition 4.5. It results that
\[
\mathcal{E}^{\text{unb}}(r) := \int_{C_{(-\infty, r)}} |\nabla u|^2 + |\nabla v|^2 + u^2 + v^2 < +\infty \quad \forall r \in \mathbb{R}
\] (34)
and
\[
\lim_{r \to -\infty} H(r) = \lim_{r \to -\infty} \int_{\Sigma_r} u^2 + v^2 = 0.
\]
Recall that \(\mathcal{E}^{\text{unb}}\) has been defined in section 2.2.

Proof. Let \(\{(u_{R_n}, v_{R_n})\}\) be the converging subsequence found in proposition 4.5, whose \(C^2_{\text{loc}}(C_\infty)\) limit is \((u, v)\). An application of corollary 2.5 on \((u_n, v_n)\), together with lemma 4.4 and the Fatou lemma, yields the estimate
\[
\mathcal{E}^{\text{unb}}(r) \leq \liminf_{n \to \infty} \int_{C_{(-\infty, r)}} (|\nabla u_n|^2 + |\nabla v_n|^2 + u_n^2 + v_n^2) \chi_{C_{(-\infty, r)}} \leq \liminf_{n \to \infty} E_{R_n}(r)
\]
\[
= \liminf_{n \to \infty} N_{R_n}(r) H_{R_n}(r) \leq \liminf_{n \to \infty} 2 \frac{H_{R_n}(1)}{e^4} e^{4r} \leq C e^{4r},
\]
which proves the (34). To complete the proof, we first note that necessarily \(\mathcal{E}^{\text{unb}}(r) \to 0\) as \(r \to -\infty\), and hence the same holds for \(\mathcal{E}^{\text{unb}}\) (which has been defined in section 2.2). Assume by contradiction that for a sequence \(r_n \to -\infty\) it results that \(H(r_n) \geq C > 0\). We define
\[
(\hat{u}_n(x, y), \hat{v}_n(x, y)) := \frac{1}{\sqrt{H(r_n)}} (u(x + r_n, y), v(x + r_n, y)).
\]
A direct computation shows that
\[
\int_{C_{(-\infty, 0)}} |
abla \hat{u}_n|^2 + |
abla \hat{v}_n|^2 \leq \int_{C_{(-\infty, 0)}} |
abla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 + 2H(r_n) \hat{u}_n \hat{v}_n = \frac{1}{H(r_n)} E^{\text{unb}}(r_n) \to 0
\]
as \(n \to \infty\). Consequently, \((\hat{u}_n, \hat{v}_n)\) tends to be a pair of constant functions of type \((\hat{u}, \hat{v})\) with \(\hat{u} = \hat{v}\) (this follows from the symmetries of \((u, v)\)). As
\[
C \int_{C_{(-\infty, 0)}} \hat{u}_n^2 \hat{v}_n^2 \leq H(r_n) \int_{C_{(-\infty, 0)}} \hat{u}_n^2 \hat{v}_n^2 \to 0,
\]
necessarily \((\hat{u}_n, \hat{v}_n) \to (0, 0)\) almost everywhere in \(C_{(-\infty, 0)}\). This is in contradiction to the fact that \(\int_{\Sigma_r} \hat{u}_n^2 + \hat{v}_n^2 = H(r_n) \geq C\). □

So far we proved that the solution \((u, v)\), found in proposition 4.5, enjoys properties (1)–(5) of theorem 1.5, and is such that \(H(r) \to 0\) as \(r \to -\infty\). The previous lemma enables us to apply the achievements of section 2.2 for \(E^{\text{unb}}\), \(H\), \(N^{\text{unb}}\) and \(\mathfrak{E}^{\text{unb}}\) (which we consider referred to the solution \((u, v)\) found in proposition 4.5), and permits us to complete the description of the growth of \((u, v)\), points (6)–(7) of theorem 1.5.

Lemma 4.8. Let \((u, v)\) be the solution found in proposition 4.5. Then \(N^{\text{unb}}(r) \to 1\) as \(r \to \infty\).

Proof. Let \(\{(u_{R_n}, v_{R_n})\}\) be the converging subsequence found in proposition 4.5, which we will simply denote by \(\{(u_n, v_n)\}\). First, arguing as in the proof of the previous lemma, we note that by the \(C^2_{\text{loc}}(C_\infty)\) convergence of \((u_n, v_n)\) to \((u, v)\) it follows that
\[
N^{\text{unb}}(r) \leq \liminf_{n \to \infty} N_{R_n}(r) \leq 2 \quad \forall r \in \mathbb{R},
\]
thanks to the Fatou lemma. This, together with the symmetries of \( (u, v) \), permits us to use lemma 2.16, which gives \( \lim_{r \to +\infty} N^\text{amb}(r) \geq 1 \). To complete the proof, it is sufficient to show that \( \lim_{r \to +\infty} N^\text{amb}(r) \leq 1 \). For any \( r > 0 \), let

\[
 f_n(r) := \int_{C_r} u_n^2 v_n^2 \frac{1}{H_{R_r}(r)}, \quad g_n(r) := \frac{\int_{\Sigma_{-R_r} \cup \Sigma_{R_r}} u_n^2 v_n^2}{H_{R_r}(r)},
\]

and let \( f \) and \( g \) the same quantities referred to the solution \( (u, v) \). Observe that \( f_n, g_n, f \) and \( g \) are continuous and nonnegative. The uniform convergence of \( (u_n, v_n) \) to \( (u, v) \) implies that \( f_n \to f \) and \( g_n \to g \), as \( n \to \infty \), uniformly on compact intervals. By definition,

\[
 f_n(r) \leq \frac{1}{N(r)} \leq 1 \quad \forall r > 0.
\]

whenever \( R_n \geq r \). We claim that \( g \in L^1(\mathbb{R}^+) \). In fact, by the monotonicity of \( H \) and proposition 2.14, it follows that

\[
 \int_0^r g(s) \, ds \leq \int_0^r \frac{\int_{\Sigma_0} u_0^2 v_0^2}{H(s)} \, ds \leq \int_{-r}^r \frac{\int_{\Sigma_0} u_0^2 v_0^2}{H(s)} \, ds \leq N^\text{amb}(r),
\]

for every \( r > 0 \). Let \( r > 0 \); it is possible to refine the computation on lemma 3.4 to obtain

\[
 N_{R_r}(r) \leq 1 + f_n(r) + \frac{\int_{C_{-R_r} \cup C_{R_r}} u_n^2 v_n^2}{H_{R_r}(r)} \leq 1 + f_n(r) + \frac{E_{R_r}(-r)}{H_{R_r}(r)}.
\]

Therefore, using again the Fatou lemma we deduce

\[
 N^\text{amb}(r) \leq \liminf_{n \to \infty} N_{R_r}(r) \leq 1 + f(r) + \liminf_{n \to \infty} \frac{E_{R_r}(-r)}{H_{R_r}(r)},
\]

and to complete the proof we will show that

\[
 \lim_{r \to +\infty} \left( f(r) + \liminf_{n \to \infty} \frac{E_{R_r}(-r)}{H_{R_r}(r)} \right) = 0. \tag{35}
\]

First, we note that

\[
 \liminf_{n \to \infty} \frac{E_{R_r}(-r)}{H_{R_r}(r)} = \liminf_{n \to \infty} \frac{N_{R_r}(-r)H_{R_r}(r)}{H_{R_r}(r)} \leq 2 \liminf_{n \to \infty} \frac{H_{R_r}(-r)}{H_{R_r}(r)}.
\]

From the \( C^2_\infty(\mathbb{C}) \) convergence of \( (u_n, v_n) \) to \( (u, v) \) it follows that

\[
 2 \liminf_{n \to \infty} \frac{H_{R_r}(-r)}{H_{R_r}(r)} = 2 \frac{H(-r)}{H(r)} \to 0 \quad \text{as } r \to +\infty
\]

where we used lemma 4.7 and the fact that \( H(r) > H(0) > 0 \) for every \( r > 0 \). For (35) it remains to prove that \( f(r) \to 0 \) as \( r \to +\infty \). Having observed that \( \lim_{r \to +\infty} N(r) \geq 1 \) and that \( g \in L^1(\mathbb{R}^+) \), it is not difficult to adapt the conclusion of the proof of lemma 3.7.

5. Systems with many components

In this section we are going to prove the existence of entire solutions with exponential growth for the \( k \) component system (2). Our construction is based on the elementary limit

\[
 \lim_{d \to +\infty} \left( 1 + \frac{z}{d} \right)^d = e^z \sin y,
\]

which shows that the harmonic function \( e^z \sin y \) can be obtained as the limit of a homogeneous harmonic polynomial. We wish to prove that the same idea applies to solutions of the system (2); there exists an entire solution to (2) having exponential growth, which can be obtained as the limit of entire solutions having algebraic growth.
5.1. Preliminary results

We recall some results contained in [2]. For $d \in \mathbb{N}/2$, let $G_d$ be the rotation of angle $\pi/d$ in counterclockwise sense.

**Theorem 5.1 (Theorem 1.6 of [2]).** Let $k \geq 2$ be a positive integer, let $d \in \mathbb{N}/2$ be such that $2d = hk$, for some $h \in \mathbb{N}$.

There exists a solution $(u_1^d, \ldots, u_k^d)$ to the system (2) which enjoys the following symmetries:

\[
\begin{align*}
  u_1^d(x, y) &= u_1^d(G_d(x, y)), \\
  u_i^d(x, y) &= u_{i+1}^d(G_d(x, y)), \\
  u_{k+1-i}^d(x, y) &= u_i^d(x, -y),
\end{align*}
\]

where we recall that indices are meant mod $k$. Moreover,

\[
\lim_{r \to +\infty} \frac{1}{r^{1+2d}} \int_{\partial B_r} \left( \sum_{i=1}^k (u_i^d)^2 \right) = b \in (0, +\infty),
\]

and

\[
\lim_{r \to +\infty} \frac{r \int_{\partial B_r} \sum_{i=1}^k \left| \nabla u_i^d \right|^2 + \sum_{1 \leq i < j \leq k} (u_i^d u_j^d)^2}{\int_{\partial B_r} \sum_{i=1}^k (u_i^d)^2} = d,
\]

where $B_r$ denotes the ball of centre 0 and radius $r$.

The solution $(u_1^d, \ldots, u_k^d)$ is modelled on the harmonic function $\Im(z^d)$, as specified by the symmetries (36). In the quoted statement, the authors modelled their construction on the functions $\Re(z^d)$ (the real part of $z^d$): it is straightforward to obtain an analogous result replacing the real part with the imaginary one (figure 1).

**Remark 5.2.** We point out that the symmetries (36) imply that $u_1^d$ is symmetric with respect to the reflection with the axis $y = \tan(\pi/(2d))x$.

For a solution $(u_1, \ldots, u_k)$ of system (2) in $\mathbb{R}^2$, we introduce the functionals

\[
E^{\text{alg}}(r; \Lambda) := \int_{B_r} \sum_{i=1}^k \left| \nabla u_i \right|^2 + \Lambda \sum_{1 \leq i < j \leq k} (u_i u_j)^2,
\]

\[
H^{\text{alg}}(r) := \frac{1}{r} \int_{\partial B_r} \sum_{i=1}^k (u_i)^2.
\]

The index alg denotes the fact that these quantities are well suited to describe the growth of $(u_1, \ldots, u_k)$ under the assumption that $(u_1, \ldots, u_k)$ has algebraic growth. In particular, as proved in lemma 2.1 of [6] and corollary A.8 of [7] for the case $k = 2$, the Almgren quotient $N^{\text{alg}}(r; 1) := E^{\text{alg}}(r; 1)/H^{\text{alg}}(r)$ is bounded in $r \in \mathbb{R}^+$ if and only if $(u_1, \ldots, u_k)$ has algebraic growth.

It is not difficult to adapt the proof of proposition 5.2 in [2] to obtain the following general result (in the sense that it holds true for an arbitrary solution of (2) in $\mathbb{R}^N$, for any dimension $N \geq 2$).

**Proposition 5.3 (see proposition 5.2 of [2]).** Let $N \geq 2$,

\[
\Lambda \in \begin{cases}
  \left[ 1, \frac{N}{N-2} \right] & \text{if } N > 2, \\
  \left( 1, +\infty \right) & \text{if } N = 2,
\end{cases}
\]

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Figure 1. In the figure we represent some of the solutions obtained in theorem 5.1. Here the number of components is set as $k = 3$: each component is drawn with a different colour. On the other hand, the periodicity (that is, how many times the patch of 3-components is replicated in the circle) is given by $h = 1$ (up, left), $h = 2$ (up, right), $h = 3$ (down, left) and $h = 4$ (down, right), respectively. As a consequence, the growth rate $d$ varies as $d = 3/2, 3, 9/2, 6$, following the same order.

and let $(u_1, \ldots, u_k)$ be a solution of (2) in $\mathbb{R}^N$; the Almgren quotient

$$N_{alg}(r; \Lambda) := \frac{E_{alg}(r; \Lambda)}{H_{alg}(r)} = \frac{r \int_{B_r} \sum_{i=1}^{k} |\nabla u_i|^2 + \Lambda \sum_{1 \leq i < j \leq k} (u_i u_j)^2}{\int_{\partial B_r} \sum_{i=1}^{k} (u_i)^2}$$

is well defined in $(0, +\infty)$ and nondecreasing in $r$.

**Proof.** We observe that

$$\frac{d}{dr} E_{alg}(r; \Lambda) = \frac{d}{dr} \left( \frac{1}{r^{N-2}} \int_{B_r} \sum_{i} |\nabla u_i|^2 + \sum_{i < j} (u_i u_j)^2 + \frac{\Lambda - 1}{r^{N-2}} \int_{\partial B_r} \sum_{i < j} (u_i u_j)^2 \right)$$

$$= \frac{2}{r^{N-2}} \int_{\partial B_r} \sum_{i} (\partial_{\nu} u_i)^2 + \frac{2}{r^{N-1}} \int_{B_r} \sum_{i < j} (u_i u_j)^2$$

$$+ \frac{(2 - N)(\Lambda - 1)}{r^{N-1}} \int_{B_r} \sum_{i < j} u_i^2 u_j^2 + \frac{\Lambda - 1}{r^{N-2}} \int_{\partial B_r} \sum_{i < j} u_i^2 u_j^2,$$
where we used equation (5.3) in [2]. Proceeding as in the proof of proposition 5.2 in [2], one obtains
\[
\frac{d}{dr} N_{\text{alg}}(r; \Lambda) \geq (2 + (\Lambda - 1)(2 - N)) \int_{\partial B_r} \sum_{i<j} u_i^2 u_j^2 \frac{H_{\text{alg}}(r)}{r^{N-1}} + \frac{(\Lambda - 1) \int_{\partial B_r} \sum_{i<j} u_i^2 u_j^2}{r^{N-2} H_{\text{alg}}(r)},
\]
which is \( \geq 0 \) by our assumption on \( \Lambda \).

**Remark 5.4.** In [2] the authors considered the case \( \Lambda = 1 \).

We work in the plane \( \mathbb{R}^2 \), so that it is possible to choose \( \Lambda = 2 \) in proposition 5.3. We denote \( E_d(\cdot; \Lambda) \) and \( H_d \) the quantities defined in (38) when referred to the functions \( (u_1^d, \ldots, u_k^d) \) defined in theorem 5.1; also, we denote \( N_d(\cdot; \Lambda) := \frac{E_d(\cdot; \Lambda)}{H_d} \). In case \( \Lambda = 2 \), we will simply write \( E_d \) and \( N_d \) to ease the notation.

**Lemma 5.5.** Let \( (u_1^d, \ldots, u_k^d) \) be defined in theorem 5.1. Then \( N_d(r) \to d \) as \( r \to +\infty \).

**Proof.** It is an easy consequence of (37) and of corollary 5.8 in [2], where it is proved that for the solution \( (u_1^d, \ldots, u_k^d) \) the following holds:
\[
\lim_{r \to +\infty} \frac{E_d(r; 2)}{r^{2d}} = \lim_{r \to +\infty} \frac{E_d(r; 1)}{r^{2d}}.
\]
Therefore,
\[
\lim_{r \to +\infty} N_d(r) = \lim_{r \to +\infty} \frac{E_d(r; 2)}{H_d(r)} = \lim_{r \to +\infty} \frac{E_d(r; 2)}{r^{2d}} \cdot \lim_{r \to +\infty} \frac{H_d(r)}{r^{2d}} = \lim_{r \to +\infty} N_d(r; 1) = d.
\]

As a consequence, the following doubling property holds true:

**Proposition 5.6 (See proposition 5.3 of [2]).** For any \( 0 < r_1 < r_2 \) the following holds:
\[
\frac{H_d(r_2)}{r_2^{2d}} \leq \frac{H_d(r_1)}{r_1^{2d}}.
\]

**Proof.** A direct computation shows that
\[
\frac{d}{dr} \log \frac{H_d(r)}{r^{2d}} = \frac{2N_d(r)}{r} - \frac{2d}{r} \leq 0;
\]
an integration gives the thesis. \( \square \)

Let us consider the scaling
\[
(u_{i,R}^d, \ldots, u_{k,R}^d) := \left( \frac{2d}{k H_d(R)} \right)^{\frac{1}{d}} (u_1^d(Rx, Ry), \ldots, u_k^d(Rx, Ry)),
\]
where \( R \) will be determined later as a function of \( d \). We see that
\[
\begin{cases}
-\Delta u_{i,R}^d = -\beta_R^d u_{i,R}^d \sum_{j \neq i} (u_{j,R}^d)^2 & \text{in } \mathbb{R}^2, \\
\int_{\partial B_1} \sum_{i=1}^k (u_{i,R}^d)^2 = \frac{2d}{k},
\end{cases}
\]
where \( \beta_R^d := k H_d(R) R^2/(2d) \).

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Remark 5.7. As a function of $R$, $\beta^d_R$ is continuous and such that $\beta^d_R \to 0$ if $R \to 0$ and $\beta^d_R \to \infty$ if $R \to \infty$.

Accordingly with our scaling, we introduce the new Almgren quotient

$$N_{d,R}(r) := \frac{E_{d,R}(r)}{H_d(r)} = \frac{r \int_{B_r} |\nabla u^d_{i,R}|^2 + 2\beta^d_R \sum_{1 \leq i < j \leq k} (u^d_{i,R} u^d_{j,R})^2}{\int_{\partial B_r} (u^d_{i,R})^2}.$$  

We point out that $N_{d,R}(r) = N_d(Rr)$, so that from lemma 5.5 and the monotonicity of $N_d$ we deduce

$$N_{d,R}(r) \leq d \quad \forall r, R > 0,$$  

for every $d$. By the symmetries, the solution $(u^d_{1,R}, \ldots, u^d_{k,R})$ is $\frac{k\pi}{d}$-periodic with respect to the angular component, thus it is convenient to restrict our attention to the cones $S^d_r := \{ (\rho, \theta) : \rho \in (0, r), \theta \in \left(0, \frac{k\pi}{d}\right) \}$ and $S^d := \{ (\rho, \theta) : \rho > 0, \theta \in \left(0, \frac{k\pi}{d}\right) \}$.

The boundary $\partial S^d_r$ can be decomposed as $\partial S^d_r = \partial_p S^d_r \cup \partial_r S^d_r$, where

$$\partial_p S^d_r := (0, r) \times \left[0, \frac{k\pi}{d}\right]$$  

and $\partial_r S^d_r := \{ r \} \times \left[0, \frac{k\pi}{d}\right]$.

Taking into account the periodicity of $(u^d_{1,R}, \ldots, u^d_{k,R})$, we note that $(u^d_{1,R}, \ldots, u^d_{k,R})$ has periodic boundary conditions on $\partial_p S^d_r$; furthermore,

$$E_{d,R}(r) = \frac{2d}{k} \int_{S^d_r} \sum_i |\nabla u^d_{i,R}|^2 + 2\beta^d_R \sum_{1 \leq i < j \leq k} (u^d_{i,R} u^d_{j,R})^2,$$

$$H_{d,R}(r) = \frac{2d}{kr} \int_{\partial S^d_r} \sum_i (u^d_{i,R})^2.$$

$$N_{d,R}(r) = \frac{r \int_{S^d_r} |\nabla u^d_{i,R}|^2 + 2\beta^d_R \sum_{1 \leq i < j \leq k} (u^d_{i,R} u^d_{j,R})^2}{\int_{\partial S^d_r} (u^d_{i,R})^2}.$$

5.2. A blow-up in a neighbourhood of $(1, 0)$

In order to pursue our strategy, we consider a further scaling

$$(\hat{u}^d_{i,R}(x, y), \ldots, \hat{u}^d_{k,R}(x, y)) = \sqrt{\frac{\beta^d_R}{d}} \left(u^d_{1,R} \left(1 + \frac{x}{d}, \frac{y}{d}\right), \ldots, u^d_{k,R} \left(1 + \frac{x}{d}, \frac{y}{d}\right)\right).$$

Accordingly, we will consider the scaled domains $\hat{S}^d = d(S^d - (1, 0))$ and $\hat{S}^d = d(S^d - (1, 0))$ and the respective boundaries. Having in mind to let $d \to \infty$, we observe that this scaling is a blow-up centred in the point $(1, 0)$. It is easy to verify that $(\hat{u}^d_{1,R}, \ldots, \hat{u}^d_{k,R})$ solves (see (39))

$$\begin{cases}
-\Delta \hat{u}^d_{i,R} = -\hat{u}^d_{i,R} \sum_{j \neq i} (\hat{u}^d_{j,R})^2 \quad \text{in } \hat{S}^d, \\
\int_{\partial_\delta \hat{S}^d} \sum_{i=1}^k (\hat{u}^d_{i,R})^2 = \frac{\beta^d_R}{d},
\end{cases}$$  

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with suitable periodic conditions on \( \partial \hat{S}^d \). A direct computation shows that from (41) it follows that

\[
N_{d,R}(r) = d \frac{r \int_{\hat{S}^d} |\nabla \hat{u}_{i,R}^d|^2 + 2 \sum_{i<j} \left( \hat{u}_{i,R}^d \hat{u}_{j,R}^d \right)^2}{\int_{\partial \hat{S}^d} \sum_i (\hat{u}_{i,R}^d)^2},
\]

where in the new coordinates

\[
r = \sqrt{\left(1 + \frac{x^2}{d}\right) + \left(\frac{y^2}{d}\right)}.
\]

We are then led to define a new Almgren quotient for the scaled functions \((\hat{u}_{1,R}^d, ..., \hat{u}_{k,R}^d)\):

\[
\hat{E}_{d,R}(r) = \frac{1}{r} \int_{\partial \hat{S}^d} \left( \sum_{i=1}^k |\nabla \hat{u}_{i,R}^d| \right)^2 + 2 \sum_{1 \leq i < j \leq k} \left( \hat{u}_{i,R}^d \hat{u}_{j,R}^d \right)^2,
\]

\[
\hat{H}_{d,R}(r) = \hat{H}_{d,R}(1) = \frac{\hat{E}_{d,R}(r)}{\hat{H}_{d,R}(r)} = \frac{1}{d} N_{d,R}(r).
\]

From equation (40), we deduce

\[
\hat{N}_{d,R}(r) \leq 1 \quad \forall r, R > 0, \forall d \in \mathbb{N}.
\]

In order to understand the behaviour of \((\hat{u}_{1,R}^d, ..., \hat{u}_{k,R}^d)\) when \(d \to \infty\), we fix \(R = R(d)\) to get a nondegeneracy condition.

**Lemma 5.8.** For every \(d \in \mathbb{N}/2\) there exists \(R_d > 0\) such that

\[
\hat{H}_{d,R_d}(1) = \int_{\partial \hat{S}^d} \sum_i (\hat{u}_{i,R_d}^d)^2 = 1.
\]

**Proof.** By (42) we know that \(\hat{H}_{d}(1) = \beta_d^0 / d\), so that we have to find \(R_d\) such that \(\beta_d^0 = d\). As observed in remark 5.7, this choice is possible.

We denote \((\hat{u}_1^d, ..., \hat{u}_k^d) := (\hat{u}_{1,R}^d, ..., \hat{u}_{k,R}^d), \hat{H}_d := \hat{H}_{d,R_d}, \hat{E}_d := \hat{E}_{d,R_d}, \hat{N}_d := \hat{N}_{d,R_d}\) and \(\beta^d := \beta_d^0\). We aim at proving that, up to a subsequence, the family \([\hat{u}_1^d, ..., \hat{u}_k^d] : d \in \mathbb{N}/2\) converges, as \(d \to +\infty\), to a solution of (2). To this aim, major difficulties arise from the fact that \(\hat{S}^d\) and \(\hat{S}^d\) depend on \(d\); in the next lemma we show that this problem can be overcome thanks to a convergence property of these domains (figure 2).

**Lemma 5.9.** For any \(r > 1\), the sets \(\hat{S}^d_r\) converge to \(\mathbb{R} \times (0, k\pi)\) as \(k \to +\infty\), in the sense that

\[
\mathbb{R} \times (0, k\pi) = \text{Int} \left( \bigcup_{n \in \mathbb{N} / 2} \bigcap_{d > n} \hat{S}^d_r \right),
\]

where for \(A \subset \mathbb{R}^2\) we mean that \(\text{Int}(A)\) denotes the inner part \(A\). Analogously,

\[
\mathbb{R} \times (0, k\pi) = \text{Int} \left( \bigcup_{n \in \mathbb{N} / 2} \bigcap_{d > n} \hat{S}^d \right) \quad \text{and} \quad (-\infty, 0) \times (0, k\pi) = \text{Int} \left( \bigcup_{n \in \mathbb{N} / 2} \bigcap_{d > n} \hat{S}^d \right).
\]
Figure 2. Visualization of the construction in lemma 5.9. In red the limiting set $\mathbb{R} \times (0, k\pi)$. In blue some of the scaled domains $\hat{S}^d_r$, for $r > 1$.

and for every $\bar{x} \in \mathbb{R}$

$$(-\infty, \bar{x}) \times (0, k\pi) = \operatorname{Int}\left(\bigcap_{n \in \mathbb{Z}} \bigcup_{d > n} \hat{S}^d_1 \right).$$

**Proof.** We prove only the first claim. Let $r > 1$.

**Step 1.** $\mathbb{R} \times (0, k\pi) \subset \bigcap_{n \in \mathbb{Z}} \bigcup_{d > n} \hat{S}^d_r$.

Let $(x, y) \in \mathbb{R} \times (0, k\pi)$. We show that for every $d \in \mathbb{N}/2$ sufficiently large $(x, y) \in \hat{S}^d_r$, that is, $(1 + x/d, y/d) \in S^d_r$, which means

$$\sqrt{\left(1 + \frac{x}{d}\right)^2 + \left(\frac{y}{d}\right)^2} < r \quad \text{and} \quad \arctan\left(\frac{y}{x + d}\right) \in \left(0, \frac{k\pi}{d}\right).$$

For the first condition it is possible to choose $d$ sufficiently large, as $r > 1$. To prove the second condition, we start by considering $d > -x$, so that $\arctan(y/(x + d)) > 0$. Now, provided $d$ is sufficiently large

$$\arctan\left(\frac{y}{x + d}\right) < \frac{k\pi}{d} \Leftrightarrow y < (x + d) \tan\left(\frac{k\pi}{d}\right).$$

Since $y < k\pi$, there exists $\varepsilon > 0$ such that $y \leq k(1 - \varepsilon)\pi$. Let $\bar{d}$ be sufficiently large so that

$$x + d > \left(1 - \frac{\varepsilon}{2}\right)d \quad \text{and} \quad \frac{d}{k\pi} \tan\left(\frac{k\pi}{d}\right) > 1 - \frac{\varepsilon}{2}$$

for every $d > \bar{d}$. Then

$$(x + d) \tan\left(\frac{k\pi}{d}\right) > \left(1 - \frac{\varepsilon}{2}\right)^2 k\pi > (1 - \varepsilon)k\pi \geq y$$

whenever $d > \bar{d}$.

**Step 2.** $\bigcap_{n \in \mathbb{Z}} \bigcup_{d > n} \hat{S}^d_r \subset \mathbb{R} \times [0, k\pi]$.

We show that $(\mathbb{R} \times [0, k\pi])^c \subset \bigcap_{n \in \mathbb{Z}} \bigcup_{d > n} \hat{S}^d_r$. If $(x, y) \notin \mathbb{R} \times [0, k\pi]$, then $y > k\pi$ or $y < 0$. We consider only the case $y > k\pi$; in such a situation

$$y > k\pi = \lim_{d \to \infty} (x + d) \tan\left(\frac{k\pi}{d}\right),$$

so that $(x, y) \notin \hat{S}^d_r$ for every $d$ sufficiently large. \[\square\]
Remark 5.10. As a consequence of the previous result, we see that
\[ \partial_r \tilde{S}^d \to \{0\} \times [0, k\pi] \quad \text{and} \quad \partial_r \tilde{S}^d_{1+\bar{x}} \to \{\bar{x}\} \times [0, k\pi] \]
for every $\bar{x} \in \mathbb{R}$.

Remark 5.11. Recall the expression of $r$ in the new variable, given by (43). For every $r > 0$ and $d \in \mathbb{N}/2$ there exists $\xi(r, d)$ such that
\[ r = 1 + \frac{\xi(r, d)}{d} \quad \Leftrightarrow \quad \xi(r, d) = d(r - 1). \]
Note that for every $(x, y) \in \partial_r \tilde{S}^d$ it results that $x < \xi(r, d)$. On the contrary, fixing $(x, y) \in \partial_r \tilde{S}^d$ there exists $\xi(d, x, y)$ such that
\[ r = \sqrt{(1 + \frac{x}{d})^2 + \left(\frac{y}{d}\right)^2} = 1 + \frac{x}{d} + \xi(d, x, y). \]
In particular, if $y = 0$ we have $\xi(d, x, 0) = 0$, while if $y > 0$, $\xi(d, x, y) \sim d^{-2}$.

We are ready to prove the convergence of $\{\hat{u}_d^1, \ldots, \hat{u}_d^k\}$ as $d \to \infty$.

Lemma 5.12. Up to a subsequence, $\{\hat{u}_d^1, \ldots, \hat{u}_d^k\}$ converges in $C^2_{\text{loc}}(C^\infty)$, as $d \to \infty$, to a nontrivial solution $(\hat{u}_1, \ldots, \hat{u}_k)$ of (2). This solution, which is $k\pi$-periodic in $y$, enjoys the symmetries
\[ \hat{u}_{i+1}(x, y) = \hat{u}_i(x, y - \pi) \quad \text{and} \quad \hat{u}_1(x, y + \frac{\pi}{2}) = \hat{u}_1(x, y - \frac{\pi}{2}). \]

Proof. From proposition 5.6 and lemma 5.8, we deduce that for any $r \geq 1$ and $d$ the inequality
\[ \frac{\tilde{H}_d(r)}{r^{2d}} = \frac{k\beta^d H_d(r)}{2^{2d}r^{2d}} \leq \frac{k\beta^d}{2^{2d}} H_d(1) = \tilde{H}_d(1) = 1 \]
holds. For every $x > 0$, let $r = 1 + x/d$; for every $d$ sufficiently large, we have
\[ \tilde{H}_d\left(1 + \frac{x}{d}\right) \leq \left(1 + \frac{x}{d}\right)^{2d} \leq 2e^{2x}. \quad (45) \]
Recalling (44) (which we apply for $R = R_d$), we deduce
\[ \tilde{E}_d\left(1 + \frac{x}{d}\right) = \tilde{N}_d\left(1 + \frac{x}{d}\right) \tilde{H}_d\left(1 + \frac{x}{d}\right) \leq 2e^{2x} \quad (46) \]
for every $d$ sufficiently large. Recall that $(\hat{u}_d^1, \ldots, \hat{u}_d^k)$ can be extended by angular periodicity in the whole plane $\mathbb{R}^2$. Let us introduce
\[ T_r^d := \{(\rho, \theta) : \rho < r, \ \theta \in \left(-\frac{\pi}{d}, (k+1)\frac{\pi}{d}\right)\} \supset S_r^d, \]
and let $\tilde{T}_d := d(T_r^d - (1, 0)) \supset \tilde{S}_r^d$. By suitably modifying the argument in lemma 5.9, it is not difficult to see that
\[ \text{Int}\left(\bigcap_{d \geq n} \bigcup_{d > n} \tilde{T}_r^d\right) = (-\infty, \bar{x}) \times (-\pi, (k+1)\pi) \]
for every $\bar{x} \in \mathbb{R}$. Hence, let $B$ an open ball contained in $\mathbb{R} \times (-\pi, (k+1)\pi)$, and let $x_B := \sup\{x : (x, y) \in B\}$, so that $B \subset (-\infty, x_B + 1) \times (-\pi, (k+1)\pi)$. Using the same argument in the proof of lemma 5.9, it is possible to show that
\[ B \subset \tilde{T}_r^d. \]

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for every $d$ sufficiently large, and by \( \text{(46)} \) and the periodicity of \( (\hat{u}_1, \ldots, \hat{u}_k) \) we deduce
\[
\int_B \sum_i |\nabla \hat{u}_i|^2 \leq 3\hat{E}_d \left( 1 + \frac{x_B + 1}{d} \right) \leq 6e^{2(x_B+1)}
\]
whenever $d$ is sufficiently large. This, together with \( \text{(45)} \), implies that \( \{ (\hat{u}_1^d, \ldots, \hat{u}_k^d) \} \) is uniformly bounded in $H^1(B)$, for every $B \subset \mathbb{R} \times (-\pi, (k+1)\pi)$. By the boundedness of the trace operator, this bound provides a uniform-in-$d$ bound on the $L^2(\partial K)$ norm for every compact $K \subset \mathbb{R} \times (-\pi, (k+1)\pi)$, which in turn, due to the subharmonicity of $u_i^d$, gives a uniform-in-$d$ bound on the $L^\infty (K)$ norm of \( \{ (\hat{u}_1^d, \ldots, \hat{u}_k^d) \} \), for every compact set $K \subset \mathbb{R} \times (-\pi, (k+1)\pi)$. The standard regularity theory for elliptic equations guarantees that when $d \to \infty$ then \( \{ (\hat{u}_1^d, \ldots, \hat{u}_k^d) \} \) converges in $C^{1,\alpha} _{\text{loc}} (\mathbb{R} \times (-\pi, (k+1)\pi))$, up to a subsequence, to a function \( (\hat{u}_1, \ldots, \hat{u}_k) \) which is a solution to (2). By the convergence and by the normalization required in lemma \( 5.8 \), we deduce that (recall also the convergence of the boundaries $\partial \hat{S}_1^d$, remark \( 5.10 \))
\[
\int_0^{k\pi} \sum_i \hat{u}_i(0, y)^2 \, dy = 1;
\]
in particular, \( (\hat{u}_1, \ldots, \hat{u}_k) \) is nontrivial. The $k\pi$-periodicity in $y$ follows directly from the convergence of the domains, lemma \( 5.9 \). By the pointwise convergence of \( (\hat{u}_1^d, \ldots, \hat{u}_k^d) \) to \( (\hat{u}_1, \ldots, \hat{u}_k) \) and by the symmetries of each function \( (\hat{u}_1^d, \ldots, \hat{u}_k^d) \) (see equation \( 36 \) and remark \( 5.2 \)) we also deduce that
\[
\hat{u}_{i+1}(x, y) = \hat{u}_i(x, y - \pi) \quad \text{and} \quad \hat{u}_1 \left( x, y + \frac{\pi}{2} \right) = \hat{u}_1 \left( x, y - \frac{\pi}{2} \right).
\]

### 5.3. Characterization of the growth of \( (\hat{u}_1, \ldots, \hat{u}_k) \)

So far we proved the existence of a solution \( (\hat{u}_1, \ldots, \hat{u}_k) \) of (2) which enjoys properties (1) and (2) of theorem \( 1.8 \). In this subsection, we are going to complete the proof of the quoted statement, showing that \( (\hat{u}_1, \ldots, \hat{u}_k) \) also enjoys properties (3)–(5). We denote as $\hat{\mathcal{E}}$, $\hat{\mathcal{E}}$, $\hat{H}$ and $\hat{N}$ the quantities $E_{\text{unb}}$, $E_{\text{unb}}$, $H$ and $N_{\text{unb}}$ introduced in section 2.2 when referred to the function \( (\hat{u}_1, \ldots, \hat{u}_k) \). First, we show that \( (\hat{u}_1, \ldots, \hat{u}_k) \) has finite energy, point (3) of theorem \( 1.8 \), and that \( \hat{H}(x) \to 0 \) as $x \to -\infty$.

**Lemma 5.13.** For every $x \in \mathbb{R}$ there holds $\hat{\mathcal{E}}(x) < +\infty$. In particular,
\[
\hat{\mathcal{E}}(x) \leq \liminf_{d \to \infty} \hat{E}_d \left( 1 + \frac{x}{d} \right) \quad \text{and} \quad \hat{\mathcal{E}}(x) \leq \liminf_{d \to \infty} \hat{E}_d \left( 1 + \frac{x}{d} \right).
\]
Furthermore, $\hat{H}(x) \to 0$ as $x \to -\infty$.

**Proof.** By the $C^2_{\text{loc}}(\mathbb{R}^2)$ convergence of \( (\hat{u}_1^d, \ldots, \hat{u}_k^d) \) to \( (\hat{u}_1, \ldots, \hat{u}_k) \) and by the convergence properties of the domains $\hat{S}_{1^d}$, lemma \( 5.9 \), we deduce
\[
\lim_{d \to \infty} \left( \sum_i |\nabla \hat{u}_i|^2 + \sum_{i<j} (\hat{u}_i \hat{u}_j)^2 \right) \chi_{\hat{S}_{1^d}} = \left( \sum_i |\nabla \hat{u}_i|^2 + \sum_{i<j} (\hat{u}_i \hat{u}_j)^2 \right) \chi_{\hat{S}_{1^d}} \quad \text{a.e. in } C_{\infty},
\]
for every $x \in \mathbb{R}$. As a consequence, we can apply the Fatou lemma obtaining
\[
\hat{\mathcal{E}}(x) \leq \liminf_{d \to \infty} \hat{E}_d \left( 1 + \frac{x}{d} \right) \leq 2e^{2x},
\]
where the uniform boundedness of $\hat{E}_d(1+x/d)$ comes from (46). To prove that $\hat{H}(x) \to 0$ as $x \to -\infty$, we can proceed with the same argument developed in lemma 4.7.

In light of the previous lemma, the monotonicity formulae proved in section 2.2 applies for $\hat{E}$, $\hat{H}$ and $\hat{N}$.

**Lemma 5.14.** $\hat{N}(x) \to 1$ holds as $x \to +\infty$.

**Proof.** By proposition 2.14, we know that $\hat{N}$ is nondecreasing in $x$, and thanks to the symmetries of $(\hat{u}_1, \ldots, \hat{u}_k)$, see lemma 5.12, lemma 2.16 implies that $\lim_{x \to +\infty} \hat{N}(x) \geq 1$. It remains to show that this limit is smaller than 1. This follows from the estimates of lemma 5.13 and from the strong convergence of $(\hat{u}_d^1, \ldots, \hat{u}_d^k) \to (\hat{u}_1, \ldots, \hat{u}_k)$, which implies that $\hat{H}_d(1+x/d) \to \hat{H}(x)$ as $d \to \infty$: therefore, for every $x \in \mathbb{R}$

$$\hat{N}(x) = \frac{\hat{E}(x)}{\hat{H}(x)} \leq \liminf_{d \to \infty} \frac{\hat{E}_d(x)}{\hat{H}_d(x)} = \liminf_{d \to \infty} \hat{N}_d(x) \leq 1,$$

where we used (44). □

In light of this achievement, we can apply corollary 2.15 to complete the proof of point (5) of theorem 1.8. The fact that $\gamma > 0$ follows by lemmas 5.14 and 2.16:

$$\lim_{r \to +\infty} \frac{\hat{H}(r)}{e^2} = \lim_{r \to +\infty} \frac{\hat{E}(r)}{e^2} \cdot \lim_{r \to +\infty} \frac{1}{\hat{N}(r)} > 0.$$

**Remark 5.15.** With a similar construction, it is possible to obtain the existence of solutions to (2) in $\mathbb{R}^2$ modelled on $\cosh x \sin y$. To do this, we can first construct solutions of (2) having algebraic growth defined outside the ball of radius 1, with homogeneous Neumann boundary conditions on $\partial B_1$. This can be done by suitably modifying the proof of theorem 1.6 in [2]. Then, performing a new blow-up in a neighbourhood of $(1, 0)$, we can obtain a solution of (2) defined in $\mathbb{R}^2_+$, with homogeneous Neumann condition on $\{x = 0\}$; this solution can be extended by even symmetry in $x$ in the whole $\mathbb{R}^2$.

## 6. Asymptotics of solutions which are periodic in one variable

**Proof of theorem 1.9.** Let us start with case (i). Since the solution $(u, v)$ is nontrivial $N(0) > 0$: in particular, from point (i) of corollary 2.15 it follows that $H(r) \to +\infty$ as $r \to +\infty$. Let us consider the shifted functions

$$(u_R(x, y), v_R(x, y)) := \frac{1}{\sqrt{H(R)}} (u(x + R, y), v(x + R, y)),$$

which solve the system

$$\begin{cases}
-\Delta u_R = -H(R)u_R v_R^2 & \text{in } C_\infty, \\
-\Delta v_R = -H(R)u_R^2 v_R & \text{in } C_\infty, \\
\int_{\Sigma_1} u_R^2 + v_R^2 = 1
\end{cases}$$

and share the same periodicity of $(u, v)$. We introduce

$$E_R(r) := \int_{C(-\infty, r)} |\nabla u_R|^2 + |\nabla v_R|^2 + 2H(R)u_R^2 v_R^2, \quad H_R(r) := \int_{\Sigma_r} u_R^2 + v_R^2 \quad \text{and} \quad N_R(r) := \frac{E_R(r)}{H_R(r)}.$$
It is easy to see that

\[ E_R(r) = \frac{1}{H(R)} E^\text{unb}(r + R) \]

\[ H_R(r) = \frac{1}{H(R)} H(r + R) \Rightarrow N_R(r) = N^\text{unb}(r + R) \]

for any \( r \) (recall that \( E^\text{unb} \) and \( N^\text{unb} \) have been defined in section 2.2). We point out that, by definition and the monotonicity of \( N^\text{unb} \), proposition 2.14, \( N_R(r) \leq N_R(\bar{r}) \) for every \( R_1 < R_2 \). Furthermore, \( N_R(r) \leq d = \lim_{R \to \infty} N(r) \) for every \( r, R \) and \( N_R(r) \to d \) as \( R \to \infty \) for every \( r \in \mathbb{R} \). Therefore, \( N_R \) tends to the constant function \( d \) in \( L^1_{\text{loc}}(\mathbb{R}) \).

Thanks to the normalization condition \( H_R(0) = 1 \) and the uniform bound \( N_R(r) \leq d \), applying corollary 2.15 (see also remark 2.17) we deduce that \( H_R(r) \) is uniformly bounded in \( R \) for every \( r > 0 \). Consequently, also \( E_R(r) \) is uniformly bounded in \( R \) for every \( r > 0 \). By means of a Poincaré inequality of type (23), we deduce that the sequence \((u_R, v_R)\) is uniformly bounded in \( H^1_{\text{loc}}(\mathbb{C}_\infty) \) and, by standard elliptic estimates, in \( L^\infty_{\text{loc}}(\mathbb{C}_\infty) \). From Theorem 2.6 of [11] (it is a local version of theorem 1.1 of [9]), we evince that the sequence \((u_R, v_R)\) converges to \((\Psi^*, \Psi^-)\), where \( \Psi \) is a nontrivial harmonic function (this is a combination of the main results in [9] and [5]). By the convergence, \( \Psi \) has to be \( 2\pi \)-periodic in \( y \).

First, we prove that \( H(r; \Psi) \to 0 \) as \( r \to -\infty \), so that the results of section 2.3 hold true for \( \Psi \). As already observed, \( N_R(r) \geq N(r) \) for every \( r \in \mathbb{R} \), for every \( R > \bar{R} \). By the expression of the logarithmic derivative of \( H_R \), see corollary 2.15 (see also remark 2.17), we have

\[ \frac{d}{dr} \log H_R(r) = 2N_R(r) \geq 2N(r) = \frac{d}{dr} \log H(r) \quad \forall r. \]

As a consequence, taking into account that \( H_R(0) = 1 \) for every \( R \), for every \( r < 0 \) it results that

\[ \frac{H_R(0)}{H_R(r)} \geq \frac{H(0)}{H(r)} \Leftrightarrow H_R(r) \geq H(r) \quad \forall R > \bar{R}. \]

Passing to the limit as \( R \to +\infty \), by the \( C^0_{\text{loc}}(\mathbb{R}^2) \) convergence of \((u_R, v_R)\) to \((\Psi^*, \Psi^-)\) it follows that \( H_R(r) \geq H(r; \Psi) \), which gives \( H(r; \Psi) \to 0 \) as \( r \to -\infty \) in light of our assumption on \((u, v)\).

Using again the expression of the logarithmic derivative of \( H_R \) and \( H(\cdot; \Psi) \), we deduce

\[ \log \frac{H_R(r_2)}{H_R(r_1)} = 2 \int_{r_1}^{r_2} N_R(s) \, ds \quad \text{and} \quad \log \frac{H(r_2; \Psi)}{H(r_1; \Psi)} = 2 \int_{r_1}^{r_2} N(s; \Psi) \, ds, \]

where \( r_1 < r_2 \). The left-hand side of the first identity converges to the left-hand side of the second identity; recalling that \( N_R \to d \) in \( L^1_{\text{loc}}(\mathbb{R}) \), we deduce

\[ \int_{r_1}^{r_2} N(s; \Psi) \, ds = \lim_{R \to +\infty} \int_{r_1}^{r_2} N_R(s) \, ds = d(r_2 - r_1) \Rightarrow \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} N(s; \Psi) \, ds = d, \]

for every \( r_1 < r_2 \). It is well known that, since \( N(\cdot; \Psi) \in L^1_{\text{loc}}(\mathbb{R}) \), the limit as \( r_2 \to r_1 \) of the left-hand side converges to \( N(r_1; \Psi) \) for almost every \( r_1 \in \mathbb{R} \). Hence, \( N(r; \Psi) = d \) for every \( r \in \mathbb{R} \). We are then in position to apply proposition 2.19:

\[ \lim_{R \to +\infty} N(R) = \lim_{R \to +\infty} N_R(0) = N(0; \Psi) = d \in \mathbb{N} \setminus \{0\}, \]

and \( \Psi(x, y) = [C_1 \cos(dy) + C_2 \sin(dy)]e^{dx} \) for some constant \( C_1, C_2 \in \mathbb{R} \).

As far as case (ii) is concerned, for the sake of simplicity we assume \( a = 0 \). One can repeat the proof with minor changes replacing \( E^\text{unb} \) and \( N^\text{unb} \) with \( E^\text{sym} \) and \( N^\text{sym} \) (which have
been defined in section 2.1). The unique nontrivial step consists in proving that in this setting

\[ H(r; \Psi) \to 0 \text{ as } r \to -\infty. \]

To this aim, we note that, as before,

\[ H_R(r) \leq H_{\bar{R}}(r) \quad \forall R > \bar{R}, \]

for every \( r > -\bar{R} \). In particular, if \( r \in (1 - \bar{R}, 0) \), by proposition 2.4 and corollary 2.5 we deduce

\[ H_R(r) \leq H_{\bar{R}}(r) = \frac{H(r + \bar{R})}{H(\bar{R})} \leq \frac{e^{2N(1)(r+\bar{R})}}{e^{2N(1)\bar{R}}} = e^{2N(1)r} \quad \forall R > \bar{R}. \]

Passing to the limit as \( R \to +\infty \), by \( C^0_{\text{loc}}(\mathbb{R}^2) \) convergence we obtain

\[ H(r; \Psi) \leq e^{2N(1)r} \quad \forall r \in (-\infty, 0), \]

which yields \( H(r; \Psi) \to 0 \) as \( r \to -\infty \).

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