Dynamics of double-well Bose–Einstein condensates subject to external Gaussian white noise

Hanlei Zheng, Yajiang Hao and Qiang Gu

Department of Physics, University of Science and Technology Beijing, Beijing 100083, People’s Republic of China
E-mail: qgu@ustb.edu.cn

Received 30 November 2012, in final form 24 January 2013
Published 26 February 2013
Online at stacks.iop.org/JPhysB/46/065301

Abstract
Dynamical properties of the Bose–Einstein condensate in a double-well potential subject to Gaussian white noise are investigated by numerically solving the time-dependent Gross–Pitaevskii equation. The Gaussian white noise is used to describe influence of the random environmental disturbance on the double-well condensate. Dynamical evolutions from three different initial states, the Josephson oscillation state, the running phase and \( \pi \)-mode macroscopic quantum self-trapping states, are considered. It is shown that the system is rather robust with respect to the weak noise whose strength is small and change rate is high. If the evolution time is sufficiently long, the weak noise will finally drive the system to evolve from high-energy states to low-energy states, but in a manner rather different from the energy-dissipation effect. In the presence of strong noise with either large strength or slow change rate, the double-well condensate may exhibit very irregular dynamical behaviours.

1. Introduction

The atomic Bose–Einstein condensate (BEC) trapped in double-well potentials builds up bosonic–Josephson junction (BJJ) [1–5]. Since it exhibits abundant quantum properties in comparison to a condensate in a single trap, the double-well condensate has already been intensively investigated theoretically in the last few years. As a BJJ, the double-well condensate can not only display dc, ac Josephson effects and the Shapiro effect, but it also exhibits a quantum nonlinear effect, called the macroscopic quantum self-trapping (MQST) [3]. On the other hand, quantum fluctuation is believed to give rise to fascinating influence on the above dynamical behaviours [6–12], such as collapse and revival of quantum oscillations [6, 7], disappearance of coherence [8, 9] and destruction of the self-trapped state [10].

Experimentally, the Josephson tunnelling and MQST in a single BJJ have been observed in 2005 [13]. Since then, more progress has been made in studying static, thermal and dynamical properties of the double-well condensate. Experimental investigation of thermal-induced phase fluctuations has been reported [14]. Measurements of the ac and dc Josephson effects in BJJ have already been realized [15]. The BJJ system has also been used to perform interference-fringe experiments [16] and to investigate the crossover from Josephson dynamics to hydrodynamics [17]. In the above theoretical studies, the double-well condensate is mainly treated as an isolated system, but actually it is coupled to a certain thermal cloud and subject to environmental distortions in experiments. Dissipation and noise effects play important roles in understanding properties of BJJs.

The dissipative effect has been studied by several groups [18–23]. It is suggested that the MQST state can be destroyed by energy dissipation [18–20]. Possible decoherence caused by dissipation is also discussed [21, 22]. It is also shown that dissipation could lead to enhancement of coherence under specific conditions [23].

Here, we consider a kind of noise effect on the double-well condensate. Noise can be classified as ‘internal noise’ or ‘external noise’ with respect to its origins [24]. Internal noise comes from the inside fluctuations of the system, including quantum and thermal fluctuations, or from the exchange symmetry of identical particles [25, 26]. On the other hand, ‘external noise’ is brought about by fluctuations which are not...
'self-originating'. It is induced by the coupling between the system and its environment. For cold atomic systems, external noise may originate from the magnetic field, laser beams or other externally applied random driving fields. This work will focus on this kind of external noise.

A number of theoretical works have been devoted to the influence of external noise on the double-well condensate [23, 27, 28]. The noise-induced dephasing [27] and phase decoherence [28] have been predicted. These studies based on the two-mode approximation mainly discussed the phase noise. The phase noise is introduced by coupling stochastic fluctuations either to the tunnel amplitude or to the number-imbalance operator. In their treatment, spatial information of the external noise is ignored.

This paper deals with noise due to fluctuations of the magnetic field or the optical potential. We shall consider the time- and space-dependent characteristics of the external noise by solving the time-dependent Gross–Pitaevskii (GP) equation numerically. In section 2, we describe a noise model in which the noise is simulated by spatially distributed stochastic potentials. The concept of Gaussian white noise and the algorithm used to solve the time-dependent GP equation for the double-well condensate are briefly described. We present in section 3 the obtained results and discuss the influence of noise on the dynamical behaviours of the system. A brief summary is given in the last section.

2. The model and method

We consider the BEC confined in a double-well potential composed of atoms of mass \( m \) with weakly repulsive interaction. Dynamics of the double-well condensate is obtained by solving the time-dependent GP equation numerically [29, 30]. We formulate the external noise by an additional potential of stochastic strength fluctuation in the space.

The external noise is modelled as the Gaussian white noise potential \( V_n(z,t) \), which is space- and time-dependent stochastically. So \( V_n(z,t) \) satisfies the following equations:

\[
(V_n(z_1,t) - V_n(z_2,t)) = 2D_0 \delta(t-t') \delta(z_1 - z_2),
\]

where \( \langle \cdot \rangle \) denotes averaging over both the space and the time. The Dirac delta function in the correlation formula makes sure \( V_n(z,t) \) is ‘white’ noise, and \( D_0 \) is the fluctuation amplitude of the noise potential. Apparently, \( D_0 \) characterizes the strength or intensity of noise and it can be controlled externally.

The noise correlation function in time at a given position is defined as \( g(t-t') = \langle V_n(z_1,t) V_n(z_2,t') \rangle_T - \langle V_n(z_1,t) \rangle_T^2 \sim \delta(t-t') \), where \( \langle \cdot \rangle_T \) means that the average is just taken over time. Under this definition, \( \langle V_n(z,t) \rangle_T \) must be zero at all given positions. In numerical calculations, \( V_n(z,t) \) is produced as a time-ordered series, \( V_n(z,t) = \sum_{n=0}^{M} V_n(z_{t+n}) / M \). \( M \) is actually a finite number and \( M \Delta t \) is the total time interval with \( \Delta t = t_{n+1} - t_n \) being the time step interval of the noise potential series. In our calculations, \( M = 100 \) is already enough to make sure \( \langle V_n(z,t) \rangle_T = 0 \). Therefore, \( \Delta t \) can be used to evaluate the velocity of noise. Faster noise corresponds to smaller \( \Delta t \).

The general three-dimensional time-dependent GP equation provides an exact and fundamental description for our research, which can be formulated as

\[
\frac{i \hbar}{2m} \frac{\partial \psi(r,t)}{\partial t} = \left[ \frac{\hbar^2}{2m} \nabla^2 + V(r) + gN|\psi(r,t)|^2 \right] \psi(r,t),
\]

where \( \psi(r,t) \) is the macroscopic wavefunction at position \( r \) and time \( t \), \( g = 4\pi \hbar^2 a/m \) is the nonlinear interaction with \( a \) being the s-wave scattering length. The external potential \( V(r,t) \) consists of the double well \( V_{dw}(r) \) and the noise potential \( V_n(r,t) \). \( V_{dw}(r) = \frac{2}{3} \left( \omega_x^2 r_x^2 + \omega_y^2 r_y^2 + \omega_z^2 r_z^2 \right) \) + \( V_0 \exp(-r^2/q_0^2) \) fixes the double-well configuration, where \( \omega \), \( q_0 \) are the trap frequencies in the direction of \( i \) \([12, 13]\) and \( V_0 \) is the barrier height. Experimentally, the BECs with repulsive interaction in a quasi-1D symmetric double-well potential can be achieved by splitting one cigar-shaped atomic cloud into two separated aligned cigars by a laser beam.

Set the double wells along the \( z \) direction, and in the \( x \) and \( y \) directions the strong confinement is exerted \( (\omega_x = \omega_y \gg \omega_z) \). Thus the original three-dimensional condensates shall be reduced into quasi-one-dimensional cigar BECs and equation (3) is reduced to a one-dimensional one. By scaling the length and energy as \( l = \sqrt{\hbar/(m \omega_z)} \) and \( \omega_z \), respectively, the dimensionless parameters are simplified as \( z = r/L \), \( \tau = t \omega_z/2 \) and \( \beta = \sqrt{\omega_x^2 + \omega_y^2}/\omega_z \). Therefore, the reduced one-dimensional GP equation can be formulated as

\[
\frac{\partial \tilde{\psi}(z,\tau)}{\partial \tau} = \left[ \frac{\partial^2}{\partial z^2} + v_{dw}(z) + 2v_n(z,\tau) + g_{1D}|\tilde{\psi}(z,\tau)|^2 \right] \tilde{\psi}(z,\tau),
\]

where \( v_{dw} = z^2 + 2v_0 \exp(-z^2/q_0^2) \), with \( v_{dw} = V_{dw}/(\hbar \omega_z) \), \( v_n = V_n/(\hbar \omega_z) \) and \( q_0 = q_0/l \). We obtain the dimensionless interaction parameter \( g_{1D} = g_{\text{N}} \beta \sqrt{\pi z} \) with \( N \) being the total atom number.

The wavefunction \( \tilde{\psi}(z,\tau) \) satisfies the normalization condition \( \int_{-\infty}^{\infty} dz |\tilde{\psi}(z,\tau)|^2 = 1 \). The fraction of atom number in the left and right well are \( n_L(\tau) = \int_{-\infty}^{0} dz |\tilde{\psi}(z,\tau)|^2 \) and \( n_R(\tau) = \int_{0}^{\infty} dz |\tilde{\psi}(z,\tau)|^2 \), respectively [30]. \( \theta_L(\tau) = \arctan \left( \int_{-\infty}^{0} dz \frac{\text{Im} \{ \tilde{\psi}(z,\tau) \} \tilde{\psi}(z,\tau)}{\int_{-\infty}^{\infty} dz |\tilde{\psi}(z,\tau)|^2} \right) \) is the phase in the left well while \( \theta_R(\tau) = \arctan \left( \int_{0}^{\infty} dz \frac{\text{Im} \{ \tilde{\psi}(z,\tau) \} \tilde{\psi}(z,\tau)}{\int_{-\infty}^{\infty} dz |\tilde{\psi}(z,\tau)|^2} \right) \) in the right well with the density \( \rho(\tau) = \psi^\ast(z,\tau) \psi(z,\tau) \).

\( \phi_+ \) and \( \phi_- \) represent the initial ground state and the first excited state wavefunctions for the condensates in the double well. Their linear combinations can be defined as the left (right) mode: \( \psi_{LR}(\tau) = \frac{\phi_+ + \phi_-}{\sqrt{2}} \). They satisfy the orthogonal condition \( \int dz \psi_{LR}(z) \psi_{LR}(z)^\ast = 0 \). The trial wavefunction for obtaining the initial state can be chosen as the superposition of \( \psi_L(\tau) \) and \( \psi_R(\tau) \) as in [30]
on a unit internal, and the dimensionless parameter \( D \) where is \( a \) using the split-step Crank–Nicolson scheme, with both the \( \phi \) \( \delta \tau \) time step is \( /\Delta_1 \theta \) [31].

The phase-space diagram of double-well BECs without noise Figure 1. Snapshots of the random Gaussian white noise potential with the noise strength \( D = 0.005 \) at the time \( \tau = 1 \) (a), \( \tau = 2 \) (b) and \( \tau = 10 \) (c), respectively.

\[
\sqrt{n_R(\tau)} e^{i\phi(z)(\tau)}.
\]

The population imbalance and relative phase at time \( \tau \) are defined as

\[
\Delta n = n_L - n_R \quad \text{and} \quad \Delta \theta = \theta_L - \theta_R.
\]

A given trial at \( \tau = 0 \) is \( \psi(z; 0) = e^{i\Delta \theta(0)} \sqrt{n_L(0)} \psi_L(z) + \sqrt{n_R(0)} \psi_R(z) \). \( \Delta n(0) = n_L(0) - n_R(0) \) represents the initial population imbalance and \( \Delta \theta(0) \) is the phase difference of condensates in the double well. The relative phase can be measured by the interference patterns of releasing the BECs from the double-well potential after different evolution times [13]. Moreover, a technique based on stimulated light scattering has been developed to detect \( \Delta \theta \) nondestructively [31].

The reduced time-dependent GP equation can be solved using the split-step Crank–Nicolson scheme, with both the space and time being discretized [32]. In our calculation, the time step is \( \delta \tau = 0.001 \) and the space step is \( \delta z = 0.01 \). The initial ground state and the first excited state wavefunctions, \( \phi_L(z) \) and \( \phi_R(z) \), can be obtained by the imaginary-time propagation. The dynamical evolution are calculated by solving equation (4) using the real-time propagation method.

The Gaussian white noise is produced numerically by the Box–Mueller algorithm [33],

\[
v_n = \sqrt{-4D \ln(a)} \cos(2\pi b),
\]

where \( a \) and \( b \) are two uniformly distributed random numbers on a unit interval, and the dimensionless parameter \( D = D_0/(\hbar \omega)^2 \). \( v_n \) varies occasionally from time to time. Figure 1 displays snapshots of the spatial form of the random potential at a different given time. Suppose that it changes \( R \) times each unit time. Then \( R \) defines a character parameter with respect to the velocity of noise and it is called the change rate of noise hereinafter. The fastest noise is the one whose step interval \( \Delta t \) just amounts to the calculation time step \( \delta \tau \). It means that the noise potential changes once per calculation step. Since the time step \( \delta \tau = 0.001 \), the change rate of the fastest noise in our study is \( R = 1000 \).

3. Results and discussions

The phase-space diagram of double-well BECs without noise has been studied based on the time-dependent GP equation [20] and the obtained results are quantitatively consistent with the two-mode model results [3]. Three typical dynamical regimes can be present. (i) The Josephson oscillation regime consists of a cluster of close orbits circling the lowest energy point \((\Delta n = 0, \Delta \theta = 0)\) or \((\Delta n = 0, \Delta \theta = 2k\pi)\) where \( k \) is an integer. (ii) The \( \pi \)-mode MQST regime consists of a cluster of close orbits circling the highest energy point \((\Delta n \neq 0, \Delta \theta = (2k \pm 1)\pi)\). (iii) The running-phase MQST regime consists of open orbits which lie between the above two regimes.

To obtain the numerical results of dynamical double-well condensate subject to the Gaussian white noise, we set the double-well potential in equation (4) to be \( v_{\text{dwell}}(z) = z^2 + 10\exp(-z^2) \) for simplicity. The interatomic interaction \( g_{1D} = 0.01 \). Under the above choice of relevant parameters, all three dynamical regimes appear in the phase diagram. First, we reproduce the phase diagram without noise, \( D = 0 \), and determine the location of each typical dynamical regime. Then we choose one initial state from each regime by setting the initial particle imbalance \( \Delta n \) and relative phase \( \Delta \theta \) appropriately.

We first consider the dynamical evolution subject to Gaussian white noise with a high change rate of \( R = 1000 \). Figure 2(a) shows the orbits presented by constant energy lines for an initial state of Josephson oscillation \((\Delta n = 0.9, \Delta \theta = 0)\) without noise \((D = 0)\). Both the population imbalance and the relative phase oscillate around the zero point \((\Delta \theta = 0, \Delta n = 0)\) during the evolution. When Gaussian white noise with the strength \( D = 0.005 \) is imposed, as illustrated in figure 2(b), the trajectory gets a little ‘fat’, but dynamical properties are not qualitatively different from the system without noise. However, the noise effect shows up apparently when \( D = 0.02 \), as seen in figure 2(c). The system evolves gradually along an inward spiral path which is still smooth at an earlier stage but then starts to zigzag after a long time. During this period, the system undergoes energy dissipation. Figure 2(d) shows that the energy dissipates more quickly.
and the system enters into a state in the black horizontal band whose evolution trajectory looks like the $\pi$-mode MQST. If the noise strength becomes stronger, the system starts $\pi$-mode-like oscillating from the very beginning, as shown in figure 3(b). One interesting phenomenon is that the trajectory may pass across the $\Delta n = 0$ line and then evolve around the down energy-maximum point ($\Delta n < 0$, $\Delta \theta = \pi$). This process will be accelerated if $D$ is increased and the system may be in a Josephson-like oscillation around the point ($\Delta n = 0$, $\Delta \theta = \pi$), as shown in figure 3(c). The energy decreases with time in an oscillatory manner. In the case of extremely strong noise, the system falls down to the irregularly energy-fluctuating state after a short period of damping.

Figure 4 displays the cases evolving initially from the $\pi$-mode MQST state ($\Delta n = 0.9$, $\Delta \theta = 0.9 \pi$). It is obvious that the $\pi$-mode MQST state is more sensitive to the noise. As shown in figure 4(b), with the participation of noise as weak as $D = 0.005$, the previous closed orbit has already been becoming very fuzzy. When the noise strength is increased to $D = 0.02$, the system gets into the running phase regime quickly. The following evolution can be regarded as a process starting from a running phase MQST state. The basic features should be similar to figure 3. The actual evolution trajectory depends on the concrete initial state. This is the reason why figures 4(c) and (d) look quite different from figures 3(c) and (d).

According to the discussions above, when the noise strength is small, the Gaussian white noise may lead to an energy dissipation effect. The noise with a larger $D$ makes the energy dissipating faster. But there are some differences in the dissipative process between the present study and the results obtained within phenomenological theories [19, 20]. The main difference lies in the evolutions from MQST states. The system damps directly towards the saddle point ($\Delta n = 0$, $\Delta \theta = \pi$), and not the energy minimum ($\Delta n = 0$, $\Delta \theta = 0$). Then it steps into the irregularly energy-fluctuating state. The phenomenological theories suggest that the energy decreases all the time until the system approaches the energy minimum point. No matter how strong the dissipation is, the system will finally damp into the energy minimum state, and then the evolution stops. The irregularly energy-fluctuating state does not appear.

We have calculated the dynamical behaviours of the double-well condensate subject to Gaussian white noise with different strength $D$ and change rate $R$. The noise effect is mainly determined by the strength $D$ when the change rate $R$ is large enough. Nevertheless, the change rate $R$ also plays an important role in understanding the dynamical evolution when $R$ is small.

Figure 5 displays a typical case of the noise with small strength ($D = 0.005$) and a slow change rate ($R = 10$). The evolution is from an initial state of the running phase MQST ($\Delta n = 0.9$, $\Delta \theta = 0.7 \pi$). Figure 5(b) looks like figure 3(b), but the evolution time is only within $\tau = 1000$,
4. Conclusion

In conclusion, we have investigated how an external Gaussian white noise affects the dynamics of condensates in a double well. Dynamical evolutions from three typical dynamical regimes of Josephson oscillation, \( \pi \)-mode MQST and running phase MQST are discussed. It is shown that the system keeps its original dynamical feature for a long time when it is subject to a noise with weak strength and a rapid change rate which we called weak noise. In this case, the noise-induced energy dissipation effect is observed. Energy of the system decreases with evolution monotonously in the Josephson oscillation regimes, and oscillatory in the MQST regimes. The difference between the results of the present study and phenomenological theory is discussed.

Either increasing the noise strength \( D \), or decreasing the noise change rate \( R \) may give rise to significant influence on dynamics of the system. If \( R \) is large enough, the influence of \( D \) is quite clear. The larger the \( D \) is, the faster the state evolves and damps. Moreover, we also figure out that the noise with large strength drives the system finally into the irregularly energy-fluctuating state, in which the population imbalance tends to be very small and the relative phase is no longer well defined. This phenomenon is different from the dissipation effect which makes the condensates evolve from the high-energy state to the lowest energy state where the particle imbalance is zero and the relative phase is \( 2k\pi \) (\( k \) is an integer). Decreasing \( R \) brings about a similar effect as increasing \( D \) in the early stage of evolution. After a period of time, the noise with a slow change rate compels the system to change irregularly, with the relative phase being destroyed. But this phenomenon happens preferentially in the large \( \Delta n \) region in the presence of the noise with a slow change rate, which is different from the effect caused by the noise with large noise strength but a fast change rate.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (grant no. 11074021 and no. 11040007) and the Fundamental Research Funds for the Central Universities of China.

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