High dimensional logistic entropy clustering

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Minimization of the (regularized) entropy of classification probabilities is a versatile class of discriminate clustering methods. The classification probabilities are usually defined through the use of some classical losses from supervised classification and the point is to avoid modelisation of the full data distribution by just optimizing the law of the labels conditioned on the observations. We give the first theoretical study of such methods, by specializing to logistic classification probabilities. We prove that if the observations are generated from a two-component isotropic Gaussian mixture, then minimizing the entropy risk over a Euclidean ball indeed allows to identify the separation vector of the mixture. Furthermore, if this separation vector is sparse, then penalizing the empirical risk by a $\ell_1$-regularization term allows to infer the separation in a high-dimensional space and to recover its support, at standard rates of sparsity problems. Our approach is based on the local convexity of the logistic entropy risk, that occurs if the separation vector is large enough, with a condition on its norm that is independent from the space dimension. This local convexity property also guarantees fast rates in a classical, low-dimensional setting.
1 Introduction

The clustering problem can be described as follows: given a measurable space \( \mathcal{X} \), a sample \((X_1, \ldots, X_n) \in \mathcal{X}^n\), and an integer \( K \geq 2 \), define a (random) labelling function \( Y : \mathcal{X} \to \{1, \ldots, K\} \). In particular, to each data \( X_i \), associate a label \( Y_i \). If the function \( Y \) is deterministic, then the task is termed “hard clustering”. If the function \( Y \) is random, the distribution of the labels \( Y(x) \), for \( x \in \mathcal{X} \), being characterized by the uplets \((P(Y(x) = 1), \ldots, P(Y(x) = K))\), then the clustering task is said to be “soft”. In the soft clustering case, a common approach - called the modelling approach - is to model the distribution of the data, typically as a mixture distribution, and to directly relate the probabilities \((P(Y(x) = 1), \ldots, P(Y(x) = K))\) to the parameters of the mixture [6]. One can then reduce to a hard clustering by assigning each point \( x \) to the maximizer of classification probabilities (or choose one at random amongst the maximizers if it is non-unique). Hard clustering algorithms include the celebrated K-means [29, 43, 31], hierachical clustering [23], spectral clustering [36] among others.

Particularly developed in the machine learning community for its flexibility when addressing complex data, the so-called “discriminative approach” to clustering amounts to model the classification probabilities \((P(Y(x) = 1), \ldots, P(Y(x) = K))\), which can be understood as the conditional probabilities of the labels with respect to the position \( x \). Proceeding this way indeed avoids the modelling of the whole distribution of data and often reduces to encode in the classification probabilities, the frontiers separating the clusters. In general, this is done through the use of classical learning losses such as the logistic, the Hinge or the Conditional Random Fields loss [14, 20]. More formally, one puts the constraint of \( P(Y(x) = k), k \in \{1, \ldots, K\} \), being proportional to \( \exp(\ell(\beta_k, x)) \), for a vector \( \beta_k \) and a loss \( \ell \). For instance the logistic loss gives classification probabilities proportional to \( \exp(w_k^T x + b_k) \) and the Hinge loss (for \( K = 2 \)) induces probabilities of a form proportional to \( \exp(-[1 - (w_k^T \varphi(x) + b_k)]^+) \) for some feature map \( \varphi \) and with \((w_1, b_1) = (-w_2, -b_2) \) in this binary case.

In addition, these losses were primarily introduced for supervised learning and in order to transfer them to the unsupervised setting, one has to define what would be a desirable (unobserved) label. Arguably,
when classifying data, one would prefer to be as sure as possible of its cluster choice. This is equivalent to saying that the maximum of classification probabilities would be as close to one as possible. Hence, a natural criterion to infer a labelling function, would be to define $\tilde{Y}$ through the probabilities $P(\tilde{Y}(x) = k) = Z_{\tilde{\beta}}^{-1}(x) \exp(\ell(\tilde{\beta}_k, x))$, with a normalizing constant $Z_{\tilde{\beta}}(x) = \sum_{k=1}^{K} \exp(\ell(\tilde{\beta}_k, x))$, such that

$$\langle \tilde{\beta}_1, ..., \tilde{\beta}_K \rangle \in \arg \max_{(\beta_1, ..., \beta_K)} \left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{Z_{\beta}(x_i)} \max_{k \in \{1, ..., K\}} [\exp(\ell(\beta_k, x_i))] \right\}. \tag{1}$$

The associated theoretical target is $P(Y_*(x) = k) = \exp(\ell(\beta_{*,k}, x))$ with,

$$\langle \beta_{*,1}, ..., \beta_{*,K} \rangle \in \arg \max_{(\beta_1, ..., \beta_K)} \left\{ \mathbb{E} \left[ \frac{1}{Z_{\beta}(X)} \max_{k \in \{1, ..., K\}} [\exp(\ell(\beta_k, X))] \right] \right\},$$

where $X$ follows the unknown - and not modeled - distribution of data.

But the maximum is not a smooth function and it may cause difficulties when trying to optimize (1). As a smooth proxy, one can try to minimize the entropy of the classification probabilities, since it achieves its minimum value when the latter probabilities are all equal to zero or one. This amounts to search for a labelling function $\hat{Y}$ satisfying $P(\hat{Y}(x) = k) = Z_{\hat{\beta}}^{-1}(x) \exp(\ell(\hat{\beta}_k, x))$ with

$$\langle \hat{\beta}_1, ..., \hat{\beta}_K \rangle \in \arg \min_{(\beta_1, ..., \beta_K)} \left\{ \frac{1}{n} \sum_{i=1}^{n} \text{Ent} \left\{ P(\hat{Y}(x_i) = 1), ..., P(\hat{Y}(x_i) = K) \right\} \right\}, \tag{2}$$

where

$$\text{Ent} \left\{ P(\hat{Y}(x_i) = 1), ..., P(\hat{Y}(x_i) = K) \right\} = \frac{1}{n} \sum_{k=1}^{K} \frac{\exp(\ell(\beta_k, x_i))}{Z_{\beta}(x_i)} \log \left( \frac{\exp(\ell(\beta_k, x_i))}{Z_{\beta}(x_i)} \right). \tag{3}$$

Often, one has to restrict the search among vectors $(\beta_1, ..., \beta_K)$ in a compact set, or to add to the entropy a regularization term encoding the complexity of the vectors $(\beta_1, ..., \beta_K)$ [20, 14]. In this second formulation, the theoretical target $(\beta_{0,1}, ..., \beta_{0,K})$ of estimation is,

The use of entropy terms in semi-supervised and unsupervised learning is indeed natural and has been the object of active research [21, 20, 14, 45, 44, 41, 27, 1, 34]. Furthermore, this approach is at the core
of some state-of-the-art deep clustering approaches [22]. Another fruitful approach in discriminative clustering consists in considering convex relaxations of some initial, untractable criteria and this methodology often comes with strong theoretical guarantees [16, 26, 4, 37, 18, 11, 10, 19, 33, 40, 13].

The starting point of our work consists in the following observation: to our knowledge, no theoretical guarantee - of the type of convergence rates - exists in the literature for (regularized) minimum entropy estimators (2). This a weakness compared to other approaches, such as convex relaxations techniques for instance. But from a practical perspective, estimators of the form of (2) have already proved to be efficient and flexible - allowing for instance feature maps embedding and the use of deep architectures - and the lack of theoretical studies needs to be filled.

We consider the unsupervised classification of a bipartite high-dimensional Gaussian mixture, with sparse means. This framework is indeed a good benchmark, since on the one hand, it is sufficiently simple to allow us to understand the nature of the target \((\beta_{0,1}, ..., \beta_{0,K})\) - with \(K = 2\) and \(\beta_{0,1} = -\beta_{0,2}\) in our bipartite framework - and to investigate the rate of convergence of estimators of the form of (2), suitably regularized by a \(\ell_1\)-penalty. On the other hand, the two-component high-dimensional Gaussian mixture has received recently at lot of attention [7, 2, 35, 28, 24, 15, 12, 3, 25, 8, 30]. Let us emphasize that our goal is not \textit{a priori} to provide a state-of-the-art method, specifically designed to solve the high-dimensional Gaussian mixture clustering, but to explore for the first time the theoretical behavior of discriminative estimators that minimize the (regularized) classification entropy and see how they can adapt to a sparse setting.

2 Some notations and definitions

Let \(a := (a_1, ..., a_d) \in \mathbb{R}^d\) and \(X\) be a random variable valued in \(\mathbb{R}^d\), with distribution \(P\). More precisely \(X := \varepsilon Z\) with \(\varepsilon \sim \text{Rad}(\frac{1}{2})\) and \(Z \sim \mathcal{N}(a, I_d)\) a Gaussian vector independent from \(\varepsilon\), with normalized variance equal to the identity matrix \(I_d\). Take \(n \in \mathbb{N}^*\), \(X^{(1)}, ..., X^{(n)}\) are observations of \(X\) independent and identically distributed according to \(P\). Our goal is to estimate the labelling function \(Y_*(x) = \text{sign}(x^t a)\),
or its opposite, which gives the same hard clustering. This amounts to estimate the separation vector \( a \). To do this, we will use an entropy criterion.

Set the logistic probability \( p_\beta (X) := 1/(1 + e^{X^T \beta}) \) where \( \beta \in \mathbb{R}^d \) and its complementary probability \( q_\beta (X) := e^{X^T \beta}/(1 + e^{X^T \beta}) \). The *logistic entropy* \( \rho_\beta \) is defined as follows, \( \rho_\beta (X) := \rho(\beta^T X) = -p_\beta (X) \log p_\beta (X) - q_\beta (X) \log q_\beta (X) \). The associated risk is \( \mathcal{R}(\beta) := \mathbb{E}[\rho_\beta (X)] \). The latter expectation will also be denoted \( P\rho_\beta \) for short. Let \( \|\cdot\|_1 \), \( \|\cdot\|_2 \) and \( \|\cdot\|_\infty \) be respectively the \( L_1 \), \( L_2 \) and \( L_\infty \)-norm, and denote \( B_1 (0, R), B_2 (0, R) \) and \( B_\infty (0, R) \) their corresponding balls centered at 0 with radius \( R \) in \( \mathbb{R}^d \). We consider the minimizer \( \beta_0 \) of the risk \( \mathcal{R}(\beta) \) over a \( L_2 \)-ball \( B_2 (0, R) \) - where the radius \( R \) will be fixed latter -. \( \beta_0 \in \arg \min_{\beta \in B_2 (0, R)} \{ \mathcal{R}(\beta) \} \), with excess risk \( \mathcal{E}(\beta, \beta_0) := \mathcal{R}(\beta) - \mathcal{R}(\beta_0) \), for \( \beta \in B_2 (0, R) \). The empirical distribution of \( X^{(1)}, ..., X^{(n)} \) is \( P_n := \frac{1}{n} \sum_{i=1}^n \delta_{X^{(i)}} \), where \( \delta_{X^{(i)}} \) is the Dirac distribution on \( X^{(i)} \), and the quantity \( \tilde{\mathcal{R}}_n (\beta) := P_n p_\beta = \frac{1}{n} \sum_{i=1}^n p_\beta (X^{(i)}) \) is the empirical counterpart of \( \mathcal{R}(\beta) \), called the empirical risk.

We denote by \( \gamma \) the probability density function of a centered standard real Gaussian variable \( \mathcal{N} (0, 1) \). \( \Phi \) is its cumulative distribution function and \( \Phi^c : t \mapsto \int_t^\infty \gamma (u) du \) the tail distribution of the density \( \gamma \). In addition, we write \( G \) the so-called Gaussian Mill’s ratio \( G (x) := \frac{\Phi^c (x)}{\gamma (x)} \). In this article \( \alpha : x \mapsto -\frac{e^x}{(1 + e^x)^2} (1 + x \frac{1-e^x}{1 + e^x}) \) and \( x_1 \) is the unique element of \( \{ x > 0 : \alpha (x) = 0 \} \), satisfying \( x_1 \in [1.54, 1.55] \).

\[ \forall u,v \in \mathbb{R}, u \wedge v := \min (u, v) \text{ and } u \lor v := \max (u, v). \]

For a vector \( \beta = (\beta_1, ..., \beta_p)^t \in \mathbb{R}^p \), we define its support as the set \( S \) of indices such that \( S = \{ i \in \{1, ..., p\}; \beta_i \neq 0 \} \). The vector \( \beta \) is said to be \( s \)-sparse if \( \text{Card}(S) \leq s \). Furthermore, for a set of indices \( I \subset \{1, ..., p\} \), we denote \( \beta^I \in \mathbb{R}^p \) the vector such that \( \beta^I_i = \beta_i \) if \( i \in I \) and \( \beta^I_j = 0 \) if \( j \notin I \).

### 3 Minimising the risk over a \( L_2 \)-ball

Recall that

\[ \beta_0 \in \arg \min_{\beta \in B_2 (0, R)} \{ \mathcal{R}(\beta) \}, \]
where the radius $R$ will be fixed later. Let us investigate the geometry of the risk $\mathcal{R}$ defined by the logistic entropy.

**Proposition 1.** The risk is symmetric, $\mathcal{R}(\beta) = \mathcal{R}(-\beta)$, and the risk value $\mathcal{R}(\beta)$ with $\|\beta\|_2 = r$ fixed is decreasing with respect to $|\beta^t a|$.

Proposition 1 states that the risk is symmetric around zero, and that its values on a sphere are increasing with respect to the distance to the line $\mathbb{R}a$. Its proof can be found in Section 5.1.

**Proposition 2.** The function $\lambda \mapsto \mathcal{R}(\lambda \beta)$ is decreasing for $\lambda \in \mathbb{R}_+$.

In Proposition 2, it is proved that the risk is decreasing on semi-lines starting at zero. For a proof of this result, see Section 5.1. From Propositions 1 and 2, we characterize the minimizers of the risk over a $L_2$-ball.

**Corollary 3.** The minimum of $\mathcal{R}(\beta)$ on $B_2(0, R)$ is reached at $\pm \beta_0$ where $\beta_0 := Ra/\|a\|_2$.

From Corollary 3, we deduce that estimating $\beta_0$ or its opposite directly gives an estimation of the best labelling function $Y_*$ for our clustering problem. A look at the proof of Propositions 1 and 2 shows that these results, and hence Corollary 3, hold true in the more general setting where the distribution of $Z$ is only assumed to be spherically symmetric.

In order to tackle the estimation of a sparse separation vector $a$, the following property will be helpful.

**Theorem 4.** Let $\beta_0 = Ra/\|a\|_2$ and let $\Lambda_{\min}$ be the smallest eigenvalue of the Hessian $d^2_{\beta_0} \mathcal{R}$ at $\beta_0$. Take a parameter $\nu = 0.95$, $R \geq \sqrt{x_1 + 0.08}$ ($R = 1.28$ for instance) and assume that $\|a\|_2 \geq 2R$, then

$$\Lambda_{\min} \geq \frac{\nu}{4} \left( \Phi^c \left( \|a\|_2 - \frac{x_1}{R} \right) - \Phi^c \left( \|a\|_2 + \frac{x_1}{R} \right) \right).$$

Theorem 4 states that if the radius $R$ and the mean vector $a$ are sufficiently large, then the risk defined by the logistic entropy is locally strongly convex around $\beta_0$. The risk is not convex over the whole $L_2$-ball $B_2(0, R)$, but this local convexity is very convenient, since it allows to deduce a quadratic growth of the excess risk pointed on $\beta_0$, as follows.
Lemma 5. Set $\beta_0$ the unique minimum of $R(\cdot)$ on $\Psi_U := \{ \beta \in B_2(0, R) : \beta^t U > 0 \}$ where $U$ is a random variable uniformly distributed on the unit $L^2$-ball. Assume that $R \geq \sqrt{x_1 + 0.08}$ and $\|a\|_2 \geq 2R$. We have

$$\inf_{\beta \in \Psi_U} \frac{E(\beta, \beta_0)}{\|\beta - \beta_0\|_2} \geq c_0 > 0$$

with

$$c_0 = L_0 \left( \frac{\|a\|_2 - R}{R} \right)^6 \exp \left( - \|a\|_2 R - 2R^2 \right)$$

for a numerical constant $L_0$ ($L_0 = 9 \times 2^{22}$ holds).

The quadratic growth of the excess risk stated in Lemma 5 will turn out to be a keystone to prove the oracle inequality for the excess risk of the minimizer of empirical risk regularized by a $\ell_1$ penalty (see Section 4). The proof of Lemma 5 is postponed to Section 5.1.

4 An oracle inequality in high dimension

Recall that $\beta_0 = Ra/\|a\|_2$ is a minimizer of the risk over the $L_2$-ball of radius $R$: $\beta_0 \in \arg\min_{\beta \in B_2(0, R)} R(\beta)$. Set $\Psi_U := \{ \beta \in B_2(0, R) : \beta^t U > 0 \}$ and where $U$ is a random variable uniformly distributed on the unit Euclidean sphere, independent from the observations. We have $\mathbb{P}(\beta_0^t U = 0) = 0$ and so $\beta_0$ or its opposite belongs to $\Psi_U$. Without loss of generality, we assume that $\beta_0 \in \Psi_U$ and analyze the situation conditionally on the choice of $U$.

We investigate the behavior of the following estimator,

$$\hat{\beta} := \arg \min_{\beta \in \Psi_U} \{ R_n(\beta) + \lambda \|\beta\|_1 \}. \quad (4)$$

Set also the empirical process $V_n(\beta) := (P_n - P)(\rho_{\beta})$. For some $T > 1$, define the event

$$\mathcal{T} := \left\{ \sup_{\beta \in B_2(0, R)} \frac{|V_n(\beta) - V_n(\beta_0)|}{\|\beta - \beta_0\|_1 \vee \lambda_0} \leq 2T\lambda_0 \right\}, \quad (5)$$
where $\lambda_0 > 0$ is to be fixed in the following theorem.

**Theorem 6.** Fix $n \geq 2$. Assume that $\beta_0$ - or equivalently $a$ - is $s$-sparse, for some integer $s \geq 1$, and denote $S$ its support. Assume also that $R = \sqrt{x_1 + 0.08}$ and $\|a\|_2 \geq 2R$. Set $M_n := \|a\|_\infty + \sqrt{2\log d + 2\log (1 + n)}$ and

$$
\lambda_0 := 3LM_n \left(5\sqrt{3\log (2d) \log n + 4}\right) n^{-1/2}.
$$

When the event $\mathcal{T}$ occurs, it holds: $\forall \lambda > 2T\lambda_0$,

$$
\mathcal{E} \left( \hat{\beta}, \beta_0 \right) + 4(\lambda - 2T\lambda_0) \left\| \hat{\beta}^{S^c} \right\|_1 \leq A_{\|a\|_2, R} s (T\lambda_0 + \lambda)^2, \tag{6}
$$

where $A_{\|a\|_2, R}$ is a constant depending only on $\|a\|_2$ and $R$. More precisely, for a numerical constant $A_0$, one can take

$$
A_{\|a\|_2, R} = A_0 \|a\|_2^8 (\|a\|_2 - R)^{-6} R^2 \|a\|_2 R + 2R^2.
$$

Furthermore, the event $\mathcal{T}$ occurs with probability at least

$$
1 - \frac{3}{4} \log \left( \frac{4R^2 n d}{L^2 M_n^2} \right) \exp \left( -21 (T - 1)^2 \log (2d) \log^2 n \right) - \frac{1}{25T^2 \log (2d) n \log^2 n}.
$$

According to Theorem 6, if the regularization parameter $\lambda$ is equal for instance to $3T\lambda_0$, then the rate of convergence of the excess risk is of the order

$$
\frac{s \log d \log^2 n \log (d \lor n)}{n},
$$

with a pre-factor that only depends on $\|a\|_2$ and $R$. Thus the estimator $\hat{\beta}$ adapts to sparsity and is able to estimate $\beta_0$ even if $d \gg n$. Furthermore, the rate of convergence of $\left\| \hat{\beta}^{S^c} \right\|_1$ would be given by

$$
\frac{s \sqrt{\log d \log^2 n \log (d \lor n)}}{n},
$$
with also a pre-factor that only depends on \( \|a\|_2 \) and \( R \). This means that if \( s \) and \( d \) are such that this rate (for a bounded \( \|a\|_2 \)) goes to zero with \( n \) growing to infinity, then the support \( S \) of \( \beta_0 \) is recovered in the sense that \( \|\hat{\beta}^{S^c}\|_1 \) goes to zero.

Note however that the dependence in \( \|a\|_2 \) is exponential in our bounds. This due to our argument of proof, which uses the local convexity of the risk around \( \beta_0 \). But when \( \|a\|_2 \) is large, the risk tends to be flat (see Theorem 4). This local convexity argument is also at the core the approach, developed in [42], to the non-convex \( \ell_1 \)-penalized loss in mixture regression (see also [9, Chapter 9]). Note that the needed lower bound on \( \|a\|_2 \) is independent from the dimension \( d \).

A careful look at the proofs also shows that when the conclusion of Lemma 5 holds, that is the excess risk dominates the square of the Euclidean distance, then Theorem 6 still holds for a general, bounded and Lipschitz loss.

It is also worth noting that in a classical, non-sparse case where the dimension is (much) smaller than the sample size, a convergence bound could also be obtained, by standard empirical process techniques. Indeed, the loss \( \rho \) is bounded and Lipschitz, so the rate of convergence of the following estimator,

\[
\tilde{\beta} \in \arg \min_{\beta \in B_2(0,R)} \left\{ \hat{R}_n(\beta) \right\},
\]

is of the order

\[
\sqrt{\frac{Rd}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\log(1/\delta)}{n},
\]

up to a numerical pre-factor and on an event of probability at least \( 1 - \delta \) for \( \delta \in (0,1) \). An important remark is that the latter rate in \( \sqrt{d/n} \) holds without any assumption on \( R \) and \( \|a\|_2 \), because the local convexity of the risk on \( \beta_0 \) - that is Theorem 4 - is not needed to prove it. If Theorem 4 furthermore holds, it is easy to see that the rate is actually \( d/n \), up to a pre-factor. Indeed, Theorem induces a so-called margin relation for the excess risk, which in turn induces a fast rate, since the loss is bounded (see for instance [32]).

Also, one can consider the adaptive selection of the regularization parameter. For this, a sensible idea
is to consider a BIC-type criterion defined with the active set of the estimators corresponding to different values of the regularization parameter.

We postpone to a forthcoming addition the practical implementation of the estimator, together with comparisons in the sparse two-component Gaussian mixture model with other available algorithms.

5 Proofs

Define the empirical process \( V_n(\beta) = (P_n - P)(\rho_\beta) \) and \( V_{n,\text{trunc}}(\beta) = (P_n - P)(\rho_\beta I_{\{G(X) \leq M_n\}}) \) where \( G(X) := \|X\|_\infty \) and note that \( \rho_\beta : \beta \mapsto \rho(\beta^t X) \) is \( L \)-lipschitz (with \( L < 2.5 \)).

5.1 Proofs of the main results

Proof of Proposition 1. Take \( X = \varepsilon Z \) where \( \varepsilon \sim \text{Rad}(1/2) \) and \( Z \sim \mathcal{N}(a, I_d) \), with \( a \in \mathbb{R}^d \) and also \( N \sim \mathcal{N}(0, 1) \). Because expression (11) of Lemma 7 is symmetric in \( X \), one has \( R(\beta) = R(-\beta) \) and

\[
R(\beta) = \mathbb{E} \left[ \log \left( 1 + e^{Z^t \beta} \right) - \frac{Z^t \beta e^{Z^t \beta}}{1 + e^{Z^t \beta}} \right].
\]

The distribution of the real-valued random variable \( Z^t \beta \) is \( \mathcal{N}(\beta^t a, \|\beta\|_2^2) \) and we assume that \( \|\beta\|_2 = r \). The criterion can be seen as a function of \( \mu := \beta^t a \) and \( r \):

\[
R(\beta) = \mathbb{E} \left[ \log \left( 1 + e^{\mu + rN} \right) - \frac{(\mu + rN) e^{\mu + rN}}{1 + e^{\mu + rN}} \right] =: R(\mu, r).
\]
Its derivative with respect to $\mu$ is:

$$
\partial_\mu R(\mu, r) = \frac{d}{d\mu} \mathbb{E} \left[ \log (1 + e^{\mu + rN}) - \frac{(\mu + rN) e^{\mu + rN}}{1 + e^{\mu + rN}} \right]
$$

$$
= \mathbb{E} \left[ \frac{d}{d\mu} \log (1 + e^{\mu + rN}) - \frac{d}{d\mu} \frac{(\mu + rN) e^{\mu + rN}}{1 + e^{\mu + rN}} \right]
$$

$$
= \mathbb{E} \left[ \frac{e^{\mu + rN}}{1 + e^{\mu + rN}} \left( \frac{e^{\mu + rN}}{1 + e^{\mu + rN}} + \frac{(\mu + rN) e^{\mu + rN}}{1 + e^{\mu + rN}} + (\mu + rN) e^{\mu + rN} \frac{-e^{\mu + rN}}{(1 + e^{\mu + rN})^2} \right) \right]
$$

$$
= \mathbb{E} \left[ \left( \frac{\mu + rN}{1 + e^{\mu + rN}} \right) \frac{e^{\mu + rN}}{1 + e^{\mu + rN}} \left( \frac{e^{\mu + rN}}{1 + e^{\mu + rN}} - 1 \right) \right]
$$

$$
\partial_\mu R(\mu, r) = -\mathbb{E} \left[ \frac{(\mu + rN) e^{\mu + rN}}{(1 + e^{\mu + rN})^2} \right].
$$

Let us define $g : x \mapsto \frac{xe^x}{(1+e^x)^2}$ so that $\partial_\mu R(\mu, r) = -\mathbb{E} [g(\mu + rN)]$. We use the lemma 8 and the fact that $g$ is odd and positive on $(0, +\infty)$ to conclude that $\mathbb{E} [g(\mu + rN)]$ has the sign of $\mu$, which gives the result. □

**Proof of Proposition 2.** Take $\beta \in \mathbb{R}^d$, there is $u \in \mathbb{R}$ such that $\beta^t a = u \|\beta\|_2$. Recall Identity (7) above, where $R$ can be seen as a function of $\mu$ and $r$ with $Z^t \beta \sim \mathcal{N}(\mu, r^2)$. Then we have

$$
\frac{\partial R(\lambda \beta)}{\partial \lambda} = \frac{\partial R(\lambda^t a, \|\lambda \beta\|_2)}{\partial \lambda} = \frac{\partial R(r u, r)}{\partial r} \|\beta\|_2.
$$

We set $\forall u \in \mathbb{R}, N_u \sim \mathcal{N}(u, 1)$ and Equation (7) gives:
\[
\frac{\partial R(ru, r)}{\partial r} = \frac{\partial}{\partial r} E \left[ \log \left( 1 + e^{ru+r\lambda_0} \right) - \frac{(ru + r\lambda_0) e^{(ru+r\lambda_0)}}{1 + e^{ru+r\lambda_0}} \right]
\]

\[
= E \left[ \frac{\partial}{\partial r} \log (1 + e^{r\lambda_0}) - \frac{\partial}{\partial r} \left( \frac{ru e^{r\lambda_0}}{1 + e^{r\lambda_0}} \right) \right]
\]

\[
= E \left[ \frac{ru e^{r\lambda_0}}{1 + e^{r\lambda_0}} - \left( \frac{ru e^{r\lambda_0}}{1 + e^{r\lambda_0}} + \frac{r\lambda_0 e^{r\lambda_0} N_0}{1 + e^{r\lambda_0}} \right) \right]
\]

\[
= E \left[ -r\lambda_0 (N_0 e^{r\lambda_0}) \right] \left( \frac{e^{r\lambda_0} - 1}{1 + e^{r\lambda_0}} \right)
\]

\[
= E \left[ -r\lambda_0^2 \frac{e^{r\lambda_0}}{(1 + e^{r\lambda_0})^2} \right] < 0.
\]

Hence \( \frac{\partial R(\lambda_0)}{\partial \lambda_0} < 0 \) as required. \(\square\)

**Proof of Theorem 4.** We make use of Equation (21) from Lemma 14: \( \forall a \in \mathbb{R}^d, R, \nu > 0, \)

\[
R \left( 1 - \left( R - \|a\|_2 + \frac{x_1 + 0.08}{R} \right) G \left( \frac{x_1}{R} + R - \|a\|_2 \right) \right) \geq (1 + \nu) \frac{e^{x_1}}{4} G \left( \|a\|_2 - \frac{x_1}{R} \right), \quad (8)
\]

where, see Section 2, \( x_1 \) is a positive numerical constant and the function \( G \) is the so-called Gaussian Mill’s ratio. By Proposition 24, we also have that \( G \) is decreasing on \( \mathbb{R} \). Hence, if Equation (8) is satisfied for some values of \( \|a\|_2, R \) and \( \nu \) such that \( \|a\|_2 - (R + (x_1 + 0.08) R^{-1}) > 0 \), then it is satisfied for any triplet \( (\|a\|_2 + h, R, \nu) \) with \( h > 0 \). In addition, we know from Lemma 26 that \( \|a\|_2 = 2R \approx 2.548, R = \sqrt{x_1 + 0.08} \approx 1.2741 \) and \( \nu = 0.95 \) make Equation (8) hold true. Consequently, it also holds true when \( \|a\|_2 \geq 2R \approx 2.548, R = \sqrt{x_1 + 0.08} \approx 1.2741 \) and \( \nu = 0.95 \).

According to Lemma 14, when Equation (8) holds, one can control from below the values of \( d_{\bar{\lambda}_0}^2 R \) (\( h, h \)).
More precisely,

\[
\Lambda_{\min} := \inf_{\|h\| = 1} \left( d^2_{\beta_0} R \right) (h, h)
\geq \inf_{\|h\| = 1} \frac{\nu}{4} \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \left( \Phi_c \left( \|a\|_2 - \frac{x_1}{R} \right) - \Phi_c \left( \|a\|_2 + \frac{x_1}{R} \right) \right)
= \frac{\nu}{4} \left( \Phi_c \left( \|a\|_2 - \frac{x_1}{R} \right) - \Phi_c \left( \|a\|_2 + \frac{x_1}{R} \right) \right) \inf_{0 \leq \eta \leq 1} \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right)
\]

because \(x_1/R \geq 1\). This proves the result. \(\square\)

**Proof of Lemma 5.** The risk \(\mathcal{R}\) admits two minima \(\beta_0\) and \(-\beta_0\) on \(B_2(0, R)\). We consider

\[
\Psi_U = \{ \beta \in B_2(0, R) : \beta^t U > 0 \},
\]

where \(U\) is a random variable uniformly distributed on the unit \(L^2\)-ball. The probability that \(U \in \beta_0^\perp\) is 0 then with probability 1 we have \(U \notin \beta_0^\perp\) and there is therefore only one vector among \(\beta_0\) and \(-\beta_0\) that satisfies \(\beta_0^t U > 0\). We call \(\beta_0\) the vector satisfying both \(\mathcal{R}(\beta_0)\) is the minimum of \(\mathcal{R}(\cdot)\) and \(\beta_0^t U > 0\).

Take \(\beta \in \Psi_U\) and let \(\varepsilon \in (0, R)\), we are about to control \(\mathcal{E}(\beta, \beta_0)\) on \(B_2(\beta_0, \varepsilon)\) and \(\{ \nu \in B_2(0, R) : \beta_0^t \nu > 0 \}\) \(B_2(\beta_0, \varepsilon)\) but these two sets may not be included \(\Psi_U\). To bypass this issue, remark that the risk \(\mathcal{R}\) is symmetric with respect to 0. Hence, in the case where \(\beta \notin \{ \nu \in B_2(0, R) : \beta_0^t \nu > 0 \}\), we will have \(\mathcal{E}(\beta, \beta_0) = \mathcal{E}(-\beta, \beta_0)\) where \(-\beta \in \{ \nu \in B_2(0, R) : \beta_0^t \nu > 0 \}\). Consequently, one can always control \(\mathcal{E}(\cdot, \beta_0)\) on \(\Psi_U\) with its values on \(\{ \nu \in B_2(0, R) : \beta_0^t \nu > 0 \}\), and without loss of generality we will focus on the control of \(\mathcal{E}(\cdot, \beta_0)\) on \(\{ \nu \in B_2(0, R) : \beta_0^t \nu > 0 \}\).

**Case 1:** \(\beta \in B_2(\beta_0, \varepsilon)\)

We know from Lemma 30 that \(\forall \beta \in B_2(\beta_0, \varepsilon)\),

\[
\frac{\mathcal{E}(\beta, \beta_0)}{e^{-\|a\|_2^2 R - R^2/2} \|\beta - \beta_0\|_2^2} \geq \frac{1}{16} \left( 1 + (\|a\|_2^2 - R^2)^2 \right) - 24 \|a\|_4^4 e^{R^2/2} e^{\|a\|_2^2} \|\beta - \beta_0\|_2\]
\]

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When $\|\beta - \beta_0\|_2 \leq \frac{1}{24} \left( \frac{1}{2} + \left( \frac{\|a\|_2 - R}{2\|a\|_2} \right)^2 \right)^{1/2}$, one has

$$\mathcal{E}(\beta, \beta_0) - \mathcal{E}(\beta_0, \beta_0) \geq \frac{1 + (\|a\|_2 - R)^2}{16}.$$ 

In particular, the latter inequality holds when

$$\varepsilon \leq \frac{1}{768} \frac{1 + (\|a\|_2 - R)^2}{\|a\|_2^2 e R^2/2} e^{-\varepsilon \|a\|_2}.$$

which is satisfied for

$$\varepsilon \leq \frac{1}{768} \frac{1 + (\|a\|_2 - R)^2}{\|a\|_2^2 e R^2/2} \exp \left( -\frac{1}{384} \frac{1 + (\|a\|_2 - R)^2}{\|a\|_2^2 e R^2/2} \|a\|_2 \right)$$

$$= \frac{1}{768} \frac{1 + (\|a\|_2 - R)^2}{\|a\|_2^4} \exp \left( -R^2/2 - \frac{1 + (\|a\|_2 - R)^2}{384 \|a\|_2^4 e R^2/2} \right) =: \varepsilon_{\text{max}}.$$

Then for all $\beta \in B_2(\beta_0, \varepsilon_{\text{max}})$, we have

$$\mathcal{E}(\beta, \beta_0) - \mathcal{E}(\beta_0, \beta_0) \geq \frac{1}{32} \left( 1 + (\|a\|_2 - R)^2 \right) e^{-\left( \frac{\|a\|_2 R - R^2}{2} \right)}.$$

Case 2: $\beta \in \{ \nu \in B_2(0, R) : \beta_0^t \nu > 0 \} \setminus B_2(\beta_0, \varepsilon_{\text{max}})$.

Lemmas 1 and 2 imply that $\forall \lambda > 1, \mathcal{E}(\lambda \beta, \beta_0) < \mathcal{E}(\beta, \beta_0)$ and

if $\nu \in \{ \mu \in B_2(0, R) : \|\mu\| = \|\beta\| \land \beta_0^t \mu > \beta_0^t \beta \}$, $\mathcal{E}(\nu, \beta_0) < \mathcal{E}(\beta, \beta_0)$.

With these two properties, we are always able to control $\mathcal{E}(\beta, \beta_0)$ with another value $\mathcal{E}(\nu, \beta_0)$ where $\nu \in B_2(\beta_0, \varepsilon_{\text{max}})$. Indeed, if $\mathbb{R} \cdot \beta$ intersects $B_2(\beta_0, \varepsilon_{\text{max}})$, there there exists $\lambda > 1$ such that $\mathcal{E}(\beta, \beta_0) > \mathcal{E}(\lambda \beta, \beta_0)$, where $\lambda \beta \in B_2(\beta_0, \varepsilon_{\text{max}})$. Otherwise, we have $\mathcal{E}(\beta, \beta_0) \geq \mathcal{E}(R \beta_0 \|\beta_0\|_2, \beta_0)$ and $\mathcal{E}(R \beta_0 \|\beta_0\|_2, \beta_0) > \mathcal{E}(\beta_{\text{inter}}, \beta_0)$ where $\beta_{\text{inter}}$ is the rotation of $R \beta_0 \|\beta_0\|_2$ towards $\beta_0$ so that $\beta_{\text{inter}}$ is at the frontier of $B_2(\beta_0, \varepsilon_{\text{max}})$.\[14\]
Moreover, $\forall \beta \in \{ \nu \in B_2 (0, R) : \beta_0^t \nu > 0 \}$ we have $\| \beta - \beta_0 \|^2 \leq 2R^2$. Consequently,

$$
\text{for all } \beta \in \{ \nu \in B_2 (0, R) : \beta_0^t \nu > 0 \} \setminus B_2 (\beta_0, \epsilon_{\text{max}}),
$$

there exists $\nu \in B_2 (\beta_0, \epsilon_{\text{max}})$ such that $\| \nu - \beta_0 \|_2 = \epsilon_{\text{max}}$ and

$$
\frac{\mathcal{E} (\beta, \beta_0)}{\| \beta - \beta_0 \|^2_2} \geq \frac{\mathcal{E} (\beta, \beta_0)}{2R^2} \geq \frac{\mathcal{E} (\nu, \beta_0)}{2R^2}.
$$

Furthermore, from Case 1 above, we have that $\forall \nu \in B_2 (\beta_0, \epsilon_{\text{max}})$ such that $\| \nu - \beta_0 \|_2 = \epsilon_{\text{max}},$

$$
\mathcal{E} (\nu, \beta_0) \geq \frac{1}{32} \left( 1 + (\| a \|_2 - R)^2 \right) e^{-\left( \| a \|_2 R - R^2 / 2 \right)} \epsilon_{\text{max}}^2.
$$

Hence, for all $\beta \in \{ \nu \in B_2 (0, R) : \beta_0^t \nu > 0 \} \setminus B_2 (\beta_0, \epsilon_{\text{max}}),$

$$
\frac{\mathcal{E} (\beta, \beta_0)}{\| \beta - \beta_0 \|^2_2} \geq \frac{\left( 1 + (\| a \|_2 - R)^2 \right) e^{-\left( \| a \|_2 R - R^2 / 2 \right)} \epsilon_{\text{max}}^2}{64R^2}.
$$

Finally, from the two cases, we get

$$
\inf_{\beta \in \{ \nu \in B_2 (0, R) : \beta_0^t \nu > 0 \}} \frac{\mathcal{E} (\beta, \beta_0)}{\| \beta - \beta_0 \|^2_2} \geq \frac{\left( 1 + (\| a \|_2 - R)^2 \right) e^{-\left( \| a \|_2 R - R^2 / 2 \right)} \epsilon_{\text{max}}^2}{64R^2}.
$$
Consequently, the result is also true when one takes the infimum over $\Psi_U$:

$$\inf_{\beta \in \Psi_U} \| \beta - \beta_0 \|^2 \geq \frac{(1 + (\|a\|_2 - R)^2)^3}{64R^2} \cdot \exp \left( - \|a\|_2 R - R^2 / 2 - R^2 - \frac{1 + (\|a\|_2 - R)^2}{384 \|a\|_2^3 e^{R^2/2}} \right) \geq \frac{((\|a\|_2 - R)^6}{9 \cdot 2^{22} \|a\|_2^3 R^2} \cdot \exp \left( - \|a\|_2 R - R^2 / 2 - R^2 - R^2 / 2 \right) \geq \frac{((\|a\|_2 - R)^6}{9 \cdot 2^{22} \|a\|_2^3 R^2} \cdot \exp \left( - \|a\|_2 R - 2R^2 \right).$$

We present now the proof of our main result, that is the oracle inequality stated in Section 4.

**Proof of Theorem 6.** We know from Lemma 10 that

$$\mathcal{E} (\hat{\beta}, \beta_0) + \lambda \| \hat{\beta} \|_1 \leq \left| V_n (\hat{\beta}) - V_n (\beta_0) \right| + \lambda \| \beta_0 \|_1 \quad (9)$$

We set ourselves in the event $\mathcal{T}$ defined in (5). It holds

$$\sup_{\beta \in B_2(0, R)} \frac{|V_n (\beta) - V_n (\beta_0)|}{\| \beta - \beta_0 \|_1 \vee \lambda_0} \leq 2T \lambda_0$$

and, as $\hat{\beta} \in B_2(0, R)$, Equation (9) gives

$$\mathcal{E} (\hat{\beta}, \beta_0) + \lambda \left\| \hat{\beta} \right\|_1 \leq 2T \lambda_0 \left\| \hat{\beta} - \beta_0 \right\|_1 \vee \lambda_0 + \lambda \| \beta_0 \|_1.$$
Case 1: $\|\hat{\beta} - \beta_0\|_1 \vee \lambda_0 = \lambda_0$. We successively have

$$
E\left(\hat{\beta}, \beta_0\right) \leq 2T\lambda_0^2 + \lambda \left(\|\beta_0\|_1 - \|\hat{\beta}\|_1\right)
\leq 2T\lambda_0^2 + \lambda \left(\|\beta_0\|_1 - \|\hat{\beta}\|_1\right)
\leq 2T\lambda_0^2 + \lambda \left(\beta_0 - \hat{\beta}\right)_1.
$$

Hence,

$$E\left(\hat{\beta}, \beta_0\right) + 2\lambda \left(\beta_0 - \hat{\beta}\right)_1 \leq 2T\lambda_0^2 + 3\lambda \left(\beta_0 - \hat{\beta}\right)_1
\leq 2T\lambda_0^2 + 3\lambda \lambda_0
\leq 2T\lambda_0^2 (T\lambda_0 + \lambda)^2.
$$

Finally, since $2(\lambda - 2T\lambda_0) \leq 2\lambda$ and $\|\hat{\beta}^{Sc}\|_1 = \|\hat{\beta}_0^{Sc} - \beta^{Sc}\|_1 \leq \|\beta_0 - \hat{\beta}\|_1$

$$E\left(\hat{\beta}, \beta_0\right) + 2(\lambda - 2T\lambda_0) \left\|\hat{\beta}^{Sc}\right\|_1 \leq 3(T\lambda_0 + \lambda)^2.$$

Case 2: $\|\hat{\beta} - \beta_0\|_1 \vee \lambda_0 = \|\hat{\beta} - \beta_0\|_1$. We have $\|\hat{\beta}\|_1 = \|\hat{\beta}^{Sc}\|_1 + \|\hat{\beta}^{Sc}\|_1$, $\|\beta_0\|_1 = \|\beta_0^{Sc}\|_1$ and $\|\hat{\beta} - \beta_0\|_1 = \|\hat{\beta}^{Sc} - \beta_0^{Sc}\|_1 + \|\hat{\beta}^{Sc}\|_1$. Consequently, it holds successively

$$E\left(\hat{\beta}, \beta_0\right) + \lambda \|\hat{\beta}\|_1 \leq 2T\lambda_0 \|\hat{\beta} - \beta_0\|_1 + \lambda \|\beta_0\|_1,$$

$$E\left(\hat{\beta}, \beta_0\right) + \lambda \|\hat{\beta}^{Sc}\|_1 + \lambda \|\hat{\beta}^{Sc}\|_1 \leq 2T\lambda_0 \|\hat{\beta}^{Sc} - \beta_0^{Sc}\|_1 + 2T\lambda_0 \|\hat{\beta}^{Sc}\|_1 + \lambda \|\beta_0^{Sc}\|_1,$$

$$E\left(\hat{\beta}, \beta_0\right) + \lambda \|\hat{\beta}^{Sc}\|_1 - 2T\lambda_0 \|\hat{\beta}^{Sc}\|_1 \leq 2T\lambda_0 \|\hat{\beta}^{Sc} - \beta_0^{Sc}\|_1 + \lambda \|\beta_0^{Sc}\|_1 - \lambda \|\hat{\beta}^{Sc}\|_1,$$

$$E\left(\hat{\beta}, \beta_0\right) + (\lambda - 2T\lambda_0) \|\hat{\beta}^{Sc}\|_1 \leq 2T\lambda_0 \|\hat{\beta}^{Sc} - \beta_0^{Sc}\|_1 + \lambda \|\beta_0^{Sc} - \hat{\beta}^{Sc}\|_1,$$

$$E\left(\hat{\beta}, \beta_0\right) + (\lambda - 2T\lambda_0) \|\hat{\beta}^{Sc}\|_1 \leq (2T\lambda_0 + \lambda) \|\beta_0 - \hat{\beta}^{Sc}\|_1.$$
Since $\beta_0 - \hat{\beta}^S$ has at most $s$ non-zero coordinates, one has $\|\beta_0 - \hat{\beta}^S\|_1 \leq \sqrt{s} \|\beta_0 - \hat{\beta}\|_2 \leq \sqrt{s} \|\beta_0 - \hat{\beta}\|_2$. Hence, for any $c_0 > 0$,

$$
E \left( \hat{\beta}, \beta_0 \right) + (\lambda - 2T\lambda_0) \left\| \hat{\beta}^S \right\|_1 \leq (T\lambda_0 + \lambda) \sqrt{\frac{s}{c_0\sqrt{c_0}}} \|\beta_0 - \hat{\beta}\|_2.
$$

Now use the fact that $\forall a, b, 2ab \leq a^2 + b^2$ to get

$$
E \left( \hat{\beta}, \beta_0 \right) + (\lambda - 2T\lambda_0) \left\| \hat{\beta}^S \right\|_1 \leq (T\lambda_0 + \lambda)^2 \frac{s}{2c_0} + \frac{c_0 \|\beta_0 - \hat{\beta}\|_2^2}{2}.
$$

So we can use Lemma 5 and have $E \left( \hat{\beta}, \beta_0 \right) \geq c_0 \|\beta_0 - \hat{\beta}\|_2^2$ where $c_0 = \frac{([[a]]_2^2R^6}{9.22 ||a||_2^2R^2} c - [[a]]_2 R - 2R^2}$. Consequently, for this choice of $c_0$,

$$
E \left( \hat{\beta}, \beta_0 \right) + (\lambda - 2T\lambda_0) \left\| \hat{\beta}^S \right\|_1 \leq (T\lambda_0 + \lambda)^2 \frac{s}{2c_0} + \frac{E \left( \hat{\beta}, \beta_0 \right)}{2},
$$

which gives

$$
E \left( \hat{\beta}, \beta_0 \right) + 2(\lambda - 2T\lambda_0) \left\| \hat{\beta}^S \right\|_1 \leq (T\lambda_0 + \lambda)^2 \frac{s}{c_0}.
$$

Finally, combining the two cases, we obtain

$$
E \left( \hat{\beta}, \beta_0 \right) + 2(\lambda - 2T\lambda_0) \left\| \hat{\beta}^S \right\|_1 \leq (T\lambda_0 + \lambda)^2 \max \left( \frac{s}{c_0}, 2 \right).
$$

The bound on the probability of the event $\mathcal{T}$ is given in Theorem 9.

\[\square\]

### 5.2 Auxiliary results

Let us first state the following basic lemma, where we compute the derivatives of the loss and its risk.
Lemma 7. With notations of section 2, it holds

\[ \rho_\beta(X) = -\log q_\beta(X) + X^t \beta.p_\beta(X) \]  
(10)

\[ \rho_\beta(X) = \log \left( 1 + e^{X^t \beta} \right) - \frac{X^t \beta e^{X^t \beta}}{1 + e^{X^t \beta}} \]  
(11)

\[ \frac{\partial \rho_\beta(X)}{\partial \beta_u} = -X_u p_\beta(X) q_\beta(X) \]  
(12)

\[ \frac{\partial q_\beta(X)}{\partial \beta_u} = X_u p_\beta(X) q_\beta(X) \]  
(13)

\[ \frac{\partial \rho_\beta(X)}{\partial \beta_u} = -X^t \beta X_u p_\beta(X) q_\beta(X) \]  
(14)

\[ \frac{\partial}{\partial \beta_v} \frac{\partial}{\partial \beta_u} \rho_\beta(X) = -X_v X_u \alpha(X^t \beta) \]  
(15)

\[ \frac{\partial}{\partial \beta_w} \frac{\partial}{\partial \beta_v} \frac{\partial}{\partial \beta_u} \rho_\beta(X) = -X_w X_v X_u \alpha'(X^t \beta) \]  
(16)

\[ (d_\beta R)(h) = \mathbb{E} \left[ -X^t \beta p_\beta(X) q_\beta(X) X^t h \right] \]  
(17)

\[ (d_\beta^2 R)(h, k) = \mathbb{E} \left[ X^t h.X^t k.\alpha(X^t \beta) \right] \]  
(18)

\[ (d_\beta^3 R)(h, k, l) = \mathbb{E} \left[ X^t h.X^t k.X^t l.\alpha'(X^t \beta) \right] \]  
(19)

Proof. Consider \( X, \beta \in \mathbb{R}^d \), \( \rho_\beta(X) \) is defined in section 2. For simplicity, \( p \) and \( q \) stand for \( p_\beta(X) \) and \( q_\beta(X) \) recall that \( p_\beta(X) = q_\beta(X) e^{-X^t \beta} \):

\[ \rho_\beta(X) = -p \log p - q \log q \]
\[ = -\left( q e^{-X^t \beta} \right) \log \left( q e^{-X^t \beta} \right) - q \log q \]
\[ = -q e^{-X^t \beta} \log q + X^t \beta q e^{-X^t \beta} - q \log q \]
\[ = -q \left( 1 + e^{-X^t \beta} \right) \log q + X^t \beta q e^{-X^t \beta} \]
\[ = -\log q + X^t \beta p \]
\[ = \log \left( 1 + e^{X^t \beta} \right) - \frac{X^t \beta e^{X^t \beta}}{1 + e^{X^t \beta}}. \]
Denote $\beta_u$ the $u$-th component of $\beta$. We have,

$$\frac{\partial p_\beta(X)}{\partial \beta_u} = \frac{\partial}{\partial \beta_u} \left[ \frac{1}{1 + e^{X^t \beta}} \right] = -\frac{X_u e^{X^t \beta}}{(1 + e^{X^t \beta})^2}$$

$$= -X_u p_\beta(X) q_\beta(X)$$

and

$$\frac{\partial q_\beta(X)}{\partial \beta_u} = \frac{\partial}{\partial \beta_u} [1 - p_\beta(X)] = -\frac{\partial p_\beta(X)}{\partial \beta_u}$$

$$= X_u p_\beta(X) q_\beta(X).$$

Secondly, we use Equation (11) to have

$$\frac{\partial p_\beta(X)}{\partial \beta_u} = -\frac{\partial \log q}{\partial \beta_u} + \frac{\partial (X^t \beta p)}{\partial \beta_u}$$

$$= -\frac{\partial q}{\partial \beta_u} + \frac{\partial (X^t \beta p)}{\partial \beta_u} p + X^t \beta \frac{\partial p}{\partial \beta_u}$$

$$= -\frac{(X_u p q)}{q} + X_u p + X^t \beta (-X_u p q)$$

$$= -X^t \beta X_u p_\beta(X) q_\beta(X).$$
The second derivatives are
\[
\frac{\partial}{\partial \beta_v} \frac{\partial}{\partial \beta_u} \rho_\beta (X) = - \frac{\partial (X \beta)}{\partial \beta_v} X_{upq} - X^t \beta X_u \frac{\partial p}{\partial \beta_v} q - X^t \beta X_u p \frac{\partial q}{\partial \beta_v} \\
= - (X_v) X_{upq} - X^t \beta X_u (-X_v pq) q - X^t \beta X_u p (X_v qp) \\
= -X_v X_u pq (1 - X^t \beta (q - p)) \\
= -X_v X_u \frac{e^{X^t \beta}}{1 + e^{X^t \beta}} \left( 1 + X^t \beta \left( \frac{1 - e^{X^t \beta}}{1 + e^{X^t \beta}} \right) \right) \\
= X_v X_u \alpha (X^t \beta) .
\]

The third derivatives are
\[
\frac{\partial}{\partial \beta_u} \frac{\partial}{\partial \beta_v} \frac{\partial}{\partial \beta_w} \rho_\beta (X) = X_v X_u \frac{\partial}{\partial \beta_w} \left[ \alpha (X^t \beta) \right] \\
= X_w X_v X_u \alpha' (X^t \beta) .
\]

As the derivatives are uniformly bounded with respect to \( \beta \), the theorem of derivation under integral can be applied and it comes that \( \forall h, k, l, \in \mathbb{R}^d \),
\[
(d_3 \mathcal{R}) (h, k, l) = \mathbb{E} \left[ X^t h . X^t k . \alpha' (X^t \beta) \right] .
\]

**Lemma 8.** Take \( r > 0 \). For any function \( g \) odd on \( \mathbb{R} \), positive on \((0, +\infty)\) and when \( U \) is a symmetric random variable with a density \( \gamma \) decreasing on \( \mathbb{R}_+ \), the quantity \( \mathbb{E} [g (\mu + rU)] \) has the sign of \( \mu \).
Proof. Take $r, \mu > 0$, $U_1$ and $U_2$ two independent copies of $U$. It holds

$$E[g(\mu + rU)] = E[g(\mu + rU_1)I_{U_1>0} + g(\mu - rU_2)I_{U_2>0}]$$

$$= E[g(\mu + rU_1)I_{U_1>0} + g(\mu - rU_2)I_{U_2>0} + g(\mu - rU_2)I_{\mu - rU_2 < U_2 < \mu}]$$

$$= E\left[g(\mu + rU_1)I_{U_1>0} + g(\mu - rU_2)I_{\mu - rU_2 < U_2 < \mu}\right]$$

Let us compute the sign of $E\left[g(\mu + rU_1)I_{U_1>0} + g(\mu - rU_2)I_{\mu - rU_2 < U_2 < \mu}\right]$. 

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\[
\begin{align*}
\mathbb{E} \left[ g(\mu + rU_1) I_{U_1 > 0} + g(\mu - rU_2) I_{U_2 < U_1} \right] \\
&= \int_0^\infty g(\mu + rx) \gamma(x) \, dx + \int_{2\mu/r}^\infty g(\mu - rx) \gamma(x) \, dx \\
&= \int_0^\infty g(\mu + rx) \gamma(x) \, dx + \int_0^\infty g\left(\mu - r\left(y + \frac{2\mu}{r}\right)\right) \gamma\left(y + \frac{2\mu}{r}\right) \, dy \\
&= \int_0^\infty g(\mu + rx) \gamma(x) \, dx + \int_0^\infty g(-ry - \mu) \gamma\left(y + \frac{2\mu}{r}\right) \, dy \\
&= \int_0^\infty g(\mu + rx) \gamma(x) \, dx + \int_0^\infty -g(\mu + ry) \gamma\left(y + \frac{2\mu}{r}\right) \, dy \\
&= \int_0^\infty g(\mu + rx) \left[ \gamma(x) - \gamma\left(x + \frac{2\mu}{r}\right) \right] \, dx \\
&> 0.
\end{align*}
\]

Let us now compute the sign of \( \mathbb{E} \left[ g(\mu - rU_1) I_{0 < U_2 \leq \mu} + g(\mu - rU_2) I_{U_2 < 2\mu} \right] \):
\[
\mathbb{E}\left[g(\mu - rU_1)I_{0 < U_2 < \frac{\mu}{r}} + g(\mu - rU_2)I_{\frac{\mu}{r} < U_2 < 2}\right] \\
= \int_0^{\frac{\mu}{r}} g(\mu - rx) \gamma(x) \, dx + \int_{\frac{\mu}{r}}^{\frac{2\mu}{r}} g\left(\mu - r\left(\frac{2\mu}{r} - y\right)\right) \gamma\left(\frac{2\mu}{r} - y\right) \, dy \\
= \int_0^{\frac{\mu}{r}} g(\mu - rx) \gamma(x) \, dx + \int_0^{\frac{\mu}{r}} g(ry - \mu) \gamma\left(\frac{2\mu}{r} - y\right) \, dy \\
= \int_0^{\frac{\mu}{r}} g(\mu - rx) \gamma(x) \, dx + \int_0^{\frac{\mu}{r}} -g(\mu - ry) \gamma\left(\frac{2\mu}{r} - y\right) \, dy \\
= \int_0^{\frac{\mu}{r}} \left[\gamma(x) - \gamma\left(\frac{2\mu}{r} - x\right)\right] \, dx \\
> 0
\]

Hence \( \mathbb{E}[g(\mu + rU)] > 0 \). If \( \mu < 0 \), then one has \( \mathbb{E}[g(\mu + rU)] = -\mathbb{E}[g(-\mu - rU)] \) and the previous result applies since \( -\mu > 0 \) and \( -U \sim U \). Thus we find that if \( \mu < 0 \), \( \mathbb{E}[g(\mu + rU)] < 0 \).

\[\square\]

**Theorem 9.** Set \( \Theta = B_2(0, R) \), \( M_n = \|a\|_\infty + \sqrt{2 \log d} + \sqrt{2 \log (1 + n)} \) and

\[
\lambda_0 = 3n^{-1/2}LM_n \left(5\sqrt{3 \log (2d) \log n + 4}\right).
\]

It holds \( \forall n \geq 2, \forall T \geq 1 \),

\[
P\left(\sup_{\beta \in \Theta} \frac{|V_n(\beta) - V_n(\beta_0)|}{\|\beta - \beta_0\|_1} > 2T\lambda_0\right) \leq \frac{3}{4} \log \left(\frac{4R^2 nd}{L^2 M_n^2}\right) \exp\left(-21(T - 1)^2 \log (2d) \log^2 n\right) + \frac{1}{25T^2 \log (2d) n \log^2 n}.
\]
Proof. First, the triangular inequality gives

\[|V_n(\beta) - V_n(\beta_0)| \leq |V_n^{\text{trunc}}(\beta) - V_n^{\text{trunc}}(\beta_0)| + |V_n^{\text{trunc}}(\beta) - V_n^{\text{trunc}}(\beta_0) - (V_n(\beta) - V_n(\beta_0))|,\]

and since \(\forall a, b, t > 0\) on has “\(a + b > 2t\)” implies “either \(a > t\) or \(b > t\)”\), the probability of interest can be controlled as follows:

\[
P\left(\sup_{\beta \in \Theta} |V_n(\beta) - V_n(\beta_0)| > 2T\lambda_0\right) \leq P\left(\sup_{\beta \in \Theta} \frac{|V_n^{\text{trunc}}(\beta) - V_n^{\text{trunc}}(\beta_0)|}{\|\beta - \beta_0\|_1 \vee \lambda_0} > T\lambda_0\right) + P\left(\sup_{\beta \in \Theta} \frac{|V_n^{\text{trunc}}(\beta) - V_n^{\text{trunc}}(\beta_0) - (V_n(\beta) - V_n(\beta_0))|}{\|\beta - \beta_0\|_1 \vee \lambda_0} > T\lambda_0\right).
\]

Apply now Lemma 40 to have:

\[
P\left(\sup_{\beta \in \Theta} |V_n(\beta) - V_n(\beta_0)| > 2T\lambda_0\right) \leq P\left(\sup_{\beta \in \Theta} \frac{|V_n^{\text{trunc}}(\beta) - V_n^{\text{trunc}}(\beta_0)|}{\|\beta - \beta_0\|_1 \vee \lambda_0} > T\lambda_0\right) + P\left(\frac{1}{n} \sum_{i=1}^{n} F\left(X^{(i)}\right) > \frac{T\lambda_0}{L}\right).
\]

From Lemmas 37 and 39, we get

\[
P\left(\sup_{\beta \in \Theta} \frac{|V_n(\beta) - V_n(\beta_0)|}{\|\beta - \beta_0\|_1 \vee \lambda_0} > 2T\lambda_0\right) \leq \frac{3}{4} \log \left(\frac{4R^2nd}{L^2M_n^2}\right) \exp\left(-21\log(2d)\log^2 n\right) + 4L\frac{M_n^2 + \|a\|_{\infty} + 1}{n^2\lambda_0^T \lambda_n^T}.
\]
Furthermore,

\[ 4L^2 \frac{M^2_n}{n^2 \lambda_0^2} T^2 + \| a \|_\infty + 1 \leq 4L^2 \frac{M^2_n}{n^2} \frac{1 + \| a \|_\infty + 1}{9L^2 M^2_n \left(5\sqrt{3 \log(2d) \log n} + 4\right)^2 T^2} \]

\[ \leq 4 \frac{1 + \| a \|_\infty + 1}{9 \times 25 (3 \log (2d) \log^2 n) n T^2} \]

\[ \leq \frac{1}{25T^2 \log (2d) n \log^2 n}. \]

Hence,

\[ P \left( \sup_{\beta \in \Theta} \frac{|V_n(\beta) - V_n(\beta_0)|}{\| \beta - \beta_0 \|_1 \vee \lambda_0} > 2T \lambda_0 \right) \leq \frac{3}{4} \log \left( \frac{4R^2 nd}{L^2 M^2_n} \right) \exp \left( -21(T - 1)^2 \log (2d) \log^2 n \right) + \frac{1}{25T^2 \log (2d) n \log^2 n}. \]

\[ \square \]

**Lemma 10.** Recall that \( \beta_0 = \arg\min_{\beta \in \Psi} \{ R(\beta) \} \) and \( \hat{\beta} := \arg\min_{\beta \in \Psi} \{ \hat{R}_n(\beta) + \lambda \| \beta \|_1 \} \). It holds

\[ E \left( \hat{\beta}, \beta_0 \right) + \lambda \| \hat{\beta} \|_1 \leq \left| V_n(\hat{\beta}) - V_n(\beta_0) \right| + \lambda \| \beta_0 \|_1. \]

**Proof.** By definition of \( \hat{\beta} \), we have:

\[ \hat{R}_n(\hat{\beta}) + \lambda \| \hat{\beta} \|_1 \leq \hat{R}_n(\beta_0) + \lambda \| \beta_0 \|_1. \]

Injecting the excess risk on both sides of the inequality gives

\[ E \left( \hat{\beta}, \beta_0 \right) + \lambda \| \hat{\beta} \|_1 \leq R(\hat{\beta}) - R(\beta_0) + \hat{R}_n(\beta_0) - \hat{R}_n(\hat{\beta}) + \lambda \| \beta_0 \|_1. \]

Then the result comes from the inequality:

\[ R(\hat{\beta}) - R(\beta_0) + \hat{R}_n(\beta_0) - \hat{R}_n(\hat{\beta}) \leq \left| V_n(\hat{\beta}) - V_n(\beta_0) \right|. \]
5.3 Some further technical lemmas

Lemma 11. Assuming $a \in \mathbb{R}^d$, $Z \sim a + N$, $N \sim \mathcal{N}(0, I_d)$, $\beta_0 = Ra/\|a\|_2$, and $h_\perp \in \beta_0^\perp$, then $h_\perp^t Z$ and $Z^t \beta_0$ are two independent Gaussian variables.

Proof. Note that $h_\perp^t a = 0$. We have

$\begin{align*}
Cov (h_\perp^t Z, Z^t \beta_0) &= E \left[ (h_\perp^t Z - E [h_\perp^t Z]) (Z^t \beta_0 - E [Z^t \beta_0]) \right] \\
&= E \left[ (h_\perp^t (a + N) - E [h_\perp^t (a + N)]) ((a + N)^t \beta_0 - E [(a + N)^t \beta_0]) \right] \\
&= E \left[ (h_\perp^t N - E [h_\perp^t N]) (a^t \beta_0 + N^t \beta_0 - E [a^t \beta_0 + N^t \beta_0]) \right] \\
&= E \left[ (h_\perp^t N - h_\perp^t E [N]) (N^t \beta_0 - E [N^t] \beta_0) \right] \\
&= h_\perp^t E [NN^t] \beta_0 \\
&= h_\perp^t \beta_0 \\
&= 0.
\end{align*}$

Lemma 12. With $Z$, $\beta_0$, $\alpha$ and $R$ usual notations and for all $h := h_\parallel + h_\perp \in \text{Vect}(\beta_0) \oplus \beta_0^\perp$ such that $\|h\|_2 = 1$ and with $\eta := \|h\|_2$, we have

$$(d_{\beta_0}^2 R) (h, h) = \eta^2 E \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_0) \right] + (1 - \eta^2) E \left[ \alpha (Z^t \beta_0) \right].$$

Proof. We computed $d_{\beta_0}^2 R$ in Equation (18) of Lemma 7. The function $\alpha$ (see Section 2) is even, so the
entries of the Hessian $d^2 R_{\beta_0}$ are

$$\forall u, v \in [1, d], \quad (d^2 R)_{u,v} = \mathbb{E} \left[ (\varepsilon Z_v)(\varepsilon Z_u) \alpha (\varepsilon Z^T \beta_0) \right]$$

$$= \mathbb{E} \left[ Z_v Z_u \alpha (Z^T \beta_0) \right].$$

Now, let us use the decomposition $h = h_\parallel + h_\perp$ and remark that $h_\parallel = \epsilon \eta \frac{\beta_0}{\| \beta_0 \|^2} = \epsilon \frac{\eta}{R} \beta_0$ with $\epsilon \in \{-1, 1\}$.

It comes

$$(d^2 R) (h, h) = \mathbb{E} \left[ (h^T Z)^2 \alpha (Z^T \beta_0) \right]$$

$$= \mathbb{E} \left[\left( h_\parallel + h_\perp \right)^T Z \right]^2 \alpha (Z^T \beta_0)$$

$$= \mathbb{E} \left[ \left( h^T Z \right)^2 + 2 h^T Z h_\perp^T Z + (h_\perp^T Z)^2 \right] \alpha (Z^T \beta_0)$$

$$= \mathbb{E} \left[ \left( h^T Z \right)^2 \alpha (Z^T \beta_0) \right] + 2 \mathbb{E} \left[ h^T Z h_\parallel^T Z \alpha (Z^T \beta_0) \right] + \mathbb{E} \left[ (h_\perp^T Z)^2 \alpha (Z^T \beta_0) \right]$$

$$= \mathbb{E} \left[ \left( \epsilon \frac{\eta}{R} \beta_0 \right)^T Z \right]^2 \alpha (Z^T \beta_0)$$

$$= \mathbb{E} \left[ \frac{1}{R^2} \left( Z^T \beta_0 \right)^2 \alpha (Z^T \beta_0) \right] + 2 \mathbb{E} \left[ h^T Z \alpha (Z^T \beta_0) \right] + \mathbb{E} \left[ (h_\perp^T Z)^2 \alpha (Z^T \beta_0) \right].$$

Also remark that $h_\perp^T Z$ and $Z^T \beta_0$ are Gaussian random variables, because $Z$ is a Gaussian vector, that are independent due to lemma 11.

$$(d^2 R) (h, h) = \eta^2 \mathbb{E} \left[ \frac{1}{R^2} \left( Z^T \beta_0 \right)^2 \alpha (Z^T \beta_0) \right] + 2 \mathbb{E} \left[ h^T Z \alpha (Z^T \beta_0) \right] + \mathbb{E} \left[ (h_\perp^T Z)^2 \alpha (Z^T \beta_0) \right]$$

$$= \eta^2 \mathbb{E} \left[ \frac{1}{R^2} \left( Z^T \beta_0 \right)^2 \alpha (Z^T \beta_0) \right] + 2 \mathbb{E} \left[ h^T Z \alpha (Z^T \beta_0) \right] \mathbb{E} \left[ h_\perp^T Z \right] + \mathbb{E} \left[ (h_\perp^T Z)^2 \right] \mathbb{E} \left[ \alpha (Z^T \beta_0) \right]$$

$$= \eta^2 \mathbb{E} \left[ \frac{1}{R^2} \left( Z^T \beta_0 \right)^2 \alpha (Z^T \beta_0) \right] + \mathbb{E} \left[ (h_\perp^T Z)^2 \right] \mathbb{E} \left[ \alpha (Z^T \beta_0) \right].$$
Note that $E \left[ (h_\perp Z)^2 \right] = E \left[ (h_\perp (a + N))^2 \right] = E \left[ (h_\perp N)^2 \right] = h_\perp^t E \left[ NN^t \right] h_\perp = 1 - \eta^2$ hence:

$$(d_{\beta_0}^2 \mathcal{R}) (h,h) = \eta^2 E \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_0) \right] + (1 - \eta^2) E \left[ \alpha (Z^t \beta_0) \right].$$

\hfill \Box

**Lemma 13.** For all $h := h_\parallel + h_\perp \in \text{Vect}(\beta_0) \oplus \beta_0^\perp$ such that $\|h\|_2 = 1$ and with $\eta := \|h\|_2$. The two following quantities

$$A := \eta^2 E \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] + (1 - \eta^2) E \left[ \alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right]$$

$$B := \eta^2 E \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_0) \mathbb{I}_{\{-x_1 < Z^t \beta_0 < x_1\}} \right] + (1 - \eta^2) E \left[ \alpha (Z^t \beta_0) \mathbb{I}_{\{-x_1 < Z^t \beta_0 < x_1\}} \right]$$

are controlled by

$$A > \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \left[ R + \left( R (\|a\|_2 - R) - \left( x_1 + \frac{8}{100} \right) \right) \right] G \left( \frac{x_1}{R} + R - \|a\|_2 \right) \gamma \left( \|a\|_2 - \frac{x_1}{R} \right) e^{-x_1},$$

$$B \leq \frac{1}{4} \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \left( \Phi^c \left( \|a\|_2 - \frac{x_1}{R} \right) - \Phi^c \left( \|a\|_2 + \frac{x_1}{R} \right) \right).$$

**Proof.** Let us first give an upper bound for the quantity $B$. Recall that, from Lemma 15 we have $-\alpha(x) \in$
\((0, \frac{1}{4})\) for \(x \in [-x_1, x_1]\). It holds

\[
B = -\eta^2 \mathbb{E} \left[ \frac{1}{R^2} \left( Z^t \beta_0 \right)^2 \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{-x_1 < Z^t \beta_0 < x_1\}} \right] - (1 - \eta^2) \mathbb{E} \left[ \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{-x_1 < Z^t \beta_0 < x_1\}} \right] \\
\leq \eta^2 \mathbb{E} \left[ \frac{1}{R^2} \left( Z^t \beta_0 \right)^2 \mathbb{I}_{\{-x_1 < Z^t \beta_0 < x_1\}} \right] + (1 - \eta^2) \mathbb{E} \left[ \frac{1}{4} \mathbb{I}_{\{-x_1 < Z^t \beta_0 < x_1\}} \right] \\
= \frac{1}{4} \left( \eta^2 x_1^2 + 1 - \eta^2 \right) \mathbb{P} \left[ -x_1 < Z^t \beta_0 < x_1 \right] \\
= \frac{1}{4} \left( \eta^2 x_1^2 + 1 - \eta^2 \right) \mathbb{P} \left[ -x_1 < \|a\|_2 < R \mathcal{N}(0, 1) < x_1 \right] \\
= \frac{1}{4} \left( \eta^2 x_1^2 + 1 - \eta^2 \right) \mathbb{P} \left[ \|a\|_2 - \frac{x_1}{R} < \mathcal{N}(0, 1) < \frac{x_1}{R} + \|a\|_2 \right] \\
= \frac{1}{4} \left( \eta^2 x_1^2 + 1 - \eta^2 \right) \left( \Phi \left( \|a\|_2 - \frac{x_1}{R} \right) - \Phi \left( \|a\|_2 + \frac{x_1}{R} \right) \right).
\]

Let us now turn to the lower bound for the quantity \(A\):

\[
A = \eta^2 \mathbb{E} \left[ \frac{1}{R^2} \left( Z^t \beta_0 \right)^2 \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] + (1 - \eta^2) \mathbb{E} \left[ \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] \\
\geq \eta^2 \mathbb{E} \left[ \frac{1}{R^2} x_1^2 \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] + (1 - \eta^2) \mathbb{E} \left[ \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] \\
= \eta^2 \frac{x_1^2}{R^2} \mathbb{E} \left[ \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] + (1 - \eta^2) \mathbb{E} \left[ \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] \\
= \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \mathbb{E} \left[ \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right].
\]

We need now to control \(\mathbb{E} \left[ \alpha \left( Z^t \beta_0 \right) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right]\) from below. We first use Lemma 19 to get:
\[
E \left[ \alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] \\
\geq \int_{x_1}^{\infty} \left( x - x_1 - \frac{8}{100} \right) e^{-x} \frac{e^{-(x/R - \|a\|_2)^2/2}}{\sqrt{2\pi R^2}} \, dx \\
= \int_{x_1}^{\infty} x e^{-x} \frac{e^{-(x/R - \|a\|_2)^2/2}}{\sqrt{2\pi R^2}} \, dx - \left( x_1 + \frac{8}{100} \right) \int_{x_1}^{\infty} e^{-x} \frac{e^{-(x/R - \|a\|_2)^2/2}}{\sqrt{2\pi R^2}} \, dx.
\]

Using the notations of Lemmas 20 and 21 we obtain,

\[
E \left[ \alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] \geq J_{\alpha,R}(1, x_1) - \left( x_1 + \frac{8}{100} \right) K_{\alpha,R}(1, x_1).
\] (20)

Hence, Lemmas 20 and 21 give:

\[
E \left[ \alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] \\
\geq R \left( 1 + (\|a\| - R) G \left( \frac{x_1}{R} + R - \|a\| \right) - \left( x_1 + \frac{8}{100} \right) \right) \gamma \left( \frac{x_1}{R} + R - \|a\| \right) e^{-x_1} - \left( x_1 + \frac{8}{100} \right) \gamma \left( \frac{x_1}{R} + R - \|a\| \right) G \left( \frac{x_1}{R} + R - \|a\| \right) e^{-x_1} \\
\geq \left[ R + \left( R (\|a\| - R) - \left( x_1 + \frac{8}{100} \right) \right) G \left( \frac{x_1}{R} + R - \|a\| \right) \right] \gamma \left( \|a\| - \frac{x_1}{R} \right) e^{-x_1}.
\]

Finally,

\[
A > \left( \eta \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \left[ R + \left( R (\|a\| - R) - \left( x_1 + \frac{8}{100} \right) \right) G \left( \frac{x_1}{R} + R - \|a\| \right) \right] \gamma \left( \|a\| - \frac{x_1}{R} \right) e^{-x_1}.
\]

\[\square\]

**Lemma 14.** Take \(a \in \mathbb{R}^d, R, \nu > 0\) and \(\beta_0 := R \frac{a}{\|a\|_2}\) if inequality

\[
R \left( 1 - \left( R - \|a\|_2 + \frac{x_1 + \frac{8}{100}}{R} \right) G \left( \frac{x_1}{R} + R - \|a\|_2 \right) \right) \geq (1 + \nu) \frac{e^{x_1}}{4} G \left( \|a\|_2 - \frac{x_1}{R} \right)
\] (21)
is true, then for all \( h := h_\parallel + h_\perp \in \text{Vect}(\beta_0) \oplus \beta_0^\perp \) such that \( \|h\|_2 = 1 \) and \( \eta := \|h\|_2 \), it also holds

\[
(d_{\beta_0}^2 \mathcal{R}) (h, h) > \frac{\nu}{4} \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \left( \Phi^c \left( \|a\|_2 - \frac{x_1}{R} \right) - \Phi^c \left( \|a\|_2 + \frac{x_1}{R} \right) \right).
\]

**Proof.** Recall that \( \beta_0 := Ra/\|a\|_2 \). We proved in Lemma 12, that \( (d_{\beta_0}^2 \mathcal{R}) (h, h) \) is given by the following formula:

\[
(d_{\beta_0}^2 \mathcal{R}) (h, h) = \eta^2 \mathbb{E} \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_R) \right] + (1 - \eta^2) \mathbb{E} [\alpha (Z^t \beta_0)].
\]

We know from Lemma 15 that \( \alpha \) is non-positive on the interval \([-x_1, x_1]\) and positive otherwise. Consequently, we study the sign of \( (d_{\beta_0}^2 \mathcal{R}) (h, h) \) on the partition \( \mathbb{R} = (-\infty, -x_1) \cup [-x_1, x_1] \cup (x_1, \infty) \):

\[
(d_{\beta_0}^2 \mathcal{R}) (h, h) = \eta^2 \mathbb{E} \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_0) \right] + (1 - \eta^2) \mathbb{E} [\alpha (Z^t \beta_0)]
\]

\[
= \eta^2 \mathbb{E} \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 > x_1\}} \right] + (1 - \eta^2) \mathbb{E} [\alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 > x_1\}}]
\]

\[
+ \eta^2 \mathbb{E} \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_0) \mathbb{I}_{\{-x_1 < Z^t \beta_0 < x_1\}} \right] + (1 - \eta^2) \mathbb{E} [\alpha (Z^t \beta_0) \mathbb{I}_{\{-x_1 < Z^t \beta_0 < x_1\}}]
\]

\[
+ \eta^2 \mathbb{E} \left[ \frac{1}{R^2} (Z^t \beta_0)^2 \alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 < x_1\}} \right] + (1 - \eta^2) \mathbb{E} [\alpha (Z^t \beta_0) \mathbb{I}_{\{Z^t \beta_0 < x_1\}}]
\]

We have found in Lemma 13 two quantities \( a > 0 \) and \( b > 0 \) such that \( A > a \) and \( b \geq B \):

\[
a := \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \left[ R + \left( R (\|a\|_2 - R) - \left( x_1 + \frac{8}{100} \right) \right) G \left( \frac{x_1}{R} + R - \|a\|_2 \right) \right] \gamma \left( \|a\|_2 - \frac{x_1}{R} \right) e^{-x_1},
\]

\[
b := \frac{1}{4} \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \left( \Phi^c \left( \|a\|_2 - \frac{x_1}{R} \right) - \Phi^c \left( \|a\|_2 + \frac{x_1}{R} \right) \right).
\]

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If \( a > (1 + \nu) b \) for some \( \nu > 0 \) then we have \( \left( d^2_{\beta_0} \mathcal{R} \right) (h, h) > A - B > a - b > \nu b \). As

\[
b < \frac{1}{4} \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \Phi^c \left( \|a\|_2 - \frac{x_1}{R} \right),
\]

the condition \( a > (1 + \nu) b \) is satisfied when these successive conditions are true:

\[
\left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \left[ R + \left[ R (\|a\|_2 - R) - \left( x_1 + \frac{8}{100} \right) \right] G \left( \frac{x_1}{R} + R - \|a\|_2 \right) \gamma \left( \|a\|_2 - \frac{x_1}{R} \right) e^{-x_1}
\]

\[
> (1 + \nu) \frac{1}{4} \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \Phi^c \left( \|a\|_2 - \frac{x_1}{R} \right)
\]

(simplify \( \left( \eta^2 \frac{x_1^2}{R^2} + 1 - \eta^2 \right) \) and \( R \) in factor in the left-hand side)

\[
R \left[ 1 - \left( R - \|a\|_2 + \frac{x_1 + \frac{8}{100}}{R} \right) G \left( \frac{x_1}{R} + R - \|a\|_2 \right) \gamma \left( \|a\|_2 - \frac{x_1}{R} \right)
\]

\[
> (1 + \nu) \frac{e^{x_1}}{4} \Phi^c \left( \|a\|_2 - \frac{x_1}{R} \right)
\]

(divide by \( \gamma \left( \|a\|_2 - \frac{x_1}{R} \right) \) and make Mill’s ratio appear)

\[
R \left( 1 - \left( R - \|a\|_2 + \frac{x_1 + \frac{8}{100}}{R} \right) G \left( \frac{x_1}{R} + R - \|a\|_2 \right) \right) \geq (1 + \nu) \frac{e^{x_1}}{4} G \left( \|a\|_2 - \frac{x_1}{R} \right).
\]

To conclude, when the latter inequality is true, one has \( \left( d^2_{\beta_0} \mathcal{R} \right) (h, h) > \nu b \).

\[\square\]

**Lemma 15.** Study of \( \alpha(x) = -\frac{e^x}{(1+e^x)^2} \left( 1 + x \frac{1-e^x}{1+e^x} \right) \). At \( x = 0 \), \( \alpha(0) = -\frac{1}{4} \) is a global minimum, \( x_{\alpha_{\max}} \) \( \in [2, 3] \) is the positive real where \( \alpha \) is maximal with value \( \alpha_{\max} \), its derivative is bounded \( \|\alpha'\|_\infty \leq 0.22 \) and by definition of \( x_1 \) (see Section 2), \( \alpha(x_1) = 0 \) with

\[
x_1 \approx 1.54340463. \tag{22}
\]
\[
\begin{array}{cccccccc}
& x & 0 & x_2 & 1 & x_1 & 2 & x_{\alpha_{\text{max}}} & \infty \\
\text{sign of } f'' & - & - & - & - & - & - & - & - \\
\text{variations of } f' & 1 & \nearrow & 0 & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & -\infty \\
\text{sign of } f' & + & + & 0 & - & - & - & - & - & - \\
\text{variations of } f & 2 & \nearrow & f(x_2) & \nearrow & 2 & \nearrow & 0 & \nearrow & 3 - e^x & \nearrow & \nearrow & \nearrow & -\infty \\
\text{sign of } f & + & + & + & + & + & 0 & - & - & - & - & - & - \\
\text{sign of } \alpha & -\frac{1}{4} & - & - & - & - & 0 & + & + & + & \alpha_{\text{max}} & + & 0 \\
\end{array}
\]

Table 1: sign and variation table of \( f \), sign table of \( \alpha \)

**Proof.** First, remark that \( \forall x > 0, \)

\[
\alpha(x) \geq 0 \iff 1 + \frac{x \left(1 - e^x\right)}{1 + e^x} \leq 0 \\
\iff x \left(1 - e^x\right) \leq -\left(1 + e^x\right) \\
\iff 1 + e^x - xe^x + x \leq 0 \\
\iff f(x) \leq 0.
\]

We study \( f : x \mapsto 1 + e^x - xe^x + x \) for \( x \in \mathbb{R}_+ \) since \( \alpha \) is even. First of all, \( f'(x) = 1 - xe^x \) and \( f''(x) = -(x + 1)e^x \) which gives the sign and variation table 1. It is obvious that there exists \( x_1 > 0 \) such that \( f(x_1) = 0 \). Set \( p_x = (1 - e^x)^{-1} \) and \( q_x = e^x (1 - e^x)^{-1} \). Note that \( p_x - q_x = \frac{1 - e^x}{1 + e^x} = -\tanh \left(\frac{x}{2}\right) \)
and \( p_x q_x = \frac{1}{4} \left((p_x + q_x)^2 - (p_x - q_x)^2\right) = \frac{1}{4} \left(1 - \tanh^2 \left(\frac{x}{2}\right)\right) \).
\[
\alpha'(x) = \frac{d}{dx} [-px q_x (1 + x (p_x - q_x))]
\]
\[
= -\frac{d}{dx} [p_x] q_x (1 + x (p_x - q_x)) - p_x \frac{d}{dx} [q_x] (1 + x (p_x - q_x)) - p_x q_x \frac{d}{dx} [1 + x (p_x - q_x)]
\]
\[
= p_x q_x q_x (1 + x (p_x - q_x)) - p_x p_x q_x (1 + x (p_x - q_x)) - p_x q_x \left[ (p_x - q_x) + x \frac{d}{dx} [(p_x - q_x)] \right]
\]
\[
= p_x q_x \left[ q_x (1 + x (p_x - q_x)) - p_x (1 + x (p_x - q_x)) - (p_x - q_x) - x (p_x q_x - p_x q_x) \right]
\]
\[
= p_x q_x \left[ - (p_x - q_x) (1 + x (p_x - q_x)) - (p_x - q_x) + 2xp_x q_x \right]
\]
\[
= p_x q_x \left[ 2xp_x q_x - (p_x - q_x) (2 + x (p_x - q_x)) \right]
\]
\[
= p_x q_x \left[ \frac{x}{2} \left( 1 - \tanh^{2} \left( \frac{x}{2} \right) \right) + \tanh \left( \frac{x}{2} \right) \left( 2 - x \tanh \left( \frac{x}{2} \right) \right) \right]
\]
\[
= p_x q_x \left[ \frac{x}{2} \left( 1 - 2 \tanh^{2} \left( \frac{x}{2} \right) \right) + 2 \tanh \left( \frac{x}{2} \right) \right]
\]
\[
= \frac{1}{4} \left( 1 - \tanh^{2} \left( \frac{x}{2} \right) \right) \left[ \frac{x}{2} \left( 1 - 3 \tanh^{2} \left( \frac{x}{2} \right) \right) + 2 \tanh \left( \frac{x}{2} \right) \right].
\]

One can see on Figure ?? that the maximum of \( \alpha \) is attained at \( 2 \leq x_{\alpha_{\max}} \leq 3 \). The function \( \alpha \) is Lipschitz and one can see graphically on Figure ?? that \( \|\alpha'\|_{\infty} \leq 0.22 \). \( \Box \)

**Lemma 16.** \( \alpha : x \mapsto -\frac{e^{x}}{(1+e^{x})^{2}} \left( 1 + x \frac{1-e^{x}}{1+e^{x}} \right) \) is concave on \([x_1, 3]\).

**Proof.** The shape of \( \alpha \) on \([x_1, 3]\) can be seen on figure 1.

We use the following compact notations: \( p = \frac{1}{1+e^{x}}, q = 1 - p, \) hence \( \alpha(x) = -pq (1 + x (p - q)) \). Recall that \( \frac{dp}{dx} = -pq, \frac{dq}{dx} = pq \) and that \( \forall x > 0, p < q \). We proved in Equation (23) that

\[
\alpha'(x) = pq [(q - p) (2 + x (p - q)) + 2xpq]
\]
\[
= p(1 - p) [(1 - 2p) (2 + x (p - q)) + 2xp(1 - p)]
\]

In this proof we will also need the variations of \( \varpi : x \mapsto 1 + x (p - q) \):

\[ \]
Figure 1: Plot of $\alpha$ on $[x_1, 3]$.

$$\varpi'(x) = \frac{d}{dx} (1 + x (p - q))$$

$$= (p - q) + x \frac{d}{dx} (p - q)$$

$$= (p - q) + x (-pq - pq)$$

$$= (p - q) - 2xpq$$

$$< 0$$

We will want the sign of $\frac{d^2 \alpha}{dx^2}$. First remark that

$$\frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = \frac{d\alpha'}{dx} \frac{1}{pq} + \alpha' \frac{d}{dx} \left[ \frac{1}{pq} \right]$$

$$= \frac{d\alpha'}{dx} \frac{1}{pq} + \alpha' \frac{-1}{(pq)^2} \frac{d}{dx} (pq)$$

$$= \alpha'' \frac{1}{pq} + \alpha' \frac{-1}{(pq)^2} (-pq + ppq)$$

$$\frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = \alpha'' \frac{1}{pq} - \alpha' \frac{1}{pq} (p - q)$$
algebraic rearrangement give $\alpha'' = p(1 - p) \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] + \alpha' (2p - 1)$. Now compute what is still missing:

\[
\frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = \frac{d}{dx} \left[ \frac{pq [(q - p)(2 + x (p - q)) + 2xpq]}{pq} \right]
\]
\[
= \frac{d}{dx} [(q - p)(2 + x (p - q)) + 2xpq] 
= \frac{d}{dx} (q - p) (2 + x (p - q)) + (q - p) \frac{d}{dx} (2 + x (p - q)) + 2 \frac{d}{dx} (xpq)
\]
\[
= (pq + pq) (2 + x (p - q)) + (q - p) [(p - q) + x (-pq - pq)] + 2 (pq - xpq + xppq)
\]
\[
= 2pq (2 + x (p - q)) + (q - p) [(p - q) - 2xpq] + 2pq (1 + x (p - q))
\]
\[
= 2pq (3 + 2x (p - q)) + (q - p) (p - q) - 2xpq (q - p)
\]
\[
= 2pq (3 + 2x (p - q)) - (q - p)^2 + 2xpq (p - q)
\]
\[
= 2pq (3 + 2x (p - q) + x (p - q)) - (q - p)^2
\]
\[
= 6pq (1 + x (p - q)) - (q - p)^2
\]
\[
= 6pq \wp(x) - (p - q)^2
\]
\[
\frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = 6p(1 - p) \wp(x) - (1 - 2p)^2
\]

- Case $x \in [x_1, 2]$:

we have $p \in [0.11, 0.18]$, hence $1 - 2p \in [0.64, 0.78]$, $x \mapsto p$ is decreasing and $p \mapsto p(1 - p)$ is increasing on this inter Vallies of interest then $p(1 - p) \in [p_{x=2}(1 - p_{x=2}), p_{x=x_1}(1 - p_{x=x_1})] \subset [0.09, 0.15]$

then $\wp$ is strictly decreasing and $\wp(x) \in [\wp(2), \wp(x_1)] \subset [-0.53, 0]$ (0 occurs because by definition $x_1$ is such that $0 = \alpha(x_1) = pq \wp(x_1))$, all intervals put together gives in case $x \in [x_1, 2]$:

\[
\alpha'(x) = p(1 - p) \left[ \frac{(1 - 2p)}{\geq 0.9} \left[ \frac{1 + x (p - q)}{\geq 0.64} \right] + 2 \frac{x_1}{\geq 1.5435} \frac{p(1 - p)}{\geq 0.09} \right] \geq 0.052
\]
\[ \alpha'(x) = p(1 - p) \left[ (1 - 2p) \left( 1 + \frac{x}{1 + x(p - q)} \right) + 2 \frac{x_1}{1 + 1 + x(p - q)} \right] \leq 0.15 \]

\[ \leq 0.19 \]

it holds

\[ \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = 2p(1 - p) \varpi(x) - \left( \frac{1 - 2p}{\varpi(x)} \right)^2 \]

\[ \leq -0.4 \]

\[ \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = 2p(1 - p) \varpi(x) - \left( \frac{1 - 2p}{\varpi(x)} \right)^2 \]

\[ \leq -0.769 \]

And finally

\[ \alpha''(x) = p(1 - p) \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] + \alpha'(x)(1 - 2p) \]

\[ \leq -0.07376 \]

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\[ \alpha''(x) = pq \left( \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] + \alpha'(x)(1 - 2p) \right) \]

\[ \leq 0.15 \quad \geq -0.769 \leq 0.19 \quad \geq -0.78 \]

\[ \geq -0.26355 \]

\( \alpha \) is concave on \([x_1, 2]\).

-We do the same in the case \(x \in [2, 2.5]\):

we have \(p \in [0.075, 0.12]\), hence \(1 - 2p \in [0.76, 0.85]\), \(x \mapsto p\) is decreasing and \(p \mapsto p(1 - p)\) is increasing on this intervals of interest then \(p(1 - p) \in [p_{x=2.5}(1 - p_{x=2.5}), p_{x=2}(1 - p_{x=2})] \subset [0.069, 0.11]\),

\(\varpi(x) \in [\varpi(3), \varpi(2)] \subset [-1.125, -0.52]\) and

\[ \alpha'(x) = p(1 - p) \left[ \frac{(1 - 2p)}{\geq 0.069} \left( \frac{1 + 1 + x (p - q)}{\geq 0.76} \right) + 2 \frac{x_1}{\geq 1.5435} \frac{p(1 - p)}{\geq 0.069} \right] \]

\[ \geq 0.008 \]

\[ \alpha'(x) = p(1 - p) \left[ \frac{(1 - 2p)}{\leq 0.12} \left( \frac{1 + 1 + x (p - q)}{\leq 0.85} \right) + 2 \frac{x_1}{\leq 1.5436} \frac{p(1 - p)}{\leq 0.12} \right] \]

\[ \leq 0.094 \]

We now have

\[ \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = 2p(1 - p) \varpi(x) - \left( \frac{1 - 2p}{\geq 0.069} \left( \frac{1 - 2p}{\geq 0.76} \right) \right)^2 \]

\[ \leq -0.64936 \leq -0.65 \]
\[
\frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = 2p(1-p) \varpi(x) - \left( \frac{1-2p}{2} \right)^2 \\
\leq 0.11 \quad \geq -0.53 \quad \geq -0.85 \quad \leq 0.7225
\]

\[
\geq -0.8391 \geq -0.84
\]

And finally

\[
\alpha''(x) = p(1-p) \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] + \alpha'(x)(1-2p)
\]

\[
\leq 0.069 \quad \leq -0.65 \quad \geq 0.008 \quad \leq -0.76
\]

\[
\leq -0.05093
\]

\[
\alpha''(x) = pq \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] + \alpha'(x)(1-2p)
\]

\[
\leq 0.11 \quad \geq -0.84 \quad \leq 0.094 \quad \geq -0.85
\]

\[
\geq -0.1723
\]

Consequently \( \alpha \) is concave on \([2, 2.5]\)

- We do the same in the case \( x \in [2.5, 3]\):

in that case \( p \in [0.047, 0.076] \), hence \( 1 - 2p \in [0.848, 0.906] \), \( x \mapsto p \) is decreasing and \( p \mapsto p(1 - p) \) is increasing on this intervalles of interest then \( p(1-p) \in [p_{x=3}(1 - p_{x=3}), p_{x=2.5}(1 - p_{x=2.5})] \subset [0.0447, 0.071] \) and \( \varpi(x) \in [\varpi(3), \varpi(2)] \subset [-1.72, -1.12] \).
\[
\alpha'(x) = p(1 - p) \left[ (1 - 2p) \left( 1 + 1 + x(p - q) \right) + 2 x_1 \ p(1 - p) \right] \\
\geq 0.0447 \left[ (1 - 2p) \left( 1 + x(p - q) \right) \right] \\
\geq -0.02113
\]

\[
\alpha'(x) = p(1 - p) \left[ (1 - 2p) \left( 1 + 1 + x(p - q) \right) + 2 x_1 \ p(1 - p) \right] \\
\leq 0.071 \left[ (1 - 2p) \left( 1 + x(p - q) \right) \right] \\
\leq 0.0079
\]

We now have

\[
\frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = 2p(1 - p) \varpi(x) - \left( 1 - 2p \right)^2 \\
\leq -0.83
\]

\[
\frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] = 2p(1 - p) \varpi(x) - \left( 1 - 2p \right)^2 \\
\geq -1.082
\]

Concerning \( \alpha'' \), since \( \alpha' \in [-0.02113, 0.0079] \), the reasoning with an interval containing 0 is a bit
different: \( p(1-p) \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] \in [-0.077, -0.039] \) and \( \alpha'(x) (1 - 2p) \in [-0.0072, 0.0191] \), consequently

\[
\alpha''(x) = p(1-p) \frac{d}{dx} \left[ \frac{\alpha'}{pq} \right] + \alpha'(x) (1 - 2p)
\]

\[
\in [-0.082, -0.0199]
\]

Consequently \( \alpha \) is concave on \([2.5, 3]\) \( \square \)

**Lemma 17.** \( \forall x \geq 3, \alpha(x) - \varphi(x) \geq xe^{-x} \left( \frac{x1+0.08-1}{x} - 4e^{-x} \right) \) where \( \alpha : x \mapsto -\frac{e^{x}}{(1+e^{x})^{2}} \left( 1 + x \frac{1-e^{x}}{1+e^{x}} \right) \) and \( \varphi : x \mapsto (x - x1 - 0.08) e^{-x} \).

**Proof.** Let us study \( \alpha - \varphi \)

\[
\alpha(x) - \varphi(x) = -\frac{e^{x}}{(1+e^{x})^{2}} \left( 1 + x \frac{1-e^{x}}{1+e^{x}} \right) - (x - x1 - 0.08) e^{-x}
\]

\[= -\frac{e^{-2x}}{(1+e^{-x})^{3}} (1 + e^{x} + x - xe^{x}) - (x - x1 - 0.08) e^{-x}
\]

\[= \frac{xe^{-2x}}{(1+e^{-x})^{3}} \left( e^{x} - \frac{e^{x}}{x} - 1 - \frac{1}{x} \right) - x \left( 1 - \frac{x1 + 0.08}{x} \right) e^{-x}
\]

\[= xe^{-x} \left[ \frac{e^{-x}}{(1+e^{-x})^{3}} \left( e^{x} - \frac{e^{x}}{x} - 1 - \frac{1}{x} \right) - \left( 1 - \frac{x1 + 0.08}{x} \right) \right]
\]

\[= xe^{-x} \left[ \frac{1}{(1+e^{-x})^{3}} \left( 1 - \frac{1}{x} - e^{-x} - \frac{e^{-x}}{x} \right) - \left( 1 - \frac{x1 + 0.08}{x} \right) \right]
\]

Set \( R(x) := \frac{1}{(1+e^{-x})^{3}} - 1 + 3e^{-x} \) and \( \delta = x1 + 0.08 \). We get
\[\alpha(x) - \varphi(x) = xe^{-x} \left[ (1 - 3e^{-x} + R(x)) \left( 1 - \frac{1}{x} - e^{-x} - \frac{e^{-x}}{x} \right) - \left( 1 - \frac{\delta}{x} \right) \right] \]
\[= xe^{-x} \left[ (-1 + \delta) \frac{1}{x} + (-1 - 3)e^{-x} + (-1 + 3) \frac{e^{-x}}{x} + 3e^{-2x} + 3 \frac{e^{-2x}}{x} + R(x) \left( 1 - \frac{1}{x} - e^{-x} - \frac{e^{-x}}{x} \right) \right] \]
\[= xe^{-x} \left[ \frac{\delta - 1}{x} - 4e^{-x} + 2 \frac{e^{-x}}{x} + 3e^{-2x} + 3 \frac{e^{-2x}}{x} + R(x) \left( 1 - \frac{1}{x} - e^{-x} - \frac{e^{-x}}{x} \right) \right] \]
\[\geq xe^{-x} \left[ \frac{\delta - 1}{x} - 4e^{-x} + R(x) \left( 1 - \frac{1}{x} - e^{-x} - \frac{e^{-x}}{x} \right) \right]. \]

Let us now discuss the sign of \( R(x) \left( 1 - \frac{1}{x} - e^{-x} - \frac{e^{-x}}{x} \right) \):
\( R'(x) = 3e^{-x} (1 + e^{-x})^{-4} - 3e^{-x} = 3e^{-x} \left( (1 + e^{-x})^{-4} - 1 \right) < 0 \), \( R \) is strictly decreasing. Since \( \lim_{x \to +\infty} R(x) = 0 \), necessarily \( R \geq 0 \). In addition, since \( x \geq 3 \),
\[1 - \frac{1}{x} - e^{-x} - \frac{e^{-x}}{x} \geq 1 - \frac{1}{3} - e^{-x} - \frac{e^{-x}}{3} = \frac{2}{3} (1 - 2e^{-x}) \geq 0.\]

Hence \( R(x) \left( 1 - \frac{1}{x} - e^{-x} - \frac{e^{-x}}{x} \right) \geq 0 \) for \( x \geq 3 \) and it comes
\[\forall x \geq 3, \alpha(x) - \varphi(x) \geq xe^{-x} \left( \frac{\delta - 1}{x} - 4e^{-x} \right). \]

\[ \square \]

**Lemma 18.** The function \( \alpha : x \mapsto -\frac{e^x}{(1+e^x)^2} \left( 1 + x \frac{1-e^x}{1+e^x} \right) \) is greater than \( \varphi : x \mapsto (x - x_1 - 0.08) e^{-x} \) on \([x_1, \infty[\). 

**Proof.** Let us prove that \( \alpha \geq \varphi \) by considering four intervals \([x_1, 2]\), \([2, 2.5]\), \([2.5, 3]\) and \([3, \infty]\). We know that \( \alpha \) is concave on \([x_1, 3]\) according to lemma 16. It is also the case of \( \varphi \) because \( \varphi''(x) = (x - x_1 - 0.08 - 2) e^{-x} \) which is negative on \([x_1, 3]\) since \( x_1 + 0.08 + 2 \approx 3.62 \). Hence, \( \alpha \) is above its geometrical chords and \( \varphi \) below its tangents on \([x_1, 3]\).

**Case 1 on \([x_1, 2]\):**
The function $\alpha$ is above $l_1 : x \mapsto \frac{\alpha(2)}{2-x_1} (x - x_1)$ and $\varphi$ is below $l_2 : x \mapsto \varphi(1.85) + \varphi'(1.85) (x - 1.85)$. And as shown on figure 2, $l_1 \geq l_2$ on $[x_1, 2]$, one can compute the first coordinate of their intersection point:

$$x_{\text{intersection}} = \frac{\varphi(1.85) + \frac{\alpha(2)}{2-x_1} x_1 - 1.85 \varphi'(1.85)}{\frac{\alpha(2)}{2-x_1} - \varphi'(1.85)}.$$ 

A numerical computation gives $x_{\text{intersection}} \approx 2.820$. The two affine functions $l_1$ and $l_2$ intersect outside the interval $[x_1, 2]$ and since at $x_1$ we have $l_2(x_1) \approx -0.00165 < 0 = l_1(x_1)$, we can conclude that on $[x_1, 2], \alpha \geq l_1 \geq l_2 \geq \varphi$.

**Case 2 on $[2, 2.5]$**:

The function $\alpha$ is above $l_1 : x \mapsto \alpha(2) + \frac{\alpha(2.5) - \alpha(2)}{2.5 - 2} (x - 2)$ and $\varphi$ is below $l_2 : x \mapsto \varphi(2.2) + \varphi'(2.2) (x - 2.2)$. $l_1 \geq l_2$ as well, one can check it with $l_1(2) \approx 0.05493 \geq 0.05450 \approx l_2(2)$ and $l_1(2.5) \approx 0.0785 \geq 0.0779 \approx l_2(2.5)$. Consequently on $[2, 2.5], \alpha \geq l_1 \geq l_2 \geq \varphi$.

**Case 3 on $[2.5, 3]$**:

The function $\alpha$ is above $l_1 : x \mapsto \alpha(2.5) + \frac{\alpha(3) - \alpha(2.5)}{3-2.5} (x - 2.5)$ and $\varphi$ is maximal at $x_1 + 1 + 0.08$ with approximate value 0.07256. And since $l_1(2.5) \approx 0.07856 \geq 0.07256$ and $l_1(3) \approx 0.07750 \geq 0.07256$, we can conclude that $l_1$ is above the maximum of $\varphi$. Hence, on $[2.5, 3], \alpha \geq l_1 \geq \varphi$.

**Case 4 on $x \in [3, \infty]$**:
Figure 3: Plot of $\alpha$ and the chord of $\varphi$ on $[2, 2.5]$.

Figure 4: Plot of $\alpha$ and the chord of $\varphi$ on $[2.5, 3]$. 
Thanks to Lemma 17, we know that \( \forall x \geq 3, \alpha(x) - \varphi(x) \geq e^{-x} (x_1 + 0.08 - 1 - 4xe^{-x}) \). Let us study the sign of \( f : x \mapsto x_1 + 0.08 - 1 - 4xe^{-x} \). For any \( x \geq 3 \),

\[
 f'(x) = -4e^{-x} + 4xe^{-x} = 4(x - 1)e^{-x} \geq 0.
\]

We also have \( f(3) \approx 0.026 > 0 \). Consequently, \( \forall x \geq 3, f(x) \geq 0 \) and \( \alpha(x) - \varphi(x) \geq 0 \) as well.

This completes the proof: \( \forall x \geq x_1, \alpha(x) - \varphi(x) \geq 0 \).

**Lemma 19.** Recall that \( \alpha = -\frac{e^x}{(1+e^x)^2} \left( 1 + x \frac{1-e^x}{1+e^x} \right) \), \( Z \sim \mathcal{N}(a, I_d) \) and \( \beta_0 = Ra/\|a\|_2, a \in \mathbb{R}^d \). We have

\[
 \mathbb{E} \left[ \alpha (Z^t \beta_0) \mathbb{1}_{\{Z^t \beta_0 \geq x_1\}} \right] \geq \int_{x_1}^{\infty} \left( x - x_1 - \frac{8}{100} \right) e^{-x} e^{-\frac{(x-R\|a\|_2)^2}{2}} \sqrt{2\pi R^2} dx. \tag{24}
\]

**Proof.** Recall that \( \alpha(x) \geq (x - x_1 - \frac{8}{100}) e^{-x} \) on \([x_1, \infty[\) according to Lemma 18. Moreover \( Z^t \beta_0 \sim \mathcal{N}(a^t \beta_0, \|\beta\|_2) \) with \( a^t \beta_0 = R \|a\|_2, \|\beta\|_2 = R \). This gives
\[
E \left[ \alpha(Z' \beta_0) \mathbb{1}_{(Z' \beta_0) > x_1} \right] = \int_{x_1}^{\infty} \alpha(x) \frac{1}{\sqrt{2\pi R^2}} \exp \left( -\frac{(x - R \|a\|)^2}{2R^2} \right) dx \\
\geq \int_{x_1}^{\infty} \left( x - x_1 - \frac{8}{100} \right) e^{-x} \frac{1}{\sqrt{2\pi R^2}} \exp \left( -\frac{(x/R - \|a\|)^2}{2} \right) dx.
\]

\[\]  

**Lemma 20.** For any \(a, z \in \mathbb{R}^d\), \(\xi \in \mathbb{R}\) and \(R > 0\), it holds

\[
J_{a, R}(\xi, z) := \int_{z}^{\infty} \left( x e^{-\xi x} - \frac{1}{\sqrt{2\pi R^2}} e^{-(x/R - \|a\|)^2/2} \right) dx = R \left( 1 + (\|a\| - \|z - R\xi\|) G \left( \frac{z}{R} + R\xi - \|a\| \right) \right) \gamma \left( \frac{z}{R} - \|a\| \right) e^{-\xi z},
\]

where \(\gamma: x \mapsto (2\pi)^{-1/2} e^{-1/2 x^2}\) is the standard Gaussian density, \(\Phi^c\) is the standard Gaussian tail function and \(G: x \mapsto \Phi^c(x)/\gamma(x)\) is the Gaussian Mill’s ratio.

**Proof.** We have

\[
J_{a, R}(\xi, z) = \int_{z}^{\infty} x e^{-\xi x} \frac{1}{\sqrt{2\pi R^2}} \exp \left( -\frac{(x - R \|a\|)^2}{2R^2} \right) dx \\
= \int_{z}^{\infty} \frac{x}{\sqrt{2\pi R^2}} \exp \left( -\frac{(x - R \|a\|)^2 + 2R^2 \xi x}{2R^2} \right) dx. \tag{25}
\]

Moreover,

\[
(x - R \|a\|)^2 + 2R^2 \xi x = (x + R^2 \xi - R \|a\|)^2 + R^2 \xi \left( 2R \|a\| - R^2 \xi \right).
\]

Hence,

\[
J_{a, R}(\xi, z) = \int_{z}^{\infty} \frac{x}{\sqrt{2\pi R^2}} \exp \left( -\frac{(x + R \xi - \|a\|)^2 + R^2 \left( 2 \|a\| - R \xi \right) R \xi}{2R^2} \right) dx \\
e^{-\frac{(2\|a\| - R \xi)^2}{4}} \int_{z}^{\infty} x \frac{1}{\sqrt{2\pi R^2}} \exp \left( -\frac{(x/R + R \xi - \|a\|)^2}{2} \right) dx \tag{26}
\]

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By the change the variable \( y = x/R + R\xi - \|a\|_2 \), we get

\[
J_{a,R} (\xi, z) = e^{-\frac{(2\|a\|_2 - R\xi)R\xi}{2}} \int_{\frac{z}{R} + R\xi - \|a\|_2}^{\infty} e^{-\frac{(y - (R\xi - \|a\|_2))}{\sqrt{2\pi}}} dy
\]

\[
= e^{-\frac{(2\|a\|_2 - R\xi)R\xi}{2}} \int_{\frac{z}{R} + R\xi - \|a\|_2}^{\infty} e^{-\frac{y^2}{2\pi}} dy
\]

\[
- R (R\xi - \|a\|_2) e^{-\frac{(2\|a\|_2 - R\xi)R\xi}{2}} \Phi^c \left( \frac{z}{R} + R\xi - \|a\|_2 \right)
\]

\[
= R \left[ 1 + (\|a\|_2 - R\xi) G \left( \frac{z}{R} + R\xi - \|a\|_2 \right) \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{R\xi(R\xi - \|a\|_2)^2}{2}}
\]

and since

\[
(2\|a\|_2 - R\xi) R\xi + (z/R + R\xi - \|a\|_2)^2 = (z/R - \|a\|_2)^2 + 2z\xi,
\]

we finally get the result.

**Lemma 21.** For any \( a, z \in \mathbb{R}^d, \xi \in \mathbb{R} \) and \( R > 0 \), it holds

\[
K_{a,R} (\xi, z) := \int_{\mathbb{R}^d} e^{-\xi x} e^{-\frac{(x/R - \|a\|_2)^2}{2R^2}} dx = \gamma \left( \frac{z}{R} - \|a\|_2 \right) G \left( \frac{z}{R} + R\xi - \|a\|_2 \right) e^{-\xi z}
\]

where \( \gamma \) is the standard Gaussian density, \( \Phi^c \) is the standard Gaussian tail function and \( G : x \mapsto \Phi^c (x) / \gamma (x) \) is the Gaussian Mill’s ratio.

**Proof.** By the same calculation as in Equation 25, we can write
\[ K_{a,R}(\xi, z) = e^{-\frac{2 \|a\|_2 \cdot R \xi}{2}} \int_z^{\infty} \frac{1}{\sqrt{2\pi} R^2} \exp\left(-\frac{(x/R + R \xi - \|a\|_2)^2}{2}\right) dx. \]

By the change the variable \( y = x/R + R \xi - \|a\|_2 \), we get

\[ K_{a,R}(\xi, z) = e^{-\frac{2 \|a\|_2 \cdot R \xi}{2}} \int_{z/R+R \xi - \|a\|_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy \]

\[ = e^{-\frac{2 \|a\|_2 \cdot R \xi}{2}} \Phi_c(z/R + R \xi - \|a\|_2) \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{2 \|a\|_2 \cdot R \xi}{2} \frac{(z/R + R \xi - \|a\|_2)^2}{2}} G(z/R + R \xi - \|a\|_2). \]

By Identity (27), it follows that

\[ K_{a,R}(\xi, z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (z/R - \|a\|_2)^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{R \xi}{2} - z \xi G(z/R + R \xi - \|a\|_2)}, \]

as expected. \( \square \)

**Lemma 22.** Set \( G(x) = \frac{\Phi_c(x)}{\gamma(x)} \) the Mill’s ratio of the standard gaussian distribution. \( G \) satisfies: \( \forall x \in \mathbb{R} \)

\[ xG^2(x) - G'(x) = 1 \]

\[ G''(x) - xG'(x) - G(x) = 0 \]

\[ G'''(x) - 2G'(x) - xG''(x) = 0 \]

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Proof. $G(x) = \frac{\Phi^c(x)}{\gamma(x)}$ and using the fact that $\frac{d\Phi^c}{dx}(x) = -\gamma(x)$ and $\gamma'(x) = -x\gamma(x)$ it comes:

$$G'(x) = \frac{-\gamma(x)\gamma'(x) - \Phi^c(x)(-x\gamma(x))}{\gamma(x)\gamma(x)}$$

$$= -1 + x\frac{\Phi^c(x)}{\gamma(x)}$$

$$= -1 + xG(x)$$

and $G' = xG - 1 \Rightarrow G'' = G + xG' \Rightarrow G''' = G' + G' + xG''$

Proposition 23. The function $G(x) = \frac{\Phi^c(x)}{\gamma(x)}$ is known as the Gaussian Mill’s ratio and $\forall x \geq 0$

$$0 < \frac{2}{x + \sqrt{x^2 + 4}} \leq G(x) \leq \frac{2}{x + \sqrt{x^2 + 4} + \frac{2}{\pi}}$$

Proof. Focus on the first inequality: the lower bound is due to [5] and the upper bound is attributed to Pollak [38] according to [17] in which one can find the inequality in the first commentar of Remark 11 p 1848.

Proposition 24. The Gaussian mill’s ratio function $G : x \mapsto \frac{\Phi^c(x)}{\gamma(x)}$ is a strictly decreasing function on $\mathbb{R}$.
Proof. We have seen in Lemma 22 that $G' = -1 + x G$. Since $G \geq 0$, it is obvious that $G' < 0$ on $]-\infty; 0]$. Furthermore, take $x > 0$, then with proposition 23 we have

$$G'(x) = -1 + x G(x)$$

$$\leq -1 + x \frac{2}{x + \sqrt{x^2 + 4 \frac{2}{\pi}}}$$

$$= \frac{2}{1 + \sqrt{1 + \frac{8}{\pi x^2}}} - 1$$

One can see that $\forall x > 0, \frac{2}{1 + \sqrt{1 + \frac{8}{\pi x^2}}} < 1$, hence $G' < 0$ everywhere on $]0, \infty[$.

Lemma 25. Define $Eq_{a,b,c,d} : 1 + a G (b - a) \geq c G (a + d)$ where $a, b, c, d \geq 0$ and $G : x \mapsto \frac{\Phi_c(x)}{\gamma(x)}$ is the Gaussian mill’s ratio where $\gamma$ and $\Phi_c$ are respectively the density and the tail function of the standard univariate gaussian. If $Eq_{a,b,c,d}$ holds true, then $\forall h > 0, Eq_{a+h,b,c,d}$ holds true.

Proof. Start with $(a, b, c, d)$ such that $Eq_{a,b,c,d}$ holds true and take $h > 0$. We proved in prop 24 that $G$ is a decreasing function, then $G (b - a) < G ((b - (a + h))$, then one has $0 \leq a G (b - a) < (a + h) G (b - (a + h))$, hence

$$1 + (a + h) G (b - (a + h)) > 1 + a G (b - a)$$

But $(a, b, c, d)$ such that $Eq_{a,b,c,d}$ holds true, therefore

$$1 + (a + h) G (b - (a + h)) > c G (a + d)$$

Finally, use again the fact that $G$ is decreasing, to have $G (a + d) > G ((a + h) + d)$ and it comes

$$1 + (a + h) G (b - (a + h)) > c G ((a + h) + d)$$

To conclude, $Eq_{a+h,b,c,d}$ holds also true.
Lemma 26. The equation \( R \left( 1 - \left( R - \|a\|_2 + \frac{x_1 + \frac{8}{R} \sqrt{x_1}}{R} \right) G\left( \frac{2}{R} + R - \|a\|_2 \right) \right) \geq (1 + \nu) \frac{e^\nu}{4} G(\|a\|_2 - \frac{x_1}{R}) \) holds true, in particular, for \( R = \sqrt{x_1 + 0.08} \approx 1.2741 \), \( \|a\|_2 = 2R = R + \frac{x_1 + 0.08}{R} \approx 2.548 \) and \( \nu = 0.95 \).

Proof. Replace the corresponding quantities to get as left side \( R \left( 1 - \left( R - \|a\|_2 + \frac{x_1 + \frac{8}{R} \sqrt{x_1}}{R} \right) G\left( \frac{2}{R} + R - \|a\|_2 \right) \right) = R = \sqrt{x_1 + 0.08} \) and as right side \((1 + \nu) \frac{e^\nu}{4} G\left( R + \frac{x_1 + \frac{8}{R} \sqrt{x_1}}{R} - \frac{x_1}{R} \right) = (1 + \nu) \frac{e^\nu}{4} G\left( \sqrt{x_1 + 0.08} + \frac{0.08}{\sqrt{x_1 + 0.08}} \right) \). Approximation show that \( \sqrt{x_1 + 0.08} \approx 1.2741 \), \( \frac{e^\nu}{4} \approx 1.1701 \), \( \sqrt{x_1 + 0.08} + \frac{0.08}{\sqrt{x_1 + 0.08}} \approx 1.3370 \) and \( G(1.337) \approx 0.5552 \). On can see it is then enough to takes \( \nu = 0.95 \) because \((1 + \nu) \frac{e^\nu}{4} G\left( \sqrt{x_1 + 0.08} + \frac{0.08}{\sqrt{x_1 + 0.08}} \right) \approx 1.2668 \) (the inequality holds true because \( 1.2741 \geq 1.2668 \)).

Lemma 27. Recall that \( (d_\beta R)(\nu) = -E\left[ X^t \beta p_\beta(X) q_\beta(X) X^t \nu \right] \) for \( \beta \in B_2(0, R) \) and \( \nu \in \mathbb{R}^d \). Assume that \( a^t \beta - 2 \|\beta\|_2^2 \geq 0 \). If \( \langle \nu, \beta \rangle \leq 0 \), it holds

\[
(d_\beta R)(\nu) \geq \frac{1}{8} e^{-(2a^t\beta - \|\beta\|_2^2)/2} \left( -\nu, \frac{\beta}{\|\beta\|_2} \right) \left( \|\beta\|_2^2 + \left( a^t \beta - \|\beta\|_2^2 \right)^2 \right).
\]

Proof. For \( \nu \in \mathbb{R}^d \), decompose it on \( \beta^\perp \oplus \text{Vect}\,(\beta) \) as \( \nu = \nu_\perp + \nu_\parallel \), and set \( \lambda_\nu := \left\langle \nu, \frac{\beta}{\|\beta\|_2} \right\rangle \) so that \( \nu_\parallel = \lambda_\nu \frac{\beta}{\|\beta\|_2} \). Recall \( X \sim \mathcal{N}(a, I_d) \) and \( Z \sim \mathcal{N}(a, I_d) \). As a consequence \( Z^t \beta \sim \mathcal{N}\left( a^t \beta, \|\beta\|_2^4 \right) \). Set also \( N \sim \mathcal{N}(0, 1) \), so that \( Z^t \beta = a^t \beta + \|\beta\|_2 N \).

We have, by symmetry in \( X \) and independence between \( Z^t \beta \) and \( Z^t \nu_\perp \),
\[(d_\beta R)(\nu) = -\mathbb{E} \left[ \beta (X) q_\beta (X) X^t \nu \right] \]
\[= -\mathbb{E} \left[ \frac{Z_\beta e^{-Z_\beta^t \beta}}{(1 + e^{-Z_\beta^t \beta})^2} Z_\beta^t (\nu_\perp + \nu_\parallel) \right] \]
\[= -\mathbb{E} \left[ \frac{Z_\beta e^{-Z_\beta^t \beta}}{(1 + e^{-Z_\beta^t \beta})^2} Z_\beta^t \nu_\perp \right] - \mathbb{E} \left[ \frac{Z_\beta e^{-Z_\beta^t \beta}}{(1 + e^{-Z_\beta^t \beta})^2} Z_\beta^t \nu_\parallel \right] \]
\[= -\mathbb{E} \left[ \frac{Z_\beta e^{-Z_\beta^t \beta}}{(1 + e^{-Z_\beta^t \beta})^2} \mathbb{E} \left[ Z_\beta^t \nu_\perp \right] \right] - \lambda_\nu_\parallel \mathbb{E} \left[ \frac{Z_\beta e^{-Z_\beta^t \beta}}{(1 + e^{-Z_\beta^t \beta})^2} Z_\beta^t \beta \right] \]
\[= -\frac{\lambda_\nu_\parallel}{\|\beta\|_2} \mathbb{E} \left[ \zeta \left( a_\nu^t \beta + \|\beta\|_2 N \right) \right], \]

where \(\zeta : x \mapsto \frac{x^2 e^x}{(1 + e^x)^2}.\) Note that the function \(\zeta\) is even and that a simple calculation gives \(\forall x > 0, \zeta(x) \geq \frac{x^2}{4} e^{-x}.\) If \(\lambda_\nu \leq 0,\)

\[-\frac{\lambda_\nu_\parallel}{\|\beta\|_2} \mathbb{E} \left[ \zeta \left( a_\nu^t \beta + \|\beta\|_2 N \right) \right] \geq -\frac{\lambda_\nu_\parallel}{\|\beta\|_2} \int_0^{+\infty} \frac{x^2}{4} e^{-x} \frac{1}{\sqrt{2\pi \|\beta\|_2^2}} \exp \left( -\frac{(x - a_\nu^t \beta)^2}{2 \|\beta\|_2^2} \right) dx.\]

Set \(N_{a,\beta} \sim \mathcal{N} \left( a_\nu^t \beta - 2 \|\beta\|_2^2, \|\beta\|_2 \right).\) This gives
\[
\mathbb{E} \left[ \zeta \left( a^t \beta + \| \beta \|^2 \right) \right] \geq \int_0^{+\infty} \frac{x^2}{4} \frac{1}{\sqrt{2\pi} \| \beta \|^2} \exp \left( - \frac{(x - a^t \beta)^2}{2 \| \beta \|^2} \right) dx \\
= \int_0^{+\infty} \frac{x^2}{4} \frac{1}{\sqrt{2\pi} \| \beta \|^2} \exp \left( - \frac{(x - (a^t \beta - \| \beta \|^2)^2 + \| \beta \|^2 2a^t \beta - \| \beta \|^2)}{2 \| \beta \|^2} \right) dx \\
eq e^{-(2a^t \beta - \| \beta \|^2)/2} \int_0^{+\infty} \frac{x^2}{4} \frac{1}{\sqrt{2\pi} \| \beta \|^2} \exp \left( - \frac{(x - (a^t \beta - 2 \| \beta \|^2))^2}{2 \| \beta \|^2} \right) dx \\
\geq \frac{1}{8} e^{-(2a^t \beta - \| \beta \|^2)/2} \mathbb{E} \left[ N_\alpha, \beta^2 \right] \\
= \frac{1}{8} e^{-(2a^t \beta - \| \beta \|^2)/2} \left( \mathbb{V} \left[ N_\alpha, \beta^2 \right] + \mathbb{E} \left[ N_\alpha, \beta^2 \right]^2 \right) \\
= \frac{1}{8} e^{-(2a^t \beta - \| \beta \|^2)/2} \left( \| \beta \|^2 + (a^t \beta - 2 \| \beta \|^2)^2 \right),
\]
where in the second inequality, we used the fact that \( a^t \beta - 2 \| \beta \|^2 \geq 0 \). Therefore

\[
(d_\beta R) (\nu) \geq - \frac{\lambda^\nu}{8 \| \beta \|^2} e^{-(2a^t \beta - \| \beta \|^2)/2} \left( \| \beta \|^2 + (a^t \beta - 2 \| \beta \|^2)^2 \right).
\]

**Definition 28.** The operator norm \( \| \cdot \|_{op} \) on the trilinear symmetric operator space with respect to \( \| \cdot \|_2 \) is defined as: for all \( T \) symmetric trilinear operator, \( \| T \|_{op} := \sup_{u \in \partial B_2(0,1)} | T(u, u, u) | \) as shown in equation (2) in both [47] and [39].

**Lemma 29.** With trilinear symmetric operator defined above, the third derivative of the risk satisfies:

\[
\forall \beta \in \mathbb{R}^d,
\]

\[
\left\| a^3 \beta + (1-t) \beta \mathcal{R} \right\|_{op} \leq 8 e^{-(a^t \beta - \| \beta \|^2)/2} \sqrt{2 (\| a \|^6 + \mathbb{E} \left[ N_0^6 \right]) \left( \| \beta \|^2 + \left[ a^t \beta - 2 \| \beta \|^2 \right]^2 \right) + \left( a^t \beta - 2 \| \beta \|^2 + 1 \right)},
\]

where \( N_0 \sim \mathcal{N} (0, 1) \).
Proof. if \( u \in \partial B_2 (0, 1) \), then for \( N \sim \mathcal{N} (0, I_d) \), \( N^t u \sim N_0 \sim \mathcal{N} (0, 1) \), and it is known that \( \mathbb{E} [||N_0||] = \sqrt{\frac{2}{\pi}} \)

and \( \mathbb{E} [||N_0||^3] = 3 \mathbb{E} [||N_0||] \mathbb{E} [||N_0||] + \mathbb{E} [||N_0||]^3 \). Owing to Equation (19) and Cauchy-Schwarz inequality, we have

\[
|d^3_\beta R (u, u, u)| = |\mathbb{E} \left[ (X^t u)^3 . \alpha' (X^t \beta) \right] |
\leq \sqrt{\mathbb{E} \left[ (X^t u)^6 \right] \mathbb{E} \left[ (\alpha' (X^t \beta))^2 \right] }.
\]

On the one hand, using the fact that \( \forall a, b > 0, \forall n \in \mathbb{N}, (a + b)^n \leq 2^{n-1} (a^n + b^n) \) we get

\[
\mathbb{E} \left[ (X^t u)^6 \right] = \mathbb{E} \left[ (a^t u + N^t u)^6 \right]
= \mathbb{E} \left[ (a^t u + N_0)^6 \right]
\leq \mathbb{E} \left[ 2^5 \left( (a^t u)^6 + N_0^6 \right) \right]
\leq 32 \left( \|a\|^6 + \mathbb{E} \left[ N_0^6 \right] \right).
\]

On the other hand, we have already proved in Lemma 15 that \( \forall x, \alpha' (x) = p_x q_x \left[ \frac{x}{2} \left( 1 - 3 \tanh^2 \left( \frac{x}{2} \right) \right) + 2 \tanh \left( \frac{x}{2} \right) \right] \).

Hence,
\[
|\alpha'(x)| \leq p_x q_x \left| \frac{x}{2} \left( 1 - 3 \tanh^2 \left( \frac{x}{2} \right) + 2 \tanh \left( \frac{x}{2} \right) \right) \right| \\
\leq p_x q_x \left| \frac{x}{2} \left( 1 - 3 \tanh^2 \left( \frac{x}{2} \right) \right| + 2 \tanh \left( \frac{x}{2} \right) \right| \\
\leq \frac{e^x}{(1 + e^x)^2} \left( \frac{x}{2} \left( 3 \tanh^2 \left( \frac{x}{2} \right) + 1 \right) + 2 \right) \\
\leq \frac{2}{(e^x/2)^2} (x + 1) \\
= 2e^{-x} (x + 1).
\]

Recall \( Z^\top \beta \sim \mathcal{N} \left( a^\top \beta, \| \beta \|_2^2 \right) \),

\[
\mathbb{E} \left[ (\alpha' (X^\top \beta))^2 \right] = \mathbb{E} \left[ (\alpha' (Z^\top \beta))^2 \right] \\
\leq \mathbb{E} \left[ 4e^{-2Z^\top \beta} (Z^\top \beta + 1)^2 \right] \\
= 4 \int_{\mathbb{R}} (x + 1)^2 e^{-2x} \frac{1}{\sqrt{2\pi \| \beta \|_2}} \exp \left( - \frac{(x - a^\top \beta)^2}{2 \| \beta \|_2^2} \right) dx. \tag{28}
\]

By denoting \( N_{a,\beta} \sim \mathcal{N} \left( a^\top \beta - 2 \| \beta \|_2^2, \| \beta \|_2^2 \right) \), we get
\[ E \left[ (\alpha' (X^T \beta))^2 \right] = 4 \int_{\mathbb{R}} (x+1)^2 \frac{1}{\sqrt{2\pi \|\beta\|_2^2}} \exp \left( -\frac{\left( x + \left( 2 \|\beta\|_2^2 - a^T \beta \right) \right)^2}{2 \|\beta\|_2^2} + \frac{2 \|\beta\|_2^2 }{2} \frac{2a^T \beta - 2 \|\beta\|_2^2}{\|\beta\|_2^2} \right) dx \]

\[ = 4e^{-2(a^T \beta - \|\beta\|_2^2)} \int_{\mathbb{R}} (x+1)^2 \frac{1}{\sqrt{2\pi \|\beta\|_2^2}} \exp \left( -\frac{(x - (a^T \beta - 2 \|\beta\|_2^2))^2}{2} \right) dx \]

\[ = 4e^{-2(a^T \beta - \|\beta\|_2^2)} E \left[ (N_{a,\beta} + 1)^2 \right] \]

\[ = 4e^{-2(a^T \beta - \|\beta\|_2^2)} (E \left[ N_{a^2,\beta}^2 \right] + 2E \left[ N_{a,\beta} \right] + 1) \]

\[ = 4e^{-2(a^T \beta - \|\beta\|_2^2)} \left( \|\beta\|_2^2 + \left[ a^T \beta - 2 \|\beta\|_2^2 \right]^2 + 2 \left[ a^T \beta - 2 \|\beta\|_2^2 \right] + 1 \right) \]

Finally, we have

\[ |d^3_\beta R(u,u,u)| \leq \sqrt{8e^{-2(a^T \beta - \|\beta\|_2^2)} \sqrt{2 \left( \|a\|_2^6 + E \left[ N_{a,\beta}^6 \right] \right) \left( \|\beta\|_2^2 + \left[ a^T \beta - 2 \|\beta\|_2^2 \right]^2 + 2 \left[ a^T \beta - 2 \|\beta\|_2^2 \right] + 1 \right)} , \]

which gives the result, according to definition 28.

\[ \square \]

**Lemma 30.** Under the condition that \( \|a\|_2 \geq 2R \), \( R = \sqrt{x_1 + 0.08} \), the excess risk \( E(\cdot, \beta_0) \) satisfies around \( \beta_0 \):

\[ \forall \beta \in B_2(\beta_0, \varepsilon) \cap B_2(0, R), \]

\[ E(\beta, \beta_0) \geq e^{-\left( \|a\|_2 R - R^2/2 \right)} \left[ \frac{1}{16} \left( 1 + \|a\|_2 - R \right)^2 \right] \left( \|\beta - \beta_0\|_2^2 - 24 \|a\|^4 e^{R^2/2} e^{\|a\|_2 \|\beta - \beta_0\|_2^2} \right) . \]

**Proof.** First note that \( E(\beta_0, \beta_0) = 0 \) by definition of \( E(\cdot, \beta_0) \). According to lemma 27 we can control \((d_\beta R)(\beta - \beta_0)\) from below.
Since $\langle \beta - \beta_0, \beta_0 \rangle \leq 0$ and $a^t \beta_0 - 2 \| \beta_0 \|^2 \geq 0$, we have

$$(d_{\beta_0} R)(\beta - \beta_0) \geq \frac{1}{8} e^{-\left(\frac{a^t \beta_0 - \| \beta_0 \|^2}{2}\right)} \left\langle \beta_0 - \beta, \frac{\beta_0}{\| \beta_0 \|^2} \right\rangle \left(\| \beta_0 \|^2 + \left(a^t \beta_0 - \| \beta_0 \|^2\right)^2 \right)$$

$$\geq \frac{1}{8} e^{-\left(\| a \|^2_2 R^2 / 2\right)} \langle \beta_0 - \beta, \beta_0 \rangle \left(1 + (\| a \|^2_2 - R^2)\right).$$

Use now Lemma 32,

$$(d_{\beta_0} R)(\beta - \beta_0) \geq \frac{1}{16} e^{-\left(\| a \|^2_2 R^2 / 2\right)} \left(1 + (\| a \|^2_2 - R^2)\right) \| \beta - \beta_0 \|^2_2$$

According to lemma 4, we can control $(d^2_{\beta} R)(\beta - \beta_0, \beta - \beta_0)$ from below:

$$(d^2_{\beta} R)(\beta - \beta_0, \beta - \beta_0) \geq \Lambda_{\min} \| \beta - \beta_0 \|^2_2 \geq 0.$$ 

In addition, we can use Lemma 29 to have

$$\int_0^1 \| (d^3_{\beta + (1-t) \beta_0} R)(\beta - \beta_0, \beta - \beta_0, \beta - \beta_0) \| dt$$

$$= \| \beta - \beta_0 \|^3_2 \int_0^1 \left| (d^3_{\beta + (1-t) \beta_0} R) \left(\frac{\beta - \beta_0}{\| \beta - \beta_0 \|^2}, \frac{\beta - \beta_0}{\| \beta - \beta_0 \|^2}, \frac{\beta - \beta_0}{\| \beta - \beta_0 \|^2}\right) \right| dt$$

$$\leq \| \beta - \beta_0 \|^3_2 \int_0^1 \| d^3_{\beta + (1-t) \beta_0} R \|_{op} dt$$

$$\leq 8 \| \beta - \beta_0 \|^3_2 \int_0^1 \exp \left(-a^t (t \beta + (1-t) \beta_0) - \| t \beta + (1-t) \beta_0 \|^2_2 \right) C_{3,a} (t \beta + (1-t) \beta_0) dt,$$

where

$$C_{3,a} : \mu \in \mathbb{R}^d \mapsto \sqrt{2 \left(\| a \|^6_2 + \mathbb{E} \left[ N_0^6 \right] \right) \left(\| \mu \|^2_2 + \left[a^t \nu - 2 \| \mu \|^2_2\right]^2 \right) + 2 \left[a^t \nu - 2 \| \mu \|^2_2 + 1\right]}.$$ 

To bound $C_{3,a}$ from above, remark that $\forall \mu \in \Psi_U$, $\| \mu \|^2_2 \leq 4 R^2$, $-2 R^2 \leq a^t \nu - 2 \| \mu \|^2_2 \leq R \| a \|^2_2 - 2 R^2 \leq$
$R\|a\|_2$ and remark also that owing to $\|a\| \geq 2R \approx 2.548$ and article [46], one has $\mathbb{E} [N^6_0] = 15$ with $N(0,1)$, so $\mathbb{E} [N^6_0] \leq \frac{1}{18} \|a\|_2^6$. Therefore, all together this leads to

$$C_{3,a}(\mu) \leq \sqrt{2 \left( \|a\|_2^6 + \mathbb{E} [N^6_0] \right)} \left( \left[ R^2 + R^2 \max (2R, \|a\|_2^2) \right] + 2 \left[ R \|a\|_2 - 2R^2 \right] + 1 \right)$$

$$\leq \sqrt{2 \left( \|a\|_2^6 + \frac{1}{18} \|a\|_2^6 \right)} \left( \left[ R^2 + R^2 \|a\|_2^2 \right] + 2 \left[ R \|a\|_2 - 2R^2 \right] + 1 \right)$$

$$\leq \sqrt{\frac{19}{9} \|a\|_2^6} \left( R^2 \|a\|_2^2 + 2R \|a\|_2 - 3R^2 + 1 \right)$$

$$\leq 3 \|a\|^4.$$

Hence

$$\int_0^1 \left| \left[ d_{t^2+(1-t)\beta_0} \mathcal{R} \right] (\beta - \beta_0, \beta - \beta_0, \beta - \beta_0) \right| dt$$

$$\leq 24 \|a\|^4 \|\beta - \beta_0\|^3 \int_0^1 \exp \left( - \left( a^t (t\beta + (1-t)\beta_0) - \|t\beta + (1-t)\beta_0\|_2^2 \right) \right) dt$$

$$\leq 24 \|a\|^4 \|\beta - \beta_0\|^3 \sup_{\|\beta - \beta_0\|_2 \leq \epsilon, 0 \leq t \leq 1} \exp \left( - \left( a^t (t\beta + (1-t)\beta_0) - \|t\beta + (1-t)\beta_0\|_2^2 \right) \right)$$

$$\leq 24 \|a\|^4 \|\beta - \beta_0\|^3 \exp \left( - \inf_{\|\beta - \beta_0\|_2 \leq \epsilon, 0 \leq t \leq 1} a^t (t\beta + (1-t)\beta_0) - \sup_{\|\beta - \beta_0\|_2 \leq \epsilon, 0 \leq t \leq 1} \|t\beta + (1-t)\beta_0\|_2^2 \right)$$

$$\leq 24 \|a\|^4 \|\beta - \beta_0\|^3 \exp \left( - a^t \beta_0 + \|a\|_2 \epsilon + R^2 \right)$$

$$\leq 24 \|a\|^4 e^{-\|a\|_2 R + R^2} e^{\|a\|_2 \epsilon + R^2} \|\beta - \beta_0\|^3.$$ 

Finally, this gives

$$\mathcal{E}(\beta, \beta_0) > e^{-\left( \|a\|_2 R^2/2 \right)} \left[ \frac{1}{16} \left( 1 + (\|a\|_2 - R)^2 \right) \|\beta - \beta_0\|_2^2 - 24 \|a\|^4 e^{R^2/2} e^{\|a\|_2} \|\beta - \beta_0\|_2^3 \right],$$

as required. □
Lemma 31. Minoration of $\Phi^c\left(\|a\|_2 - \frac{x_1}{R}\right) - \Phi^c\left(\|a\|_2 + \frac{x_1}{R}\right)$:

$$\forall a, R, x_1, \Phi^c\left(\|a\|_2 - \frac{x_1}{R}\right) - \Phi^c\left(\|a\|_2 + \frac{x_1}{R}\right) \geq 2\frac{x_1}{R} \gamma \left(\|a\|_2 + \frac{x_1}{R}\right)$$

Proof. Simple computations give

$$\Phi^c\left(\|a\|_2 - \frac{x_1}{R}\right) - \Phi^c\left(\|a\|_2 + \frac{x_1}{R}\right) = \int_{\|a\|_2 - \frac{x_1}{R}}^{\|a\|_2 + \frac{x_1}{R}} \gamma(x) d\lambda(x)$$

$$\geq \int_{\|a\|_2 - \frac{x_1}{R}}^{\|a\|_2 + \frac{x_1}{R}} \gamma\left(\|a\|_2 + \frac{x_1}{R}\right) d\lambda(x)$$

$$\geq 2\frac{x_1}{R} \gamma \left(\|a\|_2 + \frac{x_1}{R}\right)$$

Lemma 32. \(\forall \beta \in B_2(0, R), \text{ if } \beta_0 \in \partial B_2(0, R) \text{ then } \langle \beta_0 - \beta, \beta_0 \rangle \geq \frac{1}{2} \|\beta - \beta_0\|^2\).

Proof. Decompose $\beta$ as $\beta_\perp + \beta_\parallel$ on $\beta_0^\perp \oplus \text{Vect}(\beta_0)$ and note that $\exists \lambda_\beta \in [-1, 1], \beta_\parallel = \lambda_\beta \beta_0$. We have

$$\frac{\langle \beta_0 - \beta, \beta_0 \rangle}{\|\beta - \beta_0\|^2} = \frac{\langle \beta_0 - \beta_\parallel, \beta_0 \rangle}{\|\beta_\perp\|^2 + \|\beta_\parallel - \beta_0\|^2} = \frac{(1 - \lambda_\beta) R^2}{\|\beta_\perp\|^2 + (1 - \lambda_\beta)^2 R^2}$$

Furthermore, we have $\|\beta_\perp\|^2 \leq R^2$ and by pythagora’s theorem $\|\beta_\perp\|^2 \in \left[0, R^2 - \lambda_\beta^2 R^2\right]$. Therefore,

$$\frac{\langle \beta_0 - \beta, \beta_0 \rangle}{\|\beta - \beta_0\|^2} \geq \frac{(1 - \lambda_\beta) R^2}{R^2 + (1 - \lambda_\beta)^2 R^2}$$

$$\geq \frac{1 - \lambda_\beta}{1 - \lambda_\beta^2 + (1 - 2\lambda_\beta + \lambda_\beta^2)}$$

$$\geq \frac{1}{2}.$$
**Definition 33.** When one has \( P_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \) with \( x_1, \ldots, x_n \in \mathcal{X} \), define the following “empirical-\( L^2 \)-norm” as:

\[
\forall f : \mathcal{X} \rightarrow \mathbb{R}, \|f\|_{P_n} := \sqrt{\frac{1}{n} \sum_{i=1}^{n} f(X(i))}
\]

**Definition 34.** For \( \delta > 0 \), the \( \delta \)-covering number \( N(\delta, \mathcal{H}, \|\cdot\|) \) of a set \( \mathcal{H} \) is the smallest number of closed balls, with respect to \( \|\cdot\| \) with radius \( \delta \), that covers the space. The set of the centers of the balls is called a \( \delta \)-covering set. The entropy of \( \mathcal{H} \) with respect to a norm \( \|\cdot\| \) is \( H(\cdot, \mathcal{H}, \|\cdot\|) = \log N(\cdot, \mathcal{H}, \|\cdot\|) \).

**Lemma 35.** Define \( \Theta(\varepsilon) := \{ \beta \in B_2(0, R) : \|\beta - \beta_0\|_1 \leq \varepsilon \} \) and take

\[
\mathcal{H}_{\varepsilon, M_n} := \left\{ (\rho_\beta - \rho_{\beta_0}) I_{\{G \leq M_n\}} - \mathbb{E} \left[ (\rho_\beta(X) - \rho_{\beta_0}(X)) I_{\{G(X) \leq M_n\}} \right] : \beta \in \Theta(\varepsilon) \right\},
\]

where \( G(X) := \|X\|_{\infty} \). Recall that \( L \) is the Lipschitz constant of \( \rho \).

Then for all \( u > 0 \) and \( M_n > 0 \), the entropy of \( \mathcal{H}_{\varepsilon, M_n} \) with respect to the empirical-\( L^2 \)-norm \( \|\cdot\|_{P_n} \) (see definition 33) satisfies

\[
H(u, \mathcal{H}_{\varepsilon, M_n}, \|\cdot\|_{P_n}) \leq \left( \frac{4L^2 M_n^2 \varepsilon^2}{u^2} + 1 \right) \log (2d).
\]

**Proof.** Let \( \tilde{X}_1, \ldots, \tilde{X}_i \) be i.i.d copies of \( X \) and set \( \mathcal{B}_{\varepsilon, M_n} := \left\{ f_{\beta, \beta'} : X \mapsto \frac{X^t}{M_n} (\beta - \beta') I_{\{G(X) \leq M_n\}} : \beta, \beta' \in \Theta(\varepsilon) \right\} \).

One has \( \forall \beta, \beta' \in \Theta(\varepsilon), \)

\[
|\rho_\beta(X) - \rho_{\beta'}(X)| = |\rho(X^t \beta) - \rho(X^t \beta')| \leq L |X^t \beta - X^t \beta'|.
\]
With $\forall a, b > 0, (a + b)^2 \leq 2 (a^2 + b^2)$, it follows that

$$\|\rho_\beta I_{\{G(\cdot) \leq M_n\}} - \mathbb{E} [\rho_\beta I_{\{G(\cdot) \leq M_n\}}] - \rho_\beta' I_{\{G(\cdot) \leq M_n\}} + \mathbb{E} [\rho_\beta' I_{\{G(\cdot) \leq M_n\}}]\|_{P_n}^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \rho_\beta \left( X^{(i)} \right) I_{\{G(X^{(i)}) \leq M_n\}} - \mathbb{E} [\rho_\beta (X) I_{\{G(X) \leq M_n\}}] - \rho_\beta' \left( X^{(i)} \right) I_{\{G(X^{(i)}) \leq M_n\}} + \mathbb{E} [\rho_\beta' (X) I_{\{G(X) \leq M_n\}}] \right)^2$$

$$\leq \frac{2}{n} \sum_{i=1}^{n} \left( \rho_\beta \left( X^{(i)} \right) - \rho_\beta' \left( X^{(i)} \right) \right)^2 I_{\{G(X^{(i)}) \leq M_n\}} + \frac{2}{n} \sum_{i=1}^{n} \left( \mathbb{E} [\rho_\beta (X) I_{\{G(X) \leq M_n\}}] - \mathbb{E} [\rho_\beta (X) I_{\{G(X) \leq M_n\}}] \right)^2$$

$$\leq \frac{2}{n} \sum_{i=1}^{n} \left( \rho_\beta \left( X^{(i)} \right) - \rho_\beta' \left( X^{(i)} \right) \right)^2 I_{\{G(X^{(i)}) \leq M_n\}} + 2 \mathbb{E} \left[ (\rho_\beta' (X) - \rho_\beta (X))^2 I_{\{G(X) \leq M_n\}} \right].$$

Furthermore,

$$\frac{2}{n} \sum_{i=1}^{n} \left( \rho_\beta \left( X^{(i)} \right) - \rho_\beta' \left( X^{(i)} \right) \right)^2 I_{\{G(X^{(i)}) \leq M_n\}} \leq \frac{2}{n} \sum_{i=1}^{n} L^2 \left| X^{(i)\top} \beta - X^{(i)\top} \beta' \right|^2 I_{\{G(X^{(i)}) \leq M_n\}} \leq \frac{2}{n} L^2 \sum_{i=1}^{n} G \left( X^{(i)} \right)^2 \| \beta - \beta' \|^2_1 I_{\{G(X^{(i)}) \leq M_n\}} \leq 2L^2 M_n^2 \| \beta - \beta' \|^2_1.$$

One also has

$$\mathbb{E} \left[ (\rho_\beta' (X) - \rho_\beta (X))^2 I_{\{G(X) \leq M_n\}} \right] \leq L^2 M_n^2 \| \beta - \beta' \|^2_1.$$

Hence

$$\|\rho_\beta I_{\{G(\cdot) \leq M_n\}} - \mathbb{E} [\rho_\beta I_{\{G(\cdot) \leq M_n\}}] + \rho_\beta' I_{\{G(\cdot) \leq M_n\}} - \mathbb{E} [\rho_\beta' I_{\{G(\cdot) \leq M_n\}}]\|_{P_n}^2 \leq 4L^2 M_n^2 \| \beta - \beta' \|^2_1. \quad (31)$$

This relation enables us to state

$$H \left( u, \mathcal{H}_\varepsilon, M_n, \| \cdot \|_{P_n} \right) \leq H \left( \frac{u}{2LM_n^2}, \Theta (\varepsilon), \| \cdot \|_1 \right).$$

Define the convex hull of a set of vectors $\{e_j\}_{j=1}^{d}$ as $Conv \{e_j\}_{j=1}^{d} := \left\{ \sum_{j=1}^{d} v_j e_j \mid v_i \geq 0, \|v\|_1 = 1 \right\}$ and
take in particular the vectors \( \{ e_j \}_{j=1}^d \) of the canonical basis in \( \mathbb{R}^d \). Then

\[
\Theta(\varepsilon) \subset \beta_0 + \varepsilon.\text{Conv}\ \left\{ 0, \{ \pm e_j \}_{j=1}^d \right\}.
\]

Owing to the definition of \( e_j \), we have \( \forall j, \| e_j \|_1 = 1 \). so we can use Lemma 14.28 in [9] to get

\[
H(u, \Theta(\varepsilon), \| \cdot \|_1) \leq H\left(u, \beta_0 + \varepsilon.\text{Conv}\ \left\{ 0, \{ \pm e_j \}_{j=1}^d \right\}, \| \cdot \|_1\right) \\
\leq H\left(u, \frac{\varepsilon}{\varepsilon}, \text{Conv}\ \left\{ 0, \{ \pm e_j \}_{j=1}^d \right\}, \| \cdot \|_1\right) \\
\leq H\left(u, \frac{\varepsilon^2}{u^2}, \text{Conv}\ \left\{ 0, \{ \pm e_j \}_{j=1}^d \right\}, \| \cdot \|_1\right) \\
\leq \left( \frac{\varepsilon^2}{u^2} + 1 \right) \log (2d),
\]

which gives the result.

Lemma 36. Let \( \varepsilon > 0 \) and \( X^{(1)}, \ldots, X^{(i)}, \ldots, X^{(n)} \) be i.i.d. copies of \( X \). Let also

\[
\mathcal{H}_{\varepsilon,M_n} := \{ (\rho_\beta - \rho_{\beta_0}) I_{G \leq M_n} - E \left[ (\rho_\beta(X) - \rho_{\beta_0}(X)) I_{G(X) \leq M_n} \right] : \beta \in \Theta(\varepsilon) \},
\]

where \( G(X) := \| X \|_\infty \) and

\[
\Theta(\varepsilon) := \left\{ \beta \in \mathbb{R}^d : \beta \in B_2(0,R), \| \beta - \beta_0 \|_1 \leq \varepsilon \right\}.
\]

Recall that we set \( L \), the Lipschitz norm of \( \rho \). One has \( \forall T \geq 1, \forall n \geq 2, \)

\[
P\left( \sup_{\beta \in \Theta(\varepsilon)} \frac{|V_n^{\text{trunc}}(\beta) - V_n^{\text{trunc}}(\beta_0)|}{\varepsilon} \geq \frac{3LM_n T \left( 5 \sqrt{3 \log (2d) \log n + 4} \right)}{\sqrt{n}} \right) \leq \exp \left( -21 (T - 1)^2 \log (2d) \log^2 n \right).
\]

Proof. According to equation (31), \( \forall \rho_\beta \in \mathcal{H}_{\varepsilon,M_n}, \| \rho_\beta \|_{P_n} \leq 2LM_n \varepsilon =: R_n \) and \( E(\rho_\beta(X)) = 0 \). Hence,
using Lemma 35 and Definition 34, we have

\[
\log \left( 1 + N \left( u, \mathcal{H}_{\varepsilon, M_n}, \| \cdot \|_{P_n} \right) \right) \leq 1 + H \left( u, \mathcal{H}_{\varepsilon, M_n}, \| \cdot \|_{P_n} \right) \\
\leq 1 + \left( \frac{4L^2M_n^2\varepsilon^2}{u^2} + 1 \right) \log (2d) \\
\leq \left( \frac{4L^2M_n^2\varepsilon^2}{u^2} + 2 \right) \log (2d)
\]

Take \( u := 2^{-s}R_n \) where \( 0 \leq s \leq S := \min \left\{ s \geq 1 : 2^{-s} \leq \frac{4}{\sqrt{n}} \right\} \) (i.e. \( u \in \left[ \frac{2}{\sqrt{n}} R_n, R_n \right] \)), then one has \( \forall 0 \leq s \leq S \),

\[
\log \left( 1 + N \left( 2^{-s}R_n, \mathcal{H}_{\varepsilon, M_n}, \| \cdot \|_{P_n} \right) \right) \leq \left( \frac{4L^2M_n^2\varepsilon^2}{2^{-2s}R_n^2} + 2 \right) \log (2d) \\
\leq (2^{2s} + 2) \log (2d) \\
\leq 2^{2s} \left( 1 + 2^{1-2s} \right) \log (2d) \\
\leq 2^{2s} \times 3 \log (2d)
\]

Now one can apply [9, Corollary 14.4], where in our case \( A := 3 \log (2d) \). Note that \( 4 \log n \leq 3 \log_2 n \leq 5 \log n \). We get

\[
\mathbb{E} \left[ \sup_{\tilde{\rho} \in \mathcal{H}_{\varepsilon, M_n}} \left| \frac{1}{n} \sum_{i=1}^{n} \tilde{\rho}_{\beta} (X^{(i)}) \right| \right] \leq \frac{R_n}{\sqrt{n}} \left( 5\sqrt{A} \log n + 4 \right).
\]

One can apply the Massart’s concentration inequality, recalled for instance in [9, Theorem 14.2]. Then, \( \forall t > 0 \),

\[
P \left( \sup_{\tilde{\rho} \in \mathcal{H}_{\varepsilon, M_n}} \left| \frac{1}{n} \sum_{i=1}^{n} \tilde{\rho}_{\beta} (X^{(i)}) \right| \geq \frac{R_n}{\sqrt{n}} \left( 5\sqrt{A} \log n + 4 \right) + R_n t \right) \leq e^{-nt^2/8},
\]

which gives

\[
P \left( \sup_{\tilde{\rho} \in \mathcal{H}_{\varepsilon, M_n}} \left| \frac{1}{n} \sum_{i=1}^{n} \tilde{\rho}_{\beta} (X^{(i)}) \right| \geq \frac{R_n}{\sqrt{n}} \left( 5\sqrt{A} \log n + 4 \right) + R_n t \right) \leq e^{-nt^2/8}.
\]
A change of variable \( t = \frac{1}{\sqrt{n}} (T - 1) \left( 5\sqrt{A} \log n + 4 \right) \) leads to: \( \forall T \geq 1 \),

\[
P \left( \sup_{\tilde{\beta} \in \mathcal{H}, M_n} \left| \frac{1}{n} \sum_{i=1}^{n} \tilde{\beta} \left( X^{(i)} \right) \right| \geq \frac{R_n}{\sqrt{n}} T \left( 5\sqrt{A} \log n + 4 \right) \right) < \exp \left( -\frac{(T - 1)^2 \left( 5\sqrt{A} \log n + 4 \right)^2}{8} \right)
\]

Note that \( \forall \tilde{\beta} \in \mathcal{H}, M_n \),

\[
\frac{1}{n} \sum_{i=1}^{n} \tilde{\beta} \left( X^{(i)} \right) = V_{n}^{\text{trunc}} (\beta) - V_{n}^{\text{trunc}} (\beta_0).
\]

Consequently, \( \forall T \geq 1 \),

\[
P \left( \sup_{\beta \in \Theta(c)} \left| V_{n}^{\text{trunc}} (\beta) - V_{n}^{\text{trunc}} (\beta_0) \right| \geq 3LM_n \varepsilon T \left( 5\sqrt{3} \log (2d) \log n + 4 \right) \right) < \exp \left( -\frac{(3T/2 - 1)^2 \left( 5\sqrt{3} \log (2d) \log n + 4 \right)^2}{8} \right) < \exp \left( -21 (T - 1)^2 \log (2d) \log^2 n \right).
\]

\[
\text{Lemma 37.} \quad \text{Grant the notations of Lemma 36 and set } \lambda_0 := 3LM_n \left( 5\sqrt{3} \log (2d) \log n + 4 \right) n^{-1/2}. \text{ One has } \forall T \geq 1, \forall n \geq 2,
\]

\[
P \left( \sup_{\beta \in \mathcal{B}_{2}(0, R)} \left| V_{n}^{\text{trunc}} (\beta) - V_{n}^{\text{trunc}} (\beta_0) \right| \geq T \lambda_0 \right) \leq \frac{3}{4} \log \left( \frac{4R^2nd}{L^2M_n^2} \right) \exp \left( -21 (T - 1)^2 \log (2d) \log^2 n \right).
\]

\[
\text{Proof.} \quad \text{Let } \lambda_0 > 0, n \geq 2 \text{ and } T \geq 1. \text{ Let us use a peeling: define } \Theta := B_{2}(0, R) \text{ and divide it into slices as follows:}
\]

\[
\Theta_j := \{ \beta \in B_{2}(0, R) : 2^{-j-1} \leq \| \beta - \beta_0 \|_1 \leq 2^{-j} \}.
\]
Note that \( \exists j_{\inf}, j_{\sup} \in \mathbb{Z}, \exists r > 0, 2^{-j_{\sup} - 1} \leq \lambda_0 \leq 2^{-j_{\inf}} \) and \( 2^{-j_{\inf} - 1} \leq r := 2R\sqrt{d} \leq 2^{-j_{\inf}} \) with

\[
\Theta \subset \bigcup_{j=j_{\inf}}^{j_{\sup}} \Theta_j \bigcup B_1(\beta_0, 2^{-j_{\sup}})
\]

and \( \Theta \subset B_1(\beta_0, 2^{-j_{\inf} - 1}) \). One can also prove that \( j_{\inf} = \lceil -\log_2 r \rceil = \lceil -\log_2 2R\sqrt{d} \rceil \leq -1 \) because \( R, d \geq 1 \) and \( j_{\sup} = \lceil -\log_2 (\lambda_0) \rceil \). Hence

\[
P\left( \sup_{\beta \in \Theta} \frac{|V_{n,i}^{\text{trunc}}(\beta) - V_{n,i}^{\text{trunc}}(\beta_0)|}{\|\beta - \beta_0\|_1 \lor \lambda_0} \geq T\lambda_0 \right) \leq \sum_{j=j_{\inf}}^{j_{\sup}} P\left( \sup_{\beta \in \Theta_j} \frac{|V_{n,i}^{\text{trunc}}(\beta) - V_{n,i}^{\text{trunc}}(\beta_0)|}{\|\beta - \beta_0\|_1 \lor \lambda_0} \geq T\lambda_0 \right) + P\left( \sup_{\beta \in B_1(\beta_0, 2^{-j_{\sup}})} \frac{|V_{n,i}^{\text{trunc}}(\beta) - V_{n,i}^{\text{trunc}}(\beta_0)|}{\|\beta - \beta_0\|_1 \lor \lambda_0} \geq T\lambda_0 \right).
\]

Use the fact that \( \forall j \in [j_{\inf}, j_{\sup}], \lambda_0 \leq 2^{-j} \) and \( \forall \beta \in B_1(\beta_0, 2^{-j_{\sup}}), \|\beta - \beta_0\|_1 \lor \lambda_0 \leq 2^{-j_{\sup}} \):

\[
P\left( \sup_{\beta \in \Theta} \frac{|V_{n,i}^{\text{trunc}}(\beta) - V_{n,i}^{\text{trunc}}(\beta_0)|}{\|\beta - \beta_0\|_1 \lor \lambda_0} \geq T\lambda_0 \right) \leq \sum_{j=j_{\inf}}^{j_{\sup}} P\left( \sup_{\beta \in \Theta_j} \frac{|V_{n,i}^{\text{trunc}}(\beta) - V_{n,i}^{\text{trunc}}(\beta_0)|}{2^{-j}} \geq T\lambda_0 \right) + P\left( \sup_{\beta \in B_1(\beta_0, 2^{-j_{\sup}})} \frac{|V_{n,i}^{\text{trunc}}(\beta) - V_{n,i}^{\text{trunc}}(\beta_0)|}{2^{-j_{\sup}}} \geq T\lambda_0 \right).
\]

By applying Lemma 36 with \( \lambda_0 = 3LM_n \left( 5\sqrt{3\log(2d)\log n + 4} \right) n^{-1/2} \), we get

\[
P\left( \sup_{\beta \in \Theta_j} \frac{|V_{n,i}^{\text{trunc}}(\beta) - V_{n,i}^{\text{trunc}}(\beta_0)|}{2^{-j}} \geq T\lambda_0 \right) < \exp (-21 (T - 1)^2 \log (2d) \log^2 n).
\]

Then
\[
P \left( \sup_{\beta \in \Theta} \frac{V_{n, \text{trunc}}(\beta) - V_{n, \text{trunc}}(\beta_0)}{\|\beta - \beta_0\|_1} \geq T\lambda_0 \right) \leq \sum_{j = j_{\inf}}^{j_{\sup}} \exp \left( -21 (T - 1)^2 \log (2d) \log^2 n \right) \\
+ \exp \left( -21 (T - 1)^2 \log (2d) \log^2 n \right) \\
\leq (j_{\sup} - j_{\inf} + 2) \exp \left( -21 (T - 1)^2 \log (2d) \log^2 n \right).
\]

Simplify now the expression of \( j_{\sup} - j_{\inf} + 2 \),

\[
j_{\sup} - j_{\inf} + 2 = \left\lfloor \log_2 2R\sqrt{d} \right\rfloor - \left\lfloor \log_2 (\lambda_0) \right\rfloor + 2 \\
\leq \log_2 8R\sqrt{d} + 1 - \log_2 (\lambda_0) \\
\leq \log_2 \frac{16R\sqrt{d}}{\lambda_0} \\
\leq \log_2 \left( \frac{2R\sqrt{nd}}{LM_n \sqrt{3 \log (2d) \log n}} \right) \\
\leq \log_2 \left( \frac{2R\sqrt{nd}}{LM_n} \right) \\
\leq \frac{3}{2} \log \left( \frac{4R^2 nd}{L^2 M_n^2} \right) \\
\leq \frac{3}{4} \log \left( \frac{4R^2 nd}{L^2 M_n^2} \right).
\]

This finally gives the result. \( \Box \)

**Lemma 38.** With \( G(X) := \|X\|_\infty \) and \( a \) and \( X \) defined in the section Notations: if \( M_n = \|a\|_\infty + \sqrt{2 \log d} + \sqrt{2 \log n} \) then

\[
E \left( G(X) I_{\{G(X) > M_n\}} \right) \leq 2 (M_n + 1) \frac{e^{-2\sqrt{\log d \log(1+n)}}}{n}
\]

and
\begin{align*}
\mathbb{E}\left(G(X)^2 I_{\{G(X) > M_n\}}\right) & \leq 2 \left(M_n^2 + \|a\|_\infty + 1\right) \frac{e^{-2\sqrt{\log d \log(1+n)}}}{n}.
\end{align*}

**Proof.** First note that for $y \geq \|a\|_\infty$,

\begin{align*}
P( G(X) > y) & \leq P\left( \max_j |Z_j| > y - \|a\|_\infty\right) \\
& \leq dP( |Z_1| > y - \|a\|_\infty) \\
& \leq 2de^{-\frac{(y-\|a\|_\infty)^2}{2}}.
\end{align*}

Take now $M \geq \|a\|_\infty + 1$, we have

\begin{align*}
\mathbb{E}\left(G(X) I_{\{G(X) > M\}}\right) &= - \int_M^\infty y \frac{dP(G(X) > y)}{dy} dy \\
&= \left[-yP(G(X) > y)\right]_M^\infty + \int_M^\infty 1 \times P(G(X) > y) dy \\
& \leq 2Mde^{-\frac{(M-\|a\|_\infty)^2}{2}} + \int_M^\infty (y - \|a\|_\infty) \times P(G(X) > y) dy \\
& = 2Mde^{-\frac{(M-\|a\|_\infty)^2}{2}} + 2 \int_M^\infty (y - \|a\|_\infty) \times de^{-\frac{(y-\|a\|_\infty)^2}{2}} dy \\
& = 2Mde^{-\frac{(M-\|a\|_\infty)^2}{2}} + 2 \left[-de^{-\frac{(y-\|a\|_\infty)^2}{2}}\right]_M^\infty \\
& = 2Mde^{-\frac{(M-\|a\|_\infty)^2}{2}} + 2de^{-\frac{(M-\|a\|_\infty)^2}{2}} \\
& = 2(M + 1)de^{-\frac{(M-\|a\|_\infty)^2}{2}}
\end{align*}

and

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\[
\mathbb{E} \left( (G(X))^2 I_{\{G(X) > M\}} \right) = - \int_{M}^{\infty} y^2 \frac{dP(G(X) > y)}{dy} dy
\]
\[
= \left[ - y P(G(X) > y) \right]_{M}^{\infty} + \int_{M}^{\infty} y \times P(G(X) > y) dy
\]
\[
= 2M^2 d e^{-\frac{(M-1\|\|_{\infty})^2}{2}} + \int_{M}^{\infty} \left( y - \|a\|_{\infty} \right) \times P(G(X) > y) dy + \int_{M}^{\infty} \|a\|_{\infty} \times P(G(X) > y) dy
\]
\[
\leq 2M^2 d e^{-\frac{(M-1\|\|_{\infty})^2}{2}} + 2d e^{-\frac{(M-\|a\|_{\|\|})^2}{2}} + 2\|a\|_{\infty} \int_{M}^{\infty} \left( y - \|a\|_{\infty} \right) d e^{-\frac{(y-\|a\|_{\\})^2}{2}} dy
\]
\[
\leq M^2 d e^{-\frac{(M-\|a\|_{\|\|})^2}{2}} + d e^{-\frac{(M-\|a\|_{\|\|})^2}{2}} + 2\|a\|_{\infty} d e^{-\frac{(M-\|a\|_{\|\|})^2}{2}}
\]
\[
= 2 \left( M^2 + \|a\|_{\|\|} + 1 \right) d e^{-\frac{(M-\|a\|_{\|\|})^2}{2}}.
\]

Hence, for \( M_n := \|a\|_{\|\|} + \sqrt{2 \log d} + \sqrt{2 \log (1 + n)} \geq \|a\|_{\|\|} + 1 \), we have

\[
\mathbb{E} \left( (G(X) I_{\{G(X) > M_n\}} \right) \leq 2 \left( M_n + 1 \right) d e^{-\frac{(M_n-\|a\|_{\|\|})^2}{2}}
\]
\[
\leq 2 \left( M_n + 1 \right) d e^{-\frac{\left( \sqrt{2 \log d} + \sqrt{2 \log (1 + n)} \right)^2}{2}}
\]
\[
\leq 2 \left( M_n + 1 \right) d e^{-\log d - 2 \sqrt{\log d \log (1 + n)} - \log (1 + n)}
\]
\[
\leq 2 \left( M_n + 1 \right) e^{-2 \sqrt{\log d \log (1 + n)}} \frac{1 + n}{1 + n}
\]
\[
\leq 2 \left( M_n + 1 \right) e^{-2 \sqrt{\log d \log (1 + n)}} \frac{1 + n}{n}
\]

and

\[
\mathbb{E} \left( (G(X))^2 I_{\{G(X) > M_n\}} \right) \leq 2 \left( M_n^2 + \|a\|_{\|\|} + 1 \right) d e^{-\frac{(M_n-\|a\|_{\|\|})^2}{2}}
\]
\[
= 2 \left( M_n^2 + \|a\|_{\|\|} + 1 \right) e^{-2 \sqrt{\log d \log (1 + n)}} \frac{1 + n}{1 + n}
\]
\[
\leq 2 \left( M_n^2 + \|a\|_{\|\|} + 1 \right) e^{-2 \sqrt{\log d \log (1 + n)}} \frac{1 + n}{n}.
\]
Lemma 39. Assume that \( \|a\|_2 \geq 2R \approx 2.548 \). Set

\[
F(X) := G(X) I_{\{G(X) > M_n\}} + \mathbb{E} \left[ G(X) I_{\{G(X) > M_n\}} \right],
\]

where \( G(X) = \|X\|_\infty \). Moreover, take the following constants:

\[
M_n := \|a\|_\infty + \sqrt{2 \log d} + \sqrt{2 \log (1 + n)}, \quad \lambda_0 := 3LM_n n^{-1/2} \left( 5 \sqrt{3 \log (2d) \log n + 4} \right).
\]

It holds: \( \forall T > 0 \),

\[
P \left( \frac{1}{n} \sum_{i=1}^{n} F(X^{(i)}) \geq \frac{\lambda_0 T}{L} \right) \leq \frac{4 L^2 M_n^2 \|a\|_\infty + 1}{n^2}.
\]

Proof. Note that with our choice of \( \lambda_0 \), we have by Lemma 38: \( \lambda_0 T/L \geq 2 \mathbb{E} \left[ G(X) I_{\{G(X) > M_n\}} \right] \). Hence,

\[
P \left( \frac{1}{n} \sum_{i=1}^{n} F(X^{(i)}) \geq \frac{\lambda_0 T}{L} \right) = P \left( \frac{1}{n} \sum_{i=1}^{n} \left( G(X^{(i)}) I_{\{G(X^{(i)}) > M_n\}} + \mathbb{E} \left[ G(X) I_{\{G(X) > M_n\}} \right] \right) \geq \frac{\lambda_0 T}{L} \right)
\]

\[
= P \left( \frac{1}{n} \sum_{i=1}^{n} \left( G(X^{(i)}) I_{\{G(X^{(i)}) > M_n\}} - \mathbb{E} \left[ G(X) I_{\{G(X) > M_n\}} \right] \right) \geq \frac{\lambda_0 T}{L} - 2 \mathbb{E} \left[ G(X) I_{\{G(X) > M_n\}} \right] \right)
\]

\[
\leq \frac{\mathbb{V} \left( \frac{1}{n} \sum_{i=1}^{n} G(X^{(i)}) I_{\{G(X^{(i)}) > M_n\}} \right)}{\left( \frac{\lambda_0 T}{L} - 2 \mathbb{E} \left[ G(X) I_{\{G(X) > M_n\}} \right] \right)^2}
\]

\[
\leq \frac{\mathbb{E} \left( G(X)^2 I_{\{G(X) > M_n\}} \right)}{n \left( \frac{\lambda_0 T}{L} - 2 \mathbb{E} \left[ G(X) I_{\{G(X) > M_n\}} \right] \right)^2}.
\]

From Lemma 38, we get
\[
P\left( \frac{1}{n} \sum_{i=1}^{n} F(X^{(i)}) \geq \frac{\lambda_0 T}{L} \right) \leq \frac{\mathbb{E}\left( (G(X))^2 I_{(G(X)>M_n)} \right)}{n \left( \frac{\lambda_0 T}{L} - 2 \mathbb{E}\left[ G(X) I_{(G(X)>M_n)} \right] \right)^2}.
\]

\[
\leq \frac{2 \left( M_n^2 + \|a\|_{\infty} + 1 \right) e^{-2\sqrt{\log d \log(1+n)}}}{n \left( \frac{\lambda_0 T}{L} - 4 (M_n + 1) \right)^2}.
\]

\[
\leq 2L^2 M_n^2 + \|a\|_{\infty} + 1 \frac{e^{-2\sqrt{\log d \log(1+n)}}}{n^2 \lambda_0^2 T^2} (1 - 4L^2 M_n^2 + \|a\|_{\infty} + 1)^2.
\]

It holds, for \( n \geq 2 \),

\[
L \frac{M_n + 1}{n\lambda_0 T} e^{-2\sqrt{\log d \log(1+n)}} \leq \frac{L M_n + 1}{n\lambda_0}
\]

\[
= L \frac{M_n + 1}{nT} \cdot \frac{3LM_n \left( 5\sqrt{3 \log(2d) \log n + 4} \right)}{\sqrt{n}}
\]

\[
= \frac{1 + \frac{1}{M_n}}{3\sqrt{n}T \left( 5\sqrt{3 \log(2d) \log n + 4} \right)}
\]

\[
\leq \frac{1}{3\sqrt{2} \left( 5\sqrt{3 \log 2 \log 2 + 4} \right)}
\]

\[
< \frac{1}{8}.
\]

Finally, we conclude that

\[
P\left( \frac{1}{n} \sum_{i=1}^{n} F(X^{(i)}) \geq \frac{\lambda_0 T}{L} \right) < 4L^2 M_n^2 + \|a\|_{\infty} + 1 \frac{e^{-2\sqrt{\log d \log(1+n)}}}{n^2 \lambda_0^2 T^2}.
\]

\[\square\]

**Lemma 40.** Recall from Lemma 36 that \( V_n^{\text{trunc}}(\beta) = (P_n - P)(\rho \beta I_{(G \leq M_n)}) \) and \( V_n(\beta) = (P_n - P) \rho \beta \).
Recall also from Lemma 39 that $F(X) = G(X) I_{\{G(X) > M_n\}} + \mathbb{E} \left[ G(X) I_{\{G(X) > M_n\}} \right]$ with $G(X) = \|X\|_{\infty}$.

It holds true that $\forall T \geq 1$,

$$P\left( \sup_{\beta \in B_2(0,R)} \frac{|V^{\text{trunc}}_n(\beta) - V^{\text{trunc}}_n(\beta_0) - (V_n(\beta) - V_n(\beta_0))|}{\|\beta - \beta_0\|_1 \vee \lambda_0} > T\lambda_0 \right) \leq P\left( \frac{1}{n} \sum_{i=1}^{n} F\left(X^{(i)}\right) > \frac{T\lambda_0}{L} \right).$$

Proof. Basic computations and Hölder’s inequality give

$$|V^{\text{trunc}}_n(\beta) - V^{\text{trunc}}_n(\beta_0) - (V_n(\beta) - V_n(\beta_0))|$$

$$= |(P_n - P) (\rho_\beta I_{\{G > M_n\}}) - (P_n - P) (\rho_{\beta_0} I_{\{G > M_n\}})|$$

$$\leq |P_n [(\rho_\beta - \rho_{\beta_0}) I_{\{G > M_n\}}]| + |P [(\rho_\beta X - \rho_{\beta_0} X) I_{\{G(X) > M_n\}}]|$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} L \left\| X^{(i)} \right\|_1 I_{\{G(X^{(i)}) > M_n\}} + \mathbb{E} \left[ L \{X - \beta_0\} I_{\{G(X) > M_n\}} \right]$$

$$\leq L \left( \frac{1}{n} \sum_{i=1}^{n} \left\| X^{(i)} \right\|_1 \left\{ I_{\{G(X^{(i)}) > M_n\}} \|\beta - \beta_0\|_1 + \mathbb{E} \left[ \left\| X \right\|_{\infty} I_{\{G(X) > M_n\}} \right] \right\} \right)$$

$$\leq \frac{L \|\beta - \beta_0\|_1}{n} \sum_{i=1}^{n} F\left(X^{(i)}\right),$$

and the result directly follows. □

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