Hadwiger’s Conjecture and Squares of Chordal Graphs

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Abstract. Hadwiger’s conjecture states that for every graph \( G \), \( \chi(G) \leq \eta(G) \), where \( \chi(G) \) is the chromatic number and \( \eta(G) \) is the size of the largest clique minor in \( G \). In this work, we show that to prove Hadwiger’s conjecture in general, it is sufficient to prove Hadwiger’s conjecture for the class of graphs \( \mathcal{F} \) defined as follows: \( \mathcal{F} \) is the set of all graphs that can be expressed as the square graph of a split graph. Since split graphs are a subclass of chordal graphs, it is interesting to study Hadwiger’s Conjecture in the square graphs of subclasses of chordal graphs. Here, we study a simple subclass of chordal graphs, namely 2-trees and prove Hadwiger’s Conjecture for the squares of the same. In fact, we show the following stronger result: If \( G \) is the square of a 2-tree, then \( G \) has a clique minor of size \( \chi(G) \), where each branch set is a path.

Keywords: Hadwiger’s Conjecture, 2-Trees, Square Graphs, Minors

1 Introduction

The Four Color Theorem is perhaps the most famous theorem in graph theory. It states that the chromatic number of a planar graph is at most 4. The history of the development of graph theory itself is intimately linked with the attempts to solve the Four Color Conjecture. Wagner [Wag37] showed in 1937 that the four color conjecture is equivalent to the following statement: If a graph is \( K_5 \)-minor free, then it is 4-colorable. In 1943, Hadwiger [Had43] proposed the following conjecture.

Conjecture 1 (Hadwiger’s Conjecture). For \( t \geq 1 \), every graph \( G \) without a \( K_{t+1} \) minor is \( t \)-colorable.

\* Part of the work was done when this author was visiting Max Planck Institute for Informatics, Saarbrücken, Germany supported by Alexander von Humboldt Fellowship.
This conjecture if proved, would give a far reaching generalization of the 4-color theorem. Hadwiger [Had43] proved the conjecture for \( t \leq 3 \). The Four color theorem was proved by Appel and Haken [AH+, AHK+] in 1977. In view of Wagner’s theorem [Wag37], this implies that Hadwiger’s conjecture is true for \( t = 4 \). In 1993, Seymour et al. [RST93] proved Hadwiger’s conjecture for \( t = 5 \). It remains unsolved for \( t > 5 \). Kawarabayashi and Toft [KT05] showed that any graph that is \( K_7 \)-minor free and \( K_{4,4} \)-minor free is 6-colorable.

For perfect graphs Hadwiger’s conjecture is trivially true. Reed and Seymour [RS04] proved Hadwiger’s conjecture for line graphs. Belkale et al. [BC09] proved the conjecture for proper circular arc graphs. In 2008, Chudnovsky and Fradkin [CF08] published a work which generalizes both the above results: They proved Hadwiger’s conjecture for a class of graphs called quasi-line graphs, which properly contains both proper circular arc graphs and line graphs.

As far as we know, not many results are known regarding Hadwiger’s conjecture with respect to square graphs, even for the squares of well known special classes of graphs. This is surprising considering the fact that the chromatic number is well studied with respect to the squares of several special classes of graphs. For e.g., see the extensive work on Wegner’s conjecture [Weg77]. We believe that this may be due to the difficulty level involved in dealing with this problem, as we show in Theorem 1 that proving Hadwiger’s conjecture for squares of chordal graphs will also prove the conjecture for general graphs.

Hadwiger’s conjecture for powers of cycles and their complements was proved by Li and Liu [LL07]; Chandran et al. [CKR08] studied the Hadwiger number with respect to the Cartesian product of the graphs.

1.1 Our Contributions

Hadwiger’s conjecture is well-known to be a tough problem. Bollobás, Catlin and Erdős [BCE80] describe it as “one of the deepest unsolved problems in graph theory.” It could be useful if we can show that it is sufficient to concentrate on certain class of graphs. **Chordal graphs** are those graphs that have no induced cycle of length 4 or more. **Split graphs** are those graphs whose vertices can be partitioned into two sets such that one induces an independent set and the other induces a clique. Split graphs form a subclass of chordal graphs. In section 2, we show that, in order to prove the Hadwiger’s conjecture in general, it is sufficient to prove it for the class of graphs that are squares of split graphs. (See Theorem 1.)

We understand that our reduction may not really help in making Hadwiger’s conjecture easier. But it does show that the squares of split graphs captures the complexity of the general problem. For an optimistic researcher, it opens up the question of studying the Hadwiger’s conjecture on the squares of various special classes of graphs in the hope of getting some new insights about the problem.

In light of Theorem 1, it is interesting to study Hadwiger’s conjecture for squares of subclasses of chordal graphs. It can be shown that chordal graphs are exactly the class of graphs that can be constructed by starting with a clique and doing the following operation, a finite number of times: Pick a clique \( C \) in the current graph, introduce a new vertex \( v \), and make \( v \) adjacent to all the vertices in \( C \).
\textit{k-trees} are a special case of chordal graphs, where we start with a \textit{k}-clique and at each step we pick a \textit{k}-clique. Hence it is interesting to prove Hadwiger’s conjecture for squares of \textit{k}-trees. As a first step, we prove Hadwiger’s conjecture for squares of \textit{2}-trees in section 3 of this paper. (See Theorem 2.) A slightly more general class than \textit{2}-trees allows one to join a fresh vertex to a clique of size at most 2 instead of exactly 2. We remark that it is easy to extend Theorem 2 to this class of graphs.

**Structure of Branch Sets**

Although proving the Hadwiger’s conjecture requires only to show a clique minor of size at least \(\chi(G)\), it is also interesting to study the structure of branch sets forming such a clique minor. For example, in the case of graphs with independence number at most 2, Seymour proposed the following stronger conjecture [Bla05]: If \(G\) has no stable set of size 3, then \(G\) has a clique minor of size at least \(|V(G)|/2\) using only edges or single vertices as branch sets. For squares of \textit{2}-trees we show that there exists a clique minor of size \(\chi(G)\) where the branch sets forming the clique minor are paths. (See Theorem 2).

**Towards Generalizing the result of Chudnovsky and Fradkin:** Chudnovsky and Fradkin [CF08] proved that Hadwiger’s conjecture is true for quasi-line graphs. A graph \(G\) is a quasi-line graph if for every vertex \(v \in V(G)\), the set of neighbors of \(v\) in \(G\) can be expressed as the union of two cliques. A natural way to generalize concept of quasi-line graph is the following:

**Definition 1 (Generalized Quasi-line Graphs).** A graph \(G\) is a generalized quasi-line graph if for any subset \(S \subseteq V(G)\), there exists a vertex \(u \in S\) such that the neighbors of \(u\) induce union of two cliques in \(G[S]\) (the induced subgraph on \(S\)).

It is natural to consider the problem of generalizing the result of Chudnovsky and Fradkin to generalized quasi-line graphs.

**Open Problem 1** Prove Hadwiger’s conjecture for generalized quasi-line graphs.

Taking into account the difficulty level of [CF08], the above question might turn out to be difficult. Therefore it is natural to try to prove the conjecture for non-trivial subclasses of generalized quasi-line graphs.

**Observation 1** The squares of \textit{2}-trees form a subclass of generalized quasi-line graphs.

**Remark 1.** It is interesting to note that squares of \textit{2-degenerate} graphs do not form a subclass of generalized quasi-line graphs.

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4 see subsection 1.3 for the definition

5 This generalization is in the same spirit as the generalization of graphs of maximum degree \(k\) to \(k\)-degenerate graphs. A graph \(G\) is a maximum degree \(k\) graph, if every vertex has at most \(k\) neighbors. A graph \(G\) is a \(k\)-degenerate graph is for any subset \(S \subseteq V(G)\), there exists a vertex \(u \in S\), such that \(u\) has at most \(k\) neighbors in \(G[S]\). Graph classes which can be considered to be generalizations of quasi-line graphs can also be found in [KT14], for e.g. \(k\)-perfectly groupable graphs, \(k\)-simplicial graphs, \(k\)-perfectly orientable graphs etc.
Note that squares of 2-trees do not form a subclass of quasi-line graphs. Hence, by Theorem 2, we prove Hadwiger’s conjecture for a special class of generalized quasi-line graphs that is not contained in quasi-line graphs.

1.2 Future Directions

An obvious next step will be to prove Hadwiger’s Conjecture for squares of $k$-trees for fixed $k \geq 3$. It is also interesting to try to prove Hadwiger’s conjecture for squares of other special classes of graphs such as planar graphs. Another direction may be to work towards solving the open problem 1. It will be interesting to look at other non-trivial subclasses of generalized quasi-line graphs with respect to Hadwiger’s conjecture.

1.3 Preliminaries

For any graph $G$, we denote the vertices of $G$ by $V(G)$ and the edges of $G$ by $E(G)$. When we are talking about a singleton set $\{x\}$, we may abuse the notation and use $x$ for the sake of conciseness. We say disjoint vertex sets $V_1, V_2 \subseteq V(G)$ are adjacent in $G$, if there exist $v_1 \in V_1$ and $v_2 \in V_2$ such that $\{v_1, v_2\} \in E(G)$.

**Definition 2 (Square of a Graph).** For any graph $G$, the square of $G$, denoted by $G^2$, is the graph on the same vertex set as $G$, such that there is an edge between a pair of vertices $u$ and $v$ if and only if they are adjacent in $G$ or are adjacent to a common vertex in $G$.

**Definition 3 (Clique).** Any $C \subseteq V(G)$ is called a clique of $G$ if there is an edge in $G$ between every pair of vertices in $C$. We use $\omega(G)$ to denote the size of the largest clique in $G$.

**Definition 4 (Coloring, Proper Coloring, Optimal Coloring, Chromatic Number).** A coloring of graph $G$ is defined as a mapping from $V(G)$ to a set of colors. A coloring of graph $G$ is called a proper coloring if no two adjacent vertices have the same color in it. An optimal coloring of graph $G$ is any proper coloring of $G$ that minimizes the number of colors used. The Chromatic number of $G$ (denoted by $\chi(G)$) is defined as the number of colors used by an optimal coloring of $G$.

For any coloring $\mu$ of $G$ and any $S \subseteq V(G)$, we use $\mu(S)$ to denote $\{\mu(v) : v \in S\}$.

**Definition 5 (2-tree).** A 2-tree is a graph that can be constructed by starting with an edge and doing the following operation a finite number of times: Pick an edge $e = \{u, v\}$ in the current graph, introduce a new vertex $w$ and add edges $\{u, w\}$ and $\{v, w\}$.

**Definition 6 (Edge Contraction).** The operation of contraction of an edge $e = \{u, v\}$ is defined as follows: the vertices $u$ and $v$ are deleted and a new vertex $v_c$ is added to the graph. Edges are added between $v_c$ and all the vertices that were adjacent to at least one of $u$ and $v$. 
Definition 7 (Minor). A graph $H$ is called a minor of a graph $G$ if $H$ can be obtained from $G$ using any sequence of the following operations:
1. Deleting a vertex;
2. Deleting an edge;
3. Contracting an edge.

An equivalent definition of minors is as follows.

Definition 8 (Minor, Branch Sets). A graph $H$ with $V(H) = \{h_1, h_2, \ldots, h_n\}$ is said to be a minor of $G$ if there exists $S_1, S_2, \ldots, S_n \subseteq V(G)$ such that
1. for all $1 \leq i \leq n$, $G[S_i]$ is connected,
2. for all $i \neq j$, $S_i \cap S_j = \emptyset$ and
3. $S_i$ is adjacent to $S_j$ in $G$ if $\{h_i, h_j\} \in E(H)$.

The sets $S_1, S_2 \ldots S_n$ are called the branch sets of the minor $H$ of $G$.

Definition 9 (Clique Minor, Hadwiger Number). A clique minor of $G$ is defined as a minor of $G$ that is a clique. A clique minor of size $k$ of $G$ is defined as a minor of $G$ that is a $k$-clique. The Hadwiger Number of $G$ is the largest $k$ such that $G$ has a clique minor of size $k$. We denote the Hadwiger number of $G$ by $\eta(G)$.

Note that the necessary and sufficient conditions for $S_1, S_2, \ldots S_n \subseteq V(G)$ to be the branch sets of a clique minor of $G$ are that $G[S_i]$ is connected for all $i$, $S_i \cap S_j = \emptyset$ for all $i \neq j$, and $S_i$ is adjacent to $S_j$ for all $i \neq j$.

2 Reduction to Square Graphs of Split Graphs

Theorem 1. If Hadwiger’s conjecture is shown to be true for the class of graphs that can be represented as the square of some split graph, then Hadwiger’s conjecture is true for the general case.

Proof. Let $G$ be an arbitrary graph. We assume that $G$ has no isolated vertices since they do not affect chromatic number or Hadwiger’s number. We will construct a split graph $H$ from $G$, such that if Hadwiger’s conjecture is true for the square of $H$, then it is also true for $G$. By definition, the vertex set of the split graph $H$ can be split into two classes, say $C$ and $S$, where $C$ induces a clique and $S$ induces an independent set. We will make $C$ correspond to $E(G)$, in the sense that for each edge $e \in E(G)$ we have a vertex in $C$: say for $e \in E(G)$, $v_e \in C$. We make $S = V(G)$ and each vertex $x \in S$ is made adjacent to all the vertices of $C$ that correspond to the edges incident on $x$; i.e., for $v \in S$, $N_H(v) = \{v_e \in C : e$ is an edge in $G$ incident on $v\}$. Here, $N_H(v)$ denotes the neighborhood of $v$ in $H$.

In the square of $H$, the induced subgraph on $S$ is exactly the graph $G$. The reason is this: Let $x, y$ be two vertices in $S$ (i.e. $V(G)$). They are adjacent in the square of $H$ if and only if there exists a common neighbor for $x$ and $y$ in $H$. 
Clearly, this common neighbor has to be from \( C \), and therefore has to correspond to a common incident edge on \( x \) and \( y \). This is possible only if there is an edge between \( x \) and \( y \) in \( G \). This means that in the square of \( H \), \( x \) and \( y \) are adjacent if and only if \( x \) and \( y \) are adjacent in \( G \).

Also, the vertex \( x \in S \) is connected to all the vertices of \( C \) in the square of \( H \), since \( C \) is a clique in \( H \).

The chromatic number of the square of \( H \) equals \( \chi(G) + |C| \). To see that \( \chi(H^2) \leq \chi(G) + |C| \), use the following coloring for \( H^2 \): color the vertices of \( C \) with \( |C| \) different colors and then color \( S \) using an optimal coloring of \( G \). This is possible since the subgraph induced in \( H^2 \) on \( S \) is the same as \( G \). Now, suppose \( \chi(H^2) < \chi(G) + |C| \). Then, since \( C \) requires \( |C| \) different colors and they should all be different from any color in \( S \), we get that \( S \) was colored with fewer colors than \( \chi(G) \). But then we could color \( G \) using this coloring of \( S \) to get a proper coloring of \( G \) with less than \( \chi(G) \) colors, which is a contradiction.

Finally, the biggest clique minor in the square of \( H \) has exactly \( |C| + \eta(G) \) vertices. It is easy to see that a clique minor of that size exists: Consider the branch sets of \( G \) corresponding to the biggest clique minor of \( G \) in the induced subgraph on \( S \); then consider each vertex of \( C \) as a separate branch set. Clearly these branch sets produce a clique minor of size \( \eta(G) + |C| \) in the square of \( H \). Now if a larger clique minor exists, then let \( B_1, B_2, \ldots, B_k \) be the corresponding branch sets. Define \( B'_i = B_i \), if \( B_i \cap C = \emptyset \), else let \( B'_i = B_i \cap C \). It is easy to see that \( B'_1, B'_2, \ldots, B'_k \) also can produce a clique minor of size \( k \) in the square of \( H \). Thus, if \( k > |C| + \eta(G) \), then there should be more than \( \eta(G) \) branch sets that does not intersect with \( C \); which means that \( G \) has a clique minor of size greater than \( \eta(G) \), contradicting the definition of \( \eta(G) \).

Therefore, if Hadwiger’s conjecture is true for the square of \( H \), we have \( \eta(G) + |C| \geq \chi(G) + |C| \), which implies that \( \eta(G) \geq \chi(G) \). In other words, Hadwiger’s conjecture will be true for \( G \) also.

### 3 Hadwiger’s Conjecture for Square Graphs of 2-Trees

By definition, any 2-tree can be constructed by starting with an edge and doing the following operation a finite number of times: Pick an edge \( e = \{u, v\} \), introduce a new vertex \( w \) and add edges \( \{u, w\} \) and \( \{v, w\} \). We call each of these operations, a step in the construction. If \( e = \{u, v\} \) is the edge picked and \( w \) is the newly introduced vertex in a step, then we say that \( e \) is getting processed in that step and that \( w \) is a vertex-child of \( e \). We also say that each of \( \{u, w\} \) and \( \{v, w\} \) is an edge-child of \( e \) and that \( e \) is the parent of each of \( w, \{u, w\} \) and \( \{v, w\} \).

And also, \( \{u, w\} \) and \( \{v, w\} \) are called siblings of each other. Note that an edge \( e \) can be processed in more than one step. But, without loss of generality, we can assume that all the steps in which \( e \) is processed occur contiguously. Now, for each edge and vertex we define a level inductively. We define the level of the first edge and its end points to be 0. Any vertex-child or edge-child of an edge of level \( k \) is said to have level \( k + 1 \). Observe that two edges that are siblings of each other have the same level. Without loss of generality, we also assume that
the order of processing of edges follows a **breadth-first ordering**, i.e., an edge of level $i$ will be processed before an edge of level $j$, if $i < j$.

**Theorem 2.** For any 2-tree $T$, $\chi(T^2) \leq \eta(T^2)$. Moreover, $T^2$ has a clique minor of size $\chi(T^2)$ where all the branch sets are paths.

In the rest of this section, we prove Theorem 2. We will prove by induction on $\chi(T^2)$. We know that $\chi(T^2) \geq 2$. So, we take the base case as when $\chi(T^2) = 2$. Since $T^2$ has an edge, we get that $\eta(T^2) \geq 2 = \chi(T^2)$. And, since all the branch sets are singletons, they are paths. So the base case is done.

Now, consider a 2-tree $T$ with $\chi(T^2) > 2$. In the construction of $T$ as described above, let $T_i$ be the 2-tree resulting after $i$th step. Consider the step $j$ such that $\chi(T^2) = \chi(T_j^2) = \chi(T_{j-1}^2) + 1$. In the rest of the proof, we will prove that $\chi(G^2) \leq \eta(G^2)$ where $G = T_j$, and also that $G^2$ has a clique minor of size $\chi(G^2)$ where each branch set is a path.

Let $l_{\text{max}}$ be the level of the edge with the largest level in $G$. Note that the largest level of any vertex in $G$ is also $l_{\text{max}}$. Also, observe that the level of the last edge processed is $l_{\text{max}} - 1$. None of the edges that have level $l_{\text{max}}$ have been processed due to the breadth-first ordering of the processing of edges. If $l_{\text{max}} \leq 1$, then $G^2$ is a clique and hence $\chi(G^2) = \omega(G^2)$. Moreover, we have a clique minor of size $\chi(G^2)$ where each branch set is a singleton set. Hence, we can assume that $l_{\text{max}} > 1$ for the rest of the proof.

For any vertex $a$, we let $N(a)$ denote the neighbors of $a$ in $G$. $N(a)$ does not include $a$. $N[a]$ denotes $N(a) \cup a$. For a vertex set $X$, we define $N(X)$ as $\bigcup_{x \in X} N(x)$. We also define $N^2[a] = N[N[a]]$ for vertex $a$ and $N^2[X] = N[N[X]]$ for vertex set $X$. $N^2(a)$ is defined as $N^2(a) \setminus a$ for a vertex $a$ and for set $X$, $N^2(X) = N^2[X] \setminus X$.

**Lemma 1.** There exists an optimal coloring $\mu$ of $G^2$ and a vertex $p$ such that $\text{level}(p) = l_{\text{max}}$ and $p$ is the only vertex with color $\mu(p)$.

*Proof.* Let $v$ be the vertex introduced in step $j$. There exists a coloring $\mu'$ of $T_{j-1}^2$ using $\chi(G^2) - 1$ colors from the definition of $G$ and $T_{j-1}$. $\mu'$ together with a new color for $v$ gives the required coloring of $G^2$. $\square$

We fix a coloring $\mu$ and a vertex $p$ as given in lemma 1 such that the level of the vertex with smallest level in $N(p)$ is as large as possible. We call $p$ as the pivot vertex and $\mu$ as the **pivotal coloring**. From now on, when we say the color of a vertex, we mean the color of the vertex under the coloring $\mu$, unless stated otherwise.

**Lemma 2.** All colors of $\mu$ are present in $N^2(p)$ where $p$ is the pivot vertex.

*Proof.* If there was a color $c$ of $\mu$ that was not in $N^2(p)$, then we can re-color $p$ with $c$. Since $p$ was the only vertex with color $\mu(p)$, we then get a coloring of $G^2$ with $\chi(G^2) - 1$ colors which is a contradiction. $\square$
Let \{u, w\} be the parent of pivot \(p\). Since \(l_{\text{max}} > 1\), we can assume without loss of generality that there is a vertex \(t\) such that \(w\) is a child of \{u, t\}. This implies that \{u, w\} and \{w, t\} are siblings and have level equal to \(l_{\text{max}} - 1\). Also, \text{level} (\{u, t\}) = l_{\text{max}} - 2. Let \(B\) be the set of all children of \{w, t\} and \(C\) be that of \{u, w\}.

Fig. 1. The figure shows different vertex sets of 2-tree \(G\) that we use in the proof.

**Lemma 3.** For any vertex \(b \in B\), \(N(b) = \{w, t\}\). And for any vertex \(c \in C\), we have \(N(c) = \{u, w\}\).

**Proof.** The first statement is because level of edges \{b, w\} and \{b, t\} are both \(l_{\text{max}}\) and hence have not been processed. The second statement also follows similarly. \(\square\)

Let \(F\) be \((N(u) \cap N(t)) \setminus w\). Let \(C_1\) be defined as \(\{v \in N(t) | \mu(v) \in \mu(C)\}\) and \(A\) be defined as \(N(t) \setminus (B \cup F \cup C_1 \cup \{u, w\})\).

**Lemma 4.** \(\mu(A) \subseteq \mu(N(u) \setminus (C \cup F \cup \{w, t\}))\).

**Proof.** Consider a color \(c = \mu(a)\) for some \(a \in A\). Clearly, \(c \notin \mu(N^2(a))\). We know that \(c \in \mu(N^2[p])\) by lemma 2. Therefore, \(c \in \mu(N^2[p] \setminus N^2(a))\). But \(N^2[p] \setminus N^2[a] \subseteq N(u) \setminus (F \cup \{w, t\})\). Therefore \(c \in \mu(N(u) \setminus (F \cup \{w, t\}))\). Also, \(c \notin \mu(C)\) by definition of \(A\). Therefore \(c \in \mu(N(u) \setminus (C \cup F \cup \{w, t\}))\). \(\square\)
By lemma\ref{lem:happy} for each color $c \in \mu(A)$, there is a $c$-colored vertex in $N(u)$. Note that there cannot be more than one $c$-colored vertex in $N(u)$. Let $A' \subseteq N(u)$ be such that $\mu(A') = \mu(A)$. For each $a' \in A'$, let couple $(a')$ be defined as the vertex $a \in A$ with $\mu(a) = (a')$. Similarly, for each $a \in A$, let couple $(a)$ be defined as the vertex $a' \in A'$ with $\mu(a') = (a)$. Note that since $A$ and $A'$ are disjoint, a vertex and its couple are always distinct. Let $D$ be defined as $\{x \in B \mid \mu(x) \notin \mu(N(u))\}$ and let $Q = B \setminus D$. We also define $Q' = N(u) \setminus (A' \cup C \cup F \cup \{w, t\})$. Note that $A', A, F, Q', Q, D, C, C_1$ and $\{u, w, t\}$ are all disjoint with each other.

Lemma 5. If $D = \emptyset$, then $\chi(G^2) \leq \eta(G^2)$. Moreover, $\chi(G^2) = \omega(G^2)$ and hence $G^2$ has a clique minor of size $\chi(G^2)$ where each branch set is a singleton set.

Proof. If $D = \emptyset$, then $N^2[p] = N[u] \cup Q$. Therefore, $N[u] \cup Q$ has all colors of $\mu$ by lemma\ref{lem:happy}. But by definition of $Q$, $\mu(Q) \subseteq \mu(N[u])$. Therefore, $N[u]$ has all colors of $\mu$. But $N[u]$ is a clique in $G^2$. Hence, $\chi(G^2) = \omega(G^2) \leq \eta(G^2)$.

Due to lemma\ref{lem:happy} for the rest of the proof we can assume that $D \neq \emptyset$.

Lemma 6. For any $d \in D$, there is no other vertex in $N^2[p]$ with color $\mu(d)$.

Proof. Suppose there is a vertex other than $d$ in $N^2[p]$ with color $\mu(d)$. Such a vertex has to be in $N^2[p] \setminus N^2[d]$. Observe that $N^2[p] \setminus N^2[d] = Q' \cup A'$. $\mu(d) \notin \mu(Q')$ by definition of $D$. And $\mu(d) \notin \mu(A') = \mu(A)$ as $A \subseteq N^2[d]$.

Lemma 7. $\mu(Q') = \mu(Q)$.

Proof. First, we prove $\mu(Q) \subseteq \mu(Q')$. From the definition of $Q$, $\mu(Q) \subseteq \mu(N(u))$. We know that $\mu(Q) \cap \mu(N^2(Q')) = \emptyset$. Also, $\mu(Q) \cap \mu(A') = \emptyset$ as $\mu(A') = \mu(A)$. Hence, $\mu(Q) \subseteq \mu(N(u) \setminus (N^2(Q) \cup A'))). But, $N(u) \setminus (N^2(Q) \cup A') = Q'$.

Therefore, $\mu(Q) \subseteq \mu(Q')$.

Now, we will prove that $\mu(Q') \subseteq \mu(Q)$. Suppose for the sake of contradiction that there exists a vertex $e' \in Q'$ such that $\mu(e') \notin \mu(Q)$. Let $d$ be a vertex in $D$. Such a vertex exists due to lemma\ref{lem:happy}. Observe that $\mu(e') \notin N^2(d)$. This is because $N^2(d) \setminus N^2[e'] \subseteq Q \cup A \cup D \cup C_1$. $\mu(e') \notin \mu(Q)$ by assumption: $\mu(e') \notin \mu(C_1) \subseteq \mu(C)$; $\mu(e') \notin \mu(D)$ by definition of $D$ and $\mu(e') \notin \mu(A) = \mu(A')$. So we can recolor $d$ with $\mu(e')$. Now we can recolor $p$ with $\mu(d)$ due to lemma\ref{lem:happy}.

But now we get a coloring of $G^2$ with $\chi(G^2) - 1$ colors, which is a contradiction.\hfill \square

Lemma 8. If $A = \emptyset$, then $\chi(G^2) \leq \eta(G^2)$. Moreover, $\chi(G^2) = \omega(G^2)$ and hence $G^2$ has a clique minor of size $\chi(G^2)$ where each branch set is a singleton set.

Proof. If $A = \emptyset$, then $A' = \emptyset$ and hence $\mu(N^2[p]) = \mu(N[w] \cup F)$. From lemma\ref{lem:happy} we have that $|\mu(N^2[p])| = \chi(G^2)$. It is easy to see that $N[w] \cup F$ is a clique in $G^2$ and hence we get $|\mu(N[w] \cup F)| \leq \omega(G^2)$. Therefore, $\chi(G^2) = \mu(N^2[p]) = \mu(N[w] \cup F) \leq \omega(G^2) \leq \eta(G^2)$.

\hfill \square
Due to lemma 8 we assume that $A$ is not empty for the rest of the proof. Note that this also implies that $A'$ is not empty.

**Lemma 9.** $l_{\max} > 2$. (*Note that we assume here that $A'$ and $D$ are not empty.*)

**Proof.** Recall that we can assume $l_{\max} > 1$. Suppose $l_{\max} = 2$ for the sake of contradiction. Consider vertices $a' \in A'$ and $d \in D$. They exist because $A' \neq \emptyset$ and $D \neq \emptyset$ (due to lemmas 5 and 8). Since $l_{\max} = 2$, $\{u, t\}$ is the unique edge with level 0. Moreover, all vertices of $G$ should be in $N[\{u, t\}]$. We prove that there is no vertex in $N^2[a']$ which has the color as $\mu(d)$ in the coloring $\mu$. Suppose there was such a vertex $d_1$. We know $d_1 \neq d$ as $a' \notin N(u,t)$. By definition of $D$, $d_1 \notin N(u)$. Also, $d_1 \notin N(t)$ as then there will be 2 vertices in $N(t)$ with the same color. But since, every vertex has to be in $N[\{u,t\}]$, we have a contradiction. Therefore, we have that there is no vertex in $N^2[a']$ which has the color as $\mu(d)$. So, we could recolor $a'$ with color $\mu(d)$ and the coloring will still be proper. Observe that $a'$ was the only vertex in $N^2[p]$ with color $\mu(a')$ in $\mu$ because $N^2[p] \subseteq N^2[a'] \cup N^2(\text{couple}(a'))$ and $\text{couple}(a') \notin N^2[p]$. Since we replaced color of $a'$ with a different color, we can now recolor $p$ with $\mu(a')$ without violating the properness of the coloring. But now we have a proper coloring which uses fewer colors than $\mu$ which is a contradiction to the optimality of $\mu$. \hfill \Box

**Lemma 10.** $\text{level}(u) = l_{\max} - 2$.

**Proof.** Suppose $\text{level}(u) \neq l_{\max} - 2$ for the sake of contradiction. Then, since $\text{level}$ of $\{u,t\}$ is $l_{\max} - 2$, we have that $\text{level}$ of $t$ is $l_{\max} - 2$ and $\text{level}(u) < l_{\max} - 2$. Let $d$ be a vertex in $D$ which exists since $D \neq \emptyset$ (due to lemma 5). Let $\mu'$ be the coloring obtained by exchanging colors of $d$ and $p$. Note that $\mu'$ is proper because of lemma 5. Now, $\mu'$ is a coloring such that $d$ is the only vertex having color $\mu'(d)$ in $\mu'$. Note that the $\text{level}$ of the vertex with smallest $\text{level}$ in $N(d)$ is $l_{\max} - 2$. And the $\text{level}$ of the vertex with smallest $\text{level}$ in $N(p)$ is smaller than $l_{\max} - 2$ since $\text{level}(u) < l_{\max} - 2$ by our assumption. But this means that we would have selected $\mu'$ and $d$ as the pivotal coloring and pivot instead of $\mu$ and $p$, which is a contradiction. \hfill \Box

**Observation 2** Due to lemmas 11 and 9, we can assume that there is a vertex $s \in F$ such that $\{u,t\}$ is the child of $\{s,t\}$. Then, $\text{level}$ of $\{s,t\}$ is $l_{\max} - 3$. Also, $\{s,u\}$ is the sibling of $\{u,t\}$ and hence has level $l_{\max} - 2$. (see figure 4)

**Definition 10.** For any coloring $\phi$ of $G^2$ and any two colors $r$ and $g$, we define a $(\phi, r, g)$-bicolored path as any path in $G^2$ such that all the vertices in the path are colored either $r$ or $g$ under $\phi$.

**Lemma 11.** Consider vertices $a' \in A'$ and $d \in D$. There exists a $(\mu, \mu(a'), \mu(d))$-bicolored path from $a'$ to $\text{couple}(a)$ in $G^2$.

**Proof.** Let $\mu(a') = r$ and $\mu(d) = g$. Let $a = \text{couple}(a')$. Now, consider the induced (bicolored) subgraph $H$ of $G^2$ on all the vertices with colors $r$ and $g$.
in $\mu$. In particular, consider the connected component $H'$ of $H$ containing $a'$. Suppose $H'$ does not contain $a$ for the sake of contradiction. We show that then we can construct a proper coloring $\mu'$ of $G^2$, with fewer colors than $\mu$ which will be a contradiction. For all $v \in V(G) \setminus (V(H') \cup p)$, we set $\mu'(v) = \mu(v)$. For all $v \in V(H')$, we set $\mu'(v) = g$ if $\mu(v) = r$ and $\mu'(v) = g$ otherwise. In other words, for the vertices in $H'$, we exchange the colors. In particular, $\mu'(a') = g$. Finally, we set $\mu'(p) = r$. Clearly, $\mu'$ has fewer colors than $\mu$. It remains to prove that $\mu'$ is a proper coloring of $G^2$. It is easy to see that exchanging colors within $H'$ does not violate the properness of the coloring. So, we only have to prove that there is no $v \in N^2(p)$ with $\mu'(v) = \mu'(p) = r$. Suppose there was such a $v$ for the sake of contradiction. We consider two separate cases, namely, when $v \in V(H')$ and when $v \notin V(H')$. When $v \in V(H')$, $\mu(v) = g$. But then $v = d$ because by lemma $d$ is the only vertex in $N^2[p]$ with color $g$ under $\mu$. But, $d \notin V(H')$ because $a$ is the only vertex in $N^2[d]$ with color $r$ under $\mu$ and $a \notin V(H')$. Hence, we get $v \notin V(H')$ which is a contradiction in this case. Now, let us consider the case when $v \notin V(H')$. In this case $\mu(v) = r$. Observe that in the coloring $\mu$, $a'$ is the only vertex in $N^2[p]$ with color $r$ because $N^2[p] \subseteq N^2[a'] \cup N^2(a)$. But then, $v = a' \in V(H')$ which is a contradiction in this case. \qed

An edge $e_2$ is said to be the edge-descendant of edge $e_1$ if $e_2 = e_1$ or if the parent of $e_2$ is an edge-descendant of $e_1$. A vertex $v$ is said to be a vertex-descendant of edge $e$ if $v$ is the vertex-child of an edge-descendant of $e$.

**Lemma 12.** For an edge $e = \{v_1, v_2\}$ such that $\text{level}(e) = l_{\text{max}} - 2$, if $x$ is a vertex-descendant of $e$, then $N^2(x) \subseteq N\{v_1, v_2\}$.

**Proof.** Let $y$ be an arbitrary vertex in $N^2(x)$. Since $\text{level}(e) = l_{\text{max}} - 2$, there are only 2 possibilities for $x$. The first possibility is that $x$ is a vertex-child of $e$. In this case either $y$ is a vertex-child of $\{v_1, x\}$, or $y$ is a vertex-child of $\{v_2, x\}$. If $y = v_1$ or $y = v_2$ or $y \in N(v_1)$ or $y \in N(v_2)$. In each of these cases $y \in N\{v_1, v_2\}$. The second possibility for $x$ is that $x$ is a vertex-child of an edge-child of $e$. Without loss of generality assume that $x$ is the vertex-child of $\{v_1, v_3\}$ where $v_3$ is a vertex-child of $e$. But now, either $y = v_3$ or $y$ is a vertex-child of $\{v_1, v_3\}$ or $y$ is a vertex-child of $\{v_2, v_3\}$ or $y = v_1$ or $y = v_2$ or $y \in N(v_1)$. In each of these cases $y \in N\{v_1, v_2\}$. \qed

**Lemma 13.** $N^2(A' \cup Q') \subseteq N\{u, t, s\}$.

**Proof.** Consider an $x \in A' \cup Q'$. $x$ is a vertex-descendant of either $\{u, t\}$ or $\{u, s\}$. The level of $\{u, t\}$ and $\{u, s\}$ both are $l_{\text{max}} - 2$ by observation 2. If $x$ is a vertex-descendant of $\{u, t\}$, then by lemma 12, $N^2(x) \subseteq N[u, t]$. If $x$ is a vertex-descendant of $\{u, s\}$, then by lemma 12, $N^2(x) \subseteq N[u, s]$. Therefore, $N^2(x) \subseteq N\{u, t, s\}$. \qed

**Lemma 14.** If $v \in N^2(A' \cup Q')$ and $\mu(v) \in \mu(B)$, then $v \in N\{u, s\}$.

**Proof.** Consider a $v \in N^2(x)$ for some $x \in A \cup Q'$ such that $\mu(v) \in \mu(B)$. By lemma 13 we have that $v \in N\{u, t, s\}$. So, it is sufficient to prove that
For any $v \notin N[t]$. Suppose $v \in N[t]$ for the sake of contradiction. This is possible only if $v \in B$, because otherwise there are 2 vertices in $N[t]$ with the same color. But, $N^2(x) \cap B = \emptyset$, which implies $v \notin N^2(x)$, which is a contradiction. \hfill \Box

**Lemma 15.** If $v \in N^2(A' \cup Q')$ and $\mu(v) \in \mu(D)$, then $v \in N(s)$.

*Proof.* Consider such a $v$. By lemma \ref{lem:11} we have that $v \in N \{u, s\}$. But $v \notin N[u]$ by definition of $D$. \hfill \Box

Let $D'$ be defined as \{ $x \in N(s) \mid \mu(x) \in \mu(D)$ \}.

**Lemma 16.** 1. $\mu(D') = \mu(D)$ and
2. For each $d' \in D'$ and for each $a' \in A'$, there exists a $\mu, \mu(a')$, $\mu(d')$-bicolored path from $a'$ to couple$(a')$ in $G^2$ such that $d'$ is adjacent to $a'$ in this path.

*Proof.* Let $d$ be an arbitrary vertex in $D$. Let $a'_1$ and $a'_2$ be arbitrary vertices in $A'$. By lemma \ref{lem:11} there exists $\mu, \mu(a_1), \mu(d)$-bicolored path from $a'_1$ to couple$(a'_1)$ and $\mu, \mu(a_2), \mu(d)$-bicolored path from $a'_2$ to couple$(a'_2)$. Let these paths be called $P_1$ and $P_2$ respectively. Note that $P_1$ and $P_2$ each have at least 3 vertices. Let $d_1$ be the vertex adjacent to $a'_1$ in $P_1$ and $d_2$ be the vertex adjacent to $a'_2$ in $P_2$. Clearly, $\mu(d_1) = \mu(d_2) = \mu(d)$. By lemma \ref{lem:15} we know that both $d_1$ and $d_2$ are in $N(s)$. This implies $d_1 \in N[d_2]$. This together with $\mu(d_1) = \mu(d_2)$ implies $d_1 = d_2$. Hence, for each $d \in D$, there exists $d' \in N(s)$ with $\mu(d') = \mu(d)$ such that for each $a' \in A'$, there exists a $\mu, \mu(a'), \mu(d)$-bicolored path from $a'$ to couple$(a')$ and $d'$ is adjacent to $a'$ in this path. Both the statements of the lemma easily follow from the previous statement. \hfill \Box

We now extend the definition of couple for the set $D'$. For $d' \in D'$, we define couple$(d')$ as the vertex in $D$ with color the same as $d'$ in $\mu$.

**Corollary 1.** Each $d' \in D'$ is adjacent in $G^2$ to each $a' \in A'$.

**Corollary 2.** For all $d' \in D'$ and $a' \in A'$, there exists a $\mu, \mu(d'), \mu(a')$-bicolored path from $d'$ to couple$(a')$ in $G^2$.

*Proof.* Follows from lemma \ref{lem:16} because couple$(a')$ is adjacent to couple$(d')$ in $G^2$. \hfill \Box

**Definition 11. Bridging-set.** For any $k \geq 0$, $\{q_1, q_2, \ldots, q_k\} \subseteq N(s) \setminus D'$ is called a bridging-set if for each $1 \leq i \leq k$, there exists a vertex $q'_i \in Q'$ such that $\mu(q'_i) = \mu(q_i)$ and $q'_i$ is non-adjacent in $G^2$ to at least one vertex in $D' \cup \{q_1, q_2, \ldots, q_{i-1}\}$. The vertex $q'_i$ is called the bridging-partner of $q_i$ denoted by bp$(q_i)$. Also, we designate one vertex in $D' \cup \{q_1, q_2, \ldots, q_{i-1}\}$ to which $q'_i$ is non-adjacent in $G^2$ as the bridging-non-neighbor of $q'_i$ denoted by bn$(q'_i)$. If there is more than candidate, we fix one of them arbitrarily as the bridging-non-neighbor.

Note that an empty set is a bridging-set. Also, note that for any $q$ in the bridging-set, bp$(q) \neq q$ because otherwise bp$(q)$ is adjacent in $G^2$ to all vertices in $N(s)$ and hence there is no possible candidate for the bridging-non-neighbor of bp$(q)$. This contradicts definition \ref{def:11}

Let $Q_1$ be a bridging-set with maximum cardinality.
Definition 12. Bridging-sequence. For each \( v \in Q_1 \cup D' \), the bridging-sequence of \( v \) is defined as a sequence of distinct vertices \( s_1, s_2, \ldots, s_j \) where \( s_1 = v, s_j \in D' \) and for all \( 2 \leq i \leq j \), \( s_i \) is the bridging-non-neighbor of bridging-partner of \( s_{i-1} \). (From definition \[1\], it is easy to see that such a sequence should exist for all \( v \in Q_1 \cup D' \). Note that for a vertex \( d \in D' \), the bridging-sequence consist of only one vertex, that is \( d \).

Lemma 17. Let \( q \in Q_1 \), \( x = bp(q) \) and \( y = bn(x) \). If there exists \( v \in N^2(x) \) such that \( \mu(v) = \mu(y) \), then \( y \in Q_1 \) and \( v = bp(y) \).

Proof. Since \( \mu(v) = \mu(y) \in \mu(B) \), \( v \) has to be in \( N(\{s, u\}) \) by lemma \[14\]. If \( v \in N(s) \), then \( v = y \), which is not possible as \( y = bn(x) \notin N^2(x) \). Hence, \( v \in N(u) \). Clearly, this means that \( \mu(v) \notin \mu(D') \). Therefore, \( \mu(y) \notin \mu(D') \) and hence \( y \in Q_1 \). Since, the only vertex in \( N(u) \) with color \( \mu(y) \) is \( bp(y) \), we have \( v = bp(y) \).

Definition 13. Bridging-re-coloring. Given any \( z \in Q_1 \cup D' \), we define the bridging-re-coloring of \( \mu \) with respect to \( z \) (denoted by \( \psi_z \)) by the following construction:

1. For all \( x \in V(G) \), initialize \( \psi_z(x) = \mu(x) \).
2. Suppose \( s_1, s_2, \ldots, s_j \) is the bridging-sequence of \( z \). For all \( 1 \leq i < j \), set \( \psi_z(bp(s_i)) = \mu(s_{i+1}) \). (Observe that \( \forall i \neq j, \mu(s_i) \neq \mu(s_j) \) since \( s_i, s_j \in N(s) \) and hence each color is used for recoloring at most once in this step.)

Lemma 18. For all \( z \in Q_1 \cup D' \), \( \psi_z \) is an optimal coloring of \( G^2 \).

Proof. Consider a \( z \in Q_1 \cup D' \). Since \( \psi_z \) only uses colors in \( \mu \), it is clearly optimal if it is a coloring of \( G^2 \). So, we only have to prove that \( \psi_z \) is indeed a proper coloring of \( G^2 \). Let \( s_1, \ldots, s_j \) be the bridging-sequence of \( z \). Suppose \( \psi_z \) is not a proper coloring of \( G^2 \). From the construction of \( \psi_z \), this implies that there exists an \( n \in \{j - 1\} \) such that \( \psi_z(bp(s_i)) \in \psi_z(N^2(bp(s_i))) \). Let \( x = bp(s_i) \). Now, there exists a \( v \in N^2(x) \), such that \( \psi_z(v) = \psi_z(x) = \mu(s_{i+1}) \). Then it should be the case that \( \mu(v) = \mu(s_{i+1}) \) because, \( x \) is the only vertex that has color \( \mu(s_{i+1}) \) in \( \psi_z \) but has a different color in \( \mu \). Since \( s_{i+1} = bn(x) \), we get \( s_{i+1} \in Q_1 \) and \( v = bp(s_{i+1}) \) by lemma \[17\]. But \( \psi_z(bp(s_{i+1})) = \mu(s_{i+2}) \neq \mu(s_{i+1}) \) by our construction of \( \psi_z \). Hence, \( \psi_z(v) \neq \mu(s_{i+1}) \) which is a contradiction.

Lemma 19. Let \( a' \in A', q \in Q_1 \cup D' \), \( r = \mu(a') \), \( g = \mu(q) \), \( V_1 = \{x| \psi_q(x) \in \{r, g\}\} \) and \( V_2 = \{x| \mu(x) \in \{r, g\}\} \). Then, \( V_1 \subseteq V_2 \). (In fact, \( V_2 \setminus V_1 = bp(q) \)).

Proof. Follows from the construction of \( \psi_q \) using the fact that \( r, g \notin \mu(\{s_2, s_3, \ldots, s_j\}) \).

Lemma 20. For all \( q \in Q_1 \) and \( a' \in A' \), there exists a \( (\mu, \mu(a'), \mu(q))-bicolored \) path from \( a' \) to couple\((a') \) in \( G^2 \) such that \( q \) is adjacent to \( a' \) in the path.
Proof. Consider a $q \in Q_1$ and an $a' \in A'$. Let $\mu(a') = r$ and $\mu(q) = g$. Due to lemma 19, it is sufficient to prove that there exists a $(\psi_q, r, g)$-bicolored path from $a'$ to couple$(a')$ in $G^2$ such that $q$ is adjacent to $a'$ in the path. Let $a = \text{couple}(a')$. Now, consider the induced (bicolored) subgraph $H$ of $G^2$ on all the vertices with colors $r$ and $g$ under $\psi_q$. In particular, consider the connected component $H'$ of $H$ containing $a'$.

If $H'$ contains $a$, then there is a $(\psi_q, r, g)$-bicolored path from $a'$ to $a$ in $G^2$. We prove that in this path $a'$ has to be adjacent to $q$. Suppose there was a vertex $v \neq q$ that was adjacent in the path to $a'$. This means $\psi_q(v) = g$ which means $\mu(v) = g$. Then by lemma 14, $v \in N\{u, s\}$. If $v \in N(s)$ then $v = q$, which is a contradiction. Hence, $v \in N(u)$, which implies $v = bq(q)$. But, $\psi_q(bq(q)) \neq g$ by our construction of $\psi_q$. Hence, $\psi_q(v) \neq g$, which is a contradiction.

If $H'$ does not contain $a$, then we show that we can start with $\psi_q$ and recolor some vertices to get a proper coloring of $G^2$ with fewer colors than $\psi_q$. Since $\psi_q$ is an optimal coloring of $G^2$ by lemma 18, this will be a contradiction. For each vertex $v$ in $V(H')$, if $\psi_q(v) = r$ then recolor it with $g$ and otherwise recolor it with $r$. In other words, exchange colors $r$ and $g$ in $V(H')$. Finally, recolor pivot $p$ with $r$. Let the resulting assignment of colors after the recoloring be $\phi$. Recall that $p$ is the only vertex in $G$ with color $\mu(p)$ in $\mu$. By the construction of $\psi_q$, $p$ remains to be the only vertex to have color $\mu(p)$ in $\psi_q$. Hence, $\phi$ has one color less than $\psi_q$ as it does not use the color $\psi_q(p) = \mu(p)$. It only remains to prove that $\phi$ is a proper coloring of $G^2$. It is easy to see that exchanging two colors within $V(H')$ does not violate properness of the coloring. So, we now only have to prove that $r \notin \phi(N^2[p])$. Suppose for the sake of contradiction that there exists a vertex $v \in N^2(p)$ such that $\phi(v) = r$.

If $v \in V(H')$ then, $\psi_q(v) = g$. This means that $v \neq bq(q)$, by the construction of $\psi_q$. Also, $\mu(v) = g$ by lemma 19. The only vertices in $N^2[p]$ with color $g$ under $\mu$ are $bq(q)$ and one vertex in $Q$(say $q'$). But since $v \neq bq(q)$, we have $v = q'$. Now, since $a \in N^2(q')$, we get $a \in V(H')$ which is a contradiction. Hence we are left with the case when $v \notin V(H')$. Then $\psi_q(v) = r$ which implies $\mu(v) = r$. But, the only vertex in $N^2(p)$ with color $r$ is $a'$ which has color $g$ in $\psi_q$. Hence, we have a contradiction and $\phi$ is a proper coloring of $G^2$.

Corollary 3. Each $q \in Q_1$ is adjacent in $G^2$ to each $a' \in A'$.

We now extend the definition of couple to set $Q_1$. For $q \in Q_1$, let couple$(q)$ be defined as the vertex in $Q$ with the same color as $q$ in the coloring $\mu$. Then, we have the following corollary to lemma 20.

Corollary 4. For each $q \in Q_1$ and $a' \in A'$, there is a $(\mu, \mu(a'), \mu(q))$-bicolored path from $q$ to couple$(q)$.

Proof. This follows because couple$(a')$ is adjacent in $G^2$ to couple$(q)$.

Let $Q_2 = \{bq(q) \mid q \in Q_1\}$. Note that $Q_2 \subseteq Q'$ and $\mu(Q_2) = \mu(Q_1)$. Let $Q_3 = Q' \setminus Q_2$. Note that $\mu(Q_3 \cup Q_1) = \mu(Q') = \mu(Q)$.

Lemma 21. For all $q' \in Q_3$, $Q_1 \cup D' \subseteq N^2[q']$. 
Proof. Suppose there is a $z \in Q_1 \cup D'$ such that $z \notin N^2[q']$, for some $q' \in Q_3$. We will consider two separate cases.

First, consider the case when there is a vertex $q \in N(s)$ such that $\mu(q) = \mu(q')$. $q \neq q'$ because otherwise $z \in N^2[q']$. But, then $Q_1 \cup q$ is a bridging-set with $bp(q) = q'$ and $bn(q') = z$. This contradicts that $Q_1$ is a bridging-set with maximum cardinality.

Now, consider the case when $\mu(q') \notin \mu(N(s))$. Let $a'$ be some vertex in $A'$ which exists because $A'$ is not empty (due to lemma[8]). We start with the coloring $\psi_z$ and recolor $q'$ with color $\psi_z(z) = \mu(z)$, $a'$ with color $\psi_z(q') = \mu(q')$ and $p$ with color $\psi_z(a') = \mu(a')$. Let the resulting coloring be $\phi$. $\phi$ clearly has fewer colors than $\psi_z$ because it does not use the color $\psi_z(p)$. Since $\psi_z$ is an optimal coloring of $G^2$ by lemma[18] $\phi$ cannot be a proper coloring of $G^2$. Then, since $\psi_z$ is a proper coloring by lemma[18] one of the following three cases have to happen and in each case we prove a contradiction. Note that during the construction of $\phi$ from $\psi_z$, we have done 3 recolorings using $\mu(z)$, $\mu(q' = bp(q))$ and $\mu(a')$. These 3 colors are different from each other from the definitions of $Q_3, A', Q_1$ and $D'$.

1. $\exists v \in N^2(q')$ such that $\phi(v) = \phi(q') = \mu(z)$.

$q'$ is the only vertex with color $\mu(z)$ in $\phi$ that has a different color in $\psi_z$. Since $v \neq q'$, $\psi_z(v) = \phi(v) = \mu(z)$. Since, $\mu(z)$ is not a color that was recolored to some vertex during the construction of $\psi_z$, we have $\mu(v) = \psi_z(v) = \mu(z)$. Now, by lemma[14] $v \in N(u \cup s)$. If $v \in N(s)$ then $v = z$, which is a contradiction since $z \notin N^2[q']$. Hence, $v \in N(u)$, which implies $v = bp(z)$ because $bp(z)$ is the only vertex in $N(u)$ with color $\mu(z)$ under the coloring $\mu$. But, $\phi(bp(z)) = \psi_z(bp(z)) = \mu(bp(z)) \neq \mu(z)$, which is a contradiction.

2. $\exists v \in N^2(a')$ such that $\phi(v) = \phi(a') = \mu(q')$.

$a'$ is the only vertex with color $\mu(q')$ in $\phi$ that has a different color in $\psi_z$. Since $v \neq a'$, $\psi_z(v) = \phi(v) = \mu(q')$. Since, $\mu(q')$ is not a color that was recolored to some vertex during the construction of $\psi_z$, we have $\mu(v) = \psi_z(v) = \mu(q')$. Now, by lemma[14] $v \in N(u \cup s)$, $v \notin N(s)$ because $\mu(q') \notin \mu(N(s))$ by assumption. So, $v \in N(u)$ which means $v = q'$. But, $\phi(q') = \mu(z) \neq \mu(q')$ where the last non-equality follows because $\mu(q') \notin \mu(N(s))$. Hence, we have a contradiction.

3. $\exists v \in N^2(p)$ such that $\phi(v) = \phi(p) = \mu(a')$. $p$ is the only vertex with color $\mu(a')$ in $\phi$ that has a different color in $\psi_z$. Since $v \neq p$, $\psi_z(v) = \phi(v) = \mu(a')$. Since, $\mu(a')$ is not a color that was recolored to some vertex during the construction of $\psi_z$, we have $\mu(v) = \psi_z(v) = \mu(a')$. $a'$ is the only vertex in $N^2(p)$ with color $\mu(a')$ in $\mu$. But, $\phi(a') = \mu(q') \neq \mu(a')$, which is a contradiction.

$\square$

Let $a_1', a_2', \ldots, a_n'$ be the vertices in $A'$ and $z_1, z_2, \ldots, z_n$ be the vertices in $D' \cup Q_1$. If $n_a \leq n_z$, then we define vertex disjoint paths $P_1, P_2, \ldots, P_n$ such that each $P_i$ is a ($\mu, \mu(a_i'), \mu(z_i)$)-bicolored path from $a_i$ to couple($a_i$). Such a path exists for each $i$ due to lemmas[16] and [20]. If $n_z < n_a$, then we define vertex disjoint paths $P_1, P_2, \ldots, P_n$ such that each $P_i$ is a ($\mu, \mu(a_i), \mu(z_i)$)-bicolored path from $z_i$ to
couple($z_i$). These paths exist due to corollaries 2 and 4. Note that in both cases, the paths are vertex disjoint with each other because the color of the vertices in $P_i$ and $P_j$ are disjoint for $i \neq j$.

In the case when $n_a \leq n_z$, we define $B$ as the set of following branch sets: each vertex in $N[w]$ as a singleton branch set, each vertex in $F$ as a singleton branch set, and path branch sets $V(P_i)$ for $1 \leq i \leq n_a$. In the case when $n_z < n_a$, we define $B$ as the set of following branch sets: each vertex in $N[u] \setminus Q_2$ as a singleton branch set and the path branch sets $V(P_i)$ for $1 \leq i \leq n_z$. In both cases, it is easy to see that all the branch sets are connected to each other in $G^2$ and hence forms a clique minor of $G^2$. We prove this in lemma 23. Since the paths $P_i$ are vertex disjoint with each other, all branch sets in $B$ are disjoint with each other. This completes the proof of Theorem 2.

**Lemma 22.** Each pair of branch sets in $B$ are adjacent to each other in $G^2$.

**Proof.** First, let us consider the case when $n_a \leq n_z$. All vertices in $N(w)$ are adjacent in $G^2$ to each other and to vertices in $F$. All vertices in $F$ are adjacent to each other in $G^2$. Hence, the singleton branch sets are all adjacent to each other. For all $1 \leq i \leq n_a$, each vertex in $N(w) \cup F$ is adjacent in $G^2$ to $a'_i$ or couple($a'_i$). Hence each singleton branch set is adjacent to each path branch set. For all $1 \leq i, j \leq n_a$, $a'_i \in N^2(a'_j)$ and thus the branch sets $V(P_i)$ and $V(P_j)$ are adjacent. Hence, the lemma follows in this case.

Now, let us consider the case when $n_z < n_a$. All vertices in $N(u) \setminus Q_2$ are adjacent in $G^2$ to each other. Hence, the singleton branch sets are adjacent to each other. All vertices in $N(u) \setminus (Q_2 \cup Q_3 \cup A')$ are adjacent in $G^2$ to couple($z_i$) for all $1 \leq i \leq n_z$. By corollaries 3 and 4, all vertices in $A'$ are adjacent in $G^2$ to $z_i$ for all $1 \leq i \leq n_z$. By lemma 21, all vertices in $Q_3$ are adjacent in $G^2$ to $z_i$ for all $1 \leq i \leq n_z$. Hence, each singleton branch set is adjacent to each path branch set. Since $z_i \in N(s) \forall 1 \leq i \leq n_z$, we have that the path branch sets are adjacent to each other. Hence, the lemma is true in this case also.

**Lemma 23.** $|B| \geq \chi(G^2)$

**Proof.** Recall that all colors in $\mu$ are present in $\mu(N^2[p])$ by lemma 2. First, let us consider the case when $n_a \leq n_z$. All colors in $\mu(N^2[p])$ except $\mu(A)$ are present in $N(w) \cup F$, which form the singleton branch sets. But we have a path branch set $V(P_i)$ for all $1 \leq i \leq n_a$. Hence $|B| \geq |N^2[p]| - |\mu(A)| = \chi(G^2) - n_a + n_a = \chi(G^2)$.

Now, let us consider the case when $n_z < n_a$. All colors in $\mu(N^2[p])$ except $\mu(Q_1 \cup D')$ are present in $N(u) \setminus Q_2$, which form the singleton branch sets. But we have a path branch set $V(P_i)$ for all $1 \leq i \leq n_z$. Hence $|B| \geq |N^2[p]| - |\mu(Q_1 \cup D')| + n_z = \chi(G^2) - n_z + n_z = \chi(G^2)$. 

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A An example of a 2-tree with $\chi(G^2) = (1.5 - \varepsilon) \omega(G^2)$

Fig. 2. The figure shows a 2-tree $G$ with $\omega(G^2) = 2k + 5$ and $\chi(G^2) = 3k + 3$.

In this paper, we showed that for any 2-tree $G$, $G^2$ has a clique minor of size $\chi(G^2)$. It is natural to ask whether a better result is possible, i.e., whether $G^2$ has a clique of size $\chi(G^2)$\footnote{This does not mean that we are asking whether $G^2$ is perfect. In fact, it is possible to have $\chi(G^2) = \omega(G^2)$, while for some induced subgraph of $G^2$, the clique number is strictly less than chromatic number.}. This question is also relevant from the point of view of another well studied conjecture, namely Wegner’s conjecture.

\textit{Conjecture 2 (Wegner’s Conjecture}[Weg77].\textit{ For a planar graph }$G$\textit{ with maximum degree }$\Delta$,

\[
\chi(G^2) \leq 7, \quad \text{if } \Delta = 3
\]
\[
\leq \Delta + 5, \quad \text{if } 4 \leq \Delta \leq 7
\]
\[
\leq \frac{3\Delta}{2} + 1, \quad \text{if } \Delta \geq 8
\]

Lih, Wang and Zhu [LWZ03] showed that for a $K_4$-minor free graph $G$ with maximum degree $\Delta \geq 4$, $\chi(G^2) \leq \frac{3\Delta}{2} + 1$. Since 2-trees are $K_4$-minor free, the bound holds for them. Using that $\omega(G^2) \geq \Delta(G)$, we can infer that

\[
\chi(G^2) \leq \frac{3\omega(G^2)}{2} + 1 \quad \text{(1)}
\]
The question that arises now is the following: Can we improve the above bound for 2-trees. In particular, can we show that \( \chi(G^2) \leq \omega(G^2) \). This would then imply Hadwiger’s Conjecture for squares of 2-trees but would have been a much stronger result. However, the example given in figure 2 shows that such an improvement is not possible. For any \( k \geq 1 \), we can construct a graph as shown in figure 2. It is easy to verify that the chromatic number of this graph is \( 3k + 3 \) and the clique number is \( 2k + 5 \). Therefore, for this family of graphs, (1) is almost tight.