Progress and prospects of lattice supersymmetry

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Nonperturbative and Numerical Approaches to Quantum Gravity, String Theory and Holography

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Overview and plan

Significant progress currently being made in lattice studies of supersymmetric QFTs

Motivation and background

Special cases: $\mathcal{N} = 1$ super-Yang–Mills; Matrix models

Lattice $\mathcal{N} = 4$ super-Yang–Mills

- Formulation highlights
- Dimensionally reduced (2d & 3d) thermodynamics
- 4d static potential & scaling dimensions

Prospects and future directions
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Prospects and future directions

These slides: davidschaich.net/talks/2101Bangalore.pdf

Interaction encouraged — complete coverage unnecessary
Motivations

Lattice field theory promises first-principles predictions for strongly coupled supersymmetric QFTs

BSM

QFT

Holography
Formally \[ \langle O \rangle = \frac{1}{Z} \int D\Phi \ O(\Phi) \ e^{-S[\Phi]} \]

Regularize by formulating theory in finite, discrete, euclidean space-time
Background: Lattice field theory in a nutshell

Formally

\[ \langle O \rangle = \frac{1}{Z} \int D\Phi \ O(\Phi) \ e^{-S[\Phi]} \]

Regularize by formulating theory in finite, discrete, euclidean space-time

\[ \text{Gauge invariant, non-perturbative, } d\text{-dimensional} \]

Spacing between lattice sites ("a")

\[ \rightarrow \text{UV cutoff scale } 1/a \]

Remove cutoff:

\[ a \rightarrow 0 \ (L/a \rightarrow \infty) \]

Discrete \[ \rightarrow \text{continuous symmetries } \checkmark \]
Numerical lattice field theory calculations

High-performance computing
→ evaluate up to
∼ billion-dimensional integrals

(Dirac op. as ∼ $10^9 \times 10^9$ matrix)

Importance sampling Monte Carlo

Algorithms sample field configurations with probability $\frac{1}{\mathcal{Z}} e^{-S[\Phi]}$

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \; \mathcal{O}(\Phi) \; e^{-S[\Phi]} \rightarrow \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\Phi_i) \text{ with stat. uncertainty } \propto \frac{1}{\sqrt{N}}$$
Supersymmetry must be broken on the lattice

Supersymmetry is a space-time symmetry, adding spinor generators $Q^I_\alpha$ and $\overline{Q}^I_\dot{\alpha}$ to translations, rotations, boosts

\[ \{ Q^I_\alpha, \overline{Q}^I_\dot{\alpha} \} = 2\delta^{IJ} \sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu \]

broken in discrete space-time

$\rightarrow$ relevant susy-violating operators
Supersymmetry must be broken on the lattice

\[
\left\{ Q^I_\alpha, \overline{Q}^I_{\dot{\alpha}} \right\} = 2 \delta^{IJ} \sigma^\mu_{\alpha\dot{\alpha}} P_\mu \quad \text{broken in discrete space-time}
\]

\[\rightarrow \text{relevant susy-violating operators}\]

Simplifications have helped enable significant recent progress

- Avoid scalars
- Reduce dimensions
- Maximize symmetries

Scalar mass  Yukawas  Quartics  Quark mass  Gluino mass
Significant progress currently being made in lattice studies of supersymmetric QFTs

✓ Motivation and background — Questions?

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Prospects and future directions
Avoid scalars: $\mathcal{N} = 1$ super-Yang–Mills

SU$(N)$ gauge theory with single massless Majorana fermion in adjoint rep.

Straightforward lattice fermion formulations explicitly break chiral symmetry

\[ \rightarrow \text{large additive gluino mass renormalization} \]

Chiral (‘domain wall’ or ‘overlap’) lattice fermions numerically expensive
Avoid scalars: $\mathcal{N} = 1$ super-Yang–Mills

SU($N$) gauge theory with single massless Majorana fermion in adjoint rep.

1) Fine-tune gluino mass $\rightarrow$ supersymmetry in chiral continuum limit

2) Domain wall or overlap fermions
   $\rightarrow$ automatic (accidental) supersymmetry in continuum limit
Selected recent progress fine-tuning gluino mass

Scalar, pseudoscalar and fermionic partner approach degenerate supermultiplet for massless gluino

Smaller lattice spacing ‘a’ (larger $\beta$) $\rightarrow$ improved supermultiplet formation

Desy–Münster–Regensburg–Jena, arXiv:2001.09682

\[ \beta = 5.4 \]

\[ \beta = 5.6 \]

arXiv:2001.09682
Selected recent progress fine-tuning gluino mass

Measure of supersymmetry breaking from Ward identities
vanishes in chiral continuum limit, $a^2 \to 0$
Desy–Münster–Regensburg–Jena, arXiv:2003.04110

Extrapolation consistent with $O(a^2)$ discretization artifacts expected for this lattice action
Selected recent progress fine-tuning gluino mass

Alternate ‘twisted-mass’ action provides extra ‘twist angle’ parameter

→ tune this to improve approach to continuum limit

Steinhauser–Sternbeck–Wellegehausen–Wipf, arXiv:2010.00946
Recent progress with overlap $\mathcal{N} = 1$ super-Yang–Mills

$N$-order polynomial approximation to compute $\text{sign}(D) = \frac{D}{\sqrt{D \cdot D}}$ for large matrix

Piemonte–Bergner–López, arXiv:2005.02236

Bare gluino condensate from $12^4$ lattices

$N < \infty \rightarrow$ non-zero ‘residual mass’

but only multiplicative renormalization
Reduce dimensions: Supersymmetric quantum mechanics

Ultimate simplification — compactify all spatial dimensions

4d SU($N$) super-Yang–Mills (SYM) $\rightarrow$ quantum mechanics of $N \times N$ matrices

Both lattice and non-lattice numerical approaches viable

More on this topic from Jun Nishimura tomorrow

16-supercharge theories motivated by holography
Thermodynamics of maximal SYM $\leftrightarrow$ black holes in string theory
Testing holography with lattice super-Yang–Mills QM

Predict corrections to SUGRA result through large-\( N \) continuum extrapolations
Monte Carlo String/M-Theory Collaboration, \texttt{arXiv:1606.04951}

\[ 16 \leq N \leq 32 \quad \text{MMMM code parallelizes individual } N \times N \text{ matrices} \]
Recent progress: Supersymmetric mass deformation

Berenstein–Maldacena–Nastase, hep-th/0202021

Deform SYM QM while preserving maximal supersymmetry

→ more interesting features including phase transition at critical $T/\mu$

DS–Jha–Joseph, 2003.01298 & to appear

Phase diagram of critical $T/\mu$

vs. dimensionless coupling $g$

For small $g \lesssim 10^{-3}$, agree with NNLO perturbation theory

Approach leading-order holography as $g$ increases
Checkpoint

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✓ Special cases: $\mathcal{N} = 1$ SYM; Matrix models — Questions?

Lattice $\mathcal{N} = 4$ super-Yang–Mills
  • Formulation highlights
  • Dimensionally reduced (2d & 3d) thermodynamics
  • 4d static potential & scaling dimensions

Prospects and future directions
Maximize symmetries: Lattice $\mathcal{N} = 4$ super-Yang–Mills

Preserve susy sub-algebra in discrete lattice space-time

$\implies$ correct continuum limit with little or no fine tuning

Equivalent constructions from ‘topological’ twisting and dim’l deconstruction

Review:
Catterall–Kaplan–Ünsal,
arXiv:0903.4881

Need $2^d$ supersymmetries in $d$ dimensions

$d = 4 \implies \mathcal{N} = 4$ super-Yang–Mills (SYM)
$\mathcal{N} = 4$ SYM in a nutshell

Arguably simplest non-trivial 4d QFT $\rightarrow$ dualities, amplitudes, ... 

SU($N$) gauge theory with $\mathcal{N} = 4$ fermions $\Psi^I$ and 6 scalars $\Phi^{IJ}$, all massless and in adjoint rep.

**Symmetries** relate coefficients of kinetic, Yukawa and $\Phi^4$ terms

Maximal 16 supersymmetries $Q^I_\alpha$ and $\bar{Q}^I_{\dot{\alpha}}$, $I = 1, \cdots, 4$

transform under global $SU(4) \sim SO(6)$ R symmetry

Conformal $\rightarrow$ $\beta$ function is zero for all values of $\lambda = g^2 N$
Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

\[
\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
\bar{Q}_1^\dot{\alpha} & \bar{Q}_2^\dot{\alpha} & \bar{Q}_3^\dot{\alpha} & \bar{Q}_4^\dot{\alpha}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{Q}_\mu \gamma_\mu \gamma_5 + \bar{Q} \gamma_5 \\
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
\]

with $a, b = 1, \cdots, 5$

R-symmetry index $\times$ Lorentz index $\rightarrow$ reps of ‘twisted rotation group’

\[\text{SO}(4)_{\text{tw}} \equiv \text{diag} \left[ \text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R \right] \quad \text{SO}(4)_R \subset \text{SO}(6)_R\]

Change of variables $\rightarrow$ $Q$s transform with integer ‘spin’ under $\text{SO}(4)_{\text{tw}}$
Twisting $\mathcal{N} = 4$ SYM

Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

\[
\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
\bar{Q}_{1\dot{\alpha}} & \bar{Q}_{2\dot{\alpha}} & \bar{Q}_{3\dot{\alpha}} & \bar{Q}_{4\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \bar{Q}_{\mu} \gamma_\mu \gamma_5 + \bar{Q} \gamma_5 \\
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
\]

with $a, b = 1, \cdots, 5$

Discrete space-time
Can preserve closed sub-algebra

\[
\{ Q, Q \} = 2Q^2 = 0
\]
Intuitive picture — expand $4 \times 4$ matrix of supersymmetries

\[
\begin{pmatrix}
Q_1^\alpha & Q_2^\alpha & Q_3^\alpha & Q_4^\alpha \\
\overline{Q}_1^{\dot{\alpha}} & \overline{Q}_2^{\dot{\alpha}} & \overline{Q}_3^{\dot{\alpha}} & \overline{Q}_4^{\dot{\alpha}}
\end{pmatrix}
= Q + Q_\mu \gamma_\mu + Q_{\mu\nu} \gamma_\mu \gamma_\nu + \overline{Q}_\mu \gamma_\mu \gamma_5 + \overline{Q} \gamma_5
\rightarrow Q + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b
\]
with $a, b = 1, \cdots, 5$

Discrete space-time
Can preserve closed sub-algebra

\[
\{ Q, Q \} = 2Q^2 = 0
\]
Completing the twist

Fields also transform with integer spin under SO(4)$_{tw}$ — no spinors

\[ \psi \text{ and } \overline{\psi} \longrightarrow \eta, \psi_a \text{ and } \chi_{ab} \]

\[ A_\mu \text{ and } \Phi^I \longrightarrow \text{complexified gauge field } A_a \text{ and } \overline{A}_a \]

\[ \longrightarrow \text{U}(N) = \text{SU}(N) \otimes \text{U}(1) \text{ gauge theory} \]

✓ $Q$ interchanges bosonic $\leftrightarrow$ fermionic d.o.f. with $Q^2 = 0$

\[ Q A_a = \psi_a \quad Q \psi_a = 0 \]
\[ Q \chi_{ab} = -\overline{F}_{ab} \quad Q \overline{A}_a = 0 \]
\[ Q \eta = d \quad Q d = 0 \]

\[ \text{bosonic auxiliary field with e.o.m. } d = \overline{D}_a A_a \]
Lattice $\mathcal{N} = 4$ SYM

Lattice theory looks nearly the same despite breaking $Q_a$ and $Q_{ab}$

Covariant derivatives $\rightarrow$ finite difference operators

Complexified gauge fields $A_a \rightarrow$ gauge links $U_a \in \mathfrak{gl}(N, C)$

\[ Q A_a \rightarrow Q U_a = \psi_a \]

\[ Q \chi_{ab} = -\overline{F}_{ab} \]

\[ Q \eta = d \]

\[ Q \psi_a = 0 \]

\[ Q \chi_{ab} = -\overline{F}_{ab} \]

\[ Q \overline{A}_a \rightarrow Q \overline{U}_a = 0 \]

\[ Q d = 0 \]

Geometry: \( \eta \) on sites, \( \psi_a \) on links, etc.

Supersymmetric lattice action \( (QS = 0) \) from \( Q^2 \cdot = 0 \) and Bianchi identity

\[
S = \frac{N}{4\lambda_{\text{lat}}} \text{Tr} \left[ Q \left( \chi_{ab}\overline{F}_{ab} + \eta \overline{D}_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab}\overline{D}_c \chi_{de} \right]
\]
Five links in four dimensions \(\rightarrow\) \(A_4^*\) lattice

\(A_4^* \sim 4d\) analog of 2d triangular lattice

Basis vectors linearly dependent and non-orthogonal

Large \(S_5\) point group symmetry

\(S_5\) irreps precisely match onto irreps of twisted SO(4)\(_{tw}\)

\[
\begin{align*}
\psi_a & \rightarrow \psi_\mu, \ \bar{\eta} \quad \text{is} \quad 5 \rightarrow 4 \oplus 1 \\
\chi_{ab} & \rightarrow \chi_{\mu\nu}, \ \bar{\psi}_\mu \quad \text{is} \quad 10 \rightarrow 6 \oplus 4
\end{align*}
\]

\(S_5 \rightarrow SO(4)_{tw}\) in continuum limit restores \(Q_a\) and \(Q_{ab}\)
Formal formulation features

Analytic results for twisted $\mathcal{N} = 4$ SYM on $A_4^*$ lattice

- $U(N)$ gauge invariance + $Q$ + $S_5$ lattice symmetries
  $\rightarrow$ Moduli space preserved to all orders
  $\rightarrow$ One-loop lattice $\beta$ function vanishes
  $\rightarrow$ Only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$

[arXiv:1102.1725, arXiv:1306.3891, arXiv:1408.7067]

Not yet practical for numerical calculations

Must regulate zero modes and flat directions, especially in $U(1)$ sector
Two deformations stabilize lattice calculations

1) Add SU($N$) scalar potential $\propto \mu^2 \sum_a (\text{Tr} \left[ U_a \overline{U}_a \right] - N)^2$

Softly breaks susy $\rightarrow Q$-violating operators vanish $\propto \mu^2 \rightarrow 0$

Test via Ward identity violations

$Q \left[ \eta U_a \overline{U}_a \right] \neq 0$
Two deformations stabilize lattice calculations

2) Constrain $U(1)$ plaquette determinant $\sim G \sum_{a<b} (\det P_{ab} - 1)$

Implemented supersymmetrically as Fayet–Iliopoulos $D$-term potential

Test via Ward identity violations

$$Q \left[ \eta U_a \overline{U}_a \right] \neq 0$$

Log–log axes

$$\rightarrow \text{violations } \propto (a/L)^2$$
Public code for lattice $\mathcal{N} = 4$ SYM

so that the full improved action becomes

$$ S_{\text{imp}} = S_{\text{exact}}' + S_{\text{closed}} + S_{\text{soft}}' $$

$$ S_{\text{exact}}' = \frac{N}{4\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ -\mathcal{F}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}_{a}^{(+)}\psi_{b}(n) - \eta(n)\mathcal{D}_{a}^{(-)}\psi_{a}(n) \right] + \frac{1}{2} \left( \mathcal{D}_{a}^{(-)}\mathcal{U}_{a}(n) + G \sum_{a \neq b} (\det \mathcal{P}_{ab}(n) - 1) \mathbb{I}_{N} \right)^{2} - S_{\text{det}} $$

$$ S_{\text{det}} = \frac{N}{4\lambda_{\text{lat}}} G \sum_n \text{Tr} [\eta(n)] \sum_{a \neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} [\mathcal{U}_{b}^{-1}(n)\psi_{b}(n) + \mathcal{U}_{a}^{-1}(n + \hat{\mu}_{b})\psi_{a}(n + \hat{\mu}_{b})] $$

$$ S_{\text{closed}} = -\frac{N}{16\lambda_{\text{lat}}} \sum_n \text{Tr} \left[ \epsilon_{abcde} \chi_{dc}(n + \hat{\mu}_{a} + \hat{\mu}_{b} + \hat{\mu}_{c})\mathcal{D}_{c}^{(-)}\chi_{ab}(n) \right], $$

$$ S_{\text{soft}}' = \frac{N}{4\lambda_{\text{lat}}} \mu^{2} \sum_n \sum_{a} \left( \frac{1}{\mathcal{N}} \text{Tr} [\mathcal{U}_{a}(n)\mathcal{U}_{a}(n)] - 1 \right)^{2} $$

$\sim 100$ inter-node data transfers in the fermion operator — non-trivial.

Public parallel code to reduce barriers to entry: github.com/daschaich/susy

Evolved from MILC QCD code, user guide in arXiv:1410.6971
Checkpoint

Significant progress currently being made in lattice studies of supersymmetric QFTs

✓ Motivation and background

✓ Special cases: $\mathcal{N} = 1$ super-Yang–Mills; Matrix models

Lattice $\mathcal{N} = 4$ super-Yang–Mills
  ✓ Formulation highlights — Questions?
    • Dimensionally reduced (2d & 3d) thermodynamics
    • 4d static potential & scaling dimensions

Prospects and future directions
2d thermodynamics on \((r_L \times r_\beta)\) torus

Dimensionally reduce to (deconfined) 2d \(\mathcal{N} = (8, 8)\) SYM with four scalar \(Q\)

Low temperatures \(t = 1/r_\beta \longleftrightarrow\) black holes in dual supergravity

For decreasing \(r_L\) at large \(N\)

homogeneous black string (D1) \(\rightarrow\) localized black hole (D0)

\[\uparrow\]

“spatial deconfinement” signalled by Wilson line \(P_L\)
Spatial deconfinement transition signals

Peaks in Wilson line susceptibility match change in its magnitude $|PL|$, grow with size of SU($N$) gauge group, comparing $N = 6, 9, 12$

Agreement for $16 \times 4$ vs. $24 \times 6$ lattices (aspect ratio $\alpha = r_L/r_\beta = 4$)
Check holographic black hole energies

Lattice results consistent with leading expectation for sufficiently low $t \lesssim 0.4$

Similar behavior $\rightarrow$ difficult to distinguish phases

$\propto t^{3.2}$ for small-$r_L$ D0 phase

$\propto t^3$ for large-$r_L$ D1 phase

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Lattice results for 2d $\mathcal{N} = (8, 8)$ SYM phase diagram

Good agreement with bosonic QM at high temperatures

Harder to control low-temperature uncertainties (larger $N > 16$ should help)

Overall consistent with holography

Comparing multiple lattice sizes and $6 \leq N \leq 16$

Controlled extrapolations are work in progress
Similar dimensional reduction to 3d $\mathcal{N} = 8$ SYM with two scalar $Q$

Again approach leading holographic expectation $\propto t^{10/3}$ for low $t \lesssim 0.3$

Carry out continuum extrapolations for fixed aspect ratio $\alpha = 1$ and $N = 8$
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✓ Dimensionally reduced thermodynamics — Questions?

• 4d static potential & scaling dimensions

Prospects and future directions
Static potential $V(r)$ for 4d $\mathcal{N} = 4$ SYM

Static probes $\rightarrow r \times T$ Wilson loops $W(r, T) \propto e^{-V(r)T}$

Coulomb gauge trick reduces $A^*_4$ lattice complications
Static potential is Coulombic at all $\lambda$

Fits to confining $V(r) = A - C/r + \sigma r \rightarrow$ vanishing string tension $\sigma$

$\rightarrow$ Fit to just $V(r) = A - C/r$ to extract Coulomb coefficient $C(\lambda)$

$\mathcal{N} = 4$ SYM, U(2) PRELIMINARY

Discretization artifacts reduced by tree-level improved analysis
Coupling dependence of Coulomb coefficient

Continuum perturbation theory \[ C(\lambda) = \frac{\lambda}{4\pi} + O(\lambda^2) \]

Holography \[ C(\lambda) \propto \sqrt{\lambda} \text{ for } N \to \infty \text{ and } \lambda \to \infty \text{ with } \lambda \ll N \]

For \( \lambda_{\text{lat}} \leq 2 \), consistent with leading-order perturbation theory
Konishi operator scaling dimension

\( \mathcal{O}_K(x) = \sum_i \text{Tr} [\Phi^I(x)\Phi^I(x)] \) is simplest conformal primary operator

Scaling dimension \( \Delta_K(\lambda) = 2 + \gamma_K(\lambda) \) investigated through perturbation theory (& S duality), holography, conformal bootstrap

\[
C_K(r) \equiv \mathcal{O}_K(x + r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}
\]

‘SUGRA’ is 20’ op., \( \Delta_S = 2 \)

Can compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG
Konishi operator scaling dimension

Lattice scalars $\varphi(n)$ from polar decomposition $U_a(n) = e^{\varphi_a(n)} U_a(n)$

$$O^\text{lat}_K(n) = \sum_a \text{Tr} [\varphi_a(n) \varphi_a(n)] - \text{vev}$$

$$O^\text{lat}_S(n) \sim \text{Tr} [\varphi_a(n) \varphi_b(n)]$$

$$C_K(r) \equiv O_K(x + r) O_K(x) \propto r^{-2\Delta_K}$$

‘SUGRA’ is 20’ op., $\Delta_S = 2$

Can compare:
- Direct power-law decay
- Finite-size scaling
- Monte Carlo RG

\(N = 4\) SYM, U(2)

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Lattice susy overview

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Preliminary $\Delta_K$ results from Monte Carlo RG

Analyzing both $\mathcal{O}_{K}^{\text{lat}}$ and $\mathcal{O}_{S}^{\text{lat}}$

Imposing protected $\Delta_S = 2$

$\rightarrow \Delta_K(\lambda)$ looks perturbative

Systematic uncertainties from different amounts of smearing

Complication from twisting $\text{SO}(4)_R \subset \text{SO}(6)_R$

$\mathcal{O}_{K}^{\text{lat}}$ mixes with $\text{SO}(4)_R$-singlet part of $\text{SO}(6)_R$-nonsinglet $\mathcal{O}_S$

$\rightarrow$ disentangle via variational analyses
Fermion bilinear anomalous dimension

Bergner–DS, to appear

Fermion op. eigenvalues predict ‘mass’ anomalous dimension of fermion bilinear
Should vanish $\rightarrow$ test discretization and finite-volume effects in lattice calcs

Scale-dependent ‘effective anom. dim.’
due to broken conformality

Recover true critical exponent
at low energy scale $\Omega^2 \ll 1$

$0.25 \leq \lambda_{\text{lat}} \leq 2.5$, with even free theory
sensitive to lattice effects
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Prospects and future directions
Future frontier: Supersymmetric QCD

Add ‘quarks’ and squarks $\rightarrow$ investigate electric–magnetic dualities, dynamical supersymmetry breaking and more

Fine-tuning back with a vengeance

$\mathcal{O}(10)$ parameters, even using domain wall or overlap fermions [arXiv:0903.2443]
Pursuing superQCD with full fine-tuning

First step: Lattice perturbation theory as guide for future fine-tuning

Wellegehausen–Wipf, arXiv:1811.01784; Costa–Panagopoulos, arXiv:1812.06770

Alternately include only fundamental + adjoint fermions, leave scalars for future

Bergner–Piemonte, arXiv:2008.02855
Simplify superQCD: Twisted theories in 2d or 3d

Quiver construction preserves susy sub-algebra

[arXiv:1505.00467]

2-slice lattice SYM
with $U(N) \times U(F)$ gauge group

Adj. fields on each slice

Bi-fundamental in between

Decouple $U(F)$ slice

$\rightarrow U(N)$ SQCD in $d - 1$ dims.

with $F$ fund. hypermultiplets

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Dynamical susy breaking in 2d lattice superQCD

U($N$) superQCD with $F$ fundamental hypermultiplets

Observe spontaneous susy breaking only for $N > F$, as expected

Catterall–Veernala, arXiv:1505.00467

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Lattice susy overview
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Future frontier: Sign problems

Recall typical algorithms sample field configurations $\Phi$ with probability $\frac{1}{Z} e^{-S[\Phi]}$$\rightarrow \text{“sign problem” if action } S[\Phi] \text{ can be negative or complex}$

**Example: Spontaneous susy breaking needs vanishing Witten index**

Witten index is just $Z = \int \mathcal{D}\Phi \ e^{-S[\Phi]} \rightarrow \text{severe sign problem to have } Z = 0$

Alternative approaches being explored

  Joseph–Kumar, arXiv:2011.08107

  Complex Langevin to be discussed by Jun Nishimura tomorrow

  Quantum simulation to be discussed by Shailesh Chandrasekharan today
Future frontier: Sign problems

Recall typical algorithms sample field configurations $\Phi$ with probability $\frac{1}{Z} e^{-S[\Phi]}$.

→ "sign problem" if action $S[\Phi]$ can be negative or complex

Example: $\mathcal{N} = 4$ SYM has complex pfaffian $\text{pf} D = |\text{pf} D| e^{i\alpha}$

$$\langle O \rangle = \frac{1}{Z} \int [dU][d\overline{U}] \ O \ e^{-S_{B[U, \overline{U}]} \ pf D[U, \overline{U}]}$$

We phase quench $\text{pf} D \rightarrow |\text{pf} D|$, need to reweight $\langle O \rangle = \frac{\langle O e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}}$

$$\Rightarrow \langle e^{i\alpha} \rangle_{pq} = \frac{Z}{Z_{pq}} \text{ quantifies severity of sign problem}$$
$\mathcal{N} = 4$ SYM sign problem

Fix $\lambda_{\text{lat}} = g_{\text{lat}}^2 N = 0.5$

Pfaffian nearly real positive for all accessible volumes

Fix $4^4$ volume

Fluctuations increase with coupling

Signal-to-noise becomes obstruction for $\lambda_{\text{lat}} \gtrsim 4$
Recap: An exciting time for lattice supersymmetry

Significant progress currently being made in lattice studies of supersymmetric QFTs

\[ \mathcal{N} = 1 \] super-Yang–Mills avoids fine tuning from scalars

Reducing dimensions simplifies tests of holography

Preserving susy sub-algebra enables lattice \( \mathcal{N} = 4 \) SYM
\[ \longrightarrow \] thermodynamics, static potential, scaling dimensions

SuperQCD, sign problems and much more to do in the future
Thanks for your attention!  Any further questions?

Collaborators
Georg Bergner, Simon Catterall, Joel Giedt, Raghav Jha, Anosh Joseph, Angel Sherletov, Toby Wiseman

Funding and computing resources

UK Research and Innovation

Lattice susy overview

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Backup: Breakdown of Leibniz rule on the lattice

\[
\{ Q_\alpha, \overline{Q}_{\dot{\alpha}} \} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu = 2i\sigma^{\mu}_{\alpha\dot{\alpha}} \partial_\mu \text{ is problematic}
\]

\[
\implies \text{try finite difference } \partial_\phi(x) \rightarrow \Delta_\phi(x) = \frac{1}{a} [\phi(x + a) - \phi(x)]
\]

Crucial difference between \( \partial \) and \( \Delta \)

\[
\Delta [\phi \eta] = a^{-1} [\phi(x + a)\eta(x + a) - \phi(x)\eta(x)]
\]

\[
= [\Delta \phi] \eta + \phi \Delta \eta + a [\Delta \phi] \Delta \eta
\]

Full supersymmetry requires Leibniz rule \( \partial [\phi \eta] = [\partial \phi] \eta + \phi \partial \eta \)

only recovered in \( a \rightarrow 0 \) continuum limit for any local finite difference
Backup: Breakdown of Leibniz rule on the lattice

Full supersymmetry requires Leibniz rule $\partial [\phi \eta] = [\partial \phi] \eta + \phi \partial \eta$

only recovered in $a \to 0$ continuum limit for any local finite difference

Supersymmetry vs. locality ‘no-go’ theorems
by Kato–Sakamoto–So [arXiv:0803.3121] and Bergner [arXiv:0909.4791]

Complicated constructions to balance locality vs. supersymmetry
Non-ultralocal product operator $\rightarrow$ lattice Leibniz rule but not gauge invariance

D’Adda–Kawamoto–Saito, arXiv:1706.02615

Cyclic Leibniz rule $\rightarrow$ partial lattice supersymmetry but only $(0+1)d$ QM so far

Kadoh–Kamei–So, arXiv:1904.09275
Backup: Complexified gauge field from twisting

Combining $A_\mu$ and $\Phi^I \rightarrow A_a$ and $\overline{A}_a$

produces $U(N) = SU(N) \otimes U(1)$ gauge theory

Complicates lattice action but needed so that $Q A_a = \psi_a$

Further motivation: Under $SO(d)_{tw} = \text{diag}[SO(d)_{euc} \otimes SO(d)_R]$

$A_\mu \sim \text{vector} \otimes \text{scalar} = \text{vector}$

$\Phi^I \sim \text{scalar} \otimes \text{vector} = \text{vector}$

Easiest to see in 5d (then dimensionally reduce)

$A_a = A_a + i\Phi_a \rightarrow (A_\mu, \phi) + i(\Phi_\mu, \overline{\phi})$
Backup: $A_4^*$ lattice from five dimensions

Again dimensionally reduce, treating all five gauge links symmetrically

Start with hypercubic lattice
in 5d momentum space

**Symmetric** constraint \( \sum_a \partial_a = 0 \)
projects to 4d momentum space

Result is $A_4$ lattice
\[ \rightarrow \text{dual } A_4^* \text{ lattice in position space} \]
Backup: Restoration of $Q_a$ and $Q_{ab}$ supersymmetries

"$Q + \text{discrete } R_a \subset SO(4)_{tw} = Q_a$ and $Q_{ab}$"

[arXiv:1306.3891]

Test $R_a$ on Wilson loops

$\tilde{\mathcal{W}}_{ab} \equiv R_a \mathcal{W}_{ab}$

Tune coeff. $c_2$ of $d^2$ term in action for fastest restoration towards continuum limit

$N = 4 \text{ SYM, } U(2)$

$4^3 \times 12$

$6^3 \times 18$

$(L/2) \times (L/2)$

PRELIMINARY
Backup: Problem with SU($N$) flat directions

$\mu^2/\lambda_{lat}$ too small $\rightarrow U_a$ can move far from continuum form $\mathbb{I}_N + A_a$

Example: $\mu = 0.2$ and $\lambda_{lat} = 2.5$ on $8^3 \times 24$ volume

Left: Bosonic action stable $\sim 18\%$ off its supersymmetric value

Right: (Complexified) Polyakov loop wanders off to $\sim 10^9$
Backup: Problem with U(1) flat directions

Monopole condensation $\rightarrow$ confined lattice phase not present in continuum

Around the same $2\lambda_{\text{lat}} \approx 2\ldots$

**Left:** Polyakov loop falls towards zero

**Center:** Plaquette determinant falls towards zero

**Right:** Density of U(1) monopole world lines becomes non-zero
Backup: Regulating SU($N$) flat directions

Add soft $Q$-breaking scalar potential to lattice action

$$S = \frac{N}{4\lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} F_{ab} + \eta D_a U_a - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} D_c \chi_{de} + \mu^2 V \right]$$

$$V = \sum_a \left( \frac{1}{N} \text{Tr} [U_a U_a] - 1 \right)^2$$ lifts SU($N$) flat directions,

ensures $U_a = I_N + A_a$ in continuum limit

Correct continuum limit requires $\mu^2 \to 0$ to restore $Q$ and recover moduli space

Typically scale $\mu \propto 1/L$ in $L \to \infty$ continuum extrapolation
In earlier work we added another soft $Q$-breaking term

$$S_{\text{soft}} = \frac{N}{4\lambda_{\text{lat}}} \mu^2 \sum_a \left( \frac{1}{N} \text{Tr} \left[ U_a \overline{U}_a \right] - 1 \right)^2 + \kappa \sum_{a<b} |\det P_{ab} - 1|^2$$

More sensitivity to $\kappa$ than to $\mu^2$

Showing $Q$ Ward identity from bosonic action

$$\langle s_B \rangle = \frac{9N^2}{2}$$
Backup: Better regulating U(1) flat directions

\[ S = \frac{N}{4\lambda_{\text{lat}}} \left[ Q \left( \chi_{ab} \mathcal{F}_{ab} + \eta \left\{ D_a U_a + G \sum_{a < b} [\det P_{ab} - 1] \mathbb{I}_N \right\} - \frac{1}{2} \eta d \right) - \frac{1}{4} \epsilon_{abcde} \chi_{ab} \overline{D}_c \chi_{de} + \mu^2 V \right] \]

\( Q \) Ward identity violations scale \( \propto 1/N^2 \) (left) and \( \propto (a/L)^2 \) (right)

\( \sim \) effective ‘\( \mathcal{O}(a) \) improvement’ since \( Q \) forbids all dim-5 operators
Backup: Supersymmetric moduli space modification

Method to impose $Q$-invariant constraints on generic site operator $\mathcal{O}(n)$

Modify auxiliary field equations of motion $\longrightarrow$ moduli space

\[
d(n) = \overline{D}_a^{(-)} U_a(n) \quad \longrightarrow \quad d(n) = \overline{D}_a^{(-)} U_a(n) + G\mathcal{O}(n)\mathbb{I}_N
\]

However, both $U(1)$ and $\text{SU}(N) \in \mathcal{O}(n)$ over-constrains system
Backup: Dimensional reduction to 2d $\mathcal{N} = (8,8)$ SYM

Naive for now: 4d $\mathcal{N} = 4$ SYM code with $N_x = N_y = 1$

$A_4^* \rightarrow A_2^*$ (triangular) lattice

Torus **skewed** depending on $\alpha = L/N_t$

Modular transformation into fundamental domain

$\rightarrow$ some skewed tori actually rectangular

Also need to stabilize compactified links
to ensure broken center symmetries
Backup: 2d $\mathcal{N} = (8, 8)$ SYM Wilson line eigenvalues

Check ‘spatial deconfinement’ through Wilson line eigenvalue phases

**Left:** $\alpha = 2$ distributions more extended as $N$ increases $\rightarrow$ D1 black string

**Right:** $\alpha = 1/2$ distributions more compact as $N$ increases $\rightarrow$ D0 black hole

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David Schaich (Liverpool)

Lattice susy overview

ICTS Bangalore, 18 January 2021
Check ‘spatial deconfinement’ through Wilson line eigenvalue phases

Left: High-temperature $U(8)$ $8^3$ distributions more compact as $t$ increases

Right: Low-temperature $U(N)$ $12^3$ distributions more uniform as $N$ increases
Backup: Static potential is Coulombic at all $\lambda$

String tension $\sigma$ from fits to confining form $V(r) = A - C/r + \sigma r$

Slightly negative values flatten $V(r_I)$ for $r_I \lesssim L/2$

$\sigma \to 0$ as accessible range of $r_I$ increases on larger volumes
Backup: Discretization artifacts in static potential

Discretization artifacts visible at short distances where Coulomb term in $V(r) = A - C/r$ is most significant.

Danger of distorting Coulomb coefficient $C$.
Backup: Tree-level improvement

Classic trick to reduce discretization artifacts in static potential

Associate $V(r_\nu)$ data with ‘$r_i$’ from Fourier transform of gluon propagator

Recall
\[
\frac{1}{4\pi^2 r^2} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{e^{i r_\nu k_\nu}}{k^2} \quad \text{where} \quad \frac{1}{k^2} = G(k_\nu) \quad \text{in continuum}
\]

\[
A^*_4 \text{ lattice } \quad \frac{1}{r^2_\perp} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4 \hat{k}}{(2\pi)^4} \frac{\cos (i r_\nu \hat{k}_\nu)}{4 \sum_{\mu=1}^{4} \sin^2 \left( \hat{k} \cdot \hat{e}_\mu / 2 \right)}
\]

Tree-level lattice propagator from arXiv:1102.1725

$\hat{e}_\mu$ are $A^*_4$ lattice basis vectors;

momenta $\hat{k} = \frac{2\pi}{L} \sum_{\mu=1}^{4} n_\mu \hat{g}_\mu$ depend on dual basis vectors
\[ \frac{1}{r_i^2} \equiv 4\pi^2 \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\cos \left( ir_\nu \hat{k}_\nu \right)}{4 \sum_{\mu=1}^{4} \sin^2 \left( \hat{k} \cdot \hat{e}_\mu / 2 \right)} \rightarrow \text{significantly reduced discretization artifacts} \]
Backup: Scaling dimensions from MCRG stability matrix

Lattice system: \[ H = \sum_i c_i \mathcal{O}_i \] (infinite sum)

Couplings flow under RG blocking \[ \rightarrow H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)} \]

Conformal fixed point \[ \rightarrow H^* = R_b H^* \] with couplings \( c_i^* \)

Linear expansion around fixed point \[ \rightarrow \text{stability matrix} \; T_{ik}^* \]

\[
 c_i^{(n)} - c_i^* = \sum_k \frac{\partial c_i^{(n)}}{\partial c_k^{(n-1)}} \bigg|_{H^*} \left( c_k^{(n-1)} - c_k^* \right) \equiv \sum_k T_{ik}^* \left( c_k^{(n-1)} - c_k^* \right)
\]

Correlators of \( \mathcal{O}_i, \mathcal{O}_k \) \[ \rightarrow \text{elements of stability matrix} \] [Swendsen, 1979]

Eigenvalues of \( T_{ik}^* \) \[ \rightarrow \text{scaling dimensions of corresponding operators} \]
Backup: Real-space RG for lattice $\mathcal{N} = 4$ SYM

Must preserve $Q$ and $S_5$ symmetries $\iff$ geometric structure

Simple transformation constructed in arXiv:1408.7067

$$U'_a(n') = \xi U_a(n) U_a(n + \hat{\mu}_a) \quad \quad \quad \quad \eta'(n') = \eta(n)$$

$$\psi'_a(n') = \xi \left[ \psi_a(n) U_a(n + \hat{\mu}_a) + U_a(n) \psi_a(n + \hat{\mu}_a) \right] \quad \quad \quad \text{etc.}$$

Doubles lattice spacing $a \rightarrow a' = 2a$, with tunable rescaling factor $\xi$

Scalar fields from polar decomposition $U(n) = e^{\varphi(n)} U(n)$

$\implies$ shift $\varphi \rightarrow \varphi + \log \xi$ to keep blocked $U$ unitary

$Q$-preserving RG transformation needed

to show only one log. tuning to recover continuum $Q_a$ and $Q_{ab}$
Backup: Smearing for Konishi analyses

Smear to enlarge (MCRG or variational) operator basis

APE-like smearing: \[ (1 - \alpha) + \frac{\alpha}{8} \sum \nabla, \]

staples built from unitary parts of links but no final unitarization

Average plaquette stable upon smearing (right),
minimum plaquette steadily increases (left)
Spontaneous susy breaking means $\langle 0 | H | 0 \rangle > 0$ or equivalently $\langle Q \Omega \rangle \neq 0$

Twisted superQCD auxiliary field e.o.m. $\leftrightarrow$ Fayet–Iliopoulos $D$-term potential

$$d = \overline{D} a U_a + \sum_{i=1}^{F} \phi_i \overline{\phi}_i - r \mathbb{I}_N$$
$$\leftrightarrow$$

$$\text{Tr} \left[ \left( \sum_i \phi_i \overline{\phi}_i - r \mathbb{I}_N \right)^2 \right] \in H$$

Have $F \times N$ scalar vevs to zero out $N \times N$ matrix

$\rightarrow N > F$ suggests susy breaking, $\langle 0 | H | 0 \rangle > 0$ $\leftrightarrow$ $\langle Q \eta \rangle = \langle d \rangle \neq 0$