The stability of MHD Taylor-Couette flow with current-free spiral magnetic fields between conducting cylinders

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Abstract. We study the magnetorotational instability in cylindrical Taylor-Couette flow, with the (vertically unbounded) cylinders taken to be perfect conductors, and with externally imposed spiral magnetic fields. The azimuthal component of this field is generated by an axial current inside the inner cylinder, and may be slightly stronger than the axial field. We obtain an instability beyond the Rayleigh line, for Reynolds numbers of order 1000 and Hartmann numbers of order 10, and independent of the (small) magnetic Prandtl number. For experiments with $R_{\text{out}}=2 R_{\text{in}}=10$ cm and $\Omega_{\text{out}}=0.27 \Omega_{\text{in}}$, the instability appears for liquid sodium for axial fields of $\sim 20$ Gauss and axial currents of $\sim 1200$ A. For gallium the numbers are $\sim 50$ Gauss and $\sim 3200$ A. The vertical cell size is about twice the cell size known for nonmagnetic experiments.

Key words: magnetic fields – magnetohydrodynamics – general: physical data and processes

1. Motivation

A limited number of instabilities are responsible for most of the pattern formation in the Universe. Stars are formed by the Jeans instability, and the heat transport within them is driven by the Rayleigh-Bénard instability. Most high-energy radiation, however, is produced by disks around compact objects and black holes. These accretion disks must therefore be turbulent. It is now believed that this turbulence is caused by the instability of their Keplerian rotation law in the presence of weak large-scale magnetic fields. This ‘magnetorotational instability’ (MRI) was first discovered by Velikhov (1959), who considered an electrically conducting fluid between rotating cylinders with $\Omega_{\text{out}} < \Omega_{\text{in}}$. The existence of this instability has been confirmed analytically and numerically many times; see for example Balbus (2003) and Rüdiger & Hollerbach (2004) for recent reviews.

There is considerable interest in studying the magnetorotational instability in the laboratory. The simplest, and most widely studied design, is to confine a liquid metal between differentially rotating cylinders, and impose a magnetic field along the axis of the cylinders. This yields the MRI for magnetic Reynolds numbers of order 10. However, because of the extremely small magnetic Prandtl numbers $P_m(=\nu/\eta)$ of available liquid metals (see Table 1), that translates into ordinary Reynolds numbers exceeding $10^6$, which causes severe difficulties (Hollerbach & Fournier 2004). In order to overcome such problems, Hollerbach & Rüdiger (2005) proposed imposing an azimuthal field as well, with dramatic consequences, namely a reduction of the critical Reynolds number $Re_c$ from $O(10^6)$ to $O(10^3)$. These new solutions are also essentially independent of $P_m$, and are therefore ideally suited to experimental realisations in the laboratory. These results were for insulating inner and outer cylinders, in which case the axial current one must impose to generate this azimuthal magnetic field is around $2500$ A (using liquid sodium as the fluid), which is close to the upper limit of what is experimentally possible. Here we therefore consider conducting cylinders, and show that the necessary current (as well as the Reynolds number) is smaller than for insulating cylinders.

Table 1. Liquid metal material parameters in c.g.s.

|        | $\rho$ | $\nu$  | $\eta$ | $P_m$ | $\sqrt{\mu_0 \rho \eta}$ |
|--------|--------|--------|--------|-------|---------------------------|
| sodium | 0.9    | $7.1 \cdot 10^{-3}$ | $0.8 \cdot 10^4$ | $9.0 \cdot 10^{-5}$ | 8.15                      |
| gallium| 6.0    | $3.2 \cdot 10^{-3}$ | $2.1 \cdot 10^3$ | $1.5 \cdot 10^{-6}$ | 22.0                      |
2. Introduction

We consider the stability of the flow between two rotating coaxial infinitely long cylinders, in the presence of a constant axial field \( B_z \), and a current-free (within the fluid) azimuthal field \( B_\phi \). Figure 1 shows a sketch of the geometry. The fluid is incompressible, with density \( \rho \), kinematic viscosity \( \nu \) and magnetic diffusivity \( \eta \).

![Diagram of cylindrical geometry with magnetic fields](image)

Fig. 1. (online colour at www.an-journal.org) The basic geometry of the problem, consisting of two cylinders of radii \( R_{\text{in}} \) and \( R_{\text{out}} \), rotating at \( \Omega_{\text{in}} \) and \( \Omega_{\text{out}} \). \( B_z \) and \( B_\phi \) are the externally imposed magnetic fields, \( B_\phi \) by an axial current inside the inner cylinder.

By conservation of angular momentum the basic state differential rotation profile is given by

\[
\Omega = a + \frac{b}{R^2},
\]

where \( a \) and \( b \) are given by

\[
a = \frac{\mu - \eta^2}{1 - \eta^2} \Omega_{\text{in}}, \quad b = \frac{1 - \mu}{1 - \eta^2} R_{\text{in}}^2 \Omega_{\text{in}},
\]

with \( \mu = \Omega_{\text{out}}/\Omega_{\text{in}} \) and \( \eta = R_{\text{in}}/R_{\text{out}} \). \( R_{\text{in}} \) and \( R_{\text{out}} \) are the radii of the inner and outer cylinders, and \( \Omega_{\text{in}} \) and \( \Omega_{\text{out}} \) are their angular velocities.

From the azimuthal component of the induction equation one finds

\[
B_\phi = \frac{B}{R}
\]

with \( B = \beta B_0 R_{\text{in}} \). This parameter \( \beta \) therefore denotes the ratio of the toroidal field to the constant axial field \( B_z = B_0 \). This toroidal field \( B_\phi \) is maintained by an electric current running along the central axis of strength

\[
J = 5\beta B_0 R_{\text{in}}
\]

with \( J \) in Ampere, \( B_0 \) in Gauss and \( R_{\text{in}} \) in cm.

We are interested in the stability of the basic state against axisymmetric and nonaxisymmetric perturbations. The perturbed state of the flow is described by the quantities \( u'_R, \ R\Omega + u'_\phi, \ u'_z, \ B'_R, \ B_\phi + B'_\phi, \ B_0 + B'_0 \).

The solutions of the linearized MHD equations are considered in their modal representation \( F' = F'(R)\exp(i(kz + m\phi + \omega t)) \) where \( F' \) is any of \( u' \) and \( B' \). The dimensionless numbers of the problem are the magnetic Prandtl number \( Pm \), the Hartmann number \( Ha \) and the Reynolds number \( Re \),

\[
Pm = \frac{\nu}{\eta}, \quad Ha = \frac{B_0 R_0}{\sqrt{\mu_0 \rho \eta}}, \quad Re = \frac{\Omega_{\text{in}} R_{\text{in}}^2}{\nu},
\]

where \( \mu_0 \) is the permeability, and \( R_0 = (R_{\text{in}} D)^{1/2} \) with \( D = R_{\text{out}} - R_{\text{in}} \).

In order to understand the results presented here, it is useful also to consider the induction equation for the fluctuating toroidal field component,

\[
\frac{\partial B'_\phi}{\partial t} - \eta \Delta B'_\phi = \text{rot}_\phi \left( \bar{u}_\phi \times B'_\text{pol} + u'_\phi \times \bar{B}_\text{pol} + u'_\text{pol} \times \bar{B}_\phi \right),
\]

where \( u'_\text{pol} \) and \( B'_\text{pol} \) denote the poloidal components of \( u' \) and \( B' \). Chandrasekhar (1961) cancelled the last two terms on the right of this equation, and did not obtain the MRI. It occurs if only the last term in (7) is retained then the resulting Reynolds number loses its strong \( Re \propto Pm^{-1} \) dependence, and is reduced by 3 orders of magnitude (Hollerbach & Rudiger 2005, see also Table 4).

3. The equations

The equations of the problem are given in a similar form as by Rudiger & Shalybkov (2004). Lengths and wave numbers are normalized by \( R_0 \), velocities by \( \eta/R_0 \), frequencies by \( \Omega_{\text{in}} \), and magnetic fields by \( B_0 \). One then obtains

\[
\frac{dP'}{dR} + \frac{m}{R} X_2 + \frac{\mu}{R^2} u'_R + i\Re(\omega + m\Omega)u'_\phi - 2\Omega \Re u'_\phi - iHa^2kB'_R - \frac{m}{R} B'_R + 2Ha^2kB'_\phi = 0,
\]

\[
\frac{dX_2}{dR} - \left( k^2 + m^2 \right) u'_\phi - i\Re(\omega + m\Omega)u'_z + 2\frac{m}{R} u'_R - 2\Re u'_R + iHa^2kB'_\phi + iHa^2kB'_z - \frac{m}{R} P' = 0,
\]

\[
\frac{dX_3}{dR} + \frac{X_3}{R} + \left( k^2 + m^2 \right) u'_z - i\Re(\omega + m\Omega)u'_z - i\Re kP' + iHa^2kB'_\phi + iHa^2kB'_z = 0,
\]

\[
\frac{dP'_R}{dR} + \frac{u'_R}{R} + \frac{i}{R} u'_\phi + iku'_z = 0,
\]
4. Axisymmetric modes

Figure 2 shows the critical Reynolds numbers at and beyond the Rayleigh line for experiments without toroidal magnetic fields, the classical design. We note how $Re_c$ jumps abruptly from $10^4$ to $10^6$ (for liquid sodium, see Rüdiger, Schultz & Shalybkov 2003). With a toroidal field, however, the solutions are very different, as shown in Fig. 3. The toroidal field strongly modifies the extremely steep line obtained for $\beta = 0$ (Fig. 2). Its inclination is reduced and it starts at lower Reynolds numbers. For $\beta \approx 2$ we find critical Reynolds numbers of order $10^3$. This is a dramatic reduction of the values of order $10^6$ which are characteristic for $\beta = 0$.

The corresponding Hartmann numbers for these instabilities are also strongly reduced by the inclusion of a toroidal field (Fig. 4). Tables 2 and 3 show results for $\mu = 0.27$, $\beta = 0$ to 4, and $P_m = 10^{-5}$ (sodium, Table 2) and $P_m = 10^{-6}$ (gallium, Table 3). The first five columns show the nondimensional quantities indicated. The last four columns show dimensional quantities, taking $R_{in} = 5$ cm and $R_{out} = 10$ cm. Specifically, Re has been converted to $f_{in}$, the rotation frequency, in Hz, of the inner cylinder. The Hartmann number $Ha$ has been converted first to $B_{out}$, the strength, in Gauss, of the axial field. Next, the product $\beta Ha$ has been converted to the strength of the azimuthal field at $R_{in}$. Finally, $J$ denotes the axial current, in Ampere, that is required to maintain this toroidal field.

For $\beta \gtrsim 2$ we see then that the critical Reynolds numbers no longer scale as $P_m^{-1}$, as they do for $\beta = 0$, but rather become independent of $P_m$. This is the same result previously noted by Hollerbach & Rüdiger (2005) for insulating boundaries. More quantitatively, we note that for $\beta \gtrsim 2$ the required rotation rates of the inner cylinder are reduced to less than 1 Hz, the axial fields to a few tens of Gauss, and the axial current to a few tens of Ampere, in the laboratory. We conclude therefore that this new design, incorporating an axial current, is the most promising design for obtaining the MRI in a laboratory experiment.

It is important also to consider the wave numbers $k$: if the vertical cell size is too large the experiment will still run into difficulties. We note in Fig. 4 that for non-zero $\beta k$ is reduced somewhat, corresponding to a greater extent in $z$ (the vertical cell size is given by $\delta z \approx \pi/k$). The increase in cell size is not very great though, and therefore should not cause any problems.

Another interesting point in comparing $k$ for $\beta = 0$ versus $\beta > 0$ is the $P_m$-dependence, or rather the lack thereof, since $k$ is independent of $P_m$ in both cases. The details of how
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Fig. 4. The Hartmann numbers of the solutions given in Fig. 3.

Table 2. Characteristic values for \( \hat{\eta} = 0.5, \hat{\mu} = 0.27, \) and \( P_m = 10^{-5} \). In converting from nondimensional to dimensional quantities \( R_{in} = 5 \text{ cm} \) and \( R_{out} = 10 \text{ cm} \) were used, and the material properties of sodium. The magnetic fields are measured in Gauss, \( B_\phi \) denotes the toroidal field at the inner cylinder, the current \( J \) is measured in Ampere. \( \Re(\omega) \) is the real part of the Fourier frequency.

\[
\begin{array}{cccccccc}
\beta & \Re & Ha & k & \Re(\omega) & j_{in} & B_0 & B_\phi & J \\
0 & 1 \times 10^6 & 542 & 1.7 & 0 & 45 & 883 & 0 & 0 \\
1 & 3383 & 38.4 & 0.6 & 0.04 & 1.5 & 63 & 63 & 1565 \\
2 & 2383 & 14.6 & 1.3 & 0.10 & 0.11 & 24 & 48 & 1190 \\
3 & 1160 & 10.7 & 1.6 & 0.13 & 0.05 & 17 & 52 & 1308 \\
4 & 842 & 9.5 & 2.0 & 0.15 & 0.04 & 15 & 62 & 1549 \\
\end{array}
\]

this comes about are subtly different though. For the classical MRI with \( \beta = 0 \) the critical wave number is proportional to \( \Omega_{in}/V_A \), where \( V_A = B_0/\sqrt{\mu_0 \rho} \) is the Alfvén velocity. The nondimensional wave number \( k \) then becomes

\[
k \propto \frac{\Re \sqrt{P_m}}{Ha}, \tag{16}
\]

which does not depend on the magnetic Prandtl number, since \( \Re \propto P_m^{-1} \) and \( Ha \propto P_m^{-1/2} \). In contrast, for \( \beta > 0 \) \( k \) is still independent of \( P_m \), but so are \( \Re \) and \( Ha \), so that (16) cannot be the relevant balance in this case. For \( \beta > 0 \) the actual wave numbers are much greater than (16) would predict.

Finally, Fig. 5 shows the frequency \( \Re(\omega) \) of these modes. For \( \beta = 0 \) this is zero, since the modes are stationary in that case. For non-zero \( \beta \) stationary modes no longer exist; as noted also by Hollerbach & Rüdiger (2005), including a toroidal field changes the symmetries of the problem in such a way that \( \pm z \) are no longer equivalent, which inevitably means that the modes will drift one way or the other in \( z \), that is, they will be oscillatory rather than stationary. We see though that the real parts of \( \omega \) are still rather small, with the mode-oscillation time exceeding the rotation time of the inner cylinder by more than a factor of 10.

Table 3. The same as in Table 2 but for gallium (\( P_m = 10^{-6} \)).

\[
\begin{array}{cccccccc}
\beta & \Re & Ha & k & \Re(\omega) & j_{in} & B_0 & B_\phi & J \\
0 & 1 \times 10^7 & 1720 & 1.7 & 0 & 200 & 7568 & 0 & 0 \\
1 & 38250 & 39 & 0.6 & 0.04 & 0.8 & 171 & 171 & 4290 \\
2 & 2382 & 14.6 & 1.3 & 0.10 & 0.05 & 64 & 128 & 3212 \\
3 & 1160 & 10.8 & 1.7 & 0.13 & 0.02 & 48 & 143 & 3564 \\
4 & 842 & 9.5 & 2.0 & 0.15 & 0.02 & 42 & 167 & 4180 \\
\end{array}
\]

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Fig. 6. The real parts of the eigenfrequency \( \omega \) of the solutions given in Fig. 3.

5. Nonaxisymmetric modes

Hollerbach & Rüdiger considered only axisymmetric modes \( m = 0 \). These are usually the preferred modes, both in
Table 4. Insulating cylinders: Fields and axial current for \(\dot{\eta} = 0.27\) and \(\beta = 4\) \((R_{\text{in}} = 5\,\text{cm}, R_{\text{out}} = 10\,\text{cm})\). The magnetic fields are measured in Gauss, \(B_\phi\) denotes the toroidal field at the inner cylinder, the current \(J\) is measured in Ampere.

|         | \(\text{Re}\) | \(f_{\text{in}}\,[\text{Hz}]\) | \(\text{Ha}\) | \(B_z\) | \(B_\phi\) | \(J\) |
|---------|--------------|-----------------|------------|---------|----------|------|
| sodium  | 1521         | 0.07            | 16.3       | 26      | 106      | 2657 |
| gallium | 1521         | 0.03            | 16.3       | 72      | 287      | 7170 |

nonmagnetic Taylor-Couette flow, as well as in the classical \(\beta = 0\) MRI. Nevertheless, for a complete analysis the nonaxisymmetric modes with \(m > 0\) must also be considered. Figure 7 shows these results, and demonstrates that here too the critical Reynolds numbers for the onset of nonaxisymmetric modes are greater than for the onset of axisymmetric modes. Indeed, including a toroidal field reduces \(\text{Re}_c\) by far less for the nonaxisymmetric than for the axisymmetric modes. For \(\beta \gtrsim 1\) the axisymmetric modes thus occur several orders of magnitude before the nonaxisymmetric ones do. These nonaxisymmetric modes are therefore not interesting from the point of view of doing laboratory experiments.

### 6. Conclusion

The central conclusion of this work is as before in Hollerbach & Rüdiger (2005), that imposing both axial and azimuthal magnetic fields together dramatically reduces the critical Reynolds numbers required to obtain the magnetorotational instability. However, for the insulating boundary conditions considered there, the axial currents required to generate this new toroidal field were at the upper range of what could be achieved in the lab. Here we therefore considered conducting cylinders, and found that the required currents (and Reynolds numbers) are smaller. For comparison, Table 4 presents the results of Hollerbach & Rüdiger for \(\beta = 4\); we see that \(\text{Re} = 1521\) and \(\text{Ha} = 16.3\), compared with \(\text{Re} = 842\) and \(\text{Ha} = 9.5\) here (the last rows of Tables 2 and 3). Switching from insulating to conducting boundaries thus reduces both \(\text{Re}\) and \(\text{Ha}\) by almost a factor of 2, which would certainly help in achieving these toroidal fields in the lab. We conclude therefore that implementing this experiment with conducting boundaries is the most promising design for exploring the magnetorotational instability in the laboratory.

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