Commutativity in the Algorithmic Lovász Local Lemma

Vladimir Kolmogorov
Institute of Science and Technology Austria
vnk@ist.ac.at

Abstract

We consider the recent formulation of the Algorithmic Lovász Local Lemma [9, 2] for finding objects that avoid “bad features”, or “flaws”. It extends the Moser-Tardos resampling algorithm [16] to more general discrete spaces. At each step the method picks a flaw present in the current state and “resamples” it using a “resampling oracle” provided by the user. However, it is less flexible than the Moser-Tardos method since [9, 2] require a specific flaw selection rule, whereas [16] allows an arbitrary rule (and thus can potentially be implemented more efficiently).

We formulate a new “commutativity” condition, and prove that it is sufficient for an arbitrary rule to work. It also enables an efficient parallelization under an additional assumption. We then show that existing resampling oracles for perfect matchings and permutations do satisfy this condition.

Finally, we generalize the precondition in [2] (in the case of symmetric potential causality graphs). This unifies special cases that previously were treated separately.

1 Introduction

Let \( \Omega \) be a (large) set of objects and \( F \) be a set of flaws, where a flaw \( f \in F \) is some non-empty set of “bad” objects, i.e. \( f \subseteq \Omega \). Flaw \( f \) is said to be present in \( \sigma \) if \( \sigma \in f \). Let \( F_\sigma = \{ f \in F \mid \sigma \in f \} \) be the set of flaws present in \( \sigma \). Object \( \sigma \) is called flawless if \( F_\sigma = \emptyset \).

The existence of flawless objects can often be shown via a probabilistic method. First, a probability measure \( \omega \) on \( \Omega \) is introduced, then flaws in \( F \) become (bad) events that should be avoided. Proving the existence of a flawless object is now equivalent to showing that the probability of avoiding all bad events is positive. This holds if, for example, all events \( f \in F \) are independent and the probability of each \( f \) is smaller than 1. The well-known Lovász Local Lemma (LLL) [5] is a powerful tool that can handle a (limited) dependency between the events. Roughly speaking, it states that if the dependency graph is sparse enough (e.g. has a bounded degree) and the probabilities of individual bad events are sufficiently small then a flawless object is guaranteed to exist.

LLL has been the subject of intensive research, see e.g. [20] for a relatively recent survey. One of the milestone results was the constructive version of LLL by Moser and Tardos [16]. It applies to the variable model in which \( \Omega = X_1 \times \ldots \times X_n \) for some discrete sets \( X_i \), event \( f \) depends on a small subset of variables denoted as \( \text{vbl}(f) \subseteq [n] \); and two events \( f, g \) are declared to be dependent if \( \text{vbl}(f) \cap \text{vbl}(g) \neq \emptyset \). The algorithm proposed in [16] is strikingly simple: (i) sample each variable \( \sigma_i \) for \( i \in [n] \) according to its distribution; (ii) while \( F_\sigma \) is non-empty, pick an arbitrary flaw \( f \in F_\sigma \) and resample all variables \( \sigma_i \) for \( i \in \text{vbl}(f) \). Moser and Tardos proved that if the LLL condition in [5] is satisfied then the expected number of resampling is small (polynomial for most of the known applications).

The recent development has been extending algorithmic LLL beyond the variable model, and in particular to non-Cartesian spaces. The first such work was by Harris and Srinivasan [8], who
considered the space of permutations. Achlioptas and Iliopoulos [1] introduced a more abstract framework where the behaviour of the algorithm is specified by a certain multigraph. Harvey and Vondrák proposed a framework with *regenerating resampling oracles* [9], providing a more direct connection to LLL. Achlioptas and Iliopoulos [2] extended the framework to more general resampling oracles.

In this paper we study this setting from [9, 2]. It does not assume any particular structure on sets $\Omega$ and $F$. Instead, for each object $\sigma \in \Omega$ and flaw $f \in F_\sigma$ the user must provide a “resampling oracle” specified by a set of actions $A(f, \sigma) \subseteq \Omega$ that can be taken to “address” flaw $f$, and a probability distribution $\rho(\sigma' | f, \sigma)$ over $\sigma' \in A(f, \sigma)$. At each step the algorithm selects a certain flaw $f \in F_\sigma$, samples an action $\sigma' \in A(f, \sigma)$ according to $\rho(\sigma' | f, \sigma)$, and goes there. This framework captures the Moser-Tardos algorithm [16], and can also handle other scenarios such as permutations and perfect matchings (in which case $\Omega$ cannot be expressed as a Cartesian product).

One intriguing difference between the methods of [16] and [1, 9, 2] is that [16] allows an arbitrary rule for selecting a flaw $f \in F_\sigma$, whereas [1, 9, 2] require a specific rule (which depends on a permutation $\pi$ of $F$ chosen in advance). We will say that a resampling algorithm is *flexible* if it is guaranteed to work with any flaw selection rule. We argue that flexibility can lead to a much more efficient practical implementation: it is not necessary to examine all flaws in $F_\sigma$, the first found flaw will suffice. If the list of current flaws is updated dynamically then flexibility could potentially eliminate the need for a costly data structure (such as a priority queue) and thus save a factor of $\Theta(\log n)$ in the complexity. The rule may also affect the number of resamplings in practice; experimentally, the selection process matters, as noted in [20].

Achlioptas and Iliopoulos discuss flaw selection rules in [1, Section 4.3], and remark that they do not see how to accommodate arbitrary rules in their framework. It is known, however, that in special cases flexible rules can be used even beyond the variable model. Namely, through a lengthy and a complicated analysis Harris and Srinivasan [8] managed to show the correctness of a resampling algorithm for permutations, and did not make assumptions on the flaw selection rule in their proof. They also proved a better bound for the parallel version of the algorithm.

This paper aims to understand which properties of the problem enable flexibility and parallelism. Our contributions are as follows.

- We formulate a new condition that we call “commutativity”, and prove that it is sufficient for flexibility.
- We prove that it gives a better bound on the number of rounds of the parallel version of the algorithm. In particular, we show how to use commutativity for handling “partial execution logs” instead of “full execution logs” (which is required for analyzing the parallel version).
- We show that existing resampling oracles for permutations [8] and perfect matchings in complete graphs [9] are commutative. (In fact, we treat both cases in a single framework). Thus, we provide a simpler proof of the result in [8] and generalize it to other settings, in particular to perfect matchings in certain graphs (for which existing algorithms require specific rules).
- We generalize the condition in [2] for the algorithmic LLL to work (in the case of symmetric potential causality graphs). The new condition unifies special cases that were treated separately in [2].

To our knowledge, our commutativity condition captures all previously known cases when the flaw selection rules was allowed to be arbitrary.

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1The papers [1, 2] actually allowed more freedom in the choice of permutation $\pi$, e.g. it may depend on the iteration number. However, once $\pi$ has been chosen, the algorithm should still examine some “current” set of flaws and pick the lowest one with respect to $\pi$. 2
which is a function of the entire past execution history

a flaw \(f\) have on \(F\)

Definition 1. \(f\) as described in the algorithm, i.e. if two conditions hold:

2.1 Walks and the potential causality graph

in a more general case the walk can be non-Markovian.

strategy can be used. This means that flaw \(f\)-specific strategies \(\Lambda\). As stated in the introduction, our goal is to understand when an arbitrary \(\Omega\) as \(\omega\)

\(\Gamma(\cdot | f, \sigma)\). The collection of all resampling oracles will be denoted as \(\rho\). We fix some probability distribution \(\omega\) on \(\Omega\) with \(\omega(\sigma) > 0\) for all \(\sigma \in \Omega\) (it will be used later for formulating various conditions). Note that our notation is quite different from that of Harvey and Vondrák [9, 2]. The algorithm can now be stated as follows.

Algorithm 1 Random walk. Input: initial distribution \(\omega^{\text{init}}\) over \(\Omega\), strategy \(\Lambda\).

1: sample \(\sigma \in \Omega\) according to \(\omega^{\text{init}}\)
2: while \(F_{\sigma}\) non-empty do
3: select flaw \(f \in F_{\sigma}\) according to \(\Lambda\)
4: sample \(\sigma' \in A(f, \sigma)\) according to distribution \(\rho(\sigma'|f, \sigma)\), set \(\sigma \leftarrow \sigma'\).
5: end while

Clearly, if the algorithm terminates then it produces a flawless object \(\sigma\). The works [11, 9, 2] used specific strategies \(\Lambda\). As stated in the introduction, our goal is to understand when an arbitrary strategy can be used. This means that flaw \(f\) in line 3 is selected according to some distribution which is a function of the entire past execution history\(^2\). Note that if flaw \(f \in F_{\sigma}\) in line 3 depends only on \(\sigma\) then the algorithm can be viewed a random walk in a Markov chain with states \(\Omega\), while in a more general case the walk can be non-Markovian.

2.1 Walks and the potential causality graph

We say that \(\sigma \xrightarrow{f} \sigma'\) is a (valid) walk if it is possible to get from state \(\sigma\) to \(\sigma'\) by “addressing” flaw \(f\) as described in the algorithm, i.e. if two conditions hold: \(f \in F_{\sigma}\) and \(\sigma' \in A(f, \sigma)\). Whenever we write \(\sigma \xrightarrow{f} \sigma'\), we mean that it is a valid walk.

In many applications resampling oracles satisfy a special condition called atomicity [1].

Definition 1. \(\rho\) is called atomic if for any \(f \in F\) and \(\sigma' \in \Omega\) there exists at most one object \(\sigma \in \Omega\) such that \(\sigma \xrightarrow{f} \sigma'\).

Next, we need to describe “dependences” between flaws in \(F\). Let \(\sim\) be some symmetric relation on \(F\) (so that \((F, \sim)\) is an undirected graph). It is assumed to be fixed throughout the paper. For a flaw \(f \in F\) let \(\Gamma(f) = \{ g \in F \mid f \sim g\}\) be the set of neighbors of \(f\). Note, we may or may not have \(f \sim f\), and so \(\Gamma(f)\) may or may not contain \(f\). We will denote \(\Gamma^+(f) = \Gamma(f) \cup \{f\}\), and also \(\Gamma(S) = \bigcup_{f \in S} \Gamma(f)\) and \(\Gamma^+(S) = \bigcup_{f \in S} \Gamma^+(f)\) for a subset \(S \subseteq F\).

\(^2\)“Flaws” \(f\) correspond to “bad events” \(E_i\) in [9]. The distribution over \(\Omega\) was denoted in [9] as \(\mu\), the states of \(\Omega\) as \(\omega\), and the resampling oracle for the bad event \(E_i\) at state \(\omega \in \Omega\) as \(r_i(\omega)\).

\(^3\)The description of the algorithm in [10] says “pick an arbitrary violated event”. This is consistent with our definition of an “arbitrary strategy”: in the analysis Moser and Tardos mention that this selection must come from some fixed procedure (either deterministic or randomized), so that expected values are well-defined.
**Definition 2.** Undirected graph \((F, \sim)\) is called a potential causality graph for \(\rho\) if and only if it satisfies the following condition:

\[
\sigma \overset{f}{\rightarrow} \sigma' \text{ there holds } F_{\sigma'} \subseteq (F_{\sigma} - \{f\}) \cup \Gamma(f).
\]

In other words, \(\Gamma(f)\) must contain all flaws that can appear after addressing flaw \(f\) at some state. Also, \(\Gamma(f)\) must contain \(f\) if addressing \(f\) at some state can fail to eradicate \(f\).

Note that in Definition 2 we deviated slightly from [1,2]: in their analysis the potential causality graph was directed and therefore in certain cases could capture more information about \(D\). While directed graphs do matter in some applications (see examples in [1,2]), we believe that in a typical application the potential causality relation is symmetric. Using an undirected graph will be essential for incorporating commutativity.

A subset \(S \subseteq F\) will be called independent if for any distinct \(f, g \in S\) we have \(f \sim g\). (Thus, loops \(f \sim f\) in the graph \((F, \sim)\) do not affect the definition of independence). For a subset \(S \subseteq F\) we denote \(\text{Ind}(S) = \{T \subseteq S \mid T \text{ is independent}\}\).

**2.2 Commutativity**

We now formulate new conditions that will allow an arbitrary flaw selection rule to be used.

**Definition 3.** \((\rho, \sim)\) is called weakly commutative if there exists a mapping \(\text{SWAP}\) that sends any walk \(\sigma_1 \overset{f}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3\) with \(f \sim g\) to another valid walk \(\sigma_1 \overset{g}{\rightarrow} \sigma'_2 \overset{f}{\rightarrow} \sigma_3\), and this mapping is injective.

Note that in the atomic case the definition can be simplified. Namely, \((\rho, \sim)\) is weakly commutative if and only if it satisfies the following condition:

- For any walk \(\sigma_1 \overset{f}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3\) with \(f \sim g\) there exists state \(\sigma'_2 \in \Omega\) such that \(\sigma_1 \overset{g}{\rightarrow} \sigma'_2 \overset{f}{\rightarrow} \sigma_3\) is also a walk.

Indeed, by atomicity the state \(\sigma'_2\) is unique, and so mapping \(\text{SWAP}\) in Definition 3 is constructed in a natural way. This mapping is reversible and thus injective.

For several results we will also need a stronger property.

**Definition 4.** \((\rho, \sim)\) is called strongly commutative (or just commutative) if for any walk \(\tau = \sigma_1 \overset{f}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3\) with \(f \sim g\) and \(\text{SWAP}(\tau) = \sigma_1 \overset{g}{\rightarrow} \sigma'_2 \overset{f}{\rightarrow} \sigma_3\) there holds

\[
\rho(\sigma_2|f, \sigma_1)\rho(\sigma_3|g, \sigma_2) = \rho(\sigma'_2|g, \sigma_1)\rho(\sigma_3|f, \sigma'_2) \tag{1}
\]

It is straightforward to check that strong commutativity holds in the variable model of Moser and Tardos. In fact, an additional property holds: for any \(\sigma_1 \overset{f}{\rightarrow} \sigma_2 \overset{g}{\rightarrow} \sigma_3\) with \(f \sim g\) there exists exactly one state \(\sigma'_2 \in \Omega\) such that \(\sigma_1 \overset{g}{\rightarrow} \sigma'_2 \overset{f}{\rightarrow} \sigma_3\). Checking strong commutativity for non-Cartesian spaces \(\Omega\) is more involved; we refer to Section 6 for details.

**2.3 Parallel version**

We will also consider the following version of the algorithm (see Algorithm 2). It is equivalent to the parallel algorithm of Moser and Tardos [16] in the case of the variable model, and to the parallel algorithm of Harris and Srinivasan [8] in the case of permutations. It is also closely related to the “MaximalSetResample” algorithm of Harvey and Vondrák [9] (see below).

Lines 3-8 will be called a round. In some cases each round admits an efficient parallel implementation (with a polylogarithmic running time). For example, this holds in the variable model of Moser and Tardos [16]. Also, Harris and Srinivasan [8] presented an efficient implementation for permutations. Accordingly, we will be interested in the number of rounds of the algorithm.
Algorithm 2 Parallel random walk.
1: sample $\sigma \in \Omega$ according to distribution $\omega^{\text{init}}$
2: while $F_\sigma$ non-empty do
3: \hspace{1em} set $I = \emptyset$
4: \hspace{1em} while set $F_\sigma - \Gamma^+(I)$ is non-empty do
5: \hspace{2em} pick some $f \in F_\sigma - \Gamma^+(I)$
6: \hspace{2em} sample $\sigma' \in A(f, \sigma)$ according to $\rho(\sigma'|f, \sigma)$, set $\sigma \leftarrow \sigma'$.
7: \hspace{1em} set $I \leftarrow I \cup \{f\}$
8: \hspace{1em} end while
9: end while

Note, during round $r$ set $F_\sigma - \Gamma^+(I)$ in line 5 shrinks from iteration to iteration (and so flaw $f$ in line 5 satisfies $f \in F_{\sigma_r}$, where $\sigma_r$ is the state in the beginning of round $r$). This property can be easily verified using induction and Definition 2.

**π-stable strategy** Let us fix a total order $\preceq_\pi$ on $F$ defined by some permutation $\pi$ of $F$. Consider a version of Algorithm 2 where flaw $f$ in line 5 is selected as the lowest flaw in $F_\sigma - \Gamma^+(I)$ (with respect to $\preceq_\pi$). This corresponds to Algorithm 1 with a specific strategy $\Lambda$; this strategy will be called $\pi$-stable. It coincides with the MaximalSetResample algorithm of Harvey and Vondrák [9]. Although we focus on the commutative case, we will also state results for $\pi$-stable strategies since they follow automatically from the proof (which is based on the analysis of $\pi$-stable walks).

### 2.4 Algorithmic LLL conditions

In this section we formulate sufficient conditions under which a flawless object will be guaranteed to exist. The conditions involve two vectors, $\lambda$ and $\mu$. Roughly speaking, $\lambda$ characterizes resampling oracles and $\mu$ characterizes graph $(F,\sim)$.

**Definition 5.** The pair $(\rho,\sim)$ is said to satisfy Algorithmic LLL conditions if there exist vectors $\lambda, \mu \in \mathbb{R}^{|F|}$ such that

\begin{align}
\lambda_f &\geq \sum_{\sigma \in f, \sigma' \in A(f, \sigma)} \rho(\sigma'|f, \sigma) \frac{\omega(\sigma)}{\omega(\sigma')} \quad \forall f \in F, \sigma' \in \Omega \tag{2a} \\
\frac{\lambda_f}{\mu_f} &\leq \sum_{S \in \text{Ind}(\Gamma(f))} \mu(S) \leq \theta \quad \forall f \in F \tag{2b}
\end{align}

where $\theta \in (0,1)$ is some constant and $\mu(S) = \prod_{g \in S} \mu_g$.

Of course, vector $\lambda$ can be easily eliminated from (2). However, it is convenient to have it explicitly since in many cases it has a natural interpretation. Achlioptas and Iliopoulos [2] called $\lambda$ flaw charges, though instead of (2a) they used slightly stronger conditions. Namely, they considered the following cases; it is straightforward to check that in each one of them vector $\lambda$ satisfies (2a):

- The case from their earlier work [1], which in the current terminology can be described as follows: $\omega$ is a uniform distribution over $\Omega$, $\rho(\cdot|f, \sigma)$ is a uniform distribution over $A(f, \sigma)$, and $\rho$ is atomic. They then defined $\lambda_f = 1/\min_{\sigma \in f} |A(f, \sigma)|$.
- Regenerating resampling oracles of Harvey and Vondrák [9] specified by the equation

$$
\frac{1}{\omega(f)} \sum_{\sigma \in f} \rho(\sigma'|f, \sigma) \omega(\sigma) = \omega(\sigma') \quad \forall f \in F, \sigma' \in \Omega
$$
where $\omega(f) = \sum_{\sigma \in f} \omega(\sigma)$. In this case Achlioptas and Iliopoulos [2] defined $\lambda_f = \omega(f)$.

- In the general case, [2] defined flaw charges via

\[
\lambda_f = b_f \max_{\sigma \in f, \sigma' \in A(f, \sigma)} \left\{ \rho(\sigma', f, \sigma) \omega(\sigma) \right\}
\]

where $b_f = \max_{\sigma \in \Omega} \left| \{ \sigma \in f : \sigma' \in A(f, \sigma) \} \right|$.

**Remark 1.** An alternative condition that appeared in the literature (for certain $\lambda$'s) is

\[
\frac{\lambda_f}{\mu_f} \sum_{S \subseteq \Gamma(f)} \mu(S) \prod_{g \in \Gamma(f)} (1 + \mu_g) \leq \theta
\]

Clearly, (2b) is weaker than (3). We mention that (3) is analogous to the original LLL condition in [5], while [2b] corresponds to a general condition by Bissacot et al. [3] (with the matching algorithmic version by Pedgen [18] who considered the variable model of Moser and Tardos). It is known that the cluster expansion version can give better results for some applications, see e.g. [4, 17, 9].

**Shearer’s condition** Shearer [19] gave a sufficient and necessary condition for a general LLL to hold for a given dependency graph. Kolipaka and Szegedy [11] showed that this condition is sufficient for the Moser-Tardos algorithm, while Harvey and Vondrák [9] generalized the analysis to regenerating resampling oracles. We will show that the same analysis holds for the framework considered in this paper.

Consider vector $p \in \mathbb{R}^{|F|}$. For a subset $S \subseteq F$ denote $p^S = \prod_{f \in S} p_f$; this is a monomial in variables $\{p_f | f \in F\}$. Also, define polynomial $q_S$ as follows:

\[
q_S = q_S(p) = \sum_{I : S \subseteq I \in \text{Ind}(F)} (-1)^{|I|-|S|} p^I
\]

**Definition 6.** Vector $p$ is said to satisfy the Shearer’s condition if $q_S(p) \geq 0$ for all $S \subseteq F$, and $q_{\emptyset}(p) > 0$.

The pair $(\rho, \sim)$ is said to satisfy Shearer’s condition if there exist vector $p$ satisfying Shearer’s condition, vector $\lambda$ satisfying (2a), and a constant $\theta \in (0, 1)$ such that $\lambda_f \leq \theta \cdot p_f$ for all $f \in F$.

### 3 Our results

First, we state our results for the sequential version (Algorithm 1). Unless mentioned otherwise, the flaw selection strategy and the initial distribution $\omega^{\text{init}}$ are assumed to be arbitrary.

**Theorem 7.** Suppose that $(\rho, \sim)$ satisfies either condition (2) or the Shearer’s condition, and one of the following holds:

(a) Algorithm 1 uses a $\pi$-stable strategy.

(b) $(\rho, \sim)$ is weakly commutative and atomic.

(c) $(\rho, \sim)$ is strongly commutative.

Define

\[
\gamma^{\text{init}} = \max_{\sigma \in \Omega} \frac{\omega^{\text{init}}(\sigma)}{\omega(\sigma)}, \quad \text{Ind}^{\text{init}} = \begin{cases} \bigcup_{\sigma \in \text{supp}(\omega^{\text{init}})} \text{Ind}(F_\sigma) & \text{in cases (a,b)} \\ \text{Ind}(F) & \text{in the case (c)} \end{cases}
\]
where \( \text{supp}(\omega^{\text{init}}) = \{ \sigma \in \Omega | \omega^{\text{init}}(\sigma) > 0 \} \) is the support of \( \omega^{\text{init}} \). The probability that Algorithm 1 produces a flawless object in fewer than \( T + r \) steps is at least \( 1 - \theta^r \) where

\[
T = \frac{1}{\log \theta^{-1}} \left( \log \gamma^{\text{init}} + \log \sum_{R \in \text{Ind}^{\text{init}}} \mu(R) \right)
\]

and \( \mu(R) = \prod_{f \in R} \mu_f \) (in the case of condition (2)) or \( \mu(R) = \frac{q_R(\rho)}{q_\emptyset(\rho)} \) (in the case of the Shearer’s condition).

Note that part (a) of Theorem 7 is a minor variation of existing results [9, 2] (except that our precondition (2a) unifies conditions in previous works - see Section 2.4):

- Harvey and Vondrák [9] proved Theorem 7(a) in the case of regenerating oracles and distributions \( \omega^{\text{init}} = \omega \), with a slightly different expression for \( T \).
- Achlioptas and Iliopoulos [2] proved the result for the “RecursiveWalk” strategy in the special cases described in Section 2.4 (and assuming condition (2b)).

Parts (b,c) are new results.

Remark 2. The possibility of using distribution \( \omega^{\text{init}} \) which is different from \( \omega \) was first proposed by Achlioptas and Iliopoulos in [7]. Namely, they used a distribution with \( |\text{supp}(\omega^{\text{init}})| = 1 \), and later extended it to arbitrary distributions \( \omega^{\text{init}} = \omega \), with a slightly different expression for \( T \).

There is a trade-off in choosing \( \omega^{\text{init}} \): smaller \( \text{supp}(\omega^{\text{init}}) \) leads to a smaller set \( \text{Ind}^{\text{init}} \) but increases the constant \( \gamma^{\text{init}} \). It is argued in [2] that using \( \omega^{\text{init}} \neq \omega \) can be beneficial when sampling from \( \omega \) is a difficult problem, or when the number of flaws is exponentially large.

Next, we analyze the parallel version.

Theorem 8. Suppose that \((\rho, \sim)\) satisfies either condition (2) or the Shearer’s condition, and is strongly commutative. Then the probability that Algorithm 2 produces a flawless object in fewer than \( T + r \) rounds is at least \( 1 - \theta^r \) where

\[
T = \frac{1}{\log \theta^{-1}} \left( \log \gamma^{\text{init}} + \log \sum_{f \in F} \mu_f \right)
\]

where \( \gamma^{\text{init}} \) is the constant from Theorem 7 and \( \mu_f = \frac{q_f(\rho)}{q_\emptyset(\rho)} \) (in the case of the Shearer’s condition).

Our techniques The general idea of the proofs is to construct a “swapping mapping” that transforms “walks” (which are possible executions of the algorithm) to some canonical form by applying swap operations from Definition 3. Importantly, we need to make sure that the mapping is injective: this will guarantee that the sum over original walks is smaller or equal than the sum over “canonical walks”. We then upper-bound the latter sum using some standard techniques [11, 9]. We use two approaches:

1. Theorem 7(b): transforming walks to “forward stable sequences” (a forward-looking analysis). This works only in the atomic case (under the weak commutativity assumption), and can make use of the knowledge of the set \( \text{supp}(\omega^{\text{init}}) \), leading to a tighter definition of the set \( \text{Ind}^{\text{init}} \).
2. Theorems 7(c) and 8 transforming walks to “backward stable sequences” (a backward-looking analysis). This works in the non-atomic cases, but requires strong commutativity.

In this approach the “roots” of stable sequences are on the right, and have no connection to \( \omega^{\text{init}} \); this means that we must use \( \text{Ind}^{\text{init}} = \text{Ind}(F) \).

Analyzing the parallel version requires dealing with “partial execution logs” instead of “full execution logs”. It appears that this is possible only with backward sequences.

Note that previously a backward-looking analysis (with either “stable sequences” or “witness trees”) was used for the variable model of Moser and Tardos [16, 11, 18], while a forward-looking analysis was used for LLL versions on non-Cartesian spaces [1, 9, 2] and also on Cartesian spaces [7].

After the first version of this work [12] we learned about a recent book draft by Knuth [10]. He considers the variable model of Moser-Tardos, and gives an alternative proof of algorithm’s correctness which is also based on swapping arguments (justified by a technique of “coupling” two random sources, similar to [16]). We emphasize that we go beyond the variable model, in which case justifying “swapping” seems to require different techniques.

The proofs of Theorems 7 and 8 are given below in Sections 4 and 5. (A brief overview of these proofs can be found in [13]). The most technical part is probably constructing an injective swapping mapping for transforming to backward stable sequences (proved in Section 5.6). In Section 6 we describe our third result, which is a proof of strong commutativity of some existing resampling oracles. We also consider one application, namely rainbow matchings in complete graphs.

4 Proof of Theorem 7(a)

First, we will describe those results that do not involve commutativity. In particular, we will prove Theorem 7(a). Techniques used in this section are a combination of ideas from previous works [11, 1, 9, 2].

We write \( f \cong g \) for flaws \( f, g \in F \) if either \( f \sim g \) or \( f = g \) (and \( f \not\equiv g \) otherwise).

A walk of length \( t \) is a sequence \( \tau = \sigma_1 \xrightarrow{w_1} \sigma_2 \ldots \sigma_t \xrightarrow{w_t} \sigma_{t+1} \) such that \( w_i \in F_{\sigma_i} \) and \( \sigma_{i+1} \in A(w_i, \sigma_i) \) for \( i \in [t] \). Its length is denoted as \( |\tau| = t \). For such a walk we define quantity

\[
p(\tau) = \omega^{\text{init}}(\sigma_1) \cdot \prod_{i \in [t]} \rho(\sigma_{i+1}|w_i, \sigma_i)
\]

(7)

Let \( \Lambda \) be the strategy for selecting flaws used in Algorithm 1. We assume in the analysis that this strategy is deterministic, i.e. the flaw \( w_i \) in a walk \( \tau = \sigma_1 \xrightarrow{w_1} \ldots \xrightarrow{w_{i-1}} \sigma_i \xrightarrow{w_i} \sigma_{i+1} \) is uniquely determined by the previous history \( \tau_i = \sigma_1 \xrightarrow{w_1} \ldots \xrightarrow{w_{i-1}} \sigma_i \). This assumption can be made w.l.o.g.: if \( \Lambda \) is randomized (i.e. a distribution over some set of deterministic strategies) then the claim of Theorem 7 can be obtained by taking the appropriate expectation over strategies (whose number is finite for a fixed finite \( t \)). A similar argument applies to Theorem 8.

A walk \( \tau \) of length \( t \) that can be produced by Algorithm 1 with a positive probability will be called a bad \( t \)-trajectory. Equivalently, it is a walk that starts at a state \( \sigma \in \text{supp}(\omega^{\text{init}}) \) and follows strategy \( \Lambda \). Note that it goes only through flawed states (except possibly the last state). Let \( \text{Bad}(t) \) be the set of all bad \( t \)-trajectories. Clearly, for any \( \tau \in \text{Bad}(t) \) the probability that the algorithm will produce \( \tau \) equals \( p(\tau) \), as defined in (7). This gives

Proposition 9. The probability that Algorithm 1 takes \( t \) steps or more equals \( \sum_{\tau \in \text{Bad}(t)} p(\tau) \).

If \( W = w_1 \ldots w_t \) is the complete sequence of flaws in a walk \( \tau \) then we will write \( \tau = \bullet W \). If we want to indicate certain intermediate states of \( \tau \) then we will write them in square brackets at appropriate positions, e.g. \( \tau = [\sigma_1|w_1w_2|\sigma_3|w_4w_5|\sigma_6] \).
In general, a sequence of flaws will be called a word, and a sequence of flaws together with some intermediate states (such as \( [\sigma_1]w_1w_2[\sigma_3]w_4w_5[\sigma_6] \)) will be called a pattern. For a pattern \( X \) we define \( \langle X \rangle = \{ \tau \mid \tau \bullet = X \} \) to be the set of walks consistent with \( X \). The length of \( X \) (i.e. the number of flaws in it) is denoted as \( |X| \).

**Lemma 10.** For any word \( W \) and state \( \sigma \) we have

\[
\sum_{\tau \in \langle W|\sigma| \rangle} p(\tau) \leq \gamma^{\text{init}} \cdot \lambda_W \cdot \omega(\sigma) \tag{8a}
\]

\[
\sum_{\tau \in \langle W \rangle} p(\tau) \leq \gamma^{\text{init}} \cdot \lambda_W \tag{8b}
\]

where for a word \( W = w_1 \ldots w_t \) we denoted \( \lambda_W = \prod_{i \in [t]} \lambda_{w_i} \). (As described in the previous paragraph, \( \langle W|\sigma| \rangle \) is the set of walks \( \omega \) whose sequence of flaws is \( W \) and the last state is \( \sigma \)).

**Proof.** Summing [8a] over \( \sigma \in \Omega \) gives [8b], so it suffices to prove the former inequality. We use induction on the length of \( W \). If \( W \) is empty then \( \langle W[\sigma| \rangle \) contains a single walk \( \tau \) with the state \( \sigma \); we then have \( p(\tau) = \omega^{\text{init}}(\sigma) \), and the claim follows from the definition of \( \gamma^{\text{init}} \) in Theorem 7. This establishes the base case. Now consider a word \( W' = W f \) with \( f \in F \) and a state \( \sigma' \). We can write

\[
\sum_{\tau' \in \langle W'|\sigma'| \rangle} p(\tau') = \sum_{\sigma: \sigma' \in A(f, \sigma)} \sum_{\tau \in \langle W[\sigma| \rangle} p(\tau) \cdot \rho(\sigma'|f, \sigma)
\]

\[
\leq \sum_{\sigma: \sigma' \in A(f, \sigma)} \gamma^{\text{init}} \cdot \lambda_W \cdot \omega(\sigma) \cdot \rho(\sigma'|f, \sigma) \leq \gamma^{\text{init}} \cdot \lambda_W \cdot \omega(\sigma')
\]

where (a) is by the induction hypothesis, and (b) follows from [2a]. This gives the induction step, and thus concludes the proof of the lemma. \( \square \)

The following technical result will also be useful.

**Lemma 11.** Consider a walk \( \tau = \ldots[\sigma]u_1 \ldots u_k \ldots \) where \( k \geq 1 \). Suppose at least one of the following holds:

(a) \( u_k \) is not present in \( \sigma \).
(b) \( u_1 = u_k \).

Then there exists index \( i \in [k - 1] \) such that \( u_i \sim u_k \).

**Proof.** We will assume that \( \tau = \ldots \sigma_1 \overset{u_1}{\to} \sigma_2 \overset{u_2}{\to} \ldots \overset{u_{k-1}}{\to} \sigma_k \overset{u_k}{\to} \sigma_{k+1} \ldots \) where \( \sigma_1 = \sigma \).

(a) Flaw \( u_k \) is present in \( \sigma_k \) (since \( \sigma_k \overset{u_k}{\to} \sigma_{k+1} \) is a walk) but not in \( \sigma_1 \). Thus, there must exist index \( i \in [k - 1] \) such that \( u_k \) is present in \( \sigma_i \) but not in \( \sigma_i \). We know that \( \sigma_i \overset{u_i}{\to} \sigma_{i+1} \) is a valid walk. Thus, addressing \( u_i \) have caused \( u_k \) to appear, and therefore \( u_i \sim u_k \).

(b) Assume that \( u_k \) is present in \( \sigma_1 \) (otherwise the claim holds by (a)). If \( u_k \) is present in \( \sigma_2 \) then \( u_k \sim u_k \) (since addressing \( u_1 = u_k \) at state \( \sigma_1 \) did not eliminate flaw \( u_k \)). Otherwise, if \( u_k \) is not present in \( \sigma_2 \), we can apply part (a) and conclude that \( u_i \sim u_k \) for some \( i \in [2, k - 1] \). \( \square \)

### 4.1 Stable walks and stable sequences

As shown in [9], if \( \Lambda \) is a \( \pi \)-stable strategy then walks \( \tau \) that it produces have a special structure: the word \( W \) corresponding to \( \tau \) can be uniquely described by a stable sequence. This section gives all necessary definitions.
Definition 12. A sequence of sets \( \varphi = (I_1, \ldots, I_s) \) with \( s \geq 1 \) is called stable if \( I_r \in \text{Ind}(F) \) for each \( r \in [s] \) and \( I_{r+1} \subseteq \Gamma^+(I_r) \) for each \( r \in [s-1] \).

Definition 13. A word \( W = w_1 \ldots w_t \) is called stable if it can be partitioned into non-empty words as \( W = W_1 \ldots W_s \) such that flaws in each word \( W_r \) are distinct, and the sequence \( (I_1, \ldots, I_s) \) is stable where \( I_r \) is the set of flaws in \( W_r \) (for \( r \in [s] \)). If in addition each word \( W_r = w_i \ldots w_j \) satisfies \( w_i \prec_{\pi} \ldots \prec_{\pi} w_j \) then \( W \) is called \( \pi \)-stable.

A walk \( \tau = W \) is called stable (\( \pi \)-stable) if the word \( W \) is stable (\( \pi \)-stable).

It can be seen that for a stable word the partitioning in Definition 13 is unique, and can be obtained by the following algorithm. Start with one segment containing \( w_1 \), and then for \( i = 2, \ldots, t \) do the following: if there exists flaw \( w_k \) in the last segment with \( w_k \cong w_i \) then start a new segment containing \( w_i \), otherwise add \( w_i \) to the last segment. (If this algorithm is applied to an arbitrary word \( W \) then it may fail to produce a stable sequence since in the latter case, when \( w_i \) is added to the last segment \( I_r, w_i \) may not belong to \( \Gamma^+(I_{r-1}) \).

Proposition 15. Suppose that strategy \( \Lambda \) is implemented as in Algorithm 2 (with a deterministic choice in line 5). Then any \( \tau \in \text{Bad}(t) \) is a stable walk. If in addition \( \Lambda \) is a \( \pi \)-stable strategy (i.e. flaw \( f \) in line 5 is chosen as the lowest flaw in \( F \) with respect to \( \leq_{\pi} \)) then any \( \tau \in \text{Bad}(t) \) is a \( \pi \)-stable walk.

Proof. Let \( s \) be the number of rounds of Algorithm 2 that produced walk \( \tau \), and \( I_r \subseteq [s] \) be the set of flaws addressed in round \( r \in [s] \), or equivalently the set \( I \) at the end of round \( r \) (with a possible exception for \( r = s \): \( I_r \) may correspond to the “intermediate” set \( I_r \), depending on where \( \tau \) was “cut”). By this definition, we have \( \tau = W_1 \ldots W_s \) where \( W_r \) is a word containing the flaws in \( I_r \) in some order (with \( |W_r| = |I_r| \)). We will prove that \( (I_1, \ldots, I_s) \) is a stable sequence; this will imply the first claim of the proposition.

The independence of each set \( I_r \) follows directly from the construction. Consider \( r \in [2, s] \), and let \( \tau = W_1 \ldots W_{r-1} \sigma W_r \ldots \). We need to show that for each flaw \( f \) present in \( W_r \) (i.e. \( f \in I_r \)) we have \( f \in \Gamma^+(I_{r-1}) \). Suppose it is not the case. Lemma 11(a) gives that \( f \) is present in \( \sigma \). Therefore, set
$F_\sigma - \Gamma^+(I_{r-1})$ is non-empty (it contains $\sigma$). But then round $r-1$ would not have terminated at the state $\sigma$ - a contradiction.

It remains to consider the case when flaw $f$ in line 5 is chosen as the lowest flaw in $F_\sigma - \Gamma^+(I)$ with respect to $\preceq_\pi$. Consider round $r$, and let $W_r = w_i \ldots w_j$. Definition \ref{def:stabilization} and an induction argument show that during this round set $F_\sigma - \Gamma^+(I)$ in line 5 shrinks from iteration to iteration. This implies that $w_i \prec_\pi \ldots \prec_\pi w_j$, as desired. \hfill $\square$

We also need the following observation.

Lemma 16. If $\tau \overset{\bullet}{=} W$ is a $\pi$-stable walk starting at state $\sigma_1$ then $R_W \in \text{Ind}(F_{\sigma_1})$.

Proof. By definition of a stable word, set $R_W$ is independent, and corresponds to some prefix $w_1 \ldots w_k$ of the word $W$. It remains to show that for each $i \in [k]$ we have $w_i \in F_{\sigma_1}$. This follows from Lemma \ref{lemma:stabilization}(a) and the condition that $w_j \sim w_i$ for all $j \in [i - 1]$.

We now have all ingredients to prove part (a) of Theorem \ref{thm:main}. Consider Algorithm \ref{alg:stable} with a $\pi$-stable strategy. Then each walk $\tau \overset{\bullet}{=} W$ from $\text{Bad}(t)$ is $\pi$-stable, with $W \in \text{Stab}_\pi(t)$. By the definition of $\text{Bad}(t)$ and Lemma \ref{lemma:stabilization} we also know that $R_W \in \text{Ind}_{\text{init}} = \bigcup_{\sigma \in \text{supp}(\omega_{\text{init}})} \text{Ind}(F_\sigma)$. Therefore,

$$Pr[\#\text{steps} \geq t] = \sum_{\tau \in \text{Bad}(t)} p(\tau) \leq \sum_{R \in \text{Ind}_{\text{init}}} \sum_{W \in \text{Stab}_\pi(R,t)} \sum_{\tau \in \langle W \rangle} p(\tau) \leq (a) \sum_{R \in \text{Ind}_{\text{init}}} \sum_{W \in \text{Stab}_\pi(R,t)} \gamma_{\text{init}} \cdot \lambda_W \overset{(b)}{\leq} \gamma_{\text{init}} \cdot \sum_{R \in \text{Ind}_{\text{init}}} \mu(R) \cdot \theta^t = \theta^{t-T}$$

where (a) holds by Lemma \ref{lemma:stabilization} (b) holds by Theorem \ref{thm:main} and $T$ is given by the expression in \ref{eq:main}.

5 Commutativity: Proof of Theorems \ref{thm:main}(b,c) and \ref{thm:reverse}

From now on we assume that $(\rho, \sim)$ is weakly commutative. Therefore, for any walk $\tau = \ldots \sigma_1 \overset{f}{\to} \sigma_2 \overset{g}{\to} \sigma_3 \ldots$ with $f \not\equiv g$ there exists another walk $\tau' = \ldots \sigma_1 \overset{g}{\to} \sigma_2 \overset{f}{\to} \sigma_3 \ldots$ obtained from $\tau$ by applying the SWAP operator to the subwalk $\sigma_1 \overset{f}{\to} \sigma_2 \overset{g}{\to} \sigma_3$. Such operation will be called a valid swap applied to $\tau$. A mapping $\Phi$ on a set of walks that works by applying some sequence of valid swaps will be called a swapping mapping. Note that if $\tau' = \Phi(\tau)$ then the first and the last states of $\tau'$ coincide with that of $\tau$, and $\lambda_{W'} = \lambda_W$ where $\tau \overset{\bullet}{=} W$, $\tau' \overset{\bullet}{=} W'$. Furthermore, if $(\rho, \sim)$ is strongly commutative then $p(\tau') = p(\tau)$.

We now deal with the case when $\Lambda$ is an arbitrary deterministic strategy, and so walks $\tau \in \text{Bad}(t)$ are not necessarily $\pi$-stable. Our approach will be to construct a bijective swapping mapping $\Phi$ that sends walks $\tau \in \text{Bad}(t)$ to some canonical walks, namely either to $\pi$-stable walks (which will work only in the atomic case) or to the reverse of such walks (which will work in the general case).

5.1 Proof of Theorem \ref{thm:main}(b)

In this section we assume that $(\rho, \sim)$ is atomic. This gives the following observation.

Proposition 17 \ref{prop:atomic}. Walk $\tau = \sigma_1 \overset{u_1}{\to} \sigma_2 \ldots \sigma_t \overset{u_t}{\to} \sigma_{t+1}$ can be uniquely reconstructed from the sequence of flaws $w_1 \ldots w_t$ and the final state $\sigma_{t+1}$.
Proof. By atomicity, state $\sigma_i$ can be uniquely reconstructed from the flaw $w_i$ and the state $\sigma_{i+1}$. Applying this argument for $i = t, t - 1, \ldots, 1$ gives the claim.

The proposition allows us to write walks more compactly as $\tau = w_1 \ldots w_t[\sigma_{t+1}]$. Also, Lemma 10 gives for a walk $\tau = W[\sigma_{t+1}]$ that

$$p(\tau) \leq \gamma^{\text{init}} \cdot \lambda_W \cdot \omega(\sigma_{t+1}) \quad (10)$$

In Section 5.5 we will prove the following result.

**Theorem 18.** Suppose that $(\rho, \sim)$ is atomic and weakly commutative. There exists a set of $\pi$-stable walks $\text{Bad}_\pi(t)$ and a swapping mapping $\Phi : \text{Bad}(t) \rightarrow \text{Bad}_\pi(t)$ which is a bijection.

We can now prove Theorem 7(b):

$$\Pr[\#\text{steps} \geq t] = \sum_{\tau \in \text{Bad}(t)} p(\tau) \leq \sum_{\tau = W[\sigma] \in \text{Bad}(t)} \gamma^{\text{init}} \cdot \lambda_W \cdot \omega(\sigma)$$

$$\leq \sum_{\tau = W[\sigma] \in \text{Bad}_\pi(t)} \gamma^{\text{init}} \cdot \lambda_W \cdot \omega(\sigma) = \sum_{R \in \text{Ind}_{\text{init}}} \sum_{W \in \text{Stab}_\pi(R, t)} \sum_{\sigma \in \Omega} \lambda_W \cdot \omega(\sigma)$$

where in (a) we used eq. (10), (b) follows from Theorem 18 and the rest is similar to the derivation in Section 4.2.

### 5.2 Reverse stable sequences

To prove Theorem 7(c) and Theorem 8, we will use reverse stable sequences instead of forward stable sequences.

Walk $\tau$ will be called a **prefix** of a walk $\tau'$ if $\tau'$ starts with $\tau$. Walk $\tau$ is a proper prefix of $\tau'$ if in addition $\tau' \neq \tau$. A word $W$ is called a prefix of $\tau$ if $\tau = WU$ for some word $U$.

A set of walks $\mathcal{X}$ will be called valid if (i) all walks in $\mathcal{X}$ follow the same deterministic strategy (not necessarily the one used in Algorithm 1), and (ii) for any $\tau, \tau' \in \mathcal{X}$ the walk $\tau$ is not a proper prefix of $\tau'$.

For a walk $\tau$ containing flaw $f$ we define word $W^\tau_f$ as the longest prefix of $\tau$ that ends with $f$. Thus, we have $\tau = W^\tau_fU$ where $W^\tau_f = \ldots f$ and word $U$ does not contain $f$. We will also allow $f = \emptyset$; in this case we say that any walk $\tau$ contains such $f$, and define word $W^\tau_f$ so that $\tau = W^\tau_f$. Recall that for a word $W = w_1 \ldots w_t$ its reverse is denoted as $\text{REV}[W] = w_t \ldots w_1$. The following result is proven in Section 5.6.

**Theorem 19.** Fix $f \in F \cup \{\emptyset\}$, and let $\mathcal{X}_f^\pi$ be a valid set of walks containing $f$. If $(\rho, \sim)$ is weakly commutative then there exists a set of walks $\mathcal{X}_f^\pi$ and a swapping mapping $\Phi^f : \mathcal{X}_f^\pi \rightarrow \mathcal{X}_f^\pi$ which is a bijection such that

(a) for any $\tau \in \mathcal{X}_f^\pi$, the word $W = \text{REV}[W^\tau_f]$ is $\pi$-stable ($W \in \text{Stab}_\pi$), and $R_W = \{f\}$ if $f \in F$;

(b) for any word $W$ the set $\{\tau \in \mathcal{X}_f^\pi \mid \text{REV}[W^\tau_f] = W\}$ is valid.
5.3 Proof of Theorem 7(c)

In this case we have \( \text{Ind}^{\text{init}} = \text{Ind}(F) \). We will use Theorem 19 with \( f = \emptyset \) and \( \mathcal{X}^f = \text{Bad}(t) \). Part (a) gives that for any \( \tau \in \mathcal{X}^{\emptyset}_\pi \) the word \( W = \text{REV}[W] \) satisfies \( W \in \text{Stab}_\pi(R, t) \) for some \( R \in \text{Ind}(F) \). We can write

\[
Pr[\#\text{steps} \geq t] = \sum_{\tau \in \text{Bad}(t)} p(\tau) \leq \sum_{\tau \in \mathcal{X}^\emptyset_\pi} p(\tau) \leq \gamma^{1-\gamma}
\]

where in (a) we used bijectiveness of mapping \( \Phi^G \) and strong commutativity of \((\rho, \sim)\), and the rest is similar to the derivation in Section 4.2.

5.4 Analysis of the parallel version: Proof of Theorem 8

We now analyze Algorithm 2 with a deterministic choice of flaw \( f \) in line 5. Equivalently, this can be viewed as running Algorithm 1 with some deterministic strategy.

**Theorem 20.** Consider a word \( W \) and a valid set of walks \( \mathcal{X} \) such that \( W \) is a prefix of every walk in \( \mathcal{X} \). Then

\[
\sum_{\tau \in \mathcal{X}} p(\tau) \leq \gamma^{\text{init}} \cdot \lambda_W
\]

**Proof.** We use induction on \( \sum_{\tau \in \mathcal{X}} (|\tau| - |W|) \). The base case \( \sum_{\tau \in \mathcal{X}} (|\tau| - |W|) = 0 \) is straightforward: we then have \( \mathcal{X} \subseteq \langle W \rangle \), and so the claim follows from Lemma 10. Consider a valid set \( \mathcal{X} \) with \( \sum_{\tau \in \mathcal{X}} (|\tau| - |W|) \geq 1 \). Let \( \hat{\tau} \) be the longest walk in \( \mathcal{X} \), then \( |\hat{\tau}| \geq |W| + 1 \). Let \( \hat{\tau}^- \) be the proper prefix of \( \hat{\tau} \) of length \( |\hat{\tau}| - 1 \). We have \( \hat{\tau}^- \notin \mathcal{X} \) since \( \mathcal{X} \) is a valid set. Define set \( \mathcal{Y} \) as follows:

\[ \mathcal{Y} = \{ \tau \in \mathcal{X} \mid \hat{\tau}^- \text{ is a proper prefix of } \tau \} \]

By the choice of \( \hat{\tau} \) we get \( |\tau| = |\hat{\tau}| \) for all \( \tau \in \mathcal{Y} \), and so we must have \( \tau = \hat{\tau}^- \to w \to \sigma \) for some \( w \in F \) and \( \sigma \in \Omega \). Since all walks in \( \mathcal{X} \) follow the same deterministic strategy, the flaw \( w \) in the expression \( \tau = \hat{\tau}^- \to w \to \sigma \) must be the same for all \( \tau \in \mathcal{Y} \). Thus, \( \mathcal{Y} = \{ \hat{\tau}^- \to w \to \sigma \mid \sigma \in \mathcal{Y} \} \) for some set of flaws \( \mathcal{Y} \subseteq F \).

Define \( \mathcal{X}^- = (\mathcal{X} \setminus \mathcal{Y}) \cup \{ \hat{\tau}^- \} \). We have

\[
\sum_{\tau \in \mathcal{X}} p(\tau) - \sum_{\tau \in \mathcal{X}^-} p(\tau) = \sum_{\tau \in \mathcal{Y}} p(\tau) - p(\hat{\tau}^-) = p(\hat{\tau}^-) \cdot \left[ \sum_{\sigma \in \mathcal{Y}} p(\sigma | w, \hat{\sigma}) - 1 \right] \leq 0
\]

It is straightforward to check that set \( \mathcal{X}^- \) is valid, and \( W \) is a prefix of every walk in \( \mathcal{X}^- \). Using the induction hypothesis for \( \mathcal{X}^- \) and the inequality above gives the claim for \( \mathcal{X} \). \( \square \)

Consider executions of Algorithm 2 consisting of at least \( s \) rounds. For each such execution let \( \tau \) be the walk containing flaws addressed in the first \( s - 1 \) rounds and the first flaw addressed in round \( s \). Let \( \text{BadPar}(s) \) be the set of such walks \( \tau \).

We say that a word \( U = u_1 \ldots u_s \) is a *chain* of a walk \( \tau \to w_1 \ldots w_l \) if \( U \) is a subsequence of the sequence \( w_1 \ldots w_l \) and \( u_i \equiv u_{i+1} \) for \( i \in [s - 1] \).

**Proposition 21.** (a) For each \( \tau \in \text{BadPar}(s) \) the length of a longest chain in \( \tau \) equals \( s \). (b) Set \( \text{BadPar}(s) \) is valid.

**Proof.** Part (a) follows directly from Proposition 15 and the definition of a stable walk. Let us prove (b). By construction, all walks in \( \text{BadPar}(s) \) follow the same deterministic strategy used in Algorithm 2. Now consider a walk \( \tau' \in \text{BadPar}(s) \). By the definition of \( \tau' \in \text{BadPar}(s) \), any proper prefix \( \tau \) of \( \tau' \) corresponds to an execution of Algorithm 2 with at most \( r - 1 \) rounds, and so the length of a longest chain in \( \tau \) is at most \( r - 1 \). Thus, \( \tau \notin \text{BadPar}(s) \). \( \square \)
Lemma 22. For any word $w$ to word $w'$ are valid swaps. The new word now satisfies the induction hypothesis for $W$ according to $w$ forming a stable word $W$. 

Proof. It suffices to show that $\tau \in \text{BadPar}(s)$ can be transformed to a walk $\tau' = W'[\sigma]$ if and only if $W \equiv W'$. In this case we will write $\tau' \equiv \tau$; again, “$\equiv$” is an equivalence relation on the set of walks.

5.5 Proof of Theorem 18 (swapping mapping for the atomic case)

We can extend a “valid swap” operation to words in a natural way: it is a transformation of the form $W \equiv W'$ if $W'$ can be obtained from $W$ via a sequence of valid swaps. Clearly, “$\equiv$” is an equivalence relation. It can be seen that a walk $\tau \equiv W[\sigma]$ can be transformed to a walk $\tau' = W'[\sigma]$ if and only if $W \equiv W'$. In this case we will write $\tau' \equiv \tau$; again, “$\equiv$” is an equivalence relation on the set of walks.

Lemma 22. For any word $W$ there exists a sequence of valid swaps that transforms $W$ to a $\pi$-stable word.

Proof. It suffices to show that $W$ can be transformed to a stable word via valid swaps. (Transforming a stable word $W$ to a $\pi$-stable word is straightforward: if $W = W_1 \ldots W_s$ is the partition described in Definition 13 then we simply need to apply swaps inside each word $W_r$ to “sort” it according to $\leq_\pi$; any such swap will be valid by the property of $W_r$.)

Let $W = \hat{w}_1 \ldots \hat{w}_t$. We will prove by induction on $i = 1, \ldots, t$ that $W$ can be transformed via valid swaps to a word $W' = w_1 \ldots w_i$ such that the prefix $w_1 \ldots w_i$ is a stable word. The base case $i = 1$ is trivial. Suppose the claim holds for $i - 1$, and let us show it for $i \in [2, t]$.

By the induction hypothesis, $W$ can be transformed to a word $W' = w_1 \ldots w_{i-1}f \ldots w_i$ such that $w_1 \ldots w_i$ is a stable word. Let $w_1 \ldots w_{i-1} = W_1 \ldots W_s$ be the corresponding partition described in Definition 13. Let $W_r$ be the rightmost word that contains a flaw $g$ with $f \preceq g$. (If such word doesn’t exist then we set $r = 0$; thus, $r \in [0, s]$.) If $r = s$ then we can leave the word $W'$ as it is - it satisfies the induction hypothesis for $i$.

Suppose that $r \in [0, s - 1]$. Then we repeatedly swap $f$ with the left neighbor, stopping when $f$ gets between words $W_r$ and $W_{r+1}$ (or at the first position, if $r = 0$). By the definition of $r$, these are valid swaps. The new word now satisfies the induction hypothesis for $i$ (flaw $f$ will be assigned to word $W_{r+1}$).

The lemma means that there exists a swapping mapping $\Phi$ that sends any walk $\tau = W[\sigma] \in \text{Bad}(t)$ to a $\pi$-stable walk $\tau' = W'[\sigma]$. Next, we will show that any such mapping is injective on $\text{Bad}(t)$. 

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Lemma 23. Suppose that walks \( \tau, \tau' \) follow the same deterministic strategy and \( \tau \equiv \tau' \). Then \( \tau = \tau' \).

Proof. We use induction on the length of \( \tau \). The base case is trivial: if \( |\tau| = |\tau'| = 0 \) then condition \( \tau \equiv \tau' \) implies that \( \tau = \tau' = [\sigma] \) for some state \( \sigma \).

Suppose that \( \tau = fW[\sigma] \). Condition \( \tau \equiv \tau' \) means that \( \tau' \) and \( \tau \) start at the same state. Therefore, since \( \tau' \) and \( \tau \) follow the same deterministic strategy, the first flaw addressed in \( \tau' \) is the same as in \( \tau \). Thus, \( \tau' = fW'[\sigma] \). Clearly, walks \( W[\sigma] \) and \( W'[\sigma] \) follow the same deterministic strategy. We will show below that \( W[\sigma] \equiv W'[\sigma] \); this will mean that \( W[\sigma] = W'[\sigma] \) by the induction hypothesis, thus giving the desired result.

We need to show that \( W \equiv W' \). We know that \( fW \equiv fW' \), therefore there exists a sequence of words \( U^{(1)}, \ldots, U^{(k)} \) with \( U^{(1)} = fW, U^{(k)} = fW' \) such that \( U^{(i+1)} \) is obtained from \( U^{(i)} \) via a single valid swap operation. Let \( V^{(i)} \) be the word obtained from \( U^{(i)} \) by moving the first occurrence of flaw \( f \) in \( U^{(i)} \) to the first position. Thus, for each \( i \in [k] \) we have \( V^{(i)} = fW^{(k)} \) for some word \( W^{(k)} \). It can be seen that either \( V^{(i+1)} = V^{(i)} \) (if the valid swap operation applied to \( U^{(i)} \) involved the first occurrence of \( f \)) or \( V^{(i+1)} \) is obtained from \( V^{(i)} \) via a valid swap (otherwise). Thus, either \( W^{(i+1)} = W^{(i)} \) or \( W^{(i+1)} \) is obtained from \( W^{(i)} \) via a valid swap. It remains to notice that \( W^{(1)} = W \) and \( W^{(k)} = W' \). \( \square \)

We can now prove that \( \Phi \) is injective. Suppose that \( \tau, \tau' \in \text{Bad}(t) \) and \( \Phi(\tau) = \Phi(\tau') \). We have \( \tau \equiv \Phi(\tau) = \Phi(\tau') \equiv \tau' \), and so \( \tau \equiv \tau' \). From the previous lemma we obtain that \( \tau = \tau' \), which concludes the proof of Theorem 18.

5.6 Proof of Theorem 19 (swapping mapping for the non-atomic case)

Consider a word \( W = w_1 \ldots w_t \). It will be convenient to alternatively write it as \( W = w_1 \ldots w_t \) where \( w_i = (w_i, n_i) \) and \( n_i \) counts from the left which occurrence of the flaw \( w_i \) it is: \( n_i = |\{j \in [i] \mid w_j = w_i\}| \geq 1 \). Note that all elements \( w_1, \ldots, w_t \) are distinct. Tuple \( w_i \) will be called a named flaw. The flaw associated with the named flaw \( f \) will be denoted without the bold font as \( f \), i.e. \( f = (f, n_f) \).

We will denote the element of \( F \cup \{\emptyset\} \) fixed in the theorem as \( \hat{f} \) (rather than \( f \)).

For a walk \( \tau = w_1 \ldots w_t \) let us define a directed acyclic graph \( G_\tau = (V_\tau, E_\tau) \) as follows: its nodes are \( V_\tau = \{w_1, \ldots, w_t\} \), and the set \( E_\tau \) contains all edges of the form \( (f, g) \) where \( f \cong g \) and \( f \) occurs in \( \tau \) before \( g \), i.e. \( \tau = \ldots f \ldots g \ldots \).

For a walk \( \tau \) containing \( \hat{f} \) we make the following definitions:

- If \( \hat{f} \in F \), let \( W_\tau \) be the set of flaws \( g \in V_\tau \) from which node \( \hat{f} \) can be reached in \( G_\tau \), where \( \hat{f} \) is the named flaw corresponding to the rightmost occurrence of \( \hat{f} \) in \( \tau \). For \( f \in W_\tau \) let \( d_\tau(f) \) be the length of the longest path from \( f \) to \( \hat{f} \) in \( G_\tau \) plus one (so that \( d_\tau(\hat{f}) = 1 \)).
- If \( \hat{f} = \emptyset \), let \( W_\tau = V_\tau \). For \( f \in W_\tau \) let \( d_\tau(f) \) be the length of the longest path from \( f \) to a sink of \( G_\tau \) plus one (so that for any sink node \( f \in V_\tau \) we have \( d_\tau(f) = 1 \)).

Let \( W_\tau \) be the word consisting of the named flaws in \( W_\tau \) listed in the order in which they appear in \( \tau \).

It is straightforward to check that applying valid swaps to \( \tau \) does not affect graph \( G_\tau \) and set \( W_\tau \) (but may affect the order of named flaws in \( W_\tau \)). Our goal will be to apply swaps to \( \tau \) so that in the end word \( W_\tau \) becomes a prefix of \( \tau \), and furthermore this prefix equals the word \( \text{Stab}_\pi(\tau) \) which is defined below.
Definition 24. For a walk $\tau$ containing $\hat{f}$ define word $\text{Stab}_{\pi}(\tau)$ as follows. For an integer $r \geq 1$ let $I_r = \{ f \in W_r \mid d_r(f) = r \}$, and let $W_r$ be the word consisting of the named flaws in $I_r$ sorted in the decreasing order (with respect to $\leq_\pi$). Then $\text{Stab}_{\pi}(\tau) = W_s \ldots W_1$ where $s = \max\{d_r(f) \mid f \in W_r\}$.

It can be seen that we have $f \neq g$ for any distinct $f, g \in I_r$. Therefore, “sorting in the decreasing order” in the definition is a valid operation (where we assume that $f \leq_\pi g$).

Clearly, applying a valid swap to $\tau$ does not affect word $\text{Stab}_{\pi}(\tau)$ (since graph $G_\tau$ and set $W_r$ are unaffected). It is also not difficult to show the following.

Lemma 25. For any $\tau \in \mathcal{X}^f$ word $W = \text{REV}[\text{Stab}_{\pi}(\tau)]$ is $\pi$-stable. Also, $R_W = \{ \hat{f} \}$ if $\hat{f} \in F$.

Proof. Let $I_r = \{ f \mid f \in I_r \}$ be the “unnamed” version of the set $I_r$ from Definition 24. Consider $r \in [s-1]$. From the the definition of $I_r$ and $I_{r+1}$ we obtain that for every $g \in I_{r+1}$ there exists $f \in I_r$ with $(f, g) \in E_\tau$ (implying that $g \in \Gamma^+(f)$). This gives that $I_{r+1} \subseteq \Gamma^+(I_r)$. Also, if $\hat{f} \in F$ then we have $I_1 = \{ \hat{f} \}$ and so $I_1 = \{ \hat{f} \}$. Observing that $W = \text{REV}[\text{Stab}_{\pi}(\tau)] = \text{REV}[W_1] \ldots \text{REV}[W_s]$ we obtain the desired claims.

Our next goal is to show how to transform walk $\tau \in \mathcal{X}^f$ to a walk $\tau^* \triangleq \text{Stab}_{\pi}(\tau)U$. We will do this by applying swaps to swappable pairs in $\tau$.

Definition 26. Consider a walk $\tau$ containing $\hat{f}$. A pair $(f, g)$ of named flaws is called a swappable pair in $\tau$ if it can be swapped in $\tau$ (i.e. $\tau^* = \ldots fg \ldots$ and $f \neq g$) and either

(i) $(f, g) \in (V_r - W_r) \times W_r$, or

(ii) $(f, g) \in W_r \times W_r$ and their order in $\text{Stab}_{\pi}(\tau)$ is different: $\text{Stab}_{\pi}(\tau) = \ldots g \ldots f \ldots$.

The position of the rightmost swappable pair in $\tau$ will be denoted as $k(\tau)$, where the position of $(f, g)$ in $\tau$ is the number of named flaws that precede $g$ in $\tau$. If $\tau$ does not contain a swappable pair then $k(\tau) = 0$. Thus, $k(\tau) \in [0, |\tau| - 1]$.

It can be seen that the procedure of repeatedly applying swaps to swappable pairs in $\tau$ must terminate. Indeed, swapping a pair of the form (i) moves a named flaw in $W_r$ to the left, which can happen only a finite number of times. Similarly, swapping a pair of the form (ii) decreases the number of pairs $(f, g) \in \binom{X_r}{2}$ whose relative order in $W_r$ is not consistent with the relative order in $\text{Stab}_{\pi}(\tau)$, which again can happen only a finite number of times. (Note, swaps of the other type do not affect these arguments).

Lemma 27. Consider a walk $\tau^* = AU$ containing $\hat{f}$ where $A, U$ some words, and there are no swappable pairs inside $U$. Then $U = BC$ where sequence $B$ is a subsequence of $\text{Stab}_{\pi}(\tau)$, and $C$ does not contain named flaws from $W_r$.

In particular, if $|A| = 0$ and $\tau^* = U$ does not contain a swappable pair then $\tau^* \triangleq \text{Stab}_{\pi}(\tau)C$.

Proof. Let $u_1, \ldots, u_m$ be the named flaws from $W_r$ that occur in $U$ (listed in the order of their appearance in $U$). We claim that $u_1 \ldots u_m$ is a prefix of $U$. Indeed, suppose not, then $U = \ldots f u_i \ldots$ where $f \notin W_r$ and $u_i \in W_r$. This means that $(f, u_i) \notin E_\tau$, and so $f \neq u_i$. But then $(f, u_i)$ is a swappable pair in $U$ - a contradiction.

We obtained a decomposition $\tau^* = ABC$ where $B = u_1 \ldots u_m$. It remains to show that $B$ is a subsequence of $\text{Stab}_{\pi}(\tau)$. For that it suffices to prove that for any $i \in [m-1]$ the relative order of $u_i$ and $u_{i+1}$ in $B$ is the same as in $\text{Stab}_{\pi}(\tau)$. Suppose not, i.e. $\text{Stab}_{\pi}(\tau) = \ldots u_{i+1} \ldots u_i \ldots$. We must have $u_i \sim u_{i+1}$ (otherwise $(u_i, u_{i+1})$ would be a swappable pair in $\tau$, contradicting the assumption). Therefore, $(u_i, u_{i+1}) \in E_\tau$, implying that $d_\tau(u_i) > d_\tau(u_{i+1})$. Inspecting Definition 24 we conclude that $u_i$ should be to the left of $u_{i+1}$ in $\text{Stab}_{\pi}(\tau)$. We obtained a contradiction. \[\square\]
Results so far can be used to obtain a mapping $\Phi^f$ and set $X^f_0 = \Phi^f(X^f)$ that satisfy condition (a) of Theorem 19. To satisfy bijectiveness and condition (b), we need to be careful with the order in which we apply swaps to swappable pairs. We will use the following algorithm. First, let $X_0^f = X^f$, and then for $p = 0, 1, 2, \ldots$ do the following:

- Let $k = \max_{\tau \in X_p^f} k(\tau)$. If $k = 0$ then terminate.
- For each $\tau \in X_p^f$ do the following: if $k(\tau) = k$ then swap the pair $(f, g)$ at position $k$ in $\tau$, otherwise leave $\tau$ unchanged. Let $X_{p+1}^f$ be the new set of walks.

Let us fix a word $W$ (over named flaws), and define $X_p^f[W] = \{ \tau \in X_p^f | Stab_{\tau}(\tau) = W \}$ for an index $p \geq 0$. Theorem 19 will follow from the following result (note that set $X_0^f[W] \subseteq X^f$ is valid by the assumption of the theorem).

**Lemma 28.** If set $X_p^f[W]$ is valid then so is $X_{p+1}^f[W]$, and the mapping from $X_p^f[W]$ to $X_{p+1}^f[W]$ defined by the algorithm above is injective.

**Proof.** First, let us prove injectiveness. Suppose that two distinct walks $\tau^0, \tau^0 \in X_p^f[W]$ are transformed to the same walk $\tau \in X_{p+1}^f[W]$. At least one of one walks $\tau^0, \tau^0$ must have changed. Assume w.l.o.g. that $\tau \neq \tau^0$. Thus, walk $\tau^0 = AfgB$ with $f \neq g$ is transformed to a walk $\tau = AgfB$. We cannot have $\tau^0 = \tau$ (then walks $\tau, \tau$ would both be in $X_p^f[W]$, but they do not follow the same deterministic strategy - a contradiction). Thus, $\tau^0 = AfgB$, and $\tau$ was obtained from $\tau^0$ by swapping $f$ and $g$. Since the SWAP operation from Definition 3 is injective, we obtain that $\tau^0 = \tau^0$.

Now suppose that $X_{p+1}^f[W]$ is not valid. Then it contains walks $\tau, \tau$ of the form

\[
\tau = w_1 \ldots w_l f g \ldots \quad \tau = w_1 \ldots w_l f g \ldots
\]

such that $f \neq f$, and the states in $\tau$ to the left of $f$ match the corresponding states in $\tau$ to the left of $f$. Here we assume some of the symbols $f, g, f, g$ can equal $\emptyset$, which means that they don’t exist. We also assume that $f = \emptyset \Rightarrow g = \emptyset$ and $f = \emptyset \Rightarrow g = \emptyset$.

Let $\tau^0$ and $\tau^0$ be respectively the walks in $X_p^f[W]$ that were transformed to $\tau$ and $\tau$. Since $X_p^f[W]$ is valid, at least one of them must have changed. Assume w.l.o.g. that $\tau \neq \tau^0$. We know that

- $(*)$ $\tau^0$ and $\tau^0$ follow the same deterministic strategy, and they are not proper prefixes of each other;
- $(**)$ named flaws $w_1, \ldots, w_l, f$ are distinct.

We will also implicitly use the fact that the first $l + 1$ states of $\tau$ match those of $\tau$, and also that swapping adjacent flaws only affects the state between them in a deterministic way, as specified by the mapping SWAP. Four cases are possible:

- The swapped pair in $\tau$ was $(w_i, w_{i+1})$ for $i \in [\ell - 1]$, and so $\tau^0 = w_1 \ldots w_{i+1} w_i \ldots w_l f g \ldots$.
  Using $(*)$ and $(**)$, we conclude that $\tau \neq \tau^0$, and so $\tau^0 = w_1 \ldots w_{i+1} w_i \ldots w_l f g \ldots$. (Recall that swaps are applied at the same position for walks in $X_p^f$). Condition $(*)$ now gives $f = f$.

- The swapped pair in $\tau$ was $(w_l, f)$, and so $\tau^0 = w_1 \ldots w_l g f \ldots$. Using $(*)$ and $(**)$, we conclude that $\tau \neq \tau^0$, and so $\tau^0 = w_1 \ldots w_l f g \ldots$. Condition $(*)$ now gives $f = f$.

- The swapped pair in $\tau$ was $(f, g)$, and so $\tau^0 = w_1 \ldots w_l g f \ldots$ and $f \in W_{\tau^0}$. Since there are no swappable pairs in $\tau$ to the right of $(g, f)$, Lemma 27 gives that $\tau^0 \neq w_1 \ldots w_l g BC$ where $B$ is a subsequence of $Stab_{\tau}(\tau^0)$ and $C$ does not contain named flaws from $W_{\tau^0}$. Note that $B$ starts with $f$.
By property (*) we get that $\tau^0 = w_1 \ldots w_l g \ldots$. Using Lemma 27 as before, we obtain that $\tau^0 = w_1 \ldots w_l g \bar{B} \bar{C}$ where $\bar{B}$ is a subsequence of $\text{Stab}_p(\tau^0)$ and $C$ does not contain named flaws from $W_{\tau^0}$.

We now observe that $\tau^0, \bar{\tau}^0 \in X_p[W]$, i.e. $\text{Stab}_p(\tau^0) = \text{Stab}_p(\bar{\tau}^0) = W$. Therefore, $W_{\tau^0} = W_{\bar{\tau}^0}$. This fact together with the forms of $\tau^0, \bar{\tau}^0$ given above implies that $B$ is a permutation of $\bar{B}$. We also know that both $B$ and $\bar{B}$ are subsequences of $\text{Stab}_p(\tau^0) = \text{Stab}_p(\bar{\tau}^0) = W$ and all elements of $W$ are distinct. Thus, we can conclude that $B = \bar{B}$.

We showed that $\tau^0 = w_1 \ldots w_l g f \ldots$. Since $(g, f)$ is a swappable pair in $\tau^0$, it is also a swappable pair in $\bar{\tau}^0$ (at the same position as in $\tau^0$). Therefore, $\bar{\tau} = w_1 \ldots w_l f g \ldots$, and so $f = \bar{f}$.

- The swapped pair in $\tau$ was to the right of $(f, g)$. In this case condition (*) immediately gives $f = \bar{f}$.

\[\square\]

6 Constructing atomic commutative resampling oracles

Let us define a directed multigraph $D$ as follows: its set of nodes is $\Omega$, and its set of edges is the set of all valid walks $\sigma \xleftarrow{f} \sigma'$. (Each edge of $D$ is labeled by a flaw in $F$). The atomicity of $\rho$ means any state $\sigma' \in \Omega$ has at most one incoming edge in $D$ labeled by a given flaw $f \in F$.

Note that we can recover sets $A(f, \sigma)$ from $D$, since $A(f, \sigma) = \{\sigma' | \text{edge } \sigma \xleftarrow{f} \sigma' \text{ belongs to } D\}$. Therefore, resampling oracles can be uniquely reconstructed from $D$, if we assume that $\rho(\sigma'|f, \sigma)$ is a uniform distribution over $A(f, \sigma)$.

In this section we show that resampling oracles for permutations used in [8, 9] and for perfect matchings in complete graphs used in [9] are strongly commutative. To prove this, we find it easier to take an indirect approach: first, we will describe a generic route for constructing atomic multigraphs, then apply it to permutations and matchings and prove weak commutativity. We will then see that the resulting resampling procedure coincides with that in [8, 9].

The constructed multigraph will satisfy the following property: for each flaw $f$ there exists constant $A_f$ such that $|A(f, \sigma)| = A_f$ all $\sigma \in f$. Thus, $\rho(\sigma'|f, \sigma) = \frac{1}{A_f}$ for $\sigma' \in A(f, \sigma)$, and so weak commutativity will imply strong commutativity.

We start with some general observations. For an atomic multigraph $D$ let us define a mapping $\psi : F \times \Omega \to \Omega \cup \{\text{undefined}\}$ that specifies a “backward step” for $f \in F$ and $\sigma' \in \Omega$ as follows: if there exists the (unique) state $\sigma \in \Omega$ such that $\sigma \xleftarrow{f} \sigma'$ then $\psi(f, \sigma') = \sigma$, otherwise $\psi(f, \sigma') = \text{undefined}$. Note that $\psi$ satisfies the following properties:

(I) If $\psi(f, \sigma') \neq \text{undefined}$ then $\psi(f, \sigma') \subseteq f$.

(II) For any $f \in F$ and $\sigma \in f$ there exists at least one $\sigma' \in \Omega$ with $\psi(f, \sigma') = \sigma$.

It is not difficult to see that $D$ can be uniquely reconstructed from $\Omega$, $F$ and $\psi$, since for each $f \in F$ and $\sigma \in f$ we have $A(f, \sigma) = \{\sigma' \in \Omega | \psi(f, \sigma') = \sigma\}$. Furthermore, any triplet $(\Omega, F, \psi)$ specifies a valid atomic multigraph $D$, as long as $\psi$ satisfies properties (I) and (II) (and $F$ is some set of non-empty subsets of $\Omega$). Property (II), in particular, is equivalent to the condition that $A(f, \sigma)$ is non-empty for each $f \in F$ and $\sigma \in f$.

Thus, the problem of constructing the set of actions $A(f, \sigma)$ for a given $f, \sigma$ can be shifted to the problem of constructing a mapping $\psi$. Of course, after constructing $\psi$ one still needs to show that sampling from $A(f, \sigma)$ can be done efficiently.

We remark that Definition 2 of the potential causality graph can also be reformulated in terms of the mapping $\psi$, as stated below (this claim follows directly from definitions).
**Proposition 29.** Undirected graph $(F, \sim)$ is a potential causality graph for $D$ if for any $f, g \in F$ with $f \sim g$ and any $\sigma' \in g$ with $\psi(f, \sigma') \neq \text{undefined}$ we have $\psi(f, \sigma') \in g$ and $f \neq g$.

### 6.1 Matchings

We now apply the route outlined above to some matching problems. Let $G = (V, E)$ be an undirected graph with $|V| = 2n$ nodes that satisfies the following condition:

(*) If $(u', u, v, v')$ is a path in $G$ with distinct nodes then $\{u', v'\} \in E$.

We will consider the case when $\Omega$ is the set of perfect matchings in $G$ (so that each object $\sigma \in \Omega$ is a matching). We allow any flaw of the form $f_M = \{\sigma \in \Omega \mid M \subseteq \sigma\}$ where $M$ is a fixed subset of $E$. It can be assumed w.l.o.g. that $M$ is a matching (otherwise $f_M$ would be empty). Thus, $F$ can be any subset of $\{f_M \mid M \in \mathcal{M}\}$ where $\mathcal{M}$ denotes the set of matchings in $G$, with $\Omega \subseteq \mathcal{M}$. Two special cases of this framework have been considered \[8,1,9\]:

- **[P1]** $G$ is the complete graph on $2n$ vertices, so that $\Omega$ is the set of all perfect matchings of $V$.
- **[P2]** Set $V$ can be partitioned into disjoint subsets $A_1, B_1, \ldots, A_r, B_r$ such that $|A_i| = |B_i|$ for $i \in [r]$ and $E = \{\{u, v\} \mid u \in A_i, v \in B_i, i \in [r]\}$. Thus, $G$ is a union of $r$ complete bipartite graphs, and set $\Omega$ corresponds to $r$ permutations.

In fact, these are essentially the only possibilities allowed by condition (*): it can be shown that if $G$ contains at least one perfect matching then each component of $G$ is either a complete graph $K_m$ or a complete bipartite graph $K_{m,m}$. We will not need this claim, and so we leave it without a proof. Instead, we just assume that one of the two cases [P1,P2] holds (but unlike previous work, we will treat them in a unified way, relying mostly on condition (*)).

We use the following potential causality graph: $f_M \sim f_{M'}$ for $f_M, f_{M'} \in F$ if $M \cup M'$ is not a matching or $M = M'$. This graph is the same or slightly smaller than graphs used previously for cases [P1] and [P2]. (For [P2] the works \[8,9\] used a larger relation $\sim'$ instead where $f_M \sim' f_{M'}$ if $M \cup M'$ is not a matching or $M \cap M' \neq \emptyset$).

We will construct a mapping $\hat{\psi}: \mathcal{M} \times \Omega \to \Omega$ that satisfies $M \subseteq \hat{\psi}(M, \sigma)$ for any $M \in \mathcal{M}$ and $\sigma \in \Omega$. It will correspond to the mapping $\psi$ in a natural way, i.e. $\hat{\psi}(f_M, \sigma) = \hat{\psi}(M, \sigma)$ for $f_M \in F$. Clearly, such $\hat{\psi}$ will be defined everywhere on $F \times \Omega$, and will satisfy property (I).

#### Defining $\hat{\psi}$

Consider $M \in \mathcal{M}$ and $\sigma \in \Omega$. If $M$ contains a single edge $e = \{u, v\}$, then we find unique $u', v'$ with $\{u, u'\}, \{v, v'\} \in \sigma$, and set

$$\hat{\psi}(\{e\}, \sigma) = (\sigma - \{\{u, u'\}, \{v, v'\}\}) \cup \{\{u, v\}, \{u', v'\}\} \tag{12}$$

Note that if $\{u, v\} \in \sigma'$ then $\hat{\psi}(M, \sigma) = \sigma$. Otherwise nodes $u, v, u', v'$ are distinct, and we have $\hat{\psi}(M, \sigma) \in \Omega$ by the assumption (*) on the graph $G$.

Now suppose that $M = \{e_1, \ldots, e_k\}$ contains more than one edge. Then we define

$$\sigma_0 = \sigma \quad \sigma_1 = \hat{\psi}(\{e_1\}, \sigma_0) \quad \sigma_2 = \hat{\psi}(\{e_2\}, \sigma_1) \quad \ldots \quad \sigma_k = \hat{\psi}(\{e_k\}, \sigma_{k-1}) \tag{13}$$

and set $\hat{\psi}(M, \sigma) = \sigma_k$. To show that this is well-defined, we need to prove that the result does not depend on the chosen ordering of $M$. It suffices to prove this claim for $|M| = 2$, then we can use an induction argument (since any ordering of $M$ can be transformed to any other ordering via a sequence of operations that swap adjacent elements). Proving it for $|M| = 2$ can be done by inspecting all possible cases, which are visualized in Fig.\[1\] verification of the claim in each case is left to the reader.
Finally, for an object \( \sigma \) with

\[
\text{Proof.} \quad (a) \text{ In dashed lines indicate edges in } \sigma, \text{ and object } \sigma \text{ appears in the sequence the } (e_i, \sigma) \text{ the second time then we have } e_i \in \sigma_i \text{ and consequently } \sigma_i = \psi(e_i, \sigma_i). \]

(b) We claim \( \hat{\psi}(M, \psi(M, \sigma)) = \psi(M, \psi(M', \sigma)) \). Indeed, let \( M = \{e_1, \ldots, e_i\} \) and \( M' = \{e_i, \ldots, e_k\} \). Define \( \sigma_0, \sigma_1, \ldots, \sigma_k \) as in \((13)\), then \( \hat{\psi}(M, \sigma) = \psi(M, \sigma) \). Thus, it remains to show that \( \hat{\psi}(M, \sigma) = \sigma_k \). For that we need to observe that if some edge \( e_i \) appears in the sequence the \( \psi(e_i, \sigma) \) the second time then we have \( e_i \in \sigma_i \) and consequently \( \sigma_i = \psi(e_i, \sigma_i) = \sigma_k \). Thus, such \( e_i \) can be removed from the sequence without affecting the result. After removing duplicates we conclude that \( \hat{\psi}(M, \sigma) = \sigma_k \) by the definition of \( \hat{\psi} \).

In a similar way we can show that \( \hat{\psi}(M', \psi(M', \sigma)) = \psi(M', \sigma) \). This proves the claim.

(c) Let us show that conditions of Proposition \((29)\) hold. Consider flaws \( f = f_M, g = f_M' \) in \( F \) with \( f_M \sim f_M' \) and object \( \sigma \in f_M \). Condition \( f_M \neq f_M' \) holds since \( f_M \sim f_M' \), so we need to show that \( \psi(f_M, \sigma) \in f_M' \), or equivalently \( M' \subseteq \psi(M, \sigma) \).

Assume that \( M = \{e_1, \ldots, e_k\} \), and define sequence \( \sigma_0, \sigma_1, \ldots, \sigma_{k-1}, \sigma_k = \hat{\psi}(M, \sigma) \) as in \((13)\). Indeed, for the base case the claim \( M' \subseteq \sigma \) holds since \( \sigma \in f_M' \), and for the induction step we need to use the definition of \( \hat{\psi} \) and the fact that \( M \cup M' \subseteq \hat{\psi}(M, \sigma) \).

We leave verification of the induction step to the reader. \( \square \)

**Sampling from** \( A(f_M, \sigma_k) \) The general idea is to “reverse” the process in eq. \((13)\): given flaw \( f_M \) with \( M = \{e_1, \ldots, e_k\} \in M \) and object \( \sigma_k \in f_M \), we first generate possible values for \( \sigma_{k-1} \), then for \( \sigma_{k-2} \), and so on.

For a subset \( S \subseteq E \) let \( \overline{S} = \{(u, v), (v, u) \mid \{u, v\} \in A\} \) be a “directed copy” of \( S \). For an object \( \sigma \in \Omega \) and edges \( (u, v), (u', v') \in \overline{\sigma} \) define

\[
\text{Swap}_\sigma((u, v), (u', v')) = (\sigma - \{u, v\}, \{u', v'\}) \cup \{u, u'\}, \{v, v'\} \quad (14)
\]

Finally, for an object \( \sigma \in \Omega \) and an edge \( (u, v) \in \overline{\sigma} \) let us define

\[
\mathcal{N}_\sigma(u, v) = \{(u', v') \in \overline{\sigma} \mid \text{Swap}_\sigma((u, v), (u', v')) \in \Omega\} \quad (15)
\]
It can be checked that \((v,u) \in \mathcal{N}_\sigma(u,v)\) and \((u,v) \notin \mathcal{N}_\sigma(u,v)\). Furthermore, in the special cases above we have the following:

[P1] \(\mathcal{N}_\sigma(u,v) = \overline{\sigma} - \{(u,v)\}\).

[P2] If \((u,v) \in A_i \times B_i\) then \(\mathcal{N}_\sigma(u,v) = (B_i \times A_i) \cap \overline{\sigma}\).

We can now formulate the sampling algorithm (see Algorithm 3).

**Algorithm 3** Sampling from \(A(f_M,\sigma_k)\) for \(M = \{e_1,\ldots,e_k\} \subseteq \sigma_k \in \Omega\)

1: for \(i = k,k-1,\ldots,1\) do
2: choose orientation \((u,v)\) of edge \(e_i = \{u,v\}\)
3: select \((u',v') \in \mathcal{N}_{\sigma_i}(u,v) - \{e_1,\ldots,e_{i-1}\}\) uniformly at random
4: set \(\sigma_{i-1} = \text{Swap}_{\sigma_i}((u,v),(u',v'))\)
5: end for
6: return \(\sigma_0\)

Let us verify the correctness of this algorithm. Using the definitions of \(\hat{\psi}\) and \(\text{Swap}_{\sigma_i}\), the following fact can be easily checked.

**Lemma 31.** (a) Suppose that \(\{u,v\} \in \sigma_i \in \Omega\), \((u',v') \in \mathcal{N}_{\sigma_i}(u,v)\) and \(\sigma_{i-1} = \text{Swap}_{\sigma_i}((u,v),(u',v'))\). Then \(\hat{\psi}(\{u,v\},\sigma_{i-1}) = \sigma_i\).

(b) Conversely, suppose that \(\hat{\psi}(\{u,v\},\sigma_{i-1}) = \sigma_i\) for \(e_i = \{u,v\} \in E\) and \(\sigma_{i-1} \in \Omega\). Then there exists unique \((u',v') \in \mathcal{N}_{\sigma_i}(u,v)\) such that \(\sigma_{i-1} = \text{Swap}_{\sigma_i}((u,v),(u',v'))\). Furthermore, it satisfies \(\{u',v'\} \notin \sigma_{i-1} - \{e_i\}\).

Using this lemma, we can now show establish correctness of the sampling procedure. We say that two executions of Algorithm 3 are distinct if they made different choices in line 3 for some \(i \in [k]\).

**Proposition 32.** Algorithm 3 is well-defined, i.e. in line 3 we have \(e_i \in \sigma_i\). It can generate object \(\sigma_0 \in \Omega\) if and only if \(\hat{\psi}(M,\sigma_0) = \sigma_k\). Finally, distinct executions produce distinct outputs.

**Proof.** The proof will have two parts corresponding to two directions.

(a) Let \(\sigma_k,\sigma_{k-1},\ldots\) be the sequence of objects produced by the algorithm. We will show using induction on \(i = k,\ldots,1,0\) that \(\{e_1,\ldots,e_i\} \subseteq \sigma_i\) (and therefore line 3 for index \(i\) is well-defined) and \(\hat{\psi}(\{e_1,\ldots,e_k\},\sigma_i) = \sigma_k\). The base case \(i = k\) is trivial. Suppose the claim holds for \(i \in [k]\), let us show it for \(i - 1\). We have \(\{e_1,\ldots,e_i\} \subseteq \sigma_i\) by the induction hypothesis; inspecting the rule for choosing \((u',v')\), we conclude that \(\{e_1,\ldots,e_{i-1}\} \subseteq \sigma_{i-1}\). For the second claim we can write

\[
\hat{\psi}(\{e_1,\ldots,e_k\},\sigma_{i-1}) = \hat{\psi}(\{e_{i+1},\ldots,e_k\},\hat{\psi}(\{e_i\},\sigma_{i-1})) = \hat{\psi}(\{e_{i+1},\ldots,e_k\},\sigma_i) = \sigma_k
\]

where the first equality is by the definition of \(\hat{\psi}\), the second is by Lemma 31(a) and third is by the induction hypothesis. This concludes the argument.

(b) Suppose that \(\hat{\psi}(M,\sigma_0) = \sigma_k\) for \(\sigma_0 \in \Omega\) and \(M = \{e_1,\ldots,e_k\}\). Define objects \(\sigma_1,\ldots,\sigma_k\) as in (13). We claim that Algorithm 3 can replicate this sequence (in the reverse order). Indeed, by Lemma 31(b) it suffices to show that for any \((u',v') \in \mathcal{N}_{\sigma}(u,v)\) with \(\{u',v'\} \notin \sigma_{i-1} - \{e_i\}\) we also have \((u',v') \in \mathcal{N}_{\sigma}(u,v) - \{e_1,\ldots,e_{i-1}\}\). Suppose not, then \((u',v') \in \{e_1,\ldots,e_{i-1}\}\). By Proposition 30(a) we have \(\{e_1,\ldots,e_{i-1}\} \subseteq \sigma_{i-1}\), and so \(\{u',v'\} \in \sigma_{i-1}\). Thus, \(\{u',v'\} = e_i\). But \(e_i\) does not appear in \(\{e_1,\ldots,e_{i-1}\}\), and so we cannot have \((u',v') \in \{e_1,\ldots,e_{i-1}\}\) - a contradiction.

Let us now prove that the input \(M,\sigma_k\) and the output \(\sigma_0\) uniquely determine choices made during the execution (this will give the last claim of the lemma). Let \(\tilde{\sigma}_k,\ldots,\tilde{\sigma}_1,\tilde{\sigma}_0\) be the objects
produced during the execution, with $\tilde{\sigma}_k = \sigma_k$ and $\tilde{\sigma}_0 = \sigma_0$. Set $i = 1$. By Lemma 31(a) we have $\psi_i(\epsilon_i, \sigma_{i-1}) = \tilde{\sigma}_i$, implying that $\tilde{\sigma}_i = \sigma_i$ is determined uniquely. By Lemma 31(b) the choice of $(u', v')$ in line 3 for index $i$ is also determined uniquely from $\sigma_{i-1}, \sigma_i$ and $e_i$. Repeating this argument for $i = 2, \ldots, k$ (i.e. using induction) yields the claim.

We have proved that the output of Algorithm 3 is a distribution whose support is $A(f_M, \sigma_k)$. To show that this distribution is uniform, we need to observe additionally that the number of choices in line 3 for index $i$ depends on $i$ but not on the past execution history (which can be easily checked for cases [P1] and [P2]). The cardinality of $A(f_M, \sigma_k)$ is the product of these numbers over $i \in [k]$, and thus depends only on the flaw $f_M$ (more precisely, on $|M|$).

To summarize, we have constructed an atomic weakly commutative multigraph $D$, proved that Algorithm 3 samples uniformly from $A(f_M, \sigma_k)$, and the size of latter set depends only on $f_M$ (the latter implies strong commutativity). It can now be verified that the sampling procedure coincides with the procedure in 9 for perfect matchings in a complete graph (in the case [P1]), and with the procedure in 8, 1, 9 for permutations (in the case [P2]).

6.2 Application: rainbow matchings in complete graphs

We refer to 8, 1, 9 for applications of resampling oracles for permutations and perfect matchings. Here we revisit just one application, namely a rainbow matching problem. Our primary goal is to demonstrate how the choice of the distribution $\omega^{\text{init}}$ affects the bound on the expected runtime, and also compare it with the parallel version.

Let $G = (V, E)$ be a complete graph on $2n$ vertices such that each edge is assigned a color, and each color appears in at most $q$ edges. A perfect matching in $G$ is called rainbow if its edges have distinct colors. Achlioptas and Iliopoulos 1 showed that a rainbow matching exists if $q \leq \gamma n$ for some constant $\gamma < \frac{1}{255} \simeq 0.184$. Instead of (2), they used a stronger condition 1. Harvey and Vondrák 9 improved the constant to $\gamma = 0.21$ by exploiting a condition with the cluster expansion correction analogous to 2. Below we redo their calculations.

Let $F$ be the set of flaws $f_M$ such $M$ contains two vertex-disjoint edges of the same color, and assume that we use the multigraph and relation $\sim$ constructed in the previous section.

**Proposition 33.** If $\gamma = 0.21$ then condition (2) can be satisfied by setting $\mu_f = \mu = \frac{3}{4n^2}$ for $f \in F$ (if $n$ is sufficiently large).

**Proof.** Consider flaw $f_M$ where $M = \{\{v_1, v_2\}, \{v_3, v_4\}\} \in F$. For node $v \in V$ let $\Gamma(v) \subseteq F$ be the set of flaws $f_{M'}$ such that at least one of the edges in $M'$ is incident to $v$. Clearly, we have $\Gamma(f_M) \subseteq \Gamma(v_1) \cup \Gamma(v_2) \cup \Gamma(v_3) \cup \Gamma(v_4)$. Furthermore, if $S \in \text{Ind}(\Gamma(f_M))$ then for each $i \in [4]$ we have $|S \cap \Gamma(v_i)| \leq 1$ (and consequently $|S| \leq 4$). It can be seen that for each $k \in [0, 4]$ there are at most $n_k = \binom{4}{k}(2n-1)^k(q-1)^k$ subsets $S \in \text{Ind}(\Gamma(f_M))$ of size $k$. Indeed, feasible $S$ can be generated as follows: first choose a subset of $\{\Gamma(v_1), \Gamma(v_2), \Gamma(v_3), \Gamma(v_4)\}$ of size $k$ whose elements will contain a flaw in $S$ ($\binom{4}{k}$ possibilities), and then for each selected $\Gamma(v_i)$

- choose a node $v'_i$ matched to $v_i$ (at most $2n - 1$ possibilities), and
- choose an edge in $G$ of the same color as $\{v_i, v'_i\}$ (at most $q - 1$ possibilities).

Let us set value $\mu_f$ for all $f \in F$ to the same constant $\mu > 0$, then we get

$$\sum_{S \in \text{Ind}(\Gamma(f_M))} \mu(S) \leq \sum_{k=0}^{4} n_k \cdot \mu^k = (1 + (2n - 1)(q - 1)\mu)^4$$
By inspecting Algorithm 3 we can conclude that $A_f = (2n - 3)(2n - 1)$ for each $f \in F$. Thus, we get the following condition: there must exist $\mu > 0$ such that expression

$$
\theta = \frac{1}{(2n - 3)(2n - 1)\mu} \cdot (1 + (2n - 1)(q - 1)\mu)^4
$$

is a constant smaller than 1. Denote $\beta = (2n - 3)(2n - 1)\mu$, then

$$
\theta \leq \frac{1}{\beta} \cdot \left(1 + 2n \cdot (\gamma n) \cdot \frac{\beta}{4n^2} \cdot (1 + o(1))\right)^4 = \frac{(1 + \frac{1}{2}\gamma \beta + o(1))^4}{\beta}
$$

The last expression will be smaller than 1 (for a sufficiently large $n$) if $\beta = 3$ and $\gamma = 0.21$, where we used the constants from [9].

Let us now estimate the expression in (5). We have $|\Omega| = (2n - 1)!$ and $\log |\Omega| = \Theta(n \log n)$. When $\text{supp}(\omega^\text{init}) = \{\sigma^\text{init}\}$ for some $\sigma^\text{init} \in \Omega$, we get $\gamma^\text{init} = |\Omega|$ and thus $T = \Omega(n \log n)$. If, on the other hand, $\omega^\text{init} = \omega$ then we can write

$$
\sum_{R \in \bigcup_{s \in \mathbb{N}} \text{Ind}(F_s)} \mu(R) \leq \sum_{R \subseteq F} \mu(R) = \prod_{f \in F} (1 + \mu_f) = (1 + \mu)^{|F|}
$$

Observing that $|F| \leq (2n)^2 q = O(n^3)$ and $\mu = O(1/n^2)$, we obtain that $T = O(|F| \log(1 + \mu)) = O(n^3 \log(1 + O(\frac{1}{n^2}))) = O(n^3 \cdot \frac{1}{n^2}) = O(n)$. Thus, choosing $\omega^\text{init} = \omega$ leads to a better bound than initializing the algorithm with some fixed state $\sigma^\text{init}$. This may not be surprising, given that in the latter situation we need to consider the worst case. Note that a linear bound on the expected number of resampling steps has also been shown in [9].

We can also compute a bound on the expected number of rounds of the parallel version (Algorithm 2), assuming that $\omega^\text{init} = \omega$. From Theorem 8 we get

$$
T = O(\log \sum_{f \in F} \mu_f) = O(\log(|F| \cdot \mu)) = O(\log(n^3 \cdot \frac{1}{n^2})) = O(\log n)
$$

It should be noted, however, that at the moment it is not known whether a round of Algorithm 2 can be implemented efficiently (i.e. in an [expected] polylogarithmic time) for matchings in a complete graph. Such implementation has only been shown for permutations [8]. We conjecture that the technique in [8] can be extended to matchings in a complete graph, but leave this question outside the scope of this work.

### A Counting stable sequences: Proof of Theorem 14

We say that a sequence $\varphi = (I_1, \ldots, I_s)$ with $s \geq 1$ is strongly stable if

(i) $I_r \in \text{Ind}(F)$ for each $r \in [s]$,

(ii) $I_{r+1} \subseteq \Gamma(I_r)$ for each $r \in [s - 1]$, and

(iii) $I_r \neq \emptyset$ for each $r \in [2, s]$.

(Compared to Definition 12, we added condition (iii), and in (ii) replaced condition $I_{r+1} \subseteq \Gamma^+(I_r)$ with a stronger condition $I_{r+1} \subseteq \Gamma(I_r)$). For a stable word $W$ let $\varphi_W = (I_1, \ldots, I_s)$ be the corresponding stable sequence; if $W$ is empty then $\varphi_W = (\emptyset)$.

**Proposition 34.** For any $W \in \text{Stab}_n$ the sequence $\varphi_W$ is strongly stable.
Proof. We need to show that the sequence $\varphi_W = (I_1, \ldots, I_s)$ satisfies additionally property (ii). We will prove this under the assumption that there exists a walk $\tau \equiv W$. (The case when there exists a walk $\tau = \text{REV}[W]$ is completely analogous).

Let $W = W_1 \ldots W_s$ be the partitioning of $W$ given in Definition 13. It suffices to prove that if a flaw $f$ is present in adjacent segments $W_r$ and $W_{r+1}$ then $f \sim f$. Suppose not: $f \sim f$. Then by Lemma 11(b) there exists flaw $g$ between the two occurrences of $f$ with $f \sim g$. We have $W_rW_{r+1} = \ldots f \ldots g \ldots f \ldots$. We have either $g \in W_r$ or $g \in W_{r+1}$, and so we must have $f \sim g$ - a contradiction.

For a sequence $\varphi = (I_1, \ldots, I_s)$ with $s \geq 1$ let $R_\varphi = I_1$ be the first set in the sequence, and denote $|\varphi| = \sum_{r \in [s]} |I_r|$ and $\lambda_\varphi = \prod_{r \in [s]} \prod_{f \in I_r} \lambda_f$. Let $\text{Stab}$ be the set of strongly stable sequences, $\text{Stab}(R) = \{ \varphi \in \text{Stab} : R_\varphi = R \}$, $\text{Stab}(t) = \{ \varphi \in \text{Stab} : |\varphi| \geq t \}$ and $\text{Stab}(R, t) = \text{Stab}(R) \cap \text{Stab}(t)$.

Clearly, a $\pi$-stable word $W$ can be uniquely reconstructed from the corresponding sequence $\varphi_W$. Thus, $W \mapsto \varphi_W$ is an injective mapping from $\text{Stab}_\pi(R, t)$ to $\text{Stab}(R, t)$. Also, $\lambda_W = \lambda_{\varphi_W}$ for any $W \in \text{Stab}_\pi(R, t)$. This means that Theorem 14 will follow from the following result.

**Theorem 35.** Suppose that $(\rho, \sim)$ satisfies either the cluster expansion condition \[25\] or the Shearer’s condition from Definition 6. Then

$$\sum_{\varphi \in \text{Stab}(R, t)} \lambda_\varphi \leq \mu(R) \cdot \theta^t \quad \forall R \in \text{Ind}(F) \tag{16}$$

Proof. First, assume that the Shearer’s condition holds. In this case the claim has been proven in [11, 9]. To elaborate, let $p \in \mathbb{R}^{|F|}$ be the vector from Definition 6 and define

$$p_\varphi = \prod_{r \in [s]} \prod_{f \in I_r} p_f \tag{17}$$

Let $\text{Stab}'$ be the set of sequences $\phi = (I_1, \ldots, I_s)$ that satisfy conditions (i) and (ii) given in the beginning of this section. For any integer $\ell$ we can write

$$\sum_{\varphi \in \text{Stab}(R, t)} p_\varphi \leq \sum_{\varphi = (R, I_1, \ldots, I_s) \in \text{Stab}'} p_\varphi \leq \frac{q_R(p)}{q_\varnothing(p)} \quad \forall R \in \text{Ind}(F) \tag{18}$$

where the first inequality holds since for any $\varphi = (I_1, \ldots, I_s) \in \text{Stab}(R, t)$ with $s \leq \ell$ there is a corresponding sequence $\varphi' = (R, I_2, \ldots, I_s, \varnothing, \ldots, \varnothing) \in \text{Stab}'$ of length $\ell$, and the second inequality appears implicitly eq. \[2\] in [11] and as Lemma 5.10 in [9]. Taking a limit $\ell \to \infty$ in \[18\] and observing that $\mu(R) = \frac{q_R(p)}{q_\varnothing(p)}$, we get $\sum_{\varphi \in \text{Stab}(R, t)} p_\varphi \leq \mu(R)$. From definitions we have $\lambda_\varphi \leq p_\varphi \cdot \theta^t$ for any $\varphi \in \text{Stab}(R, t)$, this gives \[16\].

Now assume that condition \[2\] holds. We say that a pair of subsets $(R, S)$ is independent if $R \cap S = \varnothing$ and $R \cup S \in \text{Ind}(F)$. For such pair let $\text{Stab}(R, S, t, \ell)$ be the set of sequences of the form $\varphi = (R, I_2, \ldots, I_s)$ with $s \leq \ell$ that satisfy one of the following:

- $|s| = 1$ (i.e. $\varphi = (R)$) and $S = \varnothing$;

\footnote{Note that Bissacot et al. [3] proved that the cluster expansion condition implies Shearer’s condition, so it would suffice to prove just the second claim. However, the definition of the cluster expansion condition in [3] was slightly stronger: the sum in \[25\] was taken over $S \in \text{Ind}(\Gamma^+(f))$, while we sum over $S \in \text{Ind}(\Gamma(f))$. Due to this annoying technicality we consider the two cases separately in the proof.}

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\begin{itemize}
\item $|s| \geq 2$, $S = I_2 - \Gamma(I_1)$ and $(I_2, \ldots, I_s) \in \text{Stab}.$
\end{itemize}

It can be seen that $\text{Stab}(R, \emptyset, t, \infty) = \text{Stab}(R, t)$ for an independent set $R$.

We say that a tuple $(R, S, t, \ell)$ is valid if $(R, S)$ is an independent pair and $t \geq 0$, $\ell \geq 1$ are integers. We will prove the following for any valid tuple $(R, S, t, \ell)$:

$$\sum_{\varphi \in \text{Stab}(R,S,t,\ell)} \lambda_\varphi \leq \mu(R) \mu(S) \cdot \theta^\ell \quad (19)$$

Let us introduce a partial order $\sqsubseteq$ on tuples $(R, S, t, \ell)$ as the lexicographical order on vectors $(\ell, |R|)$ (the first component is more significant). We use induction on this partial order. The base case is $\ell = 1$. We can assume that $S = \emptyset$ and $t \leq |R|$ (otherwise $\text{Stab}(R, S, t, 1)$ is empty). In this case $\text{Stab}(R, S, t, 1)$ contains a single sequence $\varphi = (R)$, and so

$$\sum_{\varphi \in \text{Stab}(R,\emptyset,1)} \lambda_\varphi = \prod_{f \in R} \lambda_f \leq \prod_{f \in R} (\mu_f \cdot \theta) = \mu(R) \cdot \theta^{|R|} \leq \mu(R) \cdot \theta^t$$

where we used inequality $\lambda_f \leq \mu_f \cdot \theta$ that follows from condition (2b) with $S = \emptyset$.

Now consider a valid tuple $(R, S, t, \ell)$ with $\ell \geq 2$, and assume that the claim holds for lower tuples. Two cases are possible.

- $R = \emptyset$. We have a natural isomorphism between sets $\text{Stab}(\emptyset, S, t, \ell)$ and $\text{Stab}(S, \emptyset, t, \ell - 1)$, namely $\text{Stab}(\emptyset, S, t, \ell) = \{(\emptyset, I_1, \ldots, I_r) \mid (I_1, \ldots, I_r) \in \text{Stab}(S, \emptyset, t, \ell - 1)\}$. This gives

$$\sum_{\varphi \in \text{Stab}(\emptyset, S, t, \ell)} \lambda_\varphi = \sum_{\varphi \in \text{Stab}(S, \emptyset, t, \ell - 1)} \lambda_\varphi \leq \mu(S) \cdot \theta^t$$

where in the last inequality we used the induction hypothesis.

- $R \neq \emptyset$. Pick $f \in R$, and denote $R^- = R - \{f\}$. Let $\Gamma(f)$ be the set of flaws $g \in \Gamma(f)$ that (i) can occur in the second set of a sequence $\varphi \in \text{Stab}(R, S, t, \ell)$, and (ii) do belong to $\Gamma(f')$ for any $f' \in R^-$. Formally, $\Gamma(f) = \{g \in \Gamma(f) \mid g \approx f' \text{ for all } f' \in R \cup S\}$. We can write

$$\sum_{\varphi \in \text{Stab}(R,S,t,\ell)} \lambda_\varphi = \sum_{T \in \text{Ind}(\Gamma(f))} \sum_{\varphi \in \text{Stab}(R^-, S \cup T, t - 1, \ell)} \lambda_f \cdot \lambda_\varphi \leq \lambda_f \sum_{T \in \text{Ind}(\Gamma(f))} \mu(R^-) \mu(S \cup T) \cdot \theta^{t-1} \quad (a)$$

$$\leq \mu(R) \mu(S) \theta^{t-1} \cdot \frac{\lambda_f}{\mu_f} \sum_{T \in \text{Ind}(\Gamma(f))} \mu(T) \quad (b)$$

$$\leq \mu(R) \mu(S) \theta^{t-1} \cdot \theta \quad (c)$$

where (a) is by the induction hypothesis, (b) is true since $\mu(R^-) = \mu(R)/\mu_f$ and $\mu(S \cup T) = \mu(S) \mu(T)$, and (c) follows from condition (2b).

$\square$

**Acknowledgements** The author is supported by the European Research Council under the European Unions Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no 616160.
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