Topological Soliton Multiplets in 4+1 Dimensional YMCS Theory

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ABSTRACT

We generalize our results† on charged topological solitons (CTS) in 4 + 1 dimensional $SU(3)$ Yang-Mills-Chern-Simons (YMCS) theory to $SU(N)$. The $SU(N)$ multiplet structure of two classes of solitons associated with the maximal embeddings $SU(2) \times U(1)^{N-2} \subset SU(N)$ and $SO(3) \times U(1)^{N-3} \subset SU(N)$ and the vital role of the $SU(N)$ multiplet of topological currents is clarified. In the case of the first embedding one obtains a $^N C_2$-plet of CTS. In the second, for $N = 3$, one obtains neutral solitons which, though (classically) spinless, have magnetic moments. For $N \geq 4$, after modding out the above mentioned non-particulate feature, one obtains $^N C_3$ plets of CTS.

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1. Introduction

In a recent paper\(^1\) the existence of charged topological solitons (CTS) in 4+1 dimensional $SU(3)$ Yang-Mills-Chern-Simons (YMCS) theory was demonstrated. The relevance of such theories to the “syncyclonic scenario”\(^5\) (i.e. of solitonic signals of the possible existence of extra compact dimensions) and their interest due to connections with the Skyrmeon model, collective quantization of non-Abelian monopoles etc is explained in Ref.[1,4,5] and in Section 6.

In this paper we generalize our previous results to the case of $SU(N)$ and exhibit the central role of a $SU(N)$ multiplet of topological currents constructed from the CS term in determining the charge of the solitons. Thus such theories furnish a novel instance where a topological current couples “electrically” (in contrast to the “magnetic” coupling of the topological current in the case of ’t Hooft- Polyakov magnetic monopoles) to massless gauge fields.

Since, in contrast to solitons in YM-Higgs systems, the symmetry group of the vacuum is unchanged when a CTS of the sort found in Ref.[1] is present it is natural to expect that we should obtain a multiplet of solutions with respect to the unbroken gauge symmetry. We show that in fact these charged solitons occupy a triplet of the gauge group $SU(3)$ corresponding to different possible embeddings $SU(2) \times U(1) \subset SU(3)$ associated with the CTS solutions. This result can be generalized to $SU(N)$ in which case (generically called Type I) the charged solitons are associated with embeddings of $SU(2) \times U(1)^{N-2} \subset SU(N)$ and occupy $\frac{N(N-1)}{2}$plets of $SU(N)$.

We then expose a new type (Type II) of neutral soliton associated with the maximal embedding $SO(3) \subset SU(3)$ and with winding number 4 (cf. the dibaryon embedding of the Skyrmeon). For a general $SU(N)$ gauge group the relevant embeddings are $SO(3) \times U(1)^{N-3} \subset SU(N)$.

The angular momentum of both types of solitons is zero due to the orthogo-
nality of the electric and magnetic fields in the internal space. Nevertheless, the asymptotic fields are those of a non-abelian magnetic dipole. The presence of a dipole moment in the absence of an analog of spin is possible due to the extension of the field configuration i.e it is a mark of the non-particulate features of the soliton. Solutions related by certain $SU(3)$ rotations have the same charges but different magnetic moments. Likening the magnetic moment to an internal degree of freedom one may “mod it out” so as to isolate and highlight the particulate features of the topological soliton. In the case of Type II, $N = 3$ solitons this merely results in one neutral object. For higher $N$, once one mods out the action of the $SU(3)$s relating solutions with identical $SU(N)$ charges but different magnetic moments one obtains an $\frac{N(N-1)(N-2)}{6}$-plet of solitons.

This paper is organized as follows. In Section 2. we discuss the Noether and topological currents of the YMCS theory and explain their relevance to the charge and couplings of topological solitons. In Section 3. we review the analysis of Ref.[1] and simultaneously generalize it to the case $N > 3$. In Section 4. we explain how these (Type I) solitons organize into multiplets of the symmetry group $SU(N)$ of the vacuum. In Section 5. we consider the case of $SO(3) \times U(1)^{N-3} \subset SU(N)$ (type II) embeddings and explain its salient features. In Section 6. we conclude with a discussion of our results along with their connection with previous and future work.

2. Currents and Couplings

We begin with the $SU(N)$ YMCS action on $M^5$

$$S = \int d^5x (\frac{1}{2g^2} tr F_{MN}^2 + \frac{iN_f}{48\pi^2} \omega_5)$$

$$\omega_5 = \epsilon_{MNLQP} tr (\partial_M A_N (\partial_L A_P A_Q + \frac{3}{2} A_L A_P A_Q) + \frac{3}{5} A_M A_N A_L A_P A_Q)$$

The field equations are:

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\[ D_M F_{MN}^A = -\frac{N_f g_5^2}{128\pi^2} \epsilon_{NMLPQ} \; tr(\lambda^A F_{MLFPQ}) \] (2)

Our normalizations and definitions are:

\[ A_M = A_M^A \lambda^A \]\[ tr(\lambda^A \lambda^B) = 2\delta^{AB} \]

\[ F_{MN} = F_{MN}^A \lambda^A \]

\[ D_M = \partial_M + [A_M, \ldots] \]

\[ A, B, \ldots = 1, \ldots, N^2 - 1; \quad M, N, \ldots = 0, 1, 2, 3, 4 \]

\[ A, B, a, b, \ldots = 1, 2, 3, 4; \quad \mu, \nu, \ldots = 1, 2, 3 \]

\[ \{\lambda, A = 1, \ldots, N^2 - 1\} \] are the Gell-Mann matrices for SU(N). Here \( \{\lambda, A = 1, \ldots, N^2 - 1\} \) are the Gell-Mann matrices for SU(N) i.e. \( \{E_{ij}, \tilde{E}_{ij} \mid 1 \leq i < j \leq N\} \oplus \{H_{k^2-1}, k = 2, \ldots, N\} \) where \( \{E, \tilde{E}\}, H \) are N x N matrices spanning the off diagonal set and the Cartan sub-algebra (CSA) respectively:

\[ [E_{ij}]_{kl} = \delta_{i(k} \delta_{l)j} \quad [\tilde{E}_{ij}]_{kl} = -i\delta_{i[k} \delta_{l]j} \]

\[ H_{k^2-1} = \lambda^{k^2-1} = \sqrt{\frac{2}{k(k-1)}} diag(1_{k-1}, 1-k, 0_{N-k}) \] (4)

The CTS solutions of the field equations (2) found in Ref.[1] for the case of SU(3) arose essentially as winding number preserving deformations of the topological solitons of 5 dimensional YM theory (i.e instantons of 4 dimensional YM theory reinterpreted as solitons in one higher dimension) due to the addition of the CS term to the action. Before presenting the generalisation of our solution to SU(N) in Section 3 we first discuss the conserved SU(N) Noether and topological current multiplets \( \{N_M^A\}, \{T_M^A\} \).

The Noether currents which arise due to the invariance of the action (1) under global SU(N) rotations are:

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\[ N^A_M = -f^{ABC} \frac{\partial L}{\partial \partial_M A^B_N} A^C_N \]
\[ = \frac{1}{g_5^2} f^{ABC} A^B_N F^C_{NM} + \frac{\gamma}{4} \epsilon_{MNLQP} tr([\lambda^A, A_N](F_{LP} A_Q + A_Q F_{LP} - A_L A_P A_Q)) \]
\[ = (N^A_M)^Y M + (N^A_M)^{CS} \]

(5)

In addition to \( N^A_M \) there is a multiplet of conserved topological currents which plays a vital role with respect to the CTS solutions. It is easily derived as follows. Under an arbitrary variation of the gauge potential \( \delta A_M \), the CS density has variation \( \gamma = \frac{N_f}{4 \pi} \):

\[ \delta (i \gamma \omega_5) = i \gamma \epsilon_{MNLQP} tr (\frac{3}{4} F_{MN} F_{LP} \delta A_Q - \frac{1}{2} \partial_M (F_{NLAP} + A_P F_{NL} - A_N A_L A_P) \delta A_Q) \]

(6)

for a global \( SU(N) \) rotation \( \delta^A A_Q = [\frac{\lambda^A}{2i}, A_Q] \), \( \delta \omega_5 \) is identically zero. On the other hand using \( \partial_M tr(ABC) = tr D_M (ABC...) \) it is easy to show that for such a variation

\[ \delta (i \gamma \omega_5) = -\partial_M T^A_M = 0 \]
\[ T^A_M = \frac{3 \gamma}{8} \epsilon_{MNLQP} tr \lambda^A F_{NL} F_{PQ} - (N^A_M)^{CS} \]
\[ = Q^A_M - (N^A_M)^{CS} \]

(7)

\( \{T^A_M\} \) is the (identically) conserved multiplet of topological currents. As indicated the second term in (7) is precisely the negative of the contribution of the CS term to the Noether current so that the total \( SU(N) \) current is \( J^A_M = (N^A_M) + T^A_M = (N^A_M)^Y M + Q^A_M \). The field equations can thus be written in the suggestive forms

\[ \partial_M F^A_{MN} = -g_5^2 J^A_N \]

(8a)

\[ D_M F^A_{MN} = -g_5^2 Q^A_N = -g_5^2 (T^A_N + (N^A_N)^{CS}) \]

(8b)
The form (8a) shows that it is the total current (i.e. Noether plus topological) which acts as a source of the field strength. In particular the charge of a field configuration will (by definition) be \( \sim \int J_0^A d^4x \). The form (8b) of the field equations shows that the CS term’s contributions to the SU(N) current enter the YM field equations like an external current. The coupling of static soliton solutions \( \hat{A}_M \) of eqns(8) to weak, static, external potentials \( \{ \overline{A}_M \} \) clarifies this analogy. To see thus consider the expression for the field energy to leading order in the external field. With \( A_M = \hat{A}_M + \overline{A}_M \) and \( \partial_0 = 0 \) we obtain via an integration by parts

\[
\mathcal{E} = \frac{1}{g_5^2} \int d^4x (\frac{1}{4} F_{\mu\nu}^A + \frac{1}{2} F_{\mu0}^A) \\
= \hat{\mathcal{E}} + \int d^4x (\overline{A}_0^A Q_0^A + \overline{A}^A Q_\mu^A - 2 \overline{A}_\nu^A (\hat{D}_0 \hat{F}_{0\nu})^A) \\
+ \lim_{R \to \infty} \int_{S_R^3} d\Sigma_{\mu}(\overline{A}_\nu^A (\hat{F}_{\mu\nu})^A + \overline{A}_0^A (\hat{F}_{\mu0})^A) + O(A^2)
\]

where we have used the fact that \( \hat{A}_M \) satisfies eqn.(8). The 4th term in the last line of eqn.(9) is zero for the solutions we consider. The analogous argument in the absence of the Chern-Simons term would give zero to leading order in \( \{ \overline{A}_M \} \) even though the solution in that case has the fields of a dipole. Thus we see that an external potential couples to solutions of the field equations (8) couples via the total(topological plus Noether) contribution of the CS term to the current.

3. Type I or \( SU(2) \times U(1)^{N-2} \subset SU(N) \) Associated Solutions

We now present the generalization of the analysis of Ref.[1] to the case \( N > 3 \) for reference and completeness. In the absence of the CS term \( (N_f = 0) \) we have the usual self dual solutions\(^2\) in the SU(2) Euclidean sector:

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\[ A_0 = \partial_0 = 0 \]
\[ A_\mu = -\tilde{\eta}_{a\mu\nu} \frac{\lambda^a}{2} \partial_\nu \ln \Pi = i \sum_{\mu\nu} \partial_\nu \ln \Pi \]
\[ \Pi^{-1} \partial^2 \Pi = 0 \quad \Pi = 1 + \sum_{i=1}^{K} \frac{\rho_i^2}{(x-x_i)^2} \]
\[ \nu = \frac{1}{32\pi^2} \int d^4x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a = K \]

where \( \rho_i, x_i \) are the scale factors and positions of the K solitons and \( \nu = K \) the total winding number. In the above \( \{ \tilde{\eta}_{a\mu\nu} \}, \Pi \) are the anti-self-dual 't Hooft symbols, and “super potential” respectively. Fields outside the \( SU(2) \) subalgebra generated by \( \{ \lambda^a \} \) play no role. The asymptotics are those of a non-abelian magnetic dipole with zero electric charge (since \( F_{\mu0} = 0 \)). The field energy is

\[ \mathcal{E} = \frac{1}{g_5^2} \int d^4x \left( \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} F_{\mu0}^2 \right) = \frac{8\pi^2 \nu}{g_5^2} \]

We now show that these solutions become charged when the effect of the C.S. term is included. For simplicity let us work with \( \nu = K = 1 \). The field equations are:

\[ D_M F_{M\nu}^A = \frac{N_f g_5^2}{16\pi^2} \text{tr}(\lambda^A F_{\mu0} \tilde{F}_{\mu\nu}) \quad (12a) \]
\[ D_\mu F_{\mu0}^A = -\frac{N_f g_5^2}{64\pi^2} \text{tr}(\lambda^A F_{\mu\nu} \tilde{F}_{\mu\nu}) \quad (12b) \]

It is easy to check that for \( A = k^2 - 1, k = 3...N \) the ansatz of (10) gives zero for the l.h.s of eqn.(12b) while the rhs is proportional to the Pontryagin density and is hence nonzero. Thus (10) is no longer an adequate ansatz when \( N_f \neq 0 \) and we are motivated to look for a solution with the modified ansatz:

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\[ A^A_\mu = 0 \quad A \not\in \{1, 2, 3\} \]
\[ A^a_\mu = -\bar{\eta}_{a\mu\nu}\partial_\nu \ln \Pi \quad \Pi = \Pi(x^2) \]  
(14)

Spherical symmetry ensures that the r.h.s of the equation for \( D_\mu F^a_\mu \) vanishes as it must for consistency. The only nontrivial equations are then:

\[ D_\mu F^a_\mu = -\frac{\alpha\sqrt{3}}{2} \tilde{F}^b_\mu \sum_{k=3}^{N} \partial_\mu A^{k^2-1}_0 \text{tr}(\lambda^a \lambda^{k^2-1} \lambda^b) \]

\[ \partial^2 A^{k^2-1}_0 = \frac{\alpha\sqrt{3}}{8} \text{tr}(\lambda^{k^2-1} D\text{diag}(1_2, 0_{N-2})) F^a_\mu \tilde{F}^a_\mu = \frac{\alpha}{2} \sqrt{\frac{6}{k(k-1)}} \partial_\mu \tilde{\omega}_\mu \]  
(15)

Thus the Pontryagin density serves as the charge density in a Poisson equation for the electrostatic potential in the \( \{k^2 - 1, k = 3...N\} \) directions. The solution for \( A^{k^2-1}_0 \) is simply

\[ A^{k^2-1}_0(x) = -\frac{\alpha}{16\pi^2} \sqrt{\frac{6}{k(k-1)}} \int d^4 y \frac{1}{(x-y)^2} F^a_\mu(y) \tilde{F}^a_\mu(y) \]  
(16)

and also

\[ \partial_\mu A^{k^2-1}_0 = \frac{\alpha}{2} \sqrt{\frac{6}{k(k-1)}} (\tilde{\omega}_\mu + \frac{D x^\mu}{x^3}) \]  
(17)

The arbitrariness represented by D is crucial to the existence of a charged solution. It is easy to check (assuming that the deformed superpotential has the same leading behaviours as \( x \rightarrow 0 \) and \( x \rightarrow \infty \) as before) that the leading behaviour of \( \tilde{\omega}_\mu \) as \( x \rightarrow \infty \) is \( O(x^{-7}) \). Thus to obtain the electric field corresponding to the charge \( Q^{k^2-1} = -\frac{N_f}{4} \sqrt{\frac{2}{k(k-1)}} \nu \) we must choose \( D = 8 \).

The other equation is now

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\[ D_\mu F_{\mu\nu}^a = -\frac{\alpha^2}{2} \left( \sum_{k=3}^{N} \frac{6}{k(k-1)} \right) (\tilde{\omega}_\mu + \frac{D\tilde{\xi}_\mu}{x^3}) \tilde{F}_{\mu\nu}^a \]  

(18)

Due to spherical symmetry the above equation reduces to

\[
\left( \frac{\partial^2 \Pi}{\Pi} \right)' - 2\left( \frac{\Pi'}{\Pi} \right) \left( \frac{\partial^2 \Pi}{\Pi} \right) = \frac{\alpha^2}{2} \left( \sum_{k=3}^{N} \frac{6}{k(k-1)} \right) (\tilde{\omega} + \frac{D}{x^3}) \tilde{f}
\]

\[
\tilde{\omega} = -2\left( \frac{\Pi'}{\Pi} \right)^2 \left( \frac{\Pi'}{\Pi} + \frac{3}{x} \right)
\]

\[
\tilde{f} = \frac{\Pi'}{\Pi} \left( \frac{\Pi'}{\Pi} + \frac{2}{x} \right)
\]

(19)

To preserve the Pontryagin index and finiteness of energy the boundary conditions on \( \Pi \) are taken to be the same as before i.e. \( \Pi \to \frac{\rho^2}{x^2} \) as \( x \to 0 \) and \( \Pi \to 1 + \frac{\rho^2}{x^2} \) as \( x \to \infty \).

At first sight the extreme non linearitity of these equations seems intractable. One expects, however, that if one can solve it in the asymptotic regions \( x \to 0, x \to \infty \) to obtain a solution with the same winding number as for \( N_f = 0 \) then a solution which interpolates between these regions may be obtained numerically.

Evaluating the r.h.s of eqn(19) in the limit \( x \to 0 \) one finds that the leading term is \(-2\alpha^2(D-8)\sum_{k=3}^{N} \frac{6}{k(k-1)}/(x^3\rho^2)\) while the l.h.s is less singular. Hence the choice \( D = 8 \) is confirmed. With \( D = 8 \) one finds that one can solve for the unknown coefficients in the deformed superpotential

\[ \Pi_0 = 1 + \frac{\rho^2}{x^2} + a_1 x^2 + a_2 x^4 + \ldots \]  

(20)

\[ a_1 = -\frac{\alpha^2}{\rho^4} \quad a_2 = +\frac{2\alpha^2}{3\rho^6}(1 + \frac{9\alpha^2}{4\rho^2}) \]

(21)

and so on. Note that inspite of the nonlinearity of the fermion back reaction represented by the Chern-Simons term, the deformations are entirely non singular.

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Similarly in the $x \to \infty$ region

$$\Pi_\infty = 1 + \frac{\rho^2}{x^2} + \frac{b_1}{x^4} + \frac{b_2}{x^6} + \ldots$$

$$b_1 = \frac{\alpha'^2 \rho^2}{3}, \quad b_2 = \frac{\alpha'^4 \rho^2}{18}$$

A numerical integration of the equation in the intermediate region will be given elsewhere. The asymptotic behaviour of the electric field is

$$E_{\mu}^{k^2-1} = -\partial_\mu A_0^{k^2-1} = -\frac{\alpha}{2} \sqrt{\frac{6}{k(k-1)}} (\tilde{\omega}_\mu + \frac{D \tilde{x}_\mu}{x^3})$$

$$= \sqrt{\frac{6}{k(k-1)}} (\frac{4\alpha}{x^3} \tilde{x}_\mu + \frac{12\alpha \rho^4}{x^7} \tilde{x}_\mu) + O(x^{-9})$$

The electric charges are

$$Q^{k^2-1} = \frac{1}{g_5^2} \int_{S_\infty} d\Sigma_\mu E_{\mu}^{k^2-1} = -\frac{N_f}{4} \sqrt{\frac{2}{k(k-1)}}$$

The electric field near the origin is

$$E_{\mu}^{k^2-1} = -\frac{12\alpha}{\rho^4} \sqrt{\frac{6}{k(k-1)}} x_\mu + O(x^3)$$

The magnetic field is

$$F_{\mu\nu}^a = -\frac{4}{\rho^2} (\tilde{\eta}_{a\mu\nu} + 2\tilde{x}_\sigma \tilde{\eta}_{a\sigma[\mu} \tilde{x}_{\nu]}) + O(x^2) \quad x \to 0$$

$$= -\frac{4\rho^2}{x^4} (\tilde{\eta}_{a\mu\nu} + 2\tilde{x}_\sigma \tilde{\eta}_{a\sigma[\mu} \tilde{x}_{\nu]}) + O(x^{-6}) \quad x \to \infty$$

The magnetic field is thus asymptotically dipolar.

The CS term does not contribute to the expression for the field energy.
\[ E = \frac{8\pi \nu}{g_5^2} + \frac{3}{2g_5^2} \int d^4x (\frac{\partial^2 \Pi}{\Pi})^2 \]

\[ \rho_{k^2-1}(x) = -\frac{\alpha}{4} \sqrt{\frac{6}{k(k-1)}} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a \]

Thus it is the sum of the original uncharged soliton energy plus a positive definite contribution from the electrostatic energy and another due to the deviation from self-duality. The winding number is unaffected by the weak deformations exhibited in equations (20)-(23).

\[ \nu = \frac{1}{32\pi^2} \int d^4xF_{\mu\nu}^a \tilde{F}_{\mu\nu}^a = \frac{1}{16\pi^2} \int d^4x(\partial_{\mu}\tilde{\omega}_{\mu}) \]

This can be checked by transforming to a nonsingular gauge using the usual transformation \( U = (x^4 + i\tau \cdot \vec{x})/|x| \). The above solutions are parametrized by an arbitrary scale parameter \( \rho \). However the energy \( E \) is not independent of \( \rho \). Thus one expects that the present solution, if stable at all, will relax to that value of \( \rho \) which minimizes the energy. To estimate this value one may approximate \( \Pi \) by \( \Pi_0 \) and \( \Pi_\infty \) in the regions \( x \in [0, \rho] \), \( x \in [\rho, \infty] \) respectively to obtain

\[ E = \frac{8\pi^2}{g_5^2} + \frac{\beta g_b^6 N_f^4}{\rho^4} + \frac{\delta g_b^2 N_f^2}{\rho^2} \]

Where the last term is the obvious dimensional estimate for the electrostatic energy in 4 space dimensions and \( \beta, \delta \) are positive numerical constants. In the approximation we have used, it appears that our solution is unstable against growth of the free parameter \( \rho \) which will tend to \( \infty \) so as to saturate the the self-duality lower bound on the energy. Note however that the corrections to the neutral soliton

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energy are $O(g_5^2)$ and $O(g_5^6)$ thus it quite possible that quantum corrections may stabilize $\rho$ at a finite value.

Since the electric and magnetic fields are orthogonal in the internal space, the Poynting vector, and therefore also the usual (Belinfante-Bessel-Hagen) expression for the the angular momentum of the field configuration, is zero. The nonzero components of the dipole moment $\mu_{\mu\nu} = 4\rho^2\tilde{n}_{a\mu\nu}$ (which can be read off from eqn.(27)) are a signal of the non-particulate aspects of the soliton.

4. Multiplet Structure of Type I CTS

As is well known topological soliton solutions of nonlinear field equations although classical exhibit many of the features of a quantum point particle. In the present case we wish to show that the family of solutions obtained by performing a global $SU(N)$ rotation on the solution given above is analogous to the orbit of states produced by $SU(N)$ rotating an $N(N-1)/2$-plet representation charge eigenstate.

The first element of the analogy is simply the observation that the basic solution of Section 3. is analogous to a particle in an $SU(N)$ eigenstate in as much as only the CSA generators \{\(Q^{k^2-1}, k = 3...N\)\} are nonzero while the “raising and lowering” (off diagonal) charges are zero: in exact analogy with the expectation value of $SU(N)$ generators in a charge eigenstate with $T_3 = 0$. To determine what representation of $SU(N)$ our “state” is a member of, we need to determine all the distinct “eigenstates” (i.e those solutions with $Q_A = 0 \quad A \neq k^2 - 1 \quad , k = 2...N$) on the $SU(N)$ orbit obtained by $SU(N)$ rotating the solution \{\(\hat{A}_\mu\)\}. Since the field equations are covariant under such rotations every such rotation gives a solution which, however, is not in general analogous to an eigenstate since the off diagonal charge values are nonzero. In fact if we write $A'_\mu = U\hat{A}_\mu U^\dagger$ where $U \in SU(N)$ then it is easy to see that $A'_\mu$ is also a solution of the field equations with topological charge $Q' = UQ(\hat{A}_\mu)U^\dagger$ ($Q = Q^A\Lambda^A_{2i}$). We wish to determine the possible eigenstates reachable via $U \in SU(N)$. For simplicity consider the case

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$N = 3$. Writing

$$Q' = a\lambda_3 + \sqrt{3}b\lambda_8 = \sqrt{3}qU\lambda_8 U^\dagger$$

$$q = -\frac{N_f}{4\sqrt{3}}$$

$$U = \begin{pmatrix} v_{2\times 2} & w_{2\times 1} \\ x_{1\times 2} & y \end{pmatrix}$$

we wish to find the possible values of $(a,b)$. Since $detU = 1$ it follows that

$$(a^2 - b^2)b = -q^3$$

so

$$a = \pm \sqrt{\frac{b^3 - q^3}{b}}$$

Further using the unitarity of $U$ and eqn.(31) we have

$$vv^\dagger - 2ww^\dagger = \frac{1}{q} \text{diag}(a+b, -a+b)$$

$$vx^\dagger - 2wy^* = 0$$

$$xx^\dagger + yy^* = -\frac{2b}{q}$$

Since $Q$ is Hermitian $a,b$ are real. From (31) and (34) and unitarity we have

$$(w^T = (w_1, w_2))$$

$$w_1w_2^* = 0$$

$$|w_1|^2 = \frac{q - (a + b)}{3q}$$

$$|w_2|^2 = \frac{q + (a - b)}{3q}$$

Solving eqns.(32)-(35) one finds that the allowed values of $(a,b)$ are

$\{q(0,1), \quad q(\pm \frac{\sqrt{3}}{2}, -\frac{1}{2})\}$. So we see that the only charge eigenstates on the orbit are those corresponding to a $3$ or $\bar{3}$ of $SU(3)$. 

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The literal generalization of this argument to the general case is algebraically daunting. However there is an alternative more intuitive way of obtaining the above result which allows one to guess the result in the general case. In Section 3

the embedding of \(SU(2) \times U(1)^{N-2}\) in \(SU(N)\) was defined by taking the generators of \(SU(2)\) to be \(\sigma^1_{(1)} = \lambda^1 = E_{12}, \sigma^2_{(1)} = \lambda^2 = \tilde{E}_{12}, \sigma^3_{(1)} = diag(1, -1, 0_{N-2})\) and the commuting \(U(1)\)s to be generated by \(\sigma^{k^2-1}_{(1)} = H_{k^2-1}\) where we have introduced a subscript \((1)\) to label this (first) embedding. Clearly one can obtain \(NC_2 - 1\) other \((\frac{SU(N)}{SU(2) \times U(1)^{N-2}}\) related) “placements” by interchanging the role of the pair of \(N\)-plet indices \((12)\) with some other unequal ordered pair \((i < j)\) (\((13), (23)\) for \(SU(3)\) etc.). Then \(\sigma_3\) will have 1 and -1 in the \(i^{th}\) and \(j^{th}\) diagonal place and zero elsewhere, \(\sigma_8\) will have \(\sqrt{\frac{1}{3}}\) in the \(i^{th}\) and \(j^{th}\) diagonal place etc. The solutions corresponding to these options are \(A^{(i)}_{\mu} = \hat{A}_{\mu} \sigma^{(i)}_{2i}\) It is easy to calculate the topological charges for each placement (with reference to the canonical Gell-Mann basis i.e by expanding the \(\sigma^A_{(i)}\) for each placement in terms of the \(\{\lambda^A\}\) ) and confirm that they correspond precisely to the weights of a \(NC_2\)-plet of \(SU(N)\). Of course the anti-solitons occupy the conjugate representations.

5.Type II or \(SO(3) \times U(1)^{N-3} \subset SU(N)\) Associated Solutions.

As we have shown in Section 4, when the generators \(\{\sigma^a; a = 1, 2, 3\}\) in terms of which the ansatz for \(A_\mu\) is formulated receive an \(SU(2) \times U(1)^{N-2}\) embedding in \(SU(N)\) then any non-trivial solution found under the ansatz is necessarily charged. We now show that if the \(\{\sigma^a\}\) are embedded as just \(SU(2)(SO(3))\) in \(SU(3)\) then a neutral solution exists. To see this note that in the Gellmann-basis the structure constants are:

\[
\begin{align*}
f_{123} = 2f_{345} &= -2f_{367} = \frac{2}{\sqrt{3}}f_{458} = \frac{2}{\sqrt{3}}f_{678} = 1 \\
f_{147} = f_{516} = f_{246} = f_{257} &= \frac{1}{2}
\end{align*}
\]

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The first group provided us with the 3 basic placements of $SU(2) \times U(1)$. Similarly the second group provides us with the 4 basic placements of the $SO(3) \subset SU(3)$ maximal embeddings i.e $\{\sigma^a\} \equiv \{\sigma^1 = 2\lambda^1, \sigma^2 = 2\lambda^4, \sigma^3 = 2\lambda^7\} \equiv \{2\lambda^a, a = 1, 4, 7\}$ etc. Now $F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + \frac{1}{2} \epsilon^{abc} A^b_{[\mu} A^c_{\nu]}$ so if we take $A^a_{\mu} = -2\tilde{\eta}_{\alpha\mu}\partial_{\nu}\ln\Pi(x^2) \equiv 2\tilde{A}^a_{\mu}$ spherical symmetry gives $F^a_{\mu\nu}\tilde{F}^b_{\mu\nu} \sim \delta^{ab}$ so that the r.h.s. of the time component of the field equation (8) is proportional to $tr(\lambda^A\lambda^B\lambda^C) \sim tr\lambda^4 = 0$. Thus we can set $A_0 = 0$ and then the field equations reduce to $(\hat{D}_\mu\hat{F}_{\mu\nu})\tilde{a} = 0$ where $\hat{D}_\mu\hat{F} = \delta^{abc}\partial_{\mu} + \epsilon^{abc}\tilde{A}^a_{\mu}$ and these are solved by the usual self dual form of $\Pi$. Note however that since $F^a_{\mu\nu} = 2\tilde{F}^a_{\mu\nu}$ the winding number is now 4!! (recall the “dibaryon” embedding of the Skyrmeon in $SU(3)$). Thus we have obtained an electrically neutral soliton with no analog of spin at the classical level but a nonzero (non-abelian) magnetic dipole moment. Note that after collective quantization of the global gauge zero modes these solitons can pick up a charge and spin $a la$ the Skyrmeon flavor charges and spin$^9$.

When we consider $SO(3) \times U(1)^{N-3}$ embeddings in $SU(N), N \geq 4$ the solitons once more become charged. There are $N C_3$ sets-of-4 $SU(N)$ $SO(3) \times U(1)$ related embeddings analogous to the single set-of-4: $\{(147), (516), (246), (257)\}$ for $SU(3)$. This becomes obvious when we write the above example as

$\{(E_{12}, E_{13}, \tilde{E}_{23}, (E_{13}, E_{12}, E_{23}), (\tilde{E}_{12}, E_{13}, E_{23}), (\tilde{E}_{12}, \tilde{E}_{13}, \tilde{E}_{23})\}$.

As explained above in the case of $SU(3)$ the topological charges for each of these 4 choices are zero since $tr(\lambda^A\lambda^B\lambda^C) \sim tr(\lambda^A\text{diag}(1,1,1))$. For $N \geq 4$, however, one gets in place of 13 an $N \times N$ diagonal matrix with three diagonal entries equal to 1 and zero elsewhere. This matrix does not have zero trace with $\{\lambda^{k^2-1}, k = 4...N\}$ and hence all four of above embeddings have the same topological charges. Similarly one may chose $N C_3 - 1$ other sets of 3 unequal indices to fill out the set of weights of a $N C_3$ of $SU(N)$. If we mod out the action of the (different) $SU(3)$s which relate these sets-of-4 then one obtains a multiplet with the quantum numbers

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(topological charges) of a $^N C_3$-plet of $SU(N)$.

The analogy with a point particle becomes somewhat obscure in the present case. In the case of $SU(3)$ the $SO(3) \subset SU(3)$ embedding yields a soliton with zero $SU(3)$ topological charge. The angular momentum of the field configuration is zero (since the electric field and therefore the Poynting vector are zero). Thus we have a “spinless” neutral lump with a (non-Abelian) magnetic dipole moment. Clearly the presence of the dipole moment is a mark of the extension of this object. The modding out of the action of $SU(3)$ may be thought of as a “compression” to remove the non-particulate features (ie its magnetic moment) of the lump from view.

6. Discussion

In Ref[4] we suggested that an examination of the properties of a certain class of “co-winding” solitons (“syncyclons”) generically present in higher dimensional field theories (i.e defined on space-time extended beyond 4 dimensions by a compact space) is called for. Such solitons would appear as point, string or sheet vacuum defects to our three dimensional low energy eyes and might thus signal the presence of otherwise unobservable extra dimensions. The simplest paradigmatic example of such solitons is furnished by Yang-Mills (YM) theory on $M^4 \times S^1$. The basic solution is nothing but the periodic instanton or caloron$^6$ reinterpreted as a soliton on $M^4 \times S^1$. Classically, such solitons have no (non-abelian) electric charge and, in the Kaluza-Klein (KK) limit $r \gg R$ (where $R$ is the radius of $S^1$ and $r$ the 3-dimensional distance), their fields are those of a non-abelian magnetic dipole together with a scalar potential coming from the extra component of the vector potential$^7,4$.

In order to sketch a believable picture of the properties of the above generic class of solitons it is necessary to take into account the effects of coupling to fermions and the quantization of global gauge collective coordinates since these can radically change the quantum numbers labelling the soliton states due to fermion number

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fractionalization and zero point (quantum) rotation in the internal non-abelian space. In our particular example, using standard results on fermion number fractionalization in odd dimensions, one finds that the soliton picks up an abelian charge $-\frac{\nu q}{4}$ ($\nu$ the Pontryagin index) from each fermion of abelian charge $q$ (Baryon/Lepton number, Electric charge, etc). Similarly non-Abelian charge fractionalization occurs and the term in the gauge effective action which represents this effect is nothing but the 5 dimensional Chern Simons term times $N_f$. This term is an essential part of the low-momentum approximation of the fermion determinant in the gauge field background. As we have seen, inclusion of this term in the bosonic effective action yields charged topological solitons (non abelian topological “electropoles”). In order to fully determine the gauge quantum numbers of these soliton one must quantize the collective coordinates associated with global rotations in the group space. This problem bears a close analogy to that of determining the quantum numbers of the Skyrmeon. The roles of flavor and color are interchanged and the Chern-Simons term bears analogies to the WZW term which also arises by integrating out fermions coupled to the Skyrme field. However it appears that in contrast to the Skyrme case this term does not modify the Collective Quantization constraint equation which arises. Rather it only contributes a topological charge in addition to the charges that arise due to quantization. The significant generalization is that the present system is gauge invariant while the Skyrme-WZW Lagrangian has only a global (flavour) invariance. In the Skyrmeon case the effect of the WZW term in the field equations is to produce a deformation of the original soliton without obviously changing its external interactions (as opposed to its flavour and spin quantum numbers which obtain a non-trivial modification due to the contribution of the WZW term to the above mentioned Collective Quantization constraint). On the contrary, in the present case, the as we have shown the deformed solution carries (topological) $SU(N)$ electric charges which act as sources of electric fields and couple to external

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fields. In the light of the above, we expect that the low-lying soliton states may be represented as outer products of $SU(N)$ topological charge states and $SU(N)$ eigenstates coming from the Collective Quantization procedure

Charged solitons arise in YMCS-Higgs systems in 2+1 dimensions as well where the presence of the CS term allows charged vortex solutions to exist. In 2+1 dimensions the CS term makes the gauge particles massive so that the induced charge couples to a short range force carrier. The integral of the Higgs field charge density is proportional to the quantized flux (divided by 4) so that there too the electric charge is fractional and topological in nature. The existence of charged vortex solutions in 2+1 dimensions can be seen as the analog of the “Witten effect” in 4 dimensions whereby the addition of a CP violating $\theta$ term (i.e a Pontryagin density) to the YM-Higgs lagrangian results in charged monopole solutions. One therefore expects that such a conversion of “magnetic” charge neutral solitons to charged ones occurs in all odd dimensions provided the theory without the CS term has a topological soliton solution. Note however that in the present case the magnetic fields are asymptotically dipolar. Since the magnetic fields are dipolar they do not (in contrast to the situation for a single “grand unified ” monopole) obstruct the organization of the charged excitations that arise on quantization of the global gauge collective coordinates into multiplets of the symmetry group of the vacuum; thus providing a neat illustration of the “dipole evasion” discussed in references [12].

As a preliminary to the quantization of the collective coordinates of solitons in YM theory on $M^4 \times S^1$, and also because of their intrinsic interest we have studied static solutions of the YMCS system in $M^5$. The generalization of our results to $M^4 \times S^1$ and the situation in higher odd dimensions will be reported separately. We have shown that the YMCS theory in 4+1 dimensions like its be-Higgsed cousins in 2+1 dimensions exhibits fascinating solitonic behaviours.

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most remarkable and novel of these is the way in which a non-abelian multiplet structure emerges naturally, with the topological current multiplet associated with the CS term playing a vital role and furnishing a novel instance of a topological charge coupling “electrically” to massless gauge fields. The present [1,4,5] papers are but a beginning in unravelling the intricacies of their behaviour.

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