In the present paper we start the systematic explicit construction of invariant differential operators by giving explicit description of one of the main ingredients - the cuspidal parabolic subalgebras. We explicate also the maximal parabolic subalgebras, since these are also important even when they are not cuspidal. Our approach is easily generalised to the supersymmetric and quantum group settings and is necessary for applications to string theory and integrable models.

1. Introduction

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d’Alembert, Dirac, equations, (for more examples cf., e.g., [1]), to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory, (for a recent review, cf. e.g., [2]). Thus, it is important for the applications in physics to study systematically such operators.
In the present paper we start with the classical situation, with the representation theory of semisimple Lie groups, where there are lots of results by both mathematicians and physicists, cf., e.g. [3-41]. We shall follow a procedure in representation theory in which such operators appear canonically [24] and which has been generalized to the supersymmetry setting [42] and to quantum groups [43]. We should also mention that this setting is most appropriate for the classification of unitary representations of superconformal symmetry in various dimensions, [44],[45],[46],[47], for generalization to the infinite-dimensional setting [48], and is also an ingredient in the AdS/CFT correspondence, cf. [49]. (For a recent paper with more references cf. [50].)

Although the scheme was developed some time ago there is still missing explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the representations are induced.

Just in passing, we shall mention that parabolic subalgebras found applications in quantum groups, (in particular, for the quantum deformations of noncompact Lie algebras), cf. e.g., [43,51,52,53,54,55], and in integrable systems, cf. e.g., [56,57,58,59].

In the present paper the focus will be on the role of parabolic subgroups and subalgebras in representation theory. In the next section we recall the procedure of [24] and the preliminaries on parabolic subalgebras. Then, in Sections 3-11 we give the explicit classification of the cuspidal parabolic subalgebras which are the relevant ones for our purposes. The cuspidal parabolic subalgebras are also summarized in table form in an Appendix.

2. Preliminaries

2.1. General setting

Let $G$ be a noncompact semisimple Lie group. Let $K$ denote a maximal compact subgroup of $G$. Then we have an Iwasawa decomposition $G = KAN$, where $A$ is abelian simply connected, a vector subgroup of $G$, $N$ is a nilpotent simply connected subgroup of $G$ preserved by the action of $A$. Further, let $M$ be the centralizer of $A$ in $K$. Then the subgroup $P_0 = MAN$ is a minimal parabolic subgroup of $G$. A parabolic subgroup $P = M'A'N'$ is any subgroup of $G$ (including $G$ itself) which contains a minimal parabolic subgroup. The number of non-conjugate parabolic subgroups is $2^r$, where $r = \text{rank } A$, cf., e.g., [6]. Note that in general $M'$ is a reductive Lie group with structure: $M' = M_dM_sM_a$, where $M_d$ is a finite group, $M_s$ is a semisimple Lie group, $M_a$ is an abelian Lie group central in $M'$.

The importance of the parabolic subgroups stems from the fact that the representations induced from them generate all (admissible) irreducible representations of $G$ [7]. (For the role of parabolic subgroups in the construction of unitary representations we refer to [10],[13].) In fact, it is enough to use only the so-called cuspidal parabolic subgroups $P = M'A'N'$, singled out by the condition that $\text{rank } M' = \text{rank } M' \cap K$ [8],[14], so that $M'$ has discrete series representations [3]. However, often induction from a non-cuspidal parabolic is also convenient.

1 The simplest example of cuspidal parabolic subgroup is $P_0$ when $M' = M$ is compact. In all
Let \( P = M' A' N' \) be a parabolic subgroup. Let \( \nu \) be a (non-unitary) character of \( A' \), \( \nu \in \mathcal{A}^* \), where \( \mathcal{A}' \) is the Lie algebra of \( A' \). If \( P \) is cuspidal, let \( \mu \) fix a discrete series representation \( D^\mu \) of \( M' \) on the Hilbert space \( V_\mu \), or the so-called limit of a discrete series representation (cf. [23]).

Although not strictly necessary, sometimes it is convenient to induce from non-cuspidal \( P \) (especially if \( P \) is a maximal parabolic). In that case, we use any non-unitary finite-dimensional irreducible representation \( D^\mu \) of \( M' \) on the linear space \( V_\mu \).

More than this, except in the case of induction from limits of discrete series, we can always work with finite-dimensional representations \( V_\mu \) by the so-called translation. Namely, when \( P \) is non-minimal and cuspidal, then instead of the inducing discrete series representation of \( M' \) we can consider the finite-dimensional irrep of \( M' \) which lies on the same orbit of the Weyl group (in other words, has the same Casimirs).

We call the induced representation \( \chi = \text{Ind}_{G}^{G} (\mu \otimes \nu \otimes 1) \) an elementary representation of \( G \) [15]. (These are called generalized principal series representations (or limits thereof) in [23].) Their spaces of functions are:

\[
\mathcal{C}_\chi = \{ \mathcal{F} \in C^\infty (G, V_\mu) \mid \mathcal{F}(g m a) = e^{-\nu(H)} \cdot D^\mu(m^{-1}) \mathcal{F}(g) \} \tag{2.1}
\]

where \( a = \exp(H) \in A', \ H \in \mathcal{A}', \ m \in M', \ n \in N' \). The special property of \( \mathcal{C}_\chi \) is called right covariance [15],[24] (or equivariance). Because of this covariance the functions \( \mathcal{F} \) actually do not depend on the elements of the parabolic subgroup \( P = M' A' N' \).

The elementary representation (ER) \( T^\chi \) acts in \( \mathcal{C}_\chi \) as the left regular representation (LRR) by:

\[
(T^\chi(g) \mathcal{F})(g') = \mathcal{F}(g^{-1}g') , \quad g, g' \in G . \tag{2.2}
\]

One can introduce in \( \mathcal{C}_\chi \) a Fréchet space topology or complete it to a Hilbert space (cf. [6]). We shall need also the infinitesimal version of LRR:

\[
(X_L \mathcal{F})(g) = \frac{d}{dt} \mathcal{F}(\exp(-tX)g)|_{t=0} , \tag{2.3}
\]

where, \( \mathcal{F} \in \mathcal{C}_\chi, \ g \in G, \ X \in \mathfrak{g} \); then we use complex linear extension to extend the definition to a representation of \( \mathfrak{g}^\mathbb{C} \).

---

2 In general, \( \mu \) is a actually a triple \((\epsilon, \sigma, \delta)\), where \( \epsilon \) is the signature of the character of \( M_d \), \( \sigma \) gives the unitary character of \( M_a \), \( \delta \) fixes a discrete or finite-dimensional irrep of \( M_s \) on \( V_\mu \) (the latter depends only on \( \delta \)).

3 It is well known that when \( V_\mu \) is finite-dimensional \( \mathcal{C}_\chi \) can be thought of as the space of smooth sections of the homogeneous vector bundle (called also vector \( G \)-bundle) with base space \( G/P \) and fibre \( V_\mu \), (which is an associated bundle to the principal \( P \)-bundle with total space \( G \)). We shall not need this description for our purposes.
The ERs differ from the LRR (which is highly reducible) by the specific representation spaces $\mathcal{C}_\chi$. In contrast, the ERs are generically irreducible. The reducible ERs form a measure zero set in the space of the representation parameters $\mu, \nu$. (Reducibility here is topological in the sense that there exist nontrivial (closed) invariant subspace.) The irreducible components of the ERs (including the irreducible ERs) are called subrepresentations.

The other feature of the ERs which makes them important for our considerations is a highest (or lowest) weight module structure associated with them [24]. For this we shall use the right action of $G^{cl}$ (the complexification of $G$) by the standard formula:

$$(X_R F)(g) \doteq \frac{d}{dt} F(g \exp(tX))|_{t=0},$$

(2.4)

where $F \in \mathcal{C}_\chi$, $g \in G$, $X \in \mathcal{G}$; then we use complex linear extension to extend the definition to a representation of $G^{cl}$. Note that this action takes $F$ out of $\mathcal{C}_\chi$ for some $X$ but that is exactly why it is used for the construction of the intertwining differential operators.

We can show this property in all cases when $V_\mu$ is a highest weight module, e.g., the case of the minimal parabolic subalgebra and when $(M', M' \cap K)$ is a Hermitian symmetric pair. In fact, we agreed that, except when inducing from limits of discrete series, the space $V_\mu$ will be finite-dimensional.

Then $V_\mu$ has a highest weight vector $v_0$. Using this we introduce $\mathcal{G}$-valued realization $\tilde{T}_\chi$ of the space $\mathcal{C}_\chi$ by the formula:

$$\phi(g) \equiv \langle v_0, F(g) \rangle,$$

(2.5)

where $\langle, \rangle$ is the $M$-invariant scalar product in $V_\mu$. [If $M' = M_0$ is abelian or discrete then $V_\mu$ is one-dimensional and $\tilde{C}_\chi$ coincides with $\mathcal{C}_\chi$; so we set $\phi = F$.] On these functions the right action of $G^{cl}$ is defined by:

$$(X_R \phi)(g) \equiv \langle v_0, (X_R F)(g) \rangle.$$

(2.6)

Part of the main result of our paper [24] is:

**Proposition.** The functions of the $\mathcal{G}$-valued realization $\tilde{T}_\chi$ of the ER $\mathcal{C}_\chi$ satisfy:

$$X_R \phi = \Lambda(X) \cdot \phi, \quad X \in \mathcal{H}^{cl}, \quad \Lambda \in (\mathcal{H}^{cl})^* \quad (2.7a)$$

$$X_R \phi = 0, \quad X \in \mathcal{G}^{cl}_+ \quad (2.7b)$$

---

4 In the geometric language we have replaced the homogeneous vector bundle with base space $G/P$ and fibre $V_\mu$ with a line bundle again with base space $G/P$ (also associated to the principal $P$-bundle with total space $G$). The functions $\phi$ can be thought of as smooth sections of this line bundle.
where \( \Lambda = \Lambda(\chi) \) is built canonically from \( \chi \),\(^5\) \( \mathcal{G}_\pm^\mathcal{E} \) are from the standard triangular decomposition \( \mathcal{G}^\mathcal{E} = \mathcal{G}_+^\mathcal{E} \oplus \mathcal{H}^\mathcal{E} \oplus \mathcal{G}_-^\mathcal{E} \).\(^6\)

Note that conditions (2.7) are the defining conditions for the highest weight vector of a highest weight module (HWM) over \( \mathcal{G}^\mathcal{E} \) with highest weight \( \Lambda \). Of course, it is enough to impose (2.7b) for the simple root vectors \( X^+_j \).

Furthermore, special properties of a class of highest weight modules, namely, Verma modules, are immediately related with the construction of invariant differential operators.

To be more specific let us recall that a Verma module is a highest weight module \( \mathcal{V}^\Lambda \) with highest weight \( \Lambda \), induced from one-dimensional representations of the Borel subalgebra \( \mathcal{B} = \mathcal{H}^\mathcal{E} \oplus \mathcal{G}_+^\mathcal{E} \). Thus, \( \mathcal{V}^\Lambda \cong U(\mathcal{G}_-^\mathcal{E})v_0 \), where \( v_0 \) is the highest weight vector, \( U(\mathcal{G}_-^\mathcal{E}) \) is the universal enveloping algebra of \( \mathcal{G}_-^\mathcal{E} \).\(^7\) Verma modules are universal in the following sense: every irreducible HWM is isomorphic to a factor-module of the Verma module with the same highest weight.

Generically, Verma modules are irreducible, however, we shall be mostly interested in the reducible ones since these are relevant for the construction of differential equations. We recall the Bernstein-Gel’fand-Gel’fand \([5]\) criterion (for semisimple Lie algebras) according to which the Verma module \( \mathcal{V}^\Lambda \) is reducible iff

\[
2\langle \Lambda + \rho, \beta \rangle - m\langle \beta, \beta \rangle = 0 ,
\]

holds for some \( \beta \in \Delta^+, m \in \mathbb{N} \), where \( \Delta^+ \) denotes the positive roots of the root system \( (\mathcal{G}^\mathcal{E}, \mathcal{H}^\mathcal{E}) \), \( \rho \) is half the sum of the positive roots \( \Delta^+ \).

Whenever (2.8) is fulfilled there exists \([12]\) in \( \mathcal{V}^\Lambda \) a unique vector \( v_s \), called singular vector, which has the properties (2.7) of a highest weight vector with shifted weight \( \Lambda - m\beta \):

\[
\begin{align*}
X v_s &= (\Lambda - m\beta)(X) \cdot v_s , \quad X \in \mathcal{H}^\mathcal{E} , \\
X v_s &= 0 , \quad X \in \mathcal{G}_+^\mathcal{E} ,
\end{align*}
\]

(2.9a)(2.9b)

The general structure of a singular vector is \([24]\) :

\[
v_s = P_{m\beta}(X^-_1, \ldots, X^-_\ell)v_0 ,
\]

(2.10)

where \( P_{m\beta} \) is a homogeneous polynomial in its variables of degrees \( mk_i \), where \( k_i \in \mathbb{Z}_+ \) come from the decomposition of \( \beta \) into simple roots: \( \beta = \sum k_i \alpha_i \), \( \alpha_i \in \Delta_S \), the system

---

\(^5\) It contains all the information from \( \chi \), except about the character \( \epsilon \) of the finite group \( M_d \). In the case of \( G \) being a complex Lie group we need two weights to encode \( \chi \), cf. Section 3.

\(^6\) Note that we are working here with highest weight modules instead of the lowest weight modules used in \([24]\); also the weights are shifted by \( \rho \) with respect to the notation of \([24]\).

\(^7\) For more mathematically precise definition, cf. \([12]\).
of simple roots, \(X_j^-\) are the root vectors corresponding to the negative roots \((-\alpha_j)\), \(\alpha_j\) being the simple roots.

\[ \ell = \text{rank}_G \mathcal{G}^\mathbb{C} = \dim \mathcal{H}^\mathbb{C} \text{ is the (complex) rank of } \mathcal{G}^\mathbb{C} . \]

It is obvious that (2.10) satisfies (2.9a), while conditions (2.9b) fix the coefficients of \(P_{m\beta}\) up to an overall multiplicative nonzero constant.

Now we are in a position to define the differential intertwining operators for semisimple Lie groups,

Let the signature \(\chi\) of an ER be such that the corresponding \(\Lambda = \Lambda(\chi)\) satisfies (2.8) for some \(\beta \in \Delta^+\) and some \(m \in \mathbb{N}\).

Then there exists an intertwining differential operator

\[ D_{m\beta} : \tilde{T}^\chi \rightarrow \tilde{T}^{\chi'} , \]

where \(\chi'\) is such that \(\Lambda' = \Lambda'(\chi') = \Lambda - m\beta\).

The most important fact is that (2.11) is explicitly given by [24]:

\[ D_{m\beta} \varphi(g) = P_{m\beta}((X_1^-)_R, \ldots, (X_\ell^-)_R) \varphi(g) , \]

where \(P_{m\beta}\) is the same polynomial as in (2.10) and \((X_j^-)_R\) denotes the action (2.4).

One important simplification is that in order to check the intertwining properties of the operator in (2.12) it is enough to work with the infinitesimal versions of (2.1) and (2.2), i.e., work with representations of the Lie algebra. This is important for using the same approach to superalgebras and quantum groups, and to any other (infinite-dimensional) (super-)algebra with triangular decomposition.

### 2.2. Generalities on parabolic subalgebras

Let \(G\) be a real linear connected semisimple Lie group. Let \(\mathcal{G}\) be the Lie algebra of \(G\), \(\theta\) be a Cartan involution in \(\mathcal{G}\), and \(\mathcal{G} = \mathcal{K} \oplus \mathcal{P}\) be a Cartan decomposition of \(\mathcal{G}\), so that \(\theta X = X, \ X \in \mathcal{K}, \ \theta X = -X, \ X \in \mathcal{P}; \ \mathcal{K}\) is a maximal compact subalgebra of \(\mathcal{G}\). Let \(\mathcal{A}\) be a maximal subspace of \(\mathcal{P}\) which is an abelian subalgebra of \(\mathcal{G}\); \(r = \dim \mathcal{A}\) is the *split* (or *real*) rank of \(\mathcal{G}\), \(1 \leq r \leq \ell = \text{rank } \mathcal{G}\). The subalgebra \(\mathcal{A}\) is called a Cartan subspace of \(\mathcal{P}\).

Let \(\Delta_\mathcal{A}\) be the root system of the pair \((\mathcal{G}, \mathcal{A})\):

\[ \Delta_\mathcal{A} \doteq \{ \lambda \in \mathcal{A}^* \ | \ \lambda \neq 0, \ \mathcal{G}_\mathcal{A}^\lambda \neq 0 \} , \quad \mathcal{G}_\mathcal{A}^\lambda \doteq \{ X \in \mathcal{G} \ | \ [Y, X] = \lambda(Y)X , \ \forall Y \in \mathcal{A} \} . \]

The elements of \(\Delta_\mathcal{A}\) are called \(\mathcal{A}\)-restricted roots. For \(\lambda \in \Delta_\mathcal{A}\), \(\mathcal{G}_\mathcal{A}^\lambda\) are called \(\mathcal{A}\)-restricted root spaces, \(\dim_R \mathcal{G}_\mathcal{A}^\lambda \geq 1\). Next we introduce some ordering (e.g., the lexicographic

\[ A \text{ singular vector may also be written in terms of the full Cartan-Weyl basis of } \mathcal{G}^\mathbb{C}. \]

\[ \text{If } \beta \text{ is a real root, (i.e., } \beta|_{\mathcal{H}_m^\mathbb{C}} = 0, \text{ where } \mathcal{H}_m \text{ is the Cartan subalgebra of } \mathcal{M}, \text{ then some } \text{conditions are imposed on the character } \epsilon \text{ representing the finite group } M_d [17]. \]

\[ \text{The results are easily extended to real linear reductive Lie groups with a finite number of components.} \]
one) in $\Delta_A$. Accordingly the latter is split into positive and negative restricted roots: $\Delta_A = \Delta_A^+ \cup \Delta_A^-$. Now we can introduce the corresponding nilpotent subalgebras:

$$N^\pm \doteq \bigoplus_{\lambda \in \Delta_A^\pm} G_A^\lambda.$$

(2.14)

With this data we can introduce the Iwasawa decomposition of $G$:

$$G = K \oplus A \oplus N^+, \quad N = N^\pm.$$

(2.15)

Next let $M$ be the centralizer of $A$ in $K$, i.e., $M = \{ X \in K \mid [X, Y] = 0, \forall Y \in A \}$. In general $M$ is a compact reductive Lie algebra, and we can write $M = M_s \oplus M_a$, where $M_s \doteq [M, M]$ is the semisimple part of $M$, and $M_a$ is the abelian subalgebra central in $M$.

We mention also that a Cartan subalgebra $H_m$ of $M$ is given by: $H_m = H_s \oplus M_a$, where $H_s$ is a Cartan subalgebra of $M_s$. Then a Cartan subalgebra $H$ of $G$ is given by: $H = H_m \oplus A$.\[11\]

Next we recall the Bruhat decomposition [60]:

$$G = N^+ \oplus M \oplus A \oplus N^-,$$

(2.16)

and the subalgebra $P_0 \doteq M \oplus A \oplus N^-$ called a minimal parabolic subalgebra of $G$. (Note that we may take equivalently $N^+$ instead of $N^-$.)

Naturally, the $G$-subalgebras $K, A, N^\pm, M, M_s, M_a, P_0$ are the Lie algebras of the $G$-subgroups introduced in the previous subsection $K, A, N^\pm, M, M_s, M_a, P_0$, resp.

We mention an important class of real Lie algebras, the split real forms. For these we can use the same basis as for the corresponding complex simple Lie algebra $G^\mathbb{C}$, but over $\mathbb{R}$. Restricting $G \rightarrow \mathbb{R}$ one obtains the Bruhat decomposition of $G$ (with $M = 0$) from the triangular decomposition of $G^\mathbb{R} = G^+ \oplus H^\mathbb{R} \oplus G^-$, and obtains the minimal parabolic subalgebras $P_0$ from the Borel subalgebra $B = H^\mathbb{R} \oplus G^+$, (or $G^-$ instead of $G^+$). Furthermore, in this case $\dim_{\mathbb{R}} K = \dim_{\mathbb{R}} N^\pm$.

A standard parabolic subalgebra is any subalgebra $P'$ of $G$ containing $P_0$. The number of standard parabolic subalgebras, including $P_0$ and $G$, is $2^r$.

Remark: In the complex case a standard parabolic subalgebra is any subalgebra $P'$ of $G^\mathbb{C}$ containing $B$. The number of standard parabolic subalgebras, including $B$ and $G^\mathbb{C}$, is $2^\ell$, $\ell = \text{rank}_{\mathbb{C}} G$.\[\diamondsuit\]

Thus, if $r = 1$ the only nontrivial parabolic subalgebra is $P_0$.

Thus, further in this section $r > 1$.\[11\] Note that $H$ is a $\theta$-stable Cartan subalgebra of $G$ such that $H \cap P = A$. It is the most noncompact among the non-conjugate Cartan subalgebras of $G$. 7
Any standard parabolic subalgebra is of the form:
\[ \mathcal{P'} = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'^- , \tag{2.17} \]
so that \( \mathcal{M}' \supseteq \mathcal{M} \), \( \mathcal{A}' \subseteq \mathcal{A} \), \( \mathcal{N}'^- \subseteq \mathcal{N}^- \); \( \mathcal{M}' \) is the centralizer of \( \mathcal{A}' \) in \( \mathcal{G} \) (mod \( \mathcal{A}' \)); \( \mathcal{N}'^- \) is comprised from the negative root spaces of the restricted root system \( \Delta_{\mathcal{A}'} \) of \( (\mathcal{G}, \mathcal{A}') \). The decomposition (2.17) is called the Langlands decomposition of \( \mathcal{P}' \). One also has the analogue of the Bruhat decomposition (2.16):
\[ \mathcal{G} = \mathcal{N}'^+ \oplus \mathcal{A}' \oplus \mathcal{M}' \oplus \mathcal{N}'^- \], \tag{2.18} 
where \( \mathcal{N}'^+ = \theta \mathcal{N}'^- \).

The standard parabolic subalgebras may be described explicitly using the restricted simple root system \( \Delta^S_A = \Delta^+_{\mathcal{A}} \cup \Delta^-_{\mathcal{A}} \), such that if \( \lambda \in \Delta^+_{\mathcal{A}} \) (resp. \( \lambda \in \Delta^-_{\mathcal{A}} \)), one has:
\[ \lambda = \sum_{i=1}^{r} n_i \lambda_i , \quad \lambda_i \in \Delta^S_A , \quad \text{all } n_i \geq 0, \quad \text{(resp. all } n_i \leq 0 \text{)} . \tag{2.19} \]

We shall follow Warner [6], where one may find all references to the original mathematical work on parabolic subalgebras. For a short formulation one may say that the parabolic subalgebras correspond to the various subsets of \( \Delta^S \).

Clearly, \( \mathcal{N}^+(\emptyset) = 0 \), \( \mathcal{N}^+(S_r) = \mathcal{N}^+ \). Further, we need to bring (2.20) in the form (2.17). First, define \( \mathcal{G}(\Theta) \) as the algebra generated by \( \mathcal{N}^+(\Theta) \) and \( \mathcal{N}^-(\Theta) \) \( \oplus \theta \mathcal{N}^+(\Theta) \). Next, define \( \mathcal{A}(\Theta) \) as the algebra generated by \( \mathcal{A}(\Theta) \cap \mathcal{A} \), and \( \mathcal{A}_\Theta \) as the orthogonal complement (relative to the Euclidean structure of \( \mathcal{A} \)) of \( \mathcal{A}(\Theta) \) in \( \mathcal{A} \). Then \( \mathcal{A} = \mathcal{A}(\Theta) \oplus \mathcal{A}_\Theta \). Note that \( \dim \mathcal{A}(\Theta) = |\Theta| \), \( \dim \mathcal{A}_\Theta = r - |\Theta| \). Next, define:
\[ \mathcal{N}^+_\Theta \equiv \bigoplus_{\lambda \in \Delta^+_{\mathcal{A}} - \Delta^+_{\mathcal{A}_\Theta}} G^\lambda_A , \quad \mathcal{N}^-_\Theta \equiv \theta \mathcal{N}^+_\Theta . \tag{2.21} \]

Then \( \mathcal{N}^\pm = \mathcal{N}^\pm(\Theta) \oplus \mathcal{N}^\pm_\Theta \). Next, define \( \mathcal{M}_\Theta \cong \mathcal{M} \oplus \mathcal{A}(\Theta) \oplus \mathcal{N}^+(\Theta) \oplus \mathcal{N}^-(\Theta) \). Then \( \mathcal{M}_\Theta \) is the centralizer of \( \mathcal{A}_\Theta \) in \( \mathcal{G} \) (mod \( \mathcal{A}_\Theta \)). Finally, we can derive:
\[ \mathcal{P}_\Theta = \mathcal{P}_0 \oplus \mathcal{N}^+(\Theta) = \mathcal{M} \oplus \mathcal{A}(\Theta) \oplus \mathcal{N}^-_\Theta \oplus \mathcal{N}^+(\Theta) = \quad \text{(2.22)} \]
\[ = (\mathcal{M} \oplus \mathcal{A}(\Theta) \oplus \mathcal{N}^-(\Theta) \oplus \mathcal{N}^+(\Theta)) \oplus \mathcal{A}_\Theta \oplus \mathcal{N}^-_\Theta = \]
\[ = \mathcal{M}_\Theta \oplus \mathcal{A}_\Theta \oplus \mathcal{N}^-_\Theta . \]
Thus, we have rewritten explicitly the standard parabolic \( P_\Theta \) in the desired form (2.17). The associated (generalized) Bruhat decomposition (2.18) is given now explicitly as:

\[
G = N^+ \oplus P_0 = N^+_\Theta \oplus N^+(\Theta) \oplus P_0 = N^+_\Theta \oplus P_\Theta = N^+_\Theta \oplus \mathcal{M}_\Theta \oplus \mathcal{A}_\Theta \oplus N^-_\Theta .
\]  

(2.23)

Another important class are the maximal parabolic subalgebras which correspond to \( \Theta \) of the form:

\[
\Theta_j^{\max} = S_r \{ j \}, \quad 1 \leq j \leq r .
\]  

(2.24)

\[ \dim \mathcal{A}(\Theta_j^{\max}) = r - 1 , \quad \dim \mathcal{A}_{\Theta_j^{\max}} = 1. \]

**Reminder 1:** We recall for further use the fundamental result of Harish-Chandra [3] that \( G \) has discrete series representations iff \( \text{rank} G = \text{rank} \mathcal{K} \).

**Reminder 2:** We recall for further use the well known fact that \( (G, \mathcal{K}) \) is a Hermitian symmetric pair when the maximal compact subalgebra \( \mathcal{K} \) contains a \( u(1) \) factor. Then \( G \) has highest and lowest weight representations. All these algebras have discrete series representations.

3. The complex simple Lie algebras considered as real Lie algebras

Let \( G_c \) be a complex simple Lie algebra of dimension \( d \) and (complex) rank \( \ell \). We need the triangular decomposition:

\[
G_c = N^+ \oplus \mathcal{H} \oplus N^- .
\]  

(3.1)

We have \( \dim_G G_c = d \), \( \text{rank}_G G_c = \dim_G \mathcal{H} = \ell \), \( \dim_G N^\pm = (d - \ell)/2 \). Considered as real Lie algebras we have: \( \dim_R G_c = 2d \), \( \text{rank}_R G_c = \dim_R \mathcal{H} = 2\ell \), \( \dim_R \mathcal{K} = d \), \( \text{rank}_R \mathcal{K} = \ell \), \( \dim_R N^\pm = d - \ell \). Note that the maximal compact subalgebra \( \mathcal{K} \) of \( G_c \) is isomorphic to the compact real form \( G_k \) of \( G_c \).

Thus, the complex simple Lie algebras do not have discrete series representations (and highest/lowest weight representations over \( \mathbb{R} \)).

Let \( H_j \), \( j = 1, \ldots, \ell \), be a basis of \( \mathcal{H} \), i.e., \( \mathcal{H} = \text{c.l.s.} \{ H_j , j = 1, \ldots, \ell \} \), (where c.l.s. stands for complex linear span), such that each \( \text{ad}(H_j) \) has only real eigenvalues. Let \( \mathcal{A} = \mathcal{H}_R = \text{r.l.s.} \{ H_j , j = 1, \ldots, \ell \} \), where r.l.s. stands for real linear span. Then the Iwasawa decomposition of \( G_c \) is:

\[
G_c = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N} , \quad \mathcal{N} = \mathcal{N}^\pm .
\]  

(3.2)

The commutant \( \mathcal{M} \) of \( \mathcal{A} \) in \( \mathcal{K} \) is given by:

\[
\mathcal{M} = u(1) \oplus \cdots \oplus u(1) , \quad \ell \text{ factors} .
\]  

(3.3)
In fact, the basis of $\mathcal{M}$ consists of the vectors $\{i H_j, j = 1, \ldots, \ell\}$. The Bruhat decomposition of $G_c$ is:

$$G_c = N^+ \oplus \mathcal{M} \oplus \mathcal{A} \oplus N^-.$$  \hfill (3.4)

Comparing (3.1) and (3.4) we see that

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{A}. \hfill (3.5)$$

The restricted root system $(G_c, \mathcal{A})$ looks the same as the complex root system $(G_c, \mathcal{H})$, but the restricted roots have multiplicity 2, since $\dim_{\mathbb{R}} N^\pm = 2 \dim_{\mathbb{C}} N^\pm$.

Let $\Theta$ be a string subset of $S_\ell$ of length $s$. The $\mathcal{M}_\Theta$-factor of the corresponding parabolic subalgebra is:

$$\mathcal{M}_\Theta = G_s \oplus u(1) \oplus \cdots \oplus u(1), \quad \ell - s \text{ factors}, \hfill (3.6)$$

where $G_s$ is a complex simple Lie algebra of rank $s$ isomorphic to a subalgebra of $G_c$. Thus, the complex simple Lie algebras, considered as real noncompact Lie algebras, do not have non-minimal cuspidal parabolic subalgebras.

Thus, it is enough to consider elementary representations induced from the minimal parabolic subgroup $P_0 = MAN$, where $M \cong U(1) \times \cdots \times U(1), (\ell \text{ factors}), \ A \cong SO(1,1) \times \cdots \times SO(1,1), (\ell \text{ factors}), \ N \cong \exp N^\pm$.\hfill (3.5)

Thus, the signature $\chi = [\mu, \nu]$, consists of $\ell$ integer numbers $\mu_i \in \mathbb{Z}$ giving the unitary character $\mu = (\mu_1, \ldots, \mu_\ell)$ of $M$, and of $\ell$ complex numbers $\nu_i \in \mathbb{C}$ giving the character $\nu = (\nu_1, \ldots, \nu_\ell)$ of $A$, $\nu_j = \nu(H_j)$. Thus, if $H = \sum_j \sigma_j H_j$, $\sigma_j \in \mathbb{R}$, is a generic element of $\mathcal{A}$, then for the corresponding factor in (2.1) we have $e^{\nu(H)} = \exp \sum_j \sigma_j \nu_j$. Analogously, if $m = \exp i \sum_j \phi_j H_j \in M$, $\phi_j \in \mathbb{R}$, then we have $D^\mu(m^{-1}) = \exp i \sum_j \phi_j \mu_j$. Thus, the right covariance property (2.1) becomes:

$$\mathcal{F}(gman) = \exp \sum_j (\sigma_j \nu_j + i \phi_j \mu_j) \cdot \mathcal{F}(g) \hfill (3.7)$$

To relate with the general setting of the previous subsection we must introduce two weight functionals: $\Lambda, \tilde{\Lambda}$, such that $\Lambda(H_j) = \lambda_j$, $\tilde{\Lambda}(H_j) = \tilde{\lambda}_j$. Let us use (3.5) and $H = \sum_j (\sigma_j + i \phi_j) H_j \in \mathcal{H}$. Thus the elementary representations (in particular, the right covariance conditions) for a complex semisimple Lie group $G_c$ are given by:

$$\mathcal{C}_{\Lambda, \tilde{\Lambda}} = \{ \mathcal{F} \in C^\infty(G_c) | \mathcal{F}(gman) = \exp \left( \Lambda(H) + \tilde{\Lambda}(\bar{H}) \right) \cdot \mathcal{F}(g) =$$

$$= \exp \sum_j \left( (\sigma_j + i \phi_j) \lambda_j + (\sigma_j - i \phi_j) \tilde{\lambda}_j \right) \cdot \mathcal{F}(g) \hfill (3.8)$$

$$\nu_j = \lambda_j + \tilde{\lambda}_j, \quad \mu_j = \lambda_j - \tilde{\lambda}_j \in \mathbb{Z}$$

\[ \text{We should note that the minimal parabolic subgroup } P_0 \text{ is isomorphic to a Borel subgroup of } G_c, \text{ due to the obvious isomorphism between the abelian subgroup } MA \text{ and the Cartan subgroup } H \text{ of } G_c. \]
and the last condition in (3.8) stresses that we have uniqueness on the compact subgroup $M$ of the Cartan subgroup $H_c = MA$ of $G_c$.

The ERs for which $\tilde{\Lambda} = 0$ are called holomorphic, and those for which $\Lambda = 0$ are called antiholomorphic.

Thus, we see that the complex case is richer than the real one. Indeed, there are two Verma modules associated with an ER, one ’holomorphic’ $V^\Lambda$ and one ’antiholomorphic’ $V^{\tilde{\Lambda}}$. The ER is reducible when either $V^\Lambda$ or $V^{\tilde{\Lambda}}$ are reducible, i.e., when (2.8) holds for either $\Lambda$ or $\tilde{\Lambda}$.

More information can be found in [8] from where we mention some important statements: All irreducible representations of a complex semisimple Lie group are obtained as subrepresentations of the elementary representations induced from the minimal parabolic subgroup. All finite-dimensional irreps are obtained as subrepresentations when all $\lambda_j, \tilde{\lambda}_j \in \mathbb{Z}_+$. The maximal parabolic subalgebras have $M_\Theta$-factors as follows

$$M_\Theta = G_i \oplus u(1), \quad i = 1, \ldots, \ell,$$

where $G_i$ is a complex simple Lie algebra of rank $\ell - 1$ which may be obtained from $G_c$ by deleting the $i$-th node of the Dynkin diagram of $G_c$.

4. **AI : SL($n$, IR)**

In this section $G = SL(n, \mathbb{R})$, the group of invertible $n \times n$ matrices with real elements and determinant 1. Then $G = sl(n, \mathbb{R})$ and the Cartan involution is given explicitly by: $\theta X = -^t X$, where $^t X$ is the transpose of $X \in G$. Thus, $K \cong so(n)$, and is spanned by matrices (r.l.s. stands for real linear span):

$$K = r.l.s.\{X_{ij} \equiv e_{ij} - e_{ji}, \quad 1 \leq i < j \leq n\}, \quad (4.1)$$

where $e_{ij}$ are the standard matrices with only nonzero entry (=1) on the $i$-th row and $j$-th column, $(e_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$. Note that $G$ does not have discrete series representations if $n > 2$. Indeed, the rank of $sl(n, \mathbb{R})$ is $n - 1$, and the rank of its maximal compact subalgebra $so(n)$ is $[n/2]$ and the latter is smaller than $n - 1$ unless $n = 2$.

Further, the complementary space $\mathcal{P}$ is given by:

$$\mathcal{P} = r.l.s.\{Y_{ij} \equiv e_{ij} + e_{ji}, \quad 1 \leq i < j \leq n, \quad H_j \equiv e_{jj} - e_{j+1,j+1}, \quad 1 \leq j \leq n - 1\}. \quad (4.2)$$

The split rank is $r = n - 1$, and from (4.2) it is obvious that in this setting one has:

$$A = r.l.s.\{H_j, \quad 1 \leq j \leq n - 1 = r\}. \quad (4.3)$$

Since $G$ is a maximally split real form of $G^\mathbb{C} = sl(n, \mathbb{C})$, then $M = 0$, and the minimal parabolic subalgebra and the Bruhat decomposition, resp., are given as a Borel subalgebra and triangular decomposition of $G^\mathbb{C}$, but over $\mathbb{R}$:

$$G = \mathcal{N}^+ \oplus A \oplus \mathcal{N}^-, \quad \mathcal{P}_0 = A \oplus \mathcal{N}^-, \quad (4.4)$$
where $\mathcal{N}^+, \mathcal{N}^-$, resp., are upper, lower, triangular, resp.:

$$\mathcal{N}^+ = \text{r.l.s.} \{e_{ij}, \ 1 \leq i < j \leq n\}, \ \mathcal{N}^- = \text{r.l.s.} \{e_{ij}, \ 1 \leq j < i \leq n\}. \quad (4.5)$$

The simple root vectors are given explicitly by:

$$X^+_j = e_{j,j+1}, \ \ X^-_j = e_{j+1,j}, \ \ 1 \leq j \leq n - 1 = r. \quad (4.6)$$

Note that matters are arranged so that

$$[X^+_j, X^-_j] = H_j, \ \ [H_j, X^\pm_j] = \pm 2X^\pm_j, \quad (4.7)$$

and further we shall denote by $sl(2, \mathbb{R})_j$ the $sl(2, \mathbb{R})$ subalgebra of $\mathcal{G}$ spanned by $X^\pm_j, H_j$.

The parabolic subalgebras may be described by the unordered partitions of $n$.\(^{13}\) Explicitly, let $\bar{\nu} = \{\nu_1, \ldots, \nu_s\}, \ s \leq n$, be a partition of $n$: $\sum_{j=1}^s \nu_j = n$. Then the set $\Theta$ corresponding to the partition $\bar{\nu}$ and denoted by $\Theta(\bar{\nu})$ consists of the numbers of the entries $\nu_j$ that are bigger than 1:

$$\Theta(\bar{\nu}) = \{ \ j \mid \nu_j > 1 \ \} . \quad (4.8)$$

Note that in the case $s = n$ all $\nu_j$ are equal to 1 - this is the partition $\bar{\nu}_0 = \{1, \ldots, 1\}$ corresponding to the empty set: $\Theta(\bar{\nu}_0) = \emptyset$ (corresponding to the minimal parabolic). Then the factor $\mathcal{M}_{\Theta(\bar{\nu})}$ in (2.22) and (2.23) is:

$$\mathcal{M}_{\Theta(\bar{\nu})} = \bigoplus_{\nu_j > 1}^{1 \leq j \leq s} sl(\nu_j, \mathbb{R}) \bigoplus_{1 \leq j \leq s}^{s} sl(\nu_j, \mathbb{R}), \quad sl(1, \mathbb{R}) \equiv 0 \quad (4.9)$$

Certainly, some partitions give isomorphic (though non-conjugate!) $\mathcal{M}_{\Theta(\bar{\nu})}$ subalgebras. The parabolic subalgebras in these cases are called associated, and this is an equivalence relation. The parabolic subalgebras up to this equivalence relation correspond to the ordered partitions of $n$.

The most important for us cuspidal parabolic subalgebras correspond to those partitions $\bar{\nu} = \{\nu_1, \ldots, \nu_s\}$ for which $\nu_j \leq 2, \ \forall j$. Indeed, if some $\nu_j > 2$ then $\mathcal{M}_{\Theta(\bar{\nu})}$ will not have discrete series representations since it contains the factor $sl(\nu_j, \mathbb{R})$.

A more explicit description of the cuspidal cases is given as follows. It is clear that the cuspidal parabolic subalgebras are in 1-to-1 correspondence with the sequences of $r$ numbers:

$$\bar{n} = \{n_1, \ldots, n_r\} , \quad (4.10)$$

such that $n_j = 0, 1$, and if for fixed $j$ we have $n_j = 1$, then $n_{j+1} = 0$, (clearly from the latter follows also $n_{j-1} = 0$, but we shall use this notation also in other contexts). In the

\(^{13}\) The parabolic subalgebras may also be described by the various flags of $\mathbb{R}^n$, F., e.g., [6], but we shall not use this description.
language above to each \( n_j = 1 \) there is an entry \( \nu_j = 2 \) in \( \bar{\nu} \) bringing an \( sl(2, \mathbb{R}) \) factor to \( \mathcal{M}_\Theta \), i.e.,
\[
\Theta(\bar{n}) = \{ j \mid n_j = 1, \ n_{j+1} = 0 \} .
\]

(4.11)

More explicitly, the cuspidal parabolic subalgebras are given as follows:
\[
\mathcal{M}_{\Theta(\bar{n})} = \bigoplus_{1 \leq t \leq k} sl(2, \mathbb{R})_{j_t} , \ n_{j_1} = 1, \ 1 \leq j_1 < j_2 < \cdots < j_k \leq r , \ j_t < j_{t+1} - 1 .
\]

(4.12)

The corresponding \( \mathcal{A}_{\Theta(\bar{n})} \) and \( \mathcal{N}_{\Theta(\bar{n})}^\pm \) have dimensions:
\[
\dim \mathcal{A}_{\Theta(\bar{n})} = n - 1 - k , \quad \dim \mathcal{N}_{\Theta(\bar{n})}^\pm = \frac{1}{2} n(n - 1) - k ,
\]

(4.13)

where \( k = |\Theta(\bar{n})| \) was introduced in (4.12).

Note that the minimal parabolic subalgebra is obtained when all \( n_j = 0, \ \bar{n}_0 = \{0, \ldots, 0\} \), then \( \Theta(\bar{n}_0) = \emptyset, \ \mathcal{M}_{\Theta(\bar{n}_0)} = 0, \ k = 0. \)

Interlude: The number of cuspidal parabolic subalgebras of \( sl(n, \mathbb{R}), \ n \geq 2 \), including also the case \( \mathcal{P} = \mathcal{M}' = sl(n, \mathbb{R}) \) when \( n = 2 \), is equal to \( F(n+1) \), where \( F(n), n \in \mathbb{Z}_+ \), are the Fibonacci numbers.

Proof: First we recall that the Fibonacci numbers are determined through the relations
\[
F(m) = F(m - 1) + F(m - 2), \ m \in 2 + \mathbb{Z}_+ ,
\]

(4.14)

together with the boundary values: \( F(0) = 0, F(1) = 1 \). We shall count the number of sequences of \( r \) numbers \( n_i \), introduced above (\( r = n - 1 \)). Let us denote by \( N(r) \) the number of the above-described sequences. Let us divide these sequences in two groups: the first with \( n_1 = 1 \) and the others with \( n_1 = 0 \). Obviously the number of sequences with \( n_1 = 1 \) is equal to \( N(r - 2) \) since \( n_2 = 0, \) and then we are left with the above-described sequences but of \( r - 2 \) numbers. Analogously, the number of sequences with \( n_1 = 0 \) is equal to \( N(r - 1) \) since we are left with all above-described sequences of \( r - 1 \) numbers. Thus, we have proved that
\[
N(r) = N(r - 1) + N(r - 2) .
\]

This is the Fibonacci recursion relation and we have only to adjust the boundary conditions. We have \( N(1) = 2, N(2) = 3, \) i.e., \( N(r) = F(r + 2), \) or in terms of \( n = r + 1: \ N(n - 1) = F(n + 1). \)

For further use we recall that there is explicit formula for the Fibonacci numbers:
\[
F(n) = \frac{x^n - (1 - x)^n}{\sqrt{5}} = 2^{1-n} \sum_{s=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2s+1} 5^s ,
\]

(4.14)

where \( x \) is the golden ratio: \( x^2 = x + 1, \) i.e., \( x = (1 \pm \sqrt{5})/2. \)

Finally, we mention that the maximal parabolic subalgebras corresponding to \( \Theta \) from (2.24) have the following factors:
\[
\mathcal{M}_{\Theta_j} = sl(j, \mathbb{R}) \oplus sl(n - j, \mathbb{R}) , \ 1 \leq j \leq n - 1 ,
\]

(4.15)

\[
\dim \mathcal{A}_{\Theta_j} = 1, \quad \dim \mathcal{N}_{\Theta_j}^\pm = j(n - j)
\]

(Note that the cases \( j \) and \( n - j \) are isomorphic, or coinciding when \( n \) is even and \( j = \frac{1}{2} n. \) Only one of the maximal ones is cuspidal, namely, for \( \mathcal{G} = sl(4, \mathbb{R}), n = 4 \) and \( j = 2 \) we have
\[
\mathcal{M}_{\Theta_2} = sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) .
\]

(4.16)
5. \textbf{AII :} $SU^*(2n)$

The group $G = SU^*(2n), n \geq 2,$ consists of all matrices in $SL(2n, \mathcal{C})$ which commute with a real skew-symmetric matrix times the complex conjugation operator $C$:

$$SU^*(2n) \doteq \{ g \in SL(2n, \mathcal{C}) \mid J_n C g = g J_n C , \quad J_n \equiv \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \} . \quad (5.1)$$

The Lie algebra $\mathcal{G} = su^*(2n)$ is given by:

$$su^*(2n) \doteq \{ X \in sl(2n, \mathcal{C}) \mid J_n C X = X J_n C \} = \{ X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in gl(n, \mathcal{C}), \quad \text{tr} (a + \bar{a}) = 0 \} . \quad (5.2)$$

$$\dim_R \mathcal{G} = 4n^2 - 1.$$

We consider $n \geq 2$ since $su^*(2) \cong su(2),$ and we note that the case $n = 2$ (of split rank 1) will appear also below: $su^*(4) \cong so(5,1),$ cf. the corresponding Section.

The Cartan involution is given by: $\theta X = -X^\dagger.$ Thus, $\mathcal{K} \cong sp(n)$:

$$\mathcal{K} = \{ X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in gl(n, \mathcal{C}), \quad a^\dagger = -a, \quad \bar{t}b = b \} . \quad (5.3)$$

Note that $su^*(2n)$ does not have discrete series representations ($\text{rank} \mathcal{K} = n < \text{rank} su^*(2n) = 2n - 1$). The complimentary space $\mathcal{P}$ is given by:

$$\mathcal{P} = \{ X = \begin{pmatrix} a & b \\ b^\dagger & t_a \end{pmatrix} \mid a, b \in gl(n, \mathcal{C}), \quad a^\dagger = a, \quad \bar{t}b = -b, \quad \text{tr} a = 0 \} . \quad (5.4)$$

The split rank is $n - 1$ and the abelian subalgebra $\mathcal{A}$ is given explicitly by:

$$\mathcal{A} = \{ X = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a = \text{diag} (a_1, \ldots, a_n), \quad a_j \in \mathbb{R}, \quad \text{tr} a = 0 \} . \quad (5.5)$$

The subalgebras $\mathcal{N}^\pm$ which form the root spaces of the root system ($\mathcal{G}, \mathcal{A}$) are of real dimension $2n(n - 1).$ The subalgebra $\mathcal{M}$ is given by:

$$\mathcal{M} = \{ X = \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix} \mid a = i \text{diag} (\phi_1, \ldots, \phi_n), \quad \phi_j \in \mathbb{R}, \quad b = \text{diag} (b_1, \ldots, b_n), \quad b_j \in \mathcal{C} \} \cong \cong su(2) \oplus \cdots \oplus su(2), \quad n \text{ factors} . \quad (5.6)$$

\textbf{Claim:} All non-minimal parabolic subalgebras of $su^*(2n)$ are not cuspidal.

\textbf{Proof:} Necessarily $n > 2.$ Let $\Theta$ enumerate a connected string of restricted simple
roots: \( \Theta = S_{ij} = \{ i, \ldots, j \} \), where \( 1 \leq i \leq j < n \). Then the corresponding subalgebra \( M_{\Theta} \) is given by:

\[
M_{ij} = su^*(2(s+1)) \oplus su(2) \oplus \cdots \oplus su(2) \quad n-s-1 \text{ factors} \quad s \equiv j-i+1 . \quad (5.7)
\]

In general \( \Theta \) consists of such strings, each string of length \( s \) produces a factor \( su^*(2(s+1)) \), the rest of \( M_{\Theta} \) consists of \( su(2) \) factors.

The maximal parabolic subalgebras, cf. (2.24), \( 1 \leq j \leq n-1 \), contain \( M_{\Theta} \) subalgebras of the form:

\[
M_{j}^{\text{max}} = su^*(2j) \oplus su^*(2(n-j)) . \quad (5.8)
\]

(For \( j = 1 \) or \( j = n-1 \) (5.8) coincides with (5.7) for \( s = n-2 \) (and using \( su^*(2) \cong su(2) \)).

6. AIII,AIV : \( SU(p,r) \)

In this section \( G = SU(p,r) \), \( p \geq r \), which standardly is defined as follows:

\[
SU(p,r) \doteq \{ g \in GL(p+r, \mathbb{C}) \mid g^\dagger \beta_0 g = \beta_0 , \quad \beta_0 \equiv \begin{pmatrix} 1_p & 0 \\ 0 & -1_r \end{pmatrix} , \quad \det g = 1 \} , \quad (6.1)
\]

where \( g^\dagger \) is the Hermitian conjugate of \( g \). We shall use also another realization of \( G \) differing from (6.1) by unitary transformation:

\[
\beta_0 \mapsto \beta_2 \equiv \begin{pmatrix} 1_{p-r} & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix} = U \beta_0 U^{-1} , \quad U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{p-r} & 0 & 0 \\ 0 & 1_r & 1_r \\ 0 & -1_r & 1_r \end{pmatrix} . \quad (6.2)
\]

The Lie algebra \( G = su(p,r) \) is given by \( (\beta = \beta_0, \beta_2) \):

\[
su(p,r) \doteq \{ X \in gl(p+r, \mathbb{C}) \mid X^\dagger \beta + \beta X = 0 , \quad \text{tr} X = 0 \} . \quad (6.3)
\]

The Cartan involution is given explicitly by: \( \theta X = \beta X \beta \). Thus, \( K \cong u(1) \oplus su(p) \oplus su(r) \), and more explicitly is given as \( (\beta = \beta_0) \):

\[
K = \{ X = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \mid u_j^\dagger = -u_j , \quad j = 1, 2; \quad \text{tr} u_1 + \text{tr} u_2 = 0 \} . \quad (6.4)
\]

Note that \( su(p,r) \) has discrete series representations since \( \text{rank} K = 1 + \text{rank} su(p) + \text{rank} su(r) = p + r - 1 = \text{rank} su(p,r) \), and highest/lowest weight representations.

The split rank is equal to \( r \) and the abelian subalgebra \( A \) may be given explicitly by \( (\beta = \beta_2) \):

\[
A = \text{r.l.s.}\{ H_j^x \equiv e_{p-r+j,p-r+j} - e_{p+j,p+j} , \quad 1 \leq j \leq r \} . \quad (6.5)
\]

At this moment we need to consider the cases \( p = r \) and \( p > r \) separately, since the minimal parabolic subalgebras are different.
6.1. The case $SU(n, n)$, $n > 1$

In this subsection $G = SU(n, n)$. We consider $n > 1$ since $SU(1, 1) \cong SL(2, \mathbb{R})$, which was already treated.

The subalgebra $\mathcal{M} \cong u(1) \oplus \cdots \oplus u(1)$, $(n - 1)$ factors, and is explicitly given as ($\beta = \beta_2$):

$$\mathcal{M} = \{ X = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} | u = i \, \text{diag} (\phi_1, \ldots, \phi_n), \ \phi_j \in \mathbb{I} \mathbb{R} ; \ \ \text{tr} \ u = 0 \} \ . \quad (6.6)$$

The subalgebras $\mathcal{N}^\pm$ which form the root spaces of the root system $(\mathcal{G}, \mathcal{A})$ are of real dimension $n(2n-1)$. The simple root system $(\mathcal{G}, \mathcal{A})$ looks as that of the symplectic algebra $C_n$, however, the root spaces of the short roots have multiplicity 2.

Further, we choose the long root of the $C_n$ simple root system to be $\alpha_n$.

**Claim:** The nontrivial cuspidal parabolic subalgebras are given by $\Theta$ of the form:

$$\Theta_j = \{ j + 1, \ldots, n \} , \quad 1 \leq j < n \ . \quad (6.7)$$

**Proof:** First note that we exclude $j = 0$ since $\Theta_0 = S_n$. Consider now any $\Theta$ which contains a subset $S_{ij} = \{ i, \ldots, j \}$, where $1 \leq i \leq j < n$. Then the simple roots corresponding to $S_{ij}$ form a string subset $\pi_{ij}$ of the simple root system of $A_{n-1}$, but each root has multiplicity 2. Because of this multiplicity this string of simple roots will produce a subalgebra $sl(j - i + 2, \mathcal{C})$ of $\mathcal{M}_{\Theta}$. Since the simple Lie algebras $sl(n, \mathcal{C})$ do not have discrete series representations, then $\mathcal{P}_{\Theta}$ is not cuspidal. Now it remains to note that $\mathcal{P}_{\Theta_j}$ is cuspidal for all $j$ since

$$\mathcal{M}_{\Theta_j} \cong su(n - j, n - j) \oplus u(1) \oplus \cdots \oplus u(1), \ j \ \text{factors} \quad (6.8)$$

cf. the Remark above. ◊

The maximal parabolic subalgebras correspond to the sets $\Theta_{j_{\text{max}}}^\text{max}$, $j = 1, \ldots, n$, cf. (2.24). The corresponding $\mathcal{M}_{\Theta}$ subalgebras are of the form:

$$\mathcal{M}_{j_{\text{max}}}^{\text{max}} = sl(j, \mathcal{C}) \oplus su(n - j, n - j) \oplus u(1) \oplus \cdots \oplus u(1), \ j \ \text{factors} , \quad (6.9)$$

where we use the convention: $sl(1, \mathcal{C}) = 0$. Note that $\Theta_{1_{\text{max}}}^{\text{max}} = \Theta_1$ and that the only cuspidal maximal parabolic subalgebra is $\mathcal{P}_{\Theta_1}$.

The latter is also the only Heisenberg parabolic subalgebra.

6.2. The case $SU(p, r)$, $p > r \geq 1$

In this subsection $G = SU(p, r)$. We include also the case $r = 1$ although we noted that the case of split rank 1 is clear in general.

The subalgebra $\mathcal{M} \cong su(p - r) \oplus u(1) \oplus \cdots \oplus u(1)$, $(r)$ factors, and is explicitly given as ($\beta = \beta_2$):

$$\mathcal{M} = \{ X = \begin{pmatrix} u_{p-r} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} | u_{p-r}^t = -u_{p-r} , \ u = i \, \text{diag} (\phi_1, \ldots, \phi_n), \ \phi_j \in \mathbb{I} \mathbb{R} , \ \ \text{tr} \ u_{p-r} + 2\text{tr} u = 0 \} \quad (6.10)$$
The subalgebras \( \mathcal{N}^{\pm} \) which form the root spaces of the root system \((\mathbb{G}, \mathbb{A})\) are of real dimension \( r(2p - 1) \). The restricted simple root system \((\mathbb{G}, \mathbb{A})\) looks as that of the orthogonal algebra \( B_r \), however, the root spaces of the long roots have multiplicity 2, the short simple root, say \( \alpha_r \), has multiplicity \( 2(p - r) \), and there is also a root \( 2\alpha_r \) with multiplicity 1.

Similarly to the \( su(n, n) \) case one can prove that the nontrivial cuspidal parabolic subalgebras are given by \( \Theta \) of the form:

\[
\Theta_j = \{ j + 1, \ldots, r \} , \quad 1 \leq j < r , \quad r > 1 .
\]

(6.11)

The corresponding cuspidal parabolic subalgebras contain the subalgebras

\[
\mathcal{M}_{\Theta_j} \cong su(p - j, r - j) \oplus u(1) \oplus \cdots \oplus u(1) , \quad j \text{ factors} .
\]

(6.12)

The maximal parabolic subalgebras, (cf. (2.24)), contain the \( \mathcal{M}_{\Theta} \) subalgebras are of the form:

\[
\mathcal{M}_{j}^{\text{max}} = sl(j, \mathbb{C}) \oplus su(p - j, r - j) \oplus u(1) \oplus \cdots \oplus u(1) , \quad j \text{ factors} .
\]

(6.13)

Thus, the only cuspidal maximal parabolic subalgebra is \( P_{\Theta_1} \).

The latter is also the only Heisenberg parabolic subalgebra.

### 7. BDI, BDII : \( SO(p, r) \)

In this section \( G = SO(p, r) , \ p \geq r \), which standardly is defined as follows:

\[
SO(p, r) = \{ g \in SO(p + r, \mathbb{C}) \mid g^\dagger \beta_0 g = \beta_0 , \quad \beta_0 \equiv \begin{pmatrix} 1_p & 0 \\ 0 & -1_r \end{pmatrix} \} ,
\]

(7.1)

where \( g^\dagger \) is the Hermitian conjugate of \( g \). We shall use also another realization of \( G \) differing from (7.1) by unitary transformation:

\[
\beta_0 \mapsto \beta_2 \equiv \begin{pmatrix} 1_{p-r} & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{pmatrix} = U \beta_0 U^{-1} , \quad U \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{p-r} & 0 & 0 \\ 0 & 1_r & 1_r \\ 0 & -1_r & 1_r \end{pmatrix} .
\]

(7.2)

The Lie algebra \( \mathcal{G} = so(p, r) \) is given by \( (\beta = \beta_0, \beta_2) \):

\[
so(p, r) = \{ X \in so(p + r, \mathbb{C}) \mid X^\dagger \beta + \beta X = 0 \} .
\]

(7.3)

The Cartan involution is given explicitly by: \( \theta X = \beta X \beta \). Thus, \( \mathcal{K} \cong so(p) \oplus so(r) \), and more explicitly is given as \( (\beta = \beta_0) \):

\[
\mathcal{K} = \{ X = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \mid u_1 \in so(p) , \quad u_2 \in so(r) \} .
\]

(7.4)
Note that \( so(5,1) \cong su^*(4), \ so(4,2) \cong su(2,2), \ so(3,3) \cong sl(4,\mathbb{R}), \ so(4,1) \cong sp(1,1), \)
\( so(3,2) \cong sp(2,\mathbb{R}), \ so(3,1) \cong sl(2,\mathbb{C}), \ so(2,2) \cong sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}), \ so(2,1) \cong sl(2,\mathbb{R}), \)
(so\((1,1)\) is abelian). Thus, below we can restrict to \( p + r > 4 \), since the cases \( p + r = 5 \) are not treated yet.

Note that \( so(p,r) \) has discrete series representations except when both \( p,r \) are odd numbers, since then \( \text{rank } K = \text{rank } so(p) + \text{rank } so(r) = \frac{1}{2}(p + r - 2) < \text{rank } so(p,r) = \frac{1}{2}(p + r) \). It has highest/lowest weight representations when \( p \geq r = 2 \) and \( p = 2, r = 1 \).

The split rank is equal to \( r \) and the abelian subalgebra \( A \) may be given explicitly by \((\beta = \beta_2)\):

\[
A = \text{r.l.s.}\{ H_j^u \equiv e_{p-r+j,p-r+j} - e_{p+j,p+j}, \ 1 \leq j \leq r \}. \tag{7.5}
\]

The subalgebra \( \mathcal{M} \cong so(p-r) \) and is explicitly given as \((\beta = \beta_2)\):

\[
\mathcal{M} = \{ X = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} | u \in so(p-r) \} \tag{7.6}
\]

The subalgebras \( N^\pm \) which form the root spaces of the root system \((G,A)\) are of real dimension \( r(p-1) \). Except in the case \( p = r \) the restricted simple root system \((G,A)\) looks as that of the orthogonal algebra \( B_r \), however, the short simple root, say \( \alpha_r \), has multiplicity \( p-r \).

Thus, we consider first the case \( p > r > 1 \). First we note that the parabolic subalgebras given by \( \Theta_j = \{ j \}, j < r \) contain a factor: \( \mathcal{M}_{\Theta_j} = sl(2,\mathbb{R}) \oplus so(p-r) \). More generally, if \( r \notin \Theta \) then all possible cuspidal parabolic subalgebras are like those of \( sl(r,\mathbb{R}) \), adding the compact subalgebra \( so(p-r) \). Suppose now, that \( r \in \Theta \). In that case, \( \Theta \) will include a set \( \Theta_j \) of the form:

\[
\Theta_j = \{ j + 1, \ldots, r \}, \ 1 \leq j < r. \tag{7.7}
\]

That would bring a \( \mathcal{M}_{\Theta} \) factor of the form \( so(p-j,r-j) \). Thus, all possible cuspidal parabolic subalgebras are obtained for those \( j \), for which the number \((p-j)(r-j)\) is even and for fixed such \( j \) they would be like those of \( sl(j,\mathbb{R}) \), adding the non-compact subalgebra \( so(p-j,r-j) \). Clearly, if both \( p, r \) are even (odd), then also \( j \) must be even (odd), while if one of \( p, r \) is even and the other odd, i.e., \( p + r \) is odd, then \( j \) takes all values from \((7.7)\).

To be more explicit we first introduce the notation:

\[
\bar{n}_s \doteq \{ n_1, \ldots, n_s \}, \ 1 \leq s \leq r, \tag{7.8}
\]

(note that \( \bar{n}_r = \bar{n} \) from \((4.10)\)). Then we shall use the notation introduced for the \( sl(n,\mathbb{R}) \) case, namely, \( \Theta(\bar{n}_s) \) from \((4.11)\). Then the cuspidal parabolic subalgebras are given by the noncompact factors \( \mathcal{M}_{\Theta} \) from \((4.12)\):

\[
\mathcal{M}_s = \mathcal{M}_{\Theta(\bar{n}_s)} \oplus so(p-s,r-s), \begin{cases} s = 1, 2, \ldots, r-1 & p + r \text{ odd} \\
 s = 2, 4, \ldots, r-2 & p, r \text{ even} \\
 s = 1, 3, \ldots, r-2 & p, r \text{ odd} \end{cases} \tag{7.9}
\]
Next we note that we can include the case when the second factor in $\mathcal{M}_\Theta$ is compact by just extending the range of $s$ to $r$.

Thus, all cuspidal parabolic subalgebras of $so(p, r)$ in the case $p > r$ will be determined by the following $\mathcal{M}_\Theta$ subalgebras:

$$\mathcal{M}_s = \mathcal{M}_{\Theta(\bar{n}_s)} \oplus so(p-s, r-s) , \quad \begin{cases} s = 1, 2, \ldots, r & p + r \text{ odd} \\ s = 2, 4, \ldots, r & p, r \text{ even} \\ s = 1, 3, \ldots, r & p, r \text{ odd} \end{cases}$$ (7.10)

The algebras $\mathcal{M}_\Theta$ have highest/lowest weight representations only when $s = r - 2$ or $s = r$, since then the second factor is $so(p-r+2, 2)$, $so(p-r)$, resp.

Finally, we note that the maximal parabolic subalgebras corresponding to (2.24) have $\mathcal{M}_\Theta$-factors given by:

$$\mathcal{M}_{j}^{\text{max}} = sl(j, IR) \oplus so(p-j, r-j) , \quad j = 1, 2, \ldots, r .$$ (7.11)

Thus, the maximal parabolic subalgebras are cuspidal (and can be found in (7.10)) when $j = 1, 2$ and the number $(p-j)(r-j)$ is even. In addition, $\mathcal{M}_{j}^{\text{max}}$ have highest/lowest weight representations only when $r-j = 0, 2$, (or $p-j = 2$).

The case $j = 2$ is the only Heisenberg parabolic subalgebra.

Now we consider the split cases $p = r \geq 4$. (Note that the other split-real cases, i.e., when $p = r + 1$, were considered above without any peculiarities. The split cases $p = r < 4$ are not representative of the situation and were treated already: $so(3, 3) \cong sl(4, IR)$, $so(2, 2) \cong so(2, 1) \oplus so(2, 1)$, $so(1, 1)$ is not semisimple.) We accept the convention that the simple roots $\alpha_{r-1}$ and $\alpha_r$ form the fork of the $so(2r, \mathbb{R})$ simple root system, while $\alpha_{r-2}$ is the simple root connected to the simple roots $\alpha_{r-3}$, $\alpha_{r-1}$ and $\alpha_r$. Special care is needed only when $\Theta$ includes these four special roots, i.e.,

$$\Theta \supset \hat{\Theta}_s \doteq \{ s, \ldots, r \} , \quad 1 \leq s \leq r - 4 .$$ (7.12)

In these cases, we have $\mathcal{M}$ factor of the form $so(r-s, r-s)$, i.e., there will be no cuspidal parabolic if $r-s$ is odd.

For all other $\Theta$ the parabolic subalgebras would be like those of $sl(r, IR)$, when $r \notin \Theta$ or $r - 1 \notin \Theta$, or like those of $sl(r-2, IR)$ with possible addition of one or two $sl(2, IR)$ factors, (when $r - 2 \notin \Theta$). To describe the latter cases we need a modification of the notation (7.8):

$$\Theta^o(\bar{n}) = \{ j \mid n_j = 1, \quad n_{j+1} = 0 \text{ if } j \neq r - 1 \} .$$ (7.13)

Thus, the cuspidal parabolic subalgebras are determined by the following $\mathcal{M}_\Theta$ factors:

$$\mathcal{M}_\Theta = \begin{cases} \mathcal{M}_{\Theta(\bar{n}_s)} \oplus so(r-s, r-s) , & \Theta \supset \hat{\Theta}_s , \\
\mathcal{M}_{\Theta^o(\bar{n})} , & \Theta \not\supset \hat{\Theta}_s \end{cases} \begin{cases} s = 2, 4, \ldots, r-4 & r \text{ even} \\
\quad = 1, 3, \ldots, r-4 & r \text{ odd} \end{cases}$$ (7.14)
Only the second subcase, namely, $M_{\Theta^o(n)}$, has highest/lowest weight representations.

The maximal parabolic subalgebras corresponding to (2.24) have $M_{\Theta}$-factors given by:

$$M_{\max}^j = \begin{cases} 
    \text{sl}(r, \mathbb{R}) & j = r - 1, r \\
    \text{sl}(r-2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) & j = r - 2 \\
    \text{sl}(r-3, \mathbb{R}) \oplus \text{sl}(4, \mathbb{R}) & j = r - 3 \\
    \text{sl}(j, \mathbb{R}) \oplus \text{so}(r-j, r-j) & j \leq r - 4 
\end{cases} \quad (7.15)$$

Thus the maximal parabolic subalgebras which are cuspidal occur for $j = 1$ and odd $r \geq 5$, (4th case), or $j = 2$ and either $r = 4$, (2nd case), or even $r \geq 6$, (4th case):

$$M_{\max}^1 = \text{so}(r-1, r-1), \quad r = 5, 7, \ldots$$

$$M_{\max}^2 = \begin{cases} 
    \text{sl}(2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) & r = 4 \\
    \text{sl}(2, \mathbb{R}) \oplus \text{so}(r-2, r-2) & r = 6, 8, \ldots 
\end{cases} \quad (7.16)$$

Of these, only $M_{\max}^2$ for $r = 4$ has highest/lowest weight representations (it belongs to the second subcase of (7.14)).

The cases $M_{\max}^2$ are from the Heisenberg parabolic subalgebras (recall that $\text{so}(2, 2) = \text{sl}(2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R})$).

8. CI: $Sp(n, \mathbb{R})$, $n > 1$

In this section $G = Sp(n, \mathbb{R})$ - the split real form of $Sp(n, \mathbb{C})$. Both are standardly defined by:

$$Sp(n, F) \doteq \{ g \in GL(2n, F) \mid {}^t g J_n g = J_n, \quad \det g = 1 \} , \quad F = \mathbb{R}, \mathbb{C}. \quad (8.1)$$

Correspondingly, the Lie algebras are given by:

$$sp(n, F) = \{ X \in gl(2n, F) \mid {}^t X J_n + J_n X = 0 \} . \quad (8.2)$$

Note that $\dim_F sp(n, F) = n(2n+1)$. The general expression for $X \in sp(n, F)$ is

$$X = \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix}, \quad A, B, C \in gl(n, F), \quad {}^t B = B, \quad {}^t C = C. \quad (8.3)$$

A basis of the Cartan subalgebra $\mathcal{H}$ of $sp(n, \mathbb{C})$ is:

$$H_i = \begin{pmatrix} A_i & 0 \\ 0 & -A_i \end{pmatrix}, \quad i = 1, \ldots, n-1, \quad A_i = \text{diag}(0, \ldots, 0, 1, -1, 0, \ldots, 0),$$

$$H_n = \begin{pmatrix} A'_n & 0 \\ 0 & -A'_n \end{pmatrix}, \quad A'_n = (0, \ldots, 0, 2). \quad (8.4)$$

The same basis over $\mathbb{R}$ spans the subalgebra $\mathcal{A}$ of $G = sp(n, \mathbb{R})$, since $\text{rank}_F sp(n, F) = n$. Note that $sp(2, \mathbb{R}) \cong so(3, 2), \quad sp(1, \mathbb{R}) \cong sl(2, \mathbb{R})$. 

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The maximal compact subalgebra of $G = sp(n, \mathbb{R})$ is $K \cong u(n)$, thus $sp(n, \mathbb{R})$ has discrete series representations (and highest/lowest weight representations). Explicitly,

$$K = \{ X = \left( \begin{array}{cc} A & 0 \\ 0 & -tA \end{array} \right) \mid A \in u(n) \}.$$ (8.5)

The subalgebras $N^\pm$ which form the root spaces of the root system $(G, A)$ are of real dimension $n^2$.

Further, we choose the long root of the $C_n$ simple root system to be $\alpha_n$.

The parabolic subalgebras corresponding to $\Theta$ such that $n \notin \Theta$ are the same as the parabolic subalgebras of $sl(n, \mathbb{R})$. The parabolic subalgebras corresponding to $\Theta$ such that $n \in \Theta$ contain a string $\Theta' = \{ s + 1, \ldots, n \}$. This string brings in $M_\Theta$ a factor $sp(n - s, \mathbb{R})$, which has discrete series representations. Thus cuspidality depends on the rest of the possible choices and are the same as the parabolic subalgebras of $sl(j, \mathbb{R})$. Thus, we have:

$$\Theta_s = \Theta(n_{s-1}) \cup \Theta'_s, \quad s = 1, \ldots, n,$$ (8.6)

with the convention that $\Theta(\bar{n}_0) = \emptyset$, $\Theta'_n = \emptyset$. Then the $M_\Theta$-factors of the cuspidal parabolic subalgebras of $sp(n, \mathbb{R})$ are given as follows:

$$M_\Theta = M_{\Theta(n_{s-1})} \oplus sp(n - s, \mathbb{R}), \quad s = 1, \ldots, n.$$ (8.7)

The minimal parabolic subalgebra for which $M_\Theta = 0$ is obtained for $s = n$ since then $M_{\Theta(n_{s-1})}$ enumerates all cuspidal parabolic subalgebras of $sl(n, \mathbb{R})$, including the minimal case $M_\Theta = 0$.

The maximal parabolic subalgebras, cf. (2.24), $1 \leq j \leq n$, contain $M_\Theta$ subalgebras of the form:

$$M^\text{max}_j = sl(j, \mathbb{R}) \oplus sp(n - j, \mathbb{R}),$$ (8.8)

i.e., the only maximal cuspidal are those for $j = 1, 2$.

The case $j = 1$ is the only Heisenberg parabolic subalgebra.

9. CII : $Sp(p, r)$

In this section $G = Sp(p, r)$, $p \geq r$, which standardly is defined as follows:

$$Sp(p, r) \doteq \{ g \in Sp(p + r, \mathcal{D}) \mid g^\dagger \gamma_0 g = \gamma_0 \}, \quad \gamma_0 = \left( \begin{array}{cc} \beta_0 & 0 \\ 0 & \beta_0 \end{array} \right),$$ (9.1)

and correspondingly the Lie algebra $G = sp(p, r)$ is given by

$$sp(p, r) \doteq \{ X \in sp(p + r, \mathcal{D}) \mid X^\dagger \gamma_0 + \gamma_0 X = 0 \}.$$ (9.2)
The Cartan involution is given explicitly by: \( \theta X = \gamma_0 X \gamma_0 \). Thus, \( K \cong sp(p) \oplus sp(r) \), and \( G \) has discrete series representations (but not highest/lowest weight representations). More explicitly:

\[
K = \{ X = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \\ 0 & 0 & -u_1^\dagger & 0 \\ 0 & 0 & 0 & -u_2^\dagger \end{pmatrix} | u_1 \in sp(p), \ u_2 \in sp(r) \}.
\] (9.3)

The split rank is equal to \( r \) and the abelian subalgebra \( A \) may be given explicitly by:

\[
A = r.c.s. \{ H_j^{\pm} \equiv e_{p-r+j,p-r+j} - e_{p+j,p+j} - e_{2p+j,2p+j} + e_{2p+r+j,2p+r+j}, \ 1 \leq j \leq r \}.
\] (9.4)

The subalgebras \( N^\pm \) which form the root spaces of the root system \((G, A)\) are of real dimension \( r(4p - 1) \).

The subalgebra \( M \cong sp(p - r) \oplus sp(1) \oplus \cdots \oplus sp(1), \ r \) factors.

Here, just for a moment we distinguish the cases \( p > r \) and \( p = r \), since the restricted root systems are different.

When \( p > r \) the restricted simple root system \((G, A)\) looks as that of \( B_r \), however, the short simple root say, \( \alpha_r \), has multiplicity \( 4(p - r) \), the long roots have multiplicity 4.

When \( p = r > 1 \) the restricted simple root system \((G, A)\) looks as that of \( C_r \), however, the long root say, \( \alpha_r \), has multiplicity 3, the short roots have multiplicity 4. (We consider \( r > 1 \) since \( sp(1,1) \cong so(4,1) \).)

In spite of these differences from now on we can consider the two subcases together, i.e., we take \( p \geq r \).

There are two types of parabolic subalgebras depending on whether \( r \notin \Theta \) or \( r \in \Theta \).

Let \( r \notin \Theta \). Then the parabolic subalgebras are like those of \( su^*(2n) \). Let \( \Theta \) enumerate a connected string of restricted simple roots: \( \Theta = S_{ij} = \{ i, \ldots, j \} \), where \( 1 \leq i \leq j < r \). Then the corresponding subalgebra \( M_{ij} \) is given by:

\[
M_{ij} = su^*(2(s + 1)) \oplus sp(1) \oplus \cdots \oplus sp(1), \ r - s - 1 \text{ factors}, \ s \equiv j - i + 1. \] (9.5)

In general \( \Theta \) consists of such strings, each string of length \( s \) produces a factor \( su^*(2(s + 1)) \), the rest of \( M_{ij} \) consists of \( sp(1) \cong su(2) \) factors. All these parabolic subalgebras are not cuspidal.

Let \( r \in \Theta \) and consider the various strings containing \( r \):

\[
\Theta_j = \{ j + 1, \ldots, r \}, \ 1 \leq j < r. \] (9.6)

The corresponding factor in \( M_{ij} \) is given by algebra \( sp(p - j, r - j) \) which has discrete series representations. If \( \Theta \) contains in addition some other string then it would bring some \( su^*(2(s + 1)) \) factor and the corresponding \( M_{ij} \) will not have discrete series representations.
Thus the nontrivial cuspidal parabolic subalgebras are given by \( \Theta_j \) from (9.6) and the corresponding \( \mathcal{M}_\Theta \) is:

\[
\mathcal{M}_j \cong sp(p-j, r-j) \oplus sp(1) \oplus \cdots \oplus sp(1), \quad j \text{ factors}.
\] (9.7)

All these \( \mathcal{M}_j \) do not have highest/lowest weight representations. The other factors in the cuspidal parabolic subalgebras have dimensions: \( \dim \mathcal{A}_j = j, \dim \mathcal{N}_j^\pm = j(4p + 4r - 4j - 1) \). Extending the range of \( j \) we include the minimal parabolic subalgebra for \( j = r \) and the case \( \mathcal{M}' = \mathcal{P} = G \) for \( j = 0 \).

The maximal parabolic subalgebras corresponding to (2.24) have \( \mathcal{M}_\Theta \)-factors given by:

\[
\mathcal{M}_j^{\text{max}} = su^*(2j) \oplus sp(p-j, r-j), \quad 1 \leq j \leq r.
\] (9.8)

The \( \mathcal{N}_j^\pm \) factors in the maximal parabolic subalgebras have dimensions: \( \dim (\mathcal{N}_j^\pm)^{\text{max}} = j(4p + 4r - 6j + 1) \). The only cuspidal maximal parabolic subalgebra is \( \mathcal{P}_\Theta \), using \( su^*(2) \cong su(2) \cong sp(1) \) and noting that (9.7) and (9.8) coincide for \( j = 1 \).

10. DIII : \( SO^*(2n) \)

The group \( G = SO^*(2n) \) consists of all matrices in \( SO(2n, \mathcal{A}) \) which commute with a real skew-symmetric matrix times the complex conjugation operator \( C \):

\[
SO^*(2n) \doteq \{ g \in SO(2n, \mathcal{A}) \mid J_n C g = g J_n C \}.
\] (10.1)

The Lie algebra \( \mathcal{G} = so^*(2n) \) is given by:

\[
so^*(2n) \doteq \{ X \in so(2n, \mathcal{A}) \mid J_n C X = X J_n C \} = \{ X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in gl(n, \mathcal{A}), \; ^t a = -a, \; b^\dagger = b \}.
\] (10.2)

\[ \dim_R \mathcal{G} = n(2n - 1), \ \text{rank} \mathcal{G} = n. \]

Note that \( so^*(8) \cong so(6, 2), \ so^*(6) \cong su(3, 1), \ so^*(4) \cong so(3) \oplus so(2, 1), \ so^*(2) \cong so(2) \). Further, we can restrict to \( n \geq 4 \) since the other cases are not representative.

The Cartan involution is given by: \( \theta X = -X^\dagger \). Thus, \( \mathcal{K} \cong u(n) \):

\[
\mathcal{K} = \{ X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid a, b \in gl(n, \mathcal{A}), \; ^t a = -a = -\bar{a}, \; b^\dagger = b = \bar{b} \}.
\] (10.3)

and \( \mathcal{G} = so^*(2n) \) has discrete series representations (and highest/lowest weight representations). The complimentary space \( \mathcal{P} \) is given by:

\[
\mathcal{P} = \{ X = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a, b \in gl(n, \mathcal{A}), \; ^t a = -a = \bar{a}, \; b^\dagger = b = \bar{b} \}.
\] (10.4)
\[ \dim_{\mathbb{R}} \mathcal{P} = n(n-1). \] The split rank is \( r \equiv [n/2] \).

The subalgebras \( \mathcal{N}^{\pm} \) which form the root spaces of the root system \((\mathcal{G}, \mathcal{A})\) are of real dimension \( n(n-1) - [n/2] \).

Here, just for a moment we distinguish the cases \( n \) even and \( n \) odd since the subalgebras \( \mathcal{M} \) and the restricted root systems are different.

For \( n = 2r \) the split rank is equal to \( r \geq 2 \), and the restricted root system is as that of \( C_r \), but the short roots have multiplicity 4, while the long simple root \( \alpha_r \) has multiplicity 1. The subalgebra \( \mathcal{M} \cong so(3) \oplus \cdots \oplus so(3) \), \( r \) factors.

For \( n = 2r + 1 \) the split rank is equal to \( r \geq 2 \), and the restricted root system is as that of \( B_r \), but all simple roots have multiplicity 4, and there is a restricted root \( 2\alpha_r \) of multiplicity 1, where \( \alpha_r \) is the short simple root. The subalgebra \( \mathcal{M} \cong so(2) \oplus so(3) \oplus \cdots \oplus so(3) \), \( r \) factors.

In spite of these differences from now on we can consider the two subcases together.

There are two types of parabolic subalgebras depending on whether \( r \notin \Theta \) or \( r \in \Theta \).

If \( r \notin \Theta \) then the parabolic subalgebras are like those of \( su^*(2r) \), and they are not cuspidal.

Let \( r \in \Theta \) and consider the various strings containing \( r \):

\[ \Theta_j = \{ j + 1, \ldots, r \} , \quad 1 \leq j < r = [n/2] . \] (10.5)

The corresponding factor in \( \mathcal{M}_\Theta \) is given by the algebra \( so^*(2n-4j) \) which has discrete series representations. Thus, all cuspidal parabolic subalgebras are enumerated by (10.5) and are:

\[ \mathcal{M}_j = so^*(2n-4j) \oplus so(3) \oplus \cdots \oplus so(3) , \quad j \text{ factors} , \quad j = 1, \ldots, r - 1 . \] (10.6)

All these \( \mathcal{M}_j \) have highest/lowest weight representations. The other factors in the cuspidal parabolic subalgebras have dimensions: \( \dim \mathcal{A}_j = j \), \( \dim \mathcal{N}_j^{\pm} = j(4n-4j-3) \).

Extending the range of \( j \) we include the minimal parabolic case for \( j = r = [n/2] \), and the case \( \mathcal{M}' = \mathcal{P} = \mathcal{G} \) for \( j = 0 \) which is also cuspidal.

The maximal parabolic subalgebras enumerated by \( \Theta_j^{\max} \) from (2.24) have \( \mathcal{M}_\Theta \)-factors as follows:

\[ \mathcal{M}_j^{\max} = so^*(2n-4j) \oplus su^*(2j) , \quad j = 1, \ldots, r . \] (10.7)

The \( \mathcal{N}_j^{\pm} \) factors in the maximal parabolic subalgebras have dimensions: \( \dim (\mathcal{N}_j^{\pm})^{\max} = j(4n-6j-1) \). Only the case \( j = 1 \) is cuspidal, noting that \( \mathcal{M}_1^{\max} \) coincides with \( \mathcal{M}_1 \) from (10.6), \( (su^*(2) \cong su(2) \cong so(3)) \).

The case \( j = 1 \) is also the only Heisenberg parabolic subalgebra.

11. Real forms of the exceptional simple Lie algebras

We start with the real forms of the exceptional simple Lie algebras. Here we can not be so explicit with the matrix realizations. To compensate this we use the Satake diagrams.
[61],[6], which we omitted until now. A Satake diagram has a starting point the Dynkin diagram of the corresponding complex form. For a split real form it remains the same. In the other cases some dots are painted in black - these considered by themselves are Dynkin diagrams of the compact semisimple factors $\mathcal{M}$ of the minimal parabolic subalgebras. Further, there are arrows connecting some nodes which use the $\mathbb{Z}_2$ symmetry of some Dynkin diagrams. Then the reduced root systems are described by Dynkin-Satake diagrams which are obtained from the Satake diagrams by dropping the black nodes, identifying the arrow-related nodes, and adjoining all nodes in a connected Dynkin-like diagram, but in addition noting the multiplicity of the reduced roots (which is in general different from 1). More details can be seen in [6], and we have tried to make the exposition transparent (by repeating things).

11.1. $E_6'$

The split real form of $E_6$ is denoted as $E_6'$, sometimes as $E_6(\pm 6)$. The maximal compact subgroup is $K \cong sp(4)$, $\dim \mathcal{P} = 42$, $\dim \mathcal{N} = 36$. This real form does not have discrete series representations.

For a split real form the Satake diagram coincides with the Dynkin diagram of the corresponding complex Lie algebra [6].

In the present case this Dynkin-Satake diagram is taken as follows:

$$
\circ \quad \alpha_1 \quad \circ \quad \alpha_2 \quad \circ \quad \alpha_3 \quad \circ \quad \alpha_4 \quad \circ \quad \alpha_5 \quad \circ_{\alpha_6}
$$

Taking into account the above enumeration of simple roots the cuspidal parabolic subalgebras have $\mathcal{M}_\Theta$-factors as follows:

$$
\mathcal{M}_\Theta = \begin{cases} 
0 & \Theta = \emptyset, \text{ minimal} \\
sl(2, \mathbb{R})_j & \Theta = \{j\}, \quad j = 1, \ldots, 6 \\
sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k & \Theta = \{j, k\} : j + 1 < k, \quad \{j, k\} \neq \{3, 6\}; \\
sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k \oplus sl(2, \mathbb{R})_\ell & \Theta = \{j, k, \ell\}, \quad (j, k, \ell) = (1, 3, 5), (1, 4, 6), (1, 5, 6), (2, 4, 6), (2, 5, 6) \\
so(4, 4) & \Theta = \{2, 3, 4, 6\}
\end{cases}
$$

(11.2)

where $sl(2, \mathbb{R})_j$ denotes the $sl(2, \mathbb{R})$ subalgebra of $\mathcal{G}$ spanned by $X^+_j, H_j$, (using the same notation as in the Section on $sl(n, \mathbb{R})$). All these $\mathcal{M}_\Theta$, except the last case ($so(4, 4)$), have highest/lowest weight representations.

---

14 We could do this, since by being explicit we could consider simultaneously cases with different Satake diagrams.
The dimensions of the other factors are, respectively:

$$
\dim \mathcal{A}_\Theta = \begin{cases}
6 \\
5 \\
4 \\
3 \\
2
\end{cases}, \quad \dim \mathcal{N}^{\pm}_\Theta = \begin{cases}
36 \\
35 \\
34 \\
33 \\
24
\end{cases}
$$

Taking into account (2.24) the maximal parabolic subalgebras are determined by:

$$
\begin{align*}
\mathcal{M}_1^{\text{max}} &\cong \mathcal{M}_5^{\text{max}} \cong so(5, 5), \quad \dim (\mathcal{N}^{\pm}_\Theta)^{\text{max}} = 16 \\
\mathcal{M}_2^{\text{max}} &\cong \mathcal{M}_5^{\text{max}} \cong sl(5, \mathbb{R}) \oplus sl(2, \mathbb{R}), \quad \dim (\mathcal{N}^{\pm}_\Theta)^{\text{max}} = 25 \\
\mathcal{M}_3^{\text{max}} &\cong sl(3, \mathbb{R}) \oplus sl(3, \mathbb{R}) \oplus sl(2, \mathbb{R}), \quad \dim (\mathcal{N}^{\pm}_\Theta)^{\text{max}} = 29 \\
\mathcal{M}_6^{\text{max}} &\cong sl(6, \mathbb{R}), \quad \dim (\mathcal{N}^{\pm}_\Theta)^{\text{max}} = 21
\end{align*}
$$

Clearly, no maximal parabolic subalgebra is cuspidal. The case $\mathcal{M}_6^{\text{max}}$ is the only Heisenberg parabolic subalgebra.

11.2. EII : $E''_6$

Another real form of $E_6$ is denoted as $E''_6$, sometimes as $E_6^{(+2)}$. The maximal compact subgroup is $K \cong su(6) \oplus su(2)$, $\dim \mathcal{P} = 40$, $\dim \mathcal{N}^{\pm} = 36$. This real form has discrete series representations.

The split rank is equal to 4, while $\mathcal{M} \cong u(1) \oplus u(1)$.

The Satake diagram is:

$$
\begin{array}{cccccc}
\circ & \circ & \circ & \circ & \circ & \circ \\
& \alpha_6 & & & & \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\
\end{array}
$$

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the $F_4$ Dynkin diagram:

$$
\begin{array}{cccccc}
\circ & \circ & \circ & \circ & \circ \\
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\end{array}
$$

but the short roots have multiplicity 2 (the long - multiplicity 1). It is obtained from (11.5) by identifying $\alpha_1$ and $\alpha_5$ and mapping them to $\lambda_4$, identifying $\alpha_2$ and $\alpha_4$ and mapping them to $\lambda_3$, while the roots $\alpha_3, \alpha_6$ are mapped to the $F_4$-like long simple roots $\lambda_2, \lambda_1$, resp.

Using the above enumeration of $F_4$ simple roots we give the $\mathcal{M}_\Theta$-factors of all parabolic
subalgebras:

\[ \mathcal{M}_\Theta = \begin{cases} 
  u(1) \oplus u(1), & \Theta = \emptyset, \text{ minimal} \\
  sl(2, \mathbb{R})_j \oplus u(1) \oplus u(1), & \Theta = \{j\}, \ j = 1, 2 \\
  sl(2, \mathcal{C})_j \oplus u(1), & \Theta = \{j\}, \ j = 3, 4 \\
  sl(3, \mathbb{R}) \oplus u(1) \oplus u(1), & \Theta = \{1, 2\} \\
  sl(2, \mathbb{R})_1 \oplus sl(2, \mathcal{C})_j \oplus u(1), & \Theta = \{1, j\}, \ j = 3, 4 \\
  sl(4, \mathbb{R}) \oplus u(1), & \Theta = \{2, 3\} \\
  sl(2, \mathbb{R})_2 \oplus sl(2, \mathcal{C})_4 \oplus u(1), & \Theta = \{2, 4\} \\
  sl(3, \mathcal{C}), & \Theta = \{3, 4\} \\
  so(5, 3) \oplus u(1), & \Theta = \{1, 2, 3\}, \\
  sl(3, \mathbb{R}) \oplus u(1) \oplus sl(2, \mathcal{C})_4, & \Theta = \{1, 2, 4\} \\
  sl(2, \mathbb{R})_1 \oplus sl(3, \mathcal{C}), & \Theta = \{1, 3, 4\} \\
  su(3, 3), & \Theta = \{2, 3, 4\} 
\end{cases} \]

(11.7)

The dimensions of the other factors are, respectively:

\[ \dim \mathcal{A}_\Theta = \begin{cases} 
  4 & \dim \mathcal{N}_\Theta^- = 36 \\
  3 & 35 \\
  3 & 34 \\
  2 & 33 \\
  2 & 33 \\
  2 & 30 \\
  2 & 30 \\
  1 & 24 \\
  1 & 31 \\
  1 & 29 \\
  1 & 21 
\end{cases} \]

(11.8)

The maximal parabolic subalgebras are given the last four lines in the above lists corresponding to \( \Theta_j^{\text{max}}, \ j = 4, 3, 2, 1, \) cf. (2.24).

The last case with \( su(3, 3) \) is the only Heisenberg parabolic subalgebra.

The cuspidal parabolic subalgebras are those containing:

\[ \mathcal{M}_\Theta = \begin{cases} 
  u(1) \oplus u(1), & \Theta = \emptyset, \text{ minimal} \\
  sl(2, \mathbb{R})_j \oplus u(1) \oplus u(1), & \Theta = \{j\}, \ j = 1, 2 \\
  su(3, 3) \oplus u(1), & \Theta = \{1, 2, 3\} 
\end{cases} \]

(11.9)

All these \( \mathcal{M}_\Theta \), except the last case (containing \( so(4, 4) \)), have highest/lowest weight representations. The last case \( \mathcal{M}_6^{\text{max}} \) is the only Heisenberg parabolic subalgebra.

### 11.3. III : \( E_{6II}^{'''} \)

Another real form of \( E_6 \) is denoted as \( E_{6II}^{'''} \), sometimes as \( E_{6(-14)} \). The maximal compact subgroup is \( K \cong so(10) \oplus so(2), \ dim_{\mathbb{R}} \mathcal{P} = 32, \ dim_{\mathbb{R}} \mathcal{N}^\pm = 30 \). This real form has discrete series representations (and highest/lowest weight representations).
The split rank is equal to 2, while \( \mathcal{M} \cong \mathfrak{so}(6) \oplus \mathfrak{so}(2) \).

The Satake diagram is:

\[
\begin{array}{cccccc}
\circ \alpha_6 \\
\circ \alpha_1 & \bullet & \bullet & \bullet & \circ \alpha_5 \\
\end{array}
\]

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the \( B_2 \) Dynkin diagram but the long roots (incl. \( \lambda_1 \)) have multiplicity 6, while the short roots (incl. \( \lambda_2 \)) have multiplicity 8, and there are also the roots \( 2\lambda \) of multiplicity 1, where \( \lambda \) is any short root. It is obtained from (11.10) by dropping the black nodes, (they give rise to \( \mathcal{M} \)), identifying \( \alpha_1 \) and \( \alpha_5 \) and mapping them to \( \lambda_2 \), while the root \( \alpha_6 \) is mapped to the long simple root \( \lambda_1 \).

The non-minimal parabolic subalgebras are given by:

\[
\mathcal{M}_\Theta = \begin{cases} 
\mathfrak{so}(7,1) \oplus \mathfrak{so}(2), & \Theta = \{1\} \\
\mathfrak{su}(5,1), & \Theta = \{2\} 
\end{cases}, \quad \dim N_\Theta^\pm = \begin{cases} 
24 \\
21
\end{cases}
\]

Both are maximal (\( \dim A_\Theta = 1 \)) and not cuspidal.

The last case is also the only Heisenberg parabolic subalgebra.

11.4. EIV : \( E_{6}^{iv} \)

Another real form of \( E_6 \) is denoted as \( E_{6}^{iv} \), sometimes as \( E_{6(-26)} \). The maximal compact subgroup is \( K \cong f_4 \), \( \dim_\mathbb{R} \mathcal{P} = 26 \), \( \dim_\mathbb{R} \mathcal{N}^\pm = 24 \). This real form does not have discrete series representations.

The split rank is equal to 2, while \( \mathcal{M} \cong \mathfrak{so}(8) \).

The Satake diagram is:

\[
\begin{array}{cccccc}
\bullet \alpha_6 \\
\circ \alpha_1 & \bullet & \bullet & \bullet & \circ \alpha_5 \\
\end{array}
\]

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the \( A_2 \) Dynkin diagram but all roots have multiplicity 8. It is obtained from (11.12) by dropping the black nodes, while \( \alpha_1, \alpha_5 \), resp., are mapped to the \( A_2 \)-like simple roots \( \lambda_1, \lambda_2 \).

The two non-minimal parabolic subalgebras are isomorphic and given by:

\[
\mathcal{M}_\Theta = \mathfrak{so}(9,1), \quad \Theta = \{j\}, \quad j = 1, 2, \quad \dim N_\Theta^\pm = 16 .
\]

Both are maximal (\( \dim A_\Theta = 1 \)) and not cuspidal.
11.5. **E**: $E'_7$

The split real form of $E_7$ is denoted as $E'_7$, sometimes as $E_{7(+)}$. The maximal compact subgroup $K \cong su(8)$, $\dim_{\mathbb{R}} \mathcal{P} = 70$, $\dim_{\mathbb{R}} \mathcal{N}^\pm = 63$. This real form has discrete series representations.

We take the Dynkin-Satake diagram as follows:

$$
\circ \alpha_1 \quad \circ \alpha_2 \quad \circ \alpha_3 \quad \circ \alpha_4 \quad \circ \alpha_5 \quad \circ \alpha_6
$$

(11.14)

Taking into account the above enumeration of simple roots the cuspidal parabolic subalgebras have $\mathcal{M}_\Theta$-factors as follows:

$$
\mathcal{M}_\Theta = \left\{
\begin{array}{ll}
0 & \Theta = \emptyset, \text{ minimal} \\
sl(2, \mathbb{R})_j, & \Theta = \{j\}, \quad j = 1, \ldots, 7 \\
sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k, & \Theta = \{j, k\} : j + 1 < k, \\
sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k \oplus sl(2, \mathbb{R})_\ell, & \Theta = \{j, k, \ell\} = \{1, 3, 5\}, \{1, 3, 6\}, \\
sl(2, \mathbb{R})_1 \oplus sl(2, \mathbb{R})_4 \oplus sl(2, \mathbb{R})_6 \oplus sl(2, \mathbb{R})_7 & \Theta = \{2, 3, 4, 7\}, \\
sl(2, \mathbb{R})_2 \oplus sl(2, \mathbb{R})_4 \oplus sl(2, \mathbb{R})_6 \oplus sl(2, \mathbb{R})_7 & \Theta = \{2, 3, 4, 5, 6, 7\}, \\
so(4, 4), & \\
so(6, 6), & \\
\end{array}
\right.
$$

(11.15)

All these $\mathcal{M}_\Theta$, except the last two cases ($so(4, 4), so(6, 6)$), have highest/lowest weight representations.

The dimensions of the other factors are, respectively:

$$
\dim \mathcal{A}_\Theta = \left\{\begin{array}{c}7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 1\end{array}\right\}, \quad \dim \mathcal{N}_\Theta^\pm = \left\{\begin{array}{c}63 \\ 62 \\ 61 \\ 60 \\ 59 \\ 58 \\ 57 \\ 56 \\ 55 \\ 54 \\ 53 \\ 52 \\ 51 \\ 50 \\ 49 \\ 48 \\ 47 \\ 46 \\ 45 \\ 44 \\ 43 \\ 42 \\ 41 \\ 40 \\ 39 \\ 38 \\ 37 \\ 36 \\ 35 \\ 34 \\ 33 \end{array}\right\}
$$

(11.16)
Taking into account (2.24) the maximal parabolic subalgebras are determined by:

\[
\mathcal{M}_1^{\text{max}} \cong so(6, 6), \quad \dim (\mathcal{N}_\Theta^{\pm})^{\text{max}} = 33
\]
\[
\mathcal{M}_2^{\text{max}} \cong sl(6, \mathbb{R}) \oplus sl(2, \mathbb{R}), \quad \dim (\mathcal{N}_\Theta^{\pm})^{\text{max}} = 47
\]
\[
\mathcal{M}_3^{\text{max}} \cong sl(4, \mathbb{R}) \oplus sl(3, \mathbb{R}) \oplus sl(2, \mathbb{R}), \quad \dim (\mathcal{N}_\Theta^{\pm})^{\text{max}} = 53
\]
\[
\mathcal{M}_4^{\text{max}} \cong sl(5, \mathbb{R}) \oplus sl(3, \mathbb{R}), \quad \dim (\mathcal{N}_\Theta^{\pm})^{\text{max}} = 60
\]
\[
\mathcal{M}_5^{\text{max}} \cong so(5, 5) \oplus sl(2, \mathbb{R}), \quad \dim (\mathcal{N}_\Theta^{\pm})^{\text{max}} = 42
\]
\[
\mathcal{M}_6^{\text{max}} \cong E_6', \quad \dim (\mathcal{N}_\Theta^{\pm})^{\text{max}} = 27
\]
\[
\mathcal{M}_7^{\text{max}} \cong sl(7, \mathbb{R}), \quad \dim (\mathcal{N}_\Theta^{\pm})^{\text{max}} = 42
\]  

Clearly, the only maximal cuspidal parabolic subalgebra is the one containing \( \mathcal{M}_1^{\text{max}} \). The latter case is also the only Heisenberg parabolic subalgebra.

11.6. EVI : \( E_7'' \)

Another real form of \( E_7 \) is denoted as \( E_7'' \), sometimes as \( E_7(-5) \). The maximal compact subgroup is \( K \cong so(12) \oplus su(2) \), \( \dim_{\mathbb{R}} P = 64 \), \( \dim_{\mathbb{R}} N^{\pm} = 60 \). This real form has discrete series representations.

The split rank is equal to 4, while \( \mathcal{M} \cong su(2) \oplus su(2) \oplus su(2) \).

The Satake diagram is:

```
  \bullet_{\alpha_7} \\
\circ_{\alpha_1} \circ_{\alpha_2} \circ_{\alpha_3} \circ_{\alpha_4} \bullet_{\alpha_5} \circ_{\alpha_6}
```

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the \( F_4 \) Dynkin diagram, cf. (11.6), but the short roots have multiplicity 4 (the long - multiplicity 1). Going to this Dynkin-Satake diagram we drop the black nodes, (they give rise to \( \mathcal{M} \)), while \( \alpha_1, \alpha_2, \alpha_3, \alpha_5 \), are mapped to \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \), resp., of (11.6).

Using the above enumeration of \( F_4 \) simple roots we shall give the \( \mathcal{M}_\Theta \)-factors of all parabolic subalgebras:

\[
\mathcal{M}_\Theta = \begin{cases} 
\mathcal{M} = su(2) \oplus su(2) \oplus su(2), & \Theta = \emptyset, \text{ minimal} \\
sl(2, \mathbb{R})_j \oplus \mathcal{M}, & \Theta = \{j\}, \ j = 1, 2 \\
su^*(4) \oplus su(2)_{j+3}, & \Theta = \{j\}, \ j = 3, 4 \\
sl(3, \mathbb{R}) \oplus \mathcal{M}, & \Theta = \{1, 2\} \\
sl(2, \mathbb{R})_1 \oplus su^*(4) \oplus su(2)_{j+3}, & \Theta = \{1, j\}, \ j = 3, 4 \\
su(6), & \Theta = \{2, 3\} \\
so(6, 2) \oplus su(2), & \Theta = \{2, 4\} \\
su^*(6), & \Theta = \{3, 4\} \\
so(7, 3) \oplus su(2)_6, & \Theta = \{1, 2, 3\} \\
sl(3, \mathbb{R}) \oplus su^*(4) \oplus su(2)_7, & \Theta = \{1, 2, 4\} \\
sl(2, \mathbb{R})_1 \oplus su^*(6), & \Theta = \{1, 3\} \\
so^*(12), & \Theta = \{2, 3, 4\} 
\end{cases}
\]  

30
The dimensions of the other factors are, respectively:

\[
\dim \mathcal{A}_\Theta = \begin{pmatrix} 4 \\ 3 \\ 3 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \dim \mathcal{N}_\Theta^\pm = \begin{pmatrix} 60 \\ 59 \\ 56 \\ 57 \\ 55 \\ 50 \\ 55 \\ 48 \\ 42 \\ 53 \\ 47 \\ 33 \end{pmatrix}
\]

(11.20)

The maximal parabolic subalgebras are the last four in the above list corresponding to \( \Theta_j^{\text{max}} \), \( j = 4, 3, 2, 1 \), cf. (2.24).

The cuspidal parabolic subalgebras are those containing

\[
\mathcal{M}_\Theta = \begin{cases} 
\mathcal{M} = su(2) \oplus su(2) \oplus su(2), & \Theta = \emptyset, \text{ minimal} \\
sl(2, R)_j \oplus \mathcal{M}, & \Theta = \{j\}, \ j = 1, 2 \\
so(6, 2) \oplus su(2), & \Theta = \{2, 3\} \\
so^*(12), & \Theta = \{2, 3, 4\}
\end{cases}
\]

(11.21)

the last one being also maximal.

The last case is also the only Heisenberg parabolic subalgebra.

All these \( \mathcal{M}_\Theta \) have highest/lowest weight representations.

11.7. EVII : \( E_7''' \)

Another real form of \( E_7 \) is denoted as \( E_7''' \), sometimes as \( E_7(-25) \). The maximal compact subgroup is \( \mathcal{K} \cong e_6 \oplus so(2) \), \( \dim_R \mathcal{P} = 54 \), \( \dim_R \mathcal{N}_\theta^\pm = 51 \). This real form has discrete series representations (and highest/lowest weight representations).

The split rank is equal to 3, while \( \mathcal{M} \cong so(8) \).

The Satake diagram is:

\[
\begin{array}{cccccccc}
\circ & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7
\end{array}
\]

(11.22)

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the \( C_3 \) Dynkin diagram:

\[
\begin{array}{cccc}
\circ & \longrightarrow & \circ & \longrightarrow \\
\lambda_1 & \lambda_2 & \lambda_3
\end{array}
\]

(11.23)

but the short roots have multiplicity 8 (the long - multiplicity 1). Going to the \( C_3 \) diagram we drop the black nodes, (they give rise to \( \mathcal{M} \)), while \( \alpha_1, \alpha_5, \alpha_6 \) are mapped to \( \lambda_3, \lambda_2, \lambda_1 \), resp., of (11.23).
Using the above enumeration of $C_3$ simple roots we shall give the $\mathcal{M}_\Theta$-factors of all parabolic subalgebras:

$$\mathcal{M}_\Theta = \begin{cases} 
so(8) , & \Theta = \emptyset , \text{ minimal} \\
so(9,1) , & \Theta = \{j\}, \ j = 1, 2 \\
\mathfrak{sl}(2, \mathbb{R})_3 \oplus \so(8) , & \Theta = \{3\} \\
\mathfrak{e}_6^{iv} , & \Theta = \{1,2\} \\
\mathfrak{sl}(2, \mathbb{R})_3 \oplus \so(9,1) , & \Theta = \{1,3\} \\
\so(10,2) , & \Theta = \{2,3\} 
\end{cases} \quad (11.24)$$

The dimensions of the other factors are, respectively:

$$\dim \mathcal{A}_\Theta = \begin{cases} 
3 & \text{dim } \mathcal{N}_\Theta^{\pm} = 51 \\
2 & 43 \\
2 & 50 \\
1 & 27 \\
1 & 42 \\
1 & 33 
\end{cases} \quad (11.25)$$

The last three give rise to the maximal parabolic subalgebras.

The cuspidal parabolic subalgebras are those containing

$$\mathcal{M}_\Theta = \begin{cases} 
so(8) , & \Theta = \emptyset , \text{ minimal} \\
\mathfrak{sl}(2, \mathbb{R})_3 \oplus \so(8) , & \Theta = \{3\} \\
\so(10,2) , & \Theta = \{2,3\} 
\end{cases} \quad (11.26)$$

the last one being also maximal.

The last case is also the only Heisenberg parabolic subalgebra.

All these $\mathcal{M}_\Theta$ have highest/lowest weight representations.

### 11.8. EVIII : $E_8'$

The split real form of $E_8$ is denoted as $E_8'$, sometimes as $E_8^{(+8)}$. The maximal compact subgroup $K \cong \so(16)$, $\dim_{\mathbb{R}} \mathcal{P} = 128$, $\dim_{\mathbb{R}} \mathcal{N}^{\pm} = 120$. This real form has discrete series representations.

We take the Dynkin-Satake diagram as follows:

$$\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array} \quad (11.27)$$

Taking into account the above enumeration of simple roots the cuspidal parabolic sub-
algebras have $\mathcal{M}_\Theta$-factors as follows:

$$\mathcal{M}_\Theta = \begin{cases} 
0 \\
sl(2, \mathbb{R})_j, \\
sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k, \\
sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k \oplus sl(2, \mathbb{R})_\ell, \\
sl(2, \mathbb{R})_1 \oplus sl(2, \mathbb{R})_3 \oplus sl(2, \mathbb{R})_5 \oplus sl(2, \mathbb{R})_7 \\
sl(2, \mathbb{R})_1 \oplus sl(2, \mathbb{R})_4 \oplus sl(2, \mathbb{R})_6 \oplus sl(2, \mathbb{R})_8 \\
sl(2, \mathbb{R})_1 \oplus sl(2, \mathbb{R})_4 \oplus sl(2, \mathbb{R})_7 \oplus sl(2, \mathbb{R})_8 \\
sl(2, \mathbb{R})_2 \oplus sl(2, \mathbb{R})_4 \oplus sl(2, \mathbb{R})_6 \oplus sl(2, \mathbb{R})_8 \\
sl(2, \mathbb{R})_2 \oplus sl(2, \mathbb{R})_4 \oplus sl(2, \mathbb{R})_7 \oplus sl(2, \mathbb{R})_8 \\
so(4, 4), \\
so(6, 6), \\
E_7^\prime 
\end{cases}$$

\[ \Theta = \emptyset, \text{ minimal} \]
\[ \Theta = \{j\}, \quad j = 1, \ldots, 8 \]
\[ \Theta = \{j, k\} : j + 1 < k, \]
\[ \{j, k\} \neq \{3, 8\}; \quad \{j, k\} = \{7, 8\}, \]
\[ \Theta = \{j, k, \ell\} = \{1, 3, 5\}, \{1, 3, 6\}, \]
\[ \{1, 3, 7\}, \{1, 4, 6\}, \{1, 4, 7\}, \{1, 4, 8\}, \]
\[ \{1, 5, 7\}, \{1, 5, 8\}, \{1, 6, 8\}, \{2, 4, 6\}, \]
\[ \{2, 4, 7\}, \{2, 5, 7\}, \{2, 5, 8\}, \]
\[ \{2, 6, 8\}, \{2, 7, 8\}, \{3, 5, 7\}, \{4, 6, 8\}, \]
\[ \{4, 7, 8\}, \{5, 7, 8\} \]

(11.28)

All these $\mathcal{M}_\Theta$, except the last three cases ($so(4, 4), so(6, 6), E_7'$), have highest/lowest weight representations.

The dimensions of the other factors are, respectively:

$$\dim \mathcal{A}_\Theta = \begin{cases} 
8 \\
7 \\
6 \\
5 \\
4 \\
4 \\
4 \\
2 \\
1
\end{cases}, \quad \dim \mathcal{N}_\Theta^\pm = \begin{cases} 
120 \\
119 \\
118 \\
117 \\
116 \\
116 \\
116 \\
108 \\
90 \\
57
\end{cases}$$

(11.29)
Taking into account (2.24) the maximal parabolic subalgebras are determined by:

\[
\begin{align*}
\mathcal{M}_1^{\max} &\cong \text{so}(7,7), \quad \dim (\mathcal{N}_\Theta^\pm)^{\max} = 78 \\
\mathcal{M}_2^{\max} &\cong \text{sl}(7,\mathbb{R}) \oplus \text{sl}(2,\mathbb{R}), \quad \dim (\mathcal{N}_\Theta^\pm)^{\max} = 98 \\
\mathcal{M}_3^{\max} &\cong \text{sl}(5,\mathbb{R}) \oplus \text{sl}(3,\mathbb{R}) \oplus \text{sl}(2,\mathbb{R}), \quad \dim (\mathcal{N}_\Theta^\pm)^{\max} = 106 \\
\mathcal{M}_4^{\max} &\cong \text{sl}(5,\mathbb{R}) \oplus \text{sl}(4,\mathbb{R}), \quad \dim (\mathcal{N}_\Theta^\pm)^{\max} = 104 \\
\mathcal{M}_5^{\max} &\cong \text{so}(5,5) \oplus \text{sl}(3,\mathbb{R}), \quad \dim (\mathcal{N}_\Theta^\pm)^{\max} = 97 \\
\mathcal{M}_6^{\max} &\cong \text{E}_6' \oplus \text{sl}(2,\mathbb{R}), \quad \dim (\mathcal{N}_\Theta^\pm)^{\max} = 83 \\
\mathcal{M}_7^{\max} &\cong \text{E}_7', \quad \dim (\mathcal{N}_\Theta^\pm)^{\max} = 57 \\
\mathcal{M}_8^{\max} &\cong \text{sl}(8,\mathbb{R}), \quad \dim (\mathcal{N}_\Theta^\pm)^{\max} = 92
\end{align*}
\] (11.30)

Clearly, the only maximal cuspidal parabolic subalgebra is the one containing \(\mathcal{M}_7^{\max}\). The latter case is also the only Heisenberg parabolic subalgebra.

11.9. EIX : \(E_8''\)

Another real form of \(E_8\) is denoted as \(E_8''\), sometimes as \(E_8(-24)\). The maximal compact subgroup is \(K \cong e_7 \oplus su(2), \dim_\mathbb{R} P = 112, \dim_\mathbb{R} N^\pm = 51\). This real form has discrete series representations.

The split rank is equal to 4, while \(\mathcal{M} \cong \text{so}(8)\).

The Satake diagram

\[
\begin{array}{cccccccc}
\circ & \bullet & \circ & \circ & \bullet & \circ \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7
\end{array}
\] (11.31)

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the \(F_4\) Dynkin diagram, cf. (11.6), but the short roots have multiplicity 8 (the long - multiplicity 1). Going to the \(F_4\) diagram we drop the black nodes, (they give rise to \(\mathcal{M}\)), while \(\alpha_1, \alpha_5, \alpha_6, \alpha_7\) are mapped to \(\lambda_1, \lambda_3, \lambda_2, \lambda_1\), resp., of (11.6).

Using the above enumeration of \(F_4\) simple roots we shall give the \(\mathcal{M}_\Theta\)-factors of all parabolic subalgebras:

\[
\mathcal{M}_\Theta = \left\{ \begin{array}{ll}
\mathcal{M} = \text{so}(8), & \Theta = \emptyset, \text{ minimal} \\
\text{sl}(2,\mathbb{R})_j \oplus \mathcal{M}, & \Theta = \{j\}, \quad j = 1, 2 \\
\text{so}(9,1), & \Theta = \{j\}, \quad j = 3, 4 \\
\text{sl}(3,\mathbb{R}) \oplus \mathcal{M}, & \Theta = \{1, 2\} \\
\text{sl}(2,\mathbb{R})_1 \oplus \text{so}(9,1), & \Theta = \{1, j\}, \quad j = 3, 4 \\
\text{so}(10,2), & \Theta = \{2, 3\} \\
\text{sl}(2,\mathbb{R})_2 \oplus \text{so}(9,1), & \Theta = \{2, 4\} \\
e_6^{iv}, & \Theta = \{3, 4\} \\
\text{so}(11,3), & \Theta = \{1, 2, 3\}, \\
\text{sl}(3,\mathbb{R}) \oplus \text{so}(9,1), & \Theta = \{1, 2, 4\} \\
\text{sl}(2,\mathbb{R})_1 \oplus e_6^{iv}, & \Theta = \{1, 3, 4\} \\
e_7^{iv}, & \Theta = \{2, 3, 4\}
\end{array} \right. 
\] (11.32)
The dimensions of the other factors are, respectively:

\[
\begin{align*}
\dim \mathcal{A}_\Theta &= \begin{cases}
4 & \Theta = \emptyset, \text{ minimal} \\
3 & \Theta = \{j\}, j = 1, 2, 3, 4 \\
3 & \Theta = \{j, k\} = \{1, 2\}, \{3, 4\} \\
2 & \Theta = \{2\} \\
2 & \Theta = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\} \\
2 & \Theta = \{2, 3\} \\
1 & \Theta = \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \\
1 & \Theta = \{2, 3, 4\}
\end{cases}, \\
\dim \mathcal{N}_\Theta^+ &= \begin{cases}
108 & \Theta = \emptyset, \text{ minimal} \\
107 & \Theta = \{j\}, j = 1, 2, 3, 4 \\
100 & \Theta = \{j, k\} = \{1, 2\}, \{3, 4\} \\
105 & \Theta = \{2\} \\
99 & \Theta = \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \\
78 & \Theta = \{2, 3\} \\
84 & \Theta = \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \\
83 & \Theta = \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \\
57 & \Theta = \{2, 3, 4\}
\end{cases}
\end{align*}
\]

(11.33)

The maximal parabolic subalgebras are the last four in the above lists corresponding to \(\Theta_j^{\text{max}}\), \(j = 4, 3, 2, 1\), cf. (2.24).

The cuspidal ones arise from:

\[
\mathcal{M}_\Theta = \begin{cases}
\mathfrak{so}(8), & \Theta = \emptyset, \text{ minimal} \\
\mathfrak{sl}(2, \mathbb{R})_j \oplus \mathfrak{so}(8), & \Theta = \{j\}, j = 1, 2, 3, 4 \\
\mathfrak{so}(10, 2), & \Theta = \{2\} \\
\mathfrak{e}_7^\prime, & \Theta = \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}
\end{cases}
\]

(11.34)

the last one being also maximal.

The last case is also the only Heisenberg parabolic subalgebra.

All these \(\mathcal{M}_\Theta\) have highest/lowest weight representations.

11.10. FI : \(F'_4\)

The split real form of \(F'_4\) is denoted as \(F'_4\), sometimes as \(F'_4(+4)\). The maximal compact subgroup \(K \cong \mathfrak{sp}(3) \oplus \mathfrak{su}(2), \dim_{\mathbb{R}} \mathcal{P} = 28, \dim_{\mathbb{R}} \mathcal{N}^\pm = 24\). This real form has discrete series representations.

Taking into account the enumeration of simple roots as in (11.6) the parabolic subalgebras have \(\mathcal{M}_\Theta\)-factors as follows:

\[
\mathcal{M}_\Theta = \begin{cases}
0, & \Theta = \emptyset, \text{ minimal} \\
\mathfrak{sl}(2, \mathbb{R})_j, & \Theta = \{j\}, j = 1, 2, 3, 4 \\
\mathfrak{sl}(3, \mathbb{R})_j, & \Theta = \{j, k\} = \{1, 2\}, \{3, 4\} \\
\mathfrak{sl}(2, \mathbb{R})_j \oplus \mathfrak{sl}(2, \mathbb{R})_k, & \Theta = \{2\} \\
\mathfrak{sp}(2, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R}), & \Theta = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\} \\
\mathfrak{so}(4, 3), & \Theta = \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\} \\
\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}), & \Theta = \{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\} \\
\mathfrak{sp}(3, \mathbb{R}), & \Theta = \{2, 3, 4\}
\end{cases}
\]

(11.35)
The dimensions of the other factors are, respectively:

\[
\dim \mathcal{A}_\Theta = \begin{cases} 
4 \\
3 \\
2 \\
2 \\
2 \\
1 \\
1 \\
1
\end{cases}, \quad \dim \mathcal{N}_\Theta^\pm = \begin{cases} 
24 \\
23 \\
21 \\
22 \\
20 \\
15 \\
20 \\
15
\end{cases}
\] (11.36)

The maximal parabolic subalgebras are in the last three lines in the above lists corresponding to \( \Theta^\text{max}_j \), \( j = 4, 3, 2, 1 \), cf. (2.24).

The cuspidal parabolic subalgebras arise from:

\[
\mathcal{M}_\Theta = \begin{cases} 
\emptyset, & \Theta = \emptyset, \text{ minimal} \\
sl(2, \mathbb{R})_j, & \Theta = \{j\}, \quad j = 1, 2, 3, 4 \\
sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k, & \Theta = \{j, k\} = \{1, 3\}, \{1, 4\}, \{2, 4\} \\
sp(2, \mathbb{R}), & \Theta = \{2, 3\} \\
so(4, 3), & \Theta = \{1, 2, 3\} \\
sp(3, \mathbb{R}), & \Theta = \{2, 3, 4\}
\end{cases}
\] (11.37)

the last two being also maximal. All these \( \mathcal{M}_\Theta \), except the last but one, have highest/lowest weight representations.

The last case \( sp(3, \mathbb{R}) \) is the only Heisenberg parabolic subalgebra.

11.11. FII : \( F_4'' \)

Another real form of \( F_4 \) is denoted as \( F_4'' \), sometimes as \( F_4(-20) \). The maximal compact subgroup \( K \cong so(9) \), \( \dim_{\mathbb{R}} \mathcal{P} = 16 \), \( \dim_{\mathbb{R}} \mathcal{N}^\pm = 15 \). This real form has discrete series representations.

The split rank is equal to 1, while \( \mathcal{M} \cong so(7) \).

The Satake diagram is:

\[
\begin{array}{c}
\bullet \\
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{array} \quad \Rightarrow \quad 
\begin{array}{c}
\bullet \\
\circ
\end{array}
\] (11.38)

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the \( A_1 \) Dynkin diagram but the roots have multiplicity 8. Going to the \( A_1 \) Dynkin diagram we drop the black nodes, (they give rise to \( \mathcal{M} \)), while \( \alpha_4 \) becomes the \( A_1 \) diagram.

11.12. G : \( G'_2 \)

The split real form of \( G_2 \) is denoted as \( G'_2 \), sometimes as \( G_{2(+2)} \). The maximal compact subgroup \( K \cong su(2) \oplus su(2) \), \( \dim_{\mathbb{R}} \mathcal{P} = 8 \), \( \dim_{\mathbb{R}} \mathcal{N}^\pm = 6 \). This real form has discrete series representations.

The non-minimal parabolic subalgebras have \( \mathcal{M}_\Theta \)-factors as follows:

\[
\mathcal{M}_\Theta = \sl(2, \mathbb{R})_j, \quad \Theta = \{j\}, \quad j = 1, 2
\] (11.39)
They are cuspidal and maximal. All $\mathcal{M}_\Theta$ have highest/lowest weight representations. They are also Heisenberg parabolic subalgebras.

12. Summary and Outlook

In the present paper we have started the systematic explicit construction of the invariant differential operators by giving explicit description of one of the main ingredients in our setting - the cuspidal parabolic subalgebras. We explicated also the maximal parabolic subalgebras, since these are important even when they are not cuspidal. In sequels of this paper [62] we shall present the construction of the invariant differential operators and expand the scheme to the supersymmetric case, in view of applications to conformal field theory and string theory.

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## Appendix

### Table of Cuspidal Parabolic Subalgebras

| $G$ | $\mathcal{M}_\Theta$ | $\dim_\mathbb{R} A_\Theta$ | $\dim_\mathbb{R} N_\Theta^\pm$ |
|-----|---------------------|----------------------|----------------------|
| $G_\xi$ | $u(1) \oplus \cdots \oplus u(1)$ | $\ell$ | $d - \ell$ |
| $\dim G_\xi = d$ | $\ell$ factors | | |
| $\rank G_\xi = \ell$ | | | |
| $sl(n, \mathbb{R})$ | $\mathcal{M}_{\Theta(\bar{\eta})} = \bigoplus_{1 \leq t \leq k} sl(2, \mathbb{R})_{j_t}$ | $n - 1 - k$ | $\frac{1}{2}n(n-1) - k$ |
| $0 \leq k \leq \lfloor n/2 \rfloor$ | | | |
| $j_t < j_{t+1} - 1$ | | | |
| $1 \leq j_1, \ j_k \leq n - 1$ | | | |
| minimal: $k = 0, \ \mathcal{M}_\Theta = 0$ | $n - 1$ | $\frac{1}{2}n(n-1)$ | |
| $su^*(2n)$ | $su(2) \oplus \cdots \oplus su(2)$ | $n - 1$ | $2n(n-1)$ |
| $n$ factors | | | |
| $su(p, r)$ | $su(p - j, r - j) \oplus u(1) \oplus \cdots \oplus u(1)$ | $j$ | $j(2(p + r - j) - 1)$ |
| $p \geq r$ | $j$ factors, $1 \leq j < r$ | | |
| minimal: $j = r$ from above | $r$ | $r(2p - 1)$ | |
| $p > r$ | minimal: $u(1) \oplus \cdots \oplus u(1)$ | $r$ | $r(2r - 1)$ |
| $p = r$ | $r - 1$ factors | | |
| $so(p, r)$ | $\mathcal{M}_{\Theta(\bar{\eta})} \oplus so(p - s, r - s)$ | $\leq s$ | $\leq s$ |
| $p \geq r$ | $s = 1, 2, \ldots, r, \ p + r$ odd | | |
| | $s = 1, 3, \ldots, r, \ p, r$ odd | | |
| | $s = 2, 4, \ldots, r, \ p, r$ even | | |
| $p = r$ | $\mathcal{M}_{\Theta^*(\bar{\eta})}$ | | |
| $p \geq r$ | minimal: $so(p - r)$ | $r$ | $r(p - 1)$ |
| $G$               | $\mathcal{M}_\Theta$                                                                 | $\dim_{\mathbb{R}} A_\Theta$ | $\dim_{\mathbb{R}} \mathcal{N}_\Theta^\pm$ |
|------------------|--------------------------------------------------------------------------------------|--------------------------------|-----------------------------------------------|
| $sp(n, \mathbb{R})$  | $\mathcal{M}_{\Theta(n-1)} \oplus sp(n-s, \mathbb{R})$                                   | $s = 1, \ldots, n$                             | $\leq s$                                      |
|                  | minimal: $\mathcal{M}_\Theta = 0, (s = n)$                                              |                                | $n$                                           |
|                  |                                                                                       |                                | $n^2$                                         |
| $sp(p, r)$        | $sp(p-j, r-j) \oplus sp(1) \oplus \cdots \oplus sp(1)$                                   | $j$                                             | $j(4p + 4r - 4j - 1)$                        |
| $p \geq r$       | $j$ factors, $j = 1, \ldots, r$                                                      |                                |                                              |
|                  | minimal: $j = r$                                                                      |                                |                                              |
| $so^*(2n)$        | $so^*(2n-4j) \oplus so(3) \oplus \cdots \oplus so(3)$                               | $j$                                             | $j(4n - 4j - 3)$                             |
|                  | $j$ factors, $j = 1, \ldots, r \equiv [n/2]$                                          |                                |                                              |
|                  | minimal: $j = r \equiv [n/2]$                                                        |                                |                                              |
| $E_{1} \cong E_{6}'$   | $0$, minimal                                                                          | $6$                                             | $36$                                          |
|                  | $sl(2, \mathbb{R})_j$                                                                | $5$                                             | $35$                                          |
|                  | $j = 1, \ldots, 6$                                                                  |                                |                                              |
|                  | $sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k$                                   | $4$                                             | $34$                                          |
|                  | $j + 1 < k$, $\{j, k\} \neq \{3, 6\}$                                              |                                |                                              |
|                  | $sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k \oplus sl(2, \mathbb{R})_\ell$     | $3$                                             | $33$                                          |
|                  | $(j, k, \ell) = (1, 3, 5), (1, 4, 6), (1, 5, 6), (2, 4, 6), (2, 5, 6)$               |                                |                                              |
|                  | $so(4, 4)$                                                                           | $2$                                             | $24$                                          |
| $E_{II} \cong E_{6}''$  | $u(1) \oplus u(1)$, minimal                                                          | $4$                                             | $36$                                          |
|                  | $sl(2, \mathbb{R})_j \oplus u(1) \oplus u(1)$, $j = 1, 2$                          | $3$                                             | $35$                                          |
|                  | $so(4, 4) \oplus u(1)$                                                              | $1$                                             | $24$                                          |
| $E_{III} \cong E_{6}'''$ | $so(6) \oplus so(2)$                                                                | $2$                                             | $30$                                          |
|                  | $su(5, 1)$                                                                           | $1$                                             | $21$                                          |
| $E_{IV} \cong E_{6}''''$ | $so(8)$                                                                              | $2$                                             | $24$                                          |
| $G$ | $\mathcal{M}_\Theta$ | $\dim_\mathbb{R} A_\Theta$ | $\dim_\mathbb{R} N_\Theta^\pm$ |
|-----|------------------|-------------------|-------------------|
| EV  | $\cong E_7'$     | 0, minimal        | 7                 | 63                |
|     | $sl(2, \mathbb{R})_j, \ j = 1, \ldots, 7$ |                  | 6                 | 62                |
|     | $sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k$ |                  | 5                 | 61                |
|     | $j + 1 < k, \ {j, k} \neq \{3, 7\}; \ (j, k) = (6, 7)$ |                  | 4                 | 60                |
|     | $sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k \oplus sl(2, \mathbb{R})_\ell$ |                  |                  |                   |
|     | $(j, k, \ell) = (1, 3, 5), (1, 3, 6), (1, 4, 6), (1, 4, 7),$ |                  |                  |                   |
|     | $(1, 5, 7), (1, 6, 7), (2, 4, 6), (2, 4, 7), (2, 5, 7), (2, 6, 7)$ |                  |                  |                   |
|     | $so(4, 4)$ |                  | 3                 | 51                |
|     | $so(6, 6)$ |                  | 1                 | 33                |
| EVI | $\cong E_7''$   | $su(2) \oplus su(2) \oplus su(2)$, minimal | 4 | 60 |
|     | $sl(2, \mathbb{R})_j \oplus su(2) \oplus su(2) \oplus su(2), \ j = 1, 2$ | | 3 | 59 |
|     | $so(6, 2) \oplus su(2)$ | | 2 | 50 |
|     | $so^*(12)$ | | 1 | 33 |
| EVII | $\cong E_7'''$ | $so(8)$, minimal | 3 | 51 |
|      | $sl(2, \mathbb{R})_3 \oplus so(8)$ | | 2 | 50 |
|      | $so(10, 2)$ | | 1 | 33 |
| FI  | $\cong F_4'$    | 0, minimal        | 4                 | 24                |
|     | $sl(2, \mathbb{R})_j, \ j = 1, 2, 3, 4$ |                  | 3                 | 23                |
|     | $sl(2, \mathbb{R})_j \oplus sl(2, \mathbb{R})_k, \ (j, k) = (13), (14), (24)$ |                  | 2                 | 22                |
|     | $sp(2, \mathbb{R})$ |                  | 2                 | 20                |
|     | $so(4, 3)$ |                  | 1                 | 15                |
|     | $sp(3, \mathbb{R})$ | | 1 | 15 |
| FII | $\cong F_4''$   | $so(7)$           | 1                 | 15                |
| G   | $\cong G_2'$    | 0, minimal        | 2                 | 6                 |
|     | $sl(2, \mathbb{R})_j, \ j = 1, 2$ | | 1 | 5 |

40
| \( \mathcal{G} \) | \( \mathcal{M}_\Theta \) | \( \dim_{\mathbb{R}} \mathcal{A}_\Theta \) | \( \dim_{\mathbb{R}} \mathcal{N}^+_\Theta \) |
|---|---|---|---|
| EVIII \( \cong E_8' \) | 0, minimal | 8 | 120 |
| \( \text{sl}(2, \mathbb{R})_j \), \( j = 1, \ldots, 8 \) | | 7 | 119 |
| \( \text{sl}(2, \mathbb{R})_j \oplus \text{sl}(2, \mathbb{R})_k \) | | 6 | 118 |
| \( j + 1 < k, \ \{j, k\} \neq \{3, 8\} \); \( (j, k) = (7, 8) \) | | 5 | 117 |
| \( o(4, 4) \) | | 4 | 116 |
| \( o(6, 6) \) | | 2 | 108 |
| \( E_7' \) | | 1 | 57 |
| EIX \( \cong E_8'' \) | \( o(8), \text{ minimal} \) | 4 | 108 |
| \( \text{sl}(2, \mathbb{R})_j \oplus o(8), \ j = 1, 2 \) | | 3 | 107 |
| \( o(10, 2) \) | | 2 | 90 |
| \( E_7' \) | | 1 | 57 |
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