A QCD-like theory with the $Z_{N_c}$ symmetry

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We propose a QCD-like theory with the $Z_{N_c}$ symmetry. The flavor-dependent twisted boundary condition (TBC) is imposed on $N_c$ degenerate flavor quarks in the SU($N_c$) gauge theory. The QCD-like theory is useful to understand the mechanism of color confinement. Dynamics of the QCD-like theory is studied by imposing the TBC on the Polyakov-loop extended Nambu-Jona-Lasinio (PNJL) model. The TBC model is applied to two- and three-color cases. The $Z_{N_c}$ symmetry is preserved below some temperature $T_c$, but spontaneously broken above $T_c$. The color confinement below $T_c$ preserves the flavor symmetry. Above $T_c$, the flavor symmetry is broken, but the breaking is suppressed by the entanglement between the Polyakov loop and the chiral condensate. Particularly at low temperature, dynamics of the TBC model is similar to that of the PNJL model with the standard fermion boundary condition, indicating that the $Z_{N_c}$ symmetry is a good approximate concept in the latter model even if the current quark mass is small. The present prediction can be tested in future by lattice QCD, since the QCD-like theory has no sign problem.

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I. INTRODUCTION.

Understanding of the confinement mechanism is one of the most important subjects in hadron physics. According to Lattice QCD (LQCD), the system is in the confinement and chiral symmetry breaking phase at low temperature ($T$), but in the deconfinement and chiral symmetry restoration phase at high $T$. The confinement mechanism is, nevertheless, still unclear for several reasons. The main reason is that the exact symmetry is not found for the deconfinement transition and hence the order parameter is unknown. In the limit of zero current quark mass, the chiral condensate is an exact order parameter for the chiral restoration. In the limit of infinite current quark mass, on the contrary, the Polyakov loop becomes an exact order parameter for the deconfinement transition, since the $Z_{N_c}$ symmetry is exact there. For the real world in which $u$ and $d$ quarks have small current quark masses, the chiral condensate is considered to be a good order parameter, but it is not clear whether the Polyakov loop is a good order parameter. In this paper, we approach this problem by proposing a QCD-like theory with the $Z_{N_c}$ symmetry.

We start with the SU($N_c$) gauge theory with $N_f$ degenerate flavor quarks. The partition function $Z$ in Euclidean spacetime is described by

$$ Z = \int Dq D\bar{q} DA \exp[-S_0] $$

with the action

$$ S_0 = \int d^4x \left[ \sum_f \bar{q}_f (\gamma_\mu D_\mu + m_f) q_f + \frac{1}{4g^2} F_{\mu\nu}^a \right]^2, $$

where $q_f$ is the quark field with flavor $f$ and current quark mass $m_f$, $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative with the gauge field $A_\mu$, $g$ is the gauge coupling and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = F_{\mu\nu}^a T^a$ with the SU($N_c$) generator $T^a$. The temporal boundary condition for quark is

$$ q_f(x, \beta = 1/T) = -q_f(x, 0). $$

The $Z_{N_c}$ transformation changes the fermion boundary condition as

$$ q_f(x, \beta) = -\exp(-i2\pi k/N_c) q_f(x, 0) $$

for integer $k$, while the action $S_0$ keeps the original form since the $Z_{N_c}$ symmetry is the center symmetry of the gauge symmetry. The $Z_{N_c}$ symmetry thus breaks down through the fermion boundary condition in QCD.

Now we consider the SU($N$) gauge theory with $N$ degenerate flavor quarks, i.e. $N \equiv N_f = N_c$, and assume the following twisted boundary conditions (TBC):

$$ q_f(x, \beta) = -\exp(i\theta_f) q_f(x, 0) $$

for flavors $f$ labeled by integers from 1 to $N$; see Fig.1 for the twisted angles $\theta_f$. Here $\theta_f$ is an arbitrary real number in a range of $0 \leq \theta_f < 2\pi$. The action $S_0$ with the TBC is not QCD but a QCD-like theory. The QCD-like theory has the $Z_{N_c}$ symmetry, i.e. invariant under the $Z_{N_c}$ transformation. In fact, the $Z_{N_c}$ transformation changes $f$ into $f-k$, but $f-k$ can be relabeled by $f$ since $S_0$ is invariant under the relabeling. The QCD-like theory with the $Z_{N_c}$ symmetry is useful to understand the mechanism of color confinement.

When the fermion field $q_f$ is transformed by

$$ q_f \rightarrow \exp(-i\theta_f T \tau) q_f $$

for Euclidean time $\tau$, the action $S_0$ is changed into

$$ S(\theta_f) = \int d^4x \left[ \sum_f \bar{q}_f (\gamma_\mu D_\mu - \mu_f \gamma_4 + m_f) q_f + \frac{1}{4g^2} F_{\mu\nu}^a \right] $$

where $\mu_f$ is the fermion mass.
with the imaginary quark number chemical potential \( \mu_f = i T \theta_f \), while the TBC is transformed back to the standard one. The action \( S_0 \) with the TBC is thus equivalent to the action \( S(\theta_f) \) with the standard one. In the limit of \( T = 0 \), the action \( S(\theta_f) \) tends to \( S_0 \) with the \( \theta \) fixed. The QCD-like theory is thus identical with QCD at \( T = 0 \) where the Polyakov loop \( \Phi \) is zero. This indicates that in the QCD-like theory the \( \mathbb{Z}_{N_c} \) symmetry is preserved up to some temperature \( T_c \) and spontaneously broken above \( T_c \). In the QCD-like theory, the flavor-symmetry breaking is explicitly broken by the flavor-dependent TBC. As shown later, the flavor-symmetry breaking is recovered below \( T_c \). The breaking then becomes significant only above \( T_c \).

In general, the QCD partition function \( Z(T, \theta) \) with finite imaginary chemical potential \( \theta \) has the Roberge-Weiss (RW) periodicity \( \mathbb{Z}_1 \): \( Z(T, \theta) = Z(T, \theta + 2 \pi k/N_c) \) for any integer \( k \). The RW periodicity was confirmed by lattice QCD (LQCD) \([3, 12]\) and the Holographic QCD \([13]\). The RW periodicity means that \( Z(T, \theta) \) is invariant under the extended \( \mathbb{Z}_{N_c} \) transformation, i.e. the combination of the \( \mathbb{Z}_{N_c} \) transformation and the parameter transformation \( \theta \to \theta + 2 \pi k/N_c \). Actually, \( Z(T, \theta) \) is transformed into \( Z(T, \theta - 2 \pi k/N_c) \) by the \( \mathbb{Z}_{N_c} \) transformation and \( Z(T, \theta - 2 \pi k/N_c) \) is transformed back to \( Z(T, \theta) \) by the parameter transformation. The QCD partition function thus has the extended \( \mathbb{Z}_{N_c} \) symmetry, and dynamics of QCD at imaginary chemical potential is governed by the symmetry.

The extended \( \mathbb{Z}_{N_c} \) symmetry is not an internal symmetry, since the transformation includes the shift of external parameter \( \theta \). In the QCD-like theory, the shift of \( \theta \) is not necessary because of the TBC. Thus the QCD-like theory possesses the \( \mathbb{Z}_{N_c} \) symmetry as an internal symmetry, whereas QCD has the extended \( \mathbb{Z}_{N_c} \) symmetry as an external symmetry. The Polyakov-loop extended Nambu-Jona-Lasinio (PNJL) model \([2, 14, 51]\) is a good model to understand QCD at finite imaginary chemical potential \( \theta \) and hence the QCD-like theory, since the PNJL model possesses the extended \( \mathbb{Z}_{N_c} \) symmetry in the standard fermion boundary condition \([3, 2]\).

In this paper, we propose a QCD-like theory with the \( \mathbb{Z}_{N_c} \) symmetry. The theory is constructed by imposing the TBC on the SU(\( N_f \)) gauge theory with \( N_f \) degenerate flavor quarks. Dynamics of the QCD-like theory is studied concretely by imposing the TBC on the PNJL model. Two cases of \( N_c = N_f = 2 \) and 3 are mainly considered. In this paper, the PNJL model with the TBC is shortly called the TBC model, and the PNJL model with the standard boundary condition is named the standard-PNJL model. We first show that the \( \mathbb{Z}_{N_c} \) symmetry is preserved below some temperature \( T_c \), but spontaneously broken above \( T_c \). The interplay between the \( \mathbb{Z}_{N_c} \) symmetry breaking and the flavor symmetry breaking is investigated. Comparing the deconfinement transition in the TBC model with that in the standard-PNJL model, we show that the \( \mathbb{Z}_{N_c} \) symmetry is a good approximate concept in the latter model, even if the current quark mass is small. The present prediction can be checked by LQCD in future, since LQCD with the TBC is free from the sign problem.

This paper is organized as follows. The case of \( N_c = N_f = 2 \) is investigated in Sec. II and that of \( N_c = N_f = 3 \) is in Sec. III. Two interesting extensions of the TBC model are shown in Sec. III C Section IV is devoted to summary.

## II. CASE OF \( N_c = 2 \)

### A. Formalism

The two-color and two-flavor PNJL Lagrangian \([34]\) in Euclidean spacetime is

\[
\mathcal{L} = \sum_f \bar{q}_f (\gamma_\nu D_\nu - \mu_f \gamma_4 + m_f) q_f
\]

\[
-(1 - \alpha) G_8 \sum_f \sum_{a=0}^3 \left[ [\bar{q}_f \tau_a q_f]^2 + (\bar{q}_f i \gamma_5 \tau_a q_f)^2 \right]
\]

\[
+ 4 \alpha G_8 \left[ \det (\bar{q}_1 (1 + \gamma_5) q_1) + \det (\bar{q}_2 (1 - \gamma_5) q_2) \right] + \mathcal{U}(\Phi[A], \Phi[A]^*, T)
\]

with \( D_\nu = \partial_\nu + i A_\nu = \partial_\nu + i \delta_{\nu A} A_{4,A} \bar{\tau}_A^\nu \) for the gauge field \( A_\nu^\alpha \), where the \( \tau_a (\bar{\tau}_a) \) for \( a = 1, 2, 3 \) are the Pauli matrices in flavor (color) space and \( \tau_0 \) is the unit matrix in flavor space. In the NJL sector, \( (1 - \alpha) G_8 \) denotes coupling constants of scalar- and pseudoscalar-type four-quark interactions, whereas \( \alpha G_8 \) is that of the Kobayashi-Maskawa-Hoof determinant interaction \([52, 53]\). Here \( \alpha \) can vary from 0 to 1/2 for positive \( G_8 \). The \( U_A(1) \) anomaly vanishes when \( \alpha = 0 \). The Polyakov potential \( \mathcal{U} \), defined in \([14]\), is a function of the Polyakov loop \( \Phi \) and its Hermitian conjugate \( \Phi^* \). The parameter \( m_f (\mu_f) \) stands for the current quark mass (the chemical potential) for each flavor. Here we set \( m_0 \equiv m_u = m_d \).

In the PNJL model, the gauge field \( A_\mu \) is treated as a homogeneous and static background field \([16, 54]\). In the case...
of $N_c = 2$, the Polyakov-loop $\Phi$ and its conjugate $\Phi^*$ are determined in Euclidean spacetime by

$$\Phi = \frac{1}{2} r_c(L), \quad \Phi^* = \frac{1}{2} r_c(\bar{L}), \quad (9)$$

where $L = \exp(iA_4/T)$ with $A_4 = iA_0$. In the Polyakov-gauge, $A_4$ is diagonal in color space, i.e., $A_4/T = \text{diag}(\phi_1, \phi_2)$ for the $\phi_i$ satisfying $\phi_1 + \phi_2 = 0$. This leads to

$$\Phi = \frac{1}{2}(e^{i\phi_1} + e^{i\phi_2}) = \frac{1}{2}(e^{i\phi_1} + e^{-i\phi_1}) = \cos(\phi_1),$$

$$\Phi^* = \frac{1}{2}(e^{-i\phi_1} + e^{i\phi_2}) = \frac{1}{3}(e^{-i\phi_1} + e^{i\phi_1}) = \cos(\phi_1), \quad (10)$$

indicating that $\Phi$ is real. For the Polyakov-loop potential $U$, we use

$$U = -bT[2Ae^{-a/T} \Phi^2 + \log(1 - \Phi^2)] \quad (11)$$

proposed in Ref. [34], where $a = 858.1 \text{ MeV}$ and $b^{1/3} = 210.5 \text{ MeV}$. The Polyakov potential yields the second-order deconfinement phase transition at $T_c = 270 \text{ MeV}$ in the pure gauge theory.

Now we consider the imaginary chemical potential $\mu_f = i\theta_f/T$, where the twisted angles $\theta_f$ are real. Making the mean-field approximation (MFA) and the path integral over the quark fields in the PNJL partition function $Z_{PNJL}$, one can obtain the thermodynamic potential (per unit volume) as

$$\Omega = -T \ln(Z_{PNJL})/V $$

$$= -2 \sum_{f=u,d} \sum_{c=1,2} \int \frac{d^3p}{(2\pi)^3} \left[ E_f + \frac{1}{\beta} \ln \left[ 1 + e^{\beta E_f} \right] \right]$$

$$+ \frac{1}{\beta} \ln \left[ 1 + e^{-\beta E_f} \right]$$

$$+ U(\sigma, a_0) + U(\Phi, T), \quad (12)$$

where $E_f^{\pm}(p) = E_f(p) \pm \mu_f$ for $E_f(p) = \sqrt{p^2 + M_f^2}$,

$$M_u = m_0 - 2G_s(\sigma + \zeta a_0), \quad (13)$$

$$M_d = m_0 - 2G_s(\sigma - \zeta a_0), \quad (14)$$

$$U = G_s \zeta \left[ \sigma^2 + \zeta a_0^2 \right], \quad (15)$$

$$\zeta = 1 - 2\alpha, \quad \sigma = \langle \bar{u}u + \bar{d}d \rangle \text{ and } a_0 = \langle \bar{u}u - \bar{d}d \rangle. \text{ Here only the flavor-diagonal scalar condensates are taken. On the right-hand side of (12) only the first term is regularized by the three-dimensional momentum cutoff $\Lambda$ [14, 17], since it diverges.}$$

The variables, $X = (\Phi, \Phi^*, \sigma, a_0)$, are determined by the stationary conditions

$$\partial \Omega/\partial X = 0. \quad (16)$$

Solutions $X(T, \theta_f)$ of the conditions do not give a global minimum of $\Omega$ necessarily, when the solutions are inserted back to (12). There is a possibility that they yield a local minimum or even a maximum. We have then checked that the solutions yield a global minimum.

Following Ref. [34], we take the parameter set of $m_0 = 5.4 \text{ MeV}$, $\Lambda = 657 \text{ MeV}$ and $G_s = 7.23 \text{ GeV}^2$ that yield $-(\bar{u}u)^{1/3} = 218 \text{ MeV}$, the pion decay constant $f_\pi = 75.4 \text{ MeV}$ and the pion mass $m_\pi = 140 \text{ MeV}$ at vacuum.

Taking the summation over color indices in (12) leads to

$$\Omega = -2 \sum_{f=u,d} \int \frac{d^3p}{(2\pi)^3} \left[ 2E_f + \frac{1}{\beta} \ln \left[ 1 + C_{2,1}(p)e^{\theta_f} + C_{2,2}(p)e^{-i\theta_f} \right] \right]$$

$$+ U(\sigma, a_0) + U(\Phi, T), \quad (17)$$

where

$$C_{2,1}(p) = 2\Phi e^{-\beta E_f}, \quad (18)$$

$$C_{2,2}(p) = e^{-2\beta E_f}.$$
loop $\Phi$ is an approximate order parameter of the color confinement, while the chiral condensate $\sigma$ is an approximate order parameter of the chiral transition. The flavor symmetry breaking is described by the isovector condensate $a_0$. We mainly consider the $U_A(1)$ symmetric case by taking $a = 0$.

Figure 2 shows $T$ dependence of $\Phi$ and $\sigma$ calculated with the standard-PNJL model, where $\sigma$ is normalized by $\sigma_0 \equiv \sigma(T = 0, \mu_f = 0)$. Both $\sigma$ and $\Phi$ are finite for any $T$, since there is no exact chiral and $\mathbb{Z}_2$ symmetry. As $T$ increases, $\sigma$ decreases gradually, while $\Phi$ increases smoothly. The chiral and deconfinement transitions are thus crossover. Here $a_0$ is zero at any $T$, since the flavor symmetry is not broken.

Now we consider the TBC model with the $\mathbb{Z}_2$ symmetry. The Polyakov loop $\Phi$ is an exact order parameter of the color confinement. When $\Phi \neq 0$, there are two $\mathbb{Z}_2$ vacua. The vacuum with positive $\Phi$ is taken in this paper.

First we analyze the case

\[
(\theta_u, \theta_d) = (-\pi/2, \pi/2)
\]

(21)
corresponding to $\theta_1 = -\pi/2$ in the TBC of (5); see the right panel of Fig. 3 for the twisted angles. In this case, the flavor symmetry is not broken by the TBC, because

\[
\Omega(\theta_u, \theta_d) = \Omega(-\theta_u, -\theta_d) = \Omega(\theta_d, \theta_u),
\]

(22)
where the first and second equalities are obtained by the charge-conjugation and (21), respectively.

Figure 4 shows $\sigma$ and $\Phi$ as a function of $T$; note that $a_0$ is zero for any $T$ because of the flavor symmetry. The Polyakov loop $\Phi$ is zero up to $T \equiv T_c \approx 260$ MeV, but finite above $T_c$. The $\mathbb{Z}_2$ symmetry is thus preserved exactly below $T_c$, but spontaneously broken above $T_c$. The deconfinement phase transition is second-order, since $\Phi$ has no jump at $T = T_c$. Meanwhile, the chiral transition is crossover. There is no qualitative difference between the standard-PNJL model and the TBC model with $\theta_1 = -\pi/2$ for the deconfinement and chiral transitions, although the order of the deconfinement transition becomes stronger by the exact $\mathbb{Z}_2$ symmetry.

In Fig. 5 the color state factors $C_{2,1}(p = 0)$ and $C_{2,2}(p = 0)$ are drawn as a function of $T$. The one-quark state $C_{2,1}(p = 0)$ vanishes below $T_c$, because of $\Phi = 0$. Above $T_c$, on the contrary, the system is dominated by the one-color state, although the two-quark state $C_{2,2}$ remains there.

The delay of the chiral restoration at higher $T$ can be under-
stood as follows. Taking the flavor summation in (12) leads to

\[
\Omega = -2 \sum_{e=1}^{\alpha} \int \frac{d^3p}{(2\pi)^3} \left[ N_c E_f \right.
\]

\[
+ \frac{1}{\beta} \ln \left[ 1 + F_{2,1}(p)e^{-\phi_e} + F_{2,2}(p)e^{-2\phi_e} \right]
\]

\[
+ \frac{1}{\beta} \ln \left[ 1 + F_{2,1}^*(p)e^{\phi_e} + F_{2,2}^*(p)e^{2\phi_e} \right]
\]

\[
\left. + U(\sigma, a_0) + U(\Phi, T), \right]
\]

(23)

where

\[
F_{2,1}(p) = e^{i\theta_u} e^{-\beta E_u} + e^{i\theta_d} e^{-\beta E_d},
\]

\[
F_{2,2}(p) = e^{i(\theta_u + \theta_d)} e^{-\beta (E_u + E_d)}. \tag{24}
\]

Since \( \theta_u = \theta_1 \) and \( \theta_d = \theta_1 + \pi \), Eq. (24) is reduced to

\[
F_{2,1}(p) = e^{i\theta_1} \left( z_{2,1} e^{-\beta E_u} + z_{2,2} e^{-\beta E_d} \right),
\]

\[
F_{2,2}(p) = -e^{i2\theta_1} e^{-\beta (E_u + E_d)},
\]

(25)

where \( z_{2,1} = 1 \) and \( z_{2,2} = -1 \) are elements of the \( \mathbb{Z}_2 \) group.

In the case of \( (\theta_u, \theta_d) = (-\pi/2, \pi/2) \), the flavor symmetry is not broken, so that \( E = E_u = E_d \). In this situation, \( F_{2,1} \) and \( F_{2,2} \) are further reduced to

\[
F_{2,1}(p) = -i (z_{2,1} + z_{2,2}) e^{-\beta E} = 0
\]

\[
F_{2,2}(p) = e^{-2i\theta_1} e^{-\beta E}. \tag{26}
\]

The thermodynamic system thus has no \( F_{2,1} \) but finite \( F_{2,2} \). This means that u- and d-quarks are statistically in the same state. The chiral condensate \( \sigma \) has weak \( T \) dependence, since the two-quark state factor \( F_{2,2} \) is strongly suppressed by the factor \( \exp(-2\beta E) \). Eventually, the chiral restoration becomes much slower in the TBC model. This slow restoration is true also for the case of \( N_c = 3 \) and \( N_f = 2 \) \([39]\), although the \( \mathbb{Z}_3 \) symmetry is not exact in the case.

Next we analyze the case

\[
(\theta_u, \theta_d) = (0, \pi)
\]

(27)

corresponding to \( \theta_1 = 0 \) in the TBC of \([5]\); see the left panel of Fig. 4 for the twisted angles. Figure 5 presents \( T \) dependence of \( a_0 \) and \( \Phi \). In this case, the flavor symmetry is explicitly broken by the TBC. The second-order deconfinement phase transition occurs at \( T = T_c \approx 235 \text{MeV} \). Below \( T_c \), \( a_0 \) and \( \Phi \) are zero, indicating that the flavor symmetry is restored by the color confinement. Above \( T_c \), both \( a_0 \) and \( \Phi \) become finite, indicating that the flavor and \( \mathbb{Z}_2 \) symmetries break simultaneously. At high \( T \) where the flavor symmetry breaking is strong, \( \sigma \) is getting large with respect to increasing \( T \). This behavior is quite different from the corresponding behavior of \( \sigma \) in the standard-PNJL model.

Figure 7 shows \( T \) dependence of the constituent quark masses in the case of \( \alpha = 0 \) and \( (\theta_u, \theta_d) = (0, \pi) \). The solid (dashed) line represents \( u \) (d) quark mass.

In Fig. 8 the color state factors \( C_{2,1}(p = 0) \) and \( C_{2,2}(p = 0) \) are plotted for u-quark as a function of \( T \). Below \( T_c \), only the two-quark state \( C_{2,2} \) remains. Above \( T_c \), the system is dominated by the one-quark state \( C_{2,1} \).

Figure 9 shows \( T \) dependence of \( a_0 \) and \( \Phi \) in the case of \( \alpha = 0.2 \). The \( T \) dependence is similar to that in the case of \( \alpha = 0 \), although \( T_c \approx 265 \text{MeV} \) in the former and \( 235 \text{MeV} \) in the latter. Comparing Fig. 9 with Fig. 6 one can see explicitly that \( T_c \) becomes larger as \( \alpha \) increases. The \( U_A(1) \) anomaly thus delays the spontaneous breaking of the \( \mathbb{Z}_2 \) symmetry and hence that of the flavor-symmetry breaking.

III. CASE OF \( N_c = 3 \)

A. Formalism

The present formulation for \( N_c = N_f = 3 \) is parallel to that for \( N_c = N_f = 2 \) shown in Sec. 11A. The PNJL Lagrangian...
The Polyakov-loop $\Phi$ and its conjugate $\Phi^*$ are determined by

$$\Phi = \frac{1}{3} tr_c(L), \quad \Phi^* = \frac{1}{3} tr_c(\bar{L}),$$  \hspace{1cm} (29)$$

where $L = \exp(i A_4/T)$ with $A_4/T = \text{diag}(\phi_r, \phi_g, \phi_b)$. Noting that $\phi_r + \phi_g + \phi_b = 0$, one can obtain

$$\Phi = \frac{1}{3} (e^{i\phi_r} + e^{i\phi_g} + e^{i\phi_b}),$$

$$\Phi^* = \frac{1}{3} (e^{-i\phi_r} + e^{-i\phi_g} + e^{-i\phi_b}).$$  \hspace{1cm} (30)

We take the Polyakov potential of Ref. [20]:

$$U = T^4 \left[ -\frac{a(T)}{2} \Phi \Phi^* + b(T) \ln(1 - 6\Phi \Phi^* + 4(\Phi^3 + \Phi^*)^2) - 3(\Phi \Phi^*)^2 \right],$$ \hspace{1cm} (31)

$$a(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2, \quad b(T) = b_3 \left( \frac{T_0}{T} \right)^3.$$ \hspace{1cm} (32)

Parameters of $U$ are determined to reproduce LQCD data at finite $T$ in the pure gauge limit. The parameters except $T_0$ are summarized in Table I. The Polyakov potential yields the first-order deconfinement phase transition at $T = T_0$ in the pure gauge theory [54–55]. The original value of $T_0$ is 270 MeV determined from the pure gauge LQCD data, but the PNJL model with this value of $T_0$ yields a larger value of the pseudocritical temperature $T_c$ at zero chemical potential than $T_c \approx 160$ MeV predicted by full LQCD [56–58]. We then rescale $T_0$ to 195 MeV to reproduce $T_c = 160$ MeV [59].

| $a_0$  | $a_1$  | $a_2$  | $b_3$  |
|-------|-------|-------|-------|
| 3.51  | -2.47 | 15.2  | -1.75 |

**TABLE I:** Summary of the parameter set in the Polyakov-potential sector determined in Ref. [20]. All parameters are dimensionless.

Now we consider the flavor-dependent imaginary chemical potential $\mu_f = i \theta_f T$. The thermodynamic potential (per volume) based on the mean-field approximation is [42]

$$\Omega = -2 \sum_{f=u,d,s,c=r,g,b} \sum_{\sigma_c} \int \frac{d^3 p}{(2\pi)^3} \left[ E_f + \frac{1}{\beta} \ln \left[ 1 + e^{i \theta_f} e^{i \mu_f} e^{-\beta E_f} \right] + \frac{1}{\beta} \ln \left[ 1 + e^{-i \theta_f} e^{-i \mu_f} e^{-\beta E_f} \right] + U(\sigma_u, \sigma_d, \sigma_s) + U(\Phi, \Phi^*, T) \right]$$ \hspace{1cm} (33)
with \( \sigma_f = \langle \bar{q}_f q_f \rangle \) and \( E_f = \sqrt{p^2 + M_f^2} \) for \( f = u, d, s \), where the three-dimensional cutoff is taken for the momentum integration in the vacuum term \([43]\). The dynamical quark masses \( M_f \) are defined by

\[
M_f = m_f - 4G_S \sigma_f + 2G_D \sigma_f \sigma_{f''}
\]

(34)

for \( f \neq f' \) and \( f \neq f'' \) and \( f'' \neq f''' \). The mesonic potential \( U(\sigma_u, \sigma_d, \sigma_s) \) are obtained by

\[
U(\sigma_u, \sigma_d, \sigma_s) = \sum_{f = u, d, s} 2G_S \sigma_f^2 - 4G_D \sigma_u \sigma_d \sigma_s.
\]

(35)

For the 2+1 flavor system with \( m_u = m_d = m_l \), the PNJL model has five parameters \( (G_S, G_D, m_l, m_s, \Lambda) \). A typical set is obtained in Ref. \([59]\). The parameter set is fitted to empirical values of \( \eta' \)-meson mass and \( \pi \)-meson mass and \( \pi \)-meson decay constant at vacuum. In the present paper, we set \( m_s \) to \( m_l \) in the parameter set of Ref. \([59]\), since we consider the three degenerate flavor system with \( m_0 \equiv m_l = m_s \). The parameter set is shown in Table II:

| \( m_0 \) (MeV) | \( \Lambda \) (MeV) | \( G_S A^2 \) | \( G_D A^3 \) |
|-----------------|-----------------|-----------|-----------|
| 5.5             | 602.3           | 1.835     | 12.36     |

TABLE II: Summary of the parameter set in the NJL sector. All the parameters except \( m_0 \) are the same as in Ref. \([59]\).

Taking the color summation in \([33]\) leads to

\[
\Omega = -2 \sum_{f = u, d, s} \int \frac{d^3 p}{(2\pi)^3} \left[ N_c E_f + \frac{1}{\beta} \ln(1 + C_{3,1}(p)e^{i\theta_f}) + C_{3,2}(p)e^{2i\theta_f} + C_{3,3}(p)e^{3i\theta_f} \right] + U(\sigma_u, \sigma_d, \sigma_s) + U(\Phi, \Phi^*, T),
\]

(36)

where

\[
C_{3,1}(p) = 3\Phi e^{-\beta E_f},
\]

\[
C_{3,2}(p) = 3\Phi^* e^{-2\beta E_f},
\]

\[
C_{3,3}(p) = e^{-3\beta E_f}.
\]

One can find that \( \Omega \) has the RW periodicity,

\[
\Omega(\theta_u, \theta_d, \theta_s) = \Omega(\theta_f + 2k\pi/3),
\]

(38)

making the \( Z_3 \) transformation,

\[
\Phi \rightarrow e^{-2\pi k/3}\Phi, \quad \Phi^* \rightarrow e^{2\pi k/3}\Phi^*,
\]

(39)

in \( \Omega \). In the case of \( (\theta_u, \theta_d, \theta_s) = (\theta_1, \theta_1 + 2\pi/3, \theta_1 + 4\pi/3) \), \( \Omega \) is invariant under the \( Z_3 \) transformation, indicating that \( \Omega \) possesses the \( Z_3 \) symmetry. When \( \Phi \) vanishes, the color confinement \( (C_{3,1} = C_{3,2} = 0) \) occurs and thereby \( \Omega \) has the flavor symmetry \( (E_u = E_d = E_s) \) since the factors \( e^{\pm 3i\theta_f} \) have no flavor dependence in \([56]\). The flavor symmetry is thus preserved by the color confinement \( (\Phi = 0) \) also for the case of \( N_c = 3 \).

B. Numerical results

First we consider the standard fermion boundary condition by setting \( \theta_u = \theta_d = \theta_s = 0 \). In this case, the \( \sigma_f \) are degenerate and hence \( \sigma \equiv (\sigma_u + \sigma_d + \sigma_s)/3 = \sigma_f \). Figure 10 shows \( T \) dependence of \( \sigma \) and \( \Phi \). Both the chiral restoration and the deconfinement transition are crossover, although the former transition is a bit slower than the latter \([50]\).

![Figure 10: T dependence of σ and Φ at θ_u = θ_d = θ_s = 0. σ is normalized by σ0.](image_url)

Next we consider the TBC model by taking two cases of \( (\theta_u, \theta_d, \theta_s) = (0, 2\pi/3, 4\pi/3) \) and \( (-\pi, -\pi/3, \pi/3) \) that correspond to the left and right panels in Fig. 11 respectively. The charge conjugation yields the relation

\[
\Omega(\theta_u, \theta_d, \theta_s) = \Omega(-\theta_u, -\theta_d, -\theta_s) = \Omega(\theta_u, \theta_s, \theta_d)
\]

(40)

for the two cases. Thus s-quark is symmetric with d-quark in these cases. Because of the \( Z_3 \) symmetry, there are three \( Z_3 \) vacua when \( \Phi \neq 0 \). We then take the solution in which a phase of \( \Phi \) lies in a range of \(-\pi/3 \leq \phi < \pi/3\). In the solution, \( \Phi \) is found to be real.

Figure 12 shows \( T \) dependence of several physical quantities in the case of \( (\theta_u, \theta_d, \theta_s) = (\pi, -\pi/3, \pi/3) \). The order parameters \( \Phi, \sigma \) and \( \Phi_0 = (\sigma_u - \sigma_d) = \sigma_u - \sigma_s \) are plotted in panel (a). The first-order deconfinement transition takes place at \( T = T_c \approx 195 \) MeV. Below \( T_c \), \( \Phi_0 \) and \( \Phi \) are zero. The flavor symmetry is thus preserved by the color confinement. Above \( T_c \), \( \Phi_0 \) and \( \Phi \) become finite, indicating that the flavor and \( Z_3 \) symmetries break simultaneously.

For the constituent quark masses \( M_f \) shown in panel (b), all the \( M_f \) are degenerate below \( T_c \). Above \( T_c \), \( M_u \) becomes heavier whereas two of the three, \( M_d \) and \( M_s \), are degenerate and becomes lighter. The increase of \( M_u \) makes the chiral restoration slower. In panel (c), the absolute values of the
color-state factors $C_{3,1}$, $C_{3,2}$ and $C_{3,3}$ are plotted at $p = 0$. Below $T_c$, $C_{3,3}$ is small but finite, whereas $C_{3,1} = C_{3,2} = 0$. Above $T_c$, the system is dominated by the one-quark state $C_{3,1}$.

Here the case of $(\theta_u, \theta_d, \theta_s) = (0, 2\pi/3, 4\pi/3)$ is considered briefly. As shown in Fig. [13] below $T_c \approx 195\text{MeV}$ physical quantities have the same properties as those in the previous case. The difference between the two cases appears above $T_c$. Particularly for $M_f$, it is found that $M_d = M_s > M_u$ in the present case, while $M_d = M_s < M_u$ in the previous case. Thus both $d$- and $s$-quarks becomes heavier as $T$ increases from $T_c$. This property makes the chiral restoration even slower in the present case.

C. Two extensions of the TBC model

In this subsection, we extend the TBC model in two directions.

As the first extension, we use the entanglement PNJL (EPNJL) model [45, 50] instead of the PNJL model. A possible origin of the four-quark vertex $G_S$ is a gluon exchange between quarks and its higher-order diagrams. If the gluon field $A_g$ has a vacuum expectation value $\langle A_g \rangle$, $A_g$ is coupled to $\langle A_g \rangle$ and hence to $\Phi$ through $L$ [60]. This effect allows $G_S$ to depend on $\Phi$, namely $G_S = G_{\Phi}(\Phi)$ [60]. It is expected that $\Phi$ dependence of $G_{\Phi}(\Phi)$ will be determined in future by accurate methods such as the exact renormalization group method [60–62]. In this paper, however, we simply assume the following $G_{\Phi}(\Phi)$ by respecting the chiral symmetry, the charge-conjugation symmetry [37] and the extended $\mathbb{Z}_3$ symmetry [2]:

$$G_{\Phi}(\Phi) = G_S[1 - \alpha_1 \Phi \Phi^* - \alpha_2 (\Phi^3 + \Phi^*^3)]. \quad (41)$$

The PNJL model with the entanglement vertex [41] is called the EPNJL model [45, 50]. In principle, $G_D$ can depend on $\Phi$, too. However, $\Phi$-dependence of $G_D$ yields qualitatively the same effect on the phase diagram as that of $G_S$ [50]. We then neglect $\Phi$-dependence of $G_D$, following Ref. [50].

The parameters $\alpha_1$ and $\alpha_2$ in [41] are so determined as to reproduce two results of LQCD at finite $T$. The first is a result of 2+1 flavor LQCD at $\mu = 0$ [63] that the chiral transition is crossover at the physical point. The second is a result of degenerate three-flavor LQCD at $\theta = \pi$ [10] that the order of the RW endpoint is first-order for small and large quark masses but second-order for intermediate quark masses. The parameter set $(\alpha_1, \alpha_2)$ satisfying these conditions is located in the triangle region [50]

$$\{-1.5\alpha_1 + 0.3 < \alpha_2 < -0.86\alpha_1 + 0.32, \alpha_2 > 0\}. \quad (42)$$

As a typical example, we take $\alpha_1 = 0.25$ and $\alpha_2 = 0.1$, following Ref. [50] and rescale $T_0$ to $150\text{MeV}$ [50].

Figure [14] shows $T$ dependence of $\sigma$, $a_0$ and $\Phi$ calculated with the EPNJL model for (a) $(\theta_u, \theta_d, \theta_s) = (0, 0, 0)$ and (b)
Fig. 13: $T$ dependence of (a) order parameters $\sigma$, $a_0$, $\Phi$, (b) constituent quark masses $M_f$, (c) color-state factors $C_{3,1}, C_{3,2}, C_{3,3}$ at $p = 0$ in the case of $(\theta_u, \theta_d, \theta_s) = (0, 2\pi/3, 4\pi/3)$. Here $\sigma$ and $a_0$ are normalized by $\sigma_0$. Note that $M_d = M_s$, $\sigma < 0$ and $a_0 \geq 0$.

$(\theta_u, \theta_d, \theta_s) = (0, 2\pi/3, 4\pi/3)$. In panel (a), the chiral restoration and the deconfinement transition are first-order, because of the small current quark mass ($5.5$ MeV) and the strong correlation between $\sigma_f$ and $\Phi$ [50]. In panel (b), the TBC model with the entanglement vertex yields similar $T$ dependence to the EPNJL model with the standard quark boundary condition for the chiral restoration and the deconfinement transition, although the flavor symmetry is broken above $T_c$.

As the second extension of the TBC model with $N_f = N_c$, one can consider the TBC model with $N_f = lN_c$ for any positive integer $l$. It is obvious that the TBC model with $N_f = lN_c$ has the $Z_{N_c}$ symmetry, if the twisted angles $\theta_f$ are properly ordered; for example,

$$\theta_f = \theta_1 + 2\pi(f - 1)/N_f,$$

or

$$\theta_f = \theta_1 + 2\pi(f - 1)/N_c,$$

for $f = 1, 2, \cdots, N_f$.

Fig. 15: Twisted factors $e^{i\theta_f}$ on a unit circle in the complex plane in the case of $N_c = 3$, $N_f = 6$ and $\theta_1 = \pi/6$. In the left and right panels, the $e^{i\theta_f}$ are obtained by (43) and (44), respectively.
Let us consider the case of $N_c = 3$, $N_f = 6$ and $\theta_1 = \pi/6$. In Fig. 15 consider the left and right panels show the twisted angles defined by [43] and [44], respectively. Here we take the right-panel case as an example. The thermodynamic potential $\Omega$ has the same form as [36], except the flavor summation is taken from $f = 1$ to 6. It is straightforward to show that $\Omega$ has the RW periodicity and the $Z_3$ symmetry. We take the same parameter set as in the case of $N_f = N_c = 3$, except $G_c$ is taken as $G_S = G_{S,3} - G_{D,3} \sigma_f (0)/2 = 2.226$ GeV$^2$, where $G_{S,3}$ and $G_{D,3}$ mean $G_S$ and $G_D$ in the case of $N_c = N_f = 3$, respectively, and $\sigma_f (0)$ stands for $\sigma_f$ at $T = 0$ and $\theta_f = 0$ in the case of $N_c = N_f = 3$. We keep the Polyakov potential $\mathcal{U}$ of [32], but neglect the KMT determinant interaction just for simplicity.

Figure 6 presents $T$ dependence of $\sigma_f$ and $\Phi$ for the right-panel case of Fig. 15. Below $T_c \approx 190$ MeV, the flavor symmetry ($\sigma_1 = \sigma_2 = \cdots = \sigma_6$) is preserved by the color confinement ($\Phi = 0$). Above $T_c$, the flavor and $Z_3$ symmetries break simultaneously. The flavor symmetry breaking is partial because $f = 1$ is symmetric with $f = 4$, $f = 3$ with $f = 6$, and $f = 2$ with $f = 5$. As a consequence of this property, the $\sigma_f$ split into three doublets.

![Fig. 16: $T$ dependence of $\sigma_f$ and $\Phi$ in the right-panel case of Fig. 15. The solid, dashed and dot-dashed lines represent $\sigma_1 = \sigma_4$, $\sigma_3 = \sigma_6$ and $\sigma_2 = \sigma_5$, respectively, whereas the dotted line corresponds to $\Phi$.](image)

### IV. SUMMARY

We have proposed a QCD-like theory with the $Z_{N_c}$ symmetry. The QCD-like theory is constructed by imposing the flavor-dependent twisted boundary condition [5] on the SU($N_c$) gauge theory with $N_c$ degenerate flavor quarks. Dynamics of the QCD-like theory has been studied by imposing the TBC on the PNJL model. The TBC model has the $Z_{N_c}$ symmetry and hence the Polyakov loop becomes an exact order parameter of the deconfinement transition. The TBC model is a good model to investigate the mechanism of color confinement.

For both cases of $N_f = N_c = 2$ and 3, the Polyakov loop is zero up to some temperature $T_c$, but becomes finite above $T_c$. The $Z_{N_c}$ symmetry is thus preserved below $T_c$, but spontaneously broken above $T_c$. Below $T_c$, the color confinement preserves the flavor symmetry. Above $T_c$, meanwhile, the flavor symmetry is broken explicitly by the TBC. The flavor-symmetry breaking makes the chiral restoration slower, but the entanglement interaction between $\sigma$ and $\Phi$ makes the restoration faster. The entanglement interaction thus suppresses the flavor symmetry breaking. In the standard-PNJL model with degenerate flavor quarks, $\Phi$ becomes finite but small at $T$ lower than the pseudo-critical temperature, while the flavor symmetry is preserved. Dynamics of the TBC model is thus similar to that of the standard-PNJL model below $T_c$. The similarity is relatively worse above $T_c$, but it is improved by the entanglement interaction. One can then expect that QCD with the approximate $Z_3$ symmetry is similar to the QCD-like theory with the $Z_3$ symmetry and hence that the $Z_3$ symmetry is a good approximate concept in QCD, even if the current quark mass is small.

The model prediction mentioned above can be tested with LQCD, since LQCD with the TBC has no sign problem. The QCD-like theory is useful to understand the mechanism of color confinement, since the $Z_{N_c}$ symmetry is exact. For example, it is quite interesting to see $T$ dependence of the potential between $q$ and $\bar{q}$ in LQCD with the TBC.

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