OPTIMAL CONTROL OF SURFACE SHAPE

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Abstract. Controlling the shapes of surfaces provides a novel way to direct self-assembly of colloidal particles on those surfaces and may be useful for material design. This motivates the investigation of an optimal control problem for surface shape in this paper. Specifically, we consider an objective (tracking) functional for surface shape with the prescribed mean curvature equation in graph form as a state constraint. The control variable is the prescribed curvature. We prove existence of an optimal control, and using improved regularity estimates, we show sufficient differentiability to make sense of the first order optimality conditions. We also give a gradient based optimization method for both the continuous and discrete (finite element) formulations of the problem. Moreover, we provide error estimates for the state variable and adjoint state. Several numerical results are shown to illustrate the method.

Résumé. ...

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INTRODUCTION

Directed and self-assembly of micro and nano-structures is a growing research area with applications in material design \cite{11,23,25}. Controlling surface geometry can be beneficial for directing the assembly of micro-structures (colloidal particles) \cite{16}. This is because there is a coupling between the geometry of surfaces/interfaces and the arrangements of charged colloidal particles, or polymers, on those curved surfaces \cite{15,27}; in particular, the presence of defects can seriously affect the surface geometry \cite{15,16} and vice-versa. Moreover, experimental techniques have been developed for creating “custom shapes” (from swell gels) by encoding a desired surface metric \cite{26}.

With the above motivation, we investigate an optimal PDE control problem which controls the surface shape by prescribing the mean curvature. We consider an open, bounded, $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$ for an embedded surface in $\mathbb{R}^{n+1}$, with boundary of $\Omega$ denoted by $\partial \Omega$ and $n \geq 1$. If $X_1$ and $X_2$ are two Banach spaces, then $X_1 \hookrightarrow X_2$ and $X_1 \subset\subset X_2$ denote the continuous and compact embeddings of $X_1$ in $X_2$ respectively. $W_p^1(\Omega)$, $1 \leq p \leq \infty$ defines the standard Sobolev space with corresponding norm $\|\cdot\|_{W_p^1(\Omega)}$. Moreover, $W_p^1(\Omega)$ indicates the Sobolev space with zero trace and $W_p^{-1}(\Omega)$ is the canonical dual of $\tilde{W}_p^1(\Omega)$, $1 \leq p < \infty$, such that $1/p + 1/p' = 1$. We denote by $\lesssim$ the inequality $\leq C$ with a constant independent of the quantities of interest.

Then we are interested in

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\[
\inf \mathcal{J} (y,u) := \frac{1}{2} \| y - y_d \|^2_{L^2(\Omega)} + \frac{\alpha}{2} \| u \|^2_{L^2(\Omega)} \quad \text{over } y - v \in \dot{W}^{-1}_\infty (\Omega), u \in U_{ad}. \tag{1}
\]

subject to
\[
- \text{div} \ \nabla y = u \quad \text{in } \Omega. \tag{2}
\]

The second order nonlinear operator in (2) describes the mean curvature in graph form, where \( y \) is the height function, and \( Q(y) = (1 + |\nabla y|^2)^{1/2} \) denotes the surface measure. Finally, \( u \) belongs to the set of admissible controls given in Definition 1.11. In principle, either \( u \) or \( v \) (boundary value) may act as control variable, but in this work we will assume that \( v \) is a fixed given function and \( u \) is the control variable.

Toward this end, we remark that the mean curvature operator in (2) is only locally coercive \cite{17} P. 104 and control of such operators in full generality \cite{2} has not been dealt with before. The closest approach is in \cite{3,4} where they study the control of a Laplace free boundary problem with surface tension effect for \( n = 1 \). This amounts to solving a Laplace equation in bulk which is a subset of (2). In Theorem 1.9, we discretize all the quantities using piecewise linear finite elements. For \( n = 2 \) and using the continuity estimate, we directly get an a priori finite element estimate for the state equations from \cite{29}. Invoking the discrete inf-sup conditions, we derive an a priori error estimate for the adjoint equations. We extend a projection argument from \cite{4} Theorem 6.1], which in conjunction with second order sufficient conditions gives us a quasi-optimal a priori error estimate for the control and optimal if control is discretized using piecewise constant finite elements.

1. State Equations

1.1. Weak solution

A few words are in order regarding the existence of solution to the state equation (2). For a Lipschitz domain \( \Omega \) and \( v \) in \( L^p(\Omega) \), Giaquinta in \cite{12} gives a necessary and sufficient condition for the existence of solution \( y \) in the space of bounded variation (BV). In Theorem 1.1, we state another existence result from \cite{8} P. 351] which essentially says that if \( v \) is slightly more regular then \( y \) is more regular as well.

**Theorem 1.1 (\( W^1_2 \) state).** Let \( \Omega \) be Lipschitz and \( v \in W^1_2 (\Omega) \), then there exists an open set \( U_1 \subset W^{-1}_\infty (\Omega) \) with \( 0 \in U_1 \), and for every \( u \in U_1 \) there exists a unique solution \( y - v \in \dot{W}^1_1 (\Omega) \) solving (2).

Theorem 1.1 further implies that for a given \( u \in U_1 \), there exists a unique \( y - v \in \dot{W}^1_1 (\Omega) \) satisfying the state equation (2) in variational form
\[
B(y,w) = \varphi(w) \quad \text{for all } w \in \dot{W}^1_1 (\Omega), \tag{4}
\]
where \( \mathcal{B}(y, w) := \int_{\Omega} \frac{\nabla y}{Q(y)} \nabla w \) and \( \varphi(w) := \langle u, w \rangle_{W^{-1}(\Omega), W^{1}(\Omega)} \).

**Remark 1.2.** We remark that for the existence of solutions in \( W^{1}_{\infty}(\Omega) \), the standard PDE theory for linear equations only require the data \( u \) to be in \( W^{1}_{\infty}(\Omega)^{\ast} \) [1, Theorem 2.2] but given \( u \in U_{1} \) further belongs to \( W^{-1}_{\infty}(\Omega) \subset W^{1}_{\infty}(\Omega)^{*} \) and is therefore more regular. It might be possible to exploit this fact to prove that for \( v \in \tilde{W}_{\infty}^{1}(\Omega) \), the solution \( y - v \in \tilde{W}_{\infty}^{1}(\Omega) \). For this to be true our approach in Theorem 1.6 requires \( \Delta \) (Laplacian) operator to be isomorphism from \( W^{1}_{\infty}(\Omega) \) to \( W^{-1}_{\infty}(\Omega) \).

We first rewrite \([4]\) using a nonlinear operator: find \( y - v \in \tilde{W}_{\infty}^{1}(\Omega) \) satisfying

\[
\langle \mathcal{N}(y, u), w \rangle_{W^{-1}_{\infty}(\Omega), W^{1}_{\infty}(\Omega)} := \mathcal{B}(y, w) - \varphi(w) = 0 \quad \text{for all } w \in \tilde{W}^{1}_{\infty}(\Omega),
\]

where \( \langle \cdot, \cdot \rangle_{W^{-1}_{\infty}(\Omega), W^{1}_{\infty}(\Omega)} \) indicates the duality pairing.

### 1.2. Differentiability of \( \mathcal{N} \)

Next we will study some differentiability properties of \( \mathcal{N} \), for the case when \( v \in \tilde{W}_{\infty}^{1}(\Omega) \).

**Lemma 1.3.** If \( v \in \tilde{W}_{\infty}^{1}(\Omega) \), then for every \( u \in U_{1} \), the operator \( \mathcal{N}(\cdot, u) : v \oplus \tilde{W}_{\infty}^{1}(\Omega) \to W^{-1}_{\infty}(\Omega) \) is twice Fréchet differentiable with respect to \( y \) and the first order Fréchet derivative at \( y \) \( v \oplus \tilde{W}_{\infty}^{1}(\Omega) \) satisfies

\[
\langle D_{y}\mathcal{N}(y, u)(h), w \rangle_{W^{-1}_{\infty}(\Omega), W^{1}_{\infty}(\Omega)} = \left\langle \left( I - \frac{\nabla y \nabla y^{T}}{Q(y)^{2}} \right) \frac{\nabla h}{Q(y)}, \nabla w \right\rangle_{L^{\infty}(\Omega), L^{1}(\Omega)}.
\]

Moreover both the first and second order derivatives are Lipschitz continuous.

**Proof.** The derivation of \( D_{y}\mathcal{N} \) is straightforward, so is omitted. We begin by first showing that \( Q : v \oplus \tilde{W}_{\infty}^{1}(\Omega) \to L^{\infty}(\Omega) \) is Fréchet differentiable. Let \( y \in v \oplus \tilde{W}_{\infty}^{1}(\Omega) \) and \( h \in \tilde{W}_{\infty}^{1}(\Omega) \) (note: \( y + h \in v \oplus \tilde{W}_{\infty}^{1}(\Omega) \)). To this end we need to show that for every \( \epsilon > 0 \) there exists a \( \delta > 0 \), such that for \( \|h\|_{\tilde{W}_{\infty}^{1}(\Omega)} < \delta \)

\[
\frac{\|Q(y + h) - Q(y) - D_{y}Q(y)(h)\|_{L^{\infty}(\Omega)}}{\|h\|_{\tilde{W}_{\infty}^{1}(\Omega)}} < \epsilon.
\]

Define the residual \( R_{1} = Q(y + h) - Q(y) - D_{y}Q(y)(h) \). Using an algebraic manipulation, we get

\[
Q(y + h) - Q(y) = \frac{\nabla(2y + h) \cdot \nabla h}{Q(y + h) + Q(y)},
\]

whence

\[
R_{1} = \left( \frac{\nabla(2y + h)}{Q(y + h) + Q(y)} - \frac{\nabla y}{Q(y)} \right) \cdot \nabla h = \frac{(Q(y) - Q(y + h)) \nabla y + Q(y) \nabla h}{Q(y) (Q(y + h) + Q(y))} \cdot \nabla h.
\]

Invoking the \( L^{\infty} \) norm and using the necessary regularity of the underlying terms, we deduce

\[
\|R_{1}\|_{L^{\infty}(\Omega)} \leq \left( \|Q(y) - Q(y + h)\|_{L^{\infty}(\Omega)} + \|h\|_{\tilde{W}_{\infty}^{1}(\Omega)} \right) \|h\|_{\tilde{W}_{\infty}^{1}(\Omega)}.
\]

It only remains to show that \( Q \) is a Lipschitz continuous function. In view of \([1]\), for \( y, z \in v \oplus \tilde{W}_{\infty}^{1}(\Omega), y \neq z \) we get

\[
\|Q(y) - Q(z)\|_{L^{\infty}(\Omega)} \leq \left\| \frac{\nabla(y + z)}{Q(y) + Q(z)} \right\|_{L^{\infty}(\Omega)} \|y - z\|_{\tilde{W}_{\infty}^{1}(\Omega)} \leq \|y - z\|_{\tilde{W}_{\infty}^{1}(\Omega)}.
\]

(7)
Some manipulation gives

\begin{align*}
\langle R_2, w \rangle_{W^{-1}_\infty(\Omega), W^1_\infty(\Omega)} = & \left\langle \left( \frac{\nabla (y + h)}{Q(y + h)} - \frac{\nabla y}{Q(y)} - \left( I - \frac{\nabla y \nabla y^T}{Q(y)^2} \right) \frac{\nabla h}{Q(y)} \right), \nabla w \right\rangle_{L^\infty(\Omega), L^1(\Omega)}.
\end{align*}

Some manipulation gives

\[
\tilde{R}_2 = \nabla (y + h) \left( \frac{1}{Q(y + h)} - \frac{1}{Q(y)} \right) + \frac{\nabla h}{Q(y)} - \left( I - \frac{\nabla y \nabla y^T}{Q(y)^2} \right) \frac{\nabla h}{Q(y)}
\]

\[
= -\nabla (y + h) \left( \frac{Q(y + h) - Q(y)}{Q(y + h) Q(y)} \right) + \frac{\nabla h}{Q(y)} - \left( I - \frac{\nabla y \nabla y^T}{Q(y)^2} \right) \frac{\nabla h}{Q(y)}
\]

\[
= -\nabla (y + h) \left( \frac{\nabla y \nabla y^T}{Q(y)^2} \frac{\nabla h}{Q(y)} \right) + \frac{\nabla h}{Q(y)} - \left( I - \frac{\nabla y \nabla y^T}{Q(y)^2} \right) \frac{\nabla h}{Q(y)}.
\]

Computing the \( L^\infty \) norm then yields

\[
\left\| \tilde{R}_2 \right\|_{L^\infty(\Omega)} \leq O(\|h\|_{W^{-1}_\infty(\Omega)})^2 - \nabla y \left( \frac{\nabla y \cdot \nabla h}{Q(y + h) Q^2(y)} \right) + \frac{\nabla h}{Q(y)} - \left( I - \frac{\nabla y \nabla y^T}{Q(y)^2} \right) \frac{\nabla h}{Q(y)} \leq O(\|h\|_{W^{-1}_\infty(\Omega)})^2,
\]

because \( Q^{-1}(y + h) - Q^{-1}(y) = O(\|h\|_{W^{-1}_\infty(\Omega)}) \). Combining with (8), we see that \( \|R_2\|_{W^{-1}_\infty(\Omega)} \leq O(\|h\|_{W^{-1}_\infty(\Omega)})^2 \) and a standard \( \varepsilon-\delta \) argument proves the Fréchet differentiability of \( N \).

To conclude the proof we need to show the Lipschitz property for \( D_y N \). Consider a fixed but arbitrary direction \( h \), and let \( y, z \in v + \dot{W}^1_\infty(\Omega) \) with \( y \neq z \), then

\[
\langle \langle D_y N(y, u), h \rangle, w \rangle_{W^{-1}_\infty(\Omega), W^1_\infty(\Omega)} = \left\langle \left( \frac{1}{Q(y)} - \frac{1}{Q(z)} \right) \nabla h, \nabla w \right\rangle_{L^\infty(\Omega), L^1(\Omega)} - \left\langle \left( \frac{\nabla y \nabla y^T}{Q(y)^2} - \frac{\nabla z \nabla z^T}{Q(z)^2} \right) \frac{\nabla h}{Q(y)} \right\rangle_{L^\infty(\Omega), L^1(\Omega)}
\]

\[
= I_1 - I_2,
\]

where \( I_1 \) is clearly Lipschitz continuous. Continuing, we have

\[
I_2 = \left\langle \left( \frac{\nabla y \left( \nabla (y - z) \right)^T}{Q(y)^3} + \frac{\nabla y \left( Q(z)^3 - Q(y)^3 + \left( \nabla (y - z) \right) Q(y)^3 \nabla z^T \right)}{Q(y)^3 Q(z)^3} \right) \nabla h, \nabla w \right\rangle_{L^\infty(\Omega), L^1(\Omega)},
\]

and using \( a^3 - b^3 = (a - b)(a^2 + ab + b^2) \) and \([9]\) we obtain

\[
\sup_{h \in W^{-1}_\infty(\Omega)} \frac{\|D_y N(y, u), h\|_{W^{-1}_\infty(\Omega)}}{\|h\|_{W^{-1}_\infty(\Omega)}} \leq 2 \|y - z\|_{W^{-1}_\infty(\Omega)},
\]

which completes the proof. The same argument can be applied to show the twice Fréchet differentiability with respect to \( y \) with Lipschitz second order derivative, the details are omitted here to avoid repetition. \( \square \)
1.3. $W^2_p(\Omega)$-Strong Solution

We remark that for $p > n$, $W^1_p(\Omega) \subset \subset L^p(\Omega)$, consequently $L^p(\Omega) \subset \subset W^{-1}_\infty(\Omega)$. Throughout this section we assume that $v \in W^2_p(\Omega)$, $p > n$. We use the following notation

$$Y := \left( v \oplus \dot{W}^1_\infty(\Omega) \right) \cap W^2_p(\Omega), \quad p > n$$

then for $y \in Y$ means $y - v \in \dot{W}^1_\infty(\cap)W^2_p(\Omega)$.

**Lemma 1.4.** Let $U_2 \subset U_1 \cap L^p(\Omega)$ be open, then for every $u \in U_2$ and $v \in W^2_p(\Omega)$, the operator $\mathcal{N}(\cdot, u) : Y \rightarrow L^p(\Omega)$ is Fréchet differentiable and the Fréchet derivative is Lipschitz continuous and is given by

$$D_y\mathcal{N}(y, u)(h) = -\text{div} \left( I - \frac{\nabla y \nabla y^T}{Q(y)^2} \right) \nabla h.$$ 

Moreover $\mathcal{N}$ is twice Fréchet differentiable with Lipschitz second order Fréchet derivative.

**Proof.** For $p > n$, $W^1_p(\Omega)$ is a Banach algebra. Using this fact the proof is the same as in Lemma 1.3. \hfill \Box

**Remark 1.5.** We recall that $L^p(\Omega) \subset \subset W^{-1}_\infty(\Omega)$, therefore in case $U_1 \cap L^p(\Omega)$ in Theorem 1.4 is an empty set, we can always choose $U_1 \subset L^p(\Omega)$ in Theorem 1.4 consequently $U_1 \cap L^p(\Omega) = U_1 \neq \emptyset$.

Next we will state the existence and uniqueness of $y \in Y$ satisfying (2). Remarkably enough we not only get the improved regularity for $y$ but also the Fréchet differentiability of the control to state map; see, [14, Section 1.4.2] for details.

**Theorem 1.6** ($\dot{W}^1_\infty(\Omega) \cap W^2_p(\Omega)$ state). Let $\Omega$ be $C^{1,1}$. If $v \in W^2_p(\Omega)$, then for every $u \in U_2$, there exists a unique solution map $\mathcal{S} : U_2 \rightarrow Y$ such that

$$\mathcal{N}(\mathcal{S}(u), u) = 0, \quad \text{for all } u \in U_2.$$ 

Furthermore, $\mathcal{S}$ is twice continuously Fréchet differentiable as a function of $u$ with first order derivative at $u \in U_2$ given by

$$D_u\mathcal{S}(u) = - \left[ D_y\mathcal{N}(y, u) \right]^{-1} \circ D_u\mathcal{N}(y, u).$$

**Proof.** To this end it is sufficient to confirm the hypothesis of the implicit function theorem [19, 2.7.2].

1. In view of Lemma 1.4, $\mathcal{N}$ is continuously Fréchet differentiable with respect to $y$ on an open subset of $W^2_p(\Omega)$.

2. At $(y_0, u_0) = (0, 0)$, using (4) we get $\mathcal{N}(y_0, u_0) = 0$.

3. $D_y\mathcal{N}(y_0, u_0)(h) = -\Delta h$, which is a Banach space isomorphism from $W^2_p(\Omega)$ to $L^p(\Omega)$ for $\Omega$ of class $C^{1,1}$; see [13, 9.15].

Using the implicit function theorem, we conclude. \hfill \Box

1.4. $W^2_p$-Continuity Estimate

Theorem 1.6 provides existence and uniqueness of the $W^2_p(\Omega)$ solution to the state equation but not the continuity estimate for the solution variable. In order to derive the a priori finite element estimate we plan to use [20], which requires such a continuity estimate in Lemma 3.3. We use the implicit function theorem [19, 2.7.2] to prove Theorem 1.6. Hence, below we will provide an alternate proof to Theorem 1.6 for the existence of unique solution using a fixed point argument. The idea is to restrict the solution set. The proof requires the boundary data $v \in W^2_p(\Omega)$ to be small and $u \in L^p(\Omega)$ to be bounded. We remark that no such smallness condition on $v$ is needed previously in Theorem 1.6.
Our point of departure is to define a solution set
\[ \mathcal{B} = \left\{ y \in Y : \| y \|_{W^2_p(\Omega)} \leq pC_\# \right\}, \tag{9} \]
with \( C_\# \geq 1 \). For a given \( y \in \mathcal{B} \) define a map \( T \) as \( T(y) = \tilde{y} \) solving
\[ - \left( Q(y)^2 I - \nabla y \nabla y^T \right) : \nabla^2 \tilde{y} = uQ(y)^3 \quad \text{in } \Omega. \tag{10} \]

This is a linearization of the state equation \( (2) \) obtained by expanding the left-hand-side of \( (2) \) and evaluating the non-linear “coefficient” at \( y \in \mathcal{B} \). This is motivated by \( \cite{2} \) (6th line after Theorem 1), but we write it directly in non-divergence form so that we can use the result from Gilbarg-Trudinger \( \cite{13} \).

**Lemma 1.7.** The coefficient matrix in \( (10) \) is uniformly positive definite.

**Proof.** Let \( b \in \mathbb{R}^n \) be arbitrary nonzero column vector with components \( b_1, \ldots, b_n \) and if we denote by \( E = Q(y)^2 I - \nabla y \nabla y^T \) the coefficient matrix in \( (10) \) then using the definition of \( Q \) we obtain
\[
\begin{align*}
b^T E b &= b^T b + \left( \nabla y^T \nabla y \right) \left( b^T b \right) - b^T \nabla y \nabla y^T b \\
&= b^T b + \left( \nabla y^T \nabla y \right) \left( b^T b \right) - \left( \nabla y^T b \right)^T \left( \nabla y^T b \right) \\
&= b^T b + \left( \sum_{i=1}^n |\partial_i y|^2 \right) \left( \sum_{j=1}^n |b_j|^2 \right) - \left| \sum_{i=1}^n \partial_i y b_i \right|^2.
\end{align*}
\]

Then the required result is a direct consequence of Cauchy-Schwarz inequality. \( \square \)

**Lemma 1.8** (range of \( T \)). If \( v \in W^2_p(\Omega) \) and \( u \in L^p(\Omega) \) satisfy
\[ C_\Omega \left( \| v \|_{W^2_p(\Omega)} + C_\# \| u \|_{L^p(\Omega)} \right) \leq pC_\#, \tag{11} \]
where \( C_\Omega \) depends on \( p \) and \( C_\# \), then \( T \) in \( (10) \) maps \( \mathcal{B} \) to \( \mathcal{B} \).

**Proof.** For a given \( y \in \mathcal{B} \), \( Q(y) \in W^1_\infty(\Omega) \), whence the right hand side in \( (10) \) belongs to \( L^p(\Omega) \). In view of \( \cite{13} \) Theorem 9.15 in conjunction with Lemma 1.7 there exists a unique \( \tilde{y} \) solving \( (10) \). Moreover \( \cite{13} \) Lemma 9.17 implies there exists a constant \( C_\Omega \geq \left( 1 + p^2 C_\#^2 \right)^{1/2} \) such that \( \tilde{y} \) satisfies the a priori estimate:
\[ \| \tilde{y} \|_{W^2_p(\Omega)} \leq C_\Omega \left( \| v \|_{W^2_p(\Omega)} + \| u \|_{L^p(\Omega)} \right) \left( \| Q(y)^2 \|_{L^\infty(\Omega)} \right), \tag{12} \]
Since \( y \in \mathcal{B} \) and \( W^2_p(\Omega) \hookrightarrow W^1_\infty(\Omega) \) for \( p > n \) with embedding constant \( C_S \) we deduce
\[ \| \tilde{y} \|_{W^2_p(\Omega)} \leq C_\Omega \left( \| v \|_{W^2_p(\Omega)} + C_\# \| u \|_{L^p(\Omega)} \right) \]
where the constant \( C_\# \) depends on \( C_\# \), \( p \) and the embedding constant \( C_S \). Choosing \( \| v \|_{W^2_p(\Omega)} \) and \( \| u \|_{L^p(\Omega)} \) such that \( (11) \) hold, we conclude that \( T \) maps \( \mathcal{B} \) to \( \mathcal{B} \). \( \square \)

\footnote{This choice of lower bound on \( C_\Omega \) will be crucial to prove the Lipschitz continuity of operator \( S \) in Lemma 1.12.}
Theorem 1.9 (fixed point). If in addition to \([11]\) \(u \in L^p(\Omega)\) and \(v \in W^2_p(\Omega)\) further satisfy
\[
C_{\#\#} \left( \|u\|_{L^p(\Omega)} + \|v\|_{W^2_p(\Omega)} \right) < 1
\]
then the map \(T : \mathbb{B} \rightarrow \mathbb{B}\) is a contraction.

Proof. Let \(\tilde{y}_1\) and \(\tilde{y}_2\), with \(\tilde{y}_1 \neq \tilde{y}_2\), solve the linearized system \([10]\) and set \(\delta \tilde{y} := \tilde{y}_1 - \tilde{y}_2\). Computing the difference between the equations satisfied by \(\tilde{y}_1\) and \(\tilde{y}_2\) and after various algebraic manipulations we deduce
\[
- \left( Q(y_2)^2 I - \nabla y_2 \nabla y_2^T \right) : \nabla \delta \tilde{y} = u \left( Q(y_1)^3 - Q(y_2)^3 \right) - \left\{ Q(y_2)^2 I - Q(y_1)^2 I + \nabla \delta y \nabla y_1^T + \nabla y_2 \nabla \delta y^T \right\} : \nabla \tilde{y}_1.
\]

Again using the Sobolev embedding theorem it is easy to check that for \(p > n\), the right hand side belongs to \(L^p(\Omega)\). Toward this end, we invoke \([13, \text{Theorem 9.15}]\) in conjunction with Lemma \(1.7\) and \([13, \text{Lemma 9.17}]\), and find there exists a constant \(C_\Omega \geq (1 + p^2 C_{\#})^{1/2}\) such that
\[
\|\delta \tilde{y}\|_{W^p_2(\Omega)} \leq C_\Omega \left( \left\| u \left( Q(y_1)^3 - Q(y_2)^3 \right) \right\|_{L^p(\Omega)} + \left\| Q(y_2)^2 - Q(y_1)^2 \right\|_{L^p(\Omega)} \right) + \left\| \nabla \delta y \nabla y_1 + \nabla y_2 \nabla \delta y^T \right\|_{L^p(\Omega)}.
\]

We further deduce
\[
\|\delta \tilde{y}\|_{W^p_2(\Omega)} \leq C_\Omega \left( \left\| u \right\|_{L^p(\Omega)} \left\| Q(y_1)^3 - Q(y_2)^3 \right\|_{L^\infty(\Omega)} + \left\| Q(y_2)^2 - Q(y_1)^2 \right\|_{L^\infty(\Omega)} \right) + \left\| \nabla \delta y \right\|_{W_{\#}(\Omega)} + |y_2|_{W_{\#}(\Omega)} \left\| \delta \tilde{y} \right\|_{W^2_2(\Omega)} \right).
\]

Next, note that \(\tilde{y}_1\) satisfies the a priori estimate \([12]\) because \(\tilde{y}_1\) solves \([10]\). Moreover, since \(Q\) is Lipschitz continuous (see proof of Lemma \(1.3\) and \(y_1, y_2 \in \mathbb{B}\), we get
\[
\|\delta \tilde{y}\|_{W^p_2(\Omega)} \leq C_{\#\#} \left( \|u\|_{L^p(\Omega)} + \|v\|_{W^2_p(\Omega)} \right) \|\delta \tilde{y}\|_{W^2_2(\Omega)},
\]
where the constant \(C_{\#\#}\) depends on \(C_\Omega, C_{\#}, p,\) and \(C_S\) where the latter is the embedding constant for \(W^2_p(\Omega)\) in \(W^1_\infty(\Omega)\) for \(p > n\). Choosing \(u\) and \(v\) such that \([13]\) hold, we get the desired contraction. \(\square\)

Remark 1.10. In order to get the upper bound \([15]\), one of the terms in \([14]\) has been estimated as follows:
\[
C_\Omega |y_1|_{W_{\#}(\Omega)} \|\tilde{y}_1\|_{W^2_2(\Omega)} \leq C_SC_\Omega |y_1|_{W^2_2(\Omega)} \|\tilde{y}_1\|_{W^2_2(\Omega)}
\]
which is due to \(W^2_p(\Omega) \subset \subset W^1_\infty(\Omega)\). Then due to \([12]\) we obtain
\[
C_\Omega |y_1|_{W_{\#}(\Omega)} \|\tilde{y}_1\|_{W^2_2(\Omega)} \leq C_SC_\Omega^2 |y_1|_{W^2_2(\Omega)} \left( \|v\|_{W^2_2(\Omega)} + \|u\|_{L^p(\Omega)} \left\| Q(y_1)^3 \right\|_{L^\infty(\Omega)} \right).
\]

Then due to \([11]\)
\[
C_\Omega |y_1|_{W_{\#}(\Omega)} \|\tilde{y}_1\|_{W^2_2(\Omega)} \leq pC_{\#}C_SC_\Omega |y_1|_{W^2_2(\Omega)},
\]
which satisfies (13) if
\[ pC_\# \Omega C_S \| y_1 \|_{W^2_p(\Omega)} < 1 \Leftrightarrow C_S \| y_1 \|_{W^2_p(\Omega)} < \frac{1}{pC_\# \Omega}. \]

Due to the fact that \( C_\Omega \geq \left( 1 + p^2 C^2_\# \right)^{1/2} \) in Theorem 1.9 we obtain
\[ C_S \| y_1 \|_{W^2_p(\Omega)} < \frac{1}{pC_\# \left( 1 + p^2 C^2_\# \right)^{1/2}}. \] \hspace{1cm} (16)

**Definition 1.11** (control sets \( U \) and \( U_{ad} \)). We define an open set
\[ U := \{ u \in L^p(\Omega) : (11) \text{ and } (13) \text{ holds} \} \cap U_2. \]
and further define a closed set of admissible controls
\[ U_{ad} := \{ u \in L^2(\Omega) : \| u \|_{L^p(\Omega)} \leq \theta, \ p > n \}, \]
where \( \theta \) is chosen such that \( U_{ad} \subset U \).

**Lemma 1.12** (S Lipschitz). Let \( u_1, u_2 \in U, \ u_1 \neq u_2, \) then
\[ \| S(u_1) - S(u_2) \|_{W^2_p(\Omega)} \leq \| u_1 - u_2 \|_{W^2_p(\Omega)} \] \hspace{1cm} (17)
\[ \| S(u_1) - S(u_2) \|_{L^p(\Omega)} \leq \| u_1 - u_2 \|_{L^p(\Omega)}. \] \hspace{1cm} (18)

**Proof.** Recall the equations satisfied by \( S(u_1) \in Y \) and \( S(u_2) \in Y \)
\[ -\text{div} \left( \frac{1}{Q(S(u_1))} \nabla S(u_1) \right) = u_1, \quad -\text{div} \left( \frac{1}{Q(S(u_2))} \nabla S(u_2) \right) = u_2. \]
On subtracting and rearranging, we obtain
\[ -\text{div} \left( \frac{1}{Q(S(u_1))} \nabla (S(u_1) - S(u_2)) \right) = \text{div} \left( \frac{1}{Q(S(u_1))} - \frac{1}{Q(S(u_2))} \right) \nabla S(u_2) + u_1 - u_2. \]
Using the characterization of \( W^{1-1}_2(\Omega) \) functions [10] P. 283, Theorem 1] and \( S(u_2) \in B \) therefore \( Q(S(u_1)) \leq \left( 1 + p^2 C^2_\# \right)^{1/2} \) we have the a priori estimate of the solution \( S(u_1) - S(u_2) \) of the above elliptic PDE:
\[ |S(u_1) - S(u_2)|_{W^2_2(\Omega)} \leq \left( 1 + p^2 C^2_\# \right)^{1/2} |S(u_2)|_{W^2_2(\Omega)} \left\| \frac{1}{Q(S(u_1))} - \frac{1}{Q(S(u_2))} \right\|_{L^2(\Omega)} + \| u_1 - u_2 \|_{W^2_p(\Omega)}. \]
Moreover for \( p > n, \ W^2_p(\Omega) \subset \subset W^1_\infty(\Omega) \) with embedding constant \( C_S \), then
\[ |S(u_1) - S(u_2)|_{W^2_2(\Omega)} \leq C_S \| S(u_2) \|_{W^2_p(\Omega)} \left( 1 + p^2 C^2_\# \right)^{1/2} |S(u_1) - S(u_2)|_{W^2_2(\Omega)} + \| u_1 - u_2 \|_{W^2_p(\Omega)}. \]
Finally due to (16) we obtain
\[ C_S \| S(u_2) \|_{W^2_2(\Omega)} \left( 1 + p^2 C^2_\# \right)^{1/2} < \frac{1}{pC_\#} < 1, \]
where the last inequality is due to the fact that \( C_\# \geq 1 \) and \( p > n \), which completes the proof. \( \Box \)
2. Optimality Conditions

Using the control to state map we can rewrite the minimization problem (1)-(2) in the following reduced form:

\[ \inf J(u) := J(S(u), u) \quad \text{over } u \in U_{ad}, \]

where

\[ J(S(u), u) = J_1(S(u)) + J_2(u), \]

with

\[ J_1(S(u)) = \frac{1}{2} \| S(u) - y_d \|_{L^2(\Omega)}^2, \quad J_2(u) = \frac{\alpha}{2} \| u \|_{L^2(\Omega)}^2. \]

2.1. Existence of Optimal control

**Theorem 2.1.** There exists an optimal control \( \pi \) solving the reduced minimization problem [19].

**Proof.** The proof is based on a minimizing sequence argument. As \( J \) is bounded below, there exists a minimizing sequence \( \{u_n\}_{n \in \mathbb{N}} \), i.e.

\[ \inf_{u \in U_{ad}} J(S(u), u) = \lim_{n \to \infty} J(S(u_n), u_n). \]

By Definition 1.11, \( U_{ad} \) is a nonempty, closed, bounded and convex subset of \( L^p(\Omega) \) which is a reflexive Banach space for \( n < p < \infty \), thus weakly sequentially compact. Consequently, we can extract a weakly convergent subsequence \( \{u_{n_k}\}_{k \in \mathbb{N}} \subset L^p(\Omega) \) i.e.

\[ u_{n_k} \rightharpoonup \pi \text{ in } L^p(\Omega), \quad \pi \in U_{ad}. \]

Thus \( \pi \) is the candidate for our optimal control.

In the sequel, we drop the index \( k \) when extracting subsequences. Using Theorem 1.9, \( S(u_n) = y_n \) satisfies the state equation [2]. Since \( Y \subset v \oplus W_{\infty}^1(\Omega) \) for \( p > n \), the Rellich-Kondrachov theorem yields a strongly convergent subsequence \( \{y_n\}_{n \in \mathbb{N}} \subset v \oplus W_{\infty}^1(\Omega) \), i.e.

\[ y_n \to \overline{y} \text{ in } v \oplus W_{\infty}^1(\Omega). \]

Note that the limit \( \overline{y} \) is the state corresponding to the control \( \pi \). This results from replacing \( \overline{y} \) with \( y_n \) in the variational equation [4] taking the limit and making use of the embedding \( L^p(\Omega) \subset W_{\infty}^{-1}(\Omega) \).

Finally, using the fact that \( J_2(u) \) is continuous in \( L^2 \) and convex, together with the strong convergence \( y_n \to \overline{y} \) in \( L^\infty(\Omega) \), it follows that \( J \) is weakly lower semicontinous, whence

\[ \inf_{u \in U_{ad}} J(u) = \liminf_{n \to \infty} \left( J_1(S(u_n)) + J_2(u_n) \right) \geq J_1(S(\pi)) + J_2(\pi) = J(\pi). \]

\[ \square \]

2.2. First Order Necessary Conditions

We recall the following result from [22].

**Lemma 2.2.** If \( \pi \in U_{ad} \) denotes an optimal control, given by Theorem 2.1, then the first order necessary optimality condition satisfied by \( \pi \) is

\[ \langle J'(\pi), u - \pi \rangle_{L^2(\Omega), L^2(\Omega)} \geq 0, \quad \forall u \in U_{ad}. \]

Since \( U_{ad} \) is closed, we need to define a suitable set of admissible directions.
Definition 2.3. Given $u \in U_{ad}$, the convex cone $C(u)$ comprises all directions $h \in L^p(\Omega)$ such that $u + th \in U_{ad}$ for some $t > 0$, i.e.,

$$C(u) := \{ h \in L^p(\Omega) : u + th \in U_{ad}, \text{ for some } t > 0 \}.$$

Theorem 2.4. If $\pi \in U_{ad}$ denotes an optimal control, then the first-order optimality conditions are given by

the state equation (2), the adjoint equation

$$- \text{div} \left( A[\bar{y}] \nabla \bar{p} \right) = \bar{y} - y_d \quad \text{in } \Omega, \quad \bar{p} = 0 \quad \text{on } \partial \Omega$$

(20)

where $A[\bar{y}] = \frac{1}{\sqrt{|\bar{y}|}} \left( I - \frac{\nabla \bar{y} \nabla \bar{y}^T}{|\bar{y}|} \right)$ and the equation for the control

$$(\bar{p} + \alpha \pi, u - \pi)_{L^2(\Omega), L^2(\Omega)} \geq 0, \quad \forall u \in U_{ad}.$$ (21)

Proof. Using Theorem 1.6, we can infer that $J$ is Fréchet differentiable, and the Fréchet derivative of $J$ at $\pi$ in a direction $h \in C(\pi)$ is

$$\left\langle J'(\pi), h \right\rangle_{L^p'(\Omega), L^p(\Omega)} = \left\langle J'_1(S(\pi)), S'(\pi)h \right\rangle_{Y^*, Y} + \left\langle J'_2(\pi), h \right\rangle_{L^p'(\Omega), L^p(\Omega)},$$

whence

$$\left\langle J'(\pi), h \right\rangle_{L^p'(\Omega), L^p(\Omega)} = \left\langle \bar{y} - y_d, S'(\pi)h \right\rangle_{L^2(\Omega), L^2(\Omega)} + \alpha \langle \pi, h \rangle_{L^2(\Omega), L^2(\Omega)}$$

$$= \left\langle S'(\pi)^* (\bar{y} - y_d), h \right\rangle_{L^2(\Omega), L^2(\Omega)} + \alpha \langle \pi, h \rangle_{L^2(\Omega), L^2(\Omega)}.$$ (20)

Recalling the expression for $S'(\pi)$ from Theorem 1.6 and the fact that $D_uN(\bar{y}, \pi) = -I$, where we have dropped the dependence of $N$ on $v$, we get

$$\left\langle J'(\pi), h \right\rangle_{L^p'(\Omega), L^p(\Omega)} = \left\langle \left[ D_vN(\bar{y}, \pi) \right]^{-*} (\bar{y} - y_d), h \right\rangle_{L^2(\Omega), L^2(\Omega)} + \alpha \langle \pi, h \rangle_{L^2(\Omega), L^2(\Omega)}.$$ (20)

Setting $\bar{p} = \left[ D_vN(\bar{y}, \pi) \right]^{-*} (\bar{y} - y_d)$, we get (20). Moreover, we see that $J'(\pi) = \bar{p} + \alpha \pi$ which yields (21). We remark that the pairing $\left\langle J'(\pi), h \right\rangle_{L^p'(\Omega), L^p(\Omega)}$ can be simply treated as the $L^2$ pairing.

Remark 2.5. In general, $J'(u) = p(y) + cu$ for an arbitrary $u \in U_{ad}$, where $y$ solves (2) with $u$ as right-hand-side, and $p(y)$ solves (20) with right-hand-side given by $y - y_d$.

Next we will generalize a result from Gilbarg-Trudinger [13, Theorem 9.15, Lemma 9.17] where the lower order coefficient are in $L^q(\Omega)$, $q > n$, instead of being in $L^\infty(\Omega)$ this result is crucial to prove the necessary regularity for the adjoint equation (20).

Lemma 2.6. If $A \in L^\infty(\Omega)^{n \times n}$, $b \in L^q(\Omega)^n$, $n < q < \infty$, then for all $f \in L^r(\Omega)$ with $1 < r \leq q$, there exists a unique $u \in W^2_q(\Omega) \cap W^1_r(\Omega)$ solving

$$-A : D^2 w - b \cdot \nabla w = f \quad \text{in } \Omega$$

$$w = 0 \quad \text{on } \partial \Omega,$$ (22)

with

$$\|u\|_{W^2_q(\Omega)} \leq C_q \|f\|_{L^r(\Omega)}.$$ (23)
Proof. We prove the result in two steps.

1. Existence and Uniqueness. As \( L^\infty(\Omega) \) is dense in \( L^q(\Omega) \), for \( b \in L^q(\Omega)^n \) there exists \( \{ b_m \}_{m \in \mathbb{N}} \subset L^\infty(\Omega)^n \) such that \( b_m \to b \) in \( L^q(\Omega)^n \). Similarly as \( C^\infty(\Omega) \) is dense in \( L^r(\Omega) \), therefore there exists \( \{ f_m \}_{m \in \mathbb{N}} \subset C^\infty(\Omega) \) such that \( f_m \to f \) in \( L^r(\Omega) \). If we consider the auxiliary problem

\[
-A : \mathcal{D}^2 w_m - b_m \cdot \nabla w_m = f_m \quad \text{in } \Omega
\]

\[
w_m = 0 \quad \text{on } \partial \Omega,
\]

using [13] Lemma 9.17, we deduce

\[
\| w_m \|_{W^2_2(\Omega)} \leq C_\Omega \| f_m \|_{L^r(\Omega)}, \quad \forall r \in (1, \infty),
\]

and the right hand side converges to \( \| f \|_{L^r(\Omega)} \). Since a unit ball in \( W^2_2(\Omega) \) is weakly compact, there exists a subsequence, still labeled \( w_m \), that converges weakly in \( W^2_2(\Omega) \) and for \( s = \frac{r}{q - r} \), strongly in \( W^1_s(\Omega) \) to a function \( w \in W^2_s(\Omega) \) and \( W^1_s(\Omega) \). It remains to show that \( w \) satisfies (22). Because

\[
\int_\Omega v (b_m \cdot \nabla w_m) \leq \| v \|_{L^r(\Omega)} \| b_m \|_{L^s(\Omega)} \| w_m \|_{W^2_2(\Omega)},
\]

we obtain

\[
\int_\Omega f v = -\int_\Omega v (A : \mathcal{D}^2 w_m + b_m \cdot \nabla w_m) \to -\int_\Omega v (A : \mathcal{D}^2 w + b \cdot \nabla w),
\]

for all \( v \in L^r(\Omega) \).

2. Continuity estimate. Rewriting (22),

\[
-A : \mathcal{D}^2 w = f + b \cdot \nabla w \quad \text{in } \Omega
\]

\[
w = 0 \quad \text{on } \partial \Omega.
\]

In view of the definition of \( s = \frac{r}{q - r} \), it immediately follows that \( f + b \cdot \nabla w \in L^r(\Omega) \), whence [13] Lemma 9.17 implies

\[
\| w \|_{W^2_2(\Omega)} \leq C_\Omega \left( \| f \|_{L^r(\Omega)} + \| b \|_{L^s(\Omega)} \| w \|_{W^2_2(\Omega)} \right).
\]

(24)

Toward this end, we will prove (23) by contradiction. Let \( \{ w_m \}_{m \in \mathbb{N}} \subset W^2_2(\Omega) \cap \bar{W}^1_1(\Omega) \) be a sequence satisfying

\[
\| w_m \|_{W^2_2(\Omega)} = 1, \quad \| f_m \|_{L^r(\Omega)} \to 0
\]

as \( m \to \infty \), where \( f_m = -A : \mathcal{D}^2 w_m - b \cdot \nabla w_m \). Since the unit ball of \( W^2_2(\Omega) \) is weakly compact, there exists a subsequence, that converges weakly in \( W^2_2(\Omega) \) and strongly in \( W^1_1(\Omega) \) to a \( w \in W^2_2(\Omega) \cap \bar{W}^1_1(\Omega) \). Therefore,

\[
\int_\Omega f_m v = -\int_\Omega v (A : \mathcal{D}^2 w_m + b \cdot \nabla w_m) \to -\int_\Omega v (A : \mathcal{D}^2 w + b \cdot \nabla w) = 0,
\]

for all \( v \in L^r(\Omega) \), whence \(-A : \mathcal{D}^2 w - b \cdot \nabla w = 0 \) and \( w = 0 \) because of uniqueness. But from (24) we deduce

\[
1 \leq C_\Omega \| b \|_{L^s(\Omega)} \| w \|_{W^1_1(\Omega)},
\]

which is a contradiction. Thus (23) holds. \( \square \)
Corollary 2.7 (regularity of adjoint). There exists a unique \( \pi \in W^{2}_{2}(\Omega) \cap W^{1}_{1}(\Omega) \). If in addition \( y_{d} \in L^{p}(\Omega) \), \( p > n \), then \( \pi \in W^{2}_{p}(\Omega) \).

Proof. Rewriting \((20)\)

\[-A[\bar{y}] : D^{2}\bar{p} - \text{div} (A[\bar{y}]) \cdot \nabla \bar{p} = \bar{y} - y_{d} \quad \text{in} \ \Omega, \quad \bar{p} = 0 \quad \text{on} \ \partial \Omega \]

Since \( \bar{y} \in W^{2}_{p}(\Omega) \), \( p > n \), therefore \( A[\bar{y}] \in W^{1}_{p}(\Omega) \), and \( \text{div} (A[\bar{y}]) \in L^{p}(\Omega) \), then invoking Lemma 2.6 we obtain the desired result.

Corollary 2.8 (regularity of optimal control). In view of \((21)\) we have

\[ \bar{u} = \begin{cases} -\frac{\bar{p}}{\|\bar{p}\|_{L^{p}(\Omega)}} & \text{if} \ \bar{p} + \alpha \bar{u} = 0 \\ -\frac{\bar{p}}{\|\bar{p}\|_{L^{p}(\Omega)}} & \text{if} \ \bar{p} + \alpha \bar{u} \neq 0. \end{cases} \]

Then invoking Corollary 2.7 and the Sobolev embedding theorem we deduce that \( \bar{u} \in W^{1}_{2}(\Omega) \) and further if \( y_{d} \in L^{p}(\Omega), \ p > n \), then \( \bar{u} \in W^{2}_{p}(\Omega) \subset \subset W^{1}_{\infty}(\Omega) \).

2.3. Second Order Sufficient Conditions

We make the following standard assumption:

\[ J''(\pi)h^{2} \geq \delta \|h\|^{2}_{L^{2}(\Omega)} \quad \forall h \in C(\pi). \quad (25) \]

We remark that it may be possible to prove \((25)\) using the technique introduced in [3] Theorem 5.7]. The key ingredients of the proof are: the continuity estimate for the first and second order Fréchet derivatives \((29)\) and \((30)\) and the continuity estimate for our state equation (see, Theorem 1.9).

Our next goal is to prove the following crucial result:

Corollary 2.9. If \((25)\) holds, then there exists a \( \delta \) such that \( h \in C(\pi) \) if \( \|h\|_{L^{p}(\Omega)} \leq \delta \), and

\[ \langle J'(u) - J'(\pi), u - \pi \rangle_{L^{2}(\Omega) \times L^{2}(\Omega)} \geq \frac{\delta}{2} \|u - \pi\|^{2}_{L^{2}(\Omega)} \quad \forall u \in \pi + C(\pi). \quad (26) \]

The proof requires a non-trivial estimate which we will prove in Lemma 2.13. Such an estimate is needed to deal with the so-called \( 2\)-norm discrepancy, we refer to [6] for further reading on the subject. We will conclude this section with a proof of Corollary 2.9.

Proposition 2.10. For every \( u \in U \) and every \( h_{1}, h_{2} \in L^{p}(\Omega) \) the first and second order Fréchet derivatives \( S'(u)h_{1} \in W^{2}_{p}(\Omega) \) and \( S''(u)h_{1}h_{2} \in W^{2}_{p}(\Omega) \) at \( S(u) \in Y \) satisfies

\[ -\text{div} (A[S(u)]\nabla S'(u)h_{1}) = h_{1} \quad \text{in} \ \Omega, \quad S'(u)h_{1} = 0 \quad \text{on} \ \partial \Omega \quad (27) \]

\[ -\text{div} (A[S(u)]\nabla S''(u)h_{1}h_{2}) = \text{div} (D_{u}A[S(u)]\langle h_{2} \rangle \nabla S'(u)h_{1}) \quad \text{in} \ \Omega, \quad S''(u)h_{1}h_{2} = 0 \quad \text{on} \ \partial \Omega \quad (28) \]

and

\[ \|S'(u)h_{1}\|_{W^{2}_{2}(\Omega)} \lesssim \|h_{1}\|_{W^{2}_{2}(\Omega)}, \quad \|S'(u)h_{1}\|_{W^{2}_{p}(\Omega)} \lesssim \|h_{1}\|_{L^{p}(\Omega)} \quad (29) \]

\[ \|S''(u)h_{1}h_{2}\|_{W^{2}_{2}(\Omega)} \lesssim \|h_{1}\|_{W^{2}_{2}(\Omega)} \|h_{2}\|_{L^{p}(\Omega)} \quad (30) \]

Proof. The derivation of \((27)\) and \((28)\) follows immediately. The first inequality in \((29)\) is due to the characterization of \( W^{2}_{p}(\Omega) \) functions [10], P. 283, Theorem 1] and the second inequality is due to Lemma 2.6. Using both of these results, in conjunction with the Sobolev embedding \( W^{2}_{p}(\Omega) \subset \subset W^{1}_{\infty}(\Omega) \) for \( p > n \), gives \((30)\). \( \square \)
Lemma 2.11 (A is Lipschitz). If \( u_1, u_2 \in U \), with \( u_1 \neq u_2 \), the map \( A : Y \to v \oplus \hat{W}_p^1(\Omega) \) in (20) satisfies
\[
\left\| A[S(u_1)] - A[S(u_2)] \right\|_{L^\infty(\Omega)} \lesssim \left\| S(u_1) - S(u_2) \right\|_{W_2^1(\Omega)},
\]
and for \( h_1 \in L^p(\Omega) \), \( S' : U \to \mathcal{L}(L^p(\Omega), Y) \)
\[
\left\| D_u (A[S(u_1)] - A[S(u_2)]) \langle h_1 \rangle \right\|_{W_2^1(\Omega)} \lesssim \left\| S'(u_1) - S'(u_2) \right\|_{W_2^1(\Omega)} \| h_1 \|_{L^2(\Omega)}.
\]

Proof. Recall \( y_1 = S(u_1) \) and \( y_2 = S(u_2) \), for simplicity we will use this notation in the proof. It is enough to show (31), the same proof works for (32) and (33). Now
\[
\left\| A[y_1] - A[y_2] \right\|_{L^\infty(\Omega)} \leq \left\| \frac{1}{Q(y_1)} - \frac{1}{Q(y_2)} \right\|_{L^\infty(\Omega)} + \left\| \nabla y_1 \nabla y_1^T Q(y_1)^3 - \nabla y_2 \nabla y_2^T Q(y_2)^3 \right\|_{L^\infty(\Omega)}.
\]

We consider each term on the right hand side separately. For the first term we invoke the Lipschitz continuity of \( Q \) from \( \mathbb{T} \) to deduce
\[
\left\| \frac{1}{Q(y_1)} - \frac{1}{Q(y_2)} \right\|_{L^\infty(\Omega)} = \left\| \frac{Q(y_2) - Q(y_1)}{Q(y_1)Q(y_2)} \right\|_{L^\infty(\Omega)} \leq |y_1 - y_2|_{W_2^1(\Omega)}.
\]

Invoking the triangle inequality on the second term leads to
\[
\left\| \frac{\nabla y_1 \nabla y_1^T Q(y_1)^3 - \nabla y_2 \nabla y_2^T Q(y_2)^3}{Q(y_1)^3} \right\|_{L^\infty(\Omega)} \leq \left\| \frac{\nabla (y_1 - y_2) \nabla y_1^T Q(y_1)^3}{Q(y_1)^3} \right\|_{L^\infty(\Omega)} + \left\| \frac{\nabla y_2 \nabla (y_1 - y_2) Q(y_2)^3}{Q(y_1)^3} \right\|_{L^\infty(\Omega)}.
\]

where \( C > 0 \) is a generic uniform constant independent of \( \Omega, y_1 \) and \( y_2 \). \(
\)

Lemma 2.12 (\( S' \) is Lipschitz). Let \( u, u_1, u_2 \in U \), and \( h_1 \in L^p(\Omega) \) then \( S' : U \to \mathcal{L}(L^p(\Omega), Y) \) satisfies
\[
\left\| (S'(u_1) - S'(u_2)) h_1 \right\|_{W_2^1(\Omega)} \leq \| u_1 - u_2 \|_{L^p(\Omega)} \| h_1 \|_{L^2(\Omega)}.
\]

Proof. Consider the system satisfied by \( (S'(u_1) + h_1) \) and \( S'(u_2) h_1 \)
\[
- \text{div} \left( A[S(u_1) + h_1] \nabla S'(u_1) h_1 \right) = h_1 \quad \text{in} \ \Omega, \quad S'(u_1) h_1 = 0 \quad \text{on} \ \partial \Omega
\]
\[
- \text{div} \left( A[S(u_2)] \nabla S'(u_2) h_1 \right) = h_1 \quad \text{in} \ \Omega, \quad S'(u_2) h_1 = 0 \quad \text{on} \ \partial \Omega.
\]

On subtracting and rearranging
\[
- \text{div} \left( A[S(u)] \nabla (S'(u) - S'(u_1) + S'(u_2) h_1) \right) = \text{div} \left( A[S(u)] - A[S(u + h_1)] \nabla S'(u + h_1) h_1 \right) \quad \text{in} \ \Omega.
\]
\[
(S'(u) - S'(u_1) + S'(u_2) h_1) h_1 = 0 \quad \text{on} \ \partial \Omega.
\]
Using the characterization of $W^{-1}_2(\Omega)$ functions [10, P. 283, Theorem 1] we deduce
\[
\left\| (S'(u) - S'(u+h)) h_1 \right\|_{W^1_2(\Omega)} \lesssim \left\| A[S(u)] - A[S(u+h)] \right\|_{L^\infty(\Omega)} \left\| S'(u+h) h_1 \right\|_{W^1_2(\Omega)}.
\]
Using (31) and (29), we obtain
\[
\left\| (S'(u) - S'(u+h)) h_1 \right\|_{W^1_2(\Omega)} \lesssim \left\| S(u) - S(u+h) \right\|_{W^1_\infty(\Omega)} \left\| h_1 \right\|_{W^{-1}_2(\Omega)}.
\]
Using (17) and $W^{-1}_2(\Omega) \to L^2(\Omega)$ we get (34). \hfill \Box

Next, we prove an auxiliary result.

**Lemma 2.13.** Let $u \in U$ and $y_d, h, h_1, h_2 \in L^p(\Omega)$, then there exist a constant $L > 0$ such that
\[
\left| J''(u + h) \langle h_1, h_2 \rangle - J''(u) \langle h_1, h_2 \rangle \right| \leq L \left( \left\| h \right\|_{L^2(\Omega)} \left\| h_2 \right\|_{L^p(\Omega)} + \left\| h \right\|_{L^p(\Omega)} \left\| h_2 \right\|_{L^2(\Omega)} \right) \left\| h_1 \right\|_{L^2(\Omega)}.
\]

**Proof.** Using the reduced cost functional (19), a simple calculation gives
\[
J''(u + h) \langle h_1, h_2 \rangle - J''(u) \langle h_1, h_2 \rangle = \int_\Omega \left( S'(u + h)^2 - S'(u)^2 \right) h_1 h_2 + \int_\Omega \left[ (S(u + h) - y_d) S''(u + h) - (S(u) - y_d) S''(u) \right] h_1 h_2 = \int_\Omega \left( S'(u + h) - S'(u) \right) h_1 \left( S'(u + h) + S'(u) \right) h_2 + \int_\Omega \left( S(u + h) - S(u) \right) S''(u + h) + (S(u) - y_d) (S''(u + h) - S''(u)) \right] h_1 h_2
\]

Using the triangle inequality and Cauchy-Schwarz, we have
\[
\left| J''(u + h) \langle h_1, h_2 \rangle - J''(u) \langle h_1, h_2 \rangle \right| \leq \left\| (S'(u + h) - S'(u)) h_1 \right\|_{L^3(\Omega)} \left\| (S'(u + h) + S'(u)) h_2 \right\|_{L^3(\Omega)} = (I) + \left\| (S(u + h) - S(u)) \right\|_{L^3(\Omega)} \left\| S''(u + h) h_1 h_2 \right\|_{L^3(\Omega)} = (II) + \left\| \int_\Omega (S(u) - y_d) (S''(u + h) - S''(u)) h_1 h_2 \right\|_{L^3(\Omega)} = (III)
\]

We will estimate each term (I) – (III) individually. In view of (34), (29)

(I) $\lesssim \left\| h \right\|_{L^1(\Omega)} \left\| h_1 \right\|_{L^2(\Omega)} \left\| h_2 \right\|_{L^2(\Omega)}$

and using (17) and (30)

(II) $\lesssim \left\| h \right\|_{L^2(\Omega)} \left\| h_1 \right\|_{L^2(\Omega)} \left\| h_2 \right\|_{L^p(\Omega)}$.

The estimate for (III) is more involved. Recall (28), namely the system satisfied by $S''(u + h) h_1 h_2$ and $S''(u) h_1 h_2$:

\[ - \text{div} \left( A[S(u + h)] \nabla S''(u + h) h_1 h_2 \right) = \text{div} \left( D_u A[S(u + h)] h_2 \nabla S'(u + h) h_1 \right) \quad \text{in} \ \Omega, \quad S''(u + h) h_1 h_2 = 0 \quad \text{on} \ \partial \Omega, \]

\[ - \text{div} \left( A[S(u)] \nabla S''(u) h_1 h_2 \right) = \text{div} \left( D_u A[S(u)] h_2 \nabla S'(u) h_1 \right) \quad \text{in} \ \Omega, \quad S''(u) h_1 h_2 = 0 \quad \text{on} \ \partial \Omega.
\]
On subtracting and rearranging, we obtain

\[
- \text{div} \left( A[S(u)] \nabla (S''(u) - S''(u + h)) h_1 h_2 \right) = \text{div} \left( (A[S(u)] - A[S(u + h)]) \nabla S''(u + h) h_1 h_2 \right) \\
+ \text{div} \left( D_u A[S(u)] (h_2) \nabla S'(u) h_1 - D_u A[S(u + h)] (h_2) \nabla S'(u + h) h_1 \right).
\]

For \( u \in U \), we denote the variable satisfying (20) by \( p \), with right hand side \( S(u) - y_d \). We further deduce

\[
(III) = \left| \int_\Omega \nabla p \cdot \left( (A[S(u)] - A[S(u + h)]) \nabla S''(u + h) h_1 h_2 \right) \right| \\
+ \left( D_u A[S(u)] (h_2) \nabla S'(u) h_1 - D_u A[S(u + h)] (h_2) \nabla S'(u + h) h_1 \right) \\
\leq \| p \|_{W^1_2(\Omega)} \| A[S(u)] - A[S(u + h)] \|_{L^2(\Omega)} \| S''(u + h) h_1 h_2 \|_{W^2_2(\Omega)} \\
+ \| p \|_{W^1_2(\Omega)} \| D_u A[S(u)] (h_2) \|_{L^2(\Omega)} \| (S'(u) - S'(u + h)) h_1 \|_{W^2_2(\Omega)} \\
+ \| p \|_{W^1_2(\Omega)} \| D_u (A[S(u)] - A[S(u + h)]) (h_2) \|_{L^2(\Omega)} \| S'(u + h) h_1 \|_{W^2_2(\Omega)}.
\]

Using (32), (17), (30), (33), (34) and (29), we obtain

\[
(III) \lesssim \| p \|_{W^1_2(\Omega)} \left( \| h \|_{L^2(\Omega)} \| h_2 \|_{L^2(\Omega)} + \| h \|_{L^2(\Omega)} \| h_2 \|_{L^2(\Omega)} \right) \| h_1 \|_{L^2(\Omega)}.
\]

\[QED\]

**Proof of Corollary 2.9.** We first prove an auxiliary result: for an arbitrary, but fixed, \( w \in U_{ad} \), there exists an \( \epsilon > 0 \) such that

\[
\mathcal{J}(u) \geq \mathcal{J}(w) + \langle \mathcal{J}'(w), u - w \rangle + \frac{\delta}{4} \| u - w \|_{L^2(\Omega)}^2 \quad \forall u \in w + \mathcal{C}(w) \text{ with } \| u - w \|_{L^2(\Omega)} \leq \epsilon.
\]

(36)

Using Taylor's theorem, there is a \( t \in (0, 1) \) such that

\[
\mathcal{J}(u) = \mathcal{J}(w) + \langle \mathcal{J}'(w), u - w \rangle + \frac{1}{2} \mathcal{J}''(tu + (1 - t)w) (u - w)^2.
\]

Then,

\[
\mathcal{J}(u) \geq \mathcal{J}(w) + \langle \mathcal{J}'(w), u - w \rangle + \frac{1}{2} \mathcal{J}''(u) (u - w)^2 + \left( \frac{1}{2} \mathcal{J}''(tu + (1 - t)w) - \frac{1}{2} \mathcal{J}''(w) \right) (u - w)^2
\]

\[
\geq \mathcal{J}(w) + \langle \mathcal{J}'(w), u - w \rangle + \frac{\delta}{2} \| u - w \|_{L^2(\Omega)}^2 - \left( \frac{1}{2} \mathcal{J}''(tu + (1 - t)w) - \frac{1}{2} \mathcal{J}''(w) \right) (u - w)^2,
\]

where the last inequality is due to (25). Finally, (35) implies

\[
\mathcal{J}(u) \geq \mathcal{J}(w) + \langle \mathcal{J}'(w), u - w \rangle + \frac{\delta}{2} \| u - w \|_{L^2(\Omega)}^2 - L \| u - w \|_{L^2(\Omega)} \| u - w \|_{L^2(\Omega)},
\]

whence, for \( \| u - w \|_{L^2(\Omega)} \leq \epsilon \) for sufficiently small \( \epsilon \), we obtain (36). Since (36) holds for all \( u \in w + \mathcal{C}(w) \), and by definition of \( \mathcal{C} \) we know \( u \in U_{ad} \), we can exchange \( u \) and \( w \) in (36) and get

\[
\mathcal{J}(w) \geq \mathcal{J}(u) + \langle \mathcal{J}'(u), w - u \rangle + \frac{\delta}{4} \| u - w \|_{L^2(\Omega)}^2.
\]

(37)
Adding \[36\] and \[37\] gives, for a fixed but arbitrary \(w \in U_{ad}\),
\[
\langle \mathcal{J}'(u) - \mathcal{J}'(w), u - w \rangle_{L^2(\Omega) \times L^2(\Omega)} \geq \frac{\delta}{2} \| u - w \|^2_{L^2(\Omega)} \quad \forall u \in w + \mathcal{C}(w) \text{ with } \| u - w \|_{L^p(\Omega)} \leq \epsilon.
\]
Setting \(w = \pi\) then yields \[26\].

### 3. Discrete Control Problem

Let \(T\) denote a geometrically conforming, quasi-uniform triangulation of the domain \(\Omega\) such that \(\overline{\Omega} = \cup_{K \in T} K\) with \(K\) closed and \(h\) the meshsize of \(T\). Consider the following finite dimensional spaces
\[
\begin{align*}
Y^h &= \left\{ y_h \in C^0(\Omega) : y_h|_K \in P_1(K), K \in T \right\}, \\
\hat{Y}^h &= Y^h \cap \hat{W}^1_\infty(\Omega), \\
U_{ad}^h &= Y^h \cap U_{ad}.
\end{align*}
\] (38)

The spaces \(U_{ad}^h, Y^h\) will be used to approximate the continuous solution of \([1]\) and \([2]\). The spaces are based on the finite dimensional space \(P_1\), which are the linear polynomials on the domain \(K\), where \(K\) is a triangle. This discretization is classical and can be found in any standard finite element book, for instance \([5, 7]\). We remark that in our numerical implementation the \(L^p\) constraints in \(U_{ad}^h\) are enforced by scaling the functions with their \(L^p\)-norm, we refer to \([4]\) for more details. For the error analysis, we shall need the following. Let \(I_h : W^1_\delta(\Omega) \to Y^h\) be the global interpolation operator, i.e. if \(r > \alpha\) then \(I_h\) is the standard Lagrange interpolation operator, otherwise it indicates the so-called Scott-Zhang interpolation operator \([21]\). Moreover, there exists a constant \(C > 0\) independent of \(h\) and \(w\), such that \(I_h\) satisfies the optimal estimate
\[
|w - I_h w|_{W^1_\delta(\Omega)} \leq C h |w|_{W^2(\Omega)}, \quad \forall w \in W^2_\delta(\Omega) \quad 1 \leq r \leq \infty.
\]

The discrete version of the continuous optimal control problem \([1]\) is
\[
\inf \mathcal{J}_h(y_h, u_h) := \frac{1}{2} \| y_h - y_d \|^2_{L^2(\Omega)} + \frac{\alpha}{2} \| u_h \|^2_{L^2(\Omega)} \quad \text{over } y_h - v \in \hat{Y}^h, \quad u_h \in U_{ad}^h,
\] (39)
subeed to \(y_h - v \in \hat{Y}_h\) solving
\[
\int_{\Omega} \nabla y_h \cdot \nabla z_h = \int_{\Omega} u_h z_h, \quad \text{for all } z \in \hat{Y}^h.
\] (40)

We remark that in \([39]\) and \([40]\), for simplicity, we have not discretized \(y_d\) and \(v\).

The discrete optimality conditions amount to the state \([40]\); the adjoint, find \(\overline{y}_h \in \hat{Y}^h\)
\[
\int_{\Omega} \nabla z_h^T A[\overline{y}_h] \nabla \overline{y}_h = \int_{\Omega} (\overline{y}_h - y_d) z_h, \quad \text{for all } z \in \hat{Y}^h,
\] (41)
where \(A[\overline{y}_h] = \frac{1}{Q(\overline{y}_h)} \left( I - \frac{\nabla \overline{y}_h \nabla \overline{y}_h^T}{Q(\overline{y}_h)} \right)\) and the discrete variational inequality for the optimal control
\[
(\overline{y}_h + \alpha \overline{u}_h, u_h - \overline{u}_h)_{L^2(\Omega), L^2(\Omega)} \geq 0, \quad \text{for all } u_h \in U_{ad}^h.
\] (42)

**Remark 3.1.** Similar to Remark 2.5, the discrete functional derivative is given by \(\mathcal{J}'_h(u_h) = p_h(y_h) + \alpha u_h\) for an arbitrary \(u_h\) in \(U_{ad}^h\), where \(y_h\) solves \([40]\) with \(u_h\) as right-hand-side, and \(p_h(y_h)\) solves \([41]\) with right-hand-side given by \(y_h - y_d\).
**Theorem 3.2** (Existence of the discrete state). If \( h_0 > 0 \) is small enough then for \( 0 < h \leq h_0 \), there exist an open set \( U^h \subset W^{-1}_\infty(\Omega) \) with \( 0 \in U^h_1 \), such that for every \( u^h \in U^h_1 \) there exists a unique solution \( y^h - v \in Y^h \) solving \((40)\).

**Proof.** The proof is similar to [8, Page 351] and is omitted here. \(\square\)

We will discuss the existence of a discrete control in Theorem 3.5, we begin by proving a preliminary but crucial estimate for the optimal control.

**Theorem 3.3** (error estimate control). If \( p(\pi_h) \) solves the continuous adjoint equation \((20)\) with control \( \pi_h \) and \( p^h(\pi_h) \) solves the discrete adjoint equation \((41)\) with control \( \pi_h \) then there exists a constant \( C > 0 \) such that

\[
\| \pi - \pi_h \|_{L^2(\Omega)} \leq C \| p(\pi_h) - p^h(\pi_h) \|_{L^2(\Omega)}.
\]

**Proof.** The proof is based on [4], we only state the key steps here. The idea is to replace \( u \) by \( \pi_h \) in \((21)\) and \( u_h \) by \( P_h \pi \) in \((42)\), where \( P_h \) is the \( L^2 \) orthogonal projection onto \( Y^h \). This gives

\[
\langle J'(\pi), \pi_h - \pi \rangle \geq 0, \quad \langle J'_h(\pi_h), P_h \pi - \pi_h \rangle \geq 0.
\]

Using \((20)\), and replacing \( u \) by \( \pi_h \), we have

\[
\frac{\delta}{2} \| \pi_h - \pi \|^2_{L^2(\Omega)} \leq \langle J'(\pi_h) - J'(\pi), \pi_h - \pi \rangle_{L^2(\Omega) \times L^2(\Omega)}.
\]

Adding and subtracting \( J'_h(\pi_h) \) followed by using first inequality in \((44)\) we obtain

\[
\frac{\delta}{2} \| \pi_h - \pi \|^2_{L^2(\Omega)} \leq \langle J'(\pi_h) - J'_h(\pi_h), \pi_h - \pi \rangle_{L^2(\Omega) \times L^2(\Omega)} + \langle J'_h(\pi_h), \pi_h - \pi \rangle_{L^2(\Omega) \times L^2(\Omega)}.
\]

Adding and subtracting \( P_h \pi \) to \( \pi_h - u \) in the second term, and using the fact that \( P_h \) is an orthogonal projection, we have \( \langle J'_h(\pi_h), P_h \pi - \pi_h \rangle = 0 \). Therefore, invoking the second inequality in \((44)\), we deduce \((43)\) from Remark 3.4 and the Cauchy-Schwarz inequality. \(\square\)

It is clear from Theorem 3.3 that in order to prove the estimate for the control we need to estimate the solution to the continuous and discrete adjoint equations but both at the discrete optimal control \( \pi_h \). In view of \((20)\) and \((41)\), we need to estimate the solution to the continuous state equation \( \pi(\pi_h) \) and the discrete state equation \( \pi_h(\pi_h) \) both at the discrete control \( \pi_h \).

If \( n = 2 \), then the estimate for the state follows directly from [20, Section 4], which is the following lemma.

**Lemma 3.4.** Let \( y(u_h) \) solve the continuous state equation \((2)\) with discrete control \( u_h \) and \( y_h(u_h) \) solve the discrete state equation \((40)\) with discrete control \( u_h \). For \( n = 2 \), there exists \( h_0 > 0 \), and a constant \( C > 0 \) such that for all \( 0 < h \leq h_0 \)

\[
|y(u_h) - y_h(u_h)|_{W^{-1}_\infty(\Omega)} \leq C h |\log h|^4.
\]

**Proof.** In view of the fact that \( y - v \in B \), the proof follows from [20, Section 4]. \(\square\)

Combining \((45)\) with \( y - v \in B \) implies that there exists a constant \( C^* > 0 \) such that

\[
|y_h(u_h)|_{W^{-1}_\infty(\Omega)} \leq C^*.
\]

**Theorem 3.5** (Existence of discrete optimal control). For \( n = 2 \), there exist \( h_0 > 0 \) such that for \( 0 < h \leq h_0 \), there exist an optimal control solving \((39)\).
Proof. The proof is based on a minimizing sequence argument similar to Theorem 2.1. However, weak convergence of a minimizing sequence \( u_{h,n} \) yields strong convergence in finite dimensional spaces. Invoking Theorem 3.2, corresponding to \( u_{h,n} \) there exists a unique state \( y_{h,n} - v \in Y^h \subset W^1_\infty(\Omega) \) satisfying (46) i.e. \( y_{h,n}(u_{h,n}) \) is bounded on the finite dimensional space \( Y^h \) and therefore has a strongly convergent subsequence. □

**Lemma 3.6** (error estimate adjoint). The following estimate holds:

\[
|p - p_h|_{W^2_\Omega} \leq C \left( h \|p\|_{W^2_\Omega} + |y - y_h|_{W^1_\infty(\Omega)} \right).
\]

Proof. Using the discrete inf-sup condition from [5] Proposition 8.6.2 and \( I_h \), we have

\[
|p_h - I_h p|_{W^2_\Omega} \leq C \sup_{z \in Y_h} \frac{\int_\Omega \nabla z^T A(y_h) \nabla (p_h - I_h p)}{|z|_{W^1_\Omega}}.
\]

In view of (41) we obtain

\[
|p_h - I_h p|_{W^2_\Omega} \leq C \sup_{z \in Y_h} \frac{\int_\Omega (y_h - y_d) z - \nabla z^T A(y_h) \nabla I_h p}{|z|_{W^1_\Omega}}
\]

\[
= C \sup_{z \in Y_h} \frac{\int_\Omega (y_h - y_d) z - (y - y_d) z + \nabla z^T A(y) \nabla p - \nabla z^T A(y_h) \nabla I_h p}{|z|_{W^1_\Omega}}
\]

where the last equality follows immediately using (20). Invoking Cauchy-Schwarz we readily obtain

\[
|p_h - I_h p|_{W^2_\Omega} \leq C \left( \|y_h - y\|_{W^{2-1}_\infty(\Omega)} + \|A(y)\|_{L^\infty(\Omega)} |p - I_h p|_{W^2_\Omega} \right.
\]

\[
\left. + \|A(y) - A(y_h)\|_{L^\infty(\Omega)} |I_h p|_{W^2_\Omega} \right).
\]

In view of Lemma 2.11 we deduce

\[
|p_h - I_h p|_{W^2_\Omega} \leq C \left( h \|p\|_{W^2_\Omega} + |y - y_h|_{W^1_\infty(\Omega)} \right).
\]

The estimate (47) follows readily using triangle inequality. □

**Corollary 3.7.** Let \( n = 2 \) and \( h_0 \) be sufficiently small. Furthermore, let \( \overline{p}(\overline{\pi}_h) \) be the solution of the continuous adjoint equation [20] and \( \overline{y}(\overline{\pi}_h) \) the solution of the continuous state equation [4] with control \( \overline{\pi}_h \). Furthermore, let \( \overline{p}_h(\overline{\pi}_h) \) be the solution of the discrete adjoint equation (41) and \( \overline{y}_h(\overline{\pi}_h) \) the solution of the discrete state equation (40) with control \( \overline{\pi}_h \). If \( h \leq h_0 \), then there is a constant \( C \geq 1 \) depending on \( \|\overline{y}\|_{W^2_\Omega}, \|\overline{y}_h\|_{W^2_\Omega}, \|y_d\|_{L^p(\Omega)}, \) such that

\[
|\overline{p}(\overline{\pi}_h) - \overline{p}_h(\overline{\pi}_h)|_{W^2_\Omega} + |\overline{y}(\overline{\pi}_h) - \overline{y}_h(\overline{\pi}_h)|_{W^2_\Omega} + \delta\|\overline{\pi} - \overline{\pi}_h\|_{L^2(\Omega)} \leq Ch \log h^4.
\]

Proof. Using the estimate

\[
\|\overline{p}(u_h) - \overline{p}_h(u_h)\|_{L^2(\Omega)} \leq C_0 |\overline{p}(u_h) - \overline{p}_h(u_h)|_{W^2_\Omega}
\]

with Lemma 3.6 in conjunction with Lemma 3.4 we deduce

\[
|\overline{p}(\overline{\pi}_h) - \overline{p}_h(\overline{\pi}_h)|_{W^2_\Omega} \leq C^* h \log h^4
\]
with constant $C^*$ having the same dependencies as $C$. This together with (43) implies the estimate for the control $\|\overline{u} - \overline{u}_h\|_{L^2(\Omega)}$. The remaining estimates follow immediately using Lemma 3.4 and Lemma 3.6 with $u = u_h = \overline{u}_h$. □

4. Numerical Examples

4.1. Setup

We present numerical examples for the discrete optimal control problem in Section 3. We solve the optimization problem using MATLAB’s optimization toolbox with an SQP method, where we provide the gradient information.

The gradient of the cost functional (39), at each iteration of the optimization algorithm, is computed by first solving the state equation (40) for $y_h$ with the control $u_h$ taken from the previous iteration. Then, the adjoint problem (41) is solved for $p_h$ using the discrete solution $y_h$. We then define the linear form (see Remark 3.1)

$$\langle J'_h(u_h), v_h \rangle_{L^2(\Omega)} = \int_{\Omega} (p_h + \alpha u_h) v_h, \quad \text{for all } v_h \in Y^h,$$

and pass the discrete gradient vector (and cost value) to MATLAB’s optimization algorithm at the current iteration. The constraint on the control $U^h_{ad}$ is handled by MATLAB’s optimization algorithm by specifying an inequality constraint on $u_h$.

The non-linear state equation is solved with Newton’s method and a direct solver (backslash); we also use a direct solver for the adjoint problem. This was all implemented in MATLAB using the FELICITY toolbox [24].

The following sections show some examples of our computational method. In all cases, we set $\alpha = 10^{-6}$ and $\theta = 20$ in the definition of $U^h_{ad}$. We chose $\theta$ large in order to make the simulation results more interesting. The first two examples are posed on a unit square domain, which technically does not satisfy the $C^{1,1}$ domain assumption. The last example is posed on a $C^\infty$ domain in the shape of a four-leaf clover.

4.2. Sine On A Square

We take $y_d$ to be a product of sine functions and set the boundary data to $v = 0$. The domain $\Omega$ is the unit square. See Figures 1 and 2 for plots of $y_d$, $\overline{y}$, $\overline{u}$, and the optimization history. This example shows that we can recover the desired surface almost exactly when the boundary condition $v$ matches $y_d$ on $\partial \Omega$.

4.3. Gaussian On A Square (Nonzero Boundary Condition)

We take $y_d$ to be a Gaussian bump and set the boundary data to $v = -0.1 \sin(\pi x) \cos(2\pi y)$. The domain $\Omega$ is the unit square. See Figures 3 and 4 for plots of $y_d$, $\overline{y}$, $\overline{u}$, and the optimization history. In this case, we impose a mismatch between the imposed boundary condition $v$ and the desired surface $y_d$. The results show that the optimization does the “best it can” by trying to match $y_d$ in the interior of $\Omega$. Note the large value of the control $\overline{u}$ at the boundary of $\Omega$ in Figure 4.

4.4. Cosine On A Clover

We take $y_d$ to be a product of cosine functions and set the boundary data to $v = 0$. The domain $\Omega$ is a four-leaf clover (smooth domain). See Figures 5 and 6 for plots of $y_d$, $\overline{y}$, $\overline{u}$, and the optimization history. This example also has a mismatch between the imposed boundary condition $v$ and $y_d$. Again, the optimal surface $\overline{y}$ matches $y_d$ well in the interior of $\Omega$, but not at the boundary. Moreover, in Figure 7 it is evident from the convergence history of the optimization algorithm that the path to the optimal control is non-trivial.
Figure 1. Left: Desired surface height \( y_d = \sin(2\pi x) \sin(2\pi y) \). Right: Actual surface height \( \bar{y} \) (after the optimization method converges). Boundary data is \( v = 0 \).

Figure 2. Left: Optimal control function \( \pi \) for \( y_d \) in Figure 1. Right: Optimization history.

5. CONCLUSION AND FUTURE WORK

The mean curvature operator is only locally-coercive, which leads to several difficulties in proving the existence of solution to the PDE. Using two approaches (i) implicit function theorem and (ii) fixed point iteration, we provide a complete second order analysis to this PDE. The fixed point approach (ii) requires a boundary data smallness condition but no such assumption in needed in (i). We handle (ii) by proving various Fréchet differentiability results and deal with (i) by proving a new result for second order elliptic PDEs in non-divergence form, where the lower order coefficients need not be bounded; for the bounded coefficient case see [13, Theorem 9.15].
Figure 3. Left: Desired surface height $y_d = 0.1 \exp \left( -((x - 0.5)^2 + (y - 0.5)^2)/0.1 \right)$. Right: Actual surface height $\overline{y}$ (after the optimization method converges). Boundary data is $v = -0.1 \sin(\pi x) \cos(2\pi y)$.

Figure 4. Left: Optimal control function $u$ for $y_d$ in Figure 3. Right: Optimization history.

By using the regularity results for the PDE, we rigorously justify the first and second order sufficient optimality conditions and further tackle the $2$-norm discrepancy in the $L^p - L^2$ pair. We remark that the standard $2$-norm discrepancy results are for the $L^\infty - L^2$ pair. We discretize the PDE using a finite element method and prove quasi-optimal error estimates for the optimal control.
Figure 5. Left: Desired surface height $y_d = 0.1 \cos(2\pi x) \cos(2\pi y)$. Right: Actual surface height $\bar{y}$ (after the optimization method converges). Boundary data is $v = 0$.

Figure 6. Left: Optimal control function $u$ for $y_d$ in Figure 5. Right: Optimization history.

There are some possible extensions of this work. The first could be boundary control. The second is where the surface tension coefficient $K \in \mathbb{R}^{n \times n}$ in the operator

$$- \text{div} K \frac{\nabla y}{Q(y)}$$

acts as an optimal control, and the right-hand-side $u$ acts as a driving force.
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