Nonlocal Reductions of a Generalized Heisenberg Ferromagnet Equation

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Abstract

We study nonlocal reductions of coupled equations in 1 + 1 dimensions of the Heisenberg ferromagnet type. The equations under consideration are completely integrable and have a Lax pair related to a linear bundle in pole gauge. We describe the integrable hierarchy of nonlinear equations related to our system in terms of generating operators. We present some special solutions associated with four distinct discrete eigenvalues of the scattering operator. Using Lax pair diagonalization method, we derive recurrence formulas for the conserved densities and find the first two simplest conserved densities.

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1 Introduction

Heisenberg ferromagnet equation (HF)

\[ S_t = S \times S_{xx} \]

for the three dimensional vector \( S(x, t) \) of unit length is one of classical soliton equations \([2, 5, 13]\). Above, "\( \times \)" stands for the standard cross product in the three dimensional Euclidean space and the subscripts denote partial derivatives with respect to the variables \( t \) and \( x \). The vector equation (1) has been subject to various generalizations, e.g. the following coupled system of nonlinear evolution equations (NEEs)

\[
\begin{align*}
    i u_t + u_{xx} + [(\varepsilon u \overline{u}_x + v \overline{v}_x)u]_x &= 0, \\
    i v_t + v_{xx} + [(\varepsilon u \overline{u}_x + v \overline{v}_x)v]_x &= 0,
\end{align*}
\]

was proposed as an integrable generalization of HF, see \([7,14,15,17,18]\). Above, overlining means complex conjugation and "i" is the imaginary unit. The dynamical fields \( u \) and \( v \) are required to satisfy the additional condition

\[ \varepsilon |u|^2 + |v|^2 = 1 \]

which generalizes in a natural way the Lax pair of (1), see \([13]\). Another possible way to generalize (1) is by constructing multidimensional NEEs. Examples of 2 + 1 dimensional generalizations of HF are given by Ishimori’s equation \([9]\)

\[ S_t = S \times (S_{xx} + S_{yy}) + u_x S_y + u_y S_x \]

\[ u_{xx} - u_{yy} = -2 S.(S_x \times S_y) \]

and Myrzakulov I equation \([11,12]\)

\[ S_t = S \times S_{xy} + u S_x, \quad u_x = -S.(S_x \times S_y) \]

In both equations above, \( S \) is a three dimensional vector of unit length, \( u \) is a real valued function and "dot" denotes the usual scalar product of vectors in the three dimensional Euclidean space.

A recent trend in the theory of integrable systems, initiated by Ablowitz and Muslaimani \([1]\), is the study of nonlocal reductions of NEEs. Nowadays, finding nonlocal
counterparts corresponding to well-known (local) NEEs enjoys an ever increasing inter-
rest, see [4, 6, 8] and references therein. Our main purpose here is to study a nonlocal
counterpart of (2), namely the following system of NEEs

\[
\begin{align*}
    iu_t(x, t) + u_{xx}(x, t) + \left\{ \varepsilon u(x, t)u_{xx}(-x, t) + v(x, t)v_{xx}(-x, t) \right\} u(x, t) & = 0, \\
    iv_t(x, t) + v_{xx}(x, t) + \left\{ \varepsilon u(x, t)u_{xx}(-x, t) + v(x, t)v_{xx}(-x, t) \right\} v(x, t) & = 0
\end{align*}
\]

for the dynamical fields \( u \) and \( v \). That coupled system is solvable through inverse scattering
transform, i.e. it has a Lax representation, soliton type solutions, an infinite number of
conservation laws etc.

The text of the preprint is organized in the following manner. In next section, we
shall introduce the Lax pair of (7), then make some remarks on the structure of the
Lax operators and their symmetries (reductions). The third section is dedicated to the
integrable hierarchy of NEEs related to the coupled system under consideration. Starting
from a general flow Lax pair, we shall show how the hierarchy can be described in terms
of recursion operators. Section 4 contains discussion on how we can integrate (7) by
using dressing method [19, 20] (Darboux transformation method). Following the algorithm
described in [17] for the case of linear bundles in pole gauge, allows us to construct a
simple class of special solutions over constant background. In Section 5 we describe
an algorithm to derive the integrals of motion of (7). For that purpose we shall apply
Lax pair diagonalization method [3] that will allow us to find a recursive formula for the
 corresponding conserved densities. Section “Conclusion” contains our final remarks and
some further discussion on our results.

2 Lax Representation

As we mentioned, the system of nonlocal equations (7) is integrable through inverse scat-
ttering transform. Here we shall pay certain attention to its Lax representation. This is
the reason why we shall briefly remind the reader the notion of (nonlocal) reduction thus
following [10, 16].

Let us introduce the Lax operators

\[
\begin{align*}
    L(\lambda) & = i\partial_x - \lambda S(x, t), \quad \lambda \in \mathbb{C}, \\
    A(\lambda) & = i\partial_t + \sum_{j=1}^{N} \lambda^j A_j(x, t), \quad N \geq 2
\end{align*}
\]

where all the coefficients \( S(x, t) \) and \( A_j(x, t), \ j = 1, 2, \ldots, N \) are some complex traceless
\( 3 \times 3 \) matrices. Let us denote by \( \mathcal{F} \) the space of all the fundamental sets of solutions to
the auxiliary linear problem

\[
i\partial_x \Psi(x, t, \lambda) - \lambda S(x, t)\Psi(x, t, \lambda) = 0.
\]

Since \( S(x, t) \) has a zero trace, any fundamental solution \( \Psi(x, t, \lambda) \) is unimodular, i.e. we
have \( \det \Psi(x, t, \lambda) = 1 \). Assume now a finite group \( G_R \) acts on \( \mathcal{F} \) as follows:

\[
\mathcal{K}_g : \Psi(x, t, \lambda) \rightarrow \tilde{\Psi}(x, t, \lambda) = K_g \left[ \Psi \left( k_g^{-1}(x, t), k_g^{-1}(\lambda) \right) \right], \quad g \in G_R
\]

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where \( \kappa_g : \mathbb{R}^2 \to \mathbb{R}^2 \) is a smooth, invertible mapping; \( k_g : \mathbb{C} \to \mathbb{C} \) is a conformal mapping and \( K_g \) is a group automorphism of the Lie group \( \text{SL}(3, \mathbb{C}) \). \( G_R \) transforms the Lax operators according to

\[
L(\lambda) \to \tilde{L}(\lambda) = \kappa_g \circ L(\lambda) \circ \kappa_g^{-1}, \\
A(\lambda) \to \tilde{A}(\lambda) = \kappa_g \circ A(\lambda) \circ \kappa_g^{-1}.
\]

(12) (13)

In view of the equality

\[
[\tilde{L}(\lambda), \tilde{A}(\lambda)] = \kappa_g \circ [L(\lambda), A(\lambda)] \circ \kappa_g^{-1}
\]

the transformed operators \( \tilde{L}(\lambda) \) and \( \tilde{A}(\lambda) \) still commute.

The requirement that (10) is \( G_R \)-invariant, implies that

\[
\tilde{L}(\lambda) \propto L(\lambda)
\]

which imposes certain symmetry condition on the matrix coefficient \( S(x, t) \). Effectively, such condition decreases (reduces) the number of independent entries of \( S(x, t) \), i.e. the dynamical fields. Similar argument holds for the coefficients of the operator \( A(\lambda) \) since those can be expressed through \( S \) and its \( x \)-derivatives. Due to all this, \( G_R \) is called reduction group while (11) or equivalently the afore-mentioned symmetries of \( S(x, t) \) and \( A_j(x, t) \), are called nonlocal reduction.

**Remark 1** The term "nonlocal" refers to the fact that \( G_R \) acts on the independent variables so the zero curvature condition leads to a NEE having nonlocal terms. In the particular case when \( \kappa_g = \text{id}_{\mathbb{R}^2} \), \( \forall g \in G_R \) reduction is called local.

In order to illustrate the above ideas, let us consider the following reduction

\[
\Psi(x, t, \lambda) \to \tilde{\Psi}(x, t, \lambda) = H \Psi(x, t, -\lambda) H, \quad H = \text{diag} (-1, 1, 1).
\]

(14)

So we require that for any \( \Psi \in \mathcal{F}, \tilde{\Psi} \), as given in (14), is another fundamental solution to (10). Since (14) involves involutions only \((H^2 = 1, 1 \text{ is the identity matrix})\), it defines an action of the group \( Z_2 \). From (14) one immediately gets

\[
HS(x, t)H = -S(x, t), \quad HA_j(x, t)H = (-1)^j A_j(x, t), \quad j = 1, 2, \ldots, N.
\]

(15)

Taking into account the explicit form of \( H \), we deduce from (15) that the matrix coefficients of the Lax pair must have the following block structure:

\[
S = \begin{pmatrix} 0 & * & * \\
* & 0 & 0 \\
* & 0 & 0 \end{pmatrix}, \quad A_{2k-1} = \begin{pmatrix} 0 & * & * \\
* & 0 & 0 \\
* & 0 & 0 \end{pmatrix}, \quad A_{2k} = \begin{pmatrix} * & 0 & 0 \\
0 & * & * \\
0 & * & * \end{pmatrix}.
\]

Let us now impose another \( Z_2 \)-reduction of the form:

\[
\Psi(x, t, \lambda) \to \tilde{\Psi}(x, t, \lambda) = \mathcal{E} \left[ \Psi^\dagger(-x, t, -\lambda) \right]^{-1} \mathcal{E}
\]

(16)

where \( \mathcal{E} = \text{diag} (1, \varepsilon, 1), \varepsilon^2 = 1 \) and the symbol "\( \dagger \)" is for Hermitian conjugation. In that case we have the following symmetry conditions:

\[
\mathcal{E} S^\dagger(-x, t) \mathcal{E} = S(x, t), \quad \mathcal{E} A_j^\dagger(-x, t) \mathcal{E} = (-1)^j A_j(x, t).
\]

(17)
The reductions (14) and (16) commute so they both can be viewed as an action of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. After taking into account (14) and (16), the matrix $S(x, t)$ can be written down as:

$$S(x, t) = \begin{pmatrix} 0 & u(x, t) & v(x, t) \\ \varepsilon u(-x, t) & 0 & 0 \\ v(-x, t) & 0 & 0 \end{pmatrix}$$

for some complex valued functions $u$ and $v$. From that point on we shall assume that $u$ and $v$ are not entirely independent functions but satisfy the following nonlocal constraint:

$$\varepsilon u(x, t) u(-x, t) + v(x, t) v(-x, t) = 1.$$

Such constraint is essential when one tries to employ inverse scattering transform to the Lax pair under consideration, see [7] for more explanations.

**Remark 2** We note that in view of (19) the matrix-valued function $S$ satisfies:

$$S^3 = S. \quad (20)$$

Hence $S$ can be put into diagonal form and its eigenvalues are $-1$, $0$ and $1$.

Let’s now restrict ourselves with the simplest nontrivial case of quadratic flow Lax pair

$$A(\lambda) = i \partial_t + \lambda A_1(x, t) + \lambda^2 A_2(x, t). \quad (21)$$

Taking into account (18) and (19), it is not hard to check that the compatibility condition $[L(\lambda), A(\lambda)] = 0$ of (8) and (21) determines the matrix coefficients of (21) to be

$$A_1(x, t) = \begin{pmatrix} 0 & a(x, t) & b(x, t) \\ -\varepsilon a(-x, t) & 0 & 0 \\ -b(-x, t) & 0 & 0 \end{pmatrix},$$

$$a(x, t) = -u_x(x, t) - iB(x, t)u(x, t), \quad b(x, t) = -iv_x(x, t) - iB(x, t)v(x, t),$$

$$B(x, t) = \varepsilon u(x, t) u_x(-x, t) + v(x, t) v_x(-x, t),$$

$$A_2(x, t) = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 2/3 - \varepsilon u(-x, t) u(x, t) & -\varepsilon u(-x, t) v(x, t) \\ 0 & -v(-x, t) u(x, t) & 2/3 - v(-x, t) v(x, t) \end{pmatrix}. \quad (22)$$

Moreover, $u$ and $v$ must solve the following coupled system of NEEs:

$$i u_t(x, t) + u_{xx}(x, t) + [B(x, t) u(x, t)]_x = 0, \quad (23)$$

$$i v_t(x, t) + v_{xx}(x, t) + [B(x, t) v(x, t)]_x = 0. \quad (24)$$

That nonlocal system represents the main object of interest in the current preprint. As it is clear from the derivation itself, (23) and (24) represent the simplest member of an infinite family of nonlocal NEEs each corresponding to an operator $A(\lambda)$ of particular degree.\(^2\) That family (hierarchy) will be described in the section to follow.

\(^1\)Compare these expressions with those in the local case, see (1), (5) and (6).

\(^2\)The reader may notice that we ignored here the case when $\deg(A(\lambda)) = 1$. We did it intentionally because the linear flow yields to the following system of linear wave equations

$$u_t + cu_x = 0, \quad v_t + cv_x = 0, \quad c \in \mathbb{R}$$

that is not interesting in the context of inverse scattering transform and its applications.
3 Integrable Hierarchies

This section is dedicated to a description of the hierarchy of integrable NEEs associated with coupled system (23) and (24). Below we shall follow ideas and methods discussed in more detail in [5, 7].

Let us consider again the general Lax operators (8) and (9) and require the conditions (15) and (17) hold true. Then \( S(x, t) \) is given by the matrix (18). We assume that the constraint (19) is satisfied by the dynamical fields \( u \) and \( v \).

Let us analyze the compatibility condition \([L(\lambda), A(\lambda)] = 0\) of (8) and (9). After comparing the coefficients before the same powers of \( \lambda \), we derive the following string of differential relations:

\[
\begin{align*}
\lambda^{N+1} : & \quad [S, A_N] = 0, \quad (25) \\
\lambda^N : & \quad i\partial_x A_N - [S, A_{N-1}] = 0, \quad (26) \\
\lambda^j : & \quad i\partial_x A_j - [S, A_{j-1}] = 0, \quad j = 2, \ldots, N-1, \quad (27) \\
\lambda : & \quad \partial_x A_1 + \partial_t S = 0 \quad (28)
\end{align*}
\]

for the coefficients of the second Lax operator \( A(\lambda) \). For any integer \( N \geq 2 \), these recurrence relations yield to some NEE. This way we have a whole family of NEEs, called integrable hierarchy. In order to resolve (25)–(28), we start from the highest degree equality. Equation (25) tells us that for the highest degree coefficient we have:

\[
A_N = \begin{cases} 
  c_N S, & N \equiv 1 \pmod{2} \\
  c_N S_1, & N \equiv 0 \pmod{2}
\end{cases} \quad (29)
\]

where \( c_N \in \mathbb{R} \) and

\[
S_1 = S^2 - \frac{2}{3} I.
\]

Further, the structure of the recurrence relations hints to the following, rather natural splitting

\[
A_l = A^a_l + A^d_l, \quad l = 1, \ldots, N-1 \quad (30)
\]

of the matrix coefficients of the second Lax operator into \( S \)-commuting term \( A^d_l \) and some remainder \( A^a_l \), see [7,15]. For the \( S \)-commuting term we have

\[
A^d_l = \begin{cases} 
  a_l S_1, & l \equiv 0 \pmod{2} \\
  a_l S, & l \equiv 1 \pmod{2}
\end{cases} \quad (31)
\]

Above \( a_l, l = 1, \ldots, N-1 \) are some scalar functions to be determined from the relations (27) and (28).

**Remark 3** Since \( S(x, t) \) is a diagonalizable matrix (see Remark 2) we can deduce that the adjoint operator \( \text{ad}_S \) is diagonalizable too — its spectrum consists of \( 0, \pm 1, \pm 2 \). This is why we can define the inverse operator of \( \text{ad}_S \) on the \( S \)-noncommuting terms \( A^a_l \) as follows [7,15]:

\[
\text{ad}^{-1}_S A^a_l = \frac{1}{4} \left( 5 \text{ad}_SA^a_l - \text{ad}^3_S A^a_l \right).
\]

Now, let us have a more detailed look of the equation (27). After substituting (30) into (27) and taking into account that \( (S_x)^d = (S_1x)^d = 0 \), we obtain

\[
(\partial_x A^a_j)^d = -\begin{cases} 
  \partial_x a_j S, & j \equiv 1 \pmod{2} \\
  \partial_x a_j S_1, & j \equiv 0 \pmod{2}
\end{cases}, \quad j = 2, \ldots, N-1 \quad (32)
\]
for the $S$-commuting part and

$$[S, A^a_{j-1}] - i (\partial_x A^a_j)^a = \begin{cases} ia_j S, & j \equiv 1 \pmod{2} \\ ia_j S_1, & j \equiv 0 \pmod{2} \end{cases} \quad (33)$$

for the not commuting one. First, we solve (32) making use of the normalization relations

$$\text{tr} S^2 = 2, \quad \text{tr} S^2_1 = \frac{2}{3}.$$

The result for $a_j$ reads:

$$a_j = c_j - \begin{cases} \frac{1}{2} \partial_x^{-1} \text{tr} S (\partial_x A_j)^a, & j \equiv 1 \pmod{2} \\ \frac{3}{2} \partial_x^{-1} \text{tr} S_1 (\partial_x A_j)^a, & j \equiv 0 \pmod{2} \end{cases} \quad (34)$$

where the symbol $\partial_x^{-1}$ stands for any right inverse of the operator of partial differentiation in variable $x$ and $c_j \in \mathbb{R}$ is an integration constant. After substituting (34) into (33), we obtain

$$A^a_{j-1} = \begin{cases} \Lambda A^a_j + ic_j \text{ad} \partial_x^{-1} S_1, & j \equiv 0 \pmod{2} \\ \Lambda A^a_j + ic_j \text{ad} S_1, & j \equiv 1 \pmod{2} \end{cases} \quad (35)$$

where $\Lambda$ is an integro-differential operator defined as follows:

$$\Lambda \overset{\text{def}}{=} \text{iad} \partial_x^{-1} \left\{ [\partial_x(\cdot)]^a - \frac{S_x}{2} \partial_x^{-1} \text{tr} S (\partial_x(\cdot))^a - \frac{3S_1}{2} \partial_x^{-1} \text{tr} S_1 (\partial_x(\cdot))^a \right\}. \quad (36)$$

The $\Lambda$-operator introduced in (35) acts on the $S$-non commuting part of $A_j$ only but its action can formally be extended on the $S$-commuting part as well by requiring

$$\Lambda S \overset{\text{def}}{=} \text{iad} S_1, \quad \Lambda S_1 \overset{\text{def}}{=} \text{iad} S_1 \partial_x^{-1} S_1.$$

Taking into account (29) and (35), one can verify that a NEE belonging to the integrable hierarchy under consideration is generated through the equation

$$\text{iad} S_1 S_t + \sum_j c_{2j} \Lambda^{2j} S_1 + \sum_j c_{2j-1} \Lambda^{2j-1} S = 0. \quad (37)$$

The operator $\Lambda^2$ bear the name generating (recursion) operator of the integrable hierarchy. It can be checked that (37) leads to system (23) and (24) after setting $N = 2$, $c_2 = -1$ and $c_1 = 0$. Thus, the system (23) and (24) is the simplest nontrivial member of the family (37) indeed.

### 4 Special Solutions

Here, we shall obtain some special solutions to the nonlocal equations (23) and (24) in explicit form. In doing this, we shall restrict ourselves with the class of trivial background solutions satisfying the boundary condition:

$$\lim_{|x| \to \infty} u(x,t) = 0, \quad \lim_{|x| \to \infty} v(x,t) = 1. \quad (38)$$
Let us introduce the commuting operators

\[ L_0(\lambda) = i\partial_x - \lambda S^{(0)}, \quad S^{(0)}(x,t) = \begin{pmatrix} 0 & u_0(x,t) & v_0(x,t) \\ \varepsilon u_0(-x,t) & 0 & 0 \\ v_0(-x,t) & 0 & 0 \end{pmatrix}, \]

\[ A_0(\lambda) = i\partial_t + \lambda A^{(0)}_1 + \lambda^2 A^{(0)}_2, \quad \lambda \in \mathbb{C}. \]

Above, \((u_0, v_0)\) is a pair of known dynamical fields solving the system \((23)\) and \((24)\) and fulfilling the additional constraint \((19)\) and the boundary condition \((38)\). Assume now that the \(3 \times 3\)-matrix valued function \(\Psi_0\) (\(\det \Psi_0(x,t,\lambda) = 1\)) solves the problems

\[ L_0(\lambda)\Psi_0(x,t,\lambda) = 0 \]

\[ A_0(\lambda)\Psi_0(x,t,\lambda) = \Psi_0(x,t,\lambda)f(\lambda) \]

to be called further in text bare problems. The polynomial

\[ f(\lambda) = -\frac{\lambda^2}{3}\text{diag}(1,-2,1), \]

appearing in the second bare problem, represents the dispersion law of the system \((23)\) and \((24)\). We refer the reader to \([15, 17]\) for more detailed explanations.

Let us denote the set of all bare fundamental solutions by \(\mathcal{F}_0\). For any \(\Psi_0 \in \mathcal{F}_0\) we construct new \(3 \times 3\)-matrix valued function \(\Psi_1 = G\Psi_0\), where the unimodular \(3 \times 3\)-matrix \(G(x,t,\lambda)\) is called dressing factor. The set of all such \(3 \times 3\)-matrix valued functions will be denoted by \(\mathcal{F}_1\). In a natural way dressing transform induces an action on the Lax operators

\[ L_0 \rightarrow L_1 = G L_0 G^{-1}, \quad A_0 \rightarrow A_1 = G A_0 G^{-1}. \]

It is easily seen that \([L_1, A_1] = 0\) holds true. For dressing transform to be useful in constructing new solutions to \((23)\) and \((24)\), we assume that any \(\Psi_1 \in \mathcal{F}_1\) is a fundamental solution to

\[ L_1(\lambda)\Psi_1(x,t,\lambda) = 0, \]

\[ A_1(\lambda)\Psi_1(x,t,\lambda) = \Psi_1(x,t,\lambda)f(\lambda), \]

where the new Lax pair reads:

\[ L_1(\lambda) = i\partial_x - \lambda S^{(1)}, \quad S^{(1)}(x,t) = \begin{pmatrix} 0 & u_1(x,t) & v_1(x,t) \\ \varepsilon u_1(-x,t) & 0 & 0 \\ v_1(-x,t) & 0 & 0 \end{pmatrix}, \]

\[ A_1(\lambda) = i\partial_t + \lambda A^{(1)}_1 + \lambda^2 A^{(1)}_2. \]

The dynamical fields \(u_1\) and \(v_1\) above are some yet unknown solutions to \((23)\) and \((24)\) to be determined from the bare dynamical fields \(u_0\) and \(v_0\).

It is not hard to convince ourselves that dressing factor fulfills the pair of partial differential equations:

\[ i\partial_t G - \lambda \left(S^{(1)} G - G S^{(0)}\right) = 0, \]

\[ i\partial_t G + \sum_{k=1,2} \lambda^k \left(A^{(1)}_k G - G A^{(0)}_k\right) = 0 \]
that directly follows from (41), (42) and (45). Imposing certain natural requirements of
regularity of dressing factor, see [17], that system leads to the interrelation

$$S^{(1)} = S_{\infty}^{0} S_{\infty}^{-1}, \quad S_{\infty}(x, t) \overset{\text{def}}{=} \lim_{|\lambda| \to \infty} \mathcal{G}(x, t, \lambda)$$

(50)

between the bare solution $S^{(0)}$ and the dressed one $S^{(1)}$, allowing one to determine $(u_1, v_1)$
from $(u_0, v_0)$.

Further on, we shall use a dressing factor that is a rational function of the form:

$$G(x, t, \lambda) = \frac{1}{1 + \sum q \left[ \frac{\lambda B_q(x, t)}{\mu_q(\lambda - \mu_q)} + \frac{\lambda H B_q(x, t) H}{\mu_q(\lambda + \mu_q)} \right]}, \quad \mu^2_q \not\in \mathbb{R}. \quad (51)$$

That form is consistent with the symmetry relations:

$$H G(x, t, -\lambda) H = G(x, t, \lambda), \quad (52)$$

$$\mathcal{E} G(x, t, -\lambda) = [G(x, t, \lambda)]^{-1} \quad (53)$$

which follow from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ reductions imposed on the bare and the dressed fundamental
solutions, see (14) and (16).

The equation (50) tells us that we can obtain $S^{(1)}$ if we know the residues of the
dressing factor (51). In order to find $B_q(x, t)$, we first consider the identity

$$G(x, t, \lambda) G^{-1} = \mathbb{I}.$$ 

Taking into account (53), the matrix inverse of (51) is given by

$$[G(x, t, \lambda)]^{-1} = \mathbb{I} + \sum_p \left[ \frac{\lambda \mathcal{E} B_p(-x, t) \mathcal{E}}{\bar{\mu}_p(\lambda + \bar{\mu}_p)} + \frac{\lambda \mathcal{E} H B_p(-x, t) H \mathcal{E}}{\mu_q(\lambda - \bar{\mu}_p)} \right]. \quad (54)$$

Generally speaking $G$ and $G^{-1}$ have different simple poles. After evaluating the left hand
side of $\lim_{\lambda \to \bar{\mu}_p} (\lambda - \bar{\mu}_p) G G^{-1} = 0$, we derive the equations

$$\left[ \mathbb{I} + \sum q \left( \frac{\bar{\mu}_p B_q(x, t)}{\mu_q(\bar{\mu}_p - \mu_q)} + \frac{\bar{\mu}_p H B_q(x, t) H}{\mu_q(\bar{\mu}_p + \mu_q)} \right) \right] \mathcal{E} H B_p(-x, t) H \mathcal{E} = 0 \quad (55)$$

for the residues $B_p$. A similar evaluation of the residues at the poles $\pm \mu_p$ and $-\bar{\mu}_p$ does
not lead to essentially new equations this is why we shall ignore them. The equation (55)
implies that the residues of $G(x, t, \lambda)$ must be singular matrices, so the factorization

$$B_p(x, t) = X_p(x, t) F_p^T(x, t) \quad (56)$$

holds for each of them. Above, $X_p(x, t)$ and $F_p(x, t)$ are two rectangular matrices and
the superscript "$T$" stands for matrix transposition. Using the factorization (56), we can reduce (54) to

$$\mathcal{E} H F_p(-x) = \bar{\mu}_p \sum_q \left( X_q(x) \frac{F_q^T(x) \mathcal{E} H F_p(-x)}{\mu_q(\mu_q - \bar{\mu}_p)} - H X_q(x) \frac{F_q^T(x) \mathcal{E} F_p(-x)}{\mu_q(\bar{\mu}_p + \mu_q)} \right) \quad (57)$$
which is viewed as a linear system for $X_q$. The linear system (57) is very easily solved when (51) has just two simple poles and $X$, $F$ are column-vectors. In that simplest case the result for $X$ is given by:

$$X(x) = \left( \frac{\mu F^T(x) \mathcal{E} H F(-x)}{\mu (\mu - \mu_1)} - \frac{\mu F^T(x) \mathcal{E} H F(-x)}{\mu (\mu + \mu_1)} \right)^{-1} \mathcal{E} H F(-x).$$

(58)

Next, we analyze the differential equation (58) rewritten as

$$\lambda S^{(1)} = i \partial_x \mathcal{G}^{-1} + \lambda \mathcal{G} S^{(0)} \mathcal{G}^{-1}.$$  \hspace{1cm} (59)

It is not hard to prove \[17\] that the rectangular matrix $F_p(x, t)$ satisfies the rather simple partial differential equation

$$i \partial_x F_p^T = -\mu_p F_p^T S^{(0)}$$

(60)

which leads us to the conclusion that

$$F_p^T(x, t) = F_p^T(t) \Psi_0(x, t, \mu_p)^{-1}.$$  \hspace{1cm} (61)

for some matrix-valued functions $F_{p,0}$ of the variable $t$ only. It can be shown \[17\] that the time evolution of $F_{p,0}$ is driven by the linear differential equation

$$i \partial_t F_{p,0} = F_{p,0}^T \Psi_0(x, t, \mu_p)^{-1} f(\mu_p)$$

(62)

where $f(\lambda)$ is the dispersion law of (23) and (24), see \[43\]. It is immediately seen from (62) that $F_{p,0}$ depends exponentially on time, therefore we just need to make the following substitution

$$F_{p,0}^T \rightarrow F_{p,0}^T e^{-if(\mu_p)t}$$

(63)

to fully recover the time dependence in all equations containing the factors $F_{p,0}$.

In order to obtain explicit solutions of the system (23) and (24) we have to pick up a convenient bare solution. In view of the boundary condition (38), a quite natural candidate for a bare solution is the following one

$$u_0(x, t) = 0, \quad v_0(x, t) = 1.$$  \hspace{1cm} (64)

For technical reasons, see \[17\], we pick up

$$\Psi_0(x, t, \lambda) = \exp(-i\lambda S_0 x) = \begin{pmatrix} \cos \lambda x & 0 & -i \sin \lambda x \\ 0 & 1 & 0 \\ -i \sin \lambda x & 0 & \cos \lambda x \end{pmatrix}$$

(65)

as a fundamental solution of the corresponding bare problems \[51\] and \[52\]. Further, we shall assume that (51) has just two simple poles $\pm \mu$. Moreover, $X$ and $F$ will be column-vectors, so we can write down

$$X = \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}, \quad F = \begin{pmatrix} F^1 \\ F^2 \\ F^3 \end{pmatrix}$$

for each of them. In view of (65) and (61) $F$ acquires the form:

$$F(x) = \begin{pmatrix} F^1_0 \cos \mu x + i F^3_0 \sin \mu x \\ F^2_0 \\ F^3_0 \cos \mu x + i F^1_0 \sin \mu x \end{pmatrix}.$$  \hspace{1cm} (66)
where we have employed the notation $F_0^T = (F_0^1, F_0^2, F_0^3)$.

It turns out the form of the dressed solution depends on whether or not the second component of the vector $F_0$ is zero. Let us consider first the case when $F_0^2 = 0$ and $F_0^1 \neq \pm F_0^3$. After recovering the time dependence in (66) as in accordance with (63), we see that $t$ appears in a common exponential factor only. So substituting (66) into (65) and (59), that factor cancels out and we obtain the following stationary, nonsingular dressed solution:

$$u_1(x) = 0,$$
$$v_1(x) = \left[ \frac{c_1 \cosh 2\kappa x - c_2 \sin 2\omega x + i(c_3 \cos 2\omega x + c_4 \sinh 2\kappa x)}{c_1 \cosh 2\kappa x + c_2 \sin 2\omega x - i(c_3 \cos 2\omega x - c_4 \sinh 2\kappa x)} \right]^2. \quad (68)$$

Above, we have introduced the additional notation

$$\omega = \text{Re} \mu, \quad \kappa = \text{Im} \mu, \quad \varphi = \frac{\arg F_0^1 - \arg F_0^3}{2}, \quad \gamma = \frac{1}{2} \ln |F_0^1/F_0^3|, \quad (69)$$
$$c_1 = \omega \sin 2\gamma, \quad c_2 = \kappa \cos 2\varphi, \quad c_3 = \kappa \cosh 2\gamma, \quad c_4 = \omega \sin 2\varphi. \quad (70)$$

Without loss of generality we can set $\omega > 0$ and $\kappa > 0$ in (69).

It is not hard to see that when $F_0^1 = \pm F_0^3$, i.e. when $\gamma = 0$ and $\varphi = 0, \pm \pi/2$, we have that $c_1 = c_4 = 0$ while $c_2 = \pm \kappa$ and $c_3 = \kappa$. As a result, (67) and (68) degenerates to the bare solution (64), which is the reason why we required that $F_0^1 \neq \pm F_0^3$.

Let us consider now the case when the second component of $F_0$ is not zero. After recovering the time evolution in (66) and setting $F_0^2 = 1$, we obtain

$$F(x, t) = \begin{pmatrix} (F_0^1 \cos \mu x + iF_0^3 \sin \mu x) \exp \left( \frac{\mu^2 t}{3} \right) \\ (F_0^1 \cos \mu x + iF_0^3 \sin \mu x) \exp \left( -\frac{\mu^2 t}{3} \right) \end{pmatrix}$$

for the column-vector $F(x, t)$. The dressed solution now is given by:

$$u_1 = \frac{[c_1 \cosh 2\kappa x - c_2 \sin 2\omega x + i(c_3 \cos 2\omega x + c_4 \sinh 2\kappa x) - \varepsilon(\omega - i\kappa)e^{2(\omega t - \xi_0)}/2]}{[c_1 \cosh 2\kappa x + c_2 \sin 2\omega x - i(c_3 \cos 2\omega x - c_4 \sinh 2\kappa x) - \varepsilon(\omega + i\kappa)e^{2(\omega t - \xi_0)}/2]^2} \times \frac{2\omega \kappa e^{-i[(\omega^2 - \kappa^2)t + \delta_0]}}{\omega - i\kappa} \left[ e^{i\varphi - \gamma \cos(\omega + i\kappa)x} + e^{-i\varphi + \gamma \sin(\omega + i\kappa)x} \right] e^{2\omega t - \xi_0} \quad (71)$$

$$v_1 = \frac{[c_1 \cosh 2\kappa x - c_2 \sin 2\omega x + i(c_3 \cos 2\omega x + c_4 \sinh 2\kappa x) - \varepsilon(\omega + i\kappa)e^{2(\omega t - \xi_0)}/2]}{[c_1 \cosh 2\kappa x + c_2 \sin 2\omega x - i(c_3 \cos 2\omega x - c_4 \sinh 2\kappa x) - \varepsilon(\omega - i\kappa)e^{2(\omega t - \xi_0)}/2]^2} \times \{ \frac{[c_1 \cosh 2\kappa x - c_2 \sin 2\omega x + i(c_3 \cos 2\omega x + c_4 \sinh 2\kappa x) - \varepsilon(\omega - i\kappa)e^{2(\omega t - \xi_0)}/2]}{\omega - i\kappa} \times (\omega - i\kappa) - 2i\kappa \varepsilon e^{2(\omega t - \xi_0)} \} \quad (72)$$

where $\gamma, \varphi$ and the coefficients $c_j, j = 1, 2, 3, 4$ are the same as in (69) and (70) respectively. The two new parameters introduced above are defined through the equalities:

$$\delta_0 = \frac{\arg F_0^1 + \arg F_0^3}{2}, \quad \xi_0 = \frac{1}{2} \ln |F_0^1/F_0^3|.$$
5 Conservation Laws

In this section we are going to view (23) and (24) as an infinite-dimensional Hamiltonian system. More specifically, we are going to demonstrate how we can derive the conservation laws of the nonlocal coupled system under consideration. For this to be done we apply a method proposed by Drinfel’d and Sokolov [3].

As already discussed in Remark 2, the matrix (18) has a simple spectrum — its eigenvalues are 0, ±1. Thus, we can put $S(x, t)$ into a constant diagonal form by applying the gauge transformation:

$$L(\lambda) \rightarrow \tilde{L}(\lambda) = [g(x, t)]^{-1}L(\lambda)g(x, t) = i\partial_x + \tilde{U}_0(x, t) - \lambda J,$$

$$A(\lambda) \rightarrow \tilde{A}(\lambda) = [g(x, t)]^{-1}A(\lambda)g(x, t) = i\partial_t + \lambda \tilde{A}_1(x, t) + \lambda^2 \tilde{A}_2(x, t),$$

$$g(x, t) = \sqrt{2} \begin{pmatrix} 1 & 0 & 0 \\ \epsilon u(-x, t) & \sqrt{2}v(x, t) & -\sqrt{2}u(-x, t) \\ v(-x, t) & -\sqrt{2}u(x, t) & \epsilon v(-x, t) \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

on the Lax operators (8) and (21). The explicit form of $\tilde{U}_0$ to be used further is given by

$$\tilde{U}_0(x, t) = \frac{i}{2} \begin{pmatrix} \mathbb{B}(x, t) & \mathbb{C}(x, t) & \mathbb{B}(x, t) \\ -\epsilon \mathbb{C}(-x, t) & -2\mathbb{B}(x, t) & -\epsilon \mathbb{C}(-x, t) \\ \mathbb{B}(x, t) & \mathbb{C}(x, t) & \mathbb{B}(x, t) \end{pmatrix}$$

where $\mathbb{C}(x, t) = \sqrt{2}[u(x, t)v_x(x, t) - v(x, t)u_x(x, t)]$ and $\mathbb{B}(x, t)$ is the same as in (22).

Let us apply Drinfel’d and Sokolov’s method of diagonalization of Lax pair to derive the integrals of motion of (23) and (24). For that purpose we introduce the gauge transform

$$P(x, t, \lambda) = I + \frac{P_1(x, t)}{\lambda} + \frac{P_2(x, t)}{\lambda^2} + \cdots$$

All the coefficients $P_l$ $(l = 1, 2, \ldots)$ appearing above are some off-diagonal 3 × 3 matrices to be determined further. As a result of the action of $P$, we have

$$\mathcal{L} = \hat{P} \bar{L} \mathcal{P} = i\partial_x - \lambda J + \mathcal{L}_0 + \frac{L_1}{\lambda} + \cdots,$$

$$\mathcal{A} = \hat{P} \bar{A} \mathcal{P} = i\partial_t + \lambda^2 A_{-2} + \lambda A_{-1} + A_0 + \frac{A_1}{\lambda} + \frac{A_2}{\lambda^2} + \cdots$$

All the coefficients $\mathcal{L}_k$, $A_k$, $k = -1, 0, 1, \ldots$ are required to be diagonal matrices. Then the compatibility condition $[\mathcal{L}, \mathcal{A}] = 0$ is equivalent to the equations

$$\partial_t \mathcal{L}_k - \partial_x A_k = 0, \quad k = 0, 1, \ldots$$

These equations mean that $\mathcal{L}_k$ are conserved densities of the system (23) and (24) while $A_k$ are the corresponding currents.

Let us now rewrite the equality (78) in the following way:

$$\tilde{L} \mathcal{P} = \mathcal{P} \mathcal{L}$$

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Since (80) should hold identically in \( \lambda \), it splits into the following recurrence relations:

\[
\begin{align*}
\lambda^0 & : \quad \tilde{U}_0 - JP_1 = \mathcal{L}_0 - P_1 J, \\
\lambda^{-1} & : \quad iP_{1,x} + \tilde{U}_0 P_1 - JP_2 = \mathcal{L}_1 + P_1 \mathcal{L}_0 - P_2 J, \\
\vdots \quad & \\
\lambda^{-k} & : \quad iP_{k,x} + \tilde{U}_0 P_k - JP_{k+1} = \mathcal{L}_k - P_{k+1} J + \sum_{m=0}^{k-1} P_{k-m} \mathcal{L}_m, \\
\vdots \quad & 
\end{align*}
\]  

(83)

In order to resolve those relations, one should split each relation into a diagonal and off-diagonal part. For example, from the first relation above one has

\[
\mathcal{L}_0 = \tilde{U}_0^d, \quad \tilde{U}_0^a = [J, P_1] \tag{84}
\]

where the superscripts \( d \) and \( a \) above denote projection onto diagonal and off-diagonal part of a matrix respectively. Taking into account the explicit form of \( \tilde{U}_0 \) (formula (76)) for \( \mathcal{L}_0 \) we have

\[
\mathcal{L}_0 = \frac{i}{2} \mathcal{B}(x,t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Thus, as a density \( J_1 \) of the first integral of motion we can choose the quantity \(-\mathcal{B}(x,t)\).

On the other hand, after inverting the commutator in the second equation in (84), one obtains

\[
P_1 = \frac{i}{2} \begin{pmatrix} 0 & \mathcal{C}(x,t) & \mathcal{B}(x,t)/2 \\ \varepsilon \mathcal{C}(-x,t) & 0 & \varepsilon \mathcal{B}(-x,t)/2 \\ -\mathcal{B}(x,t)/2 & -\mathcal{C}(x,t) & 0 \end{pmatrix}. \tag{85}
\]

Similarly, for \( \mathcal{L}_1 \) one needs to extract the diagonal part of (82). The result reads

\[
\mathcal{L}_1 = \left( \tilde{U}_0^a P_1 \right)^d. \tag{86}
\]

After substituting the expression (85) for \( P_1 \) into (86), one obtains

\[
\mathcal{L}_1 = \frac{1}{2} \left[ \mathcal{B}^2(x,t) - 4\varepsilon \mathcal{C}(x,t)\mathcal{C}(-x,t) \right] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Therefore one can pick up

\[
\mathcal{J}_2 = \mathcal{B}^2(x,t) - 4\varepsilon \mathcal{C}(x,t)\mathcal{C}(-x,t)
\]

as a second conserved density. Proceeding further, one is able to find a conserved density of arbitrary order. It is seen from (83) that \( k \)-th coefficient \( \mathcal{L}_k \) is derived from the relation

\[
\mathcal{L}_k = (\tilde{U}_0^a P_k)^d \tag{87}
\]

while \( P_k \) is obtained from the recurrence relation:

\[
P_k = \text{ad}_J^{-1} \left( i\partial_x P_{k-1} + (\tilde{U}_0 P_{k-1})^a - \sum_{l=0}^{k-2} P_{k-1-l} \mathcal{L}_l \right). 
\]

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In order to obtain the integrals of motion of the system (23) and (24), one just needs to integrate the conserved densities over the variable $x$ as given below:

$$I_k = \int_{-\infty}^{\infty} dx j_k(x,t), \quad k = 1, 2, \ldots$$

The limits of integration implies certain choice of boundary conditions imposed on the dynamical fields $u$ and $v$. Above, we have had in mind the trivial background condition as introduced in (38).

6 Conclusion

In the present preprint we have introduced and studied the nonlocal reduction (23) and (24) of a generalized Heisenberg ferromagnet equation. The integrable hierarchy corresponding to (23) and (24) has been completely described in terms of generating operators, see (37). Furthermore, we have constructed special solutions to (23) and (24) by applying dressing procedure. We have derived only special solutions related to 4 complex discrete eigenvalues of the scattering operator. However, there exist two more types of solutions: doublet solutions and quasi-rational solutions that are related to pairs of discrete eigenvalues. Constructing those types of solutions will be demonstrated elsewhere. By similar fashion, one can derive special solutions to any member of the integrable hierarchy — the only difference will be the time dependence of the solutions.

Another important issue concerns the Hamiltonian formalism and the integrals of motion for the system of nonlocal equations under consideration. By using Drinfel’d-Sokolov’s method of Lax pair diagonalization, recursive formulas to derive the conserved densities of (23) and (24) have been obtained, see (87). Generally speaking those conserved densities contain nonlocal terms which correspond to the nonlocal terms in the NEEs under consideration.

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