Topological Susceptibility to the One-Loop Order in
Chiral Perturbation Theory

Yao-Yuan Mao$^1$ and Ting-Wai Chiu$^{1,2}$

(for the TWQCD Collaboration)

$^1$ Department of Physics, and Center for Theoretical Sciences,
National Taiwan University, Taipei 10617, Taiwan

$^2$ Center for Quantum Science and Engineering,
National Taiwan University, Taipei 10617, Taiwan

Abstract

We derive the topological susceptibility to the one-loop order in the chiral effective theory of
QCD, for an arbitrary number of flavors.
I. INTRODUCTION

In Quantum Chromodynamics (QCD), the topological susceptibility ($\chi_t$) is the most crucial quantity to measure the topological charge fluctuation of the QCD vacuum, which plays an important role in breaking the $U_A(1)$ symmetry. Theoretically, $\chi_t$ is defined as

$$\chi_t = \int d^4x \langle \rho(x)\rho(0) \rangle,$$  \hspace{1cm}  (1)

where

$$\rho(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}[F_{\mu\nu}(x)F_{\lambda\sigma}(x)],$$  \hspace{1cm}  (2)

is the topological charge density expressed in term of the matrix-valued field tensor $F_{\mu\nu}$. With mild assumptions, Witten [1] and Veneziano [2] obtained a relationship between the topological susceptibility in the quenched approximation and the mass of $\eta'$ meson (flavor singlet) in unquenched QCD with $N_f$ degenerate flavors, namely,

$$\chi_t(\text{quenched}) = \frac{F^2_\pi m^2_{\eta'}}{2N_f},$$

where $F_\pi \simeq 93$ MeV, the decay constant of pion. This implies that the mass of $\eta'$ is essentially due to the axial anomaly relating to non-trivial topological charge fluctuations, which can turn out to be nonzero even in the chiral limit, unlike those of the (non-singlet) approximate Goldstone bosons.

Using the Chiral Perturbation Theory (ChPT), Leutwyler and Smilga [3, 4] obtained the following relations in the chiral limit

$$\chi_t = \Sigma \left( \frac{1}{m_u} + \frac{1}{m_d} \right)^{-1}, \quad (N_f = 2),$$  \hspace{1cm}  (3)

$$\chi_t = \Sigma \left( \frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-1}, \quad (N_f = 3),$$  \hspace{1cm}  (4)

where $m_u$, $m_d$, and $m_s$ are the quark masses, and $\Sigma$ is the chiral condensate. This implies that in the chiral limit ($m_u \to 0$) the topological susceptibility is suppressed due to internal quark loops. Most importantly, (3) and (4) provide a viable way to extract $\Sigma$ from $\chi_t$ in the chiral limit.

From (1), one obtains

$$\chi_t = \frac{\langle Q_t^2 \rangle}{\Omega}, \quad Q_t \equiv \int d^4x \rho(x),$$  \hspace{1cm}  (5)
where $\Omega$ is the volume of the system, and $Q_t$ is the topological charge (which is an integer for QCD). Thus, one can determine $\chi_t$ by counting the number of gauge configurations for each topological sector. Furthermore, we can also obtain the second normalized cumulant

$$c_4 = -\frac{1}{\Omega} \left[ \langle Q_t^4 \rangle - 3 \langle Q_t^2 \rangle^2 \right],$$  
(6)

which is related to the leading anomalous contribution to the $\eta' - \eta'$ scattering amplitude in QCD, as well as the dependence of the vacuum energy on the vacuum angle $\theta$. (For a recent review, see for example, Ref. [5] and references therein.)

However, for lattice QCD, it is difficult to extract $\rho(x)$ and $Q_t$ unambiguously from the gauge link variables, due to their rather strong fluctuations.

To circumvent this difficulty, one may consider the Atiyah-Singer index theorem [6]

$$Q_t = n_+ - n_- = \text{index}(D),$$  
(7)

where $n_\pm$ is the number of zero modes of the massless Dirac operator $D \equiv \gamma_\mu (\partial_\mu + igA_\mu)$ with $\pm$ chirality. Since $D$ is anti-Hermitian and chirally symmetric, its nonzero eigenmodes must come in complex conjugate pairs with zero chirality. Thus one can obtain the identity

$$n_+ - n_- = m \int d^4x \text{ tr}[\gamma_5(D + m)^{-1}(x,x)],$$  
(8)

by spectral decomposition, where the nonzero modes drop out due to zero chirality. In view of (7) and (8), one can regard $\rho_t(x) \equiv m_q \text{ tr}[\gamma_5(D + m_q)^{-1}(x,x)]$ as topological charge density, to replace $\rho(x)$ in the measurement of $\chi_t$.

Recently, the topological susceptibility and the second normalized cumulant have been measured in unquenched lattice QCD with exact chiral symmetry, for $N_f = 2$ and $N_f = 2 + 1$ lattice QCD with overlap fermion in a fixed topology [7, 8], and $N_f = 2 + 1$ lattice QCD with domain-wall fermion [9]. The results of topological susceptibility turn out in good agreement with the Leutwyler-Smilga relation, with the values of the chiral condensate as follows.

$$\Sigma_{\text{MS}}(2 \text{ GeV}) = [245(5)(12) \text{ MeV}]^3, \quad (N_f = 2), \quad \text{Ref. [7]},$$

$$\Sigma_{\text{MS}}(2 \text{ GeV}) = [253(4)(6) \text{ MeV}]^3, \quad (N_f = 2 + 1), \quad \text{Ref. [8]},$$

$$\Sigma_{\text{MS}}(2 \text{ GeV}) = [259(6)(9) \text{ MeV}]^3, \quad (N_f = 2 + 1), \quad \text{Ref. [9]},$$

These results assure that lattice QCD with exact chiral symmetry is the proper framework to tackle the strong interaction physics with topologically non-trivial vacuum fluctuations.
Obviously, the next task for unquenched lattice QCD with exact chiral symmetry is to determine the second normalized cumulant $c_4$ to a good precision, and to address the question how the vacuum energy depends on the vacuum angle $\theta$ and related problems. Theoretically, it is interesting to obtain an analytic expression of $c_4$ in ChPT, as well as to extend the Leutwyler-Smilga relation to the one-loop order. In this paper, we derive the topological susceptibility to the one-loop order in ChPT, for an arbitrary number of flavors.

The outline of this paper is as follows. In Section 2, we review the derivation of topological susceptibility $\chi_t$ at the tree level of ChPT, and also derive the second normalized cumulant $c_4$ at the tree level, and discuss its implications. In Section 3, we derive $\chi_t$ up to the one-loop order in ChPT for an arbitrary number of flavors. In Section 4, we conclude with some remarks, and also present the case of $2 + 1$ flavors, in which only the one-loop corrections due to the $u$ and $d$ quarks are incorporated. In the Appendix, we present a heuristic derivation of the counterpart of the Leutwyler-Smilga relation in lattice QCD with exact chiral symmetry.

II. TOPOLOGICAL SUSCEPTIBILITY AT THE TREE LEVEL OF CHPT

Before we proceed to derive $\chi_t$ to the one-loop order in ChPT, it is instructive for us to recap the derivation of $\chi_t$ at the tree level [3, 4].

The leading terms of the effective chiral lagrangian for QCD with $N_f$ flavor at $\theta = 0$ [10] are the kinetic term and the symmetry breaking term,

$$L^{(2)} = L^{(2)}_{\text{eff}} + L^{(2)}_{\text{s.b.}} = \frac{F^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{\Sigma}{2} \text{Tr}(MU^\dagger + U^\dagger M),$$

(9)

where $U(x) = \exp\{2i\phi^a(x)t^a/F_\pi\}$ is a group element of $SU(N_f)$, $M$ is the quark mass matrix, $F_\pi$ is the pion decay constant, and $\Sigma = \langle \bar{\psi}\psi \rangle_{\text{vac}}$ is the chiral condensate of the QCD vacuum.

On the other hand, the partition function of QCD in the $\theta$ vacuum can be written as

$$Z_{N_f}(\theta) = \sum_Q e^{-iQ\theta} Z_Q,$$

(10)

where

$$Z_Q = \int [dA_\mu] e^{-S_G[A_\mu]} \det \left( \gamma_\mu D_\mu + \frac{1 - \gamma^5}{2} M + \frac{1 + \gamma^5}{2} M^\dagger \right)$$

$$= \int [dA_\mu] e^{-S_G[A_\mu]} \prod_k (\lambda_k^2 + M^\dagger M) \left\{ \begin{array}{ll} \det(M^\dagger)^Q, & Q > 0, \\ \det(M)^{-Q}, & Q < 0, \end{array} \right.$$

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where $S_G$ is the action of the gauge field, and $\lambda_k$’s are non-zero eigenvalues of the massless Dirac operator $\gamma^\mu D_\mu$ in the gauge background. Thus the physical vacuum angle on which all physical quantities depend is $\theta_{\text{phys}} = \theta + \arg \det(M)$ rather than $\theta$. Also, the $\theta$-dependence of $Z_{N_f}(\theta)$ always enters through the combinations $Me^{i\theta/N_f}$ and $M^\dagger e^{-i\theta/N_f}$. It follows that for $\theta \neq 0$, the symmetry breaking term in the chiral effective lagrangian can be written as

$$L_{s.b.}^{(2)} = \sum \text{Re} \left[ \text{Tr}(Me^{i\theta/N_f} U^\dagger) \right].$$

Defining the vacuum energy density

$$\epsilon_{\text{vac}}(M, \theta) = -\frac{1}{\Omega} \log Z_{N_f}(\theta),$$

then the topological susceptibility $\chi_t$ and the second normalized cumulant $c_4$ can be expressed as

$$\chi_t = \frac{\partial^2 \epsilon_{\text{vac}}(M, \theta)}{\partial^2 \theta} \bigg|_{\theta=0},$$

$$c_4 = \frac{\partial^4 \epsilon_{\text{vac}}(M, \theta)}{\partial^4 \theta} \bigg|_{\theta=0}.$$ (14)

For small quark masses ($L \ll m_\pi^{-1}$), the unitary matrix $U$ does not depend on $x_\mu$. Thus only the symmetry-breaking term survives in (9), and the partition function becomes

$$Z_{N_f}(\theta) = \int dU \exp \left\{ \Omega \sum \text{Re} \left[ \text{Tr}(Me^{i\theta/N_f} U^\dagger) \right] \right\},$$

where $\Omega = L^3 T$ is the space-time volume. Without loss of generality, the unitary matrix $U$ can be taken to be diagonal

$$U = \text{diag} \left( e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_{N_f}} \right), \quad \sum_{j=1}^{N_f} \alpha_j = 0,$$

where the last constraint follows from the unitarity of $U$. Also, we can choose the mass matrix to be diagonal $M = \text{diag}(m_1, \ldots, m_{N_f})$. Then we have

$$\text{Re} \left[ \text{Tr}(Me^{i\theta/N_f} U^\dagger) \right] = \sum_j m_j \cos \phi_j,$$

where $\phi_j = \theta/N_f - \alpha_j$, and $\sum_j \phi_j = \theta$.

Now, we consider a sufficiently large volume $\Omega$ satisfying $m_j \Sigma \Omega \gg 1$, then the group integral in the partition function (15) is largely due to the $U$ which minimizes the minus exponent of the integrand, i.e.,

$$\min_U \left\{ -\text{Re} \left[ \text{Tr}(Me^{i\theta/N_f} U^\dagger) \right] \right\} = \min_{\phi} \left\{ - \sum_{j=1}^{N_f} m_j \cos \phi_j \right\}, \quad \sum_{j=1}^{N_f} \phi_j = \theta.$$ (16)
For \( N_f = 2 \), this amounts to minimize the function

\[-m_1 \cos(\phi_1) - m_2 \cos(\theta - \phi_1),\]

where the constraint \( \phi_1 + \phi_2 = \theta \) has been used. A simple calculation gives the minimum,

\[-\sqrt{m_1^2 + m_2^2 + 2m_1m_2 \cos \theta}.

Thus the partition function is

\[Z_{N_f}(\theta) = Z_0 \exp \left\{ \Omega \Sigma \sqrt{m_1^2 + m_2^2 + 2m_1m_2 \cos \theta} \right\}, \quad m\Sigma \Omega \gg 1,

which gives the vacuum energy

\[\epsilon_{\text{vac}}(\theta) = \epsilon_0 - \Sigma \sqrt{m_u^2 + m_d^2 + 2m_u m_d \cos \theta},\]

where \( \epsilon_0 \) is the additive normalization constant corresponding the normalization factor \( Z_0 \) in the partition function. From (17), we obtain the topological susceptibility

\[\chi_t = \frac{\partial^2 \epsilon_{\text{vac}}}{\partial \theta^2} \bigg|_{\theta=0} = \Sigma \frac{m_u m_d}{m_u + m_d}. \tag{17}\]

Furthermore, the second normalized cumulant is

\[c_4 = \frac{\partial^4 \epsilon_{\text{vac}}}{\partial \theta^4} \bigg|_{\theta=0} = -\Sigma \frac{m_u m_d}{m_u + m_d} + 3\Sigma \frac{m_u^2 m_d^2}{(m_u + m_d)^3} = -\Sigma \left( \frac{1}{m_u^3} + \frac{1}{m_d^3} \right) \left( \frac{m_u m_d}{m_u + m_d} \right)^4, \tag{18}\]

which has not been discussed explicitly in the literature. The vital observation is that the ratio of \( \chi_t \) and \( c_4 \) is

\[\frac{c_4}{\chi_t} = -1 + \frac{3m_u m_d}{(m_u + m_d)^2}, \tag{19}\]

which goes to \(-1/4\) in the isospin limit \( m_u = m_d \). This seems to rule out the dilute instanton gas/liquid model [11, 12, 13] which predicts that \( c_4/\chi_t = -1 \). Moreover, recent numerical results of \( c_4/\chi_t \) from quenched lattice QCD [14, 15, 16] and unquenched lattice QCD [8, 9] are consistent with the prediction of ChPT.

Next we turn to the case \( N_f > 2 \). Then there is no analytic solution to the minimization problem (16). However, for the purpose of obtaining the topological susceptibility, one may consider the limit of small \( \theta \) (and \( \phi_j \)'s) because \( U = \mathds{1} \) gives the minimal vacuum energy at \( \theta = 0 \). Since \( \chi_t \) only depends on the curvature of \( \epsilon_{\text{vac}}(\theta) \) around \( \theta = 0 \), this approximation
would give the exact result of $\chi_t$ (at the tree-level). To the order of $\theta^4$, the minimization problem \([16]\) becomes

$$
\min_{\phi} \left\{ - \sum_{j=1}^{N_f} m_j \cos \phi_j \right\} = \min_{\phi} \left\{ \frac{1}{2} \sum_{j=1}^{N_f} m_j \phi_j^2 - \frac{1}{24} \sum_{j=1}^{N_f} m_j \phi_j^4 \right\}, \quad \sum_{i=1}^{N_f} \phi_i = \theta.
$$

Now introducing the Lagrange multiplier $\lambda$ to incorporate the constraint $\sum_i \phi_i = \theta$, then the minimization problem amounts to solving the equation

$$
\frac{\partial}{\partial \phi_i} \left[ \frac{1}{2} \sum_{j=1}^{N_f} m_j \phi_j^2 - \frac{1}{24} \sum_{j=1}^{N_f} m_j \phi_j^4 - \lambda \left( \sum_{j=1}^{N_f} \phi_j - \theta \right) \right] = m_i \phi_i - \frac{1}{6} m_i \phi_i^3 - \lambda = 0.
$$

Setting $\phi_i = a_1 \frac{\lambda}{m_i} + a_3 \left( \frac{\lambda}{m_i} \right)^3$ (where $a_1$ and $a_3$ are parameters), and using $\sum_i \phi_i = \theta$, we can solve for $a_1$ and $a_3$, and $\phi_i$ to the order of $\theta^3$,

$$
\phi_i = \frac{\bar{m}}{m_i} \theta + \frac{\theta^3}{6} \left[ \left( \frac{\bar{m}}{m_i} \right)^3 - \sum_{j=1}^{N_f} \left( \frac{\bar{m}}{m_j} \right)^3 \right] + \mathcal{O}(\theta^5),
$$

where $\bar{m} \equiv \left( \sum_{i=1}^{N_f} m_i^{-1} \right)^{-1}$ is the “reduced mass” of the $N_f$ quark flavors. Keeping the exponent of the partition function to the order of $\theta^4$, we have

$$
Z_{N_f}(\theta) = Z_0 \exp \left\{ -\Omega \Sigma \left( \sum_{j=1}^{N_f} \frac{1}{m_j} \right)^{-1} \frac{\theta^2}{2} + \Omega \Sigma \sum_{i=1}^{N_f} m_i^{-3} \left( \sum_{j=1}^{N_f} \frac{1}{m_j} \right)^{-4} \frac{\theta^4}{24} + \mathcal{O}(\theta^6) \right\}, \quad (20)
$$

and the vacuum energy density is

$$
\epsilon_{\text{vac}}(\theta) = \epsilon_0 + \Omega \Sigma \left( \sum_{j=1}^{N_f} \frac{1}{m_j} \right)^{-1} \frac{\theta^2}{2} - \sum_{i=1}^{N_f} m_i^{-3} \left( \sum_{j=1}^{N_f} \frac{1}{m_j} \right)^{-4} \frac{\theta^4}{24} + \mathcal{O}(\theta^6).
$$

It follows that the topological susceptibility is

$$
\chi_t = \left. \frac{\partial^2 \epsilon_{\text{vac}}}{\partial \theta^2} \right|_{\theta=0} = \sum \left( \sum_{j=1}^{N_f} \frac{1}{m_j} \right)^{-1}, \quad (21)
$$

and

$$
c_4 = \left. \frac{\partial^4 \epsilon_{\text{vac}}}{\partial \theta^4} \right|_{\theta=0} = -\sum_{i=1}^{N_f} m_i^{-3} \left( \sum_{j=1}^{N_f} \frac{1}{m_j} \right)^{-4}, \quad (22)
$$

which generalize Eqs. \([17]\) and \([18]\) to an arbitrary number of flavors. In particular, for $N_f = 3$,

$$
\chi_t = \sum \left( \frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-1}, \quad (23)
$$

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and
\[ c_4 = -\Sigma \left( \frac{1}{m_u^3} + \frac{1}{m_d^3} + \frac{1}{m_s^3} \right) \left( \frac{1}{m_u + m_d + m_s} \right)^{-4}. \] (24)

Nevertheless, these two formulas seem unnatural, since the strange quark is much heavier than the up and down quarks. Thus a plausible chiral limit is to take \( m_{u,d} \to 0 \), while keeping \( m_s \) fixed. Consequently, the condensate of the strange quark \( \langle \bar{s}s \rangle \) must be different from \( \Sigma \), and it should also enter this formula. In the Appendix, we present a heuristic derivation of the counterpart of (23) in lattice QCD with exact chiral symmetry, which takes into account of the difference between \( \langle \bar{s}s \rangle \) and \( \Sigma \), as given in Eq. (52).

III. TOPOLOGICAL SUSCEPTIBILITY TO THE ONE-LOOP ORDER OF CHPT

To the one-loop order of ChPT, one has to include \( \mathcal{L}^{(4)} \) \cite{10} at the tree level as well as the one-loop contributions of \( \mathcal{L}^{(2)} \). In 1984, Gasser and Leutwyler \cite{10} considered the low-energy expansion, where both \( p \) and \( \mathcal{M} \) are assumed to be small but \( \mathcal{M}/p^2 \) can have a finite value, such that the value of \( M_p^2/p^2 \) can be fixed. In this case, the external sources \( a_{\mu} (x) \) and \( p(x) \) can be counted as order of \( \Phi \), and \( v_{\mu} (x) \) and \( s(x) - \mathcal{M} \) as order of \( \Phi^2 \). Gasser and Leutwyler showed that at the one-loop order, the chiral effective action can be written as

\[ W = W_t + W_u + W_A + \mathcal{O} (\Phi^6), \] (25)

where \( W_t \) denotes the sum of tree diagrams and tadpole contributions (of order \( \Phi^2 \)), \( W_u \) the unitarity correction (of order \( \Phi^3 \)), and \( W_A \) the anomaly contribution (of order \( \Phi^4 \)). Because the \( \theta \) dependence enters the Lagrangian only through \( \mathcal{M} \), we can count \( \chi_t \) as order of \( \Phi^2 \), thus for the evaluation of topological susceptibility to the one-loop order, and it suffices to consider \( W_t \) only.

Moreover, Gasser and Leutwyler \cite{10} showed that the pole terms due to the one-loop contributions of \( \mathcal{L}^{(2)} \) can be absorbed by the low-energy coupling constants of \( \mathcal{L}^{(4)} \), and \( W_t \) is given by \cite{10}

\[
W_t = \sum_{p} \int d^4 x \frac{F_p^2}{2} \left\{ \frac{1}{N_f} - \frac{M_p^2}{16\pi^2 F_p^2} \ln \frac{M_p^2}{\mu_{sub}^2} \right\} \sigma_{\mu p}^A \\
+ \sum_{p} \int d^4 x \frac{F_p^2}{2} \left\{ \frac{N_f}{N_f^2 - 1} - \frac{M_p^2}{16\pi^2 F_p^2} \ln \frac{M_p^2}{\mu_{sub}^2} \right\} \sigma_{\mu p}^F + \int d^4 x \mathcal{L}^{(4)},
\] (26)
where $M_P^2$’s are the squared meson masses, $\sigma_P^2$ corresponds to the kinetic term which can be dropped in the limit of small quark masses, $\sigma_P^2$ corresponds to the symmetry breaking term,

$$\sigma_P^2 = \frac{1}{8} \text{Tr} \left( \{ \lambda_P, \lambda_P^\dagger \} (\chi^\dagger U + U^\dagger \chi) \right) - M_P^2,$$

and $\mathcal{L}^{(4)}$ is just $\mathcal{L}^{(4)}$ with renormalized low-energy coupling constants,

$$\mathcal{L}^{(4)} = L_1^x \left\{ \text{Tr} [D_\mu U(D^\mu U)^\dagger] \right\}^2 + L_2^x \text{Tr} \left[ D_\mu U(D_\nu U)^\dagger \right] \text{Tr} \left[ D^\mu U(D^\nu U)^\dagger \right]$$

$$+ L_3^x \text{Tr} \left[ D_\mu U(D^\mu U)^\dagger D_\nu U(D^\nu U)^\dagger \right]$$

$$+ L_4^x \text{Tr} \left[ D_\mu U(D^\mu U)^\dagger \right] \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right)$$

$$+ L_5^x \text{Tr} \left[ D_\mu U(D^\mu U)^\dagger (\chi U^\dagger + U \chi^\dagger) \right] + L_6^x \left[ \text{Tr} \left( \chi U^\dagger + U \chi^\dagger \right) \right]^2$$

$$+ L_7^x \left[ \text{Tr} \left( \chi U^\dagger - U \chi^\dagger \right) \right]^2 + L_8^x \left( U \chi^\dagger U^\dagger + \chi U^\dagger \chi^\dagger \right)$$

$$- i L_9^x \left[ F_R^{\mu\nu} D_\mu U(D^\nu U)^\dagger + F_L^{\mu\nu} (D^\mu U)^\dagger D^\nu U \right] + L_{10}^x \text{Tr} \left( U F_R^{\mu\nu} U^\dagger F_R^{\mu\nu} \right)$$

$$+ H_1^x \text{Tr} \left( F_R^{\mu\nu} F_R^{\nu\mu} + F_L^{\mu\nu} F_L^{\nu\mu} \right) + H_2^x \text{Tr} \left( \chi \chi^\dagger \right).$$

Here $\chi = 2(\Sigma/F_\pi^2)\mathcal{M} \equiv 2B_0\mathcal{M}$, $\lambda_P$’s are the generators of $SU(N)$ in the physical basis, $\{L_i^x(\mu_{sub}), i = 1, \cdots, 10\}$ are renormalized low-energy coupling constants, and the last two contact terms (with couplings $H_1^x(\mu_{sub})$ and $H_2^x(\mu_{sub})$) are the counter terms required for renormalization of the one-loop diagrams.

For small quark masses ($L \ll m_\pi^{-1}$), the unitary matrix $U$ does not depend on $x_\mu$, thus the term involving $\sigma_P^2$ in (26) can be dropped.

Next we consider the term with $\sigma_P^2$ in (26). Using the formula

$$\sum_P \{ \lambda_P, \lambda_P^\dagger \} = \frac{4(N_f^2 - 1)}{N_f} \mathbb{I},$$

we obtain its contribution to the chiral effective lagrangian,

$$\Sigma \text{ReTr} (\mathcal{M}U^\dagger) = \frac{N_f F_\pi^2}{2(N_f^2 - 1)} \sum_P M_P^2$$

$$= \frac{\Sigma}{4F_\pi^2} \sum_P \text{ReTr} \left( \{ \lambda_P, \lambda_P^\dagger \} \mathcal{M}U^\dagger \right) \frac{M_P^2}{16\pi^2} \ln \frac{M_P^2}{\mu_{sub}^2} + \frac{M_P^2}{32\pi^2} \ln \frac{M_P^2}{\mu_{sub}^2}.$$

For small quark masses ($L \ll m_\pi^{-1}$), the unitary matrix $U$ does not depend on $x_\mu$, so only the sixth, seventh, and eighth terms in $\mathcal{L}_r^{(4)}$ are relevant to the partition function. For $\theta \neq 0$,
\[ \theta \] enters the chiral effective lagrangian through the combinations \( \mathcal{M} e^{i\theta/N_f} \) and \( \mathcal{M}^\dagger e^{-i\theta/N_f} \).

Thus these three potential terms can be written as
\[
L_6^r \left[ 4B_0 \text{ReTr}(\mathcal{M} e^{i\theta/N_f} U^\dagger) \right]^2 + L_7^r \left[ i4B_0 \text{ImTr}(\mathcal{M} e^{i\theta/N_f} U^\dagger) \right]^2 + 8L_8^r B_0^2 \text{ReTr} \left[ (\mathcal{M} e^{i\theta/N_f} U^\dagger)^2 \right].
\]

Without loss of generality, we can take \( U \) and \( \mathcal{M} \) to be diagonal,
\[
U = \text{diag} \left( e^{i\alpha_1}, e^{i\alpha_2}, \ldots, e^{i\alpha_{N_f}} \right), \quad \sum_{j=1}^{N_f} \alpha_j = 0,
\]
\[
\mathcal{M} = \text{diag}(m_1, \ldots, m_{N_f}),
\]
therefore the contributions of (30) and (31) become (dropping the terms without \( U \) dependence)
\[
\sum_{j=1}^{N_f} m_j \cos \phi_j - \sum_{j=1}^{N_f} \left\{ \lambda_P, \lambda_P^\dagger \right\}_{jj} m_j \cos \phi_j \frac{M_P^2}{16\pi^2} \ln \frac{M_P^2}{\mu_{\text{sub}}^2}
\]
\[
+ 16B_0^2 L_6^r \left( \sum_{j=1}^{N_f} m_j \cos \phi_j \right)^2 - 16B_0^2 L_7^r \left( \sum_{j=1}^{N_f} m_j \sin \phi_j \right)^2 + 8B_0^2 L_8^r \sum_{j=1}^{N_f} m_j^2 \cos 2\phi_j,
\]
where \( \phi_j = \theta/N_f - \alpha_j \), and \( \sum_j \phi_j = \theta \).

Again, we use small \( \theta \) (small \( \phi_j \)'s) approximation and keep terms up to the order of \( \phi_j^2 \), then the evaluation of the integral in the partition function in the limit \( m_j \Omega \Sigma \gg 1 \) amounts to minimizing the generating functional
\[
\min_\phi \left[ \frac{\sum_{j=1}^{N_f} m_j \phi_j^2}{2} - \frac{\sum_{j=1}^{N_f} \left\{ \lambda_P, \lambda_P^\dagger \right\}_{jj} m_j \phi_j^2}{8F^2} \frac{M_P^2}{16\pi^2} \ln \frac{M_P^2}{\mu_{\text{sub}}^2}
\]
\[
+16B_0^2 L_6^r \sum_{j=1}^{N_f} m_j \phi_j^2 + 16B_0^2 L_7^r \left( \sum_{j=1}^{N_f} m_j \phi_j \right)^2 + 16B_0^2 L_8^r \sum_{j=1}^{N_f} m_j^2 \phi_j^2 \right],
\]
with the constraint \( \sum_j \phi_j = \theta \). We introduce the Lagrange multiplier \( \lambda \) to incorporate the constraint in finding the minimum. For simplicity, we define
\[
A_j \equiv \frac{\sum_{i=1}^{N_f} m_i}{2} - \frac{\sum_{j=1}^{N_f} \left\{ \lambda_P, \lambda_P^\dagger \right\}_{jj} m_j M_P^2}{16\pi^2} \ln \frac{M_P^2}{\mu_{\text{sub}}^2} + 16B_0^2 \left( L_6^r m_j \sum_{i=1}^{N_f} m_i + L_8^r m_j^2 \right),
\]
\[
B_j \equiv 4B_0(L_j^r)^{1/2} m_j.
\]

Then the minimization problem amounts to solving the equation
\[
\frac{\partial}{\partial \phi_j} \left[ \sum_{j=1}^{N_f} A_j \phi_j^2 + \left( \sum_{j=1}^{N_f} B_j \phi_j \right)^2 - \lambda \left( \sum_{j=1}^{N_f} \phi_j - \theta \right) \right] = 0,
\]

which gives
\[ A_i \phi_i + B_i \left( \sum_{j=1}^{N_f} B_j \phi_j \right) = \frac{\lambda}{2} \]  

(33)

Defining \((T)_{ij} \equiv 2A_i \delta_{ij} + 2B_i B_j\), (33) becomes
\[ \sum_{j=1}^{N_f} (T)_{ij} \phi_j = \lambda, \quad i = 1, \ldots, N_f. \]  

(34)

Thus we can obtain \(\lambda\) using the constraint
\[ \theta = \sum_{j=1}^{N_f} \phi_j = \lambda \sum_{j=1}^{N_f} \sum_{i=1}^{N_f} (T^{-1})_{ij} \Rightarrow \lambda = \theta \left[ \sum_{j=1}^{N_f} \sum_{i=1}^{N_f} (T^{-1})_{ij} \right]^{-1}. \]  

(35)

Now multiplying Eq. (33) with \(\phi_i\) and summing over \(i\), we obtain
\[ \min_\phi \left[ \sum_{j=1}^{N_f} A_j \phi_j^2 + \left( \sum_{j=1}^{N_f} B_j \phi_j \right)^2 \right] = \frac{\lambda \theta}{2}, \]  

(36)

which can be used to simplify (32) to
\[ \frac{\lambda \theta}{2} = \frac{\theta^2}{2} \left[ \sum_{i,j=1}^{N_f} (T^{-1})_{ij} \right]^{-1}, \]  

(37)

where the last equality follows from (35). Finally, the partition function of QCD with \(N_f\) flavors to the one-loop order of ChPT in the limit \(m \Sigma \Omega \gg 1\) is equal to
\[ Z_{N_f}(\theta) = Z_0 \exp \left\{ \frac{\Omega \theta^2}{2} \left[ \sum_{i,j=1}^{N_f} (T^{-1})_{ij} \right]^{-1} \right\}, \]  

(38)

and the vacuum energy density is
\[ \epsilon_{\text{vac}}(\theta) = \epsilon_0 + \frac{\theta^2}{2} \left[ \sum_{i,j=1}^{N_f} (T^{-1})_{ij} \right]^{-1}. \]  

(39)

Thus the topological susceptibility to the one-loop order of ChPT is
\[ \chi_t = \frac{\partial^2 \epsilon_{\text{vac}}}{\partial \theta^2} \bigg|_{\theta=0} = \left[ \sum_{i,j=1}^{N_f} (T^{-1})_{ij} \right]^{-1}. \]  

(40)

To simplify the expression of topological susceptibility, we rewrite the matrix \(T\) as
\[ (T)_{ij} \equiv 2A_i \delta_{ij} + 2B_i B_j = \Sigma(M + T')_{ij}, \]  

(41)
where
\[(T')_{ij} = -\frac{1}{4F^2} \sum_{p} \left\{ \lambda P, \lambda P^\dagger \right\}_{jj} m_j \delta_{ij} \frac{M^2_P}{16\pi^2} \ln \frac{M^2_P}{\mu^2_{sub}} + K_6 \sum_{k=1}^{N_f} m_k m_j \delta_{ij} + K_7 m_i m_j, \]\n\[+ K_6 \sum_{k=1}^{N_f} m_k m_j \delta_{ij} + K_7 m_i m_j, \] (42)

and
\[K_i \equiv \frac{32B^2_i L_i(\mu_{sub})}{\Sigma} = 32 \left( \frac{\Sigma}{F^4} \right) L_i(\mu_{sub}). \] (43)

Since the eigenvalues of the real and symmetric matrix $M^{-1/2}T'M^{-1/2}$ are much less than one in the chiral limit, we can use the Taylor expansion
\[(I + M^{-1/2}T'M^{-1/2})^{-1} \approx I - M^{-1/2}T'M^{-1/2} + O(m^2), \]
and obtain
\[
\chi_t = \left[ \sum_{i,j=1}^{N_f} (T^{-1})_{ij} \right]^{-1} \approx \Sigma \bar{m} \left[ 1 + \frac{\Sigma \bar{m}}{m_i m_j} \sum_{i,j=1}^{N_f} (T')_{ij} \right] = \Sigma \bar{m} \left[ 1 - \frac{1}{4F^2} \sum_{p} \sum_{j=1}^{N_f} \left\{ \lambda P, \lambda P^\dagger \right\}_{jj} \left( \frac{\bar{m}}{m_j} \right) \frac{M^2_P}{16\pi^2} \ln \frac{M^2_P}{\mu^2_{sub}} \right. \\
+ K_6 \sum_{i=1}^{N_f} m_i + N_f (N_f K_7 + K_8) \bar{m} \right], \] (44)
where $\bar{m} \equiv \left( \sum_{i=1}^{N_f} m_i^{-1} \right)^{-1}$, and all terms proportional to $K_i^2$ or $K_i K_j$ have been dropped. Equation (44) is the main result of this paper.

For $N_f = 2$, there are three mesons, $\pi^+, \pi^0$, and $\pi^-$. If we take their masses to be the same and use (29), we obtain
\[
\chi_t = \Sigma \left( \frac{1}{m_u} + \frac{1}{m_d} \right)^{-1} \left[ 1 - \frac{3}{2F^2} \frac{M^2_{\pi \pi}}{16\pi^2} \ln \frac{M^2_{\pi \pi}}{\mu^2_{sub}} + K_6 (m_u + m_d) \right. \\
+ \left. 2(3K_7 + K_8) \frac{m_u m_d}{m_u + m_d} \right]. \] (45)

Next we turn to the case $N_f = 3$. Taking the eight pseudoscalar mesons with non-degenerate masses, we obtain
\[
\chi_t = \Sigma \bar{m} \left[ 1 - \frac{1}{2F^2} \left( \sum_{i \neq j} \left( \frac{\bar{m}}{m_i} + \frac{\bar{m}}{m_j} \right) B_0 (m_i + m_j) \right) \frac{B_0 (m_i + m_j)}{16\pi^2} \ln \frac{B_0 (m_i + m_j)}{\mu^2_{sub}} \\
+ \left( \frac{\bar{m}}{m_u} + \frac{\bar{m}}{m_d} \right) M^2_{\pi \pi} \ln \frac{M^2_{\pi \pi}}{\mu^2_{sub}} \right. \\
+ \left. \frac{1}{3} \left( \frac{\bar{m}}{m_u} + \frac{\bar{m}}{m_d} + 4 \frac{\bar{m}}{m_s} \right) \frac{M^2_{\eta \eta}}{16\pi^2} \ln \frac{M^2_{\eta \eta}}{\mu^2_{sub}} \right] \\
+ K_6 (m_u + m_d + m_s) + 3(3K_7 + K_8) \bar{m} \right]. \] (46)
where \( \bar{m} = \left( m_u^{-1} + m_d^{-1} + m_s^{-1} \right)^{-1} \), and \( B_0 = \Sigma / F_\pi^2 \).

**IV. CONCLUDING REMARK**

In this paper, we have derived the topological susceptibility to the one-loop order in ChPT, in the limit \( m\Sigma \Omega \gg 1 \), for \( N_f = 2 \) [Eq. (45)], \( N_f = 3 \) [Eq. (46)], and an arbitrary number of flavors \( N_f \) [Eq. (44)] respectively.

For \( N_f = 3 \), since the mass of the strange quark is much heavier than the masses of \( u \) and \( d \) quarks, it seems reasonable just to incorporate the one-loop corrections due to the \( u \) and \( d \) quarks. Then, for \( N_f = 2 + 1 \) (\( u \) and \( d \) quarks to the one-loop order, and \( s \) quark at the tree level), the topological susceptibility becomes

\[
\chi_t = \Sigma \left\{ \left( \frac{1}{m_u} + \frac{1}{m_d} \right) \left[ 1 + \frac{3}{2F_\pi^2} \frac{M_\pi^2}{16\pi^2} \ln \frac{M_\pi^2}{\mu_{sub}^2} - K_6(m_u + m_d) - 2(2K_7 + K_8) \frac{m_u m_d}{m_u + m_d} \right] + \frac{1}{m_s} \right\}^{-1}.
\]  

(47)

This supplements (46) for the case \( N_f = 2 + 1 \).

Now the trend of unquenched lattice QCD simulations is to include the charm quark. Thus it is also interesting to include the case \( N_f = 2 + 1 + 1 \), with both \( s \) and \( c \) quarks being kept at the tree level. Then the topological susceptibility is

\[
\chi_t = \Sigma \left\{ \left( \frac{1}{m_u} + \frac{1}{m_d} \right) \left[ 1 + \frac{3}{2F_\pi^2} \frac{M_\pi^2}{16\pi^2} \ln \frac{M_\pi^2}{\mu_{sub}^2} - K_6(m_u + m_d) - 2(2K_7 + K_8) \frac{m_u m_d}{m_u + m_d} \right] + \frac{1}{m_s} + \frac{1}{m_c} \right\}^{-1}
\]  

(48)

In view of the one-loop results of \( \chi_t \), [Eqs. (45), (46), (47), and (48)], it would be interesting to see whether the \( \chi_t \) measured in lattice QCD with exact chiral symmetry would agree with the prediction of ChPT. Most importantly, these one-loop formulas provide a viable way to determine the low-energy constants \( F_\pi, L_6, L_7, L_8 \), in addition to the chiral condensate \( \Sigma \) which has already been determined [7, 8, 9] using the formula of \( \chi_t \) at the tree level (23).

Finally, we turn to the second normalized cumulant \( c_4 \). At this moment, we only have a formula of \( c_4 \) (22) at the tree level. For \( N_f = 2 \), the ratio \( c_4 / \chi_t = -1/4 \) in the isospin limit \( (m_u = m_d) \) seems to rule out the instanton gas/liquid model which predicts that \( c_4 / \chi_t = -1 \).
Obviously, it would be interesting to derive a formula of \( c_4 \) to the next (non-vanishing) order in ChPT.

**Appendix**

In this Appendix, we present a heuristic derivation of the relationship between topological susceptibility, chiral condensate, and the quark masses, for an arbitrary number of (non-degenerate) flavors, in the framework of lattice QCD with exact chiral symmetry. Our derivation generalizes that presented in Ref. [17] with degenerate flavors.

Consider the flavor-singlet pseudoscalar \( \eta' \)

\[
\eta'(x) = \frac{1}{N_f} \sum_{i=1}^{N_f} \bar{q}_i(x)\gamma_5 q_i(x).
\]

Its correlator at zero momentum is

\[
G_{\eta'}(p = 0) = \frac{1}{\Omega} \sum_{x,y} \langle \eta'(x)\eta'(y) \rangle
\]

\[
= \frac{1}{\Omega N_f^2} \sum_{x,y} \sum_{i,j=1}^{N_f} \langle \bar{q}_i(x)\gamma_5 q_i(x)\bar{q}_j(y)\gamma_5 q_j(y) \rangle
\]

\[
= \frac{1}{\Omega Z N_f^2} \int [dU] \det D(m)e^{-S_g[U]} \times
\]

\[
\left\{ \sum_{i=1}^{N_f} \text{Tr}[(D_c + m_i)^{-1}\gamma_5 (D_c + m_i)^{-1}\gamma_5] - \left( \sum_{i=1}^{N_f} \text{Tr}[(D_c + m_i)^{-1}\gamma_5] \right)^2 \right\}
\]

\[
= \frac{1}{\Omega Z N_f^2} \int [dU] \det D(m)e^{-S_g[U]} \left\{ \sum_{i=1}^{N_f} \frac{1}{m_i} \text{Tr}(D_c + m_i)^{-1} - \left[ \sum_{i=1}^{N_f} \frac{1}{m_i}(n_+ - n_-) \right]^2 \right\}, \tag{49}
\]

where \( S_g[U] \) is the gauge action,

\[
\det D(m) = \prod_{i=1}^{N_f} \det[(D_c + m_i)(1 + rD_c)^{-1}],
\]

\[
Z = \int [dU] \det D(m)e^{-S_g[U]},
\]

and the identity

\[
\text{Tr}[(D_c + m)^{-1}\gamma_5 (D_c + m)^{-1}\gamma_5] = \frac{1}{m}\text{Tr}(D_c + m)^{-1},
\]

has been used in the last equality of (49). Here \( D_c = D(1 - rD)^{-1} \) is the chirally symmetric Dirac operator of a Ginsparg-Wilson Dirac operator \( D \) satisfying \( D\gamma_5 + \gamma_5 D = 2rD\gamma_5 D \).
Now taking the thermodynamic limit ($\Omega \to \infty$), and then the chiral limit ($m_i \to 0$), (49) gives

$$G_{\eta'}(0) = \frac{1}{N_f^2} \left( \sum_{i=1}^{N_f} \frac{1}{m_i} \right) \left\{ \Sigma - \left( \sum_{i=1}^{N_f} \frac{1}{m_i} \right) \chi_t \right\},$$

(50)

where

$$\Sigma = \lim_{m_i \to 0} \lim_{\Omega \to \infty} \frac{1}{\Omega} \langle \text{Tr}(D_c + m_i)^{-1} \rangle,$$

$$\chi_t = \lim_{m_i \to 0} \lim_{\Omega \to \infty} \frac{1}{\Omega} \langle (n_+ - n_-)^2 \rangle.$$

If $\eta'$ stays massive, then its propagator $G_{\eta'} \propto m_{\eta'}^{-2}$ must be non-singular. This implies that the coefficient of the singular factor $\left( \sum_{i=1}^{N_f} \frac{1}{m_i} \right)$ in (50) behaves like $O(m)$, i.e.,

$$\chi_t = \Sigma \left( \sum_{i=1}^{N_f} \frac{1}{m_i} \right)^{-1},$$

which agrees with the Leutwyler-Smilga relation.

For $N_f = 2 + 1$ with fixed $m_s$, (50) is modified to

$$G_{\eta'}(0) = \frac{1}{N_f^2} \left( \frac{2}{m_{u,d}} \right) \left\{ \Sigma \left( 1 + \frac{m_{u,d}}{2m_s} \langle s\bar{s} \rangle \right) - \frac{2}{m_{u,d}} \left( 1 + \frac{m_{u,d}}{2m_s} \right)^2 \chi_t \right\},$$

(51)

where

$$\langle s\bar{s} \rangle \equiv \lim_{\Omega \to \infty} \frac{1}{\Omega} \langle \text{Tr}(D_c + m_s)^{-1} \rangle.$$

In the limit $m_{u,d} \to 0$, the coefficient of the singular factor $2/m_{u,d}$ in (51) behaves like $O(m_{u,d})$, i.e.,

$$\chi_t = \left( \frac{\Sigma}{m_u} + \frac{\Sigma}{m_d} + \frac{\langle s\bar{s} \rangle}{m_s} \right) \left( \frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} \right)^{-2},$$

(52)

which provides a more physical relationship (between $\chi_t$, $\Sigma$, $\langle s\bar{s} \rangle$, and the quark masses) than the Leutwyler-Smilga relation (4), since it reduces to (4) only in the (unphysical) limit $\langle s\bar{s} \rangle = \Sigma$.

Now it is straightforward to generalize (52) to the case $N_f = 2 + 1 + 1$,

$$\chi_t = \left( \frac{\Sigma}{m_u} + \frac{\Sigma}{m_d} + \frac{\langle s\bar{s} \rangle}{m_s} + \frac{\langle \bar{c}c \rangle}{m_c} \right) \left( \frac{1}{m_u} + \frac{1}{m_d} + \frac{1}{m_s} + \frac{1}{m_c} \right)^{-2},$$

(53)

where

$$\langle \bar{c}c \rangle \equiv \lim_{\Omega \to \infty} \frac{1}{\Omega} \langle \text{Tr}(D_c + m_c)^{-1} \rangle.$$

It would be interesting to determine $\Sigma$, $\langle s\bar{s} \rangle$, and $\langle \bar{c}c \rangle$ with the data of $\chi_t$ in lattice QCD with exact chiral symmetry.
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