Bootstrap Percolation in Living Neural Networks

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Abstract Recent experimental studies of living neural networks reveal that their global activation induced by electrical stimulation can be explained using the concept of bootstrap percolation on a directed random network. The experiment consists in activating externally an initial random fraction of the neurons and observe the process of firing until its equilibrium. The final portion of neurons that are active depends in a non linear way on the initial fraction. The main result of this paper is a theorem which enables us to find the final proportion of the fired neurons, in the asymptotic case, in the case of random directed graphs with given node degrees as the model for interacting network. This gives a rigorous mathematical proof of a phenomena observed by physicists in neural networks.

Keywords Bootstrap percolation · Phase transition · Random graphs · Neural networks

1 Introduction

Recent experimental studies of living neural networks [8, 12] reveal that their global activation induced by electrical stimulation can be explained using the concept of bootstrap percolation on a directed random network. The experiment consists in activating externally an initial random fraction of the neurons and observe the process of firing until its equilibrium. The final portion of neurons that are active depends in a non linear way on the initial fraction. The main result shown by experiments is that there exists a non-zero critical value for the fraction of initially (i.e., externally) excited neurons beyond which the global activity jumps to an almost complete activation of the network, while below this critical value the firing essentially does not spread. The main result of this paper is a theorem which enables us to find the asymptotic of final proportion of the fired neurons in the case of random directed graphs with given node degrees as the model for interacting network. This gives a rigorous mathematical proof of a phenomena observed by physicists in neural networks [10].

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In that paper, Cohen et al. find this asymptotic via mean-field assumption and they compare it to simulations and experiment. The validity of the random graph approximation to metric graphs such as the experimental neural networks is discussed in [22]. Bootstrap percolation model has been used in several related applications (see for example [15, 20, 23]). The model has a rich history in statistical physics, mostly on \( G = \mathbb{Z}^d \) and finite boxes. The problem of complete occupation on \( \mathbb{Z}^2 \) was solved by van Enter in [13]. The existence of a sharp metastability threshold in \( d \)-dimensional lattices was proved by Holroyd [16]. More recently, bootstrap percolation has been studied on the random regular graph [5], random graphs with given vertex degrees [3], and also on infinite trees [4, 14].

A neural network is a group of interconnected neurons functioning as a circuit. The neural network is modeled as a directed graph [8] whose nodes are neurons connected by synapses. The total number of neurons is \( n \). Let \( G = (V, E) \) be a directed graph on the vertex set \( V = \{1, \ldots, n\} \). We write \( i \to j \) if there is a directed link from \( i \) to \( j \). The in-degree of a node \( i \), denoted by \( d_{\text{in}}(i) \) is the number of links that point into the node, i.e., the number of links \( j \to i \) for \( j \in V \). Similarly the out-degree of a node \( i \), denoted by \( d_{\text{out}}(i) \), is the number of links emanating from \( i \), the number of links \( i \to j \) for \( j \in V \).

The adjacency matrix of a directed graph \( G \) on \( n \) vertices is the \( n \times n \) matrix \( A \) with coordinates \( A_{ij} = 1 \) if \( j \to i \) and 0 otherwise.

We now give a precise description of the model we consider here. At the beginning of the process, and as a direct response to the externally applied electrical stimulus, a neuron has a probability \( \alpha \) to fire. Once a neuron has fired, it stays “on” forever. A neuron will be “on” at time \( t + 1 \) if either it was on at time \( t \) or if at least \( \Omega \) of its incoming nodes were on at time \( t \), for some \( \Omega \) fixed in the model.

We denote by \( X_t(i) \) the state of the neuron \( i \) at time \( t \): \( i \) is on if \( X_t(i) = 1 \) and off if \( X_t(i) = 0 \). At each time step \( t + 1 \), for the state of the node \( i \) we have

\[
X_{t+1}(i) = X_t(i) + (1 - X_t(i)) \mathbb{1} \left( \sum_j A_{ij} X_t(j) \geq \Omega \right),
\]

where \( \mathbb{1}(\Xi) \) denotes the indicator of an event \( \Xi \); this is 1 if \( \Xi \) holds and 0 otherwise. The dynamics is monotonic from the definition. Indeed, since a firing neuron can never turn off, we have \( X_{t+1}(i) \geq X_t(i) \). When the algorithm finishes (suppose after \( n \) time steps), then the final state of a node \( i \) will be represented by \( X(i) \): i.e., \( X(i) = 1 \) if node \( i \) is active and \( X(i) = 0 \) otherwise. Let us define \( \Phi^{(n)}(\alpha) \) as

\[
\Phi^{(n)}(\alpha) := n^{-1} \sum_{j=1}^{n} X(j).
\]

In this paper, we are interested to find \( \Phi(\alpha) \) the asymptotic value of \( \Phi^{(n)}(\alpha) \) when \( n \to \infty \) in the case of random directed graphs with arbitrary degree distribution as the underlying model for the interacting network (see for example [11, 18, 19]). Let us define \( P(j, k) \) to be the probability that a randomly chosen vertex has in-degree \( j \) and out-degree \( k \). Since every oriented edge on a directed graph must leave some vertex and enter another, \( P(j, k) \) must satisfy \( \sum_{j,k} (j - k) P(j, k) = 0 \). The next section describes this model of random digraphs.

**Notation** We consider the asymptotic case when \( n \to \infty \) and say that an event holds w.h.p. (with high probability) if it holds with probability tending to 1 as \( n \to \infty \). We shall use \( \to_p \)