How predictions of cosmological models depend on Hubble parameter data sets

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We explore recent estimations of the Hubble parameter $H$ depending on redshift $z$, which include 31 $H(z)$ data points measured from differential ages of galaxies and 26 data points, obtained with other methods. We describe these data together with Union 2.1 observations of Type Ia supernovae and observed parameters of baryon acoustic oscillations with 2 cosmological models: the standard cold dark matter model with the $\Lambda$ term ($\Lambda$CDM) and the model with generalized Chaplygin gas (GCG). For these models with different sets of $H(z)$ data we calculate two-parameter and one-parameter distributions of $\chi^2$ functions for all observed effects, estimate optimal values of model parameters and their 1$\sigma$ errors. For both considered models the results appeared to be strongly depending on a choice of Hubble parameter data sets if we use all 57 $H(z)$ data points or only 31 data points from differential ages. This strong dependence can be explained in connection with 4 $H(z)$ data points with high redshifts $z > 2$.

I. INTRODUCTION

The latest astronomical observations and their astrophysical interpretation [1] let cosmologists conclude that our Universe demonstrates accelerated expansion and it contains $\simeq 4\%$ of visible baryonic matter, about $26\%$ of cold dark matter and $\simeq 70\%$ of dark energy (DE). The visible and dark matter have properties of cold dust with close to zero pressure. However dark energy has another equation of state with large negative pressure $p_{DE}$ close to its energy density $-\rho_{DE}$ with minus sign. Such a form of matter is considered as a source of the current cosmological acceleration, it helps us to construct a model that can describe all available now observational data and restrictions [1–4].

The simplest way to modify the Einstein theory of gravitation and to include dark energy with the mentioned properties is to add the $\Lambda$ term into the Einstein equations. In this case cosmological solutions can demonstrate accelerated expansion. The resulting dynamical equations may be also obtained, if we add the dark energy component with the equation of state $p_{DE} = -\rho_{DE}$ to the usual visible matter and cold dark matter components. This cosmological model is called $\Lambda$CDM (the $\Lambda$ term with cold dark matter), it is now the most popular and usually considered as the standard model in interpretation of observational data [1–3].

However, the $\Lambda$CDM model has some problems, in particular, vague nature of dark energy and dark matter, the fine tuning problem for the small observed value of $\Lambda$ and the coincidence problem with surprising proximity of DE and matter contribution in total energy balance nowadays [5–8]. Due to these reasons cosmologists suggest a lot of alternative models (see reviews [3–7]), in particular, scenarios with nontrivial equations of state [8–11], with interaction between dark components [12–15], with $F(R)$ Lagrangian [16–18], additional space dimensions [19] and many others.

In particular, in this paper together with the $\Lambda$CDM model we consider the model with generalized Chaplygin gas (GCG) [8–11]. In this model two dark fluids — dark energy and dark matter are unified and represented as one dark component (generalized Chaplygin gas) with the following

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equation of state connecting energy density $\rho_g$ and pressure $p_g$:

$$p_g = -B \rho_g^{-\alpha}.$$  \hspace{1cm} (1)

Here $B$ and $\alpha$ are positive constants. This fluid generalizes the classical Chaplygin gas \cite{8} with the equation of state $p = \text{const}/\rho$.

For the models $\Lambda$CDM and GCG in this paper we calculate limitations on model parameters determined from available recent observations including the Type Ia supernovae data (SN Ia) from Union 2.1 satellite \cite{4}, observable parameters baryon acoustic oscillations (BAO) and we pay special attention to different data sets of the Hubble parameter estimations $H(z)$.

Type Ia supernovae are usually considered as standard candles in the Universe, because they give possibility for each event to determine its epoch and the distance (luminosity distance) to this object. Supernova is an exploding star with huge energy release, creating a shock wave on the expanding shell \cite{20}. They are observed in rather far galaxies because of their giant luminosity. All supernovae are classified in correspondence with time dependence of the their brightness (the light curve) and their spectrum. In particular, stars of Type I have hydrogen-deficient optical spectrum and they belong to Type Ia subdivision, if they also have strong absorption near the silicon line 615 nm. For Type Ia supernovae astronomers can definitely determine their luminosity distances from light curves. In this paper Sect. III we use the Union 2.1 compilation \cite{4} with 580 SN Ia.

The observable effect of baryon acoustic oscillations (BAO) is generated by acoustic waves with ions (baryons), which propagated in the relativistic plasma before the recombination epoch and stopped after the drag era corresponding to $z_d \simeq 1059.3$ \cite{1}. This effect is observed as disturbances (a bump) in the correlation function of the galaxy distribution at the sound horizon scale $r_s(z_d)$ \cite{1, 21}. In Sect. III we analyze two types of observational manifestations the BAO effect from Refs. \cite{22} - \cite{39}, in particular, estimations of the Hubble parameter $H(z)$ for different redshifts $z$ \cite{28} - \cite{39}.

The Hubble parameter $H$ is the logarithmic derivative of the scale factor $a$ with respect to time $t$, redshift $z$ is also expressed via $a$

$$H = \frac{\dot{a}}{a}, \quad z = \frac{a_0}{a} - 1 = \frac{1}{a} - 1,$$  \hspace{1cm} (2)

if we choose here and below the value $a$ nowadays: $a_0 = a(t_0) = 1$.

The Hubble parameter $H(z)$ as the function of $z$ may be estimated with different methods: in addition to the mentioned BAO effects \cite{28} - \cite{39} (26 data points) we also have the $H(z)$ data measured from differential ages of galaxies \cite{40} - \cite{46} (31 data points are tabulated Sect. III).

In this paper we compare different approaches in choosing $H(z)$ data, make calculations with all 57 $H(z)$ data points or only 31 points from differential ages and demonstrate for 2 popular cosmological models $\Lambda$CDM and GCG that predictions of optimal model parameters strongly depend on a considered Hubble parameter data set.

In Sect. II we make a brief review of the models $\Lambda$CDM and GCG and their dynamics, in Sect. III describe observational data and in Sect. IV we demonstrate and analyze the results of our calculations.

II. MODELS

For the $\Lambda$CDM model and the model with generalized Chaplygin gas (GCG) the dynamical equations are deduced from the Einstein equations for the Robertson-Walker metric with the curvature sign $k$

$$ds^2 = -dt^2 + a^2(t) \left[ (1 - kr^2)^{-1} dr^2 + r^2 d\Omega \right]$$
and may be reduced to the system

\[
\frac{3\dot{a}^2 + k}{a^2} = 8\pi G\rho + \Lambda,
\]

\[
\dot{\rho} = -3\frac{\dot{a}}{a} (\rho + p).
\]

Here the dot denotes the time derivative, \(\rho\) and \(p\) are correspondingly the energy density and pressure of all matter, \(G\) is the Newtonian gravitational constant, the constant \(\Lambda\) equals zero for the GCG model, the speed of light \(c = 1\). Eq. (4) is the consequence of the continuity condition \(\nabla_\mu T^\mu_\nu = 0\).

For both considered models we can neglect the fraction of relativistic matter (radiation and neutrinos), because the radiation-matter ratio is rather small \(\rho_r/\rho_m \simeq 3 \cdot 10^{-4}[1]\) for observable values \(z \leq 2.36\).

In the \(\Lambda\)CDM model baryons and dark matter may be considered as one component with density \(\rho = \rho_b + \rho_{dm}\) that behaves like dust because of zero pressure \(p = 0\). In this case we use the solution \(\rho/\rho_0 = (a/a_0)^{-3}\) of Eq. (4) and rewrite the Friedmann equation (3) in the form

\[
\frac{H^2}{H_0^2} = \Omega_m a^{-3} + \Omega_\Lambda + \Omega_k a^{-2} = \Omega_m (1 + z)^3 + \Omega_\Lambda + \Omega_k (1 + z)^2.
\]

We divided Eq. (3) by \(3H_0^2\), used Eq. (2) and the following notations for the present time fractions of matter, dark energy (\(\Lambda\) term) and curvature correspondingly:

\[
\Omega_m = \frac{8\pi G\rho(t_0)}{3H_0^2}, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \quad \Omega_k = -\frac{k}{H_0^2}.
\]

These values are connected by the equality

\[
\Omega_m + \Omega_\Lambda + \Omega_k = 1,
\]

resulting from Eq. (5) if we fix \(t = t_0\). Thus, in description of the mentioned observational data the \(\Lambda\)CDM model has 3 independent parameters: \(H_0, \Omega_m\) and \(\Omega_\Lambda\) (or \(\Omega_k\)).

The GCG model includes two matter components: baryons and the generalized Chaplygin gas, the common density is \(\rho = \rho_b + \rho_g\). Unlike the \(\Lambda\)CDM in the GCG model one should separately consider baryonic matter (it may include some part of cold dark matter) and introduce the corresponding fraction

\[
\Omega_b = \frac{8\pi G\rho_b(t_0)}{3H_0^2}
\]

as an additional model parameter. However in Ref. [11] we demonstrated, that results of calculations very weakly depend on \(\Omega_b\). So in this paper we consider the simplified model with one (gas) component and suppose \(\Omega_b = 0\) or \(\rho = \rho_g\). In this case one can substitute the equation of state (4) into Eq. (4), integrate it and obtain the following consequence of the Friedmann equation (3) [9–11]:

\[
\frac{H^2}{H_0^2} = \Omega_k a^{-2} + (1 - \Omega_k) \left[ B_s + (1 - B_s) a^{-3(1+\alpha)} \right]^{1/(1+\alpha)}.
\]

Here the dimensionless parameter \(B_s = B\rho_0^{1-\alpha}\) is used instead of \(B\). If we exclude the mentioned above parameter \(\Omega_b\), the GCG model will have 4 independent parameters: \(\alpha, B_s, \Omega_k\) and \(H_0\).
III. OBSERVATIONAL DATA

A. Supernovae Ia data

In Sect. [1] we briefly mentioned the observational data under investigation and here we describe details. For Type Ia Supernovae (SN Ia) we use \(N_{SN} = 580\) data points from the table [4] after the Union 2.1 satellite investigation. This compilation provides observed (estimated) values of distance moduli \(\mu_i = \mu_i^{\text{obs}}\) for redshifts \(z_i\) in the interval \(0 < z_i \leq 1.41\). We fit free parameters of our models, when compare \(\mu_i^{\text{obs}}\) with theoretical values \(\mu_i^{th}(z_i)\) of the distance moduli, which are logarithms of the luminosity distance [1, 5]:

\[
D_L(z) = \frac{c(1+z)}{H_0} S_k \left( H_0 \int_{0}^{z} \frac{dz}{H(z)} \right), \quad S_k(x) = \begin{cases} \sinh(x\sqrt{\Omega_k})/\sqrt{\Omega_k}, & \Omega_k > 0, \\ x, & \Omega_k = 0, \\ \sin(x\sqrt{|\Omega_k|})/\sqrt{|\Omega_k|}, & \Omega_k < 0. \end{cases}
\]

(9)

For a cosmological model with theoretical value \(H(z)\) [5] or [8] depending on model parameters \(p_1, p_2, \ldots\) we calculate the distance \(D_L(z)\) and the corresponding \(\chi^2\) function, that measures differences between the SN Ia observational data and predictions of a model:

\[
\chi^2_{SN}(p_1, p_2, \ldots) = \min_{H_0} \sum_{i,j=1}^{N_{SN}} \Delta \mu_i (C_{SN}^{-1})_{ij} \Delta \mu_j,
\]

(10)

where \(\Delta \mu_i = \mu_i^{th}(z_i, p_1, \ldots) - \mu_i^{\text{obs}}, C_{SN}\) is the \(580 \times 580\) covariance matrix [4]. For the Union 2.1 data [4] the standard marginalization over the nuisance parameter \(H_0\) is required [11], it is made as the minimum over \(H_0\) in the expression (10).

B. BAO data

For baryon acoustic oscillations (BAO) we take into account the values \(d_z(z_i)\) [21]

\[
d_z(z) = \frac{r_s(z_d)}{D_V(z)}, \quad D_V(z) = \left[ \frac{czD_s^2(z)}{(1+z)^2H(z)} \right]^{1/3}.
\]

(11)

They were extracted for redshifts (redshift ranges) \(z = z_i\) from a peak in the correlation function of the galaxy distribution at the comoving sound horizon scale \(r_s(z_d)\). The value \(z_d\) corresponds to decoupling of photons, for the sound horizon scale \(r_s(z_d)\) here we use the following fitting formula [11]

\[
r_s(z_d) = \frac{(r_d \cdot h)_{fid}}{h}, \quad (r_d \cdot h)_{fid} = 104.57 \text{ Mpc}, \quad h = \frac{H_0}{100 \text{ km}/(s \cdot \text{Mpc})},
\]

(12)

providing true \(h\) dependence of \(r_d\). The value \((r_d \cdot h)_{fid} = 104.57 \pm 1.44\) Mpc is the best fit for the \(\Lambda\)CDM model [11].

In our calculations we use \(N_{BAO} = 26\) BAO data points for \(d_z(z)\) [11] from Refs. 22 - 33, tabulated here in Table I. We add 9 new points from Ref. 33 to 17 ones, which were used earlier in Refs. [10, 11, 14, 15, 18]. We use the covariance matrix \(C_d\) for correlated data from Refs. 22, 25 described in detail in Ref. [11]. So the \(\chi^2\) function for the value (11) yields

\[
\chi^2_{BAO}(p_1, p_2, \ldots) = \Delta d \cdot C_d^{-1}(\Delta d)^T, \quad \Delta d_i = d_i^{\text{obs}}(z_i) - d_i^{th}(z_i).
\]

(13)
Unlike Refs. [11, 14, 15, 18] we do not use in this paper the observational value [21]

\[ A(z) = \frac{H_0 \sqrt{\Omega_m}}{cz} D_V(z), \]

because it essentially depends on \( \Omega_m \), however \( \Omega_m \) is not the model parameter for the GCG model (see Table III).

C. \( H(z) \) data

The Hubble parameter values \( H \) at certain redshifts \( z \) can be measured with two methods: (1) extraction \( H(z) \) from line-of-sight BAO data [28] – [39] including analysis of correlation functions of luminous red galaxies [28, 37], and (2) \( H(z) \) estimations from differential ages \( \Delta t \) of galaxies (DA method) [40] – [46] via Eq. (2) and the following relation:

\[ H(z) = \frac{\dot{a}}{a} = -\frac{1}{1 + z} \frac{dz}{dt} \simeq -\frac{1}{1 + z} \Delta t. \]

The maximal set with \( N_H = 57 \) recent estimations of \( H(z) \) is shown in Fig. 1 and in Table II below, it includes 31 data points measured with DA method (the left side) and 26 data points (the right side), obtained with BAO and other methods. The \( \chi^2 \) function for the \( H(z) \) data is

\[ \chi^2_H(p_1, p_2, \ldots) = \sum_{i=1}^{N_H} \frac{[H_i - H^{th}(z_i, p_1, p_2, \ldots)]^2}{\sigma^2_{H,i}}. \] (14)

In papers [14, 18] we used only \( N_H = 30 \) \( H(z) \) data points estimated from DA method to avoid additional correlation with the BAO data from Table I. This consideration should be taken into account in the present paper: in the next section we calculate separately the \( \chi^2 \) function with \( N_H = 31 \) DA data points from the left column of Table II (30 points from Refs. [14, 18] and the recent point from Ref. [46]) and compare these results with the full \( H(z) \) data from Table II with \( N_H = 57 \) data points.

In Fig. 1 the \( H(z) \) data points from Table II estimated with DA and BAO methods are shown as correspondingly red stars and cyan diamonds. The lines demonstrate the best fitted \( H(z) \)
FIG. 1: $H(z)$ data from Table III, stars and diamonds denote data points correspondingly from DA and BAO methods. The lines are the best fitted for the $\Lambda$CDM and GCG models with 57 and 31 $H(z)$ data points.

V. RESULTS OF ANALYSIS

For any cosmological model we investigate the space of its model parameters $p_1, p_2, \ldots$ (they are $\Omega_m$, $\Omega_\Lambda$, $H_0$ for the $\Lambda$CDM and $\alpha$, $B_s$, $\Omega_k$, $H_0$ for the GCG model) and search the optimal values of these parameters, which yield the most successful description of the observational data from Sect. III. To achieve this purpose, for any set of parameters $p_1, p_2, \ldots$ we use the dependence $H(z)$ (5) or (8), calculate the integral in Eq. (9), the distances $D_L = D_L^{th}(z)$ and $D_V^{th}(z)$ (11), the values $\mu^{th}$, $d_z$, the $\chi^2$ functions $\chi^2_{SN}$ (10), $\chi^2_{BAO}$ (13), $\chi^2_{H}$ (14) and the summarized function

$$\chi^2_{tot} = \chi^2_{SN} + \chi^2_{BAO} + \chi^2_{H}. \tag{15}$$

We search minima of the functions $\chi^2_{H}$ and $\chi^2_{tot}$ in the parameter spaces of a model in the two mentioned variants of the $H(z)$ data sets: with all $N_H = 57$ data points from Table III and with only $N_H = 31$ data points from Refs. [40]–[46], estimated via the DA method.

For both considered models we calculate two-parameter distributions of $\min \chi^2_{tot}$ in planes of
TABLE II: Hubble parameter values $H(z)$ with errors $\sigma_H$ from DA and BAO methods.

| DA method | BAO method |
|-----------|------------|
| $z$ | $H(z)$ | $\sigma_H$ | Refs | $z$ | $H(z)$ | $\sigma_H$ | Refs |
| 0.070 | 69 | 19.6 | [42] | 0.24 | 79.69 | 2.99 | [34] |
| 0.090 | 69 | 12 | [40] | 0.30 | 81.7 | 6.22 | [37] |
| 0.120 | 68.6 | 26.2 | [42] | 0.31 | 78.18 | 4.74 | [33] |
| 0.170 | 83 | 8 | [40] | 0.34 | 83.8 | 3.66 | [34] |
| 0.1791 | 75 | 4 | [43] | 0.35 | 82.7 | 9.1 | [28] |
| 0.1993 | 75 | 5 | [43] | 0.36 | 79.94 | 3.38 | [33] |
| 0.200 | 72.9 | 29.6 | [42] | 0.38 | 81.5 | 1.9 | [38] |
| 0.270 | 77 | 14 | [40] | 0.40 | 82.04 | 2.03 | [33] |
| 0.280 | 88.8 | 36.6 | [42] | 0.43 | 86.45 | 3.97 | [34] |
| 0.3519 | 83 | 14 | [43] | 0.44 | 82.6 | 7.8 | [35] |
| 0.3802 | 83 | 13.5 | [45] | 0.44 | 84.81 | 1.83 | [33] |
| 0.400 | 95 | 17 | [40] | 0.48 | 87.79 | 2.03 | [33] |
| 0.4004 | 77 | 10.2 | [45] | 0.51 | 90.4 | 1.9 | [38] |
| 0.4247 | 87.1 | 11.2 | [45] | 0.52 | 94.35 | 2.64 | [33] |
| 0.4497 | 92.8 | 12.9 | [45] | 0.56 | 93.34 | 2.3 | [33] |
| 0.470 | 89 | 34 | [46] | 0.57 | 87.6 | 7.8 | [29] |
| 0.4783 | 80.9 | 9 | [45] | 0.57 | 96.8 | 3.4 | [32] |
| 0.480 | 97 | 62 | [41] | 0.59 | 98.48 | 3.18 | [33] |
| 0.593 | 104 | 13 | [43] | 0.60 | 87.9 | 6.1 | [35] |
| 0.6797 | 92 | 8 | [43] | 0.61 | 97.3 | 2.1 | [38] |
| 0.7812 | 105 | 12 | [43] | 0.64 | 98.82 | 2.98 | [33] |
| 0.8754 | 125 | 17 | [43] | 0.73 | 97.3 | 7.0 | [35] |
| 0.880 | 90 | 40 | [41] | 2.30 | 224 | 8.6 | [36] |
| 0.900 | 117 | 23 | [40] | 2.33 | 224 | 8 | [39] |
| 1.037 | 154 | 20 | [43] | 2.34 | 222 | 8.5 | [31] |
| 1.300 | 168 | 17 | [40] | 2.36 | 226 | 9.3 | [30] |
| 1.363 | 160 | 33.6 | [44] | | | | |
| 1.430 | 177 | 18 | [40] | | | | |
| 1.530 | 140 | 14 | [40] | | | | |
| 1.750 | 202 | 40 | [40] | | | | |
| 1.965 | 186.5 | 50.4 | [44] | | | | |

We use this functions to determine one-parameter distributions and the corresponding likelihood functions:

$$m_{\chi^2_{tot}}(p_j) = \min_{p_k \neq j} \chi^2_{tot}(p_1, p_2, p_3, \ldots).$$

(16)

$$m_{\chi^2_{tot}}(p_j) = \frac{\chi^2_{tot}(p_1, \ldots)}{k - 1}, \quad L_{tot}(p_j) = \exp \left( - \frac{m_{\chi^2_{tot}}(p_j) - m_{abs}}{2} \right).$$

(17)

Here $m_{abs}$ is the absolute minimum of $\chi^2_{tot}$. 

Two model parameters, for example,

$$m_{\chi^2_{tot}}(p_1, p_2) = \min_{p_3, \ldots} \chi^2_{tot}(p_1, p_2, p_3, \ldots).$$
The results of these calculations for the ΛCDM model with three independent parameters Ω_m, Ω_Λ and H_0 are presented in Figs. 2, 3 and in Table III. In the top-left panel of Fig. 2 we draw the contour plots at 1σ (68.27%), 2σ (95.45%) and 3σ (99.73%) confidence level for the two-parameter distributions (16) of χ^2_{tot} in the (Ω_m, Ω_Λ) plane. The green filled contours describe the m^X_{tot}(Ω_m, Ω_Λ) function for all 57 H(z) data points, the magenta contours present the case with 31 DA H(z) data points. Here the function (16) is

\[
m^X_{tot}(Ω_m, Ω_Λ) = \min_{H_0} χ^2_{tot}(Ω_m, Ω_Λ, H_0).
\]

This distribution includes only H(z) data.

In the top-right panel of Fig. 2 we compare the mentioned contours for χ^2_{tot} (with the same colors) and the similar contours for the function χ^2_H (14), more correctly,

\[
m^X_H(Ω_m, Ω_Λ) = \min_{H_0} χ^2_H(Ω_m, Ω_Λ, H_0).
\]

This distribution includes only H(z) data.

The green circles and magenta stars in Fig. 2 denote the minimum points of m^X_{tot}(Ω_m, Ω_Λ) (and, naturally, for χ^2_{tot}) correspondingly for 57 and 31 H(z) data points. Their coordinates (the optimal values of parameters) are tabulated in Table III. In the same way, the minimum points for χ^2_H are shown in the top-right panel as the deep green square and brown hexagram.

FIG. 2: The ΛCDM model: 1σ, 2σ and 3σ contour plots for two-parameter distributions m^X_{tot}(Ω_m, Ω_Λ) are drawn in (Ω_m, Ω_Λ) plane for 57 and 31 H(z) data points in comparison with contours for min_{H_0} χ^2_H (the top-right panel). The corresponding one-parameter distributions m^X_{tot}(Ω_m) and m^X_H(Ω_m) are in the bottom panels.
In the bottom panels of Fig. 2 we compare the one-parameter distributions \( m_{tot}^{\chi^2}(\Omega_m) \) and \( m_{tot}^{\chi^2}(\Omega_m, \Omega_\Lambda) \). These distributions and the corresponding likelihood functions \( L_{tot}(\Omega_m, \Omega_\Lambda) \) determine \( \sigma \) estimates in Table III (for \( \chi^2_{tot} \)).

In Fig. 2 we see the interesting phenomenon: the optimal values of parameters \( \Omega_m, \Omega_\Lambda \) (and positions of minimum points for \( \chi^2 \)) are essentially different for the two considered cases with 57 and 31 \( H(z) \) data points. This divergence takes place for \( \chi^2_{tot} \) (the left panels in Fig. 2), for example, these estimations for \( \Omega_m \) are correspondingly \( \Omega_m = 0.282 \pm 0.021 \) and \( \Omega_m = 0.349 \pm 0.041 \) (see Table III): the last value 0.349 is beyond 2\( \sigma \) confidence level for the \( N_H = 57 \) case. However for \( \chi^2_H \) this divergence is stronger, the correspondent estimations are \( \Omega_m = 0.227^{+0.036}_{-0.041} \) (for \( N_H = 57 \)) and \( \Omega_m = 0.359^{+0.197}_{-0.232} \) (for \( N_H = 31 \)). This is natural, because the summands \( \chi^2_{SN} + \chi^2_{BAO} \) in \( \chi^2_{tot} \) moderate this effect.

**TABLE III: Optimal values and 1\( \sigma \) estimates of model parameters**

| Model   | \( \text{min} \chi^2_{tot} \) | AIC | \( H_0 \) | \( \Omega_k \) | \( \Omega_m \) | \( \Omega_\Lambda \) | other parameters |
|---------|-------------------------------|-----|----------|--------------|-------------|----------------|-------------------|
| LCDM    | 610.31                        | 616.31 | 71.35^{+0.63}_{-0.62} | -0.085 \pm 0.048 | \Omega_m = 0.282 \pm 0.021 | \Omega_\Lambda = 0.803 \pm 0.028 |
| 57 \( H(z) \) |                               |      |          |              |             |                 |                   |
| LCDM    | 588.96                        | 594.96 | 71.77^{+1.70}_{-1.69} | -0.224^{+0.085}_{-0.084} | \Omega_m = 0.349 \pm 0.041 | \Omega_\Lambda = 0.875 \pm 0.045 |
| 31 \( H(z) \) |                               |      |          |              |             |                 |                   |
| GCG     | 609.94                        | 617.94 | 71.68^{+0.82}_{-0.83} | -0.192^{+0.188}_{-0.170} | \alpha = -0.124^{+0.235}_{-0.138} | \beta = 0.705^{+0.065}_{-0.043} |
| 57 \( H(z) \) |                               |      |          |              |             |                 |                   |
| GCG     | 587.93                        | 595.93 | 70.46^{+2.16}_{-2.51} | +0.019^{+0.541}_{-0.255} | \alpha = 0.647^{+3.25}_{-0.64} | \beta = 0.826^{+0.294}_{-0.111} |
| 31 \( H(z) \) |                               |      |          |              |             |                 |                   |

Below we concentrate on the more relevant summarized function \( \chi^2_{tot} \). In Fig. 3 we present other two- and one-parameter distributions of \( \chi^2_{tot} \) and the likelihood functions for the LCDM model. In particular, in the top-right panel the contour plots for \( m_{tot}^{\chi^2}(\Omega_k, H_0) \) are shown for the cases \( N_H = 57 \) and \( N_H = 31 \) in the same notations. In these calculation we consider the curvature fraction \( \Omega_k \) as an independent parameter (together with \( \Omega_m, H_0 \)), the fraction \( \Omega_\Lambda \) is expressed via Eq. (7): \( \Omega_\Lambda = 1 - \Omega_m - \Omega_k \).

The two-parameter distributions \( m_{tot}^{\chi^2}(\Omega_m, \Omega_\Lambda) \) for \( N_H = 57 \) and \( N_H = 31 \) in the top-right panel of Figs. 2 and 3 let us calculate the one-parameter distributions \( m_{tot}^{\chi^2}(\Omega_m) \), \( m_{tot}^{\chi^2}(\Omega_\Lambda) \) and the likelihood functions \( L_{tot}(\Omega_m) \), \( L_{tot}(\Omega_\Lambda) \) shown in the middle and bottom panels of Fig. 3. The functions \( L_{tot}(H_0) \) are deduced from the two-parameter distributions in the \((\Omega_k, H_0)\) plane.

The best fitted values of \( \text{min} \chi^2_{tot} \) and the model parameters \( \Omega_m, \Omega_\Lambda, \Omega_k, H_0 \) for the LCDM model are presented in Table III for the cases \( N_H = 57 \) and \( N_H = 31 \). The \( \sigma \) errors are calculated from the correspondent likelihood functions \( L_{tot}(p_i) \). We should emphasize, that the number \( N_p \) of model parameters is essential, when we comrade different models. So we also use the Akaike information criterion \( (11, 47) \)

\[
AIC = \text{min} \chi^2_{tot} + 2N_p.
\] (19)

Here \( N_p = 3 \) for the LCDM model.

The similar estimations for the LCDM model were made in many papers, in particular, in Refs. [1, 3, 11, 47, 49] for describing the Type Ia supernovae, \( H(z) \), BAO and other data in various combinations. One can observe the following effect (connected with the described above): the estimations of \( \Omega_m, \Omega_\Lambda, \Omega_k \) and \( H_0 \) in different papers essentially depend on a chosen \( H(z) \) data set. For example, the authors of Refs. [49] used the \( \chi^2_H \) function with \( N_H = 41 \) data points from
both DA and BAO methods and calculated $\Omega_m = 0.237 \pm 0.051$, $\Omega_\Lambda = 0.66 \pm 0.20$. However, when they excluded 3 data points [30, 31, 36] with $z \geq 2.3$, they obtained the enhanced values for both parameters $\Omega_m = 0.40^{+0.18}_{-0.14}$, $\Omega_\Lambda = 0.92^{+0.34}_{-0.23}$ (compare with our results for $\chi^2_H$ in Fig. 2).

If we compare our results for the ΛCDM model with the latest Planck data [1] ($\Omega_m = 0.308 \pm 0.012$, $\Omega_\Lambda = 0.692 \pm 0.012$, $\Omega_k = -0.005^{+0.016}_{-0.017}$, $H_0 = 67.8 \pm 0.9$ km c$^{-1}$Mpc$^{-1}$), we will find some tension for $\Omega_\Lambda$, $Omega_k$ in the case $N_H = 31$ and for $H_0$ in both cases because of too low estimation of $H_0$ in Ref. [1].

The influence of a chosen $H(z)$ data set takes place not only for the ΛCDM model. One can see in Fig. 4 and in Table III that for the GCG model this influence is even more strong. In the top panels we demonstrate the contour plots for two-parameter distributions (16) of $\chi^2_{tot}$ in the $(\alpha, B_s)$ and $(\Omega_k, B_s)$ planes for the cases $N_H = 57$ (blue filled contours) and $N_H = 31$ (red contours). In particular, the two-parameter distributions (16) in the top-left panel are

$$m^\chi_{tot}(\alpha, B_s) = \min_{\Omega_k,H_0} \chi^2_{tot}(\alpha, B_s, \Omega_k, H_0).$$
The circles and stars show the points of minima for $\chi^2_{tot}$. The similar two-parameter contour plots for the GCG model in the $(\Omega_k, H_0)$ plane are drawn in Fig. 5.

The one-parameter distributions $m^\chi_{tot}(\alpha)$, $m^\chi_{tot}(B_s)$, $m^\chi_{tot}(\Omega_k)$ and the corresponding likelihood functions (17) $L_{tot}(p_i)$ are shown in the middle and bottom panels of Fig. 4.

Fig. 4 and Table III demonstrate, that for the GCG model the best fitted values of $\alpha$, $B_s$, $\Omega_k$ strongly depend on a Hubble parameter data: $N_H = 57$ (all data points) or $N_H = 31$ (only from DA method). In particular, the best fitted values $\alpha \simeq -0.124$, $\Omega_k \simeq -0.192$ for $N_H = 57$ change their signs and become $\alpha \simeq +0.647$, $\Omega_k \simeq +0.019$, if $N_H = 31$.

In Fig. 5 we compare the $\Lambda$CDM and GCG models in the plane $(\Omega_k, H_0)$ of their common parameters. For both models we draw the one-parameter distributions $m^\chi_{tot}(\Omega_k)$, $m^\chi_{tot}(H_0)$ (they help us to compare the best results min $\chi^2_{tot}$ for these models) and the likelihood functions $L_{tot}(\Omega_k)$, $L_{tot}(H_0)$.

In the top-left panel of Fig. 5 the filled contours describe the GCG model with $N_H = 57$, 

![Fig. 4: The GCG model with $N_H = 57$ (blue) and $N_H = 31$ (red): two-parameter, one-parameter distributions and likelihood functions for $\chi^2_{tot}$.](image-url)
FIG. 5: Comparison of the two-parameter distributions $\chi^2_{\text{tot}}(\Omega_k, H_0)$ for the ΛCDM and GCG models in the plane $(\Omega_k, H_0)$ of their common parameters for the cases with 57 and 31 $H(z)$ data points (the top-left panel). The corresponding one-parameter distributions are in other panels. Notations correspond to the previous figures.

other contours differ in their color. The points of minima are marked here as the circle (GCG, $N_H = 57$), the pentagram (GCG, $N_H = 31$), the square (ΛCDM, $N_H = 57$) and the hexagrams (ΛCDM, $N_H = 31$) of the corresponding color.

Fig. 5 is useful, when we want to compare predictions the ΛCDM and GCG models in the considered cases $N_H = 57$ and $N_H = 31$. The plots $\mathcal{L}_{\text{tot}}(\Omega_k)$ and $\mathcal{L}_{\text{tot}}(H_0)$ show differences of the best fitted values, the plots $m^\chi_{\text{tot}}(\Omega_k)$ and $m^\chi_{\text{tot}}(H_0)$ describe effectiveness of these models. More detailed information is tabulated in Table III.

V. CONCLUSION

In this paper we describe the observational data for Type Ia supernovae [4], BAO (Table I) and two data sets of the Hubble parameter data $H(z)$ (all $N_H = 57$ data points from Table II and only 31 data points from differential ages) with the ΛCDM model and the model with generalized Chaplygin gas (GCG).

The results are demonstrated in Table III for all models and variants of $N_H$ we calculated the minimal values of the function $\chi^2_{\text{tot}}$ [15], the results of Akaike information criterion (19) and the
best fitted values of model parameters with 1σ errors. For the GCG model we achieve the best minimal values of min $\chi^2_{tot}$, however the Akaike criterion gives advantage to the ΛCDM model, because it has the small number $N_p = 3$ of model parameters (degrees of freedom) in comparison with with $N_p = 4$ for GCG.

But the most striking result of our calculations for both models is the large difference between the best fitted values of model parameters in the cases with $N_H = 57$ $H(z)$ data points from Table II and $N_H = 31$ data points, obtained with DA method (the left hand side of Table II). For the case $N_H = 57$ these results are close to the estimations for these models in Ref. [11], because in that paper we used $H(z)$ data points from both DA and BAO methods (though there were $N_H = 38$ points).

This essential divergence between the predictions of the variants with all $N_H = 57$ and $N_H = 31$ DA data points is seen visually in Fig. 1. It may be explained and connected with 4 $H(z)$ data points [30, 31, 36, 39] with high redshifts $z \geq 2.3$. These data points, obtained with BAO method (see the right hand side of Table II) have small errors $\sigma_H$ and strongly influence on a model predictions, when we take these points into account (in the case $N_H = 57$). Otherwise, when we include only $N_H = 31$ DA data points, this effect disappears.

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