MASSEY PRODUCTS IN MAPPING TORI

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ABSTRACT. Let \( \phi : M \to M \) be a diffeomorphism of a \( C^\infty \) compact connected manifold, and \( X \) its mapping torus. There is a natural fibration \( p : X \to S^1 \), denote by \( \xi \in H^1(X, \mathbb{Z}) \) the corresponding cohomology class. Let \( \lambda \in \mathbb{C}^\ast \).

Consider the endomorphism \( \phi^k \) induced by \( \phi \) in the cohomology of \( M \) of degree \( k \), and denote by \( J_k(\lambda) \) the maximal size of its Jordan block of eigenvalue \( \lambda \). Define a representation \( \rho_\lambda : \pi_1(X) \to \mathbb{C}^\ast \); \( \rho_\lambda(g) = \lambda^{p_\phi(g)} \); let \( H^*(X, \rho_\lambda) \) be the corresponding twisted cohomology of \( X \). We prove that \( J_k(\lambda) \) is equal to the maximal length of a non-zero Massey product of the form \( \langle \xi, \ldots, \xi, a \rangle \) where \( a \in H^k(X, \rho_\lambda) \) (here the length means the number of entries of \( \xi \)).

In particular, if \( X \) is a strongly formal space (e.g. a Kähler manifold) then all the Jordan blocks of \( \phi^k \) are of size 1. If \( X \) is a formal space, then all the Jordan blocks of eigenvalue 1 are of size 1. This leads to a simple construction of formal but not strongly formal mapping tori.

The proof of the main theorem is based on the fact that the Massey products of the above form can be identified with differentials in a Massey spectral sequence, which in turn can be explicitly computed in terms of the Jordan normal form of \( \phi^k \).

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1. INTRODUCTION

The relation between non-vanishing Massey products of length 2 and the Jordan blocks of size greater than 1 was discovered in the work of M. Fernández, A. Gray, J. Morgan, [5], where it was used to prove that certain mapping tori do not admit a structure of a Kähler manifolds. In the work of G. Bazzoni, M. Fernández, V. Muñoz [1] it was proved that the existence of Jordan blocks of size 2 implies the existence of a non-zero triple Massey product of the form \( \langle \xi, \xi, a \rangle \).

The main theorem of the present paper provides systematic treatment of these phenomena, relating the length of non-zero Massey products to the size of Jordan

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blocks. The both numbers turn out to be equal to the number of the sheet where the formal deformation spectral sequence degenerates.

Another approach to the relation between the size of Jordan blocks and formality was developed by S. Papadima and A. Suciu [11], [12]. They prove in particular that if the monodromy homomorphism has Jordan blocks of size greater than 1, then the fundamental group of the mapping torus is not a formal group.

1.1. Overview of the article. The proof of the main theorem is based on the techniques developed in [7]. We begin by an overview of this paper in Section 2. The main theorem of the paper is stated and proved in Section 3. In Section 5 we present a generalization of the main theorem to the case of spaces $X$ endowed with a non-zero cohomology class $\xi \in H^1(X, \mathbb{Z})$.

2. Formal deformations and Massey spectral sequences.

Let $X$ be a connected manifold, and $\xi \in H^1(X, \mathbb{C})$ a non-zero cohomology class, and $\lambda \in \mathbb{C}^*$. There is a spectral sequence starting with $H^*(X, \rho\lambda)$ and converging to the cohomology $H^*(X, \rho\lambda)$ where $\lambda$ is a generic complex number. There are different versions of this spectral sequence in literature, see [3], [9], [10], [4], [7]. We will recall here the versions described in [7], refering to this article for details and proofs.

2.1. Massey spectral sequences. In this subsection we describe a spectral sequence with the differential defined in terms of special Massey products.

Pick $\alpha \in \mathbb{C}$ such that $e^{\alpha} = \lambda$. The cohomology $H^*(X, \rho\lambda)$ can be computed from the twisted DeRham complex $\tilde{\Omega}^*(X) = \Omega^*(X, d\tilde{\omega})$ where $d\tilde{\omega} = dw + \alpha x \wedge \omega$. Let $a \in H^*(X, \rho\lambda)$. An $r$-chain starting from $a$ is a sequence of differential forms $\omega_1, \ldots, \omega_r \in \Omega^*(X)$ such that

$$d\omega_1 = 0, \quad [\omega_1] = a, \quad d\omega_2 = x \wedge \omega_1, \quad \ldots, \quad d\omega_r = x \wedge \omega_{r-1}.$$ 

For an $r$-chain $C$ put $\partial C = x \wedge \omega_r$; this is a cocycle in $\tilde{\Omega}^*(X)$. Denote by $MZ^m_{(r)}$ the subspace of all $a \in H^*(\tilde{\Omega}^*(X))$ such that there exists an $r$-chain starting from $a$.

Denote by $MB^m_{(r)}$ the subspace of all $\beta \in H^*(\tilde{\Omega}^*(X))$ such that there exists an $(r - 1)$-chain $C = (\omega_1, \ldots, \omega_{r-1})$ with $x \wedge \omega_r$ belonging to $\beta$. It is clear that $MB^m_{(i)} \subseteq MZ^m_{(j)}$ for every $i, j$. Put

$$MH^m_{(r)} = MZ^m_{(r)} / MB^m_{(r)}.$$

In the next definition we omit the upper indices and write $MH_{(r)}, MZ_{(r)}$ etc. in order to simplify the notation.

**Definition 2.1.** Let $a \in H^*(\tilde{\Omega}^*(X))$, and $r \geq 1$. We say that the $r$-fold Massey product $\langle \xi, \ldots, \xi, a \rangle$ is defined, if $a \in MZ_{(r)}$. In this case choose any $r$-chain $(\omega_1, \ldots, \omega_r)$ starting from $a$. The cohomology class of $\partial C = x \wedge \omega_r$ is in $MZ_{(r)}$ and is well defined modulo $MB_{(r)}$. The image of $\partial C$ in $MZ_{(r)} / MB_{(r)}$ will be called the $r$-fold Massey product of $\xi$ and $a$ and denoted by

$$\langle \xi, \ldots, \xi, a \rangle \in MZ_{(r)} / MB_{(r)},$$

for $r$ times

We say that the length of this product is equal to $r$.  

Remark 2.2. Observe that the indeterminacy of this Massey product is less than the indeterminacy of a general Massey product as defined for example in [8]. Observe also that usually the length of Massey product is defined as the number of the arguments inside the brackets, so our notion of length is less by 1 than the standard one.

We obtain a homomorphism

\[ M\mathcal{H}_{(r)} \xrightarrow{\Delta_r} M\mathcal{H}_{(r)}; \quad a \mapsto \langle \xi, \ldots, \xi, a \rangle \]

Proposition 2.3. We have \( \Delta_r^n = 0 \), and the cohomology group \( H^*(M\mathcal{H}_{(r)}^*), \Delta_r \) is isomorphic to \( M\mathcal{H}_{(r+1)}^* \).

Definition 2.4. The groups \( M\mathcal{H}_{(r)}^* \) form therefore a spectral sequence, which will be called the Massey spectral sequence associated with \( (X, \xi, \lambda) \), and denoted by \( \mathcal{M}_r^* \) or just \( \mathcal{M}_r^* \) for brevity.

2.2. Formal deformation spectral sequence. Another spectral sequence is associated to a formal deformation of the twisted DeRham complex \( \tilde{\Omega}^*(X) \). Let \( \Lambda = \mathbb{C}[\![t]\!] \), consider a representation \( \hat{\gamma} : \pi_1(X) \to (\mathbb{C}[\![t]\!])^* \) defined by the formula \( \hat{\gamma}(g) = e^{\langle \alpha + t \rangle} \).

Denote by \( \mathit{C}^*(\hat{\gamma}, \hat{\gamma}) \) the complex of \( \hat{\gamma} \)-equivariant chains on \( \hat{X} \), and by \( H^*(\hat{X}, \hat{\gamma}) \) its cohomology. The exact sequence

\[ 0 \to \mathit{C}^*(\hat{\gamma}) \xrightarrow{t} \mathit{C}^*(\hat{\gamma}) \to \mathit{C}^*(x, \rho_{1}) \to 0 \]

gives rise to an exact couple

\[ \begin{array}{ccc}
H^*(\mathit{C}, \hat{\gamma}) & \xrightarrow{t} & H^*(\hat{X}, \hat{\gamma}) \\
\downarrow & & \downarrow \\
H^*(\mathit{C}, \rho_{1}) & & H^*(\hat{X}, \rho_{1})
\end{array} \]

Definition 2.5. The associated spectral sequence will be denoted by \( \mathcal{M}_r^* \) and called the formal deformation spectral sequence.

Theorem 2.6 ([7], th. 3.6). The spectral sequences \( \mathcal{M}_r^* \) and \( \mathcal{M}_r^* \) are isomorphic.

2.3. Formality and strong formality. Recall the classical notion of formality of manifolds, introduced by D. Sullivan. Let \( X \) be a \( C^\infty \) manifold, \( \Omega^*(X) \) be the differential graded algebra of differential forms on \( X \), and \( M^*(X) \) be a minimal model for \( \Omega^*(X) \). The manifold \( X \) is called formal if there is a DGA-homomorphism \( \Omega^*(X) \to H^*(X) \) inducing an isomorphism in cohomology. Compact Kähler manifolds are formal as proved in [2].

Theorem 2.7. ([7], Th. 3.14) Assume that \( X \) is a formal manifold. Then for every \( \xi \in H^1(X, \mathbb{Z}) \) the spectral sequences \( \mathcal{M}_r^* (X, \xi, 1) \) and \( \mathcal{M}_r^* (X, \xi, 1) \) degenerate in their second term. Therefore all the Massey products \( \langle \xi, \ldots, \xi, x \rangle \) of length \( \geq 2 \), where \( x \in H^*(X, \mathbb{C}) \) are equal to zero.
Assume that \( X \) is connected and denote \( \pi_1(X) \) by \( G \). Denote by \( \text{Ch}(G) \) the set of all homomorphisms \( G \to \mathbb{C}^* \). For a character \( \rho \in \text{Ch}(G) \) denote by \( E_{\rho} \) the corresponding flat vector bundle over \( X \). Put
\[
\Omega^*(X) = \bigoplus_{\rho \in \text{Ch}(G)} \Omega^*(X, E_{\rho}).
\]
The pairing \( E_{\rho} \otimes E_{\eta} \approx E_{\rho \eta} \) induces a natural structure of a differential graded algebra on the vector space \( \Omega^*(X) \).

**Definition 2.8.** A \( C^\infty \) manifold \( X \) is strongly formal if the differential graded algebra \( \Omega^*(X) \) is formal.

This notion was introduced in the paper of T. Kohno and the author [7], although it was implicit already in the paper of H. Kasuya [6]. It turns out that Kähler compact manifolds are strongly formal ([7], Th. 6.5). This is a strengthening of the formality condition, since there exist formal, but not strongly formal spaces (see H. Kasuya’s paper [6], §9, Example 1 and also [7], Remark 6.6).

**Theorem 2.9.** ([7], Th. 6.3) Assume that \( X \) is a strongly formal manifold. Then for every \( \xi \in H^1(X, \mathbb{Z}) \) and \( \lambda \in \mathbb{C}^* \) the spectral sequences \( E^*_r(X, \xi, \lambda) \) and \( H^*_r(X, \xi, \lambda) \) degenerate in their second term. Therefore all the Massey products \( \langle \xi, \ldots, \xi, x \rangle \) of length \( \geq 2 \), where \( x \in H^*(X, \rho_\lambda) \) are equal to zero.

2.4. Terminological conventions.

**Definition 2.10.** If a spectral sequence \( E^*_r \) degenerates at sheet \( m \) in degree \( k \) we write \( \sigma_k(E^*_r) = m \).

In particular, the number \( \sigma_k(E^*_r) - 1 \) equals the maximal length of a non-zero Massey product of the form \( \langle \xi, \ldots, \xi, a \rangle \) where \( a \in H^k(X, \rho_\lambda) \).

**Definition 2.11.** Let \( A : L \to L \) be a linear map of a finite-dimensional vector space \( L \) over \( \mathbb{C} \), and \( \lambda \in \mathbb{C} \). The \( A \)-invariant subspace of \( L \) corresponding to \( \lambda \) will be denoted by \( N(A, \lambda) \). The degree of nilpotency of \( (A - \lambda) | N(A, \lambda) \) will be denoted by \( \nu(A, \lambda) \). If \( N(A, \lambda) = 0 \), we set \( \nu(A, \lambda) = 0 \) by convention. Thus the number \( \nu(A, \lambda) \) is equal to the maximal size of a Jordan block of \( A \) with eigenvalue \( \lambda \).

3. Main theorem

Let \( \phi : M \to M \) be a diffeomorphism of a \( C^\infty \) compact connected manifold, denote by \( X \) its mapping torus. There is a natural fibration \( p : X \to S^1 \), denote by \( \xi \in H^1(X, \mathbb{Z}) \) the corresponding cohomology class. Let \( \lambda \in \mathbb{C}^* \). Composing the homomorphism \( \mathbb{Z} \to \mathbb{C}^* : n \mapsto e^{n \lambda} \) with the homomorphism \( p_* : \pi_1(X) \to \pi_1(S^1) \approx \mathbb{Z} \) we obtain a representation \( \rho_\lambda : \pi_1(X) \to \mathbb{C}^* \). Denote by \( H^*(X, \rho_\lambda) \) the corresponding cohomology with local coefficients. We have a natural pairing
\[
H^*(X, \mathbb{C}) \otimes H^*(X, \rho_\lambda) \to H^*(X, \rho_\lambda)
\]
and the corresponding Massey products \( \langle \xi, \ldots, \xi, x \rangle \) are defined for \( x \in H^*(X, \rho_\lambda) \) (see Section 2.1). Denote by \( \mu_k(\lambda) \) the maximal number \( r \) such that \( \langle \xi, \ldots, \xi, x \rangle \) is not equal to zero. Then \( \mu_k(\lambda) + 1 \) is the number of the sheet where the two spectral sequences discussed at the previous section degenerate. Consider the
endomorphism $\phi^k_\lambda : H^k(M, \mathbb{C}) \to H^k(M, \mathbb{C})$, and denote by $J_k(\lambda)$ the maximal size of its Jordan block of eigenvalue $\lambda$.

**Theorem 3.1.**

1) We have $J_k(\lambda) = \mu_k(\lambda)$ for every $k$ and $\lambda$.

2) If $X$ is a strongly formal space (e.g. a compact Kähler manifold) then all Jordan blocks of $\phi^*_k$ are of size 1.

3) If $X$ is a formal space, then all Jordan blocks of eigenvalue 1 are of size 1.

**Proof.** The parts 2) and 3) follow immediately from the part 1) in view of degeneracy of corresponding spectral sequences. Proceeding to the part 1), choose any $\alpha \in \mathbb{C}$ such that $e^\alpha = \lambda$ and consider the exact couple $E$ (see (1)).

Replacing $t$ by $e^{t+\alpha} - e^\alpha$ in its upper line we obtain an exact couple $E'$ isomorphic to $E$ (since the element $\lambda(e^{t} - 1)/t \in \mathbb{C}[[t]]$ is invertible). Let $L = \mathbb{C}[u, u^{-1}] \cong \mathbb{C}[\mathbb{Z}]$, we have a tautological representation $\mathbb{Z} \to L^*$, composing it with the homomorphism $p_u : \pi_1(X) \to \mathbb{Z}$ we obtain a representation $\beta : \pi_1(X) \to L^*$. The corresponding twisted cohomology of $X$ will be denoted by $H^*(X, \beta)$. We have an exact couple

$$
\begin{array}{c}
H^*(X, \beta) \\
\downarrow \downarrow \\
H^*(X, \rho_\lambda)
\end{array}
$$

(2)

**Lemma 3.2.** There is a commutative diagram of $L$-homomorphisms

$$
\begin{array}{ccc}
H^k(X, \beta) & \xrightarrow{u-\lambda} & H^k(X, \beta) \\
\downarrow \downarrow & & \downarrow \downarrow \\
H_{k-1}(M, \mathbb{C}) & \xrightarrow{\phi^*_u - \lambda} & H_{k-1}(M, \mathbb{C})
\end{array}
$$

(3)

**Proof.** Let $\overline{X}$ be the corresponding infinite cyclic covering of $X$. The simplicial chain complex $C_*(\overline{X})$ is a free finitely generated chain complex of $L$-modules. Its homology $H_*(\overline{X})$ is isomorphic to $H_*(M)$ as $L$-module (the element $u \in L$ acts as $\phi_u$ on $H_*(M)$). The universal coefficient theorem implies an $L$-module isomorphism

$$
H^k(X, \beta) \cong \text{Ext}^1_{L}(H_{k-1}(M, \mathbb{C}), L) \cong H_{k-1}(M, \mathbb{C}).
$$

The exact couple $\mathcal{D}$ is therefore isomorphic to the following one
Here \( \deg i = \deg l = 0, \deg j = 1 \). Let \( A_k = N(\phi_k^k, \lambda) \); denote by \( B_k \) the sum of all subspaces \( N(\phi_k^k, \mu) \) with \( \mu \neq \lambda \). The restriction \( (\phi_k^k - \lambda) \mid A_k \) is nilpotent of degree equal to \( J_k(\lambda) \), and the restriction \( (\phi_k^k - \lambda) \mid B_k \) is an isomorphism of \( B_k \) onto itself. The assertion of the theorem follows now from the following lemma.

**Lemma 3.3.** Let \( \mathcal{E} \) be a graded exact couple:

(5) \[
\begin{array}{c}
D \\
\downarrow \delta \\
E \\
\uparrow \tau
\end{array}
\]

with \( \deg i = \deg l = 0, \deg j = 1 \). Assume that the \( k \)-th component \( i_k : D_k \rightarrow D_k \) of the homomorphism \( i \) decomposes as

\[
\delta \oplus \tau : A \oplus B \rightarrow A \oplus B
\]

where \( \delta \) is nilpotent of degree \( m \) and \( \tau \) is injective. Then the spectral sequence \( \mathcal{E}^\ast \) degenerates at the step \( m + 1 \) in degree \( k \).

**Proof.** The proof is an easy diagram chasing. At the step \( r \) the upper line of the derived exact couple \( \mathcal{E}^r \) equals \( i : i^{-1}(D) \rightarrow i^{-1}(D) \). Observe that \( i : i^m(D) \rightarrow i^m(D) \) is injective, therefore \( d_{m+1} : E_{m+1}^k \rightarrow E_{m+1}^k \) equals 0. On the other hand the homomorphism \( i : i^{-1}(A) \rightarrow i^{-1}(A) \) equals 0, and \( i^{-1}(A) \neq 0 \), therefore the differential \( d_m \) in the module \( E_m^k \) is non-zero. \( \square \)

4. Examples

4.1. **The Heisenberg group.** A classical example of a non-formal space arises from the Heisenberg group (see [2], p. 261). Let \( N \) denote the group of all upper triangular matrices

\[
\begin{pmatrix}
1 & a & b \\
1 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\]

with real coefficients; let \( \Gamma \) be the subgroup of matrices with integer coefficients. The space \( W = N/\Gamma \) is a compact three-dimensional manifold and

(6) \[
b_1(W) = b_2(W) = 2.
\]

The group \( H^1(W, \mathbb{R}) \) is generated by elements \( x, y, \) and \( \langle x, x, y \rangle \neq 0 \).

The space \( W \) is fibred over \( \mathbb{T}^2 \) with fiber \( S^1 \). Composing this fibration with the projection \( \mathbb{T}^2 \rightarrow S^1 \) we obtain a fibration \( W \rightarrow S^1 \) with fiber \( \mathbb{T}^2 \). Thus \( W \) is a mapping torus with a monodromy \( \phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2 \). Let \( \lambda \in \mathbb{C}^* \). The Milnor exact sequence

\[
\cdots \rightarrow H_4(\mathbb{T}^2) \xrightarrow{\phi_4^{-1}} H_4(\mathbb{T}^2) \rightarrow H_4(W) \rightarrow H_{4-1}(\mathbb{T}^2) \rightarrow \cdots
\]
together with [3] shows immediately that \( \phi^1 \) has only one eigenvalue of multiplicity 2, namely 1. Moreover, \( \phi^1 \) must have a Jordan block of size 2, and this corresponds to the above non-vanishing Massey product of length 2.

### 4.2. A formal but not strongly formal mapping torus

We will briefly indicate the necessary modifications. We have an exact couple

\[
(7) \quad H^*(X, \beta) \to H^*(X, \beta) \to H^*(X, \rho_\lambda)
\]

The \( L \)-module \( H^*(X, \beta) \) is isomorphic to a direct sum \( F_0 \oplus F_1 \oplus F_2 \) where \( F_2 \) is a free finitely generated \( L \)-module, \( F_1 \) is a direct sum of cyclic modules of the form \( L/P \) with \( \{ u - \lambda \} \mid P \), and \( F_0 \) is a direct sum of cyclic modules of the form \( L/(u - \lambda)^n L \). Put \( D_0 = F_0 \), \( D_1 = F_1 \oplus F_2 \). Contrarily to the case of mapping tori, the \( L \)-module \( D_1 \) is not a finite-dimensional vector space, and the map \( \{ u - \lambda \} \mid D_1 : D_1 \to D_1 \)

The main theorem of the paper is readily carried over to a somewhat more general (but less geometrically appealing) case of arbitrary compact manifolds \( X \) endowed with a non-zero cohomology class \( \xi \in H^1(X, \mathbb{Z}) \). Let \( p : \overline{X} \to X \) be the corresponding infinite cyclic covering; \( H^k(X, L) \) is a finitely generated \( L \)-module; denote by \( T_k \) its \( L \)-torsion submodule. Then \( T_k \) is a finite dimensional vector space over \( \mathbb{C} \). (In the case when \( X \) is a mapping torus of a map \( \phi : M \to M \), we have \( T_k \approx H^k(M, \mathbb{C}) \).) Denote by \( f_k \) the automorphism \( T_k \to T_k \) induced by the generator \( u \) of the structure group of the covering \( p \).

### 5. A Generalization

The main theorem of the paper is readily carried over to a somewhat more general (but less geometrically appealing) case of arbitrary compact manifolds \( X \) endowed with a non-zero cohomology class \( \xi \in H^1(X, \mathbb{Z}) \). Let \( p : \overline{X} \to X \) be the corresponding infinite cyclic covering; \( H^k(X, L) \) is a finitely generated \( L \)-module; denote by \( T_k \) its \( L \)-torsion submodule. Then \( T_k \) is a finite dimensional vector space over \( \mathbb{C} \). (In the case when \( X \) is a mapping torus of a map \( \phi : M \to M \), we have \( T_k \approx H^k(M, \mathbb{C}) \).) Denote by \( f_k \) the automorphism \( T_k \to T_k \) induced by the generator \( u \) of the structure group of the covering \( p \).

#### Theorem 5.1

Let \( \lambda \in \mathbb{C}^* \).

1. We have \( \nu(f_k, \lambda) = \mu_k(\lambda) \).
2. If \( X \) is a strongly formal space (e.g. a compact Kähler manifold) then all the Jordan blocks of \( f_k \) are of size 1.
3. If \( X \) is a formal space, then all the Jordan blocks of eigenvalue 1 of \( f_k \) are of size 1.

**Proof.** The proof goes on the same lines as the proof of the main theorem. We will briefly indicate the necessary modifications. We have an exact couple

\[
(7) \quad H^*(X, \beta) \to H^*(X, \beta) \to H^*(X, \rho_\lambda)
\]
is not an isomorphism. However, it is injective, and as before an application of Lemma 3.3 completes the proof. □

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References

[1] G. Bazzoni, M. Fernández, V. Muñoz. Non-formal co-symplectic manifolds, Trans. Amer. Math. Soc. 367, (2015), 4459 – 4481.
[2] P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan. Real homotopy theory of Kahler manifolds, Invent. Math. 29 (1975), 245 – 274.
[3] M. Farber. Exactness of Novikov inequalities. Functionalniy Analiz i ego Prilozheniya 19, 1985 p. 49 – 59.
[4] M. Farber. Topology of closed 1-forms and their critical points, Topology. 40 (2001), p. 235 – 258.
[5] M. Fernández, A. Gray, J. Morgan. Compact symplectic manifolds with free circle actions and Massey products. Michigan Math. J. 38, 271 – 283, 1991.
[6] H. Kasuya. Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems. Journal of Differential Geometry. 93, (2013), 269-297.
[7] T. Kohno, A. Pajitnov. Novikov homology, jump loci and Massey products. Cent. Eur. J. Math. 12 (2014), 1285 – 1304.
[8] D. Kraines. Massey higher products, Trans. Amer. Math. Soc. 124 (1966), 431-449.
[9] S.P. Novikov. Bloch homology, critical points of functions and closed 1-forms Soviet Math. Dokl. 287 (1986), 1321 – 1324.
[10] A. Pajitnov. Proof of a conjecture of Novikov on homology with local coefficients over a field of finite characteristic. Soviet Math. Dokl. 37 (1988), p. 824 – 828.
[11] S. Papadima, A. Suciu. Algebraic monodromy and obstructions to formality, Forum Math. 22 (2010), 973 – 983.
[12] S. Papadima, A. Suciu. Geometric and algebraic aspects of 1-formality, Bull. Math. Soc. Sci. Math. Roumanie 52 (100) (2009), 355 – 375.