A priori estimates versus arbitrarily large solutions for fractional semi-linear elliptic equations with critical Sobolev exponent

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Abstract We study positive solutions to the fractional semi-linear elliptic equation
\[
(−Δ)^{\sigma} u = K(x) u^{\frac{n+2\sigma}{n}} \quad \text{in} \ B_2 \setminus \{0\}
\]
with an isolated singularity at the origin, where \( K \) is a positive function on \( B_2 \), the punctured ball \( B_2 \setminus \{0\} \subset \mathbb{R}^n \) with \( n \geq 2, \sigma \in (0, 1) \), and \( (−Δ)^{\sigma} \) is the fractional Laplacian. In lower dimensions, we show that for any \( K \in C^1(B_2) \), a positive solution \( u \) always satisfies that
\[
u(x) \leq C |x|^{-\left(n - 2\sigma\right)/2}
\]

near the origin. In contrast, we construct positive functions \( K \in C^1(B_2) \) in higher dimensions such that a positive solution \( u \) could be arbitrarily large near the origin. In particular, these results also apply to the prescribed boundary mean curvature equations on \( \mathbb{R}^{n+1} \).

Keywords fractional elliptic equations, boundary mean curvature equations, local estimates, large singular solutions

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1 Introduction

In this paper, we are interested in the singular positive solutions to the fractional semi-linear elliptic equation
\[
(−Δ)^{\sigma} u = K(x) u^{\frac{n+2\sigma}{n}} \quad \text{in} \ B_2 \setminus \{0\}, \quad u > 0 \quad \text{in} \ \mathbb{R}^n \setminus \{0\},
\]
where \( K \) is a positive continuous function on \( B_2 \), the punctured ball \( B_2 \setminus \{0\} \subset \mathbb{R}^n \) with \( n \geq 2, \sigma \in (0, 1) \), and \( (−Δ)^{\sigma} \) is the fractional Laplacian defined as
\[
(−Δ)^{\sigma} u(x) = C_{n,\sigma} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n + 2\sigma}} dy
\]
with $C_{n,\sigma} = \frac{2^\sigma \sigma \Gamma(\frac{n+2\sigma}{2})}{\pi^{\frac{n}{2}} \Gamma(1-\sigma)}$ and the gamma function $\Gamma$. This equation with the critical Sobolev exponent arises in the study of the fractional Yamabe problem [6,9,10] and the fractional Nirenberg problem [16,17]. More precisely, every solution $u$ of (1.1) induces a metric $g := u^{4/(n-2\sigma)}|dx|^2$ conformal to the flat metric on $\mathbb{R}^n$ such that $K(x)$ is the fractional $Q$-curvature [6] of the new metric $g$. An interesting question is the following: if a solution $u(x)$ of (1.1) has a non-removable singularity at $\{0\}$, how does it tend to infinity as $x$ approaches the origin?

This question in the Laplacian case $\sigma = 1$ was initially studied by Caffarelli et al. [3] when $K$ is identically a positive constant. They proved that every positive solution $u$ is asymptotically radially symmetric and has precise asymptotic behavior near the isolated singularity $0$. In particular, their result implies that $u$ satisfies the following local estimate near $0$:

$$u(x) \leq C|x|^{-\frac{n+2}{2}}.$$  \hspace{1cm} (1.3)

We may also see the work of Korevaar et al. [19] for a different proof in this classic case. When $K$ is a non-constant positive function, Chen and Lin [8,23] established (1.3) for positive solutions to (1.1) in the case $\sigma = 1$ under certain flatness conditions at critical points of $K$ by using the method of moving planes. Later, Taliaferro and Zhang [27,28] further explored the flatness conditions on $K$ such that any positive solution of (1.1) with $\sigma = 1$ satisfies the local estimate (1.3) via the moving sphere method.

When $\sigma \in (0,1)$ and $K$ is identically a positive constant, Caffarelli et al. [4] proved the following local estimate:

$$u(x) \leq C|x|^{-\frac{n-2\sigma}{2}}.$$  \hspace{1cm} (1.4)

for positive solutions of (1.1) near the singularity $0$ relying on the extension formulations of fractional Laplacians established by Caffarelli and Silvestre [5]. Based on this estimate, they also showed that every solution $u$ of (1.1) is asymptotically radially symmetric. Furthermore, it is natural to consider the case where $K$ is a non-constant function and ask that under what assumptions on $K$ every singular solution $u$ of (1.1) satisfies the estimate (1.4) near the origin. In a recent paper [18], Jin and Yang established local estimates for the higher-order conformal $Q$-curvature equation by studying the corresponding integral equation which, in particular, in the case $\sigma \in (0,1)$ is closely related to the fractional equation (1.1). However, as pointed out in Remark 1.8 there, the integral equation in the case $\sigma \in (0,1)$ encounters a difficulty due to the more singular properties of an integral kernel, and it was not covered in [18]. The first goal of this paper is to derive the local estimate (1.4) for the fractional equation (1.1) in lower dimensions.

We study (1.1) via the well-known extension formulations for fractional Laplacians in [5], through which we can consider a degenerate but local elliptic equation with a Neumann boundary condition in one dimension higher. To be more precise, we first introduce some notations. We use capital letters, such as $X = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, to denote points in $\mathbb{R}^{n+1}$. We denote by $B_R$ the open ball in $\mathbb{R}^{n+1}$ with radius $R$ and center $0$, by $B_R^+$ the upper half ball $B_R \cap \mathbb{R}^{n+1}_+$, and by $B_R$ the open ball in $\mathbb{R}^n$ with radius $R$ and center $0$. We also denote by $\partial B_R^+$ the flat part of $\partial B_R^+$ which is the ball $B_R$ in $\mathbb{R}^n$. Then instead of (1.1), we study

$$\begin{cases}
\text{div}(t^{1-2\sigma} \nabla U) = 0 & \text{in } B_R^+, \\
\frac{\partial U}{\partial \nu^\sigma}(x,0) = K(x)U(x,0) \frac{t^{1+2\sigma}}{t^{2\sigma}} & \text{on } \partial B_R^+ \setminus \{0\},
\end{cases}$$  \hspace{1cm} (1.5)

where

$$\frac{\partial U}{\partial \nu^\sigma}(x,0) = - \lim_{t \to 0^+} t^{1-2\sigma} \partial_t U(x,t).$$

By [5], we only need to derive a local estimate for the trace $u(x) := U(x,0)$ of a non-negative solution $U(x,t)$ of (1.5) near the origin.

When $\sigma = 1/2$, the equation (1.5) appears in the study of prescribing mean curvature on $\partial \mathbb{B}^{n+1}$ and zero scalar curvature in $\mathbb{B}^{n+1}$ (see, for example, [7,11,13,14]). In this case, the equation is without weight and thus elliptic.
We say that $U$ is a weak solution of (1.5) if $U$ is in the weighted Sobolev space $W^{1,2}(t^{1-2\sigma},B_2^+ \setminus B_2^\varepsilon)$ for any $0 < \varepsilon < 2$ and it satisfies
\[
\int_{B_2^\varepsilon} t^{1-2\sigma} \nabla U \nabla \Psi dX = \int_{\partial B_2^\varepsilon} KU^{\frac{2+2\sigma}{2}} \Psi dx
\]
for every $\Psi \in C^\infty_c((B_2^+ \cup \partial' B_2^+) \setminus \{0\})$.

Before stating our first theorem, we introduce a notation $C^\alpha(B_2)$ with $\alpha \in (0, 1]$.

**Definition 1.1.** For $\alpha \in (0, 1)$, $C^\alpha(B_2)$ is the set of all the functions $f \in C(B_2)$ satisfying
\[
|f(x) - f(y)| \leq c(|x - y|)|x - y|^\alpha
\]
for all $x, y \in B_2$, where $c(\cdot)$ is a non-negative continuous function with $c(0) = 0$. For $\alpha = 1$, $C^1(B_2)$ is the usual space $C^1(B_2)$.

Our first result is the following local estimate for non-negative solutions of (1.5) in the dimension $n = 2$ or $n = 3$.

**Theorem 1.2.** Suppose that $\sigma \in [1/2, 1)$, $n = 2$ or $n = 3$, and $K \in C^\alpha(B_2)$ is a positive function with $\alpha = (n - 2\sigma)/2$. If $\sigma = 1/2$ and $n = 3$, then we additionally suppose that $\nabla K(0) = 0$. Let $U$ be a positive weak solution of (1.5). Then there exists a constant $C > 0$ such that
\[
u(x) \leq C|x|^{-\frac{n-2\sigma}{2}}
\]
for all $x \in B_1 \setminus \{0\}$.

**Remark 1.3.** In particular, for the case $\sigma = 1/2$, the local estimate (1.7) holds for any positive solution of the boundary mean curvature equation (1.5) when $K \in C^{1/2}(B_2)$ in the dimension $n = 2$ or $K \in C^1(B_2)$ with $\nabla K(0) = 0$ in the dimension $n = 3$.

We prove Theorem 1.2 using the method of moving spheres introduced by Li and Zhu [22] (see also [4, 15, 16, 18, 21, 27, 28] for more applications of the moving sphere method).

On the other hand, supposing only that $K \in C^1(B_2)$ is a positive function satisfying $\nabla K(0) = 0$, one wonders if the estimate (1.7) holds in the dimension $n \geq 4$. Such a problem has been explored by Leung [20] and Taliaferro [26] in the Laplacian case $\sigma = 1$. More precisely, Taliaferro [26] showed the existence of positive functions $K \in C^1(B_2)$ in the dimension $n \geq 6$ such that (1.1) in the case $\sigma = 1$ has a singular solution $u$ which can be arbitrarily large near the origin. Leung [20] proved the existence of a positive Lipschitz continuous function $K$ on $B_2$ in the dimension $n \geq 5$ such that a solution $u$ of (1.1) for $\sigma = 1$ does not satisfy $u(x) = O(|x|^{-(n-2)/2})$ near the origin.

The second goal of this paper is to generalize the result of Taliaferro [26] to the fractional equation (1.1) which, in particular, implies that (1.7) does not hold when $n > 2\sigma + 3$ if we only assume $K \in C^1(B_2)$ and $\nabla K(0) = 0$.

Now we study the existence of large singular solutions to the extension equation
\[
\begin{aligned}
\text{div}(t^{1-2\sigma} \nabla U) &= 0 & \text{in } \mathbb{R}^{n+1}_+, \\
\frac{\partial U}{\partial t}(x, 0) &= K(x)U(x, 0)^{\frac{2+2\sigma}{2}} & \text{on } \partial \mathbb{R}^{n+1}_+ \setminus \{0\}.
\end{aligned}
\]

**Theorem 1.4.** Suppose that $\sigma \in (0, 1)$ and $n > 2\sigma + 3$ is an integer. Let $k : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ function which is bounded between two positive constants and satisfies $\nabla k(0) = 0$. Let $\varepsilon$ be a positive number and $\varphi : (0, 1) \to (0, \infty)$ be a continuous function. Then there exists a $C^1$ positive function $K : \mathbb{R}^n \to \mathbb{R}$ satisfying $\nabla K(0) = 0$, $K(0) = k(0)$, $K(x) = k(x)$ for $|x| \geq \varepsilon$ and
\[
K - k \in C^1(\mathbb{R}^n) < \varepsilon
\]
such that (1.8) has a positive solution $U$ whose trace satisfies
\[
u(x) \neq O(\varphi(|x|)) \quad \text{as } |x| \to 0^+.
\]
and
\[ u(x) = O(|x|^{2\sigma - n}) \quad \text{as } |x| \to \infty. \]  

(1.11)

**Remark 1.5.** When \( \sigma = 1/2 \), Theorem 1.4 indicates that there exist positive functions \( K \in C^1(\mathbb{R}^n) \) in the dimension \( n \geq 5 \) such that the boundary mean curvature equation (1.8) has a positive solution which could be arbitrarily large near the singularity 0.

In Theorem 1.4, the function \( \varphi : (0, 1) \to (0, \infty) \) is arbitrarily given and thus its values can be taken to be very large near 0. Hence, the conclusion of Theorem 1.4 that a solution \( U \) of (1.8) can be required to satisfy (1.10) is in contrast to the result of Theorem 1.2. Our basic strategy to prove Theorem 1.4 is similar to that introduced by Taliaferro [26] for \( \sigma = 1 \), but we first have to set up a framework to fit the nonlocal equation \((-\Delta)^\sigma u = K(x)u^{\frac{n+2\sigma}{n}}\) in \( \mathbb{R}^n \setminus \{0\} \), and then we extend the constructed solution for this nonlocal equation to (1.8) in one dimension higher.

The rest of this paper is organized as follows. In Section 2, we give some basic results for the standard bubble solutions and a Green’s formula on the exterior of a half ball with the Neumann boundary condition. In Sections 3 and 4, we devote to the proofs of Theorems 1.2 and 1.4, respectively.

## 2 Preliminaries

In this section, we introduce some notations and some basic results which will be used in the proof of Theorem 1.2 in the next section. We denote by \( B_R(X) \) the open ball in \( \mathbb{R}^{n+1} \) with radius \( R \) and center \( X \), by \( B_R^n(X) \) the open ball in \( \mathbb{R}^n \) with radius \( R \) and center \( x \). For simplicity, we also write \( B_R(0), B_R^n(0) \) and \( \partial R \) as \( B_R, B_R^n \) and \( \partial R \), respectively. For a set \( \Omega \subset \mathbb{R}^{n+1}_+ \) with boundary \( \partial \Omega \), we denote by \( \partial' \Omega \) the interior of \( \Omega \cap \partial \mathbb{R}^{n+1}_+ \) in \( \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+ \) and \( \partial'' \Omega = \partial \Omega \setminus \partial' \Omega \). Thus, \( \partial' B_R = B_R \) and \( \partial'' B_R = \partial B_R \cap \mathbb{R}^{n+1}_+ \).

In this section, we always assume that \( \sigma \in [1/2, 1) \) and \( n \geq 2 \) is an integer. For \( \lambda > 0, Y \in \mathbb{R}^{n+1}_+ \setminus \{0\} \) and \( y \in \mathbb{R}^n \setminus \{0\} \), we define

\[ Y^\lambda = \frac{\lambda^2 Y}{|Y|^2} \quad \text{and} \quad y^\lambda = \frac{\lambda^2 y}{|y|^2}. \]

Let \( U : \mathbb{R}^{n+1}_+ \to (0, \infty) \) and \( u : \mathbb{R}^n \to (0, \infty) \) be two functions. Then their Kelvin transformations are defined by

\[ U^\lambda(Y) = \left( \frac{\lambda}{|Y|} \right)^{n-2\sigma} U(Y^\lambda) \quad \text{and} \quad u^\lambda(y) = \left( \frac{\lambda}{|y|} \right)^{n-2\sigma} u(y^\lambda). \]

Let

\[ \tilde{w}(y) = \left( \frac{1}{1 + |y|^2} \right)^{\frac{n-2\sigma}{2}}, \]

and

\[ \tilde{W}(y, t) = \mathcal{P}_\sigma[\tilde{w}](y, t) \]

\[ = \int_{\mathbb{R}^n} \mathcal{P}_\sigma(y - z, t) \tilde{w}(z) dz \]

\[ = \gamma_{n, \sigma} \int_{\mathbb{R}^n} \left( \frac{1}{1 + |z|^2} \right)^{\frac{n+2\sigma}{2}} \left( \frac{1}{1 + |y - tz|^2} \right)^{\frac{n-2\sigma}{2}} dz, \]

where

\[ \mathcal{P}_\sigma(y, t) = \gamma_{n, \sigma} \left( \frac{t^{2\sigma}}{|y|^2 + t^2} \right)^{(n+2\sigma)/2} \]

with a constant \( \gamma_{n, \sigma} \) such that

\[ \gamma_{n, \sigma} \int_{\mathbb{R}^n} (1 + |z|^2)^{-(n+2\sigma)/2} dz = 1. \]
Then it is well known that $\tilde{W}$ satisfies
\[
\begin{aligned}
\text{div}(t^{1-2\sigma}\nabla \tilde{W}) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+,
\frac{\partial \tilde{W}}{\partial \nu}(x,0) &= \tilde{C}_{n,\sigma}\tilde{W}(x,0)^{n+2\sigma} \quad \text{on } \partial \mathbb{R}^{n+1}_+,
\end{aligned}
\]
where $\tilde{C}_{n,\sigma}$ is a positive constant given as
\[
\tilde{C}_{n,\sigma} = \frac{2\Gamma(1-\sigma)\Gamma(n\sigma+\frac{n}{2})}{\Gamma(\sigma)\Gamma(n\sigma+\frac{n}{2})}.
\] (2.2)
Furthermore, we claim the following lemma.

**Lemma 2.1.** Let $\lambda_0 = 1/2$ and $\lambda_1 = 2$. Then there exists a $C > 0$ depending only on $n$ and $\sigma$ such that
\[
\tilde{W}(Y) - \tilde{W}^{\lambda_0}(Y) \geq C(|Y| - \lambda_0)|Y|^{2\sigma-n-1} \quad \text{for } Y \in \mathbb{R}^{n+1}_+ \setminus B_0^{+}
\] (2.3)
and
\[
\frac{\partial (\tilde{W} - \tilde{W}^{\lambda_0})}{\partial \nu} > C > 0 \quad \text{on } \partial\Omega
\] (2.4)
where $\nu$ denotes the unit outer normal vector of $\partial\Omega$. Moreover, we have
\[
\tilde{W}(Y) - \tilde{W}^{\lambda_1}(Y) < 0 \quad \text{for } Y \in \mathbb{R}^{n+1}_+ \setminus \overline{B}_1^{+}.
\] (2.5)

**Proof.** By direct computation, we obtain
\[
\tilde{w}^{\lambda_0}(y) = \left(\frac{\lambda_0^2}{\lambda_1^2 + |y|^2}\right)^{\frac{n-2\sigma}{2}}.
\]
Then $\tilde{w}(y) > \tilde{w}^{\lambda_0}(y)$ for all $y \in \mathbb{R}^n \setminus B_0^{+}$. It follows that for $Y \in \mathbb{R}^{n+1}_+ \setminus \overline{B}_0^{+}$,
\[
\tilde{W}(Y) - \tilde{W}^{\lambda_0}(Y) = \gamma_{n,\sigma} \int_{\mathbb{R}^n \setminus B_0^{+}} \left(\frac{t^{2\sigma}}{|Y - x|^{n+2\sigma}} - \left(\frac{\lambda_0}{|x|}\right)^{n+2\sigma} \frac{t^{2\sigma}}{|Y - x|^{n+2\sigma}}\right) (\tilde{w}(x) - \tilde{w}^{\lambda_0}(x)) dx > 0.
\]
We also see that for $Y \in \partial\Omega \cap \mathbb{R}^{n+1}_+$,
\[
\frac{\partial (\tilde{W} - \tilde{W}^{\lambda_0})}{\partial \nu}(Y) = \gamma_{n,\sigma} (n+2\sigma) \int_{\mathbb{R}^n \setminus B_0^{+}} \frac{t^{2\sigma}(|x|^2 - \lambda_0^2)}{\lambda_0 |Y - x|^{n+2\sigma+2}} (\tilde{w}(x) - \tilde{w}^{\lambda_0}(x)) dx > 0,
\]
and for all $y \in \partial B_0^{+}$,
\[
\frac{\partial (\tilde{w} - \tilde{w}^{\lambda_0})}{\partial \nu}(y) = (n - 2\sigma) \left(\frac{1}{1 + \lambda_0^2}\right)^{\frac{n-2\sigma+2}{2}} \frac{1 - \lambda_0^2}{\lambda_0} > 0.
\]
Noting that $\tilde{W} - \tilde{W}^{\lambda_0} \in C^1_{\text{loc}}(\mathbb{R}^{n+1}_+)$, we obtain that (2.4) holds.

On the other hand, since $\tilde{W}$ is conformally invariant, i.e., $\tilde{W}(Y) = |Y|^{2\sigma-n}\tilde{W}(Y/|Y|^2)$, we have
\[
\lim_{{|Y| \to \infty}} |Y|^{n-2\sigma}(\tilde{W} - \tilde{W}^{\lambda_0})(Y) = \tilde{W}(0) - \lambda_0^{n-2\sigma}\tilde{W}(0) = 1 - \lambda_0^{n-2\sigma} > 0.
\]
Hence, there exists a $C > 0$ depending only on $n$ and $\sigma$ such that (2.3) holds. Using the similar argument, we can prove that (2.5) holds.
We define
\[ G_\lambda(Y, \eta) = n_{n, \sigma} \left( |Y - \eta|^{2\sigma - n} - \left( \frac{\lambda}{|\eta|} \right)^{n - 2\sigma} |Y - \eta|^{2\sigma - n} \right) \]  
for \( Y = (y, t) \in \mathbb{R}^{n+1}_+ \setminus \overline{B}_\lambda^y \) and \( \eta \in \mathbb{R}^n \setminus B_\lambda \), where \( n_{n, \sigma} \) satisfies
\[ n_{n, \sigma}(n - 2\sigma) \int_{\mathbb{R}^n} (1 + |z|^2)^{(2\sigma - n)/2} dz = 1. \]

Then it is elementary to check the following lemma.

**Lemma 2.2.** The function \( G_\lambda \) satisfies the following:
\begin{enumerate}[(i)]  
  \item \( G_\lambda(Y, \eta) > 0 \) when \( |Y| > \lambda \) and \( |\eta| > \lambda \).
  \item \( G_\lambda(Y, \eta) = 0 \) when \( |Y| = \lambda \) or \( |\eta| = \lambda \).
  \item \( \operatorname{div}(t^{1 - 2\sigma} \nabla Y G_\lambda) = 0 \) for \( Y = (y, t) \in \mathbb{R}^{n+1}_+ \setminus \overline{B}_\lambda^y \) and \( \eta \in \mathbb{R}^n \setminus \overline{B}_\lambda \).
\end{enumerate}

**Lemma 2.3.** Suppose that \( E \) is a smooth bounded domain of \( \mathbb{R}^n \) with \( B_{2\lambda} \subset E \) and \( q_\lambda \in C(\overline{E} \setminus B_\lambda) \). Let
\[ \Phi_\lambda(Y) = \int_{E \setminus B_\lambda} G_\lambda(Y, \eta) q_\lambda(\eta) d\eta \quad \text{for} \ Y \in \mathbb{R}^{n+1}_+ \setminus \overline{B}_\lambda^y. \]

Then the function \( \Phi_\lambda \) satisfies the following:
\begin{enumerate}[(i)]  
  \item \( \Phi_\lambda(Y) = 0 \) when \( |Y| = \lambda \).
  \item \( \Phi_\lambda \) satisfies the equation
\[ \begin{cases}  
  \operatorname{div}(t^{1 - 2\sigma} \nabla \Phi_\lambda) = 0 & \text{in } \mathbb{R}^{n+1}_+ \setminus \overline{B}_\lambda^y, \\
  \frac{\partial \Phi_\lambda}{\partial t} = q_\lambda & \text{on } E \setminus \overline{B}_\lambda. 
\end{cases} \]
\end{enumerate}

**Proof.** Part (i) follows from Lemma 2.2(ii). The first identity of (ii) follows from Lemma 2.2 and Lebesgue’s dominated convergence theorem. Now we prove the second identity of (ii). For any \( \eta \in E \setminus \overline{B_\lambda} \), there exists a \( \delta > 0 \) such that \( B_\delta(y) \subset E \setminus \overline{B_\lambda} \). Then for any \( 0 < r < \delta \),
\[ -t^{1 - 2\sigma} \partial_t \Phi_\lambda(Y) = (n - 2\sigma)n_{n, \sigma} \int_{B_r(y)} t^{2 - 2\sigma} |Y - \eta|^{2\sigma - n - 2} q_\lambda(\eta) d\eta \]
\[ + \left[ (n - 2\sigma)n_{n, \sigma} \int_{E \setminus (B_\lambda \cup B_r(y))} t^{2 - 2\sigma} |Y - \eta|^{2\sigma - n - 2} q_\lambda(\eta) d\eta \right] \]
\[ - (n - 2\sigma)n_{n, \sigma} \int_{E \setminus B_\lambda} t^{2 - 2\sigma} \left( \frac{\lambda}{|\eta|} \right)^{n - 2\sigma} |Y - \eta|^{2\sigma - n - 2} q_\lambda(\eta) d\eta \]
\[ =: I_1 + I_2. \]
Here, we have
\[ I_1 = (n - 2\sigma)n_{n, \sigma} \int_{B_r(y)} t^{2 - 2\sigma} |Y - \eta|^{2\sigma - n - 2} (q_\lambda(\eta) - q_\lambda(y)) d\eta \]
\[ + (n - 2\sigma)n_{n, \sigma} \int_{B_r(y)} t^{2 - 2\sigma} |Y - \eta|^{2\sigma - n - 2} q_\lambda(\eta) d\eta \]
\[ = (n - 2\sigma)n_{n, \sigma} \int_{B_r(y)} t^{2 - 2\sigma} |Y - \eta|^{2\sigma - n - 2} (q_\lambda(\eta) - q_\lambda(y)) d\eta \]
\[ + (n - 2\sigma)n_{n, \sigma} q_\lambda(y) \int_{B_r(y)} (1 + |z|^2)^{(2\sigma - n - 2)/2} dz \]
\[ =: I_{11} + I_{12}, \]
where \( |I_{11}| \leq \|q_\lambda - q_\lambda(y)\|_{L^\infty(B_r(y))} \) and \( \lim_{t \to 0^+} I_{12} = q_\lambda(y) \). We also have \( \lim_{t \to 0^+} |I_2| = 0 \). Consequently, we obtain
\[ \limsup_{t \to 0^+} -t^{1 - 2\sigma} \partial_t \Phi_\lambda(Y) - q_\lambda(Y) \leq \|q_\lambda - q_\lambda(y)\|_{L^\infty(B_r(y))}. \]
Since \( q_\lambda \) is continuous at \( y \), sending \( r \to 0^+ \), we can get the desired result. \( \square \)
Here, we state an estimate of $G_{\lambda}(Y, \eta)$ whose proof is elementary and so is omitted.

**Lemma 2.4.** For $\lambda < |Y| \leq 10\lambda$ and $|\eta| > \lambda$, there exists a $C > 0$ depending only on $n$ and $\sigma$ such that

$$G_{\lambda}(Y, \eta) \leq C\frac{(|Y| - \lambda)(|\eta|^2 - \lambda^2)}{\lambda|Y - \eta|^{n-2\sigma+2}}.$$  \hfill (2.7)

Finally, we also need the following maximum principle whose proof can be found in [15].

**Lemma 2.5** (See [15]). Suppose that $U \in W^{1,2}(t^{1-2\sigma}, B^+_1 \setminus B^+_\varepsilon) \cap C(B^+_1 \setminus \{0\})$ for any $0 < \varepsilon < 1$ and

$$\liminf_{Y \to 0} U(Y) > -\infty.$$

Suppose that $U$ satisfies

$$\begin{cases}
\div(t^{1-2\sigma}\nabla U) \leq 0 & \text{in } B^+_1, \\
\frac{\partial U}{\partial \nu} \geq 0 & \text{on } \partial B^+_1 \setminus \{0\}
\end{cases}$$

in the weak sense. Then

$$U(Y) \geq \inf_{\partial^\nu B^+_1} U \quad \text{for all } Y \in B^+_1 \setminus \{0\}.$$

### 3 Local estimates in lower dimensions

In this section, we prove Theorem 1.2 using the moving sphere method introduced by Li and Zhu [22].

**Proof of Theorem 1.2.** Suppose by contradiction that there exists a sequence $\{x_j\}_{j=1}^\infty \subset B_1 \setminus \{0\}$ such that

$$|x_j| \to 0 \quad \text{as } j \to \infty,$$

but

$$|x_j| \frac{n-2\sigma}{2} u(x_j) \to \infty \quad \text{as } j \to \infty.$$

**Step 1.** We claim that $\{x_j\}_{j=1}^\infty$ can be chosen as the local maximum points of $u$.

Consider

$$f_j(x) := \left(\frac{|x|}{2} - |x - x_j|\right)^{\frac{n-2\sigma}{2}} u(x) \quad \text{for } |x - x_j| \leq \frac{|x_j|}{2}.$$

Since $u$ is positive and continuous in $\overline{B_{|x_j|/2}(x_j)}$, we can find a point $\bar{x}_j \in B_{|x_j|/2}(x_j)$ such that

$$f_j(\bar{x}_j) = \max_{|x - x_j| \leq |x_j|/2} f_j(x) > 0.$$

Let $2\mu_j := |x_j|/2 - |\bar{x}_j - x_j|$. Then

$$0 < 2\mu_j \leq \frac{|x_j|}{2} \quad \text{and} \quad \frac{|x_j|}{2} - |x - x_j| \geq \mu_j, \quad \forall |x - \bar{x}_j| \leq \mu_j.$$

By the definition of $f_j$, we have

$$(2\mu_j)\frac{n-2\sigma}{2} u(\bar{x}_j) = f_j(\bar{x}_j) \geq f_j(x) \geq (\mu_j)\frac{n-2\sigma}{2} u(x), \quad \forall |x - \bar{x}_j| \leq \mu_j.$$

Hence, we have

$$2\frac{n-2\sigma}{2} u(\bar{x}_j) \geq u(x), \quad \forall |x - \bar{x}_j| \leq \mu_j. \hfill (3.1)$$

We also have

$$(2\mu_j)\frac{n-2\sigma}{2} u(\bar{x}_j) = f_j(\bar{x}_j) \geq f_j(x_j) = \left(\frac{|x_j|}{2}\right)^{\frac{n-2\sigma}{2}} u(x_j) \to \infty \quad \text{as } j \to \infty. \hfill (3.2)$$
Now we define
\[ W_j(y,t) = \frac{1}{u(x_j)} U \left( \frac{x_j + y}{u(x_j)^{-\frac{2\sigma}{n-2\sigma}}}, \frac{t}{u(x_j)^{-\frac{2\sigma}{n-2\sigma}}} \right) \] for \((y,t) \in \Xi_j\),
where
\[ \Xi_j := \left\{ (y,t) \in \mathbb{R}^{n+1} : \left( \frac{x_j + y}{u(x_j)^{-\frac{2\sigma}{n-2\sigma}}}, \frac{t}{u(x_j)^{-\frac{2\sigma}{n-2\sigma}}} \right) \in B^*_1 \right\}. \]

Let \( \overline{W}_j(y) := W_j(y,0) \). Then \( \overline{W}_j \) satisfies \( \overline{W}_j(0) = 1 \) and
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla \overline{W}_j) = 0 & \text{in } \Xi_j, \\
\frac{\partial \overline{W}_j}{\partial \nu}(y,0) = K(\bar{x}_j + u(\bar{x}_j)^{-\frac{2\sigma}{n-2\sigma}} y) \overline{W}_j^{\frac{n+2\sigma}{n-2\sigma}}(y) & \text{on } \partial' \Xi_j \setminus \{ -u(\bar{x}_j)^{-\frac{2\sigma}{n-2\sigma}} \bar{x}_j \}.
\end{cases}
\]
Moreover, it follows from (3.1) and (3.2) that
\[ \overline{W}_j(y) \leq 2^{\frac{n-2\sigma}{2}} \text{ in } B_{R_j}, \]
where
\[ R_j := \mu_j u(\bar{x}_j)^{\frac{2}{n-2\sigma}} \to \infty \text{ as } j \to \infty. \] (3.3)

By [16, Proposition 2.6], for any given \( \bar{l} > 0 \), we have
\[ 0 \leq W_j \leq C(\bar{l}) \text{ in } B_{R_j/2} \times [0, \bar{l}], \]
where \( C(\bar{l}) \) depends only on \( n, \sigma, \|K\|_{L^\infty(B_{\bar{l}})} \) and \( \bar{l} \). Since \( 1/2 \leq \sigma < 1 \), by the regularity results in [2,16], we see that after passing to a subsequence, for some non-negative function \( \overline{W} \in W^{1,2}_{\text{loc}}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+), \)
\[ C^1_{\text{loc}}(\mathbb{R}^{n+1}_+), \]
\[ \begin{cases}
\overline{W}_j \to \overline{W} & \text{weakly in } W^{1,2}_{\text{loc}}(t^{1-2\sigma}, \mathbb{R}^{n+1}_+), \\
\overline{W}_j \to \overline{W} & \text{in } C^1_{\text{loc}}(\mathbb{R}^{n+1}_+).
\end{cases} \]
Define \( \overline{W}(y) := \overline{W}(y,0) \). Moreover, \( \overline{W} \) satisfies
\[
\begin{cases}
\text{div}(t^{1-2\sigma} \nabla \overline{W}) = 0 & \text{in } \mathbb{R}^{n+1}, \\
\frac{\partial \overline{W}}{\partial \nu} = K(0)\overline{W}^{\frac{n+2\sigma}{n-2\sigma}} & \text{on } \partial\mathbb{R}^{n+1}
\end{cases}
\]
and \( \overline{W}(0) = 1 \). Without loss of generality, we may assume \( K(0) = \tilde{C}_{n,\sigma} \), which is defined in (2.2). By the Liouville theorem in [16], we have
\[ \overline{W}(y) = \left( \frac{\mu}{1 + \mu^2 |y-y_0|^2} \right)^{\frac{n-2\sigma}{2}} \]
for some \( y_0 \in \mathbb{R}^n \) and \( \mu \geq 1 \). Hence, \( \overline{W} \) has an absolute maximum value \( \mu^{\frac{n-2\sigma}{2}} \) at \( y_0 \). It implies that \( \overline{W}_j \) has a local maximum at a point \( y_j \) close to \( y_0 \) and \( \overline{W}_j(y_j) \geq \mu^{\frac{n-2\sigma}{2}}/2 \) when \( j \) is large. Let
\[ \tilde{x}_j := \bar{x}_j + u(\bar{x}_j)^{-\frac{2\sigma}{n-2\sigma}} y_j. \]
Then \( \{ \tilde{x}_j \}_{j=1}^\infty \) are local maximum points of \( u \) for large \( j \) and
\[ u(\tilde{x}_j) = u(\bar{x}_j)\overline{W}_j(y_j) \geq \mu^{\frac{n-2\sigma}{2}}/2 u(\bar{x}_j). \]
It follows from (3.3) that
\[ |\tilde{x}_j - \bar{x}_j| = u(\bar{x}_j)^{-\frac{2\sigma}{n-2\sigma}} |y_j| < \mu_j \]
when \( j \) is large, and then \( \tilde{x}_j \in B_{\mu_j}(\tilde{x}_j) \subset B_{|x_j|/2}(x_j) \). Thus we have \(|\tilde{x}_j| \geq |x_j|/2 \geq 2\mu_j \) when \( j \) is large. By (3.3), we can obtain
\[
|\tilde{x}_j|^\frac{n-2\sigma}{2} u(\tilde{x}_j) \geq \frac{(2\mu_j)^{\frac{n-2\sigma}{2}}}{4} (\mu_j)^{\frac{n-2\sigma}{2}} u(\tilde{x}_j) \to \infty \quad \text{as} \quad j \to \infty.
\]

Using the same arguments as before, after passing to a subsequence, we can prove that the function
\[
W_j(y, t) := \frac{1}{u(\tilde{x}_j)} u(y) u(\tilde{x}_j) \rightarrow \infty \quad \text{as} \quad y \to \infty.
\]
converges to \( \tilde{W} \) in \( C^{1, 2}_{\text{loc}}(\Omega) \) and weakly in \( W^{1, 2}_{0, \text{loc}}(\Omega) \), which satisfies
\[
\begin{align*}
\text{div}(t^{1-2\sigma} \nabla \tilde{W}) &= 0 \quad \text{in} \quad \Omega, \\
\frac{\partial \tilde{W}}{\partial \nu} &= K(0) \tilde{w}^{\frac{n+2\sigma}{n}} \quad \text{on} \quad \partial \Omega, \\
\max_{\bar{\Omega}} \tilde{w} &= \tilde{w}(0) = 1,
\end{align*}
\]
where \( \tilde{w}(y) := \tilde{W}(y, 0) \). By the Liouville theorem in [16],
\[
\tilde{w}(y) = \left( \frac{1}{1 + |y|^2} \right)^{\frac{n-2\sigma}{2}} \quad \text{and} \quad \tilde{W} = \mathcal{P}_\sigma[\tilde{w}],
\]
where \( \mathcal{P}_\sigma \) is the Poisson kernel given in (2.1). From now on, we consider \( x_j \) as \( \tilde{x}_j \). The claim is proved.

**Step 2.** We give some estimates for the difference between \( W_j \) and its Kelvin transformation.

Let \( L_j := u(x_j) \) and
\[
\Omega_j := \{ (y, t) \in \mathbb{R}^{n+1} : (x_j + L_j^{-\frac{2\sigma}{n}} y, L_j^{-\frac{2\sigma}{n}} t) \in B_1^+ \}.
\]
Then for large \( j \), we know that \( B_1^+ \subset \Omega_j \). For \( \lambda \in [1/2, 2] \), let
\[
W_j^\lambda(Y) := \left( \frac{\lambda}{|Y|} \right)^{n-2\sigma} W_j \left( \frac{\lambda^2 Y}{|Y|^2} \right) \quad \text{for} \quad Y \in \Omega_j \setminus B_1^+.
\]
Now we define \( W_\lambda(Y) := W_j(Y) - W_j^\lambda(Y) \) and \( w_\lambda(y) := W_\lambda(y, 0) \). Here, we omit \( j \) for simplicity. We also define
\[
K_j(y) := K(x_j + L_j^{-\frac{2\sigma}{n}} y).
\]
Then
\[
\begin{align*}
\text{div}(t^{1-2\sigma} \nabla W_\lambda) &= 0 \quad \text{in} \quad \Omega_j \setminus B_1^+, \\
\frac{\partial W_\lambda}{\partial \nu} &= \lambda w_\lambda - q_\lambda \quad \text{on} \quad \partial(\Omega_j \setminus B_1^+) \setminus \{ -L_j^{-\frac{2\sigma}{n}} x_j \},
\end{align*}
\]
where
\[
b_\lambda(y) := K_j(y) \frac{w_j(y) + \frac{2\sigma}{n}}{w_j(y) - \frac{2\sigma}{n}} \quad \text{and} \quad q_\lambda(y) := (K_j(y^*) - K_j(y)) w_\lambda^*(y) + \frac{2\sigma}{n}.
\]
Notice that \( q_\lambda \in C(\overline{\Omega_j} \setminus B_1^+) \). Moreover, we have the following estimates for the difference between \( W_j \) and \( W_j^\lambda \) when \( \lambda \) is \( 1/2 \) or 2.

**Lemma 3.1.** Let \( \lambda_0 = 1/2 \) and \( \lambda_1 = 2 \). There exist a \( c_0 > 0 \) and a \( j_0 \geq 1 \) such that for all \( j \geq j_0 \),
\[
W_j(Y) - W_j^\lambda(Y) \geq c_0 |Y| - \lambda_0 |Y|^{2\sigma - n - 1} + c_0 L_j^{-1} \lambda_0^{-1} - |Y|^{2\sigma - n} \quad \text{for all} \quad Y \in \Omega_j \setminus B_1^+.
\]

Moreover, there exists a \( Y^* \in B_1^{\lambda_1} \setminus B_1^{\lambda_0} \) such that for all \( j \geq j_0 \),
\[
W_j(Y^*) - W_j^\lambda(Y^*) \leq -c_0.
\]
Proof. Since $\tilde{W}(Y) = |Y|^{2\sigma-n}\tilde{W}(Y/|Y|^2)$, we have $\lim_{|Y|\to \infty} |Y|^{n-2\sigma}\tilde{W}(Y) = \tilde{W}(0) = 1$. Similarly, we also have $\lim_{|Y|\to \infty} |Y|^{n-2\sigma}\tilde{W}^{\lambda_0}(Y) = \lambda_0^{2\sigma-n}$. Hence, there exist a small $c_1 \in (0, \frac{1}{4})$ and a large $R > 4$ such that

$$ |\tilde{W}(Y) - |Y|^{2\sigma-n}| \leq \frac{c_1}{2} |Y|^{2\sigma-n} \quad \text{for} \ |Y| \geq R $$

and

$$ \tilde{W}^{\lambda_0}(Y) \leq (1 - 3c_1)|Y|^{2\sigma-n} \quad \text{for} \ |Y| \geq R. $$

Since $W_j$ converges to $\tilde{W}$ in $C^1_{\text{loc}}(\mathbb{R}^n_+)$, for large $j$, we obtain

$$ W_j(Y) \geq (1 - c_1)|Y|^{2\sigma-n} \quad \text{for} \ |Y| = R \quad (3.9) $$

and

$$ W_j^{\lambda_0}(Y) \leq (1 - 2c_1)|Y|^{2\sigma-n} \quad \text{for} \ |Y| \geq R. $$

Recall that for $Y \in \partial^\prime \Omega_j$,

$$ |Y| = L_j \frac{2}{\sigma} |L_j^{-\frac{2}{\sigma}} Y| \geq L_j \frac{2}{\sigma} (1 - |x_j|). $$

Thus,

$$ |Y|^{n-2\sigma}W_j(Y) \geq L_j(1 - |x_j|)^{n-2\sigma} \inf_{B_j^+ \setminus \{0\}} U \to \infty \quad \text{as} \ j \to \infty. \quad (3.10) $$

It follows from (3.9), (3.10) and Lemma 2.5 that for sufficiently large $j$, we have

$$ W_j(Y) \geq (1 - c_1)|Y|^{2\sigma-n} \quad \text{for} \ Y \in \Omega_j \setminus B_R^+. $$

Again, since $W_j$ converges to $\tilde{W}$ in $C^1_{\text{loc}}(\mathbb{R}^n_+)$, by Lemma 2.1, there exists a $c_2 > 0$ such that

$$ W_j(Y) - W_j^{\lambda_0}(Y) \geq c_2(|Y| - \lambda_0)|Y|^{2\sigma-n-1} \quad \text{for} \ Y \in B_R^+ \setminus B_{\lambda_0}^+ \quad (3.11) $$

for sufficiently large $j$.

Now we show that (3.7) holds for $Y \in \Omega_j \setminus B_R^+$. By the definition of $W_j$, there exists a $c_3 > 0$ such that

$$ W_j(Y) \geq L_j^{-1} \inf_{B_j^+ \setminus \{0\}} U \geq c_3L_j^{-1}(\lambda_0^{2\sigma-n} - |Y|^{2\sigma-n}) \quad \text{for} \ Y \in \Omega_j \setminus B_R^+. $$

Then we have

$$ W_j(Y) - W_j^{\lambda_0}(Y) = \frac{c_1}{2}W_j(Y) + \left(1 - \frac{c_1}{2}\right)W_j(Y) - W_j^{\lambda_0}(Y) $$

$$ \geq \frac{c_1c_3}{2}L_j^{-1}(\lambda_0^{2\sigma-n} - |Y|^{2\sigma-n}) + \left[1 - \frac{c_1}{2}\right](1 - c_1)](1 - c_1)\right) |Y|^{2\sigma-n} $$

$$ \geq \frac{c_1c_3}{2}L_j^{-1}(\lambda_0^{2\sigma-n} - |Y|^{2\sigma-n}) + \frac{c_1(1 + c_1)}{2} |Y|^{2\sigma-n} $$

for $Y \in \Omega_j \setminus B_R^+$. Therefore, there exists a small $c_0 > 0$ such that for large $j$,

$$ W_j(Y) - W_j^{\lambda_0}(Y) \geq c_0|Y|^{2\sigma-n} + c_0L_j^{-1}(\lambda_0^{2\sigma-n} - |Y|^{2\sigma-n}) \quad \text{for all} \ Y \in \Omega_j \setminus B_R^+. $$

This together with (3.11) implies that (3.7) holds if $c_0$ is chosen sufficiently small. Finally, (3.8) is a direct consequence of Lemma 2.1.

Letting $\tau \in (0, 1/4)$ be a constant to be specified later, we define

$$ \Pi_j = \{(y, t) \in \mathbb{R}^{n+1} : (x_j + L_j^{-\frac{2}{\sigma}} y, L_j^{-\frac{2}{\sigma}} t) \in B^{+}_{2\tau} \} \subset \Omega_j $$

and

$$ \Sigma_j = \{(y, t) \in \mathbb{R}^{n+1} : (x_j + L_j^{-\frac{2}{\sigma}} y, L_j^{-\frac{2}{\sigma}} t) \in B^{+}_{\tau} \} \subset \Pi_j. $$
Define
\[ \Phi_\lambda(Y) := \int_{\partial \Pi_j \setminus B_\lambda} G_\lambda(Y, \eta)q_\lambda(\eta) d\eta \quad \text{for } Y \in \mathbb{R}^{n+1}_+ \setminus B_\lambda^+, \] (3.12)
where \( G_\lambda(Y, \eta) \) is the Green’s function defined in (2.6) and \( q_\lambda \) is the function given in (3.6). From Lemma 2.3, we know that \( \Phi_\lambda(Y) = 0 \) when \( |Y| = \lambda \) and
\[ \begin{align*}
\text{div}(1^{1-2\sigma}\nabla \Phi_\lambda) &= 0 \quad \text{in } \Pi_j \setminus \overline{B_\lambda^+}, \\
\frac{\partial \Phi_\lambda}{\partial \nu_\sigma} &= q_\lambda \quad \text{on } \partial \Pi_j \setminus \overline{B_\lambda^+}. 
\end{align*} \] (3.13)

**Step 3.** We state some estimates for \( \Phi_\lambda \).

Since \( w_j \) converges to \( \tilde{w} \) in \( C^1_\text{loc}(\mathbb{R}^n) \), we see that for any \( \lambda \in [1/2, 2] \),
\[ 0 \leq w_j(\eta) = \left( \frac{\lambda}{|\eta|} \right)^{n-2\sigma} w_j \left( \frac{\lambda^2 \eta}{|\eta|^2} \right) \leq C|\eta|^{2\sigma-n} \quad \text{for } \eta \in \partial \Pi_j \setminus B_\lambda \] (3.14)
when \( j \) is sufficiently large, where \( C > 0 \) depends only on \( n \) and \( \sigma \).

By our assumptions on \( K \), we have
\[ |K_j(\eta^\lambda) - K_j(\eta)| \leq c(\tau) L_j^{-1}(|\eta| - \lambda)^\alpha \quad \text{for } \eta \in \partial \Pi_j \setminus B_\lambda, \] (3.15)
where \( c(\cdot) \) is a non-negative function with \( c(0) = 0 \). This together with (3.6) and (3.14) implies that
\[ |q_\lambda(\eta)| \leq c(\tau) L_j^{-1}(|\eta| - \lambda)^\alpha |\eta|^{n-2\sigma} \quad \text{for } \eta \in \partial \Pi_j \setminus B_\lambda. \] (3.16)
Consequently, we have the following estimates for \( \Phi_\lambda \).

**Lemma 3.2.** For any \( \lambda \in [1/2, 2] \), we have
\[ |\Phi_\lambda(Y)| \leq \begin{cases} 
\frac{c(\tau) L_j^{-1}(|Y| - \lambda),} {\text{if } Y \in B_\lambda^+ \setminus B_\lambda^+,} \\
\frac{c(\tau) L_j^{-1} |Y|^{2\sigma-n} \log |Y|,} {\text{if } Y \in \Sigma_j \setminus B_{4\lambda},} 
\end{cases} \]
where \( c(\cdot) \) is a non-negative function with \( c(0) = 0 \).

**Proof.** It follows from (3.16) that for \( Y \in \Sigma_j \setminus B_\lambda^+ \),
\[ |\Phi_\lambda(Y)| \leq c(\tau) L_j^{-1} \int_{\partial \Pi_j \setminus B_\lambda} G_\lambda(Y, \eta) (|\eta| - \lambda)^\alpha |\eta|^{n-2\sigma} d\eta. \]
We have the following two cases:

**Case 1.** \( \lambda < |Y| < 4\lambda \). Let \( \partial \Pi_j \setminus B_\lambda = A \cup B \cup D \), where
\[ A := \{ \eta \in \partial \Pi_j \setminus B_\lambda : |Y - \eta| < (|Y| - \lambda)/3 \}, \\
B := \{ \eta \in \partial \Pi_j \setminus B_\lambda : (|Y| - \lambda)/3 \leq |Y - \eta| \leq (|Y| - \lambda)/3 \} \text{ and } |\eta| \leq 8\lambda, \\
D := \{ \eta \in \partial \Pi_j \setminus B_\lambda : |\eta| > 8\lambda \}. \]
Since \( G_\lambda(Y, \eta) \leq C|Y - \eta|^{2\sigma-n}, 2(|Y| - \lambda)/3 \leq |\eta| - \lambda \leq 4(|Y| - \lambda)/3 \) and \( \lambda \leq |\eta| \leq 5\lambda \) for any \( \eta \in A \), we have
\[ \int_A G_\lambda(Y, \eta) (|\eta| - \lambda)^\alpha |\eta|^{n-2\sigma} d\eta \leq C(|Y| - \lambda)^\alpha \int_A |Y - \eta|^{2\sigma-n} d\eta \]
\[ \leq C(|Y| - \lambda)^\alpha \int_{\{ \eta \in \partial \Pi_j \setminus B_\lambda : |\eta - y| < (|Y| - \lambda)/3 \}} |y - \eta|^{2\sigma-n} d\eta \]
\[ \leq C(|Y| - \lambda)^{\alpha+2\sigma} \]
\[ \leq C(|Y| - \lambda). \]
By Lemma 2.4 and $|\eta| - \lambda \leq 4|Y - \eta|$ for any $\eta \in B$, we have
\[
\int_B G_\lambda(Y, \eta)(|\eta| - \lambda)\alpha|\eta|^{-n-2\sigma} d\eta \leq C(|Y| - \lambda) \int_B |Y - \eta|^{2\sigma-n-2}(|\eta| - \lambda)^{1+\alpha} d\eta \\
\leq C(|Y| - \lambda) \int_B |Y - \eta|^{2\sigma-n-1+\alpha} d\eta \\
\leq C(|Y| - \lambda).
\]

For any $\eta \in D$, we have $|Y - \eta| \geq |\eta| - |Y| \geq |\eta|/2$ and $7|\eta|/8 \leq |\eta| - \lambda \leq |\eta|$. By Lemma 2.4,
\[
\int_D G_\lambda(Y, \eta)(|\eta| - \lambda)^\alpha|\eta|^{-n-2\sigma} d\eta \leq C(|Y| - \lambda) \int_D |\eta|^{\alpha-2\sigma} d\eta \leq C(|Y| - \lambda).
\]

Consequently, we obtain the first estimate in Lemma 3.2.

**Case 2.** $Y \in \Sigma_j \setminus \mathcal{B}_{4\lambda}^+$. Let $\partial'\Pi_j \setminus B_\lambda = A_1 \cup A_2 \cup A_3 \cup A_4$, where
\[
A_1 := \{\eta \in \partial'\Pi_j \setminus B_\lambda : |\eta| < |Y|/2\}, \\
A_2 := \{\eta \in \partial'\Pi_j \setminus B_\lambda : |\eta| > 2|Y|\}, \\
A_3 := \{\eta \in \partial'\Pi_j \setminus B_\lambda : |Y - \eta| \leq |Y|/2\}, \\
A_4 := \{\eta \in \partial'\Pi_j \setminus B_\lambda : |Y - \eta| > |Y|/2 \text{ and } |\eta| \leq |\eta| \leq 2|Y|\}.
\]

Since $G_\lambda(Y, \eta) \leq C|Y - \eta|^{2\sigma-n}$ and $|Y - \eta| \geq |Y|/2$ for any $\eta \in A_1$,
\[
\int_{A_1} G_\lambda(Y, \eta)(|\eta| - \lambda)^\alpha|\eta|^{-n-2\sigma} d\eta \leq C \int_{A_1} |Y - \eta|^{2\sigma-n}(|\eta| - \lambda)^\alpha|\eta|^{-n-2\sigma} d\eta \\
\leq C|Y|^{2\sigma-n} \int_{A_1} |\eta|^{\alpha-2\sigma} d\eta
\]
\[
\leq \begin{cases} 
C|Y|^{2\sigma-n}, & \text{if } \alpha < 2\sigma, \\
C|Y|^{2\sigma-n} \log |Y|, & \text{if } \alpha = 2\sigma.
\end{cases}
\]

For any $\eta \in A_2$, $|Y - \eta| \geq |\eta| - |Y| \geq |\eta|/2$, and then
\[
\int_{A_2} G_\lambda(Y, \eta)(|\eta| - \lambda)^\alpha|\eta|^{-n-2\sigma} d\eta \leq C \int_{A_2} |Y - \eta|^{2\sigma-n} |\eta|^{\alpha-2\sigma} d\eta \\
\leq C \int_{A_2} |\eta|^{\alpha-2\sigma} d\eta \\
\leq C|Y|^{\alpha-n}.
\]

For any $\eta \in A_3$, $|\eta| \geq |Y| - |Y - \eta| \geq |Y|/2$, and then
\[
\int_{A_3} G_\lambda(Y, \eta)(|\eta| - \lambda)^\alpha|\eta|^{-n-2\sigma} d\eta \leq C \int_{A_3} |Y - \eta|^{2\sigma-n} |\eta|^{\alpha-2\sigma} d\eta \\
\leq C|Y|^{\alpha-n} \int_{A_3} |Y - \eta|^{2\sigma-n} d\eta \\
\leq C|Y|^{\alpha-n-2\sigma} \int_{A_3} |\eta - \eta|^{2\sigma-n} d\eta \\
\leq C|Y|^{\alpha-n}.
\]

We also have
\[
\int_{A_4} G_\lambda(Y, \eta)(|\eta| - \lambda)^\alpha|\eta|^{-n-2\sigma} d\eta \leq C \int_{A_4} |Y - \eta|^{2\sigma-n} |\eta|^{\alpha-2\sigma} d\eta \\
\leq C|Y|^{2\sigma-n} \int_{A_4} |\eta|^{\alpha-2\sigma} d\eta \\
\leq C|Y|^{\alpha-n}.
\]

From above, we get the second estimate in Lemma 3.2. \qed
Step 4. We use the method of moving spheres to reach a contradiction.

Inspired by [8,27,28], for \( \lambda \in [1/2, 2] \), we define

\[
A_\lambda(Y) := -c_4 L_j^{-1}(\lambda^{2\alpha-n} - |Y|^{2\alpha-n}) + \Phi_\lambda(Y), \quad Y \in \mathbb{R}_+^{n+1} \setminus B_\lambda^+, \n
\]

where \( \Phi_\lambda \) is defined as in (3.12) and the constant \( c_4 \) is given by

\[
c_4 := \frac{1}{32} \min \left\{ c_0, \inf_{\mathcal{B}_1^+ \setminus \{0\}} U \right\} > 0.
\]

Here, we recall that \( c_0 > 0 \) is defined in Lemma 3.1.

By Lemma 3.2, we can choose \( \tau \) sufficiently small such that for any \( \lambda \in [1/2, 2] \),

\[
A_\lambda(Y) < 0 \quad \text{for all } Y \in \Sigma_j \setminus \overline{B_\lambda^+}
\]

when \( j \) is sufficiently large. Then for \( Y \in \partial^\prime \Sigma_j \), we have

\[
L_j^{-\frac{2\alpha}{\alpha}} \tau \leq L_j^{-\frac{2\alpha}{\alpha}} (\tau - |x_j|) \leq |Y| \leq L_j^{-\frac{2\alpha}{\alpha}} (\tau + |x_j|) \leq 2L_j^{-\frac{2\alpha}{\alpha}} \tau
\]

for large \( j \). By Lemma 3.2 and (3.17),

\[
|A_\lambda(Y)| \leq c_4 \lambda^{2\alpha-n} L_j^{-1} \leq 2^{n-2\alpha} c_4 L_j^{-1} \leq 8c_4 L_j^{-1} \quad \text{for } Y \in \partial^\prime \Sigma_j
\]

and

\[
W_\lambda(Y) = W_j(Y) - W_j^\lambda(Y)
\]

\[
\geq L_j^{-1} \inf_{\mathcal{B}_1^+ \setminus \{0\}} U - \lambda^{n-2\alpha} |Y|^{2\alpha-n} W_j \left( \frac{\lambda^{2\alpha}}{|Y|^{\frac{2\alpha}{\alpha}}} \right)
\]

\[
\geq L_j^{-1} \inf_{\mathcal{B}_1^+ \setminus \{0\}} U - 4^{n-2\alpha} \tau^{2\alpha-n} \|W_j\|_{L^\infty(B_j^+)} L_j^{-2}
\]

\[
\geq L_j^{-1} \inf_{\mathcal{B}_1^+ \setminus \{0\}} U - 8^{n-2\alpha} \tau^{2\alpha-n} \|\tilde{W}\|_{L^\infty(B_j^+)} L_j^{-2}
\]

\[
\geq 16c_4 L_j^{-1}
\]

for \( Y \in \partial^\prime \Sigma_j \) and large \( j \). It implies that

\[
(W_\lambda + A_\lambda)(Y) \geq 16c_4 L_j^{-1} \quad \text{for } Y \in \partial^\prime \Sigma_j
\]

when \( j \) is sufficiently large.

Now, from (3.4), (3.5), (3.13) and (3.17), we have

\[
\begin{cases}
\text{div}(t^{1-2\alpha} \nabla (W_\lambda + A_\lambda)) = 0 & \text{in } \Sigma_j \setminus \overline{B_\lambda^+}, \\
\frac{\partial (W_\lambda + A_\lambda)}{\partial \nu^\alpha} \geq b_\alpha (w_\lambda + a_\lambda) & \text{on } \partial^\prime \Sigma_j \setminus \{ -L_j^{-\frac{2\alpha}{\alpha}} x_j \},
\end{cases}
\]

(3.21)

where \( w_\lambda(x) = W_\lambda(x, 0) \) and \( a_\lambda(x) = A_\lambda(x, 0) \).

For \( \lambda_0 = 1/2 \) and \( \lambda_1 = 2 \), we define

\[
\lambda^* := \sup \{ \lambda \geq \lambda_0 : W_\mu + A_\mu \geq 0 \text{ in } (\Sigma_j \setminus \overline{B_\lambda^+}) \setminus \{ -L_j^{-\frac{2\alpha}{\alpha}} x_j \} \text{ for all } \lambda_0 \leq \mu \leq \lambda \}.
\]

Then \( \lambda^* \) is well defined as \( \lambda_0 \) belongs to the above set by Lemma 3.1. From (3.8) and (3.17), we know that \( \lambda^* < \lambda_1 \). By continuity, we have

\[
W_{\lambda^*} + A_{\lambda^*} \geq 0 \quad \text{in } (\Sigma_j \setminus \overline{B_{\lambda^*}^+}) \setminus \{ -L_j^{-\frac{2\alpha}{\alpha}} x_j \}.
\]
We also have $W_{\lambda^*} + A\lambda^* = 0$ on $\partial^\nu B_{\lambda^*}$. It follows from (3.20) and the strong maximum principle that $W_{\lambda^*} + A\lambda^* > 0$ in $(\Sigma_j \setminus \overline{B_{\lambda^*}^+}) \setminus \{-L_j^{\frac{2}{n-2}} x_j\}$.

For $\delta > 0$ small which will be fixed later, by Lemma 2.5, there exists a $c = c(\delta) > 0$ such that

$$W_{\lambda^*} + A\lambda^* > c \quad \text{in} \quad (\Sigma_j \setminus \overline{B_{\lambda^*+\delta}^+}) \setminus \{-L_j^{\frac{2}{n-2}} x_j\}.$$ 

By the uniform continuity, there exists an $\varepsilon = c(\delta) \in (0, \delta)$ small such that for all $\lambda^* < \lambda < \lambda^* + \varepsilon$,

$$(W_{\lambda} + A\lambda) - (W_{\lambda^*} + A\lambda^*) > -c/2 \quad \text{in} \quad (\Sigma_j \setminus \overline{B_{\lambda^*+\delta}^+}) \setminus \{-L_j^{\frac{2}{n-2}} x_j\}.$$ 

Hence, we obtain

$$W_{\lambda} + A\lambda > c/2 \quad \text{in} \quad (\Sigma_j \setminus \overline{B_{\lambda^*+\delta}^+}) \setminus \{-L_j^{\frac{2}{n-2}} x_j\}.$$ 

Next, we make use of the narrow domain technique from [1]. Using $(W_{\lambda} + A\lambda)^- := \max\{-W_{\lambda} + A\lambda, 0\}$ as a test function in (3.21), we obtain that for any $\lambda^* < \lambda < \lambda^* + \varepsilon$,

$$\int_{B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+} t^{1-2\sigma} |\nabla (W_{\lambda} + A\lambda)^-|^2 dY$$

$$\leq -\int_{B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+} b_{\lambda}(w_{\lambda} + a\lambda)(w_{\lambda} + a\lambda)^- dy$$

$$\leq C \int_{B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+} ((w_{\lambda} + a\lambda)^-)^2 dy$$

$$\leq C \left( \int_{B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+} ((w_{\lambda} + a\lambda)^-) \frac{2\sigma}{n-2\sigma} dy \right)^{\frac{n-2\sigma}{2\sigma}} \mathcal{L}^n(B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+)\frac{2\sigma}{n-2\sigma}$$

$$\leq C \left( \int_{B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+} t^{1-2\sigma} |\nabla (W_{\lambda} + A\lambda)^-|^2 dY \right)^{\frac{n-2\sigma}{2\sigma}} \mathcal{L}^n(B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+)\frac{2\sigma}{n-2\sigma}.$$ 

We can fix $\delta > 0$ sufficiently small such that for all $\lambda^* < \lambda < \lambda^* + \varepsilon$,

$$C \mathcal{L}^n(B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+)\frac{2\sigma}{n-2\sigma} \leq 1/2.$$ 

Then

$$\nabla (W_{\lambda} + A\lambda)^- = 0 \quad \text{in} \quad B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+.$$ 

Recall that

$$(W_{\lambda} + A\lambda)^- = 0 \quad \text{on} \quad \partial^\nu B_{\lambda}^+,$$ 

and then we have

$$W_{\lambda} + A\lambda \geq 0 \quad \text{in} \quad B_{\lambda^*+\delta}^+ \setminus B_{\lambda}^+.$$ 

This implies that there exists an $\varepsilon > 0$ such that for all $\lambda^* < \lambda < \lambda^* + \varepsilon$,

$$W_{\lambda} + A\lambda \geq 0 \quad \text{in} \quad (\Sigma_j \setminus \overline{B_{\lambda^*}^+}) \setminus \{-L_j^{\frac{2}{n-2}} x_j\},$$

which contradicts the definition of $\lambda^*$. The proof of Theorem 1.2 is completed.

\section{Large singular solutions in higher dimensions}

In this section, we prove Theorem 1.4 and always assume $\sigma \in (0, 1)$. We write $a_i \sim b_i$ if the sequence $\{a_i/b_i\}_{i=1}^\infty$ is bounded between two positive constants depending only on $n$, $\sigma$, $\inf_{\mathbb{R}^n} k$ and $\sup_{\mathbb{R}^n} k$.

The definition of the fractional Laplacian (1.2) is equivalent to

$$(-\Delta)^\sigma u(x) = \frac{C_{n,\sigma}}{2} \lim_{r \to 0^+} \int_{\mathbb{R}^n \setminus B_r} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{n+2\sigma}} dy.$$ 

(4.1)
For the real number \( \gamma > 0 \), we denote by \( C^{\gamma} \) the space \( C^{[\gamma]}{\gamma'} \), where \([\gamma]\) is the greatest integer satisfying \( [\gamma] < \gamma \) and \( \gamma' = \gamma - [\gamma] \). Define

\[
L_\sigma(\mathbb{R}^n) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2\sigma}} \, dx < \infty \right\}.
\]

If \( u \in L_\sigma(\mathbb{R}^n) \cap C^{2\sigma+\gamma}(\mathbb{R}^n \setminus \{0\}) \) for some \( \varepsilon > 0 \), then \((-\Delta)^\varepsilon u(x)\) is well defined for every \( x \in \mathbb{R}^n \setminus \{0\} \) (see, e.g., [25]). To prove Theorem 1.4, we also need the following simple lemma.

**Lemma 4.1** (See [26]). Suppose \( p > 1 \), \( \{a_i\}_{i=1}^N \subset (0, \infty) \) and \( a_1 \geq a_i \) for \( 2 \leq i \leq N \). Then

\[
\frac{\sum_{i=1}^N a_i^p}{(\sum_{i=1}^N a_i)^p} \leq 1 + \frac{\sum_{i=1}^N a_i^p}{1 + p\sum_{i=1}^N a_i} < 1.
\]

The method of proving Theorem 1.4 is similar to that of Taliaferro [26] for the Laplacian case \( \sigma = 1 \), which has also been used for the higher-order conformal \( Q \)-curvature equation in our recent paper [12]. Here, we first set up a framework to fit the nonlocal equation

\[
(-\Delta)^\varepsilon u = K(x) u^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in } \mathbb{R}^n \setminus \{0\}.
\]  

(4.2)

Roughly speaking, we construct a sequence of bubbles \( \{u_i\}_{i=1}^\infty \) such that the function \( \bar{u} := \sum_{i=1}^\infty u_i \) satisfies \( \bar{u}(x) \neq \varphi(|x|) \) near the origin. Then we find a positive bounded function \( u_0 \in C^{2\sigma+\gamma}(\mathbb{R}^n \setminus \{0\}) \) by the method of sub- and super-solutions such that

\[
u := u_0 + \bar{u} \quad \text{and} \quad K := \frac{(-\Delta)^\varepsilon u}{u^{(n+2\sigma)/(n-2\sigma)}}
\]

satisfy the corresponding conclusions of Theorem 1.4 for the nonlocal equation (4.2). We point out that the bubbles \( \{u_i\}_{i=1}^\infty \) will be selected very carefully so that \( |\nabla K| \) can be continuous in the whole \( \mathbb{R}^n \). Eventually, we define \( U := \mathcal{P}_\sigma[u] \) which is the desired solution for our Theorem 1.4, where \( \mathcal{P}_\sigma \) is the Poisson kernel given in (2.1). The same arguments as in [12, 26] will be omitted. Meanwhile, we give the detailed proofs for those different from those in [12, 26]. In particular, the main differences are as follows:

- We use the method of sub- and super-solutions for the fractional Laplacian.
- We show that \( u_0 \) is a distributional solution in \( \mathbb{R}^n \) for a nonlocal equation.
- We need a slightly different estimate for \( |\nabla K| \) since \( n > 2\sigma + 3 \).

**Proof of Theorem 1.4.** The proof consists of seven steps.

**Step 1.** Preliminaries.

Without loss of generality, we can assume that \( 0 < \varepsilon < 1 \) and \( k(0) = 1 \). Since \( \nabla k(0) = 0 \), there exists a \( C^1 \) positive function \( \bar{k} : \mathbb{R}^n \to \mathbb{R} \) such that \( \bar{k} \equiv 1 \) in a small neighborhood of the origin, \( \bar{k}(x) = k(x) \) for \( |x| \geq \varepsilon \) and \( \| \bar{k} - k \|_{C^1(\mathbb{R}^n)} < \varepsilon/2 \). Replacing \( k \) by \( \bar{k} \), we can assume that \( k \equiv 1 \) in \( B_{\delta} \) for some \( 0 < \delta < \varepsilon \).

Let

\[
h(r, \lambda) = c_{n, \sigma} \left( \frac{\lambda}{\lambda^2 + r^2} \right)^{\frac{n-2\sigma}{2}}
\]

and \( \psi_\lambda(x) := h(|x|, \lambda) \) for \( \lambda > 0 \), which are the well-known bubbles to the equation

\[
(-\Delta)^\varepsilon \psi_\lambda = \psi_\lambda^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{in } \mathbb{R}^n.
\]

After some calculations, one can find that there exist \( \delta_1 \) and \( \delta_2 \) satisfying

\[
0 < \delta_2 < \delta_1 = \frac{\delta}{2} < 4,
\]

(4.3)

and for any \( |x| \leq \delta_2 \) or \( |x| \geq \delta_1 \),

\[
\frac{1}{2} \leq \frac{\psi_\lambda(x - x_1)}{\psi_\lambda(x - x_2)} < 2 \quad \text{when } |x_1| = |x_2| = \delta_1 \text{ and } 0 < \lambda \leq \delta_2.
\]

(4.4)
Recalling that \( k \) is bounded between two positive constants, we define
\[
a = \frac{1}{2} \inf_{\mathbb{R}^n} k > 0 \quad \text{and} \quad b = \sup_{\mathbb{R}^n} k > 0.
\]
(4.5)

Let \( i_0 \) be the smallest integer greater than 2 such that
\[
\frac{4a}{i_0} > \frac{2^{\frac{n+2p}{2}}}{(2a)^{\frac{n+2p}{(n-2p)}}}.
\]
(4.6)

Choose a sequence \( \{x_i\}_{i=1}^\infty \) of distinct points in \( \mathbb{R}^n \) and a sequence \( \{r_i\}_{i=1}^\infty \) of positive numbers such that
\[
|x_1| = |x_2| = \cdots = |x_{i_0}| = \delta_1, \quad \lim_{i \to \infty} |x_i| = 0,
\]
(4.7)
\[
r_1 = r_2 = \cdots = r_{i_0} = \frac{\delta_2}{2} \quad \lim_{i \to \infty} r_i = 0,
\]
(4.8)
\[
B_{4r_i}(x_i) \subset B_{\delta_2} \setminus \{0\} \quad \text{for} \ i > i_0
\]
(4.9)

and
\[
B_{2r_i}(x_i) \cap B_{2r_j}(x_j) = \emptyset \quad \text{for} \ j > i > i_0.
\]
(4.10)

In addition, we require that the union of the line segments \( x_1x_2, x_2x_3, \ldots, x_{i_0-1}x_{i_0}, x_{i_0}x_1 \) be a regular \( i_0 \)-gon. We prescribe the side length of this polygon later. From (4.3), (4.7) and (4.8), we know that
\[
B_{2r_i}(x_i) \subset B_{2\delta_1} \setminus B_{\delta_2} \quad \text{for} \ 1 \leq i \leq i_0
\]
and hence by (4.9),
\[
B_{2r_i}(x_i) \cap B_{2r_j}(x_j) = \emptyset \quad \text{for} \ 1 \leq i \leq i_0 < j.
\]
(4.11)

Define three functions \( f : [0, \infty) \times (0, \infty) \times (0, \infty) \to \mathbb{R} \) and \( M, Z : (0, 1) \times (0, \infty) \to (0, \infty) \) by
\[
f(z_1, z_2, z_3) = z_2(z_1 + z_3)^{\frac{n+2p}{n-2p}} - z_1^{\frac{n+2p}{n-2p}},
\]
\[
M(z_2, z_3) = \frac{z_2^2z_3^{\frac{n+2p}{n-2p}}}{(1 - \frac{z_2}{z_3})^{\frac{n-2p}{n}}}, \quad \text{and} \quad Z(z_2, z_3) = \frac{z_2z_3^{\frac{n-2p}{n}}}{1 - \frac{z_2}{z_3}}.
\]
(4.12)

For each fixed \((z_2, z_3) \in (0, 1) \times (0, \infty)\), the function \( f(\cdot, z_2, z_3) : [0, \infty) \to \mathbb{R} \) is strictly increasing on \([0, Z(z_2, z_3)]\) and is strictly decreasing on \([Z(z_2, z_3), \infty)\), and attains its maximum value \( M(z_2, z_3) \) at \( z_1 = Z(z_2, z_3) \).

Define \( F : [0, \infty) \times (0, \infty) \times (0, \infty) \to (0, \infty) \) by
\[
F(z_1, z_2, z_3) = \begin{cases} 
\frac{f(z_1, z_2, z_3)}{M(z_2, z_3)}, & \text{if} \ 0 < z_2 < 1 \text{ and } 0 \leq z_1 \leq Z(z_2, z_3), \\
M(z_2, z_3), & \text{if} \ 0 < z_2 < 1 \text{ and } z_1 > Z(z_2, z_3), \\
0, & \text{if} \ z_2 = 1.
\end{cases}
\]
(4.13)

It is easy to see that \( f \) and \( F \) are \( C^1 \), \( f \leq F \) and \( F \) is non-decreasing in \( z_1, z_2 \) and \( z_3 \).

**Step 2.** Select the sequences \( \{x_i\}_{i=1}^\infty \) and \( \{\lambda_i\}_{i=1}^\infty \).

Let
\[
w(x) = (2h)^{-\frac{a}{2}} h(|x|, 1) = \frac{c_{n, \sigma}}{(2h)^{\frac{n}{2}}} \left( \frac{1}{1 + |x|^2} \right)^{\frac{n-2p}{2}} \quad \text{for} \ x \in \mathbb{R}^n.
\]
(4.14)

Then we have
\[
(-\Delta)^{\sigma} w = (2h)^{\frac{n}{2}} w^{\frac{n+2p}{n}} \quad \text{in} \ \mathbb{R}^n,
\]
(4.15)
and by (4.5),
\[
\sup_{\mathbb{R}^n} w = w(0) = c_{n, \sigma}(2h)^{-\frac{a}{2}} < c_{n, \sigma}.
\]
Choose a sequence \(\{\varepsilon_i\}_{i=1}^{\infty}\) of positive numbers such that
\[\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_{i_0} \quad \text{and} \quad \varepsilon_i \leq 2^{-i} \quad \text{for} \quad i \geq 1.\] (4.15)

Now we introduce four sequences of real numbers as follows: for \(i \geq 1\), let
\[
k_i = \left(\frac{1}{2}, 1\right) \quad \text{with} \quad k_1 = k_2 = \cdots = k_{i_0},
\]
(4.16)

\[
M_i = M(k_i, 2w(0)) = \frac{k_i}{(2w(0))^{\frac{\beta}{n-2\sigma}}} = \frac{1}{(1-k_i^{\frac{2}{n-2\sigma}})^{\frac{\beta}{n-2\sigma}}},
\]
(4.17)

\[
\rho_i = \sup \left\{ \rho > 0 : \|2^\sigma(\chi_{B_{2^\sigma}(x_i)}) \| \leq \frac{w}{2^{i+1}(2w(0))^{\frac{\beta}{n-2\sigma}} M_i} \right\}
\]
(4.18)

and
\[
\lambda_i = \sup \{\lambda > 0 : \psi_\lambda(x - x_i) \leq \varepsilon_i a^{\frac{n-2\sigma}{2\sigma}} w(x) \quad \text{for} \quad |x - x_i| \geq \rho_i\},
\]
(4.19)

where \(\mathcal{I}_{2\sigma}\) is the Riesz potential operator of order \(2\sigma\) and \(\chi_{B_{2^\sigma}(x_i)}\) is the characteristic function of the ball \(B_{2^\sigma}(x_i)\). Then we have the following lemma.

**Lemma 4.2.** For \(i \geq 1\),
\[
M_i \sim \frac{1}{(1-k_i)^{\frac{\beta}{n-2\sigma}}}, \quad \rho_i^{2\sigma} \sim \frac{1}{2^i M_i} \quad \text{and} \quad \lambda_i \sim \varepsilon_i^{\frac{n-2\sigma}{2\sigma}} \rho_i^2.
\]

**Proof.** The proof is the same as that of [12, Lemma 2.2].

By Lemma 4.2, after increasing the values of \(k_i\) for certain values of \(i\) while holding \(\varepsilon_i\) fixed, we can assume for \(i \geq 1\) that
\[
M_i > 9^i, \quad \rho_i \in (0, r_i), \quad \lambda_i \in (0, \delta_2)
\]
(4.20)

and
\[
k_i^{\frac{n-2\sigma}{2}} > \frac{1 + (\frac{1}{2})^{n-2\sigma}}{1 + \frac{1}{n-2\sigma} (\frac{1}{2})^{n-2\sigma}}, \quad M_i^\beta > \max \left\{ \frac{1}{\varepsilon_i^{\frac{n-2\sigma}{2\sigma}}}, 2^i \right\}, \quad \lambda_i^\beta < \frac{\varepsilon_i^{\frac{n-2\sigma}{2\sigma}}}{2^i},
\]
(4.21)

where \(\beta = \beta(n, \sigma) \in (0, 1/6)\) is a constant to be specified later.

Notice that for \(1 \leq i \leq i_0, \rho_i\), and \(\lambda_i\) do not change as \(x_i\) moves on the sphere \(|x| = \delta_1\). Therefore, we can require that the union of the line segments \(\overline{x_0 x_1}, \overline{x_1 x_2}, \ldots, \overline{x_{i_0-1} x_{i_0}}, \overline{x_{i_0} x_{i_1}}\) be a regular \(i_0\)-gon with the side length \(4\rho_1\). Thus,
\[
\text{dist}(B_{\rho_i}(x_i), B_{\rho_j}(x_j)) \geq \rho_i + \rho_j \quad \text{for} \quad 1 \leq i < j \leq i_0.
\]
(4.22)

By (4.10), (4.11) and (4.20), the inequality (4.22) also holds for \(1 \leq i < j\).

For \(i \geq 1\), define
\[
u_i(x) := \psi_{\lambda_i}(x - x_i).
\]
Then one can check that
\[
\min_{x \in B_{2^\sigma}(x_i)} \nu_{j+1}(x) = \left(\frac{1}{3}\right)^{n-2\sigma} \nu_{j-1}(x) \quad \text{for} \quad 2 \leq j < i_0 - 1,
\]
(4.23)

and a similar inequality holds when \(j\) is 1 or \(i_0\).

Here, we also give some inequalities which will be used later. By (4.21), Lemma 4.2 and the monotonicity of \(Z\), using the same arguments as in [12, (39)], we see that for \(1 \leq j \leq i_0\),
\[
\min_{x \in B_{2^\sigma}(x_j)} Z \left(\frac{k_j^{\frac{n-2\sigma}{2}}}{k_j^{\frac{n-2\sigma}{2}}}, \sum_{i=1, i \neq j}^{i_0} \nu_i(x) \right) \geq CM_j^{(1-\varepsilon)(n-2\sigma)}.
\]
(4.24)
Thus, by increasing $k_1$ (recalling that $k_1 = k_2 = \cdots = k_{i_0}$), we have

$$
\min_{x \in B_{2\rho_i}(x_j)} Z\left( \frac{k_j^{n+2\sigma}}{2M_j}, \sum_{i=1, i \neq j}^{i_0} u_i(x) \right) > w(0) \quad \text{for } 1 \leq j \leq i_0.
$$

By Lemma 4.2,

$$
Z\left( \frac{k_j^{n+2\sigma}}{2M_j}, \frac{1}{2^{(n-2\sigma)/\sigma}} \right) \sim \frac{1}{1 - k_j^{n+2\sigma}} \frac{1}{M_j^{(1-\beta)/2}} \sim M_j^{(1-\beta)(n-2\sigma)/2} \quad \text{for } j \geq 1.
$$

Therefore, by increasing each term of the sequence $\{k_j\}_{j=1}^{\infty}$, we also have

$$
Z\left( \frac{k_j^{n+2\sigma}}{2M_j}, \frac{1}{2^{(n-2\sigma)/\sigma}} \right) > w(0) \quad \text{for } j \geq 1.
$$

Then, by (4.19), (4.5) and (4.15), we obtain for $j \geq 1$ and $|x - x_j| \geq \rho_j$ that

$$
u_j(x) \leq \varepsilon_j a^{n-2\sigma} w(0) \leq w(0) < Z\left( \frac{k_j^{n+2\sigma}}{2M_j}, \frac{1}{2^{(n-2\sigma)/\sigma}} \right).
$$

**Step 3.** Estimate the sum of the bubbles $u_i$.

By (4.19) and (4.15), we have

$$
u_i \leq \varepsilon_j a^{n-2\sigma} w \quad \text{in } \mathbb{R}^n \setminus B_{\rho_i}(x_i)
$$

and

$$
\sum_{i=1}^{\infty} \nu_i \leq a^{n-2\sigma} w \quad \text{in } \mathbb{R}^n - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i).
$$

By (4.15) and (4.28), we know that $\sum_{i=1}^{\infty} \nu_i \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Moreover, we claim the following lemma.

**Lemma 4.3.** $\sum_{i=1}^{\infty} \nu_i \in L^{n+2\sigma}(\mathbb{R}^n) \cap L_\sigma(\mathbb{R}^n)$ and satisfies

$$
(-\Delta)^\sigma \left( \sum_{i=1}^{\infty} \nu_i \right) = \sum_{i=1}^{\infty} \nu_i^{n+2\sigma} \quad \text{in } \mathbb{R}^n \setminus \{0\}.
$$

**Proof.** By (4.28) and (4.29), we have

$$
\int_{\mathbb{R}^n} \left( \sum_{i=1}^{\infty} \nu_i(x) \right)^{n+2\sigma} dx \leq \int_{\mathbb{R}^n - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i)} (a^{n-2\sigma} w)^{n+2\sigma} dx + \sum_{i=1}^{\infty} \int_{B_{\rho_i}(x_i)} (\nu_i(x) + a^{n-2\sigma} w)^{n+2\sigma} dx
$$

$$
\leq 2^{n+2\sigma} a^{n+2\sigma} \int_{\mathbb{R}^n} w(x)^{n+2\sigma} dx + 2c_{n,\sigma} a^{n+2\sigma} \sum_{i=1}^{\infty} \int_{B_{\rho_i}} \left( \frac{\lambda_i}{\lambda_i^2 + |y|^2} \right)^{n+2\sigma} dy
$$

$$
\leq C + C \sum_{i=1}^{\infty} \lambda_i^{n-2\sigma} \int_{B_{\rho_i}/\lambda_i} \left( \frac{1}{1 + |y|^2} \right)^{n+2\sigma} dy < \infty.
$$

Therefore, $\sum_{i=1}^{\infty} \nu_i \in L^{n+2\sigma}(\mathbb{R}^n)$. By Hölder’s inequality, we have $\sum_{i=1}^{\infty} \nu_i \in L_\sigma(\mathbb{R}^n)$.

Fix $x \in \mathbb{R}^n \setminus \{0\}$. Since $\sum_{i=1}^{\infty} \nu_i$ converges to $\sum_{i=1}^{\infty} \nu_i$ in $C_{\text{loc}}^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$
\left| 2 \sum_{i=1}^{N} \nu_i(x) - \sum_{i=1}^{N} \nu_i(x + y) - \sum_{i=1}^{N} \nu_i(x - y) \right| \leq C |y|^2 \quad \text{for } y \in B_{|x|/2},
$$
where $C$ is a positive constant when $N$ is sufficiently large, and
\[
|y|^{-n-2\sigma} \left| 2 \sum_{i=1}^{N} u_i(x) - \sum_{i=1}^{N} u_i(x + y) - \sum_{i=1}^{N} u_i(x - y) \right| 
\leq |y|^{-n-2\sigma} \left( 2 \sum_{i=1}^{\infty} u_i(x) + \sum_{i=1}^{\infty} u_i(x + y) + \sum_{i=1}^{\infty} u_i(x - y) \right)
\]
for $y \in \mathbb{R}^n \setminus (B_{|x|/2} \cup \{x, -x\})$. Since $\sum_{i=1}^{\infty} u_i \in L_{\sigma}(\mathbb{R}^n)$, by Lebesgue’s dominated convergence theorem, we get (4.30).

Now, by increasing $k_i$ for each $i$, we can assume that
\[
u_i(x_i) = c_{n,\sigma} \lambda_i^{-\frac{n-2\sigma}{2}} > i \varphi(|x_i|) \quad \text{for } i \geq 1 \tag{4.31}
\]
and $u_i + |\nabla u_i| < 2^{-i}$ in $\mathbb{R}^n \setminus B_{2r_i}(x_i)$, $i \geq 1$. Thus by (4.10) and (4.11),
\[u_i + |\nabla u_i| < 2^{-i} \quad \text{in } B_{2r_i}(x_i) \tag{4.32}
\]
when $i \neq j$ and either $i \geq 1$ and $j > i_0$ or $i > i_0$ and $1 \leq j \leq i_0$. Again, by increasing $k_i$ for $i > i_0$, we can force $u_i$ and $M_i$ to satisfy
\[
\sum_{i=i_0+1}^{\infty} u_i(x) < \frac{1}{2} \min_{1 \leq i \leq i_0} u_i(x) \quad \text{for } |x| = \delta_2, \tag{4.33}
\]
\[\sum_{i=i_0+1}^{\infty} u_i(x) < \frac{\bar{u}_i}{2} \quad \text{in } B_{2r_i}(x_j), \quad j > i_0 \tag{4.34}
\]
and
\[\frac{1}{M_j^{\frac{n-2\sigma}{2}}} < \min_{|x| < \delta} u_i(x) \quad \text{for } j > i_0. \tag{4.35}
\]

It follows from (4.32) and (4.28) that
\[
\sum_{i=1, i \neq j}^{\infty} u_i + u_i^{\frac{n}{2\sigma}} \leq C \quad \text{in } B_{\rho_j}(x_j), \quad j \geq 1. \tag{4.36}
\]

Similarly, by (4.32), (4.28), Lemma 4.2 and (4.6),
\[
\sum_{i=1, i \neq j}^{\infty} |\nabla u_i| + u_i^{\frac{4\sigma}{n}} |\nabla u_i| \leq \sum_{i=1, i \neq j}^{\infty} (|\nabla u_i| + u_i^{\frac{4\sigma}{n}} |\nabla u_i|) + C
\]
\[\leq C \sum_{i=1, i \neq j}^{i_0} \left( u_i \frac{1}{\rho_j} + u_i^{\frac{n+2\sigma}{2\sigma}} \frac{1}{\rho_j} \right) + C
\]
\[\leq C^{\frac{2+\varphi}{2}} M_j^{\frac{n-\varphi}{2}} \leq C^{\frac{2+\varphi}{2}} M_j^{\frac{n-\varphi}{2}} \leq C M_j^{\frac{n-\varphi}{2}} \quad \text{in } B_{\rho_j}(x_j), \quad 1 \leq j \leq i_0,
\]
and by (4.32) and (4.28),
\[
\sum_{i=1, i \neq j}^{\infty} |\nabla u_i| + u_i^{\frac{4\sigma}{n}} |\nabla u_i| \leq C \quad \text{in } B_{\rho_j}(x_j), \quad j > i_0.
\]

Thus, we get
\[
\sum_{i=1, i \neq j}^{\infty} |\nabla u_i| + u_i^{\frac{4\sigma}{n}} |\nabla u_i| \leq C M_j^{\frac{n}{2}} \quad \text{in } B_{\rho_j}(x_j), \quad j \geq 1. \tag{4.37}
\]
Step 4. Construct the correction function $u_0$.
Since $n > 2\sigma + 3$, by Lemma 4.2 and (4.20), we have
\[ \frac{1-k_i}{\rho_i} \sim \frac{2\pi M_i^{\frac{1}{2}}} {M_i^{\frac{1}{2} + \frac{\sigma}{2}}} \leq \frac{2\pi}{M_i^{\frac{1}{2}}} \leq \left( \frac{2}{3} \right) \frac{\pi}{M} \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty. \] (4.38)
Let $\eta : [0, \infty) \rightarrow [0, 1]$ be a $C^\infty$ cut-off function satisfying $\eta(t) = 1$ for $0 \leq t \leq 1$ and $\eta(t) = 0$ for $t \geq 3/2$.
Define
\[ \kappa(x) = k(x) + \sum_{i=1}^\infty (k_i - k(x))\eta_i(x) \quad \text{for} \quad x \in \mathbb{R}^n, \] (4.39)
where $\eta_i(x) := \eta(|x-x_i|/\rho_i)$. Since $\{\eta_i\}_{i=1}^\infty$ have disjoint supports contained in $B_{2\delta_1} \setminus \{0\}$, $\kappa$ is well defined. Recalling that $k = 1$ in $B_\delta$, we obtain $\kappa(0) = k(0) = 1$, $\kappa \leq k$ in $\mathbb{R}^n$ and $\kappa(x) = k(x)$ for $|x| \geq 2\delta_1$. By (4.5) and (4.16), we have
\[ \inf_{\mathbb{R}^n} \kappa \geq a. \] (4.40)
Since $k \equiv 1$ in $B_\delta$, we have
\[ \nabla \kappa(x) = \sum_{i=1}^\infty \frac{k_i - 1}{\rho_i} \eta_i' \left( \frac{|x-x_i|}{\rho_i} \right) \frac{x_i - x_i}{|x-x_i|} \quad \text{for} \quad x \in B_\delta, \] (4.41)
and it follows from (4.38) that $\kappa \in C^1(\mathbb{R}^n)$ and $\nabla \kappa(0) = 0$.
By (4.18),
\[ 0 < T_{2\sigma} \mathcal{M} < \frac{w}{2} \quad \text{in} \quad \mathbb{R}^n, \] (4.42)
where
\[ \mathcal{M}(x) := \begin{cases} (2w(0))^{\frac{n+2\sigma}{n+2\sigma}} M_i & \text{in} \ B_{\rho_i}(x_i), \ i \geq 1, \\ 0 & \text{in} \ \mathbb{R}^n - \bigcup_{i=1}^\infty B_{2\rho_i}(x_i), \\ (2w(0))^{\frac{n+2\sigma}{n+2\sigma}} M_i \left( 2 - \frac{|x-x_i|}{\rho_i} \right) & \text{in} \ B_{2\rho_i}(x_i) \setminus B_{\rho_i}(x_i), \ i \geq 1. \end{cases} \]
Since $\mathcal{M}$ is locally Lipschitz continuous in $\mathbb{R}^n \setminus \{0\}$ and $\mathcal{M} \in L^{\frac{n}{n-\sigma}}(\mathbb{R}^n)$, we have
\[ \mathbf{v} := \frac{w}{2b} + T_{2\sigma} \mathcal{M} \in C^{2\sigma+\gamma}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \] for any $\gamma \in (0, 1)$. It follows from (4.14) and (4.42) that
\[ \frac{w}{2b} < \mathbf{v} < w < c_{n, \sigma} \quad \text{in} \quad \mathbb{R}^n. \] (4.43)
Hence, we get $\mathbf{v} \in L_\sigma(\mathbb{R}^n)$. By (4.13),
\[ (-\Delta)^\sigma \mathbf{v} = (2b)^{\frac{n+2\sigma}{n+2\sigma}} w^{\frac{n+2\sigma}{n+2\sigma}} + \mathcal{M} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \] (4.44)
Define $H : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ by
\[ H(x, v) = \kappa(x) \left( v + \sum_{i=1}^\infty u_i(x) \right)^{\frac{n+2\sigma}{n+2\sigma}} - \sum_{i=1}^\infty u_i(x)^{\frac{n+2\sigma}{n-\sigma}}. \] (4.45)
Then we have
\[ H(x, v) = f(\tilde{u}(x), \kappa(x), p(x, v)), \]
where
\[ \tilde{u}(x) := \sum_{i=1}^\infty u_i(x)^{\frac{n+2\sigma}{n+2\sigma}}, \quad \text{and} \quad p(x, v) := v + \sum_{i=1}^\infty u_i(x) - \tilde{u}(x). \]
Now, applying the method of sub- and super-solutions, we can get the correction function $H(u, v) = F(\tilde{u}(x), \kappa(x), p(x, v))$.

Thus, for $0 \leq \kappa(x) < 1$, we have

$$\tilde{u}(x) \leq M(\kappa(x), p(x, v))$$

Moreover, by the definition of $F$, we see that $H(x, v) = H(x, v)$ if and only if either $\kappa(x) < 1$ and $\tilde{u}(x) \leq \tilde{Z}(\kappa(x), p(x, v))$ or $\kappa(x) \geq 1$.

For $x \in \mathbb{R}^n - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i)$ and $\kappa(x) < 1$, we have

$$\tilde{u}(x) \leq \sum_{i=1}^{\infty} u_i(x) \leq \frac{n-2\sigma}{n-\sigma} w(x) \text{ by (4.29)}$$

$$\leq \frac{w(x)k(x) \frac{n+2\sigma}{n-\sigma}}{1 - \kappa(x) \frac{n-2\sigma}{n-\sigma}} \text{ by (4.40)}$$

$$\leq \frac{p(x, w(x))k(x) \frac{n+2\sigma}{n-\sigma}}{1 - \kappa(x) \frac{n-2\sigma}{n-\sigma}}$$

$$= \tilde{Z}(\kappa(x), p(x, w(x))) \text{ by (4.12)}.$$

Hence,

$$H(x, w(x)) = H(x, v(x)) \text{ for } x \in \mathbb{R}^n - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i).$$

Thus, for $x \in (\mathbb{R}^n \setminus \{0\}) - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i)$ and $0 \leq v \leq w(x)$, we have

$$H(x, v) \leq \tilde{H}(x, w(x)) = \kappa(x) \left( w(x) + \sum_{i=1}^{\infty} u_i(x) \right)$$

$$\leq b(2w(x)) \frac{n+2\sigma}{n-\sigma} \leq (-\Delta)^{\sigma} \sigma(x) \quad \text{(4.48)}$$

by (4.29), (4.5) and (4.44).

For $x \in B_{\rho_i}(x_i)$ and $i \geq 1$, we have $\kappa(x) \equiv k_i < 1$. Hence, from (4.47), we obtain for $x \in B_{\rho_i}(x_i)$ and $0 \leq v \leq w(x)$ that

$$H(x, v) \leq M(k_i, p(x, v)) = \frac{k_i p(x, v) \frac{n+2\sigma}{n-\sigma}}{1 - k_i \frac{n-2\sigma}{n-\sigma} \frac{n+2\sigma}{n-\sigma}}$$

$$\leq M_i \left( v + \sum_{j=1}^{\infty} u_j(x) - \tilde{u}(x) \right) \frac{n+2\sigma}{n-\sigma} \leq (-\Delta)^{\sigma} \sigma(x) \quad \text{(4.49)}$$

This together with (4.48) implies that

$$H(x, v) \leq (-\Delta)^{\sigma} \sigma(x) \text{ for } x \in \mathbb{R}^n \setminus \{0\} \text{ and } 0 \leq v \leq w(x).$$

Hence, by the non-negativity of $H$, (4.43) and (4.50), we can use $v \equiv 0$ and $\sigma$ as sub- and super-solutions of the problem

$$(-\Delta)^{\sigma} u = H(x, u) \text{ in } \mathbb{R}^n \setminus \{0\}.$$

Now, applying the method of sub- and super-solutions, we can get the correction function $u_0$ by the following two lemmas.
Lemma 4.4. There exists a \( C^\sigma(\mathbb{R}^n) \cap C^{2\sigma+\gamma}(B_2 \setminus \overline{B}_1) \) solution \( v \) to
\[
\begin{cases}
(-\Delta)^\sigma v = H(x, v) & \text{in } B_2 \setminus \overline{B}_1, \\
v = 0 & \text{on } B_2^c \cup \overline{B}_1,
\end{cases}
\]
for every \( \gamma \in (0, 1) \).

Proof. Let \( \{y_i\}_{i=1}^\infty \) be the sequence of functions defined in \( \mathbb{R}^n \) by the iteration scheme
\[
\begin{cases}
(-\Delta)^\sigma (y_{i+1} - y_i) = H(x, y_i) - (-\Delta)^\sigma y_0 \geq 0 & \text{in } B_2 \setminus \overline{B}_1, \\
y_{i+1} - y_i = 0 & \text{on } B_2^c \cup \overline{B}_1,
\end{cases}
\]
where \( y_0 := 0 \). Then we have
\[
\begin{cases}
(-\Delta)^\sigma (y_{i} - y_0) = H(x, y_0) - (-\Delta)^\sigma y_0 \geq 0 & \text{in } B_2 \setminus \overline{B}_1, \\
y_i - y_0 = 0 & \text{on } B_2^c \cup \overline{B}_1.
\end{cases}
\]
Hence, \( y_i \geq y_0 = 0 \) in \( \mathbb{R}^n \) by the maximum principle. If \( y_i \geq y_{i-1} \) for \( i \geq 1 \), then \( (-\Delta)^\sigma (y_{i+1} - y_i) \geq 0 \) in \( B_2 \setminus \overline{B}_1 \) and \( y_{i+1} - y_i = 0 \) on \( B_2^c \cup \overline{B}_1 \), which implies that \( y_{i+1} \geq y_i \) in \( \mathbb{R}^n \). By induction, \( \{y_i\}_{i=1}^\infty \) is a non-decreasing sequence of functions in \( \mathbb{R}^n \). A similar application of the maximum principle for \( y_i \) and \( \tau \) yields that \( y_i \leq \tau \) in \( B_2 \setminus \overline{B}_1 \) for all \( i \geq 1 \). Consequently, \( y_i \) converges pointwise to a limit \( v \) in \( \mathbb{R}^n \).

Moreover, for every \( i \geq 1 \), by (4.43),
\[
0 \leq H(x, y_i) \leq H(x, \tau) \leq H(x, c_{n,\sigma}) < \infty \quad \text{in } B_2 \setminus \overline{B}_1.
\]

By [24, Proposition 1.1], we have \( y_i \in C^\sigma(\mathbb{R}^n) \) and
\[
\|y_i\|_{C^\sigma(\mathbb{R}^n)} \leq C \sup_{x \in \mathbb{R}^n \setminus \overline{B}_1} H(x, c_{n,\sigma}) < \infty \quad \text{for all } i \geq 1,
\]
where \( C \) is a constant depending only on \( n \) and \( \sigma \). Since \( H \in C^1(\overline{B}_2 \setminus B_1 \times [0, c_{n,\sigma}]) \), by (4.52) and [24, Proposition 1.4], \( y_i \in C^{2\sigma+\gamma}(B_2 \setminus \overline{B}_1) \) for any \( \gamma \in (0, 1) \). Moreover, for any compact set \( K \subset B_2 \setminus \overline{B}_1 \), there exists a \( C = C(K) \) such that \( \|y_i\|_{C^{2\sigma+\gamma}(K)} \leq C \) for all \( i \geq 1 \). Thus,
\[
y_i \to v \quad \text{in } C^{2\sigma+\gamma}_{\text{loc}}(B_2 \setminus \overline{B}_1).
\]

It implies that \( v \) satisfies (4.51). Recall that \( v \leq \tau \) in \( B_2 \setminus \overline{B}_1 \), and by [24, Corollary 1.6], \( v \in C^\sigma(\mathbb{R}^n) \cap C^{2\sigma+\gamma}(B_2 \setminus \overline{B}_1) \) for every \( \gamma \in (0, 1) \).

Now we can construct the desired function \( u_0 \).

Lemma 4.5. There exists a \( C^{2\sigma+\gamma}(\mathbb{R}^n \setminus \{0\}) \) solution \( u_0 \) to
\[
\begin{cases}
(-\Delta)^\sigma u_0 = H(x, u_0) & \text{in } \mathbb{R}^n \setminus \{0\}, \\
0 \leq u_0 \leq \tau \leq w & \text{in } \mathbb{R}^n \setminus \{0\}.
\end{cases}
\]

Proof. For each positive integer \( i \geq 2 \), we consider the following problem:
\[
\begin{cases}
(-\Delta)^\sigma v = H(x, v) & \text{in } B_i \setminus \overline{B}_{1/i}, \\
v = 0 & \text{on } B_i^c \cup \overline{B}_{1/i}.
\end{cases}
\]

Using the same argument as in Lemma 4.4, we see that (4.54) has a non-negative solution \( v_i \) such that \( 0 \leq v_i \leq \tau \) in \( \mathbb{R}^n \) and \( v_i \in C^\sigma(\mathbb{R}^n) \cap C^{2\sigma+\gamma}_{\text{loc}}(B_1 \setminus \overline{B}_{1/i}) \). Moreover, \( \{v_i\}_{i=2}^\infty \) has a uniform upper bound \( w \). It follows from [24, Proposition 1.4] and the standard diagonal argument that after passing to a subsequence,
\[
v_i \to u_0 \quad \text{in } C^{2\sigma+\gamma}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})
\]
Thus, \( \sigma = 1 \)

Step 5. Define the functions \( u \) and \( K \).

Define \( \bar{H} : \mathbb{R}^n \times [0, \infty) \to (0, \infty) \) by \( \bar{H}(x, v) = F(\bar{u}(x), k(x), p(x, v)) \). Then \( \bar{H} \leq H \leq \overline{H} \) since \( \kappa \leq k \). In particular,

\[
\bar{H}(x, u_0(x)) \leq H(x, u_0(x)) \leq \overline{H}(x, u_0(x)) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.
\]

(4.57)

Similar to the arguments in [12, Step 4], we see that for \( x \in \mathbb{R}^n \) and \( v \geq 0 \),

\[
\overline{H}(x, v) = f(\bar{u}(x), k(x), p(x, v)) = k(x) \left( v + \sum_{i=1}^{\infty} u_i(x) \right)^{n+2\sigma/(n-2\sigma)} - \sum_{i=1}^{\infty} u_i(x)^{n+2\sigma/(n-2\sigma)}.
\]
Moreover, define
\[ u := u_0 + \sum_{i=1}^{\infty} u_i. \]  
(4.58)

Then \( u \in L_\sigma(\mathbb{R}^n) \cap C^{2\sigma+\gamma}(\mathbb{R}^n \setminus \{0\}) \) is a positive solution of
\[ \kappa(x)u^{\frac{n+2\sigma}{n}} \leq (-\Delta)^\sigma u \leq k(x)u^{\frac{n+2\sigma}{n}} \text{ in } \mathbb{R}^n \setminus \{0\}. \]  
(4.59)

It follows from (4.31), (4.33) and (4.53) that \( u \) satisfies (1.10) and (1.11).

Define \( K : \mathbb{R}^n \to (0, \infty) \) by
\[ K(x) := \frac{(-\Delta)^\sigma u(x)}{u(x)^{\frac{n+2\sigma}{n}}} \text{ for } x \in \mathbb{R}^n \setminus \{0\} \]  
(4.60)

and \( K(0) = 1 \). Then
\[ K(x) = \frac{H(x, u_0(x)) + \sum_{i=1}^{\infty} u_i(x)^{\frac{n+2\sigma}{n}}}{u_0(x) + \sum_{i=1}^{\infty} u_i(x)^{\frac{n+2\sigma}{n}}} \text{ for } x \in \mathbb{R}^n \setminus \{0\}, \]  
(4.61)

and hence \( K \in C^1(\mathbb{R}^n \setminus \{0\}) \). It follows from (4.59) and (4.60) that
\[ \kappa(x) \leq K(x) \leq k(x) \text{ for } x \in \mathbb{R}^n \setminus \{0\}. \]  
(4.62)

Recalling that \( \kappa, k \in C^1(\mathbb{R}^n) \) and \( \kappa(0) = K(0) = k(0) = 1 \), we get \( K \in C(\mathbb{R}^n) \),
\[ \nabla \kappa(0) = \nabla K(0) = \nabla k(0) = 0 \]  
(4.63)

and
\[ \kappa(x) = K(x) = k(x) \text{ for } |x| \geq 2\delta_1. \]  
(4.64)

**Step 6.** Show that \( K \in C^1(\mathbb{R}^n) \).

We only need to show that
\[ \lim_{|x| \to 0^+} \nabla K(x) = 0. \]  
(4.65)

Let \( S = \{ x \in \mathbb{R}^n \setminus \{0\} : H(x, u_0(x)) < H(x, u_0(x)) \} \).

It follows from (4.45) and (4.61) that
\[ S = \{ x \in \mathbb{R}^n \setminus \{0\} : \kappa(x) < K(x) \}. \]  
(4.66)

By (4.62), (4.63) and (4.66), we obtain
\[ \nabla \kappa(x) = \nabla K(x) \text{ for } x \in \mathbb{R}^n \setminus S, \]  
(4.67)

and thus (4.65) holds for \( x \in (\mathbb{R}^n \setminus \{0\}) - S \). Next, we show that (4.65) holds for \( x \in S \). It follows from (4.45) and (4.46) that
\[ \begin{cases} H(x, u_0(x)) = M(\kappa(x), p_0(x)) \\ \tilde{u}(x) > Z(\kappa(x), p_0(x)) \end{cases} \text{ for } x \in S, \]  
(4.68)

where \( p_0(x) := p(x, u_0(x)) \). Since \( \kappa \geq k_j \) in \( B_{2p_j}(x_j) \), by (4.12) we have
\[ \tilde{u}(x) > Z(k_j, p_0(x)) = M_j^{\frac{n-2\sigma}{n}} p_0(x) \text{ for } x \in S \cap B_{2p_j}(x_j), \quad j \geq 1. \]  
(4.69)

For \( x' \in (\mathbb{R}^n \setminus \{0\}) - \bigcup_{i=1}^{\infty} B_{2p_i}(x_i) \), \( \kappa(x') = k(x) \). Hence, by (4.62) and (4.66) we know that \( x' \not\in S \). Consequently,
\[ S \subset \bigcup_{i=1}^{\infty} B_{2p_i}(x_i). \]  
(4.70)
It follows from the same proofs as in [12, (86) and (87)] that
\[ S \cap B_{2\rho_j}(x_j) = S \cap B_{\rho_j}(x_j) \quad \text{for } j \geq 1 \]  
(4.71)
and
\[ u_j \geq CM_j^{(1-\beta)(n-2\sigma)} \quad \text{in } S \cap B_{2\rho_j}(x_j), \quad j \geq 1. \]  
(4.72)

Recalling (4.61) and (4.68), we have
\[ K(x) = \frac{M_j \rho_0(x)^{\frac{n+\sigma}{n-2\sigma}} + \tilde{u}(x)^{\frac{n+\sigma}{n-2\sigma}}}{(\rho_0(x) + \tilde{u}(x))^{\frac{n+\sigma}{n-2\sigma}}} = \frac{M_j (\frac{\rho_0(x)}{\tilde{u}(x)})^{\frac{n+\sigma}{n-2\sigma}} + 1}{(\frac{\rho_0(x)}{\tilde{u}(x)} + 1)^{\frac{n+\sigma}{n-2\sigma}}} \quad \text{for } x \in S \cap B_{\rho_j}(x_j), \quad j \geq 1. \]

Thus,
\[ \nabla K = \frac{n + 2\sigma}{n - 2\sigma} \left( \frac{M_j (\frac{\rho_0(x)}{\tilde{u}(x)})^{\frac{n+\sigma}{n-2\sigma}} - 1}{(\frac{\rho_0(x)}{\tilde{u}(x)} + 1)^{\frac{n+\sigma}{n-2\sigma}}} \right) \left( \nabla \frac{\rho_0}{\tilde{u}}(x) \right) \quad \text{in } S \cap B_{\rho_j}(x_j), \quad j \geq 1, \]
and hence by (4.69),
\[ |\nabla K| \leq \frac{n + 2\sigma}{n - 2\sigma} \left| \nabla \frac{\rho_0}{\tilde{u}} \right| \]
\[ \leq \frac{n + 2\sigma}{n - 2\sigma} \left[ \left| \nabla \rho_0 \right| + \left| \nabla \sum_{i=1, i \neq j}^{\infty} \frac{u_i}{\tilde{u}} \right| + \left| \nabla \frac{u_j}{\tilde{u}} \right| \right] \quad \text{in } S \cap B_{\rho_j}(x_j), \quad j \geq 1. \]  
(4.73)

Using the same algebraic calculations as in [12, Step 5], we obtain that for \( j \geq 1 \) and \( x \in S \cap B_{\rho_j}(x_j) \),
\[ |\nabla K| \leq C \left( \frac{|\nabla \rho_0|}{M_j^{\frac{n+\sigma}{4\sigma-2\sigma}}} + \frac{|\nabla u_j|}{u_j^{\frac{n+\sigma}{n-2\sigma}}} + \frac{M_j^{\frac{2\sigma}{n+\sigma}}}{M_j^{\frac{n+\sigma}{4\sigma-2\sigma}}} \right). \]  
(4.74)

We now estimate the first term in (4.74). By Lemma 4.6 and [25, Proposition 2.22], there exists a function \( h \in C(B_{\rho_j}) \cap L_\sigma(\mathbb{R}^n) \) satisfying \((-\Delta)^\sigma h = 0 \) in \( B_2 \) such that
\[ u_0(x) = r_{n,\sigma} \int_{B_1} H(y, u_0(y)) \frac{1}{|x-y|^{n-2\sigma}} dy + h(x) \quad \text{for } 0 < |x| \leq 2. \]

By (4.48), (4.49) and (4.53),
\[ H(x, u_0(x)) \leq \begin{cases} 
(2w(0))^{\frac{n+2\sigma}{n-2\sigma}} M_j & \text{in } B_{\rho_j}(x_j), \quad j \geq 1, \\
(2w(0))^{\frac{n+2\sigma}{n-2\sigma}} b & \text{in } (\mathbb{R}^n \setminus \{0\}) - \bigcup_{i=1}^{\infty} B_{\rho_i}(x_i). 
\end{cases} \]

By the regularity of \( \sigma \)-harmonic functions (see, e.g., [25]), we know that \( |\nabla h(x)| < C \) for \( |x| \leq 1 \). Hence, for \( x \in B_{\rho_j}(x_j) \),
\[ |\nabla u_0(x)| \leq C \int_{B_1} \frac{H(y, u_0(y))}{|x-y|^{n-2\sigma+1}} dy + C \leq C[I_1(x) + I_2(x) + I_3(x)] + C, \]
where
\[ I_1(x) := \int_{B_{\rho_j}(x_j)} \frac{M_j}{|x-y|^{n-2\sigma+1}} dy \leq CM_j \rho_j^{2\sigma-1} \leq CM_j^{\frac{1}{2\sigma}} \quad \text{for } x \in B_{\rho_j}(x_j) \]
by Lemma 4.2, and
\[ I_2(x) := \sum_{i=1, i \neq j}^{\infty} \int_{B_{\rho_i}(x)} \frac{M_i}{|x-y|^{n-2\sigma+1}} dy \leq C \sum_{i=1, i \neq j}^{\infty} \frac{M_i \rho_i^n}{\text{dist}(B_{\rho_i}(x_i), B_{\rho_j}(x_j))^{n-2\sigma+1}} \]
\[ \leq C \sum_{i=1, i \neq j}^{\infty} \frac{\rho_i^{n-2\sigma}}{|x-y|^{n-2\sigma+1}} \leq \frac{C}{\rho_j} \sim C2^{\frac{2\sigma}{n-2\sigma}} \rho_j^{\frac{2\sigma}{n-2\sigma}} \leq CM_j^{\frac{1}{2\sigma}} \quad \text{for } x \in B_{\rho_j}(x_j) \]
by Lemma 4.2, (4.22) and (4.21), and
\[
I_3(x) := \int_{B_4 - \bigcup_{i=1}^{n-1} B_{r_i}(x_i)} \frac{1}{|x - y|^{n-2\sigma+4}} dy \leq C \quad \text{for } x \in B_{r_j}(x_j).
\]
Thus,
\[
|\nabla u_0| \leq CM_j^{\frac{1+\beta}{\alpha}} \quad \text{in } B_{r_j}(x_j), \quad j \geq 1.
\]
(4.75)

Since \( n > 2\sigma + 3 \), it follows from (4.75) that
\[
|\nabla u_0| \leq \frac{CM_j^{\frac{1+\beta}{\alpha}}}{M_j^{(1-\beta)(n-2\sigma)}} \leq \frac{C}{M_j^{\frac{1}{\alpha}}}. \quad (4.76)
\]

Finally, we estimate the second term in (4.74). Let
\[
s_j := \inf \{ s > 0 : S \cap B_{r_j}(x_j) \subset B_s(x_j) \}
\]
and \( \tilde{u}_j(s) := \psi(s, \lambda_j) \). Then \( s_j \leq r_j \) and \( \tilde{u}_j(s) = u_j(x) \) when \( |x - x_j| = s \). By (4.72), we have
\[
\tilde{u}_j(s) \geq CM_j^{\frac{(1-\beta)(n-2\sigma)}{\alpha}} \quad \text{for } 0 \leq s \leq s_j, \quad j \geq 1.
\]
It follows from Lemma 4.2 that
\[
\left( \frac{\lambda_j}{\lambda_j^2 + s^2} \right)^{2\sigma} \geq CM_j^{1-\beta} \geq C \left( \frac{\varepsilon^{2\sigma/3}}{2\lambda_j^\alpha} \right)^{1-\beta},
\]
and hence by (4.21), we see that for \( j \geq 1 \),
\[
s_j \leq C \left( \frac{\varepsilon^{2\sigma/3}}{\lambda_j^\alpha} \right)^{1-\beta} \leq C(\lambda_j^{-\beta})^{1-\beta} \lambda_j^{\frac{2\sigma}{\alpha}} \leq C(\lambda_j^{-\beta})^{1-\beta} \lambda_j^{\frac{2\sigma}{\alpha}} \leq C\lambda_j^{\frac{2\alpha(\sigma+1)}{3\sigma}}.
\]

Since \( n > 2\sigma + 3 \), we obtain for \( 0 \leq s \leq s_j \) and \( j \geq 1 \) that
\[
\frac{-\tilde{u}_j'(s)}{\tilde{u}_j(s)^2} \leq C\lambda_j^{\frac{(n-2\sigma-3)\alpha}{3\sigma}} \quad (4.77)
\]
Since \( n > 2\sigma + 3 \), we can take
\[
\beta = \frac{(n-2\sigma-3)\sigma}{3(\sigma+1)(n-2\sigma-1)} > 0.
\]
By (4.74)–(4.77) we get
\[
|\nabla K| \leq C \left( \frac{1}{M_j^{(1+\beta)(n-2\sigma-3)(n-2\sigma-1)}} + \lambda_j^{\frac{2\alpha(\sigma+1)}{3\sigma}} \right) \quad \text{in } S \cap B_{r_j}(x_j), \quad j \geq 1.
\]
(4.78)

Hence, it follows from Lemma 4.2 and (4.21), (4.70) and (4.71) that (4.65) also holds for \( x \in S \). Thus we have \( K \in C^1(\mathbb{R}^n) \).

By sufficiently increasing \( k_i \) for each \( i \geq 1 \), we can force \( \kappa \) to satisfy
\[
\|k - \kappa\|_{C^1(\mathbb{R}^n)} < \frac{\varepsilon}{4} \quad (4.79)
\]
by (4.38), (4.39), (4.41), (4.61) and (4.64). We can also force \( K \) to satisfy
\[
|\nabla(K - \kappa)| = |\nabla(K - (\kappa - k))| \leq |\nabla K| + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2} \quad \text{in } S
\]
by (4.70), (4.71) and (4.78). Thus by (4.67), $|\nabla(K - \kappa)| \leq \varepsilon/2$ in $\mathbb{R}^n$. It follows from (4.62) and (4.79) that $K$ satisfies (1.9).

**Step 7.** Extend the solution $u$ to the half space.

By Step 5, the positive function $u$ in (4.58) belongs to $L_{\sigma}(\mathbb{R}^n) \cap C^{2\sigma+\gamma}(\mathbb{R}^n \setminus \{0\})$ and satisfies

$$(-\Delta)\sigma u = K(x)u^{\frac{n+2\sigma}{n-\sigma}} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\},$$

where $K(x)$ is defined in (4.60) which is in $C^1(\mathbb{R}^n)$ according to Step 6. Moreover, it follows from (4.31), (4.33) and (4.53) that $u$ satisfies (1.10) and (1.11).

Set

$$U(y, t) := P_{\sigma}[u](y, t)$$

$$= \gamma_{n, \sigma} \int_{\mathbb{R}^n} \frac{t^{2\sigma}}{|y - x|^2 + t^2} u(x) dx = \gamma_{n, \sigma} \int_{\mathbb{R}^n} \left(\frac{1}{1 + |z|^2}\right) \frac{-2\sigma}{n+2\sigma} u(y - tz) dz.$$

Then $U(y, t)$ is well defined for any $(y, t) \in \mathbb{R}^{n+1}_+ \setminus \{0\}$ and $U \in C^2(\mathbb{R}^{n+1}_+)$. For any $R > r > 0$, let $\tau$ be a $C^\infty$ cut-off function on $\mathbb{R}^n$ such that $\tau \subseteq B_{r/2}$ and $\tau = 1$ in $B_{r/4}$. Rewrite $U = P_{\sigma}[(1 - \tau)u] + P_{\sigma}[\tau u]$. Since both $P_{\sigma}[(1 - \tau)u]$ and $P_{\sigma}[\tau u]$ belong to $W^{1,2}(l^{1-2\sigma}, B_R^+ \setminus B_r^+)$, we have $U \in W^{1,2}(l^{1-2\sigma}, B_R^+ \setminus B_r^+)$. By the extension formulation in [5], we see that for any $\Psi \in C^\infty(\mathbb{R}^{n+1}_+ \setminus \{0\})$,

$$\int_{\mathbb{R}^{n+1}_+} t^{1-2\sigma} \nabla U \nabla \Psi d y = \int_{\supp \Psi} t^{1-2\sigma} \nabla U \nabla \Psi d y = \int_{\supp \Psi \cap \mathbb{R}^n} \frac{\partial U}{\partial \nu^\sigma} d y$$

$$= \int_{\supp \Psi \cap \mathbb{R}^n} \Psi K U^{\frac{n+2\sigma}{n-2\sigma}} d y = \int_{\mathbb{R}^n} \Psi K U^{\frac{n+2\sigma}{n-2\sigma}} d y.$$

It follows that $U$ is a positive weak solution of (1.8). The proof of Theorem 1.4 is completed. $\square$

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