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Spherically symmetric classical model of an elementary particle or a black hole spacetime without central singularity had been constructed by O. B. Zaslavskii in PRD 70(2004)104017. In this model an extremal Reissner-Nordström (RN) black hole and a Bertotti-Robinson (BR) spacetime are glued at the horizon such that the inner/core spacetime is the regular BR while outside is the extremal RN. In this note we investigate the stability of such a particle / regular black hole against linear radial perturbations. The model turns out to be stable against such perturbations with a linear equation of state after the perturbation.

I. INTRODUCTION

With the advent of general relativity, the geometric theory of gravitation due to Einstein, attempts to establish geometric model of an elementary particle took start. Upon discovering a model spacetime in which the radius of spacetime had a minimum encouraged Einstein and Rosen to propose such a model of elementary particle [1]. Later on the model turned out to be interpreted as an Einstein-Rosen bridge, and in present day’s terminology as a wormhole connecting two asymptotically flat spacetimes. Initially what Einstein and Rosen called the radius of the elementary particle coincides with the throat of the wormhole [2, 3]. The two spacetimes that are connected at the throat may host black holes which are thought recently to be entangled from the view of quantum interpretation [4]. Existence of singularities at the center of black holes has been a serious obstacle in the construction of a self-consistent geometric model of a particle. From physical grounds we observe / detect no singularity at the location of a particle such as an electron. This automatically eliminates, the models of black holes as viable particle models. Next, we must have a smooth matching of the inner and outer geometries at the surface of the so called particle model with well-defined energy scale that doesn’t contradict experiments. To overcome these requirements certain boundary conditions are imposed which came to be known as the Israel’s junction conditions [5]. Such a construction was considered by Vilenkin and Fomin in [6, 7] and recently by Zaslavskii in [8] where he attempted to match different spacetimes to fit the physical requirements. Naturally the choice of the regular Minkowski space as the inner core region to external spacetimes such as Schwarzschild or Reissner-Nordström (RN) deserved a special attention. From physical grounds we observe / detect no singularity at the location of a particle such as an electron. This automatically eliminates, the models of black holes as viable particle models. Next, we must have a smooth matching of the inner and outer geometries at the surface of the so called particle model with well-defined energy scale that doesn’t contradict experiments. To overcome these requirements certain boundary conditions are imposed which came to be known as the Israel’s junction conditions [5]. Such a construction was considered by Vilenkin and Fomin in [6, 7] and recently by Zaslavskii in [8] where he attempted to match different spacetimes to fit the physical requirements. Naturally the choice of the regular Minkowski space as the inner core region to external spacetimes such as Schwarzschild or Reissner-Nordström (RN) deserved a special attention. From physical grounds we observe / detect no singularity at the location of a particle such as an electron. This automatically eliminates, the models of black holes as viable particle models. Next, we must have a smooth matching of the inner and outer geometries at the surface of the so called particle model with well-defined energy scale that doesn’t contradict experiments. To overcome these requirements certain boundary conditions are imposed which came to be known as the Israel’s junction conditions [5].

II. STABILITY OF THE ZASLAVSKII’S PARTICLE MODEL

The geometrical particle model introduced by Zaslavskii in [8] consists of the core and outer spacetimes given by

\[ ds^2_{core} = -e^{2r/r_0} dt^2 + dr^2 + r_0^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]  (1)
and

\[ ds^2_{\text{outer}} = - \left(1 - \frac{r_h}{r} \right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{r_h}{r} \right)^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \] (2)

respectively. These two regions are glued at \( r = r_0 \) on a timelike hypersurface defined by

\[ F := r - r_0(\tau) = 0 \] (3)

where \( \tau \) stands for the proper time, as described below. Straightforward calculation gives the induced metric on the shell due to \( ds^2_{\text{core}} \) as

\[ ds^2_F = -e^2 dt^2 + r_0^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (4)

and \( ds^2_{\text{outer}} \)

\[ ds^2_F = - \left(1 - \frac{r_h}{r_0} \right)^2 dt^2 + r_0^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (5)

respectively. These clearly match provided the angular coordinates of both spacetimes are identified on the shell and the coordinate time satisfies the relations

\[-e^2 dt^2_{\text{core}} + dr^2 = - \left(1 - \frac{r_h}{r_0} \right)^2 dt^2_{\text{outer}} + dr^2 = -dr^2 \] (6)

with proper time \( \tau \) on the shell. Having (6), the first fundamental form of the shell is continuous across the shell. To see the situation of the second fundamental form we start from the definition of the extrinsic curvature

\[ K_{ij} = -n_\mu \left( \frac{\partial^2 x^\mu}{\partial \xi^i \partial \xi^j} + \Gamma^\mu_{\alpha \beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right) \] (7)

in which

\[ n_\mu = \frac{1}{\sqrt{g^{\alpha \beta} \frac{\partial F}{\partial \xi^\alpha} \frac{\partial F}{\partial \xi^\beta}}} \] (8)

is the normal 4-vector which points from the shell outward for the outer-side of the shell and inward for the inner-side of the shell. Explicit calculation yields

\[ K^2_{\text{core}} = \text{diag} \left( \frac{1}{r_0^2} + r_0 \frac{\ddot{r}_0}{r_0 \sqrt{1 + \frac{\dot{r}_0^2}{r_0^2}}}, 0, 0 \right) \] (9)

while

\[ K^2_{\text{outer}} = \text{diag} \left( \frac{1 - \frac{r_h}{r_0}}{r_0^2}, \frac{r_h - r_0}{r_0^2}, \frac{\sqrt{\left(1 - \frac{r_h}{r_0} \right)^2 + \frac{\dot{r}_0^2}{r_0^2}}}{r_0}, 0 \right) \] (10)

Applying the Israel junction condition

\[-8\pi G S^i_i = \left[ K^2_i \right] - \left[ K \right] \delta^i_i \] (11)

in which the bracket implies \([X] = X_{\text{outer}} - X_{\text{core}}, K = tr K^2_i \) and \( S^i_i = \text{diag} (-\sigma, p, p) \) we obtain \((8\pi G = 1)\)

\[ \sigma = \frac{2 \sqrt{\left(1 - \frac{r_h}{r_0} \right)^2 + \frac{\dot{r}_0^2}{r_0^2}}}{r_0} \] (12)
and
\[ p = \left( \frac{1 - \frac{r}{r_0}}{\frac{r}{r_0}} \right)^2 + \frac{\dot{r}^2}{r_0^2} + \frac{\dot{r}}{r_0} + \frac{\ddot{r}}{r_0} \frac{\sqrt{\left(1 - \frac{r}{r_0}\right)^2 + \frac{\dot{r}^2}{r_0^2}}}{r_0} - \frac{1 + \dot{r}^2 + r_0\dot{r}}{r_0\sqrt{1 + \dot{r}^2}}. \] (13)

Let’s note that at the static equilibrium condition where \( r_0 = \text{const.} \geq r_h \) we get
\[ p_0 = 0 \] (14)
and
\[ \sigma_0 = -\frac{2\left(1 - \frac{r}{r_0}\right)}{r_0} \] (15)
which vanishes at \( r_0 = r_h \).

Hence we assume that the system is at equilibrium at \( r_0 = r_h \). Any radial perturbation causes \( r_0 \neq r_h \) and therefore \( p \) and \( \sigma \) are obtained by (12) and (13) which means that on the perturbed shell there would be a perfect fluid presented. Considering a linear equation of state of the form \( p = \omega \sigma \) (\( \omega = \text{const.} \)) one finds an equation of motion given by
\[ \left(1 - \frac{r}{r_0}\right)^2 + \frac{\dot{r}^2}{r_0^2} + \frac{\dot{r}}{r_0} + \frac{\ddot{r}}{r_0} \frac{\sqrt{\left(1 - \frac{r}{r_0}\right)^2 + \frac{\dot{r}^2}{r_0^2}}}{r_0} - \frac{1 + \dot{r}^2 + r_0\dot{r}}{r_0\sqrt{1 + \dot{r}^2}} = 0 \] (16)
which is a differential equation giving the behaviour of \( r_0 \) after the perturbation. This equation is not solvable analytically but one may pursue a numerical solution provided the initial conditions are known. As we consider a radial perturbation about \( r_0 = r_h \) we set the following to be the initial conditions: at \( t = 0 \), \( r_0 = r_h = 1 \) and \( \dot{r}_0 \neq 0 \). In Figs. 1 and 2 we plot the solution of Eq. (16) for \( t_0 = 0.1 \) and \(-0.2 \), respectively, and \( \omega = 0.1, 0.2, 0.4, 0.6, 0.8, 1.0 \) and 2.0. As we observe here; for a positive \( \omega \) and a small perturbation the particle remains stable. In both cases, the shell spends more time with \( r_0 > r_h \) although in Fig. 2 initially the shell is perturbed toward the center. In such a particle model, however, both the energy density \( \sigma \) and angular pressure \( p \) are negative. They both vanish at the equilibrium radius while perturbation drives the system to attain \( \sigma < 0 \) and \( p < 0 \). Assuming that the parameter \( \omega < 0 \), yields a model with \( \sigma < 0 \) and \( p > 0 \) which results unfortunately in an unstable particle model. At any cost Eq. (12) implies that the model will always have \( \sigma < 0 \).

From (16) at \( r_0 = r_h = 1 \) one finds the acceleration
\[ a = \left(1 - \frac{\omega |v|}{\sqrt{1 + v^2} - |v|}\right)|v|\sqrt{1 + v^2} \] (17)
in which \( v = \dot{r}_0 \) and \( a = \ddot{r}_0 \) both at \( r_0 = r_h = 1 \). The direction of acceleration \( a \) depends not only on the magnitude of \( v \) but also on the value of \( \omega \). Hence for the case that directions of \( v \) and \( a \) are both negative the shell is more likely to become unstable. For the negative \( \omega \), with positive initial velocity the shell is also unstable and that is why we chose \( \omega > 0 \) in Figs. 1 and 2.

### III. A GENERAL OVERVIEW OF THE PARTICLE MODEL

[FIG. 3: Metric function \( f(r) \) vs \( r \) in accordance with Eq. (30) and (31) for \( \ell^2 = 1 \) and \( R_0 = 0.5, 1.0 \) and 1.5.]

To complete our study we consider the static spherically symmetric spacetimes in general form. The line element inside and outside the shell of the particle are given by
\[ ds^2 = -f_i(r_i) \, dt_i^2 + \frac{dr_i^2}{f_i(r_i)} + r_i^2 \left( d\theta_i^2 + \sin^2 \theta_i \, d\phi_i^2 \right) \] (18)
with \( i = 1, 2 \) indicating the inside and outside, respectively. The energy momentum tensor components on the shell are found to be from \( S_{\mu} = \text{diag}(-\sigma, p, p) \)
\[ \sigma = -\frac{1}{4\pi G} \left( \frac{\sqrt{f_2(R) + \dot{R}^2} - \sqrt{f_1(R) + \dot{R}^2}}{R(\tau)} \right) \] (19)
and
\[
\rho = \frac{1}{8\pi G} \left( \frac{2\dot{R}(\tau) + f'_2(R)}{2\sqrt{f_2(R) + \dot{R}^2}} - \frac{2\dot{R}(\tau) + f'_1(R)}{2\sqrt{f_1(R) + \dot{R}^2}} + \sqrt{f_2(R) + \dot{R}^2 - \sqrt{f_1(R) + \dot{R}^2}} \right) \tau
\]

in which \( r = R(\tau) \) represents the surface of the particle. In static condition one finds
\[
\sigma_0 = -\frac{1}{4\pi G} \left( \frac{\sqrt{f_2(R_0) - \sqrt{f_1(R_0)}}}{R_0} \right),
\]

and
\[
p_0 = \frac{1}{8\pi G} \left( \frac{f'_2(R_0) - f'_1(R_0)}{2\sqrt{f_2(R_0) + f'_1(R_0) + \sqrt{f_2(R_0) - \sqrt{f_1(R_0)}}}} \right).
\]

Considering \( \sigma_0 = 0 \) imposes
\[
f_2(R_0) = f_1(R_0)
\]
while \( p_0 = 0 \) yields
\[
f'_2(R_0) = f'_1(R_0).
\]

Having these conditions satisfied, the particle model becomes physical. In [10] we have shown that the outer RN black hole can be glued to the inner de-Sitter spacetime consistently and due to that we proposed to remove the singularity of the RN black hole.

For instance, if we set \( f_1(r) = 1 - \frac{r^2}{\ell^2} \) and \( f_2(r) = \left(1 - \frac{e}{r} \right)^2 \) the latter equations become
\[
1 - \frac{R_0^2}{\ell^2} = \left(1 - \frac{e}{R_0} \right)^2
\]
and
\[
-\frac{R_0}{\ell^2} = \frac{e}{R_0} \left(1 - \frac{e}{R_0} \right)
\]
respectively. Simultaneous solution of these equations reveals that
\[
R_0 = \frac{2}{3} e
\]
which is smaller than the horizon radius i.e., \( r_h = e \). In addition to that one obtains the geometrical condition
\[
\frac{R_0^2}{\ell^2} = \frac{3}{4}
\]

One simple solution for these equations can be found by considering Taylor expansion of the outer metric about \( R_0 \) i.e.,
\[
f_2(r) \approx f_2(R_0) + f'(R_0) (r - R_0).
\]

In this case we set
\[
f_1(r) = 1 + \frac{r^2}{\ell^2}
\]
and due to (24) and (25) we find \( f_2(R_0) = f_1(R_0) = 1 + \frac{R_0^2}{\ell^2} \) and \( f'_2(R_0) = f'_1(R_0) = \frac{2R_0}{\ell^2} \) so that to the first order we find
\[
f_2(r) = 1 - \frac{R_0^2}{\ell^2} + \frac{2R_0}{\ell^2} r.
\]

The spacetime expressed by the latter metric is singular [12] at \( r = 0 \). This singularity may be naked with \( \ell^2 \geq R_0^2 \) or hidden behind a horizon for \( \ell^2 < R_0^2 \) located at
\[
r_h = \frac{R_0^2 - \ell^2}{2R_0}.
\]
As one expects for the particle model \( R_0 \geq r_h \) and one finds
\[
R_0^2 \geq -\ell^2
\]
which is trivially satisfied. Let’s add that the energy momentum tensor of the spacetime outside the shell is found to be of the perfect fluid form given by
\[
T^\nu_\mu = \text{diag} (-\rho, P_r, P_\theta, P_\phi)
\]
in which
\[
\rho = -\frac{4R_0}{r\ell^2} + \frac{R_0^2}{r^2\ell^2},
\]
\[
P_r = -\rho
\]
and
\[
P_\theta = P_\phi = \frac{2R_0}{r\ell^2}.
\]
In Fig. 3 we plot \( f(r) \) versus \( r \) for \( \ell^2 = 1 \) and \( R_0 = 0.5, 1.0 \) and 1.5 from bottom to the top. The smooth matching is clearly seen from the plots.

**IV. CONCLUSION**

We revisit the geometrical model of an elementary particle [8] with the purpose to investigate its stability and search for alternative models. The suggested model in [8] glued spacetimes BR (inside) with the extremal RN (outside). The radius of the elementary particle coincides with the horizon of the extremal RN. At the static equilibrium the shell has no surface energy-momentum, i.e.,
Upon radial perturbation the shell absorbs energy and behaves as a perfect fluid satisfying $p = \omega \sigma$. When the shell is perturbed a master equation (i.e., Eq. (16) for the general spherical symmetry configuration, given in Sec. III), governing the perturbations is derived and plotted for different initial conditions. The oscillatory behaviours in Figs. 1 and 2 support the stability of the shell and therefore the particle is stable against such perturbations. An inward motion is observed to reverse immediately outward following a small penetration of the shell below the horizon. The energy density attained by the shell in this process is negative. The particle model constructed from BR and RN can be modified by invoking other regular metrics to replace the BR. One immediate choice is the de Sitter metric to be matched with the outer RN metric. In this venture naturally the mass and charge of the RN metric are defined entirely from the geometrical parameters.

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