Quantized Frank-Wolfe: Communication-Efficient Distributed Optimization

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Abstract

How can we efficiently mitigate the overhead of gradient communications in distributed optimization? This problem is at the heart of training scalable machine learning models and has been mainly studied in the unconstrained setting. In this paper, we propose Quantized Frank-Wolfe (QFW), the first projection-free and communication-efficient algorithm for solving constrained optimization problems at scale. We consider both convex and non-convex objective functions, expressed as a finite-sum or more generally a stochastic optimization problem, and provide strong theoretical guarantees on the convergence rate of QFW. This is done by proposing quantization schemes that efficiently compress gradients while controlling the variance introduced during this process. Finally, we empirically validate the efficiency of QFW in terms of communication and the quality of returned solution against natural baselines.

1. Introduction

Frank-Wolfe (FW) algorithm (Frank and Wolfe, 1956), also known as conditional gradient or projection-free method, has recently received a lot of attention in the machine learning
As it does not need any projections while maintaining feasibility through its execution, FW is very efficient for various constrained convex (Jaggi, 2013; Garber and Hazan, 2014; Lacoste-Julien and Jaggi, 2015; Garber and Hazan, 2015; Hazan and Luo, 2016; Mokhtari et al., 2018b) and non-convex (Lacoste-Julien, 2016; Reddi et al., 2016) optimization problems. It is known that in many scenarios, projection operations are computationally prohibitive (e.g., projections onto a nuclear norm ball or onto a matroid polytope). To avoid this cost, FW replaces the projection step with solving a linear program.

In order to apply the FW algorithm to large-scale optimization and machine learning problems (e.g., training deep neural networks, SVMs, AdaBoost, experimental design, etc) parallelization is unavoidable. To this end, distributed FW variants have been proposed for specific problems, e.g., learning low-rank matrix (Zheng et al., 2018), and optimization under block-separable constraint set (Wang et al., 2016).

A significant performance bottleneck of distributed optimization methods is the cost of communicating gradients, typically handled by using a parameter-server framework. Intuitively, if each worker/processor in the distributed system transmits the entire gradient, then at least \(d\) floating-point numbers are communicated for each worker, where \(d\) is the dimension of the optimization problem. This communication cost can be a huge burden on the performance of parallel optimization algorithms (Chilimbi et al. 2014; Seide et al. 2014; Strom, 2015). To circumvent this drawback, communication-efficient parallel algorithms have recently received significant attentions. One major approach is to quantize/compress the gradients while maintaining sufficient information (De Sa et al. 2015; Abadi et al., 2016; Wen et al., 2017). For the unconstrained optimization setting, when Stochastic Gradient Descent (SGD) does not require to perform any projection, various communication-efficient distributed algorithms have been proposed, including QSGD (Alistarh et al., 2017), SIGN-SGD (Bernstein et al., 2018), and Sparsified-SGD (Stich et al., 2018).

In the constrained setting, and in particular for distributed FW algorithms, the communication-efficient versions were only studied for specific problems such as sparse learning (Bellet et al., 2015; Lafond et al., 2016). In this paper, however, we develop Quantized Frank-Wolfe (QFW), a general communication-efficient distributed FW for both convex and non-convex objective functions. We study the performance of QFW in stochastic and finite-sum optimization settings.

Table 1: Convergence rates and average communication bits in different settings, where 
\[ z_1 = \lceil \log_2((\sqrt{\pi}dT^2)^{1/2} + 1) \rceil, z_2 = \lceil \log_2((\sqrt{\pi}dT^2)^{1/2} + 1) \rceil, z_3 = \lceil \log_2((4\sqrt{\pi}dT)^{1/2} + 1) \rceil, z_4 = \lceil \log_2((4\sqrt{\pi}dT)^{1/2} + 1) \rceil. \]

| Setting   | Function                        | Rate    | Average Bits |
|-----------|---------------------------------|---------|--------------|
| stoch.    | bounded, smooth, convex         | \(O(T^{-1/3})\) | \((M + 1)(2d + 32)\) |
| stoch.    | bounded, smooth, non-convex     | \(O(T^{-1/4})\) | \((M + 1)(2d + 32)\) |
| finite-sum | bounded, smooth, convex         | \(O(\sqrt{n} \cdot T^{-1})\) | \(d(Mz_1 + z_2) + (M + 1)(d + 32)\) |
| finite-sum | bounded, smooth, non-convex     | \(O(T^{-1/2})\) | \(d(Mz_3 + z_4) + (M + 1)(d + 32)\) |

Problem formulation: The focus of this paper is on constrained optimization in two widely recognized settings: 1) stochastic and 2) finite-sum optimization. Let \(\mathcal{K} \subseteq \mathbb{R}^d\) denotes the constraint set which is assumed to be convex and compact throughout the paper. In
**Quantized Frank-Wolfe**

workers send quantized gradients to master

master decodes and aggregates the quantized gradients

master broadcasts the quantization of the aggregated gradients

workers decode the quantized signal and update locally

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**Figure 1: Algorithm Workflow**

*constrained stochastic optimization* the goal is to solve the following problem:

$$\min_{x \in K} f(x) := \min_{x \in K} \mathbb{E}_{z \sim P}[\hat{f}(x, z)], \quad (1)$$

where $x \in \mathbb{R}^d$ is the optimization variable, $Z \in \mathbb{R}^q$ is a random variable drawn from a distribution $P$, and together they determine the choice of a stochastic function $\hat{f} : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}$. In *constrained finite-sum optimization* we further assume that $P$ is a uniform distribution over $[N] = \{1, 2, \cdots, N\}$ and the goal is to solve a special case of Problem (1), namely,

$$\min_{x \in K} f(x) := \min_{x \in K} \frac{1}{N} \sum_{i=1}^{N} f_i(x), \quad (2)$$

In parallel settings, we suppose that there is a master and $M$ workers. Each worker maintains a local copy of $x_t$. At every iteration of the stochastic case, each worker has access to independent stochastic gradients of $f$; whereas in the finite-sum case, the objective function can be decomposed as

$$f(x) = \frac{\sum_{m \in [M], i \in [n]} f_{m,i}(x)}{Mn},$$

and each worker $m$ has access to the exact gradients of $f_{m,i}(x)$ for all $i \in [n]$. This way the task of computing gradients is divided among the workers. The master node aggregates local gradients from the workers, and sends the aggregated gradients back to them so that each one of them can update the model (*i.e.*, their own iterate) locally. Note that the kind of information that workers send to the master is of the form of local gradients. Thus, by transmitting quantized gradients, we can reduce the communication complexity (*i.e.*, number of transmitted bits) significantly. The workflow diagram of a distributed quantization scheme is summarized in Fig. 1. Finally, we should highlight that there is a trade-off between gradient quantization and information flow. Intuitively, more intensive quantization reduces the communication cost, but also loses more information, which may decelerate the convergence rate. The goal of **Quantized Frank-Wolfe** is to find a communication-efficient and fast-converging parallel FW algorithm in stochastic and finite-sum cases, for both constrained convex and non-convex optimization problems.

**Our contributions:** In this paper, we propose two general purpose quantization schemes that can be readily applied to distributed optimization settings. We then propose **Quantized Frank-Wolfe**, a distributed framework that handles quantization for constrained convex
and non-convex optimization problems in finite-sum and stochastic cases. It achieves a sweet trade-off between the communication complexity and convergence rate in distributed computation. The results are summarized in Table 1.

2. Gradient Quantization Schemes

As mentioned earlier, the communication cost can be reduced effectively by sending quantized gradients. In this section, we propose two quantization schemes which have different degrees of compression and information content. Depending on the specific requirements of optimization task, one might choose one over the other.

2.1 Single-Partition Quantization

Consider the gradient vector \( g \in \mathbb{R}^d \) and let \( g_i \) be the \( i \)-th coordinate of the gradient. To transmit the scalar \( g_i \), Sign Encoding Scheme sends the product of the sign of \( g_i \) and a properly chosen random variable \( b_i \in \{0, 1\} \), defined as

\[
 b_i = \begin{cases} 
 1, & \text{w.p. } \frac{|g_i|}{\|g\|_\infty}, \\
 0, & \text{w.p. } 1 - \frac{|g_i|}{\|g\|_\infty}, 
\end{cases}
\]

where \( \|g\|_\infty \) is the \( \ell_\infty \) norm of \( g \). Note that the product of \( \text{sgn}(g_i) \) and \( b_i \) belongs to the set \( \{-1, 0, 1\} \) and can be transmitted using two bits. On the receiver side, given access to the norm \( \|g\|_\infty \), one can recover the scalar \( g_i \) (in expectation) by computing \( \text{sgn}(g_i) b_i \|g\|_\infty \), since we have \( \mathbb{E}[(\text{sgn}(g_i) b_i)\|g\|_\infty |g] = g_i \). According to this observation, for each coordinate \( i \), Sign Encoding Scheme only needs to communicate the encoded scalar \( \text{sgn}(g_i) b_i \), alongside the norm \( \|g\|_\infty \). Therefore, if we define \( \text{sgn}(g) \in \mathbb{R}^d \) as a vector containing \( \text{sgn}(g_i) \)'s, and \( b \in \mathbb{R}^d \) as a vector containing random variables \( b_i \), the Sign Encoding Scheme \( \phi(g) \) is a tuple \( (\text{sgn}(g) \circ b, \|g\|_\infty) \), where \( \circ \) is the Hadamard product operator. Similarly, the corresponding decoding scheme is \( \phi'(g) = \|g\|_\infty [\text{sgn}(g) \circ b] \), which is an unbiased estimator of the gradient vector \( g \).

Since transmitting each element of the vector \( \text{sgn}(g) \circ b \) requires two bits, the total communicated bits for \( \text{sgn}(g) \circ b \) are \( 2d \). Assuming that sending the norm \( \|g\|_\infty \), which is a scalar, requires 32 bits, the overall communicated bits of Sign Encoding Scheme for each worker are \( 32 + 2d \) per round. Finally, note that the elements of \( \phi'(g) \) can only be \( \|g\|_\infty, -\|g\|_\infty, \) or 0. Thus intuitively, Sign Encoding Scheme compresses the gradient intensively, and may lead to a loss of information. In the following lemma we formally characterize the loss in terms of the variance of Sign Encoding Scheme.

**Lemma 1** (Proof in Appendix A). For any input vector \( g \), the variance of Sign Encoding Scheme is given by

\[
\text{Var}[\phi'(g)|g] = \|g\|_1 \|g\|_\infty - \|g\|_2^2. \tag{3}
\]

Lemma 1 implies that if the absolute values of the elements of the vector \( g \) are in a same range – its energy is divided among its elements in a balanced way – then the variance of Sign Encoding Scheme, namely, \( \|g\|_1 \|g\|_\infty - \|g\|_2^2 \), is small. For instance, if all the elements of the vector \( g \) are equal to each other, then the variance is zero. Conversely, if the absolute values of a few elements of \( g \) are significantly larger than the rest, then the variance becomes
large. For instance, if one of the elements of \( g \in \mathbb{R}^d \) is \( \rho \gg 1 \) and the remaining ones are 1, then the variance is \((\rho - 1)(d - 1)\).

**Remark 1.** For the probability distribution of the random variable \( b_i \), instead of \( \|g\|_\infty \), we can use other norms \( \|g\|_p \) (where \( p \geq 1 \)). But it can be verified that the \( \ell_\infty \)-norm leads to the smallest variance for Sign Encoding Scheme.

### 2.2 Multi-Partition Quantization

Now we focus on Multi-Partition Quantization Scheme which has a lower variance comparing to Sign Encoding Scheme, but at the cost of sending more bits at each round of communication. Unlike Sign Encoding Scheme that codes each element \( g_i \) into a scalar from \{-1, 0, 1\}, in s-Partition Encoding Scheme each element \( g_i \) is encoded into an element from the set \{-1, -\frac{s-1}{s}, -\frac{1}{s}, 0, \frac{1}{s}, \ldots, \frac{s-1}{s}, 1\}. To transmit the \( i \)-th element \( g_i \) of the gradient vector \( g \), s-Partition Encoding Scheme first computes the ratio \( |g_i|/\|g\|_\infty \) and finds the indicator \( l_i \in \{0, 1, \cdots, s-1\} \) for which the following inequality is satisfied

\[
\frac{l_i}{s} \leq \frac{|g_i|}{\|g\|_\infty} \leq \frac{l_i + 1}{s}.
\] (4)

After finding \( l_i \), we define the random variable \( b_i \) by the following probability distribution

\[
b_i = \begin{cases} 
  l_i/s, & \text{w.p. } 1 - \frac{|g_i|}{\|g\|_\infty} s + l_i, \\
  (l_i + 1)/s, & \text{w.p. } \frac{|g_i|}{\|g\|_\infty} s - l_i.
\end{cases}
\] (5)

Then, instead of transmitting \( g_i \), s-Partition Encoding Scheme sends the product of \( \text{sgn}(g_i) \) and the random variable \( b_i \). It can be verified that \( \mathbb{E}[b_i|g] = |g_i|/\|g\|_\infty \). So we define the corresponding decoding scheme as \( \phi'(g_i) = \|g\|_\infty \text{sgn}(g_i) b_i \) to ensure that \( \phi'(g_i) \) is an unbiased estimator of \( g_i \).

Intuitively, in s-Partition Encoding Scheme, we partition the interval \([0, \|g\|_\infty]\) into \( s \) parts with the same length and find the specific interval in which \(|g_i|\) falls, and estimate \(|g_i|\) by one of the two end points of that interval randomly. The probability for each end point is chosen according to (5) to make sure that the estimation unbiased. Note that the output of the Sign Encoding Scheme can only take values from the set \( \{\pm \|g\|_\infty, 0\} \). Hence, Sign Encoding Scheme only considers the interval \([0, \|g\|_\infty]\) and estimates \(|g_i|\) by one of the two end points, namely, 0 and \( \|g\|_\infty \). This observation implies that the Sign Encoding Scheme can be interpreted as a single-partition quantization.

In Multi-Partition Quantization, for each coordinate \( i \), we need 1 bit to transmit \( \text{sgn}(g_i) \). Moreover, since \( b_i \in \{0, \frac{1}{s}, \ldots, \frac{s-1}{s}, 1\} \), we need \( z = \log_2(s + 1) \) bits to send \( b_i \). Finally, we need 32 bits to transmit \( \|g\|_\infty \). Hence, the total number of communicated bits is \( 32 + d(z + 1) \).

One major advantage of the s-Partition Encoding Scheme is that by tuning the partition parameter \( s \) or the corresponding assigned bits \( z \), we can smoothly control the trade-off between gradient quantization and information loss, which helps distributed algorithms to attain their best performance. In the following lemma, we formally characterize the variance of the s-Partition Encoding Scheme and highlight the accuracy v.s. communication cost trade-off.
Lemma 2 (Proof in Appendix B). The variance of s-Partition Encoding Scheme $\phi$ for any vector $g \in \mathbb{R}^d$ is bounded by
\[
\text{Var}[\phi'(g)|g] \leq \frac{d}{s^2} \|g\|_\infty^2.
\]

Lemma 2 demonstrates the trade-off between the error of quantization and the communication cost for s-Partition Encoding Scheme. In a nutshell, for larger choices of $s$, the variance is smaller, which in turn results in higher communication cost.

3. Stochastic Optimization

In this section, we aim to solve the constrained stochastic optimization problem defined in (1) in a distributed fashion. In particular, we are interested in projection-free (Frank-Wolfe type) methods and execute quantization to reduce the communication cost between the workers and the master. Recall that we assume that at each round $t$, each worker $m$ has access to an unbiased estimator of the objective function gradient $\nabla f(x_t)$, which is denoted by $g^m_t(x_t)$, i.e., $\nabla f(x_t) = \mathbb{E}[g^m_t(x_t)|x_t]$. We further assume that the stochastic gradients are independent of each other.

As shown in Fig. 1, the workflow of our proposed algorithm is easy to understand. At iteration $t$, each worker $m$ first computes its local stochastic gradient $g^m_t(x_t)$. Then, it encodes $g^m_t(x_t)$ as $\Phi(g^m_t(x_t))$ – which is quantized and can be transmitted at a low communication cost – to the master. Once the master receives all the coded stochastic gradients $\{\Phi(g^m_t(x_t))\}_{m=1}^M$, it uses a proper decoding scheme to evaluate $\{\Phi'(g^m_t(x_t))\}_{m=1}^M$, which are the decoded versions of the received signals. Then, the master evaluates the average of the decoded signals which we denote by $\tilde{g}_t$, i.e., $\tilde{g}_t = (1/M) \sum_{m=1}^M \Phi'(g^m_t(x_t))$. After using a proper quantization scheme, the master broadcasts the coded signal $\Phi(\tilde{g}_t)$ to all the workers. The workers decode the received signals and use the resulted $\Phi'(\tilde{g}_t)$ vector to improve their local stochastic gradient approximation.

Note that the vector $\Phi'(\tilde{g}_t)$ is an unbiased estimator of $\nabla f(x_t)$. If we ignore the influence of quantization, $\Phi'(\tilde{g}_t)$ has a lower variance comparing to the local vector $g^m_t(x_t)$ as its computation incorporates the information of $M$ stochastic gradients. Still, if we use $g^m_t(x_t)$, instead of the actual but unavailable gradient $\nabla f(x_t)$, Frank-Wolfe may diverge Mokhtari et al. (2018b). To overcome this issue, we need to further reduce the variance caused by quantization. To do so, each worker $m$ uses a momentum local vector $\tilde{g}_t$ to update the iterates, which is defined by
\[
\tilde{g}_t \leftarrow (1 - \rho_t)\tilde{g}_{t-1} + \rho_t \Phi'(\tilde{g}_t).
\]

As the decoded vectors $\Phi'(\tilde{g}_t)$ for the workers are identical, if they all initialize the sequence $\tilde{g}_t$ in the same way, for all iterations, the local vectors $\tilde{g}_t$ for all the workers are the same. As the update of $\tilde{g}_t$ in (7) computes a weighted average of the previous stochastic gradient approximation $\tilde{g}_{t-1}$ and the updated network average stochastic gradient $\Phi'(\tilde{g}_t)$, it has a lower variance comparing to the vector $\Phi'(\tilde{g}_t)$ (note that it is not an unbiased estimator of $\nabla f(x_t)$). The key fact that allows us to prove convergence is that the estimation error $\tilde{g}_t$ will approach zero as time passes, which is formally characterized in Lemma 3 in Appendix C.
Algorithm 1 Stochastic Quantized Frank-Wolfe

1. **Input:** constraint set $\mathcal{K}$, iteration horizon $T$, initial point $x_1 \in \mathcal{K}$, $g_0 \leftarrow 0$, number of workers $M$, step sizes $\rho_t, \eta_t$

2. **Output:** $x_{T+1} \in \mathcal{K}$

3. for $t = 1$ to $T$

4. Each worker $m \in [M]$ gets an independent stochastic gradient $g^m_t(x_t)$, encodes it as $\Phi(g^m_t(x_t))$, and pushes $\Phi(g^m_t(x_t))$ to the master

5. The master decodes $\Phi(g^m_t(x_t))$ as $\Phi'(g^m_t(x_t))$, and gets the average gradient $\bar{g}_t \leftarrow \frac{1}{M} \sum_{m=1}^{M} \Phi'(g^m_t(x_t))$

6. The master encodes $\bar{g}_t$ as $\Phi(\bar{g}_t)$, and broadcasts it to all the workers

7. Each worker decodes $\Phi(\bar{g}_t)$ as $\Phi'(\bar{g}_t)$

8. Each worker updates $x_{t+1}$ locally by $\bar{g}_t \leftarrow (1-\rho_t)\tilde{g}_{t-1} + \rho_t \Phi'(\bar{g}_t), v_t \leftarrow \arg\min_{v \in \mathcal{K}} \langle v, \bar{g}_t \rangle$ and $x_{t+1} \leftarrow x_t + \eta_t(v_t - x_t)$

9. end for

10. Output $x_{T+1}$ or $x_o$, where $x_o$ is chosen from $\{x_1, x_2, \cdots, x_T\}$ uniformly at random

After computing the vector $\bar{g}_t$ based on the update in (7), workers can update their variables in the standard way, i.e., $x_{t+1} = x_t + \eta_t(v_t - x_t)$. The vector $v_t$ is defined as $v_t = \arg\min_{v \in \mathcal{K}} \langle v, \bar{g}_t \rangle$. Similar to the argument above, if the iterates of all the workers are initialized at the same point $x_1$, then for all the iterations $t \geq 1$, the local variables $x_t$ of all the workers are identical. Note that the update of $v_t$ is slightly different from that of the Frank-Wolfe method as the exact but unavailable gradient $\nabla f(x_t)$ is replaced by its stochastic approximation $\tilde{g}_t$. The full description of our proposed Stochastic Quantized Frank-Wolfe is outlined in Algorithm 1. Finally, note that we can use different quantization schemes $\Phi$ in Algorithm 1, which lead to different convergence rates and communication costs. We explore their effects empirically in our set of experiments.

3.1 Convex Optimization

In this subsection, we focus on the convergence rate of Stochastic Quantized Frank-Wolfe when applied to convex objective functions. To do so, we first make the following assumptions on the constraint set $\mathcal{K}$, the objective function $f$, the local stochastic gradients $g^m_t$, and the quantization scheme $\Phi$.

**Assumption 1.** The constraint set $\mathcal{K}$ is convex and compact. We also denote its diameter by $D = \sup_{x,y \in \mathcal{K}} \|x - y\|$.

**Assumption 2.** The objective function $f$ is convex, bounded, i.e., $\sup_{x \in \mathcal{K}} |f(x)| \leq M_0$, and $L$-smooth over $\mathcal{K}$.

**Assumption 3.** For each worker $m$ and iteration $t$, the stochastic gradient $g^m_t$ is unbiased and has a uniformly bounded variance, i.e., for all $m \in [M]$ and $t \in [T]$, $\mathbb{E}[g^m_t(x_t)t_{x_t}] = \nabla f(x_t)$, $\text{Var}[g^m_t(x_t)t_{x_t}] \leq \sigma_t^2$. 
**Assumption 4.** For any $x_t \in \mathcal{K}$, and vectors $g_t^m(x_t)$ and $\tilde{g}_t$ generated by Stochastic Quantized Frank-Wolfe, the quantization scheme $\Phi$ satisfies

\[
\begin{align*}
\mathbb{E}[\Phi'(g_t^m(x_t))]g_t^m(x_t) &= g_t^m(x_t), \\
\mathbb{E}[\|\Phi'(g_t^m(x_t)) - g_t^m(x_t)\|^2] &\leq \sigma_2^2, \\
\mathbb{E}[\Phi'(\tilde{g}_t)]\tilde{g}_t &= \tilde{g}_t, \\
\mathbb{E}[\|\Phi'(\tilde{g}_t) - \tilde{g}_t\|^2] &\leq \sigma_3^2.
\end{align*}
\]

By considering the above assumptions, in the following theorem we show the convergence rate of Stochastic Quantized Frank-Wolfe.

**Theorem 1** (Proof in Appendix D). Under Assumptions 1 to 4, if we set $\eta_t = \frac{2}{t+3}, \rho_t = \frac{2}{(t+3)^2/4}$ in Algorithm 1, then after $T$ iterations, the output $x_{T+1}$ is a feasible point, i.e., $x_{T+1} \in \mathcal{K}$, and satisfies the inequality

\[
\mathbb{E}[f(x_{T+1})] - f(x^*) \leq \frac{Q_0}{(T + 4)^{1/3}},
\]

where $Q_0 = \max\{4^{1/3} \cdot 2M_0, 2D(Q^{1/2} + LD)\}, Q = \max\{3^{2/3}\|\nabla f(x)\|^2, \frac{4(\sigma_1^2 + \sigma_2^2)}{M} + 4\sigma_3^2 + 8L^2D^2\}$, and $x^*$ is the global minimizer of $f$ on $\mathcal{K}$.

Theorem 1 shows that the suboptimality gap of Stochastic Quantized Frank-Wolfe converges to zero at a sublinear rate of $\mathcal{O}(1/T^{1/3})$. In other words, after running at most $\mathcal{O}(\epsilon^{-3})$ iterations we can find a solution that is $\epsilon$ close to the optimum. Next, we can incorporate the concrete Sign Encoding Scheme into Stochastic Quantized Frank-Wolfe. We first need the following assumption on the stochastic gradients.

**Assumption 5.** The stochastic gradients $g_t^m$ have uniformly bounded $\ell_1$ and $\ell_\infty$ norms, i.e., $\|g_t^m\|_1 \leq G_1, \|g_t^m\|_\infty \leq G_\infty$, for all $m \in [M], t \in [T]$.

**Corollary 1** (Proof in Appendix E). Under Assumptions 1 to 3 and 5, if we set $\eta_t = \frac{2}{t^2}, \rho_t = \frac{2}{(t+3)^2/4}$, and apply Sign Encoding Scheme in Algorithm 1, then after $T$ iterations, the output $x_{T+1} \in \mathcal{K}$ satisfies

\[
\mathbb{E}[f(x_{T+1})] - f(x^*) \leq \frac{Q_0}{(T + 4)^{1/3}},
\]

where $Q_0 = \max\{4^{1/3} \cdot 2M_0, 2D(Q^{1/2} + LD)\}, Q = \max\{3^{2/3}\|\nabla f(x)\|^2, \frac{4(\sigma_1^2 + G_1G_\infty)}{M} + 4G_1G_\infty + 8L^2D^2\}$, and $x^*$ is the global minimizer of $f$ on $\mathcal{K}$.

The idea of proof is quite straightforward. We want to apply Theorem 1, so we only need to calculate $\sigma_2^2, \sigma_3^2$ given the specific quantization scheme. Then we can prove the rate by Theorem 1 directly. Considering the fact that each round of communication in Sign Encoding Scheme requires $(M + 1)(32 + 2d)$ bits, the overall communication cost to find an $\epsilon$-suboptimal solution is $\mathcal{O}(Mde^{-3})$. 

3.2 Non-Convex Optimization

With slightly different parameters, Stochastic Quantized Frank-Wolfe can be applied to non-convex settings as well. In unconstrained non-convex optimization problems, the gradient norm $\|\nabla f\|$ is usually a good measure of convergence as $\|\nabla f\| \to 0$ implies convergence to a stationary point. However, in the constrained setting it is not a good benchmark and instead we need to look at the Frank-Wolfe Gap (Jaggi, 2013; Lacoste-Julien, 2016) defined as

$$G(x) = \max_{v \in \mathcal{K}} \langle v - x, -\nabla f(x) \rangle. \tag{8}$$

For constrained optimization problem (1), if a point $x$ satisfies $G(x) = 0$, then it is a first-order stationary point. Also, by definition, we have $G(x) \geq 0$, for all $x \in \mathcal{K}$.

We will analyze the convergence rate of Algorithm 1 based on the following assumption on the objective function $f$.

**Assumption 6.** The objective function $f$ is bounded, i.e., $\sup_{x \in \mathcal{K}} |f(x)| \leq M_0$, and $L$-smooth over $\mathcal{K}$.

**Theorem 2** (Proof in Appendix F). Under Assumptions 1, 3, 4 and 6, and given the iteration horizon $T$, if we set $\eta_t = \frac{1}{(T+3)^{3/4}}$, $\rho_t = \frac{2}{(T+3)^{1/2}}$ in Algorithm 1, then after $T$ iterations we have

$$\mathbb{E}[G(x_0)] \leq \frac{8M_0 + 20DQ^{1/2}/3}{(T+3)^{3/4}} + \frac{LD^2}{2(T+3)^{3/4}},$$

where $Q = \max\{2\|\nabla f(x_1)\|^2, 4(\sigma_1^2 + \sigma_2^2)M + 4\sigma_3^2 + 2L^2D^2\}$.

In other words, Theorem 2 indicates that in the non-convex setting, Stochastic Quantized Frank-Wolfe finds an $\epsilon$-first order stationary point after at most $O(\epsilon^{-4})$ iterations. This result combined with the concrete quantization method Sign Encoding Scheme leads to the following corollary.

**Corollary 2.** Under Assumptions 1, 3, 5 and 6, if we set $\eta_t = \frac{1}{(T+3)^{3/4}}$, $\rho_t = \frac{2}{(T+3)^{1/2}}$, and apply Sign Encoding Scheme in Algorithm 1, then the output $x_0 \in \mathcal{K}$ satisfies

$$\mathbb{E}[G(x_0)] \leq \frac{8M_0 + 20DQ^{1/2}/3}{(T+3)^{3/4}} + \frac{LD^2}{2(T+3)^{3/4}},$$

where $Q = \max\{2\|\nabla f(x_1)\|^2, 4(\sigma_1^2 + \sigma_2^2)M + 4G_1G_\infty + 2L^2D^2\}$.

By using Sign Encoding Scheme, each round of communication requires $(M + 1)(32 + 2d)$ bits. Therefore, to find an $\epsilon$-first order stationary point, Corollary 2 indicates that we need $O(\epsilon^{-4})$ rounds with the overall communication cost of $O(M\epsilon^{-4})$. 
4. Finite-Sum Optimization

In this section, we focus on the finite-sum problem (2) where we assume that there are \( N \) functions in total and each worker \( m \) has access to \( n = N/M \) functions \( f_{m,i} \) for \( i \in [n] \). The major difference with the stochastic setting is that we can use a more aggressive variance reduction for communicating quantized gradients. More specifically, Nguyen et al. (2017a,b, 2019) developed the StochAstic Recursive grAdient algoritHm (SARAH), a stochastic recursive gradient update framework. Recently, Fang et al. (2018) proposed Stochastic Path-Integrated Differential Estimator (SPIDER) technique, a variant of SARAH, for unconstrained optimization in centralized settings. In this paper, we properly generalize SPIDER to the constrained and distributed settings.

To do so, let us define a period parameter \( p \in \mathbb{N}^+ \). At the beginning of each period, namely, \( \text{mod}(t, p) = 1 \), each worker \( m \), computes the full average of its local gradients and sends it to the master, the master calculates the average of these \( M \) signals, i.e., the average of gradients for all the component functions, and broadcasts it to all the workers, then the workers update the gradient estimation \( \bar{g}_t \) as follows:

\[
\bar{g}_t \leftarrow \frac{1}{Mn} \sum_{m=1}^{M} \sum_{i=1}^{n} \nabla f_{m,i}(x_t) / (Mn).
\]

Note \( \bar{g}_t \) is identical for all the workers. In the remaining iterations of that period, i.e., \( \text{mod}(t, p) \neq 1 \), each worker \( m \) samples a set of local component functions, denoted as \( S_{t,m} \), of size \( S \) uniformly at random, computes the average of these local gradients and sends it to the master, the master calculates the average of the \( M \) signals and sends it to all the workers, then the workers update the gradient estimation \( g_t \) as follow:

\[
g_t \leftarrow \frac{1}{MS} \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] / (MS) + \bar{g}_{t-1}.
\]

So \( \bar{g}_t \) is still identical for all the workers. In order to incorporate quantization, each worker simply pushes the quantized version of the average gradients. Then the master decodes the quantizations, encodes the average of decoded signals in a quantized fashion, and broadcasts the quantization. Finally, each worker decodes the quantized signal and updates \( x_t \) locally. The full description of our proposed Finite-Sum Quantized Frank-Wolfe algorithm is outlined in Algorithm 2.

Compared with Stochastic Quantized Frank-Wolfe, one advantage of Finite-Sum Quantized Frank-Wolfe is that we can use different quantization schemes \( \{\Phi_{1,t}, \Phi_{2,t}\} \) at different iterations \( t \), which makes Algorithm 2 more flexible in solving various optimization problems. We will explore their effects empirically in our set of experiments.

Finally, note again that FW is very sensitive to the accuracy of gradients in order to converge. Nevertheless, we next give strong theoretical guarantees on the convergence rate of Finite-Sum Quantized Frank-Wolfe for both convex and non-convex settings while using quantized and local gradients distributed over \( M \) machines.

4.1 Convex Optimization

To analyze the convex case, we first make an assumption on the component functions.
Algorithm 2 Finite-Sum Quantized Frank-Wolfe

1: **Input:** constraint set $\mathcal{K}$, iteration horizon $T$, initial point $x_1 \in \mathcal{K}$, step sizes $\eta_t$, period parameter $p$, sample size $S$

2: **Output:** $x_{T+1} \in \mathcal{K}$

3: for $t = 1$ to $T$ do

4: if $\text{mod}(t, p) = 1$ then

5: Each worker $m$ computes and encodes the average of its local gradients $\Phi_{1,t}(\sum_{i=1}^n \nabla f_{m,i}(x_t)/n)$, and pushes it to the master;

6: The master decodes $\Phi_{1,t}(\sum_{i=1}^n \nabla f_{m,i}(x_t)/n)$ as $\Phi_{1,t}^t(\sum_{i=1}^n \nabla f_{m,i}(x_t)/n)$, then calculates the average gradient estimation $\bar{g}_t \leftarrow (1/M)\sum_{m=1}^M \Phi_{1,t}^t(\sum_{i=1}^n \nabla f_{m,i}(x_t)/n)$

7: The master encodes $\bar{g}_t$ as $\Phi_{2,t}(\bar{g}_t)$, and broadcasts it to each worker; each worker decodes $\Phi_{2,t}(\hat{g}_t)$ as $\Phi_{2,t}^t(\hat{g}_t)$, and updates $\bar{g}_t \leftarrow \Phi_{2,t}^t(\hat{g}_t)$

8: else

9: Each worker $m$ samples $S$ component functions uniformly at random, defines $S_{t,m}$ to be the sample set, and gets exact gradients $\nabla f_{m,i}(x_t), \nabla f_{m,i}(x_{t-1})$ for all $i \in S_{t,m}$

10: Each worker pushes the quantization $\Phi_{1,t}\left(\sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})]/S\right)$ to the master

11: The master decodes the quantization $\Phi_{1,t}\left(\sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})]/S\right)$, updates $\bar{g}_t \leftarrow \sum_{m=1}^M \Phi_{1,t}^t\left(\sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})]/S\right)$

12: The master encodes $\bar{g}_t$ as $\Phi_{2,t}(\bar{g}_t)$, and broadcasts it to each worker; each worker decodes $\Phi_{2,t}(\hat{g}_t)$ as $\Phi_{2,t}^t(\hat{g}_t)$, and updates $\bar{g}_t \leftarrow \Phi_{2,t}^t(\hat{g}_t) + \bar{g}_t$

13: end if

14: Each worker updates $x_{t+1}$ locally by $v_t \leftarrow \arg\min_{v \in \mathcal{K}} \langle v, \bar{g}_t \rangle$ and $x_{t+1} \leftarrow x_t + \eta_t(v_t - x_t)$

15: end for

16: Output $x_{T+1}$ (for the convex setting) or $x_o$ (for the non-convex setting), where $x_o$ is chosen from $\{x_1, x_2, \ldots, x_T\}$ uniformly at random

**Assumption 7.** The component functions $f_{m,i}$’s are convex, $L$-smooth on $\mathcal{K}$, and uniformly bounded, i.e., sup$_{x \in \mathcal{K}} |f_{m,i}(x)| \leq M_0$, for all $m \in [M], i \in [n]$.

Since functions $f_{m,i}$ are all bounded and $L$-smooth on the compact constraint set $\mathcal{K}$, their gradients $\|\nabla f_{m,i}\|_{\infty}$ are also bounded on $\mathcal{K}$. Moreover, we only have a finite number of component functions $f_{m,i}$’s, thus, there will always be a uniform bound $G_\infty > 0$ on $\|\nabla f_{m,i}\|_{\infty}$ for all $m \in [M], i \in [n]$. For simplicity of analysis, we assume an explicit upper bound $G_\infty$ in the following theorem. But this is a direct implication of the other assumptions.

**Theorem 3** (Proof in Appendix H). Let us set $\eta_t = \frac{2}{p^{1/4}}, p = \sqrt{n}, S = \sqrt{n}$, and apply $s_{1,t} = (2^{z_{1,t}} - 1)$-Partition Encoding Scheme $\phi_{1,t}$, and $s_{2,t} = (2^{z_{2,t}} - 1)$-Partition Encoding Scheme $\psi_{2,t}$ as $\Phi_{1,t}, \Phi_{2,t}$ in Algorithm 2 where $z_{1,t} = \lceil \log_2 \left( \frac{p^{d/4}S^{1/2}}{M^{1/2}} \right) \left( \frac{1}{p} \right) + 1 \rceil$, $z_{2,t} = \lceil \log_2 \left( p^{d/2}S^{1/2} \left( \frac{1}{p} \right) + 1 \right) \rceil$. Under Assumptions 1 and 7, and sup$_{x \in \mathcal{K}} |\nabla f_{m,i}(x)|_{\infty} \leq$
We first make a standard assumption on the component functions.

**Assumption 8.** Algorithm 2 can also be applied to the non-convex setting with a slight change in parameters. Theorem 3 can be derived.

Theorem 3 indicates that in convex setting, if we use the recommended quantization schemes, then Algorithm 2 can find an $\epsilon$-suboptimal with at most $Q_0$ rounds, i.e., the Linear-optimization Oracle (LO) complexity is $O(\sqrt{n}/\epsilon)$. Also, the total Incremental First-order Oracle (IFO) complexity is $\left[\frac{4n^2T^2}{M} + 1\right] + \left[\log_2(\sqrt{n}dT^{2})^{1/2} + 1\right] + (M + 1)(d + 32)$.

Similar to the stochastic case, the key part of our analysis is to bound $\|\nabla f(x_t) - \bar{g}_t\|$, which is addressed in Lemma 4 in Appendix G. Also since finite-sum optimization can be regarded as a special case of stochastic optimization, the recursive inequality for $E[f(x_t)] - f(x^*)$ can be derived directly from the proof of Theorem 1. Combining these two ingredients, Theorem 3 can be derived.

### 4.2 Non-convex Optimization

Algorithm 2 can also be applied to the non-convex setting with a slight change in parameters. We first make a standard assumption on the component functions.

**Assumption 8.** The component functions $f_{m,i}$’s are $L$-smooth on $\mathcal{K}$ and uniformly bounded, i.e., $\sup_{x \in \mathcal{K}} |f_i(x)| \leq M_0$.

**Theorem 4 (Proof in Appendix I).** Under Assumptions 1 and 8, and $\sup_{x \in \mathcal{K}} \|\nabla f_{m,i}(x)\|_\infty \leq G_\infty$, for all $m \in [M], i \in [n]$, if we set $\eta_t = T^{-1/2}, p = \sqrt{n}, S = \sqrt{n},$ and apply $s_{1,t} = (2^{z_{1,t}} - 1)$-Partition Encoding Scheme $\phi_{1,t}$ and $s_{2,t} = (2^{z_{2,t}} - 1)$-Partition Encoding Scheme $\phi_{2,t}$ in Algorithm 2, where $z_{1,t} = z_1 = \left[\log_2(\left[\frac{4\sqrt{n}dT}{M}\right]^{1/2} + 1)\right], z_{2,t} = z_2 = \left[\log_2(\left[4\sqrt{n}dT\right]^{1/2} + 1)\right]$, then the output $x_0 \in \mathcal{K}$ of Algorithm 2 satisfies

$$E[G(x_0)] \leq \frac{2M_0 + D\sqrt{L^2D^2 + 2G_\infty^2} + \frac{LD^2}{\sqrt{T}}}{\sqrt{T}}.$$
5. Experiments

We evaluate the performance of the algorithms from two aspects. The first one is how the loss (the objective function) changes with an increasing number of epochs, while the second one is the number of bits (i.e., the communication complexity) that the master and worker nodes exchange per iteration.

We use the MNIST dataset and consider a convex model and a non-convex model. The convex model consists of a two-layer fully connected neural network with no hidden layer. The output layer has 10 neurons and the log loss for multiclass classification is used. This model is equivalent to multinomial logistic regression. The non-convex model adds two hidden layers with 10 neurons. The constraint is that the $\ell_1$-norm should be at most 1.

For both the convex and non-convex models and in both the stochastic and finite-sum settings, we vary the quantization level and compute the loss after each epoch. Additionally, we compute the average number of bits exchanged by the master and the worker nodes per iteration in order to quantify the communication complexity. A total number of 20 workers are used. In the stochastic setting, each batch of a worker contains 500 images. In the finite-sum setting, each sample of a worker contains 100 images.

The performance for the convex and non-convex models is quantified by the log loss and average Frank-Wolfe gap, respectively. Recall that according to Theorems 1 to 4, the output for the convex model is $x_{T+1}$ and the output for the non-convex model (denoted by $x_o$) is chosen uniformly at random from $\{x_1, x_2, \ldots, x_T\}$. We have $E[G(x_o)] = E[\frac{1}{T} \sum_{i=1}^T G(x_i)]$. Therefore, for the non-convex model, we plot $\frac{1}{T} \sum_{i=1}^T G(x_i)$ as the Frank-Wolfe gap at the $t$-th epoch. The results for the convex model are presented in Fig. 2 while the results for the non-convex model are presented in Fig. 3. In both figures, the subfigures in the first row (Figs. 2a and 2b and Figs. 3a and 3b) show loss vs. epoch in the stochastic setting. Those in the second row (Figs. 2c and 2d and Figs. 3c and 3d) show loss vs. epoch in the finite-sum setting. The third row shows the computational complexity in both settings.

Recall that $s_1$ is the quantization level that is used when workers send their local gradients to the master and $s_2$ is used when the master broadcasts the tensor to all workers. Figs. 2a, 2c, 3a and 3c show how the loss changes if we fix $s_1$ and vary $s_2$. We can observe that increasing $s_2$ improves the convergence performance significantly. In Figs. 2a and 3a, choosing $s_2 = 7$ achieves a similar performance to the situation where all tensors are transferred in their raw form without any quantization. Similarly, in Figs. 2c and 3c, using $s_2$ recommended by Theorems 3 and 4 results in the performance almost identical to that without quantization. According to Figs. 2e and 3e, using $s_2 = 7$ is merely at the cost of a slight increase in communication complexity compared with $s_2 = 1$. In contrast, it can be seen from Figs. 2b, 2d, 3b and 3d that if one fixes $s_2 = 1$, the improvement by choosing a larger $s_1$ is limited. This suggests that it is more worthwhile to invest communication complexity and have a finer quantization in the process of broadcasting tensors from the master node to the workers. If one chooses a smaller $s_1$ (which results in a coarse quantization when the workers transfer their local gradients to the master), the noise incurred by the coarse quantization can be reduced by averaging the local gradients received from the workers. However, the noise associated with the tensor broadcast by the master cannot be mitigated.

As illustrated in Figs. 2e, 2f, 3e and 3f, the unquantized setting suffers from the highest communication complexity. A slight increase in the quantization level of $s_2$ produces a conver-
Figure 2: Illustration of results for the convex model. The quantization level $uq$ denotes unquantized, i.e., the tensors are transferred in their raw form without any quantization. The quantization level $thm$ denotes that $z_{1,t}$ and $z_{2,t}$ are chosen according to Theorem 3.
Figure 3: Illustration of results for the non-convex model. The quantization level $uq$ denotes unquantized, i.e., the tensors are transferred in their raw form without any quantization. The quantization level $thm$ denotes the proposed $z_{1,t}$ and $z_{2,t}$ in Theorem 4.
gence performance similar to the unquantized setting while preserving a low communication complexity.

6. Conclusion

In this paper, we developed Quantized Frank-Wolfe, the first general-purpose projection-free and communication-efficient framework for constrained optimization. Along with proposing various quantization schemes, Quantized Frank-Wolfe can address both convex and non-convex optimization settings in stochastic and finite-sum cases. We provided theoretical guarantees on the convergence rate of Quantized Frank-Wolfe and validated its efficiency empirically on training multinomial logistic regression and neural networks. Our theoretical results highlighted the importance of variance reduction techniques to stabilize Frank Wolfe and achieve a sweet trade-off between the communication complexity and convergence rate in distributed settings.
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Appendix A. Proof of Lemma 1

Proof. For any $g$, as we know that $E[\phi'(g)|g] = g$, the variance of $\phi'(g)$ can be written as

$$\text{Var}[\phi'(g)|g] = E[||\phi'(g) - g||^2|g]$$

$$= \sum_{i=1}^{d} E[(g_i - \text{sgn}(g_i)b_i||g||_{\infty})^2|g]$$

$$= \sum_{i=1}^{d} g_i^2 + ||g||_{\infty}^2 E[b_i^2|g] - 2g_i\text{sgn}(g_i)|g_i|$$

$$= \sum_{i=1}^{d} ||g||_{\infty}^2 E[b_i^2|g] - g_i^2,$$

where the third equality follows from $E[b_i|g] = |g_i|/||g||_{\infty}$ and $\text{sgn}(g_i)^2 = 1$. Note that based on the probability distribution of $b_i$, we can simplify the expression $E[b_i^2|g]$ as $|g_i|/||g||_{\infty}$ and write

$$\text{Var}[\phi'(g)|g] = \sum_{i=1}^{d} ||g||_{\infty}^2 E[b_i^2|g] - g_i^2,$$

which shows that the claim in (3) holds.

\[\square\]

Appendix B. Proof of Lemma 2

Proof. For any given vector $g \in \mathbb{R}^d$, according to the expression in (4), the ratio $|g_i|/||g||_{\infty}$ lies in an interval of the form $[l_i/s, (l_i + 1)/s]$ where $l_i \in \{0, 1, \ldots, s - 1\}$. Hence, for that specific $l_i$, the following inequalities

$$\frac{l_i}{s} \leq \frac{|g_i|}{||g||_{\infty}} \leq \frac{l_i + 1}{s}$$

are satisfied. Moreover, based on the probability distribution of $b_i$ we know that

$$\frac{l_i}{s} \leq b_i \leq \frac{l_i + 1}{s}.$$

Therefore, based on the inequalities in (11) and (12) we can write

$$-\frac{1}{s} \leq \frac{|g_i|}{||g||_{\infty}} - b_i \leq \frac{1}{s}$$

\[20\]
Hence, we can show that the variance of s-Partition Encoding Scheme is upper bounded by
\[
\text{Var}[\phi'(g)|g] = \mathbb{E}[\|\phi'(g) - g\|^2 | g] \\
= \sum_{i=1}^{d} \mathbb{E}[(g_i - \text{sgn}(g_i)b_i\|g\|_\infty)^2 | g] \\
= \sum_{i=1}^{d} \mathbb{E}[(|g_i| - b_i\|g\|_\infty)^2 | g] \\
= \sum_{i=1}^{d} \|g\|_\infty^2 \left( (\frac{|g_i|}{\|g\|_\infty} - b_i)^2 | g \right) \\
\leq \frac{d}{s^2} \|g\|_\infty^2,
\]
where the inequality holds due to (13).

\[\square\]

**Appendix C. Bounding \( \|\nabla f(x_t) - \bar{g}_t\| \) in Stochastic Case**

In order to upper bound \( \|\nabla f(x_t) - \bar{g}_t\| \), we need a lemma for variance reduction, which is a generalization of Lemma 2 in (Mokhtari et al., 2018a).

**Lemma 3.** Let \( \{a_t\}_{t=0}^{T} \) be a sequence of points in \( \mathbb{R}^n \) such that \( \|a_t - a_{t-1}\| \leq G/(t + s)\alpha \) for all \( 1 \leq t \leq T \), where constants \( G \geq 0 \), \( \alpha \in (0, 1] \), \( s \geq \frac{1}{\alpha} - 1 \). Let \( \{\tilde{a}_t\}_{t=0}^{T} \) be a sequence of random variables such that \( \mathbb{E}[\tilde{a}_t|\mathcal{F}_{t-1}] = a_t \) and \( \mathbb{E}[\|\tilde{a}_t - a_t\|^2|\mathcal{F}_{t-1}] \leq \sigma^2 \) for every \( t \geq 1 \), where \( \mathcal{F}_{t-1} \) is the \( \sigma \)-field generated by \( \{\tilde{a}_i\}_{i=1}^{t-1} \) and \( \mathcal{F}_0 = \emptyset \). Let \( \{d_t\}_{t=0}^{T} \) be a sequence of random variables where \( d_0 \) is fixed and subsequent \( d_t \) are obtained by the recurrence
\[
d_t = (1 - \rho_t)d_{t-1} + \rho_t \tilde{a}_t
\]
with \( \rho_t = \frac{2}{(t + s)^{2\alpha/3}} \). Then we have
\[
\mathbb{E}[\|a_t - d_t\|^2] \leq \frac{Q}{(t + s + 1)^{2\alpha/3}}
\]
where \( Q \triangleq \max\{\|a_0 - d_0\|^2(s + 1)^{2\alpha/3}, 4\sigma^2 + 2G^2\} \).

**Proof.** First, for all \( t \geq 1 \), we have \( \rho_t \geq 0 \) and
\[
\rho_t \leq \frac{2}{(1 + s)^{2\alpha/3}} \leq \frac{2}{(\frac{1}{\alpha})^{2\alpha/3}} = 1.
\]

Then we define \( \Delta_t = \|a_t - d_t\|^2 \), thus
\[
\mathbb{E}[\Delta_t | \mathcal{F}_{t-1}] = \mathbb{E}[\|\rho_t(a_t - \tilde{a}_t) + (1 - \rho_t)(a_t - a_{t-1}) + (1 - \rho_t)(a_{t-1} - d_{t-1})\|^2 | \mathcal{F}_{t-1}] \\
\leq \rho_t^2 \sigma^2 + (1 - \rho_t)^2 \frac{G^2}{(t + s)^{2\alpha}} + (1 - \rho_t)^2 \Delta_{t-1} \\
+ 2(1 - \rho_t)^2 \mathbb{E}[(a_t - a_{t-1}, a_{t-1} - d_{t-1})|\mathcal{F}_{t-1}].
\]
By Law of Total Expectation,
\[
E[\Delta_t] = E[E[\Delta_t | F_{t-1}]] \\
\leq \rho_t^2 \sigma^2 + (1 - \rho_t)^2 \frac{G^2}{(t + s)^{2\alpha}} + (1 - \rho_t)^2 E[\Delta_{t-1}] + 2(1 - \rho_t)^2 E[(a_t - a_{t-1}, a_{t-1} - d_{t-1})].
\]

Apply Young’s inequality, we have
\[
2(a_t - a_{t-1}, a_{t-1} - d_{t-1}) \leq \beta_t \|a_{t-1} - d_{t-1}\|^2 + \frac{\|a_t - a_{t-1}\|^2}{\beta_t G^2} \frac{G^2}{\beta_t(t + s)^{2\alpha}}.
\]

So if we let \( z_t = E[\Delta_t] \) and set \( \beta_t = \rho_t/2 \), we have
\[
z_t \leq \rho_t^2 \sigma^2 + (1 - \rho_t)^2 \frac{G^2}{(t + s)^{2\alpha}} + (1 - \rho_t)^2 (\beta_t z_{t-1} + \frac{G^2}{\beta_t(t + s)^{2\alpha}})
\leq \rho_t^2 \sigma^2 + (1 - \rho_t)^2 (1 + 1/\beta_t) \frac{G^2}{(t + s)^{2\alpha}} + (1 - \rho_t)^2 (1 + \beta_t) z_{t-1}
\leq \rho_t^2 \sigma^2 + (1 + 2/\rho_t) \frac{G^2}{(t + s)^{2\alpha}} + (1 - \rho_t)^2 (1 + \rho_t/2) z_{t-1}
\leq \rho_t^2 \sigma^2 + (1 + 2/\rho_t) \frac{G^2}{(t + s)^{2\alpha}} + (1 - \rho_t) z_{t-1}.
\]

The last inequality holds since \( \rho_t \in [0, 1] \) implies \( (1 - \rho_t)^2 \leq 1 \) and \( (1 - \rho_t)(1 + \rho_t/2) \leq 1 \). Now we can further simplify \( z_t \)
\[
z_t \leq (1 - \frac{2}{(t + s)^{2\alpha/3}}) z_{t-1} + \frac{4\sigma^2}{(t + s)^{4\alpha/3}} + \frac{G^2}{(t + s)^{2\alpha}} + \frac{G^2}{(t + s)^{4\alpha/3}}
\leq (1 - \frac{2}{(t + s)^{2\alpha/3}}) z_{t-1} + \frac{4\sigma^2 + 2G^2}{(t + s)^{4\alpha/3}}
\leq (1 - \frac{2}{(t + s)^{2\alpha/3}}) z_{t-1} + \frac{Q}{(t + s)^{4\alpha/3}}.
\]

Now we claim that \( z_t \leq \frac{Q}{(t + s)^{2\alpha/3}} \) for all \( t \in \{0, 1, \cdots, T\} \). We show it by induction. The statement holds for \( t = 0 \) because of the definition of \( Q \). If the statement holds for some \( t = k - 1 \), where \( k \in [T] \), then
\[
z_k \leq (1 - \frac{2}{(k + s)^{2\alpha/3}}) z_{k-1} + \frac{Q}{(k + s)^{4\alpha/3}}
\leq (1 - \frac{2}{(k + s)^{2\alpha/3}}) \frac{Q}{(k + s)^{2\alpha/3}} + \frac{Q}{(k + s)^{4\alpha/3}}
= \frac{(k + s)^{2\alpha/3} - 1}{(k + s)^{4\alpha/3}} Q.
\]
So we only need to prove that
\[
\frac{(k + s)^{2\alpha/3} - 1}{(k + s)^{4\alpha/3}} \leq \frac{1}{(k + s + 1)^{2\alpha/3}}
\]
or equivalently,
\[
[(k + s)^{2\alpha/3} - 1] \cdot (k + s + 1)^{2\alpha/3} \leq (k + s)^{4\alpha/3}.
\]

It suffices to show that
\[
(k + s + 1)^{2\alpha/3} \leq (k + s)^{2\alpha/3} + 1.
\]

Consider \( f(k) = (k + s + 1)^{2\alpha/3} - (k + s)^{2\alpha/3} - 1 \), we observe that \( f(-s) = 0 \), and \( \frac{df(k)}{dk} = \frac{2\alpha}{3}(k + s + 1)^{2\alpha/3-1} - \frac{2\alpha}{3}(k + s)^{2\alpha/3-1} < 0 \) for all \( k > -s \) since \( \alpha \in (0, 1] \). So \( f(k) \) a decreasing function on \([-s, \infty)\), thus \( f(k) \leq 0 \) for all \( k \in \mathbb{N}^+ \), which implies \((k + s + 1)^{2\alpha/3} \leq (k + s)^{2\alpha/3} + 1\).

So the statement holds for \( t = k \), and by induction, \( z_t \leq \frac{Q}{(t+1)^{2\alpha/3}} \) for all \( t \in \{0, 1, \ldots, T\} \).

**Appendix D. Proof of Theorem 1**

**Proof.** First, since \( x_{t+1} = (1 - \eta_t)x_t + \eta_t v_t \) is a convex combination of \( x_t, v_t \), and \( x_1 \in \mathcal{K}, v_t \in \mathcal{K} \) for all \( t \), we can prove \( x_t \in \mathcal{K} \), for all \( t \) by induction. So \( x_{T+1} \in \mathcal{K} \).

Then we observe that for any iteration \( t \), we have
\[
f(x_{t+1}) - f(x^*) = f(x_t + \eta_t (v_t - x_t)) - f(x^*)
\]

\[\leq f(x_t) + \eta_t (v_t - x_t, \nabla f(x_t)) + \frac{L}{2} \eta_t^2 \|v_t - x_t\|^2 - f(x^*) \]

\[= f(x_t) - f(x^*) + \eta_t \langle v_t - x_t, \tilde{g}_t \rangle + \eta_t \langle v_t - x_t, \nabla f(x_t) - \tilde{g}_t \rangle + \frac{L}{2} \eta_t^2 \|v_t - x_t\|^2 \]

\[\leq f(x_t) - f(x^*) + \eta_t \langle x^* - x_t, \tilde{g}_t \rangle + \eta_t \langle v_t - x_t, \nabla f(x_t) - \tilde{g}_t \rangle + \frac{L}{2} \eta_t^2 \|v_t - x_t\|^2 \]

\[= f(x_t) - f(x^*) + \eta_t \langle x^* - x_t, \nabla f(x_t) \rangle + \frac{L}{2} \eta_t^2 \|v_t - x_t\|^2 \]

\[\quad + \eta_t \langle v_t - x_t, \nabla f(x_t) - \tilde{g}_t \rangle + \eta_t \langle x^* - x_t, \tilde{g}_t - \nabla f(x_t) \rangle \]

\[\leq (1 - \eta_t)(f(x_t) - f(x^*)) + \eta_t \|v_t - x^*\| \|\nabla f(x_t) - \tilde{g}_t\| + \frac{L}{2} \eta_t^2 \|v_t - x_t\|^2 \]

\[\leq (1 - \eta_t)(f(x_t) - f(x^*)) + \eta_t D \|\nabla f(x_t) - \tilde{g}_t\| + \frac{L}{2} \eta_t^2 \|v_t - x_t\|^2 \]

\[\leq (1 - \eta_t)(f(x_t) - f(x^*)) + \eta_t D \|\nabla f(x_t) - \tilde{g}_t\| + \frac{L}{2} \eta_t^2 D^2 \tag{15} \]

where Inequality (a) holds because of the \( L \)-smoothness. In (b) we used the optimality of \( v_t \). Inequality (c) is due to the convexity of \( f \), and we applied the Cauchy-Schwarz inequality in (d).
Now we want to apply Lemma 3 to bound $\|\nabla f(x_t) - \tilde{g}_t\|$. By the smoothness of $f$ and $\eta_t = \frac{2}{t+\delta}$, we have

$$\|\nabla f(x_t) - \nabla f(x_{t-1})\| \leq \eta_{t-1} \|v_{t-1} - x_{t-1}\|.L \leq \frac{2LD}{t+2}.$$  

Let $\mathcal{F}_{t-1}$ be the $\sigma$-field generated by $\{\Phi'\tilde{g}_t\}_{i=1}^{t-1}$ and $\mathcal{F}_0 = \emptyset$, by the unbiasedness of the random encoding scheme $\Phi$ and the stochastic gradient $g^m_t$, we have for all $t \geq 1$

$$\mathbb{E}[\Phi'(\tilde{g}_t)|\mathcal{F}_{t-1}] = \mathbb{E}[\tilde{g}_t|\mathcal{F}_{t-1}]$$

$$= \mathbb{E}[\sum_{m=1}^{M} \Phi'(g^m_t(x_t))|\mathcal{F}_{t-1}]$$

$$= \mathbb{E}[\sum_{m=1}^{M} \frac{\Phi'(g^m_t(x_t))}{M}|\mathcal{F}_{t-1}]$$

$$= \nabla f(x_t)$$

and

$$\mathbb{E}[||\Phi'(\tilde{g}_t) - \nabla f(x_t)||^2|\mathcal{F}_{t-1}]$$

$$= \mathbb{E}[||\Phi'(\tilde{g}_t) - \tilde{g}_t + \tilde{g}_t - \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M} + \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M} - \nabla f(x_t)||^2|\mathcal{F}_{t-1}]$$

$$= \mathbb{E}[||\Phi'(\tilde{g}_t) - \tilde{g}_t + \sum_{m=1}^{M} \frac{\Phi'(g^m_t(x_t)) - \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M}}{M}|\mathcal{F}_{t-1}] + \mathbb{E}[||\sum_{m=1}^{M} \frac{g^m_t(x_t)}{M} - \nabla f(x_t)||^2|\mathcal{F}_{t-1}]$$

$$= \mathbb{E}[||\sum_{m=1}^{M} \Phi'(g^m_t(x_t)) - \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M}||^2|\mathcal{F}_{t-1}] + \mathbb{E}[||\sum_{m=1}^{M} \frac{g^m_t(x_t)}{M} - \nabla f(x_t)||^2|\mathcal{F}_{t-1}]$$

$$+ \mathbb{E}[||\Phi'(\tilde{g}_t) - \tilde{g}_t||^2|\mathcal{F}_{t-1}] + 2\mathbb{E}[\langle \Phi'(\tilde{g}_t) - \tilde{g}_t, \sum_{m=1}^{M} \frac{\Phi'(g^m_t(x_t)) - \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M}}{M} \rangle|\mathcal{F}_{t-1}]$$

$$+ 2\mathbb{E}[\langle \sum_{m=1}^{M} \frac{\Phi'(g^m_t(x_t)) - \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M}}{M}, \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M} - \nabla f(x_t) \rangle|\mathcal{F}_{t-1}]$$

By Assumptions 3 and 4, we have

$$\mathbb{E}[\sum_{m=1}^{M} \Phi'(g^m_t(x_t)) - \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M}||^2|\mathcal{F}_{t-1}] = \sum_{m=1}^{M} \mathbb{E}[||\Phi'(g^m_t(x_t)) - \frac{g^m_t(x_t)}{M}||^2|\mathcal{F}_{t-1}] \leq \frac{\sigma_2^2}{M}.$$  

$$\mathbb{E}[||\sum_{m=1}^{M} \frac{g^m_t(x_t)}{M} - \nabla f(x_t)||^2|\mathcal{F}_{t-1}] = \sum_{m=1}^{M} \text{Var}[\frac{g^m_t(x_t)}{M},|\mathcal{F}_{t-1}] \leq \frac{\sigma_1^2}{M^2}.$$  

$$\mathbb{E}[||\Phi'(\tilde{g}_t) - \tilde{g}_t||^2|\mathcal{F}_{t-1}] \leq \sigma_3^2$$

$$\mathbb{E}[\langle \Phi'(\tilde{g}_t) - \tilde{g}_t, \sum_{m=1}^{M} \frac{\Phi'(g^m_t(x_t)) - \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M}}{M} \rangle|\mathcal{F}_{t-1}] = 0.$$  

$$\mathbb{E}[\langle \sum_{m=1}^{M} \frac{\Phi'(g^m_t(x_t)) - \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M}}{M}, \sum_{m=1}^{M} \frac{g^m_t(x_t)}{M} - \nabla f(x_t) \rangle|\mathcal{F}_{t-1}] = 0.$$

24
where we have
\[ \sigma_1^2 + \frac{\sigma_1^2}{M} + \frac{\sigma_2^2}{M} + \frac{M \sigma_3^2}{M} \]

By Jensen's Inequality,

We prove it by induction. When \( t = 0 \), we have

\[ \eta_t = \frac{2}{t+3}, \]

Now apply Lemma 3 with \( \alpha = 1, G = 2LD, s = 2 > 2\sqrt{2} - 1 = 8\frac{1}{2}, \]

we have

\[ Q = \max\{3^{2/3}, 2\sqrt{2} - 1 = 8\frac{1}{2}, \sigma_1^2 = \frac{\sigma_1^2}{M} + \frac{\sigma_2^2}{M} + \frac{M \sigma_3^2}{M} \}\]

By Jensen's Inequality,

\[ \mathbb{E}[\|\nabla f(x_t) - \tilde{g}_t\|^2] \leq \frac{Q}{(t + 3)^{2/3}} \]

Now we claim that for all \( t \in [T + 1] \)

\[ \mathbb{E}[f(x_t)] - f(x^*) \leq \frac{Q_0}{(t + 3)^{1/3}} \]

where \( Q_0 = \max\{4^{1/3} \cdot 2M_0, 2D(Q^{1/2} + LD)\} \).

We prove it by induction. When \( t = 1 \), we have

\[ \frac{Q_0}{(t + 3)^{1/3}} \geq \frac{4^{1/3} \cdot 2M_0}{4^{1/3}} = 2M_0 \geq \mathbb{E}[f(x_1)] - f(x^*). \]

Now suppose that for some \( t \in [T] \), we have \( \mathbb{E}[f(x_t)] - f(x^*) \leq \frac{Q_0}{(t + 3)^{1/3}} \), then by Eq. (19), we have
\[
\mathbb{E}[f(x_{t+1})] - f(x^*) \leq (1 - \frac{2}{t+3}) \frac{Q_0}{(t+3)^{1/3}} + \frac{Q_0}{(t+3)^{4/3}}
\]
\[
= \frac{Q_0}{(t+3)^{1/3}} - \frac{Q_0}{(t+3)^{4/3}}
\]
\[
= \frac{(t+2)Q_0}{(t+3)^{4/3}}
\]
\[
\leq \frac{Q_0}{(t+4)^{1/3}}
\]
where the last inequality holds since \((t+2)^3(t+4) \leq (t+3)^4\), for all \(t \geq 1\). Therefore, we have
\[
\mathbb{E}[f(x_t)] - f(x^*) \leq \frac{Q_0}{(t+3)^{1/3}}, \text{ for all } t \in [T+1].
\]
Specifically, we have
\[
\mathbb{E}[f(x_{T+1})] - f(x^*) \leq \frac{Q_0}{(T+4)^{1/3}}.
\]

\[\square\]

Appendix E. Proof of Corollary 1

Proof. Since \(\text{sgn}(g) \circ b\) requires \(2d\) bits and \(\|g\|_\infty\) requires 32 bits, so for each \(\phi(g)\), we need \(2d + 32\) bits of communication. At Step 4 of Stochastic Quantized Frank-Wolfe, each worker \(m\) should push \(\phi(g^m_t)\) to the master, and at Step 6, the master should broadcast \(\tilde{\phi}(\tilde{g}_t)\) to all the \(M\) workers, so we need \((2d + 32) \cdot M + 2d + 32 = (M + 1)(2d + 32)\) bits per round.

In order to apply Theorem 1, we only need to prove that \(\phi\) has similar properties to Assumption 4. We have shown that the Sign Encoding Scheme \(\phi\) is unbiased. Then by Lemma 1, we have
\[
\mathbb{E}[\|\phi'(g^m_t) - g^m_t\|^2] = \mathbb{E}[\mathbb{E}[\|\phi'(g^m_t) - g^m_t\|^2 | g^m_t]] = \mathbb{E}[\|g^m_t\|_1 \|g^m_t\|_\infty - \|g^m_t\|_2^2] \leq G_1 G_\infty.
\]
Since
\[
\mathbb{E}[\|\phi'(\tilde{g}_t) - \tilde{g}_t\|^2] = \mathbb{E}[\mathbb{E}[\|\phi'(\tilde{g}_t) - \tilde{g}_t\|^2 | \tilde{g}_t]] = \mathbb{E}[\|\tilde{g}_t\|_1 \|\tilde{g}_t\|_\infty - \|\tilde{g}_t\|_2^2] \leq G_\infty \mathbb{E}[\|\tilde{g}_t\|_1]
\]
and
\[
\mathbb{E}[\|\tilde{g}_t\|_1 | g^m_t] = \mathbb{E}[\| \frac{\sum_{m=1}^M \phi'(g^m_t)}{M} | g^m_t]\]
\[
\leq \frac{\sum_{m=1}^M \mathbb{E}[\|\phi'(g^m_t)\|_1 | g^m_t]}{M}
\]
\[
= \frac{\sum_{m=1}^M \mathbb{E}[\sum_{i=1}^d |\phi'_i(g^m_t)| | g^m_t]}{M}
\]
\[
= \frac{\sum_{m=1}^M \sum_{i=1}^d |g^m_{t,i}|}{M}
\]
\[
\leq G_1,
\]
where \( \phi'_i(g^m_i) \) is the \( i \)th element of \( \phi'(g^m_i) \), \( g^m_i \) is the \( i \)th element of \( g^m_t \). So we have
\[
\mathbb{E}[\|\tilde{g}_t\|_1] = \mathbb{E}[\mathbb{E}[\|\tilde{g}_t\|_1|g^m_t]\] \leq G_1,
\]
and
\[
\mathbb{E}[\|\phi'(\tilde{g}_t) - \tilde{g}_t\|^2] \leq G_1 G_\infty.
\]
we can apply Theorem 1 with \( \sigma_2^2 = G_1 G_\infty, \sigma_3^2 = G_1 G_\infty \), then we have
\[
\mathbb{E}[f(x_{T+1})] - f(x^*) \leq \frac{Q_0}{(T + 4)^{1/3}}
\]
where \( Q_0 = \max\{4^{1/3} \cdot 2M_0, 2D(Q^{1/2} + L D)\} \), and \( Q = \max\{3^{2/3} \|\nabla f(x)\|^2, 4(a + G_1 G_\infty)^2 + 4G_1 G_\infty + 8L^2 D^2\} \).
\[\square\]

Appendix F. Proof of Theorem 2

Proof. First, since \( x_{t+1} = (1 - \eta_t)x_t + \eta_t v_t \) is a convex combination of \( x_t, v_t \), and \( x_1 \in \mathcal{K}, v_t \in \mathcal{K}, \) for all \( t \), we can prove \( x_t \in \mathcal{K} \), for all \( t \) by induction. So the output \( x_o \in \mathcal{K} \).

Note that if we define \( v'_t = \arg\min_{v \in \mathcal{K}} \langle v, \nabla f(x_t) \rangle \), then \( \mathcal{G}(x_t) = \langle v'_t - x_t, -\nabla f(x_t) \rangle = -\langle v'_t - x_t, \nabla f(x_t) \rangle \). So we have
\[
f(x_{t+1}) \overset{(a)}{=} f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2} \|x_{t+1} - x_t\|^2
\]
\[
\overset{(b)}{=} f(x_t) + \langle \nabla f(x_t), \eta_t (v_t - x_t) \rangle + \frac{L}{2} \|\eta_t (v_t - x_t)\|^2
\]
\[
\overset{(c)}{=} f(x_t) + \eta_t \langle \tilde{g}_t, v_t - x_t \rangle + \eta_t \langle \nabla f(x_t) - \tilde{g}_t, v_t - x_t \rangle + \frac{L \eta_t^2 D^2}{2}
\]
\[
\overset{(d)}{=} f(x_t) - \eta_t \mathcal{G}(x_t) + \eta_t \langle \nabla f(x_t) - \tilde{g}_t, v_t - x_t \rangle + \frac{L \eta_t^2 D^2}{2}
\]
\[
\overset{(e)}{=} f(x_t) - \eta_t \mathcal{G}(x_t) + \eta_t \|\nabla f(x_t) - \tilde{g}_t\| v_t - x_t \| + \frac{L \eta_t^2 D^2}{2},
\]
where we used the assumption that \( f \) has \( L \)-Lipschitz continuous gradient in inequality (a). Inequalities (b), (e) hold because of Assumption 1. Inequality (c) is due to the optimality of \( v_t \), and in (d), we applied the Cauchy-Schwarz inequality.
Rearrange the inequality above, we have

\[ \eta G(x_t) \leq f(x_t) - f(x_{t+1}) + \eta D\|\nabla f(x_t) - \bar{g}_t\| + \frac{L\eta^2 D^2}{2}. \tag{20} \]

Apply Eq. (20) recursively for \( t = 1, 2, \cdots, T \), and take expectations, we attain the following inequality:

\[ \sum_{t=1}^{T} \eta_t \mathbb{E}[G(x_t)] \leq f(x_1) - f(x_{T+1}) + D \sum_{t=1}^{T} \eta_t \mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] + \frac{LD^2}{2} \sum_{t=1}^{T} \eta_t^2. \tag{21} \]

Since \( f \) has \( L \)-Lipschitz continuous gradient, and \( \eta_t = (T + 3)^{-3/4} \), we have

\[ \|\nabla f(x_t) - \nabla f(x_{t-1})\| \leq \|\nabla f(x_{t-1}) - x_{t-1}\| L \]

\[ \leq \frac{LD}{(T + 3)^{3/4}} \]

\[ \leq \frac{LD}{(t + 3)^{3/4}}. \]

Combine the inequality above with Eq. (17), and apply Lemma 3 with \( \alpha = 3/4, G = LD, s = 3 = 8\frac{1}{3}/2 - 1 = 8\frac{1}{2} - 1, \sigma^2 = \frac{\sigma_1^2 + \sigma_2^2}{M}, d_t = \bar{g}_t, \) for all \( t \geq 0, a_t = \nabla f(x_t), \bar{a}_t = \Phi'(\bar{a}_t), \) for all \( t \geq 1, a_0 = a_1 = \nabla f(x_1), \) we have

\[ \mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \leq \frac{Q}{(t + 4)^{1/2}}. \]

where \( Q = \max\{2\|\nabla f(x_1)\|^2, \frac{4(\sigma_1^2 + \sigma_2^2)}{M} + 4\sigma_3^2 + 2L^2D^2\} \).

By Jensen’s Inequality,

\[ \mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] \leq \sqrt{\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2]} \leq \frac{Q^{1/2}}{(t + 4)^{1/4}}. \]

Since

\[ \sum_{t=1}^{T} \mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] \leq \sum_{t=1}^{T} \frac{Q^{1/2}}{(t + 4)^{1/4}} \]

\[ \leq \int_{0}^{T} \frac{Q^{1/2}}{(t + 4)^{1/4}} dt \]

\[ = \frac{4Q^{1/2}}{3} [(T + 4)^{3/4} - 4^{3/4}] \]

\[ \leq \frac{4Q^{1/2}}{3} (T + 3)^{3/4} \]

by Eq. (21), we have

\[ \sum_{t=1}^{T} \mathbb{E}[G(x_t)] \leq \frac{f(x_1) - f(x_{T+1})}{(T + 3)^{-3/4}} + D \sum_{t=1}^{T} \mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] + \frac{LD^2 T \cdot (T + 3)^{-3/2}}{2} \]

\[ \leq \frac{f(x_1) - f(x_{T+1})}{(T + 3)^{3/4}} + \frac{4DQ^{1/2}}{3} (T + 4)^{3/4} + \frac{LD^2}{2} T (T + 3)^{-3/4}. \]
So we have
\[\mathbb{E}[G(x_0)] = \frac{1}{T} \sum_{t=1}^T \mathbb{E}[G(x_t)]\]
\[\leq [f(x_1) - f(x_{T+1})] \frac{(T + 3)^{3/4}}{T} + 4DQ^{1/2} \frac{(T + 4)^{3/4}}{T} + \frac{LD^2}{2} (T + 3)^{-3/4}\]
\[\leq 2M_0 \frac{(T + 3)^{3/4}}{T} + 4DQ^{1/2} \frac{(T + 4)^{3/4}}{T} + \frac{LD^2}{2} (T + 3)^{-3/4}\]
\[\leq 8M_0 + 20DQ^{1/2}/3 \frac{(T + 3)^{3/4}}{T} + \frac{LD^2}{2} (T + 3)^{-3/4}.\]

\[\square\]

Appendix G. Bounding \(\|\nabla f(x_t) - \bar{g}_t\|\) in Finite-Sum Case

We now address the bound of \(\|\nabla f(x_t) - \bar{g}_t\|\), which is resolved in the following lemma.

**Lemma 4.** Under Assumption 1, if we further assume that each \(f_{m,i}\) is bounded and has \(L\)-Lipschitz continuous gradient, and \(\|\nabla f_{m,i}(x)\|_\infty \leq G_\infty\) for all \(x \in \mathcal{K}, m \in [M], i \in [n]\), where \(G_\infty\) is a positive constant, by setting \(p = \sqrt{n}, s = \sqrt{n}\) and applying the \(s_1,t = (2^{2z_1,t} - 1)\)-Partition Encoding Scheme \(\phi_{1,t}\), the \(s_2,t = (2^{2z_2,t} - 1)\)-Partition Encoding Scheme \(\phi_{2,t}\) as \(\Phi_{1,t,\Phi_{2,t}}\), we have

\[\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \leq (L^2 D^2 + 2G_\infty^2) \frac{4dpS}{\sqrt{n}} \left(\sum_{\frac{i-1}{p} \leq \lfloor \frac{i-1}{p} \rfloor + 1} \eta_t^2 + 1\right)\]

where

\[z_{1,t} = \left\lfloor \log_2 \left(\frac{4dpS}{\sum_{\frac{i-1}{p} \leq \lfloor \frac{i-1}{p} \rfloor + 1} \eta_t^2 + 1}\right)\right\rfloor,\]

\[z_{2,t} = \left\lfloor \log_2 \left(\frac{4dpS}{\sum_{\frac{i-1}{p} \leq \lfloor \frac{i-1}{p} \rfloor + 1} \eta_t^2 + 1}\right)\right\rfloor.\]

Also, this lemma looks a bit complicated because of the summation of \(\eta_t^2\). The range of the summation is just the subset which contains \(l\) and can be expressed as \(\{kp + 1, kp + 2, \ldots, (k + 1)p\}\), where \(k \in \mathbb{N}\). This is easy to understand, since intuitively, the variance is only related to the factors within the same period \(\{kp + 1, kp + 2, \cdots, (k + 1)p\}\). In practical applications, we usually have concrete values for \(\eta_t\), which will make the sum and thus the expressions look much simpler.

**Proof.** We first define an auxiliary variable \(g_t\), which is \(\nabla f(x_t)\) if \(\text{mod}(t, p) = 1\), and is set to \(\frac{1}{M_0} \sum_{m=1}^M \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] + g_{t-1}\) otherwise.

We also define \(F_{t-1}\) to be the \(\sigma\)-field generated by all the randomness before round \(t\). We note that \(F_{t-1}, x_t\) is actually determined, and we can verify that \(\mathbb{E}[g_t | F_{t-1}] = \nabla f(x_t)\)
and \(\mathbb{E}[\bar{g}_t | F_{t-1}, g_t] = g_t\), for all \(t \in [T]\). Here, with abuse of notation, \(\mathbb{E}[\cdot | g_t]\) is the conditional expectation given not only the value of \(g_t\), but also the gradients \(\nabla f_{m,i}(x_t), \nabla f_{m,i}(x_{t-1})\) for all \(i \in S_{t,m}, m \in [M]\).
Then by law of total expectation, we have

\[
\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] = \mathbb{E}[\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2|\mathcal{F}_{t-1}] = \\
= \mathbb{E}[\mathbb{E}[\|\nabla f(x_t) - g_t + g_t - \bar{g}_t\|^2|\mathcal{F}_{t-1}] = \\
= \mathbb{E}[\mathbb{E}[\|\nabla f(x_t) - g_t\|^2|\mathcal{F}_{t-1}] + \mathbb{E}[||g_t - \bar{g}_t\|\|\mathcal{F}_{t-1}] = \\
+ 2\mathbb{E}[\mathbb{E}[(\nabla f(x_t) - g_t, g_t - \bar{g}_t)|\mathcal{F}_{t-1}] = \\
= \mathbb{E}[\|\nabla f(x_t) - g_t\|^2] + \mathbb{E}[||g_t - \bar{g}_t\|^2],
\]

where the last equation holds since

\[
\mathbb{E}[(\nabla f(x_t) - g_t, g_t - \bar{g}_t)|\mathcal{F}_{t-1}] = \\
= \mathbb{E}[\mathbb{E}[(\nabla f(x_t) - g_t, g_t - \bar{g}_t)|\mathcal{F}_{t-1}, g_t]|\mathcal{F}_{t-1}] = \\
= \mathbb{E}[\mathbb{E}[g_t - \bar{g}_t|\mathcal{F}_{t-1}, g_t]|\mathcal{F}_{t-1}] = 0.
\]

Moreover, for mod\(t, p\) \(\neq 1\),

\[
\mathbb{E}[\|\nabla f(x_t) - g_t\|^2] = \\
= \mathbb{E}[\mathbb{E}[\|\nabla f(x_t) - g_t\|^2|\mathcal{F}_{t-1}] = \\
= \mathbb{E}[\|\nabla f(x_t) - \nabla f(x_{t-1}) + \nabla f(x_{t-1}) - g_{t-1} - \frac{1}{MS} \sum_{m=1}^{M} \sum_{i \in \mathcal{S}, m} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})]|\mathcal{F}_{t-1}],
\]

With abuse of notation, we have \(\mathbb{E}[\sum_{m=1}^{M} \nabla f_{m,i}(x_t)/M|\mathcal{F}_{t-1}] = \nabla f(x_t)\), and \(\mathbb{E}[\sum_{m=1}^{M} \nabla f_{m,i}(x_{t-1})/M|\mathcal{F}_{t-1}] = \nabla f(x_{t-1})\), where \(i\) actually depends on \(m\), and is sampled from \(\mathcal{S}, m\) at random. Thus

\[
\mathbb{E}[\frac{1}{MS} \sum_{m=1}^{M} \sum_{i \in \mathcal{S}, m} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})]|\mathcal{F}_{t-1}] = \nabla f(x_t) - \nabla f(x_{t-1}),
\]

and

\[
\mathbb{E}[(\nabla f(x_t) - \nabla f(x_{t-1}) - \frac{1}{MS} \sum_{m=1}^{M} \sum_{i \in \mathcal{S}, m} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})], \nabla f(x_{t-1}) - g_{t-1})|\mathcal{F}_{t-1}] = 0.
\]
So we have
\[
\mathbb{E}[\|\nabla f(x_t) - g_t\|^2] = \mathbb{E}[\mathbb{E}[\|\left(\frac{1}{M} \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] - [\nabla f(x_t) - \nabla f(x_{t-1})]\|^2 | \mathcal{F}_{t-1}] | \mathcal{F}_{t-1}] \\
+ \mathbb{E}[\mathbb{E}[\|\nabla f(x_{t-1}) - g_{t-1}\|^2 | \mathcal{F}_{t-1}]]
\]
\[
= \mathbb{E}[\operatorname{Var}\left(\frac{1}{M} \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] | \mathcal{F}_{t-1}\right)] \\
+ \mathbb{E}[\mathbb{E}[\|\nabla f(x_{t-1}) - g_{t-1}\|^2 | \mathcal{F}_{t-1}]]
\]
\[
= \frac{1}{S} \mathbb{E}[\operatorname{Var}\left(\frac{\sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})]}{M} | \mathcal{F}_{t-1}\right)] \\
+ \mathbb{E}[\mathbb{E}[\|\nabla f(x_{t-1}) - g_{t-1}\|^2 | \mathcal{F}_{t-1}]]
\]
\[
\leq \frac{1}{S} \mathbb{E}[m \mathbb{E}[[\frac{\sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})]}{M})^2 | \mathcal{F}_{t-1}]] + \mathbb{E}[\mathbb{E}[\|\nabla f(x_{t-1}) - g_{t-1}\|^2 | \mathcal{F}_{t-1}]]
\]
\[
\leq \frac{1}{S} (LD_{\eta_1})^2 + \mathbb{E}[\mathbb{E}[\|\nabla f(x_{t-1}) - g_{t-1}\|^2 | \mathcal{F}_{t-1}]]
\]
\[
= \frac{L^2 D^2 \eta_1^2}{S} + \mathbb{E}[\mathbb{E}[\|\nabla f(x_{t-1}) - g_{t-1}\|^2 | \mathcal{F}_{t-1}]]
\]

Note for any \( t \) such that \( \text{mod}(t, p) = 1 \), we have \( g_t = \nabla f(x_t) \). Therefore
\[
\mathbb{E}[\|\nabla f(x_t) - g_t\|^2] \leq \frac{L^2 D^2}{S} \sum_{[\frac{i-1}{p}] p + 1 \leq l \leq [\frac{i}{p}]} \eta_l^2.
\]

Now we turn to bound \( \mathbb{E}[\|g_t - \bar{g}_t\|^2] \). For \( \text{mod}(t, p) \neq 1 \), We have
\[
\mathbb{E}[\|g_t - \bar{g}_t\|^2] = \mathbb{E}[\mathbb{E}[\|\frac{1}{M} \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] + g_{t-1} - \phi_{2,t}(\tilde{g}_t) - \bar{g}_{t-1}\|^2 | \mathcal{F}_{t-1}, g_t]]
\]
\[
= \mathbb{E}[\mathbb{E}[\|\frac{1}{M} \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] - \phi_{2,t}(\tilde{g}_t)\|^2 | \mathcal{F}_{t-1}, g_t]] \\
+ \mathbb{E}[\mathbb{E}[\|g_{t-1} - \bar{g}_{t-1}\|^2 | \mathcal{F}_{t-1}, g_t]]
\]
\[
+ 2 \mathbb{E}[\mathbb{E}[\|\frac{1}{M} \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] - \phi_{2,t}(\tilde{g}_t), g_{t-1} - \bar{g}_{t-1} | \mathcal{F}_{t-1}, g_t]]
\]

Moreover
\[
\mathbb{E}[\phi_{2,t}(\tilde{g}_t) | \mathcal{F}_{t-1}, g_t] = \mathbb{E}[\tilde{g}_t | \mathcal{F}_{t-1}, g_t]
\]
\[
= \mathbb{E}\left[\sum_{m=1}^{M} \phi_{1,t}(S \sum_{i \in S_{t,m}} \nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})) / M | \mathcal{F}_{t-1}, g_t\right]
\]
\[
= \frac{1}{MS} \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})],
\]
and

\[ E[\|1/MS \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] - \phi_{2,t}(\tilde{g}_t)\|^2 | F_{t-1}, g_t] \]

\[ = E[\|1/MS \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] - \tilde{g}_t + \tilde{g}_t - \phi_{2,t}(\tilde{g}_t)\|^2 | F_{t-1}, g_t] \]

\[ = E[\|1/MS \sum_{m=1}^{M} \sum_{i \in S_{t,m}} [\nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})] \\
- \sum_{m=1}^{M} \phi_{1,t}(\frac{\sum_{i \in S_{t,m}} \nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})}{S})/M\|^2 | F_{t-1}, g_t] \\
+ E[\|g_t - \phi_{2,t}(\tilde{g}_t)\|^2 | F_{t-1}, g_t, \tilde{g}_t] \]

\[ \leq \frac{1}{M} \frac{d}{s_{1,t}^2} (2G_\infty)^2 + \frac{d}{s_{2,t}^2} (2G_\infty)^2 \\
- \frac{4dG_\infty^2}{M s_{1,t}^2} + \frac{4dG_\infty^2}{s_{2,t}^2}, \]

where in the inequality, we apply Lemma 2 with \( \|\sum_{i \in S_{t,m}} \nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})\|_\infty \leq 2G_\infty \) and \( \|\tilde{g}_t\|_\infty = \|\sum_{m=1}^{M} \phi_{1,t}(\frac{\sum_{i \in S_{t,m}} \nabla f_{m,i}(x_t) - \nabla f_{m,i}(x_{t-1})}{S})/M\|_\infty \leq 2G_\infty \).

Now we have for \( \text{mod}(t, p) \neq 1 \),

\[ E[\|g_t - \tilde{g}_t\|^2] \leq \frac{4dG_\infty^2}{M s_{1,t}^2} + \frac{4dG_\infty^2}{s_{2,t}^2} + E[\|g_{t-1} - \tilde{g}_{t-1}\|^2]. \]

If \( \text{mod}(t, p) = 1 \), we have

\[ E[\|g_t - \tilde{g}_t\|^2] = E[\|\nabla f(x_t) - \tilde{g}_t + \tilde{g}_t - \phi_{2,t}(\tilde{g}_t)\|^2] \]

\[ = E[E[\|\nabla f(x_t) - \sum_{m=1}^{M} \phi_{1,t}(\sum_{i=1}^{n} \nabla f_{m,i}(x_t)/n)/M\|^2 | F_{t-1}, g_t]] \\
+ E[E[\|\tilde{g}_t - \phi_{2,t}(\tilde{g}_t)\|^2 | F_{t-1}, g_t, \tilde{g}_t]] \]

\[ \leq \frac{1}{M} E[E[\sum_{i=1}^{n} \nabla f_{m,i}(x_t)/n - \phi_{1,t}(\sum_{i=1}^{n} \nabla f_{m,i}(x_t)/n)/M] \|^2 | F_{t-1}, g_t]] + \frac{d}{s_{2,t}^2} G_\infty^2 \\
\leq \frac{dG_\infty^2}{M s_{1,t}^2} + \frac{dG_\infty^2}{s_{2,t}^2}, \]

where in the inequality, we apply Lemma 2 with \( \|\sum_{i=1}^{n} \nabla f_{m,i}(x_t)/n\|_\infty \leq G_\infty \) and \( \|\tilde{g}_t\|_\infty = \|\sum_{m=1}^{M} \phi_{1,t}(\sum_{i=1}^{n} \nabla f_{m,i}(x_t)/n)/M\|_\infty \leq G_\infty \). Since for any \( t_1, t_2 \) such that \([t_1 - 1] = [t_2 - 1]\), we have
Appendix H. Proof of Theorem 3

Proof. First, since $x_{t+1} = (1 - \eta_t) x_t + \eta_t v_t$ is a convex combination of $x_t, v_t$, and $x_1 \in \mathcal{K}, v_t \in \mathcal{K}$, for all $t$, we can prove $x_t \in \mathcal{K}$, for all $t$ by induction. So $x_{T+1} \in \mathcal{K}$.

Note for $t_1, t_2$ such that $\lceil \frac{t_1}{p} \rceil = \lceil \frac{t_2}{p} \rceil$, we have $\eta_{t_1} = \eta_{t_2}$. So $s_{1,t_1} = s_{1,t_2}, s_{2,t_1} = s_{2,t_2}$, and thus $\Phi_{1,t_1} = \Phi_{1,t_2}, \Phi_{2,t_1} = \Phi_{2,t_2}$. By Lemma 4, we have

$$
\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \leq \frac{4(L^2D^2 + 2G^2_\infty)}{p^2 \lceil \frac{t}{p} \rceil^2}.
$$

So

$$
\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] \leq \sqrt{\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2]} \leq 2\sqrt{L^2D^2 + 2G^2_\infty} \frac{\eta_t}{p \lceil \frac{t}{p} \rceil}.
$$

On the other hand, by Assumption 7, $f = \sum_{m \in [M]} \frac{f_m}{M_n}$ is a bounded $L$-smooth convex function on $\mathcal{K}$, with $\sup_{x \in \mathcal{K}} |f(x)| \leq M_0$. So Eq. (15) still holds. Taking expectation on both sides, we have

$$
\mathbb{E}[f(x_{t+1})] - f(x^*) \leq (1 - \eta_t)(\mathbb{E}[f(x_t)] - f(x^*)) + \eta_t D\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] + \frac{L}{2}\eta_t^2 D^2 \\
\leq (1 - \frac{2}{p \lceil \frac{t}{p} \rceil})(\mathbb{E}[f(x_t)] - f(x^*)) + \frac{2D(2\sqrt{L^2D^2 + 2G^2_\infty + LD})}{p^2 \lceil \frac{t}{p} \rceil^2}.
$$

Quantized Frank-Wolfe

$s_{1,t_1} = s_{1,t_2}, s_{2,t_1} = s_{2,t_2}$, thus

$$
\mathbb{E}[\|g_t - \bar{g}_t\|^2] \leq \left[ \frac{4dG^2_\infty}{Ms_1^2} + \frac{4dG^2_\infty}{s_2^2} \right] (p - 1) + \frac{dG^2_\infty}{Ms_1^2} + \frac{dG^2_\infty}{s_2^2}
\leq \frac{4dpG^2_\infty}{Ms_1^2} + \frac{4dpG^2_\infty}{s_2^2}.
$$

(24)

Now combine Eqs. (22) to (24), we have

$$
\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \leq \frac{L^2D^2}{S} \sum_{1 \leq j \leq \eta_t^2 + 4dpG^2_\infty + 4dpG^2_\infty} \frac{\sqrt{\eta_t^2}}{s_2^2}.
$$

Since we set $p = \sqrt{n}, S = \sqrt{n}, s_1 = 2^{z_1} - 1 \geq \left( \frac{4dpG^2_\infty + 4dpG^2_\infty}{\sqrt{n}} \right)^{1/2}, s_2 = 2^{z_2} - 1 \geq \left( \frac{4dpG^2_\infty + 4dpG^2_\infty}{\eta_t^2} \right)^{1/2}$, we have

$$
\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \leq (L^2D^2 + 2G^2_\infty) \frac{\eta_t^2}{\sqrt{n}}.
$$
Let \( t = kp, Q = 2D(2\sqrt{L^2D^2 + 2G^2} + LD) \) where \( k \in \mathbb{N}^+ \), and apply the inequality recursively for \( p \) times, we have

\[
\mathbb{E}[f(x_{kp+1})] - f(x^*) \leq (1 - \frac{2}{pk})p(\mathbb{E}[f(x_{(k-1)p+1})] - f(x^*)) + \frac{Q}{pk^2}.
\]

Now we claim that \((1 - \frac{2}{pk})p \leq 1 - \frac{2}{pk}p + \frac{4}{p^2k^2} \frac{p(p-1)}{2} = 1 - \frac{2}{k} + \frac{2(p-1)}{pk^2}\).

The inequality holds trivially for \( p = 1 \) and \( p = 2 \). For \( p \geq 3 \), we have \( \frac{2}{pk} < 1 \).

Define function \( h(x) = (1 - x)p - 1 + px - \frac{p(p-1)}{2}x^2 \). Then for \( x \in [0, 1] \), we have \( h'(x) = p[1 - (p-1)x - (1-x)^{p-1}], h''(x) = p(p-1)((1-x)^{p-2} - 1] \leq 0 \), then \( h'(x) \leq h'(0) = 0 \). Thus \( h(x) \leq h(0) = 0 \), i.e., \((1 - x)p \leq 1 - px + \frac{p(p-1)}{2}x^2 \). Let \( x = \frac{2}{pk} \), then we have \((1 - \frac{2}{pk})p \leq 1 - \frac{2}{pk}p + \frac{p(p-1)}{2}(\frac{2}{pk})^2 = 1 - \frac{2}{k} + \frac{2(p-1)}{pk^2}\).

Consider \( k \geq 3 \), then

\[
(1 - \frac{2}{pk})p \leq 1 - \frac{2}{k} + \frac{2(p-1)}{3pk} \leq 1 - \frac{2}{k} + \frac{2}{3k} = 1 - \frac{4}{3k},
\]

and thus

\[
\mathbb{E}[f(x_{kp+1})] - f(x^*) \leq (1 - \frac{4}{3k})(\mathbb{E}[f(x_{(k-1)p+1})] - f(x^*)) + \frac{Q}{pk^2}.
\]

Define \( Q_0 = \max\{6pM_0, 3Q\} \). Then we claim \( \mathbb{E}[f(x_{kp+1})] - f(x^*) \leq \frac{Q_0}{(k+1)p} \), for all \( k \in \mathbb{N} \).

We prove this inequality by induction. For \( k = 0, 1, 2 \), we have \( \mathbb{E}[f(x_{kp+1})] - f(x^*) \leq 2M_0 \leq \frac{Q_0}{(2+1)p} \leq \frac{Q_0}{(k+1)p} \). Now suppose that for some \( k \geq 3 \), we have \( \mathbb{E}[f(x_{(k-1)p+1})] - f(x^*) \leq \frac{Q_0}{k^2p} \), then

\[
\mathbb{E}[f(x_{kp+1})] - f(x^*) \leq (1 - \frac{4}{3k})\frac{Q_0}{kp^2} + \frac{Q_0}{3pk^2} = \frac{k-1}{pk^2}Q_0 \leq \frac{Q_0}{(k+1)p},
\]

where the last inequality holds since \((k-1)(k+1) \leq k^2\).

So we have \( \mathbb{E}[f(x_{kp+1})] - f(x^*) \leq \frac{Q_0}{(k+1)p} \), for all non-negative integer \( k \leq T/p \). Let \( T = Kp \), then

\[
\mathbb{E}[f(x_{T+1})] - f(x^*) = \mathbb{E}[f(x_{Kp+1})] - f(x^*) \leq \frac{Q_0}{(K+1)p} \leq \frac{Q_0}{T}.
\]
For any $\epsilon > 0$, set $T = \frac{Q_0}{\epsilon}$, then we have $\mathbb{E}[f(x_{T+1})] - f(x^*) \leq \epsilon$. So the LO complexity is $O(1/\epsilon)$. Also note in each period, the total number of gradient call is $Mn + (p-1) \cdot M \cdot S \cdot 2 = Mn + 2MS(p-1)$, so the average cost is $[Mn + 2MS(p-1)]/p = M[3\sqrt{n} - 2]$. Thus the total IFO complexity is $M[3\sqrt{n} - 2] \frac{Q_0}{\epsilon} = O(\sqrt{n} \max\{6M_0\sqrt{n}, 3Q\}/\epsilon)$.

The communication bits per round are at most $M[d(z_{1,T} + 1) + 32] + d(z_{2,T} + 1) + 32 = d(Mz_{1,T} + z_{2,T}) + (M + 1)(d + 32) \approx d(M[\log_2((\sqrt{\pi}dT)^{1/2} + 1)] + [\log_2((\sqrt{\pi}dT)^{1/2} + 1)]) + (M + 1)(d + 32)$.

\[\square\]

**Appendix I. Proof of Theorem 4**

**Lemma 5.** Under Assumptions 1 and 8, with $\eta_t = T^{-1/2}$ and fixed $T$ in Algorithm 2, if we further assume that

$$\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \leq \frac{c^2}{T},$$

where $c$ is a positive constant, then we have $x_0 \in \mathcal{K}$ and

$$\mathbb{E}[G(x_o)] \leq \frac{2M_0 + cD + LD^2}{\sqrt{T}}.$$

Since here we already assume that $\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2]$ has an upper bound. The convergence rate can be proved by solving a recursive inequality directly. Moreover, since finite-sum optimization is a special case of stochastic gradient optimization, we can use the analysis in the proof of Theorem 2 to get the inequality.

**Proof.** First, since $x_{t+1} = (1 - \eta_t)x_t + \eta_tv_t$ is a convex combination of $x_t, v_t$, and $x_1 \in \mathcal{K}, v_t \in \mathcal{K}$, for all $t$, we can prove $x_t \in \mathcal{K}$, for all $t$ by induction. So $x_o \in \mathcal{K}$.

By Assumption 8, $f$ is also a bounded (potentially) non-convex function on $\mathcal{K}$ with $L$-Lipschitz continuous gradient. Specifically, we have $\sup_{x \in \mathcal{K}}|f(x)| \leq M_0$. So Eq. (21) still holds, i.e.,

$$\sum_{t=1}^{T} \eta_t \mathbb{E}[G(x_t)] \leq f(x_1) - f(x_{T+1}) + D \sum_{t=1}^{T} \eta_t \mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] + \frac{LD^2}{2} \sum_{t=1}^{T} \eta_t^2.$$

Since we assume that $\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \leq \frac{c^2}{T}$, we have

$$\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] \leq \sqrt{\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2]} \leq \frac{c}{\sqrt{T}}.$$

With $\eta_t = T^{-1/2}$, we then have

$$\sum_{t=1}^{T} \mathbb{E}[G(x_t)] \leq \sqrt{T}[f(x_1) - f(x_{T+1})] + D \sum_{t=1}^{T} \mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|] + \sqrt{T} \frac{LD^2}{2} \sqrt{T} \frac{T^{-1/2}}{2} T(T^{-1/2})^2$$

$$\leq 2M_0 \sqrt{T} + DT \frac{c}{\sqrt{T}} + \frac{LD^2}{2} \sqrt{T}$$

$$= (2M_0 + cD + \frac{LD^2}{2}) \sqrt{T}.$$
So
\[
\mathbb{E}[G(x_o)] = \sum_{t=1}^{T} \frac{\mathbb{E}[G(x_t)]}{T} \leq \frac{2M_0 + cD + \frac{LD^2}{2}}{\sqrt{T}}.
\]

Now we can prove Theorem 4.

**Proof.** By Lemma 5, we only need to bound \( \mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \), which can be achieved by applying Lemma 4. Note that since \( \eta_t = T^{-1/2} \) is fixed, then \( z_{1,t} = z_1, z_{2,t} = z_2 \), for all \( t \), so \( \Phi_{1,t}, \Phi_{2,t} \) will not change with \( t \).

By Lemma 4, we have
\[
\mathbb{E}[\|\nabla f(x_t) - \bar{g}_t\|^2] \leq (L^2D^2 + 2G_\infty^2) \frac{\sum_{p=0}^{T-1} (\frac{t-1}{p}+1) \eta_t^2}{\sqrt{T}} = \frac{L^2D^2 + 2G_\infty^2}{T}.
\]

By Lemma 5,
\[
\mathbb{E}[G(x_o)] \leq \frac{2M_0 + D\sqrt{L^2D^2 + 2G_\infty^2} + \frac{LD^2}{2}}{\sqrt{T}}.
\]

The average communication bits per round are \( M[d(z_1+1)+32] + d(z_2+1)+32 = d(Mz_1 + z_2) + (M + 1)(d + 32) \).

For any \( \epsilon > 0 \), set \( T = (2M_0 + D\sqrt{L^2D^2 + 2G_\infty^2} + \frac{LD^2}{2})/\epsilon^2 \), then we have \( \mathbb{E}[G(x_o)] \leq \epsilon \). So the LO complexity is \( O(1/\epsilon^2) \). Also note in each period, the total number of gradient call is \( Mn + (p - 1) \cdot M \cdot S \cdot 2 = Mn + 2MS(p-1) \), so the average cost is \( \lceil Mn + 2MS(p-1) \rceil /p = M[3\sqrt{n} - 2] \). Thus the total IFO complexity is \( M[3\sqrt{n} - 2](2M_0 + D\sqrt{L^2D^2 + 2G_\infty^2} + \frac{LD^2}{2})^2/\epsilon^2 = O((\sqrt{n}/\epsilon^2)) \).

\( \square \)