Superdiffusion revisited

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Abstract. The concept of diffusion in collisionless space plasmas like those near the magnetopause and in the geomagnetic tail is reexamined from a fundamental statistical point of view making use of the division of particle orbits into waiting orbits and break-out into ballistic motion lying at the bottom, for instance, of Lévy flights and the celebrated $\kappa$-distribution. A stringent derivation yields an anomalous diffusion coefficient increasing with time, thus describing superdiffusion. Contrary to wide belief, superdiffusion, though faster than classical, is a weak process. Absolute values of the coefficient are small due to the largeness of the anomalous collision frequency in waiting statistics compared with the infinitesimally small binary collision frequency. We provide parallel and perpendicular diffusion coefficients, determine the exponents of temporal increase, determine the power and $\kappa$ for two-dimensional diffusion by referring to published numerical particle-in-cell simulations, fix the range of permitted $\kappa$s, and construct a relation between the diffusion coefficients and the resistive scale. We furthermore find an expression for the anomalous collision frequency from the electron pseudo-viscosity in reconnection thus identifying reconnection as a localized diffusion process in satisfactory agreement with basic physical concepts.

Keywords. Diffusion, Lévy flights, $\kappa$-distribution, Reconnection

1 Introduction

Reconnection is the dominant mechanism for the plasma and magnetic field transport across magnetic boundaries which are represented by thin current sheets/layers in collisionless magnetized plasmas. Reconnection has the enormous advantage over global diffusion of being localized with the main physics of magnetic merging and diffusion – of both magnetic field and plasma – taking place in a small spatial region with other processes signing responsible for affecting the global plasma environment. Among those processes diffusion is just another one, even though, in a wider sense, the magnetic merging inside the reconnection site can, in more general terms, as well be understood as kind of a localized collisionless and much more complicated diffusive process than the common classical collisional diffusion.

On the global scale, plasma and magnetic field diffusion remain to be of substantial interest even under collisionless conditions as a heading term for the average transport process. In this sense it is widely used, for instance in diffusive transport of cosmic rays across the magnetized interplanetary, interstellar and intergalactic spaces. The diffusion is anomalous in this case, lacking the clean stochasticity of particle-particle collisions that underlies classical diffusion. It is frequently described as quasilinear diffusion resulting from wave-particle interactions and mathematically formulated in the Fokker-Planck phase space-diffusion formalism.

Unfortunately, most of the observed diffusive particle spectra (cf., e.g. Christon et al., 1989, 1991, for the most elaborate observations in near-Earth space) barely exhibit the shapes resulting from simple quasilinear phase space diffusion. They turn out of being power law both in energy and momentum space, most frequently being described best by so-called $\kappa$-distributions or superpositions of $\kappa$-distributions with different high-energy/high-momentum slopes to which the parameters $\kappa$ are directly related. The anomalous diffusive processes leading to such distributions are sometimes summarized under the term superdiffusion.

Distributions of the $\kappa$-family were first used formally for fitting observed particle fluxes in the magnetospheric tail (Vasyliunas, 1968). Theoretical attempts of justifying solar wind $\kappa$-distributions followed, invoking wave-particle interactions with inclusion of residual binary collisions (Scudder & Olbert, 1979) or genuinely collisionless particle-photon...
bath interactions (Hasegawa et al., 1985). Fundamental statistical mechanical arguments from first physical principles for generalized Lorentzians including $\kappa$-distributions as quasi-stationary states in collisionless media far from thermal equilibrium were given in (Treumann [1999a,b], Treumann & Jaroschek [2008]). Yoon et al. (2012) obtained time-asymptotic electron $\kappa$-distributions when accounting for spontaneous emission, scattering and reabsorption of plasma fluctuations. As expected, the power law index in this case, similar to that of Hasegawa et al. (1985), becomes a function of the quasi-stationary plasma wave power. This suggests that in an adequately extended statistical mechanics of such quasi-stationary states not only the particle distribution should be included but also wave dynamics as governed by the wave-kinetic equation. In such a theory particle and wave numbers will remain separately conserved, while conservation of energy and momentum will be globally only, not separately.

2 Diffusion process

It is clear that dissipation and the related diffusion in collisionless media is mediated by waves or in a wider sense by collisionless turbulence (cf., e.g., Allegri et al. [1996]). Binary collision times $\tau_c \gg \tau_a$ are very long, much longer than anomalous interaction times. As a consequence, any real non-collisional diffusion proceeds on times much shorter than classical diffusion times. For this reason one speaks of «superdiffusion». We show, however, that absolute values of the diffusion coefficient are small.

From the particle point of view the superdiffusion process can be considered as a sequence of comparably long «waiting times» when the particle is in a stationary trapped state followed by a short «breakout» into ballistic motion until the next trapping and waiting period starts (Shlesinger et al. 1987; Klafter et al. 1990). Such a particle motion is typical for Lévy flights (cf., e.g., Shlesinger et al. 1993). We have made use in an earlier publication (Treumann 1997) of these ideas to estimate the anomalous collisionless superdiffusion coefficient in the collisionless plasma at the magnetopause. Here we provide a more general discussion of the validity of the superdiffusion concept in a collisionless plasma.

Since diffusion is a real space process, we will work in the $d$-dimensions of real space. The probability of a particle to occupy a particular volume element during a process like a Lévy flight, assumed to be caused by some kind of unspecified nonlinear interaction between plasma waves and the particle, is instead best formulated in wave number space $k$ as the probability spectrum

$$p(k) \propto \exp(-ak^\alpha), \quad \alpha = 2 - 1/\kappa$$

with $a$ some constant, and $1/2 < \kappa \in \mathbb{R}$ a rational number. One immediately realizes that for $\kappa \to \infty$ the power index $\alpha = 2$ just reproduces the classical Gaussian probability spectrum and thus corresponds to the purely stochastic binary collision processes underlying classical diffusion.

The condition that $\kappa < \infty$ transforms the spectrum into the wide non-stochastic domain which includes turbulence and Lévy flights. No closed transformation back into real space is known in this case. One therefore needs to refer to some model distributions which, to certain limits, can be considered to represent such non-stochastic motions. A convenient assumption would be a simple power law distribution; this, however, is unphysical because it diverges at small scales $x$. In the particle picture it does not conserve particle number. It is thus more appropriate to refer to the $\kappa$-distribution which, however, must now be formulated in real space and not in phase space. For this purpose one introduces a correlation length $\ell$ and postulates that

$$p(\kappa|x) = A_\kappa \left(1 + \frac{x^2}{\kappa^2}\right)^{-(\kappa+1)}$$

with $\kappa \equiv (2 - \alpha)^{-1}$ obeying the above relation with the power index $\alpha$. This restricts $\alpha \leq 2$ to the domain of flatter than Gaussian probability spectra. $A_\kappa$ is just a normalization factor which takes care of the probability integrated over all space to be 1. It is easy to show that the limit $\lim_{\kappa \to \infty} p(\kappa|x)$ smoothly reproduces the Gaussian probability distribution. The spatial distribution distinguishes itself from the one-dimensional energy space distribution $p(\kappa|\epsilon) \propto [1 + \beta(\mu - \epsilon)/\kappa]^{-(\kappa+1)}$, where $\epsilon = mv^2/2$ is particle energy, $\mu$ chemical potential, and $\beta = 1/T$ temperature in energy units (cf., e.g., Treumann 1999b; Treumann & Jaroschek 2008; Yoon et al. 2012). Clearly, for the spatial model-probability distribution, reference to any chemical potential would be inappropriate. However, below we will make use of the formal equivalence between temperature $T$ and spatial correlation length square $\ell^2$ when determining the diffusion coefficient. A more stringent lower bound on $\kappa$ is set by the requirement that the random mean square

$$\langle x^2 \rangle = \int x^2 p(\kappa|x)\text{d}^d x$$

of the expectation value of the particle displacement in the diffusion process exists. Hence, for the integral to converge at large $x$, one demands $d/2 < \kappa < \infty$. For the probability power index this implies that

$$2(1 - 1/d) < \alpha \leq 2$$

In three dimensions we thus have the restriction $4/3 < \alpha < 2$, a narrow power range containing all the physics from turbulence to Gaussian stochasticity. Interestingly, large dimensionality, expected in higher field theory or also in chaotic systems, corresponds to $\alpha \lesssim 2$ and is thus difficult to distinguish from pure stochasticity when referring to power indices.

In view of the previous discussion that the diffusion process can be envisaged as consisting of a sequence of $n$ steps
bridging the time from $t = 0$ to $t = t_n$ with the particle jumping from one waiting position to another. The expectation value of the latter becomes

$$\langle x^2(n) \rangle = \int x^2 p(n|x) dx, \quad p(n) = \prod_{i=1}^{n} p(i). \quad (5)$$

We have shown (Treumann [1997]) that the $n$th expectation value is proportional to the random mean square displacement $x$ and a power of the elapsed time sequence. Here we repeat the argument in a slightly different and more transparent form. To this purpose we work in Fourier (or momentum) space $k$ where the multiplication of the probabilities is simplest, yielding from Eq. (1)

$$p(n|k) = p^n(k) \propto \exp(-ank^\alpha) \quad (6)$$

Expanding the exponential, we arrive at the following scaling for the probability of the $n$th step

$$p(n|k) \approx 1 - ank^\alpha \approx p(k'), \quad k' = kn^{1/\alpha} \quad (7)$$

Any real space coordinate therefore scales as $x \rightarrow xn^{-1/\alpha}$. For the real-space probability this implies that

$$p(n|x) dx = p(n/x) d^d x / n^{d/\alpha} \quad (8)$$

which follows from the identity $\int p(x) dx = \int p(x') dx'$. Inserted into Eq. (5) and using Eq. (3) yields (without any need for carrying out the integration) the following relation between the $n$th expectation value and the random mean square displacement:

$$\langle x^2(n) \rangle = n^{2/\alpha}(\langle x^2 \rangle) \equiv n^{2\alpha/(2\kappa - 1)}(\langle x^2 \rangle) \quad (9)$$

This relation for the $n$th displacement will be used to construct a diffusion theory that is based on the assumption of waiting statistics in collisionless interaction of particles mediated by some non-specified non-quasilinear interaction with a turbulent wave field which is responsible for kicking out the particles of their metastable state of waiting. It is a relation which has been based on the $\kappa$ probability distribution. The power index $\kappa$ has not been specified. It is reasonable to assume that $\kappa$ is not a simple constant but is determined by the turbulent wave field. Under any real conditions it will be a functional of the wave intensity and will thus depend on space and time. Presently this dependence is ignored, assumed to be much slower than the time during that the average displacement can be measured.

### 3 Diffusion coefficient

In using probability steps $n$ we have discretized the time into pieces of free flight, waiting and some kind of interaction. We may assume that in the average corresponding to the random mean square displacement this interaction is covered by a fictitious collisionless or anomalous collision frequency $\nu_a$. Since ordinary binary collision frequencies $\nu_c$ are very small, we have the scaling $\nu_a \gg \nu_c$. Both collision frequencies define a time scale, the short anomalous timescale $\tau_a^{-1} \ll \tau_c = \nu_c^{-1}$ being much less than the ordinary collision timescale $\tau_c$. The entire diffusion process takes place in a time $t \ll \tau_c$. With this scaling in mind we can replace the discrete time steps $n \rightarrow \nu_a t$ with the product of the average anomalous collision frequency and the elapsed time $t \ll \tau_c$. Then the mean square $n$th displacement becomes a function of time

$$\langle x^2(t) \rangle = \langle x^2 \rangle (\nu_a t)^{2\alpha/(2\kappa - 1)} \quad (10)$$

Since any time-dependent average spatial displacement corresponds to a diffusion process, it defines a diffusion coefficient, in this case an anomalous diffusion coefficient $D_a$. A first glance on the above expression immediately suggests that $D_a$ will turn out to be time-dependent. Before concluding, one must calculate the root mean square displacement. This can be done with the help of our model-$\kappa$ distribution referring to Eq. (6). Inserting for $p(\kappa|x)$ and performing the integration, the integrals are tabulated [Gradshteyn & Ryzhik [1965]] yielding

$$\langle x^2 \rangle = \frac{2\kappa d}{(2\kappa - d)(2\kappa + d + 2)} \ell^2 \quad (11)$$

As was expected, the root mean square displacement emerges proportional to the correlation length $\ell$ being independent of time. The time dependence of the diffusion is thus determined by the time dependence of Eq. (10). In classical diffusion, the time dependence of $\langle x^2(t) \rangle = D_c t$ is linear, and the diffusion coefficient is constant. In order to compare with classical diffusion it is necessary to extract one factor $t$ to obtain

$$D_{ak}(d,t) = \langle x^2 \rangle (\nu_a t)^{1/(2\kappa - 1)} \quad (12)$$

The square of the correlation length in this process, $\ell^2 \sim (\nu_a t)^{\kappa}/(\nu_a t)$ is given as ratio of the average square of particle speed $\nu$ (the velocity expectation value) and anomalous collision frequency. With $\nu^2 \approx 2T/m$ and eliminating $\ell^2$ from the two last expressions the anomalous diffusion coefficient assumes the final form

$$D_{ak}(d,t) = D_{ca} \frac{4\kappa d}{(2\kappa - d)(2\kappa + d + 2)} (\nu_a t)^{1/(2\kappa - 1)} \quad (13)$$

The coefficient $D_{ca} = T/m\nu_c$ in this expression is of the form of the classical time-independent diffusion coefficient which, in this case, is given in terms of particle temperature and anomalous collision frequency. Being just a dimensional factor, it is by no means comparable to the classical diffusion coefficient. Since $\nu_a \gg \nu_c$ this factor is much less than the classical diffusion coefficient which in this case would practically be free flight. Under the anomalous collisions the free flight is abruptly reduced by the much large collision frequency $\nu_a$ to a non-stochastic chaotic diffusion of time dependent mean square displacement.
The diffusion coefficient is a function of the dimensionality $d$. For the three basic dimensions the dimensional factors of the diffusion coefficients scale as

$$1: \frac{(\kappa - 1/2)(\kappa + 3/2)}{(\kappa - 1)(\kappa + 1)}: \frac{3(\kappa - 1/2)(\kappa + 3/2)}{(\kappa - 3/2)(\kappa + 5/2)}$$

(14)

4 Parallel versus transverse diffusion

While under collisionless conditions the diffusion coefficient is infinite or at least very large, corresponding to free (ballistic) flight of the particles, the anomalous diffusion coefficients are finite, much smaller in fact. In magnetized plasmas one distinguishes between the two direction parallel $\parallel$ and perpendicular $\perp$ to the magnetic field $\mathbf{B}$. There is little reason to assume that the diffusion coefficients along and perpendicular to the magnetic field would be identical. At the contrary, parallel to the magnetic field one one dimension comes into play, while perpendicular we may have two dimensional diffusion. These may or may not be completely decoupled depending on the dominating wave particle interactions. Depending on these conditions one distinguishes several cases.

4.1 Primary parallel diffusion

The parallel diffusion coefficient from Eq. (13) is one-dimensional

$$\frac{D_\parallel}{\nu a_{\parallel} t} = \frac{\kappa(\nu a_{\parallel} t)^{1/(2\kappa-1)}}{(\kappa - 1/2)(\kappa + 3/2)}$$

(15)

For $\kappa = 3/2$ it evolves with time as $D_\parallel \propto \sqrt{\nu a_{\parallel} t}$. The scaling with time thus falls into the narrow range of exponent $0 < (2\kappa - 1)^{-1} \lesssim 1/2$. Hence, the full diffusion coefficient (remember that we already extracted a factor of $t$ in order to account for the scaling difference with respect to classical diffusion!) scales with time exponent as $1 < 1+(2\kappa - 1)^{-1} \lesssim 3/2$. The anomalous collision frequency $\nu a_{\parallel}$ in this case is determined by a parallel waiting time of the particle at some location along the magnetic field before jumping ahead to another waiting position. This can be caused by particle trapping in a wave field or in electrostatic electron holes and is a complicated process.

It is usually assumed, for instance in cosmic ray physics, that some kind of parallel diffusion is the only dominant diffusion process. In this case the above parallel diffusion is primary in the sense that any perpendicular diffusion is secondary, caused by the diffusion of particles parallel to the magnetic field followed by pitchangle scattering. In the plane perpendicular to the magnetic field the particles are assumed to perform undisturbed gyro-orbits at gyro-frequency $\omega_c = eB/m$ around the magnetic field. Any finite parallel diffusion which restricts and distorts the free parallel motion then might as well kick the particles out of their gyration.

The corresponding secondary perpendicular diffusion coefficient is then conventionally given by

$$D_{a\perp} \approx D_\parallel / (1 + \omega_c^2/\nu a_{\parallel}) \approx D_\parallel (\nu a_{\perp}/\omega_c)^2$$

(16)

where the second expression holds for the realistic case $\omega_c > \nu a_{\parallel}$. On introducing the Bohm diffusion coefficient $D_B \approx T/m \omega_c$, the secondary perpendicular diffusion coefficient becomes

$$\frac{D_{a\perp}}{D_B} \approx \frac{\nu a_{\perp}}{\nu_c} \frac{\kappa(\nu a_{\perp} t)^{1/(2\kappa-1)}}{(\kappa - 1/2)(\kappa + 3/2)}$$

(17)

This diffusion coefficient is much smaller than Bohm diffusion but increases with time. For this reason it may be called super-Bohm diffusion.

4.2 Transverse diffusion

When perpendicular diffusion is independent of the parallel diffusion (as in the simulations discussed below) it is genuinely two-dimensional with diffusion coefficient

$$\frac{D_{a\perp}}{D_B} = \frac{\omega_c}{\nu a_{\perp}} \frac{\kappa(\nu a_{\perp} t)^{1/(2\kappa-1)}}{(\kappa - 1/2)(\kappa + 2)}$$

(18)

Due to the large (artificial) factor $\omega_c/\nu a_{\perp}$ on the right-hand side, this kind of diffusion might suggest that anomalous perpendicular diffusion is substantially larger than Bohm diffusion. However, this is not true, for $\omega_c$ in this expression cancels and the diffusion is determined only by the perpendicular anomalous collision frequency $\nu a_{\perp}$. The total transverse diffusion coefficient is a combination of the above secondary and the genuinely transverse anomalous diffusivities.

5 Evolution

Estimates of diffusion coefficients based on observations in space plasma are rare. They suffer from the practical impossibility of determination of particle displacements as function of time and the subsequent transition to the asymptotic state. In addition they are mostly based on quasilinear theories of particular instabilities (Sagdeev, 1966; Liewer & Krall, 1973; Huba et al., 1977; Davidson, 1978; Sagdeev, 1979; Huba et al., 1981; LaBelle & Treumann, 1988; Treumann et al., 1991; Yoon et al., 2002; Matthaeus et al., 2003; Daughton et al., 2004; Ricci et al., 2005; Roytershteyn et al., 2012; Izutsu et al., 2013) which do not properly account for any nonlinear interactions.

Estimated $\kappa$ values in Earth’s magnetosphere (Christon et al., 1991), on the other hand, range in the interval $5 < \kappa < 10$ with most probable values somewhere near $\kappa \approx 7$. A rough estimate of the scaling with time thus yields $D_{a\perp} \propto (\nu a_{\parallel} t)^{0.1}$ or otherwise, $D_{a\perp} \propto (\nu a_{\parallel} t)^{1.1}$, when including the classical scaling $\propto t$, deviating only weakly from classical diffusion and thus barely detectable with the available resolution of
Mean Perpendicular Displacement

\[ \langle (\Delta x)^2 \rangle \propto t^{1.17} \approx t^7 \]

Spacecraft instrumentation. This makes most attempts of arriving at intentionally more precise theories and more sophisticated approaches like for instance (Zank et al., 2004) obsolete. The mere observation of \(\kappa\) distributions must thus suffice to infer about the action of anomalous diffusion processes. This would be particularly interesting to know in the electron diffusion region of reconnection. Particle-in-cell simulations in two or three dimensions should pay attention to this important problem.

We may, however, refer to sufficiently high-resolution two-dimensional particle-in-cell simulations (Scholer et al., 2000) performed in order to determine the cross-magnetic field diffusion of ions near quasi-perpendicular shocks. The results of these simulations are compiled in Figure 1. The right-hand side of the figure shows one macro-particle orbit arbitrarily selected out of the large number of particles used in the simulation to determine their instantaneous displacements from the origins of their trajectories in the simulation as function of simulation time measured in units of their identical (energy-independent) gyration frequency \(\omega_{ci} = eB/m_i\) in the total magnetic field, which is the sum of the ambient and the self-consistently generated turbulent wave magnetic field. The particle shifts its position perpendicular to the magnetic field from its start point to the end point in the simulation. It is found in a slowly changing waiting position during a final jump. Such an orbit is neither adiabatic nor stochastic.

The left part of the Figure shows the average displacement, ensemble averaged over the entire particle population, as function of simulation time. After performing an initial oscillation the average displacements settle into an about smoothly increasing curve of constant slope \(\langle (\Delta x)^2 \rangle \propto t^{7/6}\). (We should note that, because of the large number of \(\sim 6.3 \times 10^6\) macro-particles used in the simulation of which \(525000\) had high energies and contribute most to the mean displacement as well as for the high time resolution, the statistical error of the measurement is smaller than the width of the line in this figure!) The slope of the final evolution of the average displacement is close but by no means identical with classical diffusion which is shown by the slope of the two straight lines in the figure. Though the deviation in the slope is small it is nevertheless substantial and statistically significant, indicating a superdiffusive process which strongly deviates from classical diffusion, a conclusion strongly in cons-

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**Fig. 1.** Two-dimensional numerical simulation results of the mean downstream perpendicular displacement of ions near a quasi-perpendicular supercritical shock (shock normal angle \(\theta = 87^\circ\), Alfvénic Mach number \(M_A = 4\) as function of simulation time (simulation data taken from Scholer et al., 2000 courtesy American Geophysical Union). Distances are measured in ion inertial lengths \(\lambda_i = c/\omega_i\), with \(\omega_i\) ion plasma frequency. **Left:** The particle displacement performs an initial damped oscillation before settling into a continuous diffusive increase at time about \(\omega_i t \sim 40\) [in units of the ion gyro frequency \(\omega_{ci}\)]. The further time-evolution deviates apparently only slightly from the classical (linear) increase of the mean displacement, following a \(\langle (\Delta x)^2 \rangle \propto (\omega_{ci} t)^{1.37}\) power law. **Right:** Late time trajectory of an arbitrary ion of the sample used. The orbit is projected into the plane perpendicular to the mean magnetic field which consists of a superposition of the ambient and wave magnetic field. The ion orbit is neither an undisturbed gyro-oscillation nor a smooth stochastic trajectory. It consists of waiting (trapped gyrating) parts and parts when the ion suddenly jumps ahead a long distance cause by some brief but intense interaction between the particle and wave spectrum. This break out of gyration is typical for rare extreme events like those in Lévy flights referred to in the present paper.
6 Transition to collisional state

Anomalous diffusion proceeds on a faster than classical time scale with time dependent diffusion coefficient which justifies the term superdiffusion. In spite of this, the factor in front of the time which determines the absolute magnitude of the diffusion is small. It consists of the product of \( \frac{\nu}{\nu_\alpha} \) and a factor depending on \( \kappa \) and the dimension \( d \). For the case of the simulations with \( d = 2 \) this factor is 3.79. Since \( \nu_\alpha \gg \nu \), it is much larger than the very small classical collision frequency, it cannot compensate for the absolute smallness of the diffusion coefficient. Anomalous diffusion is time dependent in steps of the short anomalous collision time \( \tau_\alpha \), but in absolute numbers it is small. The displacement grows on a fast scale but in small steps. Only then when, after a long time has elapsed the order of the classical collision time \( \tau_c \), then classical diffusion takes over scattering some particles to larger, some others back to much smaller displacements. This sequence is schematically shown in Figure 2.

7 Resistive scale and relation to reconnection

We may use these arguments to briefly infer about the resistive scale \( L_\nu \), a quantity frequently referred to in discussions of diffusion in presence of current flow. It plays a role in the diffusive evolution of the magnetic field which from the induction equation is given in its simplest form

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{V} \times \mathbf{B} + D_m \nabla^2 \mathbf{B}, \quad D_m = \frac{\eta}{\mu_0} = \lambda_e^2 \nu
\]

The resistive scale is defined as \( L_\nu \sim \lambda_e \nu t \) being determined through resistivity \( \eta = \nu_\alpha \omega_e \) and electron inertial length \( \lambda_e = c/\omega_e \), with plasma frequency \( \omega_e \). It is interesting to know how the resistive scale evolves with time in a nonlinearly active though collisionless medium. If we use our expression for the product \( \nu_\alpha t \) to replace \( \nu t \) it is simple matter to find

\[
\frac{L_\nu}{\lambda_e} \sim \left\{ \frac{D_{\alpha \kappa}}{D_{\kappa d}} \left( \kappa - \frac{1}{2} d \right) \left( \kappa + 1 + \frac{1}{2} d \right) \right\}^{1/2}
\]

for the resistive scale in units of \( \lambda_e \), expressed through the (time dependent) diffusion coefficient \( D_{\alpha \kappa} \).

From this relation it is obvious that the resistive scale increases with time in a collisionless turbulent plasma where diffusion is maintained by nonlinear wave particle interactions as described in this paper. \( L_\nu \) grows together with the anomalous diffusion coefficient until finally approaching its classical value when diffusion becomes collisional at time \( t \geq \tau_\alpha \).

This is an interesting observation. Small (anomalous) resistive scales imply fast magnetic diffusion as observed in collisionless systems for instance undergoing reconnection. Since in collisionless plasma there is no resistive diffusion, one concludes that any process causing diffusion will strongly reduce the resistive scale and cause comparably fast dissipation of magnetic fields; such processes will, as for example, favor reconnection. Superdiffusion is such a process. It reduces the resistive scale to small values as long as the elapsed time \( t < \tau_c \), remains far below the resistive collision time.

Though this sounds trivial, it reflects the importance of anomalous processes in collisionless plasmas. The problem consists in finding an appropriate expression for the equivalent anomalous \( \nu_\alpha \) under collisionless conditions. Observations (Treumann et al., 1990; Bale et al., 2002) do not indicate any presence of the sufficiently high wave amplitudes in collisionless reconnection.
Fig. 2. Schematic hypothetical evolution of the diffusion coefficient for the case simulated in Figure 1 until the collisional classical diffusion state would have been reached. Time is measured here in classical collision times $\nu_c^{-1}$. Left: The anomalous increase of the diffusion coefficient with time. The growth of the diffusion coefficient gradually comes to rest after the classical collision time has elapsed. The diffusion coefficient becomes constant afterwards. Right: Time evolution of the average particle displacement. Displacement increases as in Figure 1. When approaching the classical collision time scattering of particles to both larger and smaller displacements widen the displacement range and leading to a reduced in crease until the second collision time. Similar after the third collision time. Finally the increase of the displacement becomes linear in time implying classical diffusion.

required (Sagdeev, 1966) for the quasilinear generation of anomalous resistances. Numerical particle-in-cell simulations (cf., Treumann & Baumjohann, 2013 for a recent review) confirmed instead that in all cases the main driver of fast collisionless reconnection is the electron «pseudo-viscosity» implied by the presence of non-diagonal terms (Hesse & Winske, 1998; Hesse et al., 1999) in the thermally anisotropic electron pressure tensor $P_e$ which arise when expressing $P_e$ in the stationary frame of the reconnecting current layer accounting for any subtle finite gyro-radius effects in the dynamics of electrons in the inhomogeneous magnetic field of the electron diffusion region where electrons perform bouncing Speiser orbits.

We can, however, proceed to finding an expression for the anomalous collision frequency $\nu_a$ that is equivalent to electron pseudo-viscosity. From thermodynamic transport theory (Huang, 1987), there is a well known relation between the volume viscosity $\mu_v$ (or kinematic viscosity $\mu_{kin} = \mu_v/m_eN_e$, with $N_e$ the plasma density and $T_e$ the relevant electron temperature for the pressure-tensor induced equivalent anomalous collision frequency which can be used in the above derived expressions for the respective resistive scales and superdiffusion coefficients. A more precise determination requires the transformation of the electron pressure tensor into the sheet current frame and using the exact relations between the gradients of the non-diagonal pressure tensor elements and the corresponding equivalent bulk and shear viscosities. This calculation is beyond the intention of the present paper.

Considered in this spirit collisionless reconnection can be interpreted as an equivalent local anomalous though nevertheless real super-diffusion process of the magnetic field in collisionless plasma, an interpretation which, from a general physical point of view, ultimately unifies collisionless reconnection and diffusion theories in very satisfactory concordance with fundamental electrodynamics.

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