FACTORIZATION BY ELEMENTARY MATRICES, NULL-HOMOTOPY AND PRODUCTS OF EXPONENTIALS FOR INVERTIBLE MATRICES OVER RINGS

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Abstract. Let \( R \) be a commutative unital ring. A well-known factorization problem is whether any matrix in \( \text{SL}_n(R) \) is a product of elementary matrices with entries in \( R \). To solve the problem, we use two approaches based on the notion of the Bass stable rank and on construction of a null-homotopy. Special attention is given to the case, where \( R \) is a ring or Banach algebra of holomorphic functions. Also, we consider a related problem on representation of a matrix in \( \text{GL}_n(R) \) as a product of exponentials.

1. Introduction

Let \( R \) be an associative, commutative, unital ring. A well-known factorization problem is whether any matrix in \( \text{SL}_n(R) \) is a product of elementary (equivalently, unipotent) matrices with entries in \( R \). Here the elementary matrices are those which have units on the diagonal and zeros outside the diagonal, except one non-zero entry. In particular, for \( n = 3, 4, \ldots \), Suslin [20] proved that the problem is solvable for the polynomials rings \( \mathbb{C}[\mathbb{C}^m], m \geq 1 \). For \( n = 2 \), the required factorization for \( R = \mathbb{C}[\mathbb{C}^m] \) does not always exist; the first counterexample was constructed by Cohn [4].

In the present paper, we primarily consider the case, where \( R \) is a functional Banach algebra. So, let \( \mathcal{O}(\mathbb{D}) \) denote the space of holomorphic functions on the unit disk \( \mathbb{D} \) of \( \mathbb{C} \). Recall that the disk-algebra \( A(\mathbb{D}) \) consists of \( f \in \mathcal{O}(\mathbb{D}) \) extendable up to continuous functions on the closed disk \( \mathbb{D} \). The disk-algebra \( A(\mathbb{D}) \) and the space \( H^\infty(\mathbb{D}) \) of bounded holomorphic functions on \( \mathbb{D} \) may serve as good working examples for the algebras under consideration.

In fact, we propose two approaches to the factorization problem. The first one is based on construction of a null-homotopy; see Section 2. This method applies to the disk-algebra and similar algebras. The second approach is applicable to rings whose Bass stable rank is equal to one; see Section 3. This method applies, in particular, to \( H^\infty(\mathbb{D}) \).

Also, the factorization problem is closely related to the following natural question: whether a matrix \( F \in \text{GL}_n(R) \) is representable as a product of exponentials, that is, \( F = \exp G_1 \ldots \exp G_k \) with \( G_j \in M_n(R) \). For \( n = 2 \) and matrices with entries in a Banach algebra, this question was recently considered in [13]. In Section 4 we obtain results related to this question with emphasis on the case, where \( R = \mathcal{O}(\Omega) \) and \( \Omega \) is an open Riemann surface.

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2. Factorization and null-homotopy

Given \( n \geq 2 \) and an associative, commutative, unital ring \( R \), let \( E_n(R) \) denote the set of those \( n \times n \) matrices which are representable as products of elementary matrices with entries in \( R \).

For a unital commutative Banach algebra \( R \), an element \( X \in SL_n(R) \) is said to be null-homotopic if \( X \) is homotopic to the unity matrix, that is, there exists a homotopy \( X_t : [0, 1] \to SL_n(R) \) such that \( X_1 = X \) and \( X_0 \) is the unity matrix.

We will use the following theorem:

**Theorem 1 (\cite{13} §7).** Let \( A \) be a unital commutative Banach algebra and let \( X \in SL_n(A) \). The following properties are equivalent:

(i) \( X \in E_n(A) \);
(ii) \( X \) is null-homotopic.

To give an illustration of Theorem 1 consider the disk-algebra \( A(\mathbb{D}) \).

**Corollary 1.** For \( n = 2, 3, \ldots, E_n(A(\mathbb{D})) = SL_n(A(\mathbb{D})) \).

**Proof.** We have to show that \( E_n(A(\mathbb{D})) \supset SL_n(A(\mathbb{D})) \). So, assume that

\[
F = F(z) = \begin{pmatrix} f_{11}(z) & f_{1n}(z) \\ \vdots & \ddots \\ f_{n1}(z) & f_{nn}(z) \end{pmatrix} \in SL_n(A(\mathbb{D})).
\]

Define

\[
F_t(z) = F(tz) \in SL_n(A(\mathbb{D})), \quad 0 \leq t \leq 1, \ z \in \mathbb{D}.
\]

Given an \( f \in A(\mathbb{D}) \), let \( f_t(z) = f(tz), 0 \leq t \leq 1, z \in \mathbb{D} \). Observe that \( \|f_t - f\|_{A(\mathbb{D})} \to 0 \) as \( t \to 1^- \). Applying this observation to the entries of \( F_t \), we conclude that \( F \) is homotopic to the constant matrix \( F(0) \). Since \( SL_n(\mathbb{C}) \) is path-connected, the constant matrix \( F(0) \) is homotopic to the unity matrix. So, it remains to apply Theorem 1. \( \square \)

3. Factorization and Bass stable rank

3.1. Definitions. Let \( R \) be a commutative unital ring. An element \((x_1, \ldots, x_k) \in R^k \) is called unimodular if

\[
\sum_{j=1}^k x_j R = R.
\]

Let \( U_k(R) \) the set of all unimodular elements in \( R^k \).

An element \( x = (x_1, \ldots, x_{k+1}) \in U_{k+1}(R) \) is called reducible if there exists \((y_1, \ldots, y_k) \in R^k \) such that

\[
(x_1 + y_1x_{k+1}, \ldots, x_k + y_kx_{k+1}) \in U_k(R).
\]

The **Bass stable rank** of \( R \), denoted by \( \text{bsr}(R) \) and introduced in \cite{1}, is the least \( k \in \mathbb{N} \) such that every \( x \in U_{k+1}(R) \) is reducible. If there is no such \( k \in \mathbb{N} \), then we set \( \text{bsr}(R) = \infty \).

**Remark 1.** The identity \( \text{bsr}(R) = 1 \) is equivalent to the following property: For any \( x_1, x_2 \in R \) such that \( x_1 R + x_2 R = R \), there exists \( y \in R \) such that \( x_1 + yx_2 \in R^\circ \).
3.2. A sufficient condition for factorization.

**Theorem 2.** Let $R$ be a unital commutative ring and $n \geq 2$. If $\text{bsr}(R) = 1$, then $E_n(R) = \text{SL}_n(R)$.

**Proof.** First, assume that $n = 2$. Let

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \text{SL}_2(R).$$

Since $\det X = 1$, we have

$$x_{21}R + x_{11}R = R.$$

Hence, using the assumption $\text{bsr}(X) = 1$ and Remark 1, we conclude that there exists $y \in R$ such that

$$\alpha = x_{21} + yx_{11} \in R^*.$$  

(3.1)

Now, we have

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} X = \begin{pmatrix} x_{11} & x_{12} \\ \alpha & * \end{pmatrix}.$$  

Next, using (3.1) we obtain

$$\begin{pmatrix} 1 & (1 - x_{11})\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \alpha & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ \alpha & * \end{pmatrix}.$$  

Finally, we have

$$\begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ \alpha & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & x_0 \end{pmatrix}.$$  

Since the determinant of the last matrix is equal to one, we conclude that $x_0 = 1$. Therefore, the $X$ is representable as a product of four multipliers.

For $n \geq 3$, let

$$X = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \in \text{SL}_n(R).$$

Since $\det X = 1$, there exist $\alpha_1, \ldots, \alpha_n \in R$ such that $\alpha_1x_{11} + \cdots + \alpha_{n-1}x_{n-11} + \alpha_n x_{n1} = 1$. Therefore,

$$x_{n1}R + \left( \sum_{i=1}^{n-1} \alpha_ix_{i1} \right) R = R.$$  

Applying the property $\text{bsr}R = 1$, we obtain $y \in R$ such that

$$x_{n1} + y \left( \sum_{i=1}^{n-1} \alpha_ix_{i1} \right) := \alpha \in R^*.$$  

Put

$$L = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \ddots & \ddots \\ \alpha_1y & \ldots & \alpha_{n-1}y & 1 \end{pmatrix}.$$
Then
\[
LX = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n-1} \\ \alpha \end{pmatrix}.
\]

Multiplying by the upper triangular matrix
\[
U_1 = \begin{pmatrix} 1 & 0 & (1-x_{11})\alpha^{-1} \\ 1 & 0 & -x_{21}\alpha^{-1} \\ 0 & \ddots & \ddots \\ 0 & \ddots & 1 & -x_{n-11}\alpha^{-1} \\ \end{pmatrix},
\]
we obtain
\[
U_1 LX = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \alpha \end{pmatrix}.
\]

Now, put
\[
\tilde{L} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & \ddots \\ -\alpha & 0 & 1 \end{pmatrix}.
\]

We have
\[
\tilde{L} U_1 LX = \begin{pmatrix} 1 & * & * & * \\ 0 & \vdots & Y_1 \\ \end{pmatrix}.
\]

Observe that \(Y_1 \in SL_{n-1}(R)\). So, arguing by induction, we obtain
\[
\left( \prod_{i=1}^{n-1} \tilde{L}_i U_i L_i \right) X = \begin{pmatrix} 1 \\ \vdots \\ \alpha \end{pmatrix} := U
\]
or, equivalently,
\[
\left( \prod_{i=1}^{n-1} \mathcal{L}_i U_i \right) L_{n-1} X = U,
\]
where \(\mathcal{L}_i\) are lower triangular matrices. So, we conclude that every \(X \in SL_n(R)\) is a product of \(2n\) unipotent upper and lower triangular matrices. \(\square\)

**Corollary 2.** Let \(A\) be a unital commutative Banach algebra such that \(bsr(A) = 1\). If \(X \in SL_n(A)\), then \(X\) is null-homotopic.

**Proof.** It suffices to combine Theorems [1] and [2] \(\square\)
3.3. Examples of algebras $A$ with $\text{bsr}(A) = 1$.

3.3.1. Disk-algebra $A(\mathbb{D})$. By Corollary 1, $E_n(A(\mathbb{D})) = \text{SL}_n(A(\mathbb{D}))$. Theorem 2 provides a different proof of this property. Indeed, Jones, Marshall and Wolff and Corach and Suárez proved that $\text{bsr}(A(\mathbb{D})) = 1$, so Theorem 2 applies.

3.3.2. Algebra $H^\infty(\mathbb{D})$. Let $f \in H^\infty(\mathbb{D})$. If $\|f_r - f\|_\infty \to 0$ as $r \to 1^-$, then clearly $f \in A(\mathbb{D})$. So the homotopy argument used for $A(\mathbb{D})$ is not applicable to $H^\infty(\mathbb{D})$. However, Treil proved that $\text{bsr}(H^\infty(\mathbb{D})) = 1$, hence, Theorem 2 holds for $R = H^\infty(\mathbb{D})$. Also, Corollary 2 guarantees that any $F \in \text{SL}_n(H^\infty(\mathbb{D}))$ is null-homotopic.

3.3.3. Generalizations of $H^\infty(\mathbb{D})$. Tolokonnikov proved that $\text{bsr}(H^\infty(G)) = 1$ for any finitely connected open Riemann surface $G$ and for certain infinitely connected planar domains $G$ (Behrens domains). In particular, any $F \in \text{SL}_n(H^\infty(G))$ is null-homotopic. However, even in the case $G = \mathbb{D}$ the homotopy in question is not explicit. So, probably it would be interesting to give a more explicit construction of the required homotopy.

Let $T = \partial \mathbb{D}$ denote the unit circle. Given a function $f \in H^\infty(\mathbb{D})$, it is well-known that the radial limit $\lim_{r \to 1^-} f(r\zeta)$ exists for almost all $\zeta \in T$ with respect to Lebesgue measure on $T$. So, let $H^\infty(T)$ denote the space of the corresponding radial values. It is known that $\text{bsr}(H^\infty(T) + C(T))$ is an algebra, moreover, $\text{bsr}(H^\infty(T) + C(T)) = 1$; see [13].

Now, let $B$ denote a Blaschke product in $\mathbb{D}$. Then $C + BH^\infty(\mathbb{D})$ is an algebra. It is proved in [16] that $\text{bsr}(C + BH^\infty(\mathbb{D})) = 1$.

3.4. Examples of algebras $A$ with $\text{bsr}(A) > 1$.

3.4.1. Algebra $A_{\mathbb{R}}(\mathbb{D})$. Each element $f$ of the disk-algebra $A(\mathbb{D})$ has a unique representation

$$f(z) = \sum_{j=0}^{\infty} a_j z^j, \quad z \in \mathbb{D}. \tag{3.2}$$

The space $A_{\mathbb{R}}(\mathbb{D})$ consists of those $f \in A(\mathbb{D})$ for which $a_j \in \mathbb{R}$ for all $j = 0, 1, \ldots$ in (3.2). As shown in [17], $\text{bsr}(A_{\mathbb{R}}(\mathbb{D})) = 2$. Nevertheless, the following result holds.

**Proposition 1.** For $n = 2, 3, \ldots$, $E_n(A_{\mathbb{R}}(\mathbb{D})) = \text{SL}_n(A_{\mathbb{R}}(\mathbb{D}))$.

**Proof.** For a function $f \in A_{\mathbb{R}}(\mathbb{D})$, we have $f_t \in A_{\mathbb{R}}(\mathbb{D})$ or all $0 \leq t < 1$. Hence, given a matrix $F \in \text{SL}_n(A_{\mathbb{R}}(\mathbb{D}))$, we have $F_t \in \text{SL}_n(A_{\mathbb{R}}(\mathbb{D}))$, where $F_t$ is defined by $\text{bsr}(A_{\mathbb{R}}(\mathbb{D})) = 2$. Since $\|f_t - f\|_{A_{\mathbb{R}}(\mathbb{D})} \to 0$ as $t \to 1^-$, $F$ is homotopic to the constant matrix $F_0 \in \text{SL}_n(\mathbb{C})$. Hence, $F$ is homotopic to the unity matrix. Therefore, $F \in E_n(A_{\mathbb{R}}(\mathbb{D}))$ by Theorem 1.

3.4.2. Ball algebra $A(B^m)$, polydisk algebra $A(D^m)$, $m \geq 2$, and infinite polydisk algebra $A(D^\infty)$. Let $B^m$ denote the unit ball of $\mathbb{C}^m$, $m \geq 2$. The ball algebra $A(B^m)$ and the polydisk algebra $A(D^m)$ are defined analogously to the disk-algebra $A(\mathbb{D})$. By [6] Corollary 3.13,

$$\text{bsr}(A(B^m)) = \text{bsr}(A(D^m)) = \left\lceil \frac{m}{2} \right\rceil + 1, \quad m \geq 2.$$
Proposition 2. Let \( n = 2, 3, \ldots \). Then

\[
E_n(A(B^m)) = \text{SL}_n(A(B^m)), \quad m = 2, 3, \ldots, \infty,
\]

\[
E_n(A(D^m)) = \text{SL}_n(A(D^m)), \quad m = 2, 3, \ldots, \infty.
\]

Proof. It suffices to repeat the argument used in the proof of Corollary 1 or Proposition 1.

3.4.3. Algebra \( H_\infty^\infty(\mathbb{D}) \). It is proved in [17] that \( \text{bsr}(H_\infty^\infty(\mathbb{D})) = 2 \). We have not been able to determine the connected component of the identity in \( \text{SL}_n(H_\infty^\infty(\mathbb{D})) \).

Problem 1. Is any element in \( \text{SL}_n(H_\infty^\infty(\mathbb{D})) \) null-homotopic?

4. Invertible matrices as products of exponentials

Let \( R \) be a commutative unital ring. In the present section, we address the following problem: whether a matrix \( F \in \text{GL}_n(R) \) is representable as a product of exponentials, that is, \( F = \exp G_1 \ldots \exp G_k \) with \( G_j \in M_n(R) \). For \( n = 2 \) and matrices with entries in a Banach algebra, this problem was recently studied in [15].

4.1. Basic results. There is a direct relation between the problem under consideration and factorization of matrices in \( \text{GL}_n(R) \).

Lemma 1. Let \( X \in \text{SL}_n(R) \) be a unipotent upper or lower triangular matrix. Then \( X \) is an exponential.

Proof. For \( n = 2 \), we have

\[
\exp \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.
\]

Let \( n \geq 3 \). Given \( \alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots; \gamma_1, \gamma_2, \ldots \), we will find \( \alpha_1, \alpha_2, \ldots; b_1, b_2, \ldots; c_1, c_2, \ldots \) such that

\[
\begin{pmatrix}
1 & \alpha_1 & \alpha_2 & \alpha_3 & \ldots \\
1 & \beta_1 & \beta_2 & & \\
1 & \gamma_1 & & & \\
0 & 1 & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{pmatrix}
= \exp
\begin{pmatrix}
0 & a_1 & a_2 & a_3 & \ldots \\
0 & b_1 & b_2 & & \\
0 & c_1 & & & \\
0 & 0 & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{pmatrix}.
\]

Put \( a_1 = \alpha_1, b_1 = \beta_1, \ldots \). Next, we have \( a_2 = \alpha_2 - f(a_1, b_1) = \alpha_2 - f(\alpha_1, \beta_1) \). Analogously, we find \( b_2, c_2, \ldots \). To find \( \alpha_3 \), observe that \( \alpha_3 = \alpha_3 - f(\alpha_1, \alpha_2, b_1, c_2) \).

Since \( f \) depends on \( a_i, b_i, c_i \) with \( i < 3 \), we obtain \( \alpha_3 = \alpha_3 - f(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) \), and the procedure continues. So, the equation under consideration is solvable for any \( \alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots \).

Corollary 3. Assume that \( \text{SL}_n(R) = E_n(R) \) and every element in \( E_n(R) \) is a product of \( N(R) \) unipotent upper or lower triangular matrices. Then every element in \( \text{SL}_n(R) \) is a product of \( N(R) \) exponentials.
Corollary 4. Let the assumptions of Corollary 3 hold. Suppose in addition that every invertible element in \( R \) admits a logarithm. Then every \( X \in \text{GL}_n(R) \) is a product of \( \text{N}(R) \) exponentials.

Proof. Let \( X \in \text{GL}_n(R) \). So, \( \det X \in R^* \) and \( \ln \det X \) is defined. Therefore, \( \det X = f^n \) for appropriate \( f \in R^* \) and
\[
\begin{pmatrix}
 f^{-1} & 0 \\
 \ddots & \ddots \\
 0 & f^{-1}
\end{pmatrix} X \in \text{SL}_n(R).
\]

Applying Corollary 3, we obtain
\[
X = \begin{pmatrix} f & 0 \\ \vdots & \ddots \\ 0 & f \end{pmatrix} \exp Y_1 \ldots \exp Y_N
= \exp \left[ \begin{pmatrix} \ln f & 0 \\ 0 & \ddots \\ 0 & \ln f \end{pmatrix} + Y_1 \right] \exp Y_2 \ldots \exp Y_N,
\]
as required. \( \square \)

4.2. Rings of holomorphic functions on Stein spaces.

Corollary 5. Let \( \Omega \) be a Stein space of dimension \( k \) and let \( X \in \text{GL}_n(\mathcal{O}(\Omega)) \). Then there exists a number \( E(k, n) \) such that the following properties are equivalent:

(i) \( X \) is null-homotopic;
(ii) \( X \) is a product of \( E(k, n) \) exponentials.

Proof. By [10, Theorem 2.3], any null-homotopic \( F \in \text{SL}_n(\mathcal{O}(\Omega)) \) is a product of \( \text{N}(k, n) \) unipotent upper or lower triangular matrices. So, arguing as in the proof of Corollary 4, we conclude that (i) implies (ii) with \( E(k, n) \leq \text{N}(k, n) \) The reverse implication is straightforward. \( \square \)

The numbers \( \text{N}(k, n) \) are not known in general. If the dimension \( k \) of the Stein space is fixed, then the dependence of \( \text{N}(k, n) \) on the size \( n \) of the matrix is easier to handle. Certain \( K \)-theory arguments guarantee that the number of unipotent matrices needed for factorizing an element in \( \text{SL}_n(\mathcal{O}(\Omega)) \) is a non-increasing function of \( n \) (see [7]). So, as done in [3], combining the above property and results from [11], we obtain the following estimates:

\[
E(1, n) \leq \text{N}(1, n) = 4 \quad \text{for all } n,
\]
\[
E(2, n) \leq \text{N}(2, n) \leq 5 \quad \text{for all } n \geq n(k).
\]

for each \( k \), there exists \( n(k) \) such that \( E(k, n) \leq \text{N}(k, n) \leq 6 \) for all \( n \geq n(k) \).

In Section 4.4, we in fact improve on that: we show \( E(1, 2) \leq 3 \). In general, it seems that the number of exponentials \( E(k, n) \) to factorize an element in \( \text{GL}_n(\mathcal{O}(\Omega)) \) is less than the number \( \text{N}(k, n) \) needed to write an element in \( \text{SL}_n(\mathcal{O}(\Omega)) \) as a product of unipotent upper or lower triangular matrices.

Also, remark that (ii) implies (i) in Corollary 5 for any algebra \( R \) in the place of the ring of holomorphic functions. Assume that the algebra \( R \) has a topology. Then a topology on \( \text{GL}_n(R) \) is naturally induced and the implication (i)\( \Rightarrow \)(ii) means that
any product of exponentials is contained in the connected component of the identity (also known as the principal component) of \( \text{GL}_n(R) \). The reverse implication is a difficult question, even without a uniform bound on the number of exponentials.

4.3. Rings \( R \) with \( \text{bsr}(R) = 1 \). Combining Theorem 2 and Corollary 4 we recover a more general version of Theorem 7.1(3) from [15], where \( R \) is assumed to be a Banach algebra. Moreover, we obtain similar results for larger size matrices.

**Corollary 6.** Let \( R \) be a commutative unital ring, \( \text{bsr} R = 1 \), and let every \( x \in R^* \) admit a logarithm. Then every element in \( \text{GL}_2(R) \) is a product of 4 exponentials.

**Corollary 7.** Let \( R \) be a commutative unital ring, \( \text{bsr} R = 1 \), and let every \( x \in R^* \) admit a logarithm. Then every element in \( \text{GL}_n(R), \ n \geq 3 \), is a product of 6 exponentials.

**Proof.** For \( n = 3 \), it suffices to combine Theorem 2 and Corollary 4.

Now, assume that \( n \geq 4 \). Let \( ut_m \) denote the number of unipotent matrices needed to factorize any element in \( \text{SL}_m(R) \) starting with an upper triangular matrix. Theorem 20(b) in [7] says that any element in \( \text{SL}_n(R) \) is a product of 6 exponentials for

\[
 n \geq \min \left( m \left[ \frac{ut_m(R) + 1}{2} \right] \right),
\]

where the minimum is taken over all \( m \geq \text{bsr} R + 1 \). In our case the minimum is taken over \( m \geq 2 \) and the number \( ut_2(R) = 4 \) by the proof of Theorem 2. Since \( n \geq 4 \), the proof is finished. \( \square \)

Corollary 6 applies to the disk algebra and also to the rings \( \mathcal{O}(\mathbb{C}) \) and \( \mathcal{O}(\mathbb{D}) \) of holomorphic functions. Indeed, the identity \( \text{bsr}(\mathcal{O}(\Omega)) = 1 \) for an open Riemann surface follows from the strengthening of the classical Wedderburn lemma (see [19], Chapter 6, Section 3); see also [10] or [2]). However, for \( R = \mathcal{O}(\mathbb{C}) \) and \( R = \mathcal{O}(\mathbb{D}) \), the number 4 is not optimal; see Section 4.4 below. Also, it is known that the optimal number is at least 2 (see [15]). So, we arrive at the following natural question:

**Problem 2.** Is any element of \( \text{GL}_2(\mathcal{O}(\mathbb{D})) \) or \( \text{GL}_2(\mathcal{O}(\mathbb{C})) \) a product of two exponentials?

4.4. Products of 3 exponentials. In this section, we prove the following result.

**Proposition 3.** Let \( \Omega \) be an open Riemann surface. Then every element in \( \text{SL}_2(\mathcal{O}(\Omega)) \) is a product of 3 exponentials.

We will need several auxiliary results. The first theorem is a classical one [8].

**Theorem 3** (Mittag-Leffler Interpolation Theorem). Let \( \Omega \) be an open Riemann surface and let \( \{z_i\}_{i=1}^{\infty} \) be a discrete closed subset of \( \Omega \). Assume that a finite jet

\[
 J_i(z) = \sum_{j=1}^{N_i} b_j^{(i)} (z - z_i)^j
\]

is defined in some local coordinates for every point \( z_i \). Then there exists \( f \in \mathcal{O}(\Omega) \) such that

\[
 f(z) - J_i(z) = o(|z - z_i|^{N_i}) \quad \text{as} \quad z \to z_i, \ i = 1, 2, \ldots.
\]
Corollary 8. Under assumptions of Theorem 3, suppose that $b^{(i)}_0 \neq 0$ for $i = 1, 2, \ldots$. Then there exist $f, g \in \mathcal{O}(\Omega)$ such that (4.2) holds and $f = e^g$.

Proof. Let $b_0 = b^{(i)}_0$ for some $i$. Since $b_0 \neq 0$, there exists a logarithm $\ln$ in a neighborhood of $b_0$. So, $\ln$ is a local biholomorphism which induces a bijection between jets of $f$ and $g := \ln f$.

In “modern” language, the proof of Corollary 8 uses the fact that $\mathbb{C}^*$ is an Oka manifold (we refer the interested reader to [9]). Thus for any Stein manifold $X$ and an analytic subset $Y \subset X$, a (jet of) holomorphic map $f : Y \to \mathbb{C}^*$ (along $Y$) extends to a holomorphic map $f : X \to \mathbb{C}^*$ if and only if it extends continuously. The obstruction for a continuous extension is an element of the relative homology group $H_2(X, Y, Z)$. Observe that, for any discrete subset $Y$ of a 1-dimensional Stein manifold $X$, we have $H_2(X, Y, Z) = 0$ because $H_2(X, Z) = H_1(Y, Z) = 0$. This is the point where the proof of Proposition 3 below breaks down when we replace the Riemann surface $\Omega$ by a Stein manifold of higher dimension. Even a nowhere vanishing continuous function $\alpha$, as in the proof, does not exist in general.

Lemma 2. Let $\Omega$ be an open Riemann surface and $X \in \text{GL}_2(\mathcal{O}(\Omega))$. Assume that $\lambda \in \mathcal{O}^*(\Omega)$ is the double eigenvalue of $X$ and $\det X$ has a logarithm in $\mathcal{O}(\Omega)$. Then $X$ is an exponential.

Proof. We consider two cases.

Case 1: $X(z)$ is a diagonal matrix for all $z \in \Omega$.

We have

$$X(z) = \begin{pmatrix} \lambda(z) & 0 \\ 0 & \lambda(z) \end{pmatrix} = \exp \begin{pmatrix} \alpha(z) & 0 \\ 0 & \alpha(z) \end{pmatrix},$$

Case 2: $X(z)$ is not identically diagonal.

Either the first or the second line in $X(z) - \lambda(z)I$, say $(h(z), g(z))$, is not identical zero. So,

$$v_1(z) = \begin{pmatrix} -g(z) \\ h(z) \end{pmatrix}$$

is a holomorphic eigenvector for $X(z)$ except those points $z \in \Omega$ for which $v_1(z) = 0$. Construct a function $f(z) \in \mathcal{O}(\Omega)$ such that its vanishing divisor is exactly $\min(\text{ord } g, \text{ord } h)$. Then

$$v(z) = \frac{1}{f(z)} v_1(z)$$

is a holomorphic eigenvector for $X(z)$, $z \in \Omega$.

Now, choose a matrix $P(z) \in \text{GL}_2(\mathcal{O}(\Omega))$ with first column $v(z)$. Then the matrix $P^{-1}(z)X(z)P(z)$ has the following form:

$$\begin{pmatrix} \lambda(z) & \beta(z) \\ 0 & \lambda(z) \end{pmatrix} = \exp \begin{pmatrix} \frac{1}{2} \gamma(z) & \frac{\beta(z)}{\lambda(z)} \\ 0 & \frac{1}{2} \gamma(z) \end{pmatrix}$$

Thus,

$$X(z) = \exp P(z) \begin{pmatrix} \frac{1}{2} \gamma(z) & \frac{\beta(z)}{\lambda(z)} \\ 0 & \frac{1}{2} \gamma(z) \end{pmatrix} P^{-1}(z),$$

as required. \qed
Proof of Proposition 3. Let
\[ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \]
that is, \( ad - bc = 1 \). We are looking for \( \alpha \in \mathbb{R}^* \) and \( \beta \in \mathbb{R} \) such that the matrix
\[ X \begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^2a & \beta a + b \\ \alpha^2c & \beta c + d \end{pmatrix} := Y \]
has a double eigenvalue.

Case 1: \( c = 0 \). We have
\[ X = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}. \]
It suffice to observe that
\[ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} a^{-2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & b \\ 0 & a^{-1} \end{pmatrix} \]
has the double eigenvalue \( a^{-1} \).

Case 2: \( c \neq 0 \). The matrix \( Y \) has a double eigenvalue if \( 4 \det Y = (\text{tr} Y)^2 \), that is,
\[(\alpha^2a + \beta c + d)^2 = 4\alpha^2.\]
Put
\[ \beta = \frac{2\alpha - a\alpha^2 - d}{c}. \]
Clearly, \( \beta \) is a formal solution of (4.3). Below we show how to construct \( \alpha(z) = \exp(\tilde{\alpha}(z)) \in \mathcal{O}^*(\Omega) \) such that \( \beta \) is holomorphic.

Let \( \{z_i\} \subset \Omega \) be the zero set of \( c(z) \). Fix \( i \) and \( z_i \in \Omega \). Let \( c(z_i) = \cdots = c^{(k)}(z_i) = 0 \), and \( c^{(k+1)}(z_i) \neq 0 \). Observe that \( a(z_i) \neq 0 \). So, define \( \alpha(z) \), in a neighborhood of \( z_i \), as \( 1/a(z) \) up to a sufficiently high order, namely,
\[(\alpha(z)) = 1 + (z - z_i)^k h(z),\]
where \( h(z) \) is holomorphic in a neighborhood of \( z_i \). Since \( ad - bc = 1 \), we have \( 1 - ad = (z - z_i)^k g(z) \). Therefore,
\[ 2a\alpha - a^2\alpha^2 - ad = -(1 - a\alpha^2)^2 + 1 - ad \]
\[ = -(z - z_0)^{2k} h^2(z) + (z - z_0)^k g(z) \]
vansishes of order \( k \) at \( z_i \). Hence, \( 2\alpha - a\alpha^2 - d \) also vanishes of order \( k \) at \( z_i \).

So, we have constructed \( \alpha(z) \) locally as finite jets \( J_i(z) \) defined by (4.1) with \( b^{(i)}_0 \neq 0 \) in some local coordinates for every point \( z_i \), \( i = 1, 2, \ldots \). Now, Corollary 8 provides \( \tilde{\alpha} \in \mathcal{O}(\Omega) \) such that \( \alpha(z) = \exp(\tilde{\alpha}(z)) \in \mathcal{O}^*(\Omega) \) and (4.4) holds. Hence, \( \beta \) is holomorphic.

So, the matrix
\[ X \begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} := Y \]
has a double eigenvalue and \( \det Y \) admits a logarithm. Thus, applying Lemma 2, we conclude that \( Y \) is an exponential. To finish the proof of the proposition, it remains observe that
\[ \begin{pmatrix} \alpha^2 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \alpha & \beta \alpha^{-1} \\ 0 & \alpha \end{pmatrix}, \]
where both multipliers on the right hand side are exponentials. \( \square \)
Corollary 9. Let $X \in \text{GL}_2(\mathcal{O}(\Omega))$. The following properties are equivalent:

(i) $X$ is a product of 3 exponentials;
(ii) $\det X$ is an exponential;
(iii) $X$ is null-homotopic.

Proof. Clearly, (i)⇒(iii). Now, assume that $X$ is null-homotopic. Then $\det X$ is homotopic to the function $f \equiv 1$. Since $\exp : \mathbb{C} \to \mathbb{C}^\ast$ is a covering, we conclude that $\det X(z) = \exp(h(z))$ with $h \in \mathcal{O}(\Omega)$. So, (iii) implies (ii). The implication (ii)⇒(i) is standard; see, for example, the proof of Corollary 4. □

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