THE SHIFTED COUPLED CLUSTER METHOD: A NEW APPROACH TO HAMILTONIAN LATTICE GAUGE THEORIES

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Abstract

It is shown how to adapt the non-perturbative coupled cluster method of many-body theory so that it may be successfully applied to Hamiltonian lattice SU(N) gauge theories. The procedure involves first writing the wavefunctions for the vacuum and excited states in terms of linked clusters of gauge invariant excitations of the strong coupling vacuum. The fundamental approximation scheme then consists of i) a truncation of the infinite set of clusters in the wavefunctions according to their geometric size, with all larger clusters appearing in the Schrödinger equations simply discarded, ii) an expansion of the truncated wavefunctions in terms of the remaining clusters rearranged, or “shifted”, to describe gauge invariant fluctuations about their vacuum expectation values. The resulting non-linear truncated Schrödinger equations are then solved self-consistently and exactly. Results are presented for the case of SU(2) in $d = 3$ space-time dimensions.
This letter reports the first results of a new approach to Hamiltonian lattice gauge theories. The goal is to find approximate non-perturbative solutions of the Schrödinger equation for both vacuum and excited states. Space is treated as a discrete lattice in order to i) reduce field theory to \( n \to \infty \) body quantum mechanics, ii) render the theory ultraviolet finite, and iii) make trivial the imposition of Gauss’s law, which must be imposed as a subsidiary condition on the states in an \( A_0 = 0 \) gauge, which is the natural choice with a Hamiltonian approach.\(^1\)

The basic variables are the \( SU(N) \) matrices \( U(L) \) defined on each link \( L \). Conjugate “chromoelectric” fields \( E^a(L), a = 1 \ldots N^2 - 1 \), are given by\(^2\)

\[
E^a(L) = -\left(T^a U(L)\right)_{ij} \frac{\partial}{\partial U(L)_{ij}} + \left(U^\dagger(L)T^a\right)_{ij} \frac{\partial}{\partial U^\dagger(L)_{ij}}
\]

in Schrödinger representation, where \( T^a \) are the group generators. The Hamiltonian for \( SU(N) \) in \( d \) space-time dimensions, without quarks, may be written\(^2\)

\[
H(x) = a^{-1}x^{-1/2}W(x)
\]

where the dimensionless variable \( x \) is related to the coupling \( g \) by \( x = 4g^{-4}a^{2d-8} \), and

\[
W(x) = \sum_L E^a(L)^2 - x \sum_p \text{Re} U_p
\]

where \( U_p \) is the trace of the product of \( U \)'s and \( U^\dagger \)'s around the four sides of a plaquette \( p: U_p = \text{Tr}U(1)U(2)U^\dagger(3)U^\dagger(4) \). The Schrödinger equation for the “reduced” energy \( w = ax^{1/2}E \) is then

\[
W\Psi = w\Psi
\]

The continuum limit occurs at \( x \to \infty \).

We consider first perturbation theory about the strong coupling \( x = 0 \) vacuum, but emphasize that the method we shall develop applies at \( \textit{all} \) couplings \( 0 \leq x < x_c \), where \(^1\) To avoid ghosts, a non-covariant gauge is necessary, which must be \( A_0 = 0 \) to retain rotational invariance. In contrast to QED, Gauss’s law cannot be solved explicitly.\(^2\) This operation, in which the elements of the unitary matrices \( U(L) \) are treated as if they were independent, and with the indices \( i, j \) summed over, correctly reproduces the fundamental commutation relations for \( E^a(L) \) with \( U(L’), U^\dagger(L’) \) and \( E^a(L’) \).
$x_c$ is the lowest $x$ at which there is a phase transition. For $SU(N)$, it is expected that $x_c = \infty$ i.e. the continuum limit. At $x \ll 1$, the vacuum wavefunction $\Psi_0$ of, for example, the $SU(2)$ theory in $d = 3$ is

$$
\Psi_0 = 1 + x \left( \frac{1}{3} \sum_s \square \right) + x^2 \left( \frac{1}{15} \sum_{s, s'} \square \square \sum_s \square - \frac{1}{12} \sum_s \square \square - \frac{1}{17} \sum_s \left( \square \square + \square \right) + \frac{8}{357} \sum_s \left( \square \square + \square \right) \right) + x^3 \left( \frac{1}{102} \sum_s \square \square \sum_{s, s'} \square \square \sum_{s''} \square \square - \frac{1}{216} \sum_s \square \square \sum_{s, s'} \square \square \sum_s \square \square + \ldots - \frac{881}{63180} \sum_s \square \right) + \ldots
$$

(5)

where the sums are over all sites $s$ on the lattice. As shown originally by Hubbard in the case of continuum field theory, and observed by Greensite for lattice theories, the products of sums of independent excitations exponentiate:

$$
\Psi_0 = \exp S
$$

(6)

For all couplings $0 \leq x < x_c$, and for general $SU(N)$ and $d$, the exponential form (6) holds, where the function $S$ is a single sum over the sites $s$ of the lattice of all possible linearly independent “linked clusters” $C_{i,s}$ of $SU(N)$ Wilson loops:

$$
S = \sum_s \sum_i a_i C_{i,s}
$$

(7)

By “linked” is meant products of Wilson loops at fixed relative separation and orientation. The “amplitudes” $a_i$ are then the coefficients of these clusters, summed to all orders in strong coupling perturbation theory. The representation (6),(7) holds, and is exact, provided the true vacuum is not orthogonal to the strong coupling vacuum $\Psi_0(x = 0)$ i.e. up to the lowest $x$ at which there is a phase transition. Thus, for any $N$ and $d$, the Schrödinger equation for the vacuum energy is equivalent to

$$
\sum_L \left( E^a(L)^2 S + E^a(L)S \cdot E^a(L)S \right) - x \sum_p \text{Re} U_p = \sum_s w_0
$$

(8)

The wavefunctions of excited states (glueballs) with zero momentum can be written

$$
\Psi_g = G \exp S
$$

(9)
where
\[ G = \sum_s \left( b_0 + \sum_i b_i C_{i,s} \right) \] (10)
i.e. \( G \) is a similar sum\(^3\) over the lattice of linked clusters of \( SU(N) \) Wilson loops. The corresponding Schrödinger equation is then
\[ \sum_L \left( E^a(L)^2 G + 2E^a(L)S \cdot E^a(L)G \right) = (w_g - w_0)G \] (11)

The effect of the electric field operators on a given link \( L \) in (8), (11) is to recombine the clusters in \( S \) and \( G \) containing \( U(L) \) or \( U^\dagger(L) \) according to the relation for \( SU(N) \) generators \( T^a_{\alpha\beta}T^a_{\gamma\delta} = \frac{1}{2}\delta_{\alpha\gamma}\delta_{\beta\delta} - \frac{1}{2N}\delta_{\alpha\beta}\delta_{\gamma\delta} \). Some examples for general \( SU(N) \) are
\[ E^a(L)^2 = \frac{N^2-1}{2N} \quad E^a(L) \cdot E^a(L) = \frac{1}{2} - \frac{1}{2N} \]
where the dots indicate the ends of the link \( L \).

The exponential form (6) for the vacuum wavefunction is in fact a very general expression for the fully interacting ground state of a many-body system, with \( S \) a sum of linked clusters each of which describes linked excitations of a bare or unperturbed state \( \Psi_0(x=0) \). The exponentiation then gives the correct statistical weighting for unlinked excitations consisting of products of linked components. Furthermore, it guarantees that the Schrödinger equation for the vacuum and for excited states is a single sum of linked terms only, so that there are then no size extensivity problems with approximation schemes. The expression (6) is the starting point for the many-body theory coupled cluster method\(^4\). This is an intrinsically non-perturbative approach, involving the amplitudes \( a_i, b_i \) directly rather than their perturbative expansions.

In order to apply the coupled cluster method to lattice \( SU(N) \) gauge theories, the fundamental approximation in general consists of two steps:

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\(^3\) Excited states of a given spin and parity are constructed through the appropriate dependence of the \( b_i \) on the clusters' rotation/reflection orientations, left implicit in (10).

\(^4\) For an introductory review of this method, which has been widely used in nuclear and condensed matter physics and quantum chemistry, see ref. [5]. Application to lattice gauge theory was suggested by Greensite [4], but this suggestion was not followed up.
I) A truncation of the sums of clusters in $S$ and $G$ according to the clusters’ geometrical size. All clusters then formed in the Schrödinger equations with the truncated $S$ and $G$ which are larger than the given cut-off are simply discarded.

II) A Taylor expansion of $S$ and $G$ in the remaining clusters, rearranged, or “shifted”, to describe fluctuations of the individual Wilson loops about their vacuum expectation values. These expectation values are calculated self-consistently from the vacuum wavefunction.

The first step was originally suggested by Greensite [4], but in order for the method to provide a tractable, valid and efficient approximation scheme beyond the perturbative range of couplings, the second step is in general essential. The physical idea is that amplitudes in $S$ involving fluctuations on scales larger than the vacuum correlation length $\xi$, and in $G$ larger than the size $\rho$ of the glueball, are expected to be greatly reduced relative to those for fluctuations smaller than $\xi$ and $\rho$, and to decrease very rapidly with increasing size. For fixed lattice spacing, the results should therefore converge rapidly to the exact result as the dimensions of the cut-off cluster size are increased beyond $\xi$ and $\rho$. As the lattice spacing $a \to 0$, $\xi/a$ and $\rho/a$ diverge and the size in lattice units of clusters required to describe the theory must diverge correspondingly. The success of the method therefore depends on the results obtained at successsively increasing values of $x$ (i.e. decreasing $a$) showing a sufficiently rapid approach to the expected scaling behaviour that the continuum limit can be extrapolated.

We consider first the case of $SU(2)$ with a one-plaquette cut-off, for which step I is exactly soluble. For $SU(2)$, the identity $\text{Tr}A \text{Tr}B = \text{Tr}AB + \text{Tr}AB^{-1}$ for $SU(2)$ matrices $A, B$ gives linear relations between clusters e.g.

\[
\begin{align*}
\begin{array}{c}
\hline
\hline
\end{array} &= \begin{array}{c}
\hline
\hline
\end{array} + \begin{array}{c}
\hline
\hline
\end{array} \\
\begin{array}{c}
\hline
\hline
\end{array} &= 2 + \begin{array}{c}
\hline
\hline
\end{array}
\end{align*}
\]

Thus, the one-plaquette clusters may be written as “powers” of the one-plaquette single loop $C_{1,p}$. Then, with the one-plaquette cut-off $S$, writing the infinite sum of one-plaquette clusters on each $p$ as a function $S_p(C_{1,p})$

\[
S = \sum_p S_p(C_{1,p})
\]
Schrödinger’s equation for the vacuum becomes the differential equation

\[
\sum_p \left\{ \frac{dS_p}{dC_{1,p}} 3C_{1,p} + \left( \frac{d^2 S_p}{dC^2_{1,p}} + \left( \frac{dS_p}{dC_{1,p}} \right)^2 \right) \left( C_{1,p}^2 - 4 \right) - xC_{1,p} \right\} = \sum_s w_0
\]

(13)

where terms involving clusters spanning adjacent plaquettes \( p, p' \)

\[
\frac{dS_p}{dC_{1,p}} \frac{dS_{p'}}{dC_{1,p'}} E^a C_{1,p} \cdot E^a C_{1,p'}
\]

(14)

have been discarded. The solution is

\[
S_p = \ln \left( \left( 1 - \frac{1}{4} C_{1,p}^2 \right)^{-1/2} se_2 \left( -\frac{1}{2} \cos^{-1} \left( -\frac{1}{2} C_{1,p} \right) \right) \right)
\]

(15)

where \( se_2 \) is a Mathieu function \[10\]. The associated vacuum energy per site \( w_0 \) for \( d = 3 \) is plotted in fig. 1, together with Padé approximants obtained from the strong coupling series to \( O(x^{14}) \) \[11\] with which it shows very remarkable agreement at all \( x \). Furthermore, using the Feynman-Hellman theorem, the vacuum expectation value \( \langle C_1 \rangle = -dw_0/dx \), has the correct weak coupling behaviour

\[
\langle C_1 \rangle = 2 - O(x^{-1/2}) \quad x \gg 1
\]

(16)

The first excited state involves the Mathieu function \( se_4 \). The mass gap \( \Delta w_g = w_g - w_0 \) for \( d = 3 \) is shown plotted against \( x^{-1/2} \) in fig. 2, together with Padé approximants from \( O(x^{10}) \) series \[11\]. In \( d = 3 \), the \( SU(2) \) theory is superrenormalizable and so the true (reduced) mass gap \( \Delta w_g \) is expected to go to a constant at the continuum limit \( x \to \infty \).

At weak coupling, the Mathieu mass gap diverges as \( \Delta w_g = 4x^{1/2} - \frac{5}{4} + O(x^{-1/2}) \). This \( x^{1/2} \) behaviour is as expected from the uncertainty principle with a wavefunction involving excitations spanning only a finite number of lattice spacings.

In general, increasing the cut-off size beyond one plaquette or going beyond \( SU(2) \), the resulting partial differential equation for \( S_p \) is intractable. We therefore introduce the

\[5\] Eqn. (15), which is the exact solution for an \( SU(2) \) “one plaquette universe”, has been derived by many authors e.g. refs. \[6\], \[7\], and also as the \( d = 3 \) variational solution if \( \Psi \) is taken to be a product of functions of one plaquette variables \[8\]. However, the fact that it reproduces so well the results of ref. \[11\] for \( d = 3 \) has not been noticed previously. For \( d = 4 \) (for which a variational solution is not available) eqn. (15) still results in a vacuum energy which compares well with that obtained from Padé approximants derived from strong coupling series \[11\].
second step of the approximation scheme and rearrange the sums of clusters in the given cut-off $S$ and $G$ in terms of “shifted” linked clusters $C_{i,s}^\prime$:

$$S = \sum_s \sum_i a_i^\prime C_{i,s}^\prime \quad (17)$$

$$G = \sum_s \left( b_0^\prime + \sum_i b_i^\prime C_{i,s}^\prime \right) \quad (18)$$

These shifted linked clusters consist of products of Wilson loops each minus their vacuum expectation values, still at fixed relative separation and orientation e.g. for general $\text{SU}(N)$:

$$(\begin{array}{c} 1 \\ \hline \end{array})' \quad \text{with} \quad \begin{array}{c} 1 \\ \hline \end{array}' = \begin{array}{c} 1 \\ \hline \end{array} - \langle \begin{array}{c} 1 \\ \hline \end{array} \rangle, \quad \begin{array}{c} 0 \\ \hline \end{array}' = \begin{array}{c} 0 \\ \hline \end{array} - \langle \begin{array}{c} 0 \\ \hline \end{array} \rangle$$

To illustrate the method, we consider again the one-plaquette cut-off and include in $S$ and $G$ shifted one-plaquette clusters up to the $n$'th power of the shifted one-plaquette loop $C_{1,p}^\prime = C_{1,p} - \langle C_1 \rangle$. Substituting into (8) and (11), the resulting clusters spanning two plaquettes are discarded (step I) together with all “higher order” fluctuations $(C_{1,p}^\prime)^m$, $m > n$, (step II) and coefficients of the constant term and each shifted cluster $(C_{1,p}^\prime)^i$, $i = 1$ to $n$, are equated. For the vacuum, this gives a set of $n$ non-linear equations for the amplitudes $a_i^\prime$ and an expression for $w_0$, all in terms of $x$ and $\langle C_1 \rangle$. For the glueball, the procedure gives an $n \times n$ matrix equation for the $b_i^\prime$ with eigenvalues $\Delta w_g$, together with an equation for $b_0^\prime$, all in terms of the $a_i^\prime$ and $\langle C_1 \rangle$. These coupled cluster equations are then solved using numerical library routines, with $\langle C_1 \rangle$ calculated self-consistently.

The results for $w_0$ and $\Delta w_g$ obtained [12] with the one plaquette cut-off and the expansions of $S$ and $G$ truncated at $n = 1, 2, 3 \ldots$ converge rapidly to the Mathieu results at all $x$. Each successive power of $C_{1,p}^\prime$ generates the next term in the series expansions in $x$ for $x \ll 1$ and $x^{-1/2}$ for $x \gg 1$ of the Mathieu eigenvectors and eigenvalues. In particular, even at the very simplest $n = 1$ approximation the method gives the correct weak coupling leading behaviour (16) for the vacuum expectation value $\langle C_1 \rangle$.

Proceeding to larger cut-offs with $\text{SU}(2)$ in $d = 3$, we have included in $S$ and $G$ all the shifted clusters which, unshifted, occur in 2nd, 3rd and 4th order strong coupling perturbation theory (which is used solely as a guide to which terms to include - the resulting coupled cluster equations are all solved non-perturbatively). The complete set
Table 1. Vacuum energies $E_0$ for the first four orders of approximation in the shifted coupled cluster method, together with the results of Padé approximants derived from the $O(x^{14})$ strong coupling series taken from [11] (estimated errors in the last figure are given in brackets).

| $x$ | order 1       | order 2       | order 3       | order 4       | Padé [11] |
|-----|---------------|---------------|---------------|---------------|-----------|
| 0.5 | -0.106298     | -0.0843647    | -0.0827613    | -0.0822829    | -0.082156 |
| 1.0 | -0.386012     | -0.332963     | -0.322072     | -0.319032     | -0.31587  |
| 1.5 | -0.779896     | -0.707363     | -0.685324     | -0.681253     | -0.67115  |
| 2.0 | -1.25000      | -1.16553      | -1.13371      | -1.13132      | -1.1162 (2) |
| 2.5 | -1.77435      | -1.68175      | -1.64112      | -1.64197      | -1.626    |
| 3.0 | -2.33949      | -2.24077      | -2.19185      | -2.19667      | -2.183 (3) |
| 3.5 | -2.93665      | -2.83296      | -2.77600      | -2.78509      | -2.77     |
| 4.0 | -3.55974      | -3.45180      | -3.38692      | -3.40037      | -3.40 (2) |
| 4.5 | -4.20439      | -4.09265      | -4.01992      | -4.03771      | -         |
| 5.0 | -4.86730      | -4.75207      | -4.67154      | -4.69358      | -         |
| 10.0| -12.1254      | -11.9822      | -11.8255      | -11.8828      | -         |
| 20.0| -28.2197      | -28.0321      | -27.7362      | -27.8358      | -         |

of 69 linearly independent clusters occurring at 4th order are shown in the appendix. The loop expectation values are calculated self-consistently from the wavefunction using the Feynman-Hellman theorem by making the change

$$W \rightarrow W + \sum_s \sum_{\text{loops} \in S,G} \epsilon_i C_{i,s}$$

so that the $a'_i$ become functions of the $\epsilon_i$ and $x$. Then

$$\langle C_i \rangle = \frac{\partial w_0}{\partial \epsilon_i} \bigg|_{\epsilon_i=0}$$

These are then simple quantities to calculate numerically, avoiding any group integrals.

The results, obtained using a computer to carry out all the calculations [12], are given in table 1 for $w_0$ and shown plotted in fig. 3 for $\Delta w_g$. Assuming that the Padé approximants are a good guide to the exact results, the method gives excellent results for the vacuum energy $w_0$ at strong and intermediate couplings and the correct form
\[ w_0 = -2x + \mathcal{O}(x^{1/2}) \] at weak coupling. All the Wilson loop vacuum expectation values have the correct weak coupling form, as in (16). For the mass gap \( \Delta w_g \), reasonable results are obtained at intermediate couplings, before the inevitable \( x^{1/2} \) scaling sets in at weak coupling. In addition to the eigenvalues, the method also provides the wavefunctions and the cluster vacuum expectation values at all couplings. The amplitudes \( a_i, b_i \) of the most dominant of the 69 shifted clusters occurring in the wavefunctions at 4th order are shown in figs. 4 and 5 respectively. At weak coupling all the amplitudes behave as \( a'_i = a_i^{(1/2)} x^{1/2} + \mathcal{O}(\text{const.}) \), \( b'_i = b_i^{(0)} + \mathcal{O}(x^{-1/2}) \), with, for the 69 clusters occurring at 4th order, the \( a_i^{(1/2)} \) and \( b_i^{(0)} \) ranging over six orders of magnitude. For the vacuum, it is found that, at weak coupling, simple shifted clusters spanning few plaquettes and/or containing few Wilson loops dominate over those for complicated multi-loop clusters. This can be understood as a combinatorical effect in the coupled cluster equations. For the lowest (scalar) glueball, shifted clusters consisting of single Wilson loops dominate at intermediate and weak coupling. With the glueball calculation involving the diagonalization of a (large) matrix, this effect is harder to understand. No evidence is found of a phase transition at finite \( x \).

Returning to general \( SU(N) \), it can be shown \( [13] \) that the shifting procedure guarantees that, for any \( N \) and \( d \), the approximation scheme gives the correct weak coupling vacuum expectation value \( N + \mathcal{O}(x^{-1/2}) \) for a Wilson loop in \( S \). Thus, at \( x \to \infty \), the effect of the electric field operators coupling together e.g. a pair of shifted single loops becomes

\[
E^a(L)E^a(L') = \frac{1}{2} \left( \begin{array}{c} \begin{array}{c} a \end{array} \\
\begin{array}{c} a' \end{array} \end{array} \right) - \frac{1}{2N} \begin{array}{c} a \end{array} + \text{const. term} \mathcal{O}(x^{-1/2})
\]

It is straightforward to show that, in general, in the weak coupling, large \( N \) limit, only shifted single loops\( [13] \) survive in \( S \) and \( G \). Why this appears to be a good approximation at \( N = 2 \) remains unclear.

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\( ^{\text{6}} \)For \( SU(N) \), a single Wilson loop can “wrap around” itself up to \( N - 1 \) times before it becomes linearly dependent on clusters of simpler loops.
Conclusions

In general, the application of the coupled cluster method to lattice gauge theories depends crucially on the rearrangement, or shifting, of the clusters in $S$ and $G$ to describe gauge invariant fluctuations. At $x \ll 1$ the method matches on to perturbation theory\footnote{The method, without shifting, in fact provides an efficient way, avoiding any group integrals, of calculating perturbative expansions for both eigenvalues and eigenvectors due to the linked property of the Schrödinger equations.}, while at $x \gg 1$, it is always found that small, simple shifted clusters dominate in the wavefunctions over large, complicated shifted clusters due to combinatorical effects in the Schrödinger equations. Furthermore, the interpretation of the discarded terms in the Schrödinger equations as higher order fluctuations enables otherwise formidable problems involving linear dependences among clusters to be largely circumvented.

In summary, the shifted coupled cluster method provides a non-perturbative semi-analytic approach to lattice gauge theories, involving a direct physical approximation scheme, and gives non-perturbative information on both eigenvalues and eigenfunctions of the Schrödinger equations. The results presented here show (table 1) that the method converges very rapidly for the vacuum energy (assuming that the Padé results are a good guide) and should be capable of providing excellent approximations to the vacuum wavefunction. Further work is needed to establish how well it works for excited states; through a more judicious choice of clusters in $S$ and $G$, it should be possible to improve significantly upon the results presented here for the mass gap at intermediate couplings so that it can be reliably extrapolated to the weak coupling limit. The method therefore promises to provide much valuable analytic information on the behaviour of lattice gauge theories. Fuller accounts of this work will appear elsewhere\cite{13}.

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Appendix: Fourth Order (Unshifted) Clusters

\[ C_{1,s} = \quad C_{16,s} = \quad C_{34,s} = \quad C_{52,s} = \]
\[ C_{17,s} = \quad C_{35,s} = \quad C_{53,s} = \]
\[ C_{18,s} = \quad C_{36,s} = \quad C_{54,s} = \]
\[ C_{19,s} = \quad C_{37,s} = \quad C_{55,s} = \]
\[ C_{20,s} = \quad C_{38,s} = \quad C_{56,s} = \]
\[ C_{21,s} = \quad C_{39,s} = \quad C_{57,s} = \]
\[ C_{22,s} = \quad C_{40,s} = \quad C_{58,s} = \]
\[ C_{23,s} = \quad C_{41,s} = \quad C_{59,s} = \]
\[ C_{24,s} = \quad C_{42,s} = \quad C_{60,s} = \]
\[ C_{25,s} = \quad C_{43,s} = \quad C_{61,s} = \]
\[ C_{26,s} = \quad C_{44,s} = \quad C_{62,s} = \]
\[ C_{27,s} = \quad C_{45,s} = \quad C_{63,s} = \]
\[ C_{28,s} = \quad C_{46,s} = \quad C_{64,s} = \]
\[ C_{29,s} = \quad C_{47,s} = \quad C_{65,s} = \]
\[ C_{30,s} = \quad C_{48,s} = \quad C_{66,s} = \]
\[ C_{31,s} = \quad C_{49,s} = \quad C_{67,s} = \]
\[ C_{32,s} = \quad C_{50,s} = \quad C_{68,s} = \]
\[ C_{33,s} = \quad C_{51,s} = \quad C_{69,s} = \]
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Figure Captions

Fig. 1. Graph of the vacuum energy $w_0$ vs. $x$ for the exact Mathieu solution of the one-plaquette-cutoff coupled cluster equation (13), together with i) the Padé approximants derived from the $O(x^{14})$ strong coupling perturbation theory series taken from [11], and ii) the results from $O(x^2)$ and $O(x^4)$ strong coupling perturbation theory, which show that the Mathieu result is non-trivial.

Fig. 2. Graph of the mass gap $\Delta w_g$ vs. $x^{-1/2}$ for the exact Mathieu solution of the one-plaquette-cutoff coupled cluster equation, together with i) the Padé approximants derived from the $O(x^{10})$ strong coupling perturbation theory series taken from [11], and ii) the estimated bounds, shown dotted, for their extrapolation to the continuum limit $x \to \infty$, also from [11].

Fig. 3. Graph of the mass gap $\Delta w_g$ vs. $x^{-1/2}$ for the shifted coupled cluster wave-functions $\Psi_g = G \exp S$ with $S$, $G$ including the sets of clusters which, unshifted, occur at the first four orders of strong coupling perturbation theory. Also shown are the Padé approximants and continuum extrapolations derived from the $O(x^{10})$ strong coupling perturbation theory series taken from [11].

Fig. 4. Graph of the absolute values $|a'_i/a'_1|$ vs. $x$ for the amplitudes $a'_i$ of the most dominant of the 69 shifted clusters occuring in the vacuum wavefunction $\Psi_0 = \exp S$ at the 4th order approximation, where $a'_1$ is the amplitude of the shifted single plaquette loop $C'_1$. The cluster labels $i$ are shown on the right side of the figure.

Fig. 5. Graph of the amplitudes $b'_i$ vs. $x$ for the most dominant of the 69 shifted clusters occuring in the scalar glueball wavefunction $\Psi_g = G \exp S$ at the 4th order approximation. The cluster labels $i$ are shown on the right side of the figure.
Fig. 2

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--- Series Pades

\[ x^{1/2} \]

\[ B_M \]
