SMALL CORES IN 3-UNIFORM HYPERGRAPHS

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Abstract. The main result of this paper is that for any \( c > 0 \) and for large enough \( n \) if the number of edges in a 3-uniform hypergraph is at least \( cn^2 \) then there is a core (subgraph with minimum degree at least 2) on at most 15 vertices. We conjecture that our result is not sharp and 15 can be replaced by 9. Such an improvement seems to be out of reach, since it would imply the following case of a long-standing conjecture by Brown, Erdős, and Sós; if there is no set of 9 vertices that span at least 6 edges of a 3-uniform hypergraph then it is sparse.

1. Introduction

The subject of this paper is finding small cores in 3-uniform hypergraphs in terms of its number of edges. A core is a non-empty subgraph in which every vertex has degree at least two. We introduce the notation

\[
\text{core}(n, k) = \min \{ t : e(H^3_n) \geq t \Rightarrow H^3_n \text{ contains a core on at most } k \text{ vertices} \},
\]

where \( H^3_n \) is a 3-uniform hypergraph on \( n \) vertices, and \( e(H^3_n) \) is the number of edges of \( H^3_n \).

Cores are important objects in the theory of hypergraphs. One direction of research is on finding sharp thresholds for the appearance of cores in random hypergraphs, as in [11]. More surprisingly cores of hypergraphs appeared in models of protein interaction networks [2, 12]. Some algorithmic aspects of finding minimum cores were considered in [4]. Minimum cores in terms of the number of edges might be seen as a generalization of the girth problem in graphs, as the girth of a graph is the size of the smallest subgraph with minimum degree 2. For references about the girth problem and for some variations of the possible notation of girth of hypergraphs we refer to [10] Section 4, and [5] Section 4. Since the girth problem, Erdős’ girth conjecture, is open for graphs one can expect determining \( \text{core}(n, k) \) to be hard. Indeed, most of our results are not sharp and possible improvements would imply or require progress in well know conjectures.

Our main result is closely related to the following conjecture:

Conjecture 1.1 (Brown, Erdős, and Sós). For every \( \ell \geq 3 \) and \( c > 0 \) there exists an \( n_0 = n_0(c) \) such that if \( n > n_0 \) and \( e(H^3_n) \geq cn^2 \) then there exists a \( F^3_{\ell+3} \subseteq H^3_n \) such that \( e(F^3_{\ell+3}) \geq \ell \).

Using the notation of [5] the smallest \( \ell \) such that there is an \( F^3_{\ell+3} \subseteq H^3_n \) is the \((-3)\)-girth of \( H^3_n \). To this date the best bound in this direction is the bound of Sárközi and Selkow [14].

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Theorem 1.2 (Sárközi and Selkow). For every $\ell \geq 3$ and $c > 0$ there exists an $n_0 = n_0(c)$ such that if $n > n_0$ and $e(H_n^3) \geq cn^2$ then there exists a $F_{\ell + 2 + \lfloor \log_2 \ell \rfloor}^3 \subseteq H_n^3$ such that $e(F_{\ell + 2 + \lfloor \log_2 \ell \rfloor}^3) \geq \ell$.

Their proof relies on Szemerédi’s regularity lemma. In our application we are going to improve their result for the $\ell = 10$ case using a hypergraph regularity lemma of Frankl and Rödl [9]. This result might be of independent interest as this is the first improvement in the last decade in this important problem.

The $\ell = 3$ case was proved by Ruzsa and Szemerédi [13]. We are going to use the following quantitative version:

Theorem 1.3. For every $c > 0$ there exists a $\delta > 0$ such that if the number of edges is at least $cn^2$ then there exists $\delta n^3$ subgraphs $F_6^3 \subseteq H_n^3$ such that $e(F_6^3) \geq 3$.

An important part of the paper deals with the range when the size of the hypergraph is $o(n^2)$, where the relation to the Brown, Erdős, and Sós conjecture is clear, however, finding sharp bounds on core is an interesting open problem for any edge densities. We collect bounds which we were able to obtain for various ranges. These are summarized in Table I.

2. Very small $k$ (4 to 8)

The smallest possible core in a 3-uniform hypergraph is the hypergraph $K_4^3 - e$, a clique with one edge removed. (Or equivalently, the unique 3-uniform hypergraph on 4 vertices with 3 edges.) The presence of such a subgraph belongs to the family of classical Turán-type hypergraph problems. The exact Turán density is not known. The best lower bound is due to Frankl and Füredi [8], it is $2/7 = 0.2857...$ while the upper bound, $0.2871$ is due to Baber and Talbot [1].

A hypergraph on five vertices with at least four edges will contain a core of size 5 or 4. We will show an upper and a lower bound on $\text{core}(n, 5)$.

Theorem 2.1. There are constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 n^{7/3} \leq \text{core}(n, 5) \leq c_2 n^{5/2}.$$ 

Lemma 2.2. There is a constant $c > 0$ so that the edges of $K_n^3$ can be partitioned into $cn^{2/3}$ classes (hypergraphs) such that none of them contains a core of size 5 or 4.

This immediately implies the lower bound on $\text{core}(n, 5)$.

Proof of Lemma 2.2. Colour the edges independently at random using $cn^{2/3}$ colours, each with the same probability. Let us define the random variables $X_1, X_2, \ldots, X_{\binom{n}{5}}$ for every 5-tuple of vertices such that $X_i = 1$ if at least 4 of the $\binom{5}{4}$ edges have the same colour and $X_i = 0$ otherwise.

$$\text{Prob}(X_i = 1) = c'n^{-2},$$

where $c'$ goes to zero as $c$ goes to infinity. We are going to use Lovász Local Lemma [5] in the next step. The degree of the dependence graph is

$$\binom{5}{4} \binom{n - 5}{2} + \binom{5}{4}(n - 5) < 5n^2.$$
Then
\[ c'n^{-2} < \frac{1}{5en^2} \]
with the appropriate choice of \( c \), so the lemma holds. \( \square \)

When \( e(H^3_n) > n^{5/2} \), we show that a core of size at most 5 must exist. This follows from the existence of a \( K_{2,2,1}^3 \), the complete tripartite graph on 5 vertices, which is a core. The extremal problem for complete tripartite graphs was considered by Erdős in [6]. For the sake of completeness we present the simple calculations here as we will use similar arguments later. We count the number of edge pairs that intersect in two points (Figure 1a), which is
\[
\sum_{v_i \neq v_j \in H^3_n \ i < j} \binom{\deg(v_i, v_j)}{2},
\]
where \( \deg(v_i, v_j) \) is the number of edges that contain both \( v_i \) and \( v_j \). The number of edges, \( e \), is large enough so that
\[
\sum_{v_i \neq v_j \in H^3_n \ i < j} \binom{\deg(v_i, v_j)}{2} \geq \binom{n}{2} \left( \frac{e}{\binom{n}{2}} \right) \approx \frac{e^2}{n^2} > \binom{n}{3}.
\]
The inequality shows that at least two intersecting edge pairs intersect in three points, as seen in Figure 1c. This subgraph containing four edges on five points is a core, so \( \text{core}(n, 5) \leq n^{5/2} \).

**Theorem 2.3.** There are constants \( c_1 > 0 \) and \( c_2 > 0 \) such that
\[
c_1 n^2 \leq \text{core}(n, 8) \leq \text{core}(n, 7) \leq \text{core}(n, 6) \leq c_2 n^2.
\]
The upper bound follows from a calculation similar to the \( k = 5 \) case above.

**Lemma 2.4.**
\[
\text{core}(n, 6) < n^2.
\]
*Proof.* If the number of edges, \( e \), is large enough so that
\[
\sum_{v_i \neq v_j \in H^3_n \ i < j} \binom{\deg(v_i, v_j)}{2} \geq \binom{n}{2} \left( \frac{e/\binom{n}{2}}{2} \right) \approx \frac{e^2}{n^2} > \binom{n}{2},
\]
then at least two intersecting edge pairs intersect in the two degree one vertices, as seen in Figure 1b. This subgraph containing four edges on at most 6 vertices is a core, so \( \text{core}(n, 6) < n^2 \). \( \square \)
To prove the lower bound in Theorem 2.3 we give an algebraic construction for $H_{3n}^3$ with $e(H_{3n}^3) = \alpha(n^2)$ which does not contain a core of size 8 or less.

Lemma 2.5. There is a constant $c > 0$ such that

$$\text{core}(n, 8) \geq cn^2.$$ 

Proof. Let $A, B, C \cong \mathbb{Z}/p\mathbb{Z}$ for some large prime $p$, and consider the 3-partite 3-uniform hypergraph over vertices $A \cup B \cup C$ which has edges

$$\{ \{a, b, c\} | a \in A, b \in B, c \in C, a + b = c \}.$$ 

Suppose $U \subseteq A, V \subseteq B, W \subseteq C$ are nonempty subsets such that the induced subgraph on $U \cup V \cup W$ is a core. We will show that $|U| + |V| + |W| \geq 9$.

Note that we require at least 2 max$(|U|, |V|, |W|)$ edges for every vertex to have degree two, but we can only have at most min$(|U||V|, |U||W|, |V||W|)$ edges, as any two vertices uniquely determine an edge. For $|U| + |V| + |W| < 9$ this leaves us with two cases, when they all have size 2, or when two of them have size 3 and one has size 2.

Let $U = \{u_1, \ldots, u_{|U|}\}$, $V = \{v_1, \ldots, v_{|V|}\}$, and $W = \{w_1, \ldots, w_{|W|}\}$.

If $|U| = |V| = |W| = 2$, we must have 4 edges, so every $\{u_i, v_j\}$ will be contained in an edge. The expressions $u_1 + v_1, u_1 + v_2, u_2 + v_1, u_2 + v_2$ give two distinct values, so $u_1 + v_1 = u_2 + v_2$ and $u_1 + v_2 = u_2 + v_1$. Combining these expressions, $u_2 + v_1 - v_2 + v_1 = u_2 + v_2$, so $(v_1 - v_2) + (v_1 - v_2) = 0$, therefore $v_1 = v_2$ if $2 \not| p$.

If $|U| = |W| = 3$ and $|V| = 2$, we must have 6 edges, so every $\{u_i, v_j\}$ will be contained in an edge. The expressions $u_1 + v_1, u_1 + v_2, u_2 + v_1, u_2 + v_2, u_3 + v_1, u_3 + v_2$ only give three distinct values. No three of these expressions can be equal, since then $u_1, u_2, u_3$ are not all distinct. We may assume $u_1 + v_1 = u_2 + v_2$, then $u_1 + v_2 \neq u_2 + v_1$ by the $|U| = |V| = |W| = 2$ case above.

Thus $u_1 + v_2 = u_3 + v_1$, and finally $u_2 + v_1 = u_3 + v_2$. From these we get that $u_1 = u_2 + v_2 - v_1$, and further $u_2 + v_2 - v_1 + v_2 - v_1 = u_3$, finally

$$u_2 + v_1 = u_2 + v_2 - v_1 + v_2 - v_1 + v_2,$$

which implies $(v_2 - v_1) + (v_2 - v_1) = 0$, so $v_1 = v_2$ if $3 \not| p$.

The case $|V| = |W| = 3$ and $|U| = 2$ is the same as above, and if $|U| = |V| = 3$ and $|W| = 2$ then we can repeat the argument with expressions of the form $u_i + (-u_i)$. \hfill \square

3. Small $k$ (9 to 15)

We prove that if $e(H_n^3) = o(n^2)$, then $H_n^3$ contains a core of size at most 15. We conjecture that our result is not sharp and 15 can be replaced by 9. Such improvement seems to be out of reach, since it would imply the $\ell = 6$ case of the Brown, Erdős, Sós conjecture (Conjecture 1.1).

In what follows we can suppose w.l.o.g. that $H_n^3$ is tripartite, with disjoint vertex sets $V_1, V_2,$ and $V_3$ so that every edge has one vertex in each class. (A random partition would leave an edge in the tripartite graph with probability $2/9$.)

Proposition 3.1. There is a $c > 0$ such that there are graphs $H_n^3$ without a core of size 10 with $e(H_n^3) = cn^2$ for arbitrary large $n$.

If a core has 10 vertices then at least 4 of the vertices are in the same vertex class so the core has at least 8 edges. In the paper where Brown, Erdős, and Sós stated their conjecture [16] they also showed that the conjecture (if true) is sharp; for
every $\ell$ there is a $c > 0$ such that there are arbitrary large 3-uniform hypergraphs without $\ell + 2$ vertices spanning at least $\ell$ edges.

A core on 11 vertices would also contain at least 8 edges. Proving that $\text{core}(n, 11) = o(n^2)$ would again imply the Brown, Erdős, Sós conjecture for $\ell = 8$. The situation is a bit different for core size $k = 12$. The statement $\text{core}(12) = o(n^2)$ follows from the $\ell = 4$ case of the Brown, Erdős, Sós conjecture. We show the following proposition first.

In the range of core size $k \geq 12$ and edge density $O(n^{2/3}) \leq e(H^3_n)$ we can suppose w.l.o.g. that $H^3_n$ is a linear hypergraph, i.e. no edges intersect in two vertices. More precisely we show that

**Proposition 3.2.** For every $\varepsilon > 0$ there is a threshold $n_0$ such that if the number of edge-pairs intersecting in 2 vertices is at least $(1/2 + \varepsilon)n^{3/2}$ in $H^3_n$, $n \geq n_0$, then it contains core of size at most 12.

**Proof.** Let us define an auxiliary graph $G_n$ where two vertices are connected by an edge iff they are the single vertices of two edges intersecting in two vertices in $H^3_n$. This is a simple graph since a double edge would determine a core on at most 6 vertices. If graph $G_n$ contains a $C_4$ then the four defining edge-pairs in $H^3_n$ would form a core on at most 12 vertices. $C_4$-free graphs contain at most $\sim n^{3/2}/2$ edges. □

This shows that either there is a core on at most 12 vertices, or we can remove $O(n^{3/2})$ edges from $H^3_n$ to make it linear. We will assume that $H^3_n$ is a tripartite and linear hypergraph.

Let us go back to the $k = 12$ case. The first open case of Conjecture 1.1 is $\ell = 4$. It has been proved for special triple systems in [15].

**Proposition 3.3.** The Brown, Erdős, Sós conjecture with $\ell = 4$ implies $\text{core}(n, 12) = o(n^2)$.

**Proof (sketch).** It is easy to see that if the conjecture holds then for every $c > 0$ there is a $c' > 0$ such that the number of subgraphs on 7 vertices with 4 edges is at least $c'n^3$. Such graphs (tripartite and linear subgraphs) have two degree one vertices, so if $n$ is large enough then there will be two sharing the same pair of degree one vertices. The union of the two 7 vertex graph is a core as it has no degree one vertex and its size is at most 12. □

A core on 13 vertices has at least 5 vertices in one vertex class therefore it has at least 10 edges. A proof of $\text{core}(n, 13) = o(n^2)$ would imply Conjecture 1.1 for $\ell = 10$.

We were unable to prove that $\text{core}(n, 14)$ is $o(n^2)$, however we improved the previously best known bound for the Brown Erdős Sós conjecture by Selkow and Sárközy. We proved that if $H^3_n$ contains no 14 vertices spanning at least 10 edges then $e(H^3_n) = o(n^2)$. Note that for a core on 14 vertices one would need at least 10 edges (since $14 \cdot 2/3 = 10$). The bound of Selkov and Sárközy, Theorem 1.2 guarantees 9 edges only on 14 vertices which would not be enough for a core.

**Theorem 3.4.** If $H^3_n$ contains no subgraph $F^3_{14}$ such that $e(F^3_{14}) = 10$, then $e(H^3_n) = o(n^2)$. 

Our main tool in proving this theorem is a generalization of Theorem 1.3 by Frankl and Rodl [9].

**Theorem 3.5** (Removal Lemma for 3-uniform Hypergraphs). For every $c > 0$ there exists a $c' > 0$ such that if $H^3_n$ contains at least $cn^3$ pairwise edge-disjoint cliques $K^3_4$, then it contains $c'n^4$ cliques $K^4_3$.

**Proof of Theorem 3.4** Suppose the hypergraph $H^3_n$ satisfies $e(H^3_n) = cn^2$ for some $c > 0$. As before, we may assume that $H^3_n$ is linear and tripartite. By Theorem 1.3 we know that there exists $\delta n^3$ subgraphs $F^3_6$ such that $e(F^3_6) = 3$. We will construct an auxiliary hypergraph on the same vertex set as a union of cliques $K^3_4$ placed on select four vertices of these configurations of three edges on six vertices. The four vertices are the three degree one vertices plus one degree two vertex as seen in Figure 2.

The idea is that every edge in this auxiliary hypergraph will correspond to a subgraph $F^3_6$, and each edge in the cliques that we place to the same subgraph. Using Lemma 3.5 we can find a clique where each edge corresponds to different $F^3_6$ with lots of overlap, which will give us the desired core.

However, we need to tweak our hypergraph so that we will only use a subset of these $F^3_6$ configurations where no two of them intersect in three of the selected four vertices, which would interfere with our argument.

If some configurations intersect on three of the selected four vertices, only one configuration can have a degree two vertex among these three, as a configuration is uniquely determined by a degree two and two degree one vertices.

Moreover, even if a small number (5) of configurations intersect on their degree one vertices, their union will contain a subgraph on 14 vertices with at least 10 edges.

Thus we can forget about these configurations and still have a constant fraction of them remain such that no two of them intersect in three of these select four points. We can then construct our auxiliary hypergraph as a union of $\delta n^3$ pairwise edge-disjoint cliques $K^4_3$ on these four points of the remaining configurations. Lemma 3.5 tells us that this hypergraph contains $c'n^4$ cliques $K^4_3$, in particular, some cliques which do not correspond to one configuration in $H^3_n$. Instead, each must arise from a larger structure of 10 edges on 14 vertices as seen in Figure 2, which is a union of four configurations. □

The following is the first unconditional result with $o(n^2)$ edges.

**Theorem 3.6.**

$\text{core}(n, 15) = o(n^2)$. 

\[ \text{core}(n, 15) = o(n^2). \]
Proof. By Theorem 1.3, we will have $c'n^4$ pairs of subgraphs of 6 vertices with 3 edges which overlap on two degree one vertices. These will be subgraphs on 10 vertices with 6 edges, with two degree one vertices. We remove one edge from each to create $c'n^4$ subgraphs on 9 vertices with 5 edges, with three degree one vertices as seen in Figure 4. Two of these will overlap on these degree one vertices, giving us a core on 15 vertices.

4. LARGE $k$ (16 TO $\sqrt{n}$)

In this range, when the number of edges of $H^3_n$ is at least $n^{3/2} + c$ for some $c > 0$, we show that one can guarantee a core of constant size, i.e. the size of the smallest core depends on $c$ but not on $n$.

Let us define a graph $G_m$ on $m = n^2$ vertices as follows. The vertices are the (ordered) pairs of the vertices $H^3_n$. Two vertices, $(v_i, v_j)$ and $(v_s, v_t)$ are connected by an edge in $G_m$ if there is a vertex $v_r$ such that $(v_i, v_r, v_s)$ and $(v_j, v_r, v_t)$ are two distinct edges in $H^3_n$. Note that the the vertices are not necessarily distinct. By Jensen’s inequality the number of edges, $e(G_m)$, is at least the average degree in $H^3_n$ choose 2 multiplied by the number of vertices.

$$e(G_m) \geq \left( \frac{e(H^3_n)}{n} \right)n.$$
A cycle in $G_m$ is always a core in $H^3_n$. It is yet another hard problem to determine the maximal girth of a graph in terms of the number of edges. We are going to use the asymptotically best known bounds for girth $g = 2s + 1$ see e.g. in [3] (page 1264) or in [10]. If $G_m$ has girth at least $g$ then $m \gtrsim e(H^3_n)^{s/s+1}$.

Theorem 4.1. Let $s > 2$ be an integer. If $e(H^3_n) \gg n^{3/2} + 1/s$ then $H^3_n$ contains a core on at most $3(2s + 1)$ vertices.

The proof follows from the observations above. We expect that much better bounds can be obtained and there is a chance to prove bounds without using the girth inequality in the auxiliary graph.

5. Very large $k$ ($\sqrt{n}$ to $n$)

For this range of $k$ we use a stripping algorithm similar to one commonly used to find cores in random constraint satisfaction problems [11].

Any hypergraph $H^3_n$ such that $e(H^3_n) \geq n - 2$ contains a core which can be found by repeatedly removing degree one vertices. For our purposes the following randomized step is convenient: Choose $n^{3/2} / \sqrt{e(H^3_n)}$ vertices uniformly at random, and remove the rest. The probability of one edge being present is $\left(\frac{n^{3/2}}{\sqrt{e(H^3_n)}}\right)^3$, so the expected number of edges among the chosen vertices is $\left(\frac{n^{3/2}}{\sqrt{e(H^3_n)}}\right)^3 e(H^3_n)$, which is equal to the number of vertices. Thus there exists a subgraph on that many vertices with at least as many edges, so by the stripping process above the existence of a core is guaranteed.

Thus $\text{core}(n, k) \leq n^{3/2}/k^2$ in this range.

6. Summary

In the previous sections we considered the $\text{core}(n, k)$ function. One can see more details examining the following variation of core.

$\text{core}^*(n, k) = \min \{ t : e(H^3_n) \geq t \Rightarrow H^3_n \text{ contains a core on } k \text{ vertices} \}$

The difference between the two notations is that here we are looking for the threshold where the existence of a core on exactly $k$ vertices is guaranteed.

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About the $\text{core}^*(n, k)$ function

| $k$   | $\text{core}^*(n, k) \leq O(n^{5/2})$ |
|-------|----------------------------------------|
| $k = 4$ | $0.2857(\binom{n}{3}) \leq \text{core}^*(n, 4) \leq 0.2871(\binom{n}{3})$ |
| $k = 5$ | $\Omega(n^{7/3}) \leq \text{core}^*(n, 5) \leq O(n^{5/2})$ |
| $k = 6, 7, 8$ | $\Omega(n^2) \leq \text{core}^*(n, k) = O(n^2)$ |

| $k = 9$ | $\text{conj. core}^*(n, 9) = o(n^2) \implies BES(\ell = 6)$ |
| $k = 10$ | $\text{core}^*(n, 10) = \Omega(n^2)$ |
| $k = 11$ | $\text{conj. core}^*(n, 11) = o(n^2) \implies BES(\ell = 6)$ |
| $k = 12$ | $BES(\ell = 4) \implies \text{core}^*(n, 12) = o(n^2)$ |
| $k = 13$ | $\text{conj. core}^*(n, 13) = o(n^2) \implies BES(\ell = 10)$ |
| $k = 14$ | $\text{conj. core}^*(n, 14) = o(n^2)$ see Theorem 3.4 |
| $k = 15$ | $\text{core}^*(n, 15) = o(n^2)$ see Theorem 3.6 |

Table 1. $BES(\ell = t)$ means that Conjecture 1.1 holds for $\ell = t$. 

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