A posteriori error estimates of mixed discontinuous Galerkin method for the Stokes eigenvalue problem

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Abstract

In this paper, for the Stokes eigenvalue problem in d-dimensional case (d = 2, 3), we present an a posteriori error estimate of residual type of the mixed discontinuous Galerkin finite element method using $P_k - P_{k-1}$ element ($k \geq 1$). We give the a posteriori error estimators for approximate eigenpairs, prove their reliability and efficiency for eigenfunctions, and also analyze their reliability for eigenvalues. We implement adaptive calculation, and the numerical results confirm our theoretical predictions and show that our method can achieve the optimal convergence order $O(dof^{\frac{1}{d}})$.

Key words. Stokes eigenvalue problem, discontinuous Galerkin method, residual type a posteriori error estimates, adaptive algorithm.

1 Introduction

Stokes eigenvalue problem is of great importance because of their role for the stability analysis in fluid mechanics. Hence, the development of efficient numerical methods for the problem is of great interest.

Adaptive finite element methods are favored in current science and engineering computing. For a given tolerance, adaptive finite element methods require little degrees of freedom. So far, many excellent works on the a posteriori error estimates and adaptive algorithm have been summarized in previous studies (see [1–8], etc). For the Stokes eigenvalue problem, the a posteriori error estimates has received much attention. For example, [9–11] studied the a posteriori error estimates of conforming mixed method, Liu et al. [12] presented some super-convergence results and the related recovery type a posteriori error estimators for conforming mixed method, Jia et al. [13] discussed the a posteriori error estimate of low-order non-conforming finite element, Gedicke et al. [14] conducted the a posteriori error analysis for the Arnold-Winther mixed finite element method using the stress-velocity formulation, Önder Türk et al. [15] researched a stabilized finite element method for the two-field (displacement-pressure) and three-field (stress-displacement-pressure) formulations of the Stokes eigenvalue problem.

Discontinuous Galerkin finite element method (DGFEM) was first introduced by Reed and Hill [16] in 1973 and has been developed greatly (see, e.g., [17–24]). DGFEM for eigenvalue problems has also been discussed in many papers (see [25–34]). Among them Gedicke et al. [33] discussed the a posteriori error estimate for the divergence-conforming DGFEM using

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Raviart-Thomas element for velocity-pressure formulation of the Stokes eigenvalue problem on shape-regular rectangular meshes. Felipe Lepe et al. [34] analyzed symmetric and nonsymmetric discontinuous Galerkin methods for a pseudostress formulation of the Stokes eigenvalue problem. It can be seen that solving the Stokes eigenvalue problem by DGFEM has attracted extensive attention of scholars.

For the Stokes equations, it has been studied the mixed DGFEM using $\mathbb{P}_k - \mathbb{P}_{k-1}$ element (see [19, 35–38]) and $Q_k - Q_{k-1}$ element (see [39, 40, 43]), which laid a foundation for us to further study the Stokes eigenvalue problem. Based on the above work, in this paper, we study the residual type a posteriori error estimate of the mixed DGFEM using adaptive locally refined graded meshes. Felipe Lepe et al. [34] analyzed symmetric and nonsymmetric discontinuous Galerkin methods for a pseudostress formulation of the Stokes eigenvalue problem on shape-regular rectangular meshes. In Section 2, we give the a priori error estimates for the mixed DGFEM of the Stokes eigenvalue problem based on the work of [35]. In Section 3, we give the a posterior error estimators for approximate eigenpair, and use the method of [2, 5] together with the enriching operator in [41, 42] and the lifting operator in [43, 44] to prove the reliability and efficiency of the estimator for eigenfunctions. In Section 4, we implement adaptive calculation. The numerical results show that the approximate eigenvalues obtained by our method have the same accuracy as those in [14, 33] and achieve the optimal convergence order $O(dof^{-\frac{1}{2}})$, which validates that our method is effective.

The characteristic of the DGFEM discussed in this paper is that for the Stokes eigenvalue problem both in two and three-dimensional domains, it can use high-order elements so that it can not only capture smooth solutions but also achieve the optimal convergence order for local low smooth solutions (eigenfunctions have local singularity or local low smoothness) on adaptive locally refined graded meshes.

Throughout this paper, $C$ denotes a generic positive constant independent of the mesh size $h$, which may not be the same at each occurrence. We use the symbol $a \lesssim b$ to mean that $a \leq Cb$, and $a \simeq b$ to mean that $a \lesssim b$ and $b \lesssim a$. 

2 Preliminary

Consider the following Stokes eigenvalue problem:

\[
\begin{align*}
-\mu \Delta u + \nabla p &= \lambda u, & \text{in } \Omega, \\
\text{div } u &= 0, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega, \\
\end{align*}
\]

where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a bounded polyhedral domain, $u = (u_1, \ldots, u_d)$ is the velocity of the flow, $p$ is the pressure and $\mu > 0$ is the kinematic viscosity parameter of the fluid.

Note that for constant viscosity $\mu$, the velocity eigenfunctions do not change in $\mu$ and thus the eigenvalues as well as the pressure eigenfunctions scale linearly in $\mu$, i.e., the eigenpair for arbitrary constant $\mu$ is $(\mu \lambda, \mu u, \mu p)$, where $(\lambda, u, p)$ denotes the eigenpair for $\mu = 1$. Hence, in this paper we only consider the case of $\mu = 1$.

In this paper, denote by $H^p(\Omega)$ the Sobolev space on $\Omega$ of order $p \geq 0$ equipped with the norm $\| \cdot \|_{p, \Omega}$ (denoted by $\| \cdot \|_p$ for simplicity). $H^1_0(\Omega) = \{ v \in H^1(\Omega), v|_{\partial \Omega} = 0 \}$.

For $v = (v_1, \ldots, v_d) \in H^p(\Omega)^d$, denote $\| v \|_p = \sum_{i=1}^d \| v_i \|_p$. We also use the notation $(\cdot, \cdot)$ to denote the inner product in $L^2(\Omega)^d$ which is given by $(u, v) = \int_\Omega uv dx$ for $d = 1$ and $(u, v) = \int_\Omega u \cdot v dx$ for $d = 2, 3$. Define $V = H^1_0(\Omega)^d$ with the norm $\| v \|_V = (\nabla v, \nabla v)^{\frac{1}{2}}$, and define $Q = L^2_0(\Omega) = \{ q \in L^2(\Omega) : (q, 1) = 0 \}$. 

2.1
The weak formulation of (2.1) is given by: find \((\lambda, u, p) \in \mathbb{R} \times V \times Q\), \(\|u\|_0 = 1\), such that

\[
\mathcal{A}(u, v) + \mathcal{B}(v, p) = \lambda(u, v), \quad \forall v \in V
\]

\[
\mathcal{B}(u, q) = 0, \quad \forall q \in Q,
\]

where

\[
\mathcal{A}(u, v) = (\nabla u, \nabla v),
\]

\[
\mathcal{B}(v, q) = -(div v, q).
\]

The existence and uniqueness of the velocity \(u\) follow from the Lax-Milgram lemma in the space \(Z = \{v \in V : b(v, q) = 0, \forall q \in Q\}\). The stability of the pressure can be obtained by the well-known inf-sup condition (see [57]):

\[
\beta\|q\|_{L^2(\Omega)} \leq \sup_{v \in V, v \neq 0} \frac{B(v, q)}{\|v\|_V}, \forall q \in Q.
\]

Let \(\pi_h = \{\kappa\}\) be a regular partition of \(\Omega\) with the mesh diameter \(h = \max_{\kappa \in \pi_h} h_\kappa\) where \(h_\kappa\) is the diameter of element \(\kappa\). Let \(\varepsilon_h = \varepsilon_h^I \cup \varepsilon_h^S\) where \(\varepsilon_h^I\) denotes the interior faces (edges) set and \(\varepsilon_h^S\) denotes the set of faces (edges) lying on the boundary \(\partial \Omega\). We denote by \(|\kappa|\) and \(|E|\) the measure of \(\kappa\) and \(E\) in \(\varepsilon_h\), respectively. Let \((\cdot, \cdot)_\kappa\) and \((\cdot, \cdot)_E\) denote the inner product in \(L^2(\kappa)\) and \(L^2(E)\), respectively. We denote by \(\omega(\kappa)\) the union of all elements having at least one face (edge) in common with \(\kappa\), and denote by \(\omega(E)\) the union of the elements having in common with \(E\).

Define a broken Sobolev space

\[
H^1(\Omega, \pi_h) = \{v \in L^2(\Omega) : v|_\kappa \in H^1(\kappa), \forall \kappa \in \pi_h\}.
\]

For any \(E \in \varepsilon_h\), there are two simplices \(\kappa^+\) and \(\kappa^-\) such that \(E = \kappa^+ \cap \kappa^-\). Let \(n^+\) be the unit normal of \(E\) pointing from \(\kappa^+\) to \(\kappa^-\) and let \(n^- = -n^+\).

For any \(v \in H^1(\Omega, \pi_h)\), we define the jump and mean of \(v\) on \(E\) by

\[
[v] = v^+ n^+ + v^- n^-, \quad \{v\} = \frac{1}{2}(v^+ + v^-),
\]

where \(v^+ = v|_{\kappa^+}\).

For \(v \in H^1(\Omega, \pi_h)^d\), we define the jump and mean of \(v\) on \(E \in \varepsilon_h\) by

\[
[v] = v^+ \cdot n^+ + v^- \cdot n^-, \quad \{v\} = \frac{1}{2}(v^+ + v^-).
\]

We also require the full jump of vector-valued functions. For \(v \in H^1(\Omega, \pi_h)^d\), we define the full jump by

\[
[v] = v^+ \otimes n^+ + v^- \otimes n^-,
\]

where for two vectors in Cartesian coordinates \(a = (a_i)\) and \(b = (b_j)\), we define the matrix \(a \otimes b = [a_i b_j]_{1 \leq i, j \leq d}\). Similarly, for tensors \(\tau \in H^1(\Omega, \pi_h)^{d \times d}\), the jump and mean on \(E \in \varepsilon_h\) are defined as follows, respectively:

\[
[\tau] = \tau^+ n^+ + \tau^- n^-, \quad \{\tau\} = \frac{1}{2}(\tau^+ + \tau^-).
For notational convenience, we also define the jump and mean on the boundary faces \( E \in \mathcal{E}_h \) by modifying the above definitions appropriately. We use the definition of jump by understanding that \( v^- = 0 \) (similarly, \( v^- = 0 \) and \( \tau^- = 0 \)) and the definition of mean by understanding that \( v^- = v^+ \) (similarly, \( v^- = v^+ \) and \( \tau^- = \tau^+ \)).

We define the following discrete velocity and pressure spaces:

\[
V_h = \{ v_h \in L^2(\Omega)^d : v_h|_\kappa \in \mathbb{P}_k(\kappa)^d, \ \forall \kappa \in \pi_h \},
\]

\[
Q_h = \{ q_h \in Q : q_h|_\kappa \in \mathbb{P}_{k-1}(\kappa), \ \forall \kappa \in \pi_h \},
\]

where \( \mathbb{P}_k(\kappa) \) is the space of polynomials of degree less than or equal to \( k \geq 1 \) on \( \kappa \).

The DGFEM for the problem (2.1) is to find \((\lambda_h, u_h, p_h) \in \mathbb{R}^+ \times V_h \times Q_h, \|u_h\|_0 = 1\) such that

\[
A_h(u_h, v_h) + B_h(v_h, p_h) = \lambda_h(u_h, v_h), \ \forall v_h \in V_h,
\]

\[
B_h(u_h, q_h) = 0, \ \forall q_h \in Q_h,
\]

where

\[
A_h(u_h, v_h) = \sum_{\kappa \in \pi_h} \int_\kappa \nabla u_h : \nabla v_h \, dx - \sum_{E \in \mathcal{E}_h} \int_E \{ \nabla u_h \} : [v_h] \, ds
\]

\[
- \sum_{E \in \mathcal{E}_h} \int_E \{ \nabla v_h \} : [u_h] \, ds + \sum_{E \in \mathcal{E}_h} \int_E \frac{\gamma}{h_E} [u_h] : [v_h] \, ds,
\]

\[
B_h(v_h, q_h) = - \sum_{\kappa \in \pi_h} \int_\kappa \nabla q_h \cdot v_h \, dx + \sum_{E \in \mathcal{E}_h} \int_E q_h \| v_h \| \, ds.
\]

Here \( \gamma/h_E \) is the interior penalty parameter. From Remark 2.1 in [45], in the actual numerical implementations we can set \( \gamma = C_1 k^2 \) with \( C_1 = 10 \) and \( k \) is the degree of the polynomial.

Define the DG-norm as follows:

\[
\| v_h \|^2_h = \sum_{\kappa \in \pi_h} \| v_h \|^2_{0, \kappa} + \sum_{E \in \mathcal{E}_h} \int_E \frac{\gamma}{h_E} [v_h]^2 \, ds, \ \text{on} \ V_h + V;
\]

\[
\| v_h \|^2 = \| v_h \|^2 + \sum_{E \in \mathcal{E}_h} \int_E \frac{h_E}{\gamma} |\nabla v_h|^2 \, ds, \ \text{on} \ V_h + H^{1+s}(\Omega)^d. \]

Note that \( \| \cdot \| \) is equivalent to \( \| \cdot \| \) on \( V_h \).

It is easy to know that (see [19]) the following continuity and coercivity properties hold:

\[
|A_h(u_h, v_h)| \lesssim \| u_h \| \| v_h \|, \ \forall u_h, v_h \in V_h + H^{1+s}(\Omega)^d \ (s > \frac{1}{2})
\]

\[
\|u_h\|^2_h \lesssim A_h(u_h, u_h), \ \forall u_h \in V_h.
\]

From [38] we obtain the discrete inf-sup condition (the stability of the pressure):

\[
\inf_{p_h \in Q_h} \sup_{v_h \in V_h} \frac{B_h(v_h, p_h)}{|v_h|_h \|p_h\|_0} \geq \beta^*,
\]

where \( \beta^* \) is a positive constant independent of \( h \).

We consider the source problem associated with the Stokes eigenvalue problem (2.1): Given \( f \in (L^2(\Omega))^d \),

\[
\begin{cases}
-\Delta u^f + \nabla p^f = f, & \text{in} \ \Omega, \\
\text{div} u^f = 0, & \text{in} \ \Omega, \\
u^f = 0, & \text{on} \ \partial \Omega.
\end{cases}
\]
The weak formulation of (2.10) is given by: find \((u^f, p^f) \in V \times Q\) such that

\[
A(u^f, v) + B(v, p^f) = (f, v), \quad \forall v \in V, \tag{2.11}
\]

\[
B(u^f, q) = 0, \quad \forall q \in Q, \tag{2.12}
\]

and its discontinuous Galerkin finite element form are as follows: find \((u_h^f, p_h^f) \in V_h \times Q_h\) such that

\[
A_h(u_h^f, v_h) + B_h(v_h, p_h^f) = (f, v_h), \quad \forall v_h \in V_h, \tag{2.13}
\]

\[
B_h(u_h^f, q_h) = 0, \quad \forall q_h \in Q_h. \tag{2.14}
\]

We assume that the following regularity is valid: for any \(f \in L^2(\Omega)^d (d = 2, 3)\), there exists \((u^f, p^f) \in (H^{1+r}(\Omega)^d \times H^r(\Omega)) \cap (W^{2,p}(\Omega)^d \times W^{1,p}(\Omega)) \left( \frac{1}{2} < r \leq 1, p > \frac{2d}{d+1} \right)\) satisfying (2.10) and

\[
\|u^f\|_{1+r} + \|p^f\|_r \leq C\|f\|_0, \tag{2.15}
\]

where \(C\) is a positive constant independent of \(f\).

From Lemma 6.5 in [19] we can obtain the consistency of the DGFEM, that is to say, when \((u^f, p^f)\) is the solution of the source problem (2.10), there hold the following equations:

\[
A_h(u^f, v_h) + B_h(v_h, p^f) = (f, v_h), \quad \forall v_h \in V_h, \tag{2.16}
\]

\[
B_h(u^f, q_h) = 0, \quad \forall q_h \in Q_h. \tag{2.17}
\]

From (2.13)-(2.14) and (2.16)-(2.17), we have

\[
A_h(u^f - u_h^f, v_h) + B_h(v_h, p^f - p_h^f) = 0, \quad \forall v_h \in V_h, \tag{2.18}
\]

\[
B_h(u^f - u_h^f, q_h) = 0, \quad \forall q_h \in Q_h. \tag{2.19}
\]

Since (2.11)-(2.12) and (2.13)-(2.14) are both uniquely solvable for each \(f \in L^2(\Omega)^d (d = 2, 3)\) (see, e.g., Lemma 2.4 in [35], and Lemma 7 and Proposition 10 in [36]), we can define the corresponding solution operators as follows:

\[
T : L^2(\Omega)^d \to V, \quad Tf = u^f,
\]

\[
T_h : L^2(\Omega)^d \to V_h, \quad T_h f = u_h^f,
\]

\[
S : L^2(\Omega)^d \to Q, \quad S f = p^f,
\]

\[
S_h : L^2(\Omega)^d \to Q_h, \quad S_h f = p_h^f.
\]

Then (2.11)-(2.12) and (2.13)-(2.14) can be written in the following equivalent operator forms:

\[
\lambda T u = u, \quad S(\lambda u) = p, \tag{2.20}
\]

\[
\lambda_h T_h u_h = u_h, \quad S_h(\lambda_h u_h) = p_h. \tag{2.21}
\]

It is easy to know that both \(T\) and \(T_h\) are self-adjoint and completely continuous and satisfy

\[
\|Tf\|_1 + \|Sf\|_0 \lesssim \|f\|_0, \quad \|T_h f\| + \|S_h f\|_0 \lesssim \|f\|_0. \tag{2.22}
\]

From Corollary 3.3 and Theorem 4.1 in [35] we the following lemma.

**Lemma 2.1.** Assume \((u^f, p^f) \in H^{1+s}(\Omega)^d \times H^s(\Omega)\) for \(r < s \leq k\) and \(f \in H^l(\Omega)^d\) for \(0 \leq l \leq k + 1\), then

\[
\|u^f - u_h^f\|_h + \|p^f - p_h^f\|_0 \lesssim h^s(\|u^f\|_{1+s} + \|p^f\|_s) + h^{1+l}\|f\|_l. \tag{2.23}
\]
Denote \( I_h : V \cap C^0(\Omega)^d \to V_h \cap V \) as the interpolation operator, and denote \( g_h : H^s(\Omega) \to Q_h \) as the local \( L^2 \) projection operator satisfying \( g_h p|_\kappa \in \mathbb{P}_{k-1}(\kappa) \) and
\[
\int_{\kappa} (p - g_h p) v dx = 0, \quad \forall v \in \mathbb{P}_{k-1}(\kappa), \quad \forall \kappa \in \pi_h.
\]

Before estimating the error of velocity in the sense of \( L^2 \) norm, we introduce an auxiliary problem:
\[
A(\omega, v) + B(v, q) = (u^f - u_h^f, v), \quad \forall v \in V,
\]
\[
B(\omega, z) = 0, \quad \forall z \in Q. \tag{2.24}
\]

From \( (2.15) \) we have
\[
\|\omega\|_{1+r} + \|q\|_r \lesssim \|u^f - u_h^f\|_0. \tag{2.26}
\]

Referring to Theorem 6.12 in [19], by Nitsche’s technique we can deduce the following lemma.

**Lemma 2.2.** Suppose that the conditions of Lemma 2.1 and \( (2.15) \) hold, then
\[
\|u^f - u_h^f\|_0 \lesssim h^r(\|u^f - u_h^f\| + \|p^f - p_h^f\|_0). \tag{2.27}
\]

By the above error estimates of the DG method for the source problem, next we can deduce the error estimates of the DG method for the eigenvalue problem.

By \( (2.27), (2.23) \) and \( (2.15) \), we have
\[
\|T_h - T\|_0 \to 0, \quad (h \to 0). \tag{2.28}
\]

Thus, using Babuška-Osborn spectral approximation theory [46, 47], we can get (see Lemma 2.3 in [48]):

**Lemma 2.3.** Assume that the regularity estimate \( (2.15) \) is valid. Let \((\lambda, u, p)\) and \((\lambda_h, u_h, p_h)\) be the \( j \)th eigenpair of \( (2.2) - (2.3) \) and \( (2.4) - (2.5) \), respectively. Then
\[
\|u_h - u\|_0 \leq C\|(T - T_h)u\|_0, \tag{2.29}
\]
\[
\lambda_h - \lambda = \lambda^{-2}\|(T - T_h)u, u\|_2, \tag{2.30}
\]

where \(|R| \lesssim \|(T - T_h)u\|_2^3\).

From \( (2.8) \) and \( (2.9) \) we know that \( |\cdot| \) is a norm stronger than \( \|\cdot\|_h \), i.e., \( \|v\|_h \lesssim |v|_h \).

Additionally, we have
\[
|\|u - u_h\|^2 | \lesssim \|u - u_h\|_0^2 + \sum_{\kappa \in T_h} h_{\kappa}^{2r} |u - I_hu|_{1+r, \kappa}^2. \tag{2.31}
\]

In fact, from the inverse estimate, the interpolation estimate and the trace inequality, we deduce
\[
\sum_{E \in e_h} h_E \|\nabla (u - u_h)\|_{0, E}^2 \lesssim \sum_{E \in e_h} h_E \|\nabla (I_h u - u_h)\|_{0, E}^2 + \sum_{E \in e_h} h_E \|\nabla (u - I_h u)\|_{0, E}^2
\]
\[
\lesssim \sum_{\kappa \in T_h} \|\nabla (I_h u - u_h)\|_{0, \kappa}^2 + \sum_{\kappa \in T_h} (h_{\kappa}^{2r} |u - I_h u|_{0, \kappa}^2 + |u - I_h u|_{1, \kappa}^2 + h_{\kappa}^{2r} |u - I_h u|_{1+r, \kappa}^2)
\]
\[
\lesssim \|u - u_h\|_0^2 + \sum_{\kappa \in T_h} h_{\kappa}^{2r} |u - I_h u|_{1+r, \kappa}^2.
\]
By the above inequality and (2.3) we obtain (2.31).

**Theorem 2.1.** Let \((\lambda, u, p)\) and \((\lambda_h, u_h, p_h)\) be the \(j\)th eigenpair of \((2.22) - (2.23)\) and \((2.24) - (2.25)\), respectively. Assume that the regularity estimate (2.15) is valid, and \((u, p) \in H^{1+s}(\Omega)^d \times H^s(\Omega) (r \leq s \leq k)\). Then

\[
\|u - u_h\|_0 \lesssim h^r (\|u - u_h\| + \|p - p_h\|_0),
\]

\[
\|\lambda - \lambda_h\|_0 \lesssim h^{2s},
\]

\[
\|u - u_h\|_h + \|p - p_h\|_0 \lesssim h^r (\|u\|_{1+s} + \|p\|_s).
\]

**Proof.** Taking \(f = \lambda u\) in (2.21) and (2.24), then we get \(u^f = \lambda T_h u, p^f = \lambda S_h u\). Therefore, from (2.23) we have

\[
\|\lambda T_h u - \lambda T_h u\|_h + \|\lambda S u - \lambda S_h u\|_0 \lesssim h^r (\|u\|_{1+s} + \|p\|_s).
\]

From (2.16) and (2.35) we deduce

\[
((T - T_h)u, u) = A_h((T - T_h)u, Tu) + B_h((T - T_h)u, Su)
\]

\[
= A_h((T - T_h)u, Tu - T_hu) + B_h((T - T_h)u, Su - S_hu)
\]

\[
\lesssim h^{2r} (\|u\|_{1+s} + \|p\|_s)^2.
\]

Substituting (2.36) and (2.27) into (2.30) yields (2.33).

A simple calculation shows that

\[
\|u - u_h\|_h + \|\lambda T u - \lambda T_h u\|_h \leq \|\lambda_h T_h u - \lambda T_h u\|_h = \|T_h(\lambda_h u_h - \lambda u) - \lambda T_h u\|_h \lesssim \|\lambda_h u_h - \lambda u\|_0,
\]

\[
\|p - p_h\|_0 + \|\lambda S u - \lambda S_h u\|_h \leq \|\lambda_h S_h u_h - \lambda S_h u\|_h \lesssim \|\lambda_h u_h - \lambda u\|_0,
\]

thus, from (2.29), (2.30), (2.27) and the above two estimates, we deduce

\[
\|\lambda_h u_h - \lambda u\|_0 \lesssim \|\lambda - \lambda_h\| + \|u - u_h\|_0 \lesssim \|\lambda T u - \lambda T_h u\|_0
\]

\[
\lesssim h^{2r} (\|\lambda T u - \lambda T_h u\| + \|\lambda S u - \lambda S_h u\|) \lesssim h^r (\|u - u_h\| + \|p - p_h\|_0).
\]

Thus, we get (2.32). Since it is valid the relationship \(\simeq\) in (2.37) and (2.35), we get (2.34). \(\square\)

3 A posteriori error estimate for the Stokes eigenvalue problem

3.1 The a posteriori error indicator and its reliability for the eigenfunctions

Let \((\lambda_h, u_h, p_h) \in R^+ \times V_h \times Q_h\) be an eigenpair approximation. To begin with, for each element \(e \in \pi_h\) we introduce the residuals

\[
\eta_{Re}^2 = h^2 \|\lambda_h u_h + \Delta u_h - \nabla p_h\|_{0,e}^2 + \|\text{div} u_h\|_{0,e}^2,
\]

\[
\eta_{Ee}^2 = \frac{1}{2} \sum_{E \subset \partial e \setminus \partial \Omega} h_E \|[(p_h I - \nabla u_h)]\|_{0,E}^2,
\]

where \(I\) denotes the \(d \times d\) \((d = 2, 3)\) identity matrix. Next, we introduce the following estimator \(\eta_{Is}\) to measure the jump of the approximate solution \(u_h\):

\[
\eta_{Is}^2 = \sum_{E \subset \partial e, E \in \pi_h} \gamma h^2 \|\lambda_h^{-1} ||u_h||_{2,E}^2 + \sum_{E \subset \partial e, E \in \pi_h} \gamma h^{-1} ||u_h \otimes u_h||_{0,E}^2.
\]
The local error indicator is defined as
\[
\eta^2_\kappa = \eta^2_{R_h} + \eta^2_{E_h} + \eta^2_{J_h}.
\]

Finally, we introduce the global a posteriori error estimator
\[
\eta_h = \left( \sum_{\kappa \in \pi_h} \eta^2_\kappa \right)^{\frac{1}{2}}.
\]

For \( \kappa \in \pi_h \), denote \( \theta_\kappa = \text{int} \{ \bigcup_{\pi_i \neq \emptyset} \tilde{\kappa}_i, \kappa_i \in \pi_h \} \) and \( \theta_E \) is the set of all elements which share at least one node with face \( E \). Let \( \mathbf{v}^I \) be the Scott-Zhang interpolation function \([49]\), then \( \mathbf{v}^I \in \mathbf{V} \cap \mathbf{V}_h \) and
\[
\| \mathbf{v} - \mathbf{v}^I \|_{0, \kappa} \leq h_\kappa \| \nabla (\mathbf{v} - \mathbf{v}^I) \|_{0, \kappa} \lesssim h_\kappa |\mathbf{v}|_{1, \theta_\kappa}, \quad \forall \kappa \in \pi_h, \quad (3.1)
\]
\[
\| \mathbf{v} - \mathbf{v}^I \|_{0, E} \lesssim h_E^{\frac{1}{2}} |\mathbf{v}|_{1, \theta_E}, \quad \forall E \subset \partial \kappa. \quad (3.2)
\]

Denote
\[
\sum_h = \{ \tilde{\tau} \in L^2(\Omega)^{d \times d} : \tilde{\tau}|_\kappa \in \mathbb{P}_k(\kappa)^{d \times d}, \kappa \in \pi_h \}.
\]

We introduce the lifting operator \( \mathcal{L} : \mathbf{V} + \mathbf{V}_h \to \sum_h \) by
\[
\int_{\Omega} \mathcal{L}(\mathbf{v}) : \tilde{\tau} dx = \sum_{E \in \varepsilon_h} \int_{\partial E} [\tilde{\tau}] dx, \quad \forall \tilde{\tau} \in \sum_h. \quad (3.3)
\]

Moreover, from \([43, 44]\), the lifting operator has the ability property
\[
\| \mathcal{L}(\mathbf{v}) \|^2_0 \lesssim \sum_{E \in \varepsilon_h} \| h_\kappa^{-\frac{1}{2}} [\mathbf{v}] \|^2_0, \quad \forall \mathbf{v} \in \mathbf{V} + \mathbf{V}_h. \quad (3.4)
\]

Using this operator, we introduce an auxiliary bilinear form
\[
\tilde{\mathcal{A}}_h(\cdot, \cdot) : V(h) \times V(h) \to R \quad (3.5)
\]
defined by
\[
\tilde{\mathcal{A}}_h(\mathbf{w}, \mathbf{v}) = \sum_{\kappa \in \pi_h} \sum_{E \in \varepsilon_h} \int_{E} \nabla \mathbf{w} : \nabla \mathbf{v} dx - \sum_{\kappa \in \pi_h} \int_{\partial \kappa} \mathcal{L}(\mathbf{v}) : \nabla \mathbf{w} dx - \sum_{\kappa \in \pi_h} \int_{\partial \kappa} \mathcal{L}(\mathbf{w}) : \nabla \mathbf{v} dx + \sum_{E \in \varepsilon_h} \int_{E} \gamma_E [\mathbf{w}] : [\mathbf{v}] ds. \quad (3.6)
\]

Since \( \tilde{\mathcal{A}}_h = \mathcal{A}_h \) on \( \mathbf{V}_h \times \mathbf{V}_h \), the DGFEM presented in \((2.4)-(2.5)\) is equivalent to finding \((\lambda_h, \mathbf{u}_h, p_h) \in R^+ \times V_h \times Q_h\) and satisfying
\[
\tilde{\mathcal{A}}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathcal{B}_h(\mathbf{v}_h, p_h) = \lambda_h(\mathbf{u}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (3.7)
\]
\[
\mathcal{B}_h(\mathbf{u}_h, q_h) = 0, \quad \forall q_h \in Q_h.
\]

**Lemma 3.1.** Let \( (\mathbf{u}^f, p^f) \) and \( (\mathbf{u}^f_h, p^f_h) \) be the solutions of \((2.11)-(2.12)\) and \((2.13)-(2.14)\), respectively. Then
\[
\| \mathbf{u}^f - \mathbf{u}^f_h \|_h + \| p^f - p^f_h \|_0 \lesssim \sup_{0 \neq \mathbf{v} \in \mathbf{V}} \frac{\| (\mathbf{f}, \mathbf{v}) - \tilde{\mathcal{A}}_h(\mathbf{u}^f_h, \mathbf{v}) - \mathcal{B}_h(\mathbf{v}, p^f_h) \|_h}{\| \mathbf{v} \|_h} + \inf_{\mathbf{v} \in \mathbf{V}} \| \mathbf{u}^f_h - \mathbf{v} \|_h. \quad (3.8)
\]
Proof. For $\forall \bar{u} \in V$, from (2.11) we have

$$\tilde{A}_h(u^f - \bar{u}, u^f - \bar{u}) = \tilde{A}_h(u^f, u^f - \bar{u}) - \tilde{A}_h(\bar{u}, u^f - \bar{u})$$
$$= (f, u^f - \bar{u}) - B_h(u^f - \bar{u}, p^f) - \tilde{A}_h(u^f, u^f - \bar{u}) + \tilde{A}_h(u^f - \bar{u}, u^f - \bar{u}).$$

For $\forall \bar{u} \in V$, $\forall p \in Q$, we have

$$B_h(u^f - \bar{u}, p^f - \tilde{p}) = B_h(u^f - \bar{u}, p^f) - B_h(u^f - \bar{u}, p^f_h) - B_h(u^f - \bar{u}, \tilde{p} - p^f_h).$$

Suming the above two equations and taking $v = u^f - \bar{u}$, we deduce

$$\|u^f - \bar{u}\|_h + \|p^f - \tilde{p}\|_0 \lesssim \sup_{v \in V} \frac{(f, v) - \tilde{A}_h(u^f, v) - B_h(v, p^f_h)}{\|v\|_h}$$
$$+ \|u^f_h - \bar{u}\|_h + \|\tilde{p} - p^f_h\|_0, \quad \forall (\bar{u}, \tilde{p}) \in V \times Q. \quad (3.9)$$

From the triangle inequality we have

$$\|u^f - u^f_h\|_h + \|p^f - p^f_h\|_0 \lesssim \sup_{v \in V} \frac{(f, v) - \tilde{A}_h(u^f_h, v) - B_h(v, p^f_h)}{\|v\|_h}$$
$$+ \|u^f_h - \bar{u}\|_h + \|\tilde{p} - p^f_h\|_0, \quad \forall (\bar{u}, \tilde{p}) \in V \times Q. \quad (3.10)$$

Since $(\bar{u}, \tilde{p})$ is arbitrary and $\inf_{\tilde{p} \in Q} \|\tilde{p} - p^f_h\|_0 = 0$, the part $\lesssim$ in (3.8) is valid. The other part $\gtrsim$ in (3.8) is obvious.

Lemma 3.1 can be extended to the eigenvalue problems.

Theorem 3.1. Let $(\lambda_j, u_j, p_j)$ and $(\lambda_h, u_h, p_h)$ be the $j$th eigenpair of (2.2)-(2.3) and (2.4)-(2.5), respectively. Then

$$\|u - u_h\|_h + \|p - p_h\|_0 \gtrsim \sup_{0 \neq v \in V} \frac{\tilde{A}_h(u - u_h, v) + B_h(v, p - p_h)}{\|v\|_h} + \inf_{v \in V} \|u_h - v\|_h. \quad (3.12)$$

Proof. By (2.22) we deduce

$$\|u - u_h\|_h + \|p - p_h\|_0 = \|\lambda T u - \lambda_h T u_h + \lambda_h T u_h - \lambda_h T u_h\|_h$$
$$\lesssim \|\lambda T u - \lambda_h T u_h + \lambda_h S u_h - \lambda_h S u_h\|_h$$
$$\lesssim \|\lambda T u_h - \lambda_h T u_h\|_h + \|\lambda_h S u_h - \lambda_h S u_h\|_h + \|\lambda u - \lambda_h u_h\|_0.$$
From (3.11) with \( f = \lambda_h u_h \) and (3.22), we deduce

\[
|((\lambda_h u_h, v) - \tilde{A}_h(\lambda_h T_h u_h, v) - B_h(v, S_h(\lambda_h u_h)))|
\]
\[
= |\tilde{A}_h(\lambda_h T_h u_h, v) - B_h(v, S(\lambda_h u_h)) - \tilde{A}_h(\lambda_h T_h u_h, v) - B_h(v, S_h(\lambda_h u_h))|
\]
\[
= |\tilde{A}_h(\lambda_h T_h u_h - \lambda_h T_h u_h, v) + B_h(v, S(\lambda_h u_h) - S_h(\lambda_h u_h))|
\]
\[
= |\tilde{A}_h(\lambda_h T_h u_h - \lambda T_h u + u - u_h, v) + B_h(v, S(\lambda_h u_h) - S(\lambda u) + p - p_h)|
\]
\[
\leq |\tilde{A}_h(u - u_h, v) + B_h(v, p - p_h)| + C\|\lambda_h u_h - \lambda u\|_0\|v\|_h. 
\]  

(3.15)

Substituting (3.15) into (3.14), we get

\[
\|\lambda_h T u_h - \lambda S u_h - \lambda_h S u_h\|_0 
\leq \sup_{0 \neq v \in V} \frac{|\tilde{A}_h(u - u_h, v) + B_h(v, p - p_h)|}{\|v\|_h} + C\|\lambda_h u_h - \lambda u\|_0 + \inf_{v \in V} \|u_h - v\|_h . 
\]  

(3.16)

From Theorem 2.1 we know that \( \|\lambda_h u_h - \lambda u\|_0 \) is a small quantity of higher order compared with \( \|u - u_h\|_h + \|p - p_h\|_0 \). Substituting (3.16) into (3.14), the side \( \lesssim \) in (3.12) is true. The other side \( \gtrsim \) in (3.12) is obvious. □

Lemma 3.2. Under the conditions of Theorem 2.1, there holds

\[
\tilde{A}_h(u - u_h, v) + B_h(v, p - p_h) \lesssim \sum_{\kappa \in \pi_h} (\eta_{R_\kappa} + \eta_{E_\kappa} + \eta_{I_\kappa}) \|v\|_h + \|\lambda u - \lambda_h u_h\|_0 \|v\|_h, \quad \forall v \in V .
\]  

(3.17)

Proof. Using (3.7), (3.6), (2.7) and Green’s formula we deduce that

\[
\tilde{A}_h(u - u_h, v) + B_h(v, p - p_h) = \tilde{A}_h(u, v) - \tilde{A}_h(u_h, v) + B_h(v, p) - B_h(v, p_h)
\]
\[
= \lambda(u, v) - \tilde{A}_h(u_h, v) - B_h(v, p_h)
\]
\[
= \lambda \sum_{\kappa \in \pi_h} \int_{\kappa} uv dx - \sum_{\kappa \in \pi_h} \int_{\kappa} \nabla u_h : \nabla v dx + \sum_{\kappa \in \pi_h} \int_{\kappa} L(v) : \nabla u_h dx
\]
\[
+ \sum_{\kappa \in \pi_h} \int_{\kappa} L(u_h) : \nabla v dx + \sum_{\kappa \in \pi_h} \int_{\kappa} \frac{E}{\hbar} [u_h] : [v] ds + \sum_{\kappa \in \pi_h} \int_{\kappa} div u_h p_h dx
\]
\[
= \lambda \sum_{\kappa \in \pi_h} \int_{\kappa} uv dx + \sum_{\kappa \in \pi_h} \int_{\kappa} \Delta u_h \cdot v dx - \sum_{\kappa \in \pi_h} \int_{E \subset \Omega \kappa} \frac{\partial u_h}{\partial n} \cdot v ds + \sum_{\kappa \in \pi_h} \int_{\kappa} L(v) : \nabla u_h dx
\]
\[
+ \sum_{\kappa \in \pi_h} \int_{\kappa} L(u_h) : \nabla v dx - \sum_{\kappa \in \pi_h} \int_{\kappa} \nabla p_h \cdot v dx + \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa} \int_{E} p_h v \cdot n ds . 
\]  

(3.18)

By \( v' \in V \cap V_h , (2.2) , (2.3) \) and (2.4)-(2.5), we obtain

\[
\tilde{A}_h(u - u_h, v) + B_h(v, p - p_h) = \tilde{A}_h(u - u_h, v - v') + B_h(v - v', p - p_h) + \sum_{\kappa \in \pi_h} \int_{\kappa} (\lambda u - \lambda_h u_h) \cdot v' dx.
\]
Using (3.16), Cauchy-Schwarz inequality, (3.11) and (3.2), (3.18) can be written as follows:

\[
\tilde{A}_h(u - u_h, v) + B_h(v, p - p_h) = \lambda \sum_{\kappa \in \pi_h} \int_{\kappa} u(v - v^I)dx + \sum_{\kappa \in \pi_h} \int_{\kappa} \Delta u_h \cdot (v - v^I)dx - \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa} \int_{E} \frac{\partial u_h}{\partial n} \cdot (v - v^I)ds
\]

\[+ \sum_{\kappa \in \pi_h} \int_{\kappa} \mathcal{L}(v - v^I) : \nabla u_h dx + \sum_{\kappa \in \pi_h} \int_{\kappa} \mathcal{L}(u_h) : \nabla(v - v^I)dx - \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa} \int_{E} \nabla p_h \cdot (v - v^I)ds \]

\[+ \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa} \int_{E} p_h(v - v^I) \cdot nds + \sum_{\kappa \in \pi_h} \int_{\kappa} (\lambda u - \lambda_h u_h) \cdot v^I dx \]

\[= \sum_{\kappa \in \pi_h} \int_{\kappa} (\Delta u_h + \lambda_h u_h - \nabla p_h) \cdot (v - v^I)dx - \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa} \int_{E} \frac{\partial u_h}{\partial n} \cdot (v - v^I)ds \]

\[+ \sum_{\kappa \in \pi_h} \int_{\kappa} \mathcal{L}(v - v^I) : \nabla u_h dx + \sum_{\kappa \in \pi_h} \int_{\kappa} \mathcal{L}(u_h) : \nabla(v - v^I)dx \]

\[+ \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa} \int_{E} p_h(v - v^I) \cdot nds + \sum_{\kappa \in \pi_h} \int_{\kappa} (\lambda u - \lambda_h u_h) \cdot v^I dx \]

\[
\equiv B_1 + B_2 + B_3 + B_4 + B_5.
\]

Next, we will analyze each item on the right-hand side of (3.19). Using the Cauchy-Schwarz inequality and the approximation property (3.1) and (3.2), we have

\[
|B_1| \leq \sum_{\kappa \in \pi_h} \| \Delta u_h + \lambda_h u_h - \nabla p_h \|_{0, \kappa} \| v - v^I \|_{0, \kappa}
\]

\[\lesssim \sum_{\kappa \in \pi_h} h_{\kappa} \| \Delta u_h + \lambda_h u_h - \nabla p_h \|_{0, \kappa} \| v \|_{1, \theta_{\kappa}}^2 \]

\[\lesssim \left( \sum_{\kappa \in \pi_h} h_{\kappa}^2 \| \Delta u_h + \lambda_h u_h - \nabla p_h \|_{0, \kappa}^2 \right)^{\frac{1}{2}} \| v \|_h.\]

For the second term on the right-hand side of (3.19), from (3.2) we obtain

\[
|B_2| = \frac{1}{2} \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa \setminus \partial \Omega} \int_{E} \| [p_h I - \nabla u_h] \cdot (v - v^I)ds \]

\[\leq \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa \setminus \partial \Omega} \| [p_h I - \nabla u_h] \|_{0, E} C h_{E} \| v \|_{1, \theta_E} \]

\[\lesssim \left( \sum_{\kappa \in \pi_h} \sum_{E \subset \partial \kappa \setminus \partial \Omega} (h_{E}^{\frac{1}{2}} \| [p_h I - \nabla u_h] \|_{0, E})^2 \right)^{\frac{1}{2}} \| v \|_h.\]

For the third term, by the properties of the interpolation function $v^I$, we know $[v - v^I] = 0$. Therefore, from the definition of lifting operation $\mathcal{L}$ we have

\[
B_3 = \sum_{\kappa \in \pi_h} \int_{\kappa} \mathcal{L}(v - v^I) : \nabla u_h ds = \sum_{E \in \pi_h} \int_{E} \{ \nabla u_h \} : [v - v^I] ds = 0.
\]
For the fourth term, using the Cauchy-Schwarz inequality, (3.21) and (3.23) we get

\[
|B_4| \leq \left( \sum_{\kappa \in \pi_h} \| \mathcal{L}(u_h) \|_{0, \kappa}^2 \right)^{\frac{1}{2}} \left( \sum_{\kappa \in \pi_h} \| \nabla (v - v') \|_{0, \kappa}^2 \right)^{\frac{1}{2}} \leq \left( \sum_{E \in \mathcal{E}_h} \| h^{-\frac{1}{2}} [u_h] \|_{0, E}^2 \right)^{\frac{1}{2}} \left( \sum_{E \in \mathcal{E}_h} \| \nabla (v - v') \|_{0, E}^2 \right)^{\frac{1}{2}} \approx \left( \sum_{E \in \mathcal{E}_h} \| h^{-\frac{1}{2}} [u_h] \|_{0, E}^2 \right)^{\frac{1}{2}} \| v \|_h.
\]

For the last term on the right-hand side of (3.19), we have

\[
B_5 = \sum_{\kappa \in \pi_h} \int_{\kappa} (\lambda u - \lambda_b u_h) \cdot v \, dx \leq \| \lambda u - \lambda_b u_h \|_0 \| v \|_0.
\]

Substituting $B_1, B_2, B_3, B_4, B_5$ into (3.19), we obtain the desired result (3.17). \(\square\)

In [41, 42] the authors construct the enriching operator $E_h : V_h \to V_h \cap V$ by averaging and prove the following lemma.

**Lemma 3.3.** It is valid the following estimate:

\[
\| u_h - E_h u_h \| \lesssim \sum_{E \in \mathcal{E}_h} \gamma h^{-1}_E |[u_h]|_{0, E}^2 + \sum_{E \in \mathcal{E}_h} \gamma h^{-1}_E |u_h \otimes u_h|_{0, E}^2.
\]

**Theorem 3.2.** Suppose that the conditions of Theorem 2.1 hold, then there holds

\[
\| u - u_h \| + \| p - p_h \|_0 \lesssim \eta_h.
\]

**Proof.** Substituting (3.17) and (3.20) into (3.21), we obtain (3.22). \(\square\)

### 3.2 The efficiency of the indicators for eigenfunctions

This section is devoted to prove an efficiency bound for $\eta$. To prove the results, we use the bubble function technique which was introduced in [2].

Let $\kappa$ be an element of $\pi_h$. Let $b_\kappa$ and $b_\partial \kappa$ be the standard bubble function on element $\kappa$ and face $E$ ($d = 3$) or edge $E$ ($d = 2$) of $\kappa$, respectively. Then, from [2, 3, 50] we have the following results.

**Lemma 3.4.** For any vector-valued polynomial function $v_h$ on $\kappa$, there hold

\[
\| v_h \|_{0, \kappa} \lesssim \| b_\kappa v_h \|_{0, \kappa},
\]

\[
\| b_\kappa v_h \|_{0, \kappa} \lesssim \| v_h \|_{0, \kappa},
\]

\[
\| \nabla (b_\kappa v_h) \|_{0, \kappa} \lesssim h^{-1}_\kappa |v_h|_{0, \kappa}.
\]

For any vector-valued polynomial function $\sigma$ on $E$, it is valid that

\[
\| b_E \sigma \|_{0, E} \lesssim \| \sigma \|_{0, E},
\]

\[
\| \sigma \|_{0, E} \lesssim \| b^{1/2}_E \sigma \|_{0, E}.
\]

Furthermore, for each $b_E \sigma$, there exists an extension $\sigma_b \in H^1_0(\omega(E))$ satisfying $\sigma_b|_E = b_E \sigma$ and

\[
\| \sigma_b \|_{0, \kappa} \lesssim h^{-1/2}_E \| \sigma \|_{0, E}, \quad \forall \kappa \in \omega(E),
\]

\[
\| \nabla \sigma_b \|_{0, \kappa} \lesssim h^{-1/2}_E \| \sigma \|_{0, E}, \quad \forall \kappa \in \omega(E).
\]
From the above lemma and using the standard arguments (see [20], Lemma 3.13), we can prove the following local bounds.

**Lemma 3.5.** Under the conditions of Theorem 2.1, there holds
\[
\eta_{R_{\kappa}} \lesssim \|\nabla(u - u_h)\|_{0,\kappa} + \|p - p_h\|_{0,\kappa} + h_{\kappa}\|\lambda_h u_h - \lambda u\|_{0,\kappa}.
\] (3.29)

**Proof.** For any \(\kappa \in \pi_h\), define the function \(R\) and \(W\) locally by
\[
R|_{\kappa} = \lambda_h u_h + \Delta u_h - \nabla p_h \quad \text{and} \quad W|_{\kappa} = h_{\kappa}^2 R b_{\kappa}.
\]
From (3.22) and using \(\lambda u + \Delta u - \nabla p = 0\), we have
\[
h_{\kappa}^2\|R\|_{0,\kappa}^2 \lesssim \int_{\kappa} R \cdot (h_{\kappa}^2 R b_{\kappa}) dx = \int_{\kappa} (\lambda_h u_h + \Delta u_h - \nabla p_h) \cdot W dx
\]
\[
= \int_{\kappa} (\lambda_h u_h + \Delta u_h - \nabla p_h - (\lambda u + \Delta u - \nabla p)) \cdot W dx
\]
\[
= \int_{\kappa} \Delta(u_h - u) \cdot W dx - \int_{\kappa} (p_h - p) \cdot W dx + \int_{\kappa} (\lambda_h u_h - \lambda u) \cdot W dx.
\]
Using integration by parts and \(W|_{\partial \kappa} = 0\), we obtain
\[
h_{\kappa}^2\|R\|_{0,\kappa}^2 \lesssim \int_{\kappa} \nabla(u - u_h) \cdot \nabla W dx + \int_{\kappa} (p_h - p) \text{div} W dx + \int_{\kappa} (\lambda_h u_h - \lambda u) \cdot W dx.
\]
Applying Cauchy-Schwarz inequality yields
\[
h_{\kappa}^2\|R\|_{0,\kappa}^2 \lesssim (\|\nabla(u - u_h)\|_{0,\kappa} + \|p - p_h\|_{0,\kappa} + h_{\kappa}\|\lambda_h u_h - \lambda u\|_{0,\kappa})(\|\nabla W\|_{0,\kappa} + h_{\kappa}^{-1}\|W\|_{0,\kappa}).
\] (3.30)

From (3.23) and (3.24) we get
\[
\|\nabla W\|_{0,\kappa} + h_{\kappa}^{-1}\|W\|_{0,\kappa} \lesssim h_{\kappa}\|R\|_{0,\kappa}.
\]
Dividing (3.30) by \(h_{\kappa}\|R\|_{0,\kappa}\) and noting \(\|\nabla \cdot u_h\|_0 = \|\nabla \cdot (u_h - u)\|_0\), we finish the proof. \(\square\)

**Lemma 3.6.** Under the conditions of Theorem 2.1, there holds
\[
\eta_{E_{\kappa}}^2 \lesssim \|\nabla(u - u_h)\|_{0,\omega(\kappa)} + \|p - p_h\|_{0,\omega(\kappa)} + \left(\sum_{\kappa \in \omega(\kappa)} h_{\kappa}^2\|\lambda u - \lambda_h u_h\|_{0,\kappa}^2\right)^{\frac{1}{2}}.
\]

**Proof.** For any interior edge \(E \in \varepsilon_h\), let the function \(R\) and \(\Lambda\) be such that
\[
R|_E = [p_b I - \nabla u_h]_E \quad \text{and} \quad \Lambda = h_E R b_E.
\]
Using (3.20) and \([p I - \nabla u]_E = 0\), we get
\[
h_E\|R\|_{0,E}^2 \lesssim \int_E R \cdot (h_E R b_E) ds = \int_E [(p_h - p) I - \nabla(u_h - u)] \cdot \Lambda ds.
\]
Applying Green’s formula over each of the two elements of \(\omega(E)\), we derive
\[
h_E\|R\|_{0,E}^2 \lesssim \int_E [(p_h - p) I - \nabla(u_h - u)] \cdot \Lambda ds
\]
\[
= C\left(\sum_{\kappa \in \omega(E)} \int_{\kappa} (-\Delta (u - u_h) + \nabla(p - p_h)) \cdot \Lambda dx - \sum_{\kappa \in \omega(E)} \int_{\kappa} (\nabla(u - u_h) - (p - p_h) I) : \nabla \Lambda dx\right)
\]
Using $\lambda u + \Delta u - \nabla p = 0$, we obtain
\[
\begin{align*}
    h_E \| R \|_{0,E}^2 &\lesssim \sum_{\kappa \in \Omega(E)} \int_K (\lambda_h u + \Delta u_h - \nabla p_h) \cdot \Lambda dx + \sum_{\kappa \in \Omega(E)} \int_K (\lambda u - \lambda_h u_h) \cdot \Lambda dx \\
    &+ \sum_{\kappa \in \Omega(E)} \int_K (\nabla (u - u_h) + (p - p_h) I) : \nabla \Lambda dx \equiv T_1 + T_2 + T_3.
\end{align*}
\]  

(3.31)

Using Cauchy-Schwarz inequality, shape-regularity of the mesh, (3.27) and (3.28) yields
\[
\begin{align*}
    T_1 &\lesssim \left( \sum_{\kappa \in \Omega(E)} \eta_{r_{\kappa}}^2 \right)^{1/2} \left( \sum_{\kappa \in \Omega(E)} h_{\kappa}^{-2} \| A \|_{0,\kappa}^2 \right)^{1/2} \lesssim \left( \sum_{\kappa \in \Omega(E)} \eta_{r_{\kappa}}^2 \right)^{1/2} h_E^{1/2} \| R \|_{0,E}, \\
    T_2 &\lesssim \left( \sum_{\kappa \in \Omega(E)} \left( h_{\kappa}^2 \| \lambda u - \lambda_h u_h \|_{0,\kappa}^2 \right) \right)^{1/2} h_E^{1/2} \| R \|_{0,E}, \\
    T_3 &\lesssim \left( \sum_{\kappa \in \Omega(E)} \left( \| \nabla (u - u_h) \|_{0,\kappa}^2 + \| p - p_h \|_{0,\kappa}^2 \right) \right)^{1/2} h_E^{1/2} \| R \|_{0,E}.
\end{align*}
\]

Combining the above estimates of $T_1$, $T_2$ and $T_3$, dividing (3.31) by $h_E^{1/2} \| R \|_{0,E}$ and summing over all interior edges of $\kappa$, we get the desired result. 

\section*{Lemma 3.7.} Under the conditions of Theorem 2.1, there holds
\[
\eta_{r_{\kappa}}^2 = \sum_{E \subset \partial \kappa, E \in \mathcal{E}_h^b} \gamma h_{\kappa}^{-1} \| u_h - u \|_{0,E}^2 + \sum_{E \subset \partial \kappa, E \in \mathcal{E}_h^b} \gamma h_{\kappa}^{-1} \| u_h - u \|_{0,E}^2.
\]

\textbf{Proof.} For any $E \in \mathcal{E}_h^b (\Omega)$, $[\mathbf{u}] = 0$, and for any $E \in \mathcal{E}_h \cap \partial \Omega$, $\mathbf{u} \otimes \mathbf{n} = 0$. Therefore, we obtain the desired result. 

\section*{Theorem 3.3.} Suppose that the conditions of Theorem 2.1 hold. Then the a posteriori error estimator $\eta_h$ is efficient:
\[
\begin{align*}
    \eta_{r_{\kappa}}^2 &\lesssim \sum_{\kappa \in \Omega(E)} \left( \| u - u_h \|_{0,\kappa}^2 + \| p - p_h \|_{0,\kappa}^2 + h_{\kappa}^2 \| \lambda u - \lambda_h u_h \|_{0,\kappa}^2 \right), \\
    \eta_{\kappa}^2 &\lesssim \| u - u_h \|_{h}^2 + \| p - p_h \|_{h}^2 + \sum_{\kappa \in \pi_h} h_{\kappa}^2 \| \lambda u - \lambda_h u_h \|_{0,\kappa}^2.
\end{align*}
\]  

(3.32) 

(3.33)

\textbf{Proof.} The conclusions follow from a combination of Lemmas 3.5-3.7. 

\section*{3.3 The reliability of the indicators for the eigenvalues}

\section*{Lemma 3.8.} Let $(\lambda, u, p)$ and $(\lambda_h, u_h, p_h)$ be the eigenpairs of (2.2)-(2.3) and (2.4)-(2.5), respectively, then
\[
\lambda_h - \lambda = A_h (u - u_h, u - u_h) + 2B_h (u - u_h, p - p_h) - \lambda (u - u_h, u - u_h).
\]  

(3.34)

\textbf{Proof.} By the consistency formulas (2.10)-(2.17) we get
\[
\begin{align*}
    A_h (u, v_h) + B_h (v_h, p) &= \lambda (u, v_h), \quad \forall v_h \in V_h, \\
    B_h (u, q_h) &= 0, \quad \forall q_h \in Q_h.
\end{align*}
\]  

(3.35) 

(3.36)
From $[22],[23]$ with $(v,q) = (u,p)$, $[24]-[26]$ with $(v_h,q_h) = (u_h,p_h)$ and $[53]-[56]$, we deduce

\[
A_h(u - u_h, u - u_h) + 2B_h(u - u_h, p - p_h) - \lambda(u - u_h, u - u_h) = A_h(u, u) - 2A_h(u, u_h) + A_h(u_h, u_h) + 2B_h(u, p) - 2B_h(u_h, p) - 2B_h(u_h, p_h) - \lambda(u_h, u_h) + 2\lambda(u, u_h) - \lambda(u_h, u_h) = \lambda_h(u_h, u_h) - \lambda(u_h, u_h) = \lambda_h - \lambda.
\]

The proof is completed. \(\square\)

**Theorem 3.4.** Under the conditions of Theorem 2.1, there holds

\[
|\lambda - \lambda_h| \lesssim \eta^2_h + \sum_{\kappa \in T_h} h^{2r}_\kappa (|u - I_h u|_{1+r, \kappa}^2 + \|p - q_h p\|_r^2).
\]

**Proof.** Theorem 2.1 shows $\|u - u_h\|_{0,\Omega}$ is a term of higher order than $\|\|u - u_h\| + \|p - p_h\|_0$. Hence, from (3.34) and (3.21), we obtain

\[
|\lambda - \lambda_h| \lesssim \|\|u - u_h\| + \|p - p_h\|_0^2 + \sum_{E \in \varepsilon_h} h_E \|p - p_h\|_{0,E}^2.
\]

Thus, from $[23],[24]$ and $[32],[31]$ we obtain (3.37). \(\square\)

**Remark 3.1.** From Theorems 3.2 and 3.3, we know the indicator $\eta_h$ of the eigenfunction error $\|u - u_h\|_h + \|p - p_h\|_0$ is reliable and efficient up to data oscillation, so the adaptive algorithm based on the indicator can generate a good graded mesh, which makes the eigenfunction error $\|u - u_h\|_h + \|p - p_h\|_0$ can achieve the optimal convergence rate $O(dof^{-\frac{2}{d}})$. Thus, referring to $[51],[52]$ we are able to expect to get $\sum_{\kappa \in T_h} h^{2r}_\kappa (|u - I_h u|_{1+r, \kappa}^2 + \|p - q_h p\|_r^2) \lesssim dof^{-\frac{2}{d}}$

, thereby from (3.37) we have $|\lambda - \lambda_h| \lesssim dof^{-\frac{2}{d}}$. Therefore, we think that $\eta^2_h$ as the error indicator of $\lambda_h$ is reliable and efficient.

**Remark 3.2.** Assume that $\Omega$ can be subdivided into shape-regular affine meshes $\pi_h$ consisting of parallelograms $\kappa$ ($d = 2$) or parallelepipeds $\kappa$ ($d = 3$), and the discrete velocity and pressure spaces are given by

\[
V_h = \{v_h \in L^2(\Omega)^d : v_h|_\kappa \in Q_k(\kappa)^d, \forall \kappa \in \pi_h\},
\]

\[
Q_h = \{q_h \in Q : q_h|_\kappa \in Q_{k-1}(\kappa), \forall \kappa \in \pi_h\},
\]

where $Q_k(\kappa)$ denotes the space of tensor product polynomials on $\kappa$ of degree $k$ in each coordinate direction.

For the Stokes equation (2.10), Houston et al. [40] studied the a posteriori error estimation of mixed DGFEM using the above $Q_k - Q_{k-1}$ element. For the Stokes eigenvalue problem (2.1), all analysis and conclusions in this paper are valid for the mixed DGFEM using the above $Q_k - Q_{k-1}$ element.

### 4  Numerical experiments

Using the a posteriori error indicators in this paper and consulting the existing standard algorithms (see, e.g., [53]), we present the following algorithm.
Algorithm 4.1. Choose the parameter $0 < \theta < 1$.

Step 1. Pick any initial mesh $\pi_{h_0}$ with mesh size $h_0$.

Step 2. Solve (2.4)-(2.5) on $\pi_{h_0}$ for discrete solution $(\lambda_{h_0}, u_{h_0}, p_{h_0})$.

Step 3. Let $l = 0$.

Step 4. Compute the local indicator $\eta^2$.

Step 5. Construct $\hat{\pi}_{h_l} \subset \pi_{h_l}$ by Marking Strategy E.

Step 6. Refine $\pi_{h_l}$ to get a new mesh $\pi_{h_{l+1}}$ by procedure REFINE.

Step 7. Solve (2.4)-(2.5) on $\pi_{h_{l+1}}$ for discrete solution $(\lambda_{h_{l+1}}, u_{h_{l+1}}, p_{h_{l+1}})$.

Step 8. Let $l = l + 1$ and go to step 4.

Marking Strategy E.

Given parameter $0 < \theta < 1$:

Step 1. Construct a minimal subset $\hat{\pi}_{h_l} \subset \pi_{h_l}$ by selecting some elements in $\pi_{h_l}$ such that

$$\sum_{\kappa \in \hat{\pi}_{h_l}} \eta^2_\kappa \geq \theta \eta^2_{h_l}.$$  

Step 2. Mark all elements in $\hat{\pi}_{h_l}$.

The above marking strategy was introduced by Dörfler [3] (see also Morin et al. [4]).

We use the following notations in our tables:

$l$: the $l$th iteration in Algorithm 4.1.

$\lambda_{1,h_l}$: the first discrete eigenvalue at the $l$th iteration of Algorithm 4.1.

dof: the degrees of freedom at the $l$th iteration.

$: the calculation cannot proceed since the computer runs out of memory.

We carry out experiments in $d$-dimensional cases ($d=2, 3$). Our program is compiled under the package of iFEM [54] and we use internal command ‘eigs’ in MATLAB to solve matrix eigenvalue problem.

4.1 The results in two-dimensional domains

We carry out experiments on three two-dimensional domains: $\Omega_C = (-1, 1)^2 \setminus \{0 \leq x \leq 1, y = 0\}$, $\Omega_L = (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$ and $\Omega_S = (0, 1)^2$. The discrete eigenvalue problems are solved in MATLAB 2018a on a DELL PC with 1.80GHz CPU and 32GB RAM. We take $\theta = 0.5$ and initial mesh $\pi_{h_0}$ ($h_0 = \sqrt{2/16}$) for three two-dimensional domains. To compute the error of approximations of the first eigenvalue, we take $\lambda_C = 29.9168629$, $\lambda_L = 32.13269465$ and $\lambda_S = 52.344691168$ (see [33]) as the reference values for two-dimensional domains $\Omega_C$, $\Omega_L$ and $\Omega_S$, respectively.

The adaptive refined meshes and the error curves are shown in Figures 1-8. We show some adaptively refined meshes for $k = 1, 2, 3$ on the left side of Figures 1-8 from which we can see the strongly refinement towards the tip of the slit at the origin for $\Omega_C$ and $\Omega_L$ and a clear refinement near the four corners of $\Omega_S$. Furthermore, from Figures 1-8 we can see that the error curves and error indicators curves for DG methods using $P_k - P_{k-1}$ ($k = 1, 2, 3$) element are both approximately parallel to the line with slope $-k$, which indicates the error indicators are reliable and efficient and the adaptive algorithm can reach the optimal convergence order. It coincides with our theoretical results. It also can be seen from error curves that under the same dof, the approximations obtained by adaptive algorithm are more accurate than those computed on uniform meshes, and the approximations obtained by high order elements are more accurate than those computed by low order elements on both uniform meshes and adaptive meshes.

The approximations of the first eigenvalue for $\Omega_C$, $\Omega_L$ and $\Omega_S$ using $P_3 - P_2$ element are
listed in Tables 1-3. These eigenvalues have the same accuracy as those [14,33], which further proves that our method is effective.

Figure 1: Adaptive mesh after \( l=7 \) refinement times (left) and error curves (right) of the smallest eigenvalue by DGFEM using \( P_1 - P_0 \) element on \( \Omega_C \).

Figure 2: Adaptive mesh after \( l=7 \) refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using \( P_1 - P_0 \) element on \( \Omega_L \).

Figure 3: Adaptive mesh after \( l=15 \) refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using \( P_2 - P_1 \) element on \( \Omega_C \).
Figure 4: Adaptive mesh after $l=15$ refinement times (left) and error curves (right) of the smallest eigenvalue by DGFEM using $P_2 - P_1$ element on $\Omega_L$.

Figure 5: Adaptive mesh after $l=7$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_2 - P_1$ element on $\Omega_S$.

Figure 6: Adaptive mesh after $l=25$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_3 - P_2$ element on $\Omega_C$. 
Figure 7: Adaptive mesh after $l=25$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_3 - P_2$ element on $\Omega_L$.

Figure 8: Adaptive mesh after $l=5$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_3 - P_2$ element on $\Omega_S$.

Table 1: The smallest eigenvalue using $P_3 - P_2$ element on adaptive mesh on $\Omega_C$.

| $l$ | $\text{dof}$ | $\lambda_{1,h_l}$ | $l$ | $\text{dof}$ | $\lambda_{1,h_l}$ |
|-----|-------------|------------------|-----|-------------|------------------|
| 1   | 53248       | 29.950023991     | 26  | 63206       | 29.916921865     |
| 2   | 53300       | 29.957640600     | 27  | 64610       | 29.916904165     |
| 5   | 53560       | 29.917626784     | 30  | 73424       | 29.916878484     |
| 10  | 54028       | 29.917180037     | 35  | 110630      | 29.916865373     |
| 11  | 54158       | 29.917248497     | 36  | 122876      | 29.916864735     |
| 14  | 54574       | 29.917406356     | 39  | 159094      | 29.916863538     |
| 15  | 54756       | 29.917731006     | 40  | 175812      | 29.916863378     |
| 21  | 57928       | 29.917153931     | 46  | 335842      | 29.916862940     |
| 22  | 58604       | 29.917092477     | 47  | 374920      | 29.916862915     |
| 23  | 59228       | 29.917012103     | 48  | 424814      | 29.916862902     |
| 24  | 60268       | 29.916986202     | 49  | 475852      | 29.916862889     |
| 25  | 61412       | 29.916940636     | 50  | 537862      | 29.916862882     |
Table 2: The smallest eigenvalue using $\mathbb{P}_3 - \mathbb{P}_2$ element on adaptive mesh on $\Omega_L$.

| $l$ | dof  | $\lambda_{1,h_l}$ | $l$ | dof  | $\lambda_{1,h_l}$ |
|-----|------|-------------------|-----|------|-------------------|
| 1   | 39936| 32.155997914      | 27  | 53612| 32.132716405      |
| 2   | 39988| 32.148565928      | 28  | 56420| 32.132709908      |
| 3   | 40092| 32.147074075      | 29  | 60424| 32.132705093      |
| 4   | 40196| 32.141536961      | 30  | 66664| 32.132701814      |
| 5   | 40248| 32.139031080      | 31  | 75142| 32.132699385      |
| 12  | 40924| 32.134988686      | 39  | 181662| 32.132694920    |
| 14  | 41080| 32.134620945      | 40  | 205244| 32.132694843   |
| 15  | 41288| 32.134171324      | 41  | 229424| 32.132694780   |
| 23  | 48048| 32.132766985      | 49  | 616304| 32.132694655   |
| 24  | 48880| 32.132752576      | 50  | 703092| 32.132694653   |
| 25  | 50128| 32.132737367      | 51  | 796276| 32.132694652   |
| 26  | 51688| 32.132725042      |      |       |                   |

Table 3: The smallest eigenvalue using $\mathbb{P}_3 - \mathbb{P}_2$ element on adaptive mesh on $\Omega_S$.

| $l$ | dof  | $\lambda_{1,h_l}$ | $l$ | dof  | $\lambda_{1,h_l}$ |
|-----|------|-------------------|-----|------|-------------------|
| 1   | 53248| 52.3446926681     | 10  | 273780| 52.3446911721     |
| 2   | 55900| 52.3446918954     | 11  | 337324| 52.344691702      |
| 3   | 68952| 52.3446915184     | 12  | 415636| 52.344691691      |
| 4   | 81588| 52.3446913292     | 13  | 505544| 52.344691686      |
| 5   | 95316| 52.3446912380     | 14  | 610376| 52.344691684      |
| 6   | 114192| 52.3446912049    | 15  | 768560| 52.344691683      |
| 7   | 138372| 52.3446911859    | 16  | 972192| 52.344691681      |
| 8   | 170612| 52.3446911794    | 17  | 1186328| 52.344691679     |
| 9   | 220324| 52.3446911751    |      |       |                   |

4.2 The results in three-dimensional domains

We also carry out numerical experiments on two three-dimensional domains: $\Omega_1 = (0, 1)^3 \setminus \{0 \leq x \leq 0.5, 0 \leq y \leq 0.5, 0.5 \leq z \leq 1\}$ and $\Omega_2 = (0, 1)^3$. In computation we take $\theta = 0.25$ and initial mesh $\pi_{h_0}$ ($h_0 = \sqrt{3}^8$). To compute the error of the first eigenvalue for the Stokes eigenvalue problem, we choose the values $\lambda_1 = 70.98560$ and $\lambda_2 = 62.17341$ which are obtained by adaptive procedure with as much degrees of freedom as possible as the reference values for the domains $\Omega_1$ and $\Omega_2$, respectively.

The initial meshes are shown in Figures 9-11, and the adaptive refined meshes and the error curves are shown in Figures 12-17. The numerical results on adaptive mesh are listed in Tables 1-3.

From Figures 12-17 we can see that the error curves and error indicators curves for DG methods using $\mathbb{P}_k - \mathbb{P}_{k-1}$ ($k = 1, 2, 3$) element are both approximately parallel to the line with slope $-\frac{k}{2}$, which indicates the error indicators are reliable and efficient and the adaptive algorithm can reach the optimal convergence order. It coincides with our theoretical analysis. It also can be seen from the error curves that under the same $dof$, the approximations
obtained by high order elements are more accurate than those computed by low order elements on both uniform meshes and adaptive meshes.

We can also see from Table 2 that it provides an upper bound for the exact eigenvalue by $P_k - P_{k-1}$ element. Note that the numerical results in Table 5 in [55] provide a lower bound on the cubic domain by the CR element. Thus we get a range for the exact eigenvalue of Stokes eigenvalue problem on the cubic.

\[ \text{Figure 9: The initial mesh on } \Omega_1 \text{ (left) and } \Omega_2 \text{ (right).} \]

\[ \text{Figure 10: Adaptive mesh after } l=10 \text{ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using } P_1 - P_0 \text{ element on } \Omega_1 \]
Figure 11: Adaptive mesh after $l=10$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_1 - P_0$ element on $\Omega_2$.

Figure 12: Adaptive mesh after $l=14$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_2 - P_1$ element on $\Omega_1$.

Figure 13: Adaptive mesh after $l=8$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_2 - P_1$ element on $\Omega_2$. 
Figure 14: Adaptive mesh after $l=12$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_3 - P_2$ element on $\Omega_1$.

Figure 15: Adaptive mesh after $l=5$ refinement times (left) and the error curves (right) of the smallest eigenvalue by DGFEM using $P_3 - P_2$ element on $\Omega_2$. 
Table 4: The smallest eigenvalue on adaptive mesh on $\Omega_1$.

|  | $P_1 - P_0$ | $P_2 - P_1$ | $P_3 - P_2$ |
|---|---|---|---|
| $l$ | dof | $\lambda_{1,h_l}$ | dof | $\lambda_{1,h_l}$ | dof | $\lambda_{1,h_l}$ |
| 1 | 34944 | 72.17893 | 91392 | 71.39751 | 188160 | 70.94156 |
| 2 | 38532 | 72.31180 | 93432 | 71.40261 | 190680 | 70.94249 |
| 3 | 44668 | 72.37533 | 98532 | 71.38286 | 193480 | 70.94775 |
| 4 | 56108 | 72.30857 | 106148 | 71.36073 | 194740 | 70.96169 |
| 5 | 74074 | 72.22436 | 116416 | 71.33137 | 203480 | 70.96023 |
| 6 | 91416 | 72.08199 | 134844 | 71.29360 | 207060 | 70.96067 |
| 7 | 121212 | 71.85217 | 162248 | 71.26700 | 218820 | 70.96323 |
| 8 | 159224 | 71.73561 | 189244 | 71.24186 | 234500 | 70.96341 |
| 9 | 202878 | 71.64175 | 229568 | 71.20735 | 245800 | 70.96722 |
| 10 | 270738 | 71.56537 | 281928 | 71.17551 | 259840 | 70.96886 |
| 11 | 369278 | 71.49289 | 354212 | 71.14148 | 271740 | 70.96910 |
| 12 | 475852 | 71.44762 | 437512 | 71.11079 | 300860 | 70.97064 |
| 13 | 623792 | 71.35097 | 519588 | 71.08194 | 412653 | 70.98560 |
| 14 | 822172 | 71.35097 | 685780 | 71.05522 | - | - |

Table 5: The smallest eigenvalue on adaptive mesh on $\Omega_2$.

|  | $P_1 - P_0$ | $P_2 - P_1$ | $P_3 - P_2$ |
|---|---|---|---|
| $l$ | dof | $\lambda_{1,h_l}$ | dof | $\lambda_{1,h_l}$ | dof | $\lambda_{1,h_l}$ |
| 1 | 39936 | 63.68761 | 104448 | 62.28426 | 215040 | 62.17483 |
| 2 | 47892 | 63.59785 | 128520 | 62.26349 | 231840 | 62.17461 |
| 3 | 56732 | 63.51520 | 160820 | 62.24599 | 296800 | 62.17423 |
| 4 | 70044 | 63.38799 | 207944 | 62.22633 | 366520 | 62.17393 |
| 5 | 86632 | 63.26514 | 267920 | 62.21249 | 427560 | 62.17376 |
| 6 | 109252 | 63.09579 | 352648 | 62.19957 | 491060 | 62.17341 |
| 7 | 141440 | 62.91980 | 448664 | 62.18869 | - | - |
| 8 | 186004 | 62.80083 | 611864 | 62.18345 | - | - |
| 9 | 245440 | 62.71246 | - | - | - | - |
| 10 | 339248 | 62.63796 | - | - | - | - |
| 11 | 455754 | 62.56792 | - | - | - | - |
| 12 | 608998 | 62.51093 | - | - | - | - |

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