Automorphisms of Nonnormal Toric Varieties

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Abstract—Criteria for the flexibility, rigidity, and almost rigidity of nonnormal affine toric varieties are obtained. For rigid and almost rigid toric varieties, automorphism groups are explicitly calculated.

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1. INTRODUCTION

Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero. We denote the additive group of \( \mathbb{K} \) by \( \mathbb{G}_a \).

Consider an irreducible affine algebraic variety \( X \) over the field \( \mathbb{K} \). If the algebraic torus \( T \cong (\mathbb{K}^\times)^n \) acts on \( X \) with an open orbit, then \( X \) is said to be toric. The definition of a toric variety often includes the normality requirement. We do not assume \( X \) to be normal.

We study the automorphism group of the variety \( X \). In [1], the subgroup \( \text{SAut}(X) \) of special automorphisms in the group \( \text{Aut}(X) \) of regular automorphisms was introduced. By definition, the subgroup \( \text{SAut}(X) \) is generated by all algebraic subgroups isomorphic to \( \mathbb{G}_a \). A point \( x \in X \) is said to be flexible if the tangent space \( T_x X \) is generated by tangent vectors to orbits of \( \mathbb{G}_a \)-subgroups, or, equivalently, the orbit \( \text{SAut}(X)x \) is open in \( X \). If all regular points of \( X \) are flexible, then the variety \( X \) is said to be flexible. Recall that the action of a group \( G \) on a set is said to be infinitely transitive if, for any two \( m \)-tuples \( (a_1, \ldots, a_m), a_i \neq a_j, \) and \( (b_1, \ldots, b_m), b_i \neq b_j, \) of any length \( m \), there exists an element \( g \in G \) such that \( g \cdot a_i = b_i \) for all \( 1 \leq i \leq m \). The interest in flexible varieties is largely due to the following theorem.

Theorem [1, Theorem 0.1]. For an irreducible affine variety \( X \) of dimension \( \geq 2 \), the following conditions are equivalent:

(i) the group \( \text{SAut}(X) \) acts transitively on \( X^{\text{reg}} \);

(ii) the group \( \text{SAut}(X) \) acts infinitely transitively on \( X^{\text{reg}} \);

(iii) the variety \( X \) is flexible.

There are many papers devoted to the proof of the flexibility of varieties in various classes; see, e.g., [1]–[5]. One of the first examples of flexible varieties is nondegenerate (i.e., admitting only constant invertible regular functions) normal toric varieties; see [2]. On the other hand, a toric variety not satisfying the normality condition may be nonflexible. For example, the toric curve \( \{x^2 = y^3\} \) is not flexible. The main result of this paper is a criterion for the flexibility of toric varieties, not necessarily normal. The answer is given both in terms of a combinatorial description of toric varieties (Theorem 1

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and Corollaries 2 and 4) and in geometric terms (Corollary 1). Note that Theorem 1 generalizes results of [2], because in the case of a normal toric variety, it gives precisely a condition for the variety to be nondegenerate.

The flexibility of a variety $X$ corresponds to the situation in which the group $\text{SAut}(X)$ is big and acts on $X^{\text{reg}}$ transitively. The situation in which the group $\text{SAut}(X)$ is trivial is, in a certain sense, opposite. In this case, $X$ is called a \textit{rigid} variety. A characteristic feature of rigid varieties is that the automorphism group $\text{Aut}(X)$ contains a unique maximal algebraic torus; see [6, Theorem 2.1]. Sometimes this makes it possible to explicitly describe the automorphism group of $X$; see [6] and [7]. We obtain a criterion for the rigidity of an affine toric variety (Theorem 2) and an explicit description of the automorphism groups of such varieties (Theorem 3). The automorphism group of a rigid toric variety $X$ is isomorphic to a semidirect product of the torus and a discrete subgroup of $S(X)$, which we call the \textit{symmetry group of the weight monoid} of the variety $X$; see Definition 2. If $X$ admits no nonconstant invertible functions, then the group $S(X)$ is finite. Thus, the automorphism group of a rigid toric variety without nonconstant invertible functions is a finite extension of the torus.

Yet another class of toric varieties for which we describe automorphism groups is the class of affine toric almost rigid varieties. A variety $X$ is said to be \textit{almost rigid} if it is not rigid and the subgroup $\text{SAut}(X)$ of special automorphisms is Abelian. In this paper, we obtain a criterion for the almost rigidity of an affine toric variety (Theorem 4) and describe the automorphism group of an affine almost rigid toric variety as a semidirect product of three subgroups (Theorem 5).

2. \textbf{LOCALLY NILPOTENT DERIVATIONS}

Algebraic $G_a$-subgroups in the automorphism group of an affine variety $X$ are closely related to the locally nilpotent derivations (LND) of the algebra $\mathbb{K}[X]$ of regular functions. A linear map $\delta: \mathbb{K}[X] \rightarrow \mathbb{K}[X]$ is called a \textit{derivation} of the algebra $\mathbb{K}[X]$ if it satisfies the Leibniz identity

$$\delta(fg) = f\delta(g) + \delta(f)g.$$ 

A derivation $\delta$ is \textit{locally nilpotent} if, given any $f \in \mathbb{K}[X]$, there exists a positive integer $n$ for which $\delta^n(f) = 0$. Detailed information about LNDs can be found in the book [8]. For a given LND $\delta$, we can consider the following linear operator on $\mathbb{K}[X]$, which is called the \textit{exponential of $\delta$}:

$$\exp(\delta)(f) = f + \delta(f) + \frac{\delta^2(f)}{2!} + \frac{\delta^3(f)}{3!} + \cdots .$$

Since $\delta$ is locally nilpotent, the sum in the definition of its exponential is finite. The operator $\exp(\delta)$ determines an automorphism of $\mathbb{K}[X]$ and hence corresponds to an automorphism of the variety $X$. If a function $f$ lies in the kernel of an LND $\delta$, then the map $f\delta$ is an LND as well; it is called a \textit{replica} of the derivation $\delta$. Note that the constants lie in the kernel of any LND; therefore, for any LND $\delta$, the operators $t\delta$, $t \in \mathbb{K}$, are LNDs as well. Using this observation, to each LND $\delta$ we can assign the $G_a$-subgroup $H_\delta = \{\exp(t\delta) | t \in \mathbb{K}\}$ of $\text{Aut}(\mathbb{K}[X])$. Moreover, all algebraic subgroups isomorphic to $G_a$ can be obtained in this way.

Suppose given a grading of $\mathbb{K}[X]$ by an Abelian group $G$:

$$\mathbb{K}[X] = \bigoplus_{g \in G} \mathbb{K}[X]_g .$$

A derivation $\delta$ of $\mathbb{K}[X]$ is said to be $G$-\textit{homogeneous of degree $g_0$} if, for any $g \in G$ and any homogeneous $f \in \mathbb{K}[X]_g$, we have $\delta(f) \in \mathbb{K}[X]_{g+g_0}$. In the case of a toric variety $X$, the algebra $\mathbb{K}[X]$ has the natural grading by the character group of the torus. Section 4 describes the correspondence between the LNDs homogeneous with respect to this grading and the so-called Demazure roots. These derivations play the key role in what follows. If a $\mathbb{Z}$-grading of $\mathbb{K}[X]$ is fixed, then any LND $\partial$ decomposes into a sum of homogeneous derivations as $\partial = \sum_k \partial_k$, where the first and the last components $\partial_k$ and $\partial_0$ are LNDs; see [9, Sec. 3]. It follows that, given a $\mathbb{Z}^n$-grading, any LND decomposes into a sum of homogeneous derivations, the convex hull of the degrees of these derivations is a polyhedron, and the derivations corresponding to its vertices are locally nilpotent.

In what follows, we need the following lemma.
Lemma 1. The group $\text{SAut}(X)$ is commutative if and only if the kernels of all locally nilpotent derivations of the algebra $\mathbb{K}[X]$ coincide.

Proof. Suppose that the kernels of two LNDs $\delta_1$ and $\delta_2$ of the algebra $\mathbb{K}[X]$ do not coincide. We can assume that there exists an $f \in \text{Ker}(\delta_1) \setminus \text{Ker}(\delta_2)$. Let us prove that not all LNDs commute. If $\delta_1 \circ \delta_2 \neq \delta_2 \circ \delta_1$, then we are done. Otherwise, consider the two LNDs $f \delta_1$ and $\delta_2$. We have

$$(\delta_2 \circ f \delta_1)(g) = \delta_2(\delta_1(g)) + f(\delta_2 \circ \delta_1)(g) = \delta_2(\delta_1(g)) + (f \delta_1 \circ \delta_2)(g).$$

If $g$ is such that $\delta_1(g) \neq 0$, then

$$(\delta_2 \circ f \delta_1)(g) \neq (f \delta_1 \circ \delta_2)(g).$$

Thus, there exist two noncommuting LNDs. It is easy to see that the groups $\mathcal{H}_\delta$ for these LNDs do not commute either, which proves the noncommutativity of $\text{SAut}(X)$.

Now suppose that the kernels of all LNDs coincide. Consider LNDs $\delta_1$ and $\delta_2$. According to [8, Principle 12], there exist elements $f, g \in \text{Ker}(\delta_1) = \text{Ker}(\delta_2)$ for which $f \delta_1 = g \delta_2$. Hence $\delta_1$ and $\delta_2$ commute, which means that the groups $\mathcal{H}_\delta$ and $\mathcal{H}_\delta$ commute. Thus, $\text{SAut}(X)$ is generated by a family of pairwise commuting commutative subgroups. Therefore, $\text{SAut}(X)$ is commutative. This completes the proof of Lemma 1.

3. AFFINE TORIC VARIETIES

This section presents basic facts about affine toric varieties. More detailed information can be found, e.g., in the books [10] and [11]. An irreducible algebraic variety is said to be toric if the algebraic torus $T = (\mathbb{K}^*)^n$ acts on it regularly with an open orbit. We can assume the action of $T$ on $X$ to be effective. We emphasize that our definition of a toric variety $X$ does not require $X$ to be normal. We are interested in the case of an affine variety $X$. An affine toric variety $X$ is determined by a finitely generated monoid $P$ of weights of $T$-semi-invariant regular functions. Namely, let us identify the character group $T$ with the free Abelian group $M = \mathbb{Z}^n$ so that each integer vector $m \in \mathbb{Z}^n$ corresponds to the character $\chi^m$. Since the open orbit in $X$ is isomorphic to $T$, we obtain an embedding $\mathbb{K}[X] \hookrightarrow \mathbb{K}[T]$ of the algebras of regular functions. Identifying the algebra $\mathbb{K}[X]$ with its image under this embedding, we see that $\mathbb{K}[X]$ is a $P$-graded subalgebra of $\mathbb{K}[T]$:

$$\mathbb{K}[X] = \bigoplus_{m \in P} \mathbb{K}\chi^m \subset \bigoplus_{m \in M} \mathbb{K}\chi^m = \mathbb{K}[T].$$

Consider the vector space $M_\mathbb{Q} = M \otimes \mathbb{Q}$ over the field of rational numbers. The monoid $P$ generates the cone $\sigma^\vee = \mathbb{Q}_{\geq 0}P \subset M_\mathbb{Q}$. Since $P$ is finitely generated, it follows that so is the cone $\sigma^\vee$. The effectiveness of the action $T$ on $X$ is equivalent to the condition that the monoid $P$ is not contained in any proper sublattice of $M$, which implies, in particular, that $\sigma^\vee$ is not contained in any proper subspace of $M_\mathbb{Q}$. The variety $X$ is normal if and only if the monoid $P$ is saturated, i.e., $P = M \cap \sigma^\vee$. In the case where $P$ is not saturated, we refer to the monoid $P_{\text{sat}} = M \cap \sigma^\vee$ as the saturation of $P$ and to the elements of $P_{\text{sat}} \setminus P$ as the holes of $P$. The following definitions can be found in [12].

Definition 1. An element $p$ of a monoid $P$ is called a saturation point of $P$ if the shifted cone $p + \sigma^\vee$ contains no holes, i.e., $(p + \sigma^\vee) \cap M \subset P$.

A face $\tau$ of the cone $\sigma^\vee$ is said to be almost saturated if it contains a saturation point of the monoid $P$. Otherwise, $\tau$ is said to be nowhere saturated.

The following lemma is known, but we give its proof for convenience.

Lemma 2. The maximal face (the entire cone $\sigma^\vee$) is almost saturated.
**Proof.** Let \( a_1, \ldots, a_r \) be a system of generators of the monoid \( P \). The set
\[
S = \{ \lambda_1 a_1 + \cdots + \lambda_r a_r \mid \lambda_i \in \mathbb{Q}, 0 \leq \lambda_i < 1 \}
\]
is bounded; therefore, it contains only finitely many elements of \( M \). We set \( S \cap M = \{ b_1, \ldots, b_l \} \). Since the action of \( T \) on \( X \) is effective, it follows that the group generated by \( P \) coincides with \( M \). Thus, we have \( b_j = c_j - d_j \) for each \( j \) and some \( c_j \) and \( d_j \) in \( P \). Consider \( d = d_1 + \cdots + d_l \). Let us prove that
\[
(d + \sigma^\vee) \cap M \subset P.
\]
Let \( m \in \sigma^\vee \cap M \), or, equivalently, \( d + m \in (d + \sigma^\vee) \cap M \). Since the vectors \( a_1, \ldots, a_r \) generate \( \sigma^\vee \), we can represent \( m \) as a linear combination:
\[
m = \mu_1 a_1 + \cdots + \mu_r a_r, \quad \mu_i \in \mathbb{Q}, \quad \mu_i \geq 0.
\]
Denoting the integer and the fractional part of a number by \([ \cdot ]\) and \{ \cdot \}, respectively, we write
\[
m = \sum_{i=1}^r [\mu_i] a_i + \sum_{i=1}^r \{\mu_i\} a_i.
\]
Since the first summand belongs to \( M \) and \( m \in M \), it follows that the second belongs to \( M \) as well. The inequality \( \{\mu_i\} < 1 \) implies \( \sum_{i=1}^r \{\mu_i\} a_i \in S \). Hence there exists a \( k \) for which \( \sum_{i=1}^r \{\mu_i\} a_i = b_k \). We obtain
\[
d + m = \sum_{j=1}^l d_j + \sum_{i=1}^r [\mu_i] a_i + b_k = \sum_{j \neq k} d_j + \sum_{i=1}^r [\mu_i] a_i + c_k \in P.
\]
This proves Lemma 2. \( \square \)

Let \( N \) denote the lattice of one-parameter subgroups of the torus \( T \). The lattice \( N \) is dual to \( M \); these lattices are related by an integer pairing \( M \times N \rightarrow \mathbb{Z} \), which we denote by \( \langle \cdot, \cdot \rangle \). This pairing can be extended by linearity to a pairing between the vector spaces \( N_\mathbb{Q} = N \otimes \mathbb{Q} \) and \( M_\mathbb{Q} \). In the space \( N_\mathbb{Q} \), we define the cone \( \sigma \) dual to \( \sigma^\vee \) as
\[
\sigma = \{ v \in N_\mathbb{Q} \mid \forall w \in \sigma^\vee : \langle w, v \rangle \geq 0 \}.
\]
The cone \( \sigma \) is pointed (i.e., it does not contain any nontrivial subspaces) and finitely generated.

There exists a one-to-one correspondence between the \( k \)-faces of the cone \( \sigma \) and the \((n - k) \)-faces of the cone \( \sigma^\vee \). Namely, to each face \( \tau \preceq \sigma \) there corresponds the face
\[
\hat{\tau} = \tau^\perp \cap \sigma^\vee \preceq \sigma^\vee.
\]
There also exists a one-to-one correspondence between the \((n - k) \)-faces of \( \sigma^\vee \) and the \( k \)-dimensional \( T \)-orbits in \( X \); to each face \( \hat{\tau} \preceq \sigma^\vee \) there corresponds the orbit open in the zero set of the ideal
\[
I_{\hat{\tau}} = \bigoplus_{m \notin P \setminus \hat{\tau}} \mathbb{K} \chi^m.
\]
The composition of these correspondences is a one-to-one correspondence between the \( k \)-faces of the cone \( \sigma \) and the \( k \)-dimensional \( T \)-orbits. We denote the orbit corresponding to a face \( \tau \) by \( O_\tau \).

In the case of a normal variety \( X \), the codimension of the singular locus \( X^{\text{sing}} \) is at least 2, which means that the algebra of regular functions on \( X \) coincides with that on the regular locus \( X^{\text{reg}} \). In the case of a nonnormal variety, these algebras may differ. Although the variety \( X^{\text{reg}} \) is not necessarily affine, it is convenient to pass from \( X \) to \( X^{\text{reg}} \), because the algebra \( \mathbb{K}[X^{\text{reg}}] \) is simpler than \( \mathbb{K}[X] \).

**Lemma 3.** Let \( \sigma \) be the cone corresponding to an affine toric variety \( X \), and let \( \gamma \) be the cone generated by all extremal edges of \( \sigma \) except those corresponding to orbits consisting of singular points. Then the algebra of regular functions on the quasi-affine variety \( X^{\text{reg}} \) of regular points is
\[
\mathbb{K}[X^{\text{reg}}] = \bigoplus_{m \in M \cap \gamma^\vee} \mathbb{K} \chi^m.
\]
Proof. Since the variety $X^{\text{reg}}$ is $T$-invariant, we can decompose its function algebra into the direct sum of weight components. Thus, all we need is to understand which of the functions $\chi^m$ are regular on $X^{\text{reg}}$. Suppose that some function $\chi^{m_0}$ is not regular on $X^{\text{reg}}$. The set of points at which this function is not defined is $T$-invariant, i.e., consists of orbits. The removal of all orbits of codimension 2 from $X^{\text{reg}}$ does not affect the algebra of regular functions. On the open orbit, all functions $\chi^m$, $m \in M$, are regular. Therefore, the function $\chi^{m_0}$ is not defined on some $T$-orbit of codimension 1. Thus, a function $\chi^m$ is regular on $X^{\text{reg}}$ if and only if it is regular on all regular orbits of codimension 1.

The variety $X^{\text{reg}}$ is normal. According to a well-known formula given in [11, Sec. 3.3], the order of a function $\chi^m$ on a divisor $O_\rho$ equals $\langle m, v_\rho \rangle$. The regularity of $\chi^m$ on $X^{\text{reg}}$ gives the condition $\langle m, v_\rho \rangle \geq 0$ for all edges $\rho$ such that $O_\rho$ consists of regular points. These conditions determine the cone $\gamma^\vee$. This completes the proof of Lemma 3.

Sometimes the weight monoid has symmetries, which makes it possible to consider a certain discrete subgroup in the automorphism group of a toric variety. Let $\varphi : M_Q \to M_Q$ be a linear operator, and let $\varphi \in \text{GL}_n(\mathbb{Z})$, i.e., $\varphi(M) = M$. Suppose that $\varphi(P) = P$. The operator $\varphi$ induces the automorphism $\alpha(\varphi)$ of the algebra $\mathbb{K}[X]$ defined by $\alpha(\varphi)(\chi^m) = \chi^{\varphi(m)}$. Let $\psi$ be yet another linear operator in the stabilizer $\text{St}_{\text{GL}_n(\mathbb{Z})}(P)$. It is easy to see that $\alpha(\varphi \circ \psi) = \alpha(\varphi) \circ \alpha(\psi)$. Thus, $\alpha$ is an injective homomorphism from $\text{St}_{\text{GL}_n(\mathbb{Z})}(P)$ to $\text{Aut}(X)$.

Definition 2. The image of the embedding $\alpha$ is called the symmetry group of the monoid $P$ and denoted by $S(X)$.

Remark 1. Suppose that the cone $\sigma^\vee$ contains no straight lines. This is equivalent to the conditions that the cone $\sigma$ is contained in any hyperplane and that $X$ admits no nonconstant invertible functions. Then the group $S(X)$ is finite, because the operator $\varphi$ must permute primitive (i.e., having setwise coprime coordinates) vectors on the extremal edges of the cone $\sigma^\vee$. The group $S(X)$ is also finite in the case where $\sigma^\vee$ is a straight line, because $S(X)$ is then isomorphic to a subgroup of $\text{GL}_1(\mathbb{Z}) \cong \mathbb{Z}_2$.

In all other cases, $S(X)$ is an infinite discrete group. Indeed, there are two possibilities. First, $\sigma^\vee$ may coincide with the whole space $M_Q$ of dimension $n \geq 2$. It is easy to show that, in this case, $P = M$ and $S(X) = \text{GL}_n(\mathbb{Z})$. Secondly, it may happen that $\sigma^\vee$ contains a nontrivial subspace, but is not a space itself. Let $W$ denote the maximal subspace of $\sigma^\vee$. Then $L = W \cap M$ is a nonzero sublattice of $M$. Let us fix an element $v \neq 0$ of the lattice $N$. Then, for each $l \in L$, the operator $\varphi_l$ defined by $\varphi_l(m) = m + (m, v)l$ has the property $\varphi_l(P) = P$. The automorphisms $\alpha(\varphi_l)$, $l \in L$, form a subgroup in $\text{Aut}(X)$ isomorphic to $L$.

4. LNDs on Affine Toric Varieties

Let $Y$ be a normal toric variety. In [13], all $\mathbb{G}_a$-subgroups of $\text{Aut}(Y)$ normalized by the torus were described for complete toric varieties $Y$. We are interested in the case of an affine toric variety $Y$. In this case, the $\mathbb{G}_a$-subgroups of $\text{Aut}(Y)$ normalized by the torus correspond to the $M$-homogeneous LNDs of $\mathbb{K}[Y]$, which were described in [14]. We recall this classification and use it to construct $M$-homogeneous derivations for a nonnormal affine toric variety $X$.

Let $P$ and $\sigma$ be the monoid and the cone corresponding to the variety $X$. We denote the normal affine toric variety corresponding to the monoid $P_{\text{sat}}$ by $Y$. Given an edge $\rho$ of $\sigma$, by $v_\rho$ we denote the primitive integer vector along $\rho$. An element $e \in M$ is called a Demazure root of $\sigma$ corresponding to $\rho$ if $\langle e, v_\rho \rangle = -1$ and $\langle e, v_\xi \rangle \geq 0$ for any other edge $\xi$ of $\sigma$. We denote the set of all Demazure roots corresponding to an edge $\rho$ by $\mathcal{R}_\rho$. It is easy to see that $\mathcal{R}_\rho$ is nonempty and, moreover, infinite for each edge $\rho$. Every Demazure root $e \in \mathcal{R}_\rho$ determines an LND $\partial_e$ of the algebra $A = \mathbb{K}[Y] = \bigoplus_{m \in P_{\text{sat}}} \mathbb{K}\chi^m$, which is defined on homogeneous elements by

$$\partial_e(\chi^m) = \langle m, v_\rho \rangle \chi^{m+e}.$$ 

Any $M$-homogeneous LND of $A$ is proportional to $\partial_e$ for some Demazure root $e$. Every derivation $\partial_e$ is $M$-homogeneous of degree $e$. The kernel of $\partial_e$ is the subalgebra $\bigoplus_{m \in M \cap e} \mathbb{K}\chi^m$. If the subalgebra $\mathbb{K}[X] \subset A$ is $\partial_{e}$-invariant, then $\partial_{e}$ induces an LND of $\mathbb{K}[X]$, which we denote by $\delta_e$. It is easy to see that the subalgebra $\mathbb{K}[X]$ is $\partial_{e}$-invariant if and only if $(P + e) \cap P_{\text{sat}} \subset P$. 
Example 1. Consider the variety corresponding to the monoid
\[ P = \{ (a, b) \in \mathbb{Z}_\geq 0^2 \mid (a, b) \neq (1, 0) \}. \]
We have \( \sigma^\vee = \text{cone}((1, 0), (0, 1)) \) and \( \sigma = \text{cone}((1, 0), (0, 1)) \). Let \( \rho = \mathbb{Q}_\geq 0(0, 1) \). Then
\[ \mathcal{R}_\rho = \{ e_k = (k, -1) \mid k \in \mathbb{Z}_\geq 0 \}. \]
The derivation \( \delta_{e_k} \) is well defined for \( k \geq 2 \) (see Fig. 1).

![Fig. 1.](image)

The following proposition is a key to the whole paper. It gives a criterion for the existence of a well-defined derivation \( \delta_e, e \in \mathcal{R}_\rho \), for a given \( \rho \), both in terms of the regularity of codimension-1 orbits and in the combinatorial language in terms of the semigroup \( P \).

Proposition 1. Suppose given an affine toric variety \( X \) and the corresponding cone \( \sigma \). Let \( \rho \) be an edge of \( \sigma \), and let \( O_\rho \) be the corresponding orbit. Then the following conditions are equivalent:

1. the hyperface \( \tilde{\rho} \) of the cone \( \sigma^\vee \) is almost saturated;
2. there exists a Demazure root \( e \in \mathcal{R}_\rho \) of \( \sigma \) for which the corresponding derivation \( \delta_e \) of the algebra \( \mathbb{K}[X] \) is well defined;
3. the orbit \( O_\rho \) consists of regular points.

Proof. (1) \( \Rightarrow \) (2). Let \( P \) be the weight monoid of \( X \). Since the hyperface \( \tilde{\rho} \) is almost saturated, there exists a \( w \in P \cap \tilde{\rho} \) for which
\[ (w + \sigma^\vee) \cap M \subset P. \]
Let \( r \in \mathcal{R}_\rho \) be a Demazure root corresponding to the edge \( \rho \). Take an integer vector \( u \) in the relative interior of \( \tilde{\rho} \). For any integer \( k \in \mathbb{N} \) and any edge \( \xi \neq \rho \) of the cone \( \sigma \), we have
\[ \langle r + ku, v_\rho \rangle = \langle r, v_\rho \rangle + k \langle u, v_\rho \rangle = -1, \quad \langle r + ku, v_\xi \rangle = \langle r, v_\xi \rangle + k \langle u, v_\xi \rangle \geq 0. \]
Thus, \( r + ku \in \mathcal{R}_\rho \) for any positive integer \( k \). If \( k \) is large enough, then \( \langle z, v_\xi \rangle > \langle w, v_\xi \rangle \) for every \( z \in r + ku + \sigma^\vee \) and all \( \xi \neq \rho \). Therefore,
\[ (r + ku + \sigma^\vee) \cap \sigma^\vee \subset w + \sigma^\vee. \]
Given \( k \), we set \( e = r + ku \). We have
\[ (e + P) \cap P_{\text{sat}} \subset (e + \sigma^\vee) \cap \sigma^\vee \subset w + \sigma^\vee. \]
It follows that
\[ (e + P) \cap P_{\text{sat}} \subset (w + \sigma^\vee) \cap M \subset P. \]
Therefore, the formula \( \delta_e(\chi^m) = \langle m, v_\rho \rangle \chi^{m+e} \) well defines a derivation of \( \mathbb{K}[X] \).
(2) ⇒ (3). The proof of this implication is virtually the same as the proof of Lemma 14 and Theorem 6 in [4]. The derivation $\delta_e$ corresponds to the $G_\alpha$-action $\mathcal{H}_e$ on $X$. Since the ideal of functions vanishing on $O_\rho$ is
\[ I_\rho = \bigoplus_{m \in P \setminus \hat{\rho}} \mathbb{K} \chi^m \quad \text{and} \quad \ker \delta_e = \bigoplus_{m \in P \cap \hat{\rho}} \mathbb{K} \chi^m, \]
it follows that there exists a function $f = \chi^m$ such that $f \in I_\rho$ but $\delta_e(f) \notin I_\rho$. Take a point $x \in O_\rho$ for which $\delta_e(f)(x) \neq 0$. We can assume that, for each edge $\xi$ of $\sigma$ different from $\rho$, $x$ does not belong to the closure $\overline{O}_\xi$ of its orbit. Then there exists an $s \in H_e$ for which $s \cdot x \notin \overline{O}_\rho$. Since the $H_e$-orbit of $x$ is irreducible and none of the closures $\overline{O}_\xi$ contains it entirely, it follows that there exists a point $y = s \cdot x$ belonging to none of the closures $\overline{O}_\xi$, where $\xi$ ranges over all edges of the cone $\sigma$.

Therefore, $y$ belongs to the open $T$-orbit. Since the open $T$-orbit consists of regular points, the point $y$ is regular. Therefore, so is the point $x$, i.e., $O_\rho$ consists of regular points.

(3) ⇒ (1). Fix a point $x \in O_\rho$. This is a regular point of the closure $\overline{O}_\rho$. The algebra of regular functions on the variety $\overline{O}_\rho$ is
\[ \mathbb{K}[\overline{O}_\rho] = \bigoplus_{m \in P \setminus \hat{\rho}} \mathbb{K} \chi^m. \]

Let $m_x$ denote the maximal ideal of $\mathbb{K}[\overline{O}_\rho]$ corresponding to $x$, and let $M_x$ denote the maximal ideal of $\mathbb{K}[X]$ corresponding to $x$. Then $M_x = m_x \oplus I_\rho$. Therefore,
\[ M_x \mathbb{Z}^2 = m_x \mathbb{Z}^2 \oplus I_\rho/\big(I_\rho^2 + m_x I_\rho\big). \]

Since $x$ is a regular point of $X$, we have $\dim X = \dim T_x X = \dim M_x / M_x^2$. On the other hand, since $x$ is a regular point of $\overline{O}_\rho$, we have
\[ \dim \overline{O}_\rho = \dim T_x \overline{O}_\rho = \dim m_x / m_x^2. \]

Finally, since $O_\rho$ is an orbit of codimension 1, we have
\[ \dim I_\rho / \big(I_\rho^2 + m_x I_\rho\big) = 1. \]

We choose $\chi^w \in I_\rho$ so that
\[ I_\rho = \langle \chi^w \rangle \oplus \big(I_\rho^2 + m_x I_\rho\big). \]

Now we fix a system of generators $a_1, \ldots, a_r$ of the monoid $P$. We assume that $a_1, \ldots, a_l$ belong to the face $\hat{\rho}$ and $a_{l+1}, \ldots, a_r$ do not belong to this face. Suppose that the lattice $L = \mathbb{Z}(a_1, \ldots, a_l)$ does not coincide with $M \cap \rho^\perp$. Consider the lattice $\Lambda = \mathbb{Z}(a_1, \ldots, a_l, w)$. Let us prove that $I_\rho$ is contained in the subspace $W = \bigoplus_{\lambda \in \Lambda} \mathbb{K} \chi^\lambda$. For each positive integer $k$, we set
\[ V_k = \bigoplus_{m \in M : \langle m, v_\rho \rangle = k} \mathbb{K} \chi^m. \]

Let us prove by induction that $(I_\rho \cap V_k) \subset W$.

**Base of induction.** Let $k_0$ be the least number for which $V_{k_0} \cap I_\rho \neq \{0\}$. Since $I_\rho^0 \cap V_{k_0} = \{0\}$, it follows that
\[ I_\rho \cap V_{k_0} \subset \langle \chi^w \rangle \oplus m_x I_\rho. \]

Note that
\[ \{m \in \Lambda : \langle m, v_\rho \rangle = k_0\} = w + L. \]

If $I_\rho \cap V_{k_0}$ is not contained in $\bigoplus_{m \in w + L} \mathbb{K} \chi^m$, then there exists an $\alpha \in M \setminus \Lambda$ for which $\langle \alpha, v_\rho \rangle = k_0$ and
\[ S = \bigoplus_{m \in \alpha + L} \mathbb{K} \chi^m \neq \{0\}. \]
The inclusion $I_{\tilde{\rho}} \cap V_{k_0} \subset \langle \chi^m \rangle \oplus m_x I_{\tilde{\rho}}$ implies

$$m_x I_{\tilde{\rho}} \cap V_{k_0} \subset \bigoplus_{m \in \omega + L} \mathbb{K}\chi^m + m_x I_{\tilde{\rho}}.$$  

It follows that $S = m_x S$. Since the ideal $I_{\tilde{\rho}}$ is finitely generated, $S$ is a finitely generated $\mathbb{K}[\mathcal{O}_\rho]$-module. By Nakayama’s lemma, there exists a $u \in m_x$ such that $u \cdot f = f$ for all $f \in S$. However, if $u \cdot \chi^m = \chi^m$, then $u = 1$, while $1 \notin m_x$.

**Induction step.** Let $k' > k_0$ and suppose that the required assertion is true for all $k < k'$. Then

$$I_{\tilde{\rho}} \cap V_{k'} \subset I_{\tilde{\rho}}^2 \oplus m_x I_{\tilde{\rho}}.$$  

However, by the induction hypothesis, we have $I_{\tilde{\rho}}^2 \cap V_{k'} \subset W$. If $I_{\tilde{\rho}} \cap V_{k'}$ is not contained in $W$, then there exists an $\alpha \in M \setminus L$ for which $\langle \alpha, v_p \rangle = k'$ and

$$S_{k'} = \bigoplus_{m \in \alpha + L} \mathbb{K}\chi^m \neq \{0\}.$$  

On the other hand, $S_{k'} \cap I_{\tilde{\rho}}^2 = \{0\}$, which means that $S_{k'} \subset m_x I_{\tilde{\rho}}$, and hence $S_{k'} = m_x S_{k'}$. We have again obtained a contradiction to Nakayama’s lemma.

Thus, we have proved that $I_{\tilde{\rho}} \subset W$. This implies $\mathbb{K}[X] \subset W$. Since the action of $T$ on $X$ is effective, it follows that $\Lambda = M$.

Suppose that the face $\tilde{\rho}$ is nowhere saturated. Theorem 3.3 of [12] asserts that some element $b$ of the Hilbert basis of the monoid $P_{\text{sat}}$ admits no decomposition of the form

$$b = x_1 a_1 + \cdots + x_r a_r, \quad \text{where } x_i \in \mathbb{Z}, \quad x_j \geq 0, \quad j > l.$$  

Since $\Lambda = M$, we have $L = M \cap \rho^\perp$. Therefore, $b$ does not belong to the face $\tilde{\rho}$. Let $\langle b, v_p \rangle = k$. Then $V_k \cap I_{\tilde{\rho}} = \{0\}$. Therefore, $k$ is not divisible by $k_0$. Thus, $k_0 \neq 1$. On the other hand, the effectiveness of the action of $T$ on $X$ implies the existence of a $\tilde{k}$ not divisible by $k_0$ for which $V_{\tilde{k}} \cap I_{\tilde{\rho}} \neq \{0\}$. We assume that $\tilde{k}$ is the least integer with this property. Then $V_{\tilde{k}} \cap I_{\tilde{\rho}}^2 = \{0\}$. Again applying Nakayama’s lemma, we see that the inclusion $V_{\tilde{k}} \cap I_{\tilde{\rho}} \subset m_x I_{\tilde{\rho}}$ does not hold. Therefore, dim $I_{\tilde{\rho}}/(I_{\tilde{\rho}}^2 + m_x I_{\tilde{\rho}}) \geq 2$. This contradiction proves Proposition 1.

5. FLEXIBLE TORIC VARIETIES

In this section, we prove a flexibility criterion for affine toric varieties (Theorem 1) and give its reformulations in geometric and combinatorial terms (Corollaries 1, 2, and 4).

**Lemma 4.** Let $X$ be a flexible affine variety. Then the quasi-affine variety $X^{\text{reg}}$ admits only constant invertible regular functions.

**Proof.** Since $X$ is flexible, $SAut(X)$ acts on $X^{\text{reg}}$ transitively. Any automorphism takes regular points to regular ones; hence any $G_a$-action on $X$ can be restricted to $X^{\text{reg}}$. As proved in [15], every $G_a$-action on a quasi-affine variety determines an LND of the algebra of regular functions on it. However, each invertible function belongs to the kernel of any LND (see [8, Principle 1(b)]). Thus, any invertible function $f$ is $SAut(X)$-invariant. Therefore, the set $\{f = c\}$ is $SAut(X)$-invariant for any $c \in \mathbb{K}$, which contradicts flexibility for nonconstant $f$. This proves Lemma 4.

**Theorem 1.** Suppose given an affine toric variety $X$ to which a cone $\sigma$ corresponds. Let $\gamma$ be the cone generated by all edges of $\sigma$ except those consisting of singular points. The variety $X$ is flexible if and only if the cone $\gamma$ is not contained in any hyperplane.
Proof. Suppose that the cone $\gamma$ lies in a hyperplane $H \subset N_\mathbb{Q}$. Then the cone $\gamma^\vee$ contains the straight line $H^\perp$. By Lemma 3, whenever $\gamma^\vee$ contains a straight line, there exist nonconstant invertible regular functions on $X^{\text{reg}}$. By Lemma 4, the variety $X$ is not flexible.

Now suppose that the cone $\gamma$ is not contained in any hyperplane. This means that the cone $\sigma$ has $n$ edges $\rho_1, \ldots, \rho_n$ corresponding to regular orbits for which $\{v_{\rho_1}, \ldots, v_{\rho_n}\}$ is a basis in $N_\mathbb{Q}$. By Proposition 1, there exist Demazure roots $e_1, \ldots, e_n$, where $e_i \in \mathbb{R}_{\rho_i}$, for which the LNDs $\delta_{e_i}$ are well defined. Consider the point $p = (1, 1, \ldots, 1) \in T \subset X$. For the standard basis

$$\{m_1 = (1, 0, \ldots, 0), \ldots, m_n = (0, \ldots, 0, 1)\}$$

of the lattice $M$, we have $\chi^{m_i}(p) = 1$, which implies $\chi^{m}(p) = 1$ for all $m \in M$. The vectors tangent to the $\mathbb{G}_a$-orbits $\mathcal{H}_{e_i}p$ have the form $(\delta_{e_i}(\chi^{m_1})(p), \ldots, \delta_{e_i}(\chi^{m_n})(p))$. We have

$$\delta_{e_i}(\chi^{m_j})(p) = \langle m_j, v_{\rho_i} \rangle \chi^{m_j + e_i}(p) = \langle m_j, v_{\rho_i} \rangle.$$

To prove that the tangent vectors to the orbits $\mathcal{H}_{e_i}p$ at $p$ are linearly independent, we must show that the determinant of the $n \times n$ matrix with entries $\langle m_j, v_{\rho_i} \rangle$ is nonzero. This is indeed so, because $\{m_1, \ldots, m_n\}$ is a basis of $M_\mathbb{Q}$ and $\{v_{\rho_1}, \ldots, v_{\rho_n}\}$ is a basis of $N_\mathbb{Q}$.

Thus, $p$ is a flexible point of the variety $X$. Since $p$ belongs to the open $T$-orbit, the entire orbit consists of flexible points.

For any face $\tau$ of the cone $\sigma^\vee$, we can choose a primitive vector $n \in N$ so that $\tau = \sigma^\vee \cap \langle n \rangle^\perp$. This gives the grading

$$\mathbb{K}[X]_i = \bigoplus_{\langle m, n \rangle = i} \mathbb{K} \chi_m.$$

All negative homogeneous components of this grading are trivial. According to [5, Proposition 3, Corollary 1], the closure of any smooth orbit is not SAut($X$)-invariant. (The hypotheses of Proposition 3 in the paper [5] include the normality assumption on the variety, because this paper is devoted to normal varieties, but the proof does not use this assumption, and the proposition is valid for nonnormal varieties too.) Hence there exists an element of SAut($X$) which takes a point of this orbit to a point of the open orbit. This means that all regular points of $X$ are flexible, which proves Theorem 1.

Remark 2. If a variety $X$ is normal, then all orbits of codimension 1 are regular and, therefore, the cone $\gamma$ coincides with $\sigma$. In this case, Theorem 1 coincides with Theorem 2.1 of [2, theorem 2.1].

Below we give several equivalent reformulations of Theorem 1. The first of them gives a flexibility criterion for toric varieties in purely geometric terms, i.e., uses no combinatorial data of the cone corresponding to a given toric variety $X$.

Corollary 1. An affine toric variety $X$ is flexible if and only if there exists no regular function $f \in \mathbb{K}[X]$ whose zero set consists of only singular points.

Proof. The zero set of $f$ consists of only singular points if and only if $f$ is invertible on $X^{\text{reg}}$. Any nonconstant invertible function in $\mathbb{K}[X^{\text{reg}}]$ is $M$-homogeneous (see [16, Theorem 3.1]), i.e., has the form $c \chi^m$, where $c, m \neq 0$. The inverse of $c \chi^m$ is $c^{-1} \chi^{-m}$; therefore, the invertibility of $c \chi^m$ is equivalent to the existence of a straight line contained in $\gamma^\vee$. By Theorem 1, $\gamma^\vee$ contains a straight line if and only if $X$ is not flexible. This completes the proof of Corollary 1.

The second reformulation of Theorem 1, on the contrary, gives a flexibility criterion in terms of the weight monoid $P$. It is obtained by combining Proposition 1 with Theorem 1.

Corollary 2. An affine toric variety $X$ is flexible if and only if no hyperplane in the space $N_\mathbb{Q}$ contains all those edges $\rho_i$ of the cone $\sigma$ for which the face $\rho_i$ is almost saturated.

Any bounded part of the monoid $P$ has no effect on whether or not a face is almost saturated. That is, any two monoids $P$ and $P'$ which differ from each other by only finitely many elements correspond to the same cone $\sigma^\vee$, and if a face $\tau \not\subset \sigma^\vee$ is saturated for $P$, then it is also saturated for $P'$. This implies the following assertion.

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Corollary 3. If $P$ has only finitely many holes and the cone $\sigma$ is not contained in any hyperplane, then the variety $X$ is flexible.

It is easy to see that, for a two-dimensional cone, the almost saturation of both hyperfaces is equivalent to the finiteness of the number of holes. In the general case, the finiteness of the number of holes is not necessary for flexibility, because infinitely many holes can be arranged so that the hyperfaces still be almost saturated. However, we can assert that all holes of $P$ are located “along” nowhere saturated hyperfaces and faces of lower dimension.

Lemma 5. Suppose given a cone $\sigma$ and the corresponding monoid $P$. Let $\rho_1, \ldots, \rho_k$ denote the edges of $\sigma$, and let $v_1, \ldots, v_k$ be the primitive vectors along these edges. For each $d \in \mathbb{N}$, let

$$L_i(d) = \{ m \in M \cap \sigma^\vee \mid \langle m, v_i \rangle \leq d \}. $$

Then, for any fixed positive integer $s \leq k$, the following conditions are equivalent:

1. the hyperfaces $\hat{\rho}_1, \ldots, \hat{\rho}_s$ are almost saturated;

2. there exists a constant $c \in \mathbb{N}$ such that any hole $u$ of $P$ belongs either to $L_t(c)$ for some $t > s$ or to $L_i(c) \cap L_j(c)$ for some $i, j \leq s$.

Proof. (1) $\Rightarrow$ (2). By Lemma 2, $\sigma^\vee$ has a saturation point; let us denote it by $z$. We set $x_i = \langle z, v_i \rangle$ and take $d$ such that $d > x_i$ for all $i$. Suppose that $u \notin L_i(d)$ for all $i$; then $\langle u, v_i \rangle > d > \langle z, v_i \rangle$, and hence $u - z \in \sigma^\vee \cap M$. Since $z$ is a saturation point, we have $u \in P$, which contradicts the assumption that $u$ is a hole. Thus, there exists an $i$ for which $u \in L_i(d)$.

Let $i \leq s$. Then the face $\hat{\rho}_i$ has a saturation point $w$. We set $y_j = \langle w, v_j \rangle$ and take $c_i$ such that $c_i > y_j$ for all $j$. Suppose that $u \notin L_j(c_i)$ for all $j \neq i$; then $\langle u, v_j \rangle > c_i > \langle w, v_j \rangle$. Since $\langle w, v_i \rangle = 0$, we have $u - w \in \sigma^\vee \cap M$, which contradicts the assumption that $w$ is a saturation point and $u$ is a hole. Thus, there exists a $j \neq i$ for which $u \in L_j(c_i)$.

Let $c = \max\{d, c_1, \ldots, c_s\}$. Then $u \in L_i(c)$, and if $i \leq s$, then $u \in L_j(c)$ for some $j \neq i$.

(2) $\Rightarrow$ (1). Fix a positive integer $i \leq s$. Take a point $w \in \hat{\rho}_i \cap M$ such that $w \notin L_j(c)$ for all $j \neq i$. For any $u \in w + \sigma^\vee \cap M$, we have $u \notin L_j(c)$, and hence $u \in P$. This completes the proof of Lemma 5.

Lemma 5 and Corollary 2 suggest yet another reformulation of Theorem 1, which gives a flexibility criterion for a toric variety in terms of holes in the corresponding monoid.

Corollary 4. Let $X$ be an affine toric variety, and let $P$ and $\sigma$ be the corresponding monoid and cone. Then $X$ is flexible if and only if the edges of $\sigma$ can be numbered as $\rho_1, \ldots, \rho_k$ so that the following conditions hold:

1. the edges $\rho_1, \ldots, \rho_n$ are not contained in the same hyperplane;

2. there exists a positive integer $c$ such that any hole $u$ of $P$ belongs either to $L_i(c)$ for some $t > n$ or to $L_i(c) \cap L_j(c)$ for some $i, j \leq n$.

Example 2. Consider an example of a flexible toric variety with singular locus of codimension 1. Take the submonoid $P$ in $\mathbb{Z}^3$ consisting of all points $(a, b, c)$, $a, b, c \in \mathbb{Z}_{\geq 0}$, satisfying the conditions $c \leq a + b$ and $c \neq a + b - 1$. The cone $\sigma^\vee$ is given by the inequalities $a, b, c \in \mathbb{Z}_{\geq 0}$ and $c \leq a + b$; it has four hyperfaces, on each of which one inequality turns into an equality. It is easy to show that the hyperfaces $\{a = 0\}$, $\{b = 0\}$, and $\{c = 0\}$ are almost saturated, while the hyperface $\{c = a + b\}$ is nowhere saturated. By Proposition 1, the orbit of codimension 1 corresponding to the hyperface $\{c = a + b\}$ consists of singular points. The cone $\gamma$ is generated by the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$; therefore, by Theorem 1, the toric variety corresponding to $P$ is flexible.
Recall that an affine variety $X$ is said to be rigid if the special automorphism group $\text{SAut}(X)$ is trivial. This condition is equivalent to the absence of nontrivial LNDs of the algebra $\mathbb{K}[X]$. In this section, we prove a rigidity criterion for an affine toric variety $X$ and give an explicit description of the automorphism group of a rigid toric variety.

**Theorem 2.** An affine toric variety $X$ is rigid if and only if the regular locus $X^\text{reg}$ coincides with the open $T$-orbit.

**Proof.** The regular locus is open. Hence if there is a nonopen orbit on $X$ consisting of regular points and contained in the closure of another orbit, then the other orbit consists of regular points as well. Thus, there is an orbit of codimension 1 which consists of regular points. According to Proposition 1, there exists an LND $\delta_e$ of the algebra $\mathbb{K}[X]$, i.e., the variety $X$ is not rigid.

Now suppose that $X^\text{reg}$ coincides with the open $T$-orbit. Then $X^\text{reg} \cong T$, and hence the algebra $\mathbb{K}[X^\text{reg}]$ is generated by invertible functions. This means that $\mathbb{K}[X^\text{reg}]$ has no nonzero LNDs, i.e., $X^\text{reg}$ is rigid. Since any nontrivial $\mathbb{G}_a$-action on $X$ determines a nontrivial $\mathbb{G}_a$-action on $X^\text{reg}$, it follows that the variety $X$ is rigid. This completes the proof of Theorem 2.

**Remark 3.** Lemma 1 gives a rigidity criterion for a variety $X$ in terms of its weight monoid $P$. Namely, a variety $X$ is rigid if and only if none of the hyperfaces of the cone $\sigma^\check{}$ is almost saturated.

A characteristic feature of rigid varieties is that the automorphism group of any such variety has a unique maximal torus, which is a normal subgroup in $\text{Aut}(X)$; see [6, Theorem 2.1]. This allows us to explicitly describe the automorphism group of a rigid affine toric variety. Recall that we already know two subgroups of $\text{Aut}(X)$: one of them is the torus $T$, which is embedded in $\text{Aut}(X)$ because $X$ is toric, and the other one is the subgroup $S(X) \subset \text{Aut}(X)$ introduced in Definition 2.

**Theorem 3.** Let $X$ be a rigid affine toric variety. Then the automorphism group $\text{Aut}(X)$ is a semidirect product $S(X) \ltimes T$.

**Proof.** Let $\psi$ be an automorphism of $\mathbb{K}[X]$. It takes $M$-homogeneous functions to $M$-homogeneous ones, because $T$ is a normal subgroup of $\text{Aut}(X)$. Moreover, since $\psi$ preserves multiplication, it follows that there exists a linear operator $\varphi: M_\mathbb{Q} \to M_\mathbb{Q}$ such that $\varphi(P) = P$ and, given any $m \in M$, there exists a nonzero constant $\lambda$ for which $\psi(\chi^m) = \lambda\chi^{\varphi(m)}$. Recall that we introduced a homomorphism $\alpha: \text{St}_{\text{GL}_m}(\mathbb{Z})(P) \to \text{Aut}(X)$ in Sec. 3.

Consider the automorphism $\zeta = \psi \circ \alpha(\varphi)^{-1}$. We have $\zeta(\chi^m) = \lambda\chi^m$. Since $\zeta$ is an automorphism, it follows that the proportionality coefficients $\lambda$ agree with each other, so that the map $m \mapsto \lambda$ is a homomorphism $M \to \mathbb{K}^\times$. Therefore, the action of $\zeta$ on $\mathbb{K}[X]$ coincides with that of some element of the torus $T$. This proves that $S(X)$ and $T$ generate $\text{Aut}(X)$. Obviously, the intersection of $S(X)$ and $T$ is trivial. Since $T$ is a normal subgroup of $\text{Aut}(X)$, it follows that $\text{Aut}(X) \cong S(X) \ltimes T$. This completes the proof of Theorem 3.

**Example 3.** Consider the variety $X$ corresponding to the monoid $P$ consisting of all integer points $(a, b)$ with $a \geq 0$ and $b \geq 0$ except the points $(0, 2k + 1)$ and $(2k + 1, 0)$, $k \in \mathbb{Z}_{\geq 0}$ (see Fig. 2). This variety is rigid, because both of its one-dimensional faces are not saturated.

The subgroup $S(X)$ is isomorphic to $\mathbb{Z}_2$. It consists of the trivial automorphism and the automorphism reflecting the cone across the diagonal. The automorphism group of the variety $X$ is isomorphic to $\mathbb{Z}_2 \ltimes (\mathbb{K}^\times)^2$. 

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7. ALMOST RIGID TORIC VARIETIES

In this section, we study yet another class of affine toric varieties for which automorphism groups can be described. These groups are already not algebraic, but they still are semidirect products of three subgroups, each of which can be described explicitly.

Definition 3. A variety \( X \) is said to be almost rigid if it is not rigid and all LNDs of the algebra \( \mathbb{K}[X] \) pairwise commute.

We denote the commutative group \( \text{SAut}(X) \) for an almost rigid variety \( X \) by \( U(X) \). Lemma 1 implies that a variety \( X \) is almost rigid if and only if the kernels of all LNDs of \( \mathbb{K}[X] \) coincide. Given an LND \( \partial \), we can consider the commutative subgroup \( U(\partial) = \{\exp(f\partial) \mid f \in \ker \partial\} \) in the automorphism group. Some authors use the term “almost rigid variety” for a narrower class of varieties, in which all LNDs are replicas of the same derivation \( \partial \). In this case, \( U(X) = U(\partial) \). However, not all almost rigid varieties have this property; see Example 5.

Theorem 4. An affine toric variety \( X \) is almost rigid if and only if the cone \( \sigma \) has a unique edge \( \rho \) for which the hyperface \( \hat{\rho} \) is almost saturated.

Proof. If the cone \( \sigma' \) has no almost saturated hyperfaces, then, by Theorem 2, the variety \( X \) is rigid. Suppose that \( \sigma' \) has an almost saturated hyperface \( \rho \). According to Proposition 1, there exists a Demazure root \( e \in R_{\rho} \) for which the derivation \( \delta_e \) is well defined. We have

\[
\ker \delta_e = \bigoplus_{m \in \mathbb{F} \cap \hat{\rho}} \mathbb{K}\chi^m.
\]

If there are at least two almost saturated hyperfaces, then there are LNDs with different kernels, and hence \( X \) is not almost rigid. Conversely, suppose that there is only one almost saturated hyperface \( \tau \). Let \( \partial \) be an LND of \( \mathbb{K}[X] \). Then \( \partial \) decomposes into a sum of homogeneous derivations. The vertices of the polytope generated by the degrees of summands are degrees of homogeneous LNDs, i.e., Demazure roots. All of them have degree \(-1\) in the grading corresponding to the hyperface \( \tau \). Therefore, \( \partial \) has the same degree \(-1\) in this grading. Thus, the kernel of \( \partial \) contains the zero component of this grading. According to [8, Principle 11], the transcendence degree of \( \ker \partial \) is \( \dim X - 1 \). Therefore, \( \ker \partial \) coincides with the zero component. It follows that the kernels of all LNDs coincide and \( X \) is almost rigid. This completes the proof of Theorem 4.

Let \( \hat{\rho} \) be the only almost saturated hyperface of the cone \( \sigma' \). It determines the \( \mathbb{Z} \)-grading

\[
\mathbb{K}[X] = \bigoplus_{i \in \mathbb{Z}} \mathbb{K}[X]_i, \text{ where } \mathbb{K}[X]_i = \bigoplus_{(m,v_i) = i} \chi^m.
\]

The following theorem describes the automorphism group of an almost rigid toric variety.

Theorem 5. The automorphism group of an almost rigid affine toric variety \( X \) is isomorphic to the semidirect product \( (S(X) \ltimes T) \ltimes U(X) \).

Proof. Since $\sigma$ has a unique almost saturated hyperface, it follows that the cone $\gamma^\vee$ is a half-space. Therefore, the algebra $\mathbb{K}[X^\text{reg}]$ is isomorphic to $A = \mathbb{K}[x_1, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$. The restriction of each automorphism of $X$ to $X^\text{reg}$ is an automorphism of $X^\text{reg}$, and the restrictions of different automorphisms of $X$ to $X^\text{reg}$ are different; therefore, we have an embedding

$$\Theta: \text{Aut}(X) \hookrightarrow \text{Aut}(X^\text{reg}) \cong \text{Aut}(A).$$

Note that any $M$-homogeneous LND has degree $-1$ in the $\mathbb{Z}$-grading introduced before the statement of the theorem. Consider any LND $\partial$. It can be decomposed into a finite sum of $\mathbb{Z}$-homogeneous derivations. Moreover, the extreme components in this sum are locally nilpotent. If the extreme components have degree $-1$, then the initial LND is homogeneous of degree $-1$. Suppose that $\partial_k$ is an extreme component (and hence an LND) of degree $k \neq -1$ in the $\mathbb{Z}$-grading. Then $\partial_k$ can be decomposed into $M$-homogeneous components each of which has degree $k$ in the $\mathbb{Z}$-grading. But at least one of these components corresponds to a vertex of the convex hull of the degrees of summands and is an LND. Thus, we obtain an $M$-homogeneous LND of degree $k$ in the $\mathbb{Z}$-grading. This contradiction shows that $\partial$ is homogeneous of degree $-1$ in the $\mathbb{Z}$-grading.

We set

$$B = \mathbb{K}[x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}].$$

The kernel of any LND $\partial$ has the form $\mathbb{K}[X] \cap B$. Indeed, the zero component of the grading is contained in the kernel of $\partial$, because the degree of $\partial$ equals $-1$. On the other hand, since $\tilde{\partial}$ is a hyperface, it follows that the transcendence degree of $\mathbb{K}[X] \cap B$ equals $\dim X - 1$, and since this is the zero component of the grading, the addition of any element not belonging to $\mathbb{K}[X] \cap B$ increases the transcendence degree. However, according to [8, Principle 11 (e)], the transcendence degree of $\ker \partial$ equals $\dim X - 1$. Thus, $\ker \partial = \mathbb{K}[X] \cap B$. Since $\deg \partial = -1$, it follows that

$$\ker \partial^2 = \mathbb{K}[X] \cap (B \oplus x_1 B),$$

that is, the kernel of the square of any LND has the form $\mathbb{K}[X] \cap (B \oplus x_1 B)$. Therefore, for any automorphism $\varphi$ of the algebra $\mathbb{K}[X]$, we have

$$\Theta(\varphi)(x_1) = f x_1 + g, \quad \text{where} \quad f, g \in B.$$

The map $f$ is invertible, because $\varphi$ is an automorphism, and hence $f = c \chi^m$, where $m, -m \in P$ and $c \neq 0$. There exists $\beta \in S(X)$ and $\tau \in T$ for which

$$\Theta(\beta)(x_1) = \chi^{-m} x_1 \quad \text{and} \quad \Theta(\tau)(x_1) = c^{-1} x_1.$$

We have

$$\Theta(\tau \circ \beta \circ \varphi)(x_1) = x_1 + h, \quad \text{where} \quad h \in B.$$

Let $\psi = \Theta(\tau \circ \beta \circ \varphi)$. Then $\psi$ is the image of an automorphism of $X$; therefore, for each $p \in P$, the element $\psi(\chi^p) \in A$ decomposes into a linear combination of elements $\chi^m, m \in P$. Consider the map $
abla: A \to A$ such that $\nabla(x_1) = x_1$ and $\nabla$ coincides with $\psi$ on $B$. The Laurent monomials contained in $\nabla(\chi^m)$ with nonzero coefficients form a subset in the set of Laurent monomials contained in $\psi(\chi^m)$ with nonzero coefficients. Therefore, $\nabla$ induces an automorphism of $\mathbb{K}[X]$. Note that $\nabla^{-1} = (\nabla)^{-1}$; hence $\nabla$ is the image under $\Theta$ of an automorphism $\gamma \in \text{Aut}(X)$. Moreover, the automorphism $\gamma$ is induced by an automorphism in $\text{Aut}(B) \cong \text{GL}_{n-1}(\mathbb{Z}) \rtimes T$, so that $\gamma$ is a composition of elements of $S(X)$ and $T$. We have

$$\Theta(\gamma^{-1} \circ \tau \circ \beta \circ \varphi)(x_1) = x_1 + \tilde{h}, \quad \tilde{h} \in B,$$

$$\Theta(\gamma^{-1} \circ \tau \circ \beta \circ \varphi)(x_i) = x_i, \quad i \geq 2.$$

If $\tilde{h}$ contains a Laurent monomial $\chi^m$ with nonzero coefficient, then $P$ is invariant with respect to the shift by $e = m - (1, 0, \ldots, 0)$, and the LND $\delta_k$ of the algebra $\mathbb{K}[X]$ is well defined. Therefore, $\gamma^{-1} \circ \tau \circ \beta \circ \varphi$ is a composition of automorphisms in $U(X)$.

Thus, we have seen that any automorphism $\varphi \in \text{Aut}(X)$ decomposes into a composition of automorphisms in $S(X)$, $T$, and $U(X)$. Moreover, $U(X) = \text{SAut}(X)$ is a normal subgroup in $\text{Aut}(X)$,
and each of its element is unipotent. On the other hand, the discrete subgroup $S(X)$ normalizes the torus $T$. Thus, the intersection of $S(X) \ltimes T$ with $U(X)$ is trivial, and $\text{Aut}(X) = (S(X) \ltimes T) \ltimes U(X)$.

This completes the proof of Theorem 5.

**Example 4.** Consider the variety $X$ corresponding to the monoid $P$ consisting of all points $(a, b) \in \mathbb{Z}_{\geq 0}^2$ except $(2k + 1, 0)$, $k \in \mathbb{Z}_{\geq 0}$ (see Fig. 3). This variety is almost rigid, because the face $Q_{\geq 0}(0, 1)$ of the cone $\sigma^\vee$ is almost saturated, while the face $Q_{\geq 0}(1, 0)$ is nowhere saturated.

![Fig. 3.](image)

The subgroup $S(X)$ of this variety is trivial. It is easy to show that the subgroup $U(X)$ has the form $U(\delta_e)$, where $e = (-1, 1)$. The automorphism group of the variety $X$ is isomorphic to $T \ltimes U(\delta_e)$.

Let us explicitly write out the automorphisms. We have

$$\mathbb{K}[X] = \mathbb{K}[x^2, y, xy] = \mathbb{K}[u, v, w]/(uv^2 - w^2).$$

The torus $T$ acts by the rule

$$(t_1, t_2) \cdot (u, v, w) = (t_1^2 u, t_2 v, t_1 t_2 w).$$

The derivation $\delta_e$ is defined by $\delta_e(u, v, w) = (w, 0, v^2)$, and $\text{Ker} \delta_e = \mathbb{K}[v]$. As a result, we conclude that any automorphism $X$ has the form

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} t_1^2 \left( u + f(v) w + \frac{f^2(v) v^2}{2} \right) \\ t_2 v \\ t_1 t_2 (w + f(v) v^2) \end{pmatrix}, \quad t_1, t_2 \in \mathbb{K}^\times, \quad f \in \mathbb{K}[v].$$

**Example 5.** Let us remove the point $(0, 1)$ from the monoid $P$ in the preceding example. This does not affect the existence of saturated points in faces. Thus, the corresponding variety $X$ remains almost rigid (see Fig. 4).

![Fig. 4.](image)

The groups $S(X)$, $U(X)$, and $\text{Aut}(X)$ are the same as in the preceding example. However, now not all LNDs are replicas of $\delta_e$. Indeed, e.g., for $e' = (-1, 2)$, we have $\delta_{e'} = \chi^{0, 1} \delta_e$. But $\chi^{0, 1}$ is not a regular function on $X$ in this example. Thus, $U(X)$ coincides with $\mathbb{U}(\delta)$ for no LND $\delta$.

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