Group-theoretical analysis of variable coefficient nonlinear telegraph equations

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Given a class \( \mathcal{F}(\theta) \) of differential equations with arbitrary element \( \theta \), the problems of symmetry group, nonclassical symmetry and conservation law classifications are to determine for each member \( f \in \mathcal{F}(\theta) \) the structure of its Lie symmetry group \( G_f \), conditional symmetry \( Q_f \) and conservation law \( CL_f \) under some proper equivalence transformations groups.

In this paper, an extensive investigation of these three aspects is carried out for the class of variable coefficient \((1+1)\)-dimensional nonlinear telegraph equations with coefficients depending on the space variable \( f(x)u_{tt} = (g(x)H(u)u_x)_x + h(x)K(u)u_x \). The usual equivalence group and the extended one including transformations which are nonlocal with respect to arbitrary elements are first constructed. Then using the technique of variable gauges of arbitrary elements under equivalence transformations, we restrict ourselves to the symmetry group classifications for the equations with two different gauges \( g = 1 \) and \( g = h \). In order to get the ultimate classification, the method of furcate split is also used and consequently a number of new interesting nonlinear invariant models which have non-trivial invariance algebra are obtained. As an application, exact solutions for some equations which are singled out from the classification results are constructed by the classical method of Lie reduction.

The classification of nonclassical symmetries for the classes of differential equations with gauge \( g = 1 \) is discussed within the framework of singular reduction operator. This enabled to obtain some exact solutions of the nonlinear telegraph equation which are invariant under certain conditional symmetries.

Using the direct method, we also carry out two classifications of local conservation laws up to equivalence relations generated by both usual and extended equivalence groups. Equivalence with respect to these groups and correct choice of gauge coefficients of equations play the major role for simple and clear formulation of the final results.

Mathematics Subject Classifications (2000): 35L10, 35A22, 35A30

Keywords: symmetry classification, nonclassical symmetry, conservation law, equivalence group, nonlinear telegraph equation, exact solutions, symmetry analysis, Lie algebras

1 Introduction

Since the notation of continuous group was introduced by Lie at the end of 19th century, significant progress in application of symmetries to analysis of concrete nonlinear differential equations has been achieved. The classical Lie symmetries of nonlinear differential equations allow us to find explicit solutions, conservation laws, linearizing substitutions of the Hopf-Cole type, etc [13,43,44,70,73]. For most of the application the search of explicit symmetry structure of the corresponding differential equations is always a crucial step, which consist of the cornerstone
of group analysis of differential equations. Consequently, there are two interrelated problems which still remain to be solved in the traditionally group analysis of differential equations. The first problem consists of finding the maximal Lie symmetry group admitted by a given equation. The second problem is one of classifying differential equations that admit a prescribed symmetry group. The principal tool for handling both problems is the classical infinitesimal Lie method \[13,33,70,73\]. It reduces the problem to finding the corresponding Lie symmetry algebra of infinitesimal operators whose coefficients are found as solutions of some over-determined system of linear partial differential equations (PDEs). However, if the equations under study contains with arbitrary element (functions or parameters), then one has to solve an intermediate classification problem. Namely, it is necessary to describe all the possible forms of the functions involved such that this equation admits a nontrivial invariance group. Generally, the problem can be described as follows: Given a class \( \mathcal{F}(\theta) \) of differential equations with arbitrary element \( \theta \), the problem of group classification is to determine for each member \( f \in \mathcal{F}(\theta) \) the structure of its Lie symmetry group \( G_f \), or equivalently of its Lie symmetry algebra \( A_f \) under some proper equivalence transformation groups. This description is also fit for the problems of nonclassical symmetry and conservation law classification by replacing Lie symmetry with these two different notations.

Historically, the first classification of Lie symmetries was derived by Lie, he proved that a linear two-dimensional second-order PDE may admit at most a six-parameter invariance group (apart from the trivial infinite parameter symmetry group, which is due to linearity) \[62\]. The modern formulation of the problem of group classification of PDEs was suggested by Ovsiannikov in 1959 \[72\], in which he present complete group classification of a class of nonlinear heat conductivity equations by using the technique of equivalence group and direct integration. After that, the group classification of nonlinear PDEs became the subject of intensive research. A detailed survey of the work done in this area up to the beginning of the 1990’s is given in \[44\].

In the present paper we investigate a class of hyperbolic type variable coefficient \((1+1)\)-dimensional nonlinear telegraph equations of the form

\[
f(x)u_{tt} = (g(x)H(u)u_x)_x + h(x)K(u)u_x
\]

where \( f = f(x) \), \( g = g(x) \), \( h = h(x) \), \( H = H(u) \) and \( K = K(u) \) are arbitrary and sufficient smooth real-valued functions of their corresponding variables, \( f(x)g(x)H(u) \neq 0 \). In what follows, we assume that \( (H_u, K_u) \neq (0, 0) \), i.e., \( f(x) \) is a nonlinear equation. This is because the linear case of \( f(x) = const \) was studied by Lie \[62\] in his classification of linear second-order PDEs with two variables. (See also a modern treatment of this subject in \[73\].)

The study of equation \( \mathbf{1} \) is strongly stimulated not only by its intrinsic theoretical interest but also by its significant applications in Mathematics and Engineering. In fact, hyperbolic type second-order nonlinear PDEs in two independent variables are usually used to describe different types of wave propagation. They are also used in differential geometry, in various fields of hydro- and gas dynamics, chemical technology, super conductivity, crystal dislocation to mention only a few applications areas. The corresponding models are comprised by the Liouville, sine/sinh-Gordon, Goursat, d’Alembert, Tzitzeica and nonlinear telegraph equations and a couple of others. From the group-theoretical viewpoint the popularity of these models is due to the fact that they have non-trivial Lie or Lie–Bäcklund symmetry \[10,13,28,43,45,70,73,85\]. By this very reason they are either integrable by the inverse problem methods or are linearizable and completely integrable \[11,26,60\].

The investigation of Lie symmetry classification of the \((1+1)\)-dimensional hyperbolic type second-order nonlinear PDEs has a long history. Probably, Barone et al \[8\] was the first study of the following nonlinear wave equation \( u_{tt} = u_{xx} + F(u) \), by means of symmetry method, this equation was also studied by Kumei \[50\] and Pucci et al \[83\] subsequently. Motivated by a number of physical problems, Ames et al \[5,6\] investigated group properties of quasi-linear
hyperbolic equations of the form

$$u_{tt} = [f(u)u_x]_x.$$  \(2\)

Later, their investigation was generalized in [23, 46, 87] to equations of the following forms respectively

$$u_{tt} = [f(x,u)u_x]_x,$$

$$u_{tt} = [f(u)u_x + g(x,u)]_x, \quad \text{and} \quad u_{tt} = f(x,u_x)u_{xx} + g(x,u_x).$$

The alternative form of equation (2) was also investigated by Oron and Rosenau [71] and Suhubi and Bakkaloglu [86]. Arrigo [7] classified the equations

$$u_{tt} = u_{xx} + F(t,x,u,u_x)$$

by using the infinitesimal Lie method, the technique of equivalence transformations and the theory of classification of abstract low-dimensional Lie algebras. There are also some papers [19, 32, 71, 82] devoted to the group classification of the equation of the following form

$$u_{tt} = F(u_{xx}), \quad u_{tt} = F(u_x)u_{xx} + H(u_x), \quad \text{and} \quad u_{tt} + \lambda u_{xx} = g(u,u_x).$$

It is worthwhile mentioned that the constant-coefficient nonlinear telegraph equations

$$u_{tt} = (F(u_x)u_x + H(u)u_x)$$

together with its equivalent potential systems have also been studied by Bluman et al [11, 14-16]. In their a series of papers, many interesting results (especially for case of power nonlinearities) including Lie point and nonlocal symmetries classification and conservation law of the four equivalent systems were systematically investigated. Recently, Huang and Ivanova present a strong complete group classification for a class of variable coefficient (1+1)-dimensional nonlinear telegraph equations of the form [37]

$$f(x)u_{tt} = (H(u_x)u_x + K(u)u_x).$$  \(3\)

Exact solutions and classifications of conservation law with characteristics of order 0 were also investigated [37].

From the above introduction, we can see that equation (1) is different from any aforementioned ones and is a generalization of many well studied equations. What’s more, equations (1) can be used to model a wide variety of phenomena in physics, chemistry, mathematical biology etc (see Section 2 for detail). Thus there is essential interest in investigating them from a unified and group theoretical viewpoint.

In this paper, extended group analysis of class (1) is first carried out. The usual equivalence group and the extended one including transformations which are nonlocal with respect to arbitrary elements are constructed for class (1) and its subclasses. The structure of the extended equivalence group and non-trivial subgroup of (nonlocal) gauge equivalence transformations are investigated. As a result, group classification problems related to two different gauges \(g = 1\) and \(g = h\) are really solved for each form with respect to the corresponding usual and extended equivalence group. Classical Lie reduction of some classification models are described and exact solutions are obtained by using the reduction. Nonclassical symmetries classification of class (1) with the gauges \(g = 1\) is discussed within the framework of singular reduction operator. Several
nonclassical symmetries for equation form class (1) are constructed. This enabled to obtain some exact solutions of the nonlinear telegraph equation which are invariant under certain conditional symmetries. Using the most direct method, two classifications of local conservation laws up to equivalence relations are generated by both usual and extended equivalence groups. Equivalence with respect to these groups and correct choice of gauge coefficients of equations play the major role for simple and clear formulation of the final results.

Problems of group classification, except for really trivial cases, are very difficult. Generally, there are two main approaches in studying group classification problems in the literature. The first one is the algebraic methods and is based on subgroup analysis of the equivalence group associated with a class of differential equations under consideration. Its main idea rely on the description of inequivalent realizations of Lie algebras in certain set of vector fields of the equation under consideration \cite{9,93}, which was original from S. Lie \cite{44,62} and recently rediscovered by Winternitz and Zhdanov et al \cite{34,93}. The method has been applied to classifying a number of nonlinear differential equations \cite{2,9,33,34,40,42,59,60,93,95}, including the class is normalized (see \cite{81} for rigorous definitions of normalized classes and related notions). The second approach is based on the investigation of compatibility and the direct integration, up to the equivalence relation generated by the corresponding equivalence group, of determining equations implied by the infinitesimal invariance criterion \cite{73}. This method was suggested by L.V Ovsyannikov and referred as the Lie-Ovsiannikov method. This is the most applicable approach but it is efficient only for classes of a simple structure, e.g., which have a few arbitrary elements of one or two same arguments or whose equivalence groups are finite-dimensional. A number of results on group classification problems investigated within the framework of this approach are collected in \cite{13,44,73} and other books on the subject.

Recently, based on the Lie-Ovsyannikov method and the investigation of the specific compatibility of classifying conditions, Nikitin and Popovych \cite{65} developed an effective tool (we refer it as method of furcate split) for solving the group classification problem of nonlinear Schrödinger equation. In 2004, Popovych and Ivanova extended the method to complete group classification of nonlinear diffusion-convection equations by further considering the so called additional and conditional equivalence transformations \cite{78}. In 2007, Ivanova, Popovych and Sophocleous present the extended and generalized equivalence transformation group, gauging of arbitrary elements by equivalence transformations for further investigation of nonlinear diffusion-convection equations \cite{47}. Furthermore, Popovych and Ivanova et.al also extended these new group classification idea to the nonclassical symmetries \cite{57,90} and conservation law classification \cite{49,79}. Up to now, these methods and different notations have been applied to investigating a number of different symmetry group, nonclassical symmetry and conservation law classification problems \cite{17,47,49,51,57,60,65,77,79,81,89,90,92}.

However, almost all the research was concentrated on parabolic type nonlinear diffusion-convection equations and few of hyperbolic type nonlinear partial differential \cite{87,39}. Therefore, the present paper is one of new extension of the above mentioned method and different notations to this classes of equations. The results of symmetry group, nonclassical symmetry and conservation law classification of class (11) present in this work are new. Hence, these will lead to some explicit applications in Physics and Engineering.

The structure of the paper is as follows:

Some physical examples contained in class (11) is discussed in section 2.

In section 3 the complete group of usual equivalence transformations for class (11) and the extended one including transformations which are nonlocal with respect to arbitrary elements are constructed by using Lie-Ovsiannikov method and direct method. Taking into account the non-trivial subgroup of gauge equivalence transformations, we strongly simplify the solving the group classification problem to equation (11) with two different gauges $g = 1$ and $g = h$.
the gauge $g = 1$ are contained in section 4.1. Then in section 4.2, the classification of gauge $g = h$ are presented. The sketch of the proof of the obtained results are given in Section 4.3. Classification with respect to the set of point transformations are presented in 4.4. We note that for both gauges two essentially different classifications are presented: the classification with respect to the (extended) equivalence group and the classification with respect to all possible point transformations.

In Section 5, exact solutions of some classification models are given by using the method of classical Lie reduction.

After making a brief review of notation of singular reduction operator, we then carry out a preliminary analysis of nonclassical symmetry of the class (1) with the gauge $g = 1$ in section 6. As an example, we also present several reduction operators of a special nonlinear telegraph equation and constructed some non-Lie exact solutions for them.

In Section 7, the local conservation laws of these equations are exhaustively described. Using the most direct method, two classifications of local conservation laws up to equivalence relations are generated by both usual and extended equivalence groups. Equivalence with respect to these groups and correct choice of gauge coefficients of equations play the major role for simple and clear formulation of the final results.

Finally, some conclusion and discussion are given in Section 8.

In the Appendix, classification results for the gauge $g = 1$ under the usual equivalence transformation group can be found.

2 Physical examples

Class (1) is a unified form of many significant second-order hyperbolic type nonlinear PDEs in Physics, Mechanics and Engineering Science. Physical examples corresponding to the case $f(x) = g(x) = h(x) = 1$ and $K(u) = 0$ are collected in the well known paper [5, 6], which describe the flow of one-dimensional gas, longitudinal wave propagation on a moving threadline and dynamics of a finite nonlinear string and so on. In what follows, we review several important physical models related with the coefficient $f(x) \neq 0$ or $h(x)K(u) \neq 0$ [88].

Example 1. Two-conductor transmission lines telegraph equation. The waves in two-conductor transmission lines having small transverse dimensions (in comparison with the characteristic wavelength) can often be described by the telegraph equation [53]

\[ I_x = U_t, \quad I_t = F(U)U_x + G(U), \]  

(4)

where $t$ is a spatial variable and $x$ is time; $I, U, F(U)$, and $G(U)$ are respectively the current in the conductors, the voltage between the conductors, the leakage current per unit length, and the differential capacitance. The form of $F(U)$, and $G(U)$ depend both on the configuration of the conductors, and on the properties of the medium filling it.

Setting $U = u$, we obtain the telegraph equation

\[ u_{tt} = (F(u)u_x)_x + G'(u)u_x, \]  

(5)

which fall into (1).

Example 2. Longitudinal vibrations of elastic and non-homogeneous taut strings or bars. Suppose a string is taut along the $x$–axis. The equation giving the balance of momentum is

\[ \rho \omega_{tt} = T_x, \]  

(6)

where $x$ is the coordinate of the point $P$ in the present reference system, and

\[ x = x(y, t), \]
where \( y \) represents the coordinate of the corresponding point \( P_0 \) of \( P \) in the reference shape, where \( \omega = x - y \), \( \rho \) is mass per unit length and \( T \) is the tension. To equation (6) we associate the following constitutive relations already considered in [54]:

\[
T = T(\omega_x), \rho = \rho(x). \tag{7}
\]

The balance law (5), with (7), transforms to the following second order partial differential equation

\[
u_{tt} = \left[ \frac{T'(u)}{\rho(x)} u_x \right]_x, \tag{8}\]

where \( \omega_x = u \). Of course (8) is particular case of (1).

**Example 3.** Bar with variable cross section. The equation of motion of a hyperelastic homogeneous bar, whose cross sectional area is variable along the bar, is [22]

\[
\rho \omega_{tt} = T_x + \frac{S'(x)}{S(x)} T, \tag{9}\]

where \( \rho \) is the (constant) mass density, \( \omega = y - x \) is the displacement, \( y \) is the coordinate of the point \( P \) in the present reference system, \( x \) represents the coordinate of the corresponding point \( P_0 \) of \( P \) in the reference frame, \( T \) is the tension and \( S(x) \) is the cross sectional area.

Taking into account the constitutive relation \( T = T(\omega_x) \) [22], the equation (9) becomes

\[
u_{tt} = \left[ \frac{T'(u)}{\rho(x)} u_x \right]_x + \left( \frac{S'(x)}{S(x)} \right)_x \frac{S'(x)}{S(x)} T, \tag{10}\]

where \( u = \omega_x \). Obviously when \( (\frac{S'(x)}{S(x)})_x = 0 \) the equation (10) is included in (1).

**Example 4.** One dimensional propagation of visco-elastic stress waves. In [91], the following constitutive laws were adopted for a nonlinear homogeneous visco-elastic model

\[
e_t = \Phi(T) T_t + \Omega(T), \tag{11}\]

with

\[
\Phi(T) = \frac{d_0}{(1 + kc_0 T)^2}, \quad \Omega(T) = \frac{c_0 T}{1 + kc_0 T}, \quad e_t = \omega_{xt} \tag{12}\]

where \( T \) is the stress, \( \omega \) is the displacement, \( d_0 > 0, k \) and \( c_0 \) are constants. When the one-dimensional propagation of visco-elastic stress waves is investigated, combining the momentum equation

\[
\rho \omega_{tt} = T_x \tag{13}\]

with (11) and taking into account (12), we obtain

\[
T_{xx} = \left[ \frac{d_0}{(1 + kc_0 T)^2} T_t \right]_t + \frac{c_0}{(1 + kc_0 T)^2} T_t. \tag{14}\]

So, we fall into (11) when \( x \) and \( t \) are exchanged.

**Example 5.** Hyperbolic heat equation. Many models for the heat propagation with finite speed give a hyperbolic equation which can be reduced to (1). In fact, quite recently, in order to describe one-dimensional heat conduction, the following partial differential equation has been considered in [84]

\[
\theta_{tt} - \frac{q_0 a^2}{\gamma^*} \left( \frac{\delta}{\theta} + \epsilon \right)^2 \theta_{xx} - \frac{2\epsilon}{\delta + \epsilon} \theta_{tt} + \frac{1}{\gamma^*} \theta_t + \frac{2q_0 a^2}{\gamma^*} \left( \frac{\delta}{\theta} + \epsilon \right)^2 \theta_x^2 = 0, \tag{15}\]

6
where \( \theta \) denote the (absolute) temperature while \( q_0, a, \gamma^*, \delta \) and \( \epsilon \) are suitable constants. The equation (15) is based on a nonlinear model with relaxation. If we set \( \theta = 1/u \), we obtain

\[
\frac{\gamma^*}{q_0 a^2 (\delta u + \epsilon)^2} u_{tt} + \frac{1}{q_0 a^2 (\delta u + \epsilon)^2} u_t. \tag{16}
\]

This equation was obtained in [24] in the framework of Müller’s theory for heat propagation in rigid bodies.

**Example 6.** One dimensional heat propagation in a rigid body. The models like Cattaneo’s [18], describing one dimensional heat propagation in a rigid body, are governed by a nonlinear equation of the type [25]

\[
\theta_{xx} = \frac{\tau_0 C(\theta)}{\chi} \theta_t + \frac{1}{\chi} \int C(\theta) d\theta \theta_t. \tag{17}
\]

where \( C(\theta) \) is the special heat, \( \tau_0 \) is the thermal relaxation time and \( \chi \) is the thermal conductivity. The equation (16) and (17) fall into (1) when \( x \) and \( t \) are exchanged.

### 3 Equivalence transformations and choice of investigated class

In order to perform group classification of class (1), we should first find its group of equivalence transformations. The usual equivalence group \( G^\sim \) of class (1) is formed by the nondegenerate point transformations in the space of \((t,x,u,f,g,h,H,K)\), which are projectible on the space of \((t,x,u)\), i.e. they have the form

\[
(\tilde{t}, \tilde{x}, \tilde{u}) = (T_t t, T_x x, T_u u),
\]

\[
(\tilde{f}, \tilde{g}, \tilde{h}, \tilde{H}, \tilde{K}) = (T_f f, T_g g, T_h h, T_H H, T_K K)(t, x, u, f, g, h, H, K),
\]

and transform any equation from class (1) for the function \( u = u(t,x) \) with the arbitrary elements \( f, g, h, H, K \) to an equation from the same class for the function \( \tilde{u} = \tilde{u}(\tilde{t}, \tilde{x}) \) with the new arbitrary elements \( \tilde{f}, \tilde{g}, \tilde{h}, \tilde{H}, \tilde{K} \). To find the connected component of the unity of \( G^\sim \), we have to investigate Lie symmetries of the system that consists of equation (1) and some additional conditions, that is to say we must seek for an operator of the \( G^\sim \) in the form

\[
X = \tau \partial_t + \xi \partial_x + \eta \partial_u + \pi \partial_f + \varphi \partial_g + \phi \partial_h + \rho \partial_H + \theta \partial_K \tag{18}
\]

from the invariance criterion of the following system:

\[
\begin{align*}
    f(x)u_{tt} &= (g(x)H(u)u_x)_x + h(x)K(u)u_x, \\
    f_t &= f_u = 0, \quad g_t = g_u = 0, \quad h_t = h_u = 0, \quad H_t = H_x = 0, \quad K_t = K_x = 0. \tag{19}
\end{align*}
\]

Here \( u, f, g, h, H \) and \( K \) are considered as differential variables: \( u \) on the space \((t,x)\) and \( f, g, h, H, K \) on the extended space \((t,x,u)\). The coordinates \( \tau, \xi, \eta \) of the operator (18) are sought as functions of \( t, x, u \) while the coordinates \( \pi, \varphi, \phi, \rho \) and \( \theta \) are sought as functions of \( t, x, u, f, g, h, H, K \).

The invariance criterion of system (19) yields the following determining equations for \( \tau, \xi, \eta, \)
\[\tau_x = \tau_u = \xi_t = \xi_u = \eta_x = \eta_t = 0, \quad \tau_{tt} = \eta_{uu} = 0,\]
\[\pi_t = \tau_u = \pi_H = \pi_K, \quad \varphi_t = \varphi_u = \varphi_H = \varphi_K = 0,\]
\[\phi_t = \phi_u = \phi_H = \phi_K = 0, \quad \rho_t = \rho_x = \rho_u = \rho_f = \rho_g = \rho_h = \rho_K = 0,\]
\[\theta_t = \theta_x = \theta_f = \theta_g = \theta_h = 0,\]
\[\frac{\pi}{f} + 2\xi_x - 2\tau_t - \frac{\phi}{g} - \rho_H = 0, \quad \frac{\pi}{f} + 2\xi_x - 2\tau_t - \frac{\phi}{g} = \frac{\rho}{H},\]
\[g_x \rho + \left[ -g\xi_{xx} + \frac{(2\tau_t - \xi_x - \frac{\pi}{f})g_x + \phi_x + f_x\phi_f + g_x\phi_g + h_x\phi_h - \xi_x g_x}{H} \right] \]
\[+ h\theta + (2\tau_t - \xi_x - \frac{\pi}{f} + \phi) h K = 0.\]

After easy calculations from (20), we can find the connected component of the unity of \(G^\sim\) for the class \(\mathfrak{(1)}\).

**Theorem 1.** The usual equivalence transformation group \(G^\sim\) for the class \(\mathfrak{(1)}\) consists of the transformations
\[
\hat{t} = \delta_1 t + \delta_2, \quad \hat{x} = X(x), \quad \hat{u} = \delta_3 u + \delta_4,
\]
\[
\hat{f} = \frac{e_1 \delta^2}{X_x} f, \quad \hat{g} = e_1 \epsilon_2^{-1} X_x g, \quad \hat{h} = e_1 \epsilon_3^{-1} h, \quad \hat{H} = e_2 H, \quad \hat{K} = e_3 K,
\]
where \(\delta_j (j = 1, \ldots, 4)\) and \(\epsilon_i (i = 1, 2, 3)\) are arbitrary constants, \(\delta_1 \delta_3 \epsilon_1 \epsilon_2 \epsilon_3 \neq 0, \) \(X\) is an arbitrary smooth function of \(x, X_x \neq 0.\)

It is shown that class \(\mathfrak{(1)}\) admits other equivalence transformations which do not belong to \(G^\sim\) and form, together with usual equivalence transformations, an extended equivalence group. We demand these transformations to be point with respect to \((t, x, u)\). The explicit form of the new arbitrary elements \((\hat{f}, \hat{g}, \hat{h}, \hat{H}, \hat{K})\) is determined via \((t, x, u, f, g, h, H, K)\) in some nonfixed (possibly, nonlocal) way. We can construct the complete (in this sense) extended equivalence group \(\hat{G}^\sim\) of class \(\mathfrak{(1)}\) by using the direct method.

**Theorem 2.** The extended equivalence transformation group \(\hat{G}^\sim\) for the class \(\mathfrak{(1)}\) is formed by the transformations
\[
\hat{t} = \delta_1 t + \delta_2, \quad \hat{x} = X(x), \quad \hat{u} = \delta_3 u + \delta_4,
\]
\[
\hat{f} = \frac{e_1 \delta^2}{X_x} f \int e^{-\epsilon_4 f_{\hat{H}} / \hat{g}} dx, \quad \hat{g} = e_1 \epsilon_2^{-1} X_x g \int e^{-\epsilon_4 f_{\hat{H}} / \hat{g}} dx, \quad \hat{h} = e_1 \epsilon_3^{-1} h \int e^{-\epsilon_4 f_{\hat{H}} / \hat{g}} dx, \quad \hat{H} = e_2 H, \quad \hat{K} = e_3 (K + \epsilon_4 H),
\]
where \(\delta_j (j = 1, \ldots, 4)\) and \(\epsilon_i (i = 1, \ldots, 4)\) are arbitrary constants, \(\delta_1 \delta_3 \epsilon_1 \epsilon_2 \epsilon_3 \neq 0, \) \(X\) is an arbitrary smooth function of \(x, X_x \neq 0, \int \frac{h}{g} = \int \frac{h(x)}{g(x)} dx.\)

**Remark 1.** It should be noted that the existence of such equivalence transformations can be explained in many respects by features of representation of equations in the form \(\mathfrak{(1)}\). This form usually leads to an ambiguity because the same equation has an infinite series of different representations. In fact, two representations \(\mathfrak{(1)}\) with the arbitrary element tuples \((f, g, h, H, K)\) and \((\hat{f}, \hat{g}, \hat{h}, \hat{H}, \hat{K})\) determine the same equation if and only if
\[
\hat{f} = \frac{e_1 \delta^2}{X_x} f \int e^{-\epsilon_4 f_{\hat{H}} / \hat{g}} dx, \quad \hat{g} = e_1 \epsilon_2^{-1} X_x g \int e^{-\epsilon_4 f_{\hat{H}} / \hat{g}} dx, \quad \hat{h} = e_1 \epsilon_3^{-1} h \int e^{-\epsilon_4 f_{\hat{H}} / \hat{g}} dx, \quad \hat{H} = e_2 H, \quad \hat{K} = e_3 (K + \epsilon_4 H),
\]
where \(\delta_1\) and \(\epsilon_1\) \((i = 1, \ldots, 4)\) are arbitrary constants, \(\epsilon_1 \epsilon_2 \epsilon_3 \neq 0\) (the variables \(t, x\) and \(u\) do not transform!). The transformations \(\mathfrak{(21)}\) act only on arbitrary elements and do not really change.
equations. In general, transformations of such type can be considered as trivial ("gauge") equivalence transformations and form the "gauge" (normal) subgroup \( G^{e=0} \) of the extended equivalence group \( \hat{G}^\sim \). Application of "gauge" equivalence transformations is equivalent to rewrite equations in another form. In spite of really equivalence transformations, their role in group classification comes not to choice of representatives in equivalence classes but to choice of form of these representatives.

**Remark 2.** We note that transformations (21) with \( \epsilon_4 \neq 0 \) are nonlocal with respect to arbitrary elements, otherwise they belong to \( G^\sim \) and form the "gauge" (normal) subgroup \( G^{e=0} \) of the equivalence group \( G^\sim \). The factor-group \( G^\sim / G^{e=0} \) for class (1) coincides with \( G^\sim / G^{e=0} \) and can be assumed to consist of the transformations

\[
\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = X(x), \quad \tilde{u} = \delta_3 u + \delta_4, \\
\tilde{f} = \frac{\delta_2}{X_x} f, \quad \tilde{g} = X_x g, \quad \tilde{h} = h, \tilde{H} = H, \quad \tilde{K} = K,
\]

where \( \delta_j \, (j = 1, \ldots, 4) \) are arbitrary constants, \( \delta_1 \delta_3 \neq 0 \), \( X \) is an arbitrary smooth function of \( x, X_x \neq 0 \).

Based on the this idea, we can gauge the parameter-function \( g \) in equation (1) to 1. More exactly, using theorem (1) we deduce that the transformation

\[
\tilde{t} = t, \quad \tilde{x} = \int \frac{dx}{g(x)}, \quad \tilde{u} = u
\]

from \( G^\sim / G^{e=0} \) reduce equation (1) to

\[
\tilde{f}(\tilde{x})\tilde{u}_{\tilde{t}} = (H(\tilde{u})\tilde{u}_x)_{\tilde{x}} + \tilde{h}(\tilde{x})K(\tilde{u})\tilde{u}_{\tilde{x}},
\]

where \( \tilde{f}(\tilde{x}) = g(x)f(x), \tilde{g}(\tilde{x}) = 1 \) and \( \tilde{h}(\tilde{x}) = h(x) \). (Likewise any equation of form (1) can be reduced to the same form with \( \tilde{f}(\tilde{x}) = 1 \). That is why, without loss of generality we can restrict ourselves to investigation of the equation

\[
f(x)u_t = (H(u)u_x)_x + h(x)K(u)u_x,
\]

Below, we denote \( G_1^\sim \) and \( \hat{G}_1^\sim \) as the usual and extended equivalence transformation group of equation (23).

**Theorem 3.** The usual equivalence transformation group \( G^\sim_1 \) for class (23) consists of the transformations

\[
\tilde{t} = \epsilon_1 t + \epsilon_1, \quad \tilde{x} = \epsilon_5 x + \epsilon_2, \quad \tilde{u} = \epsilon_6 u + \epsilon_3, \\
\tilde{f} = \epsilon_2 \epsilon_5^{-2} \epsilon_7 f, \quad \tilde{g} = \epsilon_8^{-1} h, \quad \tilde{h} = \epsilon_7 H, \quad \tilde{K} = \epsilon_5^{-1} \epsilon_7 \epsilon_8 K,
\]

where \( \epsilon_1, \ldots, \epsilon_8 \) are arbitrary constants and \( \epsilon_1 \epsilon_3 \epsilon_5 \epsilon_7 \epsilon_8 \neq 0 \).

Note that for class (23) there also exists a non-trivial group of discrete equivalence transformations generated by four involutive transformations of alternating sign in the sets \( \{ t \}, \{ x, K \}, \{ u \}, \{ f, H, K \} \) and \( \{ h, K \} \). Class (23) admits other equivalence transformations being nonlocal with respect to arbitrary elements which do not belong to \( G^\sim_1 \). We demand for these transformations to be point with respect to \( (t, x, u) \). In such way, using the direct method we can find a generalized equivalence group \( G^\sim \) of class (1).

**Theorem 4.** The extended equivalence transformation group \( \hat{G}^\sim_1 \) for class (23) is formed by the transformations

\[
\tilde{t} = \epsilon_4 t + \epsilon_1, \quad \tilde{x} = \epsilon_5 \int e^{\epsilon_9}fh dx + \epsilon_2, \quad \tilde{u} = \epsilon_6 u + \epsilon_3, \\
\tilde{f} = \epsilon_2 \epsilon_5^{-1} \epsilon_7 f \int e^{-2\epsilon_9} f h dx, \quad \tilde{g} = \epsilon_7 \epsilon_8^{-1} h \int e^{-\epsilon_9} f h dx, \quad \tilde{h} = \epsilon_5 \epsilon_7 H, \quad \tilde{K} = \epsilon_8 (K + \epsilon_9 H),
\]

where \( \epsilon_1, \ldots, \epsilon_9 \) are arbitrary constants and \( \epsilon_1 \epsilon_3 \epsilon_5 \epsilon_7 \epsilon_9 \neq 0, \int h = \int h(x) dx \).
The group $\hat{G}_1\sim$ is a subgroup of $\hat{G}\sim$. Its transformation can be considered as from $\hat{G}\sim$, which preserves the condition $g = 1$. The transformations (21) with non-vanishing values of the parameter $\varepsilon_9$ are also nonlocal in the arbitrary element $h$. There exists a way to avoid operations with nonlocal equivalence transformations. More exactly, we can assume that the parameter-function $K$ is determined up to an additive term proportional to $H$ and subtract such term from $K$ before applying equivalence transformations (22).

At the same time, there is another possible generalization of the gauge $g = 1$ to the general case of $h$, namely the gauge $g = h$. Any equation of the form (11) can be reduced to the equation

$$f(x)u_{tt} = (h(x)H(u)u_x)_x + h(x)K(u)u_x.$$  

(25)

by the transformation $\tilde{t} = t, \tilde{x} = \int \frac{f(x)}{g(x)}dx, \tilde{u} = u$ from $G\sim/G^\sim g$.

**Theorem 5.** The usual equivalence transformation group $G^\sim_h$ for class (25) consists of the transformations

$$\tilde{t} = \varepsilon_1 t + \varepsilon_2, \quad \tilde{x} = \varepsilon_5 x + \varepsilon_6, \quad \tilde{u} = \varepsilon_3 u + \varepsilon_4,$$

$$\tilde{f} = \varepsilon_1^2 \varepsilon_5^2 \varepsilon_7 e^f, \quad \tilde{h} = \varepsilon_8 h, \quad \tilde{H} = \varepsilon_7 H, \quad \tilde{K} = \varepsilon_5^{-1} \varepsilon_7 K,$$

where $\varepsilon_1, \ldots, \varepsilon_9$ are arbitrary constants and $\varepsilon_4 \varepsilon_5 \varepsilon_6 \varepsilon_7 \varepsilon_8 \neq 0$.

**Theorem 6.** The extended equivalence transformation group $\hat{G}^\sim_h$ for class (25) is formed by the transformations

$$\tilde{t} = \varepsilon_1 t + \varepsilon_2, \quad \tilde{x} = \varepsilon_5 x + \varepsilon_6, \quad \tilde{u} = \varepsilon_3 u + \varepsilon_4,$$

$$\tilde{f} = \varepsilon_1^2 \varepsilon_5^{-1} \varepsilon_9 e^f e^g x, \quad \tilde{h} = \varepsilon_9 e_7^{-1} h e^g x, \quad \tilde{H} = \varepsilon_7 H, \quad \tilde{K} = \varepsilon_7 (K - \varepsilon_8 H),$$

(26)

where $\varepsilon_1, \ldots, \varepsilon_9$ are arbitrary constants and $\varepsilon_1 \varepsilon_3 \varepsilon_5 \varepsilon_7 \varepsilon_9 \neq 0, \int h = \int h(x) dx$.

**Remark 3.** If $H = 0$, we assume $h = 1$ for determinacy.

4 Group Classification of nonlinear telegraph equations

In this section, we will present the group classification for the class (11) with the gauges $g = 1$ and $g = h$ under the extended equivalence transformations group $\hat{G}\sim$.

Following the algorithm in [3, 4, 47, 73, 78] we are looking for an infinitesimal operator in the form

$$Q = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u$$

(27)

which corresponds to a one-parameter Lie group of local transformation and keep the equation (11) invariant. The classical infinitesimal Lie invariance criterion for equation (11) to be invariant with respect to the operator (27) read as

$$\text{pr}^{(2)} Q(\triangle) |_{\triangle = 0} = 0, \quad \triangle = f(x) u_{tt} - (g(x) H(u) u_x)_x - h(x) K(u) u_x.$$

(28)

Here $\text{pr}^{(2)} Q$ is the usual second order prolongation [70, 73] of the operator (27). Substituting the coefficients of $\text{pr}^{(2)} Q$ into (28) yields the following determining equations for $\tau$, $\xi$ and $\eta$:

$$\tau_x = \tau_u = \xi_t = \xi_u = \eta_{uu} = 0,$$

$$H(g_{ux}) + h K_{ux} = f \eta_{tt} = 0,$$

$$\frac{f}{g} H \xi - \frac{\partial g}{g} H \xi - 2 \tau_t H - \eta H_u + 2 H \xi_x = 0,$$

$$\left( g_x + 2 g_{ux} \right) H_u + \left( (2g_{ux} - \xi_{xx}) g + (2 \tau_t - \xi_t - \xi \frac{f}{g}) g_x + \xi g_{xx} \right) H + h \eta K_u$$

$$+ (2 \tau_t - \xi_t - \xi \frac{f}{g} + \xi \frac{h}{k}) h K = 0,$$

$$2 \eta_{tt} - \tau_{tt} = 0,$$

$$2 (\xi - \eta_u) H_u - 2 \tau_t H_u - \eta H_{uu} + \frac{f}{g} H_u \xi - \frac{\partial g}{g} H_u \xi + \eta_u H_u = 0.$$
Investigating the compatibility of system (29) we find that the final equation of system (29) is an identity (substituting the third equation of system (29) to the final one can yield this conclusion). With this condition, system (29) can be rewritten in the form

\[
\begin{align*}
\tau_x &= \tau_u = \xi_t = \xi_u = \eta_{uu} = 0, \\ 2(\xi_x - \tau_t) + (f_x - g_x^2)\xi = \frac{H_u}{H} \eta, \\ (g_{\xi_x})_x H + hK \eta_x - f \eta_{tt} &= 0, \\ (g_x \eta + 2g_{\eta_x})H_u + [(2\xi_{eu} - \xi_{xx})g + (2\tau_t - \xi_x - \xi f_x)g_x + \xi g_{xx}]H + h \eta K &= 0.
\end{align*}
\]

Equations (30) do not contain arbitrary elements. Integration of them yields

\[
\tau = \tau(t), \quad \xi = \xi(x), \quad \eta = \eta^0(t, x) + \eta^0(t, x), \quad \eta^1(t, x) = \frac{1}{2} \tau_t + \alpha(x).
\]

Thus, group classification of (1) reduces to solving classifying conditions (31)-(33).

Splitting system (31)-(33) with respect to the arbitrary elements and their non-vanishing derivatives gives the equations \( \tau_t = 0, \xi = 0, \eta = 0 \) for the coefficients of the operators from \( A_{\ker} \) of (1). As a result, we obtain the following assertion.

4.1 Classification under the gauge \( g = 1 \)

**Theorem 7.** The Lie algebra of the kernel of principal groups of (1) with the gauge \( g = 1 \) is \( A_{\ker} = \langle \partial_t \rangle \).

**Theorem 8.** A complete set of inequivalent equations (1) with the gauge \( g = 1 \) with respect to the transformations from \( G_1^\ast \) with \( A_{\max} \neq A_{\ker} \) is exhausted by cases given in tables 7, 8.

| N | K(u) | f(x) | h(x) | Basis of \( A_{\max} \) |
|---|---|---|---|---|
| 1 | \( \forall \) | \( \forall \) | \( \forall \) | \( \partial_t \) |
| 2a | \( e^{vz} \) | \( e^{vz} \) | 1 | \( \partial_t, pt \partial_t + 2 \partial_x \) |
| 2a’ | 1 | \( x^p \) | \( x^{-1} \) | \( \partial_t, (p + 2) \partial_t + 2x \partial_x \) |
| 3 | 1 | \( x^{-2} \) | \( x^{-1} \) | \( \partial_t, t \partial_t + x \partial_x \) |
| 4 | 0 | 1 | 1 | \( \partial_t, \partial_x, t \partial_t + x \partial_x \) |

Table 1. Case of \( \forall H(u) \) (gauge \( g = 1 \))

Here \( p \in \{0, 1\} \mod G_1^\ast \) in case 2a \( p \neq -2 \) case 2a’.

Additional equivalence transformations:
1. 2a \( (p = 0, K = 1) \rightarrow 2a \ (p = 0, K = 0) \): \( \tilde{t} = t, \tilde{x} = x + t, \tilde{u} = u \);
2. 2a’ \( (p = -2, K = 1) \rightarrow 2a’ \ (p = -2, K = 0) \): \( \tilde{t} = t, \tilde{x} = xe^z, \tilde{u} = u \).

| N | K(u) | f(x) | h(x) | Basis of \( A_{\max} \) |
|---|---|---|---|---|
| 1 | 0 | \( \forall \) | 1 | \( \partial_t, t \partial_t - 2 \partial_x \) |
| 2 | \( e^{vz} \) | \( x^p \) | \( x^{-1} \) | \( \partial_t, [p(\mu - \nu) + (\mu - 2\nu) - q\mu]\partial_t + 2(\mu - \nu)\partial_x + 2(q + 1)\partial_u \) |
| 2’ | \( e^{vz} \) | \( e^{vz} \) | \( e^{vz} \) | \( \partial_t, [p(\mu - \nu) - q\mu]\partial_t + 2(\mu - \nu)\partial_x + 2q\partial_u \) |
| 3 | \( \frac{u}{c} e^{vz} \) | 1 | 1 | \( \partial_t, 2p + q) \partial_x + 2h^{-1} \partial_x - 4p \partial_u \) |
| 4 | \( e^{vz} \) | 1 | 1 | \( \partial_t, (p + q) \partial_x + 2h^{-1} \partial_x - 4p \partial_u \) |
| 5 | 0 | \( f^2(x) \) | 1 | \( \partial_t, \partial_x, 2h \partial_x + 2\partial_u \) |
| 6a | 0 | 1 | 1 | \( \partial_t, \partial_x, \partial_x - 2 \partial_u, \partial_x + 2 \partial_u \) |
| 6b | 0 | \( x^{-3} \) | 1 | \( \partial_t, \partial_x - \partial_x, \partial_x + x \partial_u, \partial_x + 2 \partial_u \) |

Table 2. Case of \( H(u) = e^{vz} \) (gauge \( g = 1 \))
Here \((\mu, \nu) \in \{(0,1), (1,\nu)\}, \nu \neq \mu\) in cases 2 and 3; \(\mu = 1\) and \(\nu \neq 1\) in the other cases; \(q \neq -1\) in case 2 (otherwise it is subcase of the case 1.2a); \(\epsilon = \pm 1\) in case 2; \(\alpha, \beta, \gamma, \gamma_0 = \text{const}\) and

\[
f^1(x) = \exp \left\{ \int \frac{-3\beta x - 2\gamma_1 + \alpha}{\beta x^2 + \gamma_1 x + \gamma_0} dx \right\}
\]

Additional equivalence transformations:
1. \(\mathbf{8a} \rightarrow \mathbf{8b} : \tilde{t} = t \text{ sign } x, \tilde{x} = 1/x, \tilde{u} = u - \ln |x|\).

- Table 3. Case of \(H(u) = u^\nu\) (gauge \(g = 1\))

| N | \(\mu\) | \(K(u)\) | \(f(x)\) | \(h(x)\) | Basis of \(A_{\text{max}}\) |
|---|---|---|---|---|---|
| 1 | \(\neq -4\) | 0 | \(\forall\) | 1 | \(\partial_t, \mu \partial_x - 2u \partial_u\) |
| 2 | \(\forall\) | \(|u|^\nu\) | \(|x|^p\) | \(|x|^q\) | \(\partial_t, [(p - q)\mu - \nu \mu + 2\nu] \partial_t\) |
| 2* | \(\forall\) | \(|u|^\nu\) | \(e^{px}\) | \(e^{qx}\) | \(\partial_t, [(p - q)\mu - q \nu] \partial_t\) |
| 3 | \(\forall\) | \(|u|^\nu \ln |u|\) | \(h^2 e^{\int \frac{h dx}{h_x}}\) | \((h^{-1})'\nu = -2ph\) | \(\partial_t, (2p + 1) \partial_t + 2h^{-1} \partial_x - 4p \partial_u\) |
| 4 | 0 | \(\forall\) | \(h^2\) | \((h^{-1})' = 0\) | \(\partial_t, h^{-1} \partial_x\) |
| 5 | 0 | \(u\) | \(h^2 e^{\int \frac{h dx}{h_u}}\) | \((h^{-1})' = -2ph\) | \(\partial_t, t \partial_t + 2h^{-1} \partial_x - 4p \partial_u\) |
| 6 | \(\neq -4\) | 0 | \(f^3(x)\) | 1 | \(\partial_t, \partial_x, (\mu - 2\nu) \partial_t + 2(\mu - \nu) x \partial_x + 2u \partial_u\) |
| 7 | \(\neq -4, -\frac{4}{3}\) | 0 | \(|x|^{\frac{\mu+1}{\mu+3}}\) | 1 | \(\partial_t, \mu \partial_x - 2u \partial_u, \partial_x, x \partial_x\) |
| 8 | \(\neq -4, -\frac{4}{3}, -1\) | 0 | \(|x|^{\frac{\mu+1}{\mu+3}}\) | 1 | \(\partial_t, \mu \partial_x - 2u \partial_u, (\mu + 2) \partial_t - 2(\mu + 1) x \partial_x, (\mu + 1) x^2 \partial_x + xu \partial_u\) |
| 8c | \(-1\) | 0 | \(e^x\) | 1 | \(\partial_t, 2 \partial_t + u \partial_u, \partial_x - u \partial_u, t \partial_t + x \partial_x - xu \partial_u\) |
| 9 | \(-4\) | 0 | \(f^3(x)|_{\mu = -4}\) | 1 | \(\partial_t, 2t \partial_t + 3u \partial_u, \partial_x, t^2 \partial_x + tu \partial_u, 2x \partial_x - u \partial_u\) |
| 10 | \(-4\) | 0 | 1 | 1 | \(\partial_t, 2t \partial_t + 3u \partial_u, \partial_x, t \partial_t + x \partial_x, x^2 \partial_x - 3xu \partial_u\) |

Here \(\nu \neq \mu; \epsilon = \pm 1\) in case 2 (otherwise it is subcase of the case 1.2a); \(\alpha, \beta, \gamma, \gamma_0 = \text{const}\), and

\[
f^3(x) = \exp \left\{ \int \frac{-3(\mu + 4) \beta x - 2\gamma_1 + \alpha}{(\mu + 1) \beta x^2 + \gamma_1 x + \gamma_0} dx \right\}
\]

Additional equivalence transformations:
1. \(\mathbf{8a} \rightarrow \mathbf{8b} : \tilde{t} = t, \tilde{x} = 1/x, \tilde{u} = |x|^{-\frac{\mu+1}{\mu+3}} u;\)
2. \(\mathbf{8b} \rightarrow \mathbf{8c} (\mu = -1): \tilde{t} = t, \tilde{x} = x, \tilde{u} = e^x u;\)

**Remark 4.** Tables 13 are the results of classification for class 1 with the gauge \(g = 1\) with respect to the extended equivalence transformations group \(G_1^0\). The classification for class 1 with the gauge \(g = 1\) with respect to the usual equivalence transformations group \(G_1^0\) are very complicated. In some cases, the determining equations can not be solved explicitly. Therefore, We list these results as an Appendix.

**Remark 5.** The proof of theorem 8 follows directly from the analysis of the Section 4.3.

### 4.2 Classification under the gauge \(g = h\)

**Theorem 9.** The Lie algebra of the kernel of principal groups of 1 with the gauge \(g = h\) is \(A_{\text{kor}} = \langle \partial_t \rangle\).
Theorem 10. A complete set of inequivalent equations (11) with the gauge \( g = h \) with respect to the transformations from \( \hat{G}^\sim \) with \( A^{\text{max}} \neq A^\text{ker} \) is exhausted by cases given in tables 3 and 4.

Here \( p \in \{0, 1\} \mod G^\sim \) in case 2.

Table 4. Case of \( \forall H(u) \) (gauge \( g = h \))

| N | \( K(u) \) | \( f(x) \) | \( h(x) \) | Basis of \( A^{\text{max}} \) |
|---|---|---|---|---|
| 1 | | | | \( \partial_t \), \( \partial_h - 2\partial_u \) |
| 2a | \( e^{nu} \) | \(|x|^p\) | \(|x|^q\) | \( \partial_t, \partial_x, \partial_h + \partial_d - \partial_u \) |
| 3 | | \( e^{px^2+q} \) | \( e^{px^2} \) | \( \partial_t, 2\mu + 2\nu \partial_h + 2\partial_x - 4\partial_u \) |
| 4 | | | | \( \partial_t, \partial_x, (\mu - 2\nu )\partial_h + 2\partial_x - 2\partial_u \) |
| 5 | | \( f^3(x) \) | | \( \partial_t, \partial_h + 2\partial_x - 4\partial_u \) |
| 6a | \( e^{-\nu} \) | \( 1 \) | \( 1 \) | \( \partial_t, \partial_x \) |
| 6b | \( 0 \) | \( x^{-3} \) | \( 1 \) | \( \partial_t, \partial_x, \partial_h - x\partial_x, x\partial_x + x\partial_u, \partial_t - 2\partial_u \) |

Here \( \mu, \nu \in \{(0, 1), (1, \nu)\}, \nu \neq \mu \) in cases 2 and 4, \( \mu = 1 \) and \( \nu \neq 1 \) in the other cases.

Table 5. Case of \( H(u) = e^{-\nu u} \) (gauge \( g = h \))

| N | \( K(u) \) | \( f(x) \) | \( h(x) \) | Basis of \( A^{\text{max}} \) |
|---|---|---|---|---|
| 1 | \( \neq -4 \) | 0 | \( \forall \) | \( 1 \) | \( \partial_t, \mu \partial_h - 2\partial_u \) |
| 2 | \( \forall \) | \(|u|^{\nu} \) | \(|x|^p\) | \(|x|^q\) | \( \partial_t, [(p-q+1)\mu -(p-q+2)\nu ]\partial_h + 2(2\mu - \nu )x\partial_x + 2\partial_u \) |
| 3 | \( \forall \) | \(|u|^{\nu} \ln |u| \) | \( e^{px^2+q} \) | \( e^{px^2} \) | \( \partial_t, (2\mu + q)\partial_h + 2\partial_x - 2\partial_u \) |
| 4 | 0 | \( u \) | \( h^2 e^{t h} \) | \( h^{-1} \)' = -2ph | \( \partial_t, \partial_x, (\mu - 2\nu )\partial_h + 2\partial_x - 4\partial_u \) |
| 5 | 0 | \( \forall \) | \( h^2 \) | \( (h^{-1})' = 0 \) | \( \partial_t, \partial_x \) |
| 6 | \( \forall \) | \(|u|^{\nu} \) | | \( 1 \) | \( \partial_t, \partial_x, (\mu - 2\nu )\partial_h + 2(\partial_h - \partial_u \partial_x + 2\partial_u \) |
| 7 | \( \neq -4 \) | \( f^3(x) \) | | \( 1 \) | \( \partial_t, \mu \partial_h - 2\partial_u \) |
| 8a | \( \neq -4, -\frac{4}{3} \) | 0 | | \( 1 \) | \( \partial_t, \partial_x, (\mu - 2\nu )\partial_h + 2\partial_x + 4\partial_u \) |
| 8b | \( \neq -4, -\frac{4}{3}, -1 \) | 0 | | \( \frac{3\pm\sqrt{5}}{4} \) | \( \partial_t, \mu \partial_h - 2\partial_u \) |
| 8c | \( -1 \) | 0 | \( e^x \) | \( 1 \) | \( \partial_t, \partial_x + 2\partial_u, \partial_x - u\partial_u \) |
| 9 | -4 | 0 | \( f^3(x) \vert_{\mu=-4} \) | \( 1 \) | \( \partial_t, \partial_x + 2\partial_u, \partial_x - u\partial_u \) |
| 10 | -4 | 0 | | | |
| 11 | -\frac{4}{3} | 0 | | \( 1 \) | \( \partial_t, \partial_x + 2\partial_u, \partial_x + 2\partial_x \) |

Here \( \nu \neq \mu \).

In tables 3 and 4 we list all possible \( \hat{G}^\sim \)-inequivalent sets of functions \( f(x), h(x), H(u), K(u) \) and corresponding invariance algebras under the gauges \( g = 1 \) and \( g = h \) respectively. We give the same numbers for the corresponding \( (\hat{G}^\sim \)-equivalent\) cases in the gauge \( g = 1 \) and
\( g = h \). The asterisked cases from tables 2 and 3 are equivalent to the cases from tables 5 and 6 with the same numbers, where the parameter-function \( h \) takes the value \( h = x \). The similar non-asterisked cases correspond to the same cases from tables 5 and 6, where \( p' = \frac{p - q}{q + 1}, q' = \frac{q}{q + 1} \) or \( p' = -\frac{1}{p + 2} \) if \( q = p + 1 \).

In what follows, for convenience we use double numeration \( T.N \) of classification cases, where \( T \) denotes the number of the table and \( N \) the number of the case in table \( T \). The notation ‘equation \( T.N \)’ is used for the equation of the form (1) where the parameter functions take the values from the corresponding case.

The operators from tables 1–3 or 4–6 form bases of the maximal invariance algebras if the corresponding sets of the functions \( f, h, H, K \) are \( \hat{G}^- \)-inequivalent to ones with most extensive invariance algebras. For example, in case 3.1 the adduced operators have the above property iff \( f \neq f^3 \).

**Remark 6.** Case [12]a is equivalent to case [12]a’ with respect to transformation \( \bar{t} = t, \bar{x} = \ln|\bar{x}|, \bar{u} = u, \bar{H} = H, \bar{K} = K - H, \bar{p} = p + 2 \) from \( \hat{G}^- \). We adduce case [12]a’ here for the convenience of presentation of results only.

### 4.3 Proof of classification results

Now, let us use the method of furcate split [47, 65, 78] to prove the main classification theorems 8 and 10. It should be noted that it seems impossible to formulate complete results of group classification of class (1) with respect to usual equivalence group \( G^- \) in a closed form. This can be seen from the classifications for the gauge \( g = 1 \) adduced in the Appendix, while it is quite easy to solve the problem of group classification with respect to the extended equivalence group \( \hat{G}^- \).

The basic idea of the method is based on the fact that the substitution of the coefficients of any operator from the extension of \( A^\text{ker} \) into the classifying equations results in nonidentity equations for arbitrary elements (see [47, 65, 78] for more details about the method). In the problem under consideration, the procedure of looking for the possible cases mostly depends on equation (31). For any operator \( Q \in A^{\text{max}} \) equation (31) gives some equations on \( H \) of the general form

\[(au + b)H_u = cH,
\]

where \( a, b, c \) are constant. In general, for all operators from \( A^{\text{max}} \) the number \( k \) of such independent equations is not greater than 2; otherwise they form an incompatible system on \( H \). \( k \) is an invariant value for the transformations from \( \hat{G}^- \). Therefore, there exist three inequivalent cases for the value of \( k \):

1. \( k = 0 \) : \( H(u) \) is arbitrary;
2. \( k = 1 \) : \( H(u) = e^{\mu u} \) or \( H(u) = u^\mu (\mu \neq 0) \) mod \( \hat{G}^- \),
3. \( k = 2 \) : \( H(u) = 1 \) mod \( G^- \).

Furthermore, in order to provide the final presentation of classification results in a simple way, the choice of a gauge for the arbitrary elements is very important for solving the determining equations. It is more convenient to constrain the parameter-function \( g \) instead of \( f \) in class (1). The next problem is the choice between gauges of \( g \). The case \( K \in (1, H) \) and \( k \geq 1 \) is easier to be investigated in the gauge \( g = h \). In the other cases we obtain results in a simpler explicit form and in an easier way using the gauge \( g = 1 \). Let us consider these possibilities in more detail, omitting cumbersome calculations.
Case 1: $k = 0$ (the gauges $g = 1$ and $g = h$, tables 1 and 4). We first consider the case gauge $g = 1$. Since $H(u)$ arbitrary, this means that the coefficients of any operator from $A^{\text{max}}$ must satisfy \( \eta = 0 \) and

\[
2(\xi_x - \tau_t) + \frac{f_x}{f} \xi = 0, \quad (35)
\]

\[-K(\xi h)_x + H \xi_{xx} = 0. \quad (36)\]

(i) Let us suppose that $K \notin \langle 1, H \rangle$. It follows from equation (36) that $\xi_x = 0$. Therefore, equation (35) must be in the form $f_x = \mu f$ without fail. Solving this equation yields case 2.

(ii) Now let $K \in \langle 1, H \rangle$, i.e. $K = \delta \mod \hat{G}^{-1}$ where $\delta \in \{0,1\}$. Then equation (36) can be decomposed into the following ones

\[\xi_{xx} = 0, \quad \delta(\xi h)_x = 0. \quad (37)\]

Integrating of the latter equations up to $\hat{G}^{-1}$ results to cases 2, 4 of table 1.

The classifications for the gauge $g = h$ can be derived in a similar way.

Case 2: $k = 1$ (the gauges $g = 1$ and $g = h$, tables 1, 2 and 3, 4). Here $H \in \{e^{mu}, u^\mu, \mu \neq 0\}$ mod $\hat{G}^{-1}$ and there exists $Q \in A^{\text{max}}$ with $\eta \neq 0$, otherwise there is no additional extension of the maximal Lie invariance algebra in comparison with the case $k = 0$. Below we consider the gauge $g = h$ in details, the gauge $g = 1$ can be proved in a similar way. If $H = e^{mu}$ we assume $\mu = 1$.

Case 2.1: Let us investigate the first possibility $H = e^{mu}$ (table 3). Equations (31) and (34) imply $\eta_u = 0$, i.e. $\eta = \eta^1(t, x)$ and $\tau_u = 0$. Therefore, equation (33) looks like $K_u = \nu K + \lambda H$ with respect to $K$, where $\nu, b = \text{const}$, otherwise $\eta \equiv 0$.

Consider first the case $K \in \langle 1, H \rangle$. Under the above suppositions, equations (31)–(33) can be rewritten as

\[
\frac{\varphi_x}{\varphi} \xi = (2\nu - \mu)\eta^1 + 2\tau_t, \quad (38)
\]

\[\eta^1_t = \eta^1_x = 0, \quad \xi_{xx} = \tau_u = 0, \quad (39)\]

\[\xi_x = (\mu - \nu)\eta^1, \quad (\xi h)_x = -\lambda \eta^1. \quad (40)\]

Here and below $\varphi = f/h$. From equation (38) we can get $\varphi \in \{e^{qx}, |x|^p (p \neq 0), 1\}$ mod $\hat{G}^{-1}$.

For $\varphi = e^{qx}$ it follows from the determining equations (38)–(40) that $\xi_x = 0, \nu = \mu, \lambda \neq 0, \frac{h q}{h} = 2\alpha$. Thus, $h = h_0 e^{ax^2 + h_1 x} = e^{ax^2} \mod \hat{G}^{-1}, \alpha \neq 0, f = h \varphi = e^{ax^2 + qx}, K = \lambda u e^u \mod \hat{G}^{-1}$ that falls precisely into case 5.2

If $\varphi = |x|^p, r \neq 0$, then $r \xi_x/x = (2\nu - \mu)\eta^1 + 2\tau_t$. Therefore, $\xi = (\mu - \nu)\eta^1 x, (\mu - \nu)(\frac{x^2}{h} - h_1)_x = -\lambda$. Since $\mu \neq \nu$ (otherwise, $K \in \langle 1, H \rangle$) we have $\lambda = 0$ mod $\hat{G}^{-1}$. Therefore, $h = |x|^q \mod \hat{G}^{-1}$. Then $f = |x|^p, p \neq q$, and we obtain case 5.2.

Value $\varphi = 1$ results in $2\tau_t = (\mu - 2\nu)\eta^1, \xi = (\mu - \nu)\eta^1 x + \xi_0$. If $\nu = \mu$ then $\lambda = 0$ (otherwise, $K \in \langle 1, H \rangle$), $\lambda = 1 \mod \hat{G}^{-1}$, $(\frac{h}{h})_x = 2\alpha(\eta^1 = -2\alpha)$. Therefore, $h = h_0 e^{ax^2 + h_1 x} = e^{ax^2} \mod \hat{G}^{-1}, K = \lambda u e^u$ that follows to case 5.3. If $\nu \neq \mu$, then $\lambda = 0 \mod \hat{G}^{-1}$. Therefore, $h \in \{|x|^q, 1, e^{pq}\} \hat{G}^{-1}$ that yields subcases of 5.3, 5.2 and 5.4 correspondingly.

Now, we consider the case $K \in \langle 1, H \rangle, H \neq \text{const}$. In contrast to the previous case, it is more convenient to consider this case using the gauge $g = 1$. In such case $K = 0, 1 \mod \hat{G}^{-1}$.
Application of the above suppositions reduces the determining equations to the system
\[
2ξ_x + \frac{f_x}{f} ξ = μν^1 + 2τ_1, \quad η^1_{xx} = 0, \quad η^1_{1x} = φη^1_{lt},
\]
\[
(ξ_x + \frac{φ_x}{φ} ξ - τ_1)K = 0, \quad ξ_{xx} = 2μη^1_x.
\]

Note that \( η^1 = \frac{1}{2}τ_1 + α(x) \), thus we have \( η^1 = \frac{1}{2}τ_1 + βx + α_0 \) and \( ξ = μβx^2 + γ_1x + γ_0 \). Substituting these values into the first determining equation we obtain
\[
(μβx^2 + γ_1x + γ_0) f_x \equiv -3μβx + \frac{1}{2}(μ + 4)τ_1 + μα_0 - 2γ_1.
\]

This equation gives \( l \) linearly independent equations for \( f \) of form \( (α^2x^2 + α^1x + α^0) \frac{f_x}{f} = β_1x + β_0 \).

If \( l = 0 \), then \( ξ = 0, β = 0, \frac{1}{2}(μ + 4)τ_1 = -μα_0 \). Considering case \( l = 1 \), we get \( (α^2, β_1) \neq (0, 0), (α^0, β_0) \neq (0, 0) \), otherwise \( l > 1 \). At last, if \( l \geq 2 \), then \( f \in \{ 1, e^{px}, |x|^p, p \neq 0 \} \) mod \( \hat{G}_\gamma \).

Direct substitution of the above values to the determining equation for \( K = 0 \) and obvious integration leads to the cases \[\text{5.11 (case } l = 0), \text{5.13 (case } l = 1) \text{ and } \text{5.14, 5.15 (case } l = 2)\].

Classification in case \( K = 1 \) is more cumbersome, and corresponding results can be reduced to cases \[\text{5.13, 5.12 and 5.11}\].

**Case 2.2:** Consider the case \( H = u^\mu \) (table \[\text{5.1}\]). Equations \[\text{3.13} \text{ and } \text{3.14}\] imply \( η = (\frac{1}{2} \tau_1 + α(x))u = η^1(t, x)u \). Therefore, equation \[\text{3.34}\] with respect to \( K \) looks like \( uK_u = νK + λH \), where \( ν, b = \text{const} \), otherwise \( η \equiv 0 \).

Let \( K \in (1, H) \). Using the above suppositions, we can rewrite equations \[\text{3.31}, \text{3.32}\] as
\[
\frac{φ_x}{φ} ξ = (2τ - μ)η^1 + 2τ_1, \quad η^1_1 = η^1_2 = 0, \quad ξ_{xx} = τ_1τ_0 = 0, \quad ξ_x = (μ - ν)η^1, \quad (ξh_φ)_{xx} = -λη^1.
\]

From the first equation of system \[\text{3.11}\] we can get \( φ \in \{ e^{px}, |x|^p(r \neq 0), 0 \} \) mod \( \hat{G}_h \).

For \( φ = e^{px} \) it follows from the determining equations \[\text{3.11}\] that \( ξ_x = 0, ν = μ, λ \neq 0, h_φ = 2α \).

Thus, \( h = h_0 e^{αx_2 + h_1x} = e^{αx_2} \) mod \( \hat{G}_h \), \( α \neq 0 \). \( f = hφ = e^{αx_2 + qx} \), \( K = λ|u|^μ \ln |u| \) mod \( \hat{G}_h \) that falls precisely into case \[\text{6.3.3}\].

If \( φ = |x|^p \) then \( rξ/x = (2τ - μ)η^1 + 2τ_1 \). Therefore, \( ξ = (μ - ν)η^1 x, (μ - ν)(\frac{h_φ}{h})_x = -λ \).

Since \( μ \neq ν \) (otherwise, \( K \in (1, H) \)) we have \( λ = 0 \) mod \( \hat{G}_h \). Therefore, \( h = |x|^p \) mod \( \hat{G}_h \).

Then \( f = |x|^p, p \neq q \), \( K = |w|^q \), and we obtain case \[\text{6.12} \text{.}\]

Value \( φ = 1 \) results in \( 2τ_0 = (μ - 2ν)η^1, ξ = (μ - ν)η^1 x + ξ_0 \). If \( ν = μ \) then \( λ \neq 0 \) (otherwise, \( K \in (1, H), \lambda = 1 \) mod \( \hat{G}_h \), \( (\frac{h_φ}{h})_x = 2α(η^1 = -2α) \). Therefore, \( h = h_0 e^{αx_2 - h_1x} = e^{αx_2} \) mod \( \hat{G}_h \), \( K = |u|^μ \ln |u| \) that follows to case \[\text{6.3.1}\]. If \( ν \neq μ \), then \( λ = 0 \) mod \( \hat{G}_h \). Therefore, \( h \in \{ |x|^q, 1, e^{px} \} \) \( \hat{G}_h \) that yields subcases of \[\text{6.3.1, 6.6 and 6.2}\] correspondingly.

Now, we turn to the case \( K \in (1, H), H \neq \text{const} \). For convenience we also consider this case using the gauge \( g = 1 \). Hence \( K = 0, 1 \) mod \( \hat{G}_h \). Application of the above suppositions reduces the determining equations to the system
\[
2ξ_x + \frac{f_x}{f} ξ = μη^1 + 2τ_1, \quad η^1_{xx} = 0, \quad η^1_{1x} = φη^1_{lt},
\]
\[
(ξ_x + \frac{φ_x}{φ} ξ - τ_1)K = 0, \quad ξ_{xx} = 2(μ + 1)η^1_x.
\]

Solving this system and noting that \( η^1 = \frac{1}{2}τ_1 + α(x) \), we can get \( η^1 = \frac{1}{2}τ_1 + βx + α_0 \) and \( ξ = (μ + 1)βx^2 + γ_1x + γ_0 \). Substituting these values into the first determining equation and
differentiating it with respect to the variable \( t \) we obtain

\[
\begin{align*}
\frac{1}{2}(\mu + 4)\tau_{tt} &= 0, \\
(\mu + 1)\beta x^2 + \gamma_1 x + \gamma_0 \frac{\partial f}{\partial x} &= -(3\mu + 4)\beta x + \frac{1}{2}(\mu + 4)\tau_t + \mu a_0 - 2\gamma_1.
\end{align*}
\]

The first equation of the above system implies that there exist two cases should be considered: \( \tau_{tt} = 0 \) if \( \mu \neq -4 \) and \( \tau_{tt} \neq 0 \) if \( \mu = -4 \). The second equation of system (42) gives \( l \) linearly independent equations for \( f \) of form \( (\alpha^2 x^2 + \alpha^1 x + \alpha^0)\frac{\partial f}{\partial x} = \beta_1 x + \beta_0 \). If \( l = 0 \), then \( \xi = 0, \beta = 0, \frac{1}{2}(\mu + 4)\tau_t = -\mu a_0 \). Considering case \( l = 1 \), we get \( (\alpha^2, \beta_1) \neq (0, 0), (\alpha^0, \beta_0) \neq (0, 0) \), otherwise \( l > 1 \). At last, if \( l \geq 2 \), then \( f \in \{1, e^{px}, |x|^p, p \neq 0\} \mod \hat{G}_7^\sim \).

For the case \( \tau_{tt} = 0, \mu \neq -4 \), substituting the above values into the determining equation directly for \( K = 0 \) and obvious integration leads to the cases [61] (case \( l = 0 \)), [67] (case \( l = 1 \)) and [68], [68], [68], [61] (case \( l = 2 \)). The case \( \tau_{tt} \neq 0, \mu = -4 \) with \( K = 0 \) is corresponding to the results [60], [610].

The classification for \( K = 1 \) is corresponding to subcases [62], [63] and [66].

**Case 3:** \( k = 2 \) (the gauges \( g = 1 \) and \( h = h \), tables [2], [3] and [5], [6]). The assumption of two independent equations of form (31) for \( H = \text{const} \), i.e. \( H = 1 \mod G \). \( K_u \neq 0 \) (otherwise, equation (1) is linear). In what follows, we only use the gauge \( g = h \). Equations (31)–(33) can be written as

\[
\begin{align*}
2(\xi_x - \tau_t) + (\frac{f_x}{f} - \frac{h_x}{h})\xi &= 0, \\
(h\eta_x)_x + hK\eta_x - f\eta_{tt} &= 0, \\
K_u\eta + \xi_xK + \xi(\frac{h_{xx}}{h} - \xi_{xx} + 2\eta_x + (\frac{\xi}{h})_xh_x) &= 0.
\end{align*}
\]

The latter equation looks similar to \( (au + b)K_u = cK + d \) with respect to \( K \), where \( a, b, c, d = \text{const} \). Therefore, to within transformations from \( G \), \( K \) must take one of four values:

\[ K = u^\nu, \quad \nu \neq 0, 1, \quad K = \ln u, \quad K = e^u, \quad K = u. \]

Classification for these values is carried out in the way similar to the above. The obtained extensions can be entered in either table 5 or table 6. The gauge \( g = 1 \) can be proved in a similar way.

The problem of the group classification of equation (1) is exhaustively solved.

### 4.4 Classification with respect to the set of point transformations

Although we have performed the classification by using extended equivalence group, we can find in the classification results equations that some cases from tables [1], [3] or [4], [6] are equivalent with respect to point transformations which obviously do not belong to \( G \). These transformations are called additional equivalence transformations and lead to simplification of further application of group classification results (see reference [37], [47], [78] for details). The simplest way to find such additional equivalences between previously classified equations is based on the fact that equivalent equations have equivalent maximal Lie invariance algebras. Explicit formulas for pairs of point-equivalent extension cases and the corresponding additional equivalence transformations are added after the tables. One can check that there exist no other point transformations between the equations from tables [1], [3] or tables [4], [6] numbered with Arabic numbers without Roman letters and subcases “a” of each multi-case.

**Theorem 11.** Up to point transformations, a complete list of extensions of the maximal Lie invariance group of equations from class [1] is exhausted by the cases from tables [1], [3] or tables [4], [6] numbered with Arabic numbers without Roman letters and subcases “a” of each multi-case.
As one can see, the above additional equivalence transformations have multifarious structure. This displays a complexity of a structure of the set of admissible transformations. Usually the problems of finding of all possible admissible transformations are very difficult to solve, see, e.g., [52-55, 76-80]. We will try to discuss the structure of the set of admissible transformations of class (1) in a sequel paper.

5 Lie reduction and similarity solutions

In this section new Lie exact solutions for the equations from the initial class are constructed to just illustrate possible applications of the classification results obtained. We mainly perform group analysis of three classes of equations possessing nontrivial symmetry properties from the obtained classification lists by the reduction method and then apply to finding similarity solutions. For this purpose, we first construct the optimal sets of subalgebras for each kind of maximal Lie invariance algebras arising from group classification, then perform the reductions with respect to obtained subalgebras. The method of reduction with respect to subalgebras of Lie invariance algebras is well-known and quite algorithmic to use in most cases; we refer to the standard textbooks on the subject [70, 73].

We first consider the case [6] of Table 6, i.e., the equation

\[ u_{tt} = (u^\nu u_x)_x + u^\nu u_x, \]  \( (46) \)

which admits the three-dimensional Lie invariance algebra \( g \) generated by the operators

\[ Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = (\mu - 2\nu)t\partial_t + 2(\mu - \nu)x\partial_x + 2u\partial_u. \]

These operators satisfy the commutations relations

\[ [Q_1, Q_2] = 0, \quad [Q_1, Q_3] = (\mu - 2\nu)Q_1, \quad [Q_2, Q_3] = 2(\mu - \nu)Q_2. \]

An optimal set of subalgebras of the algebra \( g \) can be easily constructed with application of the standard technique [70, 73]. Another way is to take the set from [74], where optimal sets of subalgebras are listed for all three- and four-dimensional algebras. A complete list of inequivalent one-dimensional subalgebras of the algebra \( g \) is exhausted by the subalgebras \( \langle Q_1 \rangle, \langle Q_2 \rangle, \langle Q_3 \rangle, \langle Q_2 - Q_1 \rangle, \langle Q_2 + Q_1 \rangle \). This list can be reduced if we additionally use the discrete symmetry \( (t, x, v) \rightarrow (t, -x, v) \), which maps \( \langle Q_2 + Q_1 \rangle \) to \( \langle Q_2 - Q_1 \rangle \), thereby reducing the number of inequivalent subalgebras to four.

The optimal set of two-dimensional subalgebras is formed by the subalgebras \( \langle Q_3, Q_1 \rangle \), \( \langle Q_3, Q_2 \rangle \), and \( \langle Q_1, Q_2 \rangle \). Lie reduction to algebraic equations with the latter two two-dimensional subalgebra leads only to the trivial zero solution. Below we list all the other subalgebras from the optimal set as well as the corresponding ansatze and reduced equations in Table 7. Solutions of some reduced equations are adduced.

| N | Subalgebra | Ansatz | Reduced ODE |
|---|-----------|--------|-------------|
| 1 | \( \langle Q_1 \rangle \) | \( u = \varphi(\omega), \omega = x \) | \( (\varphi''\varphi_\omega + \varphi'^2\varphi_\omega = 0) \) |
| 2 | \( \langle Q_2 \rangle \) | \( u = \varphi(\omega), \omega = t \) | \( \varphi_\omega = 0 \) |
| 3 | \( \langle Q_3 \rangle \) | \( u = t^\alpha \varphi(\omega), \omega = \frac{x}{t^\beta}, \alpha = \frac{2(\mu - \nu)}{\mu - 2\nu}, \beta = \frac{2(\mu - \nu)}{\mu - 2\nu} \) | \( (\alpha - 1)\varphi + (\beta^2 - 2\alpha\beta + 2\beta)\omega\varphi_\omega + \beta^2\omega^2\varphi_{\omega\omega} - (\varphi''\varphi_\omega)\omega - \varphi'^2\varphi_\omega = 0 \) |
| 4 | \( \langle Q_2 - Q_1 \rangle \) | \( u = \varphi(\omega), \omega = x + t \) | \( (\varphi''\varphi_\omega + \varphi'^2\varphi_\omega - \varphi_{\omega\omega} = 0 \) |
| 5 | \( \langle Q_3, Q_1 \rangle \) | \( u = Cx^{\frac{1}{\nu - \mu}} \) | \( (1 + \nu)C^{\nu + 1} + (\mu - \nu)C^{\mu + 1} = 0 \) |
Two kinds of explicit solutions can be constructed for arbitrary values of $\mu$ and $\nu$: the $x$-free solution $u = c_0 + c_1 t$ and the stationary solution $u = \left( \frac{1 + \mu}{\nu - \mu} \right)^{\nu - \mu} x^{\mu - \nu}$. We can also construct two implicit solutions from the first and the fourth ODEs in Table 7:

\[ u = \varphi(x), \quad u = \psi(x + t), \]

where $\varphi$ and $\psi$ satisfy

\[
\frac{1}{\mu - \nu} \varphi^{\mu - \nu} + \frac{1}{\nu + 1} x + \int \frac{C_1}{\varphi^{\nu + 1}} \, dx = 0, \quad \frac{1}{\mu - \nu} \psi^{\mu - \nu} + \frac{1}{\nu + 1} x + \int \frac{C_2}{\psi^{\nu + 1}} \, d\omega = 0,
\]

and $\omega = x + t$, $C_1, C_2$ are arbitrary constants. Let $\mu$ and $\nu$ be particular values, we can get some number of explicit exact solutions from the above implicit solutions. For example, from the latter implicit solution, we can get four triangular function exact solutions if $\mu = 1, \nu = 2$:

\[
u = -\frac{1}{2} - \frac{\sqrt{3}}{2} \tan \left[ \frac{\sqrt{3}}{2} (x + t) \right], \]
\[
u = -\frac{1}{2} + \frac{\sqrt{2}}{2} \cot \left[ \frac{\sqrt{2}}{2} (x + t) \right], \]
\[
u = -\frac{1}{2} - \frac{\sqrt{2}}{2} \tan \left[ \frac{\sqrt{2}}{2} (x + t) \right] \pm \frac{1}{2} \sqrt{1 + \tan^2 \left[ \frac{\sqrt{2}}{2} (x + t) \right]}, \]
\[
u = -\frac{1}{2} + \frac{\sqrt{2}}{2} \cot \left[ \frac{\sqrt{2}}{2} (x + t) \right] \pm \frac{1}{2} \sqrt{1 + \cot^2 \left[ \frac{\sqrt{2}}{2} (x + t) \right]};
\]

and a rational solutions if $\mu = -1, \nu = -2$:

\[ u = x + t. \]

Lie reduction and exact solutions of ‘truly’ variable-coefficient nonlinear telegraph waves are most interesting. We consider two cases 22 and 612 i.e. equations

\[ e^{px} u_{tt} = (u^\mu u_x)_x + ee^{qx} u^\nu u_x, \quad (47) \]

\[ |x|^p u_{tt} = (|x|^q u^\mu u_x)_x + |x|^q u^\nu u_x. \quad (48) \]

For each from these cases we denote the basis symmetry operators adduced in Table 2 and 6 by $g_1 = \langle Q_1 \rangle = \partial_t$, $Q_2 = [(p - q)\mu - q\nu] \partial_t + 2(\mu - \nu) \partial_x + 2qu \partial_u$ and $g_2 = \langle Q_1 \rangle = \partial_t$, $Q_2 = [(p - q + 1)\mu - (p - q + 2)\nu] \partial_t + 2(\mu - \nu) x \partial_x + 2u \partial_u$, which are all non-commutative algebra. A complete list of inequivalent non-zero subalgebras of $g_1$ or $g_2$ is exhausted by the algebras $\langle Q_1 \rangle$, $\langle Q_2 \rangle$ and $\langle Q_1, Q_2 \rangle$.

Lie reduction of the equations (47) and (48) to ordinary differential equations (ODEs) and an algebraic equation can be respectively made with the one-dimensional subalgebra $\langle Q_1 \rangle$, $\langle Q_2 \rangle$ and the two-dimensional subalgebra $\langle Q_1, Q_2 \rangle$ which coincides with the whole algebra $g_1$ or $g_2$. The associated ansatzes and reduced equations are listed in Table 8 and 9.

Table 8. Reduced ODEs and algebraic equation for equation (47).

| N | Subalgebra | Ansatz | Reduced ODE |
|---|---|---|---|
| 1 | $\langle Q_1 \rangle$ | $u = (\varphi(\omega))^{\frac{1}{\mu - \nu}}, \omega = x$ | $\varphi_{\omega \omega} + ee^{q_{\omega \omega}} \varphi_{\omega \omega} \frac{e^{p_{\omega \omega}} - 1}{1} = 0$ if $\mu \neq -1$ |
| 2 | $\langle Q_2 \rangle$ | $u = \exp(\varphi(\omega)), \omega = x$ | $\varphi_{\omega \omega} + ee^{(v + 1)_{\omega \omega} + q_{\omega \omega} \varphi_{\omega \omega}} = 0$ if $\mu = -1$ |
| 3 | $\langle Q_1, Q_2 \rangle$ | $u = C e^{\frac{x}{\sqrt{\alpha(\omega)}}}$ | $\alpha(\alpha - 1) \varphi + (2\alpha - \beta) \varphi_{\omega} + \beta^2 \varphi_{\omega \omega} + \omega^{-p}(\varphi^\mu \varphi_{\omega})_\omega - \omega^{-(q_{\mu} + q_{\mu} \mu + p_{\mu} + p_{\mu} \mu + p_{\mu} \mu + p_{\mu} \mu + p_{\mu} \mu + p_{\mu} \mu) \omega} \varphi_{\omega \omega} = 0$ |

Table 9. Reduced ODEs and algebraic equation for equation (48).
| N | Subalgebra | Ansatz | Reduced ODE |
|---|-----------|--------|-------------|
| 1 | $\langle Q_1 \rangle$ | $u = (\varphi(\omega))^{\frac{1}{\mu+1}}, \omega = x$ | $\varphi_{\omega\omega} - \varphi_\omega^2 + q \varphi_\omega + \varphi_\omega \varphi_\omega^{\frac{1}{\mu+1}} = 0$ if $\mu \neq -1$ |
| | | $u = \exp(\varphi(\omega)), \omega = x$ | $(\varphi_\omega)^{\omega} + \omega^{\omega} e^{(\nu+1)\nu} \varphi_\omega = 0$ if $\mu = -1$ |
| 2 | $\langle Q_2 \rangle$ | $u = |\varphi(\omega)|, \omega = x|$ | $\alpha(\alpha - 1)\varphi + (\beta^2 + 2\alpha\beta - \beta)\varphi^\omega + \beta^2 \omega^2 \varphi_\omega$ |
| | | $\alpha = \frac{(p-q)(\mu - 1) + (\mu - 2q)}{(p-q)(\mu - 1) + (\mu - 2q)}, \beta = \frac{(p-q)(\mu - 1) + (\mu - 2q)}{(p-q)(\mu - 1) + (\mu - 2q)}$ | $-\omega^{-p}(\omega^q^\mu^\nu^\varphi) \omega - \omega^{\eta-p^q^\nu^{\varphi}} \varphi_\omega = 0$ |
| 3 | $\langle Q_1, Q_2 \rangle$ | $u = C x^{\frac{1}{\mu+1}}$ | $[(q - 1)(\mu - \nu) + \mu + 1] C^{\mu+1} + (\mu - \nu) C^{\nu+1} = 0$ |

Reduction to algebraic equations gives the following solutions of the initial equations (47) and (48) respectively:

$$u = \left[ q \left( \frac{\mu + 1}{\mu - \nu} \right) e^{\frac{2}{\mu + 1}}, \frac{\varphi_\omega}{\nu - \mu} \right]^{\frac{1}{\mu + 1}} e^{\frac{2}{\mu + 1}}, \quad u = \left[ q - 1 \left( \frac{\mu + 1}{\mu - \nu} \right) \right]^{\frac{1}{\mu + 1}} x^{\frac{1}{\mu + 1}}.$$

Furthermore, some of the reduced ordinary differential equations in tables 8 and 9 are the modification of the Emden-Fowler and the Lane-Emden equations [35, 75]. For example, the first equation corresponding to case 1 of table 9 are the standard Emden-Fowler equation, while the second one to case 1 of table 9 is the generalized Lane-Emden equation. Solutions of these equations are known for a number of parameter values (see e.g. [35, 75]). As a result, classes of exact solutions can be constructed for wave equations (47) and (48) for a wide set of the parameters $\mu$ and $q$. We omit these results in order to avoid a cumbersome enumeration.

6 On nonclassical symmetries

In 1969, Bluman and Cole introduced an essential generalization of Lie symmetry in the study symmetry reduction of the linear heat equation [12]. These generalized symmetries are often called nonclassical symmetries (called also conditional or $Q$-conditional symmetries) nevertheless it was not used in [12]. A precise and rigorous definition of this notion was suggested noticeably later [58] for a recent discussion on definition of nonclassical symmetries. Since then there is an explosion of research activity in the area of investigation of nonclassical symmetries of PDEs arising from different fields of physics, biology and chemistry [20,21,36,61,67,69]. Some of these works concern with nonlinear wave equations. See, for example, [27,44].

Generally speaking, there are two main features of nonclassical symmetries of differential equation difference from Lie symmetries. The first one is that they can yield solutions not obtainable from the classical Lie symmetries. The second feature is the deriving systems of determining equations for nonclassical symmetries which crucially depends on the interplay between the operators and the equations under consideration, and thus are different from the Lie symmetries. Due to these facts, when studying the general form of nonclassical symmetry operators $Q = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u (\tau, \xi) \neq (0, 0))$ for the linear heat equation $u_t = u_{xx}$, there are two essentially different cases of nonclassical symmetries should be considered: the regular case $\tau \neq 0$ and the singular case $\tau = 0$. The factorization up to the equivalence of operators gives the two respective cases for the further investigation: 1) $\tau = 1$ and 2) $\tau = 0, \xi = 1$. In particular, for the singular case the system of determining equations for nonclassical symmetries consists of a single (1+2)-dimensional nonlinear evolution equation for the unknown function $\eta$ and, therefore, is not overdetermined. The determining equation is reduced by a nonlocal transformation to the initial equation with an additional implicit independent variable which can be assumed as a parameter [29]. The linearity of the heat equation is inessential here.

Recently, based on the above discoveries, Kunzinger and Popovyč [57] raise a number of interesting questions, to wit: Is the partition of sets of nonclassical symmetry operators of the linear heat equation with the conditions of vanishing and nonvanishing coefficients of operators...
to regular and singular cases universal for any differential equations or is it appropriate only for certain classes of differential equations? What is the proper partition of sets of nonclassical symmetry operators different from the conventional one? What are possible causes for the existence of singular cases for nonclassical symmetry operators? The answer to these questions has some fundamental importance in the research of nonclassical symmetries and will make finding an optimal way of obtaining the determining equation for nonclassical symmetries become possible. They gave a detail investigation on these questions and present a novel framework of singular reduction operators to clarify the main idea [57]. Here and below, following [57] we use sometimes the shorter and more natural term ‘reduction operators’ instead of ‘nonclassical symmetry operators’ or ‘operators of nonclassical symmetry’.

They also show that for any (1+1)-dimensional evolution equation [57]

\[ u_t = H(t, x, u, u_1, ..., u_k), \quad k > 1, \quad u_k = \frac{\partial^k u}{\partial x^k}, \quad H_{u_k} \neq 0, \]

the conventional partition of the set of its reduction operators with the conditions \( \tau \neq 0 \) and \( \tau = 0 \) is natural since it coincides with the partition of the set into the singular and regular subsets. After factorizing the subsets of the reduction operator set with respect to the usual equivalence relation of reduction operators (see Definition 2 below), there exist two different cases of inequivalent reduction operators: the regular case \( \tau = 1 \) and the singular case \( \tau = 0 \) and \( \xi = 1 \), which should be investigated separately. However, this is a specific property of evolution equations which does not hold for general partial differential equations in two independent variables. In particular, they show that for the class of nonlinear wave equations

\[ u_{tt} - u_{xx} = F(u), \]

where \( F \) is an arbitrary smooth function of \( u \), which possesses two singular sets of reduction operators, singled out by the conditions \( \tau = \xi \) and \( \tau = -\xi \), and one regular set of reduction operators, associated with the condition \( \tau \neq \pm \xi \). The singular sets are mapped to each other by alternating the sign of \( x \) and hence one of them can be excluded from the consideration. After factorization with respect to the equivalence relation of vector fields, there are two cases for further study: the singular case \( \tau = \xi = 1 \) and the regular case \( \tau \neq 1, \xi = 1 \).

However, for more general nonlinear wave equations there exist no general results. Therefore, it is shown that the structures of condition symmetries of hyperbolic type nonlinear partial differential is more complicated than general evolution equation.

In what follows we extend this new framework of singular reduction operators to the (1+1)-dimensional variable coefficient nonlinear telegraph equations [1]. With the aid of the transformation \( \tilde{t} = t, \tilde{x} = \int \frac{dx}{g(x)}, \tilde{u} = u \) from \( G^r/G^r \) in section 3, we can reduce equation (1) to one which has the same form with equation (23). Thus, without loss of generality we can restrict ourselves to investigation in detail the equation

\[ f(x)u_{tt} - (H(u)u_x)_x - h(x)K(u)u_x = 0, \quad (49) \]

where \( f = f(x), h = h(x), H = H(u) \) and \( K = K(u) \) are arbitrary smooth functions of the corresponding variables, \( fH > 0 \).

For the sake of completeness, let us first review some necessary definitions and statements on nonclassical symmetries [31, 58, 90, 96] and singular reduction operator [57].

6.1 Brief review of reduction operators of differential equation

Consider an \( r \)th order differential equation \( \mathcal{L} \) of the form \( L(t, x, u_r) = 0 \) for the unknown function \( u \) of the two independent variables \( t \) and \( x \), where \( L = L[u] = L(t, x, u_r) \) is a fixed
differential function of order \( r \) and \( u_{(r)} \) denotes the set of all the derivatives of the function \( u \) with respect to \( t \) and \( x \) of order not greater than \( r \), including \( u \) as the derivative of order zero.

In order to discuss the conditional symmetries of equation \( \mathcal{L} \), we will first treat equation \( \mathcal{L} \) from a geometric point of view as an algebraic equation in the jet space \( J^r \) of order \( r \) and is identified with the manifold of its solutions in \( J^r \). \( \mathcal{L} = \{(t, x, u_{(r)}) \in J^r | L(t, x, u(r)) = 0\} \). Let \( Q \) denote the set of vector fields of the general form

\[
Q = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u, \quad (\tau, \xi) \neq (0, 0),
\]

which is a first-order differential operator on the space \( \mathbb{R}^2 \times \mathbb{R} \) with coordinates \( t, x, \) and \( u \). Then all functions invariant under \( Q \) and only such functions satisfy a first order differential equation

\[
Q[u] := \tau u_t + \xi u_x - \eta = 0
\]
called the the characteristic equation (also known as invariant surface condition).

Denote the manifold defined by the set of all the differential consequences of the characteristic equation \( Q[u] = 0 \) in \( J^r \) by \( Q_{(r)} \), i.e.,

\[
Q_{(r)} = \{(t, x, u_{(r)}) \in J^r | D_t^\alpha D_x^\beta Q[u] = 0, \quad \alpha, \beta \in \mathbb{N} \cup \{0\}, \quad \alpha + \beta < r\},
\]

where \( D_t = \partial_t + u_{\alpha+1, \beta} \partial_{u_{\alpha, \beta}} \) and \( D_x = \partial_x + u_{\alpha, \beta+1} \partial_{u_{\alpha, \beta}} \) are the operators of total differentiation with respect to the variables \( t \) and \( x \), and the variable \( u_{\alpha, \beta} \) of the jet space \( J^r \) corresponds to the derivative \( \frac{\partial^{\alpha+\beta} u}{\partial t^\alpha \partial x^\beta} \). Denote also by \( Q_{(r)} \) the standard \( r \)th prolongation of \( Q \) to the space \( J^r \):

\[
Q_{(r)} = Q + \sum_{0<\alpha+\beta\leq r} \eta^{\alpha\beta} \partial_{u_{\alpha, \beta}}, \quad \eta^{\alpha\beta} := D_t^\alpha D_x^\beta Q[u] + \tau u_{\alpha+1, \beta} + \xi u_{\alpha, \beta+1}.
\]

**Definition 1.** The differential equation \( \mathcal{L} \) is called conditionally invariant with respect to the operator \( Q \) if the relation

\[
Q_{(r)}[L(t, x, u_{(r)})]_{|_{\mathcal{L} \cap Q_{(r)}}} = 0
\]

holds, which is called the conditional invariance criterion. Then \( Q \) is called conditional symmetry (or nonclassical symmetry, \( Q \)-conditional symmetries or reduction operator) of the equation \( \mathcal{L} \).

We denote the set of reduction operators of the equation \( \mathcal{L} \) by \( Q(\mathcal{L}) \) which is a subset of \( Q \). Any Lie symmetry operator of \( \mathcal{L} \) belongs to \( Q(\mathcal{L}) \). Sometimes \( Q(\mathcal{L}) \) is exhausted by the operators equivalent to Lie symmetry ones in the sense of the following definition.

**Definition 2.** Two differential operators \( Q \) and \( \tilde{Q} \) in \( Q \) are called equivalent \( (Q \sim \tilde{Q}) \) if they differ by a multiplier which is a non-vanishing function of \( t, x \) and \( u : \tilde{Q} = \lambda Q \), where \( \lambda = \lambda(t, x, u), \lambda \neq 0 \).

Factoring \( Q \) with respect to this equivalence relation we arrive at \( Q_f \). Elements of \( Q_f \) will be identified with their representatives in \( Q \). The property of conditional invariance is compatible with this equivalence relation on \( Q \).

**Lemma 1.** If the equation \( \mathcal{L} \) is conditionally invariant with respect to the operator \( Q \) then it is conditionally invariant with respect to any operator which is equivalent to \( Q \).
In view of this lemma, we can see that $Q \in Q(L)$ and $\hat{Q} \sim Q$ imply $\hat{Q} \in Q(L)$, i.e. $Q(L)$ is closed under the equivalence relation on $Q$. Therefore, the equivalence relation on $Q$ induces a well-defined equivalence relation on $Q(L)$; and the factorization of $Q$ with respect to this equivalence relation can be naturally restricted to $Q(L)$ that results in the subset $Q_f(L)$ of $Q_f$. As in the whole set $Q_f$, we identify elements of $Q_f(L)$ with their representatives in $Q(L)$. In this approach the problem of completely describing all reduction operators for $L$ is equivalent to finding $Q_f(L)$. In fact, nonclassical symmetries should be studied up to the above equivalence relation. The elements of $Q(L)$ which are not equivalent to Lie invariance operators of $L$ will be called pure nonclassical symmetries of $L$.

The conditional invariance criterion admits the following useful reformulation [96].

**Lemma 2.** Given a differential equation $L : L[u] = 0$ of order $r$ and differential functions $\hat{L}[u]$ and $\lambda[u] \neq 0$ of an order not greater than $r$ such that $L|_{Q(r)} = \lambda\hat{L}|_{Q(r)}$, an operator $Q$ is a reduction operator of $L$ if and only if the relation $Q(\tilde{r})\hat{L}|_{L \cap Q(r)} = 0$ holds, where $\tilde{r} = \text{ord} \hat{L} \leq r$ is the order of the differential function $\hat{L}[u]$ and the manifold $\hat{L}$ is defined in $J^r$ by the equation $\hat{L}[u] = 0$.

Consider a vector field $Q$ in the form (50) and a differential function $L = L[u]$ of order $\text{ord} L = r$ (i.e., a smooth function of variables $t, x, u$ and derivatives of $u$ of orders up to $r$).

**Definition 3.** The vector field $Q$ is called **singular** for the differential function $L$ if there exists a differential function $\hat{L} = \hat{L}[u]$ of an order less than $r$ such that $L|_{Q(r)} = \hat{L}|_{Q(r)}$. Otherwise $Q$ is called a **regular** vector field for the differential function $L$. If the minimal order of differential functions whose restrictions on $Q(r)$ coincide with $L|_{Q(r)}$ equals $k$ ($k < r$) then the vector field $Q$ is said to be of **singularity co-order** $k$ for the differential function $L$. The vector field $Q$ is called **ultra-singular** for the differential function $L$ if $L|_{Q(r)} \equiv 0$.

For convenience, the singularity co-order of ultra-singular vector fields and the order of identically vanishing differential functions are defined to equal $-1$. Regular vector fields for the differential function $L$ are defined to have singularity co-order $r = \text{ord} L$. The singularity co-order of a vector field $Q$ for a differential function $L$ will be denoted by $\text{soc}_L Q$.

If $Q$ is a singular vector field for $L$ then any vector field equivalent to $Q$ is singular for $L$ with the same co-order of singularity.

We will say that a vector field $Q$ is **(strongly) singular** for a differential equation $L$ if it is singular for the differential function $L[u]$ which is the left hand side of the canonical representation $L[u] = 0$ of the equation $L$. Usually we will omit the attribute “strongly”.

Since left hand sides of differential equations are defined up to multipliers which are nonvanishing differential functions, the conditions from Definition 3 can be weakened when considering differential equations.

**Definition 4.** A vector field $Q$ is called **weakly singular** for the differential equation $L : L[u] = 0$ if there exist a differential function $\hat{L} = \hat{L}[u]$ of an order less than $r$ and a nonvanishing differential function $\lambda = \lambda[u]$ of an order not greater than $r$ such that $L|_{Q(r)} = \lambda\hat{L}|_{Q(r)}$. Otherwise $Q$ is called a **weakly regular** vector field for the differential equation $L$. If the minimal order of differential functions whose restrictions on $Q(r)$ coincide, up to nonvanishing functional multipliers, with $L|_{Q(r)}$ is equal to $k(k < r)$ then the vector field $Q$ is said to be **weakly singular of co-order** $k$ for the differential equation $L$.

The notions of ultra-singularity in the weak and the strong sense coincide. Analogous to the case of strong regularity, weakly regular vector fields for the differential equation $L$ are defined to have weak singularity co-order $r = \text{ord} L$. The weak singularity co-order of a vector field $Q$ for an equation $L$ will be denoted by $\text{wsoc}_L Q$.  

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The following statements give the positive answer.

After the factorization of the reduction operator under the usual equivalence relation of reduction operators in Definition 2 to singular and regular cases, the classification of reduction operators can be considerably enhanced and simplified by considering Lie symmetry and equivalence transformations of (classes of) equations.

Lemma 3. Any point transformation of \( t, x \) and \( u \) induces a one-to-one mapping of \( Q \) into itself. Namely, the transformation \( g : \tilde{t} = T(t,x,u), \tilde{x} = X(t,x,u), \tilde{u} = U(t,x,u) \) generates the mapping \( g_\ast : Q \rightarrow Q \) such that the operator \( g_\ast Q = \tilde{\tau} \partial_{\tilde{t}} + \tilde{\xi} \partial_{\tilde{x}} + \tilde{\eta} \partial_{\tilde{u}} \),

where \( \tilde{\tau}(\tilde{t}, \tilde{x}, \tilde{u}) = QT(t,x,u), \tilde{\xi}(\tilde{t}, \tilde{x}, \tilde{u}) = QX(t,x,u), \tilde{\eta}(\tilde{t}, \tilde{x}, \tilde{u}) = QU(t,x,u) \). If \( Q' \sim Q \) then \( g_\ast Q' \sim g_\ast Q \). Therefore, the corresponding factorized mapping \( g_f : Q_f \rightarrow Q_f \) also is well defined and bijective.

Lemma 3 results in appearing equivalence relation between operators, which differs from usual one described in Definition 2.

Definition 6. Two differential operators \( Q \) and \( \tilde{Q} \) in \( Q \) are called equivalent with respect to a group \( G \) of point transformations if there exists \( g \in G \) for which the operators \( Q \) and \( g_\ast \tilde{Q} \) are equivalent. We denote this equivalence by \( Q \sim \tilde{Q} \mod G \).

The problem of finding reduction operators is more complicated than the similar problem for Lie symmetries because the first problem is reduced to the integration of an overdetermined system of nonlinear PDEs, whereas in the case of Lie symmetries one deals with a more overdetermined system of linear PDEs. The question occurs: could we use equivalence and gauging transformations in investigation of reduction operators as we do for finding Lie symmetries? The following statements give the positive answer.

Lemma 4. Given any point transformation \( g \) of an equation \( \mathcal{L} \) to an equation \( \tilde{\mathcal{L}} \), \( g_\ast \) maps \( Q(\mathcal{L}) \) to \( Q(\tilde{\mathcal{L}}) \) bijectively. The same is true for the factorized mapping \( g_f \) from \( Q_f(\mathcal{L}) \) to \( Q_f(\tilde{\mathcal{L}}) \).

Corollary 1. Let \( G \) be the point symmetry group of an equation \( \mathcal{L} \). Then the equivalence of operators with respect to the group \( G \) generates equivalence relations in \( Q(\mathcal{L}) \) and in \( Q_f(\mathcal{L}) \).

Consider the class \( \mathcal{L}|_S \) of equations \( \mathcal{L}_\theta : L(t,x,u_{(r)},\theta) = 0 \) parameterized with the parameter-functions \( \theta = \theta(t,x,u_{(r)}) \). Here \( L \) is a fixed function of \( t,x,u_{(r)} \) and \( \theta \). The symbol \( \theta \) denotes the tuple of arbitrary (parametric) differential functions \( \theta(t,x,u_{(r)}) = (\theta^1(t,x,u_{(r)}),...,\theta^k(t,x,u_{(r)})) \) running through the set \( S \) of solutions of the system \( S(t,x,u_{(r)},\theta(q)(t,x,u_{(r)})) = 0 \). This system consists of differential equations on \( \theta \), where \( t,x \) and \( u_{(r)} \) play the role of independent variables and \( \theta(q) \) stands for the set of all the derivatives of \( \theta \) of order not greater than \( q \). In what follows we call the functions \( \theta \) arbitrary elements. Denote the point transformation group preserving the form of the equations from \( \mathcal{L}|_S \) by \( G^\sim \).

Let \( P \) denote the set of the pairs consisting of an equation \( \mathcal{L}_\theta \) from \( \mathcal{L}|_S \) and an operator \( Q \) from \( Q(\mathcal{L}_\theta) \). In view of Lemma 4 the action of transformations from the equivalence group \( G^\sim \) on \( \mathcal{L}|_S \) and \( \{Q(\mathcal{L}_\theta) | \theta \in S \} \) together with the pure equivalence relation of differential operators naturally generates an equivalence relation on \( P \).

Definition 7. Let \( \theta, \theta' \in S, Q \in \mathcal{Q}(\mathcal{L}_\theta), Q' \in \mathcal{Q}(\mathcal{L}_{\theta'}) \). The pairs (\( \mathcal{L}_\theta, Q \)) and (\( \mathcal{L}_{\theta'}, Q' \)) are called \( G^\sim \)-equivalent if there exists \( g \in G^\sim \) such that \( g \) transforms the equation \( \mathcal{L}_\theta \) to the equation \( \mathcal{L}_{\theta'} \), and \( Q' \sim g_\ast Q \).

The classification of reduction operators with respect to \( G^\sim \) will be understood as the classification in \( P \) with respect to this equivalence relation, a problem which can be investigated
similar to the usual group classification in classes of differential equations. Namely, we construct firstly the reduction operators that are defined for all values of $\theta$. Then we classify, with respect to $G^\sim$, the values of $\theta$ for which the equation $L_\theta$ admits additional reduction operators.

6.2 Singular reduction operators

Using the above notion and the procedure given by Kunzinger and Popovych in [57], we can obtain the following assertion.

**Proposition 1.** A vector field $Q = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u$ is singular for the differential function $L = f(x)u_{tt} - (H(u)u_x)_x - h(x)K(u)u_x$ if and only if $\xi^2 f(x) = \tau^2 H(u)$.

**Proof.** Suppose that $\tau \neq 0$. According to the characteristic equation $\tau u_t + \xi u_x - \eta = 0$, we can get

$$u_t = \frac{\eta}{\tau} - \frac{\xi}{\tau} u_x,$$

$$u_{tt} = \left(\frac{\eta}{\tau}\right)_t - \left(\frac{\xi}{\tau}\right)_t u_x + \left(\frac{\eta}{\tau} - \frac{\xi}{\tau}\right) u_{ux} - \left(\frac{\eta}{\tau} - \frac{\xi}{\tau}\right)_x u_x - \left(\frac{\eta}{\tau} - \frac{\xi}{\tau}\right) u_x^2.$$

Substituting the formulas of $u_{tt}$ from above formulas into $L$, we obtain a differential function

$$\tilde{L} = \left[ f(x)\left(\frac{\xi}{\tau}\right)^2 - H(u)\right] u_{xx} + f(x) \left\{ \left(\frac{\eta}{\tau}\right)_t - \left(\frac{\xi}{\tau}\right)_t u_x + \left(\frac{\eta}{\tau} - \frac{\xi}{\tau}\right) u_{ux} - \left(\frac{\eta}{\tau} - \frac{\xi}{\tau}\right)_x u_x - \left(\frac{\eta}{\tau} - \frac{\xi}{\tau}\right) u_x^2 \right\} - H u_x^2 - h K u_x.$$

According to the definition [3] of singular vector field, we have $\text{ord} \tilde{L} < 2$ if and only if $f(x)\left(\frac{\xi}{\tau}\right)^2 - H(u) = 0$. 

Therefore, for any $f, h, H$ and $K$ with $f H > 0$ the differential function $L = f(x)u_{tt} - (H(u)u_x)_x - h(x)K(u)u_x$ possesses exactly two set of singular vector fields in the reduced form, namely, $S = \{ \partial_t + \sqrt{H/f} \partial_x + \hat{\eta} \partial_u \}$ and $S^* = \{ \partial_t - \sqrt{H/f} \partial_x + \hat{\eta} \partial_u \}$, where $\hat{\eta} = \frac{\xi}{\tau}$. Any singular vector field of $L$ is equivalent to one of the above fields. Moreover, it is easy known that form Theorem [4] each equation of the form (39) admits the discrete involutive transformation $\{x, K\}$ and $\{x, h\}$. According to Corollary [1] these two transformation generates a one-to-one mapping between $S$ and $S^*$. Hence it suffices, up to equivalence of vector fields (and permutation of $x$ and $-x$), to investigate only singular reduction operators from the set $S$.

**Proposition 2.** For any variable coefficient nonlinear telegraph equations in the form (39), the differential function $L = f(x)u_{tt} - (H(u)u_x)_x - h(x)K(u)u_x$ possesses exactly one set of singular vector fields in the reduced form, namely, $S = \{ \partial_t + \sqrt{H/f} \partial_x + \hat{\eta} \partial_u \}$.

Thus taking into account the conditional invariance criterion for an equation from class (49) and the operator $\partial_t + \sqrt{H/f} \partial_x + \eta \partial_u$, we can get

**Theorem 12.** Every singular reduction operator of an equation from class (49) is equivalent to

$$Q = \partial_t + \sqrt{H(u)/f} \partial_x + \eta(t, x, u) \partial_u,$$
where the real-valued function $\eta(t,x,u)$ satisfies the determining equations

\[
(-h_xK - 2f_\eta u + \frac{1}{2}hKf_x/f + \frac{3}{4}Hf_x^2/f^2 - H_u\eta f_u/f - 2f\eta uu + \frac{1}{4}fH_u^2\eta^2/H^2
- \frac{1}{2}Hf_{xx}/f - \frac{1}{2}f\eta^2H_{uu}/H)\sqrt{H/f} - H_u\eta_x - h_\eta K_u - \frac{1}{2}H_u f_x/f - 2H_\eta x = 0, \quad (53)
- \eta_x H_u + \sqrt{H/f}(f\eta_{tt} - \eta_{xx}H - h_\eta K + 2f\eta uu + f\eta^2\eta uu) = 0.
\]

6.3 Regular reduction operators

The above investigation of singular reduction operators of nonlinear telegraph equation of the form \(49\) shows that for these equation the regular case of the natural partition of the corresponding sets of reduction operators is singled out by the conditions $\xi \neq \pm \sqrt{H(u)/f}$. After factorization with respect to the equivalence relation of vector fields, we obtain the defining conditions of regular subset of reduction operator: $\tau = 1, \xi \neq \pm \sqrt{H(u)/f}$. Hence we have

**Proposition 3.** For any variable coefficient nonlinear telegraph equations in the form \(49\), the differential function $L = f(x)u_{tt} - (H(u)u_x)_x - h(x)K(u)u_x$ possesses exactly one set of regular vector fields in the reduced form, namely, $S = \{\partial_t + \xi \partial_x + \eta \partial_u\}$ with $\xi \neq \pm \sqrt{H(u)/f}$.

Consider the conditional invariance criterion for an equation from class \(49\) and the operator $\partial_t + \xi \partial_x + \eta \partial_u$ with $\xi = \pm \sqrt{H(u)/f}$, we can get

**Theorem 13.** Every regular reduction operator of an equation from class \(49\) is equivalent to

$$Q = \partial_t + \xi(t,x)\partial_x + \eta(t,x,u)\partial_u, \quad \xi(t,x) \neq \pm \sqrt{H(u)/f}$$

where the real-valued functions $\xi(t,x), \eta(t,x,u)$ satisfy the determining equations

\[
\xi f_x H/f - \eta H_u + 2H_x \xi_x + 2f\xi_x = 0, \\
2H_u \xi_x + f\eta_{uu}\xi^2 - \eta_a H_u - \eta H_{uu} - H_\eta u u + \xi H_u f_x/f = 0, \\
-2f\xi\eta_{uu} + \xi hK f_x/f - \eta hK u - 2H_u \eta x + hK \xi_x + 2f\xi_t \xi_x - \eta \xi x K \\
-2f\xi \eta uu - 2f \xi t u - 2H_\eta u x - f \xi tt + H\xi_{xx} = 0, \\
-hK \eta x - H_\eta x x + f\eta_{tt} + 2f\eta uu + f\eta^2 \eta uu - 2f\xi_t \eta x = 0.
\]  

Solving the above system with respect to the coefficient functions $\xi, \eta, f, H, h$ and $K$ under the equivalence group $G_1^\gamma$, we can get a classification of regular reduction operator for the class \(49\). However, due to the strong nonlinearity of system \(53\), it is difficult to get an explicit classification. Hence, we omit the detail investigation for the general case and concentrate on some special cases.

**Example.** We study the regular reduction operator of equations \(49\) with $H(u) = K(u) = u, f(x) = h(x) = 1$, i.e, nonlinear telegraph equations

$$u_{tt} = (uu_x)_x + uu_x. \quad (55)$$

From table 3, it is easy to know that equation \(55\) admits three-dimensional Lie algebra $\mathfrak{g}$ of its infinitesimal Lie symmetries with a basis:

$$X_1 = \partial_t, \quad X_2 = t\partial_t - 2u\partial_u, \quad X_3 = \partial_x. \quad (56)$$

The corresponding one-parameter groups are time translations and scale transformations.
We first discuss a special case of the regular reduction operator, i.e., consider the conditional symmetry operator in the form:

\[ Q = \partial_x + \eta(t, x, u) \partial_u. \]  

(57)

With the assumptions \( \tau = 0, \xi = 1 \) the determining equations (52) for the nonlinear telegraph equation (49) are as follows:

\[
\begin{align*}
\eta_{tu} &= 0, & \eta_{uu} &= 0, \\
 f \eta_t - H_{uu} \eta^3 + H_u (\eta^2 f_x/f - 3 \eta x - 2 \eta^2 \eta_u) &= 0, \\
- H (\eta_{xx} + 2 \eta_{ux} \eta - \eta_{ux} f_x/f - \eta_x f_x/f) - K_u \eta^2 + K (h f_{xx}/f - h_x \eta - h \eta_x) &= 0.
\end{align*}
\]

(58)

From the first two equations we obtain that

\[ \eta(t, x, u) = A(x)u + B(t, x). \]

(59)

Substituting the latest equation with \( H(u) = K(u) = u, f(x) = g(x) = 1 \) into the last equation of system (58), we can see that the functions \( A(x) \) and \( B(t, x) \) satisfy the overdetermined system:

\[
\begin{align*}
A_{xx} + 5AA_x + A_x + 2A^3 + A^2 &= 0, \\
B_{xx} + 5A_x B + 3AB_x + B_x + 4A^2 B + 2AB &= 0, \quad (60)
\end{align*}
\]

The last two equations of system (60) imply the compatibility condition

\[
(14A + 4)B_x^2 + (36A^2 + 46A_x + 18A)BB_x + (2A - 37AA_x - 4A_x - 18A^3 - 5A^2)B^2 = 0
\]

(61)

obtained by cross-differential. The second equation of (60) is a differential consequence of (61) provided the equation

\[
(60A_x - 34A^2 + 34A - 8)B_x^2 - (510AA_x + 122A_x + 348A^3 + 234A^2 + 30A)BB_x + (-267A^2 - 233A^2 A_x - 135AA_x + 6A_x - 99A^3 - 32A^2 - 70A^4) = 0
\]

(62)

is satisfied.

Eliminate \( B_x \) from (61) and (62) we come to the equation

\[
B^2(272AA_x + 215 A^3 + 591 AA_x A^2 + 262 A^4 - 2 A + 10 A_x + 230 A^2 + 38 A^2)(1872 A^3
\]

\[
-32 A_x^2 + 3336 A^4 - 148608 A^5 - 194152 A_x A^3 - 11392 AA_x^2 + 115668656 A^6 A_x
\]

\[
+363896822 A^7 A_x + 11989256 A^3 A_x + 539345898 A^5 A_x^2 - 1263352 A^4 A_x
\]

\[
+5505328 A^3 A_x^2 + 107208 A^4 A_x^2 + 3740424 A^2 A_x^2 + 119385824 A_x A^4
\]

\[
-942408 A^2 A_x^2 + 301875108 A^2 A_x^2 + 45437328 AA_x^4 + 508439090 A^8 A_x
\]

\[
+979255660 A^6 A_x + 800801280 A^3 A_x + 247114530 A_x A^4 + 638696765 A^7 A_x^2
\]

\[
+695655030 A^7 A_x^2 + 337735335 A^4 A_x^3 + 59322060 A_x A^5 + 268692620 A_x A^9
\]

\[
+34451808 A^8 + 53063493 A^8 - 238410 A^6 + 5772399 A^7 + 6096 A^3 + 68668 A_x^4
\]

\[
+94806420 A^{10} + 16949160 A_x^5 + 42378980 A^{11} - 128 A^2 + 14960 A_x A^2 + 128 AA_x)
\]

(63)

If we take the third factor in (63) and the first equation for the function \( A(x) \) in system (60), then that overdetermined system for the function \( A(x) \) admits solution \( A(x) = 0 \). Hence, solving system (60) with \( A(x) = 0 \), we can obtain

\[ B(t, x) = a(t), \]

where the function \( a(t) \) is given by:

\[ \pm \int_{t_0}^{a(t)} \frac{3}{\sqrt{6a^2 + 3c_1}} \, du - t - c_2 = 0 \]

(64)
Taking a special form \( a(t) = 6/t^2 \) from (64) yield an exact explicit solution of equation (55) obtainable by solving equation (51), which is an ordinary differential equation in the variable \( x \) for the symmetries of second type, and subsequent solution of equation (55) for the 'constants' of integration actually depending on the variable \( t \):

\[
 u(t, x) = \frac{150 x + 25 c_2 + 25 c_1 t^3 - 180 \ln(t) - 36}{25 t^2},
\]

where \( c_i(i=1, 2) \) are parameters.

If we set \( B(t, x) = 0 \), then last two equations of (60) are satisfied and we arrive at an infinitesimal conditional symmetry

\[ Q = \partial_t + A(x) u \partial_u \]

with the function \( A(x) \) satisfying the ODE \( A_{xx} + 5 A A_x + A_x + 2 A^3 + A^2 = 0 \). Particular solution \( A(x) = 1/4[\tanh(1/2x) - 1] \) of the latter equation yields exact solutions of the nonlinear telegraph equation (55)

\[
 u(t, x) = [-\cosh(1/2 x)^12] \exp(-\frac{x}{4}) (c_1 t + c_2),
\]

while particular solution \( A(x) = 1/4[\coth(1/2x) - 1] \) yields the exact solution

\[
 u(t, x) = [\sinh(1/2 x)^12] \exp(-\frac{x}{4}) (c_1 t + c_2).\]

Finally, if we take the second factor \( 272 A A_x + 215 A^3 + 591 A_x A^2 + 262 A^4 - 2 A + 10 A_x + 230 A_x^2 + 38 A^2 = 0 \) in (63), which together with the first equation of (60) imply \( A(x) = -1/2 \). Thus, we arrive at an infinitesimal conditional symmetry in the form

\[ Q = \partial_t + [\frac{\alpha t + \beta}{2}] \partial_u, \]

where \( \alpha, \beta \) are arbitrary constants. Solving equation (51), we obtain an exact solutions

\[
 u(t, x) = 2 \alpha t + 2 \beta + e^{-1/2x} c_1 \text{AiryAi} \left(-1/2 \frac{2^{2/3}(\alpha t + \beta)}{\alpha^{2/3}}\right) \\
+ e^{-1/2x} c_2 \text{AiryBi} \left(-1/2 \frac{2^{2/3}(\alpha t + \beta)}{\alpha^{2/3}}\right)
\]

of equation (55), where \( \text{AiryAi}(t) \) and \( \text{AiryBi}(t) \) are the associated Airy functions of the first kind and the second kind respectively.

Now we consider the conditional symmetry operator of equation (55) in the form:

\[ Q = \partial_t + \xi(t, x) \partial_x + \eta(t, x)u \partial_u. \quad (65) \]

In this way, solving the determining equations (54) with \( H(u) = K(u) = u, f(x) = h(x) = 1 \), we can get \( \xi = \text{const}, \eta = 0. \)

From the above discussion we can arrival at

**Theorem 14.** Equation (49) with \( H(u) = K(u) = u, f(x) = h(x) = 1 \) is conditionally invariant under the following operators:

1. \( Q = \partial_x + a(t) \partial_u \);
2. \( Q = \partial_x + 1/4[\tanh(1/2x) - 1] u \partial_u \);
3. \( Q = \partial_x + 1/4[\coth(1/2x) - 1] u \partial_u \);
4. \( Q = \partial_x + [\frac{\alpha t + \beta}{2}] \partial_u \);
5. \( Q = \partial_t + \partial_x \);

where \( a(t) \) is given by equation (64).
7 Conservation laws

Apart from exact solutions, classical and nonclassical symmetry classifications, we know that another important subject of group analysis is the construction of conservation laws of (systems of) differential equations, which play an important role in mathematical physics [70]. In fact, the knowledge of conservation laws is useful in the numerical integration of partial differential equations, for example, to control numerical errors. Also, the investigation of conservation laws of the Korteweg-de Vries equation became a starting point of the discovery of a number of techniques (such as Lax pair, inverse scattering transformation, bi-Hamiltonian structures, etc.) to solve nonlinear evolution equations. The existence of a large number of conservation laws of a evolutionary partial differential equation (system) is a strong indication of its integrability. Conservation laws have also significant uses in the theory of non-classical transformations and in the theory of normal forms and asymptotic integrability.

In this section we classify local conservation laws of equations (1) with characteristics depending, at mostly, on \( t, x \) and \( u \). For classification we use the direct method described in [49,79]. To begin with, we adduce a necessary theoretical background on conservation laws, following, e.g., [49,70,79] and considering for simplicity the case of two independent variables \( t \) and \( x \). See the above references for the general case.

Let \( \mathcal{W} \) be a system \( \mathcal{W}(t, x, u) = 0 \) of \( l \) PDEs \( W^1 = 0, \ldots, W^l = 0 \) for \( m \) unknown functions \( u = (u^1, \ldots, u^m) \) of two independent variables \( t \) and \( x \). Here \( u_{(\rho)} \) denotes the set of all the partial derivatives of the functions \( u \) of order not greater than \( \rho \), including \( u \) as the derivatives of the zero order. Let \( \mathcal{W}_{(k)} \) denote the set of all algebraically independent differential consequences that have, as differential equations, orders no greater than \( k \). We identify \( \mathcal{W}_{(k)} \) with the manifold determined by \( \mathcal{W}_{(k)} \) in the jet space \( J^{(k)} \).

**Definition 8.** A conserved vector of the system \( \mathcal{W} \) is a 2-tuple \( F = (F^1(t, x, u_{(\rho)}), F^2(t, x, u_{(\rho)})) \) for which the divergence \( \text{Div} F := D_tF^1 + D_xF^2 \) vanishes for all solutions of \( \mathcal{W} \) (i.e., \( \text{Div} F|_\mathcal{W} = 0 \)).

In Definition 8 and later, \( D_t \) and \( D_x \) denotes the operator of total differentiation with respect to the variables \( t \) and \( x \) respectively. The notation \( \mathcal{V}|_\mathcal{W} \) means that values of \( \mathcal{V} \) are considered only on solutions of the system \( \mathcal{W} \).

The crucial notion of the theory of conservation laws is one of triviality and equivalence of conservation laws.

**Definition 9.** A conserved vector \( F \) is called trivial if \( F^i = \hat{F}^i + \tilde{F}^i, i = 1, 2 \), where \( \hat{F}^i \) and \( \tilde{F}^i \) are functions of \( t, x \) and derivatives of \( u \) (i.e., differential functions), \( \hat{F}^i \) vanish on the solutions of \( \mathcal{W} \), and the 2-tuple \( \hat{F} = (\hat{F}^1, \hat{F}^n) \) is a null divergence (i.e., its divergence vanishes identically).

**Definition 10.** Two conserved vectors \( F \) and \( F' \) are called equivalent if the vector-function \( F' - F \) is a trivial conserved vector.

The notion of linear dependence of conserved vectors is introduced in a similar way. Namely, a set of conserved vectors is linearly dependent iff a linear combination of them is a trivial conserved vector.

The above definitions of triviality and equivalence of conserved vectors are natural in view of the usual “empiric” definition of conservation laws of a system of differential equations as divergences of its conserved vectors, i.e. divergence expressions which vanish for all solutions of this system. For example, equivalent conserved vectors correspond to the same conservation law. However, for deeper understanding of the problem and absolutely correct calculations a more rigorous definition of conservation laws should be used.

For any system \( \mathcal{W} \) of differential equations the set \( \text{CV}(\mathcal{W}) \) of conserved vectors of its conservation laws is a linear space, and the subset \( \text{CV}_0(\mathcal{W}) \) of trivial conserved vectors is a linear
subspace in CV(W). The factor space CL(W) = CV(W)/CV_0(W) coincides with the set of equivalence classes of CV(W) with respect to the equivalence relation added in definition 10.

**Definition 11.** The elements of CL(W) are called conservation laws of the system W, and the whole factor space CL(W) is called the space of conservation laws of W.

That is why description of the set of conservation laws can be assumed as finding CL(W) that is equivalent to construction of either a basis if dim CL(W) < ∞ or a system of generatrices in the finite dimensional case. The elements of CV(W) which belong to the same equivalence class giving a conservation law F are considered as conserved vectors of this conservation law, and we will additionally identify elements from CL(W) with their representatives in CV(W). For F ∈ CV(W) and F ∈ CL(W) the notation F ∈ F will denote that F is a conserved vector corresponding to the conservation law F. In contrast to the order r_F of a conserved vector F as the maximal order of derivatives explicitly appearing in F, the order of the conservation law F is called min{r_F | F ∈ F}. Under linear dependence of conservation laws we understand linear dependence of them as elements of CL(W). Therefore, in the framework of “representative” approach conservation laws of a system W are considered as linearly dependent if there exists linear combination of their representatives, which is a trivial conserved vector.

Let the system W be totally nondegenerate [70]. Then application of the Hadamard lemma to the definition of conservation law and integrating by parts imply that the left hand side of dependence of them as elements of CL(W) as linearly dependent if there exists linear combination of their representatives, which is a trivial conserved vector.

Let the system W be totally nondegenerate [70]. Then application of the Hadamard lemma to the definition of conservation law and integrating by parts imply that the left hand side of dependence of them as elements of CL(W) as linearly dependent if there exists linear combination of their representatives, which is a trivial conserved vector.

**Proposition 4.** Any point transformation g maps a class of equations in the conserved form into itself. More exactly, the transformation g: \( \tilde{t} = t^g(t, x, u), \tilde{x} = x^g(t, x, u), \tilde{u} = u^g(t, x, u) \) prolonged to the jet space \( J^r(V) \) transforms the equation \( D_t F^1 + D_x F^2 = 0 \) to the equation \( D_{\tilde{t}} F^1_g + D_{\tilde{x}} F^2_g = 0 \). The transformed conserved vector \( F^1_g, F^2_g \) is determined by the formula

\[
F^1_g(x, u(r)) = \frac{F^1(x, u(r))D_x \tilde{t} + F^2(x, u(r))D_x \tilde{x}}{D_{\tilde{t}} D_x \tilde{x} - D_x \tilde{t} D_{\tilde{t}} \tilde{x}},
\]

\[
F^2_g(x, u(r)) = \frac{F^1(x, u(r))D_t \tilde{x} + F^2(x, u(r))D_x \tilde{x}}{D_{\tilde{t}} D_x \tilde{x} - D_x \tilde{t} D_{\tilde{t}} \tilde{x}}.
\]
Remark 7. In the case of one dependent variable \((m = 1)\) \(g\) can be a contact transformation: 
\[
t = t^g(t, x, u(1)), \quad \dot{x} = x^g(t, x, u(1)), \quad \dot{u}(1) = u^g(1)(t, x, u(1)).
\]
Similar note is true for the below statement.

Definition 13. Let \(G\) be a symmetry group of the system \(\mathcal{W}\). Two conservation laws with the conserved vectors \(F\) and \(F'\) are called \(G\)-equivalent if there exists a transformation \(g \in G\) such that the conserved vectors \(F_g\) and \(F'\) are equivalent in the sense of Definition 10.

Any transformation \(g \in G\) induces a linear one-to-one mapping \(g_*\) in \(\text{CV}(\mathcal{W})\), transforms trivial conserved vectors only to trivial ones (i.e. \(\text{CV}_0(\mathcal{W})\) is invariant with respect to \(g_*\)) and therefore induces a linear one-to-one mapping \(g_f\) in \(\text{CL}(\mathcal{W})\). It is obvious that \(g_f\) preserves linear (in)dependence of elements in \(\text{CL}(\mathcal{W})\) and maps a basis (a set of generatrices) of \(\text{CL}(\mathcal{W})\) in a basis (a set of generatrices) of the same space. In such way we can consider the \(G\)-equivalence relation of conservation laws as well-determined on \(\text{CL}(\mathcal{W})\) and use it to classify conservation laws.

Proposition 5. If system \(\mathcal{W}\) admits a one-parameter group of transformations, then the infinitesimal generator \(Q = \xi^i \partial_i + \eta^a \partial_a\) of this group can be used for construction of new conservation laws from known ones. Namely, differentiating equation (67) with respect to the parameter \(\epsilon\) and taking the value \(\epsilon = 0\), we obtain the new conserved vector

\[
\tilde{F}^i = -Q_{(r)} F^i + (D_j \xi^i) F^j - (D_j \xi^j) F^i. \tag{68}
\]

Here \(Q_{(r)}\) denotes the \(r\)-th prolongation of the operator \(Q\).

Remark 8. Formula (68) can be directly extended to generalized symmetry operators (see, for example, [4]). A similar statement for generalized symmetry operators in evolutionary form \((\xi^i = 0)\) was known earlier [13, 30]. It was used in [20] to introduce a notion of basis of conservation laws as a set which generates a whole set of conservation laws with action of generalized symmetry operators and operation of linear combination.

Proposition 6. Any point transformation \(g\) between systems \(\mathcal{W}\) and \(\tilde{\mathcal{W}}\) induces a linear one-to-one mapping \(g_*\) from \(\text{CV}(\mathcal{W})\) into \(\text{CV}(\tilde{\mathcal{W}})\), which maps \(\text{CV}_0(\mathcal{W})\) into \(\text{CV}_0(\tilde{\mathcal{W}})\) and generates a linear one-to-one mapping \(g_f\) from \(\text{CL}(\mathcal{W})\) into \(\text{CL}(\tilde{\mathcal{W}})\).

Corollary 2. Any point transformation \(g\) between systems \(\mathcal{W}\) and \(\tilde{\mathcal{W}}\) induces a linear one-to-one mapping \(\tilde{g}_f\) from \(\text{Ch}_f(\mathcal{W})\) into \(\text{Ch}_f(\tilde{\mathcal{W}})\).

Consider the class \(\mathcal{W}|_S\) of systems \(\mathcal{W}_0\):

\[
\mathcal{W}(t, x, u(\rho), \theta) = 0 \text{ parameterized with the parameter-functions } \theta = \theta(t, x, u(\rho)).
\]

Here \(\mathcal{W}\) is a fixed function of \(t, x, u(\rho)\) and \(\theta\). The symbol \(\theta\) denotes the tuple of arbitrary (parametric) differential functions \(\theta(t, x, u(\rho)) = (\theta^1(t, x, u(\rho)), ..., \theta^k(t, x, u(\rho)))\) running through the set \(S\) of solutions of the system \(\mathcal{S}(t, x, u(\rho), \theta_{(q)}(t, x, u(\rho))) = 0\). This system consists of differential equations on \(\theta\), where \(t, x, u(\rho)\) play the role of independent variables and \(\theta_{(q)}\) stands for the set of all the derivatives of \(\theta\) of order not greater than \(q\). In what follows we call the functions \(\theta\) arbitrary elements. Denote the point transformation group preserving the form of the equations from \(\mathcal{W}|_S\) by \(G^\sim = G^\sim(\mathcal{W}, S)\).

Consider the set \(P = P(\mathcal{W}, S)\) of all pairs each of which consists of a system \(\mathcal{W}_\theta\) from \(\mathcal{W}|_S\) and a conservation law \(F\) of this system. In view of Proposition 6 action of transformations from \(G^\sim\) on \(\mathcal{W}|_S\) and \(\{\text{CV}(\mathcal{W}_\theta) | \theta \in S\}\) together with the pure equivalence relation of conserved vectors naturally generates an equivalence relation on \(P\).

Definition 14. Let \(\theta, \theta' \in S, F \in \text{CL}(\mathcal{W}_\theta), F' \in \text{CL}(\mathcal{W}_{\theta'}), F \in F, F' \in F'\). The pairs \((\mathcal{W}_\theta, F)\) and \((\mathcal{W}_{\theta'}, F')\) are called \(G^\sim\)-equivalent if there exists a transformation \(g \in G^\sim\) which transforms the system \(\mathcal{W}_\theta\) to the system \(\mathcal{W}_{\theta'}\) and such that the conserved vectors \(F_g\) and \(F'\) are equivalent in the sense of Definition 10.
In such a way, classification of conservation laws with respect to $G^\sim$ will be understood as classification in $P$ with respect to the above equivalence relation. This problem can be investigated in the way that is similar to group classification in classes of systems of differential equations, especially it is formulated in terms of characteristics. Namely, we construct firstly the conservation laws that are defined for all values of the arbitrary elements. (The corresponding conserved vectors may depend on the arbitrary elements.) Then we classify, with respect to the equivalence group, arbitrary elements for each of that the system admits additional conservation laws.

For more detail and rigorous proof of the correctness of the above definitions and statements see [19][79].

In what follows, we use the most direct method described in [79] to derive the conservation law of class (1). Due to using the transformation $\hat{t} = t, \hat{x} = \int \frac{dx}{q(x)}; \hat{u} = u$ from $G^\sim / G^\sim$ in section 3 we can reduce equation (1) to one which has the same form with equation (23). Thus, without loss of generality we can restrict ourselves to investigation conservation law of the equation (23).

**Theorem 15.** A complete list of $G^\sim$-inequivalent equations (23) having nontrivial conservation laws with characteristics of the zeroth order is exhausted by ones given in table 10.

| N  | $H(u)$ | $K(u)$ | $f(x)$ | $h(x)$ | Basis conservation laws |
|----|--------|--------|--------|--------|-------------------------|
| 1  | $\forall$ | $\forall$ | $\forall$ | 1 | CL$^1$, CL$^2$ |
| 2  | 1 | $K_u \neq 0$ | $h(h^{-1})_{xx}$ | $\forall$ | CL$^3$, CL$^4$ |
| 3  | $\forall$ | $\epsilon H + 1$ | $-h_y$ | $\forall$ | CL$^3$, CL$^5$ |
| 4  | $\forall$ | $\epsilon H + 1$ | $-h_y - h_y^{-1}$ | $\forall$ | CL$^7$, CL$^8$ |
| 5  | $\forall$ | $\epsilon H + 1$ | $-h_{u_1}$ | $\forall$ | CL$^9$, CL$^{10}$, CL$^{11}$, CL$^{12}$ |
| 6  | $\forall$ | $\epsilon H$ | $\forall$ | $\forall$ | CL$^{13}$, CL$^{14}$, CL$^{15}$, CL$^{16}$ |
| 7  | 1 | 0 | $\forall$ | $\forall$ | CL$^{17}$ |

Table 10. Conservation laws of equations (23)

Here the conserved densities $F^1$ and fluxes $F^2$ of the presented conservation laws have the following forms:

- CL$^1$: $f_{ut} - (Hu_x + fK)$;
- CL$^2$: $f(tu_t - u), -t(Hu_x + fK)$;
- CL$^3$: $e^{-t}(h^{-1})_{xx}(tu_t + u) - e^{-t}[h^{-1}u_x - (h^{-1})_x u + fK]$;
- CL$^4$: $e^t(h^{-1})_{xx}(tu_t - u) - e^t[h^{-1}u_x - (h^{-1})_x u + fK]$;
- CL$^5$: $-e^{-t}e^t h_y (tu_t - u), -e^t(e^{-t}f^huy + hu)$;
- CL$^6$: $-e^{-t}e^t h_y (tu_t + u), -e^{-t}(e^{-t}f^huy + hu)$;
- CL$^7$: $-e^{-t}e^t h_y (hu + y^{-1}h)(tu_t - u), -e^t(e^{-t}f^huy - fH + yhu)$;
- CL$^8$: $-e^{-t}e^t h_y (hu + y^{-1}h)(tu_t + u), -e^{-t}(e^{-t}f^huy - fH + yhu)$;
- CL$^9$: $e^{-t}f^h [(\alpha^{11}y + \alpha^{10})u_t - (\alpha^{11}y + \alpha^{10})u], -(\alpha^{11}y + \alpha^{10})(e^{-t}f^huy + hu) + \alpha^{11}\int H$;
- CL$^{10}$: $e^{-t}f^h [(\alpha^{21}y + \alpha^{20})u_t - (\alpha^{21}y + \alpha^{20})u], -(\alpha^{21}y + \alpha^{20})(e^{-t}f^huy + hu) + \alpha^{21}\int H$;
- CL$^{11}$: $e^{-t}f^h [(\alpha^{31}y + \alpha^{30})u_t - (\alpha^{31}y + \alpha^{30})u], -(\alpha^{31}y + \alpha^{30})(e^{-t}f^huy + hu) + \alpha^{31}\int H$;
- CL$^{12}$: $e^{-t}f^h [(\alpha^{41}y + \alpha^{40})u_t - (\alpha^{41}y + \alpha^{40})u], -(\alpha^{41}y + \alpha^{40})(e^{-t}f^huy + hu) + \alpha^{41}\int H$;
- CL$^{13}$: $e^{-t}f^h fu_t, -e^{-t}f^huy$;
- CL$^{14}$: $e^{-t}f^h [tu_t - u], -e^{-t}f^huy$;
- CL$^{15}$: $e^{-t}f^h uyfu_t, -e^{-t}f^huy + fH$;
- CL$^{16}$: $e^{-t}f^h uy [tu_t - u], -e^{-t}f^huyuy + t\int H$;

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the variable \( y \) is implicitly determined by the formula \( x = \int e^{-\epsilon \int h(y)dy} dy; \epsilon, a_{ij} = \text{const}, i, j = 0, 1; (\alpha^1, \alpha^2) = (\alpha^1(t), \alpha^2(t)), k = 1, 2, 3, 4 \) is one of fundamental solution system of the system of ODEs
\[
\alpha^1_{tt} = -(a_{11} \alpha^1 + a_{01} \alpha_0), \quad \alpha^0_{tt} = -(a_{10} \alpha^1 + a_{00} \alpha_0)
\]
\( \omega = a_{01} y^2 + (a_{00} - a_{11}) y - a_{10}; \sigma = \sigma(t, x) \) is an arbitrary solution of the linear equation \( f \sigma_{tt} - \sigma_{xx} = 0 \). Hereafter \( \int H = \int H du, \int K = \int K du \). In case \( 0 < \epsilon \in \{0, 1\} \mod G_1^- \).

In Theorem 16, the classification of conservation laws is performed with respect to the usual equivalence group \( G_1^- \) and thus some obtained results are explicit. Hence, we can further formulate the classification result in an explicit form, and indeed we can split case 5 of table 10 into a number of inequivalent cases depending on values of \( a_{ij} \). At the same time, using the extended equivalence group \( \hat{G}_1^- \), we can present the result of classification in a closed and simple form with a smaller number of inequivalent equations having nontrivial conservation laws.

**Theorem 16.** A complete list of \( \hat{G}_1^- \)-inequivalent equations (23) having nontrivial conservation laws with characteristics of the zeroth order is exhausted by ones given in table 11.

| N | \( H(u) \) | \( K(u) \) | \( f(x) \) | \( h(x) \) | Basis conservation laws |
|---|---|---|---|---|---|
| 1 | \( \forall \) | \( \forall \) | \( \forall \) | 1 | \( \text{CL}^1, \text{CL}^2 \) |
| 2 | \( 1 \) | \( K_u \neq 0 \) | \( h(h^{-1})_{xx} \) | \( \forall \) | \( \text{CL}^3, \text{CL}^4 \) |
| 3 | \( \forall \) | \( 1 \) | \( -h_x \) | \( \forall \) | \( \text{CL}^5, \text{CL}^6 \) |
| 4 | \( \forall \) | \( 1 \) | \( -h_x - h_x^{-1} \) | \( \forall \) | \( \text{CL}^7, \text{CL}^8 \) |
| 5a | \( \forall \) | \( 0 \) | \( \forall \) | \( \forall \) | \( \text{CL}^9, \text{CL}^{10}, \text{CL}^{11}, \text{CL}^{12} \) |
| 5b | \( \forall \) | \( 1 \) | \( e^x \) | \( e^x \) | \( \text{CL}^{13}, \text{CL}^{14}, \text{CL}^{15}, \text{CL}^{16} \) |
| 6c | \( \forall \) | \( 1 \) | \( x^{\mu-1} \) | \( x^\mu \) | \( \text{CL}^{17}, \text{CL}^{18}, \text{CL}^{19}, \text{CL}^{20} \) |
| 6d | \( \forall \) | \( 1 \) | \( x e^{-1} \frac{\mu}{\mu-1} \) | \( \frac{1}{2} e^{-\frac{1}{2}} \) | \( \text{CL}^{21}, \text{CL}^{22}, \text{CL}^{23}, \text{CL}^{24} \) |
| 7 | \( \forall \) | \( 1 \) | \( \frac{|x-1|^{\mu-1}}{2} \) | \( \frac{|x-1|^{\mu-1}}{2} \) | \( \text{CL}^{25}, \text{CL}^{26}, \text{CL}^{27}, \text{CL}^{28} \) |
| 8 | \( \forall \) | \( 1 \) | \( e^\mu \arctan x (x^2 + 1)^{-\frac{\mu}{2}} \) | \( e^\mu \arctan x (x^2 + 1)^{-\frac{\mu}{2}} \) | \( \text{CL}^{29}, \text{CL}^{30}, \text{CL}^{31}, \text{CL}^{32} \) |
| 9 | \( 1 \) | \( 0 \) | \( \forall \) | \( \forall \) | \( \text{CL}^{33}, \text{CL}^{34}, \text{CL}^{35}, \text{CL}^{36} \) |

Here the conserved densities \( F^1 \) and fluxes \( F^2 \) of the presented conservation laws have the following forms:

**CL**\(^1\): \( fu_t, -(Hu_x + \int K) \);

**CL**\(^2\): \( f(tu_t - u), -t(Hu_x + \int K) \);

**CL**\(^3\): \( e^{-t} (h^{-1})_{xx}(u_t + u), -e^{-t} [(h^{-1}) u_x - (h^{-1}) u + \int K] \);

**CL**\(^4\): \( e^t (h^{-1})_{xx}(u_t - u), -e^t [(h^{-1}) u_x - (h^{-1}) u + \int K] \);

**CL**\(^5\): \( e^{tf} (u_t - u), -e^{tf} (Hu_x + h \int K) \);

**CL**\(^6\): \( e^{-t} f(u_t + u), -e^{-t} (Hu_x + h \int K) \);

**CL**\(^7\): \( e^{tf} (u_t - u), -e^{tf} (xHu_x - \int H + xhu) \);

**CL**\(^8\): \( e^{-t} x f(u_t + u), -e^{-t} (xHu_x - \int H + xhu) \);

**CL**\(^9\): \( fu_t, -(Hu_x) \);

**CL**\(^10\): \( f(tu_t - u), -tHu_x \);

**CL**\(^11\): \( xfu_t, -xHu_x + \int H \);
CL12: \( x f(tu_t - u), -t(xHu_x - \int H) \);
CL13: \( u_t, -H u_x - u \);
CL14: \( tu_t - u, -t(Hu_x + u) \);
CL15: \( (x + \frac{1}{2}t^2)u_t + tu, -(x - \frac{1}{2}t^2)(Hu_x + u) + \int H \);
CL16: \( (tx - \frac{1}{6}t^3)u_t - (x - \frac{1}{2}t^2)u, -(tx - \frac{1}{6}t^3)(Hu_x + u) + t \int H \);
CL17: \( e^{x}(ut \cos t + u \sin t), -\cos t(Hu_x + e^{x}u) \);
CL18: \( e^{x}(ut \sin t - u \cos t), -\sin t(Hu_x + e^{x}u) \);
CL19: \( (x \sin t + \frac{1}{2}t \cos t)e^{x}u_t - (x \cos t + \frac{1}{2}t \cos t)e^{x}u, -(x \sin t + \frac{1}{2}t \cos t)(Hu_x + e^{x}u) + \int H \);
CL20: \( [(x - \frac{1}{2}) \cos t - \frac{1}{2}t \sin t]e^{x}u_t + (x \sin t + \frac{1}{2}t \cos t)e^{x}u, -[(x - \frac{1}{2}) \cos t - \frac{1}{2}t \sin t](Hu_x + e^{x}u) + \cos t \int H \);
CL21: \( x^{\mu - 1}[ut \cos(\sqrt{\mu}t) + \sqrt{\mu}u \sin(\sqrt{\mu}t)], -\cos(\sqrt{\mu}t)(Hu_x + x^{\mu}u) \);
CL22: \( x^{\mu - 1}[ut \sin(\sqrt{\mu}t) - \sqrt{\mu}u \cos(\sqrt{\mu}t)], -\sin(\sqrt{\mu}t)(Hu_x + x^{\mu}u) \);
CL23: \( x^{\mu}[ut \cos(\sqrt{\mu} + \frac{1}{2}t) + \sqrt{\mu} + \frac{1}{2}u \sin(\sqrt{\mu} + \frac{1}{2}t)], -\cos(\sqrt{\mu} + \frac{1}{2}t)(xHu_x + x^{\mu + 1}u) + \cos(\sqrt{\mu} + \frac{1}{2}t) \int H \);
CL24: \( x^{\mu}[ut \sin(\sqrt{\mu} + \frac{1}{2}t) - \sqrt{\mu} + \frac{1}{2}u \cos(\sqrt{\mu} + \frac{1}{2}t)], -\sin(\sqrt{\mu} + \frac{1}{2}t)(xHu_x + x^{\mu + 1}u) + \sin(\sqrt{\mu} + \frac{1}{2}t) \int H \);
CL25: \( \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}(ut \cos t + u \sin t), -\cos t(xHu_x + \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}u) + \cos t \int H \);
CL26: \( \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}(ut \sin t - u \cos t), -\sin t(xHu_x + \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}u) + \sin t \int H \);
CL27: \( \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}(-\frac{1}{2}x \cos t + \sin t)u_t - (\frac{1}{2}x \cos t - \frac{1}{2}t \sin t + \cos t)u, -(\frac{1}{2}x \cos t + \sin t)(Hu_x + \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}u) - \frac{1}{2}t \cos t \int H \);
CL28: \( \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}[(\frac{1}{2}x \sin t + \frac{1}{2}x \cos t + \cos t)u_t - (\frac{1}{2}x \cos t - \sin t)u], -(\frac{1}{2}x \sin t + \frac{1}{2}x \cos t + \cos t)(Hu_x + \frac{1}{\sqrt{2}} e^{-\frac{1}{\sqrt{2}}}u) + \frac{1}{2}t \sin t + \cos t) \int H \);
CL29: \( \frac{1}{\sqrt{2}} u_t, -xHu_x - u + \int H \);
CL30: \( \frac{1}{\sqrt{2}}(tu_t - u), -t(xHu_x + u - \int H) \);
CL31: \( (\frac{1}{2}x^2 + \frac{1}{2}x^2)u_t - \frac{1}{2}x u, -\frac{1}{2}(x^2 + 1)(Hu_x + \frac{1}{2}u) + \frac{1}{2}t^2 \int H \);
CL32: \( (\frac{1}{2}x^2 + \frac{1}{2}x^2)u_t - (\frac{1}{2}x^2 + \frac{1}{2}x^2)u, -\frac{1}{6}(x^2 + 1)(Hu_x + \frac{1}{2}u) + \frac{1}{6}t^3 \int H \);
CL33: \( (x + 1)f[\sin(\sqrt{2\mu} - 1)t]u_t - \sqrt{2\mu - 1}\cos(\sqrt{2\mu} - 1)t)u, -(x + 1)\sin(\sqrt{2\mu} - 1)(Hu_x + hu) + \sin(\sqrt{2\mu} - 1) \int H \);
CL34: \( (x + 1)f[\cos(\sqrt{2\mu} - 1)t]u_t + \sqrt{2\mu - 1}\sin(\sqrt{2\mu} - 1)t)u], -(x + 1)\cos(\sqrt{2\mu} - 1)(Hu_x + hu) + \cos(\sqrt{2\mu} - 1) \int H \);
CL35: \( (x - 1)f[\sin(\sqrt{2\mu} + 1)t]u_t - \sqrt{2\mu + 1}\cos(\sqrt{2\mu} + 1)t)u], -(x - 1)\sin(\sqrt{2\mu} + 1)(Hu_x + hu) + \sin(\sqrt{2\mu} + 1) \int H \);
CL36: \( (x - 1)f[\cos(\sqrt{2\mu} + 1)t]u_t + \sqrt{2\mu + 1}\sin(\sqrt{2\mu} + 1)t)u], -(x - 1)\cos(\sqrt{2\mu} + 1)(Hu_x + hu) + \cos(\sqrt{2\mu} + 1) \int H \);
CL37: \( e^{\sqrt{\mu - 1}i}(x - i)f(u_t + \sqrt{\mu - 1}u), -e^{\sqrt{\mu - 1}i}(x - i)(Hu_x + hu) + e^{\sqrt{\mu - 1}i} \int H \);
CL38: \( e^{\sqrt{\mu - 1}i}(x - i)f(u_t - \sqrt{\mu - 1}u), -e^{\sqrt{\mu - 1}i}(x - i)(Hu_x + hu) + e^{\sqrt{\mu - 1}i} \int H \);
CL39: \( e^{\sqrt{\mu + 1}i}(x + i)f(u_t + \sqrt{\mu + 1}u), -e^{\sqrt{\mu + 1}i}(x + i)(Hu_x + hu) + e^{\sqrt{\mu + 1}i} \int H \);
CL40: \( e^{\sqrt{\mu + 1}i}(x + i)f(u_t - \sqrt{\mu + 1}u), -e^{\sqrt{\mu + 1}i}(x + i)(Hu_x + hu) + e^{\sqrt{\mu + 1}i} \int H \);
CL41: \( f(\sigma u_t - \sigma u), -\sigma u_x + \sigma_x u \);
\( \mu = \text{const}, \sigma = \sigma(t, x) \) is an arbitrary solution of the linear equation \( f \sigma t + \sigma xx = 0 \).

**Proof.** We search the first-order conservation laws for the equations from class \([23]\) in the form
\[
D_t F(t, x, u, u_t, u_x) + D_x G(t, x, u, u_t, u_x) = 0,
\]
where \( D_t \) and \( D_x \) are the operators of the total differentiation with respective to \( t \) and \( x \) correspondingly, namely, \( D_t = \partial_t + u_t \partial_u + \cdots \), \( D_x = \partial_x + u_x \partial_u + \cdots \). Substituting the expression for \( u_{tt} \) deduced from (23) into (69) and decompose the obtained equation with respect to \( u_{xt} \) and \( u_{xx} \), we obtain

\[
F_{u_x} + G_{u_t} = 0, \\
F_{u_t} \frac{H}{f} + G_{u_x} = 0, \\
F_{u_t} \frac{H_u}{f} u_x^2 + (F_{u_t} \frac{hK}{f} + G_u) u_x + F_u u_t + F_t + G_x = 0.
\]

(70)

Up to conserved vectors equivalence for the first equation of system (70), we can assume \( F_{u_x} = G_{u_t} = 0 \), which together with the second equation imply

\[
F = F^3(t, x, u)u_t + F^2(t, x, u), \quad G = -F^3(t, x, u)\frac{H}{f} u_x + G^1(t, x, u).
\]

(71)

Substituting these expression into the last equation of system (70) and splitting it with respect to the powers of \( u_x \) and \( u_t \), we obtain the system of PDEs for the functions \( F^3, F^2 \) and \( G^1 \) of the form

\[
F_{u}^3 = 0, \quad F_{u}^2 + F_t^3 = 0, \\
F^3 \frac{hK}{f} + G_u^1 - \frac{H}{f} F^3_x + F^3 \frac{f_x}{f^2} H = 0, \\
F_t^2 + G_x^1 = 0.
\]

(72)

Solving first three equations of (72) yields

\[
F^3 = F^1(t, x), \quad F^2 = -F^1 u + F^0(t, x), \quad G^1 = \left(\frac{F^1}{f}\right)_x \int H - \frac{hF^1}{f} \int K + G^0(t, x).
\]

(73)

Substituting the latter expression for \( F^2 \) and \( G^1 \) into the last equation of system (72), we can know that the major role for classification is played by a differential consequence

\[
\left(\frac{F^1}{f}\right)_{xx} H - \left(\frac{hF^1}{f}\right)_x K - F_{tt}^1 = 0.
\]

(74)

Indeed, it is the unique classifying condition for this problem. In all classification cases we obtain the equation \( F_t^0 + G_x^0 = 0 \). Therefore, up to conserved vectors equivalence we can assume \( F^0 = G^0 = 0 \), and additionally \( F^1 \neq 0 \) for conservation laws to be non-trivial. Thus, taking into account (71) and (73) the conservation density and flux can be rewritten as

\[
F = F^1(t, x) u_t - F^1 u, \quad G = -\frac{H}{f} F^1 u_x + \left(\frac{F^1}{f}\right)_x \int H - \frac{hF^1}{f} \int K.
\]

(75)

Equation (74) implies that there exist no non-trivial conservation laws in the general case. Let us classify the special values of the parameter-functions for which equation (23) possesses non-trivial conservation laws. There exist four different possibilities for values of \( H \) and \( K \):

1. \( \dim(h, H, K) = 3 \). It follows from (74) that \( F_{tt}^1 = \left(\frac{F^1}{f}\right)_{xx} = (hF^1/f)_x = 0 \) and therefore \( F^1 = \alpha(t)f/h, (1/h)x = 0, (1/x) = 0 \) i.e., obviously \( h \in \{1, x^{-1}\}, \alpha = c_1 + c_2 t \mod G_{1}^- \). Moreover, \( h = 1 \sim x = x^{-1} \mod G_{1}^\sim \) (the corresponding transformation is \( \bar{x} = \ln|x| \) and \( \bar{f} = x^2 f \), the other variables and parameter-functions are not changed). As a result, we obtain case 1.
2. $H \in \langle 1 \rangle$, $K \in \langle 1 \rangle$. Then $H = 1 \mod G_1^\sim$ and $(h F^1/f)_x = 0, F^1_{tt} - (F^1/f)_{xx} = 0$, i.e., $F^1 = \alpha(t)/f/h$, where $\alpha_{tt}/\alpha = \lambda = \text{const}$ (otherwise we have case $1^\parallel$) and so $\lambda = 1, f = h(h^{-1})_{xx} \mod G^\sim_1$ (case $2^\parallel$).

3. $H \in \langle 1 \rangle$, $K \in \langle H, 1 \rangle$. Then $K \in \langle 1 \rangle \mod G^\sim_1$ and $(F^1/f)_x = 0, F^1_{tt} = -K(hF^1/f)_x$, i.e., $F^1 = (\alpha(t)x + \alpha^0(t))f$ and $\alpha_{tt}xf + \alpha^0_tf = -K(\alpha^1(xh)_x + h^0h_x)$. For $K = 0$ we obtain case $5_k$ at once. Suppose $K \neq 0$. Then $K = 1 \mod G^\sim_1$ and the dimension $m = \dim(f, xf, h_x, (xh)_x)$ can have only the values 2 and 3.

If $m = 3$ then there exist constants $a_{ij}, b_i, i, j = 0, 1$, and a function $\theta = \theta(x)$ such that $(b_0, b_1) \neq (0, 0), \dim(f, xf, x\theta) = 3$ and $h_x = a_{00}f + a_{01}xf + b_0\theta, (xh)_x = a_{10}f + a_{11}xf + b_1\theta$. Therefore, $\alpha_{tt} = -a_{11}\alpha^1 - a_{01}\alpha^0, \alpha^0_{tt} = -a_{10}\alpha^1 - a_{00}\alpha^0, b_1\alpha^1 + b_0\alpha^0 = 0$, i.e. $\alpha^1 = c_1 e^{\lambda t} + c_2 e^{-\lambda t}, \alpha^0 = c_3 e^{\delta t} + c_4 e^{-\delta t}$ where $c_i(i = 1, \ldots, 4), \lambda, \delta = \text{const}$ and $\lambda, \delta \neq 0$ (otherwise, this case is reduced to a subcase of $1^\parallel$, hence $\lambda, \delta = 1 \mod G^\sim$. Depending on values (either vanishing or non-vanishing) of $c_1, c_2$ and $c_3, c_4$ we obtain cases $3$ and $4$ correspondingly.

If $m = 2$ then $h_x = a_{00}f + a_{01}xf, (xh)_x = a_{10}f + a_{11}xf$ for some constants $a_{ij}, i, j = 0, 1$.

Therefore, $\alpha_{tt} = -a_{11}\alpha^1 - a_{01}\alpha^0, \alpha^0_{tt} = -a_{10}\alpha^1 - a_{00}\alpha^0, f = -\omega/h, h_x/h = -(a_{01}x + a_{00})/\omega$, where $\omega = a_{01}x^2 + (a_{00} - a_{11})x - a_{10}$, i.e., $h = \omega^{-1/2} \exp(\int \frac{1}{2}(a_{00} + a_{11})\omega^{-1})$. As a result, we obtain four conservation laws with the conserved vectors

$$(\alpha^1(t)x + \alpha^{\mu_0}(t)f)u_t - (\alpha^1(t)x + \alpha^{\mu_0}(t))f u_t, -(\alpha^1(t)x + \alpha^{\mu_0}(t))(H u_x + hu) + \alpha^{1}(\int H),$$

where $(\alpha^{1\mu}, \alpha^{\mu_0}), \mu = 1, \ldots, 4$ form a fundamental set of solutions of the system $\alpha_{tt} = -a_{11}\alpha^1 - a_{01}\alpha^0, \alpha^0_{tt} = -a_{10}\alpha^1 - a_{00}\alpha^0$. Separate consideration of possible inequivalent values of the constants $a_{ij}$ leads to cases $3, 4, 5_1$ and $5_3$.

4. $H, K \in \langle 1 \rangle$. Therefore, $H = 1, K = 0 \mod G^\sim$ and $F^1_{tt} - (F^1/f)_{xx} = 0$. Let $F^1 = \sigma(t, x)f$, we have $\sigma_{xx} - f\sigma_{tt} = 0$, which corresponds case $9$.

The above conservation laws can be used for construction of potential systems, potential symmetries and potential conservation laws. We will present such analysis elsewhere.

8 Conclusion and Remarks

In summary, we have presented an enhanced classical and nonclassical symmetries and conservation laws analysis of the class of equations $\langle 1 \rangle$ in the framework of modern group analysis of differential equations.

We have performed a complete and extended symmetry group classification of the class of equations $\langle 1 \rangle$ with the two “best” gauges $g = 1$ and $g = h$. The main results on classification are collected in tables $\langle 1 \rangle \parallel 3, \langle 1 \rangle \parallel 4$ ($g = 1$) and $\langle 1 \rangle \parallel 6$ ($g = h$) where we list inequivalent cases of extensions with the corresponding Lie invariance algebras. The success in the classification and the clear presentation of the final results are relied heavily on the regular applications of four original tools presented in $[17]$, i.e., the equivalence relation with respect to the extended equivalence group instead of the usual one, the choice of true gauges, furcate split and systematic usage of additional equivalences. Among them the first two kinds of techniques (the extended equivalence group and true gauges) are of crucial importance for obtaining a closed and explicit classification list. The extended equivalence group of class $\langle 1 \rangle$ is the extension of the usual one with the non-trivial group of gauge equivalence transformations including transformations which are nonlocal in arbitrary elements. Neglecting this transformations leads to critical swelling and complication of both calculations and results. This can be seen from the classification results of equation $\langle 1 \rangle$ with the gauge $g = 1$ under the usual equivalence transformations (adduced in the Appendix).
As an application of the classification results, exact solutions of some classification models are
given by using the method of classical Lie reduction.

Nonclassical symmetries of equation (1) are discussed within the framework of singular re-
duction operator. Determining equations related the general singular and regular reduction
operators of (1) with \( g = 1 \) are given. Several reduction operators of a special nonlinear tele-
graph equation (equation (1) with \( H(u) = K(u) = u, f(x) = h(x) = 1 \) ) are present. This
enabled to obtain some exact solutions of the corresponding equation which are invariant under
these conditional symmetries.

Using the most direct method, we have also investigated two classifications of local conser-
vation laws up to equivalence relations which are generated by both usual and extended equivalence groups. Equivalence with respect to these groups and correct choice of gauge coefficients of equations play the major role for simple and clear formulation of the final results.

It should be noted that all above results of class (1) can be applied to the specific models listed
in section 2. We do not discuss here because there are nothing but tendinous computations.

One of the natural continuation for further investigation of different properties of class (1)
is to perform group classifications of the variable gauges, i.e., when the value of \( g \) (1 or \( h \))
depends on values of other arbitrary elements, in a way similar to [48]. We can also make a
further studies of non-Lie exact solutions of class (1) by means of the classification of singular
and regular reduction operators. Furthermore, the proposed classifications of local conservation
laws of class (1) can be used to find all possible inequivalent potential systems and potential
conservation laws (see [50] for detail) associated to the given system of differential equations.
These problems will be investigated in subsequent publication.

Acknowledgements

D.j. Huang express his sincerely thanks to professor Nataliya Ivanova for stimulating discussions
and correction of the results of this paper. This work was partially supported by the National
Key Basic Research Project of China under Grant No. 2010CB126600, the National Natural
Science Foundation of China under Grant No. 60873070, Shanghai Leading Academic Discipline
Project No. B114, the Postdoctoral Science Foundation of China under Grant No. 20090450067,
Shanghai Postdoctoral Science Foundation under Grant No. 09R21410600 and the Fundamental
Research Funds for the Central Universities under Grant No. WM0911004.

References

[1] Ablowitz M J, Clarkson P A, Solitons, nonlinear evolution equations and inverse scattering, Cambridge,
Cambridge University Press, 1991.

[2] Abramenko A.A., Lagno V.I., Samoilenko A.M., Group classification of nonlinear evolution equations. II.
Invariance under solvable local transformation groups, Differ. Equ. 2002, V.38, 502-509.

[3] Akhatov I.Sh., Gazizov R.K. and Ibragimov N.Kh., Group classification of equation of nonlinear filtration
Dokl. AN SSSR, 1987, V.293, 1033–1035.

[4] Akhatov I.Sh., Gazizov R.K., Ibragimov N.Kh., Nonlocal symmetries. A heuristic approach, Itogi Nauki
i Tekhniki, Current problems in mathematics. Newest results, 1989, V.34, 3–83 (Russian, translated in
J. Soviet Math., 1991, V.55, 1401–1450).

[5] Ames W.F., Nonlinear partial differential equations in engineering, V.1, New York, Academic, 1965, V.2,
New York, Academic, 1972.

[6] Ames W.F., Adams E. and Lohner R.J., Group properties of \( u_{tt} = [f(u)u_x]_x \), Int. J. Non-Linear Mech.,
1981, V.16, 439–447.

[7] Arrigo D.J., Group properties of \( u_{xx} - u_{yy}^m u_{yy} = f(u) \), Int. J. Non-Linear Mech., 1991, V.26, 619–629.

37
[8] Barone A., Esposito F., Magee C.G. and Scott A.C., Theory and applications of the sine-Gordon equation, Riv. Nuovo Cimento, 1971, V.1, 227–267.

[9] Basarab-Horwath P, Lahno V.I., Zhdanov R.Z., The structure of Lie algebras and the classification problem for partial differential equations, Acta Applicandae Mathematicae, 2001, V.69, 43–94.

[10] Bluman G. and Anco S.C., Symmetry and integration methods for differential equations, Applied Mathematical Sciences, V.154, New-York, Springer-Verlag, 2002.

[11] Bluman G., Cheviakov A.F. and Ivanova N.M., Framework for nonlocally related PDE systems and nonlocal symmetries: Extension, simplification, and examples, J. Math. Phys., 2006, V.47, 113505.

[12] Bluman G., Cole J.D., The general similarity solution of the heat equation, J. Math. Mech., 1969, V.18, 1025-1042.

[13] Bluman G., Kumei S., Symmetries and Differential Equations, Springer, New York, 1989.

[14] Bluman G. and Temuerchaolu, Comparing symmetries and conservation laws of nonlinear telegraph equations, J. Math. Phys., 2005, V.46, 073513.

[15] Bluman G. and Temuerchaolu, Conservation laws for nonlinear telegraph equations, J. Math. Anal. Appl., 2005, V.310, 459–476.

[16] Bluman G., Temuerchaolu and Sahadevan R., Local and nonlocal symmetries for nonlinear telegraph equation, J. Math. Phys., 2005, V.46, 023505.

[17] Boyko V.M. and Popovych V.O., Group classification of Galilei-invariant higher-orders equations, Proceedings of Institute of Mathematics of NAS of Ukraine, 2001, V.36, 45–50.

[18] Cattaneo C., Sulla conduzione del calore, Atti Sem. Mat. Fis. Univ. Modena, 1948, V.3, 83-101.

[19] Chikwendu S.C., Non-linear wave propagation solutions by Fourier transform perturbation, Int. J. Non-Linear Mech., 1981, V.16, 117–128.

[20] Clarkson P.A. and Mansfield E.L., Symmetry reductions and exact solutions of a class of nonlinear heat equations, Physica D, 1994, V.70, 250-288.

[21] Clarkson P.A. and Winternitz P., Nonclassical symmetry reductions for the Kadomtsev-Petviashvili equation, Physica D, 1991, V.49, 257–272.

[22] Cristescu N., Dynamic Plasticity, Northholland, Amsterdam (1967).

[23] Donato A., Similarity analysis and nonlinear wave propagation, Int. J. Non-Linear Mech., 1987, V.22, 307–314.

[24] Donato A. and Fusco D., Wave features and infinitesimal group analysis for a second order quasilinear equation in conservative form, Int. J. Non-Linear Mech., 1987, V.22, 37-46.

[25] Engelbrecht J., Nonlinear wave processes of deformation in solid, Pitman Publishing (1983).

[26] Faddeev L.D., Takhtajan L.A., Hamiltonian method in the theory of solitons, Berlin, Springer-Verlag, 1987.

[27] Foursov M.V. and Vorob’ev E.M., Solutions of the nonlinear wave equation \( u_{tt} = (u u_x)_x \) invariant under conditional symmetries, J. Phys. A: Math. Gen., 1996, V.29, 6363-6373.

[28] Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics, Dordrecht: Kluwer, (English transl.) 1993.

[29] Fushchych W. I., Shtelen W.M., Serov M.I. and Popovyč R.O., Q-conditional symmetry of the linear heat equation, Proc. Acad. Sci. Ukraine, 1992, no. 12, 28C33.

[30] Fushchych W.I., Tsyfra I.M., On a reduction and solutions of the nonlinear wave equations with broken symmetry, J. Phys. A: Math. Gen., 1987, V.20, L45–L48.

[31] Fushchych W.I. and Zhdanov R.Z., Conditional symmetry and reduction of partial differential equations, Ukr. Math. J., 1992, V.44, 970C982.
[32] Gandarias M.L., Torrisi M., and Valenti A., Symmetry classification and optimal systems of a non-linear wave equation, *Int. J. Non-Linear Mech.*, 2004, V.39, 389–398.

[33] Gagnon L., Winternitz P. Symmetry classes of variable coefficient nonlinear Schrödinger equations. *J. Phys. A: Math. Gen.*, 1993, V.26, 7061-7076.

[34] Gazeau J P, Winternitz P. Symmetries of variable coefficient Korteweg-de Vries equations. *J. Math. Phys.*, 1992, V.33, 4087-4102.

[35] Goenner H., Havas P., Exact solutions of the generalized LaneCEmden equation, *J. Math. Phys.*, 2000, V.41, 7029-7042.

[36] Grundland A.M. and Tafel J., On the existence of nonclassical symmetries of partial differential equations, *J. Math. Phys.*, 1995, V.36, 1426-1434.

[37] Huang D.J., Ivanova N M, Group analysis and exact solutions of a class of variable coefficient nonlinear telegraph equations, *J. Math. Phys.*, 2007, V.48, 073507. (23 pages)

[38] Huang D.J., Mei J.Q., Zhang H.Q., Group classification and exact solutions of a class of variable coefficient nonlinear wave equations, *Chin. Phys. Lett.*, 2009, V.26, 050202.

[39] Huang D.J., Zhou S.G., Group properties of generalized quasi-linear wave equations, *J. Math. Anal. Appl.*, 2010, V.366, 460-472.

[40] Huang D.J., and Zhang H.Q, Preliminary group classification of quasilinear third-order evolution equations, *Appl. Math. Mech.* 2009, V.30(3), 275-292.

[41] Huang Q., Lahno V., Qu C.Z. and Zhdanov R., Preliminary group classification of a class of fourth-order evolution equations, *J. Math. Phys.*, 2009, V.50, 023503.

[42] Huang Q., Qu C.Z., Zhdanov R., Nonlocal symmetries of fourth-order nonlinear evolution equations. [arXiv:0905.2033]

[43] Ibragimov N.H., Transformation groups applied to mathematical physics, *Mathematics and its Applications (Soviet Series)*, Dordrecht, D. Reidel Publishing Co., 1985.

[44] Ibragimov N.H. (Editor), Lie group analysis of differential equations — symmetries, exact solutions and conservation laws, V.1. Boca Raton, FL, CRC Press, 1994.

[45] Ibragimov N.H., Elementary Lie group analysis and ordinary differential equations, New York, Wiley, 1999.

[46] Ibragimov N.H., Torrisi M. and Valenti A., Preliminary group classification of equations $v_{tt} = f(x, v_x) v_{xx} + g(x, v_x)$, *J. Math. Phys.*, 1991, V.32, 2988–2995.

[47] Ivanova N.M., Popovych R.O. and Sophocleous C., Group analysis of variable coefficient diffusion-convection equations. I. Enhanced group classification, *Lobachevskii Journal of Mathematics*, 2010 V.31(2), 100-122. (arXiv:0710.273, [math-ph])

[48] Ivanova N.M., Popovych R.O. and Sophocleous C., Group analysis of variable coefficient diffusion-convection equations. II. Contractions and Exact Solutions. [arXiv:0710.3049] [math-ph].

[49] Ivanova N.M., Popovych R.O. and Sophocleous C., Group analysis of variable coefficient diffusion-convection equations. III. Conservation Laws. [arXiv:0710.3053] [math-ph].

[50] Ivanova N.M., Popovych R.O. and Sophocleous C., Group analysis of variable coefficient diffusion-convection equations. IV. Potential Symmetries. [arXiv:0710.4251] [math-ph].

[51] Ivanova N.M. and Sophocleous C., On the group classification of variable coefficient nonlinear diffusion–convection equations, *J. Comp. and Appl. Math.*, 2006, V.197, 322–344.

[52] Kingston J.G. and Sophocleous C., On form-preserving point transformations of partial differential equations, *J. Phys. A: Math. Gen.*, 1998, V.31, 1597–1619.

[53] Katayev I.G., Electromagnetic Shock Waves sIliffe, London, 1966.

[54] Keller J.B. and Ting Lu, Periodic vibations of systems governed by non-linear partial differential equations, *Comm. Pure Appl. Math.*, 1966, 19(4), 371–420.
[55] Kingston J.G. and Sophocleous C., Symmetries and form-preserving transformations of one-dimensional wave equations with dissipation, *Int. J. Non-Lin. Mech.*, 2001, V.36, 987–997.

[56] Kumei S., Invariance transformations, invariance group transformations and invariance groups of the sine-Gordon equations, *J. Math. Phys.*, 1975, V.16, 2461–2468.

[57] Kunzinger M. and Popovych R.O., Singular reduction operators in two dimensions, *J. Phys. A*, 2008, V.41, 505201, 24 pp., [arXiv:0808.3577](http://arxiv.org/abs/0808.3577).

[58] Kunzinger M. and Popovych R.O., Is a nonclassical symmetry a symmetry, Proceedings of 4th Workshop Group Analysis of Differential Equations and Integrability, 2009. (math-ph/0903.0821)

[59] Lahno V.I. and Zhdanov R.Z., Group classification of nonlinear wave equations, *J. Math. Phys.*, 2005, V.46, 053301.

[60] Lagno V.I., Samoilenko A.M., Group classification of nonlinear evolution equations. I. Invariance under semisimple local transformation groups, *Differ. Equ.* 2002, V.38, 384-391.

[61] Levi D. and Winternitz P., Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A: Math. Gen.* 22 (1989) 2915-2924.

[62] Lie S., On integration of a Class of Linear Partial Differential Equations by Means of Definite Integrals, *CRC Handbook of Lie Group Analysis of Differential Equations*, V.2, 473–508. (Translation by N.H. Ibragimov of Arch. for Math., Bd. VI, Heft 3, 328–308, Kristiania 1881).

[63] Lisle I.G., Equivalence transformations for classes of differential equations, Thesis, University of British Columbia, 1992 [http://www.isc.canberra.edu.au/mathstat/StaffPages/LisleDissertation.pdf](http://www.isc.canberra.edu.au/mathstat/StaffPages/LisleDissertation.pdf). (See also Lisle I.G. and Reid G.J., Symmetry classification using invariant moving frames, ORCCA Technical Report TR-00-08 (University of Western Ontario), [http://www.orcca.on.ca/TechReports/2000/TR-00-08.html](http://www.orcca.on.ca/TechReports/2000/TR-00-08.html)

[64] Meleshko S.V., Group classification of equations of two-dimensional gas motions, *Prikl. Mat. Mekh.*, 1994, V.58, 56–62 (in Russian) (translation in *J. Appl. Math. Mech.*, 58 (1994) 629–635).

[65] Nikitin A.G. and Popovych R.O., Group classification of nonlinear Schrödinger equations, *Ukr. Math. J.*, 2001 V.53, 1053–1060.

[66] Novikov S.P., Manakov S.V., Pitaevskii L.P., Zacharov V.E., Theory of Solitons, The Inverse Scattering Method, New York, Consultants Bureau, 1980.

[67] Nucci M.C., Nonclassical symmetries as special solutions of heir-equations, *J. Math. Anal. Appl.*, 2003, V.279, 168-179.

[68] Nucci M.C. and Clarkson P.A., The nonclassical method is more general than the direct method for symmetry reductions: an example of the Fitzhugh-Nagumo equation, *Phys. Lett. A*, 1992, V.164, 49-56.

[69] Nucci M.C. and Leach P.G.L., The determination of nonlocal symmetries by the technique of reduction of order, *J. Math. Anal. Appl.*, 2000, V.251, 871–884.

[70] Olver P.J., Application of Lie Groups to Differential Equations, New York, Springer-Verlag, 1986.

[71] Oron A. and Rosenau P., Some symmetries of the nonlinear heat and wave equations, *Phys. Lett. A*, 1986, V.118, 172–176.

[72] Ovsyannikov L.V., *Dokl. Akad. Nauk SSSR*, 1959, V.125, 592–595.

[73] Ovsiannikov L.V., Group analysis of differential equations, 1982, New York: Academic Press.

[74] Patera J. and Winternitz P., Subalgebras of real three- and four-dimensional Lie algebras *J. Math. Phys.*, 1977 V.18, 1449–1455.

[75] Polyanin A.D. and Zaitsev V.F., Handbook of exact solutions for ordinary differential equations, Chapman Hall/CRC, Boca Raton, 2003.

[76] Popovych R.O., Classification of admissible transformations of differential equations, *Collection of Works of Institute of Mathematics*, Kyiv, 2006, V.3, N 2, 239–254.
[77] Popovych R.O. and Cherniha R.M., Complete classification of Lie symmetries of systems of two-dimensional Laplace equations, *Proceedings of Institute of Mathematics of NAS of Ukraine*, 2001 V.36 212–221.

[78] Popovych R.O. and Ivanova N.M., New results on group classification of nonlinear diffusion-convection equations, *J. Phys. A: Math. Gen.*, 2004, V.37, 7547–7565 (math-ph/0306035).

[79] Popovych R.O. and Ivanova N.M., Hierarchy of conservation laws of diffusion–convection equations, *J. Math. Phys.*, 2005, V.46, 043502 (math-ph/0407008).

[80] Popovych R.O., Ivanova N.M. and Eshraghi H. Group classification of (1+1)-dimensional Schrödinger equations with potentials and power nonlinearities, *J. Math. Phys.*, 2004, V.45, 3049–3057 (math-ph/0311039).

[81] Popovych R.O., Kunzinger M. and Eshraghi H., Admissible point transformations and normalized classes of nonlinear Schrödinger equations, *Acta Appl. Math.*, 2010, V.109, 315-359, arXiv:math-ph/0611061

[82] Pucci E., Group analysis of the equation $u_{tt} + \lambda u_{xx} = g(u, u_x)$, *Riv. Mat. Univ. Parma*, 1987, V.12 N 4, 71–87.

[83] Pucci E. and Salvatori M.C., Group properties of a class of semilinear hyperbolic equations, *Int. J. Non-Linear Mech.*, 1986, V.21, 147–155.

[84] Rogers C., Ruggeri T., A reciprocal Bäcklund transformation: application to nonlinear hyperbolic boundary value problems, *Lett. Nuovo Cimento*, 1985, V.44, 289-296.

[85] Stephani H., Differential equation: their solution using symmetries, Cambridge, Cambridge University Press, 1994.

[86] Suhubi E.S. and Bakkaloglu A., Group properties and similarity solutions for a quasi-linear wave equation in the plane, *Int. J. Non-Linear Mech.*, 1991, V.26, 567–584.

[87] Torrisi M. and Valenti A., Group properties and invariant solutions for infinitesimal transformations of a nonlinear wave equation, *Int. J. Non-Linear Mech.*, 1985, V.20, 135–144.

[88] Torrisi M. and Valenti A., Group analysis and some solutions of a nonlinear wave equation, *Atti Sem. Mat. Fis. Univ. Modena*, 1990, V.XXXVIII, 445-458.

[89] Vaneeva O.O., Johnpillai A.G., Popovych R.O. and Sophocleous C., Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities, *J. Math. Anal. Appl.*, 2007, V.330, 1363-1386, arXiv:math-ph/0605081.

[90] Vaneeva O.O., Popovych R.O. and Sophocleous C., Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, *Acta Appl. Math.*, 2009, V.106, 1-46, arXiv:0708.3457

[91] Varley E., Seymour B., Exact solutions for large amplitude waves in dispersive and dissipative systems, *Studies in Appl. Math.*, 1985, V.72, 241-262.

[92] Vasilenko O.F. and Yehorchenko I.A., Group classification of multidimensional nonlinear wave equations *Proceedings of Institute of Mathematics of NAS of Ukraine*, 2001, V.36, 63–66.

[93] Zhdanov R.Z., Lahno V.I., Group classification of heat conductivity equations with a nonlinear source, *J. Phys. A: Math. Gen.*, 1999, V.32, 7405–7418.

[94] Zhdanov R.Z. and Lahno V.I., Group classification of the general evolution equation: Local and quasilocal symmetries, *SIGMA*, 2005, V.1, 009.

[95] Zhdanov R.Z. and Lahno V.I., Group classification of the general second-order evolution equation: semisimple invariance groups, *J. Phys. A: Math. Theor.*, 2007, V.40, 5083-5103.

[96] Zhdanov R.Z., Tayfra I.M., Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, 101-123. (arXiv:math-ph/0207023)
A  Note on classification with respect to the usual equivalence group

In tables [12][14] we list all possible $G^\sim$-inequivalent sets of functions $f(x)$, $g(x)$, $H(u)$, $K(u)$ and corresponding invariance algebras.

Table 12. Case of $\forall H(u)$

| N | $K(u)$ | $f(x)$ | $g(x)$ | Basis of $\Lambda^{\max}$ |
|---|---|---|---|---|
| 1 | $\forall$ | $\forall$ | $\forall$ | $\partial_t$ |
| 2 | $\forall$ | $|x|^\lambda$ | $x^{-1}$ | $\partial_t, \frac{1}{2}(\lambda + 2)t\partial_t + x\partial_x$ |
| 3 | $\forall$ | $1$ | $(\alpha + \beta x)^{-1}$ | $\partial_t, \partial_x, t\partial_t + x\partial_x, \partial_x, (\beta - 1)t\partial_t + \beta x\partial_x$ |
| 4 | $H + \beta$ | $(\alpha + \beta x)^{-\frac{2}{3}}$ | $(\alpha + \beta x)^{-1}$ | $\partial_t, \partial_x, (\beta - 1)t\partial_t + \beta x\partial_x$ |
| 5 | $H$ | $\forall$ | $-f'/2f + \alpha \sqrt{|f|}$ | $\partial_t, |f|^{-\frac{1}{2}}\partial_x$ |
| 6 | $H$ | $\forall$ | $-\frac{f'}{2f} + \alpha \sqrt{|f|}$ | $\partial_t, \frac{1}{2}t\partial_t + \frac{f \sqrt{|f|}}{2\sqrt{|f|}}\partial_x$ |
| 7 | $H$ | $\forall$ | $-\frac{f'}{2f}$ | $\partial_t, \frac{1}{2}t\partial_t + |\frac{1}{2}|f|^{-\frac{1}{2}}\int \frac{x|f|^3}{2f|f|}dx|\partial_x, |f|^{-\frac{1}{2}}\partial_x$ |
| 8 | 1 | 1 | 1 | $\partial_t, t\partial_t + x\partial_x$ |

Here $\lambda \neq 0 \mod G^\sim$, $\alpha, \beta \neq 0$.

Table 13. Case of $H(u) = e^{\mu u}$

| N | $\mu$ | $K(u)$ | $f(x)$ | $g(x)$ | Basis of $\Lambda^{\max}$ |
|---|---|---|---|---|---|
| 1 | 1 | $e^{\mu u}$ | $|x|^\mu$ | $|x|^\mu$ | $\partial_t, \frac{1}{2}(p - pv - 2\nu - q + 1)t\partial_t + (1 - \nu)x\partial_x + (q + 1)\partial_u$ |
| 2 | 1 | $e^{\mu u}$ | $e^{px}$ | $ee^{px}$ | $\partial_t, \frac{1}{2}(p - pv - q)t\partial_t + (1 - \nu)\partial_x + q\partial_u$ |
| 3 | 1 | 1 | $|x|^\mu$ | $|x|^\mu$ | $\partial_t, \frac{1}{2}(1 + p - q)t\partial_t + x\partial_x + (q + 1)\partial_u$ |
| 4 | 1 | 1 | $e^{px}$ | $ee^{px}$ | $\partial_t, \frac{1}{2}(p - q)t\partial_t + \partial_u + q\partial_u$ |
| 5 | 1 | $e^{\mu u} + h_1 e^u$ | $\alpha \sqrt{|f|} - \frac{f'}{2f}$ | $\frac{f'}{2f} \sqrt{|f|}$ | $\partial_t, \frac{1}{2}(1 - 2\nu)t\partial_t + (1 - \nu)\frac{f \sqrt{|f|}}{2\sqrt{|f|}}\partial_x + \partial_u$ |
| 6 | 1 | $e^{\mu u} + h_1 e^u$ | $\alpha \sqrt{|f|} - \frac{f'}{2f} \sqrt{|f|}$ | $\frac{f'}{2f} \sqrt{|f|}$ | $\partial_t, \frac{1}{2}(1 - 2\nu)t\partial_t + (1 - \nu)\frac{f \sqrt{|f|}}{2\sqrt{|f|}}\partial_x + \partial_u$ |
| 7 | 1 | $ue^{\mu u} + h_1 e^u$ | $\frac{f'}{2f} = -2(g^{-1})_x g$ | $(g^{-1})_x = \beta g$ | $\partial_t, -\frac{1}{2}\beta t\partial_t + g^{-1}\partial_x + \beta\partial_u$ |
| 8 | 1 | $e^u$ | $\forall$ | $\forall$ | $\partial_t, \frac{1}{2}t\partial_t - \partial_u$ |
| 9 | 1 | $e^u$ | $f^1(x)$ | $g^1(x)$ | $\tau = c_1 + \frac{1}{2}c_2 t, \xi = \xi^1(x), \eta = -c_2 + (\frac{1}{3} - \frac{1}{2}f)\xi^1(x)$ |
| 10 | 1 | $e^u$ | $f^2(x)$ | $g^2(x)$ | $\tau = c_1 + \frac{1}{2}c_2 t, \xi = \xi^2(x), \eta = -c_2 + 2c_1 + (\frac{1}{3} - \frac{1}{2}g^2)\xi^2(x)$ |

Here $\epsilon = \pm 1$ and $q, \alpha, \beta \neq 0$. $f^1(x), g^1(x)$ and $\xi^1(x)$ satisfy the relation

$$\Phi_{xx} + (2\Phi - \Psi)\Phi_x + \Psi_{xx} + (\Phi - 2\Psi)\Psi_x - \Phi\Psi(\Phi + \Psi) = 0, \quad \xi^1_x - \Phi_{x}\xi^1 = 0,$$

where

$$\Phi = \frac{1}{3} (2f^1_x - g^2), \quad \Psi = (\frac{f^1}{3f^1} - g^1);$$

and $f^2(x), g^2(x)$ and $\xi^2(x)$ satisfy the relation

$$\Theta_x + \Lambda_x - \Theta\Lambda - \Lambda^2 = 0, \quad \xi^2_x - \Theta\xi^2 = c_3,$$

where

$$\Theta = \frac{1}{3} (2f^2_x + g^2), \quad \Lambda = (\frac{f^2}{3f^2} - g^2).$$

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The operators from tables 12-14 form bases of the maximal invariance algebras if the corresponding sets of the functions $f$, $g$, $H$, $K$ are $G^*$-inequivalent to ones with most extensive invariance algebras. For example, in case $\mu \neq 0$, $\nu \neq 0$. Similarly, in case $\mu = 0$ the constraint set on the parameters $\mu, \nu$ and $\lambda$ coincides with the one for case $\mu \neq 0$, and $\lambda = 0$ if $\nu = 0$.

Table 14. Case of $H(u) = u^\mu$

| N  | $\mu$ | $K(u)$ | $f(x)$ | $g(x)$ | Basis of $\Lambda^{\text{max}}$ |
|----|-------|--------|--------|--------|---------------------|
| 1  | $\forall$ | $u^\mu$ | $|x|^\mu$ | $|x|^\mu$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 2  | $\forall$ | $u^\mu$ | $e^{\nu x}$ | $e^{\nu x}$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 3  | $\forall$ | $u^\mu + h_1 u^\mu$ | $\alpha \sqrt{|f|} - \frac{f'}{f}$ | $\alpha \sqrt{|f|} - \frac{f'}{f}$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 4  | $\forall$ | $u^\mu + h_1 u^\mu$ | $\alpha \sqrt{|f|} - \frac{f'}{f}$ | $\alpha \sqrt{|f|} - \frac{f'}{f}$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 5  | $\forall$ | 1 | $|x|^\nu$ | $|x|^\nu$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 6  | $\forall$ | 1 | $e^{\nu x}$ | $e^{\nu x}$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 7  | $\forall$ | $u^\mu \ln u + h_1 u^\mu$ | $\frac{f'}{f} = (g^{-1})'g$ | $(g^{-1})'g = \beta g$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 8  | $\forall$ | $u^\mu$ | $|x|^\mu$ | $|x|^\mu$ | $\partial_t, \frac{1}{2}(p+1)\partial_t + (p+1)u\partial_u$ |
| 9  | $\neq \frac{4}{3}$ | $u^\mu$ | $f^3(x)$ | $g^3(x)$ | $\tau = \frac{1}{2}(p+1)\partial_t + (p+1)u\partial_u$ |
| 10 | $\neq \frac{4}{3}$ | $f^4(x)$ | $g^4(x)$ | $\tau = \frac{1}{2}(p+1)\partial_t + (p+1)u\partial_u$ |
| 11 | $\neq \frac{4}{3}$ | $f^5(x)$ | $g^5(x)$ | $\tau = \frac{1}{2}(p+1)\partial_t + (p+1)u\partial_u$ |
| 12 | $\frac{4}{3}$ | $u^{-\frac{4}{3}}$ | $-\frac{f'}{f}$ | $\tau = \frac{1}{2}(p+1)\partial_t + (p+1)u\partial_u$ |
| 13 | 0 | $u^\mu$ | $|x|^\mu$ | $|x|^\mu$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 14 | 0 | $u^\mu$ | $e^{\nu x}$ | $e^{\nu x}$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 15 | 0 | $\ln u + h_0$ | $g^2 \exp (\int gdx)$ | $(g^{-1})'' = \beta g$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 16 | 0 | $g^2 \exp (\int gdx)$ | $(g^{-1})'' = \beta g$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 17 | 0 | $e^{\nu x} + h_0$ | $g^2 \exp (\int gdx)p$ | $(g^{-1})'' = \beta g$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |
| 18 | 0 | $u^\mu + h_0$ | $g^2 \exp (\int gdx)p$ | $(g^{-1})'' = \beta g$ | $\partial_t, \frac{1}{2}(p-q)\partial_t + (p-q+1)u\partial_u$ |

Here $\epsilon = \pm 1$ and $q, \alpha, \beta, h_0, h_1 \neq 0$. $f^3(x), g^3(x)$ and $\xi^4(x)$ satisfy the relation

$$4\mu\Phi_{3xx} + [(16\mu + 32)\Phi_3 + (7\mu + 8)\Psi_3] \Phi_{3xx} + \mu\Psi_{3xx} - [(5\mu + 8)\Phi_3 + (2\mu + 2)\Psi_3] \Psi_{3xx} + (24\mu + 32)\Phi_3^3 + (14\mu + 16)\Phi_3^2 \Psi_3 + (2\mu + 2)\Phi_3 \Psi_3^2 = 0, \quad \xi_3 = \mu\Phi_3 \xi_3 = 0,$$

where

$$\Phi_3 = -\frac{\mu}{3\mu + 4} g^3 - \frac{2(p+1)\int g^3}{(3\mu + 4) f^3}; \quad \Psi_3 = \frac{\mu}{3\mu + 4} g^3 + \frac{5\mu + 6}{(3\mu + 4) f^3};$$

$f^4(x), g^4(x)$ and $\xi^4(x)$ satisfy the relation

$$4\mu\Phi_{4xx} + \mu\Psi_{4xx} - (12\mu + 16)\Phi_3^2 - (7\mu + 8)\Phi_4 \Psi_4 - (\mu + 1)\Phi_4^2 = 0, \quad \xi_4 = \mu\Phi_4 \xi_4 = c_3,$$

where

$$\Phi_4 = -\frac{1}{3\mu + 4} g^3 - \frac{2(p+1)\int g^3}{(3\mu + 4) f^3}; \quad \Psi_4 = \frac{1}{3\mu + 4} g^3 + \frac{5\mu + 6}{(3\mu + 4) f^3};$$

$f^5(x), g^5(x)$ satisfy the relation

$$\Phi_5^2 \Phi_{5xx} - 6\Phi_5 \Phi_{5xx} + 6\Phi_{5xx}^2 = (g^2)^2 \Phi_5 \Phi_{5xx} + \Phi_5 \Phi_{5xx}^2 - 2\Phi_5^2 \Phi_{5xx} + \Phi_5^2 \Phi_{5xx}^2 = 0,$$

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where
\[ \Phi_5 = \frac{f_5^2}{f} + 2g^5; \]
and \( \xi^6(x) \) satisfies third order ordinary differential equation
\[ \xi^{6}_{xxx} - \left[ \left( \frac{f_5}{2f} \right)^2 - 2\left( \frac{f_5}{2f} \right)_x \right] \xi^6 + \left[ \left( \frac{f_5}{2f} \right)_{xx} + \left( \frac{f_5}{2f} \right)_x \right] \xi^6 = 0. \]