A densest compact planar packing
with two sizes of discs

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Abstract
We consider packings of the plane using discs of radius 1 and $r = 0.545151 \ldots$. The value of $r$ admits compact packings in which each hole in the packing is formed by three discs which are tangent to each other. We prove that the largest density possible is that of the compact packing shown in figure 1.
1 Introduction

We consider the following packing question in two dimensions. Fix a number $r < 1$. Using discs of radius 1 and $r$, what is the densest packing of the plane? We do not impose any constraint on the relative number of discs of the two sizes.

It was proved long ago that the densest packing of the plane using discs of equal radii is to put the centers of the discs on a triangular lattice. [20, 21]. (The Voronoi cells of this packing are hexagons, and this packing is often referred to as hexagonal or honeycomb.) The density of this triangular packing is $\pi/\sqrt{12}$. L. Fejes Tóth observed that if $r$ is slightly less than 1, then one cannot do any better than this packing density [8]. The interval in which it has been proved that the highest packing density is $\pi/\sqrt{12}$ was increased to $[0.906 \cdots, 1]$ by Florian [11]. Using an an idea of Boroczky [Bo], Blind [2, 3] and G. Fejes Tóth [5] independently extended it to $[0.742 \cdots, 1]$.

For smaller values of $r$ there is a rich variety of packings with densities greater than $\pi/\sqrt{12}$. (A survey of the best known packings as a function of $r$ may be found at [17].) However, the densest packing has been rigorously established only for six particular values of $r$. All six of these values of $r$ allow compact packings. A packing is said to be compact if each disc is surrounded by a ring of discs, all of which are tangent to the disc at the center. Furthermore, each disc in the ring is tangent to the two discs in the ring which are adjacent to it in the cyclic order. Heppes has proved that for six values of $r$ which allow compact packings, the largest density is attained by a particular compact packing. It has been shown that there are only nine values of $r$ that admit compact packings [16].

In this paper we consider one of the values of $r$ which admits compact packings but for which it has not been shown that a compact packing attains the largest density. The value is $r = 0.545151042 \cdots$. The exact $r$ is a root of

$$r^8 - 8r^7 - 44r^6 - 232r^5 - 482r^4 - 24r^3 + 388r^2 - 120r + 9 = 0$$

(1)

(This equation is derived in appendix C.) In the remainder of this paper $r$ will denote this particular radius. In this paper we prove that the largest packing density possible using discs of radius 1 and $r$ is that attained by the compact packing in figure 1. It has a packing density of $\delta = 0.911627478 \cdots$.

The strategy of our proof comes from a technique in classical statistical mechanics known as “m-potentials” [19]. Our strategy is similar to Heppes “cell balancing” [13]. In statistical mechanics m-potentials were introduced to deal with frustrated spin systems. In spin systems the Hamiltonian or energy function is typically a sum over translates of a local energy function. A ground state of such a system is a configuration of spins which minimizes the total energy function. One can also ask what is the minimum of a single local energy function. This may be less than the value of the local energy function in the ground state. When this happens the system is said to be frustrated. What one would like to do locally to minimize the energy cannot be done globally to simultaneously minimize all the local energy functions.
Figure 1: A densest packing with $r = 0.545\ldots$. The quadrilateral shows a unit cell for the packing.

In classical spin systems an m-potential is a local function on the spin configurations with the property that when it is summed over all translates the result is just the zero function. Thus one can add this local function to the local energy function and obtain a total energy function with the same ground states. In some problems by carefully choosing the m-potential one can obtain a system which is not frustrated. (We should note that an important class of frustrated spin systems comes from disordered systems such as spin glasses, but m-potentials have not been useful in this context)

The disc packing problem is also frustrated. Given just three discs the densest packing is to have them all touch one another. For most values of $r$ it is not possible to find a packing in which the packing locally always consists of three discs touching each other. Even when it is possible (as it is for our value of $r$), different choices for the three discs that are tangent will give different densities for the triangle formed, and it is not possible to find a packing that only uses the densest triangle. By analogy with the m-potentials, we will introduce a function on packings that is a sum over all triangles of a function of three discs. We will refer to it as a “localizing potential” since its purpose is to reduce the global problem of finding the best packing to a local problem involving only three discs. Note that for the spin problem, if we are only interested in the ground states then it suffices that the sum of the local m-potentials be non-negative. Likewise, in the packing problem it suffices that the sum of the localizing potentials be non-negative.
In this paper we only allow discs of two sizes. An interesting and presumably more complicated question is what is the largest packing density if we allow discs with any radius in \([r, 1]\). Another interesting question is what is the densest packing if we add the constraint that the ratio of the number of one type of disc to the other must converge to a given value in the limit of packing the entire plane. This question was studied non-rigorously in [15].

In the next section we explain the method of localizing potentials in detail. Our localizing potential is the sum of two parts, a “vertex localizing potential” and an “edge localizing potential.” In section 3 we define the vertex localizing potentials used in the proof of our result and show that the sum of our vertex localizing potentials is non-negative. We define the edge localizing potentials and show their sum is non-negative in section 4. Sections 5 and 6 are devoted to studying the local problem involving just three discs that comes from adding our localizing potential to the original packing problem.

## 2 Localizing potentials

Consider the centers of the discs in a large packing. The Delaunay decomposition gives a triangulation in which the vertices of the triangles are the centers of the discs. We denote the triangles by \(T_i\). Let \(A(T)\) be the area of triangle \(T\). Given a triangle \(T\), let \(\phi_0, \phi_1, \phi_2\) be the angles in \(T\) and \(r_i\) the radii of the discs at the corresponding vertices. We define

\[
D(T) = \sum_{i=0}^{2} \frac{1}{2} \phi_i r_i^2
\]

As long as the triangle is not too “flat”, \(D(T)\) is the area of the intersection of the triangle with the three discs. Even when the triangulation contains triangles for which this is not true, the sum \(\sum_i D(T_i)\) will be (up to boundary effects) the total area of the discs in the packing since the sum of the angles \(\phi\) around a vertex is always \(2\pi\). Up to boundary effects the packing density is

\[
\frac{\sum_i D(T_i)}{\sum_i A(T_i)}
\]

So we want to prove this ratio is no greater than \(\delta\). This is equivalent to

\[
\sum_i (\delta A(T_i) - D(T_i)) \geq 0
\]

We define

\[
E(T) = \delta A(T) - D(T)
\]

Heppes defines the “surplus area” to be \(A(T) - D(T)/\delta\). We refer to the quantity \(E(T)\) as the “excess” of the triangle. If it were nonnegative for every triangle we would be done. This happens to be true when one considers packings with discs of a single radius, but
it is not true in our problem. In an optimal packing, triangles have both positive and negative excess, but the sum over all triangles of the excess is zero. We must prove that for any packing the sum over the triangles of the excess is non-negative.

We want to define a function $F(T)$ on triangles with the following two properties. First, for any Delaunay decomposition we require

$$\sum_i F(T_i) \geq 0 \quad (6)$$

Second, for any triangle that can occur in a Delaunay decomposition, we require

$$E(T) - F(T) \geq 0 \quad (7)$$

If we can do this, then we are done:

$$\sum_i E(T_i) \geq \sum_i (E(T_i) - F(T_i)) \geq 0 \quad (8)$$

We refer to $F$ as a localizing potential since it reduces proving the global inequality (3) to proving the local one (7). We will prove the following theorem.

**Theorem 1** For the particular value of $r$ we are considering ($r \approx 0.545151$), there is a localizing potential $F(T)$ which satisfies inequalities (6) and (7) with $\delta$ equal to the density of the packing shown in figure 1. Thus the density of a packing consisting of discs of radius 1 and $r$ is at most $\delta$.

We parameterize triangles by their edge lengths and the radii of the discs at these vertices. We label the vertices $i = 0, 1, 2$ and we denote the length of the edge opposite vertex $i$ by $x_i$. The radius of the disc at vertex $i$ is $r_i$. The excess is then written as $E(x_0, x_1, x_2, r_0, r_1, r_2)$, and the localizing potential as $F(x_0, x_1, x_2, r_0, r_1, r_2)$. $F$ will be the sum over the three vertices of the triangle of a vertex localizing potential plus the sum over the three edges of an edge localizing potential. The vertex potential is based on the constraint that the sum of the angles around a vertex is $2\pi$. The edge potential is based on a constraint involving the signed distance from the edge to the center of the circle which circumscribes the triangle.

Consider the vertex potential $v$ for vertex 0. It depends on $x_0, x_1, x_2$ only through the angle of the triangle at vertex 0, which we denote by $\phi_0$. So we write the vertex potential as $v(\phi_0, r_0, r_1, r_2)$. The vertex potentials for the other two vertices are $v(\phi_1, r_1, r_0, r_2)$ for vertex 1 and $v(\phi_2, r_2, r_0, r_1)$ for vertex 2. We will always take $v(\phi_0, r_0, r_1, r_2)$ to be symmetric under the interchange of its last two arguments.

Now consider the edge localizing potential for the edge opposite vertex 0. It will be a function of the edge lengths $x_0, x_1, x_2$ and the radii of the discs at the endpoints of the
edge, i.e., \( r_1, r_2 \). We write it as \( e(x_0, x_1, x_2, r_1, r_2) \). It is symmetric under the simultaneous interchange of \( r_1 \) with \( r_2 \) and \( x_1 \) with \( x_2 \). The total localizing potential for our triangle is

\[
F(x_0, x_1, x_2, r_0, r_1, r_2) = v(\theta_0, r, r_1, r_2) + v(\theta_1, r_1, r, r_2) + v(\theta_2, r_2, r_0, r_1) + e(x_0, x_1, x_2, r_1, r_2) + e(x_0, x_1, x_2, r_2, r_0) + e(x_2, x_0, x_1, r_0, r_1)
\] (9)

Note that each edge in the packing has two edge localizing potentials associated with it, while each vertex has \( n \) vertex localizing potentials associated with it where \( n \) is the number of triangles in the packing that contain the vertex. Thus the sum of \( F(T) \) over all triangles can be written as the sum of the following two sums. The first is the sum over edges of the sum of the two edge localizing potentials associated with the edge. The second is the sum over vertices of the sum of the \( n \) vertex localizing potentials associated with the vertex. \((n \text{ is vertex dependent.})\) Thus to prove (6), it suffices to prove the following two conditions. First, for every disc center in the packing we have

\[
\sum_{i=1}^{n} v(\theta_i, r, r_i, r_{i+1}) \geq 0
\] (10)

where \( n \geq 3 \) is the number of triangles that have the disc center as a vertex, \( r \) is the radius of the disc, and \( r_1, r_2, \cdots, r_n \) are the radii of the discs that surround it. These are ordered in the natural way, so that one triangle has discs of radii \( r, r_1, r_2 \), the next has radii of \( r, r_2, r_3 \) and so on. \( r_{n+1} \) is defined to be \( r_1 \). \( \theta_1, \theta_2, \cdots, \theta_n \) are the angles of these triangles at the vertex. Second, for every edge in the packing we have

\[
e(x_0, x_1, x_2, r_1, r_2) + e(x_0, x_1, x_2, r_2, r_1) \geq 0
\] (11)

Here the length of the edge is \( x_0 \), and \( r_1, r_2 \) are the radii of the two discs at its endpoints. \( x_1, x_2 \) are the lengths of the other two edges in one triangle and \( x'_1, x'_2 \) are the lengths of the other two edges in the other triangle. The edge potential \( e(x_0, x_1, x_2, r_1, r_2) \) will depend on the signed distance from the “center” of the triangle to the edge opposite vertex 0. The center of the triangle is the point equidistant from the three vertices. The signed distance is positive when the center lies on the same side of the edge as the vertex opposite the edge. For two triangles in a Delaunay decomposition that share an edge, there is a constraint on the two signed distances to this common edge that will be the basis for proving (11). See section 4.

We end this section with a discussion of why we use the Delaunay triangulation rather than the FM triangulation that was used by Heppes in his proof of the optimality of six other compact packings. For most triangles the excess is positive. As we will see in the next section, it can be negative for triangles that are close to some of the triangles that appear in the compact packing of figure 1. The only other triangles with negative excess are relatively flat triangles, i.e., triangles with a large obtuse angle. The FM triangulation has the nice property that each disc is covered by the triangles that have a vertex at the
center of the disc, and so triangles cannot be too flat. However, an FM triangulation can still contain triangles which are flat enough that their excess is slightly negative. The Delaunay triangulation can contain triangles that are even flatter and so have an even more negative excess. But in the Delaunay triangulation it is possible to define an edge localizing potential that takes care of these flat triangles. This is explained in section 4.

3 The vertex localizing potentials

We begin the proof of the theorem by considering what the values of the localizing potentials must be for the triangles that appear in our densest packing. There are four triangles that appear in this packing. We will refer to the triangle that has two large discs and one small disc as the alpha triangle. It has one side of length 2 and two of length $1 + r$. We denote the angle opposite the side of length 2 by $\alpha$ and the other two angles by $\alpha'$. For this triangle the excess is negative,

$$E_\alpha \approx -0.0022743457 \quad (12)$$

We will refer to the triangle that has one large disc and two small discs as the beta triangle. It has one side of length $2r$ and two of length $1 + r$. The angle opposite the side of length $2r$ will be called $\beta$, and the other two are $\beta'$. This triangle also has negative excess.

$$E_\beta \approx -0.0017217279 \quad (13)$$

The triangle with three small discs will be called the small equilateral triangle. Its sides are of length $2r$, and its excess is positive. We denote it by

$$E_S \approx 0.0024336170 \quad (14)$$

The fourth triangle has three large discs and will be called the large equilateral triangle. Its sides are of length 2, and its excess is also positive. We denote it by

$$E_L \approx 0.0081887688 \quad (15)$$

The localizing potential must be defined so that it equals the excess for each of the four triangles. The edge localizing potential is zero for all four of the triangles. So we have the following four conditions.

$$v(\alpha, r, 1, 1) + 2v(\alpha', 1, r, 1) = E_\alpha \quad (16)$$

$$v(\beta, 1, r, r) + 2v(\beta', r, r, 1) = E_\beta \quad (17)$$

$$3v(\pi/3, r, r, r) = E_S \quad (18)$$

$$3v(\pi/3, 1, 1, 1) = E_L \quad (19)$$
The localizing potential condition (10) for a small disc requires
\[ 2v(\alpha, r, 1, 1) + 2v(\beta', r, 1, r) + v(\pi/3, r, r, r) = 0 \] (20)
and for a large disc it requires
\[ v(\beta, 1, r, r) + 4v(\alpha', 1, r, 1) + 2v(\pi/3, 1, 1, 1) = 0 \] (21)

We have found six conditions above, but these six conditions are not linearly independent. In the packing shown in figure 11 the unit cell has 6 alpha triangles, 3 beta triangles, 1 small equilateral triangle and 2 large equilateral triangles. The sum of the excesses of the triangles in a unit cell must be zero, so
\[ 6E_\alpha + 3E_\beta + E_S + 2E_L = 0 \] (22)
This implies that the above six conditions are equivalent to five linearly independent conditions.

We solve these equations by first introducing two parameters \( x \) and \( y \). We set
\[
\begin{align*}
  v(\alpha, r, 1, 1) &= xE_\alpha \\
  v(\alpha', 1, r, 1) &= \frac{1}{2}(1 - x)E_\alpha \\
  v(\beta, 1, r, r) &= yE_\beta \\
  v(\beta', r, r, 1) &= \frac{1}{2}(1 - y)E_\beta \\
  v(\pi/3, r, r, r) &= \frac{1}{3}E_S \\
  v(\pi/3, 1, 1, 1) &= \frac{1}{3}E_L
\end{align*}
\] (23)
The conditions (20) and (21) are equivalent conditions on \( x \) and \( y \). They give
\[ x2E_\alpha - yE_\beta = 2E_\alpha + \frac{2}{3}E_L = -E_\beta - \frac{1}{3}E_S \] (24)
We take \( x = 0 \), and then \( y \) is determined by the above equation. We will explain the motivation for this choice at the end of this section.

We let \( \phi_0(r_0, r_1, r_2) \) denote the angle \( \phi_0 \) in the triangle with discs of radius \( r_0, r_1, r_2 \). So it equals \( \alpha, \alpha', \beta, \beta' \) or \( \pi/3 \). Then the vertex localizing potential is
\[ v(\phi, r_0, r_1, r_2) = v(\phi_0(r_0, r_1, r_2), r_0, r_1, r_2) + m|\phi - \phi_0(r_0, r_1, r_2)| \] (25)
where \( v(\phi_0(r_0, r_1, r_2), r_0, r_1, r_2) \) is given by the above equations. So equations (23) are minima of the localizing potential, and the potential increases with slope \( \pm m \) as we move away from these minima. Note that we use the same slope, \( m \), for all the localizing potentials. We will take \( m = 0.12 \).
Next we prove that the vertex localizing potential we have defined satisfies (10). We consider a disc $D$ and must show
\[ \sum_{i=1}^{n} v(\theta_i, \rho, r_i, r_{i+1}) \geq 0 \quad (26) \]
Here $n \geq 3$ is the number of triangles with a vertex at the center of $D$, $\rho$ is the radius of $D$, and $r_1, r_2, \ldots, r_n$ are the radii of the discs that surround $D$. The angles $\theta_1, \theta_2, \ldots, \theta_n$ are the angles of these triangles at $D$. Of course, the sum of the angles around a vertex is $2\pi$. So
\[ \sum_{i=1}^{n} \theta_i = 2\pi \quad (27) \]
For $\rho = r$, let $n_{rr}$ be the number of $i$ with $r_i = r_{i+1} = r$. Let $n_{r1}$ be the number of $i$ with $r_i = r_{i+1} = 1$. And let $n_{1r}$ be the number of $i$ with one of $r_i, r_{i+1}$ equal to $r$ and one equal to 1. (Keep in mind the symmetry $v(\phi, r_0, r_1, r_2) = v(\phi, r_0, r_2, r_1)$.)
Then to prove inequality (26) for $\rho = r$ it suffices to prove
\[ n_{rr} \frac{E_s}{3} + n_{r1} \frac{1-y}{2} E_\beta + n_{r11} x E_\alpha + m|2\pi - n_{rr} \frac{\pi}{3} - n_{r1} \beta' - n_{r11} \alpha| \geq 0 \quad (28) \]
And to prove the inequality for $\rho = 1$ it suffices to show
\[ n_{1rr} y E_\beta + n_{1r1} \frac{1-x}{2} E_\alpha + n_{111} \frac{E_L}{3} + m|2\pi - n_{1rr} \beta - n_{1r1} \alpha' - n_{111} \frac{\pi}{3}| \geq 0 \quad (29) \]
where $n_{1rr}, n_{1r1}$ and $n_{111}$ are defined in the obvious way. Each choice of $n_{rr}, n_{r1}, n_{r11}$ gives a lower bound on $m$, as does each choice of $n_{1rr}, n_{1r1}, n_{111}$. Note that for $n_{rr} = 1, n_{r1} = 2, n_{r11} = 2$, the left side of (28) is zero, and for $n_{rr} = 1, n_{1r1} = 4, n_{111} = 2$, the left side of (29) is zero. These are the cases which occur in the optimal packing. The largest lower bound on $m$ that we find from the other cases is when $n_{1rr} = 4, n_{1r1} = 4, n_{111} = 0$. This case implies
\[ m \geq 0.1185912 \quad (30) \]
We take
\[ m = 0.12 \quad (31) \]
The above computation is the motivation for the choice of $x = 0$. Other choices force $m$ to be larger.
Finally, we make a modification to definition (25). We define $v(\phi, r_0, r_1, r_2)$ to be given by the above definition provided the value is less than 0.1. Otherwise we define it to be 0.1. Proving (10) for this modified function is easy. The most negative value of $v(\phi, r_0, r_1, r_2)$ is $E_\alpha \approx -0.0022743457$. So if one or more of the $v$ in (10) is 0.1, then (10) is trivially satisfied. If all the $v$ in (10) are given by (25), then the previous proof applies.
4 The edge localizing potentials

In this section we define the edge localizing potential and show that its sum over the triangles is non-negative. The edge localizing potential is only nonzero for relatively flat triangles. Such triangles can have negative excess, but in the Delaunay triangulation they will be adjacent to a triangle with positive excess. The edge localizing potential exploits this fact.

We will refer to the point equidistant to the three vertices of a triangle as the “center” of the triangle. Note that it need not lie inside the triangle. The edge potential \( e(x_0, x_1, x_2, r_1, r_2) \) will depend on the signed distance from the center of the triangle to the edge opposite vertex 0. We let \( d(x_0, x_1, x_2) \) denote this signed distance. We define it to be positive if the center lies on the same side of the edge as vertex 0, and negative if they lie on opposite sides. We give the formula for this signed distance in appendix A. We note that the signed distance for an edge is negative when the angle opposite the edge is obtuse.

Throughout this section we will consider two triangles that share an edge. The length of this common edge is \( x_0 \), and \( r_1, r_2 \) are the radii of the two discs centered at its endpoints. \( x_1, x_2 \) are the lengths of the other two edges in one triangle and \( x'_1, x'_2 \) are the lengths of the other two edges in the other triangle.

Now consider \( d(x_0, x_1, x_2) \) and \( d(x_0, x'_1, x'_2) \), the signed distances from the centers of the two triangles to their common edge. We claim that if the two triangles come from a Delaunay decomposition, then

\[
d(x_0, x_1, x_2) + d(x_0, x'_1, x'_2) \geq 0 \tag{32}
\]

This inequality is obviously not true for arbitrary triangles that share an edge. It says that if the signed distance from the center of a triangle to an edge is negative, then the signed distance from the center of the other triangle that shares this edge to the edge must be positive and greater in absolute value. Given this inequality we can take any function \( f(d, x_0, r_1, r_2) \) which is an increasing and odd function of \( d \), and let

\[
e(x_0, x_1, x_2, r_1, r_2) = f(d(x_0, x_1, x_2), x_0, r_1, r_2) \tag{33}
\]

Then (32) implies (31).

To prove (32) we use figure 2. We have drawn the two triangles so their common edge is vertical, and we have shown the bisector of this common edge with a dashed line. The two centers are each equidistant from the endpoints of this common edge. So both centers lie on the dashed line. Inequality (32) is equivalent to \( C' \) being to the right of \( C \) (or equal to \( C \)). By definition the three vertices of a triangle are equidistant to the center of the triangle. The Delaunay decomposition has the property that no other vertex of a triangle is closer to the center than these three vertices. In particular, \( C \) is closer to \( P \) than to \( P' \), and \( C' \) is closer to \( P' \) than to \( P \). It follows that \( C' \) is to the right of \( C \) (or equal to it).
Figure 2: \( C \) and \( C' \) are the centers of the left and right triangles respectively. In this example the signed distances from \( C \) and \( C' \) to the common edge are negative and positive, respectively.

A fairly simple function \( f(d, x_0, r_1, r_2) \) will suffice for our purposes. Recall that each of \( r_1 \) and \( r_2 \) is either \( r \) or 1. We define

\[
f(d, x_0, r, r) = \begin{cases} 
0, & \text{if } x_0 < 1.8 \\
0.28d, & \text{if } 1.8 \leq x_0 < 2.2 \\
0.4d, & \text{if } 2.2 \leq x_0 
\end{cases}
\] (34)

Next we define

\[
f(d, x_0, r, 1) = \begin{cases} 
0, & \text{if } x_0 < 2.32 \\
0.06d, & \text{if } 2.32 \leq x_0 
\end{cases}
\] (35)

As always, \( f(d, x_0, 1, r) = f(d, x_0, r, 1) \). Finally, we let \( f(d, x_0, 1, 1) = 0 \).

5 Local proof the localizing potential works

We now turn to the proof of (7). We will use the computer to prove this inequality for most triangles. However, for triangles that appear in the densest packing shown in figure 1 equality holds in (7). Thus for triangles close to those that appear in this densest packing inequality (7) will be close to an equality. So we can only use the computer to prove this inequality for triangles which do not lie too close to a triangle in the densest packing. In this section we will prove (7) for the triangles which are close to a densest packing triangle.
There are four triangles that appear in this densest configuration. In all of these triangles the three discs are tangent to one another. As we will see this implies that the excess $E(T)$ has a local minimum at each of these triangles. The localizing potential $V(T)$ also has a local minimum at these triangles. We must show that when we perturb one of these triangles the increase in $E(T)$ is greater than the increase in $V(T)$.

Throughout this section we will obtain bounds on quantities by bounding their partial derivatives with respect to $x_0$, $x_1$ and $x_2$. If $f(x_0, x_1, x_2)$ is a function of the three edge lengths, and we have

$$c_i \leq \frac{\partial f}{\partial x_i}(x_0, x_1, x_2) \leq d_i$$

throughout some neighborhood of $(\bar{x}_0, \bar{x}_1, \bar{x}_2)$, then

$$\sum_{i=0}^{2} c_i \Delta x_i \leq f(x_0, x_1, x_2) - f(\bar{x}_0, \bar{x}_1, \bar{x}_2) \leq \sum_{i=0}^{2} d_i \Delta x_i$$

where $\Delta x_i = x_i - \bar{x}_i$.

We first consider triangles that have discs of radius $r$ at all three vertices and which satisfy

$$2r \leq x_i \leq 2r + \epsilon, \quad i = 0, 1, 2$$

where $\epsilon > 0$ will be determined later. Since the three discs all have radius $r$, $D$ is independent of the $x_i$. Letting

$$\Delta E = E(x_0, x_1, x_2, r, r, r) - E(2r, 2r, 2r, r, r)$$

then

$$\Delta E = \Delta(\delta A - D) \geq \delta \sum_{i=0}^{2} a_{rrr}(\epsilon) \Delta x_i = a_{rrr}(\epsilon) \sum_{i=0}^{2} \Delta x_i$$

where

$$a_{rrr}(\epsilon) = \min \frac{\partial A}{\partial x_i}$$

with the min over the above set of triangles. Note that this min is independent of $i$. Inequality (77) in the appendix gives a lower bound on $a_{rrr}(\epsilon)$.

We must compare the above with the increase in the localizing potential $V(T)$. Let $\Delta \phi_i$ denote the change in the angles $\phi_i$ corresponding to changing each $x_j$ by $\Delta x_j$. Then the change in $V(T)$ is bounded by

$$\Delta V(T) \leq m \sum_{i=0}^{2} |\Delta \phi_i| \leq m \sum_{j=0}^{2} b_{rrr}(\epsilon) \Delta x_j$$

where

$$b_{rrr}(\epsilon) = \max \sum_{i=0}^{2} \left| \frac{\partial \phi_i}{\partial x_j} \right|$$
and the max is over the set of triangles. \( b^i_{rrr}(\epsilon) \) is independent of \( i \), and we denote it by \( b_{rrr}(\epsilon) \). This proves that the excess is greater than the localizing potential for all triangles satisfying (38) provided \( m \leq m_{rrr}(\epsilon) \) where

\[
m_{rrr}(\epsilon) = \delta \frac{a_{rrr}(\epsilon)}{b_{rrr}(\epsilon)} \quad (44)
\]

For triangles that have discs of radius 1 at all three vertices the estimates are very similar. We consider the set of triangles which satisfy

\[
2 \leq x_i \leq 2 + \epsilon, \quad i = 0, 1, 2 \quad (45)
\]

The constraint we obtain on \( m \) is now \( m \leq m_{111}(\epsilon) \) where

\[
m_{111}(\epsilon) = \delta \frac{a_{111}(\epsilon)}{b_{111}(\epsilon)} \quad (46)
\]

\( a_{111}(\epsilon) \) and \( b_{111}(\epsilon) \) are defined by analogy to \( a_{rrr}(\epsilon) \) and \( b_{rrr}(\epsilon) \).

Next we consider triangles with one disc of radius 1 and two of radius \( r \). We take the large disc to be at vertex 0 and consider the set of triangles given by

\[
2r \leq x_0 \leq 2r + \epsilon, \quad 1 + r \leq x_i \leq 1 + r + \epsilon, \quad i = 0, 1, 2 \quad (47)
\]

In this case \( D \) is not independent of the \( x_i \). Using the constraint \( \phi_0 + \phi_1 + \phi_2 = \pi \), we can write \( D \) as

\[
D = \frac{1}{2}(\phi_0 + r^2\phi_1 + r^2\phi_2) = \frac{\pi}{2}r^2 + \frac{1 - r^2}{2}\phi_0 \quad (48)
\]

Thus

\[
\Delta E = \Delta(\delta A - D) \geq \sum_{i=0}^{2} \delta a^i_{1rr}(\epsilon) \Delta x_i - \frac{1 - r^2}{2} \Delta \phi_0 \quad (49)
\]

where \( a^i_{1rr} \) is the min of \( \frac{\partial A}{\partial x_i} \) over the set of triangles.

Let

\[
c^i_{1rr}(\epsilon) = \min \frac{\partial \phi_0}{\partial x_i}
\]

\[
d^i_{1rr}(\epsilon) = \max \frac{\partial \phi_0}{\partial x_i}
\]

where the min and max are over the triangles satisfying (47). Note that there are no absolute values on the partial derivatives in the above. The calculations in the appendix show that on the set of triangles \( \frac{\partial \phi_0}{\partial x_i} \) is positive for \( i = 0 \) and negative for \( i = 1, 2 \). We now have

\[
\Delta E \geq \sum_{i=0}^{2} \left[ \delta a^i_{1rr}(\epsilon) - \frac{1 - r^2}{2}d^i_{1rr}(\epsilon) \right] \Delta x_i \quad (51)
\]
We bound the increase in the localizing potential $V(T)$ as before. The constraints we obtain on $m$ are $m \leq m_{111}^i(\epsilon)$ with

$$m_{111}^i(\epsilon) = \frac{\delta a_{111}^i(\epsilon) - \frac{1 - r^2}{2} d_{111}^i(\epsilon)}{b_{111}^i(\epsilon)} \quad (52)$$

Finally we consider triangles with one disc of radius $r$ and two of radius 1. We take the small disc to be at vertex 0 and consider the set of triangles given by

$$2 \leq x_0 \leq 2 + \epsilon, \quad 1 + r \leq x_i \leq 1 + r + \epsilon, \quad i = 0, 1, 2 \quad (53)$$

We now have

$$D = \frac{1}{2} (r^2 \phi_0 + \phi_1 + \phi_2) = \frac{\pi}{2} r^2 + \frac{r^2}{2} \phi_0 \quad (54)$$

Thus

$$\Delta E \geq \sum_{i=0}^{2} \delta a_{111}^i(\epsilon) \Delta x_i + \frac{1 - r^2}{2} \Delta \phi_0 \geq \sum_{i=0}^{2} \left[ \delta a_{111}^i(\epsilon) + \frac{1 - r^2}{2} c_{111}^i(\epsilon) \right] \Delta x_i \quad (55)$$

and so we obtain the constraints $m \leq m_{111}^i(\epsilon)$ with

$$m_{111}^i(\epsilon) = \frac{\delta a_{111}^i(\epsilon) + \frac{1 - r^2}{2} c_{111}^i(\epsilon)}{b_{111}^i(\epsilon)} \quad (56)$$

Using the results of appendix B we can evaluate these bounds on $m$. We will take $\epsilon = 0.001$. For comparison we also show the values for $\epsilon = 0$. We find

$$m_{rrr}(0) = 0.135463, \quad m_{rrr}(0.001) = 0.134576$$
$$m_{111}(0) = 0.455814, \quad m_{111}(0.001) = 0.454183$$
$$m_{011}(0) = 0.232960, \quad m_{011}(0.001) = 0.231100$$
$$m_{11r}(0) = 0.179205, \quad m_{11r}(0.001) = 0.178067$$
$$m_{01r}(0) = 0.264015, \quad m_{01r}(0.001) = 0.262815$$
$$m_{1r1}(0) = 0.308628, \quad m_{1r1}(0.001) = 0.306788 \quad (57)$$

where $i$ is 1 or 2. Recall that we take $m = 0.12$, so these bounds are all easily met. In fact, with $m = 0.12$ this local proof works for $\epsilon$ as large as 0.018.

6 Global proof the localizing potential works

We now prove inequality (7) for the rest of the triangles, i.e., all the triangles that are not close to those that appear in the densest configuration in figure 1. We use interval
arithmetic for this proof. This is necessary for two reasons. First, it is needed (as in all computer-assisted proofs involving real numbers) to deal with the fact that computer calculations involving real numbers are not exact. Second, even if the computer could perform exact calculations we could not prove \( (7) \) by checking one triangle at a time since there are infinitely many triangles. Instead we must work with sets of triangles. Given intervals for the three edge lengths \( x_0, x_1, x_2 \) of the triangle, we consider the set of triangles whose edge lengths belong to the respective intervals. It is straightforward to use interval arithmetic to then compute intervals for quantities such as the angles \( \phi_0, \phi_1, \phi_2 \). Interval arithmetic is designed so that the interval computed for \( \phi_i \) means that for a triangle whose edge lengths belong to their respective intervals, the value of \( \phi_i \) must belong to its interval. We note that the code for computing intervals for quantities such as \( \phi_i \) is the same as the code for computing these quantities when we have a single real value for each edge length. We need only change the data type of the variables involved from “double” to “interval” and define versions of the basic arithmetic operations (addition, multiplication, inverse cosine, etc.) for interval variables. The localizing potential involves real numbers that must be represented by intervals, and so it too is handled using interval arithmetic.

We now fix choices of \( r_0, r_1, r_2 \) and ask what triangles must be considered in the proof of \( (7) \). Since discs cannot overlap, we have the following lower bounds on the length of the edges of the triangle.

\[
\begin{align*}
x_0 & \geq r_1 + r_2 \\
x_1 & \geq r_0 + r_2 \\
x_2 & \geq r_0 + r_1
\end{align*}
\]  \( (58) \)

Recall that the center of a triangle is the point equidistant to the three vertices, and the circumradius is the radius of the circle centered at this point which contains all three vertices. In the Delaunay decomposition no disc has a center closer to the triangle’s center than the discs at its vertices. So if the circumradius is greater than \( 1 + r \), then we can add a disc of radius \( r \) to the packing by putting its center at the center of the triangle. If we assume that the packing is saturated, then this implies that every triangle has a circumradius of at most \( 1 + r \). The circumradius is greater than half the length of any side of the triangle. So we have

\[
x_i \leq 2(1 + r), \quad i = 0, 1, 2
\]  \( (59) \)

At each stage of the computer proof we have a list of parallelepipeds \( I_0 \times I_1 \times I_2 \). Such a parallelepiped represents the set of triangles with \( x_i \in I_i \). At each stage of the computer proof the union of the parallelepipeds in the list is the set of triangles for which we must still prove \( (7) \). Initially, there is one parallelepiped in the list with

\[
I_0 = [r_1 + r_2, 2(1 + r)], \quad I_1 = [r_0 + r_2, 2(1 + r)], \quad I_2 = [r_0 + r_1, 2(1 + r)]
\]  \( (60) \)
At each stage we take a parallelepiped from the list and split it into two parallelepipeds. We compute the interval for \( E(T) - F(T) \) for each of the two. If it contains no negative values for a particular parallelepiped, then we know that (7) is true for all triangles in this parallelepiped, and we can discard it. If the interval contains some negative values, we must add this parallelepiped to the list. Note that if the parallelepiped is contained in one of the parallelepipeds (38), (45), (47), or (53), then we know \( E(T) - F(T) \geq 0 \) on this parallelepiped, and so it can be discarded. In appendix A we show that if the packing is saturated then in the Delaunay triangulation the area of every triangle is at least \( 2r^3/(1 + r) \). So for a parallelepiped, if the interval we compute for the area of the triangles is entirely greater than this lower bound, then we can discard the parallelepiped. The process ends when the list of parallelepipeds for which (7) has not been proved is empty.

We compute our interval bounds in the simplest, crudest way. For example, we compute the interval bound on the angle \( \phi_i \) by simply using interval arithmetic in the equation for \( \phi_i \) in terms of the edge lengths, eq. (61). One could compute a better interval bound by explicitly maximizing and minimizing \( \phi_i \) over the parallelepiped by considering the derivatives of \( \phi_i \) with respect to the edge lengths \( x_j \). Because of our rather crude bounds, the parallelepipeds must be split a large number of times. The computer considers a total of about 26 million parallelepipeds. This takes about 30 minutes on a laptop.

### A Formulae

The vertices are labelled 0,1,2. The length of the edge opposite vertex \( i \) is \( x_i \). The radius of the disc at vertex \( i \) is \( r_i \). Each \( r_i \) can only be \( r \) or 1. The angle at vertex \( i \) is \( \phi_i \). It is given by

\[
\cos(\phi_i) = (x_j^2 + x_k^2 - x_i^2)/(2x_jx_k)
\]

where \( \{i, j, k\} = \{0, 1, 2\} \). The area of the triangle is

\[
A = \frac{1}{4}\sqrt{2x_0^2x_1^2 + 2x_0^2x_2^2 + 2x_1^2x_2^2 - x_0^4 - x_1^4 - x_2^4}
\]

Recall that by the center of a triangle we mean the center of the circle which contains the three vertices of the triangle. We will derive a formula for the signed distance from the center to each edge and a formula for the circumradius. We take the vertices of the triangle to be

\[
P_0 = (0, 0), \quad P_1 = (x_0, 0) \quad P_2 = (a, b)
\]

where \( a^2 + b^2 = x_0^2 \). Let \( P \) be the center of the triangle. The center must lie on the line \( x = x_0/2 \), so \( P = (x_0/2, d_0) \), where \( d_0 \) is the signed distance from the center to the edge opposite vertex 0.

The circumradius of the triangle, \( R \), satisfies

\[
R = ||P - P_0|| = ||P - P_1||
\]
This implies
\[ R^2 = \frac{1}{4} x_0^2 + d_0^2 = (a - x_0/2)^2 + (b - d_0)^2 \] (65)

Using \( a^2 + b^2 = x_2^2 \), this gives
\[ ax_0 + 2bd_0 = x_2^2 \] (66)

Using
\[ a = x_2 \cos(\phi_1) = \frac{x_0^2 + x_2^2 - x_1^2}{2x_0x_2} \] (67)

and
\[ \frac{1}{2} bx_0 = A \] (68)

we find
\[ d_0 = \frac{x_0(x_1^2 + x_2^2 - x_0^2)}{8A} \] (69)

Using (65), the above formula for \( d_0 \) yields a formula for the circumradius.
\[ R = \frac{x_0x_1x_2}{4A} \] (70)

For a saturated packing we can assume
\[ R \leq 1 + r \] (71)

Otherwise we could add a disc of radius \( r \) with center at the center of the triangle. Using the above formula for \( R \) this implies a lower bound on the area \( A \).
\[ A \geq \frac{x_0x_1x_2}{4(1 + r)} \] (72)

Using the trivial bound \( x_i \geq 2r \), this implies
\[ A \geq \frac{2r^3}{1 + r} \] (73)

**B Calculations for triangles with excess near zero**

We consider triangles with \( x_0 \in [x, x + \epsilon], x_1, x_2 \in [y, y + \epsilon] \).

We have
\[ 16A^2 = 2x_0^2x_1^2 + 2x_0^2x_2^2 + 2x_1^2x_2^2 - x_0^4 - x_1^4 - x_2^4 \] (74)

Given an \( i = 0, 1 \) or \( 2 \), we let \( j \) and \( k \) be the other two integers in \( \{0, 1, 2\} \). Then
\[ 8A \frac{\partial A}{\partial x_i} = x_i(x_j^2 + x_k^2 - x_i^2) \] (75)
We assume that $\epsilon$ is small enough that the set of triangles we are considering only contains acute triangles. So $x_j^2 + x_k^2 - x_i^2 > 0$. Then $\frac{\partial A}{\partial x_i} \geq 0$ for each $i$, and so $A$ is increasing in each $x_i$. So $A$ is bounded above by its value for the triangle with sides of length $x_0 = x + \epsilon$, $x_1 = x_2 = y + \epsilon$. This area is

$$A_{\text{max}} = \frac{1}{4} (x + \epsilon) \sqrt{4(y + \epsilon)^2 - (x + \epsilon)^2} \quad (76)$$

Thus we obtain the lower bounds

$$\frac{\partial A}{\partial x_0} \geq \frac{x(2y^2 - (x + \epsilon)^2)}{8A_{\text{max}}} \quad (77)$$

and for $i = 1, 2$

$$\frac{\partial A}{\partial x_i} \geq \frac{y(x^2 + y^2 - (y + \epsilon)^2)}{8A_{\text{max}}} \quad (78)$$

From (61) we have

$$\frac{\partial \phi_0}{\partial x_0} = (\sin(\phi_0))^{-1} \frac{x_0}{x_1 x_2} \quad (79)$$

$$\frac{\partial \phi_0}{\partial x_1} = -(\sin(\phi_0))^{-1} \frac{x_2^2 + x_0^2 - x_1^2}{2x_1^2 x_2} \quad (80)$$

$$\frac{\partial \phi_0}{\partial x_2} = -(\sin(\phi_0))^{-1} \frac{x_2^2 + x_0^2 - x_1^2}{2x_2^2 x_1} \quad (81)$$

Note that $\frac{\partial \phi_0}{\partial x_0}$ is always positive and the other two partials are negative for acute triangles. So $\sin(\phi_0)$ is bounded below by its value at $x_0 = x$, $x_1 = x_2 = y + \epsilon$. Let $S_0$ be the value of $\sin(\phi_0)$ at this point. And $\sin(\phi_0)$ is bounded above by its value at $x_0 = x + \epsilon$, $x_1 = x_2 = y$. Let $S'_0$ be the value of $\sin(\phi_0)$ at this point. Then we have

$$\frac{x}{(y + \epsilon)^2 S'_0} \leq \frac{\partial \phi_0}{\partial x_0} \leq \frac{x + \epsilon}{y^2 S_0} \quad (82)$$

$$- \frac{(y + \epsilon)^2 + (x + \epsilon)^2 - y^2}{2y^2 S_0} \leq \frac{\partial \phi_0}{\partial x_i} \leq - \frac{y^2 + x^2 - (y + \epsilon)^2}{2(y + \epsilon)^3 S'_0} \quad (83)$$

for $i = 1, 2$. Similarly

$$\frac{\partial \phi_1}{\partial x_1} = (\sin(\phi_1))^{-1} \frac{x_1}{x_0 x_2} \quad (84)$$

$$\frac{\partial \phi_1}{\partial x_0} = -(\sin(\phi_1))^{-1} \frac{x_0^2 + x_1^2 - x_2^2}{2x_0^2 x_2} \quad (85)$$

$$\frac{\partial \phi_1}{\partial x_2} = -(\sin(\phi_1))^{-1} \frac{x_2^2 + x_1^2 - x_0^2}{2x_2^2 x_0} \quad (86)$$
For these derivatives we only need bounds on their absolute values.

\[
\left| \frac{\partial \phi_1}{\partial x_1} \right| \leq \frac{(y + \epsilon)}{xy S_1} \tag{87}
\]

\[
\left| \frac{\partial \phi_1}{\partial x_0} \right| \leq \frac{(x + \epsilon)^2 + (y + \epsilon)^2 - y^2}{2x^2 y S_1} \tag{88}
\]

\[
\left| \frac{\partial \phi_1}{\partial x_2} \right| \leq \frac{2(y + \epsilon)^2 - x^2}{2y^2 x S_1} \tag{89}
\]

with \(S_1\) equal to the value of \(\sin(\phi_1)\) when \(x_1 = y\), \(x_0 = x + \epsilon\) and \(x_2 = y + \epsilon\).

\section*{C Equation for \(r\)}

In this appendix we show that the radius \(r\) for the compact packing is a root of an eighth degree polynomial. The angles satisfy

\[2\alpha - \beta = 120 \tag{90}\]

So

\[\cos(\alpha - \beta/2) = \cos(60) = 1/2 \tag{91}\]

Hence

\[\cos(\alpha) \cos(\beta/2) + \sin(\alpha) \sin(\beta/2) = 1/2 \tag{92}\]

This implies

\[\sin^2(\alpha) \sin^2(\beta/2) = (1/2 - \cos(\alpha) \cos(\beta/2))^2 \tag{93}\]

which is equivalent to

\[1 - \cos^2(\alpha)^2 - \cos^2(\beta/2) = \frac{1}{4} - \cos(\alpha) \cos(\beta/2) \tag{94}\]

We have

\[\cos(\alpha) = \frac{2(1 + r)^2 - 2^2}{2(1 + r)^2} = \frac{r^2 + 2r - 1}{(1 + r)^2} \tag{95}\]

and

\[\sin(\beta/2) = \frac{r}{1 + r} \tag{96}\]

\[\cos(\beta/2) = \frac{\sqrt{2r + 1}}{1 + r} \tag{97}\]

Substituting these into the above equation, some algebra gives

\[r^8 - 8r^7 - 44r^6 - 232r^5 - 482r^4 - 24r^3 + 388r^2 - 120r + 9 = 0 \tag{98}\]
By considering the packing itself, the conditions that various pairs of discs are tangent can be used to show that \( r \) also satisfies

\[
(7 + 4\sqrt{3})r^4 + (20 + 12\sqrt{3})r^3 + (6 + 4\sqrt{3})r^2 + (-20 - 4\sqrt{3})r + 3 = 0 \tag{99}
\]

To compute the density of the packing, consider the unit cell shown in figure 1. The quadrilateral has sides of length

\[
l = 1 + r + \sqrt{3}r + \sqrt{1 + 2r} \tag{100}
\]

and angles of \( \pi/2 \) and \( 2\pi/3 \). So its area is \( \sqrt{3}l^2/2 \). It covers a total of 3 large discs and 3 small discs. So the density is

\[
\delta = \frac{2\sqrt{3}\pi(1 + r^2)}{l^2} \approx 0.911627478 \tag{101}
\]

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