Constraints of the Dynamic Equations and Lagrangian required by Superposition of Field

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Abstract

A general form of the dynamical equations of field is obtained on the requirement this field is a superposable one; hence the constraint on the forms of the Lagrangians is acquired. It shows this requirement requires the continuous transformation group of the Lagrangians of field to be compact, and that All Lagrangians of elementary particles, such as leptons, quarks, photons and gluons, satisfy this requirement. The result of regarding this character as a general property of physical field is discussed.

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I. INTRODUCTION

Superposition principle of quantum states and Schrödinger equation are two postulations of non-relativistic quantum mechanics. Then some other equations, such as Klein-Gordon equation and Dirac equation, are founded in other fields of quantum theory. As the quantum field theory is used in many aspects, many kinds of Lagrangians are born to solve different kinds of problems. Restrictions to these Lagrangians are generally Lorentz covariance, gauge invariance and renormalizibility. On the other hand, superposition principle of quantum states has been fully discussed and is examined by quite many experiments. It has been accepted as a basic rule of microscopic world. In some basic processes such as creation and annihilation of particles, quantum field theory is tested to be successful under many conditions. As the wave function in quantum mechanics can also be regarded as a kind of field, we start from a superposable field, then get the universal form of the dynamical equations and Lagrangians of the field.

In fact, the property of superposition may lead to some additional restrictions on the form of equations of motion, and therefore on that of Lagrangians of dynamic systems.

In the present paper, we try to argue the constraints of Lagrangians of field caused by that the field can be superposed.

The outline of this paper is as follows. In section II, we first obtain the general form of the equations of motion starting from a field with the property of superposition. The constraints on Lagrangians are then argued in section III. All the successful Lagrangians, such as scalar field, Dirac field, QED and QCD, and some of interesting fields are discussed in section III. A brief conclusion is made in section IV.

II. CONSTRAINTS ON EQUATION OF MOTION CAUSED BY THE PROPERTY OF SUPERPOSITION

In quantum theory, the wave function, \( \psi(\vec{x}, t) \), is the probability amplitude whose absolute square indicates the probability to find a particle in position \( \vec{x} \) at time \( t \). The evolution operator \( \hat{U}(\vec{x}_2, t_2; \vec{x}_1, t_1) \) means the probability amplitude for a particle, originally in position \( \vec{x}_1 \) at time \( t_1 \), to be find in position \( \vec{x}_2 \) at time \( t_2 \). The evolution operator can be expressed in form of integral operator of propagator. After operating on \( \psi(\vec{x}_1, t_1) \) from left-hand side, it gives wave function \( \psi(\vec{x}_2, t_2) \), i.e.

\[
\hat{U}(\vec{x}_2, t_2; \vec{x}_1, t_1)\psi(\vec{x}_1, t_1)
\]
\[
\psi(\vec{x}_1, t_1) = \psi(\vec{x}_2, t_2).
\]

Now we extract the physical meaning of \( \psi(\vec{x}, t) \), treating it as a field of time-space coordinate. \( \hat{U}(\vec{x}_2, t_2; \vec{x}_1, t_1) \) still is a functional of \( \psi(\vec{x}, t) \) as before. We still call \( \psi(\vec{x}, t), \hat{U}(\vec{x}_2, t_2; \vec{x}_1, t_1) \) as wave function, evolution operator respectively for the understanding of our deduction. \( \psi(\vec{x}, t) \) is an representation of fields with the property of superposition without any physical meaning. The operation between \( \psi(\vec{x}_1, t_1) \) and \( \hat{U}(\vec{x}_2, t_2; \vec{x}_1, t_1) \) is just like (1).

The property of superposition of \( \psi(\vec{x}, t) \)

A dynamic equation of \( \hat{U}(\vec{x}_2, t_2; \vec{x}_1, t_1) \) can be expressed by

\[
\hat{F}\hat{U}(\vec{x}_2, t_2; \vec{x}_1, t_1) = 0.
\]

Let both sides of equation (2) operate on \( \psi(\vec{x}_1, t_1) \) from left-hand side. It is easy to show that the wave function satisfies the same dynamic equation to that of the evolution operator.

\[
\hat{F}\hat{U}(\vec{x}_2, t_2; \vec{x}_1, t_1)\psi(\vec{x}_1, t_1) = \hat{F}\psi(\vec{x}_2, t_2) = 0.
\]

Final state is the superposition of all the transitional states. Mathematically, the property of superposition of \( \psi(\vec{x}, t) \) can be expressed by

\[
K(\vec{x}_3, t_3; \vec{x}_1, t_1) = \int d\vec{x}_2K(\vec{x}_3, t_3; \vec{x}_2, t_2)K(\vec{x}_2, t_2; \vec{x}_1, t_1).
\]

For simplicity, we set \( t_1 = 0, t_2 = t \) and \( t_3 = t + \Delta t \). The expansion of \( K(\vec{x}_3, t + \Delta t; \vec{x}_1, 0) \) to the first order approximation can be written as

\[
K(\vec{x}_3, t + \Delta t; \vec{x}_1, 0) \approx K(\vec{x}_3, t; \vec{x}_1, 0) + \frac{\partial K(\vec{x}_3, t; \vec{x}_1, 0)}{\partial t}\Delta t.
\]

It is trivial to get

\[
\frac{\partial K(\vec{x}_3, t; \vec{x}_1)}{\partial t} = \frac{K(\vec{x}_3, t + \Delta t; \vec{x}_1) - K(\vec{x}_3, t; \vec{x}_1)}{\Delta t},
\]

where the index of initial time \( t_1 = 0 \) is omitted.

Now setting \( \vec{x}_2 = \vec{x}_3 - \Delta \vec{x} \), the integrand in equation (4) can be expanded as

\[
K(\vec{x}_3, t_3; \vec{x}_2, t_2)K(\vec{x}_2, t_2; \vec{x}_1) = \sum_{n=0}^{\infty} \frac{(-\Delta \vec{x})^n}{n!} F_n.
\]
with

\[ F_n \equiv \frac{\partial^n}{\partial x_3^n} \left[ K(\bar{x}_3 + \Delta \bar{x}, t + \Delta t; \bar{x}_3, t)K(\bar{x}_3, t; \bar{x}_1) \right]. \]

Substitute equation (4) and (7) into (6), and it follows that

\[
\frac{\partial}{\partial t} K(\bar{x}_3, t; \bar{x}_1) = \frac{1}{\Delta t} \left[ \int d(\Delta \bar{x}) \sum_{n=0}^{\infty} \frac{(-\Delta \bar{x})^n}{n!} F_n - K(\bar{x}_3, t; \bar{x}_1) \right]
\]

\[
= \frac{1}{\Delta t} \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial x_3^n} \left( \int d(\Delta \bar{x}) G(\bar{x}_3, n)K(\bar{x}_3, t; \bar{x}_1) - K(\bar{x}_3, t; \bar{x}_1) \right) \right],
\]

(8)

where \( G(\bar{x}_3, n) \), with the time index omitted, is defined as

\[ G(x_3, n) \equiv \int d(\Delta \bar{x})(-\Delta \bar{x})^n K(x_3 + \Delta \bar{x}, t + \Delta t; x_3, t). \]

Note that for \( n = 0 \), \( G(x_3, n) \) is nothing but the total probability of finding a particle in the whole space with initial space-time coordinates \((x_3, t)\). This indicates that \( \int d(\Delta x)K(x_3 + \Delta x, t + \Delta t; x_3, t) = 1 \). Therefore, equation (8) can be reduced to

\[
\frac{\partial}{\partial t} K(\bar{x}_3, t; \bar{x}_1) = \sum_{n=1}^{\infty} \frac{\partial^n}{\partial x_3^n} S_1^n K(\bar{x}_3, t; \bar{x}_1),
\]

(9)

with

\[ S_1^n = \frac{1}{n! \Delta t} G(\bar{x}_3, n), \quad n = 1, 2, 3, \cdots \]

(10)

Operate both sides of equation (9) on the wave function \( \psi(\bar{x}_1, t_1) \) from left side, and integrate over \( \bar{x}_1 \) we acquire, with substituting \( \bar{x} \) for \( \bar{x}_3 \),

\[
\frac{\partial}{\partial t} \psi(\bar{x}, t) = \sum_{n=1}^{\infty} \frac{\partial^n}{\partial x^n} S_1^n \psi(\bar{x}, t).
\]

(11)

When the second order term in (5) is considered, we can get, by following similar deductions as above,

\[
\frac{\partial^2}{\partial t^2} \psi(\bar{x}, t) = \sum_{n=1}^{\infty} \frac{\partial^n}{\partial x^n} S_2^n \psi(\bar{x}, t).
\]

(12)

For general, it can be deduced that

\[
\frac{\partial^m}{\partial t^m} \psi(\bar{x}, t) = \sum_{n=1}^{\infty} \frac{\partial^n}{\partial x^n} S_m^n \psi(\bar{x}, t),
\]

(13)
with

\[ S_n^m = \frac{m!}{n!(\Delta t)^m} G(\bar{x}_3, n), \quad n, m = 1, 2, 3, \ldots \]  \hspace{1cm} (14)

Now we get the general form of dynamic equation, due to the requirement of the property of superposition of \( \psi(\bar{x}, t) \), defined as equation (13) or the linear nestification of it for different \( m \).

### III. CONSTRAINTS ON LAGRANGIANS BY THE PROPERTY OF SUPERPOSITION

Restriction on dynamic equation will limit the forms of Lagrangians, \( \mathcal{L} \), of the system, since Euler equation

\[ \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \]  \hspace{1cm} (15)

gives the dynamic equations for a system described by \( \mathcal{L} \). After quantization, \( \psi \) becomes field operator instead of wave function. Considering the fact that the Lagrangian contains only coordinates, \( x_\mu \), and one-order differential to coordinates \( \partial/\partial x_\mu \), but not any other higher order differential terms, the general dynamic equation derived from the property of superposition, equation (13), gives a general form of Lagrangian

\[ \mathcal{L} = \sum_k C_k \psi^i (\partial_\mu \psi)^j, \quad i, j = 0, 1, 2. \]  \hspace{1cm} (16)

where \( C_k \) is coefficient of the \( k^{th} \) item, and \( i, j \) can be selected from 0, 1, 2, according to other requirements, such as dimensional requirement and Lorentz invariance. But it is forbidden that both \( i \) and \( j \) are selected to be 2. Surely many other constraints should be concluded. For example, Lorentz invariance require the Lorentz indexes should be contracted. the coefficient \( C_k \) may contain other fields, if existing, which gives the interaction between fields.

Do all the Lagrangians describing elementary fields, which have been used successfully, meet the requirement deduced from the property of superposition? We first examine the Lagrangians of scalar field, Dirac field and electromagnetic field, which correspond to fields with spin \( s = 0, \frac{1}{2}, 1 \), respectively. The Lagrangians with no interactions of these fields are as follows,

\[ \mathcal{L}_s = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 \]
The kinetic term of scalar field can be expressed in the form of equation (16) as \( C_1 = 1/2, i = 0, \) and \( j = 2. \) While the mass term can be expressed as \( C_2 = 1/2, i = 2 \) and \( j = 0. \) For Dirac field, the kinetic term, in the form of equation (16), can be expressed as \( C_1 = i \bar{\psi} \gamma^\mu, i = 0 \) and \( j = 1. \) And the mass term is \( C_2 = m \bar{\psi}, i = 1 \) and \( j = 0. \) For electromagnetic field, it corresponds to \( i = 0 \) and \( j = 2. \) So all the free fields of these elementary particles match the requirement by the property of superposition.

Let us then consider the interaction term in quantum electrodynamics (QED). The interaction between photons and electrons is

\[ \mathcal{L}_i = -e \bar{\psi} \gamma^\mu \psi A_\mu. \]  \hspace{1cm} (18)

According to equation (16), \( i = 1, j = 0 \) and \( C_k = -e \bar{\psi} \gamma^\mu A_\mu \) are chosen for electron field \( \psi. \) We surely can also choose \( i = 1, j = 0 \) and \( C_k = -e \bar{\psi} \gamma^\mu \), if we consider it as a term for photon field, \( A_\mu. \)

Now it comes to the Lagrangian of quantum chromodynamics (QCD). The classical part of QCD Lagrangian, which is invariant under local \( SU(N_c) \) (with \( N_c = 3 \) for QCD) gauge transformation, is defined as

\[ \mathcal{L} = \sum_f \bar{\psi}_f [i \gamma^\mu D_\mu - m_f] \psi_f + \mathcal{L}_g, \]  \hspace{1cm} (19)

\[ \mathcal{L}_g = -\frac{1}{4} F^{a}_{\mu \nu} F^{a \mu \nu}, \]  \hspace{1cm} (20)

where \( D_\mu \) is the covariant derivative, and \( F^{a}_{\mu \nu} \), the gluon-field strength. Similarly, it is easy to find that the first term in equation (19) satisfies the requirement of the property of superposition. With the definition of \( F^{a}_{\mu \nu} \), the pure non-Abelian gauge field part, \( \mathcal{L}_g \), can be written as

\[ \mathcal{L}_g = -\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - g (\partial_\mu A^a_\nu) f^{abc} A^{b \mu} A^{c \nu} \]

\[ -\frac{1}{4} g^2 f^{abc} f^{a' b' c'} A^{a}_\mu A^{b'}_\nu A^{c'} A^{\mu \nu \sigma}, \]  \hspace{1cm} (21)

where \( f^{abc} \) is the structure constant of gauge group and \( g \) the coupling constant. It is obvious that the first two terms, which correspond to the kinetic part and 3-gluon
interaction part respectively, agree with the form of equation (16). However, the last term, i.e., the 4-gluon interaction term in QCD, seems to break the rules described in equation (16), since it is a quartic term of non-Abelian gauge field and may brings a term of $A^4$ when $b = b'$ and $c = c'$. But this term vanishes if the coefficient of this term is zero for $b = b'$ and $c = c'$. To do this, the coefficient must be antisymmetric, which means $f^{abc} = \varepsilon_{abc}^{'d} f^{ab'c'}$. This shows that the property of superposition of field requires the continuous group of transformation be a compact Lie group. Fortunately, this is satisfied by the present theories where $SU(N)$ groups are used. It is not hard to verify that the gauge-fixing term and Faddeev-Popov term, which are introduced to quantize the classical QCD, also meet the constraint of the property of superposition. Upper fields are successful and tested in both theory and experiment. So all the successful and tested fields of elementary particle are superposable by now. Is it incidental? We don’t think so. We promote it as a basic rule that physical fields of elementary particle are superposable.

A challenge of this opinion comes from $\varphi^4$ field. It is one kind of interaction that is obvious in contradiction with (16). $\varphi^4$ interaction is widely used as an example in textbooks of quantum field theory, because it is one of the simplest interaction and is renormalizable. But this interaction violates the property of superposition of field apparently, since it corresponds to $i = 4$ and $j = 0$ if expressed in the form of equation (16). In fact, no physical particle with $\varphi^4$ interaction is found up to now.

It should be emphasized that the term introduced to a Lagrangian, in order to manifest spontaneous symmetry breaking, may violate the property of superposition of field. For example, the introduction of Higgs field, which plays an important role in Standard Model, does not meet the property of superposition. The Lagrangian of Higgs field is

$$\mathcal{L}_{\text{Higgs}} = -\frac{1}{2} (\partial_{\mu} \varphi_a)^2 - \frac{1}{2} \mu \varphi_a^2 - \frac{\lambda}{4} \varphi_a^4, \quad (22)$$

where the last two term is the self-interaction of Higgs, which intuitively guarantees the spontaneous symmetry breaking in vacuum. But the last term apparently violates the rules in equation (16) now that it is nothing but a $\varphi^4$-interaction. If Higgs exists, it shows that the spontaneous symmetry breaking of Goldstone mechanism will terminate validity of the property of superposition. But if the property of superposition of field is considered a basic and universal rule in nature, Higgs will not exist, or more precisely, the mechanism of spontaneous symmetry breaking should be re-examined. It is the most powerful challenge to the universality of the property of superposition of field of elementary particle.
IV. CONCLUSIONS

In summary, Considering a field is superposable we deduce its dynamic equation. The general form of it is given as equation (13). Considering no higher order differentials than the first order differential to 4-coordinates can be included in Lagrangian, only \( m = 1, 2 \) can be selected in equation (13); hence the general form of Lagrangian is given as equation (16). We examined the Lagrangians of elementary particles and concluded that all the successful and tested Lagrangians meet the requirement of the property of superposition. It is verified that the property of superposition requires the continuous transformation group a Lagrangian to be a compact Lie group, which is consistent with \( SU(N) \) gauge theory. In particular, we argued the relation between spontaneous symmetry breaking and the requirement of the property of superposition. It is concluded that the existence of Higgs would violates this property. On the contrary, if the property of superposition is fundamental in nature, it will forbid the existence of Higgs.

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