Large N gauge theories and AdS/CFT correspondence

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Abstract

In the first part of these lectures we will review the main aspects of large $N$ QCD and the explicit results obtained from it. Then, after a review of the properties of $\mathcal{N} = 4$ super Yang-Mills, type IIB string theory and of AdS space, we briefly discuss the Maldacena conjecture. Finally in the last part of these lectures we will discuss the finite temperature case and we show how "hadronic" quantities as the string tension, the mass gap and the topological susceptibility can be computed in this approach.

1 Introduction

Gravity is described by the Einstein’s theory of general relativity, while the other interactions are described by gauge field theories. Actually also the theory of general relativity is a gauge theory corresponding to the gauging of the space-time Poincaré group, while those that are usually called gauge theories correspond to the gauging of an internal symmetry. But apart from the fact that they are both gauge theories does it exist any deeper relation between them? Do they imply each other in a consistent quantum theory of gravity? In the framework of field theory there is no connection; they can both exist independently from each other, but any field theory involving gravity suffers from the problem of non-renormalizability. In the framework of string theory, instead, where quantum gravity makes sense, we see not only that they naturally occur together in the same theory, but also that any attempt for constructing a string theory without gravity has been until now
unsuccessful. This seems to suggest that the presence of both gravitational and
gauge interactions is perhaps unavoidable in a consistent string theory.

String theories originated from the attempt of describing the properties of strong
interactions through the construction of the dual resonance model. It became soon
clear, however, that this model in its consistent form, that later on was recognized to
correspond to the quantization of a relativistic string, contained all sort of massless
particles as gluons, gravitons and others except a massless pseudoscalar particle
corresponding to the pion that in the chiral limit is the only massless particle that
we expect in strong interactions.

Actually a model describing $\pi\pi$ scattering in a rather satisfactory way was pro-
posed by Lovelace and Shapiro [1]. According to this model the three isospin
amplitudes for pion-pion scattering are given by:

$$A^0 = \frac{3}{2} [A(s,t) + A(s,u)] - \frac{1}{2} A(t,u)$$
$$A^1 = A(s,t) - A(s,u)$$
$$A^2 = A(t,u)$$

(1.1)

where

$$A(s,t) = \beta \frac{\Gamma(1 - \alpha_s)\Gamma(1 - \alpha_t)}{\Gamma(1 - \alpha_t - \alpha_s)}$$

(1.2)

$s = -(k_1 + k_2)^2$, $t = -(k_1 - k_3)^2$ and $u = -(k_1 - k_4)^2$ are the three Mandelstam
variables that satisfy the relation: $s + t + u = -\sum_i k_i^2 = 4m^2_\pi$. The amplitudes
in eq.(1.1) provide a model for $\pi\pi$ scattering with linearly rising Regge trajectories
containing three parameters: the intercept of the $\rho$ Regge trajectory $\alpha_0$, the Regge
slope $\alpha'$ and $\beta$. The first two can be determined by imposing the Adler’s self-
consistency condition, that requires the vanishing of the amplitude when $s = t =
u = m^2_\pi$ and one of the pions is massless, and the fact that the Regge trajectory
must give the spin of the $\rho$ meson that is equal to 1 when $\sqrt{s}$ is equal to the mass
of the $\rho$ meson $m_\rho$. These two conditions determine the Regge trajectory to be:

$$\alpha_s = \frac{1}{2} \left[ 1 + \frac{s - m^2_\pi}{m^2_\rho - m^2_\pi} \right] = 0.48 + 0.885s$$

(1.3)

Having fixed the parameters of the Regge trajectory the model predicts the masses
and the couplings of the resonances that decay in $\pi\pi$ in terms of a unique parameter
$\beta$. The values obtained are in reasonable agreement with the experiments. Moreover
one can compute the $\pi\pi$ scattering lengths:

$$a_0 = 0.395\beta \quad a_2 = -0.103\beta$$

(1.4)

and one finds that their ratio is within 10% of the current algebra ratio given by
$a_0/a_2 = -7/2$. The amplitude in eq.(1.2) has exactly the same form as the four
tachyon amplitude of the Neveu-Schwarz model with the only apparently minor
difference that $\alpha_0 = 1/2$ (for $m_\pi = 0$) instead of 1 as in the Neveu-Schwarz model.
This difference, however, implies that the critical dimension of this model is \(D = 4\) and not \(D = 10\) as in the NS model. In conclusion this model seems to be a perfectly reasonable model for describing low-energy \(\pi\pi\) scattering. The problem is, however, that nobody has been able to generalize it to the multipion scattering and to get a string interpretation as instead one has done in the case of the known string models.

Because of this and also because of the presence of some features in the string model that are not shared by strong interactions as for instance the exponential instead of the power decay of the hadronic cross-section at large transverse momentum, it became clear in the middle of the seventies that string theories could not provide a theory for strong interactions, that in the meantime were successfully described in the framework of QCD, but could instead be used as a consistent way of unifying all interactions in a theory containing also quantum gravity \([2]\). It turned out in fact that all five consistent string theories in ten dimensions all unify in a way or another gravity with gauge theories. Let us remind shortly how this comes about.

The type I theory is a theory of open and closed string. Open strings have Chan-Paton gauge degrees of freedom located at the end points and, because of them, an open string theory contains the usual gauge theories. On the other hand a pure theory of open strings is not consistent by itself; non-planar loop corrections generate closed strings and a closed string theory contains gravity. Therefore in the type I theory open strings require for consistency closed strings. This implies that gravity, that is obtained in the zero slope limit \((\alpha'_c \to 0)\) of closed strings, is a necessary consequence of gauge theories, that are obtained in the zero slope limit of the open string theory \((\alpha' \to 0)\). Remember that the two slopes are related through the relation \(\alpha'_c = \alpha'/2\).

The heterotic strings is instead a theory of only closed strings that contains, however, both supergravity and gauge theories. But in this case gravity is the fundamental theory and gauge theories are obtained from it through a stringy Kaluza-Klein mechanism. The remaining consistent theories in ten dimensions are the two type II theories that at the perturbative level contain only closed strings and no gauge degrees of freedom. However, they also contain non-perturbative objects, the \(D\)-branes that are characterized by the fact that open strings can end on them. Therefore through the \(D\)-branes open strings also appear in type II theories and with them we get also gauge theories.

In conclusion all string theories contain both gravity and gauge theories and therefore those two kinds of interactions are intrinsically unified in string theories. But, since all string theories contain gravity, it seems impossible to use a string theory to describe strong interactions. In fact they are described by QCD that does not contain gravity!!

On the other hand it is known since the middle of the seventies that, if we consider a non-abelian gauge theory with gauge group \(SU(N)\) and we take the \(^t\)

\[^t\text{This can be checked by computing the coupling of the spinless particle at the level }\alpha_s = 2\text{ and seeing that it vanishes for }D = 4.\]
Hooft limit where the number of colours $N \to \infty$, while the product $g_M^2 N \equiv \lambda$ is kept fixed [3], the gauge theory simplifies because only planar diagrams survive in this limit. In the large $N$ limit the gauge invariant observables are determined by a master field [4] that satisfies a classical equation of motion. It has also been conjectured that in this limit QCD is described by a string theory; the mesons are string excitations that are free when $N \to \infty$. This idea is also supported by the experimental fact that hadrons lie on linearly rising Regge trajectories as required by a string model. The fact that the large $N$ expansion may be a good approximation also for low values of $N$ as $N = 3$ is suggested by the consistency of its predictions with some phenomenological observations as for instance the validity of the Zweig’s rule and the successful explanation of the $U(1)$-problem in the framework of the large $N$ expansion. This is also confirmed by recent lattice simulations [5]. The concrete way in which the large $N$ expansion explicitly solves the $U(1)$-problem is reviewed in Sect. [6]. The fact, however, that any consistent string theories includes necessarily gravity has led to call the string theory coming out from QCD, the QCD string because, as QCD, it should not contain gravity. Although many attempts have been made to construct a QCD string none can be considered sufficiently satisfactory. This problem has been with us for the last thirty years. In Sect. [2] we will review the large $N$ expansion in gauge theories and the various arguments that brought people to think that, for large $N$, a string theory ought to emerge from QCD. Unfortunately, although the large-$N$ expansion drastically simplifies the structure of QCD keeping only the planar diagrams, it has not yet been possible to carry it out explicitly in the case of four-dimensional gauge theories. In order to show some example in which the large $N$ expansion can be explicitly performed in Sect. [3] we discuss it in the $CP^{N-1}$ model, where it has allowed us to study several important aspects that these models share with QCD as for instance confinement and the $U(1)$ problem, and in two-dimensional QCD, where a master field picture emerges and the spectrum of mesons can be explicitly computed.

Recent studies of D branes have allowed to establish another deep connection between gravity and gauge theories. In fact, on the one hand, a system of $N$ D $p$-branes is a classical solution of the low-energy string effective action, containing gravity, dilaton and an antisymmetric R-R $(p + 1)$-form potential. The metric corresponding to a D $p$-brane in $D = 10$ is given by:

$$(ds)^2 = H^{-1/2}(y)\eta_{\alpha\beta}dx^\alpha dx^\beta + H^{1/2}(y)\delta_{ij}dy^i dy^j$$

while the dilaton and RR potential are equal to:

$$e^{-(\phi - \phi_0)} = [H(y)]^{(p-3)/4}; \quad A_{01...p} = [H(y)]^{-1}$$

where

$$H(y) = 1 + \frac{K_p N}{r^{7-p}} \quad K_p = \frac{(2\pi \sqrt{\alpha'})^{7-p}}{(7-p)\Omega_{8-p}} g_s$$
with \( r^2 \equiv y_i y^i \) and \( \Omega_q = 2\pi^{(q+1)/2}/\Gamma[(q + 1)/2] \). The indices \( \alpha \) and \( \beta \) run along the world volume of the brane, while the indices \( i \) and \( j \) run along the directions that are transverse to the brane.

On the other hand the low-energy dynamics of a system of \( N \) D \( p \)-branes is described by the non abelian version of the Born-Infeld action that is a functional of the transverse coordinates of the brane \( x^i \) and of a gauge field \( A^a \) living on the brane. Its complete form is not yet known, but for our considerations we can take it of the form suggested in Ref. 3:

\[
S_{BI} = -\tau_p^{(0)} \int d^{p+1}\xi \; e^{-\phi}STr \; \sqrt{-\det [G_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}]} \tag{1.8}
\]

The brane tension is given by:

\[
\tau_p \equiv \frac{\tau_p^{(0)}}{g_s} = \frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha'g_s} \quad g_s \equiv e^\phi \tag{1.9}
\]

where the string coupling constant \( g_s \) is identified with the value at infinity of the dilaton field. \( G_{\alpha\beta} \) and \( B_{\alpha\beta} \) are the pullbacks of the metric \( G_{\mu\nu} \) and of the two-form NS-NS potential \( B_{\mu\nu} \), while \( F_{\alpha\beta} \) is a gauge field living on the brane. \( STr \) stands for a symmetrized trace over the group matrices. In addition to the term given in eq.(1.8) the effective action for a D \( p \)-brane contains also a Wess-Zumino term that we do not need to consider here. By expanding the Born-Infeld action in powers of \( \alpha' \) we find at the second order the kinetic term for a non abelian gauge field (the \( U(N) \) matrices are normalized as \( Tr(TiT_j) = \frac{1}{2}\delta_{ij} \)):

\[
S_{BI} = -\frac{1}{4g_{YM}^2} \int d^{p+1}\xi \; F_{\mu\nu}^a F^{a\mu\nu} \quad ; \quad g_{YM}^2 = 2g_s(2\pi)^{p-2}(\alpha')^{(p-3)/2} \tag{1.10}
\]

An interesting property of the D-brane solution in eqs.(1.5) and (1.6) is that for large values of \( r \) the metric becomes flat. Therefore, being the curvature small, the classical supergravity description provides a good approximation of the D brane.

Based on the previous deep connection between gauge theories and type IIB supergravity or more in general type IIB superstring and on the fact that the metric of a D3-brane in the near-horizon limit becomes that of \( AdS_5 \times S^5 \) Maldacena [7] made the conjecture that actually the low-energy effective action of a D3-brane, that is given by \( \mathcal{N} = 4 \) super Yang-Mills theory in four dimensions, is equivalent to type IIB string theory compactified on \( AdS_5 \times S^5 \). A detailed discussion of the Maldacena conjecture is presented in Sect. 7, while Sect. 8 is devoted to general properties of anti De Sitter space and Sect. 9 to the symmetry properties of both \( \mathcal{N} = 4 \) super Yang-Mills and type IIB string theory.

The Maldacena conjecture provides for the first time a strong evidence that a string theory comes out from a gauge theory. But \( \mathcal{N} = 4 \) super Yang-Mills is in the Coulomb phase and therefore the emergence of a string has nothing to do with the confining properties of the theory. In order to get a confining theory we have to get
rid of the conformal invariance of the theory. The simplest way of doing so is by considering $\mathcal{N} = 4$ super Yang-Mills at finite temperature, i.e. by considering its euclidean version with compactified time. Since bosons have periodic and fermions anti-periodic boundary conditions, in going to finite temperature, we also break supersymmetry. In order to deal with $\mathcal{N} = 4$ super Yang-Mills at finite temperature it is necessary to consider a finite temperature version of AdS space \cite{8}. This is what we present in Sect. 8, where, following Witten \cite{9}, we actually see that we can identify two manifolds both having as boundary the compactified Minkowski four-dimensional space. It turns out that one of them is dominant at low temperature where the theory is still in the Coulomb phase, while the other one is dominant at high temperature where instead the theory is confining \cite{9}. In the latter case the Wilson loop gives a contribution proportional to the area and from it one can extract a finite string tension. In this case the theory has a new phase at high temperature characterized by confinement and by the emergence of a mass gap \cite{9}. From the point of view of type IIB supergravity this is seen as the emergence of another solution of the supergravity equations, namely the AdS black hole, that becomes dominant at high temperature, while empty AdS space is still dominant at low temperature. This is presented in sect. 8 where we also compute the Wilson loop, from which we can extract the string tension, the mass gap and more in general the discrete spectrum of glue balls.

In the final section 9 we discuss a recent proposal by Witten \cite{9} for studying four-dimensional Yang-Mills theory starting from the M-theory 5-brane solution and we compute in this approach the topological susceptibility and the string tension.

Recent and some of them very detailed reviews on the AdS/CFT conjecture can be found in Refs. \cite{10, 11, 12, 13, 14}.

2 Large $N$ QCD

In this section we discuss some diagrammatical properties of large $N$ QCD \cite{1}, we show that, unlike the perturbative expansion, the large $N$ expansion is an expansion according to the topology of the diagrams and not in powers of the coupling constant and we see that a picture in terms of an underlying string theory seems to naturally emerge from it. At the end of this section we discuss the emergence of the master field.

QCD is a gauge field theory based on the colour group $SU(3)$. It is an asymptotically free theory whose coupling constant in perturbation theory is given by:

$$\alpha_s(Q) = \frac{4\pi}{\left(\frac{11}{3}N - \frac{2}{3}N_f\right) \log \frac{Q^2}{\Lambda^2}}$$

(2.1)

where $N = 3$ and $\Lambda \sim 250\,\text{MeV}$ is the fundamental scale of QCD. At high energy ($Q^2 \gg \Lambda^2$) the coupling constant is small and therefore QCD is well described

\footnote{For a recent review of the large $N$ expansion in QCD see Ref. \cite{15}.}
by perturbation theory, but in order to study its low-energy properties as confinement, chiral symmetry breaking and the emergence of a mass gap we need non-perturbative methods. One of them consists in putting QCD on a lattice and use numerical simulations. In this way, however, we get only a numerical but not a concrete understanding of confinement based on a definite approximation. Actually, when we formulate gauge theories on a lattice, it is rather easy to compute various physical quantities in the strong coupling approximation and it is even not so difficult to compute several terms of the strong coupling expansion. For instance it is almost immediate to show that the vacuum expectation value of the Wilson loop has a leading term proportional to the area of the loop:

\[ W(I, J) = e^{-IJ \log(N g_Y^2)} \equiv e^{-IJ a^2 \sigma} \tag{2.2} \]

where we have considered a rectangular Wilson loop with sides of lengths \( Ia \) and \( Ja \) (\( a \) is the lattice spacing). According to the Wilson confinement criterium consisting in the fact that the Wilson loop is proportional to the area of the loop the behaviour found in eq.(2.2) implies that the strong coupling limit of gauge theories confines and that in this limit the string tension is given by:

\[ \sigma = \frac{1}{a^2} \log(N g_Y^2) \tag{2.3} \]

In the same paper \[16\] in which Wilson found that the Wilson loop is proportional to the area in strong coupling lattice gauge theory, it was also realized that the strong coupling expansion of the Wilson loop can be written as a sum over all surfaces as in the relativistic string model. Thus a string picture emerges from lattice gauge theory for strong coupling. However the behaviour of lattice gauge theory for strong coupling has in general nothing to do with the continuum limit of the theory, that is the one we are interested in and that is obtained instead when the lattice spacing \( a \) goes to zero corresponding, because of asymptotic freedom, to a weak coupling limit:

\[ a^2 \Lambda_0^2 = e^{-16\pi^2/(\beta_0 g_Y^2)} \]

\[ \beta_0 = \frac{11}{3} N - \frac{2}{3} N_f \tag{2.4} \]

where \( \Lambda_0 \) is the QCD scale in some normalization scheme. According to the renormalization group a physical quantity as the string tension should show the same behaviour in terms of the coupling constant as in eq.(2.4):

\[ \sigma a^2 = \left( \frac{\sigma}{\Lambda_0^2} \right) e^{-16\pi^2/(\beta_0 g_Y^2)} \tag{2.5} \]

Monte Carlo numerical simulations have shown \[17\] that confinement is indeed also a property of the weak coupling limit in which the continuum theory is supposedly recovered as one can see from the Monte Carlo data that show the exponential behaviour with the coupling constant as in eq.(2.4).

However this limit cannot be performed analytically in some approximation. Up to now it has only been possible to reach it by numerical simulations.
Since its original formulation \cite{3} the large $N$ expansion has been the most concrete possibility for reaching an analytical understanding of the non-perturbative aspects of QCD including its confinement properties. One generates a new expansion parameter by introducing $N$ instead of 3 colours. This means that we consider an $SU(N)$ instead of an $SU(3)$ gauge theory.

In order to describe the large $N$ expansion it is convenient to draw QCD Feynman diagrams in an apparently complicated notation \cite{3}: a gluon propagator is drawn as a pair of colour lines (each carrying a label going from 1 to $N$) and a quark propagator as one colour and one flavour-carrying line. When propagators are joined through vertices (also written, of course, in double-line notation) one can count, for each Feynman diagram, its dependence upon the gauge coupling $g_{YM}$, and the numbers of colour $N$ and of flavours $N_f$. In ’t Hooft’s original expansion one keeps the number of flavours $N_f$, as well as the combination $g_{YM}^2 N$, fixed as $N$ goes to infinity. The latter requirement follows from the need to keep $\Lambda_{QCD}$ fixed (see eq. (2.1)), ensuring that meson masses approach a finite limit. The requirement of keeping $N_f$ fixed is less obvious (in Nature, after all, $N < N_f$) but is crucial in order to have vanishing mesonic widths (they behave like $N_f/N$) and to establish therefore a connection with tree-level string theory. Therefore in the following we will keep $N_f$ fixed when $N \to \infty$.

By looking at specific examples it is easy to get convinced of the validity of the following general properties:

1. If one considers a correlation function of gauge invariant operators and if one looks at its dependence upon $N$ and $g_{YM}^2 N \equiv \lambda$ one can see that the dependence on $\lambda$ clearly varies with the order of the diagram, while the dependence on $N$ is only sensitive to its topological properties. Thus the large $N$ expansion selects the topology of Feynman diagrams rather than their order and can pick up, at lowest order, important non-perturbative effects.

2. Non-planar diagrams are down by a factor $1/N^2$ with respect to the planar ones.

3. Diagrams with quark loops are down by a factor $1/N$ with respect to those without quark loops.

Because of this the diagrams that dominate in the large $N$ limit are the planar ones with the minimum number of quark loops.

Let us consider a matrix element with two gauge invariant operators $J(x)$ involving bilinears of quark fields as for instance $\bar{\psi}\psi$ or $\bar{\psi}\gamma_{\mu}\psi$. The dominant connected diagrams contributing to a correlator containing two or more $J$’s are the planar ones with only one quark loop filled in all possible ways by the gluon exchanges. It is easy to see that the two and actually also the multipoint correlators are of order $N$ for large $N$

$$< J(k)J(-k) > \sim 0(N)$$

(2.6)
because the quark loop gives a factor $N$ while all the gluon exchanges give something that is constant if $\lambda$ is kept fixed. Since the diagram is planar it is easy to convince oneself that, if one cuts it, the intermediate states consist of an ordered set of partons starting from a quark and after many gluons ending on an antiquark and that each parton shares colour indices with his nearest neighbours in the chain. As a consequence the intermediate states are singlets of the gauge group and it is natural to associate them with mesons. Intermediate states with two mesons are negligible at large $N$. One can then factorize the two-point correlators in terms of a sum over meson contributions:

$$<J(k)J(-k)> = \sum_n \frac{a_n^2}{k^2 - m_n^2} \sim 0(N) \quad (2.7)$$

In the perturbative regime we can use perturbation theory where we can see that the previous correlator behaves as a logarithm of $k^2$. In order to reproduce this logarithmic behaviour, we need an infinite number of mesons. In addition, since the meson masses, being proportional to $\Lambda$, are smooth when $N \to \infty$, i.e. $m_n \sim 0(1)$, then the meson coupling, corresponding to the probability amplitude for a current to create a meson, $a_n = <0|J|n> \sim 0(\sqrt{N})$ grows up as $\sqrt{N}$. It is natural to associate intermediate states of this kind with string-like states of the form

$$|\mathcal{M}(C_{xy})> = \mathcal{M}(C_{xy})|0> \equiv \bar{\psi}(x)P e^{i \int_{C_{xy}} A_{\mu} dx_{\mu}} \psi(y)|0> \quad (2.8)$$

in which the path $C_{xy}$ can be seen as a string with quarks at its ends ($P$ denotes a path-ordered exponential and the trace is performed in group space).

Similarly, for gauge invariant correlation functions of purely gluonic sources, intermediate states at large $N_c$ have the same colour structure as:

$$|\mathcal{W}(C)> = \mathcal{W}(C)|0> \equiv Tr P \exp \left[ \oint_C dx_{\mu} A_{\mu}(x) \right] |0> \quad (2.9)$$

and are thus naturally associated with a closed string described by the path $C$.

Let us consider now a correlator involving 3 currents $J$. It will be a function of the three momenta $p, r, s$ of the three operators. It can contain three poles respectively in the variables $p^2, r^2$ and $s^2$ corresponding to the masses of the three mesons or only two poles. The terms with three poles contains three couplings $a_n$ and a 3-meson vertex. Since each $a_n \sim 0(\sqrt{N})$ and the total expression is $0(N)$ then the 3-meson vertex is $0(1/\sqrt{N})$. This means that, for $N \to \infty$ the mesons are an infinite number of stable particles. The large $N$ expansion has the nice property of separating the problem of the formation of hadrons connected to quark confinement and the generation of a mass gap from the problem of their residual interaction. Actually there are also some reasons to believe that mesons are excitations of a string. Already the representation of a meson given in eq.(2.8) is strongly reminiscent of a string. In addition, the perturbative expansion in string theory in terms of the string coupling constant $g_s$ and the large $N$ expansion of gauge
theories in powers of $1/N$ are both topological expansions in the sense that they are expansions according to the topology of respectively the string and the gauge theory diagrams. In particular the planar diagrams, that are the dominant ones in gauge theories, are pretty much reminiscent of the tree diagrams of string theory. A tree diagram for the scattering of $M$ mesons in string theory is of the order $g_s^{M-2}$, while the same amplitude for $N \to \infty$ in gauge theory is of the order $N^{-1-M/2}$.

A characteristic feature of the planar approximation, that we have already seen above, is that the intermediate states are “irreducible” colour singlet, in the sense that they cannot be split into two singlets. This is why the corresponding mesons should have exactly zero width in the large $N$ limit precisely as it is the case in tree-level string theory. Additional evidence for having a string theory coming out from QCD comes from hadron phenomenology. In fact, if mesons are excitations of strings they will lie on linearly rising Regge trajectories as experiments seem to indicate. There are also other aspects of hadron phenomenology that are well explained (also numerically) by the large-$N$ expansion as for instance the fact that even the heavy hadrons have a small width relative to their mass. Other ones are the validity of the Zweig’s rule according to which for instance the meson $\phi$ decays in $k\bar{k}$ rather than in 3 pions as favoured by the phase space and the numerical explanation of the $U(1)$ problem.

If ’t Hooft’s considerations can be easily extended from mesons made by a quark-antiquark pair to glueballs, predicting in particular their existence and narrowness, the generalization to baryons is much more subtle. This is certainly related to the fact that a baryon is, by definition, a completely antisymmetric object that one can make out of $N$ quarks. Thus, unlike the mesonic case, the baryon’s wave-function changes in an essential way with $N$ and one cannot expect the large $N$ limit to be smooth. Arguments can be given for the baryon mass to scale indeed like $N$ and therefore like $1/(1/N)$. If we identify $1/N$ with a coupling constant, such a behaviour is reminiscent of the monopole mass, and indeed, Witten (see Refs. [18, 19]) has taken up this analogy quite far. Yet, the actual relevance of large $N$ baryons for the physical nucleon remains to be proven.

In conclusion we have seen that the large $N$ expansion provides a very natural framework for discussing this QCD reinterpretation of the old Dual-String.

In the last part of this section we will briefly discuss the idea of the master field. We have seen above that, if we restrict ourselves to connected diagrams, the leading term of a correlator involving composites of the type $\bar{\psi}\psi$ is $0(N)$, while that of a correlator involving composites of the gluon is $0(N^2)$. It is, however, easy to see that disconnected diagrams are in general dominating. Therefore if we consider the Green’s function of a collection of Wilson-loop operators as the ones given in eqs.(2.8) and (2.9):

$$<0|M_1\cdots M_n|W_1\cdots W_m|0>$$  (2.10)

it is rather simple to prove that the leading large $N$ diagrams cannot have any propagator joining together two different operators. As a consequence the VEV in
eq. (2.10) becomes the product of the VEV's:

\[
<0|\mathcal{M}_1 \cdots \mathcal{M}_n \mathcal{W}_1 \cdots \mathcal{W}_m|0> = \langle 0|\mathcal{M}_1|0\rangle \cdots \langle 0|\mathcal{M}_n|0\rangle \langle 0|\mathcal{W}_1|0\rangle \cdots \langle 0|\mathcal{W}_m|0\rangle \quad (2.11)
\]

This almost trivial observation actually leads to a very deep result: the functional integral defining our correlation functions must be dominated by a single field, the so-called master field \[4\]. This result follows immediately from studying the expectation value of the square of an operator which can be equal to the square of the expectation value if and only if the quantum average is completely dominated by one path (as in the classical theory).

This powerful result gave great hope that the large $N$ limit of QCD could be solved in closed form. There is a large literature discussing the many amazing properties and equations satisfied by the master field. We do not have time to discuss it further here.

Unfortunately, none of these approaches has lead so far to an explicit expression for the large $N$ limit of four dimensional QCD: nonetheless, the idea that some kind of string must come out from QCD is still very popular. We know, however, that any string model associated with QCD cannot coincide with the usual bosonic (or super) string since these contain gravity (or supergravity), i.e. interactions which are not contained in QCD.

These problems have also brought several people to think that the QCD string is not infinitely thin but has a finite cross section. May be the relevant model is some kind of bag model with stringlike configurations.

Although no string model has yet been derived in a rigorous way from QCD, we have presented a number of indications supporting such a connection.

In conclusion, the large $N$ expansion is a very promising approach to understand non-perturbative properties of QCD as confinement, chiral symmetry breaking and the generation of a mass gap, but, although it drastically simplifies the structure of QCD keeping only the planar diagrams, it has not been possible to perform it explicitly and arrive to an explicit computation. There are also a number of indications supporting the idea that a string model is coming out from QCD. This has been, however, clashing with the fact that all consistent string models contain gravity, while QCD does not.

3 The large $N$ expansion in $CP^{N-1}$ model

We have concluded the previous section by seeing that it has not been possible to explicitly perform the large $N$ expansion in a matrix theory as QCD. We call it matrix theory because the gluon field is a matrix of $SU(N)$. In the first part of this section we study the properties of a very interesting two-dimensional vector model, called the $CP^{N-1}$ model with fermions, because, on the one hand, it has many properties in common with QCD as classical conformal invariance, existence of a
topological charge, instanton solutions, confinement and $U(1)$ anomaly and, on the other hand, it can be explicitly solved in the large $N$ limit. This model has been very useful in the past as a toy model for QCD. Although, unlike other two-dimensional models, confinement is in this model a quantum effect, the study of confinement in this model has not helped very much to understand confinement in QCD because confinement in two dimension is substantially different from confinement in four dimensions. It has instead been very useful for understanding how to solve the $U(1)$ problem in QCD.

The action of the $CP^{N-1}$ model with fermions is given by
\[
S = \int d^2x \left\{ \Box z + \bar{\psi} (\not{D} - M_B) \psi - \frac{g}{2 N_F} \left[ (\bar{\psi} \lambda^i \psi)^2 + (\bar{\psi} \gamma_5 \lambda^i \psi)^2 \right] \right\}
\] (3.1)

The scalar field $z$ has a colour index that transforms according to the fundamental representation of $SU(N)$, while the fermion field has a colour index that takes values from 1 to $N_F$ together with a flavour index that transforms according to the fundamental of $SU(N_f)$. $D_\mu$ is the covariant derivative of a $U(1)$ gauge field $A_\mu$ and the scalar field $z$ satisfies a constraint:
\[
|z|^2 = \frac{N}{2f} \quad D_\mu = \partial_\mu + \frac{2ief}{N} A_\mu
\] (3.2)

$e$ is taken to be equal to 1 in the covariant derivative for $z$.

In order to study the quantum theory of the model one must compute the generating functional for the euclidean Green’s functions given by:
\[
Z(J, \bar{J}, \eta, \bar{\eta}) = \int Dz D\bar{z} D\psi D\bar{\psi} \delta(|z|^2 - \frac{N}{2f}) \exp \left\{ -S + \int d^2x \left[ \bar{J} \cdot z + \bar{z} \cdot J + \bar{\eta} \cdot \psi + \bar{\psi} \cdot \eta \right] \right\}
\] (3.3)

One can eliminate the quartic terms for the $\psi$ field by the introduction of auxiliary fields and one can use the integral form of the $\delta$ function. In this way the action in the previous equation will contain only terms that are at most quadratic in the fields $z$ and $\psi$. Therefore the functional integral over those fields can be explicitly performed and one gets:
\[
Z(J, \bar{J}, \eta, \bar{\eta}) = \int D\alpha D\Phi_i D\Phi_5 \exp \left\{ -S_{eff} + \int d^2x \int d^2y \left[ \bar{J}(x) \Delta_B^{-1}(x,y) J(y) + \bar{\eta}(x) \Delta_F^{-1}(x,y) \eta(y) \right] \right\}
\] (3.4)

where
\[
\Delta_B = -D_\mu D_\mu + m^2 - \frac{i}{\sqrt{N}} \alpha \\
\Delta_F = \not{D} - M_B - \frac{\lambda^i}{\sqrt{N_F}} \left[ \Phi_i + \gamma_5 \Phi_5^i \right]
\] (3.5)

\(^4\)A review of this model with all relevant references can be found in Ref. 20.

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and

\[ S_{\text{eff}} = NTr \log \Delta_B - N_F Tr \log \Delta_F + \int d^2x \left[ i\sqrt{N} \frac{\alpha}{2f} + \frac{1}{2g} \left( \Phi^i \Phi_i + \Phi_5^i \Phi_5^i \right) \right] \]  \tag{3.6}

Although the effective Lagrangian is more complicated than the original microscopic Lagrangian, it has, however, the advantage of containing directly the meson fields \( \Phi \) and \( \Phi_5 \), while the "quark" \( \psi \) and the "gluon" \( z \) fields have been integrated out. The integral over the remaining composite fields cannot be performed exactly and we must develop the action for large \( N \) and \( N_F \) keeping the number of flavours \( N_f \) fixed. In addition we must also make this expansion around a minimum and such a minimum occurs for a non zero vacuum expectation value for the fields \( \Phi \) and \( \alpha \). Actually in the first eq. in (3.5) the term \( m^2 \) corresponding to the v.e.v. of \( \alpha \) has been already explicitly extracted. In the case in which the quark mass matrix is diagonal we can take the vacuum expectation values as follows:

\[ < \Phi^0 > = M_s \sqrt{N_F N_f} \quad \quad < \Phi^i > = < \Phi_5^0 > = < \Phi_5^i > = 0 \quad i \neq 0 \]  \tag{3.7}

The index zero corresponds to the singlet field. Expanding then the effective action around such a minimum we get that the leading term for large \( N \) is equal to:

\[ S^{(1)} = \sqrt{N} i\tilde{\alpha}(0) \left[ \frac{1}{2f} - \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + m^2} \right] + \]

\[ + 2\tilde{\Phi}^0 \sqrt{N_F N_f} \left[ M_s \frac{1}{2g} - M \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2 + M^2} \right] \]  \tag{3.8}

where the tilde indicates the Fourier transform

\[ \tilde{\alpha} = \int d^2x e^{-i\mathbf{p} \cdot \mathbf{x}} \alpha(x) \quad M = M_B + M_s \]  \tag{3.9}

The integrals appearing in eq.(3.8) are ultraviolet divergent. They can be regularized by the introduction of a Pauli-Villars cut-off \( \Lambda \). Then the saddle point condition \( S^{(1)} = 0 \) requires the bare coupling constants \( f \) and \( g \) to vary with \( \Lambda \) according to the equations:

\[ \frac{2\pi}{f(\Lambda)} = \log \frac{\Lambda^2}{m^2} \quad ; \quad \frac{2\pi}{g(\Lambda)} = \frac{M_s}{M} \log \frac{\Lambda^2}{M^2} \]  \tag{3.10}

which are typical of an asymptotic free theory. In addition, as also in QCD, there is a dimensional transmutation because in the quantum theory the dimensionless coupling constants \( f \) and \( g \) are traded with the two masses \( m \) and \( M \).

Having eliminated the term \( S^{(1)} \) we can consider the quadratic part of \( S_{\text{eff}} \) that is independent of \( N \) and \( N_F \) and that is given by:

\[ S^{(2)} = \frac{1}{2} \int d^2x \int d^2y \left\{ \alpha(x) \Gamma^\alpha(x-y) \alpha(y) + A_\mu \Gamma^A_{\mu
u}(x-y) A_\nu(y) + \right\} \]
\[
\Phi_i^{\Phi}(x-y)\Phi^i(y) + \Phi_i^{\Phi_5}(x-y)\Phi^i_5(y) + 2A_\mu(x)\Gamma_\mu^{\Lambda \Phi}\Phi^0_5 \tag{3.11}
\]

where the Fourier transforms of the inverse propagators are given by:

\[
\tilde{\Gamma}^\alpha = A(p; m^2) = \frac{1}{2\pi \sqrt{p^2(p^2 + 4m^2)}} \log \frac{\sqrt{p^2 + 4m^2} + \sqrt{p^2}}{\sqrt{p^2 + 4m^2} - \sqrt{p^2}} \tag{3.12}
\]

\[
\tilde{\Gamma}^A_{\mu \nu} = \left[ \delta_{\mu \nu} - \frac{p_\mu p_\nu}{p^2} \right] \left[ (p^2 + 4m^2)A(p; m^2) - \frac{1}{\pi} - \frac{N_F N_f}{N} e^2 \left( 4M^2 A(p; M^2) - \frac{1}{\pi} \right) \right] \tag{3.13}
\]

\[
\tilde{\Gamma}^{\Phi}_{ij} = \delta_{ij} \left[ \epsilon + (p^2 + 4M^2)A(p; M^2) \right] \tag{3.14}
\]

\[
\tilde{\Gamma}^{\Phi_5}_{ij} = \delta_{ij} \left[ \epsilon + p^2 A(p; M^2) \right] \tag{3.15}
\]

and

\[
\tilde{\Gamma}^{\Lambda \Phi}_{\mu} = -2\epsilon_{\mu \nu} p_\nu eM \sqrt{\frac{N_f N_F}{N}} A(p; M^2) \tag{3.16}
\]

In conclusion the leading term in $1/N$ describes a free theory of ”mesons” that are composite fields of the fundamental ”quark” and ”gluon” fields. Higher order terms in the large $N$ expansion will describe the meson interaction.

Let us now discuss the physical properties of this model. Because of asymptotic freedom its short distance properties are completely analogous to those of QCD. Unlike QCD we are able in this case to analytically study for large $N$ and $N_F$ also its low energy properties. In particular by using the low-energy expansion:

\[
A(p; m^2) \sim \frac{1}{4\pi m^2} \left[ 1 - \frac{2}{3} \frac{p^2}{4m^2} + \ldots \right] \tag{3.17}
\]

we can extract from eq.(3.11) the low energy effective Lagrangian that in the simplified case where $N_F = e = 1$ is given by:

\[
L_{\text{eff}} = \frac{1}{2} \left[ (\partial_\mu \Pi^i)^2 + m_\pi^2 (\Pi^i)^2 \right] + \frac{1}{2} \left[ (\partial_\mu \sigma^i)^2 + (m_\pi^2 + 4M^2)(\sigma^i)^2 \right] + \frac{1}{8\pi m^2} \alpha^2 + \frac{1}{24\pi m^2} F^2 + i\sqrt{\frac{2N_f}{N}} F_\pi F \cdot S \tag{3.18}
\]

where

\[
F = \epsilon_{\mu \nu} \partial_\mu A_\nu \quad \Pi^i = \frac{1}{2\sqrt{\pi} M} \Phi^i_5 \quad \Pi^0 \equiv S \quad \sigma^i = \frac{1}{2\sqrt{\pi} M} \Phi^i \tag{3.19}
\]

and

\[
F_\pi = \frac{1}{\sqrt{2\pi}} \quad m_\pi^2 = 4\pi \epsilon M^2 \tag{3.20}
\]

An important property of this model is the generation of a kinetic term for the vector field $A_\mu$ that was not present in the classical theory. In two dimension this
implies the generation of a confining linear potential with string tension equal to 
\[ \sigma = (12m^2\pi)/N. \]

The factor \(1/N\) comes from the coupling between ”coloured” states and \(A_\mu\). Another important property that is pretty much related to the previous one is the appearance of a dependence on the \(\theta\) vacuum parameter in the large \(N\) expansion. In fact if we introduce the topological charge density

\[ q(x) = \frac{1}{2\pi\sqrt{N}} F(x) \]  

(3.21)

where \(F\) is the field defined in eq.(3.19), we neglect all terms in eq.(3.18) that include mesonic fields and we add a term with the vacuum \(\theta\) parameter we are led to consider the following effective Lagrangian:

\[ L_{eff} = \frac{1}{2} \frac{\pi N}{3m^2} q^2 + i\theta q + qJ \]  

(3.22)

where \(J\) is an external source. The algebraic equation of motion for \(q\) that one gets from the previous Lagrangian is

\[ q = -\frac{3m^2}{\pi N} (J + i\theta) \]  

(3.23)

Inserting it in eq.(3.22) we get the following generating functional:

\[ Z(J, \theta) \equiv e^{-W(J, \theta)} = e^{\frac{3m^2}{2\pi N} \int d^2x (J + i\theta)^2} \]  

(3.24)

From it putting \(J = 0\) we can extract the vacuum energy

\[ E(\theta) \equiv W(\theta, 0) = \frac{3m^2}{2\pi N} \theta^2 \]  

(3.25)

the one-point function for the topological charge density

\[ \langle q(x) \rangle_\theta = i\frac{3m^2}{\pi N} \theta \]  

(3.26)

and the two-point function for \(q\)

\[ \langle q(x)q(y) \rangle = \frac{3m^2}{\pi N} \delta(x - y) \]  

(3.27)

Notice that the vacuum energy has the form

\[ E(\theta) = NF(\theta/N) \quad F(x) = x^2 \]  

(3.28)

where the factor \(N\) in front counts just the number of degrees of freedom. From eq.(3.27) we can compute the topological susceptibility:

\[ \langle q(x) \int d^2y q(y) \rangle = \frac{d^2E(\theta)}{d\theta^2} = \frac{3m^2}{\pi N} \]  

(3.29)
Another important property of this model is the presence in the effective low-energy Lagrangian in eq. (3.18) of a mixed term with the singlet axial field and the vector field that is fundamental for the resolution of the $U(1)$ problem. We do not discuss further this here since in the next section we will be discussing its resolution in QCD.

In the second part of this section we consider QCD in two dimensions ($QCD_2$) and we show that in the light cone gauge it is possible to reformulate it completely in terms of a bilocal mesonic field [21, 22]. We then show that the master field, corresponding to the vacuum expectation value of the bilocal mesonic field, is fixed in the limit of a large number of colours by a saddle point equation whose solution is equal to the fermion propagator constructed in the original paper by 't Hooft [23]. Considering then the quadratic term containing the fluctuation around the saddle point it is possible to show that the equation of motion constructed from it gives exactly the integral equation found in Ref. [23] for the mesonic spectrum.

We consider the action

$$S = \int d^2x \left\{ \frac{1}{2g_0^2}(\partial_{x-} A^a_+)^2 + i\sqrt{2}(\bar\psi^A_+ \partial_+ \psi^A_+ + \bar\psi^A_- \partial_- \psi^A_-) - m^i \bar\psi^A_- \psi^A_+ - m^i \bar\psi^A_+ \psi^A_- - A^a_+ \sqrt{2} \bar\psi^A_+ T^a_{AB} \psi^B_+ \right\} \tag{3.31}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$, $iD_{AB} = \gamma_\mu (i\partial_\mu \mathbb{I} - A^a_\mu T^a_{AB})$, $a, b = 1...N^2 - 1$ are the indices of the adjoint representation of the colour group, $A, B = 1...N$ run over the fermionic representation of the colour $SU(N)$. If we choose the gauge $A^a_\pm = 0$ and we normalize the trace over the fundamental representation to one, we can rewrite the previous action as

$$S = \int d^2x \left\{ i\sqrt{2}(\bar\psi^A_+ \partial_+ \psi^A_+ + \bar\psi^A_- \partial_- \psi^A_-) - m^i \bar\psi^A_- \psi^A_+ - m^i \bar\psi^A_+ \psi^A_- - A^a_+ \sqrt{2} \bar\psi^A_+ T^a_{AB} \psi^B_+ \right\} \tag{3.31}$$

Integrating over $A^a_\pm$ we get

$$S = \int d^2x \left\{ i\sqrt{2}(\bar\psi^A_+ \partial_+ \psi^A_+ + \bar\psi^A_- \partial_- \psi^A_-) - m^i \bar\psi^A_- \psi^A_+ - m^i \bar\psi^A_+ \psi^A_- \right\} \tag{3.31}$$

### Conventions.

$$x^\pm = x_\mp = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \quad A^\mu B_\mu = A_0 B_0 - A_1 B_1 = A_\pm B_\mp + A_\mp B_\pm$$

$$\gamma_+ = \gamma^- = \left( \begin{array}{cc} 0 & \sqrt{2} \\ 0 & 0 \end{array} \right) \quad \gamma_- = \gamma^+ = \left( \begin{array}{cc} 0 & 0 \\ \sqrt{2} & 0 \end{array} \right) \quad \gamma_5 = -\gamma_0 \gamma_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \quad P_{R,L} = \frac{1 \pm \gamma_5}{2}$$

$$\psi = \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) \quad \bar\psi = \left( \begin{array}{c} \bar\psi_- \\ \bar\psi_+ \end{array} \right)$$

$$\chi \bar\psi = -\frac{1}{\sqrt{2}} \begin{pmatrix} \bar\psi \gamma_\pm P_R \chi & \bar\psi \gamma_- \chi \\ \psi \gamma_+ \chi & \sqrt{2} \psi P_L \chi \end{pmatrix}$$

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\[-g_0^2 \int d^2 x \, d^2 y \, G(x - y) \, \bar{\psi}^A_i(x) T_{AB} \psi^B_j(y) \psi^C_j(y) T_{CD} \psi^D_j(y) = \]
\[= \int d^2 x \, \left\{ i\sqrt{2} (\bar{\psi}^A_i \partial_x \psi^A_i + \bar{\psi}^A_i \partial_x \psi^A_i) - m_i (\bar{\psi}^A_i \psi^A_i + \bar{\psi}^A_i \psi^A_i) \right\} \]
\[+ \quad g_0^2 R \int d^2 x \, d^2 y \, G(x - y) \left\{ \bar{\psi}^A_i(x) \psi^A_j(y) \right\} \]
\[- \frac{R}{N_R} \bar{\psi}^A_i(x) \bar{\psi}^A_i(x) \psi^A_j(y) \psi^A_j(y) \}\]
\[\text{(3.32)}\]

where we used \(\sum_a T_{AB}^a T_{CD}^a = x_R(\delta_{BC} \delta_{AD} + \frac{R}{N} \delta_{AB} \delta_{CD})\) valid for the fundamental representation and where \(G(x) = -\frac{1}{2} \delta(x^+) |x^-| = \int \frac{d^2 k}{(2\pi)^2} e^{ik \cdot x} \frac{1}{k^2}\).

The interaction term suggests to introduce the composite field
\[\rho^{ij}(x, y) = \sum_A \bar{\psi}^A_j(y) \gamma^5 \psi^A_i(x)\]
and its partners
\[\rho^{ij}_+(x, y) = \sum_A \bar{\psi}^A_j(y) \gamma^5 \psi^A_i(x)\]
\[\sigma^{ij}(x, y) = \sum_A \bar{\psi}^A_j(y) \gamma^0 \psi^A_i(x)\]
\[\sigma^{ij}_5(x, y) = \sum_A \bar{\psi}^A_j(y) \gamma^5 \psi^A_i(x)\]
\[\sigma^{ij}_{R,L}(x, y) = \frac{\sigma^{ij}(x, y) \pm \sigma^{ij}_5(x, y)}{\sqrt{2}}\]
\[\text{(3.34)}\]

Now we want to change variables in the functional integral and integrate over the mesonic fields \(\rho\) and \(\sigma\) instead of the original quark field \(\psi\). We need to compute the jacobian of the transformation from the \(\tilde{\psi}\), \(\psi\) to the \(\rho\), \(\sigma\) that is given by
\[J[\rho, \rho, \sigma, \sigma] = \int[D\bar{\psi}^A_i \, D\psi^A_i \, D\bar{\psi}^A_i \, D\psi^A_i]
\[
\prod_{xy} \delta[\rho^{ij}(x, y) - \sqrt{2} \sum_A \bar{\psi}^A_j(y) \psi^A_i(x)] \prod_{xy} \delta[\rho^{ij}_+(x, y) - \sqrt{2} \sum_A \bar{\psi}^A_j(y) \psi^A_i(x)]
\]
\[
\prod_{xy} \delta[\sigma^{ij}(x, y) - \sqrt{2} \sum_A \bar{\psi}^A_j(y) \psi^A_i(x)] \prod_{xy} \delta[\sigma^{ij}_5(x, y) - \sqrt{2} \sum_A \bar{\psi}^A_j(y) \psi^A_i(x)]
\]
\[= \int[D\bar{\psi}^A_i \, D\psi^A_i \, D\bar{\psi}^A_i \, D\psi^A_i][D\alpha^{ij}_+ \, D\alpha^{ij}_5 \, D\beta^{ij}_R \, D\beta^{ij}_L]
\]
\[e^{\alpha^{ij}_+(y,x)\rho^{ij}(x,y) - \sqrt{2} \sum_A \bar{\psi}^A_j(y) \psi^A_i(x)} + \alpha^{ij}_+(y,x)\rho^{ij}_+(x,y) - \sqrt{2} \sum_A \bar{\psi}^A_j(y) \psi^A_i(x)]
\]
\[e^{\beta^{ij}_L(y,x)\sigma^{ij}_L(x,y) - \sqrt{2} \sum_A \bar{\psi}^A_j(y) \psi^A_i(x)} + \beta^{ij}_L(y,x)\sigma^{ij}_L(x,y) - \sqrt{2} \sum_A \bar{\psi}^A_j(y) \psi^A_i(x)]\]
\[\text{(3.35)}\]

where the sum over the flavour and space time indices is understood.
If we introduce the matrices

\[ M = ||M_{PQ}|| = \begin{pmatrix} \beta^j_i(x, y) & \alpha^j_i(x, y) \\ \alpha^i_j(x, y) & \beta^i_j(x, y) \end{pmatrix}, \]

\[ U = ||U_{PQ}|| = \begin{pmatrix} \sigma^j_i(x, y) & \rho^j_i(x, y) \\ \rho^i_j(x, y) & \sigma^i_j(x, y) \end{pmatrix} \]

(3.36)

where \( P \equiv (xi\alpha) \) and \( Q \equiv (yj\beta) \), we can rewrite the exponent of the integrand in eq. (3.35) as

\[ J[U] = \int [d\bar{\Psi}^A d\Psi^A] dM \exp [Tr(MU) - \sqrt{2} \bar{\Psi}^A M \Psi^A] \]

(3.37)

where \( N \) is the dimension of the fermionic representation and \( Tr \equiv tr_x tr_i tr_\alpha \).

Evaluating this integral with the saddle point method we get

\[ J[U] \propto \exp[-NTr \log U] \] (3.38)

where we have neglected non leading contributions in \( N \).

If we define the matrix

\[ D = ||D_{PQ}|| = \begin{pmatrix} -m^i_j \delta^2(x - y) & i \delta^i_j \partial_x \delta^2(x - y) \\ i \delta^i_j \partial_x \delta^2(x - y) & -m^i_j \delta^2(x - y) \end{pmatrix} \] (3.39)

where \( m^i_j \equiv 1/\sqrt{2} m^i \delta^i_j \), and we rescale the master field \( U \to NU \), we can rewrite the effective action as

\[ \frac{1}{N} S_{\text{eff}} = Tr(DU + i \log U) + \frac{1}{2} g^2 \int d^2x \ d^2y \ G(x - y) \ U_{(xi1),(yj2)} \ U_{(yj1),(xi2)} \]

\[- \frac{1}{2N} g^2 R \int d^2x \ d^2y \ G(x - y) \ U_{(xi1),(xi2)} \ U_{(yj1),(yj2)} \] (3.40)

where \( g^2 = g_0^2 x_R N \).

Varying the effective action with respect to \( U_{QP} \), we get the equation for the master field, that in the leading order in \( N \) is equal to

\[ DPQ + i(U^{-1})_{PQ} + g^2 \delta_{\alpha,2} \delta_{\beta,1} G(x - y) \ U_{(xi2),(yj1)} = 0 \] (3.41)

Multiplying it with \( U \), we get immediately

\[ D U_{PQ} + i \ \mathbb{1}_{PQ} + g^2 \delta_{\alpha,2} \int d^2z \ G(x - z) \ U_{(xi1),(zk2)} \ U_{(zk1),Q} = 0 \] (3.42)
Writing explicitly these equations we find

\[ i\partial_x^\mu \rho_\mu^ij(x, y) - m^\mu \sigma_\mu^ij(x, y) + g^2 \int d^2 z \, G(x - z) \rho_\mu^i(x, z) \rho_\mu^j(z, y) + i \delta^{ij} \delta^2(x - y) = 0 \] (3.43)

\[ i\partial_x \sigma_\mu^ij(x, y) - m^\mu \rho_\mu^ij(x, y) = 0 \] (3.44)

\[ i\partial_x^\mu \rho_\mu^ij(x, y) - m^\mu \sigma_\mu^ij(x, y) + i \delta^{ij} \delta^2(x - y) = 0 \] (3.45)

\[ i\partial_x \sigma_\mu^ij(x, y) - m^\mu \rho_\mu^ij(x, y) + g^2 \int d^2 z \, G(x - z) \rho_\mu^i(x, z) \sigma_\mu^j(z, y) = 0 \] (3.46)

In particular if we eliminate \( \sigma_\mu^ij(x, y) \) from the first equation using the second one, we get the fundamental equation

\[ i\partial_x \rho_\mu^ij(x, y) + i(m \cdot m)^\mu \int d^2 z \, \delta(x^+ - z^+) \theta(x^- - z^-) \rho_\mu^j(z, y) \]

\[ + g^2 \int d^2 z \, G(x - z) \rho_\mu^i(x, z) \rho_\mu^j(z, y) + i \delta^{ij} \delta^2(x - y) = 0 \] (3.47)

In order to solve this equation it is better to pass to momentum space. Since

\[ \rho_\mu^ij(x, y) = \sqrt{2} < 0 | \sum_A \bar{\psi}_A^j(y) \psi_A^i(x) | 0 > \]

and the vacuum is translationally invariant, we need only one momentum for the Fourier transform of \( \rho_\mu^ij(x, y) \). The previous equation (3.47) becomes

\[ \left[-p_+ \delta^{ik} + \frac{(m \cdot m)^k}{p_-} + g^2 \int dk \, G(k) \, \rho_\mu^k(p - k) \right] \rho_\mu^j(p) + i \delta^{ij} = 0 \] (3.48)

and it suggests to set

\[ \rho_\mu^ij(x, y) = \int \frac{d^2 p}{(2\pi)^2} \, e^{ip \cdot (x - y)} \rho_\mu^ij(p) = \]

\[ = \delta^{ij} \int \frac{d^2 p}{(2\pi)^2} \, e^{ip \cdot (x - y)} \frac{2i p_-}{2p_+ p_- - 2(m \cdot m)^i - p_- \Gamma(p) + i\epsilon} \] (3.49)

With this substitution eq. (3.48) becomes eq. (3.39) of ref. [23]:

\[ \Gamma(p) = \frac{4g^2}{(2\pi)^2} \int \frac{d^2 k}{k_-^2} \frac{i(p_- + k_-)}{2(p + k_+) + (p + k)_- - 2(m \cdot m)^i - (p + k)_- \Gamma(p + k) + i\epsilon} \] (3.50)

The explicit solution yields

\[ \Gamma(p) = \Gamma(p_-) = \frac{g^2}{\pi} \left( \frac{\text{sgn}(p_-)}{\lambda} - \frac{1}{p_-} \right) \] (3.51)

where \( \lambda \) is an infra-red cutoff introduced in ref. [23].
Inserting eq. (3.49) in eqs. (3.44), (3.45) and (3.46), we get the Fourier transform of the master field

\[ U_0^{ij}(p) = \frac{i \delta^{ij}}{2p_+p_- - 2(m \cdot m)^i - p_- \Gamma(p) + i\epsilon} \left( \begin{array}{cc} -2m_i & 2p_- \\ 2p_+ - \Gamma(p) & -2m_i \end{array} \right) \]  

(3.52)

where \( \Gamma(p) \) given in eq. (3.51). \( U_0(p) \) is the master field of QCD2 that is identified with the vacuum expectation value of the quark propagator.

Let us now consider the mass spectrum of the theory, i.e. the fluctuations around the master field. To this purpose we write \( U = U_0 + \frac{1}{\sqrt{N}} \delta U \), and we consider the terms in the effective action that are \( O(1) \) in \( N \). They are given by the quadratic terms in the fluctuation \( \delta U \):

\[ S_{eff}^{(2)} = -\frac{i}{2} Tr(U_0^{-1} \delta U U_0^{-1} \delta U) + \frac{1}{2} g^2 \int d^2x \; d^2y \; G(x-y) \; \delta U(x_1) \; \delta U(y_1) \]  

\[ - \frac{g^2 R}{2} \int d^2x \; d^2y \; G(x-y) U_0(x_1) U_0(y_1) U_0(x_2) U_0(y_2) \]  

(3.53)

The last term in the previous equation does not depend on \( \delta U \) and therefore will be neglected. The spectrum of the theory is determined by the equation of motion for the field \( \delta U \) that is given by

\[ i \delta U^{ij}_{\alpha\beta}(x, y) = g^2 \int d^2u \; d^2v \; U_{\alpha\beta}^{jkl}(x-u) \; G(u-v) \; \delta U_{\alpha\beta}^{k}(u, v) \; U^{ij}_{0 \alpha\beta}(v-y) \]  

(3.54)

and that in Fourier space leads to

\[ \delta U^{ij}_{\alpha\beta}(r, s) = -i g^2 T^{(ij)}_{\alpha\beta}(s + \frac{r}{2}, s - \frac{r}{2}) \int \frac{d^2k}{(2\pi)^2 k^2} \; \frac{1}{\Delta^i(s + \frac{r}{2}) \Delta^j(s - \frac{r}{2})} \; \delta U_{12}^{ij}(r, s - k) \]  

(3.55)

(no sum over \( i \) and \( j \)), where

\[ \Delta^i(p) = 2p_+p_- - 2(m \cdot m)^i - p_- \Gamma(p) + i\epsilon \]  

(3.56)

and

\[ T^{(ij)}_{\alpha\beta}(p_-, q_-) = \left( \begin{array}{cc} 4p_- m^i & -4p_- q_- \\ -4m^i m^j & 4m^i q_- \end{array} \right) \]  

(3.57)

Following ’t Hooft [23], we integrate both sides of eq. (3.54) over the variable \( s_+ \) and defining the gauge invariant field\[\] [5] *

\[ \varphi^{ij}_{\alpha\beta}(r, s_-) = \int \frac{ds_+}{2\pi} \; \delta U^{ij}_{\alpha\beta}(r, s) \]  

(3.58)

* We define

\[ \delta U(x, y) = \int \frac{d^2r}{(2\pi)^2} \frac{d^2s}{(2\pi)^2} e^{i\frac{r}{r} + i\frac{s}{s}(x-y)} \delta \tilde{U}(r, s) \]  

In the following we suppress the tilde over the Fourier transformed fields.

** Notice that this is equivalent to set \( x^+ = y^+ \) in \( \delta U(x, y) \), thus obtaining a gauge invariant object. If \( x^+ \neq y^+ \) then \( U(x, y) \) is not gauge invariant under the residual gauge transformations.
we get choosing $r_- > 0$

$$
\varphi_{\alpha\beta}^{ij}(r,s_-) = g^2 \frac{T_{\alpha\beta}^{(ij)}}{4|s_- + \frac{r_-}{2}|s_- - \frac{r_-}{2}^2} \left[ \frac{M_i^2}{2|s_- + \frac{r_-}{2}|} + \frac{M_j^2}{2|s_- - \frac{r_-}{2}|} + \frac{g^2}{\pi \lambda} r_+ \right]^{-1}
$$

$$
\theta(s_- + \frac{r_-}{2})\theta(\frac{r_-}{2} - s_-) \int \frac{dk_-}{2\pi k_-^2} \varphi_{12}^{ij}(r,s_- - k_-)
$$

where

$$
M_i^2 = 2(m \cdot m)_i - \frac{g^2}{\pi}
$$

In the sector $(\alpha, \beta) = (2, 1)$ it yields the ’t Hooft equation (eq. (15) of ref. [23]) when one identifies the Fourier transform of $\rho_{ij}^{ij}(x,y)$ with $\psi(p,r)$. In the other sectors requiring the cancellation of the IR cutoff $\lambda$, we get

$$
\varphi_{\alpha\beta}^{ij}(r,s_-) = \frac{T_{\alpha\beta}^{(ij)}}{4|s_- + \frac{r_-}{2}|s_- - \frac{r_-}{2}^2} \varphi_{12}^{ij}(r,s_-)
$$

Performing the same straightforward manipulations as in ref. [23], one is led to an integral equation for the mass spectrum ($\varphi = \varphi_{12}$; we rescale $s_- = r_- (x - \frac{1}{2})$ and define $\mu^2 = 2r_+ r_-)$:

$$
\mu^2 \varphi^{ij}(x) = \left[ \frac{M_i^2}{x} + \frac{M_j^2}{(1-x)} \right] \varphi^{ij}(x) - \frac{g^2}{\pi} P \int_0^1 \frac{\varphi^{ij}(y)}{(y-x)^2} dy
$$

that is the famous ’t Hooft equation, with a discrete spectrum of eigenvalues labelled by an integer $n$ such that $\mu_n^2 \approx g^2 \pi n$, for $n \to \infty$.

In the other sectors we get the same equation for the mass spectrum, but the mesonic fields change according to

$$
\varphi_{\alpha\beta}^{ij}(x) = C_{\alpha\beta}^{(ij)}(x) \varphi^{ij}(x)
$$

with

$$
C_{\alpha\beta}^{(ij)}(x) = \left( \begin{array}{cc}
\frac{m^i_j}{(1-x)r_-} & 1 \\
\frac{m^i_j}{x(1-x)r_-} & \frac{m^i_j}{x r_-}
\end{array} \right)
$$

In conclusion in this section we have considered two two-dimensional models, the $CP^{N-1}$ model and $QCD_2$, in which the large-$N$ expansion can be explicitly done, and we have studied their properties for $N \to \infty$. In particular in the case of $QCD_2$ we have constructed the master field and the spectrum of mesons in the large-$N$ limit.
4 $U(1)$ problem

In this section we discuss the resolution of the $U(1)$ problem in the framework of the large $N$ expansion of QCD. In addition to colour gauge symmetry QCD has also a flavour symmetry. If the quark mass matrix is zero QCD is invariant under the transformations corresponding to independent $U(N_f)$ rotations of the right and left parts of the quark field:

$$\psi_L \equiv \frac{1 - \gamma_5}{2} \psi \rightarrow U_L \psi_L \quad \psi_R \equiv \frac{1 + \gamma_5}{2} \psi \rightarrow U_R \psi_R$$  \hspace{1cm} (4.1)

where both $U_R$ and $U_L$ are $U(N_f)$ matrices. This $U_L(N_f) \otimes U_R(N_f)$ symmetry of the QCD action is called chiral symmetry. In the quantum theory QCD has an anomaly given by [3]:

$$\partial_\mu J_5^\mu = 2N_f q(x) \quad q(x) = \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a$$ \hspace{1cm} (4.2)

where $q(x)$ is the topological charge density of QCD. As a consequence, QCD with massless quarks has only a $SU_L(N_f) \otimes SU_R(N_f) \otimes U_V(1)$ where $U_V(1)$ corresponds to the baryonic number conservation. In the real world chiral symmetry can only be an approximate symmetry because quarks have a non zero mass. However, if we restrict ourselves to the three light flavours, it is an approximate symmetry because their masses are small with respect to the QCD scale $\Lambda_{QCD}$. Since, however, this symmetry is not seen in the spectrum (there is no scalar particle that has approximately the same mass of the pion!), it is assumed (and this assumption is confirmed by lattice numerical simulations) that in QCD it is spontaneously broken. The vectorial $SU(N_f)$ symmetry, that is left unbroken, is instead an approximate classification symmetry for the hadrons. As a consequence of the spontaneous breaking of chiral symmetry we get that the pseudoscalar mesons are the quasi Goldstone bosons corresponding to the spontaneous breaking of chiral symmetry and their low energy interaction can be exactly computed. In particular at low energy we can neglect all hadrons except those that are massless in the chiral limit. The effective Lagrangian describing the quasi Goldstone bosons is that of the non linear $\sigma$-model:

$$L = \frac{1}{2} Tr \left( \partial_\mu U \partial_\mu U^{-1} \right) + \frac{F_{\pi}^2}{2\sqrt{2}} Tr \left( MU + M^\dagger U^\dagger \right)$$ \hspace{1cm} (4.3)

where $U$ is a $3 \times 3$ or $N_f \times N_f$ matrix containing the fields of the pseudoscalar mesons:

$$U = \frac{F_{\pi}}{\sqrt{2}} e^{i\sqrt{2}\Phi/F_{\pi}} \quad \Phi = \Pi^i \lambda^i + \frac{1}{\sqrt{N_f}} S/\sqrt{N_f} \quad Tr \left( \lambda^i \lambda^j \right) = \delta^{ij}$$ \hspace{1cm} (4.4)

**The fact that the resolution of the $U(1)$ problem is intimately connected to the existence of the axial anomaly was suggested in Ref. [2].**
while the mass matrix $M$ can be taken to be diagonal:

$$M_{ij} = \mu_i^2 \delta_{ij} \quad (4.5)$$

In the chiral limit ($M = 0$) the Lagrangian in eq.(4.3) is invariant under chiral transformations that transform $U \rightarrow U_L U_R$ and therefore cannot be the effective Lagrangian for low-energy QCD because it does not contain the $U(1)$ anomaly in eq.(4.2). In order to have the anomaly equation satisfied we must add to the previous Lagrangian terms that also include the field corresponding to the topological charge $q(x)$ appearing in the r.h.s. of the anomaly equation. On general ground we can write the following Lagrangian:

$$L = \sum_{i=0}^{\infty} L_i(U) [q(x)]^i \quad (4.6)$$

where the first term with $i = 0$ is equal to the kinetic term in eq.(4.3). Neglecting derivative terms, that are irrelevant at low energy, requiring parity conservation and imposing that the axial anomaly is reproduced (this implies that, under an axial $U(1)$ transformation with angle $\alpha$, the previous Lagrangian transforms as $L \rightarrow L - 2\alpha N_f q(x)$) we get that all terms of the sum with even indices are invariant under the complete $U(N_f) \otimes U(N_f)$ symmetry, all the terms with odd indices are vanishing except the first one that is given by

$$L_1 = \frac{i}{2} q(x) \text{Tr} \left[ \log U - \log U^{-1} \right] \quad (4.7)$$

This term precisely reproduces in the effective theory the anomaly equation. In order to have additional restrictions we need to use the large $N$ expansion. Using the arguments developed in sect. 2 it is easy to check the following behaviour with $N$:

$$F_\pi \sim 0(\sqrt{N}) \quad L_{2k} \sim 0(N^{2-2k}) \quad (4.8)$$

This means that, for large $N$, we can neglect all even terms except the lowest one. Keeping only the leading terms in the large $N$ expansion we arrive at the following Lagrangian:

$$L = L_0(U) + \frac{i}{2} q(x) \text{Tr} \left[ \log U - \log U^{-1} \right] + \frac{1}{a F_\pi^2} q^2 + \frac{F_\pi}{2 \sqrt{2}} \text{Tr} \left( MU + M^\dagger U^\dagger \right) - \theta q \quad (4.9)$$

where $a$ is an arbitrary parameter that is $0(1/N)$ for large $N$ and we have also allowed for an arbitrary $\theta$ parameter.

It is now convenient to use the algebraic equation of motion for $q(x)$ to bring eq.(4.9) in the following form $^{11}$:

$$L = L_0(U) + \frac{F_\pi}{2 \sqrt{2}} \text{Tr} \left( MU + M^\dagger U^\dagger \right) - \frac{a F_\pi^2}{4} \left[ \theta - \frac{i}{2} \text{Tr} \left( \log U - \log U^{-1} \right) \right]^2 \quad (4.10)$$

$^{11}$The resolution of the $U(1)$ problem in the framework of the large $N$ expansion was given in Ref. [25, 26]. See also Ref. [27]. The effective Lagrangian in eq.(4.10) was derived in Refs. [28, 29, 30].
Since $UU^\dagger$ is proportional to the unit matrix and the mass matrix is diagonal we can take the vacuum expectation value of $U$ to be of the following form:

$$< U_{ij} > = e^{-i\phi_i} \delta_{ij} \frac{F_\pi}{\sqrt{2}}$$  \hspace{1cm} (4.11)

where the angles $\phi_i$ are determined imposing that $< U_{ij} >$ minimizes the energy corresponding to the Lagrangian in eq.(4.10) that is given by:

$$E = \frac{aF_\pi^2}{4}(\theta - \sum_i \phi_i)^2 - \frac{F_\pi^2}{2} \sum_i \mu_i^2 \cos \phi_i$$  \hspace{1cm} (4.12)

after having used eq.(4.11). Hence they must satisfy the following equations:

$$\mu_i^2 \sin \phi_i = a(\theta - \sum_i \phi_i)$$  \hspace{1cm} (4.13)

It is convenient to work with a field $V$ whose vacuum expectation value is proportional to the unit matrix. In terms of $U$ it is given by:

$$V_{ij} = U_{ik} < U_{kj} >^{-1} \frac{F_\pi}{\sqrt{2}}$$  \hspace{1cm} (4.14)

and the Lagrangian in eq.(4.10) becomes:

$$L = L_0(V) + \frac{aF_\pi^2}{16} \left[ Tr(\log V - \log V^\dagger) \right]^2 + \frac{F_\pi^2}{2\sqrt{2}} Tr \left[ M(\theta)(V + V^\dagger - \sqrt{2}F_\pi) \right] +$$

$$+ i \frac{aF_\pi}{2\sqrt{2}}(\theta - \sum_i \phi_i) Tr \left[ \frac{F_\pi}{\sqrt{2}}(\log V - \log V^\dagger) - (V - V^\dagger) \right]$$  \hspace{1cm} (4.15)

where an inessential constant has been omitted and

$$M_{ij}(\theta) = \mu_i^2(\theta) \delta_{ij} \hspace{1cm} \mu_i^2(\theta) = \mu_i^2 \cos \phi_i$$  \hspace{1cm} (4.16)

Since we are interested only in those results that follow from current algebra we can take as we have done in eq.(4.13)

$$V = \frac{F_\pi}{\sqrt{2}} e^{i\sqrt{2}\phi/F_\pi} \hspace{1cm} \Phi = \Pi^i \lambda^i + \frac{S}{\sqrt{N_f}}$$  \hspace{1cm} (4.17)

In this case one gets the following final Lagrangian:

$$L = \frac{1}{2} Tr(\partial_{\mu}V \partial^\mu V^\dagger) - \frac{1}{2}aN_fS^2 +$$

$$+ \frac{F_\pi^2}{2} Tr \left[ M(\theta)(\cos \frac{\sqrt{2}}{F_\pi} \Phi - 1) \right] + aF_\pi^2(\theta - \sum_i \phi_i) Tr \left[ \frac{F_\pi}{\sqrt{2}} \sin \frac{\sqrt{2}}{F_\pi} \Phi - \Phi \right]$$  \hspace{1cm} (4.18)
where the term implied by the axial anomaly has generated a mass term for the singlet field $S$ that has a coefficient $0(1/N)$ for large $N$ ($a \sim 0(1/N)$).

The mass spectrum of the pseudoscalar mesons can just be obtained from the quadratic part of the Lagrangian:

$$L_2 = \frac{1}{2} Tr (\partial_\mu \Phi \partial^\mu \Phi) - \frac{a}{2} Tr(\Phi) Tr(\Phi) - \frac{1}{2} Tr \left[ M(\theta) \Phi^2 \right]$$

(4.19)

If we decompose the matrix $\Phi$ as follows:

$$\Phi_{ij} = v_i \delta_{ij} + \tilde{\Lambda}_{ij}^{\alpha \beta}$$

(4.20)

where the matrices $\tilde{\Lambda}_{ij}^{\alpha \beta}$ are the $N_f (N_f - 1)$ generators of $SU(N_f)$ that do not belong to the Cartan subalgebra and we insert it in eq.(4.19) we get the following two-point functions:

$$<\tilde{\Pi}^{\alpha \beta}(x) \tilde{\Pi}^{\gamma \delta}(y)>^{F.T.} = i \frac{\delta^{\alpha \gamma} \delta^{\beta \delta}}{p^2 - M_{\alpha \beta}^2(\theta)}$$

$$M_{\alpha \beta}^2(\theta) = \frac{1}{2} \left( \mu_{\alpha}^2(\theta) + \mu_{\beta}^2(\theta) \right)$$

(4.21)

and

$$<v_i(x) v_j(y)>^{F.T.} = i A_{ij}^{-1}(p^2)$$

(4.22)

where F.T. stands for Fouries transform, the matrix $A_{ij}^{-1}$ is the inverse of the following matrix:

$$A_{ij}(p^2) = (p^2 - \mu_i^2(\theta)) \delta_{ij} - a B_{ij}$$

(4.23)

and $B$ is a matrix having all elements equal to 1. The masses of the physical states can be obtained diagonalizing the mass matrix and are given by the following identity:

$$\det A = \prod_{i=1}^{N_f} (p^2 - M_i^2(\theta)) = \prod_{i=1}^{N_f} (p^2 - \mu_i^2) \left[ 1 - a \sum_{j=1}^{N_f} \frac{1}{p^2 - \mu_j^2(\theta)} \right]$$

(4.24)

In the chiral limit ($\mu_i \to 0$) one gets $N_f^2 - 1$ Goldstone bosons ($M_i = 0$) as expected from the spontaneous breaking of the chiral symmetry and one particle with mass:

$$M_S^2 = a N_f$$

(4.25)

Since $a \sim 1/N$ we see that the mass of the singlet is governed in the large-$N$ limit by the same factor $N_f/N$ as the coefficient of the axial anomaly in eq.(4.2). Therefore we see that the resolution of the $U(1)$ problem is intimately related to the existence of a non vanishing axial anomaly.

A numerical comparison of the spectrum predicted by the mass formula given in eq.(4.24) with the experimental values of the pseudoscalar masses has been done in Ref. [26] in the case of three flavours. In the limit where $\mu_1, \mu_2 << \mu_3$ one gets the following masses for the $\eta$ and $\eta'$:

$$M_{\pm}^2 = m_K^2 + \frac{3}{2} a \pm \frac{1}{2} \sqrt{(2m_K^2 - 2m_\pi^2 - a)^2 + 8a^2}$$

(4.26)
and the following mixing angle:

$$\tan \phi = \sqrt{2} - \frac{3}{2\sqrt{2}} \frac{m_\eta^2 - m_\pi^2}{m_K^2 - m_\pi^2}$$  \hspace{1cm} (4.27)$$

defined by the relation

$$|\eta > = \cos \phi |8 > + \sin \phi |1 >$$  \hspace{1cm} (4.28)$$

From eq.(4.26) we can use the masses of $\eta$ and $\eta'$ to determine the parameter $a$. We get $a \sim 0.24(\text{GeV})^2$. Using this value for $a$ and neglecting the square term in the square root in eq.(4.26) one gets:

$$m_\eta^2 \simeq m_K^2 + \frac{3 - 2\sqrt{2}}{2} a = 0.27(\text{GeV})^2$$  \hspace{1cm} (4.29)$$

and

$$m_{\eta'}^2 \simeq m_K^2 + \frac{3 + 2\sqrt{2}}{2} a = 0.95(\text{GeV})^2$$  \hspace{1cm} (4.30)$$

that are very close to the experimental values given respectively by $0.30(\text{GeV})^2$ and $0.92(\text{GeV})^2$. One gets also $\phi = 14^\circ$ that is very close to the experimental value $\phi = 11^\circ$. The phenomenological Lagrangian that we have used give values for the masses that are in good agreement with the experimental ones. The resolution of the $U(1)$ problem implies that the parameter $a$ must be different from zero. It can be computed in pure Yang-Mills theory by computing the following correlator:

$$\chi_t \equiv -i \int d^4y \langle q(x)q(y) \rangle_{Y.M.} = \frac{1}{2} a F_\pi^2$$  \hspace{1cm} (4.31)$$

that in the literature is known as the topological susceptibility. From the value of $a$ obtained from the spectrum of pseudoscalar mesons we get the following value for the topological susceptibility:

$$\chi_t = (180 \text{MeV})^4$$  \hspace{1cm} (4.32)$$

Lattice calculation have confirmed this result \cite{31}.

At the end of this section we want to discuss the $\theta$ dependence that follows from the effective Lagrangian in eq.(4.18). We start noticing that, if we consider the Lagrangian in eq.(4.9), we neglect the terms that come from the fermions and that therefore depend on $U$ and we add a source term ($-iJq$), we have a Lagrangian that has precisely the same structure as the corresponding one for the $CP^{N-1}$ model given in eq.(3.22). This means that also in this case we get

$$Z(J, \theta) \equiv e^{-iW'(J, \theta)} = e^{-iaF_\pi^2(\theta + iJ)^2/4}$$  \hspace{1cm} (4.33)$$

From it we can compute the vacuum energy:

$$E(\theta) = W(0, \theta) = \frac{aF_\pi^2}{4} \theta^2$$  \hspace{1cm} (4.34)$$
and of course the topological susceptibility given in eq.(4.31). Using eq.(4.25) we can recast eq.(4.34) in the form:

\[ M^2_S = \frac{2N_f}{F_\pi^2} \frac{d^2E(\theta)}{d\theta^2} \bigg|_{\theta=0} \]  

(4.35)

that is the famous Witten’s relation [23].

The dependence on the \( \theta \) parameter for the various physical quantities is obtained by solving eqs.(4.13) that minimize the vacuum energy in eq.(4.12). They cannot be solved in general analytically. They imply that physics is periodic in \( \theta \) with period equal to \( 2\pi \). In fact, if \( \phi_i = \phi_i^{(0)}(\theta, \mu_i, a) \) is a solution of eq.(4.13), then, for \( \theta \to \theta + 2\pi \), the solution can be taken for instance to be of the following form:

\[ \phi_i(\theta + 2\pi) = \phi_i + 2\pi \quad \phi_i(\theta + 2\pi) = \phi_i \quad i \neq 1 \]  

(4.36)

with no effect on the physics because the physical quantities depend on \( e^{\pm i\phi_i} \). This means in particular that the vacuum energy must be a periodic function of \( \theta \)

\[ E(\theta + 2\pi) = E(\theta) \]  

(4.37)

However this does not necessarily mean a \( 2\pi \) periodicity of each solution of eq.(4.13). In general one needs to shift from a solution to another at some particular value of \( \theta \) (typically at \( \theta = \pm \pi \)) in order to keep the minimum energy. This can be seen very clearly for instance in the case of two flavours with \( \mu_1 = \mu_2 \equiv \mu \) for \( a >> \mu \) where from eq.(4.13) one finds \( \theta = 2\phi \). Inserting it in the vacuum energy in eq.(4.12) one gets

\[ E(\theta) = -F_\pi^2 \mu^2 \sqrt{\frac{1 + \cos \theta}{2}} \]  

(4.38)

This shows that at \( \theta = \pi \) we shift from a solution to another solution in order to minimize the energy. Because of this the vacuum energy is periodic with period \( 2\pi \) and not \( 4\pi \)!!

Let us consider now the case of one flavour and assume instead that the quantity \( x \equiv a/\mu^2 \) is very small. In this limit eq.(4.13) can be solved as a power expansion in \( x \) and we get [23]:

\[ \phi = 2\pi k + x(\theta - 2\pi k) + O(2\pi k x)^2 \]  

(4.39)

where \( k \) is an integer such that \( (2\pi k x)^2 \) is small. But since \( a \sim 1/N \) for large \( N \) more and more values of \( k \) are allowed. For very large \( N \) the number of allowed values of \( k \) is proportional to \( N \). For each value of \( \theta \) only one value of \( k \) is the true vacuum, the others correspond to metastable states. If we insert the solution given in eq.(4.39) in eq.(4.12) we get for small \( x \):

\[ E(\theta) = \frac{aF_\pi^2}{4} \min_k (\theta - 2\pi k)^2 \]  

(4.40)

27
For $-\pi < \theta < \pi$, $k = 0$ corresponds to the true vacuum, but for $\theta > \pi$ then the value $k = 1$ corresponds to the true vacuum and so on. Extracting a $N^2$ factor required by large $N$ counting we can rewrite eq.(4.40) as follows:

$$E(\theta) = N^2 \frac{a F^2}{4} Min_k \left( \frac{\theta - 2\pi k}{N} \right)^2$$

(4.41)

In conclusion we have found that the vacuum energy must be a periodic function of $\theta$ (see eq.(4.37)) and at the same time must be of the form:

$$E(\theta) = N^2 CF((\theta - 2\pi k)/N) \quad F(x) = x^2$$

(4.42)

This is only possible if we get a multibranch solution of eq.(4.13).

## 5 Anti De Sitter space

In this section we give some detail about anti De Sitter space in $D \equiv n+1$ dimensions. De Sitter or anti De Sitter spaces correspond to solutions of the pure gravity equations in presence of a cosmological term. The action of pure gravity with a cosmological term is given by

$$S = -s \int d^D x \frac{1}{16\pi G_D} \sqrt{|g|} (R + \Lambda)$$

(5.1)

The factor $s$ in front of the action is $s = 1$ if we work with a Minkowski metric with mostly minus or with a euclidean metric, while $s = -1$ in the case of a Minkowski metric with mostly plus. From the previous action we can immediately derive the following eq. of motion:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \Lambda g_{\mu\nu}$$

(5.2)

that implies that the scalar curvature is a constant:

$$R = \frac{D}{2 - D} \Lambda \quad R_{\mu\nu} = \frac{\Lambda}{2 - D} g_{\mu\nu}$$

(5.3)

De Sitter space corresponds to the case $\Lambda < 0$, while anti De Sitter space corresponds to the opposite case ($\Lambda > 0$).

$AdS_{n+1}$ can be easily represented by embedding it in a flat $(n+2)$-dimensional space. Let me call $y^a \equiv (y^0, y^1, \ldots y^n, y^{n+1})$ the coordinates of the embedding space with diagonal metric equal to $\eta_{ab} = (+1, -1, \ldots -1, +1)$. Then $AdS_{n+1}$ can be shown to be the locus characterized by the following equation:

$$y^2 = y^2_0 + y^2_{n+1} - \sum_{i=1}^{n} y^2_i = b^2$$

(5.4)
where \( b \) is a constant anti De Sitter radius. It can be shown that the previous eq. implies the two eqs. in eq. (5.3) provided that we make the following identification:

\[
\Lambda = \frac{n(n - 1)}{b^2}
\]

(5.5)

Another way of representing \( \text{AdS}_{n+1} \) is through the stereographic projection:

\[
y^0 = \rho \frac{1 + x^2}{1 - x^2}, \quad y^\mu = \rho \frac{2x^\mu}{1 - x^2}, \quad \mu = 1, \ldots n + 1
\]

(5.6)

where \( x^2 = (x^1)^2 + \ldots + (x^n)^2 - (x^{n+1})^2 \). According to the previous transformation we can use the variables \( \rho \) and \( x^\mu \) instead of \( y^0 \) and \( y^\mu \) to represent the embedding space. Starting from the flat metric in the embedding space:

\[
ds^2 = (dy^0)^2 + (dy^{n+1})^2 - d(\vec{y})^2
\]

(5.7)

one can rewrite it by using the relations in eq. (5.6) and one gets

\[
ds^2 = d\rho^2 - \frac{4\rho^2}{(1 - x^2)^2} (dx)^2
\]

(5.8)

For fixed \( \rho = b \) we get the metric of \( \text{AdS}_{n+1} \):

\[
g_{\mu\nu} = \frac{4b^2}{(1 - x^2)^2} \eta_{\mu\nu}
\]

(5.9)

where the metric \( \eta_{\mu\nu} \) is with mostly plus.

Another parametrization of \( \text{AdS}_{n+1} \) is the one that appears in the near horizon limit of a D 3-brane. If we work in Minkowski space we can introduce the variables

\[
u = y^0 + iy^{n+1} \quad v = y^0 - iy^{n+1}
\]

(5.10)

while in euclidean space, corresponding to changing the sign in front of the term \((y^{n+1})^2\), we can introduce the alternative variables:

\[
u = y^0 + y^{n+1} \quad v = y^0 - y^{n+1}
\]

(5.11)

In both cases one can rewrite eq. (5.4) as follows:

\[
y^2 = uv - \vec{y}^2 = b^2
\]

(5.12)

Introducing the new variables \( b \xi^\alpha = y^\alpha / u \) for \( \alpha = 1, \ldots, n \) and inserting it in eq. (5.12) we can extract \( v \) as a function of \( \xi \) and \( u \):

\[
v = b^2 \left( \xi^2 u + \frac{1}{u} \right)
\]

(5.13)
Then from the flat embedding metric in eq.(5.7) written in terms of the variables $u$ and $v$ we get:

$$(ds)^2 = b^2 \left( \frac{du^2}{u^2} + u^2 d\xi^2 \right)$$

(5.14)

after having used the following equations:

$$dv = 2b^2 \vec{\xi} \cdot d\vec{\xi} u + b^2 \xi^2 u du - \frac{b^2}{u^2} du \quad d\vec{y} = bud\vec{\xi} + b\xi du$$

(5.15)

Anti De Sitter space has a boundary that is obtained by rescaling the variables:

$$y^\alpha \rightarrow Ry^\alpha \quad u \rightarrow Ru \quad v^\alpha \rightarrow R\vec{v}$$

(5.16)

with $R > 0$ and by taking $R \rightarrow \infty$. In this way we get that the boundary is the manifold satisfying the eq.

$$\vec{u}\vec{v} - \vec{y}^2 = 0$$

(5.17)

But since $tR$ is as good as $R$ the boundary will be described by the two equations:

$$uv - \vec{y}^2 = 0 \quad (u, v, \vec{y}) \sim (tu, tv, t\vec{y})$$

(5.18)

with $t > 0$. We can drop the second condition by just choosing $t$ in such a way that

$$\vec{y}^2 = 1 = uv = y_0^2 + y_{n+1}^2$$

(5.19)

This means that the boundary has the topology of $S^1 \times S^{n-1}$, that is the same topology of compactified Minkowski space with euclidean compactified time. The usual Minkowski space is recovered when we uncompactify the two spheres.

In the last part of this section we introduce additional parametrizations of $AdS$ space. The first one is obtained by introducing the coordinates $(z, \vec{x}) = (1/u, \vec{\xi})$. In these coordinates the $AdS$ metric in eq.(5.14) becomes

$$ds^2 = b^2 dz^2 + d\vec{x}^2$$

(5.20)

The second one corresponds to the cylinder coordinates defined by the following eqs.:

$$y_{n+1} = b \cosh \rho \cos \tau \quad ; \quad y_0 = b \cosh \rho \sin \tau$$

$$y_i = b \sinh \rho e_i$$

(5.21)

where $e_i$ stands for a unit vector in $n$ dimensions. In terms of the previous variables one gets the following metric for $AdS_{n+1}$:

$$ds^2 = -\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{n-1}^2$$

(5.22)

in Minkowski space and

$$ds^2 = d\rho^2 + \sinh^2 \rho d\Omega_n^2$$

(5.23)
in euclidean space where instead the variables used in the previous equation are defined by

\[ y_0 = b \cosh \rho \quad , \quad y_i = b \sinh \rho e_i \quad i = 1, \ldots n + 1 \] (5.24)
in terms of the embedding space coordinates. The cavity coordinates \( r \) and \( \tau \) are instead obtained from the ones in eq.\((5.20)\) through the following relations:

\[ z = \rho \cos \theta \quad ; \quad x_i = \rho \sin \theta e_i \] (5.25)

where

\[ \rho = e^\tau \quad ; \quad \cos \theta = \frac{1 - r^2}{1 + r^2} \] (5.26)

In terms of those coordinates the metric of \( AdS_{n+1} \) becomes:

\[ ds^2 = -b^2 \left( \frac{1 + r^2}{1 - r^2} \right)^2 d\tau^2 + \frac{4b^2}{(1 - r^2)^2} \left[ dr^2 + r^2 d\Omega^2_{n-1} \right] \] (5.27)

An interesting property of \( AdS \) space is that a light ray can reach its boundary in finite time. In fact from the metric in eq.\((5.27)\) a light ray is characterized by the eq.

\[ \frac{dr}{d\tau} = \frac{1 + r^2}{2} \] (5.28)

Integrating the previous equation from the points \((r = 0, \tau = 0)\) and \((r = 1, T)\) corresponding respectively to the center of \( AdS \) space at \( \tau = 0 \) and its boundary at \( \tau = T \) we get:

\[ T = 2 \int_0^1 \frac{dr}{1 + r^2} = \pi/2 \] (5.29)

This means that a light ray starting from the center of \( AdS \) space reaches its boundary and comes back to the center in a time interval equal to \( \pi \).

6 \( \mathcal{N} = 4 \) super Yang-Mills and type IIB string

In this section we review the main properties of both \( \mathcal{N} = 4 \) super Yang-Mills theory and the type IIB string.

In order to derive the Lagrangian of \( \mathcal{N} = 4 \) super Yang-Mills it is convenient to start from \( \mathcal{N} = 1 \) super Yang-Mills in ten dimensions and then dimensionally reduce it to four dimensions. If we do this we obtain the following Lagrangian for \( \mathcal{N} = 4 \) super Yang-Mills:

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \sum_{i=1}^{3} (D_\mu A_i)^a (D^\mu A_i)^a + \frac{1}{2} \sum_{i=1}^{3} (D_\mu B_i)^a (D^\mu B_i)^a - V(A_i, B_j) + \]
\[ -\frac{i}{2} (\bar{\psi}^a \gamma^\mu (D_\mu \psi)^a - \frac{g^2}{2} f^{abc} \bar{\psi}^a \alpha^i A^i \psi^c \alpha - \frac{i g}{2} f^{abc} \bar{\psi}^a \beta^j \gamma_5 B^b \psi^c \]

(6.1)

where the potential is equal to:

\[ V(A_i, B_j) = \frac{g^2}{4} f^{abc} A_i^a A_j^b f^{a f g} A_l^f A_g^l + \frac{g^2}{4} f^{abc} B_i^b B_j^c f^{a f g} B_l^f B_g^l + \frac{g^2}{2} f^{abc} A_i^b B_j^c f^{a f g} A_l^f B_g^l \]

(6.2)

It contains a gluon field, four Majorana spinors and six real scalars. They all transform according to the adjoint representation of the gauge group. The four-dimensional internal matrices \( \alpha \) and \( \beta \) satisfy the following algebra:

\[ \{ \alpha^i, \alpha^j \} = \{ \beta^i, \beta^j \} = -2 \delta^{ij} \quad [\alpha^i, \beta^j] = 0 \]  (6.3)

and

\[ [\alpha^i, \alpha^j] = -2 \epsilon^{ijk} \alpha^k \quad [\beta^i, \beta^j] = -2 \epsilon^{ijk} \beta^k \]  (6.4)

They can be chosen to be given by:

\[ \alpha^i \equiv \eta^i_{AB} = \delta^i_A \delta^A_B - \delta^i_B \delta_A^A + \epsilon_{iAB} \]  (6.5)

and

\[ \beta^i \equiv \bar{\eta}^i_{AB} = \delta^i_A \delta_B^A - \delta^i_B \delta_A^A - \epsilon_{iAB} \]  (6.6)

where \( A, B \) are four-dimensional indices and \( g \equiv g_{YM} \).

\( N = 4 \) super Yang-Mills is invariant under an internal \( SU(4) \) symmetry group that is an R-symmetry. In order to manifestly see this invariance it is convenient to introduce the field:

\[ \bar{\Phi}_{AB} = \frac{1}{2\sqrt{2}} \left[ \eta^i_{AB} A_i - \bar{\eta}^i_{AB} B_i \right] \]  (6.7)

satisfying the condition:

\[ \Phi^{AB} \equiv \frac{1}{2} \epsilon^{ABCD} \bar{\Phi}_{CD} = \Phi^*_{AB} \]  (6.8)

This antisymmetric field transforms according to the vector 6 representation of \( SO(6) \) that has the same algebra as \( SU(4) \). By rewriting the Lagrangian in eq.(6.1) in terms of \( \Phi \) and in terms of the Weyl spinors, that transform according to the 4 representation of \( SU(4) \), one gets \( \text{[32]} \):

\[ L = -\frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a} + (D_\mu \Phi_{AB})_a (D^a \Phi^{AB})_a - i \bar{\psi}^a_{\alpha A} \sigma^A_{\alpha \alpha} (D_\mu \bar{\psi}^\alpha_A)_a + \]

\[- g^2 f^{abc} \Phi^A_b \Phi^{CD} f^{a d e} \bar{\Phi}_{A B}^d \Phi_{C D}^e - g \sqrt{2} f^{a b c} \left[ \bar{\psi}^{a A} \Phi^b_{A B} \psi^b_{C A} + \bar{\psi}^a_{\alpha A} \Phi^B_{AB} \bar{\psi}^\alpha_c \right] \]  (6.9)

It is manifestly invariant under the \( SU(4) \) R-symmetry transformations:

\[ \psi^a_{\alpha A} \rightarrow U^A_B \psi^B_{\alpha A} \quad \bar{\psi}^a_{\alpha A} \rightarrow (U^*)^B_A \bar{\psi}^a_{\alpha B} \]  (6.10)

\( \text{[32]} \)This form of the Lagrangian has been written together with F. Fucito and G. Travaglini.
and
\[ \Phi^{AB} \rightarrow U^A_C \Phi^{CD} (U^T)^B_D \quad \Phi_{AB} \rightarrow (U^*)_A^C \Phi^{CD} (U^\dagger)^D_B \] (6.11)
where \( U \) is a unitary matrix \((UU^\dagger = 1)\).

Lagrangian in eq.(6.1) can also be written in the \( \mathcal{N} = 1 \) superfield formalism. One must introduce three chiral superfields \( \Phi_i \) and one obtains the following Lagrangian:

\[
\mathcal{L} = \int d^2\theta d^2\bar{\theta} \sum_{i=1}^{3} \bar{\Phi}_i e^{2\theta V} \Phi_i + \frac{1}{8\pi} Im \left[ \int d^2\theta \tau^{\alpha} W_\alpha \right] + \left[ \int d^2\theta \sqrt{2} g f^{abc} \Phi_1^a \Phi_2^b \Phi_3^c + h.c. \right] \] (6.12)

\( \mathcal{N} = 4 \) super Yang-Mills has no trace anomaly in the sense that the trace of the energy-momentum tensor is zero also in the full quantum theory. It is a finite quantum field theory. As we have seen above it is also invariant under a \( SU(4) \) R-invariance. This symmetry is not manifest in the formulation with \( \mathcal{N} = 1 \) superfields. Only a \( SU(3) \times U(1) \) symmetry is manifest in this formulation. The \( SU(3) \) corresponds to a \( SU(3) \) rotation of the three superfields \( \Phi_i \), while the \( U(1) \) acts on the four superfields as:

\[
\Phi_i(\theta) \rightarrow e^{\frac{2\alpha}{3}i} \Phi_i(\theta e^{-i\alpha}) \quad W_\alpha(\theta) \rightarrow e^{i\alpha} W_\alpha(\theta e^{-i\alpha}) \] (6.13)

According to the previous eqs. the fermions of the three superfields \( \Phi_i \) have chiral weight equal to \(-1/3\), while the fermion of the superfield \( W_\alpha \) has chiral weight equal to \(1\). This means that the sum of their chiral weights is vanishing implying that \( \mathcal{N} = 4 \) super Yang-Mills has no \( U(1) \) axial anomaly. Finally the action in eq.(6.1)

\[ \text{is also invariant under conformal supersymmetry transformations. They can be obtained by dimensionally reducing the } \mathcal{N} = 1 \text{ supersymmetry transformations in ten dimensions. Since the spinor in ten dimensions is a Weyl-Majorana spinor the ten-dimensional theory is invariant under 16 supersymmetries. On the other hand the four-dimensional quantum theory is conformal invariant and therefore it is also invariant under the same supersymmetry transformations as before, but with a supersymmetry parameter } \alpha = \gamma_\mu x^\mu \beta \text{ that is space-time dependent, while } \beta \text{ is space-time independent. They correspond to additional 16 supersymmetries and therefore we conclude that } \mathcal{N} = 4 \text{ super Yang-Mills in four dimensions is invariant under 32 supersymmetries.} \]

Finally there is a very strong evidence that \( \mathcal{N} = 4 \) super Yang-Mills is invariant under \( SL(2, Z) \) transformations that act on the complex coupling constant \( \tau \) defined in terms of the gauge coupling constant and the \( \theta \)-parameter as:

\[
\tau \rightarrow \tau' = \frac{a \tau + b}{c \tau + d} \quad \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2_{YM}} \] (6.14)

where \( a, b, c \) and \( d \) are integers satisfying the condition \( ad - bc = 1 \). For \( \theta = 0 \) the transformation in eq.(6.14) relates weak with strong coupling. The invariance
under $SL(2, Z)$ is a precise way of stating that this theory satisfies the Montonen-Olive [32] duality in the sense that it can be equivalently formulated as a theory of fundamental $W$-mesons having magnetic monopoles as solitons or as a theory of fundamental magnetic monopoles with the $W$-mesons appearing as solitons, the two formulations having essentially the same Lagrangian.

Type IIB string is a theory of closed superstrings involving both right and left movers. The right and left spinors in the R sector have the same chirality. Therefore, it is a chiral theory with no gauge and gravitational anomalies. It is a $\mathcal{N} = 2$ supersymmetric theory in ten dimensions; this means that it contains 32 supersymmetry charges. These supersymmetries are kept also when one compactifies it on the background $AdS_5 \times S_5$. In the massless bosonic sector the ten dimensional theory contains a graviton $g_{\mu \nu}$, a dilaton $\phi$, an axion field $\chi$, two 2-form potentials $B^{(1)}_{\mu \nu}$ and $B^{(2)}_{\mu \nu}$ and a 4-form potential $A_{\mu \nu \rho \sigma}$ with self-dual field strength, while the massless fermionic sector consists of two gravitinos and two dilatinos having both the same chirality. If we forget for a moment the R-R self-dual 5-form field the low-energy effective Lagrangian for type IIB theory has the following form:

$$S_{IIB} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left[ R + \frac{1}{4} Tr(\partial \mathcal{M} \partial \mathcal{M}^{-1}) - \frac{1}{12} H^T \mathcal{M} H \right]$$  \hspace{1cm} (6.15)$$

where we have combined the field strengths $H^{(1)}$, corresponding to the NS-NS potential, and $H^{(2)}$, corresponding to the R-R potential, in a two-component vector $H = dB$ and the two scalar fields in the symmetric $SL(2, R)$ matrix:

$$\mathcal{M} = e^\phi \left( \begin{array}{cc} |\lambda|^2 & \chi \\ \chi & 1 \end{array} \right)$$  \hspace{1cm} (6.16)$$

with

$$\lambda = \chi + ie^{-\phi}$$  \hspace{1cm} (6.17)$$

This action is manifestly invariant under the global $SL(2, R)$ transformation:

$$\mathcal{M} \rightarrow \Lambda \mathcal{M} \Lambda^T \hspace{1cm} H \rightarrow (\Lambda^T)^{-1} H \hspace{1cm} \Lambda = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$$  \hspace{1cm} (6.18)$$

The metric in the Einstein frame and the 4-form potential are left invariant by the $SL(2, R)$ transformation. In terms of the matrix $\Lambda$ previously defined we get that

$$\lambda \rightarrow \frac{a\lambda + b}{c\lambda + d} \hspace{1cm} H^{(1)} \rightarrow dH^{(1)} - cH^{(2)} \hspace{1cm} H^{(2)} \rightarrow -bH^{(1)} + aH^{(2)}$$  \hspace{1cm} (6.19)$$

In particular, the transformation on the matrix $\mathcal{M}$ given in eq.(6.18) implies that the quantity $\lambda$ defined in eq.(6.17) transforms exactly as $\tau$ in eq.(6.14). Although the low-energy Lagrangian is invariant under a $SL(2, R)$ symmetry it can be seen that it reduces to an $SL(2, Z)$ symmetry in the quantum theory in order to have the Dirac quantization condition satisfied.

In conclusion we have seen that both $\mathcal{N} = 4$ super Yang-Mills and type IIB string theory compactified on $AdS_5 \otimes S_5$ have the same symmetries.
7 The Maldacena conjecture

In the introduction we have seen how a system of \( N \) D 3-branes is, on the one hand, a classical solution of the supergravity equations of motion containing the metric, the dilaton and the self-dual 5-form field strength and, on the other hand, is described by the Born-Infeld action that at low energy reduces to the Yang-Mills action in eq.(1.10) with gauge group \( U(N) \) or more precisely to its \( \mathcal{N} = 4 \) supersymmetric extension. But is it possible to find a more precise connection or even an equivalence between the world volume \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory and supergravity or better superstring that is a consistent quantum theory?

The key to answer this question came from comparing \[33, 34\] the low energy absorption cross sections of massless bulk fields as graviton and dilaton computed either by using the supergravity classical solution or the Born-Infeld action. To great surprise it was found that the two calculations exactly agree \[33, 34 \]. Moreover, if one studies the range of validity of the two previous calculations, it clearly appears that the one based on supergravity is expected to give an exact information on \( \mathcal{N} = 4 \) super Yang-Mills theory for large 't Hooft coupling \((\lambda \to \infty)\) \[33\].

These properties of a system of \( N \) D 3-branes together with the observation \[7\] that it is the region around the throat of metric of the D 3-branes that is the fundamental one to connect the supergravity solution with \( \mathcal{N} = 4 \) super Yang-Mills, brought Maldacena \[7\] to conjecture that \( \mathcal{N} = 4 \) super Yang-Mills should be somehow equivalent to type IIB string theory compactified on \( AdS_5 \times S^5 \) that is in fact the metric in the throat region. In the following we will briefly sketch the main lines of his argumentation.

Let us consider the low-energy limit of the Born-Infeld action for a system of \( N \) D 3-branes and of the bulk supergravity action. Since it amounts to take the limit \( \alpha' \to 0 \) with both \( g_s \) and \( N \) fixed, in this limit we obtain the action of \( \mathcal{N} = 4 \) super Yang-Mills, that we have discussed in sect. 6 and that is the low-energy limit of the Born-Infeld action together with free gravitons. This is a consequence of the fact that both the interaction between bulk fields as for instance the graviton and that between bulk fields and those living on the brane as the Yang-Mills fields, being proportional to the Newton’s constant \( \kappa \sim (\alpha')^2 \), go to zero when \( \alpha' \to 0 \).

On the other hand if we look at the classical solution in eqs.(1.5), (1.6) and (1.7) we see that it interpolates between flat ten-dimensional Minkowski metric obtained for \( r \to \infty \) and a metric with a long throat obtained in the limit \( r \to 0 \). In particular for \( p = 3 \) the metric is non singular when \( r \to 0 \) and in this limit becomes that of \( AdS_5 \times S^5 \). More precisely this can be seen by taking the near-horizon limit of a system of \( N \) D 3-branes defined by

\[
\begin{align*}
 r & \to 0 & \alpha' & \to 0 & U & \equiv \frac{r}{\alpha'} = \text{fixed}
\end{align*}
\] (7.1)

where the Regge slope is taken to zero, while \( U \) is kept fixed. In this limit we can
neglect the factor 1 in the function $H$ in eq.(1.7) and the metric in eq.(1.5) becomes:

$$
\frac{(ds)^2}{\alpha'} \rightarrow \frac{U^2}{\sqrt{4\pi N g_s}} (dx_{3+1})^2 + \frac{\sqrt{4\pi N g_s}}{U^2} dU^2 + \sqrt{4\pi N g_s} d\Omega_5^2
$$

(7.2)

This is the metric of the manifold $AdS_5 \times S^5$ where the two radii of $AdS_5$ and $S^5$ are equal and given by:

$$
R_{AdS_5}^2 = R_{S_5}^2 = b^2 = \alpha' \sqrt{4\pi N g_s}
$$

(7.3)

that, using the relation $g_{YM}^2 = 4\pi g_s$ following from eq.(1.10) for $p = 3$, implies:

$$
\frac{b^2}{\alpha'} = \sqrt{Ng_{YM}^2}
$$

(7.4)

If we have sufficiently soft gravitons (i.e. gravitons with wave length much bigger than the radius of the throat $b$) outside the throat they cannot interact with the excitations far down in the throat as it is confirmed by the fact that their absorption cross-section is vanishing at low energy. On the other hand a string excitation far down inside the the throat, although its proper energy (the energy measured in the reference frame instantaneously at rest at $r$) diverges at low energy ($\alpha' \rightarrow 0$), being proportional to $E_p \sim 1/\sqrt{\alpha'}$, is not negligible because its energy measured in the frame of reference where the time is the one appearing in the first term of the r.h.s. of eq.(7.2) is given by:

$$
E_t \sim \frac{r}{b} E_p \sim \frac{r}{b\sqrt{\alpha'}} \sim \frac{r}{\alpha'} = U
$$

(7.5)

that is kept fixed in the limit $\alpha' \rightarrow 0$. Therefore from the point of view of the classical solution we are left with free gravitons and all the string excitations living far down inside the throat that are described by type IIB string theory compactified on $AdS_5 \times S^5$. By comparing this result with the one obtained from the Born-Infeld action Maldacena has formulated the conjecture that $\mathcal{N} = 4$ super Yang-Mills is equivalent to type IIB string theory compactified on $AdS_5 \times S^5$. The precise relation between the parameters of the gauge and string theory is given in eq.(7.3), where $N$ is equal to the number of colours in the gauge theory and to the flux of the 5-form field strength in the supergravity solution. Since the classical solution in eq.(7.2) is a good approximation when the radii of $AdS_5$ and $S^5$ are very big

$$
\frac{b^2}{\alpha'} >> 1 \implies Ng_{YM}^2 = \lambda >> 1
$$

(7.6)

in the strong coupling limit of the gauge theory we can restrict ourselves to the type IIB supergravity compactified on $AdS_5 \otimes S^5$.

In conclusion, according to the Maldacena conjecture, classical supergravity is a good approximation if $\lambda >> 1$, while in the 't Hooft limit in which $\lambda$ is kept fixed for $N \rightarrow \infty$ classical string theory is a good approximation for $\mathcal{N} = 4$ super Yang-Mills.
In the 't Hooft limit in fact string loop corrections are negligible \((g_s << 1)\) as it follows from the equation: 
\[ \lambda = 4\pi g_s N \] for \(\lambda\) fixed and \(N \to \infty\). Finally Yang-Mills perturbation theory is a good approximation when \(\lambda << 1\).

The strongest evidence for the validity of the Maldacena conjecture comes from the fact that both \(\mathcal{N} = 4\) super Yang-Mills and type IIB string compactified on \(AdS_5 \otimes S^5\) have the same symmetries. They are, in fact, both invariant under 32 supersymmetries, under the conformal group \(O(4,2)\), corresponding to the isometries of \(AdS_5\), under the \(R\)-symmetry group \(SU(4)\), corresponding to the isometries of \(S^5\) and under the Montonen-Olive duality \([12]\) based on the group \(SL(2,\mathbb{Z})\).

If the Maldacena conjecture is true, as it seems to be implied by the many positive checks of its validity, then this is the first time that a string theory is recognized to come out from a gauge theory. In particular it is important to stress that this does not contradict the fact mentioned earlier that a string theory contains gravity while the gauge theory does not, because in this case the two theories live in different spaces: IIB string theory lives on \(AdS_5 \otimes S_5\), while \(\mathcal{N} = 4\) super Yang-Mills lives on the boundary of \(AdS_5\) that is our four-dimensional Minkowski space. The equivalence between the two previous theories realizes the holographic idea \([33,34]\) that a quantum theory of gravity is supposed to satisfy. A new puzzle, however, arises in this case because we usually connect a string theory with a confining gauge theory, while instead \(\mathcal{N} = 4\) super Yang-Mills is a conformal invariant theory and therefore is in the Coulomb and not in the confining phase. The fact that \(\mathcal{N} = 4\) super Yang-Mills is in a Coulomb phase is confirmed by the calculation of the Wilson loop where a Coulomb potential between two test charges is found \([37]\).

If two theories, as the type IIB string theory compactified on \(AdS_5 \otimes S^5\) and \(\mathcal{N} = 4\) super Yang-Mills theory, are equivalent then it must be possible to specify for each field \(Q(x)\) of the boundary Minkowski theory the corresponding field \(\Phi(y)\) of the bulk string theory and to show that, when we compute corresponding correlators in the two theories, we get the same result. In particular, in the boundary theory one can easily compute the generating functional for correlators involving \(Q(x)\)

\[
Z(\Phi_0) = \langle e^{i\int d^4x \Phi_0(x)Q(x)} \rangle \quad (7.7)
\]

By taking derivatives with respect to the arbitrary source \(\Phi_0(x)\) one can compute any correlator involving the boundary field \(Q(x)\). In Refs. \([39,40]\) the recipe for computing \(Z(\Phi_0)\) in the bulk theory has been given. First of all one must identify \(\Phi_0(x)\) with the boundary value of the field \(\Phi(y)\), which lives in the bulk theory and that corresponds to the composite \(Q(x)\) of the boundary theory. Then the generating functional given in eq.\((7.7)\) can just be obtained by performing in the bulk theory the functional integral over \(\Phi\) with the restriction that its boundary value be \(\Phi_0\):

\[
Z(\Phi_0) = \int_{\Phi \rightarrow \Phi_0} D\Phi \ e^{-S[\Phi]} \quad (7.8)
\]

In computing the previous functional integral we can use classical supergravity in the regime where \(\lambda >> 1\). Otherwise for an arbitrary value of fixed \(\lambda\) for \(N \to \infty\)
we need to compute the tree diagrams of type IIB string theory compactified on AdS$_5 \otimes S^5$.

A number of bulk fields has been identified to correspond to the various gauge invariant composite fields of $\mathcal{N} = 4$ super Yang-Mills. We do not discuss this correspondence in detail here. In the following we will just describe in some detail the correspondence between the dilaton field of type IIB supergravity and the composite given by the Yang-Mills Lagrangian $F^2 \equiv F^\alpha_{\mu\nu} F^{\alpha\mu\nu}$ showing that the two-point functions that one obtains from both eqs.\((7.7)\) and \((7.8)\) are coincident \([39, 40]\).

Since $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ is a conformal invariant quantum theory and the composite field $F^2$ has dimension 4 the two-point function involving two $F^2$ fields must have the following form:

\[
<F^2(x) F^2(z) \sim \frac{N^2}{(\vec{x} - \vec{z})^8}
\]  

(7.9)

apart from an overall constant that we do not care to compute. The previous correlator can also be obtained by using the lowest order perturbation theory in $\mathcal{N} = 4$ super Yang-Mills. $\vec{x}$ denotes here a Minkowski four-vector.

In the bulk theory we only need the dilaton kinetic term in type IIB supergravity in $D = 10$ compactified on AdS$_5 \otimes S^5$. Taking into account that the volume of $S^5$ is equal to $\pi^{3/2} b^5$, where $b$ is given in eq.\((7.3)\), we need to consider the following action:

\[
S = \frac{\pi^3 b^5}{4\kappa_{10}^2} \int d^5 x \sqrt{g} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi
\]

(7.10)

where $g_{\mu\nu} = \frac{\kappa_4^2}{z^2} \delta_{\mu\nu}$ is the metric of AdS$_5$ in the so-called Poincaré coordinates given in eq.\((5.20)\). In the limit $\lambda >> 1$, where classical supergravity is a good approximation, we just need to solve the dilaton eq. of motion given by:

\[
\partial_\mu [\sqrt{g} g^{\mu\nu} \partial_\nu \Phi] = 0
\]

(7.11)

The solution of the previous equation, that is equal to $\Phi_0$ on the boundary (corresponding to the limit $z \to 0$), can be given in terms of the Green’s function:

\[
\Phi(z, \vec{x}) = \int d^4 \vec{x} \ K(z, \vec{x}; \vec{y}) \ \Phi_0(\vec{y}) \quad ; \quad K(z, \vec{x}; \vec{y}) \sim \frac{z^4}{[z^2 + (\vec{x} - \vec{y})^2]^2}
\]

(7.12)

Inserting the solution found in eq.\((7.12)\) in the classical action we get that the contribution to the classical action is entirely due to the boundary term

\[
S = \frac{\pi^3 b^8}{4\kappa_{10}^2} \int d^4 x z^{-3} \Phi_0(\vec{y}) \partial_\mu \Phi_0(\vec{y}) \sim -\frac{\pi^3 b^8}{4\kappa_{10}^2} \int d^4 \vec{x} \int d^4 \vec{y} \frac{\Phi_0(\vec{x}) \Phi_0(\vec{y})}{(\vec{x} - \vec{y})^8}
\]

(7.13)

where we have introduced a cut off $\epsilon$ at the lower limit of integration, that, however, cancels out after having inserted eq.\((7.12)\) in eq.\((7.13)\). In conclusion in the classical approximation ($\lambda >> 1$) we get

\[
Z(\Phi_0) = \exp \left[ \frac{\pi^3 b^8}{4\kappa_{10}^2} \int d^4 \vec{x} \int d^4 \vec{x}' \frac{\Phi_0(\vec{x}) \Phi_0(\vec{x}')}{(\vec{x} - \vec{x}')^8} \right]
\]

(7.14)
Taking into account eq.\((\ref{eq:7.3})\) and that \(2\kappa_{10}^2 = (2\pi)^7 g_s^2 (\alpha')^4\), from the previous equation we can get immediately the two-point function:

\[
<F^2(x)F^2(z) = \frac{\partial^2 Z(\Phi_0)}{\partial \Phi_0(x) \partial \Phi_0(z)} \sim N^2 \frac{N^2}{(x - z)^8},
\]

that agrees with the expression given in eq.\((\ref{eq:7.3})\). Notice, however, that the supergravity approximation is in general only valid for large values of \(\lambda\) (see eq.\((\ref{eq:7.6})\)), while the previous example shows that it seems to be valid for any value of \(\lambda\). This is, of course, a consequence of the conformal invariance of \(\mathcal{N} = 4\) super Yang-Mills that requires the vanishing of the contribution to the two-point function of all string corrections to the supergravity action. The same result is also true if one computes the three point function involving \(F^2\) or the two and three-point functions involving the energy-momentum tensor. Actually, using the value of \(\kappa_{10}\) given just after eq.\((\ref{eq:7.14})\) together with eq.\((\ref{eq:7.3})\), it is easy to see that the factor in front of the Einstein action or of the dilaton kinetic term is proportional to \(N^2\) and does not depend on the gauge coupling constant \(g_{YM}\). This means that the pure supergravity approximation will never give a dependence of the correlators of gauge theory on the gauge coupling constant. In order to obtain the dependence on the gauge coupling constant we need to add string corrections. Let us restrict ourselves to the pure gravity part of type IIB supergravity. The action with the first string correction containing only the metric is given by:

\[
S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ e^{-2\phi} R + \frac{(\alpha')^3}{3 \cdot 2^7 R^4} \cdot e^{-1/2\phi} f(\tau, \bar{\tau}) \right]
\]

where

\[
f(\tau, \bar{\tau}) = \sum_{(n,m) \neq (0,0)} \frac{\tau_2^{3/2}}{m + n\tau^3}, \quad \tau = \chi + ie^{-\phi}.
\]

The prime indicates that the term \((n, m) = (0, 0)\) is excluded from the sum, \(\tau_2 \equiv Im\tau\) and \(\phi\) and \(\chi\) are respectively the dilaton and the R-R scalar of type IIB string. The function \(f\) that is invariant under the \(SL(2, Z)\) transformation \(\tau \rightarrow (a\tau + b)/(c\tau + d)\), can be expanded for small values of \(e^\phi = g_s\) getting

\[
e^{-\phi/2} f(\tau, \bar{\tau}) = 2\zeta(3)e^{-2\phi} + \frac{2\pi^2}{3} + (4\pi)^{3/2} e^{-\phi/2} \sum_{M > 0} Z_M M^{1/2} \left[ e^{2\pi i M \tau} + e^{2\pi i M \bar{\tau}} \right] (1 + e^\phi/M)
\]

The first term in the r.h.s. of the previous equation comes from the tree string diagrams, the second from one-loop string corrections, while the rest is the contribution of D instantons. If one now uses the action in eq.\((\ref{eq:7.10})\) to compute the two and three-point function for the energy-momentum tensor of \(\mathcal{N} = 4\) super Yang-Mills one finds that the extra term gives no contribution, because it identically
vanishes \[42\] when we compute it in the background \(AdS_5 \otimes S^5\). But instead it gives a non trivial contribution to the four-point amplitude. In particular, since the four-point amplitude will include the function \(f\), and since in the \(AdS/CFT\) correspondence the Yang-Mills coupling constant is related to the string coupling constant through the relation: \(g_{YM}^2 = 4\pi g_s\) we immediately see from eq.\((7.18)\) that the D instanton contribution becomes the usual instanton contribution in Yang-Mills theory \[42\]. This result is confirmed by explicit calculations in \(N = 4\) super Yang-Mills \[43\]. A quick way of fixing the dependence on \(N\) and on the gauge coupling constant of the two terms present in the action in eq.\((7.16)\) is by remembering that under a rescaling of the metric by \(b^2\) we get the following relations:

\[
g_{\mu\nu} \rightarrow b^2 g_{\mu\nu}; \quad \sqrt{g} \rightarrow b^{10} \sqrt{g} \tag{7.19}\]

and

\[
R \rightarrow b^{-2} R; \quad R^4 \rightarrow b^{-8} R^4 \tag{7.20}\]

By the previous rescalings (where we have also taken into account that the integration over the sphere \(S^5\) gives an extra factor \(b^5\)) we get that the coefficient of the Einstein term in eq.\((7.16)\) is proportional to \(N^2\) as previously found, while the coefficient of the first string correction in eq.\((7.16)\) is proportional to \(N^2 (Ng_{YM}^2)^{-3/2}\) that depends explicitly on the gauge coupling constant.

We have seen that in the equivalence between \(N = 4\) super Yang-Mills and type IIB string theory compactified on \(AdS_5 \times S^5\) the Yang-Mills action \(F^2\) that has conformal dimension equal to 4 corresponds to the massless dilaton. In Ref. \[40\] it has been shown that a massive scalar field with mass equal to \(m\) corresponds in the conformal field theory to a composite operator with dimension \(\Delta\) equal to

\[
\Delta = 2 + \sqrt{4 + b^2 m^2} \tag{7.21}\]

Here we will not derive this result whose derivation can be found in Ref. \[40\], but we will only derive eq.\((7.21)\) in the limit of very large \(b m\). Let us consider a scalar field \(\Phi\) of the bulk theory with mass \(m\) that corresponds to a composite field \(F(x)\) of the boundary theory with conformal dimension \(\Delta\). The two-point function in the boundary theory is fixed apart from an overall normalization by the conformal invariance of the theory. This means that:

\[
<F(\vec{x})F(\vec{y})> \sim |\vec{x} - \vec{y}|^{-2\Delta} \tag{7.22}\]

where we have made the composite \(F(x)\) dimensionless by multiplying it with a factor \(\mu^{-\Delta}\) (\(\mu\) is a parameter with dimension of a mass). On the other hand the previous two-point function of the boundary theory is also equal to the two-point function of the bulk theory involving the corresponding field \(\Phi\):

\[
<F(\vec{x})F(\vec{y})> \sim <\Phi(\vec{x})\Phi(\vec{y})> \tag{7.23}\]

if the two points \(\vec{x}\) and \(\vec{y}\) are on the boundary of \(AdS\) space, i.e. in Minkowski four-dimensional space. But the propagator of a free particle with mass \(m\) in the bulk
theory for $m$ very large is given by the particle action computed along a geodesic that connects the two points $\vec{x}$ and $\vec{y}$. The action describing a particle moving in AdS space is given by:

$$S = m \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \quad (7.24)$$

Choosing for the AdS metric the one given in eq.(5.20) and the two points of the boundary to be at $x = \pm a$ we get:

$$S = mb \int_a^{-a} dx \frac{dz}{z} \sqrt{1 + \left(\frac{dz}{dx}\right)^2} \quad (7.25)$$

The geodesic satisfies the equation:

$$\left(\frac{dz}{dx}\right)^2 = \left(\frac{z_0}{z}\right)^2 - 1 \quad (7.26)$$

where $z_0$ is the minimum value taken by $z$. It is easy to see that the previous eq. defines a circle with center on the boundary at $x = 0$ and with radius $z_0 = a$, that connects the two points on the boundary at $x = \pm a$. When we insert the geodesic solution in the original action we get:

$$S = 2mb \int_\epsilon^{z_0} dz \frac{1}{\sqrt{1 - \left(\frac{z}{z_0}\right)^2}} \quad (7.27)$$

where we have introduced an infrared cutoff $\epsilon$ in the bulk theory. By performing the integral in the limit of small $\epsilon$ we get:

$$\langle \Phi(-a)\Phi(a) \rangle \sim e^{-S} \sim e^{-2mb \log a/\epsilon} = \left(\frac{a}{\epsilon}\right)^{-2mb} \quad (7.28)$$

From eqs.(7.22), (7.23) and (7.28) we get for large values of the mass $m$:

$$\Delta \sim mb \quad (7.29)$$

that agrees with eq.(7.21) for $bm >> 1$. In addition by comparing eqs.(7.22) and (7.28) we can see that the infrared cutoff $\epsilon$ of the bulk theory corresponds to an ultraviolet cutoff $\mu = 1/\epsilon$ of the boundary theory [44].

8 Finite temperature $\mathcal{N} = 4$ super Yang-Mills

In the previous section we have briefly seen how the Maldacena conjecture provides for the first time a very strong evidence for the appearance of a string theory in a non-perturbative gauge theory precisely realizing the ideas reviewed in sect. 2 on the large $N$ expansion in QCD and without running into the problem that a string...
theory contains gravity while the gauge theory does not. On the other hand we are immediately confronted with a new puzzle because $\mathcal{N} = 4$ super Yang-Mills is in the Coulomb phase and therefore the emergence of a string has nothing to do with the confining properties of the theory. In order to get a confining theory we have to get rid of the conformal invariance of the theory. The simplest way for doing so is by considering $\mathcal{N} = 4$ super Yang-Mills at finite temperature. But, since bosons have periodic and fermions anti-periodic boundary conditions, in going to finite temperature, we also break supersymmetry. Therefore at high temperature we expect to reduce ourselves to a non-supersymmetric gauge theory that is presumably in the same universality class as pure Yang-Mills theory in three dimensions and that is confining.

In the previous section we have seen that $\mathcal{N} = 4$ super Yang-Mills at zero temperature is related to $AdS_5$ and it is therefore natural to expect that $\mathcal{N} = 4$ super Yang-Mills at finite temperature is related to the finite temperature version of $AdS_5$ discussed in Ref. [8]. We will see that at finite temperature we need to consider, at least for very large ’t Hooft coupling where supergravity is a good approximation, two classical solutions of the supergravity equations: the first one is $AdS_5$ that we had also at zero temperature and that is dominating at low temperature, while the second one is the $AdS_5$ black hole that is instead dominating at high temperature. The high temperature case is the most interesting one because in this case we get confinement and a mass gap.

In section 5 we have seen that anti De Sitter space in euclidean uncompactified space is described by eq.(5.12). We now want to compactify the coordinate $v$. Let us restrict ourselves to $AdS^{+}_{n+1}$ where both $u, v > 0$. The manifold in eq.(5.12) is invariant under the action of a group that we call $Z$ and that acts on the coordinates as follows:

$$u \rightarrow \lambda^{-1} u \quad \quad v \rightarrow \lambda v \quad \quad y^\alpha \rightarrow y^\alpha$$  (8.1)

We can construct a compactified version of $AdS^{+}_{n+1}$ by modding out the action of the group $Z$. In this way one gets the manifold $X_1 \equiv AdS^{+}_{n+1}/Z$. The fundamental domain for the action of $Z$ on $v$ is the interval $1 \leq \frac{v}{b} \leq \lambda$. Therefore $v$ parametrizes a circle with natural coordinate

$$\frac{v}{b} = \lambda^{\theta/2\pi} \quad 0 \leq \theta \leq 2\pi$$  (8.2)

The independent variables describing this compactified version of anti De Sitter space can be taken to be $v$ and $\vec{y}$, while $u$ is given in terms of them through eq.(5.12). The manifold $X_1$ spanned by $(v, \vec{y})$ is then topologically equivalent to $S^1 \times R^n$. The boundary of $X_1$ is instead topologically equivalent to $S^1 \times S^{n-1}$. It is convenient to perform the change of variables:

$$\frac{t}{b} = \log \frac{v}{b} - \frac{1}{2} \log \left( b^2 + r^2 \right) \quad ; \quad r^2 \equiv \sum_{\alpha=1}^{n} y^2$$  (8.3)
Then the metric of \( X_1 \) becomes

\[
(ds^2)_{X_1} = dt^2 \left[ 1 + \frac{r^2}{b^2} \right] + \frac{dr^2}{1 + r^2} + r^2 d\Omega^2_{n-1} \tag{8.4}
\]

The previous metric is a compactified version of a solution of the eq. of motion given in eq.\((5.2)\), that has, however, also another solution. This is the \( AdS_{n+1} \) black hole whose metric is given by:

\[
(ds^2)_{X_2} = dt^2 \left[ 1 + \frac{r^2}{b^2} - \frac{w_n M}{r^{n-2}} \right] + \frac{dr^2}{1 + r^2} - \frac{w_n M}{r^{n-2}} + r^2 d\Omega^2_{n-1} \tag{8.5}
\]

where

\[
w_n = \frac{16\pi G_N}{(n - 1)\Omega_{n-1}} \quad \Omega_{n-1} = Vol(S^{n-1}) \tag{8.6}
\]

The horizon of the black hole corresponds to the largest root of the equation:

\[
V(r_+) \equiv 1 + \frac{r_+^2}{b^2} - \frac{w_n M}{r_+^{n-2}} = 0 \tag{8.7}
\]

\( X_2 \) is topologically equivalent to \( R^2 \times S^{n-1} \) where \( R^2 \) corresponds to the variables \((r, t)\). Its boundary has the topology of \( S^1 \times S^{n-1} \). In conclusion we have two solutions of the classical eq.\((5.2)\): the manifold \( X_1 \) corresponding to \( AdS_{n+1} \) with a compactified coordinate having the topology of \( S^1 \times R^n \) and the manifold \( X_2 \) corresponding to the anti De Sitter black hole having the topology of \( R^2 \times S^{n-1} \). Their boundary has in both cases the topology of \( S^1 \times S^{n-1} \) that is also the topology of Minkowski space with compactified time and space.

We now want to show that at the horizon of the black hole there is no singularity if \( t \) is a periodic variable with period equal to:

\[
\beta_0 \equiv \frac{1}{T} = \frac{4\pi b^2 r_+}{n r_+^2 + (n - 2)b^2} \tag{8.8}
\]

This can be easily obtained by expanding the metric around the horizon

\[
(ds^2)_{X_2} = V'(r_+)(r - r_+)dt^2 + \frac{dr^2}{V'(r_+)(r - r_+) + \ldots} \tag{8.9}
\]

where

\[
V(r) = V'(r_+)(r - r_+) + \ldots \quad V'(r_+) = \frac{n r_+^2 + (n - 2)b^2}{r_+ b^2} \tag{8.10}
\]

By introducing the new variables:

\[
z = \frac{2(r - r_+)^{1/2}}{(V'(r_+))^{1/2}} \quad \theta = \frac{1}{2}V'(r_+)t \tag{8.11}
\]

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the metric in eq.(8.9) becomes a two-dimensional flat metric in polar coordinates:

\[(ds^2)_{X_2} = dz^2 + z^2 d\theta^2\]  

(8.12)

There is no singularity if the variable \(\theta\) is periodic with period equal to \(2\pi\). Then eq.(8.11) implies that \(t\) must also be a periodic variable with period equal to \(\beta_0\) given in eq.(8.8). If we plot \(\beta_0\) as a function of \(r_+\) we see that \(\beta_0\) is vanishing for both \(r_+ = 0\) and \(\infty\) and has a maximum value equal to \((4\pi b)/(\sqrt{n(n-2)})\) for \(r_+ = b\sqrt{(n-2)/n}\). This means that \(\beta_0\) cannot be arbitrarily large and therefore the temperature cannot be arbitrarily small. We will in fact see that this solution is relevant for high temperature, while the other solution \(X_1\) is relevant at low temperature. In order to see which one of the two solutions dominates we have to compute their classical euclidean action. In both cases it is equal to:

\[I_{\text{class}} = \frac{nV_{n+1}}{8\pi G_N b^2}\]  

(8.13)

where \(V_{n+1}\) is a divergent volume. Therefore the classical action is infinite in both cases. We compute their difference by regularizing each of the contributions with a radius \(R\) and by taking the two temperatures connected by the condition:

\[\sqrt{1 + \frac{R^2}{b^2} \beta_0(X_1)} = \sqrt{1 + \frac{R^2}{b^2} - \frac{w_n M}{b^{n-2}} \beta_0}\]  

(8.14)

that relates the two periods in a coordinate invariant way. Therefore we have to compute:

\[I_2 - I_1 = \frac{n\Omega_{n-1}}{8\pi G_N b^2} \left\{ \int_0^{\beta_0} dt \int_{r_+}^R drr^{n-1} - \int_0^{\beta_0(X_1)} dt \int_0^R drr^{n-1} \right\}\]  

(8.15)

The previous integrals can be easily computed and in the limit of the cutoff \(R \to \infty\) we get a finite result:

\[I_2 - I_1 \equiv \Delta I = \frac{\Omega_{n-1}}{4G_N} \frac{b^2 r_+^{n-1} - r_+^{n+1}}{m_+^2 + (n-2)b^2}\]  

(8.16)

If we interpret \(\Delta I\) as the free energy in statistical mechanics we get that the energy

\[E = \frac{\partial \Delta I}{\partial \beta_0} = \frac{\partial \Delta I}{\partial r_+} \frac{\partial r_+}{\partial \beta_0} = \frac{1}{w_n} \left( \frac{r_+^n}{b^2} + \frac{r_+^{n-2}}{b^2} \right) = M\]  

(8.17)

is equal to the parameter \(M\) that corresponds to the mass of the black hole and that the entropy

\[S = \beta_0 E - \Delta I = \frac{\Omega_{n-1} r_+^{n-1}}{4G_N}\]  

(8.18)

is in complete agreement with the Beckenstein-Hawking expression for the entropy of a black hole given by the area of the horizon in \((n-1)\) dimensions divided by
In particular, if we consider the case \( n = 3 \) and we take the anti De Sitter radius \( b \to \infty \) we get the metric of the Schwarzschild black hole given by:

\[
(ds^2)_S = (1 - \frac{r_g}{r})dt^2 + \frac{dr^2}{(1 - \frac{r_g}{r})^2} + r^2d\Omega^2_2
\]

where \( r_g = w_3M = 2G_NM \) is the Schwarzschild radius.

For both \( X_1 \) and \( X_2 \) the radii of the boundary \( (r \to \infty) \) with the topology of \( S^1 \times S^{n-1} \) are given by:

\[
\beta = \frac{r \beta_0}{2\pi b} \quad \quad \beta' = r \quad \quad \frac{\beta}{\beta'} = \frac{\beta_0}{2\pi b}
\]

Therefore in the decompactification limit in which the topology of \( S^1 \times S^{n-1} \) becomes that of \( S^1 \times R^{n-1} \), i.e. the topology of Minkowski space with periodic euclidean time, we must take the high temperature limit \( \beta_0 \to 0 \). This limit can be obtained for both \( r_+ \to 0 \) and \( r_+ \to \infty \). We will see later on that actually the high temperature phase corresponds to the case \( r_+ \to \infty \) because this branch is dominant with respect to the other. In this limit corresponding also to the limit \( M \to \infty \), as one can see from eq.(8.17), we get:

\[
r_+ = (w_nMb^n)^{1/n} \quad \quad \beta_0 = \frac{4\pi b^2}{nr_+}
\]

When \( r_+ \to \infty \) \( (M \to \infty) \) it is convenient to introduce the new variables:

\[
r = \left(\frac{w_nM}{b^{n-2}}\right)^{1/n} \rho \quad \quad t = \left(\frac{w_nM}{b^{n-2}}\right)^{-1/n} \tau = \frac{b}{r_+} \tau
\]

In terms of them the metric in eq.(8.5) becomes:

\[
(ds^2)_{X_2} = \left(\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}\right)d\tau^2 + \frac{d\rho^2}{\frac{\rho^2}{b^2} - \frac{b^{n-2}}{\rho^{n-2}}} + \rho^2 \left(\frac{w_nM}{b^{n-2}}\right)^{2/n}d\Omega^2_{n-1}
\]

Notice that, when \( M \to \infty \), the radius of \( S^{n-1} \) becomes very large and the period of the variable \( \tau \) becomes equal to \((4\pi b)/n\).

Both solutions \( X_1 \) and \( X_2 \) contribute to the partition function and correlators of the gauge theory that in our case is \( \mathcal{N} = 4 \) super Yang-Mills at finite temperature. In general we have to sum over both of them:

\[
e^{-I} \to e^{-I_1} + e^{-I_2} = e^{-I_1} \left[ 1 + e^{-\Delta I} \right] = e^{-I_2} \left[ 1 + e^{\Delta I} \right]
\]

From eq.(8.16) we see that, when \( r_+ \) is small, then \( \Delta I > 0 \) and therefore \( X_1 \) dominates. This is the limit that describes the low temperature phase. When instead \( r_+ \to \infty \) from the same eq. we see that \( \Delta I < 0 \) and therefore the solution \( X_2 \) is dominant at high temperature. One sees also that the branch at \( r_+ \to \infty \)
is dominant with respect to the one \( r_+ \to 0 \) because at \( r_+ \to 0 \) \( I_2 = I_1 \), while at \( r_+ \to \infty \) \( I_2 < I_1 \).

Following the Maldacena conjecture we expect that \( \mathcal{N} = 4 \) super Yang-Mills at high temperature and for large ’t Hooft coupling \( (\lambda \to \infty) \) is described by the AdS5 black hole. In order to check this let us compare the entropy of \( \mathcal{N} = 4 \) super Yang-Mills with that of AdS5 black hole [45]. The entropy of the AdS5 black hole can be obtained from eq.(8.18) for \( n = 4 \). By rewriting it in terms of the temperature related to \( r_+ \) through eq.(8.21) \( (\beta_0 = 1/T) \) and introducing \( V_3 = \Omega_3 b^3 \) we get:

\[
S_{BH} = \frac{\pi^2}{2} V_3 T^3 N^2
\]  

(8.25)

where we have used eq.(7.3) and the fact that the five-dimensional Newton constant is equal to \( 16\pi G_N^{(5)} = (2\pi)^7(\alpha')^4 g_s^2 / (b^5 \Omega_5) \) with \( \Omega_5 = \pi^3 \). The factor \( b^5 \Omega_5 \) is the volume of \( S^5 \).

The entropy of \( \mathcal{N} = 4 \) super Yang-Mills can be easily computed at weak coupling where it can just be obtained by counting the bosonic and fermionic degrees of freedom. In fact by taking into account that \( \mathcal{N} = 4 \) super Yang-Mills theory has 8 bosonic and 8 fermionic massless degrees of freedom and that the entropy of each bosonic and fermionic degrees of freedom is given respectively by:

\[
S_{BOS} = \frac{2\pi^2}{15 \cdot 3} N^2 V_3 T^3 ; \quad S_{FER} = \frac{7}{8} S_{BOS} = \frac{14\pi^2}{15 \cdot 3 \cdot 8} N^2 V_3 T^3
\]  

(8.26)

we get the following entropy at weak coupling

\[
S_{YM} = \frac{2\pi^2}{3} N^2 V_3 T^3
\]  

(8.27)

It is equal to the entropy of the black hole in eq.(8.25) apart from a numerical factor \((4/3)\). The mismatch between the two results can be easily explained from the fact that one is valid for strong coupling while the other one is valid in perturbation theory [46]. In general we expect the following behaviour of the entropy with the Yang-Mills coupling constant [46]:

\[
S(N g_{YM}^2) = \frac{2\pi^2}{3} N^2 V_3 T^3 f(N g_{YM}^2)
\]  

(8.28)

where \( f(x) \) is a smooth function that is equal to \( f(0) = 1 \) at weak coupling corresponding to \( x = 0 \) and to \( f(\infty) = 3/4 \) at strong coupling. The inclusion of the first string correction [46] gives:

\[
f(N g_{YM}^2) = \frac{3}{4} + \frac{45}{32} \zeta(3)(2g_{YM}^2 N)^{-3/2}
\]  

(8.29)

while a recent two-loop calculation [17] shows that its perturbative expansion is:

\[
f(N g_{YM}^2) = 1 - \frac{3}{2\pi^2} g_{YM}^2 N
\]  

(8.30)
These results show that the function $f$ is not a constant and are consistent with $f$ being a monotonic function interpolating between $1$ at $N g_{YM}^2 = 0$ and $3/4$ at $N g_{YM}^2 = \infty$.

In the second part of this section we consider $\mathcal{N} = 4$ super Yang-Mills at high temperature and we compute various physical quantities in this theory using its correspondence with IIB supergravity compactified on $AdS_5 \times S^5$. As we have explained in Sect. 7 supergravity is a good approximation to $\mathcal{N} = 4$ super Yang-Mills in the strong coupling limit $N g_{YM}^2 \gg 1$. In particular in the following we will show that, by computing the Wilson loop and finding that it is proportional to the area, this theory confines. We will then look at the glue ball mass spectrum and we will show that in the high temperature phase a mass gap is generated.

For computing the Wilson loop it is convenient to use the following form of the black hole metric:

$$ds^2 = \left(\alpha'\right)^2 \frac{U^2}{b^2} \left[\left(1 - \frac{U^4}{U'^4}\right) dt^2 + \sum_{i=1}^{3} (dx_i)^2 \right] + \frac{b^2 dU^2}{U^2 \left(1 - \frac{U^4}{U'^4}\right)} + b^2 d\Omega_5^2$$  \hspace{1cm} (8.31)

where the variables used here are related to those used in eq.(8.23) for $n = 4$ by the equations:

$$\tau = \frac{\alpha' U_T}{b} t \quad \rho = \frac{b}{U_T} U$$  \hspace{1cm} (8.32)

In eq.(8.31) we have also added the part of the metric corresponding to the sphere $S^5$. The variable $U$ is the same as the one defined in eq.(7.1). In these new variables the period of the periodic variable $t$ is equal to:

$$\beta = \frac{1}{T} = \frac{\pi b^2}{\alpha' U_T}$$  \hspace{1cm} (8.33)

Following Refs. [37, 38] the rectangular Wilson loop in the gauge theory can be approximated in the strong coupling limit by the value of the minimal Nambu-Goto string action. The string has the world sheet in anti De Sitter space ending on the rectangular Wilson loop. The string action in anti De Sitter space is given by:

$$S = \frac{1}{2 \pi \alpha'} \int d\sigma \int d\tau \sqrt{\det(G_{MN} \partial_\alpha x^M \partial_\beta x^N)}$$  \hspace{1cm} (8.34)

where $G_{MN}$ is the metric in eq.(8.31). The Wilson loop is along the variables $x_1$ and $x_2$ and we choose the static gauge, where $x_2 = \sqrt{\alpha' \tau}$ and $x_1 = \sqrt{\alpha' \sigma}$. The finite temperature calculation has been carried out in Refs. [48, 49]. For the sake of simplicity we consider the case in which the world sheet of the string depends only on the variable $U$ where in particular $U$ depends only on one of the world sheet variables $x_1$. In this case for the action in eq.(8.34) we get the following expression:

$$S = \frac{X_2}{2 \pi} \int_{-R/2}^{R/2} dx \left\{ \frac{U^4 (\alpha')^2}{b^4} + \left[ 1 - \frac{U^4}{U'^4} \right]^{-1} \left( \frac{dU}{dx} \right)^2 \right\}^{1/2}$$  \hspace{1cm} (8.35)
where \( x \equiv x_1, X_2 \) is the length of the "temporal" side of the rectangular Wilson loop and \( R \) is the distance between the test "quarks". Since the previous Lagrangian does not explicitly depend on \( x \) the corresponding hamiltonian is a constant of motion:

\[
- H = \frac{U^4(\alpha')^2}{b^4} \left\{ \frac{U^4(\alpha')^2}{b^4} + \left[ 1 - \frac{U^4}{U_T^4} \right]^{-1} \left( \frac{dU}{dx} \right)^2 \right\}^{-1/2} = C \tag{8.36}
\]

From it we get

\[
\left( \frac{dU}{dx} \right)^2 = \frac{(\alpha')^2}{b^4} \left[ \frac{U^4(\alpha')^2}{C^2b^4} - 1 \right] \left[ U^4 - U_T^4 \right] \tag{8.37}
\]

Introducing the minimum value of \( U \), that we call \( U_0 \), corresponding, because of the symmetry of the problem, to \( x = 0 \), and that is the value for which eq.(8.37) vanishes, we can determine the constant \( C \) in terms of \( U_0 \):

\[
C^2 = \frac{U_0^4(\alpha')^2}{b^4} \tag{8.38}
\]

Integrating the differential equation in (8.37) we get

\[
\int_0^X dx = \frac{b^2}{\alpha'} \int_{U_0}^U \frac{dU}{\left[ \left( \frac{U^4}{U_0^4} - 1 \right) \left( U^4 - U_T^4 \right) \right]^{1/2}} \tag{8.39}
\]

Since the two test "quarks" in the Wilson loop are at a distance \( R \) in Minkowski space, that is the boundary of \( AdS_5 \) obtained by taking the limit \( U \to \infty \), the value \( X = R/2 \) in the l.h.s. of eq.(8.39) corresponds to \( U \to \infty \). Introducing the variable \( w = U/U_0 \) and taking this limit in the previous eq. we get:

\[
\frac{R}{2} = \frac{b^2}{\alpha'U_0} \int_1^\infty \frac{dw}{\sqrt{(w^4 - 1)(w^4 - \left( \frac{U_T}{U_0} \right)^4)}} \tag{8.40}
\]

Analogously we can compute the energy corresponding to the minimal surface that is given by:

\[
E \equiv \frac{S}{X_2} = \frac{U_0}{\pi} \int_1^\infty dw \frac{w^4}{\sqrt{(w^4 - 1)(w^4 - \left( \frac{U_T}{U_0} \right)^4)}} \tag{8.41}
\]

This quantity is divergent and can be regularized by cutting it off at a value \( U_{\text{max}}/U_0 \) and subtracting to it the self-energy of the two test "quarks" \( (U_{\text{max}} - U_T)/\pi \) corresponding to the energy of two strings stretching up to the boundary of \( AdS_5 \). The energy is now convergent and we can take the limit \( U_{\text{max}} \to \infty \) obtaining

\[
E = \frac{U_0}{\pi} \int_1^\infty dw \left\{ \frac{w^4}{\sqrt{(w^4 - 1)(w^4 - \left( \frac{U_T}{U_0} \right)^4)}} - 1 \right\} + \frac{U_T - U_0}{\pi} \tag{8.42}
\]

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The previous equation can be rewritten as:

$$
E = \frac{U_0}{\pi} \int_{1}^{\infty} dw \left\{ \sqrt{(w^4 - 1)} \frac{(w^4 - \left(\frac{U_T}{U_0}\right)^4)}{(w^4 - \left(\frac{U_T}{U_0}\right)^4)} - 1 \right\} + \frac{U_T - U_0}{\pi} + \frac{U_0^2 \alpha'}{2\pi b^2 R} \quad (8.43)
$$

From eq. (8.40) it is easy to see that $U_0 \rightarrow U_T$ when $R \rightarrow \infty$. In this limit all terms in eq. (8.43) vanish except the last one that gives a confining potential with string tension given by:

$$
\frac{E}{R} \equiv \sigma = \frac{U_0^2 \alpha'}{2\pi b^2} = \frac{\pi b^2}{2\alpha'} T^2 \quad (8.44)
$$

where we have used eq. (8.33). Using eq. (8.44) we get finally:

$$
\sigma = \frac{\pi}{2} N g_{YM}^2 T^2 \quad (8.45)
$$

In conclusion we have shown that the high temperature phase of $\mathcal{N} = 4$ super Yang-Mills is a confining one with string tension given in eq. (8.45).

In the following we discuss in some detail the fact that at finite temperature a mass gap appears. The appearance of a mass gap is usually seen by studying the two-point function involving for instance the composite $F^2$ and showing that its large distance behaviour decays exponentially with the distance:

$$
<F^2(x) F^2(0)> \sim e^{-m|x|} \quad (8.46)
$$

From this behaviour one can just read the mass $m$ of the lowest lying state that has the same quantum numbers as $F^2$. This provides the mass gap of the theory. We could just compute the mass gap from the correlator in eq. (8.46), but there is a simpler way that we are going to discuss now following Ref. [9]. If the Maldacena conjecture is right and therefore $N = 4$ super Yang-Mills is equivalent for large values of $\lambda$ to type IIB supergravity compactified on $AdS_5 \otimes S^5$, then the Hilbert spaces of these two theories must be the same. We have seen in sect. 7 that the composite and gauge invariant field $F^2$ corresponds in supergravity to the dilaton field $\phi$. In order to construct the Hilbert space in the supergravity approximation, that is relevant for constructing the two-point function involving for instance two fields $F^2$ we have to consider the classical equation of motion of the dilaton, that is the field corresponding to $F^2$, in the $AdS_5 \otimes S^5$ background and search for its solutions satisfying certain boundary conditions. Since the dilaton is massless in ten dimensions, the $\ell = 0$ mode on $S^5$ is also massless in five dimensions. This mode does not depend on the coordinates of $S^5$ and satisfies the classical equation:

$$
\partial_{\mu} [\sqrt{g} g^{\mu\nu} \partial_{\nu} \Phi] = 0 \quad (8.47)
$$

where $g_{\mu\nu}$ is the metric of $AdS_5$ that is given in eq. (8.23) for $n = 4$. We wish to look at solutions of the previous equation that are square integrable in the previous
metric and that correspond to plane waves in the boundary gauge theory, i.e. we take the dilaton in eq.(8.47) to be of the following form:

$$\Phi(\rho, x) = f(\rho) \ e^{ik \cdot x}$$

(8.48)

Following the procedure discussed in great detail in Ref. [12] it is convenient to rescale \( \tau \) by \( \tau = b^2 \tilde{\tau} \). The period of \( \tilde{\tau} \) is equal to \( \pi/b \) and after a rescaling of the variables \( x \), the metric becomes:

$$\frac{ds^2}{b^2} = (\rho^2 - b^4/\rho^2) d\tilde{\tau}^2 + \frac{d\rho^2}{\rho^2 - b^4/\rho^2} + \rho^2 \sum_{i=1}^{3} (dx^i)^2 + d\Omega_5^2$$

(8.49)

Inserting the previous metric in the dilaton equation with a dilaton field given in eq.(8.48) we get:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \rho \left( \rho^4 - b^4 \right) \partial_\rho f \right] = k^2 f$$

(8.50)

Introducing the quantity \( x = \rho^2 \) and rescaling \( x \) by \( x \rightarrow b^2 x \) we get the following equation:

$$x(x^2 - 1) \frac{d^2 f}{dx^2} + (3x^2 - 1) \frac{df}{dx} = \frac{k^2}{4b^2} f$$

(8.51)

Rescaling \( k \rightarrow b^2 k \) in order to have a quantity with the dimension of a mass we get that in terms of the rescaled variable the coefficient of the non-derivative term in eq.(8.51) becomes \( k^2 b^2/4 \). As a boundary condition we need to impose that the solution be square integrable. This means that it must satisfy the condition:

$$\int d\rho \sqrt{g} |f(\rho)|^2 < \infty$$

(8.52)

Since \( \sqrt{g} d\rho = \rho^3 d\rho = \frac{1}{2} x dx \) the previous condition implies that \( f \sim x^{-a} \) with \( a > 1 \).

Near the horizon the metric in eq.(8.49) behaves after a rescaling of \( \rho \) as:

$$ds^2 \sim \left( x - \frac{1}{x} \right) (d\tilde{\tau})^2 + \frac{dx^2}{4(x^2 - 1)}$$

(8.53)

We introduce the variable \( z \) related to \( x \) through the relation:

$$dz = \frac{dx}{2\sqrt{x^2 - 1}}$$

(8.54)

that implies \( x = \cosh(2z) \). The coordinate singularity appearing in the metric in eq.(8.53) at \( x = 1 \) corresponds in the new variable to \( z = 0 \). Then since near the horizon

$$x - \frac{1}{x} = \frac{\sinh^2 2z}{\cosh 2z} \sim 4z^2$$

(8.55)

the metric in eq.(8.53), in the near horizon limit, becomes the two-dimensional flat metric in polar coordinates

$$ds^2 = dz^2 + 4z^2 d\tilde{\tau}^2$$

(8.56)
Since the function $f(\rho)$ is only a function of the radial variable $z$ and not of the angular variable $\tilde{\tau}$, a proper boundary condition for $f$ is that it is smooth at the origin, i.e. $\frac{df}{dz} = 0$ at $z = 0$. But, since

$$\frac{df}{dz} = \frac{dx}{dz} \frac{df}{dx} = 2 \sinh 2z \frac{df}{dx} \sim 4z \frac{df}{dx} \tag{8.57}$$

one gets that the function $f$ must be regular at the horizon $x = 1$. In conclusion we must look for square integrable solutions of the dilaton classical equation in eq. (8.51) that are regular at the horizon. We can rewrite the differential equation in (8.51) as follows:

$$y'' + \left[ \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-1} \right] y' = \frac{p}{x(x^2-1)} y \tag{8.58}$$

where $f = y$. The eigenvalues of eq. (8.58) has been determined numerically in Refs. [50, 51, 52]. In the following we describe the method proposed in Ref. [52] in order to show that the spectrum of the eigenvalues of the differential equation in (8.58) is discrete and in particular that there is a mass gap. The general solution of the previous eq. can always be written as a linear combination of two independent solutions that in general have singularities at $x = 0, 1$ and $\infty$. Therefore in general a solution cannot be represented as a convergent series expansion throughout the entire physical region $1 \leq x \leq \infty$. It is possible, however, to consider expansions that are convergent in either of the two intervals $I(\infty) \equiv \{x \in \mathbb{C}|1 < x < \infty\}$ and $I(1) \equiv \{x \in \mathbb{C}|0 < x < 2\}$. They overlap in the interval $1 < x < 2$. In the first interval $I(\infty)$ following Ref. [52] the most general solution can be written in terms of the two convergent expansions:

$$y_{1}^{(\infty)} = \frac{1}{x^2} + \sum_{n=1}^{\infty} a_{n}^{(\infty)} x^{-2-n} \tag{8.59}$$

and

$$y_{2}^{(\infty)} = \frac{p}{2} \log(x) y_{1}^{(\infty)} + \sum_{n=1}^{\infty} b_{n}^{(\infty)} x^{-n} \tag{8.60}$$

while in the interval $I(1)$ can be written in terms of the following two other convergent series:

$$y_{1}^{(1)} = 1 + \sum_{n=1}^{\infty} a_{n}^{(1)} (x - 1)^n \tag{8.61}$$

and

$$y_{2}^{(1)} = \log(x - 1) y_{1}^{(1)} + \sum_{n=1}^{\infty} b_{n}^{(1)} (x - 1)^n \tag{8.62}$$

The expansion coefficients can be determined for any value of $p$ by recursion from the differential eq. (8.58). In general, however, the previous solutions or any combination of them will not satisfy both boundary conditions that we have discussed above. For
certain values of \( p \) it turns out that there exists a solution that is simultaneously proportional to \( y_1^{(\infty)} \) and to \( y_1^{(1)} \). The condition for this to happen is that their Wronskian vanishes:

\[
W(p, x) \equiv \begin{vmatrix}
y_1^{(1)} & y_1^{(\infty)} \\
dy_1^{(1)} & dy_1^{(\infty)}
\end{vmatrix} = 0 \quad (8.63)
\]

For \( 1 < x < 2 \) both series are convergent and the Wronskian can be computed and can be seen to have the following form:

\[
W(p, x) = \frac{r(p)}{x(x-1)(x+1)} \quad (8.64)
\]

The function \( r(p) \) can also be computed to any desired accuracy. The spectrum of \( p \) is determined by the zeroes of \( r(p) \). It can be seen that there is no positive or zero eigenvalue of the differential equation in eq.(8.58). Therefore a mass gap is generated together with a discrete spectrum. In particular the eigenvalue spectrum can be approximately computed using the WKB approximation and one gets [50, 53]:

\[
M^2 = -k^2 = 8\pi \left[ \Gamma(3/4) \right]^4 T^2 n(n + 1) \quad (8.65)
\]

where \( n \) is an arbitrary positive integer.

In this section we have studied the behaviour of \( \mathcal{N} = 4 \) super Yang-Mills at high temperature. Remembering that finite temperature means that one direction (the euclidean time one) is compactified along a circle of radius equal to \( 1/(2\pi T) \), then the radius becomes very small at high temperature. This means that at high temperature the theory becomes effectively three-dimensional and therefore studying the original four-dimensional theory at high temperature corresponds essentially to study a three-dimensional theory in which supersymmetry is broken and in which we expect that both the fermions and scalars get a mass of the order of the temperature. Because of this it is then natural to think that this theory reduces to a theory of pure Yang-Mills in three dimensions. The relation between the four and the three-dimensional coupling constants can be found by expanding the D 3-brane effective Born-Infeld action and keeping only the kinetic term for the gauge field as we have done in eq. (1.10). In this way for \( p = 3 \) one gets the Yang-Mills action in four dimensions with \( g_{YM4}^2 = 4\pi g_s \). Then remembering that the time direction is compactified with radius equal to \( 1/(2\pi T) \) we get the Yang-Mills action in three dimensions with coupling constant equal to:

\[
g_{YM3}^2 = g_{YM4}^2 T \quad (8.66)
\]

We have seen that the use of the supergravity approximation is allowed only in the strong coupling limit where \( N g_{YM4}^2 \equiv \lambda >> 1 \), while the three-dimensional Yang-Mills scale \( Ng_{YM3}^2 \) is obtained in the limit in which \( T \rightarrow \infty \) and \( N g_{YM4}^2 \rightarrow 0 \). In order to study this limit we have to go away from the supergravity approximation and take into account the tree diagrams of string theory. But this is unfortunately beyond our reach at present.
9 \textbf{D = 4 Yang-Mills from the M-theory 5-brane}

In the previous section we have seen how, starting from the ten-dimensional non extremal D 3-brane, one can describe strongly coupled $\mathcal{N} = 4$ super Yang-Mills at high temperature or alternatively a theory that is presumably in the same universality class of three-dimensional Yang-Mills theory. In this section we discuss a suggestion, made by Witten [9], on how to extend the previous procedure from three to four-dimensional Yang-Mills theory. The starting point in this case is not the D 3-brane of the ten-dimensional type IIB string theory, but the 5-brane of the eleven-dimensional M-theory. This solution has in the near horizon limit a metric corresponding to the manifold $\text{AdS}_7 \otimes S^4$ with the two radii given by:

$$R_{\text{AdS}_7} \equiv b = 2\ell_p(\pi N)^{1/3} \quad R_{S^4} \equiv L = \frac{b}{2} = \ell_p(\pi N)^{1/3} \quad (9.1)$$

where the 11-dimensional Planck length $\ell_p$ is related to the 11-dimensional gravitational constant by $2\kappa_{11}^2 = (2\pi)^8\ell_p^9$. Following the notation of Ref. [54] the 11-dimensional metric of the non-extremal 5-brane in the near horizon limit is given by:

$$(ds_{11})^2 = \frac{4L^2}{y^2} \frac{dy^2}{1 - \left(\frac{y_0}{y}\right)^6} + L^2d\Omega_4^2 + \frac{y^2}{L^2} \left[ \left(1 - \frac{y_0}{y^6}\right)dt^2 + \sum_{i=1}^5 dx_i^2 \right] \quad (9.2)$$

The previous variables $y$ and $t$ are related to the variables $\rho$ and $\tau$ used in eq.(8.23) by the relations:

$$y = \frac{y_0}{b} \rho \quad t = \frac{L}{y_0} \tau \quad (9.3)$$

while the parameters $y_0$ and $L$ are given in terms of the temperature and of the 11-dimensional gravitational constant by:

$$y_0 = \frac{4\pi}{3} TL^2 \quad L^9 = \frac{\kappa_{11}^2 N^3}{27\pi^3} \quad (9.4)$$

If we compactify one the 11 dimensions belonging to the world volume of the $M5$-brane we obtain a 10-dimensional theory in which the original $M$ 5-brane becomes the D 4-brane of type IIA string theory. The dilaton and the 10-dimensional metric of the D 4-brane can be obtained using the formula:

$$(ds_{11})^2 = e^{4\phi/3} \left(dx^5 + A_\mu dx^\mu\right)^2 + e^{-2\phi/3}(ds_{10})^2 \quad (9.5)$$

where what is usually called the 11th dimension is here the 5th direction.

Comparing eq.(9.5) with the 11-th dimensional metric in eq.(9.2) we get that the dilaton is given by:

$$e^\phi = g_s \left(\frac{y}{L}\right)^{3/2} \quad (9.6)$$
and the ten-dimensional metric by:

\[(ds_{10})^2 = \frac{y}{L} (ds_{11})^2 g_s^{2/3} = \]

\[= \frac{4L}{y} \left( \frac{dy^2}{1 - \left(\frac{y_0}{y}\right)^6} \right) + L y d\Omega_4^2 + \frac{y^3}{L^2} \left[ \left( 1 - \frac{y_0}{y^6} \right) dt^2 + \sum_{i=1}^4 dx_i^2 \right] \quad (9.7) \]

Notice that the dependence on the string coupling constant in the second term of
the previous equation disappears in the third term because we use variables that
are 10-dimensional. Both \(L\) and \(y_0\) in eq.\((9.7)\) are now expressed in 10-dimensional
units. Remembering that a length \(L^{(10)}\) in 10-dimensional units is related to a
length \(L^{(11)}\) in 11-dimensional ones through the formula: \(L^{(10)} = g_s^{1/3} L^{(11)}\) we get
that the quantity \(L\) in eq.\((9.7)\) is given by:

\[L = \sqrt{\alpha'} (\pi g_s N)^{1/3} \quad (9.8)\]

where \(\sqrt{\alpha'} \equiv \ell^{(11)}\) is equal to the 11-dimensional Planck constant in 11-dimensional
units.

A system of \(N\) \(D\) \(p\)-branes of type IIA theory is described at low energy by the
non-abelian version of the Born-Infeld action given by:

\[S_{BI} = \tau_p^{(0)} STr \int d^{p+1}x e^{-\phi} \sqrt{\det (G_{\alpha\beta} + B_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta})} + \]

\[+ \frac{1}{\sqrt{2\kappa_{10}^{(0)}}} \int_{p+1} \sum_{\mu-p} \mu_{p-2n} A^{(p+1-2n)} STr e^{F/(2\pi)} \quad (9.9)\]

It contains external NS-NS and R-R fields that are normalized in such a way that
the Lagrangian of the bulk theory is given by:

\[S_{\text{bulk}} = \frac{1}{2(\kappa_{10}^{(0)})^2} \int d^{10}x \sqrt{-G} \left\{ e^{-2\phi} \left[ R + 4G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} (H_3)^2 \right] - \frac{1}{2(p+2)!} H_{p+2}^2 \right\} \quad (9.10)\]

where

\[\tau_p = \frac{\tau_p^{(0)}}{g_s} = \frac{(2\pi \sqrt{\alpha'})^{1-p}}{2\pi \alpha' g_s} \quad ; \quad \mu_p = \sqrt{2\pi(2\pi \sqrt{\alpha'})^{3-p}} \quad (9.11)\]

Remember also that \(2\kappa_{10}^2 \equiv 2(\kappa_{10}^{(0)})^2 g_s^2 = (2\pi)^7 (\alpha')^4 g_s^2\). We keep in the Born-Infeld
action only the gauge field and the zero component of the 1-form potential. Then
using the formulas given in eq.\((9.11)\) and compactifying the time (temperature)
direction we can rewrite eq.\((9.9)\) for the case of a \(D\) 4-brane as follows:

\[S_{BI} = \frac{\tau_4}{T} STr \int d^4x \sqrt{\det (G_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta})} + \frac{1}{2} (2\pi \alpha')^2 \tau_4^{(0)} \int A^{(1)} STr \left( F_2^2 \right) \quad (9.12)\]
Using the relation:

\[ \int A^{(1)} STr \left( F_2^2 \right) = \frac{1}{4T} \int d^4x e^{\alpha\beta\gamma\delta} A_\alpha Tr \left( F_\beta F_\gamma F_\delta \right) \]  

(9.13)

keeping only the time component of the R-R field \( A_\alpha \) and expanding the Born-Infeld action restricting ourselves only to the quadratic term in \( F \) we get:

\[ S_{BI} = \frac{\tau_4}{T} (2\pi \alpha')^2 \int d^4x \left\{ \frac{1}{4} Tr (F^2) + \frac{1}{8} g_s A_0 Tr (\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}) \right\} \]  

(9.14)

Using the equation:

\[ \frac{\tau_4 (2\pi \alpha')^2}{T} = \frac{1}{(2\pi)^2 T \sqrt{\alpha' g_s}} \]  

(9.15)

and introducing a parameter \( \lambda \) through the equation:

\[ 2\pi \sqrt{\alpha' g_s} = \frac{\lambda}{TN} \]  

(9.16)

we can rewrite eq.(9.14) as follows:

\[ S = \frac{N}{2\pi \lambda} \int d^4x \left[ \frac{1}{4} Tr (F^2) + \frac{1}{8} g_s A_0 Tr (\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}) \right] \]  

(9.17)

From eq.(9.17) we can read the value of the Yang-Mills coupling constant and connect it with the radius \( R_5^{(10)} \) of the 5th direction in 10-dimensional units. We get:

\[ \frac{Ng_Y^2}{2\pi} = \lambda = 2\pi R_1 TN \quad R_5^{(10)} \equiv R_1 = \sqrt{\alpha' g_s} \quad \frac{1}{T} \equiv 2\pi R_2 \]  

(9.18)

or in other words:

\[ g_Y^2 = (2\pi R_1)(2\pi T) = \frac{2\pi R_1}{R_2} \]  

(9.19)

To summarize we started with the M-theory 5-brane having a six-dimensional world volume. We have then compactified two of the six directions on two radii. The first one, that we called \( R_1 \), corresponds in going from the 11-dimensional M-theory to the 10-dimensional type IIA string theory and because of this compactification the M-theory 5-brane becomes the 10-dimensional D 4-brane of type IIA theory. The second radius \( R_2 \) that from the point of view of the 5-dimensional theory, corresponding to the world volume of the D 4-brane, is related to the temperature which is also the temperature of the non-extremal black hole solution, has been introduced in order to reduce ourselves to a 4-dimensional theory gauge theory that we are interested to study. In particular in the temperature direction we are free to choose antiperiodic boundary conditions for the fermions of the theory and correspondently they will have masses equal to \((2n + 1)/R_2\) (\( n \) is an integer) that become very big in the high temperature limit \((R_2 \to 0)\). Also the scalars of the
theory will get a mass of the order $g^2YM/R_2$ and they will also become very massive in the high temperature limit. Therefore in this limit we will be left with only the gauge field and it looks plausible that such a theory is in the same universality class as pure Yang-Mills theory.

In the second part of this section we will show how to compute various observables as for instance the Wilson loop \[49\] and the topological susceptibility \[54\] in the previously defined four-dimensional theory. Let us start computing the topological susceptibility. In eq.(9.17) we see that the Yang-Mills topological charge density is coupled to the time component $g_s A_0 \equiv h$ of the R-R field $A_\mu$:

$$h \rightarrow \tilde{O}_4 = \frac{N}{2\pi \lambda} \cdot \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} Tr (F_{\mu\nu} F_{\rho\sigma})$$

(9.20)

where $\mu, \nu, \rho$ and $\sigma$ are all four-dimensional indices. Therefore in order to compute correlators involving the topological charge density as for instance the topological susceptibility, that is related to the two-point function with two operators $\tilde{O}_4$, we must look at the classical eq. of motion for $A_0$ that follows from the type IIA supergravity Lagrangian. The relevant term of the type IIA supergravity is the kinetic term for $A_\mu$, namely

$$\frac{1}{2(k^{(0)}_{10})^2} \int d^{10}x \sqrt{g} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2(k^{(0)}_{10})^2} \int d^{10}x \sqrt{g} \frac{1}{2} g^{\mu\nu} g^{00} \partial_\mu A_0 \partial_\nu A_0$$

(9.21)

The function $h \equiv g_s A_0$ is determined by solving the classical eq. that follows from the previous action:

$$\partial_\mu \left[ \sqrt{g} g^{00} g^{\mu\nu} \partial_\nu h \right] = 0$$

(9.22)

We can assume that $h$ is only a function of $y$. In fact, since we will see that the contribution to the various correlators comes from a boundary term, the dependence on the other variables will be irrelevant because they will never contribute to a total divergence. Remembering that the background metric following from eq.(9.7) is given by:

$$g^{00} = \left( \frac{L}{y} \right)^3 \frac{1}{1 - \left( \frac{w}{y_0} \right)^6} \quad g^{yy} = \frac{y}{4L} \left[ 1 - \left( \frac{y}{y_0} \right)^6 \right] \quad \sqrt{g} = 2 \left( \frac{y}{L} \right)^9$$

(9.23)

we get the following eq. of motion:

$$\partial_y \left[ \left( \frac{y}{L} \right)^7 \partial_y h \right] = 0$$

(9.24)

Integrating it with the two boundary conditions:

$$\lim_{y \rightarrow \infty} h(y) = h^\infty \quad h(y_0) = 0$$

(9.25)
we obtain the following solution [53, 54]:

$$h(y) = h^\infty \left[ 1 - \left( \frac{y_0}{y} \right)^6 \right]$$  \hspace{1cm} (9.26)

When we insert it in the action in eq.(9.21) we are left with only a surface term that cannot be neglected and is equal to:

$$S_{class.} = \frac{1}{2\kappa^2_{10}} \int d^{10}x \partial_y \left[ \frac{\sqrt{g}}{2} g^{yy} g^{00} h \partial_y h \right] =$$

$$= \frac{1}{2\kappa^2_{10}} \int d\Omega_4 L^4 \int d^4x \int dt \left( \frac{\sqrt{g}}{2} g^{yy} g^{00} \right) h \partial_y h|_{\infty}$$  \hspace{1cm} (9.27)

Using eqs.(9.23) and (9.26) we get:

$$h \partial_y h|_{\infty} = 6(h^\infty)^2 \frac{3y^6}{y^7}  \hspace{1cm} \frac{1}{2} \sqrt{g} g^{yy} g^{00}|_{\infty} = \frac{1}{4} \left( \frac{y}{L} \right)^7$$  \hspace{1cm} (9.28)

Inserting them in eq.(9.27) we see that the dependence on $y$ cancels out and we get

$$S_{class.} = \frac{1}{2\kappa^2_{10}} (h^\infty)^2 \frac{3y^6}{2L^7} \int d\Omega_4 L^4 \int d^4x \int dt$$  \hspace{1cm} (9.29)

Using the following expressions

$$\int d\Omega_4 = \frac{8\pi^2}{3}  \hspace{1cm} \int d^4x \equiv V_4  \hspace{1cm} \int dt = \frac{1}{T}$$  \hspace{1cm} (9.30)

eq.(9.29) becomes

$$S_{class.} = V_4 \frac{2\pi^2 y^6}{\kappa^2_{10} L^7} (h^\infty)^2$$  \hspace{1cm} (9.31)

The correlator involving two operators $\tilde{O}_4$ is obtained by differentiating twice $e^{-S_{class.}}$ with respect to $h^\infty$ and eliminating the volume factor. We get:

$$\int d^4x < \tilde{O}_4(x) \tilde{O}_4(0) > = \frac{4\pi^2 y^6}{\kappa^2_{10} L^3 T} = \frac{N^2 T^4 \pi^3 2^7 \lambda}{3^6}$$  \hspace{1cm} (9.32)

where we have used eqs.(9.4) and the relation $\kappa^2_{10} = \kappa^2_{11} NT/\lambda$ that follows from eq.(9.19). Finally the topological susceptibility is given by [54]:

$$\chi_t \equiv \int d^4x \left( \frac{\lambda}{2\pi N} \right)^2 \frac{1}{4} < \tilde{O}_4(x) \tilde{O}_4(0) > = \frac{8\lambda^3 T^4 \pi^4}{3^6}$$  \hspace{1cm} (9.33)

We can use the previous results to determine the behaviour of the vacuum energy of a gauge theory in terms of the vacuum $\theta$ parameter. In particular in the supergravity
approximation one can show [53] that the vacuum energy behaves precisely as given in eq. (4.41) according to the large $N$ considerations discussed in sect. 4. We have seen above that the Born-Infeld action gives a term of the following form (see eq. (9.14)):

$$
\int_V A \wedge Tr (F \wedge F)
$$

(9.34)

where $V = S^1 \times R^4$. The previous equation implies that we can introduce a $\theta$ parameter in the four-dimensional gauge theory by requiring that the integral of the abelian vector field $A$ along the compactified direction be nonzero and equal to

$$
\int_{S^1} A = \theta + 2\pi k
$$

(9.35)

where $A$ is the value of the abelian vector field in Minkowski space corresponding to the limit $y \to \infty$ and we have taken care of the fact that $\theta$ is an angular variable by extracting from it the factor $2\pi k$. The vacuum energy of the four-dimensional gauge theory can then be computed by proceeding exactly as in the calculation of the topological susceptibility and obtaining:

$$
E(\theta) = \frac{X^4}{2} Min_k (\theta + 2\pi k)^2
$$

(9.36)

in perfect agreement with eq. (4.40) obtained in the framework of the large $N$ expansion after having used eq. (4.31).

We now turn our attention to the Wilson loop, we show that it is proportional to the area and from it we extract the string tension. The calculation is very similar to the one we have done in sect. 8 for $\mathcal{N} = 4$ super Yang-Mills at finite temperature. One starts with the string action in eq. (8.34) in the metric given in eq. (9.7). Choosing the static gauge where $x_1 \equiv x = \sqrt{A} \sigma$, $x_2 = \sqrt{A} \tau$, assuming that only the anti de Sitter variable $y$ is a function of $x$ and remembering that (see eq. (9.7)):

$$
G_{xx} = G_{x_2 x_2} = \left( \frac{y}{L} \right)^3
g_{yy} = \frac{4L^2}{y} \left[ 1 - \left( \frac{y_0}{y} \right)^6 \right]^{-1}
$$

(9.37)

we get the following expression for the string action:

$$
S = \frac{X_2}{2\pi \alpha'} \int_{-R/2}^{R/2} dx \left\{ \left( \frac{U}{L^2} \right)^3 + \frac{1}{L^2} \left[ 1 - \left( \frac{y_0}{U} \right)^3 \right] \left( \frac{dU}{dx} \right)^2 \right\}^{1/2}
$$

(9.38)

where we have defined $y \equiv U^2$. Also in this case the hamiltonian is a constant of motion implying that:

$$
-H = \left( \frac{U}{L^2} \right)^3 \left\{ \left( \frac{U}{L^2} \right)^3 + \frac{1}{L^2} \left[ 1 - \left( \frac{y_0}{U} \right)^3 \right] \left( \frac{dU}{dx} \right)^2 \right\}^{-1/2} = C
$$

(9.39)
From it we get

\[
\left( \frac{dU}{dx} \right)^2 = L^2 \left[ \left( \frac{U}{L^2} \right)^3 - \left( \frac{U_0}{U} \right)^3 \right] \left[ \left( \frac{U}{U_0} \right)^3 - 1 \right] \tag{9.40}
\]

where \( U_0 \) is the minimum value of \( U \) corresponding to \( C^2 = (U_0)^3/L^6 \). Since for symmetry reason \( U = U_0 \) corresponds to \( x = 0 \) by integrating the previous equation we get:

\[
x = \frac{U_0}{L} \left( \frac{L^2}{U_0} \right)^{3/2} \int_{1}^{U/U_0} dw \sqrt{\left( w^3 - 1 \right) \left( w^3 - \left( \frac{y_0^2}{U_0} \right)^3 \right)} \tag{9.41}
\]

where we have introduced the variable \( w = U/U_0 \). If we go to the boundary, corresponding to sending \( U \to \infty \) and correspondently \( x \to R/2 \) we get:

\[
\frac{R}{2} = \frac{U_0}{L} \left( \frac{L^2}{U_0} \right)^{3/2} \int_{1}^{\infty} dw \sqrt{\left( w^3 - 1 \right) \left( w^3 - \left( \frac{y_0^2}{U_0} \right)^3 \right)} \tag{9.42}
\]

Analogously we can compute the energy corresponding to the minimal surface that is given by:

\[
E \equiv \frac{S}{X_2} = \frac{U_0}{\pi \alpha' L} \int_{1}^{\infty} dw \frac{w^3}{\sqrt{\left( w^3 - 1 \right)(w^3 - \left( \frac{y_0^2}{U_0} \right)^3)}} \tag{9.43}
\]

Also in this case the energy is divergent. It can be regularized by cutting off the integral at \( U_{\text{max}}/U_0 \) and subtracting the quantity \( (U_{\text{max}} - U_T)/(\pi \alpha' L) \) where \( U_T \equiv y_0^2 \). With the previous subtraction the integral in eq. (9.43) becomes convergent and one can integrate up to infinity getting:

\[
E = \frac{U_0}{\pi \alpha' L} \int_{1}^{\infty} dw \left\{ \frac{w^3}{\sqrt{\left( w^3 - 1 \right)(w^3 - \left( \frac{y_0^2}{U_0} \right)^3)}} - 1 \right\} + \frac{U_T - U_0}{\alpha' \pi L} \tag{9.44}
\]

In conclusion we arrive at the two equations:

\[
R = \frac{2L^2}{U_0^{1/2}} \int_{1}^{\infty} \frac{dw}{\sqrt{\left( w^3 - 1 \right)(w^3 - \left( \frac{y_0^2}{U_0} \right)^3)}} \tag{9.45}
\]

and

\[
E = \frac{U_0}{\pi \alpha' L} \int_{1}^{\infty} dw \left\{ \frac{w^3}{\sqrt{\left( w^3 - 1 \right)(w^3 - \left( \frac{y_0^2}{U_0} \right)^3)}} - 1 \right\} + \frac{U_T - U_0}{\pi \alpha' L} \tag{9.46}
\]
From eq.\((9.43)\) we see that when \(R \to \infty\) then \(U_0 \to U_T\). On the other hand eq.\((9.46)\) can be rewritten in the form:

\[
E = \frac{U_0}{\pi \alpha' L} \int_1^\infty dw \left\{ \left[ \frac{w^3 - 1}{w^3 - \left(\frac{U_T}{U_0}\right)^3} \right]^{1/2} - 1 \right\} + \frac{U_T - U_0}{\pi \alpha' L} + \frac{U_0^{3/2}}{2\pi \alpha' L^3} R \tag{9.47}
\]

Then for large \(R\) we get a confining potential with string tension given by:

\[
\frac{E}{R} \equiv \sigma = \frac{(U_T)^{3/2}}{2\pi \alpha' L^3} \tag{9.48}
\]

Using the first eq. in \((9.4)\), eq.\((9.8)\), the first eq. in \((9.18)\) and eq.\((9.16)\) we arrive at:

\[
\sigma = \frac{8\pi}{27} (g_{YM}^2 N) T^2 \tag{9.49}
\]

In this section we have described Witten’s proposal for studying Yang-Mills theory starting from the M-theory 5-brane. In particular we have computed several observables in the strong coupling limit of the gauge theory where the supergravity approximation can be applied. In order to understand large \(N\) gauge theories one would like, however, to continue the previous results from strong to weak coupling and to show that there is no other singularity except the one obtained when \(N g_{YM}^2 \to 0\), where we expect to recover the asymptotic freedom behaviour of gauge theories for the various observables. This is, however, at the moment a difficult problem to solve and some new idea seems to be needed.

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