NORMAL FORMS AND GAUGE SYMMETRIES OF LOCAL DYNAMICS

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Abstract. A systematic procedure is proposed for deriving all the gauge symmetries of the general, not necessarily variational, equations of motion. For the variational equations, this procedure reduces to the Dirac-Bergmann algorithm for the constrained Hamiltonian systems with certain extension: it remains applicable beyond the scope of Dirac’s conjecture. Even though no pairing exists between the constraints and the gauge symmetry generators in general non-variational dynamics, certain counterparts still can be identified of the first- and second-class constraints without appealing to any Poisson structure. It is shown that the general local gauge dynamics can be equivalently reformulated in an involutive normal form. The last form of dynamics always admits the BRST embedding, which does not require the classical equations to follow from any variational principle.

1. Introduction

By a local dynamical system we understand the dynamics whose true trajectories are defined by a finite system of ordinary differential equations. Given a local dynamical system, the question arises of finding all its gauge symmetries. If the equations of motion were variational, the Dirac-Bergmann algorithm [1], [2] would do the job. In fact, the Dirac-Bergmann algorithm does even more: it iteratively brings the variational equations of motion to a canonical form of the constrained Hamiltonian dynamics with a complete set of constraints on the phase-space variables. In this form, the equations of motion are totally self-contained, having no other constraints that can be derived from any compatibility condition. It is the normal form of the variational dynamics, which is exploited in physics for many different purposes. Besides the other things, this canonical form allows one to classify the constraints by grouping them into the first and second classes, and it also results in identifying the gauge symmetry generators as the Hamiltonian vector fields associated to the first-class constraints. For the general (i.e., not necessarily variational) local dynamics, no canonical form has been yet identified which might be viewed as an equivalent of the Hamiltonian dynamics with a complete set of constraints. The correspondence between the complete set of first-class constraints, including the primary and secondary ones, and the gauge
symmetries is known as the Dirac conjecture. This conjecture is not always valid even for the regular variational equations, and the counterexamples are well known \cite{2}. Beyond the scope of validity of the Dirac conjecture, no systematic procedure has been known for identifying the gauge symmetries even for the constrained Hamiltonian dynamics.

In this paper, we work out an algorithm of bringing the general (not necessarily variational) dynamics to a certain normal form. In this form, the gauge symmetries and the complete set of constraints are explicitly identified. Since no natural pairing exists between the constraints and gauge symmetries unless the dynamics are Hamiltonian, our algorithm derives them independently: the constraints are found first, and then the symmetries are identified. Despite of the fact that Dirac’s classification of constraints is conventionally defined in terms of the Poisson brackets, the notions of the first- and second-class constraints can be naturally extended to the non-Hamiltonian systems. The algorithm works equally well in the constrained Hamiltonian dynamics, finding out all the gauge symmetries, even though the Dirac conjecture does not hold for the system. So, the method may have some new impact on the well-studied area of the constrained Hamiltonian dynamics.

Below we give some introductory comments on the background of the problem and outline the contents of the paper.

Any system of ordinary differential equations can always be depressed to the first order by introducing new variables, so we start with a first-order autonomous system. Under certain regularity conditions\footnote{From the viewpoint of physics, the regularity means that the system has a definite number degrees of freedom, i.e., the equations admit the same number of independent Cauchy data in every domain of the configuration space. A more accurate formulation of the regularity conditions is given in Section 2 as well as the explanation how to bring the general regular equations to the normal form (1), (2).} the first order ODE system can be always brought (by adding new auxiliary variables, if necessary) to the following form

\begin{equation}
\dot{x}^i = V^i(x) + \lambda^\alpha Z^i_\alpha(x), \quad \alpha = 1, \ldots, l, \quad i = 1, \ldots, n, \tag{1}
\end{equation}

\begin{equation}
T_a(x) = 0, \quad a = 1, \ldots, m, \tag{2}
\end{equation}

where $x^i, \lambda^\alpha$ are the decision variables. Thus, in the regular local dynamics, it is always possible to have only two types of variables: (i) the phase space coordinates $x^i$ being subject to the normal differential equations (1) and the constraints (2), and (ii) the variables $\lambda^\alpha$ entering linearly and
without derivatives into the right hand sides of the differential equations for \( x \)'s. We call the set of equations (1), (2) the primary normal form of local dynamics. Obviously, the compatibility conditions between the differential equations (1) and the algebraic constraints (2) can result in more algebraic equations, which are called the secondary constraints. The differential consequences of equations (1), (2) are considered in Section 3.

Two particular classes of equations (1), (2) were extensively studied by two different sciences: Dirac’s analysis of the constrained Hamiltonian systems and optimal control theory. The first one deals with the case where the number of the constraints coincides with the number of \( \lambda \)'s, i.e., \( l = m \), the vector fields \( Z_\alpha \) and \( V \) are all Hamiltonian, and

\[
Z^i_\alpha = \{ x^i, T_\alpha \}, \quad V^i = \{ x^i, H \}.
\]

Here \( \{ \cdot, \cdot \} \) is a non-degenerate Poisson bracket and \( T_\alpha \) are the algebraic constraints (2). Upon this identification, the equations of motion become variational, following from the least action principle for the functional

\[
S[x, \lambda] = \int (\rho_i(x) \dot{x}^i - H(x) - \lambda^\alpha T_\alpha(x)) \, dt,
\]

where \( \rho_i dx^i \) is a symplectic potential associated to the Poisson bracket. This can be understood as a conditional extremum for the standard Hamiltonian action subject to the constraints (2), with the variables \( \lambda \) being the corresponding Lagrange multipliers. It is the variational dynamics that the Dirac-Bergmann algorithm is normally applied to. If, after applying the algorithm, the theory appears containing the first-class constraints, the solutions are not unique for the Cauchy problem and the action (4) possesses gauge symmetries.

Optimal control theory deals with another particular case of the above equations, which is opposite, in a sense, to the variational case: only the differential equations (1) are considered, with no constraints (2) imposed on \( x \)'s. In optimal control [3], [4], the \( \lambda \)'s are called the control functions. If only equations (1) are considered, the Cauchy problem for \( x \)'s will have a unique solution corresponding to every choice of the functions \( \lambda^\alpha(t) \). The control functions remain unrestricted anyhow by equations (1), then the solutions remain ambiguous for \( x^i \) unless all \( \lambda \)'s are determined by some extra requirement. The distinctions between Dirac’s constrained dynamics and optimal control theory go far beyond imposing (or not imposing) constraints and supposing (or not supposing) the existence of a symplectic structure: they profess quite opposite concepts of the ambiguities in solutions of equations (1). Optimal control theory considers the functions \( \lambda \) as
describing a background for the dynamics, making it, in fact, non-autonomous. The objective is to minimize certain function(al)s of the solutions \(x^i(t)\) (the cost functions) by varying the control functions \(\lambda(t)\). Quite opposite, the Dirac constrained dynamics consider all the solutions for \(x(t)\) and \(\lambda(t)\) equivalent to each other whenever they correspond to the same Cauchy data. In particular, a function(al) is considered as physically observable, if it evolves in the same way on every trajectory from the equivalence class, i.e., it is out of control from the viewpoint of optimal control theory. The gauge theory treats \(\lambda\) as dynamical variables, not the functions controlled from outside. From this viewpoint, the controllable values are supposed to be unobservable because the autonomous physical models must be self-contained, leaving no room for intervention of any ultramundane force that can control/optimize what we observe. With this regard, the ambiguity brought to the solutions by the undetermined multipliers \(\lambda(t)\) is to be factored out from the dynamics, first classically and then quantum-mechanically. A basic tool for such a factorization is a gauge symmetry relating equivalent trajectories. In particular, the physical observables are identified with the gauge invariant function(al)s.

By a gauge symmetry of equations (1), (2) we understand the infinitesimal transformations of the form

\[
\delta_\varepsilon x^i = \sum_{n=0}^{p} R_{(p-n)}^{i (n)} \varepsilon^n , \quad \delta_\varepsilon \lambda^\alpha = \sum_{n=0}^{p+1} U_{(p+1-n)}^{\alpha (n)} \varepsilon^n .
\]

The transformation parameters \(\varepsilon\) are arbitrary functions of time, and \((n)\varepsilon\) stands for the \(n\)-th order time derivative of \(\varepsilon\). The structure functions \(R_{(n)}^{i}\) and \(U_{(n)}^{\alpha}\) are allowed to depend on a finite number of variables \(x^i, \lambda^\alpha, \dot{\lambda}^\alpha, \ddot{\lambda}^\alpha, \ldots\). The transformation (5) is supposed to leave equations (1), (2) invariant in the sense that the variations

\[
\delta_\varepsilon \left( \dot{x}^i - V^i(x) - Z^i_\alpha(x)\dot{\lambda}^\alpha \right), \quad \delta_\varepsilon T_\alpha(x)
\]

must be proportional to the equations of motion (1), (2) for arbitrary \(\varepsilon(t)\). In what follows we use the ordinary physical terminology and refer to the values that are proportional to the equations of motion as vanishing on-shell.

Consider, for example, unconstrained equations (1) and suppose that the vector fields \(Z_\alpha\) span an integrable distribution of rank \(m\),

\[
[Z_\alpha, Z_\beta] = U_{\alpha\beta}^{\gamma}(x)Z_\gamma ,
\]
which is preserved by \( V \),

\[
[Z_\alpha, V] = V_\alpha^\beta(x)Z_\beta. 
\]

Then there are exactly \( m \) linearly independent gauge transformations of the form (5). These read

\[
\delta_\varepsilon x^i = Z_\alpha^i \varepsilon^\alpha, \quad \delta_\varepsilon \lambda^\alpha = \dot{\varepsilon}^\alpha - \left( V_\alpha^\beta + U_\beta^{\alpha \gamma} \lambda^\gamma \right) \varepsilon^\beta. 
\]

Notice that equations (1) are fully consistent, no matter what are the vector fields \( V \) and \( Z_\alpha \) involved in the right hand side. The involution relations (7), (8) are not the necessary conditions for consistency of the equations. If the distribution \( Z = \text{span}\{Z_\alpha\} \) is not integrable and/or not preserved by \( V \), the explicit form of the gauge transformation (3) has remained unknown yet, and we are going to present it in Section 4.

Given the distinctions in the main concepts/objectives of the gauge theory and optimal control, it is no surprise that the gauge symmetry remained unstudied for the general dynamical equations (1), (2) in the context of optimal control. Even though a lot of interesting non-variational gauge models arise in the context of modern higher energy physics, e.g. self-dual Yang-Mills fields, higher-spin gauge theories, Sieberg-Witten and Donaldson-Uhlenbeck-Yau equations, etc, the issue of extending the Dirac-Bergmann method to the general local dynamics has been shelved yet, because no ways were seen to quantize the non-variational dynamics. Recently, we have proposed the BRST quantization methods applicable to arbitrary non-variational dynamics [5] (see also [6] for another version of this method). Our method implies the characteristic distribution to be on-shell involutive and tangent to the constraint surface. As it has been already explained, this is not the most general case. In Section 5, we explain that any local gauge dynamics can be eventually reformulated in the involutive form, which is physically equivalent to the original one. The involutive normal form of dynamics, resulting from the extension of the Dirac-Bergman algorithm to non-variational dynamics, thus becomes a starting point for applying the deformation quantization techniques for the general dynamical systems.

### 2. Regularity conditions and the primary normal form of local dynamics

Let us start with explanation how to bring a local dynamical system to the normal form (1), (2). Any finite system of ordinary differential equations can always be depressed to the first order by introducing new variables. By further adding new variables, if necessary, the first-order ODEs
can be equivalently rewritten in the form of inhomogeneous Pfaffian equations

\[ \theta_{J_i}(x) \dot{x}^i = V_J(x), \quad J = 1, \ldots, N, \quad i = 1, \ldots, n, \]

subject to constraints

\[ T_a(x) = 0, \quad a = 1, \ldots, m. \]

For example, given second-order equations

\[ f(y, \dot{y}, \ddot{y}) = 0, \]

by adding new variables \( v \) and \( w \), these can be equivalently rewritten in the form (10), (11) as

\[ \dot{v} = w, \quad \dot{y} = v, \quad f(y, v, w) = 0. \]

Sometimes, to take into account the geometric properties of the original equations (12), it might be more appropriate to introduce the new variables \( v \), absorbing the derivatives of \( x \), as linear inhomogeneous functions of \( \dot{x} \), \( v = e(x) \dot{x} + \Gamma(x) \). This will result, however, in the equations of the form (10), (11), anyway. We call the primary constraints or just constraints both equations (11) and the functions \( T_a(x) \).

To further proceed with the general equations of motion in the constrained Pfaffian form (10), (11), we need to impose appropriate regularity conditions. The phase space of the dynamical system is supposed to be an open domain \( U \subset \mathbb{R}^n \) with linear coordinates \( \{x^i\} \). All the functions of \( x^i \) entering equations (10), (11) are supposed to be analytic.

Let us introduce some convenient notions for describing the regularity of the equations. Let \( M \) be a matrix whose elements are analytic functions on \( \mathbb{R}^n \). Given a subset \( U \subset \mathbb{R}^n \), one can define a sequence of embedded subspaces

\[ U = U^0 \supset U^1_M \supset U^2_M \supset \cdots \supset U^n_M = U'_M, \]

where

\[ U^r_M = \{ x \in U | \text{rank}M(x) \geq r \}. \]

The conditions we adopt here are not the weakest possible ones. For example, one can extend the consideration from \( \mathbb{R}^n \) to an analytic or smooth manifold \( M \), with constraints \( \{T_a(x)\} \) being a section of a vector bundle over \( M \). In this paper, however, we are going to avoid the technicalities related to less restrictive regularity conditions, for the sake of a more clear presentation of the algorithm as such.
is the set of all points in which the rank of the matrix $M$ is not less than $r$, and $m$ is the maximal value of $\text{rank}M$ on $U$. Then $U^r_M$ is an open everywhere dense subset of $U$. A minimal subset $U'_M \subset U$ will be called a \textit{a regular part of $U$ regarding $M$}. It will be denoted just by $U'$ whenever the corresponding matrix-valued function $M$ is obvious from the context. The complementary subset to the regular part, $U \setminus U'$, will be called an \textit{abnormal part} of $U$.

Denote by $\Sigma$ the common zero locus of the constraints (11), i.e.,

$$\Sigma = \{x \in U \mid T_a(x) = 0, a = 1, ..., m\}. \tag{16}$$

The primary constraints are assumed to be consistent as algebraic equations per se. If the constraints contradict each other, the entire system of the dynamical equations (1), (2) does not have any solution, and this can hardly be considered as regular dynamics. The same consistency of the constraints by themselves will always be assumed for the secondary constraints derived in the next section.

Let $U'$ be the regular part of $U$ regarding the Jacobi matrix of the constraints $J = (\partial_i T_a)$. The constraints $\{T_a\}$ are called \textit{regular} if

$$\Sigma' = U' \cap \Sigma \neq \emptyset. \tag{17}$$

In this case, $\Sigma'$ is a smooth submanifold in $U'$ with $\dim \Sigma' = n - \text{rank}J|_{U'}$.

Denote by $\Sigma''$ the regular part of $\Sigma'$ regarding the matrix $\Theta = (\theta_{ji})$ of the Pfaffian forms in (10). In accordance with the definitions above, $\Sigma'' \subset \tilde{U} = (U')^r$, where $r = \text{rank}\Theta(p)$ for $p \in \Sigma''$. Equations (10) are linear in velocities $\dot{x}^i$, with the coefficient matrix $(\theta_{ji})$ having rank greater than or equal to $r$ on $\tilde{U}$. Restricting the dynamics to $\tilde{U}$, we can solve (10) with respect to the velocities as

$$\dot{x}^i = V^i(x) + \lambda^\alpha Z^i_\alpha(x), \quad \alpha = 1, ..., l, \quad l = n - r. \tag{18}$$

Here $\lambda^\alpha(t)$ are arbitrary functions of time, the vector field $V$ is any solution to the inhomogeneous linear equations

$$(\theta_{ji} V^i - V_j)|_{\Sigma''} = 0, \tag{19}$$

and the vector fields $\{Z_\alpha\}$ span the space of solutions to the corresponding linear system

$$\theta_{ji} Z^i|_{\Sigma''} = 0. \tag{20}$$
Upon restriction to the domain $\tilde{U}$, the system of equations (18) and (11) is completely equivalent to the original equations (10) and (11). We refer to equations (18), (11) as a primary normal form of local dynamics. It is the form (1), (2) announced in Introduction. The vector field $V$ will be called a primary drift, and the vector distribution $Z = \text{span}\{Z_\alpha\}$ will be called a primary characteristic distribution. Notice that the distribution $Z$ is not required to have a constant rank on $\tilde{U}$ nor is it supposed to be involutive.

The primary normal form of equations (1), (2) implies certain equivalence relations among their ingredients. Besides the non-degenerate changes of the decision variables

$$x^i \mapsto x'^i = f^i(x), \quad \lambda^\alpha \mapsto \lambda'^\alpha = F^\alpha_\beta(x)\lambda^\beta + G^\alpha(x),$$

the system is invariant with respect to the following redefinitions:

$$T_a \mapsto T'_a = S^b_a T_b, \quad V \mapsto V' = V + T_a W^a, \quad Z_\alpha \mapsto Z'_\alpha = Z_\alpha + T_a X^a_\alpha,$$

where $(S^b_a(x))$ is an arbitrary non-degenerate matrix on $\tilde{U}$, and $W^a$ and $X^a_\alpha$ are arbitrary vector fields. So, neither the primary drift $V$, nor the characteristic distribution $Z$ is defined uniquely by the inhomogeneous Pfaffian system (10) subject to the constraints (11). We say that two dynamical systems on $\tilde{U}$ are equivalent if their primary normal forms (1), (2) are related by transformations (21), (22). In what follows, we will deal with those dynamical systems which admit at least one analytic representative in each equivalence class, and only the analytic representations will be considered.

A function (or vector field) $F$ is called trivial if it vanishes on $\Sigma'$. With account of regularity we have

$$F|_{\Sigma'} = 0 \iff F = T_a F^a$$

for some smooth functions (or vector fields) $F^a$. From this viewpoint, the automorphisms (21), (22) imply that two characteristic distributions are equivalent whenever their difference is trivial, and two drifts are equivalent whenever they coincide modulo the characteristic distribution and a trivial vector field.

As it has been already mentioned in Introduction, the general equations of motion, being brought to the primary normal form (1), (2), can have nontrivial compatibility conditions resulting in additional constraints on $x$’s and $\lambda$’s. The secondary constraints on $x$’s can be derived directly in the inhomogeneous Pfaffian form (10), (11), as it was shown by X. Gracia and J.M. Pons [9]. In
the next section, we consider the procedure of deriving the secondary constraints and fixing
the undetermined multipliers $\lambda$ by making use of the primary normal form of the equations of motion.
This procedure allows us to extend the Dirac classification of constraints to the general equations
(1), (2), without appealing to any Poisson structure.

3. Consistency conditions and secondary constraints

The consistency conditions of the primary normal equations originate from the requirement
that the integral trajectories of the flow (1), with fixed functions $\lambda(t)$, should be confined at the
regular part of the constraint surface (17). For the regular constraints this means that the time
derivatives of all the constraints must vanish on the integral trajectories whenever they intersect $\Sigma'$:

$$\dot{T}_a |_{\text{on-shell}} = (VT_a + \lambda^\alpha Z_\alpha T_a) |_{\Sigma'} = 0. $$

The expression in brackets being a trivial function in the sense of (23), we arrive at the following
linear inhomogeneous equations for $\lambda$:

$$VT_a + \lambda^\alpha Z_\alpha T_a = F_b{}^a T_b. $$

Let $\Sigma''$ be the regular part of $\Sigma'$ regarding the matrix $M = (Z_a T_b)$, and rank$M(p) = s$ for $p \in \Sigma''$. Then, according to the definition (15), $\Sigma'' \subset (U')^s_M$.

To further proceed with solving equations (25), we decompose the constraints, characteristic
distribution and undetermined multipliers in the following way:

$$T_a = (T_A, T_{\bar{a}_1}), \quad Z_a = (Z_A, Z_{\bar{a}_1}), \quad \lambda^\alpha = (\lambda^A, \lambda^{\alpha_1}),$$

where $A = 1, ..., s$ and $D = (Z_A T_B)$ is a maximal non-degenerate minor of $M$. Denote $U$ the
regular part of $(U')^s_M$ regarding $D$. Using equations (25) with $a = A$, we can express $\lambda^A$ as linear
inhomogeneous functions of $\lambda^{\alpha_1}$:

$$\lambda^A = -D^{AB} (VT_B + \lambda^{\alpha_1} Z_{\alpha_1} T_B),$$

with the matrix $(D^{AB})$ being inverse to $D$. After substitution of the determined multipliers (27)
into the remaining equations (25), we get

$$T_{\bar{a}_1}^1 + \lambda^{\alpha_1} B_{\alpha_1 \bar{a}_1} = 0.$$
where

\[(29) \quad T^1_{\alpha_1} = V T_{\alpha_1} - D^{AB}(V T_B) Z_A T_{\alpha_1}, \quad B_{\alpha_1\bar{\alpha}_1} = Z_{\alpha_1} T_{\bar{\alpha}_1} - D^{AB}(Z_{\alpha_1} T_B) Z_A T_{\bar{\alpha}_1}. \]

Obviously, the matrix \( B = (B_{\alpha_1\bar{\alpha}_1}) \) must vanish on \( \Sigma' \), since otherwise one could determine more multipliers \( \lambda \), that would contradict maximality of the non-degenerate minor \( D \). Being a trivial analytic function in \( (1)U \), the matrix \( B \) has the form \( B_{\alpha_1\bar{\alpha}_1} = W_{\alpha_1\bar{\alpha}_1} T_c \). Equations \( (28) \) should be satisfied together with the equations of primary constraints, hence the trivial terms containing \( \lambda \)'s can be omitted and we obtain the new constraints

\[(30) \quad T^1_{\alpha_1}(x) = 0. \]

So, we see that the conservation laws \( (24) \) for the primary constraints are equivalent to equations \( (27) \) determining a part of \( \lambda \)'s as specific functions of \( x \)'s, and the \( \lambda \)-independent relations \( (30) \).

Equations \( (30) \) as well as the functions \( T^1_{\alpha_1} \) themselves are called the secondary constraints of the first stage. Notice that the secondary constraints are defined modulo trivial contributions proportional to the primary constraints. Also notice that the functions \( T^1_{\alpha_1}(x) \) are not necessarily independent.

Substituting the fixed multipliers \( (27) \) into the original equations of motion \( (1), (2) \) and adding the secondary constraints, we arrive at the following set of equations:

\[(31) \quad \dot{x}^i = V^i(x) + \lambda^{\alpha_1} Z^i_{\alpha_1}(x), \quad T^1_{\alpha_1}(x) = 0, \]

where the values

\[(32) \quad V = V - D^{AB}(V T_B) Z_A, \quad Z_{\alpha_1} = Z_{\alpha_1} - D^{AB}(Z_{\alpha_1} T_B) Z_A, \quad T^1_{\alpha_1} = (T_{\alpha_1}, T^1_{\bar{\alpha}_1}). \]

are called, correspondingly, the drift, characteristic distribution, and constraints of the first stage. By construction, all these objects are well defined on the open everywhere dense domain \( (1)U \subset U \).

Let us write down the following obvious relations:

\[(33) \quad V T_A = 0, \quad V T_{\alpha_1} = T^1_{\alpha_1}, \quad Z_{\alpha_1} T_A = 0, \quad Z_{\alpha_1} T_{\alpha_1} = W_{\alpha_1\bar{\alpha}_1} T_c. \]

Along with the condition of invertibility of the matrix \( D = (Z_A T_B) \) these relations give a tip for a simple geometric interpretation of the transition from the original equations \( (1), (2) \) to the equivalent first-stage equations \( (31) \). The original surface of primary constraints \( (16) \) can be
represented (at least locally) as a transverse intersection \( \Sigma = \Sigma^\parallel \cap \Sigma^\perp \) of two surfaces \( \Sigma^\parallel \) and \( \Sigma^\perp \). These are given by

\[
\Sigma^\parallel = \{ x \in U^\parallel \mid T_a^1(x) = 0 \}, \quad \Sigma^\perp = \{ x \in U^\parallel \mid T_A(x) = 0 \}.
\]

In its turn, the original characteristic distribution \( \mathcal{Z} = \text{span}\{Z_a\} \) is decomposed into the sum \( \mathcal{Z} = \mathcal{Z}^\parallel \oplus \mathcal{Z}^\perp \) of two sub-distributions

\[
\mathcal{Z}^\parallel = \text{span}\{(1)Z_{\alpha 1}\}, \quad \mathcal{Z}^\perp = \text{span}\{Z_A\},
\]

which are called, correspondingly, the tangential and transverse. The multipliers \( \lambda^A \), being related to the transverse sub-distribution, are fixed by the conservation law of the primary constraints, whereas the multipliers \( \lambda^{\alpha_1} \) corresponding to \( \mathcal{Z}^\parallel \) still remain undetermined.

Geometrically, relations \((33)\) mean that the distribution \( \mathcal{Z}^\parallel \) is tangent to the regular part of the primary constraint surface \( \Sigma \), whereas the complementary distribution \( \mathcal{Z}^\perp \) is transverse. The tangential distribution \( \mathcal{Z}^\parallel \subset \mathcal{Z} \) is invariantly defined by the property \( \mathcal{Z}^\parallel |_{\Sigma} \subset T\Sigma \). The dimension of the transverse distribution, being complementary to the dimension of \( \mathcal{Z}^\parallel \), coincides with the number of the multipliers \((27)\) determined from the conservation requirement for the primary constraints. The primary characteristic distribution \( \mathcal{Z} \) is tangent to the surface \( \Sigma^\parallel \), being zero locus of the primary constraints not involved in determining of the multipliers. Also notice that the first-stage drift \((1)V^1\) is tangent to the primary constraint surface \( \Sigma \) on the zero locus of the secondary constraints.

Not only do the first-stage equations \((31)\) describe the same dynamics as \((1), (2)\) (the zero-stage equations), but they appear identical to equations \((1), (2)\) in form. Therefore, we can apply to \((31)\) all the above reasonings concerning the preservation of the constraints in time. Supposing the first-stage constraints to be regular, we check whether the conservation condition

\[
\frac{d}{dt}(1)T_{a_1}\big|_{\text{on-shell}} = 0
\]

implies new secondary constraints, or it further restricts the undetermined multipliers \( \lambda \). The new constraints, if any, are called the secondary constraints of the second stage and denoted by \( T_{a_2}^2 \).

We suppose that the complete set of the second-stage constraints \( T_{a_2}^2 = (T_a, T_{a_1}^1, T_{a_2}^2) \) is regular. After exclusion of the determined multipliers from the differential equations, we get the drift and the characteristic distribution of the second stage. If there are nontrivial secondary constraints among the constraints of the second stage they must conserve. So, we further proceed with
deriving consequences from the conservation of the second-stage constraints. The algorithm will continue working, defining some multipliers and redistributing the corresponding vector fields from the tangential distribution of the previous stage to the transverse distribution of the next stage, redefining the drift, and bringing the constraints of the next stage. Since the overall number of the independent regular constraints at any stage cannot exceed the dimension of the phase space, the algorithm has to stabilize after a finite number of steps. The stabilization is achieved as soon as new secondary constraints stop to appear from the consistency conditions. If the iterative procedure terminates at the \( k \)-th stage, the equations of motion take the following form:

\[
\dot{x}^i = V^i(x) + \lambda^{ak} Z_{\alpha_k}^i(x), \quad T_{ak}^{(k)} = 0.
\]

These equations are defined in an open everywhere dense domain \( U \subset U \), and the complete set of constraints

\[
T_{ak}^{(k)} = (T_a, T_{a_1}^1, \ldots, T_{a_k}^k)
\]

includes the primary and secondary constraints of all stages.

By construction, the following relations take place:

\[
VT_{ak}^{(k)} = F_{ak}^{bk} T_{bk}^{(k)}, \quad Z_{\alpha_k}^{(k)} T_{ak}^{(k)} = F_{\alpha_k ak}^{bk} T_{bk}^{(k)}
\]

for some functions \( F_{ak}^{bk} \) and \( F_{\alpha_k ak}^{bk} \). We call (37) a complete normal form of a local dynamical system. It is the form that ensures full consistency of the dynamics as it does not have any further compatibility conditions and consequences.

Several remarks can be made about the algorithm above:

- The \((n+1)\)-st step of the algorithm becomes possible whenever the constraints of \( n \)-th stage are regular. Actually, it is the constraint surface that has an invariant geometric meaning, not the constraints as functions. The same constraint surface can be defined by different \( T \)'s, and all such constraints are considered to be equivalent. If the equivalence class includes a regular representative, this representative is picked up as the set of constraints of the \( n \)-th stage. As is seen from the iterative procedure, the algorithm is not sensitive to the specific choice of regular representative.

- The \((n+1)\)-st iteration involves the ambiguity concerning the choice of a maximal non-degenerate minor \( D_n \) of the matrix \( M_n = (Z_{a_n}^{(n)} T_{a_n}^{(n+1)}) \). The different choices can result in different domains of definition \( U \) for the system of the \((n+1)\)-st stage. Notice that
the different domains $U^{(n+1)}$’s, being open everywhere dense subsets in $U$, should coincide modulo a set of measure zero. In the intersection of these almost coinciding domains of definition, the equations of motion are not sensitive to a specific choice of a maximal minor.

- When the compatibility conditions (25) are considered as a system of linear equations for $\lambda$’s, the surface $\Sigma''$ was defined as a subset of $\Sigma'$, where $\text{rank } M = s$ is maximal. This requirement of maximality can be relaxed, in principle. One can consider a subset $S \subset \Sigma'$, where $\text{rank } M = s'$ is less than $s$. According to the terminology of Section 2, $S$ belongs to the abnormal part of the constraint surface $\Sigma'$ regarding $M$. In some cases, the abnormal set $S \subset U'$ can be a smooth submanifold defined by a set of regular constraints $T^S = 0$. At every given stage, the dimension of the abnormal part of the constraint surface is less than the dimension of the regular part, so that $\dim S < \dim \Sigma$. Being restricted to $S$, the dynamics would be regular in the same sense as with the restriction to the regular part of the constraint surface. With the restriction of the dynamics to the abnormal set $S$, a lesser number $s'$ of the multipliers $\lambda$ are determined, but the greater number of the constraints appear at this stage. Thus, at any stage of the iterative procedure, there may be an ambiguity in deciding between the regular or abnormal parts of the constraint surface. The dynamics can be regular and consistent for either of these options. It is amply clear that the dimension of the complete constraint surface (38) and the number of undetermined multipliers in (37) can depend on the choice made at certain stage (the algorithm “bifurcates”). So, after applying the algorithm, which works equally well with regular and abnormal parts of the constraint surface, one can learn that the original dynamical system (1), (2) has contained in itself several different dynamics (37), with different constraints, drifts and numbers of undetermined multipliers. These dynamics never communicate and should be interpreted as different physical systems, even though the difference reveals itself only after applying the algorithm, being not explicitly visible in the primary normal form of the dynamics.

Let us comment on the geometry of the dynamical equations in the complete normal form (37). The primary characteristic distribution $Z = \text{span}\{Z_a\}$ is eventually decomposed into the direct sum

$$Z = Z_\perp \oplus Z_\parallel$$

(40)
of the transverse and tangential distributions with respect to the complete constraint surface

\[ \Sigma = \{ x \in U \mid T^{(k)}_{a_k} = 0 \} . \]

As is seen from (39), the tangential distribution \( Z_\parallel = \text{span}\{ Z^{(k)}_a \} \) preserves the complete constraints surface (41), and the multipliers \( \lambda^{\alpha_k} \) related to the basis vector fields of \( Z_\parallel \) remain arbitrary functions of time in equations (37). The distribution \( Z_\perp \simeq Z/Z_\parallel \), being a complement to \( Z_\parallel \), is transverse to the complete constraint surface \( \bar{\Sigma} \): If \( \{ Z_A \} \) is a basis in \( Z_\perp \), then

\[ \text{rank}(Z_A^{(k)} T^{(k)}_{a_k}) = \dim Z_\perp . \]

All the multipliers \( \lambda^A \) corresponding to \( Z_\perp \) are defined by the conservation conditions of the primary or secondary constraints, at one or another stage of the algorithm.

By making use of the equivalence transformations (22), the complete set of constraints can be rearranged into the union of two subsets

\[ T^{(k)}_{a_k} = (T^\parallel_a , T^\perp_A) \]

such that

\[ Z T^\parallel_a \mid_\Sigma = 0 \quad \forall Z \in Z \quad \text{and} \quad \det D \mid_\Sigma \neq 0 , \]

where \( D = (Z_A T^\perp_B) \). Denoting by \( \bar{\Sigma}_\parallel \) and \( \bar{\Sigma}_\perp \) the zero loci of the constraints \( T^\parallel \) and \( T^\perp \), respectively, we see that the complete constraint surface can be represented as the intersection of two surfaces:

\[ \bar{\Sigma} = \bar{\Sigma}_\parallel \cap \bar{\Sigma}_\perp . \]

The primary characteristic distribution \( Z \) is tangent to \( \bar{\Sigma}_\parallel \) on \( \bar{\Sigma} \) and transverse to \( \bar{\Sigma}_\perp \). Notice that \( \text{codim}\bar{\Sigma}_\perp = \dim Z_\perp \), while the dimension of the tangential distribution \( Z_\parallel \) does not correlate with the dimension of \( \bar{\Sigma}_\parallel \) in general.

The complete drift \( \bar{V} = \bar{V}^{(k)} \) defined by (37) is constructed step by step from the primary drift \( V \) by adding terms proportional to the elements of the transverse distribution. When the algorithm has terminated at the \( k \)-th stage, the drift reads

\[ \bar{V} = V - (VT^\perp_A)(D^{-1})^{AB} Z_B . \]

Notice that \( \bar{V} \) belongs to the equivalence class of the primary drift \( V \) in the sense of the equivalence relations (22).
Let us elaborate on the correspondence between the basic notions of the constrained Hamiltonian mechanics and their counterparts in the general local dynamics. The equations (37) can be viewed as an equivalent of the Dirac constrained dynamics with the complete set of primary and secondary constraints (38). The “tangential” and “transverse” constraints $T\parallel$ and $T\perp$ correspond, respectively, to the *complete* sets of the first- and second-class constraints in Dirac’s classification of the Hamiltonian constraints. The tangential and transverse primary characteristic distributions $\mathcal{Z}\parallel$ and $\mathcal{Z}\perp$ correspond to the distributions generated by the Hamiltonian vector fields of the primary first- and second-class constraints, respectively. One may also wonder about an analog for the Hamiltonian vector fields associated with the first-class constraints. These are known to generate the whole set of gauge symmetries provided that the Hamiltonian system obeys the Dirac conjecture. From the next section we will learn that the vector fields in question form the distribution

$$Z_V = Z\parallel \cup [Z\parallel, Z\parallel] \cup [Z\parallel, \bar{V}] \cup \cdots ,$$

where the dots stand for the higher iterated commutators of $Z\parallel$ and $\bar{V}$. Whether the Dirac conjecture is true or not, it is the distribution $Z_V$ that generates the gauge symmetry transformations (5) of the dynamics.

4. **Gauge symmetries**

In the previous section, we have seen that the algorithm of stabilization of the primary constraints brings the original equations of motion to the complete normal form (37), (39). In this section, we find all the gauge symmetry transformations for these equations.

To ease the notation, we omit all the sub- and superscripts referring to the final stage of the iterative procedure from the previous section. After this omission, equations (37) read

$$\dot{x}^i = V^i(x) + \lambda^\alpha Z^i_\alpha(x) , \quad T_a(x) = 0 ,$$

where the distribution $\mathcal{E} = \text{span}\{V, Z_\alpha\}$ is assumed to be tangent to the complete constraint surface $\bar{\Sigma} = \{x \in U \mid T_a(x) = 0\}$.

The infinitesimal gauge symmetry transformations (5) are sought for equations (48) in the form:

$$\delta_\varepsilon x^i = \sum_{n=0}^{p} R^{i}_{p-n} (\varepsilon) , \quad \delta_\varepsilon \lambda^\alpha = \sum_{n=0}^{p+1} U^\alpha_{p+1-n} (\varepsilon) ,$$
where \( R \)'s and \( U \)'s are some functions of \( x, \dot{\lambda}, \ddot{\lambda}, \ldots \) up to some finite order \( ^{(k)} \), and \( \varepsilon(t) \) are the transformation parameters, being arbitrary functions of \( t \). Notice that the integer \( p \), called the order of gauge transformation, is not fixed a priori. Moreover, it can always be risen by the trivial replacement \( \varepsilon \mapsto \dot{\varepsilon} \) of the gauge transformation parameter. Conversely, if \( R_p = 0 \) and \( U_{p+1} = 0 \), then the reverse change of variables \( \dot{\varepsilon} \mapsto \varepsilon \) depresses the highest order of the derivative of \( \varepsilon \) in (49). When the functions \( R_p \) and \( U_{p+1} \) do not vanish simultaneously, the highest order of the derivative cannot be depressed and we refer to corresponding transformation as undepressible.

Let us first consider the issue of gauge symmetry for the dynamical system (48) without constraints \( T_a \). The gauge invariance of the dynamics with respect to the infinitesimal transformations (49) means that the gauge variation of (48) should vanish on their solutions, i.e.,

\[
\delta_\varepsilon (\dot{x}^i - V^i - \lambda^\alpha Z_\alpha^i) |_{\text{on-shell}} = 0.
\]

Excluding the velocity \( \dot{x}^i \) in (50) with the help of (48), we arrive at the following recurrent relations for the structure functions \( R \) and \( U \) of the gauge transformation (49):

\[
R_0^i = W_0^i,
\]

\[
R_{n+1}^i = D R_n^i + W_{n+1}^i, \quad n = 0, ..., p - 1,
\]

\[
R_{p+1}^i = D R_p^i + W_{p+1}^i = 0.
\]

Here we have introduced the following abbreviations:

\[
W_n^i = U_n^\alpha Z_\alpha^i, \quad D = -\partial - [V + \lambda^\alpha Z_\alpha, \cdot],
\]

and \( \partial \) is understood as the time derivative acting only on \( \lambda \)'s, i.e.,

\[
\partial = \dot{\lambda}^\alpha \frac{\partial}{\partial \lambda^\alpha} + \ddot{\lambda}^\alpha \frac{\partial}{\partial \lambda^\alpha} + \cdots.
\]

Equation (51) implies that the coefficient at the highest derivative of \( \varepsilon \) in \( \delta_\varepsilon x^i \) is given by a linear combination of vector fields from the characteristic distribution \( Z \). Relations (52) enable one to
express all the structure functions $R$ in terms of the functions $U$ defining the gauge transformations of the undetermined multipliers $\lambda$. Namely, by solving relations (52), we find

\[(56)\]

\[R^i_n = \sum_{m=0}^{n} D^m W^i_{n-m}.\]

In particular, equation (53) takes the form

\[(57)\]

\[R^i_{p+1} \equiv \sum_{m=0}^{p+1} D^m W^i_{p+1-m} = 0.\]

The last equation on $U$’s is the only nontrivial condition to satisfy. Its solutions form a linear space, which dimension is to be computed, depending on $p$. To this end, certain regularity conditions must be imposed on the equations of motion and their domain of definition. Also, we will use some properties of and notions about the distributions, which are briefly listed below.

So far, the components of the vector fields $V, Z_\alpha$ were supposed to be analytic functions defined in some domain of linear space. In the following, to avoid specifying each time the definitional domain for the gauge transformations, we relax the condition of analiticity and utilize the algebraic setting usually adopted in control theory [3] for the dynamical equations (1). Namely, $V$ and $Z_\alpha$ are allowed to be meromorphic vector fields. As the scar set of a meromorphic function is of measure zero, our gauge transformations will be well defined in an open everywhere dense domain $\tilde{U}$ of the original phase space. It is $\tilde{U}$ that is to be considered as the definitional domain of the gauge dynamics.

Let $F$ be the field of meromorphic functions and $W$ be the space of meromorphic vector fields on $U \subset \mathbb{R}^n$. The space $W$ has both the structure of a real, infinite-dimensional Lie algebra with respect to the commutator of vector fields and the structure of $n$-dimensional vector space over $F$. We will refer to $F$-linear subspaces of $W$ as distributions. Generally, an arbitrary distribution $P \subset W$ is not closed with respect to the commutator, i.e., the commutator of two vector fields from $P$ may not belong to $P$. If $[P, P] \subset P$, then the distribution $P$ is said to be involutive. The Lie closure of $P$ is defined as the minimal involutive distribution $\bar{P} \subset W$ that contains $P$. Uniqueness of $\bar{P}$ follows immediately from the fact that the set of the involutive distributions is closed with respect to intersecting of linear spaces. Furthermore, one can inductively see that the distribution $\bar{P}$ is generated by the vector fields

\[(58)\]

\[\left[\cdots \left[u_1, u_2]\right], u_3, \cdots, u_k\right] \quad \forall u_k \in P, \quad \forall k \in \mathbb{N}.\]
For an involutive distribution $P$, $\bar{P} = P$.

By taking the multiple commutators (58), one can filter the Lie closure $P$ by the sequence of sub-distributions

\[(59)\quad \bar{P}_l = \text{span}_F \{ \ldots [u_1, u_2], u_3, \ldots], u_k] \mid \forall u_i \in P, \ k = 1, \ldots, l \}\]
such that

\[(60)\quad \bar{P} = \bar{P}_m \supset \bar{P}_{m-1} \supset \bar{P}_{m-2} \supset \cdots \supset \bar{P}_1 = P.\]

This filtration is known as the Lie flag of the distribution $P$. Clearly, the minimal integer $m \leq n - \dim P$ involved in (60) is an invariant of the distribution $P$ together with the numbers $d_k = \dim \bar{P}_k$. The integer $m$ is usually referred to as the depth of the distribution $P$ and the sequence of integers $(d_1, d_2, \ldots, d_m)$ is called the growth vector of $P$.

To check the solvability of equations (57) we need some extension of the field $F$. Along with the coordinates $x^i$ in $\mathbb{R}^n$ we introduce the infinite set of variables $\{\lambda^\alpha_k\}, \alpha = 1, \ldots, m, k \in \mathbb{N}$. Denote by $\mathcal{F}$ the field of meromorphic functions of $x^i$ and a finite number of the variables $\lambda^\alpha_k$. Replacing in the definition of $W$ the field $F$ by its extension $\mathcal{F}$, we get the $n$-dimensional vector space

\[(61)\quad \mathcal{W} = \text{span}_\mathcal{F} \left( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right).\]

The $\mathcal{F}$-linear subspaces of the vector space $\mathcal{W}$ will be called $\lambda$-distributions.

Let us turn $\mathcal{F}$ into a differential field by setting

$$\partial x^i = 0, \quad \partial \lambda^\alpha_k = \lambda^\alpha_{k+1}. $$

Clearly, this definition just mimics the definition of the time derivative (55) if one set $\lambda^\alpha_0 = \lambda^\alpha$.

The action of the operator $\partial$ can be further extended to the $\lambda$-distributions by the rule

$$\partial(aw) = (\partial a)w + a\partial w \quad \forall a \in \mathcal{F}, \ \forall w \in \mathcal{W},$$

$$\partial \left( \frac{\partial}{\partial x^i} \right) = 0.$$ 

With all the definitions above, we can start studying the gauge symmetries (49) of the system (48). Every dynamical system (48) defines and is defined by the distribution $E$ generated by the vector fields $V$ and $Z_\alpha$. Associated to $E$ is a gauge distribution $Z_V$. The latter is defined as the limit of a filtration

\[(62)\quad Z^0_V \subset Z^1_V \subset \cdots \subset Z^\infty_V = Z_V \subset \mathcal{W} \]
of involutive $\lambda$-distributions $Z^k_V$ given by

$$Z^k_V = \bigcup_{m=0}^{k} [V, ... , [V, Z]].$$

It is clear that $Z^0_V = \tilde{Z}$ and $\tilde{E} = Z_V \cup V$.

Define an $\mathbb{R}$-linear operator $D : \mathcal{W} \rightarrow \mathcal{W}$ by

$$Dw = -\partial w - [V, w] - \lambda^\alpha [Z_\alpha, w] \quad \forall w \in \mathcal{W}.$$ 

Although the operator $D$ is not $\mathcal{F}$-linear, it satisfies the following analog of the Leibnitz rule:

$$D(aw) = (Da)w + aDw \quad \forall a \in \mathcal{F}, \quad \forall w \in \mathcal{W},$$

where

$$Da = -\partial a - Va - \lambda^\alpha Z_\alpha a.$$ 

Since $D$ is a differentiation of the field $\mathcal{F}$, one can thought of $\mathcal{W}$ as a differential $\mathcal{F}$-module.

Given an element $w \in Z$, consider the sequence of elements $D^k w \in Z_V, \ k \in \mathbb{N}$. As $Z_V$ is of finite dimension, there is $p \in \mathbb{N}$ and a set of functions $a_1, \cdots, a_p \in \mathcal{F}$ such that

$$D^p w = a_1 D^{p-1} w + a_2 D^{p-2} w + \cdots + a_p w.$$ 

**Proposition 4.1.** Let $w \in Z$ satisfy equation (67), then there exist a sequence of elements $w_1, ..., w_p \in Z$ such that

$$D^p w + D^{p-1} w_1 + D^{p-2} w_2 + \cdots + w_p = 0$$

and $w_k = b_kw$ for some $b_k \in \mathcal{F}$.

The proposition is proved by induction, making use of the Leibnitz rule (65).

**Corollary 1.** For any vector field $W_0$ from the characteristic distribution $Z = \text{span}\{Z_\alpha\}$, equation (57) has a solution for some $p$. In other words, every basis vector field $Z_\alpha$ generates a gauge transformation, so that the total number of independent gauge parameters $\epsilon$ coincides with $\text{dim} \ Z$.

An element $w \in Z$ is said to have degree not higher than $p$, if it satisfies equation (68) with some (not necessarily linear independent) $w_k \in Z$. It is clear that the elements of degree not higher than $p$ form a $\lambda$-distribution $Z_p \subset \mathcal{W}$. We have the finite filtration

$$Z = Z_N \supset Z_{N-1} \supset \cdots \supset Z_1 \supset 0.$$
The numbers
\[ \delta_p = \dim Z_p - \dim Z_{p-1} \]
are called the *indices* of the characteristic distribution \( Z \).

By making use of the Euclidean metric in \( \mathbb{R}^n \), we can split the imbedding \( Z_{p-1} \subset Z_p \) as \( Z_p = Z_{p-1} \oplus Z^\perp_{p-1} \). As a result the characteristic distribution is decomposed into the direct sum
\[
Z = Z^1 \oplus Z^2 \oplus \ldots \oplus Z^N, \quad Z^p \simeq Z_p/Z_{p-1}, \quad \dim Z^p = \delta_p.
\]
In a driftless theory \( V = 0 \) and \( \delta_1 \geq 1 \) as \( D(\lambda_0^\alpha Z_\alpha) = \lambda_1^\alpha Z_\alpha \)
and equation (68) is satisfied with \( p = 1 \). The corresponding gauge transformation (49) is the time reparametrization.

We use the decomposition (69) to construct a basis of undepressible gauge transformations for the equations of motion (48). If \( \{ Z_\alpha^p \}_{\alpha=1}^{\delta_p} \) is a basis in the \( \lambda \)-distribution \( Z^p \), then the undepressible gauge transformations read
\[
(70) \quad \delta_\varepsilon x^i = \sum_{n=0}^{p-1} R^i_{(p-n-1)\alpha_p} (\varepsilon) \alpha_p, \quad \delta_\varepsilon \lambda^\alpha = \sum_{n=0}^p U_\alpha^{(n)} \alpha_p (\varepsilon) \alpha_p,
\]
where
\[
(71) \quad R^i_{(n)\alpha_p} = \sum_{m=0}^n D^m W^i_{(n-m)\alpha_p}, \quad W^i_{(n)\alpha_p} = U_\alpha^{(n)} \alpha_p Z^i_\alpha, \quad U_\alpha^{(0)} \alpha_p = \delta_\alpha^\alpha.
\]

**Proposition 4.2.** If \( w \in Z^p \), then the vector fields \( w, Dw, D^2w, \ldots, D^{p-1}w \) are linearly independent.

This is true, otherwise it would be \( w \in Z^q \) with \( q \leq p \), because of Proposition 4.1.

**Corollary 2.** The vector fields \( R^i_{(n)\alpha_p} \) entering (70) are linear independent, so that \( \delta_\varepsilon x^i \) involves all the successive derivatives of the gauge parameter up to the order \( k \).

Let us denote
\[ Z_D = \text{span}_F \{ D^m u | \forall u \in Z, m \in \mathbb{N} \}. \]

By construction, the \( \lambda \)-distribution \( Z_D \) is invariant under the action of \( D \), i.e., \( DZ_D \subset Z_D \). It turns out that the distribution \( Z_D \) is actually involutive and coincides with \( Z_V \).
Proposition 4.3. \( Z_D = Z_V \).

Corollary 3. The gauge variations (70) of \( x \)'s are spanned by \( Z_V \).

Proof. Since \( Z_D \subset Z_V \), it remains to prove the converse inclusion. Let us first consider the special case where \( V = 0 \). For a driftless system \( Z_V = \mathcal{Z} \) and we must show that \( Z_D \) contains the Lie closure of \( Z \).

Increasing, if necessary, the order of the derivatives of the gauge parameter by changing \( \varepsilon \rightarrow (k) \varepsilon \), the gauge transformations (70) can be brought to the form

\[
\delta_\varepsilon^i x^i = \varepsilon^{\alpha} Z^i_\alpha(x) + \cdots, \quad \delta_\varepsilon^\lambda \lambda^\alpha = \varepsilon^{(n+1)} \alpha + \cdots,
\]

where the dots stand for the terms with the lower order derivatives of the gauge parameter. As the transformations (72) exhaust all the gauge symmetries of the equations, they have to form an on-shell closed gauge algebra with respect to the commutator of infinitesimal transformations:

\[
[\delta_\varepsilon^1, \delta_\varepsilon^2]|_{\text{on-shell}} = \delta_\varepsilon^3,
\]

where

\[
\varepsilon_3^\gamma = \sum_{n,m} \varepsilon^{(n)}_1 \alpha f^\gamma_{\alpha \beta m}(x, \lambda_k) \varepsilon^{(m)}_2 \beta,
\]

with \( f \)'s being the structure functions of the gauge algebra. On the other hand,

\[
[\delta_\varepsilon^1, \delta_\varepsilon^2] x^i = \varepsilon^{\alpha} \{ \delta_\varepsilon^1 \varepsilon^{\beta} (Z^i_\alpha, Z^i_\beta) + \cdots, \}
\]

where the dots stand for the other bilinear combinations of the derivatives \((k) \varepsilon \). At every fixed instant of time, the derivatives of the parameters \((k) \varepsilon \) can take on arbitrary predetermined values. Then, comparing the right and left hand sides of (73), we conclude that all commutators \([Z_\alpha, Z_\beta]\) are given by linear combinations of elements from \( Z_D \). A similar analysis for the successive commutators of the gauge transformations

\[
[\delta_\varepsilon, [\delta_\varepsilon, \cdots, [\delta_\varepsilon, \delta_\varepsilon] 
\]

shows that all multiple commutators

\[
[Z_{\alpha_m}, Z_{\alpha_{m-1}}, \cdots, Z_{\alpha_2}, Z_{\alpha_1} 
\]

are also included into \( Z_D \). In other words, we see that \( \mathcal{Z} \subset Z_D \).
The general case of a non-vanishing drift, \( V \neq 0 \), can be formally reduced to the previous one by the following trick. Let us associate to equations (48) another dynamical system

\[
\dot{x}^i = eV^i(x) + \lambda^\alpha Z_\alpha(x),
\]

where \( e \) is a new undetermined multiplier. The system (76) is driftless, and hence reparametrization invariant. The characteristic distribution of (76) is generated by the vector fields \( V \) and \( Z_\alpha \). The basis of infinitesimal gauge transformations for this new system can be chosen in the following way. First of all, the system is invariant under reparametrizations. The corresponding gauge transformation reads

\[
\delta_\epsilon x^i = \epsilon \dot{x}^i = \epsilon (eV^i + \lambda^\alpha Z_\alpha), \quad \delta_\epsilon \lambda^\alpha = \epsilon \dot{\lambda}^\alpha + \epsilon \dot{\lambda}^\alpha, \quad \delta_\epsilon e = \dot{\epsilon} e + \epsilon \dot{e}.
\]

As is seen the generator of this transformation involves the vector field \( V \) and acts nontrivially on the new multiplier \( e \). Considering \( e \) as a fixed function of time, we then define the gauge transformations of the form

\[
\delta_{\epsilon_1} x^i = (\epsilon_1)^{\alpha} Z_\alpha + \ldots, \quad \delta_{\epsilon_1} \lambda^\alpha = (\epsilon_1)^{\alpha+1} + \ldots, \quad \delta_{\epsilon_1} e = 0,
\]

The existence of these transformations easily follows from Proposition 4.1. According to the general formulas (70), (71) the expansion coefficients of \( \delta_{\epsilon_1} x^i \) in the time derivatives of \( \epsilon_1 \)'s are given by linear combinations of the vector fields \( D_{\epsilon_1}^m Z_\alpha \), where

\[
D_{\epsilon_1}^m = -\partial_e - e[V, \cdot] - \lambda^\alpha[Z_\alpha, \cdot].
\]

Since the leading terms in the variations \( \delta_{\epsilon_1} x^i \) and \( \delta_{\epsilon_1} x^i \) span the entire characteristic distribution \( \mathcal{E} = \text{span}\{V, Z_\alpha\} \), the transformations (77), (78) exhaust all the gauge symmetries of (76).

We claim that the \( \epsilon \)-transformations (78) constitute an ideal of the algebra of all gauge transformation (77), (78), i.e.,

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]|_{\text{on-shell}} = \delta_{\epsilon_3}, \quad [\delta_{\epsilon_1}, \delta_{\epsilon_1}]|_{\text{on-shell}} = \delta_{\epsilon_2}.
\]

To prove this statement it is enough to apply the commutators in the l.h.s. of the relations above to \( e \). We have

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]e = 0, \quad [\delta_{\epsilon_1}, \delta_{\epsilon_1}]e = 0.
\]

Thus the reparametrization transform (77) does not contribute to the r.h.s. of (79).

\[\text{As before, we have risen here the order of the gauge transformations to a certain uniform value } n.\]
Arguments similar to those we have used in the driftless case allow one to prove that all the multiple commutators of the vector fields $V$ and $Z_\alpha$ are contained in $Z_D$. Consider, for example, the following commutator of the gauge transformations:

\begin{equation}
[\delta_\epsilon, \delta_\varepsilon]x^i = \epsilon \varepsilon^\alpha \epsilon [V, Z_\alpha]x^i + \cdots .
\end{equation}

Because of relations (79) the right hand side of (81) must be a gauge transformation of the form (78). We have

\begin{equation}
[\delta_\epsilon, \delta_\varepsilon]x^i = \delta_\varepsilon' x^i , \quad \varepsilon'^\gamma = \epsilon \varepsilon^\alpha f^\gamma_{\alpha \lambda} (x, e_s, \lambda_k) + \cdots ,
\end{equation}

where the dots stand for the other combinations of the time derivatives of $\epsilon$ and $\varepsilon^\alpha$. Comparing the coefficients at $\epsilon \varepsilon^\alpha$ in (81) and (82), we see that every commutator $\epsilon [V, Z_\alpha]$ is given by a linear combination of $D^m_\epsilon Z_\alpha$. Setting $e = 1$, we conclude that $[V, Z_\alpha] \in Z_D$.

In a similar manner, one can see that all the higher iterated commutators of $Z_\alpha$ and $V$ belong to $Z_D$.

\[ \Box \]

By construction, the structure functions (71) are meromorphic functions of $x^i$ and $\lambda^\alpha$. So, the trajectories $(x^i(t), \lambda^\alpha(t))$ can exist such that the transformations (70) are ill defined. It is easy to see, however, that the functions (71) can be made real analytic by an appropriate change of the gauge parameters:

\begin{equation}
\varepsilon^\alpha \rightarrow \tilde{\varepsilon}^\alpha = U_{\alpha \beta} \varepsilon^\beta ,
\end{equation}

with $\det(U_{\alpha \beta})$ being a nonzero element of $\mathcal{F}$. The real analytic transformations are well defined on all the trajectories.

\textit{Example.} Consider the driftless system

\begin{equation}
\dot{x}^i = \lambda^\alpha Z_\alpha^i
\end{equation}
associated to the following characteristic distribution in $\mathbb{R}^{10}$:

$$
Z_1 = \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^8} + x^3 \frac{\partial}{\partial x^9} + x^4 \frac{\partial}{\partial x^{10}},
Z_2 = \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^6} + x^4 \frac{\partial}{\partial x^7},
Z_3 = \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^5},
Z_4 = \frac{\partial}{\partial x^4}.
$$

(85)

A straightforward computation shows that the vector fields $\{Z_\alpha, [Z_\beta, Z_\gamma]\}$ span the entire tangent space of $\mathbb{R}^{10}$ and

$$
[[Z_\alpha, Z_\beta], Z_\gamma] = 0.
$$

Introduce the vector field $X_n = \lambda_n^\alpha Z_\alpha$ and the differential $D = -\partial - [X_0, \cdot]$. One can readily check that

$$
DX_0 + X_1 = 0,
D^2X_1 + 2DX_2 + X_3 = 0,
D^3X_2 + 3D^2X_3 + 3DX_4 + X_5 = 0,
D^4X_3 + 4D^3X_4 + 6D^2X_5 + 4DX_6 + X_7 = 0.
$$

(86)

These equalities are the particular cases of the general identity

$$
\sum_{k=0}^{m} C_m^k D^{m-k} x_{m+k-1} = 0.
$$

(87)

The vector fields $\{X_0, X_1, X_2, X_3\}$ form another basis in the $\lambda$-distribution span$_{\mathcal{F}}\{Z_\alpha\}$. Comparing relations (86) with (70), (71) and (57), we get the following gauge transformations for the
differential equations\textsuperscript{[81]}:
\[
\delta_{\varepsilon} x = \varepsilon_1 X_0 \\
+ \varepsilon_2 X_1 + \varepsilon_2 (DX_1 + 2 X_2) \\
+ \varepsilon_3 X_2 + \varepsilon_3 (DX_2 + 3 X_3) + \varepsilon_3 (D^2 X_2 + 3 DX_3 + 3 X_4) \\
+ \varepsilon_4 X_3 + \varepsilon_4 (DX_3 + 4 X_4) + \varepsilon_4 (D^2 X_3 + 4 DX_4 + 6 X_5) + \varepsilon_4 (D^3 X_3 + 4 D^2 X_4 + 6 DX_6 + 4 X_6)
\]
\[
\delta_{\varepsilon} \lambda = \varepsilon_1 \lambda_0 + \varepsilon_1 \lambda_1 \\
+ \varepsilon_2 \lambda_1 + 2 \varepsilon_2 \lambda_2 + \varepsilon_2 \lambda_3 \\
+ \varepsilon_3 \lambda_2 + 3 \varepsilon_3 \lambda_3 + 3 \varepsilon_3 \lambda_4 + \varepsilon_3 \lambda_5 \\
+ \varepsilon_4 \lambda_3 + 4 \varepsilon_4 \lambda_4 + 6 \varepsilon_4 \lambda_5 + 4 \varepsilon_4 \lambda_6 + \varepsilon_4 \lambda_7.
\]

Here $\lambda_0^\alpha = \lambda^\alpha$ and $\lambda_{k+1}^\alpha = \dot{\lambda}_k^\alpha$.

Consider now the general system\textsuperscript{[48]} including both the differential equations and the constraints. It follows from Proposition\textsuperscript{4.3} that all the gauge symmetries of the differential equations alone are also the symmetries of the constraints. Indeed, the distribution $\mathcal{E}$ is tangent to the complete constraint surface $\bar{\Sigma}$, hence (by the Frobenius theorem) the closure $\bar{\mathcal{E}}$ is also tangent to $\bar{\Sigma}$. This means that not only the vector fields $Z_\alpha$ and $V$ respect the constraint surface, but also all their iterated commutators do the same:
\[
XT_a = F(X)_{b}^{\bar{a}} T_b \quad \forall X \in \bar{\mathcal{E}}.
\]
On the other hand, according to Corollary 3 from Proposition\textsuperscript{4.3} the gauge variation $\delta_{\varepsilon} x^i$ is to be spanned by the vectors from the distribution $\mathcal{Z}_V \subset \bar{\mathcal{E}}$. This immediately gives $\delta_{\varepsilon} T_a|_{\Sigma} = 0$.

An important point to stress is that the number of nontrivial gauge transformations can decrease when the constraints are taken into account. The matter is that some of the structure functions\textsuperscript{5} $R_n$ and $U_n$, which are involved into the gauge transformation\textsuperscript{[49]}, can be trivial\textsuperscript{[23]} with regard to the constraints. In other words, some linear combinations of the gauge transformations can vanish identically on $\bar{\Sigma}$. Such transformations are also called trivial. All the gauge symmetry transformations are to be considered modulo trivial ones.

\textsuperscript{5}As it has been already noticed, all these functions can be chosen to be real analytic.
To formulate a systematic algorithm for constructing a basis of nontrivial and linearly independent gauge transformations in the presence of constraints we need some algebraic background. Given a constrained dynamical system in the complete normal form (48), denote by $\mathcal{R}$ the ring of analytical functions of $x^i$ and of a finite number of variables $\lambda^\alpha_k$. Let $\mathcal{I} \subset \mathcal{R}$ denote the principle ideal generated by the regular constraints $\{T_a\}$. For simplicity sake, assume that the ideal $\mathcal{I}$ is simple. Then the quotient $\mathcal{R}/\mathcal{I}$ is an integrality domain and we can form the field of fractions $\mathcal{F}_\mathcal{I} = \text{Fr}(\mathcal{R}/\mathcal{I})$. The field $\mathcal{F}_\mathcal{I}$ is a natural substitution for the field of meromorphic functions $\mathcal{F}$ in the presence of constraints. As a practical matter, it is more convenient to use the following equivalent definition of $\mathcal{F}_\mathcal{I}$. A meromorphic function $f \in \mathcal{F}$ is said to be regular if it admits a representation $f = a/b$, where $a, b \in \mathcal{R}$ and $b \notin \mathcal{I}$. The ideal $\mathcal{I}$ being simple, all the regular meromorphic functions constitute a ring $\mathcal{R}_\mathcal{I} \subset \mathcal{F}$. It is easily seen that $\mathcal{I}$ is the maximal proper ideal of $\mathcal{R}_\mathcal{I}$ and $\mathcal{F}_\mathcal{I} = \mathcal{R}_\mathcal{I}/\mathcal{I}$. Thus $\mathcal{F}_\mathcal{I}$ is just a subquotient of $\mathcal{F}$ and in all practical calculations we can replace the elements of the field $\mathcal{F}_\mathcal{I}$ by their regular representatives in $\mathcal{F}$.

Now define the $n$-dimensional vector space $\mathcal{W}_\mathcal{I}$ over $\mathcal{F}_\mathcal{I}$ as

$$\mathcal{W}_\mathcal{I} = \text{span}_{\mathcal{F}_\mathcal{I}} \left( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right).$$

Again, we can view $\mathcal{W}_\mathcal{I}$ as a subquotient of $\mathcal{W}$ and represent the vectors of $\mathcal{W}_\mathcal{I}$ by regular elements of the $\lambda$-distribution $\mathcal{W}$, i.e., those vectors of $\mathcal{W}$ whose components are regular meromorphic functions of $\mathcal{F}$.

Since the vector fields $V$ and $Z_\alpha$ entering the definition of our dynamical system (48) are assumed to be regular and linearly independent modulo constraints, we can define the $m$-dimensional subspace $\mathcal{Z}_\mathcal{I} \subset \mathcal{W}_\mathcal{I}$ as

$$\mathcal{Z}_\mathcal{I} = \text{span}_{\mathcal{F}_\mathcal{I}} (Z_1, \ldots, Z_m).$$

Using the fact that the action of the vector fields $V$ and $Z_\alpha$ preserves the constraints $T_a$, one can easily see that the $\mathbb{R}$-linear operator $D: \mathcal{W} \rightarrow \mathcal{W}$ defined by (64) induces an $\mathbb{R}$-linear operator in the subquotient $\mathcal{W}_\mathcal{I}$. We will denote the latter operator by the same symbol $D$. Similar to the

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6 An ideal $\mathcal{I} \subset \mathcal{R}$ is said to be simple if $ab \in \mathcal{I}$ implies either $a \in \mathcal{I}$ or $b \in \mathcal{I}$. In the case where $\mathcal{R}$ is the ring of analytical functions, we have the following criterion of simplicity: If the constraint surface $\Sigma \subset \mathbb{R}^n$ associated to a set of regular constraints $T_a = 0$ is connected, then the principal ideal $\mathcal{I} = \langle T_a \rangle$ is simple.
case of unconstrained system we can filter the space $\mathcal{Z}_I$ by the finite sequence of subspaces
\begin{equation}
\mathcal{Z}_I \supset \mathcal{Z}_I^{(N)} \supset \mathcal{Z}_I^{(N-1)} \supset \cdots \supset \mathcal{Z}_I^{(1)} \supset 0,
\end{equation}
where
\begin{equation}
\mathcal{Z}_I^{(p)} = \{ u \in \mathcal{Z}_I \mid D^p u \in \mathcal{Z}_I \cup D\mathcal{Z}_I \cup \cdots \cup D^{p-1}\mathcal{Z}_I \}.
\end{equation}
Using this filtration and the Euclidean metric in $\mathbb{R}^n$, we can then split $\mathcal{Z}_I$ in the direct sum of subspaces
\begin{equation}
\mathcal{Z}_I = \mathcal{Z}_I^1 \oplus \mathcal{Z}_I^2 \oplus \cdots \oplus \mathcal{Z}_I^N, \quad \mathcal{Z}_I^p \simeq \mathcal{Z}_I^{(p)}/\mathcal{Z}_I^{(p-1)}.
\end{equation}
Now a straightforward analog of Proposition 4.1 for the differential $\mathcal{F}_I$-module $\mathcal{W}_I$ states that for any $w_0 \in \mathcal{Z}_I^p$ there exists a sequence of elements $w_1, \ldots, w_p \in \mathcal{Z}_I$ such that
\begin{equation}
D^p w_0 + D^{p-1} w_1 + \cdots + D w_{p-1} + w_p = 0.
\end{equation}
Let $W_0, \ldots, W_p \in \mathcal{W}$ be regular representatives of the elements $w_0, \ldots, w_p$. Then, according to the general formulae (70), the gauge transformations read:
\begin{equation}
\delta \varepsilon^n x^i = \sum_{n=0}^{p-1} \sum_{m=0}^{p-n-1} \varepsilon^n D^m W^{i}_{p-n-m-1}, \quad \delta \varepsilon^n \lambda^\alpha = \sum_{n=1}^{p} \varepsilon^n U^\alpha_{p-n},
\end{equation}
where $W^{i}_n = U^\alpha_n Z^i_\alpha$. Let us choose a basis $\{Z_\alpha^i\}$ in every subspace $\mathcal{Z}_I^p$. To any basis element we can associate a gauge transformation of the form (95) and this yields a complete basis of undepressible gauge transformations in the presence of constraints.

We conclude this section by some remarks concerning interpretation of $\mathcal{Z}_V$ in control theory and the theory of gauge systems. Observe that the restriction $\mathcal{Z}_V|_{\bar{\Sigma}}$, being a completely integrable distribution of $\bar{\Sigma}$, endows the constraint surface with the structure of foliation $\mathcal{F}(\bar{\Sigma})$. In the context of gauge systems, the leaves of this foliation are known as gauge orbits. Two points of the constraint surface $\Sigma$ are considered equivalent if they belong to the same gauge orbit. Equivalent points define the same physical state, so that the space of all physical states of the gauge system is identified with the space of leaves $\bar{\Sigma}/\mathcal{F}(\bar{\Sigma})$. On the other hand, one of the basic concepts of control theory is the notion of attainable set. By definition, a point $y \in \bar{\Sigma}$ belongs to the attainable set of a point $x \in \bar{\Sigma}$ if one can join $y$ to $x$ by an integral curve of (48) with some fixed functions $\lambda^\alpha(t)$. In the case where $\mathcal{Z}_V$ and $\bar{\Sigma}$ are real anaclitic and $V \in \mathcal{Z}_V$, the so-called orbit theorem ensures that the attainable set of a point $x \in \bar{\Sigma}$ coincides with the gauge orbit passing through
This coincidence is not particularly surprising, since the gauge transformations, involving arbitrary functions of time as parameters, allow the (control) functions $\lambda^\alpha(t)$ to take on arbitrary values at each given instant of time. So, the doctrines of control and gauge theories are in a sense complementary: the controllable part of dynamics is non-physical, while the physical part is uncontrollable.

5. Physical observables and an involutive normal form

By definition, $t$-local values associated to a dynamical system in the complete form (48) are functions of the phase-space coordinates $x^i$, undetermined multipliers $\lambda^\alpha$ and their derivatives up to some finite order. Note that the time derivatives of $x^i$ can always be excluded with the help of the equations of motion (48). Therefore, without loss of generality, we can identify the space of $t$-local values with the space of real analytical functions $\mathcal{R}$. Among these functions, there are trivial ones (23) that vanish on the complete constraint surface $\bar{\Sigma}$. Two $t$-local values $O_1$ and $O_2$ are said to be equivalent if their difference is a trivial function. In view of the regularity assumptions, the last condition amounts to

$$O_1 \sim O_2 \iff O_1 - O_2 = F^\alpha T_\alpha.$$  

In the previous section, the space of equivalence classes was identified with the quotient $\mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by the constraints.

The infinitesimal gauge transformation (95) maps any solution of (48) to another one. Given an initial time moment $t_0$, the gauge parameters $\varepsilon$ can be chosen vanishing together with all their derivatives involved in the gauge transform. So, the initial data remain the same, while the solutions are different. The physical values should evolve in a casual way, i.e., they should take the same value on every solution originating from a given initial state. This implies that the physical values are to be on-shell invariants of the gauge transformations. As is seen from (95), the gauge variation of $\lambda$'s starts with the highest time derivative of $\varepsilon$'s and this derivative does not contribute to the gauge transformation of $x$'s. This suggests that the gauge invariant $t$-local values can depend on $\lambda$'s only through the trivial contributions. In other words, each $t$-local physical value can be represented by a function of $x^i$. In view of Proposition 4.3 the gauge transformations for the functions of $x^i$ are generated by the gauge distribution $\mathcal{Z}_V$ so that the subspace of trivial values is automatically gauge invariant. Thus we are lead to the following
definition: A physical observable of a dynamical system brought to the complete normal form (48) is the equivalence class of a phase-space function \( O(x) \) that is on-shell invariant under the action of the gauge distribution \( \mathcal{Z}_V \), i.e.,

\[
ZO|_\Sigma = 0 \quad \forall \mathcal{Z} \in \mathcal{Z}_V .
\]

For the physical observables, the equations of motion (48) reduce to the form

\[
\dot{O} = VO ,
\]

where \( V \) is the complete drift. The undetermined multipliers \( \lambda^\alpha \) drop out of these equations as the physical observables are invariant under the action of the characteristic distribution \( \mathcal{Z} \subset \mathcal{Z}_V \). Notice that the time derivative of an observable is again an observable because \([V, \mathcal{Z}_V] \subset \mathcal{Z}_V \) and the complete constraint surface \( \bar{\Sigma} \) is invariant under the action of \( V \) and \( \mathcal{Z}_V \). With initial data specified, the unique existence of the solution \( O(t), t \geq t_0 \), to equation (98) follows from two facts: (i) the undetermined multipliers are not contained in the equations and (ii) the equivalence class of \( VO \) is gauge invariant. So, the right hand side of (98) is the same for any solution \( x^i(t) \) evolving from a given initial state \( x^i(t_0) = x^i_0 \). All that confirms ones again that the definition of the physical observables provides them causal evolution.

As is seen, the following data are only needed to define the physical observables and their time evolution: the phase space \( U \), the complete constraint surface \( \Sigma \), the complete drift \( V \), and the gauge distribution \( \mathcal{Z}_V \). The quadruple \((U, \bar{\Sigma}, V, \mathcal{Z}_V)\) can always be unambiguously derived from equations of motion in their primary normal form (1), (2) following the algorithm of the previous sections. The converse is not true: different primary normal forms can result in the same quadruple \((U, \bar{\Sigma}, V, \mathcal{Z}_V)\). These differences can be much larger than just the equivalence relations (21), (22) for the primary normal form. In particular, the output can be the same, even though the algorithm has been applied to dynamical systems with characteristic distributions \( \mathcal{Z} \) and primary constraint surfaces \( \Sigma \) of different dimensions. With the dynamics in mind of physical observables, it seems reasonable to consider two dynamical systems as being equivalent whenever they have the same complete constraint surfaces, complete drifts and coinciding gauge distributions\(^7\).

\(^7\)In control theory, two affine-control systems are called feedback equivalent whenever every solution \( x(t) \) of equations (1) with every given control \( \lambda(t) \) of one system coincides with a certain solution of another one, possibly with a different control. The feedback equivalence is much more restrictive notion than the definition above: For
Among the systems that are physically equivalent to (1), (2) there is a special one whose characteristic distribution, primary constraint set, and primary drift coincide, correspondingly, with the gauge distribution, complete constraint set, and the complete drift of the original system:

\[
\dot{x}^i = \bar{V}^i(x) + \lambda^\alpha Z_\alpha^i(x), \quad T_a(x) = 0.
\]

By construction, \(Z_V = \text{span}\{Z_\alpha\}\) is tangent to the complete constraint surface \(\bar{\Sigma} = \{x \in U | T_a(x) = 0\}\). This means no compatibility conditions can arise from (99). Also, \(Z_V\) is involutive and invariant with respect to the complete drift \(\bar{V}\). Therefore, the gauge distribution for (99) coincides with the characteristic distribution. We call equations (99) the involutive normal form of local dynamics. The system (99) describes the observables defined by the same conditions (97) and having the same evolution law (98) as the physical observables associated to the original equations (1), (2). As the characteristic distribution is involutive, the gauge transformations for (99) take quite a simple form (9).

Let us comment on the involutive normal form (99) for a system whose primary normal form is the constrained Hamiltonian dynamics [1], [2]. If equations (1), (2) follow from the least action principle (4), then the primary characteristic distribution \(Z\) is spanned by the Hamiltonian vector fields for the primary constraints and the drift is the Hamiltonian vector field for the primary Hamiltonian. As it has been already explained in Section 3, the transverse constraints \(T_\perp\) correspond to the second-class constraints, the transverse distribution \(Z_\perp\) is spanned by the Hamiltonian vector fields for the primary second-class constraints, and \(Z_\parallel\) is generated by the Hamiltonian vector field associated to the primary first-class constraints. Suppose we have no transverse constraints. Then the characteristic distribution is tangent to the complete constraint surface. From the viewpoint of Dirac’s classification, this is the case of a pure first-class system. Whenever the Dirac conjecture is true\(^8\), the Hamiltonian vector fields for all the first-class constraints generate the gauge transformations and the involutive form of dynamics (99) is again variational. The corresponding action

\[
S[x, \lambda] = \int \left( \rho_i(x) \dot{x}^i - H_{\text{tot}}(x, \lambda) \right) dt,
\]

\(^8\)It is not always true, see [2] for counterexamples.
involves the total Hamiltonian $H_{\text{tot}} = H + \lambda a T_a$ given by the sum of the original Hamiltonian \( (3) \) and the linear combination of all the first-class constraints (both primary and secondary) with independent Lagrange multipliers. In this case, the gauge distribution is spanned by the Hamiltonian vector fields $X_a = \{ T_a, \cdot \}$ for the first-class constraints.

Whether the Dirac conjecture is true or not, the algorithm of Section 4 shows that the gauge distribution is included into span$\{X_a\}$ provided that the primary normal form was variational. This does not mean, however, that every Hamiltonian vector field $X_a$ should belong to the gauge distribution. In case $Z_V \neq \text{span}\{X_a\}$, the property conjectured by Dirac does not hold and the involutive form of the constrained Hamiltonian dynamics is not variational anymore. Be it as it may, the algorithm proposed in this paper will automatically separate the constraints whose Hamiltonian vector fields contribute to the gauge transformations from those which Hamiltonian vector fields do not. This allows us to systematically identify all the true gauge symmetries for any regular, constrained Hamiltonian system, even though the system does not satisfy the Dirac conjecture.

Concluding this section, let us briefly discuss the issue of equipping the involutive dynamics (99) with a certain Hamiltonian structure when the original equations of motion are not variational. The basic idea is that only the algebra of physical observables (97), not the algebra of all $t$-local values, should be equipped with the Poisson bracket. In other words, it is sufficient to require the Jacobi identity to hold only when the Poisson bracket is applied to a pair of observables. Besides, the Poisson bracket is to be compatible with the time evolution (99) in the sense that the time derivative should differentiate the bracket of two observables (not arbitrary functions) by the Leibnitz rule. In [5], we introduced a notion of weak Hamiltonian structure, which satisfies all the above properties. A similar construction was also studied in [6].

Formally, a weak Hamiltonian structure associated to an involutive dynamical system is defined by the quadruple $(T, Z, V, P)$, where $T = \{ T_a \}$ is a set of constraints, $Z = \{ Z_\alpha \}$ are the generators of a characteristic(=gauge) distribution, $V$ is a drift, and $P = P^{ij} \partial_i \wedge \partial_j$ is a weak Poisson bivector. So all the objects are polyvector fields of degree 0, 1, and 2. In terms of the Schouten bracket of polyvector fields the defining relations for a weak Hamiltonian structure read

\begin{align}
[Z_\alpha, T_a] &= A_{\alpha a}^b T_b, \\
[Z_\alpha, Z_\beta] &= B_{\alpha \beta}^\gamma Z_\gamma + T_a C_{\alpha \beta}^a, \\
[T_a, V] &= D_a^b T_b, \\
[Z_\alpha, V] &= E_{\alpha \beta}^\beta Z_\beta + T_a F_{\alpha}^a,
\end{align}
where $A, B, C, ..., N$ are some polyvector fields. Relations (101) and (102) express the fact of involutivity of the dynamical system. The first relation in (103) identifies $P$ as a weak Poisson bivector, i.e., $P$ satisfies the Jacobi identity modulo constraints and gauge symmetry generators. Then the second relation in (103) tells us that the evolution (98) generated by $V$ preserves the Poisson algebra of physical observables (i.e., $V$ is a weakly Poisson vector field). Relations (104) mean that the Hamiltonian vector fields for the constraints are spanned by the gauge generators modulo trivial terms and that the generators are weakly Poisson vector fields. As a result the trivial functions constitute the Poisson ideal $\mathcal{I} \subset \mathcal{R}$ that makes possible to speak about the Poisson algebra of physical observables $\mathcal{R}/\mathcal{I}$.

As it was shown in [5], the weak Hamiltonian structure (101-104) admits a nice BRST imbedding that generalize the usual BFV-BRST formalism for the Hamiltonian systems with first-class constraints. Starting with this BRST embedding, it is possible to construct a fully consistent deformation quantization of a weak Hamiltonian system without any reference to variational principles. The output is a weakly associative $\ast$-product inducing an associative quantum multiplication in the space of physical observables identified with a certain BRST cohomology. The construction essentially relies on the superextension of the formality theorem [6] and may be thought of as a generalization of the Kontsevich deformation quantization to the case of non-Hamiltonian gauge theories.

The “odd counterpart” of the weak Hamiltonian structure is known as a Lagrange structure [7]. The existence of the latter structure is much less restrictive for the equations of motion than the requirement to be variational. Whereas the weak Hamiltonian structure is aimed at the construction of $\ast$-product, the Lagrange structure allows one to perform the path-integral quantization of a (non-)variational dynamical system. For the variational dynamics, this quantization is shown to reduce to the standard BV quantization [7], [8]. The analysis of classical gauge symmetries performed in Sections 4, 5 may be viewed as a pre-requisite for extending the local BRST cohomology techniques [10] from the Lagrangian to non-Lagrangian gauge theories [7], [8]. Notice, however, that to make the technique explicitly covariant in field theoretical context, our analysis
should address the issues of locality in multidimensional space. These issues are beyond the scope of this work.

6. Conclusion

In this paper we propose an algorithm of bringing the general local dynamics to certain normal forms, which allow us to identify all gauge symmetries. These normal forms can serve as starting point for the BRST imbedding and deformation quantization of not necessarily variational dynamics. Let us briefly summarize the essentials of the proposed algorithm.

The algorithm becomes applicable after imposing certain regularity conditions on the local equations of motion. Then, we see that the general regular equations can be brought to the primary normal form that includes the differential equations (1) with undetermined multipliers and the phase-space constraints (2). In this form, the dynamics are defined by the three ingredients: the primary constraints, primary characteristic distribution, and primary drift. In the case of variational dynamics, this corresponds to Hamiltonian equations subject to primary constraints. The main problem we address in this paper is finding all the gauge symmetry transformations (5) for the equations of motion (1), (2). To this end, we first consider the differential consequences of the equations of motion. A basic consistency requirement is the conservation of the constraints in time. This results in a multi-step procedure of iterating secondary constraints and determining a part of the multipliers. After terminating the procedure, we are left with equations (48) - the complete normal form of local dynamics - which assume no further restrictions on the phase-space coordinates and/or undetermined multipliers.

Having the dynamical system brought to the complete normal form, one can go over to finding its gauge symmetries. We find that the gauge transformations (95) are generated by the gauge distribution $Z_V$. The gauge distribution is the Lie closure of the tangential characteristic distribution $Z$ supplemented by all its iterated commutators with the drift vector field (47). So, the gauge symmetries are explicitly identified in the same terms that define the original system (1), (2).

Then we turn to the notion of physical observables. These are understood as gauge invariants of the dynamics. As the gauge symmetry is generated by the gauge distribution $Z_V$, the observables are defined as equivalence classes (96) of phase-space functions annihilated by $Z_V$. This definition (97) ensures that the time evolution of observables (98) is casual. It also suggests to consider two
dynamical systems as equivalent to each other whenever they have coinciding gauge distributions, complete constraint sets and complete drifts. As a result, the physical observables and their time evolutions are the same for all the equivalent systems. In every equivalence class of the dynamical systems, there is a special representative whose characteristic distribution is involutive, tangent to the constraint surface, and is preserved by the drift. In this form, called the involutive normal, the gauge transformations take the most simple form \( \mathcal{D} \). It is the involutive normal form that serves as a starting point for the deformation quantization based on the concept of weak Poisson structure \( \mathcal{D} \).

**Acknowledgements.** We are thankful to E.M. Gorbatenko for illuminating discussions on regularity of analytic varieties. This work is partially supported by the RFBR grant no 06-02-17352-a and by the grant from Russian Federation President Programme of Support for Leading Scientific Schools no 871.2008.02. SLL is partially supported by the RFBR grant 08-01-00737-a.

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