Neutron drip line and the equation of state of nuclear matter

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We investigate how the neutron drip line is related to the density dependence of the symmetry energy, by using a macroscopic nuclear model that allows us to calculate nuclear masses in a way dependent on the equation of state of asymmetric nuclear matter. The neutron drip line obtained from these masses is shown to appreciably shift to a neutron-rich side in a nuclear chart as the density derivative of the symmetry energy increases. Such shift is clearly seen for light nuclei, a feature coming mainly from the surface property of neutron-rich nuclei.

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Thanks to recent developments of radioactive ion beam facilities, one might be able to experimentally probe the stability of atomic nuclei against neutron or proton drip in a nuclear chart ranging from light to superheavy nuclei. The key quantities to study the neutron (proton) drip line are the one- and two-neutron (proton) separation energies, \( S_{n(p)} \) and \( S_{2n(2p)} \), which correspond to an energy required to remove one and two neutrons (protons) from a nucleus in the ground state, respectively. Experimentally, the neutron drip line is marginally accessible only for light nuclei \cite{1}. Even beyond the neutron drip line, however, nuclei can be present in dense neutral matter. Nuclei in the crust of neutron stars are a typical example and are considered to control the thermal and electric transport properties of matter in the crust as well as the dynamics of superfluid neutron vortices, which are relevant to the observed thermal and rotational evolution of neutron stars \cite{2}. We remark that the size and shape of nuclei are shown to be controlled by the equation of state (EOS) of asymmetric nuclear matter through the density dependence of the symmetry energy \cite{3}. In this Letter, we will investigate how the density dependence of the symmetry energy in turn affects the prediction of the neutron drip line.

Theoretically, a Weizsäcker-Bethe mass formula, which is based on a view of nuclei as incompressible spherical liquid drops of uniform density \( n_0 \), provides a standard behavior of the neutron drip line. In this formula, the nuclear binding energy \( E_B \) is written as function of mass number \( A \) and charge number \( Z \) (or neutron number \( N \)) in the form

\[
- E_B = E_{\text{vol}} + E_{\text{sym}} + E_{\text{surf}} + E_{\text{Coul}},
\]

where \( E_{\text{vol}} = a_{\text{vol}} A \) is the volume energy, \( E_{\text{sym}} = a_{\text{sym}} [(N - Z)/A]^{2/3} A \) is the symmetry energy, \( E_{\text{surf}} = a_{\text{surf}} A^{2/3} \) is the surface energy, and \( E_{\text{Coul}} = a_{\text{Coul}} Z^2 / A^{1/3} \) is the Coulomb energy. Then, the one-neutron separation energy can be evaluated as

\[
S_n \approx \frac{\partial E_B}{\partial N} \bigg|_Z = -a_{\text{vol}} - a_{\text{sym}} (1 - 4x^2) - 2a_{\text{surf}} + \frac{a_{\text{Coul}} Z^2}{3 A^{4/3}} \approx \frac{a_{\text{Coul}} Z^2}{3 A^{4/3}} (2)
\]

where \( x = Z/A \) is the proton fraction. The condition \( S_n = 0 \) gives a smoothed behavior of the neutron drip line. The derived drip line is close to \( x = 0.3 \), which is basically controlled by the competition between the volume and symmetry energy terms, except in the light region of the nuclear chart. This may be a good starting point, but one needs to go beyond Eq. (2) by taking into account nonnegligible deviation of the nuclear density from \( n_0 \).

Recently, the density dependence of the symmetry energy attracts much attention because it is relevant to the isospin dependence of nuclear masses \cite{4,5} and radii \cite{6,7}, dipole resonances \cite{8,9}, and heavy-ion collisions involving neutron-rich nuclei \cite{10,11,12,13}. In predicting the neutron drip line, uncertainties in the density dependence of the symmetry energy need to be taken seriously. The important parameter characterizing the density dependence of the symmetry energy is a density symmetry coefficient \( L \), which is defined as \( L = 3 n_0 (dS/dn)_{n=n_0} \) with the symmetry energy \( S(n) \) dependent on the density \( n \) of bulk nuclear matter. Masses of extremely neutron-rich nuclei were calculated from a macroscopic nuclear model and shown to have an appreciable dependence on \( L \), which can be understood from the density and isospin dependence of the surface tension \cite{14}. Here we address how this dependence affects the neutron drip line on the nuclear chart.

We begin with a macroscopic model of nuclei \cite{15}, which was constructed in such a way as to reproduce the known global properties of stable nuclei and can be used for describing the masses and radii of unstable nuclei in a...
manner that is dependent on the EOS of nuclear matter. This model can be summarized as follows: 

(i) We set the bulk energy per nucleon as

\[ w = \frac{3\hbar^2(3\pi^2)^{2/3}}{10m_n a}(n_n^{5/3} + n_p^{5/3}) + (1 - \alpha^2)v_s(n)/n + \alpha^2v_n(n)/n, \]  

(3)

where

\[ v_s = a_1n^2 + \frac{a_2n^3}{1 + a_3n} \]  

(4)

and

\[ v_n = b_1n^2 + \frac{b_2n^3}{1 + b_3n} \]  

(5)

are the potential energy densities for symmetric nuclear matter and pure neutron matter, \( n_n \) and \( n_p \) are the neutron and proton number densities, \( n = n_n + n_p, \) \( \alpha = (n_n - n_p)/n \) is the neutron excess, and \( m_n \) is the neutron mass. A set of expressions (3)-(5) is one of the simplest that reduces to a usual expansion

\[ \alpha^2 \]  

in the limit of \( n \to n_0 \) and \( \alpha \to 0. \) Here \( w_0 \) and \( K_0 \) are the saturation energy and the incompressibility of symmetric nuclear matter, and \( S_0 = S(n = n_0). \) In the incompressible limit, \( w_0 \) and \( S_0 \) correspond to \( a_{\text{vol}} \) and \( a_{\text{sym}} \) in the mass formula. [14]. We fix \( b_3, \) which controls the EOS of matter for large neutron excess and high density, at 1.58632 fm\(^3\). This value was obtained by one of the authors [15] in such a way as to reproduce the neutron matter energy of Friedman and Pandharipande [16]. Change in this parameter would make no significant difference in the determination of the other parameters and the final results for nuclear masses. 

(ii) We write down the total energy of a nucleus of mass number \( A \) and charge number \( Z \) as a function of the density distributions \( n_n(r) \) and \( n_p(r) \) in the form

\[ E = E_b + E_g + E_C + Nm_nc^2 + Zm_pc^2, \]  

(7)

where

\[ E_b = \int d^3r n(r)w(n_s(r), n_p(r)) \]  

(8)

is the bulk energy,

\[ E_g = F_0 \int d^3r|\nabla n(r)|^2 \]  

(9)

is the gradient energy with adjustable constant \( F_0, \)

\[ E_C = \frac{\hbar^2}{2} \int d^3r \int d^3r' n_n(r)n_p(r')/|r - r'| \]  

(10)

is the Coulomb energy, and \( m_p \) is the proton mass.

(iii) For simplicity we use the following parametrization for the nucleon distributions \( n_i(r) (i = n, p): \)

\[ n_i(r) = \begin{cases} n_i^{in} \left[ \frac{1}{r^3} \right], & r < R_i, \\ 0, & r \geq R_i, \end{cases} \]  

(11)

where \( r \) is the distance from the center of the nucleus. This parametrization allows for the central density, half-density radius, and surface diffuseness for neutrons and protons separately. 

(iv) In order to construct the nuclear model in such a way as to reproduce empirical masses and radii of stable nuclei, we first extremize the binding energy with respect to the particle distributions for fixed \( A, \) five EOS parameters, and \( F_0. \) Next, for various sets of the incompressibility and the density symmetry coefficient, we obtained the remaining three EOS parameters and the gradient coefficient by fitting the calculated optimal values of charge number, mass excess, root-mean-square (rms) charge radius to empirical data for stable nuclei on the smoothed \( \beta \) stability line [17]. In the range of the parameters \( 0 < L < 160 \) MeV and \( 180 \) MeV \( \leq K_0 \leq 360 \) MeV, as long as \( K_0S_0/3n_0L \gtrsim 200 \) MeV fm\(^3\), we obtained a reasonable fitting to such data. As a result of this fitting, the parameters \( n_0, w_0, S_0, \) and \( F_0 \) are constrained as \( n_0 = 0.14\pm0.17 \) fm\(^{-3}\), \( w_0 = -16\pm1 \) MeV, \( S_0 = 25\pm40 \) MeV, and \( F_0 = 66\pm6 \) MeV fm\(^3\). The fitting gives rise to a relation nearly independent of \( K_0, \)

\[ S_0 \approx B + CL, \]  

(12)

where \( B \approx 28 \) MeV and \( C \approx 0.075. \)

We proceed to obtain the neutron and proton drip lines from the macroscopic nuclear model. For various
sets of $L$ and $K_0$, we first evaluate the binding energy $E_B$ of nuclei in the ground state by minimizing the energy [7] for fixed $N$ and $Z$. We then draw the neutron (proton) drip line by identifying nuclides at neutron (proton) drip with those neighboring to nuclides for which $S_n = E_B(Z, N) - E_B(Z, N - 1)$ ($S_p = E_B(Z, N) - E_B(Z - 1, N)$) and $S_{2n} = E_B(Z, N) - E_B(Z, N - 2)$ ($S_{2p} = E_B(Z, N) - E_B(Z - 2, N)$) are positive and beyond which at least one of them is negative. The results obtained from the two extreme EOS models denoted as EOS C ($L = 146$ MeV and $K_0 = 360$ MeV) and EOS G ($L = 5.7$ MeV and $K_0 = 180$ MeV) are shown in Fig. 1, together with the empirically known nuclides [18, 19] and the prediction from a contemporary mass formula [17]. We remark that inclusion of the condition for $S_{2n}$ and $S_{2p}$ in addition to $S_n$ and $S_p$ in drawing the drip lines makes only a little difference in the case of the present model calculations, while being essential in the case of the prediction from the mass formula because of the Wigner, shell, and even-odd terms included therein. We remark that the rms deviations of the calculated masses from the measured values [19] are about 3 MeV, which is of order the deviations obtained from a Weizsäcker-Bethe mass formula.

We find from Fig. 1 that the obtained neutron drip lines show an appreciable $L$ dependence, while the proton ones do not. This is reasonable because nuclides at neutron (proton) drip are far away from (near) $N = Z$ (see Fig. 2). The neutron drip line does shift to a neutron-rich side as $L$ increases, a feature that will be discussed later in terms of a compressible liquid-drop model. We remark that the EOS dependence of the obtained drip lines comes predominantly from $L$ because of negligible $K_0$ dependence of the calculated masses [5].

In order to see the $L$ dependence more clearly, we plot in Fig. 3 the difference in the neutron number of nuclides at neutron and proton drip between the calculations from the EOS models C and G. The difference shows only a weak dependence on $Z$ both in the case of neutron and proton drip. This indicates that the $L$ dependence can be seen more clearly for lighter nuclei. In fact, the corresponding proton fraction of nuclides at neutron drip shows a stronger dependence on $L$ for lighter nuclei, as shown in Fig. 4. This is advantageous because heavier radioisotopes are more difficult to produce in experiments.

The $L$ dependence of the neutron drip line as obtained above can be understood within the framework of a compressible liquid-drop model in which nuclei in equilibrium are allowed to have a density different from the saturation density $n_0$ of symmetric nuclear matter. By following a line of argument of Ref. [5], we first add the surface symmetry term, $\alpha_{sym} A^{2/3} [(N - Z)/A]^2$, to the mass formula [1] based on an incompressible liquid-drop model. This surface symmetry term gives rise to additional contribution,

$$\delta S_n = -\frac{2\alpha_{sym}}{3A^{1/3}} (1 - 2x)(1 + 4x),$$

(13)
to the neutron separation energy [2]. Next, we consider the density-dependent surface tension [20],

$$\sigma(n_{in}, \alpha_{in}) = \sigma_0 \left[ 1 - C_{sym} \alpha_{in}^2 + \chi \left( \frac{n_{in} - n_0}{n_0} \right) \right],$$

(14)
where $n_{in}$ and $\alpha_{in}$ are the density and neutron excess inside a liquid drop, $\sigma_0 = \sigma(n_0, 0)$, $C_{sym}$ is the surface symmetry energy coefficient, and $\chi = (n_0/\sigma_0) d\sigma/dn_{in}|_{n_{in}=n_0,\alpha_{in}=0}$. By taking a limit of vanishing compressibility, one obtains $4\pi \sigma_0 R^2 = \alpha_{surf} A^{2/3}$ and $4\pi \sigma_0 C_{sym} R^2 = -\alpha_{sym} A^{2/3}$ with the liquid-drop radius $R$. Typically, fitting to the empirical mass data
yields \( \sigma_0 \approx 1 \text{ MeV fm}^{-2} \) and \( C_{\text{sym}} = 1.5 - 2.5 \). As we shall see below, nonvanishing compressibility effectively introduces the \( L \) dependence into the surface symmetry term through the parameter \( \chi \) characterizing the density dependence of the surface tension. The value of \( \chi \) is poorly known, but likely to be positive [5]. For example, \( \chi = 4/3 \) for the Fermi gas model.

If one ignores Coulomb and surface corrections, the equilibrium density and energy per nucleon of a liquid-drop, \( n_s \) and \( w_s \), can be evaluated from Eq. (6) as

\[ w_s = w_0 + S_n \alpha^2 \tag{15} \]

and

\[ n_s = n_0 - \frac{3n_0 L}{K_0} \alpha^2 \tag{16} \]

Strictly speaking, expressions (15) and (16) are applicable only for nearly symmetric nuclear matter. We nevertheless use these expressions for the purpose of characterizing the liquid-drop properties because the typical value of \( \alpha \) along the neutron drip line is of order 0.35–0.4, considerably smaller than unity. By substituting \( n_s \), Eq. (16), into \( n_{\text{in}} \) in Eq. (14), we thus obtain

\[ \sigma(n_s, \alpha_{\text{in}}) = \sigma_0 \left[ 1 - \left( C_{\text{sym}} + \frac{3L\chi}{K_0} \right) \alpha_{\text{in}}^2 \right]. \tag{17} \]

This result allows one to identify \( a_{\text{sym}} A^{2/3} \) with \(-4\pi \sigma_0 R^2 (C_{\text{sym}} + 3L\chi/K_0)\) and hence to conclude that with increasing \( L \), \( S_n \) increases through the surface symmetry contribution [13] for \( x < 1/2 \).

This conclusion is consistent with the \( L \) dependence of the neutron drip line shown in Fig. 1 because any positive corrections \( S_n \) act to enhance the stability of nuclei against neutron emission. It is important to note that the bulk symmetry term gives rise to a negative contribution \( S_n \) for \( x < 1/2 \), as shown in Eq. (2), and that the parameter \( a_{\text{sym}} \) corresponds to the symmetry energy coefficient \( S_0 \), which in turn is related to \( L \) by the relation [12] obtained from fitting to empirical masses and charge radii of stable nuclei. Since one obtains a larger \( a_{\text{sym}} \) for larger \( L \), the effect of \( a_{\text{sym}} \) tends to decrease \( S_n \) with \( L \) and hence to facilitate neutron drip. However, this effect is relatively small compared with the above-mentioned effect of \( a_{\text{sym}} \). This is consistent with the fact that the \( L \) dependence is clearer for lighter nuclei.

In summary, we have investigated the influence of the density dependence of the symmetry energy on the drip lines by using a macroscopic nuclear model that depends explicitly on the EOS of nuclear matter. We find that an \( L \) dependence appears appreciably in the neutron drip line and it is clearer for lighter nuclei, a feature coming mainly from the surface property through the density and neutron excess dependence of the surface tension. We note that our calculations do not include even-odd or shell corrections. The even-odd corrections would play an important role in \( S_n \). In fact, the magnitude of even-odd staggering in \( S_n \) could be comparable to that of change in \( S_n \) due to uncertainties in \( L \). This implies that some kind of smoothing would be required in deriving information about the EOS from future empirical data on neutron drip. Shell corrections would further complicate such derivation, but are intriguing in the context of magicity of nuclei close to the neutron drip line [21].

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\[ \begin{align*}
[1] & \text{M. Notani et al., Phys. Rev. C 76, 044605 (2007); T. Baumann et al., Nature (London) 449, 1022 (2007).} \\
[2] & \text{C.J. Pethick and D.G. Ravenhall, Annu. Rev. Nucl. Part. Sci. 45, 429 (1995).} \\
[3] & \text{K. Oyamatsu and K. Iida, Phys. Rev. C 75, 015801 (2007).} \\
[4] & \text{P. Danielewicz and J. Lee, Nucl. Phys. A818, 36 (2009).} \\
[5] & \text{K. Oyamatsu and K. Iida, Phys. Rev. C 81, 054302 (2010).} \\
[6] & \text{K. Oyamatsu and K. Iida, Prog. Theor. Phys. 109, 631 (2003).} \\
[7] & \text{A.R. Bodmer and Q.N. Usmani, Phys. Rev. C 67, 034305 (2003).} \\
[8] & \text{M. Warda, X. Viñas, X. Roca-Maza, and M. Centelles, Phys. Rev. C 80, 024316 (2009).} \\
[9] & \text{A. Klimkiewicz et al., Phys. Rev. C 76, 051603(R) (2007).} \\
[10] & \text{L. Trippa, G. Colò, and E. Vigezzi, Phys. Rev. C 77, 061304(R) (2008).} \\
[11] & \text{D.V. Shetty, S.J. Yennello, and G.A. Souliotis, Phys. Rev. C 76, 024606 (2007).} \\
[12] & \text{B.A. Li, L.W. Chen, and C.M. Ko, Phys. Rep. 464, 113 (2008).} \\
[13] & \text{M.B. Tsang et al., Phys. Rev. Lett. 102, 122701 (2009).} \\
[14] & \text{J.M. Lattimer, Ann. Rev. Nucl. Part. Sci. 31, 337 (1981).} \\
[15] & \text{K. Oyamatsu, Nucl. Phy. A561, 431 (1993).} \\
[16] & \text{B. Friedman and V.R. Pandharipande, Nucl. Phys. A361, 502 (1981).} \\
[17] & \text{H. Koura, T. Tachibana, M. Uno, and M. Yamada, Prog. Theor. Phys. 113, 305 (2005).} \\
[18] & \text{Chart of the Nuclides 2004, compiled by T. Horiguchi et al., JNDC and Nuclear Data Center, JAERI (2005).} \\
[19] & \text{G. Audi, A.H. Wapstra, and C. Thibault, Nucl. Phys. A729, 337 (2003).} \\
[20] & \text{K. Iida and K. Oyamatsu, Phys. Rev. C 69, 037301 (2004).} \\
[21] & \text{A. Ozawa et al., Phys. Rev. Lett. 84, 5493 (2000).} 
\end{align*} \]