DIMERS IN PIECEWISE TEMPERLEY DOMAINS

MARIANNA RUSSKIKH$^{\#, \ddagger}$

Abstract. We study the large-scale behavior of the height function in the dimer model on the square lattice. Richard Kenyon has shown, that the fluctuations of the height function on Temperley discretisations of a planar domain converge in the scaling limit (as the mesh size tends to zero) to the Gaussian Free Field with Dirichlet boundary conditions. We extend Kenyon’s result to a more general class of discretisations.

Moreover, we introduce a new factorization of the coupling function in the double-dimer model into two discrete holomorphic functions, which can be viewed as discrete fermions. For Temperley discretisations with appropriate boundary modifications, Kenyon has shown that the double-dimer height function converges to a harmonic function in the scaling limit. We use the new factorization to extend the Kenyon’s result to a class of all polygonal discretisations, that are not necessarily Temperley. Furthermore, we show, that, quite surprisingly, the expectation of the double-dimer height function in the Temperley case is exactly discrete harmonic (for an appropriate choice of Laplacian) even before taking the scaling limit.

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$^{\#}$ Section de Mathématiques, Université de Genève. 2-4 rue du Lièvre, Case postale 64, 1211 Genève 4, Suisse.
$^{\ddagger}$ Chebyshev Laboratory, Department of Mathematics and Mechanics, St. Petersburg State University. 14th Line, 29b, 199178 St. Petersburg, Russia.
E-mail addresses: Marianna.Russkikh@unige.ch.
1. Introduction

The dimer model is one of the best known models of statistical physics, first introduced to model a diatomic gas. By modifying the underlying graph, it can be used to study the Ising model (see Fisher's approach). Under the name "perfect matchings", it prominently appears in theoretical computer science and combinatorics.

A dimer covering (or perfect matching) of a graph is a subset of edges that covers every vertex exactly once. The dimer model is a random covering of a given graph by dimers. We will be interested in uniform random coverings, that is, those chosen from the distribution in which all dimer configurations are equally weighted.

In this paper, we work with dimers on finite subgraphs (or domains) of the square lattice. Such a dimer covering may be viewed as a random tiling of a domain of the dual lattice by dominoes $2 \times 1$. Thurston introduced the height function of a domino tiling which uniquely assigns integer values to all vertices of the dual lattice. Moreover, a domino tiling can be reconstructed from the values of the height function. Thus, one can think of a random domino tiling as a random height function on the vertex set of the domain.

The key question in the dimer model concerns the large-scale behavior of the expectation of the height function and of its fluctuations. We are interested in studying the scaling limit of the dimer model on planar graphs as the mesh tends to zero. One of the main interesting features lies in the conformal invariance of such scaling limit.

For planar graphs, Kasteleyn [12] showed that the partition function of the dimer model can be evaluated as the determinant of a signed adjacency matrix, the Kasteleyn matrix. The local statistics for the uniform measure on dimer configurations can be computed using the inverse Kasteleyn matrix, see [17]. The latter can be viewed as a two-point function, called the coupling function [13]. The coupling function is a complex-valued discrete holomorphic function. As such, its real and imaginary parts are discrete harmonic, and the study of the local statistics of random tilings can be reduced to the study of the convergence of discrete harmonic functions.

A Temperley discretization (see Fig. 5) is a discrete domain with special boundary conditions. It is defined in section 2.4. Temperley domains correspond to Dirichlet boundary conditions or Neumann boundary conditions for the discrete harmonic components of the coupling function. Kenyon [13, 15] used this approach to prove the conformal invariance of the limiting distribution of the height function in the case of Temperley discretizations.

More precisely, if one considers Temperley discretizations of a given domain $\Omega$, Kenyon [13] showed that the limit of the expected height function is a harmonic function with boundary values depending on the direction (the argument of the tangent vector) of the boundary. In [15] Kenyon proved that, in the case of Temperley discretizations, the fluctuations of the height function converge (as the mesh size tends to zero) to the Gaussian Free Field on $\Omega$ with Dirichlet boundary conditions. In the present paper we extend Kenyon’s result to a class of Piecewise Temperley discretisations.

For more general discretizations, with domains that are not necessarily Temperley, the large-scale behavior of the expectation of the height function and its fluctuations is much more complicated, see [5, 19, 20, 25]. In particular, the exact nature of fluctuations is not established yet.

A double-dimer configuration is a union of two dimer coverings, or equivalently a set of even-length simple loops and double edges with the property that every vertex is the endpoint of exactly two edges, see Fig. 1.

Note that there are two ways to obtain a given loop (on the dual graph). This can be interpreted as a choice of the orientation of the loop, see Fig. 1. Thus, the double-dimer model can be represented as a random covering of the dual graph by the oriented loops and double edges [24]. The height function in the double-dimer model, which is the difference of height functions for two dimer configurations,
Figure 1. Two different domino tilings of the same domain can be combined into the collection of loops and double edges. Orienting the edges of the first covering from white to black, and the edges of the second one from black to white, one gets an orientation of resulting loops.

Figure 2. Left: the coverings of the domains that differ on two squares. Right: the interface between these two squares and the collection of loops and double edges is the result of the composition of the coverings.

has a simple geometric representation: if we cross a loop, then the height function changes by $+1$ or $-1$, depending on the orientation of the loop.

In the case of discretizations by Temperley domains Kenyon [14] and Dubedat [9] obtained results confirming the prediction of the convergence of the loop ensemble of the double-dimer model to the conformal loop ensemble $CLE(4)$, see [28, 30]. The loop ensemble $CLE(4)$ is a conformally invariant object: its image under any conformal mapping has the same distribution as an analogous ensemble in the image of the domain.

We will consider coverings of a pair of domains that differ by two squares, see Fig. 2. In this case, in addition to a collection of loops and double edges, the resulting composition of the coverings contains an interface (a simple path between these two squares). It is expected that the interface converges to a conformally invariant random curve $SLE(4)$ as the mesh size tends to zero, see [27].

The coupling function plays an important role in the proof of convergence of height functions. We will show that in the double-dimer model the coupling function $C(u,v)$ has a factorisation into a product of two discrete holomorphic functions $F(u)$ and $G(v)$. Moreover, we will describe the construction of the discrete integral of this product of two discrete holomorphic functions. Then for any discrete domain the expectation of the height function of the double-dimer model can be interpreted as an integral of two discrete holomorphic functions. Due to Kenyon [13], for the single-dimer model, the expectation of the height function is harmonic in the limit for approximations by Temperley domains. Using the factorization of the double-dimer coupling function we will show that the expectation of the double-dimer height function is harmonic already on the discrete level.

**Theorem 1.** The expectation of the double-dimer height function on a Temperley domain is exactly discrete leap-frog harmonic, i.e. its discrete leap-frog Laplacian, which is defined by (3.2), equals zero.

In Section 4 we will prove the convergence of average height functions in the double-dimer model to the harmonic measure for the discretizations by polygonal domains.
Figure 3. On the left pictures the coupling function restricted to the light grey squares satisfies the Dirichlet boundary conditions, and the coupling function restricted to the dark grey squares obeys Neumann boundary conditions. The picture on the right corresponds to mixed Dirichlet and Neumann boundary conditions for the coupling function.

Theorem 2. Let $\Omega$ be a polygon with $n$ sides parallel to the axes. Let $u_0$ and $v_0$ be points on straight parts of the boundary of $\Omega$. Suppose that a sequence of discrete $n$-gons $\Omega^{\delta}$ on a grid with mesh size $\delta$ approximates the polygon $\Omega$ in a proper way, and that each polygon $\Omega^{\delta}$ has at least one domino tiling. Let black and white squares $u_0^\delta$ and $v_0^\delta$ of domain $\Omega^\delta$ tend to boundary points $u_0$ and $v_0$ of the domain $\Omega$. Let $h^\delta$ be the height function of a uniform double-dimer configuration on $\Omega$. Then $\mathbb{E} h^\delta$ converges to the harmonic measure $h_{\text{m}}(\cdot, (u_0v_0))$ of the boundary arc $(u_0v_0)$ on the domain $\Omega$.

We will show the convergence of the dimer coupling function in the case of approximations by black-piecewise Temperley domains (see Fig. 13), domains which correspond to mixed Dirichlet and Neumann boundary conditions for the coupling function (see Fig. 3). Note that the coupling function $C(u,v_0)$ with fixed $v_0$ coincides with a discrete holomorphic function $F(u)$ described in Corollary 14.

Theorem 3. Let $\Omega^\delta$ be a sequence of discrete $2k$-black-piecewise Temperley domains of mesh size $\delta$ approximating a continuous domain $\Omega$. Suppose that each $\Omega^\delta$ allows a domino tiling. Then $F^\delta$ converge uniformly on compact subsets of $\Omega$ to a continuous holomorphic function $f$.

For a more precise statement, see Theorem 28. Similarly, one can show the convergence of $G^\delta$ for approximations by white-piecewise Temperley domains. Note that a polygonal domain $\Omega^\delta$ as in Theorem 2 is black-piecewise Temperley and also white-piecewise Temperley. Thus, we obtain the convergence of the double-dimer coupling function for a polygonal domain. Due to [15] the dimer height function converges to the Gaussian Free Field in the setup of Theorem 3.

Corollary 4. Let $\Omega$ be a Jordan domain with smooth boundary in $\mathbb{R}^2$. Let $\Omega^\delta$ be a black-piecewise Temperley domain approximating $\Omega$. Let $h^\delta$ be the height function of $\Omega^\delta$ and $\overline{h}^\delta$ be its expected value. Then $h^\delta - \overline{h}^\delta$ converges weakly in distribution to the Gaussian Free Field on $\Omega$ with Dirichlet boundary conditions, as $\delta$ tends to 0.

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2. Definitions and basic facts

2.1. Height function and Temperley domain. Consider a checkerboard tiling of a discrete domain $\Omega$ with unit squares. We will use grey color for the black squares in our figures. Sometimes for convenience we will distinguish between two types of black squares, in this case in the figures black squares in even rows will be represented by a light grey and those in odd rows will be dark grey.
(see Fig. 5). A domain where all corner squares are dark grey is called an odd Temperley domain. To obtain the Temperley domain one removes dark grey square adjacent to the boundary from an odd Temperley domain.

W.P.Thurston defines the height function $h$ (which is a real-valued function on the vertices of $\Omega$) as follows. Fix a vertex $z_0$ and set $h(z_0) = 0$. For every other vertex $z$ in the tiling, take an edge-path $\gamma$ from $z_0$ to $z$. The height along $\gamma$ changes by $\pm \frac{1}{4}$ if the traversed edge does not cross a domino from the tiling or by $\mp \frac{3}{4}$ otherwise: if the traversed edge has a black square on its left then the height increases by $\frac{1}{4}$ or decreases by $\frac{3}{4}$; if it has a white square on its left then it decreases by $\frac{1}{4}$ or increases by $\frac{3}{4}$, see Fig. 4. Note that for a simply connected domain, the height is independent of the choice of $\gamma$. The height function in the double-dimer model is the difference of the two height functions of corresponding to two independent uniform dimer coverings.

2.2. Kasteleyn weights and discrete holomorphic functions. Let $G$ be a bipartite graph with $n$ black and $n$ white vertices. A Kasteleyn matrix $K_G$ is an $n \times n$ weighted adjacency matrix whose rows index the black vertices and columns index the white vertices. Let us denote by $K$ Kasteleyn weights and discrete holomorphic functions.

For a given planar graph $G$, there are many ways to choose the edge-weights satisfying the Kasteleyn condition. Let us fix the following ones, which were proposed by Kenyon in [13]: put $\tau(e) = \pm 1$ for horizontal edges and $\tau(e) = \pm i$ if $e$ is a vertical edge, see Fig. 5. It is easy to check that these weights are Kasteleyn weights.

Let $\Omega$ be a discrete domain on a square lattice that has at least one domino tiling. Let $K_{\Omega}$ be a Kasteleyn matrix of this domain. Let us denote by $C_{\Omega}(u, v)$ the elements of the inverse matrix $K_{\Omega}^{-1}$, where $u$ and $v$ are black and white squares of $\Omega$. The main advantage of choosing Kasteleyn
weights as shown in Fig. 5 is the following: with this choice of weights the function $C_{\Omega}(u, v)$ is discrete holomorphic on the domain. Thus its limiting behavior can be studied using the methods of discrete complex analysis, see [13]. Following [13], we call $C_{\Omega}(u, v)$ the coupling function.

Let $F$ be a function defined on the set of black squares of the domain $\Omega$. Recall that the function $F$ is called discrete holomorphic on $\Omega$ if for any white square $v \in \Omega$ it satisfies a discrete analogue of the Cauchy-Riemann equation (see Fig. 6), and at the same time the values of the function $F$ on the set of light grey squares are real, while on the set of dark grey squares are purely imaginary. Note that real and imaginary parts of a holomorphic function are harmonic functions. It is also true on a discrete level: consider the discrete Cauchy-Riemann equations at four white neighbours of a black square $u$, then it is easy to show that $F(u) = \frac{1}{4} \sum_{i=1}^{4} F(u_i)$. Therefore a discrete leap-frog Laplacian of $F$ at $u$ equals zero (see Fig. 6). In other words, real and imaginary parts of discrete holomorphic functions are discrete harmonic functions.

We know that $K_{\Omega}^{-1} \cdot K_{\Omega} = I$, so for any white square $v_0 \in \Omega$ the function $C_{\Omega}(u, v_0)$ considered as a function of $u \in \Omega$ is discrete holomorphic on $\Omega \setminus \{v_0\}$. Therefore, the restriction of $C_{\Omega}(u, v_0)$ to one type of black squares is a discrete harmonic function everywhere except the two squares adjacent to $v_0$.

Moreover, the function $C_{\Omega}(u, v)$ satisfies the following property:

$\triangleright$ if $u$ and $v$ are adjacent squares, then $|C_{\Omega}(u, v)|$ is equal to the probability that the domino $[uv]$ is contained in a random domino tiling of $\Omega$, see [13].
For Temperley domains, each of the two discrete harmonic components of the function \( C_\Omega(u, v_0) \) has the following boundary conditions: coupling function restricted to the light grey squares (see Fig. 3), satisfies the Dirichlet boundary conditions, and coupling function restricted to the dark grey squares obeys Neumann boundary conditions.

2.3. **Even and odd cases.** A double-dimer configuration is a union of two dimer coverings, we will consider coverings of a pair of domains \( \Omega_1, \Omega_2 \) that differ by two squares, i.e. \( |\Omega_1 \Delta \Omega_2| = 2 \). Note that there are two different situations depending on whether \( \Omega := \Omega_1 \cup \Omega_2 \) contains an odd or an even number of squares. In the odd case, assume that \( \Omega \) has one more black squares than white squares. Then domains \( \Omega_1 \) and \( \Omega_2 \) are obtained from \( \Omega \) by removing black squares \( u_1 \) and \( u_2 \) adjacent to the boundary (see Fig. 3). In the even case, let \( \Omega_1 = \Omega \) and \( \Omega_2 \) is obtained from \( \Omega \) by removing black and white squares \( u_0 \) and \( v_0 \), which are adjacent to the boundary.

By definition the **double-dimer coupling function** on \( \Omega = \Omega_1 \cup \Omega_2 \) is the difference of the two dimer coupling functions on domains \( \Omega_1 \) and \( \Omega_2 \)

\[
C_{\text{dbl-d}, \Omega}(u, v) := C_{\Omega_1}(u, v) - C_{\Omega_2}(u, v).
\]

Recall that the absolute value of the coupling function is the probability that the corresponding domino is contained in a random tiling, and the determinant of the Kasteleyn matrix is equal to the number of domino tilings of our domain, so, \( |C_{\Omega \setminus \{u_0, v_0\}}(u, v)| = \left| \frac{\det(K_{\Omega \setminus \{u_0, v_0\}})}{\det(K_\Omega)} \right| \). Note that

\[
\frac{\det(K_{\Omega \setminus \{u_0, v_0\}})}{\det(K_\Omega)} = \pm K_\Omega^{-1}(u_0, v_0) \quad \text{and} \quad \frac{\det(K_{\Omega \setminus \{u, v\}})}{\det(K_\Omega)} = \pm K_\Omega^{-1}(u, v),
\]

and also

\[
\frac{\det(K_{\Omega \setminus \{u_0, v_0, u, v\}})}{\det(K_\Omega)} = \pm \det \begin{pmatrix} K_\Omega^{-1}(u_0, v_0) & K_\Omega^{-1}(u_0, v) \\ K_\Omega^{-1}(u, v_0) & K_\Omega^{-1}(u, v) \end{pmatrix}.
\]

Therefore,

\[
C_{\text{dbl-d}, \Omega}(u, v) = C_\Omega(u, v) - C_{\Omega \setminus \{u_0, v_0\}}(u, v) = \pm \frac{K_\Omega^{-1}(u_0, v) \cdot K_\Omega^{-1}(u, v)}{K_\Omega^{-1}(u_0, v_0)}.
\]

Recall that for a fixed \( v_0 \) the function \( K_\Omega^{-1}(u, v_0) \) is a discrete holomorphic function of \( u \), let us denote it by \( F_{v_0}(u) \), similarly let us define a function \( G_{u_0}(v) := K_\Omega^{-1}(u_0, v) \). So, we obtain

\[
C_{\text{dbl-d}, \Omega}(u, v) = \text{const}_{u_0, v_0} \cdot F_{v_0}(u) \cdot G_{u_0}(v),
\]

where \( \text{const}_{u_0, v_0} = \pm 1/K_\Omega^{-1}(u_0, v_0) \).

We will show that the coupling function as a product of two discrete holomorphic functions in both even and odd cases.
3. Expectation of the height function in the double-dimer model and the proof of Theorem 3.1

**Notation.** Put $\lambda = e^{i\pi/2}$ and $\bar{\lambda} = e^{-i\pi/2}$.

Consider a checkerboard tiling $C^\delta$ of $\mathbb{R}^2$ with squares, each square has side $\delta$ and centered at a lattice point of
\[
\{(\delta n, \delta m) | n, m \in \mathbb{Z}; n + m \in 2\mathbb{Z}\}
\]
(see Fig. 9). The pair $(n, m)$ is called the coordinates of a point on this lattice. Let $\Omega^\delta$ be a simply connected discrete domain composed of a finite number of squares of $C^\delta$ bounded by disjoint simple closed lattice path. Let $V^\delta$ be the vertex set of $\Omega^\delta$. We will denote by $\Phi^\delta$ the set of black squares and by $\Phi^\delta_0$ the set of white squares of $\Omega^\delta$. So, $\Omega^\delta = \Phi^\delta \cup \Phi^\delta_0$. Let the coordinates of a square be the coordinates of its center. Then we can define the sets $\Phi^\delta_0$ and $\Phi^\delta_1$ of black squares of $\Omega^\delta$ and the sets $\Phi^\delta_0$ and $\Phi^\delta_1$ of white squares by the following properties:

- $(\Phi^\delta_0)$ both coordinates are even and the sum of coordinates is divisible by 4;
- $(\Phi^\delta_1)$ both coordinates are even and the sum of coordinates is not divisible by 4;
- $(\Phi^\delta_0)$ both coordinates are odd and the sum of coordinates is not divisible by 4;
- $(\Phi^\delta_1)$ both coordinates are odd and the sum of coordinates is divisible by 4.

Define $\partial V^\delta$ be the set of vertices on the boundary. Let $\partial \Omega^\delta$ be the set of faces adjacent to $\Omega^\delta$ but not in $\Omega^\delta$. Let $\partial \Phi^\delta$ and $\partial \Phi^\delta_0$ be the sets of black and white faces of $\partial \Omega^\delta$ correspondingly. Let $\partial_{\text{int}} \Omega^\delta$ be the set of interior faces that have a common edge with boundary of $\Omega^\delta$. Similarly define sets $\partial_{\text{int}} \Phi^\delta$ and $\partial_{\text{int}} \Phi^\delta_0$ ($\partial_{\text{int}} \Omega^\delta = \partial_{\text{int}} \Phi^\delta \cup \partial_{\text{int}} \Phi^\delta_0$). Let us denote by $\Omega^\delta$ the set $\Omega^\delta \cup \partial \Omega^\delta$, define also sets $\Phi^\delta$ and $\Phi^\delta_0$, to be exact: $\Phi^\delta = \Phi^\delta_0 \cup \partial \Phi^\delta_0$ and $\Phi^\delta_0 = \Phi^\delta_0 \cup \partial \Phi^\delta_0$. In the same way we define sets $\partial_{\Phi^\delta_{0,1}}$, $\partial_{\Phi^\delta_{0,1}}$, $\partial_{\Phi^\delta_{0,1}}$, $\partial_{\Phi^\delta_{0,1}}$, $\partial_{\Phi^\delta_{0,1}}$ and $\partial_{\Phi^\delta_{0,1}}$.

Let $F^\delta : \Phi^\delta \to \mathbb{C}$ be a function. Let us define discrete operators $\partial \Phi^\delta$ and $\bar{\partial} \Phi^\delta$ by the formulas:

\[
[\partial \Phi^\delta F^\delta](v) = \frac{1}{2} \left( \frac{F^\delta(v + \delta \lambda) - F^\delta(v - \delta \lambda)}{2\delta \lambda} + \frac{F^\delta(v + \delta \lambda) - F^\delta(v - \delta \lambda)}{2\delta \lambda} \right),
\]

\[
[\bar{\partial} \Phi^\delta F^\delta](v) = \frac{1}{2} \left( \frac{F^\delta(v + \delta \lambda) - F^\delta(v - \delta \lambda)}{2\delta \lambda} + \frac{F^\delta(v + \delta \lambda) - F^\delta(v - \delta \lambda)}{2\delta \lambda} \right),
\]

where $v \in \Phi^\delta$. Note that, If $[\bar{\partial} \Phi^\delta F^\delta](v) = 0$, then two terms involved into the definition of $[\partial F^\delta]$ are equal to each other.

We can similarly define these operators for a function $G^\delta : \Phi^\delta \to \mathbb{C}$.
**Definition 6.** A function $F^\delta: \diamondsuit^\delta \to \mathbb{C}$ is called discrete holomorphic in $\Omega^\delta$ if $[\partial^\delta F^\delta](v) = 0$ for all $v \in \diamondsuit^\delta$. Also, we always assume that $F^\delta$ is real on $\hat{\diamondsuit}^\delta$ and purely imaginary on $\check{\diamondsuit}^\delta$.

A function $G^\delta: \check{\diamondsuit}^\delta \to \mathbb{C}$ is called discrete holomorphic in $\Omega^\delta$ if $[\bar{\partial}^\delta G^\delta](u) = 0$ for all $u \in \check{\diamondsuit}^\delta$. Also, we always assume that $G^\delta$ is a real part of $\lambda R$ (resp., $\bar{\lambda} R$) on $\hat{\diamondsuit}^\delta$ (resp., on $\check{\diamondsuit}^\delta$).

**Remark 7.** If a function $F^\delta: \hat{\diamondsuit}^\delta \to \mathbb{C}$ is discrete holomorphic in $\Omega^\delta$ then $[i \cdot \partial^\delta F^\delta]: \diamondsuit^\delta \to \mathbb{C}$ is a discrete holomorphic function in $\Omega^\delta \setminus \partial_{\text{int}} \Omega^\delta$. Similarly, if $G^\delta: \check{\diamondsuit}^\delta \to \mathbb{C}$ is discrete holomorphic in $\Omega^\delta$ then $\bar{\partial}^\delta G^\delta: \check{\diamondsuit}^\delta \to \mathbb{C}$ is discrete holomorphic in $\Omega^\delta \setminus \partial_{\text{int}} \Omega^\delta$.

Define discrete Laplacian of $F^\delta$ by

$$\Delta^\delta F^\delta(u) = \frac{F^\delta(u + 2\delta \lambda) + F^\delta(u - 2\delta \lambda) + F^\delta(u + 2\delta \bar{\lambda}) + F^\delta(u - 2\delta \bar{\lambda}) - 4F^\delta(u)}{4\delta^2},$$

where $u \in \diamondsuit^\delta$. Note that $\Delta^\delta F^\delta(u) = 4[\partial^\delta \bar{\partial}^\delta F^\delta](u) = 4[\bar{\partial}^\delta \partial^\delta F^\delta](u).

A function $F^\delta: \phi^\delta \to \mathbb{C}$ (resp., $G^\delta: \phi^\delta \to \mathbb{C}$) is called a discrete harmonic function in $\Omega^\delta$ if it satisfies $\Delta^\delta F^\delta(u) = 0$ for all $u \in \phi^\delta$ (resp., $\Delta^\delta G^\delta(v) = 0$ for all $v \in \phi^\delta$).
Definition 8. Let $F^δ : \Delta^δ \rightarrow \mathbb{C}$ and $G^δ : \tilde{\Delta}^δ \rightarrow \mathbb{C}$ be discrete holomorphic functions. Let us define a discrete primitive $H^δ : \mathcal{V}^δ \rightarrow \mathbb{R}$ by the equality

$$H^δ(z') - H^δ(z) = (z' - z)F^δ(u)G^δ(v),$$

(3.1)

where $u, v$ are adjacent black and white squares (correspondingly); and $z, z'$ are their common vertices, see Fig. 9.

Remark 9. It is easy to see that, if $\Omega^δ$ is simply connected, then $H^δ$ is well defined (see Fig. 10).

Let us define the discrete leap-frog Laplacian of $H^δ$ at $z \in \text{Int} \mathcal{V}^δ$ by

$$\Delta^δ H^δ(z) = \frac{1}{4\delta^2} \sum_{z' \sim z} (H^δ(z') - H^δ(z)), $$

(3.2)

where $s \in \{1, 2, 3, 4\}$, and $z'_s$ are defined as shown in Fig. 10.

Proposition 10. Let $u_-, u_+, v_z, v_y, z$ be as shown in Fig. 10, then

$$\Delta^δ H^δ(z) = \delta \cdot (\lambda[\partial^δ F^δ(v_y)](v_z)[\partial^δ G^δ](u_-) - \lambda[\partial^δ F^δ](v_y)[\partial^δ G^δ](u_+) + \lambda[\partial^δ F^δ](v_y)[\partial^δ G^δ](u_-) - \lambda[\partial^δ F^δ](v_y)[\partial^δ G^δ](u_+)).$$

(3.3)

Proof. Note that

$$4\delta^2 \Delta^δ H^δ(z) = \delta \lambda[F^δ(u_-) + 2\delta \lambda[\partial^δ F^δ](v_y)] \cdot [G^δ(v_y) + 2\delta \lambda[\partial^δ G^δ](u_+)] + \delta \lambda F^δ(u_+)G^δ(v_y) - \delta \lambda[\partial^δ F^δ(u_+)] \cdot [G^δ(v_y) - 2\delta \lambda[\partial^δ G^δ](u_-)] - \delta \lambda F^δ(u_-)G^δ(v_y) - \delta \lambda[\partial^δ F^δ(u_-)] \cdot [G^δ(v_y) - 2\delta \lambda[\partial^δ G^δ](u_-)] - \delta \lambda F^δ(u_-)G^δ(v_y) + \delta \lambda[\partial^δ F^δ(u_-)] \cdot [G^δ(v_y) + 2\delta \lambda[\partial^δ G^δ](u_+)] + \delta \lambda F^δ(u_+)G^δ(v_y).$$
One can rewrite the above formula in the following form:

\[
F^\delta(u_-) \cdot \left[ \delta \lambda G^\delta(v_3) + 2 \delta \lambda^2 \partial^\delta G^\delta(u_-) - \delta \lambda G^\delta(v_4) + \delta \lambda G^\delta(v_2) + 2 \delta^2 \lambda^2 \partial^\delta G^\delta(u_+) \right] + F^\delta(u_+) \cdot \left[ \delta \lambda G^\delta(v_4) + 2 \delta \lambda^2 \partial^\delta G^\delta(u_-) - \delta \lambda G^\delta(v_2) + \delta \lambda G^\delta(v_3) + 2 \delta^2 \lambda^2 \partial^\delta G^\delta(u_-) \right] \\
+ G^\delta(v_3) \cdot \left[ 2 \delta \lambda^2 \partial^\delta F^\delta(v_4) + 2 \delta \lambda^2 \partial^\delta F^\delta(v_3) \right] \\
- 4 \cdot \delta^3 \lambda^3 \partial^\delta F^\delta(v_4)[\partial^\delta G^\delta(v_3)](u_-) - \delta^3 \lambda^3 \partial^\delta F^\delta(v_2)[\partial^\delta G^\delta(v_3)](u_-) \\
- \delta^3 \lambda^3 \partial^\delta F^\delta(v_4)(u_-) + \delta^3 \lambda^3 \partial^\delta F^\delta(v_3)(u_-)
\]

Finally, note that \( \lambda^3 = -\lambda \) and \( \lambda^3 = -\bar{\lambda} \).

\[\square\]

**Proposition 11.** The function \( H^\delta \) has no local maxima or minima. Moreover, a value at an interior vertex can not be strictly greater than values at two of its neighbouring vertices and strictly smaller than values at two other neighbouring vertices at the same time.

**Proof.** It is enough to show that the product of all the differences is non-positive (see Fig. 11):

\[
(H^\delta(z) - H^\delta(z_1)) \cdot (H^\delta(z) - H^\delta(z_2)) \cdot (H^\delta(z) - H^\delta(z_3)) \cdot (H^\delta(z) - H^\delta(z_4)) = (-\delta \lambda)F^\delta(u_+)G^\delta(v_3) - \delta \lambda G^\delta(v_3)F^\delta(u_-) \cdot (-\delta \lambda)G^\delta(v_3)F^\delta(u_+),
\]

\[
= \delta^4 \cdot (F^\delta(u_+) \cdot G^\delta(v_3) \cdot F^\delta(u_-) \cdot G^\delta(v_3))^2 \leq 0,
\]

since \( F^\delta(u_+) \cdot F^\delta(u_-) \in i\mathbb{R} \) and \( G^\delta(v_3) \cdot G^\delta(v_3) \in \mathbb{R} \).

\[\square\]

**Remark 12.**

1. The function \( H^\delta \) satisfies the maximum principle:

\[
\max_{z \in V^\delta} H^\delta(z) = \max_{z \in \partial V^\delta} H^\delta(z).
\]

2. Also, it is easy to see that \( H^\delta \) satisfies the following non-linear equation:

\[
(H^\delta(z) - H^\delta(z_1)) \cdot (H^\delta(z) - H^\delta(z_2)) + (H^\delta(z) - H^\delta(z_3)) \cdot (H^\delta(z) - H^\delta(z_4)) = 0,
\]

where \( z, z_1, z_2, z_3, z_4 \) are defined as shown in Fig. 11.

It is worth noting that Definition 8 coincides with the definition of a primitive of the product of two s-holomorphic functions used in [32]. To see this let us divide the set vertex \( V \) into two sets \( V_0 \) and \( V_\bullet \), as it shown on Fig. 11. On the set \( V_\bullet \), the function \( H_{s-hol} \) defined below as a discrete integral of the product of two discrete s-holomorphic functions coincides with the function \( H \) defined above.

Let \( F : \bigcirc \to \mathbb{C} \) and \( G : \bigtriangledown \to \mathbb{C} \) be discrete holomorphic functions defined above. Let \( F_{s-hol} \) be a function defined as follows:

\[
\begin{cases}
F_{s-hol}(u) = F(u) & \text{if } u \in \bigcirc; \\
F_{s-hol}(u_\lambda) = \frac{\lambda}{\sqrt{2}} \cdot (F(u_\lambda) - iF(u_1)) & \text{if } u_\lambda \in \bigtriangledown_0; \\
F_{s-hol}(z) = F(u_R) + F(u_1) & \text{if } z \in V_0; \\
F_{s-hol}(u_\lambda) = \frac{\lambda}{\sqrt{2}} \cdot (F(u_R) + iF(u_1)) & \text{if } u_\lambda \in \bigtriangledown_1,
\end{cases}
\]

where \( z \in V_0 \) and \( u_1, u_\lambda, u_R, u_\lambda \) are adjacent to the vertex \( z \) squares (see Fig. 11).

Let us similarly define a function \( G_{s-hol} \):

\[
\begin{cases}
G_{s-hol}(v) = G(v) & \text{if } v \in \bigtriangledown; \\
G_{s-hol}(u_R) = \left( \frac{\lambda G(v_\lambda) + \lambda G(v_\lambda)}{\sqrt{2}} \right) & \text{if } u_R \in \bigcirc_0; \\
G_{s-hol}(z) = F(v_\lambda) + F(v_\lambda) & \text{if } z \in V_0; \\
G_{s-hol}(u_1) = i \cdot \left( \frac{\lambda G(v_\lambda) - \lambda G(v_\lambda)}{\sqrt{2}} \right) & \text{if } u_1 \in \bigtriangledown_1.
\end{cases}
\]
Note that functions $F|_{\mathcal{V}_o}$ and $G|_{\mathcal{V}_o}$ are $s$-holomorphic functions on $\mathcal{V}_o$, i.e. for each pair of white vertices $z_1^o, z_2^o$ of the same square $a$

$$\text{Proj}_{\tau(a)}[F(z_1)] = \text{Proj}_{\tau(a)}[F(z_2)],$$

where $\text{Proj}_{\tau(a)}[z] = \tau(a) \cdot \text{Re} \left[ z \cdot \bar{\tau}(a) \right]$ and $\tau(a)$ is 1, i, $\lambda$ or $\bar{\lambda}$ if the square $a$ is a square of type $\blacklozenge_a$, $\blacklozenge_1$, $\blacklozenge_3$ or $\blacklozenge_4$ correspondingly.

Let $H_{s\text{-hol}} : \mathcal{V}_o \rightarrow \mathbb{R}$ be a function defined by the equality

$$H_{s\text{-hol}}(z_2^2) - H_{s\text{-hol}}(z_1^2) = F_{s\text{-hol}}(a) \cdot G_{s\text{-hol}}(a) \cdot (z_2^2 - z_1^2),$$

where $z_1^2, z_2^2$ are two black vertices of the same square $a$. It is easy to check that

$$H_{s\text{-hol}}(z_2^2) - H_{s\text{-hol}}(z_1^2) = (H(z_2^2) - H(z^o)) + (H(z^o) - H(z_1^2)),$$

where $z^o$ is one of two white vertices of the square $a$. Note that the function $H_{s\text{-hol}}(\cdot)$ is defined up to an additive constant. One can choose the additive constant such that the function $H_{s\text{-hol}}$ coincides with the function $H|_{\mathcal{V}_\bullet}$.

### 3.3. The expectation of the double dimer height function

In the rest of section 3, we will use the square lattice with mesh size 1 rather than $\delta$. For the simplicity of notations we will not write the index $\delta$. (Later, in section 4, we are going to use notations without index for continuous objects.) We will prove that the function $H$ defined by formula (3.1) with an appropriate choice of functions $F$ and $G$ described above is the expectation of the height function for double dimers up to a multiplicative constant.

**Lemma 13.** 1. Let a domain $\Omega$ allow a domino tiling. Suppose that a discrete holomorphic function $F : \blacklozenge \rightarrow \mathbb{C}$ vanishes on $\partial \blacklozenge$. Then $F$ is identically zero.

2. Let $\Omega$ be a domain which contains $m$ white squares and $m + 1$ black squares. Let the domain have a domino tiling after removing one black square from $\partial_{\text{int}} \blacklozenge$. Then there exists a nontrivial discrete holomorphic function $F : \blacklozenge \rightarrow \mathbb{C}$, which is equal to zero on $\partial \blacklozenge$. Such a function $F$ is unique up to a multiplicative constant. Moreover $F(u) \neq 0$ for all black squares $u \in \partial_{\text{int}} \blacklozenge$ such that $\Omega \setminus u$ allows a domino tiling.

**Proof.** 1. Consider a system of linear equations with variables that correspond to values of $F$ in the black faces, and each equation means that the function $F$ is holomorphic in some white face. The number of variables is equal to the number of black faces, and the number of equations is equal to the number of white faces. So we have a linear system with a square matrix. To prove that the system has only the trivial solution it is enough to show that the determinant of the matrix is not equal to zero. Note that the absolute value of the determinant is equal to the number of the domino
tilings of $\Omega$, since the matrix is the Kasteleyn matrix of $\Omega$. Hence, if the domain has a domino tiling then the determinant is not zero. Therefore $F \equiv 0$.

2. We can consider a system of linear equations in the same way as described above. Note that in this case the number of variables is one more than the number of equations. Hence the system has a non-trivial solution. Let $F$ have the values which correspond to this solution. Let $u'$ be a square in $\partial_{\text{int}}\diamondsuit$ and let the domain $\Omega \smallsetminus u'$ have a domino tiling. Let $F$ be equal to zero at $u'$. Note that the function $F$ satisfies the conditions of the first part of the lemma, therefore $F \equiv 0$ on $\Omega$. We obtain a contradiction with a non-triviality of the solution of our system. Similarly to the proof of the first part of the lemma we can show that there is a unique discrete holomorphic function $F$ such that $F(u') = 1$. \hfill $\square$

**Corollary 14.** Let a domain $\Omega$ contain the same number of black and white squares, and let $\Omega$ allow a domino tiling. Fix a black square $u_0 \in \partial_{\text{int}}\diamondsuit_0$ and a white square $v_0 \in \partial_{\text{int}}\diamondsuit_0$ such that the domain $\Omega \smallsetminus \{u_0, v_0\}$ allows a domino tiling. Then the following holds:

1. There exists a unique function $F: \diamondsuit \to \mathbb{C}$ such that $F|_{\partial \diamondsuit} = 0$ and $F$ is discrete holomorphic everywhere in $\diamondsuit$ except at the face $v_0$ where one has $\bar{\partial} F(v_0) = \lambda$. Moreover, $F(u_0) \neq 0$.

2. Similarly, there exists a unique function $G: \hat{\diamondsuit} \to \mathbb{C}$ such that $G|_{\partial \hat{\diamondsuit}} = 0$ and $G$ is discrete holomorphic everywhere in $\hat{\diamondsuit}$ except at the face $u_0$ where one has $\bar{\partial} G(u_0) = i\nu$. Moreover, $G(v_0) \neq 0$.

**Proof.** Due to Lemma 13 the function $F$ on $\Omega \smallsetminus v_0$ is unique up to a multiplicative constant. Moreover, $F(u_0) \neq 0$ since the domain $\Omega \smallsetminus \{u_0, v_0\}$ allows a domino tiling. Therefore, $\bar{\partial} F(v_0) \neq 0$ (otherwise $F$ is identically zero due to Lemma 13). Finally, the condition $\bar{\partial} F(v_0) = \lambda$ defined the function $F$ uniquely. \hfill $\square$

In the setup of Corollary 14 we construct the function $H$ defined on the vertex set of the domain $\Omega \smallsetminus \{u_0, v_0\}$ as described in Section 3.2. So, the formula (3.1) holds for all square edges of the domain $\Omega$ except boundary edges of the squares $u_0$, $v_0$. Note that if $u_0$ and $v_0$ are not corner squares of the domain $\Omega$, then the vertex set of the domain $\Omega \smallsetminus \{u_0, v_0\}$ and the vertex set of the domain $\Omega$ are the same. Define $\partial \Omega = (u_0v_0) \cup (v_0u_0)$, see Fig. 9. It is clear that $H$ is constant on each of boundary segments. Recall that $H$ is defined up to an additive constant, which can be chosen so that $H|_{\{u_0v_0\}} \equiv 0$.

**Lemma 15.** The value of the function $H$ on the boundary segment $(u_0v_0)$ equals

$$
H|_{(u_0v_0)} = 4iG(v_0)|\bar{\partial} F(v_0) = -4iF(u_0)|\bar{\partial} G(u_0) \neq 0.
$$

**Proof.** Consider the difference between the values of the function $H$ in boundary vertices of the square $v_0$:

$$(H(z_0) - H(z_0^+)) = (H(z_2) - H(z_2^+)) + (H(z_-) - H(z_-^+)) + (H(z_3) - H(z_3^+))$$

$$= G(v_0)(-\bar{\lambda} F(u_1) - \lambda F(u_2) + \bar{\lambda} F(u_3))$$

$$= 4iG(v_0)|\bar{\partial} F(v_0),$$

where $u_1$, $u_2$, $u_3$, $z_+$, $z_-$, $z_0^+$, $z_0^-$ and $v_0$ are defined as shown in Fig. 9.

The second expression for $H|_{(u_0v_0)}$ can be obtained in a similar way. Finally, $H|_{(u_0v_0)} \neq 0$ since $G(v_0) \neq 0$. \hfill $\square$

Recall that we can think about the inverse Kasteleyn matrix $C_\Omega(u, v)$ as a function of two variables $u \in \diamondsuit$ and $v \in \hat{\diamondsuit}$. If $v \in \hat{\diamondsuit}_0$, then $C_\Omega(u, v)$ is a discrete holomorphic function of $u$, with a simple pole at $v$:

$$4\bar{\lambda}\bar{\partial}[C_\Omega(u, v)](v) = C_\Omega(v + \lambda, v) - C_\Omega(v - \lambda, v) + iC_\Omega(v + \bar{\lambda}, v) - iC_\Omega(v - \bar{\lambda}, v) = 1,$$

since the product of the Kasteleyn matrix and the inverse Kasteleyn matrix is equal to the identity matrix.
Let $\Omega' = \Omega \setminus \{u_0, v_0\}$. Recall that $C_{\text{dbl-d}, \Omega}(u, v) = C_\Omega(u, v) - C_{\Omega'}(u, v)$.

**Proposition 16.** Let $u \in \diamondsuit$ and $v \in \lozenge$, then the following identity holds

$$C_\Omega(u, v) - C_{\Omega'}(u, v) = \text{const} \cdot F(u)G(v),$$

where $\text{const} = \frac{1}{4G(v_0)}$.

**Proof.** For a fixed $\bar{v} \in \lozenge$, consider $C_\Omega(u, \bar{v}) - C_{\Omega'}(u, \bar{v})$ as a function of $u$. This function is holomorphic at all faces in $\diamondsuit \setminus v_0$. Moreover $\partial_u(C_\Omega - C_{\Omega'})(u, \bar{v}) \neq 0$, since otherwise $C_\Omega(u, \bar{v}) - C_{\Omega'}(u, \bar{v}) \equiv 0$ from Lemma 13. Hence, for fixed $\bar{v} \in \lozenge$ this difference is equal to $F(u)$ up to a multiplicative constant. So,

$$C_\Omega(u, \bar{v}) - C_{\Omega'}(u, \bar{v}) = k_1 \cdot F(u),$$

where $k_1$ depends on $\bar{v}$.

Similarly, for a fixed $\bar{u} \in \diamondsuit$, consider $C_\Omega(\bar{u}, v) - C_{\Omega'}(\bar{u}, v)$ as a function of $v$. We obtain that $C_\Omega(\bar{u}, v) - C_{\Omega'}(\bar{u}, v) = k_2 \cdot \lambda G(v)$, where $k_2$ depends on $\bar{u}$.

Therefore

$$C_\Omega(u, v) - C_{\Omega'}(u, v) = \text{const} \cdot F(u)G(v).$$

Consider $C_\Omega(u, v_0) - C_{\Omega'}(u, v_0)$ as a function of $u$. Note that

$$C_{\Omega'}(u, v_0) \equiv 0.$$

Hence

$$C_\Omega(u, v_0) = \text{const} \cdot F(u)G(v_0).$$

Recall that

$$4\partial_u[C_\Omega(u, v_0)](v_0) = \lambda.$$

Thus, $\text{const} = \frac{1}{4G(v_0)}$. \hfill \Box

**Corollary 17.** Let $h$ be the height function in the double-dimer model on the vertices of the domain $\Omega$. Then for all $z \in \mathcal{V}$ the following equality holds

$$\mathbb{E}[h(z)] = H(z) \cdot H|_{(u_0v_0)}^{-1},$$

where the value $H|_{(u_0v_0)}$ is given in Lemma 15.

**Proof.** Let $h_\Omega$ and $h_{\Omega'}$ be height functions in the dimer model on domains $\Omega$ and $\Omega'$, i.e. $h = h_{\Omega} - h_{\Omega'}$. Recall that the probability that there is a domino $[uv]$ in the domino tiling of $\Omega$ is equal to $|C_\Omega(u, v)|$. It is easy to see, that

$$\mathbb{E}[h_{\Omega}(z_1) - h_{\Omega}(z_2)] = \frac{3}{4} \cdot \mathbb{P}[uv] + (-\frac{1}{4}) \cdot (1 - \mathbb{P}[uv]),$$

where $u$, $v$ are adjacent squares; and $z_1$, $z_2$ are their common vertices. Therefore,

$$\mathbb{E}[h_{\Omega}(z_1) - h_{\Omega}(z_2)] = \mathbb{P}[uv] - \frac{1}{4} = |C_{\Omega}(u, v)| - \frac{1}{4}.$$

Similarly, $\mathbb{E}[h_{\Omega'}(z_1) - h_{\Omega'}(z_2)] = |C_{\Omega'}(u, v)| - \frac{1}{4}$.

So, $\mathbb{E}[h(z_1) - h(z_2)] = |C_{\Omega}(u, v)| - |C_{\Omega'}(u, v)|$.

Note that for $u_1$, $u_2$, $u_3$, $u_4$ and $v$ defined as shown on Fig. 10 the following equality holds:

$$1 = \mathbb{P}[u_1v] + \mathbb{P}[u_2v] + \mathbb{P}[u_3v] + \mathbb{P}[u_4v] = |C_{\Omega}(u_1, v)| + |C_{\Omega}(u_2, v)| + |C_{\Omega}(u_3, v)| + |C_{\Omega}(u_4, v)|.$$

Moreover,

$$C_\Omega(u_2, v) + iC_\Omega(u_3, v) - C_\Omega(u_4, v) - iC_\Omega(u_1, v) = 1,$$
since the product of the Kasteleyn matrix and the inverse Kasteleyn matrix is equal to the identity matrix. Therefore

\[ |C_{\Omega}(u,v)| - |C_{\Omega'}(u,v)| = \tau(uv) \cdot (C_{\Omega}(u,v) - C_{\Omega'}(u,v)), \]

where \(\tau(uv)\) is the Kasteleyn weight of the edge \((uv)\). To complete the proof it is enough to apply Proposition 16. \(\square\)

3.4. Proof of Theorem 11. We call a discrete domain an odd Temperley domain if all its corner squares are of type \(\diamond_o\). Recall that to obtain a Temperley domain one should remove a square of type \(\diamond_o\) from the set \(\partial_{int}\) from an odd Temperley domain, see Fig. 5. A Temperley domain always admits a domino tiling.

We need to adjust the notation from the previous section to this setup. Corollary 14 is stated for the case of the domain containing the same number of black and white squares. If we consider a discrete domain in which the number of black squares is greater by one than the number of white squares (see. Fig. 5), then we have some differences in definitions of functions \(F\) and \(G\). Fix two black squares \(u_1, u_2 \in \partial_{int}\) in such a way, that after removing one of them the resulting domain allows a domino tiling. Let \(u_1 \in \diamond\).

1. There exists a unique function \(F: \diamond \to \mathbb{C}\) such that \(F|_{\partial \diamond} = 0\), \(F(u_1) = 1\) and \(F\) is discrete holomorphic everywhere in \(\diamond\).
2. There exists a unique function \(G: \diamond \to \mathbb{C}\) such that \(G|_{\partial \diamond} = 0\) and \(G\) is discrete holomorphic everywhere in \(\diamond\) except at faces \(u_1, u_2\) and one has \([\partial G](u_1) = i\).

The existence and the uniqueness of functions \(F\) and \(G\) follow from Lemma 13.

**Proposition 18.** Let \(\Omega\) be an odd Temperley domain. Then the expectation of the double-dimer height function on \(\Omega\) is a discrete leap-frog harmonic function.

**Proof.** Note that Proposition 16 and Corollary 17 are still true in odd case. So, it is enough to show that \(H\) is a discrete leap-frog harmonic function. This follows directly from Proposition 10. In this case \(F\) is a discrete holomorphic function at all white squares of \(\Omega\). So, its imaginary part is a discrete harmonic function with zero boundary conditions. Therefore \(\Im F\) is identically zero, and thus the real part of \(F\) is a constant. Hence, \(\partial F\) is identically zero. \(\square\)

4. Double-dimer height function in polygonal domains

From now onwards, we will use the square lattice with mesh size \(\delta\) rather than 1. Let \(\Omega\) be a polygon in \(\mathbb{C}\) with sides parallel to vectors \(\lambda\) and \(\bar{\lambda}\). For each sufficiently small \(\delta > 0\), let \(\Omega^\delta\) be a discrete polygon approximating \(\Omega\) on the square lattice with mesh size \(\delta\).

Let us define functions \(F^\delta\) and \(G^\delta\) similarly to the previous section:

1. The function \(F^\delta\) is discrete holomorphic everywhere in \(\hat{\diamond}^\delta\) except at the face \(u_0^\delta\) where one has \([\bar{\partial}^\delta F^\delta](u_0^\delta) = \frac{1}{\delta^2}\).
2. Similarly, the function \(G^\delta\) is discrete holomorphic everywhere in \(\hat{\diamond}^\delta\) except at the face \(u_0^\delta\) where one has \([\bar{\partial}^\delta G^\delta](u_0^\delta) = \frac{1}{\delta^2}\).

Our goal is to prove the convergence of the functions \(H^\delta\) defined by the formula (3.1). Recall that this definition can be thought of as \(“H^\delta = \int^\delta \Re[F^\delta G^\delta dz]\”\). We will prove that the functions \(F^\delta\) and \(G^\delta\) converge individually.

To prove the convergence of the functions \(F^\delta\) we will consider approximations by domains \(\Omega^\delta\) with fixed colour type of the corners. We will describe this classification bellow. The limits of the functions \(F^\delta\) and \(G^\delta\) depend on the type of the corners. At the same time the limit of the functions \(H^\delta\) does not depend on the type of the corners.
We will call a corner of $\Omega^\delta$ a **convex** corner if the interior angle is $\pi/2$, and **concave** if the interior angle is $3\pi/2$. A corner is called **white** if there is a white square in the corner, and **black** if there is a black square in this corner, see Fig. 9.

**Lemma 19.** If a simply connected domain $\Omega^\delta$ contains the same number of black and white squares then
\[
\#\{\text{white convex corners}\} = \#\{\text{white concave corners}\} + 2,
\]
\[
\#\{\text{black convex corners}\} = \#\{\text{black concave corners}\} + 2.
\]

*Proof.* Note that $\pi \cdot (\#\{\text{corners}\} - 2) = \frac{\pi}{2} \cdot \#\{\text{convex corners}\} + \frac{3\pi}{2} \cdot \#\{\text{concave corners}\}$, hence
\[
\#\{\text{convex corners}\} = \#\{\text{concave corners}\} + 4.
\]

Recall that the height along boundary changes by $\pm \frac{1}{2}$; if an edge has a black square on its left then the height increases by $\frac{1}{2}$; if it has a white square on its left then it decreases by $\frac{1}{2}$. Then
\[
\#\{\text{white convex corners}\} + \#\{\text{black concave corners}\} = 
\]
\[
\#\{\text{white convex corners}\} + \#\{\text{black convex corners}\},
\]
since the height function on the boundary is well defined if the domain contains the same number of black and white squares (this is easily proved by induction on the number of black squares, starting from the case of a $2 \times 1$ rectangle). \qed

Let $\Omega^\delta$ allow a domino tiling. Let $u_0^\delta$ and $v_0^\delta$ be black and white squares in $\partial_{\text{int}} \Omega^\delta$ placed away from the corners of $\Omega^\delta$ in such a way that the domain $\Omega^\delta \setminus \{u_0^\delta, v_0^\delta\}$ allows a domino tiling. Let $\{v_k^\delta\}_{k=1}^{n-1}$ be the set of white squares located in the concave white corners of the domain $\Omega^\delta$, and let $\{v_k^\delta\}_{k=1}^{n+1}$ be the set of white squares located in the convex white corners of the domain $\Omega^\delta$, see Fig. 9. Recall that the cardinality of the latter set is greater by two than the cardinality of the former due to Lemma 19. Similarly, let $\{\tilde{u}_k^\delta\}_{k=1}^{m-1}$ be the set of black squares located in the concave black corners of the domain $\Omega^\delta$, and let $\{\tilde{u}_k^\delta\}_{k=1}^{m+1}$ be the set of black squares located in the convex black corners of the domain $\Omega^\delta$ (see Fig. 9).

4.1. **Discrete boundary value problem for the functions** $F$ and $G$. Note that for all $u^\delta \in \partial^\delta$, one has $F^\delta(u^\delta) = 0$, which can be thought of as a zero Dirichlet boundary conditions either for $\text{Re}[F^\delta]$ or for $\text{Im}[F^\delta]$. Similarly, for all $v^\delta \in \partial^\delta$, either $\text{Re}[\lambda G^\delta]$ or $\text{Re}[\lambda G^\delta]$ has zero Dirichlet boundary conditions.

**Remark 20.** The function $F^\delta$ (resp., $G^\delta$) changes boundary conditions only at white (resp., black) corners of $\Omega^\delta$.

A function on a discrete domain $\Omega^\delta$ is called **semibounded** in a subdomain $U^\delta \subset \Omega^\delta$ if either the maximum or the minimum of this function in $U^\delta$ is attained on the boundary of $U^\delta$. A function on a discrete domain $\Omega^\delta$ is called **bounded** in a subdomain $U^\delta \subset \Omega^\delta$ if both, the maximum and the minimum of this function in $U^\delta$, are attained on the boundary of $U^\delta$.

**Lemma 21.** The function $F^\delta|_{\phi_0^\delta}$ is bounded in neighbourhoods of white convex corners and semi-bounded in neighbourhoods of white concave corners.

*Proof.* Note that the function $F^\delta|_{\phi_0^\delta}$ is discrete harmonic in $\phi_0^\delta$, except at the squares of type $\phi_0^\delta$ adjacent to $\{v_k^\delta\}$ and $v_0^\delta$, where $\{v_k^\delta\}$ is the set of white squares in the white concave corners. In particular, the function $F^\delta$ is bounded in vicinities of white convex corners $\{v_k^\delta\}$, see Fig. 12. Let us consider a neighbourhood of a corner $\tilde{v}_k^\delta$. Note that in this neighbourhood the function $F^\delta|_{\phi_0^\delta}$ is discrete harmonic everywhere except at the unique black square of type $\phi_0^\delta$ adjacent to $\tilde{v}_k^\delta$. 

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Note that at this square either the maximum or the minimum of $F^\delta$ can be reached, thus $F^\delta|_{\phi_{\delta}}$ is semi-bounded near $\tilde{\phi}_{\delta,k}$.

\[ \text{Figure 12. Discrete harmonicity of the function } F^\delta \text{ together with the boundary conditions implies the following equations for } u \in \partial_{\text{int}} \Omega^\delta, \text{ see also Fig. 3.} \]

4.2. The continuous analogue of the functions $F^\delta$ and $G^\delta$. A real-valued function $F$ is called semi-bounded in a subdomain $U \subset \Omega$ if either the maximum or the minimum of the function in $U$ is attained on the boundary of $U$. A function on $\Omega$ is called bounded in a subdomain $U \subset \Omega$ if both, the maximum and the minimum of the function, are attained on the boundary of $U$.

A complex-valued function $F$ is called semi-bounded in the domain $\Omega$ if both $\Re F$ and $\Im F$ are semi-bounded functions.

**Proposition 22.** Let $\Omega$ be a simply connected Jordan domain with smooth boundary. Let $v_0$ be a boundary point which lies on a straight segment of the boundary of $\Omega$, and this segment goes to the direction $\lambda$. Let $\{v_k^*\}_{k=1}^{n+1} \cup \{\tilde{v}_k\}_{k=1}^{n-1}$ be a set of marked points on $\partial \Omega \setminus \{v_0\}$. Then there exists a unique holomorphic function $f_\Omega$ on $\Omega$ such that:

- $f_\Omega(z) = \frac{\lambda}{z-v_0} + O(1)$ in a vicinity of the point $v_0$;
- $f_\Omega$ is bounded in vicinities of the points $v_k^*$;
- $f_\Omega$ is semi-bounded in vicinities of the points $\tilde{v}_k$;
- along each boundary arc between marked points $\{v_k^*\}_{k=1}^{n+1} \cup \{\tilde{v}_k\}_{k=1}^{n-1}$, one has either $\Re[f_\Omega] = 0$ or $\Im[f_\Omega] = 0$;
- aforementioned boundary conditions change at all marked points $\tilde{v}_k$ and $v_k^*$.

**Proof.** Let $\phi$ be a conformal mapping of the domain $\Omega$ onto the upper half plane $\mathbb{H}$ such that none of the marked points and $v_0$ is mapped onto infinity. Then $f_{\mathbb{H}} := f_\Omega \circ \phi^{-1}$ is a holomorphic function on $\mathbb{H}$, which satisfies the following conditions:

1. $f_{\mathbb{H}}(w) = \frac{\lambda \phi(v_0)}{w-\phi(v_0)} + O(1)$ in a vicinity of the point $\phi(v_0)$;
2. $f_{\mathbb{H}}$ is bounded in vicinities of the points $\phi(v_k^*)$;
3. $f_{\mathbb{H}}$ is semi-bounded in vicinities of the points $\phi(\tilde{v}_k)$;
4. $f_{\mathbb{H}}$ is bounded at infinity;
5. on each segment of the real line between the points of the set $\{\phi(\tilde{v}_k)\}_{k=1}^{n-1} \cup \{\phi(v_k^*)\}_{k=1}^{n+1}$ one has either $\Re[f_{\mathbb{H}}] = 0$ or $\Im[f_{\mathbb{H}}] = 0$;
6. the function $f_{\mathbb{H}}$ changes the boundary conditions at all points $\phi(\tilde{v}_k)$ and $\phi(v_k^*)$, and only at these points.

Without lost of generality assume that $\phi(\tilde{v}_k) = 0$ for some $k$. Let us consider a function $f_{\mathbb{H}}(w^2)$ in a vicinity of zero. Let us show that $f_{\mathbb{H}}(w^2) = O(1/w)$ as $w \to 0$. The boundary conditions (5), (6) of the function $f_{\mathbb{H}}(w^2)$ allow one to extend this function to a punctured vicinity of 0 by Schwarz reflection principle.
Great Picard’s Theorem together with the semi-boundedness condition (3) implies that \( f_\Omega(w^2) \) cannot have an essential singularity at zero. So, the function \( f_\Omega(w^2) \) either is regular or has a pole at zero. This pole must be simple due to (3), and hence \( f_\Omega(w) = O\left((w - \phi(\tilde{v}_k))^{-\frac{1}{2}}\right) \) in a vicinity of \( \tilde{v}_k \).

Similarly, conditions (2), (5) and (6) imply that \( f_\Omega(w) = O\left((w - \phi(v_k^s))^\frac{1}{2}\right) \) in a vicinity of each of the points \( v_k^s \).

Consider a function

\[
f_\Omega(w) = \frac{c_\phi}{w - \phi(v_0)} \cdot \prod_{k=1}^{n-1} (w - \phi(\tilde{v}_k))^\frac{1}{2} \cdot \prod_{k=1}^{m} (w - \phi(v_k^s))^\frac{1}{2} \cdot \prod_{k=1}^{n} (w - \phi(v_k^s))^{-\frac{1}{2}},
\]

which can be extended to a bounded function in the whole plane by the Schwarz reflection principle. Hence it is a constant, and

\[
f_\Omega(w) = \frac{c_\phi}{w - \phi(v_0)} \cdot \prod_{k=1}^{n-1} (\phi(z) - \phi(v_0))^\frac{1}{2} \cdot \prod_{k=1}^{m} (\phi(z) - \phi(v_k^s))^\frac{1}{2} \cdot \prod_{k=1}^{n} (\phi(z) - \phi(v_k^s))^{-\frac{1}{2}},
\]

where the real constant \( c_\phi \) can be determined from the condition (1).

Since \( f_\Omega = f \circ \phi^{-1} \), we obtain

\[
f_\Omega(z) = \frac{c_\phi}{(\phi(z) - \phi(v_0))} \cdot \prod_{k=1}^{n-1} (\phi(z) - \phi(v_0)) \cdot \prod_{k=1}^{m} (\phi(z) - \phi(v_k^s))^\frac{1}{2} \cdot \prod_{k=1}^{n} (\phi(z) - \phi(v_k^s))^{-\frac{1}{2}}, \tag{4.1}
\]

where \( c_\phi \) is a real constant that depends on \( \phi \).

Similarly, for the set of boundary points \( \{\tilde{u}_k\}_{k=1}^{m-1} \cup \{u_k^s\}_{s=1}^{m} \) and the point \( u_0 \) on a straight segment of the boundary of \( \Omega \) parallel to vector \( \lambda \), there exists a unique holomorphic function \( g \), which satisfies conditions analogous to conditions from Proposition 22.

\[
\begin{align*}
\triangleright & \quad g_\Omega(z) = \frac{1}{z - u_0} + O(1) \text{ in a vicinity of the point } u_0; \\
\triangleright & \quad g_\Omega \text{ is bounded in vicinities of the points } u_k^s; \\
\triangleright & \quad g_\Omega \text{ is semi-bounded in vicinities of the points } \tilde{u}_k; \\
\triangleright & \quad \text{along each boundary segment between boundary points of the set } \{u_k^s\}_{k=1}^{m} \cup \{\tilde{u}_k\}_{k=1}^{m-1}, \text{ one has either } \text{Re}[\lambda g_\Omega] = 0 \text{ or } \text{Re}[\lambda g_\Omega] = 0; \\
\triangleright & \quad \text{aforementioned boundary conditions of the function } g_\Omega \text{ change at all points } \tilde{u}_k \text{ and } u_k^s.
\end{align*}
\]

This function is written as follows

\[
g_\Omega(z) = \frac{\lambda \tilde{c}_\phi}{(\phi(z) - \phi(u_0))} \cdot \prod_{k=1}^{m} (\phi(z) - \phi(u_k^s))^\frac{1}{2} \cdot \prod_{k=1}^{n} (\phi(z) - \phi(\tilde{u}_k))^{-\frac{1}{2}}, \tag{4.2}
\]

where \( \tilde{c}_\phi \) is a real constant that depends on \( \phi \).

It is worth noting that the product of the functions \( f_\Omega(z) \) and \( g_\Omega(z) \) defined by (4.1) and (4.2), respectively, does not depend on the colours of corners of \( \Omega \) (while each of \( f_\Omega(z) \), \( g_\Omega(z) \) does depend on these colours).

**Proposition 23.** Let \( \Omega \) be a polygon in \( \mathbb{C} \) with sides parallel to vectors \( \lambda \) and \( \tilde{\lambda} \). Let \( v_0 \) and \( u_0 \) be the points on the straight part of the boundary of the polygon \( \Omega \). Let \( \{v_k^s\}_{k=1}^{m} \cup \{u_k^s\}_{s=1}^{m} \) be the set of vertices of the convex corners of the polygon \( \Omega \), and \( \{\tilde{v}_k\}_{k=1}^{n-1} \cup \{\tilde{u}_s\}_{s=1}^{m} \) be the set of vertices of the concave corners of the polygon \( \Omega \). Assume that the boundary arc \( (u_0 v_0) \) contains 0.

Let functions \( f_\Omega \) and \( g_\Omega \) be defined as in Proposition 22, then the function

\[
\int_0^w \text{Re}[f_\Omega(z)g_\Omega(z)]dz
\]
is proportional to the harmonic measure $\text{hm}_\Omega(w,(v_0u_0))$ in the domain $\Omega$.

**Proof.** Let us consider the product of functions $f_\Omega(z)$ and $g_\Omega(z)$. It equals

$$f_\Omega(z) \cdot g_\Omega(z) = \frac{\lambda c_\Phi c_\tilde{\Phi}}{(\phi(z) - \phi(v_0)) \cdot (\phi(z) - \phi(u_0))} \times$$

$$\prod_{k=1}^{n+1} (\phi(z) - \phi(v_k))^{\frac{1}{2}} \cdot \prod_{k=1}^{m-1} (\phi(z) - \phi(u_k))^{\frac{1}{2}} \cdot \prod_{k=1}^{m+1} (\phi(z) - \phi(u_k^*))^{\frac{1}{2}} \cdot \prod_{k=1}^{m-1} (\phi(z) - \phi(\tilde{u}_k))^ {\frac{1}{2}}.$$ 

Let $\psi(w)$ be a conformal transformation of the upper half-plane onto the interior of a simple polygon $\Omega$, the inverse mapping to $\phi$. The Schwarz–Christoffel mapping theorem implies that

$$\psi'(w) = \lambda c_\Phi \cdot \prod_{k=1}^{n-1} (w - \phi(\tilde{v}_k))^{\frac{1}{2}} \cdot \prod_{k=1}^{m-1} (w - \phi(\tilde{u}_k))^{\frac{1}{2}} \cdot \prod_{k=1}^{m+1} (w - \phi(v_k^*))^{\frac{1}{2}} \cdot \prod_{k=1}^{m-1} (w - \phi(u_k^*))^{\frac{1}{2}},$$

where $c_\psi$ is a real constant.

Note that $\phi$ is the inverse mapping to $\psi$, so $\frac{1}{\psi'(\phi(z))} = \phi'(z)$.

Therefore

$$f(z) \cdot g(z) = \frac{\lambda c_\Phi c_\tilde{\Phi}}{\phi(z) - \phi(v_0)} \cdot \phi'(z) \cdot \phi'(z) = \frac{ic_\Phi c_\tilde{\Phi}}{\phi(v_0) - \phi(u_0)} \cdot \left( \log \left( \frac{\phi(z) - \phi(v_0)}{\phi(z) - \phi(u_0)} \right) \right)',$$

hence $\int \text{Re}[f(z)g(z)]$ is proportional to $\frac{1}{\pi} \text{Im} \log \left( \frac{\phi(z) - \phi(v_0)}{\phi(z) - \phi(u_0)} \right)$ which is the harmonic measure of $(v_0u_0)$.

□

Now to complete the proof of Theorem 2 it is enough to prove convergence of functions $F^\delta$ and $G^\delta$.

In Section 5 we will prove a more general result: the convergence of $F^\delta$ for approximations by black-piecewise Temperley domains. This special type of discrete domains is defined below in Section 5.1. Similarly, one can show the convergence of $G^\delta$ for approximations by white-piecewise Temperley domains. In the setup of Proposition 23 the polygonal approximations $\Omega^\delta$ are $2n$-black-piecewise Temperley and $2m$-white-piecewise Temperley domains at the same time.

5. **Convergence of $F^\delta$ in black-piecewise Temperley domains**

5.1. **Black-piecewise Temperley domains.** Let us fix a natural number $n$. A discrete domain is called a $2n$-black-piecewise Temperley domain if it is a domain with $n+1$ convex white corners and $n-1$ concave white corners. Consider a segment of the boundary between two neighbouring white corners; we will call such a segment a black Temperley segment. Note that all black squares
Figure 14. A path on the set $\Phi_0^\delta$ from the square $y^\delta$ to the square adjacent to $\tilde{v}_k^\delta$.

on this part of the boundary are of the same type: either they all are of type $\Phi_0^\delta$ or all of type $\Phi_1^\delta$ (see Fig. 13).

Let $\Omega$ be a bounded, simply connected Jordan domain with a piecewise-smooth boundary and $2n$ boundary marked points $v_1^\delta, \ldots, v_{n+1}^\delta, \tilde{v}_1, \ldots, \tilde{v}_{n-1}$. For sufficiently small $\delta$, we say that a $2n$-black-piecewise Temperley domain $\Omega^\delta$ approximates $\Omega$ if the boundaries of the $2n$-black-piecewise Temperley domain are within $O(\delta)$ of the boundaries of $\Omega$, and if furthermore, all convex white corners $v_k^\delta$ are within $O(\delta)$ of the set of marked points $v_k^\delta$ and all concave white corners $\tilde{v}_j^\delta$ are within $O(\delta)$ of the set of marked points $\tilde{v}_j$.

5.2. Proof of the convergence. Let $u^\delta$ be a square on the square lattice with mesh size $\delta$. By $B_r^\delta(u^\delta)$ we denote the set of squares on this lattice such that the distance from them to $u^\delta$ is less than or equal to $r$. Let $\partial B_r^\delta(u^\delta)$ be the set of boundary squares of the set $B_r^\delta(u^\delta)$.

Consider a discrete domain $\Omega^\delta$. Let $E^\delta$ be a subset of the set $\partial \Omega^\delta$. Let $\text{hm}_\Omega^\delta(x^\delta, E^\delta)$ be a discrete harmonic function in $\Omega^\delta$ such that it is equal to $\chi_{E^\delta}$ on the boundary of $\Omega^\delta$, where $\chi_{E^\delta}$ is the characteristic function of the set $E^\delta$. The function $\text{hm}_\Omega^\delta(x^\delta, E^\delta)$ is called a harmonic measure. Note that the harmonic measure is a probabilistic measure for any fixed $x^\delta \in \Omega^\delta$. Note also that the value of $\text{hm}_\Omega^\delta(x^\delta, E^\delta)$ equals to the probability that a simple random walk starting at $x$ first hits the boundary of the domain $\Omega^\delta$ on the set $E^\delta$.

Let $F^\delta$ be a discrete harmonic function in $\Omega^\delta$ defined on the set $\Omega^\delta \cup \partial \Omega^\delta$. Then it is easy to see that

$$F^\delta(x^\delta) = \sum_{y^\delta \in \partial \Omega^\delta} F^\delta(y^\delta) \cdot \text{hm}_\Omega^\delta(x^\delta, \{y^\delta\}).$$

Remark 24. From now on we assume that $\delta > 0$ and $r > 0$ are chosen so that the discrete punctured vicinity $B_{2r}^\delta(\tilde{v}_k^\delta) \setminus \{\tilde{v}_k^\delta\}$ contains neither $v_0^\delta$ nor white corner squares of $\Omega^\delta$ for all $k \in \{1, \ldots, n-1\}$.

Lemma 25. Let $x^\delta$ be a black square in the middle of one of the arcs of the set $\partial B_{r/2}^\delta(\tilde{v}_k^\delta) \cap \Phi_0^\delta$. Let $y^\delta \in \Phi_0^\delta$ be a black square on the boundary of $B_{r/2}^\delta(\tilde{v}_k^\delta)$. Let $\gamma^\delta$ be a path on the set $\Phi_0^\delta$ starting in $y^\delta$ and ending at the black square of $\Phi_0^\delta$ adjacent to $\tilde{v}_k^\delta$ (see Fig. 14). Let $\text{hm}_r^\delta(x^\delta, \gamma^\delta)$ be a harmonic measure on $B_{2r}^\delta(\tilde{v}_k^\delta) \cap \Phi_0^\delta \setminus \gamma^\delta$. Then there exists a constant $\bar{c} > 0$ that does not depend on $\delta$ such that for all $y^\delta \in \Phi_0^\delta \cap \partial B_{r/2}^\delta$, one has

$$\text{hm}_r^\delta(x^\delta, \gamma^\delta) \geq \bar{c} = \bar{c}(\Omega) > 0.$$ 

For a more general statement see Lemma 3.14 in [3].

Proof. Let us consider two gray discrete domains of width $\frac{r}{l}$, where $l$ is a large enough positive number, see Fig. 15. Let these domains contain $x^\delta$ and cross the boundary of $\Omega^\delta$. The probability
Figure 15. The probability that a random walk on a square lattice with mesh size $2\delta$ travels all the way from $x^\delta$ to the boundary of $\Omega^\delta$ inside the gray domain is separated from the zero.

that a random walk on a square lattice with mesh size $2\delta$ travels all the way from $x^\delta$ to the boundary of $\Omega^\delta$ inside the gray domain is separated from the zero.

Note that at least one of the gray domains necessarily intersects the path $\gamma^\delta$. The probability of the event that a random walk travels all the way from $x^\delta$ to the boundary of $\Omega^\delta$ inside this gray domain is less than $\text{hm}_t(x^\delta, \gamma^\delta)$.

\[ \square \]

Lemma 26. Let $M^\delta_t(r) = \max_{u^\delta \in \Omega^\delta_{r,t}} |F^\delta(u^\delta)|$, where

$$
\Omega^\delta_{r,t} = \Omega^\delta \setminus \left( \bigcup_{k=1}^{n-1} B^\delta_r(\tilde{v}^\delta_k) \cup B^\delta_t(v^\delta_0) \right).
$$

Then for some fixed sufficiently small $t > 0$, as $\delta \to 0$ we have

$$
M^\delta_t\left(\frac{r}{2}\right) \leq \frac{4}{c} \cdot M^\delta_t(r),
$$

where $c$ is the absolute constant from Lemma 25.

Proof. Note that it is enough to prove that

$$
\max_{u^\delta \in \Omega^\delta \cap \partial B^\delta_{r/2}(\tilde{v}^\delta_k)} |\text{Re}F^\delta(u^\delta)| \leq \frac{2}{c} \cdot M^\delta_t(r),
$$

for all $k \in \{1, \ldots, n-1\}$, since similarly the same inequality holds for $\text{Im}F^\delta$.

Let $y^\delta$ be the square in $\Omega^\delta \cap \partial B^\delta_{r/2}(\tilde{v}^\delta_k)$ such that

$$
|\text{Re}F^\delta(y^\delta)| = \max_{u^\delta \in \Omega^\delta \cap \partial B^\delta_{r/2}(\tilde{v}^\delta_k)} |\text{Re}F^\delta(u^\delta)|.
$$

Without lost of generality we may assume that $\text{Re}F^\delta(y^\delta) > 0$. Note that $\text{Re}F^\delta$ is a discrete harmonic function, and hence there exists a path $\gamma^\delta$ on the set $\Phi^\delta$ from $y^\delta$ to the boundary of the domain $\Omega^\delta \cap B^\delta_r(\tilde{v}^\delta_k)$ or to the square adjacent to $\tilde{v}^\delta_k$ along which the absolute value of the function $\text{Re}F^\delta$ increases. If the path $\gamma^\delta$ ends on the boundary of the domain $\Omega^\delta \cap B^\delta_r(\tilde{v}^\delta_k)$, then

$$
\max_{u^\delta \in \Omega^\delta \cap \partial B^\delta_{r/2}(\tilde{v}^\delta_k)} |\text{Re}F^\delta(u^\delta)| \leq \max_{u^\delta \in \Omega^\delta \cap \partial B^\delta_{r/2}(\tilde{v}^\delta_k)} |\text{Re}F^\delta(u^\delta)|.
$$
Assume that $\gamma^\delta$ ends at the square adjacent to $v_0^\delta$. Let $\text{hm}^\delta(\cdot, \gamma^\delta)$ be the harmonic measure in the domain $B_{2\delta}(\vec v_k^\delta) \cap \Phi_0^\delta \setminus \gamma^\delta$. Due to Lemma 25 there exists a black square $x^\delta \in \Phi_0^\delta$ on the boundary of $B_{\delta}(\vec v_k^\delta)$ such that $\text{hm}^\delta(x^\delta, \gamma^\delta) \geq \bar c > 0$. Note that

$$M^\delta_k(r) \geq \text{Re} F^\delta(x^\delta) \geq \text{Re} F^\delta(y^\delta) \cdot \text{hm}^\delta(x^\delta, \gamma^\delta) - M^\delta_k(r) \cdot (1 - \text{hm}^\delta(x^\delta, \gamma^\delta)).$$

Hence,

$$2M^\delta_k(r) \geq \text{hm}^\delta(x^\delta, \gamma^\delta) \cdot \text{Re} F^\delta(y^\delta) \geq \bar c \cdot \text{Re} F^\delta(y^\delta).$$

To complete the prove, recall that we assumed $\text{Re} F^\delta(y^\delta) = \max_{u^\delta \in \Omega^{2\delta}_{\gamma^\delta}} |\text{Re} F^\delta(u^\delta)|$. □

Let $F^\delta_{C, v_0^\delta}$ be a unique discrete holomorphic function on the whole plane $\mathbb{C} \setminus \{v_0^\delta\}$ tending to zero at infinity and such that $[\bar \partial F^\delta_{C, v_0^\delta}](v_0^\delta) = \frac{\lambda}{\delta}$. Note that $\text{Re} F^\delta_{C, v_0^\delta}$ and $\text{Im} F^\delta_{C, v_0^\delta}$ are discrete harmonic everywhere except two squares adjacent to $v_0^\delta$. It is well known that $F^\delta_{C, v_0^\delta}$ is asymptotically equal to $\frac{1}{2\pi} \cdot \frac{\lambda}{z-v_0^\delta}$ as $\delta \downarrow 0$. We need to introduce a similar function $F^\delta_{C, v_0^\delta}$ on a half-plane $\mathbb{H}^\delta$, where $\partial \mathbb{H}^\delta$ goes to the direction $\lambda$ and $v_0^\delta \in \partial \mathbb{H}^\delta \cap \gamma^\delta$. The imaginary part of $F^\delta_{C, v_0^\delta}$ equals zero on the boundary, and $[\bar \partial F^\delta_{C, v_0^\delta}](v_0^\delta) = \frac{\lambda}{\delta}$. There is a unique discrete holomorphic function with these two properties that tends to zero at infinity.

Let us consider the sum $F^\delta_{C, v_0^\delta} + F^\delta_{C, v_0^\delta+2\lambda \delta}$, where by $v_0^\delta + 2\lambda \delta$ we denote a white square on the distance $\delta$ from the square $v_0^\delta$ that does not belong to $\mathbb{H}^\delta$. This sum tends to zero at infinity, since both $F^\delta_{C, v_0^\delta}$ and $F^\delta_{C, v_0^\delta+2\lambda \delta}$ tend to zero at the infinity. Note that $F^\delta_{C, v_0^\delta+2\lambda \delta}$ is discrete holomorphic on $\mathbb{H}^\delta$, therefore $F^\delta_{C, v_0^\delta} + F^\delta_{C, v_0^\delta+2\lambda \delta}$ is holomorphic on $\mathbb{H}^\delta \setminus \{v_0^\delta\}$ and $[\bar \partial F^\delta_{C, v_0^\delta} + F^\delta_{C, v_0^\delta+2\lambda \delta}](v_0^\delta) = \frac{\lambda}{\delta}$. Finally, note that

$$\text{Im} F^\delta_{C, v_0^\delta}(u) = G^\delta(u, v_0^\delta+\lambda \delta) - G^\delta(u, v_0^\delta - \lambda \delta),$$

where $G^\delta(u, u')$ is the classical Green’s function on $\mathbb{C} \setminus \Phi_0^\delta$ satisfies $\Delta^\delta G^\delta(u, u') = \delta_{u=u'} \cdot \frac{1}{2\pi}$. The Green’s function is symmetric, therefore $\text{Im} [F^\delta_{C, v_0^\delta} + F^\delta_{C, v_0^\delta+2\lambda \delta}]$ vanishes on $\partial \mathbb{H}^\delta$. As a consequence we have $F^\delta_{C, v_0^\delta}(u) = F^\delta_{C, v_0^\delta} + F^\delta_{C, v_0^\delta+2\lambda \delta}$.

**Corollary 27.** Let

$$M^\delta_*(r) = \max_{u \in \Omega^\delta_{\gamma^\delta}} |F^\delta(u) - F^\delta_{\gamma^\delta}(u)|.$$ 

Then, for all sufficiently small $\delta$, one has

$$M^\delta_*(\frac{r}{2}) \leq \frac{4}{c} \cdot M^\delta_*(r) + C_*,$$

where $C_*$ is an absolute constant and $\bar c$ is the constant from Lemma 25.

**Proof.** Note that $F^\delta_{\gamma^\delta}$ is uniformly bounded away from $v_0^\delta$ and it vanishes on $\partial \mathbb{H}^\delta$, hence $F^\delta - F^\delta_{\gamma^\delta}$ is uniformly bounded on $\partial \Omega^\delta$. Moreover, function $F^\delta - F^\delta_{\gamma^\delta}$ is discrete holomorphic on $\Omega^\delta$, in particular it is discrete holomorphic on $B^\delta_{\gamma^\delta}(v_0^\delta) \cap \Omega^\delta$ and vanishes on $\partial \Omega^\delta \cap B^\delta_{\gamma^\delta}(v_0^\delta)$. Therefore the statement of Lemma 26 is valid for $F^\delta - F^\delta_{\gamma^\delta}$ and $t = 0$. □

We are now in the position to prove the convergence of $F^\delta$. Note that $F^\delta$ can be thought of as defined in polygonal representation of $\Omega^\delta$ by some standard continuation procedure, linear on edges and multilinear inside faces. Then we have the following
Theorem 28. Let $\Omega^\delta$ be a sequence of discrete $2k$-black-piecewise Temperley domains of mesh size $\delta$ approximating a continuous domain $\Omega$. Suppose that each $\Omega^\delta$ allows a domino tiling. Let the sets of white corner squares $\{v_k^\delta\}_{k=1}^{n+1}$ and $\{\tilde{v}_k^\delta\}_{k=1}^{n-1}$ approximate the sets of boundary points $\{v_k\}_{k=1}^{n+1}$ and $\{\tilde{v}_k\}_{k=1}^{n-1}$ correspondingly, and let $v_0^\delta$ approximate a boundary point $v_0$, which lies on a straight segment of the boundary of $\Omega$. Then $F^\delta$ converges uniformly on compact subsets of $\Omega$ to a continuous holomorphic function $f_\Omega$, where $f_\Omega$ is defined as in Proposition 22.

In the following proof we use the idea described in [2] (proof of Theorem 2.16).

Proof. The first case: suppose that $M^\delta(r)$ is bounded for all $r > 0$, as $\delta \to 0$. Corollary 27 implies that discrete holomorphic functions $F^\delta - F^\delta_r$ are uniformly bounded, and therefore equicontinuous due to Harnack principle on compact subsets of $\Omega$. Thus, due to the Arzela–Ascoli theorem, the family $F^\delta - F^\delta_r$ is precompact and hence converges along a subsequence to some holomorphic function $\bar{f}$ uniformly on compact subsets of $\Omega$. Note that $F^\delta_r \Rightarrow f_\Omega$, where $f_\Omega(z) = \frac{1}{\pi} \cdot \log \frac{\lambda}{z - v_0}$. Let $f_\Omega := \bar{f} - f_\Omega$, then $F^\delta \Rightarrow f_\Omega$, i.e. $\Re F^\delta \Rightarrow \Re f_\Omega$ and $\Im F^\delta \Rightarrow \Im f_\Omega$. Since a discrete solution of Dirichlet problem converges to its continuous counterpart up to the boundary, the boundary conditions for the functions $F^\delta$ yield the same boundary conditions for their limit. Thus, the function $f_\Omega$ solves the boundary value problem described in Proposition 22, therefore it is determined uniquely. This implies that all convergent subsequences of the family $\{F^\delta\}$ have the same limit and thus the whole family converges to $f_\Omega$.

The second case: suppose that $M^\delta(r)$ tends to infinity along a subsequence as $\delta \to 0$ for some $r > 0$. Let us show that this is impossible. Consider a discrete holomorphic function $\bar{F}^\delta = \frac{F^\delta - F^\delta_r}{M^\delta(r)}$.

Using the same arguments as above, we can show that the family $\bar{F}^\delta$ converges to some holomorphic function $f_\Omega$. Note that the limit is bounded near $v_0$, since $F^\delta - F^\delta_r$ is discrete holomorphic and bounded near $v_0$. Also, note that $\frac{F^\delta_r}{M^\delta(r)}$ tends to zero away from $v_0$. Therefore, as in the previous case, the limit satisfies all boundary conditions described in Proposition 22, except the first one: the behaviour near the point $v_0$. The only function satisfying these properties is zero.

Suppose that there exists a sequence of squares $\{u^\delta_{\text{inner}}\}$ converging to $u_{\text{inner}} \in \Omega$ such that

$$\Re \bar{F}^\delta(u^\delta_{\text{inner}}) > \text{const}_\Omega > 0.$$  \hfill (5.1)

Then we have $f_\Omega(u_{\text{inner}}) > 0$, which contradicts the fact that $f_\Omega$ vanishes on $\Omega$, and therefore the second case is impossible.

To complete the proof let us show the existence of the sequence $\{u^\delta_{\text{inner}}\}$. Let $u^\delta_{\text{max}}$ be chosen so that $1 = \sup_{u^\delta \in \Omega^\delta} |\bar{F}^\delta(u^\delta)| = |\bar{F}^\delta(u^\delta_{\text{max}})|$. Assume that $u^\delta_{\text{max}} \in \phi^\delta_0$, i.e. $|\bar{F}^\delta(u^\delta_{\text{max}})| = |\Re \bar{F}^\delta(u^\delta_{\text{max}})|$.

Without loss of generality we may assume that $\Re \bar{F}^\delta(u^\delta_{\text{max}}) > 0$. Let $\lim_{\delta \to 0} u^\delta_{\text{max}} = \Omega_{\gamma} = \Omega \setminus \left( \bigcup_{k=1}^{n-1} B_r(\tilde{v}_k) \right)$. The discrete maximum principle implies that $\lim_{\delta \to 0} u^\delta_{\text{max}} \subset \partial B_r(\tilde{v}_k)$.

Note that $\Re \bar{F}^\delta$ is a discrete harmonic function, and hence there exists a path $\gamma^\delta$ on the set $\phi^\delta_0$ from $u^\delta_{\text{max}}$ to the boundary of the domain $\Omega^\delta$ or to the square adjacent to $\tilde{v}_k^\delta$ along which the absolute value of the function $\Re \bar{F}^\delta$ increases. The boundary conditions together with the fact that the limit function vanishes imply that $\gamma^\delta$ goes along a subarc $N^\delta_k \subset \partial \Omega^\delta$ where $\Re F^\delta$ has Neumann boundary condition and ends at the square adjacent to $\tilde{v}_k^\delta$.

Assume that $B_r(\tilde{v}_k) \cap \Omega$ is connected, the other case is treated similarly. Denote by $U^\delta$ the discrete subdomain of $B^\delta_r(\tilde{v}_k) \cap \Omega^\delta$ that is bounded by the subarc of $\partial \Omega^\delta$ where $\Re F^\delta$ has Dirichlet boundary condition, the path $\gamma^\delta \cap \Omega^\delta$ and the arc $\partial B^\delta_r(\tilde{v}_k) \cap \Omega^\delta$. Note that $U^\delta$ converges to $B_r(\tilde{v}_k) \cap \Omega$. 

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The absolute value of $\text{Re}\tilde{F}_*^\delta$ is bounded by $\epsilon_\delta$ away from the pieces of the boundary of $\Omega^\delta$ where $\text{Re}\tilde{F}_*^\delta$ has Neumann boundary conditions. Note that the function $\text{Re}\tilde{F}_*^\delta$ is semi-bounded in a vicinity of the point $\tilde{v}_k^\delta$, therefore near the boundary $\text{Re}\tilde{F}_*^\delta > -c$, where $c > 0$ is a constant. Let $u_{\text{inner}}^\delta \in U^\delta$ be a black square in the middle of one of the arcs of the set $\partial B_{r/2}(\tilde{v}_k^\delta) \cap C^\delta$. Then

$$\text{Re}\tilde{F}_*^\delta(u^\delta) \geq -\epsilon_\delta \cdot 1 + (-c) \cdot \text{hm}_{U^\delta} (u^\delta, \text{vicinity of } N^\delta_k) \cap (\partial B_{r/2}(\tilde{v}_k^\delta) \cap \Omega^\delta) + 1 \cdot \text{hm}_{U^\delta} (u^\delta, \gamma^\delta \cap \partial U^\delta).$$

Due to Lemma 25 we have $\text{hm}_{U^\delta} (u^\delta, \gamma^\delta \cap \partial U^\delta) > \text{const}(U) > 0$. Note that $\epsilon_\delta$ tends to zero as $\delta \to 0$. Also, $\text{hm}_{U^\delta} (u^\delta, \text{vicinity of } N^\delta_k) \cap (\partial B_{r/2}(\tilde{v}_k^\delta) \cap \Omega^\delta)$ tends to zero as $\delta \to 0$. Hence we construct a sequence of squares $u_{\text{inner}}^\delta$ converging to $u_{\text{inner}} \in \Omega$ such that (5.1) holds.

\section*{Appendix A. Single dimer model and the Gaussian Free Field}

In [15] Kenyon proved that a scaling limit of the height function in the dimer model on Temperley domains is the Gaussian Free Field. Our goal in this section is to prove Corollary 4, i.e. to show

$$hm_{U^\delta} \rightarrow 0$$

as $\delta \rightarrow 0$. To this end, we use the technique described in Section 3. In particular, we can write these functions in the following way

$$f_0(z, w) = \prod_{k=1}^{n+1} \left( z - v_k^\delta \right)^{\frac{1}{2}} \cdot \prod_{k=1}^{n-1} \left( z - \tilde{v}_k \right)^{-\frac{1}{2}} \cdot \left( \frac{s(w)}{z - w} + \frac{s(w)}{z - w} \right),$$

where $s(z) = z^2 - 1$ and $\tilde{v}_k$ are the boundary vertices.

\begin{align*}
\frac{\partial}{\partial \delta} \text{Im} \left[ C\Omega(\cdot, v') \right](v) &= 0, \quad v' \in \Omega, \quad v \neq v' \\
\frac{\partial}{\partial \delta} \text{Im} \left[ C\Omega(\cdot, v') \right](v) &= 0, \quad \forall v \in \Omega, \quad v \neq v'
\end{align*}

A.1. Boundary conditions for the coupling function. For a fixed $v' \in \Omega$, the function $C\Omega(u, v')$ is discrete holomorphic as a function of $u$, with a simple pole at $v'$:

\begin{align*}
C\Omega(\cdot, v')|_{\partial \Omega} &= 0, \\
C\Omega(\cdot, v')|_{\partial \Omega} &= 1, \\
\frac{\partial}{\partial v} C\Omega(\cdot, v') &= 0, \quad \forall v \in \Omega, \quad v \neq v'
\end{align*}

Therefore in a $2n$-black-piecewise Temperley domain, for a fixed $v' \in \Omega$, the boundary conditions of the coupling function $C\Omega(u, v')$ as a function of $u$ change at all white corners, and there are $2n$ parts of the boundary with either $\text{Re}[C\Omega(\cdot, v')] = 0$ or $\text{Im}[C\Omega(\cdot, v')] = 0$.

In other words, a black-piecewise Temperley domain corresponds to mixed Dirichlet and Neumann boundary conditions for the discrete harmonic components of the coupling function. Recall that in a Temperley domain we have a simple boundary conditions, namely, $\text{Im}[C\Omega(u, v')] = 0$ for all boundary squares $u$, in other words $\text{Im}[C\Omega(u, v')]$ as a function of $u$ has Dirichlet boundary conditions.

A.2. Asymptotic values of the coupling function. Following [13], we define two functions $f_0(z_1, z_2)$ and $f_1(z_1, z_2)$. For a fixed $z_2$,

\begin{itemize}
\item the function $f_0(z_1, z_2)$ is analytic as a function of $z_1$, has a simple pole of residue $1/\pi$ at $z_1 = z_2$, and no other poles on $\Gamma$;
\item $f_0(\cdot, z_2)$ is bounded in the vicinity of the points $v_k^*$;
\item the function $f_0(\cdot, z_2)$ is semi-bounded in the vicinity of the points $\tilde{v}_k$;
\item on each segment into which points from the set $\{\tilde{v}_k\}_{k=1}^{n-1} \cup \{v_k^*\}_{k=1}^{n+1}$ split the boundary, we have either $\text{Re}[f_0(\cdot, z_2)] = 0$ or $\text{Im}[f_0(\cdot, z_2)] = 0$;
\item the boundary conditions of the function $f_0(\cdot, z_2)$ change at all points $\tilde{v}_k, v_k^*$.
\end{itemize}

The function $f_1(z_1, z_2)$ has the same definition, except for a difference in the boundary conditions: if on a segment between two points from the set $\{\tilde{v}_k\}_{k=1}^{n-1} \cup \{v_k^*\}_{k=1}^{n+1}$ we have $\text{Re}[f_0(\cdot, z_2)] = 0$ (or $\text{Im}[f_0(\cdot, z_2)] = 0$), then on that segment $\text{Im}[f_1(\cdot, z_2)] = 0$ (or $\text{Re}[f_1(\cdot, z_2)] = 0$). The existence and uniqueness of such functions can be shown using the technique described in Section 3. In particular, we can write these functions in the following way

$$f_0(z, w) = \prod_{k=1}^{n+1} \left( z - v_k^\delta \right)^{\frac{1}{2}} \cdot \prod_{k=1}^{n-1} \left( z - \tilde{v}_k \right)^{-\frac{1}{2}} \cdot \left( \frac{s(w)}{z - w} + \frac{s(w)}{z - w} \right),$$

where $s(z) = z^2 - 1$ and $\tilde{v}_k$ are the boundary vertices.
\[ f_1(z,w) = \prod_{k=1}^{n+1} (z-v_k^*)^{\frac{1}{2}} \cdot \prod_{k=1}^{n-1} (z-\tilde{v}_k)^{-\frac{1}{2}} \cdot \left( \frac{s(w) - s(w)}{z-w} \right), \]

where \( s(w) = \prod_{k=1}^{n+1} (w-v_k^*)^{-\frac{1}{2}} \cdot \prod_{k=1}^{n-1} (w-\tilde{v}_k)^{\frac{1}{2}} \).

**Theorem 29.** Let \( \Omega \) be a bounded, simply connected domain in \( \mathbb{C} \) with \( k \) marked points. Assume that a sequence of discrete \( k \)-black-piecewise Temperley domains \( \Omega^\delta \) on a grid with mesh size \( \delta \) approximates the domain \( \Omega \), and each domain \( \Omega^\delta \) has at least one domino tiling. Let a sequence of white squares \( v^\delta \) approximates a point \( v \in \Omega \). Then the coupling function \( \frac{1}{\delta} C_{\Omega^\delta}(u,v) \) satisfies the following asymptotics:

for \( v^\delta \in \hat{0}_0 \):

\[ \frac{1}{\delta} C_{\Omega^\delta}(u,v^\delta) - \frac{2}{\lambda} \cdot F_{C,v^\delta}(u) = f_0(u,v) - \frac{1}{\pi(u-v)} + o(1); \]

if \( v^\delta \in \hat{1}_1 \), then

\[ \frac{1}{\delta} C_{\Omega^\delta}(u,v^\delta) - \frac{2}{\lambda} \cdot F_{C,v^\delta}(u) = f_1(u,v) - \frac{1}{\pi(u-v)} + o(1), \]

where \( F_{C,v^\delta}(u) \) is defined in Section 5.2.

Proof. Recall that \( F_{C}(z,v^\delta) \) is asymptotically equal \( \frac{1}{2\pi} \cdot \frac{\lambda}{z-v_0} \) as \( \delta \downarrow 0 \). Now, to obtain the result use the techniques described in Section 5.2. \( \square \)

A.3. **Sketch of the proof of Corollary 4.** In [15] Kenyon proved convergence of the height function on Temperley domains to the Gaussian free field. To obtain the same result for black-piecewise Temperley domains it is enough to show that the limits of moments of height function in Temperley case and black-piecewise Temperley case are the same.

Due to [13] one can obtain the following result for black-piecewise Temperley approximations. Let \( f_+(z,w) = f_0(z,w) + f_1(z,w) \) and \( f_-(z,w) = f_0(z,w) - f_1(z,w) \).

**Proposition 30.** Let \( \gamma_1, \ldots, \gamma_m \) be a collection of pairwise disjoint paths running from the boundary of \( \Omega \) to \( z_1, \ldots, z_m \) respectively. Let \( h(z_i) \) denote the height function at a point in black-piecewise Temperley domain \( \Omega^\delta \) lying within \( O(\delta) \) of \( z_i \). Then

\[
\lim_{\delta \to 0} \mathbb{E}[(h(z_1) - \mathbb{E}[h(z_1)]) \cdots (h(z_m) - \mathbb{E}[h(z_m)])] = \sum_{\epsilon_1, \ldots, \epsilon_m \in \{-1,1\}} \epsilon_1 \cdots \epsilon_m \int_{\gamma_1} \cdots \int_{\gamma_m} \det \left( F_{\epsilon_i,\epsilon_j}(z_i, z_j) \right) dz_1^{(\epsilon_1)} \cdots dz_m^{(\epsilon_m)},
\]

where \( dz_j = dz_j \) and \( dz_j^{(-1)} = dz_{\overline{j}} \), and

\[
F_{\epsilon_i,\epsilon_j}(z_i, z_j) = \begin{cases} 
0 & i = j \\
\frac{2}{z-w} & (\epsilon_i, \epsilon_j) = (1,1) \\
\epsilon_i s(w) & (\epsilon_i, \epsilon_j) = (1,-1) \\
\epsilon_i s(z) & (\epsilon_i, \epsilon_j) = (-1,1) \\
\frac{2}{z-w} & (\epsilon_i, \epsilon_j) = (-1,-1).
\end{cases}
\]

Recall that in Temperley case [13] one has \( f_+(z,w) = \frac{2}{z-w} \) and \( f_-(z,w) = \frac{2}{z-w} \). In black-piecewise Temperley case we have

\[
\begin{align*}
&f_+(z,w) = \frac{2}{z-w} \frac{s(w)}{s(z)} \\
&f_-(z,w) = \frac{2}{z-w} \frac{s(w)}{s(z)}
\end{align*}
\]
where the function $s(w)$ is defined in the previous section.

One can easily check that the following lemma holds:

**Lemma 31.** Let $\epsilon_1, \ldots, \epsilon_m \in \{-1,1\}$. Let us define function $S_{\epsilon_i,\epsilon_j}(z,w)$ as follows:

$$
S_{\epsilon_i,\epsilon_j}(z,w) = \begin{cases}
0 & i = j \\
\frac{s(w)}{s(z)} & (\epsilon_i, \epsilon_j) = (1,1) \\
\frac{s(w)}{s(z)} & (\epsilon_i, \epsilon_j) = (-1,1) \\
\frac{s(w)}{s(z)} & (\epsilon_i, \epsilon_j) = (1,-1) \\
\frac{s(w)}{s(z)} & (\epsilon_i, \epsilon_j) = (-1,-1).
\end{cases}
$$

Then

$$
S_{\alpha(1),\epsilon_1}(z_{\alpha(1)},z_1) \cdot \ldots \cdot S_{\alpha(m),\epsilon_m}(z_{\alpha(m)},z_m) = \begin{cases}
1 & \alpha(i) \neq i \ \forall i \in \{1,2,\ldots,m\} \\
0 & \text{otherwise},
\end{cases}
$$

where $\alpha$ is a permutation of the set $\{1,2,\ldots,m\}$.

**Corollary 32.** The limits of moments of height function in Temperley case and black-piecewise Temperley case are the same.

In [15] Kenyon showed that:

**Proposition 33.** Let $\Omega$ be a Jordan domain with smooth boundary. Let $z_1, \ldots, z_m$ (with $m$ even) be distinct points of $\Omega$. Let $\Omega^{\delta}$ be a Temperley approximation of $\Omega$ and $h_{\Omega^{\delta}}$ be a height function in domain $\Omega^{\delta}$. Then

$$
\lim_{\delta \to 0} \left[ \mathbb{E}\left( (h(z_1) - \mathbb{E}[h(z_1)]) \cdot \ldots \cdot (h(z_m) - \mathbb{E}[h(z_m)]) \right) \right] = \\
\left( \frac{-16}{\pi} \right)^{m/2} \sum_{\text{pairings } \alpha} g_D(z_{\alpha(1)},z_{\alpha(2)}) \cdot \ldots \cdot g_D(z_{\alpha(m-1)},z_{\alpha(m)}),
$$

where $g_D$ is a Green’s function with Dirichlet boundary conditions on $\Omega$.

By Corollary 32 this proposition holds for black-piecewise Temperley domains. And the following lemma completes the proof of Corollary 4:

**Lemma 34 (II).** A sequence of multidimensional random variables whose moments converge to the moments of a Gaussian, converges itself to a Gaussian.

**A.4. Generalisation to isoradial graphs.** In this section we will discuss the result of Theorem 28 for the dimer model on isoradial graphs. A rhombic lattice (or isoradial graph) was introduced by Duffin [10] as a large family of graphs, where discretizations of Laplace and Cauchy-Riemann operators can be defined similar to that for the square lattice. The isoradial graphs is the largest known class of graphs where classical complex analysis results have discrete analogs, see [3]. A lot of planar graphs admit isoradial embeddings [21]. Discrete complex analysis allows to obtain results for two-dimensional lattice models on isoradial graphs, notably the Ising [23 4] and dimer [18 6 8 22] models.

Let $\Gamma$ be an isoradial graph, i.e. a planar graph in which each face is inscribed into a circle of a common radius $\delta$. Also one can thing about $\delta$ as a mesh size of the isoradial graph. Suppose that all circle centres are inside the corresponding faces, then the dual graph $\Gamma^*$ is also isoradial with the same radius. The rhombic lattice is the graph on a couple $\Lambda$ of two vertex sets $\Gamma$ and $\Gamma^*$ (see Fig. 16). We will use the following assumption (see [3])

(♠) the rhombi angles are uniformly bounded from 0 and $\pi$. 

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Isoradial dimer model was introduced by Kenyon [18]. A dimer configuration in this setup is a perfect matching of the bipartite graph $\Omega^\delta$. The vertex set of $\Omega^\delta$ obtained as a superposition of $\Lambda$ (two types of black vertices) and rhombi centers (white vertices), and there is an edge between black and white vertices if the black vertex and corresponding rhombi are adjacent (see Fig. 16). Note that $\Omega^\delta$ is an isoradial graph, where each face is inscribed into a circle of radius $\frac{\delta}{2}$, for more details see [8].

We will call a white vertex on the boundary of $\Omega^\delta$ a corner if it is adjacent to two boundary black vertices of different types. We can define as before convex and concave white corners, see Fig. 16. Note that Lemma [19] holds also in isoradial case. An isoradial graph $\Omega^\delta$ is called a 2n-black-piecewise Temperley graph if it has $n+1$ convex white corners and $n-1$ concave white corners, see Fig. 16.

Now we can formulate the similar result for isoradial graphs analogous to Theorem 28.

**Theorem 35.** Let $\Omega^\delta$ be a sequence of isoradial 2k-black-piecewise Temperley graphs approximating a continuous domain $\Omega$. Assume that each $\Omega^\delta$ allows a perfect matching. Let the sets of white boundary vertex $\{v^*_{\delta, k}\}_{k=1}^{n+1}$ and $\{\tilde{v}_{\delta, k}\}_{k=1}^{n-1}$ approximate the sets of boundary points $\{v^*_{k}\}_{k=1}^{n+1}$ and $\{\tilde{v}_{k}\}_{k=1}^{n-1}$ correspondingly, and let $v^*_{\delta, 0}$ approximate a point boundary point $v^*_{0}$ which lies on a straight segment of the boundary of $\Omega$. Then $F^\delta_{iso}$ converges uniformly on compact subsets of $\Omega$ to a continuous holomorphic function $f_{\Omega}$, where $f_{\Omega}$ is defined as in Proposition 22.

The proof mimics the proof of Theorem 28 using the approach described in [3].

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