A geometrical representation of the quantum information metric in the gauge/gravity correspondence

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Abstract

We study a geometrical representation of the quantum information metric in the gauge/gravity correspondence. We consider the quantum information metric that measures the distance between the ground states of two theories on the field theory side, one of which is obtained by perturbing the other. We show that the information metric is represented by a back reaction to the volume of a codimension-2 surface on the gravity side if the unperturbed field theory possesses the Poincare symmetry.

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1 Introduction

It has recently been recognized that the quantum information theory seems to play a crucial role \cite{1} in gaining deep understanding of the gauge/gravity correspondence \cite{2-4}. For instance, the connection of the quantum information metric with the bulk geometry has been investigated in \cite{5-14}.

In \cite{5}, we found a universal formula that represents the quantum information metric in terms of a back reaction to a geometrical quantity in the bulk. We considered a CFT and a theory obtained by perturbing the CFT by a primary operator, and calculated the quantum information metric that measures the distance between the ground states of the two theories. We showed that the quantum information metric is represented by a back reaction to the volume of a codimension-2 surface. This is universal in the sense that it holds for perturbations by scalar, vector and tensor operators.

In this letter, we push forward with the above project. We show that the above formula holds also for a field theory, which is not necessarily a CFT and has a gravity dual whose background geometry is not necessarily AdS. We introduce a covariant calculation on the gravity side, which allows us to derive a condition that must be satisfied by the dual geometry in order that the quantum information metric can be represented by the back reaction to the volume of the codimension-2 surface. We find that the condition implies that the original field theory possesses the Poincare invariance.

2 On-shell action and Einstein equation

In this section, we consider a back reaction caused by a scalar field in a background geometry in the bulk. The background geometry is supposed to be dual to a field theory on the boundary which is perturbed by an operator corresponding to the scalar field in the bulk. We represent the on-shell action for the scalar field in terms of the back reaction to the background geometry. As seen in the next section, the information metric that measures the distance between the ground states of the original and perturbed field theories corresponds to the on-shell action for the scalar field. Thus, we eventually obtain a formula that represents the information metric in terms of the back reaction to the background geometry.

The coordinates in the \((d+1)\)-dimensional bulk spacetime are denoted by \(x^\mu = (z, x^i) = \)
where $i = 0, \ldots, d - 1$, $a = 1, \ldots, d - 1$ and $\tau = x^0$. The spacetime metric takes the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{zz} dz^2 + g_{ij} dx^i dx^j,$$

where $g_{\mu\nu}$ is expanded around a background $\hat{g}_{\mu\nu}$ as

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \tilde{g}_{\mu\nu}$$

with $\tilde{g}_{\mu\nu}$ a perturbation. A gauge condition $\hat{g}_{zi} = \tilde{g}_{zi} = \tilde{g}_{zz} = 0$ is imposed. We assume that

$$\partial_i \hat{g}_{\mu\nu} = 0$$

and that the spacetime is asymptotically $\hat{g}_{zz} \to 1/z^2$, $\hat{g}_{ij} \to 1/z^2$, $\tilde{g}_{ij} \to 0$ as $z \to 0$.

The gravity action consists of the Einstein-Hilbert action and the Gibbons-Hawking term:

$$S_g = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} \left( -R + 2\Lambda \right) + S_{GH},$$

where the bulk cosmological constant $\Lambda$ is given by $\Lambda = -\frac{d(d-1)}{2}$. The matter action is given by

$$S_M = \int d^{d+1}x \sqrt{g} \left( \frac{1}{2} \partial_\mu \Psi \partial^\mu \Psi + U_1(\Psi) + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + U_2(\Phi) \right),$$

where $\Psi$ is a scalar field that gives the background metric $\hat{g}$, and $\Phi$ is a perturbation that gives a back reaction corresponding to $\tilde{g}_{\mu\nu}$ in (2.2). In what follows, we keep the contribution up to the second order in $\Phi$.

The Einstein equations are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}.$$  

We substitute (2.2) into (2.6) and expand (2.6) in terms of $\tilde{g}_{\mu\nu}$. Then, the zeroth and first orders read

$$\hat{R}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} \hat{R} + \Lambda \hat{g}_{\mu\nu} = 8\pi G_N \hat{T}_{\mu\nu},$$

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} + \Lambda \tilde{g}_{\mu\nu} = 8\pi G_N \tilde{T}_{\mu\nu},$$

The Riemann curvature $R^\mu_{\nu\rho\sigma}$ is defined by $R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\nu\lambda} \Gamma^\lambda_{\rho\sigma}$ and the Ricci tensor is defined by $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$. 

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respectively, where
\[ \hat{T}_{\mu\nu} = \partial_\mu \Psi \partial_\nu \Psi - \hat{g}_{\mu\nu} \left( \frac{1}{2} \partial_\rho \Psi \partial^\rho \Psi + U_1(\Psi) \right), \]  
(2.9)
\[ \tilde{T}_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \tilde{g}_{\mu\nu} \left( \frac{1}{2} \partial_\rho \Phi \partial^\rho \Phi + \frac{1}{2} U''_2(0)\Phi^2 \right) - \tilde{g}_{\mu\nu} \left( \frac{1}{2} \partial_\rho \Psi \partial^\rho \Psi + U_1(\Psi) \right) \]  
(2.10)
with \( U''_2(\Phi) = \partial^2 U_2(\Phi)/\partial \Phi^2 \).

Because of (2.3), there exist \( d \) Killing vectors corresponding to translations in the \((d+1)\)-dimensional background spacetime. We denote one of those by \( \xi^\mu \), which satisfies
\[ \hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu = 0, \]
\[ L_\xi \Psi = 0. \]  
(2.11)
We can further assume that the invariance under the translation corresponding to \( \xi^\mu \) is preserved by the perturbation:
\[ L_\xi \tilde{g}_{\mu\nu} = 0, \quad L_\xi \Phi = 0. \]  
(2.12)

We contract \( \xi^\mu \) with the Einstein equations (2.8) as
\[ \frac{\xi_\mu \xi^\nu}{\xi^2} \left( \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} \tilde{g}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} + \Lambda \tilde{g}_{\mu\nu} \right) = 8\pi G_N \frac{\xi_\mu \xi^\nu}{\xi^2} \tilde{T}_{\mu\nu}, \]  
(2.13)
where \( \xi^2 = \tilde{g}_{\mu\nu} \xi^\mu \xi^\nu \) and expand \( \tilde{R}_{\mu\nu} \) to the first order in \( \tilde{g}_{\mu\nu} \) as
\[ \tilde{R}_{\mu\nu} = \frac{1}{2} \tilde{g}^{\rho\sigma} \left( \hat{\nabla}_\rho \hat{\nabla}_\sigma \tilde{g}_{\mu\nu} + \hat{\nabla}_\rho \hat{\nabla}_\mu \tilde{g}_{\nu\sigma} + \hat{\nabla}_\sigma \hat{\nabla}_\nu \tilde{g}_{\rho\mu} - \hat{\nabla}_\rho \hat{\nabla}_\nu \tilde{g}_{\sigma\mu} - \hat{\nabla}_\sigma \hat{\nabla}_\mu \tilde{g}_{\rho\nu} \right). \]  
(2.14)

By using (2.7), (2.11), (2.12) and (2.14), we obtain from (2.13)
\[ \frac{1}{2} \hat{\nabla}_\rho \left\{ \frac{1}{\xi^2} \hat{\nabla}^\mu (\xi^2) \tilde{g}^{\rho\sigma} \tilde{g}_{\mu\sigma} - \hat{\nabla}^\mu \left( \frac{\xi_\rho \xi^\mu}{\xi^2} \tilde{g}_{\rho\sigma} \hat{\nabla}_\mu + \frac{2}{\xi^2} \hat{\nabla}_\rho (\xi_\mu \xi^\nu) \tilde{g}_{\mu\nu} \right) \right\} \]
\[ + \frac{1}{\xi^2} \hat{\nabla}_\rho \left( \frac{2}{\xi^2} \right) \xi_\mu \xi^\nu \tilde{g}_{\mu\nu} - \frac{1}{\xi^2} \hat{\nabla}_\rho (\xi_\mu \xi^\nu) \tilde{g}_{\mu\nu} \right) + \frac{1}{\xi^2} \left\{ \tilde{g}^{\mu\nu} \hat{\nabla}_\rho \tilde{g}_{\mu\nu} - \tilde{g}^{\mu\nu} \hat{\nabla}_\rho \tilde{g}_{\mu\nu} \right\} \]
\[ - \frac{1}{2} \hat{\nabla}_\rho \left( \frac{1}{\xi^2} \right) \xi_\mu \xi^\nu \tilde{g}_{\mu\nu} - \frac{1}{\xi^2} \hat{\nabla}_\rho (\xi_\mu \xi^\nu) \tilde{g}_{\mu\nu} + \frac{1}{\xi^2} \left\{ \hat{\nabla}_\rho \left( \frac{1}{\xi^2} \right) \tilde{g}^{\mu\nu} - \frac{2}{\xi^2} \hat{\nabla}_\rho \tilde{g}_{\mu\nu} \right\} \tilde{g}_{\mu\nu} \]
\[ + \frac{1}{\xi^2} \hat{\nabla}_\rho \xi_\sigma \hat{\nabla}_\mu \xi_\rho \tilde{g}_{\mu\sigma} - \frac{2}{\xi^2} \hat{\nabla}_\rho \xi_\mu \tilde{g}_{\rho\nu} \right\} \tilde{g}_{\mu\nu} + 
\frac{1}{4} \left\{ \hat{\nabla}_\rho \left( \frac{1}{\xi^2} \right) \tilde{g}^{\rho\sigma} - \frac{2}{\xi^2} \hat{\nabla}_\rho \xi_\sigma \tilde{g}_{\rho\sigma} \right\} \tilde{g}_{\mu\nu} \]
\[ = -4\pi G_N \left( \partial_\mu \Phi \partial^\mu \Phi + U''_2(0)\Phi^2 \right). \]  
(2.15)
By using the equation of motion for $\Phi$, we see that the RHS of (2.15) is a total derivative term, $-4\pi G_N \nabla_\mu \left( \Phi \nabla^\mu \Phi \right)$. The third and fourth lines in the LHS are calculated as

$$
\frac{1}{2} \xi^a \partial_z \left( \frac{\tilde{g}_{a\beta}}{\xi^2} \right) \left\{ \frac{1}{\xi^2} \partial^z \xi^2 \tilde{g}^{\nu\beta} \tilde{g}_{\nu\mu} + \partial^z \tilde{g}_{\nu\sigma} \tilde{g}^{\beta\sigma} \xi^2 \tilde{g}_{\mu\nu} - \partial^z \tilde{g}_{\sigma\nu} \tilde{g}^{\beta\sigma} \xi^2 \tilde{g}_{\mu\nu} \right. \\
\left. + \frac{1}{2} \partial^z \tilde{g}_{\lambda\nu} \tilde{g}^{\beta\lambda} \xi^2 \tilde{g}_{\mu\nu} \right\}.
$$

(2.16)

Thus, (2.15) reduces to

$$
\frac{1}{2} \nabla_\rho \left( \frac{1}{\xi^2} \nabla^\rho (\xi^2) \tilde{g}^{\mu\sigma} \tilde{g}_{\mu\sigma} - \nabla^\rho \left( \frac{\xi^2 \xi^\rho}{\xi^2} \tilde{g}_{\mu\nu} \right) - \frac{\xi^2 \xi^\rho}{\xi^2} \tilde{g}^{\mu\nu} \nabla_\sigma \tilde{g}_{\mu\nu} + \frac{2}{\xi^2} \nabla^\rho (\xi^2) \tilde{g}^{\mu\nu} \tilde{g}_{\mu\nu} + \partial^z \tilde{g}^{\mu\nu} \tilde{g}_{\mu\nu} - \tilde{g}^{\mu\nu} \tilde{g}^{\rho\sigma} \nabla_\nu \tilde{g}_{\mu\sigma} \right) \right)

= -4\pi G_N \nabla_\mu \left( \Phi \nabla^\mu \Phi \right).
$$

(2.17)

(2.18)

Then, in order for the LHS to be total derivative terms, the above expression must vanish. This leads us to impose a condition

$$
\xi^\mu \partial_z \left( \frac{\tilde{g}_{\mu\nu}}{\xi^2} \right) = 0.
$$

(2.19)

We assume that the boundary where the dual field theory lives is specified by $z = \epsilon$. Then, integrating both sides of (2.18) over the bulk yields

$$
\frac{1}{2} \int d^d x \sqrt{\tilde{g}} \left\{ \tilde{g}^{zz} \partial_z \left( \left( \tilde{g}^{ij} - \frac{\xi^i \xi^j}{\xi^2} \right) \tilde{g}_{ij} \right) - \frac{\tilde{g}^{zz}}{2} \left( \partial_z \tilde{g}^{ij} + \frac{1}{\xi^2} \partial_z \xi^2 \tilde{g}^{ij} \right) \tilde{g}_{ij} \right\} = 8\pi G_N S_{\Phi \text{ on-shell}},
$$

(2.20)

where (2.1) is used, and the on-shell action for $\Phi$ in the RHS is given by

$$
S_{\Phi \text{ on-shell}} = -\frac{1}{2} \int_{z=\epsilon} d^d x \sqrt{g} \tilde{g}^{zz} \Phi \partial_z \Phi.
$$

Hereafter, we identify the direction of $\xi^i$ with that of $x^0$ such that $\xi^2 = 0$, $\xi^0 = 1$ and $\xi^a = 0$. We make an ADM-like decomposition of the metric $g_{ij}$ in $d$ dimensions as

$$
g_{ij} = \left( \begin{array}{cc} \frac{N_a}{N} & -\frac{N_b}{N} \\ -\frac{N_b}{N} & \gamma_{ab} + \frac{N_a N_b}{N^2} \end{array} \right), \quad \tilde{g}_{ij} = \left( \begin{array}{cc} N^2 + N_a N_b & N_a \\ N_a & \gamma_{ab} \end{array} \right),
$$

(2.21)

where $a, b = 1, \ldots, d - 1$. Expanding $N$, $N_a$ and $\gamma_{ab}$ around the background as $N = \hat{N} + \tilde{N}$, $N_a = \hat{N}_a + \tilde{N}_a$ and $\gamma_{ab} = \hat{\gamma}_{ab} + \tilde{\gamma}_{ab}$, leads to

$$
\hat{g}_{ij} = \left( \begin{array}{cc} 1 & -\frac{\hat{N}_a}{N^2} \\ -\frac{\hat{N}_a}{N^2} & \hat{\gamma}_{ab} \end{array} \right), \quad \tilde{g}_{ij} = \left( \begin{array}{cc} 2 \frac{\hat{N}_a N_b}{N^3} - \frac{\hat{N}_a}{N^2} & \tilde{\gamma}_{ab} - 2 \frac{\hat{N}_a N_b}{N^2} \tilde{N} + \frac{\hat{N}_a N_b}{N^2} + \frac{\hat{N}_a \tilde{N}_b}{N^2} \\ -\frac{\hat{N}_a N_b}{N^2} + \frac{\hat{N}_a \tilde{N}_b}{N^2} & 2 \frac{\hat{N}_a N_b}{N^3} - \frac{\hat{N}_a}{N^2} \end{array} \right).
$$

(2.22)
Note that $\xi^2$ and $\sqrt{g}$ are given by

$$\xi^2 = \frac{1}{N^2}, \quad \sqrt{g} = \sqrt{\frac{N^2 + N^a N_a}{N^2}} \sqrt{g_{zz} \sqrt{\bar{\gamma}}}$$  \hspace{1cm} (2.23)$$

and that (2.17) is equivalent to

$$\partial_z \bar{N}_a = 0 . \hspace{1cm} (2.24)$$

By substituting (2.23) and (2.24) into (2.19), we obtain

$$8\pi G_N S_{\Phi \text{ on-shell}} = \frac{1}{2} \int_{z=\epsilon} d^d x \sqrt{\frac{N^2 + N^a N_a}{N^2 \sqrt{g_{zz}}}} \sqrt{\bar{\gamma}} \left\{ \partial_z (\bar{\gamma}^{ab} \bar{\gamma}_{ab}) - \frac{1}{2} \left( \partial_z \bar{\gamma}^{ab} - \frac{1}{N^2} \partial_z \bar{N}^2 \bar{\gamma}^{ab} \right) \bar{\gamma}_{ab} \right\} . \hspace{1cm} (2.25)$$

The boundary specified by $z = \epsilon$ where the dual field theory lives is a codimension-2 hyperplane perpendicular to $\xi^i$. The volume of hyperplane is given up to the first order in $\bar{\gamma}_{ab}$ as

$$V = \int_{z=\epsilon} \, d^{d-1} x \, \sqrt{\bar{\gamma}} = \int_{z=\epsilon} \, d^{d-1} x \, \sqrt{\bar{\gamma}} (1 + \frac{1}{2} \bar{\gamma}^{ab} \bar{\gamma}_{ab}) . \hspace{1cm} (2.26)$$

We subtract the zeroth order contribution from this and obtain

$$\delta V = \frac{1}{2} \int_{z=\epsilon} \, d^{d-1} x \, \sqrt{\bar{\gamma}} \bar{\gamma}^{ab} \bar{\gamma}_{ab} . \hspace{1cm} (2.27)$$

We consider the $z$ derivative of $\delta V$:

$$\delta V' = \frac{1}{2} \int_{z=\epsilon} \, d^{d-1} x \partial_z (\sqrt{\bar{\gamma}}) \, \bar{\gamma}^{ab} \bar{\gamma}_{ab} + \delta v' , \hspace{1cm} (2.28)$$

$$\delta v' = \frac{1}{2} \int_{z=\epsilon} \, d^{d-1} x \sqrt{\bar{\gamma}} \, \partial_z \left( \bar{\gamma}^{ab} \bar{\gamma}_{ab} \right) , \hspace{1cm} (2.29)$$

where the prime represents the $z$-derivative. Here the first term and the second term in the RHS of (2.28) represent the canonical scaling contribution and a nontrivial scaling contribution, respectively. Then, (2.25) is rewritten as

$$8\pi G_N S_{\Phi \text{ on-shell}} = \int d\tau \sqrt{\frac{N^2 + N^a N_a}{N^2 \sqrt{g_{zz}}}} \left\{ \delta v' - \frac{1}{4} \int_{z=\epsilon} \, d^{d-1} x \sqrt{\bar{\gamma}} \left( \partial_z \bar{\gamma}^{ab} - \frac{1}{N^2} \partial_z \bar{N}^2 \bar{\gamma}^{ab} \right) \bar{\gamma}_{ab} \right\} . \hspace{1cm} (2.30)$$
Here we require the second term in the RHS of (2.30) to vanish. Then, $\hat{\gamma}_{ab}$ is determined as

$$\hat{\gamma}_{ab} = \frac{1}{N^2} C_{ab}$$  \hspace{1cm} (2.31)

with $C_{ab}$ being a constant tensor. Furthermore, (2.24) implies that $\hat{g}_{ij}$ is expressed as

$$\hat{g}_{ij} = \frac{1}{N^2} D_{ij}$$  \hspace{1cm} (2.32)

with $D_{ij}$ being a constant tensor.

$D_{ij}$ is diagonalizable so that $\hat{N}_a$ can be set to zero and we redefine $g_{zz} \rightarrow \frac{1}{z^2}, g_{ij} \rightarrow \frac{g_{ij}}{z^2}$. Thus, from (2.30), we obtain

$$S_{\Phi \text{ on-shell}} = \frac{T}{8\pi G_N} \delta v'$$  \hspace{1cm} (2.33)

where we have used

$$\frac{1}{N} = \frac{\hat{g}_{\tau\tau}}{z}$$  \hspace{1cm} (2.34)

and

$$T = \int d\tau \hat{g}_{\tau\tau}.$$  \hspace{1cm} (2.35)

The AdS metric satisfies (2.32) so that we can take

$$\hat{N} = z, \hat{N}_a = 0, \hat{\gamma}_{ab} = \frac{1}{z^2} \delta_{ab},$$  \hspace{1cm} (2.36)

and we reproduce the result in [5].

(2.32) indicates that the background spacetime has $d$-dimensional Poincare invariance. This implies that the original dual field theory also has it.

The background spacetime $\hat{g}_{\mu\nu}(z)$ satisfying (2.32) and the background matter field $\Psi(z)$ are determined by the equations of motion

$$(d - 1) \left( \frac{f''}{f} - \frac{f'^2}{f^2} + \frac{1}{z} \frac{f'}{f} \right) = -16\pi G_N \partial_z \Psi \partial_z \Psi,$$  \hspace{1cm} (2.37)

$$z^2 \Psi'' - (d - 1)z \Psi' + \frac{d}{2} \frac{f'}{f} z^2 \Psi' = \frac{\partial U_1(\Psi)}{\partial \Psi},$$  \hspace{1cm} (2.38)

where we define $\frac{1}{N^2} = \frac{f(z)}{z^2}$. 

6
As an example of solutions, we consider the GPPZ flow \cite{15} which is dual to 4-dimensional $\mathcal{N} = 1$ super Yang-Mills theory, where the scalar field has a non-zero VEV. In this case, the potential $U_1(\Psi)$ is given by

$$U_1(\Psi) = -\frac{3}{32\pi G_N} \left[ -5 + \left( \cosh \left( \frac{4\sqrt{\pi G_N}}{\sqrt{3}} \Psi \right) \right)^2 + \cosh \left( \frac{4\sqrt{\pi G_N}}{\sqrt{3}} \Psi \right) \right], \quad (2.39)$$

and the solution to (2.37) and (2.38) is given by

$$\Psi = \frac{\sqrt{3}}{4\sqrt{\pi G_N}} \log \frac{1+z}{1-z}, \quad f(z) = 1 - z^2. \quad (2.40)$$

## 3 Information metric for a dual operator to bulk scalar field

In this section, we introduce the quantum information metric and show that the one for the original and perturbed theories on the field theory side is represented by the geometrical quantity $\delta v$ in (2.33).

We consider a field theory defined by a Lagrangian density $L_0$ on $d$-dimensional Euclidean spacetime whose coordinates are $x^i$ ($i = 0, \ldots, d - 1$), where $x^0 \equiv \tau$ is viewed as the Euclidean time. We also consider another field theory with a Lagrangian density $L$ obtained by perturbing the theory as

$$L = L_0 + \phi(0)(\vec{x})\mathcal{O}(x), \quad (3.1)$$

where $\mathcal{O}(x)$ is a scalar operator and $\phi(0)(\vec{x})$ is a source independent of the time $\tau$.

We denote the ground states of the theories $L_0$ and $L$ by $|\Omega_0\rangle$ and $|\Omega\rangle$, respectively. Then, the inner product $\langle \Omega|\Omega_0\rangle$ is given by a path integration

$$\langle \Omega|\Omega_0\rangle = (ZZ_0)^{-1/2} \int D\phi \exp \left[ - \int d^{d-1}x \left( \int_0^\tau d\tau L_0 + \int_0^\tau d\tau L \right) \right], \quad (3.2)$$

where $Z_0$ and $Z$ is the partition functions of the theories $L_0$ and $L$, respectively. We assume that $\langle \mathcal{O}(x) \rangle_0 = 0$ and the time reversal symmetry: $\langle \mathcal{O}(\tau, \vec{x})\mathcal{O}(\tau', \vec{x}') \rangle_0 = \langle \mathcal{O}(-\tau, \vec{x})\mathcal{O}(-\tau', \vec{x}') \rangle_0$, where

$$\langle \cdots \rangle_0 = \frac{1}{Z_0} \int D\phi \cdots \exp \left[ - \int d^d x \ L_0 \right]. \quad (3.3)$$
The information metric $\mathcal{G}$ that measures the distance between the ground states of the two theories is obtained by expanding $\langle \Omega | \Omega_0 \rangle$ up to the second order in $\phi_{(0)}$:

$$\mathcal{G} = \frac{1}{T} (1 - \langle \Omega' | \Omega \rangle) = \frac{1}{2T} \int_0^\infty d\tau \int_{-\infty}^0 d\tau' \int d^{d-1}x \int d^{d-1}x' \langle \delta \mathcal{L}(\tau, x) \delta \mathcal{L}(\tau', x') \rangle_0$$

$$= \frac{1}{2T} \int_0^\infty d\tau \int_{-\infty}^0 d\tau' \int d^{d-1}x \int d^{d-1}x' \phi_{(0)}(\vec{x}) \langle \mathcal{O}(\tau, \vec{x}) \mathcal{O}(\tau', \vec{x}') \rangle_0 \phi_{(0)}(\vec{x}') ,$$

where $T$ is the volume of the time direction. Here we assume that we make an appropriate regularization for the two point function of $\mathcal{O}$ to suppress a divergence occurring at $\tau = \tau' = 0$. In the case in which $\mathcal{L}_0$ is a CFT with $\mathcal{O}$ a primary operator, a regularization is given in section 3 of [5].

We apply the above result to the case in the previous section: the theory $\mathcal{L}_0$ possesses a gravity dual corresponding to the background geometry with the $d$-dimensional Poincare invariance, and the operator $\mathcal{O}$ corresponds to a scalar field $\Phi$, which coincides with $\phi_{(0)}$ on the boundary. The $\tau$-independence of $\phi_{(0)}$ is consistent with (2.12). We consider a situation where the classical approximation on the gravity side is valid. By using the GKP-Witten relation

$$\langle \exp \left[ - \int d^d x \phi_{(0)}(x) \mathcal{O}(x) \right] \rangle = \exp \{ - S_{\Phi \text{ on--shell}} \} ,$$

we can show that

$$S_{\Phi \text{ on--shell}} = -4T \mathcal{G} .$$

Thus, by using (2.33), we obtain a formula,

$$\mathcal{G} = - \frac{1}{32\pi G_N} \delta \nu' .$$

This formula was obtained in [3] in the case where a CFT is perturbed by a primary operator $\mathcal{O}$ on the field theory side and the background geometry on the gravity side is given by the AdS.

4 Conclusion

In this letter, we considered a field theory that has a gravity dual, and perturbed it by an operator which corresponds to a scalar field in the bulk. We performed a covariant calculation
to find the condition that must be satisfied by the bulk geometry in order that the on-shell action for the scalar field is represented by a back reaction to the volume of a codimension-2 surface. The condition implies the Poincare invariance of the original field theory. We saw that the quantum information metric that measures the distance between the ground states of the original and perturbed theories is represented by the on-shell action. While we considered only a perturbation by a scalar field, we should obtain the same results for perturbations by vector and tensor fields. Thus, we conclude that the universal formula in [5] that represents the quantum information metric in terms of the back reaction to the volume of the codimension-2 surface is extended to the case of a general gauge/gravity correspondence if the above condition is satisfied. It is interesting to elucidate what is represented by the extra terms in the RHS of (2.30) which we have if we do not impose the condition of the Poincare invariance. We hope that our result leads us to gain deeper understanding of the relationship between quantum information and quantum geometry.

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