Centralizers of $C^1$-contractions of the half line

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Abstract: A subgroup $G \subset \text{Diff}_+^1([0,1])$ is $C^1$-close to the identity if there is a sequence $h_n \in \text{Diff}_+^1([0,1])$ such that the conjugates $h_n gh_n^{-1}$ tend to the identity for the $C^1$-topology, for every $g \in G$. This is equivalent to the fact that $G$ can be embedded in the $C^1$-centralizer of a $C^1$-contraction of $[0, +\infty)$ (see [Fa] and Theorem 1.1).

We first describe the topological dynamics of groups $C^1$-close to the identity. Then, we show that the class of groups $C^1$-close to the identity is invariant under some natural dynamical and algebraic extensions. As a consequence, we can describe a large class of groups $G \subset \text{Diff}_+^1([0,1])$ whose topological dynamics implies that they are $C^1$-close to the identity.

This allows us to show that the free group $F_2$ admits faithful actions which are $C^1$-close to the identity. In particular, the $C^1$-centralizer of a $C^1$-contraction may contain free groups.

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1 Introduction

1.1 Groups $C^1$-close to the identity and centralizers of contractions

The main motivation of this paper is the study of centralizers of the $C^1$-contractions of the half line $[0, +\infty)$. A diffeomorphism $f$ of $[0, +\infty)$ is called a contraction if $f(x) < x$ for every $x \neq 0$. Unless it is explicitly indicated, a contraction will now refer to a $C^1$-diffeomorphism. When $f$ is at least $C^2$, Szekeres, in [Sz] (see also [Sc]), proved that $f$ is the time-one map of the flow of a $C^1$-vector field $X$, and Kopell’s Lemma (see [Ko]) implies that the $C^1$-centralizer of $f$ is precisely the flow $\{X_t, t \in \mathbb{R}\}$ of the Szekeres vector field. When $f$ is only required to be $C^1$, Szekeres result does not hold anymore and neither does Kopell’s Lemma. Actually, the $C^1$-centralizer of a $C^1$-contraction $f$ may be very different according to $f$. Generically it is trivial (i.e. equal to $\{f^n, n \in \mathbb{Z}\}$, see [Lo]) but it can also be very large (non abelian and non countable). We will see that there are at the same time many algebraic and dynamical restrictions on the possible groups, but also a large variety of dynamical properties which allows a group to be included in a centralizer of a contraction.

In [Fa], we consider groups $G$ of diffeomorphisms of a segment $I \subset (0,1)$. We say that $G$ is embeddable in the centralizer of a contraction if there exists a contraction $f$ of $[0, +\infty)$ and a subgroup $\tilde{G}$ of the $C^1$-centralizer of $f$ which induces $G$ by restriction to $I$. [Fa] Theorem 3] shows that $G$ is embeddable in the centralizer of a contraction if and only if there is a $C^1$-continuous path of diffeomorphisms $h_t \in \text{Diff}_+^1(I)$ such that $h_t gh_t^{-1}$ tends to the identity for every $g \in G$.

As a direct consequence one deduces that, if $G$ is embeddable in the centralizer of a contraction, then $G$ is also embeddable in the centralizer of a diffeomorphism $f$ of $[0,1]$ without fixed point in $(0,1)$.

Finally, an argument by A. Navas proves that:

Theorem 1.1. Let $I$ be a compact segment. Given a group $G \subset \text{Diff}_+^1(I)$, the two following properties are equivalent:

- there is a $C^1$-continuous path of diffeomorphisms $h_t \in \text{Diff}_+^1(I)$, $t \in [0,1)$, such that $h_t gh_t^{-1}$ tends to the identity in the $C^1$-topology when $t$ tends to $1$, for every $g \in G$;
- there is a sequence of diffeomorphisms $h_n \in \text{Diff}_+^1(I)$, $n \in \mathbb{N}$, such that $h_n gh_n^{-1}$ tends to the identity in the $C^1$-topology, for every $g \in G$ as $n \to +\infty$.

This fact is not trivial at all and is specific to the identity map: [Fa] provides examples of pairs of diffeomorphisms $f, g \in \text{Diff}^1([0,1])$ such that there are sequences $h_n \in \text{Diff}^1([0,1])$ leading $g$ to $f.$
by conjugacy (that is \( h_n g h_n^{-1} \xrightarrow{C_1} f \)), but such that there is no continuous path \( h_t \) leading \( g \) to \( f \) by conjugacy. The proof of Theorem 1.1 is presented in Section 2.

Therefore, we have four equivalent notions which induce a well defined class of subgroups \( G \) of \( Diff^1_+(\mathbb{R}) \):
- \( G \) is embeddable in the centralizer of a contraction;
- \( G \) is embeddable in the centralizer of a diffeomorphism \( f \in Diff^1_+ \) without fixed point in the interior of \( I \);
- there exists a path of diffeomorphisms \( h_t \in Diff^1_+ \) leading by conjugacy every \( g \in G \) to the identity;
- there exists a sequence of diffeomorphisms \( h_n \in Diff^1_+ \) leading by conjugacy every \( g \in G \) to the identity.

**Definition 1.1.** Let \( I \) be a segment. We say that a group \( G \subset Diff^1_+(\mathbb{R}) \) is **\( C^1 \)-close to the identity** if it satisfies one of the four equivalent properties above, that is, for instance, if there is a sequence of diffeomorphisms \( h_n \in Diff^1_+ \) such that for every \( g \in G \)

\[
h_n g h_n^{-1} \xrightarrow{n \to \infty} id.
\]

We denote by \( C^1_{id}(I) \) the class consisting of these groups; when \( I = [0,1] \) we simply denote it by \( C^1_{id} \).

The aim of this paper is to describe this class of groups \( C^1 \)-close to identity, up to group isomorphisms, up to topological conjugacy, and/or up to smooth conjugacy.

In other words, we try to answer to the following questions:

**Question 1.** What group \( G \) admits a faithful representation \( \varphi : G \to Diff^1(\mathbb{R}) \) so that \( \varphi(G) \) is \( C^1 \)-close to the identity?

As a partial answer, [Mc] implies that Baumslag-Solitar groups cannot be \( C^1 \)-close to the identity. In contrast, Theorem 1.9 shows that the free group \( \mathbb{F}_2 \) admits actions \( C^1 \)-close to the identity.

**Question 2.** What is the topological dynamics of a group \( C^1 \)-close to the identity? In other words, given a group \( G \subset Homeo_+ ([0,1]) \), under what hypotheses does there exist \( h \in Homeo_+ ([0,1]) \) such that \( h G h^{-1} \) is contained in \( C^1_{id} \subset Diff^1_+ ([0,1]) \)?

Theorem 1.7 presents a large class of groups \( G \subset Diff^1_+ ([0,1]) \), called **elementary groups**, whose topological dynamics implies that they are \( C^1 \)-close to the identity. Thus, every group \( G' \subset Diff^1_+ ([0,1]) \) topologically conjugated to \( G \) is \( C^1 \)-close to the identity.

**Question 3.** Given a group \( G \subset Diff^1_+ ([0,1]) \), under what conditions does it belong to \( C^1_{id} \)?

Let us illustrate this question by another which is more precise. An immediate obstruction for a group \( G \subset Diff^1_+ ([0,1]) \) to be \( C^1 \)-close to the identity is the existence of a hyperbolic fixed point for an element \( g \in G \). We will say that \( G \) is **without hyperbolic fixed points** if, for every \( g \in G \) and every \( x \in Fix(g) \), \( Dg(x) = 1 \). A. Navas suggested that it could be a necessary and sufficient condition, maybe for finitely presented groups. [EA] proved that this is true for cyclic groups.

**Question 4.** Let \( G \) be a subgroup of \( Diff^1_+ ([0,1]) \) (maybe assuming countable, or finitely presented, or any other natural hypothesis). We wonder if

\[
G \text{ without hyperbolic fixed point } \iff C^1 \text{-close to identity}
\]

We denote by \( C^1_{nonhyp} \) the class of groups \( G \subset Diff^1_+ ([0,1]) \) without hyperbolic fixed points. For now we know:

\[
C^1_{id} \subset C^1_{nonhyp}
\]

The rest of the introduction expounds the statements of the results above, and proposes some directions for answering these questions.
1.2 Structure results : description of the topological dynamics

This section gives necessary conditions on the topological dynamics of a group $G \subset \text{Diff}^1_+(\mathbb{R})$, for $G$ being $C^1$-close to the identity.

Given a group $G \subset \text{Homeo}^+(\mathbb{R})$, a pair of successive fixed points of $G$ is a pair $\{a, b\}$ such that $(a, b)$ is a connected component of $[0, 1] \setminus \text{Fix}(g)$ for some $g \in G$. One says that two pairs of successive fixed points $\{a, b\}$ and $\{c, d\}$ are linked if $(a, b) \cap \{c, d\}$ or $(c, d) \cap \{a, b\}$ consists in exactly one point.

**Definition 1.2.** A group $G \subset \text{Homeo}^+(\mathbb{R})$ is without linked fixed points if there are no linked pairs of successive fixed points.

The main topological restriction for a group $G \subset \text{Diff}^1_+(\mathbb{R})$ to be $C^1$-close to identity is:

**Theorem 1.2.** If $G \subset \text{Diff}^1_+(\mathbb{R})$ is $C^1$-close to the identity, then $G$ is without linked fixed points.

Using completely different methods due to A. Navas, one proves a slightly stronger result:

**Theorem 1.3.** ([Na]) Any group $G \subset \text{Diff}^1_+(\mathbb{R})$ without hyperbolic fixed point is without linked fixed points.

Thus, the intervals of successive fixed points form a nested family. As a consequence, we will see that the family of pairs of successive fixed points is at most countable (Proposition 1.9).

Another consequence of being without linked fixed points is that, for every interval $(a, b)$ of successive fixed points and any $g \in G$, either $g([a, b]) = [a, b]$ or $g([a, b]) \cap (a, b) = \emptyset$. Considering the stabilizers $G_{[a, b]}$ of the segments $[a, b]$, this provides a stratified description of the dynamics of $G$, as stated in Theorem 1.4 below.

**Theorem 1.4.** Let $G \subset \text{Homeo}_+(\mathbb{R})$ be a group without linked fixed point, and $\{a, b\}$ be a pair of successive fixed points.

Then:
- for every $g \in G$, either $g([a, b]) = [a, b]$ or $g((a, b)) \cap (a, b) = \emptyset$. We denote by $G_{[a, b]}$ the stabilizer of $[a, b]$.
- there is a morphism $\tau_{a, b} : G_{[a, b]} \to \mathbb{R}$ whose kernel is precisely the set of $g \in G_{[a, b]}$ having fixed points in $(a, b)$, and which is positive at $g \in G_{[a, b]}$ if and only if $g(x) - x > 0$ on $(a, b)$.
- The union of the minimal sets of the action of $G_{[a, b]}$ on $(a, b)$ is a non-empty closed subset $\Lambda_{a, b}$ on which the action of $G_{[a, b]}$ is semi-conjugated to the group of translations $\tau_{a, b}(G_{[a, b]})$.
- The elements of the kernel of the map induce the identity map on $\Lambda_{a, b}$.

The morphism $\tau_{a, b}$ is unique up to multiplication by a positive number.

The morphism $\tau_{a, b}$ is called a relative translation number.

For describing the dynamics of $G$ we are lead to consider:
- the nested configuration of the intervals of successive fixed points;
- for each interval $(a, b)$ of successive fixed points, the relative translation number $\tau_{a, b}$ and the set $\Lambda_{a, b}$, union of the minimal sets in $(a, b)$. Each connected component $I$ of $(a, b) \setminus \Lambda_{a, b}$ is
  - either a wandering interval
  - or an interval of successive fixed points,
  - or else it may be the union of an increasing sequence of intervals of successive fixed points.

In the first case, one can stop the study. In both last cases, one considers the restriction to $I$ of the stabilizer $G_I$ : it is a group without linked fixed points, so one may proceed the study.

1.3 Completion of a group without linked fixed points

One difficulty for classifying the groups $C^1$-close to the identity it that each element $g \in G$ may have infinitely many pairs of successive fixed points. For bypassing this difficulty, we enrich the group $G$ so that, for every $g \in G$ and any pair $\{a, b\}$ of successive fixed points of $g$, the group $G$ contains the diffeomorphism $g_{a, b}$ which coincides with $g$ on $[a, b]$ and with the identity out of $[a, b]$. The diffeomorphism $g_{a, b}$ has a unique pair of successive fixed point. Such a diffeomorphism will be called simple. Every element $g \in G$ can be seen as an infinite product of the simple elements $g_{a, b}$, for all the pairs $\{a, b\}$ of successive fixed points of $g$. 
More precisely, given \( g, h \in \text{Homeo}_+(\mathbb{R}) \), we say that \( h \) is induced by \( g \) if \( g \) and \( h \) coincide on the support of \( h \). We say that a group \( G \subseteq \text{Homeo}_+(\mathbb{R}) \) without linked fixed points is complete if it contains any homeomorphism \( h \) induced by an element \( g \in G \).

Corollary 5.10 shows that every group \( C^1 \)-close to the identity is a subgroup of a complete group \( C^1 \)-close to the identity. Analogous results hold for groups of homeomorphisms without linked fixed points, or for groups of diffeomorphisms without hyperbolic fixed points: Proposition 5.3 associates to each group \( G \) without linked fixed points its completion \( \hat{G} \) which is the smallest complete group without linked fixed points containing \( G \). The families of intervals of successive fixed points of \( \hat{G} \) and \( G \) are the same, and Corollary 5.6 states that the morphisms of translation numbers associated to each interval of successive fixed points also coincide for \( G \) and \( \hat{G} \).

1.3.1 Totally rational groups and topological basis

Let us present here a problem, coming from an unsuccessful attempt of us for classifying groups \( C^1 \)-close to the identity.

We say that a group \( G \) without linked fixed points is totally rational if, for any successive fixed points \( \{a, b\} \), the image of the translation number \( \tau_{a,b}(G[a,b]) \) is a cyclic (monogene) group (i.e. a group of the form \( \mathbb{Z} \)). Proposition 5.3 and Corollary 5.6 imply that the completion of a totally rational group is totally rational, allowing us to consider complete totally rational groups.

Given a complete totally rational group \( G \), let us call a topological basis any family \( \{f_i\}_{i \in \mathcal{I}} \) of elements \( f_i \in G \) such that:
- for every \( i \in \mathcal{I} \), \( f_i \) has a unique pair \( \{a_i, b_i\} \) of successive fixed points; thus \( [a_i, b_i] \) is the support of \( f_i \).
- for every \( i \in \mathcal{I} \), let \( \tau_i \) be the relative translation number associated to \( \{a_i, b_i\} \). As \( G \) is totally rational, its image is a cyclic group. We require that \( \tau_i(f_i) \) is a generator of the image of \( \tau_i \).
- for every pair \( \{a, b\} \) of successive fixed points of \( G \), there is a unique \( i \in \mathcal{I} \) such that \( \{a_i, b_i\} \) and \( \{a, b\} \) are in the same \( G \)-orbit.
- for every pair \( \{a, b\} \) of successive fixed points of \( G \), there is an element \( g \) of the subgroup \( \langle f_i, i \in \mathcal{I} \rangle \) generated by the \( f_i \) such that \( \{a, b\} \) is a pair of successive fixed points of \( g \).

The family of intervals of successive fixed points is countable (Proposition 4.9), so that any countable \( \mathcal{I} \) is possible. We can get rather easily the first three items of the definition: just choose one interval contained in the support of \( f_i \).

Question 5. Given a complete totally rational group \( G \subseteq \text{Homeo}_+(\mathbb{R}) \) without linked fixed points, does \( G \) admit a topological basis?

If the answer is negative, same question with the more restrictive assumption that \( G \subseteq \text{Diff}^1_+([0, 1]) \) is \( C^1 \)-close to the identity.

Our intuition is that every element \( g \in G \) is determined by its intervals of successive fixed points and on each of them, by the value of the corresponding translation number, which provides a coordinate in the topological basis. However, even assuming the existence of a topological basis, there remain many issues before making rigorous our intuition. In particular:

Question 6. Let \( G \subseteq \text{Homeo}_+(\mathbb{R}) \) without linked fixed points, and assume \( G \) admits a topological basis \( \{f_i\}_{i \in \mathcal{I}} \). Is \( G \) contained in the \( C^0 \)-closure of the group generated by the \( f_i \)?

Same question, in the \( C^1 \)-topology, with the more restrictive assumption that \( G \subseteq \text{Diff}^1_+([0, 1]) \) is \( C^1 \)-close to the identity.

1.4 Realisation results

1.4.1 Invariance of \( C^1_{id} \) by some extensions.

Let us first present two results, enlighting some invariance of the class \( C^1_{id} \) by some natural extensions.

\footnote{1. Consider the group generated by a family \( g_i \in \text{Diff}^1_+([0, 1]) \), so that the support of \( g_i \) consists in one interval contained in a fundamental domain of \( g_i \). Now consider the family \( f_i = g_{i+1}g_i^{-1} \); this family satisfies the three first hypotheses but not the last one.}
Theorem 1.5. Let $G_n \subset \text{Diff}_+^1([0,1])$, $n \in \mathbb{N}$, be an increasing sequence of subgroups:

$$\forall n \in \mathbb{N}, G_n \subset G_{n+1}.$$

Assume that every $G_n$ is finitely generated and is $C^1$-close to the identity.

Then $G = \bigcup_{n \in \mathbb{N}} G_n \subset \text{Diff}_+^1([0,1])$ is $C^1$-close to the identity.

We don’t know if Theorem 1.5 holds for uncountable groups $G_n$.

The technical heart of this paper consists in proving:

Theorem 1.6. Consider a segment $I \subset (0,1)$. Let $G \subset \text{Diff}_+^1(I)$ be a group $C^1$-close to the identity and $f \in \text{Diff}_+^1([0,1])$ be a diffeomorphism such that

$$f(I) \cap I = \emptyset$$

(in other words, $I$ is contained in the interior of a fundamental domain of $f$).

Then the group $<f, G>$ generated by $f$ and $G$ is $C^1$-close to the identity.

Actually, we will prove in Theorem 8.1 a slightly stronger version, where $I$ is not assumed to be contained in the interior of a fundamental domain of $f$, but only contained in a fundamental domain. This weaker hypothesis requires extra technical conditions.

Remark 1.3. Under the hypotheses of Theorem 1.6, any conjugates $f^i G f^{-i}$ and $f^j G f^{-j}$, $i \neq j$, have disjoint supports and therefore commute. Actually, the group $<f, G>$ generated by $f$ and $G$ is isomorphic to

$$<G, f> = \left( \bigoplus_{\mathbb{Z}} G \right) \times \mathbb{Z}$$

where the factor $\mathbb{Z}$ is generated by $f$ and acts on $(\bigoplus_{\mathbb{Z}} G)$ by conjugacy as a shift of the $G$ factors (see Lemma 7.12).

1.4.2 Elementary groups and fundamental systems

We now define a class of subgroups $G \subset \text{Diff}_+^1([0,1])$, called elementary groups, whose topological dynamics implies that they are $C^1$-close to the identity: every group $G'$ topologically conjugated to $G$ is $C^1$-close to the identity.

Definition 1.4. A collection $(f_n)_{n \in \mathbb{N}}$, $f_n \in \text{Diff}_+^1(\mathbb{R})$ is called a fundamental system if for every $n \in \mathbb{N}$, $\mathbb{R} \setminus \text{Fix}(f_n)$ consists in a unique connected component with compact closure $S_n$ (called the support of $f_n$) and for every $n \in \mathbb{N}$ there is a fundamental domain $I_n$ of $f_n$ such that, for every $i, j \in \mathbb{N}$ we have the following property:

- either $S_i \subset I_j$
- or $S_j \subset I_i$
- or else $f_i$ and $f_j$ have supports with disjoint interiors

$$\hat{S}_i \cap \hat{S}_j = \emptyset$$

A group $G \subset \text{Diff}_+^1([0,1])$ is said to be an elementary group if it is generated by a fundamental system supported in $[0,1]$.

Remark 1.5. Every group $G' \subset \text{Diff}_+^1([0,1])$ topologically conjugated to an elementary group is an elementary group.

Our main result is:

Theorem 1.7. Every elementary group $G \subset \text{Diff}_+^1([0,1])$ is $C^1$-close to the identity.

Theorem 1.7 is obtained as a consequence of Theorem 8.1 (the stronger version of Theorem 1.6 above) and Theorem 1.5: by Theorem 1.5 one only needs to consider groups generated by a finite fundamental system, and for these groups, Theorem 8.1 enables us to argue by induction on the cardinal of the fundamental system.

The groups contained in an elementary group are very specific. In particular, we can prove:

Proposition 1.6. Every finitely generated group contained in an elementary group is solvable.
1.5 Examples and counter examples

A simple solvable (non nilpotent) group is the Baumslag-Solitar group whose presentation is

\[ B(1, n) = \langle a, b | aba^{-1} = b^n \rangle, \]

where \( n \) is an integer such that \(|n| > 1\). This group has a very natural affine action on \( \mathbb{R} \) (\( a \) acts as the homothety of ratio \( n \) and \( b \) is a translation) and on the circle (by identifying the affine group to a subgroup of \( PSL(2, \mathbb{R}) \)) and therefore on the segment \([0, 1]\) (by opening the fixed point \( \infty \) of the circle).

In particular, \( B(1, n) \) has analytic actions on \([0, 1]\). These analytic actions have been classified in [BW]. All these actions have linked pairs of successive fixed points, and therefore are not \( C^1 \)-close to the identity. On the other hand, \( B(1, n) \) admits \( C^0 \)-actions without linked fixed points, but [BMNR] shows that these \( C^0 \)-actions cannot be \( C^1 \).

Indeed, [Mc] shows the following Theorem, which holds in any dimension:

**Theorem 1.8 (A. McCarthy).** For every \( n, |n| > 1 \), and any compact manifold \( M \), there is a \( C^1 \)-neighborhood \( U_n \) of the identity in \( Diff^1(M) \) such that every morphism \( \rho : B(1, n) \to Diff^1(M) \) with \( \rho(a) \in U_n \) and \( \rho(b) \in U_n \) satisfies \( \rho(b) = id \).

In particular, no subgroup of \( C^1_{id} \) is isomorphic to \( B(1, n) \). Actually, [Mc] considers a more general class of groups called *abelian by cyclic*:

\[ G_A = \langle a, b_1, \ldots, b_k | b_ib_j = b_jb_i, \forall i, j \text{ and } fg_if^{-1} = g_1^{a_{i1}} \cdots g_k^{a_{ij}}, \forall i > \rangle, \]

where \( A = (a_{i,j}) \in GL(k, \mathbb{R}) \) has integer entries. [Mc] shows that, if \( A \) has no eigenvalue of modulus 1, then there is no faithful action on compact manifolds such that the generators belong to a neighborhood \( U_A \) of the identity in \( Diff^1(M) \). In particular, no subgroup in \( C^1_{id} \) is isomorphic to \( G_A \). Actually, [BMNR] showed recently that, for every faithfull \( C^1 \)-action of the group \( G_A \) on \([0, 1]\), the diffeomorphism associated to \( a \) admits a hyperbolic fixed point whose derivative is an eigenvalue of the matrix \( A \). Consequently, no group in \( C^1_{nonhyp} \) is isomorphic to \( G_A \).

This shows that there are algebraic obstructions for a group admitting faithful actions on \([0, 1]\) to be \( C^1 \)-close to identity. It is natural to ask if \( C^1_{id} \) contains free groups. The answer is positive as stated below:

**Theorem 1.9.** There is a group \( G \subset Diff^1([0, 1]) \), \( C^1 \)-close to the identity (and totally rational), such that \( G \) is isomorphic to the free group \( \mathbb{F}_2 \).

The proof of Theorem 1.9 consists in building a group \( G_\omega = \langle f_\omega, g_\omega \rangle \subset Diff^1_+([0, 1]) \), for all the reduced words \( \omega \), so that \( G_\omega \) is \( C^1 \) close to the identity and the pair \( \{f_\omega, g_\omega\} \) does not satify the relation \( \omega \). Then, we glue together the groups \( G_\omega \) in a way so that their supports are pairwize disjoint. Theorem 1.6 is our main tool for building the groups \( G_\omega \).

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### 2 Isotopies versus sequences of conjugacy to identity.

#### 2.1 The cohomological equation and the proof of Theorem 1.1

The aim of the section is the proof (suggested by A. Navas) of Theorem 1.1. Let us first start with the following observation:

**Lemma 2.1.** Consider \( f \in Diff^1_+([0, 1]) \) and a sequence \( \{h_n\}_{n \in \mathbb{N}} \) with \( h_n \in Diff^1_+([0, 1]) \). Then

\[
\left( h_n f h_n^{-1} \xrightarrow{C^1} Id \right) \Leftrightarrow \left( \log Dh_n(f(x)) - \log Dh_n(x) \xrightarrow{\text{unif}} - \log Df(x) \right)
\]
Proof: Just notice that the right term means that \( \log D(h_n f h_n^{-1}(h(x))) \) converges uniformly to 0, that is, \( D(h_n f h_n^{-1}) \) converges uniformly to 1. This implies that \( h_n f h_n^{-1} \) is \( C^1 \)-close to an isometry of \([0, 1]\), that is, to the identity map. \(\square\)

A straightforward calculation implies:

**Corollary 2.2.** Assume that \( \psi_n : [0, 1] \to \mathbb{R} \) is a sequence of continuous maps satisfying

\[
\psi_n(f(x)) - \psi_n(x) \xrightarrow{\text{uniform}} - \log Df(x).
\]

Then \( h_n : [0, 1] \to \mathbb{R} \) defined by

\[
h_n(x) = \frac{\int_0^x e^{\psi_n(t)} \, dt}{\int_0^1 e^{\psi_n(t)} \, dt}
\]

is a sequence of diffeomorphisms of \([0, 1]\) such that \( h_n f h_n^{-1} \xrightarrow{C^1} \text{Id.} \)

For any continuous map \( \psi : [0, 1] \to \mathbb{R} \), we will denote by \( h_\psi \in Diff^1_+(\mathbb{R}) \) the diffeomorphism defined as above, that is:

\[
h_\psi(x) = \frac{\int_0^x e^{\psi(t)} \, dt}{\int_0^1 e^{\psi(t)} \, dt}
\]

Notice that:

- \( \psi \mapsto h_\psi \) is a continuous map from \( C^0([0, 1], \mathbb{R}) \) to \( Diff^1_+(\mathbb{R}) \).
- \( h_{\log Df} = g \) for every \( g \in Diff^1_+(\mathbb{R}) \).

Consequently, finding diffeomorphisms \( h_n \) conjugating \( f \) \( C^1 \)-close to the identity is equivalent to find almost solutions of the cohomological equation. The advantage of this approach is that the cohomological equation is a linear equation. As a consequence, convex sums of almost solutions are still almost solutions.

**Lemma 2.3.** Assume that \( \psi_n : [0, 1] \to \mathbb{R} \) is a sequence of continuous maps satisfying

\[
\psi_n(f(x)) - \psi_n(x) \xrightarrow{\text{uniform}} - \log Df(x).
\]

Let \( \psi_n, t \in [0, +\infty) \) be defined as follows:

- if \( t = n \in \mathbb{N} \), then \( \psi_t = \psi_n \);
- if \( t \in (n, n+1) \), then \( \psi_t(x) = (n + 1 - t)\psi_n(x) + (t - n)\psi_{n+1}(x) \).

Then \( \{h_t = h_{\psi_t} \}_{t \in [0, +\infty)} \) is a continuous path of diffeomorphisms satisfying

\[
h_t f h_t^{-1} \xrightarrow{C^1_{t \to +\infty}} \text{id.}
\]

**Proof of Theorem 1.1:** Let \( G \) be a group and assume that \( h_n \) is a sequence of diffeomorphisms such that \( h_n g h_n^{-1} \xrightarrow{C^1} \text{id} \) for every \( g \in G \). Let \( \psi_n \) denote \( \log Dg \) and define \( \psi_t, t \in [0, +\infty) \), as in Lemma 2.3 as convex sums of the \( \psi_n \). One denotes \( h_t = h_{\psi_t} \). Notice that, for \( t = n \in \mathbb{N} \), one has \( h_t = h_n \), so that the notation is coherent. Now, for every \( g \in G \), one has:

\[
h_t g h_t^{-1} \xrightarrow{C^1_{t \to +\infty}} \text{id.}
\]

**2.2 Increasing union of groups \( C^1 \)-close to the identity:** proof of Theorem 1.5

**Proof of Theorem 1.5:** Let \( \{G_n\}_{n \in \mathbb{N}} \) be an increasing sequence of finitely generated groups \( C^1 \)-close to the identity. For every \( n \in \mathbb{N} \), let \( k_n \) be an integer such that \( S_n = \{g_1, \ldots, g_{k_n}\} \) is a system of generators of \( G_n \).

We denote \( E_n = \bigcup_{i \leq n} S_i \). As \( G_n \) is \( C^1 \)-close to the identity, there is a sequence \( h_{n,i} \in Diff^1_+(\mathbb{R}) \) such that

\[
\forall g \in G_n, \quad h_{n,i} g h_{n,i}^{-1} \xrightarrow{C^1_{i \to +\infty}} \text{id.}
\]
Fix a sequence $\varepsilon_n > 0$ such that $\varepsilon_n \rightarrow 0$. For every $n$, there is $i(n)$ so that

$$\forall g \in \mathcal{E}_n, \quad \|h_n,i(n)g h_n^{-1,i(n)} - id\|_1 < \varepsilon_n.$$  

As a straightforward consequence, for every $g \in G = \bigcup_{n \in \mathbb{N}} G_n$ one has

$$h_n,i(n)g h_n^{-1,i(n)} \xrightarrow{n \to \infty} id.$$  

According to Theorem 3.1 this is equivalent to the fact that $G$ is $C^1$-close to the identity, ending the proof.  

As any countable group is an increasing union of finitely generated groups, we easily deduce:

**Corollary 2.4.** If $\{G_n\}_{n \in \mathbb{N}}$ is an increasing sequence of countable groups and if $G_n \in C^1_{id}$ for all $n$, then

$$G = \bigcup_{n \in \mathbb{N}} G_n \in C^1_{id}.$$  

### 3 Relative translation numbers for groups $C^1$-close to identity

The aim of this section is to give a direct proof of Theorem 1.4 in the case of groups of diffeomorphisms $C^1$-close to identity. This proof was essentially done (and never written) by the first author with S. Crovisier and A. Wilkinson in 2003. Section 4.2 presents the (later) proof due to A. Navas, with completely different arguments, for groups of homeomorphisms without linked fixed points.

Let us restate Theorem 3.1 in the setting of groups $G \in C^1_{id}$:

**Theorem 3.1.** Let $G \subset Diff^1([0,1])$ be a subgroup $C^1$-close to identity, $f$ in $G$, and $I$ a connected component of $[0,1] \setminus Fix(f)$. Assume $(f(x) - x) > 0$ for $x \in I$. Then:

- for every $g \in G$, either $g(I) = I$ or $g(I) \cap I = \emptyset$.
- Let denote $G_I = \{g \in G \mid g(I) = I\}$ the stabilizer of $I$. There is a unique group morphism $\tau_{f,I} : G_I \to \mathbb{R}$ with the following properties:
  - the kernel $\text{Ker}(\tau_{f,I})$ is precisely the set of elements $g \in G$ having a fixed point in $I$.
  - $(g \in G_I$ and $\tau_{f,I}(g) = 0) \iff Fix(g) \cap I \neq \emptyset$
  - $\tau_{f,I}$ is increasing : given any $g, h \in G_I$, assume there is $x \in I, g(x) \geq h(x) \implies \tau_{f,I}(g) \geq \tau_{f,I}(h)$.
  - $\tau_{f,I}(f) = 1$.

Theorem 3.1 is the aim of this Section 3.

### 3.1 Background on diffeomorphisms $C^1$-close to identity and compositions

According to [Bo0] Lemma 4.3.B.1 one has:

**Lemma 3.1.** Let $M$ be a compact manifold. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, there is $\delta > 0$ so that, for any $f \in Diff^1(M)$ whose $C^1$-distance to identity is less than $\delta$, for every $x \in M$, for any $y$ with $\|y - x\| < n\|f(x) - x\|$, one has:

$$\|(f(y) - y) - (f(x) - x)\| < \varepsilon\|f(x) - x\|.$$  

As a consequence [Bo0] shows:

**Lemma 3.2.** [Bo0] Lemma 4.3.D.1 Let $M$ be a compact manifold. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, there is $\delta > 0$ so that, for any $f_1, \ldots, f_n \in Diff^1(M)$ whose $C^1$-distance to identity is less than $\delta$, for every $x \in M$, one has:

$$\|(f_n \cdots f_1(x) - x) - \sum_i (f_i(x) - x)\| < \varepsilon \sup_i\|f_i(x) - x\|.$$  

3.2 Translation numbers

In this section, $G \subseteq Diff^1_+(\mathbb{R})$ denotes a group $C^1$-close to the identity. Thus, there is a $C^1$-continuous path $h_t \in Diff[0,1]$, $t \in [0,1)$, so that $h_t h_t^{-1}$ $C^1$-tends to id as $t \to 0$, for every $g \in G$. For every element $g$ of $G$, we will denote $g_t = h_t h_t^{-1}$. For every point $x \in [0,1]$ we denote $x_t = h_t(x)$.

As a direct consequence of Lemma 3.2 one gets:

**Lemma 3.3.** Let $g \in G$ and $x \in [0,1]$ so that $g(x) \neq x$. Then, for every $n \in \mathbb{Z}$, one has:

$$\lim_{t \to 1} \frac{g^n(x_t) - x_t}{g(x_t) - x_t} = n$$

One deduces

**Corollary 3.4.** Consider $f, g \in G$ and $x \in [0,1]$ so that $f(x) > x$. Assume that there are $n > 0$ and $m \in \mathbb{Z}$ so that

$$g^n(x) \in [f^m(x), f^{m+1}(x)].$$

Then, for every $t$ close enough to 1 one has:

$$\frac{g_t(x_t) - x_t}{f(x_t) - x_t} \in \left[\frac{m-1}{n}, \frac{m+2}{n}\right].$$

Analogous statements hold in the case $f(x) < x$ or $n < 0$.

**Proof:** The statement is immediately satisfied if $g(x) = x$. We assume now $g(x) \neq x$.

The conjugacy by $h_t$ preserves the order. Thus, by assumption, one has

$$\frac{g^n_t(x_t) - x_t}{f^m(x_t) - x_t} \in \left[\frac{m}{n}, \frac{m+1}{n}\right].$$

For every $t$ we denote $\alpha_t$ so that $\frac{g^n(x_t)-x_t}{f^m(x_t)-x_t} = \alpha_t \frac{m}{n} \frac{n(x_t)-x_t}{f(x_t)-x_t}$. Lemma 3.3 implies that $\alpha_t$ tends to 1 as $t \to 1$.

Thus

$$\frac{g_t(x_t) - x_t}{f(x_t) - x_t} \in \left[\frac{m}{n}, \frac{m+1}{n}\right].$$

For $t \to 1$, Lemma 3.3 implies that $\frac{m}{n} \frac{f^{m+1}(x_t)-x_t}{f^m(x_t)-x_t}$ tends to $\frac{m+1}{n}$. One concludes by choosing $t$ large enough so that $[\alpha_t^{-1} \frac{m}{n}, \alpha_t^{-1} \frac{m+1}{n}] \subset (\frac{m-1}{n}, \frac{m+2}{n})$.

And conversely:

**Corollary 3.5.** Consider $f, g \in G$ and $x \in [0,1]$ so that $f(x) > x$ and assume there are $n, m \in \mathbb{Z}$, $n > 0$, and a sequence $t_i \to 1$, $i \in \mathbb{N}$ so that

$$\frac{g_{t_i}(x_{t_i}) - x_{t_i}}{f(x_{t_i}) - x_{t_i}} \in \left[\frac{m}{n}, \frac{m+1}{n}\right].$$

Then $g^n(x) \in [f^{m-1}(x), f^{m+2}(x)]$

Analogous statements hold in the case $f(x) < x$ or $n < 0$. The proof of Corollary 3.5 follows from the same estimates as Corollary 3.4 and is left to the reader.

One deduces

**Corollary 3.6.** Consider $f \in G$ and $x$ so that $f(x) \neq x$. Then

1. For any $g \in G$ the ratio $\frac{g(x_t)-x_t}{f(x_t)-x_t}$ as a limit $\tau_f(g,x) \in \mathbb{R} \cup \{-\infty, +\infty\}$ as $t \to 1$.
2. $\tau_{f^{-1}}(g,x) = \tau_f(g^{-1},x) = -\tau_f(g,x)$ with the convention $-\infty = +\infty$.
3. $\tau_f(g,x) \in \{-\infty, +\infty\}$ if and only if $f$ has a fixed point in $(x,g(x))$ or $(g(x),x)$ (according to the sign of $g(x) - x$).
4. let denote $G_{x,f} = \{ g \in G, \tau_f(g,x) \in \mathbb{R} \}$. Then $G_{x,f}$ is a subgroup of $G$ containing $f$ and $\tau_f(g,x): G_{x,f} \to \mathbb{R}$ is a morphism of groups sending $f$ to $1$.

5. - if $\tau_f(g,x) \in \mathbb{R}^*$ then $\tau_g(f,x) = \frac{1}{\tau_f(g,x)}$.
- $\tau_f(g,x) \in \{-\infty, +\infty\} \iff \tau_g(f,x) = 0$

6. if $\tau_f(g,x) \in \mathbb{R}^*$, then for every $h \in G$ one has
- $\tau_f(h,x) \in \{-\infty, +\infty\} \iff \tau_g(h,x) \in \{-\infty, +\infty\}$
- $\tau_f(h,x) \in \mathbb{R} \Rightarrow \tau_g(h,x) = \tau_g(f,x)\tau_f(h,x)$.

7. $\tau_f(g,x) = 0 \iff g$ has a fixed point in $[x,f(x)]$.

**Proof:** First notice that the sign of $\tau_f(g,x)$ is determined by the sign of $(f(x) - x)(g(x) - x)$.

For the first item, assume that there is a sequence $t_i$ so that $\frac{g_i(x_i) - x_i}{f_i(x_i) - x_i}$ is bounded, and hence, up to consider a subsequence, converges to some $\tau \in \mathbb{R}$. Then Corollary 4.3 implies that any rational estimate of $\tau$:

$$\frac{m}{n} < \tau < \frac{m+1}{n}$$

leads to a dynamical estimate of $g^n(x)$. Now Corollary 4.3 implies that $\frac{g_i(x_i) - x_i}{f_i(x_i) - x_i}$ belongs to $[\frac{m-2}{n}, \frac{m+3}{n}]$ for any $t$ close enough to $1$.

By considering finer rational estimates, one easily deduces that $\frac{g_i(x_i) - x_i}{f_i(x_i) - x_i}$ converges to $\tau$. This concludes the proof of item 1.

Item 2) is a direct consequence of Lemma 4.3. Item 3) is a direct consequence of Corollaries 4.4 and 4.5. Assuming $g(x) > x$ and $f(x) > x$, notice that if $f$ has no fixed points in $[x,g(x)]$ then there is $n > 0$ with $f^n(x) > g(x)$ and therefore Corollary 4.4 $\frac{g_i(x_i) - x_i}{f_i(x_i) - x_i}$ is bounded by $n+1$ for any $t$ close to $1$.

Item 4) is a direct consequence of Lemma 4.2. Items 5) and 6) are easy consequences of the definition. Item 7) is a direct consequence of items 3) and 5).

**Lemma 3.7.** Consider $f \in G$ and $x \not\in \text{Fix}(f)$. Let $I$ be the connected component of $[0,1] \setminus \text{Fix}(f)$ containing $x$. Then,

- for every $g \in G$ and any $y \in I$ one has
  $$\tau_f(g,x) = \tau_f(g,y).$$

We will denote it $\tau_{f,I}(g)$

- $\tau_{f,I}(g) \in \mathbb{R} \iff g(I) = I$, that is, $g$ belongs to the stabilizer $G_I$ of $I$.
- $\tau_{f,I}: G_I \to \mathbb{R}$ is a morphism of group sending $f$ to $1$.
- If $\tau_{f,I}(g) \in \{-\infty, +\infty\}$ then $g(I)$ and $I$ are disjoint.

**Proof:** If $\tau_f(g,x) = 0$, then Corollary 3.6 asserts that $g$ as a fixed point in the interior of $I$. One easily deduces that $\tau_f(g,y) = 0$ for every $y \in I$. Therefore, Corollary 3.6 implies that $[y, f(y)) \cap \text{Fix}(g) \neq \emptyset$ for every $y \in I$. This implies in particular that the end points of $I$ are fixed points of $g$, so that $g \in G_I$.

Assume now that $\tau_f(g,x) \in \mathbb{R}^*$. Thus $g$ has no fixed point in $I$. Let $J$ be the connected component of $x$ in $[0,1] \setminus \text{Fix}(g)$. We have seen $\tau_g(f,x) = \frac{1}{\tau_f(g,x)} \in \mathbb{R}^*$, so that $f$ has no fixed points in $J$. One concludes $I = J$, implying that $g \in G_I$.

Furthermore, Lemma 4.4 implies that, for $t$ close enough to $1$, $g_i(y_t) - y_t$ and $f_i(y_t) - x_t$ are almost constant on several fundamental domains (of $g_t$ and $f_t$, respectively), the error term being small compared with $g_t(x_t) - x_t$ and $f_t(x_t) - x_t$ respectively. As, by hypothesis, $g_t(x_t) - x_t$ and $f_t(x_t) - x_t$ remains in bounded ratio as $t \to 1$, one gets that the error term is small with respect to both. As a consequence, one gets that $\tau_g(y_t) = \lim_{t \to 1} \frac{g_i(y_t) - y_t}{f_i(y_t) - y_t}$ is locally constant, hence is constant on $I$. Finally, assume that $\tau_f(g,x) \in \{-\infty, +\infty\}$. This is equivalent to $\tau_g(f,x) = 0$. Let $J$ denote the connected component of $x$ in $[0,1] \setminus \text{Fix}(g)$. Then we have seen that each fundamental domain of $g$ in $J$ contains a fixed point of $f$. Notice that the extremities of $J$ are disjoint from $I$, when at least one of the extremities of $I$ is contained in $J$. One easily concludes that $I \subset J$.

Now let denote $(a,b) = I$. Up to replace $g$ by $g^{-1}$, let us assume $g(x) - x > 0$ so that $g(y) - y > 0$ for every $y \in J$. Now for every small $\varepsilon > 0$, $[a + \varepsilon, g(a + \varepsilon)]$ contains a fixed point of $f$; as there are no fixed points of $f$ between $a$ and $b$ one deduces $g(a + \varepsilon) \geq b$. As a consequence, $g(I) \cap I = \emptyset$, ending the proof.

We conclude the construction of the relative translation number $\tau_{f,I}$, and the proof of Theorem 5.1, by showing
Lemma 3.8. Let $f \in G$ and $x$ so that $f(x) > x$. Let $I$ be the component of $x$ in $[0,1] \setminus Fix(f)$. Then, for $g,g'$ in the stabilizer $G_I$, one has:

$$\exists y \in Ig(y) \geq g'(y) \Rightarrow \tau_{f,I}(g) \geq \tau_{f,I}(g')$$

Proof: Just notice than $\frac{g(t)-y}{f(t)-y} \geq \frac{g(t)-y}{f(t)-y}$ for every $t$. \qed

3.3 Groups of diffeomorphisms without linked fixed points

Remark 3.9. A group $G \subset Diff^1_+(\mathbb{I})$ is without linked fixed points if and only if, given any $f,g \in G$, and given $[a,b]$ and $[c,d]$ be connected components of $[0,1] \setminus Fix(f)$ and $[0,1] \setminus Fix(g)$, respectively, then

either $[a,b] \cap [c,d] = \emptyset$ or $[a,b] = [c,d]$ or $[a,b] \subset [c,d]$ or $[a,b] \supset [c,d]$

Next straightforward lemma gives another formulation of being without linked fixed points:

Lemma 3.10. A group $G \subset Diff^1_+(\mathbb{I})$ is without linked fixed points if and only if given $\{a,b\}$ and $\{c,d\}$ be successive fixed points of $f \in G$ and $g \in G$, respectively, then

either $\{c,d\} \cap [a,b] = \emptyset$ or $[c,d] \subset [a,b]$

Theorem 3.2. Let $G \subset Diff^1_+(\mathbb{I})$ be a subgroup $C^1$-close to identity. Then $G$ is without linked fixed points.

Proof: Let $f,g \in G$, and $I$ and $J$ be connected components of $[0,1] \setminus Fix(f)$ and $[0,1] \setminus Fix(g)$ respectively. Assume that one end point of $J$ belongs to $I$. This implies that $\tau_{J,I}(g) = 0$. Therefore, according to Corollary 3.6, $g$ has fixed points in every fundamental domains of $f$ in $I$. This implies that $J$ is contained in the interior of $I$.

If no end point of $I$, $J$ belongs to the other interval, this implies that either $I = J$ or $I \cap J = \emptyset$. \qed

4 Groups without linked fixed points

4.1 Groups without linked fixed points and without crossing

In [Na], Navas defines the notion of groups without crossing as follows:

Definition 4.1. One says that a group $G \subset Homeo_+(\mathbb{I})$ of homeomorphisms is without crossing if given any $f,g \in G$, one has the following property:

Let $\{a,b\}$ be a connected components of $[0,1] \setminus Fix(f)$ Then, $\{g(a),g(b)\} \cap (a,b) = \emptyset$

These two notions are equivalent:

Lemma 4.2. A group $G \subset Homeo_+(\mathbb{I})$ is without crossing if and only if it has no linked fixed points.

Proof: Assume first that $G$ admits crossing: thus there is $f \in G$ and $a,b$ be successive fixed point of $f$, and $g \in G$ so that $g(b) \in (a,b)$. Let $c,d$ be the successive fixed points of $g$ so that $b \in (c,d)$. Then $b \in (c,d)$ but $[a,b] \not\subseteq (c,d)$ so that $G$ is not without linked fixed points.

Assume now that $g$ has no fixed points in $[a,b]$. Then $gf^{-1}$ admits $c = g(a) < a$ and $d = g(b) \in (a,b)$ as successive fixed points so that $[a,b]$ and $[c,d]$ are linked pairs of successive fixed points.

Conversely assume that $[a,b]$ and $[c,d]$ are linked pairs of fixed points, of $f \in G$ and $g \in G$, respectively. Up to reverse the orientation or to exchange the role of $f$ and $g$, one may assume that $d \in (a,b)$ and $c \not\in (a,b)$. Up to exchange $f$ with $f^{-1}$, one may assume $f(d) < d$. Therefore $c \leq a < f(d) < d$ so that $f(d) \in (c,d) : G$ admits a crossing. \qed
4.2 Dynamics of groups without crossing

In this section $G \subset \text{Homeo}_+([0,1])$ is a group without crossing (or equivalently, without linked fixed points). We fix an element $f \in G$ and a connected component $I$ of $[0,1] \setminus \text{Fix}(f)$. As $G$ is without linked fixed point, for any $g \in G$

- either $g(I) \cap I = \emptyset$,
- or $g(I) = I$, that is, $g$ belongs to the stabilizer $G_I$ of $I$.

In this section, we consider the dynamics of $G_I$ in restriction to $I$.

As $f$ has no fixed points (by assumption) on $I$, every orbit $G_I(y)$, $y \in I$ meets a (compact) fundamental domain $[x, f(x)]$ of $f$. One easily deduces (using Zorn lemma) that

**Lemma 4.3.** The action of $G_I$ on $I$ admits minimal closed sets.

Furthermore, the following classical result explains what are the possibilities :

**Lemma 4.4.** Let $H \subset \text{homeo}_+(\mathbb{R})$ be a group and assume there if $h \in H$ without fixed point. Then $H$ admits minimal sets, the union of the minimal sets is closed, and

- either there is a unique minimal set which is either $\mathbb{R}$ or the product $C \times \mathbb{Z}$ where $C$ is a Cantor set.
- or there is $h \in H$ without fixed points so that every minimal of $G$ is exactly one orbit of $g$.

Let denote by $U \subset I$ the union of the open sets $I \setminus \text{Fix}(g)$, for $g \in G$ with $\text{Fix}(g) \cap I \neq \emptyset$. That is :

$$U = \bigcup_{\{g \in G, \text{Fix}(g) \cap I \neq \emptyset}\}} I \setminus \text{Fix}(g).$$

$U$ is an open set as union of open sets. The no linked fixed points property implies that each connected component of $U$ is the union of an increasing sequence of connected components of $I \setminus \text{Fix}(g_n)$; as a consequence, each connected component of $U$ is contained in a fundamental domain $[x, f(x)]$, $x \in I$.

**Remark 4.5.** If $g$ has a fixed point $x$ in $I$, then it has another fixed point in $(x, f(x)]$; otherwise, the next fixed point of $g$ will be so that $(x, y)$ contains $f(x)$ and thus $G$ has a crossing.

Thus, $g$ has fixed points in every fundamental domain of $f$ in $I$.

One deduces that

**Lemma 4.6.** $\Lambda = I \setminus U$ is a nonempty closed subset invariant by $G_I$. Furthermore, for every $g \in G_I$ with fixed points in $I$, the restriction of $g$ to $\Lambda$ is the identity map.

$$\text{Fix}(g) \cap I \neq \emptyset \iff g|_{\Lambda} = id_{\Lambda}.$$  

**Proof :** Let $J$ be a component of $U$. If $g$ is the identity on $J$, in particular the extremities of $J$ are fixed point of $g$. Assume now that $x \in J$ is not fixed for $g$ and consider the pair $\{a, b\}$ of successive fixed points of $g$ around $x$. By construction of $U$, $(a, b) \subset J$. One deduces easily that the extremities of $J$ are fixed points of $g$. Thus $g$ is the identity map on the boundary $\partial U$. Finally, every point $x \in I \setminus \text{Fix}(g)$ is contained in a connected component of $U$, so that $g$ is the identity map on $\Lambda$, as announced. \hfill $\Box$

Let $G_0^I$ be the set of elements $g \in G$ so that $\text{Fix}(g) \cap I \neq \emptyset$. One deduces that

**Lemma 4.7.** $G_0^I$ is a normal subgroup of $G_I$. Furthermore, $\Gamma = G_I/G_0^I$ induces a group of homeomorphisms of $\Lambda$ whose action is free (the non trivial element have no fixed points).

As a consequence one gets

**Lemma 4.8.** Assume that the action of $G_I$ on $I$ has a unique minimal $\mathcal{M}$ set which is either $I$ or $C \times \mathbb{Z}$. Then, there is an increasing continuous map from $\mathcal{M}$ to $\mathbb{R}$ which induces a semi-conjugacy of the action of $\Gamma$ on $\mathcal{M}$ with a dense group of translation of $\mathbb{R}$.

Otherwise, $\Gamma$ is monogene, and its action on $\Lambda$ is conjugated to a translation.

**Proof :** If the minimal set is $I$ itself, this implies that the elements of $G_I$ have no fixed points in $I$. Therefore, Hölder theorem implies that the action is conjugated to a dense translations group.

Assume now that the action of $G_I$ on $I$ has a unique minimal set $\mathcal{M}$ homeomorphic to $C \times \mathbb{Z}$. By collapsing each closure of connected component of the complement of the minimal on a point, one defines
a projection of $I$ to an interval, which is surjective on $\mathcal{M}$. The action passes to the quotient and defines a free minimal action on the quotient interval: this quotient action is therefore conjugated to a dense translations group.

When a minimal set $\mathcal{M}_0$ is the orbit of an element $g \in G$, every minimal set is an orbit of $g$; the union of the minimal sets is a closed subset $K$ on which the dynamics is generated by $g$. In this case, the minimal sets are precisely the orbits of $g$ which are invariant by the whole group. If the orbit of $x$ for $g$ is not a minimal of the action, there is $h \in G_I$ so that $x \neq h(x)$ but $x$ and $h(x)$ are in the same fundamental domain for $g$ hence belong to the same connected component of $I \setminus K$; this implies that $h$ leaves invariant this component, hence has fixed points in $I$. This shows that $x$ belongs to the open set $U$. In other words, we have shown that $\Lambda = K$. As $g$ is a generator of the action on $K$, one deduces that $g$ is a generator of $\Gamma$, ending the proof.

As a consequence we proved Theorem 3.4 which provides for groups of homeomorphisms without crossing (or without linked fixed points) the same dynamical description as for group of diffeomorphisms $C^1$-close to identity, given by Theorem 3.3.

4.3 Countable family of intervals

Proposition 4.9. Let $G \subset \text{Homeo}_+([0,1])$ be a subgroup without crossing. Then, the family of pairs of successive fixed points of the elements of $G$ is at most countable.

Proof: Let $G$ be a group without crossing and let $\mathcal{P}_n$ be the set of pairs of succesive fixed points $\{a, b\}$ with $|a - b| \geq \frac{1}{n}$. For proving the lemma it is enough to prove that $\mathcal{P}_n$ is at most countable, for every $n \in \mathbb{N} \setminus \{0\}$.

Let $\{a, b\} \in \mathcal{P}_n$ be such a pair and $g \in G$ so that $\{a, b\}$ are successive fixed points of $g$. Up to replace $g$ by $g^{-1}$, one assumes that $g(x) - x > 0$ for $x \in (a, b)$.

Let $\varepsilon > 0$ so that $0 < g(x) - x < \frac{1}{n}$ for $x \in (a, a + \varepsilon)$ and denote $c = g^{-1}(a + \varepsilon)$.

Claim 1. Any pair $\{p, q\} \in \mathcal{P}_n$ is disjoint from $(a, c]$.

Proof: Assume $\{p, q\}$ meets $(a, c]$. As $G$ is without crossing and as $\{a, b\}$ is a pair of successive fixed points for $g$, the pair $\{p, q\}$ is contained in a fundamental domain of $g$ that is in $[p, g(p)]$. However our choice of $c$ implies that $|p - q| < \frac{1}{n}$ contradicting the fact that $\{p, q\} \in \mathcal{P}_n$.

Now, to every pair $\{a, b\}$, one can associated the largest open interval $J_{a,b} = (a, d)$ so that $(a, d)$ is disjoint from any pair $\{p, q\} \in \mathcal{P}_n$. In other words, $d = \inf\{p > a, 3q \text{ so that } \{p, q\} \in \mathcal{P}_n\}$. The claim asserts that this open interval $J_{a,b}$ is not empty (that is, $d > a$).

By construction, if $\{a_1,b_1\}, \{a_2,b_2\}$ are two distinct pairs in $\mathcal{P}_n$, then $J_{a_1,b_1} \cap J_{a_2,b_2} = \emptyset$. Now, any family of disjoint open intervals is countable, concluding.

4.4 A characterization of crossing, and entropy

In the next section we will show that groups of diffeomorphisms admitting a crossing have hyperbolic fixed points. The main step for the proof is the next lemma, which provides a dynamical characterization of the existence of crossings:

Lemma 4.10. Consider $G \subset \text{Homeo}_+([0,1])$. Assume that $G$ admits crossings. Then there are $h_1, h_2 \in G$ and a segment $I \subset [0,1]$ so that $h_1(I)$ and $h_2(I)$ are disjoint segments contained in $I$.

As a direct consequence, the topological entropy of the semi group generated by $h_1^{-1}$ and $h_2^{-1}$ is log 2.

Proof: According to Lemma 3.10 there are $f, g \in G$ and successive fixed points $\{a, b\}$ and $\{c, d\}$ of $f, g$ respectively, so that (up to reverse the orientation, and to exchange $f$ with $g$) $b \in (c, d)$ but $(a, b) \not\subseteq (c, d)$, that is $a < c < b < d$.

Up to exchange $f$ with $f^{-1}$ and $g$ with $g^{-1}$, on may assume that $f(x) - x > 0$ on $(a, b)$ and $g(x) - x > 0$ on $(c, d)$.

Consider $x_0 \in (a, c)$, and fix $I = [x_0, b]$. Then for $n > 0$ large $f^n(I)$ is a small segment in $I$ arbitrarily close to $b$. Then, for positive $m$, $g^{-m}f^n(I)$ form an infinite collection of disjoint segments contained in $(c, b)$ hence contained in the interior of $I$.

\hfill $\Box$
4.5 Group without hyperbolic fixed point

Recall that a group $G \subset Diff^1_+(\{0,1\})$ is \textit{without hyperbolic fixed points} if, for every $g \in G$, one has $Dg(x) = 1$ for every $x \in Fix(g)$. The aim of this section is to prove Theorem 1.3 that is, if $G$ is without hyperbolic fixed points, then it is without crossing.

We present here a proof of A. Navas. Let us start by stating two lemmas.

**Lemma 4.11.** Let $I$ be a segment and $f, g: I \to I$ be diffeomorphisms onto their images, and so that $f(I) \cap g(I) = \emptyset$. Then there is an infinite sequence $\omega_i, i \in \mathbb{N}, \omega_i \in \{f, g\}$ so that

$$\limsup \frac{1}{n} \log \ell(\omega_{n-1}\omega_{n-2} \cdots \omega_0(I)) < 0$$

where $\ell$ denotes the length.

The proof of Lemma 4.11 is postponed at the end of the section.

**Lemma 4.12.** Let $f, g \in Diff^1([0,1])$ be two $C^1$-diffeomorphisms and assume there is a segment $I$ for which there is an infinite word $\omega_i, i \in \mathbb{N}, \omega_i \in \{f, g\}$ so that

$$\limsup \frac{1}{n} \log \ell(\omega_{n-1}\omega_{n-2} \cdots \omega_0(I)) < 0$$

Then for every $t \in I$ one has

$$\limsup \frac{1}{n} \log |D((\omega_{n-1}\omega_{n-2} \cdots \omega_0)(x))| = \limsup \frac{1}{n} \log \ell(\omega_{n-1}\omega_{n-2} \cdots \omega_0(I)) < 0$$

**Proof:** This lemma looks like a distortion control, which does not exist in the $C^1$-setting. However the proof is an easy consequence of the uniform continuity of $Df$ and $Dg$: the length of $\omega_{n-1}\omega_{n-2} \cdots \omega_0(I)$ tends to 0, by assumptions.

Therefore

$$\log |D(\omega_n)(\omega_{n-1}\omega_{n-2} \cdots \omega_0(x))| - \log \frac{\ell(\omega_{n-1}\omega_{n-2} \cdots \omega_0(I))}{\ell(\omega_{n-1}\omega_{n-2} \cdots \omega_0(I))} \xrightarrow{\text{unif}} 0, \text{ for } n \to \infty$$

One concludes by noticing that

$$\frac{1}{n} \log |D((\omega_{n-1}\omega_{n-2} \cdots \omega_0)(x))| = \frac{1}{n} \sum_{0}^{n-1} \log |(D\omega_i)(\omega_{i-1} \cdots \omega_0(x))|,$$

and

$$\frac{1}{n} \log \ell(\omega_{n-1}\omega_{n-2} \cdots \omega_0(I)) = \frac{1}{n} \left( \log \ell(I) + \sum_{1}^{n-1} \frac{\ell(\omega_i \cdots \omega_0(I))}{\ell(\omega_{i-1} \cdots \omega_0(I))} \right)$$

Before giving the proof of Lemma 4.11 let us conclude the proof of Theorem 1.3

**Proof of Theorem 1.3:** Let $G \subset Diff^1_+(\{0,1\})$ be a group with a crossing. According to Lemma 4.10 there are $f, g \in G$ and a segment $I \subset [0,1]$ so that $f(I)$ and $g(I)$ are disjoint segments contained in $I$.

According to Lemma 4.11 there is a word $\omega \in \{f, g\}$ so that the size of the segments $\omega_{n-1} \cdots \omega_0(I)$ decreases exponentially. Then, Lemma 4.12 implies that for every $x \in I$ one has $|D\omega_{n-1} \cdots \omega_0(x)| < 1$. As $\omega_{n-1} \cdots \omega_0(I) \subset I$ there is a fixed point in $I$ and this fixed point has derivative < 1, so that it is hyperbolic.

This implies that $G$ is not without hyperbolic fixed point, concluding the proof.

**Proof of Lemma 4.11:**

For any $n \in \mathbb{N}$ let $\Omega_n = \{f, g\}^n$ be the set of words $(\omega_i)_{i \in \{0,\ldots,n-1\}}$ of length $n$, with letters $\omega_i \in \{f, g\}$. In particular the cardinal of $\Omega_n$ is

$$\# \Omega_n = 2^n$$

As the intervals $\omega_{n-1} \cdots \omega_0(I)$ are pairwise disjoint, one expects that the length is in general no much more than $\frac{1}{n^\ell(I)}$. 

Let denote
\[ B_n = \left\{ (\omega_i)_{i \in \{0, \ldots, n-1\}} \in \Omega_n | \ell(\omega_{n-1} \cdots \omega_0(I)) \geq \ell(I) \cdot \left(\frac{2}{3}\right)^n \right\}. \]

A simple calculation shows
\[ |B_n| \leq \left(\frac{3}{2}\right)^n. \]

Thus \( \frac{|B_n|}{\#\Omega_n} \leq \left(\frac{3}{4}\right)^n \).

Choose \( 0 < \varepsilon < 1 \) and \( T > 0 \) so that \( \sum_1^{\infty} \left(\frac{3}{4}\right)^{nT} < \varepsilon \).

Let denote
\[ B_n^T = \{ (\omega_i)_{i \in \{0, \ldots, nT-1\}} \in \Omega_{nT} | \exists i > 0, (\omega_i) \in B_i \}. \]

A simple calculation shows
\[ \frac{|B_n^T|}{\#\Omega_{nT}} \leq \sum_1^{n-1} \left(\frac{3}{4}\right)^{Tn}. \]

Let \( \Omega_\infty = \{ f, g \}^\mathbb{N} \) be the set of infinite words in letters \( f, g \). It's a Cantor set. Consider
\[ G_n^T = \{ (\omega_i)_{i \in \mathbb{N}} \in \Omega_\infty | (\omega_i)_{i \in \{0, \ldots, nT\}} \notin B_n^T \} \]

and
\[ G_\infty^T = \{ (\omega_i)_{i \in \mathbb{N}} \in \Omega_\infty | \forall n > 0, (\omega_i)_{i \in \{0, \ldots, nT\}} \notin B_n^T \} \]

Then \( G_n^T \) is a decreasing sequence of compact subsets of \( \Omega_\infty \) and
\[ G_\infty^T = \bigcap_{n>0} G_n^T. \]

The fact that \( \frac{|B_n^T|}{\#\Omega_{nT}} < 1 \) implies that \( G_n^T \) is not empty. One deduces that \( G_\infty^T \) is not empty (as a decreasing sequence of non-empty compact sets).

One concludes by noticing that, for every word \( (\omega_i)_{i \in \mathbb{N}} \in G_\infty^T \) one has
\[ \limsup \frac{1}{n} \log \ell(\omega_{n-1} \cdots \omega_0)(I) \leq \log\left(\frac{2}{3}\right). \]

\[ \square \]

**Remark 4.13.** The proof above give much more : if one endows \( \Omega_\infty \) with the measure whose weight on each cylinder of length \( n \) is \( \frac{1}{2^n} \), then for almost every word \( (\omega_i) \) in \( \Omega_\infty \) the exponential rate of decreasing of the length is upper bounded by \( -\log 2 \).

**5 Completion of a group without crossing**

**5.1 Complete group without crossing**

Consider a homeomorphism \( g \in \text{Homeo}_+([0,1]) \). We say that a homeomorphism \( h \in \text{Homeo}_+([0,1]) \) is **induced** by \( g \) if for every \( x \in [0,1] \), one has
\[ h(x) \in \{ x, g(x) \} \]

In other words, \( h \) is obtained from \( g \) in replacing \( g \) by the identity map in the union of some connected components of \( [0,1] \setminus \text{Fix}(g) \). More precisely, one easily check :

**Lemma 5.1.** Given \( g \in \text{Homeo}_+([0,1]) \), a homeomorphism \( h \) is induced by \( g \) if and only if there is a family \( I \) of connected components of \( [0,1] \setminus \text{Fix}(g) \) so that \( h \) is the map \( g_I \) defined as follows :
- if \( x \in \bigcup_{I \in I} I \), \( g_I(x) = g(x) \)
- otherwise \( g_I(x) = x \).
Definition 5.2. Let $G \subset \text{Homeo}_+([0,1])$ be a group without crossing. One says that $G$ is complete if, for any $g \in G$, any homeomorphism induced by $g$ belongs to $G$.

The aim of this section is to show

Proposition 5.3. Any group $G$ without crossing is contained in a complete group without crossing.

Let $G \subset \text{Homeo}_+([0,1])$ be a group without crossing. We will denote by $I(G) \subset \text{Homeo}_+([0,1])$ the group generated by all the elements $h$ induced by elements $g \in G$.

Remark 5.4. For every $h \in I(G)$ and $x \in [0,1]$ there is $g \in G$ with $h(x) = g(x)$. In other words, $I(G)$ and $G$ have the same orbits.

Lemma 5.5. Assume $G$ is without crossing. If \{a, b\} are successive fixed points for some $h \in I(G)$ then \{a, b\} are successive fixed points for some $g \in G$.

Proof: Assume, by absurd, that there is a pair of successive fixed points \{a, b\} for an element $h \in I(G)$, but which is not of successive fixed points for $G$. So, $h$ can be written as $h = h_n \cdots h_1$ where $h_i$ is induced by $g_i \in G$. We choose the pair \{a, b\} so that $n$ is the smallest possible. Fix a point $x \in (a, b)$.

For every $i \in \{1, \ldots, n\}$, let $a_i$ be the largest fixed points of $g_i$ less than or equal to $x$, and $b_i$ the smallest larger than or equal to $x$. If $g_i(x) \neq x$, then $\{a_i, b_i\}$ are successive fixed points of $g_i$ (otherwise, $a_i = b_i = x$). As $h(x) \neq x$, there is at least one index $i$ for which $g_i(x) \neq x$.

As $G$ is without crossing, the intervals $(a_i, b_i)$ are totally ordered for the inclusion. The union $I = \bigcup_i (a_i, b_i)$ is invariant by all the $g_i$: indeed, either $I$ is an interval of successive fixed points of $g_i$ or $g_i$ has a fixed point in $I$, and then belongs to the stabilizer of $I$. One deduces $I$ is fixed by all the $h_i$. As a consequence one gets $(a, b) \subset I$.

Notice that there is $j \in \{1, \ldots, n\}$ so that $I = (a_j, b_j)$. Let $i_1, \ldots, i_k$ be the set of indices so that $I = (a_{i_j}, b_{i_j})$. If there is $i$ so that $h_i = id$ on $I$, then $n$ was not the minimum number. As the indices $i_j$ have been chosen so that $I$ is a component of $[0,1] \setminus \text{Fix}(g_i)$, this implies that $h_i = g_i$ on $I$. Thus, one gets the same interval \{a, b\} if we substitute the $h_{i_j}$ by $g_{i_j}$. Thus we will now assume $h_{i_j} = g_{i_j}$.

One considers now the group generated by the restriction of the $g_i$ to $I$. It is a group without crossing and the restrictions of the $h_i$ to $I$ are induced by $g_i$. We consider a minimal set $M \subset I$ of the action of the group generated by the $g_i$ on $I$: every $g_i$, $i \notin \{i_1, \ldots, i_k\}$, has fixed points in $I$; according to Lemma 4.6, $g_i$ induces the identity map on $M$. Therefore the same happens for $h_i$, $i \notin \{i_1, \ldots, i_k\}$. So the action of $h$ on $M$ is the same as $g = g_n \cdots g_1$.

Let us now consider the translation number $\tau$ relative to $g_i$ on $I$. Then $\tau(g)$ is the sum of the $\tau(g_{i_j})$.

First assume that $\tau(g) \neq 0$. Thus the orbits of $g$ on the minimal $M$ goes from one extremities of $I$ to the other, and so do the orbit of $h$ on $M$. In particular, $I = \{a, b\}$, so that \{a, b\} is a pair of successive fixed point of $g_i$, contradicting the definition of \{a, b\}.

Thus $\tau(g) = 0$. This implies $k > 1$. Now we will use the fact that, if $f \in G$ and $h$ is induced by $g \in G$, then $fh^{-1}$ is induced by $fgf^{-1}$ in $G$. By using inductively the elementary fact $f(f^{-1}hf) = hf$, we can rewrite the word

$$h_n \cdots h_{n_k+1}g_{n_k}h_{n_k-1} \cdots h_{n_1+1}g_{n_1}h_{n_1-1} \cdots h_1 = g_{n_k} \cdots g_{n_1} \tilde{h}_{n-k} \cdots \tilde{h}_1$$

where the $\tilde{h}_i$ are induced by elements of $G$. However, $\tilde{g} = g_{n_k} \cdots g_{n_1}$ belongs to $G$. Thus, one can rewrite this word has $\tilde{g} \tilde{h}_{n-k} \cdots \tilde{h}_1$, which has only $n - k + 1 < n$ letters. This contradicts the fact that $n$ was chosen realizing the minimum.

Corollary 5.6. Assume that the group $G$ is without crossing. Then $I(G)$ is without crossing.

Let \{a, b\} be a pair of successive fixed points, for some $f \in G$. Then, the image of the translation number associated to \{a, b\} is the same for $G$ and $I(G)$:

$$\tau_{f,[a,b]}(G) = \tau_{f,[a,b]}(I(G)) \subset \mathbb{R}$$

Proof: Groups without crossing are the groups without linked fixed points. This property is a property of the sets of pairs of successive fixed points. According to Lemma 5.5, the pairs of successive fixed points are the same for $G$ and for $I(G)$, concluding.

Let now \{a, b\} be a pair of successive fixed points for some $g \in G$. Consider the stabilizer $I(G)_{[a,b]}$.

As $I(G)$ is without crossing, the translation number relative to $g$ extends on $I(G)_{[a,b]}$. Furthermore, the action of $G_{[a,b]}$ on $(a, b)$ admits a minimal set $M$. For every $x \in [0,1]$ and every $h \in I(G)$ there is $g \in G$ so that $h(x) = g(x)$. This implies that
Claim 2. The minimal set $\mathcal{M}$ is invariant under the action of $I(G)_{(a,b)}$

**Proof:** if $h \in I(G)$ and $x \in \mathcal{M}$, and if $h(x) \in (a,b)$, then there is $g \in G$ with $g(x) = h(x)$; then $g(x) \in \mathcal{M}$ (because $\mathcal{M}$ is invariant by $G_{(a,b)}$). One concludes $h(x) \in \mathcal{M}$. Thus the minimal set $\mathcal{M}$ is invariant by $I(G)_{(a,b)}$. \qed

Now the action of $I(G)_{(a,b)}/\text{Ker}(\tau_{g,(a,b)})$ is a free action. For every $h \in I(G)$ and $x \in \mathcal{M}$ there is $g \in G$ with $g(x) = h(x)$; this implies that $g$ and $h$ coincide on $\mathcal{M}$, and thus $\tau_{f,(a,b)}(h) = \tau_{f,(a,b)}(g)$, concluding. \qed

We don’t know if, in general, $I(G)$ is a complete group, that is, if $I(I(G)) = I(G)$. For this reason, let us denote $I^n(G)$ defined as $I^{n+1}(G) = I(I^n(G))$. The sequence $I^n(G)$ is an increasing sequence of groups. We denote

$$I^\infty(G) = \bigcup_{n \in \mathbb{N}} I^n(G)$$

Next Lemma ends the proof of Proposition 5.3:

**Lemma 5.7.** For every $G \subset \text{Homeo}_+(\mathbb{R})$ without crossing, $I^\infty(G)$ is a complete group without crossing. Furthermore:

- the orbits of $I^\infty(G)$ and of $G$ are equal;
- any pair of successive fixed points $\{a,b\}$ of $I^\infty(G)$ are a pair of successive fixed points of $G$;
- for any pair $\{a,b\}$ of successive fixed points of some $f \in G$, the images $\tau_{f,[a,b]}(G)$ and $\tau_{f,[a,b]}(I^\infty(G))$ are equal;
- any complete group without crossing containing $G$ contains $I^\infty(G)$.

**Proof:** The unique non-trivial point is that $I^\infty(G)$ is complete. For that, it is enough to show that, if $g \in I^\infty(G)$ then every homeomorphism $h$ induced from $g$ also belongs to $I^\infty(G)$. Notice that there is $n$ so that $g \in I^n(G)$; thus $h \in I^{n+1}(G)$, concluding. \qed

The group $I^\infty(G)$ is called the **completion** of $G$.

### 5.2 Completion of groups $C^1$-close to identity or without hyperbolic fixed points.

The aim of this section is to prove that the completion of groups $C^1$-close to identity or without hyperbolic fixed points are $C^1$-close to identity or without fixed points, respectively. Notice that:

**Remark 5.8.** For any sequence $\mathcal{H} = \{h_n, n \in \mathbb{N}\}$ of diffeomorphisms of $[0,1]$, let $G_\mathcal{H}$ be the set of diffeomorphisms so that $h_nh^{-1} \to \text{id}$ as $n \to \infty$. Then $G_\mathcal{H}$ is a group $C^1$-close to identity.

**Lemma 5.9.** Consider a group $G$, $C^1$-close to the identity, and $h_n$ a sequence of diffeomorphisms so that $h_nh^{-1} \to \text{id}$ for every $g \in G$. Consider an element $g \in G$, and a family $\mathcal{I}$ of connected components of $[0,1] \setminus \text{Fix}(g)$.

Then, the induced map $g_\mathcal{I}$ (equal to $g$ on the components in $\mathcal{I}$ and equal to $\text{id}$ out of these components), is a $C^1$ diffeomorphism of $[0,1]$. Furthermore, $h_nh_\mathcal{I}h^{-1} \to \text{id}$ as $n \to \infty$. In other words, $g_\mathcal{I} \in G_\mathcal{H}$.

**Proof:** First notice that $g$ has no hyperbolic fixed point: the derivative of $g$ is 1 at each extremities of the components of $\mathcal{I}$. One deduces that $g_\mathcal{I}$ is a diffeomorphism.

Then, $h_nh_\mathcal{I}h^{-1}$ is induced by $h_nh^{-1}$. Therefore its $C^1$-distance to identity is smaller that the distance from $h_nh^{-1}$ to identity. \qed

**Corollary 5.10.** Consider a group $G$, $C^1$-close to identity, and a sequence $\mathcal{H} = \{h_n\}$ of diffeomorphisms so that $h_nh^{-1} \to \text{id}$ for every $g \in G$. Then the completion $I^\infty(G)$ is contained in $G_\mathcal{H}$. In particular, $I^\infty(G)$ is $C^1$-close to identity.

**Lemma 5.11.** If $G \subset \text{Diff}^+_1([0,1])$ is a group without hyperbolic fixed points, then the completion $I^\infty(G)$ is contained in $\text{Diff}^+_1([0,1])$ and is without hyperbolic fixed points.
Proof : It is enough to show that \( I(G) \) is a group of diffeomorphism without hyperbolic fixed points.

As seen before, as the elements \( g \in G \) are diffeomorphism without hyperbolic fixed points, every induced map is a diffeomorphism. It remains to show that \( I(G) \) is without hyperbolic fixed points.

Assume that \( a \) is a hyperbolic fixed point of an element \( h \in I(G) \). Then \( a \) is an isolated fixed point of \( h \). Let \( b \) be the next fixed point, that is \( \{a, b\} \) is a pair of successive fixed points of \( h \). According to Lemma \( \ref{lem:hyperbolic_fixed_points} \) there is \( g \in G \) so that \( \{a, b\} \) are successive fixed points of \( g \). Thus \( \tau_{g,(a,b)}(h) \) is well defined and finite. However, the derivative \( g'(a) \) is 1 because \( G \) is without hyperbolic fixed points. One easily deduces that \( h'(a) = 1 \) (otherwise, \( \tau_{g,(a,b)}(h) \) is infinite), contradicting the hypothesis. \( \Box \)

5.3 Algebraic presentation : specific subgroups

The finitely generated groups \( C^1 \)-close to identity may be complicated. However the complete groups without crossing admits special subgroups with a simple presentation.

An element in \( G \) is called \textit{simple} if \( [0,1] \setminus \text{Fix}(g) \) consist in a unique interval whose closure is the support of \( g \) and denoted \( \text{supp}(g) \).

Remark 5.12. 1. Let \( G \subset \text{Homeo}_+(\,0,1\,\) be a complete group without crossing. Each \( g \in G \) is the limit, for the \( C^0 \)-topology, of the products of the induced simple elements \( g_1 \) where \( I \) covers the set of connected components of \([0,1] \setminus \text{Fix}(g)\).

2. Let \( G \subset \text{Diff}^+_1([0,1]) \) be a complete group without hyperbolic fixed points. Each \( g \in G \) is the limit, for the \( C^1 \)-topology, of the products of the induced simple elements \( g_1 \) where \( I \) covers the set of connected components of \([0,1] \setminus \text{Fix}(g)\).

Let \( G \subset \text{Homeo}_+(\,0,1\,) \) be a group without crossing. According to Theorem \( \ref{th:limit_set} \) given any pairs \( f, g \) simple elements of \( G \), one has one of the following possibilities :

- either the supports \( \text{supp}(f) \) and \( \text{supp}(g) \) have disjoint interiors
- or the supports are equal
- or else the support of one of the diffeomorphism is contained in a fundamental domain of the other.

The group generated by \( f \) and \( g \) depends essentially on these 3 configurations and admits a simple presentation if \( \text{supp}(f) \neq \text{supp}(g) \).

Proposition 5.13. Let \( G \) be group without crossing let \( g_1 \) and \( g_2 \) be two simple elements of \( G \). Then

- if \( \text{Int}(\text{supp}(g_1) \cap \text{supp}(g_2)) = \emptyset \) then \( g_1 \) and \( g_2 \) commute :
  \[
  \langle g_1, g_2 \rangle = \mathbb{Z}^2.
  \]

- if \( \text{supp}(g_2) \subset \text{Int}(\text{supp}(g_1)) \), the group \( \langle g_1, g_2 \rangle \) admits as unique relations the fact that the conjugates of \( g_1 g_2 g_1^{-1}, i \in \mathbb{Z} \), pairwise commute. More precisely
  \[
  \langle g_1, g_2 \rangle = \left( \bigoplus_{\mathbb{Z}} \mathbb{Z} \right) \rtimes \mathbb{Z}
  \]

  where \( \mathbb{Z} \) acts by conjugacy on \( \left( \bigoplus_{\mathbb{Z}} \mathbb{Z} \right) \) as a shift of the \( \mathbb{Z} \) factors.

Démonstration. We just need to prove the second point. If \( \text{supp}(g) \subset \text{Int}(\text{supp}(f)) \), then the images by \( f^i \) of \( \text{supp}(g) \) are pairwise disjoint. This proves that the \( g_i = f^i g f^{-i}, i \in \mathbb{Z} \), pairwise commute. This allows us to define a morphism

\[
\varphi : \left( \bigoplus_{\mathbb{Z}} \mathbb{Z} \right) \rtimes \mathbb{Z} = \langle a, b | [a^i b a^{-i}, a^j b a^{-j}], i, j \in \mathbb{Z} \rangle \rightarrow \langle f, g \rangle,
\]

so that \( \varphi(a) = f \) and \( \varphi(b) = g \). It remains to show that \( \varphi \) is injective. For that, notice that every element of the group can be written as \( a^i b_{i_1} \ldots b_{i_k} \) with pairwise distinct \( b_{i_k} = a^{i_k} b a^{-i_k} \). The image is \( f^i g_{i_1}^{\beta_1} \ldots g_{i_k}^{\beta_k} \). The translation number relative to \( f \) is \( i \) so that the element is not the identity unless \( i = 0 \).

In that case the element is \( g_{i_1}^{\beta_1} \ldots g_{i_k}^{\beta_k} \) which vanishes only if all the \( \beta_i \) vanish, ending the proof. \( \Box \)
Lemma 5.14. 1. Let $I, J$ be two segments with disjoint interiors. Consider subgroups $H, K \subseteq \text{Homeo}_+(\{0, 1\})$, supported on $I$ and $J$, respectively. Then the group generated by $H$ and $K$ is isomorphic to $H \oplus K$.

2. Consider $f \in G$ and $I$ a fundamental domain of $f$. Let $H \subseteq \text{Homeo}_+(\{0, 1\})$ be a subgroup of homeomorphisms supported in $I$.

Then the group generated by $H$ and $f$ is

$$< H, f > = \left( \bigoplus_{\mathbb{Z}} H \right) \times \mathbb{Z}$$

where the factor $\mathbb{Z}$ is generated by $f$ and acts on $(\bigoplus_{\mathbb{Z}} H)$ by conjugacy as a shift of the $H$ factors.

6 Elementary groups

In this section we define rules for a family $S \subseteq \text{Homeo}_+(0, 1]$ so that the generated group is without crossing and admits $S$ as a topological basis.

6.1 Fundamental systems and elementary groups

Recall that $f \in \text{Homeo}_+(\{0, 1\})$ is called simple if $[0, 1] \setminus \text{Fix}(f)$ has a unique connected component (whose closure is the support $\text{supp}(f)$). A simple homeomorphism $f$ is called positive is $f(x) \geq x$ for all $x \in [0, 1]$.

Let $\text{Int}([0, 1])$ denote the set of segments of $[0, 1]$.

Definition 6.1. Consider a family $S = \{f, S_f, I_f\} \subseteq \text{Homeo}_+(\{0, 1\}) \times \text{Int}([0, 1]) \times \text{Int}([0, 1])$. One says that $S$ is a fundamental system if:

- for any $(f, S_f, I_f) \in S$, $f$ is a simple positive homeomorphism, $S_f = \text{supp}(f)$ and $I_f \subseteq \text{supp}(f)$ is a fundamental domain of $f$.
- for any distinct $(f', S_{f'}, I_{f'}) \neq (g, S_g, I_g) \in S$ one has
  - either $S_{f'}$ and $S_g$ have disjoint interiors
  - or $S_{f'} \subseteq I_g$ or else $S_g \subseteq I_{f'}$

The aim of this section is:

Proposition 6.2. Let $S$ be a fundamental system. Then, the group $G \subseteq \text{Homeo}_+(\{0, 1\})$ generated by the elements of $S$ is without crossing, and totally rational.

Lemma 6.3. Let $S$ be a fundamental system and $G \subseteq \text{Homeo}_+(\{0, 1\})$ be the group generated by $S$. Assume that $\{a, b\}$ is a pair of successive fixed points of an element of $G$. Then there are $f \in G$ and $(g, S_g, I_g) \in S$ so that $[a, b] = f(S_g)$.

Proof: Assume it is not the case, and consider a pair $\{a, b\}$ of successive fixed points which are not in the orbit of some $S_g$. Let $h = f_n^\pm \cdots f_1^\pm, f_i \in S$, having $\{a, b\}$ as a pair of successive fixed points. We chose $\{a, b\}$ and $h$ so that $a$ is minimal for these properties.

Notice that $(a, b)$ is not disjoint from all the supports of the $S_{f_i}$, otherwise $h$ would be the identity on $[a, b]$ contradicting the definition of $a, b$.

Claim 3. Consider $x \in (a, b)$ so that $h(x) \neq x$. The supports $S_{f_j}$ containing $x$ are totally ordered by the inclusion, by definition of a fundamental system. Let $i$ such that $S_{f_i}$ is the largest support $S_{f_j}, j = 1 \ldots n$, containing $x$.

Then $S_{f_i}$ contains $[a, b]$.

Proof: For every $j$, one has $f_j(S_{f_i}) = S_{f_i}$, because the support of $f_j$ is either disjoint from $S_{f_i}$ or contained in it. Therefore the end points of $S_{f_i}$ are fixed points of all the $f_j$, hence of $h$. The definition of successive fixed points of $h$ implies that $[a, b] \subseteq S_{f_i}$, concluding.

One deduces

Claim 4. $S_{f_i} \subseteq S_{f_j}$ for every $j$. 

Proof: If $S_{f_j}$ is not contained in $S_{f_i}$, then $f_j$ is the identity map on $S_{f_i}$. Furthermore, $S_{f_i}$ is invariant under all the $f_k$, and contains $[a,b]$. One deduces that $f_i$ is the identity map on the orbit of $[a,b]$ under the subgroup generated by the $f_k$, $k = 1 \ldots n$. Thus $\{a,b\}$ is still a pair of successive fixed points of the word obtained by deleting the letter $f_j$. This contradicts the minimality of $n$. \hfill $\square$

Now we split $\{1, \ldots n\} = A \coprod B$ where $A$ are the indices $j$ so that $f_j = f_i$ and $B$ the other indices, so that, for $j \in B$ one has

$$S_{f_j} \subset I_{f_i}.$$ 

Every element $f_j$, $j \in B$, acts as the identity on the orbit for $f_i$ of $\partial I_{f_i}$. One deduces

Claim 5. Let $\alpha$ denote the sum of the coefficient $\pm 1$ of the $f_k$, $k \in A$. Then $\alpha = 0$.

Proof: Notice that, $h = f_i^{\alpha}$ on the orbit of $\partial I_{f_i}$. If $\alpha \neq 0$, this implies that $h$ has no fixed point on $S_{f_i}$. Thus $\{a,b\} = S_{f_i}$, contradicting the definition of $\{a,b\}$. \hfill $\square$

Now, one can rewrite $h$ as a product of conjugated of $f_j$, $j \in B$ by some power $f_i^{\beta_j}$. In other words there are $\beta_j \in \mathbb{Z}$, $j \in B$, so that

$$h = \prod_{j \in B} f_i^{-\beta_j} f_j f_i^{\beta_j}.$$ 

Each of these conjugates $f_i^{\beta_j} f_j f_i^{-\beta_j}$ is supported in $f_i^{\beta_j}(S_j)$ which is contained in the fundamental domain $f_i^{\beta_j}(I_{f_i}).$

If all the $\beta_j$ are not equal, then the conjugates corresponding to different $\beta_j$ have disjoint interior of support, hence commutes. Furthermore, $\{a,b\}$ is contained in one of these intervals. One does not change the pair of successive fixed point by deleting the terms corresponding to the other $\beta_j$. Once again, if the $\beta_j$ are not constant, one gets a smaller word, contradicting the minimality of $n$.

We can now assume that all the $\beta_j$ are equal to some $\beta$. So, $h$ is the conjugate by $f_i^\beta$ of the product of the $f_j$, $j \in B$. So, $\{a,b\}$ is the image by $f_i^\beta$ of a pair $\{c,d\}$ of successive fixed points of the product of the $f_j$, $j \in B$. As the cardinal of $B$ is strictly smaller than $n$, the minimality of $n$ implies that $\{c,d\}$ is the image by an element of $G$ of one of the $S_{g_i}$, $g \in S$. One deduces the same property for $\{a,b\} = f_i^\beta(\{c,d\})$, getting a contraction with the definition of $\{a,b\}$. \hfill $\square$

If the group $G$ admits a crossing, this means by definition that there is a pair of successive fixed points $\{a,b\}$ and an element $g \in G$ with $g(\{a,b\}) \cap \{a,b\} \neq \emptyset$. According to Lemma 6.3 one can assume that $(a,b)$ is some $S_f$ with $f \in S$. One concludes the proof of Proposition 6.2 by showing :

Lemma 6.4. Let $S$ be a fundamental system and $G \subset \text{Homeo}_+(\{0,1\})$ be the group generated by the element of $S$. Given any $f \in S$ and any $g \in G$ either $g(S_f)$ and $S_f$ have disjoint interior or $g(S_f) = S_f$.

Proof: Assume, arguing by absurd, that it is not the case and consider

$$g = g_{a}^{m} \cdots g_{1}^{n}, g_i \in S, \text{ and } f \in S$$ so that :

- $g_{i+1} \neq g_i$ for every $i \in \{1, \ldots n-1\}$,
- $g(S_f) \neq S_f$,
- and $g(S_f)$ intersects the interior of $S_f$.

We chose $g, f$ so that $\sum_{i=1}^{n} |\alpha_i|$ is minimal for these properties.

If $S_{g_i} \subset S_f$ or if $S_{g_i}$ and $S_f$ have disjoint interiors, then $g_i(S_f) = S_f$. In that case, one may delete $g_i$ contradicting the minimality of $n$.

Thus, $S_f \subset I_{g_i}$, by definition of a fundamental system. So, $g_{a}^{\alpha_i}(S_f) \subset g_{a}^{\alpha_i}(I_{g_i})$ and its interior is disjoint from $I_{g_i}$ but is contained in $g_{i+1}$. $S_f \subset I_{g_{i+1}}$, by definition of a fundamental system. So, $g_{a}^{\alpha_{i+1}}(S_f) \subset g_{a}^{\alpha_{i+1}}(I_{g_{i+1}})$ and its interior is disjoint from $I_{g_{i+1}}$ but is contained in $g_{i+2}$. $S_f \subset I_{g_{i+2}}$ as $g_{i+2} \neq g_{i+1}$, one gets that either $S_{g_i}$ is of interior disjoint from $S_{g_{i+1}}$, or $S_{g_{i+1}} \subset I_{g_{i+2}}$. In both cases, $g_2$ is the identity map on $g_2^{\alpha_i}(g_f)$. Thus, one may delete $g_2$, contradicting the minimality of the word.

So $S_{g_{i+1}} \subset I_{g_{i+2}}$ and $g_{a}^{\alpha_{i+1}}(S_f) \subset g_{a}^{\alpha_{i+1}}(I_{g_{i+2}}) \subset S_{g_{i+2}}$. In particular, $g_{a}^{\alpha_{i+2}}(S_f)$ is disjoint from $S_f$.

An easy induction proves that $S_{g_{i \cdot + 1}} \subset I_{g_{i \cdot + 2}}$ and $g_{a}^{\alpha_{i \cdot + 1}} \cdots g_{a}^{\alpha_{i \cdot + 2}}(S_f)$ is disjoint from $S_f$ for every $i$, concluding. \hfill $\square$

Definition 6.5. A group $G \subset \text{Homeo}_+(\{0,1\})$ is called an elementary group if it is generated by a fundamental system.

Remark 6.6. If $H$ is topologically conjugate to an elementary group $G$, then $H$ is an elementary group.
6.2 The topological dynamics and the topology of the fundamental systems

Next proposition explains that a elementary group, generated by a finite fundamental system, is topologically determined by the topological configuration of the intervals \((S_f, I_f)\) of the fundamental systems.

**Proposition 6.7.** Let \(S\) and \(\Sigma\) be two finite fundamental systems and let \(G\) and \(\Gamma\) be the groups generated by \(S\) and \(\Sigma\), respectively. Assume that there is a homeomorphism \(h: [0, 1] \rightarrow [0, 1]\) and a bijection \(\varphi: S \rightarrow \Sigma\), \(\varphi(f, S_f, I_f) = (\varphi(f), S_{\varphi(f)}, I_{\varphi(f)})\) so that for every \(f\),

\[
S_{\varphi(f)} = h(S_f) \quad \text{and} \quad I_{\varphi(f)} = h(I_f)
\]

Then there is a homeomorphism \(\tilde{h}: [0, 1] \rightarrow [0, 1]\) conjugating \(G\) to \(\Gamma\):

\[
\Gamma = \{ \tilde{h}gh^{-1}, g \in G \}.
\]

More precisely for every \((f, S_f, I_f) \in S\), \(\tilde{h}f\tilde{h}^{-1} = \varphi(f)\).

**Proof:** We argue by induction on the cardinal of \(S\) and \(\Sigma\). If the cardinal is 1, one just notes that any to positive simple homeomorphism of \([0, 1]\) are topologically conjugates.

We assume now that the statement has been proved for any fundamental system of cardinal less or equal than \(n\) and we consider fundamental systems \(S\) and \(\Sigma\) of cardinal \(n + 1\).

Let us write \(S = \{(f, S_f, I_f)\} \cup \tilde{S}\) where \(S_f\) is a maximal interval in the (nested) family of supports, and \(\Sigma = \{(\phi, S_\phi, I_\phi)\} \cup \tilde{\Sigma}\) where \(\phi = \varphi(f)\).

We consider \(h_1\) so that \(h_1gh_1^{-1} = \varphi(g)\) for \(g \in S\). Notice that \(h_1(S_g) = h(S_g) = S_{\varphi(g)}\) for \(g \in S\). As a consequence, there is a homeomorphism \(h_2\) which coincides with \(h_1\) on the union \(S_\tilde{S}\) of the supports \(S_g\), \(g \in S\), and with \(h\) out of this union.

Notice that \(S_\tilde{S} \cap S_f \subset I_f\). Therefore \(h_2(I_f) = h(I_f) = I_\phi\).

Thus there is a unique homeomorphisms \(h_3: S_f \rightarrow S_\phi\) conjugating \(f\) with \(\phi\) and coinciding with \(h_2\) on \(I_f\).

The announced homeomorphism \(\tilde{h}\) is the homeomorphism which coincides with \(h_3\) on \(S_f\), and with \(h_2\) out of \(S_f\).

\(\square\)

6.3 Elementary groups of diffeomorphisms

If the homeomorphisms of a fundamental system are diffeomorphisms, the corresponding elementary group will be a group of diffeomorphisms.

**Proposition 6.8.** Any elementary group \(G \subset \text{Diff}_+([0, 1])\) is without hyperbolic fixed points.

**Proof:** We consider a fundamental system \(S = \{(f, S_f, I_f)\}\), with \(f \in \text{Diff}_+([0, 1])\). Note that every diffeomorphism \(f\) in \(S\) is, by definition of fundamental system, a simple diffeomorphism, and therefore has no hyperbolic fixed points. Let \(G\) be the group generated by \(S\).

Every \(g \in G\) can be written as \(g = f_{a_1}^{\alpha_1} \ldots f_1^{\alpha_1} \in G\), \(f_i \in S\), \(f_{i+1} \neq f_i\), \(\alpha_i \in \mathbb{Z}\) (this presentation of \(g\) may be not unique). Arguing by absurd we assume that there is \(g \in G\) having hyperbolic fixed points. We chose \(g\) so that \(\sum |\alpha_i|\) is the minimal number with this property.

An hyperbolic fixed point is isolated, so that it belongs to a pair of successive fixed points \(\{a, b\}\) (let assume it is \(a\)).

We consider \(f_i\) so that the support \(S_{f_i}\) is the largest of the supports \(S_{f_j}\) containing \(a, b\). Then \(S_{f_i}\) is invariant by all the \(f_j\). If some \(S_{f_j}\) is not contained in \(S_{f_i}\), one may delete the letter \(f_j\) contradicting the minimality of \(\sum |\alpha_i|\).

We consider the sum \(\alpha\) of the \(\alpha_j\) for which \(f_j = f_i\) (recall that \(S_{f_i}\) is the largest support). If \(\alpha \neq 0\) then \(g\) coincide with \(f_i^{\alpha}\) on the orbit of \(\partial(I_{f_i})\). One deduces that \(\{a, b\} = \partial S_{f_i}\) and that the derivative of \(g\) at each end point is the same as the one of \(f_i^{\alpha}\) which is 1 (because \(f_i\) is simple).

So \(\alpha = 0\). This allows us to rewrite \(g\) as the product of conjugates of the \(f_j \neq f_i\) by powers \(f_i^{\beta_j}\). These diffeomorphisms are supported on \(f_i^{\beta_j}(I_{f_i})\) which have disjoint interiors. Exactly as in the proof of Lemma 6.3, one deduces that, if the \(\beta_j\) are not all equal, then one may delete some of the \(f_i^{\alpha_j}\) contradicting the minimality of \(\sum |\alpha_i|\).
So the $\beta_j$ are all equal to some $\beta$ and one gets that $g$ is the conjugate by $f_1^\beta$ of the product of the $f_j^\alpha$, $f_j \neq f_1$. However, having a hyperbolic fixed point is invariant by conjugacy, so that one deduces that the product of the $f_j^\alpha$, $f_j \neq f_1$ has a hyperbolic fixed point. This contradicts, once again, the minimality of $\sum |\alpha_i|$. \hfill $\square$

One of the main result of this paper consists in proving that every elementary group of diffeomorphisms of $[0,1]$ is indeed $C^1$-close to identity. This will be the aim of Section 7.

7 Free group

The aim of this section is to prove Theorem 1.9 (assuming Theorem 1.6), that we recall below.

**Theorem 7.1.** There is a subgroup $G \subset Diff^1([0,1])$, $C^1$-close to the identity, and isomorphic to the free group $F^2$.

Before that let us notice that the elementary groups do not contain any free group $F^2$.

7.1 Finitely generated subgroups of elementary groups are solvable : proof of Proposition refp.solvable2

In contrast we notice that elementary groups do not contain free groups:

**Proposition 7.1.** Any elementary group $G$ generated by a finite fundamental system $S$ is solvable, with length bounded by the cardinal of $S$.

Recall that a group is **solvable** if the sequence $G_1 = [G,G], \ldots, G_{n+1} = [G_n,G_n]$ statifies that there is $k$ so that $G_k = \{1\}$. The infimum of such $k$ is the length $\ell(G)$ of the solvable group $G$.

As a direct corollary one gets

**Corollary 7.2.** For any fundamental system $G$, the group $G$ generated by $G$ does not contain any non cyclic free group.

**Proof of Corollary 7.2:** Assume there is a subgroup $< f,g > \subset G$ isomorphic to $F_2$. Notice that $f$ and $g$ are written as finite words in the generators in $G$ of $G$. Thus the group $< f,g >$ is a subgroup of a group generated by a finite fundamental system. This group is solvable, hence does not contain any subgroup isomorphic to $F_2$, contradicting the hypothesis. \hfill $\square$

Notice that our argument above proved Proposition 1.6.

We start the proof of Proposition 7.1 by showing:

**Lemma 7.3.** Fix and integer $k$. Assume that $\{G_i\}_{i \in \mathbb{N}}$ is a sequence of solvable groups of length $\ell(G_i) \leq k$. Then the abelian product

$$G = \bigoplus_{i \in \mathbb{N}} G_i$$

is solvable with length $\ell(G) \leq k$.

**Proof:** Just notice that $[G,G] = \bigoplus_{i \in \mathbb{N}} [G_i,G_i]$. \hfill $\square$

**Proof of Proposition 7.1:** We present a proof by induction on the cardinal of $S$. If this cardinal is 1, the group is a cyclic abelian group, hence is solvable of length 1. We assume now that Proposition 7.1 is proved for fundamental systems of cardinal less or equal to $n$. Let $S = \{(f_i,S_i,I_i), i \in \{1,\ldots,n+1\}\}$ be a fundamental system. Up to re-index the $f_i$, one may assume that $S_{n+1}$ is maximal for the inclusion among the $S_i$.

Let $A \subset \{1,\ldots,n\}$ denote the set of indices $i$ for which $S_i$ and $S_{n+1}$ have disjoint interiors, and $B = \{1,\ldots,n\} \setminus A$ is the set of indices $j$ for which $S_j$ is contained in $I_{n+1}$.

First assume that $A$ is not empty. Then $S_A = \{(f_i,S_i,I_i), i \in A\}$ and $S' = \{(f_j,S_j,I_j), j \in B \cup \{n+1\}\}$ are fundamental systems with cardinal $\leq n$. Let $G_A$ and $G'$ the elementary groups generated by $S_A$ and $S'$, respectively. They are solvable groups of length bounded by $n$. Therefore, $G = G_A \oplus G'$ is solvable of length bounded by $n$, according to Lemma 7.3 which concludes the proof in this case.
We assume now that $A$ is empty so that $B = \{1, \ldots, n\}$. Let denote $G_B$ the group generated by the fundamental system $\mathcal{S}_B = \{(f_j, S_j, I_j), i \in B\}$. One easily shows that $[G, G]$ is contained in $\bigoplus_{i \in \mathbb{Z}} G_{B,i}$ where $G_{B,i} = f_{n+1}^{i}G_Bf_{n+1}^{-i}$. These groups are solvable of length $\ell(G_B) \leq n$ by our induction hypothesis, so that $G$ is solvable of length $\ell(G) \leq n + 1$ according to Lemma 7.3.

\section{Proof of Theorem 1.9}

Let $A = \{a_i, i \in \mathbb{N}\}$ be a countable set called alphabet. Let say that a word $\omega = \{\omega_i\}$ in $n$ letters of the alphabet $A$ is universal (among the groups in $C_1^1$) if $\omega(f_1, \ldots, f_n) = id$ for any $f_1, \ldots, f_n \in Diff^1_1([0,1])$ for which the group $< f_1, \ldots, f_n >$ is $C^1$-close to the identity.

The length of such a word is $n$. The word is reduced if $\omega_{i+1}\omega_i \neq 1$, and cyclically reduced if furthermore $\omega_n\omega_1 \neq 1$.

We will prove:

**Proposition 7.4.** There is no universal reduced non-trivial words.

Let us deduce the Theorem 1.9 from Proposition 7.4.

**Proof:** Assume there is a universal reduced word. In particular there is no universal word in $2$ letters. Therefore, for any word $\omega$ in $2$ letters there is a pair $f_\omega, g_\omega$ so that the group $< f_\omega, g_\omega >$ is $C^1$-close to the identity and $\omega(f_\omega, g_\omega) \neq id$.

One fix a sequence $I_\omega \subset [0,1]$ of pairwise disjoint segments. For every $\omega$, one choose diffeomorphisms $\tilde{f}_\omega, \tilde{g}_\omega$ supported on $I_\omega$, smoothly conjugated to $(f_\omega, g_\omega)$, and so that $\tilde{f}_\omega, \tilde{g}_\omega$ tend uniformly to the identity when the length of $\omega$ tends to $\infty$ (that is possible because $< f_\omega, g_\omega >$ is $C^1$-close to the identity). One defines $f$ and $g$ as being $\tilde{f}_\omega, \tilde{g}_\omega$ on $I_\omega$ and the identity map out of the union of the $I_\omega$. One easily check that $f$ and $g$ are homeomorphisms. Now, $f$ and $g$ are diffeomorphisms because $\tilde{f}_\omega, \tilde{g}_\omega$ tend uniformly to the identity.

Finally the group $< f, g >$ is isotopic to identity and $\omega(f, g) \neq id$ for every reduced word $\omega$. \hfill \Box

\subsection{No universal relation: proof of Proposition 7.4}

Notice that, if $\omega$ is a universal word, then for each letter $a_i \in \mathcal{A}$, the sum of the coefficients in $a_i$ of the $\omega_i$ vanishes. Otherwise, $\omega(id, \ldots, id, f, id, \ldots, id)$ would be different from id, for $f \neq id$ at the $i$th position, contradicting the universality of the relation $\omega$.

Consider a letter appearing in $\omega$, say $\omega_1$. Then, one can write $\omega$ as a product of conjugates of the other letters by powers of the letter corresponding to $\omega_1$. One replace each $\omega_i a_i \omega^{-1}_1$ by a new letter denoted $b_{i,j}$. One gets a reduced words $\varphi(\omega)$ of length less or equal than $n-2$ (in the alphabet $B = \{b_{i,j}\}$).

Notice that a reduced word of length $\leq 2$ cannot be universal. One concludes the proof of Proposition 7.4 and therefore of Theorem 1.9 by proving

**Lemma 7.5.** If $\omega$ is universal then $\varphi(\omega)$ is universal.

**Proof:** Assume that $\varphi(\omega)$ is not universal. Therefore, there is an interval $I$ and $h_{i,j} \in Diff^1(I)$ so that $\varphi(\omega)(h_{i,j}) \neq id$ and the group generated by the $h_{i,j}$ is $C^1$-close to the identity.

One considers $f \in Diff^1([0,1])$, without hyperbolic fixed points, and so that $I$ is contained in the interior of a fundamental domain of $f$. According to Theorem 1.6, the group generated by $f$ and the $h_{i,j}$ is $C^1$-close to the identity.

One denotes by $g_i$ the diffeomorphisms which coincides with $f^{-j}h_{i,j}f^j$ on $f^{-j}(I)$, for every $j$ for which $h_{i,j}$ is defined, and the identity out of the $f^{-j}(I)$. The group generated by the $g_i$ is $C^1$-close to the identity.

Now $\omega(f, \{g_i\})$ is a diffeomorphism which coincides with $\varphi(\omega)(h_{i,j})$ in restriction to $I$, hence is not the identity map, contradicting the hypothesis on $\omega$. \hfill \Box
8 Group extensions in the class $C^1_{id}$.

The aim of this section is to prove Theorem 8.1: given a group $G \subset \text{Diff}^1_+(\{0, 1\}, C^1)$-close to the identity and supported in the interior of a fundamental domain of a diffeomorphism $f \in \text{Diff}^1_+(\{0, 1\})$, without hyperbolic fixed points, the group $\langle f, G \rangle$ is $C^1$-close to the identity. We will prove her a slightly stronger version which will be used in the proof of Theorem 8.1:

**Theorem 8.1.** Let $f \in \text{Diff}^1_+(\{0, 1\})$ be a diffeomorphism without hyperbolic fixed points and $G \subset \text{Diff}^1_+(\{0, 1\})$ be a group supported on a fundamental domain $[x_0, f(x_0)]$. Assume that there is a $C^1$-continuous path $h_t \in \text{Diff}^1_+(\{0, 1\}), t \in [0, 1)$, so that

- for every $g \in G$
  
  $h_t g h_t^{-1} \xrightarrow{C^1} \text{Id}$,

  and

- $h_t$ is supported on $[x_0, f(x_0)]$.

Then the group $\langle f, G \rangle$ generated by $f$ and $G$ is $C^1$-close to the identity.

More precisely, there is a $C^1$-continuous path $H_t \in \text{Diff}^1_+(\{0, 1\}), t \in [0, 1)$, so that

- for every $g \in \langle f, G \rangle$
  
  $H_t g H_t^{-1} \xrightarrow{C^1} \text{Id}$,

  and

- $H_t$ is supported on the support of $f$ and $\text{DH}_t(0) = \text{DH}_t(1) = 1$.

Notice that Theorem 8.1 follows directly from Theorem 8.4: if the support of $G$ is contained in the interior of the fundamental domain $(x_0, f(x_0))$ then given $h^0_t \in \text{Diff}^1(\text{supp}(G), t \in [0, 1)$ realizing an isotopy by conjugacy of $G$ to the identity, one easily build another isotopy by conjugacy $h_t \in \text{Diff}^1(\{0, 1\})$, supported on $(x_0, f(x_0))$.

Theorem 8.1 is the main technical result of this paper. The proof is the aim of the whole section. Let us first present a sketch of proof.

8.1 Sketch of proof

The proof uses strongly arguments in [F²] which build explicitly conjugacies from a given diffeomorphisms to a neighborhood of the identity. Here we will need to come back to the contraction of [F¹] for getting a simultaneous conjugacy. For this reason it will be sometimes practical to use the following notation

**Notation 1.** Given $f, g \in \text{Diff}^1_+(\{0, 1\})$ and $(h_t)_{t \in [0, 1]}$, $h_t \in \text{Diff}^1_+(\{0, 1\})$, a $C^1$-continuous path of diffeomorphisms, the notation

$f \sim h_t g$

means that $(h_t f h_t^{-1})_{t \in [0, 1]}$ is an isotopy by conjugacy from $f$ to $g$, that is:

$h_t f h_t^{-1} \xrightarrow{C^1} t \to 1 g$.

One will write $f \sim g$ if there exists an isotopy by conjugacy from $f$ to $g$.

**Sketch of proof of Theorem 8.1** : Let $G$ be a group $C^1$-close to the identity, supported in a fundamental domain $[a, b]$ of $f \in \text{Diff}^1_+(\{0, 1\})$.

Let $(h_t)_{t \in [0, 1]}$ be a continuous path of $C^1$-diffeomorphisms realizing an isotopy by conjugacy from $G$ to $id$ and such that $h_t$ has derivative equal to $1$ at $a$ and $b$. We will extend these diffeomorphisms $(h_t)_{t \in [0, 1]}$ to diffeomorphisms $\hat{h}_t$ of $[0, 1]$ in such a way that $h_t f h_t^{-1} \xrightarrow{C^1} f$ (Lemma 8.3: in other word, for $t \to 1$, the extensions $\hat{h}_t$ almost commute with $f$).

In this way, the impact induced on $f$ by the isotopy by conjugacy from $G$ to $id$ will be slight.

Let $(\varphi_t)_{t \in [0, 1]}$ be a continuous path of $C^1$-diffeomorphisms for which $\varphi_t f \varphi_t^{-1} \xrightarrow{C^1} id$ (the existence of $(\varphi_t)_{t \in [0, 1]}$ is given [F²]); assume that one can choose $\varphi_t$ so that, furthermore, $\varphi_t$ is affine on $[a, b]$ or all $t \in [0, 1)$. Under this assumption, the conjugacy by $(\varphi_t)_{t \in [0, 1]}$ will not affect the $C^1$-distance of $h_t g h_t^{-1}$
to the identity for \( g \in G \), and we can compose the two isotopies, by first conjugating by \( \tilde{h}_t \) and then by \( \varphi_t : \)

\[ \forall g \in <G, f >, g \sim_{\varphi_t \circ \tilde{h}_t} \text{id}. \]

Indeed, we will ensure the existence of such \( \varphi_t \) affine on the support of \( G \) when the support of \( G \) is contained in the interior of a fundamental domain of \( f \). When the support of \( G \) is precisely one fundamental domain, we will weaken slightly this assumption, ensuring that the logarithm of the derivative of \( \varphi_t \) is equicontinuous on \([a, b]\) (Lemma \[S.3\]). This will ensure that conjugating by \( \varphi_t \) will have a bounded effect on the \( C^1 \)-distance of \( h_t \circ \tilde{h}_t^{-1} \) to the identity for \( g \in G \).

\[ \square \]

As explained in this sketch of proof the two main tools for Theorem \[10\] are Lemmas \[S.3\] and \[S.4\] stated in next sections.

8.2 Background from \([Fa]\)

We rewrite \([Fa]\) Proposition 9 in a form which will be more convenient here.

If \( h_0 \) is a diffeomorphism supported in a fundamental domain of the diffeomorphism \( f \), one gets a homeomorphism \( h \) commuting with \( f \) by defining the restriction \( h_n = h|_{I_n}, \ I_n = f^n(h_0) \) as \( h_n = f_{n-1}h_{n-1}f_{n-1}^{-1} \) where \( f_n \) is the restriction \( f|_{I_n} \). The homeomorphism \( h \) will be a diffeomorphism if and only if \( h_n \) tends to the identity map in the \( C^1 \) topology as \( n \to \pm \infty \). That is not the case in general. However, if one only wants that \( f \) and \( h \) almost commute, then one is allowed to modify slightly the induction process, and one can do it guaranteeing this convergence to the identity. That was the aim of \([Fa]\). In Proposition \[S.1\] and \[S.2\] below, we renormalized the intervals \( I_n \) so that then \( f_n \) appear as diffeomorphisms of \([0, 1]\).

**Proposition 8.1.** Let \((f_n)_{n \in \mathbb{N}} \in [0,1] \times \mathbb{N} \) be a sequence of diffeomorphisms of \([0, 1]\) such that \((f_n)_{n \in \mathbb{N}} \) converges to the identity in the \( C^1 \)-topology when \( n \) tends to infinity.

Let \((h_0)_{n \in \mathbb{N}} \) be a \( C^0 \) diffeomorphisms of \([0, 1]\) such that, \( Dh_0(0) = 1 = Dh_0(1) \). Fix \( \varepsilon > 0 \).

Then, there exists a sequence \((\psi_n)_{n \in \mathbb{N}} \) of \( C^1 \)-diffeomorphisms of \([0, 1]\) such that:

- \( D\psi_n(0) = 1 = D\psi_n(1) \) for all \( n \in \mathbb{N} \);
- \( \|f_n - \psi_n - f_n\|_1 < \varepsilon \) for all \( n \in \mathbb{N} \);
- the sequence \((h_n)_{n \in \mathbb{N}} \) of \( C^1 \)-diffeomorphisms of \([0, 1]\) , defined by induction as \( h_0 = id \) and \( h_n = f_{n-1} \psi_{n-1} h_{n-1} f_{n-1}^{-1} \) if \( n \in \mathbb{N}^* \) satisfies:

\[ \exists N > 0, \forall n \geq N, h_n = id. \]

For Theorem \[9.1\] we will need the version with parameters, also due to \([Fa]\), of this proposition. Let us state it below:

**Proposition 8.2.** Let \((f_t,n)_{(t,n) \in [0, 1] \times \mathbb{N}} \) be a collection of diffeomorphisms such that:

- for all \( n, (f_{t,n})_{(t,n) \in [0, 1]} \) is a \( C^1 \)-continuous path in \( Diff_+^1([0, 1]) \) so that \( Df_{t,n}(0) = 1 = Df_{t,n}(1) \) do not depend on \( t \in [0, 1] \);
- for all \( t \in [0, 1] \), \((f_{t,n})_{n \in \mathbb{N}} \) converges to the identity in the \( C^1 \)-topology, when \( n \) tends to infinity.

Let \((h_{t,0})_{(t,n) \in [0, 1]} \) be a \( C^1 \) continuous path of diffeomorphisms of \([0, 1]\) such that, for all \( t \in [0, 1] \), one has \( Dh_{t,0}(0) = 1 = Dh_{t,0}(1) \). Let \((\varepsilon_t)_{(t,n) \in [0, 1]} \) be continuous path of strictly positive real numbers.

Then, there exists a collection \((\psi_{t,n})_{(t,n) \in [0, 1] \times \mathbb{N}} \) of \( C^1 \)-diffeomorphisms of \([0, 1]\) such that:

- \( D\psi_{t,n}(0) = 1 = D\psi_{t,n}(1) \) for all \( (t,n) \in [0, 1] \times \mathbb{N} \);
- \( \|f_{t,n} - \psi_{t,n} - f_{t,n}\|_1 < \varepsilon_t \) for all \( (t,n) \in [0, 1] \times \mathbb{N} \);
- Let \((h_{t,n})_{n \in \mathbb{N}, t \in [0, 1]} \) be the collection of \( C^1 \)-diffeomorphisms of \([0, 1]\) , defined by induction from \( h_{t,0} \) by

\[ h_{t,n} = f_{t,n-1} \psi_{t,n-1} h_{t,n-1} f_{t,n-1}^{-1} \] if \( n \in \mathbb{N}^* \).

Then, for every \( t \in [0, 1] \), there is \( N_t > 0 \), increasing with \( t \in [0, 1] \), so that

\[ h_{t,n} = f_{t,n-1} \psi_{t,n-1} h_{t,n-1} f_{t,n-1}^{-1} = id, \quad \forall n \geq N_t. \]

- \((\psi_{t,n})_{(t,n) \in [0, 1]} \) is \( C^1 \)-continuous for all \( n \in \mathbb{N} \);
- \((h_{t,n})_{(t,n) \in [0, 1]} \) is \( C^1 \)-continuous for all \( n \in \mathbb{N} \).
8.3 Diffeomorphisms almost commuting with \( f \) and prescribed in a fundamental domain

**Lemma 8.3.** Let us consider \( f \in \text{Diff}^1_+(\{0,1\}) \), \( I = [x, f(x)] \) a fundamental domain of \( f \), \((a, b)\) the connected component of \([0, 1] \setminus \text{Fix}(f)\) containing \( x \), and \((h_t)_{t \in [0, 1)}\) a \( C^1\)-continuous path of \( C^1\)-diffeomorphisms of \([0, 1]\) supported on \( I \).

Then, for all continuous paths \((\varepsilon_t)_{t \in [0, 1)}\) of strictly positive real numbers, there exists a \( C^1\)-continuous path \((\tilde{h}_t)_{t \in [0, 1)}\) of \( C^1\)-diffeomorphisms of \([0, 1]\) such that, for all \( t \in [0, 1) \), :
- \( \tilde{h}_t \) coincides with the identity map on a neighbourhood of \( a \) and \( b \);
- the support of \( \tilde{h}_t \) is contained in the orbit for \( f \) of the support of \( h_t \):
  \[ \text{Supp}(\tilde{h}_t) \subset \bigcup_{n \in \mathbb{Z}} f^n(\text{Supp}(h_t)); \]
- \( \tilde{h}_t \) coincides with \( h_t \) on the fundamental domain \( I \):
  \[ \tilde{h}_t|_I = h_t; \]
- \( \| \tilde{h}_t f \tilde{h}_t^{-1} - f \|_1 < \varepsilon_t \).

The proof of Lemma 8.3 consists in pushing \( h_t \) by \( f \) in the iterates \( f^n(I) \) of the fundamental domain. In each of these fundamental domains, one applies a small perturbation so that the diffeomorphism obtained in \( f^n(I) \) becomes closer to \( id \).

**Proof of Lemma 8.3:**

Let \((\varepsilon_t)_{t \in [0, 1)}\) be a continuous path of strictly positive real numbers converging to 0. We denote by \((f_n)_{n \in \mathbb{N}}\) the sequence of \( C^1\)-diffeomorphisms of \([0, 1]\) defined by : for all \( n \in \mathbb{N} \), \( f_n \) is the normalization of the diffeomorphism \( f \mid [f^n(x), f^{n+1}(x)] \); that is : \( f_n \) is obtained by conjugating \( f \mid [f^n(x), f^{n+1}(x)] \) by the affine maps from \([f^n(x), f^{n+1}(x)]\) and \([f^{n+1}(x), f^{n+2}(x)]\) to \([0, 1]\). Notice that, as \( f \) is \( C^1 \), the sequence \((f_n)_{n \in \mathbb{N}}\) converges to \( id \) when \( n \) tends to \( \infty \), with respect to the \( C^1\)-topology.

One considers then \( h_{t,0} \) as being the normalization of \( h_t \mid [x, f(x)] \) on the interval \([0, 1]\). In particular, the equality \( Dh_{t,0}(0) = 1 = Dh_{t,0}(1) \) is satisfied.

Proposition 8.2 asserts that there exists a collection \((\psi_{t,n})_{t,n \in [0,1) \times \mathbb{N}}\) of diffeomorphisms of \([0, 1]\) such that :
- \((\psi_{t,n})_{t \in [0, 1)}\) is a \( C^1\)-continuous path for all \( n \in \mathbb{N} \);
- for all \((t, n) \in [0, 1) \times \mathbb{N} \), \( D\psi_{t,n}(0) = 1 = D\psi_{t,n}(1) \);
- for all \((t, n) \in [0, 1) \times \mathbb{N} \), one has : \( \| f_n \circ \psi_{t,n} - f_n \|_1 < \varepsilon_t \);
- for all \( t \in [0, 1) \), the sequence of diffeomorphisms of \([0, 1]\) defined by :
  \[ \begin{cases}
  h_{t,0} = f_n, & \text{for all } n \in \mathbb{N} \\
  h_{t,n} = f_{n-1} \psi_{t,n-1} f_{n-1}^{-1} h_{t,n-1} f_{n-1} & \text{for all } n \in \mathbb{N}^* \end{cases} \]
  is stationary, equal to \( id \) for all \( n \in \mathbb{N} \) great enough.
- for all \( n \in \mathbb{N} \), the path \((h_{t,n})_{t \in [0,1)}\) is a \( C^1\)-continuous path.

One gets a similar result and a similar \( C^1\)-continuous collection \((h_{t,n})_{(t,n) \in [0,1) \times (-\mathbb{N})}\) by considering the negative iterates of \( f \).

Consider now the \( C^1\)-continuous path \((h_{t})_{t \in [0, 1)}\) of \( C^1\)-diffeomorphisms of \([0, 1]\) defined by :
- \( h_t \mid [f^n(x), f^{n+1}(x)] \) is conjugated to \( h_{t,n} \) by the affine map from \([f^n(x), f^{n+1}(x)]\) to \([0, 1]\), for all \( n \in \mathbb{Z} \);
- \( h_t = id \) on the complement of \( \bigcup_{n \in \mathbb{Z}} f^n([x, f(x)]) \).

By straightforward calculations using \( \| f_n \circ \psi_{t,n} - f_n \|_1 < \varepsilon_t \) one gets :
\[ \| D(h_{t,n+1} f_n h_{t,n}^{-1}) - Df_n \|_1 < \varepsilon_t, \]
from which follows \( \| h_{t,n+1} f_n h_{t,n}^{-1} - f_n \|_1 < \varepsilon_t \), thus \( \| h_t f h_t^{-1} - f \|_1 < \varepsilon_t \), concluding the proof.

\[ \square \]

8.4 Conjugacy to the identity prescribed in a fundamental domain

The aim of this section is :
Lemma 8.4. Let \( f \in Diff^1_+(\{0,1\}) \) be a diffeomorphism without hyperbolic fixed point, and let \([a,b],\ b=f(a)\), be a fundamental domain of \( f \). There exists a \( C^1\)-continuous path \( (\alpha_t)_{t \in [0,1)} \) of \( C^1\)-diffeomorphisms of \([0,1]\) so that:

- \( D\alpha_t(0) = 1 = D\alpha_t(1) \) for all \( t \in [0,1) \).
- \( (\alpha_t)_{t \in [0,1)} \) has equicontinuous Log-derivative on \([a,b]\):
  \[
  \forall \varepsilon > 0, \exists \delta > 0, \forall t \in [0, 1), \forall x, y \in [a, b] \text{ so that } |x - y| \leq \delta \text{ one has :}
  |\log D\alpha_t(x) - \log D\alpha_t(y)| < \varepsilon
  \]
- \( f \sim_{\alpha_t} id \)

The proof is a variation on the proof of the main result in \([Fa]\):

Theorem 8.2 \([Fa]\). Given \( f \in Diff^1_+(\{0,1\}) \) without fixed points in \((0,1)\), given any continuous pathes \(0 < a_t < b_t < 1, \ t \in [0,1)\), given any \( C^1\)-continuous path \( g_t \in [0,1)\), where \( g_t \in Diff^1_+(\{0,1\}) \) is a diffeomorphism without fixed point in \((0,1)\) which coincides with \( f \) on \([0, a_t]\) and on \([b_t, 1]\), there is a \( C^1\)-continuous path \( h_t \in Diff^1_+(\{0,1\}), \ t \in [0,1)\), so that, for every \( t \), \( h_t \) coincides with the identity on a neighborhood of \( 0 \) and of \( 1 \), and the \( C^1 \) distance \( \|h_tfh_t^{-1} - g_t\|_1 \) tends to 0 as \( t \to 1 \).

Let us sketch the proof of \([Fa]\), so that we will explain the modification we need here. 

Sketch of proof of \([Fa]\) : As \( f \) and \( g_t \) coincide on \([0, a_t]\), there is a unique diffeomorphism \( \hat{h}_t \) of \((0,1)\) which is the identity map in a neighborhood of \( 0 \), and conjugating the restriction \( f|_{[0,1)} \) to \( g_t|_{(0,1)} \), \([Fa]\) chooses \( h_t \) so that it coincides with \( \hat{h}_t \) out of an arbitrarily small neighborhood of \( 1 \). The idea is that, in a neighborhood of \( 1 \), \( f \) and \( g_t \) coincide so that \( h_t \) commute with \( f \). One concludes as in the proof of Lemma 8.3 : by using Proposition 8.2 one can modify \( \hat{h}_t \) slowly in the successive fundamental domains of \( f \) in order to get a diffeomorphism \( h_t \) coinciding with \( \hat{h}_t \) out of a small neighborhood of \( 1 \), with the identity map in a smaller neighborhood of \( 1 \) and almost commuting with \( f \) on \([b_t, 1]\).

\[
\square
\]

Let us now modify slightly the proof of \([Fa]\). Consider point \( x_t, y_t \in [a_t, b_t] \) varying continuously with \( t \in [0,1)\).

Let \( \varphi_t : [x_t, f(x_t)] \to [y_t, g_t(y_t)] \), \( t \in [0,1) \) be a \( C^1\)-continuous path of diffeomorphisms satisfying

\[
D\varphi_t(f(x_t))Df(x_t) = Dg_t(y_t)D\varphi_t(x_t).
\]

Then there is a unique \( C^1\)-diffeomorphism \( \tilde{h}_t : (0,1) \to (0,1) \) conjugating \( f \) to \( g_t \) and coinciding with \( \varphi_t \) on \([x_t, f(x_t)]\). As before, as \( f \) and \( g_t \) coincide on \([0, a_t]\) and \([b_t, 1]\), one gets that \( h_t \) commutes with \( f \) in a neighborhood of \( 0 \) and of \( 1 \). One concludes as before : one can modify \( \tilde{h}_t \) slowly in the successive fundamental domains of \( f \) and \( f^{-1} \) in order to get a diffeomorphism \( h_t \) coinciding with \( \tilde{h}_t \) out of a small neighborhood of \( 0 \) and of \( 1 \), with the identity map in a smaller neighborhoods of \( 0 \) and \( 1 \) and almost commuting with \( f \) on \([0, a_t]\) and on \([b_t, 1]\).

Summarizing, this proves : 

Theorem 8.3. \([Fa]\)

- given \( f \in Diff^1_+(\{0,1\}) \) without fixed points in \((0,1)\) so that \( f(x) - x > 0 \) on \((0,1)\)
- given any continuous pathes \( 0 < a_t < b_t < 1, \ t \in [0,1)\)
- given any continuous pathes \( x_t, y_t \in [a_t, b_t], \ t \in [0,1)\)
- given any \( C^1\)-continuous path \( g_t \in Diff^1_+(\{0,1\}) \) is a diffeomorphism without fixed point in \((0,1)\) so that \( g_t \) coincides with the identity on \([0, a_t]\) and on \([b_t, 1]\)
- given any \( C^1\)-continuous path \( \varphi_t : [x_t, f(x_t)] \to [y_t, g_t(y_t)] \) so that
  \[
  D\varphi_t(f(x_t))Df(x_t) = Dg_t(y_t)D\varphi_t(x_t).
  \]

Then there is a \( C^1\)-continuous path \( h_t \in Diff^1_+(\{0,1\}), \ t \in [0,1) \), so that

- for every \( t \), the diffeomorphism \( h_t \) coincides with the identity on a neighborhood of \( 0 \) and of \( 1 \)
- \( h_t \) coincides with \( \varphi_t \) on \([x_t, f(x_t)]\)
- the \( C^1 \) distance \( \|h_tfh_t^{-1} - g_t\|_1 \) tends to 0 as \( t \to 1 \).

According to Theorem 8.3 Lemma 8.4 is now a direct consequence of the following lemma :
Lemma 8.5. Let \( f \in Diff^1([0,1]) \) without hyperbolic fixed points, without fixed points in \((0,1)\) so that \( f(x) - x > 0 \) on \((0,1)\), and \([a,b], b = f(a), \) be a fundamental domain of \( f \). Then:
- there is a \( C^1 \)-continuous path \( g_t, t \in [0,1], g_t \in Diff^1([0,1]), \) so that:
  - \( g_t \) is without fixed point in \((0,1)\)
  - \( g_t \) are continuous paths \( 0 < a_t < b_t < 1 \) so that \( g_t \) coincides with \( f \) on \([0,a_t] \cup [b_t,1]\)
  - \( g_t \xrightarrow{t \to 1} \text{id} \)
- there is a \( C^1 \)-continuous path of diffeomorphisms \( \varphi_t : [a,f(a)] \to [a,g_t(a)] \) so that

\[
D\varphi_t(f(a))Df(a) = Dg_t(a)D\varphi_t(a),
\]
- and \( (\varphi_t)_{t \in [0,1]} \) has equicontinuous Log-derivative on \([a,f(a)]\).

Proof of Lemma 8.4: Lemma 8.3 and Theorem 8.3 imply that there is \( h_t \) so that \( h_t f h_t^{-1} \) is \( C^1 \)-asymptotic to the isotopy \( g_t \) which tends to the identity. Furthermore, \( h_t \) coincides with \( \varphi_t \) on \([a,f(a)]\), hence has equicontinuous Log-derivative on the fundamental domain \([a,f(a)]\), ending the proof.

Lemma 8.6. Consider \( f \in Diff^1([0,1]) \) without hyperbolic fixed points, without fixed points in \((0,1)\) so that \( f(x) - x > 0 \) on \((0,1)\), and a fundamental domain \([c,d = f(c)]\) of \( f \).

Let \( c < d_t \leq d \) be a continuous path so that \( d_t \to c \) as \( t \to 1 \) and \( a_t > 0 \) be a continuous path with \( a_t \to 1 \) as \( t \to 1 \).

Then there is a \( C^1 \) continuous path of diffeomorphisms \( g_t \in Diff^1([0,1]) \), there are continuous pathes \( 0 < a_t < b_t < 1 \) so that:
- \( g_t(x) > x \) for \( x \in (0,1) \)
- \( g_t \) coincides with \( f \) on \([0,a_t] \cup [b_t,1]\)
- \( g_t(c) = d_t \)
- \( Dg_t(c) = \alpha_t \)
- \( g_t \xrightarrow{t \to 1} \text{id} \)

Hint for a proof: In other words, \( g_t \) is an isotopy of \( g_0 \) to the identity map, without creating new fixed points, prescribing the image and the derivative at a point \( c \) and requiring that \( g_t \) coincides with \( f \) in a small neighborhood of 0 and 1. That is possible because we require that the image \( d_t = g_t(c) \) tends to \( c \), that the derivative \( \alpha_t = Dg_t(c) \) tends to \( 1 \), and because \( Df(0) = Df(1) = 1 \) so that \( f \) is arbitrarily \( C^1 \)-close to the identity map in sufficiently small neighborhoods of \( 0 \) and \( 1 \).

Lemma 8.7. Consider \( f \in Diff^1([0,1]) \) without hyperbolic fixed points, without fixed points in \((0,1)\), so that \( f(x) - x > 0 \) on \((0,1)\). Let \([c,d], d = f(c)\) be a fundamental domain of \( f \). Then, there is a \( C^1 \)-continuous path of diffeomorphisms \( \varphi_t : [c,d] \to [c,\varphi_t(d)], t \in [0,1], \) so that
- \( \varphi_t(d) \to c \) as \( t \to 1 \);
- Let us denote \( \alpha_t = \frac{Df(c)D\varphi_t(d)}{D\varphi_t(c)} \). Then \( \alpha_t \to 1 \) as \( t \to 1 \);
- \( \log(D\varphi_t), t \in [0,1], \) is equicontinuous.

Proof: Notice that adding a constant to a function does not change the equicontinuity properties. As a consequence, one can compose each \( \varphi_t \) by some affine map without changing the equicontinuity of the family \( \log(D\varphi_t) \); furthermore composing by an affine map does not change the ratio \( \frac{Df(c)D\varphi_t(d)}{D\varphi_t(c)} \). In other words, the first item is for free.

Now, we choose some \( \varphi_t \_0 \) so that \( \frac{Df(c)D\varphi_t(d)}{D\varphi_t(c)} = 1 \), and for \( t > t_0 \) one chooses \( \varphi_t \) as being the composition of \( \varphi_{t_0} \) by some affine map.

\( \square \)

8.5 Conjugacy by an equicontinuous Log-derivative map

Lemma 8.8. Let \( (\alpha_t)_{t \in [0,1]} \) be a \( C^1 \)-continuous path of \( C^1 \)-diffeomorphisms of \([0,1]\) with equicontinuous Log derivative : \( \{\log D\alpha_t\}_{t \in [0,1]} \) is equicontinuous.
Then for every \( \eta > 0 \), there is \( \varepsilon > 0 \) such that, for all \( g \in \text{Diff}_+^1([0,1]) \) satisfying \( \|g - \text{id}\|_1 < \varepsilon \), one has:

\[
\|\alpha_t g \alpha_t^{-1} - \text{id}\|_1 < \eta \quad \text{for all} \quad t \in [0,1).
\]

In particular, if \((g_t)_{t \in [0,1]}\) is a path of diffeomorphisms converging to \text{id} when \( t \) tends to 1, then

\[
\alpha_t g \alpha_t^{-1} \xrightarrow{t \to 1} \text{id}.
\]

**Proof:** Consider \( x \in [0,1] \) and \( y = \alpha_t^{-1}(x) \). Then

\[
D(\alpha_t g \alpha_t^{-1}) = D\alpha_t(g(y)) \cdot Dg(y).
\]

By assumption, \( |Dg(y) - 1| < \varepsilon \). Therefore, it is enough to check that

\[
\log\left(\frac{D\alpha_t(g(y))}{D\alpha_t(y)}\right) = \log D\alpha_t(g(y)) - \log D\alpha_t(y)
\]

is uniformly bounded in function of \( \varepsilon \), and that this bound tends to 0 as \( \varepsilon \to 0 \). Notice that \( |g(y) - y| < \varepsilon \).

Thus, the equicontinuity of \( \log D\alpha \) provides the uniform bound of \( \log\left(\frac{D\alpha_t(g(y))}{D\alpha_t(y)}\right) \) in function of \( \varepsilon \). \( \square \)

### 8.6 Isotopy by conjugacy to the identity and perturbations

**Definition 8.9.** Let \( \varepsilon_t > 0 \) and \( \eta_t > 0 \), \( t \in [0,1] \) be continuous paths with \( \varepsilon_t \to 0 \) and \( \eta_t \to 0 \) as \( t \to 1 \).

A \( C^1 \)-continuous path \((\psi_t)_{t \in [0,1]}\), \( \psi_t \in \text{Diff}_+^1([0,1]) \) is an \((\varepsilon_t)_{t \in [0,1]}\)-robust isotopy by conjugacy of speed \((\eta_t)_{t \in [0,1]}\) from \( f \) to \text{id} if, for all continuous path \((g_t)_{t \in [0,1]}\) satisfying \( \|g_t - f\|_1 < \varepsilon_t \), one has:

\[
\|\psi_t g_t \psi_t^{-1} - \text{id}\|_1 \to 0 \quad \text{as} \quad t \to 1.
\]

**Lemma 8.10.** Let \( f, (\varphi_t)_{t \in [0,1]} \) be \( C^1 \)-diffeomorphisms of \( \mathbb{R} \) such that \( \|\varphi_t \varphi_t^{-1} - \text{id}\|_1 \to 0 \) as \( t \to 1 \).

For all continuous path \((\varepsilon_t)_{t \in [0,1]}\) of strictly positive real numbers converging to 0, there exist a continuous path \((\eta_t)_{t \in [0,1]}\) of strictly positive real numbers converging to 0 and a continuous map \( r : [0,1] \to [0,1] \), satisfying \( r(0) = 0, r(t) \xrightarrow{t \to 1} 1 \) such that, \( (\psi_t = \varphi_t r(t))_{t \in [0,1]} \) is an \((\varepsilon_t)_{t \in [0,1]}\)-robust isotopy by conjugacy of speed \((\eta_t)_{t \in [0,1]}\) from \( f \) to \text{id}.

We split the proof in two lemmas. The first ones just states that any isotopy by conjugacy is \( \varepsilon_t \) robust, if one chooses \( \varepsilon_t > 0 \) small enough.

**Lemma 8.11.** Consider \( f \in \text{Diff}_+^1([0,1]) \) and a \( C^1 \)-continuous path \((\varphi_t)_{t \in [0,1]}\), \( \varphi_0 = \text{id} \), so that \( \|\varphi_t \varphi_t^{-1} - \text{id}\|_1 \xrightarrow{t \to 1} 0 \).

Denote \( \mu_t = 2 \cdot \|\varphi_t \varphi_t^{-1} - \text{id}\|_1 \). Then there is a continuous path \( \nu_t > 0 \) so that for all continuous path \((g_t)_{t \in [0,1]}\) satisfying \( \|g_t - f\|_1 < \nu_t \), one has:

\[
\|\varphi_t g_t \varphi_t^{-1} - \text{id}\|_1 < \mu_t.
\]

In other words, the isotopy by conjugacy \( \varphi_t \) is \( \nu_t \)-robust of speed \( \mu_t \).

**Sketch of proof:** For every \( t \in [0,1] \), one needs to bound \( D\varphi_t(g(x)) - D\varphi_t(f(x)) \), for \( |g(x) - f(x)| < \nu_t \), uniformly in \( x \in [0,1] \), by a constant \( \mu_t \) depending in a simple continuous way on \( \mu_t \). As \( D\varphi_t \) is bounded on \([0,1]\), one essentially needs to bound (uniformly in \( x \)) \( D\varphi_t(g(x)) - D\varphi_t(f(x)) \), for \( |g(x) - f(x)| < \nu_t \). In other words, \( \nu_t \) depends strongly on the continuity modulus \( \delta_t \) of \( \varphi_t \) for the constant \( \mu_t \), where \( \delta_t = |x - y| < \delta_t \Rightarrow D\varphi_t(x) - D\varphi_t(y) < \mu_t \cdot \max_{x \in [0,1]} |D\varphi_t(x)| \).

The unique difficulty is to choose \( 0 < \nu_t < \delta_t \) depending continuously on \( t \in [0,1] \). This is possible because the modulus of continuity of a continuous function (of a compact metric set) associated to a given constant, depends lower-semi-continuously on the function. One concludes by noticing that, given a strictly positive lower semi continuous map \( \delta_t : [0,1] \to \mathbb{R} \), there is a strictly positive function \( 0 < \nu_t < \delta_t \).

\( \square \)

For getting Lemma 8.10 from Lemma 8.11, one just needs to apply the following simple observation
Lemma 8.12. Let $\varepsilon_t > 0$ and $\nu_t > 0$, $t \in [0,1)$, be continuous paths so that $\varepsilon_t \to 0$. Then, there is a continuous map $r: [0,1) \to [0,1]$, $r(0) = 0$ and $r(t) \to 1$ as $t \to 1$, and there is $0 \leq t_0 < 1$ so that, for every $t_0 \leq t < 1$, one has:

$$\nu_{r(t)} > \varepsilon_t.$$ 

Proof of Lemma 8.10: Choose $\mu_t, \nu_t > 0$ given by Lemma 8.11 and $r(t)$ and $t_0$ given by Lemma 8.12.

Thus, for all continuous path $(g_t)_{t \in [0,1)}$ satisfying $\|g_t - f\| < \varepsilon_t$, for every $t \geq t_0$ one has

$$\|\phi_{r(t)}g_t\phi_{r(t)}^{-1} - \text{id}\|_1 < \nu_{r(t)}.$$

Notice that $t \to \nu_{r(t)}$ is continuous and tends to 0 as $t \to 1$.

In other words, the choice $\eta_t = \mu_{r(t)}$ is convenient for $t \geq t_0$. One extend such $\eta_t$ for $t \in [0, t_0]$ by a simple compactness argument. More precisely: one chooses $\eta_t$, $t \in [0,1)$ so that:

- for every $t \in [0,1)$, $\eta_t \geq \mu_{r(t)}$
- $\eta_t = \mu_{r(t)}$ for every $t$ close enough to 1.
- for every $t \in [0, t_0)$,

$$\eta_t \geq \max_{x \in [0,1]} Df(x) + \max_{t \in [0, t_0], x \in [0,1]} D\phi_{r(t)}(x) + \max_{t \in [0, t_0], x \in [0,1]} D\phi_{r(t)}^{-1}(x)$$

$t \to \eta_t$ is continuous.

For this choice of $\eta$, $\phi_{r(t)}$ is $\varepsilon_t$-robust with speed $\eta_t$, concluding the proof.

8.7 Group extensions in the class $C^1_{id}$: proof of Theorem 1.6

We are now ready to prove Theorem 1.6:

Proof of Theorem 1.6: Let $f$ be a $C^1$-diffeomorphism of $[0,1]$, without hyperbolic fixed points and $I = [x, f(x)]$ a fundamental domain of $f$. Let $G \subset \text{Diff}^1([0,1])$ be a group of diffeomorphisms whose supports are included in $I$. Assume $G$ is $C^1$-close to $\text{id}$; more precisely, we assume that there is a $C^1$-continuous path of diffeomorphisms $h_t$, $t \in [0,1)$, supported on $I$, which realizes an isotopy by conjugacy of the elements of $G$ to the identity. One will prove that the group $(G, f)$, generated by $f$ and the elements of $G$, is $C^1$-close to the identity and admits an isotopy by conjugacy to the identity $H_t, t \in [0,1)$, so that $D(H_t)(0) = D(H_t(1)) = 1$. Indeed, $H_t$ will coincide with the identity in small neighborhoods of 0 and 1.

One begins by extending the path $(h_t)_{t \in [0,1]}$ to $[0,1]$ by Lemma 8.3 in such a way that $\|h_t f(h_t)^{-1} - f\|_1 < \varepsilon_t$, where $(\varepsilon_t)_{t \in [0,1)}$ is some continuous path of strictly positive real numbers converging to 0, and $h_t$ coincide with the identity in a neighborhood of 0 and 1.

As explained in Section 8.4 and from Lemma 8.10 one can choose an $(\varepsilon_t)_{t \in [0,1)}$-robust isotopy $(\alpha_t)_{t \in [0,1)}$ from $f$ to $\text{id}$ which has equicontinuous Log-derivative, and so that $\alpha_t$ coincides with the identity map in small neighborhoods of 0 and 1.

Then, by definition of an $(\varepsilon_t)_{t \in [0,1)}$-robust isotopy, one has:

$$\|\alpha_t h_t f(h_t)^{-1} \alpha_t^{-1} - \text{id}\|_1 \to 0, \quad t \to 1$$

and, from Lemma 8.8 one has also:

$$\forall g \in G, \|\alpha_t h_t gh_t^{-1} \alpha_t^{-1} - \text{id}\|_1 \to 0, \quad t \to 1$$

Thus $H_t = \alpha_t h_t$ is the announced isotopy by conjugacy of $<f,G>$ to the identity.

9 Isotopy to the identity of groups generated by a fundamental system

The aim of this section is to prove Theorem 1.7: any group $G$ of diffeomorphisms of $[0,1]$ generated by a fundamental system is $C^1$-close to the identity. We will prove a slightly stronger version: the diffeomorphisms $f_n$ are not assumed to be simple.
**Theorem 9.1.** Let \((f_n)_{n \in \mathbb{N}}\) be a collection of \(C^1\)-diffeomorphisms of \(\mathbb{R}\) with compact support and without hyperbolic fixed point and, for each \(n \in \mathbb{N}\), let \(I_n\) be a given fundamental domain of \(f_n\) such that:

for all \(i < n\),
- either \(\text{Supp}(f_n) \subset I_i\);
- or \(\text{Supp}(f_i) \subset I_n\);
- or \(\text{Supp}(f_n) \cap \text{Supp}(f_i) = \emptyset\)

(where \(\text{Supp}(f)\) denotes the interior of the support of \(f\)).

Then the group \((f_n, n \in \mathbb{N})\) generated by \((f_n)_{n \in \mathbb{N}}\) is isotopic by conjugacy to the identity.

Let \(f_n, n \in \mathbb{N}\), be a collection of diffeomorphisms satisfying the hypotheses of Theorem 9.1 and let denote by \(G\) the group generated by the \(f_n\). Therefore, \(G\) is the increasing union of the groups \(G_n = \langle f_0, \ldots, f_n \rangle\). According to Theorem 1.5 if all the \(G_n\) are \(C^1\)-close to the identity, \(G\) is \(C^1\)-close to the identity.

Therefore, Theorem 9.1 is a straightforward consequence of Theorem 1.5 with the following finite version of Theorem 9.1:

**Theorem 9.2.** Let \(N > 0\) be an integer and \((f_n)_{n \in \{0, \ldots, N\}}\) be a collection of \(C^1\)-diffeomorphisms of \([0, 1]\) without hyperbolic fixed point and, for each \(n \in \mathbb{N}\), let \(I_n\) be a given fundamental domain of \(f_n\) such that:

for all \(i < n\),
- either \(\text{Supp}(f_n) \subset I_i\);
- or \(\text{Supp}(f_i) \subset I_n\);
- or \(\text{Supp}(f_n) \cap \text{Supp}(f_i) = \emptyset\)

(where \(\text{Supp}(f)\) denotes the interior of the support of \(f\)).

Then the group \((f_0, \ldots, f_N)\) is \(C^1\)-close to the identity. More precisely there is a \(C^1\)-continuous family \(\{h_t\}_{t \in [0, 1]}, h_t \in \text{Diff}^1([0, 1]), \text{ supported on } \bigcup_{i=0}^{N} \text{Supp}(f_i), \text{ so that } Dh_t(0) = Dh_t(1) = 1 \text{ and } \forall i \in \{0, \ldots, N\}, f_i \xrightarrow{h_t} \text{id}\)

### 9.1 Proof of Theorem 9.2

One proves Theorem 9.2 by induction on \(N\). For \(N = 0\), this is precisely the main result of [5]. Assume now that Theorem 9.2 is proved for \(N \geq 0\); we will prove it for \(N + 1\).

Let \(f_0, \ldots, f_{N+1}\) be diffeomorphisms satisfying the hypotheses of Theorem 9.2. The supports \(S_i, S_j\) of \(f_i, f_j, i \neq j\) either have disjoint interiors or are included one in a fundamental domain of the other. Consider \(I \subset \{0, \ldots, N+1\}\) be the indices for which \(S_i\) is maximal for the inclusion.

First assume that \(I\) contains more than 1 element.

Then, for every \(i \in I\) the collection \(\{f_j, S_j \subset S_i\}\) satisfies the hypotheses of Theorem 9.2 and contains strictly less element than \(N + 1\). Therefore, the induction hypothesis provides continuous paths \(h_t^i, t \in [0, 1]\), supported on \(S_i\), realizing an isotopy of all the \(f_j\) with \(S_j \subset S_i\) to the identity, and so that the derivatives at 0 and 1 are equal to 1. One defines the announced family \(h_t\) as coinciding with \(h_t^i\) on \(S_i\), \(i \in I\).

Assume now that \(I\) contains a unique element.

Up to change the indexation, one may assume that \(I = \{N+1\}\). Thus, the group \(G_N = \langle f_0, \ldots, f_N \rangle\) is supported in the fundamental domain \(I_{N+1}\) of \(f_{N+1}\).

By the induction hypothesis, there is a \(C^1\)-continuous path \(h_t^N, t \in [0, 1]\), supported on \(I_{N+1}\) and realizing an isotopy by conjugacy of the elements of \(G_N\) to the identity.

Thus \(G_N\) and \(f_{N+1}\) satisfy the hypotheses of Theorem 9.1 which provides the announced path \(h_t^{N+1}\), concluding the proof.

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