HOMOTOPY THEORY OF PRESHEAVES OF SIMPLICIAL GROUPOIDS

ZHI-MING LUO

Abstract. We show that the category of presheaves of simplicial groupoids on a site \( C \) is a right proper simplicial model category. We define \( G \)-torsor of presheaf of 2-groupoids \( G \), presheaf of simplicial groups \( G \) and presheaf of simplicial groupoids \( G \) on a site \( C \) and classify \( G \)-torsors by the homotopy classes \( [\ast, \mathcal{W} G] \).

1. Introduction

The techniques of homotopy theory have been extremely fruitful in other areas of mathematics such as algebraic K-theory \[11\] and algebraic geometry \[24\]. The goal of this paper is to develop the machinery needed to do homotopy theory in category of presheaves of simplicial groupoids on any small site \( C \), and to connect this homotopy theory to the ordinary homotopy theory of spaces.

Axiomatic homotopy theory is a natural extension of the ordinary homotopy theory for topological spaces to other categories. It comes from two systems of axioms. One is K.Brown’s axioms for a category of fibrant objects \[2\], \[8\, Section I.9.\]; the other one is Quillen’s axioms for a closed model category \[25\], \[26\], \[8\, Section II.1.\].

Quillen’s axioms imply Brown’s axioms. But on the level of sheaves (presheaves), there are two homotopy theories. One is the local theory which was constructed by Brown \[2\] for the category of simplicial sheaves on a topological space, and by Jardine \[11\], \[9\] for the category of simplicial sheaves and simplicial presheaves on any site. The other one is the global theory for the corresponding categories, which was developed by Brown and Gersten \[3\], Joyal \[15\] and Jardine \[9\], respectively. The local theory satisfies Brown’s axioms, the global theory satisfies Quillen’s axioms. The two theories are distinct, since it is not true that every local fibration is a global fibration \[9\].

This paper is based on the Quillen closed model category axioms.

The central theorem of simplicial homotopy theory asserts that the category \( S \) of simplicial sets, equipped with three classes of morphisms, namely cofibrations, fibrations and weak equivalences, has a closed model structure \[25\]. Mathematicians have found many categories with closed model structures. For example, the category of simplicial groupoids (Dwyer-Kan \[6\], \[8\, Section V.7.\]), the category of 2-groupoids (Moerdijk-Svensson \[23\]), the category of simplicial presheaves (Jardine \[9\]), the category of simplicial sheaves (Joyal \[15\]) and so on. Crans \[5\] uses

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adjoint functors to prove that the categories of sheaves of 2-groupoids, of bisimplicial sheaves and of simplicial sheaves of groupoids have closed model structures according to the well-known closed model category of simplicial sheaves.

In Section 2 we use techniques based on Jardine’s paper [9], to prove that the categories of presheaves of simplicial groupoids and of presheaves of 2-groupoids on any site $C$ are right proper simplicial model categories (Theorems 2.4, 2.6, 2.7, 2.9, 2.10 and 2.11). Both Crans [5] and Joyal-Tierney [17] obtain Quillen closed model structures on the category of sheaves of simplicial groupoids in the general sense (i.e., the simplicial groupoids are groupoid objects in simplicial sets, not just groupoids enriched over simplicial sets; in our case we restrict the simplicial groupoids to be groupoids enriched in simplicial sets). They use the classifying space functor $B$ to define the weak equivalences whereas we use the functor $W$.

Section 3 studies the classification of $G$-torsors for various (pre)sheaves $G$. The definition of $G$-torsor and the relations between $G$-torsors and the homotopy classes $[*,BG]$ are well-known for sheaves of groups and groupoids (Sections 3.1 and 3.2). We extend the definition of $G$-torsors to presheaves of 2-groupoids, simplicial groups and simplicial groupoids $G$ on any site $C$ in Sections 3.3, 3.4 and 3.5. After we define the category $Tor(*,G)$ of all $G$-torsors we obtain similar classifying results for $G$-torsors: $\pi_0 Tor(*,G) \cong [*,WG]$ (Theorem 3.7, Corollary 3.12 and Theorem 3.23). All these definitions and results for (pre)sheaves of groups, groupoids, 2-groupoids and simplicial groups are special cases of those for presheaves of simplicial groupoids. Joyal-Tierney [18] obtain similar results for sheaves of simplicial groupoids $G$, but their definition of $G$-torsor is different from ours; our definition is much more flexible.

2. Presheaves of simplicial groupoids

2.1. Presheaves of simplicial groupoids. Suppose that $C$ is a simplicial object in the category of small categories. Write $E_C$ for the category constructed using the following variant of the Grothendieck construction.

The set of objects of $E_C$ consists of all pairs $(x,n)$ with $x \in C_n$. A morphism $(f,\theta) : (x,m) \to (y,n)$ is a pair consisting of an ordinal number map $\theta : m \to n$ and a morphism $f : x \to \theta^*y$ of $C_m$. Composition is defined in the natural way. There is an obvious forgetful functor $\pi : E_C \to \Delta$ which takes values in the ordinal number category $\Delta$.

The segment category $Seg(n)$ of subintervals $[j,n]$ of $n = [0,n]$ has as objects the intervals $[j,n] = \{j,j+1,\ldots,n\}$ and as morphisms the inclusions of intervals. $Seg(n)$ can be identified with the opposite $n^{op}$ via the functor $[j,n] \mapsto j$. There is a functor $c_n : n^{op} \to \Delta$ which is defined on objects by $j \mapsto n - j$, and which sends morphisms to inclusions.

An $n$-cocycle taking values in the simplicial category $C$ is a functor $X : n^{op} \to E_C$ which is a lifting of $c_n$ in the sense that the diagram of functors

$$
\begin{array}{ccc}
\Delta & \xrightarrow{c_n} & n^{op} \\
\downarrow & & \downarrow \\
E_C & \xrightarrow{X} & \leftarrow \pi
\end{array}
$$
commutes. This is a generalization of the definition of an \( n \)-cocycle taking values in a groupoid enriched in simplicial sets, in view of the identification of the categories \( \text{Seg}(n) \) and \( n^{op} \).

The \( n \)-cocycle \( X : n^{op} \to EC \) is otherwise described as a string of arrows
\[
(x_0, n) \leftarrow (x_1, n - 1) \leftarrow \cdots \leftarrow (x_n, 0)
\]
each of which has the form \((\alpha_i, d^n_i)\), with \( \alpha_i : x_{n-i} \to d^n_0(x_{n-i-1}) \). This means that the string consists of objects \( x_i \in C_{n-i} \) and morphisms \( x_i \to d^n_0(x_{i-1}) \).

Every ordinal number map \( \theta : m \to n \) induces a commutative diagram
\[
\begin{array}{c}
m - i \\
\theta_i \\
n - \theta(i)
\end{array} \xrightarrow{\cong} \begin{array}{c}[i, m] \\
\theta \\
[\theta(i), n]
\end{array} \xrightarrow{\cong} m
\]
and there is a corresponding diagram
\[
\begin{array}{c}
\theta^*_0 x_{\theta(0)}, m \\
(1, \theta_0) \\
(x_{\theta(0)}, n - \theta(0))
\end{array} \xleftarrow{(1, \theta_1)} \begin{array}{c}(\theta^*_1 x_{\theta(1)}, m - 1) \\
(1, \theta_1) \\
(x_{\theta(1)}, n - \theta(1))
\end{array} \xleftarrow{(1, \theta_m)} \begin{array}{c}(\theta^*_m x_{\theta(m)}, 0) \\
(1, \theta_m) \\
(x_{\theta(m)}, n - \theta(m))
\end{array}
\]
The string on top is denoted by \( \theta^* X \).

In this way, a simplicial set \( WC \) is defined, with \( WC_n \) given by the set of \( n \)-cocycles in \( C \). The functoriality follows from the relations
\[
\theta_{\tau(i)} \tau(i) = (\theta \tau)_i
\]
associated to composeable ordinal number maps
\[
k \xrightarrow{\tau} m \xrightarrow{\theta} n.
\]

There is an obvious function
\[
j : dBC_n = (BC_n)_n \to WC_n
\]
which sends a string
\[
x_0 \xleftarrow{x_1} \cdots \xleftarrow{x_n}
\]
to the cocycle consisting of the object \( x_0 \in C_n \) and the morphisms
\[
d^n_0 \alpha_{n-j} : d^n_0 x_j \to d^n_0 x_{j-1},
\]
or rather to the string
\[
(x_0, n) \leftarrow (d^n_0 x_1, n - 1) \leftarrow \cdots \leftarrow (d^n_0 x_n, 0)
\]
in the Grothendieck construction \( EC \).

Suppose that \( \theta : m \to n \) is an ordinal number map. One checks that the composite
\[
dBC_n \xrightarrow{j} WC_n \xrightarrow{\theta^*} WC_m
\]
sends the string of arrows \( x_0 \leftarrow x_1 \leftarrow \cdots \leftarrow x_n \) in \( C_n \) to the string
\[
(\theta^*_0 d^n_0 x_{\theta(0)}, m) \leftarrow (\theta^*_1 d^n_0 x_{\theta(1)}, m - 1) \leftarrow \cdots \leftarrow (\theta^*_m d^n_0 x_{\theta(m)}, 0)
\]
while the composite
\[ dBC_n \xrightarrow{\theta^*} dBC_m \xrightarrow{j} WC_m \]
sends that same string in \( C_n \) to the string
\[ (\theta^* x_{\theta(0)}, m) \leftarrow (d_0^m \theta^* x_{\theta(1)}, m-1) \leftarrow \cdots \leftarrow (d_0^m \theta^* x_{\theta(m)}, 0). \]

Since \( \theta^* d_0^m (\theta^* x_{(\theta(i))}) = d_0^i \theta^* x_{\theta(i)} \) (from above diagram of \([i, m], [\theta(i), n]\) and \( m, n \)), it follows that the maps \( j \) respect the simplicial structures.

A simplicial groupoid \( G \) (or simplicial group \( G \)) is a simplicial object in the category of small groupoids (or groups), so the functor \( \overline{W} \) can act on it and produce a simplicial set \( \overline{W}G \) (see [8, Section V.4 and V.7]).

**Lemma 2.1.** Suppose that \( G \) is a simplicial groupoid (groupoid enriched in simplicial sets). Then the map \( j : dBG \to \overline{WG} \) is a weak equivalence.

**Proof.** Since both functors \( dB \) and \( \overline{W} \) are right adjoint functors, they preserve limits and products, so they preserve homotopy equivalences as well [8, pp.303,304]. They also preserve disjoint unions. If \( H \) is a simplicial group, the map \( j : dBH \to \overline{WH} \) classifies the \( H \)-bundle \( dEH \to dBH \), and so \( j \) is a weak equivalence for simplicial groups. Every simplicial groupoid \( G \) is homotopy equivalent to a disjoint union of simplicial groups. \( \square \)

**Lemma 2.2.** Suppose that \( H \) is a simplicial groupoid. Then the following statements hold:

1) \( dBH \) is a Kan complex.
2) \( \overline{WH} \) is a Kan complex.

**Proof.** Every simplicial groupoid \( H \) contains a strong deformation retraction \( \bigsqcup \pi_0 H(x, x) \), where the vertices \( x \) are indexed over a set of representatives of \( \pi_0 H \). Suppose that \( X \) denotes either \( dB \) or \( \overline{W} \). Then every lifting problem
\[ \begin{array}{ccc}
\land^n_k & \rightarrow & X(H) \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & \bigsqcup X(H(x, x))
\end{array} \]
is isomorphic to a lifting problem
\[ \begin{array}{ccc}
\land^n_k & \rightarrow & X(H) \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & \bigsqcup X(H(x, x))
\end{array} \]

It follows that \( X(H) \) is a Kan complex for all simplicial groupoids \( H \) if and only if \( X(G) \) is a Kan complex for all simplicial groups \( G \).

In the first case, the bisimplicial set \( BG \) consists of connected simplicial sets \( BG_n \) in each horizontal degree \( n \), and therefore satisfies the \( \pi^*_K \)-Kan condition [8, Lemma IV.4.2.]. In vertical degree \( k \), \( BG_{*,k} = G^{xk} \), which is a Kan complex since all simplicial groups are Kan complexes. It follows from Lemma IV.4.8 of [8] that \( dBG \) is a Kan complex.

For the second statement, we know that \( \overline{WG} \) is a Kan complex if \( G \) is a simplicial group, by standard theory [8, Corollary V.6.8]. \( \square \)
Let $\mathcal{C}$ be a fixed small Grothendieck site. $s\text{Gd}\text{Pre}(\mathcal{C})$ is the category of presheaves of simplicial groupoids on $\mathcal{C}$; its objects are the contravariant functors from $\mathcal{C}$ to the category $s\text{Gd}$ of simplicial groupoids, and its morphisms are natural transformations.

If we see a presheaf of simplicial groupoids $G$ as a simplicial object in the category of presheaves of groupoids, then $W G$ is a simplicial presheaf. That means there is a functor $W : s\text{Gd}\text{Pre}(\mathcal{C}) \to S\text{Pre}(\mathcal{C})$.

Recall the adjunction between the loop groupoid functor $G : S \to s\text{Gd}$ and the universal cocycle functor $W$ [8, Lemma V.7.7]. By applying these functors sectionwise to simplicial presheaves and presheaves of simplicial groupoids, one obtains functors $G : S\text{Pre}(\mathcal{C}) \rightleftarrows s\text{Gd}\text{Pre}(\mathcal{C}) : W$.

So there is

**Proposition 2.3.** The functor $G : S\text{Pre}(\mathcal{C}) \to s\text{Gd}\text{Pre}(\mathcal{C})$ is left adjoint to the functor $W$.

We recall the definition of a closed model category structure on a category $D$. Closed model categories are an abstract setting in which to do homotopy theory.

A *Quillen closed model category* $D$ is a category which is equipped with three classes of morphisms, called cofibrations, fibrations and weak equivalences which together satisfy the following axioms [25], [26], [8]:

**CM1:** The category $D$ is closed under all finite limits and colimits.

**CM2:** Suppose that the following diagram commutes in $D$:

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow{h} & & \downarrow{f} \\
Z & \xleftarrow{\phantom{f}} & Y
\end{array}
$$

If any two of $f$, $g$ and $h$ are weak equivalences, then so is the third.

**CM3:** If $f$ is a retract of $g$ and $g$ is a weak equivalence, fibration or cofibration, then so is $f$.

**CM4:** Suppose that we are given a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{i} & X \\
\downarrow{p} & & \downarrow{p} \\
V & \xrightarrow{\phantom{p}} & Y
\end{array}
$$

where $i$ is a cofibration and $p$ is a fibration. Then the lifting exists, making the diagram commute, if either $i$ or $p$ is also a weak equivalence.

**CM5:** Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where $p$ is a fibration and $i$ is a trivial cofibration, and

(b) $f = q \cdot j$ where $q$ is a trivial fibration and $j$ is a cofibration.

An object $X$ is called *cofibrant* if the map from the initial object $\emptyset$, to $X$ is a cofibration. An object $X$ is called *fibrant* if the map from $X$ to the final object $\ast$, is a fibration. The category obtained from $D$ by formally inverting the weak equivalences is called the *homotopy category* associated to $D$, and denoted $Ho(D)$.
A category $\mathcal{D}$ is a simplicial category if there is a mapping space functor

$$\text{Hom}_\mathcal{D}(\cdot, \cdot) : \mathcal{D}^{\text{op}} \times \mathcal{D} \to \mathcal{S}$$

with the properties that for $A$ and $B$ objects in $\mathcal{D}$

1. $\text{Hom}_\mathcal{D}(A, B)_0 = \text{hom}_\mathcal{D}(A, B)$;
2. the functor $\text{Hom}_\mathcal{D}(A, \cdot) : \mathcal{D} \to \mathcal{S}$ has a left adjoint $A \otimes \cdot : \mathcal{S} \to \mathcal{D}$ which is associative in the sense that there is an isomorphism

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L,$$

natural in $A \in \mathcal{D}$ and $K, L \in \mathcal{S}$;
3. The functor $\text{Hom}_\mathcal{D}(\cdot, B) : \mathcal{D}^{\text{op}} \to \mathcal{S}$ has a left adjoint $\text{hom}_\mathcal{D}(\cdot, B) : \mathcal{S} \to \mathcal{D}^{\text{op}}$.

A simplicial model category $\mathcal{D}$ is both a closed model category and a simplicial category which satisfies the following axiom:

**SM7** Suppose $j : A \to B$ is a cofibration and $q : X \to Y$ is a fibration. Then

$$\text{Hom}_\mathcal{D}(B, X) \xrightarrow{\tau \cdot q} \text{Hom}_\mathcal{D}(A, X) \times_{\text{Hom}_\mathcal{D}(A, Y)} \text{Hom}_\mathcal{D}(B, Y)$$

is a fibration of simplicial sets, which is trivial if $j$ or $q$ is trivial.

A right proper closed model category $\mathcal{D}$ is a closed model category such that: **P1** the class of weak equivalences is closed under base change by fibrations. In other words, axiom **P1** says that, given a pullback diagram

$$\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow & & \downarrow p \\
Z & \xrightarrow{g} & W
\end{array}$$

of $\mathcal{D}$ with $p$ a fibration, if $g$ is a weak equivalence then so is $g_*$.

The category $\mathcal{S}\text{Pre}(\mathcal{C})$ of simplicial presheaves has a Quillen closed model structure [9], hence we can give some definitions in the category $\text{sGdPre}(\mathcal{C})$.

A map $f : X \to Y$ in the category $\text{sGdPre}(\mathcal{C})$ is said to be a fibration if the induced map $W(f) : WX \to WY$ is a global fibration in the category $\mathcal{S}\text{Pre}(\mathcal{C})$ in the sense of [9].

A map $g : Z \to U$ in the category $\text{sGdPre}(\mathcal{C})$ is said to be a weak equivalence if the induced map $W(g) : WZ \to WU$ is a topological weak equivalence in the category $\mathcal{S}\text{Pre}(\mathcal{C})$ in the sense of [9].

A cofibration in the category $\text{sGdPre}(\mathcal{C})$ is a map of presheaves of simplicial groupoids which has the left lifting property with respect to all both fibrations and weak equivalences.

**Theorem 2.4.** The category $\text{sGdPre}(\mathcal{C})$, with the classes of fibrations, weak equivalences and cofibrations as defined above, satisfies the axioms for a closed model category.

**Proof.** See [10].
Remark 2.5. Crans [5] and Joyal-Tierney [17] provide two different Quillen closed model structures on the category of sheaves of simplicial groupoids in the general sense (i.e., the simplicial groupoids are groupoid objects in simplicial sets, not just groupoids enriched) (see the comment in the introduction of [17]). They use the classifying space functor $B$ to define the weak equivalences whereas we use the functor $W$.

Let $X$ be a presheaf of simplicial groupoids, and let $K$ be a simplicial set. The presheaf of simplicial groupoids $X \otimes K$ is defined at $U \in C$ by

$$X \otimes K(U) = X(U) \otimes K,$$

The presheaf of simplicial groupoids $X^K$ is defined at $U \in C$ by

$$X^K(U) = \text{hom}(K, X(U)),$$

where $\text{hom}(K, X(U))$ is a simplicial groupoid.

For presheaves of simplicial groupoids $G$ and $H$, define a simplicial set $\text{Hom}(G, H)$ by requiring that the $n$-simplices be maps of presheaves of simplicial groupoids of the form $G \otimes \Delta^n \to H$.

Let $i : K \to L$ be a cofibration in $S$ and $q : U \to V$ be a fibration in $sGdPre(C)$, then the map

$$U^L \to V^L \times_{V^K} U^K$$

is a fibration, which is trivial if either $i$ or $q$ is trivial, since when we apply the functor $W$ we get a similar map in $SPre(C)$ and the category $SPre(C)$ is a simplicial model category.

Suppose that $p : G \to H$ is a fibration and that $i : X \to Y$ is a cofibration of presheaves of simplicial groupoids, then

$$\text{Hom}(Y, G) \to \text{Hom}(X, G) \times_{\text{Hom}(X, H)} \text{Hom}(Y, H)$$

is a fibration of simplicial sets, which is trivial if $i$ or $p$ is trivial by [8, Proposition II.3.13].

The mapping space functor satisfies the axiom $\text{SM7}$, so there is

Theorem 2.6. The category $sGdPre(C)$ is a simplicial model category.

Theorem 2.7. The category $sGdPre(C)$ is right proper.

Proof. Given a pullback diagram in $sGdPre(C)$

$$
\begin{array}{ccc}
X & \xrightarrow{g_*} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{p} & W
\end{array}
$$

with $p$ a fibration and $g$ a weak equivalence, there exists a pullback diagram in $SPre(C)$

$$
\begin{array}{ccc}
WX & \xrightarrow{Wg_*} & WY \\
\downarrow & & \downarrow \\
WZ & \xrightarrow{Wp} & WW
\end{array}
$$
W preserves fibrations and weak equivalences, hence Wp is a fibration and Wg is a weak equivalence in SPre(C). SPre(C) is proper, so Wg is a weak equivalence as well, hence the map g∗ is a weak equivalence. So the axiom P1 holds. □

2.2. Presheaves of 2-groupoids. 2-GpdPre(C) is the category of presheaves of 2-groupoids on C; its objects are the contravariant functors from C to the category 2-Gpd of 2-groupoids, and its morphisms are natural transformations.

Recall the adjunction [20]:

\[ \pi : sGd \rightleftarrows 2 - Gpd : B \]

By applying these functors sectionwise to presheaves of simplicial groupoids and presheaves of 2-groupoids, one obtains functors

\[ \pi : sGdPre(C) \rightleftarrows 2 - GpdPre(C) : B \]

and there is

Proposition 2.8. The functor \( \pi : sGdPre(C) \to 2 - GpdPre(C) \) is left adjoint to the functor B.

The category sGdPre(C) of presheaves of simplicial groupoids has a Quillen closed model structure (see Section 2.1), hence we can give some definitions in the category 2-GpdPre(C).

A map f : X → Y in the category 2-GpdPre(C) is said to be a fibration if the induced map B(f) : BX → BY is a fibration in the category sGdPre(C).

A map g : Z → U in the category 2-GpdPre(C) is said to be a weak equivalence if the induced map B(g) : BZ → BU is a weak equivalence in the category sGdPre(C).

A cofibration in the category 2-GpdPre(C) is a map of presheaves of 2-groupoids which has the left lifting property with respect to all fibrations and weak equivalences.

Theorem 2.9. The category 2-GpdPre(C), with the classes of fibrations, weak equivalences and cofibrations as defined above, satisfies the axioms for a closed model category.

Proof. See [20] or [19]. □

Let X be a presheaf of 2-groupoids, and let K be a simplicial set. The presheaf of 2-groupoids X ⊗ K is defined at U ∈ C by

\[ X \otimes K(U) = X(U) \otimes K, \]

The presheaf of 2-groupoids X^K is defined at U ∈ C by

\[ X^K(U) = (X(U))^K. \]

For presheaves of 2-groupoids G and H, define a simplicial set Hom(G, H) by requiring that the n-simplices be maps of presheaves of 2-groupoids of the form G ⊗ △^n → H.

Let i : K → L be a cofibration in S and q : U → V be a fibration in 2-GpdPre(C), then the map

\[ U^L \overset{(q \circ i^*)}{\longrightarrow} V^L \times_{V^K} U^K \]
is a fibration, which is trivial if either $i$ or $q$ is trivial, since apply the functor $B$ to get a similar map in $sGdPre(\mathcal{C})$ and the category $sGdPre(\mathcal{C})$ is a simplicial model category.

Suppose that $p : G \to H$ is a fibration and that $i : X \to Y$ is a cofibration of presheaves of 2-groupoids, then

$$\text{Hom}(Y, G) \to \text{Hom}(X, G) \times_{\text{Hom}(X, H)} \text{Hom}(Y, H)$$

is a fibration of simplicial sets, which is trivial if $i$ or $p$ is trivial by [8, Proposition II.3.13].

The mapping space functor satisfies the axiom $\text{SM7}$, so there is

Theorem 2.10. The category $2\text{-GpdPre}(\mathcal{C})$ is a simplicial model category.

Theorem 2.11. The category $2\text{-GpdPre}(\mathcal{C})$ is right proper.

Proof. The functor $B$ preserves pullbacks and the category $sGdPre(\mathcal{C})$ is right proper (Theorem 2.7).

\[\Box\]

3. Classification of torsors

3.1. Torsors for sheaves of groups. Suppose that $G$ is a sheaf of groups on $\mathcal{C}$. A (right) $G$-torsor is a non-empty sheaf $E$ (meaning $E \to 1$ is surjective) equipped with a free (right) $G$-action $a : E \times G \to E$ which is transitive [16], [12], [22].

The non-abelian cohomology object $H^1(\mathcal{C}, G)$ is the set of isomorphism classes of $G$-torsors on the site $\mathcal{C}$, as usual.

It is shown in [10], [12]:

Theorem 3.1. There is an isomorphism

$$[*, BG] \cong H^1(\mathcal{C}, G)$$

for any sheaf of groups $G$ on any Grothendieck site $\mathcal{C}$.

Remark 3.2. The sheaf of groupoids Tor($G$) consisting of groupoids Tor($G|_U$) of all $G|_U$-torsors over $U \in \mathcal{C}$ is a canonical gerbe [7, 4]. If $\mathcal{C}$ has a terminal object $*$ and $\mathcal{G}$ is a $G$-gerbe such that $\mathcal{G}(*) \neq \emptyset$, then the $G$-torsors are bijective to the objects of the groupoid $\mathcal{G}(*)$ up to a unique isomorphism [4 Proposition 5.2.5], i.e., the isomorphism classes of $G$-torsors are bijective to the isomorphism classes of objects of groupoid $\mathcal{G}(*)$, so $[*, BG]$ is also in a natural bijection with the set of isomorphism classes of objects of $\mathcal{G}(*)$.

Remark 3.3. The central theorem of non-abelian cohomology asserts that the second cohomology group $H^2(\mathcal{C}, A)$ of the sheaf of abelian groups $A$ on the site $\mathcal{C}$ is isomorphic to the group of equivalence classes of $A$-gerbes over $\mathcal{C}$ [7, 4].

3.2. Torsors for sheaves of groupoids. Let $\mathcal{E}$ denote the topos $\text{Shv}(\mathcal{C})$ of sheaves on the site $\mathcal{C}$. We can see a sheaf of groupoids $G$ as a reflexive graph in $\mathcal{E}$ [10]:

$$\begin{array}{c}
G_0 \xrightarrow{s} G_1 \xrightarrow{t} G_0,
\end{array}$$

where $st = ts = id$, provided with an associative composition $c : G_1 \times_{G_0} G_1 \to G_1$, for which the elements of $G_0$ are units (via $u$), and each element of $G_1$ is invertible.

There are two kinds of definitions of $G$-torsor of sheaves of groupoids $G$. One is in the sense of [10], the other one is in the sense of [12]. We can show that they
are equivalent. The definitions of $G$-torsors in following sections are based on the definition in [12].

A (right) $G$-torsor in the sense of [16] is a non-empty sheaf $E$ over $G_0$ in $\mathcal{E}$ equipped with a free (right) action $a : E \times_{G_0} G_1 \to E$ which is transitive, where $E \times_{G_0} G_1$ is the pullback:

$$
\begin{array}{ccc}
E \times_{G_0} G_1 & \xrightarrow{\pi_1} & E \\
\downarrow{\pi_2} & & \downarrow{f} \\
G_1 & \xrightarrow{t} & G_0
\end{array}
$$

In other words, if we denote by $E$ the groupoid given by the top of the diagram (cf. [17]):

$$
\begin{array}{ccc}
E \times_{G_0} G_1 & \xrightarrow{a} & E \\
\downarrow{\pi_2} & & \downarrow{f} \\
G_1 & \xrightarrow{s} & G_0
\end{array}
$$

then the $G$-torsor $E$ is a sheaf such that the sheaf of groupoids $E$ is trivial and locally connected.

A $G$-torsor in the sense of [12] is a simplicial sheaf map $Y \to BG$ such that $Y$ is a homotopy colimit of some functor $X$ and the canonical map $Y \to *$ is a local weak equivalence.

We can construct a sequence of pullbacks

$$
\begin{array}{cccc}
\vdots & \xrightarrow{E \times_{G_0} G_1 \times_{G_0} G_1 \cdots \times_{G_0} G_1} & \cdots & E \times_{G_0} G_1 \times_{G_0} G_1 \\
\downarrow & & \downarrow \pi_2 & \downarrow{f} \\
\vdots & \xrightarrow{G_1 \times_{G_0} G_1 \cdots \times_{G_0} G_1} & \cdots & G_1 \times_{G_0} G_1
\end{array}
$$

Let $Y$ be the simplicial sheaf with sheaf $Y_n = E \times_{G_0} G_1 \times_{G_0} G_1 \cdots \times_{G_0} G_1$ (n factors of $G_1$, so $Y_0 = E$) and the natural structure maps. Thus the above diagram is a simplicial sheaf map $\pi : Y \to BG$ such that all diagrams

$$
\begin{array}{ccc}
Y_n & \xrightarrow{0^*} & Y_0 \\
\downarrow{\pi} & & \downarrow{f} \\
BG_n & \xrightarrow{0^*} & BG_0
\end{array}
$$

are pullbacks, where $0^*$ is the map corresponding to the ordinal number map $0 : 0 \to n$ which picks out the object 0.

So $Y$ has the homotopy colimit structure $Y \cong EX$ for some sheaf-valued functor $X : G \to \text{Shv}(C)$ defined on the small sheaf of groupoids $G$ over the site $C$ [12]. Since $Y$ is obtained from $BG$ by pullbacks, $Y$ is the nerve $EX$ of the sheaf of groupoids:

$$
\begin{array}{ccc}
E & \xrightarrow{u} & E \times_{G_0} G_1 \\
\downarrow{\pi_1} & & \downarrow{\pi_1} \\
\end{array}
$$

where $u : e \to (e, id_{f(e)})$. Since $G$ acts transitively on the sheaf $Y_0 = E$, $Y$ has only one component; $G$ acts freely on the sheaf $Y_0 = E$, so there is only an identity
map from every object in $Y$ to itself. So the canonical map $Y \to \ast$ is a local weak equivalence.

Every $G$-torsor $E$ in the sense of [16] uniquely determines a simplicial sheaf map $Y \to BG$ such that the map satisfies the above pullback condition and the canonical map $Y \to \ast$ is a locally weak equivalence. The simplicial sheaf map $Y \to BG$ is just a $G$-torsor in the sense of [12].

On the other hand, for each $G$-torsor $Y \to BG$ in the sense of [12] the simplicial sheaf $Y$ is formed by pullbacks

\[
\begin{array}{c}
Y_n \\
\downarrow \pi \\
BG_n
\end{array} \xrightarrow{0^*} \begin{array}{c}
Y_0 \\
f \\
BG_0
\end{array}
\]

such that the diagram

\[
\begin{array}{c}
Y_0 \times_{G_0} G_1 \\
\downarrow \pi_2 \\
G_1
\end{array} \xrightarrow{d_1 = s} \begin{array}{c}
Y_0 \\
f \\
G_0
\end{array}
\]

commutes, hence $d_1 : Y_0 \times_{G_0} G_1 \to Y_0$ is an action. $Y \to \ast$ is a local weak equivalence, so the action is free and transitive and $Y_0$ is a $G$-torsor in the sense of [16].

It’s obvious that there is a bijection between sets of the two kinds of $G$-torsors. So the definitions of $G$-torsor in the sense of [16] and [12] are equivalent.

Take the special case $G_0 = \ast$, then the sheaf of groups $G$ is a special sheaf of groupoids

\[
\begin{array}{c}
\ast \\
\xrightarrow{u} G \\
\xrightarrow{s} \ast
\end{array}
\]

A $G$-torsor is a $G$-space $Y_0$ such that

\[
Y \cong EG \times_G Y_0 \simeq \ast
\]

which is equivalent to the above definition.

The set of $G$-torsors and natural transformations between them forms a category which will be denoted by $\text{Tors}(\ast, G)$. This category is a groupoid because every morphism of $G$-torsors is an isomorphism [12]. Denote the path components of $\text{Tors}(\ast, G)$ by $\pi_0 \text{Tors}(\ast, G)$.

It is shown in [12]

**Theorem 3.4.** The function $\pi_0 \text{Tors}(\ast, G) \to [\ast, BG]$ is a natural bijection for any sheaf of groupoids $G$ on any Grothendieck site $\mathcal{C}$.

**Remark 3.5.** As pointed out by Jardine [13], there is a mistake in the proof of [12]. Jardine corrected it in [13]. The correction will also appear in the proof of Theorem 3.23.

**Remark 3.6.** Let $H(\mathcal{C}, G)$ denote the category of $G$-torsors and $H^1(\mathcal{C}, G)$ the set of connected components of $H(\mathcal{C}, G)$ [16]. Thus Theorem 3.4 has same form as Theorem 3.1.
3.3. Torsors for presheaves of 2-groupoids. Let $\mathbf{2Sets}$ denote the 2-category in which the objects are the sets, the morphisms are the maps between sets, the 2-arrows are the maps $X \times I \to Y$ where $X$ and $Y$ are two sets and $I$ is the 2-point set $\{0, 1\}$. So there is a unique 2-arrow between each pair of parallel 1-arrows. In the language of internal categories, $\mathbf{2Sets}$ is a category enriched in categories. Let $\mathbf{2Sets}_0$, $\mathbf{2Sets}_1$ and $\mathbf{2Sets}_2$ denote the classes of objects, 1-arrows and 2-arrows in $\mathbf{2Sets}$, respectively. The underlying category $\mathbf{2Sets}_1 \rightarrow \mathbf{2Sets}_0$ of $\mathbf{2Sets}$ is just the ordinary category $\mathbf{Sets}$ of sets, $\mathbf{2Sets}_2 \rightarrow \mathbf{2Sets}_0$ is the category in which the objects are the sets, the morphisms between objects $X$ and $Y$ are the homotopy maps $X \times I \to Y$.

Similarly, let $\mathbf{2Pre}(\mathcal{C})$ denote the 2-category in which the objects are the presheaves, the morphisms are the maps between presheaves, the 2-arrows are the maps $X \times I \to Y$ where $X$ and $Y$ are two presheaves and $I$ is the constant presheaf of 2-point set $\{0, 1\}$.

Suppose that $G$ is a 2-groupoid. For any set-valued 2-functor $X : G \to \mathbf{2Sets}$, we define its homotopy colimit $\operatorname{holim}_G X$ as the simplicial set with $n$-simplices:

$$\bigsqcup_{(a_0, a_1, \ldots, a_n)} X(a_0) \times BG(a_0, a_1)_0 \times BG(a_1, a_2)_1 \times \cdots \times BG(a_{n-1}, a_n)_n$$

where $(a_0, \ldots, a_n) \in \operatorname{Ob}(G)^{n+1}$. $G(a_i, a_{i+1})$, $i = 0, 1, \ldots, n-1$ are the groupoids of morphisms from $a_i$ to $a_{i+1}$, $BG(a_i, a_{i+1})$, $i = 0, 1, \ldots, n-1$ is the set of all $i$-simplices in the classifying space $BG(a_i, a_{i+1})$.

It follows that the category of 2-functor $X : G \to \mathbf{2Sets}$ and 2-natural transformations is equivalent to the category of simplicial set maps $\pi : Y \to \mathcal{W}G$ such that $Y$ is the homotopy colimit of $X$, with fibrewise maps over $\mathcal{W}G$ as morphisms.

This equivalence is natural, and therefore gives an internal description in category of presheaves of presheaf-valued 2-functor $X : G \to \mathbf{2Pre}(\mathcal{C})$ defined on a small presheaf of 2-groupoids $G$ over a site $\mathcal{C}$.

Suppose that $G$ is a presheaf of 2-groupoids on a small site $\mathcal{C}$. Analogous to the definition in the case of sheaf of groupoids, when $G$ is a presheaf of 2-groupoids, we define a $G$-torsor to be a simplicial presheaf map $Y \to \mathcal{W}G$ such that $Y$ is a homotopy colimit $\operatorname{holim}_G X$ and the canonical map $Y \to *$ is a local weak equivalence.

When we explicitly express the homotopy colimit $\operatorname{holim}_G X$ by Moore complex, it’s obvious that it satisfies a pullback condition

$$\cdots \longrightarrow E \times_{G_0} G_1 \times_{G_0} G_2 \longrightarrow E \times_{G_0} G_1 \overset{\pi_2}{\longrightarrow} E \quad \overset{\pi_1}{\longrightarrow} \quad E$$

$$\cdots \longrightarrow G_1 \times_{G_0} G_2 \longrightarrow G_1 \quad \overset{f}{\longrightarrow} \quad G_0$$

In fact, this diagram is a simplicial presheaf map $Y \to \mathcal{W}G$ which satisfies the pullback condition. The map $Y \to *$ is a local weak equivalence, so the presheaf $Y_0 = E$ is non-empty over $G_0$ and the action $a : E \times_{G_0} G_1 \to E$ is transitive, $G_2$
also acts freely and transitively on $E \times_{G_0} G_1$. Denote by $E$ the 2-groupoid:

$$
\begin{array}{c}
E \times_{G_0} G_2 \\
\downarrow t \\
E \times_{G_0} G_1 \\
\downarrow t \\
E
\end{array}
$$

Then $Y = \overline{W}E$.

The set of $G$-torsors and natural transformations between them forms a category. We denote it by $\text{Tors}(\ast, G)$. An argument similar to that in [12] shows that the morphisms between torsors are isomorphisms, so the category $\text{Tors}(\ast, G)$ is a groupoid. Denote the path components of $\text{Tors}(\ast, G)$ by $\pi_0 \text{Tors}(\ast, G)$.

The set $[\ast, \overline{W}G]$ of maps from $\ast$ to $\overline{W}G$ in $\text{Ho}(\text{SPre}(\mathcal{C}))$ may be described as a filtered colimit by Verdier hypercovering characterization

$$
[\ast, \overline{W}G] \cong \lim_{\to V} \pi(V, \overline{W}G)
$$

indexed over simplicial homotopy classes represented by locally trivial fibrations (hypercovers) $V \to \ast$, where $\pi(\ast )$ indicates simplicial homotopy classes of maps.

When $G$ is a 2-groupoid, $BG$ is a simplicial groupoid and a fibrant object in the category $s\text{Gd}$ since $BG_0$ is just the forgetful groupoid $G_0$ of the 2-groupoid $G$, the map $BG \to \ast$ has the path lifting property; and the map $BG(x, x) \to \ast$ is a fibration of simplicial sets since $BG(x, x) = B(G(x, x))$ is the classifying space of the groupoid $G(x, x)$. The functor $\overline{W} : s\text{Gd} \to \text{S}$ preserves fibrations and weak equivalences (Theorem V.7.8.(2), [8]), So $\overline{W}G$ is a fibrant object in the category $\text{S}$.

When $G$ is a presheaf of 2-groupoids, $\overline{W}G$ is a locally fibrant object in the category $\text{SPre}(\mathcal{C})$.

All $G$-torsors $Y \to \overline{W}G$ are of the form $\overline{W}f : \overline{W}I \to \overline{W}G$ where $f : I \to G$ is a 2-functor defined on a 2-groupoid $I$ which is trivial in the sense that $I(x, x) \simeq \ast$ the terminal object of the category of presheaves of groupoids $\text{GpdPre}(\mathcal{C})$ for all local choices of objects $x$ and $I$ is locally connected. Then $Y = \overline{W}I$ is locally fibrant, and $Y \to \ast$ is a (local) weak equivalence. So $Y$ is a locally trivial fibrant object, i.e., $Y$ is a hypercover of $\ast$.

Every $G$-toral $Y \to \overline{W}G$ has within it a hypercover $Y \to \ast$, and every morphism $Y \to Y'$ of torsors is a morphism of hypercovers.

Send the $G$-toral $f : Y \to \overline{W}G$ to its homotopy class $[f] : Y \to \overline{W}G$. $\ast$ is the colimit of all its hypercovers, so $[f]$ has a unique factorization

$$
\begin{array}{c}
Y \\
\downarrow {\ [f]} \\
\downarrow {\ [f']} \\
\ast
\end{array}
$$

where $[f']$ is the homotopy class of maps $\ast \to \overline{W}G$. When we send $f$ to $[f']$ we have a map

$$\text{Ob Tors}(\ast, G) \to [\ast, \overline{W}G]$$
For any two $G$-torsors $Y \to \mathbb{W}G$ and $Y' \to \mathbb{W}G$ in same component of category $\text{Tors}(\ast, G)$, there is an isomorphism $i : Y \to Y'$ such that the diagrams commute. The two torsors $f_1 : Y \to \mathbb{W}G$ and $f_2 : Y' \to \mathbb{W}G$ map into the same morphism $[f'] : \ast \to \mathbb{W}G$, so the above map can factor through a function

$$
\pi_0 \text{Tors}(\ast, G) \to [\ast, \mathbb{W}G]
$$

Following Theorem 14 in [12], we have

**Theorem 3.7.** The function $\pi_0 \text{Tors}(\ast, G) \to [\ast, \mathbb{W}G]$ is a natural bijection for any presheaf of 2-groupoids $G$ on any Grothendieck site $\mathcal{C}$.

**Proof.** We have constructed the map $\varphi : \pi_0 \text{Tors}(\ast, G) \to [\ast, \mathbb{W}G]$. Now we need find its inverse function.

For a 2-groupoid $G$ and any object $x_0 \in G$, we define the comma 2-category $G \downarrow x_0$ to be the 2-category in which the objects are the morphisms $x \to x_0$, the morphisms (1-arrows) and the 2-arrows are the commutative diagrams

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow \quad & & \downarrow \\
  x_0 & & y
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad
\begin{array}{ccc}
  x & \overset{id}{\xrightarrow{\quad}} & y \\
  \downarrow \quad & & \downarrow \\
  x_0 & & y
\end{array}
\]

respectively, where $\xrightarrow{id} y$ is the identity 2-arrow. In fact, $G \downarrow x_0$ is a 2-groupoid. Since there is only one morphism between any two objects in $G \downarrow x_0$ and all 2-arrows are identities, the 2-groupoid $G \downarrow x_0$ is a trivial 2-groupoid, so $\mathbb{W}(G \downarrow x_0)$ is contractible.

Suppose given a 2-functor $f : I \to G$, where $I$ is a trivial 2-groupoid (i.e., $I$ has only one 1-cell between any two objects and only identity 2-cell from a 1-cell to itself). We define the comma 2-groupoid $f \downarrow x_0$ to be the 2-groupoid in which the objects are the pairs $(x, f(x) \to x_0), x \in I$, the morphisms (1-arrows) and the 2-arrows are the commutative diagrams

\[
\begin{array}{ccc}
  x & \xrightarrow{f(x)} & f(y) \\
  \downarrow \quad & & \downarrow \\
  x_0 & & y
\end{array}
\quad \quad \quad \quad \quad \quad \quad \quad
\begin{array}{ccc}
  x & \overset{id}{\xrightarrow{\quad}} & f(y) \\
  \downarrow \quad & & \downarrow \\
  x_0 & & y
\end{array}
\]

respectively. $I$ is trivial, so is each component of $f \downarrow x_0$. Thus $\mathbb{W}(f \downarrow x_0)$ is weakly equivalent to the constant simplicial set $\pi_0 \mathbb{W}(f \downarrow x_0)$.

Given a hypercover $V \to \ast$, the simplicial presheaf map $V \to \mathbb{W}G$ is of the form $\mathbb{W}f : \mathbb{W}I \to \mathbb{W}G$ where $f : I \to G$ is a presheaf-valued 2-functor defined on the trivial presheaf of 2-groupoids $I$. 
Make the homotopy colimit construction

\[
\begin{array}{c}
\mathbb{W}I(U) \xrightarrow{\alpha} d(\bigsqcup_{x_0, \ldots, x_n} \mathbb{W}(f \downarrow x_0)) \xrightarrow{\beta} d(\bigsqcup_{x_0, \ldots, x_n} \pi_0 \mathbb{W}(f \downarrow x_0)) \\
\mathbb{W}G(U) \xrightarrow{\alpha} d(\bigsqcup_{x_0, \ldots, x_n} \mathbb{W}(G(U) \downarrow x_0)) \xrightarrow{\beta} d(\bigsqcup_{x_0, \ldots, x_n} *)
\end{array}
\]

on the presheaf level in each section, where \(x_0 \to \cdots \to x_n\) is an \(n\)-simplex of simplicial set \(\mathbb{W}(G(U))\).

In the simplicial set \(d(\bigsqcup_{x_0, \ldots, x_n} \mathbb{W}(f \downarrow x_0))\), there is only one 1-simplex between any two objects \((y, f(y) \to x)\) and \((y', f(y') \to x')\) where \(y, y' \in I, x, x' \in G\)

\[
\begin{array}{c}
y \\
\downarrow i; \\
y'
\end{array}
\quad
\begin{array}{c}
f(y) \quad \longrightarrow \quad x \\
\downarrow f(i) \\
f(y') \quad \longrightarrow \quad x'
\end{array}
\]

All 2-simplices are the identities 2-arrows in \(I\), and so on. So \(d(\bigsqcup_{x_0, \ldots, x_n} \mathbb{W}(f \downarrow x_0))\) is equivalent to \(*\). The top map \(\alpha\) is the standard weak equivalence associated to the simplicial map \(f : \mathbb{W}I(U) \to \mathbb{W}G(U)\), and the top map \(\beta\) is also a weak equivalence since \(\mathbb{W}(f \downarrow x_0)\) is weak equivalent to constant simplicial set \(\pi_0 \mathbb{W}(f \downarrow x_0)\). So \(d(\bigsqcup_{x_0, \ldots, x_n} \pi_0 \mathbb{W}(f \downarrow x_0))\) is equivalent to \(*\). \(d(\bigsqcup_{x_0, \ldots, x_n} *)\) just is the simplicial set \(\mathbb{W}(G(U))\), \(d(\bigsqcup_{x_0, \ldots, x_n} \pi_0 \mathbb{W}(f \downarrow x_0))\) is the homotopy colimit over \(G\), so the presheaf map \(h(f)\) is a \(G\)-torsor.

When \(f : \mathbb{W}I \to \mathbb{W}G\) is a \(G\)-torsor, \(\mathbb{W}I\) is the homotopy colimit of presheaves over \(G\) for some 2-functor \(X\) such that \(\mathbb{W}I = \text{holim}_G X\) and there is a natural isomorphism \(\pi_0 \mathbb{W}(f \downarrow x_0) \cong X(x_0)\), so the map \(h(f)\) is canonically isomorphic to \(f\).

Any homotopy \(\mathbb{W}I \times \Delta^1 \to \mathbb{W}G\) can extend to a map \(\mathbb{W}I \times \mathbb{W}G(1) \to \mathbb{W}G\) where \(G(1)\) is the trivial 2-groupoid on two objects. If \(I\) is locally connected and trivial, then so is \(I \times G(1)\). Corresponding to the diagram

\[
\begin{array}{ccc}
\mathbb{W}I & \xrightarrow{f} & \mathbb{W}G \\
\downarrow d' & & \downarrow g \\
\mathbb{W}(I \times G(1)) & \xrightarrow{g} & \mathbb{W}G
\end{array}
\]
there is a diagram
\[ d(\bigsqcup_{x_0 \to \cdots \to x_n} \pi_0 W(f \downarrow x_0)) \]
\[ \begin{array}{ccc}
    & & h(f) \\
    d'' & \downarrow & \downarrow h(g) \\
    d(\bigsqcup_{x_0 \to \cdots \to x_n} \pi_0 W(g \downarrow x_0)) & \rightarrow & d(\bigsqcup_{x_0 \to \cdots \to x_n} \pi_0 W(f' \downarrow x_0)) \\
    d' & \downarrow h(f') & \\
    d(\bigsqcup_{x_0 \to \cdots \to x_n} \pi_0 W(f' \downarrow x_0)) & \rightarrow & d(\bigsqcup_{x_0 \to \cdots \to x_n} \pi_0 W(f \downarrow x_0)) \\
\end{array} \]

So any homotopy \( f \simeq f' \) determines torsors \( h(f) \) and \( h(f') \) which are in the same component of \( \text{Tors}(\ast, G) \).

For any homotopy map class in the set \([\ast, \overline{WG}]\), it has a representative \([f] : V \to \overline{WG}\). Suppose that it has other representative \([f'] : V' \to \overline{WG}\). All hypercovers over \( \ast \) form a filtered category. So there exists hypercover \( V'' \) and arrows \( V \to V'' \) and \( V' \to V'' \) [21] such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{f} & \overline{WG} \\
\downarrow & & \downarrow \\
V'' & \xrightarrow{f'} & V' \\
\end{array}
\]

commutes. By same argument as above, the torsors \( h(f) \) and \( h(f') \) are in the same component of \( \text{Tors}(\ast, G) \).

There is a well-defined function
\[ \psi : [\ast, \overline{WG}] \to \pi_0 \text{Tors}(\ast, G) \]
sending the homotopy map class \([f]\) to the \( G \)-torsor \( h(f) \).

For any element in \( \pi_0 \text{Tors}(\ast, G) \), take a representative \( f : Y \to \overline{WG} \), \([f]\) is the representative of its image in \([\ast, \overline{WG}]\), \( h(f) \) is isomorphic to \( f \), so the composite function \( \psi \circ \pi_0 \text{Tors}(\ast, G) \to [\ast, \overline{WG}] \to \pi_0 \text{Tors}(\ast, G) \) is identity.

In the above homotopy colimit diagram, the bottom \( \beta \) is also a weak equivalence because each summand \( \overline{W}(G(U) \downarrow x_0) \) is contractible. The simplicial set
\[ X(U) = d(\bigsqcup_{x_0 \to \cdots \to x_n} \overline{W}(G(U) \downarrow x_0)) \]
consists of strings \((y, x)\) of \((n\text{-})\)arrows
\[ y_0 \to \cdots \to y_n \to x_0 \to \cdots \to x_n \]
of length \(2n + 1\). In fact, \( y_0 \to \cdots \to y_n \) and \( x_0 \to \cdots \to x_n \) are two \( n \)-simplices in \( \overline{W}(G(U)) \). The map \( \alpha \) sends this simplex to the string \( y_0 \to \cdots \to y_n \), while \( \beta \) maps it to the string \( x_0 \to \cdots \to x_n \). The \( n \)-simplices of the simplicial set \( X(U) \) can be identified with the simplicial maps \( \triangle^n \ast \triangle^n \to \overline{W}(G(U)) \) defined on the join \( \triangle^n \ast \triangle^n \), and with simplicial structure maps induced by precomposition with maps \( \theta \ast \theta : \triangle^m \ast \triangle^m \to \triangle^n \ast \triangle^n \). The maps \( \alpha \) and \( \beta \) are induced by the inclusions \( \triangle^n \to \triangle^n \ast \triangle^n \) of the left and right standard \( n \)-simplex respectively.
The \( n \)-simplex in \( X(U) \) can be expressed by diagram
\[
\begin{array}{c}
\downarrow \\
\end{array}
\]
Then maps \( \alpha \) and \( \beta \) are homotopic maps. This construction is natural, hence the two maps represent the same morphism in the homotopy category of simplicial presheaves.

So the two composition maps
\[
d(\bigsqcup_{x_0 \to \cdots \to x_n} \mathcal{W}(f \downarrow x_0)) \to d(\bigsqcup_{x_0 \to \cdots \to x_n} \mathcal{W}(G(U) \downarrow x_0)) \to d(\bigsqcup_{x_0 \to \cdots \to x_n} *)
\]
and
\[
d(\bigsqcup_{x_0 \to \cdots \to x_n} \mathcal{W}(f \downarrow x_0)) \to d(\bigsqcup_{x_0 \to \cdots \to x_n} \mathcal{W}(G(U) \downarrow x_0)) \to d(\bigsqcup_{x_0 \to \cdots \to x_n} *)
\]
are homotopic. Denote this homotopy class as \([g]\).

\[
d(\bigsqcup_{x_0 \to \cdots \to x_n} \mathcal{W}(f \downarrow x_0)) \text{ can be written as } \mathcal{W}H \text{ for some 2-groupoid } H, \text{ hence it’s fibrant, and it’s weak equivalent to } *, \text{ then the simplicial presheaf } d(\bigsqcup_{x_0 \to \cdots \to x_n} \mathcal{W}(f \downarrow x_0)) \text{ is a hypercover over } * \text{. By the above homotopy colimit diagram, } [f], [g] \text{ and } [h(f)] \text{ represent the same element in } [*, \mathcal{W}G].
\]
The composition function \( \varphi \psi : [*, \mathcal{W}G] \to \pi_0 \text{Tors}(*, G) \to [*, \mathcal{W}G] \) sends \([f]\) to \([h(f)]\), so it’s identity. \( \Box \)

3.4. **Torsors for presheaves of simplicial groups.** Suppose that \( G \) is a presheaf of simplicial groups on a small site \( \mathcal{C} \).

Let \( X \) be a simplicial presheaf. \( G \) acts on \( X \) if there is a morphism of simplicial presheaves
\[
\mu : G \times X \to X
\]
so that the following diagrams commute:
\[
\begin{array}{c}
m \times 1 \\
\end{array}
\]
and
\[
\begin{array}{c}
i \\
\end{array}
\]
where \( m \) is the multiplication in \( G \) and \( i(X) = (e, X) \) on each level of each section.

Such a simplicial presheaf \( X \) is called a *simplicial \( G \)-presheaf*. Let \( \text{SPre}(\mathcal{C})_G \) be the category of simplicial \( G \)-presheaves.

The forgetful functor \( \text{SPre}(\mathcal{C})_G \to \text{SPre}(\mathcal{C}) \) has a left adjoint given by
\[
X \mapsto G \times X
\]
It’s easy to prove that there is a closed model structure on the category \( \text{SPre}(\mathcal{C})_G \), where a map \( f : X \to Y \) of simplicial \( G \)-presheaves is a fibration (respectively weak equivalence) if the underlying map of simplicial presheaves is a global fibration (respectively local weak equivalence) and a cofibration is a map which has the left
lifting property with respect to all both fibrations and weak equivalences (cf. Theorem V.2.3).

A principal $G$-bundle (or principal $G$-fibration) $f : E \to B$ is a local fibration in $\mathbf{SPre}(\mathcal{C})_G$ so that

1. $B$ has trivial $G$-action;
2. $E$ is a cofibrant simplicial $G$-presheaf, and
3. the induced map $E/G \to B$ is an isomorphism.

**Lemma 3.8.** Suppose that $X$ is a cofibrant simplicial $G$-presheaf. Then $G$ acts freely on $X$ in all sections.

**Proof.** Suppose that the functor $L_U : \mathbf{S} \to \mathbf{SPre}(\mathcal{C})$ is the functor $?_U$ in [9], the left adjoint functor of the $U$-section functor $X \to X(U)$:

$$L_U(Y)(V) = \bigsqcup_{\varphi : V \to U} Y.$$

The maps $G \times L_U \partial \Delta^n \to G \times L_U \Delta^n$ generate the cofibrations of the category of simplicial $G$-presheaves, and any pushout

$$
\begin{array}{c}
G \times L_U \partial \Delta^n \\
\downarrow \downarrow \\
G \times L_U \Delta^n \\
\downarrow \\
Z \\
\downarrow \\
W
\end{array}
$$

has the effect of adding some freely generated $G(U)$-space to $Z(U)$ for each $U \in \mathcal{C}$. The cofibration $\emptyset \to X$ has a factorization

$$
\begin{array}{c}
\emptyset \\
\downarrow \\
V \\
\downarrow \pi \\
X
\end{array}
$$

where $\pi$ is a trivial fibration and the map $\emptyset \to V$ is a transfinite colimit of pushouts of the above form. It follows that $G$ acts freely on $V$ in all sections. But then, by a standard argument $X$ is a retract of $V$ (since $\pi$ is a trivial fibration), so that $G$ acts freely on $X$ in all sections. \hfill \Box

Now for the converse:

**Lemma 3.9.** Suppose that $G$ acts freely on the simplicial $G$-presheaf $Y$ in all sections. Then $Y$ is cofibrant.

**Proof.** Consider the partially ordered set of all cofibrant subobjects $K \subset Y$, where $K \leq L$ if there is a $G$-cofibration $K \subset L$ which respects the inclusions into $Y$. This poset is non-empty since $G(x) \subset Y$ is a cofibrant subobject of $Y$ by the condition that $G$ acts freely on $Y$ in all sections for any simplex $x \in Y(U)$. If

$$K_1 \leq K_2 \leq \cdots$$

is a totally ordered collection of cofibrant subobjects, the $K_\infty = \cup K_i$ is cofibrant and all inclusions $K_i \subset K_\infty$ are cofibrations. It follows that the poset of cofibrant subobjects of $Y$ is inductively ordered.
Zorn’s lemma therefore asserts that the poset of cofibrant subobjects of $Y$ has maximal elements. Pick such a maximal subobject $M$ and assume that $M \neq Y$. Then there is a simplex $x \in Y(U) \smallsetminus M(U)$ of minimal dimension, and the diagram

\[
\begin{array}{ccc}
G(\langle x \rangle) \cap M & \to & M \\
\downarrow & & \downarrow \\
G(\langle x \rangle) & \to & G(\langle x \rangle) \cup M
\end{array}
\]

is a pushout of $G$-subcomplexes of $Y$. But

\[
G(\langle \partial x \rangle) = G(\langle x \rangle) \cap M
\]

where $\langle \partial x \rangle$ is the subcomplex generated by the faces of $x$. It follows that the inclusion map $M \to G(\langle x \rangle) \cup M$ is a cofibration of simplicial $G$-presheaves, which contradicts the maximality of $M$ if $M \neq Y$. Thus, $Y$ is cofibrant. \hfill $\Box$

Hence a simplicial $G$-presheaf $X$ is cofibrant if and only if $G$ acts freely on it in all sections.

For every cofibrant simplicial $G$-presheaf $X$ the canonical map $X(U) \to X(U)/G(U)$ is a fibration of simplicial sets \cite[Corollary V.2.7]{[8]} for any $U \in \mathcal{C}$. Then $X \to X/G$ is a local fibration \cite[Corollary 1.8]{[9]} where $X/G$ is the simplicial presheaf $U \mapsto X(U)/G(U)$. So the canonical map $X \to X/G$ is a principal $G$-bundle. Any $G$-bundle $f : E \to B$ is isomorphic to a quotient map

\[
q : X \to X/G
\]

where $X \in \text{SPre}(\mathcal{C})_G$ is cofibrant. That means that cofibrant simplicial $G$-presheaves $X$ correspond to principal $G$-bundles $X \to X/G$.

A $G$-torsor on $\mathcal{C}$ is a cofibrant simplicial $G$-presheaf $X$ such that the canonical map $X/G \to \ast$ is a hypercover. A map $f : X \to Y$ of $G$-torsors is just a $G$-equivariant map of simplicial presheaves. Write $\textbf{G-tors}$ for the corresponding category and $\pi_0(\textbf{G-tors})$ for the corresponding path components of $\textbf{G-tors}$.

Choose a factorization

\[
\emptyset \xrightarrow{i} E \to BG
\]

where $i$ is a cofibration and $\pi$ is a trivial fibration in the category of simplicial $G$-presheaves $\text{SPre}(\mathcal{C})_G$. Write $BG = EG/G$. Observe that $BG$ is locally fibrant, since it is the presheaf of Kan complexes (cf. Lemma V.3.7. \cite{[8]}).

Note that $EG$ is unique up to equivariant homotopy equivalence, so $q : EG \to BG$ is unique up to homotopy equivalence.

Suppose that $U \to BG$ is a map of simplicial presheaves, where $U \to \ast$ is a hypercover. Then the pullback $U \times_{BG} EG$ has a free $G$-action and is therefore a cofibrant simplicial $G$-presheaf. The projection map $U \times_{BG} EG \to U$ is the quotient by the $G$-action and $U \to \ast$ is a hypercover, so that $U \times_{BG} EG$ is a $G$-torsor.

Further, any string of morphisms

\[
U_1 \to U_2 \to BG
\]

where $U_1 \to U_2$ is a map of hypercovers induces a morphism

\[
U_1 \times_{BG} EG \to U_2 \times_{BG} EG
\]
of $G$-torsors in the obvious way.

**Remark 3.10.** In the standard theory $WG$ is a cofibrant simplicial $G$-presheaf, but the map $WG \to \ast$ is only a (trivial) local fibration. Find a factorization

$$
\begin{array}{ccc}
WG & \xrightarrow{j} & EG \\
\downarrow & & \downarrow \pi \\
\ast & & \ast
\end{array}
$$

such that $\pi$ is a trivial fibration and $j$ is a trivial cofibration in the category of simplicial $G$-presheaves $\text{SPre}(\mathcal{C})_G$. Then the induced comparison of principal $G$-bundles

$$
\begin{array}{ccc}
WG & \xrightarrow{j} & EG \\
\downarrow & & \downarrow \pi \\
WG & \xrightarrow{j_*} & BG
\end{array}
$$

induces a local weak equivalence $j_* : \overline{WG} \to BG$ such that $j_*$ has a homotopy inverse $j^* : BG \to \overline{WG}$ where $j^*$ is a local weak equivalence as well. Similarly, if $U \to \overline{WG}$ is a map of simplicial presheaves, where $U \to \ast$ is a hypercover, then $U \times_{\overline{WG}} WG$ is also a $G$-torsor.

A similar argument works for the diagonal map $d(EG) \to d(BG)$ induced by the standard bisimplicial presheaf map $EG \to BG$, because the diagonal map is a principal $G$-fibration and the object $d(EG)$ is weak equivalence to a point. It follows that $d(BG) \simeq BG$ for the two different senses of $BG$.

A simplicial presheaf $X$ is called *projective fibrant* if the map $X \to \ast$ has the right lifting property with respect to all maps $L_U \wedge_k \Delta^n \to L_U \Delta^n$, $U \in \mathcal{C}$. It’s obvious that the simplicial presheaf $BG$ is projective fibrant. In effect, the map $EG \to BG$ is a principal $G$-bundle and hence a surjective Kan fibration in each section, and $EG$ is globally fibrant, and hence a Kan complex in each section, so that $BG$ is a Kan complex in each section. The simplicial presheaf $\overline{WG}$ is projective fibrant as well.

Say that a presheaf of simplicial groupoids $H$ is projective fibrant if the object $\overline{WH}$ is projective fibrant.

More generally, one says that the map $p : G \to H$ is a projective fibration of presheaves of simplicial groupoids if the induced map $\overline{WG} \to \overline{WH}$ is a projective fibration. This is equivalent to the assertion that the map $p$ has the right lifting property with respect to all maps $G(L_U \wedge_k \Delta^n) \to G(L_U \Delta^n)$, $U \in \mathcal{C}$.

Every presheaf of simplicial groupoids $H$ has a *projective fibrant model* $i : H \to H_f$, in the sense that the map $i$ is a sectionwise weak equivalence which has the left lifting property with respect to all projective fibrations, and $H_f$ is projective fibrant. This is a consequence of the obvious small object argument.

Suppose once again that $G$ is a presheaf of simplicial groups. Then $G$ is projective fibrant by a standard argument. Suppose that given a homotopy $h : U \times \Delta^1 \to \overline{WG}$ where $U$ is a hypercover. Then the induced map $h_* : G(U \times \Delta^1) \to G$ has a
factorization

\[
\begin{array}{ccc}
G(U \times \Delta^1) & \xrightarrow{h_*} & G \\
\downarrow_{j} & & \downarrow_{\pi} \\
H & & \\
\end{array}
\]

where the map \( j : G(U \times \Delta^1) \to H \) is a projective fibrant model for \( G(U \times \Delta^1) \) in the category of presheaves of simplicial groupoids. \( j \) has the left lifting property with respect to all projective fibrations, \( G \) is projective fibrant, so the lifting \( \pi \) exists. It follows that \( \mathbb{W}H \to \ast \) is a hypercover since it’s sectionwise weak equivalence and fibration, hence it’s a locally trivial fibration, and there is a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{hd^0} & \mathbb{W}G \\
\downarrow_{hd^1} & & \downarrow_{\pi} \\
\mathbb{W}H & \xrightarrow{j_*} & \mathbb{W}G \\
\end{array}
\]

In particular, the principal \( G \)-fibrations over \( U \) which are induced by the maps \( hd^1 \) and \( hd^0 \) are in the same component of the \( G \)-torsor category.

For the composite map \( U \xrightarrow{f} BG \xrightarrow{j_*} \mathbb{W}G \), the principal \( G \)-bundles \( U \times_{BG} EG \) and \( U \times_{\mathbb{W}G} WG \), which are induced by the maps \( f \) and \( j^*f \) respectively, are in the same component of the \( G \)-torsor category. If \( f, g : U \to BG \) are homotopic, so are \( j_*f, j^*g : U \to \mathbb{W}G \). Hence the \( G \)-torsors which are induced by \( f \) and \( g \) are in same component of \( G \)-tors.

It follows that there is a well defined function

\[
\psi_G : [\ast, BG] = \lim_{\overset{\to}{U}} \pi(U, BG) \to \pi_0(G - \text{tors})
\]

which is given by sending the naive homotopy class of a map \( U \to BG \) defined on a hypercover \( U \to \ast \) to the path component of the object \( U \times_{BG} EG \).

Suppose that \( X \) is a \( G \)-torsor. Then \( X \) is a cofibrant simplicial \( G \)-presheaf so that the lifting \( \phi \) exists in the diagram

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{\phi} & EG \\
\downarrow_{\pi} & & \downarrow_{\pi} \\
X & \xrightarrow{\ast} & \\
\end{array}
\]

since \( \pi \) is a trivial fibration. Moreover, any two such liftings are homotopic. Make a fixed choice of lifting \( \phi_X \) for all \( G \)-torsors \( X \), and write \( \phi_X : X/G \to BG \) for the corresponding induced map.

Suppose that \( f : X \to Y \) is a morphism of \( G \)-torsors. Then \( \phi_Y f \) is a second possible choice for \( \phi_X \) and so \( \phi_Y f \) and \( \phi_X \) are (naively) equivariantly homotopic. It follows that the maps \( \phi_{X_*} \) and \( \phi_{Y_*} f_* \) are naively homotopic. The assignment

\[
X \mapsto [\phi_{X_*}] \in \pi(X/G, BG)
\]
therefore determines a well-defined function
\[ \varphi_G : \pi_0(\mathbf{G} - \text{tors}) \to \lim_{\mathbf{U} \to \ast} \pi(\mathbf{U}, BG). \]

It’s obvious that both maps \( \varphi_G \cdot \psi_G \) and \( \psi_G \cdot \varphi_G \) are identity maps.

We have therefore shown

**Theorem 3.11.** The function \( \psi_G : \ast, BG \to \pi_0(\mathbf{G} - \text{tors}) \) is a bijection for all presheaves of simplicial groups \( G \) on a Grothendieck site \( \mathcal{C} \).

Note that the theorem is independent of the choice of the cofibrant model \( EG \) for the point.

**Corollary 3.12.** Suppose that \( G \) is a presheaf of simplicial groups on a Grothendieck site \( \mathcal{C} \). Then there is a bijection
\[ \ast, W\mathbf{G} \cong \pi_0(\mathbf{G} - \text{tors}) \]

**Remark 3.13.** Every trivial cofibration \( i : A \to B \) of simplicial \( G \)-presheaves induces a trivial cofibration \( i_* : A/G \to B/G \). In effect, \( i \) has the left lifting property with respect to all global fibration \( p : X \to Y \) of simplicial presheaves with trivial \( G \)-action.

Suppose that \( X \) is a cofibrant simplicial \( G \)-presheaf such that the induced map \( X/G \to \ast \) is a weak equivalence. Find a trivial cofibration \( j : X \to \tilde{X} \) in the category of simplicial \( G \)-presheaves such that \( \tilde{X} \) is fibrant. Then the induced map \( j_* : X/G \to X/G \) is a trivial cofibration of simplicial \( G \)-presheaves, and \( X/G \) is locally fibrant so the map \( X/G \to \ast \) is a hypercover. Write \( G - \text{tors}_0 \) for the category of cofibrant simplicial \( G \)-presheaves \( X \) such that \( X/G \to \ast \) is a local weak equivalence. Then the inclusion
\[ G - \text{tors} \subset G - \text{tors}_0 \]
induces an isomorphism
\[ \pi_0(G - \text{tors}) \cong \pi_0(G - \text{tors}_0). \]

**Remark 3.14.** Write \( G - \text{tors}_1 \) for the category of simplicial \( G \)-presheaves \( Y \) such that the canonical map \( d(EG \times_G Y) \to \ast \) is a local weak equivalence, where \( d(X) \) denotes the diagonal of a bisimplicial object \( X \). Then there is an inclusion
\[ G - \text{tors}_0 \subset G - \text{tors}_1 \]
since the canonical map \( d(EG \times_G Z) \to Z/G \) is a weak equivalence if \( Z \) is cofibrant.

On the other hand, if \( X \) is a simplicial \( G \)-presheaf such that \( d(EG \times_G X) \to \ast \) is a weak equivalence, there is a trivial fibration \( Z \to X \) of simplicial \( G \)-presheaves such that \( Z \) is cofibrant. The induced map \( d(EG \times_G Z) \to d(EG \times_G X) \) is a local weak equivalence, so that \( Z \) is an object of \( G - \text{tors}_0 \). It follows that there is an isomorphism
\[ \pi_0(G - \text{tors}_0) \cong \pi_0(G - \text{tors}_1) \]

If one defines \( G \)-torsors on \( \mathcal{C} \) as simplicial presheaves \( X \) with principal \( G \)-action such that the canonical map \( d(EG \times_G X) \simeq X/G \to \ast \) is a weak equivalence, the similar bijection of Theorem 3.11 exists.
3.5. **Torsors for presheaves of simplicial groupoids.** First of all, we extend the results in Section 3.4.

Write $\text{Triv}/X$ for the category whose objects are all simplicial presheaf morphisms $W \to X$ such that the map $W \to *$ is a local weak equivalence. Observe that there is a function

$$\psi_X : \pi_0(\text{Triv}/X) \to [*, X]$$

which is defined by associating to an object $W \to X$ the composite

$$* \leftarrow W \rightarrow X$$

in the homotopy category.

Note that there are corresponding constructions for any object $X$ of a model category $M$.

**Lemma 3.15.** Suppose that $M$ is a right proper model category, and suppose that the map $f : X \to Y$ is a weak equivalence. Then the induced map

$$f_* : \pi_0(\text{Triv}/X) \to \pi_0(\text{Triv}/Y)$$

is a bijection.

**Proof.** The function $f_*$ is induced by a functor which is defined by associating to the object $W \to X$ the composite

$$W \rightarrow X \xrightarrow{f} Y.$$ 

Suppose that $U \to Y$ is an object of $\text{Triv}/Y$. Choose a factorization

$$U \xrightarrow{j} V \xrightarrow{p} Y,$$

where $j$ is a trivial cofibration and $p$ is a fibration. Form the pullback

$$X \times_Y V \xrightarrow{f'} V \xrightarrow{p} Y.$$ 

Then the map $f'$ is a weak equivalence by the properness assumption, so that the projection $X \times_Y V \to X$ is an object of $\text{Triv}/X$. Observe that the path component of this object is independent of the choice made, and is independent of the choice of representative for the path component of $U \to Y$. It follows that there is a well-defined function

$$g : \pi_0(\text{Triv}/Y) \to \pi_0(\text{Triv}/X).$$

The composite functions $g \cdot f_*$ and $f_* \cdot g$ are both identities. $\square$

**Lemma 3.16.** Suppose that $Y$ is an object of a right proper model category $M$ in which the terminal object $*$ is cofibrant. Then the function

$$\psi_Y : \pi_0(\text{Triv}/Y) \to [*, Y]$$

is a bijection.
Proof. Suppose first of all that $Y$ is fibrant. Then the function
$$\pi(\ast, Y) \to [\ast, Y]$$
is a bijection since $\ast$ is cofibrant. Here, $\pi(\ast, Y)$ denotes homotopy classes of maps with respect to a fixed cylinder object $I$ of $\ast$. If two maps $f, g : \ast \to Y$ are homotopic, then there is a diagram

![Diagram](https://example.com/diagram.png)

Then the morphisms $d_0$ and $d_1$ are weak equivalences, so that $f$ and $g$ are in the same path component of $\text{Triv}/Y$. It follows that there is a well defined function
$$\phi : \pi(\ast, Y) \to \pi_0(\text{Triv}/Y)$$
and the diagram

![Diagram](https://example.com/diagram.png)

commutes. Finally, if $U \to Y$ is an object of $\text{Triv}/Y$, there is a factorization

![Diagram](https://example.com/diagram.png)

where $j$ is a trivial cofibration and $p$ is a trivial fibration. The fibration $p$ has a section $s : \ast \to V$ since $\ast$ is cofibrant, and the map $U \to Y$ extends to a map $V \to Y$ since $j$ is a trivial cofibration and $Y$ is fibrant. It follows that the function $\phi$ is surjective, and is therefore a bijection.

The map $\psi_Y$ is therefore a bijection if $Y$ is fibrant. The general case follows from Lemma 3.15. □

Lemma 3.17. Suppose that $G$ is a presheaf of simplicial groups. Then there is a bijection
$$[\ast, BG] \cong \pi_0(G - \text{tors}_0).$$

Proof. We establish the existence of a bijection
$$\pi_0(\text{Triv}/BG) \cong \pi_0(G - \text{tors}_0).$$

First of all, there is a functor $\text{Triv}/BG \to G - \text{tors}_0$ which is defined by associating the cofibrant simplicial $G$-presheaf $X \times_{BG} EG$ to the object $X \to BG$ of $\text{Triv}/BG$.

Suppose that $Z$ is a cofibrant simplicial $G$-presheaf such that $Z/G \to \ast$ is a local weak equivalence. Then there is a $G$-equivariant map $Z \to EG$ and an induced map $Z/G \to BG$. The class of the object $Z/G \to BG$ in $\pi_0(\text{Triv}/BG)$ is independent of the choices that have been made: any two $G$-equivariant maps $Z \to EG$ are
naively homotopic and so the induced maps \( Z/G \rightarrow BG \) are naively homotopic and hence represent the same path component of \( \pi_0(\text{Triv}/BG) \). It follows that there is a well-defined function

\[
\pi_0(G - \text{tors} \rightarrow \pi_0(\text{Triv}/BG)
\]

and this function is the inverse of the function in \( \pi_0 \) which is induced by the functor of the previous paragraph.

\[\square\]

**Corollary 3.18.** There is a bijection

\[\left[\ast, BG\right] \cong \pi_0(G - \text{tors}) \]

Recall that the objects of the category \( G - \text{tors} \) are simplicial \( G \)-presheaves \( Z \) such that \( d(EG \times G Z) \rightarrow \ast \) is a local weak equivalence.

**Remark 3.19.** If \( G \) is a sheaf of groups, then a \( G \)-torsor \( X \) is naturally a member of \( G - \text{tors} \) after identification of \( X \) with a constant simplicial \( G \)-sheaf, and in this way the category \( G - \text{tors} \) of ordinary \( G \)-torsors imbeds in \( G - \text{tors} \), and the induced function

\[
\pi_0(G - \text{tors}) \rightarrow \pi_0(G - \text{tors})
\]

is a bijection.

Note that \( EG \) is a cofibrant simplicial \( G \)-sheaf such that \( EG \rightarrow \ast \) is a global fibration and \( BG = EG/G \). Write \( NG \) for the nerve of \( G \) in this instance. Recall that there is a weak equivalence \( NG \rightarrow BG \).

Write \( C(U) \) for the Čech resolution associated to a covering \( U \rightarrow \ast \). The objects of the category \( \text{cov}/NG \) are the morphisms \( C(U) \rightarrow NG \), and the morphisms are just commutative diagrams. There is an obvious inclusion functor \( \text{cov}/NG \subset \text{Triv}/NG \). Observe that there are bijections

\[
\pi_0(\text{cov}/NG) \cong \pi_0(\text{Triv}/NG) \cong \pi_0(\text{Triv}/BG)
\]

Now it’s well known that the set of naive homotopy classes \( \pi(C(U), NG) \) is isomorphic to the set of isomorphism classes of \( G \)-torsors which trivialize over \( U \), and that the set of isomorphism classes of \( G \)-torsors is isomorphic to

\[
\lim_{\overset{\longrightarrow}{U}} \pi(C(U), NG),
\]

where the colimit is indexed over the coverings \( U \rightarrow \ast \). If two maps \( C(U) \rightarrow NG \) are homotopic, the homotopy \( C(U) \times \Delta^1 \rightarrow NG \) factors through the nerve \( N\pi(C(U) \times \Delta^1) \), which is itself a Čech resolution \( C(V) \) for some covering \( V \rightarrow \ast \). It follows that if two maps \( C(U) \rightarrow NG \) are homotopic, then they represent the same element of \( \pi_0(\text{cov}/NG) \), and there is a well-defined function

\[
\lim_{\overset{\longrightarrow}{U}} \pi(C(U), NG) \rightarrow \pi_0(\text{cov}/NG).
\]

This function is a bijection, with an obvious inverse. The function (1) is therefore isomorphic to the composite isomorphism (2).

The category \( sGdPre(C) \) (or \( sGdShv(C) \)) of (pre)sheaves of simplicial groupoids is right proper (Theorem 2.7). The terminal object \( \ast \) is cofibrant in \( sGdPre(C) \) (or \( sGdShv(C) \)).
Lemma 3.20. There is a natural diagram of bijections

$$\pi_0(\text{Triv}/G) \xrightarrow{dB} \pi_0(\text{Triv}/dBG)$$

for all objects $G \in s\text{GdPre}(C)$.

Proof. The proof is essentially trivial, and follows from the existence of the diagram of bijections

$$[\ast, G] \xrightarrow{dB} [\ast, dBG]$$

$$[\ast, W\text{G}] \xrightarrow{j} [\ast, W\text{G}]$$

together with Lemma 3.16. □

A (weakly) simplicial category $\mathcal{A}$ is a simplicial object in the category of categories having a discrete simplicial class of objects; in other words, a (weakly) simplicial category $\mathcal{A}$ is a category enriched in simplicial sets. The simplicial groupoid is a weakly simplicial category. The full simplicial set category $\mathbf{S}$ with the function complexes $\text{Hom}(X, Y)$ is also a weakly simplicial category.

The simplicial set of morphisms from $A$ to $B$ in a weakly simplicial category $\mathcal{A}$ is denoted by $\mathcal{A}(A,B)$; the corresponding set of $n$-simplices $\mathcal{A}(A,B)_n$ is the set of morphisms from $A$ to $B$ in the category at the level $n$.

A simplicial functor $f : \mathcal{A} \to \mathcal{B}$ is a morphism of (weakly) simplicial categories. This means that $f$ consists of a function $f : \text{Ob}(\mathcal{A}) \to \text{Ob}(\mathcal{B})$ and simplicial set maps $f : \mathcal{A}(A,B) \to \mathcal{B}(f(A),f(B))$ which respect identities and compositions at all levels.

A natural transformation $\eta : f \to g$ of simplicial functors $f,g : \mathcal{A} \to \mathcal{B}$ consists of morphisms

$$\eta_A : f(A) \to g(A)$$

in $\text{hom}(f(A),g(A)) = \mathcal{B}(f(A),g(A))_0$, one for each object $A$ of $\mathcal{A}$, such that the following diagram of simplicial set maps commutes

$$\mathcal{A}(A,B) \xrightarrow{f} \mathcal{B}(f(A),f(B))$$

$$\mathcal{B}(g(A),g(B)) \xrightarrow{\eta_A} \mathcal{B}(f(A),g(B))$$

for each pair of objects $A, B$ of $\mathcal{A}$.

Suppose that $C$ is a simplicial category, the simplicial functor taking values in simplicial sets $X : C \to \mathbf{S}$ gives rise to a bisimplicial set $BE_C X$ with simplicial set

$$\bigcup_{(a_0,a_1,\ldots,a_n)} X(a_0) \times C(a_0,a_1) \times \cdots \times C(a_{n-1},a_n)$$

in horizontal degree $n$. In vertical degree $m$, it is the simplicial set $\text{holim}_C m X_m$. 
The homotopy colimit of the simplicial functor $X$ is the diagonal $d(BCX)$; one usually writes $\text{holim}_G X = d(BCX)$.

$E_C X$ is a translation simplicial category. In effect, each of $m$-simplex simplicial functors $X_m : C_m \to \text{Sets}$ gives rise to a translation category $E_{C_m} X_m$ having objects $(i, x)$ with $i$ an object of $C_m$ (or $C$) and $x \in X_m(i)$, and with morphisms $\alpha : (i, x) \to (j, y)$ where $\alpha : i \to j$ is a morphism of $C_m$ such that $X_m(\alpha)(x) = y$.

Then the nerve $BE_{C_m} X_m$ is the homotopy colimit $\text{holim}_{C_m} X_m$. Furthermore, the data is simplicial in $n$, so the simplicial object $BE_C X$ is indeed a bisimplicial set.

Suppose that $G$ is a simplicial group and that $X$ is a simplicial set admitting a left $G$-action. Then the functor $X : G \to \text{S}$ sending the unique object $*$ to the simplicial set $X$ is a simplicial functor. The Borel construction $d(EG \times_G X)$ is the homotopy colimit $\text{holim}_G X$.

When $G$ is a presheaf of simplicial groups on a small site $C$, define $G$-torsors as the simplicial functors taking values in simplicial presheaves $X : G \to \text{SPre}(C)$ such that $\text{holim}_G X \to *$ is a locally weak equivalence, then the category $G$-tors of such $G$-torsors coincides with the category $G - \text{tors}_1$ in Section 3.4.

**Lemma 3.21.** Suppose that $C$ is a category enriched in simplicial sets and that $X : C \to \text{S}$ is a simplicial functor taking values in simplicial sets. Suppose that all arrows $a \to b$ of $C_0$ induce weak equivalences $X(a) \to X(b)$. Then the map $X(a) \to F_\alpha$ taking values in the homotopy fibre over $a$ of the diagonal simplicial set map $\text{holim}_C X \to d(BC)$ is a weak equivalence.

**Proof.** For the pullback diagram of bisimplicial sets:

$$
\begin{array}{ccc}
\bigcup \limits_{a \to a_0, a_1, \ldots, a_n \in BC_n} X(a) & \longrightarrow & \bigcup \limits_{(a_0, a_1, \ldots, a_n)} X(a_0) \times C(a_0, a_1) \times \cdots \times C(a_{n-1}, a_n) \\
\downarrow & & \downarrow \pi \\
* & \longrightarrow & BC = \bigcup \limits_{(a_0, a_1, \ldots, a_n)} C(a_0, a_1) \times \cdots \times C(a_{n-1}, a_n)
\end{array}
$$

applying the diagonal functor $d$ one obtains the pullback diagram (since $d$ has a left adjoint $d^* \text{ [S] p. 220}$) then it preserves pullback):

$$
\begin{array}{ccc}
X(a) & \longrightarrow & \text{holim}_G X \\
\downarrow & & \downarrow d(\pi) \\
* & \longrightarrow & d(BC)
\end{array}
$$

Consider all bisimplices $\sigma : \Delta^{r,s} \to BC$ of $BC$, form the pullback diagram

$$
\begin{array}{ccc}
\pi^{-1}(\sigma) & \longrightarrow & \bigcup \limits_{(a_0, a_1, \ldots, a_n)} X(a_0) \times C(a_0, a_1) \times \cdots \times C(a_{n-1}, a_n) \\
\downarrow & & \downarrow \pi \\
\Delta^{r,s} & \longrightarrow & BC
\end{array}
$$

in the category of bisimplicial sets. Since $\Delta^{0,0} = *$, then $X(a) = d\pi^{-1}(a)$.

The bisimplices $\Delta^{m,n} \to BC$ of $BC$ are the objects of the category of bisimplices of $BC$, denoted by $\Delta^{\times 2} \downarrow BC$. A morphism $\sigma \to \tau$ of this category is a commutative
diagram of bisimplicial set maps

\[
\begin{array}{c}
\Delta^{r,s} \\
\downarrow \sigma \\
\downarrow \tau \\
\downarrow \\
BC \\
\end{array}
\]

\[
\Delta^{m,n} \\
\end{array}
\]

The assignment \( \sigma \mapsto \pi^{-1}(\sigma) \) defines a functor
\[
\pi^{-1} : \Delta^{\times 2} \downarrow BC \to S^2.
\]

This Lemma follows from Lemma IV.5.7 in [8] if we can show that each morphism of bisimplices

\[
\begin{array}{c}
\Delta^{r,s} \\
\downarrow \sigma \\
\downarrow \tau \\
\downarrow \\
BC \\
\end{array}
\]

induces a weak equivalence
\[
(\zeta_1, \zeta_2)_* : d\pi^{-1}(\tau) \to d\pi^{-1}(\sigma)
\]

Since the argument in p. 246 of [8] shows that if the pullback diagram
\[
\begin{array}{c}
d\pi^{-1}(a) \\
\downarrow \pi \\
\ast \\
B(\Delta^{\times 2} \downarrow BC) \\
\end{array}
\]

of simplicial sets is homotopy cartesian, then the pullback diagram
\[
\begin{array}{c}
X(a) \\
\downarrow \pi \\
\ast \\
d(BC) \\
\end{array}
\]

is homotopy cartesian.

It is sufficient to show that the bisimplicial set map \( \pi^{-1}(\tau)(\zeta_1, \zeta_2)_* \pi^{-1}(\sigma) \) is a pointwise weak equivalence by Proposition IV.1.7 in [8].

Every bisimplex \( \sigma : \Delta^{k,l} \to BC \) is determined by a string of arrows
\[
\sigma : a_0 \overset{\alpha_1}{\longrightarrow} a_1 \overset{\alpha_2}{\longrightarrow} a_2 \cdots \overset{\alpha_k}{\longrightarrow} a_k
\]
of length \( k \) in \( C_l \), where \( C_l \) is the category in simplicial degree \( l \) in the simplicial category \( C \). In horizontal degree \( n \), this bisimplex determines a simplicial set map
\[
\bigsqcup_{\gamma : n \to k} \Delta^l \to BC_n = \bigsqcup_{(c_0, c_1, \cdots, c_n)} C(c_0, c_1) \times \cdots \times C(c_{n-1}, c_n).
\]
On the summand corresponding to $\gamma : n \to k$, this map restricts to the composite

$$\gamma^*(\sigma) : \Delta^l \to C(a_{\gamma(0)}, a_{\gamma(1)}) \times \cdots \times C(a_{\gamma(n-1)}, a_{\gamma(n)}) \to BC_n.$$ 

The simplicial set $(\operatorname{holim}_\gamma C X)_n$ in horizontal degree $n$ has the form

$$(\operatorname{holim}_\gamma C X)_n = \bigsqcup_{(c_0, c_1, \ldots, c_n)} X(c_0) \times C(c_0, c_1) \times \cdots \times C(c_{n-1}, c_n)$$

It follows that (in horizontal degree $n$) there is an identification

$$(\pi^{-1}(\sigma))_n = \bigsqcup_{\gamma : n \to k} X(a_{\gamma(0)}) \times \Delta^l.$$ 

It’s obvious that any map $(1, \theta) : \Delta^{k,r} \to \Delta^{k,l}$ induces the simplicial set map

$$\bigsqcup_{\gamma : n \to k} X(a_{\gamma(0)}) \times \Delta^r \to \bigsqcup_{\gamma : n \to k} X(a_{\gamma(0)}) \times \Delta^l$$

in horizontal degree $n$ which is specified on summands by

$$1 \times \theta : X(a_{\gamma(0)}) \times \Delta^r \to X(a_{\gamma(0)}) \times \Delta^l$$

such a map is plainly a weak equivalence, and hence induces a pointwise weak equivalence

$$\pi^{-1}(\sigma(1 \times \theta)) \to \pi^{-1}(\sigma).$$

In particular, any vertex $\Delta^0 \to \Delta^l$ determines a weak equivalence

$$\bigsqcup_{\gamma : n \to k} X(a_{\gamma(0)}) \to \bigsqcup_{\gamma : n \to k} X(a_{\gamma(0)}) \times \Delta^l$$

Any bisimplicial set map $(\zeta_1, \zeta_2) : \Delta^{r,s} \to \Delta^{k,l}$ and any choice of vertex $v : \Delta^0 \to \Delta^l$ together induce a commutative diagram of bisimplicial set maps

$$\xymatrix{ \Delta^{r,0} \ar[r]^{(1, v)} \ar[d]_{(\zeta_1, 1)} & \Delta^{r,s} \ar[d]^{(\zeta_1, \zeta_2)} \\ \Delta^{k,0} \ar[r]_{(1, \zeta_2(v))} & \Delta^{k,l} }$$

It therefore suffices to assume that all diagrams of bisimplices

$$\xymatrix{ \Delta^{r,0} \ar[r]^{\tau} \ar[d]_{\theta} & BC \ar[d]^{\sigma} \\ \Delta^{k,0} }$$

induce pointwise weak equivalence

$$\pi^{-1}(\tau) \xrightarrow{\theta^*} \pi^{-1}(\sigma).$$
The simplicial functor $X : C \to S$ restricts to an ordinary functor $X_0 : C_0 \to S$ via the identification of the category $C_0$ with a discrete simplicial subobject of the simplicial category $C$. There is a pullback diagram

$$\begin{align*}
\text{holim}_{C_0} X_0 & \quad \longrightarrow \quad \text{holim}_C X \\
\downarrow & \quad \quad \downarrow \\
BC_0 & \quad \longrightarrow \quad BC
\end{align*}$$

and all bisimplices $\Delta^{k,0} \to BC$ factor through the inclusion $BC_0 \to BC$. Each morphism $a \to b$ of $C_0$ induces a weak equivalence

$$X_0(a) = X(a) \overset{\sim}{\longrightarrow} X(b) = X_0(b)$$

It therefore follows from the standard argument for ordinary functors taking values in simplicial sets that all induced maps

$$\pi^{-1}(\tau) \overset{\theta_*}{\longrightarrow} \pi^{-1}(\sigma)$$

are weak equivalences of simplicial sets.

**Corollary 3.22.** Suppose that $G$ is a simplicial groupoid (groupoid enriched in simplicial sets), and that $X : G \to S$ is a simplicial functor taking values in simplicial sets. Then the map $X(a) \to F_a$ taking values in the homotopy fibre over $a$ of the diagonal simplicial set map $\text{holim}_a X \to d(BG)$ is a weak equivalence.

**Proof.** For any arrow $a \to b$ of $G_0$ it has an inverse arrow $b \to a$, hence the induced simplicial map $X(a) \to X(b)$ has an inverse map $X(b) \to X(a)$, that means $X(a) \to X(b)$ is a weak equivalence. \hfill \Box

Suppose that $H$ is an object of the category $s\text{Gd}$ of simplicial groupoids and let $f : U \to H$ be a morphism of $s\text{Gd}$. Take $a \in \text{Ob}(H)$ and write $f \downarrow a$ for the simplicial category given in degree $n$ by the comma category $f_n \downarrow a$ for the simplicial category in the general senses, it is not category enriched in simplicial sets since its simplicial class of objects isn’t discrete). Then the functors $H_n \to \text{cat}$ given by $a \mapsto f_n \downarrow a$ assemble to give a bisimplicial functor $B(f \downarrow) : H \to S^2$ with $a \mapsto B(f \downarrow a)$ taking values in bisimplicial sets. It follows that the assignment $a \mapsto dB(f \downarrow a)$ defines a simplicial functor taking values in simplicial sets. The simplicial sets $dB(f \downarrow a)$ therefore become identified with the homotopy fibres $F_a$ of the diagonal simplicial set map

$$\text{holim}_H dB(f \downarrow) \overset{d\pi}{\longrightarrow} dBH$$

by the Corollary 3.22. In “horizontal degree” $n$, this map can be identified with the projection

$$dB(f \downarrow a_0) \times H(a_0, a_1) \times \cdots \times H(a_{n-1}, a_n) \to H(a_0, a_1) \times \cdots \times H(a_{n-1}, a_n).$$

The forgetful functors $f_n \downarrow a \to U_n$ also assemble to define a diagonal weak equivalence

$$\text{holim}_H B(f \downarrow) \overset{\omega}{\longrightarrow} BU$$

of trisimplicial sets (here we see $\text{holim}_H B(f \downarrow)$ as the trisimplicial set giving rise to the homotopy colimit). This is a consequence of a standard result of Quillen, applied in each degree: Suppose that $f : C \to D$ is a functor of small categories.
Then the canonical map $\text{holim}_{D} B(f \downarrow) \to BC$ is a weak equivalence. That means the simplicial set map

$$\text{holim}_{H} dB(f \downarrow) \to dB$$

is a weak equivalence.

A torsor for a presheaf of simplicial groupoids $G$ is a simplicial functor $X : G \to \text{SPre} (C)$ taking values in simplicial presheaves such that the associated simplicial presheaf $\text{holim}_{G} X$ is weakly equivalent to a point. A morphism $X \to Y$ of $G$-torsors is a natural transformation of simplicial functors. Insofar as $X$ can be locally identified with the homotopy fibre of the canonical map $\text{holim}_{G} X \to dBG$, a map $X \to Y$ of $G$-torsors restricts to (pointwise) weak equivalences $X|_{U} \to Y|_{U}$ on all sites $\mathcal{C} \downarrow U$ for which $X(U)$ is non-empty.

Write $G\text{-Tors}$ for the category of $G$-torsors, and let $\pi_{0}(G - \text{Tors})$ denote its set of path components. There is a well-defined function

$$\phi : \pi_{0}(G - \text{Tors}) \to \pi_{0}(\text{Triv}/dBG)$$

which is induced by associating a $G$-torsor $X$ the element represented by the map $\text{holim}_{G} X \to dBG$.

There is a function

$$\psi : \pi_{0}(\text{Triv}/G) \cong \pi_{0}(\text{Triv}/dBG) \to \pi_{0}(G - \text{Tors})$$

which is defined as follows. Let $f : U \to G$ be an object of $\text{Triv}/G$ and perform the construction as above sectionwise to form the diagram

$$
\begin{array}{ccc}
\text{dBU} & \xrightarrow{\cong} & \text{holim}_{G} dB(f \downarrow) \\
\text{dBf} \downarrow & & \downarrow f_{*} \\
\text{dBG} & \xrightarrow{\alpha} & \text{holim}_{G} dB(G \downarrow) \\
\end{array}
$$

Then the simplicial $G$-functor $a \mapsto dB(f \downarrow a)$ is a $G$-torsor since $dBU \to *$ is a weak equivalence. This construction is functorial and defines the function $\psi$.

Note that $\text{holim}_{G} dB(G \downarrow)$ is the simplicial set consisting of strings $(b, a)$ of arrows $b_{0} \to b_{1} \to \cdots \to b_{n} \to a_{0} \to a_{1} \to \cdots \to a_{n}$ of length $2n + 1$ in $G_{n}$, and the map $\alpha$ takes this string to the string $b_{0} \to b_{1} \to \cdots \to b_{n}$ while $\beta$ maps this element to the string $a_{0} \to a_{1} \to \cdots \to a_{n}$, note that $\beta$ is just the projection map $d\pi$. The $n$-simplices of $\text{holim}_{G} dB(G \downarrow)$ can therefore be identified with functors $n \star n \to G_{n}$ defined on the poset join $n \star n$, and with simplicial structure maps induced by precomposition with maps $\theta \star \theta : m \star m \to n \star n$. The maps $\alpha$ and $\beta$ are induced by the inclusions $n \to n \star n$ of the left and right substrings of length $n$ respectively.

There is a poset map $h_{n} : n \times 1 \to n \star n$ which is defined by

$$(i, \epsilon) \mapsto \left\{ \begin{array}{ll} i & \text{if } \epsilon = 0, \\
 n + i & \text{if } \epsilon = 1. \end{array} \right.$$
As a picture, \( h_n \) is the diagram

\[
\begin{array}{cccccc}
    & b_0 & \rightarrow & b_1 & \rightarrow & \cdots & \rightarrow & b_n \\
    \downarrow & & & & & & & \\
    a_0 & \rightarrow & a_1 & \rightarrow & \cdots & \rightarrow & a_n \\
\end{array}
\]

The maps \( h_n \) are natural in ordinal number \( n \). It follows that the composites

\[
\Delta^n \times \Delta^1 \xrightarrow{h_n} B(n \ast n) \xrightarrow{(b,a)} dBG
\]

together define a simplicial set map \( \text{holim}_{G} dB(G \downarrow) \times \Delta^1 \rightarrow dBG \) from \( \alpha \) to \( \beta \).

This construction is natural in all simplicial groupoids, and so the map \( s \) associated to \( \beta \) is homotopic (where \( \beta \cdot f_* \) is just the projection map \( \text{detr} : \text{holim}_{G} dB(f \downarrow) \rightarrow dBG \)).

It follows that the canonical map \( \beta \cdot f_* \) and the original map \( dBf : dB_{G} \rightarrow dBG \) represent the same element of \( \pi_0(\text{Triv}/dBG) \), and so the composite \( \phi \cdot \psi \) is the identity function.

If \( X \) is a \( G \)-torsor, then the canonical map \( \text{holim}_{G} X \rightarrow dBG \) is induced by a simplicial groupoid map \( f : E_GX \rightarrow G \) (where \( E_GX \) is the simplicial groupoid in which the level \( n \) groupoid is the translation category for the functor \( X_n : G_n \rightarrow \text{Set} \) in each degree \( n \)). The comma category \( f_{n \downarrow}a \) has objects \( ((b,x), b \rightarrow a) \) where \( a, b \in G_n, x \in X_n(b) \) and morphisms \( \alpha : ((b,x), b \rightarrow a) \rightarrow ((c,y), c \rightarrow a) \) where \( \alpha : b \rightarrow c \) is a morphism of \( G_n \) such that \( X_n(\alpha)(x) = y \) and the composite with \( c \rightarrow a \) is the map \( b \rightarrow a \). The \( n \)-simplices in \( \text{dBG}(f \downarrow a) \) is:

\[
((b_0,x_0), b_0 \rightarrow a) \rightarrow ((b_0,x_1), b_1 \rightarrow a) \rightarrow \cdots \rightarrow ((b_n,x_n), b_n \rightarrow a)
\]

the \( n \)-simplices in \( B(f_{n \downarrow}a) \). There is a \( G \)-natural function \( \text{dBG}(f \downarrow a) \rightarrow X_n(a) \) sending the \( n \)-simplex to \( X_n(\beta)(x_0) \) where \( \beta : b_0 \rightarrow a \). For every \( n \)-simplex \( x \) in \( X_n(a) \) its preimages are connected, hence these functions induce a map \( \text{dBG}(f \downarrow a) \rightarrow X(a) \) for all \( a \) which is a weak equivalence, and hence determines a map of simplicial functors

\[
\text{dBG}(f \downarrow) \rightarrow X
\]

\( X \) is a \( G \)-torsor, then \( \text{holim}_{G} X \) is weakly equivalent to a point, so is \( \text{holim}_{G} dB(f \downarrow) \), that means \( \text{dBG}(f \downarrow) \) is a \( G \)-torsor as well, hence the above map is a map of \( G \)-torsors. It follows that the composite \( \psi \cdot \phi \) is the identity function. We have therefore proved the following

**Theorem 3.23.** The natural function

\[
\phi : \pi_0(\text{G - Tors}) \rightarrow \pi_0(\text{Triv}/dBG) \cong \ast, \text{W}G
\]

is a bijection for each presheaf of simplicial groupoids \( G \) on any Grothendieck site \( C \).

**Proof.** The displayed isomorphism follows from Lemma 3.20 and Lemma 3.16. The proof that \( \phi \) is a bijection is displayed above.

**Remark 3.24.** The definition of \( G \)-torsors and the bijection in Theorem 2.4 are available for the sheaf of simplicial groupoids \( G \). If \( G \) is a presheaf of simplicial groupoids and \( L^2G \) is its associated sheaf, then \( \text{W}L^2G \) is the simplicial sheaf associated to \( \text{W}G \), and the map \( \text{W}G \rightarrow \text{W}L^2G \) is a weak equivalence, and all we’re interested in is the invariant \( \ast, \text{W}G \) = \( \ast, \text{W}L^2G \).
Same arguments are valid in Section 3.3 and 3.4.

**Remark 3.25.** Joyal-Tierney [18] obtain similar result for the sheaves of simplicial groupoids $G$, but their definition of $G$-torsors is different from ours, our definition is much more flexible.

**Theorem 3.7** is a special case of **Theorem 3.23**.

**Example 3.26.** When a Grothendieck site $C$ is the trivial category $\ast$, a presheaf of simplicial groupoids $G$ over $C$ is just an ordinary simplicial groupoids. Thus $\pi_0(G - \text{Tors}) \cong [\ast, W G] \cong [\ast, G] \cong \pi_0 G$, so the set of path components of $G$-torsors is bijective to the set of path components of the simplicial groupoid $G$.

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Department of Mathematics, University of Western Ontario, London, Ontario, Canada

N6A 5B7

E-mail address: zluo@uwo.ca