COMPLEXITY OF QUANTUM CIRCUITS VIA SENSITIVITY, MAGIC, AND COHERENCE

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ABSTRACT. Quantum circuit complexity—a measure of the minimum number of gates needed to implement a given unitary transformation—is a fundamental concept in quantum computation, with widespread applications ranging from determining the running time of quantum algorithms to understanding the physics of black holes. In this work, we study the complexity of quantum circuits using the notions of sensitivity, average sensitivity (also called influence), magic, and coherence. We characterize the set of unitaries with vanishing sensitivity and show that it coincides with the family of matchgates. Since matchgates are tractable quantum circuits, we have proved that sensitivity is necessary for a quantum speedup. As magic is another measure to quantify quantum advantage, it is interesting to understand the relation between magic and sensitivity. We do this by introducing a quantum version of the Fourier entropy-influence relation. Our results are pivotal for understanding the role of sensitivity, magic, and coherence in quantum computation.

CONTENTS

1. Introduction 2
2. Sensitivity and circuit complexity 5
3. Quantum Fourier entropy and influence 18
4. Magic and circuit complexity 23
5. Coherence and circuit complexity 28
6. Concluding remarks 30
Acknowledgments 31
Appendix A. OTOCs 31
Appendix B. Boolean Fourier entropy-influence conjecture 36
Appendix C. Discrete Wigner function and symplectic Fourier transformation 37
References 38

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1. Introduction

A central problem in the field of quantum information and computation is to compute the complexity required to implement a target unitary operation $U$. One usually defines this to be the minimum number of basic gates needed to synthesize $U$ from some initial fiducial state $|1\rangle - |3\rangle$. To determine the so-called quantum circuit complexity of a given unitary operation, a closely related concept, called the circuit cost, was proposed and investigated in a series of seminal papers by Nielsen et al. [4–7]. Surprisingly, the circuit cost, defined as the minimal geodesic distance between the target unitary operation and the identity operation in some curved geometry, was shown to provide a useful lower bound for the quantum circuit complexity [5,6].

In more recent years, the quantum circuit complexity, as well as the circuit cost, was shown to also play an important role in the domain of high-energy physics [8–12]. For example, its evolution was found to exhibit identical patterns to how the geometry hidden inside black hole horizons evolves. Further studies have also investigated the circuit complexity in the context of quantum field theories [13–15], including conformal field theory [16, 17] and topological quantum field theory [18]. Recently, Brown and Susskind argue that the property of possessing less-than-maximal entropy, or uncomplexity, could be thought of as a resource for quantum computation [8]. This was supported by Yung Halpern et al. who present a resource theory of quantum uncomplexity [19]. Furthermore, a connection between quantum entanglement and quantum circuit complexity was revealed by Eisert, who proved that the entangling power of a unitary transformation provides a lower bound for its circuit cost [20].

Let us summarize the main ideas we present in this paper, which we will describe in more detail in §1.1. In this paper, we study the quantum circuit complexity of quantum circuits via their sensitivities, magic, and coherence. The first property, namely sensitivity, is a measure of complexity that plays an important role in the analysis of Boolean functions [21, 22] and can be applied to a range of topics, including the circuit complexity of Boolean circuits [23–25], error-correcting codes [26], and quantum query complexity [27]. A fundamental result in circuit complexity is that the average sensitivity, also called the influence, of constant-depth Boolean circuits is bounded above by the depth and the number of gates in the circuit [23, 24]. While the notion of influence has been generalized to describe quantum Boolean functions [28], considerably little is hitherto known about the connection between the sensitivity (or influence) and the circuit complexity of a quantum circuit. In this regard, our first result provides an upper bound on the circuit sensitivity—a measure of sensitivity for unitary transformations—of a quantum circuit by its circuit cost.

Secondly, we characterize unitaries with zero circuit sensitivity, which we call stable unitaries. We generalize the definition of sensitivity to Clifford algebras, where we use the noise operator defined by Carlen and Lieb [29]. We find that stable gates in this case are exactly matchgates, a well-known family of tractable quantum circuits [30–36]. This provides a new understanding of matchgates via sensitivity. Our result also implies that sensitivity is necessary for a quantum
computational advantage; for a more extended discussion, see Remark 2. In addition, we show a relation between average scrambling and the average sensitivity. Magic is another important resource in quantum computation, which characterizes how far away a quantum state (or gate) is from the set of stabilizer states (or gates). The Gottesman-Knill theorem [37] states that stabilizer circuits comprising Clifford unitaries and stabilizer inputs and measurements can be simulated efficiently on a classical computer. Hence, magic is necessary to realize a quantum advantage [38–42]. Magic measures have been used to bound the classical simulation time in quantum computation [43–50], and also in condensed matter physics [51]. However, the relationship between magic and the complexity of quantum circuits has so far largely been unexplored.

To reveal the connection between magic and circuit complexity, we implement two different approaches. The first approach (see §4.1) uses consequences of the quantum Fourier entropy-influence relation and conjecture, which shows the relation between magic and sensitivity. It can be summarized by the set of inferences diagrammed here:

\[ \text{magic-sensitivity relation} + \text{sensitivity-complexity relation} \]

\[ \text{QEFI} \quad \text{magic-complexity relation} \]

Depending on whether one takes a proven result or a conjectured bound, one arrives at an uninteresting or interesting result, respectively. The classical Fourier entropy-influence conjecture was proposed by Friedgut and Kalai [52], and has many useful implications in the analysis of Boolean functions and computational learning theory. For example, if the Fourier entropy-influence conjecture holds, then it implies the existence of a polynomial-time agnostic learning algorithm for disjunctive normal forms (DNFs) [53].

The second method (see §4.2) we take here is to exhibit the connection between magic and circuit cost directly by introducing the magic rate and magic power. Magic power quantifies the incremental magic by the circuit, while the magic rate quantifies the small incremental magic in infinitesimal time.

Finally, we show the connection between coherence and circuit complexity for quantum circuits. Quantum coherence, which arises from superposition, plays a fundamental role in quantum mechanics. The recent significant developments in quantum thermodynamics [54, 55] and quantum biology [56–58] have shown that coherence can be a very useful resource at the nanoscale. This has led to the development of the resource theory of coherence [59–64]. However, thus far, little is known about the connection between coherence and circuit complexity. In this paper, we address this gap and provide a lower bound on the circuit cost by the power of coherence in the circuit.

The rest of the paper is structured as follows. In §1.1, we summarize the main results of our work. In §2, we investigate the connection between circuit complexity and circuit sensitivity and propose a new interpretation of matchgates in terms of sensitivity. In §3, we consider the relationship between quantum Fourier
entropy and influence. In §4, we study the connection between magic and the circuit cost of quantum circuits. In §5, we study the connection between coherence and the circuit cost of quantum circuits.

1.1. **Main results.** We start by summarizing three of our main results concerning lower bounds on quantum circuit complexity in terms of average sensitivity, magic, and coherence. Here, the complexity of a quantum circuit is taken to be the circuit cost introduced by Nielsen et al.:

**Definition 1** (Nielsen et al. [6]). Let $U \in SU(d^n)$ be a unitary operation and $h_1, \ldots, h_m$ be traceless Hermitian operators that are supported on 2 qudits and normalized as $\|h_i\|_\infty = 1$. The circuit cost of $U$, with respect to $h_1, \ldots, h_m$, is defined as

$$\text{Cost}(U) := \inf \int_0^1 \sum_{j=1}^m |r_j(s)| ds,$$

where the infimum above is taken over all continuous functions $r_j : [0, 1] \to \mathbb{R}$ satisfying

$$U = \mathcal{P} \exp \left(-i \int_0^1 H(s) ds \right),$$

and

$$H(s) = \sum_{j=1}^m r_j(s) h_j,$$

where $\mathcal{P}$ denotes the path-ordering operator.

The theorem below, which gives lower bounds for the circuit cost, collects Theorems 12, 43 and 50 in one place:

**Theorem 2 (Results on Circuit Complexity).** The circuit cost of a quantum circuit $U \in SU(d^n)$ is lower bounded as follows:

$$\text{Cost}(U) \geq c \max \left\{ \text{CiS}[U], \frac{\mathcal{M}[U]}{d^2}, \frac{\mathcal{C}_r(U)}{\log(d)} \right\},$$

where $c$ is a universal constant independent of $d$ and $n$. The quantities $\text{CiS}[U]$ ($\mathcal{M}[U]$, $\mathcal{C}_r(U)$, respectively), defined formally in (13) ((51), (64), respectively), quantify the sensitivity (magic, coherence, respectively) of quantum circuits. Note that here and throughout this paper, the logarithm is taken to be of base 2.

We also define the circuit sensitivity $\text{CiS}^G$ for any unitary in terms of the generators of the Clifford algebra, yielding a new understanding for matchgates (see Theorem 20 for more details):

**Theorem 3 (Matchgates via Sensitivity).** A unitary $U$ satisfies $\text{CiS}^G[U] = 0$ if and only if it is a matchgate.

Matchgates are a well-known family of tractable circuits, and our result shows that $\text{CiS}^G$ could also be used to serve as a measure of non-Gaussianity (noting that matchgates are also called Gaussian operations).
To show the connection between magic and influence (or non-Gaussianity quantified by influence), we also prove the following statement (an informal version of Theorem 23):

**Theorem 4 (Quantum Fourier Entropy-Influence Relation).** For any linear n-qudit operator $O$ with $\|O\|_2 = 1$, we have

$$H[O] \leq c[\log n + \log d]I[O] + h[P_O[\bar{0}]],$$

where $h(x) := -x \log x - (1 - x) \log(1 - x)$ is the binary entropy and $c$ is a universal constant.

2. Sensitivity and circuit complexity

Given the $n$-qudit system $\mathcal{H} = (\mathbb{C}^d)^\otimes n$, the inner product between two operators $A$ and $B$ on $\mathcal{H}$ is defined as $\langle A, B \rangle = \frac{1}{d^n} \text{Tr}[A^\dagger B]$, and the $l_2$ norm induced by the inner product is defined by $\|A\|_2 := \sqrt{\langle A, A \rangle}$. More generally, for $p \geq 1$, the $l_p$ norm is defined as $\|A\|_p = \left(\frac{1}{d^n} \text{Tr}[|A|^p]\right)^{1/p}$ with $|A| = \sqrt{A^\dagger A}$. Taking $V := \mathbb{Z}_d \times \mathbb{Z}_d$, the set of generalized Pauli operators is

$$\mathcal{P}_n = \{ P_{\bar{a}} : P_{\bar{a}} = \otimes_i P_{a_i} \}_{\bar{a} \in V^n},$$

where $P_{a_i} = X^{s_i}Z^{t_i}$ for any $a_i = (s_i, t_i) \in V$. Here, the qudit Pauli $X$ and $Z$ are the shift and clock operators, respectively, defined by $X |j\rangle = |j + 1 \mod d\rangle$ and $Z |j\rangle = \exp(2ij\pi/d) |j\rangle$, respectively. Let us define $P_O[\bar{a}]$ for any $\bar{a} \in V^n$ as

$$P_O[\bar{a}] = \frac{1}{d^{2n}} \text{Tr}[OP_{\bar{a}}]^2, \forall \bar{a} \in V^n. \tag{6}$$

Note that the condition $\|O\|_2 = 1$ is equivalent to saying that $\{ P_O[\bar{a}] \}_{\bar{a}}$ is a probability distribution over $V^n$.

2.1. Influence.

**Definition 5 (Montanaro and Osborne [28]).** Given a linear operator $O$, the local influence at the $j$-th qudit is defined as

$$I_j[O] = \sum_{\bar{a}: a_j \neq (0,0)} P_O[\bar{a}], \tag{7}$$

and the total influence is defined as the sum of all the local influences:

$$I[O] = \sum_{j \in [n]} I_j[O]. \tag{8}$$

With the assumption that $P_O$ in (6) is a probability distribution, the local influence and total influence can be rewritten, respectively, as

$$I_j[O] = \sum_{\bar{a}: a_j \neq (0,0)} P_O[\bar{a}] = \mathbb{E}_{\bar{a} \sim P_O} |a_j|, \tag{9}$$

$$I[O] = \sum_{\bar{a} \in V^n} |\text{supp}(\bar{a})|P_O[\bar{a}] = \mathbb{E}_{\bar{a} \sim P_O} |\bar{a}|, \tag{10}$$

where $|a_j| = 1$ if $a_j = (0,0)$ and 0 otherwise; $\text{supp}(\bar{a})$ (the support of $\bar{a}$) denotes the set of indices $i$ for which $a_i \neq 0$; and $|\bar{a}| := |\text{supp}(\bar{a})|$.,
Note that it is easy to see that the influence can be used to quantify the sensitivity of the single-qudit depolarizing channel \( D_\gamma(\cdot) = (1 - \gamma)(\cdot) + \gamma \text{Tr}[\cdot] \mathbb{I}/d \) as follows
\[
\frac{\partial}{\partial \gamma} \left\| D_\gamma^{(j)}[O] \right\|_2^2 \bigg|_{\gamma=0} = -2I_j[O],
\]
where \( D_\gamma^{(j)} \) denotes the depolarizing channel acting on the \( j \)-th qudit. This implies that
\[
\frac{\partial}{\partial \gamma} \left\| D_\gamma^{\otimes n}[O] \right\|_2^2 \bigg|_{\gamma=0} = -2I[O].
\]
Hence, influence is an average version of sensitivity with respect to depolarizing noise. Note that the notion of influence, \( I_j(O) \) and \( I(O) \), could be applied to quantum states \( |\psi\rangle \) by setting \( O = \sqrt{d^n} |\psi\rangle\langle\psi| \) to ensure that the corresponding probability distribution \( P_O \) defined in (6) sums to 1.

2.2. Circuit sensitivity and complexity.

**Definition 6 (Circuit Sensitivity).** For a unitary \( U \), the circuit sensitivity \( \text{CiS}[U] \) is the change of influence caused by \( U \), defined as
\[
\text{CiS}[U] = \max_{O: \|O\|_2 = 1} \left| I[OUU^\dagger] - I[O] \right|.
\]

First, let us present a basic lemma of circuit sensitivity, which indicates that in the maximization in (13), it suffices to just consider traceless operators:

**Lemma 7.** The circuit sensitivity equals
\[
\text{CiS}[U] = \max_{O: \|O\|_2 = 1, \text{Tr}[O] = 0} \left| I[OUU^\dagger] - I[O] \right|,
\]
that is, it suffices to just consider a maximization over all traceless operators with \( \|O\|_2 = 1 \).

**Proof.** First, \( P_O[\vec{0}] \) defined in (6) is unitarily invariant. Hence, if \( \text{Tr}[O] \neq 0 \), let us define a new operator \( O' \) as
\[
O' = \frac{1}{\sqrt{1 - P_O[\vec{0}]}} \left( O - \frac{\text{Tr}[O]}{d^n} I \right).
\]
Then \( O' \) satisfies the conditions \( \text{Tr}[O'] = 0 \) and \( \|O'\|_2 = 1 \). Also,
\[
I[O'] = \frac{1}{1 - P_O[\vec{0}]} I[O], \quad I[OUO'^\dagger] = \frac{1}{1 - P_O[\vec{0}]} I[OUU^\dagger].
\]
Hence, we have
\[
I[OUO'^\dagger] - I[O'] = \frac{1}{1 - P_O[\vec{0}]} (I[OUU^\dagger] - I[O]).
\]
Therefore, the maximum must be attained by traceless operators. \( \square \)
Now, let us consider the \( n \)-qudit Hamiltonian acting nontrivially on a \( k \)-qudit subsystem. We prove here a simple upper bound on the total change of the total influence \( I \) through unitary evolution.

**Proposition 8 (Small Total Circuit Sensitivity).** Given an \( n \)-qudit system with a Hamiltonian \( H \) acting nontrivially on a \( k \)-qudit subsystem, the total change of influence induced by the unitary \( U_t = e^{-itH} \) is bounded from above by \( k \):

\[
\text{CiS}[U_t] \leq k.
\]

**Proof.** Since \( H \) acts on only a \( k \)-qudit subsystem, there exists a subset \( S \) of size \( k \) such that \( H = H_S \otimes I_{S^c} \) and \( U_t = U_S \otimes I_{S^c} \). Due to the subadditivity of the circuit sensitivity under tensorization (Proposition 14), \( \text{CiS}[U_t] \leq \text{CiS}[U_S] \leq k \). □

Now, let us introduce the influence rate to quantify the change of influence in an infinitesimally small time interval. This will be used to prove the connection between circuit sensitivity and circuit complexity.

**Definition 9 (Influence Rate).** Given an \( n \)-qudit Hamiltonian \( H \) and a linear operator \( O \) with \( \|O\|_2 = 1 \), the influence rate of the unitary \( U_t = e^{-itH} \) acting on \( O \) is defined as follows

\[
R_I(H,O) = \frac{dI[U_tOU_t^\dagger]}{dt} \bigg|_{t=0},
\]

which can be used to quantify small incremental influence for a given unitary evolution.

By a direct calculation, we have the following explicit form of the influence rate:

\[
R_I(H,O) = \frac{i}{d^n} \sum_{\vec{a} \in V^n} |\vec{a}| \left( \text{Tr}[[O,H]P_{\vec{a}}]\text{Tr} [OP_{\vec{a}}^\dagger] + \text{Tr} \left[[O,H]P_{\vec{a}}^\dagger\right]\text{Tr} [OP_{\vec{a}}] \right).
\]

(17)

First, let us provide an upper bound on the influence rate.

**Lemma 10.** Given an \( n \)-qudit system with a Hamiltonian \( H \) and a linear operator \( O \) with \( \|O\|_2 = 1 \), we have

\[
|R_I(H,O)| \leq 4n \|H\|_\infty,
\]

(18)

where \( \|H\|_\infty \) denotes the operator norm.

**Proof.** Since \( |\vec{a}| \leq n \), the Schwarz inequality yields

\[
\frac{1}{d^{2n}} \sum_{\vec{a} \in V^n} |\vec{a}| \left| \text{Tr}[[O,H]P_{\vec{a}}]\text{Tr} [OP_{\vec{a}}^\dagger] \right| \leq n \frac{1}{d^{2n}} \sum_{\vec{a} \in V^n} \left| \text{Tr}[[O,H]P_{\vec{a}}]\text{Tr} [OP_{\vec{a}}^\dagger] \right| \leq 2n \|O\|_2 \|O\|_2 \leq 2n \|H\|_\infty,
\]

where the last inequality comes from the Hölder inequality and the fact that \( \|O\|_2 = 1 \). Similarly, we can prove that

\[
\frac{1}{d^{2n}} \sum_{\vec{a} \in V^n} |\vec{a}| \left| \text{Tr} \left[[O,H]P_{\vec{a}}^\dagger\right]\text{Tr} [O_{\vec{a}}] \right| \leq 2n \|H\|_\infty.
\]
Hence, by the expression of influence rate in (17), we have
\[ |R_I(H, O)| \leq 4n \|H\|_\infty. \]
\[ \square \]

Let us provide an upper bound on the influence rate for the unitary generated by a local Hamiltonian.

**Theorem 11 (Small Incremental Influence).** Given an n-qudit system with the Hamiltonian \( H \) acting on a k-qudit subsystem, and a linear operator \( O \) with unit norm \( \|O\|_2 = 1 \), one has
\[ |R_I(H, O)| \leq 4k \|H\|_\infty. \]

**Proof.** Since \( H \) acts on a k-qudit subsystem, there exists a subset \( S \) of size \( k \) such that \( H = H_S \otimes I_{S^c} \). Define \( O^{(1)}_{\bar{b}} \) on \( (\mathbb{C}^d)^S \) for \( \bar{b} \in V^S \) by
\[ O^{(1)}_{\bar{b}} = \frac{1}{d^{n-k}} \text{Tr}_S[OP_{\bar{b}}]. \]

Also define \( O^{(2)}_\bar{c} \) on \( (\mathbb{C}^d)^S \) for any \( \bar{c} \in V^{S^c} \) as
\[ O^{(2)}_\bar{c} = \frac{1}{d^{n-k}} \text{Tr}_{S^c}[OP_{\bar{c}}]. \]

Note that \( \sum_{\bar{b} \in V^S} \|O^{(1)}_{\bar{b}}\|_2^2 = \sum_{\bar{c} \in V^{S^c}} \|O^{(2)}_{\bar{c}}\|_2^2 = 1 \). Defining \( A_{\bar{b}} = O^{(1)}_{\bar{b}} / \|O^{(1)}_{\bar{b}}\|_2 \) and \( B_{\bar{c}} = O^{(2)}_{\bar{c}} / \|O^{(2)}_{\bar{c}}\|_2 \), we get that \( I[O] \) can be written as
\[ I[O] = \sum_{\bar{c} \in V^{S^c}} \|O^{(2)}_{\bar{c}}\|_2^2 I[B_{\bar{c}}] + \sum_{\bar{b} \in V^S} \|O^{(1)}_{\bar{b}}\|_2^2 I[A_{\bar{b}}]. \]

Hence,
\[ I[U_i O U_i^+] = \sum_{\bar{c} \in V^{S^c}} \|O^{(2)}_{\bar{c}}\|_2^2 I[U_i B_{\bar{c}} U_i^+] + \sum_{\bar{b} \in V^S} \|O^{(1)}_{\bar{b}}\|_2^2 I[A_{\bar{b}}], \]
and so
\[ R_I(H, O) = \sum_{\bar{c} \in V^{S^c}} \left( \|O^{(2)}_{\bar{c}}\|_2^2 R_I(H_S, B_{\bar{c}}) \right). \]

Since both \( H_S \) and \( B_{\bar{c}} \) for any \( \bar{c} \in V^{S^c} \) act on a k-qudit subsystem, we have
\[ R_I(H_S, B_{\bar{c}}) \leq 4k \|H_S\|_\infty, \]
by Lemma 10. Therefore, we obtain
\[ |R_I(H, O)| \leq \sum_{\bar{c} \in V^{S^c}} \left( \|O^{(2)}_{\bar{c}}\|_2^2 |R_I(H_S, B_{\bar{c}})| \right) \leq 4k \|H_S\|_\infty, \]
as claimed. \[ \square \]
Here, we use circuit sensitivity to quantify the average sensitivity of a quantum circuit. In classical Boolean circuits, the average sensitivity of the circuit plays an important role in lower bounding the complexity of a circuit \cite{23–25}. Hence, a natural question is: what is the connection between circuit sensitivity and circuit complexity for quantum circuits? Here, we use the circuit cost defined in \cite{6} to quantify the complexity of quantum circuits. Our next result establishes a connection between the circuit sensitivity and the circuit cost of a quantum circuit.

**Theorem 12 (Circuit Sensitivity Lower Bounds Circuit Cost).** The circuit cost of a quantum circuit $U \in SU(d^n)$ is lower bounded by the circuit sensitivity as follows

$$\text{Cost}(U) \geq \frac{1}{8} \text{CiS}[U]. \quad (22)$$

**Proof.** The proof follows the same idea as that in \cite{20, 65}. First, let us take a Trotter decomposition of $U$ such that for arbitrarily small $\varepsilon > 0$,

$$\|U - V_N\|_{\infty} \leq \varepsilon,$$

where $V_N$ is defined as follows

$$V_N := \prod_{t=1}^{N} W_t,$$

$$W_t := \exp\left(-\frac{i}{N} \sum_{j=1}^{m} r_j \left(\frac{t}{N}\right) h_j\right).$$

and

$$W_t^{(l)} = \lim_{l \to \infty} W_t^{(l)},$$

$$W_t^{(l)} := \left(W_t^{1/l} \cdots W_t^{1/l}\right)^l,$$

$$W_{t,j} := \exp\left(-\frac{i}{N} r_j \left(\frac{t}{N}\right) h_j\right).$$

Let us define $O_t = W_t O_{t-1} W_t^\dagger$ with $O_0 = O$. Then by applying $W_t$, we have

$$I[O_t] - I(O_{t-1}) = I\left(W_t O_{t-1} W_t^\dagger\right) - I(O_{t-1})$$

$$= \lim_{l \to \infty} I\left(W_t^{(l)} O_{t-1} W_t^{(l)}\right) - I(O_{t-1})$$

$$\leq \frac{1}{N} \sum_{j=1}^{m} 8 \left| r_j \left(\frac{t}{N}\right) h_j\right|$$

$$= \frac{8}{N} \sum_{j=1}^{m} \left| r_j \left(\frac{t}{N}\right) h_j\right|,$$
where the inequality above follows from Theorem 11 for \( k = 2 \). Taking the summation over all \( t \), we have
\[
I(UOU^\dagger) - I(O) \leq \frac{8}{N} \sum_{t=1}^{N} \sum_{j=1}^{m} |r_j \left( \frac{t}{N} \right)|.
\]

Since the circuit cost can be expressed as
\[
\text{Cost}(U) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \sum_{j=1}^{m} |r_j \left( \frac{t}{N} \right)|,
\]
we have
\[
I(UOU^\dagger) - I(O) \leq 8 \text{Cost}(U),
\]
which completes the proof of the theorem. □

2.3. Stable unitaries. Here we characterize quantum circuits with zero circuit sensitivity and provide a complete characterization of such unitaries.

**Definition 13.** An \( n \)-qudit unitary (or gate or circuit) \( U \) is stable if \( \text{CiS}[U] = 0 \).

Here, to characterize the stable unitaries, we need to consider weight-1 Pauli operators, i.e. \( P_{\vec{a}} \) with \( |\vec{a}| = 1 \).

**Proposition 14.** The circuit sensitivity satisfies the following three properties:

1. An \( n \)-qudit unitary \( U \) is stable if and only if for any weight-1 Pauli operator \( O \), both \( UOU^\dagger \) and \( U^\dagger OU \) can be written as a linear combination of weight-1 Pauli operators.
2. \( \text{CiS}[V_2UV_1] = \text{CiS}[U] \) for any unitary \( V_1 \) and any stable unitary \( V_2 \).
3. \( \text{CiS} \) is subadditive under multiplication and tensorization:
   \[
   \text{CiS}[UV] \leq \text{CiS}[U] + \text{CiS}[V], \quad \text{CiS}[U \otimes V] \leq \text{CiS}[U] + \text{CiS}[V].
   \]

**Proof.**

1. If \( \text{CiS}[U] = 0 \), for any weight-1 Pauli operator \( O \),
   \[
   I[UOU^\dagger] = I[O] = 1.
   \]

Hence, \( UOU^\dagger \) can be written as a linear combination of weight-1 Pauli operators. Similarly, \( U^\dagger OU \) can be written as a linear combination of weight-1 Pauli operators.

On the other hand, if it holds that for any weight-1 Pauli operator \( O \), both \( UOU^\dagger \) and \( U^\dagger OU \) can be written as a linear combination of weight-1 Pauli operators, then \( UP_aU^\dagger \) and \( UP_a^\dagger U \) can be written as a linear combination of Pauli operators with weights less than \( |\vec{a}| \). Hence, we have
\[
\text{Tr} \left[ P_b^\dagger UP_a U^\dagger \right] \neq 0 \text{ only if } |\vec{a}| = |\vec{b}|.
\]

Let us define the transition matrix \( T_U \) as follows
\[
T_U[\vec{b}, \vec{a}] = \frac{1}{d^n} \text{Tr} \left[ P_b^\dagger U P_a U^\dagger \right],
\]
for any \( \vec{a}, \vec{b} \in V^n \). It is easy to see that \( T_U \) is a unitary matrix. Here, due to the condition (24), the unitary matrix can be decomposed as

\[
T_U = \bigoplus_{k=0}^{n} T^{(k)}_U,
\]

where \( T^{(k)}_U \) is a \(((d^2 - 1)^k \times (d^2 - 1)^k)\) unitary matrix for any \( 0 \leq k \leq n \), defined by \( T^{(k)}_U[\vec{b}, \vec{a}] = \frac{1}{d^{2k}} \text{Tr} \left[ P^\dagger_b U P_d U^\dagger \right] \) for any \( \vec{a}, \vec{b} \) with \(|\vec{a}| = |\vec{b}| = k\). Hence,

\[
\text{Tr} \left[ P^\dagger_b U O U^\dagger \right] = \sum_{\vec{a}:|\vec{a}|=|\vec{b}|} T^{(k)}_U[\vec{b}, \vec{a}] \text{Tr} \left[ OP_{\vec{a}} \right],
\]

and therefore,

\[
\sum_{\vec{b}:|\vec{b}|=k} P_{UOU^\dagger}[\vec{b}] = \sum_{\vec{b}:|\vec{b}|=k} P_O[\vec{b}],
\]

for any \( 0 \leq k \leq n \). This implies that \( I[UOU^\dagger] = I[O] \). Similarly,

\[
\sum_{\vec{b}:|\vec{b}|=k} P_{U^\dagger OU}[\vec{b}] = \sum_{\vec{b}:|\vec{b}|=k} P_O[\vec{b}],
\]

and \( I[U^\dagger OU] = I[O] \). Therefore, \( \text{CiS}[U] = 0 \).

(2) This statement follows directly from the definition.

(3) Subadditivity under multiplication comes directly from the triangle inequality:

\[
\text{CiS}[UV] \leq \max_{O:|O|=1} \left| I[UVOV^\dagger U^\dagger] - I[VOV^\dagger] \right| + \max_{O:|O|=1} \left| I[VOV^\dagger] - I[O] \right|.
\]

Hence, to prove the subadditivity under tensorization, we only need to prove that \( \text{CiS}[U \otimes I] \leq \text{CiS}[U] \). Let us assume that \( U \) acts only on the \( k \)-qudit subsystem \( S \) with \( k \leq n \). Similarly to the proof of Theorem 11, let us define \( O^{(1)}_{\vec{b}} \) on \((C^d)^{S'}\) for any \( \vec{b} \in V^S \) as (20) and \( A_{\vec{b}} = O^{(1)}_{\vec{b}} / \left\| O^{(1)}_{\vec{c}} \right\|_2 \). Define \( O^{(2)}_{\vec{c}} \) on \((C^d)^{S'}\) for any \( \vec{c} \in V^{S'} \) as (21) and \( B_{\vec{c}} = O^{(2)}_{\vec{c}} / \left\| O^{(2)}_{\vec{c}} \right\|_2 \), so \( I[O] \) can be written as

\[
I[O] = \sum_{\vec{c} \in V^{S'}} \left\| O^{2}_{\vec{c}} \right\|_2^2 I[B_{\vec{c}}] + \sum_{\vec{b} \in V^S} \left\| O^{1}_{\vec{b}} \right\|_2^2 I[A_{\vec{b}}].
\]

Similarly, \( I[U \otimes IOU^\dagger \otimes I] \) can be written as

\[
I[U \otimes IOU^\dagger \otimes I] = \sum_{\vec{c} \in V^{S'}} \left\| O^{(2)}_{\vec{c}} \right\|_2^2 I[UB_{\vec{c}}U^\dagger] + \sum_{\vec{b} \in V^S} \left\| O^{(1)}_{\vec{b}} \right\|_2^2 I[A_{\vec{b}}].
\]
Hence

$$|I[U \otimes IOU^\dagger \otimes I] - I[O]| \leq \sum_{\bar{c} \in V^S} \left\| O^\dagger_{\bar{c}} \right\|^2_2 \left\| I[U B \bar{c} U^\dagger] - I[B \bar{c}] \right\|_2$$

$$\leq \text{CiS}[U] \sum_{\bar{c} \in V^S} \left\| O^\dagger_{\bar{c}} \right\|^2_2$$

$$= \text{CiS}[U],$$

where we infer the second inequality from the definition of CiS. The last equality comes from the fact that $\sum_{\bar{c} \in V^S} \left\| O^\dagger_{\bar{c}} \right\|^2_2 = 1$. \hfill \Box

We give two examples of stable unitaries. In fact, all stable unitaries can be generated by these two types of unitaries.

1. A Kronecker product of single-qudit unitaries, $\bigotimes_{i=1}^n U_i$.
2. Swap gates, i.e. the unitary mapping $|\psi\rangle |\phi\rangle \mapsto |\phi\rangle |\psi\rangle$.

**Proposition 15.** The set of stable unitaries is generated by the single-qudit unitaries and the swap unitaries.

**Proof.** Given an $n$-qudit stable unitary $U$, let us consider its action on $X_1$, where $X_i$ denotes the Pauli operator $X$ acting on the $i$-th qudit. Since $U$ has zero circuit sensitivity, we have

$$UX_1U^\dagger = \sum_{i \in A} \alpha_i Q^X_i + \sum_{i \in B} \beta_i Q^Y_i + \sum_{i \in C} \gamma_i Q^Z_i,$$

where $Q^X_i$ is written as $Q^X_i = \sum_{j=1}^{d-1} c_{ij} X^j$ with at least one coefficient $c_{ij} \neq 0$, and $A$ is the set of all indices $i$ such that $\alpha_i \neq 0$. The quantities $Q^Y_i$ and $C$ are similarly defined. Moreover, $Q^Y_i$ is defined as $Q^Y_i = \sum_{j,k=1}^{d-1} c_{ijk} X^j Z^k$ with at least one coefficient $c_{ijk} \neq 0$, and $B$ is the set of all indices $i$ for which $\beta_i \neq 0$. Since $(UX_1U^\dagger)^2 = I$, we have $|A| \leq 1$, $|B| \leq 1$ and $|C| \leq 1$. The first inequality holds because if $|A| \geq 2$, then there exists two indices $i \neq j$ such that $(UX_1U^\dagger)^2$ must contain some term $Q^X_i \otimes Q^Y_j$, which contradicts with the fact that $(UX_1U^\dagger)^2 = I$. Hence, we can simplify $UX_1U^\dagger$ as

$$UX_1U^\dagger = \alpha_i Q^X_i + \beta_j Q^Y_j + \gamma_k Q^Z_k.$$

Since $(UX_1U^\dagger)^2 = I$, we have $i = j = k$. This holds because if $j \neq i$, then $(UX_1U^\dagger)^2$ must contain the term $Q^X_i \otimes Q^Y_j$. Hence, we have

$$UX_1U^\dagger = \alpha_i Q^X_i + \beta_i Q^Y_j + \gamma Q^Z_i.$$

Similarly, we have

$$UZ_1U^\dagger = \alpha_j Q^X_j + \beta_i Q^Y_j + \gamma Q^Z_j.$$

If $i \neq j$, then $[UX_1U^\dagger, UZ_1U^\dagger] = 0$, that is, $[X_1, Z_1] = 0$, which is impossible. Therefore $i = j$, i.e., there exists a local unitary $V$ such that for any $d \times d$ matrix $A$, $UA_1 \otimes I_{n-1} U^\dagger = A_1' \otimes I_{n-1} = VA_i V^\dagger I_{n-1}$. Hence

$$V^\dagger \text{SWAP}_1 U = I_1 \otimes V_2,$$
where $SWAP_{ij}$ is the swap unitary between 1 and $i$, and $V_2$ has zero circuit sensitivity on $n - 1$ qudits. By repeating the above process, we get that $U$ can be generated by the local unitaries and swap unitaries. □

Stable unitaries also preserve multipartite entanglement, where the entanglement is quantified by the average Rényi-2 entanglement entropy:

$$\bar{S}^{(2)}(\rho) = \mathbb{E} S^{(2)}(\rho_A),$$

where $\mathbb{E} := \frac{1}{2^n} \sum_{A \subset [n]}$ denotes the expectation over subsets $A \subset [n]$; $S^{(2)}(\rho_A) = -\log \text{Tr} \left[ \rho_A^2 \right]$ denotes the Rényi-2 entanglement entropy; and $\rho_A$ denotes the reduced state of $\rho$ on the subset $A$.

**Corollary 16.** Stable unitaries cannot increase the entanglement measure $\bar{S}^{(2)}$.

**Proof.** It is easy to verify that both local unitaries and swap unitaries will not change this average entanglement Rényi-2 entropy, so the corollary follows from the proposition. □

This shows that sensitivity is necessary for a quantum computational advantage.

**Corollary 17.** Given an $n$-qudit product state $\bigotimes_{i=1}^n \rho_i$ as input, a stable quantum circuit $U$, and a single-qudit measurement set $\{N, I - N\}$, the outcome probability can be classically simulated in $\text{poly}(n, d)$ time.

**Proof.** Since the stable quantum circuit can be generated by local unitaries and swap gates, such quantum circuits with product input states and local measurements can be simulated efficiently on a classical computer. □

2.4. **Matchgates are Gaussian stable gates.** In this section, we define variants of influence and circuit sensitivity, called Gaussian influence and Gaussian circuit sensitivity and show that matchgates have vanishing circuit sensitivity. We will show that Gaussian circuit sensitivity is necessary for a quantum computational advantage and that it provides a good measure to quantify the non-Gaussianity of quantum circuits. Let us consider the influence based on the generators of a Clifford algebra for an $n$-qubit system. First, we introduce $2n$ Hermitian operators $\gamma_i$ which satisfy the Clifford algebra relations

$$\{ \gamma_i, \gamma_j \} = 2 \delta_{i,j} I, \ \forall \ i, j = 1, \ldots, 2n.$$  \hspace{1cm} (25)

Any linear operator can be expressed as a polynomial of degree at most $2n$ as follows

$$O = \sum_{S \subset [2n]} O_S \gamma^S,$$  \hspace{1cm} (26)

where $\gamma^S = \prod_{i \in S} \gamma_i$. Then

$$\mathbb{E}_{S \sim U} \left| \text{Tr} \left[ (\gamma^S)^\dagger O \right] \right|^2 = \|O\|^2_2.$$  \hspace{1cm} (27)
Here the $\mathbb{E}_{S \sim U}$ denotes the expectation taken over all $S \subset [2n]$ with respect to the uniform distribution, that is, $\mathbb{E}_{S \sim U} = \frac{1}{2^{2n}} \sum_{S \subset [2n]}$. Hence, $\|O\|_2 = 1$ leads to a probability distribution over $S$, which is defined as follows

$$P^G_O[S] = \left( \frac{1}{2^{2n}} \right) \left| \mathrm{Tr} \left( \gamma^S \right)^\dagger O \right|^2, \quad \forall S \subset [2n]. \quad (28)$$

Matchgates are an important family of tractable circuits, first proposed by Valiant in the context of counting problems [30]. Later, they were generalized to free fermionic quantum circuits, which are generated by a quadratic Hamiltonian in terms of Clifford generators $\{ \gamma_i \}$, i.e., $H = \sum_{i,j} \ h_{ij} \gamma_i \gamma_j$ [34]. One important fact concerning matchgates (which are also called Gaussian gates) is that for each generator $\gamma_i$, $U \gamma_i U^\dagger$ and $U^\dagger \gamma_i U$ can always be written as linear combinations of $\gamma_i$ [34].

We provide a new interpretation of matchgates via sensitivity, showing that they are the only unitaries which cannot change the influence. To obtain this result, define the influence with respect to the generators of the Clifford algebra $\{ \gamma_i \}_i$; we call this the Gaussian influence, to distinguish it from the previous definition.

**Definition 18 (Gaussian Influence).** Given a linear operator $O$, the local influence at the $j$-th qudit is

$$I^G_{\gamma_j}[O] = \sum_{S \ni j \in S} P^G_O[S], \quad (29)$$

and the total influence is the sum of all the local influences,

$$I^G_O = \sum_{j \in [2n]} I^G_{\gamma_j}[O] = \sum_{S \subset [2n]} |S| P^G_O[S]. \quad (30)$$

Consider the Markov semigroup $P_t$ introduced by Carlen and Lieb in [29],

$$P_t(\gamma^S) = e^{-t |S|} \gamma^S. \quad (31)$$

Given an operator $O$, we have

$$\frac{\partial}{\partial t} \| P_t(O) \|_2^2 \bigg|_{t=0} = - \sum_{S \subset [2n]} |S| |P^G_O[S]| = -I^G_O. \quad (32)$$

**Remark 1.** There is no obvious relationship between the Pauli weight and the Gaussian weight. In particular, there exist operators whose Pauli weight is 1 and Gaussian weight is $n$, and also operators whose Gaussian weight is 1 and Pauli weight is $n$. Consequently, there is no obvious relationship between the total influence $I$ and the total Gaussian influence $I^G$.

Here, let us define the circuit sensitivity of a unitary with respect to $I^G$.

**Definition 19.** Given a unitary $U$, let us define the Gaussian circuit sensitivity $\text{CiS}^G$ as the change of influence caused by the unitary evolution,

$$\text{CiS}^G[U] = \max_{O: \|O\|_2 = 1} \left| I^G[OUU^\dagger] - I^G[O] \right|. \quad (33)$$

We say that $U$ is Gaussian stable if $\text{CiS}^G[U] = 0$. 


Theorem 20. The Gaussian circuit sensitivity of an $n$-qudit unitary $U$ satisfies the following three properties:

1. The unitary $U$ is Gaussian stable if and only if $U$ is a matchgate.
2. $\text{CiS}^G[V_2UV_1] = \text{CiS}^G[U]$ for any unitary $V_1$ and matchgate $V_2$.
3. $\text{CiS}^G$ is subadditive under multiplication,

$$\text{CiS}^G[UV] \leq \text{CiS}^G[U] + \text{CiS}^G[V].$$

Proof:

(1) On one hand, if $\text{CiS}^G[U] = 0$, then for any generator $\gamma_i$,

$$I^G[U \gamma_i U^\dagger] = I^G[\gamma_i] = 1.$$

Hence, $U \gamma_i U^\dagger$ can be written as $\sum_j c_j \gamma_j$. Similarly, $U^\dagger \gamma_i U$ can be written as a linear combination of $\{ \gamma_j \}_j$.

On the other hand, if for any generator $\gamma_i$, both $U \gamma_i U^\dagger$ and $U^\dagger \gamma_i U$ can be written as a linear combination of $\{ \gamma_j \}_j$, then $U \gamma_i U^\dagger$ and $U^\dagger \gamma_i U$ can be written as a linear combination of $\{ \gamma_{S'} : S' \subset [2n], \mid S' \mid \leq \mid S \mid \}$. Hence, we have

$$\text{Tr} \left[ (\gamma_{S'})^\dagger U \gamma_{S} U^\dagger \right] \neq 0 \text{ only if } |S'| = |S|.$$

(34)

Let us define the transition matrix $T_U$ as follows

$$T_U[S_1, S_2] = \frac{1}{2^n} \text{Tr} \left[ (\gamma_{S_1})^\dagger U \gamma_{S} U^\dagger \right],$$

for any $S_1, S_2 \subset [2n]$. It is easy to see that $T_U$ is a unitary matrix. Here, due to condition (35), the unitary matrix can be decomposed as

$$T_U = \bigoplus_{k=0}^{2n} T_U^{(k)},$$

where $T_U^{(k)}$ is a $(2^n) \times (2^n)$ unitary matrix for any $0 \leq k \leq 2n$, and defined as

$$T_U^{(k)}[S_1, S_2] = \frac{1}{2^n} \text{Tr} \left[ (\gamma_{S_1})^\dagger U \gamma_{S} U^\dagger \right] \text{ for any } S_1, S_2 \text{ with } |S_1| = |S_2| = k.$$

Hence,

$$\text{Tr} \left[ (\gamma_{S_1})^\dagger UOU^\dagger \right] = \sum_{S_2 : |S_2| = |S_1|} T_U^{(|S_1|)}[S_1, S_2] \text{Tr} \left[ (\gamma_{S_2})^\dagger O \right],$$

and therefore,

$$\sum_{S_1 : |S_1| = k} P^G_{OU^\dagger}[S_1] = \sum_{S_1 : |S_1| = k} P^G_O[S_1],$$

for any $0 \leq k \leq n$. This implies that $I^G[UOU^\dagger] = I^G[O]$. Similarly,

$$\sum_{S_1 : |S_1| = k} P^G_{U^\dagger OU}[S_1] = \sum_{S_1 : |S_1| = k} P^G_O[S_1],$$

and we have $I^G[U^\dagger OU] = I^G[O]$. Therefore, $\text{CiS}^G[U] = 0$.

(2) It follows directly from the definition.

(3) It follows directly from the triangle inequality. □
Since matchgates can be simulated efficiently on a classical computer, the Gaussian stable gates cannot yield a quantum advantage. From this we infer that Gaussian sensitivity is necessary for a quantum computational advantage. Since matchgates are sometimes called Gaussian operations and Gaussian circuit sensitivity can be used as a measure to quantify the non-Gaussian nature of quantum circuits, the Gaussian circuit sensitivity shows how "non-matchgate" a circuit is.

The set of stable gates (i.e., $\text{CiS} = 0$) and Gaussian stable gates (i.e., $\text{CiS}^G = 0$) are quite different. For example, for an $n$-qubit system, the SWAP gates are stable but not Gaussian stable; on the other hand, the nearest neighbor (n.n.) $G(Z,X)$ gate is Gaussian stable, but not stable. Here the gate $G(Z,X)$ is defined as

$$G(Z,X) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$ 

Complementing this, we remark that a single-qubit unitary acting on the first qubit $U_1$ lies in the overlap of the two sets. We illustrate this in Figure 1.

Remark 2. Here we consider the sensitivity of quantum circuits with respect to noise, where we define the stable gates (or circuits) as the gates with zero sensitivity. The circuit sensitivity (or influence) may be used to quantify the classical simulation time, a question we plan to study in the future.

In classical computation, algorithmic stability is one of the fundamental properties of a classical algorithm, and it plays an important role in computational learning theory. For example, it gives insight into the differential privacy of randomized algorithms [66, 67], into the generalization error of learning algorithms [68, 69], and so on. This implies that algorithmic stability is useful to understand learning. Hence, one defines quantum algorithmic stability via influence (or circuit sensitivity) for quantum algorithms or circuits as a generalization of the classical theory. One can then study its application in quantum differential
privacy [70, 71] and in understanding the generalization error of quantum machine learning [72–78]. Besides, the stable gates (or circuits) can be efficiently simulated on a classical computer, which shows that stability may not imply a quantum speedup.

In summary, there appears to be a trade-off between quantum computational speedup and the capability of generalization in quantum machine learning.

2.5. Quantifying scrambling by influence on average case. Here we clarify the relationship between influence and scrambling. Information scrambling measures the delocalization of quantum information by chaotic evolution. Scrambling prevents one from determining the initial conditions that precede chaotic evolution through the use of local measurements. One well-known measure of scrambling is the out-of-time-ordered commutator (OTOC). This is defined as the Hilbert-Schmidt norm of the commutator between two initially commuting local Pauli strings after one operator evolves under the action of a unitary. Scrambling refers to the speed of growth of the OTOC. Mathematically, the OTOC is defined by

\[ C(t) = \frac{1}{2} \|[O_D(t), O_A]\|^2 = 1 - \langle O_D(t)O_AO_D(t)O_A \rangle, \tag{36} \]

where \( O_D(t) := U_t O_D U_t^\dagger \), the expectation value \( \langle \cdot \rangle \) is taken with respect to the \( n \)-qubit maximally mixed state \( \mathbb{I}/d^n \), and \( A, D \) denote two disjoint subregions of the \( n \)-qudit system. For simplicity, we take the local dimension to be \( d = 2 \), i.e., the systems we consider are qubit systems.

If we restrict the regions \( A \) and \( D \) to be 1-qubit systems, then the average OTOC over all possible positions for \( A \) can be expressed in terms of the influence of \( O_D(t) \). Without loss of generality, let us assume that the region \( D \) is taken to be the \( n \)-th qubit.

**Proposition 21 (Average OTOC-Influence Relation).** If the region \( D \) is the \( n \)-th qubit, then

\[ \mathbb{E}_A \mathbb{E}_{O_A} \langle O_D(t)O_AO_D(t)O_A \rangle = 1 - \frac{d^2}{d^2 - 1} \frac{1}{n-1} \sum_{j=1}^{n-1} I_j [O_D(t)], \tag{37} \]

where \( \mathbb{E}_A \) denotes the average over all positions \( j \in [n-1] \) such that \( O_A \) initially commutes with \( O_D \), and \( \mathbb{E}_{O_A} \) denotes the average over all local non-identity Pauli operators on position \( A \).

**Proof.** Since \( \langle O_D(t)O_AO_D(t)O_A \rangle \) can be written as the linear combination of the terms \( \langle P_{\vec{a}} P_{\vec{b}} P_{\vec{c}} \rangle \) with \( \vec{a}, \vec{b} \in V^n \) and \( P_{\vec{c}} \) being the local non-identity Pauli operator on the \( j \)-th qubit, we first consider the average of \( \langle P_{\vec{a}} P_{\vec{c}_j} P_{\vec{b}} P_{\vec{c}_j} \rangle \) with \( P_{\vec{c}_j} \) taking
on all non-identity Pauli operators uniformly,
\[
\mathbb{E}_{P_j} \langle P_a P_j P_b P_j \rangle = \frac{1}{d^2 - 1} \sum_{j=(s,t) \in V \setminus \{(0,0)\}} \langle P_a P_j P_b P_j \rangle
\]
\[
= \delta_{\vec{a},\vec{b}} \frac{1}{d^2 - 1} (d^2 \delta_{a_j,0} - 1)
\]
\[
= \delta_{\vec{a},\vec{b}} \left( 1 - |a_j| \frac{d^2}{d^2 - 1} \right).
\]
Hence, any \(O_D(t)\) can be written as
\[
\mathbb{E}_A \mathbb{E}_{O_A} \langle O_D(t) O_A O_D(t) O_A \rangle = \frac{1}{n-1} \sum_{j=1}^{n-1} \sum_{\vec{a} \in V^n} \left( 1 - |a_j| \frac{d^2}{d^2 - 1} \right) P_{O_D}[\vec{a}]
\]
\[
= 1 - \frac{d^2}{d^2 - 1} \cdot \frac{1}{n-1} \sum_{\vec{a} \in V^n} \left[ \sum_{j=1}^{n-1} |a_j| \right] P_{O_D}[\vec{a}]
\]
\[
= 1 - \frac{d^2}{d^2 - 1} \cdot \frac{1}{n-1} \sum_{\vec{a} \in V^n} \sum_{j=1}^{n-1} |a_j| P_{O_D}[\vec{a}]
\]
\[
= 1 - \frac{d^2}{d^2 - 1} \cdot \frac{1}{n-1} \sum_{j=1}^{n-1} I_j[O_D(t)],
\]
where \(I_j\) is defined as \(I_j[O] = \sum_{a_j \neq 0} P_{O}[\vec{a}]\).

Proposition 21 ensures that the average OTOC tends to
\[
1 - \frac{d^2}{d^2 - 1} \cdot \frac{1}{n-1} \sum_{j=1}^{n-1} I_j[O_D(t)] \to 1 - \frac{d^2}{d^2 - 1} \cdot \frac{1}{n} I[O_D(t)] \text{ as } n \to \infty.
\]
This provides the relations between scrambling and the total influence. Aside from the OTOC, higher-order OTOCs, such as the 8-point correlator, can also be related to the total influence on average (See Appendix A).

3. Quantum Fourier entropy and influence

Here, we define the quantum Fourier entropy \(H[O]\) and show its relationship with the influence \(I[O]\). We shall show that the quantum Fourier entropy can be used as a measure of magic in quantum circuits, which we call the “magic entropy”. In addition, we use results on quantum Fourier entropy and influence to obtain the relations between magic and sensitivity (or Gaussian sensitivity).

3.1. Quantum Fourier entropy-influence relation and conjecture.

**Definition 22 (Quantum Fourier Entropy and Min-entropy).** Given a linear \(n\)-qudit operator \(O\) with \(\|O\|_2 = 1\), the quantum Fourier entropy \(H[O]\) is
\[
H[O] = H[P_O] = - \sum_{\vec{a} \in V^n} P_O[\vec{a}] \log P_O[\vec{a}],
\]
with \( \{ P_O[\vec{a}] \} \) being the probability distribution defined in (6). The quantum Fourier min-entropy \( H_\infty[O] \) is

\[
H_\infty[O] = H_\infty[P_O] = \min_{\vec{a} \in V^n} \log \frac{1}{P_O[\vec{a}]}.
\]

One can also define the quantum Fourier Rényi entropy as

\[
H_\alpha[O] = H_\alpha[P_O] = \frac{1}{1 - \alpha} \log \left( \sum_{\vec{a}} P_O^{\alpha}[\vec{a}] \right).
\]

In the study of classical Boolean functions, Friedgut and Kalai proposed the now well-known Fourier entropy-influence conjecture [52]. Another well-known, but weak, conjecture is the Fourier min-entropy-influence conjecture. Appendix B provides a brief introduction to the Fourier entropy-influence conjecture for Boolean functions.

**Theorem 23 (Weak QFEI).** For any linear operator \( O \) on an \( n \)-qudit system with \( \| O \|_2 = 1 \), we have

\[
H[O] \leq c[\log n + \log d] I[O] + h[P_O[\vec{0}]],
\]

where \( h(x) := -x \log x - (1 - x) \log(1 - x) \) is the binary entropy and \( c \) is a universal constant. Here, \( c \) can be taken to be 2.

**Proof.** Let us define a new probability distribution \( \{ W_k[O] \}_k \) on the set \([n]\) as follows

\[
W_k[O] = \sum_{\vec{a} : |\vec{a}| = k} P_O[\vec{a}].
\]

Therefore, the total influence \( I[O] \) can be rewritten as

\[
I[O] = \sum_{\vec{a} \in V^n} |\vec{a}| P_O[\vec{a}] = \sum_k k W_k[O].
\]

Hence, the quantum Fourier entropy can be written as

\[
H[O] = \sum_{\vec{a} \in V^n} P_O[\vec{a}] \log \frac{1}{P_O[\vec{a}]}
\]

\[
= \sum_{\vec{a} \in V^n} P_O[\vec{a}] \left( \log \frac{W_{|\vec{a}|}[O]}{P_O[\vec{a}]} + \log \frac{1}{W_{|\vec{a}|}[O]} \right)
\]

\[
= \sum_{\vec{a} \in V^n} P_O[\vec{a}] \log \frac{W_{|\vec{a}|}[O]}{P_O[\vec{a}]} + \sum_{\vec{a} \in V^n} P_O[\vec{a}] \log \frac{1}{W_{|\vec{a}|}[O]}
\]

\[
= \sum_k W_k[O] \sum_{\vec{a} : |\vec{a}| = k} \frac{P_O[\vec{a}]}{W_k[O]} \log \frac{W_k[O]}{P_O[\vec{a}]} + \sum_k W_k[O] \log \frac{1}{W_k[O]}.
\]
Note that if \( W_k[O] \neq 0 \), then \( \frac{P_0[\bar{a}]}{W_k[O]} \) is a probability distribution on the set \( S_k = \{ \bar{a} \in V^n : |\bar{a}| = k \} \). Hence
\[
\sum_{\bar{a}:|\bar{a}|=k} \frac{P_0[\bar{a}]}{W_k[O]} \log \frac{W_k[O]}{P_0[\bar{a}]} \leq \log |S_k| \leq \log \left( \binom{n}{k} (d^2 - 1)^k \right) \leq k(\log n + \log(d^2 - 1)).
\]

Therefore, we have
\[
\sum_k W_k[O] \sum_{\bar{a}:|\bar{a}|=k} \frac{P_0[\bar{a}]}{W_k[O]} \log \frac{W_k[O]}{P_0[\bar{a}]} \leq \sum_k W_k[O]k(\log n + \log(d^2 - 1)) = [\log n + \log(d^2 - 1)]I[O].
\]

Next, let us prove that \( \sum_k W_k[O] \log \frac{1}{W_k[O]} \leq I[O] + h(P_0[\bar{0}]) \). First, if \( \text{Tr}[O] = 0 \), then \( H[O] \leq I[O] \). This comes from the positivity of the relative entropy between the probability distributions \( \bar{W} = \{ W_k[O] \} \) and \( \bar{p} = \{ p_k \} \), with \( p_k = 2^{-k} \) for \( 1 \leq k \leq n \) and \( p_0 = 2^{-n} \), which can be expressed as
\[
D(\bar{W}||\bar{p}) = \sum_{k=0}^{n} W_k[O] \log \frac{W_k[O]}{p_k[O]} = \sum_k kW_k[O] + \sum_k W_k[O] \log W_k[O] \geq 0.
\]

If \( \text{Tr}[O] \neq 0 \), let us us define a new operator
\[
O' = \frac{1}{1 - W_0[O]} \sum_{\bar{a} \neq 0} O_\bar{a} \bar{a}.
\]

Then for this new operator \( O' \), we have
\[
H[O'] \leq I[O'],
\]
and
\[
I[O] = (1 - W_0[O])I[O'].
\]

Hence,
\[
\sum_k W_k[O] \log \frac{1}{W_k[O]}
= W_0[O] \log \frac{1}{W_0[O]} + \sum_{k \geq 1} W_k[O] \log \frac{1}{W_k[O]}
= W_0[O] \log \frac{1}{W_0[O]} + \sum_{k \geq 1} (1 - W_0[O]) W_k[O'] \log \frac{1}{(1 - W_0[O]) W_k[O']}
= W_0[O] \log \frac{1}{W_0[O]} + (1 - W_0[O]) \log \frac{1}{(1 - W_0[O])} + (1 - W_0[O]) \sum_{k \geq 1} W_k[O'] \log \frac{1}{W_k[O']}
\leq h(W_0[O]) + (1 - W_0)I[O']
= I[O] + h(P_0[\bar{0}]),
\]
where $h$ denotes the binary entropy $h(x) = -x \log x - (1 - x) \log(1 - x)$. This completes the proof of the theorem.

Now, let us consider the quantum Fourier entropy-influence conjecture on qubit systems, which improves upon Theorem 23.

**Conjecture 24 (Quantum Fourier Entropy-Influence Conjecture).** Given a Hermitian operator $O$ on $n$-qubit systems with $O^2 = I$,

$$H[O] \leq cI[O], \quad (42)$$

where the constant $c$ is independent of $n$.

**Proposition 25 (QFEI Implies FEI).** If QFEI is true for Hermitian operators $O$ on $n$-qubit system with $O^2 = I$, then FEI is also true.

**Proof.** Consider any function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ with the corresponding Fourier expansion $f(x) = \sum_{S \subseteq [n]} \hat{f}(S)x_S$. Let us define the following observable

$$O_f = \sum_{S \subseteq [n]} \hat{f}(S)X^S.$$

where $X^S := \prod_{i \in S} X_i$ and $X_i$ is the Pauli $X$ operator on the $i$-th qubit. $O_f$ is a Hermitian operator with $O_f^2 = I$. Note that

$$O_f |x\rangle = f(x) |x\rangle, \quad \forall x \in \{-1, 1\}^n,$$

where $|\pm 1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and $|0\rangle, |1\rangle$ are the eigenstates of the Pauli $Z$ operator. Hence $\langle O_f, P_a \rangle = \hat{f}(S)$ when $P_a = X^S$, and $\langle O_f, P_a \rangle = 0$ otherwise. That is,

$$H[f] = H[O_f],
I[f] = I[O_f].$$

This completes the proof of the proposition.

Similarly, QFMEI is a quantum generalization of FMEI.

**Proposition 26 (QFMEI Implies FMEI).** If QFMEI is true for all quantum Boolean functions, FMEI is also true.

**Proof.** The proof follows the same lines as the proof of Proposition 25.

3.2. **Magic entropy-circuit sensitivity relation.** Magic is an important resource in quantum computation, as a quantum circuit without magic provides no quantum advantage. The Gottesman-Knill theorem states that Clifford unitaries with stabilizer states and Pauli measurements can be efficiently simulated on a classical computer [37, 79]. Here, a Clifford unitary is defined as a unitary which maps a Pauli operator to a Pauli operator. Since any Pauli operator is generated by the product of weight-1 Pauli operators, the Clifford unitaries are precisely those unitaries which map any weight-1 Pauli operator to a Pauli operator. For a non-Clifford unitary, an important task is to quantify the amount of magic in the unitary. Here, we introduce a new concept, which we call the magic entropy.
**Definition 27 (Magic Entropy).** Given a unitary $U$, the magic entropy $M[U]$ is

$$M[U] := \max_{O: \text{weight-1 Pauli}} H[UOU^\dagger]. \quad (43)$$

Since the quantum Fourier entropy of any weight-1 Pauli is always 0, the magic entropy can also be written as follows

$$M[U] = \max_{O: \text{weight-1 Pauli}} (H[UOU^\dagger] - H[O]), \quad (44)$$

which also quantifies the change of quantum Fourier entropy on weight-1 Pauli operators.

**Proposition 28.** The magic entropy $M[U]$ satisfies the following three properties:

1. **Faithfulness:** $M[U] \geq 0$, and $M[U] = 0$ if and only if $U$ is a Clifford unitary.
2. **Invariance under multiplication by Clifford unitaries:** $M[UV] = M[U]$ for any Clifford unitary $V$.
3. **Maximization under tensorization:** $M[U_1 \otimes U_2] = \max \{M[U_1], M[U_2]\}$ for any unitaries $U_1$ and $U_2$.

**Proof.** These properties follow directly from the definition of magic entropy. \(\square\)

**Example 1.** Let us consider a widely-used single-qubit non-Clifford $T$ gate, which is defined as $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$. The magic entropy of $T$ is $M[T] = 1$.

Based on the relations between quantum Fourier entropy and influence in §3.1, we can obtain the connection between magic entropy and circuit sensitivity.

**Proposition 29 (Magic-Sensitivity Relation).** Given an $n$-qudit unitary $U$, the magic entropy and circuit sensitivity satisfy the following relation:

$$M[U] \leq c[\log n + \log d](\text{CiS}[U] + 1). \quad (45)$$

**Proof.** Based on Theorem 23, we have

$$H[UOU^\dagger] \leq c[\log n + \log d]I[UOU^\dagger].$$

Besides, as $I[O] = 1$ for a weight-1 Pauli operator $O$, we have

$$I[UOU^\dagger] \leq \text{CiS}[U] + I[O] = \text{CiS}[U] + 1.$$  

Thus

$$H[UOU^\dagger] \leq c[\log n + \log d](\text{CiS}[U] + 1),$$

for any weight-1 Pauli operator $O$. \(\square\)

**Proposition 30.** If the QFEI conjecture holds for an $n$-qubit system, then for any $n$-qubit unitary $U$,

$$M[U] \leq c(\text{CiS}[U] + 1). \quad (46)$$

**Proof.** The proof is similar to that for Proposition 29. \(\square\)
Since the Gaussian influence $I^G$ has properties similar to the influence $I$, we can get the following connection between quantum Fourier entropy and Gaussian influence by a similar proof, which we call the weak Quantum Fourier entropy-Gaussian influence relation (QFEGI). Hence, it also implies the connection between magic entropy and Gaussian circuit sensitivity for quantum circuits.

**Theorem 31 (Weak QFEGI).** For any linear operator $O$ on an $n$-qubit system with $\|O\|_2 = 1$, we have

$$H[O] \leq c \log(2n) I^G[O] + h[P_0[\bar{0}]],$$

where $h(x) := -x \log x - (1 - x) \log(1 - x)$ is the binary entropy and $c$ is a universal constant.

**Proof.** The proof is similar to that of Theorem 23, so we omit it here. □

**Proposition 32 (Magic-Gaussian Sensitivity Relation).** Given an $n$-qudit unitary $U$, the magic entropy and Gaussian circuit sensitivity satisfy the following relation

$$M[U] \leq c (\log 2n) (\mathrm{CiS}^G[U] + 1).$$

**Proof.** The proof is similar to that of Theorem 29, so we omit it here. □

### 4. Magic and Circuit Complexity

#### 4.1. A lower bound on circuit cost from magic-influence relation

As the influence of a unitary evolution can provide a lower bound on the circuit cost, the magic-influence relation directly implies a lower bound on the circuit cost by the amount of magic.

**Proposition 33.** The circuit cost of a unitary $U \in \text{SU}(d^n)$ satisfies the following lower bound given by the magic entropy

$$\text{Cost}(U) + 1 \geq \frac{1}{c_d \log n} M[U].$$

**Proof.** This is because

$$M[U] \leq c_d \log(n)(\mathrm{CiS}[U] + 1) \leq c_d \log(n)(\text{Cost}(U) + 1),$$

where the first inequality comes from Proposition 29, and the second inequality comes from Theorem 12. □

**Proposition 34.** If the QFEI conjecture holds for $n$-qubit systems, then the circuit cost of a unitary $U \in \text{SU}(2^n)$ satisfies the following bound

$$\text{Cost}(U) + 1 \geq \frac{1}{c} M[U].$$

**Proof.** This is because

$$M[U] \leq c(\mathrm{CiS}[U] + 1) \leq c(\text{Cost}(U) + 1),$$

where the first inequality comes from Proposition 30, and the second inequality comes from Theorem 12. □
4.2. A lower bound on circuit cost by magic power. In subsection §4.1, we obtain a lower bound on the circuit cost based on the magic-influence relation. This lower bound has a log\(n\) factor, which can be removed under the quantum Fourier entropy-influence conjecture. In this subsection, our goal is to get rid of the log\(n\) factor without the conjecture. First, let us introduce another concept called magic power, which is a generalization of magic entropy.

**Definition 35 (Magic Power).** Given a unitary \(U\), the magic power \(\mathcal{M}[U]\) is the maximal magic generated by \(U\),

\[
\mathcal{M}[U] = \max_{O: \|O\|_2 = 1} \left| H[UOU^\dagger] - H[O] \right|.
\]

It is easy to see that the magic power satisfies \(\mathcal{M}[U] \geq M[U]\). Let us first discuss some properties of the magic power.

**Lemma 36.** The magic power equals

\[
\mathcal{M}[U] = \max_{O: \|O\|_2 = 1, \text{Tr}[O] = 0} \left| H[UOU^\dagger] - H[O] \right|,
\]

that is, the maximization is taken over all traceless operators with \(\|O\|_2 = 1\).

**Proof.** Let us define a new operator \(O'\):

\[
O' = \frac{1}{\sqrt{1 - P_O[0]}} \left( O - \frac{\text{Tr}[O]}{d^n} I \right).
\]

If \(\text{Tr}[O] \neq 0\), then \(O'\) satisfies the condition \(\text{Tr}[O'] = 0\) and \(\|O'\|_2 = 1\). Since \(P_{UOU^\dagger}[0] = P_O[0]\), \(H[UOU^\dagger]\) and \(H[O]\) can be rewritten as

\[
H[O] = h \left[ P_O[0] \right] + (1 - P_O[0]) M[O'],
\]

\[
H[UOU^\dagger] = h \left[ P_O[0] \right] + (1 - P_O[0]) M[UO'U^\dagger].
\]

Hence we have

\[
H[UOU^\dagger] - H[O] = (1 - P_O[0])(H[UO'U^\dagger] - H[O']).
\]

Therefore, the maximization is obtained from traceless operators. \(\square\)

**Proposition 37.** The magic power \(\mathcal{M}[U]\) satisfies the following three properties:

1. Magic power is faithful: \(\mathcal{M}[U] \geq 0\), and \(\mathcal{M}[U] = 0\) if and only if \(U\) is a Clifford unitary.
2. Magic power is invariant under multiplication by Cliffords: \(\mathcal{M}[V_2UV_1] = \mathcal{M}[U]\) for any unitary \(V_1\) and Clifford unitary \(V_2\).
3. Magic power is subadditive under multiplication and tensorization:

\[
\mathcal{M}[UV] \leq \mathcal{M}[U] + \mathcal{M}[V], \quad \mathcal{M}[U \otimes V] \leq \mathcal{M}[U] + \mathcal{M}[V].
\]

**Proof.**

(1) \(\mathcal{M}[U] \geq 0\) comes directly from the definition of \(\mathcal{M}[U]\). If \(\mathcal{M}[U] = 0\), it implies that \(H[U_P^{}P_U^\dagger] = 0\) for any Pauli operator \(P_a\), that is \(U_P^{}P_U^\dagger\) is a Pauli operator. Hence the unitary \(U\) is a Clifford unitary. If \(U\) is a Clifford unitary—i.e. if
$U$ always maps Pauli operators to Pauli operators—then the probability distribution $\{P_{UOU}\}$ is equivalent to $\{P_{O}\}$ up to some permutation. Hence $H[UOU] = H[O]$.

(2) This follows directly from the definition of $\mathcal{M}[U]$.

(3) Subadditivity under multiplication comes directly from the triangle inequality, that is

$$\mathcal{M}[UV] \leq \max_{O: \|O\|_2 = 1} \left| H[UVOV^\dagger U^\dagger] - H[VOV^\dagger] \right| + \max_{O: \|O\|_2 = 1} \left| H[VOV^\dagger] - H[O] \right| = \mathcal{M}[U] + \mathcal{M}[V].$$

Hence, to prove subadditivity under tensorization, we only need to prove that $\mathcal{M}[U \otimes I] \leq \mathcal{M}[U]$. Let us assume that $U$ acts on only a $k$-qudit subsystem $S$ with $k \leq n$. Let us define $O_\tilde{c}$ on $(\mathbb{C}^d)^S$ for any $\tilde{c} \in V^S$ as follows

$$O_\tilde{c} = \frac{1}{d^{n-k}} \text{Tr}_{S^c}[O \tilde{c}],$$

and it is easy to verify that $\sum_{\tilde{c} \in V^S} \|O_\tilde{c}\|^2_2 = 1$. Defining $B_\tilde{c} = O_\tilde{c} / \|O_\tilde{c}\|_2$, we get that $H[O]$ can be written as

$$H[O] = \sum_{\tilde{c}} \|O_\tilde{c}\|^2_2 H[B_\tilde{c}] - \sum_{\tilde{c}} \|O_\tilde{c}\|^2_2 \log \|O_\tilde{c}\|^2_2.$$

Similarly,

$$H[U \otimes IOU^\dagger \otimes I] = -\sum_{\tilde{c}} \|O_\tilde{c}\|^2_2 H[UB_\tilde{c}U^\dagger] - \sum_{\tilde{c}} \|O_\tilde{c}\|^2_2 \log \|O_\tilde{c}\|^2_2.$$

Hence

$$\left| H[U \otimes IOU^\dagger \otimes I] - H[O] \right| \leq \sum_{\tilde{c}} \|O_\tilde{c}\|^2_2 \left| H[UB_\tilde{c}U^\dagger] - H[B_\tilde{c}] \right| \leq \mathcal{M}[U].$$

Hence, we obtain the result. \qed

**Example 2.** By a simple calculation, the magic power of a $T$ gate is $\mathcal{M}[T] = 1$. Moreover, for $n$ copies the $T$ gate, namely $T^{\otimes n}$, its magic power is $\mathcal{M}[T^{\otimes n}] = n$, whereas its magic entropy $M[T^{\otimes n}] = 1$, which follows directly from the maximization of magic entropy under tensorization. This example illustrates that magic power may be much larger than magic entropy for the same unitary.

We now introduce the magic rate, which can be used to quantify small incremental magic for a given unitary evolution.

**Definition 38 (Magic Rate).** Given an $n$-qudit Hermitian Hamiltonian $H$ and a linear operator $O$ with $\|O\|_2 = 1$, the magic rate of the unitary $U_t = e^{-itH}$ acting on $O$ is

$$R_M(H, O) = \left. \left( \frac{d}{dt} H[U_tOU_t^\dagger] \right) \right|_{t=0}. \quad (55)$$
First, let us provide an analytic formula for the magic rate by a direct calculation as follows,

\[ R_M(H, O) = \frac{i}{d^{2n}} \sum_{\alpha \in \mathbb{V}^n} \left( \text{Tr}[[O, H]P_\alpha] \text{Tr} \left[ OP^\dagger_\alpha \right] \log P_\alpha[\alpha] \right. \]
\[ + \text{Tr}[[O, H]P^\dagger_\alpha] \text{Tr} [OP_\alpha] \log P_\alpha[\alpha] \right). \]

**Lemma 39.** Consider the function \( g(x) = x(\log x)^2 \) with \( x \in [0, 1] \). Then \( 0 \leq g(x) \leq g(e^{-2}) = (2\log e)^2/e^2 \) for \( x \in [0, 1] \). Moreover, \( g(x) \) is increasing on \([0, e^{-2}]\) and decreasing on \([e^{-2}, 1]\).

**Proof.** This lemma follows from elementary calculus. See Fig. 2 for a plot of the function \( g(x) \).

![Figure 2](image)

**FIGURE 2.** A plot of the function \( g(x) = x(\log x)^2 \) for \( x \in [0, 1] \), where the logarithm is taken to be of base 2. The maximum value of \( g(x) \) is \( g(e^{-2}) \approx 1.1267 \), which occurs at \( x = e^{-2} \approx 0.135 \). The function \( g(x) \) vanishes at both \( x = 0 \) and \( x = 1 \). In addition, it is increasing on \([0, e^{-2}]\) and decreasing on \([e^{-2}, 1]\).

**Lemma 40.** Given an \( n \)-qudit Hamiltonian \( H \) and a linear operator \( O \) with \( \|O\|_2 = 1 \), we have
\[ |R_M(H, O)| \leq 8d^n \|H\|_\infty \log(e)/e. \] (56)

**Proof.** The Schwarz inequality yields
\[ \frac{1}{d^{2n}} \sum_{\alpha \in \mathbb{V}^n} |\text{Tr}[[O, H]P_\alpha]| \text{Tr} \left[ OP^\dagger_\alpha \right] \log P_\alpha[\alpha] \]
\[ \leq \|H, O\|_2 \left( \sum_{\alpha \in \mathbb{V}^n} P_\alpha[\alpha] \log^2 P_\alpha[\alpha] \right)^{1/2} \]
\[ \leq \|H, O\|_2 (d^{2n} g(e^{-1}))^{1/2} \]
\[ \leq 2d^n \|H\|_\infty \sqrt{g(e^{-2})}, \]
where the second inequality comes from the fact that $g(x) \leq g(e^{-2})$ and the last inequality comes from the Hölder inequality. Similarly,

$$\frac{1}{d^{2n}} \sum_{\vec{a} \in V_n} |\text{Tr} \left[ (O,H)P_\vec{a}^2 \right]| \text{Tr} \left[ O \log P_\vec{a} \right] \leq 2d^n \|H\|_\infty \sqrt{g(e^{-2})}. \quad (57)$$

Therefore, we get the bound in (56). □

**Theorem 41 (Small Incremental Magic).** Given an $n$-qudit system with the Hamiltonian $H$ acting on a $k$-qudit subsystem, and a linear operator $O$ with $\|O\|_2 = 1$, one has

$$|R_M(H,O)| \leq 8d^k \|H\|_{\infty} \log(e)/e. \quad (58)$$

**Proof.** Since $H$ acts on a $k$-qudit subsystem, there exists a subset $S$ of size $k$ such that $H = H_S \otimes I_{S^c}$. Define $O_{\vec{c}}$ on $(\mathbb{C}^d)^S$ for $\vec{c} \in V^S$ by

$$O_{\vec{c}} = \frac{1}{d^{n-k}} \text{Tr}_{S^c}[O_{\vec{c}}].$$

Note that $\sum_{\vec{c} \in V^S} \|O_{\vec{c}}\|_2^2 = 1$. Define $B_{\vec{c}} = O_{\vec{c}}/\|O_{\vec{c}}\|_2$. Then, $H[O]$ can be written as

$$H[O] = \sum_{\vec{c}} \|O_{\vec{c}}\|_2^2 H[U_\vec{c}U_\vec{c}^\dagger] - \sum_{\vec{c}} \|O_{\vec{c}}\|_2^2 \log \|O_{\vec{c}}\|_2^2.$$

Hence,

$$R_M(O,H) = \sum_{\vec{c}} \|O_{\vec{c}}\|_2^2 R_M(B_{\vec{c}},H_S).$$

Then, by Lemma 40, we have $|R_M(B_{\vec{c}},H_S)| \leq 4d^k \|H_S\|_{\infty}$. Therefore, we have

$$|R_M(O,H)| \leq \sum_{\vec{c}} \|O_{\vec{c}}\|_2^2 |R_M(B_{\vec{c}},H_S)| \leq 4d^k \|H\|_{\infty}. \quad (59)$$

□

In (58), the dependence on the local dimension $d$ occurs as $O(d^k)$. In §5, the connection between coherence and circuit complexity is studied, where we show that the dependence on the local dimension is $O(k \log d)$. This suggests that a similar bound may also hold for magic.

**Conjecture 42.** Given an $n$-qudit system with the Hamiltonian $H$ acting on a $k$-qudit subsystem,

$$|R_M(H,O)| \leq c k \log(d) \|H\|_{\infty}, \quad (60)$$

where $c$ is a constant independent of $k,d,$ and $n$.

**Theorem 43 (Magic power bounds the circuit cost).** The circuit cost of a quantum circuit $U \in \text{SU}(d^n)$ is lower bounded by the magic power as follows

$$\text{Cost}(U) \geq \frac{e}{8d^2 \log(e)} \mathcal{M}[U]. \quad (61)$$

**Proof.** The proof is almost the same as that of Theorem 12, which we omit here. □
Corollary 44. If Conjecture 42 holds, then the circuit cost of a quantum circuit $U \in SU(d^n)$ is lower bounded by the magic power as follows

$$\text{Cost}(U) \geq \frac{c}{\log d} \mathcal{M}[U].$$

Proof. The proof is almost the same as that of Theorem 43, which we omit here. □

5. COHERENCE AND CIRCUIT COMPLEXITY

First, let us recall the basic concepts in the resource theory of coherence. Given a fixed reference basis $\mathcal{B} = \{ |i\rangle \}_i$, any state which is diagonal in the reference basis is called an incoherent state. The set of all incoherent states is denoted as $\mathcal{I}$. To quantify the coherence in a state, we need to define a coherence measure. Examples of such measures include the $l_1$ norm coherence and relative entropy of coherence [60]. In this work, we focus on the relative entropy of coherence, which is defined as follows:

$$C_r(\rho) = S(\Delta(\rho)) - S(\rho),$$

where $S(\rho) := -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy of $\rho$ and $\Delta(\cdot) := \sum_i |i\rangle \langle i| |i\rangle |i\rangle$ is the completely dephasing channel. This allows us to define the cohering power for a unitary evolution $U$ as:

$$C_r(U) = \max_{\rho \in \mathcal{D}(C^d \otimes \cdots \otimes C^d)} |C_r(U \rho U^\dagger) - C_r(\rho)|.$$  

Proposition 47. Given an $n$-qudit system with a Hamiltonian $H$ and an $n$-qudit quantum state $\rho \in \mathcal{D}(C^d \otimes \cdots \otimes C^d)$, the coherence rate satisfies the following bound

$$|R_C(H, \rho)| \leq 4 \|H\|_\infty D_{\max}(\rho \| \Delta(\rho)),$$

where $D_{\max}$ is the maximal relative entropy defined as

$$D_{\max}(\rho \| \sigma) = \log \min \{ \lambda : \rho \leq \lambda \sigma \}.$$ 

Proof. To prove this result, we need the following lemma.
Lemma 48. (Mariën et al. [65]) Given two positive operators $A$ and $B$ with $A \leq B$ and $\text{Tr}[B] = 1$, there exists a universal constant $c$ such that

$$\text{Tr}[[A, \log B]] \leq 4h(p),$$

where $p = \text{Tr}[A]$, and $h(p) = -p \log p - (1 - p) \log(1 - p)$. Here, $c$ can be taken to be 4 [80].

The proof of Proposition 47 is a corollary of the above lemma by taking $A = 2^{-D_{\text{max}}(\rho \| \Delta(\rho))} \rho$ and $B = \Delta(\rho)$. □

Theorem 49. Given an $n$-qudit system with the Hamiltonian $H$ acting a $k$-qudit subsystem and an $n$-qudit quantum state $\rho \in D(\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d)$, we have

$$|R_C(H, \rho)| \leq 4 \|H\|_o^k \log(d).$$

Proof. Since $H$ acts on a $k$-qudit subsystem, there exists a subset $S \subset [n]$ with $|S| = k$ such that $H = H_S \otimes I_{S^c}$. Based on Lemma 46, we have

$$R_C(H, \rho) = -i \sum_{\vec{z} \in [d]^n} \langle \vec{z} | H_S \otimes I_{S^c} | \vec{z} \rangle \log p(\vec{z}).$$

Let us decompose $|\vec{z}\rangle = |\vec{x}\rangle |\vec{y}\rangle$, where $\vec{x} \in [d]^S$ and $\vec{y} \in [d]^{S^c}$. Then we have

$$R_C(H, \rho) = -i \sum_{\vec{x} \in [d]^S, \vec{y} \in [d]^{S^c}} \langle \vec{x} | H_S \otimes I_{S^c} | \vec{x} \rangle \log \text{Tr}[\rho |\vec{x}\rangle \langle \vec{x}| \otimes |\vec{y}\rangle \langle \vec{y}|].$$

Now, let us define a set of $k$-qudit states $\{\rho_{\vec{y}}\}_{\vec{y}}$ as follows

$$\rho_{\vec{y}} := \frac{\text{Tr}_{S^c}[\rho |\vec{y}\rangle \langle \vec{y}|_{S^c}]}{p_{\vec{y}}},$$

for any $\vec{y} \in [d]^{S^c}$, where the probability $p_{\vec{y}}$ is defined as

$$p_{\vec{y}} = \text{Tr}[\rho |\vec{y}\rangle \langle \vec{y}|_{S^c} \otimes I_S].$$

Note that $\sum_{\vec{y}} p_{\vec{y}} = 1$. Hence, $R_C(H, \rho)$ can be rewritten as

$$R_C(H, \rho) = -i \sum_{\vec{x} \in [d]^S} \sum_{\vec{y} \in [d]^{S^c}} \langle \vec{x} | H_S | \vec{x} \rangle p_{\vec{y}} \log(\text{Tr}[\rho_{\vec{y}} |\vec{x}\rangle \langle \vec{x}|] p_{\vec{y}}),$$

$$= -i \sum_{\vec{x} \in [d]^S} \sum_{\vec{y} \in [d]^{S^c}} \langle \vec{x} | H_S | \vec{x} \rangle p_{\vec{y}} \log \text{Tr}[\rho_{\vec{y}} |\vec{x}\rangle \langle \vec{x}|]$$

$$- i \sum_{\vec{x} \in [d]^S} \sum_{\vec{y} \in [d]^{S^c}} \langle \vec{x} | H_S | \vec{x} \rangle p_{\vec{y}} \log p_{\vec{y}}.$$
Therefore,
\[ R_C(H, \rho) = -i \sum_{\tilde{x} \in [d]^S} \sum_{\tilde{y} \in [d]^S} \langle \tilde{x} | [H_S, \rho_{\tilde{y}}] | \tilde{x} \rangle p_{\tilde{y}} \log \text{Tr} \left[ \rho_{\tilde{y}} | \tilde{x} \rangle \langle \tilde{x} | \right] \]
\[ = \sum_{\tilde{y} \in [d]^S} p_{\tilde{y}} \left( -i \sum_{\tilde{x} \in [d]^S} \langle \tilde{x} | [H_S, \rho_{\tilde{y}}] | \tilde{x} \rangle \log \text{Tr} \left[ \rho_{\tilde{y}} | \tilde{x} \rangle \langle \tilde{x} | \right] \right) \]
\[ = \sum_{\tilde{y} \in [d]^S} p_{\tilde{y}} R_C(H, \rho_{\tilde{y}}). \]

By Proposition 47, we have
\[ |R_C(H, \rho)| \leq 4 \sum_{\tilde{y} \in [d]^S} p_{\tilde{y}} \|H_S\|_\infty D_{\text{max}}(\rho_{\tilde{y}} \| \Delta(\rho_{\tilde{y}})). \]

Since \( \rho_{\tilde{y}} \) is a quantum state on a \( k \)-qudit system, \( D_{\text{max}}(\rho_{\tilde{y}} \| \Delta(\rho_{\tilde{y}})) \leq k \log(d) \).

Hence, we have
\[ |R_C(H, \rho)| \leq 4k \|H_S\|_\infty \log(d). \]

\[ \square \]

**Theorem 50.** [Cohering power lower bounds the circuit cost] The circuit cost of a quantum circuit \( U \in SU(d^n) \) is lower bounded by the cohering power as follows
\[ \text{Cost}(U) \geq \frac{1}{8 \log(d)} C_r(U). \quad (71) \]

**Proof.** The proof is the same as that in Theorem 12, which we omit here. \( \square \)

### 6. Concluding Remarks

In this work, we investigated the connection between circuit complexity and influence, magic, and coherence in quantum circuits. Our main result is a lower bound on the circuit complexity by the circuit sensitivity, magic power, and cohering power of the circuit.

We provided a characterization of scrambling in quantum circuits by the average sensitivity. We gave a characterization of unitaries with zero circuit sensitivity and showed that such unitaries can be efficiently simulated on a classical computer. In other words, circuits consisting of just these unitaries can yield no quantum advantage. In this regard, our result provides a new understanding of matchgates via sensitivity. This raises the following interesting question: does the sensitivity of a quantum circuit determine the classical simulation time of the circuit? This is a question we leave for future work. Moreover, it will be interesting to develop a framework of quantum algorithmic stability based on sensitivity and apply it to quantum differential privacy and generalization capability of quantum machine learning.

Finally, we also defined a quantum version of the Fourier entropy-influence conjecture, and applied it to establishing a connection between circuit complexity and magic. If the quantum Fourier entropy-influence conjecture is true, then we can infer that the classical conjecture also holds.
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APPENDIX A. OTOCs

**Lemma 51** (4-point correlator, weight m). If the region $D$ is the last $k$-th qubit, then

$$
\mathbb{E}_A \mathbb{E}_{O_A} \langle O_D(t) O_A O_D(t) O_A \rangle = \frac{1}{(n-k)^m} \sum_{j=0}^{m} \left( -\frac{4}{3} \right)^j \binom{n-k-j}{m-j} I^{(j)}_{[n-k]}[O_D(t)],
$$

where $\mathbb{E}_A$ denotes the average over all of the size-$m$ subsets $A \in [n-k]$ so that $O_A$ commutes with $O_D$ at the beginning, $\mathbb{E}_{O_A}$ denotes the average over all local Pauli operators with weight $m$ on position $A$, and $I^{(j)}_{[n-k]}[O_D(t)]$ is defined as

$$
I^{(j)}_{[n-k]}[O_D(t)] = \sum_{S \subseteq [n-k], |S| = j} I_S [O_D(t)],
$$

$$
I_S [O_D(t)] = \sum_{S \subseteq \text{supp} (\vec{a})} P_{O_D(t)}[\vec{a}],
$$

**Proof.** Let $S$ be a subset of $[n-k]$ with $|S| = m$. The average of all the weight-$m$ Pauli operators with support on $S$ is equal to

$$
\mathbb{E}_{PS} \langle P_a P_j P_b P_j \rangle = \left( -\frac{1}{3} \right)^{|\text{supp}(\vec{a}) \cap S|} \delta_{\vec{a} \vec{b}}.
$$

Hence,

\begin{align*}
\mathbb{E}_A \mathbb{E}_{O_A} \langle O_D(t) O_A O_D(t) O_A \rangle & = \frac{1}{(n-k)^m} \sum_{S \subseteq [n-k]} \sum_{\vec{a}} \left( -\frac{1}{3} \right)^{|\text{supp}(\vec{a}) \cap S|} P_{O_D(t)}[\vec{a}] \\
& = \frac{1}{(n-k)^m} \sum_{S \subseteq [n-k]} \sum_{\vec{a}} \left( -\frac{1}{3} \right)^{|\text{supp}(\vec{a}) \cap S|} P_{O_D(t)}[\vec{a}] \\
& = \frac{1}{3^m} \frac{1}{(n-k)^m} \sum_{\vec{a}} \sum_{S \subseteq [n-k]} \sum_{j=0}^{m} \left( -\frac{1}{3} \right)^j \binom{|\text{supp}(\vec{a}) \cap [n-k]|}{j} P_{O_D(t)}[\vec{a}] \\
& = \frac{1}{3^m} \frac{1}{(n-k)^m} \sum_{\vec{a}} 3^m \sum_{j=0}^{m} \left( -\frac{1}{3} \right)^j \binom{|\text{supp}(\vec{a}) \cap [n-k]|}{j} P_{O_D(t)}[\vec{a}] \\
& \times \left( n-k - |\text{supp}(\vec{a}) \cap [n-k]| \right) P_{O_D(t)}[\vec{a}]
\end{align*}
Let us introduce the Krawtchouk polynomial $K_m(x; n, q)$, which is defined as follows:

$$K_m(x; n, q) = \sum_{j=0}^{m} (-1)^j (q - 1)^{m-j} \binom{x}{j} \binom{n-x}{m-j}. \quad (76)$$

This can be rewritten as

$$K_m(x; n, q) = \sum_{j=0}^{m} (-q)^j (q - 1)^{m-j} \binom{n-j}{m-j} \binom{x}{j}. \quad (77)$$

Then the above equation equals

$$\mathbb{E}_A \mathbb{E}_{O_A} (O_D(t) O_A O_D(t) O_A)$$

$$= \frac{1}{3^m} \frac{1}{\binom{n-k}{m}} \sum_{\tilde{a}} K_m(|\text{supp}(\tilde{a}) \cap [n-k]; n-k, 4) P_{O_D(t)}[\tilde{a}]$$

$$= \frac{1}{3^m} \frac{1}{\binom{n-k}{m}} \sum_{\tilde{a}} \sum_{j=0}^{m} (-4)^j 3^{m-j} \binom{n-k-j}{m-j} \left(\binom{\text{supp}(\tilde{a}) \cap [n-k]}{j} P_{O_D(t)}[\tilde{a}] \right)$$

$$= \frac{1}{\binom{n-k}{m}} \sum_{j=0}^{m} \left( -\frac{4}{3} \right)^j \binom{n-k-j}{m-j} \sum_{\tilde{a} : |\text{supp}(\tilde{a}) \cap [n-k]| \geq j} \left(\binom{\text{supp}(\tilde{a}) \cap [n-k]}{j} P_{O_D(t)}[\tilde{a}] \right)$$

$$= \frac{1}{\binom{n-k}{m}} \sum_{j=0}^{m} \left( -\frac{4}{3} \right)^j \binom{n-k-j}{m-j} \sum_{S : S \subseteq [n-k], |S| = j} \sum_{\tilde{a} \cap [S]} P_{O_D(t)}[\tilde{a}]$$

$$= \frac{1}{\binom{n-k}{m}} \sum_{j=0}^{m} \left( -\frac{4}{3} \right)^j \binom{n-k-j}{m-j} I_{j}^{(n-k)}[O_D(t)].$$

Lemma 52 (8-point correlator). If the region $D$ is the last $k$-th qubit, then

$$\mathbb{E}_A \mathbb{E}_{O_A} (O_D(t) O_A O_D(t) O_A O_D(t) O_A O_D(t) O_A)$$

$$= \|O_D(t) \ast O_D(t)\|_2^2 \left[ 1 - \frac{4}{3} \frac{1}{n-k} \sum_{j \in [n-k]} I_j [O_D(t) \ast O_D(t)] \right], \quad (78)$$

where $\mathbb{E}_A$ denotes the average over all of the positions $j \in [n-k]$ so that $O_A$ commutes with $O_D$ at the beginning, and $\mathbb{E}_{O_A}$ denotes the average over all local non-identity Pauli operators on position $D$. The convolution $O_D(t) \ast O_D(t)$ is defined in (92).
Proof. Let us express the operator $O = \sum \hat{f}(\vec{a})P_{\vec{a}}$. Then, the correspond generalized Wigner function $f$ is defined as follows

$$f(\vec{x}) = \sum_{\vec{a}} \hat{f}(\vec{a})(-1)^{\langle \vec{x}, \vec{a} \rangle}, \quad (79)$$

where the inner product $\langle \cdot, \cdot \rangle_\times$ denotes the symplectic inner product. (See Appendix C for a brief introduction of the generalized Wigner function and symplectic Fourier transformation.)

Let us first consider the average of $\langle P_a P_b P_c P_d P_j \rangle$. It is easy to verify that

$$\mathbb{E}_{P_j} \langle P_a P_j P_b P_c P_d P_j \rangle$$
$$= \frac{1}{3} \left[ \langle P_a X_j P_b X_j P_c X_j P_d X_j \rangle + \langle P_a Y_j P_b Y_j P_c Y_j P_d Y_j \rangle + \langle P_a Z_j P_b Z_j P_c Z_j P_d Z_j \rangle \right]$$
$$= \frac{1}{3} (4 \delta_{b_j+d_j,0} - 1) \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d},\vec{0}}$$
$$= \left[ 1 - \frac{4}{3} |b_j + d_j| \right] \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d},\vec{0}}.$$

Therefore, we have

$$\mathbb{E}_A \mathbb{E}_{O_A} \langle O_D(t) O_A O_D(t) O_A O_D(t) O_A O_D(t) O_A \rangle$$
$$= \frac{1}{n-k} \sum_{j \in [n-k]} \sum_{\vec{d}, \vec{a}, \vec{b}, \vec{c}, \vec{d}} \left[ 1 - \frac{4}{3} |b_j + d_j| \right] \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d},\vec{0}} \hat{f}_{O_D(\vec{a})} \hat{f}_{O_D(\vec{b})} \hat{f}_{O_D(\vec{c})} \hat{f}_{O_D(\vec{d})}$$
$$= \frac{1}{n-k} \sum_{j \in [n-k]} \sum_{\vec{d}, \vec{a}, \vec{b}, \vec{c}, \vec{d}} \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d},\vec{0}} \hat{f}_{O_D(\vec{a})} \hat{f}_{O_D(\vec{b})} \hat{f}_{O_D(\vec{c})} \hat{f}_{O_D(\vec{d})}$$
$$- \frac{4}{3} \frac{1}{n-k} \sum_{j \in [n-k]} \sum_{\vec{d}, \vec{a}, \vec{b}, \vec{c}, \vec{d}} |b_j + d_j| \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d},\vec{0}} \hat{f}_{O_D(\vec{a})} \hat{f}_{O_D(\vec{b})} \hat{f}_{O_D(\vec{c})} \hat{f}_{O_D(\vec{d})}$$
$$= \sum_{\vec{a}, \vec{b}, \vec{c}, \vec{d}} \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d},\vec{0}} \hat{f}_{O_D(\vec{a})} \hat{f}_{O_D(\vec{b})} \hat{f}_{O_D(\vec{c})} \hat{f}_{O_D(\vec{d})}$$
$$- \frac{4}{3} \frac{1}{n-k} \sum_{j \in [n-k]} \sum_{\vec{d}, \vec{a}, \vec{b}, \vec{c}, \vec{d}} |b_j + d_j| \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d},\vec{0}} \hat{f}_{O_D(\vec{a})} \hat{f}_{O_D(\vec{b})} \hat{f}_{O_D(\vec{c})} \hat{f}_{O_D(\vec{d})}.$$
Let us compute the two terms separately. First,
\[
\sum_{\vec{a},\vec{b},\vec{c},\vec{d}} \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d}} \hat{f}_{\vec{O}_D}(\vec{a}) \hat{f}_{\vec{O}_D}(\vec{b}) \hat{f}_{\vec{O}_D}(\vec{c}) \hat{f}_{\vec{O}_D}(\vec{d})
\]
\[
= \sum_{\vec{a},\vec{b},\vec{c}} \left[ \sum_{\vec{d}} \hat{f}_{\vec{O}_D}(\vec{a}) \hat{f}_{\vec{O}_D}(\vec{b}) \right] \left[ \sum_{\vec{d}} \hat{f}_{\vec{O}_D}(\vec{c}) \hat{f}_{\vec{O}_D}(\vec{d}) \right]
\]
\[
= \sum_{\vec{b}} \left( \mathbb{E}_{\vec{d}} |f_{\vec{O}_D}(\vec{a})|^2 (-1)^{\langle \vec{a},\vec{b} \rangle} \right)^2
\]
\[
= \mathbb{E}_{\vec{d}} |f_{\vec{O}_D}(\vec{a})|^4
\]
\[
= \|O_D * O_D\|^2_2,
\]
where the convolution $O * P$ satisfies
\[
f_{O * P} = f_O f_P.
\]
(80)

Besides,
\[
\frac{1}{n-k} \sum_{j \in [n-k] \setminus \vec{a},\vec{b},\vec{c},\vec{d}} |b_j + d_j| \delta_{\vec{a}+\vec{b}+\vec{c}+\vec{d}} \hat{f}_{\vec{O}_D}(\vec{a}) \hat{f}_{\vec{O}_D}(\vec{b}) \hat{f}_{\vec{O}_D}(\vec{c}) \hat{f}_{\vec{O}_D}(\vec{d})
\]
\[
= \frac{1}{n-k} \sum_{\vec{a},\vec{b},\vec{c},\vec{d}} \left[ \mathbb{E}_{\vec{d}} |f_{\vec{O}_D}(\vec{a})|^2 (-1)^{\langle \vec{a},\vec{b} \rangle} \right]^2
\]
where it is easy to verify that
\[
I_j[O * O] = \sum_{\vec{c},\vec{d} \neq \vec{c}_j} \mathbb{E}_{\vec{d}} |f_{\vec{O}_D}(\vec{a})|^2 (-1)^{\langle \vec{a},\vec{c} \rangle}.
\]
(81)

\[\square\]

**Lemma 53** (8-point correlator, weight $m$). If the region $D$ is the last $k$-th qubit, then
\[
\mathbb{E}_A \mathbb{E}_{O_A} \langle O_D(t) O_A O_D(t) O_A (O_D(t) O_A O_D(t)) O_A \rangle
\]
\[
= \|O_D(t) * O_D(t)\|^2_2 \frac{1}{(n-k)^2} \sum_{j=0}^{m} \left( \begin{array}{c} n-k-j \\ m-j \end{array} \right) f_{[n-k]}^{(j)} (O_D(t) * O_D(t)),
\]
(82)
where $\mathbb{E}_A$ denotes the average over all of the size-$m$ subsets $A \in [n-k]$ so that $O_A$ commutes with $O_D$ at the beginning. $\mathbb{E}_{O_A}$ denotes the average over all local Pauli operators with weight $m$ on position $A$, and $t^{(j)}_{[n-k]}[O_D(t) \ast O_D(t)]$ is defined as above.

**Proof.** Since $O = \sum_{a} \hat{f}(\bar{a}) P_a$, let us first consider the average of $\langle P_a P_b P_c P_d P_e P_f \rangle$. It is easy to verify that

$$\mathbb{E}_{P_a} \langle P_a P_b P_c P_d P_e P_f \rangle = \left( -\frac{1}{3} \right)^{|\text{supp}(\bar{b}+\bar{d}) \cap S|} \delta_{\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}}.$$  

Hence, we have

$$\mathbb{E}_A \mathbb{E}_{O_A} \langle O_D(t) O_A O_D(t) O_A O_D(t) O_A O_D(t) O_A \rangle = \frac{1}{(n-k)} \sum_{S \subset [n-k]} \sum_{\bar{a},\bar{b},\bar{c},\bar{d}} \left( -\frac{1}{3} \right)^{|\text{supp}(\bar{b}+\bar{d}) \cap S|} \delta_{\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}}.$$

$$= \frac{1}{3^m} \left( \sum_{\bar{a},\bar{b},\bar{c},\bar{d}} 3^m \left( -\frac{1}{3} \right)^{|\text{supp}(\bar{b}+\bar{d}) \cap S|} \delta_{\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}} \right) \times \hat{f}_{O_D}(\bar{a}) \hat{f}_{O_D}(\bar{b}) \hat{f}_{O_D}(\bar{c}) \hat{f}_{O_D}(\bar{d})$$

$$= \frac{1}{3^m} \left( \sum_{\bar{a},\bar{b},\bar{c},\bar{d}} 3^m \left( -\frac{1}{3} \right)^{|\text{supp}(\bar{b}+\bar{d}) \cap S|} \delta_{\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}} \right) \times \left( \frac{m-n-k}{m-j} \right) \times \hat{f}_{O_D}(\bar{a}) \hat{f}_{O_D}(\bar{b}) \hat{f}_{O_D}(\bar{c}) \hat{f}_{O_D}(\bar{d})$$

$$= \frac{1}{3^m} \left( \sum_{\bar{a},\bar{b},\bar{c},\bar{d}} 3^m \left( -\frac{1}{3} \right)^{|\text{supp}(\bar{b}+\bar{d}) \cap S|} \delta_{\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}} \right) \times \left( \frac{m-n-k}{m-j} \right) \times \hat{f}_{O_D}(\bar{a}) \hat{f}_{O_D}(\bar{b}) \hat{f}_{O_D}(\bar{c}) \hat{f}_{O_D}(\bar{d})$$

$$= \frac{1}{3^m} \left( \sum_{\bar{a},\bar{b},\bar{c},\bar{d}} 3^m \left( -\frac{1}{3} \right)^{|\text{supp}(\bar{b}+\bar{d}) \cap S|} \delta_{\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}} \right) \times \left( \frac{m-n-k}{m-j} \right) \times \hat{f}_{O_D}(\bar{a}) \hat{f}_{O_D}(\bar{b}) \hat{f}_{O_D}(\bar{c}) \hat{f}_{O_D}(\bar{d})$$

Since

$$\hat{f}(\bar{a}) \hat{f}_{O_D}(\bar{a}+\bar{b}) \hat{f}_{O_D}(\bar{a}+\bar{c}) \hat{f}_{O_D}(\bar{a}+\bar{d}) \hat{f}_{O_D}(\bar{a}+\bar{e}) \hat{f}_{O_D}(\bar{a}+\bar{f}) = \left( \mathbb{E}_{\bar{a}} |f(\bar{a})|^2 (-1)^{\bar{a} \cdot \bar{c}} \right)^2,$$  

(84)
we have

\[
\mathbb{E}_A \mathbb{E}_O \langle OD(t) O_A O_D(t) O_A O_D(t) O_A O_D(t) O_A \rangle
= \frac{1}{3^m} \frac{1}{m} \sum_{k} K_m([n-k] \cap \text{supp}(\bar{c}); n-k, 4)
\times \sum_{\bar{a}, \bar{b}} \hat{f}_{O_D}(\bar{a}) \hat{f}_{O_D}(\bar{a} + \bar{b}) \hat{f}_{O_D}(\bar{a} + \bar{c}) \hat{f}_{O_D}(\bar{a} + \bar{b} + \bar{c})
\]

\[
= \frac{1}{3^m} \frac{1}{m} \sum_{k} K_m([n-k] \cap \text{supp}(\bar{c}); n-k, 4) \left( \mathbb{E}_A |f(\bar{a})|^2 (-1)^{\langle \bar{a}, \bar{c} \rangle} \right)^2
\]

\[
= \frac{1}{3^m} \frac{1}{m} \sum_{k} \sum_{j=0}^m (-4)^j 3^{m-j} \left( \binom{n-k-j}{m-j} \right) |\hat{f}_{O_D}(\bar{c})|^2
\]

\[
= \| O_D(t) * O_D(t) \|^2 \frac{1}{3^m} \sum_{k} \sum_{j=0}^m (-4)^j 3^{m-j} \left( \binom{n-k-j}{m-j} \right) I_j^{[n-k]}[O_D(t) * O_D(t)].
\]

\[
\square
\]

APPENDIX B. BOOLEAN FOURIER ENTROPY-INFLUENCE CONJECTURE

In this section, we give a brief introduction to the Fourier entropy-influence conjecture for Boolean functions. Boolean functions, defined as functions \{-1, 1\}^n \rightarrow \{-1, 1\} (or \mathbb{R}), are a basic object in theoretical computer science. The inner product between Boolean functions is defined as

\[
\langle f, g \rangle := \mathbb{E}_x f(x)g(x),
\]

where \( \mathbb{E}_x := \frac{1}{2^n} \sum_{x \in \{\pm\}^n} \). Each Boolean function \( f \) has the following Fourier expansion

\[
f(x) = \sum_{S \subseteq [n]} \hat{f}(S) x_S,
\]

where the parity functions \( x_S := \prod_{i \in S} x_i \), and the Fourier coefficients \( \hat{f}(S) = \langle f, x_S \rangle = \mathbb{E}_{x \in \{-1,1\}^n} f(x) x_S \). Parseval’s identity tells us that

\[
\mathbb{E}_{x \in \{\pm\}^n} f(x)^2 = \sum_S \hat{f}(S)^2.
\]

Let us define the discrete derivative \( D_j[f] \) as \( D_j[f](x) = (f(x) - f(x \oplus e_j))/2 \), where \( x \oplus e_j \) denotes the flip from \( x_j \) to \(-x_j\). Then the \( j \)-th local influence \( I_j \) is defined as the \( l_2 \) norm of the discrete derivative \( D_j[f] \):

\[
I_j[f] = \mathbb{E}_{x \in \{\pm\}^n} |D_j[f](x)|^2,
\]

which can also be written as

\[
I_j[f] = \sum_{S : j \in S} \hat{f}(S)^2 |S|,
\]
where $|S|$ denotes the size of the subset $S$. The total influence of the Boolean function is defined as $I[f] = \sum_{j \in [n]} I_j[f]$, which can also be written as

$$I[f] = \sum_{S \subseteq [n]} \hat{f}(S)^2 |S|.$$ 

Assume that $\|f\|_2 = 1$. Then, $\sum_S \hat{f}(S)^2 = 1$ and the Fourier entropy of the Boolean function $f$ is defined as

$$H[f] = \sum_{S \subseteq [n]} |\hat{f}(S)|^2 \log \frac{1}{|\hat{f}(S)|^2},$$

and the min Fourier entropy $H_{\infty}$ is defined as

$$H_{\infty}[f] = \min_{S \subseteq [n]} \log \frac{1}{|\hat{f}(S)|^2}.$$ 

One of most important open problems in the analysis of Boolean functions is proving the Fourier entropy-influence (FEI) conjecture that was proposed by Friedgut and Kalai [52].

Conjecture 54 (FEI conjecture). There exists a universal constant $c$ such that, for all $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$,

$$H[f] \leq c I[f].$$

(85)

A natural extension of the FEI conjecture is the following Fourier min-entropy-influence conjecture, which follows from the fact that $H_{\min}[f] \leq H[f]$.

Conjecture 55 (FMEI conjecture). There exists a universal constant $c$ such that, for all $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$,

$$H_{\min}[f] \leq c I[f].$$

(86)

Although both the FEI and FMEI conjectures remain open, several significant steps have been made to prove these conjectures; see [81–88].

APPENDIX C. DISCRETE WIGNER FUNCTION AND SYMPLECTIC FOURIER TRANSFORMATION

We introduce some basics on the Fourier analysis of the discrete Wigner function. The discrete Wigner function was proposed for the odd-dimensional case, and one well-known result for odd-dimensional discrete Wigner functions is the discrete Hudson theorem, which states that any given pure state is a stabilizer state if and only if its Wigner function is nonnegative [89]. Here, we generalize the definition of the discrete Wigner function to the qubit case, where the discrete Hudson theorem may not hold.

Let us define the generalized phase point operator as follows

$$A_{\vec{a}} = \sum_{\vec{b}} P_{\vec{a}}(-1)^{\langle \vec{a}, \vec{b} \rangle} s,$$

(87)
where \( P_{\vec{b}} \) is an \( n \)-qubit Pauli operator and \( \langle \vec{a}, \vec{b} \rangle_s \) denotes the symplectic inner product. Hence, given an observable \( O \) (or a quantum state), the (generalized) discrete Wigner function \( f \) is defined as follows

\[
f_O(\vec{a}) = \langle O, A_{\vec{a}} \rangle,
\]
which can also be written as follows

\[
f_O(\vec{a}) = \sum_{\vec{b}} \langle P_{\vec{b}}, O \rangle (-1)^{\langle \vec{a}, \vec{b} \rangle_s} = \sum_{\vec{b}} O_{\vec{b}} (-1)^{\langle \vec{a}, \vec{b} \rangle_s}.
\]

Hence, the Pauli coefficient \( O_{\vec{b}} \) is the symplectic Fourier transform of the discrete Wigner function, i.e.,

\[
\hat{f}_O(\vec{b}) = \mathbb{E}_{\vec{a}} f_O(\vec{a}) (-1)^{\langle \vec{a}, \vec{b} \rangle_s}.
\]

Parseval’s identity tells us that

\[
\mathbb{E}_{\vec{a}} f(\vec{a})^2 = \sum_{\vec{b}} \hat{f}_O(\vec{b})^2 = \sum_{\vec{b}} |O_{\vec{b}}|^2.
\]

To consider the higher-order OTOC, we need to use the convolution of two observables. We define the convolution of two observables \( O_1 \) and \( O_2 \) as follows

\[
f_{O_1 * O_2} = f_{O_1} f_{O_2}.
\]

Hence

\[
\hat{f}_{O_1 * O_2}(\vec{b}) = \mathbb{E}_{\vec{a}} f_{O_1}(\vec{a}) f_{O_2}(\vec{a}) (-1)^{\langle \vec{a}, \vec{b} \rangle_s}.
\]

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