Boundary expansions of complete conformal metrics with negative Ricci curvatures

Yue Wang

Received: 16 May 2017 / Accepted: 31 March 2021 / Published online: 29 June 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract

We study the boundary behaviors of a complete conformal metric which solves the $\sigma_k$-Ricci problem on the interior of a manifold with boundary. We establish asymptotic expansions and also $C^1$ and $C^2$ estimates for this metric multiplied by the square of the distance in a small neighborhood of the boundary.

1 Introduction

Let $(M^n, \partial M, g)$ be a smooth Riemannian manifold with boundary and $1 \leq k \leq n$. We consider the following problem:

$$\sigma_k[g^{-1}Ric(e^{2u}g)] = (n-1)^k C^n_k e^{2ku} \quad \text{in } M \setminus \partial M, \quad (1.1)$$

$$u = \infty \quad \text{on } \partial M, \quad (1.2)$$

where $C^n_k = \binom{n}{k}$, $Ric(e^{2u}g)$ is the Ricci curvature of the conformal metric $e^{2u}g$, and $\sigma_k(A)$ is the $k$-th elementary symmetric polynomial in the eigenvalues of the symmetric matrix $A$. Let $\Gamma^+_k$ be the connected component of the set $\{\sigma_k > 0\}$ which contains the positive definite cone.

Gursky, Streets, and Warren [17] proved that (1.1) and (1.2) admit a unique solution $u \in C^\infty(M \setminus \partial M)$ with an additional requirement that $-Ric(e^{2u}g) \in \Gamma^+_k$. Moreover, $e^{2u}g$ is a complete metric and

$$\lim_{x \to \partial M} (u + \log d) = 0, \quad (1.3)$$

where $d$ is the distance to $\partial M$. By comparing (1.1) with the equation in Theorem 1.4 [17], we note that a constant $(n-1)^k C^n_k$ is inserted in the right-hand side of (1.1). With the newly inserted constant factor, the constant term in the expansion (1.3) is zero. See calculation in Lemma 2.1 and conclusion in Theorem 1.1. Refer to Theorem 1.4 [17]. By solving this...
Dirichlet problem with “infinite boundary data”, they produced complete metrics with constant $\sigma_k$-curvature on manifolds with boundary. The case $k = 1$ of their result appeared in [3] and the existence of a complete metric of negative Ricci curvature with constant $\sigma_k$-Ricci curvature was shown in [16] with the assumption that the given background metric already has a negative Ricci curvature. Other related results on compact manifolds have been shown in [5, 8, 11, 18], and [22].

In this paper, we study further expansions of $u$ near the boundary. For brevity, we consider the case that $g$ is the standard Euclidean metric. Assume $\Omega \subseteq \mathbb{R}^n$ is a bounded smooth domain, for $n \geq 3$. For $f \in C^2(\Omega)$, define a symmetric matrix $A(f)$ by

$$A(f) = (n - 2)\nabla^2 f + \Delta f I_{n \times n} + (n - 2)(|\nabla f|^2 I_{n \times n} - \nabla f \otimes \nabla f),$$

where $I_{n \times n}$ is the identity $n \times n$ matrix. We are led to the following problem:

$$\sigma_k(A(u)) = (n - 1)^k C_n e^{2k u} \text{ in } \Omega, \quad u = \infty \text{ on } \partial \Omega,$$

with the additional requirement that $A(u) \in \Gamma_k^+$. Set

$$e^{2u} = w^{4 \frac{n}{n-2}}.$$

For $k = 1$, (1.5) and (1.6) are reduced to the following more familiar form:

$$\Delta w = \frac{1}{4} n(n - 2) w^{\frac{n+2}{n-2}} \text{ in } \Omega, \quad w = \infty \text{ on } \partial \Omega.$$

Loewner and Nirenberg [21] proved the existence of the unique positive solution of (1.7) and Aviles and McOwen [3] proved the same result for the corresponding equation in general manifolds. Andersson, Chruściel and Friedrich [2] and Mazzeo [23] established the polyhomogeneous expansions for the solutions. Graham [15] studied the renormalized volume expansion. He identified the first two renormalized volume coefficients and the information contained in the anomaly, namely, the difference of the renormalized volumes corresponding to different choices of conformal representatives, and proved the conformal invariance of the energy, the coefficient of the log-term in the volume expansion.

We now present our main results for (1.5) and (1.6). As in (1.3), we denote by $d(x)$ the distance function in $\Omega$ to $\partial \Omega$ with respect to the Euclidean metric and set

$$D_\delta = \{x \in \Omega | d(x) \leq \delta\}. \quad (1.8)$$

If $\partial \Omega$ is smooth, then $d$ is smooth in a sufficiently small neighborhood of $\partial \Omega$.

We have the following result for the expansions of $u + \log d$ up to the first log-term.

**Theorem 1.1** Assume that $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, for $n \geq 3$, and that $u$ is the solution of (1.5)–(1.6). Then, for $\delta_2$ sufficiently small,

$$|u(x) + \log d(x) - c_1(y(x))d(x) - \cdots - c_{n-1}(y(x))d^{n-1}(x) - c_{n,1}(y(x))d^n(x)\log d(x) - C d^n(x) \leq 0 \text{ in } D_{\delta_2}, \quad (1.9)$$

where $y(x)$ is the unique point on $\partial \Omega$ such that $|x - y(x)| = d(x)$, $C$ and $\delta_2$ are positive constants depending only on $\Omega$, $n$ and $k$, and $c_1, \cdots, c_{n-1}$ and $c_{n,1}$ are smooth functions on $\partial \Omega$. 
We note that \( c_1(y(x)), \ldots, c_{n-1}(y(x)) \) and \( c_{n,1}(y(x)) \) will be given by (2.20), (2.23), and (2.24). We can simply write (1.9) as
\[
|u + \log d - c_1 \circ y d - \cdots - c_{n-1} \circ y d^{n-1} - c_{n,1} \circ y d^n \log d| \leq Cd^n \quad \text{in } D_{\delta_2}.
\]

Next, we derive the \( C^1 \) and \( C^2 \) estimates for \( u + \log d \).

**Theorem 1.2** Assume that \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \), for \( n \geq 3 \), and that \( u \) is the solution of (1.5)–(1.6). Then,
\[
|\nabla (u + \log d - c_1 \circ y d)| \leq Cd^{n-\alpha} \quad \text{in } D_{\delta_3},
\]
where \( C \) and \( \delta_3 \) are positive constants depending only on \( \Omega \), \( n \) and \( k \), \( c_1 \circ y \) is the function in (2.20), \( \alpha = 1/2 \) when \( n = 3 \) and \( \alpha = 1 \) when \( n \geq 4 \).

**Theorem 1.3** Assume that \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \), for \( n \geq 8 \), and that \( u \) is the solution of (1.5)-(1.6). Then,
\[
|\nabla^2 (u + \log d - c_1 \circ y d)| \leq C \quad \text{in } D_{\delta_4},
\]
where \( C \) and \( \delta_4 \) are positive constants depending only on \( \Omega \), \( n \) and \( k \), and \( c_1 \circ y \) is the function in (2.20).

The paper is organized as follows. Since \( \sigma_k(A(u)) - (n-1)^k C_n e^{2k\alpha} \) is orthogonal invariant, we use principle coordinate system in our argument near the boundary. In Sect. 2, we prove the boundary expansion of \( u + \log d \). In Sect. 3 and 4, we derive the \( C^1 \) and \( C^2 \) estimates for \( u + \log d \), respectively. We provide a brief discussion of principle coordinates in the Appendix.

We would like to thank Professor Matthew Gursky for suggesting the problem and explaining the paper [17] and for his stimulating ideas. The paper was initiated while the author was visiting University of Notre Dame in 2015. We would also like to thank Professor Qing Han for helpful discussions. Last but not least, we would like to thank the editor and the anonymous referee for valuable suggestions.

## 2 Boundary expansions

In this section, we will prove Theorem 1.1. Specifically, we derive the boundary expansion near boundary involving all local terms by the maximum principle.

Consider a bounded smooth domain \( \Omega \) in \( \mathbb{R}^n \). We denote by \( \kappa_1, \ldots, \kappa_{n-1} \) the principal curvatures of \( \partial \Omega \) and set
\[
H_{\partial \Omega} = \kappa_1 + \cdots + \kappa_{n-1}.
\]
Note that we do not divide the sum above by \( n-1 \).

Let \( d \) be the distance function to \( \partial \Omega \). For any \( f \in C^2(\Omega) \), let \( A(f) \) be the matrix defined in (1.4). Consider the operators
\[
F(f) = \sigma_k(\lambda(A(f))) - (n-1)^k C_n e^{2kf}, \quad (2.1)
\]
and
\[
\tilde{F}(f) = F(f) d^{2k}. \quad (2.2)
\]
Hence, we write
\[
\tilde{F}(f) = \sigma_k(\lambda(\tilde{A}(f))) - d^{2k}(n - 1)^k C_n^k e^{2kf},
\]
where the matrix \( \tilde{A}(f) \) is given by
\[
\tilde{A}(f) = d^2 A(f).
\]

By [17], there exists a unique solution \( u \in C^\infty(\Omega) \) of (1.5)-(1.6). Then, \( F(u) = 0 \) in \( \Omega \). Theorem 1.1 concerns of the expansion of such a solution \( u \) up to a certain order near the boundary.

The proof of Theorem 1.1 consists of two steps. In the first step, we identify a function \( v \) near the boundary such that \( \tilde{F}(v) \) vanishes up to a certain order near the boundary. In the second step, we prove that such a \( v \) provides a good approximation of \( u \) up to the desired order.

In the following, for any \( x \in D_\delta \) with \( \delta \) small, we always denote by \( y(x) \) the unique point on \( \partial \Omega \) such that \( |x - y(x)| = d(x) \). We also denote by \( C \) a positive constant depending only on \( \Omega \), \( n \) and \( k \).

**Lemma 2.1** Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^n \), for \( n \geq 3 \). Then, there exist smooth functions \( c_1, \ldots, c_{n-1}, \) and \( c_{n,1} \) on \( \partial \Omega \) such that the function \( v \) defined by
\[
v(x) = -\log d(x) + c_1(y(x))d(x) + \cdots + c_{n-1}(y(x))d^{n-1}(x) + c_{n,1}(y(x)) d^n(x) \log d(x),
\]
satisfies, for any constant \( \tau \in (0, 1) \),
\[
\tilde{F}(v) = O(d^{n+\tau}).
\]

Note that there is no zeroth-order term in the expansion of \( v \) in (2.5).

**Proof** For smooth functions \( c_0, \ldots, c_{n-1}, \) and \( c_{n,1} \) on \( \partial \Omega \) to be determined, set, for \( i = 0, 1, \ldots, n-1 \),
\[
v_i(x) = -\log d(x) + c_0(y(x)) + c_1(y(x))d(x) + \cdots + c_i(y(x))d^i(x),
\]
and for \( i = n \),
\[
v_n(x) = -\log d(x) + c_0(y(x)) + c_1(y(x))d(x) + \cdots + c_{n-1}(y(x))d^{n-1}(x) + c_{n,1}(y(x))d^n(x) \log d(x)
\]
in \( D_\delta \) with \( \delta \) small.

We will determine \( c_0, \ldots, c_{n-1}, \) and \( c_{n,1} \) successively and finally prove
\[
\tilde{F}(v_n) = O(d^{n+\tau}).
\]

For any \( x_0 \in D_\delta \), we will carry out our computations in the principle coordinate \( (x', x_n) \) at \( y_0 = y(x_0) \). In this coordinate, \( x_0 \) is given by \( (0, x_n) \).

**Step 0.** We first determine \( c_0 \). To this end, we will expand \( \tilde{F}(v_0) \) in terms of \( d \) and identify the zeroth-order term in the expansion of \( \tilde{F}(v_0) \).

Recall that
\[
\tilde{F}(v_0) = \sigma_k(\lambda(\tilde{A}(v_0))) - d^{2k}(n - 1)^k C_n^k e^{2kv_0}.
\]
Since $\tilde{F}(v_0)$ is orthogonal invariant, we compute in the principle coordinate $(x', x_n)$ at $y_0 = y(x_0)$. By (2.4), for $m = 0, 1, \cdots, n$, $\tilde{A}(v_m) = (\tilde{A}_{ij}(v_m))$ satisfies

$$
\tilde{A}_{ij}^m \equiv \tilde{A}_{ij}(v_m) = d^2 A_{ij}(v_m) = d^2 [-(n-2)v_m \delta_{ij} + (n-2)(|\nabla v_m|^2 \delta_{ij} - \delta_i v_m \delta_j v_m)].
$$

(2.9)

We denote $\tilde{A}_{ij}(v_m)$ by $\tilde{A}_{ij}^m$ in this proof for convenience.

For the second term $d^{2k}(n-1)^k c_n^k e^{2k\lambda_0}$ in $\tilde{F}(v_0)$, we have

$$
d^{2k}(n-1)^k c_n^k e^{2k\lambda_0} = (n-1)^k c_n^k e^{2k\lambda_0}.
$$

(2.10)

To compute the first term $\sigma_k(\lambda(\tilde{A}(v_0)))$ in $\tilde{F}(v_0)$, we denote by $a, b \in \{1, \cdots, n-1\}$ the subscripts different from $n$ and write $d_i, d_{ij}$ for $\delta_i d, \delta_{ij} d$, respectively.

We compute at the point $(x', x_n) = (0, x_n)$. By (5.1) and (5.2), we have in principle coordinates, for $a \neq b \in \{1, \cdots, n-1\}$,

$$
d_{aa}(0, x_n) = -\kappa_a - \kappa_a^2 d(0, x_n) - \kappa_a^3 d^2(0, x_n) + O(d^3),
$$

$$
d_{an}(0, x_n) = d_{ab}(0, x_n) = d_{nn}(0, x_n) = 0,
$$

$$
d_{a}(0, x_n) = 1, d_{a}(0, x_n) = 0,
$$

(2.11)

where $\kappa_1, \cdots, \kappa_{n-1}$ are principal curvatures of $\partial \Omega$ at $y_0$. Next, for any smooth function $f$ defined on $\partial \Omega$, we note

$$
\nabla f(y(x)) \nabla d(x) = 0,
$$

(2.12)

which is straight by observing that for any $y_1 \in \partial \Omega$,

$$
f(y(x)) = f(y_1), \text{ for } x = y_1 + v(y_1)d(x),
$$

where $v(y_1)$ is the unit inner normal vector at $y_1$. Therefore, for smooth function $c_0(y(x))$ to be determined, we have

$$
(c_0 \circ y)_a(0, x_n) = 0, (c_0 \circ y)_{nn}(0, x_n) = 0.
$$

Hence, at $(0, x_n)$,

$$
(v_0)_n d = -1, \quad (v_0)_a d = (c_0 \circ y)_a d,
$$

$$
(v_0)_{nn} d^2 = 1, \quad (v_0)_{na} d^2 = (c_0 \circ y)_{na} d^2 = O(d^2),
$$

$$
(v_0)_{ab} d^2 = \kappa_a \delta_{ab} d + \kappa_a^2 \delta_{ab} d^2 + (c_0 \circ y)_{ab} d^2 + O(d^3).
$$

(2.13)

A straightforward calculation yields

$$
\tilde{A}_{aa}^0(0, x_n) = (n-1) + O(d),
$$

$$
\tilde{A}_{an}^0(0, x_n) = \tilde{A}_{ab}^0(0, x_n) = O(d),
$$

$$
\tilde{A}_{nn}^0(0, x_n) = (n-1) + O(d).
$$

(2.14)

Hence,

$$
\sigma_k(\lambda(\tilde{A}(v_0))(0, x_n) = C_n^k(n-1)^k + O(d).
$$

By combining with (2.10), we have, at $(0, x_n)$,

$$
\tilde{F}(v_0) = C_n^k(n-1)^k - (n-1)^k C_n^k e^{2k\lambda_0} + O(d).
$$
Therefore, we take
\[ c_0 \equiv 0 \quad \text{on } \partial \Omega, \]
and conclude
\[ \tilde{F}(v_0) = O(d). \]

**Step 1-Step 2.** Next, we compute \( c_1 \) and \( c_2 \) successively. To this end, we need to identify terms involving \( d \) and \( d^2 \) in the expansion of \( \tilde{F}(v_2) \) successively. Note that \( c_0 = 0 \).

Let \( v(y_1) \) be the unit inner normal vector to \( \partial \Omega \). For any smooth function \( f \) defined on \( \partial \Omega \), we have, for any \( y_1 \in \partial \Omega \) and \( x = y_1 + v(y_1) d(x) \),
\[ f(y(x)) = f(y_1). \]

Hence,
\[ \partial^k_n (f \circ y)(0, x_n) = 0. \quad (2.15) \]

By (2.11), at \((0, x_n)\), we have, for \( a \neq b \in \{1, \cdots, n - 1\} \),
\begin{align*}
(v_2)_a d &= -1 + c_1 \circ y d + 2c_2 \circ y d^2 + O(d^{a+2}), \\
(v_2)_{nn} d^2 &= 1 + 2c_2 \circ y d^2 + O(d^{a+2}), \\
(v_2)_{na} d^2 &= (c_1 \circ y)_{a} d^2 + O(d^{a+2}), \\
(v_2)_{aa} d^2 &= (c_1 \circ y)_{a} d^2 + O(d^{2+a}), \\
(v_2)_{aab} d^2 &= \kappa_a d + \kappa^2_a d^2 - \kappa_a c_1 \circ y d^2 + O(d^{a+2}), \\
(v_2)_{abbd} d^2 &= O(d^{a+2}),
\end{align*}
for any \( \alpha \in (0, 1) \), where the terms containing \( (c_1 \circ y)_{an} \) are included in \( O(d^{a+2}) \).

For the second term \( d^{2k} (n - 1)^k c_n^k e^{2kv_2} \) in \( \tilde{F}(v_2) \), we have
\begin{align*}
d^{2k} (n - 1)^k c_n^k e^{2kv_2} &= (n - 1)^k c_n^k (1 + 2k c_1 \circ y d + 2k c_2 \circ y d^2 + 2k^2 (c_1 \circ y)^2 d^2) \\
&+ O(d^{2+a}). \quad (2.17)
\end{align*}

By (2.16), a straightforward calculation at \((0, x_n)\) yields
\begin{align*}
\Delta v_2 d^2 &= 1 + H_{\partial \Omega} d + 2c_2 \circ y d^2 + \sum_a \kappa_a^2 d^2 - H_{\partial \Omega} c_1 \circ y d^2 + O(d^{a+2}), \\
|\nabla v_2| d^2 &= 1 - 2 c_1 \circ y d - 4 c_2 \circ y d^2 + (c_1 \circ y)^2 d^2 + O(d^{a+2}), \quad (2.18)
\end{align*}
where \( H_{\partial \Omega} = H_{\partial \Omega}(y_0) \) since we are computing on the line \((0, x_n)\). Another straightforward calculation at \((0, x_n)\) yields
\begin{align*}
\tilde{A}_{aa} &= (n - 1) + (n - 2) \kappa_a d + H_{\partial \Omega} d - 2(n - 2) c_1 \circ y d + O(d^2), \\
\tilde{A}_{an} &= \tilde{A}_{ab} = O(d^2), \\
\tilde{A}_{nn} &= (n - 1) + H_{\partial \Omega} d + O(d^2). \quad (2.19)
\end{align*}

Note that the lowest order term in the expansion of \( \tilde{A}_{ii} \) at \((0, x_n)\) is the constant \( n - 1 \). Hence,
\begin{align*}
\sigma_k (\lambda(\tilde{A}(v_2)))(0, x_n) &= C_n^k (n - 1)^k + C_{n-1}^k (n - 1)^{k-1} [(2n - 2) H_{\partial \Omega} d - 2(n - 2)(n - 1) c_1 \circ y d] + O(d^2).
\end{align*}
Recall
\[ \tilde{F}(v_2) = \sigma_k(\lambda(\tilde{A}(v_2))) - d^{2k}(n - 1)^k C_n^k e^{2kv_2}. \]

By combining with (2.17), we have, at (0, \( x_1 \)),
\[ \tilde{F}(v_2) = C_n^k(n - 1)^k \left( \frac{2k}{n} H_{\beta\Omega} d - 2(n - 2) \frac{k}{n} c_1 \circ y d + 1 - 1 - 2k c_1 \circ y d \right) + O(d^2). \]

Note that \( c_1 \circ y \) has a nonzero coefficient. Therefore, we take
\[ c_1 = \frac{1}{2(n - 1)} H_{\beta\Omega} \quad \text{on } \partial\Omega. \]

By straightforward calculation, we find that
\[ \tilde{F}(v_2) = O(d^2). \]

Now we are ready to compute \( c_2 \). We identify the term involving \( d^2 \) in the expansion of \( \tilde{F}(v_2) \).

By straightforward calculation, we find that \( c_2 \circ y \) has a nonzero coefficient. Then, by substituting the value of \( c_1 \circ y \) into (2.16), (2.17) and (2.19), we can proceed similarly and take \( c_2 \) such that \( \tilde{F}(v_2) = O(d^{2+\tau}) \). The expression of \( c_2 \) is written in Proposition 2.3.

By the expressions of \( c_1 \) and \( c_2 \), we can see \( c_1 \) and \( c_2 \) are smooth functions on \( \partial\Omega \) and \( c_1 \circ y, c_2 \circ y \) are smooth functions in \( D_\delta \) with \( \delta \) small.

Step 1. Next, we compute \( c_i \), for \( i = 3, \ldots, n - 1 \). Take \( c_0, \ldots, c_{i-1} \) already obtained in Step 0 to Step \( i - 1 \). We will demonstrate that we will be able to compute \( c_i \) in terms of \( c_1, \ldots, c_{i-1} \) and their derivatives without providing an explicit expression. We will also prove that \( c_j \) is a smooth function on \( \partial\Omega \).

We use an induction argument to make it clear. Assume \( c_j, j = 0, \ldots, i - 1 \), already obtained are smooth functions on \( \partial\Omega \) depending only on \( n, k \) and the local geometry of \( \partial\Omega \) and \( c_j \circ y, j = 0, \ldots, i - 1 \) are smooth functions in \( D_\delta \) with \( \delta \) small. Note, if \( i = 3 \), this assumption holds by Step 0 to Step 2.

To compute \( c_i \) we need to identify the term involving \( d^i \) in the expansion of \( \tilde{F}(v_i) \).

Note the derivatives of \( c_i \circ y \) only appear in the terms involving \( d^l, l > i \) and the term involving \( d^j, j \leq i \) is determined only by \( c_0, \ldots, c_j \). Then we compute to see how \( c_i \circ y d^i \) appear in the expansion of \( \tilde{F}(v_i) \).

For the second term in \( \tilde{F}(v_i) \), we have
\[ d^{2k}(n - 1)^k C_n^k e^{2kv_i} = \cdots + (n - 1)^k C_n^k 2k c_i \circ y d^i + \cdots. \]

By (5.1) and (5.2), a straightforward calculation yields, at \( (0, x_n) \),
\[ (v_i)_{nn} d^2 = 1 + \cdots + i(i - 1)c_i \circ y d^i + \cdots, \]
\[ (v_i)_n d = -1 + \cdots + ic_i \circ y d^i + \cdots. \]

Hence, at \( (0, x_n) \),
\[ \tilde{A}^i_{aa} = \cdots + i(i - 1)c_i \circ y d^i - 2(n - 2)i c_i \circ y d^i + \cdots = \cdots + (i - 2n + 3)i c_i \circ y d^i + \cdots, \]
\[ \tilde{A}^i_{nn} = \cdots + (n - 1)i(i - 1)c_i \circ y d^i + \cdots. \]
Note the lowest order term in the expansion of $\tilde{A}_{ij}$, $l \in \{1, \ldots, n\}$ is the constant $n - 1$ by (2.19). Hence, at $(0, x_n)$,

$$\tilde{F}(v_i) = \sigma_k(\lambda(\tilde{A}(v_i))) - d^{2k}(n - 1)^kC^k_i e^{2k\nu_i}$$

$$= \cdots + (n - 1)^kC^k_{n-1}(n - 1)^k(i - 2n + 3)i c_i \circ y \, d^i$$

$$+ C^k_{n-1}(n - 1)^k(i - 1)c_i \circ y \, d^i - (n - 1)^kC^k_n2kci \circ y \, d^i + \cdots$$

$$= \cdots + 2(i - n)(i + 1)(n - 1)^kC^k_{n-1}c_i \circ y \, d^i + \cdots .$$

(2.21)

Therefore, the coefficient of $c_i \circ y(0, x_n)$ is not zero.

After computing on $(0, x_n)$, we compute in $N(y_0) \subset \mathbb{R}^n$, a small neighborhood of $y_0$. By smoothness of $d(x)$ and the assumption that $c_j \circ y, j = 0, \ldots, i - 1$ are smooth functions in $D_\delta$, we have, in $N(y_0) \cap \Omega$, the coefficient before $c_i \circ y$ is not zero.

Next, we consider other terms involving $d^j$ in the expansion of $\tilde{F}(v_i)$.

The derivatives of $c_i \circ y$ only appear in the terms involving $d^j, j > i$. Hence, except $c_i \circ y d^i$, only $c_j \circ y, j = 1, \ldots, i - 1$ and their derivatives appear in the term involving $d^i$. The calculation is more lengthy than that in Step 0 to Step 2. We only point out how we deal with terms like $(c_i \circ y)_{pq}, l = 1, \ldots, i - 1, p, q \in \{1, \ldots, n\}$. In fact, since $h = (c_l \circ y)_{pq}$ is a smooth function of $x$, we can expand it at $y(x)$ by the distance function $d(x)$. Then for any $m \in \mathbb{N}_+$,

$$h(x) = h(y(x)) + h_1(y(x))d(x) + \cdots + h_m(y(x))d^m + O(d^{m+1}(x)),$$

(2.22)

where $h_m(y(x))$ depends on local coordinates.

In sum, since $c_i \circ y d^i$ has a nonzero coefficient in $N(y_0) \cap \Omega$, by substituting the smooth functions $c_0 \circ y, c_1 \circ y, \ldots, c_i \circ y$, we can compute and determine a smooth function $(c_i \circ y)^{y_0}$ in $N(y_0) \cap \Omega$ such that $\tilde{F}(v_i) = O(d^{l+\tau}), \tau \in (0, 1)$. Thus, $(c_l \circ y)^{y_0}$ can be expressed as $(c_l \circ y)^{y_0} = G_i^{y_0}(c_1 \circ y, c_2 \circ y, \ldots, c_i \circ y, y)$, where $G_i^{y_0}$ is a smooth function obtained from the known functions $c_l \circ y$, $1 \leq l \leq i - 1$, and their derivatives. Correspondingly, we obtain a smooth function $v_l^{y_0}$ in $N(y_0) \cap \Omega$.

Now we provide a brief discussion here to show that $c_l$ is a smooth function on $\partial \Omega$.

By the same method, for any $x_1 \in N(y_0) \cap \Omega$, in the principle coordinate at $y_1 = y(x_1)$, we can obtain a smooth function $(c_l \circ y)^{y_1}(x)$. Since $v_l^{y_0}$ is smooth in $N(y_0) \cap \Omega$, we can expand $\tilde{F}(v_l^{y_0})$ is orthogonal invariant, we have

$$(c_l \circ y)^{y_0}(x_1) = (c_l \circ y)^{y_1}(x_1).$$

Since $x_0$ is arbitrarily chosen in $D_\delta$ and $x_1$ is arbitrarily chosen in $N(y_0) \cap \Omega$, we obtain a smooth function $c_l \circ y(x)$ defined globally in $D_\delta$, which is locally given by

$$(c_l \circ y)(x) = (c_l \circ y)^{y_0}(x), \quad x \in N(y_0) \cap \Omega.$$

Hence, $c_l \circ y$ can be expressed as

$$c_l \circ y = G_i(c_1 \circ y, c_2 \circ y, \ldots, c_{i-1} \circ y),$$

(2.23)

where $G_i$ is a smooth function obtained from $c_j \circ y, 1 \leq j \leq i - 1$, and their derivatives. Therefore, $c_i$ depends only on $n, k$ and the local geometry of $\partial \Omega$.

We point out that although the derivatives $(c_l \circ y)_{pq}$ are not defined globally, when we put back these derivatives into $\tilde{F}(v_{l})$, it turns out a globally defined quantity because $c_1 \circ y, c_2 \circ y, \ldots, c_{i-1} \circ y$ are defined globally and smooth, $\tilde{F}(v_{l})$ is orthogonal invariant and we can expand $\tilde{F}(v_{l})$ as a polynomial of $d(x)$. 

\begin{flushright}
\textcopyright \ Springer
\end{flushright}
Step n. Now we determine $c_{n,1}$.

In (2.21), $c_l o y d^l$ has a coefficient $(i - n)$ which is nonzero for $i \neq n$. Hence, we cannot compute out a term $c_n o y d^n$. This can be resolved by including a term $d^n \log d$ in the expansion. Therefore we set $v_n$ in the form (2.8).

In the principle coordinate at $y_0 \in \partial \Omega$, a straightforward calculation yields the coefficient of $d^n \log d$ in $\bar{F}(v_n)$ equals 0 and the term involving $d^n$ in $\bar{F}(v_n)$ is determined by $c_0, \cdots, c_{n-1}$, and $c_{n,1}$. By straightforward computation, the coefficient of $c_{n,1}$ is nonzero. Then proceeding similarly as in the previous steps, by requiring the coefficient of $d^n$ in $\bar{F}(v_n)$ to be zero, we obtain

$$c_{n,1} o y = G_n(c_1 o y, c_2 o y, \cdots, c_{n-1} o y),$$

(2.24)

where $G_n$ is a smooth function obtained from $c_l o y$, $1 \leq l \leq n - 1$, and their derivatives. As explained at the end of Step i, $c_{n,1}$ depends only on $n, k$ and the local geometry of $\partial \Omega$.

Finally, since the term involving $d^i$ in $\bar{F}(v_n)$ is determined only by $c_0, \cdots, c_i$, for $i \leq n-1$, and the term involving $d^n$ in $\bar{F}(v_n)$ is determined by $c_0, \cdots, c_{n-1}$ and $c_{n,1}$, we have, by the choice of $c_0, \cdots, c_{n-1}$ and $c_{n,1}$, $\bar{F}(v_n) = O(d^{n+\tau})$.

**Remark 2.2** (1) The functions $c_1, \cdots, c_{n-1}$ and $c_{n,1}$ defined by $c_1(y(x)), \cdots, c_{n-1}(y(x))$ and $c_{n,1}(y(x))$ in (2.20), (2.23) and (2.24) are functions on $\partial \Omega$. They are the coefficients of the so-called local terms, since they can be expressed explicitly. However, we cannot calculate out $c_n$ just by local geometry of $\partial \Omega$, i.e. $c_n$ is a global term, as explained in the proof.

(2) We point out that $c_0 \equiv 0$ is because a constant $(n - 1)^k C_n^k$ is inserted in the right-hand side of (1.1) and this constant is chosen just for this purpose.

(3) There is no $k$ in the expression of $c_1$ (See (2.20)), which is different from the expression of $c_2$ where $k$ appears. We will see the expression of $c_2$ in Proposition 2.3, which shows $c_2$ can be expressed as the sum of two parts, the first part independent of $k$ and the second part a good geometric quantity multiplied by $k$.

(4) There is a term $c_{n,1} d^n \log d$ in our expansion. We have shown the computability of $c_{n,1}$. However, if the readers want to see whether $c_{n,1}$ equals zero or for what kind of $\partial \Omega$, $n, k$, it equals zero, the readers can calculate it out by themselves following the computing procedure that we provided. Of course, the computing procedure will be very lengthy when $n$ becomes very large.

**Proposition 2.3** The function $c_2$ in (2.5) has the following expression:

$$c_2 = \frac{n}{6(n-2)} \left\{ \left( \frac{-3n + 2}{4n(n-1)} - \frac{n^3 - 3n - n^2 + 4}{2n(n-1)^2} \right) H^2_{\partial \Omega} + \left( \frac{2}{n} + \frac{(n-2)^2}{2n(n-1)^3} \right) |^{\circ} \Pi |^2 \right\} + k \left( \frac{(n-2)}{12(n-1)^3} \right)^{\circ} |^{\circ} \Pi |^2,$$

(2.25)

where $\Pi$ is the second fundamental form and $^{\circ} \Pi$ is the trace-free second fundamental form, i.e.,

$$^{\circ} \Pi = \Pi - \frac{1}{n-1} H_{\partial \Omega} g,$$

where $g$ is the Euclidean metric in our case and $H_{\partial \Omega} = k_1 + \cdots + k_{n-1}$.
Proof By (2.20) and the calculation described in Step 1-Step 2 in the proof of Lemma 2.1, we have
\[
\frac{6}{n}(n-2)c_2 = I + \hat{T},
\]
where
\[
I = -\frac{1}{(n-1)^3n} \left\{ \sum_{a<b} \left( \frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_a \right) - \frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_b \right\} + (n-2)\kappa_b + (n-1)H_{\partial\Omega}^2,
\]
\[
\hat{T} = \frac{k}{(n-1)^3n} \left\{ \sum_{a<b} \left( \frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_a \right) - \frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_b \right\} + (n-2)\kappa_b + (n-1)H_{\partial\Omega}^2.
\]
Set
\[
D = \sum_{a<b} \left( \frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_a \right) - \frac{1}{n-1} H_{\partial\Omega} + (n-2)\kappa_b + (n-1)H_{\partial\Omega}^2.
\]
Then,
\[
D = \frac{n^3 - 3n - n^2 + 4}{2(n-1)} H_{\partial\Omega}^2 - \frac{(n-2)^2}{2} \|\Pi\|^2,
\]
\[
\|\Pi\|^2 = \sum_a (\kappa_a - \frac{1}{n-1} H_{\partial\Omega})^2 = \|\Pi\|^2 - \frac{1}{n-1} H_{\partial\Omega}^2,
\]
and
\[
D - \frac{(n-1)n}{2} H_{\partial\Omega}^2 = - \frac{(n-2)^2}{2} (\|\Pi\|^2 - \frac{1}{n-1} H_{\partial\Omega}^2) = - \frac{(n-2)^2}{2} \|\Pi\|^2.
\]
Hence, we have (2.25). \qed

Before deriving the boundary expansion of $u$, we show a version of the maximum principle, which will be of use to us.

Theorem 2.4 Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$, for $n \geq 3$. Suppose $u$ and $v$ are smooth sub- and super-solutions, respectively, to (1.5)-(1.6) in $\Omega$ and $A(u), A(v) \in \Gamma_k^+$. If for any $p \in \partial\Omega$, $\lim_{x \to p}(u - v) \leq 0$ exists and $\lim_{x \to \partial\Omega}(u - v) \leq 0$, then $u \leq v$ in $\Omega$.

Proof Suppose that $u > v$ somewhere in $\Omega$. Let $C$ be the maximum of $u - v$, which is attained at some point $x_0 \in \Omega$. Then, $w = u - C$ is a strict sub-solution to (2.1). Hence at the point $x_0$, we have $w(x_0) = v(x_0)$ and $F(w)(x_0) > F(v)(x_0)$. Then,
\[
\sigma_k(\lambda(A(w)))(x_0) > \sigma_k(\lambda(A(v)))(x_0).
\]
However, $v \geq w$ near $x_0$. Therefore, we have $dw(x_0) = dv(x_0)$ and $(v - w)_ij(x_0) \geq 0$, and hence $A(w)(x_0) \leq A(v)(x_0)$. We use Lemma 3.1 in [18] and then obtain
\[
\sigma_k(\lambda(A(w)))(x_0) \leq \sigma_k(\lambda(A(v)))(x_0).
\]
This leads to a contradiction. \qed
According to Theorem 1.4 [17], the solution $u$ to (1.5)-(1.6) has the decay estimate (1.3). Now, we prove that the decay rate is actually $O(d)$.

**Lemma 2.5** Assume that $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, for $n \geq 3$ and that $u$ is the solution of (1.5)-(1.6). Then,

$$|u + \log d| \leq Cd \text{ in } D_{\delta_1},$$  \hspace{1cm} (2.26)

where $C$ and $\delta_1$ are positive constants depending only on $\Omega$, $n$ and $k$.

**Proof** By (1.3), we can take a small positive constant $\epsilon$ to be determined and then a small enough positive constant $\delta_0$ depending on $\epsilon$ such that

$$|u + \log d| \leq \epsilon \text{ in } D_{\delta_0}. \hspace{1cm} (2.27)$$

Set

$$\phi = -\log d + Cd.$$

Take a small positive constant $\delta_1 < \delta_0$ to be determined and set

$$C = \frac{\epsilon}{\delta_1}. \hspace{1cm} (2.28)$$

Then by (2.27), (2.28) and (1.3), we have

$$u \leq -\log d + Cd \text{ on } \partial D_{\delta_1},$$

and

$$Cd \leq \epsilon \text{ in } D_{\delta_1}. \hspace{1cm} (2.29)$$

We use principle coordinates in $D_{\delta_1}$ due to the orthogonal invariance of $F(f)$, $f \in C^2(D_{\delta_1})$. By (2.19) and (2.29), we have, when $\epsilon \ll 1$ and $\delta_1$ are small,

$$F(\phi) = \frac{1}{d^{2k}}(\sigma_k(\lambda(d^2A(\phi)))) - d^{2k}(n - 1)^k C_n^k e^{2k\phi}$$

$$= \frac{1}{d^{2k}}[(n - 1)^{k-1} C_n^{k-1}((2(n - 1))H_{\partial\Omega} - 2(n - 2)(n - 1)Cd]$$

$$- (n - 1)^k C_n^k 2kCd + O(\epsilon Cd)].$$

Hence, when $\delta_1$ and $\epsilon$ are small enough and thus $C$ big enough, we have $F(\phi) < 0$ in $D_{\delta_1}$. Here, by the definition of $C$ in (2.28), we know that the choices of $\delta_1$ and $\epsilon$ are independent. On the other hand, by (2.19) and (2.29), when $\delta_1$ and $\epsilon$ are small enough, we have

$$\sigma_k(\lambda(A(\phi))) = \frac{1}{d^{2k}}(\sigma_k(\lambda(d^2A(\phi)))) > 0.$$

Obviously, $A(\phi) \in \Gamma^+$. Therefore, by using Theorem 2.4, we have $u \leq \phi = -\log d + Cd$ in $D_{\delta_1}$. Similarly, we can prove $u \geq -\log d - Cd$ in $D_{\delta_1}$. \hfill \Box

Now, we can derive the boundary expansion of $u$ involving all local terms.

**Proof of Theorem 1.1** Take $\delta_2$ small to be determined such that $\delta_2 \leq \delta_1$, where $\delta_1$ is as in Lemma 2.5. Consider in $D_{\delta_2}$. For any fixed $\epsilon \in (0, 1)$, set

$$A = 2C\delta_2^{1-n},$$

$$q = n + \epsilon. \hspace{1cm} (2.30)$$
where $C$ is a large enough constant depending on the constant in (2.26) and $\partial \Omega$, $n$, $k$. By the definition of $A$, when $\delta_1$ is small,
\[
Ad^n - Ad^q \geq \frac{A}{2}d^n \geq Cd \quad \text{on } \partial D_{\delta_2},
\]
and
\[
Ad^n \leq 2C\delta_2 \quad \text{in } D_{\delta_2}. \tag{2.31}
\]
Hence, for a positive constant $\mu \ll 1$ to be determined, we can choose $\delta_2$ small such that
\[
Ad^n \leq \mu. \tag{2.32}
\]
Next, set
\[
\varphi = Ad^n - Ad^q,
\]
and
\[
\overline{v} = v + \varphi, \quad v = v - \varphi.
\]
where $c_1, \ldots, c_{n-1}$, and $c_{n,1}$ are the functions on $\partial \Omega$ and $v$ is defined in (1.9). Then by (2.6) and (2.32), a straightforward calculation yields, in $D_{\delta_2}$,
\[
\overline{F}(\overline{v}) = -C_{n-1}^k (n - 1)^k (2\varepsilon)(n + 1 + \varepsilon)Ad^{n+\varepsilon} + O(\mu Ad^{n+\varepsilon}). \tag{2.33}
\]
Choose $\delta_2$ small enough and thus $\mu$ small by (2.31) and $A$ large by (2.30). Then, $\overline{F}(\overline{v}) < 0$ and therefore $F(\overline{v}) = \frac{1}{d^n} \overline{F}(\overline{v}) < 0$ in $D_{\delta_2}$. Next by (2.32), we have $A(\overline{v}) \in \Gamma^+_k$, if $\delta_2$ is small. By the maximum principle Theorem 2.4, $u \leq \overline{v}$ in $D_{\delta_2}$. Similarly, we have $u \geq \overline{v}$ in $D_{\delta_2}$. Hence, we have the desired result. \qed

**Remark 2.6** $|u - v| \leq Cd^n$ is an estimate up to the order of the first global term and there is a logarithmic term in $v$. In fact, logarithmic terms appear in many problems, such as the singular Yamabe problem in [2,21] and [23], the complex Monge-Ampère equations in [9,10] and [14], and the asymptotically hyperbolic Einstein metrics in [1,4,6] and [19]. Fefferman [10] first observed that logarithmic terms should appear in the expansion.

### 3 The $C^1$-Estimates

In this section, we prove $C^1$ estimate for $u + \log d$ in a sufficiently small neighborhood of $\partial \Omega$ where $u$ is the solution to (1.5)–(1.6). In our argument, we sometimes use notation $O(g)$ where $g$ is a nonnegative function, which means if $h = O(g)$, then $|h| \leq Cg$ for some constant $C$ depending only on $\Omega$, $n$ and $k$ but independent of $g$.

**Lemma 3.1** Assume that $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, for $n \geq 3$, and that $u$ is the solution of (1.5)-(1.6). Then
\[
|\nabla (u + \log d - c_1 \circ y d)| \leq C \quad \text{in } D_{\delta_3},
\]
where $c_1 \circ y$ is the function in (2.20), and $C$ and $\delta_3$ are positive constants depending only on $\Omega$, $n$ and $k$. \qed
Proof} Take $\delta_2$ as the constant in Theorem 1.1 and $\tilde{c}_1$, $\psi \in C^\infty(\Omega)$ satisfying
\[ \tilde{c}_1 = c_1 \circ y, \psi = d \text{ in } D_{\frac{1}{2}\delta_2}, \]
and
\[ \psi \geq \frac{1}{2}\delta_2, \text{ in } \Omega \setminus D_{\frac{1}{2}\delta_2}, \]
where $c_1 \circ y$ is the function as given in (2.20). Set
\[ w = u + \log \psi - \tilde{c}_1 \psi. \]
We will prove for some $C_0 > 1$,
\[ \frac{|w|}{\psi^2} \leq C_0 \text{ in } \Omega. \tag{3.1} \]
First, by Theorem 1.1, we know (3.1) holds in $D_{\delta_2/2}$. Next, take
\[ j_1 = -\log\left(\frac{1}{2}\delta_2\right) + C\delta_2, \quad j_2 = -\log\left(\frac{1}{2}\delta_2\right) - C\delta_2. \]
By Remark 4.10 in [17], for $i = 1, 2$, respectively, we can solve
\[ F(u_{ji}) = 0 \text{ in } \Omega \setminus D_{\frac{1}{2}\delta_2}, \tag{3.2} \]
\[ u_{ji} = j_i \text{ on } \partial(\Omega \setminus D_{\frac{1}{2}\delta_2}). \tag{3.3} \]
By maximum principle and Lemma 2.5, we obtain
\[ u_{j_2} \leq u \leq u_{j_1} \text{ in } \Omega \setminus D_{\frac{1}{2}\delta_2}. \]
Hence, (3.1) holds in $\Omega \setminus D_{\frac{1}{2}\delta_2}$.

We rewrite the equation (1.5) as
\[ \sigma_k(\psi^2(\overline{A}(w - \log \psi + \tilde{c}_1 \psi))) = e^{2k\tilde{c}_1\psi} \left(\frac{n-1}{n-2}\right)^k C_n^k e^{2kw} = e^{2k\tilde{c}_1\psi} \beta_n \kappa e^{2kw} \text{ in } \Omega, \tag{3.4} \]
where under any Euclidean orthogonal coordinate,
\[ (\overline{A}(u))_{ij} = \partial_{ij}u + \frac{1}{n-2}\Delta u\delta_{ij} + |\nabla u|^2\delta_{ij} - \partial_i u \partial_j u. \tag{3.5} \]
Next we will use the definition and properties of Newton transformation. See [7,12,18,24].
We denote the $(k-1)$-Newton transformation associated with $\psi^2(\overline{A}(w - \log \psi + \tilde{c}_1 \psi))$ as $T_{k-1} \equiv T$, which is positive since $\psi^2 \overline{A} \in \Gamma_k^+$. In particular, if $A_{ij}$ are the components of a symmetric matrix $A$, then the $q$th Newton transformation associated with $A$ is
\[ T_q(A)^i_j = \frac{1}{q!}\delta_{ji}A^i_{1j}A^j_{1i} \cdots A^j_{iq}. \]
Here $\delta_{ji}A^i_{1j}A^j_{1i} \cdots A^j_{iq}$ is the generalized Kronecker delta symbol. We frequently use the following properties of $T_{k-1}(A)$:
\[ T_{k-1}(A)_{ij}A_{ij} = k\sigma_k(A); \]
\[ trT_{k-1}(A) = (n - k + 1)\sigma_{k-1}(A); \tag{3.6} \]
\[ \partial_m(\sigma_k(A)) = T_{k-1}(A)_{ij} \partial_m(A_{ij}). \]
Set
\[ Q_{ij} = T_{ij} + \frac{1}{n-2}T_{li}\delta_{ij}. \]
There is a summation in $l$. Then, $Q_{ij}$ is positive definite. Set

$$
\phi(s) = \frac{1}{p^2 (3C_0)^p} (2C_0 + s)^p,
$$

for some $p$ large to be determined and $C_0$ as in (3.1). Then,

$$
\frac{1}{p^2} \geq \phi(s) > 0 \quad \text{for any } s \in [-C_0, C_0].
$$

Set

$$
h = (1 + \frac{|\nabla w|^2}{2}) e^{\phi(\frac{w}{\psi^2})} = ve^{\phi(\frac{w}{\psi^2})}.
$$

We will prove, for some constant $C$,

$$
|h|_{L^\infty(\Omega)} \leq C.
$$

This implies the desired result.

First, for any point $y_0 \in \partial \Omega$, take a principal coordinate system $(x', x_n)$ at $y_0$ (See Appendix or [13]), with the unit inner normal vector $v$ in the $x_n$-direction. By Theorem 1.1, we know $w \equiv 0$ on $\partial \Omega$ and $w \leq Cd^2$ in $D_{\delta_2}$. Hence, $\nabla w(y_0) = 0$ and

$$
|\frac{\partial w}{\partial v}(y_0)| = |\lim_{d \to 0} w(0, d) - 0 - 0| = 0.
$$

Hence, $\nabla w(y_0) = 0$, implying $|h(y_0)| \leq C$.

Thus, without loss of generality, we can assume that the maximum of $h$ attains at a point $x_0 \in \Omega$. The proof is inspired by [18]. Assume $|\nabla w(x_0)|$ is sufficiently large. Otherwise the conclusion is immediate. All the calculation below is at the point $x_0$. For brevity, we write

$$
s = \frac{w}{\psi^2}.
$$

Differentiate $h$ twice. Since $Q_{ij}$ is positive definite, we have

$$
h_i = 0, \quad Q_{ij} h_{ij} \frac{\psi^4}{v e^\phi} \leq 0.
$$

Hence,

$$
\psi^4 Q_{ij} w_{ij} w_i = -v \phi'(s)(\frac{w}{\psi^2})_i, \quad (3.7)
$$

and

$$
\psi^4 Q_{ij} \psi_4 = (\phi''(s) - (\phi'(s))^2) Q_{ij} (\frac{w}{\psi^2})_i (\frac{w}{\psi^2})_j \psi^4 + \phi'(s) Q_{ij} (\frac{w}{\psi^2})_i \psi^4 \leq 0. \quad (3.8)
$$

By (3.1), we have

$$
\psi^4 \partial_i (\frac{w}{\psi^2}) \partial_j (\frac{w}{\psi^2}) = w_i w_j + O(|\nabla w|\psi) + \frac{4w^2 \psi_j}{\psi^2},
$$

$$
\psi^4 \partial_{ij} (\frac{w}{\psi^2}) = w_{ij} \psi^2 + O(|\nabla w|\psi + \psi^2).
$$
We will prove later $\phi''(s) - (\phi'(s))^2 > 0$. Then, (3.8) reduces to
\[
0 \geq \frac{1}{v} Q_{ij}(w_i w_j)^4 + (\phi''(s) - (\phi'(s))^2) Q_{ij}(w_i w_j + O(|\nabla w| \psi + 1)) \\
+ \phi'(s) Q_{ij}(w_i w_j^2 + O(|\nabla w| \psi + 1)).
\] (3.9)

By the properties in (3.6), we have
\[
Q_{ij}(w_i w_j)^2 = T_{ij}(\psi^2(\omega - \log \psi + \tilde{c}_1 \psi)_{ij} - \psi^2 |\nabla w|^2 \delta_{ij} \\
+ \psi^2 \partial_i w \partial_j w + O(|\nabla w| \psi + 1)) \\
= k \beta_{n,k} e^{2k\tilde{c}_1 \psi} e^{2k w} \\
+ T_{ij}(-\psi^2 |\nabla w|^2 \delta_{ij} + \psi^2 \partial_i w \partial_j w + O(|\nabla w| \psi + 1)).
\] (3.10)

Next, by applying $\partial_m$ to (3.4), we obtain
\[
T_{ij}(2\psi \partial_m \psi (\partial_i w + \frac{1}{n-2} \Delta w \delta_{ij} + |\nabla w|^2 \partial_i w \partial_j w) \\
+ \psi^2 (\partial_i \omega_m w + \frac{1}{n-2} \Delta w m \delta_{ij} - 2v \phi'(s) \partial_m (\frac{w}{\psi^2}) \delta_{ij} - \partial_i w \partial_j m w - \partial_i m w \partial_j w) \\
- 2\partial_i m w (\partial_i \psi - \partial_i (\tilde{c}_1 \psi) \psi^2 \delta_{ij} + 2\partial_j m w (\partial_j \psi - \partial_j (\tilde{c}_1 \psi) \psi^2) + O(1 + |\nabla w|)) \\
= 2k \beta_{n,k} e^{2k\tilde{c}_1 \psi} e^{2k w} \partial_m w + 2k \beta_{n,k} e^{2k\tilde{c}_1 \psi} e^{2k w} \partial_m (\tilde{c}_1 \psi).
\] (3.11)

We multiply (3.11) by $\frac{1}{v} \psi^2 \partial_m w$ and sum over $m$. Then by (3.6) and (3.7), we get
\[
\frac{1}{v} Q_{ij}(w_i w_j)^4 = 2 \frac{k \beta_{n,k} e^{2k\tilde{c}_1 \psi} e^{2k w} |\nabla w|^2 \psi^2 + O(1) \\
+ T_{ij}(2\phi'(s) \psi^2 |\nabla w|^2 \delta_{ij} - 2\phi'(s) \psi^2 w_i w_j + O(1 + |\nabla w| \phi'(s))).
\] (3.12)

Note $0 < \phi'\left(\frac{w}{\psi^2}\right), \phi''\left(\frac{w}{\psi^2}\right) < 1$ and substitute (3.10), (3.12) and
\[
Q_{ij} w_i w_j = T_{ij} w_i w_j + \frac{1}{n-2} T_{ij} |\nabla w|^2
\]
into (3.9). Then, we have
\[
0 \geq O(1) + T_{ij} ((\phi''(s) - (\phi'(s))^2) w_i w_j + (\phi''(s) - (\phi'(s))^2) \frac{1}{n-2} |\nabla w|^2 \delta_{ij} \\
+ 2\phi'(s) \psi^2 |\nabla w|^2 \delta_{ij} - 2\phi'(s) \psi^2 w_i w_j - \phi'(s) \psi^2 |\nabla w|^2 \delta_{ij} + \phi'(s) \psi^2 w_i w_j \\
+ O(1 + |\nabla w|)) \\
= O(1) + T_{ij} ((\phi''(s) - (\phi'(s))^2 - \phi'(s) \psi^2) w_i w_j \\
+ (\frac{1}{n-2} \phi''(s) - \frac{1}{n-2} (\phi'(s))^2 + \phi'(s) \psi^2) |\nabla w|^2 \delta_{ij} + O(1 + |\nabla w|))
\]

By the expression of $\phi$, we have, for a large constant $C$,
\[
\phi'\left(\frac{w}{\psi^2}\right) > \frac{1}{\rho 3 \rho C_0},
\]
and

$$\phi''\left(\frac{w}{\psi^2}\right) - (\phi')^2\left(\frac{w}{\psi^2}\right) - C\phi'\left(\frac{w}{\psi^2}\right) > \frac{1}{p3pC_0^2}(p - 1 - \frac{1}{p} - 3CC_0).$$

Fix \( p \) large enough. Then, we have, for some positive \( \epsilon \),

$$C \geq \epsilon T_{ij}w_iw_j + T_{ij}(2\epsilon |\nabla w|^2 \delta_{ij} + O(1 + |\nabla w|))$$

$$\geq \epsilon T_{ij}w_iw_j + T_{ij}(\epsilon |\nabla w|^2 \delta_{ij} + O(1)\delta_{ij}),$$

where we used the fact \( |T_{ij}|^2 \leq T_{ii}T_{jj} \). Take \( B \) large to be determined. (In fact, \( B \) is determined based on the quantities obtained in Case 2.) There are two cases to be discussed.

Case 1 The matrix \( \epsilon |\nabla w|^2 \delta_{ij} + O(1)\delta_{ij} \) has an eigenvalue less than \( B \). In this case, the gradient estimate is immediate.

Case 2 The matrix \( \epsilon |\nabla w|^2 \delta_{ij} + O(1)\delta_{ij} \) has all eigenvalues bigger than \( B \). By absorbing lower order terms, we have

$$C \geq \epsilon T_{ij}w_iw_j + T_{ii}B \geq T_{ii}.$$

By (3.6), we have for \( k \geq 2, \sigma_{k-1} \leq C \), independent of \( B \). Then by Proposition 4.2 in [18] and [20], (3.6) and the positive lower bound for \( \sigma_k \), we have the eigenvalues are bounded and hence, combining the positive lower bound for \( \sigma_k \) and (3.6), \( T_{ii} \geq \frac{1}{p} \) where \( C \) is independent of \( B \). (See explanation after Proposition 4.2 in [18].) Therefore, we can fix \( B \) large enough to get a contradiction. For \( k = 1 \), since \( (T_{ii})_{ij} = \delta_{ij} \), (3.13) gives the gradient bound. (See explanation after Proposition 4.2 in [18].)

Then, we have \( |\nabla w|^2(x_0) \leq C \). This finishes the proof. \( \square \)

We now improve Lemma 3.1 under the same assumption.

**Proof of Theorem 1.2** Take \( \alpha \) as in Theorem 1.2, \( \psi \) as in the proof of Lemma 3.1 and \( \tilde{c}_1, \ldots, \tilde{c}_{n,1} \in C^\infty(\Omega) \) satisfying

$$\tilde{c}_1 = c_1 \circ y, \ldots, \tilde{c}_{n,1} = c_{n,1} \circ y \text{ in } D_{\delta_2}/2,$$

where \( c_i \circ y, i = 1, \ldots, n - 1, \) and \( c_{n,1} \circ y \) are functions as in (2.20), (2.23) and (2.24) and we rewrite the constant \( \delta_3 \) in Lemma 3.1 as \( \delta_2 \).

Set

$$f = \tilde{c}_1\psi + \cdots + \tilde{c}_{n,1}\psi^n \log \psi,$$

and

$$w = u + \log \psi - f.$$

First, we will prove, for some \( C_0 > 1 \),

$$\frac{|w|}{\psi^{2+2\alpha}} \leq C_0 \text{ in } \Omega.$$

By Theorem 1.1, (3.15) holds in \( D_{\delta_2}/2 \). We point out that, in order to apply Theorem 1.1, we require \( 2 + 2\alpha \leq n \), which results in the choice of \( \alpha \) in the statement of Theorem 1.2. Next, using \( u_{j_1} \) and \( u_{j_2} \), obtained in the proof of Lemma 3.1, we know that (3.15) holds in \( \Omega \setminus D_{\delta_2}/2 \).

We rewrite the equation (1.5) as

$$\sigma_k(\psi^2(\overline{A}(w - \log \psi + f))) = e^{2kf} \left(\frac{n-1}{n-2}\right) \beta_{n,k} e^{2kw} \text{ in } \Omega,$$

(3.16)
where \((\overline{A}(u))_{ij}\) is as in (3.5) and \(f\) is as in (3.14). We use \(T_{k-1} \doteq T\) for \((k-1)\)-Newton transformation associated with \(\psi^2\overline{A}(w - \log \psi + f)\), which is positive since \(\psi^2\overline{A} \in \Gamma_k^+\).

Set
\[
Q_{ij} = T_{ij} + \frac{1}{n - 2} T_{ll} \delta_{ij}.
\]
Then, \(Q_{ij}\) is positive definite by \[18\]. By the properties in (3.6), we have
\[
Q_{ij}(w_{ij} \psi^{2}) = T_{ij}(\psi^{2}(w - \log \psi + f)_{ij} - \psi^{2}\Delta_{ij} + \psi^{2}\partial_{i} w \partial_{j} w + O(\|\nabla w\|\psi + 1)).
\]
and hence
\[
Q_{ij} w_{ij} \geq T_{ij}(-\|\nabla w\|^2 \delta_{ij} + \partial_{i} w \partial_{j} w + O(\|\nabla w\|\psi + 1)). \quad (3.17)
\]
Set
\[
\phi(s) = \left\{ \begin{array}{ll}
\frac{1}{p^2 (3C_0)^{p}} (2C_0 + s)^{p}, & \text{for some } p \text{ large to be determined and } C_0 \text{ in } (3.15),
\end{array} \right.
\]
for any \(s \in [-C_0, C_0]\). Then,
\[
\frac{1}{p^2} \geq \phi(s) > 0
\]
Set
\[
h = (1 + \frac{1}{2} \|\nabla (\frac{w}{\psi^{a}})\|^2) e^{\phi(\frac{w}{\psi^{2+2\alpha}})} \doteq ve^{\phi(\frac{w}{\psi^{2+2\alpha}})}.
\]
We will prove, for some constant \(C\),
\[
|h|_{L^\infty(\Omega)} \leq C.
\]
This would imply the desired conclusion.

First, for an arbitrary point \(y_0 \in \partial \Omega\), take a principal coordinate system \((x', x_n)\) at \(y_0\) and we can argue similarly as in the proof of Lemma 3.1. Note that \(w\) defined in this proof satisfies \(\frac{w}{\psi^{a}} \equiv 0 \text{ on } \partial \Omega\) and \(\frac{w}{\psi^{a}} \leq Cd^{2+\alpha} \text{ in } D_{\delta_2}\). Then, \(\nabla_x(\frac{w}{\psi^{a}})(y_0) = 0 \text{ on } \partial \Omega\) and
\[
|\frac{\partial}{\partial \nu} (\frac{w}{\psi^{a}})(y_0)| = |\lim_{d \to 0} \frac{(\frac{w}{\psi^{a}})(0, d) - 0}{d - 0}| = 0.
\]
Hence, \(\nabla(\frac{w}{\psi^{a}})(y_0) = 0\), implying \(|h(y_0)| \leq C\).

Thus, without loss of generality, we can assume that the maximum of \(h\) attains at a point \(x_0 \in \Omega\). The proof is inspired by [18]. Take \(A\) large to be determined. Without loss of generality, we assume \(|\nabla(\frac{w}{\psi^{a}})(x_0)| \geq A\) is sufficiently large. Otherwise the conclusion is obvious. All calculation below is at \(x_0\). For brevity, we write
\[
s = \frac{w}{\psi^{2+2\alpha}}.
\]
By differentiating \(h\) once, we have \(h_i = 0\) and hence
\[
(\frac{w}{\psi^{a}})_{li}(\frac{w}{\psi^{a}}) = -v\phi'(s)(\frac{w}{\psi^{2+2\alpha}})_{li}. \quad (3.18)
\]
Using (3.15), we have
\[
(w)_{i} = \frac{w_{i}}{\psi^{\alpha}} + O(\psi^{\alpha+1}),
\]
\[
∂_{ij}(w)_{i} = \frac{w_{ij}}{\psi^{\alpha}} + O(\frac{1}{\psi^{\alpha+1}}),
\]
\[
∂_{ij}(w)_{2\alpha+2} = \frac{w_{ij}}{\psi^{2\alpha+2}} + O(\frac{1}{\psi^{2\alpha+3}} + \frac{1}{\psi^{2}}).
\]

Apply ∂_{m} to (3.16) and then by Lemma 3.1, we have
\[
Q_{ij}\left(\frac{w}{\psi^{2}}\right)_{ijm} = T_{ij}\{-2(w)_{im}\frac{w}{\psi^{\alpha}}\psi_{ij} + (\frac{w}{\psi^{\alpha}})\psi_{ij} + (\frac{w}{\psi^{\alpha}})im\frac{w}{\psi^{\alpha}}i\psi_{ij}
\]
\[
+ (\frac{w}{\psi^{\alpha}})imO(\frac{1}{\psi}) + 0(|∇(\frac{w}{\psi^{\alpha}})|\frac{1}{\psi^{2}} + \frac{1}{\psi^{3+\alpha}} + |∇(\frac{w}{\psi^{\alpha}})|^2\frac{1}{\psi^{1-\alpha}})
\]
\[
+ Q_{ij}w_{ij}O(\frac{1}{\psi^{1+\alpha}}) + O(\frac{1}{\psi^{3+\alpha}} + |∇(\frac{w}{\psi^{\alpha}})|\frac{1}{\psi^{2}}).
\]

(3.19)

Next, differentiate h one more time. Since Q_{ij} is positive definite, we have 0 ≥ Q_{ij}h_{ij} \frac{1}{\psi} and hence
\[
0 \geq \frac{1}{v}Q_{ij}(\frac{w}{\psi^{2}})_{ij} + (\phi''(s) - (\phi'(s))^2)Q_{ij}(\frac{w}{\psi^{2+2\alpha}})_{i} \frac{w}{\psi^{2+2\alpha}}_{j}
\]
\[
+ \phi'(s)Q_{ij}(\frac{w}{\psi^{2+2\alpha}})_{ij}.
\]

(3.20)

We sum (3.19) with \frac{1}{v}(\frac{w}{\psi^{2}})m. Note 0 < \phi'(\frac{w}{\psi^{2}}), \phi''(\frac{w}{\psi^{2}}) < 1. Then,
\[
\frac{1}{v}Q_{ij}(\frac{w}{\psi^{2}})_{ijm} = T_{ij}\{2\phi'(s)\psi^{2+2\alpha}(|∇(\frac{w}{\psi^{2+2\alpha}})|^2\psi^{ij} - (\frac{w}{\psi^{2+2\alpha}})_{i}(\frac{w}{\psi^{2+2\alpha}})_{j})
\]
\[
+ \phi'(s)|∇(\frac{w}{\psi^{2+2\alpha}})|O(\frac{1}{\psi^{1+\alpha}}) + O(\frac{1}{\psi^{3+\alpha}} + \frac{1}{\psi^{3+\alpha}} + |∇(\frac{w}{\psi^{\alpha}})|\frac{1}{\psi^{1-\alpha}})
\]
\[
+ Q_{ij}w_{ij}O(\frac{1}{\psi^{1+\alpha}}) + O(\frac{1}{\psi^{3+\alpha}} + \frac{1}{\psi^{2}}).
\]

(3.21)

Note \phi'(\frac{w}{\psi^{2}}) > \frac{1}{\psi^{3+\alpha}}C_{\psi} and we will prove later \phi''(s) - (\phi'(s))^2 > 0. Then by (3.17), (3.21) and Lemma 3.1, (3.20) reduces
\[
0 \geq O(\frac{1}{\psi^{3+\alpha}} + \frac{1}{\psi^{2}})
\]
\[
+ T_{ij}\{((\phi''(s) - (\phi'(s))^2)\frac{1}{n-2} + 2\phi'(s)\psi^{2+2\alpha}|∇(\frac{w}{\psi^{2+2\alpha}})|^2\psi^{ij}
\]
\[
+ ((\phi''(s) - (\phi'(s))^2) - 2\phi'(s)\psi^{2+2\alpha})(\frac{w}{\psi^{2+2\alpha}})_{i}(\frac{w}{\psi^{2+2\alpha}})_{j}
\]
\[
+ \phi'(s)|∇(\frac{w}{\psi^{2+2\alpha}})|O(\frac{1}{\psi^{1+\alpha}}) + O(\frac{1}{\psi^{4+2\alpha}})\}.
\]

(3.22)
Multiply (3.22) by $\psi^{4+2\alpha}$. By
\[
\left( \frac{w}{\psi^{2+2\alpha}} \right)_i = \left( \frac{w}{\psi^2} \right)_i \left( \frac{1}{\psi^{2+\alpha}} \right) + O\left( \frac{1}{\psi} \right),
\]
we have
\[
0 \geq O(1) + T_{ij} \left\{ \left( \phi''(s) - \left( \phi'(s) \right)^2 \right) \frac{1}{n-2} + 2\phi'(s)\psi^\alpha|\nabla \left( \frac{w}{\psi^\alpha} \right) |^2 \delta_{ij} 
\right.
\]
\[
+ \left( \left( \phi''(s) - \left( \phi'(s) \right)^2 \right) - 2\phi'(s)\psi^\alpha \right) \left( \frac{w}{\psi^\alpha} \right)_i \left( \frac{w}{\psi^\alpha} \right)_j 
\tag{3.23}
\]
\[
+ \left| \nabla \left( \frac{w}{\psi^\alpha} \right) \right| O(1) + O(1) \right\}.
\]
By the expression of $\phi$, for a large constant $C$, we have
\[
\phi'(\frac{w}{\psi^2}) > \frac{1}{p^{3p} C_0},
\]
\[
\phi''(\frac{w}{\psi^2}) - (\phi')^2(\frac{w}{\psi^2}) - C\phi'(\frac{w}{\psi^2}) > \frac{1}{p^{3p} C_0} (p - 1 - \frac{1}{p} - 3CC_0).
\]
Fix $p$ large enough. Then, we have, for some positive $\epsilon$,
\[
C \geq \epsilon T_{ij}(\frac{w}{\psi^\alpha})_i(\frac{w}{\psi^\alpha})_j + T_{ij} \left( 2\epsilon \left| \nabla \left( \frac{w}{\psi^\alpha} \right) \right|^2 + O(1) \right) \delta_{ij},
\]
where we used the fact $|T_{ij}|^2 \leq T_{ii} T_{jj}$. Take $B$ large to be determined and we consider two cases.

Case 1. If the matrix
\[
2\epsilon \left| \nabla \left( \frac{w}{\psi^\alpha} \right) \right|^2 \delta_{ij} + O(1) \delta_{ij}
\]
has an eigenvalue less than $B$, then the gradient estimate is immediate.

Case 2. Otherwise, absorbing lower order terms, we have
\[
C \geq \epsilon T_{ij}(\frac{w}{\psi^\alpha})_i(\frac{w}{\psi^\alpha})_j + BT_{ii}.
\]
We argue similarly as in the proof of Lemma 3.1. Then we have $|\nabla (\frac{w}{\psi^\alpha})|(\chi_0) \leq C$. $\square$

Remark 3.2 We emphasize again that the validity of (3.15) requires a relation of $\alpha$ and $n$. In fact, for a general $\alpha \geq \frac{1}{2}$ and $w$ defined above, when $n \geq 2 + 2\alpha$, we have
\[
|\frac{w}{\psi^{2+2\alpha}}| \leq C, \quad |\nabla \frac{w}{\psi^\alpha}| \leq C.
\]

4 The $C^2$-Estimates

In this section, we derive estimates of second derivatives.

Proof of Theorem 1.3 Take $w$, $\psi$ and $f$ as defined in the proof of Theorem 1.2. This proof is divided into two steps.
Step 1 We will prove that there exists a constant $C$, depending only on $\partial \Omega$, $n$ and $k$, such that

$$\Delta w \geq -C \quad \text{in } \Omega.$$  

We proceed to prove this in $D_{\delta_3/2}$, where $\delta_3$ is the constant in Theorem 1.2. The proof in $\Omega \setminus D_{\delta_3/2}$ is similar but easier.

By (1.5)–(1.6), Theorem 1.1 and noting $\psi = d$ in $D_{1/2}$, where $\delta_3$ is the constant in Theorem 1.2, we have, in $D_{1/2}$,

$$n \left( \frac{C_{kn}}{(C_n^k)^{1/k}} \lambda \left( \frac{1}{n-2} A(u) \right) \right) = \frac{1}{d^2} \left( \frac{n(n-1)}{n-2} \right) e^{2(w+f)}$$

$$= \frac{1}{d^2} \left( \frac{n(n-1)}{n-2} \right) (1 + 2c_1 \circ y d + O(d^2)).$$  

By the expression of $A(u)$ in (1.4), a straightforward calculation yields, in $D_{1/2}$,

$$\sigma_k \left( \lambda \left( \frac{1}{n-2} A(u) \right) \right) = (1 + \frac{n}{n-2}) \Delta u + (n-1)|\nabla u|^2$$

$$= (1 + \frac{n}{n-2}) \Delta w + \frac{1}{d^2} \left( \frac{n(n-1)}{n-2} \right) + \frac{c_1 \circ y 2n(n-1)}{d} + O(1).$$

(4.2)

where we used the fact that $-\Delta d = H_{\partial \Omega} \circ y + O(d)$ and the definition of $c_1 \circ y$ in (2.20).

By Maclaurin’s inequality, we have

$$\sigma_k \left( \lambda \left( \frac{1}{n-2} A(u) \right) \right) \geq \frac{n}{(C_n^k)^{1/k}} \sigma_k^{1/k} \lambda \left( \frac{1}{n-2} A(u) \right).$$

By combining with (4.1) and (4.2) and by a straightforward calculation, we have $\Delta w \geq -C$.

Step 2 Next, we will prove

$$\max_{y \in \mathbb{S}^{n-1}, p \in \Omega} \partial_p \partial_y w \leq C,$$

where $C$ is a positive constant depending only on $\Omega$, $n$ and $k$. The proof of this step is inspired by [18].

First, assume $n \geq 2 + 2\alpha$, for some $\alpha \geq 1$. Later on, we will take $\alpha = 3$ but we write it in the present form to demonstrate why we choose $\alpha = 3$. By Remark 3.2, we have

$$|\frac{w}{\psi^{2+2\alpha}}| \leq C, \quad |\frac{\nabla w}{\psi^\alpha}| \leq C.$$  

(4.3)

Hence, $\nabla w \equiv 0$ on $\partial \Omega$. Moreover, in principal coordinates at any boundary point $y_0$ with $e_n$ as the unit inner normal vector to $\partial \Omega$ at $y_0$, we have

$$\nabla_{x'} \nabla w(y_0) = 0,$$

and

$$|\nabla_n \nabla w(y_0)| \leq \lim_{d \to 0} \left| \frac{Cd - 0}{d} \right| = C.$$

Therefore, we obtain

$$|\nabla^2 w|_{L^\infty(\partial \Omega)} \leq C.$$  

(4.4)
Next, set
\[ h(p, \gamma) = \partial_\gamma \partial_\gamma w(p) + \Lambda \frac{\nabla w|^2}{\psi 2\alpha} (p) \quad \text{for} \quad (p, \gamma) \in \Omega \times \mathbb{S}^{n-1}, \quad (4.5) \]
where \( \Lambda \) is a constant to be determined. We will prove
\[ |h|_{L^\infty(\Omega \times \mathbb{S}^{n-1})} \leq C, \]
which implies the conclusion in Step 2 by (4.3).

Without loss of generality, we assume that the maximum of \( h \) attains at \((p, \gamma) \in \Omega \times \mathbb{S}^{n-1}\).
Otherwise, by (4.4), the conclusion is immediate. Then, by rotating coordinates at \( p \), we may assume \( \frac{\partial}{\partial x_1} = \gamma \). Set
\[ \tilde{h}(x) = h(x, \frac{\partial}{\partial x_1}) = w_{11} + \Lambda \frac{\nabla w|^2}{\psi 2\alpha}. \]
Without loss of generality, we can assume \( w_{11}(p) \geq 1 \). Otherwise, the desired result is immediate. Since \( p \) is the maximum point of \( \tilde{h} \), we have, at \( p \),
\[ 0 = \partial_i \tilde{h} = w_{11i} + \Lambda \frac{2w_kw_{ki}}{\psi 2\alpha} - 2\alpha \Lambda \frac{\psi_i \nabla w|^2}{\psi 2\alpha + 1}, \quad (4.6) \]
and
\[ 0 \geq \partial_{ij} \tilde{h} = w_{11ij} + \Lambda \frac{2w_kw_{ki}}{\psi 2\alpha} - 4\alpha \Lambda \frac{w_kw_k\psi_i}{\psi 2\alpha + 1} - 4\alpha \Lambda \frac{w_kw_k\psi_j}{\psi 2\alpha + 1} - 2\alpha \Lambda \frac{\psi_i \nabla w|^2}{\psi 2\alpha + 1} + 2(2\alpha + 1) \Lambda \frac{\psi_ij \nabla w|^2}{\psi 2\alpha + 2} + \Lambda \frac{2w_kw_{kij}}{\psi 2\alpha}. \quad (4.7) \]
All the calculation below is at \( p \). Recall from Sect. 3 that
\[ Q_{ij} = T_{ij} + \frac{1}{n-2} T_{il} \delta_{ij}. \]
Since \( Q_{ij} \) is positive definite, using (4.3) and (4.7), we have, at \( p \),
\[ 0 \geq Q_{ij} w_{11ij} + \Lambda \frac{2}{\psi 2\alpha} Q_{ij} w_{ki} w_{ki} + \Lambda O \left( \frac{1}{\psi \alpha} \right) Q_{ij} w_{kij} + \Lambda T_{ij} \delta_{ij} O \left( \frac{1}{\psi^2} + \frac{|\nabla w|^2}{\psi \alpha + 1} \right). \quad (4.8) \]
We write \( \overline{A}_{ij} = \overline{A}_{ij}(u) \) for convenience.

First, we consider the term \( Q_{ij} w_{kij} \). By (3.6) and (4.3), differentiating (3.16) with respect to \( x_m \), we have
\[ \frac{\partial_m (L.H.S.)}{\partial_m (R.H.S.)} = \partial_m \psi \frac{\partial}{\partial \gamma} ( \psi \overline{A}_{ij} ) = T_{ij} (2\psi \psi_m \overline{A}_{ij} + \psi^2 (\overline{A}_{ij})_m) = \psi^2 T_{ij} (\overline{A}_{ij})_m + k \sigma_k \psi \frac{2}{\psi}, \]
\[ \partial_m (R.H.S.) = \partial_m \psi (2k (f + w) \beta_{n,k}) = O(1). \]
Hence,
\[ T_{ij} (\overline{A}_{ij})_m = O\left( \frac{1}{\psi^3} \right). \quad (4.9) \]
On the other hand, substituting \( u = - \log \psi + w + f \) in \( \overline{A}_{ij}(u) \) and then by (4.3), we have
\[ Q_{ij} w_{ijm} = O\left( \frac{1}{\psi^3} \right) (1 + T_{ij} \delta_{ij}) + O\left( \frac{|\nabla^2 w|^2}{\psi} \right) T_{ij} \delta_{ij}. \quad (4.10) \]
Next, we consider the term $Q_{ij}w_{11ij}$. Set $\sigma = (\sigma_k)^{1/k}$. Then

$$\sigma(\lambda(\psi^2 A_{ij})) = e^{2(f+w)\beta_{n,k}}.$$  \hfill (4.11)

Differentiate (4.11) twice with respect to $x_1$ and compare the R.H.S. with the L.H.S. By (4.3) and the concavity of $\sigma$, we have

$$T_{ij}(\psi^2 A_{ij})_{11} \geq -C - Cw_{11}.$$  \hfill (4.12)

Substituting $u = -\log \psi + w + f$ in $A_{ij}(u)$ and then by (3.6), (4.3), (4.6) and (4.9), we have

$$Q_{ij}w_{11ij} \geq -C - Cw_{11}.$$  \hfill (4.13)

Substitute (4.10) and (4.13) into (4.8). Then, multiply (4.8) by $\psi^{3+\alpha}$. By (4.3), we have

$$0 \geq -C - C\psi^{\alpha+1}w_{11} + T_{ij} \delta_{ij} \left( \frac{1}{\psi^{\alpha-3}} \sum_{i,k} |w_{ki}|^2 + O((1 + \Lambda) + \psi^{\alpha+3} |\nabla^2 w|^2) \right)$$

$$+ (1 + \Lambda)|\nabla^2 w| \psi^2 + 2\Lambda \psi^{\alpha-3} T_{ij} w_{kj} w_{ki}.$$  \hfill (4.14)

Choose $\Lambda$ large enough and, without loss of generality, we may assume $\sum_{i,k} |w_{ki}|^2$ is large and much larger than $\Lambda$. Then we have, for a positive constant $c$,

$$C + C\psi^{\alpha+1}w_{11} \geq \sigma_{k-1}(\psi^2 A_{ij}) c \sum_{i,k} |w_{ki}|^2 \geq \sigma_{k-1}(\psi^2 A_{ij}) c w_{11}^2.$$  \hfill (4.14)

On the other hand, by Maclaurin’s inequality, we have

$$\sigma_{k-1} \geq \left( \frac{\sigma_k}{k} \right)^{\frac{k-1}{k}} \left( \frac{n}{k-1} \right).$$

Note that, for some positive $c_0$,

$$\sigma_k(\psi^2 A_{ij}) = e^{2k(f+w)}\beta_{n,k} > c_0,$$

where we used (4.3) and the definition of $f$. Then, we have, for some positive $c_1$,

$$\sigma_{k-1}(\psi^2 A_{ij}) > c_1.$$  \hfill (4.14)

Hence, (4.14) implies, for some positive constant $\epsilon_0$,

$$C + Cw_{11} \geq \epsilon_0 w_{11}^2.$$  \hfill (4.14)

Then, we draw the conclusion $w_{11} \leq C$ and finish the proof in Step 2.

Combining the two steps, we have the desired conclusion. \hfill \Box

5. Appendix

Principle coordinates are discussed in details in [13]. In this appendix, we collect several results from [13] most relevant to our study.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial \Omega$ and $d$ be the distance function to $\partial \Omega$. Set $\Gamma_{\mu} = \{ x \in \Omega | d(x) < \mu \}$ for sufficiently small $\mu$. 

\textcopyright Springer
According to Lemma 14.16 [13], for each point \( x \in \Gamma_\mu \), there exists a unique point \( y = y(x) \in \partial \Omega \) such that \( |x - y(x)| = d(x) \). The points \( x \) and \( y \) are related by \( x = y(x) + v(y(x))d(x) \), where \( v \) is the unit inner normal vector to \( \partial \Omega \).

Next, let \( x_0 \in \Gamma_\mu \) and \( y_0 \in \partial \Omega \) be related by \( |x_0 - y_0| = d(x_0) \). According to Lemma 14.17 [13], in terms of a principle coordinate system at \( y_0 \), we have

\[
D^2d(x_0) = \text{diag}\left( \frac{-\kappa_1}{1 - \kappa_1 d}, \cdots, \frac{-\kappa_{n-1}}{1 - \kappa_{n-1} d}, 0 \right),
\]

(5.1)

where \( \kappa_1, \cdots, \kappa_{n-1} \) are the principle curvatures of \( \partial \Omega \) at \( y_0 \). By the proof of Lemma 14.17 [13], we also have

\[
Dd(x_0) = v(y_0) = (0, 0, \cdots, 1).
\]

(5.2)

References

1. Anderson, M.: Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds. Adv. Math. 179, 205–249 (2003)
2. Andersson, L., Chruściel, P.T., Friedrich, H.: On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein’s field equations. Comm. Math. Phys. 149, 587–612 (1992)
3. Aviles, P., McOwen, R.C.: Complete conformal metrics with negative scalar curvature in compact Riemannian manifolds. Duke Math. J. 56, 395–398 (1988)
4. Biquard, O., Herzlich, M.: Analyse sur un demi-espace hyperbolique et poly-homogeneite locale, arXiv:1002.4106
5. Brooks, R.: A construction of metrics of negative Ricci curvature. J. Diff. Geom. 29(1), 85–94 (1989)
6. Chruściel, P., Delay, E., Lee, J., Skinner, D.: Boundary regularity of conformally compact Einstein metrics. J. Diff. Geom. 69, 111–136 (2005)
7. Caffarelli, L., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. Acta Math. 155(3–4), 261–301 (1985)
8. Chang, A., Gursky, M., Yang, P.: An a priori estimate for a fully nonlinear equation on four-manifolds. J. Anal. Math. 87, 151–186 (2002)
9. Cheng, S.-Y., Yau, S.-T.: On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman’s equation. Comm. Pure Appl. Math. 33, 507–544 (1980)
10. Fefferman, C.: Monge-Ampère equation, the Bergman kernel, and geometry of pseudoconvex domains. Ann. Math. 103, 395–416 (1976)
11. Zhiyong Gao, L., Yau, S.-T.: The existence of negatively Ricci curved metrics on three-manifolds. Invent. Math. 85(3), 637–652 (1986)
12. Garding, L.: An inequality for hyperbolic polynomials. J. Math. Mech. 8, 957–965 (1959)
13. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (1983)
14. Lee, J., Melrose, R.: Boundary behavior of the complex Monge-Ampère equation. Acta Math. 148, 159–192 (1982)
15. Robin Graham, C.: Volume renormalization for singular Yamabe metrics, arXiv:1606.00069
16. Guan, B.: Complete conformal metrics of negative Ricci curvature on compact manifolds with boundary. Int. Math. Res. Not. (2008)
17. Gursky, M., Streets, J., Warren, M.: Existence of complete conformal metrics of negative Ricci curvature on manifolds with boundary. Calc. Var. Partial. Differ. Equ. 41(1), 21–43 (2011)
18. Gursky, M., Viaclovsky, J.: Fully nonlinear equations on manifolds with negative curvature. Indiana Univ. Math. J. 52(2), 399–419 (2003)
19. Helliwell, D.: Boundary regularity for conformally compact Einstein metrics in even dimensions. Comm. P.D.E. 33, 842–880 (2008)
20. Yau, Y.L.: Some existence results for fully nonlinear elliptic equations of Monge-Ampère type. Comm. Pure Appl. Math. 43(2), 233271 (1990)
21. Lohkamp, J.: Metrics of negative Ricci curvature. Ann. of Math. (2) 140(3), 655–683 (1994)
22. Mazzeo, R.: Regularity for the singular Yamabe problem. Indiana Univ. Math. J. 40, 1277–1299 (1991)
24. Robert, C.: Reilly, On the Hessian of a function and the curvatures of its graph. Michigan Math. J. 20, 373–383 (1973)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.