Constructing the Primitive Roots of Prime Powers

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Abstract

We use only addition and multiplication to construct the primitive roots of $p^{k+1}$ from the primitive roots of $p^k$, where $p$ is an odd prime and $k \geq 2$.

1 Introduction

There is a well-known result (Lemma 2) which gives a construction for all the primitive roots of a given positive integer $n$ in terms of any one of them. This construction is based on exponentiation.

In contrast, Niven et al. take a different approach to constructing primitive roots. Their result (Lemma 4) gives an explicit construction of all the primitive roots of $p^2$ from the primitive roots of $p$, where $p$ is any prime. This alleviates the need for taking a single primitive root of $p^2$ and exponentiating it in order to get the remaining primitive roots. It also has the added benefit of shedding light on how the primitive roots of $p^2$ are arranged, in terms of the primitive roots of $p$. Furthermore, every primitive root constructed requires only a single addition and multiplication, so the construction is easy to do by hand.

In this article, we show how to construct the primitive roots of $p^{k+1}$ from the primitive roots of $p^k$, where $k \geq 2$, and $p$ is an odd prime. The construction builds on the above result of Niven et al. [3], and allows us to construct the primitive roots of arbitrary prime powers in terms of the primitive roots of the primes.

2 Preliminaries

Let $n$ be a positive integer, and suppose that $a$ is relatively prime to $n$. The \textit{order of $a$ mod $n$} is the smallest positive integer $m$ such that $a^m \equiv 1 \pmod{n}$, and we write $m = \text{ord}_n(a)$. The following result is used throughout this article.

\footnote{Powers of the even prime 2 do not have primitive roots in general. For nice proofs of this result, see [4] and [5].}
Lemma 1 ([2, 3]) If \( a \) and \( n \) are relatively prime, then \( a^k \equiv 1 \pmod{n} \) if and only if \( \text{ord}_n(a) \) divides \( k \).

Now Euler’s theorem says that \( a^{\varphi(n)} \equiv 1 \pmod{n} \), where \( \varphi(n) \) denotes the Euler totient function. It follows from Lemma 1 that \( \text{ord}_n(a) \) is always a positive divisor of \( \varphi(n) \), and hence cannot exceed \( \varphi(n) \). In fact \( \text{ord}_n(a) \) can actually reach \( \varphi(n) \), and if it does then \( a \) is called a primitive root of \( n \). For more information about orders and primitive roots, see [1], [2], [3], [4], [5], and [6].

It is easy to verify that every integer congruent to \( a \mod{n} \) has the same order as \( a \), so it suffices to consider single representatives from every congruence class of integers mod \( n \) which are relatively prime to \( n \). Niven et al. [3] call any such set of representatives a reduced residue system mod \( n \), and it is easy to verify that there are always \( \varphi(n) \) elements in any one of these sets. The reduced residue system we will be using in this article is the standard one, given by the set

\[
\{1 \leq a \leq n : \gcd(a, n) = 1\}. \tag{1}
\]

We refer to the number of primitive roots found in the set (1) above as the number of primitive roots of \( n \). It turns out that if \( n \) has primitive roots, then the number of primitive roots of \( n \) is exactly \( \varphi(\varphi(n)) \). This is a direct consequence of the following result.

Lemma 2 ([6, Corollary 8.4.1]) If \( g \) is a primitive root of \( n \), then the primitive roots of \( n \) are given by the numbers \( g^k \), where \( k \) is relatively prime to \( \varphi(n) \).

Note that Lemma 2 is precisely the result alluded to in Section 1 which allows us to construct all the primitive roots of \( n \) in terms of one of them.

The only positive integers which have primitive roots are 1, 2, 4, \( p^k \), and \( 2p^k \), where \( p \) is any odd prime and \( k \geq 1 \). Together with Lemma 2 the following classic results allow us to construct the primitive roots of all the numbers that have them in terms of the primitive roots of the primes.

Lemma 3 ([3, 5, 6])

1. If \( g \) is a primitive root of a prime \( p \), then at least one of (but not necessarily both of) \( g \) and \( g + p \) is a primitive root of \( p^2 \).

2. If \( p \) is any odd prime and \( g \) is a primitive root of \( p^2 \), then \( g \) is a primitive root of \( p^k \) for all \( k \geq 2 \).

3. If \( p \) is an odd prime, \( k \geq 1 \), and \( g \) is a primitive root of \( p^k \), then the odd number out of \( g \) and \( g + p^k \) is a primitive root of \( 2p^k \).

Part 1 of Lemma 3 is proved in [5, page 179] and [6, Theorem 8.9]. Part 2 is proved in [3, page 102] and [5, page 179], and Part 3 is proved in [6, Theorem 8.14]. Note that Parts 1 and 2 only give partial constructions of the primitive roots of prime powers; Lemma 2 then gives the remaining primitive roots.
3 An alternate construction

In this section we give an alternate construction by Niven et al. of the primitive roots of \( p^2 \). We also state the main theorem of this article, which gives a construction of the primitive roots of \( p^{k+1} \) for \( k \geq 2 \). We illustrate both of these results with a running example.

**Lemma 4 (Niven et al. [3, Theorem 2.39])** If \( p \) is prime and \( g \) is a primitive root of \( p \), then \( g + tp \) is a primitive root of \( p^2 \) for all values \( 0 \leq t \leq p - 1 \) except one. This exceptional value of \( t \) is given by the formula

\[
t = \frac{1 - g^{p-1}}{p} \left( (p - 1)g^{p-2} \right)^{-1},
\]

where \( t \) is understood to be reduced mod \( p \) if necessary, and \( ((p - 1)g^{p-2})^{-1} \) denotes the multiplicative inverse of \( (p - 1)g^{p-2} \), mod \( p \). Furthermore, all the primitive roots of \( p^2 \) can be constructed this way.

Let us use Lemma 4 to construct the primitive roots of \( 3^2 = 9 \). It is easily verified that 2 is the only (incongruent) primitive root of the prime 3, and applying Lemma 4 gives that \( 2 + 3t \) is a primitive root of \( 3^2 \) for all \( 0 \leq t \leq 2 \), except for

\[
t = \frac{1 - 3^{3-1}}{3} \left( (3 - 1)2^{3-2} \right)^{-1} = \frac{-3}{3} (4)^{-1} = -1 \cdot 1 = -1 \equiv 2 \pmod{3}.
\]

Thus the numbers

\[
2 + 3 \cdot 0 = 2 \quad \text{and} \quad 2 + 3 \cdot 1 = 5
\]

are primitive roots of \( 3^2 \), and \( 2 + 3 \cdot 2 = 8 \) is not a primitive root. Furthermore, as 2 is the only primitive root of 3, Lemma 4 implies that 2 and 5 are the only primitive roots of \( 3^2 \).

That was a very simple example; it is not at all hard to compute the primitive roots of \( 3^2 = 9 \) directly. Thus we give a slightly more sophisticated example: a construction of the primitive roots of \( 5^2 = 25 \). It is easy to verify that 2 and 3 are the only primitive roots of the prime 5. By Lemma 4 it follows that the numbers

\[
2 + 5t_1 \quad (0 \leq t_1 \leq 4)
\]

\[
3 + 5t_2 \quad (0 \leq t_2 \leq 4)
\]

are primitive roots of \( 5^2 = 25 \) for all \( 0 \leq t_1, t_2 \leq 4 \), except for the values

\[
t_1 = \frac{1 - 5^{5-1}}{5} \left( (5 - 1)2^{5-2} \right)^{-1} = \frac{-15}{5} \cdot (32)^{-1} = -3 \cdot 3 \equiv 1 \pmod{5}
\]

and

\[
t_2 = \frac{1 - 3^{5-1}}{5} \left( (5 - 1)3^{5-2} \right)^{-1} = \frac{-80}{5} \cdot (108)^{-1} = -16 \cdot 2 \equiv 3 \pmod{5}.
\]
Thus the following numbers are all primitive roots of \(5^2\):

\[
\begin{align*}
2 + 5 \cdot 0 &= 2 & 3 + 5 \cdot 0 &= 3 \\
2 + 5 \cdot 2 &= 12 & 3 + 5 \cdot 1 &= 8 \\
2 + 5 \cdot 3 &= 17 & 3 + 5 \cdot 2 &= 13 \\
2 + 5 \cdot 4 &= 22 & 3 + 5 \cdot 4 &= 23
\end{align*}
\]

As 2 and 3 are the only primitive roots of 5, it follows by Lemma \(\text{I}\) that we have constructed all the primitive roots of \(5^2\).

The main theorem of this article extends the construction of primitive roots given in Lemma \(\text{I}\) to arbitrary powers of an odd prime.

**Theorem 1** If \(p\) is an odd prime, \(k \geq 2\), and \(g\) is a primitive root of \(p^k\), then \(g + tp^k\) is a primitive root of \(p^{k+1}\) for all \(0 \leq t \leq p - 1\). Furthermore, all the primitive roots of \(p^{k+1}\) can be constructed this way.

As an illustration of Theorem \(\text{I}\), let us construct the primitive roots of \(3^3 = 27\) from the primitive roots of \(3^2 = 9\). We saw above that the primitive roots of \(3^2\) are 2 and 5. It follows by Theorem \(\text{I}\) that the numbers

\[
\begin{align*}
2 + 3^2 \cdot 0 &= 2 & 5 + 3^2 \cdot 0 &= 5 \\
2 + 3^2 \cdot 1 &= 11 & 5 + 3^2 \cdot 1 &= 14 \\
2 + 3^2 \cdot 2 &= 20 & 5 + 3^2 \cdot 2 &= 23
\end{align*}
\]

are all primitive roots of \(3^3\)—no exceptions. Furthermore, as 2 and 5 are the only primitive roots of \(3^2\), Theorem \(\text{I}\) implies that the numbers above make up all the primitive roots of \(3^3\). This can be verified by direct calculation.

Let us go one step further. We now know that the primitive roots of \(3^3\) are 2, 5, 11, 14, 20, and 23. Applying Theorem \(\text{I}\) again, it follows that the numbers

\[
\begin{align*}
2 + 3^3 \cdot 0 &= 2 & 2 + 3^3 \cdot 1 &= 29 & 2 + 3^3 \cdot 2 &= 56 \\
5 + 3^3 \cdot 0 &= 5 & 5 + 3^3 \cdot 1 &= 32 & 5 + 3^3 \cdot 2 &= 59 \\
11 + 3^3 \cdot 0 &= 11 & 11 + 3^3 \cdot 1 &= 38 & 11 + 3^3 \cdot 2 &= 65 \\
14 + 3^3 \cdot 0 &= 14 & 14 + 3^3 \cdot 1 &= 41 & 14 + 3^3 \cdot 2 &= 68 \\
20 + 3^3 \cdot 0 &= 20 & 20 + 3^3 \cdot 1 &= 47 & 20 + 3^3 \cdot 2 &= 74 \\
23 + 3^3 \cdot 0 &= 23 & 23 + 3^3 \cdot 1 &= 50 & 23 + 3^3 \cdot 2 &= 77
\end{align*}
\]

are all primitive roots of \(3^4 = 81\). Furthermore, Theorem \(\text{I}\) implies that these are all the primitive roots of \(3^4\). This can be checked by a direct calculation.

Finally, to complete the picture we recall a method which allows us to construct the primitive roots of \(2p^k\) from \(p^k\), where \(p\) is any odd prime and \(k \geq 1\).

**Lemma 5** \((\text{II, page 173})\) Let \(p\) be an odd prime, let \(k \geq 1\), and suppose that \(g\) is a primitive root of \(p^k\). Then \(g\) is a primitive root of \(2p^k\) if \(g\) is odd, and \(g + p^k\) is a primitive root of \(2p^k\) if \(g\) is even. Furthermore, every primitive root of \(2p^k\) can be constructed in this way.
From Lemma 5 it is clear that all the odd primitive roots of \(3^4\) are primitive roots of \(2 \cdot 3^4 = 162\), and adding \(3^4\) to the even primitive roots of \(3^4\) will give the remaining primitive roots of \(2 \cdot 3^4\). Thus the numbers

\[
\begin{align*}
5 & \quad 2 + 3^4 = 83 \\
11 & \quad 14 + 3^4 = 95 \\
23 & \quad 20 + 3^4 = 101 \\
29 & \quad 32 + 3^4 = 113 \\
41 & \quad 38 + 3^4 = 119 \\
47 & \quad 50 + 3^4 = 131 \\
59 & \quad 56 + 3^4 = 137 \\
65 & \quad 68 + 3^4 = 149 \\
77 & \quad 74 + 3^4 = 155
\end{align*}
\]

are all primitive roots of \(2 \cdot 3^4\), and Lemma 5 implies that there are no others.

4 Hensel’s Lemma

In order to complete the proof of Lemma 4, Niven et al. use a result known as Hensel’s Lemma. Hensel’s Lemma is used to show that there is exactly one value in the range \(0 \leq t \leq p - 1\) such that \(g + tp\) is not a primitive root of \(p^2\) (where \(g\) is a primitive root of the prime \(p\)). Furthermore, Hensel’s Lemma gives the formula (2) for this exceptional \(t\), and it is hard to see how this formula could arise by more direct means.

Hensel’s Lemma originated in the theory of \(p\)-adic numbers, but it also has an equivalent form in the theory of polynomial congruences. We will be using the latter form. In this section we recall some basic definitions from the theory of polynomial congruences, and we state Hensel’s Lemma. For more information about polynomial congruences and the corresponding version of Hensel’s Lemma, see [3] and [5].

Let \(f(x)\) be a polynomial with integer coefficients. Then a **polynomial congruence** is a congruence of the form

\[
f(x) \equiv 0 \pmod{n},
\]

where \(n\) is called the **modulus** of the congruence. An integer \(x_0\) satisfying \(f(x_0) \equiv 0 \pmod{n}\) is called a **solution** of the congruence (3).

In this article we will only be interested in polynomial congruences of prime power moduli. All of these congruences have the form

\[
f(x) \equiv 0 \pmod{p^k},
\]

where \(p\) is prime and \(k \geq 1\). Now it turns out (see [5] for details) that all the solutions of

\[
f(x) \equiv 0 \pmod{p^{k+1}}
\]

are all primitive roots of \(2 \cdot 3^4\), and Lemma 5 implies that there are no others.
can be constructed from the solutions of (4). Specifically, if $x_1$ is a solution of (5) then we can always write
\[ x_1 = x_0 + tp^k, \]
where $x_0$ is some solution to (4) and the integer $t$ is yet to be determined. Following [3], we say that the solution $x_1$ in (6) lies above the solution $x_0$, and that $x_0$ lifts to $x_1$.

**Lemma 6 (Hensel’s Lemma [3, 5])** Let $f(x)$ be a polynomial with integer coefficients, and suppose that $p$ is prime and that $k \geq 1$. Let $x_0$ be a solution to $f(x) \equiv 0 \pmod{p^k}$. Then exactly one of the following occurs:

1. If $f'(x_0) \not\equiv 0 \pmod{p}$ then $x_0$ lifts to exactly one solution $x_1$ to $f(x) \equiv 0 \pmod{p^{k+1}}$. This solution is given by $x_1 = x_0 + tp^k$, where
   \[ t = \frac{f(x_0)}{p^k} (f'(x_0))^{-1}. \]
   Here $t$ is understood to be reduced mod $p$ if necessary, and $(f'(x_0))^{-1}$ stands for the multiplicative inverse of $f'(x_0)$, mod $p$.

2. If $f'(x_0) \equiv 0 \pmod{p}$ and $x_0$ is a solution of $f(x) \equiv 0 \pmod{p^{k+1}}$, then $x_0$ lifts to $x_1 = x_0 + tp^k$ for all integers $0 \leq t \leq p - 1$. Thus $x_0$ lifts to $p$ distinct solutions of $f(x) \equiv 0 \pmod{p^{k+1}}$.

3. Finally, if $f'(x_0) \equiv 0 \pmod{p}$ but $x_0$ is not a solution of $f(x) \equiv 0 \pmod{p^{k+1}}$, then $x_0$ does not lift to any solutions of $f(x) \equiv 0 \pmod{p^{k+1}}$. Thus if $f(x) \equiv 0 \pmod{p^{k+1}}$ has any solutions at all, then they do not lie above $x_0$.

**5 Proof of the Main Theorem**

**Proof of Theorem**

First we show that for any $0 \leq t \leq p-1$, either $g + tp^k$ is a primitive root of $p^{k+1}$, or $\text{ord}_{p^{k+1}}(g + tp^k) = \varphi(p^k)$. Set $h = \text{ord}_{p^{k+1}}(g + tp^k)$. Then $(g + tp^k)^h \equiv 1 \pmod{p^{k+1}}$, and so $(g + tp^k)^h \equiv 1 \pmod{p^k}$. Now $g \equiv g + tp^k \pmod{p^k}$, so
\[ g^h \equiv (g + tp^k)^h \equiv 1 \pmod{p^k}. \]

This means that $h$ is a multiple of $\text{ord}_{p^k}(g)$, which is $\varphi(p^k)$ since $g$ is a primitive root of $p^k$. Hence we can write
\[ h = y\varphi(p^k) \]
for some positive integer $y$.

On the other hand, $h$ was defined to be the order of $g + tp^k$, mod $p^{k+1}$. Thus $h$ is a divisor of $\varphi(p^{k+1}) = p\varphi(p^k)$, so
\[ p\varphi(p^k) = xh \]

(8)
for some positive integer $x$. Putting Equations (7) and (8) together give that

$$p\varphi(p^k) = xh = xy\varphi(p^k),$$

which implies that

$$xy = p.$$ 

Since $p$ is prime and $y$ is positive, this implies that either $y = p$ or $y = 1$.

Together with (7), this information gives that either $h = p\varphi(p^k) = \varphi(p^{k+1})$ or $h = \varphi(p^k)$. In the former case we have that $g + tp^k$ is a primitive root of $p^{k+1}$, and in the latter case we have that $\text{ord}_{p^{k+1}}(g + tp^k) = \varphi(p^k)$.

Thus, to prove Theorem 1 it suffices to show that $\text{ord}_{p^{k+1}}(g + tp^k) \neq \varphi(p^k)$ for all $0 \leq t \leq p - 1$. Suppose that there was some $0 \leq t \leq p - 1$ such that $\text{ord}_{p^{k+1}}(g + tp^k) = \varphi(p^k)$. Then

$$(g + tp^k)^{\varphi(p^k)} \equiv 1 \pmod{p^{k+1}},$$

so $g + tp^k$ is a solution of the congruence $f(x) \equiv 0 \pmod{p^{k+1}}$, where $f(x) = x^{\varphi(p^k)} - 1$. As $g$ is a primitive root of $p^k$, $g$ is a solution of $f(x) \equiv 0 \pmod{p^k}$, and hence $g$ lifts to $g + tp^k$. Now $f'(x) = x^{\varphi(p^k) - 1}$, so we have that

$$f'(g) = \varphi(p^k) g^{\varphi(p^k) - 1} = (p^k - p^{k-1}) g^{\varphi(p^k) - 1} = p(p^{k-1} - p^{k-2}) g^{\varphi(p^k) - 1} \equiv 0 \pmod{p}.$$ 

Notice that the above fails if $k = 1$. Thus it is imperative that we have $k \geq 2$.

Finally, since $g$ is a primitive root of $p^2$, it follows from Part 2 of Lemma 3 that $g$ is a primitive root of $p^{k+1}$. Hence

$$g^{\varphi(p^k)} \not\equiv 1 \pmod{p^{k+1}},$$

since $\varphi(p^k) < \varphi(p^{k+1}) = \text{ord}_{p^{k+1}}(g)$. Thus $g$ is not a solution of $f(x) \equiv 0 \pmod{p^{k+1}}$, so Part 3 of Hensel’s Lemma applies. It follows that $g$ does not lift to any solutions of $f(x) \equiv 0 \pmod{p^{k+1}}$, which is a contradiction.

Therefore $\text{ord}_{p^{k+1}}(g + tp^k) \neq \varphi(p^k)$ for all $0 \leq t \leq p - 1$, so $g + tp^k$ is a primitive root of $p^{k+1}$ for all $0 \leq t \leq p - 1$. It only remains to show that we can construct all the primitive roots of $p^{k+1}$ in this way. As $t$ ranges from 0 to $p - 1$, $tp^k$ runs through all the multiples of $p^k$ from 0 to $(p - 1)p^k$. As all of these are distinct mod $p^{k+1}$, it follows that the numbers $g + tp^k$ are distinct mod $p^{k+1}$ for all $0 \leq t \leq p - 1$. Furthermore, if $g_1$ and $g_2$ are any primitive roots of $p^k$, and $0 \leq t_1, t_2 \leq p - 1$, then $g_1 + t_1p^k \equiv g_2 + t_2p^k \pmod{p^{k+1}}$ implies that $g_1 + t_1p^k \equiv g_2 + t_2p^k \pmod{p^k}$, and hence

$$g_1 - g_2 \equiv (t_1 - t_2)p^k \equiv 0 \pmod{p^k},$$

that is, $g_1 \equiv g_2 \pmod{p^k}$.
It follows that as \( g \) ranges through all the \( \varphi(p^k) \) primitive roots of \( p^k \), and as \( t \) ranges independently from 0 to \( p - 1 \), there will be \( p\varphi(p^k) \) distinct constructed primitive roots of \( p^{k+1} \). As it can be verified that \( p\varphi(p^k) = \varphi(p^{k+1}) \), it follows that we have constructed all the primitive roots of \( p^{k+1} \) exactly once. \[Q.E.D.\]

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