THE (2, 3)-GENERATION OF THE SPECIAL LINEAR GROUPS
OVER FINITE FIELDS

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ABSTRACT. We complete the classification of the finite special linear groups
$SL_n(q)$ which are (2, 3)-generated, i.e., which are generated by an involution
and an element of order 3. This also gives the classification of the finite simple
groups $PSL_n(q)$ which are (2, 3)-generated.

1. Introduction

It is a well known fact that every finite simple group can be generated by a pair
of suitable elements: for alternating groups this is a classical result of Miller [11],
for groups of Lie type it is due to Steinberg [17] and for sporadic groups it was
proved by Aschbacher and Guralnick [1]. A more difficult problem is to find for
a finite nonabelian (quasi)simple group $G$ the minimum prime $r$, if it exists, such
that $G$ is (2, $r$)-generated, i.e. such that $G$ can be generated by two elements of
respective orders 2 and $r$. We denote such minimum prime $r$ by $\omega(G)$ (setting
$\omega(G) = \infty$ if $G$ is not (2, $r$)-generated for any prime $r$). Since groups generated by
two involutions are dihedral, we must have $\omega(G) \geq 3$.

Miller himself proved that $\omega(\text{Alt}(n)) = 3$ if $n = 5$ or $n \geq 9$, while it is easy
to verify that $\omega(\text{Alt}(n)) = 5$ if $n = 6, 7, 8$. The special linear groups were firstly
considered in [21], where Tamburini showed that $\omega(SL_n(q)) = 3$ for all $n \geq 25$ and
all prime power $q$. Woldar [29] proved that all simple sporadic groups are (2, 3)-
generated, except for $M_{11}$, $M_{22}$, $M_{23}$ and McL, for which $\omega(G) = 5$. As proved
by Lübeck and Malle [9], all simple exceptional groups are (2, 3)-generated with
the only exception of the Suzuki groups $Sz(2^{2m+1})$, for which Suzuki himself [18]
proved that $\omega(Sz(2^{2m+1})) = 5$.

Hence, we are left to consider the finite simple classical groups. A key result
for such groups is due to Liebeck and Shalev, who proved in [8] that, apart from
the infinite families $\text{PSp}_4(2^m)$ and $\text{PSp}_4(3^m)$, all finite simple classical groups are
(2, 3)-generated with a finite number of exceptions. So, the problem of finding the
exact value of $\omega(G)$ reduces to classifying the exceptions to the Liebeck and Shalev’
theorem. However, their result relies on probabilistic methods and does not provide
any estimates on the number or the distribution of such exceptions. We remark
that King proved in [6] that $\omega(G) \neq \infty$ for all finite simple classical groups $G$, but
in general the problem of computing the exact value of $\omega(G)$ is still wide open (see
[14] for a recent survey on this topic).

In this paper we consider the projective special linear groups $PSL_n(q)$. Many
authors, such as Di Martino, Macbeath Tabakov, Tamburini and Vavilov, already

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We observe that the $(2,3)$-generation of $\text{SL}_n(q)$. Summarizing their results we have the following list of $(2,3)$-generated groups:

- $(i)$ $\text{PSL}_2(q)$ if $q \neq 9$ (see [10]);
- $(ii)$ $\text{SL}_3(q)$ if $q \neq 4$ (see [15]);
- $(iii)$ $\text{SL}_4(q)$ if $q \neq 2$ (see [16]);
- $(iv)$ $\text{SL}_n(q)$ if $5 \leq n \leq 11$ (see [13, 20, 19, 4, 5]);
- $(v)$ $\text{SL}_n(q)$ if $n \geq 13$ (see [22]);
- $(vi)$ $\text{SL}_n(q)$ if $n \geq 5$ and $q \neq 9$ is odd (see [2, 3]).

We observe that the $(2,3)$-generation of $\text{SL}_n(q)$ clearly implies the $(2,3)$-generation of $\text{PSL}_n(q)$.

Here, using a constructive approach as in many of the above papers and in particular the permutational method illustrated in [22], we solve the last remaining case, i.e. we prove the $(2,3)$-generation of $\text{SL}_{12}(q)$, obtaining the following classification.

**Theorem 1.1.** The groups $\text{PSL}_2(q)$ are $(2,3)$-generated for any prime power $q$, except when $q = 9$. The groups $\text{SL}_n(q)$ are $(2,3)$-generated for any prime power $q$ and any integer $n \geq 3$, except when $(n,q) \in \{(3,4),(4,2)\}$.

Observe that $\varpi(G) = 5$ if $G \in \{\text{PSL}_2(9) \cong \text{Alt}(6), \text{SL}_3(4), \text{PSL}_4(4), \text{PSL}_4(2) \cong \text{Alt}(8)\}$. Clearly $\text{SL}_2(q)$ cannot be $(2,r)$-generated when $q$ is odd, as the unique involution is the central one.

Regarding the $(2,3)$-generation of the other finite classical groups, we recall that only partial results are available, mainly concerning small or high dimensions, see [12, 15, 16, 23, 24, 25].

Finally, we recall that the infinite groups $\text{PSL}_n(\mathbb{Z})$ are $(2,3)$-generated if and only if either $n = 2$ or $n \geq 5$, and that the groups $\text{SL}_n(\mathbb{Z})$ are $(2,3)$-generated if and only if $n \geq 5$ (see [22, 26, 27, 28]).

2. The $(2,3)$-Generation of $\text{SL}_{12}(q)$

Let $q = p^n$, where $p$ is a prime and let $\mathbb{F}_q$ be the field of $q$ elements. Let $V$ be a 12-dimensional $\mathbb{F}_q$-space, that we identify with the row vectors of $\mathbb{F}_{q}^{12}$. Let $C = \{e_1, e_2, \ldots, e_{12}\}$ be the canonical basis of $V$. For any element $\sigma \in \text{Alt}(C)$, we write $g = g_\sigma$ to denote the permutation matrix $g \in \text{SL}_{12}(q)$ corresponding to $\sigma$ with respect to $C$. This allows us to consider $\text{Alt}(C)$ as a subgroup of $\text{SL}_{12}(q)$.

Now, let

(1) \[ y = (e_1, e_2, e_3)(e_4, e_5, e_6)(e_7, e_8, e_9)(e_{10}, e_{11}, e_{12}) \]

and let $x$ be the matrix, written with respect to $C$, such that:

- $(a)$ $x$ swaps $e_1$ and $e_8$;
- $(b)$ $e_2x = -e_2$ and $e_5x = e_5$;
- $(c)$ $x$ swaps $e_{3i}$ and $e_{3i+1}$ for all $1 \leq i < 3$;
- $(d)$ $x$ acts on $(e_{11}, e_{12})$ as the matrix $\begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}$ with $t \in \mathbb{F}_q$.

Clearly $x$ and $y$ have orders, respectively, 2 and 3, and

(2) \[ H = \langle x, y \rangle \]

is a subgroup of $\text{SL}_{12}(q)$.

First of all we prove the following.

**Lemma 2.1.** If $p \neq 5$, then the group $H$ contains $\text{Alt}(C)$. 
Proposition 2.3. gives Theorem 1.1. Results on the \( SL_{12} \) and \( SL_{12}^p \) are chosen in such a way that \( SL_{12} \) acts on \( \langle x, y \rangle \) as the involution \( E \). Since, by Lemma 2.1, \( Alt(5) \) is \( SL_{12} \)-generated. So, we have to prove that \( SL_{12} \) is \( SL_{12}^p \). Hence, we can now prove the following proposition that, combined with the known results on the \( (2,3) \)-generation of \( SL_n(q) \) described in the Introduction, immediately gives Theorem 1.1.

Lemma 2.2. Let \( t \neq 0, 2 \) be such that \( \mathbb{F}_q = \mathbb{F}_q(t) \). Then, the normal closure \( N \) of the involution \( w = I_5 - 2E_{5,5} + tE_{5,4} \) under \( Alt(5) \) is \( \langle SL_5(q), \text{diag}(-1, 1, 1, 1, 1) \rangle \).

We can now prove the following proposition that, combined with the known results on the \( (2,3) \)-generation of \( SL_n(q) \) described in the Introduction, immediately gives Theorem 1.1.

Proposition 2.3. For all primes \( p \neq 5 \) and all integers \( a \geq 1 \), the groups \( SL_{12}(p^a) \) are \( (2,3) \)-generated.

Proof. Set \( q = p^a \). Let \( H = \langle x, y \rangle \) be as in (2), where the element \( t \in \mathbb{F}_q \) in \( x \) is chosen in such a way that \( t \neq 0, 2 \) and \( \mathbb{F}_q(t) = \mathbb{F}_q \). As already observed, \( H \leq SL_{12}(q) \). So, we have to prove that \( SL_{12}(q) \leq H \).

First, consider the element \( g = (e_1, e_8)(e_9, e_{10}) \in Alt(5) \). Then \( w = gx \) acts on \( \langle e_8, \ldots, e_{12} \rangle \) as the involution \( I_5 - 2E_{5,5} + tE_{5,4} \). By Lemma 2.2, we get that \( SL_5(q) \) is contained in \( K = \langle w, Alt(\{e_8, \ldots, e_{12}\}) \rangle \). It follows that \( T = \langle K', Alt(5) \rangle \) is \( SL_{12}(q) \). Since, by Lemma 2.1, \( Alt(5) \) is a subgroup of \( H \) we have \( T \leq H \), whence \( H = SL_{12}(q) \).

For sake of completeness, using the permutational method we now prove the \( (2,3) \)-generation of \( SL_{12}(5)^a \) for all \( a \geq 1 \).

Let \( \tilde{y} = y \) be as in (1) and let \( \tilde{x} \) be the matrix, written with respect to \( \mathcal{C} \), such that:

(a) \( e_1 \tilde{x} = -e_1, e_5 \tilde{x} = e_5 \) and \( e_8 \tilde{x} = e_8 \);
(b) \( \tilde{x} \) swaps \( e_{3i} \) and \( e_{3i+1} \) for \( i = 2, 3 \);
(c) \( \tilde{x} \) acts on \( \langle e_2, e_3, e_4 \rangle \) as the involution \( x_3 = \begin{pmatrix} 3 & 3 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 3 \end{pmatrix} \);
(d) \( \tilde{x} \) acts on \( \langle e_{11}, e_{12} \rangle \) as the matrix \( \begin{pmatrix} 1 & 0 & 0 \\ t & -1 & 0 \\ 0 & t & -1 \end{pmatrix} \) with \( t \in \mathbb{F}_q \).
Also in this case, \( \tilde{x} \) and \( \tilde{y} \) have orders, respectively, 2 and 3, and
\[
(3) \quad \tilde{H} = \langle \tilde{x}, \tilde{y} \rangle
\]
is a subgroup of \( SL_{12}(q) \).

**Lemma 2.4.** The group \( \tilde{H} \) contains \( Alt(C) \).

**Proof.** Let \( \tilde{c} = [\tilde{x}, \tilde{y}] \) and define \( \tilde{\gamma} = \tilde{c}^{12} \) and \( \tilde{\delta} = \tilde{\gamma}^{9^2} \). We firstly observe that both \( \tilde{\gamma} \) and \( \tilde{\delta} \) fix the decomposition \( V = \langle e_1, \ldots, e_8 \rangle \oplus \langle e_9 \rangle \oplus \cdots \oplus \langle e_{12} \rangle \). Since \( \tilde{\gamma}^3 \tilde{\delta}^2 \) and \( \tilde{\gamma}^3 \tilde{\delta}^3 \) have orders, respectively, 313 and 19531, we obtain that \( K = \langle \tilde{\gamma}, \tilde{\delta} \rangle \) coincides with the subgroup \( \left\{ \begin{pmatrix} A & 0 \\ 0 & I_4 \end{pmatrix} : A \in SL_8(5) \right\} \cong SL_8(5) \) (use, for instance, [7]). In particular, \( K \) contains the elements \( g_1 = \text{diag}(1, x_3, I_8) \), \( g_2 = (e_1, e_2, e_3, e_4, e_5, e_6, e_7) \) and \( g_3 = (e_6, e_7, e_8) \). Now, as \( g_3 \tilde{x} e^{-1} g_3^9 \tilde{x} = (e_4, e_7, e_8) \), we obtain that \( \tilde{H} \) contains the subgroup \( \langle g_2, g_3^9 \tilde{x}, \tilde{y} \rangle = Alt(C) \). \( \square \)

**Corollary 2.5.** For all integers \( a \geq 1 \), the groups \( SL_{12}(5^a) \) are \((2,3)\)-generated.

**Proof.** It suffices to repeat the proof of Proposition 2.3 using \( \tilde{x}, \tilde{y}, \tilde{H} \) instead of \( x, y, H \), respectively, and defining \( w = g \tilde{x} \), where \( g = (e_6, e_7)(e_9, e_{10}) \). \( \square \)

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5

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