Symmetries of massless vertex operators in $AdS_5 \times S^5$

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**Abstract**

The worldsheet sigma-model of the superstring in $AdS_5 \times S^5$ has a one-parameter family of flat connections parametrized by the spectral parameter. The corresponding Wilson line is not BRST invariant for an open contour, because the BRST transformation leads to boundary terms. These boundary terms define a cohomological complex associated to the endpoint of the contour. We study the cohomology of this complex for Wilson lines in some infinite-dimensional representations. We find that for these representations the cohomology is nontrivial at the ghost number 2. This implies that it is possible to define a BRST invariant open Wilson line. The central point in the construction is the existence of massless vertex operators transforming exactly covariantly under the action of the global symmetry group. In flat space massless vertices transform covariantly up to adding BRST exact terms, but in AdS it is possible to define vertices so that they transform exactly covariantly.
1 Introduction

Nonlocal conserved charges play the central role in quantum integrability [1]. For the superstring in $AdS_5 \times S^5$ their existence was proven in the classical sigma-model in [2] using the Green-Schwarz-Metsaev-Tseytlin formalism, and in [3, 4, 5] using the pure spinor formalism. The existence of the nonlocal conserved charges at the quantum level was proven in [5, 6].

1.1 Open Wilson lines on the worldsheet

1.1.1 Wilson lines and BRST operator

The most important feature of the string worldsheet $\sigma$-model (besides the conformal invariance) is the existence of the BRST structure. The physically meaningful constructions should respect the action of the BRST operator $Q_{\text{BRST}}$. The nonlocal conserved charges of [3, 4, 5] are only BRST invariant up to the boundary terms. To be more precise, we need to introduce the transfer matrix which is the generating function of the nonlocal conserved charges. For an open contour $C_{AB}$ connecting points $A$ and $B$ on the string worldsheet:

\[ \begin{array}{c}
A \\
\end{array} \quad \longrightarrow \quad \begin{array}{c}
B \\
\end{array} \]

we define the transfer matrix:

\[ T_\rho[C_{AB}] = P \exp \left( - \int_A^B J[z] \right) \] (1)

where $C_{AB}$ is a contour connecting points $A$ and $B$ on the string worldsheet, and $\rho$ is a representation of $g = psu(2,2|4)$. The currents $J[z]$ depend on the spectral parameter $z$ and the representation $\rho$. The transfer matrix is a function of the spectral parameter $z$. It is the generating function of the conserved charges. It can be also though of as the Wilson line operator corresponding to the flat connection $J[z]$ on the string worldsheet. In this paper we will use both expressions: “transfer matrix” and “Wilson line”, and understand them as synonyms.

The BRST variation of the Wilson line $T_\rho[C_{AB}]$ results in the boundary
terms. Using the notations of [7]:

\[ \varepsilon Q_{BRST} T[C_A^B](z) = \left( \frac{1}{z^3} \varepsilon \lambda_3(B) + z \varepsilon \lambda_1(B) \right) T[C_A^B](z) - \\
T[C_A^B](z) \left( \frac{1}{z^3} \varepsilon \lambda_3(A) + z \varepsilon \lambda_1(A) \right) \]

(2)

This equation is very interesting. Nontrivial boundary terms in (2) allow to “bootstrap” at least at the classical level the structure of \( r-s \) matrices, see Section 7 of [7]. On the other hand, these boundary terms present a problem: the nonlocal conserved charges are not physical quantities, at least not in an obvious sense. (Because they are not in the kernel of \( Q_{BRST} \).) What should we do with them?

1.1.2 The plugs

A natural thing to try is to find an operator which when inserted at the endpoint of the Wilson line would make it \( Q_{BRST} \)-closed. This is somewhat analogous to the open Wilson line in QCD. The expression \( P \exp \int_A^B A_\mu dx^\mu \) is unphysical, because it is not gauge invariant. The physical quantity is:

\[ \overline{\psi}(B) \left( P \exp \int_A^B A_\mu dx^\mu \right) \psi(A) \]

(3)

We may call \( \psi(A) \) and \( \overline{\psi}(B) \) “the plugs” because they fix “leaking boundary terms” in gauge transformations, or in BRST transformations. Can we find similar plugs for the Wilson line on the string worldsheet in \( AdS_5 \times S^5 \)? In this paper we will report a progress in this direction.

1.1.3 Wilson lines in infinite-dimensional representations

Remember that the Wilson line depends on a choice of representation; we have to choose a representation \( \rho \) of \( psu(2, 2|4) \). Consider the space of states of the linearized supergravity multiplet in \( AdS_5 \times S^5 \). It splits into the direct sum of infinitely many infinite-dimensional irreducible representations of \( psu(2, 2|4) \), each corresponding to a BPS state. We will argue that when \( \rho \) is one of those infinite-dimensional BPS representations, then there is a suitable plug of the ghost number 2. This is closely related to the vertex operators for the massless states in \( AdS_5 \times S^5 \). In fact we will relate the BRST
cohomology complex corresponding to the endpoint of the Wilson line to the
BRST complex corresponding to the vertex operators. This is essentially
an example of the Frobenius reciprocity. The main nontrivial point is the
construction of the vertex operators transforming \textit{strictly covariantly}
under the global supersymmetries of $AdS_5 \times S^5$. This is different from flat space
where massless vertices transform covariantly only up to BRST exact terms.

Wilson lines in infinite-dimensional representations played an important
role in the integrable context in [8] in the construction of the Q-operator.
They also played an important role in the AdS/CFT context in [9] for the
interpretation of the YM Feynman diagrams in the string worldsheet the-
ory.

1.2 Representation theory interpretation of the SUGRA spectrum

Consider an infinite-dimensional irreducible representation $\mathcal{H}$ of $psu(2,2|4)$. It is natural to ask the question:

When does $\mathcal{H}$ appear in the decomposition of the space of solutions
of the linearlized Type IIB SUGRA equations in $AdS_5 \times S^5$? (4)

The results of our paper imply that this answer can be answered directly in
terms of the structure of $\mathcal{H}$, as a representation of $psu(2,2|4)$. Namely, given
$\mathcal{H}$, we consider the following complex:

$$
\ldots \longrightarrow \mathcal{H}' \otimes \mathfrak{g}_5 \mathcal{P}^n \xrightarrow{Q_{\text{endpoint}}} \mathcal{H}' \otimes \mathfrak{g}_5 \mathcal{P}^{n+1} \longrightarrow \ldots
$$

defined entirely in terms of $\mathcal{H}$ — see Section 7.2. Then, we claim that
the multiplicity of $\mathcal{H}$ in the space of linearized SUGRA solutions is equal to
the dimension of the second cohomology $H^2(Q_{\text{endpoint}}, \mathcal{H}' \otimes \mathfrak{g}_5 \mathcal{P}^\bullet)$ of this
complex. (We only claim this when $\mathcal{H}$ has “high enough” spin on $S^5$; see the
end of Section 4.5.)

1.3 The plan of the paper

Most of the paper is about massless vertex operators in $AdS_5 \times S^5$. In Section
2 we give a geometrical definition of vertex operators using the representation
of $AdS_5 \times S^5$ as the coset space $G/H$. In Section 3 we explain what it means
for the vertex to be strictly covariant, and then in Sections 4 and 5 prove the existence of such covariant vertices. In Section 6 we discuss the flat space limit of our construction. (Although it is impossible to construct the strictly covariant vertex in the flat space, but nevertheless the construction in $AdS_5 \times S^5$ has a well-defined flat space limit, which does transforms strictly covariantly, but only under a subgroup $SO(1, 4) \times SO(5) \subset SO(1, 9)$.) In Section 7 we explain how the covariant vertex plugs the endpoint of the Wilson line. In Section 8 we present some consequences of our construction; we explain how to prepare the vertex operator depending on the spectral parameter.

1.4 Notations

The algebra of supersymmetries of $AdS_5 \times S^5$ has a $\mathbb{Z}_4$ grading:

$$\text{psu}(2, 2|4) = g = g_\bar{0} + g_\bar{1} + g_2 + g_3$$  \hspace{1cm} (6)

We denote $Ug$ the universal enveloping algebra of $g$.

Let $g$ denote the group element of $Ug$, i.e. an element of the group $PSU(2, 2|4)$:

$$g = e^\omega e^{\theta_L} e^{\theta_R} e^x$$  \hspace{1cm} (7)

where $\omega \in g_\bar{0}$, $\theta_L \in g_3$, $\theta_R \in g_1$ and $x \in g_2$. So defined $\theta_{L,R}$ and $x$ are coordinates of the super-$AdS_5 \times S^5$. The generators of $g = psu(2, 2|4)$ are the same as in [22, 7]:

$$t^a_\alpha \in g_3, \quad t^1_\alpha \in g_1, \quad t^2_\mu \in g_2, \quad t^0_{\mu\nu} \in g_0$$  \hspace{1cm} (8)

For a vector space $L$ we will denote $L'$ the dual vector space. In particular, the space of states is denoted $\mathcal{H}$ and the space of linear functionals on the states is denoted $\mathcal{H}'$. We will mostly consider the linear functionals which are the values of some supergravity fields (such as the Ramond-Ramond field strength) at a fixed point in $AdS_5 \times S^5$. These could be also thought of as non-normalizable elements of $\mathcal{H}$, “delta-functions type of states” in $\mathcal{H}$.

For an even vector space $L$ we denote $\Lambda^n L$ the space of antisymmetric tensors. For an odd vector space $\Lambda^n L$ will stand for symmetric tensors.

Example: $g_3$ is an odd vector space, and $g_3'$ is the dual space. Therefore $\Lambda^5 g_3'$ is identified with the fifth order polynomials of some bosonic spinor variable $\lambda^a$. 

4
2 Massless vertex operators as functions on the group manifold

Massless vertex operators in $AdS_5 \times S^5$ were introduced in [10].

Using the group theory language, we can define the vertex operator as a collection of functions $V_{\alpha\beta}(g)$, $V_{\dot{\alpha}\dot{\beta}}(g)$ and $V_{\dot{\alpha}\dot{\beta}}(g)$ of $g \in PSU(2,2|4)$ subject to the condition of $g_0$-covariance, which says that for any $h \in SO(1,4) \times SO(5)$ we should get:

$$V_{\alpha\beta}(hg) = h_\alpha^{\alpha'} h_\beta^{\beta'} V_{\alpha'\beta'}(g)$$ (9)
$$V_{\dot{\alpha}\dot{\beta}}(hg) = h_\alpha^{\alpha'} h_\beta^{\beta'} V_{\dot{\alpha}'\dot{\beta}'}(g)$$ (10)
$$V_{\dot{\alpha}\dot{\beta}}(hg) = h_\alpha^{\dot{\alpha}'} h_\beta^{\dot{\beta}'} V_{\dot{\alpha}'\dot{\beta}'}(g)$$ (11)

Here $h_\alpha^{\alpha'}$ and $h_\beta^{\beta'}$ are the matrix elements of $h \in G_0$ acting on $g_3$ and $g_1$ respectively.

For a tensor field $\varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(g)$ we introduce the covariant derivatives:

$$T_3^\alpha \varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(g) = \frac{d}{ds} \mid_{s=0} \varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(e^{-st_3^\alpha} g)$$

$$T_m^\alpha \varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(g) = \frac{d}{ds} \mid_{s=0} \varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(e^{-st_m^\alpha} g)$$ (12)

$$T_1^\alpha \varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(g) = \frac{d}{ds} \mid_{s=0} \varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(e^{-st_1^\alpha} g)$$

$$T_0^{[mn]} \varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(g) = \frac{d}{ds} \mid_{s=0} \varphi_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(e^{-st_0^{[mn]} g})$$

Collectively:

$$T_\bar{\alpha} \bar{\varphi}_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(g) = \frac{d}{ds} \mid_{s=0} \bar{\varphi}_{\alpha_1...\alpha_m \dot{\beta}_1...\dot{\beta}_n}(e^{-st_{\bar{\alpha}} g})$$ (13)

Note that the covariant derivatives $T_{\bar{\alpha}}$ for $\bar{\alpha} \neq \bar{0}$ satisfy the condition of $g_0$-covariance, for example:

$$T_3^\alpha V_{\beta\gamma}(hg) = h_\alpha^{\alpha'} h_\beta^{\beta'} h_\gamma^{\gamma'} T_3^\alpha V_{\beta'\gamma'}(g)$$ (14)
Also, the $g_0$-covariance condition (9) — (11) can be formulated as the following explicit expression for the covariant derivative $\mathcal{T}^0_{[\mu\nu]}$ along $g_0$:

$$\mathcal{T}^0_{[\mu\nu]} V_{\beta\gamma}(g) = \frac{d}{ds} \bigg|_{s=0} V_{\beta\gamma}(e^{-st_{[\mu\nu]}^0} g) = f_{[\mu\nu]\beta} V_{\beta'\gamma}(g) + f_{[\mu\nu]\gamma'} V_{\beta'\gamma}(g) \quad (15)$$

The condition that the vertex operator is $Q$-closed can be written as follows:

$$\mathcal{T}^3_{(\alpha} V_{\beta\gamma)} = f_{(\alpha\beta} \gamma m S_{\beta) m} \quad (16)$$

$$\mathcal{T}^3_{(\alpha} V_{\beta\gamma)} = f_{(\alpha\beta} \gamma m S_{\beta) m} \quad (17)$$

$$\mathcal{T}^3_{(\alpha} V_{\beta\gamma)} + \mathcal{T}^1_{(\gamma} V_{\alpha\beta)} = f_{\alpha\beta} \gamma m A_{\gamma m} \quad (18)$$

where $A$ and $S$ are defined by these equations.

The gauge transformations are:

$$\delta_{\Phi,\tilde{\Phi}} V_{\alpha\beta} = \mathcal{T}^3_{(\alpha} \Phi_{\beta)} \quad (20)$$

$$\delta_{\Phi,\tilde{\Phi}} V_{\alpha\beta} = \mathcal{T}^3_{(\alpha} \tilde{\Phi}_{\beta)} \quad (21)$$

$$\delta_{\Phi,\tilde{\Phi}} V_{\alpha\beta} = \mathcal{T}^3_{(\alpha} \tilde{\Phi}_{\beta)} + \mathcal{T}^1_{(\alpha} \Phi_{\beta)} \quad (22)$$

where $\Phi$ and $\tilde{\Phi}$ are the parameters of the gauge transformations.

The BRST operator is:

$$Q = Q_L + Q_R = \lambda^\alpha \mathcal{T}^3_{(\alpha} + \bar{\lambda}^{\hat{\alpha}} \mathcal{T}^1_{(\hat{\alpha} \quad (23)$$

Our definition of the vertex is slightly weaker than the definition of [10]. The definition of [10] requires that:

$$V_{\alpha\beta} = 0 \quad \text{and} \quad V_{\alpha\beta} = 0 \quad (24)$$

the only nonzero component of the vertex remains $V_{\alpha\beta}$. In fact $V_{\alpha\beta}$ and $V_{\alpha\beta}$ are always $Q$-exact. Therefore the condition (21) can always be satisfied by adding to the vertex something $Q_{BRST}$-exact. In this sense the components $V_{\alpha\beta}$ and $V_{\alpha\beta}$ can always be “gauged away”. But we want the covariant vertex operator. We suspect that it might be impossible to gauge away $V_{\alpha\beta}$ and $V_{\alpha\beta}$ in a covariant way. This is the reason why we prefer to allow these components in the definition of the vertex operator.
3 Covariant vertex: the definition

3.1 Vertices and states

In string theory vertex operators represent states. Vertex operators are functions \( V(x, \theta, \lambda) \). The global symmetries act on \( x \) and \( \theta \), and therefore act on vertex operators.

On the other hand, the global symmetries act on the space of states. Therefore the action of the global symmetry group on states should agree with the action on vertex operators. Naively, this would imply that if \( V_\Psi(x, \theta, \lambda) \) is a vertex operator corresponding to the state \( \Psi \) then

\[
V_{g\Psi}(x, \theta, \lambda) = V_\Psi(gx, g\theta, \lambda)
\]  

But in fact this formula, generally speaking, holds only up to BRST-trivial corrections (terms which are \( Q_{BRST} \) of something).

Note that \( V_\Psi(x, \theta, \lambda) \) is not defined unambiguously, because we could add to it BRST-exact terms and get physically equivalent vertex. We will prove that in \( AdS_5 \times S^5 \) it is possible to use this freedom in the definition of \( V_\Psi \) and choose \( V_\Psi \) so that it transforms covariantly, as in (25). In our proof we will use the fact that vertices corresponding to supergravity states exist. This was proven in [10]. Given the existence of the vertex, we will prove that it is always possible to correct it by a BRST-exact expression, if necessary, to get a covariant vertex.

3.2 Examples of covariant vertices: AdS radius and \( \beta \)-deformation

The first example of the covariant vertex was given in [11]. It was shown that the zero mode dilaton vertex is given by the expression \( \frac{1}{2} \) which is independent of \( x \) and \( \theta \):

\[
V(x, \theta, \lambda) = \text{Str}(\lambda_3 \lambda_1)
\]  

This operator plays the central role in [11]. The corresponding marginal deformation of the action changes the radius of \( AdS_5 \times S^5 \). The radius is invariant under the global symmetries, therefore in this case covariance means invariance; the vertex (26) is invariant under the global symmetries. This is

\footnote{Using the notations of [11]: \( \eta_{a\dot{a}} \lambda^a \dot{\lambda}^\dot{a} \)}
related to the fact that the action of the pure spinor superstring in \( AdS_5 \times S^5 \) is exactly invariant under the global symmetries (while the action in flat space is invariant only up to adding a total derivative). Our construction can be considered a generalization of \((26)\) for fields with nontrivial dependence on \(x\) and \(\theta\).

The second example\(^2\) is:

\[
V_{ab}^{\beta} = (g^{-1}\varepsilon(\lambda_3 - \lambda_1)g)_a (g^{-1}\varepsilon'(\lambda_3 - \lambda_1)g)_b \tag{27}
\]

Here the indices \(a\) and \(b\) enumerate the adjoint representation of \(psu(2,2|4)\). Notice that \((27)\) is antisymmetric under the exchange of \(a\) and \(b\). Therefore this vertex transforms in the antisymmetric product of two adjoint representations of \(psu(2,2|4)\). This antisymmetric product splits into two irreducible components. The first component is the adjoint representation. But the part of \((27)\) belonging to the adjoint representation is actually \(Q_{BRST}\)-exact:

\[
\varepsilon^a V_{ab}^{\beta} = \varepsilon Q_{BRST} (g^{-1}\varepsilon'(\lambda_3 + \lambda_1)g)_c \tag{28}
\]

(Notice that this formula played an important role in Section 6 of [5].)

The deformation of the action corresponding to \((27)\) follows from the standard descent procedure. Let us denote:

\[
\Lambda_a(\varepsilon) = (g^{-1}\varepsilon(\lambda_3 - \lambda_1)g)_a \tag{29}
\]

This is the ghost number 1 cocycle corresponding to the local conserved currents, see also Appendix A. It corresponds to the local conserved currents in the following sense:

\[
d\Lambda_a(\varepsilon) = \varepsilon Q (j_a) \tag{30}
\]

where \(j_{a\pm}(\tau^+,\tau^-)\) is the density of the local conserved charge corresponding to the global symmetries. Therefore:

\[
d(\Lambda_a(\varepsilon)\Lambda_b(\varepsilon')) = 2\varepsilon Q j_{[a} \Lambda_{b]}(\varepsilon') \tag{31}
\]

and:

\[
d(j_{[a} \Lambda_{b]}(\varepsilon)) = -\varepsilon Q (j_{[a} \wedge j_{b]}) \tag{32}
\]

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\(^2\)Note in revised version: a more detailed discussion of \(V_{ab}^{\beta}\) will be presented in the forthcoming paper with O. Bedoya, L. Beviláqua, and V.O. Rivelles
We conclude that for any constant antisymmetric matrix $B^{ab}$ we can infinitesimally deform the worldsheet action as follows:

$$S \rightarrow S + B^{ab} \int j_a \wedge j_b$$ (33)

Consider for example $B_{ab}$ in the directions of $S^5$. We get:

$$S \rightarrow S + B^{kl}[mn] \left( \int X_k dX_l \wedge X_m dX_n + \ldots \right)$$ (34)

where $X_j$ describes the embedding of $S^5$ into $\mathbb{R}^6$ and dots denote $\theta$-dependent terms. These $\theta$-dependent terms appear because $j_a$ includes $\theta$. Eq. (34) corresponds to the marginal deformations of the $\mathcal{N} = 4$ Yang-Mills known as $\beta$-deformations [12], as follows from their quantum numbers.

The subspace $g \subset g \wedge g$ corresponds to $B$ of the following form:

$$B^{[kl][mn]} = \delta^{km} A^{ln} - \delta^{lm} A^{kn} + \delta^{ln} A^{km} - \delta^{kn} A^{lm}$$ (35)

where $A^{mn}$ is antisymmetric matrix; then the corresponding deformation of the Lagrangian is a total derivative $d(A^{mn} X_m dX_n)$. The complementary space has real dimension 90, it corresponds to the representation $45\mathbf{C}$ of $so(6)$. This is the expected quantum numbers of the linearized $\beta$-deformation, cp. Section 3.1 of [13] and references therein. It was observed in [13] that some of these deformations are obstructed when we pass from the linearized supergravity equations to the nonlinear equations. Not all of the deformations (33) can be extended to the solutions of the nonlinear supergravity equations as solutions constant in $AdS_5$ directions, but only those which satisfy some nonlinear equations on $B^{ab}$. If these nonlinear equations are not satisfied, then the nonlinear solutions will have “resonant terms” and because of these resonant terms will not be periodic in the global time of $AdS_5$.

### 3.3 Universal vertex

Suppose that we are looking at the massless states transforming in some representation $\mathcal{H}$ of the global symmetry group $PSU(2,2|4)$. For every state

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3Similar phenomenon for “fast moving strings” was discussed in [14] [15]. Generally speaking, deviations from periodicity in the global time of AdS correspond to something like anomalous dimension. In this case it shows that the beta function of the deformed theory is actually nonzero at the higher order in the deformation, unless if additional (cubic) constraints are imposed on the deformation parameter.

9
\[ \Psi \in \mathcal{H} \] we have the corresponding vertex operator \( \mathcal{V}(\Psi) \). As we discussed, \( \mathcal{V}(\Psi) \) consists of the components: \( \mathcal{V}_{\alpha\beta}(\Psi) \), \( \mathcal{V}_{\dot{\alpha}\dot{\beta}}(\Psi) \) and \( \mathcal{V}_{\dot{\alpha}\dot{\beta}}(\Psi) \). We write \( \mathcal{V} \) instead of \( V \) to stress that \( \mathcal{V} \) is a function of \( \Psi \). Mathematically it would be more appropriate to call it “a linear operator from the space of states to the space of vertex operators”. We will call \( \mathcal{V} \) the “universal vertex” for the representation \( \mathcal{H} \) because it is a uniform definition of vertex operators for all states in \( \mathcal{H} \):

\[ \mathcal{V} : \mathcal{H} \longrightarrow \text{(functions of } x, \theta, \lambda) \]

The global symmetry group \( G = PSU(2,2|4) \) acts on both space of states and space of vertex operators. It acts on the space of states by definition, because it is the global symmetry group of the theory. It also acts on the space of vertex operators. The action on the space of vertex operators may seem obvious, but we would like to spell it out explicitly because we feel that some confusion is possible. A vertex operator has components \( \mathcal{V}_{\alpha\beta}, \mathcal{V}_{\dot{\alpha}\dot{\beta}} \) and \( \mathcal{V}_{\dot{\alpha}\dot{\beta}} \) which are all functions of the group element \( g \), i.e. \( \mathcal{V}_{\alpha\beta}(g), \mathcal{V}_{\dot{\alpha}\dot{\beta}}(g) \) and \( \mathcal{V}_{\dot{\alpha}\dot{\beta}}(g) \) satisfying the conditions of \( g_0 \)-covariance (9) — (11). Then, the action of the global symmetry transformation \( g' \in PSU(2,2|4) \) is defined as follows:

\[
\begin{align*}
(g' \cdot \mathcal{V}_{\alpha\beta})(g) &= \mathcal{V}_{\alpha\beta}(gg') \\
(g' \cdot \mathcal{V}_{\dot{\alpha}\dot{\beta}})(g) &= \mathcal{V}_{\dot{\alpha}\dot{\beta}}(gg') \quad (36) \\
(g' \cdot \mathcal{V}_{\dot{\alpha}\dot{\beta}})(g) &= \mathcal{V}_{\dot{\alpha}\dot{\beta}}(gg')
\end{align*}
\]

Because \( g' \) hits \( g \) on the right, this action of the global symmetries is manifestly consistent with the conditions of \( g_0 \)-covariance (9) — (11) and also commutes with the covariant derivatives (12).

This defines the action of \( G \) on \( \mathcal{V}(\Psi) \) for any fixed \( \Psi \); the expression:

\[ g'.(\mathcal{V}(\Psi)) \]

is defined by (36):

\[
( g'.(\mathcal{V}(\Psi)) )(g) = ( \mathcal{V}(\Psi) )(gg') \quad (38)
\]

### 3.4 Covariant universal vertex

It is natural to ask, if it is true that (37) is equal to this:

\[ \mathcal{V}(g'\Psi) \]

(39)
In other words, if it is true or not that:

\[
(\mathcal{V}(\Psi))(gg') \overset{?}{=} (\mathcal{V}(g'\Psi))(g)
\]

This is not automatically true. What is automatically true is this statement:

\[
(\mathcal{V}(\Psi))(gg') = (\mathcal{V}(g'\Psi))(g) + Q_{BRST}(\text{smth})
\]

Remember that vertex operators are defined modulo BRST-exact expressions. The question is, can we choose a representative for \(\mathcal{V}(\Psi)\) in the equivalence class of \(\mathcal{V}(\Psi) \simeq \mathcal{V}(\Psi) + Q_{BRST}(\text{smth})\), “uniformly in \(\Psi\)”, so that (40) is true?

The answer to this question is “no” in flat space, but “yes” in AdS. It turns out that in \(AdS_5 \times S^5\) it is possible to choose the vertex operator to be covariant.

Let us introduce the notation for the action of the global symmetries (compare to (13)):

\[
t^\alpha_A f(g) = \frac{d}{ds} f(g e^{st^\alpha_A})
\]

Definition: The covariant vertex is a superfield

\[
\mathcal{V}(\Psi) = \lambda^\alpha \chi^\beta \nu_{\alpha\beta}(\Psi) + \bar{\lambda}^{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} \nu_{\dot{\alpha}\dot{\beta}}(\Psi) + \bar{\lambda}^{\dot{\alpha}} \bar{\chi}^{\dot{\beta}} \nu_{\dot{\alpha}\dot{\beta}}(\Psi)
\]

depending linearly on the state \(\Psi\), and such that:

1. It is annihilated by \(\mathcal{Q}\):

\[
(\lambda^\alpha \mathcal{T}_\alpha + \bar{\lambda}^{\dot{\alpha}} \mathcal{T}_{\dot{\alpha}})\mathcal{V}(\Psi) = 0
\]

and is not \(\mathcal{Q}\)-exact, and

2. the action of the global symmetry on \(\mathcal{V}\) as a function of \(g\) agrees with the action of the global symmetry on the space of states:

\[
t^\alpha \mathcal{V}(\Psi) = \mathcal{V}(t^\alpha \Psi)
\]

\(^4\)“Automatically true” means true under the assumption that the vertex operators exist. The existence was proven in [10].
4 Taylor series for the vertex.

Let us study vertex operators for states which are not necessarily normalizable. In other words, let us forget about the boundary conditions near the boundary of AdS, and study the supergravity states which are not necessarily normalizable. Moreover, let us pick a point in $\text{AdS}_5 \times S^5$ and consider the Taylor expansion of the supergravity fields around this point. Let us not worry about the convergence of the Taylor series. Just study the supergravity equations, BRST cohomology etc. on formal Taylor series. The question of convergence, and the question of the behaviour at spatial infinity, can be studied later.

For the study of the Taylor series the mathematical notion of the *coinduced representation* is useful.

4.1 A review of coinduced representations

Let us study the supergravity fields around a point in $\text{AdS}_5 \times S^5$ corresponding to the unit $1 \in G$. If we do not insist on convergence, then the space of supergravity fields around a point can be replaced by a more algebraic notion, the so-called *coinduced representation* [10].

For a representation $V$ of $G_0$, we define the *coinduced representation* $\text{Coind}_{g_0}^g V$, in the following way:

**Definition:**

the space of linear functions $f$ from the universal enveloping algebra $Ug$ to $V$, which satisfy the condition of $g_0$-invariance:

$$f(x\xi) = \rho(x)f(\xi) \text{ for any } x \in g_0, \xi \in Ug$$

is called the *coinduced representation* and denoted $\text{Coind}_{g_0}^g V$.

The mathematical notation for such functions is:

$$f \in \text{Hom}_{g_0}(Ug, V)$$ (46)

Here “$\text{Hom}(A, B)$” means the space of linear maps from $A$ to $B$, and the subindex $g_0$ means $g_0$-invariant functions: $f(x\xi) - \rho(x)f(\xi) = 0$. 

12
The action of $g$ on this space is defined by the formula:

$$x.f(\xi) = f(\xi x), \text{ for } x \in g, \: \xi \in Ug$$  \hspace{1cm} (47)

To summarize:

$$\text{Coind}_g^g V = \text{Hom}_g(Ug, V)$$

We will now explain that $\text{Coind}_g^g V$ encodes the Taylor series of various tensor fields on $AdS_5 \times S^5$, where $V$ is the representation of $g_0$ corresponding to the type of the tensor.

Let us start with the trivial representation $V = C$ of $g_0$. In this case we should get scalar fields on AdS. The correspondence between the elements of $\text{Hom}_{g_0}(Ug, C)$ and the scalar fields on AdS goes as follows. Given a scalar field $\phi(g)$, the corresponding element $f \in \text{Hom}_{g_0}(Ug, C)$ is given by the formula:

$$f(x_1x_2 \cdots x_n) = x_1.x_2 \cdots x_n.\phi(1) \hspace{1cm} (48)$$

where

$$x_1.x_2 \cdots x_n.\phi(g) = \left. \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_n} \right|_{t_1=\cdots=t_n=0} \phi(ge^{\epsilon_1x_1}e^{\epsilon_2x_2} \cdots e^{\epsilon_nx_n})$$

In this case the $g_0$-invariance condition says that $f(x\xi) = 0$ for any $x \in g_0$, and this is indeed satisfied for $f$ defined in (48) because $\phi(e^x) = \phi(1)$ for any $x \in g_0$ because $\phi$ is well defined on $G/Go$.

There is also a map going in the opposite direction. Namely, given $f$ a linear function from $Ug$ to $C$ we define the corresponding scalar field $\phi(g)$ as follows:

$$\phi(g) = f(g)$$  \hspace{1cm} (49)

Note that on the right hand side we treat $g$ as a group element\footnote{The $\xi \in Ug$ is called group element if it is of the form $e^x$ for some $x \in g$} of $Ug$.

We have just explained why for $V = C$ the space $\text{Hom}_{g_0}(Ug, V)$ encodes the Taylor coefficients of the scalar function on AdS; for general $V$ a similar construction shows that $\text{Hom}_{g_0}(Ug, V)$ encodes the Taylor coefficients of the tensor field with indices transforming in the representation $V$ of $g_0$.\footnote{The $\xi \in Ug$ is called group element if it is of the form $e^x$ for some $x \in g$}
4.2 Pure spinors

To describe the linearized SUGRA in $AdS_5 \times S^5$ we need two pure spinor variables $\lambda_3 \in g_3$ and $\lambda_1 \in g_1$ satisfying the constraints:

$$\{\lambda_3, \lambda_3\} = \{\lambda_1, \lambda_1\} = 0$$  \hspace{1cm} (50)

We will consider various types of vertex operators, which are homogeneous polynomials in $\lambda_3$ and $\lambda_1$. Note that (50) are invariant under the action of $g_0$. Therefore the polynomials of $\lambda_3$ and $\lambda_1$ form a representation of $g_0$. We will introduce the notation for such polynomials:

**Definition:** We denote $P^{(m,n)}$ the space of polynomials of $\lambda_3$ and $\lambda_1$ which have the degree $m$ in $\lambda_3$ and $n$ in $\lambda_1$.

We will define the polynomials of $\lambda$ by specifying their coefficients, which are elements of $\Lambda^m g_3' \otimes \Lambda^n g_1'$ (see Section 1.4 for notations). We have to “discard” those polynomials which are identically zero because of the pure spinor constraints (50).

As a trivial example, let us consider the quadratic polynomials of $\lambda_3$. The coefficients belong to $\Lambda^2 g_3'$. Let us denote $t^3_\alpha$ the basis vectors of $g_3'$, such that:

$$\langle t^\alpha_3, t^\beta_3 \rangle = \delta^\alpha_\beta$$

The space $\Lambda^2 g_3$ consists of expressions of the form $U_{\alpha\beta} t^\alpha_3 \otimes t^\beta_3$ where $U_{\alpha\beta} = U_{\beta\alpha}$. As an example of the polynomial which is identically zero, take $f^{\mu\nu}_{\alpha\beta} t^\alpha_3 \otimes t^\beta_3$ where $f^{\mu\nu}_{\alpha\beta}$ is the structure constants defined by $\{t^3, t^3\} = f^{\mu\nu}_{\alpha\beta} t^\mu_\alpha$. Such a polynomial is identically zero because of the pure spinor constraint (50):

$$f^{\mu\nu}_{\alpha\beta} \lambda^\alpha_3 \lambda^\beta_3 = 0$$

To summarize:

$$P^{(m,n)} = (\Lambda^m g_3'/(\Lambda^m g_3')_{null}) \otimes (\Lambda^n g_1'/(\Lambda^n g_1')_{null})$$  \hspace{1cm} (51)

where $(\Lambda^m g_3')_{null}$ denotes a subspace of $\Lambda^m g_3'$ corresponding to those polynomials on $g_3$ which vanish identically on $\lambda_3$ because of the pure spinor constraint.
We will also introduce:

\[ P^i = \bigoplus_{m+n=l} P^{(m,n)} \]  

(52)

Note that \( P^{(m,n)} \) and \( P^i \) are representations of \( g_0 \), but not of \( g \). The construction of coinduced representation is used to build the representations of \( g \) from these spaces.

### 4.3 Vertex as an element of Coind\(_{g_0}^g \mathcal{P}^2\)

In this section we consider the Taylor series of the vertex operator and do not bother about the convergence and the behaviour near the boundary. Then the vertex operator can be considered an element of the coinduced representation:

\[ \mathcal{V}(\Psi) \in \text{Coind}_{g_0}^g \mathcal{P}^2 \]  

(53)

We would like to discuss vertex operators “uniformly” for all vectors \( \Psi \in \mathcal{H} \). We will therefore introduce the “universal” vertex operator:

\[ \mathcal{V} \in \text{Hom}_\mathbb{C}(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^2) \]  

(54)

In other words, we have a linear function on the Hilbert space \( \mathcal{H} \) which to every vector \( \Psi \in \mathcal{H} \) associates the corresponding vertex operator:

\[ \mathcal{V} : \Psi \mapsto \mathcal{V}(\Psi) \in \text{Coind}_{g_0}^g \mathcal{P}^2 \]  

(55)

Given a state \( \Psi \in \mathcal{H} \) we get \( \mathcal{V}(\Psi) \) — an element of Coind\(_{g_0}^g \mathcal{P}^2\). This means, by definition, that for every \( \Psi \), the object \( \mathcal{V}(\Psi) \) is a linear map from \( Ug \) to \( \mathcal{P}^2 \) satisfying the \( g_0 \)-invariance condition:

\[ \mathcal{V}(\Psi)(x\xi) = \rho(x) \mathcal{V}(\Psi)(\xi) \text{ for any } x \in g_0, \xi \in Ug \]  

(56)

Given such \( \mathcal{V}(\Psi) \), how do we construct the “usual” vertex operator? As an element of Coind\(_{g_0}^g \mathcal{P}^2\) our \( \mathcal{V}(\Psi) \) is a function of \( \xi \in Ug \) with values in \( \mathcal{P}^2 \). Let us evaluate this function on a group element \( \xi = g = e^x \), where \( x \in g \). We get \( \mathcal{V}(\Psi)(g) \) — an element from \( \mathcal{P}^2 \), i.e. a quadratic polynomial in \( \lambda_3 \) and \( \lambda_1 \). The “usual” vertex operator is just the evaluation of this polynomial:

\[ V_{\Psi}(g, \lambda) = \mathcal{V}(\Psi)(g)(\lambda) \]  

(57)
4.4 Action of the BRST operator

The BRST complex is:

\[
\cdots \xrightarrow{Q_{BRST}} \text{Hom}_C(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^n) \xrightarrow{Q_{BRST}} \text{Hom}_C(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^{n+1}) \xrightarrow{Q_{BRST}} \cdots
\]  

(58)

The BRST operator acts on the universal vertex \(V(\Psi)\) in the following way:

\[
(Q_{BRST} V)(\Psi)(\xi)(\lambda) = V(\Psi)(\lambda_3 \xi + \lambda_1 \xi)(\lambda)
\]  

(59)

Note that \(Q_{BRST} V\) is an element of \(\text{Hom}_C(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^2)\). In terms of the “usual” vertex \(V_\Psi(g, \lambda)\) defined by (57) we get:

\[
Q_{BRST} V_\Psi(g, \lambda) = \left. \frac{d}{dt} \right|_{t=0} V_\Psi(e^{t(\lambda_3 + \lambda_1)} g, \lambda)
\]  

(60)

4.5 Covariant universal vertex

**Statement of covariance**  Note that in Eq. (54) we use the notation \(\text{Hom}_C\) rather than \(\text{Hom}_g\). There is no apriori reason why \(V\) would respect the action of \(g\). But in the next section we will see that under some conditions on \(\mathcal{H}\), it is possible to choose the universal vertex operator which does respect the global symmetry. We will call it the *covariant* universal vertex:

\[
V \in \text{Hom}_g(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^2)
\]  

(61)

Given Eq. (57) this implies:

\[
V(\Psi)(\xi x) + V(x \Psi)(\xi) = 0
\]  

(62)

**Condition on \(\mathcal{H}\): sufficiently high spin**  The conditions on \(\mathcal{H}\) are the following. Consider \(\mathcal{H}\) as a representation of \(so(6) \subset g\) — the symmetry algebra of \(S^5\). As a representation of \(so(6)\), \(\mathcal{H}\) is the direct sum of infinitely many finite-dimensional representations of \(so(6)\). We request that the minimal value of the quadratic Casimir of \(so(6)\) on \(\mathcal{H}\) be sufficiently high.

5 Existence of the covariant vertex

In this Section we will use some facts about the Lie algebra cohomology which we learned mostly from [16, 17, 18]. See Chapter 3 §6 of [19] for a very brief summary.
5.1 Brief summary

The physical states correspond to the cohomology of $Q_{BRST}$ at the ghost number 2, therefore:

$$\text{Hom}_g(\mathcal{H}, H^2(Q_{BRST}, \text{Coind}[^g\mathcal{P}]^*)) = \text{C}^d(\mathcal{H})$$

(63)

where $d(\mathcal{H})$ is the multiplicity of $\mathcal{H}$ (how many times $\mathcal{H}$ enters in the SUGRA spectrum on $AdS_5 \times S^5$). We will argue that the second cohomology of the BRST operator can be calculated using the covariant subcomplex. In other words,

$$\text{Hom}_g(\mathcal{H}, H^2(Q_{BRST}, \text{Coind}[^g\mathcal{P}]^*)) = H^2(Q_{BRST}, \text{Hom}_g(\mathcal{H}, \text{Coind}[^g\mathcal{P}]^*)))$$

(64)

(Notice that $\text{Hom}_g(\mathcal{H}, \text{Coind}[^g\mathcal{P}]^*)$ is the covariant subcomplex.) To prove Eq. (64) we rewrite it in the following form:

$$H^0(\ g\ , \ \text{Hom}_C(\mathcal{H}, H^2(Q_{BRST}, \text{Coind}[^g\mathcal{P}]^*))) = H^2(\ Q_{BRST}\ , \ H^0(\ g\ , \ \text{Hom}_C(\mathcal{H}, \text{Coind}[^g\mathcal{P}]^*)))$$

(65)

Here we have used the fact that for any representation $L$ of the Lie algebra $\mathfrak{g}$ the zeroth cohomology group $H^0(\mathfrak{g}, L)$ equals the space of invariants $\text{Inv}_g L$. In particular, for two representations $\mathcal{A}$ and $\mathcal{B}$, $H^0(\ g\ , \ \text{Hom}_C(\mathcal{A}, \mathcal{B}) = \text{Hom}_g(\mathcal{A}, \mathcal{B})$.

The idea of the proof of (65) is to note that the left and the right hand side of (65) are two different second approximations to calculating the cohomology of the “total” differential $Q_{BRST} + Q_{Lie}$. Therefore the equality of the left hand side and the right hand side follows if we prove that the second approximation is actually exact.

5.2 Bicomplex and spectral sequence

Let us start by fixing some universal vertex (not necessarily covariant):

$$\mathcal{V} : \mathcal{H} \rightarrow \text{Coind}[^g\mathcal{P}]^2$$

(66)

At this point we do not require that this vertex is covariant; it is apriori an element of $\text{Hom}_C(\mathcal{H}, \text{Coind}[^g\mathcal{P}]^2)$ rather than $\text{Hom}_g(\mathcal{H}, \text{Coind}[^g\mathcal{P}]^2)$. We will
introduce the Lie algebra BRST operator of $g$. For each generator $t_i$ of $g$ we introduce the corresponding ghost $c_i$, and define:

$$Q_{\text{Lie}} = c^i t_i - \frac{1}{2} f^k_{ij} c^i c^j \frac{\partial}{\partial c^k}$$

(67)

We will consider the action of $Q_{\text{Lie}}$ on expressions polynomial in $c^i$. The polynomials of $c^i$ are specified by their coefficients; in degree $l$ the coefficients live in $\Lambda^l g^*$. Therefore $Q_{\text{Lie}}$ acts on the vertex operator as follows:

$$\text{Hom}_C(\mathcal{H}, \text{Coind}_g^P \mathcal{P}^2) \xrightarrow{Q_{\text{Lie}}} \text{Hom}_C(\mathcal{H}, \text{Coind}_g^P \mathcal{P}^2) \otimes g$$

(68)

We will consider the bicomplex with the differential $Q_{\text{tot}}$:

$$Q_{\text{tot}} = Q_{\text{BRST}} + Q_{\text{Lie}}$$

(69)

To prove the existence of the covariant vertex we will consider the spectral sequence computing the cohomology of this bicomplex. There are two ways to construct the spectral sequence. One can first calculate the cohomology of $Q_{\text{BRST}}$ and then consider $Q_{\text{Lie}}$ as a perturbation. The other way is to first calculate the cohomology of $Q_{\text{Lie}}$ and then consider $Q_{\text{BRST}}$ as a perturbation. These two ways of calculating the cohomology of $Q_{\text{tot}}$ should give the same result. We will see that this implies the existence of the covariant vertex. We will now consider the two methods in turn.

**First $Q_{\text{BRST}}$ then $Q_{\text{Lie}}$** The first term of the spectral sequence has:

$$E_1^{p,q} = H_{Q_{\text{BRST}}}^p(\text{Hom}_C(\mathcal{H}, \text{Coind}_g^P \mathcal{P}^*) \otimes \Lambda^p g')$$

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}$$

(70)

where $\Lambda^p g'$ stands for the $c$-ghosts; an element of $E_1^{p,q}$ is schematically $\lambda^q c^p$. The differential in the first term is $d_1 = Q_{\text{Lie}}$. The second term is:

$$E_2^{p,q} = H^p( \mathcal{G}, H_{Q_{\text{BRST}}}^q(\text{Hom}_C(\mathcal{H}, \text{Coind}_g^P \mathcal{P}^*)) )$$

$$d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$$

(71)

The higher differentials are of the type $d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$.

**First $Q_{\text{Lie}}$ then $Q_{\text{BRST}}$** The first term is:

$$E_1^{p,q} = H^p( \mathcal{G}, \text{Hom}_C(\mathcal{H}, \text{Coind}_g^P \mathcal{P}^q) )$$

$$\tilde{d}_1 : \tilde{E}_1^{p,q} \to \tilde{E}_1^{p,q+1}$$

(72)

where $\tilde{d}_1 = Q_{\text{BRST}}$. The higher differentials are of the type $\tilde{d}_r : \tilde{E}_r^{p,q} \to \tilde{E}_{r+1-r,q+r}$. 
Figure 1: The first page of $E$ and $\tilde{E}$; arrows denote $d_1$ and $\tilde{d}_1$. The Lie algebra ghost number (the number of $c$'s) increases in the horizontal direction, while the BRST ghost number (the number of $\lambda_3$ plus the number of $\lambda_1$) in the vertical direction.

**Existence of the covariant vertex** First of all, we want to show that $E_{\infty}^{0,2} = E_1^{0,2}$. The first observation is that by definition $d_1 : E_1^{0,2} \to E_1^{1,2}$ is zero. This is because the vertex is **covariant up to BRST-exact correction** (see Eq. (41)). Therefore $E_1^{0,2} = E_2^{0,2}$. Also, we will show (for $H$ with large enough spin) that $E_2^{2,1} = E_3^{3,0} = 0$. This implies that $E_\infty^{0,2} = E_1^{0,2}$.

Then we remember the relation between $E_\infty^{0,2}$ and $H^2(Q_{\text{tot}})$, which is the following. The space $H^2(Q_{\text{tot}}) = E^2$ has a filtration, corresponding to the number of the $c$-ghosts. Namely, $F^p E^2$ consists of expressions containing at least $p$ $c$-ghosts. Then $E_\infty^{0,2} = E^2/F^1 E^2 = H^2(Q_{\text{tot}})/F^1 H^2(Q_{\text{tot}})$. To summarize:

$$E_1^{0,2} = \text{[unintegrated vertices]}$$
$$E_1^{0,2} = H^2(Q_{\text{tot}})/F^1 H^2(Q_{\text{tot}})$$

On the other hand we will show that $\tilde{E}_1^{1,1} = \tilde{E}_1^{2,0} = 0$ (see Eqs. (87), (90)) and also that $\tilde{E}_1^{1,0} = 0$ (similar to (90)). This implies that:

$$H^2(Q_{\text{tot}}) = \frac{\text{Ker } \tilde{d}_1 : \tilde{E}_1^{0,2} \to \tilde{E}_1^{0,3}}{\text{Im } \tilde{d}_1 : \tilde{E}_1^{0,1} \to \tilde{E}_1^{0,2}}$$

We are now ready to prove the existence of the covariant vertex. Notice that $\tilde{E}_1^{0,2} = H^0(\mathbf{g}, \text{Hom}_C(H, \text{Coind}_g^{P^g}))$ is the space of functions $f_{\Psi}(x, \theta, \lambda)$, parametrized by $\Psi \in H$, transforming covariantly under $\mathbf{g}$. This means
that $\tilde{E}_1^0$ is the covariant subcomplex of the BRST complex. (The subspace consisting of the covariant expressions.) And Eq. (75) shows that:

$$H^2(Q_{tot})$$

is the second cohomology of the covariant subcomplex \quad (76)

Now the comparison of (73), (74) and (76) shows that the cohomology of $Q_{BRST}$ can be calculated using the covariant subcomplex. In fact, if the representation $H$ has large enough momentum in $S^5$, then $F^1 E^2$ is zero (because already $E_2^{1,1}$ and $E_2^{2,0}$ are zero). This means that the factor space on the right hand side of (74) is just $H^2(Q_{tot})$.

This means that there is a covariant choice of the vertex. In the rest of this section we will prove the required vanishing theorems and explain
explicitly how the non-covariant vertex can be modified into the covariant one.

5.3 The descent

Since the vertex transforms covariantly up to BRST-exact terms, we must have:

\[ Q_{\text{Lie}} \mathcal{V} = Q_{\text{BRST}} \mathcal{W} \]  

(77)

where

\[ \mathcal{W} \in \text{Hom}_{\mathcal{C}}(\mathcal{H}, \text{Coind}_{\mathcal{g}_0} \mathcal{P}^1) \otimes \mathcal{g} \]  

(78)

Note that \( Q_{\text{Lie}} \mathcal{W} \) is \( Q_{\text{BRST}} \)-closed and has ghost number 1:

\[ Q_{\text{Lie}} \mathcal{W} \in \text{Hom}_{\mathcal{C}}(\mathcal{H}, \text{Coind}_{\mathcal{g}_0} \mathcal{P}^1) \otimes \Lambda^2 \mathcal{g} \]  

(79)

We want to argue that there exists \( \mathcal{U} \) such that \( Q_{\text{Lie}} \mathcal{W} = Q_{\text{BRST}} \mathcal{U} \). More precisely: note that we are free to add to \( \mathcal{W} \) something in the kernel of \( Q_{\text{BRST}} \); we want to prove that it is possible to use this freedom and choose \( \mathcal{W} \) so that there exists \( \mathcal{U} \) such that \( Q_{\text{Lie}} \mathcal{W} = Q_{\text{BRST}} \mathcal{U} \). An obstacle to this would be a nonzero \( d_2 \mathcal{V} \) where

\[
d_2 : \quad H^0(\mathcal{g}, H^2_{Q_{\text{BRST}}} (\text{Hom}_{\mathcal{C}}(\mathcal{H}, \text{Coind}_{\mathcal{g}_0} \mathcal{P}^1)) \to \quad \to H^2(\mathcal{g}, H^1_{Q_{\text{BRST}}} (\text{Hom}_{\mathcal{C}}(\mathcal{H}, \text{Coind}_{\mathcal{g}_0} \mathcal{P}^1))))
\]  

(80)

We want to argue that the space \( H^2(\mathcal{g}, H^1_{Q_{\text{BRST}}} (\text{Hom}_{\mathcal{C}}(\mathcal{H}, \text{Coind}_{\mathcal{g}_0} \mathcal{C}))) \) on the right hand side is zero. Note that the BRST cohomology in ghost number 1 corresponds to local conserved charges. But the only conserved charges are the global symmetries \( psu(2,2|4) \), and those transform in the adjoint representation \( \mathcal{g} \) of \( \mathcal{g} \). This means that on the right hand side of (80) we have:

\[ H^2(\mathcal{g}, \text{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{g})) \]  

(81)

This cohomology group is zero. Indeed, we can compute it using the Serre-Hochschild spectral sequence of \( g_{\text{even}} \subset g \). Already the first term of this spectral sequence consists of the following spaces, which are all zero:

\[
\text{Hom}_{g_{\text{even}}} (\Lambda^2 g_{\text{odd}}, \text{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{g})) , \quad H^1(g_{\text{even}}, \text{Hom}_{\mathcal{C}}(g_{\text{odd}} \otimes \mathcal{H}, \mathcal{g})) , \quad H^2(g_{\text{even}}, \text{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{g}))
\]  

(82)

\[ ^{6}\text{See Appendix for the classification of the vertex operators of the ghost number 1 in the covariant BRST complex.} \]
Note that $g_{\text{even}} = g_A \oplus g_S$ where $g_A = so(2,4)$ and $g_S = so(6)$. Consider the corresponding Casimir operators $\Delta_A$ and $\Delta_S$. For the cohomology to be nonzero, we need both of them zero, but $\Delta_S$ is positive definite at least for $\mathcal{H}$ with large enough momenta. (Note also that $\mathcal{H}$ is an infinite dimensional irreducible representation of $g$, so there are no invariants in its tensor product with powers of $g$.)

Therefore $Q_{\text{Lie}} W = Q_{\text{BRST}} U$ for some $U$. In other words $d_2 V$ is zero, and we can proceed with computing $d_3$.

Consider $Z = Q_{\text{Lie}} U$. Note that $Z$ has zero pure spinor ghost number, and $Q_{\text{BRST}} Z = 0$. Since $Z$ is of ghost number 0, this implies that $Z$ is a constant; it does not contain any $x$ or $\theta$. Also, we could have added a constant to $U$ without affecting $Q_{\text{BRST}} U$; therefore $Z$ by itself is not very well defined by our construction. What is well defined is $Z$ modulo the image of $Q_{\text{Lie}}$:

$$[Z] \in H^3_{Q_{\text{Lie}}} (\mathcal{H}' \otimes \Lambda^* g) = H^3(g, \mathcal{H}')$$  \hspace{1cm} (83)

\section{5.4 The ascent}

But the Lie algebra cohomology group $H^3(g, \mathcal{H}')$ is zero:

$$H^3(g, \mathcal{H}') = 0$$  \hspace{1cm} (84)

22
One can see that it is zero from the Serre-Hochschild spectral sequence corresponding to $g_{\text{even}} \subset g$. Already the first term of this spectral sequence consists of the following spaces, which are all zero:

$$H^0(g_{\text{even}}, \text{Hom}_{\mathbb{C}}(\Lambda^2 g_{\text{odd}}, \mathcal{H}')) = H^1(g_{\text{even}}, \text{Hom}_{\mathbb{C}}(\Lambda^2 g_{\text{odd}}, \mathcal{H}')) =$$

$$= H^2(g_{\text{even}}, \text{Hom}_{\mathbb{C}}(g_{\text{odd}}, \mathcal{H}')) = H^3(g_{\text{even}}, \mathcal{H}') = 0 \quad (85)$$

The vanishing of these cohomologies can be proven as follows. Note that $g_{\text{even}}$ splits into $g_A = \mathfrak{so}(2, 4)$ and $g_S = \mathfrak{so}(6)$. For the cohomology to be nontrivial, both $\Delta_A$ and $\Delta_S$ should be zero. But $-\Delta_S$ is positive definite.

Therefore we can remove $Z$ by modifying $U$, adding to $U$ a constant term $-\Delta U$ so that the modified $U - \Delta U$ has $Q_{\text{Lie}}(U - \Delta U) = 0$. (Note that adding the constant term does not change the image of $U$ under $Q_{\text{BRST}}$.) Is it possible to find such $U'$ that $U - \Delta U = Q_{\text{Lie}}U'$? The answer is “yes”, because

$$H^2(g, \text{Hom}_{\mathbb{C}}(\mathcal{H}, \text{Coind}_{g_{0}} g_{0}, C)) = 0 \quad (86)$$

This can be proven using the Shapiro’s lemma (Proposition 6.8 and Theorem 6.9 from [16]):

$$H^2(g, \text{Hom}_{\mathbb{C}}(\mathcal{H}, \text{Coind}_{g_{0}} g_{0}, C)) =$$

$$= \text{Ext}^2_{g}(\mathcal{H}, \text{Coind}_{g_{0}} g_{0}, C) =$$

$$= \text{Ext}^2_{g_{0}}(\mathcal{H}|_{g_{0}}, C) =$$

$$H^2(g_{0}, \mathcal{H}'|_{g_{0}}) \quad (87)$$

Note that $g_{0} = \mathfrak{so}(1, 4) \oplus \mathfrak{so}(5)$. We want to prove that $H^2(g_{0}, \mathcal{H}'|_{g_{0}}) = 0$. The space $\mathcal{H}'$ consists of functionals on the space of states. Since we work in the vicinity of the fixed point $x_0 \in AdS_5 \times S^5$ our $\mathcal{H}'$ is generated by the values of various supergravity fields at the point $x_0$. For example the Ramond-Ramond field strength $H_{ijk}(x_0)$ and its derivatives. Under the action of $\mathfrak{so}(1, 4) \oplus \mathfrak{so}(5)$ this space splits into infinitely many finite-dimensional representations. For example $\partial_i H_{kjl}(x_0)$ lives in $(\text{Vect} \otimes \Lambda^3 \text{Vect})_0$ where $\text{Vect}$ is the vector representation of $\mathfrak{so}(1, 4) \oplus \mathfrak{so}(5)$ and index 0 means that the contraction $g^{ij} \partial_i H_{jkl}$ is zero. It follows from the general theory of Lie algebra cohomology that $H^2$ of $\mathfrak{so}(1, 4) \oplus \mathfrak{so}(5)$ with coefficients in any finite-dimensional representation is zero.

7this is an over-simplification; in fact one has to add also the expression of the form $g^{pq} g^{qr} F_{iklpq} H_{jp'q'}$, for the contraction to be zero; the Ramond-Ramond 5-form $F_{iklpq}$ is nonzero in the AdS background.
These arguments imply that $U'$ is in the image of $Q_{\text{Lie}}$. We can modify $W$ by adding to it:

$$\Delta W = Q_{\text{BRST}}Q_{\text{Lie}}^{-1}U'$$

(88)

Then we have:

$$Q_{\text{Lie}}(W + \Delta W) = 0$$

(89)

Now we use:

$$H^1(\mathfrak{g}, \text{Hom}_C(\mathcal{H}, \text{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}_0} \mathfrak{g}_{\text{odd}})) =$$

$$= H^1(\mathfrak{g}_0, \mathcal{H}'|_{\mathfrak{g}_0} \otimes_C \mathfrak{g}_{\text{odd}}) = 0$$

(90)

Therefore $W + \Delta W$ is in the image of $Q_{\text{Lie}}$.

Now the modified vertex:

$$\mathcal{V} + Q_{\text{BRST}}Q_{\text{Lie}}^{-1}(W + \Delta W)$$

(91)

is covariant.

Our procedure could perhaps be summarized as follows:

$$\mathcal{V}_{\text{covariant}} = \mathcal{V} + Q_{\text{BRST}}Q_{\text{Lie}}^{-1}(W + Q_{\text{BRST}}Q_{\text{Lie}}^{-1}(U - Q_{\text{Lie}}^{-1}Z))$$

(92)

where

$$W = Q_{\text{BRST}}Q_{\text{Lie}}\mathcal{V}$$

$$U = Q_{\text{BRST}}Q_{\text{Lie}}W$$

$$Z = Q_{\text{Lie}}U$$ (does not depend on $x, \theta$)

6 How the descent procedure works in the flat space limit.

In flat space it is impossible to choose a covariant vertex, because of the nontrivial cohomology

$$Z \in H^3(\text{super-Poincare algebra}, C)$$

(93)

which represents the NSNS 3-form field strength.

But one can satisfy a weaker covariance condition. Note that in flat space the generators of the Lorentz subalgebra $so(1,9)$ of the Poincare algebra can not be obtained as commutators of other generators. Therefore it is consistent
to require the covariance under all translations and supersymmetries, but
only some rotations. In particular, it turns out that we can choose a vertex
covariant under:

\[ \text{SP}_{\text{small}} = \{ \text{translations, supersymmetries, and } so(1, 4) \oplus so(5) \subset so(1, 9) \} \]  

This is a subalgebra of the super-Poincare algebra:

\[ \text{SP}_{\text{small}} \subset \text{SP} \]  

(94)
corresponding to the split of the space-time:

\[ \mathbb{R}^{1+9} = \mathbb{R}^{1+4}_A \times \mathbb{R}^5_S \]  

(95)
We will say that the ten spacetime directions split into 1 + 4 A-directions
and 5 S-directions (the letters A and S stand for the AdS and the sphere).

Let us now explain how the diagram of Fig. 4 works in flat space.

6.1 Maxwell field

Instead of considering Fig. 4 literally let us study the similar diagram
for the supersymmetric Maxwell field (rather than supergravity). This is a toy
model; the supersymmetric Maxwell field in flat space is “one half of the
supergravity field”. The “usual” (non-covariant) vertex operator is of the
form:

\[ V(x, \theta) = (\lambda \Gamma^\mu \theta) a_\mu + (\lambda \Gamma^\mu \theta)(\psi \Gamma^\nu \theta) - \frac{1}{4}(\theta \Gamma^{\mu\nu\rho\sigma})(\lambda \Gamma^\rho \theta) \partial_\mu a_\nu + \ldots \]  

(97)
where \( a_\mu = a_\mu(x) \) and \( \psi = \psi(x) \) are the vector potential and the photino.

6.1.1 Action of the Poincare algebra

Let us first try to understand if it is possible to choose the vertex covariant
under the even Poincare algebra. The vertex operator (97) involves the gauge
field \( a_\mu \). Because of the gauge invariance the gauge field is not in one to one
correspondence with the physical states. The physical states are described
by \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \), not by \( a_\mu \). To describe \( a_\mu \) in terms of \( f_{\mu\nu} \), let us break
the translational symmetries by choosing a point \( 0 \) in space-time. Then we
can write, in the vicinity of the chosen point:

\[ a = \iota_E \frac{1}{\mathcal{L}_E} f \]  

(98)
where $E = x^\mu \frac{\partial}{\partial x^\mu}$ and $\epsilon_E$ and $\mathcal{L}_E$ are the corresponding $\epsilon$ and Lie derivative. For example, $\mathcal{L}_E(xdx) = 2xdx$ and $\epsilon_E dx = x$. Note that Eq. (98) is one particular way to choose a vector potential with the field strength $f$.

Let us therefore replace $\epsilon$ with $\epsilon_E$. This breaks the translation symmetry, since the gauge (98) depended on a choice of point $x = 0$. Can we restore the translational symmetry? Let us introduce the operator $Q_{\text{Lie}}$ acting on the physical vertex operators in the following way:

$$Q_{\text{Lie}} V^I = e^\mu (t_{\mu I} V^I - t^I_{\mu J} V^J)$$

Here the index $I$ runs over an infinite set enumerating the basis vectors of $\mathcal{H}$, and $t_{\mu I}$ are the generators of translations $\frac{\partial}{\partial x^\mu}$. Fermionic parameters $e^\mu$ are the Lie-algebraic ghosts of the translation algebra. This operator $Q_{\text{Lie}}$ measures the deviation of the vertex operator from transforming covariantly under the action of the global shift. We observe that $Q_{\text{Lie}} V$ is $d$ of something:

Let us calculate $Q_{\text{Lie}}$ of this “something”:

$$Q_{\text{Lie}} \left( \epsilon_E \frac{1}{\mathcal{L}_E} f \right) = \left[ \mathcal{L}_c , \epsilon_E \frac{1}{\mathcal{L}_E} \right] f = \left( \epsilon_c \frac{1}{\mathcal{L}_E} \epsilon_E \frac{1}{\mathcal{L}_E} \frac{1}{\mathcal{L}_E} \right) f = d \left( \frac{1}{\mathcal{L}_E(\mathcal{L}_E + 1)} \epsilon_E \epsilon_c f \right)$$

(100)

Expanding $f$ in Taylor series around $x = 0$ and taking into account that $df = 0$ we can see that (101) is equal to:

$$\frac{1}{2} \epsilon_c^2 f(0)$$

(102)

We should stress that (101) is equal to the constant (independent of $x$) expression (102). In other words, the only term in the Taylor expansion of (101) around the point $x = 0$ is the constant term. One can see it, for example,
because the Lie derivative $\mathcal{L}_E$ of (101) vanishes. This can be seen from the identity $\mathcal{L}_E \iota^2_c f + 2 \iota_E \iota_c \mathcal{L}_c f = 0$ which follows from $df = 0$.

Notice that if we started with some other point $x_0$ (not the origin), then \(102\) would change by $Q_{\text{Lie}}$ of something. For example, an infinitesimal shift by $y$ would change \(102\) by the $Q_{\text{Lie}}$-exact expression:

$$\frac{1}{2} \iota^2_c y^\rho \partial_\rho f(0) = y^\rho c^\mu c^\nu \partial_\mu f_{\nu\rho}(0) = -c^\mu c^\nu \partial_\mu f_{\nu\rho} y^\rho = Q_{\text{Lie}}(f_{\nu\rho} c^\nu y^\rho) \quad (103)$$

(This is a manifestation of the general fact, that a Lie algebra acts trivially in its cohomology.)

The $Q_{\text{Lie}}$-cohomology class of:

$$\iota^2_c f(0) = f_{\mu\nu}(0) c^\mu c^\nu \quad (104)$$

is the obstacle for defining $a$ such that $f = da$ in a covariant way.

We have so far discussed only the action of shifts. The expression \(104\) as we defined it represents the cohomology class of the algebra of translations $\mathbb{R}^{1+9}$. But we can also think of it as a cocycle of the Poincare algebra. Indeed, $f_{\mu\nu}$ transforms covariantly under rotations and boosts and therefore \(104\) is closed under the $Q_{\text{Lie}}$ of the full Poincare algebra.

The $Q_{\text{Lie}}$ of the full Poincare algebra is the sum of $Q_{\text{Lie}}^{\text{translations}}$ of translations $\mathbb{R}^{1+9}$ and $Q_{\text{Lie}}^{\text{Lorentz}}$ of rotations and boosts. Expression \(104\) is in the kernel of $Q_{\text{Lie}}^{\text{translations}}$ by our construction, and more explicitly because $f$ is a closed form. But it is also in the kernel of $Q_{\text{Lie}}^{\text{Lorentz}}$ because $f$ transforms covariantly under rotations and boosts.

Another question is whether or not \(104\) is exact. One can see that this is not exact as a cocycle of the full Poincare algebra, in the following way. Let $P$ stand for the Poincare algebra. We have:

$$f_{\mu\nu}(0) c^\mu c^\nu \in H^2(P, \mathcal{H}') \quad (105)$$

Notice that the space of states of the gauge field contains a proper subspace closed under the action of the Poincare algebra. (In other words, it is not an irreducible representation.) This subspace consists of those gauge fields which have a constant field strength: $f_{\mu\nu}(x) = f_{\mu\nu}(0)$. Let us call this subspace $\mathcal{H}_{\text{zero-modes}}$:

$$\mathcal{H}_{\text{zero-modes}} \subset \mathcal{H} \quad (106)$$
Therefore there is a projection
\[ \mathcal{H}' \to (\mathcal{H}_{\text{zero-modes}})' \] (107)
This projection naturally acts on the cocycles of \( \mathbb{P} \) with values in \( \mathcal{H}' \), and therefore on the cohomology groups:
\[ \mathcal{H}' \otimes \Lambda \mathbb{P} \to (\mathcal{H}_{\text{zero-modes}})' \otimes \Lambda \mathbb{P} \]
\[ H^\bullet(\mathbb{P}, \mathcal{H}') \to H^\bullet(\mathbb{P}, (\mathcal{H}_{\text{zero-modes}})') \] (108)
It is straightforward to see that the projection of (104) to \((\mathcal{H}_{\text{zero-modes}})' \otimes \Lambda \mathbb{P}\) is automatically a nonzero cohomology class.

Indeed \( \mathcal{H}'_{\text{zero-modes}} \) transforms as antisymmetric rank 2 tensor of the Lorentz algebra, and trivially under translations. Therefore the cohomology complex of the Poincare algebra with coefficients in \( \mathcal{H}'_{\text{zero-modes}} \) is equivalent to the cohomology complex of the Lorentz algebra with coefficients in \( \Lambda^2 \mathbb{R}^{1+9} \otimes \Lambda \mathbb{R}^{1+9}; \) the projection of (104) is in \( H^0(\text{Lorentz}, (\Lambda^2 \mathbb{R}^{1+9} \otimes \Lambda^2 \mathbb{R}^{1+9})_{\text{inv}}) \).

This implies that (104) represents a nontrivial cohomology class in \( H^2(\mathbb{P}, \mathcal{H}') \). This is what prevents us from choosing the vertex covariant with respect to the Poincare algebra.

### 6.1.2 The obstacle (104) vanishes after we break \( \mathbb{P} \) to \( \mathbb{P}_{\text{small}} \)

Let us start with introducing some notations. For any vector \( v^\mu \) we denote \( \overline{v} \) the vector with the components:
\[ \overline{v}^\mu = \begin{cases} v^\mu & \text{if } \mu \in \{0, 1, \ldots, 4\} \\ -v^\mu & \text{if } \mu \in \{5, \ldots, 9\} \end{cases} \] (109)
Also introduce:
\[ 2\Delta_S = \partial_\mu \overline{v}_\mu - \overline{v}_\mu \partial_\mu = 2 \sum_{i=5, \ldots, 9} \left( \frac{\partial}{\partial x^i} \right)^2 \] (110)

What happens if we do not require the invariance under the full Poincare algebra \( \mathbb{P} \), but only under the \( \mathbb{P}_{\text{small}} \) of (94)? Then we can restrict ourselves to the subspace of \( \mathcal{H} \) where \( -\Delta_S \) is a fixed positive number. On this subspace, it is possible to express \( a_\mu \) in terms of \( f_{\mu\nu} \) in a \( \mathbb{P}_{\text{small}} \)-covariant way. Let us choose the covariant gauge:
\[ \partial^\mu a_\mu = 0 \] (111)
and fix the residual gauge transformations with the additional “axial” gauge
gauge condition:
\[ \overline{\partial}^\mu a_\mu = 0 \]  
(112)
where \( \overline{\partial} \) is introduced as in (109). In the gauge (112) we can express the
gauge field \( a_\mu \) in terms of the gauge field strength \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \):
\[ a_\mu = \frac{1}{2\Delta S} \partial_\nu f_{\mu\nu} \]  
(113)
Now we have two different expressions for the vector potential, Eq. (98)
and Eq. (113). The difference between these two expressions is a gauge
transformation. Let us use a “diacritical” mark to distinguish (113) from
(98):
\[ \hat{a}_\mu = \frac{1}{2\Delta S} \partial_\nu f_{\mu\nu} \text{ vs. } a_\mu = \left( \iota_E \frac{1}{\mathcal{L}_E} f \right)_\mu \]  
(114)
Note that \( \hat{a} \) is \( \mathbf{P}_{\text{small}} \)-covariant:
\[ Q_{\text{Lie}} \hat{a} = 0 \]  
(115)
while \( a \) is not — see Eq. (100). Also, for every gauge field we can calculate
\( \hat{a}(0) \). This is a functional of the gauge field, \( \text{i.e.} \) an element of \( \mathcal{H}' \). If we
calculate its \( Q_{\text{Lie}} \) as a cochain with values in \( \mathcal{H}' \) we get:
\[ Q_{\text{Lie}}( \hat{a}_\mu(0) ) = -Q_{\text{Lie}}(\iota_{\mathcal{L}_E} \hat{a}_\mu(0)) \]  
(116)
Now let us return to Eqs. (100), (101) and (102). On the subspace \( -\Delta_S = \text{const} > 0 \) the cohomology class of (102) trivializes:
\[ e^{\mu_\nu} f_{\mu\nu}(0) = Q_{\text{Lie}}(\iota_{\mathcal{L}_E} \hat{a}(0)) \]  
(117)
This is analogous to Eq. (84) of Section 5.3. Therefore the same arguments
as we presented in Section 5.3 should imply that
\[ \frac{1}{\mathcal{L}_E(\mathcal{L}_E + 1)} \iota_E \iota_c f(x) - \iota_c \hat{a}(0) = Q_{\text{Lie}}(\text{something}) \]  
(118)
Indeed we have:
\[ \frac{1}{\mathcal{L}_E(\mathcal{L}_E + 1)} \iota_E \iota_c f(x) - \iota_c \hat{a}(0) = \]  
(119)
\[ = Q_{\text{Lie}} \left( -\iota_E \left( \hat{a}(0) + \frac{1}{\mathcal{L}_E}(\hat{a} - \hat{a}(0)) \right) \right) \]
Also notice that:

\[
d \left( t_E \left( \dot{a}(0) + \frac{1}{L_E} (\dot{a} - \dot{a}(0)) \right) \right) = \dot{a} - a \quad (120)
\]

This means that the correction of the vector potential:

\[
a \rightarrow \dot{a} \quad (121)
\]

is completely analogous to the correction of the vertex operator described in Section 5.3. It turns the non-covariant expression \( a_\mu dx^\mu \) into the covariant expression \( \dot{a}_\mu dx^\mu \).

### 6.1.3 Action of the supersymmetry

Let us now study the action of the supersymmetry generators. Let us consider the part of the \( Q_{\text{Lie}} \) involving the supersymmetry generators. We have:

\[
Q_{\text{Lie}} = Q_{\text{Lie}}^{(x, \theta)} + Q_{\text{Lie}}^H
\]

\[
Q_{\text{Lie}}^{(x, \theta)} = \xi^\alpha \frac{\partial}{\partial \theta^\alpha} - \xi^\alpha \Gamma^\mu_{\alpha \beta} \theta^\beta \frac{\partial}{\partial x^\mu} \quad (123)
\]

\[
Q_{\text{Lie}}^H = \xi^\alpha t_\alpha \quad (124)
\]

\[
Q_{\text{BRST}} = \lambda^\alpha \frac{\partial}{\partial \theta^\alpha} + \lambda^\alpha \Gamma^\mu_{\alpha \beta} \theta^\beta \frac{\partial}{\partial x^\mu} \quad (125)
\]

Here \( t_\alpha \) is the supersymmetry transformation acting on the space of states \( \mathcal{H} \) and therefore (after fixing the gauge!) on \( a_\mu \) and \( \psi \). We write only the part of \( Q_{\text{Lie}} \) corresponding to the super-translations; \( \xi^\alpha \) are the bosonic ghosts corresponding to the super-translations. To make \( t_\alpha \) act on \( a \) and \( \psi \) we have to choose the gauge.

With this notation, let us first of all present \( Q_{\text{Lie}}^{(x, \theta)} \) acting on the vertex
operator \( \Pi \) in the following form:

\[
\varepsilon'^{x, \theta} Q_{\text{Lie}} \varepsilon V = (\varepsilon \Lambda \Gamma^\mu \varepsilon' \xi) a_\mu - (\varepsilon' \xi \Gamma^\nu \theta)(\varepsilon \Lambda \Gamma^\mu \theta) \partial_\nu a_\mu - \\
- \frac{1}{2} (\varepsilon' \xi \Gamma^{\mu \rho} \theta)(\varepsilon \Lambda \Gamma^\rho \theta) \partial_{[\mu} a_{\nu]} - \frac{1}{4} (\theta \Gamma^{\mu \rho \theta})(\varepsilon \Lambda \Gamma^\rho \theta)(\varepsilon' \xi) \partial_{[\mu} a_{\nu]} + \\
+(\varepsilon \Lambda \Gamma^\mu \varepsilon' \xi)(\psi \Gamma^\mu) + (\varepsilon \Lambda \Gamma^\mu \theta)(\psi \Gamma^\mu \varepsilon' \xi) + \ldots = \\
= - \frac{3}{2} (\varepsilon \Lambda \Gamma^\mu \theta) \left( (\varepsilon' \xi \Gamma^\mu) + \frac{1}{2 \Delta S} \partial_\mu (\varepsilon' \xi \Gamma^\rho \partial_\rho \psi) \right) - \\
- \frac{2}{3} (\varepsilon \Lambda \Gamma^\rho \theta)(\varepsilon' \xi \Gamma^{\mu \rho} \Gamma^\theta \theta) \partial_{[\mu} a_{\nu]} + \ldots \\
+ \varepsilon Q_{\text{BRST}} \left( \frac{1}{2} (\theta \Gamma^\mu \varepsilon' \xi) (\psi \Gamma^\mu \theta) + \frac{3}{2 \Delta S} (\varepsilon' \xi \Gamma^\rho \partial_\mu \psi) + \\
(\theta \Gamma^\mu \varepsilon' \xi) a_\mu - \frac{1}{12} (\theta \Gamma^{\mu \rho \theta})(\theta \Gamma^\rho \varepsilon' \xi) \partial_{[\mu} a_{\nu]} + \ldots \right) \\
(126)
\]

This implies that

\[
\varepsilon'^{x, \theta} Q_{\text{Lie}} a_\mu = - \frac{3}{2} \left( (\varepsilon' \xi \Gamma^\mu) + \partial_\mu \frac{1}{2 \Delta S} (\varepsilon' \xi \Gamma^\rho \partial_\rho \psi) \right) \\
(127)
\]

\[
\varepsilon'^{x, \theta} Q_{\text{Lie}} \psi = - \frac{2}{3} \varepsilon' \xi \Gamma^{\mu \nu} \partial_{[\mu} a_{\nu]} \\
(128)
\]

Therefore we have indeed:

\[
\varepsilon'^{x, \theta} Q_{\text{Lie}} \varepsilon V = \varepsilon Q_{\text{BRST}} \left( (\theta \Gamma^\mu \varepsilon' \xi) a_\mu + \frac{3}{2 \Delta S} (\varepsilon' \xi \Gamma^\rho \partial_\mu \psi) + \ldots \right) \\
(129)
\]

On the right hand side \( Q_{\text{BRST}} \) is taken of the expression which is \( Q_{\text{Lie}} \)-exact:

\[
(\theta \Gamma^\mu \varepsilon' \xi) a_\mu + \frac{3}{2 \Delta S} (\varepsilon' \xi \Gamma^\rho \partial_\mu \psi) + \ldots = \\
= \varepsilon Q_{\text{Lie}} \left( \frac{3}{2 \Delta S} (\theta \Gamma^\rho \partial_\mu \psi) - \frac{1}{2 \Delta S} (\theta \Gamma^{\mu \rho \sigma}) \partial_\mu \partial_\rho a_\sigma + \ldots \right) \\
(130)
\]

Then we have:

\[
Q_{\text{BRST}} \left( \frac{3}{2 \Delta S} (\theta \Gamma^\rho \partial_\mu \psi) - \frac{1}{2 \Delta S} (\theta \Gamma^{\mu \rho \sigma}) \partial_\mu \partial_\rho a_\sigma + \ldots \right) = \\
= \frac{1}{2 \Delta S} \left( \frac{3}{2} (\theta \Gamma^\rho \partial_\mu \psi) + \frac{3}{2} (\theta \Gamma^\nu \lambda)(\theta \Gamma^\rho \partial_\nu \partial_\mu \psi) - (\lambda \Gamma^{\mu \rho \sigma}) \partial_\mu \partial_\rho a_\sigma + \ldots \right) \\
(131)
\]

31
This means that the following vertex operator:

\[
\tilde{V} = -\frac{3}{2\Delta_S} (\lambda \Gamma^\mu \partial_\mu \psi) + (\lambda \Gamma^\mu \theta) a_\mu + \frac{1}{2\Delta_S} (\lambda \Gamma^\mu \Gamma^{\nu \sigma} \theta) \partial_\mu \partial_\nu a_\sigma + \ldots
\]

transforms covariantly under the odd shifts. This can be also understood as follows:

\[
\tilde{V} = -\frac{3}{2\Delta_S} (\lambda \Gamma^\mu \partial_\mu \psi) + (\lambda \Gamma^\rho \Gamma^{\mu \nu} \theta) \frac{1}{2\Delta_S} \partial_\rho \partial_\mu a_\sigma + \ldots
\]

Now we recognize what it is:

\[
\tilde{V}(x, \theta, \lambda) = \frac{1}{2\Delta_S} \lambda^\alpha \Gamma^\mu_{\alpha \beta} \partial_\mu W^\beta(x, \theta)
\]

where \( W^\alpha(x, \theta) \) is the superfield related to the Maxwell superfield \( A_\alpha(x, \theta) \) by the chain of transformations:

\[
\mathcal{T}_{(\alpha A_\beta)} = \frac{1}{2} \Gamma^\mu_{\alpha \beta} A_\mu
\]

\[
\mathcal{T}_\alpha A_\mu - \mathcal{T}_\mu A_\alpha = g_{\mu \nu} \Gamma^\nu_{\alpha \beta} W^\beta
\]

See [20] for a recent discussion of \( W^\alpha \).

### 6.2 Supergravity field.

In flat space the supergravity fields split into the product of the left and the right component; the left and right components are essentially free Maxwell fields. The bispinor field is defined as follows:

\[
P^{\alpha \dot{\beta}} = W^\alpha_L W^\dot{\beta}_R
\]

where \( W^\alpha \) is the field strength superfield of the free Maxwell theory; the \( \theta = 0 \) component of \( W^\alpha \) is the gaugino \( \psi^\alpha \).

This bispinor field is a linear combination of the RR field strengths contracted with the gamma-matrices [21]:

\[
P^{\alpha \dot{\beta}} = F_{lmnp} e^l_\alpha e^m_\beta e^n_\epsilon e^p_\gamma (\Gamma^{abcde})^{\alpha \dot{\beta}} +
\]

\[
+ a_3 F_{lmn} e^l_\alpha e^m_\beta e^n_\epsilon (\Gamma^{abc})^{\alpha \dot{\beta}} + a_1 F_{l} e^l_\alpha (\Gamma^{\alpha})^{\alpha \dot{\beta}}
\]

\(^8\)I want to thank Yuri Aisaka for discussions about \( W^\alpha \).
where $a_3$ and $a_1$ are some numeric coefficients which we do not need. The supersymmetry variations of $P$ is given by this equation:

\begin{align}
\delta^\beta_\alpha P^{\beta\dot{\beta}} &= \delta^\beta_\alpha C^{\beta\dot{\beta}} + (\Gamma_{mn})^\beta_\dot{\alpha} C^{\dot{\beta}mn} \\
\delta^\dot{\beta}_\dot{\alpha} P^{\beta\dot{\beta}} &= \delta^\dot{\beta}_\dot{\alpha} C^{\beta\dot{\beta}} + (\Gamma_{mn})^\dot{\beta}_\dot{\alpha} C^{\beta\dot{\beta}mn}
\end{align}

(139, 140)

where $C^\beta$ is a combination of the left dilatino $\chi$, and the left gravitino field strength $\partial_{[m}\bar{\psi}_{n]}$, and $C^{\dot{\beta}}$ is a combination of the corresponding right fields $\tilde{\chi}$ and $\partial_{[m}\tilde{\bar{\psi}}_{n]}$.

The $\mathbf{Sp}_{small}$-covariant vertex in flat space is the product of two expressions of the form (134):

\[ \tilde{V} = (\lambda \Gamma^\mu \Delta^{-2}_S \partial_\mu \partial_\nu P \bar{T}^{\nu} \tilde{\lambda}) \]  (141)

This is BRST equivalent to:

\[ \tilde{V}' = (\lambda \Gamma^\mu \Delta^{-1}_S P \Gamma_\mu \tilde{\lambda}) \]  (142)

6.3 Relation between the covariant vertex in $AdS_5 \times S^5$ and the flat space expressions (141), (142).

The construction of Section 5 implies that the $PSU(2,2|4)$-covariant vertex operator exists in $AdS_5 \times S^5$. This construction gives (141) when applied in the flat space limit. (We have demonstrated this for the Maxwell field, but the free supergravity vertex in flat space is just the product of “left” and “right” Maxwell vertices.) Therefore both (141) and the BRST-equivalent (142) should be the flat space limits of some covariant vertices in $AdS_5 \times S^5$.

Notice that (142) reduces to (26) in the zero momentum limit, except for the overall normalization factor $\frac{1}{\Delta_S}$ which becomes singular on the zero mode.

However we were not able to write explicit expressions in terms of the supergravity fields in AdS space which would explicitly generalize the flat space formulas (141) or (142).

\footnote{I want to thank N. Berkovits for suggesting this simplified form, and many other useful comments}
7 Covariant vertex and the endpoint of the Wilson line

7.1 The BRST complex of the endpoint

Consider the Wilson line operator corresponding to a semi-infinite contour going from infinity to some point $B$ on the string worldsheet:

\[ B \allowbreak \quad \rho \allowbreak \quad \infty \]

in some representation $\rho$ of $psu(2, 2|4)$. Consider the action of $Q_{BRST}$ on this operator. If we neglect the contribution of the boundary terms at infinity, then the BRST variation is:

\[ \varepsilon Q_{BRST} T[C_B^\infty](z) = \left( \frac{1}{z} \varepsilon \lambda^3_3(B) \rho(t^3_3) + z \varepsilon \lambda^1_1(B) \rho(t^1_1) \right) T[C_B^\infty](z) \tag{143} \]

(See Section 2.2 of [22] and Section 7 of [7] for a discussion of this formula.)

Let us fix some vector $\psi$ in the representation $\rho$ “at infinity”; then this expression:

\[ T[C_B^\infty](z) \psi \]

is a vector in the representation space of $\rho$. Pick a vector $v$ in the dual space, and evaluate it on (144):

\[ v \left( T[C_B^\infty](z) \psi \right) \in C \tag{145} \]

This gives a number. Consider vectors $v$ depending on the pure spinors $\lambda_3, \lambda_1$ and the spectral parameter $z$. Then Eq. (143) can be regarded as defining the action of $Q_{BRST}$ on $v$:

\[ Q_{\text{endpoint}} v = \left( \frac{1}{z} \lambda^3_3 \rho(t^3_3) + z \lambda^1_1 \rho(t^1_1) \right) v \tag{146} \]

This defines the BRST complex of the endpoint. The $n$-cochains of this complex are elements

\[ v \in \text{Hom}_{\mathfrak{g}_0} \left( \begin{bmatrix} \text{linear space of the representation } \rho \text{ in which we evaluate Wilson line} \end{bmatrix} , \mathcal{P}^n \right) \]
where $\mathcal{P}^n$ is defined in Section 4.2 and the differential is (146). The “plugs” which we introduced in Section 1.1.2 are the cohomologies of this complex. Unfortunately we do not know a general classification of the cohomologies of this complex for a general representation $\rho$.

7.2 The BRST complex of the Wilson line endpoint is isomorphic to the BRST complex of covariant vertices

We will now consider the special case where $\rho$ is the representation of $\text{psu}(2,2|4)$ on the space of states $\mathcal{H}$ of the BPS multiplet. In this case we will relate the BRST complex of the endpoint (146) to the BRST complex of covariant supergravity vertices.

7.2.1 Frobenius reciprocity

Let us remember the general structure of the covariant vertex from Section 4.5:
\[
\mathcal{V} \in \text{Hom}_g(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^n) = \text{Hom}_g(\mathcal{H}, \text{Hom}_{g_0}(\mathcal{U}_g, \mathcal{P}^n))
\]

Here $n = 2$ for the supergravity vertex, but we want to consider the whole BRST complex so we keep $n$. Let us evaluate $\mathcal{V}$ on the unit of the group:
\[
\text{Hom}_g(\mathcal{H}, \text{Hom}_{g_0}(\mathcal{U}_g, \mathcal{P}^n)) \ni \mathcal{V} \quad \text{evaluate on } 1 \in \mathcal{U}_g
\]

This defines a correspondence between covariant vertices $\mathcal{V}$ and vectors in $\text{Hom}_{g_0}(\mathcal{H}, \mathcal{P}^n)$:
\[
\text{Hom}_g(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^n) \ni \mathcal{V} \quad \text{evaluate on } 1 \rightarrow v \in \text{Hom}_{g_0}(\mathcal{H}, \mathcal{P}^n) \quad (148)
\]

Notice that $\mathcal{V}$ is a function of $x, \theta$ while $v$ is essentially its value at $x = \theta = 0$. Nevertheless, the correspondence (148) is a one-to-one correspondence between the elements of $\text{Hom}_g(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^n)$ and the elements of $\text{Hom}_{g_0}(\mathcal{H}, \mathcal{P}^n)$. Indeed, the symmetry under $g$:
\[
\text{Hom}_g(\mathcal{H}, \text{Coind}_{g_0}^g \mathcal{P}^n)
\]
allows to relate $\mathcal{V}(\Psi)(g)$ to $\mathcal{V}(g^{-1}\Psi)(1)$, see Eq. (62). In other words, if we know the value of the covariant vertex at the point $x = \theta = 0$ for all states $\Psi$, then because of the global symmetry we know the covariant vertex everywhere (for arbitrary $x$ and $\theta$). This construction is an example of the Frobenius reciprocity:

$$\text{Hom}_g(\mathcal{H}, \text{Coind}^{g_0}_g L) \cong \text{Hom}_{g_0}(\mathcal{H}^{g_0}, L)$$

which is true for any representation $L$ of $g_0$; in our case $L = \mathcal{P}^n$.

To summarize, given the covariant vertex $\mathcal{V}$, we define $v \in \text{Hom}_{g_0}(\mathcal{H}, \mathcal{P}^n)$ by saying that the value of $v$ on $\Psi \in \mathcal{H}$ is:

$$v(\Psi) = \mathcal{V}(\Psi)(1) \quad (149)$$

7.2.2 The action of $Q_{\text{BRST}}$

Note that both the left hand side and the right hand side of (149) are elements of $\mathcal{P}^n$ i.e. polynomials of $\lambda_3$ and $\lambda_1$. We can evaluate them on $\lambda$:

$$v(\Psi)(\lambda_3, \lambda_1) = \mathcal{V}(\Psi)(1)(\lambda_3, \lambda_1) \quad (150)$$

This is a quadratic polynomial in $\lambda_3$ and $\lambda_1$. Eqs. (62) and (59) imply that the action of $Q_{\text{BRST}}$ on the covariant vertex corresponds to the following action on $v$:

$$(Q_{\text{BRST}}v)(\Psi)(\lambda_3, \lambda_1) = -v(\lambda_3 \Psi + \lambda_1 \Psi)(\lambda_3, \lambda_1) \quad (151)$$

This formula for $Q_{\text{BRST}}$ can be interpreted in the following way. The space $\text{Hom}_C(\mathcal{H}, \mathcal{P}^2)$ is obviously a representation of $\mathfrak{g} = \text{psu}(2,2|4)$, just because $\mathcal{H}$ is by definition a representation of $\mathfrak{g}$. (The $\mathcal{P}^2$ part just “goes along for the ride”.) Let us denote this representation $\rho$ (the action of $x \in \mathfrak{g}$ on $v$ is $\rho(x)v$). Then (151) implies that the action of $Q_{\text{BRST}}$ on $\text{Hom}_g(\mathcal{H}, \text{Coind}^{g_0}_g \mathcal{P}^2)$ corresponds to the action of the nilpotent operator $Q_{\text{endpoint}}$ on $v$ defined by this formula:

$$Q_{\text{BRST}}v(\lambda_3, \lambda_1) = (\lambda_3^3 \rho(t^3_\alpha) + \lambda_1^1 \rho(t^1_\alpha))v(\lambda_3, \lambda_1) \quad (152)$$

This is identical to $Q_{\text{endpoint}}$ of (146) at $z = 1$. We conclude that:
\( v \) represents a cohomology class \( H^2(Q_{\text{endpoint}}) \) of the following complex:

\[
\ldots \longrightarrow \mathcal{H}' \otimes_{\mathfrak{g}_0} \mathcal{P}^n \overset{Q_{\text{endpoint}}}{\longrightarrow} \mathcal{H}' \otimes_{\mathfrak{g}_0} \mathcal{P}^{n+1} \longrightarrow \ldots
\]  

(153)

In other words, the BRST complex on covariant massless vertices (independent of derivatives) is equivalent to the endpoint complex \( \text{Hom}_{\mathfrak{g}_0}(\mathcal{H}, \mathcal{P}^\bullet) \).

7.2.3 Including the spectral parameter \( z \) corresponds to rescaling the pure spinors.

We have demonstrated that the complex (152) is equivalent to (146) at \( z = 1 \). But in fact (146) at \( z = 1 \) is equivalent to (146) at \( z \neq 1 \) by rescaling of \( \lambda_3 \) and \( \lambda_1 \). In other words, the map

\[
v \mapsto v'
\]

\[
v'(\lambda_3, \lambda_1) = v(z^{-1}\lambda_3, z\lambda_1)
\]  

(154)

is the equivalence of the complex (146) at \( z = 1 \) and the same complex at \( z \neq 1 \).

7.3 Endpoint BRST complex and the Lie algebra cohomology

There is a relation between the endpoint cohomology and the cohomology of the positive-frequency part of the loop algebra of \( psl(4|4) \).

Consider the algebra formed by the positive frequency \( \mathbb{Z}_4 \)-twisted loops with values in \( psl(4|4) \). We will denote is \( L_+ \mathfrak{g} \). The cohomology complex is generated by the ghosts \( c_{-k}^a \), where \( k \in \{1, 2, 3, \ldots\} \) and \( a \) the enumerates the adjoint representation of \( psl(4|4) \). We have \( c_{-1}^\alpha, c_{-2}^{m}, c_{-3}^{\dot{\alpha}}, c_{-4}^{[mn]}, c_{-5}^{\alpha}, \text{etc.} \) The “energy” operator \( L_0 \) counts the lower indices, for example:

\[
L_0 c_{-3}^{\dot{\alpha}} = -3 c_{-3}^{\dot{\alpha}}, \quad L_0 c_{-1}^{\alpha} c_{-4}^{[mn]} = -5 c_{-1}^{\alpha} c_{-4}^{[mn]}
\]

Notice that \( L_0 \) is a symmetry of the cohomology complex. Another symmetry is the \( c \)-ghost number (the number of letters \( c \)). Let \( H^p(L_+ \mathfrak{g}, \mathcal{C}) \) denote the cohomology group with \( L_0 = q \) and ghost number \( p \). The first cohomology
group $H^1(L_+g, C)$ is generated by $e^a_{-1}$ (and therefore has $L_0 = -1$). Some of other nontrivial cohomology groups are:

\[ H^2_{-2}(L_+g, C) : \quad X_{\alpha\beta} c_{-1}^\alpha c_{-1}^\beta, \quad f_m^{\alpha\beta} X_{\alpha\beta} = 0 \quad (155) \]

\[ H^3_{-3}(L_+g, C) : \quad X_{\alpha\beta\gamma} c_{-1}^\alpha c_{-1}^\beta c_{-1}^\gamma, \quad f_m^{\alpha\beta\gamma} X_{\alpha\beta\gamma} = 0 \quad (156) \]

Generally speaking, $H^k_{-k}$ is generated by the expressions of the form:

\[ H^k_{-k}(L_+g, C) : \quad X_{\alpha_1 \cdots \alpha_k} c_{-1}^{\alpha_1} \cdots c_{-1}^{\alpha_k}, \quad f_m^{\alpha_1 \alpha_2} X_{\alpha_1 \alpha_2 \cdots \alpha_k} = 0 \quad (157) \]

But $H^k_{-k}$ is not all of the cohomology, for example there is nontrivial \( H^2_{-4} \). The pure spinor cohomology should be identified with the part of the $L_+g$ cohomology with “energy” equal to minus the ghost number, i.e. $H^k_{-k}$.

8 Application: vertex operators depending on the spectral parameter

8.1 How to introduce the spectral parameter into the vertex operator

In flat infinite space massless vertex operators have the form:

\[ V_k(x, \theta) = p(\lambda, \theta) e^{ikX} \quad (158) \]

where $p$ is some polynomial of $\lambda$ and $\theta$. We can write

\[ X(\tau^+, \tau^-) = X_L(\tau^+) + X_R(\tau^-) \quad (159) \]

where

\[ X_L = \int_{-\infty}^{(\tau^+, \tau^-)} d\tau^+ \partial_+ X \quad \text{and} \quad X_R = \int_{-\infty}^{(\tau^+, \tau^-)} d\tau^- \partial_- X \quad (160) \]

Therefore there is a generalization of (158):

\[ V_{k_L,k_R}(x, \theta) = p(\lambda, \theta) e^{ik_L X_L + ik_R X_R} \quad (161) \]

This generalization is only well defined in flat space on those worldsheets which do not have handles, or in toroidal compactifications with appropriate

\[ ^{10} \text{this can be seen from the comparison of the character of the } H^*(L_+g, C) \text{ with the character of} \]
integrality conditions on $k_L$ and $k_R$. We can formally consider (161), for example on an infinite worldsheet without handles, if we neglect boundary effects.

We will now argue that there is a partial analogue of (161) in $AdS_5 \times S^5$.

Given a state $\Psi \in \mathcal{H}$ we can prepare a nonlocal $z$-dependent covariant vertex operator, in the following way. Consider the transfer matrix $T_{(\infty)}^{(\tau^+,\tau^-)}(z)$ from infinity to the point $(\tau^+, \tau^-)$ on the worldsheet. Let us fix a vector $\Psi^{(\infty)} \in \mathcal{H}$, and consider:

$$V_{\Psi^{(\infty)}}(\tau^+, \tau^- | z) = v \left( T_{(\infty)}^{(\tau^+,\tau^-)}(z) \Psi^{(\infty)} \right) ( z^{-1} \lambda_3(\tau^+, \tau^-) , z \lambda_1(\tau^+, \tau^-) )$$  (162)

So defined $V_{\Psi^{(\infty)}}(\tau^+, \tau^- | z)$ is analogous to:

$$e^{ikz-1}X_L + ikzX_R$$  (163)

In particular for $z = 1$ we get $T_{(\infty)}^{(\tau^+,\tau^-)}(1) = g(\tau^+, \tau^-)g(\infty)^{-1}$ and therefore:

$$V_{\Psi^{(\infty)}}(\tau^+, \tau^- | 1) = v \left( g(\tau^+, \tau^-)g(\infty)^{-1} \Psi^{(\infty)} \right) ( \lambda(\tau^+, \tau^-) )$$  (164)

This formula gives us back the covariant vertex for the state $\Psi$ if we identify:

$$g(\infty)^{-1} \Psi^{(\infty)} = \Psi$$  (165)

### 8.2 Is there a 2-point vertex operator?

For any $u \in \mathcal{H}'$ we define $u^\dagger$ as a non-normalizable vector in the space of states, characterized by the formula:

$$u(\Psi) = (u^\dagger, \Psi) \text{ for any } \Psi \in \mathcal{H}$$  (166)

where $(,)$ is the hermitean scalar product in $\mathcal{H}$. Note that $u^\dagger$ strictly speaking does not belong to $\mathcal{H}$ because it is not normalizable. For example, one dimensional quantum mechanics has $\mathcal{H} = L^2(\mathbb{R})$ — the space of square integrable functions of one variable, with the norm $||f||^2 = \int dx |f(x)|^2$. The dual space $\mathcal{H}'$ is the space of generalized functions; if $u \in \mathcal{H}'$ is defined by the formula $u(f) = f(0)$ then $u^\dagger$ is a delta-function $\delta(x)$.

Using these notations we can define the two-point vertex operator:

$$\Psi \quad u^\dagger$$
\[ V^{2pt}(\tau_1^+, \tau_1^-), (\tau_2^+, \tau_2^-) = v \left( \frac{T}{\tau_2} \right) \left( \lambda(\tau_2^+, \tau_2^-) \right) \left( \lambda(\tau_1^+, \tau_1^-) \right) \] (167)

However, we conjecture that this 2-point vertex operator is in fact BRST exact. Indeed, although we have not checked it explicitly, it should be true that the derivative of \[ V^{2pt}(\tau_1^+, \tau_1^-), (\tau_2^+, \tau_2^-) \] with respect to \( \tau_1 \) is \( Q_{BRST} \)-exact. Therefore, up to BRST-exact terms this vertex is independent of \( \tau_1 \) and \( \tau_2 \). On the other hand, when \( \tau_1 \rightarrow \tau_2 \) we get a local vertex operator of the ghost number 4. There is no \( psu(2,2|4) \)-invariant cohomology at the ghost number 4. This implies that (167) is BRST exact.

### 9 Conclusions

In this paper we introduced a family of \( z \)-dependent vertex operators \( \{162\} \) parametrized by a choice of the BPS representation of \( psu(2,2|4) \). Schematically, these vertex operators have a form:

\[ V_{\Psi,\infty}(\tau^+, \tau^-|z) = \left\langle \text{plug}(\tau^+, \tau^-|z) \left| P \exp \left( -\int_{\infty}^{(\tau^+, \tau^-)} J[z] \right) \left| \Psi,\infty \right. \right. \right\rangle \] (168)

This expression is strictly speaking not BRST invariant, because of the boundary term at infinity. Indeed we have put \( \Psi,\infty \) an arbitrary vector from \( \mathcal{H} \), and this is generally speaking not a valid plug. We assume that we can neglect this boundary term because it is at infinity\(^{11} \). We can consider \( V_{\Psi,\infty}(\tau^+, \tau^-|z) \) locally near the point \((\tau^+, \tau^-)\). Notice that \( \left\langle \text{plug}(\tau^+, \tau^-) \right| \) is a \( \lambda \)-dependent vector in the dual space to \( \mathcal{H} \). (In fact \( \left\langle \text{plug}(\tau^+, \tau^-) \right| \) depends on \( \tau^+ \) and \( \tau^- \) through \( \lambda(\tau^+, \tau^-) \).) We can also think of \( \left\langle \text{plug}(\tau^+, \tau^-) \right| \) as an element of \( \mathcal{H} \), but then we have to remember that it is not normalizable; it is a \( \delta \)-function type of state, rather than a proper wave packet. Note that for a fixed \( \lambda \), our \( \left\langle \text{plug} \right| \) is a fixed vector in \( \mathcal{H}' \). In other words, for every BPS representation \( \mathcal{H} \) we have a map, which takes a pair of pure spinors and transforms them into a vector in the space of BPS states:

\[
\begin{array}{c}
\text{pure spinors} \\
\lambda_3, \lambda_1 \\
\rightarrow
\end{array}
\begin{array}{c}
a \text{non-normalizable} \\
\text{vector in} \mathcal{H} \text{ which we call} \langle \text{plug} \rangle
\end{array}
\] (169)

\(^{11}\)an attempt to bring the second endpoint from infinity is described in Section 8.2

40
It would be interesting to describe this map explicitly. The non-normalizable vector in \( \mathcal{H} \) on the right hand side of (169) is obviously not invariant under \( \text{psu}(2,2|4) \) (it belongs to an irreducible representation). But it transforms covariantly under \( \text{so}(1,4) \oplus \text{so}(5) \subset \text{psu}(2,2|4) \), in the sense that the action of \( \text{so}(1,4) \oplus \text{so}(5) \) on the right hand side of (169) agrees with the action of \( \text{so}(1,4) \oplus \text{so}(5) \) on the left hand side of (169).

Another way of thinking about \( \langle \text{plug} \rangle \) is in terms of the cohomology of the operator:

\[
\frac{1}{z} \lambda_3^{\alpha} t_3^\alpha + z \lambda_1^{\dot{\alpha}} t_1^{\dot{\alpha}}
\]

(170)

acting on the BPS representation \( \mathcal{H} \) (more precisely, the \( g_0 \)-invariant tensor product of \( \mathcal{H}' \) with the space of polynomials of \( \lambda_3, \lambda_1 \)). Our results imply that the second cohomology of this operator is nontrivial\(^{12}\), represented by the cocycle (169).

Notice that this provides a purely representation-theoretic characterization of the linearized SUGRA spectrum on \( \text{AdS}_5 \times S^5 \). Indeed, the question of the existence of the excitation transforming in the representation \( \mathcal{H} \) is reduced to the calculation of the cohomology of the operator (170), which is defined in terms of the generators \( t_a \) of the representation \( \mathcal{H} \).

There is also another example of a plug, a plug of the ghost number 1. Consider the Wilson line in the adjoint representation. The cohomology of (170) in the adjoint representation is nontrivial and is represented by:

\[
\frac{1}{z} \lambda_3^{\alpha} t_3^\alpha - z \lambda_1^{\dot{\alpha}} t_1^{\dot{\alpha}}
\]

(171)

This is obviously a \( \lambda \)-dependent vector in the adjoint representation, of the ghost number 1. One can verify that this is annihilated by (170); note the relative minus sign of the second term in (171). Therefore we can take (171) as a plug, and consider:

\[
V(\tau^+, \tau^- | z) = \text{Str} \left( \frac{1}{z} \lambda_3^{\alpha} t_3^\alpha - z \lambda_1^{\dot{\alpha}} t_1^{\dot{\alpha}} \right) \text{P exp} \left( - \int_{-\infty}^{\tau^+, \tau^-} J[z] \right) \Psi^{(\infty)}
\]

At \( z = 1 \) the corresponding integrated vertex operator is the density of the local conserved charge \( \text{Str}( (j_+ d\tau^+ - j_- d\tau^-) \Psi^{(\infty)} ) \). We will prove in Appendix A that (171) is the only example of the endpoint cohomology at ghost number

\(^{12}\) A similar (but different) cohomology problem was considered in [24, 25, 26].
1. In particular, there is no nontrivial cohomology for representations other than the adjoint.

With these notations the 2-point vertex \(^{(167)}\) reads:

\[
V^{2-\text{pt}}((\tau_2^+, \tau_2^-), (\tau_1^+, \tau_1^-)|z) =
= \left\langle \text{plug}(\tau_2^+, \tau_2^-) \left| P\exp \left( - \int_{(\tau_1^+, \tau_1^-)} \left( \tau_2^+ \cdot \tau_2^- \right) \right) \right| \text{plug}(\tau_1^+, \tau_1^-) \right\rangle
\]  
\hspace{1cm} (172)

(But as we discussed at the end of Section \(8.2\) this must be BRST exact.)

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**A Coomology at ghost number one**

In this section we will prove that the only cohomology at ghost number 1 are the global \(psu(2,2|4)\) conserved charges.

**A.1 Global conserved charges and BRST cohomology**

The conserved charges are the descendants of the cohomology classes of the ghost number 1. The “standard” local conserved charges correspond to the
global symmetries $PSU(2,2|4)$. They descend from the following operator:

$$\text{Ad}(g). (\lambda_3^\alpha t_\alpha^3 - \lambda_1^\alpha t_\alpha^1)$$  \hfill (173)

In other words, we have the following cohomology class of the ghost number one in the adjoint representation of $g$:

$$\lambda_3^\alpha t_\alpha^3 - \lambda_1^\alpha t_\alpha^1$$  \hfill (174)

In this section we will prove that there are no nontrivial cohomology classes of the ghost number 1 in representations other than the adjoint.

### A.2 Cohomology classes at ghost number 1: the defining equations.

Fix a representation $\mathcal{F}$ of $g = psu(2,2|4)$. We will assume two things about $\mathcal{F}$:

- as a representation of $g$ it is irreducible
- as a representation of $g_{\text{even}}$ it is completely reducible, i.e. decomposes into the direct sum of irreducible representations

We will write a representative of the cohomology class in the following way:

$$\lambda_3^\alpha V_\alpha + \lambda_1^\alpha \tilde{V}_{\bar{\alpha}}$$  \hfill (175)

The condition of $g_0$-invariance says that $V_\alpha$ and $\tilde{V}_{\bar{\alpha}}$ should define intertwining operators of $g_0$:

$$V \in \text{Hom}_{g_0}(g_3, \mathcal{F}),$$  \hfill (166)

$$\tilde{V} \in \text{Hom}_{g_0}(g_1, \mathcal{F})$$  \hfill (177)

In other words:

$$t^0_{[\rho\sigma]} V_\alpha = f_{[\rho\sigma][\alpha}^\beta V_\beta$$  \hfill (178)

$$t^0_{[\rho\sigma]} \tilde{V}_{\bar{\alpha}} = f_{[\rho\sigma][\bar{\alpha}}^\beta \tilde{V}_{\bar{\beta}}$$  \hfill (179)
— the conditions of \(g_0\)-covariance. The conditions for being annihilated by \(Q\) are:

\[

t_\alpha^3 V_\beta + t_\beta^3 V_\alpha = f_{\alpha\beta}^\mu A_\mu \\
\tag{180}
\]

\[

t_\alpha^1 \dot{V}_\beta + t_\beta^1 \dot{V}_\alpha = f_{\alpha\beta;\dot{\mu}} \dot{A}_\mu \\
\tag{181}
\]

\[

t_\alpha^3 \dot{V}_\beta + t_\beta^1 V_\alpha = 0 \\
\tag{182}
\]

For example, the class (174) is represented by:

\[
V_\alpha = t_\alpha^3, \quad \dot{V}_\alpha = -t_\alpha^1 \\
A_\mu = -\dot{A}_\mu = 2t_\mu^2 \\
\tag{183}
\]

We consider the solutions of (180) trivial if they are of the form:

\[
V_\alpha = t_\alpha^3 \Phi , \quad \dot{V}_\alpha = t_\alpha^1 \Phi \\
\tag{184}
\]

where \(t_0^{[\mu \nu]} \Phi = 0\).

We want to prove the following:

**Theorem:** Nontrivial solutions to Eqs. (180–182) exist only when \(\mathcal{F}\) is the adjoint representation of \(g\), and are given by (183) up to adding a trivial solution. There are no other nontrivial solutions.

We will now proceed to prove this.

### A.3 Cohomology classes at ghost number 1: consequences of the defining equations

Acting on (180) by \(t_1^\beta\) we get:

\[
\left( f_{\dot{\beta} [\rho \sigma]} t_0^{[\rho \sigma]} V_\beta - t_\alpha^3 t_\beta^1 V_\beta \right) + (\alpha \leftrightarrow \beta) = f_{\alpha\beta}^\mu t_\beta^1 A_\mu \\
\tag{185}
\]

This with Eqs. (178) and (182) implies:

\[
\left( f_{\dot{\beta} [\rho \sigma]} f_{[\rho \sigma] \beta} \gamma^V \gamma + (\alpha \leftrightarrow \beta) \right) + f_{\alpha\beta}^\mu t_\mu^2 \dot{V}_\beta = f_{\alpha\beta}^\mu t_\beta^1 A_\mu \\
\tag{186}
\]

This with the Jacobi identity for \(ff\) implies:

\[
f_{\mu \beta}^\alpha V_\alpha = t_\mu^2 \dot{V}_\beta - t_\beta^1 A_\mu \\
\tag{187}
\]

44
Similarly we have:

\[ f_{\mu\beta} \dot{\tilde{V}}_\alpha = t_2^{\mu} V_\beta - t_3^\beta \tilde{A}_\mu \]  \hspace{1cm} (188)

Let us act on (187) by \( f^{\dot{\gamma}\beta} \nu \dot{t}_3^\gamma \):

\[ f^{\dot{\gamma}\beta} \nu f_{\mu\beta} \alpha \dot{t}_3^\beta V_\alpha = f^{\dot{\gamma}\beta} \nu \dot{t}_3^\beta t_2^{\nu} \tilde{V}_\beta - f^{\dot{\gamma}\beta} \nu \dot{t}_3^\beta t_1^{\beta} A_\mu = \]  \hspace{1cm} (189)

\[ = f^{\dot{\gamma}\beta} \nu f_{\dot{\gamma}\mu} \alpha t_3^{\alpha} \tilde{V}_\beta + \frac{1}{2} f^{\dot{\gamma}\beta} \nu f_{\dot{\gamma}\beta} \lambda t_3^\mu \tilde{A}_\lambda - \frac{1}{2} f^{\dot{\gamma}\beta} \nu f_{\dot{\gamma}\beta} \lambda t_2^2 A_\mu \]

This and (182) implies:

\[ t_2^{\mu} \tilde{A}_\nu - t_2^{\nu} A_\mu = 0 \]  \hspace{1cm} (190)

Let us denote: \( B_\mu = A_\mu + \tilde{A}_\mu \). We have:

\[ t_2^{\mu} B_\nu = 0 \]  \hspace{1cm} (191)

Note that the gauge transformation

\[ \delta V_\alpha = t_3^\alpha \Phi \quad , \quad \delta \tilde{V}_\alpha = t_1^\alpha \Phi \]  \hspace{1cm} (192)

where \( \Phi \) is \( g_0 \)-invariant leads to:

\[ \delta B_\mu = t_2^\mu \Phi \]  \hspace{1cm} (193)

Therefore we should think of \( B_\mu \) as an element of \( H^1(g, g_{\text{even}}, \mathcal{F}) \). But this cohomology group is zero because \( H^1(g, \mathcal{F}) = 0 \) (notice that the Serre-Hochschild spectral sequence for \( g_{\text{even}} \subset g \) has \( E_2^{p,0} = H^p(g, g_{\text{even}}, \mathcal{F}) \) and \( d_2 \) acts from \( E_2^{p,q} \) to \( E_2^{p+2,q-1} \)). Therefore we should be able to gauge away \( B_\mu \).

Let us therefore assume:

\[ A_\mu = -\tilde{A}_\mu \]  \hspace{1cm} (194)

Note that this equation and (190) implies:

\[ t_2^{\mu} A_\nu + t_2^{\nu} A_\mu = 0 \]  \hspace{1cm} (195)

Therefore we can rewrite (187) and (188) as follows:

\[ f_{\mu\beta} ^\alpha V_\alpha = t_2^{\mu} \tilde{V}_\beta - t_1^\beta A_\mu \]  \hspace{1cm} (196)

\[ f_{\mu\beta} ^\dot{\alpha} \tilde{V}_\dot{\alpha} = t_2^{\mu} V_\beta + t_3^\beta A_\mu \]  \hspace{1cm} (197)

Let us define \( F_{\mu\nu} \) by the following equation:

\[ F_{\rho\sigma} = t_2^{\rho} A_{\sigma} \]  \hspace{1cm} (198)
Eq. (195) implies that $F_{\rho\sigma}$ is antisymmetric: $F_{\rho\sigma} = -F_{\sigma\rho}$. We get:

\begin{align*}
t^2_{\lambda} F_{\mu\nu} &= t^2_{\lambda} t^2_{\mu} A_{\nu} = f_{\lambda\mu} \lbrack \rho \sigma \rbrack t^0_{\lbrack \rho \sigma \rbrack} A_{\nu} + t^2_{\mu} t^2_{\lambda} A_{\nu} = \\
&= f_{\lambda\mu} [\rho \sigma] f_{[\rho\sigma][\nu]}^{\kappa} A_{\kappa} - t^2_{\mu \nu} t^2_{\lambda} A_{\lambda} = \\
&= -\frac{1}{2} f_{\mu\nu} [\rho \sigma] f_{[\rho\sigma]\lambda}^{\kappa} A_{\kappa} - \frac{1}{2} f_{\mu\nu} [\rho \sigma] f_{[\rho\sigma]\lambda}^{\kappa} A_{\kappa} = \\
&= -f_{\mu\nu} [\rho \sigma] f_{[\rho\sigma]\lambda}^{\kappa} A_{\kappa}
\end{align*}

(199)

This implies, first of all, that $F_{\mu\nu}$ is zero when $\mu$ is tangent to $S^5$ and $\nu$ is tangent to $AdS_5$. Therefore $F_{\mu\nu}$ is completely encoded in terms of $G_{[\mu\nu]}$ which is defined by this equation:

\begin{equation}
F_{\mu\nu} = f_{\mu\nu} \lbrack \rho \sigma \rbrack G_{[\rho \sigma]}
\end{equation}

(200)

Eq. (199) implies for $G_{[\rho\sigma]}$:

\begin{equation}
t^2_{\lambda} G_{[\mu\nu]} = f_{\lambda\mu\nu}^{\kappa} A_{\kappa}
\end{equation}

(201)

This means that the linear space formed by $A_{\mu}$ and $G_{[\mu\nu]}$ is closed under the action of $g_{even}$, and is in fact the adjoint representation of $g_{even}$ (where $A_{\mu}$ corresponds to $2t_{\mu}$ and $G_{[\mu\nu]}$ corresponds to $2t_{\lbrack \mu\nu \rbrack}$). This is already close to what we wanted to prove. But we have to also tame the expressions of this form:

\begin{equation}
t^3_{\alpha} t^1_{\beta} t^1_{\gamma} G_{[\mu\nu]} , \ t^1_{\alpha} t^3_{\beta} t^3_{\gamma} G_{[\mu\nu]} , \ etc.
\end{equation}

(202)

For this purpose, let us use (196) and (197) in this expression:

\begin{align*}
t^3_{\alpha} t^1_{\beta} t^1_{\gamma} A_{\mu} - t^1_{\beta} t^3_{\alpha} A_{\mu} = \\
= t^3_{\alpha}(t^2_{\mu} \tilde{V}_{\beta} - f_{\mu\beta}^{\gamma} V_{\gamma}) + t^1_{\beta}(t^2_{\mu} V_{\alpha} - f_{\mu\alpha}^{\gamma} \tilde{V}_{\gamma}) = \\
= -f_{\mu\alpha}^{\gamma}(t^1_{\gamma} \tilde{V}_{\beta} + t^1_{\beta} V_{\gamma}) - f_{\mu\beta}^{\gamma}(t^3_{\alpha} V_{\gamma} + t^3_{\gamma} V_{\alpha}) = \\
= f_{\beta\mu}^{\gamma} f_{\alpha\gamma}^{\nu} A_{\nu} - f_{\alpha\mu}^{\gamma} f_{\gamma\beta}^{\nu} A_{\nu}
\end{align*}

(203)

On the other hand, the combination $t^3_{\alpha} t^1_{\beta} A_{\mu} + t^1_{\beta} t^3_{\alpha} A_{\mu}$ can be calculated using the $g_{0}$-invariance. This implies:

\begin{equation}
t^3_{\alpha} t^1_{\beta} A_{\mu} = f_{\beta\mu}^{\gamma} f_{\alpha\gamma}^{\nu} A_{\nu}
\end{equation}

(204)
We will also use this:

\[ t_1^{\alpha} F_{\mu\nu} = t_1^{\alpha} t_2^\mu A_\nu = \frac{1}{c} f_{\mu}^{\gamma\delta} t_1^\alpha t_3^\gamma A_\nu = \]

\[ = \frac{1}{c} f_{\mu}^{\gamma\delta} f_{\alpha\gamma}^{[\rho\sigma]} t_3^\delta A_\nu - \frac{1}{c} f_{\mu}^{\gamma\delta} t_3^\gamma t_1^\alpha A_\nu = \]

\[ = \frac{1}{c} (fff)(t^3 A) \quad (205) \]

Here we used the schematic notation \((fff)\) for a product of three structure constants with some indices contracted, and \(c\) is determined from \(f_{\mu}^{\alpha\beta} f_{\alpha\beta}^{\nu} = c\delta_\nu^\mu\). The subspace of \(\mathcal{F}\) generated by acting on \(A\) and \(G\) by finitely many \(t^3\) and \(t^1\) is finite-dimensional. Indeed, using (204) and (205) we can prove that it is generated as a linear space by expressions of the form:

\[ t^{[\alpha_1}^{\cdots} t^{\alpha_k]} A_\mu , \quad t^{[\alpha_1}^{\cdots} t^{3\alpha_k]} A_\mu \quad (k \geq 0) \quad (206) \]

and

\[ G_{[\mu\nu]} \quad (207) \]

where the square brackets stand for the antisymmetrization of the indices (for example \(t^{3}_{[\alpha_1} t^{3}_{\alpha_k]} A_\mu \) stands for \((t^{3}_{\alpha_1} t^{3}_{\alpha_k]} - t^{3}_{\alpha_k} t^{3}_{\alpha_1]} A_\mu \)). Eqs. (204) and (205) imply that this subspace is closed under the action of \(g\). Because \(\mathcal{F}\) is assumed to be irreducible, we conclude that \(\mathcal{F}\) is generated by (206), (207). Because of the antisymmetrization of the indices of \(t^3\) and \(t^1\) there are only finitely many linearly independent expressions of the form (206). This proves that \(\mathcal{F}\) is a finite-dimensional space.

The subspace in \(\mathcal{F}\) generated by:

\[ A_\mu , \quad G_{[\mu\nu]} , \quad t_\alpha A_\mu , \quad t_\alpha A_\mu \]

is closed under \(g_{\text{even}}\). Let us denote this space \(\mathcal{L}\). Obviously \(\mathcal{L} \subset \mathcal{F}\).

**Theorem.** \(\mathcal{L} = \mathcal{F}\).

**Proof.** One can see that for any element \(v\) of \(\mathcal{H}\) (i.e. a finite linear combination of expressions of the form (206)) there is a number \(p\) such that for any \(q > p\) and any \(q\) elements \(\xi_1, \ldots, \xi_q\) of \(g_{\text{even}}\) we get:

\[ \xi_1 \cdots \xi_q \ v \in \mathcal{L} \quad (209) \]

Indeed, let us consider for example acting by \(\xi \in g_2\) on expressions of the form \(t^{3}_{[\alpha_1} \cdots t^{3}_{\alpha_k]} A_\mu\). Let us define the degree of such an expression by the
following formula:

\[
\deg t_{[\alpha_1}^{\alpha_k]} A_\mu = \deg t_{[\dot{\alpha}_1}^{\dot{\alpha}_k]} A_\mu = k \quad (210)
\]

More precisely, we introduce a filtration of $\mathcal{H}$ saying that $F^k\mathcal{H}$ consists of all the elements of $\mathcal{H}$ which can be written as linear combinations of the expressions of the form (206) of the degree less or equal $k$. Using Eqs. (204) and (205) we derive that for $k > 1$:

\[
t^2_{\mu} F^k\mathcal{H} \subset F^{k-1}\mathcal{H} \quad (211)
\]

This implies (209). Because of the assumption that $\mathcal{F}$ is completely reducible as a representation of $g_{\text{even}}$, Eq. (209) implies that $\mathcal{L} = \mathcal{F}$.

We conclude that $\mathcal{F}$ is in fact generated by expressions (208). Note that the linear space generated by (208) consists of the even subspace generated by $A_\mu$ and $G_{[\mu\nu]}$, and odd subspace generated by $t_\alpha A_\mu, t_{\dot{\alpha}} A_\mu$. The even subspace is the same as in the adjoint representation. Therefore the odd space should be also the same.

This proves that $\mathcal{H}$ is the adjoint representation of $g = psu(2,2|4)$.

### A.4 Relaxing the requirement that $\mathcal{F}$ is irreducible

We have argued that the subspace generated by (206) in fact coincides with $\mathcal{F}$, based on $\mathcal{F}$ being an irreducible representation of $g$. This requirement can be replaced with the requirement that $H^1(g, \mathcal{F}) = 0$. Suppose that (206) generate a smaller subspace $\mathcal{F}_{A+G} \subset \mathcal{F}$. Let us denote $v$ the projection of $V$ on $\mathcal{F}/\mathcal{F}_{A+G}$:

\[
v = V \mod \mathcal{F}_{A+G} \quad (212)
\]

Then (181) implies:

\[
t^3_\alpha v_\beta + t^3_\beta v_\alpha = t^1_\alpha \tilde{v}_\beta + t^1_\beta \tilde{v}_\alpha = t^3_\alpha \tilde{v}_\beta + t^1_\beta v_\alpha = 0 \quad (213)
\]

These equations imply that $(v_\alpha, \tilde{v}_\beta)$ actually form the spinor representation of $g_{\text{even}}$:

\[
t_m v_\alpha = f_{m\alpha}^\beta \tilde{v}_\beta \quad (214)
\]

\[
t_m \tilde{v}_\alpha = f_{m\dot{\alpha}}^\beta v_\beta \quad (215)
\]

48
We will now explain, using (214) and (215), that $v$ can be gauged away if $H^1(g, g_{\text{even}}, \mathcal{F}) = 0$. And this cohomology group vanishes if $H^1(g, \mathcal{F})$ vanishes.

More generally, let us consider the subcomplex of the BRST complex covariant under $g_{\text{even}}$. This means consider the chains which satisfy:

$$t^2 m v_{\alpha_1...\alpha_m \beta_1...\beta_n} = f_{m \alpha_1}^{\alpha'_1} v_{\alpha'_1...\alpha_m \beta_1...\beta_n} + \ldots$$

(216)

On this complex the BRST operator coincides with the Lie cohomology $Q$-operator in the relative complex $H^\bullet(g, g_{\text{even}}, \mathcal{F})$. The relative cohomology is related to $H^\bullet(g, \mathcal{F})$ by the Serre-Hochschild spectral sequence. Namely

$$H^p(g, h, \mathcal{F}) = E^{p,0}_2$$

(217)

The differential $d_r$ acts from $E^{p,q}_r$ to $E^{p+r,q+1-r}_r$. In particular: $E^{1,0}_r$ is related to $E^{1+r,1-r}_r$.

$$E^{1-r,-1+r}_r \xrightarrow{d_r} E^{1,0}_r \xrightarrow{d_r} E^{1+r,1-r}_r$$

(218)

This means that $E^{1,0}_2$ cannot cancel with anything and therefore $H^1(g, h, \mathcal{F}) \neq 0$ implies that $H^1(g, \mathcal{F}) \neq 0$. Therefore, if $H^1(g, \mathcal{F})$ vanishes, then $H^1(g, h, \mathcal{F})$ should also vanish.

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