TOWARDS A FURTHER UNDERSTANDING OF THE
DYNAMICS IN THE EXCITATORY NNLIF NEURON MODEL:
BLOW-UP AND GLOBAL EXISTENCE

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Abstract. The Nonlinear Noisy Leaky Integrate and Fire (NNLIF) model
is widely used to describe the dynamics of neural networks after a diffusive
approximation of the mean-field limit of a stochastic differential equation. In
previous works, many qualitative results were obtained: global existence in the
inhibitory case, finite-time blow-up in the excitatory case, convergence towards
stationary states in the weak connectivity regime. In this article, we refine
some of these results in order to foster the understanding of the model. We
prove with deterministic tools that blow-up is systematic in highly connected
excitatory networks. Then, we show that a relatively weak control on the firing
rate suffices to obtain global-in-time existence of classical solutions.

1. Introduction. Since the biologically plausible mathematical models in neuro-
science tend to be far too complex to be tackled numerically or analytically, re-
searchers in computational neurosciences proposed simpler models aimed at encom-
passing the qualitative complexity of neural networks in a tractable framework. A
popular approach is to study mean field differential models arising from stochastic
differential equations. Over the last decade, many partial differential models were
studied: Fokker-Planck equations with Poisson discharges [29]; population density
models of integrate and fire neurons with jumps [18] and Fokker-Planck equations
including conductance variables [33, 32], which are formal mean-field limits of inte-
grate and fire networks assuming Poisson approximations [34, 15, 16]; time elapsed
models [30, 31, 26], recently derived as mean-field limits of Hawkes processes [13, 12];
the mean-field McKean-Vlasov equations [25] related to the behaviour of Fitzhugh-
Nagumo neurons [19], etc.

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We consider the following PDE model (see [1, 2, 3] for its derivation): 

\[
\frac{\partial p}{\partial t}(v, t) + \frac{\partial}{\partial v}[( -v + bN(t))p(v, t)] - a\frac{\partial^2 p}{\partial v^2}(v, t) = N(t)\delta(v - V_R), \quad v \leq V_F, \quad (1)
\]

where we denote by \( \delta \) the Dirac mass at point 0. For the sake of clearness, we will often write \( \delta_{V_R} = \delta(v - V_R) \). The firing rate \( N \) of the network is given by 

\[
N(t) = -a\frac{\partial p}{\partial v}(V_F, t) \geq 0. \quad (2)
\]

The parameter \( a > 0 \) is the diffusion coefficient and \( b \) is the connectivity parameter. If \( b \) is positive, the neural network is average-excitatory; if \( b \) is negative, the network is average-inhibitory.

The PDE (1) is completed with initial and boundary conditions

\[
p(v, 0) = p^0(v) \geq 0, \quad \int_{-\infty}^{V_F} p^0(v)dv = 1 \quad \text{and} \quad p(V_F, t) = p(-\infty, t) = 0. \quad (3)
\]

The deterministic NNLIF model arises as the probability density of the following non-linear stochastic mean-field equation:

\[
X_t = X_0 + \int_0^t f(X_s)ds + \alpha\mathbb{E}[M_t] + \sigma B_t - M_t, \quad (4)
\]

where \( \sigma > 0, \alpha \in \mathbb{R} \) are parameters, \((B_t)_{t \geq 0}\) is a standard Brownian motion in \( \mathbb{R} \), \( f : \mathbb{R} \to \mathbb{R} \) is a Lipschitz continuous function and \( X_0 < 1 \) almost surely. The process \( M_t \) counts the number of times \( X_t \) hits the threshold \( V_F = 1 \) before time \( t \) and \( \mathbb{E}[M_t] \) represents the expected number of times the threshold is reached before \( t \). This stochastic process was at first studied by Delarue, Inglis, Rubenthaler and Tanré in [15, 16]; the process \( X_t \) was meant to describe the potential of a typical neuron in an infinite mean-field network. As we further explain below this type of equation was investigated in a very similar mathematical setting in [20, 21, 28, 27] with the aim of modelling large portfolio credit problems or systems of banks with mutual exposures in mathematical finance.

When \( f(x) = -x \), according to [15][Lemma 4.2, (iii)], the density \( p(v, t)dv = \mathbb{P}(X_t \in dv) \) is a solution in the sense of distributions to (1)-(2)-(3) with \( b = \alpha, \ a = \frac{\sigma^2}{2} \), \( V_F = 1, \ V_R = 0 \), \( N(t) = \frac{d}{dt}\mathbb{E}[M_t] \), \( p^0(v)dv = \mathbb{P}(X_0 \in dv) \). The connection between problems (1)-(2)-(3) and (4) has been the subject of recent studies: [24, 7].

The aim of this article is to investigate two aspects of the system (1)-(2)-(3).

The first one concerns finite-time blow-up. More precisely, our goal is to better understand which combinations of the initial condition \( p^0 \) and the connectivity strength \( b \) give rise to blow-up events. A first result was obtained by Cáceres, Carrillo and Perthame in [4][Theorem 2.2] with deterministic techniques. They proved that in the excitatory case \( b > 0 \), finite-time blow-up occurs in at least two cases: if the initial condition is concentrated enough around the firing potential \( V_F \) depending on the value of \( b > 0 \) or if \( b \) is large enough depending on the initial condition \( p^0 \).
We prove here that if we choose \(b\) large enough, then blow-up occurs independently of the initial condition \(p^0\). Note that finite-time blow-up was also studied in the stochastic system (4). Hambly, Ledger and Søjmark proved in [21][Theorem 1.1] that for all initial condition \(X_0\) there exists a large enough connectivity \(\alpha\) such that the solution blows-up in finite time; their bound on \(\alpha\) depends only on the first moment of the law of \(X_0\).

There is another interesting question, which we are not addressing in this article: is it possible to extend the solutions after a blow-up event? Carrillo, González, Gualdani and Schonbek proved in [10][Theorem 1.1] that for all initial condition \(X_0\) there exists a large enough connectivity \(\alpha\) such that the solution blows-up in finite time; their bound on \(\alpha\) depends only on the first moment of the law of \(X_0\).

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Secondly, we investigate global-in-time existence of classical solutions for system (5) in the excitatory case \((b > 0)\). As we said above, given \(b > 0\) there are blowing-up solutions for initial conditions \(p^0(\cdot)\) concentrated enough around \(V_F\) ([4][Theorem 2.2]). Hence, we cannot hope to find a criterion for global-in-time existence independently of the initial condition. In this article, we prove that given an initial condition \(p^0\), we can find a value \(b^*(p^0)\) such that if \(0 < b < b^*\), then the classical solution is global-in-time (see Theorem 4.5). Such a result was obtained for the associated stochastic equation (4) in [15][Theorem 2.4]: for any initial condition \(X_0 = x_0 < 1\), there exists \(\alpha^*(x_0) \in [0, 1]\) such that for all \(\alpha \in [0, \alpha^*(\cdot)\), there exists a global-in-time solution of (4) such that \(t \mapsto E[M_t]\) is \(C^1\) on \(\mathbb{R}_+\) (i.e. the firing rate is continuous). Note that in [27, 21], the firing rate \(N(t)\) is in \(L^2_{loc}\) until
the first time of explosion ($t^* := \text{sup}\{t > 0 \mid N(t) < +\infty\} = +\infty$) in order to apply the result of [10][Theorem 1.1] which states that $T^*$ is the maximal time of existence. To do this, we proceed in two steps. First, we prove uniform a priori estimates on the firing rate $N(t)$ in $L^q$, $q > 2$. The strategy is to construct new entropy estimates inspired from the method the authors of [11][Theorem 3.1] used to prove uniform $L^2$ estimates on $N(t)$. Second, we recast the problem (1)-(2)-(3) into a Stefan-like free boundary problem via a change of variables as proposed in [10], which allows us to obtain an implicit Duhamel formula on $N(t)$: we use this formula to lift our $L^q$ estimates ($q > 2$) to $L^\infty$ estimates. Let us mention that $q > 2$ is necessary to conduct our proof (see Remark 4).

To complete the state of the art on this type of equations, note that other variants of the classical NNLIF system were proposed and studied. In [6], Cáceres and Perthame considered the NNLIF system with a refractory period. In [9], Cáceres and Schneider proposed a general system with excitatory-inhibitory coupling, a refractory state and a synaptic delay (the transmission of pulses in not instantaneous any more). Numerical simulations are provided that show stable periodic solutions in the delayed case. Cáceres, Roux, Salort and Schneider ([8]) proved some results on the NNLIF equation with synaptic delay: global-in-time existence for all smooth enough initial data for both positive and negative values of $b$, convergence towards stationary states for small connectivities $b$, $L^2$ estimates on the firing rate. Some papers proposed suitable numerical methods for NNLIF type systems: [5], [35], [22], [36].

This article is organised as follows. In Section 2, we introduce the definitions of solution we will use throughout the article. Section 3 focuses on finite-time blow-up: we prove that for $b$ large enough, there is no global-in-time solution to (3) – even in the weak sense – independently of the initial datum. Section 4 is about global-in-time existence of the solutions of (5): we first derive, in Subsection 4.1, uniform $L^q$ estimates on the firing rate $N$ for small enough connectivities $b$, $L^2$ estimates on the firing rate. Some papers proposed suitable numerical methods for NNLIF type systems: [5], [35], [22], [36].

2. Notions of solution. We are interested in the classical NNLIF system which writes as follows. Let $V_R, V_F \in \mathbb{R}$ such that $V_R < V_F$. Let $b \in \mathbb{R}$, $a \in \mathbb{R}^*_+$. We study the system

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial v} \left[ (-v + bN(t))p \right] - a \frac{\partial^2 p}{\partial v^2} = \delta_{V_R} N(t),$$

$$N(t) = -a \frac{\partial p}{\partial v}(V_F, t), \quad p(V_F, t) = 0, \quad p(-\infty, t) = 0,$$

$$p(v, 0) = p^0(v) \geq 0, \quad \int_{-\infty}^{V_F} p^0(v) dv = 1.$$
We define both classical and weak solutions for this system, following [10] and [4]:

**Definition 2.1.** We say that $\mathbf{(p, N)}$ is a classical (fast-decreasing) solution, of system (5) on $[0, T^*[, T^* \in \mathbb{R}_+ \cup \{+\infty\}$ if

- $p \in \mathcal{C}^0([-\infty, V_F]) \times [0, T^*[ \cap \mathcal{C}^2,1\big([-\infty, V_R[ \cup V_R, V_F]) \times [0, T^*]\big)$
- $L^\infty\big([0, T^*[, L^2,1\big([-\infty, V_F])\big)$
- Functions $p$ and $N$ satisfy (5) in the classical sense on $[-\infty, V_R[ \cup V_R, V_F]$ and in the sense of distributions in $[-\infty, V_F]$;
- $\forall t \in [0, T^*[\,, \forall Q \in \mathbb{R}[X], \lim_{v \to -\infty} Q(v)p(v,t) = 0$ and $\lim_{v \to -\infty} Q(v)\frac{\partial p}{\partial v}(v,t) = 0$ (fast-decreasing).

And, for weak solutions:

**Definition 2.2.** Let $T^* \in \mathbb{R}_+ \cup \{+\infty\}$. Let $p \in L^\infty([0, T^*[, L^2,1\big([-\infty, V_F])\big)$ and $N \in L^1_{\text{loc,+}}([0, T^*[^\,].$ The pair $(p, N)$ is said to be a weak solution of (5) if for every test function $\phi \in \mathcal{C}^\infty([-\infty, V_F])$ such that for almost every $t \in [0, T^*[\,$

$$
\left(\phi^\cdot p^\cdot, \frac{\partial^2 \phi}{\partial v^2}\right) + \partial_v^\phi \left(1 + \frac{\partial \phi}{\partial v}\right) \in \left(L^1([-\infty, V_F])\right)^3,
$$

we have

$$
\frac{d}{dt} \int_{-\infty}^{V_F} p(v,t)\phi(v)dv = \int_{-\infty}^{V_F} \left( -v + bN(t) \frac{\partial \phi}{\partial v} + a \frac{\partial^2 \phi}{\partial v^2} \right) p(v,t)dv
+ N(t) \left( \phi(V_R) - \phi(V_F) \right),
$$

where the time derivative is to be taken in the sense of distributions and where $\lim_{t \to 0} \|p(\cdot, t) - p^0(\cdot)\|_{L^1} = 0$.

A weak solution is said to be fast-decreasing if for all polynomial function $Q \in \mathbb{R}[X]$, for almost every $t \in [0, T^*[\,$ we have $\lim_{v \to -\infty} Q(v)p(v,t) = 0$.

**Remark 1.** If we apply the definition of weak solution with the test function $\phi = 1$, we obtain for $t \in \mathbb{R}_+$,

$$
\int_{-\infty}^{V_F} p(v,t)dv = \int_{-\infty}^{V_F} p^0(v)dv = 1.
$$

For the sake of brevity in our results, we will refer to the following regularity assumptions on initial data.

**Assumptions 1** (Initial data for classical solutions). $p^0 \in \mathcal{C}^0([-\infty, V_F]) \cap \mathcal{C}^1([-\infty, V_R[ \cup V_R, V_F]) \cap L^1([-\infty, V_F])$ is non-negative, fast decreasing at $-\infty$ and $p^0(V_F) = 0$. $\frac{dp^0}{dv}$ admits finite left and right limits at $V_R$ and is fast decreasing at $-\infty$.

To sum up, the following is known ([10],[4],[11]) on the solutions of the deterministic problem (5):

- For any initial datum satisfying Assumptions 1, there exists a unique local-in-time classical solution of (5) and its maximal time of existence satisfies $T^* = \sup\{t > 0 | N(t) < +\infty\}$.
- The shape of stationary states is known (see (13) below).
• In the inhibitory or in the linear case \((b \leq 0)\), \(T^* = +\infty\), there exists a unique stationary state and the solutions are uniformly bounded; the firing rate \(N\) is also uniformly bounded. There exists a constant \(C > 0\) such that if \(-C < b \leq 0\), the solution converges exponentially fast towards the stationary state \((p_\infty, N_\infty)\).

• In the excitatory case \((b > 0)\), there exists \(p^0\) such that no weak solution can be global-in-time. If \(b\) is large enough, there is no stationary state. If \(b\) is small enough, there is a unique stationary state; if \(b\) is small enough regarding \(p^0\), the solution converges (up to its time of existence) exponentially fast towards the unique stationary state. For intermediate values of \(b\), two or more stationary states can coexist.

3. Blow-up for all initial conditions when \(b\) is large enough. We extend here the result of [4] on finite-time blow-up in the NNLIF system which states that for every positive \(b\) there exist blowing-up solutions from initial data \(p^0(\cdot)\) concentrated enough around \(V_F\). We prove that for large enough positive values of \(b\) all solutions blow-up in finite time regardless of the initial repartition of their mass.

The main idea is to use an evolution equation for the linear moment in order to control the quantity of mass that stays near \(V_F\): we prove that if we wait for a long enough time \(t_0\), some quantity of the initial mass \(\|p^0\|_1 = 1\) by hypothesis) will stay between \(V_R\) and \(V_F\) and this quantity is independent from the initial condition \(p^0\). Then, we use the method of [4] and we prove, thanks to our control on the mass between \(V_R\) and \(V_F\), that the exponential moment is unbounded, thus reaching a contradiction with mass conservation.

**Theorem 3.1.** We have

- If \(V_F \leq 0\), then if \(b \geq V_F - V_R\), no fast-decreasing weak solution of (5) can be global in time.
- If \(V_F > 0\), then if

\[
b > \max \left( V_F - V_R, \frac{V_F}{4} \inf_{y \in [1,\sqrt{1+\frac{1}{b}}]} \left(1 + y\right)^2 \left(1 - e^{-4\frac{V_F - V_R}{V_F - y^2/4}}\right) e^{\frac{2}{y-1}}\right),
\]

no fast-decreasing weak solution of (5) can be global in time.

**Proof.** Assume \((p, N)\) is a global in time fast-decreasing weak solution. We denote

\[
M_\phi(t) = \int_{-\infty}^{V_F} (V_F - v)p(v,t)dv
\]

and we use the definition of weak solution with the test function \(\phi(v) = V_F - v\); it yields

\[
\frac{d}{dt}M_\phi(t) = V_F - M_\phi(t) + N(t)(V_F - V_R - b).
\]

If \(b \geq V_F - V_R\), \(\frac{d}{dt}M_\phi(t) \leq V_F - M_\phi(t)\), that is to say

\[
\frac{d}{dt} \left( M_\phi(t) - V_F \right) \leq - \left( M_\phi(t) - V_F \right).
\]

Using Grönwall’s lemma, we deduce

\[
M_\phi(t) - V_F \leq \left( M_\phi(0) - V_F \right) e^{-t}.
\]
If $M_\phi(0) \leq V_F$, then $M_\phi(t) \leq V_F$ for all $t \in \mathbb{R}_+^*$ and otherwise we have the exponential convergence of $M_\phi(t)$ towards $V_F$. Thus, we have

\[ \forall \varepsilon \in \mathbb{R}_+^*, \exists t_0 \in \mathbb{R}_+^* \text{ such that } \forall t \in [t_0, +\infty[, M_\phi(t) \leq V_F + \varepsilon. \]

- Assume $V_F < 0$:
  
  There exists $\varepsilon \in \mathbb{R}_+^*$ small enough such that $V_F + \varepsilon < 0$ and then the quantity $M_\phi(t)$ becomes negative after the time $t_0$, which is a contradiction. Therefore, no weak solution can be global in time in this case.

- Assume $V_F \geq 0$:
  
  We have, for all $\eta \in \mathbb{R}_+^*$, $\forall t \in [t_0, +\infty]$, 
  
  \[ \eta \int_{-\infty}^{V_F-\eta} p(v, t)dv \leq \int_{-\infty}^{V_F-\eta} (V_F - v)p(v, t)dv \leq M_\phi(t) \leq V_F + \varepsilon, \]
  
  and by choosing $\eta = (V_F + \varepsilon)\xi$, where $\xi \in ]1, +\infty[\text{ is a yet unchosen value, we obtain}$
  
  \[ \forall t \in [t_0, +\infty], \int_{-\infty}^{V_F-\eta} p(v, t)dv \leq \frac{V_F + \varepsilon}{\eta} < \frac{1}{\xi}. \]
  
  Since $p$ is a probability density on $]-\infty, V_F[\text{ we have,}$
  
  \[ \int_{V_F-\eta}^{V_F} p(v, t)dv \geq \frac{\xi - 1}{\xi}. \]

Let’s take now the test function defined by $\psi(v) = e^{\mu v}$. We denote

\[ M_\mu(t) = \int_{-\infty}^{V_F} p(v, t)e^{\mu v}dv. \]

The previous computations yield,

\[ \forall t \in [t_0, +\infty], \quad M_\mu(t) \geq \int_{V_F-\eta}^{V_F} e^{\mu v}p(v, t)dv \geq e^{\mu(V_F-\eta)} \int_{V_F-\eta}^{V_F} p(v, t)dv \geq \frac{\xi - 1}{\xi} e^{\mu(V_F-\eta)}. \]

We write the weak solution definition for $\psi$:

\[ \frac{d}{dt} M_\mu(t) = \mu \int_{-\infty}^{V_F} (-v + bN(t))e^{\mu v}p(v, t)dv + a\mu^2 M_\mu(t) + N(t)(e^{\mu V_R} - e^{\mu V_F}), \]

which gives the following bound, for $t > t_0$:

\[ \frac{d}{dt} M_\mu(t) \geq -V_F \mu M_\mu(t) + \mu bN(t)M_\mu(t) + a\mu^2 M_\mu(t) + N(t)(e^{\mu V_R} - e^{\mu V_F}). \]

The later writes

\[ \frac{d}{dt} M_\mu(t) \geq N(t) \left( b\mu M_\mu(t) + e^{\mu V_R} - e^{\mu V_F} \right) + \mu(a\mu - V_F)M_\mu(t). \]

Thus, we have,

\[ \frac{d}{dt} M_\mu(t) \geq N(t) \left( b\mu \frac{\xi - 1}{\xi} e^{\mu(V_F-\eta)} + e^{\mu V_R} - e^{\mu V_F} \right) + \mu(a\mu - V_F)M_\mu(t). \]

If $b$ satisfies

\[ b > \frac{\xi}{\xi - 1} \frac{e^{\mu V_F} - e^{\mu V_R}}{\mu e^{\mu(V_F-\eta)}}, \]

we have,

\[ M_\mu(t) \geq \frac{\xi}{\xi - 1} \frac{e^{\mu V_F} - e^{\mu V_R}}{\mu e^{\mu(V_F-\eta)}}, \]

Therefore, $M_\mu(t)$ is a weak solution of

\[ \frac{d}{dt} M_\mu(t) = -V_F \mu M_\mu(t) + \mu bN(t)M_\mu(t) + a\mu^2 M_\mu(t) + N(t)(e^{\mu V_R} - e^{\mu V_F}). \]
we then have
\[ \forall t \in [t_0, +\infty[ , \quad \frac{d}{dt} M_\mu(t) \geq \mu(a_\mu - V_F) M_\mu(t). \]

If we also assume that \( \mu > \frac{V_F}{a} \), by Grönwall’s lemma and since \( M_\mu(t_0) > 0 \), we obtain
\[ \lim_{t \to +\infty} M_\mu(t) = +\infty, \]
which is a contradiction with the inequality
\[ \forall t \in \mathbb{R}_+, \quad M_\mu(t) < e^{\mu V_F} \int_{-\infty}^{V_F} p(v,t) dv = e^{\mu V_F}. \]

Therefore, if \( b \) satisfies condition (8) for some \( \varepsilon > 0 \), the solution cannot be global in time. As \( \varepsilon \in \mathbb{R}_+ \) can be arbitrarily small, if
\[ b > \frac{\xi}{\xi - 1} \frac{e^{\mu V_F} - e^{\mu V_R}}{\mu e^{\mu V_F}} e^{\mu V_F} \xi, \]
then there is no global-in-time fast-decreasing weak solution.

Taking the infimum on \( \xi \in ]1, +\infty[ \) and \( \mu \in ]\frac{V_F}{a}, +\infty[ \) in the right-hand side of (9) and applying Lemma A.1 of the appendix, we get the final result.

**Remark 2.** When \( V_F \leq 0 \) the bound is optimal; indeed, for \( b < V_F - V_R \) there exists at least one stationary state, which constitutes a global in time fast-decreasing solution. In the case \( V_F > 0 \), the optimal bound can be strictly above \( V_F - V_R \) because for some parameters there exist stationary states in the case \( b > V_F - V_R \) (see [4][Theorem 3.1]). Yet, we probably have a sub-optimal lower bound for \( b \) because of the limitations of our method. Finding the optimal bound and linking it to the existence of stationary states is an interesting open question.

**Remark 3.** In the article [4], the authors study the number of stationary states depending on the parameters. They find that there is no stationary state under the condition
\[ b > \max \left( 2(V_F - V_R), 2V_F \int_0^{+\infty} \left( \frac{e^{\varepsilon V_F}}{\varepsilon} - \frac{e^{\varepsilon V_R}}{\varepsilon} \right) e^{-s^2} ds \right). \]

We did a numerical quantitative comparison (see Figure 1) between (10) and the lower bound for \( b \) we provide in Theorem 3.1. Indeed, if every solution blows-up in finite time, there can be no stationary state. The numerical simulations indicate that our bound is better when \( a \) or \( V_F - V_R \) are large.

4. **Global-in-time existence in excitatory networks.** In this section, we prove global-in-time existence in an excitatory network for a wide class of initial data. To do so, we first improve the \( L^2 \) estimates on the firing rate in [11][Theorem 3.1] into \( L^q \) estimates. Then, we take advantage of the \( L^q \) estimates (we choose \( q = 3 \) for clarity), \( q > 2 \), and we obtain global-in-time existence for appropriate parameters and initial data by following ideas of [10].
Figure 1. Comparison between the lower bound (10) in [4] and the lower bound in Theorem 3.1 for non-existence of stationary states, for different values of $a$. The new bound is always better when $V_F - V_R$ is large enough. We set $V_R = -1$ for convenience but it does not impact the results.

4.1. A priori $L^q$ estimates for the firing rate. For the sake of clarity, we assume in all this section that

$$0 < V_R < V_F.$$  \hfill (11)

This can be done without loss of generality by using the rescaling $\bar{p}(v, t) = \alpha p(\beta v + \gamma t, t)$. $\bar{N}(t) = \frac{\alpha}{\beta} N(t)$, with $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$.

The authors of [11][Theorem 3.1] proved uniform $L^2$ estimates on the firing rate:

$$\int_0^T N(t)^2 dt \leq C(1 + T).$$

In the excitatory case $b > 0$, $\eta = \eta(p^0) > 0$ depends on the initial datum $p^0$. Making more smallness hypotheses on $b$, they used these $L^2$ estimates in order to prove convergence to the unique (unique because $b$ is then small enough) stationary state via an entropy method and a Poincaré-like inequality. As explained in Remark 4 bellow, these $L^2$ estimates are not sufficient to prove global-in-time existence; the convergence to stationary state result in [11] is only valid up to an unknown time of existence. Thus we extend the result to $L^q$ estimates under similar hypotheses. More precisely, we prove:

**Theorem 4.1.** Let $b_1 \in \mathbb{R}^*_+$ be such that there exists a stationary state $(p_1^\infty, N_1^\infty)$. Let $V_M \in [V_R, V_F]$: denote

$$S_q(b_1, V_M) = \int_{V_M}^{V_F} \frac{p_0(v)^q}{p_1^\infty(v)^{q-1}} dv.$$

For every exponent $q \in \mathbb{N}$, $q \geq 2$, for every initial condition $p^0$ such that $S_q(b_1, V_M) < +\infty$, for every fast-decreasing classical solution of (5) from $p^0$. 


• There exists a constant $C_q \in \mathbb{R}_+^*$ depending only on $V_F, q, b_1$ and a (thus independent from $S_q(b_1, V_M)$) and a constant $T \in \mathbb{R}_+$ which depends only on $V_M$ and $S_q(b_1, V_M)$ such that for every interval $I \subset [T, +\infty[$ and for all $b \in ]-\infty, 0[$,
\[
\int_I N(t)^q \, dt \leq C_q (1 + |I|).
\]
• Let $b \in \mathbb{R}_+^*$. If $b$ is small enough regarding $S_q(b_1, V_M)$ and $V_M$, there exists a constant $C_q \in \mathbb{R}_+^*$ depending only on $V_F, q, a, b_1, V_M$ and $S_q(b_1, V_M)$ such that for every interval $I \subset \mathbb{R}_+$,
\[
\int_I N(t)^q \, dt \leq C_q (1 + |I|).
\]

Let us first make some comments on the term
\[
S_q(b_1, V_M) = \int_{V_M}^{V_F} \frac{p^0(v)^q}{p^1_\infty(v)^{q-1}} \, dv
\]
and the technical hypothesis $S_q(b_1, V_M) < +\infty$.

• The main idea is that these uniform estimates on the firing rate $N(t)$ depend only upon the initial repartition of mass in a neighbourhood of $V_F$: if we choose any parameter $V_M$ such that $V_R < V_M < V_F$, the proof of Theorem 4.1 only uses the equations on $[V_M, V_F]$ and the shape of the initial condition $p^0$ in $]-\infty, V_M]$ doesn’t have any impact on the results. Hence, if $b < 0$ or if $b > 0$ is small enough regarding $S_q(b_1, V_M)$ and $V_M$, we can control uniformly the $L^q$ norm of the firing rate.

• The technical hypothesis $S_q(b_1, V_M) < +\infty$ is necessary to apply our method because we study at some point an ODE whose initial condition is close to $S_q(b_1, V_M)$. Thus, we need this initial quantity to be finite because otherwise the computations don’t make sense. It is not a restrictive hypothesis as a sufficient condition to have it is simply (see computation (39) below): $p^0 \in L^\infty([-\infty, V_F[)$ and
\[
\limsup_{v \to V_F} \frac{p^0(v)}{V_F - v} < +\infty.
\]

Note that the rate of decay at the boundary is an important matter for this type of equation (see remark 6 below).

Then, let us recall from [4][Section 3.] that the stationary states $(p_\infty, N_\infty)$ satisfy
\[
\frac{\partial}{\partial v} \left[ (v + bN_\infty)p_\infty \right] - a \frac{\partial^2 p_\infty}{\partial v^2} = \delta_{V_R} N_\infty, \tag{12}
\]
\[
N_\infty = -a \frac{\partial p_\infty}{\partial v}(V_F), \quad p_\infty(V_F) = p_\infty(-\infty) = 0,
\]
and are of the form
\[
p_\infty(v) = \frac{N_\infty}{a} e^{\frac{v-bN_\infty}{2a}} \int_{\max(v, V_R)}^{V_F} e^{\frac{w-bN_\infty}{2a}} \, dw. \tag{13}
\]
where $N_\infty$ is a solution of a fixed-point equation defined and studied in [4].

Let $(p^1_\infty, N^1_\infty)$ be a stationary state associated to a parameter $b_1 > 0$. Let $(p, N)$ be the unique classical solution associated with the parameter $b$ and the suitable initial condition $p^0$. We denote
\[
H(v, t) = \frac{p(v, t)}{p_\infty(v)} \quad \text{and} \quad W(v, t) = p^1_\infty(v) H(v, t)^q = \frac{p(v, t)^q}{p^1_\infty(v)^{q-1}}.
\]
We will also use the following cutoff function

\[ \forall v \in ]-\infty, V_F[, \quad \gamma(v) = e^{\frac{v}{\beta^2} (V_F - \alpha)^2} \text{ if } v > \alpha, \quad \gamma(v) = 0 \text{ otherwise}, \quad (14) \]

where \( \beta = (V_F - \alpha)^2 \) and \( \alpha \in ]V_R, V_F[. \) Function \( \gamma \) has some handy properties one can find in [11][Lemma 3.4]:

**Lemma 4.2** (Carrillo, Perthame, Salort, Smets). Let \( 0 < V_R < V_F \) and let \( \alpha \in ]V_R, V_F[. \) Then \( \gamma \) defined in (14) is a positive increasing function on \( ]\alpha, V_F[ \) and the following properties hold:

1. \( \lim_{v \to V_F} \frac{\gamma(v)}{\gamma(\alpha)} = 0; \)
2. There exists a constant \( C > 0 \) such that \( \gamma'^2 + \gamma''^2 + \gamma'''^2 \leq C \gamma; \)
3. There exists a constant \( \eta \in ]\alpha, V_F[ \) such that \( \gamma''(\eta) = 0 \) and for all \( v \in ]\eta, V_F[, \) \( \gamma''(v) \leq 0. \)

The following technical lemma is proved in the appendix. The proof is purely computational.

**Lemma 4.3.** Let \((p, N)\) a fast-decreasing classical solution of (5) for a parameter \( b \in \mathbb{R} \) and let \( b_1 \in \mathbb{R}^*_+ \) such that there exists at least one stationary state \((p^1_\infty, N^1_\infty)\) for equation (5). Then, with the notations above,

\[
\frac{d}{dt} \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv \\
\leq \int_{-\infty}^{V_F} \left( -v + bN(t) \right) W(v, t) \gamma'(v) dv + a \frac{\partial W}{\partial v}(V_F, t) \gamma(V_F) - a W(V_F, t) \gamma'(V_F)
\]

\[
- a(q-1) \int_{-\infty}^{V_F} p^1_\infty(v) H(v, t)^q - 2 \left( \frac{\partial H}{\partial v}(v, t) \right)^2 \gamma(v) dv + a \int_{-\infty}^{V_F} W(v, t) \gamma''(v) dv
\]

\[
- (q-1)(bN(t) - b_1 N^1_\infty) \int_{-\infty}^{V_F} \frac{\partial p^1_\infty}{\partial v}(v) H(v, t)^q \gamma(v) dv. \quad (15)
\]

We now prove another technical lemma, this proof is inspired from the proof of Theorem 3.1 in [11].

**Lemma 4.4.** Let \( \alpha \in ]V_R, V_F[. \) If \( \alpha \) is close enough to \( V_F, \)

- If \( b \leq 0, \) there exist constants \( C_1, C_2 \in \mathbb{R}^*_+ \) depending only on \( \alpha, V_F, q, b_1 \)

\[
\frac{d}{dt} \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv \leq -\frac{\gamma(V_F)}{N^1_\infty q^{-1}} N(t)^q
\]

\[
+ (q-1)(bN(t) - C_1) \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv + C_2 \int_{-\infty}^{V_F} H(v, t)^q - 2 p^1_\infty(v) dv. \quad (16)
\]

- If \( b > 0, \) there exist constants \( C_3, C_4, C_5, C_6, C_7 \in \mathbb{R}^*_+ \) depending only on \( \alpha, V_F, q, b_1 \)

\[
\frac{d}{dt} \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv \leq -\frac{\gamma(V_F)}{N^1_\infty q^{-1}} N(t)^q \quad (17)
\]
\[+ \beta N(t)^q \left(C_3 + C_4 \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv + C_5 \int_{\alpha}^{V_F} H(v,t)^{q-2}p_{\infty}^1(v)dv\right)\]
\[+ C_6 \int_{\alpha}^{V_F} H(v,t)^{q-2}p_{\infty}^1(v)dv - C_7 \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv. \quad (18)\]

**Proof.** We distinguish two cases depending upon the sign of \(b\).

- **Inhibitory case:**

Looking at the expression of \(p_{\infty}^1\) in formula (13), we can see that if \(\alpha\) is close enough to \(V_F\), \(p_{\infty}^1\) is of class \(C^\infty\) on \(\alpha, V_F\) and we have

\[\forall v \in [\alpha, V_F], \quad \frac{\partial p_{\infty}^1}{\partial v}(v) \leq -p_{\infty}^1(v).\]

Since \(b \leq 0\), we have

\[-(bN(t)-b_1N_{\infty}^1) \int_{-\infty}^{V_F} \frac{\partial p_{\infty}^1}{\partial v}(v)H(v,t)^q\gamma(v)dv \leq (bN(t)-b_1N_{\infty}^1) \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv.\]

Moreover, since \(\gamma\) is increasing, \(b\) is non-positive and \(\alpha\) is positive, we have

\[\int_{-\infty}^{V_F} (-v + bN(t))W(v,t)\gamma'(v)dv \leq 0. \quad (19)\]

Using boundary conditions and the facts that \(\partial_v p(V_F, t) = \lim_{v \to V_F} \frac{-p_{\infty}(v)}{V_F-v}\) and \(\partial_v p_{\infty}^1(V_F) = \lim_{v \to V_F} \frac{-p_{\infty}^1(v)}{V_F-v} > 0\), we compute

\[W(V_F, t) = \lim_{v \to V_F} \left(\frac{p(v,t)^{q-1}}{p_{\infty}(v)^{q-1}p(v,t)}\right) = \lim_{v \to V_F} \left(\frac{-p(v,t)}{V_F-v} - \frac{p_{\infty}^1(v)}{V_F-v}\right) \left(\frac{p(v,t)}{p_{\infty}(v)}\right)^{q-1} = \left(\frac{\partial_v p(V_F, t)}{\partial_v p_{\infty}(V_F)}\right) \left(\frac{p(v,t)}{p_{\infty}(v)}\right)^{q-1} p(V_F, t) = 0. \quad (21)\]

and

\[-a \frac{\partial W}{\partial v}(V_F, t)\]
\[= -a \lim_{v \to V_F} \frac{q\frac{\partial p}{\partial v}(v,t)p(v,t)^{q-1}p_{\infty}^1(v)^{q-1} - (q-1)\frac{\partial p_{\infty}^1}{\partial v}(v)p_{\infty}^1(v)^{q-2}p(v,t)^q}{p_{\infty}(v)^{2(q-1)}}\]
\[= qN(t)^qN_{\infty}^{q-1} - (q-1)N(t)^qN_{\infty}^{q-1}N_{\infty}^{q-1} 2(q-1) = \frac{N(t)^q}{N_{\infty}^{q-1}}. \quad (22)\]

The function \(\gamma\) being non-negative and its second order derivative being non-positive on an interval \([\eta, V_F]\) for some \(\eta \in [\alpha, V_F]\) (see Lemma 4.2), we deduce from (15), (19), (20), (21) and (22) the inequality

\[\frac{d}{dt} \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv \]
\[\leq -\frac{N(t)^q}{N_{\infty}^{q-1}} \gamma(V_F) - aq(q-1) \int_{-\infty}^{V_F} H(v,t)^{q-2}p_{\infty}^1(v) \left(\frac{\partial H}{\partial v}(v,t)\right)^2 \gamma(v)dv \quad (23)\]
\[+ a \int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv + (q-1)(bN(t) - b_1N_{\infty}^1) \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv.,\]
Now, let’s control the term containing $\gamma''$. We compute, using integration by parts and $\gamma(\eta) = 0$ (Lemma 4.2),

\[
\int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv = \int_{-\infty}^{\eta} H^{q-1}\frac{\partial}{\partial v} \left( \int_{-\infty}^{v} p(w,t)dw \right) \gamma'' dv
\]

\[
= - \int_{-\infty}^{\eta} \frac{\partial H^{q-1}}{\partial v} \left( \int_{-\infty}^{v} p(w,t)dw \right) \gamma'' dv - \int_{-\infty}^{\eta} H^{q-1} \left( \int_{-\infty}^{v} p(w,t)dw \right) \gamma'' dv
\]

\[
= - (q - 1) \int_{-\infty}^{\eta} \frac{\partial H}{\partial v} H^{q-2} \left( \int_{-\infty}^{v} p(w,t)dw \right) \gamma'' dv
\]

Moreover, since $p$ is a probability density,

\[
\left| \int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv \right| \leq (q - 1) \int_{-\infty}^{\eta} \left| \frac{\partial H}{\partial v} \right| H^{q-2} |\gamma''(v)| dv + \int_{-\infty}^{\eta} H^{q-1} |\gamma'''(v)| dv.
\]

If $\alpha$ is close enough to $V_F$, there exists $\tilde{C} \in \mathbb{R}^*_+$ such that $p^1_{\infty} \geq \tilde{C}$ on $[\alpha, \eta]$, and with Peter and Paul inequality,

\[
\left| \int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv \right| \leq \varepsilon \tilde{C} (q - 1) \int_{-\infty}^{\eta} H^{q-2} \left( \frac{\partial H}{\partial v} \right)^2 p^1_{\infty} |\gamma''|^2 dv + \frac{\tilde{C}(q - 1)}{\varepsilon} \int_{-\infty}^{\eta} H^{q-1} p^1_{\infty} dv,
\]

There exists $C \in \mathbb{R}^*_+$ such that $\gamma'' \leq C_\gamma$ and $\gamma''' \leq C_\gamma$ (Lemma 4.2); thus there exists $\tilde{C} \in \mathbb{R}^*_+$ such that

\[
\left| \int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv \right| \leq \varepsilon \tilde{C} (q - 1) \int_{-\infty}^{\eta} H^{q-2} (v,t)p^1_{\infty}(v) \left( \frac{\partial H}{\partial v} (v,t) \right)^2 \gamma(v) dv
\]

\[
+ \varepsilon \tilde{C} \int_{-\infty}^{\eta} W(v,t)\gamma(v)dv + \frac{2\tilde{C}(q - 1)}{\varepsilon} \int_{-\infty}^{\eta} H^{q-2}(v,t)p^1_{\infty}(v) dv.
\]

We have eventually

\[
\left| \int_{-\infty}^{\eta} W(v,t)\gamma''(v)dv \right| \leq \varepsilon \tilde{C} (q - 1) \int_{-\infty}^{V_F} H^{q-2}(v,t)p^1_{\infty}(v) \left( \frac{\partial H}{\partial v} (v,t) \right)^2 \gamma(v) dv
\]

\[
+ \varepsilon \tilde{C} \int_{-\infty}^{V_F} W(v,t)\gamma(v)dv + \frac{2\tilde{C}(q - 1)}{\varepsilon} \int_{-\infty}^{V_F} H^{q-2}(v,t)p^1_{\infty}(v) dv.
\]
Combining (23) and (24), and choosing \( \varepsilon \) small enough, there exist constants \( C_1, C_2 \in \mathbb{R}_+^* \) depending only on \( \gamma, q, a \) and \( b_1 \) such that

\[
\frac{d}{dt} \int_{-\infty}^{V_F} W(v, t)\gamma(v)dv \leq -\frac{\gamma(V_F)}{N_1 q \tau} N(t) q + (q - 1)(bN(t) - C_1) \int_{-\infty}^{V_F} W(v, t)\gamma(v)dv
\]

\[
+ C_2 \int_{\alpha}^{V_F} H(v, t)q^{-2}p_\infty^1(v)dv. \tag{25}
\]

**Excitatory case:**

Like in the previous case, we have

\[
\int_{-\infty}^{V_F} -vW(v, t)\gamma'(v)dv \leq 0
\]

and, for \( \alpha \) close enough to \( V_F \),

\[
b_1 N_1 \int_{-\infty}^{V_F} \frac{\partial p_\infty^1}{\partial v}(v)H(v, t)q\gamma(v)dv \leq -b_1 N_1 \int_{-\infty}^{V_F} W(v, t)\gamma(v)dv.
\]

The term containing \( \gamma'' \) can be handled the same way as before. There remain the terms

\[
J_1(t) = bN(t) \int_{-\infty}^{V_F} W(v, t)\gamma'(v)dv, \quad J_2(t) = -bN(t) \int_{-\infty}^{V_F} \frac{\partial p_\infty^1}{\partial v}(v)H(v, t)q\gamma(v)dv.
\]

In order to obtain a bound for \( J_1 \), we use a partition of unity: by properties of \( \gamma \) (Lemma 4.2), there exists \( \omega \in [\alpha, V_F] \) such that \( \gamma' \equiv \gamma \) on \( [\omega, V_F] \); we choose two functions \( \gamma_1, \gamma_2 \in C^\infty([ -\infty, V_F]) \) and a value \( \varepsilon_1 \in \mathbb{R}_+^* \) small enough such that \( \gamma_1 + \gamma_2 = 1 \) and

- \( \gamma_1 \) is non-increasing on \( [ -\infty, V_F] \) and \( \gamma_1 \equiv 0 \) on \( [\omega + \varepsilon_1, V_F] \);
- \( \gamma_2 \) is non-decreasing on \( [ -\infty, V_F] \) and \( \gamma_2 \equiv 0 \) on \( [ -\infty, \omega] \).

Then,

\[
\int_{-\infty}^{V_F} W(v, t)\gamma'(v)dv = \int_{-\infty}^{V_F} W(v, t)\gamma'(v)\gamma_1(v)dv + \int_{-\infty}^{V_F} W(v, t)\gamma'(v)\gamma_2(v)dv.
\]

Since \( \gamma_2 = 0 \) on \( [ -\infty, \omega] \), \( \gamma_2(v) \leq 1 \) for all \( v \in [ -\infty, V_F] \) and \( \gamma' \leq \gamma \) on \( [\omega, V_F] \),

\[
\int_{-\infty}^{V_F} W(v, t)\gamma'(v)\gamma_2(v)dv = \int_{\omega}^{V_F} W(v, t)\gamma'(v)\gamma_2(v)dv
\]

\[
\leq \int_{\omega}^{V_F} W(v, t)\gamma(v)dv \leq \int_{-\infty}^{V_F} W(v, t)\gamma(v)dv. \tag{26}
\]

Using the inequality \( jk \leq \frac{\varepsilon}{q} j^q + \frac{q - 1}{q \varepsilon^{q - 1}} k^\frac{q}{q - 1} \), we obtain for all \( \varepsilon \in \mathbb{R}_+^* \),

\[
bN(t) \int_{-\infty}^{V_F} W(v, t)\gamma'(v)\gamma_2(v)dv \leq \left( \frac{(q - 1)\varepsilon^\frac{q}{q - 1}}{q} + \frac{bN(t)^q}{q \varepsilon^{q - 1}} \right) \int_{-\infty}^{V_F} W(v, t)\gamma(v)dv.
\]
We do for the term $\gamma_1$ what we did previously unto the term $\gamma''$, which yields
\[
\int_{-\infty}^{V_F} W(v, t)\gamma'(v)\gamma_1(v)dv = \int_{-\infty}^{V_F} H^{q-1}\frac{\partial}{\partial v}\left(\int_{-\infty}^{v} p(w, t)dw\right)\gamma'(v)\gamma_1(v)dv
\]
\[
= -\int_{-\infty}^{V_F} \left(\int_{-\infty}^{v} p(w, t)dw\right)\left(\frac{\partial H^{q-1}}{\partial v}\gamma'' \gamma_1 + H^{q-1} (\gamma'(\gamma_1)')\right)dv
\]

Moreover, since $p$ is a probability density,
\[
\left|\int_{-\infty}^{V_F} W(v, t)\gamma'(v)\gamma_1(v)dv\right| \leq (q-1)\int_{-\infty}^{V_F} \left|\frac{\partial H}{\partial v}\right| H^{q-2} |\gamma'(v)\gamma_1(v)| dv
\]
\[
+ \int_{-\infty}^{V_F} H^{q-1} |(\gamma'(\gamma_1)'(v)| dv.
\]

If $\alpha$ is close enough to $V_F$, there exists like before $C_{\omega}$ such that $p_1^{\omega}$ > $C_{\omega}$ on $[\alpha, \omega + \epsilon_1]$, and $\gamma'(v)^2$ < $C_{\gamma}(v)$. Then, there exists $C_2 \in \mathbb{R}^*_+$ such that, by Peter and Paul inequality,
\[
bN(t) \int_{-\infty}^{V_F} \left|\frac{\partial H}{\partial v}\right| H^{q-2} |\gamma'' \gamma_1| dv \leq \bar{C}_2 \epsilon \int_{-\infty}^{V_F} H^{q-2} p_1^{\omega} \left(\frac{\partial H}{\partial v}\right)^2 \gamma dv
\]
\[
+ \frac{\bar{C}_2}{\epsilon} b^2 N(t)^2 \int_{\omega}^{\omega + \epsilon_1} H^{q-2} p_1^{\omega} dv. \tag{27}
\]

Hence, using again $jk \leq \frac{\epsilon}{q} j^q + \frac{q-1}{q \epsilon - q^{-1}}$, there exists $\bar{C}_3, s \in \mathbb{R}^*_+$ such that
\[
bN(t) \int_{-\infty}^{V_F} \left|\frac{\partial H}{\partial v}\right| H^{q-2} |\gamma'' \gamma_1| dv \leq \bar{C}_3 \epsilon \int_{-\infty}^{V_F} H^{q-2} p_1^{\omega} \left(\frac{\partial H}{\partial v}\right)^2 \gamma dv
\]
\[
+ \frac{\bar{C}_3}{\epsilon} \bar{C}_6 \epsilon^s + \frac{q}{q} b^2 N(t)^2 \int_{\omega}^{V_F} H^{q-2} p_1^{\omega} dv. \tag{28}
\]

Besides, using the properties of $\gamma$ and $\gamma_1$, we have
\[
|\gamma'(\gamma_1)'(v)|^2 \leq (|\gamma''(v)|\gamma_1(v) + |\gamma'(v)\gamma_1'(v)|)^2
\]
\[
\leq \gamma''(v)^2 \gamma_1^2 + 2|\gamma''(v)|\gamma_1(v)\gamma_1'(v)|\gamma_1'(v) + \gamma'(v)^2 \gamma_1'(v)^2
\]
\[
= \gamma''(v)^2 \gamma_1^2 + (\gamma''(v)^2 + \gamma'(v)^2) \gamma_1'^2 + \gamma'(v)^2 \gamma_1'^2
\]
\[
\leq C(1 + 2 \gamma_1'^2) + \gamma'(v)^2 \gamma_1'^2. \tag{29}
\]

The method used to prove (28) combined with (29) gives that there exist $\bar{C}_4, \bar{C}_5, \bar{C}_6, s_3, s_4 \in \mathbb{R}^*_+$ such that
\[
bN(t) \int_{-\infty}^{V_F} H^{q-1} |(\gamma'(\gamma_1)'(v)| dv
\]
\[
\leq \bar{C}_4 \epsilon \int_{-\infty}^{V_F} W\gamma dv + \left(\bar{C}_5 \epsilon^{s_3} + \bar{C}_6 \epsilon^{s_4} b^q N(t)^q\right) \int_{\omega}^{\omega + \epsilon_1} H^{q-2} p_1^{\omega} dv. \tag{30}
\]
Collecting the previous results, we get the following bound for $J_1$: there exist $C_7, C_8, s_3, s_6 \in \mathbb{R}_+^*$ such that

$$J_1(t) \leq \epsilon C_7 \left( \int_{-\infty}^{V_F} H^{g-2} p_\infty^1 \left( \frac{\partial H}{\partial v} \right)^2 \gamma dv + \int_{-\infty}^{V_F} W \gamma dv \right) + \frac{C_8 b^q N(t)^q}{\epsilon^{s_3}} \int_{-\infty}^{V_F} W \gamma dv$$

$$+ \frac{C_9 (1 + b^q N(t)^q)}{\epsilon^{s_6}} \int_{\alpha}^{V_F} H^{g-2} p_\infty^1 dv. \quad (31)$$

Let’s now bound $J_2$. Integration by parts yields

$$\int_{-\infty}^{V_F} \frac{\partial H}{\partial v} (v) H(v, t)^{q} \gamma(v) dv = - \int_{-\infty}^{V_F} W(v, t) \gamma'(v) dv - q \int_{-\infty}^{V_F} \frac{\partial H}{\partial v} H^{q-1} p_\infty^1 \gamma dv.$$

The term

$$bN(t) \int_{-\infty}^{V_F} W(v, t) \gamma'(v) dv$$

can be controlled by inequality (31). The term

$$bN(t)q \int_{-\infty}^{V_F} \frac{\partial H}{\partial v} H^{q-1} p_\infty^1 \gamma dv$$

can be bounded, thanks to inequality Peter and Paul, which gives

$$bN(t)q \int_{-\infty}^{V_F} \left| \frac{\partial H}{\partial v} \right| H^{q-1} p_\infty^1 \gamma dv \leq \epsilon C_{10} \int_{-\infty}^{V_F} H^{q-1} p_\infty^1 \gamma dv$$

$$+ \frac{C_{11} b^2 N(t)^2}{\epsilon} \int_{-\infty}^{V_F} W \gamma dv,$$

where $C_{10}, C_{11} \in \mathbb{R}_+^*$. The previous method yields the following bound for $J_2$: there exist $C_{12}, C_{13}, C_{14}, s_7, s_8 \in \mathbb{R}_+^*$ such that

$$J_1(t)$$

$$\leq \epsilon C_{12} \left( \int_{-\infty}^{V_F} H^{g-2} p_\infty^1 \left( \frac{\partial H}{\partial v} \right)^2 \gamma dv + \int_{-\infty}^{V_F} W \gamma dv \right) + \frac{C_{13} b^q N(t)^q}{\epsilon^{s_7}} \int_{-\infty}^{V_F} W \gamma dv$$

$$+ \frac{C_{14} (1 + b^q N(t)^q)}{\epsilon^{s_8}} \int_{\alpha}^{V_F} H^{g-2} p_\infty^1 dv. \quad (32)$$

Combining (15), (31) and (32), and choosing $\epsilon$ small enough, we obtain that there exist $C_3, C_4, C_5, C_6, C_7 \in \mathbb{R}_+^*$ depending only on $\gamma, q, b_1, a$ and $N_1^1$ such that

$$\frac{d}{dt} \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv \leq - \gamma(V_F) \frac{\gamma(V_F)}{N_1^1} - N(t)^q$$

$$+ b^q N(t)^q \left( C_3 + C_4 \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv + C_5 \int_{-\infty}^{V_F} H(v, t)^{q-2} p_\infty^1 (v) dv \right)$$

$$+ C_6 \int_{\alpha}^{V_F} H(v, t)^{q-2} p_\infty^1 (v) dv - C_7 \int_{-\infty}^{V_F} W(v, t) \gamma(v) dv. \quad (33)$$

We are now able to prove Theorem 4.1. We proceed by a kind of bootstrap method.
Proof. We are going to use Lemma (4.4). Let’s denote for clarity
\[ H(v, t) = \frac{p(v, t)}{p^1_\infty(v)}, \quad W(q; v, t) = p^1_\infty(v)H(v, t)^q = \frac{p(v, t)^q}{p^1_\infty(v)^{q-1}}, \]
\[ \gamma(\alpha; v) = 1_{[\alpha, V_F]}(v)e^{q(\alpha - V_F)} \quad \text{and} \quad I_{q, \alpha}(t) = \int_{-\infty}^{V_F} W(q; v, t)\gamma(\alpha; v)dv. \]

Assume that for a natural number \( q \geq 2 \) and a real number \( \alpha_{q-2} \in [V_R, V_F] \) close enough to \( V_F \) to apply Lemma 4.4 we have a constant \( K_{q-2} \) such that
\[ \int_{\alpha_{q-2}}^{V_F} H(v, t)^{q-2}p^1_\infty(v)dv \leq K_{q-2}. \] (34)

Then, Lemma (4.4) reduces in the inhibitory case to

\[ \frac{d}{dt}I_{q, \alpha_{q-2}}(t) \leq -\frac{\gamma(\alpha_{q-2}; V_F)}{N_1^{q-1}}N(t)^q + (q-1)(bN(t) - C_1)I_{q, \alpha_{q-2}}(t) + C_2K_{q-2}. \] (35)

Let us proceed as in the proof of [11][Theorem 3.1]. By assumptions we have \( I_{q, \alpha_{q-2}}(0) < +\infty \). Since \( b \leq 0 \), (35) implies \( I_{q, \alpha_{q-2}}'(t) \leq C_2K_{q-2} + (q-1)C_1I_{q, \alpha_{q-2}}(t) \).

Thus, assuming \( t \) is large enough depending on \( I_{q, \alpha_{q-2}}(0) \), we have
\[ I_{q, \alpha_{q-2}}(t) \leq \frac{2C_2K_{q-2}}{(q-1)C_1}. \]
Hence, integrating (35) in time, we get
\[ \int_J N(t)^qdt \leq C_q(1 + |J|), \]
with \( C_q \in \mathbb{R}_+ \) a constant and for all interval \( J \subset [T_0, +\infty] \) with \( T_0 \) depending upon \( I_{q, \alpha_{q-2}}(0) \).

In the excitatory case, Lemma (4.4) and (34) yield

\[ \frac{d}{dt}I_{q, \alpha_{q-2}}(t) \leq N(t)^q \left( b^q(C_3 + C_5K_{q-2}) + b^qC_4I_{q, \alpha_{q-2}}(t) - \frac{\gamma(\alpha_{q-2}; V_F)}{N_1^{q-1}} \right) + C_6K_{q-2} - C_7I_{q, \alpha_{q-2}}(t). \] (36)

Define
\[ L = \max\left(I_{q, \alpha_{q-2}}(0), \frac{C_qK_{q-2}}{C_7}\right). \]

If \( b \) is small enough to have
\[ b^q(C_3 + C_5K_{q-2}) + b^qC_4L - \frac{\gamma(\alpha_{q-2}; V_F)}{N_1^{q-1}} < 0, \]
then for all \( t > 0 \), \( I_{q, \alpha_{q-2}}(t) \leq L \) and integrating (36) between 0 and \( t \) we get
\[ \int_0^t N(t)^qdt \leq C(1 + t), \]
with \( C_q \in \mathbb{R}_+ \) a constant. As a by-product, we have uniform boundedness of \( I_{q, \alpha_{q-2}}(t) \) in both inhibitory and excitatory cases and hence there exists a constant \( K_q > 0 \) such that
\[ \int_{\alpha_{q-2}}^{V_F} W(q; v, t)\gamma(\alpha_{q-2}; v)dv = \int_{\alpha_{q-2}}^{V_F} H(v, t)^q p^1_\infty(v)\gamma(\alpha_{q-2}; v)dv \leq K_q. \] (37)
By choosing a new value \( \alpha_q \in [V_R, V_F] \) such that \( \alpha_{q-2} < \alpha_q \), we have the property (34) for this new \( \alpha_q \) because there exists a constant \( C \) such that
\[
\int_{\alpha_q}^{V_F} H(v, t)^q p_1^1(v) dv \leq C \int_{\alpha_{q-2}}^{V_F} H(v, t)^q p_1^1(v) \gamma(\alpha_{q-2}; v) dv \leq C K_q =: K_q.
\]
Therefore, we can easily prove (34) by induction: for \( q = 2 \), the result is obvious because
\[
\int_{\alpha_0}^{V_F} H^{q-2}(v, t)p_1^1(v) dv = \int_{\alpha_0}^{V_F} p_1^1(v) dv \leq \int_{-\infty}^{V_F} p_1^1(v) dv = 1;
\]
for \( q = 3 \), the result holds because of mass conservation of \( p \):
\[
\int_{\alpha_1}^{V_F} H^{q-2}(v, t)p_1^1(v) dv = \int_{\alpha_1}^{V_F} \frac{p(v, t)}{p_1^1(v)} p_1^1(v) dv = \int_{\alpha_1}^{V_F} p(v, t) dv \leq \int_{-\infty}^{V_F} p(v, t) dv = 1.
\]
Then, induction works if we choose an increasing sequence of numbers \( \alpha_q < V_F, q \in \mathbb{N} \).

4.2. **Global-in-time existence.** We now use uniform \( L^3 \) estimates in order to prove global-in-time existence for small enough excitatory connectivities. More precisely, we prove:

**Theorem 4.5.** Let \( b_1 \in \mathbb{R}_+^* \) be such that there exists a stationary state \((p_1^1, N_1^1)\). Let \( V_M \in [V_R, V_F] \); denote
\[
S_3(b_1, V_M) = \int_{V_M}^{V_F} \frac{p_0^3(v)}{p_1^1(v)^2} dv.
\]
If \( p_0 \) satisfies Assumptions 1, there exists \( b^* \in \mathbb{R}_+^* \) depending only upon \( S_3(b_1, V_M) \) and \( V_M \) such that for all \( b \in [0, b^*] \), there exists a unique global-in-time classical solution \((p, N)\) of (5).

In [10][Proposition 4.3], the authors use an \( L^\infty \) bound on \( N \) to obtain global-in-time existence of classical solutions in the inhibitory case \( b < 0 \). We are going to adapt their method in the excitatory case \( b > 0 \).

While constructing local-in-time solutions to (5), the authors of [10] introduce the change of variables
\[
y = e^t v, \quad \tau = \frac{1}{2}(e^{2t} - 1) \quad \text{or equivalently} \quad v = \frac{y}{\sqrt{2\tau + 1}}, \quad t = \frac{1}{2} \log(2\tau + 1).
\]
It allows us to write (5) as an equivalent free-boundary Stefan-like problem on the time-dependent domain \([-\infty, s(\tau)]\) where the free boundary \( s(\tau) \) evolves depending on the solution up to time \( \tau \) (see it’s expression in (38)). As in [10], we assume without loss of generality (using the same rescaling as in hypothesis (11) that \( V_F = 0, a = 1 \) and \(-v + bN(t)\) is replaced by \(-v + b_0 + bN(t)\), \( b_0 \in \mathbb{R} \). We denote
\[
w(y, \tau) = \alpha(\tau)p\left(y\alpha(\tau), -\ln(\alpha(\tau))\right), \quad \alpha(\tau) = \frac{1}{\sqrt{2\tau + 1}},
\]
\[
M(\tau) = \alpha(\tau)^2 N(t) = -\frac{dw}{dy}(0, \tau),
\]
or equivalently
\[
p(v, t) = e^t w \left(e^t v, \frac{1}{2}(e^{2t} - 1)\right).
\]
We then make the change of variable

\[ x = y - b_0(\sqrt{1 + 2\tau} - 1) - b \int_0^\tau \frac{M(s)}{\alpha(s)} \, ds. \]

with the associated new unknown \( u(x, \tau) = w(y, \tau) \). The pair \((u, M)\) is then a solution to

\[
\begin{align*}
\frac{\partial u}{\partial \tau}(x, \tau) &= \frac{\partial^2 u}{\partial y^2}(x, \tau) + M(\tau) \delta_{s(\tau)} + \frac{\nu_R}{\alpha(\tau)}(x) & x \in [-\infty, s(\tau)], \tau \in \mathbb{R}_+ \\
M(\tau) &= -\frac{\partial u}{\partial x}(s(\tau), \tau) & \tau \in \mathbb{R}_+ \\
s(\tau) &= s(0) - b_0(\sqrt{1 + 2\tau} - 1) - b \int_0^\tau \frac{M(s)}{\alpha(s)} \, ds & \tau \in \mathbb{R}_+ \\
u(-\infty, \tau) &= u(s(\tau), \tau) = 0 & \tau \in \mathbb{R}_+ \\
u(x, 0) &= u^0(x) & x \in [-\infty, s(0)].
\end{align*}
\]

(38)

For this system, we recall the notion of classical solution introduced in [10].

**Definition 4.6** (Classical solutions for the Stefan-like problem). Assume \( u_0 \) satisfies Assumptions 1. We say that \( u \) is a classical solution of (38) with initial datum \( u^0 \) on the interval \( J = [0, T] \) or \( J = [0, T] \), for a given \( T > 0 \) if

1. \( M \) is a continuous function on \( J \), \( u \) is continuous in the region \( \{(x, \tau): -\infty < x \leq s(\tau), \tau \in J\} \) and for all \( \tau \in J \), \( u(\cdot, \tau) \in L^1([-\infty, s(\tau)]) \).
2. If we denote \( s_1(\tau) := s(\tau) + \frac{\nu_R}{\alpha(\tau)} \), then \( \partial_x u \) and \( \partial_x u \) are continuous in the region \( \{(x, \tau): x \in [-\infty, s(\tau)], s(\tau) \in \{s_1(\tau)\}, \tau \in J \} \).
3. \( \partial_x u(s_1(\tau)^+, \tau), \partial_x u(s_1(\tau)^-, \tau), \partial_x u(s(\tau)^-, \tau) \) are well defined and \( \partial_x u \) vanishes at \( -\infty \).
4. Equations (38) are satisfied in the sense of distributions in the region \( \{(x, \tau): -\infty < x \leq s(\tau), \tau \in J\} \) and pointwise (in the classical sense of a function’s derivative) in \( \{(x, \tau): x \in [-\infty, s(\tau)], s(\tau) \in \{s_1(\tau)\}, \tau \in J \} \).

The article [10][Theorem 4.2] provides a criterion for global-in-time existence:

**Theorem 4.7** (Carrillo, Gonzales, Gualdani, Schonbeck). Assume \( u_0 \) satisfies Assumptions 1. There exists a unique maximal classical solution of (38) in the sense of Definition 4.6 and its maximal time of existence \( T^* \) satisfies

\[ T^* = \sup\{\tau \in \mathbb{R}_+ \mid M(\tau) < +\infty\}. \]

We are now able to prove Theorem 4.5:

**Proof.** First, note that by Assumptions 1, \( p^0 \) has a derivative on the left of \( V_F \), and thus

\[
\lim_{v \to V_F^-} \frac{p^0(v)}{p^\infty(v)} = \lim_{v \to V_F^-} \left( \frac{\frac{p^0(v)}{p^\infty(v)}}{\frac{p^\infty(v)}{p^\infty(v)}} \right)^2 p^0(v) = \left( \frac{\frac{dp^0}{dv}(V_F^-)}{N1} \right)^2 p^0(V_F) = 0. \]

(39)

Hence, since \( p^0 \) is \( C^1 \) on \( |V_R, V_F| \) and \( p^\infty > 0 \) on \( |V_R, V_F| \), we have \( S_3(b_1, V_M) < +\infty \), which is a technical hypothesis of Theorem 4.1.

We follow the ideas of the proof of global-in-time existence in the inhibitory case in the article [10][Proposition 4.3 and Theorem 4.4]. We derive the equivalent free boundary Stefan-like problem as we recall above and we assume the local-in-time
solution of (38) from initial datum $u^0$ exists on the maximal time interval $[0, \tau_0[$. There exists a value $\varepsilon \in \mathbb{R}^*_+$, chosen small enough for what follows, such that,

$$U_{\tau_0 - \varepsilon} = \sup_{x \in [-\infty, s(\tau_0 - \varepsilon)]} \left| \frac{du}{dx}(x, \tau_0 - \varepsilon) \right| < +\infty.$$  

We are going to prove that there exists $K_\infty \in \mathbb{R}^*_+$ such that

$$\sup_{\tau \in [\tau_0 - \varepsilon, \tau_0]} M(\tau) < K_\infty.$$  

Then, Theorem 4.7 gives the existence of the solution on $[0, \tau_0 + \varepsilon[$. If $\varepsilon$ does not depend on $U_{\tau_0 - \varepsilon}$, we can repeat the argument and we have global-in-time existence for the solution.

For the sake of clarity, we assume without loss of generality that $\tau_0 - \varepsilon = 0$. We are thus going to find a bound for $M$ on $[0, \tau_0[$ with respect to $\tau_0$, for $\tau_0$ small enough. We recall first the expression of the heat kernel:

$$G(x, \tau, \xi, \eta) = \frac{1}{\sqrt{4\pi(\tau - \eta)}} e^{-\frac{(x-\xi)^2}{4(\tau - \eta)}}.$$  

It is proved in [10], sections 2 and 3 that, recalling that $t = \frac{1}{2} \log(2\tau + 1)$, the continuous function $M$ satisfies

$$M(\tau) = \alpha(\tau)^2 N(t) = e^{-2t} N(t)$$

and for all $\tau \in [0, \tau_0[$,

$$M(\tau) = -2 \int_{-\infty}^{s(0)} G(s(\tau), \tau, \xi, 0) \frac{du^0}{dx}(\xi) d\xi + 2 \int_0^\tau M(\eta) \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta), \eta) d\eta - 2 \int_0^\tau M(\eta) \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta) d\eta. \quad (40)$$

We denote,

$$\Phi(\tau) = \sup_{s \in [0, \tau]} M(s).$$

• Firstly, by properties of $G$,

$$2 \int_{-\infty}^{s(0)} G(s(\tau), \tau, \xi, 0) \frac{du^0}{dx}(\xi) d\xi \leq \left\| \frac{du^0}{dx} \right\|_{\infty}. \quad (41)$$

Applying Hölder inequality, we obtain

$$2 \int_0^\tau M(\eta) \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta), \eta) d\eta \leq 2 \left( \int_0^\tau M(\eta)^3 d\eta \right)^{\frac{1}{3}} \left( \int_0^\tau \left| \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta), \eta) \right|^2 d\eta \right)^{\frac{1}{2}}, \quad (42)$$

and

$$2 \int_0^\tau M(\eta) \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta) d\eta \leq 2 \left( \int_0^\tau M(\eta)^3 d\eta \right)^{\frac{1}{3}} \left( \int_0^\tau \left| \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta) + \frac{V_R}{\alpha(\eta)}, \eta) \right|^2 d\eta \right)^{\frac{1}{2}}. \quad (43)$$
Then, by Theorem 4.1, there exists a constant \( C \in \mathbb{R}_+^* \) such that
\[
\int_0^\tau M(\eta)^3 d\eta = \int_0^\tau \frac{1}{2} e^{(\tau^2 - 1)} e^{-6\tau} N(s)^3 ds \leq C(1 + \frac{1}{2}(e^{2\tau} - 1)) = C(1 + \tau),
\]
which yields
\[
\left( \int_0^\tau M(\eta)^3 d\eta \right)^{\frac{1}{3}} \leq C^\frac{1}{3}(1 + \tau_0)^{\frac{1}{3}}.
\] (45)

We come back to the bound (42). We have
\[
s(\tau) = s(0) - b_0(\sqrt{2\tau + 1} - 1) - b \int_0^\tau \frac{M(s)}{\alpha(s)} ds.
\]
Since \( \alpha^{-1} \) is 1-Lipschitz,
\[
|s(\tau) - s(\eta)| \leq |b_0|(\sqrt{2\tau + 1} - \sqrt{2\eta + 1}) + b \int_\eta^\tau M(s)\sqrt{2s + 1} ds
\leq (|b_0| + b\sqrt{2\tau_0 + 1}\Phi(\tau))|\tau - \eta|.
\]
Moreover, we compute
\[
\frac{\partial}{\partial x}G(x, \tau, \xi, \eta) = -\frac{1}{2\sqrt{4\pi}(\tau - \eta)^{\frac{3}{2}}} e^{-\frac{(x - \xi)^2}{4(\tau - \eta)}},
\]
and with the previous bound, we have
\[
\int_0^\tau \left| \frac{\partial}{\partial x}G(s(\tau), \tau, s(\eta), \eta) \right|^{\frac{3}{2}} d\eta = \frac{1}{8\pi^{\frac{3}{2}}} \int_0^\tau \frac{(s(\tau) - s(\eta))^{\frac{3}{2}}}{(\tau - \eta)^{\frac{3}{2}}} e^{-\frac{3(x(s(\tau)) - s(\eta))^2}{4(\tau - \eta)}} d\eta \leq \frac{|b_0| + b\sqrt{2\tau_0 + 1}\Phi(\tau)}{8\pi^{\frac{3}{2}}} \int_0^\tau \frac{1}{|\tau - \eta|^{\frac{3}{2}}} d\eta = \frac{|b_0| + b\sqrt{2\tau_0 + 1}\Phi(\tau)}{6\pi^{\frac{3}{2}}} \tau^{\frac{3}{2}}.
\] (46)

Hence,
\[
\left| 2 \int_0^\tau M(\eta) \frac{\partial}{\partial x}G(s(\tau), \tau, s(\eta), \eta) d\eta \right| \leq 2C^\frac{1}{2}(1 + \tau_0)^{\frac{1}{2}} \frac{|b_0| + b\sqrt{2\tau_0 + 1}\Phi(\tau)}{6\pi^{\frac{3}{2}}} \tau^{\frac{3}{2}}.
\]

and applying the inequality \( (a_1 + a_2)^{\frac{3}{2}} \leq 2^{\frac{3}{2}}|a_1|^{\frac{3}{2}} + 2^{\frac{3}{2}}|a_2|^{\frac{3}{2}} \), we come to
\[
\left| 2 \int_0^\tau M(\eta) \frac{\partial}{\partial x}G(s(\tau), \tau, s(\eta), \eta) d\eta \right| \leq \frac{2^{\frac{3}{2}}|b_0|^{\frac{3}{2}}C^{\frac{1}{2}}(1 + \tau_0)^{\frac{1}{2}}}{3\pi^{\frac{3}{2}}} \tau_0^{\frac{3}{2}}
+ \frac{2^{\frac{3}{2}}b^{\frac{3}{2}}(2\tau_0 + 1)^{\frac{3}{2}}C^{\frac{1}{2}}(1 + \tau_0)^{\frac{1}{2}}}{3\pi^{\frac{3}{2}}} \tau_0^{\frac{3}{2}} \Phi(\tau)^{\frac{3}{2}}.
\] (47)

We now come back to the bound (43). Using (45) and Hölder’s inequality, we have
\[
|s(\tau) - s(\eta)| - \frac{V_R}{\alpha(\tau)} \geq \frac{|V_R|}{\alpha(\tau)} - |s(\tau) - s(\eta)|
\geq |V_R| - b_0\tau_0 - b \int_\eta^\tau \frac{M(s)}{\alpha(s)} ds
\geq |V_R| - b_0\tau_0 - bC^\frac{1}{2}(1 + \tau_0)^{\frac{1}{2}}(2\tau_0 + 1)^{\frac{1}{2}} \tau_0^{\frac{3}{2}}.
\] (48)
If \(\tau_0\) is small enough regarding \(b_0\) and \(C\), we have \(|V_R| - b_0\tau_0 - bC^\frac{1}{2}(1 + \tau_0)^\frac{1}{2}(2\tau_0 + 1)^\frac{3}{2}\tau_0^2 > 0\). We denote \(K(\tau_0) = |V_R| - b_0\tau_0 - bC^\frac{1}{2}(1 + \tau_0)^\frac{1}{2}(2\tau_0 + 1)^\frac{3}{2}\tau_0^2\). We use inequality \(ye^{-y^2} \leq e^{-\frac{y^2}{2}}\) in order to obtain
\[
\left| \frac{\partial}{\partial x} G(x, \tau, \xi, \eta) \right| \leq \frac{1}{\sqrt{4\pi(\tau - \eta)}} e^{-\frac{\eta^2 - \xi^2}{8(\tau - \eta)}}. \tag{49}
\]
Using the change of variable \(z = \frac{\sqrt{3}K(c)}{4\sqrt{\pi - \eta}}\), we compute:
\[
\int_0^\tau \left| \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta)) + \frac{V_R}{\alpha(\eta)}, \eta \right|^2 d\eta \\
\leq \frac{1}{(4\pi)^\frac{1}{2}} \int_0^\tau \frac{1}{(\tau - \eta)^\frac{1}{2}} e^{-\frac{3K(\tau_0)^2}{16(\tau - \eta)}} d\eta \\
\leq \frac{1}{(4\pi)^\frac{1}{2}} \int_0^\tau \frac{1}{(\tau - \eta)^\frac{1}{2}} e^{-\frac{3K(\tau_0)^2}{16(\tau - \eta)}} d\eta \\
= \frac{\sqrt{2}}{\sqrt{3}(\pi)^\frac{1}{2}K(\tau_0)} \int_0^{+\infty} e^{-z^2} dz.
\]
Hence,
\[
2 \int_0^\tau M(\eta) \frac{\partial}{\partial x} G(s(\tau), \tau, s(\eta)) + \frac{V_R}{\alpha(\eta)}, \eta d\eta \\
\leq C^\frac{1}{2}(1 + \tau_0)^\frac{1}{2} \frac{2^\frac{1}{2}}{3^\frac{1}{2}\sqrt{\pi}K(\tau_0)^\frac{1}{2}} \left( \int_0^{+\infty} e^{-z^2} dz \right)^\frac{\frac{1}{2}}{\frac{1}{2}}. \tag{50}
\]
- Collecting inequalities (41), (47) and (50), we derive the following bound for \(M\) on \([0, \tau_0]\),
\[
M(\tau) \leq 2 \left\| \frac{du^0}{dx} \right\|_\infty + \frac{2^\frac{1}{2}|b_0|^2 C^\frac{1}{2}(1 + \tau_0)^\frac{1}{2}}{3^\frac{1}{2}\pi^\frac{1}{2}} \tau_0^\frac{1}{2} + \frac{2^\frac{1}{2}b^2 (2\tau_0 + 1)^\frac{1}{2} C^\frac{1}{2}(1 + \tau_0)^\frac{1}{2}}{3^\frac{1}{2}\pi^\frac{1}{2}} \tau_0^\frac{1}{2} \Phi(\tau) \frac{1}{2} \Phi(\tau)^\frac{1}{2} \\
+ \frac{2^\frac{1}{2}C^\frac{1}{2}(1 + \tau_0)^\frac{1}{2}}{3^\frac{1}{2}\sqrt{\pi}K(\tau_0)^\frac{1}{2}} \left( \int_0^{+\infty} e^{-z^2} dz \right)^\frac{\frac{1}{2}}{\frac{1}{2}}. \tag{51}
\]
As a consequence, there exists \(\Lambda \in \mathbb{R}_+^\ast\) depending only on \(b_0, b, \frac{du}{dx} u^0\) and \(V_R\) such that, if \(\tau_0\) is chosen small enough with respect to \(b\),
\[
M(\tau) \leq \Lambda + \frac{1}{2} \Phi(\tau)^\frac{1}{2}.
\]
Denote \(\Psi(\tau) = \max(\Phi(\tau), 1); \) we have \(\Phi(\tau)^\frac{1}{2} \leq \Psi(\tau)^\frac{1}{2} \leq \Psi(\tau)\). Taking the supremum on both sides of the inequality and the maximum with 1, we obtain \(\Psi(\tau) \leq 2\Lambda\), which in turn implies
\[
M(\tau) \leq 2\Lambda. \tag{52}
\]
\[\square\]

**Remark 4.** In order to prove this result, we needed a bound on \(N\) in \(L^3\); the \(L^2\) estimate proved in article [11] was not enough. Indeed, in bounds (42) and (43),
we have $L^3$ norms of $M$ in the right-hand side. Applying Hölder inequality with exponent 2 instead of 3 would lead to the integral

$$\int_0^\tau \frac{(s(\tau) - s(\eta))^2}{(\tau - \eta)^3} e^{-\frac{(s(\tau) - s(\eta))^2}{4(\tau - \eta)}} d\eta,$$

in computation (46), and then the following diverging integral would appear:

$$\int_0^\tau \frac{1}{|\tau - \eta|} d\eta.$$

However, with the $L^3$ estimate of $M$, we have instead in (46) the converging integral

$$\int_0^\tau \frac{1}{|\tau - \eta|^\frac{3}{4}} d\eta.$$

Remark 5. The convergence to stationary state proved in [11][Theorem 3.5] with the help of $L^2$ bounds on $N$ is a priori and thus only valid up to an unknown time of existence; global-in-time existence is not granted. Moreover, this result demands severe restrictions on the size of $b$: the non-linearity has to be weak enough in order to apply the Entropy method and the Poincaré-like inequality. In our result, we only demand $b$ to be small with respect to $S_3(b_1, V_M)$ and $V_M$. Although we have no proof of it, we may have proved global-in-time existence for parameters where two or more stationary states coexist, which is impossible in a priori convergence results of [11][Theorem 2.1 and Theorem 3.5].

Remark 6. The assumptions on the initial datum $p^0$ are similar to what can be found in the early probabilistic literature. Namely, in the articles [15, 16] and in the note [14][Lemma 2.1], the authors use the working assumption that law $\nu$ of $X_0$ is differentiable at $V_F$ and satisfies

$$\forall x \in [V_F - \varepsilon, V_F], \quad \nu(dx) \leq \beta (V_F - x)dx,$$

(53)

for some $\beta, \varepsilon > 0$. Assumptions 1 could be weakened merely to (53) with technical work, since more regularity is granted after any non-zero period of time (see [21][Proposition 2.1]).

However, note that the question of the decay of $p^0$ near $V_F$ is a subtle and important one, since the short-time regularity of the solutions depends upon the boundary decay of the initial condition. In [24], the authors assume the initial density to be 0 in a vicinity of $V_F$; in [27], the result is proved under the hypothesis that the initial density decays like $o(x^\beta)$ near the origin ($V_F = 0$ in this work). The general results of [21] which allowed to fully prove propagation of chaos for the underlying particle system rely on the assumption that the initial density decays like $O(x^\beta)$ for some $0 < \beta < 1$. It can be seen in [21][Theorem 1.7] that the exponent $\beta$ has an effect upon the behaviour of solutions in short time.

5. Conclusion. In this article, we covered two aspects of the NNLIF model in the excitatory case: finite-time blow-up and global-in-time existence.

First, we proved systematic blow-up in the high connectivity case, hence upgrading previous results about blow-up and stationary states non-existence ([4]). Our bound for $b$ doesn’t seem to be optimal though, and the obtention of an optimal bound is an open problem. It is reasonable to think that as soon as there is no stationary state, every solution blows-up in finite time. In the case $V_F \leq 0$ our bound is optimal. Note that our bound in the case $V_F > 0$ converges to the bound for $V_F = 0$ when $V_F$ goes to 0.
Then, we upgraded previously obtained uniform $L^2$ estimates on the firing rate into uniform $L^q$ estimates at a fairly reasonable price in terms of hypotheses. Using $L^3$ estimates, we proved global-in-time existence when $b > 0$ is small enough regarding $p^0$. Previous deterministic results were a priori convergence to stationary states in cases when the stationary state is unique ($b > 0$ very small). Although we do make smallness hypotheses on $b$, we don’t demand existence of a unique stationary state in our global existence theorem. The result had been obtained for the stochastic counterpart (4) in [15] and other works ([20, 21, 28, 27]) focused on global solvability for problem (4) in the broader setting of physical solutions were solutions can continue after a blow-up event.

We didn’t address two difficult questions: non-existence of periodic solutions for the non-delayed NNLIF model and convergence to stationary states for large and negative values of $b$. These two questions are linked, since there is numerical evidence for periodic solutions in the NNLIF model with synaptic delay when $b < 0$ and $|b|$ is large. All attempts to find periodic solutions in the non-delayed NNLIF model failed, which may be an indicator of unconditional convergence towards stationary states in the inhibitory case. Yet, there is no efficient tool to prove such a conjecture, because general entropy methods break in the high connectivity regime. More broadly, there is a need for the development of new methods in order to tackle the remaining questions around NNLIF-type models, as they still contain complexity we can scarcely fathom with numerical exploration (see [22] for the most recent numerical insights).

Appendix A. Technical results. We first prove the following technical result which is used at the end of the proof of Theorem 3.1.

**Lemma A.1.** Let $V_F, a \geq 0$ and $V_R < V_F$. Then,

$$\inf_{x, \mu \in ]1, +\infty[} \frac{\xi}{\mu} \frac{e^{\mu V_F} - e^{\mu V_R}}{e^{\mu V_F}} e^{\mu V_F \xi} = \frac{V_F}{4} \inf_{y \in ]1, \sqrt{1 + \frac{4}{\mu V_F}}[} (1 + y)^2 \left(1 - e^{-\frac{V_F}{\mu V_F} \frac{1}{y^2 - 1}}\right) e^{\frac{y^2}{2}}. \quad (54)$$

**Proof.** We study the auxiliary function $f : ]1, +\infty[ \rightarrow \mathbb{R}^*_+$ defined by

$$f(\xi) = \frac{\xi}{\xi - 1} \frac{e^{\mu V_F} - e^{\mu V_R}}{e^{\mu V_F}} e^{\mu V_F \xi}.$$ 

If $V_F = 0$ and $V_R = 0$, its minimum is $\frac{1}{\mu}(1 - e^{\mu V_R})$, and this quantity goes to 0 when $\mu$ goes to infinity and the right-hand side of (54) is also 0; hence the result. Otherwise, we compute

$$f'(\xi) = \frac{e^{\mu V_F \xi}}{\xi - 1} \left[ V_F \xi - \frac{1}{\mu(\xi - 1)} \right] \frac{e^{\mu V_F} - e^{\mu V_R}}{e^{\mu V_F}}.$$  

This derivative is negative on ]1, $\xi$] and positive on ]$\xi$, +\infty[, where

$$\xi = \frac{1}{2} \left(1 + \sqrt{1 + \frac{4}{\mu V_F}}\right),$$ 

$$\bar{\xi} = \frac{1}{2} \left(1 + \sqrt{1 + \frac{4}{\mu V_F}}\right).$$
and $f$ reaches its global minimum for $\xi = \bar{\xi}$, and that minimum is

$$J(\mu) = \frac{1}{2} \left( \frac{1}{4} + \frac{1}{\sqrt{\nu F}} \right) e^{\mu V_F - \mu V_R} e^{\mu V_F \frac{1}{2} \left( 1 + \frac{1}{\sqrt{1 + \frac{4}{\mu V_F}} \right)}$$

With the change of variable $x = \frac{V_F \mu}{4}$, it's equivalent to finding, for $x$ in $[\frac{4}{a}, +\infty[$,

the minimum of

$$K(x) = \frac{V_F}{4} \left( 1 + \sqrt{1 + \frac{1}{x}} \right)^2 \left( 1 - e^{-4 \frac{V_F - V_R}{\nu F} x} \right) e^{2x \left( 1 + \sqrt{1 + \frac{1}{x}} \right)}$$

With the change of variable $y = \sqrt{1 + \frac{1}{x}}$, it's equivalent to finding, for $y$ in $[1, \sqrt{1 + \frac{a}{4}}]$, the minimum of

$$L(y) = \frac{V_F}{4}(1 + y)^2 \left( 1 - e^{-4 \frac{V_F - V_R}{\nu F} \frac{1}{y^2} x} \right) e^{2x}.$$

Then we prove the technical lemma 4.3 stated above.

**Proof of Lemma 4.3.** We first compute the partial derivatives of $W$:

$$\frac{\partial W}{\partial t} = qH^{q-1} \frac{\partial p}{\partial t} = -qH^{q-1} \frac{\partial}{\partial v}(-v + bN(t)p) + aqH^{q-1} \frac{\partial^2 p}{\partial v^2} + qH^{q-1} N(t)\delta_{V_R}, \quad (55)$$

and

$$\frac{\partial H}{\partial v} = \frac{1}{p_\infty^1} \left( \frac{\partial p}{\partial v} - H \frac{\partial p_\infty^1}{\partial v} \right),$$

$$\frac{\partial W}{\partial v} = qH^{q-1} \frac{\partial p}{\partial v} - (q - 1)H^q \frac{\partial p_\infty^1}{\partial v},$$

$$\frac{\partial^2 W}{\partial v^2} = q(q - 1) \frac{\partial H}{\partial v} H^q \frac{\partial^2 p}{\partial v^2} + qH^{q-1} \frac{\partial^2 p}{\partial v^2} - (q - 1)H^q \frac{\partial H}{\partial v} \frac{\partial p_\infty^1}{\partial v}$$

Thus, using the equation (12) for $p_\infty^1$,

$$aqH^{q-1} \frac{\partial^2 p}{\partial v^2} = a \frac{\partial^2 W}{\partial v^2} - aq(q - 1)p_\infty^1 H^{q-2} \left( \frac{\partial H}{\partial v} \right)^2$$

$$+ (q - 1)H^q \left( -v + b_1 N_\infty^1 \frac{\partial p_\infty^1}{\partial v} - p_\infty^1 - N_\infty^1 \delta_{V_R} \right). \quad (56)$$
Moreover,
\[
\frac{\partial}{\partial v} \left[ (-v + bN(t))W \right] = \frac{\partial}{\partial v} \left[ (-v + bN) PH^{q-1} \right] = H^{q-1} \frac{\partial}{\partial v} \left[ (-v + bN(t))p \right] + \left( -v + bN(t) \right) PH^{q-2} \frac{\partial H}{\partial v} \\
= H^{q-1} \frac{\partial}{\partial v} \left[ (-v + bN(t))p \right] + (q-1)H^{q-1} \left( -v + bN(t) \right) \frac{\partial p}{\partial v} \\
- (q-1)H^{q} \left( -v + bN(t) \right) \frac{\partial p_{\infty}^1}{\partial v}.
\]

Hence,
\[
- qH^{q-1} \frac{\partial}{\partial v} \left[ (-v + bN(t))p \right] = - \frac{\partial}{\partial v} \left[ (-v + bN(t))W \right] \\
- (q-1)H^{q} \left( -v + bN(t) \right) \frac{\partial p_{\infty}^1}{\partial v} - p_{\infty}^1. \tag{57}
\]

Putting (56) and (57) into (55), we obtain
\[
\frac{\partial W}{\partial t} + \frac{\partial}{\partial v} \left[ (-v + bN(t))W \right] - a \frac{\partial^2 W}{\partial v^2} + aq(q-1)p_{\infty}^1 H^{q-2} \left( \frac{\partial H}{\partial v} \right)^2 \\
+ (q-1)(bN(t) - b_{1}N_{\infty}^1)H^{q} \frac{\partial p_{\infty}^1}{\partial v} = N_{\infty}^1 \left( q \frac{N(t)}{N_{\infty}^1} - (q-1)H \right) H^{q-1} \delta_{V_{R}}. \tag{58}
\]

We then multiply this identity by \( \gamma \) and we integrate; since the support of \( \gamma \) is included in \([\alpha, V_{F}]\), with \( \alpha > V_{R} > 0 \) by hypothesis (11) made without loss of generality, we have the result by integration by parts.

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