Flat surfaces along swallowtails

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Abstract

We consider developable surfaces along the singular set of a swallowtail which are considered to be flat approximations of the swallowtail. For the study of singularities of such developable surfaces, we introduce the notion of Darboux frames along swallowtails and invariants. As a by-product, we give a new example of a frontal which is locally homeomorphic to a swallowtail.

1 Introduction

Recently, there appeared several articles concerning on differential geometry of singular surfaces in the Euclidean 3-space [4, 5, 18, 19, 20, 23, 26, 27, 29]. Wave fronts and frontals are particularly interesting singular surfaces which always have normal directions even along singularities. Surfaces which have only cuspidal edges and swallowtails as singularities are the generic wave fronts in the Euclidean 3-space. In this paper we consider a developable surfaces along the singular locus of a swallowtail surface in the Euclidean 3-space, and a singular point of a frontal surface which has the similar properties to a swallowtail. Such a developable surface is called a developable surface along swallowtail, (or a singular point of a frontal surface which have a similar properties to a swallowtail). Actually there are infinitely many developable surfaces along the singular locus of the swallowtail. Since a frontal surface has the normal direction at any point (even at a singular point), we focus on typical two developable surfaces along it. One of them is a developable surface which is tangent to the swallowtail surface and another one is normal to it. These two developable surfaces are considered to be flat approximations of the swallowtail along the singular locus of it. We investigate the singularities of these developable surfaces and induce new invariants for the swallowtail. For the purpose, we introduce the notion of Darboux frames along swallowtails which is analogous to the notion

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of Darboux frames along curves on regular surfaces (cf. \cite{7, 8, 15}). Since the Darboux frame along a swallowtail is orthonormal frame, we can obtain the structure equation and the invariants (cf. equation (2.6)). We show that these invariants are related to the invariants which are known as basic invariants of the swallowtail in \cite{19, 20, 26}. By using the Darboux frame, we can directly and instinctively understand geometric properties of the swallowtail. Moreover, if one of the three basic invariants is constantly equal to zero, we have special developable surfaces.

The similar investigation for cuspidal edges has been done in \cite{16}. This paper is not only a kind of continuous investigation of \cite{16} but also gives a new example of a frontal which is locally homeomorphic to a swallowtail as a by-product (cf. Example 4.5). We only know a cupsidal crosscap as such an example so far as we know.

## 2 Preliminaries

### 2.1 Preliminaries on frontals

The precise definition of the swallowtail (surface) is given as follows: The unit cotangent bundle $T^*_u\mathbb{R}^3$ of $\mathbb{R}^3$ has the canonical contact structure and can be identified with the unit tangent bundle $T_1\mathbb{R}^3$. Let $\alpha$ denote the canonical contact form on it. A map $i : M \rightarrow T_1\mathbb{R}^3$ is said to be isotropic if the pull-back $i^*\alpha$ vanishes identically. We call the image of $\pi \circ i$ the wave front set of $i$, where $\pi : T_1\mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the canonical projection and we denote it by $W(i)$. Moreover, $i$ is called the Legendrian lift of $W(i)$. With this framework, we define the notion of fronts as follows: A map-germ $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ is called a frontal if there exists a unit vector field (called unit normal of $f$) $\nu$ of $\mathbb{R}^3$ along $f$ such that $L = (f, \nu) : (\mathbb{R}^2, 0) \rightarrow (T_1\mathbb{R}^3, 0)$ is an isotropic map by an identification $T_1\mathbb{R}^3 = \mathbb{R}^3 \times S^2$, where $S^2$ is the unit sphere in $\mathbb{R}^3$ (cf. \cite{1}, see also \cite{17}). A frontal $f$ is a front if the above $L$ can be taken as an immersion. A point $q \in (\mathbb{R}^2, 0)$ is a singular point if $f$ is not an immersion at $q$. A map $f : M \rightarrow N$ between 2-dimensional manifold $M$ and 3-dimensional manifold $N$ is called a frontal (respectively, a front) if for any $p \in M$, the map-germ $f$ at $p$ is a frontal (respectively, a front). A singular point $p$ of a map $f$ is called a cuspidal edge if the map-germ $f$ at $p$ is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^2, v^3)$ at 0, and a singular point $p$ is called a swallowtail if the map-germ $f$ at $p$ is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, 4v^3 + 2uv, 3v^4 + uv^2)$ at 0. (Two map-germs $f_1, f_2 : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^m, 0)$ are $\mathcal{A}$-equivalent if there exist diffeomorphisms $S : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ and $T : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^m, 0)$ such that $f_2 \circ S = T \circ f_1$.) Therefore if the singular point $p$ of $f$ is a swallowtail, then $f$ at $p$ is a front. Furthermore, cuspidal edges and swallowtails are two types of generic singularities of fronts. Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a frontal and $\nu$ its unit normal. Let $\lambda$ be a function which is a non-zero functional multiplication of the function

$$\det(f_u, f_v, \nu)$$

for some coordinate system $(u, v)$, and $( )_u = \partial/\partial u$, $( )_v = \partial/\partial v$. A singular point $p$ of $f$ is called non-degenerate if $d\lambda(p) \neq 0$. Let 0 be a non-degenerate singular point of $f$. 

2
Then the set of singular points $S(f)$ is a regular curve, we take a parameterization $\gamma(t)$ ($\gamma(0) = 0$) of it. We set $\tilde{\gamma} = f \circ \gamma$ and call $\tilde{\gamma}$ the singular locus. One can show that there exists a vector field $\eta$ along $\gamma$, such that

$$ \ker df_{\gamma(t)} = \langle \eta(t) \rangle_{\mathbb{R}}. $$

Set

$$ \varphi(t) = \det(\gamma'(t), \eta(t)). \quad (2.1) $$

Here, we denote $' = d/dt$. A non-degenerate singular point $0$ is the first kind if $\varphi(0) \neq 0$. A non-degenerate singular point $0$ is the second kind if $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. We remark that if $f$ is a front, then the singular point of the first kind is the cuspidal edge, and the singular point of the second kind is the swallowtail \cite{17}. The following criteria for cuspidal edge and swallowtail are known.

**Fact 2.1.** Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a front, and $0$ a non-degenerate singularity. Then the followings are equivalent:

- $0$ is cuspidal edge (respectively, swallowtail),
- $\varphi(0) \neq 0$ (respectively, $\varphi(0) = 0$, $\varphi'(0) \neq 0$),
- $\eta\lambda(0, 0) \neq 0$ (respectively, $\eta\lambda(0, 0) = 0$, $\eta\eta\lambda(0, 0) \neq 0$).

On the other hand, a developable surface is known to be a frontal, so that the normal direction is well-defined at any point. We say that a developable surface is an osculating developable surface along $f$ if it contains the singular set of $f$ such that the normal direction of the developable surface coincides with the normal direction of $f$ at any point of the singular set. We also say that a developable surface is a normal developable surface along $f$ if it contains the singular set of $f$ such that the normal direction of the developable surface belongs to the tangent plane of $f$ at any point of the singular set, where the tangent plane of $f$ at $\gamma(t)$ is $\nu(\gamma(t))$. In this paper, we study the geometric properties of a non-degenerate singular point of a frontal $f$ using these two developable surfaces along $f$. In particular, we show that the singular values of those developable surfaces characterize some geometric properties of $f$.

### 2.2 Frames on non-degenerate singularities of frontals

First, we show the following lemma to take a frame along a singular curve.

**Lemma 2.2.** If $0$ is a singular point of the second kind of $f$. Then $\tilde{\gamma} = f \circ \gamma$ satisfies

$$ \det(\tilde{\gamma}'',(0), \tilde{\gamma}''(0), \nu(0)) \neq 0, $$

where $\gamma$ is a parameterization of $S(f)$ near $0$. In particular $\tilde{\gamma}$ is the $3/2$-cusp at $0$.

Here, a $3/2$-cusp is a map-germ $(\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ which is $A$-equivalent to $t \mapsto (t^2, t^3, 0)$ at $0$. Taking a coordinate system $(u, v)$ satisfying $S(f) = \{(u, v) | v = 0\}$. Then one can
take a null vector field \( \eta(u) = \partial_u + \varepsilon(u) \partial_v, \varepsilon(0) = 0, \varepsilon'(0) \neq 0 \). If we take such a coordinate system \((u, v)\), we set \( t = \partial/\partial u \) or \( t = d/du \) in what follows. Since our consideration in this paper is local, we assume that

\[
\varepsilon'(0) > 0
\]

(2.2)

by changing the coordinate \((u, v)\) to \((u, -v)\) if necessary. Then \( df(\eta) = 0 \) on the \( u\)-axis, so that there exists a vector valued function \( g \) such that

\[
\begin{align*}
    f_u(u, v) + \varepsilon(u) f_v(u, v) &= vg(u, v). \quad (2.3)
\end{align*}
\]

Then differentiating (2.3), we get

\[
\begin{align*}
    f_{uu}(u, v) + \varepsilon'(u) f_v(u, v) + \varepsilon(u) f_{uv}(u, v) &= vg_u(u, v), \\
    f_{uv}(u, v) + \varepsilon(u) f_{vu}(u, v) &= g(u, v) + vg_u(u, v), \\
    f_{uuv}(u, v) + \varepsilon''(u) f_v(u, v) + 2\varepsilon'(u) f_{uv}(u, v) + \varepsilon(u) f_{uuv}(u, v) &= vg_{uu}(u, v).
\end{align*}
\]

(2.4)

On the \( u\)-axis, it holds that

\[
\begin{align*}
    f_{uu}(u, 0) + \varepsilon'(u) f_v(u, 0) + \varepsilon(u) f_{uv}(u, 0) &= 0, \\
    f_{uv}(u, 0) + \varepsilon(u) f_{vu}(u, 0) &= g(u, 0), \\
    f_{uuv}(u, 0) + \varepsilon''(u) f_v(u, 0) + 2\varepsilon'(u) f_{uv}(u, 0) + \varepsilon(u) f_{uuv}(u, 0) &= 0.
\end{align*}
\]

(2.5)

Proof of Lemma 2.2. Since the assertion does not depend on the choice of coordinate systems, we take a coordinate system \((u, v)\) and take a null vector field as above. Then \( \det(\hat{\gamma}''(0), \hat{\gamma}'''(0), \nu(0)) = \det(f_{uu}, f_{uuv}, \nu)(0, 0) \). On the other hand,

\[
\begin{align*}
    \eta \eta \lambda(0) &= \det(\eta f_u, f_v, \nu)(0) + 2 \det(\eta f_u, \eta f_v, \nu)(0) + 2 \det(\eta f_u, f_v, \nu)(0) \\
    &= \det(f_{uu} + \varepsilon' f_{uv}, f_v, \nu)(0) + 2 \det(f_{uu}, f_{uv}, \nu)(0) + 2 \det(f_{uu}, f_v, \nu)(0) \\
    &= \frac{1}{2\varepsilon'} \det(f_{uu}, f_{uu}, \nu)(0) - \frac{1}{\varepsilon'} \det(f_{uu}, f_{uuv}, \nu)(0) \\
    &= -\frac{3}{2\varepsilon'} \det(f_{uu}, f_{uuv}, \nu)(0).
\end{align*}
\]

One can easily show that 0 is a singular point of the second kind if and only if \( d\lambda \neq 0, \eta \lambda = 0 \) and \( \eta \eta \lambda \neq 0 \) at \((0, 0)\). Thus the assertion follows.

Let 0 be a singular point of the second kind of a frontal \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \). By Lemma 2.2, \( \det(\hat{\gamma}'', \hat{\gamma}''', \nu) \neq 0 \) holds, we take a parameter of \( \gamma \) satisfying \( \det(\hat{\gamma}'', \hat{\gamma}''', \nu) > 0 \). Again by Lemma 2.2, \( \hat{\gamma}''(0) \neq 0 \), the tangent line of \( \hat{\gamma}(u) = f \circ \gamma(u) \) at 0 is well-defined. Set a unit vector field \( \mathbf{e}(u) \) along \( \gamma \) such that \( \mathbf{e}(u) \) is tangent to \( \hat{\gamma} \) if \( u \neq 0 \) which satisfies

\[
\mathbf{e}(0) = \lim_{u \to 0} \frac{\hat{\gamma}'(u)}{||\hat{\gamma}'(u)||}.
\]

We set

\[
\nu(u) = \nu \circ \gamma(u) \quad \text{and} \quad \mathbf{b}(u) = -\mathbf{e}(u) \times \nu(u).
\]
Then \( \{e, b, \nu\} \) forms a positive orthonormal frame along \( \gamma \). We have the following Frenet-Serret type formula:

\[
\begin{align*}
\mathbf{e}'(u) &= \tilde{\kappa}_g(u)b(u) + \tilde{\kappa}_\nu(u)\nu(u), \\
\mathbf{b}'(u) &= -\tilde{\kappa}_g(u)e(u) + \tilde{\kappa}_t(u)\nu(u), \\
\nu'(u) &= -\tilde{\kappa}_\nu(u)e(u) - \tilde{\kappa}_t(u)b(u).
\end{align*}
\] (2.6)

Note that the above invariants depend on the choice of parameter. These invariants can be written by using known invariants.

**Proposition 2.3.** The invariants \( \tilde{\kappa}_g, \tilde{\kappa}_\nu \) and \( \tilde{\kappa}_t \) satisfy

\[
\tilde{\kappa}_g = |\alpha|\kappa_s, \quad \tilde{\kappa}_\nu = \alpha \kappa_\nu, \quad \tilde{\kappa}_t = \alpha \kappa_t,
\] (2.7)

where

\[
\alpha(u) = \text{sgn}(u)|\gamma'(u)|,
\]

and which is a \( C^\infty \) function. Here \( \kappa_s \) is the singular curvature (26), \( \kappa_\nu \) is the limiting normal curvature (20) and \( \kappa_t \) is the cuspidal torsion (19).

We define \( \kappa_g(u) = \text{sgn}(\alpha(u))\kappa_s(u) \), and call the geodesic curvature. Then \( \tilde{\kappa}_g = \alpha \kappa_g \).

We take the coordinate system \( (u, v) \) satisfying \( S(f) = \{v = 0\} \). Setting \( \hat{u} = u, \hat{v} = \int_0^u |f_v(u, v)| \, dv \), one can see the coordinate system \( (\hat{u}, \hat{v}) \) satisfies \( S(f) = \{\hat{v} = 0\} \) and \( |f_v(\hat{u}, \hat{v})| = 1 \). Let \( (u, v) \) be a coordinate system satisfying \( S(f) = \{v = 0\} \) and \( |f_v(u, v)| = 1 \). We take the null vector field

\[
\eta(u) = \partial_u + \varepsilon(u)\partial_v \quad (\varepsilon(0) = 0, \, \varepsilon'(0) > 0)
\]
as above. Then \( \varepsilon = \alpha \). Since 0 is non-degenerate, \( \lambda_v \neq 0 \). Thus \( \det(g, f_v, \nu) \neq 0 \) at 0.

**Lemma 2.4.** Under the above settings, we have

\[
f_v(u, 0) = -e(u), \quad f_v \times g/|f_v \times g| = \nu(u).
\]

**Proof.** We remark that by the assumption (2.2), \( \text{sgn}(u)\varepsilon(u) > 0 \). By (2.3),

\[
e(0) = \lim_{u \to +0} \frac{f_u(u, 0)}{|f_u(u, 0)|} = \lim_{u \to +0} \frac{-\varepsilon(u)}{\varepsilon(u)} f_v(u, 0) = -f_v(0, 0),
\]

and hence it holds that \( e(u) = -f_v(u, 0) \). Next, since \( \eta f = vg \),

\[
\nu(u) = \pm \frac{f_v \times g}{|f_v \times g|}(u, 0). \quad (2.8)
\]

On the other hand, by (2.5),

\[
\det(f_{uu}, f_{uuv}, f_v \times g) = \det(-\varepsilon' f_v, -(-\varepsilon'' f_v + 2\varepsilon' f_u v), f_v \times g) = 2(\varepsilon')^2 \det(f_v, g, f_v \times g),
\]

we see that the \( \pm \) sign in (2.8) should be +. \( \Box \)
Proof of Proposition 2.3. Let \((u, v)\) be a coordinate system just after Proposition 2.3. We see
\[
\tilde{\kappa}_g = \langle \epsilon', b \rangle = - \langle f_{uu}, b \rangle = \det(f_{uu}, \epsilon, \nu) = - \det(f_v, f_{uv}, \nu).
\]
By the definition of the singular curvature ([20 (1.7)]), and by (2.5), we have
\[
\kappa_s = \text{sgn}(\epsilon) \frac{\det(f_u, f_{uu}, \nu)}{|f_u|^3} = \text{sgn}(\epsilon) \frac{\det(f_v, f_{uv}, \nu)}{|\epsilon|}. \tag{2.3}
\]
Moreover,
\[
\eta \lambda = \eta \det(f_u, f_v, \nu) = \det(\eta f_u, f_v, \nu) + \det(f_u, \eta f_v, \nu) = \det(-\epsilon f_v, g, \nu) = -\epsilon
\]
holds on the \(u\)-axis. Thus \(\kappa_s = - \frac{\det(f_v, f_{uv}, \nu)}{|\epsilon|}\), and we have \(\tilde{\kappa}_g = |\epsilon| \kappa_s\).

Next we consider \(\tilde{\kappa}_v\). By Lemma 2.4 and (2.5),
\[
\tilde{\kappa}_v = \langle \epsilon', \nu \rangle = - \langle f_{uu}, \nu \rangle = - \langle g - \epsilon f_{vv}, \nu \rangle = \epsilon \langle f_{vv}, \nu \rangle.
\]
On the other hand, by the definition of the limiting normal curvature ([20 (3.11)]),
\[
\kappa_v = \frac{\langle f_{uu}, \nu \rangle}{|f_u|^2} = \frac{\langle \epsilon^2 f_{vv}, \nu \rangle}{|\epsilon|^2} = |\epsilon| \langle f_{vv}, \nu \rangle
\]
holds, thus we have \(\tilde{\kappa}_v = \epsilon \kappa_v\). Finally, by Lemma 2.4 and (2.5),
\[
\tilde{\kappa}_t = \langle b', \nu \rangle = \det(e, \nu, \nu') \frac{\det(f_v, f_u \times g, (f_u \times g)_a)}{|f_v \times g|^2} \tag{2.9}
\]
where we used the formula \(\det(a \times b, a \times c, d) = \det(a, b, c) \langle a, d \rangle\) \((a, b, c, d \in \mathbb{R}^3)\). On the other hand, by the definition of the cuspidal torsion ([19 (5.1)]), (2.3) and (2.5),
\[
\kappa_t = \frac{\det(f_u, \eta f_v, \eta f_u)}{|f_u \times \eta f|^2} \frac{\det(f_u, \eta f_v, f_{uu}) \langle f_u, \eta f \rangle}{|f_u|^2 |f_u \times \eta f|^2}
\]
holds, thus we have \(\tilde{\kappa}_t = \epsilon \kappa_t\).
Remark 2.5. Our formula (2.6) depends on the choice of the parameter. Usually the Frenet-Serret formula is written by the arclength parameter. However, in our case the arclength parameter is not differentiable. In [28], the half arc-length parameter for plane cusp was introduced. It is also well-defined and differentiable in our case. If we take the half arc-length parameter \( h \), then the function \( \alpha \) in (2.7) is equal to \( h \).

Remark 2.6. We remark that invariants of frames along curves in \( \mathbb{R}^3 \) with singularities (framed singular curves) are studied in [10]. The frame considered in the present paper is constructed by using the normal vector of given frontal. So our invariants are related to the geometry of the frontal. See [10] for the study of the properties of invariants of framed singular curves itself.

3 Developable surfaces along singular set

Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) be a frontal and \( \nu \) its unit normal, and let 0 be a singular point of the second kind. Throughout in this section, we take the coordinate system \((u, v)\) near 0 satisfying \( S(f) = \{ v = 0 \} \). Let \( \{e, b, \nu\} \) be the Darboux frame defined in Subsection 2.2. In this section, following [7, 15, 16], we consider developable surfaces along \( S(f) \).

Developable surfaces along curves with singularities are considered in [10]. See [6, 25] for basic notions for ruled surfaces, and [12, 13, 14] for singularities of ruled surfaces.

3.1 Osculating developable surfaces

We assume that \((\tilde{\kappa}_\nu(u), \tilde{\kappa}_t(u)) \neq (0, 0)\) in a small neighborhood of 0. By (2.3), \( \tilde{\kappa}_\nu(0) = 0 \), this assumption is equivalent to \( \tilde{\kappa}_t(0) \neq 0 \). Under this assumption, we define a ruled surface \( OD_f : I \times \mathbb{R} \to \mathbb{R}^3 \) by

\[
OD_f(u, t) = f(u, 0) + tD_\nu(u) \left( \frac{\tilde{\kappa}_t(u)e(u) - \tilde{\kappa}_\nu(u)b(u)}{\sqrt{\tilde{\kappa}_t(u)^2 + \tilde{\kappa}_\nu(u)^2}} \right),
\]

and call an osculating developable surface along \( f \). Set

\[
\hat{\delta}_o = \tilde{\kappa}_g((\tilde{\kappa}_\nu)^2 + (\tilde{\kappa}_t)^2) - \tilde{\kappa}_t(\tilde{\kappa}_\nu)' + (\tilde{\kappa}_t)'\tilde{\kappa}_\nu,
\]

where \( ' = \frac{d}{du} \). By (2.6), we see

\[
\overline{D_\nu} = \frac{\hat{\delta}_o}{(\tilde{\kappa}_1^2 + \tilde{\kappa}_2^2)^{3/2}}(\tilde{\kappa}_\nu e + \tilde{\kappa}_t b)
\]

and \( \det(\hat{\gamma}', \overline{D_\nu}, \overline{D_\nu}) = 0 \), it holds that \( OD_f \) is a developable surface. Setting \( \lambda_o = \frac{\hat{\delta}_o t + \tilde{\kappa}_\nu \hat{\delta}_o (\tilde{\kappa}_\nu^2 + \tilde{\kappa}_t^2)^{1/2}}{\delta_o} \), it holds that \( S(OD_f) = \{ \lambda_o(u, t) = 0 \} \). If \( \delta_o(0) = 0 \), then all the points on the ruling passing through \( \hat{\gamma}(0) \) are singular value. When \( \delta_o \neq 0 \), we set

\[
t_o(u) = \frac{-\tilde{\kappa}_\nu \hat{\delta}_o (\tilde{\kappa}_\nu^2 + \tilde{\kappa}_t^2)^{1/2}}{\delta_o},
\]
and \( s_o(u) = OD_f(u, t_o(u)) \). Then

\[
s_o = \gamma - \frac{\langle \gamma', D_o \rangle}{\langle D_o, D_o \rangle} D_o
\]

holds, and thus \( s_o(u) \) is the striction curve (cf. [6, Section 17.3]) of \( OD_f \). In this case, we have

\[
t_o' = -\frac{\tilde{\sigma}_o(\tilde{\kappa}_v^2 + \tilde{\kappa}_t^2) + \tilde{\varepsilon}_t \tilde{\kappa}_v \tilde{\delta}_o^2}{\sqrt{\tilde{\kappa}_v^2 + \tilde{\kappa}_t^2 \tilde{\delta}_o^2}}, \quad s_o' = -\frac{\tilde{\sigma}_o(\tilde{\kappa}_v e - \tilde{\kappa}_t b)}{\tilde{\delta}_o},
\]

where

\[
\tilde{\sigma}_o = -\varepsilon_\tilde{\kappa}_v \tilde{\delta}_o' - (\varepsilon' \tilde{\kappa}_v + \varepsilon \tilde{\kappa}_g \tilde{\kappa}_t - 2 \varepsilon \tilde{\kappa}_v') \tilde{\delta}_o
\]

\[
= \tilde{\kappa}_v \varepsilon' \left( \tilde{\kappa}_g (\tilde{\kappa}_v^2 + \tilde{\kappa}_t^2) - \tilde{\kappa}_t \tilde{\kappa}_v' + \tilde{\kappa}_v \tilde{\kappa}_t' \right) - \varepsilon \left( \tilde{\kappa}_t (2(\tilde{\kappa}_v')^2 - \tilde{\kappa}_v \tilde{\kappa}_v'' + \tilde{\kappa}_v (-2\tilde{\kappa}_v' \tilde{\kappa}_v' + \tilde{\kappa}_v \tilde{\kappa}_v'')
\]

\[
+ \tilde{\kappa}_v (\tilde{\kappa}_v^2 + \tilde{\kappa}_t^2) \tilde{\kappa}_v' + 3 \tilde{\kappa}_t (-\tilde{\kappa}_v \tilde{\kappa}_v' + \tilde{\kappa}_v \tilde{\kappa}_v') \tilde{\kappa}_g + \tilde{\kappa}_t (\tilde{\kappa}_v^2 + \tilde{\kappa}_t^2) \tilde{\kappa}_g).
\]

We have the following characterization of singularities of the osculating developable surface using \( \tilde{\delta}_o \) and \( \tilde{\sigma}_o \). Except for \( u = 0 \), singular points are cuspedal edges, we stick to our consideration to \( u = 0 \). We have \( t_o(0) = 0 \),

\[
\tilde{\delta}_o = \tilde{\kappa}_t (\tilde{\kappa}_g \tilde{\kappa}_t - \tilde{\kappa}_v'), \quad \tilde{\delta}_o' = \tilde{\kappa}_t (2 \tilde{\kappa}_g \tilde{\kappa}_t' + \tilde{\kappa}_g \tilde{\kappa}_t'') \text{ at } u = 0
\]

and

\[
\tilde{\sigma}_o = 0, \quad \tilde{\sigma}_o' = \tilde{\kappa}_t \varepsilon' (\tilde{\kappa}_g \tilde{\kappa}_t - 3 \tilde{\kappa}_v') (\tilde{\kappa}_g \tilde{\kappa}_t - \tilde{\kappa}_v') \text{ at } u = 0.
\]

**Theorem 3.1.** We assume that \( \tilde{\kappa}_t(0) \neq 0 \). If \( OD_f \) satisfies \( \tilde{\delta}_o(0) \neq 0 \), namely, \( \tilde{\kappa}_g(0) \tilde{\kappa}_t(0) - \tilde{\kappa}_v'(0) \neq 0 \), then the singular point \( (0,0) \) of \( OD_f \) is

(1) never be a cuspidal edge,

(2) swallowtail if and only if \( \tilde{\kappa}_g(0) \tilde{\kappa}_t(0) - 3 \tilde{\kappa}_v'(0) \neq 0 \).

If \( OD_f \) satisfies \( \tilde{\delta}_o(0) = 0 \) (namely, \( \tilde{\kappa}_g(0) \tilde{\kappa}_t(0) - \tilde{\kappa}_v'(0) = 0 \), then the singular point \( (0,0) \) of \( OD_f \) is

(3) cuspidal beaks if and only if \( \tilde{\kappa}_t \tilde{\kappa}_g^2 + 2 \tilde{\kappa}_g \tilde{\kappa}_t' - \tilde{\kappa}_v''(0) \neq 0 \), and \( \tilde{\kappa}_v'(0) \neq 0 \).

**Proof.** By (3.3), and by a calculation, the null vector field \( \eta_o \) of \( OD_f \) is

\[
\eta_o = (\tilde{\kappa}_v^2 + \tilde{\kappa}_t^2) \partial_u - \varepsilon \tilde{\kappa}_t (\tilde{\kappa}_v^2 + \tilde{\kappa}_t^2)^{1/2} \partial_t.
\]

Since \( \nu \) is a unit normal vector to \( OD_f \) and \( \eta_o \nu(0) \neq 0 \), \( OD_f \) is front at \( (0,0) \). The function \( \varphi \) in (2.1) is

\[
\varphi(u) = \det \left( \begin{array}{cc} \tilde{\kappa}_v^2 + \tilde{\kappa}_t^2 & \tilde{\kappa}_v' \\ \tilde{\kappa}_v' & \tilde{\kappa}_t' \end{array} \right) = \frac{\tilde{\sigma}_o(\tilde{\kappa}_v^2 + \tilde{\kappa}_t^2)^{3/2}}{\tilde{\delta}_o^2}.
\]

8
Since \( \varepsilon(0) = 0 \), the condition \( \varphi(0) = 0 \) is equivalent to \( \tilde{\sigma}_0(0) = 0 \), and the condition \( \varphi(0) = \varphi'(0) = 0 \) is equivalent to \( \tilde{\sigma}_0(0) = \tilde{\sigma}'_0(0) = 0 \). Since always \( \tilde{\sigma}_0(0) = 0 \) holds, we have the assertion (1). By (3.7), when \( \tilde{\delta}_0(0) \neq 0 \), then \( \tilde{\sigma}'_0(0) \neq 0 \) is equivalent to \( \hat{\kappa}_g(0)\hat{\kappa}_t(0) - 3\hat{\kappa}'_g(0) \neq 0 \). Thus we have the assertion (2). If \( \tilde{\delta}_0(0) = 0 \), then \( d\lambda_0 = 0 \) holds, and since \( (\lambda_0)_{ut} = 0 \), \( \text{det Hess} \lambda_u(0,0) < 0 \) is equivalent to \( (\lambda_0)_{ut} \neq 0 \), and it is equivalent to \( \tilde{\delta}'_0(0) \neq 0 \). Moreover, \( \eta_0 \eta_0 \lambda_u(0,0) \neq 0 \) is equivalent to \( \tilde{\kappa}'_t(0) \neq 0 \). Thus we have the assertion (3). \( \square \)

### 3.2 Normal developable surfaces

We assume that \( (\hat{\kappa}_t(u), \hat{\kappa}_g(u)) \neq (0,0) \). Under this assumption, we define a ruled surface \( ND_f : I \times \mathbb{R} \rightarrow \mathbb{R}^3 \) by

\[
ND_f(u,t) = f(u,0) + t\tilde{D}_n(u) \quad \left( \frac{\tilde{D}_n(u)}{\sqrt{\hat{\kappa}_t(u)^2 + \hat{\kappa}_g(u)^2}} = \frac{\tilde{\kappa}_t(u)e(u) + \tilde{\kappa}_g(u)\nu(u)}{\sqrt{\hat{\kappa}_t(u)^2 + \hat{\kappa}_g(u)^2}} \right).
\]

and call a normal developable surface along \( f \). Set

\[
\tilde{\delta}_n = \tilde{\kappa}_t(\hat{\kappa}_t^2 + \hat{\kappa}_t'^2) - \tilde{\kappa}_g\hat{\kappa}_t' + \tilde{\kappa}_t\hat{\kappa}_g'.
\]

By (2.6), we see

\[
\tilde{D}'_n = \frac{\tilde{\delta}_n}{(\hat{\kappa}_t^2 + \hat{\kappa}_t'^2)^{3/2}} (-\tilde{\kappa}_g e + \tilde{\kappa}_t \nu),
\]

and \( \text{det} (\tilde{\gamma}', \tilde{D}_n, \tilde{D}_n') = 0 \), it holds that \( ND_f \) is a developable surface. Setting \( \lambda_n = -\tilde{\delta}_n t + \tilde{\kappa}_g \varepsilon(\hat{\kappa}_t^2 + \hat{\kappa}_t'^2)^{1/2} \), it holds that \( S(ND_f) = \{ \lambda_n(u,t) = 0 \} \). If \( \tilde{\delta}_n(0) = 0 \), then all the points on the ruling passing through \( \tilde{\gamma}(0) \) are singular value. When \( \tilde{\delta}_n \neq 0 \), we set

\[
t_n(u) = \frac{\tilde{\kappa}_g \varepsilon(\hat{\kappa}_g^2 + \hat{\kappa}_t'^2)^{1/2}}{\tilde{\delta}_n},
\]

and \( s_n = ND_f(u, t_n(u)) \). Then

\[
s_n = \tilde{\gamma} - \frac{\langle \tilde{\gamma}', \tilde{D}_n' \rangle}{\langle \tilde{D}_n, \tilde{D}_n' \rangle} \tilde{D}_n
\]

holds, and thus \( s_n(u) \) is the striction curve of \( ND_f \). In this case, we have

\[
t'_n = \frac{\sigma_n(\hat{\kappa}_t^2 + \hat{\kappa}_t'^2) - \varepsilon \hat{\kappa}_t \delta_n^2}{\sqrt{\hat{\kappa}_t^2 + \hat{\kappa}_t'^2}} \frac{\hat{\kappa}_t e + \hat{\kappa}_g \nu}{\delta_n^2},
\]

where

\[
\tilde{\sigma}_n = -\varepsilon \tilde{\kappa}_g \tilde{\delta}_n + (\varepsilon' \tilde{\kappa}_g + \varepsilon \tilde{\kappa}_g \tilde{\kappa}_t + 2\varepsilon \tilde{\kappa}_t^2) \tilde{\delta}_n
\]

\[
= \tilde{\kappa}_g \varepsilon \left( \tilde{\delta}_n(\hat{\kappa}_g^2 + \hat{\kappa}_t'^2) + \tilde{\kappa}_g \tilde{\kappa}_t' + \tilde{\kappa}_g \hat{\kappa}_t' + \varepsilon \left( \tilde{\kappa}_t(2\hat{\kappa}_g^2 - \tilde{\kappa}_g \hat{\kappa}_t'^2) + \tilde{\kappa}_g (-2\tilde{\kappa}_g' \hat{\kappa}_t' + \tilde{\kappa}_g \hat{\kappa}_t''') + \hat{\kappa}_g (\hat{\kappa}_g^2 + \hat{\kappa}_t'^2) \hat{\kappa}_t' \right) \right).
\]
Similarly to the case of the osculating developable surface we have the following characterization of singularities of the normal developable surfaces. We also note that \( t_n(0) = 0 \),

\[
\tilde{\delta}_n = \tilde{\kappa}_g'\tilde{\kappa}_t - \tilde{\kappa}_g\tilde{\kappa}_t', \quad \tilde{\delta}'_n = \tilde{\kappa}_g(\tilde{\kappa}_g^2 + \tilde{\kappa}_t^2) + \tilde{\kappa}_g''\tilde{\kappa}_t - \tilde{\kappa}_g\tilde{\kappa}_t'' \quad \text{at} \quad u = 0 \tag{3.13}
\]

and

\[
\tilde{\sigma}_n = \varepsilon'\tilde{\kappa}_g(\tilde{\kappa}_g'\tilde{\kappa}_t - \tilde{\kappa}_g\tilde{\kappa}_t'), \quad \tilde{\sigma}'_n = (\tilde{\kappa}_g'\tilde{\kappa}_t - \tilde{\kappa}_g\tilde{\kappa}_t')(3\varepsilon'\tilde{\kappa}_g' + \varepsilon''\tilde{\kappa}_g) \quad \text{at} \quad u = 0. \tag{3.14}
\]

**Theorem 3.2.** We assume that \((\tilde{\kappa}_t(0), \tilde{\kappa}_g(0)) \neq (0, 0)\). If \( ND_f \) satisfies \( \tilde{\delta}_n(0) \neq 0 \), (namely, \( \tilde{\kappa}_g'(0)\tilde{\kappa}_t(0) - \tilde{\kappa}_g(0)\tilde{\kappa}_t'(0) \neq 0 \)), then the singular point \((0, 0)\) of \( ND_f \) is

1. cuspoidal edge if and only if \( \tilde{\kappa}_g \neq 0 \) holds,
2. swallowtail if and only if \( \tilde{\kappa}_g = 0 \) holds.

If \( ND_f \) satisfies \( \tilde{\delta}_n(0) = 0 \) (namely, \( \tilde{\kappa}_g'(0)\tilde{\kappa}_t(0) - \tilde{\kappa}_g(0)\tilde{\kappa}_t'(0) = 0 \)), then the singular point \((0, 0)\) of \( ND_f \) is not a cuspoidal beaks.

**Proof.** By (3.3), and a calculation, the null vector field \( \eta_n \) of \( ND_f \) is

\[
\eta_n = (\tilde{\kappa}_g^2 + \tilde{\kappa}_t^2)\partial_u - \varepsilon\tilde{\kappa}_t(\tilde{\kappa}_g^2 + \tilde{\kappa}_t^2)^{1/2}\partial_t.
\]

Since \( b \) is a unit normal vector to \( ND_f \) and \( \eta_n b(0) \neq 0 \), \( ND_f \) is front at \((0, 0)\). The function \( \varphi \) in (2.1) is

\[
\varphi(u) = \det \left( \begin{array}{c} 1 \\ \tilde{\kappa}_g^2 + \tilde{\kappa}_t^2 \\ -\varepsilon\tilde{\kappa}_t(\tilde{\kappa}_g^2 + \tilde{\kappa}_t^2)^{1/2} \end{array} \right) = -\frac{\tilde{\sigma}_n(\tilde{\kappa}_g^2 + \tilde{\kappa}_t^2)^{3/2}}{\delta_n^2}. \tag{3.15}
\]

Since \( \varepsilon(0) = 0 \), the condition \( \varphi(0) = 0 \) is equivalent to \( \tilde{\sigma}_n(0) = (\varepsilon'\tilde{\kappa}_g(\tilde{\kappa}_t\tilde{\kappa}_g' - \tilde{\kappa}_g\tilde{\kappa}_t'))(0) = 0 \), and the condition \( \varphi(0) = \varphi'(0) = 0 \) is equivalent to \( \tilde{\sigma}_n(0) = 0 \), and \( \tilde{\sigma}'_n = (\tilde{\kappa}_g\tilde{\kappa}_g' - \tilde{\kappa}_g\tilde{\kappa}_t')(3\varepsilon'\tilde{\kappa}_g' + \varepsilon''\tilde{\kappa}_g)(0) = 0 \). Thus the assertions (1) and (2) hold. Since \( (\lambda_n)_u = \varepsilon'\tilde{\kappa}_g(\tilde{\kappa}_g^2 + \tilde{\kappa}_t^2)^{1/2} \), \( (\lambda_n)_t = \tilde{\kappa}_t\tilde{\kappa}_g' - \tilde{\kappa}_g\tilde{\kappa}_t' \) holds, \( d\lambda_n(0, 0) = 0 \) under the assumption \( (\tilde{\kappa}_t(0), \tilde{\kappa}_g(0)) \neq (0, 0) \) is equivalent to \( \tilde{\kappa}_g(0) = \tilde{\kappa}_g'(0) = 0 \). A necessary condition that \( 0 \) is a cuspoidal beaks is \( \lambda''_n(0, 0) \neq 0 \). However it does not hold under the condition \( \tilde{\kappa}_g(0) = \tilde{\kappa}_g'(0) = 0 \). Thus we have the last assertion.

Here we give two examples.

**Example 3.3** (Standard swallowtail). The standard swallowtail is \( f : (u, v) \mapsto (u, 4v^3 + 2uv, 3v^4 + uv^2) \). The normal developable surface \( ND_f \) can be written as Figure 1. Since \( f \) is a tangent developable surface, \( OD_f \) coincides with \( f \).

**Example 3.4.** Let us set

\[
f(u, v) = \left( v + \frac{u^2}{2} - \frac{u^2v}{2} - \frac{u^4}{8}, \frac{u^3}{3} + uv, \frac{v^2}{2} \right). \tag{3.16}
\]

The osculating and normal developable surfaces of \( f \) are drawn in Figures 2 and 3.
4 Special swallowtails

In this section we consider the case when the singular values of $OD_f$ and $ND_f$ are special. In particular, the empty set and a point. Namely, we study the cases $OD_f$ and $ND_f$ are a cylinder or a cone. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a frontal and $0$ a singular point of the second kind. Let $\{e, b, \nu\}$ be the Darboux frame defined in Subsection 2.2.

We now define the notion of contour edges. For a unit vector $k \in S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$, we say that $S(f)$ is the tangential contour edge of the orthogonal projection with the direction $k$ if

$$S(f) = \{(u, 0) \in (\mathbb{R}^2, 0) \mid \langle \nu(u), k \rangle = 0\}.$$ 

We also say that $S(f)$ is the normal contour edge of the orthogonal projection with the direction $k$ if

$$S(f) = \{(u, 0) \in (\mathbb{R}^2, 0) \mid \langle b(u), k \rangle = 0\}.$$ 

Moreover, for a point $c \in \mathbb{R}^3$, we say that $S(f)$ is the tangential contour edge of the central projection (respectively, normal contour edge of the central projection) with the center $c$
if
\[ S(f) = \{(u,0) \in (\mathbb{R}^2,0) \mid \langle f(u,0) - c, \nu(u) \rangle = 0 \}. \]
(respectively, \( S(f) = \{(u,0) \in (\mathbb{R}^2,0) \mid \langle f(u,0) - c, b(u) \rangle = 0 \}. \))

For a regular surface, the notion of contour edges corresponds to the notion of contour generators [2].

### 4.1 Osculating developable is a cylinder or a cone

**Theorem 4.1.** With the same notations as the previous sections, we have the following:

(A) Suppose that \( \tilde{\kappa}_2^2 + \tilde{\kappa}_1^2 \neq 0 \). Then the following properties are equivalent:

1. \( OD_f \) is a cylinder,
2. \( \tilde{\delta}_o \equiv 0 \),
3. \( \nu \) is a part of a great circle in \( S^2 \).
4. \( S(f) \) is a tangential contour edge with respect to an orthogonal projection.
5. \( D_o \) is a constant vector.

(B) Suppose that \( \tilde{\kappa}_2^2 + \tilde{\kappa}_1^2 \neq 0 \). Then the following properties are equivalent:

1. \( ND_f \) is a cylinder,
2. \( \tilde{\delta}_n(u) \equiv 0 \),
3. \( b \) is a part of a great circle in \( S^2 \),
4. \( S(f) \) is a normal contour edge with respect to an orthogonal projection.
5. \( D_n \) is a constant vector.

**Proof.** We show the assertions (A). By (3.2), we see that the equivalency of (1), (2) and (5). The condition \( \tilde{\kappa}_2^2 + \tilde{\kappa}_1^2 \neq 0 \) means that \( \nu \) is a non-singular spherical curve. Moreover, since \( \nu'' = (\tilde{\kappa}_g \tilde{\kappa}_t - \tilde{\kappa}_g \tilde{\kappa}_t) e + (-\tilde{\kappa}_v \tilde{\kappa}_g - \tilde{\kappa}_v \tilde{\kappa}_t) b \) and by (3.11), we see that \( \det(\nu, \nu', \nu'') = \tilde{\delta}_o \). This implies that the geodesic curvature of \( \nu \) is \( \tilde{\delta}_o (\tilde{\kappa}_2^2 + \tilde{\kappa}_1^2)^{-3/2} \), and it shows that the equivalency of (2) and (3). We assume (5). Then \( D_o(u) \) is a constant vector \( D_o \). Thus \( \langle \nu(u), D_o \rangle = 0 \) for any \( u \). This implies that \( S(f) \) is a tangential contour edge with respect to \( D_o \), and it implies (4). Conversely, we assume (4). Then there exists a vector \( k \) such that \( \langle \nu(u), k \rangle = 0 \) holds for any \( u \). This implies that \( \nu(u) \) belongs to the normal plane of \( k \) passing through the origin, and it implies (3). Thus the assertion of (A) holds. One can show the assertion of (B) by the same method to the proof of (A) using (3.9) and (3.10) instead of (3.1) and (3.2).

We also have the following theorem.
Theorem 4.2. With the same notations as above, we have the following:
(A) Suppose that $\tilde{\kappa}_f^2 + \tilde{\kappa}_g^2 \neq 0$ and $\tilde{\delta}_n \neq 0$ for any $u \in I$. Then the following properties are equivalent:

1. $OD_f$ is a cone,
2. $\tilde{\sigma}_o \equiv 0$,
3. $S(f)$ is a tangential contour edge with respect to a central projection.
4. $s_o$ is a constant vector.

(B) Suppose that $\tilde{\kappa}_f^2 + \tilde{\kappa}_g^2 \neq 0$ and $\tilde{\delta}_n \neq 0$ for any $u \in I$. Then the following properties are equivalent:

1. $ND_f$ is a cone,
2. $\tilde{\sigma}_n \equiv 0$,
3. $S(f)$ is a normal contour edge with respect to a central projection.
4. $s_n$ is a constant vector.

Proof. By (3.5), we see that the equivalency of (1), (2) and (4). We assume (2). Then $OD_f$ is a cone.

Conversely, we assume (3). Then there exists a vector $c$ such that $\langle f(u, 0) - c, \nu(u) \rangle \equiv 0$. By (3.4), $s_o(u)$ is a constant vector for any $u$. We set $c = s_o(u)$. Then by (3.4), $f(u, 0) - c$ is parallel to $D_o(u)$. Thus $\langle f(u, 0) - c, \nu(u) \rangle = \langle D_o(u), \nu(u) \rangle = 0$ holds for any $u$. This implies (3).

On the other hand, by (2.6), the three vectors $\nu(u), \nu'(u), \nu''(u)$ are linearly independent if and only if $\delta_o(u) \neq 0$. Hence

$$\langle s_o(u) - c, \nu(u) \rangle \equiv \langle s_o(u) - c, \nu'(u) \rangle \equiv \langle s_o(u) - c, \nu''(u) \rangle \equiv 0$$

implies $s_o(u) - c \equiv 0$, and this implies (I). Thus the assertion of (A) holds. One can show the assertion of (B) by the same method to the proof of (A) using (3.12) instead of (3.5).

Let us consider a cylinder $c_g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ and a cone $c_o : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, and consider and a Whitney cusp $f_w : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Here, a Whitney cusp is a map germ which is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^3 + uv)$. Then each $c_o \circ f_w : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ and $c_o \circ f_w : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is a frontal and each 0 is a singularity of the second kind. Thus $OD_{c_o \circ f_w}$ is a cylinder, and $OD_{c_o \circ f_w}$ is a cone. These examples are a kind of trivial examples. Here we give non-trivial examples.
**Example 4.3.** Let us set

\[
f(u, v) = \left( -\frac{u^2}{2} + v, \frac{u^3}{3} - uv, \frac{u^4}{8} + \frac{1}{2} \left( \frac{u^2}{2} - v \right)^2 - \frac{u^2v}{2} \right).
\]

Then we see that \( S(f) = \{ v = 0 \} \) and \( \tilde{\delta}_n(u) = 0 \). Thus \( OD_f \) is a cylinder. The figure of this example is given in Figure 4.

![Figure 4: Swallowtail in Example 4.3 with its osculating developable and both surfaces](image)

Let us consider a plane \( p : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) and a sphere \( s : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \), and consider and a Whitney cusp \( f_w : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0) \). Then each \( p \circ f_w : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) and \( s \circ f_w : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) is a frontal and each 0 is a singularity of the second kind. Thus \( ND_{c} \circ f_w \) is a cylinder, and \( ND_{c} \circ f_w \) is a cone. We say that \( f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0) \) is a Whitney frontal if it is \( A \)-equivalent to \((u, v) \mapsto (u, v^2, 0)\) or \((u, v) \mapsto (u, v^3 + uv, 0)\). Then \( p \circ f_w \) and \( s \circ f_w \) are Whitney frontals. These examples are a kind of trivial examples. Here we give non-trivial examples

**Example 4.4.** Let us set

\[
f(u, v) = \left( -\frac{u^2}{2} + v, \frac{u^3}{3} - uv, \frac{u^4}{8} - \frac{u^2v}{2} \right).
\]

Then we see that \( S(f) = \{ v = 0 \} \) and \( \tilde{\delta}_n(u) = 0 \). Thus \( ND_f \) is a cylinder. The figure of this example is given in Figure 5.

![Figure 5: Swallowtail in Example 4.4 with its normal developable and both surfaces](image)

We also give another example of a singular point whose normal developable is a cylinder. It is not a swallowtail but a singular point of the second kind.
**Example 4.5.** Let us set

\[ f(u, v) = (u^2 - v, -u^3 + 3u(u^2 - v), (2u^2 - v)^3v^3). \]

Then we see that \( f \) is a frontal and \( 0 \) is a singular point of the second kind but is not a swallowtail. We also see that \( S(f) = \{ v = 0 \} \) and \( \delta_5(u) = 0 \). Thus \( NDf \) is a cylinder. The figure of this example is given in Figure 6. In this case, the image is locally homeomorphic to the swallowtail. We say that \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0) \) is a *quasi-swallowtail* if it is a frontal, \( 0 \) is a singular point of the second kind and the image is homeomorphic to the swallowtail. Then this example is the quasi-swallowtail.

![Figure 6: Surface in Example 4.5 with its normal developable and both surfaces](image)

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