Lengths of edges in carrier graphs

Michael Siler

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Abstract

We show that if $X$ is a minimal length carrier graph in a hyperbolic 3-manifold, $M$, then if $X$ contains a sufficiently short edge, it must contain a short circuit, as well. The meaning of “short” depends only on the rank of $\pi_1(M)$. We also expand the class of manifolds which are known to have minimal length carrier graphs.

Carrier graphs were introduced as a tool to study hyperbolic 3-manifolds by Matthew White in [12], where he proved that an upper bound on the rank of $\pi_1$ of a closed hyperbolic 3-manifold gives an upper bound on its minimal injectivity radius. They were later used by Souto in [11] and then Biringer in [2] to show that the rank of $\pi_1$ equals the Heegaard genus for large classes of 3-manifolds fibering over the circle. Namazi and Souto used carrier graphs to prove that rank equals genus for two handlebodies glued together via a sufficiently large power of a pseudo-Anosov map ($\mathbb{S}$). Biringer and Souto also used them to prove that there are only finitely many closed hyperbolic 3-manifolds with a given upper bound on rank and lower bounds on injectivity radius and first eigenvalue of the Laplacian ($\mathbb{S}$).

Despite this interest in carrier graphs, relatively little is known about their apparently very nice geometry. For instance, there does not seem to be a single concrete example of a minimal length carrier graph in the literature. It seems likely that a more thorough understanding of their geometry would yield more results about hyperbolic 3-manifolds, especially regarding their rank. Our main result, Theorem 2, says roughly that for a hyperbolic 3-manifold $M$ with, if a minimal length carrier graph has a very short edge, then the graph must have a short circuit. More precisely, we show

**Theorem 2.** Suppose $M$ is a hyperbolic 3-manifold and $f : X \to M$ is a minimal length carrier graph. Let $k = \text{rank } \pi_1(M)$ and assume $k > 1$. For every $r > 0$, there exists $l > 0$ such that if every circuit in $X$ has length

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greater than \( r \), then every edge in \( X \) has length at least \( l \). The value of \( l \) depends only on \( r \) and \( k \).

In addition, we emphasize that such an \( l \) can be defined explicitly in terms of \( r \).

In Section 1, we give basic definitions and lemmas regarding carrier graphs. We also present a theorem generalizing White’s result in [12] that closed hyperbolic 3-manifolds have minimal length carrier graphs. In Section 2, we prove Theorem 2 which has as an immediate corollary that a lower bound on the injectivity radius of \( M \) gives a lower bound on the length of any edge in a minimal length carrier graph for \( M \). Finally, in Section 3 we give an example to show that the main result is stronger than this corollary.

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1 Carrier Graphs

For the remainder of the article, \( M \) will be a hyperbolic 3-manifold. We fix an identification of \( M \) with \( \mathbb{H}^3/\Gamma \), for some discrete, torsion-free group \( \Gamma < \text{Isom}^+(\mathbb{H}^3) \). A carrier graph is a finite, trivalent graph \( X \) along with a map \( f : X \to M \), with \( f_* : \pi_1(X) \to \pi_1(M) \) surjective. We will assume that all carrier graphs have rank \( \pi_1(X) = \text{rank} \pi_1(M) \). As pointed out in [12], the fact that \( X \) has the minimal possible rank implies that simple closed curves in \( X \) map essentially under \( f \). This is because a simple closed curve in \( X \) is part of a minimal cardinality generating set of \( \pi_1(X) \), so if it died in \( M \), we could break the curve and get a \( \pi_1 \)-surjective map of a lower rank graph into \( M \).

We define the length of an edge \( e \) of \( X \) (relative to the map \( f \)) as the length of the path \( f|_e \). Then \( \text{len}_f(X) \) is the sum of lengths of the edges of \( X \). A minimal length carrier graph is a carrier graph \( f \) such that for all carrier graphs \( g \), \( \text{len}_f(X) \leq \text{len}_g(X) \).

**Theorem** (White [12]). If \( M \) is a closed, hyperbolic 3-manifold, then \( M \) has a minimal length carrier graph. In addition, if \( f : X \to M \) is a minimal length carrier graph for any hyperbolic 3-manifold \( M \) (closed or not), then:

1. the edges of \( X \) map to geodesics;
2. every edge of $X$ has positive length; in particular, the images of vertices are still trivalent;

3. edges adjacent to the same vertex meet at an angle of $2\pi/3$.

Our main result is only useful when minimal length carrier graphs actually exist, so we will extend White’s proof of the existence of minimal length carrier graphs to a larger class of 3-manifolds. For definitions of compression body and NP-end, see [4] and [7], respectively. An NP-end is essentially a topological end of $M \setminus \{\text{cusps of } M\}$.

**Theorem 1.** Let $M$ be an orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is finitely generated and nonabelian. If $M$ does not contain a minimal length carrier graph, then it has a compact core which is a compression body $C$.

Furthermore, the topological end of $M$ corresponding to the positive boundary of $C$ contains a $\pi_1$-surjective simply degenerate NP-end.

**Proof.** Let $l$ be the infimum of the lengths of all carrier graphs for $M$ and let $f_i : X_i \to M$ be a sequence of carrier graphs with geodesic edges and $\lim_{i \to \infty} \text{len}_{f_i}(X_i) = l$. In [12], White applies the Arzelà-Ascoli theorem to such a sequence and shows that the resulting limit is a minimal length carrier graph. For his argument to work, it is sufficient for there to be a compact set $K$ containing $f_i(X_i)$ for all $i$. Suppose that $M$ does not have a minimal length carrier graph and thus, that this condition does not hold for any subsequence of $\{f_i\}$.

Because it converges, the sequence $\{\text{len}_{f_i}(X_i)\}$ has an upper bound $L$. There is a compact submanifold $C \subset M$ for which the inclusion map is a homotopy equivalence (see [10]), and if fact, by the topological version of the tameness theorem ([1], [5]), we can pick $C$ so that $M \setminus C = \partial C \times [0, \infty)$. The radius $L$ neighborhood $C_L$ of $C$ is also compact, and so for some $i_0$, $f_{i_0}(X_{i_0}) \not\subset C_L$. Since $\text{len}_{f_{i_0}}(X_{i_0}) \leq L$, we have that $f_{i_0}(X_{i_0}) \cap C = \emptyset$. Thus, $f_{i_0}(X_{i_0})$ is contained in $S \times [0, \infty)$ for some component $S$ of $\partial C$. Since $f_{i_0}$ is a carrier graph, it follows that the map $\pi_1(S) \to \pi_1(C)$ induced by inclusion is surjective, and in particular, the map $\pi_1(\partial C) \to \pi_1(C)$ is surjective. Since $C$ is compact and has a $\pi_1$-surjective boundary component, $C$ must be a compression body. For a proof of this well-known fact, see [4] Lemma 2.2.2.

Unless $C \cong \text{surface} \times I$, it has only one $\pi_1$-surjective boundary component $S$, and so $f_i(X_i)$ is eventually contained in the end $S \times [0, \infty)$. If $C \cong \text{surface} \times I$, then it is possible that $f_i(X_i)$ lies in each of the two ends infinitely often. In that case, we will assume that we have passed to a subsequence that lies entirely in one end $S \times [0, \infty)$.

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Let $M_0$ be the manifold obtained by removing standard neighborhoods of the cusps of $M$. Using the upper halfspace model of hyperbolic space, each cusp has a neighborhood isometric to the horoball $\{(x, y, z) \in \mathbb{H}^3 | z \geq Z\}$ modulo a group of translations, where $Z > 0$ depends on the cusp. There is a vertical projection to the horosphere $\{(\ast, \ast, Z) \in \mathbb{H}^3\}$ that descends to a retraction $\rho : M \to M_0$. The composition of $\rho$ with any $f_i$ is still a carrier graph for $M$. Since $f_i$ cannot map entirely into a cusp (because then $\pi_1(M)$ would be abelian) and $\text{len}_{f_i}(X_i) < L$, there is an upper bound for the depth of $f_i(X_i)$, i.e. the $z$-coordinate, in any cusp. This, combined with the upper bound on the length of $f_i(X_i)$, implies an upper bound on the length of $\rho \circ f_i$. In addition, a point in a cusp with distance $d$ from the boundary of the cusp gets moved a distance $d$ by $\rho$. Since no point of $f_i(X_i)$ can have depth greater than $L$ in a cusp, $\rho(f_i(X_i))$ is contained in a radius $L$ neighborhood of $f_i(X_i)$. Thus, $\{\rho \circ f_i\}$ is a sequence of bounded length carrier graphs that eventually leaves every compact set, and $\rho \circ f_i$ misses fixed neighborhoods of every cusp. This implies that $\rho(f_i(X_i))$ is contained in an NP-end for large $i$.

We now have that the end corresponding to the positive boundary of $C$ contains an NP-end, which carries $\pi_1(M)$. Let $E$ be an NP-end that contains $\rho(f_i(X_i))$ for infinitely many $i$. According to the tameness theorem, there are two possibilities for the geometry of $E$: it is geometrically finite or simply degenerate. Suppose $E$ is geometrically finite. Then $E$ has a flaring geometry. In particular, if $E = S' \times [0, \infty)$, then it is easy to show that the injectivity radius at any point away from any cusps in $S' \times [t, \infty)$, goes to infinity as $t \to \infty$. Since $\rho(f_i(X_i))$ is not contained in any standard cusp neighborhood, contains nontrivial loops, and is exiting $E$, the length of the carrier graph $\rho \circ f_i$ must being going to infinity. This contradicts these graphs having bounded length. Hence, $E$ is simply degenerate.

We will assume from now on that $f : X \to M$ is a minimal length carrier graph.

Suppose two geodesic segments in $\mathbb{H}^3$ with the same length meet at a shared endpoint with angle $\varphi < 2\pi/3$. We can replace the edges with a triod (with the same endpoints) as shown in Figure 1. A little hyperbolic trigonometry shows (with the edge length labels from the figure) that $2c > 2b + a$, so the triod is shorter. One can apply this procedure to a pair of geodesic edges that share an endpoint in a carrier graph. Generally, such a pair of edges will not be the same length, so we instead apply it to two “edge segments” (i.e. a subarc of an edge). Note that if an edge has both endpoints at the same vertex, we may apply the procedure with edge segments from
the same edge, each with length up to half that of the full edge. This shortening procedure is the key tool that White uses to prove points 2 and 3 in the theorem above. We will need a stronger statement. Let $Sh(c, \varphi)$ be the reduction in length after performing the shortening procedure on edge segments of length $c$ meeting at angle $\varphi$. If $\varphi$ is understood, we will use the notation $Sh(c)$. With some hyperbolic trigonometry, one can see that $a$ and $b$ are functions of $c$ and $\varphi$. So $Sh(c, \varphi) = 2c - 2b(c, \varphi) - a(c, \varphi)$.

**Lemma 1.** For a fixed length $c$, if $\varphi < 2\pi/3$, $Sh(c, \varphi)$ is a strictly decreasing function of $\varphi$. For a fixed angle $\varphi < 2\pi/3$, $Sh(c, \varphi)$ is a strictly increasing function of $c$.

Hence, smaller angles and longer edges produce a greater reduction in length in the shortening procedure.

**Proof.** Fixing $c$, we need to show that $2b + a$ is an increasing function of $\varphi$. We know $c$ and its opposite angle, so the hyperbolic law of sines immediately shows that $b$ is an increasing function of $\varphi$. Given $b$, we can compute $a$. Thus, treating $b$ as a function of $\varphi$ and $a$ as a function of $b$, we have:

$\frac{d}{d\varphi} (2b + a) = \frac{2db}{d\varphi} + \frac{da}{db} \frac{db}{d\varphi}$.

So it suffices to show that $\frac{da}{db} > -2$. Given that the angle opposite $c$ is $2\pi/3$, the hyperbolic law of cosines says $\cosh c = \cosh a \cosh b + \frac{1}{2} \sinh a \sinh b$. Implicit differentiation with respect to $b$ yields

$0 = \sinh(a)a' \cosh(b) + \cosh(a) \sinh(b) + \frac{1}{2} \cosh(a)a' \sinh(b) + \frac{1}{2} \sinh(a) \cosh(b)$

$0 = a' \sinh(a) \cosh(b) + \frac{1}{2} \cosh(a) \sinh(b)) + \cosh(a) \sinh(b) + \frac{1}{2} \sinh(a) \cosh(b)$

$a' = -\frac{\cosh(a) \sinh(b) + \frac{1}{2} \sinh(a) \cosh(b)}{\sinh(a) \cosh(b) + \frac{1}{2} \cosh(a) \sinh(b)}$
We want \( a' > -2 \), which is equivalent to

\[
\cosh(a) \sinh(b) + \frac{1}{2} \sinh(a) \cosh(b) < 2 \sinh(a) \cosh(b) + \cosh(a) \sinh(b)
\]

\[
\frac{1}{2} \sinh(a) \cosh(b) < 2 \sinh(a) \cosh(b)
\]

which is true if \( a \neq 0 \). However, in the shortening procedure, \( a = 0 \) if and only if \( \varphi = 2\pi/3 \); hence, the result follows.

Now fix \( 0 < \varphi < 2\pi/3 \). Pick \( c_1 \) and \( c_2 \) with \( 0 < c_1 < c_2 \). For \( i = 1, 2 \), let \( a_i = a(c_i, \varphi) \) and \( b_i = b(c_i, \varphi) \). Using some hyperbolic trigonometry, one can explicitly write down formulas for \( a \) and \( b \) as functions of \( c \) and \( \varphi \). By (rather tediously) differentiating them, it is not hard to prove that \( a \) and \( b \) are increasing functions of \( c \), so \( a_1 < a_2 \) and \( b_1 < b_2 \). Let \( a' = a_2 - a_1 \) and \( c' = c_2 - c_1 \). We wish to show that \( 2c_2 - 2b_2 - a_2 > 2c_1 - 2b_1 - a_1 \). This is equivalent to \( c' + b_1 > b_2 + \frac{1}{2} a' \). Figure 1 shows two symmetric triangles which each give the relationship between \( a \), \( b \) and \( c \). Figure 2 shows the corresponding triangles for \( a_i \), \( b_i \) and \( c_i \) for \( i = 1, 2 \) on top of each other.

The quadrilateral \( wxyz \) can be split into two triangles by inserting the diagonal \([x,y]\). By the triangle inequality, we have \( c' + b_1 > \text{len}([x,y]) \). The line segments \([x,y]\), \([y,z]\) and \([x,z]\) form a triangle in which the angle opposite \([x,y]\) is \( 2\pi/3 \). As noted above, the shortening procedure works because of the observation that in such a triangle,

\[
2 \text{len}([x,y]) - 2 \text{len}([y,z]) - \text{len}([x,z]) > 0.
\]
Since \( \text{len}(y, z) = b_2 \) and \( \text{len}([x, z]) = a' \), it follows that \( c' + b_1 > b_2 + \frac{1}{2}a' \), which was our goal.

We now give two simple lemmas needed for the main theorem.

**Lemma 2.** Fix any \( \varphi < 2\pi/3 \). There exists \( z > 0 \) and \( s_0 > 0 \) such that if \( s < s_0 \) and \( c/s > z \), then \( Sh(c) > s \).

**Proof.** Suppose the following claim is true: there exists \( c_0 > 0 \) and \( y > 0 \) such that for all \( c \leq c_0 \), \( \frac{Sh(c)}{c} \geq \frac{1}{z} \). Let \( z = \frac{1}{y} \); then for any positive \( s \) and \( c \) with \( c \leq c_0 \) and \( c/s > z \),

\[
\frac{Sh(c)}{c} \geq \frac{1}{z} = \frac{c}{z} \geq s
\]

Let \( s_0 = c_0/z \), and pick \( s \) with \( 0 < s < s_0 \) and \( c \) with \( c/s > z \). Let \( c' = zs \). Then \( c' < c_0 \) and \( c'/s = z \); so \( Sh(c') \geq s \). Since \( c > c' \), by Lemma 1, \( Sh(c) > Sh(c') \geq s \).

To prove the claim, note that by continuity, it suffices to show that \( \lim_{c \to 0} \frac{Sh(c)}{c} > 0 \). This limit is, by definition, the derivative of \( Sh \) evaluated at 0. One can, without too much difficulty, write \( b \) and \( a \) explicitly as functions of \( c \) (and \( \varphi \)), then take their derivatives at 0 to get that the derivative of \( Sh \) evaluated at \( c = 0 \) is

\[
2 - \frac{3}{2} - \frac{1}{2} \sqrt{4 - 3B^2}
\]

where \( B = \frac{2}{\sqrt{3}} \sin \left( \frac{\varphi}{2} \right) \). This quantity is easily seen to be positive. Showing that \( Sh(c) > 0 \) for \( c > 0 \) by this method is much harder, which is why Lemma 1 uses a geometric argument for that case.

**Lemma 3.** Let \( X \) be a finite, metric graph (e.g. a minimal length carrier graph). For any \( m, l_0 > 0 \), if \( X \) has an edge \( e \) of length less than \( l_0 \), then \( e \) is contained in a connected subgraph \( S \) with the following properties:

1. For every edge \( e' \) adjacent to \( S \), but not contained in \( S \),
   \[
   \frac{\text{len}(e')}{\text{len}(S)} > m;
   \]

2. If \( |S| \) is the number of edges in \( S \), then
   \[
   \text{len}(S) < l_0(m + 1)^{|S| - 1}.
   \]

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Proof. We will build $S$ inductively, one edge at a time, so that at each step, the second condition is satisfied. We will stop once the first condition is also satisfied. Let $e_1$ be an edge of $X$ with $\text{len}(e_1) < l_0$, and let $S_1 = e_1$. Note that $S_1$ satisfies the second property of the lemma. Having defined $S_i$ (satisfying property 2), if the first condition of the lemma holds for $S_i$, then set $S = S_i$ and stop. Otherwise, let $e_{i+1}$ be an edge adjacent to $S_i$, but not in it, with $\text{len}(e_{i+1}) \leq m \cdot \text{len}(S_i)$. Set $S_{i+1} = S_i \cup e_{i+1}$. Note that $\text{len}(S_{i+1}) \leq \text{len}(S_i) + m \cdot \text{len}(S_i) = (m+1) \text{len}(S_i) < (m+1)l_0(m+1)^i - 1$. So $S_{i+1}$ satisfies the second property, as well. Since $X$ has finitely many edges, this process must eventually stop, yielding $S$. \qed

2 Main result

Recall the statement of the main theorem:

**Theorem 2.** Suppose $M$ is a hyperbolic 3-manifold and $f : X \to M$ is a minimal length carrier graph. Let $k = \text{rank} \pi_1(M)$ and assume $k > 1$. For every $r > 0$, there exists $l > 0$ such that if every circuit in $X$ has length greater than $r$, then every edge in $X$ has length at least $l$. The value of $l$ depends only on $r$ and $k$.

By a circuit in a graph, we mean a simple, closed curve. Note that each circuit in $X$ represents an element of a minimal cardinality generating set for $\pi_1(X)$, and thus for $\pi_1(M)$. We will refer to an element of a minimal cardinality generating set as a basis element. Thus, to meet the criterion that every circuit in $X$ has length greater than $r$, it suffices for every basis element of $M$ to have length greater than $r$.

Proof. Let $k = \text{rank} \pi_1(X)$. For $\varphi = \cos^{-1}(-1/3)$, let $z$ and $s_0$ be as in Lemma 2. Choose $m > 0$ big enough so that $(1/2)(m-2)/(4k-5) > z$. Let $r > 0$, suppose that every circuit in $X$ has length greater than $r$, and set

$$l = \min \left\{ \frac{s_0}{(4k-5)(m+1)^{2k-4}}, \frac{r}{(m+1)^{3k-4}} \right\}.$$

Suppose $X$ has an edge $e$ with $\text{len}_f(e) < l$. Lemma 3 says that $e$ is contained in a subgraph $S \subset X$ with $\text{len}(S) < l(m+1)^{|S|-1}$, where $|S|$
Figure 3: Collapsing a tree. Dashed lines are $S$, dotted lines are $C$, solid lines are part of $R$.

is the number of edges in $S$, and for every edge $e'$ that touches $S$ but is not contained in it, $\text{len}(e') > m\text{len}(S)$. For brevity, let $L = \text{len}(S)$. It is easy to see that $X$ has exactly $3k - 3$ edges; so $|S| \leq 3k - 3$ and hence, $L < l(m+1)^{3k-4} \leq r$. Since $X$ has no circuits with length less than $r$, $S$ must be a tree. Then $|S| \leq (\# \text{ vertices in } X) - 1 = 2k - 3$, and $L < l(m+1)^{2k-4}$.

We are going to create a new carrier graph $Y$ that will be homeomorphic to the graph obtained from $X$ by collapsing $S$ to a point. We first construct the abstract graph $Y$. Let $R$ be the complement of the interior of $S$ in $X$. Then $R \cap S$ consists of the vertices of $S$ that have at least one adjacent edge not contained in $S$. We are going to modify $R$ by “splitting” each vertex in $R \cap S$ that has two edges from $R$ attached to it. More precisely, let $v$ be such a vertex and let $e_1$ and $e_2$ be the edges in $R$ that are attached to it. Suppose that $e_1$ and $e_2$ are distinct edges and let $w_i$ be the other endpoint of $e_i$. Remove $v$, $e_1$ and $e_2$ from $R$. For $i = 1, 2$, add in a new vertex $v_i$ and an edge $e_i'$ connecting $w_i$ and $v_i$. If $e_1 = e_2$, then remove $e_1$ and $v$ and replace them with two new vertices $v_1$ and $v_2$ and an edge connecting them. In either case, we will say that $v_1$ and $v_2$ were split from $v$. Call the modified graph $R'$, and set $Y = R' \cup C$. Note that $Y$ will have some vertices of valence 2 exactly at the points in $R' \cap C$. See Figure 3.

We now describe a map $g : Y \to M$. The graph $R$ is a quotient of $R'$ obtained by identifying pairs of vertices in $R'$ split from the same vertex in $R$. On $R'$, define $g$ to be the composition of $f$ with the quotient map $R' \to R$. Fix a nonvertex point $p \in S$, let $p'$ be the cone point of $C$, and set $g(p') = f(p)$. If $[v', p'] \subset C$ is an edge in $C$, let $v$ be the endpoint in $R$ that
came from. There is a unique, injective path \([v, p]\) in \(S\) from \(v\) to \(p\). Let \(g\) map the edge \([v', p']\) to the path \(f|_{[v, p]}\). There will be some valence two vertices in \(Y\) coming from the endpoints of \(R\). We will treat the two edges attached to such a vertex as a single edge, and though we may still refer to these endpoints, they will not be considered vertices. The map \(g: Y \to M\) is still a carrier graph.

Notice that the length (with respect to \(g\)) of any edge in \(C\) is less than or equal to \(L\). The number of such edges is at most twice the number of vertices in \(S\), and the number of vertices in \(X\) is \(2k - 2\). Thus, \(\text{len}_g(C) \leq (4k - 4)L\).

Since \(Y\) was formed by replacing \(S\) with \(C\), \(\text{len}_g(Y) \leq \text{len}_f(X) + (4k - 5)L\).

Our goal will be to show that we can apply the shortening procedure to \(Y\) at the point \(h(p')\) to reduce its length by more than \((4k - 5)L\), thereby making it shorter than \(X\). This will contradict \(X\) being a minimal length carrier graph and therefore, will show that \(X\) cannot contain the short loop \(e\).

Corollary 7.2 of [6] says that if \(Q_1, \ldots, Q_n \in \mathbb{H}^3\) are distinct from \(P \in \mathbb{H}^3\), then

\[
\sum_{1 \leq i < j \leq n} \cos \angle(Q_i, P, Q_j) \geq -n/2.
\]

Since \(S\) is a tree and has at least one edge, \(p'\) has valence at least four. If we lift \(h\) on a small neighborhood of \(p'\) to \(\mathbb{H}^3\) and apply this corollary to any four edges, we get that there are two edges attached to \(p'\) with angle at most \(\cos^{-1}(-1/3)\) between them. Let \(\eta_1\) and \(\eta_2\) be two such edges and let \(\varphi \leq \cos^{-1}(-1/3)\) be the angle between them. Note that we could have \(\eta_1 = \eta_2\), if both endpoints from this edge are at \(p'\).

We can get a lower bound for the lengths of the edges attached to \(p'\). Let \(e_0\) be an edge attached to \(p'\) and suppose \(e_0\) has only one endpoint at \(p'\). Then \(e_0\) can be written as \(e_0 = e_1 \cup e_2\), where \(e_1 \subset R'\) and \(e_2 \subset C\). From the construction of \(Y\) and \(g\), we see that there is some edge \(e_1^* \subset X\) that touches \(S\) but is not contained in it, such that \(g(e_1) = f(e_1^*)\). Hence, \(\text{len}_g(e_1) = \text{len}_f(e_1^*) > mL\). Also, since \(e_2\) is an edge in \(C\), \(\text{len}_g(e_2) \leq L\).

Under \(h\), the image of \(e_0\) comes from straightening \(g(e_1 \cup e_2)\) into a geodesic. Applying the triangle inequality, we get that \(\text{len}_h(e_0) \geq (m - 1)L\). Now suppose \(e_0\) has both endpoints at \(p'\). The argument here is similar: we write
$e_0 = e_1 \cup e_2 \cup e_3$, where $e_1$ and $e_3$ are edges in $C$ and have length at most $L$, and $e_2$ is in $R'$ and has length at least $mL$. With two applications of the triangle inequality, we get $\text{len}_h(e_0) \geq (m - 2)L$.

We are going to apply the shortening procedure to the edges $\eta_1$ and $\eta_2$. If these are distinct edges, then we can use an edge segments of length $\min\{\text{len}(\eta_1), \text{len}(\eta_2)\} \geq (m - 1)L$ from each. If $\eta_1 = \eta_2$, then each edge segment can use up to half of the edge. Thus, we can use edge segments of length at least $(1/2)(m - 2)L$. Let $c$ be the length of the longest edge segments in $\eta_1$ and $\eta_2$ that we can do the shortening procedure on. We have $c \geq (1/2)(m - 2)L$. The reduction in the length of $Y$ coming from doing the shortening procedure on $\eta_1$ and $\eta_2$ is $Sh(c, \varphi)$. From Lemma 1 we have

$$Sh(c, \varphi) \geq Sh((1/2)(m - 2)L, \cos^{-1}(-1/3)) > (4k - 5)L.$$ 

The last inequality comes from Lemma 2 since, by our choices of $m$ and $L$,

$$\frac{(1/2)(m - 2)L}{(4k - 5)L} > z$$

and

$$(4k - 5)L < (4k - 5)l(m + 1)^{2k-4} \leq s_0.$$ 

After the shortening procedure we will have a carrier graph shorter than the minimal length carrier graph $X$, which is a contradiction. This implies that no edge of $X$ can have length less than $l$. \hfill \Box

3 Short loops, long generators

An immediate corollary of Theorem 2 is that a lower bound on injectivity radius gives a lower bound on the lengths of edges in the minimal length carrier graph, since there are no basis elements with length less than twice the injectivity radius. It is possible, a priori, that Theorem 2 does not give any more information than that corollary. In other words, it may be that if $M$ has a small injectivity radius, then it must have a short basis element, too. We now give an example of a sequence of closed manifolds for which the injectivity radius goes to zero, but for which there is a lower bound on the length of a basis element and, thus, a lower bound on the length of any edge in their minimal length carrier graphs. This example was suggested independently by Ian Agol, Ian Biringer, and Juan Souto.
Let $H$ be a genus 2 handlebody, and let $DH$ be the double of $H$. Let 
$\gamma \subset \partial H$ be an essential, separating, simple, closed curve sufficiently com-
pli cated to make $H \setminus \gamma$ have a complete hyperbolic structure, in which a 
neighborhood of $\gamma$ is a cusp. Then $M_\infty = DH \setminus \gamma$ is a cusped, finite vol-
ume, hyperbolic 3-manifold. Let $M_n = H \cup g^n H$, where $g : \partial H \to \partial H$
is Dehn twist along $\gamma$. Notice that $M_n$ can be obtained from $DH$ by $1/n$
Dehn surgery on a neighborhood of $\gamma$; call the core of the filling torus $\gamma_n$.
It is clear that the map $g^n$ acts trivially on $H_1(\partial H)$, so one can easily check
that $H_1(M_n) = \mathbb{Z}^2$. Since there is an obvious genus 2 Heegaard splitting of
$M_n$, we have rank $\pi_1(M_n) = 2$. According to Thurston’s hyperbolic Dehn
filling theorem (see [3]), the sequence $M_n$ converges geometrically to $M_\infty$.
Thus, the minimal injectivity radius of $M_n$ approaches zero. Also, for large
enough $n$, any sufficiently short (nontrivial) curve must be contained in the
Margulis tube around $\gamma_n$, so that in $\pi_1(M)$, it represents a power of $\gamma_n$.
Hence, we only need to show that $\gamma_n$ cannot be a basis element. Note that
$\gamma_n$ and $\gamma$ are freely homotopic in $M_n$, and $\gamma$ is trivial in $H_1(M_n)$, since it
is a separating curve in $\partial H$. Any generating pair for $\pi_1(M_n)$ must descend
to a pair of generators for $H_1(M_n) = \mathbb{Z}^2$, so neither $\gamma_n$ nor any sufficiently
short curve can be a basis element.

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Department of Mathematics, Statistics, and Computer Science (M/C 249), University of Illinois at Chicago, 851 S. Morgan St., Chicago, IL 60607-7045

*E-mail address: wsiler2@uic.edu*